Tensors and Their Applications

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Certificate of Originality

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Abstract

In recent several decades, tensors have more and more important applications in both mathematical field and physical field. This thesis devotes to tensors and their applications in several research areas. These applications include positive (semi)-definiteness of structure tensors (hypermatrices), strong ellipticity condition for elasticity tensors and tensor representation theory in physics. In details, these three topics are:

1. finding a new class of positive (semi-)definiteness tensors and verifying their properties;
2. constructing a kind of elasticity tensor with a special structure such that the strong ellipticity condition can be verified more easily;
3. presenting an irreducible isotropic function basis of a third order three-dimensional symmetric tensor and proposing a minimal isotropic integrity basis and an irreducible isotropic function basis of a Hall tensor.

For the first topic, motivated by a kind of positive definite test matrix, the Moler matrix, we introduce a new class of positive semi-definite tensor, the MO tensor, which is a generalization of the Moler matrix. We pay our attention to two special cases of the MO tensors: the essential MO tensor and the Sup-MO tensor. Both of them are proved to be positive definite. Especially, the definition of the Sup-MO tensor is based on the concepts of the MO value, the MO set and the Sup-MO value

which are all defined in this work. Furthermore, an essential MO tensor is also a completely positive tensor. Furthermore, the properties of the H-eigenvalues of the Sup-MO tensor are presented. We show that the smallest H-eigenvalue of a Sup-MO tensor is positive and approaches to zero as its dimension tends to infinity.

In the second topic, we focus on the verification for the strong ellipticity of a fourth order elasticity tensor. The problem of verifying the strong ellipticity is converted to an optimization problem of verifying the M-positive semi-definiteness of a partially symmetric tensor. Hence, a kind of tensors which satisfy the strong ellipticity condition is proposed. The elasticity $\mathscr{M}$-tensor is constructed with respect to the M-eigenvalues of elasticity tensors. After proposing a Perron-Frobenius-type theorem for M-spectral radii of the nonnegative elasticity tensors, we are able to show that any nonsingular elasticity $\mathscr{M}$-tensor satisfies the strong ellipticity condition. Furthermore, several equivalent definitions for nonsingular elasticity $\mathscr{M}$-tensors are established in this topic.

In the last topic, we turn our attention to tensor representation theory in the physical field. An isotropic irreducible function basis with 11 invariants of a third order three-dimensional symmetric tensor, which is a proper subset of the Olive-Auffray minimal integrity basis of that tensor, are presented. This result is essential to further investigation for the irreducible function basis of higher order tensors. What is more, the representations of the Hall tensor are also investigated. The Hall tensor, which comes from the Hall effect, an important magnetic effect observed in electric conductors and semiconductors, is a third order three-dimensional tensor whose first two indices are skew-symmetric. We build a connection between its hemitropic and isotropic invariants and invariants of a second order three-dimensional tensor via the third order permutation tensor, i.e., the Levi-Civita tensor. Then, a minimal integrity basis with 10 isotropic invariants for the Hall tensor is proposed and it is proved to be an irreducible function basis for that Hall tensor as well.

Publications Arising From This Thesis

The results in this thesis are based on the following articles written by the author during her study period at the Department of Applied Mathematics, The Hong Kong Polytechnic University as a Ph.D. candidate:

1. Z. Chen, J. Liu, L. Qi, Q. Zheng, and W. Zou. An irreducible function basis of isotropic invariants of a third order three-dimensional symmetric tensor. *Journal of Mathematical Physics*, **59**(8):081703, 2018.

2. W. Ding, J. Liu, L. Qi, and H. Yan. Elasticity $\mathscr{M}$-tensors and the strong ellipticity condition. *arXiv Preprint, arXiv: 1705.09911v2*, 2019.

3. J. Liu, W. Ding, L. Qi, and W. Zou. Isotropic polynomial invariants of Hall tensor. *Applied Mathematics and Mechanics*, **39**(12): 1845-1856, 2018.

4. Y. Xu, J. Liu, and L. Qi. A new class of positive semi-definite tensors. *Journal of Industrial and Management Optimization*, Doi: 10.3934/jimo.2018186, 2018.

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Contents

List of Notations

$\mathbb{R}$	real number field
$\mathbb{R}^n$	real number field with dimension n
$\mathbb{R}^n_+$	nonnegative real vector field with dimension n
$\mathbb{R}^n_{++}$	positive real vector field with dimension n
$\mathbb{R}^{m\times n}$	$m-$by$-n$ matrix on the real number field
$\mathscr{A}, \mathscr{B}, \mathscr{C} \ldots$	tensors
$\mathscr{O}$	zero tensor
$\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \ldots$	second order tensors
$\boldsymbol{I}$	second order identity tensor
$\mathbf{A}, \mathbf{B}, \mathbf{C} \ldots$	matrices
$\mathbf{O}$	zero matrix
$\mathbf{x}, \mathbf{y}, \mathbf{u} \ldots$	vectors
$\mathbf{0}$	zero vector
$\mathbf{x}^{[m]}$	vector whose ith component is x_i^m
$\mathbb{T}_{m,n}$	the set of all the real mth order n-dimensional tensors
$\mathbb{S}_{m,n}$	the set of all the real mth order n-dimensional symmetric tensors
$\mathbf{e}$	vector whose components are all 1
$\mathbf{e_i}$	the ith coordinate vector in $\mathbb{R}^n$

$\mathbb{E}_{4,n}$ — the set of all fourth order n-dimensional tensors whose components satisfy: $a_{ijkl} = a_{jikl} = a_{ijlk} = a_{klij}$

$\mathscr{E}$ — the identity element in $\mathbb{E}_{4,n}$ whose components satisfy: $e_{ijkl} = \begin{cases} 1, & \text{if } i = j \text{ and } k = l \\ 0, & \text{otherwise} \end{cases}$, which is also called the elasticity identity tensor

$\boldsymbol{\varepsilon}$ — permutation tensor(i.e., Levi-Civita tensor) with components ε_{ijk} in three dimensions

$\boldsymbol{Q}$ — orthogonal tensor with components q_{ij}

$\langle \boldsymbol{Q} \rangle \boldsymbol{A}$ — a second order tensor $\boldsymbol{A}$ under an orthogonal transformation

$\mathscr{H}$ — a Hall tensor with components h_{ijk}

$\boldsymbol{\epsilon}_i \otimes \boldsymbol{\epsilon}_j \otimes \boldsymbol{\epsilon}_k$ — orthonormal base in three dimensions

$\mathrm{tr}\, \boldsymbol{A}$, $\boldsymbol{A}^\top$ — trace and transpose, respectively, of a second order tensor $\boldsymbol{A}$

$\det \boldsymbol{A}$ — determinant of a second order tensor $\boldsymbol{A}$

$\boldsymbol{\varepsilon}\mathscr{H}$ — second order tensor with components $\varepsilon_{kli} h_{klj}$ in three dimensions

$\boldsymbol{\varepsilon}\boldsymbol{A}$ — third order tensor with components $\varepsilon_{ijl} a_{lk}$ in three dimensions

Chapter 1

Introduction

1.1 Background

The concept of tensors originally came from the works on differential geometry of Carl Friedrich Gauss, Bernhard Riemann, Elwin Bruno Christoffel and so on, in the nineteenth century [60]. Around the early 20th century, Gregorio Ricci-Curbastro, Tullio Levi-Civita, etc., further investigated and analyzed tensors [66]. Especially, in 1916, the great scientist Albert Einstein, applied a mathematical discipline on the tensor analysis in the study of general relativity. Until now, tensor analysis has already played a significant role in theoretical physics [71], continuum mechanics [26], fluid dynamics [84], and many other fields in science [1, 4, 48].

As geometric objects, tensors are able to describe linear or multi-linear relations between geometric scalars, vectors, and other tensors. In physics, a tensor is a physical quantity whose physical property is independent from coordinate system changes. When a coordinate basis is given, the representation of a tensor is an organized multidimensional array of numerical values. In this case, tensors can be regarded as hypermatrices mathematically, which means that tensors can be treated as a higher order generalization of a matrix in the mathematics field. Hence, the terminology of "tensor" is applied both for tensors as physical quantities and multidimensional arrays (hypermatrices). In this thesis, we treated tensors as hypermatrices in Chapter

2, and as physical quantities in Chapters 3 and 4.

Recent decades, researchers in the field of both mathematics and physics pay high attention to tensors. On one hand, from a mathematical point of view, tensors have wide applications in automatic control [57], spectral hypergraph theory [11, 21, 59], higher order Markov chains [19, 45], polynomial optimization [64], multi-linear systems [10], etc. In 2017, Qi and Luo published the book *Tensor Analysis: Spectral Theory and Special Tensors* [62]. In the book, they mainly concerned about tensor eigenvalues, the applications on hypergraph theory, different special kinds of tensors which are positive (semi-)definite tensors, etc. On the other hand, in the physical field, the applications of tensors include liquid crystal study [16], piezoelectric effects [32], solid mechanics [7] and so on. Qi, Chen and Chen also published a book *Tensor Eigenvalues and Their Applications* in 2018 [60]. More applications in the field of both hypermatrices and physics are discussed in this book.

1.2 Positive Semi-Definiteness of Tensors

As multi-dimensional array, an mth order n-dimensional tensor $\mathscr{A}$ is denoted as

$$\mathscr{A}_{m,n} = (a_{i_1 \cdots i_m}), i_j \in [n], j \in [m],$$

where $[n] := \{1, \cdots, n\}$, and $[m] := \{1, \cdots, m\}$. The set of all the real mth order n-dimensional tensors is denoted as $\mathbb{T}_{m,n}$. When all of its components $a_{i_1 \cdots i_m}$'s are invariant under any permutation of its indices, it is called a symmetric tensor. The set of all the real symmetric mth order n-dimensional tensors is denoted as $\mathbb{S}_{m,n}$. When m is even, an mth order n-dimensional symmetric tensor $\mathscr{A}$ defines an mth degree homogeneous polynomial $f(\mathbf{x}), \mathbf{x} \in \mathbb{R}^n$ as follows,

$$f(\mathbf{x}) \equiv \mathscr{A}\mathbf{x}^m := \sum_{i_1, \cdots, i_m=1}^{n} a_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m}. \tag{1.1}$$

The positive definiteness of this homogeneous polynomial form Eq.(1.1) plays a pivotal role in the stochastic process, automatic control, magnetic resonance imaging, and so on. We say that if $f(\mathbf{x})$ is positive definite (positive semi-definite), then the symmetric tensor $\mathscr{A}$ is positive definite (positive semi-definite)[58]. Hence, the problem of positive (semi-)definiteness of the homogeneous polynomial form Eq.(1.1) is converted to the problem of positive (semi-)definiteness of the symmetric tensor $\mathscr{A}$.

Definition 1.1. *Let m be even and $\mathscr{A} \in \mathbb{S}_{m,n}$. (1) The tensor $\mathscr{A}$ is called positive semidefinite (**PSD**) if $f(\mathbf{x}) = \mathscr{A}\mathbf{x}^m \geqslant 0, \forall \mathbf{x} \in \mathbb{R}^n$; (2) The tensor $\mathscr{A}$ is called positive definite (**PD**) if $f(\mathbf{x}) = \mathscr{A}\mathbf{x}^m > 0, \forall \mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0}$.*

However, it is NP-hard to identify that whether a given general even order symmetric tensor is positive semi-definite or not by Hillar and Lim[36]. On the other hand, during the research of this topic, researchers have found many interesting results. For example, if an even order symmetric tensor has some special structures, then it will be easily identified that whether it is positive semi-definite or not, or there are some checkable conditions to prove such a tensor is a PSD or PD tensor.

Motivated by the study of positive definiteness of $\mathscr{A}$, Qi introduced the concepts of eigenvalues, H-eigenvalues, Z-eigenvalues, E-eigenvalues of an even order symmetric tensor $\mathscr{A}$ in 2005 [58]. Now we recall the definitions of these definitions in [58]. Assume that $\mathbf{x} = (x_1, x_2, \cdots, x_n)^\top \in \mathbb{R}^n$, then $\mathbf{x}^{[m]}$ is a vector in $\mathbb{R}^n$ denoted as $\mathbf{x}^{[m]} = (x_1^m, x_2^m, \cdots, x_n^m)^\top$.

Definition 1.2. [58] *Let $\mathscr{A} \in \mathbb{T}_{m,n}, \lambda \in \mathbb{C}$. If λ and a nonzero vector $\mathbf{x} \in \mathbb{C}^n$ are the solutions of the following polynomial equation:*

$$\mathscr{A}\mathbf{x}^{m-1} = \lambda\mathbf{x}^{[m-1]}, \tag{1.2}$$

then we call $\lambda \in \mathbb{C}$ an eigenvalue of $\mathscr{A}$ and $\mathbf{x}$ an **eigenvector** *of $\mathscr{A}$ associated with the eigenvalue λ. When an eigenvalue of $\mathscr{A}$ has a real eigenvector $\mathbf{x}$, it is called an*

H-eigenvalue *of* $\mathscr{A}$. *A real eigenvector which is associated with an H-eigenvalue is called an* **H-eigenvector**.

Definition 1.3. [58] *Let* $\mathscr{A} \in \mathbb{S}_{m,n}$ *and* $\lambda \in \mathbb{C}$. *If there exists a complex vector* $\mathbf{x}$ *such that*

$$\begin{cases} \mathscr{A}\mathbf{x}^{m-1} = \lambda\mathbf{x}, \\ \quad \mathbf{x}^\top\mathbf{x} = 1, \end{cases} \tag{1.3}$$

then we call that λ *is an* **E-eigenvalue** *of* $\mathcal{A}$, *and* $\mathbf{x}$ *is an* **E-eigenvector** *of the tensor* $\mathscr{A}$ *associated with the E-eigenvalue* λ. *When an E-eigenvalue has a real E-eigenvector, it is called a* **Z-eigenvalue** *and call the real E-eigenvector is called a* **Z-eigenvector**.

Furthermore, the set of all the eigenvalues of tensor $\mathscr{A}$ is called the spectrum of $\mathscr{A}$, and the spectral radius of $\mathscr{A}$ is the largest modulus of the elements in the spectrum of $\mathscr{A}$ which is denoted as $\rho(\mathscr{A})$. The spectral theory of tensors are closely related with the positive (semi-)definiteness of tensors.

It has been shown there that an even order symmetric tensor is positive (semi-)definite if and only if all of its H-eigenvalues or Z-eigenvalues are positive (non-negative) [58]. Various easily checkable positive (semi-)definite tensors with special structures have been discovered consequently, e.g., SOS tensors, even order symmetric diagonally dominated tensors, even order symmetric B-tensors, even order symmetric M-tensors and so on [14, 42, 58, 63, 91].

Especially, a kind of structured tensors is completely positive tensors. It has strong connections with nonnegative tensor factorization which make it a useful tool in polynomial optimization problems, data analysis and so on [25]. It was first introduced in [65] in 2014. The definition is recalled as follows,

Definition 1.4. [65] *Let* $\mathscr{A} \in \mathbb{S}_{m,n}$. *If there exist an integer* r *and some* $\mathbf{u}^{(k)} \in \mathbb{R}^n_+$, $k \in$

$[r]$ *such that*

$$\mathscr{A} = \sum_{k=1}^{r} (\mathbf{u}^{(k)})^m,$$

then $\mathscr{A}$ *is called a* **completely positive tensor***.*

An even order completely positive tensor is a positive semi-definite tensor. All of its H-eigenvalues and Z-eigenvalues are nonnegative. Furthermore, in [47], Luo and Qi proved that Pascale tensors and Lehmer tensors are completely positive tensors.

Due to the significance of positive (semi-)definiteness and spectral theory of tensors, a new kind of positive (semi-)definite tensors are investigated in Chapter 2 of this thesis.

1.3 The Strong Ellipticity Condition of Elasticity Tensors

In 1678, Hooke's law was first published in the work [37] by Robert Hooke, a British Physicist. It shows that the force needed to extend or compress a spring by some distance has linear relations with that distance. In the modern elasticity theory, Hooke's law has been generalized. It states that the strain tensor $\boldsymbol{S}$ of an elastic object or material whose components are s_{ik} is pressure-dependent with the stress applied to it. Under a certain coordinate system $\boldsymbol{\epsilon}_i \otimes \boldsymbol{\epsilon}_j \otimes \boldsymbol{\epsilon}_k$, denote the stress tensor as $\boldsymbol{D}$ whose components are d_{jl}. Due to the multiple independent components of the strain tensor and stress tensor, the "proportional coefficient" is a real tensor instead of a real number. This tensor is called the elasticity tensor, a fourth order three-dimensional tensor. Denoted the elasticity tensor as $\mathscr{G}$ whose components are g_{ijkl}. In general, Hooke's law can be presented as follows,

$$\boldsymbol{S} = \mathscr{G}\boldsymbol{D},$$

i.e.,

$$s_{ik} = \sum_{k,l=1}^{3} g_{ijkl} d_{jl}, i, j, k, l \in \{1, 2, 3\}.$$

Because of the symmetric property of tensors $\boldsymbol{S}$ and $\boldsymbol{D}$, the elasticity tensor $\mathscr{G}$ has the following symmetric properties:

$$g_{ijkl} = g_{kjil} = g_{ilkj} = g_{jilk}, i, j, k, l \in \{1, 2, 3\}. \tag{1.4}$$

Hence, there are 21 independent components in the elasticity tensor which are related to elastic moduli.

One of the most important research topics is the strong ellipticity condition in the elasticity theory. The strong ellipticity condition guarantees the existence of solutions of basic boundary-value problems of elastostatics and thus ensures an elastic material to satisfy some mechanical properties [60]. Therefore, identify whether the strong ellipticity holds or not for a given material is an important problem in mechanics [28]. The **strong ellipticity condition** (SE-condition)for an elasticity tensor can be stated by

$$\mathscr{G}\mathbf{x}^2\mathbf{y}^2 := \sum_{i,j,k,l=1}^{n} g_{ijkl} x_i x_j y_k y_l > 0 \tag{1.5}$$

for any nonzero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$. In the 1970s, necessary and sufficient conditions for strong ellipticity of the equations governing finite plane equilibrium deformations of a compressible hyperelastic solid were proposed by Knowles and Sternberg [40, 41]. From [40] and [41], the "strong ellipticity condition" has been a common term in mechanics. Then, Simpson and Spector extended their works to the special case using the representation theorem for copositive matrices in [73]. In the 1990s, some reformulations of the strong ellipticity condition were established by Rosakis [70] and Wang and Aron [82] as well. Moreover, Walton and Wilber [80] provided sufficient conditions for strong ellipticity of a general class of anisotropic hyperelastic materials. Those conditions require the first partial derivatives of the reduced-stored energy function to satisfy several inequalities and the second partial derivatives to satisfy a convexity condition. Recently, some sufficient and necessary conditions for the strong ellipticity of certain classes of anisotropic linearly elastic materials were given by Chiriţă, Danescu, and Ciarletta[20] and Zubov and Rudev [96]. Gourgiotis and Bigoni [27] also investigated the strong ellipticity of materials under extreme mechanical anisotropy.

Besides the aforementioned literature, there is another approach to deal with the SE-condition. For a given fourth order n dimensional elasticity tensor $\mathscr{G} = (g_{ijkl})$, there exists a one to one correspondence between it and a partially symmetric tensor $\mathscr{A}$ which is symmetric with respect to the first two indices and the last two indices, respectively.

$$a_{ijkl} = a_{jikl} = a_{ijlk} = a_{klij}, i, j, k, l \in [n]. \tag{1.6}$$

The correspondence can be shown as

$$\mathscr{G}\mathbf{x}^2\mathbf{y}^2 = \mathscr{A}\mathbf{x}^2\mathbf{y}^2, \quad \text{for any} \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n. \tag{1.7}$$

Denote the set of all fourth order n-dimensional tensors satisfying symmetric property Eq.(1.6) as $\mathbb{E}_{4,n}$. According to Eq.(1.7), the problem of identifying SE-condition is equivalent to identifying the positive global minimal value of the following optimization problem:

$$\begin{array}{ll} \min & \mathscr{A}\mathbf{x}^2\mathbf{y}^2, \\ \text{s.t.} & \mathbf{x}^\top\mathbf{x} = 1, \ \mathbf{y}^\top\mathbf{y} = 1. \end{array} \tag{1.8}$$

In 2009, Qi, Dai, and Han [61] proved a necessary and sufficient condition of the SE-condition by introducing M-eigenvalues for tensors that belong to $\mathbb{E}_{4,n}$. Here, “M” stands for “mechanics”. They showed that the SE-condition of an elasticity tensor holds if and only if all the M-eigenvalues of its corresponding tensor that belongs to $\mathbb{E}_{4,n}$ are positive. For convenience, we say that the SE-condition of an elasticity tensor holds if and only if all the M-eigenvalues of the elasticity tensor are positive. Furthermore, in [30], Han, Dai, and Qi linked the strong ellipticity condition to the rank-one positive definiteness of three second order tensors, three fourth order tensors, and a sixth order tensor. In [83], Wang, Qi, and Zhang proposed a practical power method for computing the largest M-eigenvalue of any elasticity tensor which can be applied to the verification of the strong ellipticity. Recently, the M-eigenvalues of fourth order elasticity tensors and related algorithms are generalized to higher order cases by Huang and Qi [39]. Chang, Qi, and Zhou [13] defined another type of "eigenvalues" for elasticity tensors which are called the singular values that can also be a necessary and sufficient condition for the strong ellipticity.

Therefore, it is meaningful to study the sufficient and necessary conditions for the SE-condition of the elasticity tensor. In Chapter 3, we will give some conditions for verifying the SE-condition of the elasticity tensors with special structures.

1.4 Tensor Representation Theory

Tensor representation theory plays a fundamental and important role in theoretical and applied physics, especially in mechanics. This topic focuses on tensor functions and tensor invariants which are extremely useful for constructing the constitutive equations of physical relations. The complete and irreducible basis with isotropic invariants could predict the available nonlinear constitutive theories by the formulation of energy term. Because irreducible representations for tensor-valued functions can be immediately yielded from known irreducible representations for invariants [93], the studies of isotropic function basis have most priority.

In physical field, a **constitutive equation** is used for showing the intrinsic relation between two physical quantities which is specific to a substance or material, and approximates the response of that substance or material to external stimuli, usually as applied forces or fields. For example, Hooke's law which we have mentioned in Section 1.3 is a constitutive equation. The intrinsic properties of materials, e.g., symmetric property, are precisely expressed in the constitutive equations by the representation of tensor functions. A **tensor function** is a function whose agencies are tensors and whose values are scalars or tensors, and tensor polynomials are special tensor functions whose components are polynomial functions of the components of the tensor agencies. For mathematic convenience, researchers always assume that the constitutive equations are all polynomial equations. According to the work of Pipkin and Wineman [56] in 1963 and that of Wineman and Pipkin [86] in 1964, we know that complete representations for tensor polynomials can be regarded as complete representations for tensor functions.

It is well known that a constitutive law should have invariance since the physical relations will not change when the coordinate system was changed. Therefore, it requires that a constitutive equation should be constructed by invariants. Hence, the complete and irreducible representation for invariant tensor functions is the key to modeling nonlinear constitutive equations. We call a scalar-valued tensor function as an **isotropic invariant** if it is invariant under any orthogonal transformations. We call it as a **hemitropic invariant** if it is invariant only under rotations. In general, we mainly consider two kinds of representation basis: the minimal integrity basis and the irreducible function basis.

For example, denote $\boldsymbol{T}$ as a second order three-dimensional symmetric tensor, $\operatorname{tr}\boldsymbol{T}$, $\operatorname{tr}\boldsymbol{T}^2$ and $\operatorname{tr}\boldsymbol{T}^3$ are its three isotropic invariants. Furthermore, the set of these three invariants is not only a minimal integrity basis but also an irreducible function basis of the second order three-dimensional symmetric tensor $\boldsymbol{T}$, which means that the set of these three invariants is a complete and irreducible representation of $\boldsymbol{T}$. Hence, every isotropic invariants of $\boldsymbol{T}$ is expressible by a function of $\operatorname{tr}\boldsymbol{T}$, $\operatorname{tr}\boldsymbol{T}^2$ and $\operatorname{tr}\boldsymbol{T}^3$.

Perhaps the modern development of tensor representation theory may be traced back to the great mathematician Hermann Weyls book [85]. It was first published in 1939. Then, from the works of Rivlin in 1948 [69, 68], the general methods for nonlinear constitutive equations and applications of tensor representation theory in continuum mechanics were beginning to be investigated. Since 1955, after the work of Rivlin and Ericksen [67], there has been an extensive development on the theory of tensor function representations. In the following decades, lots of researchers made their efforts on this topic [51, 55, 74, 76, 77, 81, 86, 92, 93, 94, 95]. Especially, in 1994, Zheng gave a survey paper on this topic [93]. He summarized the literature of tensor representation theory in details.

Most of the development of tensor representation theory after 1994 focused on minimal integrity bases of isotropic invariants of third and fourth order three-dimensional tensors [8, 53, 54, 75, 94, 95]. Boehler, Kirillov and Onat [8] studied the polynomial basis of anisotropic invariants of the elasticity tensor in 1994. The tensor function representations involving tensors of orders higher than two were investigated by Zheng and Betten [95] and Zheng [94] in 1995 and 1996, respectively. Furthermore, Smith and Bao [75] presented the minimal integrity bases of isotropic invariants for third and fourth order three-dimensional symmetric and traceless tensors in 1997. Even though a minimal integrity basis for a fourth order three-dimensional symmetric and traceless tensor has already been proposed by Boehler, Kirillov and Onat [8] in 1994, the minimal integrity basis given by Smith and Bao [75] for the same tensor is slightly different.

Meanwhile, the tensor representation theory is closely related to the classical invariant theory in algebraic geometry [2, 35, 53, 79]. One of the most well-known approaches for computing the complete invariant basis was first introduced by Hilbert[35]. In 2014, Olive and Auffray [53] proposed an integrity basis with thirteen isotropic invariants of a (completely) symmetric third order three-dimensional tensor. Olive claimed that this integrity basis is a minimal integrity basis in [52][p. 1409]. Moreover, the minimal integrity basis with 297 invariants of the fourth order elasticity tensors successfully obtained via the approaches from the algebraic geometry by Olive, Kolev, and Auffray[54] in 2017.

Very recently, in 2018, Prof. Qi's research group has made some new progress on this topic. Two isotropic irreducible functional bases of a fourth order three-dimensional symmetric and traceless tensor was presented by Chen, Chen, Qi and Zou [17], the minimal integrity basis and irreducible function basis of a two-dimensional Eshelby tensor were proposed by Ming, Zhang and Chen [49], and Chen, Hu, Qi and Zou [15] showed that any minimal integrity basis of a third order three-dimensional symmetric and traceless tensor is also an irreducible function basis of that tensor.

