Structured Tensors: Theory and Applications

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Ph.D

The Hong Kong Polytechnic University

2016

The Hong Kong Polytechnic University

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A thesis submitted in partial fulfilment of the requirements

for the degree of Doctor of Philosophy

March 2016

Certificate of Originality

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______________________________(Signed)

CHEN Haibin (Name of student)

Dedicate to my parents.

Abstract

The thesis is devoted to studying spectral properties and positive semi-definiteness of several kinds of structured tensors. Furthermore, the SOS (sum-of-squares) tensor decomposition of structured tensors in the literature are established. Five topics are considered:

1. Positive definiteness and semi-definiteness of even order Cauchy tensors.
2. Generalized Cauchy tensors and Hankel tensors.
3. Some spectral properties of odd-bipartite Z-Tensors and their absolute tensors.
4. SOS tensor decomposition and applications.
5. Positive semi-definiteness and extremal H-eigenvalues of extended essentially non-negative tensors.

For topic 1, motivated by symmetric Cauchy matrices, we define symmetric Cauchy tensors and their generating vectors in this thesis. An even order symmetric Cauchy tensor is positive semi-definite if and only if its generating vector is positive. An even order symmetric Cauchy tensor is positive definite if and only if its generating vector has positive and mutually distinct entries. This extends Fiedler's result for symmetric Cauchy matrices to symmetric Cauchy tensors. Then, it is proven that the positive semi-definiteness character of an even order symmetric Cauchy tensor

can be equivalently checked by the monotone increasing property of a homogeneous polynomial related to the Cauchy tensor. The homogeneous polynomial is strictly monotone increasing in the non-negative orthant of the Euclidean space when the even order symmetric Cauchy tensor is positive definite. At last, bounds of the largest H-eigenvalue of a positive semi-definite symmetric Cauchy tensor are given and several spectral properties on Z-eigenvalues of odd order symmetric Cauchy tensors are shown. We also establish that all the H-eigenvalues of non-negative Cauchy tensors are non-negative. Further questions on Cauchy tensors are raised.

For topic 2, we present various new results on generalized Cauchy tensors and Hankel tensors. We first introduce the concept of generalized Cauchy tensors which extends Cauchy tensors in the current literature, and provide several conditions characterizing positive semi-definiteness of generalized Cauchy tensors with nonzero entries. Furthermore, we prove that all even order generalized Cauchy tensors with positive entries are completely positive tensors, which means every such that generalized Cauchy tensor can be decomposed as the sum of non-negative rank-1 tensors. Secondly, we present new mathematical properties of Hankel tensors. We prove that an even order Hankel tensor is Vandermonde positive semi-definite if and only if its associated plane tensor is positive semi-definite. We also show that, if the Vandermonde rank of a Hankel tensor $\mathcal{A}$ is less than the dimension of the underlying space, then positive semi-definiteness of $\mathcal{A}$ is equivalent to the fact that $\mathcal{A}$ is a complete Hankel tensor, and so, is further equivalent to the SOS tensor decomposition property of $\mathcal{A}$. Thirdly, we introduce a new class of structured tensors called Cauchy-Hankel tensors, which is a special case of Cauchy tensors and Hankel tensors simultaneously. Sufficient and necessary conditions are established for an even order Cauchy-Hankel tensor to be positive definite.

For topic 3, stimulated by odd-bipartite and even-bipartite hypergraphs, we define odd-bipartite (weakly odd-bipartite) and even-bipartite (weakly even-bipartite) tensors. It is verified that all even order odd-bipartite tensors are irreducible tensors, while all even-bipartite tensors are reducible no matter the parity of the order. Based on properties of odd-bipartite tensors, we study the relationship between the largest H-eigenvalue of a symmetric Z-tensor with non-negative diagonal elements, and the largest H-eigenvalue of absolute tensor of that Z-tensor. When the order is even and the symmetric Z-tensor is weakly irreducible, we prove that the largest H-eigenvalue of the Z-tensor and the largest H-eigenvalue of the absolute tensor of that Z-tensor are equal, if and only if the Z-tensor is weakly odd-bipartite. Examples show the authenticity of the conclusions. Then, we prove that a symmetric Z-tensor with non-negative diagonal entries and the absolute tensor of the Z-tensor are diagonal similar, if and only if the Z-tensor has even order and it is weakly odd-bipartite. After that, it is proved that, when an even order symmetric Z-tensor with non-negative diagonal entries is weakly irreducible, the equality of the spectrum of the Z-tensor and the spectrum of absolute tensor of that Z-tensor, can be characterized by the equality of their spectral radii.

For topic 4, we examine structured tensors which have SOS tensor decomposition, and study the SOS-rank of SOS tensor decomposition. We first show that several classes of even order symmetric structured tensors available in the literature have SOS tensor decomposition. These include positive Cauchy tensors, weakly diagonally dominated tensors, B_0-tensors, double B-tensors, quasi-double B_0-tensors, MB_0-tensors, H-tensors, absolute tensors of positive semi-definite Z-tensors and extended Z-tensors. We also examine the SOS-rank of SOS tensor decomposition and the SOS-width for SOS tensor cones. The SOS-rank provides the minimal number of squares in the SOS tensor decomposition, and, for a given SOS tensor cone, its SOS-width is the maximum possible SOS-rank for all the tensors in this cone. We first deduce an upper bound for general tensors that have SOS decomposition and the SOS-width for general SOS tensor cone using the known results in the literature of polynomial theory. Then, we provide an explicit sharper estimate for the SOS-rank of SOS tensor decomposition with bounded exponent and identify the SOS-width for the tensor cone consisting of all tensors with bounded exponent that have SOS decompositions. Finally, as applications, we show how the SOS tensor decomposition can be used to compute the minimum H-eigenvalue of an even order symmetric extended Z-tensor and test the positive definiteness of an associated multivariate form. Numerical experiments are also provided to show the efficiency of the proposed numerical methods ranging from small size to large size numerical examples.

For topic 5, we study positive semi-definiteness and extremal H-eigenvalues of extended essentially non-negative tensors. We first prove that checking positive semi-definiteness of a symmetric extended essentially non-negative tensor is equivalent to checking positive semi-definiteness of all its condensed subtensors. Then, we prove that, for a symmetric positive semi-definite extended essentially non-negative tensor, it has a sum-of-squares (SOS) tensor decomposition if each positive off-diagonal element corresponds to an SOS term in the homogeneous polynomial of the tensor. Using this result, we can compute the minimum H-eigenvalue of such kinds of extended essentially non-negative tensors. Then, for general symmetric even order extended essentially non-negative tensors, we show that the largest H-eigenvalue of the tensor is equivalent to the optimal value of an SOS programming problem. As an application, we show this approach can be used to check co-positivity of symmetric extended Z-tensors. Numerical experiments are given to show the efficiency of the proposed methods.

Underlying papers

This thesis is based on the following five papers written by the author during the period of stay at the Department of Applied Mathematics, The Hong Kong Polytechnic University as a graduate student:

1. Haibin Chen, Liqun Qi, *Positive definiteness and semi-definiteness of even order symmetric Cauchy tensors*, Journal of Industrial Management and Optimization, 11(4) (2015) 1263-1274.

2. Haibin Chen, Guoyin Li, Liqun Qi, *Further results on Cauchy tensors and Hankel tensors*, Applied Mathematics and Computation, 275 (2016) 50-62.

3. Haibin Chen, Liqun Qi, *Spectral Properties of Odd-Bipartite Z-Tensors and Their Absolute Tensors*, Frontiers of Mathematics in China, 11(3) (2016) 539-556.

4. Haibin Chen, Guoyin Li, Liqun Qi, *SOS Tensor Decomposition: Theory and Applications*, to appear in: Communications in Mathematical Sciences.

5. Haibin Chen, Yannan Chen, Guoyin Li, Liqun Qi, *Positive Semi-definiteness and Extremal H-eigenvalues of Extended Essentially Nonnegative Tensors*, arXiv:1511.02328(2015).

In addition, the following is a list of other papers written by the author during the period of his Ph.D study.

1. Haibin Chen, Zhenghai Huang, Liqun Qi, *Copositivity Detection of Tensors: Theory and Algorithm*, arXiv:1603.01823(2016).

2. Haibin Chen, Liqun Qi, Yingsheng Song, *Sufficient Tensors and Tensor Complementarity Problem*, preprint.

Acknowledgements

The endeavor of carrying out research is a fascinatingly non-isolated activity. I am grateful to the several individuals who have supported me in various ways during the PhD program and I would like to hereby acknowledge their assistance.

First and foremost, I wish to express my great thanks to my supervisor, Prof. Liqun Qi, for his enlightening guidance, invaluable discussions and insightful ideas throughout the years. Especially, I would like to show my great appreciation to Prof. Qi, for his strong encouragement and patience during my study career. What I have benefited most from him is the rigorous and diligent attitude to scientific research.

Then, I will show my gratitude to my co-supervisor Dr. Xun Li, who has been a constant source of inspiration at the forefront of my PhD experience. Furthermore, I will show my great thanks to Prof. Yiju Wang, Prof. Zhenghai Huang, Porf. Yisheng Song, Dr. Guoyin Li and Dr. Yannan Chen for their support and help during the research process in The Hong Kong Polytechnic University.

Finally, I would like to express my special thanks to my family for their love, encouragement and support.

Contents

List of Notations

$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{N}$	set of natural numbers
$\mathbb{R}^n$	set of n-dimensional real vectors
$\mathbb{R}^n_+$	non-negative orthant of $\mathbb{R}^n$
$\mathbb{R}^{m\times n}$	set of $m \times n$ real matrices
$\mathcal{A}, \mathcal{B}$	real tensors
$\lvert\mathcal{A}\rvert$	absolute tensor of $\mathcal{A}$
$S_{m,n}$	set of symmetric tensors with order m dimension n
A, B	real matrices
$\mathbf{x}, \mathbf{y}$	n dimensional real/complex vectors
$\mathbf{e_i}$	the ith coordinate vector in $\mathbb{R}^n$
$\mathbf{0}$	vector with zero entries
$\mathcal{I}$	the identity tensor
I_n	identity matrix of dimension n

Chapter 1

Introduction

1.1 Background

The concept of tensors was introduced by Gauss, Riemann and Christoffel, etc., in the 19th century in the study of differential geometry. In the very beginning of the 20th century, Ricci, Levi-Civita, etc., further developed tensor analysis as a mathematical discipline. It was Einstein who applied tensor analysis in his study of general relativity in 1916. This made tensor analysis an important tool in theoretical physics, continuum mechanics and many other areas of science and engineering [28, 41, 59, 104].

A tensor is a multidimensional array and it is a physical quantity which is independent from co-ordinate system changes. More formally, mth-order tensor is an element of the tensor product of m vector spaces, each of which has its own coordinate system. This notion of tensors is not to be confused with tensors in physics and engineering (such as stress tensors) [69], which are generally referred to as tensor fields in mathematics [96]. A zero order tensor is a scalar. A first order tensor is a vector and a second-order tensor is a matrix, and tensors of order three or higher are called higher-order tensors.

Recently, more and more researchers have paid attentions to tensor problems and several interesting and important research directions are hot in numerical multilinear algebra, such as spectral theory of tensors, spectral hypergraph theory, structure property of special tensors, tensor decomposition and so on.

Positive semi-definiteness and spectral properties of tensors are important topics in tensor computation and multilinear algebra. Since the early work of [78] and [62], a lot of researchers have devoted themselves to the study of spectral properties of tensors in the past several years [4, 5, 6, 12, 11, 18, 32, 43, 47, 60, 73, 109]. Tensor eigenvalue problems have wide applications in polynomial optimization [71], spectral hypergraph theory [18, 109, 80], high-order Markov chains [70], signal processing [84], and imaging science [88]. On the other hand, positive semi-definiteness is an important structure property of tensors and it has many applications in optimal control, magnetic resonance imaging and spectral hypergraph theory [78, 80, 88, 89]. It is known that the problem of determining whether a given general even order symmetric tensor is positive semi-definite or not is NP-hard [35]. On the other hand, for some special structured tensors, it has been shown that either they are positive semi-definite or positive definite in the even order symmetric case, or there are easily checkable conditions to identify such tensors are positive semi-definite or not.

1.2 Structured tensors

Structured tensors mean tensors with special structure. In recent years, several kinds of structured tensors have been studied such as non-negative tensors [5, 7, 27, 38, 51, 60, 63, 70, 89, 79, 87], Hankel tensors [11, 10, 20, 81], complete positive tensors [87], co-positive tensors [79], Hilbert tensors [100], P-tensors [99, 109], B-tensors [82], diagonally dominant tensors, H-tensors [43, 54] and so on. Furthermore, researchers not only established results on spectral theory and positive semi-definiteness property of structured tensors, but also gave some important applications of structured tensors in stochastic process and data fitting. In the thesis, we will explore more in detail about non-negative tensors, Hankel tensors, B-tensors and H-tensors.

1.2.1 Non-negative tensors

One of the important structured tensor classes is the class of non-negative tensors, that is, tensors with non-negative entries. The non-negative tensors arise naturally in spectral hypergraph theory and high-order Markov chain theory. Recently, a lot of theoretical conclusions and efficient numerical schemes have been proposed for non-negative tensors.

The Perron-Frobenius theorem is a fundamental result for non-negative matrices. It has been widely used not only in mathematics but also in various fields of science and technology, such as economics, operational research, and page rank in the internet; for more information, see [62, 78, 83, 86]. Chang, Pearson, and Zhang generalized this theorem to the class of non-negative tensors recently [5]. The Perron-Frobenius theorem for non-negative tensors is related to measuring higher order connectivity in linked objects [61] and hypergraphs [21]. Later, Yang and Yang gave further results for the Perron-Frobenius theorem for non-negative tensors and some other results from non-negative matrices are generalized [110, 90]. Friedland, Gaubert and Han [27] pointed out that the Perron-Frobenius theorem for non-negative tensors has a very close link with the Perron-Frobenius theorem for homogeneous monotone maps. They introduced weakly irreducible non-negative tensors and established the Perron-Frobenius theorem for such tensors.

Based on a Perron-Frobenius type theorem for non-negative tensors [5], Ng, Qi, and Zhou proposed an iterative method to find the largest eigenvalue of an irreducible non-negative tensor [70]. The NQZ method in [70] is efficient but it is not always convergent for irreducible non-negative tensors. Later on, Chang, Pearson and Zhang [7] introduced primitive tensors which is a subclass of irreducible non-negative tensors, and established the convergence of the NQZ method for primitive tensors. Moreover, Liu, Zhou, and Ibrahim [63] modified the NQZ method such that the modified algorithm is always convergent for finding the largest eigenvalue of an irreducible non-negative tensor. Recently, Zhang and Qi [113] established the linear convergence of the NQZ method for essentially positive tensors. Zhang, Qi and Xu [115] established the linear convergence of the LZI method for weakly positive tensors. More recently, numerical method is also presented to calculate the maximum eigenvalue for non-negative tensors without the irreducible assumption by using a partion technique [36]. Furthermore, some variational principles for Z-eigenvalues of non-negative tensors are presented in [8].

1.2.2 Hankel tensors

Hankel tensors were introduced by Papy, De Lathauwer and Van Huffel in [75] in the context of the harmonic retrieval problem, which is at the heart of many signal processing problems. In [2], Badeau and Boyer proposed fast higher-order singular value decomposition (HOSVD) for third order Hankel tensors.

The comprehensive spectral theory and positive semi-definiteness of Hankel tensors and some other applications were studied in [20, 81, 11]. Hankel tensors are symmetric tensors. In [81], positive semi-definite Hankel tensors were studied. Each Hankel tensor is associated with an Hankel matrix. If that Hankel matrix is positive semi-definite, then the Hankel tensor is called a strong Hankel tensor. It was proved that an even order strong Hankel tensor is positive semi-definite. A symmetric tensor is a Hankel tensor if and only if it has a Vandermonde decomposition. If the coefficients of that Vandermonde decomposition are non-negative, then the Hankel tensor is called a complete Hankel tensor. It was proved that an even order complete Hankel tensor is also positive semi-definite. An example of a positive semi-definite Hankel tensor, which is neither strong nor complete Hankel tensor was also given in [81].

Recently, the SOS (sum-of-squares) tensor decomposition of even order Hankel tensors and applications are studied in [56, 55, 10]. Tensors with SOS decomposition are positive semi-definite symmetric tensors, but not vice versa. The problem for determining an even order symmetric tensor has SOS decomposition or not is equivalent to solving a semi-infinite linear programming problem, which can be done in polynomial time. On the other hand, the problem for determining an even order symmetric tensor is positive semi-definite or not is NP-hard. In [56], Li et al. studied SOS-Hankel tensors. Currently, there are two known positive semi-definite Hankel tensor classes: even order complete Hankel tensors and even order strong Hankel tensors. It is shown that complete Hankel tensors are strong Hankel tensors, and even order strong Hankel tensors are SOS-Hankel tensors [56]. Moreover, several examples of positive semi-definite Hankel tensors are given, which are not strong Hankel tensors. However, all of them are still SOS-Hankel tensors. Does there exist a positive semi-definite non-SOS-Hankel tensor? The answer to this question remains open. If the answer to this question is no, then the problem for determining an even order Hankel tensor is positive semi-definite or not is solvable in polynomial-time.

1.2.3 B-tensors

B-tensors are a special class of structured tensors that are natural generalization of B-matrices. Meaningful and interesting conclusions about symmetric B-tensors can be found in [82, 99, 109].

In [82], Qi et al. used a new technique to prove that an even order symmetric B-tensor is positive definite. It is shown that a symmetric B-tensor can always be decomposed to the sum of a strictly diagonally dominated symmetric M-tensor and several positive multiples of partially all one tensors, and a symmetric B_0-tensor can always be decomposed to the sum of a diagonally dominated symmetric M-tensor and several positive multiples of partially all one tensors. Even order partially all one tensors are positive semi-definite. An even order diagonally dominated symmetric tensor is positive semi-definite, and an even order strictly diagonally dominated symmetric tensor is positive definite. Therefore, when the order is even, all symmetric B-tensors are positive definite, and the corresponding symmetric B_0-tensors are positive semi-definite [82]. Hence, the condition provided in [82], gives an easily checkable sufficient condition for positive definite and semi-definite tensors.

After that, motivated by notion of B-tensors, several kinds of structured tensors such as double B-tensors, quasi-double B-tensors and MB-tensors, are defined and applied in the location of real eigenvalues [52, 53, 13].

1.2.4 H-tensors

H-tensors were first defined in [19], and it is proved that all H-tensors have quasi-strictly diagonally dominant property. Then, much more properties of H-tensors were further studied in [43, 54], where the authors in [54] referred nonsingular H-tensors simply as H-tensors and the authors in [43] referred nonsingular H-tensors as strong H-tensors. In [54], the authors proved that if a given tensor is an even order symmetric strong H-tensor with positive diagonal entries, then the tensor is a positive definite tensor. In [43], it is proved that a symmetric H-tensor with non-negative diagonal entries is positive semi-definite, which implies that H-tensors are useful in checking the positive semi-definiteness of homogeneous polynomials.

Very recently, Wang et al. studied the bounds for the Z-spectral radius of nonsingular H-tensors [107], and numerical examples illustrate that the bounds are sharper than known bounds.

1.3 Summary of contributions of the thesis

The original contributions of this thesis are as follows:

- Several necessary and sufficient conditions for an even order Cauchy tensor to be positive semi-definite are given. Some properties of positive semi-definite Cauchy tensors are presented. Inequalities about the largest H-eigenvalue and the smallest H-eigenvalue of Cauchy tensors are shown. Then, some spectral properties on Z-eigenvalues of odd order Cauchy tensors are shown. Furthermore, properties of generalized Cauchy tensors and some new properties of Hankel tensors are provided.

- Odd-bipartite and even-bipartite tensors are defined in this paper. Using this notions, the relation between the largest H-eigenvalue of a Z-tensor with non-negative diagonal elements, and the largest H-eigenvalue of the Z-tensor's absolute tensor are studied. Sufficient and necessary conditions for the equality of these largest H-eigenvalues are given when the Z-tensor has even order. For the odd order case, sufficient conditions are presented. On the other side, relation between spectral sets of an even order symmetric Z-tensor with non-negative diagonal entries and its absolute tensor are studied.

- The SOS tensor decomposition of various kinds of structured tensors is studied in the even order symmetric case. These include positive Cauchy tensors, weakly diagonally dominated tensors, B_0-tensors, double B-tensors, quasi-double B_0-tensors, MB_0-tensors, H-tensors, absolute tensors of positive semi-definite Z-tensors and extended Z-tensors. The SOS-rank of SOS tensor decomposition and the SOS-width for SOS tensor cones are also examined. In particular, an explicit sharp estimate is provided for SOS-rank of tensors with bounded exponent and SOS-width for the tensor cone consisting of all such tensors with bounded exponent that have SOS decomposition. Then, applications for the SOS decomposition of extended Z-tensors are presented.

- The notion of essentially non-negative tensor is generalized to a more general form i.e. extended essentially non-negative tensor. Positive semi-definiteness and SOS tensor decomposition of symmetric essentially nonnegative tensors are studied. Then, by SOS optimization technique, the extremal H-eigenvalues of a symmetric even order extended essentially non-negative tensor can be computed by solving an SOS optimization problem. Numerical examples illustrate the significance. An important application is presented that is checking the co-positivity of symmetric tensors with even or odd orders.

1.4 Organization of the thesis

The thesis is structured as follows.

- Chap. 2 We will first recall some basic notions of tensors are given such as H-eigenvalues, Z-eigenvalues and positive semi-definite tensors. Then, we introduce the notion of Vandermonde positive semi-definite tensors, which is a special class of positive semi-definite tensors. At last, some basic results about homogeneous polynomials are presented.

- Chap. 3 Motivated by symmetric Cauchy matrices, we define symmetric Cauchy tensors and their generating vectors in this paper. Hilbert tensors are symmetric Cauchy tensors. An even order symmetric Cauchy tensor is positive semi-definite if and only if its generating vector is positive. An even order symmetric Cauchy tensor is positive definite if and only if its generating vector has positive and mutually distinct entries. This extends Fiedler's result for symmetric Cauchy matrices to symmetric Cauchy tensors. Then, it is proven that the positive semi-definiteness character of an even order symmetric Cauchy tensor can be equivalently checked by the monotone increasing property of a homogeneous polynomial related to the Cauchy tensor. The homogeneous polynomial is strictly monotone increasing in the non-negative orthant of the Euclidean space when the even order symmetric Cauchy tensor is positive definite. At last, bounds of the largest H-eigenvalue of a positive semi-definite symmetric Cauchy tensor are given and several spectral properties on Z-eigenvalues of odd order symmetric Cauchy tensors are shown. We also establish that all the H-eigenvalues of non-negative Cauchy tensors are non-negative. Further questions on Cauchy tensors are raised.

- Chap. 4 We first introduce the concept of generalized Cauchy tensors which extends Cauchy tensors to a more general form, and provide several conditions characterizing positive semi-definiteness of generalized Cauchy tensors with nonzero entries. Furthermore, we prove that all even order generalized Cauchy tensors with positive entries are completely positive tensors, which means every such that generalized Cauchy tensor can be decomposed as the sum of non-negative rank-1 tensors. Secondly, we present new mathematical properties of Hankel tensors. We prove that an even order Hankel tensor is Vandermonde positive semi-definite if and only if its associated plane tensor is positive semi-definite. We also show that, if the Vandermonde rank of a Hankel tensor $\mathcal{A}$ is less than the dimension of the underlying space, then positive semi-definiteness of $\mathcal{A}$ is equivalent to the fact that $\mathcal{A}$ is a complete Hankel tensor, and so, is further equivalent to the SOS property of $\mathcal{A}$. Thirdly, we introduce a new class of structured tensors called Cauchy-Hankel tensors, which is a special case of Cauchy tensors and Hankel tensors simultaneously. Sufficient and necessary conditions are established for an even order Cauchy-Hankel tensor to be positive definite.

- Chap. 5 Stimulated by odd-bipartite and even-bipartite hypergraphs, we define odd-bipartite (weakly odd- bipartie) and even-bipartite (weakly even-bipartite) tensors. It is verified that all even order odd-bipartite tensors are irreducible tensors, while all even-bipartite tensors are reducible no matter the parity of the order. Based on properties of odd-bipartite tensors, we study the relationship between the largest H-eigenvalue of a Z-tensor with non-negative diagonal elements, and the largest H-eigenvalue of absolute tensor of that Z-tensor. When the order is even and the Z-tensor is weakly irreducible, we prove that the largest H-eigenvalue of the Z-tensor and the largest H-eigenvalue of the absolute tensor of that Z-tensor are equal, if and only if the Z-tensor is weakly odd-bipartite. Examples show the authenticity of the conclusions. Then, we prove that a symmetric Z-tensor with non-negative diagonal entries and the absolute tensor of the Z-tensor are diagonal similar, if and only if the Z-tensor has even order and it is weakly odd-bipartite. After that, it is proved that, when an even order symmetric Z-tensor with non-negative diagonal entries is weakly irreducible, the equality of the spectrum of the Z-tensor and the spectrum of absolute tensor of that Z-tensor, can be characterized by the equality of their spectral radii.

- Chap. 6 We examine structured tensors which have sum-of-squares (SOS) tensor decomposition, and study the SOS-rank of SOS tensor decomposition. We first show that several classes of even order symmetric structured tensors available in the literature have SOS tensor decomposition. These include positive Cauchy tensors, weakly diagonally dominated tensors, B_0-tensors, double B-tensors, quasi-double B_0-tensors, MB_0-tensors, H-tensors, absolute tensors of positive semi-definite Z-tensors and extended Z-tensors. We also examine the SOS-rank of SOS tensor decomposition and the SOS-width for SOS tensor cones. The SOS-rank provides the minimal number of squares in the SOS tensor decomposition, and, for a given SOS tensor cone, its SOS-width is the maximum possible SOS-rank for all the tensors in this cone. We first deduce an upper bound for general tensors that have SOS decomposition and the SOS-width for general SOS tensor cone using the known results in the literature of polynomial theory. Then, we provide an explicit sharper estimate for the SOS-rank of SOS tensor decomposition with bounded exponent and identify the SOS-width for the tensor cone consisting of all tensors with bounded exponent that have SOS decompositions. Finally, as applications, we show how the SOS tensor decomposition can be used to compute the minimum H-eigenvalue of an even order symmetric extended Z-tensor and test the positive definiteness of an associated multivariate form. Numerical experiments are also provided to show the efficiency of the proposed numerical methods ranging from small size to large size numerical examples.

- Chap. 7 We study positive semi-definiteness and extremal H-eigenvalues of extended essentially non-negative tensors. We first prove that checking positive semi-definiteness of a symmetric extended essentially non-negative tensor is equivalent to checking positive semi-definiteness of all its condensed subtensors. Then, we prove that, for a symmetric positive semi-definite extended essentially non-negative tensor, it has a sum-of-squares (SOS) tensor decomposition if each positive off-diagonal element corresponds to an SOS term in the homogeneous polynomial of the tensor. Using this result, we can compute the minimum H-eigenvalue of such kinds of extended essentially non-negative tensors. Then, for general symmetric even order extended essentially non-negative tensors, we show that the largest H-eigenvalue of the tensor is equivalent to the optimal value of an SOS programming problem. As an application, we show this approach can be used to check co-positivity of symmetric extended Z-tensors. Numerical experiments are given to show the efficiency of the proposed methods.

- Chap. 8 Some final remarks and future work are listed in this section.

Chapter 2

Preliminaries

A real tensor with order m and dimension n is defined by $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$, $i_j \in [n]$, $j \in [m]$. If the entries $a_{i_1 i_2 \cdots i_m}$ are invariant under any permutation of the subscripts, then tensor $\mathcal{A}$ is called symmetric tensor. Let $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$. The two forms below will be used in the following analysis frequently:

$$\mathcal{A}\mathbf{x}^{m-1} = \left(\sum_{i_2, i_3, \cdots, i_m = 1}^{n} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \right)_{i=1}^{n};$$

$$\mathcal{A}\mathbf{x}^{m} = \sum_{i_1, i_2, \cdots, i_m = 1}^{n} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}.$$

In this paper, we always consider real symmetric tensors. The identity tensor $\mathcal{I}$ with order m and dimension n is given by $\mathcal{I}_{i_1 \cdots i_m} = 1$ if $i_1 = \cdots = i_m$ and $\mathcal{I}_{i_1 \cdots i_m} = 0$ otherwise.

We first fix some symbols and recall some basic facts. Let $m, n \in \mathbb{N}$. Consider $S_{m,n} := \{\mathcal{A} : \mathcal{A}$ is an mth-order n-dimensional symmetric tensor$\}$. Clearly, $S_{m,n}$ is a vector space under the addition and multiplication defined as below: for any $t \in \mathbb{R}$, $\mathcal{A} = (a_{i_1 \cdots i_m})_{1 \leqslant i_1, \cdots, i_m \leqslant n}$ and $\mathcal{B} = (b_{i_1 \cdots i_m})_{1 \leqslant i_1, \cdots, i_m \leqslant n}$,

$$\mathcal{A} + \mathcal{B} = (a_{i_1 \cdots i_m} + b_{i_1 \cdots i_m})_{1 \leqslant i_1, \cdots, i_m \leqslant n} \text{ and } t\mathcal{A} = (t a_{i_1 \cdots i_m})_{1 \leqslant i_1, \cdots, i_m \leqslant n}.$$

For each $\mathcal{A}, \mathcal{B} \in S_{m,n}$, we define the inner product by

$$\langle \mathcal{A}, \mathcal{B} \rangle := \sum_{i_1, \cdots, i_m = 1}^{n} a_{i_1 \cdots i_m} b_{i_1 \cdots i_m}.$$

The corresponding norm is defined by $\|\mathcal{A}\| = (\langle \mathcal{A}, \mathcal{A} \rangle)^{1/2} = \left(\sum_{i_1, \cdots, i_m = 1}^{n} (a_{i_1 \cdots i_m})^2 \right)^{1/2}$.

For a vector $\mathbf{x} \in \mathbb{R}^n$, we use x_i to denote its ith component. Moreover, for a vector $\mathbf{x} \in \mathbb{R}^n$, we use $\mathbf{x}^m$ to denote the mth-order n-dimensional symmetric rank one tensor induced by $\mathbf{x}$, i.e.,

$$(\mathbf{x}^m)_{i_1 i_2 \cdots i_m} = x_{i_1} x_{i_2} \cdots x_{i_m}, \ \forall\, i_1, \cdots, i_m \in \{1, \cdots, n\}.$$

Suppose $\mathbf{x} = (x_1, x_2, \cdots, x_n)$, $\mathbf{y} = (y_1, y_2, \cdots, y_n)$. Then $\mathbf{x} \geqslant \mathbf{y}$ ($\mathbf{x} \leqslant \mathbf{y}$) means $x_i \geqslant y_i$ ($x_i \leqslant y_i$) for all $i \in [n]$. If both $\mathcal{A} = (a_{i_1 \cdots i_m})_{1 \leqslant i_j \leqslant n}$ and $\mathcal{B} = (b_{i_1 \cdots i_m})_{1 \leqslant i_j \leqslant n}$, $j = 1, \cdots, m$, are tensors, then $\mathcal{A} \geqslant \mathcal{B}$ ($\mathcal{A} \leqslant \mathcal{B}$) means $a_{i_1 \cdots i_m} \geqslant b_{i_1 \cdots i_m}$ ($a_{i_1 \cdots i_m} \leqslant b_{i_1 \cdots i_m}$) for all $i_1, \cdots, i_m \in [n]$.

2.1 H-eigenvalue and Z-eigenvalue of tensors

We now recall the definitions of eigenvalues and eigenvectors for a tensor [78, 62].

Definition 2.1. *Let $\mathbb{C}$ be the complex field. Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be an order m dimension n tensor. A pair $(\lambda, \mathbf{x}) \in \mathbb{C} \times (\mathbb{C}^n \backslash \{\mathbf{0}\})$ is called an eigenvalue-eigenvector pair of tensor $\mathcal{A}$, if they satisfy*

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]},$$

where $\mathcal{A}\mathbf{x}^{m-1}$ and $\mathbf{x}^{[m-1]}$ are all n dimensional column vectors given by

$$\mathcal{A}\mathbf{x}^{m-1} = \left(\sum_{i_2, \cdots, i_m = 1}^{n} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \right)_{1 \leqslant i \leqslant n}$$

and $\mathbf{x}^{[m-1]} = (x_1^{m-1}, \ldots, x_n^{m-1})^T \in \mathbb{C}^n$.

If the eigenvalue λ and the eigenvector $\mathbf{x}$ are real, then λ is called an H-eigenvalue of $\mathcal{A}$ and $\mathbf{x}$ is its corresponding H-eigenvector.

Definition 2.2. *Let $\mathcal{A}$ be a symmetric tensor with order m and dimension n. We say $\lambda \in \mathbb{R}$ is a Z-eigenvalue of $\mathcal{A}$ and $\mathbf{x} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ is an Z-eigenvector corresponding to λ if $(\mathbf{x}, \lambda)$ satisfies*

$$\begin{cases} \mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}, \\ \mathbf{x}^T \mathbf{x} = 1. \end{cases}$$

The definitions of Z-eigenvalue and H-eigenvalue were introduced by Qi in [78]. Independently, Lim [62] also gave the definitions via a variational approach and established an interesting Perron-Frobenius theorem for tensors with non-negative entries. From [78] and [6], both Z-eigenvalues and H-eigenvalues for an even order symmetric tensor always exist. Moreover, from the definitions, we can see that finding an H-eigenvalue of a symmetric tensor is equivalent to solving a homogeneous polynomial equation while calculating a Z-eigenvalue is equivalent to solving non-homogeneous polynomial equations. In general, the behaviors of Z-eigenvalues and H-eigenvalues can be quite different. For example, a diagonal symmetric tensor $\mathcal{A}$ has exactly n many H-eigenvalues and may have more than n Z-eigenvalues (for more details see [78]). Recently, a lot of researchers have devoted themselves to the study of eigenvalue problems of symmetric tensors and have found important applications in diverse areas including spectral hypergraph theory [78, 58], dynamical control [72], medical image science [57, 88] and signal processing [49].

The spectral radius of tensor $\mathcal{A}$ is denoted by

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{A}\}.$$

All eigenvalues of tensor $\mathcal{A}$ construct the spectrum denoted by $Spec(\mathcal{A})$.

Next, we present two fundamental results about eigenvalues of tensors (see [78]), which will be much used in the sequel.

Lemma 2.1. *Suppose that $\mathcal{T} = a(\mathcal{B} + b\mathcal{I})$, where a and b are two real numbers. Then μ is an eigenvalue (H-eigenvalue) of tensor $\mathcal{T}$ if and only if $\mu = a(\lambda + b)$, where λ is an eigenvalue (H-eigenvalue) of tensor $\mathcal{B}$. In this case, they have the same eigenvectors (H-eigenvectors).*

Lemma 2.2. *Let $\mathcal{A}$ be a symmetric tensor with order m and dimension n. Suppose that the minimum H-eigenvalue and maximum H-eigenvalue of $\mathcal{A}$ are denoted by $\lambda_{min}(\mathcal{A})$ and $\lambda_{max}(\mathcal{A})$ respectively. Then, we have*

$$\lambda_{min}(\mathcal{A}) = \min_{\mathbf{x}\neq\mathbf{0}} \frac{\mathcal{A}\mathbf{x}^m}{\|\mathbf{x}\|_m^m} = \min_{\|\mathbf{x}\|_m=1} \mathcal{A}\mathbf{x}^m, \quad \lambda_{max}(\mathcal{A}) = \max_{\mathbf{x}\neq\mathbf{0}} \frac{\mathcal{A}\mathbf{x}^m}{\|\mathbf{x}\|_m^m} = \max_{\|\mathbf{x}\|_m=1} \mathcal{A}\mathbf{x}^m,$$

where $\|\mathbf{x}\|_m = (\sum_{i=1}^n |x_i|^m)^{\frac{1}{m}}$.

2.2 Positive semi-definite tensors

We first note that an m-th order n-dimensional symmetric tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$, uniquely defines an m-th degree homogeneous polynomial $f_{\mathcal{A}}(\mathbf{x})$ on $\mathbb{R}^n$: for all $\mathbf{x} = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$,

$$f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m = \sum_{i_1,i_2,\cdots,i_m\in[n]} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}. \tag{2.1}$$

Conversely, any m-th degree homogeneous polynomial function $f(\mathbf{x})$ on $\mathbb{R}^n$ also uniquely corresponds a symmetric tensor. Furthermore, an even order tensor $\mathcal{A}$ is called positive semi-definite (positive definite) if $f_{\mathcal{A}}(\mathbf{x}) \geqslant 0$ ($f_{\mathcal{A}}(\mathbf{x}) > 0$) for all $\mathbf{x} \in \mathbb{R}^n$ ($\mathbf{x} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$).

Denote $\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \geqslant \mathbf{0}\}$. If $\mathcal{A}\mathbf{x}^m \geqslant 0$ for all $\mathbf{x} \in \mathbb{R}^n_+$, then $\mathcal{A}$ is called co-positive. Positive semi-definite tensors are co-positive tensors, but the converse maybe not true in general. From the definition, it is easy to see that, for a positive semi-definite tensor, its order m must be an even number. Therefore, in the following analysis, we always assume the order of the tensor is even when we consider a positive semi-definite tensor. An important fact which will be used frequently later on is that an even order symmetric tensor is positive semi-definite (definite) if and only if all H-eigenvalues of the tensor are non-negative (positive).

We call $\mathbf{u} \in \mathbb{R}^n$ a Vandermonde vector if $\mathbf{u} = (1, \mu, \mu^2, \cdots, \mu^{n-1})^T \in \mathbb{R}^n$ for some $\mu \in \mathbb{R}$. If $\mathcal{A}\mathbf{u}^m \geqslant 0$ for all Vandermonde vectors $\mathbf{u} \in \mathbb{R}^n$, then we say that tensor $\mathcal{A}$ is Vandermonde positive semi-definite. It's obvious that positive semi-definite tensors are always Vandermonde positive semi-definite, but not vice versa.

2.3 SOS tensor decomposition

Tensor decomposition is an important research area, and it has found numerous applications in data mining [44, 46, 45], computational neuroscience [16, 25], and statistical learning for latent variable models [1]. An important class of tensor decomposition is sum-of-squares (SOS) tensor decomposition.

Suppose $\mathcal{A}$ is a symmetric tensor with order m and dimension n. Let $f_{\mathcal{A}}(\mathbf{x})$ be the homogeneous polynomial corresponding tensor $\mathcal{A}$ such as in (2.1). If $f_{\mathcal{A}}(\mathbf{x})$ is a sums-of-squares (SOS) polynomial, then we say $\mathcal{A}$ has an SOS tensor decomposition (or an SOS decomposition, for simplicity). It is clear that a tensor with SOS decomposition and an SOS polynomial must have even degree. If a given tensor has SOS decomposition, then the tensor is positive semi-definite, but not vice versa. Next, we recall a useful lemma which provides a test for verifying whether a homogeneous polynomial is a sums-of-squares polynomial or not. To do this, we introduce some basic notions.

