A MODEL OF COALITION REWARD PROGRAMS

JINGMAI WANG

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The Hong Kong Polytechnic University

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The Hong Kong Polytechnic University
Department of Logistics and Maritime Studies

A Model of Coalition Reward Programs

Jingmai Wang

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Abstract

A coalition reward program is a joint reward program that links multiple brands in a partnership, such that consumers who make a purchase from a brand in the coalition earn a reward that is redeemable across the brands. However, despite the increasing prevalence of such programs, there have been some failures and setbacks in recent years. One famous example is the Plenti loyalty program, which announced its demise only three years after its launch. Plenti linked a remarkably wide variety of brands, including Macy's, Chili's, and AT&T, most of which now offer proprietary reward programs. This leads to the following questions: why did these brands return to proprietary reward programs? Which type of program is more effective for building customer loyalty and increasing firm profit: a proprietary reward program or a coalition reward program? Moreover, as coalition reward programs typically differ greatly in size, how can the optimal size be determined, and how does the size of a coalition affect its effectiveness?

This thesis will study the optimal design of coalition reward programs, whether joining a coalition benefits a firm, and how the size affects its effectiveness. To this end, we will start with the simplest case: a coalition reward program consisting of n independent and symmetric firms, each of which sells a non-durable product to infinitesimal customers over an infinite horizon. Time is continuous, and a customer visits each firm following a Poisson process at a rate λ. Thus, a customer visits the coalition at a rate $n\lambda$. Any purchase from a firm in the coalition earns the customer a reward that can be redeemed across the firms before the reward expires. The coalition manager chooses the price, the reward, the expiration term, and the size n to maximize the program's long-term average revenue.

We will use continuous-time dynamic programming to analyze the customer's decision problem for a given coalition program, in which the state represents whether the customer holds a valid reward when she visits the coalition. We will analyze the coalition's optimal decisions based on the customer's optimal behavior, and compare the coalition's profit with that of the proprietary reward program. Finally, we will investigate how to determine the optimal size. Overall, this thesis will study coalition reward programs from the perspective of operations management by determining the optimal design of the programs and developing an improved understanding of the rationale and effectiveness of such programs.

Keywords: coalition reward program, coalition size, consumer discounting, price discrimination mechanism, dynamic programming.

Publications during MPhil Study

1. Niu, Baozhuang, Lingyun Chen, and **Jingmai Wang***. "Ad valorem tariff vs. specific tariff: Quality-differentiated e-tailers' profitability and social welfare in cross-border e-commerce". *Omega* 108 (2022): 102584.

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Chapter 1

Introduction

1.1 Research Background

A coalition reward program joins multiple brands in a partnership, such that a consumer's purchase from any brand in the coalition earns the consumer a reward that can be redeemed in a future purchase across the brands. The concept of coalition reward programs is not new, but the recent development of digital marketing has led to brands in various sectors showing substantially increased interest in such programs. For example, a recent survey by McKinsey & Company (2020) shows that more than 40 percent of Financial Time's Top 25 Brands of 2019 have engaged in coalition reward programs for various reasons, such as to drive consumer engagement, to enhance brand awareness, and to deliver a better consumer experience.

However, despite the increasing prevalence of coalition reward programs, there have been some failures and setbacks in recent years. One famous example is the Plenti loyalty program, which was launched by American Express in 2015 but abruptly ended in 2018. Plenti brought a remarkably wide variety of brands together, including Macy's, Chili's, and AT&T, most of which now offer proprietary reward programs. Other examples include those difficulties encountered by Aeroplan in Canada and

Nectar in the UK. Hence, a question naturally arises: why did these brands return to proprietary reward programs? Alternatively, which type of reward program is more effective for building customer loyalty and increasing brands' profit –a coalition reward program or a proprietary reward program? These questions have sparked endless debate in practice and academia; however, very few studies build an analytical framework to examine the design and effectiveness of coalition reward programs. This thesis will build an analytical framework to obtain findings that will aid in the design of next-generation coalition reward programs and encourage more research in this critical area of management science.

In particular, the size of a coalition reward program may affect the program's effectiveness. Intuitively, the more brands partnered in a coalition, the more consumers who will purchase a product from the coalition and thus earn and redeem the reward (especially when the reward has an expiry date), and thus the greater the benefit each brand can accrue from being a partner in the coalition. As a result, a larger coalition should be most favored by brands, provided that there is no competition between the brands in the coalition. However, there is a large difference between the sizes of coalitions in practice. Payback, for example, consists of more than 600 online partners and 35 retail companies in Germany, despite the fact that Mike Hughes, CEO of the loyalty solution provider Exchange Solutions, argues that "a more focused coalition loyalty program with a small number of participants that are aligned around the frequency and repeat purchases provides a much more compelling value proposition". Given this controversy in practice, a natural question arises: how can the optimal size of a coalition reward program be determined, and how does this size affect the success of the program? To the best of our knowledge, no literature has answered these critical questions. Our research fills this gap.

In this thesis, we will explicitly model repeated interactions between the firms in a coalition reward program and their consumers over time, study the optimal design of the program, investigate whether joining the coalition benefits the firm, and identify how to determine the optimal size of the coalition. The specific research questions that we will answer are as follows.

1. What are the optimal retail price, reward size, and expiration term in a coalition reward program?
2. Does a coalition reward program generate higher profits than a proprietary reward program?
3. What impact does a coalition reward program have on consumer behavior–can it simultaneously increase consumer surplus and firm profit?
4. What is the optimal size of a coalition reward program, i.e., the size that results in the highest profit for its partners?

This thesis will generate several long-term benefits for businesses and consumers. For example, our findings will assist firms that are planning to launch promotions in deciding whether to offer individual promotions or joint promotions via a coalition reward program. In addition, our findings will assist managers of coalition reward programs in designing the optimal policies for their programs, such as the number of partners in coalitions. This thesis investigates coalition reward programs from the perspective of operations management and contributes to the literature on both coalitions and reward programs.

1.2 The Layout of the Thesis

The remainder of the thesis proceeds as follows. Chapter 2 reviews the related literature on consumer behavior, coalition reward programs, coalitions, and reward programs. Chapter 3 presents the structure of a coalition reward program, market composition, and the consumer's decision problem. Based on the customer's behavior, we study the firm's profitability and the coalition's optimal decisions in the setting of exogenous and endogenous expiration term in Chapter 4. Chapter 5 compares firm profit and consumer surplus with those of the no reward program and proprietary reward program. To check the robustness of our main results, we consider three extensions in Chapter 6. In Chapter 7, we summarize this thesis and discuss future research opportunities. All the proofs are in the appendix.

Chapter 2

Literature Review

The literature review chapter presents a review of research on consumer behavior, coalition reward programs, coalitions, and reward programs. There are four major sections. Section 2.1 examines the studies on pricing and strategic consumer behavior. Section 2.2 investigates both empirical and analytical research on coalition reward programs. Section 2.3 reviews the studies in the field of operations management and economics. Section 2.4 highlights operations management and marketing literature in the reward program setting.

2.1 Pricing and Strategic Consumer Behavior

This thesis considers the optimal design of a coalition reward program with pricing decisions in the presence of forward-looking consumers who rationally anticipate the expected value of the reward in their purchases. Therefore, it is related to the literature on pricing and strategic consumer behavior. The research in this stream abounds and we refer readers to Shen and Su (2007), Aviv and Pazgal (2008), and Elmaghraby et al. (2008) for a comprehensive review. Most of the papers in this stream consider consumers' purchase timing decisions, and consumers make at most one purchase. In contrast, our thesis captures the fact that consumers interact with the firms in the coalition over a long time horizon and make repeat purchases.

2.2 Studies on Coalition Reward Programs

2.2.1 Empirical Research

Much of the existing literature on coalition reward programs is empirical study with a focus on either the impact of coalition reward programs on consumer's purchase and redemption behaviors or the benefit a firm gains from joining a coalition. For example, Stourm et al. (2017) use a dataset from a European coalition reward program to show how the reward offered by partners in a coalition influences consumers' purchase in other stores, and how such influence is affected by differences in stores' redemption policies, overlap in product categories, and geographic distance between stores. Note that in such a coalition, some firms compete by selling products in the same category, while others may complement each other. Similarly, Taylor and Dong (2020) use a dataset from a major European credit card issuer to analyze how the spatial evolution of a coalition reward program influences card usage and redemption activities. They find that the credit card revenue increases substantially if the coalition reduces the size to only include those key branch locations. Dorotic et al. (2011) use the longitudinal data of five prominent retailers in a coalition to investigate the joint promotion and cross-retailer effects. Dorotic et al. (2021) analyze the evolution of consumer purchases across 33 partners from 16 industry sectors in which some partners are competitors while others may be seen as complementary or neutral, and find that cannibalization and synergistic effects co-exist in a partnership loyalty program. Schumann et al. (2014) investigate the impact of a service failure by one partner on consumers' response to other firms in the coalition. None of the above studies analytically examine the design of coalition reward programs or the relationship between coalition size and the firm's profit in the coalition. These key questions will be answered by this work.

2.2.2 Analytical Research

One exception is Gardete and Lattin (2018), who analytically explore the profitability of coalition reward programs in a specific setting: each of two firms competing in one market forms a coalition loyalty program with one of two firms in a different market. However, their model setup and assumptions are distinct from ours. Specifically, they consider a one-period model in which consumers require at most one unit of each product sold in two markets. The reward earned by a purchase from a firm in one market can only be redeemed in a purchase from the other firm in the coalition, which is in a different market. In contrast, we will consider an infinite horizon model in which consumers make repeat purchases from firms in a coalition, and a reward can be redeemed in a future purchase across the firms. In addition, in their model, the profitability is due to each firm in the coalition leveraging its partner's market power to charge higher prices. In contrast, our coalition reward program allows consumers to earn and redeem the reward more rapidly, leading to the increased usage of rewards and thus possible higher profits for firms in a coalition. Finally, they assume that there are only two firms in each market, whereas our model considers a coalition comprising n firms to determine how coalition size affects the partner's profitability.

2.3 Studies on Coalitions

2.3.1 Operations Management Literature on Coalitions

There is a stream of operations management literature on coalitions (but not in the reward program setting). Axelrod et al. (1995) investigate a firm's incentive to join the competing alliances and theorize that the utility of a firm increases with the size of an alliance and decreases due to the presence of competitors in the alliance. Granot and Sošić (2005) examine under what conditions, in terms of product substitutability, three retailers form a three-member alliance, a two-member alliance, or no alliance. Nagarajan and Sošić (2007) consider a two-stage game in which n firms form coalitions in the first stage, which make price and inventory decisions and compete against one another in the second stage. They provide conditions under which the coalitions are stable. Interestingly, they also examine the impact of the size n of the market and the degree of competition on prices, inventory levels, and market structure. Nagarajan and Sošić (2009) study a decentralized assembly system in which n suppliers form alliances to sell complementary components to a downstream assembler. A different setting in An et al. (2015) identifies the conditions under which it is beneficial for a farmer to join the aggregation. All of the above studies consider either substitutability or complementarity between firms in a coalition or competition between coalitions. In contrast, we will focus on independent firms that may be in different industries and determine why they would form a coalition reward program. We will also analyze how the number of partners in a coalition affects the profitability of its partners.

2.3.2 Economics Literature on Coalitions

Coalition has also been extensively studied in the economics literature, most of which focuses on whether the equilibrium exists in a coalition or the stability of a coalition (Che 1994, Eaton and Eswaran 1997, Greenberg and Weber 1993). Yi (1997) establishes a unified framework to characterize and compare stable coalition structures under different coalition formation rules. Bloch (1995) characterizes the structure of associations when symmetric firms are Cournot competitors and finds that the associations formed are asymmetric and inefficient. Konishia and Ray (2003) consider coalition formation as an ongoing, dynamic process and study the existence of the equilibrium. Xue (1998) identifies the coalitions that are likely to form among rational and farsighted individuals and also analyzes the stability of the coalitions. In contrast, we neither look into the stability of coalitions nor model the formation as a dynamic process, but focus on the incentives of independent sellers in different industries to join a coalition reward program, the impact of the coalition on consumer welfare, and the impact of the size on the firm's profitability in a coalition.

2.4 Studies on Reward Programs

There has been much research on reward programs in both operations management and marketing literature. Chung et al. (2018) show that reward sales have a nontrivial impact on the seller's optimal dynamic pricing policy. Chun and Ovchinnikov (2019) show that the switch from a frequency-based reward to a revenue-based one in airlines can create a win-win situation. Sun and Zhang (2019) study a cash reward program with a finite expiration term and show that the program can be used to implement price discrimination among consumers with heterogeneous shopping frequency and valuation. Liu et al. (2021) analyze "Buy X, Get One Free" proprietary reward programs and their impact on consumer behavior and firm profit. In addition, there are many empirical studies that examine the effects of reward programs on sales, consumer retention, and firm profitability, and the results are mixed. For example, Sharp and Sharp (1997) find no evidence that reward programs increase aggregate-level profits; likewise, Liu (2007) finds no effect of reward programs on consumers who are already buying heavily from the seller. On the other hand, Lewis (2004) and Lal and Bell (2003) show that reward programs do increase sales and profits for grocery retailers. All the above works consider proprietary reward programs. In this thesis, we analyze the design of coalition reward programs and compare them with proprietary reward programs to determine firms' incentive to join a coalition.

Chapter 3

Model Setup

3.1 Structure of a Coalition Reward Program

We model a coalition reward program consisting of n independent firms, each of which sells a non-durable product to a population of infinitesimal customers over an infinite time horizon. In this model, time is continuous. The market size is normalized to 1, and the marginal cost of the product is normalized to 0. We first examine the simplest case, in which the n firms are symmetric in terms of customer's arrival rate and customer's valuation of the products.[1] A customer visits each firm in the coalition following a Poisson process with a rate λ, which is exogenous and can be interpreted as the degree to which the customer needs the product. Therefore, a customer visits the coalition with a rate $n\lambda$. Any purchase from a firm in the coalition earns the customer a reward r, which is redeemable across the firms before its expiration. The expiration term is assumed to be exponentially distributed with rate μ.[2] The coalition

[1]Since we focus on the incentives of independent firms to join a coalition reward program, we single out the asymmetric setting. We have to say that this assumption is a little bit restrictive, so Section 6.3 considers asymmetric firms in the coalition to get richer insights.

[2]This assumption is required to ensure that this continuous-time model is analytically tractable. One may wonder why we do not adopt a discrete-time model. The reason is that the number of

reward program is defined by a quartet of variables (p, r, μ, n): retail price p, reward size r, expiration rate μ, and coalition size n. The price p is assumed to be constant over time, and the size n is a positive integer. The coalition chooses the quartet to maximize its long-term average revenue. A natural assumption is that $r \leq p$; that is, the value of the reward is less than the price paid by customers for a product.

3.2 The Consumer's Decision Problem

We first formulate the decision problem of a generic customer with parameter (v, λ) in a given coalition program (p, r, μ, n). We assume that the customer discounts the future surplus with a rate δ ($\delta = 0$ corresponds to the case where the customer does not discount). Hence, the customer's objective is to maximize the total discounted surplus.

The customer's decision problem can be modeled as a continuous-time infinite horizon discounted reward dynamic program. Let $i \in \{0, 1\}$ denote the state, where $i = 0$ (1) denotes that the customer does not (does) hold a valid reward when she visits a firm in the coalition. Upon arrival, the customer observes the current state and decides whether to make a purchase. We consider the discrete-time Markov chain embedded in the continuous-time semi-Markov process and apply uniformization by taking the maximum transition rate to be $\nu = \delta + n\lambda + \mu$ (Puterman 1994). Uniformization is a well-known technique used to formulate continuous-time dynamic programs and simplify the analysis of the resulting dynamic programming model. The transition rates in different states for a continuous Markov chain usually differ. For the continuous-

the firms in the coalition that the consumer visits in each discrete time period follow a binomial distribution, which defies rigorous subsequent analysis. However, to check the robustness of our findings, Section 6.2 considers a discrete approximation (by assuming that the probability of visiting each firm is too small such that the probability of visiting more than one firm can be ignored) in which the expiration term is a constant.

time Markov chain in this model, there are two possible transitions in state 1: the consumer visits the coalition (with rate $n\lambda$) and makes a purchase, the reward expires (with rate μ). Hence, the total transition rate is $n\lambda + \mu + \delta$ (as δ is the customer's discounting rate). However, there is only one possible transition when the state is 0: the consumer visits the coalition (with rate $n\lambda$) and makes a purchase. Hence, the total transition rate is $n\lambda + \delta$. The key idea of uniformization is to add fictitious transitions such that the total transition rate is the same in all states. Here, by taking the total transition rate ν, we add a fictitious transition with rate μ in state 0. Note that the fictitious transition returns to the same state.

Let $u(\cdot)$ be the value function, which denotes the maximum surplus earned by the customer. Then, the optimality equations are given by

$$u(1) = \frac{n\lambda}{\nu} \max\Big\{v - p + r + u(1), u(1)\Big\} + \frac{\mu}{\nu} u(0), \tag{3.1}$$

$$u(0) = \frac{n\lambda}{\nu} \max\Big\{v - p + u(1), u(0)\Big\} + \frac{\mu}{\nu} u(0). \tag{3.2}$$

When the current state is 1 and the customer makes a purchase upon her arrival, she pays the price $p - r$ and earns a new reward, yielding a surplus of $v - p + r + u(1)$; otherwise, she remains in state 1. If the reward expires before her arrival, then she moves to state 0. Similarly, when the current state is 0 and the customer makes a purchase upon her arrival, she pays the price p and earns a new reward, yielding a surplus of $v - p + u(1)$; otherwise, she remains in state 0. The second term on the right-hand side of equation (3.2) is a fictitious transition that returns to the same state.

We assume that the customer always makes a purchase in state 0 when she visits a firm in the coalition. That is, $v - p + u(1) \geq u(0)$. Otherwise, the customer will not stay in the market over the long run and therefore does not contribute to a firm's revenue. The dynamic programming model admits an explicit solution given in the following proposition.

Proposition 1. *If*

$$v - p + \frac{n\lambda}{\nu} r \geq 0, \tag{3.3}$$

then it is optimal for a (v, λ)-consumer to make a purchase whenever she visits a firm in the coalition, and a solution to the optimality equations (3.1)–(3.2) *is given by*

$$u(1) = \frac{n\lambda}{\delta}\Big\{v - p + \frac{\delta + n\lambda}{\nu} r\Big\}, \quad u(0) = \frac{n\lambda}{\delta}\Big\{v - p + \frac{n\lambda}{\nu} r\Big\}.$$

Condition (3.3) is intuitive. If the sum of the customer's valuation v and the expected value of the reward $\frac{n\lambda}{\nu} r$ is greater than the price p, then the customer will make a purchase.

The state of a consumer who makes a purchase whenever she visits a firm in the coalition follows a Markov chain, as illustrated in Figure 3.1. We write down the

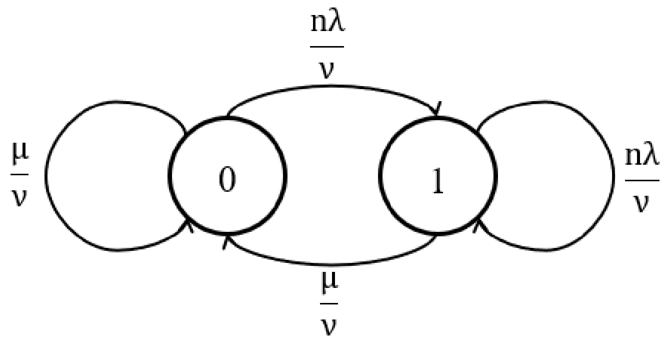


Figure 3.1: State transition diagram.

balance equations from the state transition diagram and solve these equations for the stationary probabilities of each state. Let q_0 and q_1 denote the stationary probability of states 0 and 1, respectively. We obtain $q_0 = \frac{\mu}{n\lambda + \mu}$ and $q_1 = \frac{n\lambda}{n\lambda+\mu}$. Therefore, if $p \leq v + \frac{n\lambda}{\nu} r$, a generic customer's profit contribution is

$$n\lambda\big\{q_0 p + q_1(p - r)\big\} = n\lambda\Big(p - \frac{n\lambda}{n\lambda + \mu} r\Big).$$

Chapter 4

Coalition's Decision Problem

Customers differ in their product valuation. A proportion α of customers are high-valuation customers with valuation v_H, whereas the rest are low-valuation customers with valuation $v_L < v_H$. Customers are also heterogeneous in their shopping frequency: a proportion β of customers are frequent customers with a rate λ_F; the rest are infrequent customers with a rate λ_I, where $0 \leq \lambda_I < \lambda_F \leq 1$. Thus, customers differ in two dimensions: product valuation and shopping frequency. In practice, customers' valuation and shopping frequency are usually correlated; for example, homemakers who visit the grocery store every day tend to have a lower valuation than office ladies who visit the store once a week. To model the correlation between these two dimensions, we assume that a proportion γ of all customers are high-valuation frequent customers. Consequently, the proportion $\alpha - \gamma$ are high-valuation infrequent customers, the proportion $\beta - \gamma$ are low-valuation frequent customers, and the proportion $1 - \alpha - \beta + \gamma$ are low-valuation infrequent customers. The case where $\gamma < \alpha\beta$ corresponds to the scenario in which there is a negative correlation between valuation and shopping frequency.