In Chapter 4, we first present an eleven invariant isotropic function basis of a third order three-dimensional symmetric tensor and show that this function basis is indeed an irreducible function basis of a third order three-dimensional symmetric tensor. It is the first time to give an irreducible function basis of isotropic invariants of a third order three-dimensional symmetric tensor. Then we propose a minimal isotropic integrity basis with 10 invariants for the Hall tensor, a special kind of third order three-dimensional tensors, and prove that this minimal integrity basis is also an irreducible isotropic function basis of it. These results are significant to the further research of irreducible function bases of higher order tensors.

1.5 Organization of the Thesis

In this section, we will give a brief introduce of the contributions in this thesis. The results in this thesis are all based on the works of [18], [22], [46] and [88]. These works were organized as follows,

1. In Chapter 2, motivated by the test matrices, Moler matrices, we define a new

class of tensors, the MO tensors. As a natural generalization of Moler matrices, their positive semi-definiteness are shown in this chapter. MO tensors, which include Sup-MO tensors and essential MO tensors with the concept of MO value, are well studied. Meanwhile, some related properties are proved in this chapter.

2. In Chapter 3, we first briefly introduce the strong ellipticity condition and its relationship with different types of positive definiteness of the elasticity tensors. As preparation for defining the elasticity $\mathscr{M}$-tensor, we study the M-spectral radius of nonnegative elasticity tensors then. Inspired by the M-tensors, we introduce a kind of elasticity tensors with special structure, the elasticity $\mathscr{M}$-tensors and the nonsingular elasticity $\mathscr{M}$-tensors. In the next, we prove that they are M-positive (semi-)definite tensors and present other equivalent definitions for nonsingular elasticity $\mathscr{M}$-tensors.

3. In Chapter 4, firstly, we introduce some basic concepts and definitions in the tensor representation theory. Secondly, an isotropic function basis with eleven invariants of a third order three-dimensional symmetric tensor is presented, and we prove that it is an irreducible function basis of a third order three-dimensional symmetric tensor. Thirdly, we investigate the minimal integrity basis and irreducible function basis for the Hall tensor. We discover a connection between the invariants of a Hall tensor and that of a second order three-dimensional tensor, which is quite important for constructing the complete and irreducible representations for the Hall tensor.

4. In the final chapter, we briefly summarize the conclusions and list some future work.

Chapter 2

A New Class of Positive Semi-Definite Tensors: MO-Tensors

In Section 1.2, we mentioned that the positive semi-definiteness of some tensors with special structures is easy to be checked. Hence, in this chapter, we will introduce a new class of positive semi-definite tensors which is a natural generalization of a special kind of positive definite matrices, the Moler Matrices. Moler matrix is a kind of test matrix and it is positive definite. In 2016, two kinds of positive semi-definite test matrices, the Pascal matrices and the Lehmer matrices, have been extended to tensors by Luo and Qi [47]. In [47], they extended the Pascal matrix to the Pascal tensor and generalized Pascal tensor, and extended the Lehmer matrix to the Lehmer tensor and generalized Lehmer tensor. They showed that these two special kinds of tensors are easily checkable and proved that they are completely positive tensors.

Inspired by their work on generalizing test matrices to tensors, we pay our attention to another kind of test matrices, the Moler matrix which is a positive definite symmetric matrix. Especially, one of its eigenvalues is quite small, and it is usually used for testing eigenvalue computations. Furthermore, the smallest eigenvalue of a Moler matrix approaches 0, when the dimension of the Moler matrix $n \to +\infty$. Motivated by the good properties of Moler matrix, we define a new class of positive semi-definite tensors. Because there have been M-tensors already, we call this new class of tensor MO tensor, which comes from the first two letters of Moler.

In this chapter, we will first give a review on some basic definitions, theorems, and lemmas which are useful in the following contents in preliminaries. And we will also give a proof for good properties of the Moler matrices. Then we will define MO tensors, which include Sup-MO tensors and essential MO tensors with the concept of MO value, and prove some related properties in the Section 2.2. Finally, we will give some final remarks on future work.

2.1 Preliminaries

Since some properties of a matrix have connections with its dimension, we denote $\mathbf{A}(n)$ as an n-dimensional matrix in this chapter. Furthermore, there will be some conclusions of an mth order n-dimensional tensor that have relations with its order and dimension, we denote $\mathscr{A}(m,n)$ as an mth order n-dimensional tensor in this chapter as well. At first, we will recall the definition for the Moler matrices as follows,

Definition 2.1 ([50]). *Let $\mathbf{A}(n) \in \mathbb{R}^{n\times n}$. $\mathbf{A}(n)$ is called the n-dimensional* **Moler matrix** *if*

$$\mathbf{A}(n)_{i,j} = \begin{cases} i, & i = j \\ \min\{i,j\} - 2, & i \neq j \end{cases}.$$

In the following proposition, we prove that the Moler matrix is a positive definite matrix, and its smallest eigenvalue approaches 0 in decreasing as its dimension tends to infinity. The symbol "$\searrow$" means "approach down to".

Proposition 2.1. *Let $\mathbf{A}(n) \in \mathbb{R}^{n\times n}$ be an n-dimensional Moler matrix. (1) $\mathbf{A}(n)$ is positive definite; (2) Let $\lambda_{min}(\mathbf{A}(n))$ be the smallest eigenvalue of $\mathbf{A}(n)$. Then*

$\lambda_{min}(\mathbf{A}(n)) \searrow 0$.

Proof. (1) Note that $\mathbf{A}(n) = LL^\top$, where

$$L_{i,j} = \begin{cases} 1, & i = j \\ -1, & i > j \\ 0, & i < j \end{cases}.$$

Hence, $\mathbf{A}(n)$ is positive definite.

(2) It is obvious that $0 < \lambda_{min}(\mathbf{A}(n+1)) \leqslant \lambda_{min}(\mathbf{A}(n))$. Assume that $\mathbf{x} = \left(1, \frac{1}{2}, \frac{1}{2^2}, \cdots, \frac{1}{2^{n-1}}\right)^\top$. We have $L^\top \mathbf{x} = \begin{pmatrix} \frac{1}{2^{n-1}} \\ \vdots \\ \frac{1}{2^{n-1}} \end{pmatrix}$,

$$\frac{\mathbf{x}^\top \mathbf{A}(n)\mathbf{x}}{\mathbf{x}^\top \mathbf{x}} = \frac{3n}{4^n - 1} \geqslant \lambda_{min}(\mathbf{A}(n)),$$

which means that $\lambda_{min}(\mathbf{A}(n)) \searrow 0$, when $n \to +\infty$. $\square$

In Section 1.2, we have already briefly introduced the definitions of the PSD tensors, H-eigenvalues and completely positive tensors in Definitions 1.1, 1.2 and 1.4. Before we extend the Moler matrices to tensors, we also need some properties of H-eigenvalues for preparation.

Theorem 2.1. [58] *Let $\mathscr{A}(m,n) \in \mathbb{S}_{m,n}$, and m be even. Then $\mathscr{A}(m,n)$ is positive definite (positive semi-definite) if and only if all of the H-eigenvalues of $\mathscr{A}(m,n)$ are positive (non-negative). Furthermore, we have*

1. $\lambda_{min}(\mathscr{A}(m,n)) = \min \frac{\mathscr{A}(m,n)\mathbf{x}^m}{\|\mathbf{x}\|_m^m}$,

2. $\lambda_{min}(\mathscr{A}(m,n)) = \min\{\mathscr{A}(m,n)\mathbf{x}^m : \|\mathbf{x}\|_m = 1\}$,

where $\mathbf{x} \in \mathbb{R}^n$, and $\|\mathbf{x}\|_m = (\sum_{i=1}^n |x_i|^m)^{\frac{1}{m}}$.

2.2 MO Tensors and Their Properties

At first in this section, we will introduce concepts of the **MO value**, the **MO set** and the **Sup-MO value**. Then, we will present a new class of positive semi-definite tensors.

Definition 2.2. *Let m be an even number. (1) $\alpha(m)$ is called as the* **MO value***, if $\mathscr{A}(m,n) := \mathscr{M}(m,n) - \alpha(m)\mathscr{N}(m,n)$ is positive semi-definite for any n, where*

$$\mathscr{M}(m,n)_{i_1,i_2,\cdots,i_m} = \begin{cases} i_1, & i_1 = i_2 = \cdots = i_m \\ \min\{i_1, i_2, \cdots, i_m\}, & else \end{cases}, \tag{2.1}$$

$$\mathscr{N}(m,n)_{i_1,i_2,\cdots,i_m} = \begin{cases} 0, & i_1 = i_2 = \cdots = i_m \\ 1, & else \end{cases}. \tag{2.2}$$

We call the set of all MO values as the **MO set** *$\Omega(m)$; (2) We call $\alpha^*(m) = \sup\{\Omega(m)\}$ as the* **Sup-MO value***. We also define* **Sub-MO value** *$\alpha_*(m)$ of $\Omega(m)$ as $\alpha_*(m) = \inf\{\Omega(m)\}$.*

What is noteworthy is that $\alpha(m)$ is a parameter only related to m. Therefore, when we consider exploring its properties, it is necessary to prove that these properties still hold when $n \to \infty$. Here, we mainly focus on the properties of $\alpha^*(m)$. Based on the aforementioned concepts, MO tensors and Sup-MO tensors are defined as follows.

Definition 2.3. *Let m be an even number. (1) $\mathscr{A}(m,n) \in \mathbb{S}_{m,n}$ is called an* **MO tensor***, if*

$$\mathscr{A}(m,n) = \mathscr{M}(m,n) - \alpha(m)\mathscr{N}(m,n),$$

where $\mathscr{M}(m,n)$ and $\mathscr{N}(m,n)$ are defined in Eq.(2.1) and Eq.(2.2), respectively, and $\alpha(m) \in \Omega(m)$. (2) $\mathscr{A}(m,n) \in \mathbb{S}_{m,n}$ is called a **Sup-MO tensor***, if*

$$\mathscr{A}(m,n) = \mathscr{M}(m,n) - \alpha^*(m)\mathscr{N}(m,n).$$

It is worth noting that the Definitions 2.2 and 2.3 rely on the definition of positive (semi-)definiteness of the tensor. Since there is no definition of positive (semi-)definiteness of an odd order tensor, there are no definitions for the MO value, MO set, Sup-MO value, MO tensor and Sup-MO tensor when the order m is odd.

The following theorem shows that the Sup-MO tensor $\mathscr{A}(n,m)$ can be reduced to the Moler matrix $\mathbf{A}(n)$ when $m=2$.

Theorem 2.2. *Let $\Omega(m)$ be the MO set, $\mathscr{A}(m,n) \in \mathbb{S}_{m,n}$ be a Sup-MO tensor. When $m=2$, we have $\alpha^*(2) = 2 = \max\{\Omega(2)\}$.*

Proof. According to the property of the Moler matrix, we get $2 \in \Omega(2)$. Next, we need to prove that 2 is the Sup-MO value in this case. If $\alpha^*(2) > 2$, then there exists an $\alpha \in (2, \alpha^*(2)) \bigcap \Omega(2)$ such that

$$\mathscr{M}(2,n) - \alpha \mathscr{N}(2,n) = \mathscr{M}(2,n) - 2\mathscr{N}(2,n) - (\alpha-2)\mathscr{N}(2,n),$$

where $\mathscr{M}(m,n)$ and $\mathscr{N}(m,n)$ are defined in Eq.(2.1) and Eq.(2.2), respectively.

Let $\mathbf{x} = \left(1, \frac{1}{2}, \cdots, \frac{1}{2^{n-1}}\right)^\top \in \mathbb{R}^n$,

$$\mathbf{x}^\top(\mathscr{M}(2,n) - \alpha\mathscr{N}(2,n))\mathbf{x} = \mathbf{x}^\top(\mathscr{M}(2,n) - 2\mathscr{N}(2,n))\mathbf{x} - (\alpha-2)\mathbf{x}^\top\mathscr{N}(2,n)\mathbf{x}.$$

Due to

$$\mathbf{x}^\top(\mathscr{M}(2,n) - 2\mathscr{N}(2,n))\mathbf{x} = \frac{3n}{4^n-1}, \text{ and } \mathbf{x}^\top\mathscr{N}(2,n)\mathbf{x} = \frac{8}{3} - \frac{4}{2^{n-1}} + \frac{4}{3(4^{n-1})},$$

when $n \to +\infty$, we have

$$\mathbf{x}^\top(\mathscr{M}(2,n) - \alpha\mathscr{N}(2,n))\mathbf{x} < 0,$$

which contradicts with $\alpha \in \Omega(2)$. Hence $\alpha^*(2) = 2$. $\square$

In the following work, we discuss a special class of MO tensors, the essential MO tensors. The following theorem presents the relationship between the essential MO tensors and the completely positive tensors. Then we give some properties of Sup-MO values in MO tensors.

Definition 2.4. *Let $\mathscr{A}(m,n) \in \mathbb{S}_{m,n}$. $\mathscr{A}(m,n)$ is called the mth order n-dimensional* **essential MO tensor** *if*

$$\mathscr{A}(m,n)_{i_1,\cdots,i_m} = \begin{cases} i_1, & i_1 = i_2 = \cdots = i_m \\ \min\{i_1, i_2, \cdots, i_m\} - 1, & otherwise \end{cases}.$$

Theorem 2.3. *Let $\mathscr{A}(m,n)$ be an mth order n-dimensional essential MO tensor. For all n and even m, it is positive definite. Moreover, it is a completely positive tensor for all n and m, as $\mathscr{A}(m,n) = \sum\limits_{i=1}^{n} \mathbf{e}_i^m + \sum\limits_{i=2}^{n} \mathbf{r}_i^m$, where $(\mathbf{e}_i)_j = \begin{cases} 1, & j = i \\ 0, & otherwise \end{cases}$, $(\mathbf{r}_i)_j = \begin{cases} 1, & j \geqslant i \\ 0, & otherwise \end{cases}$.*

Proof. Let $\mathscr{B}(m,n)_{i_1,\cdots,i_m} = \begin{cases} 1, & i_1 = i_2 = \cdots = i_m = 1 \\ 1, & i_1, i_2, \cdots, i_m \geqslant 2 \\ 0, & otherwise \end{cases}$.

Because $\mathscr{B}(m,n) = (1,0,\cdots,0)^m + (0,1,\cdots,1)^m$, $\mathscr{B}(m,n)$ is a complete positive tensor. Let $\mathscr{C}(m,n) = \mathscr{A}(m,n) - \mathscr{B}(m,n)$. Then

$$\mathscr{C}(m,n) = \begin{cases} i_1 - 1, & i_1 = i_2 = \cdots = i_m \geqslant 2 \\ 0, & \min\{i_1, i_2, \cdots, i_m\} = 1 \\ \min\{i_1, i_2, \cdots, i_m\} - 2, & otherwise \end{cases}.$$

Let $\mathscr{A}(m, n-1)_{i_1,i_2,\cdots,i_m} = \mathscr{C}(m,n)_{i_1+1,i_2+1,\cdots,i_m+1}, i_j \in [n-1], j \in [m]$. Then $\mathscr{A}(m,n-1)$ is an mth order $(n-1)$-dimensional essential MO tensor. Furthermore, if $\mathscr{A}(m,n-1)$ is the completely positive tensor, then $\mathscr{A}(m,n)$ is a completely positive tensor.

By the same way, we could get $\mathscr{A}(m,i), i \in [n]$, are all essential MO tensors. When $n = 1$, $\mathscr{A}(m,1)$ equals the positive number 1. By induction, we have that $\mathscr{A}(m,n)$ is a completely positive tensor and also a positive definite tensor. $\square$

Theorem 2.3 also means that the MO set is a nonempty set for all n and even m. Hence, following corollaries are given.

Corollary 2.1. *(1) Let $\Omega(m)$ be the MO set, and $\mathscr{M}(m,n), \mathscr{N}(m,n)$ be defined in Eq.(2.1) and Eq.(2.2), respectively. For all n and even m, 1 is always an MO value, i.e. $1 \in \Omega(m)$.*

(2) For all n and even m, $\mathscr{M}(m,n) - \alpha\mathscr{N}(m,n)$ is always completely positive while $\alpha \in [0,1]$.

Proof. (1) Since $\mathscr{M}(m,n) - \mathscr{N}(m,n)$ is an essential MO tensor for all n and even m, we know that 1 is an MO value.

(2) Since $\mathscr{N}(m,n) = \mathbf{e}^m - \sum_{i=1}^{n} \mathbf{e}_i^m$,

$$\mathscr{M}(m,n) = \sum_{i=1}^{n} \mathbf{e}_i^m + \sum_{i=2}^{n} \mathbf{r}_i^m + \mathbf{e}^m - \sum_{i=1}^{n} \mathbf{e}_i^m = \sum_{i=2}^{n} \mathbf{r}_i^m + \mathbf{e}^m,$$

where $\mathbf{e} = (1, 1, \cdots, 1)^\top$. Hence, if $\alpha \in [0,1]$,

$$\mathscr{M}(m,n) - \alpha\mathscr{N}(m,n) = \sum_{i=2}^{n} \mathbf{r}_i^m + (1-\alpha)\mathbf{e}^m + \alpha \sum_{i=1}^{n} \mathbf{e}_i^m$$

is completely positive. $\square$

According to the definition of Sub-MO value and Corollary 2.1, there is a small result for the Sub-MO value $\alpha_*(m)$.

Corollary 2.2. *$\alpha_*(m)$ exists, and $-\frac{1}{2} \leqslant \alpha_*(m) \leqslant 0$.*

Proof. From Corollary 2.1(2), $\alpha_*(m) \leqslant 0$. Let $\mathbf{x} = (1, -1, 0, \cdots, 0)^\top$, when $\alpha < -\frac{1}{2}$,

$$(\mathscr{M}(m,n) - \alpha\mathscr{N}(m,n))\mathbf{x}^m = 1 + 2\alpha < 0,$$

therefore, $\alpha_*(m) = \inf\{\Omega(m)\}$ exists, and $\alpha_*(m) \geqslant -\frac{1}{2}$. □

The following proposition is one of the most important contents in this chapter. Some interesting properties of the Sup-MO values and Mo set are investigated here.

Proposition 2.2. *Let $\Omega(m)$ be the MO set, m be even and $\mathscr{M}(m,n), \mathscr{N}(m,n)$ be defined in Eq.(2.1) and Eq.(2.2), respectively. Then,*

(1) for any $\alpha_1(m), \alpha_2(m) \in \Omega(m)$, $[\alpha_1(m), \alpha_2(m)] \subseteq \Omega(m)$.

(2) $1 < \alpha^*(m) \leqslant 2$.

(3) $\alpha^*(m) = \max\{\Omega(m)\}$.

(4) $\alpha^*(m) \searrow 1$, when $m \to +\infty$.

Proof. (1) It is obvious.

(2) Since $1 \in \Omega(m)$ for all even m, $\Omega(m) \neq \varnothing$. Considering the tensor $\mathscr{M}(m,n) - 2\mathscr{N}(m,n)$ and $\mathbf{x} = \left(1, \frac{1}{2}, \cdots, \frac{1}{2^{n-1}}\right)^\top \in \mathbb{R}^n$. Then

$$(\mathscr{M}(m,n) - 2\mathscr{N}(m,n))\mathbf{x}^m = 2\sum_{i=1}^{n}(\mathbf{e}_i^\top\mathbf{x})^m + \sum_{i=2}^{n}(\mathbf{r}_i^\top\mathbf{x})^m - (\mathbf{e}^\top\mathbf{x})^m,$$

where $\mathbf{e} = (1, \cdots, 1)^\top$. When $m \geqslant 4$ and $n \geqslant 2$, we have

$$\sum_{i=2}^{n}(\mathbf{r}_i^\top\mathbf{x})^m \leqslant \sum_{i=1}^{n-1}\frac{1}{(2^m)^i} \leqslant \frac{2^m}{2^m - 1} \leqslant \frac{16}{15},$$

$$\sum_{i=1}^{n}(\mathbf{e}_i^\top\mathbf{x})^m = \sum_{i=1}^{n}\frac{1}{(2^m)^{i-1}} \leqslant \frac{2^m}{2^m - 1} \leqslant \frac{16}{15},$$

and

$$(\mathbf{e}^\top \mathbf{x})^m = (\sum_{i=1}^{n} \frac{1}{2^{i-1}})^m = \left(\frac{1 - \dfrac{1}{2^n}}{1 - \dfrac{1}{2}} \right)^m \geqslant \left(\frac{3}{2}\right)^m.$$

Then

$$(\mathscr{M}(m,n) - 2\mathscr{N}(m,n))\mathbf{x}^m \leqslant \frac{48}{15} - \left(\frac{3}{2}\right)^m < 0.$$

Therefore, for all even m, 2 is an upper bound of $\Omega(m)$. Therefore, $\alpha^*(m)$ exists and $\alpha^*(m) \leqslant 2$.

Next, we prove $\alpha^*(m) > 1$. Let

$$\mathscr{U}(m,n) = \mathscr{M}(m,n) - \mathscr{N}(m,n),$$

$$\mathscr{V}(m,n;\beta) = \mathscr{U}(m,n) - \beta\mathscr{N}(m,n),$$

and $\beta = \alpha - 1$. The first thing is to prove that there exists $\beta \in (0,1)$ such that

$$\mathscr{V}(m,n;\beta)\mathbf{x}^m = (1+\beta)\sum_{i=1}^{n}(\mathbf{e}_i^\top \mathbf{x})^m + \sum_{i=2}^{n}(\mathbf{r}_i^\top \mathbf{x})^m - \beta(\mathbf{e}^\top \mathbf{x})^m \geqslant 0,$$

for all $n \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$ satisfying $||\mathbf{x}||_m = 1$.

If $\mathbf{e}^\top \mathbf{x} = 0$, then $\mathscr{V}(m,n;\beta)\mathbf{x}^m \geqslant 0$ for all $\beta \in [0,1]$. If $\mathbf{e}^\top \mathbf{x} \neq 0$, then there exist $\mathbf{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ and $\mathbf{z} = (z_1, z_2, \cdots, z_n) \in \mathbb{R}^n$ such that

$$y_1 = \mathbf{e}^\top \mathbf{x}, y_i = \mathbf{r}_i^\top \mathbf{x}, i = 2, \cdots, n,$$

and

$$z_i = \frac{y_i}{y_1}, i = 1, \cdots, n.$$

Then $z_1 = 1$ and

$$\mathscr{V}(m,n;\beta)\mathbf{x}^m = y_1^m \left[(1+\beta)\left(\sum_{i=1}^{n-1}(z_i - z_{i+1})^m + z_n^m \right) + \sum_{i=2}^{n} z_i^m - \beta \right].$$

Let

$$g_{m,n}(\mathbf{z},\beta) = (1+\beta)\left(\sum_{i=1}^{n-1}(z_i - z_{i+1})^m + z_n^m\right) + \sum_{i=2}^{n} z_i^m, \tag{2.3}$$

and

$$f_{m,n}(\beta) \ = \min_{\mathbf{z}\in\mathbb{R}^n, z_1=1} \ g_{m,n}(\mathbf{z},\beta). \tag{2.4}$$

We have

$$0 \leqslant f_{m,n+1}(\beta) \leqslant f_{m,n}(\beta), \text{for all } \beta \in [0,1].$$

Hence,

$$f_m(\beta) = \lim_{n\to+\infty} f_{m,n}(\beta) \tag{2.5}$$

exists for all $\beta \in [0,1]$.

However, if there exists $\beta \in [0,1]$ such that $f_m(\beta) = 0$, then by the definition of $f_{m,n}(\beta)$, there exists $\mathbf{z}^* \in \mathbb{R}^n$ with $z_1^* = 1$ such that $(1-z_2^*)^m < \varepsilon$ and $z_2^* < \varepsilon$, for any $\varepsilon > 0$, which is impossible. Hence, $f_m(\beta) > 0$. Furthermore, $f_m(0) > 0$.

What is more, when $m \geqslant 4$, assume that $\mathbf{z}^* = \left(1, \frac{1}{2}, 0, \cdots, 0\right)^\top$, we get

$$g_{m,n}(\mathbf{z}^*, 1) = \frac{5}{2^m}.$$

Thus $f_m(1) \leqslant f_{m,n}(1) \leqslant \dfrac{5}{2^m} < 1$.

Additionally, since m is even,

$$g_{m,n}(\mathbf{z},\beta_1) \leqslant g_{m,n}(\mathbf{z},\beta_2), \text{for all } \mathbf{z} \in \mathbb{R}^n, 0 \leqslant \beta_1 \leqslant \beta_2.$$

Then $f_{m,n}(\beta_1) \leqslant f_{m,n}(\beta_2)$ and $f_m(\beta_1) \leqslant f_m(\beta_2)$, which means that $f_m(\beta)$ is a non-decreasing function of β on $[0,1]$.

Denote $f_m(\beta^+)$ and $f_m(\beta^-)$ as the right-hand and left-hand limit on $\beta \in (0,1)$, $f_m(0^+)$, and $f_m(1^-)$ as the right-hand limit on 0 and left-hand limit on 1 of the function $f_m(\beta)$, respectively. Now we will prove that $f_m(\beta)$ is a continuous function of β on $(0,1)$.

Since $f_m(\beta)$ is a nondecreasing function, for any $\beta \in (0,1)$, $f_m(\beta^+)$, $f_m(\beta^-)$, $f_m(0^+)$, $f_m(1^-)$ exist and $f_m(\beta^+) \geqslant f_m(\beta^-)$. Assuming that $f_m(\beta^+) > f_m(\beta^-)$, and $\delta = \frac{f_m(\beta^+)-f_m(\beta^-)}{2} > 0$, for $0 < \beta_1 < \beta < \beta_2$, there exists N^*, when $n > N^*$, we have

$$\begin{aligned} f_{m,n}(\beta_1) &\leqslant f_m(\beta_1) + \frac{f_m(\beta^+) - f_m(\beta^-)}{2} \\ &\leqslant f_m(\beta^-) + \frac{f_m(\beta^+) - f_m(\beta^-)}{2} \\ &= \frac{f_m(\beta^+) + f_m(\beta^-)}{2}, \end{aligned}$$

and

$$f_{m,n}(\beta_2) \geqslant f_m(\beta_2) \geqslant f_m(\beta^+).$$

Therefore, when $n > N^*$,

$$f_{m,n}(\beta_2) - f_{m,n}(\beta_1) \geqslant \frac{f_m(\beta^+) - f_m(\beta^-)}{2} = \delta > 0.$$

Because $g_{m,n}(\mathbf{z},\beta)$ is continuous in β, and the level set of $g_{m,n}(\mathbf{z},\beta)$ is bounded, according to Proposition 4.4 in [9], $f_{m,n}(\beta)$ is continuous in β. When $\beta_1 \to \beta, \beta_2 \to \beta$, we have that

$$f_{m,n}(\beta^+) - f_{m,n}(\beta^-) \geqslant \frac{f_m(\beta^+) - f_m(\beta^-)}{2} = \delta > 0,$$

which is a contradiction. Hence, $f_m(\beta)$ is continuous in β on $(0,1)$.

