THE HONG KONG POLYTECHNIC UNIVERSITY
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A FRAMEWORK FOR MODELING UNCERTAIN RELATIONSHIPS BETWEEN SPATIAL OBJECTS IN GIS BASED ON FUZZY TOPOLOGY

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A thesis submitted in partial fulfilment of the requirements for the Degree of
Doctor of Philosophy

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CERTIFICATE OF ORIGINALITY

I hereby declare that this thesis is my own work and that, to the best of my knowledge and belief, it reproduces no material previously published or written, nor material that has been accepted for the award of any other degree or diploma, except where due acknowledgement has been made in the text.

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Abstract

Boundaries of spatial objects in geographical information systems (GIS) may be vague or fuzzy and the classical set theory which is based on crisp boundary. Since the crisp set-based description may not match to what in the real world, and lead a wrong description in GIS and corresponding spatial analysis. As the fuzzy theory gives us another way of representing objects in GIS, we consider fuzzy sets and investigate corresponding fuzzy topological relations.

The objectives of the thesis as follows:

- To solve the existing problems of 9-intersection model by introducing several useful intersection models to describe the relations of point to point, point to line, point to region, line to line, line to region and region to region respectively.
- To the handle the vague or fuzzy models by investigating the topological relations in the cases of point to point, point to line, point to region, line to line, line to region and region to region respectively.
- To develop models for quantitatively compute uncertainty topological relations between spatial objects in GIS.
- To develop qualitative fuzzy topological relations under several invariant properties which including quasi-coincidence, connective and others.
- To apply the developed fuzzy topology to image processing.

The main focus of this thesis is mainly on modeling topological relations between spatial objects in GIS. Several issues on modeling topological relations between spatial objects are addressed, which including (a) defining topological relations and fuzzy GIS elements; (b) proving that topological relations between spatial objects are shape dependent; (c) modeling topological relations between spatial objects by using the concepts of quasi-coincidence and quasi-difference in fuzzy topology theory; (d) creating the computable fuzzy topology for practically implementing these conceptual topological relations in a computer environment.

The first issue is giving a new definition of the topological relations between two spatial objects which actually is an extended model for topological relations between two spatial objects. For this issue, we have found out that the number of topological

relations between the two sets is not as simple as finite; actually, it is infinite and can be approximated by a sequence of matrices. Moreover, as point, line and region (polygon) are the basic elements in GIS, thus we define them based on a fuzzy set.

Topology is normally considered as shape independent of spatial objects. This may not necessarily be true in describing relations between spatial objects in GIS. We present a proof that the topological relations between spatial objects are dependent on the shape of spatial objects. That is, topological relations of non-convex sets cannot be deformed to the topological relations of convex sets. The significant theoretical value of this finding is that topology of spatial objects are shape dependent. This indicates that when we describe topological relations between spatial objects in GIS, both topology and the shape of objects need to be considered.

There are two theoretical issues on modeling topological relations between spatial objects.

The first one is using the concepts of quasi-coincidence and quasi-difference to distinguish the topological relations between fuzzy objects and to indicate the effect of one fuzzy object on another in a fuzzy topology. Secondly, based on the developed computational fuzzy topology, methods for computing the fuzzy topological relations of spatial objects are proposed in this issue. For modeling the topological relations between spatial objects, the concepts of a bound on the intersection of the boundary and interior, and the boundary and exterior are defined based on the computational fuzzy topology. Furthermore, the qualitative measures for the intersections are specified based on the α-cut induced fuzzy topology, which are $(A_\alpha \wedge \partial A)(x) < 1-\alpha$ and $((A^c)_\alpha \wedge \partial A)(x) < 1-\alpha$. For computing the topological relations between spatial objects, the intersection concept and the integration method are applied. A computational 9-intersection model is thus developed. The computational topological relations between spatial objects are defined based on the ratio of the area/volume of the *meet* of two fuzzy spatial objects to the *join* of two fuzzy spatial objects. This is a step ahead of the existing topological relations models: from a conceptual definition of topological relations to the computable definition of topological relations. As a result, the quantitative values of topological relations can be computed.

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CHAPTER ONE
INTRODUCTION

1.1 Overview

Geographic information system (GIS) is widely applied in many different fields including map-making, site selection, emergency response planning, simulating environmental effects and others. In design and analysis of geographic entities, it is discovered that many spatial objects are vague and fuzzy. For example, the boundaries of urban areas, boundaries of states (Blakemore, 1984) clouds of gas and habitats of particular plants are vague or fuzzy (Cohn and Gotts, 1996). It seems that the original design and development of GIS, which is based on the assumption that the measurements on spatial objects (such as rivers, roads, trees and buildings) are error free and thus is not suitable for these applications. Therefore, there is a need to fundamentally enhance the basis of existing GIS by further coping with the vague/fuzzy modeling and the corresponding fuzzy topological relations description between spatial objects in GIS.

Topological relations between spatial objects are fundamental information in GIS, along with positional and attribute information. Information on topological relations can be used for spatial queries (e.g. asking queries for a user), spatial analyses, data quality control (e.g., checking for topological consistency), and others. Topological relations can be crisp or fuzzy depending on the certainty or uncertainty of spatial objects and the nature of their relations. When concerning spatial objects are uncertain, or their relations are not certain, the issue of uncertain topological relations emerges. There are many uncertain relations that need to be modeled among spatial objects in GIS. For example, the land covered by two kinds of vegetation: grasses and forests. Topological relation between grass and forest are uncertain. In this type of spatial analysis, it is essential to understand the uncertain topological relations between spatial objects.

Research on topological relations between spatial objects has attracted a great deal of attention in the past few decades. Many more studies have examined the topological relations between crisp spatial objects. White (1980) introduced the algebraic topological models for spatial objects. Allen (1983) identified 13 topological relations between two temporal intervals.

The major research outcomes on topological relations between regions have included, for example, 4-intersection models (Egenhofer, 1991; Winter, 2000) and 9-intersection models (Egenhofer, 1991; Cohn and Gotts, 1996; Clementini and Di Felice, 1996; Smith, 1996; Shi and Guo, 1999; Tang and Kainz, 2002; Tang *et al*, 2003, 2004, 2005).

The models (Egenhofer, 1993; Winter, 2000; Cohn and Gotts, 1996; Clementini and Di Felice, 1996; Smith, 1996; Tang and Kainz, 2002; Tang *et al*, 2003, 2004, 2005) were developed under the concepts of interior, boundary, and exterior which makes it potentially possible to model uncertainty relations between spatial objects conceptually. However, it is still difficult to implement these concepts (such as interior, boundary, and exterior) in a computer system, even with a membership function of a fuzzy set (or uncertain spatial object). Therefore, at this stage, while the conceptual framework on modeling topological relations between spatial objects is well developed (Egenhofer, 1993; Kainz *et al*, 1993; Mark and Egenhofer, 1994; Egenhofer and Mark, 1995; Cohn and Gotts, 1996; Clementini and Di Felice, 1996; Smith, 1996; Winter, 2000; Tang and Kainz, 2002; Tang *et al*, 2003, 2004, 2005), mechanisms that can be used to implement and operate these conceptual models are lacking.

1.2 Application areas of topological relations in GIS

Topological relations can be used in a GIS in the following three areas: for relation-related spatial query, for detecting logical inconsistent errors, to facilitate spatial relations related analysis. A topology-based GIS can be used to detect topological error, and can partially remove topological error. For example, we can make sure whether two lines are overshoot or undershoot, polygons are closed and the nodes of a polygon are properly labeled.

Query and reasoning

Query is a basic function of GIS, which is used to answer simple questions. For examples, "how many house are there within 100 meters around a particular point", "which city is the closest to Hong Kong" or "how far from building A to building B along certain path (see Figure 1.1, http://www.centamap.com/cent/index.htm)". Those simple questions arise the basic properties of the operation of queries, which include measurement of objects, like length, area, or shape and relation between objects. Moreover, some queries may involve transformation of spatial object, which may involve the operation on geometric and topological relations. Therefore, relations (geometric or topological) should be modeled carefully in GIS.

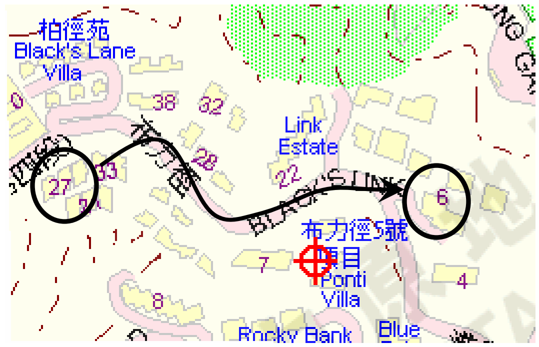
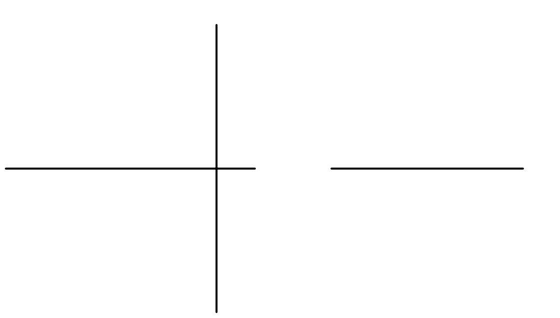


Figure 1.1: How far from building 27 to building 6 along Black's Link in Hong Kong

Topological consistency checking

Topological inconsistency errors may often occur in digitizing. For example, when two line segments that are supposed to meet each other at a node, but it doesn't. This kind of topological error is called overshoot or undershoot (Figure 1.2(a)). Another example is that a polygon may not be closed (Figure 1.2(b)). As topological errors violate the topological relations that has been well packed in the database. Many topological errors can be corrected automatically by operating certain programs. Therefore, it is important to store useful topological relations between spatial objects for the checking.



(1) (2)

Figure 1.2(a): Digitizing errors: overshoot (1) and undershoot (2) respectively

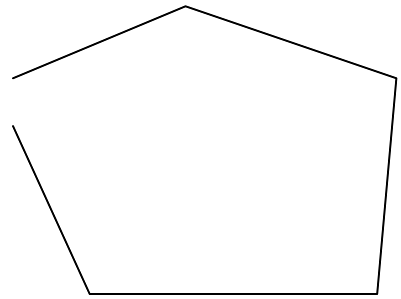


Figure 1.2(b): Digitizing errors: unclosed polygon

Modeling topological relations

Spatial objects may gradually change their location, orientation, shape and size over time. For example, the size of an island may be changing from time to time due to the effect of tide. In Hong Kong, Sha Chau is an island located in the west of Tuen Mun, due to the effect of tide, it is a single island in low tide and it will become two islands in high tide. To predict the topological relations between these two islands is related to the cognition of topological relations between two spatial objects (the two islands). In high tide, the topological relation between two islands is disjoint. In low tide, the topological relation between two islands is connected (see Figure 1.3, the base map was based on the data from the website http://www.centamap.com/cent/index.htm). Therefore, a set of topological relations with respect to spatial objects are important to model those changes, which each relation should be invariant under homeomorphism (rotation, scaling and translation are several examples). A useful set of topological relations between spatial objects allows us to predict the change of spatial objects.

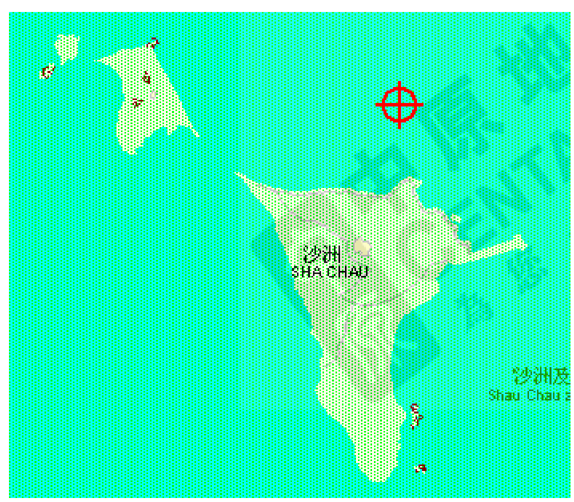


Figure 1.3: The topological relation between two islands is changing from time to time.

1.3 Methodology

The topological relations between two spatial objects are the fundamental properties in GIS. The techniques to study the topological relations between two spatial objects in this research will include fuzzy topology and probability theory. In the past, research on the representing topological relations between regions has concentrated on four streams. The first is the extending the 9-intersection model and which mainly focus on formalizing two-dimensional topological relations (Egenhofer, 1993; Cohn and Gotts, 1996; Smith, 1996). The second is the Voronoi-based 9-intersection model (Chen *et al*, 2001). The third is the representing topological relations for higher-dimensional space. The fourth is the formulation of topological relations and computable for handling easily.

On the other hand, in practical GIS cases, besides the region to region case, there are many other special cases to be modeled, for example the topological relations between line and region, regions with holes etc. For a crisp line and region, several existing intersection models state that a line segment in two-dimensional (2D) space have non-empty interior. But actually, a line should have an empty interior in 2D space, while it has non-empty interior in one-dimensional (1D) space. Therefore, when talking about the intersection relations, we should clearly state the space it belongs to. That means in the language of mathematical, the embedding of a line into a two-dimensional space should be considered.

1.3.1 Extended model of topological relations between crisp GIS objects

We first extend the presents models for describing topological relations between crisp in GIS (Egenhofer 1993; Cohn and Gotts, 1996; Smith, 1996; Chen *et al*, 2001). Which we first provide a new definition of the topological relations between two spatial objects, which is an extension of the traditional definition based on empty and non-empty under homeomorphic mapping. Based on this new definition, which includes topology of the object itself and several topological properties, we have uncovered a sequence of topological relations between two convex sets.

There are a number of new findings from this study. Among them, two major findings are: (a) the number of topological relations between the two sets is not as simple as finite; actually, it is infinite and can be approximated by a sequence of matrices, and (b) the topological relations between two sets are dependent on the shape of the sets themselves.

1.3.2 Modeling fuzzy topological relations in GIS

The boundaries of many objects in GIS can be vague or fuzzy (Leung and Chen, 1967; Leung and Yan, 1997, Shi and Guo 1999, Winter 1998, 2000) and the classical set theory (Apostol, 1974; Steven 1964), which is based on a crisp boundary, may not be suitable for handling these problems. An incorrect interpretation of fuzzy/vague GIS objects may cause not only loss of information, but also leading a wrong description of reality in GIS. The tide makes it difficult to determine the boundary of a sea and there are many problems if we want to determine the boundary of the Pacific Ocean. Due to the effect of the tide, some islands in the Ocean may appear and disappear from time to time. We may also have problem to describe these islands in a crisp and static GIS. By using a traditional GIS to describe objects may lead inaccuracy and even mistake for both single objects and relations between these two objects. As a result, problems will arise in GIS queries, analyses and the final decisions making. Therefore, the classical set theory (Wang, Hall and Subaryono, 1990) may not be a suitable basic tool for describing objects in GIS. Alternatively, fuzzy set theory provides a useful solution to the description of uncertain objects in GIS. The fuzzy topology can be applied to describe and quantify the fuzzy topological relations in GIS.

On modeling fuzzy topological relations between uncertain objects in GIS, the quasi-coincidence and quasi-difference, which are used to (a) distinguish the topological relations between fuzzy objects and (b) indicate the effect of one fuzzy object to the others. Geometrically, features in GIS can be classified as point features, linear features and polygon or region features. In this dissertation, we first introduce several basic concepts in fuzzy topology, which will be used in this study. This is followed by several definitions of fuzzy points, fuzzy lines and fuzzy regions for GIS objects. Next, the level of one fuzzy object affected the other is modeled based on the sum and difference of the membership functions that are the quasi-coincidence and quasi-difference respectively. Finally, an applicable example of using quasi-coincidence and quasi-difference based on the new definitions of fuzzy point, line and polygon are given.

1.3.3 Computable fuzzy topological space in GIS

Many management, analysis and display of spatial information on spatial objects in GIS are based on regions. Actually, the boundary of a spatial object may be vague or fuzzy (such as the boundary urban region) and the classical set theories that are based on a crisp boundary may not be suitable for handling these problems. It can be expected that we will be confused if we want to determine the boundary and the interior of the Pacific Ocean. As the ordinary topology can only give Boolean answer, it is obvious that it will cause the loss of information in making decision. Therefore, the classical set theories (Wang, Hall and Subaryono, 1990) may not be a good tool in GIS. As fuzzy theory provides us with a lot of useful information in the representation of GIS, we can apply fuzzy sets and investigate its topological relations for better modeling objects, especially uncertain objects and their relations in GIS.

Even we can successfully find out a membership function of a spatial object based on observations, we need to further quantify the topological relations and store them in a computer. Based on the 9-intersection models, the models (Egenhofer, 1993; Winter, 2000; Cohn and Gotts, 1996; Clementini and Di Felice, 1996; Smith, 1996; Tang, 2002) mainly have conceptual definitions of interior, boundary and exterior. That means we do not have the formulae to compute the value of interior, boundary and exterior. Therefore, it is difficult to implement these definitions in a computer. It seems that a particular and useful fuzzy topological space need to be developed which can help to compute the value of the interior, boundary and exterior of a spatial object once the membership function is known. The 9-intersection and other topological models can thus be implemented in the computer environment.

In this regards, we propose a computable fuzzy topological space, which is useful in GIS. In this aspect, we first definite two new operators, interior and closure operators. Then, these operators will be used to further define a computable fuzzy topological space, which will used to compute the interior, boundary and closure for fuzzy spatial objects in GIS.

1.3.4 Qualitative and quantitative fuzzy topological relations under several invariant properties

The fuzzy topological relations are elementary relations in the study of topological relations between spatial objects in GIS. Many researchers have developed their models in this area based on these relations (Egenhofer, 1991; Cohn and Gotts, 1996; Clementini and Di Felice, 1996; Smith, 1996; Shi and Guo, 1999; Tang and Kainz, 2002).

The properties of topological spaces that are preserved under homeomorphic mappings are called the topological invariants of the spaces. To study the topological relations, we need to first investigate the properties of a fuzzy mapping, especially homeomorphic mapping. The topological relations are an invariant under homeomorphic mappings. With these, we can thus guarantee the properties that will remain unchanged in a GIS transformation, such as the maintenance of topological consistency when digitizing a map or transferring a map from a system to another. Moreover, among these topological relations, we can extract useful topological relations, which commonly exist in GIS.

1.4 Structure of the thesis

The rest of this thesis is organized as follows. Chapter 2 review the point set topology, fuzzy topological theory, fuzzy mapping and explain the topological relations in a mathematic point of view.

Chapter 3 gives a review and an analysis on the development of uncertainty relations between spatial objects in GIS. The review includes several models of topological relations in GIS, which are Egenhofer's 4-intersection model, 9-intersection model, Clementini and Di Felice's 9-intersection model, Cohn and Gotts's 'Egg-Yolk' model, Tang and Kainz's 9-intersection model and Chen, Li, Li and Gold's Voronoi-based 9-intersection model.

Chapter 4 first gives an analysis on the Egenhofer's 9-intersection model. Then, based on a new definition of topological relations between two sets, the extended topological relations model between convex sets is descried in detail. Finally, it extends the proposed model from convex region to the non-convex region.

Chapter 5 gives a mathematical proof to show that the number of components in the intersection of the interior of two convex spatial regions in two-dimensional space is at most one, while the number of components can be more than one if they are not convex. That is the topological relations between spatial objects are dependent on the shape of spatial objects and hence the topological relations of non-convex sets cannot be deformed to the topological relations of convex sets.

Chapter 6 first gives several basic definitions (point, line and region) and basic properties of fuzzy elements in GIS that will be applied to model topological relations between spatial objects in GIS. Secondly several the basic properties of fuzzy topological space and mappings are described. Thirdly, quasi-coincidence and quasi-difference, which are used (a) to distinguish the topological relations between fuzzy objects and (b) to indicate the effect of one fuzzy object on another in a fuzzy topology, are adopted for the development. Finally, an applicable example of using quasi-coincidence and quasi-difference based on the new definitions of fuzzy point, line and polygon are given.

In chapter 7, we present a development of computational fuzzy topological space, which is based on the interior operator and closure operator. These operators are further defined as a coherent fuzzy topological space -- the complement of the open set is the closed set and vice versa; where the open set and closed set are defined by interior and closure operators – two level cuts. The elementary components of fuzzy topology for spatial objects – interior, boundary and exterior – are thus computed based on the computational fuzzy topological space. An example of calculating the interior, boundary, and exterior of

Mikania micrantha based on the aerial photographs of the Hong Kong countryside is provided in order to demonstrate the application of the theoretical development. Practically, the developed computational fuzzy topological space is applicable for computing the values of fuzzy topological relations, such as defined concepts by the 9-intersection model.

In chapter 8, based on the developed computational fuzzy topological space, the methods for computing the fuzzy topological relations of spatial objects are proposed. Specifically, the following areas are covered: (a) the homeomorphic invariants of the fuzzy topology are proposed; (b) the connectivity based on the newly developed fuzzy topology is defined; and (c) the fuzzy topological relations between simple fuzzy regions in GIS are modeled. For modeling the topological relations between spatial objects, the concepts of a bound on the intersection of the boundary and interior, and the boundary and exterior are defined based on the computational fuzzy topological space. Furthermore, the qualitative measures for the intersections are specified based on the α-cut induced fuzzy topological space, which are $(A_\alpha \wedge \partial A)(x) < 1 - \alpha$ and $((A^c)_\alpha \wedge \partial A)(x) < 1 - \alpha$.

For computing the topological relations between spatial objects, the intersection concepts and the integration method are applied, and a computational 9-intersection model is thus developed in chapter 9. The computational topological relations between spatial objects are defined based on the ratio of the area/volume of the *meet* of two fuzzy spatial objects to the *join* of two fuzzy spatial objects. This is a step further to the existing topological relations models: from a conceptual definition of topological relations to computable definition of topological relations. As a result, the quantitative values of topological relations can be calculated.

Chapter 10 presents conclusions, discussions and further research work, which includes how to use the fuzzy topological theory on modeling uncertainty topological relations in GIS.

The research procedure is sketched in Figure 1.4.

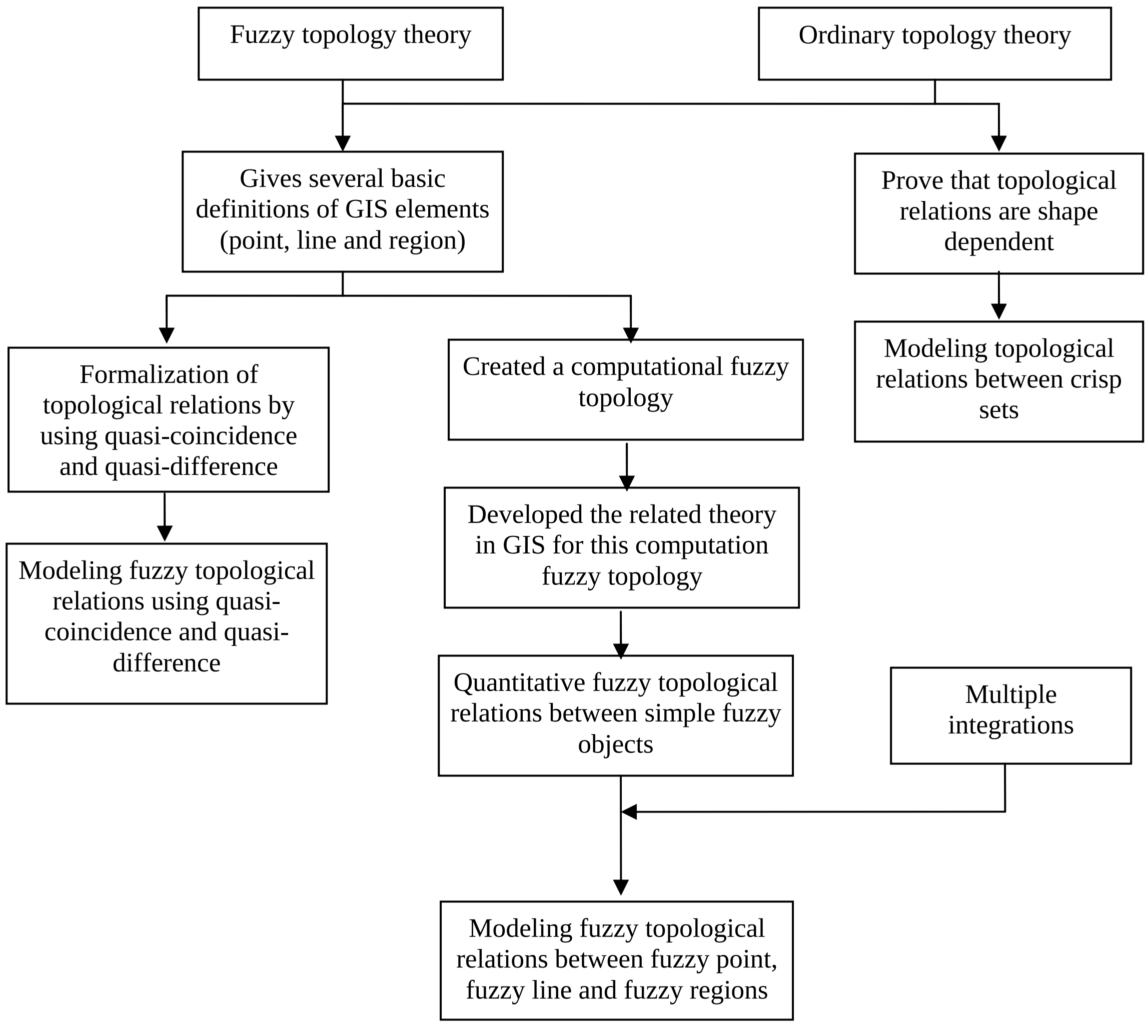


Figure 1.4: Research procedures

CHAPTER TWO
MATHEMATICAL FOUNDATION

2.1 Point set topological

The development of a coherent, mathematical theory of spatial relations to handle geographic applications is an important job (Egenhofer, 1993). Since mathematical theory can give a strong support on the geographic application. Thus, point-set topology (Steven, 1964; Apostol, 1974; Bredon, 1993) provides a useful tool to study the topological relations between spatial objects in GIS objects. In this chapter, several models are based on the theorem of point-set topology. Therefore, let us first review several related definitions and theorems of point-set topology.

2.1.1 The definition of topological space

A topological space is a set X with a collection of subsets of X called “open” sets, such that:

(1) the intersection of the two open sets is open;
(2) the union of any collection of open sets is open; and
(3) the empty set ϕ and whole space X are open.

Moreover, a subset C of X is called “closed” if its complement X\C is open.

2.1.2 The definitions of interior, closure and boundary

Definition 2.1: If X is a topological space and $A \subset X$, then the largest open set U contained in A is called the “interior” of A in X and denoted by A^{o}

Definition 2.2: If X is a topological space and $A \subset X$, then the smallest closed set F containing A is called the “closure” of A in X and denoted by $\overline{A}$.

Definition 2.3: If X is a topological space and $A \subset X$, then the set of the elements in X but not in A is called the complement of A and denoted by A^{c}.

Definition 2.4: If X is a topological space and $A \subset X$, then the boundary of A is defined to be $\partial A = \overline{A} \cap \overline{A^c}$.

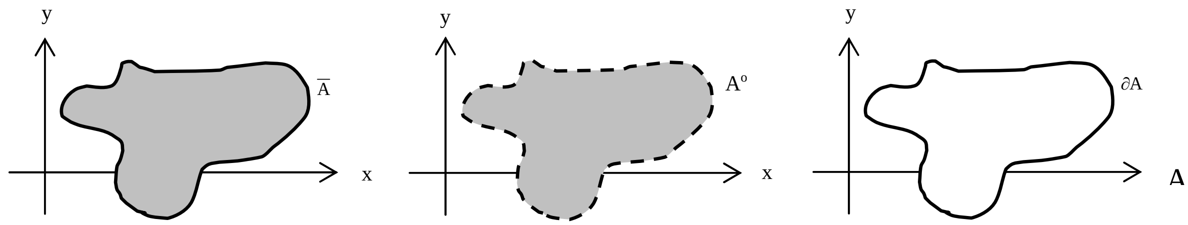


Figure 2.1(a): The closure of A in $\mathbf{R}^2$ Figure 2.1(b): The interior of A in $\mathbf{R}^2$ Figure 2.1(c): The boundary of A in $\mathbf{R}^2$

Figure 2.1(a) shows the closure of A in $\mathbf{R}^2$, Figure 2.1(b) shows the interior of A in $\mathbf{R}^2$, and Figure 2.1(c) shows the boundary of A in $\mathbf{R}^2$.

2.1.3 Several basic properties of point set topology

The properties of topological spaces that are preserved under homeomorphism are called the topological invariants of the spaces. Connectivity, compactness and first fundamental group are several fundamental topological invariants. As these invariants are invariant under bi-continuous mappings (homeomorphisms), studying these invariants can help us to understand the (topological) relations of spatial objects. Thus, in this chapter, we will discuss topological relations based on these invariants.

Definition 2.5: If X and Y are two topological spaces and $f : X \to Y$ is a mapping, then f is said to be continuous if $f^{-1}(U)$ is open for each open set $U \subset Y$.

Definition 2.6: If X and Y are two topological spaces and $f : X \to Y$ is a function, then f is said to be a homeomorphism if both f and f^{-1} are continuous.

Definition 2.7: The topological properties that are preserved under homeomorphism are called homeomorphic invariant.

Definition 2.8: A topological space X is called connected if it is not the disjointed union of two nonempty open subsets.

Definition 2.9: A topological space X is called compact if every open covering of X has a finite subcover.

Proposition 2.10: Compactness, connectivity and first fundamental group are homeomorphic invariant.

Q.E.D.

2.2 Fuzzy topological theory

Fuzzy sets are the basic element of fuzzy topology. Zadeh (1965) introduced the concept of the fuzzy set. Fuzzy theory has been developed since 1996, and the theory of fuzzy topology (Zadeh, 1965, Chang, 1968; Wong, 1974; Wu and Zheng, 1991; Liu and Luo, 1997) has been developed based on the fuzzy set. Topological relations are one of the concerns in modeling spatial objects, besides their geometric and attribute aspects. Uncertain topological relationships need to be modeled due to the existence of the indeterminate and uncertain boundaries between spatial objects in GIS. Fuzzy topological theory can potentially be applied to the modeling of fuzzy topological relations among spatial objects. The followings are several definitions and basic properties of fuzzy sets in GIS that will be used later.

Definition 2.11 (fuzzy subset): Let X be a nonempty ordinary set, I be the closed interval [0, 1], which actually is a complete lattice.

An I-fuzzy subset on X is a mapping (called the membership function of A) $\mu_A : X \rightarrow I$, i.e. the family of all the [0, 1]-fuzzy or I-fuzzy subsets on X is just I^X consisting of all the mappings form X to I. I^X here is called an I-fuzzy topological space, X is called the carrier domain of each I-fuzzy subset on it, and I is called the value domain of each I-fuzzy subset on X. $A \in I^X$ is called a crisp subset on X, if the image of the mapping is the subset of $\{0, 1\} \subset I$.

Definition 2.12 (rules of set relations): Let A and B be two fuzzy sets in X with membership functions $\mu_A(x)$ and $\mu_B(x)$ respectively. Then

(i) A = B iff $\mu_A(x) = \mu_B(x)$ for all x in X.

(ii) $A \leq B$ iff $\mu_A(x) \leq \mu_B(x)$ for all x in X.

(iii) C = $A \vee B$ iff $\mu_C(x) = \max[\mu_A(x),\ \mu_B(x)]$ for all x in X.

(iv) D = $A \wedge B$ iff $\mu_D(x) = \min[\mu_A(x),\ \mu_B(x)]$ for all x in X.

(v) E = X\A iff $\mu_E(x) = 1 - \mu_A(x)$ for all x in X.

In the following, we introduce the fuzzy topology, and which will be a base for the description of uncertain relations between objects in GIS. Fuzzy topology is an extension of ordinary topology that fuses two structures, order structure and topological structure. Furthermore, the fuzzy interior, boundary and exterior play an important role in the uncertain relations between GIS objects. Therefore, we have to define those items with a very clear concept.

Definition 2.13 (fuzzy topological space): Let X be a non-empty ordinary sets, I be a I-lattice, $\delta \subset I^x$. δ is called a I-fuzzy topology on X, and (I^x, δ) (or (X, I^X, δ) for detail; or I^x for short) is called an I-fuzzy topological space (I-fts), if δ satisfies the following conditions:

(i) $0, 1 \in \delta$;

(ii) If $A, B \in \delta$, then $A \wedge B \in \delta$,

(iii) Let $\{A_i : i \in J\} \subset \delta$, where J is an index set, then $\vee_{i \in J} A_i \in \delta$.

The elements in δ are called open elements and the elements in the complement of δ are called closed elements.

Definition 2.14 (interior and closure): For any fuzzy set A, we defined

(i) the interior of A as the join of all the open subset contained in A, denoted by A^{o}, i.e. $A^{o} = \vee\{B \in \delta \; : \; B \leq A\}$.

(ii) the closure of A as the meet of all the closed subset containing A, denoted by $\overline{A}$, i.e. $\overline{A} = \wedge\{B \in \delta' \; : \; B \geq A\}$.

Definition 2.15 (fuzzy complement): For any fuzzy set A, we defined the complements of A by $A^{c}(x) = 1 - A(x)$.

Definition 2.16 (fuzzy boundary): The boundary of a fuzzy set A is defined as $\partial A = \overline{A} \wedge \overline{A^{c}}$.

Theorem 2.17 (properties of interior): Let (I^{X}, δ) be an L-fts. Then

(i) $0^{o} = 0$, $1^{o} = 1$. (The empty and the whole set are open.)

(ii) $A^{o} \leq A$.

(iii) $A^{o\,o} = A^{o}$.

(iv) $A \leq B \;\Rightarrow\; A^{o} \leq B^{o}$.

(v) $(A \wedge B)^{o} = A^{o} \wedge B^{o}$.

Proof: (i) and (ii) are by the definition of interior.

(iii): Since $A^{o\,o}$ is the largest open contained in A^{o} and A^{o} itself is open, hence $A^{o\,o} = A^{o}$.

(v): by (vi) $(A \wedge B)^{o} \leq A^{o}$ and $\leq B^{o}$, therefore $(A \wedge B)^{o} \leq A^{o} \wedge B^{o}$. On the other hand $A^{o} \wedge B^{o} \leq A \wedge B$ which $A^{o} \wedge B^{o}$ is open contained in $A \wedge B$, hence $A^{o} \wedge B^{o}$ must be contained in the largest open set $(A \wedge B)^{o}$. Thus $(A \wedge B)^{o} = A^{o} \wedge B^{o}$.

Q.E.D.

Theorem 2.18 (properties of closure): Let (I^X, δ) be an I-fts. Then

(i) $\overline{0} = 0$, $\overline{1} = 1$. (The empty and the whole set are closed.)

(ii) $A \le \overline{A}$.

(iii) $\overline{\overline{A}} = \overline{A}$.

(iv) $A \le B \;\Rightarrow\; \overline{A} \le \overline{B}$.

(v) $\overline{A \vee B} = \overline{A} \vee \overline{B}$.

Proof: (i) and (ii) are by the definition of closure.

(iii): Since $\overline{\overline{A}}$ is the smallest closed set containing $\overline{A}$ and $\overline{A}$ itself is closed, hence $\overline{\overline{A}} = \overline{A}$.

(v): by (iv) $\overline{A} \le \overline{A \vee B}$ and $\overline{B} \le \overline{A \vee B}$, so $\overline{A} \vee \overline{B} \le \overline{A \vee B}$. On the other hand, since $\overline{A}$ and $\overline{B}$ are closed set containing A and B respectively, $\overline{A} \vee \overline{B}$ is a closed set containing $A \vee B$. As $\overline{A \vee B}$ is the smallest closed containing $A \vee B$, hence $\overline{A \vee B} \le \overline{A} \vee \overline{B}$. Thus $\overline{A \vee B} = \overline{A} \vee \overline{B}$.

Q.E.D.

Theorem 2.17 and 2.18 allow us make a generalization and introduce the following two concepts, which are the new direction of defining the fuzzy topological space.

Definition 2.19 (Closure operator): An operator $\alpha : I^x \to I^x$ is a fuzzy closure operator if the following conditions are satisfied:

(i) $\alpha(0) = 0$,

(ii) $A \le \alpha(A)$, for all $A \in I^x$,

(iii) $\alpha(A \vee B) = \alpha(A) \vee \alpha(B)$,

(iv) $\alpha(\alpha(A)) = \alpha(A)$, for all $A \in I^x$.

Definition 2.20 (Interior operator): An operator $\alpha : I^x \to I^x$ is a fuzzy interior operator if the following conditions are satisfied:

(i) $\alpha(1) = 1$,

(ii) $\alpha(A) \leq A$, for all $A \in I^x$,

(iii) $\alpha(A \wedge B) = \alpha(A) \wedge \alpha(B)$,

(iv) $\alpha(\alpha(A)) = \alpha(A)$, for all $A \in I^x$.

2.3 Fuzzy mapping

In GIS, we also concern about the change of certain spatial objects. For example what would happen to a map from a computer system to another computer system? What properties will be unchanged (invariant) under transformation and so on? Hence, the about definitions and theories are the fundamental tools to investigate those properties. Thus, properties of topological spaces that are preserved under homeomorphic mappings are called topological invariants of the spaces. To study the topological relations, we need to study the properties of a fuzzy mapping firstly, especially the homeomorphic mapping. The topological relations are invariants under homeomorphic mappings. Thus, we can guarantee the unchanged properties in GIS transformation such as maintaining of topological consistency when digitizing a map or transferring a map from a system to another system. The followings are several basic definitions and theories of fuzzy mapping.

Definition 2.21 (fuzzy mapping): Let I^X, I^Y be I-fuzzy topological spaces, $f : X \to Y$ be an ordinary mapping. Based on $f : X \to Y$, define I-fuzzy mapping $f^{\to} : I^X \to I^Y$ and its I-fuzzy reverse mapping $f^{\leftarrow} : I^Y \to I^X$ by

$$f^{\to} : I^X \to I^Y,\ f^{\to}(A)(y) = \vee\{A(x) : x \in X, f(x) = y\},\ \forall A \in I^X,\ \forall y \in Y,$$

$$f^{\leftarrow} : I^Y \to I^X,\ f^{\leftarrow}(B)(x) = B(f(x)),\ \forall B \in I^Y,\ \forall x \in X.$$

Proposition 2.22: Let I^X, I^Y be I-fuzzy topological spaces, $f : X \to Y$ be an ordinary mapping. $A \in I^X$ and $B \in I^Y$.

(i) $f^{\leftarrow} f^{\to}(A) \geq A$ and $f^{\to} f^{\leftarrow}(B) \geq B$.

(ii) $f^{\leftarrow}(B') = f^{\leftarrow}(B)'$.

(iii) If A(x) = 1, then $f^{\to}(A)(y) = 1$.

(iv) If B(y) = α, then $f^{\leftarrow}(B)(x) = \alpha$.

Proof: (i) By definition $f^{\leftarrow} f^{\to}(A)(x) = f^{\to}(A)(f(x)) = \vee\{A(x) : x \in X, f(x) = y\} \geq A(x)$.

By definition $f^{\to} f^{\leftarrow}(B)(y) = \vee\{f^{\leftarrow}(B)(x) : x \in X, f(x) = y\}$

$$= \begin{cases} B(f(x)) & \text{if } y = f(x) \\ 0 & \text{if no } x \text{ such that } f(x) = y \end{cases}.$$

(ii) $f^{\leftarrow}(B')(x) = B'(f(x)) = 1 - B(f(x)) = 1 - f^{\leftarrow}(B)(x) = f^{\leftarrow}(B)'(x)$.

(iii) By definition $f^{\to}(A)(y) = \vee\{(A)(x) : x \in X, f(x) = y\} = 1$.

(iv) By definition $f^{\leftarrow}(B)(x) = B(f(x)) = B(y) = \alpha$.

Q.E.D.

Theorem 2.23: Let I^X, I^Y be I-fuzzy topological spaces, $f : X \to Y$ be an ordinary mapping. Then for any $a \in I$ and any $A \in I^X$, we have $f^{\to}(aA) = af^{\to}(A)$.

Proof: $f^{\to}(aA)(y) = \vee\{(aA)(x) : x \in X, f(x) = y\}$

$$= \vee\{a \wedge (A(x)) : x \in X, f(x) = y\}$$

$$= a \wedge \vee\{(A(x)) : x \in X, f(x) = y\}$$

$$= a \wedge (f^{\to}(A)(y))$$

$$= af^{\to}(A)(y).$$

So $f^{\to}(aA) = af^{\to}(A)$

Q.E.D.

Proposition 2.24: Let I^X , I^Y be I-fuzzy topological spaces, $A \subset X$, $B \subset Y$, $f : X \to Y$ be an ordinary mapping. Then

(i) $f^{\to}(\chi_A) = \chi_{f[A]}$.

(ii) $f^{\leftarrow}(\chi_B) = \chi_{f^{-1}(B)}$.

Proof: (i)

$$\begin{aligned} f^{\to}(\chi_A)(y) &= \vee\{\chi_A(x) : x \in X, f(x) = y\} \\ &= \begin{cases} 1 & \text{if } \; y = f(x) \\ 0 & \text{otherwise} \end{cases} \\ &= \chi_{f[A]}(y). \end{aligned}$$

(ii)

$$\begin{aligned} f^{\leftarrow}(\chi_B)(x) &= \chi_B(f(x)). \\ &= \begin{cases} 1 & \text{if } \; f(x) \in B \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } \; x \in f^{-1}(B) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Q.E.D.

Definition 2.25 (injective mapping): Let (I^X, δ), (I^Y, μ) be I-fts's, $f^{\to} : (I^X, \delta) \to (I^Y, \mu)$ is called an I-fuzzy injective mapping. If $f^{\to}(A) = f^{\to}(B)$, then A = B.

