OPTIMAL INVESTMENT PROBLEMS OVER A FINITE TIME HORIZON

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Certificate of Originality

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______________________________(Signed)

ZHOU Wenxin (Name of student)

Dedicate to my parents.

Abstract

This thesis is concerned with optimal investment problems over a finite time horizon. The value function is constructed and the corresponding Hamilton-Jacobi-Bellman (HJB) equation can be derived by applying dynamic programming. In this thesis, we derive the properties of the strategy as well as the boundary and terminal line. We also discuss the optimal stopping time with multi-assets. The main contents of this thesis are divided into three parts.

In the first part, we study an optimal consumption investment model with uncertain exit time. The value function is not only the expectation of utility of the price of assets on maturity date, but also the expected utility produced in the whole process. Using the method of partial differential equation (PDE), we prove the smoothness of the value function without specifying a particular utility function, where a non-smooth and non-concave situation is considered. Some restrictions are imposed on the problem. The continuity of the optimal strategy and some properties of the boundary and terminal line are derived.

In the second part, we discuss the above problem with constraints. The value function can be characterized by two types of second-order partial differential equations in different regions. One is a fully nonlinear equation, and the other is a linear equation. We construct an approximation problem to make the equations satisfy the parabolic condition. Using the method of partial differential equation, we prove the existence, uniqueness and regularity of the solution to the original problem via the approximation problem. We derive the properties of the free boundary line and ascertain its end point.

In the third part, we consider the optimal stopping time for investors to leave the financial market among multi-assets to obtain maximum profit. The utility function is considered as a quadratic form. Two models are researched respectively in this part. One is with a normal utility function, and the other is based on a Logarithmic utility-maximization objective. A two-stage problem is formulated. The main problem is a nonstandard optimal stopping time problem. Using the method of stochastic analysis, we turn it into a standard one. The subproblem with control variable in the drift and volatility terms is solved via stochastic control method. Numerical examples are also presented accordingly to illustrate the efficiency of the theoretical results.

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Contents

Notation

$S_{i,t}$	: the price of risky asset i at time t
$S_{0,t}$	: the price of risk free asset at time t
r_t	: the return rate of risk free asset at time t
$\mu_{i,t}$	: the excess appreciation rate of risky asset i at time t
$b_{i,t}$	: the appreciation rate of risky asset i at time t
$\sigma_{i,t}$	: the volatility vector of risky asset i at time t
W	: the standard Brownian motion
$\pi_{i,t}$	: the holding amount of risky asset i at time t
X_t	: the wealth of the investor at time t
τ	: the uncertain exit time
∇A	: the gradient of matrix A
A'	: the transpose of matrix A

Chapter 1

Introduction

1.1 Background and Literature Review

With the explosion of economic growth, wealth has been accumulated in the hands of people. As illiquid wealth cannot generate more money, it is crucial to invest them into various kinds of activities so as to realize the accession of wealth and to reduce the impact of inflation. Types of investment can be generalized as physical investment and capital investment. Due to the fact that the investment products are diversified, it becomes a problem to choose the appropriate combination of products in order to maximize the terminal wealth without taking unbearable risks after an investment period, which is the so-called portfolio selection problem.

Researchers have tried a lot to construct a pure mathematical model to describe the portfolio selection problem. The work of Markowitz (1952) lays a firm foundation and establishes the mean-variance framework, which has been the core of a majority of researches in finance ever since. In the mean-variance framework, the objective of investors is to select an optimal portfolio which can balance the gains and risks in the whole process, where gains are expressed as the expectation of final return, and risks are denoted by variances. Pratt (1964) considered utility functions for money as a measure of risk aversion, where the elasticity of marginal utility is of great importance in deciding the risk tolerance. Samuelson (1969) generalized the one-period case in Markowitz (1952) to a multi-period model, where the investment and consumption lie on the whole lifetime. Merton (1969, 1971, 1973) established the theory of a continuous-time case, where geometric Brownian motion is used to describe the motion law of risky assets. A type of utility function called hyperbolic absolute risk aversion (HARA) utility is also integrated into this model and an Hamilton-Jacobi-Bellman (HJB) equation is derived by the principle of dynamic programming. The HARA utility is a type of risk aversion which indicates that the risk tolerance is linear to wealth. Merton (1971) describe the formulation of the investment problem as follows. Consider that the consumption is denoted by N, the wealth is denoted by X, the utility function is denoted by U, the time is denoted by t and the "bequest" function is denoted by Q. Then the problem to choose the optimal portfolio for an investment with consumption is formulated as follows:

$$\max \mathbb{E}_0\left[\int_0^T U\big(N(t),t\big)dt + Q\big(X(T),T\big)\right],$$

$$\text{s.t.} \quad X(0) = X_0.$$

The classic article of Black and Scholes (1973) settles the foundation of option pricing theory by using geometric Brownian motion to describe the stock prices.

The groundbreaking works have accelerated the development of researches in financial field. Although most papers analyze HJB equation using stochastic analysis, martingale theory, the dual method and original differential equations, few of them solve the problem according to the theory of partial differential equation (PDE).

Consider the investment of a financial company, where a self-finance mode is utilized during the allocation of wealth. The income of the company comes from the return of its investment, and the consumption of the company is used on paying dividends. Investment is usually divided into two classes. First, companies will spend a portion of wealth into risk-free assets, such as regular bank account or government loan. The income is always in direct proportion to the cost of investment. Second, investment in risky assets is the major part such as purchasing risky securities and shares. Multiple kinds of investment consumption problems are discussed by using the Itô's lemma to analyze the prices, wealth and consumption, and the corresponding HJB equation can be derived. In a continuous-time case, when a model possesses Markov property, an HJB equation is derived, converting stochastic optimal control problems into partial differential equations or corresponding variational inequality problems. The Itô's lemma is introduced as follows. When a process X_t satisfies the stochastic differential equation:

$$dX_t = \mu_t dt + \sigma_t dW_t,$$

where W_t is a Wiener process, μ_t and σ_t are parameters. Consider a twice differentiable scalar function $f(t, x)$, then the following equation can be obtained:

$$df(t, X_t) = \left(f_t + \mu_t f_x + \frac{1}{2}\sigma_t^2 f_{xx} \right) dt + \sigma_t f_x dW_t.$$

The HJB equation is a type of partial derivative equation and it is vital in optimal control theory. The solution to the HJB equation is a value function with minimum cost according to some specific dynamic system and cost function. Bertsekas (2005) introduces some basic ideas. Consider an continuous-time optimal control problem during $[0, T]$:

$$\begin{aligned} \min_{\pi} \quad & h\big(x(T)\big) + \int_0^T g\big(x(t), \pi(t)\big) dt \\ \text{s.t.} \quad & \frac{dx(t)}{dt} = f\big(x(t), \pi(t)\big), \quad 0 \le t \le T, \\ & x(0) \, \text{is given}, \end{aligned}$$

where g is the cost function, h is a function expressing the final state, $x(t)$ is the system state and $\pi(t)$ is the admissible control. Here all f, g and h are assumed to be continuously differentiable. After dividing the time horizon into N pieces with equal length of $\delta = \frac{T}{N}$, we get the approximate discrete-time control problem as

$$\begin{aligned} \min_{\pi} \quad & h\big(x(N\delta)\big) + \sum_{k=0}^{N-1} g\big(x(k\delta), \pi(k\delta)\big) \cdot \delta, \\ \text{s.t.} \quad & x\big((k+1)\delta\big) = x(k\delta) + f\big(x(k\delta), \pi(k\delta)\big) \cdot \delta, \\ & k = 0, 1, \cdots, N. \end{aligned}$$

In order to derive the HJB equation, we consider the optimal cost-to-go function, which is also known as the value function. Let $\tilde{J}^*(t,x)$ denote the optimal cost-to-go at time t and let x denote the state in the discrete-time version. According to dynamic programming, we derive

$$\begin{aligned} &\tilde{J}^*(k\delta, x) = \min_{\pi} \Big[g(x,\pi)\delta + \tilde{J}^*\big((k+1)\delta, x + f(x,\pi)\delta\big)\Big], \quad k = 0, 1, \cdots, N-1, \\ &\tilde{J}^*(N\delta, x) = h(x). \end{aligned}$$

Let $J^*(t,x)$ be the optimal cost-to-go for the continuous problem, then the following HJB equation for $J^*(t,x)$ is satisfied,

$$0 = \min_{\pi} \big[g(x,\pi) + \nabla_t J^*(t,x) + \nabla_x J^*(t,x)' f(x,\pi) \big], \quad \forall t, x,$$

where the boundary condition is

$$J^*(T,x) = h(x).$$

We use the principle of optimality to obtain the equation, which is the basic idea in dynamic programming. Bellman (1957) state that an optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

In order to solve the HJB equation, the problem can be tackled backwards in time. If we can get the explicit solution of of the HJB equation, then the optimal control policy can be obtained. However, in most cases, the HJB equation is not tractable.

There is a wide range of applications for HJB equations. In the financial field, the related work includes Asmussen and Taksar (1997), Højgaard and Taksar (1998) and Zariphopoulou (1994). A classical solution in a bounded domain can be derived under certain circumstances for HJB equations. The HJB equation usually corresponds to different partial differential equations in different regions according to control constraints. The solution of the HJB equation is governed by the so-called "principle of smooth fit". Illustrate an optimal stopping problem as follows:

$$V(x) = \sup_{\tau} \mathbb{E}_x \big[G(X_\tau) \big],$$

where V is the value function, G is the differentiable gain function, X is a diffusion process and τ denotes the uncertain exit time. "Principle of smooth fit" states that the optimal stopping time τ^* which separates the holding region C from the exit region D satisfies the property that $V'(\tau^*) = G'(\tau^*)$. In the case with an infinite time horizon, the general solutions in each region are expressed with some unknown constant parameters. Taksar (2000) and Asmussen et al. (2000) show that the unknown parameters can be determined by using "principle of smooth fit" at free boundary points and by analyzing conditions of fixed boundary points. For the problem with a finite time horizon, it is hard to derive its explicit solution. This calls for studying the corresponding solutions and the free boundaries. The free boundary often represents the switching curve determined by points between two regions arising from two types of different policies. The properties of the solution play an important role in financial decision-making. Carpenter (2000) researched the dynamic investment problem of a risk averse manager compensated with a call option. The value function is maximizing the expected utility of payoff at the terminal date. Thus, the utility function as the price of the total assets is not concave. However, the author still proves the concavity of the value function. In addition, he derives the expression of solution using the method of martingales. It should be noted that the expression can also be obtained by making dual (Frenchel-Legendre) transformation of the HJB equation and then solving a linear problem. Pliska (1986), Karatzas et al. (1987) and Cox and Huang (1989) made dual transformation directly on the value function. However, for problems of limited time, it is hard to find the explicit solution to the corresponding Barenblatt parabolic partial differential equation. Guan and Yi (2014, 2016), and Han and Yi (2015) studied on this topic.

Consider a portfolio selection problem with uncertain exit time. Choosing a proper time point to stop investment is one of the most important things for investors to make maximum profit. Since the highest return will be unknown until the end of time horizon, it is natural to set a more realistic objective, which is to minimize the difference between the return of the stopping time and the maximum return over the whole time horizon. In the field of mathematical finance, this problem is always formulated to an optimal stopping problem which has important applications and has been well developed in the past decades, especially mixed with stochastic dynamic system. Dayanik and Karatzas (2003) investigated the optimal stopping problems for one dimensional diffusions and show how to reduce the discounted optimal stopping problem for an arbitrary diffusion process to an undiscounted one for standard Brownian motion. For a stock selling model, Shiryaev et al. (2008) addressed the optimal stopping issue in an equity market by considering the relative error and a log-normal price process. Du Toit et al. (2009) used the geometric Brownian motion assumption of stock price as Shiryaev et al. (2008) considered the optimal stopping problem for stochastic differential equations with random coefficients. Dai and Zhong (2012) provided a PDE approach to characterize the resulting free boundary corresponding to the optimal selling strategy. Wu et al. (2018) chose an optimal point when the investor stops the investment among multi-assets. Li et al. (2017) sought for the optimal exit time based on a logarithmic utility-maximization objective .

In the financial market, investors need to select a portfolio among various assets. An optimal consumption and investment under short-selling prohibition was studied in Xu and Shreve (1992a,b). A consumption-portfolio selection problem and an optimal stopping problem was mixed in Choi et al. (2004) and the investor's decision to switch from active portfolio management to passive management was discussed.

1.2 Contributions and Organization

In Chapter 2, the mathematical description of financial markets is presented, where the risk free and risky assets are expressed as different differential equations respectively. In addition, some prepositive knowledge and useful lemmas are shown for further use in following chapters.

In Chapter 3, a class of optimal investment problem is studied in finite horizon. Based on the model described in Carpenter (2000), we consider the case where there is an uncertain exit time under some deterministic distribution, which forces the investors to leave the market with some utilities as compensation. Thus, there will be an integral term on time appearing in the definition of the value function. The dual equation of the corresponding HJB equation is quasi-linear due to this integral term, giving rise to the consequences that the expression of the solution cannot be obtained. However, we can still use the technique of PDE to study the properties of the solution and get the optimal trading strategies.

Consider the condition that the investment will generate utilities in the whole process, but not just on the maturity date. We will not specify a particular utility function except for some general restrictions such as the growth condition. Under these restrictions, we can prove that the value function is smooth, strictly increasing and strictly concave on current assets. The optimal portfolios on risky assets are continuous. The condition that the utility function is concave is not required in this part. Moreover, when it approaches the terminal investment date and the asset price belongs to the con-concave region, the investors tend to prefer risks to increase the volatility of the assets process. In addition, when the price of assets becomes small, the optimal portfolio on the risky assets approaches to zero, and the value function increases rapidly.

Using the technique of PDE, we research a class of investment problems in general cases, and give the proof of existence and smoothness of the value function. We also make use of the method of dual transformation to study the HJB equation.

The organization of this chapter is shown below. After presenting the mathematical model, we discuss the terminal condition of the value function when the utility function is not concave. Then we derive the HJB equation and construct a fully nonlinear problem. By making dual transformation, we convert the problem into a new quasi-linear PDE problem, and get the existence and properties of the solution. Moreover, the optimal investment strategy is given by the solution, and the properties of the strategy on boundaries are studied. An example is illustrated where the exit time is uncertain.

In Chapter 4, we continue to study the problem over finite time horizon with constraints. For a classic option pricing model, the portion of risky investment is certain, making HJB equation a linear equation, and the expression of solution is easy to derive. Under an investment-consumption model, the risk capital is a controllable variable with no upper bound. Thus a fully nonlinear equation is derived, which can be converted to a linear equation by Legendre transformation. Considering that in real financial market, the borrowing limit of an investor depends on the corresponding total assets at present. We assume the amount of risky assets is controllable and depends on the upper bound of current total assets in a functional form. When the amount of risky assets does not reach this upper bound, we derive a fully nonlinear equation, while when the amount reaches the boundary, we get a quasi-linear equation. Thus, it is a free-boundary problem, where there is usually no explicit expression of solutions.

In order to study the existence and uniqueness of solution to the original problem, we rewrite the equations as one fully nonlinear equation and construct an approximation problem. Using comparison principle, we obtain the properties of the approximation problem and finally prove the properties of value function. This Barenblatt parabolic equation is singular on the left boundary. We estimate the value function and its partial derivatives, and then derive the uniqueness and upper bound of the free boundary. In order to study the smoothness of the free boundary, we construct a function to prove the continuity, and ascertain its end point.

The content of this chapter is organized as follows. After presenting the mathematical formulation of this model, we define the value function, then the related HJB equation is derived and discussed. We construct the approximation problems about the value function and its derivatives, discuss the existence and uniqueness of the solution to the approximation problems and the original problem, and derive the existence and smoothness of the free boundary.

In Chapter 5, we analyze the right time for an investor to stop the investment among multi assets over a given time horizon. It means that before determining the optimal stopping time, a portfolio problem need to be solved. There are two models introduced in this chapter. We formulate the problem into a two-stage problem. The main problem is a stopping problem but not a standard one due to the non-adapted term in the objective function. The subproblem is an optimal control problem with a given terminal payoff where the control variables involving in the drift and volatility terms of the dynamic system. After deriving the optimal portfolio of the sub-problem, we substitute it to the main problem. And then transform the non-adapted stopping problem to a standard one by stochastic analysis. Simply speaking, we face an optimal stopping problem with a utility-maximization objective and with more general drift and volatility coefficients. We consider the utility function of a quadratic form instead of a relative error criterion in Shiryaev et al. (2008). Therefore the maximum wealth can be zero. In addition, the involvement of multi assets makes our model more general compared with those of Du Toit et al. (2009) and Dai and Zhong (2012). The process of the multi assets is similar to Xu and Shreve (1992a,b). All these make our analysis more realistic and meaningful.

This chapter is organized as follows. We formulate the problem to a two-stage model and transform it into an equivalent optimal stopping problem. A numerical example is presented to demonstrate the theoretical results.

Chapter 2

Preliminary

In this chapter, we introduce the basic concepts in the financial market, and present the differential equations which describe the properties of risk-free and risky assets. Some useful lemmas are also introduced for further use.

2.1 Model Formulation

Consider a financial market with a fixed filtered complete probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$. A standard $\mathcal{F}_t$-adapted one-dimensional Brownian motion $\{W_t, t \geq 0\}$ is defined on the space, where the boundary condition $W(0) = 0$ and the terminal time $T > 0$ are given. Let $L^2_{\mathcal{F}}(0, T; \mathbb{R})$ denote the set of all $\mathbb{R}$-valued, $\mathcal{F}_t$-progressively measurable stochastic processes $f(t)$ satisfying $\mathbb{E}\int_0^T |f(t)|^2 dt < +\infty$.

The manager operates in a complete, arbitrage-free, continuous-time financial market consisting of a riskless asset with instantaneous interest rate r and n risky assets. The risky asset prices S_i are governed by the stochastic differential equations

$$\frac{dS_{i,t}}{S_{i,t}} = (r + \mu_i)dt + \sigma_i dW_t^j, \quad \text{for } i = 1, 2, \cdots, n, \tag{2.1.1}$$

where the interest rate r, the excess appreciation rates μ_i, and the volatility vectors σ_i are constants, W is a standard n-dimensional Brownian motion. In addition, the covariance matrix $\sigma\sigma'$ is strongly nondegenerate.

A trading strategy for the manager is an n-dimensional process π_t, where $\pi_{i,t}$ is the holding amount of the i-th risky asset in the portfolio at time t. An admissible trading strategy π_t must be progressively measurable with respect to $\{\mathcal{F}_t\}$ such that $X_t \geq 0$. Note that $X_t = \pi_{0,t} + \sum_{i=1}^n \pi_{i,t}$, where $\pi_{0,t}$ is the amount invested in the money. Hence, the wealth X_t evolves according to

$$\begin{cases} dX_s = (rX_s + \mu'\pi_s)ds + \pi_s'\sigma dW_s, \quad s \geq t, \\ X_t = x. \end{cases} \tag{2.1.2}$$

2.2 Basic Knowledge

We provide some basic knowledge prepositive to this thesis, and present some useful lemmas in this part.

2.2.1 Parabolic distance

Recall that $\mathbb{R}^n$ is the n-dimensional Euclidean space, the point on which is denoted by $x = (x_1, x_2, \cdots, x_n)$. By introducing a time variable t, denote the point on the constructed $n+1$-dimensional space $\mathbb{R}^{n+1}$ as $X = (x, t_X)$. We now introduce the distance in $\mathbb{R}^{n+1}$. $\delta(X, Y)$ is called the parabolic distance if

$$\delta(X, Y) = \max\Big\{|x - y|, |t_X - t_Y|^{\frac{1}{2}}\Big\}.$$

Let $Q_R(X)$ represent a ball with the center of X and radius of R regarding the parabolic distance $\delta(X, Y)$, i.e.,

$$\begin{aligned} Q_R(X) &= \Big\{Y \in \mathbb{R}^{n+1} | \delta(X, Y) < R\Big\} \\ &= B_R(x) \times (t_X - R^2, t_X + R^2), \end{aligned}$$

where $B_R(x)$ denotes an n-dimensional ball with the center of x and radius of R. Let D be a bounded region in $\mathbb{R}^{n+1}$. For any random $X \in D$, denote $D(X,r) = D \cap Q_r(X)$. Denote $d = \text{diam}(D)$, which is the diameter of D in regard to $\delta(X,Y)$. Now we introduce some spaces.

Definition 2.2.1. (Morrey space) *For* $1 \leq p < \infty$, $\theta \geq 0$, *let* $L^{p,\theta}(D;\delta)$ *denote the normed linear space composed by all functions* u, *where*

$$\|u\|_{L^{p,\theta}(D;\delta)} := \left\{ \sup_{\substack{X \in D \\ d \geq \rho > 0}} |D(X,\rho)|^{-\theta} \int_{D(X,\rho)} |u(Y)|^p \mathrm{d}Y \right\}^{\frac{1}{p}} < \infty.$$

Here $\|u\|_{L^{p,\theta}(D;\delta)}$ *is the norm of the space.*

Definition 2.2.2. (Campanato space) *For* $p \geq 1$, $\theta \geq 0$, *let* $\mathfrak{L}^{p,\theta}(D;\delta)$ *denote the normed linear space composed by all functions* u, *where*

$$[u]_{\mathfrak{L}^{p,\theta}(D;\delta)} := \left\{ \sup_{\substack{X \in D \\ d \geq \rho > 0}} |D(X,\rho)|^{-\theta} \int_{D(X,\rho)} |u(Y) - u_{X,\rho}|^p \mathrm{d}Y \right\}^{\frac{1}{p}} < \infty.$$

The norm is defined by

$$\|u\|_{\mathfrak{L}^{p,\theta}(D;\delta)} := \left\{ \|u\|^p_{L^p(D)} + [u]^p_{\mathfrak{L}^{p,\theta}(D;\delta)} \right\}^{\frac{1}{p}}.$$

Here $u_{X,\rho}$ *denotes the integral mean of* u *on* $D(X,\rho)$, *i.e.,*

$$u_{X,\rho} = |D(X,\rho)|^{-1} \int_{D(X,\rho)} u(Y) \mathrm{d}Y.$$

2.2.2 Hölder space

For $0 < \alpha \leq 1$, let $C^\alpha(\overline{D};\delta)$ denote the normed linear space composed by all functions u, where

$$[u]_{\alpha;D} := \sup_{\substack{X\in D\\ d\geq\rho>0}} \frac{|u(X)-u(Y)|}{\delta(X,Y)^\alpha} < \infty.$$

The norm is defined by

$$|u|_{\alpha;D} = \sup_D |u| + [u]_{\alpha;D}.$$

In order to describe spaces of higher orders, we introduce the definition of semi-norm as follows:

$$|u|_{0;Q_T} = \sup_{Q_T} |u|,$$

$$[u]_{\alpha;Q_T} = \sup_{\substack{X,Y\in Q_T\\ X\neq Y}} \frac{|u(X)-u(Y)|}{\delta(X,Y)^\alpha}, \quad 0<\alpha<1,$$

$$[u]^t_{\alpha;Q_T} = \sup_{\substack{x\in\Omega. t\neq r\\ t,r\in[o,T]}} \frac{|u(x,t)-u(x,\tau)|}{|t-\tau|^\alpha}, \quad 0<\alpha<1.$$

The semi-norm of higher derivatives is also introduced as

$$[u]_{k+\alpha;Q_T} = \begin{cases} \sum\limits_{r+2s=k} [D^s_t D^r_x u]_\alpha, & k \text{ is even},\\ \sum\limits_{r+2s=k} [D^s_t D^r_x u]_\alpha + \sum\limits_{r+2s=k-1} [D^s_t D^r_x u]^t_{\frac{1+\alpha}{2}}, & k \text{ is odd}. \end{cases}$$

The linear space composed by functions u in $C(\overline{Q}_T)$ is denoted by $C^{k+\alpha,\frac{k+\alpha}{2}}(\overline{Q}_T)$ or $C^{k+\alpha}(\overline{Q}_T;\delta)$, where

$$|u|_{k+\alpha;Q_T} := \sum_{0\leq r+2s\leq k} [D^s_t D^r_x u]_{0;Q_T} + [u]_{k+\alpha;Q_T} < \infty.$$

After introducing the norm $|u|_{k+\alpha;Q_T}$, it will turn into a Banach space.

For a positive integer l, and $1 \le p < \infty$, denote

$$\|u\|_{W_p^{l,\frac{l}{2}}(Q_T)} = \begin{cases} \left\{ \sum\limits_{0 \le r+2s \le l} \|D_t^s D_x^r u\|_{L^p(Q_T)}^p \right\}^{\frac{1}{p}}, & l \text{ is even}, \\ \left\{ \sum\limits_{0 \le r+2s \le l} \|D_t^s D_x^r u\|_{L^p(Q_T)}^p + \sum\limits_{0 \le r+2s \le l-1} [D_t^s D_x^r u]_{L_{p,t}^{\frac{1}{2}}(Q_T)}^p \right\}^{\frac{1}{p}}, & l \text{ is odd}. \end{cases}$$

Let $W_p^{l,\frac{l}{2}}(Q_T)$ denote the normed linear space composed by functions u, where

$$\|u\|_{W_p^{l,\frac{l}{2}}(Q_T)} < \infty.$$

Let Ω be a bounded region in $\mathbb{R}^n$, and $Q_T = \Omega \times (0, T]$. We now introduce some concepts about weak solution.