For all $\mathbf{x} \in \mathbb{R}^n$, consider a homogeneous polynomial $f(\mathbf{x}) = \sum_{\alpha} f_{\alpha}\mathbf{x}^{\alpha}$ with degree m (m is an even number), where $\alpha = (\alpha_1, \cdots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$, $\mathbf{x}^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $|\alpha| := \sum_{i=1}^n \alpha_i = m$. Let $f_{m,i}$ be the coefficient associated with x_i^m. Let $\mathbf{e_i}$ be the ith unit vector and let

$$\Omega_f = \{\alpha = (\alpha_1, \cdots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n : f_\alpha \neq 0 \text{ and } \alpha \neq m\,\mathbf{e_i},\ i = 1, \cdots, n\}. \quad (2.2)$$

Then, f can be decomposed as $f(\mathbf{x}) = \sum_{i=1}^n f_{m,i} x_i^m + \sum_{\alpha \in \Omega_f} f_\alpha \mathbf{x}^\alpha$. Recall that $2\mathbb{N}$ denotes the set consisting of all the even numbers. Define

$$\hat{f}(\mathbf{x}) = \sum_{i=1}^n f_{m,i} x_i^m - \sum_{\alpha \in \Delta_f} |f_\alpha| \mathbf{x}^\alpha,$$

where

$$\Delta_f := \{\alpha = (\alpha_1, \cdots, \alpha_n) \in \Omega_f : f_\alpha < 0 \text{ or } \alpha \notin (2\mathbb{N} \cup \{0\})^n\}. \quad (2.3)$$

Lemma 2.3. *[23, Corollary 2.8] Let f be a homogeneous polynomial of degree m, where m is an even number. If $\hat{f}$ is a polynomial which always takes non-negative values, then f is a sums-of-squares polynomial.*

Chapter 3

Positive definiteness and semi-definiteness of even order Cauchy tensors

A Cauchy matrix (maybe not square) is an $m \times n$ structure matrix assigned to $m+n$ parameters $x_1, x_2, \cdots, x_m, y_1, \cdots, y_n$ as follows: [74]

$$C = \left[\frac{1}{x_i + y_j} \right], \ i \in [m], j \in [n]. \tag{3.1}$$

The Cauchy matrix has been studied and applied in algorithm designing [26, 30, 33]. When $x_i = y_i$ in (3.1), it is a real symmetric Cauchy matrix. Stimulated by the notion of symmetric Cauchy matrices, we give the following definition.

Definition 3.1. *Let vector* $\mathbf{c} = (c_1, c_2, \cdots, c_n) \in \mathbb{R}^n$. *Suppose that a real tensor* $\mathcal{C} = (c_{i_1 i_2 \cdots i_m})$ *is defined by*

$$c_{i_1 i_2 \cdots i_m} = \frac{1}{c_{i_1} + c_{i_2} + \cdots + c_{i_m}}, \quad j \in [m], \ i_j \in [n].$$

Then, we say that $\mathcal{C}$ *is an order* m *dimension* n *symmetric Cauchy tensor and the vector* $\mathbf{c} \in \mathbb{R}^n$ *is called the generating vector of* $\mathcal{C}$.

We should point out that, in Definition 3.1, for any m elements $c_{i_1}, c_{i_2}, \cdots, c_{i_m}$

in generating vector $\mathbf{c}$, it satisfies

$$c_{i_1} + c_{i_2} + \cdots + c_{i_m} \neq 0,$$

which implies that $c_i \neq 0$, $i \in [n]$.

By Definition 3.1, a dimension $n \times n$ real symmetric Cauchy matrix is an order 2 dimension n real symmetric Cauchy tensor. It is easy to check that every principal subtensors of a symmetric Cauchy tensor is a symmetric Cauchy tensor with a generating vector being a subvector of the generating vector of the original symmetric Cauchy tensor. In this chapter, we always consider mth order n dimensional real symmetric Cauchy tensors. Hence, it can be called Cauchy tensors for simplicity.

Cauchy tensors belong to structured tensors, and they have close relationships with Hankel tensors and Hilbert tensors. Suppose Cauchy tensor $\mathcal{C}$ and its generating vector $\mathbf{c}$ are defined as in Definition 3.1. If

$$c_{i_1} + c_{i_2} + \cdots + c_{i_m} \equiv c_{j_1} + c_{j_2} + \cdots + c_{j_m}$$

whenever

$$i_1 + i_2 + \cdots + i_m = j_1 + j_2 + \cdots + j_m,$$

then Cauchy tensor $\mathcal{C}$ is a Hankel tensor in the sense of [80]. In general, a symmetric Cauchy tensor is not a Hankel tensor. If entries of $\mathbf{c}$ are defined such that

$$c_i = i - 1 + \frac{1}{m},\ i \in [n],$$

then Cauchy tensor $\mathcal{C}$ is a Hilbert tensor according to [100].

3.1 Positive semi-definite Cauchy tensors

In this section, we will give some sufficient and necessary conditions for even order Cauchy tensors to be positive semi-definite or positive definite. Some conditions are extended naturally from the Cauchy matrix case.

Theorem 3.1. *Assume a Cauchy tensor $\mathcal{C}$ is of even order. Let $\mathbf{c} \in \mathbb{R}^n$ be the generating vector of $\mathcal{C}$. Then Cauchy tensor $\mathcal{C}$ is positive semi-definite if and only if $\mathbf{c} > 0$.*

Proof. For necessity, suppose that an even order Cauchy tensor $\mathcal{C}$ is positive semi-definite. It is easy to check that all composites of generating vector $\mathbf{c}$ are positive since

$$\mathcal{C}\mathbf{e_i}^m = \frac{1}{mc_i} \geqslant 0, \ i \in [n]$$

where $\mathbf{e_i}$ is the ith coordinate vector of $\mathbb{R}^n$. So, $c_i > 0$ for all $i \in [n]$, which means $\mathbf{c} > 0$.

On the other hand, assume that $\mathbf{c} > 0$. For any $\mathbf{x} \in \mathbb{R}^n$, it holds that

$$\begin{aligned} \mathcal{C}\mathbf{x}^m &= \textstyle\sum_{i_1,\cdots,i_m \in [n]} c_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m} \\ &= \textstyle\sum_{i_1,\cdots,i_m \in [n]} \frac{x_{i_1} x_{i_2} \cdots x_{i_m}}{c_{i_1} + c_{i_2} + \cdots + c_{i_m}} \\ &= \textstyle\sum_{i_1,\cdots,i_m \in [n]} \displaystyle\int_0^1 t^{c_{i_1} + c_{i_2} + \cdots + c_{i_m} - 1} x_{i_1} x_{i_2} \cdots x_{i_m} dt \\ &= \int_0^1 \left(\sum_{i \in [n]} t^{c_i - \frac{1}{m}} x_i \right)^m dt \\ &\geqslant 0. \end{aligned}$$

Here the last inequality follows that m is even. By the arbitrariness of $\mathbf{x}$, we know that Cauchy tensor $\mathcal{C}$ is positive semi-definite and the desired result holds. □

Corollary 3.1. *Assume that even order Cauchy tensor $\mathcal{C}$ and its generating vector $\mathbf{c} \in \mathbb{R}^n$ are defined as in Theorem 3.1. Then Cauchy tensor $\mathcal{C}$ is negative semi-definite if and only if $\mathbf{c} < 0$.*

Corollary 3.2. *Assume that even order Cauchy tensor $\mathcal{C}$ and its generating vector $\mathbf{c} \in \mathbb{R}^n$ are defined as in Theorem 3.1. Then Cauchy tensor $\mathcal{C}$ is not positive semi-definite if and only if there exist at least one negative element in $\mathbf{c}$.*

From the results about H-eigenvalues and Z-eigenvalues in [78], we have the following result.

Corollary 3.3. *Assume that even order Cauchy tensor $\mathcal{C}$ and its generating vector $\mathbf{c} \in \mathbb{R}^n$ are defined as in Theorem 3.1. If $\mathbf{c} > 0$, then all the H-eigenvalues and Z-eigenvalues of Cauchy tensor $\mathcal{C}$ are non-negative.*

Theorem 3.2. *Assume even order Cauchy tensor $\mathcal{C}$ has generating vector $\mathbf{c} = (c_1, c_2, \cdots, c_n) \in \mathbb{R}^n$. Suppose $c_1, c_2, \cdots, c_n$ are positive and mutually distinct. Then Cauchy tensor $\mathcal{C}$ is positive definite.*

Proof. For the sake of simplicity, without loss of generality, assume that

$$0 < c_1 < c_2 < \cdots < c_n.$$

Since $\mathbf{c} > 0$ and $c_1, c_2, \cdots, c_n$ are mutually distinct. From Theorem 3.1, we know that Cauchy tensor $\mathcal{C}$ is positive semi-definite.

We prove by contradiction that Cauchy tensor $\mathcal{C}$ is positive definite when the conditions of this theorem hold. Assume there exists a nonzero vector $\mathbf{x} \in \mathbb{R}^n$ such that

$$\mathcal{C}\mathbf{x}^m = 0.$$

By the proof of Theorem 3.1, one has

$$\int_0^1 \left(\sum_{i \in [n]} t^{c_i - \frac{1}{m}} x_i \right)^m dt = 0,$$

which means

$$\sum_{i \in [n]} t^{c_i - \frac{1}{m}} x_i \equiv 0, \ t \in [0, 1].$$

Thus

$$x_1 + t^{c_2 - c_1} x_2 + \cdots + t^{c_n - c_1} x_n \equiv 0, \ t \in (0, 1].$$

By continuity and the fact that $c_1, c_2, \cdots, c_n$ are mutually distinct, it holds that

$$x_1 = 0$$

and

$$x_2 + t^{c_3 - c_2} x_3 + \cdots + t^{c_n - c_2} x_n \equiv 0,\ t \in (0, 1].$$

Repeat the process above. We obtain

$$x_1 = x_2 = \cdots = x_n = 0,$$

which is a contradiction with $\mathbf{x} \neq 0$. So, for all nonzero vectors $\mathbf{x} \in \mathbb{R}^n$, it holds $\mathcal{C}\mathbf{x}^m > 0$ and $\mathcal{C}$ is positive definite. □

From this theorem, we easily have the following corollary, which was first proved in [100].

Corollary 3.4. *An even order Hilbert tensor is positive definite.*

From Theorem A of [24], we know that a symmetric Cauchy matrix

$$C = \left[\frac{1}{c_i + c_j}\right]$$

is positive definite if and only if all the c_i's are positive and mutually distinct. In fact, the theorem below shows that conditions in Theorem 3.2 is also a sufficient and necessary condition, which is a natural extension of Theorem A of [24], by Fielder.

Theorem 3.3. *Let even order Cauchy tensor $\mathcal{C}$ and its generating vector $\mathbf{c}$ be defined as in Theorem 3.2. Then, Cauchy tensor $\mathcal{C}$ is positive definite if and only if the elements of generating vector are positive and mutually distinct.*

Proof. By Theorem 3.2, we only need to prove the "only if" part of this theorem. Suppose that Cauchy tensor $\mathcal{C}$ is positive definite. Firstly, by Theorem 3.1, we know that

$$c_i > 0,\ i \in [n].$$

We now prove by contradiction that c_i's are mutually distinct. Suppose that two elements of $\mathbf{c}$ are equal. Without loss of generality, assume $c_1 = c_2 = a > 0$. Let $\mathbf{x} \in \mathbb{R}^n$ be a vector with elements $x_1 = 1, x_2 = -1$ and $x_i = 0$ for the others. Then, one has

$$\begin{aligned}
\mathcal{C}\mathbf{x}^m &= \textstyle\sum_{i_1,\cdots,i_m\in[n]} c_{i_1i_2\cdots i_m} x_{i_1}x_{i_2}\cdots x_{i_m} \\
&= \textstyle\sum_{i_1,\cdots,i_m\in[n]} \frac{x_{i_1}x_{i_2}\cdots x_{i_m}}{c_{i_1}+c_{i_2}+\cdots+c_{i_m}} \\
&= \frac{1}{ma}\textstyle\sum_{i_1,\cdots,i_m\in[2]} x_{i_1}x_{i_2}\cdots x_{i_m} \\
&= \frac{1}{ma}\left[(-1)^m + m(-1)^{m-1} + \frac{m!}{2!(m-2)!}(-1)^{m-2} + \cdots + (-1)^{m-m}\right] \\
&= \frac{1}{ma}\left[1+(-1)\right]^m \\
&= 0,
\end{aligned}$$

where we get a contradiction with the assumption that Cauchy tensor $\mathcal{C}$ is positive definite. Thus, elements of generating vector $\mathbf{c}$ are mutually distinct and the desired result follows. □

We denote the homogeneous polynomial $\mathcal{C}\mathbf{x}^m$ as

$$f(\mathbf{x}) = \mathcal{C}\mathbf{x}^m = \sum_{i_1,\cdots,i_m\in[n]} c_{i_1i_2\cdots i_m} x_{i_1}x_{i_2}\cdots x_{i_m}.$$

For all $\mathbf{x}, \mathbf{y} \in X \subseteq \mathbb{R}^n$, if $f(\mathbf{x}) \geqslant f(\mathbf{y})$ when $\mathbf{x} \geqslant \mathbf{y}(\mathbf{x} \leqslant \mathbf{y})$, we say that $f(\mathbf{x})$ is monotone increasing (monotone decreasing respectively) in X. If $f(\mathbf{x}) > f(\mathbf{y})$ when $\mathbf{x} \geqslant \mathbf{y}$, $\mathbf{x} \neq \mathbf{y}(\mathbf{x} \leqslant \mathbf{y}$, $\mathbf{x} \neq \mathbf{y})$, we say that $f(\mathbf{x})$ is strictly monotone increasing (strict monotone decreasing respectively) in X.

The following conclusion means that the positive semi-definite property of a Cauchy tensor is equivalent to the monotonicity of a homogeneous polynomial respected to the Cauchy tensor in $\mathbb{R}^n_+$.

Theorem 3.4. *Let $\mathcal{C}$ be an even order Cauchy tensor with generating vector $\mathbf{c}$. Then, $\mathcal{C}$ is positive semi-definite if and only if $f(\mathbf{x})$ is monotone increasing in $\mathbb{R}^n_+$.*

Proof. For sufficiency, let $\mathbf{x} = \mathbf{e_i}, \mathbf{y} = \mathbf{0}$ and $\mathbf{x} \geqslant \mathbf{y}$. Then we have

$$\frac{1}{mc_i} = \mathcal{C}\mathbf{x}^m = f(\mathbf{x}) \geqslant f(\mathbf{y}) = \mathcal{C}\mathbf{y}^m = 0,$$

which implies that $c_i > 0$ for $i \in [n]$. By Theorem 3.1, it holds that Cauchy tensor $\mathcal{C}$ is positive semi-definite.

For necessary conditions, suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$ and $\mathbf{x} \geqslant \mathbf{y}$. Then, we know that

$$\begin{array}{rl} f(\mathbf{x}) - f(\mathbf{y}) = & \mathcal{C}\mathbf{x}^m - \mathcal{C}\mathbf{y}^m \\ = & \sum_{i_1,\cdots,i_m \in [n]} c_{i_1 i_2 \cdots i_m}(x_{i_1} x_{i_2} \cdots x_{i_m} - y_{i_1} y_{i_2} \cdots y_{i_m}) \\ = & \sum_{i_1,\cdots,i_m \in [n]} \frac{x_{i_1} x_{i_2} \cdots x_{i_m} - y_{i_1} y_{i_2} \cdots y_{i_m}}{c_{i_1} + c_{i_2} + \cdots + c_{i_m}} \\ \geqslant & 0. \end{array}$$

Here, the last inequality follows that $\mathbf{x} \geqslant \mathbf{y}$ and the fact that $c_i > 0$, for $i \in [n]$, which means that $f(\mathbf{x})$ is monotone increasing in $\mathbb{R}^n_+$ and the desired result holds. □

Lemma 3.1. *Let $\mathcal{C}$ be an even order Cauchy tensor with generating vector $\mathbf{c}$. Suppose $\mathcal{C}$ is positive definite. Then the homogeneous polynomial $f(\mathbf{x})$ is strictly monotone increasing in $\mathbb{R}^n_+$.*

Proof. From the condition that $\mathcal{C}$ is positive definite, by Theorem 3.3, we have

$$c_i > 0, \ i \in [n],$$

where scalars $c_i, \ i \in [n]$ are entries of generating vector $\mathbf{c}$. For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$ satisfying that $\mathbf{x} \geqslant \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$, there exists index $i \in [n]$ such that

$$x_i > y_i \geqslant 0.$$

Then, it holds that

$$\begin{array}{rl} f(\mathbf{x}) - f(\mathbf{y}) = & \mathcal{C}\mathbf{x}^m - \mathcal{C}\mathbf{y}^m \\ = & \sum_{i_1,\cdots,i_m \in [n],(i_1,i_2,\cdots,i_m) \neq (i,i,\cdots,i)} c_{i_1 i_2 \cdots i_m}(x_{i_1} x_{i_2} \cdots x_{i_m} - y_{i_1} y_{i_2} \cdots y_{i_m}) \\ & + c_{ii\cdots i}(x_i^m - y_i^m) \\ = & \sum_{i_1,\cdots,i_m \in [n],(i_1,i_2,\cdots,i_m) \neq (i,i,\cdots,i)} \frac{x_{i_1} x_{i_2} \cdots x_{i_m} - y_{i_1} y_{i_2} \cdots y_{i_m}}{c_{i_1} + c_{i_2} + \cdots + c_{i_m}} \\ & + \frac{1}{mc_i}(x_i^m - y_i^m) \\ > & 0, \end{array}$$

which implies that the homogeneous polynomial $f(\mathbf{x})$ is strictly monotone increasing in $\mathbb{R}^n_+$. $\square$

Now, we give an example to show that the strictly monotone increasing property for the polynomial $f(\mathbf{x})$ is only a necessary condition for the positive definiteness property of Cauchy tensor $\mathcal{C}$ but not a sufficient condition.

Example 3.1. *Let $\mathcal{C} = (c_{i_1i_2i_3i_4})$ be a Cauchy tensor with order 4 dimension 3, and with generating vector $\mathbf{c} = (1, 1, 1)$. Then,*

$$c_{i_1i_2i_3i_4} = \frac{1}{4},\ \ i_1, i_2, i_3, i_4 \in [3]$$

and the homogeneous polynomial

$$f(\mathbf{x}) = \mathcal{C}\mathbf{x}^4 = \frac{1}{4} \sum_{i_1,i_2,i_3,i_4\in[3]} x_{i_1}x_{i_2}x_{i_3}x_{i_4}.$$

By direct computation, we know that $f(\mathbf{x})$ is strictly monotone increasing in $\mathbb{R}^3_+$. From Theorem 3.3, Cauchy tensor $\mathcal{C}$ is not positive definite. $\square$

Let r_i denote the sum of the ith row elements of Cauchy tensor $\mathcal{C}$, which can be written such that

$$r_i = \sum_{i_2,\cdots,i_m\in[n]} \frac{1}{c_i + c_{i_2} + \cdots + c_{i_m}},\ \ i \in [n],$$

where $\mathbf{c} = (c_1, \cdots, c_n)$ is the generating vector of Cauchy tensor $\mathcal{C}$. Suppose

$$R = \max_{1\leqslant i\leqslant n} r_i,\ \ r = \min_{1\leqslant i\leqslant n} r_i.$$

If Cauchy tensor $\mathcal{C}$ is positive semi-definite, by Theorem 3.1, it is easy to check that

$$R = \sum_{i_2,\cdots,i_m\in[n]} \frac{1}{\underline{a} + c_{i_2} + \cdots + c_{i_m}},\ \ r = \sum_{i_2,\cdots,i_m\in[n]} \frac{1}{\bar{a} + c_{i_2} + \cdots + c_{i_m}},$$

where $\underline{a} = \min_{1 \leqslant i \leqslant n} c_i, \ \bar{a} = \max_{1 \leqslant i \leqslant n} c_i$.

Now, before giving the next conclusion, we give the definition of irreducible tensors, which will be used in the sequel. The following definition is consistent with [5] and [80] respectively.

Definition 3.2. *For a tensor $\mathcal{T}$ with order m and dimension n. We say that $\mathcal{T}$ is reducible if there is a nonempty proper index subset $I \subset [n]$ such that*

$$t_{i_1 i_2 \cdots i_m} = 0, \ \forall \ i_1 \in I, \ \forall \ i_2, i_3, \cdots, i_m \notin I.$$

Otherwise we say that $\mathcal{T}$ is irreducible.

Theorem 3.5. *Let $\mathcal{C}$ be a positive semi-definite even order tensor with generating vector $\mathbf{c} \in \mathbb{R}^n$. Suppose $\mathbf{x} \in \mathbb{R}^n$ is the eigenvector of $\mathcal{C}$ corresponding to $\rho(\mathcal{C})$. Assume*

$$x_{\bar{i}} = \max_{1 \leqslant i \leqslant n} x_i, \ x_{\underline{i}} = \min_{1 \leqslant i \leqslant n} x_i. \tag{3.2}$$

Then, $R = r_{\bar{i}}, \ r = r_{\underline{i}}$.

Proof. Since Cauchy tensor $\mathcal{C}$ is positive semi-definite, from Theorem 3.1, all elements of $\mathcal{C}$ and $\mathbf{c}$ are positive. By Definition 3.2, we know that $\mathcal{C}$ is irreducible. Thus $\mathbf{x} > 0$ from Theorem 1.4 of [5]. Without loss of generality, suppose

$$R = r_l, \ r = r_s.$$

By the analysis before this theorem, it holds that

$$\bar{a} = \max_{1 \leqslant i \leqslant n} c_i = c_s, \ \underline{a} = \min_{1 \leqslant i \leqslant n} c_i = c_l.$$

On the other side, by Definition 2.1, we have

$$\begin{array}{rcl} \rho(\mathcal{C}) x_{\bar{i}}^{m-1} & = & (\mathcal{C}\mathbf{x}^{m-1})_{\bar{i}} \\ & = & \sum_{i_2, \cdots, i_m \in [n]} c_{\bar{i} i_2 i_3 \cdots i_m} x_{i_2} \cdots x_{i_m} \\ & = & \sum_{i_2, \cdots, i_m \in [n]} \frac{x_{i_2} \cdots x_{i_m}}{c_{\bar{i}} + c_{i_2} + \cdots + c_{i_m}} \\ & \leqslant & \sum_{i_2, \cdots, i_m \in [n]} \frac{x_{i_2} \cdots x_{i_m}}{\underline{a} + c_{i_2} + \cdots + c_{i_m}} \\ & = & (\mathcal{C}\mathbf{x}^{m-1})_l \\ & = & \rho(\mathcal{C}) x_l^{m-1}, \end{array}$$

which implies that

$$x_{\bar{i}} = x_l$$

So we can take $\bar{i} = l$ and $R = r_{\bar{i}}$ holds. Similarly, by Definition 2.1 and (3.2), one has

$$\begin{array}{rcl}\rho(\mathcal{C})x_{\underline{i}}^{m-1} & = & (\mathcal{C}\mathbf{x}^{m-1})_{\underline{i}} \\ & = & \sum_{i_2,\cdots,i_m\in[n]} c_{\underline{i}i_2i_3\cdots i_m} x_{i_2}\cdots x_{i_m} \\ & = & \sum_{i_2,\cdots,i_m\in[n]} \frac{x_{i_2}\cdots x_{i_m}}{c_{\underline{i}}+c_{i_2}+\cdots+c_{i_m}} \\ & \geqslant & \sum_{i_2,\cdots,i_m\in[n]} \frac{x_{i_2}\cdots x_{i_m}}{\bar{a}+c_{i_2}+\cdots+c_{i_m}} \\ & = & (\mathcal{C}\mathbf{x}^{m-1})_s \\ & = & \rho(\mathcal{C})x_s^{m-1},\end{array}$$

which means that $x_{\underline{i}} = x_s$ and we can take $\underline{i} = s$. Thus $r = r_{\underline{i}}$ and the desired results follows. □

Theorem 3.6. *Suppose an even order Cauchy tensor $\mathcal{C}$ has positive generating vector $\mathbf{c} \in \mathbb{R}^n$. Then $\mathcal{C}$ is positive definite if and only if $r_1, r_2, \cdots, r_n$ are mutually distinct.*

Proof. By conditions, all elements of $\mathbf{c}$ are positive, so it is obvious that $r_1, r_2, \cdots, r_n$ are mutually distinct if and only if $c_1, c_2, \cdots, c_n$ are mutually distinct. By Theorem 3.3, the desired conclusion follows. □

3.2 Inequalities for Cauchy tensors

In this section, we give several inequalities about the largest and the smallest H-eigenvalues of Cauchy tensors. The bounds for the largest H-eigenvalues are given for positive semi-definite Cauchy tensors. Moreover, properties of Z-eigenvalues and Z-eigenvectors of odd order Cauchy tensors are also shown.

It should be noted that a real symmetric tensor always has Z-eigenvalues and an even order real symmetric tensor always has H-eigenvalues [78]. We denote the largest and smallest H-eigenvalues of Cauchy tensor $\mathcal{C}$ by λ_{max} and λ_{min} respectively. When $\mathcal{C}$ is a positive semi-definite Cauchy tensor, then by the Perron-Frobenius theory of non-negative tensors [5], we have

$$\lambda_{max} = \rho(\mathcal{C}).$$

Lemma 3.2. *Assume $\mathcal{C}$ is a Cauchy tensor with generating vector $\mathbf{c}$. If the entries of $\mathbf{c} = (c_1, c_2, \cdots, c_n)$ have different signs, then,*

$$\lambda_{min} \leqslant \frac{1}{m \max\{c_i |\ c_i < 0, i \in [n]\}} < 0 < \frac{1}{m \min\{c_i |\ c_i > 0, i \in [n]\}} \leqslant \lambda_{max}.$$

Proof. From Theorem 5 of [78], we have

$$\lambda_{max} = \max \left\{ \mathcal{C}\mathbf{x}^m |\ \sum_{i \in [n]} x_i^m = 1,\ \mathbf{x} \in \mathbb{R}^n \right\}$$

and

$$\lambda_{min} = \min \left\{ \mathcal{C}\mathbf{x}^m |\ \sum_{i \in [n]} x_i^m = 1,\ \mathbf{x} \in \mathbb{R}^n \right\}.$$

Combining this with the fact that

$$\mathcal{C}\mathbf{e_i}^m = \frac{1}{mc_i},\ i \in [n],$$

we have the conclusion of the lemma. □

Let r, R, $\bar{a}$ and $\underline{a}$ be defined as in Section 3.1. We have the following result.

Theorem 3.7. *Assume even order Cauchy tensor $\mathcal{C}$ has generating vector $\mathbf{c} = (c_1, c_2, \cdots, c_n)$. Suppose $\mathbf{c} > 0$ and at least two elements of $\mathbf{c}$ are different. Then*

$$r + \frac{1}{m\bar{a}} \left(\sqrt{\frac{R}{r}} - 1 \right) < \lambda_{max} < R - \frac{1}{m\bar{a}} \left(1 - \sqrt{\frac{r}{R}} \right).$$

Proof. Suppose $\mathbf{x} \in \mathbb{R}^n$ is the eigenvector of $\mathcal{C}$ corresponding to λ_{max}. By conditions, Cauchy tensor $\mathcal{C}$ is an irreducible non-negative tensor and it follows $\mathbf{x} > 0$ from Theorem 1.4 of [5]. Without loss of generality, let $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and suppose $0 < x_i \leqslant 1,\ i \in [n]$ such that

$$x_s = \min_{i\in[n]} x_i > 0,\ x_l = \max_{j\in[n]} x_j = 1. \tag{3.3}$$

By Theorem 3.5, we have

$$R = r_l,\ r = r_s$$

and $R > r$ since at least two entries of $\mathbf{c}$ are not equal.

On the other side, by the definition of eigenvalues, from (3.3), one has

$$\begin{aligned} \lambda_{max} x_s^{m-1} &= (\mathcal{C}\mathbf{x}^{m-1})_s \\ &= \textstyle\sum_{i_2,\cdots,i_m\in[n]} c_{si_2\cdots i_m} x_{i_2} x_{i_3} \cdots x_m \\ &\leqslant \textstyle\sum_{i_2,\cdots,i_m\in[n]} c_{si_2\cdots i_m} \\ &= r, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \lambda_{max} &= \lambda_{max} x_l^{m-1} \\ &= (\mathcal{C}\mathbf{x}^{m-1})_l \\ &= \textstyle\sum_{i_2,\cdots,i_m\in[n]} c_{li_2\cdots i_m} x_{i_2} x_{i_3} \cdots x_m \\ &\geqslant x_s^{m-1} \textstyle\sum_{i_2,\cdots,i_m\in[n]} c_{li_2\cdots i_m} \\ &= R x_s^{m-1}. \end{aligned} \tag{3.5}$$

Thus, by (3.4) and (3.5), we have

$$0 < x_s^{m-1} \leqslant \frac{\lambda_{max}}{R} \leqslant \frac{r}{x_s^{m-1} R},$$

which can be written as

$$0 < x_s^{m-1} \leqslant \sqrt{\frac{r}{R}}.$$

Combining this with (3.3), we obtain

$$\begin{array}{rcl}\lambda_{max} &=& \lambda_{max}x_l^{m-1}\\ &=& (\mathcal{C}\mathbf{x}^{m-1})_l\\ &=& \sum_{i_2,\cdots,i_m\in[n],(l,i_2,\cdots,i_m)\neq(l,s,\cdots,s)} c_{li_2\cdots i_m}x_{i_2}x_{i_3}\cdots x_m + c_{ls\cdots s}x_s^{m-1}\\ &<& \sum_{i_2,\cdots,i_m\in[n]} c_{li_2\cdots i_m} - c_{ls\cdots s} + c_{ls\cdots s}\sqrt{\frac{r}{R}}\\ &<& R - \frac{1}{m\bar{a}}(1-\sqrt{\frac{r}{R}}),\end{array}$$

and

$$\begin{array}{rcl}\lambda_{max} &=& \frac{1}{x_s^{m-1}}\sum_{i_2,\cdots,i_m\in[n]} c_{si_2\cdots i_m}x_{i_2}x_{i_3}\cdots x_m\\ &=& \frac{c_{sl\cdots l}x_l^{m-1}}{x_s^{m-1}} + \frac{1}{x_s^{m-1}}\sum_{i_2,\cdots,i_m\in[n],(i_2i_3\cdots i_m)\neq(ll\cdots l)} c_{si_2\cdots i_m}x_{i_2}x_{i_3}\cdots x_m\\ &=& \frac{c_{sl\cdots l}}{x_s^{m-1}} + \frac{1}{x_s^{m-1}}\sum_{i_2,\cdots,i_m\in[n],(i_2i_3\cdots i_m)\neq(ll\cdots l)} c_{si_2\cdots i_m}x_{i_2}x_{i_3}\cdots x_m\\ &>& \sqrt{\frac{R}{r}}c_{sll\cdots l} + r - c_{sll\cdots l}\\ &>& r + \frac{1}{m\bar{a}}(\sqrt{\frac{R}{r}}-1),\end{array}$$

from which we get the desired inequalities. □

Next, we will give several spectral properties for odd order Cauchy tensors.

Theorem 3.8. *Suppose an order m dimension n Cauchy tensor $\mathcal{C}$ has generating vector $\mathbf{c}$. Let m be odd and $\mathbf{c} > 0$. Assume $\lambda \in \mathbb{R}$ is a Z-eigenvalue of $\mathcal{C}$ with Z-eigenvector $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. If Z-eigenvalue $\lambda > 0$, then $\mathbf{x} \geqslant 0$; if Z-eigenvalue $\lambda < 0$, then $\mathbf{x} \leqslant 0$.*

Proof. By the condition $\mathbf{c} > 0$, we know that all entries of Cauchy tensor $\mathcal{C}$ are positive. By definitions of Z-eigenvalue and Z-eigenvector, for any $i \in [n]$, we have that

$$\begin{array}{rcl}\lambda x_i &=& (\mathcal{C}\mathbf{x}^{m-1})_i\\ &=& \sum_{i_2,i_3,\cdots,i_m\in[n]} c_{ii_2\cdots i_m}x_{i_2}x_{i_3}\cdots x_m\\ &=& \sum_{i_2,i_3,\cdots,i_m\in[n]} \frac{x_{i_2}x_{i_3}\cdots x_m}{c_i+c_{i_2}+\cdots+c_{i_m}}\\ &=& \sum_{i_2,i_3,\cdots,i_m\in[n]} \int_0^1 t^{c_i+c_{i_2}+\cdots+c_{i_m}-1}x_{i_2}x_{i_3}\cdots x_m dt\\ &=& \int_0^1 t^{c_i-\frac{1}{m}}(\sum_{j\in[n]} t^{c_j-\frac{1}{m}}x_j)^{m-1}dt.\end{array} \tag{3.6}$$

Since m is odd, by (3.6), one has

$$\lambda x_i \geqslant 0, \quad \text{for} \quad i \in [n],$$

which implies that $\mathbf{x} \geqslant \mathbf{0}$ when $\lambda > 0$ and $\mathbf{x} \leqslant \mathbf{0}$ when $\lambda < 0$. □

Theorem 3.9. *Suppose a Cauchy tensor $\mathcal{C}$ and its generating vector $\mathbf{c}$ are defined as in Theorem 3.8. If all entries of $\mathbf{c}$ are mutually distinct, then $\mathcal{C}$ has no zero Z-eigenvalue.*

Proof. By conditions, since entries of generating vector $\mathbf{c}$ are mutually distinct, without loss of generality, suppose

$$0 < c_1 < c_2 < \cdots < c_n.$$

We prove the result by contradiction. Suppose $\mathcal{C}$ has Z-eigenvalue $\lambda = 0$ with Z-eigenvector $\mathbf{x} \in \mathbb{R}^n$. Then, by (3.6), for any $i \in [n]$, we have

$$\int_0^1 t^{c_i - \frac{1}{m}} \left(\sum_{j \in [n]} t^{c_j - \frac{1}{m}} x_j \right)^{m-1} dt \equiv 0.$$

From properties of integration, one has

$$t^{c_i - \frac{1}{m}} \left(\sum_{j \in [n]} t^{c_j - \frac{1}{m}} x_j \right)^{m-1} \equiv 0, \ t \in [0, 1],$$

i.e.,

$$t^{c_i - \frac{1}{m}} \left(t^{c_1 - \frac{1}{m}} x_1 + t^{c_2 - \frac{1}{m}} x_2 + \cdots + t^{c_n - \frac{1}{m}} x_n \right) \equiv 0, \ t \in [0, 1]. \tag{3.7}$$

By (3.7), we obtain

$$t^{c_1 - \frac{1}{m}} x_1 + t^{c_2 - \frac{1}{m}} x_2 + \cdots + t^{c_n - \frac{1}{m}} x_n \equiv 0, \ t \in (0, 1],$$

which implies that

$$x_1 + t^{c_2-c_1}x_2 + \cdots + t^{c_n-c_1}x_n \equiv 0,\ t \in (0,1]. \tag{3.8}$$

Since $c_1, c_2, \cdots, c_m$ are mutually distinct, by the continuity property of operators and (3.8), it follows that

$$x_1 = 0.$$

Thus, the equation (3.8) can be written as

$$t^{c_2-c_1}x_2 + \cdots + t^{c_n-c_1}x_n \equiv 0,\ t \in (0,1],$$

which is equivalent to

$$x_2 + t^{c_3-c_2}x_3 + \cdots + t^{c_n-c_2}x_n \equiv 0,\ t \in (0,1].$$

By the continuity property, we have $x_2 = 0$. Repeating the process above, we get

$$x_1 = x_2 = \cdots = x_n = 0,$$

which is contradicting with the fact that $\mathbf{x}$ is a Z-eigenvector corresponding to $\lambda = 0$. The desired conclusion follows. □

Next, we have the following theorem on H-eigenvalues of non-negative Cauchy tensors. By [110], we know that each non-negative symmetric tensor has at least one H-eigenvalue, which is the largest modulus of its eigenvalues. Here, for non-negative Cauchy tensors, all the H-eigenvalues must be non-negative.

Theorem 3.10. *Let $\mathcal{C}$ be a non-negative Cauchy tensor with order m dimension n. Let $\mathbf{c} = (c_1, c_2, \cdots, c_n)^T$ be the generating vector of tensor $\mathcal{C}$. Then all H-eigenvalues of Cauchy tensor $\mathcal{C}$ are non-negative.*

Proof. In the case where m is even, since $\mathcal{C}$ is non-negative and by the definition of a Cauchy tensor, we have $c_i > 0$, $i = 1, \cdots, n$. By Theorem 3.1, we know that $\mathcal{C}$ is positive semi-definite. Then, Theorem 5 of [78] gives us that all H-eigenvalues of $\mathcal{C}$ are non-negative.

We now consider the case where m is odd. Let λ be an arbitrary H-eigenvalue of $\mathcal{C}$ with an H-eigenvector $\mathbf{x} \neq 0$. By the definition of H-eigenvalue, it holds that

$$\begin{aligned}
\lambda x_i^{m-1} =& (\mathcal{C}\mathbf{x}^{m-1})_i \\
=& \sum_{i_2,\cdots,i_m=1}^{n} \frac{x_{i_2}x_{i_3}\cdots x_{i_m}}{c_i + c_{i_2} + \cdots + c_{i_m}} \\
=& \sum_{i_2,\cdots,i_m=1}^{n} \left(\int_0^1 t^{c_i+c_{i_2}+\cdots+c_{i_m}-1} x_{i_2}x_{i_3}\cdots x_{i_m} dt \right) \\
=& \int_0^1 \left(\sum_{i_2,\cdots,i_m=1}^{n} t^{c_i+c_{i_2}+\cdots+c_{i_m}-1} x_{i_2}x_{i_3}\cdots x_{i_m} \right) dt \\
=& \int_0^1 \left(\sum_{j=1}^{n} t^{c_j+\frac{c_i-1}{m-1}} x_j \right)^{m-1} dt.
\end{aligned}$$

This implies that $\lambda \geqslant 0$ since m is odd. Thus, the desired result holds. $\square$

Now, we give an example to verify the result of Theorem 3.10. Here, we only show the non-negativity of H-eigenvalues for an odd order non-negative Cauchy tensor since all even order non-negative Cauchy tensors are always positive semi-definite [78].

Example 3.2. *Let $\mathcal{C} = (c_{i_1i_2i_3})$ be a non-negative Cauchy tensor with generating vector $\mathbf{c} = (1, 1, 2)$. Then, it has entries such that*

$$c_{111} = c_{222} = \frac{1}{3},\ c_{333} = \frac{1}{6},\ c_{112} = c_{121} = c_{211} = \frac{1}{3},\ c_{113} = c_{131} = c_{311} = \frac{1}{4},$$

$$c_{122} = c_{221} = c_{212} = \frac{1}{3},\ c_{133} = c_{331} = c_{313} = \frac{1}{5},\ c_{223} = c_{232} = c_{322} = \frac{1}{4},$$

$$c_{233} = c_{332} = c_{323} = \frac{1}{5},\ c_{123} = c_{132} = c_{312} = c_{321} = c_{231} = c_{213} = \frac{1}{4}.$$

By Definition 2.1, to get all H-eigenvalues of $\mathcal{C}$ is equivalent to solving the following system:

$$\begin{cases} \frac{1}{3}x_1^2 + \frac{2}{3}x_1x_2 + \frac{1}{2}x_1x_3 + \frac{1}{3}x_2^2 + \frac{1}{5}x_3^2 + \frac{1}{2}x_2x_3 = \lambda x_1^2 \\ \frac{1}{3}x_2^2 + \frac{2}{3}x_1x_2 + \frac{1}{2}x_2x_3 + \frac{1}{3}x_1^2 + \frac{1}{5}x_3^2 + \frac{1}{2}x_1x_3 = \lambda x_2^2 \\ \frac{1}{6}x_3^2 + \frac{2}{5}x_1x_3 + \frac{2}{5}x_2x_3 + \frac{1}{4}x_2^2 + \frac{1}{4}x_1^2 + \frac{1}{2}x_1x_2 = \lambda x_3^2 \end{cases} \tag{3.9}$$

Since Cauchy tensor $\mathcal{C}$ is non-negative, it always has H-eigenvalue i.e. the above system at least has a solution $\lambda \in \mathbb{R}$. Next, we will prove that λ may not be negative. Without loss of generality, choose one equation from (3.9) such that

$$\frac{1}{3}x_1^2 + \frac{2}{3}x_1x_2 + \frac{1}{2}x_1x_3 + \frac{1}{3}x_2^2 + \frac{1}{5}x_3^2 + \frac{1}{2}x_2x_3 = \lambda x_1^2. \tag{3.10}$$

It is east to see that the left of the equality in (3.10) is quadratic form and the corresponded symmetric matrix is

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}.$$

By direct computation, we obtain that the matrix is positive semi-definite, which implies that

$$\frac{1}{3}x_1^2 + \frac{2}{3}x_1x_2 + \frac{1}{2}x_1x_3 + \frac{1}{3}x_2^2 + \frac{1}{5}x_3^2 + \frac{1}{2}x_2x_3 \geqslant 0, \ \forall \ x \in \mathbb{R}^3.$$

Thus, all H-eigenvalues of Cauchy tensor $\mathcal{C}$ are non-negative. □

3.3 Final remarks

In this chapter, we give several necessary and sufficient conditions for an even order Cauchy tensor to be positive semi-definite. Some properties of positive semi-definite Cauchy tensors are presented. Furthermore, inequalities about the largest H-eigenvalue and the smallest H-eigenvalue of Cauchy tensors are shown. At last, some spectral properties on Z-eigenvalues and H-eigenvlaues of odd order Cauchy tensors are shown.