To determine the profit of the coalition reward program from the four homogeneous customer segments, we analyze the relationship between the price p, $v_H + \frac{n\lambda_F}{\nu_F} r$,

$v_H + \frac{n\lambda_I}{\nu_I}r$, $v_L + \frac{n\lambda_F}{\nu_F}r$, and $v_L + \frac{n\lambda_I}{\nu_I}r$,[1] which determine the customer's purchase behavior in each segment. Hence, there are several cases to examine when analyzing this relationship. By optimizing and comparing the average profit of the coalition in each case, we can find the optimal quartet (p, r, μ, n). It should be pointed out that the coalition's total profit is evenly allocated to its members due to the symmetry among the firms. Section 4.1 analyzes coalition's decision problem with an exogenous expiration date. Section 4.2 investigates coalition's decision problem with an endogenous expiration date, while Section 4.3 studies coalition's decision problem with both an endogenous expiration date and an endogenous coalition size.

4.1 Exogenous Expiration Term

Due to competition pressure or industry norms, the expiration term μ is often exogenous in practice, so this section examines the case with an exogenous μ.

Note that when the firm does not adopt any reward programs and sets a price v_L, all consumers make a purchase, yielding a profit of

$$\pi_1 = \big(\beta\lambda_F + (1-\beta)\lambda_I\big)v_L.$$

When the firm does not adopt any reward programs and sets a price v_H, only high-valuation consumers make a purchase, yielding a profit of

$$\pi_2 = \big(\gamma\lambda_F + (\alpha-\gamma)\lambda_I\big)v_H.$$

A coalition reward program is adopted only if the firm profit in a coalition is higher than that when the firm does not adopt any reward programs. The following lemma provides a necessary condition for the firm to join a coalition reward program.

Lemma 1. *For any fixed size n and exogenous μ, if*

$$(\alpha-\gamma)\lambda_I\Big(\lambda_F\mu - \lambda_I(\delta+\mu)\Big)(n\lambda_F+\mu) > \beta\lambda_F^2\delta(n\lambda_I+\mu), \tag{4.1}$$

[1] With a slight abuse of notation here, we denote $\nu_F = \delta + n\lambda_F + \mu$ and $\nu_I = \delta + n\lambda_I + \mu$.

then the optimal price, reward, and the average profit in the coalition reward program are

$$p^* = v_L + \frac{n\lambda_F}{\delta + n\lambda_F + \mu} r^*,$$

$$r^* = \min\Big\{\frac{(v_H - v_L)(\delta + n\lambda_F + \mu)(\delta + n\lambda_I + \mu)}{n(\lambda_F - \lambda_I)(\delta + \mu)}, \frac{(\delta + \mu + n\lambda_F)}{\delta + \mu} v_L\Big\},$$

$$\pi^* = \beta\lambda_F\Big(v_L + \frac{n\lambda_F}{\delta + n\lambda_F + \mu} r^* - \frac{n\lambda_F}{n\lambda_F + \mu} r^*\Big) + (\alpha - \gamma)\lambda_I\Big(v_L + \frac{n\lambda_F}{\delta + n\lambda_F + \mu} r^* - \frac{n\lambda_I}{n\lambda_I + \mu} r^*\Big).$$

Otherwise, it is never optimal to adopt the coalition reward program.

The coalition reward program is offered only if condition (4.1) holds. Note that condition (4.1) is derived by

$$\beta\lambda_F\Big(\frac{n\lambda_F}{\delta + n\lambda_F + \mu} - \frac{n\lambda_F}{n\lambda_F + \mu}\Big) + (\alpha - \gamma)\lambda_I\Big(\frac{n\lambda_F}{\delta + n\lambda_F + \mu} - \frac{n\lambda_I}{n\lambda_I + \mu}\Big) > 0, \quad (4.2)$$

The left-hand side of which is the coefficient of the reward r in the coalition's profit. Only if this coefficient is positive, the optimal reward r^* is non-zero. Note that $\frac{n\lambda_F}{\delta+n\lambda_F+\mu} r$ is frequent consumers' perceived value of the reward and $\frac{n\lambda_F}{n\lambda_F+\mu} r$ is frequent customers' materialized value of the reward, so $\Big(\frac{n\lambda_F}{\delta+n\lambda_F+\mu} - \frac{n\lambda_F}{n\lambda_F+\mu}\Big) r$ denotes frequent customers' value difference between perceived value and materialized value, which is clearly negative. As a result, only if $\frac{n\lambda_F}{\delta+n\lambda_F+\mu} - \frac{n\lambda_I}{n\lambda_I+\mu}$ is positive, inequality (4.2) holds. Moreover, the fraction of frequent consumers β cannot be too large; otherwise, inequality (4.2) does not hold. This is because frequent consumers are more likely to redeem the reward, so if there are too many frequent consumers, the reward will be redeemed extensively, hurting the firm's profit.

Note that an optimally designed coalition reward program must exclude LI consumers; otherwise, it is dominated by the full market coverage with price v_L and no reward programs. The price is set to ensure that both LF and HI consumers make a purchase; HF consumers make a purchase as long as LF consumers do. The highest prices acceptable to LF and HI consumers are $v_L + \frac{n\lambda_F}{\delta+n\lambda_F+\mu} r$ and $v_H + \frac{n\lambda_I}{\delta+n\lambda_I+\mu} r$, where

$\frac{n\lambda_F}{\delta+n\lambda_F+\mu}r$ and $\frac{n\lambda_I}{\delta+n\lambda_I+\mu}r$ are the perceived value of the reward to LF and HI consumers, respectively. Therefore, the optimal price must take the minimum between these two. Given that consumers discount the value of the reward, the perceived value of the reward is always lower than its materialized value, so the firm has an incentive to offer a relatively lower reward, rendering $v_L+\frac{n\lambda_F}{\delta+n\lambda_F+\mu}r$ smaller than $v_H+\frac{n\lambda_I}{\delta+n\lambda_I+\mu}r$. Consequently, the optimal price is $v_L+\frac{n\lambda_F}{\delta+n\lambda_F+\mu}r$. One can verify that the profit when the price equals $v_H+\frac{n\lambda_I}{\delta+n\lambda_I+\mu}r$ is always dominated by that when the price equals $v_L+\frac{n\lambda_F}{\delta+n\lambda_F+\mu}r$.

Under the optimally designed coalition reward program, HF and LF customers pay the same effective price $p-\frac{n\lambda_F}{n\lambda_F+\mu}r$, lower than the effective price $p-\frac{n\lambda_I}{n\lambda_I+\mu}r$ paid by HI customers. In other words, the coalition reward program can price discriminate consumers based on their shopping frequency rather than their valuation. Intuitively, the more frequently the consumer makes a purchase, the higher chance that she could earn and redeem a reward, and thus a lower effective price that she pays in the long term.

Next, we analyze how the average profit of the coalition changes with respect to the size n. We first introduce a notation $\hat{n}$ defined as follows:

$$\hat{n}=\frac{\left(1-\frac{\lambda_I}{\lambda_F}\sqrt{\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]}}\right)\mu}{\lambda_I\left(\sqrt{\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]}}-1\right)},$$

which is a critical threshold to the relationship between the coalition's average profit and size n. Note that $\hat{n}$ is positive only when

$$1<\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]}<\left(\frac{\lambda_F}{\lambda_I}\right)^2,$$

which requires an intermediate ratio between the fraction of HI consumers $(\alpha-\gamma)$ and frequent consumers (β).

Proposition 2 (The Relationship Between Profit and Size n)**.** *Suppose condition* (4.1) *holds.*

(a) *Suppose either* $\lambda_F v_L \le \lambda_I v_H$ *or* $\lambda_F v_L > \lambda_I v_H$ *but* $n < \frac{(v_H - v_L)(\delta+\mu)}{v_L\lambda_F - v_H\lambda_I}$, *then*

$$p^* = r^* = \frac{(\delta + \mu + n\lambda_F)}{\delta + \mu} v_L,$$
$$\pi^* = \beta\lambda_F v_L + (\alpha - \gamma)\lambda_I v_L - \beta\lambda_F \frac{n\lambda_F\delta}{n\lambda_F + \mu}\frac{v_L}{\delta + \mu} + (\alpha - \gamma)\lambda_I \frac{n[\lambda_F\mu - \lambda_I(\delta + \mu)]}{n\lambda_I + \mu}\frac{v_L}{\delta + \mu}.$$

The profit increases in n.

(b) *Suppose* $\lambda_F v_L > \lambda_I v_H$ *and* $n \ge \frac{(v_H - v_L)(\delta+\mu)}{v_L\lambda_F - v_H\lambda_I}$, *then*

$$p^* = \frac{\lambda_F(\delta + \mu + n\lambda_I)v_H - \lambda_I(\delta + \mu + n\lambda_F)v_L}{(\lambda_F - \lambda_I)(\delta + \mu)},$$

$$r^* = \frac{(v_H - v_L)(\delta + n\lambda_F + \mu)(\delta + n\lambda_I + \mu)}{n(\lambda_F - \lambda_I)(\delta + \mu)},$$

$$\pi^* = \beta\lambda_F v_L + (\alpha - \gamma)\lambda_I v_L - \beta\lambda_F \frac{\delta + n\lambda_I + \mu}{n\lambda_F + \mu}\frac{\lambda_F\delta(v_H - v_L)}{(\delta + \mu)(\lambda_F - \lambda_I)}$$
$$+ (\alpha - \gamma)\lambda_I \frac{[\lambda_F\mu - \lambda_I(\delta + \mu)](v_H - v_L)}{(\delta + \mu)(\lambda_F - \lambda_I)}\frac{\delta + n\lambda_I + \mu}{n\lambda_I + \mu}.$$

(i) *If* $1 < \frac{(\alpha-\gamma)[\lambda_F\mu - \lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu) - \lambda_I\mu]} < \left(\frac{\lambda_F}{\lambda_I}\right)^2$, *then the profit increases in* n *when* $n < \hat{n}$ *and decreases in* n *otherwise.*

(ii) *If* $\frac{(\alpha-\gamma)[\lambda_F\mu - \lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu) - \lambda_I\mu]} \le 1$, *the profit increases in* n.

(iii) *If* $\frac{(\alpha-\gamma)[\lambda_F\mu - \lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu) - \lambda_I\mu]} \ge \left(\frac{\lambda_F}{\lambda_I}\right)^2$, *the profit decreases in* n.

Proposition 2 characterizes the relationship between the coalition's average profit and the size n. Intuitively, as the size n increases, the probability of using the reward before expiration increases, enabling the coalition to collect more profit from customers by adjusting the decisions on p and r accordingly. Moreover, as the size n increases, the probability that customers visit the coalition and make a purchase also increases; the earlier customers make a purchase, the less their surplus is discounted. Therefore, the average profit should increase in the size n. However, as n increases, the heterogeneity in shopping frequency is less prominent, and thus the discrimination power of the coalition reward program is reduced. Consider one extreme case that the size n reaches infinity and then it is impossible to distinguish frequent and infrequent consumers as they all make purchases extensively. In this regard, the profit decreases in the size n. To summarize, how the profit changes with the size n depends on the magnitude of the positive and negative forces.

Part (a) says that when the values of v_L and λ_F are low or the size n is small, the optimal reward can be as high as the price. This is because, with a low valuation v_L and a low shopping frequency λ_F, LF consumers have a low incentive to make a purchase, and thus the firm needs to offer a high reward to attract them to make a purchase. On the other hand, when the size n is small, the breakage rate of the reward is relatively high (the probability of using the reward before expiration is low), so the firm also needs to offer a high reward to attract consumers to make a purchase. However, even if a high reward (as high as the price) is offered, due to the low values of v_L, λ_F, and n, the highest acceptable price to LF consumers is still far below that to HI consumers ($v_L + \frac{n\lambda_F}{\delta+n\lambda_F+\mu}r < v_H + \frac{n\lambda_I}{\delta+n\lambda_I+\mu}r$), so the firm can only set the price $v_L + \frac{n\lambda_F}{\delta+n\lambda_F+\mu}r$, ensuring LF consumers purchase but leaving a large amount of consumer surplus uncollected from HI consumers. In such an imbalanced market , as the size n increases, the coalition reward program collects more surplus from HI consumers, and the price discrimination power is not weakened. The reason is as follows. Note that the effective prices paid by frequent consumers and HI consumers are

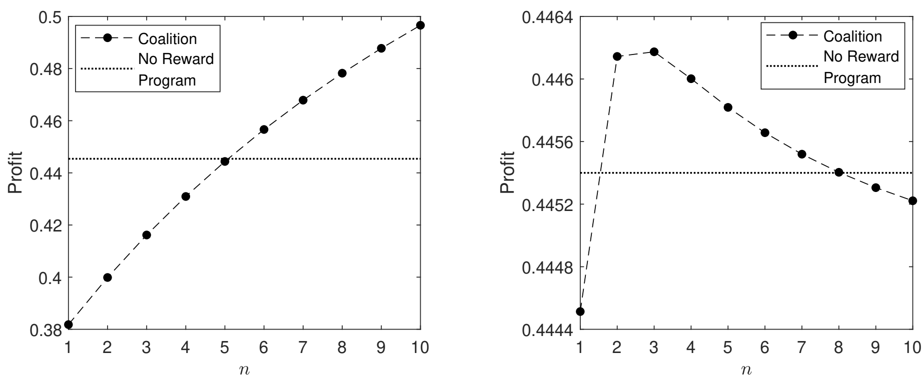
$$p - \frac{n\lambda_F}{n\lambda_F + \mu}r = \frac{\mu}{n\lambda_F + \mu}p = \frac{n\lambda_F + \mu + \delta}{n\lambda_F + \mu}\frac{\mu}{\delta + \mu}v_L,$$

$$p - \frac{n\lambda_I}{n\lambda_I + \mu}r = \frac{\mu}{n\lambda_I + \mu}p = \frac{n\lambda_F + \mu + \delta}{n\lambda_I + \mu}\frac{\mu}{\delta + \mu}v_L,$$

respectively. Clearly, the first effective price is smaller than the second one. Moreover, as n goes to infinity, the effective price paid by frequent consumers is decreasing, whereas the effective price paid by HI consumers is increasing, indicating that the gap between the effective prices paid by these two groups of consumers is growing. That is, the price discrimination power of the coalition reward program gets reinforced as the size n increases. This explains why the profit increases in n in this case.

When the size n is large, the probability of using the reward before expiration is high, and thus the firm does not have to offer an attractive reward r to consumers, rather, the optimal reward r is set to make the two possible prices $v_L + \frac{n\lambda_F}{\delta+n\lambda_F+\mu}r$ and $v_H + \frac{n\lambda_I}{\delta+n\lambda_I+\mu}r$ equal to each other, striking a better balance between the profits collected from LF consumers and HI consumers. In this case, the profit firstly increases and then decreases in n. The fact that the profit decreases in n is mainly because the discrimination power is undermined as n increases in this more balanced market. Note that the effective prices paid by frequent consumers and HI consumers are $p - \frac{n\lambda_F}{n\lambda_F+\mu}r$ and $p - \frac{n\lambda_I}{n\lambda_I+\mu}r$, respectively, where the former price is smaller than the latter one. However, as n goes to infinity, the difference $(\frac{n\lambda_F}{n\lambda_F+\mu} - \frac{n\lambda_I}{n\lambda_I+\mu})$ between the probability that each group of consumers redeems the reward before expiration is diminishing, indicating that the price discrimination power of the coalition reward program is weakened as the size n increases. On the other hand, as n increases, the materialized value of the reward for HI consumers $(\frac{n\lambda_I}{n\lambda_I+\mu}r)$ increases faster than the materialized value of the reward for frequent consumers $(\frac{n\lambda_F}{n\lambda_F+\mu}r)$, because HI consumers' probability of using the reward before expiration increases faster than that of the frequent consumers. This makes it possible that the price p increases slower than $\frac{n\lambda_I}{n\lambda_I+\mu}r$ but faster than $\frac{n\lambda_F}{n\lambda_F+\mu}r$. Consequently, the effective price paid by HI consumers $(p - \frac{n\lambda_I}{n\lambda_I+\mu}r)$ is decreasing in n, while the effective price paid by frequent consumers $(p - \frac{n\lambda_F}{n\lambda_F+\mu}r)$ is increasing in n, also implying that the effective price difference between these two groups is shrinking. Therefore, if HI consumers account for a sufficiently large proportion such that the decreasing effect dominates the increasing effect (as the condition in Proposition 2(b)-(iii)), the profit would decrease in the size n. Otherwise, the profit would increase. This explains why the optimal profit may decrease in the size n when n is large enough in this more balanced market.

Figure 4.1 depicts how the profit changes with respect to the coalition size n. We set $(v_H, v_L, \lambda_F, \lambda_I, \delta) = (20, 10, 0.1, 0.03, 0.01)$ and $(\alpha, \beta, \gamma) = (0.6, 0.2, 0.061)$. We not only characterize the optimal average profit of the coalition for each n but also depict the optimal profit when the firm does not adopt any reward programs for comparison. Figure 4.1(a) assumes a larger value of μ, corresponding to a shorter expiration date. As n increases but remains small, the positive effect (e.g., reduction of reward breakage rate) is very strong, and therefore, the profit keeps increasing as more firms join the coalition. One can verify that when $\mu = 0.5$, the parameter set satisfies the condition in Proposition 2(a). Figure 4.1(b) assumes a smaller value of μ, corresponding to a longer expiration date, under which the negative effect (eroding the discrimination power) is more prominent when the coalition size reaches a certain threshold. Therefore, after achieving the maximum when $n = 3$, the profit decreases as n increases. One can verify that when $\mu = 0.03$, the parameter set satisfies the condition in Proposition 2(b).



(a) $\mu = 0.5$ (b) $\mu = 0.03$

Figure 4.1: The optimal average profit of the coalition with respect to the size n.

Proposition 3 (The Optimal Size). *Suppose condition* (4.1) *holds.*

(a) Suppose $\lambda_F v_L \leq \lambda_I v_H$, then the profit keeps increasing in n. Moreover, one can check

$$\lim_{n\to\infty} \pi^* = \frac{\mu}{\mu+\delta}\Big\{\beta\lambda_F v_L + (\alpha-\gamma)\lambda_F v_L\Big\} < \frac{\mu}{\mu+\delta}\Big\{\beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_H\Big\}.$$

(b) Suppose $\lambda_F v_L > \lambda_I v_H$.

(i) If $1 < \frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]} < \left(\frac{\lambda_F}{\lambda_I}\right)^2$, *then the optimal size* $n^* = \hat{n}$ *if* $\hat{n} \geq \frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F-v_H\lambda_I}$ *and* $n^* = \frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F-v_H\lambda_I}$ *otherwise.*

(ii) If $\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]} \leq 1$, *then the profit keeps increasing in* n.