Definition 2.2.3. *Let $V_2(Q_T)$ denote the normed linear space composed by functions u, where*

$$\|u\|_{V_2(Q_T)} := \left\{ \operatorname*{ess\,sup}_{0<t<T} \|u(\cdot,t)\|_{L^2(\Omega)}^2 + \|D_x u\|_{L^2(Q_T)}^2 \right\}^{\frac{1}{2}} < \infty.$$

Let $V_2^{1,0}(Q_T)$ denote the normed linear space composed by functions u in $V_2(Q_T)$, where

$$\lim_{h \to 0} \|u(\cdot, t+h) - u(\cdot, t)\|_{L^2(\Omega)} = 0, \quad t, t+h \in [0, T].$$

Afterwards, let $\mathring{V}_2(Q_T)$, $\mathring{V}_2^{1,0}(Q_T)$, $\mathring{W}_2^{1,1}(Q_T)$ denote the spaces composed by functions u where $u(\cdot,t)|_{\partial\Omega} = 0$, a.e. $t \in (0,T)$ in $V_2(Q_T)$, $V_2^{1,0}(Q_T)$ and $W_2^{1,1}(Q_T)$ respectively.

Definition 2.2.4. *Consider a parabolic equation*

$$u_t - D_j(a^{ij} D_i u + d^j u) + b^i D_i u + cu = f - D_i f^i.$$

Denote

$$Lu = -D_j(a^{ij}D_iu + d^ju) + b^iD_iu + cu.$$

The function $u \in V_2(Q_T)$ *is a weak solution, if* $\forall t \in (0,T),\ \varphi \in \mathring{W}_2^{1,1}(Q_T),\ \varphi(x,0) = 0,$ *and*

$$(u(\cdot,t),\varphi(\cdot,t)) - \int_0^t (u,\varphi_t)\mathrm{d}t + \int_0^t (Lu,\varphi)\mathrm{d}t = \int_0^t [(f,\varphi) + (f^i, D_i\varphi)]\mathrm{d}t,$$

where

$$(u,v) = \int_\Omega u(x,t)v(x,t)\mathrm{d}x.$$

2.2.3 Some useful lemmas

Now we will introduce the comparison principle for parabolic equations.

Lemma 2.2.5. *Suppose* F *is a continuous function in* $\Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \times S_N$. *It is assumed that for all* $x \in \Omega$, $t \in [0,T]$, $r \in \mathbb{R}$, $p \in \mathbb{R}^N$, $M,\ \widehat{M} \in S_N$, *we have*

$$M \le \widehat{M} \Rightarrow F(x,t,r,p,q,M) \le F(x,t,r,p,q,\widehat{M}).$$

Let $u_1,\ u_2 \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ *satisfy*

$$\begin{cases} D_tu_1 - F(x,t,u_1,D_iu_1,D_{ij}u_1) \ge D_tu_2 - F(x,t,u_2,D_iu_2,D_{ij}u_2) \text{ in } Q_T, \\ u_1 \ge u_2(\text{ or } \dfrac{\partial u_1}{\partial n} \ge \dfrac{\partial u_2}{\partial n}) \text{ on } S_T = \partial\Omega \times [0,T], \\ u_1(x,0) \ge u_2(x,0), \end{cases}$$

where $\dfrac{\partial u}{\partial n}$ *is the gradient on outer normal direction of* Ω. *Then we have* $u_1 \ge u_2$ *in* Q_T.

Now we will introduce the Schauder estimate and C^α estimate for parabolic equations.

Lemma 2.2.6. (Schauder estimate) *For parabolic equation*

$$\begin{cases} u_t - a^{ij}(x,t)D_{ij}u + b^i(x,t)D_iu + c(x,t)u = f(x,t) \text{ in } Q_T, \\ u = g(x,t) \text{ on } S_T = \partial\Omega \times [0,T], \\ u(x,0) = \varphi(x), \end{cases}$$

where $a^{ij}(x,t)$ satisfies that for $\Lambda > \lambda > 0$, we have

$$\lambda|\xi|^2 \le a^{ij}(x,t)\xi_i\xi_j \le \Lambda|\xi|^2, \quad \forall (x,t) \in Q_T, \quad \xi \in \mathbb{R}^n.$$

(a) *Global estimate. Conditions are as follows:* (i) $a^{ij}, b^i, c \in C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)$*;* (ii) $\partial\Omega \in C^{2,\alpha}$, $g \in C^{2+\alpha,1+\frac{\alpha}{2}}(S_T)$*;* (iii) $u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ *satisfies the equation and initial boundary value and compatibility conditions. Then the conclusion is*

$$|u|_{2+\alpha,Q_T} \le C\Big(|f|_{\alpha,Q_T} + |\varphi|_{2+\alpha,\Omega} + |u|_{0,Q_T} + |g|_{2+\alpha,S_T}\Big),$$

where C depends on $n,\ \alpha,\ \Lambda,\ \lambda,\ |a^{ij},\ b^i,\ c|_{\alpha,Q_T}$, and $\partial\Omega$.

(b) *Interior estimate. Conditions are as follows:* (i) $a^{ij}, b^i, c \in C^{\alpha,\frac{\alpha}{2}}(Q_T)$*;* (ii) $Q \subset\subset Q_T$*;* (iii) $u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ *satisfies the equation. Then the conclusion is*

$$|u|_{2+\alpha,Q} \le C\Big(|f|_{\alpha,Q_T} + |u|_{0,Q_T}\Big),$$

where C depends on $n,\ \alpha,\ \Lambda,\ \lambda,\ |a^{ij},\ b^i,\ c|_{\alpha,Q_T}$, and $\operatorname{dist}(Q, \partial_p Q_T)$.

(c) *Maximum norm estimate. Conditions are as follows:* (i) $u \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ *is the solution to the equation;* (ii) *Coefficients are continuous and bounded;* (iii) $c(x,t) \ge -C_0,\ C_0 > 0$*. Then the conclusion is*

$$|u|_{0,Q_T} \le e^{C_0T}\Big(\sup_{\partial_pQ_T}|u| + T\sup_{Q_T}|f|\Big).$$

Combining the global estimate and the maximum norm estimate, we derive

$$|u|_{2+\alpha,Q_T} \le C\Big(|f|_{\alpha,Q_T} + |\varphi|_{2+\alpha,\Omega} + |g|_{2+\alpha,S_T}\Big).$$

Lemma 2.2.7. (C^α estimate) *For parabolic equation*

$$u_t - D_j(a^{ij}D_iu + d^ju) + (b^iD_iu + cu) = f - D_if^i,$$

where $a^{ij}(x)$ satisfies for $\Lambda > \lambda > 0$, we have

$$\lambda|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2, \quad \forall (x,t) \in Q_T, \quad \xi \in \mathbb{R}^n.$$

(a) *Global estimate. Conditions are as follows:* (i) $a^{ij}, b^i, d^i, c \in L^\infty(\Omega)$*;* (ii) $f \in L^{\frac{(n+2)p}{n+2+p}}(Q_T)$*,* $f^i \in L^p(Q_T)$*;* (iii) $\mathbb{R}^n \backslash \Omega$ *is a region of type (A), i.e., there exists* $A \in (0,1)$*,* $\rho_0 > 0$*, such that* $|B_\rho(x_0)\backslash\Omega| \ge A|B_\rho(x_0)|$*,* $\forall x_0 \in \partial\Omega$*,* $\rho \in (0,\rho_0]$*;* (iv) $u \in C^\varepsilon(\partial_pQ_T, \delta)$*;* (v) u *is a weak solution to the equation. Then the conclusion is*

$$|u|_{\beta,Q_T} \le C\Big(|u|_{0,Q_T} + [u]_{\varepsilon,\partial_pQ_T} + |f|_{L^{\frac{(n+2)p}{n+2+p}}(Q_T)} + \sum_i |f^i|_{L^p(Q_T)}\Big),$$

where C, β depend on n, Λ, λ, p, ε, Ω, and *L^∞ norm of coefficients.*

(b) *Interior estimate. Conditions are as follows:* (i) $a^{ij}, b^i, d^i, c \in L^\infty(\Omega)$*;* (ii) $f \in L^{\frac{(n+2)p}{n+2+p}}(Q_T)$*,* $f^i \in L^p(Q_T)$*;* (iii) $Q \subset\subset Q_T$*;* (iv) u *is a weak solution to the equation. Then the conclusion is*

$$|u|_{\beta,Q} \le C\Big(|u|_{0,Q_T} + |f|_{L^{\frac{(n+2)p}{n+2+p}}(Q_T)} + \sum_i |f^i|_{L^p(Q_T)}\Big),$$

where C, β depend on n, Λ, λ, p, Ω, L^∞ norm of coefficients, and $\mathrm{dist}(Q, \partial_pQ_T)$.

(c) *Maximum norm estimate. Conditions are as follows:* (i) $a^{ij}, b^i, d^i, c \in L^\infty(Q_T)$*;*

(ii) $f \in L^{\frac{(n+2)p}{n+2+p}}(Q_T)$, $f^i \in L^p(Q_T)$, $p > n+2$; (iii) $c - D_i d^i \geq -C_0$ *in* Q_T; (iv) u *is a weak solution to the equation. Then the conclusion is*

$$\sup_{Q_T} |u| \leq \sup_{\partial_P Q_T} |u| + C\Big(|f|_{L^{\frac{(n+2)p}{n+2+p}}(Q_T)} + \sum_i |f^i|_{L^p(Q_T)} \Big),$$

where C *depends on* n, Λ, λ, p, T, and L^∞ *norm of coefficients.*

Chapter 3

Optimal Investment Problems over a Finite Time Horizon

In this chapter, we study a class of optimal investment problems in finite horizon. We discuss the terminal condition of the value function when the utility function is not concave. We derive the HJB equation, construct a fully nonlinear problem, and convert it into a new quasi-linear PDE problem by making dual transformation. We prove the existence and properties of the solution, and the optimal investment strategy is given. The properties of the strategy on boundaries are studied. An example is illustrated where the exit time is uncertain.

3.1 Formulation of HJB Equations

In the general framework, the dynamic problem is to choose an admissible trading strategy π_s $(t \leq s \leq T)$ to maximize

$$V(x,t) = \sup_{\pi} \mathbb{E}\left[\int_t^T f(X_s,s)ds + g(X_T)\right], \tag{3.1.1}$$

where $f(x,t)$ and $g(x)$ are non-negative continuous functions defined in $\Omega_T = \{(x,t) : x > 0,\ 0 < t < T\}$, and are increasing in x.

When $X_t = 0$, in order to keep $X_s \geq 0$, we get that $\pi_s = 0$ and $X_s \equiv 0,\ t \leq s \leq T$. Thus, we obtain a left boundary condition

$$V(0,t) = \int_t^T f(0,s)ds + g(0). \tag{3.1.2}$$

In order to make (4.1.1) a well defined function (a finite function), some constraints should be imposed on $f(x,t)$ and $g(x,t)$. Without loss of generality, we suppose that:

Condition I: There is a $\gamma \in (0,1)$ and an $M > 0$ such that for all $x,\ y \geq 0$, we get

$$\begin{cases} |g(x) - g(y)| \leq \dfrac{M}{\gamma}|x-y|^\gamma, \\ |f(x,t) - f(y,t)| \leq \dfrac{M}{\gamma}|x-y|^\gamma, \end{cases} \tag{3.1.3}$$

which also imply the growth condition that

$$\begin{cases} g(x) \leq g(0) + \dfrac{M}{\gamma}x^\gamma, \\ f(x,t) \leq f(0,t) + \dfrac{M}{\gamma}x^\gamma. \end{cases} \tag{3.1.4}$$

Condition II: The limit condition is shown as follows,

$$\lim_{x\to+\infty} g(x) = +\infty. \tag{3.1.5}$$

3.1.1 The Case that $g(x)$ is Non-concave

When $g(x)$ is non-concave, denote $\varphi(x)$ as its concave hull, i.e., $\varphi(x)$ is the minimal concave function not less than $g(x)$ (See Figure 3.1).

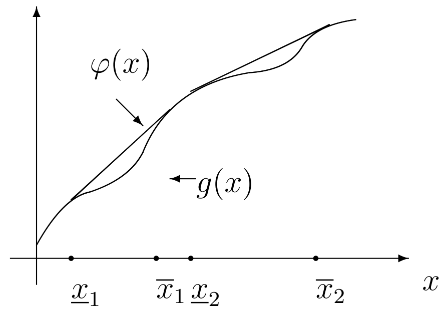


Fig 3.1 $\varphi(x)$.

Since $g(x)$ is an increasing and continuous function, $\varphi(x)$ will be increasing and continuous. Thus, $\{x > 0|\varphi(x) > g(x)\}$ is an open set which can be written as (in general case)

$$\{\varphi(x) > g(x)\} = \bigcup_{m=1}^{\infty} (\underline{x}_m, \overline{x}_m), \tag{3.1.6}$$

where $\{(\underline{x}_m, \overline{x}_m)\}_{m=1}^{\infty}$ are countable disjoint open intervals. In these intervals, $\varphi(x)$ is a linear function.

Since the portfolio π_t is unconstrained, we point out that the terminal condition of $V(x,t)$ should be $\varphi(x)$ but not $g(x)$. In fact, in a short time, the behavior of the asset price is like a martingale. When time approaches the terminal date and the current asset price x is located in $(\underline{x}_m, \overline{x}_m)(m \in \mathbb{Z})$, the investor could adopt such a strategy that he/she will buy sufficient risky assets and then X_s will rapidly touch $\underline{x}_m$ or $\overline{x}_m$ (with probability approximately equal to $\dfrac{x - \underline{x}_m}{\overline{x}_m - \underline{x}_m}$ and $\dfrac{\overline{x}_m - x}{\overline{x}_m - \underline{x}_m}$, respectively), so that the contribution of $\mathbb{E}[g(X_T)]$ to the value function is approximate to

$$\frac{x - \underline{x}_m}{\overline{x}_m - \underline{x}_m} g(\underline{x}_m) + \frac{\overline{x}_m - x}{\overline{x}_m - \underline{x}_m} g(\overline{x}_m) = \varphi(x).$$

Therefore, the value function is not less than $\varphi(x)$ near the terminal date. Under this idea, we could prove the following theorem.

Theorem 3.1.1. *The behavior of the value function near the terminal date is shown below:*

$$\lim_{t\to T-} V(x,t) = \varphi(x). \tag{3.1.7}$$

Proof. The proof of Theorem 3.1.1 can be accomplished by proving $\limsup_{t\to T-} V(x,t) \leq \varphi(x)$ and $\liminf_{t\to T-} V(x,t) \geq \varphi(x)$. We begin to prove the two inequalities respectively.

(i) Proof of the first inequality.

Define

$$\zeta_s = e^{-(r+\frac{1}{2}\mu'(\sigma'\sigma)^{-1}\mu)s-\mu'\sigma^{-1}W_s},$$

then we get

$$d\zeta_s = \zeta_s[-rds - \mu'\sigma^{-1}dW_s],$$

and

$$\begin{aligned} d(\zeta_s X_s) &= \zeta_s dX_s + X_s d\zeta_s + d\zeta_s dX_s \\ &= \zeta_s[(rX_s + \mu'\pi_s)ds + \pi_s'\sigma dW_s - rX_s ds - \mu'\sigma^{-1}X_s dW_s - (\mu'\sigma^{-1})(\pi_s'\sigma)'ds] \\ &= \zeta_s[\pi_s'\sigma - \mu'\sigma^{-1}X_s]dW_s. \end{aligned} \tag{3.1.8}$$

Thus, $\zeta_s X_s$ is a martingale. For any admissible π, by Jensen's inequality, we have

$$\mathbb{E}\Big[\varphi\Big(\frac{\zeta_T}{\zeta_t}X_T\Big)\Big] \leq \varphi\Big(\mathbb{E}\Big[\frac{\zeta_T}{\zeta_t}X_T\Big]\Big) = \varphi(x).$$

Then

$$\limsup_{t\to T-}\sup_{\pi}\mathbb{E}\Big[\varphi\Big(\frac{\zeta_T}{\zeta_t}X_T\Big)\Big] \leq \varphi(x). \tag{3.1.9}$$

We come to prove

$$\lim_{t\to T-}\sup_{\pi}\mathbb{E}\left[\left|\varphi(X_T)-\varphi\Big(\frac{\zeta_T}{\zeta_t}X_\tau\Big)\right|\right]=0. \tag{3.1.10}$$

It is not hard to see from (3.1.3) that for all $0<y<x$,

$$|\varphi(x)-\varphi(y)|\le C|x-y|^\gamma.$$

Indeed, by (3.1.3), we derive

$$g(x)\le g(y)+C|x-y|^\gamma\le\varphi(y)+C|x-y|^\gamma.$$

Since $\varphi(y)+C|x-y|^\gamma$ is concave on x for any fixed y, we obtain

$$\varphi(x)\le\varphi(y)+C|x-y|^\gamma.$$

Thus, for any admissible π, we get

$$\mathbb{E}\left[\left|\varphi(X_T)-\varphi\Big(\frac{\zeta_T}{\zeta_t}X_T\Big)\right|\right]\le C\mathbb{E}\left[\Big(\frac{\zeta_T}{\zeta_t}X_T\Big)^\gamma\left|\frac{\zeta_t}{\zeta_T}-1\right|^\gamma\right].$$

Using Hölder inequality, we obtain

$$\begin{aligned}\mathbb{E}\left[\left|\varphi(X_T)-\varphi\Big(\frac{\zeta_T}{\zeta_t}X_T\Big)\right|\right] &\le C\left[\mathbb{E}\Big(\frac{\zeta_T}{\zeta_t}X_T\Big)\right]^\gamma\left(\mathbb{E}\left[\left|\frac{\zeta_t}{\zeta_T}-1\right|^{\frac{\gamma}{1-\gamma}}\right]\right)^{1-\gamma}\\ &\le Cx^\gamma\left(\mathbb{E}\left[\left|\frac{\zeta_t}{\zeta_T}-1\right|^{\frac{\gamma}{1-\gamma}}\right]\right)^{1-\gamma}.\end{aligned}$$

Hence,

$$\lim_{t\to T-}\sup_{\pi}\mathbb{E}\left[\left|\varphi(X_T)-\varphi\Big(\frac{\zeta_T}{\zeta_t}X_T\Big)\right|\right]\le Cx^\gamma\lim_{t\to T-}\left(\mathbb{E}\left[\left|\frac{\zeta_t}{\zeta_T}-1\right|^{\frac{\gamma}{1-\gamma}}\right]\right)^{1-\gamma}=0.$$

Meanwhile, we turn to prove

$$\lim_{t\to T-}\sup_{\pi}\mathbb{E}\left[\int_t^T f_T(X_s,s)ds\right]=0. \tag{3.1.11}$$

Using (3.1.4), we have

$$\begin{aligned}
\mathbb{E}\Big[\int_t^T f_T(X_s,s)ds\Big] &\leq \mathbb{E}\Big[\int_t^T CX_s^\gamma ds\Big] + \mathbb{E}\Big[\int_t^T f_T(0,s)ds\Big] \\
&\leq C\int_t^T \Big(\mathbb{E}\Big[\Big|\frac{\zeta_s}{\zeta_t}X_s\Big|\Big]\Big)^\gamma \Big(\mathbb{E}\Big[\Big|\frac{\zeta_t}{\zeta_s}\Big|^{\frac{\gamma}{1-\gamma}}\Big]\Big)^{1-\gamma} ds + \mathbb{E}\Big[\int_t^T f_T(0,s)ds\Big] \\
&= C\int_t^T x^\gamma \Big(\mathbb{E}\Big[\Big|\frac{\zeta_t}{\zeta_s}\Big|^{\frac{\gamma}{1-\gamma}}\Big]\Big)^{1-\gamma} ds + \int_t^T f_T(0,s)ds.
\end{aligned}$$

Note that the right hand side of the last equality above is independent of π. Let $t \to T-$ and we have (3.1.11).

Therefore, by (3.1.9), (3.1.10) and (3.1.11), we get

$$\begin{aligned}
\limsup_{t\to T-} V(x,t) &= \limsup_{t\to T-}\sup_\pi \mathbb{E}\Big[\int_t^T f_T(X_s,s)ds + g(X_T)\Big] \\
&= \limsup_{t\to T-}\sup_\pi \mathbb{E}[g(X_T)] \\
&\leq \limsup_{t\to T-}\sup_\pi \mathbb{E}[\varphi(X_\tau)] \\
&\leq \limsup_{t\to T-}\sup_\pi \mathbb{E}\Big[\varphi\Big(\frac{\zeta_\tau}{\zeta_t}X_\tau\Big)\Big] + \lim_{t\to T-}\sup_\pi \mathbb{E}\Big[\Big|\varphi(X_\tau) - \varphi\Big(\frac{\zeta_\tau}{\zeta_t}X_\tau\Big)\Big|\Big] \\
&\leq \varphi(x).
\end{aligned}$$

Thus, we prove that $\limsup_{t\to T-} V(x,t) \leq \varphi(x)$.

(ii) Proof of the second inequality.

For fixed $t < T$, if $x \in \{\varphi(x) = g(x)\}$, set $\pi = 0$ and we can get

$$V(x,t) \geq g(x) = \varphi(x).$$

Thus, $\liminf_{t\to T-} V(x,t) \geq \varphi(x)$.

Otherwise, if $x \in (\underline{x}_m, \overline{x}_m)$ for an $m \in \mathbb{Z}$, we choose π_s to make the coefficient of (3.1.8) satisfying

$$\frac{\zeta_s}{\zeta_t}[\pi_s'\sigma - \mu'\sigma^{-1}X_s] = (\pi_s^N)' := N\chi_{\left\{\underline{x}_m < \frac{\zeta_s}{\zeta_t}X_s < \overline{x}_m\right\}} I_n', \quad \forall\, N > 0,$$

where I_n is an n-dimensional unit column vector. Let $X_s^N = \dfrac{\zeta_s}{\zeta_t}X_s$. Then using (3.1.8) we have

$$dX_s^N = (\pi_s^N)'dW_s, \quad t \leq s \leq T.$$

It is not hard to obtain that

$$\underline{x}_m \leq X_s^N \leq \overline{x}_m, \quad t \leq s \leq T.$$

Since

$$\begin{aligned}\{\underline{x}_m < X_s^N < \overline{x}_m\} &= \{\underline{x}_m < X_s^N = x + NI_n'(W_s - W_t) < \overline{x}_m,\ t \leq s \leq T\} \\ &\subset \{\underline{x}_m < x + NI_n'(W_T - W_t) < \overline{x}_m\},\end{aligned}$$

we derive

$$\mathbb{P}(\underline{x}_m < X_T^N < \overline{x}_m) \leq \mathbb{P}(\underline{x}_m < x + NI_n'(W_T - W_t) < \overline{x}_m) \to 0, \quad N \to \infty.$$

Then we get

$$\underline{x}_m\mathbb{P}(X_T^N = \underline{x}_m) + \overline{x}_m\mathbb{P}(X_T^N = \overline{x}_m) \to \mathbb{E}X_T^N = x, \quad N \to \infty.$$

Therefore,

$$\lim_{N\to\infty} \mathbb{P}(X_T^N = \underline{x}_m) = \frac{\overline{x}_m - x}{\overline{x}_m - \underline{x}_m}, \quad \lim_{N\to\infty} \mathbb{P}(X_T^N = \overline{x}_m) = \frac{x - \underline{x}_m}{\overline{x}_m - \underline{x}_m}.$$

As a result,

$$\lim_{N\to\infty} \mathbb{E}[g(X_T^N)] = \frac{\overline{x}_m - x}{\overline{x}_m - \underline{x}_m} g(\underline{x}_m) + \frac{x - \underline{x}_m}{\overline{x}_m - \underline{x}_m} g(\overline{x}_m) = \varphi(x).$$

Thus,

$$\sup_{\pi} \mathbb{E}\Big[g\Big(\frac{\zeta_T}{\zeta_t} X_T\Big)\Big] \geq \lim_{N\to\infty} \mathbb{E}[g(X_T^N)] = \varphi(x).$$

Meanwhile, similar to (3.1.10), we have

$$\lim_{t\to T-} \sup_{\pi} \mathbb{E}\Big[\Big|g(X_T) - g\Big(\frac{\zeta_T}{\zeta_t} X_T\Big)\Big|\Big] = 0.$$

Hence,

$$\begin{aligned}
\liminf_{t\to T-} V(x,t) &\geq \liminf_{t\to T-} \sup_{\pi} \mathbb{E}\Big[g(X_T)\Big] \\
&\geq \liminf_{t\to T-} \sup_{\pi} \mathbb{E}\Big[g\Big(\frac{\zeta_T}{\zeta_t} X_T\Big)\Big] - \lim_{t\to T-} \sup_{\pi} \mathbb{E}\Big[\Big|g(X_T) - g\Big(\frac{\zeta_T}{\zeta_t} X_T\Big)\Big|\Big] \\
&\geq \varphi(x).
\end{aligned}$$

Then we prove that $\liminf_{t\to T-} V(x,t) \geq \varphi(x)$.

As a result, we prove Theorem 3.1.1. □

3.1.2 The HJB Equation

Using the theory of viscosity solution in differential equations, we obtain the following HJB equation

$$-V_t - \max_{\pi} \Big[\frac{1}{2}(\pi'\sigma\sigma'\pi)V_{xx} + \mu'\pi V_x\Big] - rxV_x = f(x,t) \quad \text{in} \quad \Omega_T \qquad (3.1.12)$$

in viscosity sense (See Crandall and Lions (1983), Lions (1983), Fleming and Soner (1992)). Then we will prove that the solution of equation (3.1.12) under boundary condition (3.1.2) and terminal condition (3.1.7) belongs to $C^{2,1}(\Omega_T)\bigcap C(\overline{\Omega_T})$.

Since $f(x,t)$ may not be smooth, concave or convex, we give an easing restriction that

Condition III. $f(x,t)$ is differentiable w.r.t x almost everywhere in Ω_T and there exists the following decomposition

$$f_x(x,t) = P(x,t) - Q(x,t), \quad a.e., \tag{3.1.13}$$

where $P(x,t)$ and $Q(x,t)$ are locally bounded and increasing in x, namely $f_x(x,t)$ is a bounded variation function for all $t < T$. In other words, $f(x,t)$ can be decomposed into a convex (in x) function and a concave (in x) function.