However, there are still some questions that we are not sure now. The Cauchy matrix can be combined with many other structured matrices to form new structured matrices such as Cauchy-Toeplitz matrix and Cauchy-Hankel matrix [97, 105, 106]. Can we get the type of Cauchy-Toeplitz tensors and Cauchy-Hankel tensors? If so, how about their spectral properties? What are the necessary and sufficient conditions for their positive semi-definiteness?

Chapter 4

Generalized Cauchy tensors and Hankel tensors

Among the various structured tensors we mentioned in the previous chapters, in this chapter, we mainly study Cauchy tensors and Hankel tensors, where further results about these two classes of tensors are given. The symmetric Cauchy tensors were defined and analyzed in chapter 3. In the following discussion, we simply refer it as Cauchy tensors instead of symmetric Cauchy tensors. One of the nice properties of a Cauchy tensor is that its positive semi-definiteness (or positive definiteness) can be easily verified by the sign of the associated generating vectors. In fact, it was proved in Theorem 3.1 and Theorem 3.2, that an even order Cauchy tensor is positive semi-definite if and only if each of entries of its generating vector is positive, and an even order Cauchy tensor is positive definite if and only if each entries of its generating vector is positive and mutually distinct.

Hankel tensors arise from signal processing and data fitting [2, 20, 75]. As far as we know, the definition of Hankel tensor was first introduced in [75]. Recently, some easily verifiable structured tensors related to Hankel tensors were also introduced in [81]. These structured tensors include strong Hankel tensors, complete Hankel tensors and the associated plane tensors that correspond to underlying Hankel tensors. It was proved that if a Hankel tensor is co-positive or an even order Hankel tensor is positive semi-definite, then the associated plane tensor is co-positive or positive semi-definite respectively [81]. Furthermore, results on positive semi-definiteness of even order strong and complete Hankel tensors were given. However, the relationship between strong Hankel tensors and complete Hankel tensors was not provided in [81]. Later, in [56], it was shown that complete Hankel tensors are strong Hankel tensors; while the converse is, in general, not true.

4.1 Generalized Cauchy tensors

Now, given two vectors $\mathbf{c} = (c_1, c_2, \cdots, c_n)^T$, $\mathbf{d} = (d_1, d_2, \cdots, d_n)^T \in \mathbb{R}^n$. Consider the **generalized Cauchy tensor** $\mathcal{C} = (c_{i_1 i_2 \cdots i_m})$ with order m dimension n, where

$$c_{i_1 i_2 \cdots i_m} = \frac{d_{i_1} d_{i_2} \cdots d_{i_m}}{c_{i_1} + c_{i_2} + \cdots + c_{i_m}}, \ i_j \in [n], \ j \in [m].$$

For the sake of simplicity, we call vectors $\mathbf{c}$, $\mathbf{d}$ the generating vectors of the generalized Cauchy tensor $\mathcal{C}$. In the special case when $d_i = 1$, $i \in [n]$, a generalized Cauchy tensor reduces to a Cauchy tensor defined in Definition 3.1. In the case when $m = 2$, a generalized Cauchy tensor collapses to a symmetric generalized Cauchy matrix [74]. We also note that every rank-one tensor with the form $\mathbf{u}^m$ for some $\mathbf{u} \in \mathbb{R}^n$ is, in particular, a generalized Cauchy tensor.

Define Cauchy tensor $\overline{\mathcal{C}} = (\overline{c}_{i_1, i_2, \cdots i_m})$ where

$$\overline{c}_{i_1, i_2, \cdots i_m} = \frac{1}{c_{i_1} + c_{i_2} + \cdots + c_{i_m}}, \ i_j = 1, \cdots, n, \ j = 1, \cdots, m.$$

It is easy to see for any $\mathbf{x} \in \mathbb{R}^n$, we have

$$\mathcal{C}\mathbf{x}^m \equiv \overline{\mathcal{C}}\mathbf{y}^m,$$

where $\mathbf{y} \in \mathbb{R}^n$ with $y_i = d_i x_i$ for $i = 1, \cdots, n$. By Theorems 3.1 and Theorem 3.2 in chapter 3, one may easily conclude that the generalized Cauchy tensor $\mathcal{C}$ is positive semi-definite if and only if $d_i = 0, c_i \neq 0$ or $d_i \neq 0, c_i > 0$, $i \in [n]$ and $\mathcal{C}$ is positive definite if and only if $c_1, c_2, \cdots, c_n$ are positive real numbers and mutually distinct, and $d_i \neq 0$, $i = 1, \cdots, n$.

In this section, we mainly characterize SOS tensor decomposition and completely positiveness of even order generalized Cauchy tensors with nonzero entries. Before giving the main results, we briefly recall the definitions of SOS tensor decomposition and completely positive tensors.

SOS tensor decomposition is first introduced in [37]. The definition of SOS decomposition relies on the celebrated concept of SOS polynomials, which is a fundamental concept in polynomial optimization theory [38, 37, 48, 50, 95]. Assume $\mathcal{A}$ is a symmetric tensor with order m and dimension n. Let $m = 2k$ be an even number. If

$$f(\mathbf{x}) = \mathcal{A}\mathbf{x}^m, \ \mathbf{x} \in \mathbb{R}^n$$

can be decomposed to the sum of squares of polynomials of degree k, then f is called a **sum-of-squares (SOS) polynomial**, and we say the corresponding symmetric tensor $\mathcal{A}$ has an **SOS tensor decomposition** [37]. From the definition, any tensor with SOS decomposition is positive semi-definite. On the other hand, the converse is not true, in general [37, 38]. The importance of studying SOS decomposition is that the problem for determining an even order symmetric tensor is an SOS tensor or not is equivalent to solving a semi-infinite linear programming problem, which can be done in polynomial time; while determining the positive semi-definiteness of a symmetric tensor is, in general, NP-hard. Interestingly, it was recently shown in [37] that for a so-called Z-tensor $\mathcal{A}$ where the off-diagonal elements are all non-positive, $\mathcal{A}$ is positive semi-definite if and only if it has SOS decomposition.

Tensor $\mathcal{A}$ is called a **completely decomposable tensor** if there are vectors $\mathbf{x_j} \in \mathbb{R}^n$, $j \in [r]$ such that $\mathcal{A}$ can be written as sums of rank-one tensors generated by the vector $\mathbf{x_j}$, that is,

$$\mathcal{A} = \sum_{j\in[r]} \mathbf{x_j}^m.$$

If $\mathbf{x_j} \in \mathbb{R}^n_+$ for all $j \in [r]$, then $\mathcal{A}$ is called a **completely positive tensor** [87]. It was shown that a strongly symmetric, hierarchically dominated non-negative tensor is a completely positive tensor [87]. It can be directly verified that all even order completely positive tensors have SOS decomposition, and so, are also positive semi-definite tensors. We note that verifying a tensor $\mathcal{A}$ is a completely decomposable or not, and finding its explicit rank one decomposition are highly nontrivial. This topic has attracted a lot of researchers and many important work has been established along this direction. For detailed discussions, see [17, 45, 87] and the reference therein.

We now characterize the SOS decomposition and complete decomposability for even order generalized Cauchy tensors with nonzero entries.

Theorem 4.1. *Let $\mathcal{C}$ be a generalized Cauchy tensor with even order m and dimension n. Let $\mathbf{c} = (c_1, \cdots, c_n)^T \in \mathbb{R}^n$ and $\mathbf{d} = (d_1, \cdots, d_n)^T \in \mathbb{R}^n$ be the generating vectors of $\mathcal{C}$. Assume $d_i \neq 0,\ i \in [n]$. Then, the following statements are equivalent:*

(i) *the generalized Cauchy tensor $\mathcal{C}$ is a completely decomposable tensor;*

(ii) *the generalized Cauchy tensor $\mathcal{C}$ has SOS decomposition;*

(iii) *the generalized Cauchy tensor $\mathcal{C}$ is positive semi-definite;*

(iv) $c_i > 0$, $i \in [n]$.

Proof. Since m is even, by the definitions of completely decomposable tensor, SOS tensor and positive semi-definite tensor, we can easily obtain (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii).

[(iii) $\Rightarrow$ (iv)] Let $\mathcal{C}$ be an even order generalized Cauchy tensor which is positive semi-definite. Then

$$\mathcal{C}\mathbf{e_i}^m = \frac{d_i^m}{mc_i} \geqslant 0.$$

So $c_i > 0$ for all $i \in [n]$.

[(iv) $\Rightarrow$ (i)] Suppose that $c_i > 0$, $i \in [n]$. Then, for any $\mathbf{x} \in \mathbb{R}^n$,

$$\begin{aligned} f(\mathbf{x}) = \mathcal{C}\mathbf{x}^m &= \sum_{i_1,i_2,\cdots,i_m=1}^{n} \frac{d_{i_1}d_{i_2}\cdots d_{i_m}}{c_{i_1}+c_{i_2}+\cdots+c_{i_m}} x_{i_1}x_{i_2}\cdots x_{i_m} \\ &= \sum_{i_1,i_2,\cdots,i_m=1}^{n} \left(\int_0^1 t^{c_{i_1}+c_{i_2}+\cdots+c_{i_m}-1} d_{i_1}d_{i_2}\cdots d_{i_m} x_{i_1}x_{i_2}\cdots x_{i_m} dt \right) \\ &= \int_0^1 \left(\sum_{i_1,i_2,\cdots,i_m=1}^{n} t^{c_{i_1}+c_{i_2}+\cdots+c_{i_m}-1} d_{i_1}d_{i_2}\cdots d_{i_m} x_{i_1}x_{i_2}\cdots x_{i_m} \right) dt \\ &= \int_0^1 \left(\sum_{i=1}^{n} t^{c_i-\frac{1}{m}} d_i x_i \right)^m dt. \end{aligned} \tag{4.1}$$

By the definition of Riemann integral, we have

$$\mathcal{C}\mathbf{x}^m = \lim_{k\to\infty} \sum_{j=1}^{k} \frac{\left(\sum_{i=1}^{n} (\frac{j}{k})^{c_i-\frac{1}{m}} d_i x_i \right)^m}{k}.$$

Let $\mathcal{C}_k$ be the symmetric tensor such that

$$\begin{aligned} \mathcal{C}_k\mathbf{x}^m &= \sum_{j=1}^{k} \frac{\left(\sum_{i=1}^{n} (\frac{j}{k})^{c_i-\frac{1}{m}} d_i x_i \right)^m}{k} \\ &= \sum_{j=1}^{k} \left(\sum_{i=1}^{n} \frac{(\frac{j}{k})^{c_i-\frac{1}{m}} d_i}{k^{\frac{1}{m}}} x_i \right)^m \\ &= \sum_{j=1}^{k} \left(\langle \mathbf{u^j}, \mathbf{x} \rangle \right)^m, \end{aligned} \tag{4.2}$$

where

$$\mathbf{u^j} = \left(\frac{(\frac{j}{k})^{c_1 - \frac{1}{m}} d_1}{k^{\frac{1}{m}}}, \cdots, \frac{(\frac{j}{k})^{c_n - \frac{1}{m}} d_n}{k^{\frac{1}{m}}} \right) \in \mathbb{R}^n, \ j = 1, \cdots, k. \tag{4.3}$$

Let $\mathrm{CD}_{m,n}$ denote the set consisting of all completely decomposable tensor with order m and dimension n. From [56, Theorem 1], $\mathrm{CD}_{m,n}$ is a closed convex cone when m is even. It then follows that $\mathcal{C} = \lim_{k\to\infty} \mathcal{C}_k$ is also a completely decomposable tensor. $\square$

Next, we provide a sufficient and necessary condition for the complete positivity of a generalized Cauchy tensor with nonzero entries, in terms of its generating vectors.

Theorem 4.2. *Let $\mathcal{C}$ be a generalized Cauchy tensor defined as in Theorem 4.1 with generating vectors $\mathbf{c} = (c_1, \cdots, c_n)^T \in \mathbb{R}^n$ and $\mathbf{d} = (d_1, \cdots, d_n)^T \in \mathbb{R}^n$. Assume $d_i \neq 0$, $i \in [n]$. Then $\mathcal{C}$ is a completely positive tensor if and only if $c_i > 0$ and $d_i > 0$, $i \in [n]$.*

Proof. For necessary condition, suppose that $\mathcal{C}$ is a completely positive tensor. Then, for any vector $\mathbf{x} \in \mathbb{R}^n_+$, we must have $\mathcal{C}\mathbf{x}^m \geqslant 0$. So, $\mathcal{C}\mathbf{e_i}^m = \frac{d_i^m}{mc_i} \geqslant 0$. This implies that $c_i > 0$, $i \in [n]$. To finish the proof, we only need to show $d_i > 0$, $i \in [n]$. To see this, we proceed by the method of contradiction and suppose that

$$I_- := \{i \in \{1, \cdots, n\} : d_i < 0\} \neq \emptyset.$$

Denote r to be the cardinality of I_-. Without loss of generality, we assume that $I_- = \{1, \cdots, r\}$. Then, $d_1 < 0$ and $d_{r+1} > 0$, and hence, the $(r+1, 1, \cdots, 1)^{\text{th}}$ entry of $\mathcal{C}$ satisfies

$$\mathcal{C}_{r+1\ 1\ \cdots\ 1} = \frac{d_{r+1} d_1^{m-1}}{c_{r+1} + (m-1)c_1} < 0.$$

Note that each entry of a completely positive tensor must be a non-negative number. This makes contradiction, and hence, the necessary condition follows.

To prove the sufficient condition, from (4.1)-(4.2), we know that

$$\mathcal{C}\mathbf{x}^m = \lim_{k\to\infty} \sum_{j=1}^{k} \left(\langle \mathbf{u}^{\mathbf{j}}, \mathbf{x}\rangle\right)^m .$$

As $c_i > 0$ and $d_i > 0$, $i \in [n]$, (4.3) implies that $\mathbf{u}^{\mathbf{j}} \in \mathbb{R}^n_+$, $j \in [k]$. So each $\mathcal{C}_k$ is a completely positive tensor. Let $\mathrm{CP}_{m,n}$ denote the set consisting of all completely positive tensors with order m and dimension n. From [87], $\mathrm{CP}_{m,n}$ is a closed convex cone for any $m, n \in \mathbb{N}$. It then follows that $\mathcal{C} = \lim_{k\to\infty} \mathcal{C}_k$ is also a completely positive tensor. $\square$

Let $\mathcal{A} = (a_{i_1\cdots i_m})$ and $\mathcal{B} = (b_{i_1\cdots i_m})$ be two real tensors with order m and dimension n. Then their Hadamard product is a real order m dimension n tensor

$$\mathcal{A} \circ \mathcal{B} = (a_{i_1\cdots i_m} b_{i_1\cdots i_m}).$$

From Proposition 1 of [87], we know that the Hadamard product of two completely positive tensors is also a completely positive tensor. So, we have the following conclusion.

Corollary 4.1. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two positive semi-definite Cauchy tensors. Then the Hadamard product $\mathcal{C}_1 \circ \mathcal{C}_2$ is also positive semi-definite.*

4.2 Further properties on Hankel tensors

Hankel tensors arise from signal processing and some other applications [2, 20, 75, 81]. Recall that an order m dimension n tensor $\mathcal{A} = (a_{i_1 i_2\cdots i_m})$ is called a Hankel tensor if there is a vector $\mathbf{v} = (v_0, v_1, \cdots, v_{(n-1)m})^T$ such that

$$a_{i_1 i_2\cdots i_m} = v_{i_1+i_2+\cdots+i_m-m},\ \forall\ i_1, i_2, \cdots, i_m \in [n]. \tag{4.4}$$

Such a vector $\mathbf{v}$ is called the generating vector of Hankel tensor $\mathcal{A}$.

For any $k \in \mathbb{N}$, let $s(k, m, n)$ be the number of distinct sets of indices $(i_1, i_2, \cdots, i_m)$, $i_j \in [n], j \in [m]$ such that $i_1 + i_2 + \cdots + i_m - m = k$. For example, $s(0, m, n) = 1, s(1, m, n) = m, s(2, m, n) = \frac{m(m+1)}{2}$. Suppose $\mathcal{P} = (p_{i_1 i_2 \cdots i_{(n-1)m}})$ is an order $(n-1)m$ dimension 2 tensor defined by

$$p_{i_1 i_2 \cdots i_{(n-1)m}} = \frac{s(k, m, n) v_k}{\binom{(n-1)m}{k}},$$

where $k = i_1 + i_2 + \cdots + i_{(n-1)m} - (n-1)m$. Then tensor $\mathcal{P}$ is called the **associated plane tensor** of Hankel tensor $\mathcal{A}$. When $n = 2$, it is obvious that $\mathcal{P} = \mathcal{A}$.

In [81], it was proved that, if a Hankel tensor is co-positive, then its associated plane tensor $\mathcal{P}$ is co-positive and the associated plane tensor is positive semi-definite if the Hankel tensor is positive semi-definite. Since the associated plane tensor $\mathcal{P}$ has dimension 2, we can use the Z-eigenvalue method in [85] to check its positive semi-definiteness (alternatively, noting that any 2-dimensional symmetric tensor is positive semi-definite if and only if it has SOS tensor decomposition, we can also verify the positive semi-definiteness of the associated plane tensor by solving a semi-definite programming problem). Thus, the positive semi-definiteness of the associated plane tensor is a checkable necessary condition for the positive semi-definiteness of even order Hankel tensors (see more discussion in [81]). This naturally raises the following questions: can these necessary conditions be also sufficient? If not, are there any concrete counter-examples?

We first present a result stating that the positive semi-definiteness of the associated plane tensor is equivalent to the Vandermonde positive semi-definiteness of the original Hankel tensor.

Theorem 4.3. *Let $\mathcal{A}$ be a Hankel tensor defined as in (4.4) with an even order m. Then, the associated plane tensor $\mathcal{P}$ is positive semi-definite if and only if $\mathcal{A}$ is Vandermonde positive semi-definite.*

Proof. For necessary condition, let $\mathbf{u} = (1, \mu, \mu^2, \cdots, \mu^{n-1})^T \in \mathbb{R}^n$ be an arbitrary Vandermonde vector. If $\mu = 0$, then we have

$$\mathcal{A}\mathbf{u}^m = \sum_{i_1, i_2, \cdots, i_m \in [n]} a_{i_1 i_2 \cdots i_m} u_{i_1} u_{i_2} \cdots u_{i_m} = v_0. \tag{4.5}$$

By our assumption, for $\mathbf{y} = (1, 0)^T \in \mathbb{R}^2$, it follows that

$$\mathcal{P}\mathbf{y}^{(n-1)m} = \sum_{i_1, i_2, \cdots, i_{(n-1)m} \in [2]} p_{i_1 i_2 \cdots i_{(n-1)m}} y_{i_1} y_{i_2} \cdots y_{i_{(n-1)m}} = v_0 \geqslant 0.$$

Combining this with (4.5), we obtain

$$\mathcal{A}\mathbf{u}^m \geqslant 0. \tag{4.6}$$

If $\mu \neq 0$, there exist $y_1, y_2 \in \mathbb{R}\backslash\{0\}$ such that $\mu = \frac{y_2}{y_1}$. Let $\mathbf{y} = (y_1, y_2)^T \in \mathbb{R}^2$. Then, we have

$$\begin{aligned}
\mathcal{P}\mathbf{y}^{(n-1)m} = & \sum_{i_1, i_2, \cdots, i_{(n-1)m} \in [2]} p_{i_1 i_2 \cdots i_{(n-1)m}} y_{i_1} y_{i_2} \cdots y_{i_{(n-1)m}} \\
= & y_1^{(n-1)m} \sum_{k=0}^{(n-1)m} \binom{(n-1)m}{k} \frac{s(k, m, n) v_k}{\binom{(n-1)m}{k}} \mu^k \\
= & y_1^{(n-1)m} \mathcal{A}\mathbf{u}^m \\
\geqslant & 0.
\end{aligned}$$

By (4.6) and the fact that m is even, for all Vandermonde vectors $\mathbf{u} \in \mathbb{R}^n$, it follows that

$$\mathcal{A}\mathbf{u}^m \geqslant 0,$$

which implies Hankel tensor $\mathcal{A}$ is Vandermonde positive semi-definite.

For sufficiency, let $\mathbf{y} = (y_1, y_2)^T \in \mathbb{R}^2$. We now verify that $\mathcal{P}\mathbf{y}^{(n-1)m} \geqslant 0$. To see this, we first consider the case where $y_1 \neq 0$. In this case, let $\mathbf{u} = (1, \mu, \mu^2, \cdots, \mu^{n-1})^T \in \mathbb{R}^n$, where $\mu = \frac{y_2}{y_1}$. From the analysis above, we have

$$\mathcal{P}\mathbf{y}^{(n-1)m} = y_1^{(n-1)m} \mathcal{A}\mathbf{u}^m \geqslant 0 \tag{4.7}$$

since m is even and $\mathcal{A}$ is Vandermonde positive semi-definite.

On the other hand, if $\mathbf{y} = (y_1, y_2)^T \in \mathbb{R}^2$ with $y_1 = 0$, then we let $\mathbf{y}_\epsilon = (\epsilon, y_2)^T \in \mathbb{R}^2$ and $\mathbf{u} = (1, \mu, \mu^2, \cdots, \mu^{n-1})^T \in \mathbb{R}^n$, where $\mu = \frac{y_2}{\epsilon}$, $\epsilon > 0$. By (4.7), we have

$$\mathcal{P}{\mathbf{y}_\epsilon}^{(n-1)m} = \epsilon^{(n-1)m}\mathcal{A}\mathbf{u}^m \geqslant 0.$$

Combining this with the fact that $\epsilon \mapsto \mathcal{P}{\mathbf{y}_\epsilon}^{(n-1)m}$ is a continuous mapping, it follows that

$$\mathcal{P}\mathbf{y}^{(n-1)m} = \lim_{\epsilon \to 0} \mathcal{P}{\mathbf{y}_\epsilon}^{(n-1)m} \geqslant 0.$$

This then implies that plane tensor $\mathcal{P}$ is positive semi-definite and the desired result holds. $\square$

Below, we provide an example illustrating that a Hankel tensor which is Vandermonde positive semi-definite need not to be positive semi-definite. This example together with Theorem 4.3, also implies that the positive semi-definiteness of the associate plane tensor is not sufficient for positive semi-definiteness of the Hankel tensor.

Example 4.1. *Let $\mathcal{A}$ be a Hankel tensor with order $m = 4$ and dimension $n = 3$. Let the generating vector of $\mathcal{A}$ be $v_0 = 1, v_1 = -1, v_2 = 1$ and $v_3 = v_4 = \cdots = v_8 = 0$. So, for any $\mathbf{u} = (1, \mu, \mu^2)^T \in \mathbb{R}^3$,*

$$\begin{aligned}
\mathcal{A}\mathbf{u}^4 = & \sum_{i_1,i_2,i_3,i_4 \in [3]} a_{i_1 i_2 i_3 i_4} u_{i_1} u_{i_2} u_{i_3} u_{i_4} \\
= & \sum_{k=0}^{k=8} s(k, 4, 3) v_k \mu^k \\
= & v_0 + 4v_1\mu + 10v_2\mu^2 \\
= & 1 - 4\mu + 10\mu^2 \geqslant 0
\end{aligned}$$

for all $\mu \in \mathbb{R}$. By Theorem 4.3, we know that the associated plane tensor $\mathcal{P}$ is positive semi-definite. We now verify that $\mathcal{A}$ is not positive semi-definite. To see this, let $\mathbf{x} = (1, 1, -1)^T$, then,

$$\begin{aligned}
\mathcal{A}\mathbf{x}^4 = & \sum_{i_1,i_2,i_3,i_4 \in [3]} v_{i_1+i_2+i_3+i_4-4} x_{i_1} x_{i_2} x_{i_3} x_{i_4} \\
= & v_0 x_1^4 + 4v_1 x_1^3 x_2 + v_2(6x_2^2 x_1^2 + 4x_1^3 x_3) \\
= & x_1^4 - 4x_1^3 x_2 + (6x_2^2 x_1^2 + 4x_1^3 x_3) \\
= & 1 - 4 + 6 - 4 = -1 < 0,
\end{aligned}$$

which implies that Hankel tensor $\mathcal{A}$ is not positive semi-definite. □

The following example shows that the the co-positivity of the associated plane tensor is also not sufficient for the co-positivity of the Hankel tensor, in general.

Example 4.2. *Let $\mathcal{A}$ be a Hankel tensor with order $m = 4$ and dimension $n = 3$. Let the generating vector of $\mathcal{A}$ be $v_0 = 1, v_1 = -1, v_2 = \frac{1}{2}$, $v_3 = v_4 = \cdots = v_8 = 0$. Let $\mathbf{x} = (1, \frac{1}{2}, 0)^T$. Then, we have*

$$\mathcal{A}\mathbf{x}^4 = -\frac{1}{4} < 0,$$

which implies that Hankel tensor $\mathcal{A}$ is not co-positive. On the other hand, it holds that

$$\mathcal{A}\mathbf{u}^4 = 1 - 4\mu + 5\mu^2 \geqslant 0$$

for any Vandermonde vector $\mathbf{u} = (1, \mu, \mu^2)^T \in \mathbb{R}^3$. By Theorem 4.3, the associated plane tensor $\mathcal{P}$ is positive semi-definite. Thus, $\mathcal{P}$ is co-positive. □

A special class of Hankel tensor is the complete Hankel tensors. To recall the definition of a complete Hankel tensor, we note that, for a Hankel tensor $\mathcal{A}$ with order m dimension n, if

$$\mathcal{A} = \sum_{k=1}^{r} \alpha_k (\mathbf{u_k})^m, \tag{4.8}$$

where $\alpha_k \in \mathbb{R}$, $\alpha_k \neq 0$, $\mathbf{u_k} = (1, \mu_k, \mu_k^2, \cdots, \mu_k^{n-1})^T \in \mathbb{R}^n$, $k = 1, 2, \cdots, r$, for some $\mu_i \neq \mu_j$ for $i \neq j$, then, we say $\mathcal{A}$ has a **Vandermonde decomposition**. The corresponding vector $\mathbf{u_k}$, $k = 1, \cdots, r$ are called **Vandermonde vectors** and the minimum value of r is called **Vandermonde rank** of $\mathcal{A}$ [81]. From Theorem 3 of [81], we know that $\mathcal{A}$ is a Hankel tensor if and only if it has a Vandermonde decomposition (4.8). If $\alpha_k > 0$ for $k \in [r]$ in (4.8), then $\mathcal{A}$ is called a complete Hankel tensor.

In [81], it is proved that an even order complete Hankel tensor is positive semi-definite. Moreover, examples were also presented to show that the converse is, in general, not true. Here, in the following theorem, we show that if the Vandermonde rank of a Hankel tensor $\mathcal{A}$ is less than the dimension of the underlying space, then positive semi-definiteness of $\mathcal{A}$ is equivalent to the fact that $\mathcal{A}$ is a complete Hankel tensor, and so, is further equivalent to the SOS decomposition property of $\mathcal{A}$.

Theorem 4.4. *Let $\mathcal{A}$ be a Hankel tensor with an even order m. Assume that the Hankel tensor $\mathcal{A}$ has Vandermonde decomposition (4.8) with the Vandermonde rank r satisfies $r \leqslant n$. Then, the following statements are equivalent:*

(i) *$\mathcal{A}$ is a positive semi-definite tensor;*

(ii) *$\mathcal{A}$ is a complete Hankel tensor.*

(iii) *$\mathcal{A}$ has SOS tensor decomposition;*

Proof. We first note that the implications [(ii)] $\Rightarrow$ [(iii)] and [(iii)] $\Rightarrow$ [(i)] are direct consequences from the definitions. Thus, to see the conclusion, we only need to prove [(i)] $\Rightarrow$ [(ii)]. To do this, we proceed by the method of contradiction and assume that there exists at least one coefficient α_i in (4.8) which is negative. Without loss of generality, we assume that $\alpha_1 < 0$. For any $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$, then we have

$$\begin{aligned}\mathcal{A}\mathbf{x}^m =& \sum_{k=1}^{r} \alpha_k ({\mathbf{u_k}}^T \mathbf{x})^m \\ =& \alpha_1({\mathbf{u_1}}^T\mathbf{x})^m + \alpha_2({\mathbf{u_2}}^T\mathbf{x})^m + \cdots + \alpha_r({\mathbf{u_r}}^T\mathbf{x})^m.\end{aligned} \tag{4.9}$$

Consider the following two homogeneous linear equation systems

$$A\mathbf{x} = \mathbf{0}, \quad B\mathbf{x} = \mathbf{0},$$

where

$$A = \begin{pmatrix} 1 & \mu_1 & \mu_1^2 & \cdots & \mu_1^{n-1} \\ 1 & \mu_2 & \mu_2^2 & \cdots & \mu_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \mu_r & \mu_r^2 & \cdots & \mu_r^{n-1} \end{pmatrix}, \quad B = \begin{pmatrix} 1 & \mu_2 & \mu_2^2 & \cdots & \mu_2^{n-1} \\ 1 & \mu_3 & \mu_3^2 & \cdots & \mu_3^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \mu_r & \mu_r^2 & \cdots & \mu_r^{n-1} \end{pmatrix}.$$

By conditions $r \leqslant n$, it is easy to get

$$Rank(A) = r \leqslant n, \quad Rank(B) = r - 1 < n,$$

which imply that there is vector $\mathbf{x_0} \in \mathbb{R}^n$, $\mathbf{x_0} \neq \mathbf{0}$ such that

$$A\mathbf{x_0} \neq \mathbf{0}, \quad B\mathbf{x_0} = \mathbf{0}.$$

Here, $Rank(A)$ denotes the rank of matrix A. So, it holds that

$${\mathbf{u_1}}^T\mathbf{x_0} \neq 0, \quad {\mathbf{u_i}}^T\mathbf{x_0} = 0, \quad i \in \{2, 3, \cdots, r\}.$$

Combining this with (4.9), we have

$$\mathcal{A}{\mathbf{x_0}}^m = \alpha_1({\mathbf{u_1}}^T\mathbf{x_0})^m < 0$$

since m is even. However, this contradicts to the fact that $\mathcal{A}$ is positive semi-definite. Thus, all coefficients in (4.8) are positive and $\mathcal{A}$ is a complete Hankel tensor. $\square$

An interesting consequence of Theorem 4.4 is as follows: a necessary condition for a PNS (positive semi-definite but not with SOS tensor decomposition) Hankel tensor $\mathcal{A}$ is that the Vandermonde rank r of the Hankel tensor $\mathcal{A}$ is strictly larger than the dimension n of the underlying space. We note that searching for a PNS Hankel tensor is a non-trivial task and is related to Hilbert's 17th question. Recently, some extensive study for PNS Hankel tensor has been initialed in [55].

Next, we provide some necessary conditions for the positive semi-definiteness of a Hankel tensor $\mathcal{A}$ in terms of the sign properties of the coefficients of its Vandermonde decomposition.

Proposition 4.1. *Let $\mathcal{A}$ be a Hankel tensor with the Vandermonde decomposition (4.8). Suppose that $\mathcal{A}$ is positive semi-definite. Then,*

(i) the coefficients of the Vandermonde decomposition satisfy

$$\alpha_1 + \alpha_2 + \cdots + \alpha_r \geqslant 0;$$

(ii) if $r > n$, then the total number of positive coefficients of the Vandermonde decomposition is greater than or equal to n;

(iii) if $r \leqslant n$, then all coefficients of the Vandermonde decomposition are positive.

Proof. (i) Since $\mathcal{A}$ is positive semi-definite, so we have

$$\mathcal{A}\mathbf{e_1}^m = \sum_{i=1}^{r} \alpha_i (\mathbf{u_i^T e_1})^m = \alpha_1 + \alpha_2 + \cdots + \alpha_r \geqslant 0.$$

(ii) Denote the total number of positive coefficients in (4.8) by t. Without loss of generality, let

$$\alpha_i > 0,\ i \in [t];\ \alpha_j < 0,\ j \in \{t+1, t+2, \cdots, r\}.$$

We proceed by the method of contradiction and suppose that $t < n$. If $t = 0$, we can easily get a contradiction because $\mathcal{A}$ is positive semi-definite. If $0 < t < n$, consider the following two linear equation systems

$$A\mathbf{x} = \mathbf{0} \tag{4.10}$$

and

$$B\mathbf{x} = \mathbf{0}, \tag{4.11}$$

where

$$A = \begin{pmatrix} 1 & \mu_1 & \mu_1^2 & \cdots & \mu_1^{n-1} \\ 1 & \mu_2 & \mu_2^2 & \cdots & \mu_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \mu_t & \mu_t^2 & \cdots & \mu_t^{n-1} \end{pmatrix}, \quad B = \begin{pmatrix} 1 & \mu_1 & \mu_1^2 & \cdots & \mu_1^{n-1} \\ 1 & \mu_2 & \mu_2^2 & \cdots & \mu_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \mu_r & \mu_r^2 & \cdots & \mu_r^{n-1} \end{pmatrix}.$$

Noting that $Rank(A) = t < n$ and $Rank(B) = n$, basic linear algebra theory implies that the system (4.10) has nonzero solutions and system (4.11) has only zero solution. Thus, there exists $\bar{\mathbf{x}} \in \mathbb{R}^n, \bar{\mathbf{x}} \neq 0$ such that

$$\mathbf{u_i^T}\bar{\mathbf{x}} = 0, \quad i \in [t] \text{ and } (\mathbf{u_{t+1}^T}\bar{\mathbf{x}}, \cdots, \mathbf{u_r^T}\bar{\mathbf{x}})^T \neq \mathbf{0}.$$

Note that the order m is an even number (as $\mathcal{A}$ is positive semi-definite). This implies that

$$\mathcal{A}\bar{\mathbf{x}}^m = \sum_{j=t+1}^{r} \alpha_j(\mathbf{u_j^T}\bar{\mathbf{x}})^m < 0.$$

This contradicts with the fact that $\mathcal{A}$ is positive semi-definite. Then we get $t \geqslant n$.

(iii) If $r \leqslant n$, then the conclusion is a direct result of Theorem 4.4. □

The following example shows that $r \leqslant n$ in Theorem 4.4 is necessary and the results (i), (ii) of Proposition 4.1 are not sufficient.

Example 4.3. *Let $\mathcal{A}$ be 4th order 2 dimension Hankel tensor with Vandermonde decomposition such that*

$$\mathcal{A} = (a_{i_1i_2i_3i_4}) = \mathbf{x}^4 + \mathbf{y}^4 - \mathbf{z}^4, \tag{4.12}$$

where $\mathbf{x} = (1,0), \mathbf{y} = (1,1), \mathbf{z} = (1,-1)$ *are Vandermonde vectors in* $\mathbb{R}^2$*. We first prove that the Vandermonde rank of* $\mathcal{A}$ *is* $r = 3$*.*

(I) Suppose $\mathcal{A} = \alpha \mathbf{u}^4,\ \alpha \in \mathbb{R}, \alpha \neq 0, \mathbf{u} = (1,\mu) \in \mathbb{R}^2$*. Then, by (4.12), we obtain*

$$a_{1111} = \alpha = 1,\ a_{1112} = \alpha\mu = 2,\ a_{1122} = \alpha\mu^2 = 0,$$

which are contradictive equations. So, the Vandermonde rank of $\mathcal{A}$ *satisfies* $r \geqslant 2$*.*

(II) Suppose $\mathcal{A} = \alpha_1 \mathbf{u_1}^4 + \alpha_2 \mathbf{u_2}^4$*, where* $\mathbf{u_1} = (1,\mu_1), \mathbf{u_2} = (1,\mu_2) \in \mathbb{R}^2, \mu_1 \neq \mu_2$*, and* α_1, α_2 *are nonzero real numbers. Then, by (4.12), we have the following system*

$$\begin{cases} \alpha_1 + \alpha_2 = 1, & (1) \\ \alpha_1\mu_1 + \alpha_2\mu_2 = 2, & (2) \\ \alpha_1\mu_1^2 + \alpha_2\mu_2^2 = 0, & (3) \\ \alpha_1\mu_1^3 + \alpha_2\mu_2^3 = 2, & (4) \\ \alpha_1\mu_1^4 + \alpha_2\mu_2^4 = 0. & (5) \end{cases}$$

We first prove that $\mu_1 \neq 1, \mu_2 \neq 1$*. By contradiction, if* $\mu_1 = 1$*, then by (2) (4), we have*

$$\alpha_2\mu_2(\mu_2^2 - 1) = 0,$$

which implies that $\mu_2 = 0$ *or* $\mu_2 = -1$ *(*μ_2 *can not be 1 since* $\mu_1 \neq \mu_2$*). If* $\mu_1 = 1, \mu_2 = 0$*, we get a contradiction from (2) and (3); if* $\mu_1 = 1, \mu_2 = -1$*, we get another contradiction from (1) and (3). Thus,* $\mu_1 \neq 1$*. Similarly, we can prove that* $\mu_2 \neq 1$*.*

On the other hand, by (2),(3),(4),(5), it holds that

$$\alpha_1\mu_1(\mu_1^2 - 1) = \alpha_2\mu_2(1 - \mu_2^2) \tag{4.13}$$

and

$$\alpha_1\mu_1^2(\mu_1^2 - 1) = \alpha_2\mu_2^2(1 - \mu_2^2). \tag{4.14}$$

By (4.13), (4.14), it follows that $\mu_1 = \mu_2$*, which is a contradiction. Thus, the Vanderonde rank of the Hankel tensor* $\mathcal{A}$ *is* $r = 3$*.*

For this Hankel tensor $\mathcal{A}$, it is easy to check that conditions (i), (ii) in Proposition 4.1 hold. But $\mathcal{A}$ is not positive semi-definite since that

$$\mathcal{A}\mathbf{x_0}^4 = -15 < 0, \ \mathbf{x_0} = (1, -1) \in \mathbb{R}^2.$$

$\square$

4.3 Properties of Cauchy-Hankel tensors

In the literature, there is an important class of structured matrices called Cauchy-Hankel matrices which is closely related with Cauchy matrices and Hankel matrices simultaneously [29, 98, 97]. A matrix A is called a Cauchy-Hankel matrices if it can be formulated as

$$A = \left(\frac{1}{g + h(i+j)}\right)_{i,j=1}^{n},$$

where g and h are real constants such that $h \neq 0$ and $\frac{g}{h}$ is not an integer [3].

As a natural extension of Cauchy-Hankel matrix, a tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ with order m and dimension n is called a **Cauchy-Hankel tensor** if

$$a_{i_1 i_2 \cdots i_m} = \frac{1}{g + h(i_1 + i_2 + \cdots + i_m)}, \ i_j \in [n], \ j \in [m], \tag{4.15}$$

where $g, h \neq 0 \in \mathbb{R}$ and $\frac{g}{h}$ is not an integer.