Definition 2.26 (surjective mapping): Let (I^X, δ) , (I^Y, μ) be I-fts's, $f^{\to} : (I^X, \delta) \to (I^Y, \mu)$ is called an I-fuzzy surjective mapping if for all $B \in \mu$. Then there exists $A \in \delta$ such that $B = f^{\to}(A)$.

Definition 2.27 (bijective mapping): Let (I^X, δ), (I^Y, μ) be I-fts's, $f^{\to} : (I^X, \delta) \to (I^Y, \mu)$ is called an I-fuzzy bijective mapping if $f^{\to} : (I^X, \delta) \to (I^Y, \mu)$ is both injective and surjective.

Theorem 2.28: Let I^X, I^Y be I-fuzzy spaces, $f : X \to Y$ be an ordinary mapping. Then $f^{\to} : (I^X, \delta) \to (I^Y, \mu)$ is bijective if and only if $f : X \to Y$ is bijective.

Q.E.D.

Definition 2.29 (fuzzy continuous mapping): Let (I^X, δ), (I^Y, μ) be I-fts's, $f^{\to} : (I^X, \delta) \to (I^Y, \mu)$ be an I-fuzzy continuous mapping if $f^{\leftarrow}$ maps every open subset in (I^Y, μ) as an open subset in (I^X, δ), i.e. for all $U \in \mu$, $f^{\leftarrow}(\mu) \in \delta$.

Definition 2.30 (fuzzy homeomorphism): Let (I^X, δ), (I^Y, μ) be I-fts's, $f^{\to} : (I^X, \delta) \to (I^Y, \mu)$ is called an I-fuzzy homeomorphism, if it is bijective continuous and open.

Definition 2.31 (stratification): Let (I^X, δ) be an L-fts, μ be the I-fuzzy topology on X generated by $\delta \cup \{\underline{a} : a \in I\}$, then μ is called the stratification of δ, and call (I^X, μ) the stratification of (I^X, δ). Where $\underline{a}(x) = aX(x) = a \wedge X(x)$.

Proposition 2.32: Stratification is preserved by I-fuzzy reverse mapping. That is if $f^{\to} : (I^X, \delta) \to (I^Y, \mu)$ is contiuous and (I^X, μ) is stratified, then (I^X, δ) is stratified.

Q.E.D.

Proposition 2.33: Let (I^X, δ), (I^X, μ) be I-fts's, $f^{\to} : (I^X, \delta) \to (I^Y, \mu)$ an I-fuzzy continuous mapping, δ_o and μ_o be the stratification of δ and μ respectively. Then $f^{\to} : (I^X, \delta_o) \to (I^Y, \mu_o)$ is continuous. This proposition tells us the continuous between certain of topologies can be extended to its stratification automatically.

Q.E.D.

Definition 2.34 (connectness): Let (I^X, δ) be an I-fts, A, $B \in I^X$. A and B are called separated, if $\overline{A} \wedge B = A \wedge \overline{B} = 0$.

A is called connected, if there does not exist separated C, $D \in I^X \setminus \{0\}$ such that $A = C \vee D$. Call $\left(I^X, \delta\right)$ is connected, if the largest I-fuzzy subset 1 is connected.

In general topological space X is connected if and only if X itself is the unique non-empty open-and-closed subset in X (equivalent to the original definition of connected set in general topology) (Steven, 1964). This is true because any other non-empty open-and-closed subset will generate a pair of separated subsets with itself and its complementary set and their union is just X. But in I-fuzzy topology, the meet of a subset and its "dual form" – pseudo-complementary set need not be 0. So the parallel conclusion does not hold in I-fuzzy topology (Liu and Luo, 1997).

2.4 Topological relations from a mathematic point of view

Both the nine-intersection model and the Voronoi-based nine-intersection model are actually trying to classify the intersection relations of two objects in the sense of the topological relations. Usually a relation consists in associating objects of one kind with objects of another kind. The following is the definition of relation in the language of mathematics.

Definition 2.35: Let A and B be two sets. A relation, denoted by "~", from set A to set B is an ordered triple (A, B, G), where $G \subset A \times B$; G is called the graph of "~"; A and B are respectively called the set of departure and the set of destination of "~".

Definition 2.36: Let P be a set and let "~" be a relation on P.

(1) ~ is called reflexive, if a ~ a for every $a \in P$.

(2) ~ is called symmetric, if a ~ b, then b ~ a for every $a, b \in P$.

(3) ~ is called transitive, if a ~ b and b ~ c, then a ~ c for every $a, b, c \in P$.

Definition 2.37: Let P be a set and let "~" be a relation on P. If <P, "~"> has the properties of (1), (2) and (3), then <P, "~"> is called an equivalent relation.

Actually, the definition 2.37 is just the partition of P.

Example 2.38: let

$$P = R^2 \setminus \{(x, y) : y = 0\} \text{ (the real plane except the x-axis)}$$

and let the relation be two points having the same class if and only if "the y-coordinates of these two points have the same sign." This is an equivalent relation, as

(1) Point (x, y) itself the y-coordinates must have the same sign as itself. That means that the relation is reflexive.

(2) (x_1, y_1) have relations with (x_2, y_2), meaning that the sign of y_1 and y_2 must be the same. Thus, it equivalent to (x_2, y_2) having a relation with (x_1, y_1). That means that the relation is symmetric.

(3) If (x_1, y_1) have a relation with (x_2, y_2), and (x_2, y_2) have a relation with (x_3, y_3), this means that y_1, y_2 and y_3 have the same sign. In particular, y_1 and y_3 have the same sign. Thus, (x_1, y_1) have a relation with (x_3, y_3). That means that the relation is transitive.

The result of this relation on $\Re^2 \setminus \{(x, y) : y = 0\}$ is simply the partitioning of $\Re^2 \setminus \{(x, y) : y = 0\}$ into two parts: one part is above the x-axis and the other part is below the x-axis (see Figure 2.2).

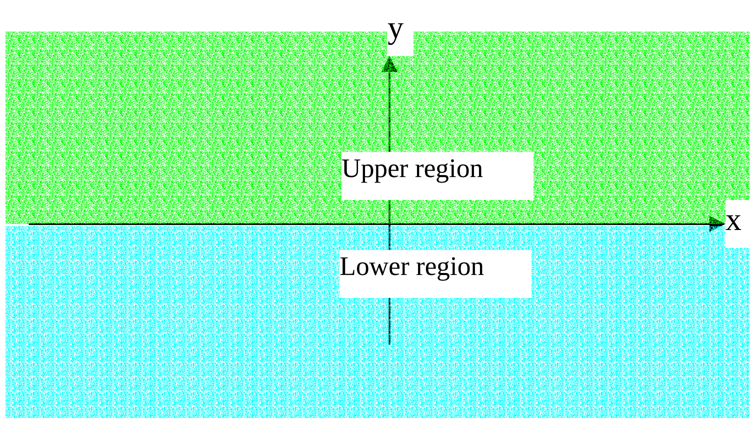


Figure 2.2: The equivalent class in Example 2.38 divides the plane into two parts

CHAPTER THREE
A REVIEW OF UNCERTAINTY RELATIONS BETWEEN SPATIAL OBJECTS IN GIS

Topological relations are the fundamental properties for the spatial query and analysis in GIS. The topological relations between crisp spatial objects have been developed. Based on the 4-intersection and the ordinary point set theory, Egenhofer and Franzosa (1991) gave the topological relations between two spatial regions in 2-D. In 1993, Egenhofer gave an extension on the 4-intersection model by introducing the 9-intersection model. Later, Egenhofer, Clementini and Di Felice (1994) gave an extension of the topological relations between spatial objects in 2-D with arbitrary holes.

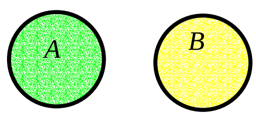
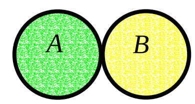
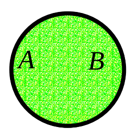
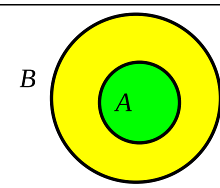
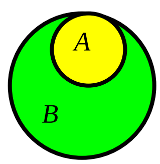
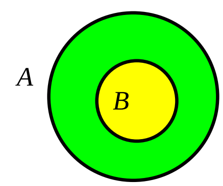
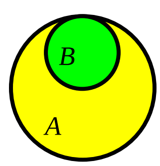
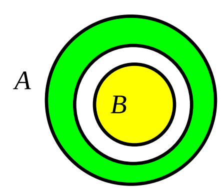
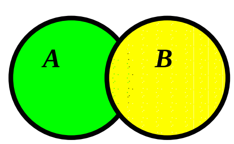
Based on the 9-intersection, Cohn and Gotts (1996) gave 46 topological relations between two regions with indeterminate boundaries. While Clementini and Di Felice (1996) gave 44 topological relations between two regions with indeterminate boundaries. However how to formalize the topological relations between fuzzy regions it needs more investigation. By using the 9-intersection matrix and the fuzzy theory, there are 44 relations between two simply fuzzy regions were given (Tang and Kainz, 2002). Based on the Voronoi diagram, Li *et al* (1999, 2002) and Chen *et al* (2001) gave a Voronoi-based 9-intersection model. In their paper, the problems of the 4- and 9-intersection models were discussed and a new 9-intersection model, called Voronoi-based 9-intersection model we given.

3.1 Egenhofer's 4-intersection model

Based on point-set topology, a spatial object A can be divided into two parts, interior and boundary, denoted by A^o and ∂A respectively. The topological relations between objects can be described by the four intersections of interiors and boundaries, denoted by $\begin{pmatrix} \partial A \cap \partial B & \partial A \cap B^o \\ A^o \cap \partial B & A^o \cap B^o \end{pmatrix}$. By considering the invariance property, empty and non-empty of each intersection, there are 16 possible relations between objects in 2-dimensional space

(Egenhofer, 1991). If just talking about the topological spatial relations between polygonal areas in the plane, there are nine relations between two spatial regions and it summarizes in the Table 3.1. “0” means empty and “1” means non-empty. The detail discussion will be hold on chapter 4.

Table 3.1: Nine topological relations between two spatial regions from Egenhofer.

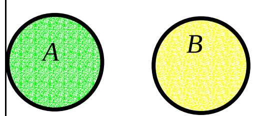
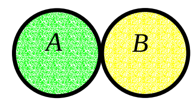
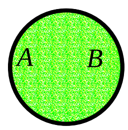
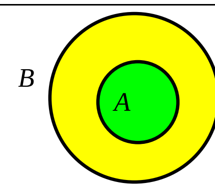
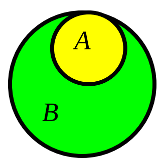
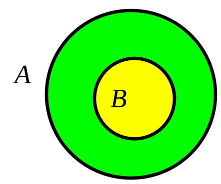
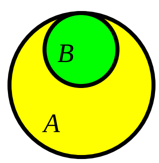
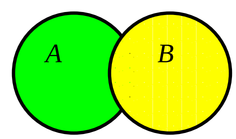
$\begin{pmatrix} \partial A \cap \partial B & \partial A \cap B^{o} \\ A^{o} \cap \partial B & A^{o} \cap B^{o} \end{pmatrix}$	Illustration diagrams	$\begin{pmatrix} \partial A \cap \partial B & \partial A \cap B^{o} \\ A^{o} \cap \partial B & A^{o} \cap B^{o} \end{pmatrix}$	Illustration diagrams
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ A and B are disjoint		$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ A and B touch	
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ A equals B		$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ B contains A	
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ B covers A		$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ A contains B	
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ A covers B		$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ A and B overlap with disjoint boundaries	
$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ A and B overlap with intersecting boundaries			

3.2 Egenhofer's 9-intersection model

Based on point-set topology, and embed set A into $\mathbf{R}^2$, then the set A can be divided into three parts in $\mathbf{R}^2$,interior (A^o), boundary (∂A) and exterior (A^c). This gives an extension from the 4-intersection model to a new intersection model by considering the intersection of interior (A^o), boundary (∂A) and exterior (A^c) with interior (B^o), boundary (∂B) and exterior (B^c). Based on intersections of these six parts, nine combinations of intersection, $\begin{pmatrix} A^o \cap B^o & \partial A \cap B^o & A^c \cap B^o \\ A^o \cap \partial B & \partial A \cap \partial B & A^c \cap \partial B \\ A^o \cap B^c & \partial A \cap B^c & A^c \cap B^c \end{pmatrix}$, which can be used to represent the topological relations between spatial objects (Egenhofer, 1993).

By considering the invariance property, empty and non-empty of each component, there are 512 possible relations between spatial objects. From the 512 possible cases, Egenhofer claims that there are only eight can be realized if the objects are spatial regions in $\mathbf{R}^2$ (Egenhofer, 1993). They are disjoint, contains, inside, equal, meet, covers, coveredBy and overlap respectively. Table 3.2 shows these eight relations between two spatial regions and the detail of this model will be discussed in chapter 4.

Table 3.2: Egenhofer's 9-intersection model with eight relations between two spatial regions

$\begin{pmatrix} A^o \cap B^o & \partial A \cap B^o & A^c \cap B^o \\ A^o \cap \partial B & \partial A \cap \partial B & A^c \cap \partial B \\ A^o \cap B^c & \partial A \cap B^c & A^c \cap B^c \end{pmatrix}$		$\begin{pmatrix} A^o \cap B^o & \partial A \cap B^o & A^c \cap B^o \\ A^o \cap \partial B & \partial A \cap \partial B & A^c \cap \partial B \\ A^o \cap B^c & \partial A \cap B^c & A^c \cap B^c \end{pmatrix}$	
$\begin{pmatrix} \phi & \phi & \sim\phi \\ \phi & \phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$ Disjoint		$\begin{pmatrix} \phi & \phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$ Meet	
$\begin{pmatrix} \sim\phi & \phi & \phi \\ \phi & \sim\phi & \phi \\ \phi & \phi & \sim\phi \end{pmatrix}$ Equal		$\begin{pmatrix} \sim\phi & \phi & \phi \\ \sim\phi & \phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$ Inside	
$\begin{pmatrix} \sim\phi & \phi & \phi \\ \sim\phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$ CoveredBy		$\begin{pmatrix} \sim\phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$ Contains	
$\begin{pmatrix} \sim\phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$ Covers		$\begin{pmatrix} \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$ Overlap	

3.3 Algebric Model

In dealing with indeterminate boundaries, based on Egenhofer's 9-intersection model

$$\begin{pmatrix} A^o \cap B^o & \partial A \cap B^o & A^c \cap B^o \\ A^o \cap \partial B & \partial A \cap \partial B & A^c \cap \partial B \\ A^o \cap B^c & \partial A \cap B^c & A^c \cap B^c \end{pmatrix},$$

Clementini and Di Felice (1996) defined a region with a broad boundary, by using two simple regions, denoted

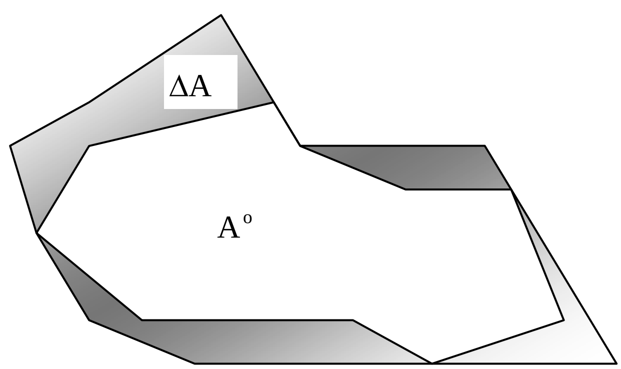


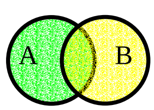
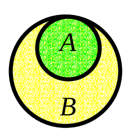
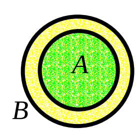
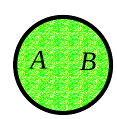
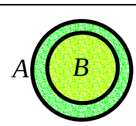
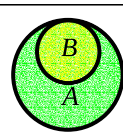
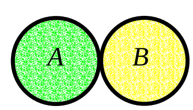
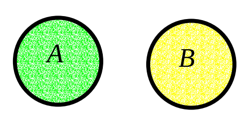
Figure 3.1: Region A with broad boundary

by ΔA. More precisely, the broad boundary is a simple connected subset of $\mathbf{R}^2$ with a hole. Figure 3.1 shows that the shaded region is a broad boundary of A. Based on the empty and non-empty invariance, Clementini and Di Felice's algebric model, $\begin{pmatrix} A^o \cap B^o & \Delta A \cap B^o & A^c \cap B^o \\ A^o \cap \Delta B & \Delta A \cap \Delta B & A^c \cap \Delta B \\ A^o \cap B^c & \Delta A \cap B^c & A^c \cap B^c \end{pmatrix}$, gave total 44 relations between two spatial regions with broad boundary.

3.4 'Egg-Yolk' Model

When dealing with non-exact spatial objects and the vagueness/fuzziness spatial objects, Cohn and Gotts (1996) suggest using two concentric sub-regions, indicating degree of "membership" in a vague/fuzzy region, which "yolk" represents the precise part and "egg" represents the vague/fuzzy part. Based on the RCC (Region connection calculus) theory (Randell 1992), eight base relations can be defined. They are DC (Disconnected), EC (Externally Connected), PO (Partially Overlapping), TPP (Tangential Proper Part), NTPP (Non-tangential Proper Part), EQ (Equal), NTPPI (Non-tangential Proper Part Inverse) and TPPI (Tangential Proper Part Inverse) respectively (see Table 3.3).

Table 3.3: RCC relations between two regions

PO(*A*, *B*)	TPP(*A*, *B*)	NTTP(*A*, *B*)	EQ(*A*, *B*)
			
NTPPI(*A*, *B*)	TPPI(*A*, *B*)	EC(*A*, *B*)	DC(*A*, *B*)
			

The egg-yolk model is an extension of RCC theory into the vague/fuzzy region. 46 relations can be identified to represent conservatively defined limits on the possible "complete crispings" or precise versions of a vague region.

3.5 Tang's 9-intersection model between simple fuzzy regions

To investigate the topological relations between fuzzy regions, Tang and Kainz (2002) decompose a fuzzy set A into several topological parts, they are:

(i) the core, $A^{\oplus}$, which is the fuzzy interior part with value equal to one;

(ii) the c-boundary, $\partial^{c}A$, the fuzzy subset of ∂A with $\partial A(x) = \overline{A}(x)$.

(iii) b-closure, $A^{\perp}$, the fuzzy subset of $\overline{A}$ with $\overline{A}(x) > \partial A(x)$.

(iv) the outer, $A^{=}$, the fuzzy complement of A with value equal to one.

Then formalize a 9-intesection matrix and a 4*4 intersection matrix, which is based on the different topological parts of two fuzzy sets. For the 9-intersection matrix, $\begin{pmatrix} A^{\oplus} \wedge B^{\oplus} & A^{\oplus} \wedge \ell B & A^{\oplus} \wedge B^{=} \\ \ell A \wedge B^{\oplus} & \ell A \wedge \ell B & \ell A \wedge B^{=} \\ A^{=} \wedge B^{\oplus} & A^{=} \wedge \ell B & A^{=} \wedge B^{=} \end{pmatrix}$, the core ($A^{\oplus}$), fringe ($\ell A$) and outer ($A^{=}$) are adopted to formalize, where the fringe is the union of c-boundary and b-closure. There are total 44 relations between two simply fuzzy regions. For the 4*4 intersection matrix, $\begin{pmatrix} A^{\oplus} \wedge B^{\oplus} & A^{\oplus} \wedge B^{\perp} & A^{\oplus} \wedge \partial^{c}B & A^{\oplus} \wedge B^{=} \\ A^{\perp} \wedge B^{\oplus} & A^{\perp} \wedge B^{\perp} & A^{\perp} \wedge \partial^{c}B & A^{\perp} \wedge B^{=} \\ \partial^{c}A \wedge B^{\oplus} & \partial^{c}A \wedge B^{\perp} & \partial^{c}A \wedge \partial^{c}B & \partial^{c}A \wedge B^{=} \\ A^{=} \wedge B^{\oplus} & A^{=} \wedge B^{\perp} & A^{=} \wedge \partial^{c}B & A^{=} \wedge B^{=} \end{pmatrix}$, the core ($A^{\oplus}$), c-boundary ($\partial^{c}A$), b-closure ($A^{\perp}$) and outer ($A^{=}$) are adopted to formalize the fuzzy topological relations between spatial objects. More relations can be got in this model.

3.6 Voronoi-based 9-intersection model

Let $\{P_1, P_2, \ldots, P_n\}$ be a set of n points in $\mathbf{R}^2$, then the region

$$V(P_i) = \{x \in R^2 : \mathrm{dis}(x, P_i) \le \mathrm{dis}(x, P_j),\ j \ne i,\ j = 1, 2, \ldots, n\},$$

where dis(x, P_i) is the distance of x and P_i. V(P_i) is called a Voronoi regions of P_i. The set $\{P_1, P_2, \ldots, P_n\}$ induced a set of Voronoi regions $\{V(P_1), V(P_2), \ldots, V(P_n)\}$, which form a non-overlapping tessellation of $\mathbf{R}^2$. Furthermore, all spatial objects, point, line and region, in $\mathbf{R}^2$ can define a set of Voronoi regions. Based on the Egenhofer's 9-intersection model, a Voronoi-based 9-intersection model is developed (Chen, Li and Gold, 2001) by replacing the complement of an object with its Voronoi regions that is $\begin{pmatrix} A^o \cap B^o & \partial A \cap B^o & A^v \cap B^o \\ A^o \cap \partial B & \partial A \cap \partial B & A^v \cap \partial B \\ A^o \cap B^v & \partial A \cap B^v & A^v \cap B^v \end{pmatrix}$, where A^v denoted by the Voronoi region of A. In this model, topological relations between area-area, line-area, area-point, line-line, line-point and point-point are as follow: (see Table 3.4)

Table 3.4: Number of topological relations by using V9I

Cases		Number
AA	(Area-Area)	13
LL	(Line-Line)	8
LA	(Line-Area)	13
PP	(Point-Point)	3
PL	(Point-Line)	4
PA	(Point-Area)	5

3.7 Analysis of these models

Actually, there are many limitations on the existence models. The models (Egenhofer, 1991, 1993; Winter, 2000; Cohn and Gotts, 1996; Clementini and Di Felice, 1996; Smith, 1996; Tang and Kainz, 2002; Tang, Kainz and Yu, 2003; Tang, 2004) only give a framework to conceptually describe the topological relations between two regions.

Computation of topological relations is very essential for practically implementing them in a GIS. With the aim of computing and implementing the 9-intersection model and other topological models for real world GIS software and applications, our research work is aimed at firstly defining a computational fuzzy topological space to compute interior, boundary and exterior, which is based on the two operators, interior operator and closure operator. Then apply the 9-intersection integration models to give a list of qualitative fuzzy topological relations between simple fuzzy regions in GIS.

CHAPTER FOUR
EXTENDED MODEL OF TOPOLOGICAL RELATIONS BETWEEN OBJECTS IN GIS

This chapter presents an extended model for describing topological relations between two sets (spatial objects) in GIS. First, based on the definition of the topological relations between two objects, we uncover a sequence of topological relations between two convex sets. Second, an extended model for topological relations between two sets is proposed based on the new definition. The topological relations between two convex sets are expressed as a sequence of 4×4 matrices, which are the topological properties of $A^o \cap B^o$, $A^o \setminus B$, $B^o \setminus A$, $\partial A \cap \partial B$. Moreover, the model is also extended for handling the properties of the topological relations between two non-convex sets, where the factor of first fundamental group is added to $A \cup B$ handle these complex relations. There is a new finding from this study, which is the number of topological relations between two sets is not as simple as finite; actually, it is infinite and can be approximated by a sequence of matrices.

4.1 An analysis of the nine-intersection model

Egenhofer and Herring (1991) decomposed any region A into three parts: interior, boundary and exterior, denoted by A^o, ∂A and A^c, respectively. The nine-intersection model for the topological relations between two non-empty regions, A and B, was then defined as follows:

$$\begin{pmatrix} A^o \cap B^o & \partial A \cap B^o & A^c \cap B^o \\ A^o \cap \partial B & \partial A \cap \partial B & A^c \cap \partial B \\ A^o \cap B^c & \partial A \cap B^c & A^c \cap B^c \end{pmatrix}$$

(nine-intersection model for the topological relations)

Considering the values of empty and non-empty, there are eight topological relations between two non-empty regions. In fact, this model contains an insufficiency and we will illustrate this as follow.

4.1.1 The topologies of a line segment and a region in $\mathbf{R}^2$

The topology of a line segment in **R** is the collection of all open sets in one-dimensional (1D) space. For example, in the Figure 4.1(a) the open interval (0, 1) is an open set in 1D. Here, in Figure 4.1(a), the round bracket indicates that the interval does include the points 0 and 1. The topology of a region in $\mathbf{R}^2$ is the collection of all open sets in 2D. For example, in Figure 4.1(b), the open disc

$$D(0, 1) = \{(x, y) \in R^2 : x^2 + y^2 < 1\}$$

is an open set in 2D. Here, the dot circle line indicates that the boundary is not included.

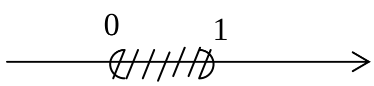


Figure 4.1(a): Open set in **R**

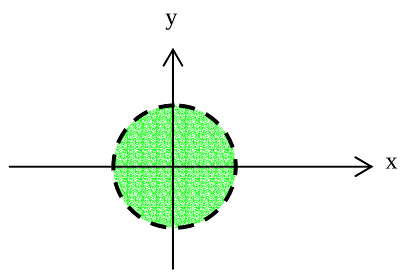


Figure 4.1(b): Open set in $\mathbf{R}^2$

4.1.2 Line segment in $\mathbf{R}^2$

In this section, we try to illustrate the topology of a line segment. The line segment in $\mathbf{R}^2$ can be described as an **embedding** of a connected interval from **R** to $\mathbf{R}^2$, which does not have intersection; i.e.,

$$\alpha : [0, 1] \rightarrow \Re^2,$$

where [0, 1] is a closed interval in **R** and $\alpha(t_1) \neq \alpha(t_2)$ for all $t_1, t_2 \in [0, 1]$ (see Figure 4.2). Thus, the induced topology of a line segment in $\mathbf{R}^2$ is the collection of all open intervals in **R**. In other words, references to the interior, boundary and exterior of a line segment will be to the topology in **R**.

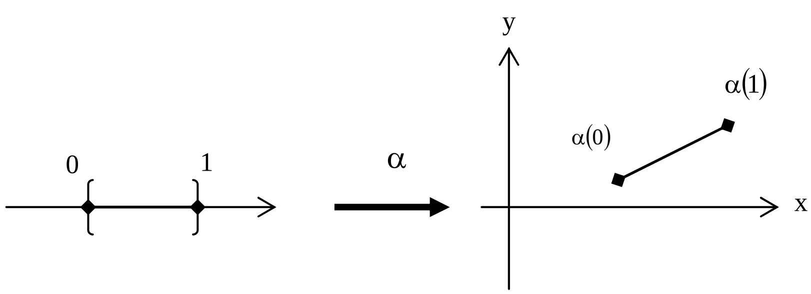


Figure 4.2: Line segment in $\mathbf{R}^2$

4.1.3 Line in R^2 has empty interior

Definition 4.1: Let A be a set in $\mathbf{R}^2$ and let $x \in A$; x is then called an interior element of A if there exists an small open disc D(x, r) in $\mathbf{R}^2$, such that $D(x, r) \subset A$.

In Figure 4.3(a), point P_1 is an interior element of A, while points P_2 and P_3 are not interior elements of A.

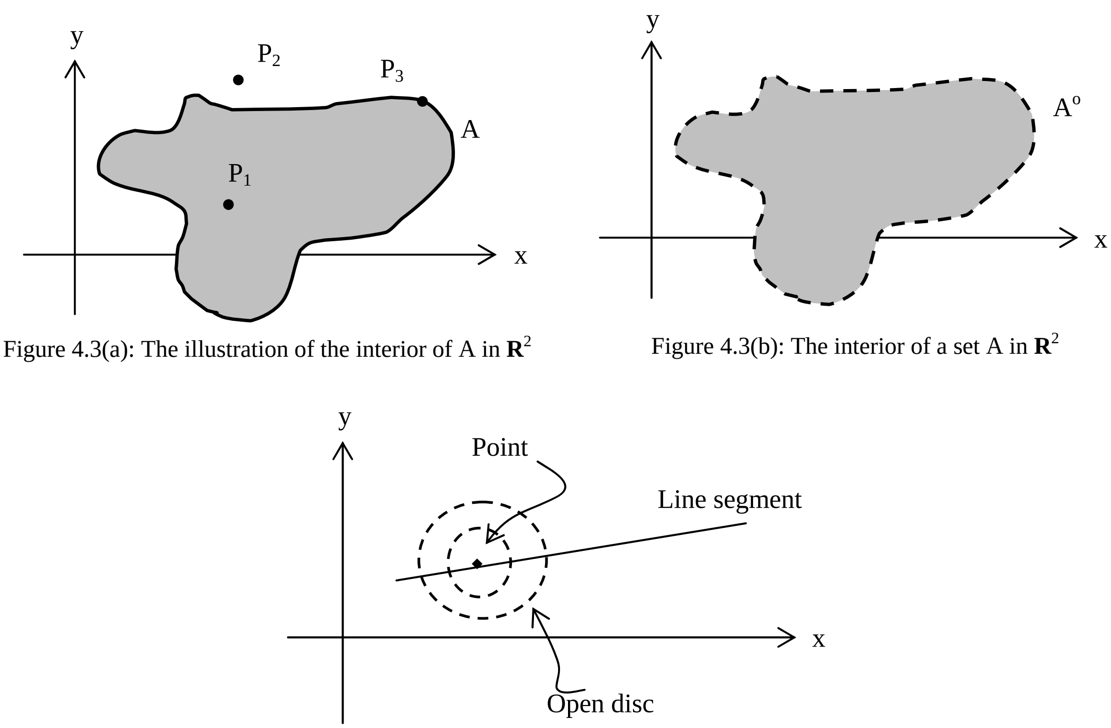


Figure 4.3(a): The illustration of the interior of A in $\mathbf{R}^2$

Figure 4.3(b): The interior of a set A in $\mathbf{R}^2$

Figure 4.3(c): The interior of line in $\mathbf{R}^2$ is empty

Definition 4.2: The interior of a set A in $\mathbf{R}^2$ is defined by the collection of all interior elements in A, denoted by A^o. As in Figure 4.3(b), the interior of A is just the region of A excluded the boundary.

A line segment in $\mathbf{R}^2$ has an empty interior (Michael, 1995). Indeed, we can pick up an arbitrary point in that line segment, for whatever a small open disc with this point as the center, must contain some points not within this line segment. Thus, by the definition of interior, a line segment in $\mathbf{R}^2$ has an empty interior. One may see this in Figure 4.3(c). As the interior of a line segment in $\mathbf{R}^2$ is empty, it might be inappropriate to say that the intersection between the interior of a line segment and a region is non-empty, as we did in the case of the nine-intersection model. That is the matrix $\begin{pmatrix} A^{o} \cap L^{o} & \partial A \cap L^{o} & A^{c} \cap L^{o} \\ A^{o} \cap \partial L & \partial A \cap \partial L & A^{c} \cap \partial L \\ A^{o} \cap L^{c} & \partial A \cap L^{c} & A^{c} \cap L^{c} \end{pmatrix}$ should have all zero in the first row, where L represents a line segment in $\mathbf{R}^2$.

4.2 The extended model for topological relations between convex objects

After analyses the 9-intersection models, here we try to propose a model for the topological relations between convex objects which actually an extension of the present models for GIS.

4.2.1 The subspace topology

Let X be a topological space and $A \subset X$; this subset A then inherits a topology from the space X in a natural way.

Definition 4.3: Let X be a topological space and $A \subset X$; the relative topology or the subspace topology on A is then the collection of intersections of A itself with open sets of X. With this topology, A is called a subspace of X.

4.2.2 The definition of topological components of A and B

Nine-intersection models only mentioned empty and non-empty regions. In fact, the topological components of two regions possess many other homeomorphic invariance; for example, the compactness, connectivity, first fundamental group, etc. Therefore, the scope of the topological relations between two regions can be extended to these invariants, connectivity, compactness and first fundamental group, instead of to the empty and non-empty properties (invariants) only.

Definition 4.4: Let $A, B \subset \Re^{n}$, with the topology induced by the usual Euclidean metric. Then, the set of all finite compositions of the interior operator, closure operator and complementary operator, where empty composition, as a finite one, is defined as the identity operator, are called **topological parts** of A and B.

All of the **topological parts** of A and B are finite compositions of the interior operator and closure operator, and are complementary. Thus, the **disjoint topological parts** $A^o \cap B^o$, $A^o \cap \partial B$, $\partial A \cap B^o$, $\partial A \cap \partial B$, $A^o \setminus B$, $B^o \setminus A$, $\partial A \setminus B$, $\partial B \setminus A$ and $A^c \cap B^c$, plus $A \cup B$, can generate all of the other **topological parts**.

Definition 4.5: Let $A, B \subset R^n$, with the topology induced by the usual Euclidean metric. The components of the topological part are then called the **topological components**.

In Figure 4.4, the topological part of $\partial A \cap \partial B$ contains two components: **component 1 and component 2**.

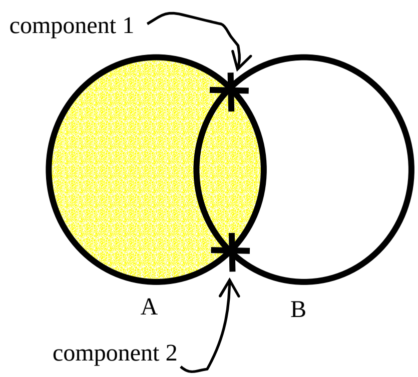


Figure 4.4. The topological part of $\partial A \cap \partial B$ contains two components

Lemma 4.6: Let $A \subset X$ and $f: X \to Y$ be a homeomorphism; then, f preserves the intersection, union, interior, exterior, closure and boundary. These are stated as:

(i) $f(A \cap B) = f(A) \cap f(B)$,

(ii) $f(A \cup B) = f(A) \cup f(B)$,

(iii) $f(A^o) = f(A)^o$,

(iv) $f(A^c) = f(A)^c$,

(v) $f(\overline{A}) = \overline{f(A)}$, and

(vi) $f(\partial A) = \partial f(A)$.

Proof: (i) and (ii) are the general facts for bijective mappings.

(iii): Since $A^o \subset A$, $\Rightarrow$ $f(A^o) \subset f(A)$ $\Rightarrow$ $f(A^o) \subset f(A)^o$. On the other hand, since $f(A)^o \subset f(A)$ $\Rightarrow$ $f^{-1}(f(A)^o) \subset A$ $\Rightarrow$ $f^{-1}(f(A)^o) \subset A^o$ $\Rightarrow$ $f(A)^o \subset f(A^o)$.

(iv): $y \in f(A)^c \Rightarrow y \notin f(A) \Rightarrow \exists\ x \notin A$, such that f(x) = y, and the former is in $f(A^c)$. Hence, $y \in f(A^c)$. Thus, $f(A)^c \subset f(A^c)$.
On the other hand, since f is a homeomorphism, we can replace the role of A by f(A) and f by f^{-1}. We can get $f^{-1}(f(A)^c) \supset f^{-1}(f(A))^c = A^c \Rightarrow f(A)^c \supset f(A^c)$.

(v): $A \subset \overline{A}$ $\Rightarrow$ $f(A) \subset f(\overline{A})$ $\Rightarrow$ $\overline{f(A)} \subset \overline{f(\overline{A})}$ $\Rightarrow$ $\overline{f(A)} \subset f(\overline{A})$. On the other hand, since $\overline{f(\overline{A})}$ is the smallest closed set, it contains closed set $f(\overline{A})$. Thus, $\overline{f(\overline{A})} \supset f(\overline{A}) \Rightarrow \overline{f(A)} \supset \overline{f(\overline{A})} \supset f(\overline{A}) \Rightarrow \overline{f(A)} \supset f(\overline{A})$.

(vi): $f(\partial A) = f\left(\overline{A^c} \cap \overline{A}\right) = f\left(\overline{A^c}\right) \cap f(\overline{A}) = \overline{f(A^c)} \cap \overline{f(A)} = \overline{f(A)^c} \cap \overline{f(A)} = \partial f(A)$.
The first equality is by the definition, the second is the fact of general bijective function, the third is by (iii), the forth is by (ii) and the last is by definition again.

Q.E.D.

Theorem 4.7: Let $A, B \subset X$ and $f : X \to Y$ be a homeomorphism; the following are then true.

(i) $f(A^o \cap B^o) = f(A^o) \cap f(B^o) = f(A)^o \cap f(B)^o$,

(ii) $f(A^o \cap \partial B) = f(A)^o \cap \partial f(B)$ and $f(\partial A \cap B^o) = \partial f(A) \cap f(B)^o$,

(iii) $f(\partial A \cap \partial B) = \partial f(A) \cap \partial f(B)$,

(iv) $f(A^o \setminus B) = f(A)^o \setminus f(B)$ and $f(B^o \setminus A) = f(B)^o \setminus f(A)$,

(v) $f(\partial A \setminus B) = \partial f(A) \setminus f(B)$ and $f(\partial B \setminus A) = \partial f(B) \setminus f(A)$,

(vi) $f\left(A^c \cap B^c\right) = f(A)^c \cap f(B)^c$,

(vii) $f(A \cup B) = f(A) \cup f(B)$.

Proof: The above theorem is easily deduced from Lemma 4.6.

Q.E.D.

Remark 1: The above theorem tells us that the topological parts of $A^o \cap B^o$, $A^o \cap \partial B$, $\partial A \cap B^o$, $\partial A \cap \partial B$, $A^o \setminus B$, $B^o \setminus A$, $\partial A \setminus B$, $\partial B \setminus A$, $A^c \cap B^c$ and $A \cup B$ are invariant under homeomorphism.

Definition 4.8: Let $A, B \subset R^n$, with the topology induced by the usual Euclidean metric. The topological components with homeomorphic invariants are then called the **homeomorphic invariant topological components**. $A^o \cap B^o$, $A^o \cap \partial B$, $\partial A \cap B^o$, $\partial A \cap \partial B$, $A^o \setminus B$, $B^o \setminus A$, $\partial A \setminus B$, $\partial B \setminus A$, $A^c \cap B^c$ and $A \cup B$ are **homeomorphic invariant topological components**.

4.2.3 The definition of the topological relations between two sets

Definition 4.9: The topological relations between the two sets A and B is the topological properties of all of the homeomorphic invariant topological components (or topological components for short) of A and B.

The topological properties of each topological component are described by a sequence of numbers. This sequence of numbers is a series of topological properties; i.e., (Subspace topology, number of components, compactness, the first fundamental group, etc.).

1. **Subspace topology:** -1 means empty, 0 means just a single point, 1 means the usual topology of R, 2 means the usual topology of R^2, etc.
2. **Number of components:** 0 means no intersection, 1 means one component, 2 means 2 components, etc.
3. **Compactness:** 0 means not a compact component, 1 means a compact component.

4. **The first fundamental group:** 0 means the trivial group, 1 means Z, 2 means $Z \times Z$, etc.

Here, we try to explain the meaning of these numbers. In Figure 4.5, if both A and B have non-empty interior sets in R^2 and connected with no holes, and contain one component only, then the topological properties of $A^o \cap B^o$ in the aspects of "Topology, number of components, compactness and the first fundamental group" is $(2, 1, 0, 0)$. This means that the topology of $A^o \cap B^o$ is R^2, the number of components is 1, $A^o \cap B^o$ is not compact and the first fundamental group is the trivial group.

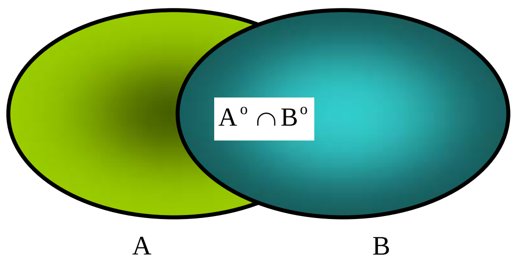


Figure 4.5: There is only one component in $A^o \cap B^o$

Remark 2: The topologies of A and B may not be same type of features as in GIS. For example, if A is a line segment and B is a non-empty region, then the topology of A means an open interval in **R**, denoted by 1. The topology of B means an open disc in $\mathbf{R}^2$, denoted by 2.

4.2.4 Assumption

In the application of GIS, we need to make several assumptions either about the phenomena of the real world or about the limited nature of the theorem.

(i) In the real world, all spatial objects are closed and bounded. Thus, all objects are assumed to be bounded and closed.

(ii) We also assume that non-empty interior regions are regular closed; i.e., $A = \overline{A^o}$.

(iii) The spatial objects are assumed to be connected.

(iv) The spatial objects do not contain any holes.

(v) The non-empty interior regions will be assumed to be convex.

Figure 4.6(a) is the case of convex to convex, while Figure 4.6(b) is the case of convex to non-convex.

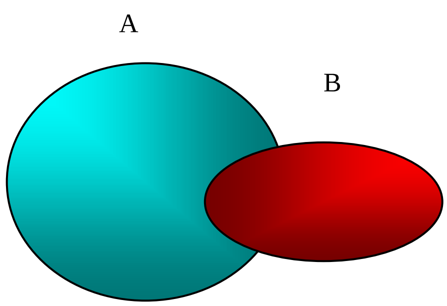


Figure 4.6 (a): Convex to convex case

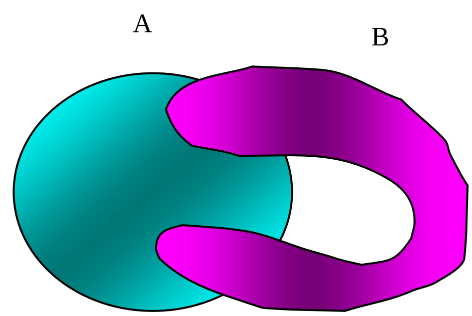


Figure 4.6 (b): Convex to non-convex case

(vi) In the case of topological relations between convex to convex, we will consider the disjoint topological parts $A^o \cap B^o$, $A^o \cap \partial B$, $\partial A \cap B^o$, $\partial A \cap \partial B$, $A^o \setminus B$, $B^o \setminus A$, $\partial A \setminus B$, $\partial B \setminus A$, $A^c \cap B^c$ and $A \cup B$ only. Indeed, the other topological parts are composed of these parts.