Finally,

$$\partial(f_{m,n}(\beta))_\beta = \left\{ \sum_{i=1}^{n-1} (z_i^* - z_{i+1}^*)^m + (z_n^*)^m \right\},$$

where $\mathbf{z}^* \in \arg\min_{\mathbf{z}\in\mathbb{R}^n, z_1=1} g_{m,n}(\mathbf{z},\beta)$. Noting that $f_{m,n}(\beta) < 1$, for any $\beta \in [0,1]$, there exists $N > 0$ such that $||\eta_{m,n}|| < 1$, for all $\eta_{m,n} \in \partial(f_{m,n}(\beta))_\beta$ and $||\eta_m|| < 1$, for all $\eta_m \in \partial(f_m(\beta))_\beta$, when $n > N$.

Hence, for all $\mathbf{z} \in \mathbb{R}^n$ with $z_1 = 1$, there is only one $\beta^*(m) \in (0,1)$ satisfying $f_m(\beta^*(m)) = \beta^*(m)$. Besides that, when $0 < \beta \leqslant \beta^*(m)$, we have

$$g_{m,n}(\mathbf{z},\beta) \geqslant f_{m,n}(\beta) \geqslant f_m(\beta) \geqslant \beta.$$

When $\beta^*(m) \leqslant \beta < 1$, we have $f_m(\beta) \leqslant \beta$. This means that there exists $1 > \beta > 0$ satisfying

$$\mathscr{V}(m,n;\beta)\mathbf{x}^m \geqslant 0,$$

and there exists $\alpha = 1 + \beta > 1$, satisfying

$$\mathscr{M}(m,n) - \alpha\mathscr{N}(m,n) \succeq 0.$$

Thus $\alpha^*(m) > 1$.

Additionally, we prove that $\beta^*(m) = \alpha^*(m) - 1$. Obviously, $\alpha^*(m) - 1 \geqslant \beta^*(m)$. If $\alpha^*(m) - 1 > \beta^*(m)$, then there exists β_1 such that $\beta^*(m) < \beta_1 < \alpha^*(m) - 1$. Since $f_m(\beta_1) < \beta_1$, according to the definition of $f_m(\beta)$, there exists $N > 0$ such that $f_{m,n}(\beta_1) - \beta_1 < 0$, when $n > N$. Hence, $\mathscr{U}(m,n) - \beta_1\mathscr{N}(m,n) \not\succeq 0$, which is a contradiction. Thus $\beta^*(m) = \alpha^*(m) - 1$.

(3) From Theorem 2.2, when $m = 2$, $\alpha^*(2) = \max\{\Omega(2)\}$. According to (2), when $m \geqslant 4$, $\mathscr{M}(m,n) - \alpha^*(m)\mathscr{N}(m,n) \succeq 0$. It means that $\alpha^*(m) \in \Omega(m)$. Therefore, $\alpha^*(m) = \max\{\Omega(m)\}$, for all even m.

(4) Since $0 < f_{m,n}(\beta) < 1, \beta \in [0,1]$, when m is even and $m \geqslant 4$, there exists a $\mathbf{z}^* \in \arg\min_{\mathbf{z}\in\mathbb{R}^n, z_1=1} g_{m,n}(\mathbf{z},\beta)$ satisfying

$$(z_i^* - z_{i+1}^*)^m \leqslant 1, i = 1, \cdots, n-1, (z_i^*)^m \leqslant 1, i = 2, \cdots, n.$$

Thus

$$(z_i^* - z_{i+1}^*)^{m+2} \leqslant (z_i^* - z_{i+1}^*)^m, i = 1, \cdots, n-1;$$
$$(z_i^*)^{m+2} \leqslant (z_i^*)^m, i = 2, \cdots, n.$$

Hence, $f_{m,n+2}(\beta) \leqslant f_{m,n}(\beta)$, and there exists $N > 0$ such that $f_{m+2}(\beta) \leqslant f_m(\beta)$, , when $n > N$. By the aforementioned definition of $\beta^*(m)$, without losing generality, assuming that $0 \leqslant \beta \leqslant \beta^*(m)$, we obtain

$$f_{m-2}(\beta) \geqslant f_m(\beta) \geqslant \beta.$$

Therefore, $\beta^*(m-2) \geqslant \beta^*(m)$, which means $\alpha^*(m-2) \geqslant \alpha^*(m)$.

Since $\beta^*(m) \geqslant 0$ and $\beta^*(m)$ is nonincreasing, $\beta^* = \lim\limits_{m\to+\infty} \beta^*(m)$ exists. If $\beta^* \neq 0$, then there exists an $N > 0$, when $n > N$ and $m > N$, we have

$$f_{m,n}(\beta^*(m)) \geqslant f_m(\beta^*(m)) = \beta^*(m) \geqslant \beta^* > 0.$$

However, when $n > N$ and $m > N$, $f_{m,n}(\beta^*(m)) \to 0$, which is a contradiction. Hence, $\beta^* = 0$ and $\lim\limits_{m\to+\infty} \alpha^*(m) = 1$. $\square$

For computing the Sup-MO value $\alpha^*(m)$, we can achieve the goal by using a simple algorithm as follows,

Algorithm 2.1 (Computing $\alpha^*(m)$)**.**

S1: Let $\beta^0 = 1, \beta^1 = \beta_0^1 = 1, n = 1, k = 0, \varepsilon > 0,$ *m be an even number .*

S2: Solve the Problem (2.4) for getting $f_{m,n}(\beta_k^n)$.

S3: If $f_{m,n}(\beta_k^n) - \beta_k^n < -\varepsilon$, *then* $\beta_{k+1}^n = \frac{\beta_k^n}{2}$;

If $f_{m,n}(\beta_k^n) - \beta_k^n > \varepsilon$, $\beta_{k+1} = \frac{\beta_k^n + 1}{2}$, $k = k+1$, *go to S2;*

Else denote $\beta^n = \beta_k^n$.

S4: If $|\beta^n - \beta^{n-1}| < \varepsilon$*, stop and output* $\beta^n + 1$*;*

else $n = n + 1$*,* $k = 0$*, go to S2.*

It is not difficult to compute $f_{m,n}(\beta)$ since (2.4) is a convex problem. According to Algorithm 2.1, the numerical solutions for some $\alpha^*(m)$'s are:

$$\alpha^*(4) = 1.1429,\ \alpha^*(6) = 1.0323,\ \alpha^*(8) = 1.0079.$$

In the final work of this chapter, a property of the minimal eigenvalue of the Sup-MO tensor is proposed. Hence, the positive definiteness of Sup-MO tensors is proved.

Theorem 2.4. *Let* $\Omega(m)$ *be an MO set. For all even* $m \geqslant 4$ *and* $\alpha^*(m) = \max\{\Omega(m)\}$, $\mathscr{A}(m, n; \alpha^*(m))$ *is a Sup-MO tensor, i.e.,*

$$\mathscr{A}_{i_1,\cdots,i_m}(m, n; \alpha^*(m)) = \mathscr{M}(m, n) - \alpha^*(m)\mathscr{N}(m, n)$$
$$= \begin{cases} i_1, & i_1 = i_2 = \cdots = i_m, \\ \min\{i_1, i_2, \cdots, i_m\} - \alpha^*(m), & \textit{otherwise}. \end{cases}$$

Denote $\lambda_{min}(\mathscr{A}(m, n; \alpha^*(m)))$ *as the smallest eigenvalue of* $\mathscr{A}(m, n; \alpha^*(m))$*, we have* $\lambda_{min}(\mathscr{A}(m, n; \alpha^*(m)))$ *strictly decreases to* 0*, when* $n \to \infty$*. Therefore, we know that* $\mathscr{A}(m, n; \alpha^*(m))$ *is positive definite.*

Proof. According to Theorem 2.1, we can see that $\lambda_{min}(\mathscr{A}(m, n; \alpha^*(m)))$ decreases in n, for all even m. In the following, we show that it is strictly decreasing to 0. $g_{m,n}(\mathbf{z}, \beta)$, $f_{m,n}(\beta)$ and $f_m(\beta)$ are defined as Eqs. (2.3), (2.4) and (2.5), respectively.

Since $1 > f_m(\beta^*(m)) = \beta^*(m) > 0$, $f_{m,n}(\beta^*(m)) \to \beta^*(m)$. Assume that $\mathbf{z}^* \in \arg\min_{\mathbf{z}\in\mathbb{R}^n, z_1=1} g_{m,n}(\mathbf{z}, \beta^*(m))$. Because $z_1^* = 1$, $||\mathbf{z}^*||_m \geqslant 1$. Let $w_i^* = z_i^* - z_{i+1}^*, i = 1, \cdots, n-1$, $w_n^* = z_n^*$. Thus, when $m \geqslant 4$,

$$\beta^*(m) \leqslant g_{m,n}(\mathbf{z}^*, \beta^*(m)) = f_{m,n}(\beta^*) \leqslant f_{m,n}(1) < \frac{5}{2^m} < 1.$$

By the definition of $g_{m,n}(\mathbf{z}^*, \beta^*(m))$, $\sum_{i=2}^{n}(z_i^*)^m \leqslant \frac{5}{2^m} < 1$, which means that $|z_2^*| \leqslant \frac{5^{\frac{1}{m}}}{2} < 1$. Thus $w_1^* = 1 - z_2^* \geqslant 1 - \frac{5^{\frac{1}{m}}}{2}$. Hence, when $m \geqslant 4$, $||\mathbf{w}^*||_m \geqslant 1 - \frac{5^{\frac{1}{m}}}{2}$.

According to Theorem 2.1,

$$0 \leqslant \lambda_{min}(\mathscr{A}(m,n;\alpha^*(m))) \leqslant \frac{\mathscr{A}(m,n;\alpha^*(m))(\mathbf{w}^*)^m}{||\mathbf{w}^*||_m^m}.$$

By the definition of $\mathbf{w}^*$, $\mathscr{A}(m,n;\alpha^*(m))(\mathbf{w}^*)^m \to 0$, when $n \to +\infty$. Due to $||\mathbf{w}^*||_m \geqslant 1 - \frac{5^{\frac{1}{m}}}{2}$, we get that $\lambda_{min}(\mathscr{A}(m,n;\alpha^*(m))) \to 0$, when $n \to \infty$.

Next, we prove that the decreasing of $\lambda_{min}(\mathscr{A}(m,n;\alpha^*(m)))$ is strict. Consider the following program:

$$\begin{aligned} \min \quad & \mathscr{A}(m,n;\alpha^*(m))\mathbf{x}^m \\ s.t. \quad & ||\mathbf{x}||_m = 1. \end{aligned}$$

Then its KKT conditions are

$$\begin{cases} \mathscr{A}(m,n;\alpha^*(m))\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]} \\ ||\mathbf{x}||_m = 1. \end{cases} \tag{2.6}$$

The smallest solution $\lambda_{m,n}$ and the corresponding vector $\mathbf{x} \in \mathbb{R}^n$ of above program are the smallest H-eigenvalue and H-eigenvector of $\mathscr{A}(m,n;\alpha^*(m))$. If $\lambda_{m,n} = \lambda_{m,n+1}$ for some n, then there exist $\mathbf{x} \in \mathbb{R}^n$ and $\bar{\mathbf{x}} \in \mathbb{R}^{n+1}$ with $\bar{\mathbf{x}} = (\mathbf{x}^\top, 0)^\top$ satisfying

$$\mathscr{A}(m,n;\alpha^*(m))\mathbf{x}^{m-1} = \lambda_{m,n}\mathbf{x}^{[m-1]},$$

$$\mathscr{A}(m,n+1;\alpha^*(m))\bar{\mathbf{x}}^{m-1} = \lambda_{m,n+1}\bar{\mathbf{x}}^{[m-1]}.$$

Hence,

$$\sum_{i_2,\cdots,i_m=1}^{n+1} \mathscr{A}(m,n+1;\alpha^*(m))_{n+1,i_2,\cdots,i_m}\bar{x}_{i_2}\cdots\bar{x}_{i_m} = \lambda_{m,n+1}\bar{x}_{n+1}^{m-1}.$$

Because $\bar{\mathbf{x}} = (\mathbf{x}^\top, 0)^\top$, the above equation is

$$\sum_{i_2,\cdots,i_m=1}^{n} \mathscr{A}(m, n+1; \alpha^*(m))_{n+1,i_2,\cdots,i_m} x_{i_2} \cdots x_{i_m} = 0.$$

Since

$$\sum_{i_2,\cdots,i_m=1}^{n} \mathscr{A}(m, n; \alpha^*(m))_{n,i_2,\cdots,i_m} x_{i_2} \cdots x_{i_m} = \lambda_{m,n} x_n^{m-1},$$

we have

$$\begin{aligned} &\sum_{i_2,\cdots,i_m=1}^{n} \mathscr{A}(m, n; \alpha^*(m))_{n,i_2,\cdots,i_m} x_{i_2} \cdots x_{i_m} \\ -&\sum_{i_2,\cdots,i_m=1}^{n} \mathscr{A}(m, n+1; \alpha^*(m))_{n+1,i_2,\cdots,i_m} x_{i_2} \cdots x_{i_m} \\ =&\alpha^*(m) x_n^{m-1} = \lambda_{m,n} x_n^{m-1}. \end{aligned}$$

According to the above proof, $\alpha^*(m) > 1 > \lambda_{m,n}$. Therefore, $x_n = 0$. By the same discussion, we get $\mathbf{x} = 0$, which contradicts with $||\mathbf{x}||_m = 1$.

Thus, $\lambda_{min}(\mathscr{A}(m, n; \alpha^*(m)))$ strictly decreases. Finally, together with Corollary 2.1, we have $\mathscr{A}(m, n; \alpha^*(m))$ is positive definite. □

2.3 Final Remarks

In this chapter, we generalize the Moler matrix to the MO tensor by introducing the concepts of the MO values and the MO set. Then we mainly investigate two special cases of the MO tensor which are the Sup-MO tensor and the essential MO tensor. We show that an even order essential MO tensor is a completely positive tensor and positive definite. Then, some related properties of the Sup-MO value of an even order Sup-MO tensor are proposed. Moreover, the positive definiteness of an even order Sup-MO tensor is proved, because the minimal H-eigenvalue of the Sup-MO tensor strictly decreases to 0, when $n \to \infty$. However, there are still some further questions remain for the MO tensor, the Sup-MO tensor and the Sub-MO tensor:

1. Are the Sup-MO tensors SOS (sum-of-squares) tensors? An SOS tensor has good structure for verifying the positive definiteness. From the definition of SOS tensors [14], an SOS tensor is a PSD tensor, but not vice versa. This theory can be traced back to David Hilbert [34]. We have tested some Sup-MO tensors randomly, and found that they were SOS tensors. It is an interesting question for future study.

2. We have not verified whether $\alpha_*(m)$ can be reached or not. That is why we do not know whether $\Omega(m)$ is compact or not. If the $\Omega(m)$ is compact, how to get the length of the MO set $\Omega(m)$ is also a challenging work. It is necessary to explore the continuous properties of $\alpha_*(m)$ and $\Omega(m)$.

3. Since the good properties of the Moler matrices make them good test matrices for the eigensystems and linear equations, we are not sure that if the Sup-MO tensor can also be a good candidate for testing in some tensor computation software packages or not.

Chapter 3

Strong Ellipticity Condition for the Elasticity Tensors

In Section 1.3, we have briefly introduced the strong ellipticity condition of an elasticity tensor. We have converted the problem of verifying the SE-condition to identifying the minimal value of an optimization problem, the Problem (1.8) in Section 1.3. Due to the complexity of calculating the minimal value of Problem (1.8), we try to construct some sufficient and necessary condition for solving this problem by introducing some special structures for the tensors belongs to $\mathbb{E}_{4,n}$.

It is worth noting that the symmetric M-matrices, which is also called the Stieltjes matrices, are an essential kind of the positive semi-definite matrices used in many disciplines in engineering and science, such as numerical solutions of partial differential equations, the Markov chains, linear systems of equations, the graph theory, the queueing theory and so on [6]. In 2014, the higher order $\mathscr{M}$-tensors are introduced by Zhang, Qi, and Zhou [91]. They also showed the positive semi-definiteness of an even order symmetric $\mathscr{M}$-tensor. Ding, Qi, and Wei [23] established several equivalent definitions of nonsingular $\mathscr{M}$-tensors. It should be noted that, in some literature, the nonsingular $\mathscr{M}$-tensor is called the strong $\mathscr{M}$-tensor as well. In 2016, Ding and Wei [24] presented that, for any polynomial system of equations, if the right-hand side is a positive vector and its coefficient tensor is a nonsingular $\mathscr{M}$-tensor, then there exists a unique positive solution. They also presented an iterative algorithm to solve such systems. Moreover, other numerical methods have been proposed by Han [31], Xie, Jin, and Wei [87] and Li, Xie, and Xu [44]. Recently, in 2018, for the M-tensor equations, Bai, He, Ling, and Zhou [3] and Li, Guan, and Wang [43] considered their nonnegative solutions. In fact, the above $\mathscr{M}$-structure is defined in relation to the tensor eigenvalues introduced by Qi [58]. Considering [62, Chapter 2], a tensor is said to be M-positive (semi-)definite if its M-eigenvalues are all positive (or nonnegative). Hence, motivated by the above $\mathscr{M}$-structure, we will define the elasticity $\mathscr{M}$-tensors with respect to the M-eigenvalues, which will be proved to be M-positive semi-definite. Subsequently, we are able to find a large kind of tensors which satisfy the strong ellipticity condition.

In this chapter, we will first introduce the basic knowledge of several types of positive (semi-)definiteness of the elasticity tensors which are related to the SE-condition. Second, before defining the elasticity $\mathscr{M}$-tensors in the following, we will study the M-spectral radius of nonnegative elasticity tensors for preparation. Third, the nonsingular elasticity $\mathscr{M}$-tensors and the elasticity $\mathscr{M}$-tensors will be introduced, their M-positive (semi-)definiteness will be proved and we will propose some equivalent definitions for elasticity $\mathscr{M}$-tensors. Finally, we will also give the conclusion remarks.

3.1 SE-Condition and Positive Semi-Definiteness

Denote $\mathbb{E}_{4,n}$ as the set of all fourth order n-dimensional tensors who satisfy (1.6) as follows,

$$a_{ijkl} = a_{jikl} = a_{ijlk} = a_{klij}, i, j, k, l \in [n].$$

At first, we recall the equivalent optimization problem (1.8) in Section 1.3:

$$\begin{array}{ll} \min & \mathscr{A}\mathbf{x}^2\mathbf{y}^2, \\ \text{s.t.} & \mathbf{x}^\top\mathbf{x} = 1,\ \mathbf{y}^\top\mathbf{y} = 1. \end{array}$$

The KKT condition [5] of the minimization problem (1.8) is written as

$$\begin{cases} \mathscr{A}\mathbf{x}\mathbf{y}^2 = \lambda\mathbf{x}, \\ \mathscr{A}\mathbf{x}^2\mathbf{y} = \lambda\mathbf{y}, \\ \mathbf{x}^\top\mathbf{x} = 1,\ \mathbf{y}^\top\mathbf{y} = 1, \end{cases} \tag{3.1}$$

where $(\mathscr{A}\mathbf{x}\mathbf{y}^2)_i := \sum_{j,k,l=1}^n a_{ijkl}x_jy_ky_l$ and $(\mathscr{A}\mathbf{x}^2\mathbf{y})_l := \sum_{i,j,k=1}^n a_{ijkl}x_ix_jy_k$.

In this formulation, two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ are defined as an **M-eigenvalue** and a pair of corresponding **M-eigenvectors** of $\mathscr{A}$, respectively, by Qi, Dai, and Han in [61]. Therefore, a tensor satisfying the SE-condition is also called an **M-positive definite (M-PD)** tensor [62]. Similarly with Definition 1.1, the positive definiteness for an even order symmetric tensor, we say that a tensor $\mathscr{A} \in \mathbb{E}_{4,n}$ is **M-positive semi-definite (M-PSD)** if $\mathscr{A}\mathbf{x}^2\mathbf{y}^2 \geqslant 0$ for any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ [62]. The following theorem shows that the M-positive definiteness is treated equivalently to the positivity of a tensor's M-eigenvalues.

Theorem 3.1. [61] *A tensor in $\mathbb{E}_{4,n}$ is M-positive definite if and only if all of its M-eigenvalues are positive; A tensor in $\mathbb{E}_{4,n}$ is M-positive semi-definite if and only if all of its M-eigenvalues are nonnegative.*

Here, it is necessary to define a special tensor $\mathscr{E} \in \mathbb{E}_{4,n}$ by

$$e_{ijkl} = \begin{cases} 1, & \text{if } i = j \text{ and } k = l, \\ 0, & \text{otherwise,} \end{cases}$$

which serves as an identity element in $\mathbb{E}_{4,n}$. We call it the **identity tensor** in this work. When $n = 3$, the components of the identity tensor $\mathscr{E}$ are

$$e_{1111} = e_{1122} = e_{1133} = e_{2211} = e_{2222} = e_{2233} = e_{3311} = e_{3322} = e_{3333} = 1,$$

and others are 0. We can verify that

$$\mathscr{E}\mathbf{x}\mathbf{y}^2 = \mathbf{x}(\mathbf{y}^\top\mathbf{y}),\ \mathscr{E}\mathbf{x}^2\mathbf{y} = (\mathbf{x}^\top\mathbf{x})\mathbf{y},$$

and

$$\mathscr{E}\mathbf{x}^2\mathbf{y}^2 = (\mathbf{x}^\top\mathbf{x})(\mathbf{y}^\top\mathbf{y}).$$

Hence, we have the following homogeneous definition for M-eigenvalues:

$$\begin{cases} \mathscr{A}\mathbf{x}\mathbf{y}^2 = \lambda\mathscr{E}\mathbf{x}\mathbf{y}^2, \\ \mathscr{A}\mathbf{x}^2\mathbf{y} = \lambda\mathscr{E}\mathbf{x}^2\mathbf{y}. \end{cases} \tag{3.2}$$

By comparing (3.1) and (3.2), we can see that if the triplet $(\lambda, \mathbf{x}, \mathbf{y})$ satisfies (3.1) then $(\lambda, \alpha\mathbf{x}, \beta\mathbf{y})$ satisfies (3.2) for any nonzero real scalar α, β. Noted that (3.2) is exactly the KKT condition of the following minimization problem:

$$\begin{array}{ll} \min & \mathscr{A}\mathbf{x}^2\mathbf{y}^2, \\ \text{s.t.} & (\mathbf{x}^\top\mathbf{x})(\mathbf{y}^\top\mathbf{y}) = 1, \end{array} \tag{3.3}$$

whose global optimal value being positive is able to guarantee the SE-condition. The following proposition is an observation from the definition of the identity tensor.

Proposition 3.1. *Let $\mathscr{A} \in \mathbb{E}_{4,n}$. Assume that $\mathscr{B} = \alpha(\mathscr{A} + \beta\mathscr{E})$, where α, β are two real scalars. Then μ is an M-eigenvalue of $\mathscr{B}$ if and only if $\mu = \alpha(\lambda + \beta)$, where λ is an M-eigenvalue of $\mathscr{A}$. Furthermore, λ and μ correspond to the same M-eigenvectors.*

Proof. On the one hand, according to the definition of M-eigenvalue, if μ is an M-eigenvalue of $\mathscr{B}$, then $\mathscr{B}\mathbf{x}\mathbf{y}^2 = \mu\mathbf{x}$ and $\mathscr{B}\mathbf{x}\mathbf{y}^2 = \mathscr{B}\mathbf{x}^2\mathbf{y} = \mu\mathbf{y}$, where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are the corresponding M-eigenvectors. When $\alpha = 0$, the result is obvious. When $\alpha \neq 0$,

$$\alpha\mathscr{A}\mathbf{x}^2\mathbf{y} + \alpha\beta\mathbf{y} = \alpha\mathscr{A}\mathbf{x}^2\mathbf{y} + \alpha\beta\mathscr{E}\mathbf{x}^2\mathbf{y} = \alpha(\mathscr{A} + \beta\mathscr{E})\mathbf{x}^2\mathbf{y} = \mu\mathbf{y}$$

implies that

$$\mathscr{A}\mathbf{x}^2\mathbf{y} = (\alpha^{-1}\mu - \beta)\mathbf{y}.$$

Similarly, we have

$$\mathscr{A}\mathbf{x}\mathbf{y}^2 = (\alpha^{-1}\mu - \beta)\mathbf{x}.$$

Thus $\lambda = \alpha^{-1}\mu - \beta$ is the M-eigenvalue for $\mathscr{A}$ corresponding to $\mathbf{x}$ and $\mathbf{y}$.