It is obvious that a Cauchy-Hankel tensor is a symmetric tensor. From Definition 3.1, we know that a Cauchy-Hankel tensor defined by (4.15) is a Cauchy tensor with generating vector

$$\mathbf{c} = (\frac{g}{m} + h, \frac{g}{m} + 2h, \cdots, \frac{g}{m} + nh)^T \in \mathbb{R}^n,$$

and it is a Hankel tensor [75, 81] at the same time with

$$v_k = \frac{1}{g + h(k+m)}, \ k \in \{0, 1, 2, \cdots, (n-1)m\}.$$

Theorem 4.5. *Let $\mathcal{A}$ be a Cauchy-Hankel tensor defined as in (4.15) with even order m. Then, $\mathcal{A}$ is positive definite if and only if*

$$g + mh > 0, \quad g + nmh > 0.$$

Proof. For necessary condition, since $\mathcal{A}$ is positive definite, so we have

$$\mathcal{A}\mathbf{e_1}^m = \frac{1}{g+mh} > 0, \quad \mathcal{A}\mathbf{e_n}^m = \frac{1}{g+mnh} > 0,$$

and the desired results hold.

For sufficiency, since

$$g + mh > 0, \quad g + nmh > 0,$$

it follows that

$$g + smh > 0, \quad \forall\ s \in \{1, 2, \cdots, n\}.$$

Combining Theorem 3.3 and the fact that

$$g + imh \neq g + jmh, \quad \forall\ i, j \in [n],\ i \neq j,$$

we know that $\mathcal{A}$ is positive definite and the desired result follows. $\square$

Next, we define the homogeneous polynomial $f(\mathbf{x})$ as below

$$f(\mathbf{x}) = \mathcal{A}\mathbf{x}^m = \sum_{i_1, i_2, \cdots, i_m \in [n]} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m},$$

for $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$. Let $\mathbf{x}, \mathbf{y} \in X \subseteq \mathbb{R}^n$. If $f(\mathbf{x}) \geqslant f(\mathbf{y})$ for any $\mathbf{x} \geqslant \mathbf{y}$($\mathbf{x} \leqslant \mathbf{y}$ respectively), then we say $f(\mathbf{x})$ is monotonically increasing (monotonically decreasing respectively) in X. If $f(\mathbf{x}) > f(\mathbf{y})$ for any $\mathbf{x} \geqslant \mathbf{y}, \mathbf{x} \neq \mathbf{y}$($\mathbf{x} \leqslant \mathbf{y}, \mathbf{x} \neq \mathbf{y}$ respectively), then we say $f(\mathbf{x})$ is strict monotonically increasing (strict monotonically decreasing respectively) in X.

When $\mathcal{A}$ is a Cauchy tensor with even order, it has been proved that $f(\mathbf{x})$ is strict monotonically increasing in $\mathbb{R}^n_+$ if the Cauchy tensor $\mathcal{A}$ is positive definite; while the converse need not to be true (see chapter 3). For even order Cauchy-Hankel tensors, we have the following conclusion, which is stronger than the corresponded conclusion listed in chapter 3.

Theorem 4.6. *Let $\mathcal{A}$ be a Cauchy-Hankel tensor defined as in (4.5) with an even order m. Then, $\mathcal{A}$ is positive definite if and only if $f(\mathbf{x}) = \mathcal{A}\mathbf{x}^m$ is strict monotonically increasing in $\mathbb{R}^n_+$.*

Proof. For the only if part, suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$, $\mathbf{x} \geqslant \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$, which means that there exists at least one subscript i satisfying $x_i > y_i$. Then, we have

$$\begin{aligned} f(\mathbf{x}) - f(\mathbf{y}) =& \mathcal{A}\mathbf{x}^m - \mathcal{A}\mathbf{y}^m \\ =& \sum_{i_1, i_2, \cdots, i_m \in [n]} \frac{x_{i_1} x_{i_2} \cdots x_{i_m} - y_{i_1} y_{i_2} \cdots y_{i_m}}{g + h(i_1 + i_2 + \cdots + i_m)} \\ =& \frac{x_i^m - y_i^m}{g + imh} + \sum_{i_1 i_2 \cdots i_m \neq ii \cdots i} \frac{x_{i_1} x_{i_2} \cdots x_{i_m} - y_{i_1} y_{i_2} \cdots y_{i_m}}{g + h(i_1 + i_2 + \cdots + i_m)}. \end{aligned}$$

Since $\mathcal{A}$ is positive definite, by Theorem 4.5, we obtain

$$g + kmh > 0, \quad \forall\ k \in [n].$$

So, we obtain

$$\frac{x_i^m - y_i^m}{g + imh} > 0$$

and

$$\sum_{i_1 i_2 \cdots i_m \neq ii \cdots i} \frac{x_{i_1} x_{i_2} \cdots x_{i_m} - y_{i_1} y_{i_2} \cdots y_{i_m}}{g + h(i_1 + i_2 + \cdots + i_m)} \geqslant 0.$$

Thus, we have

$$f(\mathbf{x}) - f(\mathbf{y}) > 0,$$

which implies that $f(\mathbf{x})$ is strict monotonically increasing in $\mathbb{R}^n_+$.

For the if part, note that $\mathbf{e_i} \in \mathbb{R}^n_+$ and $\mathbf{e_i} \geqslant \mathbf{0}$, $\mathbf{e_i} \neq \mathbf{0}$, $i = 1, n$. It then follows that

$$f(\mathbf{e_1}) - f(\mathbf{0}) = \mathcal{A}\mathbf{e_1}^m = \frac{1}{g + mh} > 0$$

and

$$f(\mathbf{e_n}) - f(\mathbf{0}) = \mathcal{A}\mathbf{e_n}^m = \frac{1}{g + nmh} > 0.$$

By Theorem 4.5, we know that Cauchy-Hankel tensor $\mathcal{A}$ is positive definite and the desired results hold. □

Theorem 4.5 and Theorem 4.6 provide a convenient checkable condition to verify the positive definiteness of the Cauchy-Hankel tensor, and the strict monotonicity of the multivariate polynomial corresponding to the tensor. Here, we present several examples to show the efficiency of the theory conclusions.

Example 4.4. *Suppose $\mathcal{A} = (a_{i_1 i_2 i_3 i_4})$ is a Cauchy-Hankel tensor such that*

$$a_{i_1 i_2 i_3 i_4} = \frac{1}{9 - 2(i_1 + i_2 + i_3 + i_4)}, \ i_j \in [3], j \in [4].$$

Here, it takes $g = 9, h = -2$ and $m = 4, n = 3$. Since $g + mh > 0, g + mnh < 0$, tensor $\mathcal{A}$ is not positive definite and strict monotonically increasing in $\mathbb{R}^n_+$. In fact, it holds that

$$\mathcal{A}\mathbf{e_2}^4 = -\frac{1}{7}, \ \mathcal{A}\mathbf{e_2}^4 < \mathcal{A}\mathbf{0}^4.$$

□

Example 4.5. *Let $\mathcal{A} = (a_{i_1 i_2 i_3 i_4 i_5 i_6})$ be a tensor such that*

$$a_{i_1 i_2 i_3 i_4} = \frac{1}{100 - 3(i_1 + i_2 + i_3 + i_4 + i_5 + i_6)}, \ i_j \in [4], j \in [6].$$

By Theorem 4.5, Theorem 4.6 and the fact that

$$g + mh = 100 - 6 \cdot 3 = 72 > 0, \;\; g + mnh = 100 - 6 \cdot 4 \cdot 3 = 18 > 0,$$

$\mathcal{A}$ *is positive definite and* $\mathcal{A}\mathbf{x}^m$ *is strict monotonically increasing.* $\square$

4.4 Final remarks

In this chapter, we present various new results on Cauchy tensors and Hankel tensors which complements the existing literature. Firstly, we show that generalized positive semi-definite Cauchy tensors with nonzero entries have SOS tensor decomposition. Furthermore, sufficient and necessary conditions are given to guarantee an even order generalized Cauchy tensor is a completely positive tensor. The nonnegativity of H-eigenvalues of non-negative Cauchy tensors are also established. For Hankel tensors, we prove that it is Vandermonde positive semi-definite if and only if the associated plane tensor is positive semi-definite. We also show that, if the Vandermonde rank of a Hankel tensor $\mathcal{A}$ is less than the dimension of the underlying space, then positive semi-definiteness of $\mathcal{A}$ is equivalent to the fact that $\mathcal{A}$ is a complete Hankel tensor, and so, is further equivalent to the SOS tensor decomposition of $\mathcal{A}$. Finally, properties of Cauchy-Hankel tensors are also given.

Chapter 5

Spectral properties of odd-bipartite Z-tensors and their absolute tensors

Since the pioneer work of [78] and [62], a lot of researchers have devoted themselves to the study of spectral properties of tensors in the past several years [4, 5, 6, 12, 11, 18, 32, 43, 47, 60, 73, 109]. The main difficulty in tensor problems is that they are generally nonlinear. Because of the difficulties in studying the properties of a general tensor, researchers focus on some structured tensors. Z-tensors are an important class of structured tensors and have been well studied [19, 68, 116]. They are closely related with spectral graph theory, the stationary distribution of Markov chains and the convergence of iterative methods for linear systems.

Recently, in [40], Hu et al. considered the largest Laplacian H-eigenvalue and the largest signless Laplacian H-eigenvalue of a k-uniform connected hypergraph. When the order is even and the hypergraph is odd-bipartite, they proved that the largest Laplacian H-eigenvalue and the largest signless Laplacian H-eigenvalue are equal. For the odd order case, it is proved that the largest Laplacian H-eigenvalue is strictly less than the largest signless Laplacian H-eigenvalue [40]. Later, Shao et al. [94] gave several spectral characterizations of the connected odd-bipartite hypergraphs. They proved that the spectrum of the Laplacian tensor and the spectrun of the signless Laplacian tensor of an uniform hypergraph are equal if and only if the hypergraph is an even order connected odd-bipartite hypergraph. Since the Laplacian tensor is a special case of Z-tensors and the signless Laplacian tensor is a special case of the absolute tensors of Z-tensors, questions comes naturally: what is the relation between the largest H-eigenvalue of a general Z-tensor, and the largest H-eigenvalue of the Z-tensor's absolute tensor? What is the relation between spectrums of a general Z-tensor and its absolute tensor? These constitute main motivations of the paper.

5.1 Odd-bipartite and even-bipartite tensors

In this section, we first define odd-bipartite tensors and even-bipartite tensors. Then, some special characteristics of this kinds of tensors are shown.

Definition 5.1. *Assume $\mathcal{A} = (a_{i_1 \cdots i_m})$ is an tensor with order m and dimension n. If there is a nonempty proper index subset $V \subset [n]$ such that*

$$a_{i_1 \cdots i_m} \neq 0, \ \textit{when } |V \cap \{i_1, \cdots, i_m\}| \textit{ is odd}$$

and $a_{i_1 \cdots i_m} = 0$ for the others, then $\mathcal{A}$ is called an odd-bipartite tensor corresponding to set V or $\mathcal{A}$ is odd-bipartite for simple.

Here, we should note that indices of an edge $\{i_1, \cdots, i_m\}$ in hypergraph [39] are different from each other, which is a notable distinction to general tensors. So, in this article, we define that $|V \cap \{i_1, \cdots, i_m\}|$ is the number of indices $V \cap \{i_1, \cdots, i_m\}$, and duplicate indices should be calculated. For example, suppose $V = \{1, 2, 3\}$ and $\mathcal{A}$ is a 4th order 6 dimensional tensor, then

$$|V \cap \{1, 1, 3, 3\}| = 4, \ |V \cap \{1, 2, 3, 5\}| = 3, \ |V \cap \{4, 6, 4, 5\}| = 0.$$

Definition 5.2. *Assume $\mathcal{A} = (a_{i_1 \cdots i_m})$ is a tensor with order m and dimension n. $\mathcal{A}$ is called weakly odd-bipartite if there is a nonempty proper index subset $V \subset [n]$ such that*

$$a_{i_1 \cdots i_m} = 0, \text{ when } |V \cap \{i_1, \cdots, i_m\}| \text{ is even.}$$

From Definitions 5.1 and 5.2, even-bipartite and weakly even-bipartite tensors can be defined similarly. Furthermore, we can easily prove that, if $\mathcal{A}$ is odd-bipartite (even-bipartite, respectively), then $\mathcal{A}$ is weakly odd-bipartite (weakly even-bipartite respectively), but not vice versa. For example, suppose $\mathcal{A}$ is a 3rd order 2 dimensional tensor with entries such that

$$a_{222} = 1 \text{ and } a_{i_1 i_2 i_3} = 0$$

for the others. It is easy to check that $\mathcal{A}$ is weakly odd-bipartite corresponding to the index set $V = \{2\}$ but not odd-bipartite corresponding to $\{1\}$ or $\{2\}$.

When m is odd, for all $i_1, i_2, \cdots, i_m \in [n]$ and a nonempty proper index subset $V \subset [n]$, it holds that $|\{i_1, i_2, \cdots, i_m\} \cap V|$ is odd if and only if $|\{i_1, i_2, \cdots, i_m\} \cap \bar{V}|$ is even, where $\bar{V} = [n] \backslash V$. So, by Definitions 5.1 and 5.2, we can readily obtain the following conclusion.

Lemma 5.1. *Let $\mathcal{A}$ be a tensor with order m and dimension n. Assume m is odd. Then, $\mathcal{A}$ is odd-bipartite (or weakly odd-bipartite respectively) corresponding to nonempty proper index subset $V \subset [n]$ if and only if $\mathcal{A}$ is even-bipartite (or weakly even-bipartite respectively) corresponding to the nonempty proper index subset $\bar{V} = [n] \backslash V$.*

Irreducible tensors are a class of important and useful tensors, which have been repeatedly used in Perron Frobenius Theorem for non-negative tensors [5, 110, 111]. Next, we will study the relation between irreducible tensors and odd-bipartite tensors. To do this, we first list the corresponding definition below, which comes from [5].

Definition 5.3. *For a tensor $\mathcal{T}$ with order m and dimension n. We call $\mathcal{T}$ is reducible if there is a nonempty proper index subset $V \subset [n]$ such that*

$$t_{i_1 i_2 \cdots i_m} = 0, \ \forall \ i_1 \in V, \ \forall \ i_2, i_3, \cdots, i_m \notin V.$$

Otherwise we call $\mathcal{T}$ is irreducible.

Theorem 5.1. *Let m be even. Assume tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$ with order m and dimension n is odd-bipartite. Then $\mathcal{A}$ is irreducible.*

Proof. Since $\mathcal{A}$ is odd-bipartite, there exists a nonempty proper index subset $V \subset [n]$ satisfying

$$a_{i_1 \cdots i_m} \neq 0, \text{ when the number } |V \cap \{i_1, \cdots, i_m\}| \text{ is odd.} \tag{5.1}$$

By contradiction, suppose $\mathcal{A} = (a_{i_1 \cdots i_m})$ is reducible, then there is a nonempty proper index subset $V_1 \subset [n]$ such that

$$a_{i_1 \cdots i_m} = 0, \ \forall \ i_1 \in V_1, \ \forall \ i_2, \cdots, i_m \notin V_1. \tag{5.2}$$

We will break the proof into four cases. (i) If $V_1 \subseteq V$, let $i_1 \in V_1, i_2, \cdots, i_m \notin V$. Here, several indices in $i_2, \cdots, i_m$ may equal to each other when the number of elements in $[n] \backslash V$ is strictly less than $m - 1$. Then, by (5.2) we have

$$a_{i_1 \cdots i_m} = 0,$$

which contradicts (5.1) since $|V \cap \{i_1, \cdots, i_m\}| = 1$ is odd.

(ii) If $V \subseteq V_1$, let $i_1 \in V, i_2, \cdots, i_m \notin V_1$. Then, by (5.2) one has

$$a_{i_1 \cdots i_m} = 0,$$

which is a contradiction with (5.1).

(iii) If $V \cap V_1 \neq \varnothing$ and neither $V \subseteq V_1$ nor $V_1 \subseteq V$, let $i_1 \in V_1 \backslash V,\ i_2,\ \cdots, i_m \in V \backslash V_1$. Then it follows that

$$a_{i_1 \cdots i_m} = 0,$$

which also contradicts (5.1), since $|V \cap \{i_1, \cdots, i_m\}| = m - 1$ is a odd number.

(iv) If $V \cap V_1 = \varnothing$, let $i_1 \in V_1, i_2, \cdots, i_m \in V$. By Definition 5.3, we have

$$a_{i_1 \cdots i_m} = 0.$$

Since $|V \cap \{i_1, \cdots, i_m\}| = m - 1$ be odd, by (5.1), one has

$$a_{i_1 \cdots i_m} \neq 0,$$

which is a contradiction. All in all, we know that $\mathcal{A}$ can not be reducible and the desired results follows. □

If a tensor $\mathcal{A}$ is even-bipartite, no matter the order of $\mathcal{A}$ is odd or even, we have the following result.

Theorem 5.2. *Assume tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$ with order m and dimension n is even-bipartite corresponding to a nonempty proper index subset $V \subseteq [n]$. Then $\mathcal{A}$ is reducible corresponding to V.*

Proof. By definitions of reducible tensors and even-bipartite tensors, the conclusion obviously holds. □

Suppose an even order Z-tensor and its absolute tensor are defined such that,

$$\mathcal{A} = \mathcal{D} - \mathcal{C}, \quad |\mathcal{A}| = \mathcal{D} + \mathcal{C}, \tag{5.3}$$

where $\mathcal{D}$ is an non-negative diagonal tensor and $\mathcal{C}$ is an non-negative tensor with zero diagonal entries. From Theorem 5.2, if $\mathcal{C}$ is odd-bipartite, then tensors $\mathcal{A}$ and $|\mathcal{A}|$ are irreducible. Combining this with Theorem 3.1 of [27] we have the following result.

Corollary 5.1. *Let m be even. Suppose tensor $\mathcal{A} = \mathcal{D} - \mathcal{C}$ with order m and dimension n is defined as in (5.3). Then, $\mathcal{A}$ and its absolute tensor $|\mathcal{A}|$ are all weakly irreducible if non-negative tensor $\mathcal{C}$ is odd-bipartite.*

By the Perron-Frobenius theorem on non-negative tensors in [5] and by Theorem 4.1 of [27], the following result follows.

Corollary 5.2. *Let m be even. Assume tensor $\mathcal{A}$ is defined as in Corollary 5.1. If $\mathcal{C}$ is odd-bipartite, the largest H-eigenvalue of $|\mathcal{A}|$ is $\rho(|\mathcal{A}|)$. Furthermore, there exists a positive n dimensional eigenvector $\mathbf{x} \in \mathbb{R}^n$ such that*

$$|\mathcal{A}|\mathbf{x}^{m-1} = \rho(|\mathcal{A}|)\mathbf{x}^{[m-1]}.$$

5.2 Relation between the largest H-eigenvalues of a Z-tensor and its absolute tensor

In this section, suppose an order m dimension n Z-tensor $\mathcal{A}$ with non-negative diagonal elements has format

$$\mathcal{A} = \mathcal{D} - \mathcal{C}, \tag{5.4}$$

where $\mathcal{D}$ is an non-negative diagonal tensor and $\mathcal{C}$ is an non-negative tensor with zero diagonal elements. So the absolute format of $\mathcal{A}$ is $|\mathcal{A}| = \mathcal{D} + \mathcal{C}$. In the following analysis, entries of $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{D}$ are always defined as below

$$\mathcal{A} = (a_{i_1 \cdots i_m}),\ \mathcal{C} = (c_{i_1 \cdots i_m}),\ \mathcal{D} = (d_{i_1 \cdots i_m}),\ i_1, i_2, \cdots, i_m \in [n].$$

For the sake of simple, let $d_{ii \cdots i} = d_i$, $i \in [n]$.

During this part, we mainly study the relationship between the largest H-eigenvalue of a Z-tensor $\mathcal{A}$ in (5.4), and the largest H-eigenvalue of the absolute tensor of $\mathcal{A}$. Sufficient and necessary conditions or sufficient conditions to guarantee the equality of these largest H-eigenvalues are shown. It should be noted that all even order non-negative tensors always have H-eigenvalues [110]. To proceed, we make an assumption in advance, all tensors considered in this part always have H-eigenvalues.

The largest H-eigenvalues of $\mathcal{A}$ and $|\mathcal{A}|$ are denoted by $\lambda(\mathcal{A})$ and $\lambda(|\mathcal{A}|)$ respectively. From Corollary 5.2, we know that $\lambda(|\mathcal{A}|) = \rho(|\mathcal{A}|)$.

Theorem 5.3. *Let m be even. Suppose $\mathcal{A} = \mathcal{D} - \mathcal{C}$ is defined as (5.4). Then,*

$$\lambda(\mathcal{A}) = \lambda(|\mathcal{A}|)$$

if $\mathcal{C}$ is odd-bipartite.

Proof. By Lemma 13 of [80], we have

$$\lambda(\mathcal{A}) \leqslant \rho(\mathcal{A}) \leqslant \rho(|\mathcal{A}|) = \lambda(|\mathcal{A}|).$$

Thus, in order to prove the conclusion, we only need to prove

$$\lambda(|\mathcal{A}|) \leqslant \lambda(\mathcal{A}).$$

Since $\mathcal{C}$ is odd-bipartite, there exists a nonempty proper index subset $V \subset [n]$ satisfying

$$c_{i_1 \cdots i_m} \neq 0, \text{ if } |V \cap \{i_1, \cdots, i_m\}| \text{ is odd},$$

and $c_{i_1 \cdots i_m} = 0$ for the others. So, for all entries of $\mathcal{A}$, it follows that

$$a_{i_1 \cdots i_m} \neq 0, \text{ if } |V \cap \{i_1, \cdots, i_m\}| \text{ is odd},$$

and $a_{i_1 \cdots i_m} = 0$ for the others except the diagonal entries $a_{ii \cdots i}$, $i \in [n]$. By Theorem 5.2, we know that $\mathcal{C}$, $\mathcal{A}$ and $|\mathcal{A}|$ are all irreducible tensors. From Theorem 4.1 of [27] and Definition 2.1, there is a vector $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} > \mathbf{0}$ satisfying

$$|\mathcal{A}|\mathbf{x}^{m-1} = \lambda(|\mathcal{A}|)\mathbf{x}^{[m-1]}.$$

Suppose $\mathbf{y} \in \mathbb{R}^n$ be defined such that $y_i = x_i$ whenever $i \in V$ and $y_i = -x_i$ for the others. When $i \in V$, we have

$$
\begin{aligned}
(\mathcal{A}\mathbf{y}^{m-1})_i &= \left[(\mathcal{D}-\mathcal{C})\mathbf{y}^{m-1}\right]_i \\
&= d_i y_i^{m-1} - \textstyle\sum_{i_2,\cdots,i_m\in[n]} c_{ii_2\cdots i_m} y_{i_2}\cdots y_{i_m} \\
&= d_i y_i^{m-1} - \textstyle\sum_{i_2,\cdots,i_m\in[n] \quad |V\cap\{i,i_2,\cdots,i_m\}| \text{ is odd}} c_{ii_2\cdots i_m} y_{i_2}\cdots y_{i_m} \\
&= d_i x_i^{m-1} + \textstyle\sum_{i_2,\cdots,i_m\in[n] \quad |V\cap\{i,i_2,\cdots,i_m\}| \text{ is odd}} c_{ii_2\cdots i_m} x_{i_2}\cdots x_{i_m} \\
&= \left[(\mathcal{D}+\mathcal{C})\mathbf{x}^{m-1}\right]_i \\
&= \lambda(|\mathcal{A}|) x_i^{m-1} \\
&= \lambda(|\mathcal{A}|) y_i^{m-1}.
\end{aligned}
\tag{5.5}
$$

Here the fourth equality follows the fact that m is even and exactly odd number indices take negative values for each $\{i_2,\cdots,i_m\}\subseteq[n]$. When $i\notin V$, we have

$$
\begin{aligned}
(\mathcal{A}\mathbf{y}^{m-1})_i &= \left[(\mathcal{D}-\mathcal{C})\mathbf{y}^{m-1}\right]_i \\
&= d_i y_i^{m-1} - \textstyle\sum_{i_2,\cdots,i_m\in[n]} c_{ii_2\cdots i_m} y_{i_2}\cdots y_{i_m} \\
&= d_i y_i^{m-1} - \textstyle\sum_{i_2,\cdots,i_m\in[n] \quad |V\cap\{i,i_2,\cdots,i_m\}| \text{ is odd}} c_{ii_2\cdots i_m} y_{i_2}\cdots y_{i_m} \\
&= -d_i x_i^{m-1} - \textstyle\sum_{i_2,\cdots,i_m\in[n] \quad |V\cap\{i,i_2,\cdots,i_m\}| \text{ is odd}} c_{ii_2\cdots i_m} x_{i_2}\cdots x_{i_m} \\
&= -\left[(\mathcal{D}+\mathcal{C})x^{m-1}\right]_i \\
&= -\lambda(|\mathcal{A}|) x_i^{m-1} \\
&= \lambda(|\mathcal{A}|) y_i^{m-1}.
\end{aligned}
\tag{5.6}
$$

Here the fourth equality follows the fact that m is even and exactly even number indices take negative values for each $\{i_2,\cdots,i_m\}\subseteq[n]$. The last equality of (5.6) follows from the definition of $y_i = -x_i$ when $i\notin V$. Thus, by (5.5), (5.6) and Definition 2.1, $\lambda(|\mathcal{A}|)$ is a H-eigenvalue of $\mathcal{A}$ with H-eigenvector $\mathbf{y}$. So, we have

$$
\lambda(|\mathcal{A}|)\leqslant\lambda(\mathcal{A}),
$$

and the desired result follows. □

Here, in the proof of Theorem 5.3, odd-bipartite property of $\mathcal{C}$ guarantees that $|\mathcal{A}|$ has a positive H-eigenvector. Actually, if the H-eigenvector is non-negative, one can obtain the same result. Before proving this, we first cite an useful conclusion from [110].

Lemma 5.2. *If $\mathcal{A}$ is a non-negative tensor with order m and dimension n, then $\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ with a non-negative eigenvector $\mathbf{y}\neq 0$.*

Theorem 5.4. *Let m be even. Suppose $\mathcal{A}$ is defined as in Theorem 5.3. If $\mathcal{C}$ is weakly odd-bipartite, then it holds that*

$$\lambda(\mathcal{A}) = \lambda(|\mathcal{A}|).$$

Proof. Since tensor $\mathcal{C}$ is weakly odd-bipartite, so there is a nonempty proper index subset $V \subseteq [n]$ such that

$$c_{i_1 \cdots i_m} = 0, \text{ when } |\{i_1, \cdots, i_m\} \cap V| \text{ is even},$$

and $|\{i_1, \cdots, i_m\} \cap V|$ must be an odd number for nonzero entries $c_{i_1 \cdots i_m} \neq 0$, $i_1, \cdots, i_m \in [n]$.

On the other hand, by Lemma 5.2, there is a non-negative H-eigenvector $\mathbf{x} \geqslant 0$ of $|\mathcal{A}|$ corresponding to $\lambda(|\mathcal{A}|)$. Suppose vector $\mathbf{y} \in \mathbb{R}^n$ be defined such that $y_i = x_i$ whenever $i \in V$ and $y_i = -x_i$ for the others. Then, the remaining process is similar with the proof of Theorem 5.3. $\square$

Now, we will give an example to show that the conditions in Theorem 5.4 is not necessary. For example, suppose 4th order 2 dimensional tensor $\mathcal{A}$ with entries such that

$$a_{1111} = a_{2222} = 1, \ a_{1122} = -1,$$

and $a_{i_1 i_2 i_3 i_4} = 0$ for the others. After calculating the largest H-eigenvalues of $\mathcal{A}$ and $|\mathcal{A}|$, we obtain

$$\lambda(\mathcal{A}) = \lambda(|\mathcal{A}|) = 1.$$

But, the non-negative tensor $\mathcal{C}$ is not weakly odd-bipartite corresponding to any nonempty proper index subset of $\{1, 2\}$. In the following, sufficient and necessary conditions for the equality of the two largest H-eigenvalues are presented, and it is proved that the necessity of the Theorem 5.4 holds when the non-negative tensor $\mathcal{C}$ is weakly irreducible. Before doing this, we cite a definition from [80].

Definition 5.4. *Assume that $\mathcal{T}$ is a tensor with order m and dimension n. Construct a graph $\hat{G} = (\hat{V}, \hat{E})$, where $\hat{V} = \cup_{j=1}^{d} V_j$ and V_j are subsets of $\{1, 2, \cdots, n\}$ for $j = 1, \cdots, d$. Suppose that $i_j \in V_j, i_l \in V_l, j \neq l$. $(i_j, i_l) \in \hat{E}$ if and only if $t_{i_1 i_2 \cdots i_m} \neq 0$ for some $m-2$ indices $\{i_1, \cdots, i_m\} \backslash \{i_j, i_l\}$. Then, tensor $\mathcal{T}$ is called weakly irreducible if $\hat{G}$ is connected.*

As observed in [27], an irreducible tensor must be always weakly irreducible.

Theorem 5.5. *Let $\mathcal{A}$ be defined as in Theorem 5.4. Assume $\mathcal{C}$ is weakly irreducible. Then,*

$$\lambda(\mathcal{A}) = \lambda(|\mathcal{A}|),$$

if and only if $\mathcal{C}$ is weakly odd-bipartite.

Proof. The sufficient condition has been proved in Theorem 5.4, and we only need to prove the necessary part.

Suppose $\mathbf{x} \in \mathbb{R}^n$ is an H-eigenvector of $\mathcal{A}$ corresponding to $\lambda(\mathcal{A})$ such that $\sum_{i=1}^{n} x_i^m = 1$. Assume $\mathbf{y} \in \mathbb{R}^n$ be defined by $y_i = |x_i|$, for $i \in [n]$. Since m is even, one has $\sum_{i=1}^{n} y_i^m = 1$. By Lemma 3.1 of [57], we have

$$\begin{array}{rl} \lambda(\mathcal{A}) = & \mathcal{A}\mathbf{x}^m = (\mathcal{D} - \mathcal{C})\mathbf{x}^m \\ = & \sum_{i=1}^{n} d_i x_i^m - \sum_{i_1, \cdots, i_m \in [n]} c_{i_1 i_2 \cdots i_m} x_{i_1} \cdots x_{i_m} \\ \leqslant & \sum_{i=1}^{n} d_i y_i^m + \sum_{i_1, \cdots, i_m \in [n]} c_{i_1 i_2 \cdots i_m} y_{i_1} \cdots y_{i_m} \\ = & (\mathcal{D} + \mathcal{C})\mathbf{y}^m \leqslant \lambda(|\mathcal{A}|). \end{array} \tag{5.7}$$

Hence, by the fact that $\lambda(\mathcal{A}) = \lambda(|\mathcal{A}|)$, all inequalities in equation (5.7) should be equalities, which implies that $\mathbf{y}$ is a H-eigenvector of $|\mathcal{A}|$ corresponding to $\lambda(|\mathcal{A}|)$. Since $\mathcal{C}$ is weakly irreducible, $|\mathcal{A}|$ is also weakly irreducible. According to Theorem 4.1 of [27], it holds that $\mathbf{y} > 0$ i.e., all elements in $\mathbf{y}$ are positive. Let $V = \{i \in [n] |\ x_i > 0\}$ and $\bar{V} = \{i \in [n] |\ x_i < 0\}$. Then $V \cup \bar{V} = [n]$. By (5.7), we obtain

$$\sum_{i_1, \cdots, i_m \in [n]} c_{i_1 i_2 \cdots i_m} (|x_{i_1}| \cdots |x_{i_m}| + x_{i_1} \cdots x_{i_m}) = 0,$$

which implies that

$$c_{i_1i_2\cdots i_m}(|x_{i_1}|\cdots|x_{i_m}|+x_{i_1}\cdots x_{i_m})=0,$$

for all $i_1,i_2,\cdots,i_m\in[n]$ since $\mathcal{C}$ is non-negative. When $|\{i_1,i_2,\cdots,i_m\}\cap V|$ is even, we have

$$|x_{i_1}|\cdots|x_{i_m}|+x_{i_1}\cdots x_{i_m}>0,$$

which implies $c_{i_1i_2\cdots i_m}=0$. When $|\{i_1,i_2,\cdots,i_m\}\cap V|$ is odd, we have

$$|x_{i_1}|\cdots|x_{i_m}|+x_{i_1}\cdots x_{i_m}=0.$$

In this case, the value $c_{i_1i_2\cdots i_m}$ may be zero or may not be zero . Thus, from Definition 5.2, it follows that $\mathcal{C}$ is weakly odd-bipartite corresponding to set V and the desired conclusion holds. □

Next, we study the relationship between a Z-tensor and its absolute tensor in the odd order case. In [40], Hu et al. proved that the largest H-eigenvalue of an odd order Laplacian tensor is always strictly less than the largest H-eigenvalue of an signless Laplacian tensor corresponded to the Laplacian tensor. By definitions of Laplacian tensor and signless Laplacian tensor in connected hypergraphs, we know that their diagonal entries are positive, and subscripts of each nonzero element are mutually distinct. However, general Z-tensors (5.4) may not possess those advantages. Hence, for a general odd order Z-tensor (5.4), the largest H-eigenvalue of $\mathcal{A}$ may not be strictly less than the largest H-eigenvalue of $|\mathcal{A}|$ when the order is odd.

The following example shows that the largest H-eigenvalues of a Z-tensor (5.4) and its absolute tensor are equal.

Example 5.1. *Let $\mathcal{A}$ be a 5th order 3 dimensional tensor. Its entries are given by*

$$a_{11111}=a_{22222}=a_{33333}=1,\ a_{11122}=a_{22233}=-1$$

and $a_{i_1i_2i_3i_4i_5} = 0$ *for the others. Then the H-eigenvalue problems for* $\mathcal{A}$ *and* $|\mathcal{A}|$ *are*

$$\begin{cases} x_1^4 - x_1^2 x_2^2 = \lambda x_1^4, \\ x_2^4 - x_2^2 x_3^2 = \lambda x_2^4, \\ x_3^4 = \lambda x_3^4, \end{cases}$$

and

$$\begin{cases} x_1^4 + x_1^2 x_2^2 = \lambda x_1^4, \\ x_2^4 + x_2^2 x_3^2 = \lambda x_2^4, \\ x_3^4 = \lambda x_3^4. \end{cases}$$

After calculating these equation sets, we know that $\lambda(\mathcal{A}) = \lambda(|\mathcal{A}|) = 1$.

□

Theorem 5.6. *Let* A *be defined as (5.4). Assume* m *is odd. Suppose* $\mathcal{C}$ *is weakly odd-bipartite corresponding to a nonempty proper index subset* $V \subseteq [n]$. *If for all* $i \in V$, *it satisfies*

$$c_{ii_2i_3\cdots i_m} = 0, \ \forall \ i_2, i_3, \cdots, i_m \in [n],$$

then $\lambda(\mathcal{A}) = \lambda(|\mathcal{A}|)$.

Proof. By the analysis in Theorems 5.3-5.5, from Lemma 13 of [80] and Corollary 5.2, it follows that

$$\lambda(\mathcal{A}) \leqslant \rho(\mathcal{A}) \leqslant \rho(|\mathcal{A}|) = \lambda(|\mathcal{A}|).$$

Thus, we only need to prove

$$\lambda(|\mathcal{A}|) \leqslant \lambda(\mathcal{A}).$$

Let $\mathbf{x} \in \mathbb{R}^n$ be a non-negative H-eigenvector of $|\mathcal{A}|$ corresponding to $\lambda(|\mathcal{A}|)$. So, for all $i \in [n]$, we have

$$(|\mathcal{A}|\mathbf{x}^{m-1})_i = [(\mathcal{D} + \mathcal{C})\mathbf{x}^{m-1}]_i = \lambda(|\mathcal{A}|)x_i^{m-1}. \tag{5.8}$$

Suppose $\mathbf{y} \in \mathbb{R}^n$ be defined as $y_i = -x_i,\ i \in V$ and $y_i = x_i,\ i \notin V$. By conditions, $\mathcal{C}$ is weakly odd-bipartite corresponding to subset V, which means

$$c_{i_1 i_2 i_3 \cdots i_m} = 0,\ i_1, i_2, \cdots, i_m \in [n]$$

when $|\{i_1, i_2, i_3, \cdots, i_m\} \cap V|$ is even. Then, for all $i \in [n]$, one has

$$\begin{array}{rcl}(\mathcal{A}\mathbf{y}^{m-1})_i &=& [(\mathcal{D}-\mathcal{C})\mathbf{y}^{m-1}]_i \\ &=& d_i y_i^{m-1} - \sum_{i_2,\cdots,i_m \in [n] \quad |V \cap \{i, i_2, \cdots, i_m\}| \text{ is odd}} c_{i i_2 \cdots i_m} y_{i_2} \cdots y_{i_m} \\ &=& d_i x_i^{m-1} - \sum_{i_2,\cdots,i_m \in [n] \quad |V \cap \{i, i_2, \cdots, i_m\}| \text{ is odd}} c_{i i_2 \cdots i_m} y_{i_2} \cdots y_{i_m},\end{array} \tag{5.9}$$

where the third equality follows $m-1$ is even and $y_i^{m-1} = x_i^{m-1}$. When $i \in V$, by the fact that $c_{i i_2 i_3 \cdots i_m} = 0,\ i_2, i_3, \cdots, i_m \in [n]$, and by (5.8), (5.9), we have

$$\begin{array}{rcl}(\mathcal{A}\mathbf{y}^{m-1})_i &=& [(\mathcal{D}-\mathcal{C})\mathbf{y}^{m-1}]_i \\ &=& d_i y_i^{m-1} - \sum_{i_2,\cdots,i_m \in [n]} c_{i i_2 \cdots i_m} y_{i_2} \cdots y_{i_m} \\ &=& d_i x_i^{m-1} = \lambda(|\mathcal{A}|) x_i^{m-1} \\ &=& \lambda(|\mathcal{A}|) y_i^{m-1}.\end{array} \tag{5.10}$$

Similarly, when $i \notin V$, it holds that

$$\begin{array}{rcl}(\mathcal{A}\mathbf{y}^{m-1})_i &=& [(\mathcal{D}-\mathcal{C})\mathbf{y}^{m-1}]_i \\ &=& d_i y_i^{m-1} - \sum_{i_2,\cdots,i_m \in [n] \quad |V \cap \{i, i_2, \cdots, i_m\}| \text{ is odd}} c_{i i_2 \cdots i_m} y_{i_2} \cdots y_{i_m} \\ &=& d_i x_i^{m-1} + \sum_{i_2,\cdots,i_m \in [n] \quad |V \cap \{i, i_2, \cdots, i_m\}| \text{ is odd}} c_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \\ &=& d_i x_i^{m-1} + (\mathcal{C}\mathbf{x}^{m-1})_i \\ &=& [(\mathcal{D}+\mathcal{C})\mathbf{x}^{m-1}]_i \\ &=& (|\mathcal{A}|\mathbf{x}^{m-1})_i = \lambda(|\mathcal{A}|) x_i^{m-1} \\ &=& \lambda(|\mathcal{A}|) y_i^{m-1},\end{array} \tag{5.11}$$

where the third equality follows the fact that m is odd and exactly odd indices take negative values. By (5.10) and (5.11), we know that $\lambda(|\mathcal{A}|)$ is a H-eigenvalue of $\mathcal{A}$. Hence, we have $\lambda(|\mathcal{A}|) \leqslant \lambda(\mathcal{A})$ and the desired result follows. $\square$

Now, we present a example to verify the authenticity of Theorem 5.6.