(iii) If $\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]} \geq \left(\frac{\lambda_F}{\lambda_I}\right)^2$, *then* $n^* = \frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F-v_H\lambda_I}$.

Moreover, one can check

$$\pi^*(n^*) < \frac{\mu}{\mu+\delta}\Big\{\beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_H\Big\}.$$

Proposition 3 is an immediate result of Proposition 2. In the imbalanced market, the profit keeps increasing in the size n, so the coalition should absorb the partners as many as possible. In the balanced market, there exists an intermediate optimal size for the coalition, given that a small size cannot alleviate the negative effect of consumer discounting and high reward breakage rate, while a large size can dilute the discrimination power of the coalition reward program. Therefore, the coalition should not absorb any new partners once its size reaches a certain threshold. Proposition 3 indicates that market composition is crucially important for the coalition managers to design an optimal coalition reward program. In particular, the coalition managers should be aware of the heterogeneity in customers' product valuation as well as their shopping frequency to determine the potential revenue contribution of each type of consumers. If the potential revenue contribution of low-valuation frequent consumers is less than that of high-valuation infrequent consumers, the market is imbalanced and the coalition managers should not set any caps on the size of the coalition. Otherwise, the market is more balanced and a cap on the coalition size is necessary.

In addition, independent of the relationship between $\lambda_F v_L$ and $\lambda_I v_H$, Proposition 3 shows an upper bound of the profit that the coalition reward program can achieve. As discussed earlier, the coalition reward program can price discriminate consumers based on their shopping probabilities; HF and LF consumers pay the same effective price less than v_L; HI consumers pay an effective price less than v_H, yielding an immediate profit upper bound $\beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_H$. In addition, consumers also discount their future surplus with a rate δ. Taking the discounting factor into account gives the upper bound in Proposition 3.

4.2 Endogenous Expiration Term

This section analyzes the problem with an endogenous expiration term for a fixed coalition size. We first introduce a notation $\hat{\mu}$ that is critical to the relationship between the optimal profit and the expiration term. Define $\hat{\mu}$ such that

$$\beta\lambda_F^2\delta(n\lambda_F+2\hat{\mu}+\delta)(n\lambda_I+\hat{\mu})^2-(\alpha-\gamma)\lambda_I\left[\lambda_F\hat{\mu}^2-\lambda_I(\delta+\hat{\mu})^2-n\lambda_F\lambda_I\delta\right](n\lambda_F+\hat{\mu})^2=0.$$

Note that the left-hand side of the above equation is positive when μ is sufficiently small. Note also that both the first term and the second term increase in μ; however, the second term increases faster than the first one, and the left-hand side is negative when μ is sufficiently large, which implies the existence and uniqueness of $\hat{\mu}$.

Proposition 4. *Suppose condition* (4.1) *holds. For any fixed n, the optimal profit increases in μ if $\mu < \max\{\hat{\mu}, \frac{(v_L\lambda_F-v_H\lambda_I)n}{v_H-v_L}-\delta\}$ and decreases in μ otherwise.*

Proposition 4 implies that an intermediate expiration term is optimal. If μ is too small (the expiration date is too long), the breakage rate of the reward will be too low for both frequent and infrequent consumers such that it is difficult to distinguish these two types of consumers, which dilutes the price discrimination power of the coalition reward program. As μ increases, the expiration date decreases, reinforcing the discrimination power of the coalition reward program. Hence, the optimal profit increases in μ when $\mu < \hat{\mu}$. However, if μ is too large (the expiration date is too short), the coalition needs to decrease the price or raise the reward to entice consumers to make a purchase, undermining the firm's profit. Therefore, the optimal profit decreases in μ when $\mu \geq \hat{\mu}$.

One may wonder how the optimal expiration date changes with respect to the size n.

Proposition 5. *Suppose condition* (4.1) *holds. If δ is sufficiently small (i.e., $\delta n \lambda_F \leq 2\mu^2$), then the optimal expiration rate μ^* increases in n. That is, the larger the size n, the shorter the expiration date.*

Proposition 5 characterizes the relationship between the optimal expiration date and the size n. Intuitively, the larger the size n, the more likely that the consumer visits the coalition and makes a purchase, and hence the more likely that she earns and redeems the reward. Therefore, the coalition should decrease the expiration date to balance the expected value of the reward. This result tells the coalition managers that adjusting the expiration date of the reward can be a powerful way to boost reward redemption and drive more engagement.

This result confirms our intuition that the expiration date of the coalition's reward is supposed to be shortened as the coalition expands its scope by absorbing new members. However, in practice, the coalition managers should not adjust the expiration date too frequently even if the coalition expands fast, because frequent adjustment of the reward terms will confuse the consumers (on the expected value of the reward) and discourage consumer engagement. Instead, the coalition managers may consider adjusting the expiration date only when there is a substantial change in the number of partners in the coalition.

4.3 Endogenous Expiration Term and Endogenous Coalition Size

This section investigates the problem with both an endogenous expiration term and an endogenous coalition size. That is, the expiration term and coalition size decisions are jointly made by the coalition manager, to maximize the optimal average profit. We conduct extensive numerical experiments to illustrate the coalition's optimal decision policy.

Similar to Figure 4.1, we set $(v_H, v_L, \lambda_F, \lambda_I, \delta) = (20, 10, 0.1, 0.03, 0.01)$ and $(\alpha, \beta, \gamma) = (0.6, 0.2, 0.061)$. First, we fix each size n to find the corresponding optimal expiration term. Then, by comparing the optimal average profit under all different $(n,\ \mu)$, we can choose an effective and practical combination of the expiration term and coalition size.

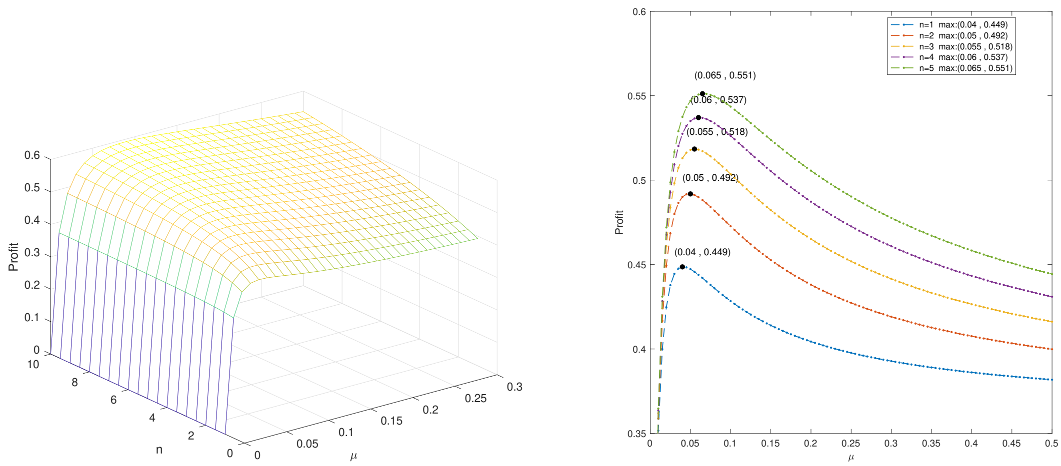


Figure 4.2: The optimal average profit of the coalition with respect to the expiration term μ and the size n.

Figure 4.2 confirms our findings in Proposition 5. When the coalition size increases by absorbing new partners, the expiration term of the coalition's reward can be set shorter. To some extent, a short expiration date creates an emergency, giving consumers the motivation to redeem their reward earlier before they lose it. Given that consumers discount future surpluses, such a configuration of the reward expiration date makes the coalition more profitable. In reality, the coalition manager are often constrained by limited resources (e.g., the expiration term and coalition size decisions should satisfy condition (4.1)). It may be hard to set both optimal expiration term and coalition size. Therefore, the coalition manager should keep fine-tuning with new constraints in the dynamic business world.

Chapter 5

Comparison

To shed light on the adoption of coalition reward programs, this chapter compares the coalition reward program with that when the firm does not adopt any reward programs and the proprietary reward program in terms of both firm profit and consumer surplus.

5.1 Comparison with No Reward Programs

This section discusses and compares the coalition reward program with no reward programs. Recall that when the firm does not adopt any reward programs and sets a price v_L, the firm earns a profit of $\pi_1 = \big(\beta\lambda_F + (1-\beta)\lambda_I\big)v_L$. When the firm does not adopt any reward programs and sets a price v_H, the firm collects a profit of $\pi_2 = \big(\gamma\lambda_F + (\alpha-\gamma)\lambda_I\big)v_H$. The following proposition compares the coalition's profit with that when the firm does not adopt any reward programs.

Proposition 6. *Suppose condition* (4.1) *holds. For any fixed n and μ,*

(a) If $\gamma \geq \alpha\beta$, then $\pi^* \leq \min\{\pi_1, \pi_2\}$. That is, if product valuation and shopping frequency are positively correlated, then coalition reward programs cannot bring

a higher profit than that when the firm does not adopt any reward programs.

(b) If $1-\alpha-\beta+\gamma$ *is sufficiently small, then* $\pi^* \geq \pi_1$.

(c) If $\beta-\gamma$ *is sufficiently large, then* $\pi^* \geq \pi_2$.

Proposition 6(a) implies that a negative correlation between product valuation and shopping frequency is necessary for the coalition reward program to be profitable. Compared with the full market coverage with no reward programs, the coalition can charge a higher effective price to HI consumers but LI consumers do not make a purchase. On the other hand, compared with the partial market coverage with no reward programs, the coalition can attract LF consumers to make a purchase but HF consumers pay a lower effective price. Therefore, for the coalition program to be profitable, HI and LF (LI and HF) consumers need to account for a large (small) proportion of the population, implying a negative correlation between valuation and shopping frequency. Proposition 6(b)-(c) argues that if γ is sufficiently small (which implies a strong negative correlation), then the coalition program is more profitable than that without no reward programs.

Although this negative correlation argument is also valid for the proprietary reward program (see Proposition 2 in Sun and Zhang (2019)), our result demonstrates that simply linking multiple brands in a partnership cannot guarantee the profitability of the coalition. The brands can benefit from joining a coalition only if the correlation between valuation and shopping frequency is negative in the aggregate consumer composition. The coalition managers should pay attention to the consumer composition for each brand and then decide whether admitting a single brand can improve the overall market composition.

One may also take it for granted that the brand bringing more consumers should acquire more revenue than the others in the coalition. However, this negative correlation result, which probably holds for asymmetric firms, implies that among asymmetric brands (in terms of the shopping frequency), the one bringing the largest number of consumers may not contribute the most to the coalition. For example, the consumers may have a strong positive correlation between valuation and shopping frequency for the brand with a higher consumer shopping frequency (higher values of λ_F and λ_I). Therefore, the coalition's profit should not be allocated to the brands simply based on their consumers' shopping frequency. We believe that the question of how to allocate the coalition's profit among asymmetric brands is an interesting topic and deserves a separate study.

Table 5.1 and 5.2 characterize consumer welfare of each segment in the imbalanced market and balanced market, respectively, under three scenarios: the firm joins a coalition reward program, the firm does not adopt any reward program and sets a price v_L (sale at v_L), and the firm does not adopt any reward program and sets a price v_H (sale at v_H). No matter the market is imbalanced or balanced, the effective

	Coalition	v.s.	Sale at v_L	Sale at v_H
High-valuation frequent	$v_H - \frac{\mu(\delta+n\lambda_F+\mu)}{(n\lambda_F+\mu)(\delta+\mu)}v_L$	$>$	$v_H - v_L$	0
Low-valuation frequent	$v_L - \frac{\mu(\delta+n\lambda_F+\mu)}{(n\lambda_F+\mu)(\delta+\mu)}v_L$	$>$	0	0
High-valuation infrequent	$v_H - \frac{\mu(\delta+n\lambda_F+\mu)}{(n\lambda_I+\mu)(\delta+\mu)}v_L$	$<$	$v_H - v_L$	0
Low-valuation infrequent	0	$=$	0	0

Table 5.1: Consumer welfare of each segment when $v_L\lambda_F \leq v_H\lambda_I$.

price paid by HF and LF consumers is lower than that of sale at v_L, thereby leading to a higher surplus. The effective price paid by HI consumers is higher than v_L, thereby leading to a lower surplus. LI consumers do not make a purchase, resulting in a zero surplus which equals that of sale at v_L. In fact, the effective price paid by LF consumers cannot be greater than v_L; otherwise, LF consumers will not make a purchase, in which case the coalition's profit will be dominated by sale at v_H. Similarly, the effective price paid by HI consumers has to be higher than v_L; otherwise, all the three segments of consumers contribute no more than v_L, in which case the coalition's profit will be dominated by sale at v_L.

	Coalition	v.s.	Sale at v_L	Sale at v_H
High-valuation frequent	$(v_H - v_L)\left[1 + \frac{\delta+n\lambda_I+\mu}{n\lambda_F+\mu}\frac{\lambda_F\delta}{(\delta+\mu)(\lambda_F-\lambda_I)}\right]$	$>$	$v_H - v_L$	0
Low-valuation frequent	$(v_H - v_L)\frac{\delta+n\lambda_I+\mu}{n\lambda_F+\mu}\frac{\lambda_F\delta}{(\delta+\mu)(\lambda_F-\lambda_I)}$	$>$	0	0
High-valuation infrequent	$(v_H - v_L)\left[1 - \frac{\delta+n\lambda_I+\mu}{n\lambda_I+\mu}\frac{\lambda_F\mu-\lambda_I(\delta+\mu)}{(\delta+\mu)(\lambda_F-\lambda_I)}\right]$	$<$	$v_H - v_L$	0
Low-valuation infrequent	0	$=$	0	0

Table 5.2: Consumer welfare of each segment when $v_L\lambda_F > v_H\lambda_I$.

Combine Tables 5.1 and 5.2.

Given that HF and LF consumers earn a higher surplus and HI consumers earn a lower surplus, one may wonder if the aggregate consumer surplus is higher or lower than sale at v_L. The following proposition compares the aggregate consumer surplus among the three scenarios.

Proposition 7. *For any fixed size and exogenous expiration date, the aggregate consumer surplus in coalition reward programs is higher than that when the firm does not adopt any reward programs and sets a price v_H, but lower than that when the firm does not adopt any reward programs and sets a price v_L.*

Note that in both an imbalanced and balanced market, when the firm does not adopt any reward programs and sells to high-valuation consumers only with a price v_H, consumers in each segment obtain a zero surplus, so the aggregate consumer surplus is also zero in this scenario. Note also that the consumer welfare of each segment in coalition reward programs is nonnegative because otherwise, consumers will not make a purchase. Clearly, the coalition reward program brings a higher aggregate consumer surplus. However, compared with the scenario when the firm sells at the price v_L

with no reward programs, the coalition reward program brings a lower aggregate consumer surplus. Recall that the necessary condition (4.2) for adopting coalition reward programs requires a small fraction of frequent consumers and a large fraction of HI consumers. Therefore, the surplus increment earned by frequent consumers could be dominated by the surplus loss of HI consumers.

Combine Propositions 6 and 7.

According to Propositions 6 and 7, when there is a positive correlation between valuation and shopping frequency, a lose-lose situation could occur if the firm switches from no reward programs and sales at v_L to joining a coalition reward program, given that the firm cannot benefit and the aggregate consumer surplus is hurt as well. However, when there is a negative correlation between valuation and shopping frequency, a win-win situation could emerge if the firm switches from no reward program and sales at v_H to joining a coalition reward program, given that both the firm's profit and the aggregate consumer surplus could be boosted. This result indicates that market composition is crucially important to the success of coalition reward programs not only from the perspective of brand profit but also from the perspective of social welfare.

5.2 Comparison with Proprietary Reward Programs

This subsection discusses and compares coalition reward programs with proprietary reward programs. To facilitate our discussion, let $\pi^c(n^*)$ denote the optimal average profit of the coalition with an endogenized size n, $\pi^c(n)$ denote the optimal average profit of the coalition with an exogenous size n, and π^p denote the optimal profit of the proprietary reward program. Moreover, to make a meaningful comparison, we assume that the size of the coalition reward program is at least two, i.e., $n^* \geq 2$; otherwise, $\pi^c(n^*)$ always (weakly) dominates π^p which equals $\pi^c(1)$, a special case of the coalition reward program.

Similar to condition (4.1) for coalition reward programs, if

$$(\alpha-\gamma)\lambda_I\big(\lambda_F\mu-\lambda_I(\delta+\mu)\big)(\lambda_F+\mu)>\beta\lambda_F^2\delta(\lambda_I+\mu) \tag{5.1}$$

does not hold, then it is never optimal to adopt the proprietary reward program, compared to the firm's profit without any reward programs. The following proposition compares these two types of reward programs.

Proposition 8. *Suppose conditions* (4.1) *and* (5.1) *hold. For any exogenous* μ,

(a) if $(v_H-v_L)(\delta+\mu)\geq v_L\lambda_F-v_H\lambda_I$, *then* $\pi^c(n^*)\geq\pi^p$;

(b) suppose $(v_H-v_L)(\delta+\mu)<v_L\lambda_F-v_H\lambda_I$. *If* $\hat{n}>\frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F-v_H\lambda_I}$ *or* $\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]}<1$, *then* $\pi^c(n^*)\geq\pi^p$.

   (i) In particular, if $\hat{n}>\frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F-v_H\lambda_I}$ *and* $\frac{(\alpha-\gamma)\lambda_I[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta\lambda_F[\lambda_F(\delta+\mu)-\lambda_I\mu]}>\frac{\lambda_I+\mu}{\lambda_F+\mu}$, *then there exists a threshold* n' *such that* $\pi^c(n)<\pi^p$ *if* $n>n'$;

(c) suppose $(v_H-v_L)(\delta+\mu)<v_L\lambda_F-v_H\lambda_I$. *If* $\hat{n}<\frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F-v_H\lambda_I}$ *or* $\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]}>(\frac{\lambda_F}{\lambda_I})^2$, *then* $\pi^c(n^*)<\pi^p$.

Compared with proprietary reward programs, the effect of coalition reward programs on firm profit is two-fold. On the one hand, due to the increased shopping frequency of each consumer, consumer surplus is less discounted, so the firm in the coalition can extract more surplus from consumers. Note that the positive effect of decreased breakage rate on the firm's profit in a coalition disappears if the expiration dates for both proprietary and coalition reward programs are optimized. On the other hand, as n increases, the heterogeneity in shopping frequency between frequent and infrequent consumers is less prominent, which undermines the price discrimination power of the coalition reward program. Therefore, the comparison between proprietary reward programs and coalition reward programs depends on the magnitude of the positive and negative effects.

When the positive effect of relieving consumer discounting dominates the negative effect of weakening discrimination power, the coalition reward program beats the proprietary reward program. Proposition 8(a)-(b) identify the conditions under which the coalition reward program brings a higher profit than the proprietary reward program with an exogenous expiration date. Noteworthy, as long as the optimal expiration date of the proprietary reward program satisfies the conditions in Proposition 8(a)-(b), $\pi^c(n^*) \geq \pi^p$ still holds even when the expiration dates for both proprietary and coalition reward programs are optimized. However, Proposition 8(b)-(i) shows that if the coalition reward program is not properly designed, for example, an inadequate number of partners are admitted into the coalition, then the coalition reward program could generate a lower profit than the proprietary reward program, indicating that an adequate size of the coalition is crucially important to the success of a coalition reward program.

If the consumer's discounting factor is sufficiently small such that the positive effect of relieving consumer discounting is dominated by the negative effect of weakening discrimination power, then the proprietary reward program may beat the coalition reward program. Proposition 8(c) specifies the conditions under which the optimally designed proprietary reward program brings a higher profit than an optimally designed coalition reward program under the same exogenous expiration date. Again, if the optimal expiration date of the coalition reward program satisfies the conditions in Proposition 8(c), then the proprietary reward program still performs better than the coalition even when the expiration dates for both types of reward programs are optimized.