From the definition in (4.1.1), we easily see that V is increasing in x. From equation (3.1.12), we know that the solution of it must be concave. Otherwise, the maximum in the equation will be infinite. So we seek for the solution of (3.1.2) satisfying

$$V_x > 0, \quad V_{xx} < 0, \quad x > 0, \quad 0 < t < T. \tag{3.1.14}$$

Note that the gradient of $\pi'\sigma\sigma'\pi$ with respect to π is

$$\nabla_\pi(\pi'\sigma\sigma'\pi) = 2\sigma\sigma'\pi.$$

Hence,

$$\pi^* = -(\sigma\sigma')^{-1}\mu\frac{V_x}{V_{xx}}.$$

Define $a^2 = \mu'(\sigma\sigma')^{-1}\mu$, thus, we obtain the following initial-boundary problem

$$\begin{cases} -V_t + \dfrac{a^2}{2}\dfrac{V_x^2}{V_{xx}} - rxV_x = f(x,t), \\ V(0,t) = \displaystyle\int_t^T f_T(0,s)ds + g(0), \quad 0 < t < T, \\ V(x,T-) = \varphi(x), \quad x > 0. \end{cases} \tag{3.1.15}$$

We will show that this problem has a (unique) solution $\widehat{V}(x,t) \in C^{2,1}(\Omega_T) \bigcap C(\overline{\Omega_T})$ which satisfies (3.1.14) under **Condition I–III**.

3.2 Main Results

In this section, we will show the main results of this chapter. We will prove the solvability of the initial-boundary problem, and study the behavior of optimal strategy near the boundary and terminal line.

3.2.1 The Solvability of (3.1.15)

Since (3.1.15) is a fully nonlinear PDE problem, the parabolic condition is difficult to verify directly. However, we are able to transform it to a quasi-linear equation through the dual (Fenchel-Legendre) transformation. We will first introduce some knowledge of dual transformation, and then derive the dual equation. Although the derivation depends on some a-priori assumptions on the solution. But we will rigorously prove the existence and uniqueness of solution to the dual problem, and study the related properties. Then we are able to construct the solution of problem (3.1.15) by inverse transformation. Under the verification theorem, we can get that it is the value function defined in (3.1.1). Thus, we could construct the optimal investment strategies by the solution of problem (3.1.15).

The dual transformation of $\varphi(x)$

Firstly, we introduce the concept of dual transformation.

Definition 3.2.1. *If $u : (0, +\infty) \to \mathbb{R}$ is increasing and concave on $(0, +\infty)$, then the dual transformation is the function $\widetilde{u} : (0, +\infty) \to \mathbb{R} \cup \{+\infty\}$ such that*

$$\widetilde{u}(y) = \sup_{x>0}(u(x) - xy), \quad y > 0.$$

The next proposition collects some results used in this section.

Proposition 3.2.2. *$\widetilde{u}$ is a decreasing function, and is convex on $(0, +\infty)$. We have the conjugate relation*

$$u(x) = \inf_{y>0}(\widetilde{u}(y) + xy), \quad x > 0.$$

Denote $dom(\widetilde{u}) = \{y > 0 : \widetilde{u}(y) < +\infty\}$. Suppose one of the two following equivalent conditions is satisfied:

(i) u is differentiable on $(0, +\infty)$,

(ii) $\widetilde{u}$ is strictly convex on $int(dom(\widetilde{u}))$,

then the derivative u' is a mapping from $(0, +\infty)$ into $int(dom(\widetilde{u})) \neq \emptyset$ and we have

$$u'(x) = \arg\min_{y\geq 0}(\widetilde{u}(y) + xy), \quad \forall x > 0.$$

Moreover, we can define $\widetilde{u}'(y\pm) = \lim_{z\to y\pm} \dfrac{\widetilde{u}(z) - \widetilde{u}(y)}{z - y}$, then

$$\widetilde{u}'(y-) \leq \widetilde{u}'(y+) \leq 0, \quad \forall y \in dom(\widetilde{u}),$$

and

$$\arg\max_{x\geq 0}(u(x) - xy) = \{x \geq 0 : u'(x) = y\} = [-\widetilde{u}'(y+), -\widetilde{u}'(y-)], \quad \forall y \in dom(\widetilde{u}).$$

If we further suppose that u is strictly concave, then $\widetilde{u}$ is differentiable with

$$\widetilde{u}'(y) = -(u')^{-1}(y).$$

Finally, under the additional conditions

$$u'(0) = +\infty, \quad u'(+\infty) = 0,$$

we have $int(dom(\widetilde{u})) = dom(\widetilde{u}) = (0, +\infty)$.

Proof. See Appendix B of Pham (2009). □

Now, let us define the duel transformation of $\varphi(x)$ as

$$\psi(y) = \sup_{x>0}(\varphi(x) - xy), \quad y > 0.$$

(see Figure 3.2.1)

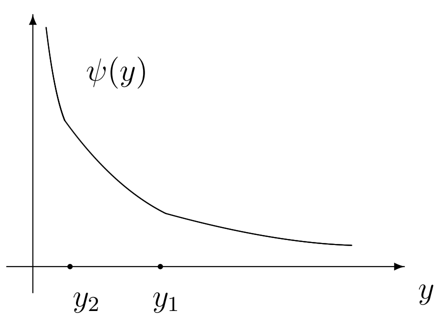


Fig 3.2.1 $\psi(y)$.

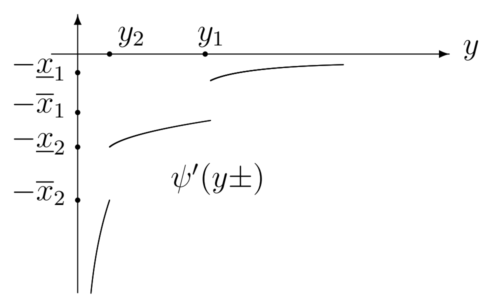


Fig 3.2.2 $\psi'(y\pm)$.

Then, by Proposition 3.2.2, $\psi(y)$ is a decreasing and convex function and

$$\varphi(x) = \inf_{y>0}(\psi(y) + xy).$$

Due to the fact that $\varphi(x)$ is not strictly concave, thus $\psi(y)$ is not continuously differentiable. However, since $\psi(y)$ is convex, we can define

$$\psi'(y\pm) = \lim_{z \to y\pm} \frac{\psi(z) - \psi(y)}{z - y}.$$

Corresponding to the description of $\varphi(x)$ in (3.1.6), we can define

$$y_m = \varphi'(x), \quad x \in (\underline{x}_m, \overline{x}_m), \quad m = 1, 2, ...,$$

and we have

$$\psi'(y_m+) = -\underline{x}_m, \quad \psi'(y_m-) = -\overline{x}_m, \quad m = 1, 2, ...$$

(see Figure 3.2.2).

On the other hand, by (3.1.4), we derive

$$\psi(y) = \sup_{x>0}(\varphi(x) - xy) \leq \sup_{x>0}\Big(g(0) + M\frac{1}{\gamma}x^{\gamma} - xy\Big) = g(0) + M^{\frac{1}{1-\gamma}}\frac{1-\gamma}{\gamma}y^{\frac{\gamma}{\gamma-1}} \tag{3.2.1}$$

Due to (3.1.5), we obtain

$$\psi(y) = \sup_{x>0}\big(\varphi(x) - xy\big) \geq \varphi\Big(\frac{1}{y}\Big) - 1 \to +\infty. \tag{3.2.2}$$

We will use these results later.

The dual problem of (3.1.15)

Now we define a dual transformation of $V(x,t)$. For any $t \in (0, T)$, define

$$v(y,t) = \sup_{x>0}\Big(V(x,t) - xy\Big), \quad y > 0. \tag{3.2.3}$$

Firstly, we take a-priori assumption that $V(x,t)$ is twice continuous differentiable in x and

$$V_x(0+,t) = +\infty, \quad V_x(+\infty,t) = 0. \tag{3.2.4}$$

Then the optimal x to fixed $y > 0$ satisfies

$$\partial_x\Big(V(x,t) - xy\Big) = V_x(x,t) - y = 0.$$

Define a transformation

$$x = I(y,t) := \Big(V_x(\cdot,t)\Big)^{-1}(y). \tag{3.2.5}$$

Owing to (3.1.14) and (3.2.4), $I(y,t)$ is continuously decreasing in y and range onto $(0,+\infty)$. Thus,

$$v(y,t) = V(I(y,t),t) - I(y,t)y. \tag{3.2.6}$$

It follows from (3.2.6) that we have

$$v_y(y,t) = V_x(I(y,t),t)I_y(y,t) - yI_y(y,t) - I(y,t) = -I(y,t), \tag{3.2.7}$$

$$v_{yy}(y,t) = -I_y(y,t) = \frac{-1}{V_{xx}(I(y,t),t)}, \tag{3.2.8}$$

$$v_t(y,t) = V_t(I(y,t),t) + V_x(I(y,t),t)I_t(y,t) - yI_t(y,t) = V_t(I(y,t),t).$$

Thus, for any $y > 0$, set $x = I(y,t)$, then problem (3.1.15) yields

$$\begin{cases} -v_t - \dfrac{a^2}{2}y^2 v_{yy} + ryv_y = f(-v_y,t), \quad y>0,\ 0<t<T, \\ v(y,T-) = \psi(y), \quad y>0. \end{cases} \tag{3.2.9}$$

The solvability of problem (3.2.9)

Theorem 3.2.3. *Problem* (3.2.9) *has a solution* $v \in C^{2,1}(\Omega_T) \bigcap C(\Omega_T \bigcup \{t = T\})$ *satisfying (in* Ω_T*)*

$$\int_t^T f(0,s)ds + \psi(y) \le v(y,t) \le \int_t^T f(0,s)ds + g(0) \tag{3.2.10}$$

$$+M^{\frac{1}{1-\gamma}} e^{A(T-t)} \frac{1-\gamma}{\gamma} y^{\frac{\gamma}{\gamma-1}},$$

$$v_y(y,t) < 0, \tag{3.2.11}$$

$$v_{yy}(y,t) > 0, \tag{3.2.12}$$

where $A = \frac{a^2}{2} \frac{\gamma}{(1-\gamma)^2} + \frac{\gamma r + 1}{1-\gamma}$.

Proof. By the theorem of quasi-linear equation, since $f(x,t)$ is Hölder continuous, we could obtain a solution $v \in C^{2,1}(\Omega_T) \bigcap C(\Omega_T \bigcup \{t = T\})$ to (3.2.9) (See Lieberman (1996); Oleinik (1973)). Now we give the proof of estimate (3.2.10)-(3.2.12). Denote

$$w(y,t) = \int_t^T f(0,s)ds + \psi(y),$$

then

$$w_t(y,t) = -f(0,t), \quad w_y(y,t) \le 0, \quad w_{yy}(y,t) \ge 0.$$

Thus,

$$-w_t - \frac{a^2}{2} y^2 w_{yy} + ryw_y - f(-w_y,t) \le f(0,t) - f(-w_y,t) \le 0,$$

and $w(y,T) = \psi(y)$. Using the comparison principle (see Friedman (1975)), we know that w is a sub-solution of (3.2.9).

On the other hand, denote

$$W(y,t)=\int_t^T f(0,s)ds+g(0)+M^{\frac{1}{1-\gamma}}e^{A(T-t)}\frac{1-\gamma}{\gamma}y^{\frac{\gamma}{\gamma-1}},$$

then we derive

$$\begin{aligned}
&-W_t-\frac{a^2}{2}y^2W_{yy}+ryW_y-f(-W_y,t)\\
\geq\ & f(0,t)+M^{\frac{1}{1-\gamma}}e^{A(T-t)}y^{\frac{\gamma}{\gamma-1}}\Big[\frac{1-\gamma}{\gamma}A-\frac{a^2}{2}\frac{1}{1-\gamma}-r\Big]-f\Big(M^{\frac{1}{1-\gamma}}e^{A(T-t)}y^{\frac{1}{\gamma-1}},t\Big)\\
\geq\ & f(0,t)+M^{\frac{1}{1-\gamma}}e^{A(T-t)}y^{\frac{\gamma}{\gamma-1}}\Big[\frac{1-\gamma}{\gamma}A-\frac{a^2}{2}\frac{1}{1-\gamma}-r\Big]-f(0,t)-M^{\frac{1}{1-\gamma}}\frac{1}{\gamma}e^{A\gamma(T-t)}y^{\frac{\gamma}{\gamma-1}}\\
\geq\ & M^{\frac{1}{1-\gamma}}e^{A(T-t)}y^{\frac{\gamma}{\gamma-1}}\Big[\frac{1-\gamma}{\gamma}A-\frac{a^2}{2}\frac{1}{1-\gamma}-r-\frac{1}{\gamma}\Big]\\
\geq\ & 0,
\end{aligned}$$

where the second inequality is derived due to (3.1.3). By (3.2.1), we can obtain

$$w(y,T)=g(0)+M^{\frac{1}{1-\gamma}}\frac{1-\gamma}{\gamma}y^{\frac{\gamma}{\gamma-1}}\geq\psi(y).$$

By comparison principle, W is a super-solution of (3.2.9).

Since $\psi(y)$ is decreasing, using maximum principle, we could prove $v_y\leq 0$. Differentiate the equation in (3.2.9) w.r.t. y, we have

$$-\partial_t v_y-\frac{a^2}{2}y^2\partial_{yy}v_y+(r-a^2)y\partial_y v_y+rv_y+f_x(-v_y,t)v_{yy}=0. \tag{3.2.13}$$

Using the strong maximum principle, we get $v_y<0$.

Moreover, differentiate the equation (3.2.13) w.r.t. y, we get

$$-\partial_t v_{yy}-\partial_y\Big[\frac{a^2}{2}y^2\partial_y v_{yy}\Big]+(r-a^2)y\partial_y v_{yy}+(2r-a^2)v_{yy}+\partial_y\Big[f_x(-v_y,t)v_{yy}\Big]=0.$$

Using (3.1.4), we obtain

$$-\partial_t v_{yy} - \partial_y\Big[\frac{a^2}{2}y^2\partial_y v_{yy} + Q(-v_y,t)v_{yy}\Big] + (r-a^2)y\partial_y v_{yy} + (2r-a^2)v_{yy} = -\partial_y\Big[P(-v_y,t)v_{yy}\Big].$$

We change it to

$$-\partial_t v_{yy} - \partial_y\Big[\frac{a^2}{2}y^2\partial_y v_{yy} + Q(-v_y,t)v_{yy}\Big] + [(r-a^2)+P(-v_y,t)]y\partial_y v_{yy} + (2r-a^2)v_{yy}$$

$$= -\partial_y\Big[P(-v_y,t)v_{yy}\Big] + P(-v_y,t)\partial_y v_{yy}.$$

Regard it as a linear PDE on v_{yy} with divergence form. Define a parabolic operator $\mathcal{L}$ as

$$\mathcal{L}u := -\partial_t u - \partial_y\Big[\frac{a^2}{2}y^2\partial_y u + Q(-v_y,t)u\Big] + [(r-a^2)+P(-v_y,t)]y\partial_y u + (2r-a^2)u,$$

then we have

$$\mathcal{L}v_{yy} = -\partial_y\Big[P(-v_y,t)v_{yy}\Big] + P(-v_y,t)\partial_y v_{yy}.$$

Since $Q(x,t)$ is increasing in x and $-v_y(y,t)$ is decreasing in y, then $Q(-v_y(y,t),t)$ is decreasing in y. Thus,

$$-\partial_y Q(-v_y,t) \geq 0$$

in weak sense and then we can use maximum principle on $\mathcal{L}$. The same argument yields $-\partial_y P(-v_y,t) \geq 0$, then we get

$$-\partial_y\Big[P(-v_y,t)v_{yy}\Big] + P(-v_y,t)\partial_y v_{yy} = -\Big[\partial_y P(-v_y,t)\Big]v_{yy} \geq 0$$

in weak sense. Therefore,

$$\mathcal{L}v_{yy} \geq 0$$

in weak sense. Since $\psi(y)$ is convex, using the strong maximum principle with divergence form (see Lieberman (1996)), we have $v_{yy} > 0$.

□

Lemma 3.2.4. *The limit condition of v_y is shown below as,*

$$\lim_{y \to 0+} v_y(y,t) = -\infty, \quad 0 < t < T, \tag{3.2.14}$$

$$\lim_{y \to +\infty} v_y(y,t) = 0, \quad 0 < t < T, \tag{3.2.15}$$

$$\lim_{y \to +\infty} y v_y(y,t) = 0, \quad 0 < t < T. \tag{3.2.16}$$

Proof. For any $t \in (0,T)$, it is not hard to see that $\lim\limits_{y \to 0+} v(y,t) \geq \lim\limits_{y \to 0+} \psi(y) = +\infty$. By $v_{yy} > 0$, for some fixed $y_0 > 0$, we derive

$$v_y(y,t) \leq \frac{v(y_0,t) - v(y,t)}{y_0 - y} \to -\infty, \quad y \to 0+.$$

Hence, we prove (3.2.14).

Owing to $v_{yy} > 0$, for any $y > 0$, we have

$$v_y(y,t) \geq \frac{v(y,t) - v\left(\frac{y}{2}, t\right)}{\frac{y}{2}}.$$

Using (3.2.10), we get

$$\begin{aligned} v_y(y,t) \quad &\geq \quad \frac{\psi(y) - g(0) - M^{\frac{1}{1-\gamma}} e^{A(T-t)} \frac{1-\gamma}{\gamma} \left(\frac{y}{2}\right)^{\frac{\gamma}{\gamma-1}}}{\frac{y}{2}} \\ &\geq \quad -C y^{\frac{1}{\gamma-1}} \to 0, \quad y \to +\infty, \end{aligned}$$

where the last inequality above is derived due to (3.2.1). Furthermore,

$$yv_y(y,t) \quad \geq \quad -Cy^{\frac{\gamma}{\gamma-1}} \to 0, \quad y \to +\infty.$$

Combine the result with $v_y < 0$, we obtain (3.2.15) and (3.2.16).

□

The solution of problem (3.1.15)

Now, we set

$$\widehat{V}(x,t) = \inf_{y>0} \Big(v(y,t) + xy \Big). \tag{3.2.17}$$

We come to prove that $\widehat{V}(x,t)$ defined in (3.2.17) is the solution of problem (3.1.15).

According to (3.2.12), (3.2.14) and (3.2.15), we derive

$$y^* = \arg\min_{y>0} \Big(v(y,t) + xy \Big) = J(x,t) := \Big(v_y(\cdot,t) \Big)^{-1}(-x), \quad \forall x > 0,\ 0 < t < T,$$

and

$$\widehat{V}(x,t) = v\Big(J(x,t), t \Big) + xJ(x,t), \tag{3.2.18}$$

where $J(x,t) \in C\Big((0,+\infty) \times (0,T) \Big)$ is decreasing in x.

Lemma 3.2.5. *The limit condition of J is shown as*

$$\lim_{x \to 0+} J(x,t) = +\infty, \quad 0 < t < T, \tag{3.2.19}$$

$$\lim_{x \to 0+} xJ(x,t) = 0, \quad 0 < t < T, \tag{3.2.20}$$

$$\lim_{x \to +\infty} J(x,t) = 0, \quad 0 < t < T. \tag{3.2.21}$$

Proof. (3.2.19), (3.2.20) and (3.2.21) can be derived from (3.2.15), (3.2.16) and (3.2.14), respectively. □

Theorem 3.2.6. *$\widehat{V} \in C^{2,1}(\Omega_T)\bigcap C(\overline{\Omega_T})$ is the solution of (3.1.15) which satisfies*

$$\widehat{V}_x > 0, \quad \widehat{V}_{xx} < 0, \quad (x,t) \in \Omega_T, \tag{3.2.22}$$

and

$$\lim_{x\to 0+} \widehat{V}_x(x,t) = +\infty, \quad \lim_{x\to +\infty} \widehat{V}_x(x,t) = 0, \quad 0 < t < T. \tag{3.2.23}$$

Proof. By (3.2.18), we get

$$\widehat{V}_x(x,t) = v_y(J(x,t),t)J_x(x,t) + xJ_x(x,t) + J(x,t) = J(x,t) > 0, \tag{3.2.24}$$

$$\widehat{V}_{xx}(x,t) = J_x(x,t) = \partial_x\Big[\Big(v_y(\cdot,t)\Big)^{-1}(x)\Big] = \frac{-1}{v_{yy}\Big(J(x,t),t\Big)} < 0, \tag{3.2.25}$$

$$\begin{aligned}\widehat{V}_t(x,t) &= v_y\Big(J(x,t),t\Big)J_t(x,t) + v_t\Big(J(x,t),t\Big) + xJ_t(x,t) \\ &= v_t\Big(J(x,t),t\Big).\end{aligned} \tag{3.2.26}$$

Then we derive $\widehat{V}(x,t) \in C^{2,1}(\Omega_T)$ and

$$\Big(-\widehat{V}_t + \frac{a^2}{2}\frac{\widehat{V}_x^2}{\widehat{V}_{xx}} - rx\widehat{V}_x + \beta\widehat{V}\Big)(x,t) = \Big(-v_t - \frac{a^2}{2}y^2 v_{yy} - (\beta - r)yv_y + \beta v\Big)\Big(J(x,t),t\Big) = 0.$$

Next, we verify the boundary and terminal conditions. Due to (3.2.19), (3.2.20) and (3.2.10), we have

$$\lim_{x\to 0+} \widehat{V}(x,t) = \lim_{x\to 0+}\Big[v\Big(J(x,t),t\Big) + xJ(x,t)\Big] = \lim_{y\to+\infty} v(y,t) = \int_t^T f_T(0,t) + g(0), \quad t \in (0,T).$$

Then $\widehat{V}$ meets the boundary condition in (3.1.15).

According to (3.2.10), which implies $v(y,t) \geq \psi(y)$, we derive

$$\widehat{V}(x,t) = \inf_{y>0} \Big(v(y,t) + xy \Big) \geq \inf_{y>0} \Big(\psi(y) + xy \Big) = \varphi(x).$$

On the other hand, we obtain

$$\widehat{V}(x,t) = \inf_{y>0} \Big(v(y,t) + xy \Big) \leq v\Big(\varphi'(x), t\Big) + x\varphi'(x).$$

Let $t \to T-$, we get

$$\limsup_{t\to T-} \widehat{V}(x,t) \leq \lim_{t\to T-} v\Big(\varphi(x), t\Big) + x\varphi(x) = \psi\Big(\varphi'(x)\Big) + x\varphi'(x) = \varphi(x). \quad (3.2.27)$$

Then $\widehat{V}$ meets the terminal condition in (3.1.15).

(Actually, since $\varphi'(x)$ is continuous and $\lim_{t\to T-} v_y(y) = \psi(y)$ is locally uniform to y, the limit in (3.2.27) will be locally uniform to x.) □

The optimal portfolio in risky assets

Corollary 3.2.7. *The optimal portfolio in risky assets π_t^* is a continuous vector function of the current asset x and the current time t, which can be expressed as*

$$\widehat{\pi}(x,t) = -(\sigma\sigma')^{-1}\mu \frac{\widehat{V}_x(x,t)}{\widehat{V}_{xx}(x,t)}.$$

Thus, $\pi_s^* = \widehat{\pi}(X_s, s)$, $t \leq s \leq T$.

Proof. Here, we give the proof of the verification theorem. Before that, we introduce the dynamic programming principle (see Pham (2009)) to (3.1.1). For any stopping time θ,

$$V(x,t) = \sup_{\pi} \mathbb{E}\Big[\int_t^{T\wedge\theta} f(x,s)ds + V(X_{T\wedge\theta}, T\wedge\theta) \Big].$$

Set $\theta = T - \varepsilon$, then we have

$$V(x,t) = \sup_{\pi} \mathbb{E}\Big[\int_t^{T-\varepsilon} f(x,s)ds + V(X_{T-\varepsilon}, T-\varepsilon)\Big].$$

Let $\varepsilon \to 0$ to the above, by using (3.1.7), we get

$$V(x,t) = \sup_{\pi} \mathbb{E}\Big[\int_t^{T} f(X_s,s)ds + \varphi(X_T)\Big], \quad t < T. \tag{3.2.28}$$

That means definition (3.1.1) is equivalent to (3.2.28), where $g(x)$ is replaced by $\varphi(x)$.