Example 5.2. *Set a 5th order 3 dimensional tensor $\mathcal{A}$ such that*

$$a_{11111} = 1,\ a_{22222} = 1,\ a_{33333} = 3,\ a_{11333} = -1,\ a_{22333} = -2$$

and $a_{i_1i_2i_3i_4} = 0$ *for the others. Let* $V = \{3\}$*. Then* $\mathcal{C}$ *is weakly odd-bipartite corresponding to the set* V *and* $c_{3i_2i_3i_4i_5} = 0,\ \forall\ i_2, i_3, i_4, i_5 \in [3]$*.*

The H*-eigenvalue problems for* $\mathcal{A}$ *and* $|\mathcal{A}|$ *are to solve*

$$\begin{cases} x_1^4 - x_1x_3^3 = \lambda x_1^4, \\ x_2^4 - 2x_2x_3^3 = \lambda x_2^4, \\ 3x_3^4 = \lambda x_3^4, \end{cases}$$

and

$$\begin{cases} x_1^4 + x_1x_3^3 = \lambda x_1^4, \\ x_2^4 + 2x_2x_3^3 = \lambda x_2^4, \\ 3x_3^4 = \lambda x_3^4. \end{cases}$$

After calculating the largest H*-eigenvalues of* $\mathcal{A}$ *and* $|\mathcal{A}|$*, we obtain*

$$\lambda(\mathcal{A}) = \lambda(|\mathcal{A}|) = 3.$$

□

The next example shows that the conditions in Theorem 5.6 are not necessary.

Example 5.3. *Let* $\mathcal{A}$ *be a 5th order 3 dimensional tensor. Its entries are given by*

$$a_{11111} = 1,\ a_{22222} = 2,\ a_{33333} = 4,\ a_{11122} = a_{11333} = -1,\ a_{22233} = -2$$

and $a_{i_1i_2i_3i_4i_5} = 0$ *for the others. Then the* H*-eigenvalue problems for* $\mathcal{A}$ *and* $|\mathcal{A}|$ *are*

$$\begin{cases} x_1^4 - x_1^2x_2^2 - x_1x_3^3 = \lambda x_1^4, \\ 2x_2^4 - 2x_2^2x_3^2 = \lambda x_2^4, \\ 4x_3^4 = \lambda x_3^4, \end{cases}$$

and

$$\begin{cases} x_1^4 + x_1^2x_2^2 + x_1x_3^3 = \lambda x_1^4, \\ 2x_2^4 + 2x_2^2x_3^2 = \lambda x_2^4, \\ 4x_3^4 = \lambda x_3^4. \end{cases}$$

After calculating these equation sets, we know that $\lambda(\mathcal{A}) = \lambda(|\mathcal{A}|) = 4$, but the non-negative tensor $\mathcal{C}$ is not weakly odd-bipartite corresponding to any nonempty proper index subset of $\{1, 2, 3\}$.

$\square$

By Lemma 5.1 and Theorem 5.6, we have the following conclusion.

Corollary 5.3. *Let A be defined as in (5.4). Assume m is odd. Suppose $\mathcal{C}$ is weakly even-bipartite corresponding to a nonempty proper index subset $V \subseteq [n]$. If for all $i \notin V$, it satisfies*

$$c_{ii_2i_3\cdots i_m} = 0,\ \forall\ i_2, i_3, \cdots, i_m \in [n],$$

then $\lambda(\mathcal{A}) = \lambda(|\mathcal{A}|)$.

5.3 Relation between spectrums of a symmetric Z-tensor and its absolute tensor

In this section, we will study the relation between the spectrum of an even order symmetric Z-tensor with non-negative diagonal entries, and the spectrum of the absolute tensor of the Z-tensor. It is proved that, if the symmetric Z-tensor is weakly irreducible and odd-bipartite, then the two spectral sets equal. Furthermore, for an weakly irreducible symmetric Z-tensor with non-negative diagonal entries, we show that the spectral sets of the Z-tensor and its absolute tensor equal if and only if their spectral radii equal. Before proving the conclusion, we firstly cite the definition of diagonal similar tensors [93], which is useful in the following analysis.

Definition 5.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be two order $m \geqslant 2$ dimension n tensors. If there exists a nonsingular diagonal matrix P of dimension n such that $\mathcal{B} = P^{-(m-1)}\mathcal{A}P$, then $\mathcal{A}$ and $\mathcal{B}$ are called diagonal similar.*

Here, tensor $\mathcal{B} = P^{-(m-1)}\mathcal{A}P$ is defined by

$$b_{i_1 i_2 \cdots i_m} = \sum_{j_1, j_2, \cdots, j_m \in [n]} a_{j_1 j_2 \cdots j_m} p_{i_1 j_1}^{m-1} p_{j_2 i_2} \cdots p_{j_m i_m}, \; i_1, i_2, \cdots, i_m \in [n].$$

Theorem 5.7. *Assume order m dimension n symmetric Z-tensor $\mathcal{A}$ is defined as in (5.4). Suppose $\mathcal{C}$ is weakly irreducible. Then, $\mathcal{A}$ and $|\mathcal{A}|$ are diagonal similar if and only if m is even and $\mathcal{C}$ is weakly odd-bipartite.*

Proof. For necessary, from Definition 5.5, we know that there is a nonsingular diagonal matrix P satisfying

$$\mathcal{A} = P^{-(m-1)}|\mathcal{A}|P,$$

i.e.,

$$\mathcal{D} - \mathcal{C} = P^{-(m-1)}(\mathcal{D} + \mathcal{C})P.$$

Since $\mathcal{D} = P^{-(m-1)}\mathcal{D}P$, we have

$$-\mathcal{C} = P^{-(m-1)}\mathcal{C}P,$$

which implies that

$$-c_{i_1 i_2 \cdots i_m} = c_{i_1 i_2 \cdots i_m} p_{i_1 i_1}^{-(m-1)} p_{i_2 i_2} \cdots p_{i_m i_m}. \tag{5.12}$$

If $p_{11} = p_{22} = \cdots = p_{nn}$, by (5.12), we get $\mathcal{C} = 0$, which is a contradiction to the fact that $\mathcal{C}$ is weakly irreducible. So there are at least two distinct diagonal entries in P.

When $c_{i_1 i_2 \cdots i_m} \neq 0$, by (5.12), one has

$$-p_{i_1 i_1}^m = p_{i_1 i_1} p_{i_2 i_2} \cdots p_{i_m i_m}. \tag{5.13}$$

By (5.13), and by the fact that $\mathcal{C}$ is weakly irreducible, we obtain

$$p_{ii}^m = p_{jj}^m, \quad i, j \in [n],$$

which implies that m is even and

$$V = \{i \in [n] \mid p_{ii} < 0\} \neq \emptyset, \quad \tilde{V} = \{i \in [n] \mid p_{ii} > 0\} \neq \emptyset.$$

Combining this with (5.12)-(5.13), we know that

$$c_{i_1 i_2 \cdots i_m} = 0, \quad when\ |\{i_1, i_2, \cdots, i_m\} \cap V|\ is\ even.$$

Thus, tensor $\mathcal{C}$ is weakly odd-bipartite corresponding to V and the only if part holds.

For the if part, without loss of generality, suppose $\mathcal{C}$ is weakly odd-bipartite corresponding to $\Omega \subset [n]$. Let P be a diagonal matrix with i-th diagonal entries being -1 when $i \in \Omega$ and 1 when $i \notin \Omega$. By a direct computation, one has

$$\mathcal{A} = P^{-(m-1)} |\mathcal{A}| P.$$

Apparently, P is a nonsingular diagonal matrix. From Definition 5.5, it follows that $\mathcal{A}$ and $|\mathcal{A}|$ are diagonal similar. □

It should be noted that diagonal similar tensors have the same characteristic polynomials, and thus they have the same spectrum (see Theorem 2.1 of [93]), which is similar to the matrix case.

Corollary 5.4. *Assume tensor $\mathcal{A}$ is defined as in Theorem 5.7. Let m be even. Suppose $\mathcal{C}$ is odd-bipartite. Then $Spec(\mathcal{A}) = Spec(|\mathcal{A}|)$.*

Now, we first introduce an useful lemma from [111].

Lemma 5.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two order m dimension n tensors with $|\mathcal{B}| \leqslant \mathcal{A}$. Then*

(1) $\rho(\mathcal{B}) \leqslant \rho(\mathcal{A})$.

(2) Furthermore, if $\mathcal{A}$ is weakly irreducible and $\rho(\mathcal{B}) = \rho(\mathcal{A})$, where $\lambda = \rho(\mathcal{A})e^{i\psi}$ is an eigenvalue of $\mathcal{B}$ with an eigenvector $\mathbf{y}$, then,

(i) all the components of $\mathbf{y}$ are nonzero;

(ii) let $U = diag(y_1/|y_1|, \cdots, y_n/|y_n|)$ be a nonsingular diagonal matrix, we have $\mathcal{B} = e^{i\psi} U^{-(m-1)} \mathcal{A} U$.

Theorem 5.8. *Assume order m dimension n symmetric Z-tensor $\mathcal{A}$ is defined as in (5.4). If $\mathcal{C}$ is weakly irreducible, then $\rho(\mathcal{A}) = \rho(|\mathcal{A}|)$ if and only if $Spec(\mathcal{A}) = Spec(|\mathcal{A}|)$.*

Proof. The sufficient condition is obvious. Now, we prove the only if part. Suppose $\lambda = \rho(|\mathcal{A}|)e^{i\psi}$ is an eigenvalue of $\mathcal{A}$. Since $\mathcal{C}$ is weakly irreducible, from Lemma 5.3, we know that there exists a nonsingular diagonal matrix P such that

$$\mathcal{A} = e^{i\psi} P^{-(m-1)} |\mathcal{A}| P, \tag{5.14}$$

which means

$$\mathcal{D} - \mathcal{C} = e^{i\psi} P^{-(m-1)} (\mathcal{D} + \mathcal{C}) P. \tag{5.15}$$

By the fact that all diagonal elements of $\mathcal{C}$ equal zero, by (5.15), one has

$$\mathcal{D} = e^{i\psi} P^{-(m-1)} \mathcal{D} P = e^{i\psi} \mathcal{D},$$

which implies $e^{i\psi} = 1$. So, by Definition 5.5 and (5.14), we know that $\mathcal{A}$ and $|\mathcal{A}|$ are diagonal similar tensors. Thus, from Theorem 2.3 of [93], it holds that $Spec(\mathcal{A}) = Spect(|\mathcal{A}|)$. □

5.4 Final remarks

Odd-bipartite and even-bipartite tensors are defined in this chapter. Using this, we studied the relation between the largest H-eigenvalue of a Z-tensor with non-negative diagonal elements, and the largest H-eigenvalue of the Z-tensor's absolute tensor. Sufficient and necessary conditions for the equality of these largest H-eigenvalues are given when the Z-tensor has even order. For the odd order case, sufficient conditions are presented. Examples are given to verify the authenticity of the conclusions. On the other side, relation between spectral sets of an even order symmetric Z-tensor with non-negative diagonal entries and its absolute tensor are studied.

In this paper, we only study the case of H-eigenvalues of Z-tensors. Do Z-eigenvalues of Z-tensors also hold in such case? This may be an interesting work in the future.

Chapter 6

SOS tensor decomposition and applications

Tensor decomposition is an important research area, and it has found numerous applications in data mining [44, 46, 45], computational neuroscience [16, 25], and statistical learning for latent variable models [1]. An important class of tensor decomposition is sum-of-squares (SOS) tensor decomposition. It is known that to determine a given even order symmetric tensor is positive semi-definite or not is an NP-hard problem in general. On the other hand, an interesting feature of SOS tensor decomposition is checking whether a given even order symmetric tensor has SOS decomposition or not can be verified by solving a semi-definite programming problem (see for example [37]), and hence, can be validated efficiently. SOS tensor decomposition has a close connection with SOS polynomials, and SOS polynomials are very important in polynomial theory [14, 15, 31, 34, 77, 92] and polynomial optimization [42, 48, 50, 51, 76, 95]. It is known that an even order symmetric tensor having SOS decomposition is positive semi-definite, but the converse is not true in general. Recently, a few classes of structured tensors such as B-tensors [82] and diagonally dominated tensors [78], have been shown to be positive semi-definite in the even order symmetric case. It then raises a natural and interesting question: Will these structured tensors admit an SOS decomposition? Providing an answer for this question is important because this will enrich the theory of SOS tensor decomposition, achieve a better understanding for these structured tensors, and lead to efficient numerical methods for solving problems involving these structured tensors.

In this chapter, we will provide clear answers for the above theoretical question and providing applications on important numerical problems involving structured tensors.

6.1 SOS tensor cone and its dual cone

In this part, we study the cone consisting of all tensors that have SOS tensor decomposition, and its dual cone [67]. We use $\mathrm{SOS}_{m,n}$ to denote the cone consisting of all order m and dimension n tensors, which have SOS decomposition. The following simple lemma from [37] gives some basic properties of $\mathrm{SOS}_{m,n}$.

Lemma 6.1. (cf. [37]) *Let $m, n \in \mathbb{N}$ and m be an even number. Then,* $\mathrm{SOS}_{m,n}$ *is a closed convex cone with dimension at most $I(m,n) = \binom{n+m-1}{m}$.*

For a closed convex cone C, we recall that the dual cone of C in $S_{m,n}$ is denoted by $C^{\oplus}$ and defined by $C^{\oplus} = \{\mathcal{A} \in S_{m,n} : \langle \mathcal{A}, \mathcal{C} \rangle \geqslant 0 \text{ for all } \mathcal{C} \in C\}$. Let $\mathcal{M} = (m_{i_1 i_2 \cdots i_m}) \in S_{m,n}$. We also define the symmetric tensor $\mathrm{sym}(\mathcal{M} \otimes \mathcal{M}) \in S_{2m,n}$ by

$$\mathrm{sym}(\mathcal{M} \otimes \mathcal{M})\mathbf{x}^{2m} = (\mathcal{M}\mathbf{x}^m)^2 = \sum_{1 \leqslant i_1, \cdots, i_m, j_1, \cdots, j_m \leqslant n} m_{i_1 \cdots i_m} m_{j_1 \cdots j_m} x_{i_1} \cdots x_{i_m} x_{j_1} \cdots x_{j_m}.$$

Moreover, in the case where the degree $m = 2$, $\mathrm{SOS}_{2,n}$ and its dual cone are equal, and both reduce to the cone of positive semi-definite $(n \times n)$ matrices. Therefore, to avoid triviality, we consider the duality of the SOS tensor cone $\mathrm{SOS}_{m,n}$ in the case where m is an even number with $m \geqslant 4$.

Proposition 6.1. (Duality between tensor cones) *Let $n \in \mathbb{N}$ and m be an even number with $m \geqslant 4$. Then, we have* $\mathrm{SOS}_{m,n}^{\oplus} = \{\mathcal{A} \in S_{m,n} : \langle \mathcal{A}, \mathrm{sym}(\mathcal{M} \otimes \mathcal{M}) \rangle \geqslant 0,\ \forall\ \mathcal{M} \in S_{\frac{m}{2},n}\}$ *and* $\mathrm{SOS}_{m,n} \nsubseteq \mathrm{SOS}_{m,n}^{\oplus}$.

Proof. We define $\mathrm{SOS}^h_{m,n}$ to be the cone consisting of all mth order n dimensional symmetric tensors such that $f_{\mathcal{A}}(\mathbf{x}) := \langle \mathcal{A}, \mathbf{x}^m \rangle$ is a polynomial which can be written as sums of finitely many homogeneous polynomials. We now see that indeed $\mathrm{SOS}^h_{m,n} = \mathrm{SOS}_{m,n}$. Clearly, $\mathrm{SOS}^h_{m,n} \subseteq \mathrm{SOS}_{m,n}$. To see the reverse inclusion, we let $\mathcal{A} \in \mathrm{SOS}_{m,n}$. Then, there exists $l \in \mathbb{N}$ and $f_1, \cdots, f_l$ are real polynomials with degree at most $\frac{m}{2}$ such that $\langle \mathcal{A}, \mathbf{x}^m \rangle = \sum_{i=1}^l f_i(\mathbf{x})^2$. In particular, for all $t \geqslant 0$, we have

$$t^m \langle \mathcal{A}, \mathbf{x}^m \rangle = \langle \mathcal{A}, (t\mathbf{x})^m \rangle = \sum_{i=1}^l f_i(t\mathbf{x})^2$$

Dividing t^m on both sides and letting $t \to +\infty$, we see that $\langle \mathcal{A}, \mathbf{x}^m \rangle = \sum_{i=1}^l f_{i,\frac{m}{2}}(\mathbf{x})^2$, where $f_{i,\frac{m}{2}}$ is the $\frac{m}{2}$th-power term of f_i, $i = 1, \cdots, l$. This shows that $\mathcal{A} \in \mathrm{SOS}^h_{m,n}$. Thus, we have $\mathrm{SOS}^h_{m,n} = \mathrm{SOS}_{m,n}$. It then follows that

$$\begin{aligned}
\left(\mathrm{SOS}_{m,n}\right)^{\oplus} = \left(\mathrm{SOS}^h_{m,n}\right)^{\oplus} &= \{\mathcal{A} \in S_{m,n} : \langle \mathcal{A}, \mathcal{C} \rangle \geqslant 0 \text{ for all } \mathcal{C} \in \mathrm{SOS}^h_{m,n}\} \\
&= \{\mathcal{A} \in S_{m,n} : \langle \mathcal{A}, \mathcal{C} \rangle \geqslant 0 \text{ for all } \mathcal{C} = \sum_{i=1}^l \mathrm{sym}(\mathcal{M}_i \otimes \mathcal{M}_i), \\
&\qquad \mathcal{M}_i \in S_{\frac{m}{2},n}, i = 1, \cdots, l\} \\
&= \{\mathcal{A} \in S_{m,n} : \langle \mathcal{A}, \mathrm{sym}(\mathcal{M} \otimes \mathcal{M}) \rangle \geqslant 0 \text{ for all } \mathcal{M} \in S_{\frac{m}{2},n}\}.
\end{aligned}$$

We now show that $\mathrm{SOS}_{m,n} \subsetneqq \mathrm{SOS}^{\oplus}_{m,n}$ if $m \geqslant 4$. Let $f(\mathbf{x}) = x_1^4 + x_2^4 + \frac{1}{4}x_3^4 + 6x_1^2x_2^2 + 6x_1^2x_3^2 + 6x_2^2x_3^2$ and let $\mathcal{A} \in S_{4,3}$ be such that $\mathcal{A}\mathbf{x}^4 = f(\mathbf{x})$. Then, $\mathcal{A}$ has an SOS decomposition and $\mathcal{A}_{1,1,1,1} = \mathcal{A}_{2,2,2,2} = 1$, $\mathcal{A}_{3,3,3,3} = \frac{1}{4}$, $\mathcal{A}_{1,1,3,3} = \mathcal{A}_{1,1,2,2} = \mathcal{A}_{2,2,3,3} = 1$. We now see that $\mathcal{A} \notin \mathrm{SOS}^{\oplus}_{m,n}$. To see this, we only need to find $M \in S_{2,3}$ such that $\langle \mathcal{A}, \mathrm{sym}(M \otimes M) \rangle < 0$. To see this, let $M = \mathrm{diag}(1, 1, -4)$. Then, $\mathrm{sym}(M \otimes M)\mathbf{x}^4 = (\mathbf{x}^T M \mathbf{x})^2 = (x_1^2 + x_2^2 - 4x_3^2)^2$. Direct verification shows that $\mathrm{sym}(M \otimes M)\mathbf{x}^4 = x_1^4 + x_2^4 + 16x_3^4 + 2x_1^2x_2^2 - 8x_1^2x_3^2 - 8x_2^2x_3^2$. So, $\mathrm{sym}(M \otimes M)_{1,1,1,1} = \mathrm{sym}(M \otimes M)_{2,2,2,2} = 1$, $\mathrm{sym}(M \otimes M)_{3,3,3,3} = 16$, $\mathrm{sym}(M \otimes M)_{1,1,2,2} = \frac{1}{3}$,

$\mathrm{sym}(M \otimes M)_{1,1,3,3} = \mathrm{sym}(M \otimes M)_{2,2,3,3} = -\frac{4}{3}$. Therefore,

$$\langle \mathcal{A}, \mathrm{sym}(M \otimes M) \rangle = 1+1+\frac{1}{4}\cdot 16+6\left(1\cdot\frac{1}{3}\right)+6\left(1\cdot\left(-\frac{4}{3}\right)\right)+6\left(1\cdot\left(-\frac{4}{3}\right)\right) = -8 < 0,$$

and the desired results hold. $\square$

Question: It is known from polynomial optimization (see [50, Proposition 4.9] or [48]) that the dual cone of the cone consisting of all sums-of-squares polynomials (possibly nonhomogeneous) is the moment cone (that is, all the sequence whose associated moment matrix is positive semi-definite). Can we link the dual cone of $\mathrm{SOS}_{m,n}$ to the moment matrix? Can the membership problem of $\mathrm{SOS}_{m,n}^{\oplus}$ be solvable in polynomial time?

6.2 SOS tensor decomposition of several classes of structured tensors

In this section, we examine the SOS tensor decomposition of several classes of symmetric even order structured tensors, such as weakly diagonally dominated tensors, B_0-tensors, double B-tensors, quasi-double B_0-tensors, MB_0-tensors, H-tensors, absolute tensors of positive semi-definite Z-tensors and extended Z-tensors.

6.2.1 Even order symmetric weakly diagonally dominated tensors have SOS decompositions

In this section, we establish that even order symmetric weakly diagonally dominated tensors have SOS decompositions. Firstly, we give the definition of weakly diagonally dominated tensors. To do this, we introduce an index set $\Delta_{\mathcal{A}}$ associated with a tensor $\mathcal{A}$. Now, let $\mathcal{A}$ be a tensor with order m and dimension n, and let $f_{\mathcal{A}}$ be its associated homogeneous polynomial such that $f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m$. We then define the index set $\Delta_{\mathcal{A}}$ as Δ_f with $f = f_{\mathcal{A}}$, as given as in (2.3).

Definition 6.1. *We say mth order n dimensional tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ is a* **diagonally dominated tensor** *if, for each $i = 1, \cdots, n$,*

$$a_{ii\cdots i} \geqslant \sum_{(i_2,\cdots,i_m)\neq(i\cdots i)} |a_{ii_2\cdots i_m}|.$$

We say $\mathcal{A}$ is a **weakly diagonally dominated tensor** *if, for each $i = 1, \cdots, n$,*

$$a_{ii\cdots i} \geqslant \sum_{\substack{(i_2\cdots i_m)\neq(i\cdots i),\\ (i,i_2\cdots,i_m)\in\Delta_{\mathcal{A}}}} |a_{ii_2\cdots i_m}|.$$

Clearly, any diagonally dominated tensor is a weakly diagonally dominated tensor. However, the converse is, in general, not true.

Theorem 6.1. *Let $\mathcal{A}$ be a symmetric weakly diagonally dominated tensor with order m and dimension n. Suppose that m is even. Then, $\mathcal{A}$ has an SOS tensor decomposition.*

Proof. Denote $I = \{(i, \cdots, i) \mid 1 \leqslant i \leqslant n\}$. Let $\mathbf{x} \in \mathbb{R}^n$. Then,

$$\begin{aligned}
\mathcal{A}\mathbf{x}^m &= \sum_{i=1}^{n} a_{ii\cdots i} x_i^m + \sum_{(i_1,\cdots,i_m)\notin I} a_{i_1 i_2\cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m} \\
&= \sum_{i=1}^{n} \left(a_{ii\cdots i} - \sum_{\substack{(i_2\cdots i_m)\neq(i\cdots i)\\ (i,i_2\cdots,i_m)\in\Delta_{\mathcal{A}}}} |a_{ii_2\cdots i_m}| \right) x_i^m + \\
&\quad \sum_{i=1}^{n} \sum_{\substack{(i_2\cdots i_m)\neq(i\cdots i)\\ (i,i_2\cdots,i_m)\in\Delta_{\mathcal{A}}}} |a_{ii_2\cdots i_m}| x_i^m + \sum_{(i_1,\cdots,i_m)\notin I} a_{i_1 i_2\cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}
\end{aligned}$$

$$
\begin{aligned}
= & \sum_{i=1}^{n}\left(a_{ii\cdots i}-\sum_{\substack{(i_2\cdots i_m)\neq(i\cdots i)\\(i,i_2\cdots,i_m)\in\Delta_{\mathcal{A}}}}|a_{ii_2\cdots i_m}|\right)x_i^m \\
& +\sum_{i=1}^{n}\sum_{\substack{(i_2\cdots i_m)\neq(i\cdots i)\\(i,i_2\cdots,i_m)\in\Delta_{\mathcal{A}}}}|a_{ii_2\cdots i_m}|x_i^m+\sum_{i=1}^{n}\sum_{\substack{(i_2\cdots i_m)\neq(i\cdots i)\\(i,i_2\cdots,i_m)\in\Delta_{\mathcal{A}}}}a_{ii_2\cdots i_m}x_ix_{i_2}\cdots x_{i_m} \\
& +\sum_{i=1}^{n}\sum_{\substack{(i_2\cdots i_m)\neq(i\cdots i)\\(i,i_2\cdots,i_m)\notin\Delta_{\mathcal{A}}}}a_{ii_2\cdots i_m}x_ix_{i_2}\cdots x_{i_m}
\end{aligned}
$$

Define

$$
h(\mathbf{x})=\sum_{i=1}^{n}\sum_{\substack{(i_2\cdots i_m)\neq(i\cdots i)\\(i,i_2\cdots,i_m)\in\Delta_{\mathcal{A}}}}|a_{ii_2\cdots i_m}|x_i^m+\sum_{i=1}^{n}\sum_{\substack{(i_2\cdots i_m)\neq(i\cdots i)\\(i,i_2\cdots,i_m)\in\Delta_{\mathcal{A}}}}a_{ii_2\cdots i_m}x_ix_{i_2}\cdots x_{i_m}.
$$

We now show that h is a sums-of-squares polynomial.

To see h is indeed sums-of-squares, from Lemma 2.3, it suffices to show that

$$
\hat{h}(\mathbf{x}):=\sum_{i=1}^{n}\sum_{\substack{(i_2\cdots i_m)\neq(i\cdots i)\\(i,i_2\cdots,i_m)\in\Delta_{\mathcal{A}}}}|a_{ii_2\cdots i_m}|x_i^m-\sum_{i=1}^{n}\sum_{\substack{(i_2\cdots i_m)\neq(i\cdots i)\\(i,i_2\cdots,i_m)\in\Delta_{\mathcal{A}}}}|a_{ii_2\cdots i_m}|x_ix_{i_2}\cdots x_{i_m}
$$

is a polynomial which always takes non-negative values. As $\hat{h}$ is a homogeneous polynomial with degree m on $\mathbb{R}^n$, let $\hat{\mathcal{H}}$ be a symmetric tensor with order m and dimension n such that $\hat{h}(\mathbf{x})=\hat{\mathcal{H}}\mathbf{x}^m$. Since $\mathcal{A}$ is symmetric, the nonzero entries of $\hat{\mathcal{H}}$ are the same as the corresponding entries of $\mathcal{A}$. Now, let λ be an arbitrary H-eigenvalue of $\hat{\mathcal{H}}$, from the Gershgorin Theorem for eigenvalues of tensors [78], we have

$$
\left|\lambda-\sum_{\substack{(i_2\cdots i_m)\neq(i\cdots i)\\(i,i_2\cdots,i_m)\in\Delta_{\mathcal{A}}}}|a_{ii_2\cdots i_m}|\right|\leqslant\sum_{\substack{(i_2\cdots i_m)\neq(i\cdots i)\\(i,i_2\cdots,i_m)\in\Delta_{\mathcal{A}}}}|a_{ii_2\cdots i_m}|.
$$

So, we must have $\lambda \geqslant 0$. This shows that all H-eigenvalues of $\hat{\mathcal{H}}$ must be non-negative, and so, $\hat{\mathcal{H}}$ is positive semi-definite [78]. Thus, $\hat{h}$ is a polynomial which always takes non-negative values.

Now, as $\mathcal{A}$ is a weakly diagonally dominated tensor and m is even,

$$\sum_{i=1}^{n} \left(a_{ii\cdots i} - \sum_{\substack{(i_2\cdots i_m)\neq(i\cdots i) \\ (i,i_2\cdots,i_m)\in\Delta_{\mathcal{A}}}} |a_{ii_2\cdots i_m}| \right) x_i^m$$

is an SOS polynomial. Moreover, from the definition of $\Delta_{\mathcal{A}}$, for each $(i_1 \cdots i_m) \notin \Delta_{\mathcal{A}}$, $a_{i_1\cdots i_m} \geqslant 0$ and $x_{i_1} \cdots x_{i_m}$ is a squares term. Then,

$$\sum_{i=1}^{n} \sum_{\substack{(i_2\cdots i_m)\neq(i\cdots i) \\ (i,i_2\cdots,i_m)\notin\Delta_{\mathcal{A}}}} a_{ii_2\cdots i_m} x_i x_{i_2} \cdots x_{i_m}$$

is also a sums-of-square polynomial. Thus, $\mathcal{A}$ has an SOS tensor decomposition. $\square$

As a diagonally dominated tensor is weakly diagonally dominated, the following corollary follows immediately.

Corollary 6.1. *Let $\mathcal{A}$ be a symmetric diagonally dominated tensor with even order m and dimension n. Then, $\mathcal{A}$ has an SOS tensor decomposition.*

6.2.2 The absolute tensor of an even order symmetric positive semi-definite Z-tensor has an SOS decomposition

Let $\mathcal{A}$ be an order m dimension n tensor. If all off-diagonal elements of $\mathcal{A}$ are non-positive, then $\mathcal{A}$ is called a Z-tensor [116]. A Z-tensor $\mathcal{A} = (a_{i_1,\ldots,i_m})$ can be written as

$$\mathcal{A} = \mathcal{D} - \mathcal{C}, \tag{6.1}$$

where $\mathcal{D}$ is a diagonal tensor where its ith diagonal elements equals $a_{ii\ldots i}$, $i = 1, \ldots, n$, and $\mathcal{C}$ is a non-negative tensor (or a tensor with non-negative entries) such that diagonal entries all equal to zero. We now define the absolute tensor of $\mathcal{A}$ by

$$|\mathcal{A}| = |\mathcal{D}| + \mathcal{C}.$$

Note that all even order symmetric positive semi-definite Z-tensors have SOS decompositions [37, 38], a natural interesting question would be: do all absolute tensors of even order symmetric positive semi-definite Z-tensors have SOS tensor decompositions? Below, we provide an answer for this question.

Theorem 6.2. *Let $\mathcal{A}$ be a symmetric Z-tensor with even order m and dimension n defined as in (6.1). If $\mathcal{A}$ is positive semi-definite, then $|\mathcal{A}|$ has an SOS tensor decomposition.*

Proof. Let $\mathcal{A} = (a_{i_1 \dots i_m})$ be a symmetric positive semi-definite Z-tensor. From (6.1), we have $\mathcal{A} = \mathcal{D} - \mathcal{C}$, where $\mathcal{D}$ is a diagonal tensor where the diagonal entries of $\mathcal{D}$ is $d_i := a_{i \dots i}, i \in [n]$ and $\mathcal{C} = (c_{i_1 i_2 \cdots i_m})$ is a non-negative tensor with zero diagonal entries. Define three index sets as follows:

$$\begin{aligned}
I =& \{(i_1, i_2, \cdots, i_m) \in [n]^m \mid i_1 = i_2 = \cdots = i_m\};\\
\Omega =& \{(i_1, i_2, \cdots, i_m) \in [n]^m \mid c_{i_1 i_2 \cdots i_m} \neq 0 \ \ \textit{and} \ \ (i_1, i_2, \cdots, i_m) \notin I\};\\
\Delta =& \{(i_1, i_2, \cdots, i_m) \in \Omega \mid c_{i_1 i_2 \cdots i_m} > 0 \ \ \textbf{or}\\
& \quad \text{at least one index in } (i_1, i_2, \cdots, i_m) \text{ exists odd times}\}.
\end{aligned}$$

Let $f(\mathbf{x}) = |\mathcal{A}|\mathbf{x}^m$ and define a polynomial $\hat{f}$ by

$$\hat{f}(\mathbf{x}) = \sum_{i=1}^{n} d_i x_i^m - \sum_{(i_1, i_2, \cdots, i_m) \in \Delta} |c_{i_1 i_2 \cdots i_m}| x_{i_1} x_{i_2} \cdots x_{i_m}.$$

From Lemma 2.3, to see polynomial $f(\mathbf{x}) = |\mathcal{A}|\mathbf{x}^m$ is a sums-of-squares polynomial, we only need to show that $\hat{f}$ always takes non-negative value. To see this, as $\mathcal{A}$ is positive semi-definite, we have $d_i \geqslant 0$. Since $c_{i_1 i_2 \cdots i_m} \geqslant 0$, $i_j \in [n]$, $j \in [m]$, it follows that

$$\begin{aligned}
\hat{f}(\mathbf{x}) =& \sum_{i=1}^{n} d_i x_i^m - \sum_{(i_1,i_2,\cdots,i_m)\in\Delta} c_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m} \\
=& \sum_{i=1}^{n} d_i x_i^m - \sum_{(i_1,i_2,\cdots,i_m)\in\Omega} c_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m} + \sum_{(i_1,i_2,\cdots,i_m)\in\Omega\backslash\Delta} c_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m} \\
=& \mathcal{A}\mathbf{x}^m + \sum_{(i_1,i_2,\cdots,i_m)\in\Omega\backslash\Delta} c_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m} \\
\geqslant& 0.
\end{aligned}$$

Here, the last inequality follows from the fact that m is even, $\mathcal{A}$ is positive semi-definite and $x_{i_1} x_{i_2} \cdots x_{i_m}$ is a square term if $(i_1, i_2, \cdots, i_m) \in \Omega\backslash\Delta$. Thus, the desired result follows. □

6.2.3 SOS tensor decomposition for even order symmetric extended Z-tensors

In this subsection, we introduce a new class of symmetric tensor which extends symmetric Z-tensors to the cases where the off-diagonal elements can be positive, and examine its SOS tensor decomposition.

Let f be a polynomial on $\mathbb{R}^n$ with degree m. Let $f_{m,i}$ be the coefficient of f associated with x_i^m, $i \in [n]$. We say f is an extended Z-polynomial if there exist $s \in \mathbb{N}$ with $s \leqslant n$ and index sets $\Gamma_l \subseteq \{1, \cdots, n\}$, $l = 1, \cdots, s$ with $\bigcup_{l=1}^{s} \Gamma_l = \{1, \cdots, n\}$ and $\Gamma_{l_1} \cap \Gamma_{l_2} = \varnothing$ for all $l_1 \neq l_2$ such that

$$f(\mathbf{x}) = \sum_{i=1}^{n} f_{m,i} x_i^m + \sum_{l=1}^{s} \sum_{\alpha_l \in \Omega_l} f_{\alpha_l} \mathbf{x}^{\alpha_l},$$

where

$$\Omega_l = \{\alpha \in ([n] \cup \{0\})^n : \ |\alpha| = m, \mathbf{x}^\alpha = x_{i_1} x_{i_2} \cdots x_{i_m}, \{i_1, \cdots, i_m\} \subseteq \Gamma_l,$$
$$\text{and} \ \ \alpha \neq m\mathbf{e_i}, \ i = 1, \cdots, n\}$$

for each $l = 1, \cdots, s$ and either one of the following two conditions holds:

(1) $f_{\alpha_l} = 0$ for all but one $\alpha_l \in \Omega_l$;

(2) $f_{\alpha_l} \leqslant 0$ for all $\alpha_l \in \Omega_l$.

We now say a symmetric tensor $\mathcal{A}$ is an **extended Z-tensor** if its associated polynomial $f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m$ is a an extended Z-polynomial.

From the definition, it is clear that any Z-tensor is an extended Z-tensor with $s = 1$ and $\Gamma_1 = \{1, \cdots, n\}$. On the other hand, an extended Z-tensor allows a few elements of the off-diagonal elements to be positive, and so, an extended Z-tensor need not to be a Z-tensor. For example, consider a symmetric tensor $\mathcal{A}$ where its associated polynomial $f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m = x_1^6 + x_2^6 + x_3^6 + x_4^6 + 4x_1^3x_2^3 + 6x_3^2x_4^4$. It can be easily see that $\mathcal{A}$ is an extended Z-tensor but not a Z-tensor (as there are positive off-diagonal elements). In [66], partially Z-tensors are introduced. There is no direct relation between these two concepts, except that both of them contain Z-tensors. But they do have intersection which is larger than the set of all Z-tensors. Actually, the example just discussed is not a Z-tensor, but it is an extended Z-tensor and a partially Z-tensor as well.

We now see that any positive semi-definite extended Z-tensor has an SOS tensor decomposition. To achieve this, we recall the following useful lemma, which provides us a simple criterion for determining whether a homogeneous polynomial with only one mixed term is a sum of squares polynomial or not.

Lemma 6.2. *[23] Let $b_1, b_2, \cdots, b_n \geqslant 0$ and $d \in \mathbb{N}$. Let $a_1, a_2, \cdots, a_n \in \mathbb{N}$ be such* *that* $\sum_{i=1}^{n} = 2d$. *Consider the homogeneous polynomial* $f(\mathbf{x})$ *defined by*

$$f(\mathbf{x}) = b_1 x_1^{2d} + \cdots + b_n x_n^{2d} - \mu x_1^{a_1} \cdots x_n^{a_n}.$$

Let $\mu_0 = 2d \prod_{a_i \neq 0, 1 \leqslant i \leqslant n} (\frac{b_i}{a_i})^{\frac{a_i}{2d}}$. *Then, the following statements are equivalent:*

(i) f *is a non-negative polynomial i.e.* $f(\mathbf{x}) \geqslant 0$ *for all* $\mathbf{x} \in \mathbb{R}^n$*;*

(ii) *either* $|\mu| \leqslant \mu_0$ *or* $\mu < \mu_0$ *and all* a_i *are even;*

(iii) f *is an SOS polynomial.*

Theorem 6.3. *Let* $\mathcal{A}$ *be an even order positive semi-definite extended* Z*-tensor. Then,* $\mathcal{A}$ *has an SOS tensor decomposition.*

Proof. Let $f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m$. As $\mathcal{A}$ is a positive semi-definite symmetric extended Z-tensor, there exist $s \in \mathbb{N}$ and index sets $\Gamma_l \subseteq \{1, \cdots, n\}$, $l = 1, \cdots, s$ with $\bigcup_{l=1}^{s} \Gamma_l = \{1, \cdots, n\}$ and $\Gamma_{l_1} \cap \Gamma_{l_2} = \varnothing$ for all $l_1 \neq l_2$ such that for all $\mathbf{x} \in \mathbb{R}^n$

$$f(\mathbf{x}) = \sum_{i=1}^{n} f_{m,i} x_i^m + \sum_{l=1}^{s} \sum_{\alpha_l \in \Omega_l} f_{\alpha_l} \mathbf{x}^{\alpha_l}$$

such that, for each $l = 1, \cdots, s$, either one of the following two condition holds: (1) $f_{\alpha_l} = 0$ for all but one $\alpha_l \in \Omega_l$; (2) $f_{\alpha_l} \leqslant 0$ for all $\alpha_l \in \Omega_l$. Define, for each $l = 1, \cdots, s$,

$$h_l(\mathbf{x}) := \sum_{i \in \Gamma_l} f_{m,i} x_i^m + \sum_{\alpha_l \in \Omega_l} f_{\alpha_l} \mathbf{x}^{\alpha_l}.$$

It follows that each h_l is an extended Z-polynomial. Moreover, from the construction, $\sum_{l=1}^{s} h_l = f_{\mathcal{A}}$ and so, $\inf_{\mathbf{x} \in \mathbb{R}^n} \sum_{l=1}^{s} h_l(\mathbf{x}) = 0$. Note that each h_l is also a homogeneous polynomial, and hence $\inf_{\mathbf{x} \in \mathbb{R}^n} h_l(\mathbf{x}) \leqslant 0$. Noting that each h_l is indeed a polynomial on $(x_i)_{i \in \Gamma_l}$, $\bigcup_{l=1}^{s} \Gamma_l = \{1, \cdots, n\}$ and $\Gamma_{l_1} \cap \Gamma_{l_2} = \varnothing$ for all $l_1 \neq l_2$, we have $\inf_{\mathbf{x} \in \mathbb{R}^n} \sum_{l=1}^{s} h_l(\mathbf{x}) = \sum_{l=1}^{s} \inf_{\mathbf{x} \in \mathbb{R}^n} h_l(\mathbf{x})$. This enforces that $\inf_{\mathbf{x} \in \mathbb{R}^n} h_l(\mathbf{x}) = 0$.