On the other hand, when $\mu = \alpha(\lambda + \beta)$ and λ is an M-eigenvalue of $\mathscr{A}$ corresponding to the M-eigenvectors $\mathbf{x}$ and $\mathbf{y}$, it can get

$$\mu\mathbf{y} = \alpha(\lambda+\beta)\mathbf{y} = \alpha\lambda\mathbf{y} + \alpha\beta\mathbf{y} = \alpha\mathscr{A}\mathbf{x}^2\mathbf{y} + \alpha\beta\mathscr{E}\mathbf{x}^2\mathbf{y} = \alpha(\mathscr{A}+\beta\mathscr{E})\mathbf{x}^2\mathbf{y} = \mathscr{B}\mathbf{x}^2\mathbf{y},$$

$$\mu\mathbf{x} = \alpha(\lambda+\beta)\mathbf{x} = \alpha\lambda\mathbf{x} + \alpha\beta\mathbf{x} = \alpha\mathscr{A}\mathbf{x}\mathbf{y}^2 + \alpha\beta\mathscr{E}\mathbf{x}\mathbf{y}^2 = \alpha(\mathscr{A}+\beta\mathscr{E})\mathbf{x}\mathbf{y}^2 = \mathscr{B}\mathbf{x}\mathbf{y}^2.$$

Hence, μ is an M-eigenvalue of $\mathscr{B}$ corresponding to the same M-eigenvectors $\mathbf{x}$ and $\mathbf{y}$. $\square$

There is another kind of the positive semi-definiteness of the tensors in $\mathbb{E}_{4,n}$. Usually, there are two ways to unfold a tensor in $\mathbb{E}_{4,n}$ into n^2-by-n^2 matrices:

$$\text{(i)}\ \ \mathbf{A}_x = \begin{bmatrix} \mathbf{A}_x^{(1,1)} & \mathbf{A}_x^{(1,2)} & \cdots & \mathbf{A}_x^{(1,n)} \\ \mathbf{A}_x^{(2,1)} & \mathbf{A}_x^{(2,2)} & \cdots & \mathbf{A}_x^{(2,n)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_x^{(n,1)} & \mathbf{A}_x^{(n,2)} & \cdots & \mathbf{A}_x^{(n,n)} \end{bmatrix} \in \mathbb{R}^{n^2\times n^2},$$

$$\text{(ii)}\ \ \mathbf{A}_y = \begin{bmatrix} \mathbf{A}_y^{(1,1)} & \mathbf{A}_y^{(1,2)} & \cdots & \mathbf{A}_y^{(1,n)} \\ \mathbf{A}_y^{(2,1)} & \mathbf{A}_y^{(2,2)} & \cdots & \mathbf{A}_y^{(2,n)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_y^{(n,1)} & \mathbf{A}_y^{(n,2)} & \cdots & \mathbf{A}_y^{(n,n)} \end{bmatrix} \in \mathbb{R}^{n^2\times n^2},$$

where $\mathbf{A}_x^{(k,l)} := \mathscr{A}(:,:,k,l)(k,l=1,\cdots,n)$ and $\mathbf{A}_y^{(i,j)} := \mathscr{A}(i,j,:,:)(i,j=1,\cdots,n)$. Note that $\mathbf{A}_x$ and $\mathbf{A}_y$ are permutation similar to each other, i.e., there exists a permutation matrix $\mathbf{P}$ such that $\mathbf{A}_x = \mathbf{P}^\top\mathbf{A}_y\mathbf{P}$. Then $\mathscr{A}$ is M-PD or M-PSD if its corresponding matrix $\mathbf{A}_x$ (or equivalently $\mathbf{A}_y$) is PD or PSD, respectively. This can be proved by noticing that

$$\mathscr{A}\mathbf{x}^2\mathbf{y}^2 = (\mathbf{y}\otimes\mathbf{x})^\top\mathbf{A}_x(\mathbf{y}\otimes\mathbf{x}) = (\mathbf{x}\otimes\mathbf{y})^\top\mathbf{A}_y(\mathbf{x}\otimes\mathbf{y}),$$

where $\otimes$ denotes the Kronecker product [38]. Thus we call $\mathscr{A}$ S-positive (semi)definite if $\mathbf{A}_x$ or $\mathbf{A}_y$ is positive (semi-)definite, and call the eigenvalues of $\mathbf{A}_x$ or $\mathbf{A}_y$ the S-eigenvalues of $\mathscr{A}$. In fact, the S-positive definiteness is a sufficient condition for the M-positive definiteness, but the converse is not true. A counterexample is shown as follows.

Example 3.1. *Consider the case* $n = 3$. *Let* $\mathscr{A} \in \mathbb{E}_{4,3}$ *be defined by*

$$a_{1111} = a_{2222} = a_{3333} = 2, \quad a_{1221} = a_{2121} = a_{2112} = a_{1212} = 1,$$

and all other entries equal to zero. Then we have

$$\mathscr{A}\mathbf{x}^2\mathbf{y}^2 = 2(x_1y_1 + x_2y_2)^2 + 2x_3^2y_3^2,$$

thus $\mathscr{A}$ *is M-PSD apparently, while the unfolding matrix*

$$\mathbf{A}_x = \mathbf{A}_y = \left[\begin{array}{ccc|ccc|ccc} 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{array}\right]$$

is not positive semi-definite.

Next, several more notations will be introduced for convenience. Let $\mathscr{A} \in \mathbb{E}_{4,n}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Two n-by-n matrices $\mathscr{A}\mathbf{x}^2 \in \mathbb{R}^{n\times n}$ and $\mathscr{A}\mathbf{y}^2 \in \mathbb{R}^{n\times n}$ are defined by

$$(\mathscr{A}\mathbf{x}^2)_{kl} := \sum_{i,j=1}^{n} a_{ijkl}x_ix_j, \quad k, l = 1, 2, \ldots, n,$$

$$(\mathscr{A}\mathbf{y}^2)_{ij} := \sum_{k,l=1}^{n} a_{ijkl}y_ky_l, \quad i, j = 1, 2, \ldots, n.$$

We note that

$$\mathscr{A}\mathbf{x}^2 = \begin{bmatrix} \mathbf{x}^\top \mathbf{A}_x^{(1,1)}\mathbf{x} & \mathbf{x}^\top \mathbf{A}_x^{(1,2)}\mathbf{x} & \cdots & \mathbf{x}^\top \mathbf{A}_x^{(1,n)}\mathbf{x} \\ \mathbf{x}^\top \mathbf{A}_x^{(2,1)}\mathbf{x} & \mathbf{x}^\top \mathbf{A}_x^{(2,2)}\mathbf{x} & \cdots & \mathbf{x}^\top \mathbf{A}_x^{(2,n)}\mathbf{x} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{x}^\top \mathbf{A}_x^{(n,1)}\mathbf{x} & \mathbf{x}^\top \mathbf{A}_x^{(n,2)}\mathbf{x} & \cdots & \mathbf{x}^\top \mathbf{A}_x^{(n,n)}\mathbf{x} \end{bmatrix},$$

and

$$\mathscr{A}\mathbf{y}^2 = \begin{bmatrix} \mathbf{y}^\top \mathbf{A}_y^{(1,1)}\mathbf{y} & \mathbf{y}^\top \mathbf{A}_y^{(1,2)}\mathbf{y} & \cdots & \mathbf{y}^\top \mathbf{A}_y^{(1,n)}\mathbf{y} \\ \mathbf{y}^\top \mathbf{A}_y^{(2,1)}\mathbf{y} & \mathbf{y}^\top \mathbf{A}_y^{(2,2)}\mathbf{y} & \cdots & \mathbf{y}^\top \mathbf{A}_y^{(2,n)}\mathbf{y} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{y}^\top \mathbf{A}_y^{(n,1)}\mathbf{y} & \mathbf{y}^\top \mathbf{A}_y^{(n,2)}\mathbf{y} & \cdots & \mathbf{y}^\top \mathbf{A}_y^{(n,n)}\mathbf{y} \end{bmatrix}.$$

Furthermore, it is straightforward to verify that

$$\begin{aligned} &\mathscr{A}\mathbf{x}^2\mathbf{y}^2 = \mathbf{y}^\top(\mathscr{A}\mathbf{x}^2)\mathbf{y} = \mathbf{x}^\top(\mathscr{A}\mathbf{y}^2)\mathbf{x}, \\ &\mathscr{A}\mathbf{x}^2\mathbf{y} = (\mathscr{A}\mathbf{x}^2)\mathbf{y}, \quad \mathscr{A}\mathbf{x}\mathbf{y}^2 = (\mathscr{A}\mathbf{y}^2)\mathbf{x}. \end{aligned} \tag{3.4}$$

The symmetries in $\mathscr{A}$ imply that both $\mathscr{A}\mathbf{x}^2$ and $\mathscr{A}\mathbf{y}^2$ are symmetric matrix. According to Eq.(3.4), we can prove the following necessary and sufficient condition for the M-positive (semi-)definiteness.

Proposition 3.2. *Let $\mathscr{A} \in \mathbb{E}_{4,n}$. Then $\mathscr{A}$ is M-positive definite or M-positive semi-definite if and only if the matrix $\mathscr{A}\mathbf{x}^2$ (or $\mathscr{A}\mathbf{y}^2$) is positive definite or positive semi-definite for each nonzero $\mathbf{x} \in \mathbb{R}^n$ (or $\mathbf{y} \in \mathbb{R}^n$), respectively.*

Proof. On one side, if $\mathscr{A}$ is M-PD, then for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n\backslash\{\mathbf{0}\}$, $\mathscr{A}\mathbf{x}^2\mathbf{y}^2 > 0$. It means that $\mathbf{y}^\top(\mathscr{A}\mathbf{x}^2)\mathbf{y} > 0$, for any nonzero $\mathbf{y} \in \mathbb{R}^n$. Hence, the matrix $\mathscr{A}\mathbf{x}^2$ is positive definite for each nonzero $\mathbf{x} \in \mathbb{R}^n$. Similarity, $\mathscr{A}\mathbf{y}^2$ is positive definite for each nonzero $\mathbf{y} \in \mathbb{R}^n$. On the other side, when $\mathscr{A}\mathbf{x}^2$ (or $\mathscr{A}\mathbf{y}^2$) is positive definite for each nonzero $\mathbf{x} \in \mathbb{R}^n$ (or $\mathbf{y} \in \mathbb{R}^n$), it has $\mathbf{y}^\top(\mathscr{A}\mathbf{x}^2)\mathbf{y} > 0$ (or $\mathbf{x}^\top(\mathscr{A}\mathbf{y}^2)\mathbf{x} > 0$) for any nonzero $\mathbf{y} \in \mathbb{R}^n$ (or $\mathbf{x} \in \mathbb{R}^n$), which means that $\mathscr{A}$ is M-positive definite.
Similarity, $\mathscr{A}$ is M-positive semi-definite if and only if the matrix $\mathscr{A}\mathbf{x}^2$ (or $\mathscr{A}\mathbf{y}^2$) is positive semi-definite for each $\mathbf{x} \in \mathbb{R}^n$ (or $\mathbf{y} \in \mathbb{R}^n$), respectively. □

Generally speaking, the above necessary and sufficient condition is still as hard as the SE-condition to check. However, it motivates some checkable sufficient conditions. Therefore, we will introduce some sufficient conditions to verify the SE-condition.

3.2 Nonnegative Elasticity Tensors

There is a well-known result about nonnegative matrices called the Perron-Frobenius theorem [6], which states that the spectral radius of any nonnegative matrix is an eigenvalue with a nonnegative eigenvector and the eigenvector is positive and unique if the matrix is irreducible. In the past decades, the Perron-Frobenius theorem has been extended to higher order tensors by Chang, Pearson, and Zhang [12] and Yang and Yang [89, 90]. One may refer to [62, Chapter 3] for a whole picture of the nonnegative tensor theory. We will also obtain similar results for nonnegative elasticity tensors in this section.

From the discussions in Section 3.1, we have variational forms of the extremal M-eigenvalues. Let $\mathscr{B} \in \mathbb{E}_{4,n}$. Denote $\lambda_{\max}(\mathscr{B})$ and $\lambda_{\min}(\mathscr{B})$ as the maximal and the minimal M-eigenvalues of $\mathscr{B}$, respectively. Then

$$\begin{aligned} \lambda_{\max}(\mathscr{B}) &= \max\left\{\mathscr{B}\mathbf{x}^2\mathbf{y}^2 : \mathbf{x},\mathbf{y} \in \mathbb{R}^n,\ \mathbf{x}^\top\mathbf{x} = \mathbf{y}^\top\mathbf{y} = 1\right\}, \\ \lambda_{\min}(\mathscr{B}) &= \min\left\{\mathscr{B}\mathbf{x}^2\mathbf{y}^2 : \mathbf{x},\mathbf{y} \in \mathbb{R}^n,\ \mathbf{x}^\top\mathbf{x} = \mathbf{y}^\top\mathbf{y} = 1\right\}. \end{aligned} \tag{3.5}$$

The maximal absolute value of all the M-eigenvalues is called the **M-spectral radius** of a tensor in $\mathbb{E}_{4,n}$, denoted by $\rho_M(\cdot)$. Apparently, the M-spectral radius is equal to the greater one of the absolute values of the maximal and the minimal M-eigenvalues. The following theorem reveals that $\rho_M(\mathscr{B}) = \lambda_{\max}(\mathscr{B})$ when $\mathscr{B} \in \mathbb{E}_{4,n}$ is nonnegative.

Theorem 3.2. *The M-spectral radius of any nonnegative tensor in $\mathbb{E}_{4,n}$ is exactly its greatest M-eigenvalue. Furthermore, there is a pair of nonnegative M-eigenvectors corresponding to the M-spectral radius.*

Proof. It is enough to show that $\lambda_{\max}(\mathscr{B}) \geqslant \left|\lambda_{\min}(\mathscr{B})\right|$ for proving the first statement. For convenience, denote λ_1 and λ_2 as the maximal and the minimal M-eigenvalues of $\mathscr{B}$ respectively, and $(\mathbf{x}_1, \mathbf{y}_1)$ and $(\mathbf{x}_2, \mathbf{y}_2)$ are the corresponding M-eigenvectors. According to Eq. (3.5), we know that $\lambda_1 = \mathscr{B}\mathbf{x}_1^2\mathbf{y}_1^2$ and $\lambda_2 = \mathscr{B}\mathbf{x}_2^2\mathbf{y}_2^2$. Then employing the nonnegativity of the entries of $\mathscr{B}$, we have

$$|\lambda_2| = \left|\mathscr{B}\mathbf{x}_2^2\mathbf{y}_2^2\right| \leqslant \mathscr{B}|\mathbf{x}_2|^2|\mathbf{y}_2|^2 \leqslant \lambda_1.$$

Next, we consider the eigenvectors of the M-spectral radius. Assume that $\mathbf{x}_1$ or $\mathbf{y}_1$ is not a nonnegative vector. Then we also have

$$\lambda_1 = \mathscr{B}\mathbf{x}_1^2\mathbf{y}_1^2 \leqslant \mathscr{B}|\mathbf{x}_1|^2|\mathbf{y}_1|^2 \leqslant \lambda_1,$$

thus $\mathscr{B}|\mathbf{x}_1|^2|\mathbf{y}_1|^2 = \lambda_1$. Therefore, $\big(|\mathbf{x}_1|, |\mathbf{y}_1|\big)$ is also a pair of M-eigenvectors corresponding to λ_1, which is nonnegative. □

Theorem 3.2 can be regarded as the weak Perron-Frobenius theorem for the tensors in $\mathbb{E}_{4,n}$. Combining Theorem 3.2 and Eq. (3.5), we have the following corollary, which shrinks the feasible domain in Eq. (3.5).

Corollary 3.1. *Let $\mathscr{B} \in \mathbb{E}_{4,n}$. If $\mathscr{B}$ is nonnegative, then*

$$\rho(\mathscr{B}) = \max\left\{\mathscr{B}\mathbf{x}^2\mathbf{y}^2 : \mathbf{x}, \mathbf{y} \in \mathbb{R}_+^n,\ \mathbf{x}^\top\mathbf{x} = \mathbf{y}^\top\mathbf{y} = 1\right\}.$$

Corollary 3.2. *Let $\mathscr{B} \in \mathbb{E}_{4,n}$ be nonnegative. Then we have $\rho(\mathscr{B}) = 0$ if and only if $\mathscr{B} = \mathscr{O}$, where $\mathscr{O}$ is a zero tensor.*

Proof. On the one hand, if $\mathscr{B}$ is a zero tensor, then $\rho(\mathscr{B}) = 0$.

On the other hand, if $\rho(\mathscr{B}) = 0$, then we have $\mathscr{B}\mathbf{x}^2\mathbf{y}^2 = 0$, for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. This means that, for any $\mathbf{y} \in \mathbb{R}^n$, we can get $\mathbf{y}^\top(\mathscr{B}\mathbf{x}^2)\mathbf{y} = 0$. Hence, for any $\mathbf{x} \in \mathbb{R}^n$ and fixed $k, l \in \{1, 2, \cdots, n\}$,

$$\mathscr{B}\mathbf{x}^2 = \sum_{i,j=1}^{n} b_{ijkl}x_ix_j = \mathbf{O},$$

where $\mathbf{O}$ is a zero matrix.

Assume that $\mathbf{x} = \mathbf{e}_i$ ($i = 1, \cdots, n$), where $\mathbf{e}_i$ is the unit vectors whose i-th component is 1 and others are zero, we have $b_{iikl} = 0$ for all fixed $k, l \in \{1, 2, \cdots, n\}$. Furthermore, when $\mathbf{x} = \mathbf{e}_i + \mathbf{e}_j$ ($i \neq j$, and $i, j = 1, \cdots, n$), we have

$$b_{iikl} + 2b_{ijkl} + b_{jjkl} = 0$$

for all fixed $k, l \in \{1, 2, \cdots, n\}$. Thus we have $b_{ijkl} = 0$ for all fixed $k, l \in \{1, 2, \cdots, n\}$. This means that $\mathscr{B} = \mathscr{O}$ when $\rho(\mathscr{B}) = 0$.

□

Chang, Qi, and Zhou [13] also studied the strong ellipticity for nonnegative elasticity tensors. They introduced the singular values of a tensor $\mathscr{B} \in \mathbb{E}_{4,n}$ as

$$\begin{cases} \mathscr{B}\mathbf{x}\mathbf{y}^2 = \sigma \mathbf{x}^{[3]}, \\ \mathscr{B}\mathbf{x}^2\mathbf{y} = \sigma \mathbf{y}^{[3]}, \end{cases}$$

and they also investigated the Perron-Frobenius theorem for the singular values. Nevertheless, it is hard to find an identity tensor similar to the tensor $\mathscr{E}$ in our case, thus we may not be able to define a kind of $\mathscr{M}$-tensors with respect to their singular values. However, they introduced the definition for irreducibility of the elasticity tensors in [13]. Recall the notations of $\mathbf{A}_x^{(k,l)}$ ($k, l = 1, \cdots, n$) and $\mathbf{A}_y^{(i,j)}$ ($i, j = 1, \cdots, n$) in Section 3.1. Let $\mathscr{B} \in \mathbb{E}_{4,n}$ be nonnegative. If all the $n \times n$ matrices $\mathbf{B}_x^{(k,k)}$ ($k = 1, \cdots, n$) and $\mathbf{B}_y^{(i,i)}$ ($i = 1, \cdots, n$) are irreducible matrices, then the nonnegative elasticity tensor $\mathscr{B}$ is called **irreducible**[13]. With the irreducibility of an nonnegative elasticity tensor, a useful lemma can be proved.

Lemma 3.1. *Let $\mathscr{B} \in \mathbb{E}_{4,n}$. If $\mathscr{B}$ is nonnegative and irreducible, then there is a pair of positive M-eigenvectors corresponding to its M-spectral radius, i.e.,*

$$\rho(\mathscr{B}) = \max\left\{\mathscr{B}\mathbf{x}^2\mathbf{y}^2 : \mathbf{x}, \mathbf{y} \in \mathbb{R}^n_{++}, \mathbf{x}^\top \mathbf{x} = \mathbf{y}^\top \mathbf{y} = 1\right\},$$

where $\mathbb{R}^n_{++}$ is positive real vector field with dimension n.

Proof. Since $\mathscr{B}$ is nonnegative, from Theorem 3.2, there exists a pair of nonnegative M-eigenvectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$ corresponding to its M-spectral radius $\rho(\mathscr{B})$. Moreover, we have

$$\mathscr{B}\mathbf{x}^2 = \sum_{i,j=1}^{n} \mathbf{B}_y^{(i,j)} x_i x_j \geqslant \sum_{i=1}^{n} \mathbf{B}_y^{(i,i)} x_i x_i.$$

Due to the nonnegativity of $\mathbf{x}$, there exists an $i_0 \in \{1, \cdots, n\}$ such that $x_{i_0} > 0$. Hence,

$$\mathscr{B}\mathbf{x}^2 \geqslant \sum_{i=1}^{n} \mathbf{B}_y^{(i,i)} x_i x_i \geqslant \mathbf{B}_y^{(i_0,i_0)} x_{i_0} x_{i_0}.$$

Since $\mathscr{B}$ is irreducible, $\mathbf{B}_y^{(i,i)}$ $(i \in \{1, \cdots, n\})$ are irreducible matrices. Thus, $\mathscr{B}\mathbf{x}^2$ is also irreducible. Hence, the corresponding M-eigenvector $\mathbf{y}$ is positive such that

$$\mathscr{B}\mathbf{x}^2\mathbf{y} = \rho(\mathscr{B})\mathbf{y}.$$

Similarity, $\mathscr{B}\mathbf{y}^2$ is irreducible, and the corresponding M-eigenvector $\mathbf{x}$ is positive such that $\mathscr{B}\mathbf{x}\mathbf{y}^2 = \rho(\mathscr{B})\mathbf{x}$. In the summery, we have

$$\rho(\mathscr{B}) = \max\left\{\mathscr{B}\mathbf{x}^2\mathbf{y}^2 : \mathbf{x}, \mathbf{y} \in \mathbb{R}^n_{++}, \ \mathbf{x}^\top\mathbf{x} = \mathbf{y}^\top\mathbf{y} = 1\right\}.$$

□

However, when $\mathscr{B}$ is nonnegative and irreducible, not all positive M-eigenvectors are corresponding to its M-spectral radius, such as following example:

Example 3.2. *Let $\mathscr{B} \in \mathbb{E}_{4,2}$ be defined by*

$$b_{1111} = 4, \ b_{1122} = b_{2211} = 10, \ b_{2222} = 2,$$

$$b_{1112} = b_{1121} = b_{1211} = b_{2111} = 1,$$

$$b_{1212} = b_{1221} = b_{2112} = b_{2121} = 1,$$

and

$$b_{1222} = b_{2122} = b_{2212} = b_{2221} = 2.$$

It is a nonnegative irreducible elasticity tensor. By computing its M-eigenvalues and corresponding M-eigenvectors, we have $\lambda_{max} = 10.9075$, *and the M-eigenvectors are*

$$\mathbf{x}_1 = (0.2936, 0.9560)^\top, \mathbf{y}_1 = (0.9442, 0.3294)^\top,$$

and

$$\mathbf{x}_2 = (0.9442, 0.3294)^\top, \mathbf{y}_2 = (0.2936, 0.9560)^\top.$$

Furthermore, the second max M-eigenvalue is $\lambda_{2nd-max} = 10.5$. *The corresponding M-eigenvectors are* $\mathbf{x} = (0.7071, 0.7071)^\top$, $\mathbf{y} = (0.7071, 0.7071)^\top$. *Hence, not all the positive M-eigenvectors are corresponding to its M-spectral radius.*

3.3 Elasticity $\mathscr{M}$-Tensors

Recall that the identity tensor $\mathscr{E}$ is defined by $e_{iikk} = 1$ and other entries being zero. Let $\mathscr{A} \in \mathbb{E}_{4,n}$. Accordingly, we call the entries a_{iikk} $(i, k = 1, 2, \ldots, n)$ diagonal, and other entries are called off-diagonal. Obviously, the diagonal entries of an M-positive definite tensor must be positive, and the ones of an M-positive semi-definite tensor must be non-negative. It is worth noting that the diagonal entries of $\mathscr{A}$ also lie on the diagonal of its unfolding matrix.

A tensor in $\mathbb{E}_{4,n}$ is called an elasticity $\mathscr{Z}$-tensor if all its off-diagonal entries are non-positive. If $\mathscr{A} \in \mathbb{E}_{4,n}$ is an elasticity $\mathscr{Z}$-tensor, then we can always write it as $\mathscr{A} = s\mathscr{E} - \mathscr{B}$, where $\mathscr{B}$ is a nonnegative tensor in $\mathbb{E}_{4,n}$. Such partition of an elasticity $\mathscr{E}$-tensor is not unique. If a tensor $\mathscr{A} \in \mathbb{E}_{4,n}$ can be written as $\mathscr{A} = s\mathscr{E} - \mathscr{B}$ satisfying that $\mathscr{B} \in \mathbb{E}_{4,n}$ is nonnegative and $s \geqslant \rho_M(\mathscr{B})$, then we call $\mathscr{A}$ an **elasticity $\mathscr{M}$-tensor**. Furthermore, if $s > \rho_M(\mathscr{B})$, then we call $\mathscr{A}$ a **nonsingular elasticity $\mathscr{M}$-tensor**.

Theorem 3.3. *Let $\mathscr{A} \in \mathbb{E}_{4,n}$ be an elasticity $\mathscr{Z}$-tensor. Then $\mathscr{A}$ is a nonsingular elasticity $\mathscr{M}$-tensor if and only if $\alpha > \rho_M(\alpha\mathscr{E} - \mathscr{A})$, where $\alpha = \max\{a_{iikk} : i,k = 1,2,\ldots,n\}$.*

Proof. The "if" part is obvious by the partition $\mathscr{A} = \alpha\mathscr{E} - (\alpha\mathscr{E} - \mathscr{A})$. Thus we focus on the "only if" part. If $\mathscr{A}$ is a nonsingular elasticity $\mathscr{M}$-tensor, then it can be written as $\mathscr{A} = s\mathscr{E} - \mathscr{B}$ satisfying that $\mathscr{B} \in \mathbb{E}_{4,n}$ is nonnegative and $s > \rho_M(\mathscr{B})$. Denote $\beta = \min\{b_{iikk} : i,k = 1,2,\ldots,n\}$, then $\alpha = s - \beta$. Moreover, we can also write $\alpha\mathscr{E} - \mathscr{A} = \mathscr{B} - \beta\mathscr{E}$, thus $\rho_M(\alpha\mathscr{E} - \mathscr{A}) = \rho_M(\mathscr{B}) - \beta$. Therefore, $s > \rho_M(\mathscr{B})$ implies that $\alpha > \rho_M(\alpha\mathscr{E} - \mathscr{A})$. $\square$

The above theorem is a simple but useful observation. We can utilize this theorem to prove the following proposition, which reveals that any elasticity $\mathscr{M}$-tensor is the limit of a series of nonsingular elasticity $\mathscr{M}$-tensors. Hence, we may omit the proofs of following results for general elasticity $\mathscr{M}$-tensors, since it can be verified by taking limits of the results for nonsingular elasticity $\mathscr{M}$-tensors.

Proposition 3.3. *$\mathscr{A} \in \mathbb{E}_{4,n}$ is an elasticity $\mathscr{M}$-tensor if and only if $\mathscr{A} + t\mathscr{E}$ is a nonsingular elasticity $\mathscr{M}$-tensor for any $t > 0$.*

Proof. Since $\mathscr{A}$ is an elasticity $\mathscr{M}$-tensor, there exists a nonnegative elasticity tensor $\mathscr{B}$ with $s \geqslant \rho_M(\mathscr{B})$ such that $\mathscr{A} = s\mathscr{E} - \mathscr{B}$. Then for any $t > 0$, we have $\mathscr{A} + t\mathscr{E} = (s+t)\mathscr{E} - \mathscr{B}$. Clearly, $s + t > \rho(\mathscr{B})$, which implies that $\mathscr{A} + t\mathscr{E}$ is a nonsingular elasticity $\mathscr{M}$-tensor.