We will prove that $\widehat{V}(x,t)$ constructed by (3.2.17) satisfies (3.2.28). Fix $x > 0$ and $t < T$, for any admissible π_s, let X_s satisfies (2.1.2), i.e.,

$$\begin{cases} dX_s = (r_s X_s + \mu' \pi_s)ds + \pi_s' \sigma_s dW_s, \quad s \geq t, \\ X_t = x. \end{cases}$$

Since $\widehat{V}(x,t) \in C^{2,1}(\Omega_T) \bigcap C(\overline{\Omega_T})$, we can use Itô formula and get

$$\begin{aligned}
&\mathbb{E}\Big[\widehat{V}(X_{T-\varepsilon}, T-\varepsilon) - \widehat{V}(x,t)\Big] \\
= \;& \mathbb{E}\Big[\int_t^{T-\varepsilon} \Big[\widehat{V}_t + \frac{1}{2}(\pi'\sigma\sigma'\pi)\widehat{V}_{xx} + \mu'\widehat{\pi}\widehat{V}_x - rx\widehat{V}_x\Big](X_s,s)ds\Big] \\
\leq \;& -\mathbb{E}\Big[\int_t^{T-\varepsilon} \Big[-\widehat{V}_t - \max_{\pi}\Big(\frac{1}{2}(\pi'\sigma\sigma'\pi)\widehat{V}_{xx} + \mu'\widehat{\pi}\widehat{V}_x\Big) - rx\widehat{V}_x\Big](X_s,s)ds\Big] \\
= \;& -\mathbb{E}\Big[\int_t^{T-\varepsilon} f(X_s,s)ds\Big].
\end{aligned}$$

Let $\varepsilon \to 0$, we have

$$\widehat{V}(x,t) \geq \mathbb{E}\Big[\int_t^{T} f(X_s,s)ds + \varphi(X_T)\Big].$$

Therefore,

$$\widehat{V}(x,t) \geq \sup_{\pi} \mathbb{E}\Big[\int_t^T f(X_s,s)ds + \varphi(X_T)\Big].$$

On the other hand, define

$$\widehat{\pi}(x,t) = -(\sigma\sigma')^{-1}\mu\frac{\widehat{V}_x(x,t)}{\widehat{V}_{xx}(x,t)}.$$

Let $\widehat{X}_s$ be the solution of the following SDE

$$\begin{cases} dX_s = (rX_s + \mu'\widehat{\pi}(X_s,s))ds + \widehat{\pi}'(X_s,s)\sigma dW_s, \quad s \geq t, \\ X_t = x. \end{cases}$$

By Itô's formula, we derive

$$\begin{aligned}
\mathbb{E}\Big[\widehat{V}(x,T-\varepsilon) - \widehat{V}(x,t)\Big] &= \mathbb{E}\Big[\int_t^{T-\varepsilon}\Big[\widehat{V}_t + \frac{1}{2}(\widehat{\pi}'\sigma\sigma'\widehat{\pi})\widehat{V}_{xx} + \mu'\widehat{\pi}\widehat{V}_x - rx\widehat{V}_x\Big](\widehat{X}_s,s)ds\Big] \\
&= \mathbb{E}\Big[\int_t^{T-\varepsilon}\Big[\widehat{V}_t - \frac{a^2}{2}\frac{\widehat{V}_x}{\widehat{V}_{xx}} + rx\widehat{V}_x\Big](\widehat{X}_s,s)ds\Big] \\
&= -\mathbb{E}\Big[\int_t^{T-\varepsilon} f(\widehat{X}_s,s)ds\Big].
\end{aligned}$$

Let $\varepsilon \to 0$, we have

$$\widehat{V}(x,t) = \mathbb{E}\Big[\int_t^T f(\widehat{X}_s,s)ds + \varphi(\widehat{X}_T)\Big].$$

Thus,

$$\widehat{V}(x,t) \leq \sup_{\pi} \mathbb{E}\Big[\int_t^T f(X_s,s)ds + \varphi(X_T)\Big].$$

□

3.2.2 Behavior of Strategy near Boundary and Terminal Line

In this part, we will study the behavior of π^* near $x = 0$ and $t = T$.

Lemma 3.2.8. *The limit condition of v when $y \to +\infty$ is shown below as*

$$\lim_{y\to+\infty} y^2 v_{yy}(y,t) = 0, \quad 0 < t < T. \tag{3.2.29}$$

$$\lim_{y\to+\infty} v_t(y,t) = -f(0,t), \quad 0 < t < T. \tag{3.2.30}$$

Proof. Let $z = \ln y$, $u(z,t) = v(y,t) - \varphi(0) - \int_t^T f_T(0,s)ds$. Thus,

$$v_t = u_t - f(0,t), \quad yv_y = u_z, \quad y^2 v_{yy} = u_{zz} - u_z.$$

Then, by (3.2.9), we get

$$\begin{cases} -u_t - \frac{a^2}{2}u_{zz} + \left(r + \frac{a^2}{2}\right)u_z = f(-e^{-z}u_z, t) - f(0,t), & -\infty < z < +\infty,\ 0 < t < T, \\ u(z, T-) = \psi(e^z) - \varphi(0), & -\infty < z < +\infty. \end{cases} \tag{3.2.31}$$

According to (3.2.10) and (3.2.16), we know that

$$\lim_{z\to+\infty} u(z,t) = 0, \quad 0 < t < T, \tag{3.2.32}$$

$$\lim_{z\to+\infty} u_z(z,t) = 0, \quad 0 < t < T. \tag{3.2.33}$$

In order to prove (3.2.29) and (3.2.30), we only need to prove

$$\lim_{z\to+\infty} u_{zz}(z,t) = 0, \quad 0 < t < T, \tag{3.2.34}$$

$$\lim_{z\to+\infty} u_t(z,t) = 0, \quad 0 < t < T. \tag{3.2.35}$$

Set $Q_\delta^{b,c} = \{(z,t)|b < z < c,\ \delta < t < T-\delta\}$. For any $c > b > 0$, we apply $W_p^{2,1}$ interior estimate. For $p > 3$, we obtain

$$|u|_{W_p^{2,1}\left(Q_{\delta/2}^{n-1,n+2}\right)} \leq C\Big(|u|_{L_p\left(Q_{\delta/3}^{n-2,n+3}\right)} + |f(-e^{-z}u_z,t) - f(0,t)|_{L_p\left(Q_{\delta/3}^{n-2,n+3}\right)}\Big),$$

where C is independent of n. Using (3.2.32), (3.2.33) and the Höder continuity of $f(x,t)$, we have

$$\lim_{n\to+\infty} |u|_{W_p^{2,1}\left(Q_{\delta/2}^{n-1,n+2}\right)} = 0.$$

By Sobolev embedding theorem, we get $\lim\limits_{n\to+\infty} |u_z|_{\alpha,\, Q_{\delta/2}^{n-1,n+2}} = 0$, $(0 < \alpha < 1 - \dfrac{3}{p})$. Applying Schauder interior estimate, we get

$$|u|_{2+\alpha,\, Q_\delta^{n,n+1}} \leq C\Big(|u|_{0,\, Q_{\delta/2}^{n-1,n+2}} + |f(-e^{-z}u_z,t) - f(0,t)|_{\alpha,\, Q_{\delta/2}^{n-1,n+2}}\Big).$$

Thus we have $\lim\limits_{n\to+\infty} |u|_{2+\alpha,\, Q_\delta^{n,n+1}} = 0$, which implies (3.2.34) and (3.2.35). □

Theorem 3.2.9. *The limit condition of $\widehat{\pi}$ near $x = 0$ is shown below as*

$$\lim_{x\to 0+} \widehat{\pi}(x,t) = \lim_{x\to 0+}\left[-(\sigma\sigma')^{-1}\mu\frac{\widehat{V}_x(x,t)}{\widehat{V}_{xx}(x,t)}\right] = 0, \quad 0 < t < T. \tag{3.2.36}$$

Proof. By (3.2.29), we derive

$$\lim_{x\to 0+}\frac{\widehat{V}_x(x,t)}{\widehat{V}_{xx}(x,t)} = \lim_{y\to+\infty}\Big(-yv_{yy}(y,t)\Big) = \lim_{y\to+\infty}\frac{1}{y}\lim_{y\to+\infty}\Big(-y^2v_{yy}(y,t)\Big) = 0.$$

□

Lemma 3.2.10. *The limit condition of v_y near $t = T$ is shown below as*

$$\psi'(y-) \leq \liminf_{t\to T-} v_y(y,t) \leq \limsup_{t\to T-} v_y(y,t) \leq \psi'(y+), \quad y > 0. \tag{3.2.37}$$

Proof. For fixed $y > 0$, if (3.2.37) is not true, then there exists a sequence $\{t_n\}_{n\in\mathbb{Z}^+}$ such that $\lim\limits_{n\to\infty} t_n = T$, and

$$\theta := \lim_{n\to\infty} v_y(y,t_n) > \psi'(y+) \quad (\text{or } < \psi'(y-)).$$

Note that

$$v(y,t_n) = \widehat{V}\Big(-v_y(y,t_n),t_n\Big) + v_y(y,t_n)y.$$

Let $n \to \infty$, we have

$$\lim_{n\to\infty} v(y,t_n) = \varphi(-\theta) + \theta y < \max_{x\geq 0}\Big(\varphi(x) - xy\Big) = \psi(y),$$

which contradicts with $\lim\limits_{y\to T-} v(y,t) = \psi(y)$. Thus, we prove the lemma. □

Lemma 3.2.11. *The limit condition of V_x near $t = T$ is shown below as*

$$\lim_{t\to T-} V_x(x,t) = \varphi'(x) = y_m = \frac{\varphi(\overline{x}_m) - \varphi(\underline{x}_m)}{\overline{x}_m - \underline{x}_m}, \quad x \in (\underline{x}_m, \overline{x}_m),\ m \in \mathbb{Z}. \tag{3.2.38}$$

Proof. For $x \in (\underline{x}_m, \overline{x}_m)$, due to Lemma 3.2.10, for any small $\varepsilon > 0$, we derive

$$\begin{aligned}
\liminf_{t\to T-} v_y(y_m+\varepsilon,t) &\geq \psi'\Big((y_m+\varepsilon)-\Big) \geq \psi'(y_m+) \\
&= -\underline{x}_m > -x > -\overline{x}_m \\
&= \psi'(y_m-) \geq \psi'\Big((y_m-\varepsilon)+\Big) \geq \limsup_{t\to T-} v_y(y_m-\varepsilon,t).
\end{aligned}$$

When t is sufficiently close to T, we obtain

$$v_y(y_m + \varepsilon, t) > -x > v_y(y_m - \varepsilon, t).$$

Due to the fact that v_y is increasing to y, we get

$$y_m + \varepsilon > \Big(v_y(\cdot, t)\Big)^{-1}(-x) > y_m - \varepsilon,$$

namely

$$y_m + \varepsilon > V_x(x, t) > y_m - \varepsilon.$$

Thus,

$$y_m + \varepsilon \geq \limsup_{t \to T-} V_x(x, t) \geq \liminf_{t \to T-} V_x(x, t) \geq y_m - \varepsilon.$$

Since ε is arbitrary, we prove (3.2.38).

□

Theorem 3.2.12. *When $t \to T-$, we get $\dfrac{1}{\widehat{\pi}(x,t)} \to 0$ in $L^1([b,c])$ for any fixed $[b,c] \subset (\underline{x}_m, \overline{x}_m)$.*

Proof. Owing to Lemma 3.2.11, we can prove that

$$\begin{aligned} \int_b^c \frac{1}{\widehat{\pi}(x,t)} dx &= \frac{-1}{(\sigma\sigma')^{-1}\mu} \int_b^c \frac{\widehat{V}_{xx}(x,t)}{\widehat{V}_x(x,t)} dx \\ &= \frac{-1}{(\sigma\sigma')^{-1}\mu} \Big[\ln\Big(\widehat{V}_x(c,t)\Big) - \ln\Big(\widehat{V}_x(b,t)\Big) \Big] \to 0, \quad t \to T- . \end{aligned}$$

□

3.3 An Example: Carpenter's Model with Uncontrollable Exit Time

The wealth of the manager when he/she leaves the market is the addition of the payoff of a call option on the assets and a constant $K > 0$, which represents the personal wealth and the fixed compensation. Suppose that the strike price $b > 0$ is postulated as a constant, then the wealth is denoted as

$$W_\tau = (X_\tau - b)^+ + K,$$

where τ is the exit time.

The manager will choose an investment policy to maximize his/her expected utility of wealth at any future possible exit time. The utility function U which shows the behavior of the risk-averse manager, is strictly increasing and strictly concave. It can be expressed as

$$U(W) = \frac{1}{\gamma} W^\gamma$$

with $0 < \gamma < 1$.

We suppose that there is an exit time and the investor may be forced to leave the financial market by some uncontrollable reasons. At any present time t, the exit time denoted by τ is usually supposed as a random variable under exponential distribution with mean value $1/\lambda$, and it is assumed to be independent of $\{\mathcal{F}_t\}$. We can get that

$$P(\tau \leq s) = \begin{cases} 1 - e^{-\lambda(s-t)}, & s < T; \\ e^{-\lambda(T-t)}, & s = T. \end{cases}$$

So the value function is

$$
\begin{aligned}
V_d(x,t) &= \sup_{\pi} Ee^{-\beta(T-t)}g(X_\tau) \\
&= \sup_{\pi} E\Big[\int_t^T \lambda e^{-(\beta+\lambda)(s-t)}g(X_s)ds + e^{-(\beta+\lambda)(T-t)}g(X_T)\Big],
\end{aligned}
$$

where $\beta > 0$ is the discounted factor, and

$$
g(x) := U\Big[(x-b)^+ + K\Big] = \frac{1}{\gamma}\Big((x-b)^+ + K\Big)^\gamma.
$$

Denote $\varphi(x)$ as its concave hull, see Figure 6.1.

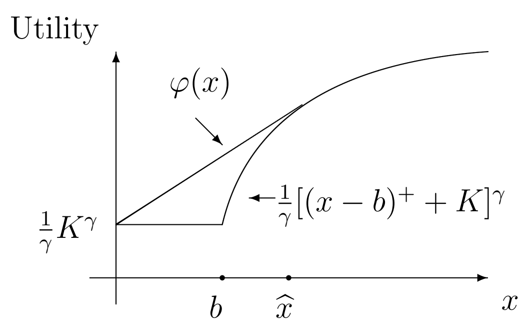


Fig 3.1 $\varphi(x)$.

Let $V(x,t) = e^{(\beta+\lambda)(T-t)}V_d(x,t)$, then

$$
V(x,t) = \sup_{\pi} E\Big[\int_t^T \lambda e^{-(\beta+\lambda)(s-T)}g(X_s)ds + g(X_T)\Big].
$$

Note that

$$
g(x) \leq \frac{1}{\gamma}(x^\gamma + K^\gamma),
$$

and

$$
g(x) = \Big[\frac{1}{\gamma}\Big((x-b)^+ + K\Big)^\gamma - K^{\gamma-1}(x-b)^+\Big] - \Big[-K^{\gamma-1}(x-b)^+\Big],
$$

where $\frac{1}{\gamma}\Big((x-b)^+ + K\Big)^\gamma - K^{\gamma-1}(x-b)^+$ and $-K^{\gamma-1}(x-b)^+$ are concave functions w.r.t. x. Therefore, $g(x)$ and $f(x,t) = e^{-(\beta+\lambda)(T-t)}g(x)$ satisfy (3.1.3)-(3.1.4) and (3.1.13). Thus, $V(x,t) \in C^{2,1}(\Omega_T)\bigcap C(\overline{\Omega_T})$ and the optimal portfolio vector π_t^* converges to 0 when $x \to 0$.

Chapter 4

Optimal Investment Problems with Constraints over a Finite Time Horizon

In this chapter, we study the investment problems with constraints in a finite time horizon. We define the value function and derive the related HJB equation. We construct the approximation problems about the value function and its derivatives, discuss the existence and uniqueness of the solution to the approximation problems and the original problem, and derive the existence and smoothness of the free boundary.

4.1 Formulation of HJB Equations

The market consists of two continuously traded securities. One is a risk-free bank account and the other is a risky stock.

In this chapter, we assume that the portfolio π_s satisfies

$$\pi_s \leq kX_s + b, \quad s \in [t, T],$$

where $k,\ b > 0$ are deterministic constants. Define an admissible investment set as

$$\Pi_t := \Big\{ \pi_s \in L^2_{\mathcal{F}}\big([t, T]; \mathbb{R}\big) : \pi_s \leq kX_s + b,\ X_s \geq 0 \Big\}.$$

Under the utility $U(x) := \dfrac{x^{1-\gamma}}{1-\gamma}$, $\gamma > 0$, $\gamma \neq 1$, the agent's objective is to find an admissible portfolio $\pi(\cdot)$, among all admissible portfolios such that

$$V(x,t) = \sup_{\pi \in \Pi_t} \mathbb{E}\left[e^{-\beta(T-t)} \frac{X_T^{1-\gamma}}{1-\gamma} \middle| X_t = x \right], \tag{4.1.1}$$

where $\beta > 0$ is the discounted rate. We just need to study the case with $\beta = 0$, since we can take the transformation of $\widehat{V} = e^{\beta(T-t)} V$.

If there is no restriction for π_t, the explicit solution of (4.1.1) can be expressed by

$$\overline{V} := e^{\rho(T-t)} \frac{x^{1-\gamma}}{1-\gamma},$$

where $\rho := \dfrac{\mu^2}{2\sigma^2} \dfrac{1-\gamma}{\gamma} + r(1-\gamma)$. In this case, the optimal investment is $\overline{\pi}_t := \kappa X_t$, where $\kappa := \dfrac{\mu}{\sigma^2 \gamma}$.

When $b = 0$, it is obvious to derive the solution of (4.1.1) as

$$\underline{V} := e^{\eta(T-t)} \frac{x^{1-\gamma}}{1-\gamma},$$

where $\eta := -\dfrac{\sigma^2}{2} \underline{k}^2 (1-\gamma)\gamma + (\mu \underline{k} + r)(1-\gamma)$. In this case, the optimal investment is $\underline{\pi}_t := \underline{k} X_t$, where $\underline{k} := \min\{\kappa, k\}$.

Hence, we can get an upper bound and a lower bound on V, where

$$\underline{V} \leq V \leq \overline{V}. \tag{4.1.2}$$

If $k \geq \kappa$, then $\underline{V} = \overline{V}$. This means $V = \overline{V}$. Thus, we only need to discuss the case with $k < \kappa$.

Due to the fact that the utility function $U(x) = \dfrac{x^{1-\gamma}}{1-\gamma}$ is increasing and concave, we know that V is also increasing and concave with respect to x. Thus $V_x(\cdot, t)$ is finite almost everywhere for each t. Moreover, according to the concavity property, we get

$$\frac{V(\lambda x,t)-V(x,t)}{(\lambda-1)x}\leq V_x(x,t)\leq \frac{V(x,t)-V\left(\frac{x}{2},t\right)}{\frac{x}{2}},$$

for $\lambda>1$. Using (4.1.2), we choose λ large enough to satisfy a growth condition on V_x so that

$$\underline{C}x^{-\gamma}\leq V_x\leq \overline{C}x^{-\gamma}, \tag{4.1.3}$$

for $0<\underline{C}<\overline{C}$, where $\underline{C}$ and $\overline{C}$ are independent of x.

4.1.1 Related Equations

Applying dynamic programming principle, we derive the following HJB equation with terminal-boundary condition

$$\begin{cases} -V_t-\max\limits_{0\leq\pi\leq kx+b}\left(\frac{1}{2}\sigma^2\pi^2V_{xx}+\mu\pi V_x\right)-rxV_x=0, & x>0,\ 0<t<T,\\ V_x(0+,t)=+\infty, & 0<t<T,\\ V(x,T)=\dfrac{x^{1-\gamma}}{1-\gamma}, & x>0.\end{cases} \tag{4.1.4}$$

Since V is increasing in x, we have

$$V_x\geq 0. \tag{4.1.5}$$

This leads to the optimal strategy to (4.1.4) satisfying (4.1.5) as follows:

$$\pi^*=\operatorname*{argmax}_{0\leq\pi\leq kx+b}\left(\frac{1}{2}\sigma^2\pi^2V_{xx}+\mu\pi V_x\right)=\begin{cases}-\dfrac{\mu}{\sigma^2}\dfrac{V_x}{V_{xx}}, & \text{if } V_{xx}<0 \text{ and } -\dfrac{\mu}{\sigma^2}\dfrac{V_x}{V_{xx}}<kx+b,\\ kx+b, & \text{if } V_{xx}\geq 0 \text{ or } -\dfrac{\mu}{\sigma^2}\dfrac{V_x}{V_{xx}}\geq kx+b.\end{cases}$$

Then the equation in (4.1.4) can be rewritten as

$$\begin{cases} -V_t + \dfrac{\mu^2}{2\sigma^2}\dfrac{V_x^2}{V_{xx}} - rxV_x = 0, & \text{if } V_{xx} < 0 \text{ and } -\dfrac{\mu}{\sigma^2}\dfrac{V_x}{V_{xx}} < kx + b, \\ -V_t - \dfrac{1}{2}\sigma^2(kx+b)^2 V_{xx} - \mu(kx+b)V_x - rxV_x = 0, & \text{if } V_{xx} \geq 0 \text{ or } -\dfrac{\mu}{\sigma^2}\dfrac{V_x}{V_{xx}} \geq kx + b, \end{cases} \tag{4.1.6}$$

which is a free boundary problem.

For convenience, we define the function

$$A(z,x) := \begin{cases} -\dfrac{\mu}{\sigma^2}z, & \text{if } 0 < -\dfrac{\mu}{\sigma^2}z < kx + b, \\ kx + b, & \text{if } -\dfrac{\mu}{\sigma^2}z \geq kx + b \quad \text{or} \quad z > 0. \end{cases}$$

Then $\pi^* = A\Big(\dfrac{V_x}{V_{xx}}, x\Big)$. The derivatives and boundary conditions can be presented by

$$A_z(z,x) = \begin{cases} -\dfrac{\mu}{\sigma^2}, & \text{if } 0 < -\dfrac{\mu}{\sigma^2}z < kx + b, \\ 0, & \text{if } -\dfrac{\mu}{\sigma^2}z > kx + b \quad \text{or} \quad z > 0; \end{cases}$$

$$A_x(z,x) = \begin{cases} 0, & \text{if } 0 < -\dfrac{\mu}{\sigma^2}z < kx + b, \\ k, & \text{if } -\dfrac{\mu}{\sigma^2}z > kx + b \quad \text{or} \quad z > 0; \end{cases}$$

$$A(\pm\infty, x) := \lim_{z\to\pm\infty} A(z,x) = kx + b;$$

$$A_z(\pm\infty, x) := \lim_{z\to\pm\infty} A_z(z,x) = 0;$$

$$A_x(\pm\infty, x) := \lim_{z\to\pm\infty} A_x(z,x) = k.$$

Thus, $A \in C\big([-\infty,+\infty] \times (0,+\infty)\big)$ and $A_z,\ A_x \in L^\infty\big([-\infty,+\infty] \times (0,+\infty)\big)$. Denote

$$G(u,v,x) := A\Big(\frac{u}{v}, x\Big).$$

This function is Lipschitz continuous in $[\varepsilon, +\infty) \times (-\infty, +\infty) \times [0, L]$ for any fixed $\varepsilon,\ L > 0$, since

$$|G_u(u,v,x)| = \Big|A_z\Big(\frac{u}{v}, x\Big)\frac{1}{v}\Big| = \begin{cases} -\dfrac{\mu}{\sigma^2}\dfrac{1}{v} = -\dfrac{\mu}{\sigma^2}\dfrac{u}{v}\dfrac{1}{u} \le \dfrac{kL+b}{\varepsilon}, & \text{if } 0 < -\dfrac{\mu}{\sigma^2}\dfrac{u}{v} < kx+b, \\ 0, & \text{if } -\dfrac{\mu}{\sigma^2}\dfrac{u}{v} > kx+b \text{ or } v > 0; \end{cases}$$

$$|G_v(u,v,x)| = \Big|-A_z\Big(\frac{u}{v}, x\Big)\frac{u}{v^2}\Big| = \begin{cases} \dfrac{\mu}{\sigma^2}\dfrac{u}{v^2} = \dfrac{\sigma^2}{\mu}(-\dfrac{\mu}{\sigma^2}\dfrac{u}{v})^2\dfrac{1}{u} \le \dfrac{\sigma^2}{\mu}\dfrac{(kL+b)^2}{\varepsilon}, & \\ \qquad \text{if } 0 < -\dfrac{\mu}{\sigma^2}\dfrac{u}{v} < kx+b, & \\ 0, & \text{if } -\dfrac{\mu}{\sigma^2}\dfrac{u}{v} > kx+b \text{ or } v > 0. \end{cases}$$

Now, (4.1.4) can be rewritten as a fully nonlinear equation problem

$$\begin{cases} -V_t - \dfrac{1}{2}\sigma^2 A^2\Big(\dfrac{V_x}{V_{xx}}, x\Big)V_{xx} - \mu A\Big(\dfrac{V_x}{V_{xx}}, x\Big)V_x - rxV_x = 0, \quad \text{in } (0,+\infty)\times[0,T], \\ V_x(0+,t) = +\infty, \quad 0 < t < T, \\ V(x,T) = \dfrac{x^{1-\gamma}}{1-\gamma}, \quad x > 0. \end{cases} \tag{4.1.7}$$

Define the following operator

$$\mathcal{L}V := \frac{1}{2}\sigma^2 A^2\Big(\frac{V_x}{V_{xx}}, x\Big)V_{xx} + \mu A\Big(\frac{V_x}{V_{xx}}, x\Big)V_x + rxV_x.$$

Then we derive the following equations from (4.1.6)

$$\begin{cases} -V_t - \dfrac{\mu^2}{2\sigma^2}\Big(\dfrac{V_x}{V_{xx}}\Big)^2 V_{xxx} + \dfrac{\mu^2}{\sigma^2}V_x - rxV_{xx} - rV_x = 0, \\ \qquad\qquad \text{if } V_{xx} < 0 \text{ and } -\dfrac{\mu}{\sigma^2}\dfrac{V_x}{V_{xx}} < kx+b, \\ -V_t - \dfrac{1}{2}\sigma^2(kx+b)^2 V_{xxx} - (\mu+\sigma^2 k)(kx+b)V_{xx} - rxV_{xx} - rV_x = 0, \\ \qquad\qquad \text{if } V_{xx} \ge 0 \text{ or } -\dfrac{\mu}{\sigma^2}\dfrac{V_x}{V_{xx}} \ge kx+b, \end{cases}$$

which can be merged into

$$-V_t - \frac{1}{2}\sigma^2 A^2\Big(\frac{V_x}{V_{xx}}, x\Big)V_{xxx} - (\mu + \sigma^2 k)A\Big(\frac{V_x}{V_{xx}}, x\Big)V_{xx} - rxV_{xx} - (\mu k + r)V_x = 0.$$

Further, we define

$$\mathcal{T}W := \frac{1}{2}\sigma^2 A^2\Big(\frac{W}{W_x}, x\Big)W_{xx} + (\mu + \sigma^2 k)A\Big(\frac{W}{W_x}, x\Big)W_x + rxW_x + (\mu k + r)W. \tag{4.1.8}$$

Hence, we have

$$\partial_x(\mathcal{L}V) = \mathcal{T}V_x,$$

This means $W = V_x$. Then we obtain

$$\begin{cases} -W_t - \mathcal{T}W = 0, & \text{in } (0, +\infty) \times [0, T], \\ W(0+, t) = +\infty, & 0 < t < T, \\ W(x, T) = x^{-\gamma}, & x > 0. \end{cases} \tag{4.1.9}$$

4.2 Main Results

In this section, we give the main results of this problem.