(vii) All topological components in same topological part will be assumed to have the same subspace topology.

(viii) The order of topological components will not be considered. As there are only finite entries to represent the topological relations, it is hard to represent the order.

4.2.5 The useful topological parts

In this section, we first investigate the topological relations between two convex non-empty interior sets in $\mathbf{R}^2$. We will assume that the sets A and B are connected convex non-empty interior sets in $\mathbf{R}^2$. We try to discover useful topological parts via "Topology, number of components, compactness, the first fundamental group," by using their topological properties.

(i) All topological components of A and B, $A^o \cap B^o$, $A^o \cap \partial B$, $\partial A \cap B^o$, $\partial A \cap \partial B$, $A^o \setminus B$, $B^o \setminus A$, $\partial A \setminus B$, $\partial B \setminus A$, $A^c \cap B^c$ and $A \cup B$, belong to the trivial group. This means that, in the case of convex to convex, we do not need to consider the factor of the first fundamental group.

Proof: $A^o \cap B^o$, $A^o \cap \partial B$, $\partial A \cap B^o$, $\partial A \cap \partial B$, $A^o \setminus B$, $B^o \setminus A$, $\partial A \setminus B$, $\partial B \setminus A$, $A^c \cap B^c$ and $A \cup B$ do not have holes. Therefore, the topological components of A and B are part of the trivial group.

Q.E.D.

(ii) If the topologies of A and B are given (point or line or region), then the compactness of the topological components of $A^o \cap B^o$, $A^o \cap \partial B$, $\partial A \cap B^o$, $\partial A \cap \partial B$, $A^o \setminus B$, $B^o \setminus A$, $\partial A \setminus B$, $\partial B \setminus A$, $A^c \cap B^c$ and $A \cup B$ have been determined. For example, if A and B are both non-empty interior connected convex sets in R^2, $A^o \setminus B$ must be open; hence, it is not compact. This fact tells us that if we consider the cases of region to region and region to line separately, we do not need to consider the compactness of each topological component.

(iii) As $\left(A^o \cap B^o\right) \subset A^o$, B^o is open and the topology of $A^o \cap B^o$ is either zero or the minimum number of the topology of A^o and B^o. This fact tells us that if we consider the cases of region to region and region to line separately, we do not need to consider the topology of each topological component.

(iv) All of the numbers of the topological components $A^c \cap B^c$ and $A \cup B$ must be one. This means that in the case of convex to convex, the components $A^c \cap B^c$ and $A \cup B$ are useless.

Proof: $A \cup B$ has only one component, is connected and does not contain holes. By the Jordan curve theorem, the boundary of $A \cup B$ divides the $\mathbf{R}^2$ into two parts: one is $(A \cup B)^o$ which is open and bounded; the other is $A^c \cap B^c$ which is open and unbounded and has only one component.

Q.E.D.

(v) In $\mathbf{R}^2$, the topology of $\partial A \cap \partial B$ is either empty, zero or one.

Proof: The dimension of $\partial A \cap \partial B$ is either empty, zero or one, as is the topology.

Q.E.D.

(vi) The number of components of $\partial A \setminus B^o$ and $\partial B \cap A^o$ depend on the number of components of $A^o \setminus B$, while $\partial B \setminus A^o$ and $\partial A \cap B^o$ depend on $B^o \setminus A$. In other words, $|A^o \setminus B| = |\partial A \setminus B^o| = |\partial B \cap A^o|$ and $|B^o \setminus A| = |\partial B \setminus A^o| = |\partial A \cap B^o|$.

This tells us that among the topological parts $A^o \cap \partial B$, $\partial A \cap B^o$, $A^o \setminus B$, $B^o \setminus A$, $\partial A \setminus B$, $\partial B \setminus A$, the useful topological parts are only $A \setminus B^o$ and $B \setminus A^o$.

Proof: Since the boundary of each component of $A^o \setminus B$ contains two components, one is $\partial A \setminus B^o$, the other is $\partial B \cap A^o$.

Q.E.D.

To investigate the case of convex region to convex region, the above facts tell us that the only four useful topological components are $A^o \cap B^o$, $A^o \setminus B$, $B^o \setminus A$ and $\partial A \cap \partial B$. We will consider these four and create a new four-intersection model; i.e., $\begin{pmatrix} A^o \cap B^o & A^o \setminus B \\ B^o \setminus A & \partial A \cap \partial B \end{pmatrix}$, which by (i) to (vi), we only need to consider the connectivity of each entry.

Remark 3: In general, the subspace topology of A and B may not be same. Thus, the notation A^o and B^o may not be same topology. For example, if A is a non-empty region, its topology is the collection of all open discs in $\mathbf{R}^2$. If B is just a line segment, its topology is the collection of all open intervals in R.

4.2.6 The number of components of useful topological parts

The topological parts $A^o \cap B^o$, $A^o \setminus B$, $B^o \setminus A$ and $\partial A \cap \partial B$ are used in the case of convex region to convex region. The following are several useful facts to investigate the topological relations between two non-empty convex regions.

(v) The number of components of $A^o \cap B^o$ is at most 1.

Proof: Suppose $A^o \cap B^o$ is non-empty. Since both A and B are convex, then $A^o \cap B^o$ also convex and any two points on $A^o \cap B^o$ can be joined by a line segment such that the whole segment lie on $A^o \cap B^o$. This proves that $A^o \cap B^o$ is path-connected and hence is connected.

Q.E.D.

(vi) If A and B are both non-empty interiors, then the number of components of $(A \cup B) \setminus (A \cap B)$ depends on the number of components of $\partial A \cap \partial B$. In other words, if $|\partial A \cap \partial B| \geq 2$ and $|A^o \setminus B| < \infty$, $|B^o \setminus A| < \infty$ and $|\partial A \cap \partial B| < \infty$, then $|A^o \setminus B| + |B^o \setminus A| = |\partial A \cap \partial B|$.

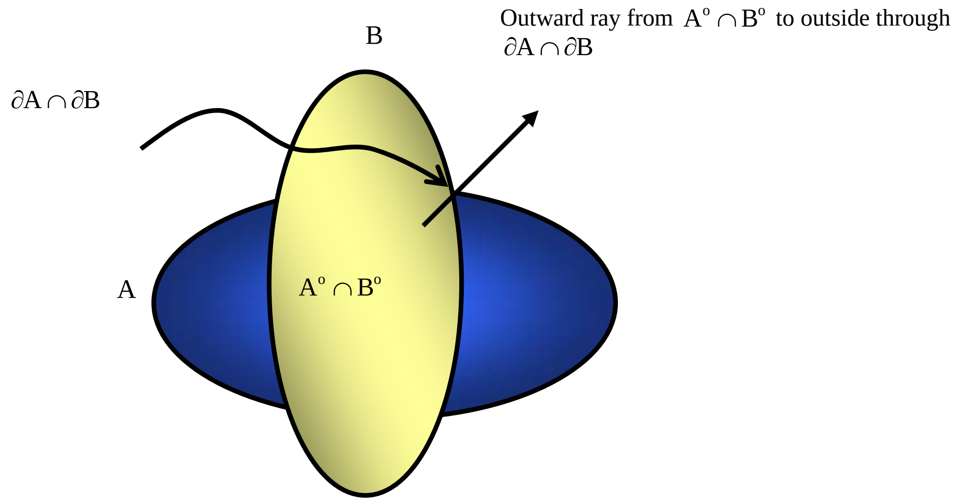


Figure 4.7: Illustrate each component of $\partial A \cap \partial B$ corresponds one left component of $(A \cup B) \setminus (A \cap B)$

Proof: As the fact that $(\partial A \cap B^o) \cup (\partial B \cap A^o) = (A \cup B) \setminus (A \cap B)$, and $\partial A \cap B^o$, $\partial B \cap A^o$ are disjointed open sets, hence $|(\partial A \cap B^o)| + |(\partial B \cap A^o)| = |(A \cup B) \setminus (A \cap B)|$. This means that we need to prove $|\partial A \cap \partial B| = |A \cup B \setminus A \cap B|$.

If $|\partial A \cap \partial B| \geq 2$, each component of $\partial A \cap \partial B$ corresponds to two components of $(A \cup B) \setminus (A \cap B)$. One is on the left-hand side of the outward ray from $A^o \cap B^o$ to the outside through $\partial A \cap \partial B$, and the other is on the right of the outward ray from $A^o \cap B^o$ to outside through $\partial A \cap \partial B$. Therefore, each component of $\partial A \cap \partial B$ corresponds to one left component of $(A \cup B) \setminus (A \cap B)$. This proves that $|\partial A \cap \partial B| \leq |(A \cup B) \setminus (A \cap B)|$.

On the other hand, each component of $(A \cup B) \setminus (A \cap B)$ corresponds to two components of $\partial A \cap \partial B$ (see Figure 4.7). One is on the left and the other is on the right. So, each component of $A \cup B \setminus A \cap B$ corresponds to one right component of $\partial A \cap \partial B$. This proves that $|\partial A \cap \partial B| \geq |(A \cup B) \setminus (A \cap B)|$.

Q.E.D.

(vii) The number of components of $A^o \setminus B$, $B^o \setminus A$ and $\partial A \cap \partial B$ can be infinity.

Proof: $S = \left\{ \begin{pmatrix} 1 & 0 \\ n & n \end{pmatrix} \ : \ n \geq 3 \right\}$ is a sequence of topological relations.

Q.E.D.

(viii) Based on the results in sections 4.25 and 4.26, we have the following sequence of topological relations, which are all the topological relations between two convex sets can be represented by the matrix $\begin{pmatrix} A^o \cap B^o & A^o \setminus B \\ B^o \setminus A & \partial A \cap \partial B \end{pmatrix}$ (see Table 4.1).

Table 4.1: Topological relations between two convex sets are represented by the matrix $\begin{pmatrix} A^o \cap B^o & A^o \setminus B \\ B^o \setminus A & \partial A \cap \partial B \end{pmatrix}$.

	$H = \left\{ \begin{pmatrix} 1 & m \\ n & n+m \end{pmatrix} : n, m \in Z^+ \right\}$ **Boundary Crossing**	
$L = \left\{ \begin{pmatrix} 1 & m \\ 0 & m \end{pmatrix} : m \in Z^+ \right\}$ **Internal Tangent**		$R = \left\{ \begin{pmatrix} 1 & 0 \\ n & n \end{pmatrix} : n \in Z^+ \right\}$ **Internal Tangent**
	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ **Disjoint**	
	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ **External Tangent**	
$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ **Inside**	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **Equal**	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ **Contain**
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ **Cover and Tangent at one**		$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ **CoveredBy and Tangent at one**
$\begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$ **Cover and Tangent at two**		$\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ **CoveredBy and Tangent at two**

4.3 Several special topological relations

Geometrically, GIS features can be classified as points, lines and polygons. Thus, the topological relations between point-to-point, point-to-line, point-to-polygon, line-to-line, line-to-polygon and polygon-to-polygon should be described completely. A detailed description is given below.

4.3.1 The topological relations between a line segment to a convex region

In this sub-section, the regions are assumed to be simply connected, do not contain any holes and to be convex. As we mentioned previously, the interior of a line segment is empty. Based on the Egenhofer's nine-intersection model, many of the topological relations between the line segment and the region cannot be identified. Thus, we should consider another model. Since the topology of a line segment is in **R**, we try to define a line segment in the following way. Let P and Q be the end points of a line segment, and define a map $\alpha : [0,1] \rightarrow \Re^2$ by $\alpha(t) = P + t(Q - P)$, where [0, 1] is a closed interval in R and $\alpha(t_1) \neq \alpha(t_2)$ for all $t_1, t_2 \in [0,1]$.

Now, we define $\alpha^o = \alpha((0,1))$, $\partial\alpha = \alpha(0) \cup \alpha(1)$ and $\alpha^c = \Re^2 - \alpha([0,1])$, where (0, 1) and [0, 1] are open and closed intervals in **R**. On the other hand, we decompose any region A into three parts, interior, boundary and exterior; denoted by A^o, ∂A and A^c, respectively. Hence, we also define the topological relations between a line segment and a convex non-empty interior region as follows:

$$\begin{pmatrix} A^o \cap \alpha^o & A^o \setminus \alpha \\ \alpha^o \setminus A & \partial A \cap \partial\alpha \end{pmatrix}$$

(Topological relations between a line and a region)

Based on this model, there are a total of nine topological relations between a line segment and a convex non-empty interior region. Indeed,

1. the number of components of $A^o \cap \alpha^o$ is either zero or one.
2. the number of components of $A^o \setminus \alpha$, $\alpha^o \setminus A$ and $\partial A \cap \partial\alpha$ is at most two.

3. $A^o \setminus \alpha$ and $\alpha^o \setminus A$ cannot be both 2.
4. $A^o \setminus \alpha$ will not be zero.

We will use this fact to determine the total number of topological relations between a line segment and a convex region.

i. If $|A^o \cap \alpha^o| = 0$, then $|A^o \setminus \alpha| = 1$, $\alpha^o \setminus A \neq \phi$ and $|\partial A \cap \partial\alpha| \leq 1$.

ii. If $|A^o \cap \alpha^o| = 1$, then

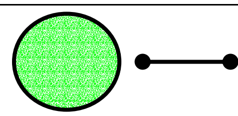
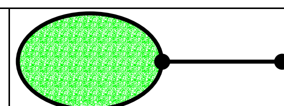
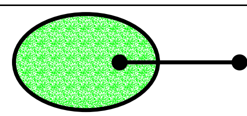
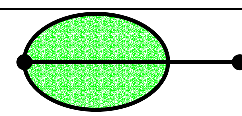
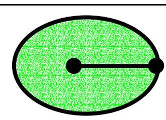
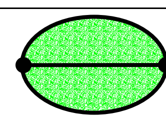
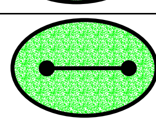
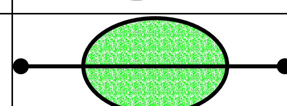
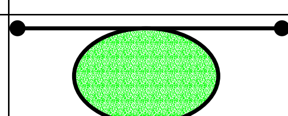
  I. If $|\partial A \cap \partial\alpha| = 2$, then $|A^o \setminus \alpha| = 2$, $|\alpha^o \setminus A| = 1$.

  II. If $|\partial A \cap \partial\alpha| = 1$, then either $|A^o \setminus \alpha| = 2$ and $|\alpha^o \setminus A| = 1$ or $|A^o \setminus \alpha| = 1$ and $\alpha^o \setminus A = \phi$.

  III. If $|\partial A \cap \partial\alpha| = 0$, then either $|A^o \setminus \alpha| = 1$ and $|\alpha^o \setminus A| = 1$ or $|A^o \setminus \alpha| = 1$ and $\alpha^o \setminus A = \phi$ or $|A^o \setminus \alpha| = 2$ and $|\alpha^o \setminus A| = 2$.

The above fact can be illustrated by Table 4.2:

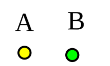
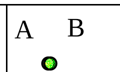
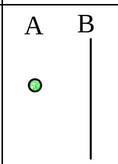
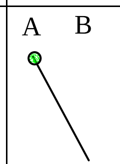
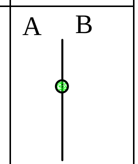
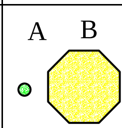
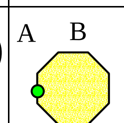
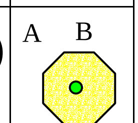
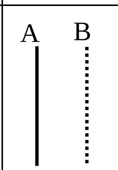
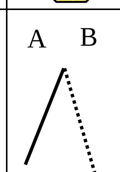
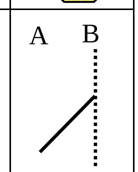
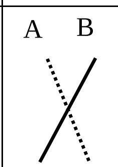
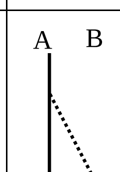
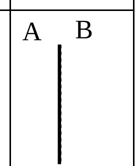
Table 4.2: The total number of topological relations between a line segment to a convex region

1. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$		2. $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	
3. $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$		4. $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$	
5. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$		6. $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$	
7. $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$		8. $\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$	
9. $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$			

4.3.2 The topological relations between point, line and polygon

The cases of the line-to-polygon and polygon-to-polygon are very complicated and are described above. Therefore, we will just give the topological relations of the cases point-to-point, point-to-line, point-to-polygon and line-to-line in this section. The definition of a point in $\mathbf{R}^2$ is just a coordinate in $\mathbf{R}^2$. The definition of a line segment has been defined in section 4.3.1, and here we will adopt this definition. Based on this definition, we have the following results (see Table 4.3):

Table 4.3: The topological relations between point, line and convex polygon

point-to-point	1. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$		2. $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$			
point-to-line	1. $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$		2. $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$		3. $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$	
point-to-polygon	1. $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus (0)$		2. $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \oplus (0)$		3. $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus (1)$	
line-to-line	1. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$		2. $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$		3. $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$	
	4. $\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$		5. $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$		5. $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$	

4.4 Modeling the case of non-convex regions

To investigate the case of non-convex regions, we will base our examination on the properties of the convex case. Based on the only four useful topological parts $A^o \cap B^o$, $A^o \setminus B$, $B^o \setminus A$ and $\partial A \cap \partial B$, the factor of the fundamental group of $A \cup B$ will be considered. It is called the four-intersection- π_1 model; i.e.,

$\begin{pmatrix} A^o \cap B^o & A^o \setminus B \\ B^o \setminus A & \partial A \cap \partial B \end{pmatrix} \oplus \pi_1(A \cup B)$. In this section, we try to discuss some properties of the four-intersection-π_1 model. In Figure 4.8, the topological relations of A and B can be represented by $\begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix} \oplus (1)$.

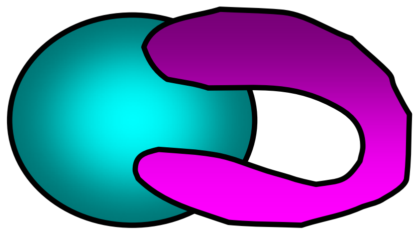


Figure 4.8: The first fundamental group of $A \cup B$ is Z and represented by 1

The following are several properties of the non-convex case that can be represented by this four-intersection-π_1 model.

(i) If the representative number of $\pi_1 = 0$; i.e., it is a trivial group, then this will reduce the case convex to convex.

Proof: If $\pi_1 = 0$, then $A \cup B$ does not contain any holes and we apply similar arguments in section 4.2 that we can obtain the desired results.

Q.E.D.

(ii) If both A and B are non-empty interior sets in $\mathbf{R}^2$, then there are two kinds of topological components of $A^c \cap B^c$. One and only one component is unbounded, denoted by $U_{A^c \cap B^c}$. The others are bounded, denoted by $B_{A^c \cap B^c}$.

(iii) The boundary of each bounded component of $A^c \cap B^c$ consists of two parts, $\partial A \setminus B$ and $\partial B \setminus A$. Moreover, $|\partial A \setminus B| = |\partial B \setminus A|$ and $|\partial A \cap \partial B| = 2 \times |\partial B \setminus A|$.

Proof: Since each bounded component of $A^c \cap B^c$ is bounded by two topological parts, one is $\partial A \setminus B$ and one is $\partial B \setminus A$. In addition, each component of each part corresponds to two components of $\partial A \cap \partial B$.

Q.E.D.

(iv) In R^2, $\left|B_{A^c \cap B^c}\right|$ = the representative number of π_1. This tells us that we don't need to consider the factor of $A^c \cap B^c$.

Proof: By (ii) we can see that each component of the bounded component of $A^c \cap B^c$ is a hole of $A \cup B$. Each hole distributes one of the representative numbers of π_1.

Q.E.D.

(v) Let $\pi_1 = k > 0$; then $|\partial A \cap \partial B| \geq 2k$. This is the corollary of (iii) and (iv).

(vi) Let $\{B_1, B_2, B_3, ..., B_n\}$ be the set of all bounded components of $A^c \cap B^c$. Let p_i be the number of components of $\partial A \cap \partial B$. If $|\partial A \cap \partial B| \geq 2$ and $|A^o \setminus B| < \infty$, $|B^o \setminus A| < \infty$, $|\partial A \cap \partial B| < \infty$ and $\left|B_{A^c \cap B^c}\right| < \infty$, then

$$|A \setminus B^o| + |B \setminus A^o| = |\partial A \cap \partial B| - s,$$

where $\left|B_{A^c \cap B^c}\right| < s \leq \max\{p_i\} \times \left|B_{A^c \cap B^c}\right|$ (see Figure 4.9).

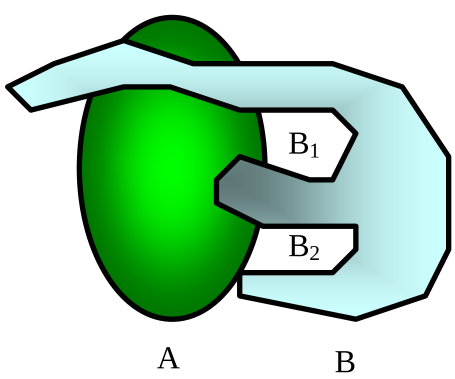


Figure 4.9: Shows that $|A^o \setminus B| + |B^o \setminus A| = 4$, $|\partial A \cap \partial B| = 7$, bounded components of $A^c \cap B^c = 2$, $\max\{p_i\} = 2$ and $s = 3$.

Proof: Let $\{B_1, B_2, B_3, ..., B_n\}$ be the set of all bounded components of $A^c \cap B^c$. Let $C = \bigcup_{i=1}^{n} B_i$, then the topological relations between $A \cup C$ and $B \cup C$ reduces to the case of convex regions. Indeed, the first fundamental group of $(A \cup C) \cup (B \cup C)$ is a trivial group.

Hence, we have

$$\left|(A \cup C) \setminus (B \cup C)^o\right| + \left|(B \cup C) \setminus (A \cup C)^o\right| = \left|\partial(A \cup C) \cap \partial(B \cup C)\right|.$$

But

$$\left|(A \cup C) \setminus (B \cup C)^o\right| = \left|A \setminus B^o\right|,$$

$$\left|(B \cup C) \setminus (A \cup C)^o\right| = \left|B \setminus A^o\right|$$

and $\quad \left|\partial(A \cup C) \cap \partial(B \cup C)\right| \geq \left|\partial A \cap \partial B\right| - \max\{p_i\} \times \text{Bounded components of } A^c \cap B^c$

and

$$\left|\partial(A \cup C) \cap \partial(B \cup C)\right| < \left|\partial A \cap \partial B\right| - \text{Bounded components of } A^c \cap B^c.$$

Q.E.D.

(vii) For the four-intersection-π_1 model, $\begin{pmatrix} A^o \cap B^o & A^o \setminus B \\ B^o \setminus A & \partial A \cap \partial B \end{pmatrix} \oplus \pi_1(A \cup B)$, we can obtain the topological relations between two non-empty interior regions, which can be represented by the following set of matrices (see Table 4.4):

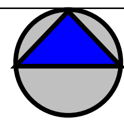
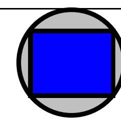
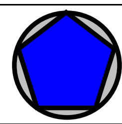
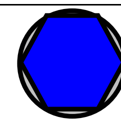
Table 4.4: Topological relations between two non-empty interior regions are represented by $\begin{pmatrix} A^o \cap B^o & A^o \setminus B \\ B^o \setminus A & \partial A \cap \partial B \end{pmatrix} \oplus \pi_1(A \cup B)$.

<table>
<tr><td colspan="3">$H = \left\{ \begin{pmatrix} p & n \\ m & n+m+s \end{pmatrix} \oplus (k) : m, n, p, k \in Z^+ \text{ and } k \le s \le \max\{p_i\} \times k \right\}$
Boundary Crossing</td></tr>
<tr><td colspan="2">$L = \left\{ \begin{pmatrix} 1 & m \\ 0 & m \end{pmatrix} : m \in Z^+ \right\}$
Internal Tangent</td><td>$R = \left\{ \begin{pmatrix} 1 & 0 \\ n & n \end{pmatrix} : n \in Z^+ \right\}$
Internal Tangent</td></tr>
<tr><td></td><td>$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
Disjoint</td><td></td></tr>
<tr><td></td><td>$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
External Tangent</td><td></td></tr>
<tr><td>$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$
Inside</td><td>$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
Equal</td><td>$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$
Contain</td></tr>
<tr><td>$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$
Cover and Tangent at one</td><td></td><td>$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
CoveredBy and Tangent at one</td></tr>
<tr><td>$\begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$
Cover and Tangent at two</td><td></td><td>$\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$
CoveredBy and Tangent at two</td></tr>
</table>

4.5 The significance of the new model

We have pointed out that the topological relation between two sets is not finite. In fact, it is infinite and can be approximated by a sequence of matrices. Here, we want to construct a sequence of topological relations. The following is a sequence of different topological relations between two convex non-empty regions. We actually try to construct a kind of sequence of topological relations, where the topological relations are equivalent to those between a circle and a regular polygon, with a regular polygon internally tangent to a circle, $S = \left\{ \begin{pmatrix} 1 & 0 \\ n & n \end{pmatrix} \ : \ n \geq 3 \right\}$. Those topological relations can be illustrated by Table 4.5.

Table 4.5: Infinite sequence of topological relations between two convex regions

1. $\begin{pmatrix} 1 & 0 \\ 3 & 3 \end{pmatrix}$		2. $\begin{pmatrix} 1 & 0 \\ 4 & 4 \end{pmatrix}$	
3. $\begin{pmatrix} 1 & 0 \\ 5 & 5 \end{pmatrix}$		4. $\begin{pmatrix} 1 & 0 \\ 6 & 6 \end{pmatrix}$	
: : :	: : :	: : :	: : :

In addition, with this new model, many more relations beyond the topological relations can be distinguished. For example, the diagram in Figure 4.10 represents a piece of land with two kinds of vegetation: grasses and forests. Topological relation between grass and forest can be described by $\begin{pmatrix} 1 & 0 \\ 3 & 3 \end{pmatrix} \oplus (0)$ or just $\begin{pmatrix} 1 & 0 \\ 3 & 3 \end{pmatrix}$.

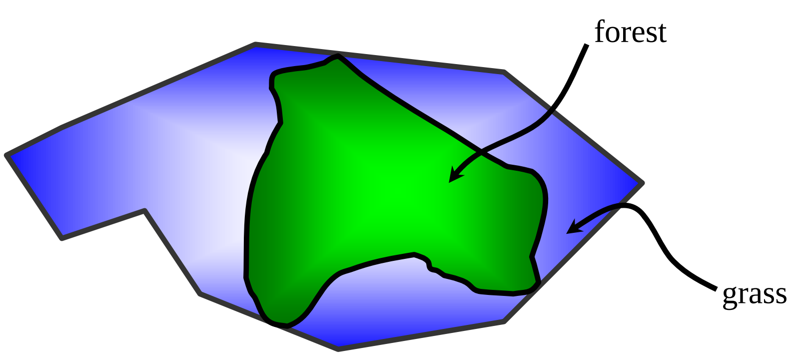


Figure 4.10: Topological relation between grass and forest

4.6 Summary

Thus far, many articles have discussed the topological relations between crisp spatial objects. Based on the four-intersection model and the ordinary point set topology, Egenhofer and Franzosa gave the topological relations between two spatial regions in a two-dimensional (2D) space. Based on the nine-intersection model, Cohn and Gotts (1996) found 46 topological relations between two regions with indeterminate boundaries, while Clementini and Di Felice found 44 topological relations between two regions with indeterminate boundaries. However, there exists a common insufficiency in the existing models. In fact, a line should have an empty interior in 2D space, although it is not in one-dimensional (1D) space.

On the other hand, there are many topological properties and it is not sufficient to simply consider the empty and non-empty invariants. Actually, many other topological properties, such as connectivity, compactness, first fundamental group and subspace topology, can help to distinguish the topological relations in the use of GIS. In this dissertation, we have considered such invariants and give a model of topological relations that actually is an infinite sequence of numbers or matrices.

In this chapter, a framework for describing topological relations between two spatial objects has been presented. This is based on the topological properties of the topological components. By considering these components (or properties), we can obtain a sequence of topological relations, which is infinity. The proposed framework can give a theoretical basis for the design and implementation of a GIS. The proposed solution can represent the topological relations between any two arbitrary objects without holes and connected sets.

The proposed solution has both advantages and disadvantages, and the following is a summary of the analysis on the framework. The pros of the solution include:

- In the case of convex to convex, we need to consider four parts only. They are $A^o \cap B^o$, $A^o \setminus B$, $B^o \setminus A$ and $\partial A \cap \partial B$.

- Many topological relations between set A and set B can be well separated by using this model.
- The abstract concepts of topological properties are represented by numbers only, so it is easy to systemize.
- If the topological properties of set A and set B are also considered, they can distinguish the relations of complex spatial entities such as spatial objects with holes.
- If we consider all topological properties on each topological component, we can distinguish much more topological relations.
- As the definition of topological relations is abstract, we can extend this definition to a higher dimensional space, for example to a three-dimensional space.

On the other hand, there are some disadvantages to this model. For example:

- The case of disjoint cannot be well separated useless the metric is introduced.
- The order of the intrinsic properties of the topological relations cannot be separated due to the assumption made for the model.
- Different topological components in the same topological part may be different subspace topologies. But this model cannot separate these relations.
- In this study, we have coped the topological parts $A^o \cap B^o$, $A^o \cap \partial B$, $\partial A \cap B^o$, $\partial A \cap \partial B$, $A^o \setminus B$, $B^o \setminus A$, $\partial A \setminus B$, $\partial B \setminus A$, $A^c \cap B^c$ and $A \cup B$. In fact, these parts can be further extended by considering other possible cases, and there may be a suitable set of topological parts should be chosen.
- Since the number of topological relations in this chapter is infinite, it still need some work on how to implement of a GIS.

In describing objects in GIS, both sets A and B are assumed to be bounded. If the number of components of $\partial A \cap \partial B$ is large, sets A and B will tend to be approximately equal. This means that the tail of $\left\{ \begin{pmatrix} 1 & 0 \\ n & n \end{pmatrix} : n \in Z^+ \right\}$, $\left\{ \begin{pmatrix} 1 & m \\ 0 & m \end{pmatrix} : m \in Z^+ \right\}$ and

$\left\{ \begin{pmatrix} p & n \\ m & n+m-s \end{pmatrix} \oplus (k) : m, n, p, k \in Z^{+} \text{ and } k \le s \le 2k \right\}$ can be neglected. Therefore, the number of topological relations between two sets can be approximated by the finite.

The new model is purely about topological properties and does not involve any metric properties. Thus, another extension of this work is to introduce metric properties into this proposed model. In particular, the cases of disjoint can be well resolved with such an extension. This study will serve as a basis for describing topological relations between two fuzzy regions.

CHAPTER FIVE
TOPOLOGICAL RELATIONS DEPENDENT ON THE SHAPE OF SPATIAL OBJECTS

GIS are mainly composed of four types of information: spatial, temporal, attribute, and topological relations. Information on topological relations are fundamental in GIS, and is mainly used for (a) spatial queries and analyses; and for (b) data quality checking. Many studies have been devoted to the development of topological relations in GIS.

A topological space is a set X with a collection of subsets of X called “open” sets, such that: (a) the intersection of the two open sets is open; (b) the union of any collection of open sets is open; and (c) the empty set ϕ and whole space X are open. Moreover, a subset C of X is called “closed” if its complement X\C is open.

The properties of topological spaces that are preserved under homeomorphism are called the topological invariants of the spaces. Connectivity, compactness, and the first fundamental group are several fundamental topological invariants. As these invariants are invariant under bi-continuous mappings (homeomorphisms), studying these invariants can help us to understand the topological relations between spatial objects.

From a mathematic point of view, the topological relation is actually an equivalence relation (reflexive, symmetric, and transitive), which is simply the partitioning of the relation between two spatial objects into different partitions.

Based on the properties of topological relations that are invariant under topological transformation, such as translation, scaling, and rotation (Egenhofer, 1989), Egenhofer and Franzosa (1994) gave the following definition. Let A_1, $B_1 \subset X$ and A_2, $B_2 \subset Y$. The topological relation between A_1 and B_1 is then equivalent to the topological relation between A_2 and B_2 if there exists a **homeomorphic map** $f: X \rightarrow Y$ such that $f(A_1) = A_2$ and $f(B_1) = B_2$. When examining this definition carefully, we discover that the topological relations are invariant under a homeomorphic map, but that the spatial objects themselves are not necessary invariant under the homeomorphic map. This means that the shape of the spatial objects themselves will affect the topological relations. In this chapter, we will give a mathematical proof to show that topological relations between spatial objects actually depend on the shapes of the objects themselves.

The logic follows of this research in this chapter is shown in Figure 5.1.

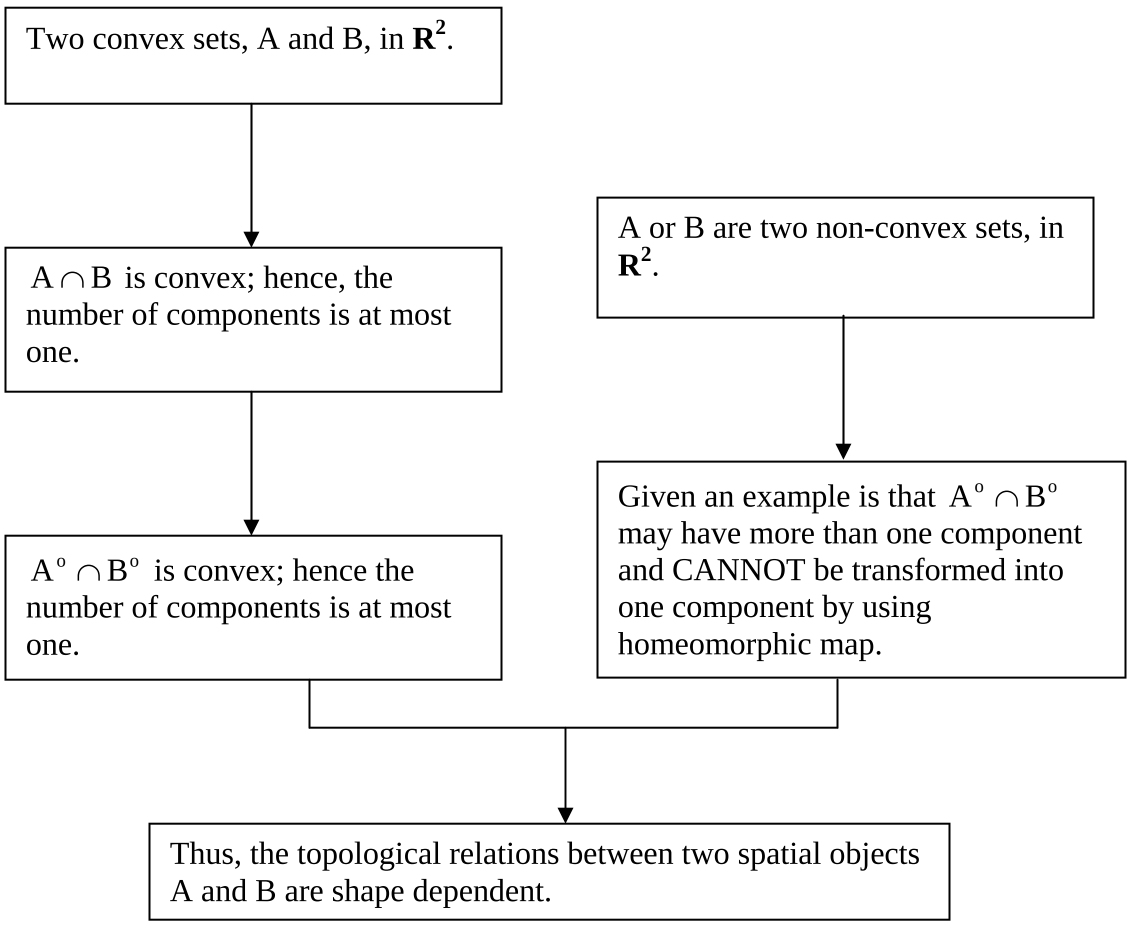


Figure 5.1: The structure of this chapter.

5.1 The number of components of $A^o \cap B^o$

Topology is a tool to study certain geometric problems, which does not depend on the exact shape of the objects, but rather on the way they are connected to each other. In the aspect of topological relations between two objects A and B, the topology surely does not depend on the exact shape of $A \cup B$, but may depend on the shape of A or B individually! In this chapter, we try to explain this fact and point out that the topological relation depends on the shape of the individual spatial objects involved.

In this section, we first illustrate why the topological relations between spatial objects depends on the shape of the spatial objects themselves. We will then provide a mathematical proof to show the basic difference in topological relations between convex sets and non-convex sets. A homeomorphic map $f: X \to Y$ preserves the intersection, interior, exterior, closure, and boundary, as $f(A \cap B) = f(A) \cap f(B)$, $f(A^o) = f(A)^o$, $f(A^c) = f(A)^c$, $f(\overline{A}) = \overline{f(A)}$ and $f(\partial A) = \partial f(A)$. Thus, the number of components of $A^o \cap B^o$, $A^o \cap \partial B$, $\partial A \cap B^o$, $\partial A \cap \partial B$, $A^c \cap B^c$, …, is preserved by homeomorphism. By the definition of topological relations between spatial objects (Egenhofer and Franzosa, 1994), the number of components of $A^o \cap B^o$, $A^o \cap \partial B$, $\partial A \cap B^o$, $\partial A \cap \partial B$, $A^c \cap B^c$, … is an invariant property of a topological relation.

In this section, unless specified, we will assume that all of the convex sets in R^n have a non-empty interior in R^n. The structure of the proof is shown in Figure 5.2.

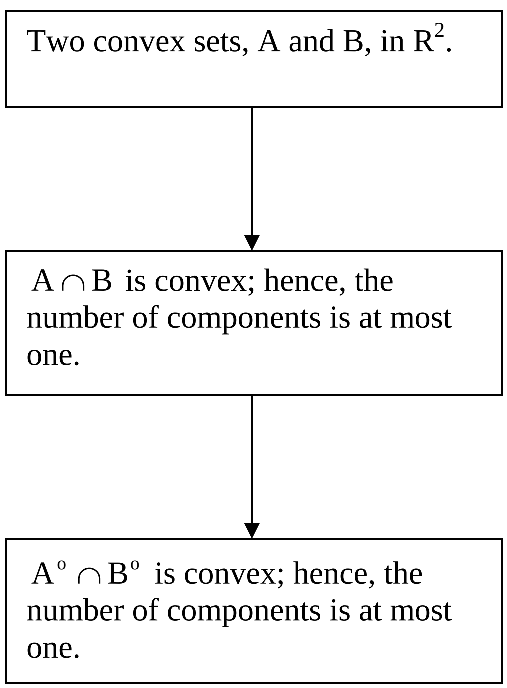


Figure 5.2: The logic follow of the proof in this section

Definition 5.1: A convex non-empty set in $\mathbf{R}^n$ is a set $A \subset \Re^n$ with the property that whenever points $p, q \in A$, the line segment between p and q is contained in A.

Figure 5.3(a) shows a convex set in $\mathbf{R}^2$. As any two points p and q are in A, the line segment by joining these two points are still in this set A. Figure 5.3(b) shows a non-convex set B in $\mathbf{R}^2$. There exist two points at some point in the line segment, created by joining these two points lying outside set B.

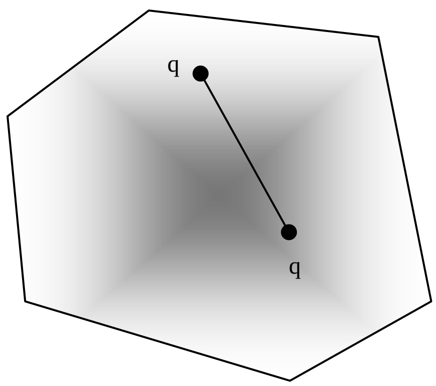


Figure 5.3(a): A convex set A in $\mathbf{R}^2$

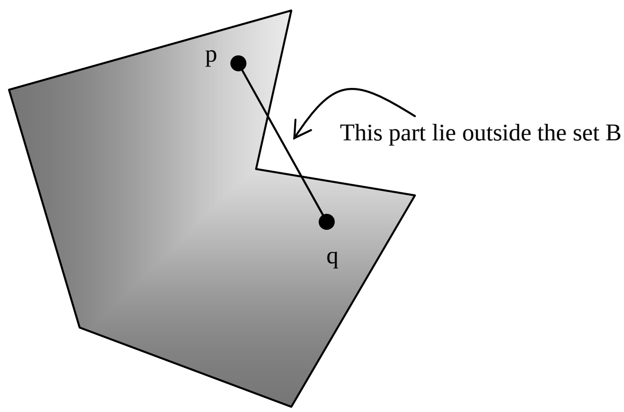


Figure 5.3(b): A non-convex set B in $\mathbf{R}^2$

Proposition 5.2: If $A \subset \Re^n$ is a non-empty interior, a connected convex closed set, then any ray from any one interior point intersects ∂A at a maximum of one point. Figure 5.4(a) shows that a ray from the interior point of A can only have one point intersecting with the boundary of A.