In particular, each h_l is a polynomial which takes non-negative values. We now see that h_l, $1 \leqslant l \leqslant s$, are SOS polynomial. Indeed, if $f_{\alpha_l} = 0$ for all but one $\alpha_l \in \Omega_l$, then h_l is a homogeneous polynomial with only a mixed term, and so, Lemma 6.2 implies that h_l is a SOS polynomial. On the other hand, if $f_{\alpha_l} \leqslant 0$ for all $\alpha_l \in \Omega_l$, h_l corresponds to a Z-tensor, and so, h_l is also a SOS polynomial in this case because any positive semi-definite Z-tensor has an SOS tensor decomposition [37]. Thus, $f_{\mathcal{A}} = \sum_{l=1}^{s} h_l$ is also a SOS polynomial, and hence the conclusion follows. □

Remark 6.1. *A close inspection of the above proof indicates that we indeed shows that the associated polynomial $f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m$ satisfies $f_{\mathcal{A}} = \sum_{l=1}^{s} h_l$ where each h_l is an SOS polynomial in $(x_i)_{i \in \Gamma_l}$.*

6.2.4 Even order symmetric B_0-tensors have SOS decompositions

In this part, we show that even order symmetric B_0 tensors have SOS tensor decompositions. Recall that a tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ with order m and dimension n is called a B_0-tensor [82] if

$$\sum_{i_2,\cdots,i_m=1}^{n} a_{i i_2 \cdots i_m} \geqslant 0$$

and

$$\frac{1}{n^{m-1}} \sum_{i_2,\cdots,i_m=1}^{n} a_{i i_2 \cdots i_m} \geqslant a_{i j_2 \cdots j_m} \text{ for all } (j_2, \cdots, j_m) \neq (i, \cdots, i).$$

To establish that a B_0-tensor has an SOS tensor decomposition, we first present the SOS tensor decomposition of the all-one-tensor. We say $\mathcal{E}$ is an all-one-tensor if with each of its elements of $\mathcal{E}$ is equal to one.

Lemma 6.3. *Let $\mathcal{E}$ be an even order all-one-tensor. Then, $\mathcal{E}$ has an SOS tensor decomposition.*

Proof. Let $\mathcal{E} = (e_{i_1 i_2 \cdots i_m})$ be an all-one-tensor with even order m and dimension n. For all $\mathbf{x} \in \mathbb{R}^n$, one has

$$\begin{aligned}
\mathcal{E}\mathbf{x}^m =& \sum_{i_1, i_2, \cdots, i_m \in [n]} e_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m} \\
=& \sum_{i_1, i_2, \cdots, i_m \in [n]} x_{i_1} x_{i_2} \cdots x_{i_m} \\
=& (x_1 + x_2 + \cdots + x_n)^m \\
\geqslant& 0,
\end{aligned}$$

which implies that $\mathcal{E}$ has an SOS tensor decomposition. $\square$

Let $J \subset [n]$. $\mathcal{E}^J$ is called a partially all-one-tensor if its elements are defined such that $e_{i_1 i_2 \cdots i_m} = 1$, $i_1, i_2, \cdots, i_m \in J$ and $e_{i_1 i_2 \cdots i_m} = 0$ for the others. Similar to Lemma 6.3, it is easy to check that all even order partially all-one-tensors have SOS decompositions.

We also need the following characterization of B_0-tensors established in [82].

Lemma 6.4. *Suppose that $\mathcal{A}$ is a B_0-tensor with order m and dimension n. Then either $\mathcal{A}$ is a diagonally dominated symmetric M-tensor itself, or we have*

$$\mathcal{A} = \mathcal{M} + \sum_{k=1}^{s} h_k \mathcal{E}^{J_k},$$

where $\mathcal{M}$ is a diagonally dominated symmetric M-tensor, s is a positive integer, $h_k > 0$ and $J_k \subseteq \{1, \cdots, n\}$, for $k = 1, \cdots, s$, and $J_k \cap J_l = \varnothing$, for $k \neq l$.

From Theorem 6.1, Lemma 6.3 and Lemma 6.4, we have the following result.

Theorem 6.4. *All even order symmetric B_0-tensors have SOS tensor decompositions.*

Before we move on to the next part, we note that, stimulated by B_0-tensors in [82], symmetric double B-tensors, symmetric quasi-double B_0-tensors and symmetric MB_0-tensors have been studied in [52, 53]. Below, we briefly explain that, using a similar method of proof as above, these three classes of tensors all have SOS decompositions. To do this, let us recall the definitions of these three classes of tensors.

For a real symmetric tensor $\mathcal{B} = (b_{i_1 i_2 \cdots i_m})$ with order m and dimension n, denote

$$\beta_i(\mathcal{B}) = \max_{j_2,\cdots,j_m \in [n],(i,j_2,\cdots,j_m)\notin I} \{0, b_{ij_2\cdots j_m}\};$$

$$\Delta_i(\mathcal{B}) = \sum_{j_2,\cdots,j_m \in [n],(i,j_2,\cdots,j_m)\notin I} (\beta_i(\mathcal{B}) - b_{ij_2\cdots j_m});$$

$$\Delta_j^i(\mathcal{B}) = \Delta_j(\mathcal{B}) - (\beta_j(\mathcal{B}) - b_{jii\cdots i}),\ i \neq j.$$

As defined in [52, Definition 3], $\mathcal{B}$ is called a double B-tensor if, $b_{ii\cdots i} > \beta_i(\mathcal{B})$, for all $i \in [n]$ and for all $i, j \in [n], i \neq j$ such that

$$b_{ii\cdots i} - \beta_i(\mathcal{B}) \geqslant \Delta_i(\mathcal{B})$$

and

$$(b_{ii\cdots i} - \beta_i(\mathcal{B}))(b_{jj\cdots j} - \beta_j(\mathcal{B})) > \Delta_i(\mathcal{B})\Delta_j(\mathcal{B}).$$

If $b_{ii\cdots i} > \beta_i(\mathcal{B})$, for all $i \in [n]$ and

$$(b_{ii\cdots i} - \beta_i(\mathcal{B}))(b_{jj\cdots j} - \beta_j(\mathcal{B}) - \Delta_j^i(\mathcal{B})) \geqslant (\beta_j(\mathcal{B}) - b_{ji\cdots i})\Delta_i(\mathcal{B}),$$

then tensor $\mathcal{B}$ is called a quasi-double B_0-tensor (see Definition 2 of [53]).

Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ such that

$$a_{i_1 i_2 \cdots i_m} = b_{i_1 i_2 \cdots i_m} - \beta_{i_1}(\mathcal{B}),\ \text{for all}\ \ i_1 \in [n].$$

If $\mathcal{A}$ is an M-tensor, then B is called an MB_0-tensor (see Definition 3 of [53]). It was shown in [53] that all quasi-double B_0-tensors are MB_0-tensors.

In [52], Li et al. proved that, for any symmetric double B-tensor $\mathcal{B}$, either $\mathcal{B}$ is a doubly strictly diagonally dominated (DSDD) Z-tensor, or $\mathcal{B}$ can be decomposed to the sum of a DSDD Z-tensor and several positive multiples of partially all-one-tensors (see Theorem 6 of [52]). From Theorem 4 of [52], we know that an even order symmetric DSDD Z-tensor is positive definite. This together with the fact that any positive semi-definite Z-tensor has an SOS tensor decomposition [37] implies that any even order symmetric double B-tensor $\mathcal{B}$ has an SOS tensor decomposition. Moreover, from Theorem 7 of [53], we know that, for any symmetric MB_0-tensor, it is either an M-tensor itself or it can be decomposed as the sum of an M-tensor and several positive multiples of partially all-one-tensors. As even order symmetric M-tensors are positive semi-definite Z-tensors [116] which have, in particular, SOS decomposition, we see that any even order symmetric MB_0 tensor also has an SOS tensor decomposition. Combining these and noting that any quasi-double B_0-tensor is an MB_0-tensor, we arrive at the following conclusion.

Theorem 6.5. *Even order symmetric double B-tensors, even order symmetric quasi double B_0-tensors and even order symmetric MB_0-tensors all have SOS tensor decompositions.*

6.2.5 Even order symmetric H-tensors with non-negative diagonal elements have SOS decompositions

In this part, we show that any even order symmetric H-tensor with non-negative diagonal elements has an SOS tensor decomposition. Recall that an mth order n dimensional tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$, it's comparison tensor is defined by $\mathcal{M}(\mathcal{A}) = (m_{i_1 i_2 \cdots i_m})$ such that

$$m_{ii\cdots i} = |a_{ii\cdots i}|, \quad \text{and} \quad m_{i_1 i_2 \cdots i_m} = -|a_{i_1 i_2 \cdots i_m}|,$$

for all $i, i_1, \cdots, i_m \in [n], (i_1, i_2, \cdots, i_m) \notin I$. Then, tensor $\mathcal{A}$ is called an H-tensor [19] if there exists a tensor $\mathcal{Z}$ with non-negative entries such that $\mathcal{M}(\mathcal{A}) = s\mathcal{I} - \mathcal{Z}$ and $s \geqslant \rho(\mathcal{Z})$, where $\mathcal{I}$ is the identity tensor and $\rho(\mathcal{Z})$ is the spectral radius of $\mathcal{Z}$ defined as the maximum of modulus of all eigenvalues of $\mathcal{Z}$. If $s > \rho(\mathcal{Z})$, then $\mathcal{A}$ is called a nonsingular H-tensor. A characterization for nonsingular H-tensors was given in [19] which states $\mathcal{A}$ is a nonsingular H-tensor if and only if there exists an enteritis positive vector $\mathbf{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ such that

$$|a_{ii\cdots i}| y_i^{m-1} > \sum_{(i,i_2,\cdots,i_m)\notin I} |a_{ii_2\cdots i_m}| y_{i_2} y_{i_3} \cdots y_{i_m}, \quad \forall\, i \in [n].$$

We note that the above definitions were first introduced in [19]. These were further examined in [43, 54] where the authors in [54] referred nonsingular H-tensors simply as H-tensors and the authors in [43] referred nonsingular H-tensors as strong H-tensors.

Theorem 6.6. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be a symmetric H-tensor with even order m dimension n. Suppose that all the diagonal elements of $\mathcal{A}$ are non-negative. Then, $\mathcal{A}$ has an SOS tensor decomposition.*

Proof. We first show that any nonsingular H-tensor with positive diagonal elements has an SOS tensor decomposition. Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be a nonsingular H-tensor with even order m dimension n such that $a_{ii\cdots i} > 0$, $i \in [n]$. Then, there exists a vector $\mathbf{y} = (y_1, \cdots, y_n)^T \in \mathbb{R}^n$ with $y_i > 0$, $i = 1, \cdots, n$, such that

$$a_{ii\cdots i} y_i^{m-1} > \sum_{(i,i_2,\cdots,i_m)\notin I} |a_{ii_2\cdots i_m}| y_{i_2} y_{i_3} \cdots y_{i_m}, \ \forall\, i \in [n]. \tag{6.2}$$

To prove the conclusion, by Lemma 2.3, we only need to prove

$$\hat{f}_{\mathcal{A}}(\mathbf{x}) = \sum_{i\in[n]} a_{ii\cdots i} x_i^m - \sum_{(i_1,i_2,\cdots,i_m)\in\Delta_{\mathcal{A}}} |a_{i_1 i_2 \cdots i_m}| x_{i_1} x_{i_2} \cdots x_{i_m} \geqslant 0, \ \forall\, \mathbf{x} \in \mathbb{R}^n.$$

From (6.2), we know that

$$
\begin{aligned}
\hat{f}_{\mathcal{A}}(\mathbf{x}) \geqslant & \sum_{i\in[n]}\left(\sum_{(i,i_2,\cdots,i_m)\notin I}|a_{ii_2\cdots i_m}|y_i^{1-m}y_{i_2}y_{i_3}\cdots y_{i_m}x_i^m\right) \\
& - \sum_{(i_1,i_2,\cdots,i_m)\in\Delta_{\mathcal{A}}}|a_{i_1i_2\cdots i_m}|x_{i_1}x_{i_2}\cdots x_{i_m}.
\end{aligned}
\tag{6.3}
$$

Here, for any fixed tuple $(i_1^0, i_2^0, \cdots, i_m^0) \in \Delta_{\mathcal{A}}$, assume $(i_1^0, i_2^0, \cdots, i_m^0)$ is constituted by k distinct indices $j_1^0, j_2^0, \cdots, j_k^0$, $k \leqslant m$, which appear $s_1, s_2, \cdots, s_k$ times in $(i_1^0, i_2^0, \cdots, i_m^0)$ respectively, $s_l \in [m], l \in [k]$. Then, one has $s_1 + s_2 + \cdots + s_k = m$. Without loss of generality, we denote $a = |a_{i_1^0 i_2^0\cdots i_m^0}| > 0$. Let $\pi(i_1^0, i_2^0, \cdots, i_m^0)$ be the set consisting of all permutations of $(i_1^0, i_2^0, \cdots, i_m^0)$. So, on the right side of (6.3), there are some terms corresponding to the fixed tuple $(i_1^0, i_2^0, \cdots, i_m^0)$ such that

$$
\begin{aligned}
& \sum_{(j_1^0, i_2, \cdots, i_m) \in \pi(i_1^0, i_2^0, \cdots, i_m^0)} |a_{j_1^0 i_2 \cdots i_m}| y_{j_1^0}^{1-m} y_{i_2} y_{i_3} \cdots y_{i_m} x_{j_1^0}^m \\
& + \sum_{(j_2^0, i_2, \cdots, i_m) \in \pi(i_1^0, i_2^0, \cdots, i_m^0)} |a_{j_2^0 i_2 \cdots i_m}| y_{j_2^0}^{1-m} y_{i_2} y_{i_3} \cdots y_{i_m} x_{j_2^0}^m \\
& + \cdots \\
& + \sum_{(j_k^0, i_2, \cdots, i_m) \in \pi(i_1^0, i_2^0, \cdots, i_m^0)} |a_{j_k^0 i_2 \cdots i_m}| y_{j_k^0}^{1-m} y_{i_2} y_{i_3} \cdots y_{i_m} x_{j_k^0}^m \\
& - \sum_{(i_1, i_2, \cdots, i_m) \in \pi(i_1^0, i_2^0, \cdots, i_m^0)} |a_{i_1 i_2 \cdots i_m}| x_{i_1} x_{i_2} \cdots x_{i_m} \\
= \; & \binom{m-1}{s_1 - 1} \binom{m - s_1}{s_2} \cdots \binom{m - s_1 - s_2 \cdots - s_{k-1}}{s_k} a y_{j_1^0}^{s_1 - m} y_{j_2^0}^{s_2} \cdots y_{j_k^0}^{s_k} x_{j_1^0}^m \\
& + \binom{m-1}{s_2 - 1} \binom{m - s_2}{s_1} \cdots \binom{m - s_1 - s_2 \cdots - s_{k-1}}{s_k} a y_{j_2^0}^{s_2 - m} y_{j_1^0}^{s_1} y_{j_3^0}^{s_3} \cdots y_{j_k^0}^{s_k} x_{j_2^0}^m \\
& + \cdots\cdots \\
& + \binom{m-1}{s_k - 1} \binom{m - s_k}{s_1} \cdots \binom{m - s_k - s_1 \cdots - s_{k-2}}{s_{k-1}} a y_{j_k^0}^{s_k - m} y_{j_1^0}^{s_1} \cdots y_{j_{k-1}^0}^{s_{k-1}} x_{j_k^0}^m \\
& - \binom{m}{s_1} \binom{m - s_1}{s_2} \binom{m - s_1 - s_2}{s_3} \cdots \binom{m - s_1 - s_2 \cdots - s_{k-1}}{s_k} a x_{j_1^0}^{s_1} x_{j_2^0}^{s_2} \cdots x_{j_k^0}^{s_k} \\
= \; & \frac{(m-1)! a y_{j_1^0}^{s_1} y_{j_2^0}^{s_2} \cdots y_{j_k^0}^{s_k}}{s_1! s_2! \cdots s_k!} \left[s_1 \left(\frac{x_{j_1^0}}{y_{j_1^0}} \right)^m + s_2 \left(\frac{x_{j_2^0}}{y_{j_2^0}} \right)^m + \cdots + s_k \left(\frac{x_{j_k^0}}{y_{j_k^0}} \right)^m \right. \\
& \left. - m \left(\frac{x_{j_1^0}}{y_{j_1^0}} \right)^{s_1} \left(\frac{x_{j_2^0}}{y_{j_2^0}} \right)^{s_2} \cdots \left(\frac{x_{j_k^0}}{y_{j_k^0}} \right)^{s_k} \right] \\
\geqslant \; & 0,
\end{aligned}
$$

where the last inequality follows the arithmetic-geometric inequality and the fact $\mathbf{y} > \mathbf{0}$. Thus, each tuple $(i_1, i_2, \cdots, i_m) \in \Delta_{\mathcal{A}}$ corresponds to a non-negative value on the right side of (6.3), which implies that $\hat{f}(\mathbf{x}) \geqslant 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Hence, by Lemma 2.3, $\mathcal{A}$ has an SOS tensor decomposition.

Now, let $\mathcal{A}$ be a general H-tensor with non-negative diagonal elements. Then, for each $\epsilon > 0$, $\mathcal{A}_\epsilon := \mathcal{A} + \epsilon\mathcal{I}$ is a nonsingular H-tensor with positive diagonal elements. Thus, $\mathcal{A}_\epsilon \to \mathcal{A}$, and for each $\epsilon > 0$, $\mathcal{A}_\epsilon$ has an SOS tensor decomposition. As $\mathrm{SOS}_{m,n}$ is a closed convex cone, we see that $\mathcal{A}$ also has an SOS tensor decomposition and the desired results follows. □

6.3 The SOS-rank of SOS tensor decomposition

In this section, we study the SOS-rank of SOS tensor decomposition. Let us formally define the SOS-rank of SOS tensor decomposition as follows. Let $\mathcal{A}$ be a tensor with even order m and dimension n. Suppose $\mathcal{A}$ has a SOS tensor decomposition. As shown in Proposition 6.1, $\mathrm{SOS}_{m,n} = \mathrm{SOS}^h_{m,n}$ where $\mathrm{SOS}_{m,n}$ is the SOS tensor cone and $\mathrm{SOS}^h_{m,n}$ is the cone consisting of all mth-order n-dimensional symmetric tensors such that $f_{\mathcal{A}}(\mathbf{x}) := \langle \mathcal{A}, \mathbf{x}^m \rangle$ is a polynomial which can be written as sums of finitely many *homogeneous polynomials*. Thus, there exists $r \in \mathbb{N}$ such that the homogeneous polynomial $f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m$ can be decomposed by

$$f_{\mathcal{A}}(\mathbf{x}) = f_1^2(\mathbf{x}) + f_2^2(\mathbf{x}) + \cdots + f_r^2(\mathbf{x}), \ \forall \ \mathbf{x} \in \mathbb{R}^n,$$

where $f_i(\mathbf{x}), \ i \in [r]$ are homogeneous polynomials with degree $\frac{m}{2}$. The minimum value r is called the **SOS-rank** of $\mathcal{A}$, and is denoted by $\mathrm{SOSrank}(\mathcal{A})$.

Let C be a convex cone in the SOS tensor cone, that is C is a convex cone such that $C \subseteq \mathrm{SOS}_{m,n}$. We define the **SOS-width** of the convex cone C by

$$\text{SOS-width}(C) = \sup\{\mathrm{SOSrank}(\mathcal{A}) : \mathcal{A} \in C\}.$$

Here, we do not care about the minimum of the SOS-rank of all the possible tensors in the cone C as it will be always zero. Recall that it was shown by Choi et al. in [15, Theorem 4.4] that, an SOS homogeneous polynomial can be decomposed as sums of at most Λ many squares of homogeneous polynomials where

$$\Lambda = \frac{\sqrt{1+8a}-1}{2} \text{ and } a = \binom{n+m-1}{m}. \tag{6.4}$$

This immediately gives us that

Proposition 6.2. *Let $\mathcal{A}$ be a tensor with even order m and dimension n, $m, n \in \mathbb{N}$. Suppose $\mathcal{A}$ has an SOS tensor decomposition. Then, its SOS-rank satisfies* $\mathrm{SOSrank}(\mathcal{A}) \leqslant \Lambda$, *where Λ is given in (6.4). In particular,* $\text{SOS-width}(\mathrm{SOS}_{m,n}) \leqslant \Lambda$.

In the matrix case, that is, $m = 2$, the upper bound Λ equals the dimension n of the symmetric tensor which is tight in this case. On the other hand, in general, the upper bound is of the order $n^{m/2}$ and need not to be tight. However, for a class of structured tensors with bounded exponent (BD-tensors) that have SOS decompositions, we show that their SOS-rank is less or equal to the dimension n which is significantly smaller than the upper bound in the above proposition. Moreover, in this case, the SOS-width of the associated BD-tensor cone can be determined explicitly. To do this, let us recall the definition of polynomials with bounded exponent and define the BD-tensors. Let $e \in \mathbb{N}$. Recall that f is said to be a degree m homogeneous polynomials on $\mathbb{R}^n$ with bounded exponent e if

$$f(\mathbf{x}) = \sum_{\alpha} f_\alpha \mathbf{x}^\alpha = \sum_{\alpha} f_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

where $0 \leqslant \alpha_j \leqslant e$ and $\sum_{j=1}^n \alpha_j = m$. We note that degree 4 homogeneous polynomials on $\mathbb{R}^n$ with bounded exponent 2 is nothing but the bi-quadratic forms in dimension n. Let us denote $\mathrm{BD}_{m,n}^e$ to be the set consists of all degree m homogeneous polynomials on $\mathbb{R}^n$ with bounded exponent e.

An interesting result for characterizing when a positive semi-definite (PDF) homogeneous polynomial with bounded exponent has SOS tensor decomposition was established in [14] and can be stated as follows.

Lemma 6.5. *Let $n \in \mathbb{N}$ with $n \geqslant 3$. Suppose e, m are even numbers and $m \geqslant 4$.*

(1) *If $n \geqslant 4$, then $\mathrm{BD}^e_{m,n} \cap \mathrm{PSD}_{m,n} \subseteq \mathrm{SOS}_{m,n}$ if and only if $m \geqslant en - 2$;*

(2) *If $n = 3$, then $\mathrm{BD}^e_{m,n} \cap \mathrm{PSD}_{m,n} \subseteq \mathrm{SOS}_{m,n}$ if and only if $m = 4$ or $m \geqslant 3e - 4$.*

Now, we say a symmetric tensor $\mathcal{A}$ is a BD-tensor with order m, dimension n and exponent e if $f(\mathbf{x}) = \mathcal{A}\mathbf{x}^m$ is a degree m homogeneous polynomial on $\mathbb{R}^n$ with bounded exponent e. We also define $\mathrm{BD}^e_{m,n}$ to be the set consisting of all symmetric BD-tensors with order m, dimension n and exponent e. It is clear that $\mathrm{BD}^e_{m,n}$ is a convex cone.

Theorem 6.7. *Let $n \in \mathbb{N}$ with $n \geqslant 3$. Suppose e, m are even numbers and $m \geqslant 4$. Let $\mathcal{A}$ be a BD-tensor with order m, dimension n and exponent e. Suppose that $\mathcal{A}$ has an SOS tensor decomposition. Then, we have $\mathrm{SOSrank}(\mathcal{A}) \leqslant n$. Moreover, we have*

$$\text{SOS-width}(\mathrm{BD}^e_{m,n} \cap \mathrm{SOS}_{m,n}) = \begin{cases} 1 & \textit{if } m = en \\ n & \textit{otherwise.} \end{cases}$$

Proof. As $\mathcal{A}$ is a BD-tensor and it has SOS decomposition, the preceding lemma implies that either (i) $n \geqslant 4$ and $m \geqslant en - 2$ (ii) $n = 3$ and $m = 4$ and (iii) $n = 3$ and $m \geqslant 3e - 4$. We now divide the discussion into these three cases.

Suppose that Case (i) holds, i.e., $n \geqslant 4$ and $m \geqslant en - 2$. From the construction, we have $m \leqslant en$. If $m = en$, then $\mathcal{A}$ has the form $ax_1^e \cdots x_n^e$. Here, $a \geqslant 0$ because $\mathcal{A}$ has SOS decomposition and e is an even number. In this case, $\mathrm{SOSrank}(\mathcal{A}) = 1$. Now, let $m = en - 2$. Then,

$$\mathcal{A}\mathbf{x}^m = x_1^e \cdots x_n^e \left(\sum_{(i,j) \in F} a_{ij} x_i^{-1} x_j^{-1} \right),$$

for some $a_{ij} \in \mathbb{R}$, $(i,j) \in F$ and for some $F \subseteq \{1, \cdots, n\} \times \{1, \cdots, n\}$. As e is an even number and $\mathcal{A}$ has SOS decomposition, we have

$$\sum_{(i,j)\in F} a_{ij} x_i^{-1} x_j^{-1} \geqslant 0 \text{ for all } x_i \neq 0 \text{ and } x_j \neq 0.$$

Thus, by continuity, $Q(t_1, \cdots, t_n) = \sum_{(i,j)\in F} a_{ij} t_i t_j$ is a positive semi-definite quadratic form, and so, is at most sums of n many squares of linear functions in $t_1, \cdots, t_n$. Let $Q(t_1, \cdots, t_n) = \sum_{k=1}^n \left[q_k(t_1, \cdots, t_n)\right]^2$ where q_k are linear functions. Then,

$$\mathcal{A}\mathbf{x}^m = x_1^e \cdots x_n^e \left(\sum_{i=1}^n \left[q_k(x_1^{-1}, \cdots, x_n^{-1})\right]^2 \right) = \sum_{i=1}^n \left(x_1^e \cdots x_n^e \left[q_k(x_1^{-1}, \cdots, x_n^{-1})\right]^2 \right),$$

Note that

$$x_1^e \cdots x_n^e \left[q_k(x_1^{-1}, \cdots, x_n^{-1})\right]^2 = \left[x_1^{\frac{e}{2}} \cdots x_n^{\frac{e}{2}} q_k(x_1^{-1}, \cdots, x_n^{-1}) \right]^2$$

is a square. Thus, $\mathrm{SOSrank}(\mathcal{A}) \leqslant n$ in this case.

Suppose that Case (ii) holds, i.e., $n = 3$ and $m = 4$. Then by Hilbert's theorem [34], $\mathrm{SOSrank}(\mathcal{A}) \leqslant 3 = n$.

Suppose that Case (iii) holds, i.e., $n = 3$ and $m \geqslant 3e - 4$. In the case of $m = en - 2 = 3e - 2$ and $m = en = 3e$, using similar argument as in the Case (i), we see that the conclusion follows. The only remaining case is when $m = 3e - 4$. In this case, as $\mathcal{A}$ is a BD-tensor with order m, dimension 3 and exponent e and $\mathcal{A}$ has SOS decomposition, we have

$$\mathcal{A}\mathbf{x}^m = x_1^e x_2^e x_3^e G(x_1^{-1}, x_2^{-1}, x_3^{-1}),$$

where G is a positive semi-definite form and is of 3 dimension and degree 4. It then from Hilbert's theorem [34] that $G(t_1, t_2, t_3)$ can be expressed as at most the sum of 3 squares of 3-dimensional quadratic forms. Thus, using similar line of argument as in Case (i) and noting that $e \geqslant 4$ (as $m = 3e - 4$ and $m \geqslant 4$), we have $\mathrm{SOSrank}(\mathcal{A}) \leqslant n = 3$.

Combining these three cases, we see that $\mathrm{SOSrank}(\mathcal{A}) \leqslant n$, and $\mathrm{SOSrank}(\mathcal{A}) = 1$ if $m = en$. In particular, we have $\text{SOS-width}(\mathrm{BD}^e_{m,n} \cap \mathrm{SOS}_{m,n}) \leqslant n$, and $\text{SOS-width}(\mathrm{BD}^e_{m,n} \cap \mathrm{SOS}_{m,n}) = 1$ if $m = en$. To see the conclusion, we consider the homogeneous polynomial

$$f_0(\mathbf{x}) = \begin{cases} x_1^e \cdots x_n^e (\sum_{i=1}^n x_i^{-2}) & \text{if } n \geqslant 3 \text{ and } m = en - 2 \\ x_1^2 x_2^2 + x_2^2 x_3^2 + x_3^2 x_1^2 & \text{if } n = 3 \text{ and } m = 4 \\ x_1^e x_2^e x_3^e (x_1^{-2} x_2^{-2} + x_2^{-2} x_3^{-2} + x_3^{-2} x_1^{-2}) & \text{if } n = 3 \text{ and } m = 3e - 4 \end{cases}$$

and its associated BD-tensor $\mathcal{A}_0$ such that $f_0(\mathbf{x}) = \mathcal{A}_0 \mathbf{x}^m$. It can be directly verified that

$$\mathrm{SOSrank}(\mathcal{A}_0) = \begin{cases} n & \text{if } n \geqslant 3 \text{ and } m = en - 2, \\ 3 & \text{if } n = 3 \text{ and } m = 4, \\ 3 & \text{if } n = 3 \text{ and } m = 3e - 4. \end{cases}$$

For example, in the case $n \geqslant 3$ and $m = en - 2$, to see $\mathrm{SOSrank}(\mathcal{A}_0) = n$, we only need to show $\mathrm{SOSrank}(\mathcal{A}_0) \geqslant n$. Suppose on the contrary that $\mathrm{SOSrank}(\mathcal{A}_0) \leqslant n-1$. Then, there exists $r \leqslant n-1$ and homogeneous polynomial f_i with degree $m/2 = \frac{e}{2}n-1$ such that

$$x_1^e \cdots x_n^e \left(\sum_{i=1}^n x_i^{-2} \right) = \sum_{i=1}^r f_i(\mathbf{x})^2.$$

This implies that for each $\mathbf{x} = (x_1, \cdots, x_n)$ with $x_i \neq 0$, $i = 1, \cdots, n$

$$\sum_{i=1}^n x_i^{-2} = \sum_{i=1}^r \left[\frac{f_i(\mathbf{x})}{x_1^{\frac{e}{2}} \cdots x_n^{\frac{e}{2}}} \right]^2$$

Letting $t_i = x_i^{-1}$, by continuity, we see that the quadratic form $\sum_{i=1}^n t_i^2$ can be written as a sum of at most r many squares of rational functions in $(t_1, , \cdots, t_n)$. Then, the Cassels-Pfister's Theorem [22, Theorem 17.3] (see also [22, Corollary 17.6]), implies that the quadratic form $\sum_{i=1}^n t_i^2$ can be written as a sum of at most r many sums of squares of polynomial functions in $(t_1, , \cdots, t_n)$, which is impossible.

In the case $n = 3$ and $m = 4$, we only need to show $\mathrm{SOSrank}(\mathcal{A}_0) \geqslant 3$. Suppose on the contrary that $\mathrm{SOSrank}(\mathcal{A}_0) \leqslant 2$. Then, there exist $a_i, b_i, c_i, d_i, e_i, f_i \in \mathbb{R}$, $i = 1, 2$, such that

$$\begin{aligned} x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_1^2 &= (a_1x_1^2 + b_1x_2^2 + c_1x_3^2 + d_1x_1x_2 + e_1x_1x_3 + f_1x_2x_3)^2 \\ &\quad +(a_2x_1^2 + b_2x_2^2 + c_2x_3^2 + d_2x_1x_2 + e_2x_1x_3 + f_2x_2x_3)^2. \end{aligned}$$

Comparing with the coefficients gives us that $a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 0$ and

$$\begin{cases} d_1^2 + d_2^2 = 1 \\ e_1^2 + e_2^2 = 1 \\ f_1^2 + f_2^2 = 1 \\ d_1e_1 + d_2e_2 = 0 \\ d_1f_1 + d_2f_2 = 0 \\ e_1f_1 + e_2f_2 = 0. \end{cases}$$

From the last three equations, we see that one of $d_1, d_2, e_1, e_2, f_1, f_2$ must be zero. Let us assume say $d_1 = 0$. Then, the first equation shows $d_2 = \pm 1$ and hence, $e_2 = 0$ (by the fourth equation). This implies that $e_1 = \pm 1$ and $f_2 = 0$. Again, we have $f_1 = \pm 1$ and hence

$$e_1f_1 + e_2f_2 = (\pm 1)(\pm 1) + 0 = \pm 1 \neq 0.$$

This leads to a contradiction.

For the last case, suppose again by contradiction that $\mathrm{SOSrank}(\mathcal{A}_0) \leqslant 2$. Then, there exist two homogeneous polynomial f_i with degree $m/2 = \frac{3e}{2} - 2$ such that

$$x_1^e x_2^e x_3^e (x_1^{-2}x_2^{-2} + x_2^{-2}x_3^{-2} + x_3^{-2}x_1^{-2}) = \sum_{i=1}^{2} f_i(\mathbf{x})^2.$$

This implies that for each $\mathbf{x} = (x_1, \cdots, x_n)$ with $x_i \neq 0$, $i = 1, \cdots, n$

$$x_1^{-2}x_2^{-2} + x_2^{-2}x_3^{-2} + x_3^{-2}x_1^{-2} = \sum_{i=1}^{2} \left[\frac{f_i(\mathbf{x})}{x_1^{\frac{e}{2}} \cdots x_3^{\frac{e}{2}}} \right]^2$$

Letting $t_i = x_i^{-1}$, using a similar line argument in the case $m = en - 2$, we see that the polynomial $t_1^2 t_2^2 + t_2^2 t_3^2 + t_3^2 t_1^2$ can be written as sums of 2 squares of polynomials in (t_1, t_2, t_3). This is impossible by the preceding case. Therefore, the conclusion follows. $\square$

Below, let us mention that calculating the exact SOS-rank of SOS tensor decomposition is not a trivial task even for the identity tensor, and this relates to some open question in algebraic geometry in the literature. To explain this, we recall that the identity tensor $\mathcal{I}$ with order m and dimension n is given by $\mathcal{I}_{i_1 \cdots i_m} = 1$ if $i_1 = \cdots = i_m$ and $\mathcal{I}_{i_1 \cdots i_m} = 0$ otherwise. The identity tensor $\mathcal{I}$ induces the polynomial $f_{\mathcal{I}}(\mathbf{x}) = \mathcal{I}\mathbf{x}^m = x_1^m + \cdots + x_n^m$. It is clear that, $\mathcal{I}$ has an SOS tensor decomposition when m is even and the corresponding SOS-rank of $\mathcal{I}$ is less than or equal to n. It was conjectured by Reznick [91] that $f_{\mathcal{I}}(\mathbf{x})$ cannot be written as sums of $(n-1)$ many squares, that is, $\mathrm{SOSrank}(\mathcal{I}) = n$. The positive answer for this conjecture in the special case of $m = n = 4$ was provided in [108, 112]. On the other hand, the answer for this conjecture in the general case is still open to the best of our knowledge. Moreover, this conjecture relates to another conjecture of Reznick [91] in the same paper where he showed that the polynomial $f_R(\mathbf{x}) = x_1^n + \cdots + x_n^n - n x_1 \cdots x_n$ can be written as sums of $(n-1)$ many squares whenever $n = 2^k$ for some $k \in \mathbb{N}$, and he conjectured that the estimate of the numbers of squares is sharp. Indeed, he also showed that this conjecture is true whenever the previous conjecture of "$f_{\mathcal{I}}(\mathbf{x})$ cannot be written as sums of $(n-1)$ many squares" is true.

6.4 Applications

In this section, we provide some applications for the SOS tensor decomposition of the structure tensors such as finding the minimum H-eigenvalue of an even order extended Z-tensor and testing the positive definiteness of a multivariate form. We also provide some numerical examples/experiments to support the theoretical findings. Throughout this section, all numerical experiments are performed on a desktop, with 3.47 GHz quad-core Intel E5620 Xeon 64-bit CPUs and 4 GB RAM, equipped with Matlab 2015.

Finding the minimum eigenvalue of a tensor is an important topic in tensor computation and multilinear algebra, and has found numerous applications including automatic control and image processing [78]. Recently, it was shown that the minimum H-eigenvalue of an even order symmetric Z-tensor [38, 37] can be found by solving a sums-of-squares optimization problem, which can be equivalently reformulated as a semi-definite programming problem, and so, can be solved efficiently. In [38], some upper and lower estimates for the minimum H-eigenvalue of general symmetric tensors with even order are provided via sums-of-squares programming problems. Examples show that the estimate can be sharp in some cases.

On the other hand, it was unknown in [37, 38] that whether similar results can continue to hold for some classes of symmetric tensors which are not Z-tensors, that is, for symmetric tensors with possible positive entries on the off-diagonal elements. In this section, as applications of the derived SOS decomposition of structured tensors, we show that the class of even order symmetric extended Z-tensor serves as one such class. To present the conclusion, the following Lemma plays an important role in our later analysis.

Lemma 6.6 ([78])**.** *Let $\mathcal{A}$ be a symmetric tensor with even order m and dimension n. Denote the minimum H-eigenvalue of $\mathcal{A}$ by $\lambda_{min}(\mathcal{A})$. Then, we have*

$$\lambda_{min}(\mathcal{A}) = \min_{\mathbf{x}\neq\mathbf{0}} \frac{\mathcal{A}\mathbf{x}^m}{\|\mathbf{x}\|_m^m} = \min_{\|\mathbf{x}\|_m=1} \mathcal{A}\mathbf{x}^m, \tag{6.5}$$

where $\|\mathbf{x}\|_m = (\sum_{i=1}^n |x_i|^m)^{\frac{1}{m}}$.

Theorem 6.8. (Finding the minimum H-eigenvalue of an even order symmetric extended Z-tensor) *Let m be an even number. Let $\mathcal{A}$ be a symmetric extended Z-tensor with order m and dimension n. Then, we have*

$$\lambda_{\min}(\mathcal{A}) \quad = \quad \max_{\mu, r \in \mathbb{R}} \{\mu : f_{\mathcal{A}}(\mathbf{x}) - r(\|\mathbf{x}\|_m^m - 1) - \mu \in \Sigma_m^2[\mathbf{x}]\},$$

where $f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m$ and $\Sigma_m^2[\mathbf{x}]$ is the set of all SOS polynomials with degree at most m.