4.2.1 Approximation method

If we regard equations in (4.1.7) and (4.1.9) as linear equations, coefficients of the second order term will not have positive lower bounds, i.e. (4.1.7) and (4.1.9) will not satisfy the parabolic condition. Therefore, we define

$$\mathcal{L}_\varepsilon V := \mathcal{L}V + \frac{\varepsilon^2}{2}V_{xx}$$

and

$$\mathcal{T}_\varepsilon W := \mathcal{T}W + \frac{\varepsilon^2}{2}W_{xx}.$$

Consider the following approximation problem of (4.1.9) in the bounded domain $Q_\varepsilon := \left(\varepsilon, \dfrac{1}{\varepsilon}\right) \times [0, T]$ as

$$\begin{cases} -W_t^\varepsilon - \mathcal{T}_\varepsilon W^\varepsilon = 0, & \text{in } Q_\varepsilon, \\ W^\varepsilon(\varepsilon, t) = \varepsilon^{-\gamma}, & 0 < t < T, \\ W^\varepsilon(\frac{1}{\varepsilon}, t) = \varepsilon^{\gamma}, & 0 < t < T, \\ W^\varepsilon(x, T) = x^{-\gamma}, & \varepsilon < x < \dfrac{1}{\varepsilon}. \end{cases} \tag{4.2.1}$$

Now, we prove the existence, uniqueness and regularity of the solution to problem (4.1.7) via the above approximation problem (4.2.1).

First, we introduce the lemma below.

Lemma 4.2.1. *There exists a unique solution* $W^\varepsilon \in C^{2,1}(Q_\varepsilon) \bigcap C(\overline{Q_\varepsilon})$ *of problem* (4.2.1)*. Moreover, it satisfies*

$$e^{-N(T-t)} x^{-\gamma} \le W^\varepsilon \le e^{M(T-t)} 2^\gamma (x+\varepsilon)^{-\gamma}, \tag{4.2.2}$$

where $M := \dfrac{\mu^2(\gamma+1)}{2\sigma^2\gamma} + (\mu k + r) + \dfrac{1}{2}\gamma(\gamma+1)$, $N := \dfrac{(\mu+\sigma^2 k)^2}{2\sigma^2}\dfrac{\gamma}{\gamma+1} + r\gamma - (\mu k + r)$.

Proof. Using the theorem of quasi-linear equation, we obtain the existence of solution in $C^{2,1}(Q_\varepsilon) \bigcap C(\overline{Q_\varepsilon})$ (see Lieberman (1996) or Oleinik (1973)). Denote

$$\phi(x,t) := e^{-N(T-t)} x^{-\gamma}.$$

Then we obtain the following results

$$\phi > 0, \quad \phi_x < 0, \quad \phi_{xx} > 0.$$

Furthermore, we have

$$
\begin{aligned}
-\phi_t - \mathcal{T}_\varepsilon \phi =& -\phi_t - \mathcal{T}_\varepsilon \phi \\
=& -\phi_t - \frac{1}{2}\sigma^2 A^2\Big(\frac{\phi}{\phi_x}, x\Big)\phi_{xx} - \frac{\varepsilon^2}{2}\phi_{xx} - (\mu + \sigma^2 k) A\Big(\frac{\phi}{\phi_x}, x\Big)\phi_x \\
& - rx\phi_x - (\mu k + r)\phi \\
\leq& -\phi_t + \max_{a \in \mathbb{R}} \Big(-\frac{1}{2}\sigma^2 a^2 \phi_{xx} - (\mu + \sigma^2 k) a \phi_x \Big) - rx\phi_x - (\mu k + r)\phi \\
\leq& -\phi_t + \frac{(\mu + \sigma^2 k)^2 \phi_x^2}{2\sigma^2 \phi_{xx}} - rx\phi_x - (\mu k + r)\phi \\
=& e^{-N(T-t)} x^{-\gamma} \Big[-N + \frac{(\mu + \sigma^2 k)^2}{2\sigma^2} \frac{\gamma}{\gamma + 1} + r\gamma - (\mu k + r) \Big] \\
=& 0.
\end{aligned}
$$

Since $\phi \leq W^\varepsilon$ in $\partial_p Q_\varepsilon$ ($\partial_p Q_\varepsilon := \Big\{x = \varepsilon\Big\} \bigcup \Big\{x = \frac{1}{\varepsilon}\Big\} \bigcup \Big\{t = T\Big\}$ is the parabolic boundary of Q_ε), we can obtain the first inequality in (4.2.2) using the comparison principle to the quasi-linear equation (see Friedman (1964) or Oleinik (1973)).

Similarly, denote

$$\Phi(x, t) := e^{M(T-t)} 2^\gamma (x + \varepsilon)^{-\gamma},$$

and the corresponding derivatives are given by

$$\Phi > 0, \quad \Phi_x < 0, \quad \Phi_{xx} > 0, \quad A\Big(\frac{\Phi}{\Phi_x}\Big) \leq \frac{\mu}{\sigma^2}\Big|\frac{\Phi}{\Phi_x}\Big|.$$

Moreover, we have

$$
\begin{aligned}
-\Phi_t - \mathcal{T}_\varepsilon \Phi =& - \Phi_t - \mathcal{T}_\varepsilon \Phi \\
=& - \Phi_t - \frac{1}{2}\sigma^2 A^2\Big(\frac{\Phi}{\Phi_x}, x\Big)\Phi_{xx} - \frac{\varepsilon^2}{2}\Phi_{xx} - (\mu + \sigma^2 k) A\Big(\frac{\Phi}{\Phi_x}, x\Big)\Phi_x \\
& - rx\Phi_x - (\mu k + r)\Phi \\
\geq & - \Phi_t - \frac{\mu^2}{2\sigma^2}\Big(\frac{\Phi}{\Phi_x}\Big)^2 \Phi_{xx} - \frac{\varepsilon^2}{2}\Phi_{xx} - (\mu k + r)\Phi \\
\geq & e^{M(T-t)} 2^\gamma \Big[(x+\varepsilon)^{-\gamma}\Big(M - \frac{\mu^2(\gamma+1)}{2\sigma^2\gamma} - (\mu k + r)\Big) - \frac{\varepsilon^2}{2}\gamma(\gamma+1)(x+\varepsilon)^{-\gamma-2}\Big] \\
\geq & e^{M(T-t)} 2^\gamma (x+\varepsilon)^{-\gamma}\Big(M - \frac{\mu^2(\gamma+1)}{2\sigma^2\gamma} - (\mu k + r) - \frac{1}{2}\gamma(\gamma+1)\Big) \\
=& 0.
\end{aligned}
$$

Due to the fact that $2^\gamma(x+\varepsilon)^{-\gamma} \geq x^{-\gamma}$, $\forall x \geq \varepsilon$, we get $\Phi \geq W^\varepsilon$ in $\partial_p Q_\varepsilon$. According to the comparison principle to quasi-linear equation, the second inequality in (4.2.2) holds. □

Next, we derive the following results.

Proposition 4.2.2. *For $\varepsilon > 0$ and $\gamma > 0$, we have*

$$W_x^\varepsilon \leq 0. \tag{4.2.3}$$

Proof. First of all, we claim that

$$W_x^\varepsilon(\varepsilon, t) \leq 0, \quad W_x^\varepsilon\Big(\frac{1}{\varepsilon}, t\Big) \leq 0.$$

Note that $\varepsilon^{-\gamma}$ is a constant super-solution to problem (4.2.1). When $W^\varepsilon(\varepsilon, t) = \varepsilon^{-\gamma}$, we have $W_x^\varepsilon(\varepsilon, t) \leq 0$. Similar discussions will lead to $W_x^\varepsilon\Big(\frac{1}{\varepsilon}, t\Big) \leq 0$. Since $W_x^\varepsilon(x, T) = -\gamma x^{-\gamma-1} < 0$, (4.2.3) is true on the parabolic boundary of Q_ε.

Taking derivation to the equation in (4.2.1), we obtain the following equation on W_x^ε in the divergence form as follows

$$-\partial_t W_x^\varepsilon - \partial_x\Big[\Big(\frac{\sigma^2}{2}A^2\Big(\frac{W^\varepsilon}{W_x^\varepsilon},x\Big)+\frac{\varepsilon^2}{2}\Big)\partial_x W_x^\varepsilon\Big] - \Big[(\mu+\sigma^2 k)A\Big(\frac{W^\varepsilon}{W_x^\varepsilon},x\Big)+rx\Big]\partial_x W_x^\varepsilon$$

$$-(\mu+\sigma^2 k)\Big[A_z\Big(\frac{W^\varepsilon}{W_x^\varepsilon},x\Big)\Big(1-\frac{W^\varepsilon W_{xx}^\varepsilon}{(W_x^\varepsilon)^2}\Big)+A_x\Big(\frac{W^\varepsilon}{W_x^\varepsilon},x\Big)\Big]W_x^\varepsilon-(\mu k+2r)W_x^\varepsilon=0.$$

By simple calculation, we get

$$-\partial_t W_x^\varepsilon - \partial_x\Big[\Big(\frac{\sigma^2}{2}A^2\Big(\frac{W^\varepsilon}{W_x^\varepsilon},x\Big)+\frac{\varepsilon^2}{2}\Big)\partial_x W_x^\varepsilon\Big]$$

$$-\Big[(\mu+\sigma^2 k)A\Big(\frac{W^\varepsilon}{W_x^\varepsilon},x\Big)-(\mu+\sigma^2 k)A_z\Big(\frac{W^\varepsilon}{W_x^\varepsilon},x\Big)\frac{W^\varepsilon}{W_x^\varepsilon}+rx\Big]\partial_x W_x^\varepsilon$$

$$-\Big[(\mu+\sigma^2 k)\Big(A_z\Big(\frac{W^\varepsilon}{W_x^\varepsilon},x\Big)+A_x\Big(\frac{W^\varepsilon}{W_x^\varepsilon},x\Big)\Big)+(\mu k+2r)\Big]W_x^\varepsilon=0. \quad (4.2.4)$$

Note that $A\Big(\frac{W^\varepsilon}{W_x^\varepsilon},x\Big)$, $A_z\Big(\frac{W^\varepsilon}{W_x^\varepsilon},x\Big)\frac{W^\varepsilon}{W_x^\varepsilon}$ and $A_x\Big(\frac{W^\varepsilon}{W_x^\varepsilon},x\Big)$ are bounded. Using the maximum principle of divergence form (see Friedman (1964) or Oleinik (1973)), we obtain the desired result (4.2.3). □

Proposition 4.2.3. *The inequality is derived as follows,*

$$W_t^\varepsilon \le NW^\varepsilon. \quad (4.2.5)$$

Proof. Denote $w(x,t) := e^{N(T-t)}W^\varepsilon(x,t)$ and $\overline{w}(x,t) = w(x,t-h)$. Then both w and $\overline{w}$ satisfy the same following equation

$$-w_t - \mathcal{T}_\varepsilon w - Nw = 0.$$

The first inequality in (4.2.2) yields $w \ge x^\gamma$, which implies

$$w(x,T) = x^\gamma \le w(x,T-h) = \overline{w}(x,T).$$

Since $w(\varepsilon,t) = e^{N(T-t)}\varepsilon^{-\gamma}$ and $w\Big(\frac{1}{\varepsilon},t\Big) = e^{N(T-t)}\varepsilon^{\gamma}$ are decreasing in t, we have $\overline{w} \geq w$ in $\partial_p Q_\varepsilon$. Therefore, by comparison principle, we get $\overline{w} \geq w$ in Q_ε, i.e. $w := e^{N(T-t)}W^\varepsilon(x,t)$ is decreasing in t, which implies the desired result (4.2.5). □

According to the above propositions, we can get the following lemmas.

Lemma 4.2.4. *For any $d > a > 0$, there exists $\delta > 0$, which is independent of ε (but depends on a, b), such that*

$$A\Big(\frac{W^\varepsilon}{W^\varepsilon_x}, x\Big) \geq \delta, \quad (x,t) \in [a,d] \times [0,T].$$

By Lemma 4.2.1, we know that W^ε has the uniform positive lower bound in $[a,d] \times [0,T]$. Hence, we only need to prove the following result.

Lemma 4.2.5. *For any $a > d > 0$, there exists $C > 0$ which is independent of ε, such that*

$$W^\varepsilon_x \geq -C \quad in \quad [a,d] \times [0,T]. \tag{4.2.6}$$

Proof. Define

$$\mathcal{S}_\varepsilon = \Big\{ -\frac{\mu}{\sigma^2}\frac{W^\varepsilon}{W^\varepsilon_x} < kx + b\Big\} \bigcap \Big\{W^\varepsilon_x < 0\Big\},$$

$$\mathcal{R}_\varepsilon = \Big\{ -\frac{\mu}{\sigma^2}\frac{W^\varepsilon}{W^\varepsilon_x} \geq kx + b\Big\} \bigcup \Big\{W^\varepsilon_x \geq 0\Big\}.$$

It is obvious that (4.2.6) holds in $\mathcal{R}_\varepsilon \bigcap \big([a,d]\times[0,T]\big)$. Now, we focus on $\mathcal{S}_\varepsilon \bigcap \big([a,d]\times [0,T]\big)$. Denote $\theta := \frac{\mu^2}{\sigma^2}$. By the PDE equation of (4.2.1), we have

$$-W^\varepsilon_t - \frac{\theta}{2}\Big(\frac{W^\varepsilon}{W^\varepsilon_x}\Big)^2 W^\varepsilon_{xx} - \frac{\varepsilon^2}{2}W^\varepsilon_{xx} - rxW^\varepsilon_x + (\theta - r)W^\varepsilon = 0, \quad (x,t) \in \mathcal{S}_\varepsilon.$$

Note that $\partial_x\Big(\frac{W^\varepsilon}{W^\varepsilon_x}\Big)=1-\frac{W^\varepsilon}{(W^\varepsilon_x)^2}W^\varepsilon_{xx}$ and $W^\varepsilon_{xx}=-\frac{(W^\varepsilon_x)^2}{W^\varepsilon}\Big[\partial_x\Big(\frac{W^\varepsilon}{W^\varepsilon_x}\Big)-1\Big]$. Then we obtain

$$-\frac{W^\varepsilon_t}{W^\varepsilon}+\frac{1}{2}\Big[\theta+\varepsilon^2\Big(\frac{W^\varepsilon_x}{W^\varepsilon}\Big)^2\Big]\partial_x\Big(\frac{W^\varepsilon}{W^\varepsilon_x}\Big)-\frac{1}{2}\Big[\theta+\varepsilon^2\Big(\frac{W^\varepsilon_x}{W^\varepsilon}\Big)^2\Big]-rx\frac{W^\varepsilon_x}{W^\varepsilon}+(\theta-r)=0,\quad (x,t)\in\mathcal{S}_\varepsilon.$$

Using(4.2.3) and (4.2.5), we get

$$\partial_x\Big(\frac{W^\varepsilon}{W^\varepsilon_x}\Big)\le\frac{2(N-\theta+r)+\Big[\theta+\varepsilon^2\Big(\frac{W^\varepsilon_x}{W^\varepsilon}\Big)^2\Big]}{\theta+\varepsilon^2\Big(\frac{W^\varepsilon_x}{W^\varepsilon}\Big)^2},\quad (x,t)\in\mathcal{S}_\varepsilon. \tag{4.2.7}$$

Let $\lambda:=\max\Big\{1,\frac{2(N+r)-\theta}{\theta}\Big\}$. Then

$$\partial_x\Big(\frac{W^\varepsilon}{W^\varepsilon_x}\Big)\le\lambda,\quad (x,t)\in\mathcal{S}_\varepsilon.$$

Thus,

$$\begin{aligned}\partial_x\Big(\frac{(W^\varepsilon)^{-\lambda}}{W^\varepsilon_x}\Big)&=\partial_x\Big((W^\varepsilon)^{-(\lambda+1)}\frac{W^\varepsilon}{W^\varepsilon_x}\Big)\\&=(W^\varepsilon)^{-(\lambda+1)}\Big[\partial_x\Big(\frac{W^\varepsilon}{W^\varepsilon_x}\Big)-(\lambda+1)\Big]\\&\le-(W^\varepsilon)^{-(\lambda+1)},\quad (x,t)\in\mathcal{S}_\varepsilon.\end{aligned}$$

According to estimation (4.2.2), there exist two constants $C_2>C_1>0$ independent of ε such that

$$C_1\le W^\varepsilon\le C_2,\quad (x,t)\in\Big[\frac{a}{2},d\Big]\times[0,T].$$

Hence, we obtain

$$\partial_x\Big(\frac{(W^\varepsilon)^{-\lambda}}{W^\varepsilon_x}\Big)\le-C_2^{-(\lambda+1)},\quad (x,t)\in\mathcal{S}_\varepsilon\bigcap\Big[\frac{a}{2},d\Big]\times[0,T]. \tag{4.2.8}$$

For any $(x,t) \in \mathcal{S}_\varepsilon \bigcap \Big([a,d]\times[0,T]\Big)$, let $y = \sup\Big\{z \in \Big(\frac{a}{2}, x\Big)\Big|(z,t) \in \mathcal{R}_\varepsilon\Big\}$, then we obtain $\Big\{(z,t)\Big|y < z < x\Big\} \subset \mathcal{S}_\varepsilon$. If $y = \frac{a}{2}$, i.e., $\Big\{(z,t)\Big|\frac{a}{2} < z < x\Big\} \subset \mathcal{S}_\varepsilon$, due to (4.2.8), we get

$$\Big(\frac{(W^\varepsilon)^{-\lambda}}{W^\varepsilon_x}\Big)(x,t) \le \Big(\frac{(W^\varepsilon)^{-\lambda}}{W^\varepsilon_x}\Big)\Big(\frac{a}{2},t\Big) - \Big(x - \frac{a}{2}\Big)C_2^{-(\lambda+1)} \le \Big(x-\frac{a}{2}\Big)C_2^{-(\lambda+1)} \le -\frac{a}{2}C_2^{-(\lambda+1)}.$$

Therefore,

$$W^\varepsilon_x(x,t) \ge -\frac{2}{a}C_2^{(\lambda+1)}(W^\varepsilon)^{-\lambda}(x,t) \ge -\frac{2}{a}C_2^{(\lambda+1)}C_1^{-\lambda},$$

which implies (4.2.6). Otherwise, if $y > \frac{a}{2}$, due to (4.2.8), we obtain

$$\begin{aligned}\Big(\frac{(W^\varepsilon)^{-\lambda}}{W^\varepsilon_x}\Big)(x,t) \le \Big(\frac{(W^\varepsilon)^{-\lambda}}{W^\varepsilon_x}\Big)(y,t) = \Big(\frac{W^\varepsilon}{W^\varepsilon_x}\frac{1}{(W^\varepsilon)^{\lambda+1}}\Big)(y,t)\\ = -\frac{\sigma^2}{\mu}\Big(ky+b\Big)\frac{1}{(W^\varepsilon)^{\lambda+1}(y,t)} \le -\frac{\sigma^2}{\mu}k\frac{a}{2}\frac{1}{C_2^{\lambda+1}},\end{aligned}$$

which also implies (4.2.6).

□

Now, suppose W^ε is the solution of (4.2.1), and define

$$V^\varepsilon(x,t) = \int_1^x W^\varepsilon(y,t)\mathrm{d}y + \int_t^T h_\varepsilon(t)\mathrm{d}t + \frac{1}{1-\gamma}, \tag{4.2.9}$$

where

$$h_\varepsilon(t) := \Big(\frac{1}{2}\sigma^2 A^2\Big(\frac{W^\varepsilon}{W^\varepsilon_x}, x\Big)W^\varepsilon_x + \mu A\Big(\frac{W^\varepsilon}{W^\varepsilon_x}, x\Big)W^\varepsilon + rxW^\varepsilon\Big)(1,t).$$

Then $V^\varepsilon_x = W^\varepsilon$. Hence, we have

$$\partial_x(-V^\varepsilon_t - \mathcal{L}_\varepsilon V^\varepsilon) = -W^\varepsilon_t - \mathcal{T}_\varepsilon W^\varepsilon = 0.$$

Moreover, note that

$$(-V_t^\varepsilon - \mathcal{L}_\varepsilon V^\varepsilon)(1,t) = 0,$$

we derive

$$(-V_t^\varepsilon - \mathcal{L}_\varepsilon V^\varepsilon)(x,t) = (-V_t^\varepsilon - \mathcal{L}_\varepsilon V^\varepsilon)(1,t) + \int_1^x \partial_x(-V_t^\varepsilon - \mathcal{L}_\varepsilon V^\varepsilon)(y,t)\mathrm{d}y = 0.$$

Therefore, V^ε satisfies the following equation.

$$\begin{cases} -V_t^\varepsilon - \mathcal{L}_\varepsilon V^\varepsilon = 0 \quad \text{in} \quad Q_\varepsilon, \\ V^\varepsilon(x,T) = \dfrac{x^{1-\gamma}}{1-\gamma}, \quad \varepsilon < x < \dfrac{1}{\varepsilon}. \end{cases} \tag{4.2.10}$$

Lemma 4.2.6. *There exists $0 < \alpha < 1$ such that, for any $[a,d] \subset (0,+\infty)$,*

$$|V^\varepsilon|_{C^{3+\alpha,\frac{3+\alpha}{2}}([a,d]\times[0,T])} \le C, \tag{4.2.11}$$

where C is independent of ε.

Proof. Note that W^ε is uniformly bounded in any bounded region $[a,d]\times[0,T] \subset (0,+\infty)\times[0,T]$. Since the coefficients of the second derivative of (4.2.1) have uniform positive upper and lower bounds which are independent of ε in $[a,d]\times[0,T]$, i.e., (4.2.1) satisfies the uniform parabolic condition in $[a,d]\times[0,T]$. Taking $C^{\alpha,\frac{\alpha}{2}}$ interior estimate (see Friedman (1964) or Lieberman (1996)), we obtain

$$|W^\varepsilon|_{C^{\alpha,\frac{\alpha}{2}}([a,d]\times[0,T])} \le C, \tag{4.2.12}$$

where C is independent of ε. Using $C^{\alpha,\frac{\alpha}{2}}$ interior estimate to (4.2.4) yields

$$|W_x^\varepsilon|_{C^{\alpha,\frac{\alpha}{2}}([a,d]\times[0,T])} \le C. \tag{4.2.13}$$

Therefore, according to the definition in (4.2.9), V^ε is uniformly bounded in $[a,d]\times[0,T]$. Using $C^{\alpha,\frac{\alpha}{2}}$ interior estimate to (4.2.10), we derive

$$|V^\varepsilon|_{C^{\alpha,\frac{\alpha}{2}}([a,d]\times[0,T])} \le C.$$

According to (4.2.12), (4.2.13) and the equation in (4.2.10), we have

$$|V_t^\varepsilon|_{C^{\alpha,\frac{\alpha}{2}}([a,d]\times[0,T])} \le C.$$

Hence, $|V^\varepsilon|_{C^{2+\alpha,1+\frac{\alpha}{2}}([a,d]\times[0,T])}$ is uniformly bounded. Furthermore, taking Schauder interior estimate to (4.2.1) (seeFriedman (1964) or Lieberman (1996)), we have

$$|V_x^\varepsilon|_{C^{2+\alpha,1+\frac{\alpha}{2}}([a,d]\times[0,T])} \le C,$$

which implies the desired result (4.2.11). □

4.2.2 Existence and uniqueness of solution to the original problem

Based on analysis in the above sections, we can obtain the following theorem.

Theorem 4.2.7. *There exists a unique solution $V \in C^{3,2}\big((0,+\infty)\times[0,T]\big)$ of problem* (4.1.4)*. Moreover, it satisfies*

$$e^{-N(T-t)}\frac{x^{1-\gamma}}{1-\gamma} - C_T \le V \le e^{M(T-t)}2^\gamma\frac{x^{1-\gamma}}{1-\gamma} + C_T, \tag{4.2.14}$$

$$e^{-N(T-t)}x^{-\gamma} \le V_x \le e^{M(T-t)}x^{-\gamma}, \tag{4.2.15}$$

$$V_{xx} \le 0, \tag{4.2.16}$$

$$V_{xt} \le NV_x, \tag{4.2.17}$$

where M and N are defined in Lemma 4.2.1, $C_T > 0$ only depends on T.

Proof. By Lemma 4.2.6, problem (4.2.10) has at least one solution $V^\varepsilon \in C^{3+\alpha,\frac{3+\alpha}{2}}\left(\left[\varepsilon,\frac{1}{\varepsilon}\right]\times [0,T]\right)$, such that for any region $Q=[a,d]\times[0,T]\subset(0,+\infty)\times[0,T]$, there exists a subsequence, which is denoted by V^ε, satisfying $V^\varepsilon \to V$ in $C^{3,\frac{3}{2}}(Q)$. Therefore, V satisfies the equation and the terminal condition of (4.1.7).