Let us examine the counter-intuitive result in Proposition 8(c) in detail. In the more balanced market, recall that the effective prices paid by frequent consumers and HI consumers are $p - \frac{n\lambda_F}{n\lambda_F+\mu}r$ and $p - \frac{n\lambda_I}{n\lambda_I+\mu}r$, respectively, where $p - \frac{n\lambda_F}{n\lambda_F+\mu}r < p - \frac{n\lambda_I}{n\lambda_I+\mu}r$. Moreover, as n increases, the materialized value of the reward for HI consumers ($\frac{n\lambda_I}{n\lambda_I+\mu}r$) increases faster than the materialized value of the reward for frequent consumers ($\frac{n\lambda_F}{n\lambda_F+\mu}r$), because HI consumers' probability of using the reward before expiration increases faster than that of the frequent consumers. This makes it possible that the price p increases slower than $\frac{n\lambda_I}{n\lambda_I+\mu}r$ but faster than $\frac{n\lambda_F}{n\lambda_F+\mu}r$. Consequently, the effective price paid by HI consumers ($p - \frac{n\lambda_I}{n\lambda_I+\mu}r$) is decreasing in n, while the effective price paid by frequent consumers ($p - \frac{n\lambda_F}{n\lambda_F+\mu}r$) is increasing in n, implying that the effective price difference between these two groups is shrinking. Therefore, if HI consumers account for a larger proportion such that the decreasing effect dominates the increasing effect, then the coalition reward program is weaker than the proprietary reward program in terms of the price discrimination power, and thus generates a lower profit. This explains the conditions in Proposition 8(c).

Noteworthy, it is easier to satisfy condition (4.1) than condition (5.1). Therefore, compared with that when the firm does not adopt any reward programs, if offering proprietary reward programs cannot bring a higher profit (e.g., condition (5.1) is not satisfied but condition (4.1) holds), then the firms may consider offering joint promotions via a coalition reward program to boost their profit.

Finally, if conditions (4.1) and (5.1) hold, the social welfare (the sum of firm profit and aggregate consumer welfare) is equal between proprietary and coalition reward programs because all consumers except LI consumers make a purchase in both types of reward programs. Therefore, a higher profit is paired with lower consumer welfare. On the flip side, a lower profit is paired with higher consumer welfare.

Chapter 6

Extension

We consider three extensions in this chapter. First, we consider the model without discounting factor, to investigate the impact of customer discounting on our findings. Second, we perform a discrete approximation to relax some assumptions in the continuous-time model, e.g., the exponential distribution of the expiration term, to check the robustness of our results. In the discrete-time model, the expiration date will be set as a constant. Finally, we consider asymmetric firms, which will help us understand whether a given firm that wants to form a coalition with others should choose to partner with a firm that has a higher or lower arrival rate than itself. Not surprisingly, some of our main results are consistent with the base model.

6.1 The Model without Consumer Discounting

Our main model assumes that consumers discount future surplus with a rate δ. Let $\delta = 0$. We rule out the effect of consumer discounting. By comparison, we investigate the impact of customer discounting on our findings.

Lemma 2. *For any fixed size n and exogenous μ, the optimal price, reward, and the* *average profit in the coalition reward program are*

$$
\begin{aligned}
p^* &= v_L + \frac{n\lambda_F}{n\lambda_F + \mu} r^*, \\
r^* &= \min\left\{\frac{(v_H - v_L)(n\lambda_F + \mu)(n\lambda_I + \mu)}{n\mu(\lambda_F - \lambda_I)}, \frac{(\mu + n\lambda_F)}{\mu} v_L\right\}, \\
\pi^* &= \beta\lambda_F v_L + (\alpha - \gamma)\lambda_I \min\left\{v_H, \frac{n\lambda_F + \mu}{n\lambda_I + \mu} v_L\right\}.
\end{aligned}
$$

It is worth noting that, the firm always has incentives to offer a nontrivial reward. In Section 4.1, however, condition 4.1 is necessary for the firm to join a coalition reward program. That is, consumer discounting deters the firm's participation motivation to some extent. Consumers who discount future surpluses have a lower perceived value of the reward. To engage consumer purchases, a larger reward is needed, which hurts the firm's profit.

Proposition 9. *The Relationship Between Profit and Size n without Discounting.*

(i) If $\lambda_F v_L \leq \lambda_I v_H$ or $\lambda_F v_L > \lambda_I v_H$ but $n < \frac{\mu(v_H - v_L)}{v_L\lambda_F - v_H\lambda_I}$, then $\pi^* = \beta\lambda_F v_L + (\alpha - \gamma)\lambda_I \frac{n\lambda_F + \mu}{n\lambda_I + \mu} v_L$, which is increasing in n.

(ii) If $\lambda_F v_L > \lambda_I v_H$ and $n \geq \frac{\mu(v_H - v_L)}{v_L\lambda_F - v_H\lambda_I}$, then $\pi^* = \beta\lambda_F v_L + (\alpha - \gamma)\lambda_I v_H$, which is independent of n.

That is,

(a) Suppose $\lambda_F v_L \leq \lambda_I v_H$, then the profit is increasing in n. Moreover, one can check

$$\lim_{n\to\infty} \pi^* = \beta\lambda_F v_L + (\alpha - \gamma)\lambda_F v_L < \beta\lambda_F v_L + (\alpha - \gamma)\lambda_I v_H.$$

(b) Suppose $\lambda_F v_L > \lambda_I v_H$, then the profit first increases in n and then becomes constant for all $n \geq \frac{\mu(v_H - v_L)}{v_L\lambda_F - v_H\lambda_I}$.

Analogous to Proposition3(a), the firm's long-run average profit increases as the coalition absorbs more partners in the imbalanced market. When the values of v_L and λ_F are low, the price discrimination power between HI and LF consumers becomes more prominent as n increases. However, different from Proposition3(b), the negative force of consumer discounting vanishes in the balanced market. When n is large, the expansion of the coalition size makes no difference to the firm's long-run average profits. In that case, the optimal reward r is set to keep a balance between the profits collected from LF consumers and HI consumers (i.e., $v_L + \frac{n\lambda_F}{n\lambda_F+\mu}r = v_H + \frac{n\lambda_I}{n\lambda_I+\mu}r$). In the long run, frequent consumers paid the price at v_L while HI consumers paid the price at v_H, which is independent of the number of partners in the coalition.

6.2 A Discrete Approximation

In the continuous-time model, we suggest the exponential distribution of the expiration term. In practice, coalitions may set a fixed expiration term for easy operation. Hence, we perform a discrete approximation to relax this assumption, where the expiration date can be set as a constant in the discrete-time model.

We consider a coalition reward program consisting of n independent and substitute firms. In each period, the customer visits each firm in the coalition with a probability λ (we assume λ is small such that $\lambda \ll 1$). Even though there is a positive probability that the customer visits more than one firm, the probability is quite small and will be ignored in our analysis. Therefore, such an arrival process can be the binomial approximation to the Poisson process and we can assume that the customer visits one retailer only, which is quite common in literature (Lautenbacher and Stidham (1999)).

We assume that consumer discounts the future surplus with a per-period discount factor δ, where $\delta \in [0, 1)$. Now the consumer's problem can be modeled as a discrete-time infinite horizon discounted reward dynamic programming. The optimality equations are given by

$$u(i) = \begin{cases} n\lambda \max\{v - p + r + \delta u(K), \delta u(i-1)\} + (1-n\lambda)\delta u(i-1), & \text{if} \quad i = 1, 2, ...K, \\ n\lambda \max\{v - p + \delta u(K), \delta u(0)\} + (1-n\lambda)\delta u(0), & \text{if} \quad i = 0. \end{cases} \tag{6.1}$$

Let $G(i,\lambda) = \delta n\lambda \frac{1-\delta^i(1-n\lambda)^i}{1-\delta(1-n\lambda)}$. Similarly, if $v - p + G(K,\lambda)r \geq 0$, then it is optimal for a (v,λ)-consumer to make a purchase whenever she visits a firm in the coalition. Let q_0 and q_i denote the stationary probability of states 0 and i. We obtain $q_0 = (1-n\lambda)^K$ and $q_i = n\lambda(1-n\lambda)^{K-i}$. Then, a generic customer's profit contribution is

$$n\lambda\Big(q_0 p + (1-q_0)(p-r)\Big) = n\lambda\Big(p - [1-(1-n\lambda)^K]r\Big).$$

Proposition 10 (The Relationship Between Profit and Size n). *Suppose*

$$(\alpha-\gamma)\lambda_I\Big[(1-n\lambda_I)^K - 1 + G(K,\lambda_F)\Big] > \beta\lambda_F\Big[1-(1-n\lambda_F)^K - G(K,\lambda_F)\Big] \tag{6.2}$$

holds.

(a) Suppose $v_H G(K,\lambda_F) - v_L G(K,\lambda_I) \leq v_H - v_L$, then

$$p^* = r^* = \frac{v_L}{1-G(K,\lambda_F)},$$
$$\pi^* = \beta\lambda_F \frac{(1-n\lambda_F)^K}{1-G(K,\lambda_F)} v_L + (\alpha-\gamma)\lambda_I \frac{(1-n\lambda_I)^K}{1-G(K,\lambda_F)} v_L.$$

(b) Suppose $v_H G(K,\lambda_F) - v_L G(K,\lambda_I) > v_H - v_L$, then

$$p^* = \frac{G(K,\lambda_F)v_H - G(K,\lambda_I)v_L}{G(K,\lambda_F) - G(K,\lambda_I)},$$

$$r^* = \frac{v_H - v_L}{G(K,\lambda_F) - G(K,\lambda_I)},$$

$$\pi^* = \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_F \frac{[1-(1-n\lambda_F)^K] - G(K,\lambda_F)}{G(K,\lambda_F) - G(K,\lambda_I)}(v_H - v_L)$$

$$+ (\alpha-\gamma)\lambda_I \frac{G(K,\lambda_F) - [1-(1-n\lambda_I)^K]}{G(K,\lambda_F) - G(K,\lambda_I)}(v_H - v_L).$$

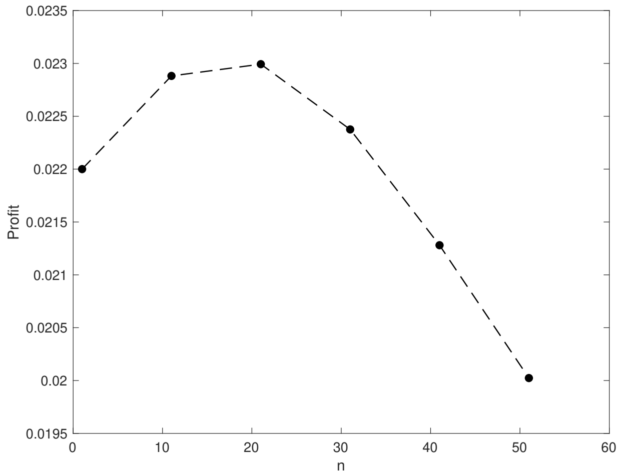


Figure 6.1: The optimal average profit of the coalition with respect to the size n in the discrete-time model.

Figure 6.1 depicts the profit changes with respect to the coalition size n. We set $(v_H, v_L, \lambda_F, \lambda_I, \delta, K) = (20, 10, 0.01, 0.003, 0.9, 12)$ and $(\alpha, \beta, \gamma) = (0.6, 0.2, 0.061)$. Under such conditions, we show that the positive and negative forces of the size n still exist. More specifically, the coalition size has an opposite impact on the price-discrimination power of the reward program in different markets. That is, our main results in the continuous-time model carry over to the discrete-time model.

6.3 Asymmetric Firms

Our main model assumes that the coalition reward program consists of n symmetric firms. In practice, however, each firm in a coalition can bring different number of consumers. For example, a firm with a lower price tends to have higher foot traffic. To check the robustness of our results, we consider asymmetric firms in terms of the arrival rate. For analytical tractability, we assume that there are only two firms (1 and 2), each with different arrival rates $\lambda_1 > \lambda_2$. A natural assumption is that

$p_1 < p_2$.

We suppose that the consumer makes a purchase whenever she visits a firm in the coalition in state 1, and only purchases the product with a lower price in state 0. Then, a generic customer's profit contribution is

$$\begin{aligned}
&\frac{\lambda_1}{\lambda_1+\lambda_2+\mu}(p_1-r)+\frac{\lambda_2}{\lambda_1+\lambda_2+\mu}(p_2-r)+\frac{\mu\lambda_1}{(\lambda_1+\lambda_2+\mu)(\lambda_1+\lambda_2)}p_1\\
=&\frac{\lambda_1}{\lambda_1+\lambda_2}p_1+\frac{\lambda_2}{\lambda_1+\lambda_2+\mu}p_2-\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+\mu}r.
\end{aligned}$$

Recall that the consumer purchase whenever she visits firm 1 or firm 2 in the coalition when the two firms are symmetric. In that case, a generic customer's profit contribution is

$$\begin{aligned}
&\frac{\lambda_1}{\lambda_1+\lambda_2+\mu}(p_1-r)+\frac{\lambda_2}{\lambda_1+\lambda_2+\mu}(p_2-r)\\
&\quad+\frac{\mu\lambda_1}{(\lambda_1+\lambda_2+\mu)(\lambda_1+\lambda_2)}p_1+\frac{\mu\lambda_2}{(\lambda_1+\lambda_2+\mu)(\lambda_1+\lambda_2)}p_2\\
=&\frac{\lambda_1}{\lambda_1+\lambda_2}p_1+\frac{\lambda_2}{\lambda_1+\lambda_2}p_2-\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+\mu}r.
\end{aligned}$$

By comparison, we find that, the coalition incurs a loss (i.e., $\frac{\mu\lambda_2}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_2+\mu)}p_2$) due to the competition between the two firms. Facing asymmetric firms, the consumer' purchase behavior alters when she doesn't hold a valid reward upon her arrival in the coalition. In the long run, firm 2 (with a lower arrival rate) contributes less to the coalition.

It is obvious that taking asymmetry into consideration complicates the model significantly. Since firms can differentiate pricing and thus, consumers' purchase behavior becomes diversified. In general, a firm with a higher λ engages more consumers in the coalition reward program, and then, consumers may be attracted by a firm with a lower λ due to reward instead of a low-price. However, the contribution of each firm doesn't linearly dependent on the arrival rate. It would be interesting to further investigate these problems in an explicit way.

Chapter 7

Conclusions and Suggestions for Future Research

7.1 Conclusion

Our thesis makes two main contributions, which will not only be theoretically interesting but also provide important practical implications to the coalition managers. On the one hand, the effect of the coalition size is prevalent yet little understood in prior literature. Our research fills up the gap by developing an analytical model. Meanwhile, we characterize the time value of rewards by assuming that consumers discount future surpluses at a continuous rate. To the best of our knowledge, we are the first to incorporate consumer discounting and coalition size as the critical features of coalition reward programs. On the other hand, our work suggests that the effectiveness of the coalition reward program is influenced by the market composition. Therefore, the coalition manager should pay attention to consumer purchase behavior. Based on this, the coalition manager can differentiate consumers via the optimal design of the coalition reward program, and thus maximize loyalty and profitability. More importantly, we reveal that simply increases the number of partners in the coalition reward program can be counterproductive. Hence, coalition managers should be more prudent in the partnership portfolio and consumer segmentation.

7.2 Future Research

This thesis has a few limitations that result in several possible future research directions. First, we can further investigate asymmetric firms in a coalition. For example, we can determine under what conditions it is beneficial for a specific firm to join the coalition, how the profit should be distributed between the asymmetric firms, and who will benefit more from joining the coalition. Second, we can incorporate the redemption threshold and reward accumulation in the coalition reward program, which are widely used in the retail industry. We leave these studies for future research.

Appendices

Proofs of the Results in the Main Text

Proof of Proposition 1

We first show if $v - p + u(1) \geq u(0)$, then $v - p + r + u(1) \geq u(1)$. Suppose for a contradiction that $v - p + r + u(1) < u(1)$. Then,

$$u(1) - u(0) = \frac{n\lambda}{\nu}\Big\{u(1) - [v - p + u(1)]\Big\} = \frac{n\lambda}{\nu}(-v + p) > 0,$$

from which we obtain $v < p$. It follows immediately that

$$v - p + u(1) - u(0) = v - p + \frac{n\lambda}{\nu}(-v + p) = \frac{\beta + \mu}{\nu}(v - p) < 0,$$

contradicting our supposition. Hence, it must be the case that $v - p + r + u(1) \geq u(1)$.

Suppose $v - p + u(1) \geq u(0)$. Then, equations (3.1) and (3.2) reduce to

$$u(1) = \frac{n\lambda}{\nu}\Big\{v - p + r + u(1)\Big\} + \frac{\mu}{\nu}u(0),$$

$$u(0) = \frac{n\lambda}{\nu}\Big\{v - p + u(1)\Big\} + \frac{\mu}{\nu}u(0).$$

Solving the above set of equations yields that

$$u(1) = \frac{n\lambda}{\delta}\Big\{v - p + \frac{\delta + n\lambda}{\nu}r\Big\},$$

$$u(0) = \frac{n\lambda}{\delta}\Big\{v - p + \frac{n\lambda}{\nu}r\Big\}.$$

Putting $u(1)$ and $u(0)$ back to the supposition $v - p + u(1) \geq u(0)$ gives that

$$v - p + \frac{n\lambda}{\nu} r \geq 0,$$

under which the consumer will make a purchase whenever she visits a firm in the coalition. This completes the proof.

Proof of Lemma 1

To determine the profit of the coalition reward program from the four homogeneous customer segments, we analyze the relationship between the price p, $v_H + \frac{n\lambda_F}{\nu_F} r$, $v_H + \frac{n\lambda_I}{\nu_I} r$, $v_L + \frac{n\lambda_F}{\nu_F} r$, and $v_L + \frac{n\lambda_I}{\nu_I} r$ to determine the customer's purchase behavior in each segment. Recall that $\nu_F = \delta + n\lambda_F + \mu$ and $\nu_I = \delta + n\lambda_I + \mu$. One can check $v_L + \dfrac{n\lambda_I}{\nu_I} r < \min\left\{v_L + \dfrac{n\lambda_F}{\nu_F} r, v_H + \dfrac{n\lambda_I}{\nu_I} r\right\} < \max\left\{v_L + \dfrac{n\lambda_F}{\nu_F} r, v_H + \dfrac{n\lambda_I}{\nu_I} r\right\} < v_H + \dfrac{n\lambda_F}{\nu_F} r$. There are six cases.

<u>Case 1:</u> $p \leq v_L + \frac{n\lambda_I}{\nu_I} r < \min\left\{v_L + \frac{n\lambda_F}{\nu_F} r, v_H + \frac{n\lambda_I}{\nu_I} r\right\} < \max\left\{v_L + \frac{n\lambda_F}{\nu_F} r, v_H + \frac{n\lambda_I}{\nu_I} r\right\} < v_H + \frac{n\lambda_F}{\nu_F} r$.

All consumers make a purchase. The optimization problem becomes

$$\beta\lambda_F\Big(p - \frac{n\lambda_F}{n\lambda_F + \mu} r\Big) + (1-\beta)\lambda_I\Big(p - \frac{n\lambda_I}{n\lambda_I + \mu} r\Big)$$

$$s.t.\ p \leq v_L + \frac{n\lambda_I}{n\lambda_I + \mu + \delta} r$$

$$p \geq r.$$

Clearly,

$$p^* = v_L + \frac{n\lambda_I}{n\lambda_I + \mu + \delta} r^*,$$

$$\begin{aligned}\pi^* &= \beta\lambda_F\Big(v_L + \frac{n\lambda_I}{n\lambda_I + \mu + \delta} r^* - \frac{n\lambda_F}{n\lambda_F + \mu} r^*\Big) + (1-\beta)\lambda_I\Big(v_L + \frac{n\lambda_I}{n\lambda_I + \mu + \delta} r^* - \frac{n\lambda_I}{n\lambda_I + \mu} r^*\Big) \\ &< \beta\lambda_F v_L + (1-\beta)\lambda_I v_L \\ &= \pi_1,\end{aligned}$$

where the above inequality holds because

$$\frac{n\lambda_I}{n\lambda_I+\mu+\delta} < \frac{n\lambda_I}{n\lambda_I+\mu} < \frac{n\lambda_F}{n\lambda_F+\mu}.$$

Hence, it is never optimal to adopt the coalition reward program in this case.