Proof. Consider the following problem

$$(P) \quad \min\{\mathcal{A}\mathbf{x}^m : \|\mathbf{x}\|_m^m = 1\}$$

and denote its global minimizer by $\mathbf{a} = (a_1, \cdots, a_n)^T \in \mathbb{R}^n$. Clearly, $\sum_{i=1}^n a_i^m = 1$. Then, $\lambda_{\min}(\mathcal{A}) = f_{\mathcal{A}}(\mathbf{a}) = \mathcal{A}\mathbf{a}^m$. It follows that for all $\mathbf{x} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$

$$\begin{aligned} f_{\mathcal{A}}(\mathbf{x}) - \lambda_{\min}(\mathcal{A}) \sum_{i=1}^n x_i^m &= f_{\mathcal{A}}(\mathbf{x}) - f_{\mathcal{A}}(\mathbf{a}) \sum_{i=1}^n x_i^m \\ &= \sum_{i=1}^n x_i^m \left(f_{\mathcal{A}}(\frac{\mathbf{x}}{(\sum_{i=1}^n x_i^m)^{\frac{1}{m}}}) - f_{\mathcal{A}}(\mathbf{a}) \right) \geqslant 0, \end{aligned}$$

where the last inequality holds as m is even and $\bar{\mathbf{x}} = \frac{\mathbf{x}}{(\sum_{i=1}^n x_i^m)^{\frac{1}{m}}}$ belongs to the feasible set of (P). This shows that $g(\mathbf{x}) := f_{\mathcal{A}}(\mathbf{x}) - \lambda_{\min}(\mathcal{A}) \sum_{i=1}^n x_i^m$ is a homogeneous polynomial which always take non-negative values. As $\mathcal{A}$ is an extended Z-tensor, there exist $s \in \mathbb{N}$ and index sets $\Gamma_l \subseteq \{1, \cdots, n\}$, $l = 1, \cdots, s$ with $\bigcup_{l=1}^s \Gamma_l = \{1, \cdots, n\}$ and $\Gamma_{l_1} \cap \Gamma_{l_2} = \varnothing$ such that for all $\mathbf{x} \in \mathbb{R}^n$

$$f_{\mathcal{A}}(\mathbf{x}) = \sum_{i=1}^n f_{m,i} x_i^m + \sum_{l=1}^s \sum_{\alpha_l \in \Omega_l} f_{\alpha_l} \mathbf{x}^{\alpha_l} \tag{6.6}$$

such that, for each $l = 1, \cdots, s$, either one of the following two condition holds: (1) $f_{\alpha_l} = 0$ for all but one $\alpha_l \in \Omega_l$; (2) $f_{\alpha_l} \leqslant 0$ for all $\alpha_l \in \Omega_l$. Thus,

$$g(\mathbf{x}) = \sum_{i=1}^n (f_{m,i} - \lambda_{\min}(\mathcal{A})) x_i^m + \sum_{l=1}^s \sum_{\alpha_l \in \Omega_l} f_{\alpha_l} \mathbf{x}^{\alpha_l},$$

is an extended Z-polynomial which always takes non-negative values. Let $\mathcal{B}$ be a symmetric tensor such that $g(\mathbf{x}) = \mathcal{B}\mathbf{x}^m$. Then, $\mathcal{B}$ is a positive semi-definite extended Z-tensor and so is SOS by Theorem 6.3. Thus, $g(\mathbf{x})$ is an SOS polynomial with degree m. Note that $g(\mathbf{x}) = f_{\mathcal{A}}(\mathbf{x}) - \lambda_{\min}(\mathcal{A})\sum_{i=1}^n x_i^m = f_{\mathcal{A}}(\mathbf{x}) - \lambda_{\min}(\mathcal{A})(\sum_{i=1}^n x_i^m - 1) - \lambda_{\min}(\mathcal{A})$. This shows that

$$\lambda_{\min}(\mathcal{A}) \leqslant \max_{\mu,r\in\mathbb{R}}\{\mu : f_{\mathcal{A}}(\mathbf{x}) - r(\|\mathbf{x}\|_m^m - 1) - \mu \in \Sigma_m^2[\mathbf{x}]\}.$$

To see the reverse inequality, take any (μ, r) with $f_{\mathcal{A}}(\mathbf{x}) - r(\|\mathbf{x}\|_m^m - 1) - \mu \in \Sigma_m^2[\mathbf{x}]$. Then, for all $\mathbf{x} \in \mathbb{R}^n$,

$$f_{\mathcal{A}}(\mathbf{x}) - r(\|\mathbf{x}\|_m^m - 1) - \mu \geqslant 0.$$

This shows that $r \geqslant \mu$ and $f_{\mathcal{A}}(\mathbf{x}) \geqslant r\|\mathbf{x}\|_m^m$ for all $\mathbf{x} \in \mathbb{R}^n$. This shows that $\lambda_{\min}(\mathcal{A}) \geqslant r \geqslant \mu$, and so, the conclusion follows. $\square$

Remark 6.2. *Let $\mathcal{A}$ be an extended Z-tensor. As in (6.6), its associated polynomial $f_{\mathcal{A}}$ can be written as $f_{\mathcal{A}}(\mathbf{x}) = \sum_{i=1}^n f_{m,i}x_i^m + \sum_{l=1}^s \sum_{\alpha_l\in\Omega_l} f_{\alpha_l}\mathbf{x}^{\alpha_l}$. Then, Remark 6.1 implies that*

$$\begin{aligned}
\lambda_{\min}(\mathcal{A}) &= \max_{\mu,r\in\mathbb{R}}\{\mu : f_{\mathcal{A}}(\mathbf{x}) - r(\|\mathbf{x}\|_m^m - 1) - \mu \in \Sigma_m^2[\mathbf{x}]\} \\
&= \max_{\mu,r\in\mathbb{R}}\{\mu : f_{\mathcal{A}}(\mathbf{x}) - r\|\mathbf{x}\|_m^m \in \Sigma_m^2[\mathbf{x}], r - \mu \geqslant 0\} \\
&= \max_{\mu,r\in\mathbb{R}}\{\mu : \sum_{i\in\Gamma_l} f_{m,i}x_i^m + \sum_{\alpha_l\in\Omega_l} f_{\alpha_l}\mathbf{x}^{\alpha_l} - r\|\mathbf{x}^{(l)}\|_m^m \in \Sigma_m^2[\mathbf{x}^{(l)}], l = 1, \cdots, s \\
&\qquad\qquad r - \mu \geqslant 0\}
\end{aligned}$$

where, for each $l = 1, \cdots, s$, $\mathbf{x}^{(l)} = (x_i)_{i\in\Gamma_l}$ and $\Sigma_m^2[\mathbf{x}^{(l)}]$ is the set of all SOS polynomials in $\mathbf{x}^{(l)}$.

As explained in [37, 38], the sums-of-squares problem

$$\max_{\mu,r\in\mathbb{R}}\{\mu : f_{\mathcal{A}}(\mathbf{x}) - r(\|\mathbf{x}\|_m^m - 1) - \mu \in \Sigma_m^2[\mathbf{x}]\}$$

can be equivalently rewritten as a semi-definite programming problem (SDP), and so, can be solved efficiently. Indeed, this conversion can be done by using the commonly used Matlab Toolbox YALMIP [64, 65]. On the other hand, the size of the equivalent SDP problem of the relaxation problem increase dramatically when the dimension/order of the tensor increases. For example, as illustrate in Table 1, for a 4th-order 50-dimensional tensor, the equivalent SDP problem has 1326 variables and 316251 constraints. Fortunately, a robust SDP software (SDPNAL [117]) has been established recently which enables us to solve large-scale SDP (dimension up to 5000 and number of constraint of the SDP up to 1 million). This enables us to find the minimum H-eigenvalue for medium-size tensor. Later on, we will explain how to use SDPNAL together with the observation in Remark 6.2 to find the minimum H-eigenvalue for large-size tensor.

We first illustrate how to compute the minimum H-eigenvalue of an extended Z-tensor $\mathcal{A}$ using the above sums-of-squares problem via Matlab Toolbox YALMIP [64, 65] via two small-size problems. We will show the performance of the method for various larger-size problem later.

Example 6.1. *Consider the symmetric tensor $\mathcal{A}$ with order 6 and dimension 4 where*

$$\mathcal{A}_{111111} = \mathcal{A}_{222222} = \mathcal{A}_{333333} = \mathcal{A}_{444444} = 1,$$

$$\mathcal{A}_{i_1\cdots i_6} = \frac{1}{5},\ \textit{for all } (i_1, \cdots, i_6) = \sigma(1,1,1,2,2,2),$$

$$\mathcal{A}_{i_1\cdots i_6} = \frac{2}{5},\ \textit{for all } (i_1, \cdots, i_6) = \sigma(3,3,4,4,4,4),$$

and $\mathcal{A}_{i_1\cdots i_6} = 0$ otherwise. Here $\sigma(i_1, \cdots, i_6)$ denotes all the possible permutation of $(i_1, \cdots, i_6)$. The associated polynomial

$$f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m = x_1^6 + x_2^6 + x_3^6 + x_4^6 + 4x_1^3x_2^3 + 6x_3^2x_4^4$$

is an extended Z-polynomial. So, $\mathcal{A}$ is an extended Z- tensor. It can be easily verified that $\mathcal{A}$ is not a Z-tensor.

To compute its minimum H-eigenvalue, we note that the corresponding sums-of-squares optimization problem reads

$$\max_{\mu,r\in\mathbb{R}}\{\mu : f_{\mathcal{A}}(\mathbf{x}) - r(\|\mathbf{x}\|_6^6 - 1) - \mu \in \Sigma_6^2[\mathbf{x}]\}.$$

Convert this sums-of-squares optimization problem into a semi-definite programming problem using the Matlab Toolbox YALMIP *[64, 65], and solve it by using the SDP software* SDPNAL *we obtain that $\lambda_{\min}(\mathcal{A}) = -1$. The simple code using* YALMIP *is appended as follows:*

```
sdpsettings('solver','sdpnal')
sdpvar x1 x2 x3 x4 r mu
f = x1^6+x2^6+x3^6+x4^6+4*x1^3*x2^3+6*x3^2*x4^4;
g = [(x1^6+x2^6+x3^6+x4^6)-1];
F = [sos(f-mu-r*g)];
solvesos(F,-mu,[],[r;mu])
```

Moreover, note from the geometric mean inequality that $|x_1^3x_2^3| = (x_1^6)^{\frac{1}{2}}(x_2^6)^{\frac{1}{2}} \leqslant \frac{1}{2}x_1^6 + \frac{1}{2}x_1^6$. It follows that

$$f_{\mathcal{A}}(\mathbf{x}) + \|\mathbf{x}\|_6^6 = 2x_1^6 + 2x_2^6 + 2x_3^6 + 2x_4^6 + 4x_1^3x_2^3 + 6x_3^2x_4^4 \geqslant 0 \textit{ for all } \mathbf{x} \in \mathbb{R}^n.$$

On the other hand, consider $\bar{\mathbf{x}} = (\sqrt[6]{\frac{1}{2}}, -\sqrt[6]{\frac{1}{2}}, 0, 0)$. We see that $f_{\mathcal{A}}(\bar{\mathbf{x}}) + \|\bar{\mathbf{x}}\|_6^6 = 0$. This shows that $\lambda_{\min}(\mathcal{A}) = \min\{f_{\mathcal{A}}(\mathbf{x}) : \|\mathbf{x}\|_6 = 1\} = -1$. This verifies the correctness of our computed minimum H-eigenvalue. $\square$

Example 6.2. *Let $\alpha, \beta \in \mathbb{R}$ and consider the symmetric tensor $\mathcal{A}$ with order 6 and dimension 4 where*

$$\mathcal{A}_{111111} = \mathcal{A}_{222222} = \mathcal{A}_{333333} = \mathcal{A}_{444444} = 1,$$

$$\mathcal{A}_{i_1\cdots i_6} = \alpha, \text{ for all } (i_1, \cdots, i_6) = \sigma(1,1,1,2,2,2),$$

$$\mathcal{A}_{i_1\cdots i_6} = \beta, \text{ for all } (i_1, \cdots, i_6) = \sigma(3,3,3,4,4,4),$$

and $\mathcal{A}_{i_1\cdots i_6} = 0$ *otherwise. Here* $\sigma(i_1, \cdots, i_6)$ *denotes all the possible permutation of* $(i_1, \cdots, i_6)$*. The associated polynomial*

$$f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m = x_1^6 + x_2^6 + x_3^6 + x_4^6 + 20\alpha\, x_1^3 x_2^3 + 20\beta\, x_3^3 x_4^3$$

is an extended Z*-polynomial. So,* $\mathcal{A}$ *is an extended* Z*- tensor. It can be easily verified that if either* $\alpha > 0$ *or* $\beta > 0$*, then* $\mathcal{A}$ *is not a* Z*-tensor.*

To compute its minimum H*-eigenvalue, we randomly generate 100 instance of* $(\alpha, \beta) \in [-5, 5] \times [-5, 5]$*. For each* (α, β)*, we convert the corresponding sums-of-squares optimization problem*

$$\max_{\mu, r \in \mathbb{R}} \{\mu : f_{\mathcal{A}}(\mathbf{x}) - r(\|\mathbf{x}\|_6^6 - 1) - \mu \in \Sigma_6^2[\mathbf{x}]\}$$

into a semi-definite programming problem using the Matlab Toolbox YALMIP *[64, 65], and solve it by using the SDP software* SDPNAL*. We then compare the computed minimum* H*-eigenvalue with the true minimum* H*-eigenvalue of* $\mathcal{A}$*. Indeed, similar to the preceding example, we can verify that* $\lambda_{\min}(\mathcal{A}) = m(\alpha, \beta)$ *where*

$$m(\alpha, \beta) := \begin{cases} 1 - 10|\alpha| & \text{if } \ |\alpha| \geqslant |\beta|, \\ 1 - 10|\beta| & \text{if } \ |\alpha| < |\beta|. \end{cases}$$

For all the 100 *generated* (α, β)*, the maximum difference of the computed* H*-minimum eigenvalue and the true* H*-minimum eigenvalue is* $6.2039e - 05$. □

Medium-size examples

We now consider a few medium-size examples which involves symmetric extended Z-tensor with order up to 30 or dimension up to 60 .

Example 6.3. *Let $m = 10k$ with $k \in \mathbb{N}$. Consider the symmetric tensor $\mathcal{A}$ with order m and dimension 4 where*

$$\mathcal{A}_{1\cdots 1} = \mathcal{A}_{2\cdots 2} = \mathcal{A}_{3\cdots 3} = \mathcal{A}_{4\cdots 4} = 1,$$

$$\mathcal{A}_{i_1\cdots i_m} = \alpha, \text{ for all } (i_1, \cdots, i_m) = \sigma(\underbrace{1, \cdots, 1}_{m/2}, \underbrace{2, \cdots, 2}_{m/2}),$$

$$\mathcal{A}_{i_1\cdots i_m} = \beta, \text{ for all } (i_1, \cdots, i_m) = \sigma(\underbrace{3, \cdots, 3}_{m/5}, \underbrace{4, \cdots, 4}_{4m/5}),$$

$$\mathcal{A}_{i_1\cdots i_m} = \beta, \text{ for all } (i_1, \cdots, i_m) = \sigma(\underbrace{3, \cdots, 3}_{4m/5}, \underbrace{4, \cdots, 4}_{m/5}),$$

with $\alpha = 2\binom{m}{m/2}^{-1}$ and $\beta = -\binom{m}{m/5}^{-1}$, and $\mathcal{A}_{i_1\cdots i_m} = 0$ otherwise. Here $\sigma(i_1, \cdots, i_m)$ denotes all the possible permutation of $(i_1, \cdots, i_m)$. The associated polynomial

$$f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m = x_1^m + x_2^m + x_3^m + x_4^m + 2x_1^{\frac{m}{2}} x_2^{\frac{m}{2}} - x_3^{\frac{m}{5}} x_4^{\frac{4m}{5}} - x_3^{\frac{4m}{5}} x_4^{\frac{m}{5}},$$

is an extended Z-polynomial. So, $\mathcal{A}$ is an extended Z-tensor. It can be easily verified that $\mathcal{A}$ is not a Z-tensor. Moreover, using weighted geometric mean inequality, we can directly verify that the true minimum H-eigenvalue is 0.

We compute the minimum H-eigenvalue by solving the corresponding sums-of-squares problem for the case $m = 20, 30$, and compare with the true minimum H-eigenvalue. The results are summarized in Table 1. $\square$

Example 6.4. *Let $n = 4k$ with $k \in \mathbb{N}$. Consider the symmetric tensor $\mathcal{A}$ with order 4 and dimension n where*

$$\mathcal{A}_{1111} = \mathcal{A}_{2222} = \cdots = \mathcal{A}_{nnnn} = n,$$

$$\mathcal{A}_{i_1 i_2 i_3 i_4} = \frac{1}{6}, \text{ for all } (i_1, i_2, , i_3, i_4) = \sigma(4i-3, 4i-2, 4i-1, 4i), i = 1, \cdots, \frac{n}{4},$$

and $\mathcal{A}_{i_1i_2i_3i_4} = 0$ *otherwise. Here* $\sigma(i_1, \cdots, i_4)$ *denotes all the possible permutation of* $(i_1, \cdots, i_4)$. *The associated polynomial*

$$f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m = n(x_1^4 + \cdots + x_n^4) + 4\sum_{i=1}^{n/4} x_{4i-3}\, x_{4i-2}\, x_{4i-1}\, x_{4i}$$

is an extended Z*-polynomial. So,* $\mathcal{A}$ *is an extended* Z*-tensor. It can be easily verified that* $\mathcal{A}$ *is not a* Z*-tensor. Moreover, using geometric mean inequality, we can directly verify that the true minimum* H*-eigenvalue is* $n-1$.

We compute the minimum H*-eigenvalue by solving the corresponding sums-of-squares problem for the case* $n = 20, 40, 50, 60$, *and compare with the true minimum* H*-eigenvalue. The results are summarized in Table 1.*

Table 6.1: Test results for medium size tensors

Problem	m	n	NV	NC	Computed eigenvalue	True eigenvalue	Time (YAL.)	Time (SDP.)
Ex. 6.3	20	4	1001	1001	-1.7634e-09	0	11.9487	0.5700
Ex. 6.3	30	4	3876	6936	1.1382e-12	0	198.8141	8.2700
Ex. 6.4	4	20	231	10626	19.0000	19	4.6951	0.4763
Ex. 6.4	4	40	861	135751	39.0000	39	440.8231	1.7727
Ex. 6.4	4	50	1326	316251	49.0000	49	2365.9043	5.1109
Ex. 6.4	4	60	1891	635376	59.0000	59	9322.0631	50.2934

The table above summarizes the numerical results of Example 5.3 and Example 5.4 where we compute the minimum H-eigenvalue by first converting the corresponding sums-of-squares problem to an SDP problem using YALMIP and solving this S-DP problem using SDPNAL. We observe that, for all the above numerical examples, the minimum H-eigenvalues can be found successfully for medium-size tensors. In particular, the data of the above table are explained as follows.

- m: the order of the symmetric tensor,

- n: the dimension of the symmetric tensor,
- NV: the number of variables of the equivalent SDP problem,
- NC: the number of constraints in the equivalent SDP problem,
- Computed eigenvalue: the calculated minimum H-eigenvalue,
- True eigenvalue: the true minimum H-eigenvalue
- Time (YALMIP): the CPU-time for converting the sums-of-squares problem to SDP (measured in seconds).
- Time (SDPNAL): the CPU-time for solving SDP via SDPNAL (measured in seconds).

6.4.1 Large size examples

Finally, we illustrate with an example that using SDPNAL together with the observation in Remark 6.2 enables us to solve some large size tensors (dimension up to 2000).

As one can observed in Table 1, most of the time are occupied in YALMIP in converting the sums-of-squares problem into an SDP problem. This process involves matching up the coefficients of all the involved $\binom{m+n-1}{m}$ many monomials, and so, can be time-consuming. On the other hand, by using the sums-of-squares problem discussed in Remark 6.2 and letting $k = \max_{1\leqslant l\leqslant s} |\Gamma_l|$, the corresponding process only involves $s\binom{m+k-1}{m}$ many monomials which is much smaller than $\binom{m+n-1}{m}$ when s is large and k is small. For example, as in Example 5.4, we can set $s = n/4$, $k = 4$ and $m = 4$, and so, $s\binom{m+k-1}{m}$ is of the order n; while $\binom{m+n-1}{m} = \binom{n+3}{4}$ which is of the order n^4.

The following table summarizes the numerical results of Example 6.4 with dimension from 500 to 2000, where we compute the minimum H-eigenvalue by first converting the corresponding sums-of-squares problem discussed in Remark 6.2 to an SDP problem using YALMIP and solving this SDP problem using SDPNAL. We observe that, for all the instances, the minimum H-eigenvalues can be found successfully. The meaning of the data are the same as in Table 1.

Table 6.2: Test results for large size tensors

Problem	m	n	NV	NC	Computed eigenvalue	True eigenvalue	Time(YAL.)	Time(SDP.)
Ex. 6.4	4	500	1250	1375	499.0000	499	4.6299	6.8295
Ex. 6.4	4	1000	2500	2750	999.0000	999	8.8298	66.5566
Ex. 6.4	4	2000	5000	5500	1999.0000	1999	20.9729	563.6903

6.4.2 Testing positive definiteness of a multivariate form

For a multivariate form $\mathcal{A}\mathbf{x}^m$, we say it is positive definite if $\mathcal{A}\mathbf{x}^m > 0$ for all $\mathbf{x} \neq \mathbf{0}$. Testing positive definiteness of a multivariate form $\mathcal{A}\mathbf{x}^m$ is an important problem in the stability study of nonlinear autonomous systems via Lyapunov's direct method in automatic control [78]. Researchers in automatic control have studied the conditions of such positive definiteness intensively. However, for $n \geqslant 3$ and $m \geqslant 4$, this is, in general, a hard problem in mathematics. Recently, some efficient methods based on eigenvalues of tensors were proposed to solve the problem in the case where $m = 4$ [72].

In this part, we show that testing positive definiteness of a multivariate form $\mathcal{A}\mathbf{x}^m$ where $\mathcal{A}$ is an extended Z-tensor can be computed by sums-of-squares problem via Theorem 6.8. Indeed, a direct consequence of Theorem 6.8 and Lemma 6.6 give us the following useful test:

Corollary 6.2. *Let $\mathcal{A}$ be an extended Z-tensor. Then, the associated multivariate* *form $\mathcal{A}\mathbf{x}^m$ is positive definite if and only if*

$$\max_{\mu,r\in\mathbb{R}}\{\mu : f_{\mathcal{A}}(\mathbf{x}) - r(\|\mathbf{x}\|_m^m - 1) - \mu \in \Sigma_m^2[\mathbf{x}]\} > 0,$$

where $f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m$ and $\Sigma_m^2[\mathbf{x}]$ is the set of all SOS polynomials with degree at most m.

We now use the above corollary to test the positive definiteness of extended Z-tensors. To do this, we first generate 100 extended Z-tensors as numerical examples. These extended Z-tensors are randomly generated by the following procedure.

Procedure 1

(i) Given (m, n, s, k, M) with m is an even number and $n = sk$, where n and m are the dimension and the order of the randomly generated tensor, respectively, and M is a large positive constant.

(ii) Randomly generate a random positive integer L and a partition of the index set $\{1, \cdots, n\}$, $\{\Gamma_1, \cdots, \Gamma_s\}$, such that $|\Gamma_i| = k$, $i = 1, \cdots, s$ and $\Gamma_i \cap \Gamma_{i'} = \emptyset$ for all $i \neq i'$. For each $i = 1, \cdots, s-1$, generate a random multi-index $(l_1^i, \cdots, l_m^i)$ with $l_j^i \in \Gamma_i$, $j = 1, \cdots, m$ and a random number $\bar{a}_{l_1^i \cdots l_m^i} \in [0, 1]$. Generate one randomly mth-order k-dimensional symmetric tensor $\mathcal{B}$, such that all elements of $\mathcal{B}$ are in the interval $[0, 1]$.

(iii) We define extended Z-tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ such that

$$a_{i_1\cdots i_m} = \begin{cases} (-1)^L M & \text{if } \ i_1 = \cdots = i_m = i \text{ for all } i = 1, \cdots, n, \\ \bar{a}_{l_1^i \cdots l_m^i} & \text{if } \ (i_1, \cdots, i_m) = \sigma(l_1^i, \cdots, l_m^i), \ l_1^i, \cdots, l_m^i \in \Gamma_i, i \in [s-1], \\ -\mathcal{B}_{i_1\cdots i_m} & \text{if } \ i_1, \cdots, i_m \in \Gamma_s, \\ 0 & \text{othewise.} \end{cases}$$

Here $\sigma(i_1, \cdots, i_m)$ denotes all the possible permutation of $(i_1, \cdots, i_m)$.

From the construction of $\mathcal{A}$, it can be verified that $\mathcal{A}$ is an extended Z-tensor. Let $f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m$. We then solve the sums-of-squares problem

$$\max_{\mu,r\in\mathbb{R}}\{\mu : f_{\mathcal{A}}(\mathbf{x}) - r(\|\mathbf{x}\|_m^m - 1) - \mu \in \Sigma_m^2[\mathbf{x}]\}$$

and use the preceding corollary to determine whether $\mathcal{A}\mathbf{x}^m$ is a positive definite multivariate form or not. Here, to speed up the algorithm, as we did for the large size tensors, we first convert the sums-of-squares problem into an SDP by using Remark 6.2 and YALMIP. Then, we solve the equivalent SDP by using the software SDPNAL. The correctness can be verified by looking at the randomly generated positive number L. Indeed, from the construction, if L is an even number and M is a large positive number, the diagonal elements will strictly dominate the sum of the off-diagonal elements, and so, $\mathcal{A}\mathbf{x}^m$ is a positive definite multivariate form. On the other hand, if L is an odd number, then the diagonal elements will be negative, and so, $\mathcal{A}\mathbf{x}^m$ is not a positive definite multivariate form in this case.

The following table summarize the results for the correctness of testing the positive definiteness of a multivariate form generated by an extended Z-tensor. As we can see the results, in our numerical experiment, all the 100 randomly generated instance has been correctly identified.

m	n	s	k	M	PD	NPD	Correctness
4	20	4	5	100	48	52	100%
4	25	5	5	100	46	54	100%
4	40	4	10	100	52	48	100%
4	60	4	15	100	45	55	100%
4	100	4	25	100	44	56	100%

6.5 Final remarks

In this chapter, we establish SOS tensor decomposition of various even order symmetric structured tensors available in the current literature. These include weakly diagonally dominated tensors, B_0-tensors, double B-tensors, quasi-double B_0-tensors,

MB_0-tensors, H-tensors, absolute tensors of positive semi-definite Z-tensors and extended Z-tensors. We also examine the SOS-rank of SOS tensor decomposition and the SOS-width for SOS tensor cones. In particular, we provide an explicit sharp estimate for SOS-rank of tensors with bounded exponent and SOS-width for the tensor cone consisting of all such tensors with bounded exponent that have SOS decomposition. Finally, applications for the SOS decomposition of extended Z-tensors are provided and several numerical experiments illustrate the significance.

Below, we raise some open questions which might be interesting for future work:

Question 1: Can we evaluate the SOS-rank of symmetric B_0-tensors?

Question 2: Can we evaluate the SOS-rank of symmetric Z-tensors?

Question 3: Can we evaluate the SOS-rank of symmetric diagonally dominated tensors?

Question 4: Can we use the techniques in Section 5 to find the minimum H-eigenvalue of an even order symmetric structured tensors other than the extended Z-tensors?

Chapter 7

Positive semi-definiteness and extremal H-eigenvalues of extended essentially non-negative tensors

In this chapter, we study a new class of structured tensors named extended essentially non-negative tensors, which are extensions of the class of essentially non-negative tensors [38, 114]. The extended essentially non-negative tensors allow the off-diagonal elements can have negative values. We then show that its largest and smallest H-eigenvalues can be found by using polynomial optimization techniques under suitable conditions.

One of the important structured tensor classes is the class of non-negative tensors, that is, tensors with non-negative entries. The non-negative tensors arise naturally in spectral hypergraph theory and high-order Markov chain theory. Recently, efficient numerical schemes have been proposed to calculate the maximum eigenvalue based on a Perron-Frobenius type theorem for non-negative tensors [70]. Recently, Hu et al. [38] and Zhang et al. [114], studied a more general class called essentially non-negative tensors which means all the off-diagonal elements of the underlying tensor are non-negative. Hu et al. showed that the largest H-eigenvalue of an even order essentially non-negative tensor can be found by solving a sum of squares (SOS) polynomial optimization problem, which can be equivalently reformulated as a semi-definite linear programming problem [76]. Then, using this technique, two different upper bounds for the maximum H-eigenvalue of general even order symmetric tensors are provided [38]. Now, one question can be raised naturally: whether similar results still hold for some classes of tensors which are not essentially non-negative tensors, that is, for tensors with possible negative entries on the off-diagonal elements. This is the main motivation of this part.

7.1 Positive semi-definiteness of symmetric extended essentially non-negative tensors

In this section, we formally define extended essentially non-negative tensors and examine their positive semi-definiteness. Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be a symmetric even order tensor. We say $\mathcal{A}$ is an **extended essentially non-negative tensor** if there exist $s \in \mathbb{N}$ with $s \leqslant n$ and index sets $\Gamma_l \subseteq [n]$, $l \in [s]$ with $\bigcup_{l=1}^{s} \Gamma_l = [n]$ such that

(i) $\Gamma_{l_1} \cap \Gamma_{l_2} = \varnothing$, $l_1 \neq l_2$;

(ii) for any $l_1 \neq l_2$, $a_{i_1 i_2 \cdots i_m} = 0$ if $\{i_1, i_2, \cdots, i_m\} \cap \Gamma_{l_1} \neq \varnothing$ and $\{i_1, i_2, \cdots, i_m\} \cap \Gamma_{l_2} \neq \varnothing$;

(iii) for each $l \in [s]$, either one of the following two conditions holds:

(1) the off-diagonal entries $a_{i_1 i_2 \cdots i_m} = 0$ for all but possible permutations of one fixed $(i_1, i_2, \cdots, i_m) \in \Gamma_l^m$;

(2) the off-diagonal entries $a_{i_1 i_2 \cdots i_m} \geqslant 0$ for all $(i_1, i_2, \cdots, i_m) \in \Gamma_l^m$.

In the following analysis, we always denote $\mathcal{A}(\Gamma_l) = (a^l_{i_1 i_2 \cdots i_m})$, $l \in [s]$ to be the condensed subtensors of $\mathcal{A}$ where the corresponding entries are given by

$$a^l_{i_1 i_2 \cdots i_m} = a_{i_1 i_2 \cdots i_m}, \ i_1, i_2, \cdots, i_m \in \Gamma_l,$$

and $a^l_{i_1 i_2 \cdots i_m} = 0$ otherwise. So it holds that $\mathcal{A} = \sum_{l=1}^{s} \mathcal{A}(\Gamma_l)$.

Theorem 7.1. *Let m be even. Assume $\mathcal{A}$ is a symmetric even order extended essentially non-negative tensor with order m and dimension n. Then, $\mathcal{A}$ is positive semi-definite if and only if its condensed subtensors $\mathcal{A}(\Gamma_1), \mathcal{A}(\Gamma_2), \cdots, \mathcal{A}(\Gamma_s)$ are all positive semi-definite.*

Proof. Suppose that all the condensed subtensors are positive semi-definite. Then, the sufficiency part follows easily as

$$\mathcal{A}\mathbf{x}^m = \sum_{l=1}^{s} \mathcal{A}(\Gamma_l)\mathbf{x}^m \geqslant 0, \ \forall \ \mathbf{x} \in \mathbb{R}^n.$$

To prove the necessary conditions, without loss of generality, we assume that $\mathcal{A}(\Gamma_1)$ is not positive semi-definite. Then, there is $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq \mathbf{0}$ satisfying

$$\mathcal{A}(\Gamma_1)\mathbf{x}^m = \sum_{i_j \in [n], j \in [m]} a^1_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m} = \sum_{i_1, \cdots, i_m \in \Gamma_1} a_{i_1 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m} < 0.$$

Let $\mathbf{y} = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$ be defined in a way such that $y_i = x_i$ when $i \in \Gamma_1$, and $y_i = 0$ otherwise. It is easy to check that $\mathbf{y} \neq \mathbf{0}$ and

$$\mathcal{A}(\Gamma_1)\mathbf{y}^m = \mathcal{A}(\Gamma_1)\mathbf{x}^m < 0. \tag{7.1}$$

By the definition of condensed subtensor, we know that

$$\mathcal{A}(\Gamma_l)\mathbf{y}^m = \sum_{i_1, \cdots, i_m \in \Gamma_l} a_{i_1 \cdots i_m} y_{i_1} y_{i_2} \cdots y_{i_m} = 0, \ l = 2, 3, \cdots, s. \tag{7.2}$$

Thus, by (7.1) and (7.2), we obtain

$$\mathcal{A}\mathbf{y}^m = \sum_{l=1}^{s} \mathcal{A}(\Gamma_l)\mathbf{y}^m = \mathcal{A}(\Gamma_1)\mathbf{y}^m < 0,$$

which contradicts the fact that $\mathcal{A}$ is positive semi-definite. So, all condensed subtensors of $\mathcal{A}$ are positive semi-definite and the desired results hold. $\square$

Remark 7.1. *From Theorem 7.1, checking the positive semi-definiteness of a given symmetric extended essentially non-negative tensor is equivalent to checking the positive semi-definiteness of all its condensed subtensors. Note that each subtensors* $\mathcal{A}(\Gamma_l)$ *only has nonzero values if the indices* $(i_1, \ldots, i_m) \in \Gamma_l$*. So, the positive semi-definiteness of the subtensor* $\mathcal{A}(\Gamma_l)$ *is equivalent to the positive semi-definiteness of a tensor with dimension* $|\Gamma_l|$ *which is smaller comparing with the original tensor. So, from the computation point of view, it is often much easier to check the positive semi-definiteness of the condensed subtensor than the original extended essentially non-negative tensor.*

Theorem 7.2. *Let* m *be even. Assume* $\mathcal{A}$ *is a symmetric even order extended essentially non-negative tensor with order* m *and dimension* n*. Then,* $\mathcal{A}$ *is positive definite if and only if its condensed subtensors* $\mathcal{A}(\Gamma_l), l \in [s]$ *satisfy that*

$$\mathcal{A}(\Gamma_l)\mathbf{x}^m > 0, \text{ for all } \mathbf{x} \in \mathbb{R}^n \text{ such that } \exists\ i \in \Gamma_l,\ x_i \neq 0. \tag{7.3}$$

Proof. For sufficient conditions, if $\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0}$, then there exist $l_1 \in [s]$ and at least one $i \in \Gamma_{l_1}$ such that $x_i \neq 0$. By (7.3) and Theorem 7.1, we have

$$\mathcal{A}\mathbf{x}^m = \sum_{l=1}^{s} \mathcal{A}(\Gamma_l)\mathbf{x}^m \geqslant \mathcal{A}(\Gamma_{l_1})\mathbf{x}^m > 0,$$

which implies that $\mathcal{A}$ is positive definite by the arbitrariness of $\mathbf{x} \in \mathbb{R}^n$.

To prove the necessary conditions, without loss of generality, suppose that there is $\mathbf{x} \in \mathbb{R}^n$ with at least one $i \in \Gamma_1$ such that $x_i \neq 0$ and $\mathcal{A}(\Gamma_1)\mathbf{x}^m \leqslant 0$. Let $\mathbf{y} \in \mathbb{R}^n$ be defined by $y_i = x_i$ when $i \in \Gamma_1$ and $y_i = 0$ for the others. Then $\mathbf{y} \neq \mathbf{0}$ and it follows that

$$\mathcal{A}(\Gamma_1)\mathbf{y}^m = \mathcal{A}(\Gamma_1)\mathbf{x}^m \leqslant 0$$

and $\mathcal{A}(\Gamma_l)\mathbf{y}^m = 0$ for $l = 2, 3, \cdots, s$. Thus, we know that

$$\mathcal{A}\mathbf{y}^m = \sum_{l=1}^{s} \mathcal{A}(\Gamma_l)\mathbf{y}^m = \mathcal{A}(\Gamma_1)\mathbf{y}^m \leqslant 0,$$

which is contradict with the fact that $\mathcal{A}$ is positive definite and the desired results hold. $\square$

To present the next conclusion, we first recall the notion of Z-tensors, which have been studied in [19, 116]. Z-tensor is a tensor with non-positive off-diagonal entries and positive semi-definite Z-tensors are called M-tensors.

Theorem 7.3. *Let m be even. Assume $\mathcal{A}$ is a symmetric even order extended essentially non-negative tensor with order m and dimension n. If $\mathcal{A}$ is positive semi-definite, then $\mathcal{A}$ is a non-negative tensor or M-tensor, or the sum of a non-negative tensor and an M-tensor.*

Proof. From the definition of an extended essentially non-negative tensor, if all off-diagonal entries of $\mathcal{A}$ are non-negative, and by the positive semi-definiteness of $\mathcal{A}$, we have

$$a_{ii\cdots i} = \mathcal{A}\mathbf{e}_i^m \geqslant 0, \ \forall \ i \in [n],$$

which implies that $\mathcal{A}$ is non-negative. If all off-diagonal entries of $\mathcal{A}$ is non-positive, then $\mathcal{A}$ is a Z-tensor and then $\mathcal{A}$ is an M-tensor since $\mathcal{A}$ is positive semi-definite.

Now, suppose that $\mathcal{A}$ has non-negative off-diagonal entries and non-positive off-diagonal entries at the same time. Denote the condensed subtensors of $\mathcal{A}$ to be $\mathcal{A}(\Gamma_l)$, $l \in [s]$. Then $\mathcal{A} = \sum_{l=1}^{s} \mathcal{A}(\Gamma_l)$. From the structure of the extended essentially non-negative tensor, without loss of generality, assume $\mathcal{A}(\Gamma_1), \mathcal{A}(\Gamma_2), \cdots, \mathcal{A}(\Gamma_t)$ have non-negative off-diagonal entries, and $\mathcal{A}(\Gamma_{t+1}), \mathcal{A}(\Gamma_{t+2}), \cdots, \mathcal{A}(\Gamma_s)$ have nonpositive off-diagonal entries. Since $\mathcal{A}$ is positive semi-definite, by Theorem 7.1 and the definition of M-tensor it follow that

$$\mathcal{A}(\Gamma_1) + \mathcal{A}(\Gamma_2) + \cdots + \mathcal{A}(\Gamma_t)$$

is a non-negative tensor and

$$\mathcal{A}(\Gamma_{t+1}) + \mathcal{A}(\Gamma_{t+2}) + \cdots + \mathcal{A}(\Gamma_s)$$

is an M-tensor. Thus, the desired results hold. $\square$

An eigenvalue method was provided in Theorem 5 of [78] to check the positive semi-definiteness of a symmetric even order tensor. So, a question raises naturally: can we compute the H-eigenvalues of a symmetric extended essentially non-negative tensor through its condensed subtensors? The following theorem shows the relationship between H-eigenvalues of a symmetric extended essentially non-negative tensor and its condensed subtensors.

Theorem 7.4. *Let $\mathcal{A}$ be a symmetric extended essentially non-negative tensor with order m and dimension n. Then, we have the following result:*

(i) *all nonzero H-eigenvalues of the condensed subtensors of $\mathcal{A}$ are H-eigenvalues of $\mathcal{A}$;*

(ii) *each H-eigenvalue of $\mathcal{A}$ is the H-eigenvalue of some condensed subtensors of $\mathcal{A}$.*

Proof. Let $\mathcal{A}(\Gamma_1), \mathcal{A}(\Gamma_2), \cdots, \mathcal{A}(\Gamma_s)$ be the condensed subtensors of $\mathcal{A}$.

(i) For any $t \in [s]$, let $\lambda \neq 0$ be an H-eigenvalue of $\mathcal{A}(\Gamma_t)$. Then, by Definition 2.1, there is $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq \mathbf{0}$ such that

$$\mathcal{A}(\Gamma_t)\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]}.$$

Since $\lambda \neq 0$, from the notion of condensed subtensor, we obtain that $x_i = 0$ for all $i \notin \Gamma_t$. So, it follows that

$$\mathcal{A}\mathbf{x}^{m-1} = \sum_{l=1}^{s} \mathcal{A}(\Gamma_l)\mathbf{x}^{m-1} = \mathcal{A}(\Gamma_t)\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]},$$

which means that λ is an H-eigenvalue of $\mathcal{A}$.