Taking derivative for the PDE equation in (4.1.7) with respect to t, we obtain the following equation

$$-\partial_t V_t - \frac{1}{2}\sigma^2 A^2\Big(\frac{V_x}{V_{xx}}, x\Big)\partial_{xx}V_t - \mu A\Big(\frac{V_x}{V_{xx}}, x\Big)\partial_x V_t - rx\partial_x V_t = 0.$$

Since $V \in C^{3+\alpha,\frac{3+\alpha}{2}}\big((0,+\infty)\times[0,T]\big)$ and $A\Big(\frac{V_x}{V_{xx}}, x\Big)$ belongs to $C^{\alpha,\frac{\alpha}{2}}$ with positive upper and lower bounds in any bounded region contained in $(0,+\infty)\times[0,T]$, we obtain $V_t \in C^{2,1}\big((0,+\infty)\times[0,T]\big)$ using Schauder interior estimate. Therefore, we get $V \in C^{3,2}\big((0,+\infty)\times[0,T]\big)$.

It follows from (4.2.2) and (4.2.5) that we have (4.2.15) and (4.2.17). Also, we derive (4.2.14) from (4.2.9) using estimation (4.2.15) together with the boundedness of $|V^\varepsilon|_{C^{3,\frac{3}{2}}(Q)}$.

Finally, we prove its uniqueness. Suppose that $V_1,\ V_2 \in C^{2,1}((0,+\infty)\times[0,T])$ are two solutions to problem (4.1.4) under growth condition:

$$|V_i| \le C(x^{1-\gamma}+1), \quad i = 1,2, \tag{4.2.18}$$

for some large constant $C > 0$.

Define the barrier function

$$\Phi^L := 4e^{\beta(T-t)}C\frac{x^2+1}{L} \quad \text{in } [0,L]\times[0,T],$$

where $\beta > 0$ is undetermined. Note that

$$\begin{aligned}&-\partial_t\Phi^L - \sup_{0\le\pi\le kx+b}\Big(\frac{1}{2}\sigma^2\pi^2\partial_{xx}\Phi^L + \mu\pi\partial_x\Phi^L\Big) - rx\partial_x\Phi^L\\ =&\frac{4e^{\beta(T-t)}C}{L}\Big(\beta(x^2+1) - \sigma^2(kx+b)^2 - 2\mu(kx+b)x - 2rx\Big).\end{aligned}$$

We choose β large enough to get

$$-\partial_t \Phi^L - \sup_{0\le\pi\le kx+b}\Big(\frac{1}{2}\sigma^2\pi^2\partial_{xx}\Phi^L + \mu\pi\partial_x\Phi^L\Big) - rx\partial_x\Phi^L \ge 0.$$

Introducing $V_2^\varepsilon(x,t) := V_2(x+\varepsilon,t)$, we derive

$$\begin{aligned}
&-\partial_t V_2^\varepsilon - \sup_{0\le\pi\le kx+b}\Big(\frac{1}{2}\sigma^2\pi^2\partial_{xx}V_2^\varepsilon + \mu\pi\partial_x V_2^\varepsilon\Big) - rx\partial_x V_2^\varepsilon \\
\ge& -\partial_t V_2^\varepsilon - \sup_{0\le\pi\le k(x+\varepsilon)+b}\Big(\frac{1}{2}\sigma^2\pi^2\partial_{xx}V_2^\varepsilon + \mu\pi\partial_x V_2^\varepsilon\Big) - r(x+\varepsilon)\partial_x V_2^\varepsilon \\
=&0.
\end{aligned}$$

According to the equation

$$-\partial_t V_1 - \sup_{0\le\pi\le kx+b}\Big(\frac{1}{2}\sigma^2\pi^2\partial_{xx}V_1 + \mu\pi\partial_x V_1\Big) - rx\partial_x V_1 = 0,$$

we obtain

$$\begin{aligned}
&-\partial_t\Big(V_1 - V_2^\varepsilon - \Phi^L\Big) - \sup_{0\le\pi\le kx+b}\Big[\frac{1}{2}\sigma^2\pi^2\partial_{xx}\Big(V_1 - V_2^\varepsilon - \Phi^L\Big) \\
&\qquad + \mu\pi\partial_x\Big(V_1 - V_2^\varepsilon - \Phi^L\Big)\Big] - rx\partial_x\Big(V_1 - V_2^\varepsilon - \Phi^L\Big) \le 0.
\end{aligned}$$

Moreover,

$$(V_1 - V_2^\varepsilon - \Phi^L)(x,T) = \Big(\frac{x^{1-\gamma}}{1-\gamma} - \frac{(x+\varepsilon)^{1-\gamma}}{1-\gamma} - \Phi^L(x,T)\Big) \le 0,$$

and owing to (4.2.18) and $\partial_x V_1(0+,t) = +\infty$, we have $\Big(V_1 - V_2^\varepsilon - \Phi^L\Big)(L,t) \le 0$ and $\partial_x\Big(V_1 - V_2^\varepsilon - \Phi^L\Big)(0+,t) = +\infty \ge 0$, respectively.

Applying the maximum principle, we get $V_1 - V_2^\varepsilon - \Phi^L \le 0$ in $(0,L]\times[0,T]$. For the fixed point $(x,t)\in(0,+\infty)\times[0,T]$, we choose L satisfying $x < L$ to get $\Big(V_1 - V_2^\varepsilon - \Phi^L\Big)(x,t) \le 0$. Taking $L\to+\infty$ and $\varepsilon\to 0$, we have $V_1 \le V_2$.

□

4.2.3 The existence and smoothness of free boundary under a special case

Define

$$\mathcal{S} := \{\pi^* < kx + b\}, \quad \mathcal{R} := \{\pi^* \geq kx + b\},$$

where $\pi^* := \min\left\{ -\frac{\mu}{\sigma^2}\frac{V_x}{V_{xx}},\ kx+b \right\}$. We discuss the existence of these two regions.

Lemma 4.2.8. *For any $t \in [0,T]$,*

$$\lim_{x\to 0} \pi^*(x,t) = 0. \tag{4.2.19}$$

Proof. We first claim that

$$\liminf_{x\to 0} \pi^*(x,t) = 0. \tag{4.2.20}$$

If not, there exists $t_0 \in [0,T]$ and $\delta > 0$, such that

$$-\frac{\mu}{\sigma^2}\frac{V_x}{V_{xx}}(x,t_0) \geq \pi^*(x,t_0) \geq \delta, \quad x \in (0,\delta).$$

Note that

$$\ln\big(V_x(\delta,t_0)\big) - \ln\big(V_x(x,t_0)\big) = \int_x^\delta \frac{V_{xx}}{V_x}(y,t_0)\mathrm{d}y \geq -\frac{\mu}{\sigma^2\delta}(\delta - x), \quad x \in (0,\delta).$$

Using the first inequality in (4.2.15), we derive

$$\ln\big(V_x(\delta,t_0)\big) + N(T-t) + \gamma \ln x \geq -\frac{\mu}{\sigma^2\delta}(\delta - x), \quad x \in (0,\delta).$$

Taking $x \to 0+$, we get a contradiction that $-\infty \geq -\frac{\mu}{\sigma^2}$. Therefore, (4.2.20) holds.

Taking $\varepsilon \to 0$ to (4.2.7), we obtain

$$\partial_x\Big(\frac{V_x}{V_{xx}}\Big) = \partial_x\Big(\frac{W}{W_x}\Big) \leq \frac{2(N+r)-\theta}{\theta} < +\infty \quad \text{in} \quad \mathcal{S}.$$

Thus, we have

$$\lim_{x\to 0}\pi^*(x,t) = \liminf_{x\to 0}\pi^*(x,t) = 0.$$

$\square$

Lemma 4.2.9. *If* $k \geq \kappa = \dfrac{\mu}{\sigma^2\gamma}$*, we have*

$$\pi^*(x,t) < kx + b, \quad \forall x > 0,\ 0 \leq t \leq T.$$

Otherwise, if $k < \kappa = \dfrac{\mu}{\sigma^2\gamma}$*, for any* $t \in [0,T]$*, we have*

$$\big\{x > 0 | \pi^*(x,t) = kx + b\big\} \neq \emptyset. \tag{4.2.21}$$

Proof. If $k \geq \kappa = \dfrac{\mu}{\sigma^2\gamma}$, according to the discussion above, $\pi^*(x,t) = \kappa x < kx + b$.

Now, consider the case of $k < \kappa = \dfrac{\mu}{\sigma^2\gamma}$, we come to prove (4.2.21). If not, there exists $t_0 \in [0,T]$, such that $\pi^*(x,t_0) < kx + b$ for all $x > 0$, i.e.,

$$-\frac{\mu}{\sigma^2}\frac{V_x}{V_{xx}}(x,t_0) < kx + b, \quad x > 0.$$

Note that

$$\begin{aligned}\ln\big(V_x(x,t_0)\big) - \ln\big(V_x(1,t_0)\big) &= \int_1^x \frac{V_{xx}}{V_x}(y,t_0)\mathrm{d}y \\ &< -\frac{\mu}{\sigma^2}\int_1^x \frac{1}{ky+b}\mathrm{d}y = -\frac{\mu}{\sigma^2 k}\Big(\ln(kx+b) - \ln(k+b)\Big).\end{aligned}$$

Using the first inequality in (4.2.15), we have

$$-N(T-t) - \gamma\ln x - \ln\big(V_x(1,t_0)\big) < -\frac{\mu}{\sigma^2 k}\Big(\ln(kx+b) - \ln(k+b)\Big),$$

which contradicts with $\gamma < \dfrac{\mu}{\sigma^2 k}$ when we take $x \to +\infty$. $\square$

Now, define the free boundary line

$$g(t) := \inf\{x > 0 | \pi^*(x,t) = kx + b\}.$$

Due to (4.2.19) and (4.2.21), we obtain $0 < g(t) < +\infty$ when $k < \kappa = \dfrac{\mu}{\sigma^2\gamma}$.

Theorem 4.2.10. *If*

$$q := -\frac{\gamma}{\gamma+1}\frac{1}{\sigma^2}(\mu+\sigma^2 k)^2 + \frac{\mu}{\sigma^2}(\mu+\sigma^2 k) - 2r\gamma > 0$$

(which implies $k < \kappa$), the free boundary line $g(t)$ is unique, i.e.,

$$\mathcal{S} = \big\{x < g(t)\big\}, \quad \mathcal{R} = \big\{x \geq g(t)\big\}. \tag{4.2.22}$$

Moreover, we have

$$0 < g(t) \leq \frac{\mu b}{q},$$

$g(t) \in C^1([0,T])$ and $g(T) = \dfrac{b}{\dfrac{\mu}{\sigma^2\gamma} - k} > 0.$

Proof. Taking $\varepsilon \to 0$ for (4.2.7), we have

$$\partial_x\Big(\frac{V_x}{V_{xx}}\Big) = \partial_x\Big(\frac{W}{W_x}\Big) \leq \frac{2N + 2r - \theta}{\theta}$$

$$= \frac{\dfrac{(\mu+\sigma^2 k)^2}{\sigma^2}\dfrac{\gamma}{\gamma+1} + 2r\gamma - 2(\mu k + r) + 2r - \dfrac{\mu^2}{\sigma^2}}{\dfrac{\mu^2}{\sigma^2}} \quad \text{in} \quad \mathcal{S}.$$

Then

$$\partial_x\Big(-\frac{\mu}{\sigma^2}\frac{V_x}{V_{xx}} - (kx+b)\Big) \geq -\frac{\mu}{\sigma^2}\frac{\dfrac{(\mu+\sigma^2 k)^2}{\sigma^2}\dfrac{\gamma}{\gamma+1} + 2r\gamma - 2(\mu k + r) + 2r - \dfrac{\mu^2}{\sigma^2}}{\dfrac{\mu^2}{\sigma^2}} - k$$

$$= \frac{q}{\mu} > 0 \quad \text{in} \quad \mathcal{S},$$

which implies $\{x \geq g(t)\} \subset \mathcal{R}$, and will imply (4.2.22).

What is more, define $h(x,t) := -\frac{\mu}{\sigma^2}\frac{V_x}{V_{xx}} - (kx+b)$. Since $h(g(t),t) = 0$, $h(0+,t) = -b$ and $\partial_x h(x,t) \geq \frac{1}{\mu} q > 0$ when $x < g(t)$, we have

$$b = h(g(t),t) - h(0+,t) \geq \frac{q}{\mu} g(t),$$

which implies $g(t) \leq \frac{\mu b}{q}$.

Define

$$J := -V_t - \frac{\mu}{2}(kx+b)V_x - rxV_x,$$

we first claim that

$$\pi^* < kx + b \Leftrightarrow J < 0. \tag{4.2.23}$$

Indeed, $\pi^*(x,t) < kx + b$ implies $-\frac{\mu}{\sigma^2}\frac{V_x}{V_{xx}} < kx + b$. According to the first equation in (4.1.6), we have

$$J = -\frac{\mu^2}{2\sigma^2}\frac{V_x^2}{V_{xx}} - \frac{\mu}{2}(kx+b)V_x < 0.$$

On the other hand, $\pi^*(x,t) = kx + b$ implies $-\frac{\mu}{\sigma^2}\frac{V_x}{V_{xx}} \geq kx + b$. It follows from the second equation in (4.1.6) that we have

$$J = \frac{1}{2}\sigma^2(kx+b)^2 V_{xx} + \frac{\mu}{2}(kx+b)V_x \geq 0.$$

Therefore, (4.2.23) holds.

Now, we prove the continuity of $g(t)$. By contradiction, if $g(t)$ is discontinuous at the point $t_0 \in [0,T]$, i.e.,

$$x_1 := \liminf_{t \to t_0} g(t) < x_2 := \limsup_{t \to t_0} g(t), \tag{4.2.24}$$

by the continuity of J, we have $J(x, t_0) = 0,\ \forall x \in [x_1, x_2]$. Hence, $J_x = 0,\ \forall x \in [x_1, x_2]$. On the other hand, when $-\frac{\mu}{\sigma^2}\frac{V_x}{V_{xx}}(x, t_0) = kx + b$, we have

$$
\begin{aligned}
J_x &= -V_{xt} - \frac{\mu}{2}(kx+b)V_{xx} - \frac{\mu}{2}kV_x - rxV_{xx} - rV_x \\
&= -V_{xt} + \frac{\mu^2}{2\sigma^2}V_x - \frac{\mu}{2}kV_x - rxV_{xx} - rV_x \\
&\geq -NV_x + \frac{\mu^2}{2\sigma^2}V_x - \frac{\mu}{2}kV_x - rV_x \\
&= \Big(-N + \frac{\mu^2}{2\sigma^2} - \frac{\mu}{2}k - r\Big)V_x = \frac{1}{2}qV_x > 0, \qquad (4.2.25)
\end{aligned}
$$

where the first inequality follows (4.2.17) and (4.2.16). Thus, (4.2.24) is impossible. To conclude, $g(t) \in C\big([0, T]\big)$.

Now, we prove $g(t) \in C^1\big([0, T]\big)$. Note that

$$J\big(g(t), t\big) = 0, \quad t \in [0, T].$$

Since $V \in C^{3,2}\big((0, +\infty) \times [0, T]\big)$, we have $J \in C^{1,1}\big((0, +\infty) \times [0, T]\big)$. Therefore,

$$J_x\big(g(t), t\big)g'(t) + J_t\big(g(t), t\big) = 0, \quad t \in [0, T].$$

Note that (4.2.25) implies $J_x\big(g(t), t\big) \geq \frac{1}{2}qV_x > 0$. Then we can derive

$$g'(t) = \frac{J_t\big(g(t), t\big)}{J_x\big(g(t), t\big)} \in C\big([0, T]\big).$$

This means $g(t) \in C^1\big([0, T]\big)$.

Finally, we ascertain $g(T)$. According to the terminal condition $V(x, T) = \frac{x^{1-\gamma}}{1-\gamma}$, we obtain $-\frac{\mu}{\sigma^2}\frac{V_x}{V_{xx}}(x,T) = \frac{\mu}{\sigma^2}\frac{1}{\gamma}x$. Thus, $g(T)$ is the root of the equation

$$\frac{\mu}{\sigma^2}\frac{1}{\gamma}x = kx + b.$$

Then we have $g(T) = \dfrac{b}{\dfrac{\mu}{\sigma^2\gamma} - k}$. □

Chapter 5

Optimal Stopping Time with Risky Assets

In this chapter, we discuss the problems with optimal stopping time. The investor is expected to maximize her personal utilities and to minimize the difference between the realized return at the stopping point and her potentially maximum return. The utility function of a quadratic form is considered. Two models are introduced in this chapter. The details are shown below.

5.1 Running Maximum

Consider the case where there are more than one risky assets in the market. There is a capital market in which $m+1$ basic securities (or assets) are traded continuously. One of the securities is a risk-free asset, whose price follows

$$\begin{cases} dS_{0,t} = rS_{0,t}dt, \quad t \geq 0, \\ S_{0,0} = s_0 > 0, \end{cases} \tag{5.1.1}$$

where $r > 0$ is the interest rate. The other m securities are risky assets, whose prices follow

$$\begin{cases} dS_{i,t} = S_{i,t}\Big\{b_i dt + \sum_{j=1}^{m} \sigma_{ij} dW_t^j\Big\}, \quad t \geq 0, \\ S_{i,0} = s_i > 0, \quad i = 1, 2, \cdots, m, \end{cases} \tag{5.1.2}$$

where $b := (b_1, b_2, \cdots, b_m)'$ is the appreciation rate, W^j is the j-th dimensional Brownian motion with $W^j(0) = 0$, $\sigma := (\sigma_{ij})_{m\times m}$ is the volatility, and the diffusion matrix $\sigma'\sigma$ is nondegenerate.

Suppose that an agent enters the market with an initial wealth $x_0 > 0$. The total wealth at time $s \in [0, \widehat{T}]$ is denoted by $x(s)$ and he can stop the investment at any point before the pre-specified date $\widehat{T} > 0$. The trading of assets is self-financed and takes place continuously. The transaction cost and consumptions are ignored in this paper. Let $\pi(s) := (\pi_1(s), \pi_2(s), \cdots, \pi_m(s))'$ be a portfolio of the agent at time s, where $\pi_i(s), \quad i = 1, 2\cdots, m$, is the value in the i-th asset. Then the amount of wealth invested in the risk-free asset is $x(s) - \mathbf{1}'\pi(s)$. Here $\mathbf{1}$ is the m-dimensional column vector whose entries are all 1. Therefore $x(\cdot)$ satisfies

$$\begin{cases} dx(s) = \left\{ rx(s) + \sum_{i=1}^{m}(b_i - r)\pi_i(s) \right\} ds + \sum_{j=1}^{m}\sum_{i=1}^{m}\sigma_{ij}\pi_i(s)dW^j(s), \quad 0 \le s \le \hat{T}, \\ x(0) = x_0, \end{cases} \tag{5.1.3}$$

Define the running maximum wealth process

$$M(s) = \max_{0\le u\le s} x(u), \;\; s \ge 0.$$

5.2 Model 1

In this section, we study a right time for an investor to stop the investment among multi-assets over a given investment horizon so as to obtain maximum profit. We formulate it into a two-stage problem. The main problem is not a standard optimal stopping problem due to the non-adapted term in the objective function and we turn it to a standard one by stochastic analysis. The subproblem with control variable in the drift and volatility terms is solved first via stochastic control method. A numerical example is presented to illustrate the efficiency of the theoretical results.

5.2.1 Model Formulation

The investor's objective is to choose an optimal portfolio and to determine the right time to stop investment. We can formulate it to the following optimal stopping problem among multi assets:

$$\min_{0 \le \hat{\tau} \le \widehat{T}} \quad \mathbb{E}\big[x(\hat{\tau}) - M(\widehat{T})\big]^2 \tag{5.2.1}$$

$$\text{subject to} \quad \begin{cases} \displaystyle\max_{\pi(\cdot)} & \mathbb{E}\left[\dfrac{(x(\widehat{T}))^\gamma}{\gamma}\right], \\ \text{subject to} & \big(x(\cdot), \pi(\cdot)\big) \text{ satisfy (5.1.3)}, \end{cases} \tag{5.2.2}$$

where $0 < \gamma < 1$.

Note that the above two-stage problem setting is very insightful. In this problem setting, the investor expects to maximize his personal utility and to minimize the difference between the realized return at the stopping point and his potentially maximum return. The criterion here we choose is minimizing a quadratic form $\mathbb{E}\big[x(\hat{\tau}) - M(\widehat{T})\big]^2$ over $[0, \hat{\hat{T}}]$ where the maximum wealth $M(T)$ can be zero. One can also use linear form or log form in Dai and Zhong (2012), Du Toit et al. (2009) and Shiryaev et al. (2008). Different performance measures will typically yield different results, and it is up to the investor to decide which performance measure is most appropriate to him. The problem is also not a standard optimal stopping problem due to the non-$\mathcal{F}_t$-adapted term $M(\hat{T})$. We will turn it to a standard one by time-change technique. Since the wealth process involves the control variable in the drift and volatility terms, we need to derive the optimal portfolio first.

5.2.2 An equivalent optimal stopping problem

Before solving the optimal stopping problem, we deal with the subproblem (5) via stochastic control method. Since the payoff is homogeneous, we conjecture the following value function

$$V^1(s,x) = c(s)\frac{x^\gamma}{\gamma}$$

which satisfies the Hamilton-Jacobi-Bellman (HJB) function in Karatzas et al. (1998),

$$V_s^1(s,x) + \max_{\pi(\cdot)}\left\{[rx + \pi'(b - r\mathbf{1})]V_x^1(s,x) + \frac{1}{2}\pi'\sigma\sigma'\pi V_{xx}^1\right\} = 0.$$

It follows that the optimal portfolio is

$$\hat{\pi}(s) = -\frac{V_x^1(s,x)}{V_{xx}^1(s,x)}(\sigma\sigma')^{-1}(b - r\mathbf{1})x(s).$$

By simple calculation we can derive

$$\hat{\pi}(s) = \frac{1}{1-\gamma}(\sigma\sigma')^{-1}(b - r\mathbf{1})x(s). \tag{5.2.3}$$

Substituting (5.3.3) into (5.3.2) yields the wealth process $x(\cdot)$ without the control variable in the drift and volatility terms

$$\begin{cases} dx(s) = x(s)\left\{\left(r + \frac{1}{1-\gamma}\theta'\theta\right)ds + \frac{1}{1-\gamma}\theta' dW(s)\right\}, \\ x(0) = x_0, \end{cases} \tag{5.2.4}$$

where $\theta = \sigma^{-1}(b - r\mathbf{1})$.

By virtue of a time-change technique, there exists a one-dimensional standard Brownian motion $B(s)$, $s \geq 0$, on $(\Omega, \mathcal{F}, P)$ such that

$$\frac{1}{1-\gamma}\theta' W(s) = B(\beta(s)), \quad 0 \leq s \leq \hat{T},$$

where $\beta(s) := \dfrac{1}{(1-\gamma)^2}\theta'\theta$.