<u>Case 2:</u> $v_L + \frac{n\lambda_I}{\nu_I} r < p \le \min\left\{v_L + \frac{n\lambda_F}{\nu_F} r, v_H + \frac{n\lambda_I}{\nu_I} r\right\} < \max\left\{v_L + \frac{n\lambda_F}{\nu_F} r, v_H + \frac{n\lambda_I}{\nu_I} r\right\} < v_H + \frac{n\lambda_F}{\nu_F} r$.

All consumers but LI segment make a purchase. The optimization problem becomes

$$\beta\lambda_F\Big(p - \frac{n\lambda_F}{n\lambda_F+\mu} r\Big) + (\alpha-\gamma)\lambda_I\Big(p - \frac{n\lambda_I}{n\lambda_I+\mu} r\Big)$$

$$s.t.\ v_L + \frac{n\lambda_I}{n\lambda_I+\mu+\delta} r < p \le \min\Big\{v_L + \frac{n\lambda_F}{n\lambda_F+\mu+\delta} r, v_H + \frac{n\lambda_I}{n\lambda_I+\mu+\delta} r\Big\}$$

$$p \ge r.$$

Clearly, $p^* = \min\left\{v_L + \frac{n\lambda_F}{n\lambda_F+\mu+\delta} r, v_H + \frac{n\lambda_I}{n\lambda_I+\mu+\delta} r\right\}$. We have two subcases.

<u>Subcase 1:</u> Suppose

$$v_L + \frac{n\lambda_F}{n\lambda_F+\mu+\delta} r \ge v_H + \frac{n\lambda_I}{n\lambda_I+\mu+\delta} r.$$

It follows that $p^* = v_H + \frac{n\lambda_I}{n\lambda_I+\mu+\delta} r$. Reorganizing the supposition yields $r \ge \frac{(v_H-v_L)(\delta+n\lambda_F+\mu)(\delta+n\lambda_I+\mu)}{n(\lambda_F-\lambda_I)(\delta+\mu)}$. Moreover, $p \ge r$ gives $r \le \frac{\delta+n\lambda_I+\mu}{\delta+\mu} v_H$. The profit in this case becomes

$$\beta\lambda_F\Big[v_H + (\frac{n\lambda_I}{n\lambda_I+\mu+\delta} - \frac{n\lambda_F}{n\lambda_F+\mu})r\Big] + (\alpha-\gamma)\lambda_I\Big[v_H + (\frac{n\lambda_I}{n\lambda_I+\mu+\delta} - \frac{n\lambda_I}{n\lambda_I+\mu})r\Big].$$

Note that $\frac{n\lambda_I}{n\lambda_I+\mu+\delta} < \frac{n\lambda_I}{n\lambda_I+\mu} < \frac{n\lambda_F}{n\lambda_F+\mu}$, so the profit decreases in r, and thus

$$r^* = \frac{(v_H-v_L)(\delta+n\lambda_F+\mu)(\delta+n\lambda_I+\mu)}{n(\lambda_F-\lambda_I)(\delta+\mu)},$$

under which

$$v_L + \frac{n\lambda_F}{n\lambda_F+\mu+\delta} r^* = v_H + \frac{n\lambda_I}{n\lambda_I+\mu+\delta} r^*.$$

Hence, the profit in Subcase 1 is dominated by that in Subcase 2.

Subcase 2: Suppose

$$v_L + \frac{n\lambda_F}{n\lambda_F+\mu+\delta} r \le v_H + \frac{n\lambda_I}{n\lambda_I+\mu+\delta} r.$$

It follows that $p^* = v_L + \frac{n\lambda_F}{n\lambda_F+\mu+\delta} r$. Reorganizing the supposition yields $r \le \frac{(v_H-v_L)(\delta+n\lambda_F+\mu)(\delta+n\lambda_I+\mu)}{n(\lambda_F-\lambda_I)(\delta+\mu)}$. Moreover, $p \ge r$ gives $r \le \frac{\delta+n\lambda_F+\mu}{\delta+\mu} v_L$. So,

$$r \le \min\Big\{\frac{(v_H-v_L)(\delta+n\lambda_F+\mu)(\delta+n\lambda_I+\mu)}{n(\lambda_F-\lambda_I)(\delta+\mu)}, \frac{\delta+n\lambda_F+\mu}{\delta+\mu} v_L\Big\}. \tag{A1}$$

The profit in this case becomes

$$\beta\lambda_F\Big[v_L + (\frac{n\lambda_F}{n\lambda_F+\mu+\delta} - \frac{n\lambda_F}{n\lambda_F+\mu})r\Big] + (\alpha-\gamma)\lambda_I\Big[v_L + (\frac{n\lambda_F}{n\lambda_F+\mu+\delta} - \frac{n\lambda_I}{n\lambda_I+\mu})r\Big].$$

Note that if and only if

$$\beta\lambda_F(\frac{n\lambda_F}{n\lambda_F+\mu+\delta} - \frac{n\lambda_F}{n\lambda_F+\mu}) + (\alpha-\gamma)(\frac{n\lambda_F}{n\lambda_F+\mu+\delta} - \frac{n\lambda_I}{n\lambda_I+\mu}) > 0, \tag{A2}$$

$r^* \neq 0$. Moreover, by (A1), we have

$$r = \min\Big\{\frac{(v_H-v_L)(\delta+n\lambda_F+\mu)(\delta+n\lambda_I+\mu)}{n(\lambda_F-\lambda_I)(\delta+\mu)}, \frac{\delta+n\lambda_F+\mu}{\delta+\mu} v_L\Big\}.$$

Reorganizing inequality (A2) gives condition (4.1) in Lemma 1.

Case 3: $v_L + \frac{n\lambda_I}{\nu_I} r < v_L + \frac{n\lambda_F}{\nu_F} r < p \le v_H + \frac{n\lambda_I}{\nu_I} r < v_H + \frac{n\lambda_F}{\nu_F} r.$

Only high-valuation consumers make a purchase. The optimization problem becomes

$$\gamma\lambda_F\Big(p - \frac{n\lambda_F}{n\lambda_F+\mu} r\Big) + (\alpha-\gamma)\lambda_I\Big(p - \frac{n\lambda_I}{n\lambda_I+\mu} r\Big)$$

$$s.t.\ v_L + \frac{n\lambda_F}{n\lambda_F+\mu+\delta} r < p \le v_H + \frac{n\lambda_I}{n\lambda_I+\mu+\delta} r$$

$$p \ge r.$$

Clearly,

$$\begin{aligned} p^* &= v_H + \frac{n\lambda_I}{n\lambda_I+\mu+\delta} r^*, \\ \pi^* &= \gamma\lambda_F\Big(v_H + \frac{n\lambda_I}{n\lambda_I+\mu+\delta} r^* - \frac{n\lambda_F}{n\lambda_F+\mu} r^*\Big) + (\alpha-\gamma)\lambda_I\Big(v_H + \frac{n\lambda_I}{n\lambda_I+\mu+\delta} r^* - \frac{n\lambda_I}{n\lambda_I+\mu} r^*\Big) \\ &< \gamma\lambda_F v_H + (\alpha-\gamma)\lambda_I v_H \\ &= \pi_2, \end{aligned}$$

where the above inequality holds because

$$\frac{n\lambda_I}{n\lambda_I+\mu+\delta} < \frac{n\lambda_I}{n\lambda_I+\mu} < \frac{n\lambda_F}{n\lambda_F+\mu}.$$

Hence, it is never optimal to adopt the coalition reward program in this case.

Case 4: $v_L + \frac{n\lambda_I}{\nu_I}r < v_H + \frac{n\lambda_I}{\nu_I}r < p \leq v_L + \frac{n\lambda_F}{\nu_F}r < v_H + \frac{n\lambda_F}{\nu_F}r.$

Only frequent consumers make a purchase. The optimization problem becomes

$$\begin{aligned}
&\beta\lambda_F\Big(p - \frac{n\lambda_F}{n\lambda_F+\mu}r\Big)\\
s.t.\ &v_H + \frac{n\lambda_I}{n\lambda_I+\mu+\delta}r < p \leq v_L + \frac{n\lambda_F}{n\lambda_F+\mu+\delta}r\\
&p \geq r.
\end{aligned}$$

Clearly,

$$\begin{aligned}
p^* &= v_L + \frac{n\lambda_F}{n\lambda_F+\mu+\delta}r^*,\\
\pi^* &= \beta\lambda_F\Big[v_L + (\frac{n\lambda_F}{n\lambda_F+\mu+\delta} - \frac{n\lambda_F}{n\lambda_F+\mu})r^*\Big] < \beta\lambda_F v_L < \pi_1.
\end{aligned}$$

Hence, it is never optimal to adopt the coalition reward program in this case.

Case 5: $v_L + \frac{n\lambda_I}{\nu_I}r < \min\Big\{v_L + \frac{n\lambda_F}{\nu_F}r, v_H + \frac{n\lambda_I}{\nu_I}r\Big\} < \max\Big\{v_L + \frac{n\lambda_F}{\nu_F}r, v_H + \frac{n\lambda_I}{\nu_I}r\Big\} < p \leq v_H + \frac{n\lambda_F}{\nu_F}r.$

Only high-valuation frequent consumers make a purchase. The optimization problem becomes

$$\begin{aligned}
&\gamma\lambda_F\Big(p - \frac{n\lambda_F}{n\lambda_F+\mu}r\Big)\\
s.t.\ &p \leq v_H + \frac{n\lambda_F}{n\lambda_F+\mu+\delta}r.\\
&p \geq r.
\end{aligned}$$

Clearly,

$$p^* = v_H + \frac{n\lambda_F}{n\lambda_F + \mu + \delta} r^*,$$

$$\pi^* = \gamma\lambda_F \Big[v_H + (\frac{n\lambda_F}{n\lambda_F + \mu + \delta} - \frac{n\lambda_F}{n\lambda_F + \mu}) r^* \Big] < \gamma\lambda_F v_H < \pi_2.$$

Hence, it is never optimal to adopt the coalition reward program in this case.

Case 6: $v_L + \frac{n\lambda_I}{\nu_I} r < \min\Big\{ v_L + \frac{n\lambda_F}{\nu_F} r, v_H + \frac{n\lambda_I}{\nu_I} r \Big\} < \max\Big\{ v_L + \frac{n\lambda_F}{\nu_F} r, v_H + \frac{n\lambda_I}{\nu_I} r \Big\} < v_H + \frac{n\lambda_F}{\nu_F} r < p.$

No consumers make a purchase. Clearly, it is not optimal to adopt the coalition reward program in this case.

Taking the above six cases into consideration, we see that only Subcase 2 in Case 2 is possibly optimal. This completes the proof.

Proof of Proposition 2

Part (a): One can check when either $\lambda_F v_L \leq \lambda_I v_H$ or $\lambda_F v_L > \lambda_I v_H$ but $n \leq \frac{(v_H - v_L)(\delta + \mu)}{v_L \lambda_F - v_H \lambda_I}$ holds,

$$\frac{\delta + n\lambda_F + \mu}{\delta + \mu} v_L \leq \frac{(v_H - v_L)(\delta + n\lambda_F + \mu)(\delta + n\lambda_I + \mu)}{n(\lambda_F - \lambda_I)(\delta + \mu)}.$$

Hence, according to Lemma 1,

$$r^* = \frac{\delta + n\lambda_F + \mu}{\delta + \mu} v_L,$$

$$p^* = v_L + \frac{n\lambda_F}{n\lambda_F + \delta + \mu} \frac{\delta + n\lambda_F + \mu}{\delta + \mu} v_L = \frac{\delta + \mu + n\lambda_F}{\delta + \mu} v_L = r^*,$$

$$\begin{aligned}\pi^* &= \beta\lambda_F \Big[v_L + (\frac{n\lambda_F}{n\lambda_F + \mu + \delta} - \frac{n\lambda_F}{n\lambda_F + \mu}) \frac{\delta + n\lambda_F + \mu}{\delta + \mu} v_L \Big] \\ &\quad + (\alpha - \gamma)\lambda_I \Big[v_L + (\frac{n\lambda_F}{n\lambda_F + \mu + \delta} - \frac{n\lambda_I}{n\lambda_I + \mu}) \frac{\delta + n\lambda_F + \mu}{\delta + \mu} v_L \Big] \\ &= \beta\lambda_F v_L + (\alpha - \gamma)\lambda_I v_L - \beta\lambda_F \frac{n\lambda_F \delta}{n\lambda_F + \mu} \frac{v_L}{\delta + \mu} + (\alpha - \gamma)\lambda_I \frac{n[\lambda_F \mu - \lambda_I(\delta + \mu)]}{n\lambda_I + \mu} \frac{v_L}{\delta + \mu}.\end{aligned}$$

Taking derivative to π^* yields

$$\begin{aligned}
\frac{d\pi^*}{dn} &= -\frac{v_L}{\delta+\mu}\Big\{\beta\lambda_F\frac{\lambda_F\delta\mu}{(n\lambda_F+\mu)^2} - (\alpha-\gamma)\lambda_I\frac{\mu[\lambda_F\mu-\lambda_I(\delta+\mu)]}{(n\lambda_I+\mu)^2}\Big\} \\
&= -\frac{v_L\mu}{\delta+\mu}\frac{\beta\lambda_F^2\delta(n\lambda_I+\mu)^2-(\alpha-\gamma)\lambda_I[\lambda_F\mu-\lambda_I(\delta+\mu)](n\lambda_F+\mu)^2}{(n\lambda_F+\mu)^2(n\lambda_I+\mu)^2} \\
&= \frac{v_L\mu}{\delta+\mu}\frac{(\alpha-\gamma)\lambda_I[\lambda_F\mu-\lambda_I(\delta+\mu)](n\lambda_F+\mu)^2-\beta\lambda_F^2\delta(n\lambda_I+\mu)^2}{(n\lambda_F+\mu)^2(n\lambda_I+\mu)^2} \\
&= \frac{v_L\mu}{\delta+\mu}\frac{(\alpha-\gamma)\lambda_I[\lambda_F\mu-\lambda_I(\delta+\mu)](n\lambda_F+\mu)-\beta\lambda_F^2\delta(n\lambda_I+\mu)\frac{n\lambda_I+\mu}{n\lambda_F+\mu}}{(n\lambda_F+\mu)(n\lambda_I+\mu)^2} \\
&> \frac{v_L\mu}{\delta+\mu}\frac{(\alpha-\gamma)\lambda_I[\lambda_F\mu-\lambda_I(\delta+\mu)](n\lambda_F+\mu)-\beta\lambda_F^2\delta(n\lambda_I+\mu)}{(n\lambda_F+\mu)(n\lambda_I+\mu)^2} \\
&> 0,
\end{aligned}$$

where the last inequality holds because of condition (4.1). Hence, the profit π^* increases in n in this case.

Part (b): One can check when $\lambda_F v_L > \lambda_I v_H$ and $n \geq \frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F-v_H\lambda_I}$ hold,

$$\frac{\delta+n\lambda_F+\mu}{\delta+\mu}v_L \geq \frac{(v_H-v_L)(\delta+n\lambda_F+\mu)(\delta+n\lambda_I+\mu)}{n(\lambda_F-\lambda_I)(\delta+\mu)}.$$

Hence, according to Lemma 1,

$$\begin{aligned}
r^* &= \frac{(v_H-v_L)(\delta+n\lambda_F+\mu)(\delta+n\lambda_I+\mu)}{n(\lambda_F-\lambda_I)(\delta+\mu)}, \\
p^* &= v_L + \frac{n\lambda_F}{n\lambda_F+\delta+\mu}\frac{(v_H-v_L)(\delta+n\lambda_F+\mu)(\delta+n\lambda_I+\mu)}{n(\lambda_F-\lambda_I)(\delta+\mu)} \\
&= \frac{\lambda_F(\delta+\mu+n\lambda_I)v_H-\lambda_I(\delta+\mu+n\lambda_F)v_L}{(\lambda_F-\lambda_I)(\delta+\mu)}, \\
\pi^* &= \beta\lambda_F\Big[v_L+\big(\frac{n\lambda_F}{n\lambda_F+\mu+\delta}-\frac{n\lambda_F}{n\lambda_F+\mu}\big)\frac{(v_H-v_L)(\delta+n\lambda_F+\mu)(\delta+n\lambda_I+\mu)}{n(\lambda_F-\lambda_I)(\delta+\mu)}\Big] \\
&\quad + (\alpha-\gamma)\lambda_I\Big[v_L+\big(\frac{n\lambda_F}{n\lambda_F+\mu+\delta}-\frac{n\lambda_I}{n\lambda_I+\mu}\big)\frac{(v_H-v_L)(\delta+n\lambda_F+\mu)(\delta+n\lambda_I+\mu)}{n(\lambda_F-\lambda_I)(\delta+\mu)}\Big] \\
&= \beta\lambda_F v_L+(\alpha-\gamma)\lambda_I v_L-\beta\lambda_F\frac{\delta+n\lambda_I+\mu}{n\lambda_F+\mu}\frac{\lambda_F\delta(v_H-v_L)}{(\delta+\mu)(\lambda_F-\lambda_I)} \\
&\quad + (\alpha-\gamma)\lambda_I\frac{[\lambda_F\mu-\lambda_I(\delta+\mu)](v_H-v_L)}{(\delta+\mu)(\lambda_F-\lambda_I)}\frac{\delta+n\lambda_I+\mu}{n\lambda_I+\mu}.
\end{aligned}$$

Taking derivative to π^* yields

$$\begin{aligned}\frac{d\pi^*}{dn} &= -\frac{v_H - v_L}{(\lambda_F - \lambda_I)(\delta + \mu)}\Big\{\beta\lambda_F^2\delta\frac{\lambda_I\mu - \lambda_F(\delta+\mu)}{(n\lambda_F + \mu)^2} - (\alpha - \gamma)\lambda_I[\lambda_F\mu - \lambda_I(\delta+\mu)]\frac{-\lambda_I\delta}{(n\lambda_I + \mu)^2}\Big\} \\ &= \frac{(v_H - v_L)\delta}{(\lambda_F - \lambda_I)(\delta + \mu)}\Big\{\beta\lambda_F^2\frac{\lambda_F(\delta+\mu) - \lambda_I\mu}{(n\lambda_F + \mu)^2} - (\alpha - \gamma)\lambda_I^2\frac{\lambda_F\mu - \lambda_I(\delta+\mu)}{(n\lambda_I + \mu)^2}\Big\}.\end{aligned}$$

(i) Suppose $1 < \frac{(\alpha-\gamma)[\lambda_F\mu - \lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu) - \lambda_I\mu]} < \left(\frac{\lambda_F}{\lambda_I}\right)^2$, then the definition of $\hat{n}$ implies that $\hat{n}$ is positive. One can check that if $n = \hat{n}$, $\frac{d\pi^*}{dn} = 0$; if $n < \hat{n}$, $\frac{d\pi^*}{dn} > 0$; if $n > \hat{n}$, $\frac{d\pi^*}{dn} < 0$. Therefore, the profit π^* first increases and then decreases in n.

(ii) Suppose $\frac{(\alpha-\gamma)[\lambda_F\mu - \lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu) - \lambda_I\mu]} \leq 1$. Then,

$$(\alpha - \gamma)[\lambda_F\mu - \lambda_I(\delta+\mu)] \leq \beta[\lambda_F(\delta+\mu) - \lambda_I\mu].$$

Note that

$$\frac{\lambda_I^2}{(n\lambda_I + \mu)^2} - \frac{\lambda_F^2}{(n\lambda_F + \mu)^2} = \frac{2n\lambda_F\lambda_I\mu(\lambda_I - \lambda_F) + (\lambda_I^2 - \lambda_F^2)\mu^2}{(n\lambda_I + \mu)^2(n\lambda_F + \mu)^2} < 0,$$

so,

$$(\alpha - \gamma)\lambda_I^2\frac{\lambda_F\mu - \lambda_I(\delta+\mu)}{(n\lambda_I + \mu)^2} \leq \beta\lambda_F^2\frac{\lambda_F(\delta+\mu) - \lambda_I\mu}{(n\lambda_F + \mu)^2}.$$

Hence, $\frac{d\pi^*}{dn} \geq 0$ for any n.