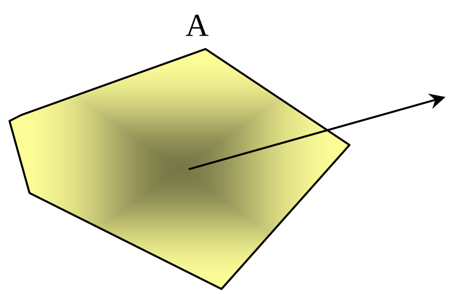


Figure 5.4(a): A ray from the interior of convex set can only have one point intersecting with the boundary of A

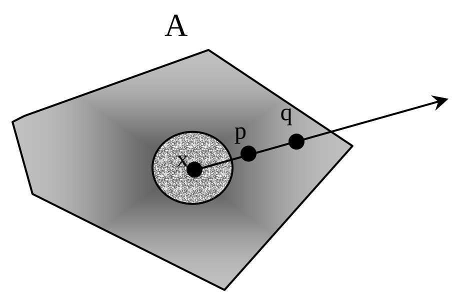


Figure 5.4(b): line segments from points in B_n to q

Proof: Suppose R is a ray from point x and let p, $q \in R \cap A$, with neither $p \neq x$ nor $q \neq x$. Suppose q is further than p from point x. Now we want to prove p is in A^o. Thus, if q is in the boundary of A or if the ray does not meet the boundary of A, since x lies in the interior of A, there exists a small enough n-ball B_n such that $B_n \subset A^o$. Then, let *C* be the union of all line segments from points in B_n to q (see Figure 5.4(b)). Thus, point q is in the interior of *C*, and *C* is contained completely in A, since A is a convex set. Hence, p is in A^o.

Q.E.D.

Proposition 5.3: If $A \subset \Re^n$ is a non-empty interior, connected convex compact set, then $\partial A \cong S^{n-1}$, where $S^{n-1} = \{x \in \Re^n : \| x \| = 1\}$ is called an n-ball and $\| x \|$ is the usual Euclidean norm of x.

Proof: Pick a point p in A^o, then define $f : \partial A \to S^{n-1}$ by $f(x) = \frac{x - p}{\| x - p \|}$, which is a well-defined mapping and is continuous and bijective. Hence, f is a homoeomorphism and $\partial A \cong S^{n-1}$.

Q.E.D.

Remark 1: Any non-empty interior, connected convex compact set is homeomorphic to an n-ball. This means that the topological properties of a non-empty interior convex compact set are the same as those of an n-ball. But the topological relations between two non-empty interior, connected convex compact sets cannot be isomorphic to the topological relations between two n-balls.

Remark 2: Proposition 5.3 gives us a good tool to study the topological relations between two non-empty interior convex compact sets.

Proposition 5.4: Let A be a non-empty interior, connected convex closed set; A^o is then a connected convex set.

Proof: Let there be p, q in A^o; then there exist two open balls, B_1 and B_2, in A^o with centers p and q, respectively. Let *C* be the union of all line segments joining these two balls (see Figure 5.5). Obviously, the line segment pq is in $C^o \subset A$, which implies that $C^o \subset A^o$.

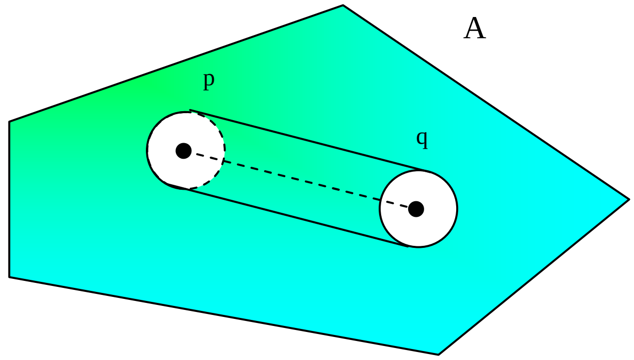


Figure 5.5: The union of all segment joining these two balls

Q.E.D.

Proposition 5.5: Let A and B be two non-empty interior, connected convex compact sets; then, $A \cap B$ is either an empty set or a connected convex set.

Proof: If $A \cap B$ is non-empty, A and B are convex compact. For any points p, q $\in A \cap B$, we have the line segment joining these two points lying on $A \cap B$. Hence, $A \cap B$ is a connected convex compact set.

Q.E.D.

Proposition 5.6: Let A and B be two non-empty interior, connected convex compact sets; then, $A^o \cap B^o$ is either an empty or a non-empty connected convex set. Hence, the number of components of $A^o \cap B^o$ is at most one.

Proof: If $A^o \cap B^o$ is non-empty, A and B are convex compact. Then, by proposition 5.5, $A \cap B$ is a connected convex compact set. By proposition 5.4, $(A \cap B)^o = A^o \cap B^o$ is either an empty or non-empty connected convex set.

Q.E.D.

5.2 Examples of shape depending on topological relations

The following are several examples of topological relations between objects that are dependent on the shape of the objects themselves.

5.2.1 The topological relation between two discs

It is not difficult to count the total number of topological relationships between two discs, since the boundaries of these two discs can be described by the following formulae:

$$(x-h_1)^2+(y-k_1)^2=r_1^{\,2} \qquad \text{(i)}$$

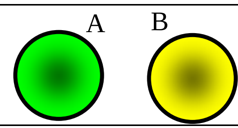
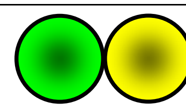
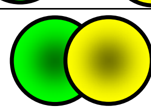
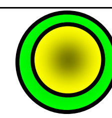
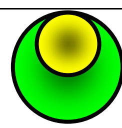
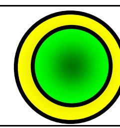
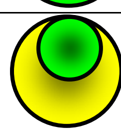
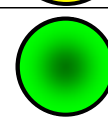
$$(x-h_2)^2+(y-k_2)^2=r_2^{\,2}. \qquad \text{(ii)}$$

We can obtain the following two results by solving equations (i) and (ii).

1. *The number of components of* $\partial A\cap\partial B$ *is at most two, and the numbers 0, 1, 2 are possible.*
2. *The number of components of* $A^o\setminus B$, $B^o\setminus A$ *is at most one each.*

Then, combining these two facts and the previous facts, the total number of topological relationships between two circular discs is 8. These relations are all listed in Table 5.1:

Table 5.1. The total number of topological relationships between two circles

1. $\begin{pmatrix}0&1\\1&0\end{pmatrix}$		2. $\begin{pmatrix}0&1\\1&1\end{pmatrix}$	
3. $\begin{pmatrix}1&1\\1&2\end{pmatrix}$		4. $\begin{pmatrix}1&1\\0&0\end{pmatrix}$	
5. $\begin{pmatrix}1&1\\0&1\end{pmatrix}$		6. $\begin{pmatrix}1&0\\1&0\end{pmatrix}$	
7. $\begin{pmatrix}1&0\\1&1\end{pmatrix}$		8. $\begin{pmatrix}1&0\\0&1\end{pmatrix}$	

The total number of topological relations between two circular discs is same as that of the Egenhofer's 9-intersection model. This is because Egenhofer's 9-intersection model actually describes the topological relations between two circular discs.

5.2.2 *The topological relationships between two ellipse-shaped sets*

To count the total number of topological relationships between two ellipse-shaped sets is also an easy job, since the boundaries of these two discs can be described by the following formulae:

$$\frac{(x-h_1)^2}{a_1^{\ 2}}+\frac{(y-k_1)^2}{b_1^{\ 2}}=1 \qquad \text{(iii)}$$

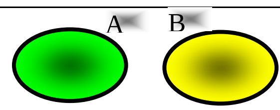
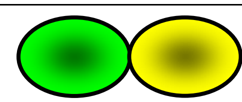
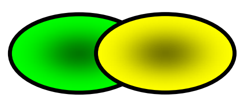
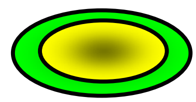
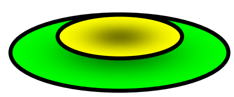
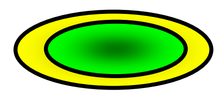
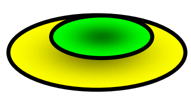
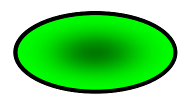
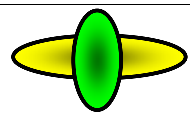
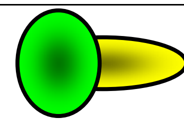
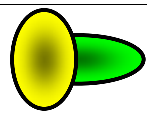
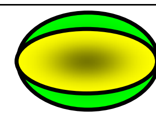
$$\frac{(x-h_2)^2}{a_2^{\ 2}}+\frac{(y-k_2)^2}{b_2^{\ 2}}=1. \qquad \text{(iv)}$$

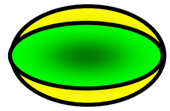
We can obtain the following two results by solving equations (iii) and (iv).

1. *The number of components of* $\partial A \cap \partial B$ *is at most four, and the numbers 0, 1, 2, 3, 4 are possible.*
2. *The number of components of* $A^o \setminus B$, $B^o \setminus A$ *is at most two each.*

These two facts are then combined with the previous facts, and we get a total of 13 topological relationships between two ellipses. These relations are all listed in Table 5.2:

Table 5.2. The total number of topological relationships between two ellipses

1. $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$		2. $\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$	
3. $\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}$		4. $\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}$	
5. $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$		6. $\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$	
7. $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$		8. $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	
9. $\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}$		10. $\begin{pmatrix}1 & 1\\ 2 & 3\end{pmatrix}$	
11. $\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}$		12. $\begin{pmatrix}1 & 2\\ 0 & 2\end{pmatrix}$	

13. $\begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$		

The total number of topological relations between two ellipses is 13, which is different from the 8 topological relations between two circular discs described above. This is because topological relations between spatial objects are shape dependent.

5.3 Summary

The homeomorphic map $f: X \rightarrow Y$ preserves the intersection, interior, exterior, closure, and boundary as $f(A \cap B) = f(A) \cap f(B)$, $f(A^{o}) = f(A)^{o}$, $f(A^{c}) = f(A)^{c}$, $f(\overline{A}) = \overline{f(A)}$ and $f(\partial A) = \partial f(A)$. Thus, the number of components of $A^{o} \cap B^{o}$, $A^{o} \cap \partial B$, $\partial A \cap B^{o}$, $\partial A \cap \partial B$, $A^{o} \setminus B$, $B^{o} \setminus A$, $\partial A \setminus B$, $\partial B \setminus A$, $A^{c} \cap B^{c}$, $A \cup B$, … is preserved by homeomorphism. By the definition of the topological relations between spatial objects (Egenhofer and Franzosa; 1994), the number of components of $A^{o} \cap B^{o}$, $A^{o} \cap \partial B$, $\partial A \cap B^{o}$, $\partial A \cap \partial B$, $A^{o} \setminus B$, $B^{o} \setminus A$, $\partial A \setminus B$, $\partial B \setminus A$, $A^{c} \cap B^{c}$, $A \cup B$, … is an invariant property of the topological relations.

We have tried to illustrate that the number of components of $A^{o} \cap B^{o}$ can be more than one, but any homeomorphism (topological transformation) cannot reduce its number. The topological relations between spatial objects can be thought of as rubber, and the topological transformation can be thought as changing its shape by shrinking and stretching. But some laws must be obeyed, such as that the number of holes in a spatial object cannot be changed, the number of components of $A^{o} \cap B^{o}$, $A^{o} \cap \partial B$, $\partial A \cap B^{o}$, $\partial A \cap \partial B$, $A^{o} \setminus B$, $B^{o} \setminus A$, $\partial A \setminus B$, $\partial B \setminus A$, $A^{c} \cap B^{c}$, $A \cup B$, … cannot be changed, and so on. Figure 5.6 illustrates that the topological relation of the diagram cannot be modeled by convex sets. Indeed, the number of components of $A^{o} \cap B^{o}$ in the case of between non-convex set can be more than one. (see Figure 5.6(b))

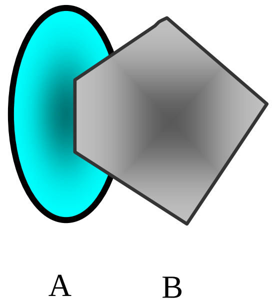


Figure 5.6(a): The topological relations between convex sets, the number of components of $A^{o} \cap B^{o}$ is at most one

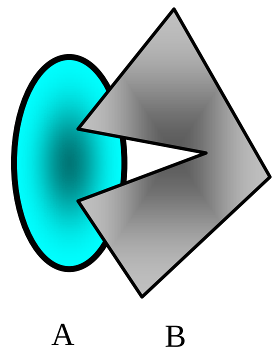


Figure 5.6(b): The topological relations between non-convex sets, the number of components of $A^{o} \cap B^{o}$ can be more than one

The topological relations between two objects A and B do not depend on the global shape. That is, the shape of $A \cup B$ will not be preserved under a topological transformation, such as translation, scaling, and rotation. But the relations will be affected by the shape of the spatial objects themselves. That is, in Figure 5.6(b), set B cannot be deformed to a convex set while keeping the convexity of set A.

In this chapter, we have given a mathematical proof to show that the number of components in the intersection of the interior of two convex spatial regions in two-dimensional space is at most two, while the number of components can be more than one if they are not convex. Therefore, the topological relations between spatial objects cannot be modeled by convex sets only, since the number of components in spatial objects is an invariant property of topological relations. This also point out why the model of topological relations between convex sets is different from the model of topological relations between non-convex sets in Chapter 4.

CHAPTER SIX

MODELING TOPOLOGICAL RELATIONS BETWEEN UNCERTAIN OBJECTS IN GIS

GIS were designed and developed as tools for the management, analysis, and display of spatial information. Due to the fact that the spatial objects in GIS contain uncertainties, such as random errors in measuring spatial objects, or vagueness/fuzziness in interpreting the boundaries of natural objects. Therefore, there is a need to enhance existing GIS by further coping with the uncertainties in spatial objects and the topological relations between uncertain spatial objects. Thus, the classical set theory (Steven, 1964), which is based on a crisp boundary, may not be fully suitable for handling such problems of uncertainty (Wang, Hall and Subaryono, 1990). Fuzzy set theory and fuzzy topological theory, on the other hand, provides a useful tool to describe the uncertainty of single objects in GIS.

This chapter presents a study on modeling fuzzy topological relations between uncertain objects in GIS. Quasi-coincidence and quasi-difference, which are used (a) to distinguish the topological relations between fuzzy objects and (b) to indicate the effect of one fuzzy object on another in a fuzzy topology, are adopted for the development. Geometrically, features in GIS can be classified as point features, linear features and polygon or region features. In this chapter, we first introduce several basic concepts in fuzzy topology that will be used in this study. This is followed by several definitions of fuzzy points, fuzzy lines and fuzzy regions for GIS objects. Next, the level at which one fuzzy object affects the other is modeled based on the sum and difference of the membership functions that are quasi-coincident and quasi-different, respectively. Finally, an applicable example of using quasi-coincidence and quasi-difference based on the new definitions of fuzzy point, line and polygon are given.

6.1 Fuzzy definition for GIS elements

Fuzzy sets are the basic element of a fuzzy topology. The followings are several definitions and basic properties of fuzzy sets in GIS that will be applied. As we mentioned before, point, line and region (polygon) are the basic elements in GIS. We now define them based on a fuzzy set. The definition of a fuzzy point is adopted from Fuzzy Topology (Liu and Luo, 1997) and the simple fuzzy line and fuzzy region are defined in this study. Actually, the definition of a fuzzy region have been discussed by Schneider (1999) and Tang *et al* (2002). However, the simple fuzzy line and fuzzy region in this thesis are defined which is based on the real applications.

Fuzzy point

Definition 6.1 (fuzzy point): An I-fuzzy point on X is an I-fuzzy subset $x_a \in I^X$ defined as

$$x_a(y) = \begin{cases} a & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}.$$

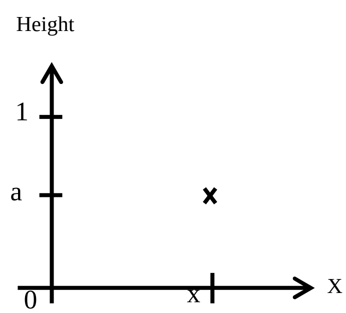


Figure 6.1: Fuzzy point

Simple fuzzy line

Definition 6.2: The line in X (or $\mathbf{R}^2$) can be described as an **embedding** of a connected interval from R to X (or $\mathbf{R}^2$), which does not have intersection, i.e.

$$\alpha : [0,1] \rightarrow X \ \text{(or } \mathbf{R}^2\text{)},$$

where [0, 1] is a closed interval in **R** and $\alpha(t_1) \neq \alpha(t_2)$ for all $t_1 \neq t_2$, $t_1, t_2 \in [0,1]$. (see Figure 6.2)

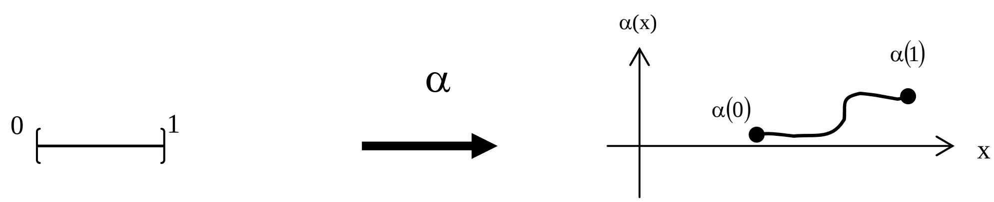


Figure 6.2: Line in $\mathbf{R}^2$ can be thought as a mapping from the interval [0, 1] to $\mathbf{R}^2$

Remark 1:

(i) The condition $\alpha(t_1) \neq \alpha(t_2)$ for all $t_1, t_2 \in [0,1]$ makes sure that this line does not intersect itself.

(ii) The line in the above definition can be a poly-line, i.e. it can be several line segments joined together one by one.

Definition 6.3 (simple fuzzy line): I-fuzzy subset $\ell \in I^X$ is called a simple I-fuzzy line if supp(ℓ) is a line in X (or $\mathbf{R}^2$). (see Figure 6.3)

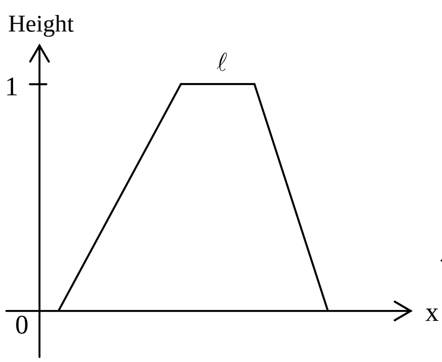


Figure 6.3: A simple fuzzy line

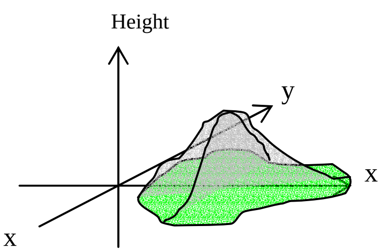


Figure 6.4 (a): A fuzzy set in $\mathbf{R}^2$

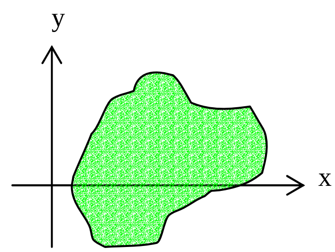


Figure 6.4 (b): The support of A in $\mathbf{R}^2$

<u>Fuzzy region</u>

Definition 6.4 (fuzzy region in X (or $\mathbf{R}^2$)**):** A fuzzy set A in X (or $\mathbf{R}^2$) is called a fuzzy region if supp(A) has non-empty interior in the sense of an ordinary topology. (see Figure 6.4)

Definition 6.5: Let I^X be an I-fuzzy topological space, $A \in I^X$, $a \in I$. Define an I-fuzzy subset aA by

$$aA(x) = a \wedge A(x)$$

Called a layer of A, or the a-layer of A. (see Figure 6.5)

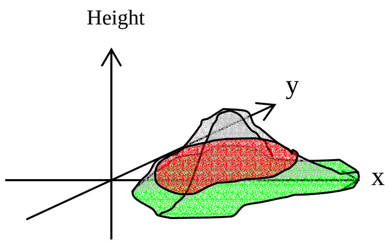


Figure 6.5 (a): The fuzzy set A

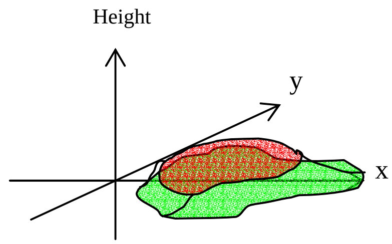


Figure 6.5 (b): a-layer of A

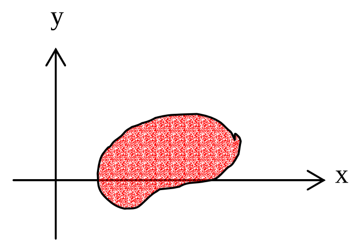


Figure 6.5 (c): The top projection of a-layer of A

Remark 2: The a-layer of a fuzzy set refers to shrinking down all the membership values of a fuzzy set that are greater than a to be value equal to a. In GIS, it is a useful operator to see which is less than a particular value within the range of uncertainties.

Definition 6.6 (a-level of fuzzy set): Let I^X be an I-fuzzy space, $A \in I^X$, $a \in I$. Define the a-level of A as the ordinary set

$$A_{[a]} = \{x \in X : A(x) \geq a\};$$

$$A_{(a)} = \{x \in X : A(x) > a\};$$

$$A^{[a]} = \{x \in X : A(x) \leq a\};$$

$$A^{(a)} = \{x \in X : A(x) < a\}.$$

Proposition 6.7: Let A be a I-fuzzy subset on X, then $A = \bigvee_{a \in I} aA_{[a]}$.

Proof: For all $x \in X$, $aA_{[a]}(x) = \begin{cases} A(x) & \text{if } x \in A_{(a)} \\ a & \text{if } A(x) = a \\ 0 & \text{if } x \notin A_{[a]} \end{cases}$, therefore, $\bigvee_{a \in I} aA_{[a]}(x) = A(x)$.

Q.E.D.

Remark 3: $A = \bigvee_{a\in I} aA_{[a]}$ can be illustrated in Figure 6.6.

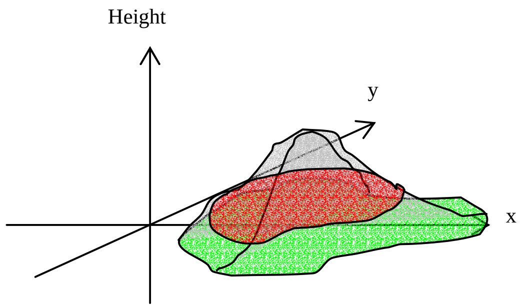


Figure 6.6 (a): The fuzzy set A

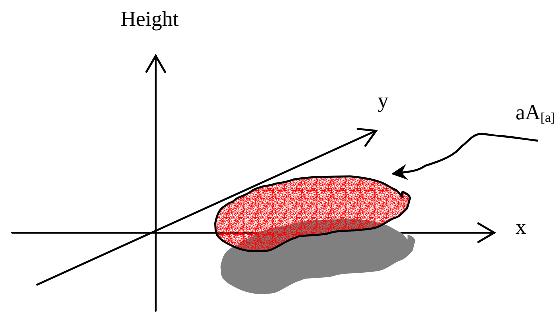


Figure 6.6 (b): The fuzzy set $aA_{[a]}$

6.2 Quasi-coincidence

The concept of quasi-coincidence is a kind of partition of fuzzy sets via the sum of their membership functions. As the membership function of a fuzzy set can be abstract, it can be used to interpret a very deep meaning, such as the density of SARS diseases or the ability of immunity and so on. The sum of the membership values can be used to interpret how closeness of their relations is. Therefore, it is very useful in GIS.

The concept of quasi-coincidence provides a stable fundamental neighborhood's structural description to a fuzzy topological relations between objects in GIS. The concept of quasi-coincidence is supported by the following definitions.

Definition 6.8 (disjoint): Two fuzzy sets A and B are totally disjoint if $A \wedge B = 0$.

Definition 6.9 (quasi-coincident): Let A and B be two Fuzzy sets in I^X. We say A quasi-coincident with B (write $A_{\hat{q}}B$) at x if A(x) + B(x) > 1. Denoted by $A\hat{q}B = \{x \in X: A(x) + B(x) > 1\}$.

The quasi-coincident set is an ordinary set and just the collection of all x in X with the properties of A(x) + B(x) > 1. Thus we can definite the quasi-coincident fuzzy sets as follows:

$$\min(A \wedge B) = \begin{cases} A(x) \wedge B(x) & \text{if} \quad A(x) + B(x) > 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and}$$

$$\text{mean}(A \wedge B) = \begin{cases} \dfrac{A(x) + B(x)}{2} & \text{if} \quad A(x) + B(x) > 1 \\ 0 & \text{otherwise} \end{cases}.$$

Example 6.10: Let two fuzzy sets

$$A(x) = \begin{cases} -\dfrac{1}{13}(\mid x-2 \mid - \mid x+2 \mid + 16) & \text{if} \quad \dfrac{-17}{2} \le x \le \dfrac{15}{2} \\ 0 & \text{otherwise} \end{cases} \quad \text{and}$$

$$B(x) = \begin{cases} -\dfrac{1}{13}(\mid x-6 \mid - \mid x-2 \mid + 16) & \text{if} \quad \dfrac{-9}{2} \le x \le \dfrac{23}{2} \\ 0 & \text{otherwise} \end{cases}.$$

Then the crisp set A⩓B is shown in Figure 6.7(a), while the two corresponding fuzzy sets $\min(A \wedge B)$ and $\text{mean}(A \wedge B)$ are shown in the Figure 6.7(b) and Figure 6.7(c) respectively.

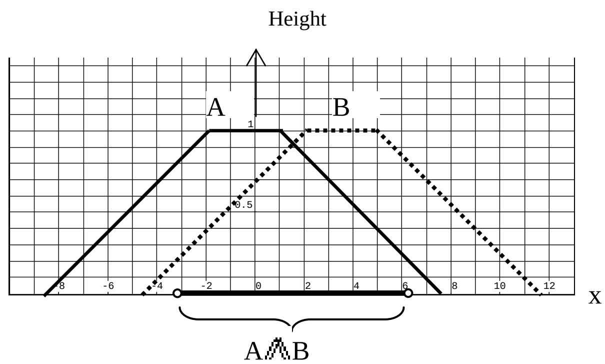


Figure 6.7 (a): The set of A⩕B

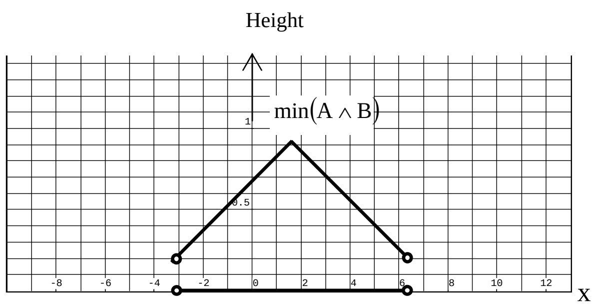


Figure 6.7 (b): The set of $\min(A \wedge B)$

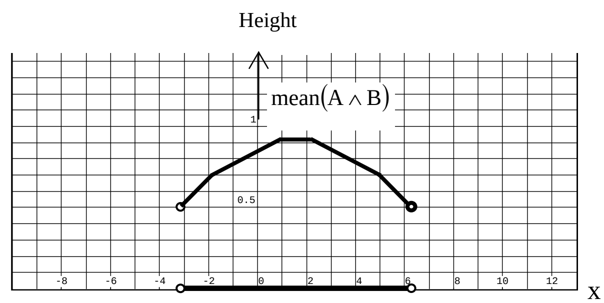


Figure 6.7 (c): The set of $\operatorname{mean}(A \wedge B)$

Proposition 6.11: $\min(A \wedge B) \le A \wedge B$.

Proof: If $A(x)+B(x)>1$, $\min(A \wedge B)=A(x)\wedge B(x)$. If $A(x)+B(x)\le 1$, $\min(A \wedge B)=0 \le A(x)\wedge B(x)$.

Q.E.D.

Proposition 6.12: Let $(I^X,\ \delta)$ be an I-fts', $A, B, C \in I^X$, $\{A_t : t \in T\} \subset I^X$, $x \in X, a \in I \setminus \{0\}$. Then

(i) A⩕B = B⩕A.

(ii) A quasi-coincident with B at x $\Leftrightarrow$ B quasi-coincident with A at x $\Leftrightarrow$ $x \in$ A⩕B $\Leftrightarrow$ $x \in$ B⩕A.

(iii) If $A \le B$, then A⩕C $\subset$ B⩕C.

(iv) A⩕$\vee_{t\in T} A_t = \cup_{t \in T}$ A ⩕A_t.

Proof: it is just the direct verification.

Q.E.D.

Remark 4: Proposition 6.12 (i) to (iv) give several properties of quasi-coincidence.

Proposition 6.13: Let (I^X, δ) and (I^Y, μ) be two I-fts', $A, B \in I^X$, $C, D \in I^Y$, $f : X \to Y$ be an ordinary mapping. Then

(i) $A_{\hat{q}} f^{\leftarrow}(C) \Leftrightarrow f^{\rightarrow}(A)_{\hat{q}} C$.

(ii) $A_{\hat{q}} B \Rightarrow f^{\rightarrow}(A)_{\hat{q}} f^{\rightarrow}(B)$.

(iii) $f^{\leftarrow}(C)_{\hat{q}} f^{\leftarrow}(D) \Rightarrow C_{\hat{q}} D$.

Proof: (i): If $A_{\hat{q}} f^{\leftarrow}(C)$ then there exists $x \in X$ such that $A(x) + f^{\leftarrow}(C)(x) > 1$. Let f(x) = y. we have $f^{\rightarrow}(A)(y) + C(y) \geq A(x) + f^{\leftarrow}(C)(x) > 1$.

Conversely, if $f^{\rightarrow}(A)(y) + C(y) > 1$, then $C(y) > 1 - f^{\rightarrow}(A)(y) = f^{\rightarrow}(A)'(y) = f^{\rightarrow}(A')(y)$ $\Rightarrow$ $f^{\leftarrow}C(x) > f^{\leftarrow}f^{\rightarrow}(A')(y) \geq A'(x)$ $\Rightarrow$ $f^{\leftarrow}C(x) > A'(x)$ $\Rightarrow$ $f^{\leftarrow}C(x) > 1 - A(x)$ $\Rightarrow$ $A(x) + f^{\leftarrow}C(x) > 1$.

(ii): Let f(x) = y, $x \in A_{\hat{q}} B$ $\Rightarrow$ $f^{\rightarrow}(A)(y) + f^{\rightarrow}(B)(y) \geq A(x) + B(x) > 1$ $\Rightarrow$ $f^{\rightarrow}(A)_{\hat{q}} f^{\rightarrow}(B)$.

(iii): $x \in f^{\leftarrow}(C)_{\hat{q}} f^{\leftarrow}(D)$ $\Rightarrow$ $C(f(x)) + D(f(x)) = f^{\leftarrow}(C)(x) + f^{\leftarrow}(D)(x) > 1$ $\Rightarrow$ $C_{\hat{q}} D$.

Q.E.D.

Remark 5: Proposition 6.13 states several invariants of quasi-coincidence under mapping. Of course, it is invariant under homeomorphism. Moreover, if f is a homeomorphic map, we have the proposition below.

Proposition 6.14: Let (I^X, δ) and (I^Y, μ) be two I-fts', $A, B \in I^X$, $C, D \in I^Y$, $f : X \to Y$ be a homeomorphic map. Then

(i) $A_{\hat{q}} f^{\leftarrow}(C) \Leftrightarrow f^{\rightarrow}(A)_{\hat{q}} C$.

(ii) $A_{\hat{q}} B \Leftrightarrow f^{\rightarrow}(A)_{\hat{q}} f^{\rightarrow}(B)$.

(iii) $f^{\leftarrow}(C)_{\hat{q}} f^{\leftarrow}(D) \Leftrightarrow C_{\hat{q}} D$.

Q.E.D.

Definition 6.15: Define the fuzzy set $A_{\frac{1}{2}}$ by $A_{\frac{1}{2}}(x) = \begin{cases} A(x) & \text{if } A(x) > \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$.

Proposition 6.16: Let $A \in I^X$ and $f^{\rightarrow} : (I^X, \delta) \rightarrow (I^Y, \mu)$ be an I-fuzzy homeomorphism, then $f^{\rightarrow}\left(A_{\frac{1}{2}}\right) = f^{\rightarrow}(A)_{\frac{1}{2}}$, i.e. the set $A_{\frac{1}{2}}$ is invariant under homeomorphism.

Proof: Since $f^{\rightarrow} : (I^X, \delta) \rightarrow (I^Y, \mu)$ is an I-fuzzy homeomorphism, $\forall\, y \in Y$, there exists unique $x_o \in X$ such that $f(x_o) = y$. Thus $f^{\rightarrow}(A)(y) = \vee\{A(x) : x \in X, f(x) = y\} = A(x_o)$.

$$\text{Since } f^{\rightarrow}\left(A_{\frac{1}{2}}\right)(y) = \begin{cases} f^{\rightarrow}\left(A_{\frac{1}{2}}\right)(y) & \text{if } A(x_o) > \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} A_{\frac{1}{2}}(x_o) & \text{if } A(x_o) > \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} A(x_o) & \text{if } A(x_o) > \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}.$$

$$\text{On the other hand, } f^{\rightarrow}(A)_{\frac{1}{2}}(y) = \begin{cases} f^{\rightarrow}(A)(y) & \text{if } f^{\rightarrow}(A)(y) > \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} A(x_o) & \text{if } A(x_o) > \frac{1}{2}. \\ 0 & \text{otherwise} \end{cases}$$

Therefore, $f^{\rightarrow}\left(A_{\frac{1}{2}}\right)(y) = f^{\rightarrow}(A)_{\frac{1}{2}}(y)$, for all $y \in Y$. Hence $f^{\rightarrow}\left(A_{\frac{1}{2}}\right) = f^{\rightarrow}(A)_{\frac{1}{2}}$.

Q.E.D.

Proposition 6.17: The quasi-coincident, $A \curlywedge B$, set is divided into two parts: one is $A(x) > \frac{1}{2}$ and $B(x) > \frac{1}{2}$ denoted by $\min(A \wedge B)^{(A \,\&\, B > 0.5)}$ and the other is $A(x) \leq \frac{1}{2}$ or $B(x) \leq \frac{1}{2}$ denoted by $\min(A \wedge B)^{(A \text{ or } B \leq 0.5)}$.

Proof: $A \curlywedge B = \{x \in X : A(x) + B(x) > 1\}$. If $A(x) > \frac{1}{2}$ and $B(x) > \frac{1}{2}$, then $A(x) + B(x) > \frac{1}{2}$. If $A(x) \leq \frac{1}{2}$ or $B(x) \leq \frac{1}{2}$, then one of A and B must be greater than 0.5.

Q.E.D.

Remark 6:

(i) Since the set $A \curlywedge B$ is invariant under homeomorphisms, it is a homeomorphic invariant topological component (or topological components for short). Thus, we can use this direction to partition the set $A \curlywedge B$ into several homeomorphic invariant parts, we can guarantee the unchanged properties in a GIS transformation.

(ii) In GIS, many reports are tried to study the empty and non-empty of homeomorphic components so that it can give fuzzy topological relations between two spatial objects in GIS. Thus we can see the homeomorphic invariant is very important. Furthermore, we not merely study the empty and non-empty properties, but also other potentially useful topological properties of these components.

Proposition 6.18: For $A(x) > \frac{1}{2}$ and $B(x) > \frac{1}{2}$, we have $A \wedge\!\!\wedge B^c \subset A \wedge\!\!\wedge B$ and $A^c \wedge\!\!\wedge B \subset A \wedge\!\!\wedge B$. This proposition tells us that the fuzzy sets $A(x) > \frac{1}{2}$ and $B(x) > \frac{1}{2}$ can be further divided into three parts: $A \wedge\!\!\wedge B^c$, $A^c \wedge\!\!\wedge B$ and $\{x \in X$: $A(x) > \frac{1}{2}$ and $B(x) > \frac{1}{2}$ and $A(x) = B(x)\}$. Denoted by $\min(A \wedge B^c)^{(A \,\&\, B > 0.5)}_{(A+B>1)}$, $\min(A^c \wedge B)^{(A \,\&\, B > 0.5)}_{(A+B>1)}$ and $\{A = B\}$.

Proof: For $A(x) > \frac{1}{2}$ and $B(x) > \frac{1}{2}$, we have $B^c(x) \le \frac{1}{2}$.

Hence if $A(x) > B(x)$, i.e. $A \wedge\!\!\wedge B^c$, then $A(x) > B(x) > B^c(x)$, i.e. $A \wedge\!\!\wedge B$. Thus we have $A \wedge\!\!\wedge B^c \subset A \wedge\!\!\wedge B$.

Similarly, we can prove $A^c \wedge\!\!\wedge B \subset A \wedge\!\!\wedge B$.

Q.E.D.

Proposition 6.19: $A \wedge\!\!\wedge B^c$, $A^c \wedge\!\!\wedge B$ and $\{A = B\}$ are invariant under homeomorphic fuzzy mapping.

Proof: By proposition 6.14 (ii).

Q.E.D.

So far, we have decomposed the fuzzy set $A \wedge B$ into several homeomorphic invariant parts, $A \wedge B_{(A+B \le 1)}$, $\min\,(A \wedge B)^{(A \text{ or } B \le 0.5)}_{(A+B>1)}$, $\min\,(A' \wedge B)^{(A \,\&\, B > 0.5)}_{(A+B>1)}$, $\{A = B\}$ and $\min\,(A \wedge B')^{(A \,\&\, B > 0.5)}_{(A+B>1)}$. Figure 6.8 shows the logic diagram of the decomposition of $A \wedge B$ while Figure 6.9 illustrates the structure of the decomposition of $A \wedge B$ through example 6.10.

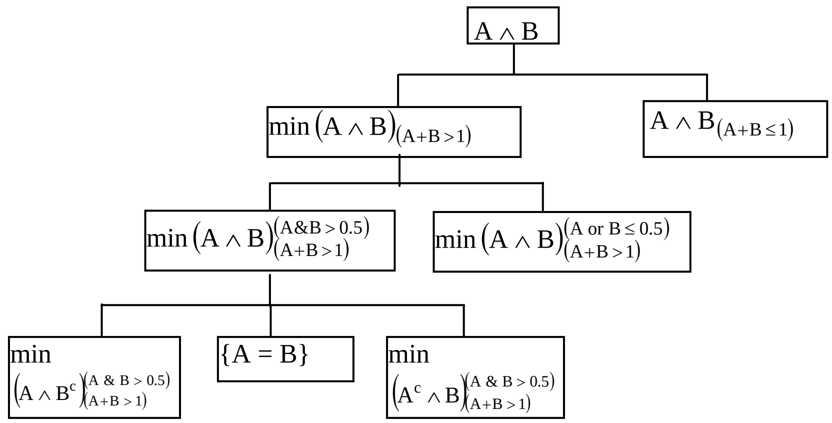


Figure 6.8: The logic diagram of the decomposition of $A \wedge B$

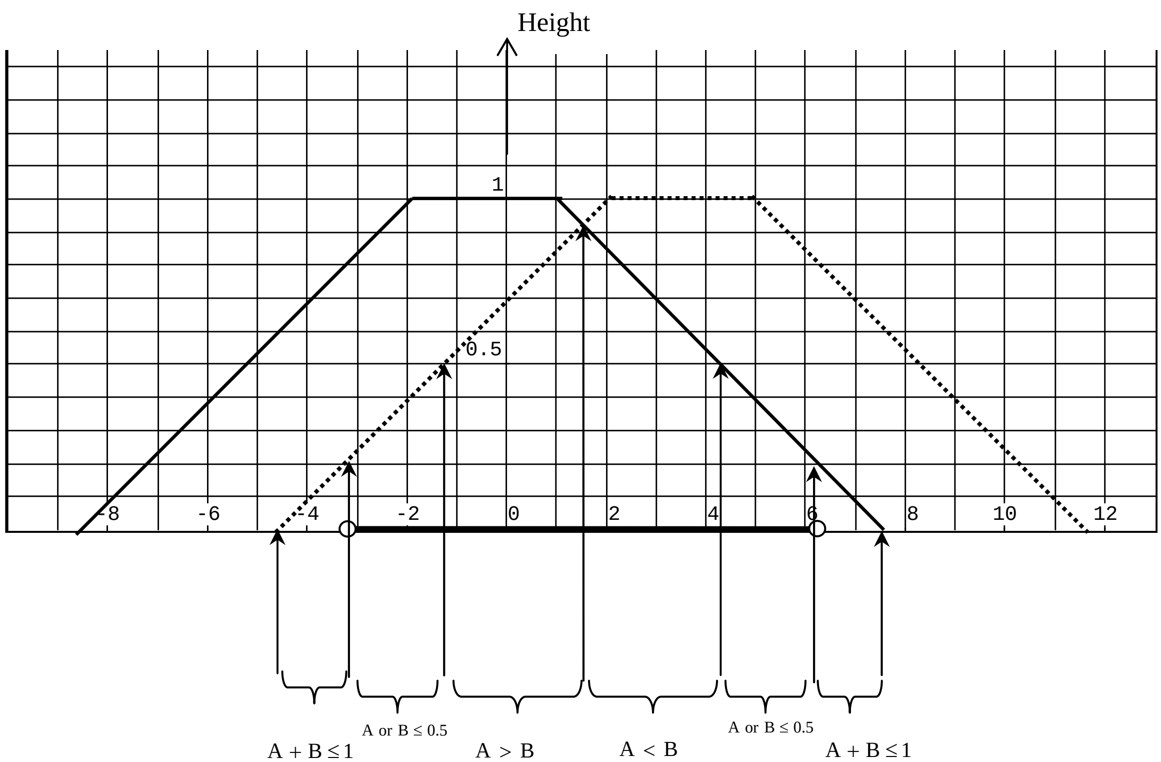


Figure 6.9: An illustration of the structure of the decomposition of $A \wedge B$

Figure 6.10 illustrates the topological relations with the concept of a quasi-coincidence, A⩕B, in $\mathbf{R}^2$. The region enclosed by a dash line is the quasi-coincidence in different cases.