Set $t := \dfrac{1}{(1-\gamma)^2}\theta'\theta$, equation (5.3.4) is equivalent to

$$\begin{cases} dx(t) = x(t)\big\{\mu dt + dB(t)\big\}, \\ x(0) = x_0, \end{cases} \tag{5.2.5}$$

where $\mu = \dfrac{r}{\theta'\theta}(1-\gamma)^2 + 1 - \gamma$. Thus, problem (5.3.1) is equivalent to

$$\min_{0\le\tau\le T} \mathbb{E}\big[x(\tau) - M(T)\big]^2 \tag{5.2.6}$$

where $T = \dfrac{1}{(1-\gamma)^2}\theta'\theta\widehat{T}$. This is still not a standard optimal stopping problem because the term $M(T)$ is not $\mathcal{F}_t$ adapted. We use the same approach as in Shiryaev et al. (2008) to get around it. The value function in Yong and Zhou (1999) associated with problem (5.3.6) is

$$\begin{aligned} V(t,x,M) &= \min_{t\le\tau\le T} \mathbb{E}\big[(x(\tau) - M(T))^2 | \mathcal{F}_t\big] \\ &= \min_{t\le\tau\le T} \mathbb{E}\big[x(\tau)^2 - 2x(\tau)M(T) + M(T)^2 | \mathcal{F}_t\big] \\ &= \min_{t\le\tau\le T} \mathbb{E}\big[x(\tau)^2 - 2x(\tau)\mathbb{E}[M(T)|\mathcal{F}_\tau] + \mathbb{E}[M(T)^2|\mathcal{F}_\tau]\big|\mathcal{F}_t\big]. \end{aligned} \tag{5.2.7}$$

Defining $\nu := \mu - \dfrac{1}{2}$, we rewrite (see Steele (2012))

$$x(t) := x(0)\exp(\nu t + B(t)), \quad M(t) := x(0)\exp\Big(\max_{0\le u\le t}(\nu u + B(u))\Big).$$

Denote $\psi(t, x(t), M(t)) = \mathbb{E}[M(T)|\mathcal{F}_t]$ and $\phi(t, x(t), M(t)) = \mathbb{E}[M(T)^2|\mathcal{F}_t]$. Then we have

$$
\begin{aligned}
&\psi(t,x(t),M(t)) = \mathbb{E}(M(T)|\mathcal{F}_t)\\
=\ & \mathbb{E}\Big[x(0)\exp\Big(\max_{0\le u\le T}(\nu u + B(u))\Big)\Big|\mathcal{F}_t\Big]\\
=\ & \mathbb{E}\Big[x(0)\exp\Big(\max\Big\{\max_{0\le u\le t}(\nu u + B(u)), \max_{t\le u\le T}(\nu u + B(u))\Big\}\Big)\Big|\mathcal{F}_t\Big]\\
=\ & \mathbb{E}\Big[x(0)\exp\Big(\max\Big\{\max_{0\le u\le t}(\nu u + B(u)), (\nu t + B(t)) + \max_{0\le u\le T-t}(\nu u + B(u))\Big\}\Big)\Big|\mathcal{F}_t\Big]\\
=\ & \mathbb{E}\Big[x(t)\exp\Big(\max\Big\{\max_{0\le u\le t}(\nu u + B(u)) - (\nu t + B(t)), \max_{0\le u\le T-t}(\nu u + B(u))\Big\}\Big)\Big|\mathcal{F}_t\Big]\\
=\ & \mathbb{E}\Big[x(t)\exp\Big(\max\Big\{y, \max_{0\le u\le T-t}(\nu u + B(u))\Big\}\Big)\Big|y = \max_{0\le u\le t}(\nu u + B(u)) - (\nu t + B(t))\Big]\\
=\ & x(t)G_1\Big(t, \ln\Big(\frac{M(t)}{x(t)}\Big)\Big),
\end{aligned}
\tag{5.2.8}
$$

where

$$
G_1(t,y) = \mathbb{E}\Big[\exp\Big(\max\Big\{y, \max_{0\le u\le T-t}(\nu u + B(u))\Big\}\Big)\Big], \quad (t,y)\in[0,T]\times[0,\infty)
$$

and

$$
\begin{aligned}
&\phi(t,x(t),M(t)) = \mathbb{E}(M(T)^2|\mathcal{F}_t)\\
=\ & \mathbb{E}\Big[x(0)^2\exp\Big(\big(\max_{0\le u\le T}(\nu u + B(u))\big)^2\Big)\Big|\mathcal{F}_t\Big]\\
=\ & \mathbb{E}\Big[x(0)^2\exp\Big(\Big(\max\Big\{\max_{0\le u\le t}(\nu u + B(u)), \max_{t\le u\le T}(\nu u + B(u))\Big\}\Big)^2\Big)\Big|\mathcal{F}_t\Big]\\
=\ & \mathbb{E}\Big[x(0)^2\exp\Big(\Big(\max\Big\{\max_{0\le u\le t}(\nu u + B(u)), (\nu t + B(t)) + \max_{0\le u\le T-t}(\nu u + B(u))\Big\}\Big)^2\Big)\Big|\mathcal{F}_t\Big]\\
=\ & \mathbb{E}\Big[x(t)^2\exp\Big(\Big(\max\Big\{\max_{0\le u\le t}(\nu u + B(u)) - (\nu t + B(t)), \max_{0\le u\le T-t}(\nu u + B(u))\Big\}\Big)^2\Big)\Big|\mathcal{F}_t\Big]\\
=\ & \mathbb{E}\Big[x(t)^2\exp\Big(\Big(\max\Big\{y, \max_{0\le u\le T-t}(\nu u + B(u))\Big\}\Big)^2\Big)\Big|y = \max_{0\le u\le t}(\nu u + B(u)) - (\nu t + B(t))\Big]\\
=\ & x(t)^2G_2\Big(t, \ln\Big(\frac{M(t)}{x(t)}\Big)\Big),
\end{aligned}
\tag{5.2.9}
$$

where

$$G_2(t,y) = \mathbb{E}\Big[\exp\Big(\Big(\max\Big\{y, \max_{0\le u\le T-t}(\nu u + B(u))\Big\}\Big)^2\Big)\Big], \quad (t,y)\in[0,T]\times[0,\infty).$$

It follows (5.3.7) that

$$V(t,x,M) = \min_{t\le\tau\le T}\mathbb{E}\big[x(\tau)^2 - 2x(\tau)\psi(\tau,x(\tau),M(\tau)) + \phi(\tau,x(\tau),M(\tau))|\mathcal{F}_t\big] \quad (5.2.10)$$

satisfies the free boundary partial differential equation form (see Karatzas et al. (1998))

$$\begin{cases} \max\{\mathscr{L}V, V - x^2 + 2x\psi - \phi\} = 0, \\ V_M(t,M,M) = 0, \\ V(T,x,M) = (x-M)^2, \end{cases} \quad (5.2.11)$$

where the operator $\mathscr{L}$ is defined by

$$\mathscr{L}f(t,x,M) = f_t(t,x,M) + \mu x f_x(t,x,M) + \frac{1}{2}x^2 f_{xx}(t,x,M).$$

Since the value function $V(t,x,M)$ (5.3.10) is homogenous, let

$$U(t,\ln z) = V(t,1,z), \quad 0\le t\le T, \quad z\ge 1, \quad (5.2.12)$$

we have

$$V(t,x,M) = x^2 V\Big(t,1,\frac{M}{x}\Big) = x^2 U\Big(t,\ln\Big(\frac{M}{x}\Big)\Big), \quad 0\le t\le T, \quad 0 < x\le M.$$

According to Equation (5.3.10) and expressions of G_1 and G_2, we have

$$\begin{aligned} V(t,x,M) &= \min_{t\le\tau\le T}\mathbb{E}\big[x(\tau)^2 - 2x(\tau)\psi(\tau,x(\tau),M(\tau)) + \phi(\tau,x(\tau),M(\tau))\big|\mathcal{F}_t\big] \\ &= \min_{t\le\tau\le T}\mathbb{E}\Big[x(\tau)^2 - 2x(\tau)^2 G_1\Big(\tau,\ln\Big(\frac{M(\tau)}{x(\tau)}\Big)\Big) + x(\tau)^2 G_2\Big(\tau,\ln\Big(\frac{M(\tau)}{x(\tau)}\Big)\Big)\Big|\mathcal{F}_t\Big] \\ &= \min_{t\le\tau\le T}\mathbb{E}\Big[x(\tau)^2\Big(1 - 2G_1\Big(\tau,\ln\Big(\frac{M(\tau)}{x(\tau)}\Big)\Big) + G_2\Big(\tau,\ln\Big(\frac{M(\tau)}{x(\tau)}\Big)\Big)\Big)\Big|\mathcal{F}_t\Big] \\ &= \min_{t\le\tau\le T}\mathbb{E}\Big[x(\tau)^2 G\Big(\tau,\ln\Big(\frac{M(\tau)}{x(\tau)}\Big)\Big)\Big|\mathcal{F}_t\Big], \end{aligned}$$

(5.2.13)

where $G(t,y) = 1 - 2G_1(t,y) + G_2(t,y)$. Please refer to 5.4 for the explicit form of $G_1(t,y)$ and $G_2(t,y)$. The proofs are similar to those in Shiryaev et al. (2008).

Equation (5.3.12) implies that equation (5.3.6) is equivalent to a standard optimal stopping problem with a terminal payoff G and an underlying (adapted) state process

$$Y(t) = \ln\Big(\frac{M(t)}{x(t)}\Big), \quad Y(0) = 0.$$

Following the standard techniques from the theory of optimal stopping for Markov Processes (see e.g. Peskir and Shiryaev (2006)) we consider the problem below

$$U(t,y) = \inf_{\tau \in \mathcal{T}_{T-t}} \mathbb{E}_{t,y}[G(t+\tau, Y(t+\tau))],$$

where $Y(t) = y$ under the probability $\mathbb{P}_{t,y}$ with $(t,y) \in [0,T] \times [0,\infty)$ given and fixed, and $\mathcal{T}_s$ in general denotes the set of all $\mathcal{F}_\tau$-stopping times, $\tau \in [0,s]$ for $s > 0$.

In fact, U satisfies the following dynamic programming equation (or variational inequalities)

$$\begin{cases} \max & \{\widehat{\mathscr{L}}U, U-G\} = 0, \quad (t,y) \in [0,T] \times [0,\infty), \\ \text{subject to} & U_y(t,0+) = 0, \qquad t \in [0,T), \\ & U(T,y) = G(T,y), \quad y \in (0,\infty), \end{cases} \tag{5.2.14}$$

where the operator $\widehat{\mathscr{L}}$ is defined by

$$\widehat{\mathscr{L}}f(t,y) = f_t(t,y) - (\nu+2)f_y(t,y) + \frac{1}{2}f_{yy}(t,y) + 2(\nu+1)f(t,y).$$

Hence, the original problem is transferred into finding U. Since $x(\cdot)$ has stationary independent increments and $Y(\cdot)$ is a Markovian process, we rewrite

$$U(t,y) = \inf_{0 \le \tau \le T-t} \mathbb{E}[G(t+\tau, Y^y(\tau))], \tag{5.2.15}$$

where $Y(\cdot)$ under $\mathbb{P}$ is explicitly given as

$$Y^y(t) = y \vee \ln\left(\frac{M(t)}{x(t)}\right), \quad t \geq 0.$$

Applying the theory of optimal stopping, we derive the following region in which the investor may sell the shares he holds, given the pre-determined relationship between his target return and the expected maximum return.

Theorem 5.2.1. *For the optimal stopping problem (5.2.15), the holding region is*

$$C = \{(t, y) \in [0, T] \times [0, \infty) : U(t, y) < G(t, y)\},$$

while the selling region is

$$D = \{(t, y) \in [0, T] \times [0, \infty) : U(t, y) = G(t, y)\}.$$

Also, an optimal selling time is

$$\tau^* = \inf\left\{t \in [0, T] : \left(t, \ln\left(\frac{M(t)}{x(t)}\right)\right) \in D\right\}.$$

5.2.3 Numerical Results

In this section, we give one numerical example to in which we change the value of the parameter γ. The main steps are as follows:

step 1 Give all the parameters γ, b and σ in the model (4);

step 2 Compute the optimal portfolio of sub-problem (5) by (6);

step 3 Construct the equation (17) and discretize it;

step 4 Draw the regions based on Theorem 3.1.

We solve the mathematical formulation given in equation (5.3.13) via the finite difference approach by imposing a uniform grid on the (t, y) domain. A Crank-Nicolson scheme is adopted for the discretization of the partial differential equation and the semi-infinite interval for y is truncated at a sufficiently large value of y. The derivative boundary condition is discretized using a forward difference approximation. For the results shown below, we take the grid spacing to be 0.005 for y and 0.001 for t dimensions.

Let $m = 3$. The interest rate of the bond and the appreciation rate of the m stocks are $r = 0.05$ and $(b_1, b_2, b_3)' = (0.1,\ 0.12,\ 0.15)'$, respectively, and the volatility matrix is

$$\sigma = \begin{bmatrix} 0.3000 & 0 & 0 \\ 0.2000 & 0.3464 & 0 \\ 0.2500 & 0.1443 & 0.4082 \end{bmatrix}.$$

Then

$$\theta := \sigma^{-1}(b_1 - r, b_2 - r, b_3 - r)' = (0.1667,\ 0.1058,\ 0.1055)'.$$

Using Theorem 5.3.1 and the parameter value of γ ranging between 0.7 and 0.9, we observe that the selling region decreases as the value of γ increases, as shown by the following 6 figures.

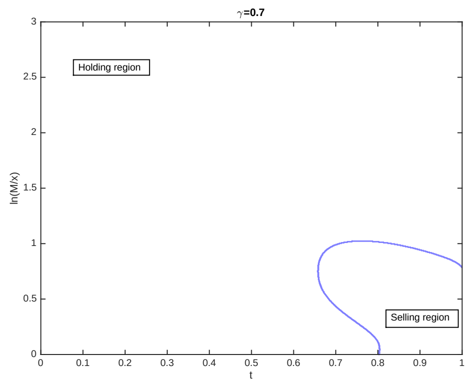


Figure 1: $\gamma = 0.7$

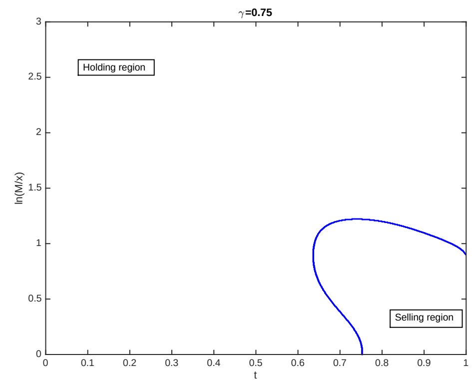
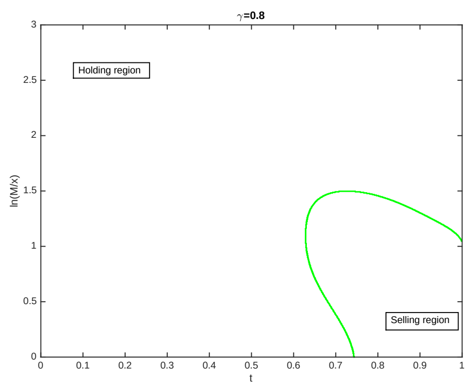


Figure 2: $\gamma = 0.75$



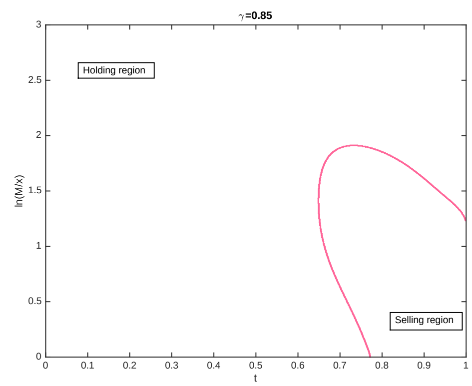


Figure 3: $\gamma = 0.8$

Figure 4: $\gamma = 0.85$

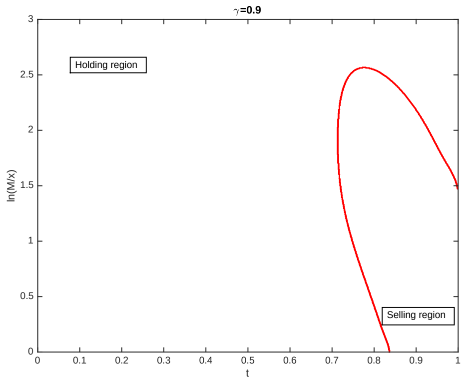


Figure 5: $\gamma = 0.9$

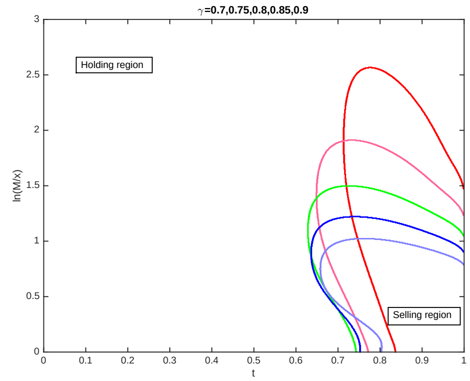


Figure 6: different γ

5.3 Model 2

In this section, we study the right time for an investor to stop the investment over a given investment horizon so as to obtain as close to the highest possible wealth as possible, according to a Logarithmic utility-maximization objective involving the portfolio in the drift and volatility terms. The problem is formulated as an optimal stopping problem, although it is non-standard in the sense that the maximum wealth involved is not adapted to the information generated over time. By delicate stochastic analysis, the problem is converted to a standard optimal stopping one involving adapted processes. Numerical examples shed light on the efficiency of the theoretical results.

5.3.1 Model Formulation

Assume that an investor can stop investment at any point before a pre-specified date $\widehat{T} > 0$. The question is to choose an optimal portfolio and to determine the right time to stop investment. The main objective of this study is to determine conditions for which the investor should sell her shares. Ideally, the investor would like to exit when the value is highest, which is at time s, such that $x(s) = \alpha M(\widehat{T})$. More generally, the investor may have an investment target that is a fraction of (or possibly equal to) the maximum value, $\alpha M(\widehat{T})$, where $0 < \alpha \leq 1$. With this objective, we assume that the investor chooses an exit time to minimize the mean squared difference between exit value and investment target value. We formulate it to the following optimal stopping problem:

$$\min_{0 \leq \hat{\tau} \leq \widehat{T}} \quad \mathbb{E}\big[x(\hat{\tau}) - \alpha M(\widehat{T})\big]^2, \tag{5.3.1}$$

$$\text{subject to} \quad \begin{cases} \displaystyle\max_{\pi(\cdot)} \quad \mathbb{E}\Big[\ln(x(\widehat{T}))\Big], \\ \text{subject to} \quad (x(\cdot), \pi(\cdot)) \;\; \text{satisfy (5.1.3)}. \end{cases} \tag{5.3.2}$$

Note that the above two-stage problem setting is very insightful. It is more realistic than those addressed in Shiryaev, Xu and Zhou (2008) since m-dimensional financial assets are considered and the drift and volatility terms involving the portfolio.

5.3.2 An equivalent optimal stopping problem

Before further developing techniques derived in Shiryaev, Xu and Zhou (2008), we know the optimal portfolio of sub-problem (5.3.2) via stochastic control method

$$\hat{\pi}(s) \equiv (\hat{\pi}_1(s), \hat{\pi}_2(s), \cdots, \hat{\pi}_m(s))' = (\sigma\sigma')^{-1}(b - r\mathbf{1})x(s), \tag{5.3.3}$$

where $\mathbf{1} = (1, 1, \cdots, 1)'$ is an m-dimensional column vector.

Substituting (5.3.3) into (5.3.2) yields the wealth process $x(\cdot)$ without the control variable in the drift and volatility terms

$$\begin{cases} dx(s) = x(s)\big\{(r+|\theta|^2)ds + \theta' dW(s)\big\}, \\ x(0) = x_0, \end{cases} \tag{5.3.4}$$

where $\theta = \sigma^{-1}(b - r\mathbf{1})$.

This is similar to the case in Shiryaev, Xu and Zhou (2008), but it is more mathematically complex. By virtue of a time-change technique, there exists a one-dimensional standard Brownian motion $B(s)$, $s \geq 0$, on $(\Omega, \mathcal{F}, P)$ such that

$$\theta' W(s) = B(\beta(s)), \;\; 0 \leq s \leq \hat{T},$$

where $\beta(s) := |\theta|^2 s$.

Set $t := |\theta|^2 s$, equation (5.3.4) is equivalent to

$$\begin{cases} dx(t) = x(t)\big\{\mu dt + dB(t)\big\}, \\ x(0) = x_0, \end{cases} \tag{5.3.5}$$

where $\mu = \dfrac{r}{|\theta|^2} + 1$. Thus, the problem (5.3.1) is equivalent to

$$\min_{0 \leq \tau \leq T} \mathbb{E}\big[x(\tau) - \alpha M(T)\big]^2 \tag{5.3.6}$$

over $\tau \in \mathcal{T}$, the set of all $\mathcal{F}_t$-stopping time $\tau \in [0,T]$, where $T = |\theta|^2 \widehat{T}$. Consequently, the value function associated with problem (5.3.6) is

$$\begin{aligned} V(t,x,M) &= \min_{t \leq \tau \leq T} \mathbb{E}\big[(x(\tau) - \alpha M(T))^2 |\mathcal{F}_t\big] \\ &= \min_{t \leq \tau \leq T} \mathbb{E}\big[x(\tau)^2 - 2\alpha x(\tau) M(T) + \alpha^2 M(T)^2 |\mathcal{F}_t\big] \\ &= \min_{t \leq \tau \leq T} \mathbb{E}\Big[x(\tau)^2 - 2\alpha x(\tau)\mathbb{E}[M(T)|\mathcal{F}_\tau] + \alpha^2 \mathbb{E}[M(T)^2|\mathcal{F}_\tau]\Big|\mathcal{F}_t\Big]. \end{aligned} \tag{5.3.7}$$

Defining $\nu := \mu - \dfrac{1}{2}$, we rewrite

$$x(t) := x(0)\exp(\nu t + B(t)), \quad M(t) := x(0)\exp\Big(\max_{0 \leq u \leq t}(\nu u + B(u))\Big).$$

Denote $\psi(t, x(t), M(t)) = \mathbb{E}[M(T)|\mathcal{F}_t]$ and $\phi(t, x(t), M(t)) = \mathbb{E}[M(T)^2|\mathcal{F}_t]$. Then

$$
\begin{aligned}
&\psi(t, x(t), M(t)) = \mathbb{E}[M(T)|\mathcal{F}_t] \\
= \ & \mathbb{E}\Big[x(0)\exp\Big(\max_{0\le u\le T}(\nu u + B(u))\Big)\Big|\mathcal{F}_t\Big] \\
= \ & \mathbb{E}\Big[x(0)\exp\Big(\max\Big\{\max_{0\le u\le t}(\nu u + B(u)), \max_{t\le u\le T}(\nu u + B(u))\Big\}\Big)\Big|\mathcal{F}_t\Big] \\
= \ & \mathbb{E}\Big[x(0)\exp\Big(\max\Big\{\max_{0\le u\le t}(\nu u + B(u)), (\nu t + B(t)) + \max_{0\le u\le T-t}(\nu u + B(u))\Big\}\Big)\Big|\mathcal{F}_t\Big] \\
= \ & \mathbb{E}\Big[x(t)\exp\Big(\max\Big\{\max_{0\le u\le t}(\nu u + B(u)) - (\nu t + B(t)), \max_{0\le u\le T-t}(\nu u + B(u))\Big\}\Big)\Big|\mathcal{F}_t\Big] \\
= \ & \mathbb{E}\Big[x(t)\exp\Big(\max\Big\{y, \max_{0\le u\le T-t}(\nu u + B(u))\Big\}\Big)\Big|y = \max_{0\le u\le t}(\nu u + B(u)) - (\nu t + B(t))\Big] \\
= \ & x(t)G_1\Big(t, \ln\Big(\frac{M(t)}{x(t)}\Big)\Big),
\end{aligned}
\tag{5.3.8}
$$

where

$$
G_1(t, y) = \mathbb{E}\Big[\exp\Big(\max\Big\{y, \max_{0\le u\le T-t}(\nu u + B(u))\Big\}\Big)\Big], \quad (t, y) \in [0, T] \times [0, \infty)
$$

and

$$
\begin{aligned}
&\phi(t, x(t), M(t)) = \mathbb{E}[M(T)^2|\mathcal{F}_t] \\
= \ & \mathbb{E}\Big[x(0)^2\exp\Big(\big(\max_{0\le u\le T}(\nu u + B(u))\big)^2\Big)\Big|\mathcal{F}_t\Big] \\
= \ & \mathbb{E}\Big[x(0)^2\exp\Big(\Big(\max\Big\{\max_{0\le u\le t}(\nu u + B(u)), \max_{t\le u\le T}(\nu u + B(u))\Big\}\Big)^2\Big)\Big|\mathcal{F}_t\Big] \\
= \ & \mathbb{E}\Big[x(0)^2\exp\Big(\Big(\max\Big\{\max_{0\le u\le t}(\nu u + B(u)), (\nu t + B(t)) + \max_{0\le u\le T-t}(\nu u + B(u))\Big\}\Big)^2\Big)\Big|\mathcal{F}_t\Big] \\
= \ & \mathbb{E}\Big[x(t)^2\exp\Big(\Big(\max\Big\{\max_{0\le u\le t}(\nu u + B(u)) - (\nu t + B(t)), \max_{0\le u\le T-t}(\nu u + B(u))\Big\}\Big)^2\Big)\Big|\mathcal{F}_t\Big] \\
= \ & \mathbb{E}\Big[x(t)^2\exp\Big(\Big(\max\Big\{y, \max_{0\le u\le T-t}(\nu u + B(u))\Big\}\Big)^2\Big)\Big|y = \max_{0\le u\le t}(\nu u + B(u)) - (\nu t + B(t))\Big] \\
= \ & x(t)^2G_2\Big(t, \ln\Big(\frac{M(t)}{x(t)}\Big)\Big),
\end{aligned}
\tag{5.3.9}
$$

where

$$G_2(t,y) = \mathbb{E}\Big[\exp\Big(\Big(\max\Big\{y, \max_{0\le u\le T-t}(\nu u + B(u))\Big\}\Big)^2\Big)\Big], \quad (t,y)\in[0,T]\times[0,\infty).$$

It follows (5.3.7) that

$$V(t,x,M) = \min_{t\le\tau\le T}\mathbb{E}\big[x(\tau)^2 - 2\alpha x(\tau)\psi(\tau,x(\tau),M(\tau)) + \alpha^2\phi(\tau,x(\tau),M(\tau))|\mathcal{F}_t\big], \tag{5.3.10}$$

which is governed by

$$\begin{cases} \max\{\mathscr{L}V, V - x^2 + 2\alpha x\psi - \alpha^2\phi\} = 0, \\ V_M(t,M,M) = 0, \\ V(T,x,M) = (x-\alpha M)^2, \end{cases} \tag{5.3.11}$$

where the operator $\mathscr{L}$ is defined by

$$\mathscr{L}f(t,x,M) = f_t(t,x,M) + \mu x f_x(t,x,M) + \frac{1}{2}x^2 f_{xx}(t,x,M).$$

The value function $V(t,x,M)$ satisfies

$$V(t,\lambda x,\lambda M) = \lambda^2 V(t,x,M),$$

because scaling both $x(t)$ and $M(t)$ by the same positive constant at a time t prior to the terminal time T results in the payoff $(x(T)-\alpha M(T))^2$ being scaled by the same constant. In particular, if

$$U(t,\ln z) = V(t,1,z), \quad 0\le t\le T, \quad z\ge 1,$$

then we may determine $V(t,x,M)$ as

$$V(t,x,M) = x^2 V\Big(t,1,\frac{M}{x}\Big) = x^2 U\Big(t,\ln\Big(\frac{M}{x}\Big)\Big), \quad 0\le t\le T, \quad 0 < x\le M.$$

According to equation (5.3.10) and expressions of G_1 and G_2, we have

$$
\begin{aligned}
V(t,x,M) &= \min_{t\le\tau\le T} \mathbb{E}\big[x(\tau)^2 - 2\alpha x(\tau)\psi(\tau,x(\tau),M(\tau)) + \alpha^2\phi(\tau,x(\tau),M(\tau))\big|\mathcal{F}_t\big] \\
&= \min_{t\le\tau\le T} \mathbb{E}\Big[x(\tau)^2 - 2\alpha x(\tau)^2 G_1\Big(\tau, \ln\Big(\frac{M(\tau)}{x(\tau)}\Big)\Big) + \alpha^2 x(\tau)^2 G_2\Big(\tau, \ln\Big(\frac{M(\tau)}{x(\tau)}\Big)\Big)\Big|\mathcal{F}_t\Big] \\
&= \min_{t\le\tau\le T} \mathbb{E}\Big[x(\tau)^2\Big(1 - 2\alpha G_1\Big(\tau, \ln\Big(\frac{M(\tau)}{x(\tau)}\Big)\Big) + \alpha^2 G_2\Big(\tau, \ln\Big(\frac{M(\tau)}{x(\tau)}\Big)\Big)\Big)\Big|\mathcal{F}_t\Big] \\
&= \min_{t\le\tau\le T} \mathbb{E}\Big[x(\tau)^2 G\Big(\tau, \ln\Big(\frac{M(\tau)}{x(\tau)}\Big)\Big)\Big|\mathcal{F}_t\Big],
\end{aligned}
\tag{5.3.12}
$$

where $G(t,y) = 1 - 2\alpha G_1(t,y) + \alpha^2 G_2(t,y)$.