(iii) Suppose $\frac{(\alpha-\gamma)[\lambda_F\mu - \lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu) - \lambda_I\mu]} \geq \left(\frac{\lambda_F}{\lambda_I}\right)^2$. Then,

$$(\alpha - \gamma)\lambda_I^2[\lambda_F\mu - \lambda_I(\delta+\mu)] \geq \beta\lambda_F^2[\lambda_F(\delta+\mu) - \lambda_I\mu].$$

Note that $\frac{1}{(n\lambda_I+\mu)^2} > \frac{1}{(n\lambda_F+\mu)^2}$ so,

$$(\alpha - \gamma)\lambda_I^2\frac{\lambda_F\mu - \lambda_I(\delta+\mu)}{(n\lambda_I + \mu)^2} > \beta\lambda_F^2\frac{\lambda_F(\delta+\mu) - \lambda_I\mu}{(n\lambda_F + \mu)^2}.$$

Hence, $\frac{d\pi^*}{dn} < 0$ for any n. This completes the proof.

Proof of Proposition 3

Part (a): Suppose $\lambda_F v_L \leq \lambda_I v_H$. Proposition 2(a) tells us that

$$\pi^* = \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_F \frac{n\lambda_F\delta}{n\lambda_F+\mu}\frac{v_L}{\delta+\mu} + (\alpha-\gamma)\lambda_I \frac{n[\lambda_F\mu - \lambda_I(\delta+\mu)]}{n\lambda_I+\mu}\frac{v_L}{\delta+\mu}$$
$$= \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_F \frac{\lambda_F\delta}{\lambda_F+\frac{\mu}{n}}\frac{v_L}{\delta+\mu} + (\alpha-\gamma)\lambda_I \frac{\lambda_F\mu - \lambda_I(\delta+\mu)}{\lambda_I+\frac{\mu}{n}}\frac{v_L}{\delta+\mu}.$$

So,

$$\lim_{n\to\infty}\pi^* = \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_F \frac{v_L\delta}{\delta+\mu} + (\alpha-\gamma)\lambda_I \frac{\lambda_F\mu - \lambda_I(\delta+\mu)}{\lambda_I}\frac{v_L}{\delta+\mu}$$
$$= \beta\lambda_F v_L \frac{\mu}{\delta+\mu} + (\alpha-\gamma)v_L\lambda_F\frac{\mu}{\delta+\mu}$$
$$< \beta\lambda_F v_L \frac{\mu}{\delta+\mu} + (\alpha-\gamma)v_H\lambda_I\frac{\mu}{\delta+\mu} \qquad [\text{by } v_L\lambda_F \leq v_H\lambda_I]$$
$$= \frac{\mu}{\mu+\delta}\Big(\beta\lambda_F v_L + (\alpha-\gamma)v_H\lambda_I\Big).$$

Part (b): Suppose $\lambda_F v_L > \lambda_I v_H$. Note that the profit in Proposition 2(a) is equal to that in Proposition 2(b) when $n = \frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F - v_H\lambda_I}$.

If $1 < \frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]} < \left(\frac{\lambda_F}{\lambda_I}\right)^2$, then $\hat{n}$ is positive. By Proposition 2, if $\hat{n} \geq \frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F - v_H\lambda_I}$, then $n^* = \hat{n}$. If $\hat{n} < \frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F - v_H\lambda_I}$, then $n^* = \frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F - v_H\lambda_I}$.

If $\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]} \leq 1$, then the profit keeps increasing in n for any n.

If $\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]} \geq \left(\frac{\lambda_F}{\lambda_I}\right)^2$, the profit first increases in n when $n \leq \frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F - v_H\lambda_I}$ and decreases in n otherwise, so, the optimal size $n^* = \frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F - v_H\lambda_I}$.

This completes the proof.

Proof of Proposition 4

Case 1: Suppose $v_L\lambda_F - v_H\lambda_I > 0$.

Subcase 1: Suppose $\mu < \frac{(v_L\lambda_F - v_H\lambda_I)n}{v_H - v_L} - \delta$. Note that it is equivalent to $n \geq \frac{(v_H - v_L)(\delta+\mu)}{v_L\lambda_F - v_H\lambda_I}$. Then, according to Proposition 2(b), the profit, denoted by π_3^1, is as follows,

$$\pi_3^1 = \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_F \frac{\delta + n\lambda_I + \mu}{n\lambda_F + \mu} \frac{\lambda_F \delta (v_H - v_L)}{(\delta+\mu)(\lambda_F - \lambda_I)}$$

$$+ (\alpha-\gamma)\lambda_I \frac{[\lambda_F\mu - \lambda_I(\delta+\mu)](v_H - v_L)}{(\delta+\mu)(\lambda_F - \lambda_I)} \frac{\delta + n\lambda_I + \mu}{n\lambda_I + \mu}.$$

One can verify that $\frac{\delta + n\lambda_I + \mu}{(n\lambda_F + \mu)(\delta+\mu)}$ is decreasing in μ and $\frac{\lambda_F\mu - \lambda_I(\delta+\mu)}{\delta+\mu}\frac{\delta+n\lambda_I+\mu}{n\lambda_I+\mu}$ is increasing in μ. Therefore, the profit π_3^1 is increasing in μ.

Subcase 2: Suppose $\mu \geq \frac{(v_L\lambda_F - v_H\lambda_I)n}{v_H - v_L} - \delta$. Hence, according to Proposition 2(a), the profit, denoted by π_3^2, is as follows,

$$\pi_3^2 = \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_F \frac{n\lambda_F\delta}{n\lambda_F + \mu}\frac{v_L}{\delta+\mu} + (\alpha-\gamma)\lambda_I \frac{n[\lambda_F\mu - \lambda_I(\delta+\mu)]}{n\lambda_I + \mu}\frac{v_L}{\delta+\mu}.$$

Taking derivative with respect to μ yields

$$\begin{aligned}\frac{d\pi_3^2}{d\mu} &= \beta\lambda_F n\lambda_F\delta v_L \frac{n\lambda_F + 2\mu + \delta}{(n\lambda_F+\mu)^2(\delta+\mu)^2} + (\alpha-\gamma)\lambda_I n v_L \frac{\lambda_F\delta n\lambda_I - \lambda_F\mu^2 + \lambda_I(\delta+\mu)^2}{(n\lambda_I+\mu)^2(\delta+\mu)^2}\\ &= \frac{nv_L}{(\delta+\mu)^2}\Big\{\frac{\beta\lambda_F^2\delta(n\lambda_F+2\mu+\delta)(n\lambda_I+\mu)^2 - (\alpha-\gamma)\lambda_I\big[\lambda_F\mu^2 - \lambda_I(\delta+\mu)^2 - n\lambda_F\lambda_I\delta\big](n\lambda_F+\mu)^2}{(n\lambda_F+\mu)^2(n\lambda_I+\mu)^2}\Big\}.\end{aligned}$$

Let $f(\mu) = \beta\lambda_F^2\delta(n\lambda_F + 2\mu + \delta)(n\lambda_I+\mu)^2 - (\alpha-\gamma)\lambda_I\big[\lambda_F\mu^2 - \lambda_I(\delta+\mu)^2 - n\lambda_F\lambda_I\delta\big]$. One can verify that $f(0) = \beta\lambda_F^2\delta(n\lambda_F+\delta)(n\lambda_I)^2 - (\alpha-\gamma)\lambda_I\big[-\lambda_I(\delta)^2 - n\lambda_F\lambda_I\delta\big] > 0$ and $f(+\infty) < 0$.

Taking the first derivative with respect to μ yields
$f'(\mu) = 2\{\beta\lambda_F^2\delta(n\lambda_I+\mu)\big[(n\lambda_F+2\mu+\delta)+(n\lambda_I+\mu)\big] - (\alpha-\gamma)\lambda_I(n\lambda_F+\mu)\big[\lambda_F\mu^2 - \lambda_I(\delta+\mu)^2 - n\lambda_F\lambda_I\delta + (n\lambda_F+\mu)[\lambda_F\mu - \lambda_I(\delta+\mu)]\big]\}$.
One can verify that $f'(0) = 2\{\beta\lambda_F^2\delta n\lambda_I\big[(n\lambda_F+\delta+n\lambda_I)\big] - (\alpha-\gamma)\lambda_I n\lambda_F(-\lambda_I(\delta)^2 - n\lambda_F\lambda_I\delta)\} > 0$ and $f'(+\infty) < 0$.

Taking the second derivative with respect to μ yields
$f"(\mu) = 2\{\beta\lambda_F^2\delta(n\lambda_F+6\mu+\delta) + (\alpha-\gamma)\lambda_I^2\big[\delta^2 + 6\delta\mu + 6\mu^2 + n\lambda_F(n\lambda_F+6\mu+5\delta)\big] - \lambda_I\lambda_F\big[(\alpha-\gamma)6\mu^2 + n\lambda_F[(\alpha-\gamma)(6\mu+n\lambda_F) - \beta4\delta)\big]\}$.

One can verify that $f^{"}(0) = 2\{\beta\lambda_F^2\delta(n\lambda_F+\delta) + (\alpha-\gamma)\lambda_I^2\big[\delta^2 + n\lambda_F(n\lambda_F+5\delta)\big] - \lambda_I\lambda_F n\lambda_F\big[(\alpha-\gamma)n\lambda_F - \beta 4\delta\big]\}$ and $f^{"}(+\infty) < 0$.

Taking the third derivative with respect to μ yields
$f^{(3)}(\mu) = 12\{\beta\lambda_F^2\delta - (\alpha-\gamma)\lambda_I\big[(n\lambda_F+2\mu)(\lambda_F-\lambda_I) - \delta\lambda_I\big]\}$.
Note that $f^{(3)}(\mu)$ is monotonically decreasing in μ and $f^{(3)}(0) = 12\{\beta\lambda_F^2\delta - (\alpha-\gamma)\lambda_I\big[n\lambda_F(\lambda_F-\lambda_I) - \delta\lambda_I\big]\}$.

(i) Suppose $f^{(3)}(0) \leq 0$ holds. For all $\mu > 0$, $f^{(3)}(\mu) < 0$, which implies that $f^{"}(\mu)$ is decreasing in μ. Suppose $f^{"}(0) \leq 0$ also holds, then $f^{"}(\mu) < 0$ implies that $f'(\mu)$ is decreasing in μ.
Note that $f'(0) > 0$ and $f'(+\infty) < 0$. There must exist and only exist one point (say $\dot{\mu}$) where $f'(\dot{\mu}) = 0$. Then, $f'(\mu)$ is increasing in μ if $\mu < \dot{\mu}$ and $f'(\mu)$ is decreasing in μ otherwise.
Note that $f^(0) > 0$ and $f^(+ \infty) < 0$. There must exist and only exist one point (say $\hat{\mu}$) where $f(\hat{\mu}) = 0$. Therefore, $f(\mu)$ is increasing in μ if $\mu < \hat{\mu}$ and $f(\mu)$ is decreasing in μ otherwise.

(ii) Suppose $f^{(3)}(0) \leq 0$ holds. For all $\mu > 0$, $f^{(3)}(\mu) < 0$, which implies that $f^{"}(\mu)$ is decreasing in μ. Suppose $f^{"}(0) \leq 0$ doesn't hold, i.e., $f^{"}(0) > 0$. Thus, $f'(\mu)$ first increases and then decreases in μ.
Note that $f'(0) > 0$ and $f'(+\infty) < 0$. There must exist and only exist one point (say $\dot{\mu}$) where $f'(\dot{\mu}) = 0$. Then, $f'(\mu)$ is increasing in μ if $\mu < \dot{\mu}$ and $f'(\mu)$ is decreasing in μ otherwise.
Note that $f^(0) > 0$ and $f^(+ \infty) < 0$. There must exist and only exist one point (say $\hat{\mu}$) where $f(\hat{\mu}) = 0$. Therefore, $f(\mu)$ is increasing in μ if $\mu < \hat{\mu}$ and $f(\mu)$ is decreasing in μ otherwise.

(iii) Suppose $f^{(3)}(0) \leq 0$ doesn't hold, i.e., $f^{(3)}(0) > 0$. Thus, $f^{"}(\mu)$ first increases and then decreases in μ. Suppose $f^{"}(0) \leq 0$ doesn't hold, i.e., $f^{"}(0) > 0$. Thus, $f'(\mu)$ first increases and then decreases in μ.

Note that $f'(0) > 0$ and $f'(+\infty) < 0$. There must exist and only exist one point (say $\dot{\mu}$) where $f'(\dot{\mu}) = 0$. Then, $f'(\mu)$ is increasing in μ if $\mu < \dot{\mu}$ and $f'(\mu)$ is decreasing in μ otherwise.
Note that $f^{(}0) > 0$ and $f^{(} + \infty) < 0$. There must exist and only exist one point (say $\hat{\mu}$) where $f(\hat{\mu}) = 0$. Therefore, $f(\mu)$ is increasing in μ if $\mu < \hat{\mu}$ and $f(\mu)$ is decreasing in μ otherwise.

One can check that $f^{(3)}(0) \leq 0$ must hold when $f^{"}(0) \leq 0$. Therefore, taking the above three scenarios into consideration, we show the existence and uniqueness of $\hat{\mu}$.

By the definition of $\hat{\mu}$, we have $\frac{d\pi_3^2}{d\mu} > 0$ when $\mu < \hat{\mu}$ and $\frac{d\pi_3^2}{d\mu} < 0$ otherwise. Hence, if $\hat{\mu} \geq \frac{(v_L\lambda_F - v_H\lambda_I)n}{v_H - v_L} - \delta$, then the profit π_3^2 increases in μ when $\mu < \hat{\mu}$ and decreases in μ otherwise. If $\hat{\mu} < \frac{(v_L\lambda_F - v_H\lambda_I)n}{v_H - v_L} - \delta$, then the profit π_3^2 decreases in μ.

Note that $\pi_3^1 = \pi_3^2$ when $\mu = \frac{(v_L\lambda_F - v_H\lambda_I)n}{v_H - v_L} - \delta$, so combining Subcases 1 and 2 establishes that the optimal profit increases in μ when $\mu < \max\{\hat{\mu}, \frac{(v_L\lambda_F - v_H\lambda_I)n}{v_H - v_L} - \delta\}$ and decreases in μ otherwise.

Case 2: Suppose $v_L\lambda_F - v_H\lambda_I \leq 0$.

In this case, the profit is equal to π_3^2. Hence, the subsequent analysis follows the same as Subcase 2 in Case 1. Because $\hat{\mu} \geq \frac{(v_L\lambda_F - v_H\lambda_I)n}{v_H - v_L} - \delta$ in this case, the profit increases in μ when $\mu < \hat{\mu}$ and decreases in μ otherwise. This completes the proof.

Proof of Proposition 5

It suffices to show $\hat{\mu}(n)$ increases in n. Note that $\hat{\mu}(n)$ is derived by letting $\frac{d\pi_3^2}{d\mu} = 0$. We expect to show π_3^2 is supermodular in (μ, n). Recall that

$$\pi_3^2 = \beta\lambda_F v_L\Big(1 - \frac{n\lambda_F}{n\lambda_F + \mu}\frac{\delta}{\delta + \mu}\Big) + (\alpha - \gamma)\lambda_I\Big(v_L + \frac{n[\lambda_F\mu - \lambda_I(\delta + \mu)]}{(n\lambda_I + \mu)(\delta + \mu)}\Big).$$

For the first term, taking derivative with respect to μ yields

$$\beta\lambda_F v_L n\lambda_F\delta\frac{n\lambda_F + 2\mu + \delta}{(n\lambda_F + \mu)^2(\delta + \mu)^2}.$$

Then, we continue to take derivative with respect to n and obtain

$$\frac{\beta\lambda_F^2\delta v_L}{(\delta+\mu)^2}\frac{1}{(n\lambda_F+\mu)^4}\Big\{(2n\lambda_F+2\mu+\delta)(n\lambda_F+\mu)^2-2(n^2\lambda_F+2n\mu+n\delta)(n\lambda_F+\mu)\lambda_F\Big\}$$

$$=\frac{\beta\lambda_F^2\delta v_L}{(\delta+\mu)^2}\frac{1}{(n\lambda_F+\mu)^4}\Big\{2n\mu^2\lambda_F+2\mu^3+\delta\mu^2-\delta n^2\lambda_F^2\Big\}$$

$$>0,$$

where the last inequality holds because δ is sufficiently small (i.e., $\delta n\lambda_F \le 2\mu^2$). This indicates that the first term in π_3^2 is supermodular in (μ, n).

For the second term, taking derivative with respect to μ yields

$$(\alpha-\gamma)\lambda_I v_L n\frac{\lambda_F\delta n\lambda_I-\lambda_F\mu^2+\lambda_I(\delta+\mu)^2}{(n\lambda_I+\mu)^2(\delta+\mu)^2}.$$

Then, we continue to take derivative with respect to n and obtain

$$\frac{(\alpha-\gamma)\lambda_I v_L}{(\delta+\mu)^2(n\lambda_I+\mu)^4}\Big\{\Big[2n\lambda_F\lambda_I\delta-\lambda_F\mu^2+\lambda_I(\delta+\mu)^2\Big](n\lambda_I+\mu)^2 \tag{A3}$$
$$-\Big[n^2\lambda_F\lambda_I\delta-n\lambda_F\mu^2+n\lambda_I(\delta+\mu)^2\Big]2\lambda_I(n\lambda_I+\mu)\Big\}$$

$$=\frac{(\alpha-\gamma)\lambda_I v_L}{(\delta+\mu)^2(n\lambda_I+\mu)^3}\Big\{2n\lambda_F\lambda_I\delta\mu+n\lambda_F\lambda_I\mu^2+\mu\lambda_I(\delta+\mu)^2-\lambda_F\mu^3-n\lambda_I^2(\delta+\mu)^2\Big\}$$

$$=\frac{(\alpha-\gamma)\lambda_I v_L}{(\delta+\mu)^2(n\lambda_I+\mu)^3}\Big\{2n\lambda_F\lambda_I\delta\mu+(n\lambda_I-\mu)[\lambda_F\mu^2-(\delta+\mu)^2\lambda_I]\Big\}. \tag{A4}$$

Now, let us take a detour to derive some properties that will be used to show (A4) ≥ 0. One can check that $\lambda_F\delta n\lambda_I-\lambda_F\mu^2+\lambda_I(\delta+\mu)^2$ (i) decreases in μ; (2) is positive when μ is sufficiently small; (3) is negative when μ is sufficiently large. Letting $\lambda_F\delta n\lambda_I-\lambda_F\mu^2+\lambda_I(\delta+\mu)^2=0$ gives

$$\tilde{\mu}=\frac{\lambda_I\delta+\sqrt{\lambda_I^2\delta^2+(\lambda_F-\lambda_I)(\lambda_F\delta n\lambda_I+\lambda\delta^2)}}{\lambda_F-\lambda_I}.$$

Therefore, the second term in π_3^2 increases in μ when $\mu<\tilde{\mu}$ and decreases in μ otherwise. According to the definition of $\hat{\mu}$, we must have $\hat{\mu}>\tilde{\mu}$. Moreover, for any $\mu>\hat{\mu}$, we have $\lambda_F\delta n\lambda_I-\lambda_F\mu^2+\lambda_I(\delta+\mu)^2<0$, which implies

$$\lambda_F\mu^2-(\delta+\mu)^2\lambda_I>\lambda_F\delta n\lambda_I. \tag{A5}$$

Putting (A5) back to (A4) yields that for any $\mu > \tilde{\mu}$,

$$\begin{aligned}(\text{A4}) &\geq \frac{(\alpha-\gamma)\lambda_I v_L}{(\delta+\mu)^2(n\lambda_I+\mu)^3}\Big\{2n\lambda_F\lambda_I\delta\mu + (n\lambda_I-\mu)n\lambda_F\lambda_I\delta\Big\} \\ &= \frac{(\alpha-\gamma)\lambda_I v_L}{(\delta+\mu)^2(n\lambda_I+\mu)^3}\Big\{n\lambda_F\lambda_I\delta(\mu+n\lambda_I)\Big\} \geq 0,\end{aligned}$$

indicating that the second term in π_3^2 is supermodular in (μ, n) for any $\mu \geq \tilde{\mu}$.