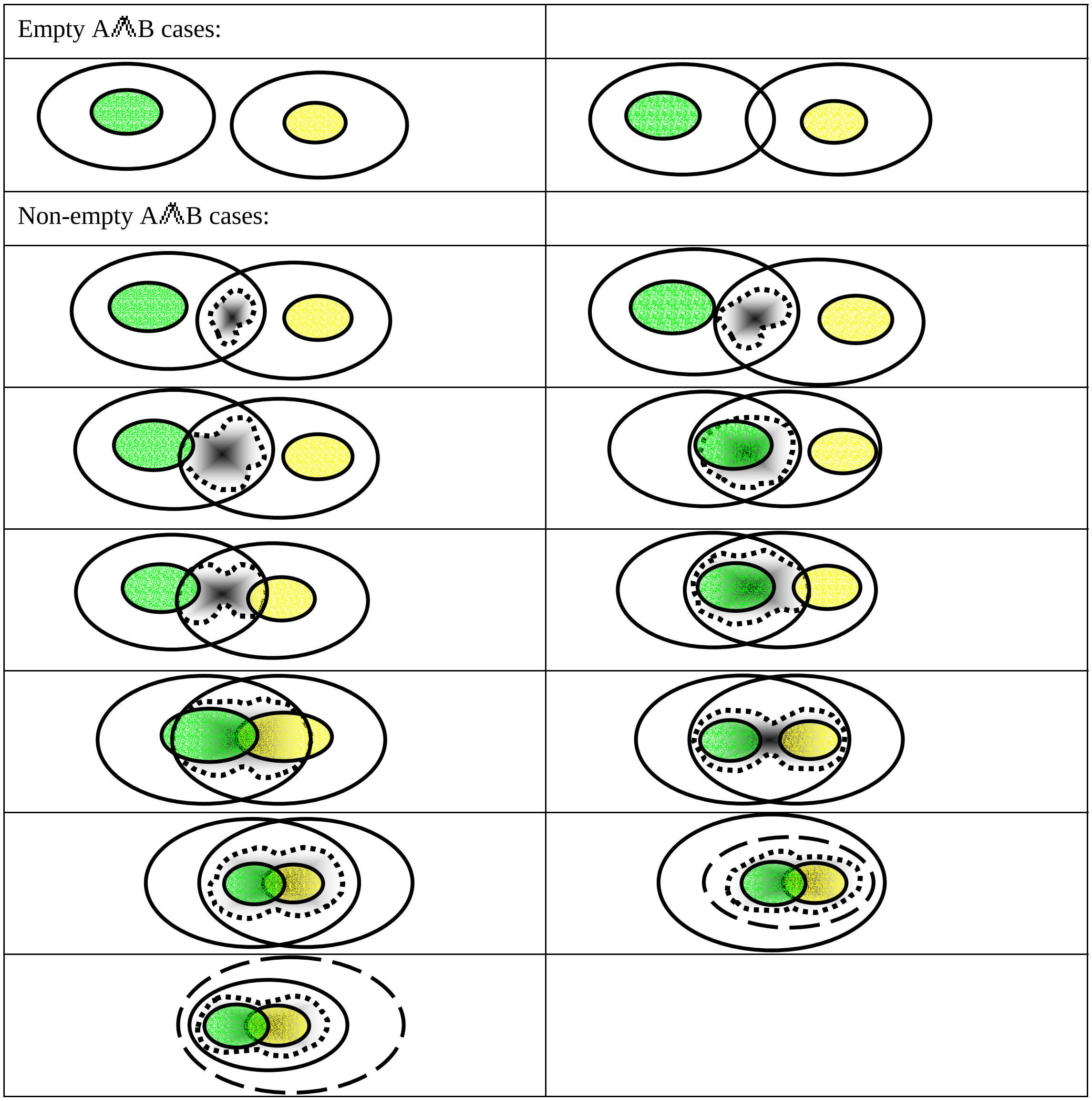


Figure 6.10: Different cases of quasi-coincident fuzzy topological relations between two objects in GIS

Figure 6.11 illustrates the topological relations with the concept of quasi-coincidence, $A \wedge\!\!\wedge B^c$, in $\mathbf{R}^2$. The region enclosed by a dot line is the quasi-coincidence in different cases.

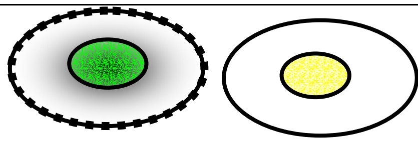
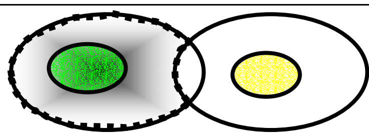
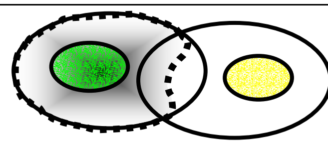
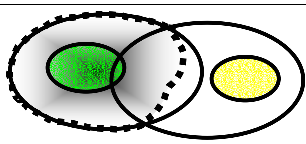
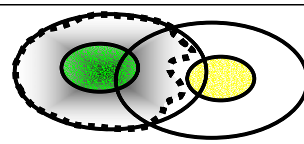
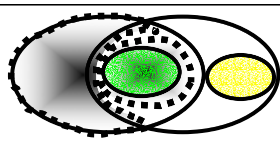
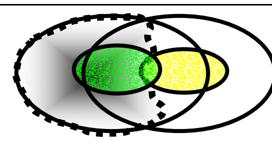
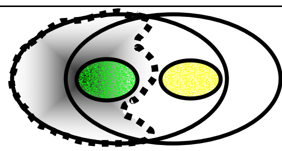
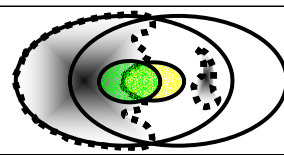
Disjoint, then $A \wedge\!\!\wedge B^c$ is equal supp(A):	
	
Non-empty $A \wedge\!\!\wedge B^c$ cases:	
	
	
	
	

Figure 6.11: Different cases of quasi-coincident fuzzy topological relations between two objects in GIS

6.3 Quasi-difference

The concept of quasi-difference is a kind of partition of fuzzy sets via the difference of their membership functions. The difference can be used to compare the fuzzy value of two fuzzy sets for analyzing which part is higher than the other and which part is lower than the other.

Definition 6.20 (quasi-difference): Let A and B be two Fuzzy sets in I^x. Define the quasi-difference of A and B, denoted by A\\B, as

$$A\backslash\backslash B = \vee \left\{x_\lambda \in M(\downarrow A): B(x) = 0\right\} \vee \vee \left\{x_\lambda \in M(\downarrow A): \lambda \text{ not} \geq B(x) > 0\right\}.$$

Where the definitions of $\downarrow A$ and $M(\downarrow A)$ one can refer to Liu and Luo (1997).

Definition 6.21: Let A and B be two Fuzzy sets in I^x. We define $A\backslash\backslash B_o = \vee \left\{x_\lambda \in M(\downarrow A): B(x) = 0\right\}$. (see Figure 6.12)

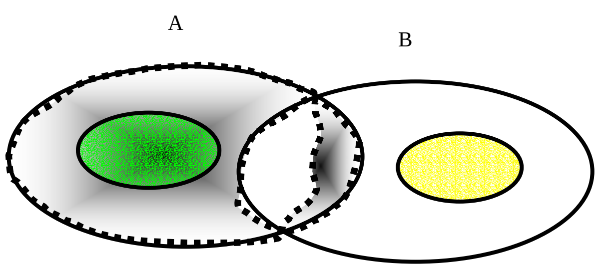


Figure 6.12 (a): The part enclosed by a dot line is the quasi-difference of A and B, A\\B

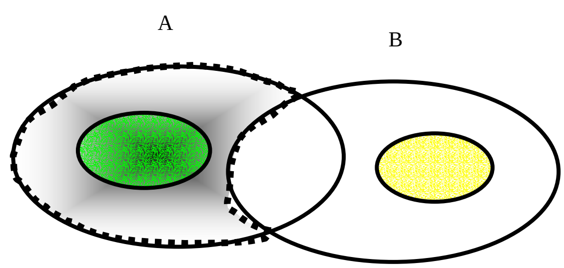


Figure 6.12 (b): The part enclosed by a dot line is the quasi-difference of A and B, $A\backslash\backslash B_o$

Proposition 6.22: Let I^X be an I-fuzzy space, A, B, C in I^X, $\{A_t : t \in T\} \subset I^X$, $x_a \in Pt(I^X)$. Then the following conclusions are held:

(i) $A\backslash\backslash B \leq A$.

(ii) $A\backslash\backslash \underline{0} = A$.

(iii) $1_L \notin M(I) \Rightarrow A\backslash\backslash 1 = A$.

(iv) $A \leq B \Rightarrow A\backslash\backslash C \leq B\backslash\backslash C$.

(v) $\left(\vee_{t\in T} A_t\right)\backslash\backslash C = \vee_{t\in T} (A_t\backslash\backslash C)$.

(vi) $\forall x \in \text{supp}(B),\ B(x) > A(x) \Rightarrow A\backslash\backslash B = A$.

(vii) $A \wedge B = 0 \Rightarrow A\backslash\backslash B = A$.

(viii) x_a is not quasi-coincidence with A', then $A\backslash\backslash x_a = A$.

(ix) supp(B) = supp(C), $B \le C$, then $A\backslash\backslash B \le A\backslash\backslash C$.

Q.E.D.

Remark 7: In the definition of quasi-difference of I-fuzzy sets, if replacing I by {0, 1}, then this concept will become our crisp set difference.

Lemma 6.23: Let $A \in I^x$ and $f^{\rightarrow} : (I^x, \delta) \rightarrow (I^y, \mu)$ be an I-fuzzy homeomorphism, and f(x) = y, then $f^{\rightarrow}(A)(y) = 0$ if and only if A(x) = 0.

Proof: $f^{\rightarrow}(A)(y) = 0 \Rightarrow f^{\leftarrow}(f^{\rightarrow}(A))(x) = 0 \Rightarrow$ A(x) = 0.

On the other hand, $f^{\rightarrow}(A)(y) = \vee\{A(x) : f(x) = y\} = A(x) = 0$.

Q.E.D.

Proposition 6.24: Let $A \in I^x$ and $f^{\rightarrow} : (I^x, \delta) \rightarrow (I^y, \mu)$ be an I-fuzzy homeomorphism, then $f^{\rightarrow}(A\backslash\backslash B_o) = f^{\rightarrow}(A)\backslash\backslash f^{\rightarrow}(B)_o$, i.e. the set $A\backslash\backslash B_o = \vee\{x_\lambda \in M(\downarrow A) : B(x) = 0\}$ is invariant under homeomorphic mappings.

Proof: Since $f^{\rightarrow}(A\backslash\backslash B_o) = \begin{cases} f^{\rightarrow}(A)(y) & \text{if } B(x) = 0 \\ 0 & \text{otherwise} \end{cases}$, but $f^{\rightarrow}(A)\backslash\backslash f^{\rightarrow}(B_o) = \begin{cases} f^{\rightarrow}(A)(y) & \text{if } f^{\rightarrow}(B)(y) = 0 \\ 0 & \text{otherwise} \end{cases}$. By lemma 6.23, there is one-one corresponding value of B(x) = 0 and $f^{\rightarrow}(B)(y) = 0$, so $f^{\rightarrow}(A\backslash\backslash B_o) = f^{\rightarrow}(A)\backslash\backslash f^{\rightarrow}(B)_o$.

Q.E.D.

By using the above results, the fuzzy set $A \vee B$ can be decomposed into several homeomorphic invariant parts, $A \backslash\backslash B_o$, and $B \backslash\backslash A_{o.}$ On $A \mathbin{\wedge\!\!\!\wedge} B$, we have $(A(x) > \frac{1}{2}$ and $B(x) > \frac{1}{2})$, and $(A(x) \le \frac{1}{2}$ or $B(x) \le \frac{1}{2})$. On $A(x) > \frac{1}{2}$ and $B(x) > \frac{1}{2}$, we have three more parts, which are $\{A = B\}$, $\min\left(A \wedge B^c\right)_{(A+B>1)}^{(A \,\&\, B>0.5)}$ and $\min\left(A^c \wedge B\right)_{(A+B>1)}^{(A \,\&\, B>0.5)}$.

Combining the previous works and the above results, we have decomposed the fuzzy set $A \vee B$ into several homeomorphic invariant parts,

(1) $A \backslash\backslash B_o$;

(2) $B \backslash\backslash A_o$;

(3) $A \wedge B_{(A+B \le 1)}$;

(4) $\min\left(A \wedge B\right)_{(A+B>1)}^{(A \text{ or } B \le 0.5)}$;

(5) $\min\left(A^c \wedge B\right)_{(A+B>1)}^{(A \,\&\, B>0.5)}$;

(6) $\{A = B\}$ and

(7) $\min\left(A \wedge B^c\right)_{(A+B>1)}^{(A \,\&\, B>0.5)}$.

Figure 6.13 shows the logic diagram of the decomposition of $A \vee B$ while Figure 6.14 illustrates the structure of the decomposition of $A \vee B$ through example 6.10.

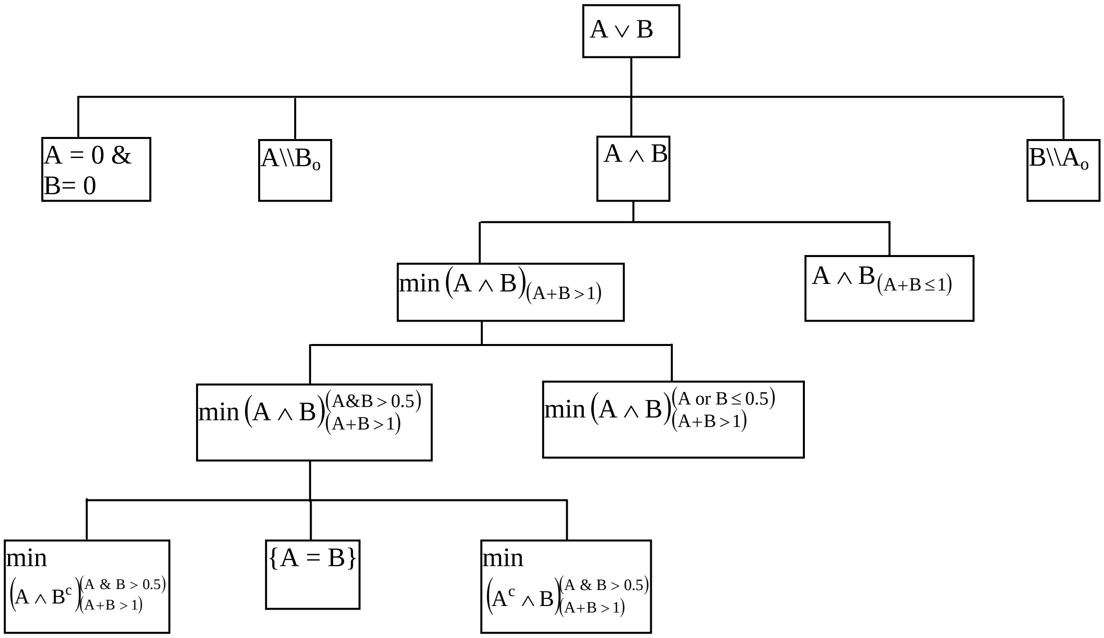


Figure 6.13: The logic diagram of the decomposition of $A \vee B$

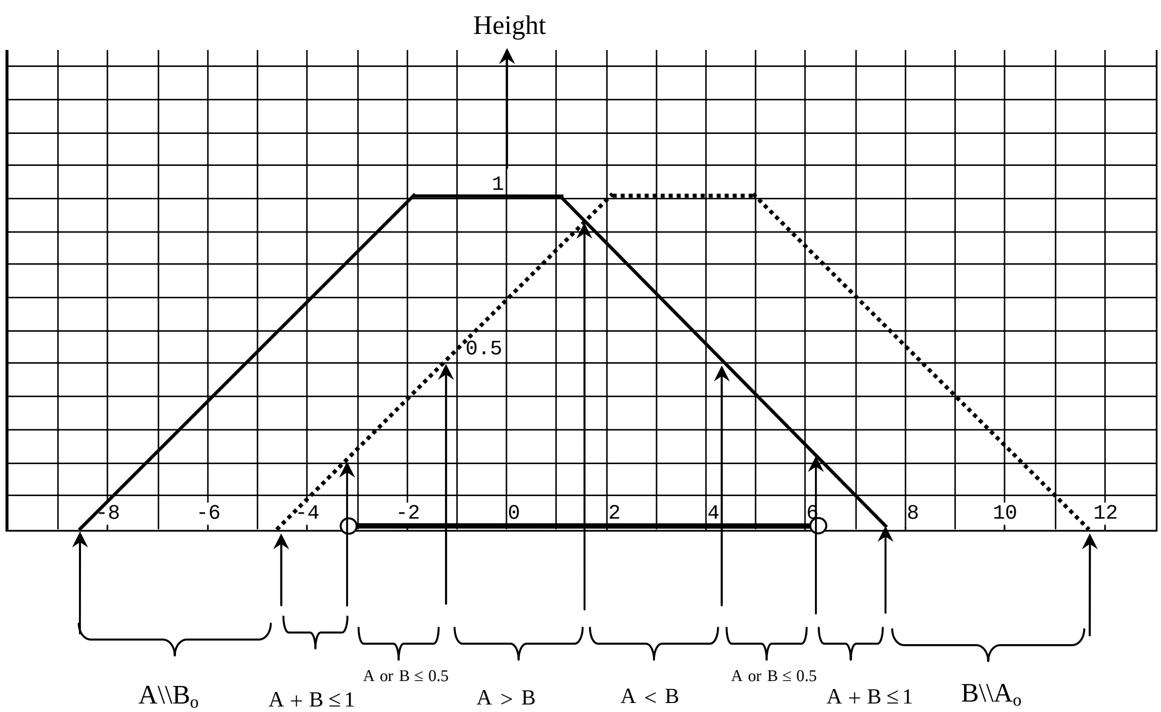


Figure 6.14: An illustration of the structure of the decomposition of $A \vee B$

Figure 6.15 illustrates the topological relations with the concept of quasi-difference, A\\B, in $\mathbf{R}^2$. The region enclosed by a dot line is the quasi-coincidence in different cases.

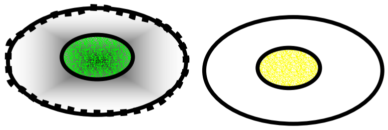
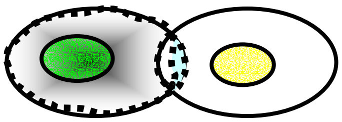
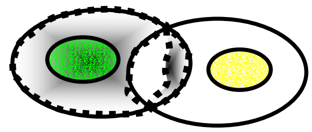
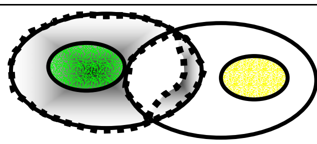
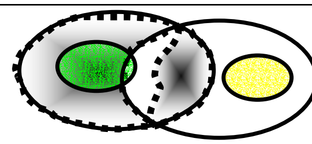
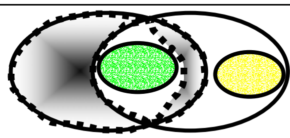
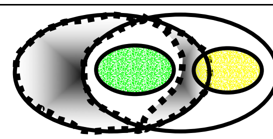
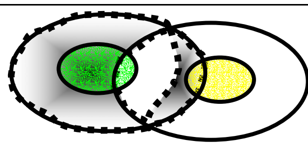
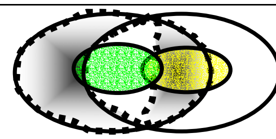
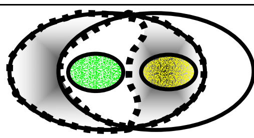
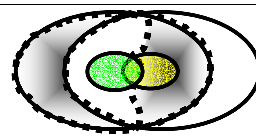
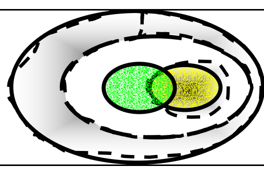
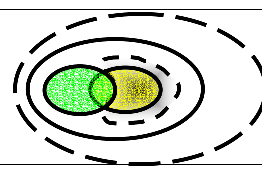
Disjoint, then A\\B is equal supp(A):	
	
Non-empty A\\B cases:	
	
	
	
	
	
	

Figure 6.15: Different cases of quasi-different fuzzy topological relations between two objects in GIS

6.4 Topological relations between two fuzzy set in $\mathbf{R}^2$

Topological relations are the fundamental properties for the spatial analysis in GIS. Based on the 4-intersection and the ordinary point set theory, Egenhofer and Franzosa (1991) gave the topological relations between two spatial regions in the two-dimensional space (2-D). Later, Egenhofer, Clementini and Di Felice (1994) gave an extension of the topological relations between spatial objects in 2-D with arbitrary holes. Shi and Guo (1999) described uncertain relations between objects in GIS.

Based on the 9-intersection, Cohn and Gotts (1996) gave 46 topological relations between two regions with indeterminate boundaries. While Clementini and Di Felice (1996) gave 44 topological relations between two regions with indeterminate boundaries. By using the 9-intersection matrix and the fuzzy theory, there are 44 relations between two simply fuzzy regions (Tang and Kainz, 2002). The topological relations between ordinary sets are not as simple as finite (Liu and Shi, 2003). Therefore, the topological relations between fuzzy sets in $\mathbf{R}^2$ will not as simple as finite either. A basic solution to solve the fuzzy relations between fuzzy sets is to investigate its homeomorphic invariant (Wu and Zheng, 1991), and then to classify the relations by using its invariant properties. Thus, it is guaranteed that the topological relations will not be changed under series of homeomorphic mappings.

Definition 6.25 (topological relations): The topological relations between two fuzzy sets A and B are the topological properties of all the homeomorphic invariants topological components of A and B.

Here, we classify the topological relations of fuzzy sets A and B by using the results in sections 6.2 and 6.3. The target topological components of A and B will be

(1) $A\backslash\backslash B_o$;

(2) $B\backslash\backslash A_o$;

(3) $A \wedge B_{(A+B \leq 1)}$;

(4) $\min\left(A \wedge B\right)_{(A+B>1)}^{(A \text{ or } B \leq 0.5)}$;

(5) $\min\left(A^c \wedge B\right)_{(A+B>1)}^{(A \,\&\, B>0.5)}$;

(6) {A = B} and

(7) $\min\left(A \wedge B^c\right)_{(A+B>1)}^{(A \,\&\, B>0.5)}$.

Indeed, those components are homeomorphic invariant topological components that have been proved in section 6.2 and 6.3.

The components $A \wedge B_{(A+B\leq 1)}$, $\min\left(A \wedge B\right)_{(A+B>1)}^{(A \text{ or } B\leq 0.5)}$, $\min\left(A \wedge B^c\right)_{(A+B>1)}^{(A \,\&\, B>0.5)}$, $\min\left(A^c \wedge B\right)_{(A+B>1)}^{(A \,\&\, B>0.5)}$ and {A = B} can be used to classify the depth of the relation between two fuzzy sets. The components A\\$B_o$ and B\\$A_o$ can be used to indicate the depth of independent part of each fuzzy set. Thus the topological relations between the fuzzy sets A and B can be described by the topological properties (empty and non-empty, subspace properties, connectivity, compactness and etc) of the 7-tuple,

$$(A\backslash\backslash B_o,\ B\backslash\backslash A_o,\ A \wedge B_{(A+B\leq 1)},\ \min\left(A \wedge B\right)_{(A+B>1)}^{(A \text{ or } B\leq 0.5)},\ \min\left(A \wedge B^c\right)_{(A+B>1)}^{(A \,\&\, B>0.5)},\ \min\left(A^c \wedge B\right)_{(A+B>1)}^{(A \,\&\, B>0.5)},\ A = B).$$

Furthermore, by providing formulae of two fuzzy sets, we can calculate their relations through the above 7-tupled topological components.

The following propositions are several elementary properties of the above 7-tupled topological components.

Proposition 6.26: If A\\$B_o$ is empty, then supp(B) $\subset$ supp(A). By symmetry, we also have, if B\\$A_o$, then supp(A) $\subset$ supp(B).

Proposition 6.27: $A \wedge B_{(A+B\leq 1)}$, $\min\left(A \wedge B\right)_{(A+B>1)}^{(A \text{ or } B\leq 0.5)}$, $\min\left(A \wedge B^c\right)_{(A+B>1)}^{(A \,\&\, B>0.5)}$, $\min\left(A^c \wedge B\right)_{(A+B>1)}^{(A \,\&\, B>0.5)}$ and {A = B} all are empty, if and only if $A \wedge B$ is empty; hence, A and B are disjoint.

Proposition 6.28: One of $A \wedge B_{(A+B \leq 1)}$, $\min\left(A \wedge B\right)_{(A+B>1)}^{(A \text{ or } B \leq 0.5)}$, $\min\left(A \wedge B^c\right)_{(A+B>1)}^{(A \,\&\, B>0.5)}$, $\min\left(A^c \wedge B\right)_{(A+B>1)}^{(A \,\&\, B>0.5)}$ or $\{A = B\}$ is non-empty, if and only if $A \wedge B$ is non-empty; hence, A and B are not disjoint.

6.5 An applicable Example

In order to demonstrate the applicability of the proposed solutions for modeling fuzzy topological relations to the real world's GIS problems, here we use the concept of quasi-coincident to detect the effect of SARS (Severe Acute Respiratory Syndrome) disease's distribution to people in a community – a spatially analyzed problem for a GIS. The objectives of this applicable study are:

- To trace the path of an infected person and investigate the effect of this person to the community;
- To investigate the effect of a certain infected region to its neighboring regions by using fuzzy topology; and
- To detect whether a person within a region is safe or not by using fuzzy topology.

6.5.1 Background

SARS has been one of the most serious diseases, which threatens the lives and health of people in many areas of the world, such as Hong Kong, Beijing, Guangdong Province, Singapore and Canada etc. This disease affected the daily life seriously and killed nearly 300 people in Hong Kong within a few months. One of the problems faced to GIS professionals is to detect the spatially distributional patterns of the disease within a community.

Symptoms SARS

Experts believe that the SARS virus is stable in faeces (and urine) at room temperature for at least one to two days. Virus is more stable (up to 4 days) in stool from diarrhoea patients than in normal stool, in where it could only be found for up to 6 hours. The disease spreads from person to person. It often begins with a high fever, headache and sore throat. Other possible symptoms include loss of appetite, confusion, rash and diarrhea. Not everyone has reacted at the same way. The WHO (The World Health Organization) said that doctors are on the lookout for those symptoms with:

- a fever over 38 C; and
- cough, shortness of breath, difficulty breathing.

<u>The way of catching SARS</u>

It is likely that infection takes place through droplets of body fluids - produced by sneezing or coughing. An official report into a mass outbreak in Amoy Garden in Hong Kong concluded that the virus had spread through a sewage pipe. The WHO is not ruling out the possibility that it may also be transmitted when people touch objects such as lift buttons. Airlines insist that an infected person cannot spread the virus throughout an aircraft. However, the WHO states that people sitting within two rows may be at risk.

6.5.2 Data Collection

Precise data on SARS cases plays an important role for further analysis and decisions making. Basically, we need to collect the following data sets:

(a) The base map of the analyzing target region;

(b) The number of new infected buildings for the target region each day;

(c) The number of deaths for the target region each day;

(d) The number of patients in ICU for the target region each day;

(e) For each building of the target region, the number of new infected bodies or the target region each day; and

(f) For each building of the target region, the number of infected bodies for the target region each day.

The base map was based on the data from the website http://www.centamap.com/cent/index.htm. Unfortunately, the detailed information on items from (b) to (f) are still not available for general public at this time. Therefore, we simulated these data for this study only.

6.5.3 Fuzzy topology for SARS analysis

To investigate the effect of a person to a community (Fuzzy line to region)

Let A(x(t)) be the membership function of the trace of an **infected person** (Actually, it is a fuzzy path and stands for the **density** of the SARS virus of the trace of an infected person.). The value equal to zero means the infected body does not contain any diseases and the value equal to one means he/she has very high density of virus. The path of this infected body will affect the community for a long time once he/she traces the public place. Let B(x(t)) be a community (a region), which represents **the condition of certain people or the situation of this point x at time t**, etc, and can be for example the population density function. When B(x(t)) is equal to zero, it means the location is in a very good condition or is a very safe place. On the other hand, one means the location is in a very bad condition or is very dangerous. Therefore, we can investigate the effect of an infected body to the community by using the concept of quasi-coincidence. For a person at a particular location x(t), the case $A(x(t)) + B(x(t)) > 1$ means he/she has a very high probability to be infected. Otherwise, the person has a relative low probability to be infected.

To investigate the effect of an infected region to its neighbor (R to R)

Let A(x(t)) be the membership function (**The density of this virus** which is dependent on the time factor.) of certain virus within a certain region. The value, zero, means the region does not contain any disease and the value, one, means it has very high density of virus. The virus will affect people within the region for a long time once a certain public place has such virus. In normal case, the virus will steady in a public place for a long time and then disappear. It will disappear very soon after the process of sterilization is under taking. Let B(x(t)) be a neighbor region. B(x(t)) represents the **condition of certain people** or the **situation of this point** etc (It is a fuzzy set and is dependent on the time factor), and can be for example the population density function. B(x(t)) is represented as a membership value at each point x and at time t. When B(x(t)) is equal to zero, it means the location is in a very good condition or is a very safe place. On the other hand, one means the location is in a very bad condition or is very dangerous. Therefore, we can investigate the effect of an infected region to its neighbor by using the concept of quasi-coincidence. For a person at a particular location x(t), the case A(x(t)) + B(x(t)) > 1 means he/she has a very high probability to be infected. Otherwise, the person has a relatively low probability to be infected.

To detect whether a person within a region is safe

Figure 6.16 presents the residential regions seriously affected by SARS during the period include two residential areas: Amoy Garden with serious SARS spread at earlier time and Lower Ngau Tau Kok Estate – a neighboring region in danger – potentially to be affected very soon. The target of this modeling is to study the effect of SARS spread from Amoy Garden to the Lower Ngau Tau Kok Estate. On top of the digital map, there is a grid with two values for each of the grids: the upper one (A(x(t))) stands for the **density** of the SARS virus from Amoy Garden, and the lower one (B(x(t))) represents the **conditions** of the Lower Ngau Tau Kok Estate. Both fuzzy sets A(x(t)) and B(x(t)) are dependent on the time factor.

From Figure 5.16, we can see that the Block E is the most seriously infected building (with very high A(x(t)) value, such 0.9), and this trend is steadily reduced when the distance is far from Block E (the A(x(t) value decreased from 0.78 to 0.65, 0.45 etc). On the other hand, the membership function (B(x(t))) for Lower Ngau Tau Kok Estate decreased the center of the Estate with its boundaries, from 0.50, to 0.45, 0.25 etc.

By applying the concept of quasi-coincidence (some relations of quasi-coincidence are illustrated in Figure 6.10 and Figure 6.11), those areas with a high risk to be infected are the areas fulfilled the following condition:

$$A(x(t)) + B(x(t)) > 1$$

The areas fulfilled this condition is indicated by a closed dash line. This means that the people living within this area have a comparatively higher risk to be infected from the SARS spread of the Amoy Garden.

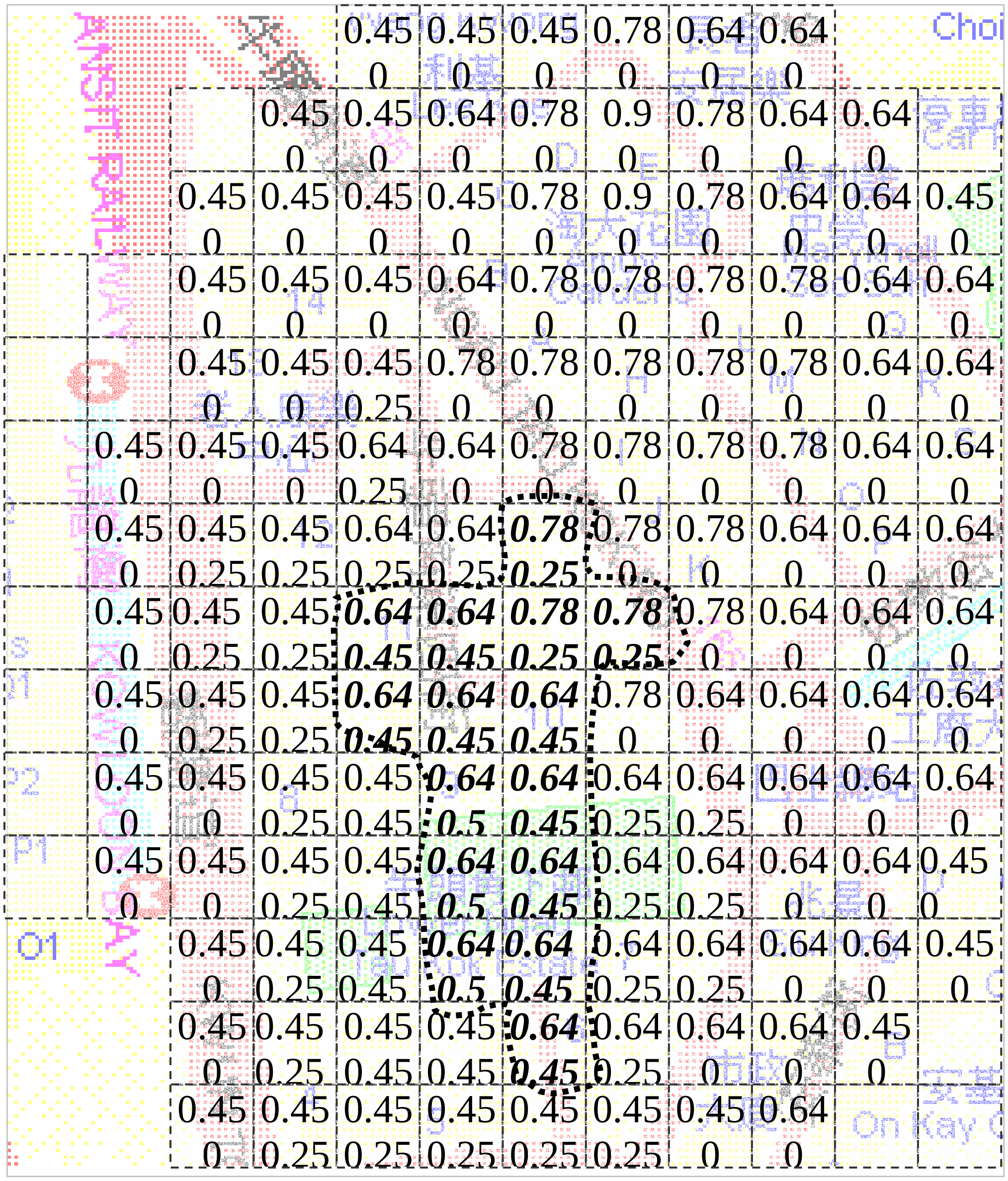


Figure 6.16: The effect of Amoy Garden to Lower Ngau Tau Kok Estate can be computed by the concept of quasi-coincidence

6.6 Summary

Topological relations between objects with indeterminate boundaries have been investigated for several years. Based on the 9-intersection, Cohn and Gotts (1996) gave 46 topological relations between two regions with indeterminate boundaries while Clementini and Di Felice (1996) gave 44 topological relations between two regions with indeterminate boundaries. The usual technique handling these kinds of problems is based on empty and non-empty invariance.

Actually, there are many other useful topological invariants that can be used to study the topological relations between fuzzy sets. In this study, we proposed to use the quasi-coincidence and quasi-difference to indicate fuzzy neighborhood relations between fuzzy objects in GIS.

In this chapter, we first give a basic definition of GIS elements based on fuzzy topology -- fuzzy point, simple fuzzy line and fuzzy region. Then a framework for describing topological relations between two fuzzy objects has been presented. This is based on the quasi-coincidence and quasi-difference. By applying these two concepts, we can obtain a 7-tupled topological relation:

$$(A\backslash\backslash B_o,\ B\backslash\backslash A_o,\ A \wedge B_{(A+B \le 1)},\ \min(A \wedge B)^{(A \text{ or } B \le 0.5)}_{(A+B>1)},\ \min(A \wedge B^c)^{(A \,\&\, B > 0.5)}_{(A+B>1)},\ \min(A^c \wedge B)^{(A \,\&\, B > 0.5)}_{(A+B>1)},\ A = B).$$

This 7-tuple can be immediately used in GIS to (a) describe fuzzy topological relations between two spatial objects, and furthermore (b) to quantify the effect of one fuzzy object to the other fuzzy objects, which is a step further to the traditional fuzzy topological modeling – description only. The proposed solution can describe topological relations between any two arbitrary fuzzy objects without any constrains.

With the quantified relations based on the quasi-coincidence and quasi-difference, we can describe the interaction between fuzzy objects to enable us to study many real worlds' problems, for instance to calculate the effect of SARS spread from one region to another. The example of SARS, has verified several important functions of the 7-tuple, which firstly enable us to study the effect of a fuzzy object to the other fuzzy object. Secondly, the level of topological relations between two fuzzy objects can be quantified by the components of the 7-tupled topological relations.

There are several things on what we have achieved. The first is that the information of the overlapped part of two uncertain objects is described by five topological invariant components, $A \wedge B_{(A+B \le 1)}$, $\min\left(A \wedge B\right)^{(A \text{ or } B \le 0.5)}_{(A+B>1)}$, $\min\left(A \wedge B^c\right)^{(A \,\&\, B > 0.5)}_{(A+B>1)}$, $\min\left(A^c \wedge B\right)^{(A \,\&\, B > 0.5)}_{(A+B>1)}$ and $\{A = B\}$. Thus a clear picture is given to the overlapped part of two uncertain objects. The second achievement is that the information on one fuzzy object is not affected by the other fuzzy object ($A \backslash\backslash B_o$ and $B \backslash\backslash A_o$). Thirdly, for any two given fuzzy objects with memberships function, the 7-tupled topological relations' components can be easily computed.

The pros of the proposed solution include:

- The concepts of quasi-coincidence and quasi-difference are easy to understand.
- The quasi-coincidence and quasi-difference can be directly applied to GIS. Therefore, the theorems and properties of these two concepts can be directly transferred to be a part of the theories for GIS.
- The level structure between two fuzzy objects can be expressed by the concept of quasi-coincidence and quasi-difference.
- The abstract concepts of fuzzy topological properties are represented by functions only, so it is easy to implement.
- The quasi-coincidence and quasi-difference can help to distinguish the relations of arbitrary fuzzy objects without any constrains.

On the other hand, there are also some rooms for further improvement of the research presented in this study, such as,

- Theoretically, this model is quite completed. However, the solution needs to be further adjusted for different cases and for complicated cases in the real world.
- More fuzzy concepts and invariants should be integrated, so that the model becomes more practical and completed.

CHAPTER SEVEN

A FUZZY TOPOLOGICAL SPACE FOR COMPUTING THE INTERIOR, BOUNDARY, AND EXTERIOR OF SPATIAL OBJECTS QUANTITATIVELY IN GIS

In this study, we take the further step of developing a theory for modeling uncertain relations between spatial objects. Specifically, we intend to develop a computational fuzzy topological space, which can potentially be used for computing uncertain spatial relations between spatial objects quantitatively in either GIS, remote sensing, or other areas related to spatial objects.

There are two stages for modeling fuzzy topological relations among spatial objects: (a) to define and describe spatial relations qualitatively and (b) to compute the fuzzy topological relations quantitatively. For the first stage of modeling fuzzy spatial relations, a number of models have been developed (Egenhofer, 1993; Winter, 2000; Cohn and Gotts, 1996; Clementini and Di Felice, 1996; Smith, 1996; Tang and Kainz, 2002; Tang, Kainz and Yu, 2003; Tang, 2004; Tang *et al*, 2005), which can provide a conceptual definition of uncertain topological relations between spatial objects – based on descriptions of the interior, boundary, and exterior of spatial objects in GIS. As for the second stage of the modeling of uncertain topological relations, we need to further develop methods to compute the quantitative values of these topological relations, for instance to compute the membership values of the interior, boundary, and exterior of a spatial object based on fuzzy membership functions. However, the quantitative computation of fuzzy topological values for uncertain relations is still an open issue. Therefore, the aim of this research was identified as developing a fuzzy topological theory that can be used to compute the quantitative values of topological relations. As a result the topological models developed in the previous researches can be practically implemented in a GIS.

In order to develop a useful method for computing the topological relations of the existing topological models (such as the 9-intersection or other topological relations models) and which can be implemented in real-world GIS software, our research is organized into two phases within a comprehensive framework. The aim in the first phase is to define a computational fuzzy topological space to compute the interior, boundary and exterior of spatial objects, which is based on the two operators, the interior operator and closure operator. This part is presented in this chapter. The aim of the second phase is to apply the developed fuzzy topological space to topological relations models (such as the 9-intersection model) to quantitatively computing the fuzzy topological relations between simple fuzzy spatial objects in GIS.

In order to achieve the aim of the first phase of development (which will present in this chapter), the following research developments are presented in this chapter: defining two new operators – interior and closure, which are based on two kinds of level cuts; further defining a computational fuzzy topological space, which will be used to compute the interior, boundary, and closure of fuzzy spatial objects in GIS; and providing an example to show how to use this computational fuzzy topological space to calculate the interior, boundary, and exterior for a real-world data set.