Conversely, if $\mathscr{A} + t\mathscr{E}$ is a nonsingular elasticity $\mathscr{M}$-tensor for any $t > 0$, then by the previous theorem we have $\alpha_t > \rho_M\big(\alpha_t\mathscr{E} - (\mathscr{A} + t\mathscr{E})\big)$, where α_t is the greatest diagonal entry of $\mathscr{A} + t\mathscr{E}$. Denote α as the largest diagonal entry of $\mathscr{A}$. Then $\alpha_t = \alpha + t$, thus $\alpha + t > \rho_M(\alpha\mathscr{E} - \mathscr{A})$ for any $t > 0$. When t approaches 0, it can be concluded that $\alpha \geqslant \rho_M(\alpha\mathscr{E} - \mathscr{A})$, which implies that $\mathscr{A}$ is an elasticity $\mathscr{M}$-tensor. $\square$

It is well-known that a symmetric nonsingular M-matrix is positive definite [6]. The same statement was also proved for symmetric nonsingular $\mathscr{M}$-tensors in [91]. Moreover, we shall show that a nonsingular elasticity $\mathscr{M}$-tensor is M-positive definite thus satisfies the strong ellipticity condition. In this spirit, we find a class of structured tensors that satisfies the strong ellipticity condition.

Theorem 3.4. *Let $\mathscr{A} \in \mathbb{E}_{4,n}$ be an elasticity $\mathscr{Z}$-tensor. Then $\mathscr{A}$ is a nonsingular elasticity $\mathscr{M}$-tensor if and only if $\mathscr{A}$ is M-positive definite; and $\mathscr{A}$ is an elasticity $\mathscr{M}$-tensor if and only if $\mathscr{A}$ is M-positive semidefinite.*

Proof. Denote $\mathscr{A} = s\mathscr{E} - \mathscr{B}$, where $\mathscr{B}$ is nonnegative.

If $\mathscr{A}$ is a nonsingular elasticity $\mathscr{M}$-tensor, then $s > \rho_M(\mathscr{B})$. By Eq. (3.5), we have $s > \mathscr{B}\mathbf{x}^2\mathbf{y}^2$ for all $\mathbf{x}^\top\mathbf{x} = \mathbf{y}^\top\mathbf{y} = 1$. Recall that $\mathscr{E}\mathbf{x}^2\mathbf{y}^2 = (\mathbf{x}^\top\mathbf{x})(\mathbf{y}^\top\mathbf{y})$. Then $s\mathscr{E}\mathbf{x}^2\mathbf{y}^2 > \mathscr{B}\mathbf{x}^2\mathbf{y}^2$, which is equivalent to $\mathscr{A}\mathbf{x}^2\mathbf{y}^2 > 0$ for all $\mathbf{x}^\top\mathbf{x} = \mathbf{y}^\top\mathbf{y} = 1$. Therefore $\mathscr{A}$ is M-positive definite.

On the other hand, suppose that $\mathscr{A}$ is M-positive definite, i.e., $\mathscr{A}\mathbf{x}^2\mathbf{y}^2 > 0$ for all $\mathbf{x}^\top\mathbf{x} = \mathbf{y}^\top\mathbf{y} = 1$. Then similarly we have $s = s\mathscr{E}\mathbf{x}^2\mathbf{y}^2 > \mathscr{B}\mathbf{x}^2\mathbf{y}^2$ for all $\mathbf{x}^\top\mathbf{x} = \mathbf{y}^\top\mathbf{y} = 1$. We know from Eq. (3.5) that $s > \rho_M(\mathscr{B})$, i.e, $\mathscr{A}$ is a nonsingular elasticity $\mathscr{M}$-tensor.

The result for general elasticity $\mathscr{M}$-tensors can be proved similarly. □

The following equivalent definitions for elasticity $\mathscr{M}$-tensors is straightforward corollary of Proposition 3.1, Lemma 3.1 and Theorem 3.4.

Corollary 3.3. *Let $\mathscr{A} \in \mathbb{E}_{4,n}$ be an elasticity $\mathscr{Z}$-tensor.*

1. *$\mathscr{A}$ is an (nonsingular) elasticity $\mathscr{M}$-tensor if and only if*

$$\min\left\{\mathscr{A}\mathbf{x}^2\mathbf{y}^2 : \mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+,\ \mathbf{x}^\top\mathbf{x} = \mathbf{y}^\top\mathbf{y} = 1\right\} \geqslant 0 (> 0).$$

2. *Further assume that $\mathscr{A}$ is irreducible. $\mathscr{A}$ is an (nonsingular) elasticity $\mathscr{M}$-tensor if and only if*

$$\min\left\{\mathscr{A}\mathbf{x}^2\mathbf{y}^2 : \mathbf{x},\mathbf{y} \in \mathbb{R}^n_{++},\ \mathbf{x}^\top\mathbf{x} = \mathbf{y}^\top\mathbf{y} = 1\right\} \geqslant 0 (> 0).$$

Recall that the S-eigenvalues of a tensor in $\mathbb{E}_{4,n}$ are defined by the eigenvalues of its unfolding matrices $\mathbf{A}_x$ and $\mathbf{A}_y$. Of course, we can also define $\mathscr{M}$-tensors with respect to S-eigenvalues, which coincide with those tensors $\mathscr{A}$ whose unfolding matrices $\mathbf{A}_x$ and $\mathbf{A}_y$ are $\mathbf{M}$-matrices. In this case, $\mathscr{A}$ is also M-positive semidefinite since $\mathbf{A}_x$ and $\mathbf{A}_y$ are positive semidefinite matrices. However, the converse may still not hold, when $\mathscr{A}$ is an elasticity $\mathscr{M}$-tensor, as shown by the following example.

Example 3.3. *Consider the case $n = 2$. Let $\mathscr{A} \in \mathbb{E}_{4,2}$ be defined by*

$$\begin{array}{lll} a_{1111} = 13, & a_{1122} = 2, & a_{2211} = 2, \\ a_{2222} = 12, & a_{1112} = -2, & a_{1211} = -2, \\ a_{1212} = -4, & a_{1222} = -1, & a_{2212} = -1. \end{array}$$

By our calculations with Mathematica, $\mathscr{A}$ has six M-eigenvalues:

$$13.4163,\ 12.1118,\ 11.2036,\ 6.1778,\ 0.2442,\ \textit{and}\ 0.1964.$$

The minimal M-eigenvalue of $\mathscr{A}$ is 0.1964, *which is positive. Thus $\mathscr{A}$ is a nonsingular elasticity $\mathscr{M}$-tensor by Theorem 3.4. Nonetheless, the unfolding matrices of $\mathscr{A}$ are*

$$\mathbf{A}_x = \mathbf{A}_y = \begin{bmatrix} 13 & -2 & -2 & -4 \\ -2 & 2 & -4 & -1 \\ -2 & -4 & 2 & -1 \\ -4 & -1 & -1 & 12 \end{bmatrix},$$

with four eigenvalues: -2.8331, 6.0000, 9.2221, *and* 16.6110. *There is a negative eigenvalue, which implies that $\mathbf{A}_x$ and $\mathbf{A}_y$ are not positive semidefinite and thus not* $\mathbf{M}$*-matrices.*

We now provide some equivalent definitions of nonsingular elasticity $\mathscr{M}$-tensors, which serve as verification conditions. Recall the definitions of the two n-by-n matrices $\mathscr{A}\mathbf{x}^2$ and $\mathscr{A}\mathbf{y}^2$ in Section 3.1. The next theorem shows that these two matrices admit the same structures with the original elasticity tensor.

Theorem 3.5. *Let $\mathscr{A} \in \mathbb{E}_{4,n}$ be an elasticity $\mathscr{Z}$-tensor. Then $\mathscr{A}$ is a nonsingular elasticity $\mathscr{M}$-tensor if and only if $\mathscr{A}\mathbf{x}^2$ is a nonsingular $\mathbf{M}$-matrix for each $\mathbf{x} \geqslant \mathbf{0}$; $\mathscr{A}$ is an elasticity $\mathscr{M}$-tensor if and only if $\mathscr{A}\mathbf{x}^2$ is an $\mathbf{M}$-matrix for each $\mathbf{x} \geqslant \mathbf{0}$.*

Proof. Suppose that $\mathscr{A}$ is a nonsingular elasticity $\mathscr{M}$-tensor. Then we know by Eq. (3.4) that $\mathscr{A}\mathbf{x}^2$ is positive definite for each $\mathbf{x} \in \mathbb{R}^n$ since $\mathscr{A}$ is M-positive definite. Another simple observation is that $\mathscr{A}\mathbf{x}^2$ is a $\mathbf{Z}$-matrix for each $\mathbf{x} \geqslant \mathbf{0}$ when $\mathscr{A}$ is an elasticity $\mathscr{Z}$-tensor. Thus $\mathscr{A}\mathbf{x}^2$ is a positive definite $\mathbf{Z}$-matrix for each $\mathbf{x} \geqslant \mathbf{0}$. From the equivalent definitions of nonsingular $\mathbf{M}$-matrices [6], it can be concluded that $\mathscr{A}\mathbf{x}^2$ is a nonsingular $\mathbf{M}$-matrix for each $\mathbf{x} \geqslant \mathbf{0}$.

Conversely, if $\mathscr{A}\mathbf{x}^2$ is a nonsingular $\mathbf{M}$-matrix for each $\mathbf{x} \geqslant \mathbf{0}$, then $\mathscr{A}\mathbf{x}^2$ is always positive definite. That is, $\mathscr{A}\mathbf{x}^2\mathbf{y}^2 = \mathbf{y}^\top \mathscr{A}\mathbf{x}^2\mathbf{y} > 0$ for each $\mathbf{x} \geqslant \mathbf{0}$ and $\mathbf{y} \in \mathbb{R}^n$. Write $\mathscr{A} = s\mathscr{E} - \mathscr{B}$, where $\mathscr{B}$ is nonnegative. Then $s > \mathscr{B}\mathbf{x}^2\mathbf{y}^2$ for each $\mathbf{x}, \mathbf{y} \geqslant \mathbf{0}$ satisfying $\mathbf{x}^\top\mathbf{x} = \mathbf{y}^\top\mathbf{y} = 1$. Hence, Corollary 3.1 tells that $s > \rho_M(\mathscr{B})$, i.e., $\mathscr{A}$ is a nonsingular elasticity $\mathscr{M}$-tensor. □

Similarly, we have a parallel result for $\mathscr{A}\mathbf{y}^2$.

Theorem 3.6. *Let $\mathscr{A} \in \mathbb{E}_{4,n}$ be an elasticity $\mathscr{Z}$-tensor. Then $\mathscr{A}$ is a nonsingular elasticity $\mathscr{M}$-tensor if and only if $\mathscr{A}\mathbf{y}^2$ is a nonsingular $\mathbf{M}$-matrix for each $\mathbf{y} \geqslant 0$; $\mathscr{A}$ is an elasticity $\mathscr{M}$-tensor if and only if $\mathscr{A}\mathbf{y}^2$ is an $\mathbf{M}$-matrix for each $\mathbf{y} \geqslant 0$.*

There is a well-known equivalent definition for nonsingular $\mathbf{M}$-matrices called semi-positivity. That is, a $\mathbf{Z}$-matrix $\mathbf{A}$ is a nonsingular $\mathbf{M}$-matrix if and only if there exits a positive (or equivalently nonnegative) vector $\mathbf{x}$ such that $\mathbf{A}\mathbf{x}$ is also a positive vector. Ding, Qi, and Wei [23] proved that this also holds for nonsingular $\mathscr{M}$-tensors. The semi-positivity is essential to verify whether a tensor is a nonsingular $\mathscr{M}$-tensor and is also important for solving the polynomial systems of equations with $\mathscr{M}$-tensors [24]. Combining the semi-positivity of nonsingular **M**-matrices and Theorems 3.5 and 3.6, we have the following equivalent conditions for nonsingular elasticity $\mathscr{M}$-tensors immediately.

Theorem 3.7. *Let $\mathscr{A} \in \mathbb{E}_{4,n}$ be an elasticity $\mathscr{Z}$-tensor. The following conditions are equivalent:*

1. *$\mathscr{A}$ is a nonsingular elasticity $\mathscr{M}$-tensor.*
2. *For each $\mathbf{x} \geqslant \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$, there exists $\mathbf{y} > \mathbf{0}$ such that $\mathscr{A}\mathbf{x}^2\mathbf{y} > \mathbf{0}$.*
3. *For each $\mathbf{x} \geqslant \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$, there exists $\mathbf{y} \geqslant \mathbf{0}$ such that $\mathscr{A}\mathbf{x}^2\mathbf{y} > \mathbf{0}$.*
4. *For each $\mathbf{y} \geqslant \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$, there exists $\mathbf{x} > \mathbf{0}$ such that $\mathscr{A}\mathbf{x}\mathbf{y}^2 > \mathbf{0}$.*
5. *For each $\mathbf{y} \geqslant \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$, there exists $\mathbf{x} \geqslant \mathbf{0}$ such that $\mathscr{A}\mathbf{x}\mathbf{y}^2 > \mathbf{0}$.*

A matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is called strictly diagonally dominant if

$$|a_{ii}| > \sum_{j\neq i} |a_{ij}|, \quad i = 1, 2, \ldots, n.$$

Condition (2) in Theorem 3.7 states that for each nonnegative vector $\mathbf{x}$, there exists a positive vector $\mathbf{y}$ such that $\mathscr{A}\mathbf{x}^2\mathbf{y} = \mathscr{A}\mathbf{x}^2\mathbf{y} > \mathbf{0}$. Denote a diagonal matrix $\mathbf{D}$ with $d_{ii} = y_i$ for $i = 1, 2, \ldots, n$ and $\widetilde{\mathbf{A}} := (\mathscr{A}\mathbf{x}^2)\mathbf{D}$. When $\mathscr{A}$ be an elasticity $\mathscr{Z}$-tensor, the matrix $\widetilde{\mathbf{A}}$ is also a **Z**-matrix. Thus we have

$$|\widetilde{a}_{ii}| - \sum_{j\neq i} |\widetilde{a}_{ij}| = \widetilde{a}_{ii} + \sum_{j\neq i} \widetilde{a}_{ij} = (\mathscr{A}\mathbf{x}^2\mathbf{y})_i > 0, \quad i = 1, 2, \ldots, n,$$

which implies that $\widetilde{\mathbf{A}}$ is strictly diagonally dominant. Applying the above discussion, we can prove the following corollary of Theorem 3.7.

Corollary 3.4. *Let $\mathscr{A} \in \mathbb{E}_{4,n}$ be an elasticity $\mathscr{Z}$-tensor. The following conditions are equivalent:*

1. *$\mathscr{A}$ is a nonsingular elasticity $\mathscr{M}$-tensor.*
2. *For each $\mathbf{x} \geqslant \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$, there exists a positive diagonal matrix $\mathbf{D}$ such that $\mathbf{D}(\mathscr{A}\mathbf{x}^2)\mathbf{D}$ is strictly diagonally dominant.*
3. *For each $\mathbf{y} \geqslant \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$, there exists a positive diagonal matrix $\mathbf{D}$ such that $\mathbf{D}(\mathscr{A}\mathbf{y}^2)\mathbf{D}$ is strictly diagonally dominant.*

3.4 Final Remarks

In this chapter, we have established several sufficient and necessary conditions for the strong ellipticity (M-positive definiteness) of general elasticity tensors. At first, we briefly introduced two types of positive semi-definiteness which have connections with the strong ellipticity condition for the elasticity tensors. We mainly discuss the M-positive semi-definiteness in Section 3.1.

Next, we consider the properties for nonnegative elasticity tensors. A Perron-Frobenius type theorem for M-spectral radii of a nonnegative elasticity tensor has been proposed in Section 3.2. Then we investigate a class of tensors satisfying the SE-condition, the elasticity $\mathscr{M}$-tensor. Combining Theorems 3.3 – 3.7 and Corollarys 3.3, 3.4, we summarize the equivalent definitions for nonsingular elasticity $\mathscr{M}$-tensors given in this work. Let $\mathscr{A} \in \mathbb{E}_{4,n}$ be an elasticity $\mathscr{Z}$-tensor. The following conditions are equivalent:

1. $\mathscr{A}$ is a nonsingular elasticity $\mathscr{M}$-tensor.
2. $\mathscr{A}$ is M-positive definite, i.e., $\mathscr{A}\mathbf{x}^2\mathbf{y}^2 > 0$ for all nonzero $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
3. $\min\left\{\mathscr{A}\mathbf{x}^2\mathbf{y}^2 : \mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+, \mathbf{x}^\top\mathbf{x} = \mathbf{y}^\top\mathbf{y} = 1\right\} > 0$.

4. All the M-eigenvalues of $\mathscr{A}$ are positive.

5. $\alpha > \rho_M(\alpha\mathscr{E} - \mathscr{A})$, where $\alpha = \max\left\{a_{iikk} : i, k = 1, 2, \ldots, n\right\}$.

6. For each $\mathbf{x} \geqslant \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$, $\mathscr{A}\mathbf{x}^2$ is a nonsingular $\mathbf{M}$-matrix.

7. For each $\mathbf{x} \geqslant \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$, there exists $\mathbf{y} > \mathbf{0}$ such that $\mathscr{A}\mathbf{x}^2\mathbf{y} > \mathbf{0}$.

8. For each $\mathbf{x} \geqslant \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$, there exists $\mathbf{y} \geqslant \mathbf{0}$ such that $\mathscr{A}\mathbf{x}^2\mathbf{y} > \mathbf{0}$.

9. For each $\mathbf{x} \geqslant \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$, there exists a positive diagonal matrix $\mathbf{D}$ such that $\mathbf{D}(\mathscr{A}\mathbf{x}^2)\mathbf{D}$ is strictly diagonally dominant.

10. For each $\mathbf{y} \geqslant \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$, $\mathscr{A}\mathbf{y}^2$ is a nonsingular $\mathbf{M}$-matrix.

11. For each $\mathbf{y} \geqslant \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$, there exists $\mathbf{x} > \mathbf{0}$ such that $\mathscr{A}\mathbf{x}\mathbf{y}^2 > \mathbf{0}$.

12. For each $\mathbf{y} \geqslant \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$, there exists $\mathbf{x} \geqslant \mathbf{0}$ such that $\mathscr{A}\mathbf{x}\mathbf{y}^2 > \mathbf{0}$.

13. For each $\mathbf{y} \geqslant \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$, there exists a positive diagonal matrix $\mathbf{D}$ such that $\mathbf{D}(\mathscr{A}\mathbf{y}^2)\mathbf{D}$ is strictly diagonally dominant.

However, there are still some open questions. One of the most important questions is that could we design some algorithms for the nonnegative elasticity tensors or elasticity $\mathscr{M}$-tensors such that all the M-eigenvalues are able to be calculated effectively? Hence, we can verify the SE-condition for the elasticity tensors with those special structures easily.

Chapter 4

Tensor Invariants

It is well-known that tensor is one of the fundamental tools in the physical area. In Section 1.4, we have briefly introduced the importance of the tensor representation theory which is a topic that focuses on the tensor invariants. Hence, in this chapter, our main goal is to investigate the representations for two kinds of special tensors in physics: the third order three-dimensional symmetric tensors and the third order three-dimensional Hall tensors.

For convenience, in this chapter, the summation convention, the Einstein notation, is used. If an index repeated twice in a single term and is not otherwise defined, then it means that this term is summed up with respect to this index from 1 to 3. For instance, denote $\boldsymbol{G}$ as any second order tensor with components g_{ij}, after an orthogonal transformation under an orthogonal tensor $\boldsymbol{Q}$, we have

$$\langle \boldsymbol{Q} \rangle \boldsymbol{G} = \boldsymbol{Q}\boldsymbol{G}\boldsymbol{Q}^\top,$$

and its components are

$$(\boldsymbol{Q}\boldsymbol{G}\boldsymbol{Q}^\top)_{rs} = \sum_{i,j=1}^{3} q_{ri}q_{sj}g_{ij} := q_{ri}q_{sj}g_{ij}.$$

In this chapter, we will first give some basic concepts in the tensor representation theory, such as the isotropic invariant, the hemitropic invariant, the integrity basis, the function basis and so on. Then, we will give an irreducible function basis for the third order three-dimensional symmetric tensors in the next section. Furthermore, we study another kind of important third order three-dimensional tensors in physics, the Hall tensor. We build a connection between a Hall tensor and a second order three-dimensional tensor which leads us to find a minimal integrity basis and an irreducible function basis for that Hall tensor.

4.1 Basic Concepts in Tensor Representation Theory

Assume that $\mathscr{A}$ is an mth order tensor represented by $a_{i_1 i_2 \cdots i_m}$ under a certain orthonormal coordinate $\boldsymbol{\epsilon}_i \otimes \boldsymbol{\epsilon}_j \otimes \boldsymbol{\epsilon}_k$. We call a scalar-valued tensor function $f(\mathscr{A})$ an **isotropic invariant** of $\mathscr{A}$ if it is invariant under any orthogonal transformations, including reflections and rotations, i.e.,

$$f(\langle \boldsymbol{Q} \rangle \mathscr{A}) = f(\mathscr{A}),$$

or equivalently expressed by

$$f(q_{i_1 j_1} q_{i_2 j_2} \ldots q_{i_m j_m} a_{j_1 j_2 \ldots j_m}) = f(a_{i_1 i_2 \ldots i_m}),$$

where $\boldsymbol{Q}$ is a second order n-dimensional orthogonal tensor ($\boldsymbol{Q}^\top \boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{Q}^\top = \boldsymbol{I}$) with components q_{ij}. If $f(\mathscr{A})$ is only invariant under rotations, i.e., $f(\langle \boldsymbol{Q} \rangle \mathscr{A}) = f(\mathscr{A})$ for any orthogonal tensor $\boldsymbol{Q}$ with $\det \boldsymbol{Q} = 1$, then it is called a **hemitropic invariant** of tensor $\mathscr{A}$ [93]. Furthermore, if $f(\mathscr{A})$ is a polynomial, then it is called a polynomial invariant of $\mathscr{A}$. In Section 1.4, we have mentioned that tensor functions and constitutive laws can all be assumed as polynomials. Hence, in this work, invariants always stand for polynomial invariants unless specific remarks are made there.

For any second order tensor, since it keeps unaltered under the central inversion $-\boldsymbol{I}$, the isotropic invariants and the hemitropic invariants are equivalent [93]. Nevertheless, any isotropic polynomial invariant of a third order tensor has to be the summation of several even order degree polynomials.

Now we briefly review the definitions and properties of (minimal) integrity bases and (irreducible) function bases of a tensor.

Definition 4.1. [93] *Let $\Psi = \{\psi_1, \psi_2, \ldots, \psi_r\}$ be a set of isotropic (or hemitropic, respectively) invariants of a tensor $\mathscr{A}$.*

1. *Ψ is said to be* **polynomial irreducible** *if none of $\psi_1, \psi_2, \ldots, \psi_r$ can be expressed by a polynomial of the remainders.*

2. *Ψ is called an isotropic (or hemitropic, respectively)* **integrity basis** *if any isotropic (or hemitropic, respectively) invariant of $\mathscr{A}$ is expressible by a polynomial of $\psi_1, \psi_2, \ldots, \psi_r$.*

3. *Ψ is called an isotropic (or hemitropic, respectively)* **minimal integrity basis** *if it is polynomial irreducible and an isotropic (or hemitropic, respectively) integrity basis.*

Definition 4.2. [93] *Let $\Psi = \{\psi_1, \psi_2, \ldots, \psi_r\}$ be a set of isotropic (or hemitropic, respectively) invariants of a tensor $\mathscr{A}$.*

1. *An invariant in Ψ is said to be* **functionally irreducible** *if it cannot be expressed by a single-valued function of the remainders, Ψ is said to be functionally irreducible if all of $\psi_1, \psi_2, \ldots, \psi_r$ are functionally irreducible.*

2. *Ψ is called an isotropic (or hemitropic, respectively)* **function basis** *if any isotropic (or hemitropic, respectively) invariant of $\mathscr{A}$ is expressible by a function of $\psi_1, \psi_2, \ldots, \psi_r$.*

3. *Ψ is called an isotropic (or hemitropic, respectively)* **irreducible function**

basis *if it is functionally irreducible and is an isotropic (or hemitropic, respectively) function basis.*

According to Definitions 4.1 and 4.2, it is straightforward to demonstrate that an isotropic (or hemitropic, respectively) integrity basis is an isotropic (or hemitropic, respectively) function basis, but the converse is not true in general. Hence, the number of invariants in an isotropic (or hemitropic, respectively) irreducible function basis is less than or equivalent with the number of invariants in an isotropic (or hemitropic, respectively) minimal integrity basis. For example, in next section, we will prove the number of the irreducible function basis of a third order symmetric traceless tensor is 11 which is less than 13 which is the number of invariants in its minimal integrity basis [18].

Particularly, Olive, Kolev and Auffry have proved that the number of invariants of each degree in an isotropic (or hemitropic, respectively) minimal integrity basis is always fixed [54]. Nevertheless, it is still unclear whether the number of invariants of an irreducible function basis is fixed.

Unfortunately, the number of invariants in a minimal integrity basis of a tensor sometimes can be quite big. For instance, the number of minimal integrity basis for an elasticity tensor is 297 [54]. From an experimental point of view, it is impossible to detect all the values of the invariants in such a big minimal integrity basis of a tensor. However, as we mentioned before, a minimal integrity basis for a tensor is also a function basis, the number of invariants in an irreducible function basis consisting of polynomial invariants is less than or equal to that of a minimal integrity basis. Hence, it is meaningful to study the irreducible function basis of a tensor. This also motivates us to study the irreducible function basis for the third order three-dimensional symmetric tensor and the Hall tensor in the following sections.

4.2 Irreducible Function Basis for a Third Order Three-Dimensional Symmetric Tensor

4.2.1 Previous work

In this subsection, we will recall some previous work for some types of the third order three-dimensional tensors in the representation theory. These works include Smith and Bao's minimal integrity basis result for a third order three-dimensional symmetric and traceless tensor [75] , the consequent result of Chen, Hu, Qi and Zou [15] which has confirmed that Smith and Bao's minimal integrity basis is also an irreducible function basis, and the minimal integrity basis result for a third order three-dimensional symmetric tensor of Olive and Auffray [53].