Therefore, π_3^2 is supermodular in (μ, n) for any $\mu \geq \tilde{\mu}$. Because $\hat{\mu} > \tilde{\mu}$, it follows immediately that $\hat{\mu}(n)$ increases in n. This completes the proof.

Proof of Proposition 6

Part (a): We first show an upper bound of π^*.

Suppose $v_L\lambda_F \leq v_H\lambda_I$. By Proposition 2(a),

$$\begin{aligned}\pi^* &= \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_F\frac{n\lambda_F\delta}{n\lambda_F+\mu}\frac{v_L}{\delta+\mu} + (\alpha-\gamma)\lambda_I\frac{n[\lambda_F\mu-\lambda_I(\delta+\mu)]}{n\lambda_I+\mu}\frac{v_L}{\delta+\mu} \\ &< \frac{\mu}{\delta+\mu}\Big\{\beta\lambda_F v_L + (\alpha-\gamma)\lambda_F v_L\Big\} \\ &< \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_H,\end{aligned}$$

where the first inequality holds because π^* increases in n and the second inequality holds because $v_L\lambda_F \leq v_H\lambda_I$.

Suppose $v_L\lambda_F > v_H\lambda_I$. By Proposition 2(b),

$$\begin{aligned}\pi^* = \beta\lambda_F\Big\{v_L - \frac{\delta+n\lambda_I+\mu}{n\lambda_F+\mu}\frac{\lambda_F\delta(v_H-v_L)}{(\delta+\mu)(\lambda_F-\lambda_I)}\Big\} + (\alpha-\gamma)\lambda_I\Big\{v_L \\ + \frac{[\lambda_F\mu-\lambda_I(\delta+\mu)](v_H-v_L)}{(\delta+\mu)(\lambda_F-\lambda_I)}\frac{\delta+n\lambda_I+\mu}{n\lambda_I+\mu}\Big\} \\ < \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I[v_L+(v_H-v_L)] \\ = \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_H,\end{aligned}$$

where the last inequality holds because

$$[\lambda_F\mu - \lambda_I(\delta+\mu)](\delta+n\lambda_I+\mu) - (\delta+\mu)(\lambda_F-\lambda_I)(n\lambda_I+\mu) = -\lambda_F\delta n\lambda_I - \lambda_I(\delta+\mu)\delta < 0.$$

Now, we show $\pi^* \leq \min\{\pi_1, \pi_2\}$ when $\gamma \geq \alpha\beta$.

$$\begin{aligned}\pi^* - \pi_1 <& \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_H - [\beta\lambda_F + (1-\beta)\lambda_I]v_L = [(\alpha-\gamma)v_H - (1-\beta)v_L]\lambda_I \\ <& [\alpha(1-\beta)v_H - (1-\beta)v_L]\lambda_I = (1-\beta)(\alpha v_H - v_L)\lambda_I.\end{aligned}$$

Also,

$$\pi^* - \pi_2 < \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_H - [\gamma\lambda_F + (\alpha-\gamma)\lambda_I]v_H = [\beta v_L - \gamma v_H]\lambda_F < \beta(v_L - \alpha v_H)\lambda_F.$$

Observe that if $\alpha v_H \geq v_L$, then $\pi^* < \pi_2$. Otherwise, $\pi^* < \pi_1$. Hence, $\pi^* \leq \min\{\pi_1, \pi_2\}$. This completes the proof of Part (a).

Part (b): Suppose $v_L\lambda_F \leq v_H\lambda_I$. We have

$$\begin{aligned}&\pi^* - \pi_1 \\ =& \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_F\frac{n\lambda_F\delta}{n\lambda_F+\mu}\frac{v_L}{\delta+\mu} + (\alpha-\gamma)\lambda_I\frac{n[\lambda_F\mu - \lambda_I(\delta+\mu)]}{n\lambda_I+\mu}\frac{v_L}{\delta+\mu} \qquad \text{(A6)} \\ &- [\beta\lambda_F + (1-\beta)\lambda_I]v_L \\ =& -(1-\alpha-\beta+\gamma)\lambda_I v_L + \Big(-\beta\lambda_F\frac{\lambda_F\delta}{n\lambda_F+\mu} + (\alpha-\gamma)\lambda_I\frac{\lambda_F\mu - \lambda_I(\delta+\mu)}{n\lambda_I+\mu}\Big)\frac{nv_L}{\delta+\mu}. \qquad \text{(A7)}\end{aligned}$$

Note that the second term in (A6) is strictly positive because of condition (4.1), so if $1-\alpha-\beta+\gamma$ is sufficiently small, then (A6) is positive, and thus $\pi^* \geq \pi_1$ in this case.

Suppose $v_L\lambda_F > v_H\lambda_I$. We have

$$\begin{aligned}
&\pi^* - \pi_1 \\
=&\beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_F \frac{\delta + n\lambda_I + \mu}{n\lambda_F + \mu}\frac{\lambda_F\delta(v_H - v_L)}{(\delta+\mu)(\lambda_F - \lambda_I)} \\
&+ (\alpha-\gamma)\lambda_I \frac{[\lambda_F\mu - \lambda_I(\delta+\mu)](v_H - v_L)}{(\delta+\mu)(\lambda_F-\lambda_I)}\frac{\delta + n\lambda_I + \mu}{n\lambda_I + \mu} - \beta\lambda_F v_L - (1-\beta)\lambda_I v_L \\
=& -(1-\alpha-\beta+\gamma)\lambda_I v_L + \Big(-\beta\lambda_F \frac{\lambda_F\delta}{n\lambda_F+\mu} + (\alpha-\gamma)\lambda_I \frac{\lambda_F\mu - \lambda_I(\delta+\mu)}{n\lambda_I+\mu}\Big)\frac{(\delta+n\lambda_I+\mu)(v_H-v_L)}{(\delta+\mu)(\lambda_F-\lambda_I)}.
\end{aligned} \tag{A8}$$

Similarly, the second term in (A8) is strictly positive, so if $1-\alpha-\beta+\gamma$ is sufficiently small, then $\pi^* \geq \pi_1$ in this case. This completes the proof of Part (b).

Part (c): The proof of Part (c) follows a similar approach as that of Part (b), and thus is omitted.

Proof of Proposition 7

When the firm does not adopt any reward programs and sets a price v_H, the aggregate consumer surplus is 0. Note that consumer welfare of each segment in coalition reward programs is nonnegative, because otherwise, consumers will not make a purchase. Tables 5.1 and 5.2 show that the aggregate consumer surplus in coalition reward programs is strictly positive. Therefore, coalition reward programs bring a higher aggregate consumer surplus than that when the firm does not adopt any reward programs and sets a price v_H. Now, we focus on the comparison between the aggregate consumer surplus in coalition reward programs and that when the firm does not adopt any reward programs and sets a price v_L. We have two cases.

Case 1: Suppose $v_L\lambda_F \leq v_H\lambda_I$.

According to Table 5.1, the comparison reduces to

$$
\begin{aligned}
&\gamma\lambda_F\Big\{v_H - \frac{\mu(\delta+n\lambda_F+\mu)}{(n\lambda_F+\mu)(\delta+\mu)}v_L - (v_H - v_L)\Big\} + (\beta-\gamma)\lambda_F\Big\{v_L - \frac{\mu(\delta+n\lambda_F+\mu)}{(n\lambda_F+\mu)(\delta+\mu)}v_L\Big\}\\
&\qquad + (\alpha-\gamma)\lambda_I\Big\{v_H - \frac{\mu(\delta+n\lambda_F+\mu)}{(n\lambda_I+\mu)(\delta+\mu)}v_L - (v_H - v_L)\Big\}\\
=&\beta\lambda_F\Big\{v_L - \frac{\mu(\delta+n\lambda_F+\mu)}{(n\lambda_F+\mu)(\delta+\mu)}v_L\Big\} + (\alpha-\gamma)\lambda_I\Big\{v_L - \frac{\mu(\delta+n\lambda_F+\mu)}{(n\lambda_I+\mu)(\delta+\mu)}v_L\Big\}\\
=&\frac{nv_L}{\delta+\mu}\Big\{\beta\lambda_F\frac{\lambda_F\delta}{n\lambda_F+\mu} - (\alpha-\gamma)\lambda_I\frac{\lambda_F\mu-\lambda_I(\delta+\mu)}{n\lambda_I+\mu}\Big\}\\
<&0,
\end{aligned}
$$

where the last inequality holds because of condition (4.1).

<u>Case 2:</u> Suppose $v_L\lambda_F > v_H\lambda_I$.

According to Table 5.2, the comparison reduces to

$$
\begin{aligned}
&\gamma\lambda_F\Big\{(v_H - v_L)\Big[1+\frac{\delta+n\lambda_I+\mu}{n\lambda_F+\mu}\frac{\lambda_F\delta}{(\delta+\mu)(\lambda_F-\lambda_I)}\Big] - (v_H - v_L)\Big\}\\
&\qquad + (\beta-\gamma)\lambda_F\Big\{(v_H - v_L)\frac{\delta+n\lambda_I+\mu}{n\lambda_F+\mu}\frac{\lambda_F\delta}{(\delta+\mu)(\lambda_F-\lambda_I)}\Big\}\\
&\qquad + (\alpha-\gamma)\lambda_I\Big\{(v_H - v_L)\Big[1-\frac{\delta+n\lambda_I+\mu}{n\lambda_I+\mu}\frac{\lambda_F\mu-\lambda_I(\delta+\mu)}{(\delta+\mu)(\lambda_F-\lambda_I)}\Big] - (v_H - v_L)\Big\}\\
=&\beta\lambda_F\Big\{(v_H - v_L)\frac{\delta+n\lambda_I+\mu}{n\lambda_F+\mu}\frac{\lambda_F\delta}{(\delta+\mu)(\lambda_F-\lambda_I)}\Big\}\\
&\qquad - (\alpha-\gamma)\lambda_I\Big\{(v_H - v_L)\frac{\delta+n\lambda_I+\mu}{n\lambda_I+\mu}\frac{\lambda_F\mu-\lambda_I(\delta+\mu)}{(\delta+\mu)(\lambda_F-\lambda_I)}\Big\}\\
=&(v_H - v_L)\frac{\delta+n\lambda_I+\mu}{(\delta+\mu)(\lambda_F-\lambda_I)}\Big\{\beta\lambda_F\frac{\lambda_F\delta}{n\lambda_F+\mu} - (\alpha-\gamma)\lambda_I\frac{\lambda_F\mu-\lambda_I(\delta+\mu)}{n\lambda_I+\mu}\Big\}\\
<&0,
\end{aligned}
$$

where the last inequality holds because of condition (4.1). This completes the proof.

Proof of Proposition 8

Case 1: Suppose $v_L\lambda_F \leq v_H\lambda_I$. Proposition 2(a) tells us that

$$\pi^c(n) = \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_F \frac{n\lambda_F\delta}{n\lambda_F+\mu}\frac{v_L}{\delta+\mu} + (\alpha-\gamma)\lambda_I \frac{n[\lambda_F\mu - \lambda_I(\delta+\mu)]}{n\lambda_I+\mu}\frac{v_L}{\delta+\mu},$$

$$\pi^p = \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_F \frac{\lambda_F\delta}{\lambda_F+\mu}\frac{v_L}{\delta+\mu} + (\alpha-\gamma)\lambda_I \frac{\lambda_F\mu - \lambda_I(\delta+\mu)}{\lambda_I+\mu}\frac{v_L}{\delta+\mu}.$$

Observe that $\pi^p = \pi^c(1)$. Since $\pi^c(n)$ increases in n, it follows immediately that $\pi^c(n^*) \geq \pi^p$ for any exogenous μ.

Case 2: Suppose $v_L\lambda_F > v_H\lambda_I$. We have two subcases.

Subcase 1: Suppose $(v_H - v_L)(\delta+\mu) \geq v_L\lambda_F - v_H\lambda_I$. Then, Proposition 2(a) tells us that

$$\pi^p = \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_F \frac{\lambda_F\delta}{\lambda_F+\mu}\frac{v_L}{\delta+\mu} + (\alpha-\gamma)\lambda_I \frac{\lambda_F\mu - \lambda_I(\delta+\mu)}{\lambda_I+\mu}\frac{v_L}{\delta+\mu}.$$

While, if $n < \frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F - v_H\lambda_I}$, then

$$\pi^c(n) = \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_F \frac{n\lambda_F\delta}{n\lambda_F+\mu}\frac{v_L}{\delta+\mu} + (\alpha-\gamma)\lambda_I \frac{n[\lambda_F\mu - \lambda_I(\delta+\mu)]}{n\lambda_I+\mu}\frac{v_L}{\delta+\mu}.$$

Otherwise,

$$\begin{aligned}\pi^c(n) = \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_F \frac{\delta+n\lambda_I+\mu}{n\lambda_F+\mu}\frac{\lambda_F\delta(v_H-v_L)}{(\delta+\mu)(\lambda_F-\lambda_I)} \\ + (\alpha-\gamma)\lambda_I \frac{[\lambda_F\mu - \lambda_I(\delta+\mu)](v_H-v_L)}{(\delta+\mu)(\lambda_F-\lambda_I)}\frac{\delta+n\lambda_I+\mu}{n\lambda_I+\mu}.\end{aligned}$$

Since $\pi^p = \pi^c(1)$ and $\pi^c(n)$ increases in n when $n < \frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F - v_H\lambda_I}$, it follows immediately that $\pi^c(n^*) \geq \pi^p$ for any exogenous μ. Combining Case 1 and Subcase 1 in Case 2 establishes Part (a).

Subcase 2: Suppose $(v_H - v_L)(\delta+\mu) < v_L\lambda_F - v_H\lambda_I$. Then, Proposition 2(b) tells us that

$$\pi^c(n) = \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_F \frac{\delta + n\lambda_I + \mu}{n\lambda_F + \mu} \frac{\lambda_F \delta(v_H - v_L)}{(\delta+\mu)(\lambda_F - \lambda_I)}$$
$$+ (\alpha-\gamma)\lambda_I \frac{[\lambda_F\mu - \lambda_I(\delta+\mu)](v_H - v_L)}{(\delta+\mu)(\lambda_F-\lambda_I)} \frac{\delta + n\lambda_I + \mu}{n\lambda_I + \mu},$$
$$\pi^p = \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_F \frac{\delta + \lambda_I + \mu}{\lambda_F + \mu} \frac{\lambda_F \delta(v_H - v_L)}{(\delta+\mu)(\lambda_F - \lambda_I)}$$
$$+ (\alpha-\gamma)\lambda_I \frac{[\lambda_F\mu - \lambda_I(\delta+\mu)](v_H - v_L)}{(\delta+\mu)(\lambda_F-\lambda_I)} \frac{\delta + \lambda_I + \mu}{\lambda_I + \mu}.$$

Observe that $\pi^p = \pi^c(1)$.

If $\hat{n} \geq \frac{(v_H - v_L)(\delta+\mu)}{v_L\lambda_F - v_H\lambda_I}$, then $\pi^c(n)$ increases in n when $n < \hat{n}$. If $\frac{(\alpha-\gamma)[\lambda_F\mu - \lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]} < 1$, then $\pi^c(n)$ always increases in n. In both scenarios, $\pi^c(n^*) \geq \pi^p$. This establishes the first part of Part (b).

If $\hat{n} < \frac{(v_H - v_L)(\delta+\mu)}{v_L\lambda_F - v_H\lambda_I}$ or $\frac{(\alpha-\gamma)[\lambda_F\mu - \lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]} > (\frac{\lambda_F}{\lambda_I})^2$, then $\pi^c(n)$ always decreases in n. So, $\pi^c(n^*) < \pi^p$. This establishes Part (c).

Finally, we show the second part of Part (b): if $\hat{n} > \frac{(v_H - v_L)(\delta+\mu)}{v_L\lambda_F - v_H\lambda_I}$ and $\frac{(\alpha-\gamma)\lambda_I[\lambda_F\mu - \lambda_I(\delta+\mu)]}{\beta\lambda_F[\lambda_F(\delta+\mu)-\lambda_I\mu]} > \frac{\lambda_I+\mu}{\lambda_F+\mu}$, then there exists a threshold n' such that $\pi^c(n) < \pi^p$ if $n > n'$. Note that $\pi^p = \pi^c(1)$ and $\pi^c(n)$ increases in n when $n \geq \hat{n}$ and then decreases, it suffices to show $\lim_{n\to\infty} \pi^c(n) < \pi^p$. One can verify

$$\lim_{n\to\infty} \pi^c(n) = \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_F \frac{\lambda_I}{\lambda_F} \frac{\lambda_F \delta(v_H - v_L)}{(\delta+\mu)(\lambda_F - \lambda_I)}$$
$$+ (\alpha-\gamma)\lambda_I \frac{[\lambda_F\mu - \lambda_I(\delta+\mu)](v_H - v_L)}{(\delta+\mu)(\lambda_F-\lambda_I)}.$$

Thus,

$$\begin{aligned}
&\pi^p - \lim_{n\to\infty} \pi^c(n) \\
=&\frac{\delta(v_H - v_L)}{(\lambda_F - \lambda_I)(\delta+\mu)} \left\{ (\alpha-\gamma)\lambda_I \frac{\lambda_F\mu - \lambda_I(\delta+\mu)}{\lambda_I+\mu} - \beta\lambda_F \frac{\lambda_F(\delta+\mu) - \lambda_I\mu}{\lambda_F+\mu} \right\} \\
>&0,
\end{aligned}$$

where the last inequality holds because $\frac{(\alpha-\gamma)\lambda_I[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta\lambda_F[\lambda_F(\delta+\mu)-\lambda_I\mu]} > \frac{\lambda_I+\mu}{\lambda_F+\mu}$. This establishes the second part of Part (b) and completes the proof.

Proofs and Additional Analysis in the Extension

The model without consumer discounting

Our main model assumes that consumers discount future surplus with a rate δ. Let $\delta = 0$. We rule out the effect of consumer discounting. By comparison, we investigate the impact of customer discounting on our findings.

The objective of the consumer is to maximize the long-run average consumer surplus. The consumer's problem can be modeled as an infinite-horizon average-reward dynamic programming. The consumer's value function is composed of two parts, where g^* represents the optimal average per period consumer surplus, $h(\cdot)$ represents the bias function. The optimality equations are given by

$$g^* + h(1) = \frac{n\lambda}{n\lambda+\mu}\max\left\{v-p+r+h(1), h(1)\right\} + \frac{\mu}{n\lambda+\mu}h(0), \tag{A9}$$

$$g^* + h(0) = \frac{n\lambda}{n\lambda+\mu}\max\left\{v-p+h(1), h(0)\right\} + \frac{\mu}{n\lambda+\mu}h(0). \tag{A10}$$

Note that the solution for $h(\cdot)$ is not unique. By fixing $h(0) = 0$, we derive a solution to the optimal average consumer surplus g^* and the bias function $h(1)$. We summarize the results in the following lemma.

Lemma A1. *If*

$$v - p + \frac{n\lambda}{n\lambda+\mu}r \geq 0, \tag{A11}$$

then it is optimal for a (v, λ)-consumer to make a purchase whenever she visits a firm *in the coalition, and a solution to the optimality equations* (A9)–(A10) *is given by*

$$g^* = \frac{n\lambda}{n\lambda + \mu}(v - p + \frac{n\lambda}{n\lambda + \mu} r), \quad h(1) = \frac{n\lambda}{n\lambda + \mu} r.$$

The following lemma provides the optimal design of a coalition reward program.