In order to achieve the aim of the second phase of development (which will present in chapter 8 and chapter 9), the following research will be conducted and presented in a separate chapters: defining the connectivity of the new fuzzy topological space; modeling homeomorphic invariants of this new fuzzy topological space – that is, the topological relations that will not be changed under homeomorphic mapping; and, finally, giving a list of qualitative fuzzy topological relations between simple fuzzy spatial objects in GIS.

Every interior or closure operator can actually define a fuzzy topology (Liu and Luo, 1997) separately. Based on this understanding, we can define a fuzzy topological space in which the interior and closure operators are defined by a suitable level cutting. As a result, the interior, boundary, and closure of spatial objects in GIS can be computed by using these two kinds of level cuttings (two operators on a fuzzy set). The topological relations models, such as the 9-intersection models, can thus be implemented in a GIS environment. Figure 7.1 summarizes the two-phase research development framework. Moreover, Figure 7.2 shows the logical flow of this chapter.

<u>Phase One</u>

To construct a computable fuzzy topological space that can effectively implement the interior, boundary, and exterior in a computer environment.

↓

<u>Phase Two</u>

To apply the developed fuzzy topological space for topological models, such as the 9-intersection model, in order to quantitatively compute topological relations in GIS.

Figure 7.1: A summary of the two-phase research development plan

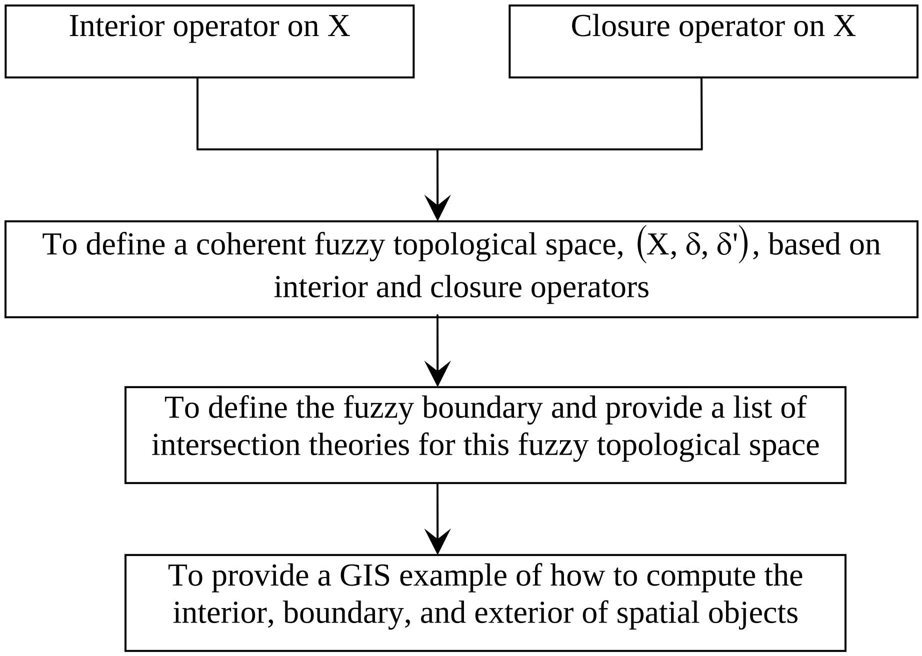


Figure 7.2: The logical flow of this chapter

7.1 Fuzzy topological space induced by interior and closure operators

Recall the definition of interior operator is that an operator $\alpha : I^x \to I^x$ is a fuzzy interior operator if the following conditions are satisfied: (i) $\alpha(1) = 1$; (ii) $\alpha(A) \le A$, for all $A \in I^x$; (iii) $\alpha(A \wedge B) = \alpha(A) \wedge \alpha(B)$, for all $A,\ B \in I^x$; (iv) $\alpha(\alpha(A)) = \alpha(A)$, for all $A \in I^x$. The definition of closure operator is that An operator $\alpha : I^x \to I^x$ is a fuzzy closure operator if the following conditions are satisfied: (i) $\alpha(0) = 0$; (ii) $A \le \alpha(A)$, for all $A \in I^x$; (iii) $\alpha(A \vee B) = \alpha(A) \vee \alpha(B)$, for all $A,\ B \in I^x$; (iv) $\alpha(\alpha(A)) = \alpha(A)$, for all $A \in I^x$.

Remark 1: With regard to the closure operator, we can consider this operator to be a machine that works on fuzzy sets. The condition $\alpha(0) = 0$ means that when the input is empty fuzzy set, the output is also an empty fuzzy set (see Figure 7.3). The condition $A \le \alpha(A)$ means that the operator will enlarge the inputted fuzzy set. The condition

$\alpha(A \vee B) = \alpha(A) \vee \alpha(B)$ means that the operator will be not affected by the join relation. The last condition $\alpha(\alpha(A)) = \alpha(A)$ means that operating the fuzzy set two times is equal to one time. The interior operator has similar characteristics.

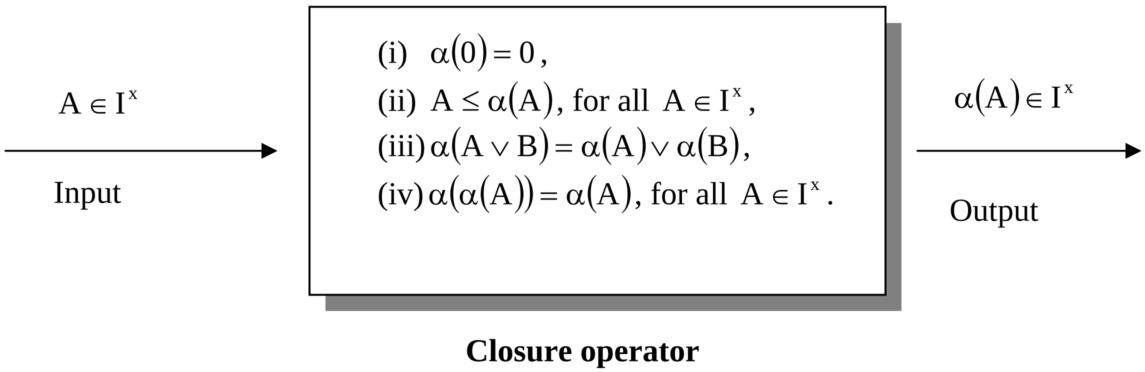


Figure 7.3: The concept of the closure operator

Here we try to via this direction to define two operators, one is interior and the other is closure operator. We want these two operators can manipulate with each other so that they can further define a same fuzzy topological space, which the open sets are the image of the interior operator and the closed sets are the image of the closure operator. On the same time, we also want the complement of an open set of interior operator, is exactly the closed set of closure operator.

We know that each interior operator corresponds to one fuzzy topological space and each closure operator corresponds to one fuzzy topological space (Liu and Luo, 1997) (see Figure 7.4). In general, if we define two operators, interior and closure, separately, they will define two fuzzy topologies, respectively. These two topologies may not cohere with each other. That is, the open set defined by the interior operator may not be the complement of the closed set, which is defined by the closure operator. Therefore, we try to further define two operators, one is an interior operator and the other is a closure operator target, to define a computable fuzzy topological spaces respectively. We want these two operators to be able to cohere with each other so that they can define the same fuzzy topological space, in which the open sets are the image of the interior operator and the closed sets are the image of the closure operator. At the same time, we want the complement of an open set to be a closed set (see Figure 7.5).

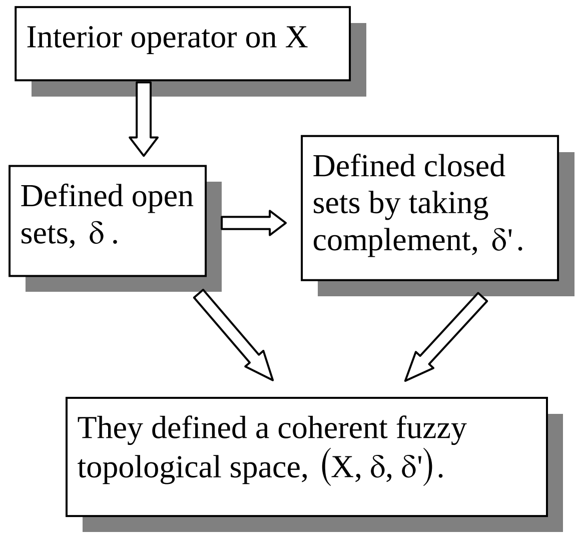


Figure 7.4(a): Fuzzy topological space defined by the interior operator

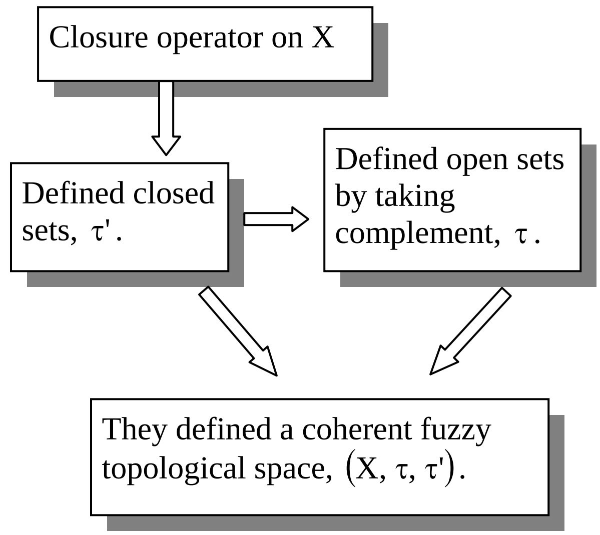


Figure 7.4(b): Fuzzy topological space defined by the closure operator

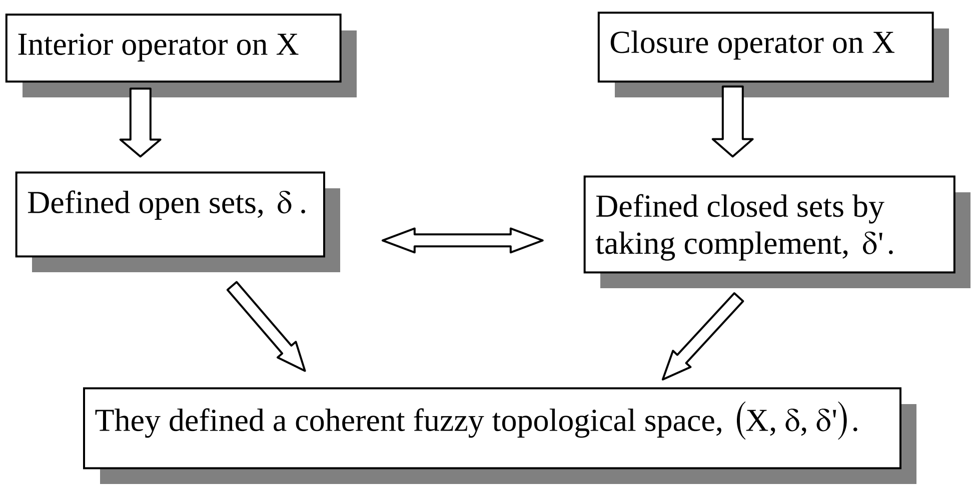


Figure 7.5: The interior operator and closure operator are defined by a coherent fuzzy topological space

The following two definitions are about the interior and closure operators, which are coherent with each other in defining a fuzzy topological space.

Definition 7.1 (Interior and Closure operator): Let A be a fuzzy set in $[0,\ 1]^x = I^x$. For any fixed $\alpha \in [0,\ 1]$, define two operators on $[0,\ 1]^x = I^x$ as follows:

$$A \xrightarrow{\bar{\alpha}} A^{\alpha} \in I^x \qquad \text{and}$$

$$A \xrightarrow{\underline{\alpha}} A_{\alpha} \in I^x \qquad ,$$

where the fuzzy sets A^{α} and A_{α} in X are defined by:

$$A^{\alpha}(x) = \begin{cases} 1 & \text{if } A(x) \geq \alpha \\ A(x) & \text{if } A(x) < \alpha \end{cases} \qquad \text{and}$$

$$A_{\alpha}(x) = \begin{cases} A(x) & \text{if } A(x) > \alpha \\ 0 & \text{if } A(x) \leq \alpha \end{cases}$$

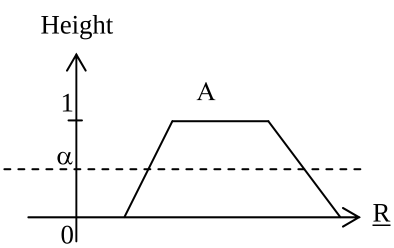


Figure 7.6(a): The fuzzy set A in **R**

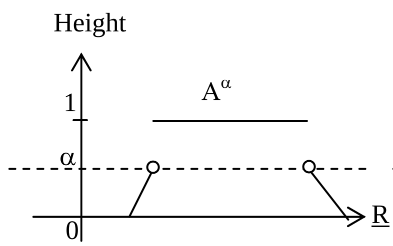


Figure 7.6(b): The fuzzy set A^{α} in **R**

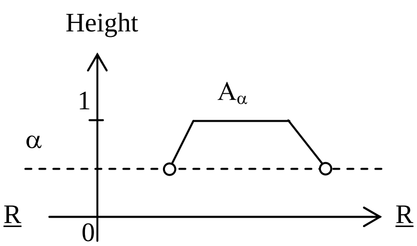


Figure 7.6(c): The fuzzy set A_{α} in **R**

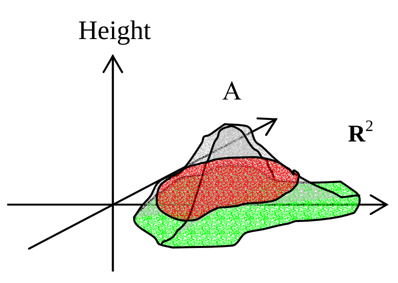


Figure 7.7(a): The fuzzy set A in $\mathbf{R}^2$

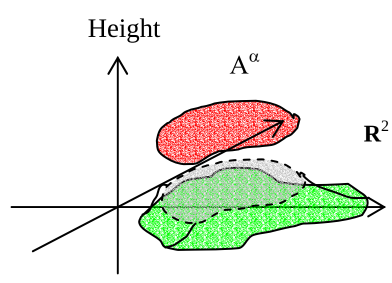


Figure 7.7(b): The fuzzy set A^{α} in $\mathbf{R}^2$

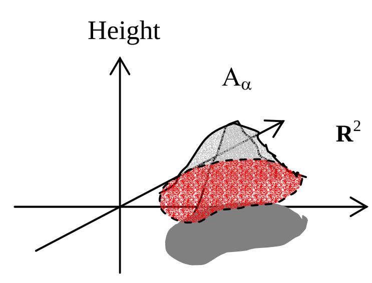


Figure 7.7(c): The fuzzy set A_{α} in $\mathbf{R}^2$

Remark 2: The geometric interpretation of the closure operator is that it raises up all fuzzy membership values greater than α to one. The geometric interpretation of the interior operator is that it cuts all fuzzy membership values that are less than or equal to α (see Figure 7.6 and Figure 7.7). **The following proposition is several important properties of these two operators. It will be used later to prove that they further define a new fuzzy topological space**.

Proposition 7.2: Let A, B, and A_i ($i \in \Lambda$) be fuzzy sets of I^X. Then the following hold for all $\alpha \in [0, 1]$;

(i) $0^{\alpha} = 0 = 0_{\alpha}$ and $1^{\alpha} = 1 = 1_{\alpha}$;

(ii) $A_{\alpha} \le A \le A^{\alpha}$;

(iii) $A \le B \Rightarrow A^{\alpha} \le B^{\alpha}$ and $A_{\alpha} \le B_{\alpha}$;

(iv) $(A^{\alpha})^{\alpha} = A^{\alpha}$ and $(A_{\alpha})_{\alpha} = A_{\alpha}$;

(v) $(A^{\alpha})^{c} = (A^{c})_{1-\alpha}$ and $(A_{\alpha})^{c} = (A^{c})^{1-\alpha}$;

(vi) $\alpha \le \beta \Rightarrow A^{\beta} \le A^{\alpha}$ and $(A^{\alpha})^{\beta} = A^{\beta}$;

(vii) $\alpha \le \beta \Rightarrow A_{\beta} \le A_{\alpha}$ and $(A_{\beta})_{\alpha} = A_{\beta}$;

(viii) If Λ is finite, then $\left(\vee_{i \in \Lambda} A_i\right)^{\alpha} = \vee_{i \in \Lambda} (A_i)^{\alpha}$ and $\left(\wedge_{i \in \Lambda} A_i\right)_{\alpha} = \wedge_{i \in \Lambda} (A_i)_{\alpha}$;

(ix) $\left(\wedge_{i \in \Lambda} A_i\right)^{\alpha} = \wedge_{i \in \Lambda} \left(A_i^{\alpha}\right)$ and $\left(\vee_{i \in \Lambda} A_i\right)_{\alpha} = \vee_{i \in \Lambda} \left(A_{i\alpha}\right)$;

(x) $A_{\alpha} \le A \le A^{1-\alpha}$.

Proof: (i) and (ii) are trivial facts. The geometric interpretation of $A_{\alpha} \le A$ is that A_{α} is exactly equal to the cutting of the tail of A. The geometric interpretation of $A \le A^{\alpha}$ is that A is exactly equal to A^{α} unless the raising part (see Figure 7.8).

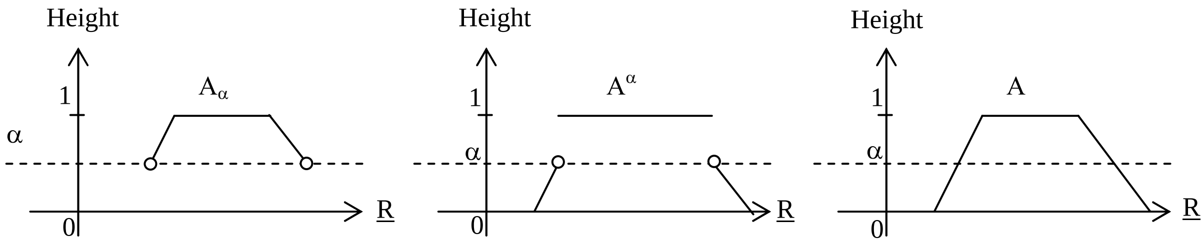


Figure 7.8: An illustration of $A_\alpha \le A \le A^\alpha$

(iii): For all $x \in X$,

$A(x) \ge \alpha \;\Rightarrow\; B(x) \ge \alpha$. So $\{x \in X : A(x) \ge \alpha\} \subseteq \{x \in X : B(x) \ge \alpha\}$.

$A(x) < \alpha \;\Rightarrow\; A(x) \le \min(B(x),\ 1)$.

Combine these two we have $A^\alpha(x) \le B^\alpha(x)$ for all $x \in X$.

So $A^\alpha \le B^\alpha$.

Similarly, we have $A_\alpha \le B_\alpha$.

(iv): For all $x \in X$,

$A(x) \ge \alpha \;\Rightarrow\; A^\alpha(x) = 1 \;\Rightarrow\; A^\alpha(x) \ge \alpha \;\Rightarrow (A^\alpha)^\alpha(x) = 1$.

$A(x) < \alpha \;\Rightarrow\; A(x) = A^\alpha(x) < \alpha \;\Rightarrow\; (A^\alpha)^\alpha(x) = A^\alpha(x) = A(x)$.

So $(A^\alpha)^\alpha = A^\alpha$.

Similarly, we have $(A_\alpha)_\alpha = A_\alpha$.

The geometric interpretation of $(A^\alpha)^\alpha = A^\alpha$ is that $(A^\alpha)^\alpha$ cannot have any extra raising part after raised (see Figure 7.9(a)). The geometric interpretation of $(A_\alpha)_\alpha = A_\alpha$ is that $(A_\alpha)_\alpha$ cannot be cutting after cutting (see Figure 7.9(b)).

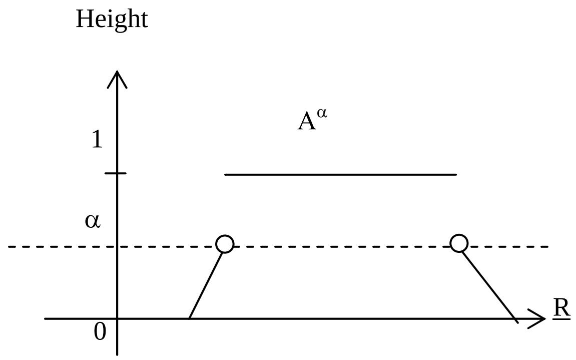


Figure 7.9(a): geometric interpretation of $(A^\alpha)^\alpha = A^\alpha$

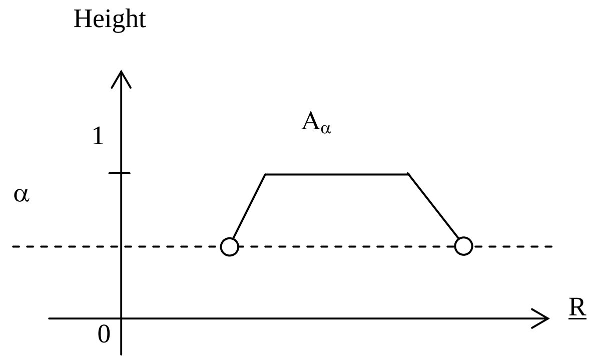


Figure 7.9(b): geometric interpretation of $(A_\alpha)_\alpha = A_\alpha$

(v): $(A^\alpha)^c(x) = (1 - A^\alpha)(x) = 1 - A^\alpha(x) = \begin{cases} 0 & \text{if } A(x) \geq \alpha \\ 1 - A(x) & \text{if } A(x) < \alpha \end{cases}$ and

$$(A^c)_{1-\alpha}(x) = \begin{cases} A^c(x) & \text{if } A^c(x) > 1-\alpha \\ 0 & \text{if } A^c(x) \leq 1-\alpha \end{cases}$$

$$= \begin{cases} 1 - A(x) & \text{if } 1 - A(x) > 1-\alpha \\ 0 & \text{if } 1 - A(x) \leq 1-\alpha \end{cases}$$

$$= \begin{cases} 1 - A(x) & \text{if } A(x) < \alpha \\ 0 & \text{if } A(x) \geq \alpha \end{cases}.$$

Hence $(A^\alpha)^c = (A^c)_{1-\alpha}$. The fact $(A_\alpha)^c = (A^c)^{1-\alpha}$ is similar.

The result $(A^\alpha)^c = (A^c)_{1-\alpha}$ allows us to define the complement of a closed set is open makes sense while $(A_\alpha)^c = (A^c)^{1-\alpha}$ do the complement of a open set is closed makes sense.

(vi): If $\alpha \leq \beta$, then $A(x) \geq \beta \Rightarrow A(x) \geq \alpha$. Hence $A^\beta(x) = 1 \Rightarrow A^\alpha(x) = 1$. So, if $A(x) \geq \alpha$ then $A^\beta(x) \leq A^\alpha(x)$.

For second statement, if $A(x) < \alpha$, then $A(x) < \beta$, hence $A^\beta(x) = A^\alpha(x) = A(x)$.

For all x in X, $A^\alpha(x) \leq (A^\alpha)^\beta(x) \leq (A^\alpha)^\alpha(x) = A^\alpha(x)$. Thus $(A^\alpha)^\beta = A^\beta$.

(vii): Similar to (vi).

(viii): $A_i(x) \geq \alpha$ for some $i \in \Lambda \Leftrightarrow \underset{i\in\Lambda}{\vee} A_i(x) \geq \alpha$.

$\therefore \quad A_i^{\alpha}(x) = 1 \Leftrightarrow \left(\underset{i\in\Lambda}{\vee} A_i\right)^{\alpha}(x) = 1$.

Therefore, $\underset{i\in\Lambda}{\vee}\left(A_i^{\alpha}(x)\right) = \left(\underset{i\in\Lambda}{\vee} A_i\right)^{\alpha}(x)$ for $\underset{i\in\Lambda}{\vee} A_i(x) \geq \alpha$.

On the other hand, $A_i(x) < \alpha$ for all $i \in \Lambda \Leftrightarrow \underset{i\in\Lambda}{\vee} A_i(x) < \alpha$.

Therefore, $A_i(x) < \alpha$ for all $i \in \Lambda \Rightarrow \underset{i\in\Lambda}{\vee}\left(A_i^{\alpha}(x)\right) = \underset{i\in\Lambda}{\vee} A_i(x)$ and;

$\underset{i\in\Lambda}{\vee} A_i(x) < \alpha \Rightarrow \left(\underset{i\in\Lambda}{\vee} A_i\right)^{\alpha}(x) = \underset{i\in\Lambda}{\vee} A_i(x)$.

Hence $\underset{i\in\Lambda}{\vee}\left(A_i^{\alpha}(x)\right) = \left(\underset{i\in\Lambda}{\vee} A_i\right)^{\alpha}(x)$ for all x, i.e. $\underset{i\in\Lambda}{\vee}\left(A_i^{\alpha}\right) = \left(\underset{i\in\Lambda}{\vee} A_i\right)^{\alpha}$.

For the second statement, $\left(\underset{i\in\Lambda}{\vee}\left(A_i^{\alpha}\right)\right)^c = \underset{i\in\Lambda}{\wedge}\left(A_i^{\alpha}\right)^c = \underset{i\in\Lambda}{\wedge}\left(A_i^{c}\right)_{1-\alpha}$ and $\left(\left(\underset{i\in\Lambda}{\vee} A_i\right)^{\alpha}\right)^c = \left(\left(\underset{i\in\Lambda}{\vee} A_i\right)^c\right)_{1-\alpha} = \left(\underset{i\in\Lambda}{\wedge} A_i^{c}\right)_{1-\alpha}$. Therefore, $\underset{i\in\Lambda}{\wedge}\left(A_i^{c}\right)_{1-\alpha} = \left(\underset{i\in\Lambda}{\wedge} A_i^{c}\right)_{1-\alpha}$ and replace A_i^c and $1-\alpha$ by A_i and α respectively, then we get the desired result.

The results of (viii) makes both finite union of closed sets is closed and finite intersection of open sets is open to be well defined.

(ix): $\underset{i\in\Lambda}{\wedge} A_i(x) < \alpha \Rightarrow A_i(x) < \alpha$ for some $i \in \Lambda \Rightarrow A_i^{\alpha}(x) = A_i(x) < \alpha$ for some $i \in \Lambda \Rightarrow \underset{i\in\Lambda}{\wedge}\left(A_i^{\alpha}(x)\right) = \underset{i\in\Lambda}{\wedge} A_i(x) = \left(\underset{i\in\Lambda}{\wedge} A_i\right)^{\alpha}(x)$.

$\underset{i\in\Lambda}{\wedge} A_i(x) \geq \alpha \Rightarrow A_i(x) \geq \alpha$ for all $i \in \Lambda \Rightarrow A_i^{\alpha}(x) = 1$ for all $i \in \Lambda \Rightarrow \underset{i\in\Lambda}{\wedge}\left(A_i^{\alpha}(x)\right) = 1$.

On the other hand, $\underset{i\in\Lambda}{\wedge} A_i(x) \geq \alpha \Rightarrow \left(\underset{i\in\Lambda}{\wedge} A_i\right)^{\alpha}(x) = 1$.

This proved $\left(\underset{i\in\Lambda}{\wedge} A_i\right)^{\alpha} = \underset{i\in\Lambda}{\wedge}\left(A_i^{\alpha}\right)$.

$\underset{i\in\Lambda}{\vee} A_i > \alpha \;\Rightarrow\; A_i(x) > \alpha$ for some $i \in \Lambda \;\Rightarrow\; A_{i\,\alpha}(x) = A_i(x) > \alpha$ for some $i \in \Lambda$

$\Rightarrow\; \underset{i\in\Lambda}{\vee}\left(A_{i\,\alpha}(x)\right) = \underset{i\in\Lambda}{\vee} A_i(x) = \left(\underset{i\in\Lambda}{\vee} A_i\right)_\alpha(x)$.

$\underset{i\in\Lambda}{\vee} A_i \le \alpha \;\Rightarrow\; A_i(x) \le \alpha$ for all $i \in \Lambda \;\Rightarrow\; A_{i\,\alpha}(x) = 0$ for all $i \in \Lambda \;\Rightarrow$

$\underset{i\in\Lambda}{\vee}\left(A_{i\,\alpha}(x)\right) = 0 = \left(\underset{i\in\Lambda}{\vee} A_i\right)_\alpha(x)$.

This proved $\left(\underset{i\in\Lambda}{\vee} A_i\right)_\alpha = \underset{i\in\Lambda}{\vee}\left(A_{i\,\alpha}\right)$.

The results of (ix) make arbitrary intersection of close sets is closed and arbitrary union of open sets is open to be well defined.

(x): For all $x \in X$, $A_\alpha(x) \le A(x)$ and $A(x) \le A^{1-\alpha}(x)$.

Q.E.D.

Proposition 7.3: The mappings $\overline{\alpha}$ and $\underline{\alpha}$ are closure and interior operator respectively.

Proof: For $\overline{\alpha}$, we have to check the following definitions:

(i) $\overline{\alpha}(1) = 1$,

(ii) $\overline{\alpha}(A) \le A$, for all $A \in I^x$,

(iii) $\overline{\alpha}(A \wedge B) = \overline{\alpha}(A) \wedge \overline{\alpha}(B)$, for all $A,\ B \in I^x$.

(iv) $\overline{\alpha}\left(\overline{\alpha}(A)\right) = \overline{\alpha}(A)$, for all $A \in I^x$.

For $\underline{\alpha}$, we have to check the definitions:

(i) $\underline{\alpha}(0) = 0$,

(ii) $A \le \underline{\alpha}(A)$, for all $A \in I^x$,

(iii) $\underline{\alpha}(A \vee B) = \underline{\alpha}(A) \vee \underline{\alpha}(B)$, for all $A,\ B \in I^x$.

(iv) $\underline{\alpha}\left(\underline{\alpha}(A)\right) = \underline{\alpha}(A)$, for all $A \in I^x$.

Q.E.D.

Remark 3: According to the above proposition, for any fuzzy set $A \in I^x$, $A \in I^x$ is close if and only if $\overline{\alpha}(A) = A$. That is, the fuzzy topological space induced by $\overline{\alpha}$ is the collection of $\tau^{\alpha} = \{A^{\alpha} : A \in I^{X}\}$. On the other hand, $A \in I^{x}$ is open if and only if $\underline{\alpha}(A) = A$. That is, the fuzzy topological space induced by $\underline{\alpha}$ is the collection of $\tau_{\alpha} = \{A_{\alpha} : A \in I^{X}\}$. According to Liu and Luo (1997), the family of all the fuzzy topologies on X is one-one corresponding with the family of all interior and closure operators, respectively. But it seems that these two operators only define two fuzzy topologies separately, and they do not yet cohere with each other.

Remark 4: The results in proposition 7.2 allow us to define a new fuzzy topological space. Indeed, a fuzzy topological space (I^{x}, δ) on X satisfies the conditions (a) 0, $1 \in \delta$; (b) if A, $B \in \delta$, then $A \wedge B \in \delta$, (c) let $\{A_{i} : i \in J\} \subset \delta$, where J is an index set, then $\vee_{i \in J} A_{i} \in \delta$. The elements in δ are called open elements and the elements in the complement of T are called closed elements. The result in proposition 7.2(i) allows (a) to be defined, proposition 7.2(viii) allows (b) to be defined, and proposition 7.2(ix) allows (c) to be defined. Moreover, 7.2(v) makes the interior and closure operators coherent with each other. We will give a process and proof in proposition 7.5 such that these two operators actually defined a coherent fuzzy topological space.

Remark 5: Pascali and Ajmal (1997) also defined two similar operators, interior and closure operators. In their definitions, these two operators may not further define a coherent fuzzy topology, and they give a necessary and sufficient condition for these two operators to be coherent with each other. Those are the necessary and sufficient conditions for these two operators to define an identical fuzzy topological space.

Remark 6: The α-cut operators are not necessary defining a fuzzy topological spaces (Pascali and Ajmal, 1997).

Definition 7.4: For and $1 \geq \alpha > 0$, define $\tau^{\alpha} = \{A^{\alpha} : A \in I^{x}\}$ and $\tau_{\alpha} = \{A_{\alpha} : A \in I^{x}\}$, which the former set is closed while the later set is open. In the previous chapters, we use the notations A^{o} and $\overline{A}$ to denote the open set and closed respectively. Here, since the open set and closed are dependent on the α cutting. Therefore, A^{α} and A_{α} are the open set and closed which are induced by interior and closure operators respectively.

Proposition 7.5: The triple tuple $\left(X, \tau_{\alpha}, \tau^{1-\alpha}\right)$ is a fts, where τ_{α} is the open sets and $\tau^{1-\alpha}$ is the closed sets which satisfy $\left(A_{\alpha}\right)^{c} = \left(A^{c}\right)^{1-\alpha}$, i.e. the complement of τ_{α} is the collection of closed sets. This proposition shows how combine these two α cutting and then from a fuzzy topology.

Proof: From the result in proposition 7.3, we get the three axioms of fuzzy topological space.

(1) $0^{1-\alpha} = 0 = 0_{\alpha}$ and $1^{1-\alpha} = 1 = 1_{\alpha}$ show that 0, 1 are both elements of τ_{α} and $\tau^{1-\alpha}$.

(2) $\left(\underset{i\in\Lambda}{\wedge} A_i\right)_{\alpha} = \underset{i\in\Lambda}{\wedge}\left(A_i\right)_{\alpha}$, where Λ is finite, shows that the finite intersection of δ is also in δ.

(3) Finally, $\left(\underset{i\in\Lambda}{\vee} A_i\right)_{\alpha} = \underset{i\in\Lambda}{\vee}\left(A_i\right)_{\alpha}$ shows that the union of δ is also in δ.

Q.E.D.

Proposition 7.6: Let A, B and A_i ($i \in \Lambda$) be fuzzy sets of I^X. Then the following hold for all $\alpha \in [0, 1]$;

(i) A is open if and only if the ordinary set $S = \{x : \alpha \geq A(x) > 0\}$ is empty if and only if either $A(x) > \alpha$ or $A(x) = 0$, for all $x \in X$.

(ii) $A(x) \leq \alpha$, for all $x \in X$, if and only if A has empty interior.

(iii) A is closed if and only if the ordinary set $S = \{x : 1 > A(x) \geq 1 - \alpha\}$ is empty iff either $A(x) = 1$ or $A(x) < \alpha$.

Proof:

(i): A is open if and only if $A = A_{\alpha}$ if and only if $A(x) = A_{\alpha}(x)$ for all $x \in X$ if and only if the ordinary set $S = \{x : \alpha \geq A(x) > 0\}$ is empty if and only if either $A(x) > \alpha$ or $A(x) = 0$, for all $x \in X$.

(ii): Trivial.

(iii): By definition.

Q.E.D.

So far, we have achieved our first stage, in which we defined two operators, one the interior and the other the closure operator. The two operators cohere with each other. The first entry of the triple $\left(X, \tau_\alpha, \tau^{1-\alpha}\right)$ index is the base set, the second entry index is the close set and the third entry index is the open set. By proposition 7.3(v), we get the manipulation of the interior and closure operator.

Remark 7: The fuzzy line and fuzzy point may have a non-empty interior. For ordinary topological space, as we know that points and lines in two-dimensional space have an empty interior, when talking about topological relations, this has caused confusion in the past. Therefore, we should treat these cases carefully. For the new fuzzy topological space, the following example tells us that we do not need to worry about such cases. The main reason for this is that this fuzzy topological space is defined by leveling only. Here, we do not consider the neighborhood structure at the same level. We only talk about the leveling neighborhood structure. In this case, even with regard to a point or a line segment with a membership function, the interior is still non-empty. The following is an example.

Example 1: Let $\alpha = \frac{1}{4}$ and define a fuzzy line $L : R^2 \to [0, 1]$ by

$$L(x, y) = \begin{cases} x + y & \text{if } x = y \text{ and } 0 < x, y \le \frac{1}{2} \\ 2 - x - y & \text{if } x = y \text{ and } \frac{1}{2} < x, y < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Then $\left\{(x, y) : \frac{1}{8} < x, y < \frac{7}{8}\right\}$ is the interior of L (see Figure 7.10).

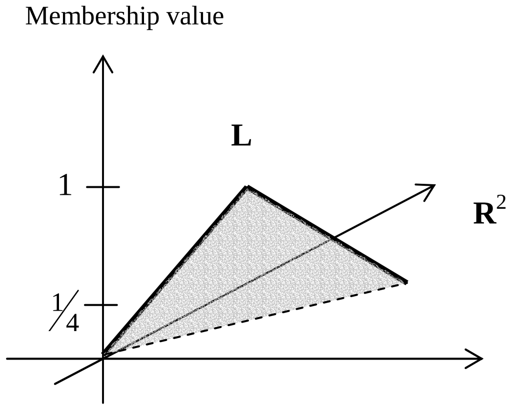


Figure 7.10(a): The fuzzy line L in $\mathbf{R}^2$.

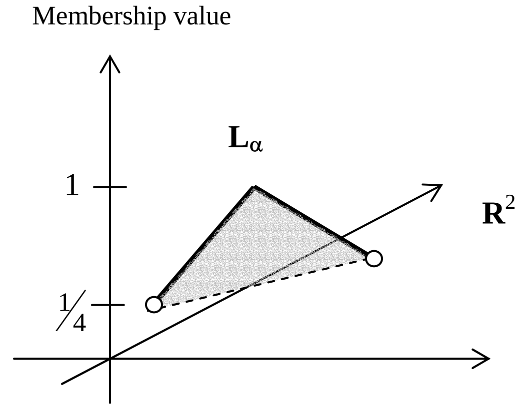


Figure 7.10(b): The fuzzy line in $\mathbf{R}^2$ with a non-empty interior L_α.

Proposition 7.7: If $A^{1-\alpha}(x) = A(x) < 1$, then $(A^c)_\alpha(x) = A^c(x)$.

Proof: $A^{1-\alpha}(x) = A(x) < 1 \Rightarrow 0 < A(x) < 1-\alpha \Rightarrow A^c(x) > \alpha \Rightarrow (A^c)_\alpha(x) = A^c(x)$.

Q.E.D.

Proposition 7.8: If $A_\alpha(x) = A(x) > 0$, then $(A^c)^{1-\alpha}(x) = A^c(x)$.

Proof: $A_\alpha(x) > 0 \Rightarrow A(x) > \alpha \Rightarrow 1 - A(x) < 1-\alpha \Rightarrow A^c(x) < 1-\alpha \Rightarrow (A^c)^{1-\alpha}(x) = A^c(x)$.

Q.E.D.

7.2 Fuzzy boundary and the intersection theory

In ordinary topological space, when we define a topological space, the boundary of a set A is defined as the intersection of the closure of A with the closure of the complement of A. That is, $\partial A = A^o \wedge (A^c)^o$. On the other hand, it has an equivalent definition; that is, $\partial A = \overline{A} - A^o$. Unfortunately, it is no longer true in fuzzy topology (Liu and Lao, 1997; Tang and Kainz, 2002; Wu and Zheng, 1991). However, to be consistent with the previous studies, we adopt the former as the definition of fuzzy boundary.

Definition 7.9 (Fuzzy boundary): For $1 > \alpha > 0$, define the boundary of a fuzzy set A as $\partial A = A^{1-\alpha} \wedge (A^c)^{1-\alpha}$.

Proposition 7.10: $\partial A = A^{1-\alpha} \wedge (A_\alpha)^c$.

Proof: By proposition 7.2(v).

Q.E.D.

Proposition 7.11: $\partial A(x) = 0$ if and only if $A(x) = 1$ or $A(x) = 0$. Hence $\partial A = \phi$ iff A is a crisp set.

Proof: $\partial A(x) = 0 \iff$ if $A^{1-\alpha}(x) = 0$ or $(A_\alpha)^c(x) = 0 \iff A(x) = 0$ or $A_\alpha(x) = 1 \iff$ $A(x) = 0$ or $A(x) = 1$.

$\partial A = \phi \iff \partial A(x) = 0$ for all $x \in X \iff A(x) = 0$ or $A(x) = 1$ for all $x \in X \iff$ A is a crisp set.

Q.E.D.

Proposition 7.12: If $\alpha < \frac{1}{2}$, then $\partial A(x) < 1 - \alpha$.

Proof: As $\alpha < \frac{1}{2} \Rightarrow \alpha < 1 - \alpha$.

If $A^{1-\alpha}(x) \geq 1 - \alpha \Rightarrow A(x) \geq 1 - \alpha \Rightarrow 1 - A(x) \leq \alpha < 1 - \alpha \Rightarrow A^c(x) < 1 - \alpha \Rightarrow$ $(A^c)^{1-\alpha}(x) = A^c(x) < 1 - \alpha$.

If $(A^c)^{1-\alpha}(x) \geq 1 - \alpha \Rightarrow A^c(x) \geq 1 - \alpha \Rightarrow 1 - A(x) \geq 1 - \alpha \Rightarrow A(x) \leq \alpha < 1 - \alpha \Rightarrow$ $A^{1-\alpha}(x) < 1 - \alpha$.

Q.E.D.

Remark 8: Form proposition 7.12, when $\alpha < 1 - \alpha$ (or $\alpha < \frac{1}{2}$), we can see that the fuzzy value of the boundary is less than one. Figure 7.11(a) shows that when $\alpha < 1 - \alpha$, the value of the boundary is less than α. But for $\alpha > 1-\alpha$, the fuzzy value of the boundary may be one (see Figure 7.11(b)).