In 1997, Smith and Bao [75] presented a minimal integrity basis of an irreducible third order three-dimensional tensor. An irreducible tensor in the physical field means that this tensor is not only symmetric but also traceless. Denote $\mathscr{G}$ as any third order tensor. It is said to be traceless if the traces of all slices of its representations are 0. In [16], they proved that the traceless property of tensors is preserved under orthogonal transformations. The following theorem is from the work of Smith and Bao [75] and shows the details of the minimal integrity basis.

Theorem 4.1. [75] *Let $\mathscr{D}$ be an irreducible (i.e., symmetric and traceless) third order three-dimensional tensor. Denote $v_p := d_{ijk}d_{ij\ell}d_{k\ell p}$, and*

$$\begin{array}{ll} I_2 := d_{ijk}d_{ijk}, & I_4 := d_{ijk}d_{ij\ell}d_{pqk}d_{pq\ell}, \\ I_6 := v_iv_i, & I_{10} := d_{ijk}v_iv_jv_k. \end{array} \tag{4.1}$$

Then the set of all the invariants in (4.1), i.e., $\Theta_{irr} := \{I_2, I_4, I_6, I_{10}\}$, is a minimal integrity basis of $\mathscr{D}$.

We can see that this minimal integrity basis contains an invariant with degree 2 (I_2), an invariant with degree 4 (I_4), an invariant with degree 6 (I_6) and an invariant with degree 10 (I_{10}), which are all even degree polynomials of the components of $\mathscr{D}$. Very recently, Chen, Hu, Qi and Zou [15] proved the following theorem to show the irreducible function basis of $\mathscr{D}$.

Theorem 4.2. [15] *Under the notation of Theorem 4.1, the Smith-Bao minimal integrity basis $\Theta_{irr} = \{I_2, I_4, I_6, I_{10}\}$ is also an irreducible function basis of $\mathscr{D}$.*

According to [94], a third order three-dimensional symmetric tensor $\mathscr{A}$ can be decomposed into a vector $\mathbf{u}$ and a third order three-dimensional symmetric and traceless tensor $\mathscr{D}$, with

$$u_i = a_{i\ell\ell}$$

and

$$d_{ijk} = a_{ijk} - \frac{1}{5}\left(u_k\delta_{ij} + u_j\delta_{ik} + u_i\delta_{jk}\right),$$

where $\delta_{pq} = 1$ if $p = q$ and $\delta_{pq} = 0$ if $p \neq q$.

Then, in 2014, Olive and Auffray [53] presented an integrity basis for a third order three-dimensional symmetric tensor in the following theorem.

Theorem 4.3. [53] *Let $\mathscr{A}$ be a third order three-dimensional symmetric tensor with the above decomposition. The following thirteen invariants*

$$\begin{array}{ll}
I_2 := d_{ijk}d_{ijk}, & J_2 := u_iu_i, \\
I_4 := d_{ijk}d_{ij\ell}d_{pqk}d_{pq\ell}, & J_4 := d_{ijk}u_kd_{ij\ell}u_\ell, \\
K_4 := d_{ijk}d_{ij\ell}d_{k\ell p}u_p, & L_4 := d_{ijk}u_ku_ju_i, \\
I_6 := v_iv_i, & J_6 := d_{ijk}d_{ij\ell}u_kd_{\ell pq}u_pu_q, \\
K_6 := v_kw_k, & L_6 := d_{ijk}d_{ij\ell}u_kv_\ell, \\
M_6 := d_{ijk}d_{pqk}u_iu_ju_pu_q, & I_8 := d_{ijk}d_{ij\ell}u_kd_{pq\ell}d_{pqr}v_r, \\
I_{10} := d_{ijk}v_iv_jv_k, &
\end{array} \tag{4.2}$$

where $v_p := d_{ijk}d_{ij\ell}d_{k\ell p}$ and $w_k := d_{ijk}u_iu_j$, form an integrity basis

$$\Theta^{(1)}_{sym} := \{I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, K_6, L_6, M_6, I_8, I_{10}\}$$

of $\mathscr{A}$.

In the integrity basis $\Theta^{(1)}_{sym}$ of $\mathscr{A}$, there are two invariants with degree 2, four invariants with degree 4, five invariants with degree 6, one invariant with degree 8 and one invariant with degree 10. Because an integrity basis is always a function basis, we are able to start from the Olive-Auffray integrity basis to find an irreducible function basis of $\mathscr{A}$.

4.2.2 An eleven invariant function basis

Denote

$$\Theta^{(2)}_{sym} = \{I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6, M_6, I_{10}\},$$

where $I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6, M_6, I_{10}$ are defined in Eq. (4.2). Now we will show that $\Theta^{(2)}_{sym}$ is a function basis of the third order three-dimensional symmetric tensor $\mathscr{A}$ in this subsection. Note that the set $\Theta^{(2)}_{sym}$ is obtained by dropping K_6 and I_8 from the Olive-Auffray integrity basis $\Theta^{(1)}_{sym}$ in Theorem 4.3. Hence, we need to show that K_6 and I_8 can be represented by a single-valued function of those remained invariants. But first, we prove the following proposition.

Proposition 4.1. *In the Olive-Auffray integrity basis, we have*

$$2I_2J_2 - 3J_4 \geqslant 0,$$

where equality holds if and only if either $\mathscr{D} = \mathscr{O}$ *or* $\mathbf{u} = \mathbf{0}$.

Proof. According to the definition in Theorem 4.3, if either $\mathscr{D} = \mathscr{O}$ or $\mathbf{u} = \mathbf{0}$, we have $I_2J_2 = 0$ and $J_4 = 0$. Hence $2I_2J_2 - 3J_4 = 0$ in this case.

Then, consider the optimization problem

$$\min\{2I_2J_2 - 3J_4 : d_{ijk}d_{ijk} = 1, u_iu_i = 1.\},$$

where the variables are the seven independent components of $\mathscr{D}$ and the three components of $\mathbf{u}$. Utilizing GloptiPoly 3 [33] and SeDuMi [78], we compute the minimum value of this optimization problem is 0.2, where the minimizer is

$$\begin{array}{ccc} d_{111} = 0.2829, & d_{112} = d_{113} = 0, & d_{122} = -0.2828, \\ d_{123} = -0.2450, & d_{222} = 0, & d_{223} = -0.2828, \\ u_1 = -0.4471, & u_2 = -0.7746, & u_3 = -0.4474. \end{array}$$

Therefore, the minimum value is positive. This implies that if $2I_2J_2 - 3J_4 = 0$ then either $\mathscr{D} = \mathscr{O}$ or $\mathbf{u} = \mathbf{0}$. □

Before we prove the following theorem, we need to mention a phenomenon in the representation theory. Sometimes, there exists some function relations among the elements of an integrity basis or a function basis. These relations are called syzygies. If we could find any single-valued function for an invariant in a function basis of a tensor from these syzygies, then we can make sure that the invariant does not belong to the irreducible function basis of the tensor. Then we could prove the following theorem.

Theorem 4.4. *The eleven invariant set*

$$\Theta_{sym}^{(2)} = \{I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6, M_6, I_{10}\}$$

is a function basis of the third order three-dimensional symmetric tensor $\mathscr{A}$.

Proof. Consider all possible tenth degree powers and products of these thirteen invariants $I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, K_6, L_6, M_6, I_8, I_{10}$ in the Olive-Auffray minimal integrity basis $\Theta_{sym}^{(1)}$ of $\mathscr{A}$. We find linear relations among these tenth degree powers and products. Thus, we have two syzygy relations among these thirteen invariants as follows.

$$\begin{aligned} 6J_2I_8 = {} & -I_2^2J_2K_4 - I_2^3L_4 + 3I_2I_4L_4 - 3I_2J_4K_4 + 4J_2I_4K_4 \\ & + 2I_2^2J_6 + 3I_2J_2L_6 - 3L_4I_6 - 6I_4J_6 + 3J_4L_6 + 6K_4K_6, \end{aligned} \tag{4.3}$$

and

$$2I_2J_2K_6 + I_2^2J_2J_4 - I_2J_4^2 + 2I_2K_4L_4 + 3J_2K_4^2 - 2J_2I_4J_4$$
$$+ J_2^2I_6 - 2I_2^2M_6 - 12K_4J_6 + 6L_4L_6 + 6I_4M_6 - 3J_4K_6 = 0.$$

i.e.,

$$\begin{aligned}(2I_2J_2 - 3J_4)K_6 = &-I_2^2J_2J_4 + I_2J_4^2 - 2I_2K_4L_4 - 3J_2K_4^2 + 2J_2I_4J_4 \\ &- J_2^2I_6 + 2I_2^2M_6 + 12K_4J_6 - 6L_4L_6 - 6I_4M_6.\end{aligned} \tag{4.4}$$

We begin with using the syzygy relation (4.3). If $\mathbf{u} = \mathbf{0}$, then $J_2 = u_iu_i = 0$, and the right-hand side of (4.3) equals zero. In this case, we have

$$I_8 = d_{ijk}d_{ij\ell}u_kd_{pq\ell}d_{pqr}v_r = 0,$$

where $v_p := d_{ijk}d_{ij\ell}d_{k\ell p}$. If $\mathbf{u} \neq \mathbf{0}$, then $J_2 = u_iu_i \neq 0$. By the syzygy relation (4.3), we have

$$I_8 = -\frac{1}{6}I_2^2K_4 + \frac{2}{3}I_4K_4 + \frac{1}{2}I_2L_6 + \frac{1}{6J_2}(-I_2^3L_4 + 3I_2I_4L_4$$
$$- 3I_2J_4K_4 + 2I_2^2J_6 - 3L_4I_6 - 6I_4J_6 + 3J_4L_6 + 6K_4K_6).$$

Then I_8 is a single-valued function of $I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, K_6, L_6, M_6$ and I_{10}.

We now use the syzygy relation (4.4). If $2I_2J_2 - 3J_4 = 0$, by Proposition 4.1, either $\mathscr{D} = \mathscr{O}$ or $\mathbf{u} = \mathbf{0}$. It implies that $K_6 = 0$. Note in this case, the right-hand side of (4.4) is also equal to zero. If $2I_2J_2 - 3J_4 \neq 0$, we have

$$K_6 = \frac{1}{2I_2J_2 - 3J_4}(-I_2^2J_2J_4 + I_2J_4^2 - 2I_2K_4L_4 - 3J_2K_4^2$$
$$+ 2J_2I_4J_4 - J_2^2I_6 + 2I_2^2M_6 + 12K_4J_6 - 6L_4L_6 - 6I_4M_6).$$

This shows that K_6 is a function of $I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6, M_6$ and I_{10}.

Hence, $\Theta_{sym}^{(2)} = \{I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6, M_6, I_{10}\}$ is a function basis of the third order three-dimensional symmetric tensor $\mathscr{A}$. □

4.2.3 An irreducible function basis for a third order three-dimensional symmetric tensor

In order to show that $\Theta_{sym}^{(2)} = \{I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6, M_6, I_{10}\}$ is an irreducible function basis of the third order three-dimensional symmetric tensor $\mathscr{A}$, we should prove that each of these eleven invariants is not a function of the other ten invariants.

For proving that each of K_4, L_4, J_6 and L_6 is not a function of the ten other invariants in this function basis, we need the following proposition.

Proposition 4.2. *We have the following four conclusions.*

(a) If there is a third order three-dimensional tensor $\mathscr{A}$ such that $K_4 = L_4 = J_6 = 0$ but $L_6 \neq 0$, then L_6 is not a function of $I_2, J_2, I_4, J_4, K_4, L_4,\ I_6, J_6, M_6$ and I_{10}.

(b) If there is a third order three-dimensional tensor $\mathscr{A}$ such that $K_4 = L_4 = L_6 = 0$ but $J_6 \neq 0$, then J_6 is not a function of $I_2, J_2, I_4, J_4, K_4, L_4,\ I_6, L_6, M_6$ and I_{10}.

(c) If there is a third order three-dimensional tensor $\mathscr{A}$ such that $K_4 = J_6 = L_6 = 0$ but $L_4 \neq 0$, then L_4 is not a function of $I_2, J_2, I_4, J_4, K_4,\ I_6, J_6, L_6, M_6$ and I_{10}.

(d) If there is a third order three-dimensional tensor $\mathscr{A}$ such that $L_4 = J_6 = L_6 = 0$ but $K_4 \neq 0$, then K_4 is not a function of $I_2, J_2, I_4, J_4, L_4,\ I_6, J_6, L_6, M_6$ and I_{10}.

Proof. There is an observation from the definition of invariants $I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6, M_6$ and I_{10}. We find that, in the invariants K_4, L_4, J_6 and L_6, the degrees of components from $\mathbf{u}$ are odd and the degrees of components from $\mathscr{D}$ are even. Meanwhile, in the invariants $I_2, J_2, I_4, J_4, I_6, M_6$ and I_{10}, the degrees of components both from $\mathbf{u}$ and from $\mathscr{D}$ are even. Hence, when we keep $\mathscr{D}$ unchanged but change $\mathbf{u}$ to $-\mathbf{u}$, invariants $I_2, J_2, I_4, J_4, I_6, M_6$ and I_{10} will be unchanged, while K_4, L_4, J_6 and L_6 change their signs.

For conclusion (a), if there is a third order three-dimensional tensor $\mathscr{A}$ such that $K_4 = L_4 = J_6 = 0$ but $L_6 \neq 0$, we may keep $\mathscr{D}$ unchanged but change $\mathbf{u}$ to $-\mathbf{u}$, then $I_2, J_2, I_4, J_4, I_6, M_6$ and I_{10} are unchanged, K_4, L_4 and J_6 are still zeros, but L_6 changes its sign and value as it is not zero. It implies that L_6 is not a function of $I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, M_6$ and I_{10}. The other three conclusions (b), (c) and (d) can be proved similarly. □

Now we can present the most important theorem of this section.

Theorem 4.5. *For any given third order three-dimensional symmetric tensor $\mathscr{A}$, the eleven invariant set $\Theta_{sym}^{(2)} = \{I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6, M_6, I_{10}\}$ is an irreducible function basis of $\mathscr{A}$.*

Proof. By Theorem 4.4, $\Theta_{sym}^{(2)}$ is a function basis of $\mathscr{A}$. Now we need to show that each of these eleven invariants is not a function of the ten other invariants. We will divide the proof into three parts.

Part (i). In this part, we present that each of I_2, I_4, I_6, I_{10} and J_2 is not a function of the other ten invariants. The first four invariants belong to the irreducible function basis Θ_{irr} of the symmetric and traceless tensor $\mathscr{D}$. The fifth invariant J_2 forms an irreducible function basis of the vector $\mathbf{u}$. Applying this property, it is a direct observation that each of them is not a function of the other ten invariants easily.

By Theorem 4.2, $\Theta_{irr} = \{I_2, I_4, I_6, I_{10}\}$ is an irreducible function basis of $\mathscr{D}$. This implies that each of these four invariants is not a function of the other three invariants. Hence, each of these four invariants is not a function of the ten other invariants of $\Theta_{sym}^{(2)}$.

Assume that $\mathscr{D} = \mathscr{O}$, and $\mathbf{u} \neq \mathbf{u}'$ such that $u_i u_i \neq u'_i u'_i$. Then J_2 takes two different values but the other ten invariants $I_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6, M_6$ and I_{10} are all zero. It shows that J_2 is not a function of the ten other invariants $I_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6, M_6$ and I_{10}.

Part (ii). In this part, we present that each of K_4, L_4, J_6 and L_6 is not a function of the ten other invariants. We use Proposition 4.2 to attain this result.

First, we show that L_6 is not a function of the other ten invariants. Denote $a_{111}, a_{112}, a_{113}, a_{122}, a_{123}, a_{133}, a_{222}, a_{223}, a_{233}$ and a_{333} as the representatives of the components of $\mathscr{A}$. If the values of these ten components are fixed, then the other components of $\mathscr{A}$ are also fixed by symmetry. Let

$$a_{111} = \frac{3}{5},\ a_{122} = \frac{6}{5},\ a_{133} = -\frac{4}{5},\ a_{223} = \frac{1}{2},\ a_{333} = -\frac{1}{2},$$

and

$$a_{112} = a_{113} = a_{123} = a_{222} = a_{233} = 0.$$

Then we have

$$K_4 = L_4 = J_6 = 0 \text{ and } L_6 = -2,$$

to make sure that we can use Proposition 4.2 (a). The values of the other invariants are: $I_2 = 7, J_2 = 1, I_4 = \frac{37}{2},\ J_4 = 2,\ I_6 = 4,\ M_6 = 0, I_{10} = 4$. According to Proposition 4.2 (a), L_6 is not a function of $I_2, J_2, I_4, J_4, K_4, L_4,\ I_6, J_6, M_6$ and I_{10}.

Then we show that J_6 is not a function of the other ten invariants. Let

$$a_{111} = \frac{1}{6}\sqrt{\frac{1}{2}(149 - \sqrt{313})} - \frac{18(-215 + 7\sqrt{313})}{5\sqrt{8053043 - 308071\sqrt{313}}},$$

$$a_{112} = \frac{121(2963 - 103\sqrt{313})}{10(-215 + 7\sqrt{313})}\sqrt{\frac{298 - 2\sqrt{313}}{648164815 - 26977811\sqrt{313}}},$$

$$a_{113} = \frac{3966519 - 219867\sqrt{313}}{5\sqrt{648164815 - 26977811\sqrt{313}}(-215 + 7\sqrt{313})},$$

$$a_{122} = -\frac{6(-215 + 7\sqrt{313})}{5\sqrt{8053043 - 308071\sqrt{313}}}, \qquad a_{123} = 1,$$

$$a_{133} = -\frac{1}{6}\sqrt{\frac{1}{2}(149 - \sqrt{313})} - \frac{6(-215 + 7\sqrt{313})}{5\sqrt{8053043 - 308071\sqrt{313}}},$$

$$a_{222} = \frac{363(2963 - 103\sqrt{313})}{10(-215 + 7\sqrt{313})}\sqrt{\frac{298 - 2\sqrt{313}}{648164815 - 26977811\sqrt{313}}},$$

$$a_{223} = 1 + \frac{3966519 - 219867\sqrt{313}}{5\sqrt{648164815 - 26977811\sqrt{313}}(-215 + 7\sqrt{313})},$$

$$a_{233} = \frac{121(2963 - 103\sqrt{313})}{10(-215 + 7\sqrt{313})}\sqrt{\frac{298 - 2\sqrt{313}}{648164815 - 26977811\sqrt{313}}},$$

$$a_{333} = -1 + \frac{3(3966519 - 219867\sqrt{313})}{5\sqrt{648164815 - 26977811\sqrt{313}}(-215 + 7\sqrt{313})}.$$

These values imply $K_4 = L_4 = L_6 = 0$ and $J_6 \neq 0$. Except that $a_{123} = 1$, the approximate digit values of the other independent components are as follows:

$$a_{111} = 1.554,\ a_{112} = -0.1877,\ a_{113} = -0.01287,$$

$$a_{122} = 0.06780,\ a_{133} = -1.283,\ a_{222} = -0.5631,$$

$$a_{223} = 0.9871,\ a_{233} = -0.1877,\ a_{333} = -1.039.$$

Then we have

$$K_4 = L_4 = L_6 = 0 \text{ and } J_6 = 0.5112,$$

satisfying the condition of Proposition 4.2 (b). The values of the other invariants are $I_2 = 17.29, J_2 = 1, I_4 = 132.6, J_4 = 2.547, I_6 = 83.81, M_6 = 0.1687$ and $I_{10} = -831$. According to Proposition 4.2 (b), J_6 is not a function of $I_2, J_2, I_4, J_4, K_4, L_4, I_6, L_6, M_6$ and I_{10}.

Then we show that L_4 is not a function of the other ten invariants. With the same method, we can find a symmetric third order three-dimensional tensor $\mathscr{A}$ such that $K_4 = J_6 = L_6 = 0$ and $L_4 \neq 0$. In details, except that $a_{123} = 1$, the approximate digit values of the other independent components are as follows:

$$\begin{array}{lll} a_{111} = 1.0358, & a_{112} = 0.06373, & a_{113} = -0.06357, \\ a_{122} = 1.8269, & a_{133} = -1.9697, & a_{222} = 0.1912, \\ a_{223} = 0.9364, & a_{233} = 0.06373, & a_{333} = -1.1907. \end{array}$$

Then we have $K_4 = J_6 = L_6 = 0$ and $L_4 = -0.3843$, satisfying the condition of Proposition 4.2 (c). We also have $I_2 = 32.2465$, $J_2 = 1$, $I_4 = 394.69$, $J_4 = 9.1213$, $I_6 = 509.67$, $M_6 = 3.2506$ and $I_{10} = 17825.1$. By Proposition 4.2 (c), L_4 is not a function of $I_2, J_2, I_4, J_4, K_4,\ I_6, J_6, L_6, M_6$ and I_{10}.

We further show that K_4 is not a function of the other ten invariants. Let

$$\begin{array}{lll}
a_{111} = \frac{3}{5\sqrt{2}}, & a_{112} = \frac{\sqrt{3}}{10}, & a_{113} = \frac{1}{10}, \\
a_{122} = \frac{4\sqrt{2}}{15} - \frac{1}{\sqrt{3}}, & a_{123} = \frac{1}{3} + \frac{1}{\sqrt{6}}, & a_{133} = -\frac{\sqrt{2}}{15} + \frac{1}{\sqrt{3}}, \\
a_{222} = \frac{3\sqrt{3}}{10}, & a_{223} = -\frac{9}{10}, & a_{233} = \frac{\sqrt{3}}{10}, \\
a_{333} = \frac{13}{10}. & &
\end{array}$$

Then we get $L_4 = J_6 = L_6 = 0$ and $K_4 = \frac{8}{9}$, satisfying the condition of Proposition 4.2 (d). We also have

$$\begin{array}{llll}
I_2 = 8, & J_2 = \frac{3}{2}, & I_4 = \frac{88}{3}, & J_4 = \frac{8}{3}, \\
I_6 = \frac{64}{9}, & M_6 = \frac{11}{9}, & I_{10} = \frac{11776}{729}. &
\end{array}$$

According to Proposition 4.2 (d), K_4 is not a function of $I_2, J_2, I_4, J_4, L_4, I_6, J_6, L_6, M_6$ and I_{10}.

Part (iii). Since we cannot use Proposition 4.2 to show that each of M_6 and J_4 is not a function of the ten other invariants. We will use another tactics in this part. We try to find a tensor $\mathscr{A}$ there such that $K_4 = L_4 = J_6 = L_6 = 0$ to reduce the influence of these four invariants. Then we change some components of $\mathscr{A}$ such that K_4, L_4, J_6 and L_6 remain at zero, the value of M_6 or J_4 is changed and the values of the remaining other six invariants are unchanged.

We first show that M_6 is not a function of the ten other invariants. Denote $u_1 = 5a, u_2 = 5b, u_3 = 5c$, $d_{123} = d$ and the other six independent components of $\mathscr{D}$ be zeros. Assume that $a = b = 0$ and $c = d = 1$. Then

$$I_2 = 6, J_2 = 25, I_4 = 12, J_4 = 50, K_4 = L_4 = I_6 = J_6 = L_6 = I_{10} = 0 \text{ and } M_6 = 0.$$

Assume that $a = b = \frac{\sqrt{2}}{2}$, $c = 0$ and $d = 1$. We still have

$$I_2 = 6, J_2 = 25, I_4 = 12, J_4 = 50, K_4 = L_4 = I_6 = J_6 = L_6 = I_{10} = 0,$$

but $M_6 = 625$. Hence, M_6 is not a function of $I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6$ and I_{10}.

Finally, we prove that J_4 is not a function of the ten other invariants. Let

$$\begin{array}{lll} a_{111} = \frac{3}{5}\cos\theta, & a_{112} = \frac{1}{5}\sin\theta, & a_{113} = 0, \\ a_{122} = \frac{1}{5}\cos\theta, & a_{123} = 1, & a_{133} = \frac{1}{5}\cos\theta, \\ a_{222} = \frac{3}{5}\sin\theta, & a_{223} = 1, & a_{233} = \frac{1}{5}\sin\theta \\ a_{333} = -1. & & \end{array}$$

Then we have $K_4 = L_4 = J_6 = L_6 = 0$. We also have $I_2 = 10,\ J_2 = 1,\ I_4 = 44,\ I_6 = 16,\ I_{10} = -64$, and

$$J_4(\theta) = 2 + 4\cos\theta\sin\theta + 2\sin^2\theta,$$

$$M_6(\theta) = \sin^2\theta(2\cos\theta + \sin^2\theta).$$

Clearly, $J_4(\frac{3}{4}\pi) = 1$, $M_6(\frac{3}{4}\pi) = \frac{1}{4}$, $M_6(0) = 0$ and $M_6(\frac{\pi}{4}) = \frac{9}{4}$. Since $M_6(\theta)$ is continuous in the interval $[0, \frac{\pi}{4}]$, there exists $\theta_0 \in (0, \frac{\pi}{4})$ such that $M_6(\theta_0) = M_6(\frac{3}{4}\pi) = \frac{1}{4}$. On the other hand, we have

$$J_4'(\theta) = 4\cos(2\theta) + 2\sin(2\theta) \geqslant 0, \qquad \forall\theta \in \left[0, \frac{\pi}{4}\right].$$

It follows that $J_4(\theta_0) \geqslant J_4(0) = 2 > J_4(\frac{3}{4}\pi) = 1$. Thus, J_4 is not a function of I_2, J_2, I_4, K_4, L_4, I_6, J_6, L_6, M_6 and I_{10}.

Combining the results of all these three parts, each of these eleven invariants is not a function of the ten other invariants. Therefore, this eleven invariant set $\Theta^{(2)}_{sym} = \{I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6, M_6, I_{10}\}$ is indeed an irreducible function basis of $\mathscr{A}$. □

The strategy of Part (i) and the first part of Part (iii) of this proof to show that each of I_2, J_2, I_4, I_6, M_6, I_{10} is not a function of the other ten invariants may follow the idea of [55]. For self-sufficiency and completeness of this thesis, we give this part of the proof directly.

4.3 Representations for the Hall Tensor

Except for the third order three-dimensional symmetric tensors in physics, there are some other special third order three-dimensional tensors, such as the piezoelectric tensors, Hall tensors, and so on. It is worth noting that the Hall tensor, which comes from the Hall effect, plays an important role in physics. Since there is no work about the integrity basis and function basis of the Hall tensor, we will work on this problem for the Hall tensor in this section.