Lemma A2. *For any fixed size n and exogenous μ, the optimal price, reward, and the average profit in the coalition reward program are*

$$p^* = v_L + \frac{n\lambda_F}{n\lambda_F + \mu} r^*,$$

$$r^* = \min\Big\{\frac{(v_H - v_L)(n\lambda_F + \mu)(n\lambda_I + \mu)}{n\mu(\lambda_F - \lambda_I)}, \frac{(\mu + n\lambda_F)}{\mu} v_L\Big\},$$

$$\pi^* = \beta\lambda_F v_L + (\alpha - \gamma)\lambda_I \min\Big\{v_H, \frac{n\lambda_F + \mu}{n\lambda_I + \mu} v_L\Big\}.$$

It won't be worse for the firm to join the coalition reward program without discounting. In Section 4.1, however, condition 4.1 is necessary for the firm to join a coalition reward program. That is, consumer discounting deters the firm's motivation to offer a reward. Consumers who discount future surpluses have a lower perceived value of the reward. To engage consumer purchases, a larger reward is needed, which hurts the firm's profit.

Proposition A1. *The Relationship Between Profit and Size n without Discounting.*

(i) If $\lambda_F v_L \le \lambda_I v_H$ or $\lambda_F v_L > \lambda_I v_H$ but $n < \frac{\mu(v_H - v_L)}{v_L\lambda_F - v_H\lambda_I}$, then $\pi^* = \beta\lambda_F v_L + (\alpha - \gamma)\lambda_I \frac{n\lambda_F + \mu}{n\lambda_I + \mu} v_L$.
The profit increases in n.

(ii) If $\lambda_F v_L > \lambda_I v_H$ and $n \ge \frac{\mu(v_H - v_L)}{v_L\lambda_F - v_H\lambda_I}$, then $\pi^* = \beta\lambda_F v_L + (\alpha - \gamma)\lambda_I v_H$.
The profit is independent of n.

That is,

(a) Suppose $\lambda_F v_L \leq \lambda_I v_H$*, then the profit is increasing in* n*. Moreover, one can check*

$$\lim_{n\to\infty} \pi^* = \beta\lambda_F v_L + (\alpha-\gamma)\lambda_F v_L < \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_H.$$

(b) Suppose $\lambda_F v_L > \lambda_I v_H$*, then the profit first increases in* n *and then becomes constant for all* $n \geq \frac{\mu(v_H-v_L)}{v_L\lambda_F - v_H\lambda_I}$*.*

Lemma A3. *The Relationship Between Profit and Expiration Rate* μ*.*

(i) If $\mu < \frac{n(v_L\lambda_F - v_H\lambda_I)}{v_H-v_L}$*, then* $\pi^* = \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_H$*.*
The profit is independent of μ*.*

(ii) If $\mu \geq \frac{n(v_L\lambda_F - v_H\lambda_I)}{v_H-v_L}$*, then* $\pi^* = \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I \frac{n\lambda_F+\mu}{n\lambda_I+\mu} v_L$*.*
The profit decreases in μ*.*

Then,

(a) Suppose $\lambda_F v_L \leq \lambda_I v_H$*, the profit decreases in* μ*.*

(b) Suppose $\lambda_F v_L > \lambda_I v_H$*, the profit is first independent of* μ *and then decreases in* μ *for all* $\mu \geq \frac{n(v_L\lambda_F - v_H\lambda_I)}{v_H-v_L}$*.*

Proposition A2 (Comparison with No Reward Programs)**.** *For any fixed* n *and* μ*,*

(a) If $\gamma \geq \alpha\beta$*, then* $\pi^* \leq \min\{\pi_1, \pi_2\}$*. That is, if product valuation and shopping frequency are positively correlated, then coalition reward programs cannot bring a higher profit than that when the firm does not adopt any reward programs.*

(b) If $(\alpha-\gamma)v_H > (1-\beta)v_L$*, then* $\pi^* \geq \pi_1$*.*

(c) If $\beta v_L > \gamma v_H$*, then* $\pi^* \geq \pi_2$*.*

Proposition A3 (Comparison with Proprietary Reward Programs)**.** *For any exogenous* μ*, it won't be worse for the firm to join the coalition reward program.*

A discrete approximation

In the continuous-time model, we suggest the exponential distribution of the expiration term. We perform a discrete approximation to relax this assumption, where the expiration date can be set as a constant in the discrete-time model.

We consider a coalition reward program consisting of n independent and substitute firms. In each period, the customer visits each firm in the coalition with a probability λ (we assume λ is small such that $\lambda \ll 1$). The probability that the customer visits at least one of the firms is $1-(1-\lambda)^n$. The probability that the customer visits only one of the firms with the probability $n\lambda(1-\lambda)^{n-1}$. The number of firms that the customer visits in the coalition is distributed Binomial (n,λ).

$$
\begin{aligned}
P(N(\Delta)=1) &= e^{-\lambda\Delta}\lambda\Delta \\
&= \lambda\Delta(1-\lambda\Delta+\frac{\lambda^2}{2}\Delta^2-\cdots) \qquad (Taylor \quad Series) \\
&= \lambda\Delta+(-\lambda^2\Delta^2+\frac{\lambda^3}{2}\Delta^3-\cdots) \\
&= \lambda\Delta+o(\Delta).
\end{aligned}
$$

Note that $o(\Delta)$ shows a function that is negligible compared to Δ as $\Delta \to 0$. That is, even though there is a positive probability that the customer visits more than one firm, the probability is quite small and will be ignored in our analysis. Such an arrival process can be the binomial approximation to the Poisson process and we can assume that the customer visits one retailer only, which is quite common in literature (Lautenbacher and Stidham 1999).

We assume that consumer discounts the future surplus with a per-period discount factor δ, where $\delta \in [0,1)$. The objective of the consumer is to maximize the discounted surplus. The consumer's problem can be modeled as an infinite-horizon discounted-reward dynamic programming. The optimality equations are given by

$$u(i) = \begin{cases} n\lambda \max\{v - p + r + \delta u(K), \delta u(i-1)\} + (1 - n\lambda)\delta u(i-1), & \text{if} \quad i = 1, 2, ...K, \\ n\lambda \max\{v - p + \delta u(K), \delta u(0)\} + (1 - n\lambda)\delta u(0), & \text{if} \quad i = 0. \end{cases} \tag{A12}$$

Lemma A4. *Let* $G(i, \lambda) = \delta n\lambda \frac{1-\delta^i(1-n\lambda)^i}{1-\delta(1-n\lambda)}$. *If*

$$v - p + G(K, \lambda)r \geq 0, \tag{A13}$$

then it is optimal for a (v, λ)*-consumer to make a purchase whenever she visits a firm in the coalition, and a solution to the optimality equations* (A12) *is given by*

$$u(i) = \frac{n\lambda}{1-\delta}\Big\{v - p + G(K, \lambda)r\Big\} + \frac{G(i, \lambda)}{\delta}r, \quad u(0) = \frac{n\lambda}{1-\delta}\Big\{v - p + G(K, \lambda)r\Big\}.$$

Let q_0 and q_i denote the stationary probability of states 0 and i. We obtain $q_0 = (1-n\lambda)^K$ and $q_i = n\lambda(1-n\lambda)^{K-i}$. Therefore, if $p \leq v+G(K, \lambda)r$, a generic customer's profit contribution is $n\lambda\Big(q_0 p + (1 - q_0)(p - r)\Big) = n\lambda\Big(p - [1 - (1 - n\lambda)^K]r\Big)$.

Lemma A5. *For any fixed size* n *and fixed expiration term* K, *if*

$$(\alpha - \gamma)\lambda_I\Big[(1 - n\lambda_I)^K - 1 + G(K, \lambda_F)\Big] > \beta\lambda_F\Big[1 - (1 - n\lambda_F)^K - G(K, \lambda_F)\Big], \tag{A14}$$

then the optimal price, reward, and the average profit in the coalition reward program are

$$p^* = v_L + G(K, \lambda_F)r^*,$$

$$r^* = \min\Big\{\frac{v_H - v_L}{G(K, \lambda_F) - G(K, \lambda_I)}, \frac{v_L}{1 - G(K, \lambda_F)}\Big\},$$

$$\begin{aligned} \pi^* = {} & \beta\lambda_F\Big(v_L + G(K, \lambda_F)r^* - [1 - (1 - n\lambda_F)^K]r^*\Big) \\ & + (\alpha - \gamma)\lambda_I\Big(v_L + G(K, \lambda_F)r^* - [1 - (1 - n\lambda_I)^K]r^*\Big). \end{aligned}$$

Otherwise, it is never optimal to adopt the coalition reward program.

Proposition A4 (The Relationship Between Profit and Size n)**.** *Suppose condition* (A14) *holds.*

(a) Suppose $v_H G(K,\lambda_F) - v_L G(K,\lambda_I) \le v_H - v_L$*, then*

$$\begin{aligned} p^* = r^* &= \frac{v_L}{1-G(K,\lambda_F)}, \\ \pi^* &= \beta\lambda_F \frac{[1-n\lambda_F)^K}{1-G(K,\lambda_F)} v_L + (\alpha-\gamma)\lambda_I \frac{1-n\lambda_I)^K}{1-G(K,\lambda_F)} v_L. \end{aligned}$$

(b) Suppose $v_H G(K,\lambda_F) - v_L G(K,\lambda_I) > v_H - v_L$*, then*

$$\begin{aligned} p^* &= \frac{G(K,\lambda_F)v_H - G(K,\lambda_I)v_L}{G(K,\lambda_F)-G(K,\lambda_I)}, \\ r^* &= \frac{v_H - v_L}{G(K,\lambda_F)-G(K,\lambda_I)}, \\ \pi^* &= \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_F \frac{[1-(1-n\lambda_F)^K]-G(K,\lambda_F)}{G(K,\lambda_F)-G(K,\lambda_I)}(v_H - v_L) \\ &\quad + (\alpha-\gamma)\lambda_I \frac{G(K,\lambda_F)-[1-(1-n\lambda_I)^K]}{G(K,\lambda_F)-G(K,\lambda_I)}(v_H - v_L). \end{aligned}$$

Asymmetric firms

Our main model assumes that the coalition reward program consists of n symmetric firms. In practice, however, each firm in a coalition can bring different number of consumers. For example, a firm with a lower price tends to have higher foot traffic. To check the robustness of our results, we consider asymmetric firms in terms of their arrival rate. For analytical tractability, we assume that there are only two firms (1 and 2), each with different arrival rates $\lambda_1 > \lambda_2$. A natural assumption is that $p_1 < p_2$. Then, we can determine under what conditions it is beneficial for firm 1 (2) to join the coalition, how the profit should be distributed between the two asymmetric firms, and who will benefit more from joining the coalition. This will help us understand whether a given firm that wants to form a coalition with others should choose to partner with a firm that has a higher or lower λ than itself.

Note that the consumer visits the coalition reward program with the probability $\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+\mu}$. Then, according to the law of large numbers, she visits firm 1 upon her arrival with the probability $\frac{\lambda_1}{\lambda_1+\lambda_2}$. Hence, the probability that the consumer visits firm 1 before the reward has expired is $\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+\mu} \times \frac{\lambda_1}{\lambda_1+\lambda_2} = \frac{\lambda_1}{\lambda_1+\lambda_2+\mu}$. Similarly, the probability that the consumer visits firm 2 before the reward has expired is $\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+\mu} \times \frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{\lambda_2}{\lambda_1+\lambda_2+\mu}$. The probability that the consumer visits firm 1 while he has no available reward is $\frac{\mu}{\lambda_1+\lambda_2+\mu} \times \frac{\lambda_1}{\lambda_1+\lambda_2} = \frac{\mu\lambda_1}{(\lambda_1+\lambda_2+\mu)(\lambda_1+\lambda_2)}$. The probability that the consumer visits firm 2 while he has no available reward is $\frac{\mu}{\lambda_1+\lambda_2+\mu} \times \frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{\mu\lambda_2}{(\lambda_1+\lambda_2+\mu)(\lambda_1+\lambda_2)}$.

Let the maximum transition rate to be $\nu_2 = \delta + \lambda_1 + \lambda_2 + \mu$. Then, the optimality equations are given by

$$u(1) = \frac{\lambda_1}{\nu_2} \max\Big\{v - p_1 + r + u(1), u(1)\Big\} + \frac{\lambda_2}{\nu_2} \max\Big\{v - p_2 + r + u(1), u(1)\Big\} + \frac{\mu}{\nu_2} u(0), \tag{A15}$$

$$u(0) = \frac{\lambda_1}{\nu_2} \max\Big\{v - p_1 + u(1), u(0)\Big\} + \frac{\lambda_2}{\nu_2} \max\Big\{v - p_2 + u(1), u(0)\Big\} + \frac{\mu}{\nu_2} u(0). \tag{A16}$$

We assume that the customer always purchases the product with a lower price in state 0 when she visits the coalition. That is, $v - p_1 + u(1) \le u(0)$ and $v - p_2 + u(1) \ge u(0)$. According to Proposition 1, we get an explicit solution to the dynamic proposition.

Lemma A6. *If*

$$p_1 \le v + u(1) - u(0) \le p_2 \le v + r, \tag{A17}$$

then it is optimal for a (v, λ)-consumer to make a purchase whenever she visits a firm in the coalition in state 1, and only purchases the product with a lower price in state 0. A solution to the optimality equations (A15)–(A16) *is given by*

$$u(1) = \frac{\lambda_1}{\delta}\Big\{\frac{\nu_2}{\nu_2 - \lambda_2} v - p_1 - \frac{\lambda_2}{\nu_2 - \lambda_2} p_2 + \frac{\lambda_1 + \lambda_2}{\nu_2 - \lambda_2} r\Big\},$$
$$u(0) = \frac{\lambda_1}{\delta}\Big\{\frac{\lambda_1\nu_2 + \lambda_2\delta}{\lambda_1(\nu_2 - \lambda_2)} v - p_1 - \frac{\lambda_2(\lambda_1 + \delta)}{\lambda_1(\nu_2 - \lambda_2)} p_2 + \frac{(\lambda_1 + \delta))(\lambda_1 + \lambda_2)}{\lambda_1(\nu_2 - \lambda_2)} r\Big\}.$$

In this case, a generic customer's profit contribution is

$$
\begin{aligned}
&\frac{\lambda_1}{\lambda_1+\lambda_2+\mu}(p_1-r)+\frac{\lambda_2}{\lambda_1+\lambda_2+\mu}(p_2-r)+\frac{\mu\lambda_1}{(\lambda_1+\lambda_2+\mu)(\lambda_1+\lambda_2)}p_1\\
=&\frac{\lambda_1}{\lambda_1+\lambda_2}p_1+\frac{\lambda_2}{\lambda_1+\lambda_2+\mu}p_2-\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+\mu}r.
\end{aligned}
$$

Note that the consumer purchase whenever she visits firm 1 or firm 2 in the coalition when the two firms are symmetric. In that case, a generic customer's profit contribution is

$$
\begin{aligned}
&\frac{\lambda_1}{\lambda_1+\lambda_2+\mu}(p_1-r)+\frac{\lambda_2}{\lambda_1+\lambda_2+\mu}(p_2-r)\\
&\quad+\frac{\mu\lambda_1}{(\lambda_1+\lambda_2+\mu)(\lambda_1+\lambda_2)}p_1+\frac{\mu\lambda_2}{(\lambda_1+\lambda_2+\mu)(\lambda_1+\lambda_2)}p_2\\
=&\frac{\lambda_1}{\lambda_1+\lambda_2}p_1+\frac{\lambda_2}{\lambda_1+\lambda_2}p_2-\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+\mu}r.
\end{aligned}
$$

The loss $\frac{\mu\lambda_2}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_2+\mu)}p_2$ is due to the competition between two firms when they are asymmetric. The consumer' purchase behavior alters when she doesn't hold a valid reward upon her arrival in the coalition.

Lemma A7. *For any fixed size n and exogenous μ,*

$$
\begin{aligned}
\pi_1^A=&\beta\lambda_{1,F}\Big(\frac{\lambda_{1,F}}{\lambda_{1,F}+\lambda_{2,F}}p_1+\frac{\lambda_{1,F}}{\lambda_{1,F}+\lambda_{2,F}+\mu}p_2-\frac{\lambda_{1,F}+\lambda_{2,F}}{\lambda_{1,F}+\lambda_{2,F}+\mu}r\Big)\\
&+(\alpha-\gamma)\lambda_I\Big(\frac{\lambda_{1,I}}{\lambda_{1,I}+\lambda_{2,I}}p_1+\frac{\lambda_{2,I}}{\lambda_{1,I}+\lambda_{2,I}+\mu}p_2-\frac{\lambda_{1,I}+\lambda_{2,I}}{\lambda_{1,I}+\lambda_{2,I}+\mu}r\Big).\\
\pi_2^A=&\beta\lambda_{2,F}\Big(\frac{\lambda_{1,F}}{\lambda_{1,F}+\lambda_{2,F}}p_1+\frac{\lambda_{2,F}}{\lambda_{1,F}+\lambda_{2,F}+\mu}p_2-\frac{\lambda_{1,F}+\lambda_{2,F}}{\lambda_{1,F}+\lambda_{2,F}+\mu}r\Big)\\
&+(\alpha-\gamma)\lambda_{2,I}\Big(\frac{\lambda_{1,I}}{\lambda_{1,I}+\lambda_{2,I}}p_1+\frac{\lambda_{2,I}}{\lambda_{1,I}+\lambda_{2,I}+\mu}p_2-\frac{\lambda_{1,I}+\lambda_{2,I}}{\lambda_{1,I}+\lambda_{2,I}+\mu}r\Big).
\end{aligned}
$$

Let $\pi_1^p(\pi_2^p)$ denote the optimal profit if firm 1(2) adopts the proprietary reward program.

Lemma A8. *(a) Suppose either $\lambda_F v_L \leq \lambda_I v_H$ or or $\lambda_F v_L > \lambda_I v_H$ but $(v_H - v_L)(\delta + \mu) \geq v_L\lambda_F - v_H\lambda_I$, then*

$$\pi_1^p = \beta\lambda_{1,F}v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_{1,F}\frac{\lambda_{1,F}\delta}{\lambda_{1,F}+\mu}\frac{v_L}{\delta+\mu}$$

$$+ (\alpha-\gamma)\lambda_I\frac{\lambda_{1,F}\mu - \lambda_I(\delta+\mu)}{\lambda_I+\mu}\frac{v_L}{\delta+\mu}.$$

$$\pi_2^p = \beta\lambda_{2,F}v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_{2,F}\frac{\lambda_{2,F}\delta}{\lambda_{2,F}+\mu}\frac{v_L}{\delta+\mu}$$

$$+ (\alpha-\gamma)\lambda_I\frac{\lambda_{2,F}\mu - \lambda_I(\delta+\mu)}{\lambda_I+\mu}\frac{v_L}{\delta+\mu}.$$

(b) Suppose $v_L\lambda_F > v_H\lambda_I$ and $(v_H - v_L)(\delta+\mu) < v_L\lambda_F - v_H\lambda_I$, then

$$\pi_1^p = \beta\lambda_{1,F}v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_{1,F}\frac{\delta+\lambda_I+\mu}{\lambda_{1,F}+\mu}\frac{\lambda_{1,F}\delta(v_H-v_L)}{(\delta+\mu)(\lambda_{1,F}-\lambda_I)}$$

$$+ (\alpha-\gamma)\lambda_I\frac{[\lambda_{1,F}\mu - \lambda_I(\delta+\mu)](v_H-v_L)}{(\delta+\mu)(\lambda_{1,F}-\lambda_I)}\frac{\delta+\lambda_I+\mu}{\lambda_I+\mu}.$$

$$\pi_2^p = \beta\lambda_{2,F}v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_{2,F}\frac{\delta+\lambda_I+\mu}{\lambda_{2,F}+\mu}\frac{\lambda_{2,F}\delta(v_H-v_L)}{(\delta+\mu)(\lambda_{2,F}-\lambda_I)}$$

$$+ (\alpha-\gamma)\lambda_I\frac{[\lambda_{2,F}\mu - \lambda_I(\delta+\mu)](v_H-v_L)}{(\delta+\mu)(\lambda_{2,F}-\lambda_I)}\frac{\delta+\lambda_I+\mu}{\lambda_I+\mu}.$$

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