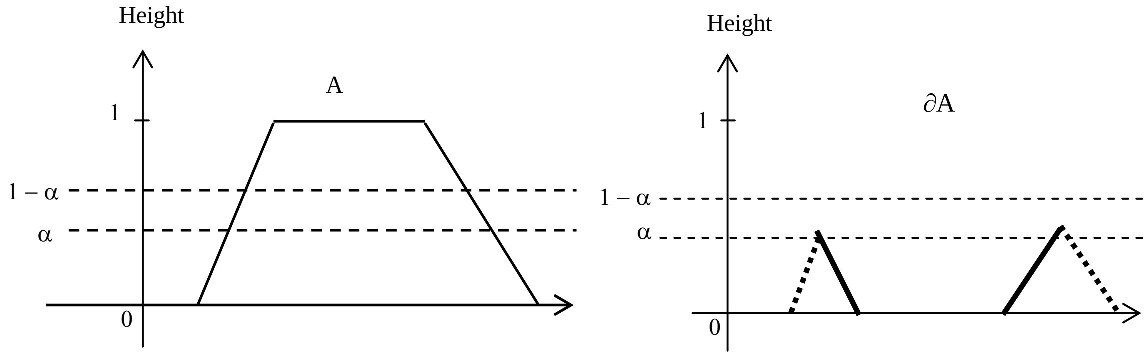


Figure 7.11(a): The fuzzy boundary of A for $\alpha < \frac{1}{2}$

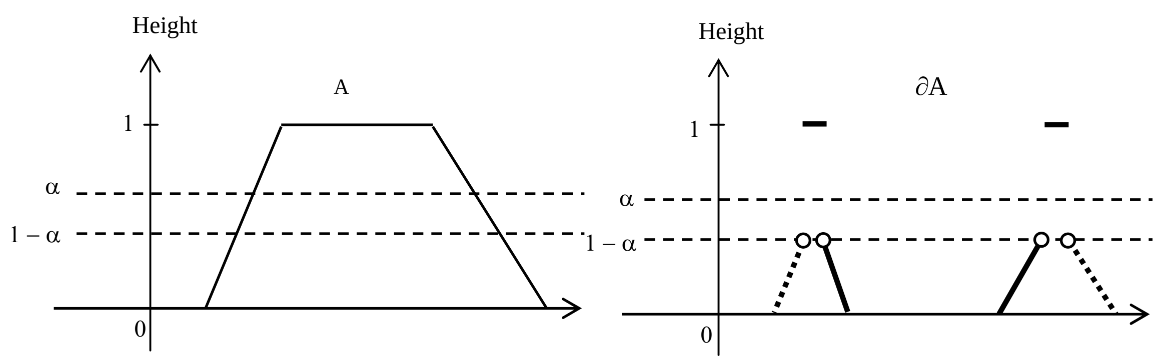


Figure 7.11(b): The fuzzy boundary of A for $\alpha > \frac{1}{2}$

Proposition 7.13: If $1 > \alpha \geq \frac{1}{2}$, then the meet of the interior and the interior of the complement is empty, i.e. $A_\alpha \wedge \left(A^c\right)_\alpha = \phi$.

Proof: $A_\alpha(x) \neq 0$ if and only if $A(x) > \alpha$, $\Rightarrow$ $1 - A(x) < 1 - \alpha$ $\Rightarrow$ $A^c(x) < 1 - \alpha \leq \alpha$ $\Rightarrow$ $\left(A^c\right)_\alpha(x) = 0$.

Q.E.D.

Proposition 7.14: $\left(A_\alpha \wedge \partial A\right)(x) < 1 - \alpha$ and $\left(\left(A^c\right)_\alpha \wedge \partial A\right)(x) < 1 - \alpha$.

Proof: If $A_\alpha(x) \neq 0 \Rightarrow A(x) > \alpha \Rightarrow 1 - A(x) < 1 - \alpha \Rightarrow A^c(x) < 1 - \alpha \Rightarrow$ $(A^c)^{1-\alpha}(x) < 1 - \alpha \Rightarrow \left(A_\alpha \wedge A^{1-\alpha} \wedge (A^c)^{1-\alpha}\right)(x) < 1 - \alpha \Rightarrow (A_\alpha \wedge \partial A)(x) < 1 - \alpha$.

The other statement is similar.

Q.E.D.

Remark 9: From propositions 7.13 and 7.14, we can see that in fuzzy topological space, fuzzy set $A \in I^X$ can be decomposed into three components, interior (A_α), boundary (∂A), and exterior ($(A^c)_\alpha$) and that they are not disjointed. Only the interior and exterior are disjointed (i.e., $A_\alpha \wedge (A^c)_\alpha = \phi$). The intersections of the interior and boundary, and the exterior and boundary are bounded above by $1 - \alpha$. That is, $(A_\alpha \wedge \partial A)(x) < 1 - \alpha$ and $((A^c)_\alpha \wedge \partial A)(x) < 1 - \alpha$. The former approaches (Cohn and Gotts, 1996; Clementini and Di Felice, 1996; Tang and Kainz, 2002) assume that such intersections are empty. This is one way in which our result is different from them. Figure 7.12 illustrates the supported structure of the relations between the interior, boundary, and exterior.

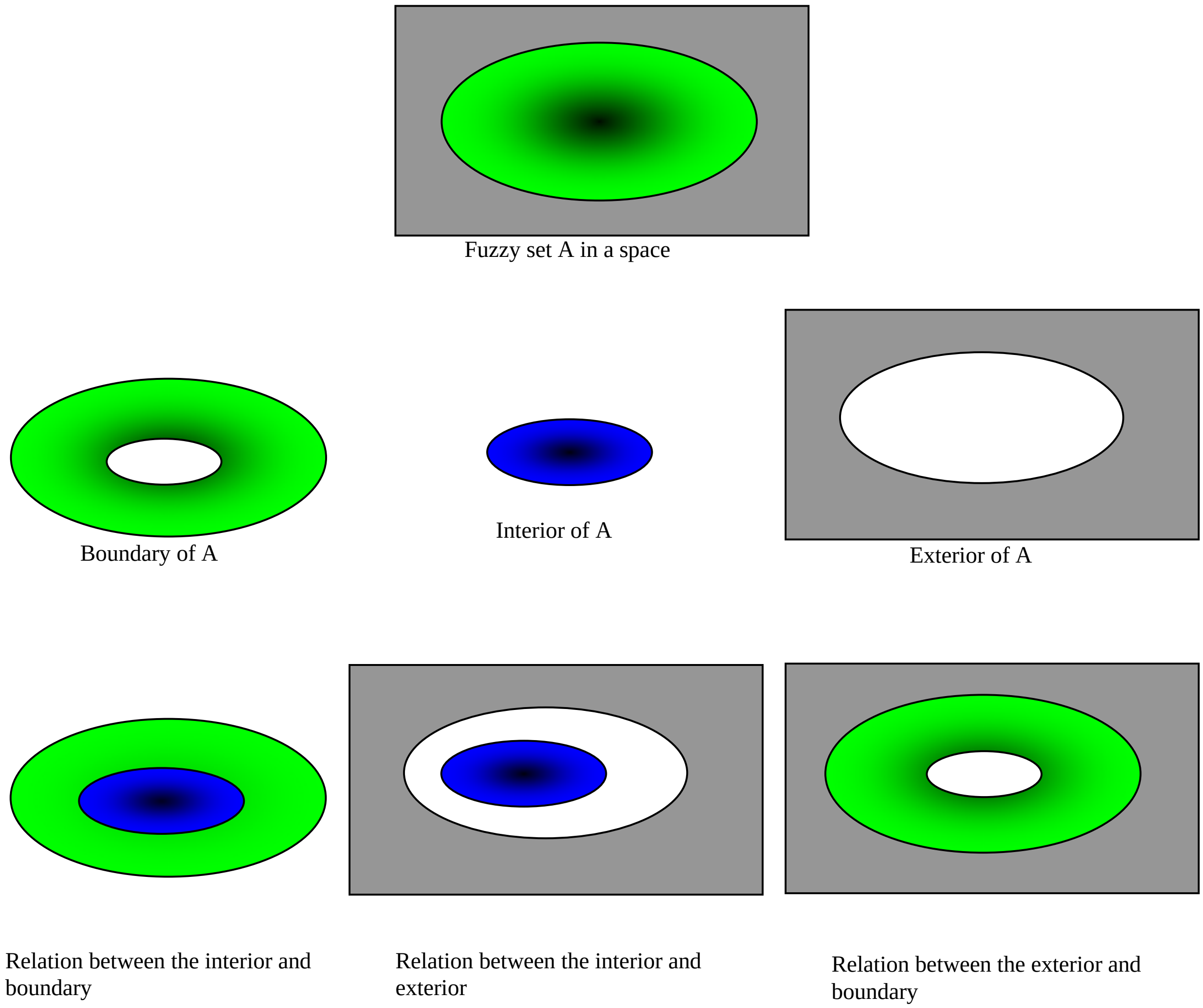


Figure 7.12: Relations between the interior, boundary, and exterior

Before giving a real data example, let us first give a simple example to illustrate the change in relationships between the interior and boundary with different values of α. In this example, we can see that when a small value of α is chosen, large values of the intersection (interior with boundary) will obtain. On the other hand, when a large value of α is chosen, the intersection will have small values (interior with boundary).

Example 2: Define a fuzzy set $A : R^2 \to [0,1]$ by

$$A(x, y) = \begin{cases} 1 & \text{if } x^2 + y^2 < 0.5 \\ \exp(-x^2 - y^2) & \text{if } 0.5 \le x^2 + y^2 \le 2 \\ 0 & \text{otherwise} \end{cases}.$$

Then $$A^c(x, y) = \begin{cases} 0 & \text{if } x^2 + y^2 < 0.5 \\ 1 - \exp(-x^2 - y^2) & \text{if } 0.5 \le x^2 + y^2 \le 2 \\ 1 & \text{otherwise} \end{cases}.$$

For $\alpha = 0.3$ (see Figure 7.13(a)),

Interior	$A_{0.3}(x, y) = \begin{cases} 1 & \text{if } x^2 + y^2 < 0.5 \\ \exp(-x^2 - y^2) & \text{if } 0.5 \le x^2 + y^2 \le -\ln(0.3) \\ 0 & \text{otherwise} \end{cases}$
Boundary	$\partial A(x, y) = A^{0.7} \wedge (A_{0.3})^c(x, y) = \begin{cases} \exp(-x^2 - y^2) & \text{if } 0.5 \le x^2 + y^2 \le -\ln(0.5) \\ 1 - \exp(-x^2 - y^2) & \text{if } -\ln(0.5) < x^2 + y^2 < -\ln(0.3) \\ 0 & \text{otherwise} \end{cases}$

For $\alpha = 0.6$ (see Figure 7.13(b)),

Interior	$A_{0.6}(x, y) = \begin{cases} 1 & \text{if } x^2 + y^2 < 0.5 \\ 0 & \text{otherwise} \end{cases}$
Boundary	$\partial A(x, y) = A^{0.4} \wedge (A_{0.6})^c(x, y) = \begin{cases} \exp(-x^2 - y^2) & \text{if } 0.5 \le x^2 + y^2 \le -\ln(0.5) \\ 1 - \exp(-x^2 - y^2) & \text{if } -\ln(0.5) < x^2 + y^2 < -\ln(0.3) \\ 0 & \text{otherwise} \end{cases}$

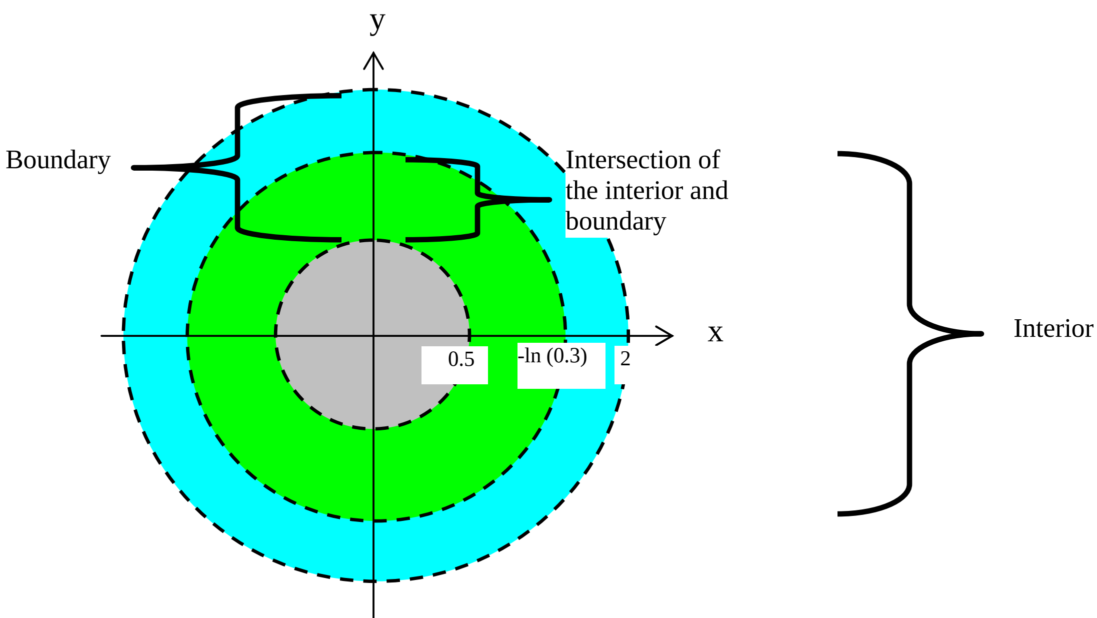


Figure 7.13(a): If a small value is given for α , we get a large value for the intersection of the interior and boundary

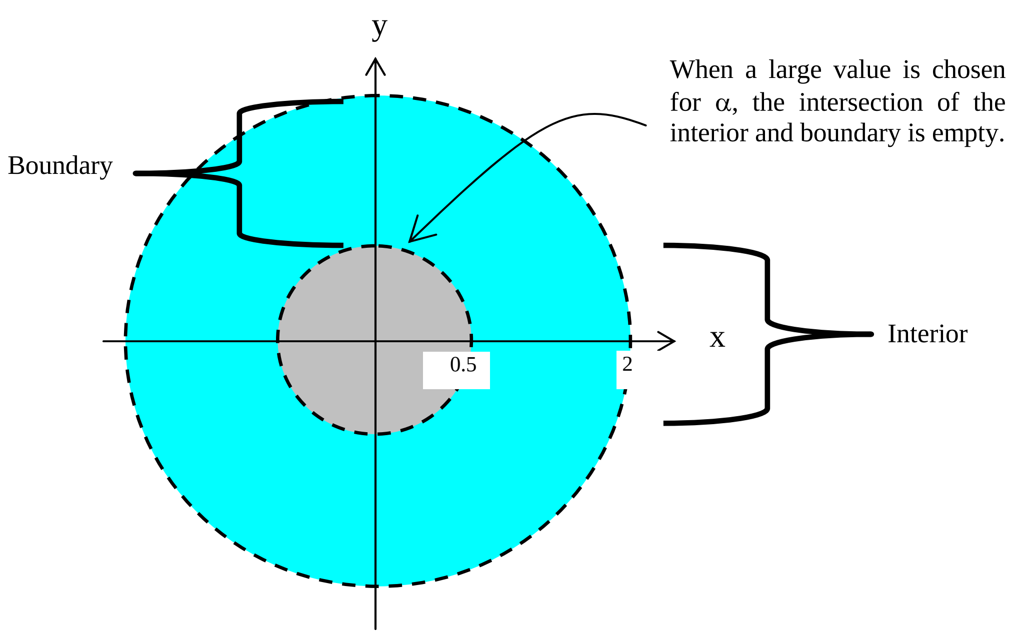


Figure 7.13(b): If a large value is given for α , we get a smaller value for the intersection of interior and boundary

7.3 Example of the calculation of interior and boundary

For a general fuzzy topological space, even if we have the membership function of a fuzzy set, we only get the abstract definitions of interior, boundary, and closure rather the than formula for computing them. With these definitions, we cannot practically use these abstract definitions for topological calculations, or for other applications. For the fuzzy topological space induced by these two operators, the interior operator and closure operator are defined based on formulae. That is, if we have the formula of a fuzzy set, then the interior and closure of this fuzzy set can be computed by using the natural definitions of these two operators. Hence, the other parts that included the boundary and exterior can be computed directly. In this section, we try to demonstrate how to compute the interior, closure, and boundary of spatial objects for real GIS data, by using an example.

7.3.1 Introduction

Mikania micrantha (see Figure 7.14) has caused serious problems for plants. It is an exotic plant commonly found in the Hong Kong countryside, and its fast-growing characteristic poses a serious threat to local plants. With no natural enemies in Hong Kong, it grows very quickly, covering the trees and preventing them from absorbing sufficient quantities of sunlight, and competing with other plants for water and nutrients. This eventually leads to the death of the host trees, which has happened in many areas in Hong Kong (Shi, Dai and Liu, 2003).

Figure 7.14: Mikania micrantha in Hong Kong

7.3.2 The objectives of this study

The aim of this study is to determine the level of the effect of each area affected by *Mikania micrantha*, based on infrared photos. Each area affected by *Mikania micrantha* is digitized and the size is recorded. The level of any area affected by *Mikania micrantha* is marked by a percentage level within the interval [0, 1]. Then, for a particular α, we try to determine the interior, closure, and boundary of each area affected by *Mikania micrantha*. For the interior, if the value is closer to 1, this means that the effect is high in relation to the overall affected area. If the value is closer to zero, this means that the effect is low in relation to the overall affected areas.

The following are the objectives of this study:

(ii) To detect the areas affected by *Mikania micrantha* using the aerial photo interpretation-based approach.

(iii) To classify the fuzzy interior, boundary, and exterior of the areas affected by *Mikania micrantha* using the new model.

7.3.3 Methodology

The *Mikania micrantha* areas shown in an aerial photo are available from the previous study. Some of the analytical and statistical results from that study can also be used for this study.

(i) Each aerial photo is viewed as a fuzzy topological space.

(ii) Each area affected by *Mikania micrantha* in the aerial photo is viewed as a fuzzy set/point in the fuzzy topological space.

(iii) The size of each area affected by *Mikania micrantha* is used to calculate the fuzzy value of the fuzzy sets.

(iv) The fuzzy value of each area affected by *Mikania micrantha* is defined as:

$$\begin{cases} \dfrac{\log(\text{Area of certain affected area})}{\log(\text{Total area of affected area})} & \text{if } \dfrac{\log(\bullet)}{\log(*)} > 0 \\ 0 & \text{otherwise} \end{cases}$$

, which is a well-defined mapping from the interval $[1, \infty)$ to the interval $[0, 1]$.

(v) The fuzzy interior and boundary will be computed for α equal to 0.3, 0.45, and 0.6, respectively.

7.3.4 Results

Each *Mikania micrantha* area has an identity number and its boundary has been digitized on the aerial photos. The blue polygons with identity numbers (see Figure 7.15) are the areas affected by the *Mikania micrantha*. Table 7.1 shows the size of each area affected by *Mikania micrantha* on an aerial photo.

Figure 7.15: The polygons of the areas affected by *Mikania micrantha*

Table 7.1: The size of each area affected by *Mikania micrantha*

ID	Area (m^2)	ID	Area (m^2)	ID	Area (m^2)
1	7.97	19	60.5	37	195.46
2	8.17	20	61.97	38	265.48
3	8.8	21	63.86	39	293.35
4	10.23	22	73.64	40	312.6
5	10.37	23	73.83	41	315.02
6	15.32	24	76.67	42	343.49
7	17.09	25	77.1	43	349.76
8	17.52	26	82.68	44	388.28
9	24.75	27	85.58	45	401.55
10	25.12	28	87.16	46	403.61
11	28	29	93.8	47	498.05
12	31.69	30	104.38	48	564.57
13	31.92	31	105.35	49	629.68
14	36.75	32	135.05	50	774.58
15	37,83	33	142.95	51	786.1
16	42.46	34	155.6	52	855.94
17	53.36	35	184.86	53	1014.44
18	57.1	36	192.21		
				Total	10713.58
				Average	202.14

Table 7.2: The values of the fuzzy exterior, fuzzy interior, and fuzzy boundary with different values of α

			α = 0.3	α = 0.3	α = 0.3	α = 0.45	α = 0.45	α = 0.45	α = 0.6	α = 0.6	α = 0.6
ID	AREA (sq.m)	Fuzzy Value	exterior	interior	boundary	exterior	interior	boundary	exterior	interior	boundary
1	7.97	0.22	0.78	0.00	0.22	0.78	0.00	0.22	0.78	0.00	0.22
2	8.17	0.23	0.77	0.00	0.23	0.77	0.00	0.23	0.77	0.00	0.23
3	8.80	0.23	0.77	0.00	0.23	0.77	0.00	0.23	0.77	0.00	0.23
4	10.23	0.25	0.75	0.00	0.25	0.75	0.00	0.25	0.75	0.00	0.25
5	10.37	0.25	0.75	0.00	0.25	0.75	0.00	0.25	0.75	0.00	0.25
6	15.32	0.29	0.71	0.00	0.29	0.71	0.00	0.29	0.71	0.00	0.29
7	17.09	0.31	0.69	0.31	0.31	0.69	0.00	0.31	0.69	0.00	0.31
8	17.52	0.31	0.69	0.31	0.31	0.69	0.00	0.31	0.69	0.00	0.31
9	24.75	0.35	0.65	0.35	0.35	0.65	0.00	0.35	0.65	0.00	0.35
10	25.12	0.35	0.65	0.35	0.35	0.65	0.00	0.35	0.65	0.00	0.35
11	28.00	0.36	0.64	0.36	0.36	0.64	0.00	0.36	0.64	0.00	0.36
12	31.69	0.37	0.63	0.37	0.37	0.63	0.00	0.37	0.63	0.00	0.37
13	31.92	0.37	0.63	0.37	0.37	0.63	0.00	0.37	0.63	0.00	0.37
14	36.75	0.39	0.61	0.39	0.39	0.61	0.00	0.39	0.61	0.00	0.39
15	37.83	0.39	0.61	0.39	0.39	0.61	0.00	0.39	0.61	0.00	0.39
16	42.46	0.40	0.60	0.40	0.40	0.60	0.00	0.40	0	0.00	1.00
17	53.36	0.43	0.57	0.43	0.43	0.57	0.00	0.43	0	0.00	1.00
18	57.10	0.44	0.56	0.44	0.44	0.56	0.00	0.44	0	0.00	1.00
19	60.50	0.44	0.56	0.44	0.44	0.56	0.00	0.44	0	0.00	1.00
20	61.97	0.44	0.56	0.44	0.44	0.56	0.00	0.44	0	0.00	1.00
21	63.86	0.45	0.55	0.45	0.45	0.55	0.45	0.45	0	0.00	1.00
22	73.64	0.46	0.54	0.46	0.46	0.54	0.46	0.46	0	0.00	1.00
23	73.83	0.46	0.54	0.46	0.46	0.54	0.46	0.46	0	0.00	1.00
24	76.67	0.47	0.53	0.47	0.47	0.53	0.47	0.47	0	0.00	1.00
25	77.10	0.47	0.53	0.47	0.47	0.53	0.47	0.47	0	0.00	1.00
26	82.68	0.48	0.52	0.48	0.48	0.52	0.48	0.48	0	0.00	1.00
27	85.58	0.48	0.52	0.48	0.48	0.52	0.48	0.48	0	0.00	1.00
28	87.16	0.48	0.52	0.48	0.48	0.52	0.48	0.48	0	0.00	1.00
29	93.80	0.49	0.51	0.49	0.49	0.51	0.49	0.49	0	0.00	1.00
30	104.38	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0	0.00	1.00
31	105.35	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0	0.00	1.00
32	135.05	0.53	0.47	0.53	0.47	0.47	0.53	0.47	0	0.00	1.00
33	142.95	0.53	0.47	0.53	0.47	0.47	0.53	0.47	0	0.00	1.00
34	155.60	0.54	0.46	0.54	0.46	0.46	0.54	0.46	0	0.00	1.00
35	184.86	0.56	0.44	0.56	0.44	0	0.56	0.44	0	0.00	1.00
36	192.21	0.57	0.43	0.57	0.43	0	0.57	0.43	0	0.00	1.00
37	195.46	0.57	0.43	0.57	0.43	0	0.57	0.43	0	0.00	1.00
38	265.48	0.60	0.40	0.60	0.40	0	0.60	0.40	0	0.00	1.00
39	293.35	0.61	0.39	0.61	0.39	0	0.61	0.39	0	0.61	0.39
40	312.60	0.62	0.38	0.62	0.38	0	0.62	0.38	0	0.62	0.38
41	315.02	0.62	0.38	0.62	0.38	0	0.62	0.38	0	0.62	0.38
42	343.49	0.63	0.37	0.63	0.37	0	0.63	0.37	0	0.63	0.37
43	349.76	0.63	0.37	0.63	0.37	0	0.63	0.37	0	0.63	0.37
44	388.28	0.64	0.36	0.64	0.36	0	0.64	0.36	0	0.64	0.36
45	401.55	0.65	0.35	0.65	0.35	0	0.65	0.35	0	0.65	0.35
46	403.61	0.65	0.35	0.65	0.35	0	0.65	0.35	0	0.65	0.35
47	498.05	0.67	0.33	0.67	0.33	0	0.67	0.33	0	0.67	0.33
48	564.57	0.68	0.32	0.68	0.32	0	0.68	0.32	0	0.68	0.32
49	629.68	0.69	0.31	0.69	0.31	0	0.69	0.31	0	0.69	0.31
50	774.58	0.72	0	0.72	0.28	0	0.72	0.28	0	0.72	0.28
51	786.10	0.72	0	0.72	0.28	0	0.72	0.28	0	0.72	0.28
52	855.94	0.73	0	0.73	0.27	0	0.73	0.27	0	0.73	0.27
53	1014.44	0.75	0	0.75	0.25	0	0.75	0.25	0	0.75	0.25

Table 7.2 shows that for different α, the values of the fuzzy exterior, fuzzy interior, and fuzzy boundary are different. From the table, we can see that the larger the value of α, the smaller the size of the interior. When $\alpha = 0.3$, ID numbers greater than 6 have non-zero values; when $\alpha = 0.45$, ID numbers greater than 20 have non-zero values; when $\alpha = 0.6$, ID numbers greater than 38 have non-zero values. These facts show that the relation between α and the size of the closure are directly proportional, while the relation between α and the size of interior are inversely proportional. Actually, this new model can be used to classify the fuzzy interior, boundary, and exterior of fuzzy spatial objects. Classifying the fuzzy interior, boundary, and exterior of the areas affected by *Mikania micrantha* in this chapter is a potential application in GIS.

7.4 Summary

Topological relations between indeterminate boundaries have been investigated for several years. Based on the 9-intersection, Cohn and Gotts (1996) give 46 topological relations between two regions with indeterminate boundaries while Clementini and Di Felice give 44 topological relations between two regions with indeterminate boundaries. Those models can only give a conceptual definition of topological relations. It is because their models are based on a general fuzzy topological space. Even we have the membership function of a fuzzy set, we only get the abstract definitions of interior, boundary and closure. But the membership functions of interior, boundary and closure cannot be implemented in a computer. For the fuzzy topological space induced by these two operators, as interior operator and closure operator are defined based on algorithms. That is if we have the formula of a fuzzy set, then interior and closure of this fuzzy set can be computed by using the natural definitions of these two operators. Hence, the other parts that included boundary and exterior can be computed directly.

7.4.1 Discussion on the newly developed fuzzy topological model

The computational fuzzy topological model introduced in this chapter provides a solution to quantitatively compute the topological relations between spatial objects. This is a step ahead of the topological models developed in the past. This model not only provides conceptual definitions, but also quantitative descriptions of the topological relations between spatial objects.

Based on a general fuzzy topological space, even with the formula of the membership function of a fuzzy set, we can only obtain abstract definitions of the interior, boundary, and closure. However, the explicit membership functions of the interior, boundary, and closure cannot be obtained. On the other hand, the fuzzy topological space induced by these two operators (interior operator and closure operator) is defined based on algorithms. As a result, if we have the membership function (or *formula*) of a fuzzy set, then the interior and closure of this fuzzy set can be computed by using the natural definitions of these two operators. Hence, the boundary and exterior can be computed directly.

The structure of a general fuzzy topological space is still very abstract, while the structure of the induced fuzzy topological space, $\left(X, \tau_{\alpha}, \tau^{1-\alpha}\right)$, in this chapter is relatively straightforward. The simple fuzzy topological space can actually enrich information in GIS, which includes classifying fuzzy spatial objects, error control, calculating fuzzy topological relations, and so forth.

Geometrically, GIS features can be classified as point, line, and polygon. In ordinary topological space, the point and line in two-dimensional space have an empty interior, and the 9-intersection model may not fit the case of point and line topological relations (Egenhofer and Franzosa, 1991). Therefore, we need to treat their topological relation separately. Both the fuzzy point and fuzzy line in fuzzy two-dimensional space may have a non-empty interior (see example 1 in Figure 7.10). As a result, the 9-intersection model can be directly applied to the cases of fuzzy point and fuzzy line modeling.

When applying fuzzy topology in GIS, it is inappropriate to assume that the membership values of the interior is equal to one and the membership values of boundary is between 0 and 1 exclusively. Indeed, when $\alpha > \frac{1}{2}$, the interior value may not be one and the boundary value may be one (see Remark 8).

The former approaches (Cohn and Gotts, 1996; Smith, 1996; Tang and Kainz, 2002) have introduced the concept of fuzzy topology into GIS. We can thus apply the computational fuzzy topological space to calculate the interior, boundary, and exterior of fuzzy spatial objects. An example of classifying the fuzzy interior, boundary, and exterior of the areas affected by *Mikania micrantha* is provided in this chapter. Moreover, this topological space is a computable fuzzy topological space and the fuzzy theory can be applied directly. As a result, the developed theory and the practical operation are linked very closely. Therefore, the potential applications of fuzzy topological space, presented in this chapter, are very wide. For, example, Liu, *et al*. (2005) applied fuzzy topological space for image segmentation and classification.

GIS is used to model, retrieve, and analyze spatial objects with the inherent structure in space. Some of these inherent structures actually can be described by topological structures (Egenhofer and Franzosa, 1991). Originally, GIS was designed based on the assumption that the measurements on spatial objects in GIS are error free. However, this assumption may not always be true due to the vagueness/ fuzziness of spatial objects. For example, uncertainty among the boundary region between urban and rural areas is difficult to handle by traditional GIS. Fuzzy topological space can potentially be used to describe such inherent structures in GIS.

Fuzzy topological space is dependent on the α used in leveling cuts. Different values of α generate different fuzzy topologies and may have different topological structures. Therefore, we can generate suitable fuzzy topological space by adjusting the value of α; the generated fuzzy topological space can thus match the cases of the application concerned. Moreover, different values of α can provide a multi-directional spatial analysis in GIS. For example, different values of α provide different values of interior, boundary, and exterior. An optimal value of α can be obtained by investigating these fuzzy topologies. More information can be generated for spatial queries by applying fuzzy topology.

7.4.2 Concluding remarks

In this chapter, we presented a research outcome – computational fuzzy topological space, which is based on the interior operator and closure operator and they are further defined as a coherent fuzzy topological space – where the complement of the open set is the closed set and vice versa. Here, the open set and closed set are defined by the interior and closure operators – two level cuts. The elementary components of fuzzy topological space for spatial objects – interior, boundary, and exterior – are thus able to be computed based on computational fuzzy topological space. Furthermore, this research outcome provides a new dimension in studies of topological relations between spatial objects.

Topological relations between spatial objects with indeterminate boundaries have been investigated for several years. The existing models, such as the 9-intersection, can only provide conceptual definitions and qualitative description on topological relations. On the other hand, the computational fuzzy topological space induced by the interior operator and closure operator in this study provides a solution to practically compute the values of topological relations, and thus can easily be implemented in a computer environment. That is, if we have the formula of the membership function of a fuzzy set, the interior and closure of this fuzzy set can be computed by using the natural definitions of these two operators. As a result, we not only can obtain information on the fuzzy topological relations between two objects (such as the value of the interior, boundary, and exterior), but also a quantitative level of these topological relations (such as the total number of topological relations between simple fuzzy regions) can also be provided based on the computational fuzzy topological space developed in this study.

Fuzzy topological relations between uncertain objects can be used for fuzzy spatial queries, fuzzy spatial analyzes, and other questions, just as topological relations between spatial objects are used for spatial queries and spatial analyses. The existing topological models, such as the 9-intersection model and other models on topological relations, can thus be practically implemented in a GIS and used for computing fuzzy topologies, based on the computational fuzzy topological space solution developed in this study.

CHAPTER EIGHT

COMPUTING THE FUZZY TOPOLOGICAL RELATIONS OF SPATIAL OBJECTS BASED ON INDUCED FUZZY TOPOLOGICAL SPACE

Topological relations between spatial objects are fundamental information used in GIS, along with positional and attribute information. Information on topological relations can be used for spatial queries, spatial analyses, data quality control (e.g., checking for topological consistency), and others. For modeling the topological relations between spatial objects, the concepts of a bound on the intersection of the boundary and interior, and the boundary and exterior are defined in this chapter based on the newly developed computational fuzzy topological space. Furthermore, the qualitative measures for the intersections are specified based on the α-cut induced fuzzy topology, which are $(A_\alpha \wedge \partial A)(x) < 1-\alpha$ and $((A^c)_\alpha \wedge \partial A)(x) < 1-\alpha$. Specifically, the following areas are covered: (a) the homeomorphic invariants of the fuzzy topological space are proposed; and (b) the connectivity of the newly developed fuzzy topology is defined. The work of providing fuzzy topological relations between simple fuzzy spatial objects will be presented in chapter 9.

8.1 Preserving properties of computational fuzzy topological space

The properties of topological spaces that are preserved under homeomorphic mappings are called the topological invariants of the spaces. To study the topological relations, we need to first investigate the properties of a fuzzy mapping, especially homeomorphic mapping. The topological relations are invariants under homeomorphic mappings. With these, we can thus guarantee the properties that will remain unchanged in a GIS transformation, such as the maintenance of topological consistency when digitizing a map or transferring a map from a system to another system.

In the coming two sections, we would like to develop the preserving properties of the computational fuzzy topological space and the connectivity of this fuzzy topological space in GIS. The main objective of this section is to prove the open and closed sets that are preserved by fuzzy mapping and fuzzy reverse mapping. Furthermore, the connectivity of a fuzzy topological space is elementary in any study of the topological relations between spatial objects in GIS. Therefore, the properties of connection in the new induced fuzzy topological space will be studied in section 8.2. Recall that let (I^X, δ), (I^Y, μ) be I-fts's, $f^{\rightarrow}: (I^X, \delta) \rightarrow (I^Y, \mu)$ is called an I-fuzzy homeomorphism, if it is bijective, continuous, and open.

Proposition 8.1: Let $A \in I^X$, $B \in I^Y$, let (I^X, δ), (I^Y, μ) be I-fts's induced by the interior operator and closure operator, $f^{\rightarrow}: (I^X, \delta) \rightarrow (I^Y, \mu)$ and $f^{\leftarrow}: (I^Y, \mu) \rightarrow (I^X, \delta)$. The following then holds:

(i) $f^{\rightarrow}(A_\alpha) = [f^{\rightarrow}(A)]_\alpha$

(ii) $f^{\rightarrow}(A^{1-\alpha}) = [f^{\rightarrow}(A)]^{1-\alpha}$

(iii) $f^{\leftarrow}(B_\alpha) = [f^{\leftarrow}(B)]_\alpha$

(iv) $f^{\leftarrow}(B^{1-\alpha}) = [f^{\leftarrow}(B)]^{1-\alpha}$

Proof:

(i) For all $y \in Y$, if $\{f^{-1}(y)\} = \phi$, the result is obvious as both sides are zero.

Suppose $\{f^{-1}(y)\} \neq \phi$.

If there exists $x_o \in \{f^{-1}(y)\}$ such that $A(x_o) > \alpha$, then $f^{\rightarrow}(A_\alpha)(y) = \bigvee_{x \in \{f^{-1}(y)\}} A(x) > \alpha$ and $[f^{\rightarrow}(A)]_\alpha\ (y) = \bigvee_{x \in \{f^{-1}(y)\}} A(x) > \alpha$.

If for all $x \in \{f^{-1}(y)\}$ such that $A(x) \leq \alpha$, then $f^{\rightarrow}(A_\alpha)(y) = 0$ and $f^{\rightarrow}(A)(y) = \bigvee_{x \in \{f^{-1}(y)\}} A(x) \leq \alpha$, i.e. $[f^{\rightarrow}(A)]_\alpha\ (y) = 0$.

Thus, we have $f^{\rightarrow}(A_\alpha)(y) = [f^{\rightarrow}(A)]_\alpha(y)$ for all $y \in Y$.

Hence, $f^{\rightarrow}(A_\alpha) = [f^{\rightarrow}(A)]_\alpha$.

(ii) For all $y \in Y$, if $\{f^{-1}(y)\} = \phi$, the result is obvious as both sides are zero.

Suppose $\{f^{-1}(y)\} \neq \phi$.

If there exists $x_o \in \{f^{-1}(y)\}$ such that $A(x_o) \geq 1-\alpha$, then

$$f^{\to}(A^{1-\alpha})(y) = \bigvee_{x \in \{f^{-1}(y)\}} A^{1-\alpha}(x) = 1 \text{ and } [f^{\to}(A)]^{1-\alpha}(y) = \bigvee_{x \in \{f^{-1}(y)\}} A(x) = 1.$$

If for all $x \in \{f^{-1}(y)\}$ such that $A(x) < 1-\alpha$, then

$$\begin{aligned} f^{\to}(A^{1-\alpha})(y) &= \bigvee_{x \in \{f^{-1}(y)\}} A^{1-\alpha}(x) \\ &= \bigvee_{x \in \{f^{-1}(y)\}} A(x) \\ &= \begin{cases} 1 & \text{if } \vee A(x) < 1-\alpha \\ \bigvee_{x \in \{f^{-1}(y)\}} A(x) & \text{if } \vee A(x) = 1-\alpha \end{cases} \end{aligned}$$

$$\begin{aligned} f^{\to}(A)(y) &= \bigvee_{x \in \{f^{-1}(y)\}} A(x) \\ &= \begin{cases} 1 & \text{if } \vee A(x) < 1-\alpha \\ \bigvee_{x \in \{f^{-1}(y)\}} A(x) & \text{if } \vee A(x) = 1-\alpha \end{cases}. \end{aligned}$$

Hence, $[f^{\to}(A)]^{1-\alpha}(y) = \begin{cases} 1 & \text{if } \vee A(x) < 1-\alpha \\ \bigvee_{x \in \{f^{-1}(y)\}} A(x) & \text{if } \vee A(x) = 1-\alpha \end{cases}.$

(iii) $$f^{\leftarrow}(B_\alpha)(x) = B_\alpha(f(x)) = \begin{cases} B(f(x)) & \text{if } B(f(x)) > \alpha \\ 0 & \text{if } B(f(x)) \leq \alpha \end{cases}.$$

$$\begin{aligned} [f^{\leftarrow}(B)]_\alpha(x) &= \begin{cases} f^{\leftarrow}(B)(x) & \text{if } f^{\leftarrow}(B)(x) > \alpha \\ 0 & \text{if } f^{\leftarrow}(B)(x) \leq \alpha \end{cases} \\ &= \begin{cases} B(f(x)) & \text{if } B(f(x)) > \alpha \\ 0 & \text{if } B(f(x)) \leq \alpha \end{cases} \end{aligned}$$

(iv) $$f^{\leftarrow}(B^{1-\alpha})(x) = B^{1-\alpha}(f(x)) = \begin{cases} 1 & \text{if } B(f(x)) \geq 1-\alpha \\ B(f(x)) & \text{if } B(f(x)) < 1-\alpha \end{cases}$$

$$\left[f^{\leftarrow}(B)\right]^{1-\alpha}(x) = \begin{cases} 1 & \text{if } f^{\leftarrow}(B)(x) \geq 1-\alpha \\ f^{\leftarrow}(B)(x) & \text{if } f^{\leftarrow}(B)(x) < 1-\alpha \end{cases}$$

$$= \begin{cases} 1 & \text{if } B(f(x)) \geq 1-\alpha \\ B(f(x)) & \text{if } B(f(x)) < 1-\alpha \end{cases}$$

Q.E.D.

Remark 1: This proposition can be interpreted as both $f^{\rightarrow}$ and $f^{\leftarrow}$ presevering open and closed sets. It means that $f^{\rightarrow} : (I^X, \delta) \rightarrow (I^Y, \mu)$ is bi-continuous.

Example 1: This example verifies that $f^{\rightarrow}(A_\alpha) = \left[f^{\rightarrow}(A)\right]_\alpha$ by a simple mapping.

Define $f : X = \{a_1, a_2, a_3, a_4\} \rightarrow Y = \{b_1, b_2, b_3, b_4\}$ by $f(a_1) = b_1$, $f(a_2) = b_1$, $f(a_3) = b_3$, $f(a_4) = b_4$. Suppose $A \in I^X$ such that $A(a_1) = \frac{3}{4}$, $A(a_2) = \frac{1}{4}$, $A(a_3) = \frac{1}{2}$, $A(a_4) = 1$. Let $\alpha = 0.7$.

By definition, $f^{\rightarrow}(A)(b_1) = A(a_1) \vee A(a_2) = \frac{3}{4}$, $f^{\rightarrow}(A)(b_2) = 0$, $f^{\rightarrow}(A)(b_3) = \frac{1}{2}$ and $f^{\rightarrow}(A)(b_4) = 1$. Hence, $\left[f^{\rightarrow}(A)\right]_{0.7}(b_1) = \frac{3}{4}$, $\left[f^{\rightarrow}(A)\right]_{0.7}(b_2) = 0$, $\left[f^{\rightarrow}(A)\right]_{0.7}(b_3) = 0$ and $\left[f^{\rightarrow}(A)\right]_{0.7}(b_4) = 1$.

On the other hand, $A_{0.7}(a_1) = \frac{3}{4}$, $A_{0.7}(a_2) = 0$, $A_{0.7}(a_3) = 0$, $A_{0.7}(a_4) = 1$. Hence, $f^{\rightarrow}(A_{0.7})(b_1) = A_{0.7}(a_1) \vee A_{0.7}(a_2) = \frac{3}{4}$, $f^{\rightarrow}(A_{0.7})(b_2) = 0$, $f^{\rightarrow}(A_{0.7})(b_3) = 0$ and $f^{\rightarrow}(A_{0.7})(b_4) = 1$.