It is well-known that the Hall effect is an essential magnetic effect observed in electric semiconductors and conductors [32]. This effect was named after Edwin Hall who discovered it in 1879 [29]. When an electric current density **J** is flowing through a plate and the plate is simultaneously immersed in a magnetic field **F** with a component transverse to the current, the electric field strength **E** is proportional to current density and magnetic field strength

$$E_i = h_{ijk} J_j F_k,$$

where the third order tensor $\mathscr{H}$ with components h_{ijk} is called the **Hall tensor** [32]. Since the Onsager relation for transport processes with time reversal is valid, we know that the components of the Hall tensor under any orthonormal basis satisfy $h_{ijk} = -h_{jik}$ for all $i, j, k = 1, 2, 3$. The Hall tensor is quite significant for describing the electromagnetic induction. Hence, it is vital to investigate the minimal integrity basis and irreducible function basis for the Hall tensor. Meanwhile, in physics, there are other tensors which are third order three-dimensional tensors whose first two indices are antisymmetric. For example, the tensors in the Faraday effect [32].

In this section, we are devoted to the invariants of the Hall tensors. We will build a connection between the invariants of a Hall tensor and that of a second order tensor, which is useful for the subsequent contents. Moreover, a minimal isotropic integrity basis with 10 isotropic invariants of the Hall tensor will be proposed. Then, we will prove that the minimal integrity basis with 10 invariants of the Hall tensor is also its irreducible function basis. Different from Section 4.2, definitions for hemitropic and isotropic invariants which are mentioned at the beginning of this chapter play an important role in this part.

4.3.1 Connection between the Hall tensor and the second order three-dimensional tensor

Denote $\mathscr{H}$ as a Hall tensor represented by h_{ijk} under an orthogonal basis $\boldsymbol{\epsilon}_i \otimes \boldsymbol{\epsilon}_j \otimes \boldsymbol{\epsilon}_k$. Define a second order tensor $\boldsymbol{A}$ accordingly, with components a_{ij} under this orthogonal basis, by the tensor product operation

$$\boldsymbol{A} := \tfrac{1}{2}\boldsymbol{\varepsilon}\mathscr{H},$$

or equivalently

$$(a_{ij})\boldsymbol{\epsilon}_i \otimes \boldsymbol{\epsilon}_j = (\tfrac{1}{2}\varepsilon_{kli}h_{klj})\boldsymbol{\epsilon}_i \otimes \boldsymbol{\epsilon}_j,$$

where $\boldsymbol{\varepsilon}$ is the third order Levi-Civita tensor. The third order three-dimensional Levi-Civita tensor is also called the permutation tensor. Its components are presented as

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1, & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ 0, & \text{otherwise} \end{cases},$$

i.e., $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$, $\varepsilon_{321} = \varepsilon_{213} = \varepsilon_{132} = -1$ and others are 0.

Conversely, the Hall tensor can also be expressed with this second order tensor by

$$\mathscr{H} := \boldsymbol{\varepsilon}\boldsymbol{A},$$

or equivalently

$$(h_{ijk})\boldsymbol{\epsilon}_i \otimes \boldsymbol{\epsilon}_j \otimes \boldsymbol{\epsilon}_k = (\varepsilon_{ijl} a_{lk})\boldsymbol{\epsilon}_i \otimes \boldsymbol{\epsilon}_j \otimes \boldsymbol{\epsilon}_k.$$

Due to the anti-symmetric property of the first two indices of the components in a Hall tensor, we know that there are nine independent components in a Hall tensor $\mathscr{H}$. Without loss of generality, denote the nine independent components of the Hall tensor $\mathscr{H}$ as:

$$h_{121}, h_{122}, h_{123}, h_{131}, h_{132}, h_{133}, h_{231}, h_{232} \text{ and } h_{233}.$$

Thus, under a right-handed certain coordinate, the representation of the associated second order tensor can be mathematically written into a matrix form:

$$\begin{pmatrix} h_{231} & h_{232} & h_{233} \\ -h_{131} & -h_{132} & -h_{133} \\ h_{121} & h_{122} & h_{123} \end{pmatrix} \in \mathbb{R}^{3\times 3}.$$

Hence, we are able to prove the following theorem which reveals the connection between the invariants of the Hall tensor and the ones of its associated second order tensor.

Theorem 4.6. *Assume that $\mathscr{H}$ is a Hall tensor with components h_{ijk}. $\boldsymbol{A}(\mathscr{H})$ is used for denoting the associated second order tensor of $\mathscr{H}$.*

1. *Any isotropic invariant of $\mathscr{H}$ is an isotropic invariant of $\boldsymbol{A}(\mathscr{H})$;*

2. *Any isotropic invariant of $\boldsymbol{A}(\mathscr{H})$ with even degree is an isotropic invariant of $\mathscr{H}$, and any isotropic invariant of $\boldsymbol{A}(\mathscr{H})$ with odd degree is a hemitropic invariant of $\mathscr{H}$.*

Proof. (1) An isotropic invariant $f(\mathscr{H})$ of the Hall tensor $\mathscr{H}$ is also a polynomial function of its associated second order tensor $\boldsymbol{A}(\mathscr{H})$, denoted by $g(\boldsymbol{A}) := f(\boldsymbol{\varepsilon}\boldsymbol{A})$.

Thus, we need to prove that $g(\boldsymbol{A})$ is an isotropic invariant of $\boldsymbol{A}(\mathscr{H})$. Denote $\boldsymbol{Q}$ as any orthogonal tensor. By the definition of isotropic invariants, we have

$$f(\langle \boldsymbol{Q} \rangle \mathscr{H}) = f(\mathscr{H}) = f(\boldsymbol{\varepsilon A}) = g(\boldsymbol{A}).$$

Utilizing the equality $\langle \boldsymbol{Q} \rangle \boldsymbol{\varepsilon} = (\det \boldsymbol{Q})\boldsymbol{\varepsilon}$, then

$$\begin{aligned} f(\langle \boldsymbol{Q} \rangle \mathscr{H}) &= f(\langle \boldsymbol{Q} \rangle (\boldsymbol{\varepsilon A})) = f(\langle \boldsymbol{Q} \rangle \boldsymbol{\varepsilon} \langle \boldsymbol{Q} \rangle \boldsymbol{A}) \\ &= f((\det \boldsymbol{Q})\boldsymbol{\varepsilon} \langle \boldsymbol{Q} \rangle \boldsymbol{A}) = g((\det \boldsymbol{Q})\langle \boldsymbol{Q} \rangle \boldsymbol{A}). \end{aligned} \tag{4.5}$$

Since an isotropic invariant of a third order tensor must be an even function, we have

$$g((\det \boldsymbol{Q})\langle \boldsymbol{Q} \rangle \boldsymbol{A}) = g(\langle \boldsymbol{Q} \rangle \boldsymbol{A}).$$

Hence, $g(\langle \boldsymbol{Q} \rangle \boldsymbol{A}) = g(\boldsymbol{A})$, i.e., $g(\boldsymbol{A})$ is an isotropic invariant of $\boldsymbol{A}$.

(2) Denote $g(\boldsymbol{A})$ as an invariant of $\boldsymbol{A}(\mathscr{H})$. It is also a polynomial of the Hall tensor $\mathscr{H}$ which can be denoted by $f(\mathscr{H}) := g(\frac{1}{2}\boldsymbol{\varepsilon}\mathscr{H})$. For any orthogonal tensor $\boldsymbol{Q}$, since $g(\boldsymbol{A})$ is an invariant, we get

$$g(\langle \boldsymbol{Q} \rangle \boldsymbol{A}) = g(\boldsymbol{A}) = g(\boldsymbol{\varepsilon}\mathscr{H}) = f(\mathscr{H}).$$

Recall that $\langle \boldsymbol{Q} \rangle \boldsymbol{\varepsilon} = (\det \boldsymbol{Q})\boldsymbol{\varepsilon}$. Then

$$\begin{aligned} g(\langle \boldsymbol{Q} \rangle \boldsymbol{A}) &= g(\langle \boldsymbol{Q} \rangle (\boldsymbol{\varepsilon}\mathscr{H})) = g(\langle \boldsymbol{Q} \rangle \boldsymbol{\varepsilon} \langle \boldsymbol{Q} \rangle \mathscr{H}) \\ &= g((\det \boldsymbol{Q})\boldsymbol{\varepsilon} \langle \boldsymbol{Q} \rangle \mathscr{H} = f((\det \boldsymbol{Q})\langle \boldsymbol{Q} \rangle \mathscr{H}). \end{aligned} \tag{4.6}$$

Hence, on one hand, when $g(\boldsymbol{A})$ is an invariant of even degree, we have $f(\langle \boldsymbol{Q} \rangle \mathscr{H}) = f(\mathscr{H})$ for any orthogonal tensor $\boldsymbol{Q}$. That is, $f(\mathscr{H})$ is an isotropic invariant of the Hall tensor $\mathscr{H}$. On the other hand, when $g(\boldsymbol{A})$ is an invariant of odd degree, only for orthogonal tensor $\boldsymbol{Q}$ satisfying $\det \boldsymbol{Q} = 1$, it holds that $f(\langle \boldsymbol{Q} \rangle \mathscr{H}) = f(\mathscr{H})$, which means that $f(\mathscr{H})$ is a hemitropic invariant of the Hall tensor $\mathscr{H}$. The proof is completed. $\square$

4.3.2 The minimal integrity basis of the Hall tensor

According to Theorem 4.6, we are able to construct an integrity basis for the Hall tensor from the integrity basis of its associated second order tensor. For the associated second order tensor $\boldsymbol{A}(\mathscr{H})$, we can split it into $\boldsymbol{A}(\mathscr{H}) = \boldsymbol{T} + \boldsymbol{W}$, where $\boldsymbol{T}$ is a symmetric tensor with components $t_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$ and $\boldsymbol{W}$ is a skew-symmetric tensor with components $w_{ij} = \frac{1}{2}(a_{ij} - a_{ji})$. It is well-known that following 7 invariants $\operatorname{tr}\boldsymbol{T}$, $\operatorname{tr}\boldsymbol{T}^2$, $\operatorname{tr}\boldsymbol{T}^3$, $\operatorname{tr}\boldsymbol{W}^2$, $\operatorname{tr}\boldsymbol{T}\boldsymbol{W}^2$, $\operatorname{tr}\boldsymbol{T}^2\boldsymbol{W}^2$ and $\operatorname{tr}\boldsymbol{T}^2\boldsymbol{W}^2\boldsymbol{T}\boldsymbol{W}$ form a minimal integrity basis of $\boldsymbol{A}(\mathscr{H})$ and also an irreducible function basis as well. Then we denote the invariants of $\boldsymbol{A}(\mathscr{H})$ as follows:

$$\begin{array}{llll} I_1 := \operatorname{tr}\boldsymbol{T}, & I_2 := \operatorname{tr}\boldsymbol{T}^2, & J_2 := \operatorname{tr}\boldsymbol{W}^2, & I_3 := \operatorname{tr}\boldsymbol{T}^3, \\ J_3 := \operatorname{tr}\boldsymbol{T}\boldsymbol{W}^2, & I_4 := \operatorname{tr}\boldsymbol{T}^2\boldsymbol{W}^2, & I_6 := \operatorname{tr}\boldsymbol{T}^2\boldsymbol{W}^2\boldsymbol{T}\boldsymbol{W}. & \end{array} \tag{4.7}$$

These invariants contain one invariant with degree 1, two invariants with degree 2, two invariants with degree 3, one invariant with degree 4 and one invariant with degree 6. The following theorem presents a method to obtain a minimal integrity basis of $\mathscr{H}$ from this particular minimal integrity basis of $\boldsymbol{A}(\mathscr{H})$.

Theorem 4.7. *Let $\mathscr{H}$ be a Hall tensor with components h_{ijk}, and $\boldsymbol{A}(\mathscr{H})$ be its associated second order tensor with components a_{ij}. Denote $K_2 := I_1^2$, $J_4 := I_1 I_3$, $K_4 := I_1 J_3$, $J_6 := I_3^2$, $K_6 := J_3^2$ and $L_6 := I_3 J_3$. Then the invariant set*

$$\Theta_{hall} := \{I_2, J_2, K_2, I_4, J_4, K_4, I_6, J_6, K_6, L_6\} \tag{4.8}$$

is a minimal integrity basis of $\mathscr{H}$. Here, $\{I_1, I_2, J_2, I_3, J_3, I_4, I_6\}$ are defined in Eq. (4.7).

Proof. According to Theorem 4.6, any isotropic invariant of $\mathscr{H}$ is also an invariant of $\boldsymbol{A}(\mathscr{H})$, which means that any isotropic invariant of $\mathscr{H}$ can be expressed by a polynomial denoted as $p(I_1, I_2, J_2, I_3, J_3, I_4, I_6)$. Furthermore, any isotropic invariant of an even order tensor consists of several even degree monomials. Each even degree monomial containing I_1, I_3, J_3 should be a polynomial of $I_1^2, I_1I_3, I_1J_3, I_3^2, I_3J_3$ and J_3^2. Hence, the isotropic invariant $p(I_1, I_2, J_2, I_3, J_3, I_4, I_6)$ can also be written into a polynomial of the invariants in Θ_{hall}. That is, Eq. (4.8) is an integrity basis of $\mathscr{H}$.

Next, we need to verify the polynomial irreducibility of this integrity basis. There is a natural observation that these isotropic invariants are homogenous polynomials of the 9 independent components in the Hall tensor $\mathscr{H}$. A similar method as the approach proposed by Chen et al.[15] is employed in this part. The process can be divided into three parts:

(I) There are exactly three isotropic invariants with degree 2, i.e., I_2, J_2, K_2, in this integrity basis. Take I_2 for example. If it is not polynomial irreducible with the other 9 invariants in this basis, then there exists a linear combination of the other two degree-2 invariants J_2, K_2. Therefore, if I_2, J_2, K_2 are polynomial irreducible, then the unique triple of (c_1, c_2, c_3) such that

$$c_1I_2 + c_2J_2 + c_3K_2 = 0 \tag{4.9}$$

is $c_1 = c_2 = c_3 = 0$. In fact, as we mentioned before, Eq. (4.9) is a polynomial of the 9 independent components in the Hall tensor. Without loss of generality, denote a point $\mathbf{y} = (h_{121}, h_{122}, h_{123}, h_{131}, h_{132}, h_{133}, h_{231}, h_{232}, h_{233}) \in \mathbb{R}^9$, where $\mathbb{R}^9$ is the real number field with dimension 9, then we can say that Eq. (4.9) is a polynomial of $\mathbf{y}$. Note that Eq. (4.9) holds for an arbitrary Hall tensor. Now, when we generate n points $\mathbf{y_1}, \cdots, \mathbf{y_n} \in \mathbb{R}^9$, c_1, c_2, c_3 must be the solution to the linear system of equations

$$\begin{pmatrix} I_2(\mathbf{y_1}) & J_2(\mathbf{y_1}) & K_2(\mathbf{y_1}) \\ I_2(\mathbf{y_2}) & J_2(\mathbf{y_2}) & K_2(\mathbf{y_2}) \\ \vdots & \vdots & \vdots \\ I_2(\mathbf{y_n}) & J_2(\mathbf{y_n}) & K_2(\mathbf{y_n}) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4.10}$$

The coefficient matrix of System (4.10) is denoted by M_2, and denote $r(M_2)$ as the rank of the coefficient matrix M_2. Then $r(M_2)$ reveals the number of polynomial irreducible invariants in these three isotropic invariants. Take $n = 3$ and

- $\mathbf{y_1} = (-2, 3, 5, 0, -5, -4, -5, 2, -2)$,
- $\mathbf{y_2} = (-3, 0, 1, 1, 2, -4, 3, 0, 3)$,
- $\mathbf{y_3} = (-2, 0, -1, 2, 1, -3, 5, 2, 3)$.

By numerical calculations, we can determine that $r(M_2) = 3$. Hence, the only solution for System (4.10) is $c_1 = c_2 = c_3 = 0$, which implies that these three invariants with degree 2 are polynomial irreducible.

(II) For the invariants with degree 4, consider the following linear equation

$$\begin{aligned} &c_1(I_2)^2 + c_2(J_2)^2 + c_3(K_2)^2 + c_4 I_2 J_2 + c_5 I_2 K_2 + c_6 J_2 K_2 \\ &+ c_7 I_4 + c_8 J_4 + c_9 K_4 = 0, \end{aligned} \tag{4.11}$$

where $c_1, \cdots, c_9$ are scalars. If there exists a unique $(c_1, c_2, \dots, c_9) = (0, 0, \dots, 0)$ such that (4.11) holds for any Hall tensor, then all the three degree-4 invariants I_4, J_4, K_4 are polynomial irreducible. We also generate n points $\mathbf{y_1}, \cdots, \mathbf{y_n} \in \mathbb{R}^9$ and consider the following linear system:

$$\begin{pmatrix} I_2^2(\mathbf{y_1}) & \cdots & I_4(\mathbf{y_1}) & J_4(\mathbf{y_1}) & K_4(\mathbf{y_1}) \\ I_2^2(\mathbf{y_2}) & \cdots & I_4(\mathbf{y_2}) & J_4(\mathbf{y_2}) & K_4(\mathbf{y_2}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ I_2^2(\mathbf{y_n}) & \cdots & I_4(\mathbf{y_n}) & J_4(\mathbf{y_n}) & K_4(\mathbf{y_n}) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4.12}$$

The coefficient matrix of System (4.12) is denoted by M_4. Take $n = 9$ and

- $\mathbf{y_1} = (4, 1, -3, 1, -4, -2, -1, 0, -5)$,
- $\mathbf{y_2} = (1, 5, 4, 0, -1, -5, -3, 5, -2)$,
- $\mathbf{y_3} = (-4, 4, -4, 1, -5, -2, 2, 3, 4)$,
- $\mathbf{y_4} = (-4, -5, 5, 5, -2, 3, 5, -1, 2)$,

- $\mathbf{y_5} = (0, 4, 3, 3, 1, -2, 3, 5, -4)$,
- $\mathbf{y_6} = (5, -3, 3, 3, -4, -2, 3, 5, -5)$,
- $\mathbf{y_7} = (-3, -2, 2, 4, -4, 1, 4, 2, 0)$,
- $\mathbf{y_8} = (-5, -3, 4, -1, 1, -2, -2, -3, 0)$,
- $\mathbf{y_9} = (0, -2, -2, 1, 5, 3, 4, 0, 0)$.

We can verify that the rank of M_4 is $r(M_4) = 9$, which implies that these three degree-4 invariants cannot be polynomial represented by other invariants with degree-4 and degree-2.

(III) Similarly, in the case of degree 6, the verification linear equation is

$$c_1(I_2)^3 + c_2(J_2)^3 + \cdots + c_{19}K_2K_4 + c_{20}I_6 + \cdots + c_{23}L_6 = 0. \tag{4.13}$$

Hence, we generate n points $\mathbf{y_1}, \cdots, \mathbf{y_n} \in \mathbb{R}^9$. Consider a linear system similar to system (4.12). Its coefficient matrix can be denoted as M_6, and its rank is denoted by $r(M_6)$. Take $n = 23$ and

- $\mathbf{y_1} = (3, -5, 1, 4, 2, 3, 3, 1, -3)$,
- $\mathbf{y_2} = (-5, -1, 2, -5, -2, 3, 3, 4, -1)$,
- $\mathbf{y_3} = (-4, 2, 1, -3, -2, -2, 1, 4, -1)$,
- $\mathbf{y_4} = (-2, 0, 3, 2, -2, -2, -5, 5, 2)$,
- $\mathbf{y_5} = (-2, -5, -5, -4, 3, -5, -3, 2, -3)$,
- $\mathbf{y_6} = (5, -4, 1, 3, -4, 1, -1, 4, 0)$,
- $\mathbf{y_7} = (-3, 3, 5, -3, -3, 1, 2, -2, -3)$,
- $\mathbf{y_8} = (2, 2, -5, 4, 4, -1, -5, 4, -5)$,
- $\mathbf{y_9} = (-2, -1, 2, 3, -2, -1, -2, -2, 5)$,

- $\mathbf{y_{10}} = (-4, -3, -4, -2, -5, -5, 5, -2, -3)$,
- $\mathbf{y_{11}} = (3, 2, -2, -5, 5, -3, 0, -2, -5)$,
- $\mathbf{y_{12}} = (4, -4, -1, 4, -4, 0, 1, 3, -1)$,
- $\mathbf{y_{13}} = (3, 0, -5, 0, 2, -5, -5, 4, 1)$,
- $\mathbf{y_{14}} = (-4, 5, -5, 2, -1, -4, -5, -2, -5)$,
- $\mathbf{y_{15}} = (2, -5, -5, 5, 0, 2, 2, 3, 4)$,
- $\mathbf{y_{16}} = (1, 4, 4, -1, -5, -3, 4, -5, 1)$,
- $\mathbf{y_{17}} = (-2, 5, -5, 1, -2, 1, 0, -5, 4)$,
- $\mathbf{y_{18}} = (0, -4, -5, 0, -5, -2, -2, -2, 2)$,
- $\mathbf{y_{19}} = (1, 2, 1, -1, 3, -4, -5, 4, 5)$,
- $\mathbf{y_{20}} = (3, -3, 1, -3, -5, 3, 5, 1, 1)$,
- $\mathbf{y_{21}} = (0, -1, 3, 0, -3, 5, 3, 0, 3)$,
- $\mathbf{y_{22}} = (1, -5, -4, -1, 0, -1, -5, -5, 2)$,
- $\mathbf{y_{23}} = (-4, -2, 3, 4, 5, -3, 4, 3, 3)$.

Then $r(M6) = 23$, which implies that these four invariants with degree 6 are polynomial irreducible in the integrity basis. Noted that in all parts of this procedure, the points $\mathbf{y}$ are not unique and fixed.

Therefore, we have presented that Θ_{hall} in (4.8) is a minimal integrity basis of $\mathscr{H}$. $\square$

In the above discussion, we fix the inducing initial, i.e., a particular minimal integrity basis of the second order tensor. However, the minimal integrity basis is not unique in general. We can also construct another minimal integrity basis for a Hall tensor from another minimal integrity basis of the associated second order tensor, denoted by

$$\{\tilde{I}_1, \tilde{I}_2, \tilde{J}_2, \tilde{I}_3, \tilde{J}_3, \tilde{I}_4, \tilde{I}_6\}.$$

Construct another integrity basis $\tilde{\Theta}_{hall} := \{\tilde{I}_2, \tilde{J}_2, \tilde{K}_2, \tilde{I}_4, \tilde{J}_4, \tilde{K}_4, \tilde{I}_6, \tilde{J}_6, \tilde{K}_6, \tilde{L}_6\}$ of the Hall tensor in the same way, where

$$\tilde{K}_2 := \tilde{I}_1^2,\ \tilde{J}_4 := \tilde{I}_1\tilde{I}_3,\ \tilde{K}_4 := \tilde{I}_1\tilde{J}_3,\ \tilde{J}_6 := \tilde{I}_3^2,\ \tilde{K}_6 := \tilde{J}_3^2 \text{ and } \tilde{L}_6 := \tilde{I}_3\tilde{J}_3.$$

Because the integrity basis has already got the same number of invariants as the minimal integrity basis (4.8), it must also be a minimal integrity basis. Therefore, we have the following corollary.

Corollary 4.1. *Denote $\mathscr{H}$ as a Hall tensor with components h_{ijk}, and $\boldsymbol{A}(\mathscr{H})$ as its associated second order tensor with components a_{ij}. Let $\{\tilde{I}_1, \tilde{I}_2, \tilde{J}_2, \tilde{I}_3, \tilde{J}_3, \tilde{I}_4, \tilde{I}_6\}$ be any minimal integrity basis of the second order tensor $\boldsymbol{A}(\mathscr{H})$. Denote*

$$\tilde{\Theta}_{hall} := \{\tilde{I}_2, \tilde{J}_2, \tilde{K}_2, \tilde{I}_4, \tilde{J}_4, \tilde{K}_4, \tilde{I}_6, \tilde{J}_6, \tilde{K}_6, \tilde{L}_6\}$$

with $\tilde{K}_2 := \tilde{I}_1^2$, $\tilde{J}_4 := \tilde{I}_1\tilde{I}_3$, $\tilde{K}_4 := \tilde{I}_1\tilde{J}_3$, $\tilde{J}_6 := \tilde{I}_3^2$, $\tilde{K}_6 := \tilde{J}_3^2$ and $\tilde{L}_6 := \tilde{I}_3\tilde{J}_3$. Then $\tilde{\Theta}_{hall}$ is a minimal integrity basis of the Hall tensor $\mathscr{H}$.

4.3.3 Irreducible function basis for the Hall tensor

In this subsection, we shall prove that the minimal integrity basis given in Subsection 4.3.2 is also an irreducible function basis of the Hall tensor $\mathcal{K}$. According to the approach proposed by Pennisi and Trovato[55] in 1987, to present a given function basis of a tensor is functionally irreducible, for each invariant in this basis, we should find two different sets of independent variables in the tensor, denoted by V and V', such that this invariant takes different values in V and V' while all the remainders are the same in V and V'. The following theorem is proved in this spirit.

Theorem 4.8. *The set $\Theta_{hall} = \{I_2, J_2, K_2, I_4, J_4, K_4, I_6, J_6, K_6, L_6\}$ is an irreducible function basis of the Hall tensor, where $I_2, J_2, K_2, I_4, J_4, K_4, I_6, J_6, K_6$ and L_6 are defined in Theorem 4.7 and Eq. (4.7).*

Proof. According to the Definitions 4.1 and 4.2, we know that an integrity basis of a tensor is a function basis of the tensor. Since we have proved in Subsection 4.3.2 that these ten invariants form a minimal integrity basis of the Hall tensor, this basis is also a function basis.

Denote

$$V = \{h_{121}, h_{122}, h_{123}, h_{131}, h_{132}, h_{133}, h_{231}, h_{232}, h_{233}\}$$

and

$$V' = \{h'_{121}, h'_{122}, h'_{123}, h'_{131}, h'_{132}, h'_{133}, h'_{231}, h'_{232}, h'_{233}\}$$

as two different sets of independent variables of the Hall tensor $\mathscr{H}$. Then we will find ten pairs of $\{V, V'\}$ to prove that all the ten isotropic invariants in Θ_{hall} is functionally irreducible.

1. For I_2 in V, let $h_{121} = h_{122} = h_{123} = h_{132} = h_{133} = h_{231} = h_{233} = 0$, $h_{131} = -1$ and $h_{232} = 1$.

   Then in V', let $h'_{121} = h'_{122} = h'_{123} = h'_{132} = h'_{133} = h'_{231} = h'_{233} = 0$, $h'_{131} = -2$ and $h'_{232} = 2$.