Equation (5.3.12) implies that equation (5.3.6) is equivalent to a standard optimal stopping problem with a terminal payoff G and an underlying (adapted) state process

$$
Y(t) = \ln\Big(\frac{M(t)}{x(t)}\Big), \quad Y(0) = 0.
$$

Following the dynamic programming approach we consider the problem below

$$
U(t,y) = \inf_{\tau\in\mathcal{T}_{T-t}} \mathbb{E}_{t,y}[G(t+\tau, Y(t+\tau))],
$$

where $Y(t) = y$ under the probability $\mathbb{P}_{t,x}$ with $(t,y) \in [0,T]\times[0,\infty)$ given and fixed, and $\mathcal{T}_s$ in general denotes the set of all $\mathcal{F}$-stopping times $\tau \in [0,s]$ for $s > 0$.

In fact, U satisfies the following dynamic programming equation (or variational inequalities)

$$
\begin{cases}
\max & \{\widehat{\mathscr{L}}U, U - G\} = 0, \quad (t,y)\in[0,T]\times[0,\infty), \\
\text{subject to} & U_y(t,0+) = 0, \qquad t\in[0,T), \\
 & U(T,y) = G(T,y), \quad y\in(0,\infty),
\end{cases}
\tag{5.3.13}
$$

where the operator $\widehat{\mathscr{L}}$ is defined by

$$
\widehat{\mathscr{L}}f(t,y) = f_t(t,y) - (\nu+2)f_y(t,y) + \frac{1}{2}f_{yy}(t,y) + 2(\nu+1)f(t,y).
$$

Hence, the original problem is transferred into finding U. Since $x(\cdot)$ has stationary independent increments and $Y(\cdot)$ is a Markovian process, we rewrite

$$U(t,y) = \inf_{0\le\tau\le T-t} \mathbb{E}[G(t+\tau, Y^y(\tau))],$$

where $Y(\cdot)$ under $\mathbb{P}$ is explicitly given as

$$Y^y(t) = y \vee \ln\left(\frac{M(t)}{x(t)}\right), \quad t \ge 0.$$

Theoretically, we have derived a region in which the venture capitalist may sell the shares they hold, given the pre-determined relationship between her target return and the expected maximum return.

Theorem 5.3.1. *The holding region is*

$$C = \{(t,y) \in [0,T] \times [0,\infty) : U(t,y) < G(t,y)\},$$

while the exit region is

$$D = \{(t,y) \in [0,T] \times [0,\infty) : U(t,y) = G(t,y)\}.$$

Also, an optimal exit time is

$$\tau^* = \inf\left\{t \in [0,T] : \left(t, \ln\left(\frac{M(t)}{x(t)}\right)\right) \in D\right\}.$$

5.3.3 Numerical Results

To investigate comparative statics, we present one numerical example in which we change the value of the parameter α. Following the standard approach for estimating the above problem via the finite difference approach, we solve the mathematical formulation given in equation (5.3.13) by imposing a uniform grid on the (t,y) domain.

A Crank-Nicolson scheme is adopted for the discretization of the partial differential equation and the semi-infinite interval for y is truncated at a sufficiently large value of y. The derivative boundary condition is discretized using a forward difference approximation. For the results shown below, we take the grid spacing to be 0.005 for y and 0.001 for t dimensions.

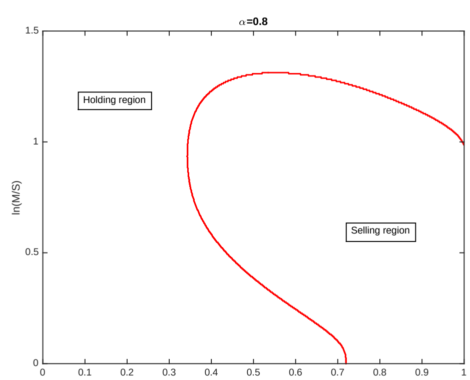
Let $m = 3$. The interest rate of the bond and the appreciation rate of the m stocks are $r = 0.05$ and $(b_1, b_2, b_3)' = (0.1,\ 0.12,\ 0.15)'$, respectively, and the volatility matrix is

$$\sigma = \begin{bmatrix} 0.3000 & 0 & 0 \\ 0.2000 & 0.3464 & 0 \\ 0.2500 & 0.1443 & 0.4082 \end{bmatrix}.$$

Then

$$\theta := \sigma^{-1}(b_1 - r, b_2 - r, b_3 - r)' = (0.1667,\ 0.1058,\ 0.1055)'.$$

Using Theorem 5.3.1 and the parameter value of α ranging between 0.8 and 1, we observe that the exit region decreases as the value of α increases, as shown by the combined picture at the right-bottom corner of Figure.



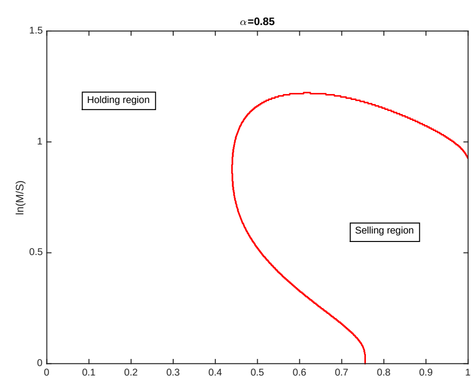


Figure 1: $\alpha = 0.8$

Figure 2: $\alpha = 0.85$

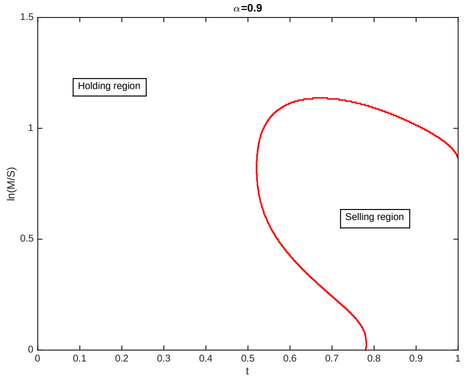


Figure 3: $\alpha = 0.9$

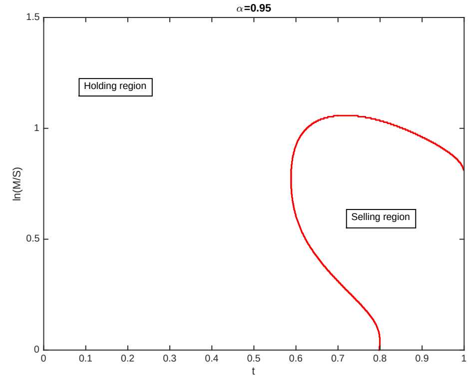


Figure 4: $\alpha = 0.95$

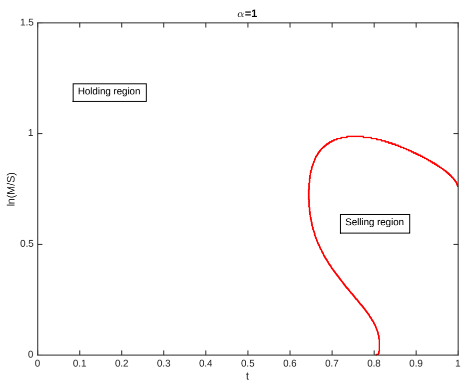


Figure 5: $\alpha = 1$

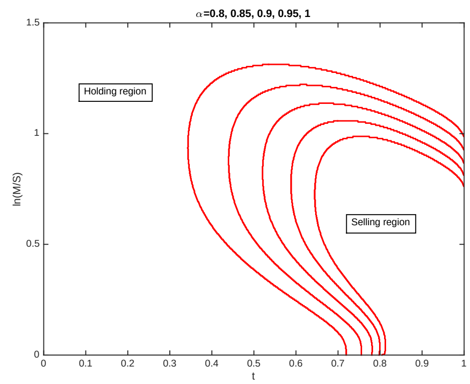


Figure 6: different α

5.4 Expression of Function G_1 and G_2

In this part, we derive the expression of functions G_1 and G_2.

5.4.1 Expression of Function G_1

We now derive the explicit expression of the function G_1, defined by

$$\begin{aligned} G_1(t,y) &= \mathbb{E}\Big[\exp\Big(\max\Big\{y, \max_{0\le u\le T-t}(\nu u + B(u))\Big\}\Big)\Big] \\ &= \int_y^\infty e^z \mathrm{d}\mathbb{P}\Big(\max_{0\le u\le T-t}(\nu u + B(u)) \le z\Big) + e^y \mathbb{P}\Big(\max_{0\le u\le T-t}(\nu u + B(u)) \le y\Big). \end{aligned}$$

Note that

$$\mathbb{P}\Big(\max_{0\le u\le T-t}(\nu u + B(u)) \le z\Big) = \Phi\Big(\frac{z-\nu(T-t)}{\sqrt{T-t}}\Big) - e^{2\nu z}\Phi\Big(\frac{-z-\nu(T-t)}{\sqrt{T-t}}\Big).$$

According to the standard normal distribution, we have

$$\begin{aligned}\int_y^\infty e^z \mathrm{d}\Phi\Big(\frac{z-\nu(T-t)}{\sqrt{T-t}}\Big) &= \int_y^\infty e^z \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(z-\nu(T-t))^2}{2(T-t)}}\mathrm{d}z \\ &= e^{(\nu+\frac{1}{2})(T-t)}\Big[1-\Phi\Big(\frac{y-(\nu+1)(T-t)}{\sqrt{T-t}}\Big)\Big].\end{aligned}$$

Assume that $\nu \neq -\frac{1}{2}$. Then

$$\begin{aligned}&\int_y^\infty e^z \mathrm{d}\Big[e^{2\nu z}\Phi\Big(\frac{-z-\nu(T-t)}{\sqrt{T-t}}\Big)\Big] \\ =& \int_y^\infty 2\nu e^{(1+2\nu)z}\Phi\Big(\frac{-z-\nu(T-t)}{\sqrt{T-t}}\Big)\mathrm{d}z + \int_y^\infty e^{(1+2\nu)z}\mathrm{d}\Phi\Big(\frac{-z-\nu(T-t)}{\sqrt{T-t}}\Big) \\ =& -\frac{2\nu}{1+2\nu}e^{(1+2\nu)y}\Phi\Big(\frac{-y-\nu(T-t)}{\sqrt{T-t}}\Big) - \frac{1}{1+2\nu}e^{(\nu+\frac{1}{2})(T-t)}\Big[1-\Phi\Big(\frac{y-(\nu+1)(T-t)}{\sqrt{T-t}}\Big)\Big].\end{aligned}$$

Thus

$$\begin{aligned}G_1(t,y) =& e^y\Phi\Big(\frac{y-\nu(T-t)}{\sqrt{T-t}}\Big) - \frac{1}{1+2\nu}e^{(1+2\nu)y}\Phi\Big(\frac{-y-\nu(T-t)}{\sqrt{T-t}}\Big) \\ &+\frac{2(1+\nu)}{1+2\nu}e^{(\nu+\frac{1}{2})(T-t)}\Big[1-\Phi\Big(\frac{y-(\nu+1)(T-t)}{\sqrt{T-t}}\Big)\Big].\end{aligned}$$

In addition, note that when $\nu = -\frac{1}{2}$,

$$
\begin{aligned}
& \int_y^\infty e^z \mathrm{d}\Big[e^{2\nu z}\Phi\Big(\frac{-z-\nu(T-t)}{\sqrt{T-t}}\Big)\Big] \\
= & \int_y^\infty e^z \mathrm{d}\Big[e^{-z}\Phi\Big(\frac{-z-\nu(T-t)}{\sqrt{T-t}}\Big)\Big] \\
= & -\int_y^\infty \Phi\Big(\frac{-z-\nu(T-t)}{\sqrt{T-t}}\Big)\mathrm{d}z + \int_y^\infty \mathrm{d}\Phi\Big(\frac{-z-\nu(T-t)}{\sqrt{T-t}}\Big) \\
= & \; y\Phi\Big(\frac{-y-\nu(T-t)}{\sqrt{T-t}}\Big) - \frac{\sqrt{T-t}}{\sqrt{2\pi}}e^{-\frac{(x+\nu(T-t))^2}{2(T-t)}} + \nu(T-t)\Big[1-\Phi\Big(\frac{y+\nu(T-t)}{\sqrt{T-t}}\Big)\Big] \\
& -\Phi\Big(\tfrac{-y-\nu(T-t)}{\sqrt{T-t}}\Big).
\end{aligned}
$$

Thus

$$
\begin{aligned}
G_1(t,y) = \; & 1-\Phi\Big(\frac{y-(\nu+1)(T-t)}{\sqrt{T-t}}\Big) - y\Phi\Big(\frac{-y-\nu(T-t)}{\sqrt{T-t}}\Big) + \frac{\sqrt{T-t}}{\sqrt{2\pi}}e^{-\frac{(y+\nu(T-t))^2}{2(T-t)}} \\
& -\nu(T-t)\Big[1-\Phi\Big(\frac{y+\nu(T-t)}{\sqrt{T-t}}\Big)\Big] + e^y\Phi\Big(\frac{y-\nu(T-t)}{\sqrt{T-t}}\Big).
\end{aligned}
$$

5.4.2 Expression of Function G_2

We now derive the explicit expression of the function G_2, defined by

$$
\begin{aligned}
G_2(t,y) = \; & \mathbb{E}\Big[\exp\Big(\max\Big\{y, \max_{0\le u\le T-t}(\nu u + B(u))\Big\}\Big)^2\Big] \\
= \; & \int_y^\infty e^{2z}\mathrm{d}\mathbb{P}\Big(\max_{0\le u\le T-t}(\nu u+B(u)) \le z\Big) + e^{2y}\mathbb{P}\Big(\max_{0\le u\le T-t}(\nu u+B(u))\le y\Big).
\end{aligned}
$$

Note that

$$
\mathbb{P}\Big(\max_{0\le u\le T-t}(\nu u+B(u))\le z\Big) = \Phi\Big(\frac{z-\nu(T-t)}{\sqrt{T-t}}\Big) - e^{2\nu z}\Phi\Big(\frac{-z-\nu(T-t)}{\sqrt{T-t}}\Big).
$$

According to the standard normal distribution, we have

$$\begin{aligned}\int_y^{\infty} e^{2z}\mathrm{d}\Phi\Big(\frac{z-\nu(T-t)}{\sqrt{T-t}}\Big) &= \int_y^{\infty} e^{2z}\frac{1}{\sqrt{2\pi(T-t)}}e^{-\frac{(z-\nu(T-t))^2}{2(T-t)}}\mathrm{d}z \\ &= e^{2(\nu+1)(T-t)}\Big[1-\Phi\Big(\frac{y-(\nu+2)(T-t)}{\sqrt{T-t}}\Big)\Big].\end{aligned}$$

Assume that $\nu \neq -1$. Then

$$\begin{aligned}&\int_y^{\infty} e^{2z}\mathrm{d}\Big[e^{2\nu z}\Phi\Big(\frac{-z-\nu(T-t)}{\sqrt{T-t}}\Big)\Big] \\ =& \int_y^{\infty} 2\nu e^{2(1+\nu)z}\Phi\Big(\frac{-z-\nu(T-t)}{\sqrt{T-t}}\Big)\mathrm{d}z + \int_x^{\infty} e^{2(1+\nu)z}\mathrm{d}\Phi\Big(\frac{-z-\nu(T-t)}{\sqrt{T-t}}\Big) \\ =& -\frac{\nu}{1+\nu}e^{2(1+\nu)y}\Phi\Big(\frac{-y-\nu(T-t)}{\sqrt{T-t}}\Big) - \frac{1}{1+\nu}e^{2(\nu+1)(T-t)}\Big[1-\Phi\Big(\frac{y-(\nu+2)(T-t)}{\sqrt{T-t}}\Big)\Big].\end{aligned}$$

Thus

$$\begin{aligned}G_2(t,y) =& e^{2y}\Phi\Big(\frac{y-\nu(T-t)}{\sqrt{T-t}}\Big) - \frac{1}{1+2\nu}e^{2(1+\nu)y}\Phi\Big(\frac{-y-\nu(T-t)}{\sqrt{T-t}}\Big) \\ &+\frac{2+\nu}{1+\nu}e^{2(\nu+1)(T-t)}\Big[1-\Phi\Big(\frac{y-(\nu+2)(T-t)}{\sqrt{T-t}}\Big)\Big].\end{aligned}$$

Also, note that when $\nu = -1$,

$$\begin{aligned}&\int_y^{\infty} e^{2z}\mathrm{d}\Big[e^{2\nu z}\Phi\Big(\frac{-z-\nu(T-t)}{\sqrt{T-t}}\Big)\Big] \\ =& \int_y^{\infty} e^{2z}\mathrm{d}\Big[e^{-2z}\Phi\Big(\frac{-z-\nu(T-t)}{\sqrt{T-t}}\Big)\Big] \\ =& -2\int_y^{\infty} \Phi\Big(\frac{-z-\nu(T-t)}{\sqrt{T-t}}\Big)\mathrm{d}z + \int_y^{\infty}\mathrm{d}\Phi\Big(\frac{-z-\nu(T-t)}{\sqrt{T-t}}\Big) \\ =& 2y\Phi\Big(\frac{-y-\nu(T-t)}{\sqrt{T-t}}\Big) - \frac{2\sqrt{T-t}}{\sqrt{2\pi}}e^{-\frac{(y+\nu(T-t))^2}{2(T-t)}} + 2\nu(T-t)\Big[1-\Phi\Big(\frac{y+\nu(T-t)}{\sqrt{T-t}}\Big)\Big] \\ &-\Phi\Big(\frac{-y-\nu(T-t)}{\sqrt{T-t}}\Big).\end{aligned}$$

Thus

$$
\begin{aligned}
G_2(t,y) = \; & 1-\Phi\Big(\frac{y-(\nu+2)(T-t)}{\sqrt{T-t}}\Big)-2y\Phi\Big(\frac{-y-\nu(T-t)}{\sqrt{T-t}}\Big)+\frac{2\sqrt{T-t}}{\sqrt{2\pi}}e^{-\frac{(y+\nu(T-t))^2}{2(T-t)}} \\
& -2\nu(T-t)\Big[1-\Phi\Big(\frac{y+\nu(T-t)}{\sqrt{T-t}}\Big)\Big]+e^{2y}\Phi\Big(\frac{y-\nu(T-t)}{\sqrt{T-t}}\Big).
\end{aligned}
$$

Chapter 6

Conclusion

In this thesis, we investigate the optimal investment problems over a finite time horizon. We construct the value function and derive the corresponding HJB equation. The properties of trading strategy are studied in the thesis, and the problem of optimal stopping time is also discussed. We study an optimal consumption investment model with uncertain exit time and also discuss the case with constraints. The value function is not only the expectation of utility of the price of assets on maturity date, but also the expected utility produced in the whole process. By using the method of partial differential equation, we prove some properties of the problem. Meanwhile, the behavior of free boundary line is also researched. The problem of optimal stopping time is also studied. We formulate a two-stage problem. By Using the method of stochastic analysis, we turn the nonstandard main problem into a standard one. By using stochastic control method, we solve the subproblem. Numerical examples are given respectively.

Bibliography

Asmussen, S. and Taksar, M. (1997), "Controlled diffusion models for optimal dividend pay-out," *Insurance: Mathematics and Economics*, 20, 1–15.

Asmussen, S., Højgaard, B., and Taksar, M. (2000), "Optimal risk control and dividend distribution policies. Example of excess-of loss reinsurance for an insurance corporation," *Finance and Stochastics*, 4, 299–324.

Bellman, R. (1957), *Dynamic programming*, Princeton University Press.

Bertsekas, D. P. (2005), *Dynamic programming and optimal control*, vol. 1, Athena scientific Belmont, MA.

Black, F. and Scholes, M. (1973), "The pricing of options and corporate liabilities," *Journal of political economy*, 81, 637–654.

Carpenter, J. N. (2000), "Does option compensation increase managerial risk appetite?" *The journal of finance*, 55, 2311–2331.

Choi, K. J., Koo, H. K., and Kwak, D. Y. (2004), "Optimal stopping of active portfolio management," *ANNALS OF ECONOMICS AND FINANCE.*, 5, 119–152.

Cox, J. C. and Huang, C.-f. (1989), "Optimal consumption and portfolio policies when asset prices follow a diffusion process," *Journal of economic theory*, 49, 33–83.

Dai, M. and Zhong, Y. (2012), "Optimal stock selling/buying strategy with reference to the ultimate average," *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics*, 22, 165–184.

Dayanik, S. and Karatzas, I. (2003), "On the optimal stopping problem for one-dimensional diffusions," *Stochastic processes and their applications*, 107, 173–212.

Du Toit, J., Peskir, G., et al. (2009), "Selling a stock at the ultimate maximum," *The Annals of Applied Probability*, 19, 983–1014.

Friedman, A. (1964), "Partial differential equations of parabolic type," *Englewood Cliffs, NJ.*

Friedman, A. (1975), "Parabolic variational inequalities in one space dimension and smoothness of the free boundary," *Journal of Functional Analysis*, 18, 151–176.

Guan, C. and Yi, F. (2014), "A free boundary problem arising from a stochastic optimal control model with bounded dividend rate," *Stochastic Analysis and Applications*, 32, 742–760.

Guan, C. and Yi, F. (2016), "A free boundary problem arising from a stochastic optimal control model under controllable risk," *Journal of Differential Equations*, 260, 4845–4870.

Han, X. and Yi, F. (2015), "Equity value, bankruptcy, and optimal dividend policy with finite maturity-variational inequality approach with discontinuous coefficient," *Journal of Inequalities and Applications*, 2015, 67.

Højgaard, B. and Taksar, M. (1998), "Optimal proportional reinsurance policies for diffusion models with transaction costs," *Insurance: Mathematics and Economics*, 22, 41–51.

Karatzas, I., Lehoczky, J. P., and Shreve, S. E. (1987), "Optimal portfolio and consumption decisions for a small investor on a finite horizon," *SIAM journal on control and optimization*, 25, 1557–1586.

Karatzas, I., Shreve, S. E., Karatzas, I., and Shreve, S. E. (1998), *Methods of mathematical finance*, vol. 39, Springer.

Li, X., Wu, X., and Zhou, W. (2017), "Optimal stopping investment in a logarithmic utility-based portfolio selection problem," *Financial Innovation*, 3, 28.

Lieberman, G. M. (1996), *Second order parabolic differential equations*, World scientific.

Markowitz, H. (1952), "Portfolio selection," *The journal of finance*, 7, 77–91.

Merton, R. (1971), "Optimum consumption and portfolio rules in a continuous-time model," *Journal of Economic Theory*, 3, 373–413.

Merton, R. C. (1969), "Lifetime portfolio selection under uncertainty: The continuous-time case," *The review of Economics and Statistics*, pp. 247–257.

Merton, R. C. (1973), "Theory of rational option pricing," *The Bell Journal of economics and management science*, pp. 141–183.

Oleinik, O. (1973), "Second order equations with non-negative characteristic form," *American Mathematical Society.*

Peskir, G. and Shiryaev, A. (2006), *Optimal stopping and free-boundary problems*, Springer.

Pham, H. (2009), *Continuous-time stochastic control and optimization with financial applications*, vol. 61, Springer Science & Business Media.

Pliska, S. R. (1986), "A stochastic calculus model of continuous trading: optimal portfolios," *Mathematics of Operations Research*, 11, 371–382.

Pratt, J. W. (1964), "Risk Aversion in the Small and in the Large," *Econometrica*, 32, 122–136.

Samuelson, P. A. (1969), "Lifetime Portfolio Selection By Dynamic Stochastic Programming," *The Review of Economics and Statistics*, 51, 239–246.

Shiryaev, A., Xu, Z., and Zhou, X. Y. (2008), "Thou shalt buy and hold," *Quantitative finance*, 8, 765–776.

Steele, J. M. (2012), *Stochastic calculus and financial applications*, vol. 45, Springer Science & Business Media.

Taksar, M. I. (2000), "Optimal risk and dividend distribution control models for an insurance company," *Mathematical methods of operations research*, 51, 1–42.

Wu, X., Vong, S., and Zhou, W. (2018), "Optimal Stopping Time of a Portfolio Selection Problem with Multi-assets," *Journal of the Operations Research Society of China*, pp. 1–17.

Xu, G.-L. and Shreve, S. E. (1992a), "A duality method for optimal consumption and investment under short-selling prohibition. I. General market coefficients," *The Annals of Applied Probability*, pp. 87–112.

Xu, G.-L. and Shreve, S. E. (1992b), "A duality method for optimal consumption and investment under short-selling prohibition. II. Constant market coefficients," *The Annals of Applied Probability*, pp. 314–328.

Yong, J. and Zhou, X. Y. (1999), *Stochastic controls: Hamiltonian systems and HJB equations*, vol. 43, Springer Science & Business Media.

Zariphopoulou, T. (1994), "Consumption-investment models with constraints," *SIAM Journal on Control and Optimization*, 32, 59–85.