(ii) Let λ be any H-eigenvalue of $\mathcal{A}$ with H-eigenvector $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq \mathbf{0}$, So, there is at least one $t \in [s]$ and one $i \in \Gamma_t$ such that $x_i \neq 0$. Let $\mathbf{y} \in \mathbb{R}^n$ be defined by $y_i = x_i$, $i \in \Gamma_t$ and $y_i = 0$ otherwise. Thus $\mathbf{y} \neq \mathbf{0}$. When $i \notin \Gamma_t$, it follows that

$$(\mathcal{A}(\Gamma_t)\mathbf{y}^{m-1})_i = 0 = \lambda y_i^{m-1}. \tag{7.4}$$

When $i \in \Gamma_t$, we have

$$\begin{aligned}
(\mathcal{A}(\Gamma_t)\mathbf{y}^{m-1})_i &= \sum_{i_2,\cdots,i_m \in \Gamma_t} a_{ii_2\cdots i_m} y_{i_2} y_{i_3} \cdots y_{i_m} \\
&= \sum_{i_2,\cdots,i_m \in \Gamma_t} a_{ii_2\cdots i_m} x_{i_2} x_{i_3} \cdots x_{i_m} \\
&= \sum_{i_2,\cdots,i_m \in [n]} a_{ii_2\cdots i_m} x_{i_2} x_{i_3} \cdots x_{i_m} \\
&= (\mathcal{A}\mathbf{x}^{m-1})_i = \lambda x_i^{m-1} \\
&= \lambda y_i^{m-1}.
\end{aligned} \tag{7.5}$$

From (7.4) and (7.5), we know that λ is an H-eigenvalue of $\mathcal{A}(\Gamma_t)$ with H-eigenvector $\mathbf{y}$ and the desired conclusions hold. □

Remark 7.2. *The conclusion of Theorem 7.4 means that the spectrum of a symmetric extended essentially non-negative tensor is a subset of the union of spectral sets of all its condensed subtensors. In the symmetric even order case, if the condensed subtensors of a extended essentially non-negative tensor do not have negative H-eigenvalues, then the extended essentially non-negative tensor is positive semi-definite.*

7.2 The SOS tensor decomposition of symmetric extended essentially non-negative tensors and its applications

In this section, we study the SOS tensor decomposition of symmetric even order extended essentially non-negative tensors. Sufficient conditions are given to guarantee the SOS tensor decomposition of a given symmetric extended essential non-negative tensor and, as a application, we show that the derived SOS tensor decomposition can be used to compute the minimum H-eigenvalue of a given symmetric even order extended essentially non-negative tensor.

Theorem 7.5. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be a symmetric extended essentially non-negative tensor with even order m and dimension n. Suppose $\mathcal{A}$ is positive semi-definite. For $\mathbf{x} \in \mathbb{R}^n$, if each positive off-diagonal element corresponds to an SOS term i.e.*

$$a_{i_1 i_2 \cdots i_m} > 0 \ \Rightarrow \ x_{i_1} x_{i_2} \cdots x_{i_m} \in \Sigma_m^2[\mathbf{x}], \tag{7.6}$$

then $\mathcal{A}$ has an SOS tensor decomposition.

Proof. For any $\mathbf{x} \in \mathbb{R}^n$, to prove the result, we only need to prove that

$$f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m \in \Sigma_m^2[\mathbf{x}]. \tag{7.7}$$

By Theorem 7.3, we prove the conclusion from the three cases below.

(1) If $\mathcal{A}$ is a non-negative tensor, by (7.6), we know that (7.7) holds since m is even.

(2) If $\mathcal{A}$ is an M-tensor, note that all even order symmetric positive semi-definite Z-tensors or M-tensors have SOS decompositions [37, 38], so equation (7.7) holds.

(3) If $\mathcal{A} = \mathcal{B} + \mathcal{C}$, where $\mathcal{B}$ is non-negative and $\mathcal{C}$ is an M-tensor, from Theorem 7.1, if follows that $\mathcal{B}$ and $\mathcal{C}$ are all positive semi-definite tensors. By **(1)** and **(2)**, we have (7.7) holds since

$$f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m = \mathcal{B}\mathbf{x}^m + \mathcal{C}\mathbf{x}^m \in \Sigma_m^2[\mathbf{x}].$$

Thus, the desired conclusions follows. □

Now, we compute the minimum H-eigenvalue of an extended essentially non-negative tensor defined as in Theorem 7.1 via sum-of-squares polynomial technique, which has been much applied in optimization theory [38, 42, 48, 50].

Theorem 7.6. *Suppose $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ is a symmetric extended essentially non-negative tensor with even order m and dimension n. Let $f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m$. For $\mathbf{x} \in \mathbb{R}^n$, if each positive off-diagonal element corresponds to an SOS term i.e.*

$$a_{i_1 i_2 \cdots i_m} > 0 \;\Rightarrow\; x_{i_1} x_{i_2} \cdots x_{i_m} \in \Sigma_m^2[\mathbf{x}].$$

Then, it holds that

$$\lambda_{\min}(\mathcal{A}) \;=\; \max_{\mu, r \in \mathbb{R}} \left\{ \mu : f_{\mathcal{A}}(\mathbf{x}) - r(\|\mathbf{x}\|_m^m - 1) - \mu \in \Sigma_m^2[\mathbf{x}] \right\}.$$

Proof. By Lemma 2.2, consider the following optimization problem

$$\min\{\mathcal{A}\mathbf{x}^m : \|\mathbf{x}\|_m^m = 1\}$$

and denote its global minimizer by $\mathbf{x}^* = (x_1^*, \cdots, x_n^*)^T \in \mathbb{R}^n$. Then, $\lambda_{\min}(\mathcal{A}) = f_{\mathcal{A}}(\mathbf{x}^*) = \mathcal{A}\mathbf{x}^{*m}$. For all $\mathbf{x} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$, it follows that

$$f_{\mathcal{A}}(\mathbf{x}) - \lambda_{\min}(\mathcal{A})\|\mathbf{x}\|_m^m = f_{\mathcal{A}}(\mathbf{x}) - f_{\mathcal{A}}(\mathbf{x}^*)\|\mathbf{x}\|_m^m = \|\mathbf{x}\|_m^m \left(f_{\mathcal{A}}\left(\frac{\mathbf{x}}{\|\mathbf{x}\|_m}\right) - f_{\mathcal{A}}(\mathbf{x}^*) \right) \geqslant 0. \tag{7.8}$$

Since $\mathcal{A}$ is an extended essentially non-negative tensor such that (7.6) holds, the tensor corresponding to $f_{\mathcal{A}}(\mathbf{x}) - \lambda_{\min}(\mathcal{A})\|\mathbf{x}\|_m^m$ is also an extended essentially non-negative tensor satisfying (7.6). Then, from (7.8) and Theorem 7.5, we know that

$$f_{\mathcal{A}}(\mathbf{x}) - \lambda_{\min}(\mathcal{A})(\|\mathbf{x}\|_m^m - 1) - \lambda_{\min}(\mathcal{A}) \in \Sigma_m^2[\mathbf{x}],$$

which implies that

$$\lambda_{\min}(\mathcal{A}) \leqslant \max_{\mu, r \in \mathbb{R}} \left\{\mu : f_{\mathcal{A}}(\mathbf{x}) - r(\|\mathbf{x}\|_m^m - 1) - \mu \in \Sigma_m^2[\mathbf{x}]\right\}.$$

To see the reverse inequality, take any (μ, r) with $f_{\mathcal{A}}(\mathbf{x}) - r(\|\mathbf{x}\|_m^m - 1) - \mu \in \Sigma_m^2[\mathbf{x}]$. Then, for all $\mathbf{x} \in \mathbb{R}^n$,

$$f_{\mathcal{A}}(\mathbf{x}) - r(\|\mathbf{x}\|_m^m - 1) - \mu \geqslant 0.$$

This shows that $r \geqslant \mu$ and $f_{\mathcal{A}}(\mathbf{x}) \geqslant r\|\mathbf{x}\|_m^m$ for all $\mathbf{x} \in \mathbb{R}^n$. This shows that $\lambda_{\min}(\mathcal{A}) \geqslant r \geqslant \mu$, and so, the conclusion follows. $\square$

We now obtain the following result as a direct corollary of Theorem 7.6 and Theorem 5 of [78].

Corollary 7.1. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be a symmetric extended essentially non-negative tensor with even order m and dimension n. Let $f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m$. For $\mathbf{x} \in \mathbb{R}^n$, if each positive off-diagonal element corresponds to an SOS term i.e.*

$$a_{i_1 i_2 \cdots i_m} > 0 \;\Rightarrow\; x_{i_1} x_{i_2} \cdots x_{i_m} \in \Sigma_m^2[\mathbf{x}].$$

Then, $\mathcal{A}$ is positive semi-definite if and only if

$$\max_{\mu, r \in \mathbb{R}} \left\{\mu : f_{\mathcal{A}}(\mathbf{x}) - r(\|\mathbf{x}\|_m^m - 1) - \mu \in \Sigma_m^2[\mathbf{x}]\right\} \geqslant 0.$$

Next, we present several numerical examples to illustrate how to compute the minimum H-eigenvalue of a symmetric extended essentially non-negative tensor $\mathcal{A}$ using the above sum-of-squares problem via Matlab Toolbox YALMIP [64, 65].

Example 7.1. *Let $\mathcal{A}$ be a symmetric tensor with order 6 dimension 4 such that*

$$a_{111111} = a_{222222} = \frac{11}{4}, \quad a_{333333} = a_{444444} = 1,$$

and

$$a_{\pi(1,1,1,2,2,2)} = -\frac{3}{4}, \quad a_{\pi(3,3,4,4,4,4)} = \frac{2}{5},$$

and $a_{i_1 \cdots i_6} = 0$ otherwise, where $\pi(i_1, \cdots, i_6)$ denotes all permutations of $i_1, \cdots, i_6$. Then, $\mathcal{A}$ is a symmetric extended essentially non-negative tensor and the associated polynomial

$$f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^6 = \frac{11}{4}x_1^6 + \frac{11}{4}x_2^6 + x_3^6 + x_4^6 - \frac{15}{2}x_1^3x_2^3 + 6x_3^2x_4^4.$$

It can be easily verified that $\mathcal{A}$ is not an essentially non-negative tensor.

To compute its minimum H-eigenvalue, we note that the corresponding sums-of-squares optimization problem reads

$$\max_{\mu, r \in \mathbb{R}} \{\mu : f_{\mathcal{A}}(\mathbf{x}) - r(\|\mathbf{x}\|_6^6 - 1) - \mu \in \Sigma_6^2[\mathbf{x}]\}.$$

Convert this sums-of-squares optimization problem into a semi-definite programming problem using the Matlab Toolbox YALMIP *[64, 65], and solve it by using the SDP software* SDPNAL *we obtain that $\lambda_{\min}(\mathcal{A}) = -1$. The simple code using* YALMIP *is appended as follows:*

```
sdpsettings('solver','sdpnal')
sdpvar x1 x2 x3 x4 r mu
f = 11/4*x1^6+11/4*x2^6+x3^6+x4^6-15/2*x1^3*x2^3+6*x3^2*x4^4;
g = [(x1^6+x2^6+x3^6+x4^6)-1];
F = [sos(f-mu-r*g)];
solvesos(F,-mu,[],[r;mu])
```

Moreover, note from the geometric mean inequality that $|x_1^3x_2^3| = (x_1^6)^{\frac{1}{2}}(x_2^6)^{\frac{1}{2}} \leqslant \frac{1}{2}x_1^6 + \frac{1}{2}x_2^6$. *It follows that*

$$f_{\mathcal{A}}(\mathbf{x}) + \|\mathbf{x}\|_6^6 = \mathcal{A}\mathbf{x}^m = \frac{15}{4}x_1^6 + \frac{15}{4}x_2^6 + x_3^6 + x_4^6 - \frac{15}{2}x_1^3x_2^3 + 6x_3^2x_4^4 \geqslant 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n.$$

On the other hand, consider $\bar{\mathbf{x}} = (\sqrt[6]{\frac{1}{2}}, -\sqrt[6]{\frac{1}{2}}, 0, 0)$. *We see that* $f_{\mathcal{A}}(\bar{\mathbf{x}}) + \|\bar{\mathbf{x}}\|_6^6 = 0$. *This shows that* $\lambda_{\min}(\mathcal{A}) = \min\{f_{\mathcal{A}}(\mathbf{x}) : \|\mathbf{x}\|_6 = 1\} = -1$. *This verifies the correctness of our computed minimum H-eigenvalue.*

Example 7.2. *Let* $m = 20$. *Consider the symmetric tensor* $\mathcal{A}$ *with order* m *and dimension* 4 *such that*

$$a_{1\cdots 1} = a_{2\cdots 2} = a_{3\cdots 3} = a_{4\cdots 4} = 1,$$

$$a_{\pi(\underbrace{1,\cdots,1}_{m/2},\underbrace{2,\cdots,2}_{m/2})} = -\frac{2}{\binom{m}{m/2}},$$

$$a_{\pi(\underbrace{3,\cdots,3}_{m/5},\underbrace{4,\cdots,4}_{4m/5})} = \frac{1}{\binom{m}{m/5}},$$

$$a_{\pi(\underbrace{3,\cdots,3}_{4m/5},\underbrace{4,\cdots,4}_{m/5})} = \frac{1}{\binom{m}{m/5}},$$

and $a_{i_1\cdots i_m} = 0$ *otherwise, where* $\pi(i_1, \cdots, i_m)$ *denotes all permutations of* $i_1, \cdots, i_m$. *Then* $\mathcal{A}$ *is a symmetric extended essentially non-negative tensor and the associated polynomial*

$$f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m = x_1^{20} + x_2^{20} + x_3^{20} + x_4^{20} - 2x_1^{10}x_2^{10} + x_3^4x_4^{16} + x_3^{16}x_4^4.$$

It can be easily verified that $\mathcal{A}$ *is not a essentially non-negative tensor. Moreover, using geometric mean inequality, we can directly verify that the true minimum H-eigenvalue of* $\mathcal{A}$ *is* 0.

We compute the minimum H-eigenvalue by solving the corresponding sums-of-squares problem and we get the computed H-eigenvalue is 7.4108e-09.

7.3 The largest H-eigenvalue of symmetric even order extended essentially non-negative tensors

In this section, we study the largest H-eigenvalue for general even order symmetric extended essentially non-negative tensors. To proceed, we recall an useful result about symmetric extended Z-tensors, which were studied in last chapter. In fact, if tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ is an extended essentially non-negative tensor, then $-\mathcal{A} = (-a_{i_1 i_2 \cdots i_m})$ is an extended Z-tensor.

Theorem 7.7. *Let $\mathcal{A}$ be a symmetric extended essentially non-negative tensor with even order m and dimension n. For $\mathbf{x} \in \mathbb{R}^n$, suppose $f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m$. Then, it holds that*

$$\lambda_{max}(\mathcal{A}) = \min_{t \in \mathbb{R}, \mu \in \mathbb{R}} \{t \mid t - f_{\mathcal{A}}(\mathbf{x}) + \mu(\|\mathbf{x}\|_m^m - 1) \in \Sigma_m^2[\mathbf{x}]\}. \tag{7.9}$$

Proof. By Lemma 2.2, there is $\mathbf{x}_0 \in \mathbb{R}^n$, $\|\mathbf{x}_0\|_m = 1$ satisfying

$$\lambda_{max}(\mathcal{A}) = f_{\mathcal{A}}(\mathbf{x}_0) = \max_{\|\mathbf{x}\|_m = 1} \mathcal{A}\mathbf{x}^m.$$

Assume $t = \mu = \lambda_{max}(\mathcal{A})$, then we have that

$$t - f_{\mathcal{A}}(\mathbf{x}) + \mu(\|\mathbf{x}\|_m^m - 1) = \|\mathbf{x}\|_m^m \left(-f(\frac{\mathbf{x}}{\|\mathbf{x}\|_m}) + \lambda_{max}(\mathcal{A}) \right) \geqslant 0. \tag{7.10}$$

Since $\mathcal{A}$ is an extended essentially non-negative tensor, $\mathcal{A} - \lambda_{max}(\mathcal{A})\mathcal{I}$ is an extended essentially non-negative tensor. So, $-\mathcal{A} + \lambda_{max}(\mathcal{A})\mathcal{I}$ is an extended Z-tensor. From (5.2) and Theorem 6.3, we know that

$$-f_{\mathcal{A}}(\mathbf{x}) + \mu\|\mathbf{x}\|_m^m \in \Sigma_m^2[\mathbf{x}]. \tag{7.11}$$

Let t^* be the optimal value of the optimization problem (7.9). Then, by (7.10) and (7.11), we obtain that

$$t^* \leqslant \lambda_{max}(\mathcal{A}).$$

On the other hand, for all $\mathbf{x} \in \mathbb{R}^n$, take any (t, μ) with $t - f(\mathbf{x}) + \mu(\|\mathbf{x}\|_m^m - 1) \in \Sigma_m^2[\mathbf{x}]$. Then, it holds that

$$t - f_{\mathcal{A}}(\mathbf{x}) + \mu(\|\mathbf{x}\|_m^m - 1) \geqslant 0, \quad \forall\ \mathbf{x} \in \mathbb{R}^n,$$

which implies that $t \geqslant \mu$ and $\mu\|\mathbf{x}\|_m^m \geqslant f_{\mathcal{A}}(\mathbf{x})$. Thus, we know that $t^* \geqslant f_{\mathcal{A}}(\mathbf{x})$, for all $\mathbf{x} \in \mathbb{R}^n$, $\|\mathbf{x}\|_m^m = 1$ and the desired result follows. $\square$

Next, we present an example to verify the preceding result.

Example 7.3. *Suppose $\mathcal{A}$ is a symmetric tensor with even order m and dimension 4 such that*

$$a_{11\cdots 1} = a_{22\cdots 2} = a_{33\cdots 3} = a_{44\cdots 4} = -1,$$

$$a_{\pi(\underbrace{1,\cdots,1}_{\frac{m}{2}},\underbrace{2,\cdots,2}_{\frac{m}{2}})} = \alpha,\ a_{\pi(\underbrace{3,\cdots,3}_{\frac{m}{2}},\underbrace{4,\cdots,4}_{\frac{m}{2}})} = \beta, \quad \alpha, \beta \in [-5, 5],\ \alpha\beta \neq 0,$$

and $a_{i_1\cdots i_m} = 0$ otherwise, where $\pi(i_1, \cdots, i_m)$ denotes any permutation of $i_1, \cdots, i_m$. So, $\mathcal{A}$ is an extended essentially non-negative tensor and the associated polynomial is

$$f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m = -x_1^m - x_2^m - x_3^m - x_4^m + 20\alpha\, x_1^{\frac{m}{2}} x_2^{\frac{m}{2}} + 20\beta\, x_3^{\frac{m}{2}} x_4^{\frac{m}{2}}.$$

Since $|x_i^{\frac{m}{2}} x_j^{\frac{m}{2}}| \leqslant \frac{x_i^m + x_j^m}{2}$, for all $\mathbf{x} \in \mathbb{R}^n$ and $\|x\|_m^m = 1$, we know that

$$\begin{aligned} f_{\mathcal{A}}(\mathbf{x}) &= -x_1^m - x_2^m - x_3^m - x_4^m + 20\alpha\, x_1^{\frac{m}{2}} x_2^{\frac{m}{2}} + 20\beta\, x_3^{\frac{m}{2}} x_4^{\frac{m}{2}} \\ &\leqslant \begin{cases} -1 & \alpha < 0,\ \beta < 0,\ \frac{m}{2} \text{ is even} \\ \max\{10\alpha, 10\beta\} - 1 & \alpha\beta < 0,\ \frac{m}{2} \text{is even} \\ \max\{10|\alpha|, 10|\beta|\} - 1 & \text{otherwise.} \end{cases} \end{aligned}$$

From Lemma 2.2, by a direct computation, it is easy to check that the largest H-eigenvalue of $\mathcal{A}$ is

$$\lambda_{max}(\mathcal{A}) = \begin{cases} -1 & \alpha < 0,\ \beta < 0,\ \frac{m}{2} \text{ is even} \\ \max\{10\alpha, 10\beta, 0\} - 1 & \alpha\beta < 0,\ \frac{m}{2} \text{ is even} \\ \max\{10|\alpha|, 10|\beta|\} - 1 & \text{otherwise.} \end{cases}$$

According to Theorem 7.7, we compute the largest H-eigenvalue using the Matlab Toolbox YALMIP *[64, 65] and the computed largest H-eigenvalue and the true largest H-eigenvalue are listed in the following table.*

m	n	α	β	computed eigenvalue	true eigenvalue
6	4	4.7060	3.6693	46.0599	46.0600
6	4	-3.2314	4.5738	44.7384	44.7380
10	4	4.7977	-2.1518	46.9765	46.9770
10	4	4.0307	-2.8181	39.3069	39.3070
16	4	-3.6955	-0.9794	-1	-1
16	4	-3.9077	4.7379	46.3788	46.3790
20	4	3.1472	4.0579	39.5792	39.5790
20	4	-4.2465	-1.6186	-1	-1

7.4 Applications to testing the co-positivity of symmetric extended Z-tensors

The definition of co-positive tensors was introduced in [79]. It is a natural extension of the definition of co-positive matrices. Recently, co-positive tensors found important applications in the tensor complementarity problem [9, 102, 101]. Che, Qi and Wei [9] showed that the tensor complementarity problem defined by a strictly co-positive tensor has a nonempty and compact solution set. Song and Qi [102] proved that a real tensor is strictly semi-positive if and only if the corresponding tensor complementarity problem has a unique solution for any non-negative vector and a real tensor is semi-positive if and only if the corresponding tensor complementarity problem has a unique solution for any positive vector. It was shown there that a real symmetric tensor is a (strictly) semi-positive tensor if and only if it is (strictly) co-positive. Song and Qi [101] further presented global error bound analysis for the tensor complementarity problem defined by a strictly semi-positive tensor. Thus, co-positive and strictly co-positive tensors play an important role in the tensor complementarity problem.

A tensor $\mathcal{A}$ with order m and dimension n is **co-positive** if and only if $\mathcal{A}\mathbf{x}^m \geqslant 0, \forall\ \mathbf{x} \in \mathbb{R}^n_+$, and $\mathcal{A}$ is **strictly co-positive** if and only if $\mathcal{A}\mathbf{x}^m > 0, \forall\ \mathbf{x} \in \mathbb{R}^n_+\backslash\{\mathbf{0}\}$. Recall that positive semi-definite tensors are co-positive tensors and positive definite tensors are strictly co-positive tensors. Generally speaking, it is NP-hard to verify the co-positivity of a symmetric tensor.

In this section, as an application of Theorem 7.7, we will test the co-positvity of symmetric extended Z-tensors (odd or even order). Numerical experiments are also presented to verify the efficiency of our conclusion. By the way, in subsection 7.4.1, we first present an answer to the following question, which is left behind in [79]:

Question: When the order m is odd, does a co-positive tensor $\mathcal{A}$ always have an H-eigenvalue?

7.4.1 H-eigenvalues of odd order co-positive tensors

We first provide an example of a co-positive tensor with odd order where its H-eigenvalue does not exists

Example 7.4. (An odd order co-positive tensor without H-eigenvalues) *Let us consider the following symmetric tensor with order* 3 *and dimension* 2 *with*

$$\mathcal{A}_{111} = 10,\ \mathcal{A}_{222} = 4,$$

and

$$\mathcal{A}_{112} = \mathcal{A}_{121} = \mathcal{A}_{211} = -\sqrt{3},\ \textit{and}\ \mathcal{A}_{221} = \mathcal{A}_{212} = \mathcal{A}_{122} = \sqrt{3}.$$

Then, it can be verified that

$$\mathcal{A}\mathbf{x}^3 = 10x_1^3 + 4x_2^3 - 3\sqrt{3}x_1^2x_2 + 3\sqrt{3}x_1x_2^2.$$

From the geometric inequality, for any $x_1, x_2 \geqslant 0$,

$$10x_1^3 + 4x_2^3 = \frac{2}{3}(\sqrt[3]{15}x_1)^3 + \frac{1}{3}(\sqrt[3]{12}x_2)^3 \geqslant (\sqrt[3]{15}x_1)^2(\sqrt[3]{12}x_2) \geqslant 3\sqrt{3}x_1^2x_2.$$

This shows that

$$\mathcal{A}\mathbf{x}^3 \geqslant 0 \text{ for all } \mathbf{x} \in \mathbb{R}^2_+,$$

and so, $\mathcal{A}$ is co-positive.

We now see that $\mathcal{A}$ does not have any H-eigenvalue. To see this, we proceed by the method of contradiction and suppose that there exists an H-eigenpair $(\mathbf{x}, \lambda) \in (\mathbb{R}^2 \backslash \{\mathbf{0}\}) \times \mathbb{R}$ of $\mathcal{A}$. Then, by Definition 2.1, we have

$$\begin{cases} 10x_1^2 - 2\sqrt{3}x_1x_2 + \sqrt{3}x_2^2 = \lambda x_1^2 \\ -\sqrt{3}x_1^2 + 2\sqrt{3}x_1x_2 + 4x_2^2 = \lambda x_2^2 \end{cases}$$

Clearly, if $x_2 = 0$ then $x_1 = 0$, which is impossible. If $x_2 \neq 0$, by dividing x_2^2 on both sides, we see that the following equations have a real solution $(z, \lambda) \in \mathbb{R}^2$ with $z \neq 0$:

$$\begin{cases} (10 - \lambda)z^2 - 2\sqrt{3}z + \sqrt{3} = 0 \\ -\sqrt{3}z^2 + 2\sqrt{3}z + (4 - \lambda) = 0 \end{cases}$$

By looking at the determinant of these two quadratic equations, we have $12 - 4\sqrt{3}(10 - \lambda) \geqslant 0$ and $12 + 4\sqrt{3}(4 - \lambda) \geqslant 0$, and so,

$$\lambda \geqslant 10 - \sqrt{3} \text{ and } \lambda \leqslant 4 + \sqrt{3}$$

which is impossible.

$\square$

Next, in the non-degenerated case, we prove that any co-positive tensor always has a non-negative H-eigenvalue with non-negative H-eigenvector. To do this, we first define a polynomial with order $2m$ in the following way

$$h(\mathbf{x}) = \sum_{i_1,\ldots,i_m=1}^{n} a_{i_1\ldots i_m} x_{i_1}^2 \cdots x_{i_m}^2,$$

where $\mathcal{A} = (a_{i_1\ldots i_m})$ is a symmetric tensor with order m and dimension n. Denote its associated symmetric tensor by $\mathcal{A}_h$, that is,

$$h(\mathbf{x}) = \mathcal{A}_h \mathbf{x}^{2m} \text{ for all } \mathbf{x} \in \mathbb{R}^n \tag{7.12}$$

Proposition 7.1. *Consider a co-positive symmetric tensor $\mathcal{A}$ with order m (odd or even) and dimension n. Suppose that $\mathcal{A}$ is non-degenerated in the sense that there exists an H-eigenvector $\mathbf{x}$ of $\mathcal{A}_h$ such that the following implication holds:*

$$\mathbf{x} \in \mathbb{R}^n_+ \backslash \{\mathbf{0}\},\ x_i = 0 \ \Rightarrow\ (\mathcal{A}\mathbf{x}^{m-1})_i = 0.$$

Then, $\mathcal{A}$ has a non-negative H-eigenvalue with non-negative eigenvector.

Proof. Let $\mathcal{A}_h$ be defined as in (7.12). As $\mathcal{A}$ is co-positive, we know that

$$\mathcal{A}_h\mathbf{x}^{2m} = \sum_{i_1,\ldots,i_m=1}^{n} a_{i_1\ldots i_m} x_{i_1}^2 \cdots x_{i_m}^2 \geqslant 0,\ \forall\ \mathbf{x} \in \mathbb{R}^n,$$

which implies that $\mathcal{A}_h$ is positive semi-definite. Combining this with the fact that $\mathcal{A}_h$ is a tensor with even order, $\mathcal{A}_h$ has at least one H-eigenvalue and all the H-eigenvalues must be non-negative [78]. Let $(\bar{\mathbf{x}}, \bar{\lambda}) \in (\mathbb{R}^n \backslash \{\mathbf{0}\}) \times \mathbb{R}$ be an H-eigenpair of $\mathcal{A}_h$. Then, $\bar{\lambda} \geqslant 0$. Let $f(\mathbf{x}) = \mathcal{A}\mathbf{x}^m$ Note that

$$h(\mathbf{x}) = \mathcal{A}_h\mathbf{x}^{2m} = f(x_1^2, \ldots, x_n^2), \quad \forall\ \mathbf{x} \in \mathbb{R}^n.$$

This shows that for all $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\forall\ i \in [n]$, it holds that

$$2m\big(\mathcal{A}_h\mathbf{x}^{2m-1}\big)_i = \big(\nabla h(\mathbf{x})\big)_i = 2\big(\nabla f(x_1^2, \ldots, x_n^2)\big)_i x_i.$$

Now, as $(\bar{\mathbf{x}}, \bar{\lambda}) \in (\mathbb{R}^n \backslash \{\mathbf{0}\}) \times \mathbb{R}$ is an H-eigenpair of $\mathcal{A}_h$, we have

$$\frac{1}{m}\big(\nabla f(\bar{x}_1^2, \ldots, \bar{x}_n^2)\big)_i \bar{x}_i = \big(\mathcal{A}_h\bar{\mathbf{x}}^{2m-1}\big)_i = \bar{\lambda}\bar{x}_i^{2m-1},\ \forall\ i \in [n],$$

which means that

$$\big(\nabla f(\bar{x}_1^2, \ldots, \bar{x}_n^2)\big)_i = \ m\bar{\lambda}\bar{x}_i^{2m-2} \quad \text{if } x_i \neq 0,\ i \in [n].$$

Now, let $\mathbf{z} = (\bar{x}_1^2, \ldots, \bar{x}_n^2) \in \mathbb{R}^n_+ \backslash \{\mathbf{0}\}$, it follows from the non-degenerated condition that

$$\begin{aligned}\big(\mathcal{A}\mathbf{z}^{m-1}\big)_i = \big(\frac{1}{m}\nabla f(\mathbf{z})\big)_i \quad &= \quad \begin{cases} \bar{\lambda}\bar{z}_i^{m-1} & \text{if } \bar{z}_i = \bar{x}_i^2 \neq 0 \\ 0 & \text{if } \bar{z}_i = \bar{x}_i^2 = 0. \end{cases} \\ &= \quad \bar{\lambda}\bar{z}_i^{m-1}.\end{aligned}$$

From Definition 2.1, we obtain that $(\mathbf{z}, \bar{\lambda})$ is an H-eigenpair of $\mathcal{A}$. Thus, the conclusion follows. $\square$

7.4.2 Testing the co-positivity of symmetric extended Z-tensors

Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be a symmetric tensor with order m dimension n. Then $\mathcal{A}$ is co-positive if and only if

$$\mathcal{A}\mathbf{x}^m = \sum_{i_1,\ldots,i_m=1}^{n} a_{i_1\ldots i_m} x_{i_1} \cdots x_{i_m} \geqslant 0, \ \forall\, \mathbf{x} \in \mathbb{R}^n_+,$$

which is equivalent to

$$h(\mathbf{x}) = \mathcal{A}_h \mathbf{x}^{2m} = \sum_{i_1,\ldots,i_m=1}^{n} a_{i_1\ldots i_m} x_{i_1}^2 \cdots x_{i_m}^2 \geqslant 0, \ \forall\, \mathbf{x} \in \mathbb{R}^n, \tag{7.13}$$

where $\mathcal{A}_h$ is a symmetric tensor with order $2m$ and dimension n defined as in (7.12).

In particular, if $\mathcal{A}$ is a symmetric extended Z-tensor (odd or even order), then $\mathcal{A}_h$ is an even order extended Z-tensor. Thus, $-\mathcal{A}_h$ is an even order extended essentially non-negative tensor. Let $f(\mathbf{x}) = -\mathcal{A}_h \mathbf{x}^{2m}$. Then, by Theorem 7.7 and (7.13), we have the following corollary.

Corollary 7.2. *Let $\mathcal{A}$ be a symmetric extended Z-tensor with order m and dimension n. For $\mathbf{x} \in \mathbb{R}^n$, suppose $\mathcal{A}_h$ and $f(\mathbf{x})$ are defined as above. Then, $\mathcal{A}$ is co-positive if and only if*

$$\min_{t\in\mathbb{R},\mu\in\mathbb{R}} \{t \mid t - f(\mathbf{x}) + \mu(\|\mathbf{x}\|_{2m}^{2m} - 1) \in \Sigma^2_{2m}[\mathbf{x}]\} \leqslant 0. \tag{7.14}$$

We now use the above corollary to test the co-positivity of symmetric extended Z-tensors with order m and dimension n. The concrete process is listed below.

Procedure

(i) Given (m, n, s, k, M) with m is an even number and $n = sk$, where n and m are the dimension and the order of the randomly generated tensor, respectively, and M is a large positive constant.

(ii) Randomly generate a partition of the index set $\{1, \cdots, n\}$, $\{\Gamma_1, \cdots, \Gamma_s\}$, such that $|\Gamma_i| = k$, $i = 1, \cdots, s$ and $\Gamma_i \cap \Gamma_{i'} = \emptyset$ for all $i \neq i'$. For each $i = 1, \cdots, s-1$, generate a random multi-index $(l_1^i, \cdots, l_m^i)$ with $l_j^i \in \Gamma_i$, $j = 1, \cdots, m$ and a random number $\bar{a}_{l_1^i \cdots l_m^i} \in [0, 1]$. Generate one randomly mth-order k-dimensional symmetric tensor $\mathcal{B}$, such that all elements of $\mathcal{B}$ are in the interval $[0, 1]$.

(iii) We define extended Z-tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ such that

$$a_{i_1 \cdots i_m} = \begin{cases} M & \text{if} \quad i_1 = \cdots = i_m = i \text{ for all } i = 1, \cdots, n, \\ \bar{a}_{l_1^i \cdots l_m^i} & \text{if} \quad (i_1, \cdots, i_m) = \sigma(l_1^i, \cdots, l_m^i) \text{ with } l_1^i, \cdots, l_m^i \in \Gamma_i, i = 1, \cdots, s-1, \\ -\mathcal{B}_{i_1 \cdots i_m} & \text{if} \quad i_1, \cdots, i_m \in \Gamma_s, \\ 0 & \text{othewise.} \end{cases}$$

Here $\sigma(i_1, \cdots, i_m)$ denotes all the possible permutation of $(i_1, \cdots, i_m)$.

(iv) Let $\mathcal{A}_h = (a^h_{i_1 i_2 \cdots i_{2m}})$ be a extended Z-tensor with order $2m$ and dimension n such that

$$a^h_{\sigma(i_1 i_1 i_2 i_2 \cdots i_m i_m)} = a_{i_1 i_2 \cdots i_m}, \ \forall \ i_1, i_2, \cdots, i_m \in [n],$$

and $a^h_{i_1 i_2 \cdots i_{2m}} = 0$ otherwise.

(v) Suppose $f(\mathbf{x}) = -\mathcal{A}_h \mathbf{x}^{2m}$, $\mathbf{x} \in \mathbb{R}^n$. Then solve the SOS programming problem (7.14) by Matlab Toolbox YALMIP [64, 65] and SeDuMi [103].

Table 7.1 summarizes the results for the percent of co-positivity of symmetric extended Z-tensors which is generated by the above procedure. We perform 100 tests for fourth order and sixth order symmetric extended Z-tensors. Obviously, for fixed order m and dimension n, the percent of co-positive extended Z-tensors increase as the parameter M grow. If M is large enough, the extended Z-tensor is positive definite, so it must be co-positive.

$(m,n,s,k) = (3,12,4,3)$					
M	3	4	5	6	7
Copositivity	4%	31%	70%	95%	100%
$(m,n,s,k) = (4,9,3,3)$					
M	10	12	14	16	18
Copositivity	9%	28%	59%	82%	97%
$(m,n,s,k) = (5,6,2,3)$					
M	30	35	40	45	50
Copositivity	7%	18%	46%	78%	95%
$(m,n,s,k) = (6,6,2,3)$					
M	80	100	120	140	160
Copositivity	3%	16%	57%	86%	98%

Table 7.1: The percentage of copositive instances of randomly generated extended Z-tensors.

7.5 Final remarks

In this chapter, we extend the essentially non-negative tensor to a more general form. Positive semi-definiteness and SOS tensor decomposition of symmetric essentially non-negative tensors are studied. Then, by SOS optimization technique, the extremal H-eigenvalues of a symmetric even order extended essentially non-negative tensor can be computed by solving an SOS optimization problem. Numerical examples illustrate the significance. At last, an important application is presented that is checking the co-positivity of symmetric tensors with even or odd orders.

Here, the extended essentially non-negative tensors considered are all with even order. So, can we compute the extremal H-eigenvalues of odd order symmetric extended essentially non-negative tensors? This may be interesting in the future work.

Chapter 8

Conclusions and future work

The purpose of this paper is to study structure properties and spectral properties of structured tensors in the literature. Several new class of structured tensors are defined, which are natural extensions of matrix. Furthermore, some numerical examples and applications are provided to verify the theoretical conclusions.

8.1 Conclusions of the paper

The main content of the article are listed below:

- Cauchy tensors and generalized Cauchy tensors are defined. Several necessary and sufficient conditions for an even order Cauchy tensor to be positive semi-definite or positive definite are given. SOS tensor decomposition property and completely positivity property of generalized Cauchy tensors are studied. Furthermore, H-spectral properties and Z-spectral properties of these two class of tensors and some new properties of Hankel tensors are presented.

- To study the relationship of the largest H-eigenvalues between symmetric Z-tensors and their absolute tensors, we define odd-bipartite and even-bipartite tensors in this paper. Using this notions, sufficient and necessary conditions for the equality of these largest H-eigenvalues are given when the Z-tensor has even order. For the odd order case, sufficient conditions are presented. On the other side, relation between spectral sets of an even order symmetric Z-tensor with non-negative diagonal entries and its absolute tensor are studied.

- The SOS tensor decomposition property is established for various even order symmetric structured tensors available in the current literature. In particular, an explicit sharp estimate is provided for SOS-rank of tensors with bounded exponent and SOS-width for the tensor cone consisting of all such tensors with bounded exponent that have SOS decomposition. Then, applications for the SOS decomposition of extended Z-tensors are presented.

- We study the extended essentially non-negative tensor, which are general forms of essentially non-negative tensor. Positive semi-definiteness and SOS tensor decomposition of symmetric essentially non-negative tensors are studied. Then, by SOS optimization technique, the extremal H-eigenvalues of a symmetric even order extended essentially non-negative tensor can be computed by solving an SOS optimization problem. Numerical examples illustrate the significance. An important application is presented that is checking the co-positivity of symmetric tensors with even or odd orders.

8.2 Future works

Although many results about structured tensors are provided in this paper, there are still some questions that we are not sure now. Now, we list some questions here, which may be interesting in the future.

- Can we get the type of Cauchy-Toeplitz tensors? If so, how about their spectral properties? What are the necessary and sufficient conditions for their positive semi-definiteness?

- In Chapter 5, we only study the relationships of H-eigenvalues of Z-tensors and their absolute tensors, do Z- eigenvalues of Z-tensors also hold in such case? Are there some sufficient and necessary conditions to guarantee the equality of those two largest Z-eigenvalues?

- Can we evaluate the SOS-rank of symmetric B_0-tensors?

- Can we evaluate the SOS-rank of symmetric Z-tensors?

- Can we evaluate the SOS-rank of symmetric diagonally dominated tensors?

- Can we use the techniques in Chapter 6 to find the minimum H- eigenvalue of an even order symmetric structured tensors other than the extended Z-tensors?

- In Chapter 7, the extended essentially non-negative tensors considered are all with even order. So, can we compute the extremal H-eigenvalues of odd order symmetric extended essentially non-negative tensors?

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