   We have that $I_2 = 2$ and $I'_2 = 8$, while other invariants: $\{J_2, K_2, I_4, J_4, K_4, I_6, J_6, K_6, L_6\}$, and $\{J'_2, K'_2, I'_4, J'_4, K'_4, I'_6, J'_6, K'_6, L'_6\}$ are all equal to 0. It means that I_2 is functionally irreducible in the function basis Θ_{hall}.

2. For J_2, in V, let $h_{121} = h_{122} = h_{123} = h_{132} = h_{133} = h_{231} = h_{233} = 0$, $h_{131} = 1$ and $h_{232} = 1$. Meanwhile in V', let all the variables be 0.

   We have that $J_2 = 2$ and $J'_2 = 0$, while other invariants: $\{I_2, K_2, I_4, J_4, K_4, I_6, J_6, K_6, L_6\}$, and $\{I'_2, K'_2, I'_4, J'_4, K'_4, I'_6, J'_6, K'_6, L'_6\}$ all equals 0. This means that J_2 is functionally irreducible in the function basis Θ_{hall}.

3. For K_2, in V, let $h_{121} = h_{122} = h_{131} = h_{133} = h_{232} = h_{233} = 0$, $h_{123} = -\sqrt{\frac{2+\sqrt[3]{4}}{2}}$, $h_{132} = 0$ and $h_{231} = \sqrt{\frac{2+\sqrt[3]{4}}{2}}$.

In V', let $h'_{121} = h'_{122} = h'_{131} = h'_{133} = h'_{232} = h'_{233} = 0$, $h'_{123} = 1, h'_{132} = \sqrt[3]{2}$ and $h'_{231} = 1$.

We have $K_2 = 0$. It is not equal to $K'_2 = (2 - \sqrt[3]{2})^2$, while other invariants: $I_2 = I'_2 = 2 + \sqrt[3]{4}$ and $J_2 = J'_2 = I_4 = I'_4 = J_4 = J'_4 = K_4 = K'_4 = I_6 = I'_6 = J_6 = J'_6 = K_6 = K'_6 = L_6 = L'_6 = 0$. This means that K_2 is functionally irreducible.

4. For I_4, in V, let $h_{121} = -2$, $h_{122} = 0$, $h_{123} = 1$, $h_{131} = 1$, $h_{132} = 1$, $h_{133} = 0$, $h_{231} = 0$, $h_{232} = 1$ and $h_{233} = 2$.

   In V', let $h'_{121} = -\sqrt{3}$, $h'_{122} = -\sqrt{2}$, $h'_{123} = 1$, $h'_{131} = 0$, $h'_{132} = 1$, $h'_{133} = -\sqrt{2}$, $h'_{231} = 0$, $h'_{232} = 0$ and $h'_{233} = \sqrt{3}$.

   We have $I_4 = 5$. It is not equal to $I'_4 = 7$, while $I_2 = I'_2 = 2$, $J_2 = J'_2 = 10$, $K_6 = K'_6 = 9$ and others are all equal to 0. This means that I_4 is functionally irreducible.

5. For J_4, assume that $s = 4 + \sqrt{14}$, and $t = 4 - \sqrt{14}$.

   In V, let $h_{121} = h_{122} = h_{131} = h_{133} = h_{232} = h_{233} = 0$, $h_{123} = 1, h_{132} = 1$ and $h_{231} = \frac{\sqrt[3]{2t}}{2} + \sqrt[3]{\frac{s}{4}}$.

   In V', let $h'_{121} = h'_{122} = h'_{131} = h'_{133} = h'_{232} = h'_{233} = h'_{231} = 0$, and

$$h'_{123} = 2 - \frac{\sqrt[3]{2t}}{2} - \frac{\sqrt[3]{2s}}{2} - \frac{\sqrt[6]{2}}{2}\sqrt{2\sqrt[3]{4} + 8\sqrt[3]{t} + \sqrt[3]{2t^2} + 8\sqrt[3]{s} + \sqrt[3]{2s^2}},$$

$$h'_{132} = -1 + \frac{\sqrt[3]{2t}}{4} + \frac{\sqrt[3]{2s}}{4} - \frac{\sqrt[6]{2}}{4}\sqrt{2\sqrt[3]{4} + 8\sqrt[3]{t} + \sqrt[3]{2t^2} + 8\sqrt[3]{s} + \sqrt[3]{2s^2}}.$$

   We have

$$J_4 = -J'_4 = \frac{3}{8}\left(-4 + \sqrt[3]{2t} + \sqrt[3]{2s}\right)\left(\sqrt[3]{2t} + \sqrt[3]{2s}\right).$$

Meanwhile,

$$\begin{array}{lll} I_2 & = I_2' & = 2 + \frac{\left(\sqrt[3]{2t}+\sqrt[3]{2s}\right)^2}{4}, \\ K_2 & = K_2' & = \frac{1}{4}\left(-4+\sqrt[3]{2t}+\sqrt[3]{2s}\right)^2, \\ J_6 & = J_6' & = \frac{9}{16}\left(\sqrt[3]{2t}+\sqrt[3]{2s}\right)^2, \end{array}$$

and others are all equal to 0. This reveals that J_4 is functionally irreducible.

6. For K_4, in V, let

$$\begin{array}{lll} h_{121} = -\frac{1}{2}\sqrt{\frac{-12+6\sqrt[3]{9}}{16-3\sqrt[3]{3}-3\sqrt[3]{9}}}, & h_{122} = \frac{1}{2}, & h_{123} = -1, \\ h_{131} = 0, & h_{132} = -\frac{\sqrt[3]{9}}{2}, & h_{133} = \frac{1}{2}, \\ h_{231} = -\frac{1}{2}, & h_{232} = 0, & h_{233} = \frac{1}{2}\sqrt{\frac{-12+6\sqrt[3]{9}}{16-3\sqrt[3]{3}-3\sqrt[3]{9}}}. \end{array}$$

In V', let

$$\begin{array}{ll} h'_{121} = 0, & h'_{122} = -\frac{\sqrt[6]{3}}{2}\sqrt{\frac{9+5\sqrt[3]{3}-6\sqrt[3]{9}}{16-3\sqrt[3]{3}-3\sqrt[3]{9}}}, \\ h'_{123} = 1, & h'_{131} = -\frac{1}{2}\sqrt{\frac{22-12\sqrt[3]{3}-2\sqrt[3]{9}}{16-3\sqrt[3]{3}-3\sqrt[3]{9}}}, \\ h'_{132} = \frac{\sqrt[3]{9}}{2}, & h'_{133} = -\frac{\sqrt[6]{3}}{2}\sqrt{\frac{9+5\sqrt[3]{3}-6\sqrt[3]{9}}{16-3\sqrt[3]{3}-3\sqrt[3]{9}}}, \\ h'_{231} = \frac{1}{2}, & h'_{232} = -\frac{1}{2}\sqrt{\frac{22-12\sqrt[3]{3}-2\sqrt[3]{9}}{16-3\sqrt[3]{3}-3\sqrt[3]{9}}}, \\ h'_{233} = 0. & \end{array}$$

We have

$$K_4 = -K_4' = \frac{6+21\sqrt[3]{3}-17\sqrt[3]{9}}{-256+48\sqrt[3]{3}+48\sqrt[3]{9}}.$$

Meanwhile,

$$\begin{array}{lll} I_2 & = I_2' & = \frac{5+3\sqrt[3]{3}}{4}, \\ J_2 & = J_2' & = \frac{4-3\sqrt[3]{3}+3\sqrt[3]{9}}{-32+6\sqrt[3]{3}+6\sqrt[3]{9}}, \\ K_2 & = K_2' & = \frac{(-3+\sqrt[3]{9})^2}{4}, \\ I_4 & = I_4' & = \frac{-23+36\sqrt[3]{3}+9\sqrt[3]{9}}{-256+48\sqrt[3]{3}+48\sqrt[3]{9}}, \\ K_6 & = K_6' & = \frac{-47+78\sqrt[3]{3}-31\sqrt[3]{9}}{64(-16+3\sqrt[3]{3}+3\sqrt[3]{9})^2}, \end{array}$$

and others are all equal to 0. This shows that K_4 is functionally irreducible.

7. For I_6, in V, let $h_{121} = -1$, $h_{122} = -1$, $h_{123} = 1$, $h_{131} = 1$, $h_{132} = 1$, $h_{133} = -1$, $h_{231} = 0$, $h_{232} = 1$ and $h_{233} = 1$.

   In V', let $h'_{121} = -1$, $h'_{122} = -1$, $h'_{123} = 1$, $h'_{131} = -1$, $h'_{132} = 1$, $h'_{133} = -1$, $h'_{231} = 0$, $h'_{232} = -1$ and $h'_{233} = 1$.

   We have
   $$I_6 = -I'_6 = 2.$$
   At the same time, $I_2 = I'_2 = 2$, $J_2 = J'_2 = 6$, $I_4 = I'_4 = 4$, and others are all equal to 0. This shows that I_6 is functionally irreducible.

8. For J_6, in V, let $h_{121} = h_{122} = h_{131} = h_{133} = h_{232} = h_{233} = 0$, $h_{123} = -\sqrt{3}\sqrt[3]{2}$, $h_{132} = 0$ and $h_{231} = \sqrt{3}\sqrt[3]{2}$.

   In V', let $h'_{121} = h'_{122} = h'_{131} = h'_{133} = h'_{232} = h'_{233} = 0$, $h'_{123} = \sqrt[3]{2}$, $h'_{132} = 2\sqrt[3]{2}$ and $h'_{231} = \sqrt[3]{2}$. We have
   $$J_6 = 0 \neq J'_6 = 144.$$
   At the same time, $I_2 = I'_2 = 6\sqrt[3]{4}$, and others are all equal to 0. This reveals that J_6 is functionally irreducible.

9. For K_6, in V, let $h_{121} = \frac{1}{2}$, $h_{122} = 1$, $h_{123} = 0$, $h_{131} = \frac{3}{2}$, $h_{132} = 0$, $h_{133} = 1$, $h_{231} = 0$, $h_{232} = \frac{3}{2}$ and $h_{233} = \frac{1}{2}$.

   In V', let $h'_{121} = -\frac{1}{2}$, $h'_{122} = \frac{1}{2}$, $h'_{123} = 0$, $h'_{131} = \sqrt{3}$, $h'_{132} = 0$, $h'_{133} = \frac{1}{2}$, $h'_{231} = 0$, $h'_{232} = \sqrt{3}$ and $h'_{233} = -\frac{1}{2}$.

   We have
   $$K_6 = \frac{9}{4} \neq K'_6 = \frac{3}{4}.$$
   At the same time, $I_2 = I'_2 = \frac{1}{2}$, $J_2 = J'_2 = -\frac{13}{2}$, $I_4 = I'_4 = -\frac{13}{16}$ and others are all equal to 0. This reveals that K_6 is functionally irreducible.

10. For L_6, in V, let $h_{121} = -1$, $h_{122} = \frac{1}{2}$, $h_{123} = -1$, $h_{131} = 0$, $h_{132} = 2$, $h_{133} = \frac{1}{2}$, $h_{231} = 3$, $h_{232} = 0$ and $h_{233} = 1$.

    In V', let $h'_{121} = 0$, $h'_{122} = -\frac{1}{2}\sqrt{\frac{5}{2}}$, $h'_{123} = 1$, $h'_{131} = -\frac{1}{2}\sqrt{\frac{5}{2}}$, $h'_{132} = -2$, $h'_{133} = -\frac{1}{2}\sqrt{\frac{5}{2}}$, $h'_{231} = -3$, $h'_{232} = -\frac{1}{2}\sqrt{\frac{5}{2}}$ and $h'_{233} = 0$.

    We have
    $$L_6 = -L'_6 = -\frac{45}{2}.$$

    At the same time, $I_2 = I'_2 = 14$, $J_2 = J'_2 = -\frac{5}{2}$, $I_4 = I'_4 = -\frac{45}{4}$, $J_6 = J'_6 = 324$, $K_6 = K'_6 = \frac{25}{16}$ and others equal 0. This reveals that L_6 is functionally irreducible.

Hence, this minimal integrity basis $\Theta_{hall} = \{I_2, J_2, K_2, I_4, J_4, K_4, I_6, J_6, K_6, L_6\}$ is also an irreducible function basis of the Hall tensor $\mathscr{H}$. □

In the above proof, the counter examples V and V' in the cases (1), (2), (4) and (7) are based on related sets in Pennisi and Trovato[55], while the examples V and V' in the case (5) are discussed with Dr. Yannan Chen.

4.4 Final Remarks

4.4.1 Significance of Section 4.2

The result in Section 4.2 is significant for the future investigation of irreducible function bases of higher order tensors. On one hand, it is the first result on irreducible function bases of a third order three-dimensional symmetric tensor. On the other hand, we need to consider that there are still at least three syzygy relations among these eleven invariants in $\Theta^{(2)}_{sym}$, see Eq.(4.14-4.16) in the following. This reveals that an irreducible function basis which is constructed by polynomial invariants may not be algebraically minimal which means that the basis consists of polynomial invariants and there is no algebraic relations in these invariants [72]. This point is observed since there are still some syzygy relations among these eleven invariants.

Consider all possible sixteenth degree powers or products that can be constructed from the eleven invariants in $\Theta^{(2)}_{sym}$: $I_2, J_2, I_4, J_4, K_4, L_4, I_6, J_6, L_6, M_6$ and I_{10}. Find linear relations among these sixteenth-degree powers or products. Then we find the following three syzygy relations among these eleven invariants as follows.

$$\begin{aligned}
&\frac{4}{9}I_2^3J_2^3K_4+\frac{2}{9}I_2^4J_2^2L_4+\frac{4}{3}I_2^3J_2J_4L_4-\frac{8}{9}I_2J_2^3I_4K_4-\frac{4}{9}I_2^2J_2^2I_4L_4-\frac{4}{3}I_2^2J_2^2J_4K_4\\
&-2I_2^2J_4^2L_4+2I_2^2K_4L_4^2+2J_2^2K_4^3+4I_2J_2J_4^2K_4+5I_2J_2K_4^2L_4-4I_2J_2I_4J_4L_4\\
&-\frac{4}{3}I_2^3J_2^2J_6+\frac{2}{3}J_2^3K_4I_6+\frac{1}{3}I_2J_2^2L_4I_6+\frac{8}{3}I_2J_2^2I_4J_6+\frac{4}{3}I_2^2J_2J_4J_6-2I_2^3L_4M_6\\
&+J_2J_4L_4I_6-16I_2K_4L_4J_6-14J_2K_4^2J_6+6I_2L_4^2L_6+4J_2K_4L_4L_6+6I_2I_4L_4M_6\\
&-2I_2J_4K_4M_6+4J_2I_4K_4M_6+4I_2^2J_6M_6-2J_2^2I_6J_6-4I_2J_2L_6M_6-12I_4J_6M_6\\
&+6J_4L_6M_6+24K_4J_6^2-12L_4J_6L_6-4J_4^3K_4+4I_4J_4^2L_4-J_4K_4^2L_4=0,
\end{aligned}\tag{4.14}$$

$$\begin{aligned}
&2I_2^3J_2^3J_4-4I_2J_2^3I_4J_4-6J_2^3J_4I_6-9I_2^2J_2^2J_4^2+18J_2^2I_4J_4^2+9J_4^4+36I_2J_2J_6^2-54J_4J_6^2\\
&-48I_2J_2^2K_4J_6+144J_2J_4K_4J_6+12I_2J_2^3K_4^2-36J_2^2J_4K_4^2-24I_2^2J_2L_4J_6+36I_2J_4L_4J_6\\
&+12I_2^2J_2^2K_4L_4-18I_2J_2J_4K_4L_4-18J_4^2K_4L_4+6I_2^3J_2L_4^2-6I_2J_2I_4L_4^2-9I_2^2J_4L_4^2\\
&+9I_4J_4L_4^2-36J_2J_4L_4L_6-6I_2^3J_2^2M_6+12I_2J_2^2I_4M_6+9J_2^2I_6M_6+36I_2^2J_2J_4M_6\\
&-72J_2I_4J_4M_6-18I_2J_4^2M_6-108K_4J_6M_6+27J_2K_4^2M_6+18I_2K_4L_4M_6+54L_4L_6M_6\\
&-18I_2^2M_6^2+54I_4M_6^2=0
\end{aligned}\tag{4.15}$$

and

$$\frac{1}{18}I_2^5J_2^3 - \frac{2}{9}I_2^3J_2^3I_4 + \frac{2}{9}I_2J_2^3I_4^2 + \frac{1}{12}I_2^2J_2^3I_6 - \frac{1}{6}J_2^3I_4I_6 - \frac{1}{6}I_2^4J_2^2J_4 + \frac{1}{3}I_2^2J_2^2I_4J_4$$
$$+\frac{1}{2}I_2J_2^2J_4I_6 + \frac{1}{2}I_2^3J_2J_4^2 - I_2J_2I_4J_4^2 - \frac{3}{4}J_2J_4^2I_6 - \frac{1}{2}I_2^2J_4^3 + I_4J_4^3 - I_2^2J_2K_4J_6$$
$$+2J_2I_4K_4J_6 + \frac{1}{4}I_2^2J_2^2K_4^2 - \frac{1}{2}J_2^2I_4K_4^2 + \frac{3}{2}I_2J_2J_4K_4^2 - \frac{9}{4}J_4^2K_4^2 + \frac{1}{2}I_2^3J_2K_4L_4$$
$$-I_2J_2I_4K_4L_4 - \frac{1}{2}I_2^2J_4K_4L_4 + I_4J_4K_4L_4 + 2I_2J_2J_6L_6 - 3J_4J_6L_6 - 2I_2J_2^2K_4L_6$$
$$+3J_2J_4K_4L_6 - \frac{1}{2}I_2^2J_2L_4L_6 - J_2I_4L_4L_6 + \frac{3}{2}I_2J_4L_4L_6 - \frac{1}{6}I_2^4J_2M_6 + \frac{5}{6}I_2^2J_2I_4M_6$$
$$-J_2I_4^2M_6 - I_2J_2I_6M_6 + \frac{3}{2}J_4I_6M_6 = 0. \tag{4.16}$$

Since these three syzygy relations cannot reveal any single-valued function relation of any of these eleven invariants, with respect to other ten invariants, none of the invariants in $\Theta^{(2)}_{sym}$ can be dropped from these three syzygies.

The second point is meaningful to the further research for irreducible function bases of higher order tensors. It is also verified by the study for the irreducible function basis of a fourth order three-dimensional symmetric and traceless tensor. There are still five syzygy relations among the nine invariants in the Smith-Bao minimal integrity basis of a fourth order three-dimensional symmetric and traceless tensor[15, 72]. What is worse, these five syzygy relations are not so well-structured like Eq. (4.3) and (4.4), but even more complicated than Eq. (4.14)-(4.16). Nevertheless, the Smith-Bao minimal integrity basis has been proved to be an irreducible function basis of the fourth order three-dimensional symmetric and traceless tensor in [17] by Chen, Chen, Qi and Zou very recently. Moreover, any minimal integrity basis of a third order three-dimensional symmetric and traceless tensor is indeed an irreducible function basis of that tensor, which was proved in [15]. Hence, it reveals that a minimal integrity basis can still have possibility to be an irreducible function basis even with several syzygy relations.

In Subsection 4.2.3, we show that the eleven invariant function basis is indeed an irreducible function basis, by presenting that each of these eleven invariants is not a function of other ten invariants based on the method proposed by Pennisi and Trovato [55]. In fact, we divide the proof into three parts. In Part (I), we prove that each of the five invariants I_2, I_4, I_6, I_{10} and J_2, which form the irreducible function bases of the composition tensors $\mathscr{D}$ and $\mathbf{u}$, respectively, is not a function of the ten other invariants. In Part (II), we utilize Proposition 4.2 to show that neither of K_4, L_4, L_4 and J_6 is a function of the ten other invariants. In Part (III), we use another strategy to prove that each of the remaining two invariants M_6 and J_4 is not a function of the ten other invariants. Such tactics may be also instructive for the further investigate of irreducible function bases of higher order tensors.

4.4.2 Conclusions for Section 4.3

In Section 4.3, we mainly investigate isotropic invariants of the Hall tensor. For this purpose, we build a connection between the invariants of the Hall tensor $\mathscr{H}$ and the ones of its associated second order tensor $\boldsymbol{A}(\mathscr{H})$. $\boldsymbol{A}(\mathscr{H})$ can be decomposed into a second order symmetric tensor $\boldsymbol{T}$ and a second order skew-symmetric tensor $\boldsymbol{W}$. We know that $\{I_1 := \operatorname{tr}\boldsymbol{T}, I_2 := \operatorname{tr}\boldsymbol{T}^2, J_2 := \operatorname{tr}\boldsymbol{W}^2, I_3 := \operatorname{tr}\boldsymbol{T}^3, J_3 := \operatorname{tr}\boldsymbol{T}\boldsymbol{W}^2, I_4 := \operatorname{tr}\boldsymbol{T}^2\boldsymbol{W}^2, I_6 := \operatorname{tr}\boldsymbol{T}^2\boldsymbol{W}^2\boldsymbol{T}\boldsymbol{W}\}$ is the minimal integrity basis of $\boldsymbol{A}(\mathcal{K})$ as in the previous work. It is also an irreducible function basis of $\boldsymbol{A}(\mathcal{K})$. We reveal the following statements in this section:

1. $\Theta_{hall} = \{I_1^2, I_2, J_2, I_4, I_1I_3, I_1J_3, I_6, I_3^2, J_3^2, I_3J_3\}$ is an isotropic minimal integrity basis of the Hall tensor $\mathscr{H}$.

2. $\Theta_{hall} = \{I_1^2, I_2, J_2, I_4, I_1I_3, I_1J_3, I_6, I_3^2, J_3^2, I_3J_3\}$ is also an isotropic irreducible function basis of the Hall tensor $\mathscr{H}$ as well.

Not only for this particular selection, we can also begin with any minimal integrity basis of the second order tensor and use the same method to construct an isotropic invariant basis for the Hall tensor. We also prove in this work that such basis of the Hall tensor is a minimal integrity basis. A further question is whether there exists an irreducible function basis consisting of less than ten polynomial invariants.

Chapter 5

Conclusions and Further Research

Our purpose of this thesis is to investigate the tensor applications in different research areas. As a hypermatrix in mathematics, the positive definiteness of structured tensors has a strong connection with the positive definiteness of a homogeneous polynomial form Eq.(1.1). Hence, we introduce a new class of positive semi-definite tensors, the MO tensors, according to the special structure of the Moler matrix. As a physical quantity, the elasticity tensor plays an important role in mechanics. Since it is difficult to verify the strong ellipticity condition for an elasticity tensor, we establish the elasticity $\mathscr{M}$-tensor and nonsingular elasticity $\mathscr{M}$-tensor whose SE-condition are easy to be verified. For studying the further application in physics, we are attracted by the tensor representation theory. In this field, the representations for the third order three-dimensional symmetric tensor and the Hall tensor are investigated. To be specific, we have the following conclusions of this thesis:

1. In Chapter 2, we introduce concepts of the MO value, the MO set and the MO tensor for extending the Moler matrices to tensors. We mainly focus on the Sup-MO tensor and the essential MO tensor which are proved to have good properties. The Sup-MO tensors are proved to be positive definite tensors and the essential MO tensors are proved to be completely positive tensors. It is worthy to note that the smallest H-eigenvalue of the Sup-MO tensor is strictly

decreasing to 0 when its dimension n tends to infinity. This property is quite similar to that of the Moler matrix in Proposition 2.1. This implies that the Sup-MO tensors may be a good candidate for test tensors just as the role of the Moler matrices.

2. In Chapter 3, we first introduce two kinds of positive semi-definiteness: M-positive semi-definiteness and S-positive semi-definiteness, which all have connections with the strong ellipticity condition for the elasticity tensors. Then, a Perron-Frobenius type theorem for the nonnegative elasticity tensors is given for preparation. The uppermost contribution in this chapter is that we establish the elasticity $\mathscr{M}$-tensor and the nonsingular elasticity $\mathscr{M}$-tensor satisfying the SE-condition. These are inspired by the structures of the M-tensors and nonsingular M-tensors. Similar to the nonsingular M-tensors, we are able to construct several equivalent definitions for the nonsingular elasticity $\mathscr{M}$-tensor as well. These equivalent definitions give us more choices to verify a nonsingular elasticity $\mathscr{M}$-tensor that satisfies the SE-condition.

3. In Chapter 4, we first propose an irreducible isotropic function basis with 11 invariants for a third order three-dimensional symmetric tensor, which is also the first result of irreducible function bases for the third order three-dimensional symmetric tensor. Since there are at least 3 syzygy relations among those 11 invariants, see Eq. (4.14)-(4.16), it also reveals that there may still exist some algebraic relations among the invariants in an irreducible function basis of a tensor. Then we propose a minimal isotropic integrity basis with 10 isotropic invariants for the third order three-dimensional Hall tensor with the help of the minimal integrity basis of a second order three-dimensional tensor. What is more, we show that this minimal integrity basis is also an irreducible function basis of the Hall tensor.

However, there are still some further questions on these topics. Here, we list some questions that may be interesting in the future.

1. In Chapter 2, we also mentioned a concept called the Sub-MO value $\alpha_*(m)$. Whether the Sub-MO value can be reached or not, is still a question for future study. It decides that the MO set is compact or not. Furthermore, if the MO set is compact, could we get the length of the MO set?

2. We also have no idea about whether the Sup-MO tensors are SOS tensors or not. And how to utilize the Sup-MO tensors as a testing role in some tensor computation software packages is also an interesting problem.

3. Even though we have several equivalent definitions for the nonsingular elasticity $\mathscr{M}$-tensor, we will try to design some algorithms based on these equivalent conditions for verifying the SE-condition more quickly in the future.

4. In Chapter 4, we have proposed the two irreducible function bases for two kinds of third order three-dimensional tensors, respectively. However, our method comes from the work of Pennisi and Trovato in 1987 [55]. This method is only effective for the tensors with very few independent components. When the number of independent components in a tensor increase, we are not able to verify the irreducible function basis for the tensor by this method. When the number of independent components in a tensor increases, the number of invariants in a representation of the tensor will increase rapidly. For example, for a third order three-dimensional piezoelectric tensors with 18 independent components, its minimal isotropic integrity basis contains around 30000 invariants, it is impossible to find counterexamples to show every invariant is functionally irreducible. Hence, how to construct an irreducible function basis for large quantities of invariants is still a meaningful topic in future work.

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