Thus, we have $f^{\rightarrow}(A_\alpha)(y) = \left[f^{\rightarrow}(A)\right]_\alpha(y)$ for all $y \in Y$; hence, $f^{\rightarrow}(A_\alpha) = \left[f^{\rightarrow}(A)\right]_\alpha$.

Theorem 8.2: Let $A \in I^X$, $B \in I^Y$, let (I^X, δ), (I^Y, μ) be I-fts's induced by an interior operator and closure operator. Then, $f^{\rightarrow} : (I^X, \delta) \rightarrow (I^Y, \mu)$ is an I-fuzzy homeomorphism if and only if $f : X \rightarrow Y$ is a bijective mapping.

8.2 Connectivity of spatial objects in GIS based on fuzzy topological space

Fuzzy topology can be applied to describe and analyze the structure of neighborhood and leveling of spaces. Connectivity is a preserving property of fuzzy topolgy. The usual definition of the connection of fuzzy subset, A, in fuzzy topological space is that A cannot be separated by two non-zero open or closed fuzzy sets, called open connected and closed connected, respectively. As the natural character of fuzzy topological space, this kind of connection also contains two types of structures – neighborhood and leveling, respectively. In GIS, the connectivity of spatial objects depends on the neighbourhood structure of the objects themselves, rather than on the leveling structure. Thus, the ordinary definition of the connection of fuzzy topological space may not be very suitable for describing relations between spatial objects in GIS. In I^R, Figure 8.1 shows two spatial fuzzy objects in GIS that are considered to be connected. Figure 8.2 shows two spatial fuzzy objects in GIS that are considered to be disconnected.

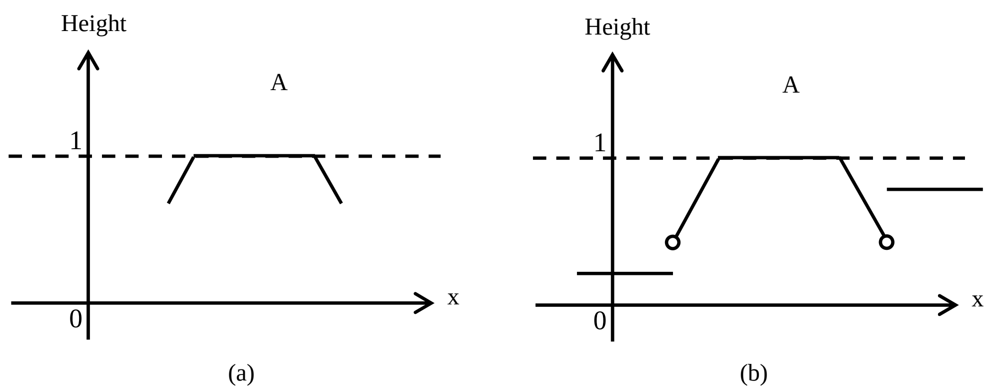


Figure 8.1: Two examples of connected fuzzy spatial objects in **R**

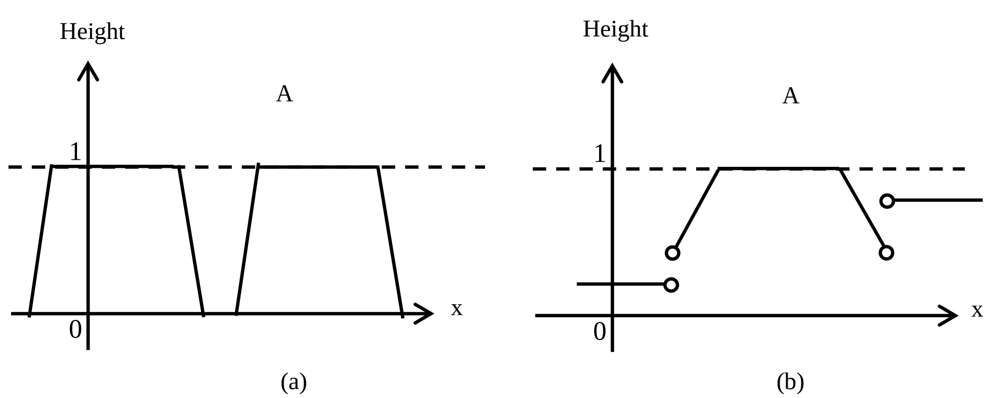


Figure 8.2: Two examples of disconnected fuzzy spatial objects in **R**

As a result, it is necessary to develop a new definition of connectivity due to the fact that the existing definition of connectivity in fuzzy topological space is not applicable to GIS. The concept of connectivity in GIS is that whether the spatial object is connected to the other spatial object only in the sense of background space simply involves the concept of neighborhood rather than the concept leveling. Based on this understanding, among the two fuzzy spatial objects in Figure 8.1, one is fuzzy connected (the example of Figure 8.1(a)) while the other is not fuzzy connected (the example 1 of Figure 8.1(b)). On the other hand, in GIS we often consider the connectivity based on its support sets. According to this, both examples of fuzzy spatial objects in Figure 8.1 are considered as connected in GIS. Based on this fact, the concept of connectivity in GIS should be defined based on the background topological space.

The background set X also has its topology, therefore, we may let β be a topology of X and (X, β) be this background topological space. Thus, we have two kinds of notations: (a) the fuzzy topological space (X, I^X, δ); and (b) its background topological space (X, β). These two topologies may not be related. But under certain assumptions, there are many nice results about their relations (Martin, 1980; Liu and Luo, 1997; Luo, 1988). We denoted this topological space by (X, I^X, δ, β).

Definition 8.3 (support of A): Support(A) or Supp(A) is equal to the set $\{x \in X: A(x) > 0\}$. The closure of Supp(A) in background topological space is denoted by $\overline{\mathrm{Supp}(A)}$.

Definition 8.4 (supported connected fuzzy set): Let (X, I^X, δ) be an I-fts and (X, β) be its background topological space, $A, B \in I^X$. A and B are called supported separated, if

$$\overline{\mathrm{Supp}(A)} \cap \mathrm{Supp}(B) = \mathrm{Supp}(A) \cap \overline{\mathrm{Supp}(B)} = \phi.$$

A is called supported connected in (X, I^X, δ, β), if there does not exist supported separated $C, D \in I^X \setminus \{0\}$ such that $A = C \vee D$ and $\mathrm{Supp}(A) = \mathrm{Supp}(C) \cup \mathrm{Supp}(D)$.

Definition 8.5 (Supported connected component): Let (X, I^X, δ) be an I-fts and (X, β) be its background topological space, $A \in I^X$. A is called a supported connected component of (X, I^X, δ, β), if A is a maximal supported connected subset in (X, I^X, δ, β); i.e., if $B \in I^X$ is a supported connected component and $B \geq A$, then B = A.

Proposition 8.6: Let (X, I^X, δ) be an I-fts and (X, β) be its background topological space, $A \in I^X$. Every fuzzy point in A belongs to one and only one supported component of A.

Proposition 8.7: Let (X, I^X, δ) be an I-fts and (X, β) be its background topological space. Then, different supported components of (X, I^X, δ, β) are separated.

Theorem 8.8: Let (X, I^X, δ, β) and (Y, I^Y, μ, γ) be two I-fuzzy topological spaces, $f : X \to Y$ an ordinary continuous mapping. If $A \in I^X$ is a supported connected fuzzy set, then $f^{\to}(A)$ is also a supported connected fuzzy set.

Proof: Suppose $A \in I^X$ is a supported connected fuzzy set. Since $f : X \to Y$ is continuous, f(Supp(A)) is connected. The next step is to prove that f(Supp(A)) = Supp($f^{\to}(A)$). If this is true, we can conclude that $f^{\to}(A)$ is a supported connected fuzzy set. But by definition, f(Supp(A)) = $\{f(x) : A(x) > 0\}$ and $f^{\to}(A)(y) = \begin{cases} \vee\{A(x)\} & \text{if } \; x \in f^{-1}(y) \\ 0 & \text{otherwise} \end{cases}$. Thus, Supp($f^{\to}(A)$) = $\{y = f(x) : \vee A(x) > 0\}$ $= \{y = f(x) : A(x) > 0\}$.

Q.E.D.

Remark 2: From Theorem 8.8, we can see that the supported connectivity of a fuzzy topological space (X, I^X, δ, β) does not depend on the topology (X, I^X, δ), but only on its background topology (X, β). This makes it easy to model fuzzy topological relations in spatial querying and analysis.

8.3 Modeling simple fuzzy objects in GIS

We need to first model simple object in GIS before we can model the fuzzy topological relations between the objects. The basic simple objects in GIS may include simple points, simple line segments, and simple regions. In the crisp case, simple points, lines, and regions have been discussed widely (Egenhofer and Franzosa, 1991; Clementini and Di Felice, 1996). In the fuzzy case, Tang and Kainz (2002) have given a definition of a simple fuzzy region.

In order to give a generic framework on the number of topological relations between these simple spatial objects; simple fuzzy points, simple fuzzy line segments, and simple fuzzy regions are defined as follows.

Definition 8.9 (fuzzy point): An I-fuzzy point on X is an I-fuzzy subset $x_a \in I^X$, defined as: $x_a(y) = \begin{cases} a & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$.

Definition 8.10 (Crisp line segment in X): Let P and Q be two points in X. The line segment joining PQ is defined as the image of a map $\alpha : [0,1] \to X$ by $\alpha(t) = P + t(Q - P)$, where [0, 1] is a closed interval in **R**.

Definition 8.11 (Crisp line in X): The line in X (or $\mathbf{R}^2$) can be described as an embedding of a connected interval from R to X (or $\mathbf{R}^2$), which does not have an intersection, i.e.:

$$\alpha : [0,1] \to R^2 \text{ (or X)},$$

where [0, 1] is a closed interval in **R** and $\alpha(t_1) \neq \alpha(t_2)$ for all $t_1 \neq t_2$, $t_1, t_2 \in [0,1]$.

Definition 8.12 (Simple fuzzy line segment): The simple fuzzy line segment (L) is a fuzzy subset in X with

(i) for any $\alpha \in (0, 1)$, the fuzzy line L_α (the interior of fuzzy line segment L) is a supported connected line segment (i.e., a crisp line segment in the background topological space) in the background topological space and

(ii) $\partial L \left(= L_\alpha \wedge L^{1-\alpha}\right)$ has at most two supported connected components.

Definition 8.13 (Simple fuzzy line): A fuzzy subset in X is called a simple fuzzy line (L) if L is a supported connected line in the background topological space (i.e., a crisp line in the background topological space).

Remark 3: Geometrically, GIS features can be classified as points, lines, and polygons. Actually, a simple fuzzy line segment is the basic element of a fuzzy line. Indeed, any fuzzy line can be represented by a composition of simple fuzzy line segments in GIS. Moreover, simple fuzzy points, simple fuzzy line segments, and simple fuzzy regions are defined and serve to model topological relations between spatial objects. Figure 8.3 shows a simple fuzzy line segment while Figure 8.4 shows a non-simple fuzzy line segment.

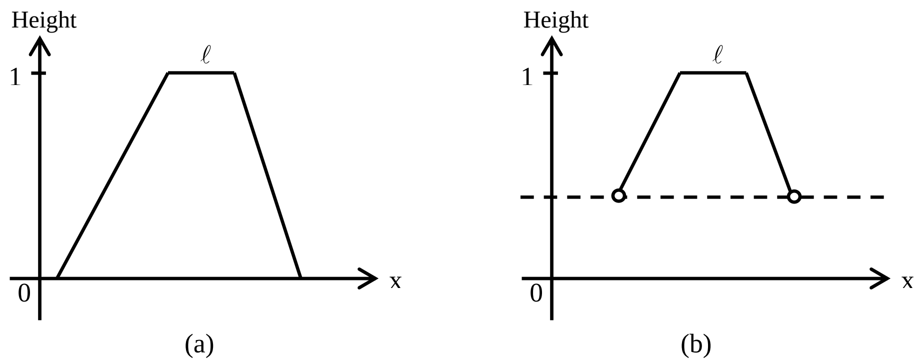
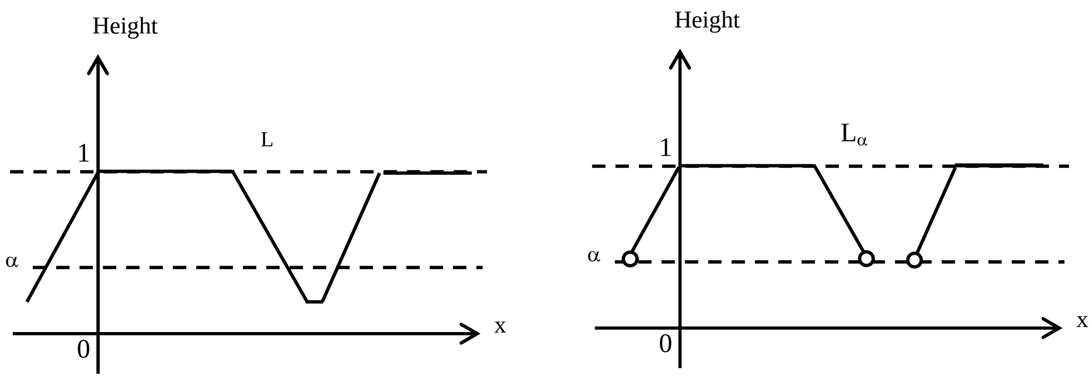


Figure 8.3: A simple fuzzy line segment



(a) (b)

Figure 8.4: A non-simple fuzzy line segment

Definition 8.14 (Fuzzy region in X): A fuzzy set A in X is called a fuzzy region if supp(A) has a non-empty interior in the background topological space.

Definition 8.15 (Simple fuzzy region): A simple fuzzy region is a fuzzy region in X with

(i) for any $\alpha \in (0,1)$, the fuzzy set A_α and $\partial A\left(= A_\alpha \wedge A^{1-\alpha}\right)$ are two supported connected regular bounded open sets in the background topological space.

(ii) in the background topological space, any outward normal from Supp(A_α) must meet Supp(∂A) and have only one component.

Remark 4: Figure 8.5 shows a simple fuzzy region, since for any $\alpha \in (0,1)$, the fuzzy set A_α and $\partial A\left(= A_\alpha \wedge A^{1-\alpha}\right)$ are two supported connected, regular bounded open sets in the background topological space. Any outward normal from Supp(A_α) must meet Supp(∂A) and have only one component.

Figure 8.6 shows a non-simple fuzzy region, since for some $\alpha_0 \in (0,1)$, the fuzzy set A_α is not a supported connected set (which has two components).

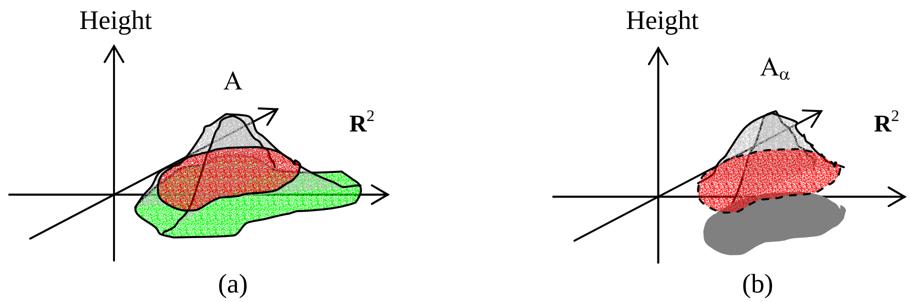


Figure 8.5: A simple fuzzy region

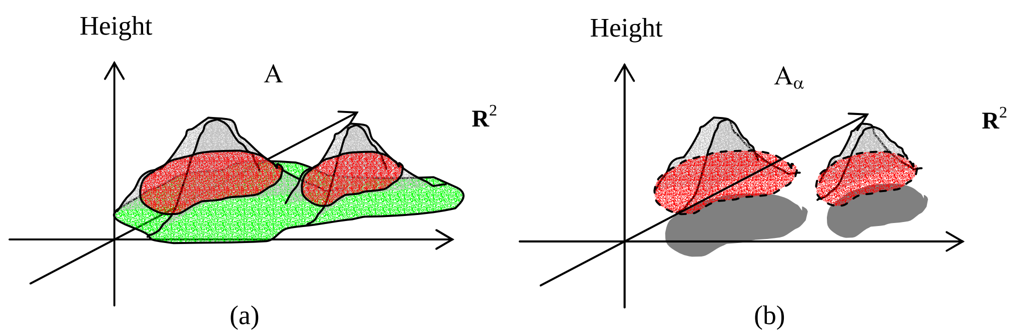


Figure 8.6: A non-simple fuzzy region

Proposition 8.16: A simple fuzzy set (region or line) is supported by a connected fuzzy set.

Proof: If not, Supp(A) has two supported connected components. Let $x_1 \in \text{Supp}(A_1)$ and $x_2 \in \text{Supp}(A_2)$ be these two supported connected components. Take $\alpha_o = \frac{1}{2}\min\{A(x_1), A(x_2)\}$; then, $\text{Supp}(A_{\alpha_o})$ has two supported connected components.

Q.E.D.

Proposition 8.17: Let (X, I^X, δ, β) and (Y, I^Y, μ, γ) be two I-fuzzy spaces, $f : X \to Y$ an ordinary continuous mapping. If $A \in I^X$ is simple fuzzy set, then $f^{\to}(A)$ is also simple fuzzy set.

Q.E.D.

8.4 Summary

Fuzzy topological relations are elementary relations in studying the topological relations between spatial objects in GIS, especially for uncertain spatial objects in GIS. In this chapter, we presented a study on developing method for computing the fuzzy topological relations of spatial objects, based on the recently developed computational fuzzy topological space. Our contributions include the following: (a) proposing the homeomorphic invariants of the fuzzy topological space; (b) defining the connectivity based on the newly fuzzy topological space; and (c) modeling the simple fuzzy objects in GIS.

The preserving properties and the connectivity of the newly developed fuzzy topological space, based on which the topological relations are invariants under homeomorphic mappings, were studied. With such a development, we can guarantee the unchanged properties in a GIS transformation, such as the maintenance of topological consistency in transferring a map from one system to another.

Besides the above theoretical developments, we can start to study the number of topological relations between spatial objects. That is based on this work and we will focus on investigating the quantitatively topological relations in next chapter, i.e. topological relations among simple fuzzy region to simple fuzzy region; simple fuzzy regions to simple line segments, simple fuzzy regions to fuzzy points, simple fuzzy line segments to simple fuzzy line segments, and simple fuzzy line segments to fuzzy points.

CHAPTER NINE
QUANTITATIVE FUZZY TOPOLOGICAL RELATIONS BETWEEN SIMPLE FUZZY OBJECTS

In the practical application of GIS, except the region to region case, there are many other special cases, for example the topological relations between line and region. Moreover, as we mentioned before, point, line and region (polygon) are the basic elements in GIS. Therefore, modeling the topological relations between fuzzy region to fuzzy line, fuzzy region to fuzzy point, fuzzy line to fuzzy line, fuzzy line to fuzzy point are important. Therefore, the result of topological relations between (1) simple fuzzy regions; (2) simple fuzzy region to simple fuzzy line segment; (3) simple fuzzy region to simple fuzzy point; (4) simple fuzzy line segment to simple fuzzy line segment; (5) simple fuzzy line segment to simple fuzzy points; and (6) simple fuzzy point to simple fuzzy point in $\mathbf{R}^2$ are presented in this chapter. Moreover, in this chapter, we have discovered that (a) the topological relations between simple fuzzy regions are 44; (b) the topological relations between simple fuzzy region to line segment are 16; (c) the topological relations between simple fuzzy line segments are 46; (d) 3 topological relations between simple fuzzy region to fuzzy point and simple fuzzy line segment to fuzzy point. These will form the basis of the full set topological relations between spatial objects.

9.1 Quantitative fuzzy topological relations between simple fuzzy regions

In this section, we now provide a method for determining fuzzy topological relations between spatial objects. The framework for quantify the spatial relations in the method for computing the quantitative fuzzy topological relations between spatial objects has been newly developed in this study of the nine-intersection model. The framework of the fuzzy topological relations between two objects A and B is defined as follows:

$$\begin{pmatrix} \int_X A_\alpha \wedge B_\alpha \, dV & \int_X \partial A \wedge B_\alpha \, dV & \int_X \left(A^c\right)_\alpha \wedge B_\alpha \, dV \\ \int_X A_\alpha \wedge \partial B \, dV & \int_X \partial A \wedge \partial B \, dV & \int_X \left(A^c\right)_\alpha \wedge \partial B \, dV \\ \int_X A_\alpha \wedge \left(B^c\right)_\alpha \, dV & \int_X \partial A \wedge \left(B^c\right)_\alpha \, dV & \int_X \left(A^c\right)_\alpha \wedge \left(B^c\right)_\alpha \, dV \end{pmatrix}.$$

First, we should explain the meaning of the notations $\int_X A_\alpha \wedge B_\alpha \, dV$ and $\int_X \partial A \wedge \partial B \, dV$ etc., which define the quantitative relations between spatial objects.

Definition 9.1: For any two fuzzy spatial objects (fuzzy sets) A, $B \in I^X$, defined

$$\int_X A \wedge B \, dV = \frac{\int_X A \wedge B(x) \, dx}{\int_X A \vee B(x) \, dx}.$$

The geometric meaning of $\int_X A \wedge B \, dV$ is illustrated in Figure 9.1, which is the ratio of the area (or volume) of the *meet* of two fuzzy spatial objects to the *join* of two fuzzy spatial objects. In previously studying the topological relations between spatial objects, researchers (Egenhofer, 1991; Cohn and Gotts, 1996; Clementini and Di Felice, 1996; Smith, 1996; Shi and Guo, 1999; Tang and Kainz, 2002) used the intersection of sets to give quantitative relations. Here, we use volume ratio (the volume of the intersection to the volume of the meet of two fuzzy spatial objects (see Figure 9.1). Obviously, the former models can only give a local quantitative value (or Boolean value) at different points, while the method in this model can provide a global quantitative value to each spatial object.

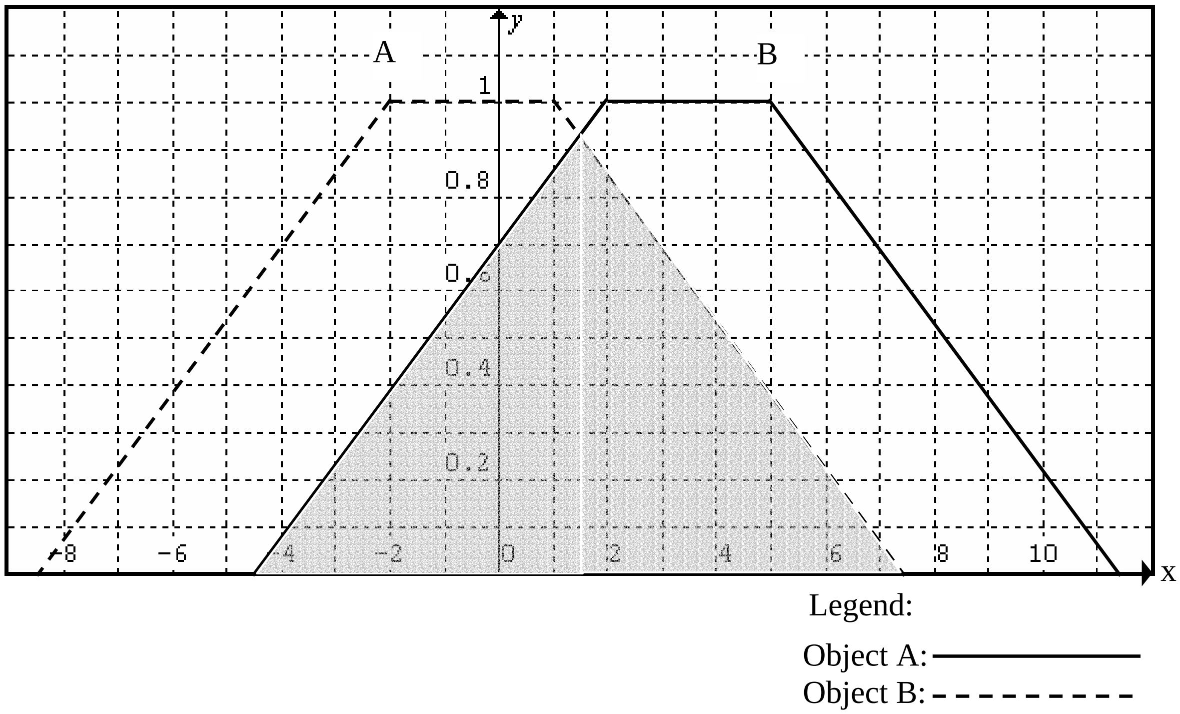


Figure 9.1: The geometric meaning of integrations

Since for any spatial object $A \in I^X$, the three components, interior (A_α), boundary (∂A), and exterior ($\left(A^c\right)_\alpha$) are not disjointed (see Figure 9.1), there is double counting of the integration. For example, $\int_X A_\alpha \wedge B_\alpha \, dV$ and $\int_X \partial A \wedge \partial B \, dV$ will be double counted on the part $\partial A \wedge B_\alpha$ and $\partial B \wedge A_\alpha$. But we have the bound of the overlapping part; that is, for any α, $(A_\alpha \wedge \partial A)(x) < 1-\alpha$ and $\left(\left(A^c\right)_\alpha \wedge \partial A\right)(x) < 1-\alpha$; and for $\alpha \geq \frac{1}{2}$, $A_\alpha \wedge \left(A^c\right)_\alpha = \phi$, respectively. This means that we can control the size of the overlapping between the interior to the boundary and the exterior to the boundary by choosing a large value for α. We can see that if the value of α is very close to zero, the uncertainty is very large, while if the value of α is very close to one, the uncertainty is very small. Figure 9.2 shows how the sizes of $\mathrm{Supp}(A_\alpha \wedge \partial A)$ and $\mathrm{Supp}\left(\left(A^c\right)_\alpha \wedge \partial A\right)$ change. That is, when a larger α is chosen, smaller values of $A_\alpha \wedge \partial A$ and $\left(A^c\right)_\alpha \wedge \partial A$ are obtained; when a smaller α is chosen, larger values of $A_\alpha \wedge \partial A$ and $\left(A^c\right)_\alpha \wedge \partial A$ are obtained.

Moreover, if the topology is discrete, for a suitable α, the uncertainty of spatial objects can be controlled to zero. Indeed, an α is chosen such that $A_\alpha \wedge \partial A = 0$ and $(A^c)_\alpha \wedge \partial A = 0$. Therefore, in modeling topological relations between simple spatial regions, we may neglect the effect of $A_\alpha \wedge \partial A$ and $(A^c)_\alpha \wedge \partial A$.

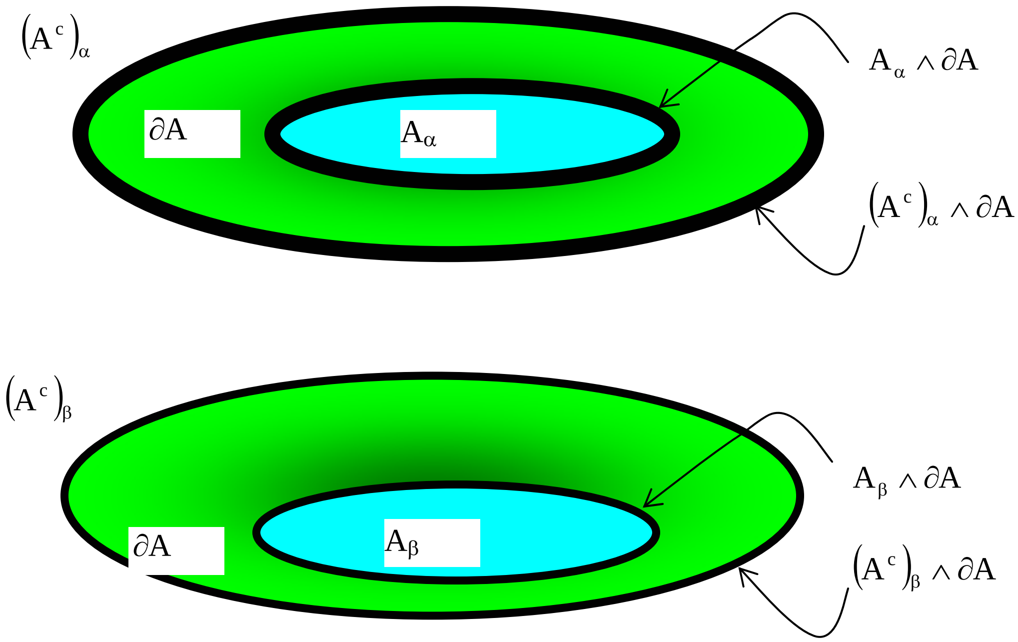


Figure 9.2: When a smaller α is chosen, larger $A_\alpha \wedge \partial A$ and $(A^c)_\alpha \wedge \partial A$ are obtained, as indicated in figure 9.2 (a); when a larger α is chosen, smaller $A_\alpha \wedge \partial A$ and $(A^c)_\alpha \wedge \partial A$ are obtained, as indicated in figure 9.2 (b).

Identification by a 3×3 ***integration matrix***

By directly using the value of zero and non-zero, the 3×3 integration matrix

$$\begin{pmatrix} \int_X A_\alpha \wedge B_\alpha \, dV & \int_X \partial A \wedge B_\alpha \, dV & \int_X (A^c)_\alpha \wedge B_\alpha \, dV \\ \int_X A_\alpha \wedge \partial B \, dV & \int_X \partial A \wedge \partial B \, dV & \int_X (A^c)_\alpha \wedge \partial B \, dV \\ \int_X A_\alpha \wedge (B^c)_\alpha \, dV & \int_X \partial A \wedge (B^c)_\alpha \, dV & \int_X (A^c)_\alpha \wedge (B^c)_\alpha \, dV \end{pmatrix}$$

gives a total of $2^9 = 512$ different cases of topological relations between two simple fuzzy regions. However, for a simple fuzzy region in $\mathbf{R}^2$, it is not possible for all of these topological relations to occur.

(i) Let A and B be two simple fuzzy regions in $\mathbf{R}^2$. The 3×3 integration matrix will then satisfy the following conditions. Due to the fact that simple fuzzy region are bounded, $\int_X \left(A^c\right)_\alpha \wedge \left(B^c\right)_\alpha \, dV$ is non-zero for all cases.

(ii) Each part of A (A_α, ∂A and $\left(A^c\right)_\alpha$) must intersect with at least one part of B (B_α, ∂B and $\left(B^c\right)_\alpha$), and vice versa.

(iii) If $\int_X A_\alpha \wedge B_\alpha \, dV$ and $\int_X A_\alpha \wedge \left(B^c\right)_\alpha \, dV$ are non-zero, then $\int_X A_\alpha \wedge \partial B \, dV$ must be non-zero, and vice versa.

(iv) If $\int_X \partial A \wedge \partial B \, dV$ is zero, then either $\int_X \left(A^c\right)_\alpha \wedge \partial B \, dV$ or $\int_X \partial A \wedge \left(B^c\right)_\alpha \, dV$ is non-zero.

(v) If both $\int_X A_\alpha \wedge \partial B \, dV$ and $\int_X \partial A \wedge B_\alpha \, dV$ are non-zero, then $\int_X \partial A \wedge \partial B \, dV$ must be non-zero.

(vi) If $\int_X A_\alpha \wedge \left(B^c\right)_\alpha \, dV$ is non-zero, then $\int_X \partial A \wedge \left(B^c\right)_\alpha \, dV$ must be non-zero, and vice versa.

(vii) If $\int_X A_\alpha \wedge B_\alpha \, dV$ is zero and $\int_X \partial A \wedge B_\alpha \, dV$ is non-zero, then $\int_X \partial A \wedge \partial B \, dV$ must be non-zero, and vice versa.

(viii) If $\int_X A_\alpha \wedge B_\alpha \, dV$ and $\int_X A_\alpha \wedge \partial B \, dV$ are non-zero, then $\int_X \partial A \wedge B_\alpha \, dV$ and $\int_X \partial A \wedge \partial B \, dV$ are non-zero, and vice versa.

(ix) If $\int_X A_\alpha \wedge B_\alpha \, dV$ and $\int_X A_\alpha \wedge \partial B \, dV$ are zero, then $\int_X A_\alpha \wedge \left(B^c\right)_\alpha \, dV$ and $\int_X \partial A \wedge \left(B^c\right)_\alpha \, dV$ are non-zero, and vice versa.

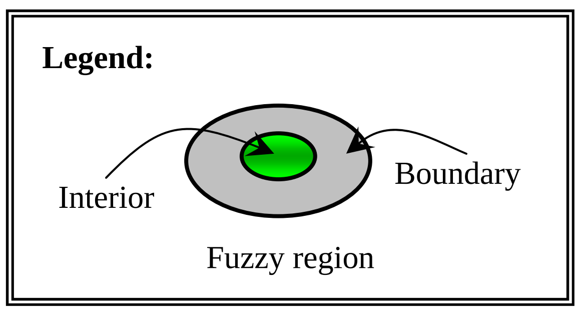
(x) If $\int_X \partial A \wedge B_\alpha \, dV$ and $\int_X \partial A \wedge \left(B^c\right)_\alpha \, dV$ are non-zero, then $\int_X \partial A \wedge \partial B \, dV$ must be non-zero, and vice versa.

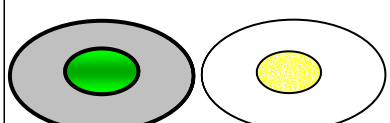
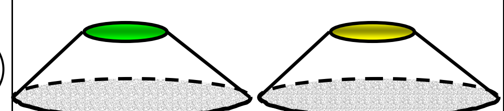
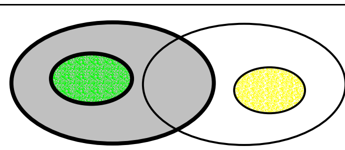
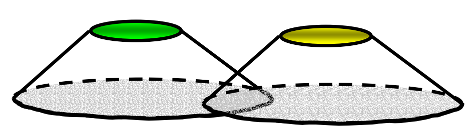
(xi) If $\int_X A_\alpha \wedge B_\alpha \, dV$ and $\int_X \partial A \wedge B_\alpha \, dV$ are non-zero, then $\int_X (A^c)_\alpha \wedge B_\alpha \, dV$ is non-zero, and vice versa.

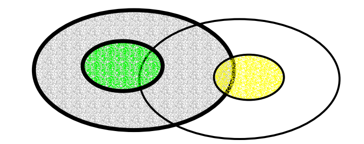
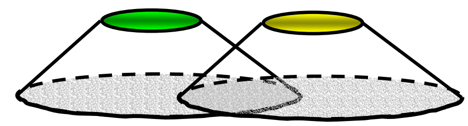
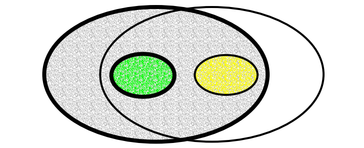
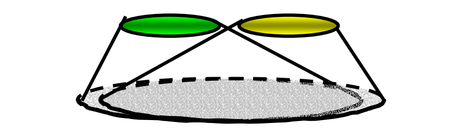
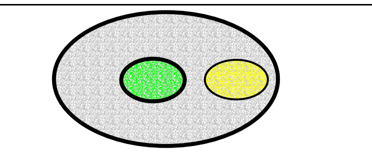
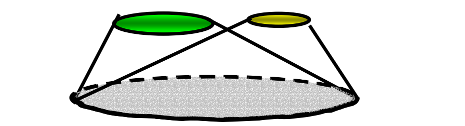
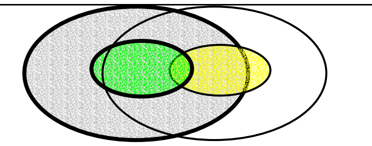
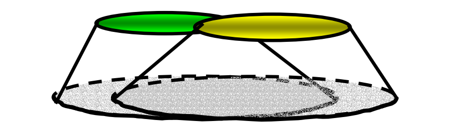
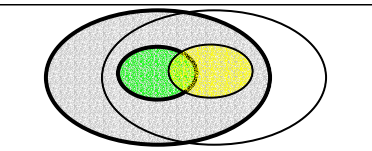
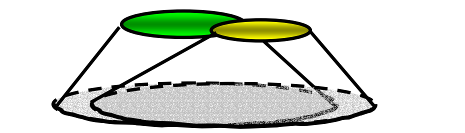
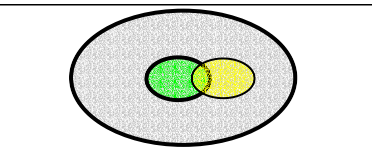
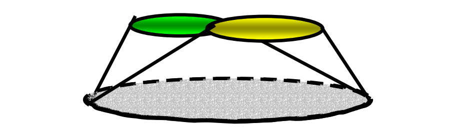
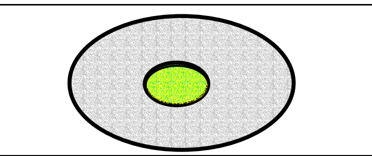
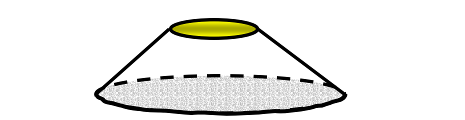
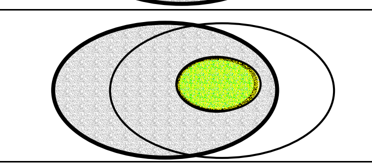
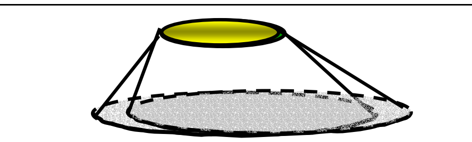
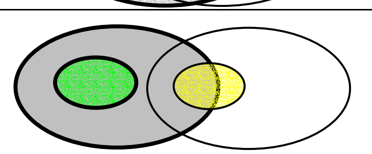
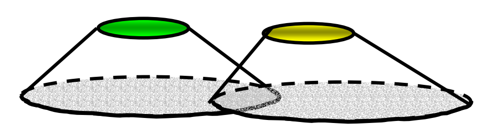
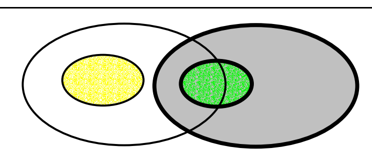
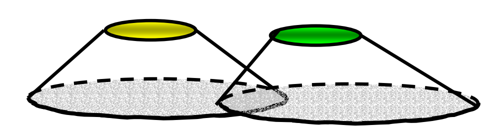
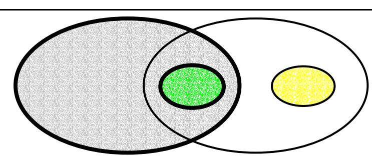
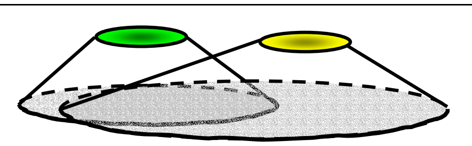
Based on the above conditions, 44 relations between simple fuzzy regions in $\mathbf{R}^2$ have been identified by using the 3×3 integration matrix. This result is similar to the two results from previous studies (Clementini and Di Felice, 1996) and (Tang and Kainz, 2002). However the method used here is totally different. In fact, the method proposed in this study is a generalization of the previous models. This will be discussed in detail in the coming section. The 44 relations between two simple fuzzy regions are listed in Table 9.1, which include the value of the matrix, the support view, and membership vale of the fuzzy topological relations between two simple fuzzy regions.

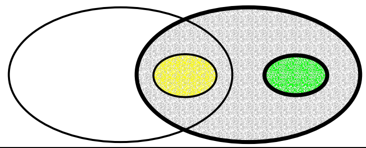
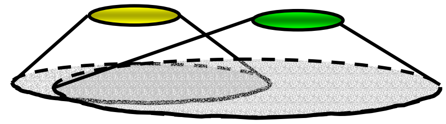
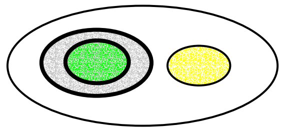
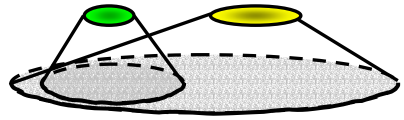
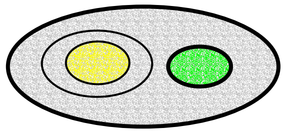
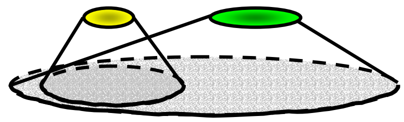
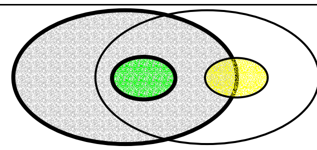
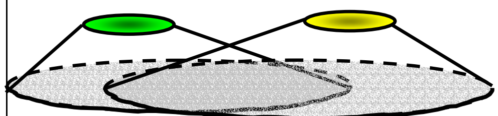
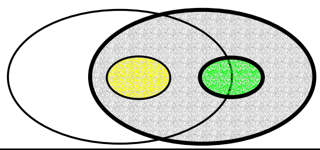
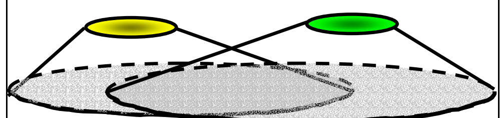
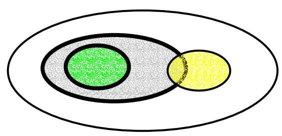
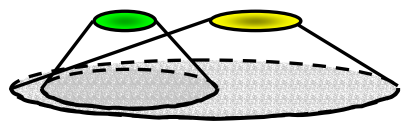
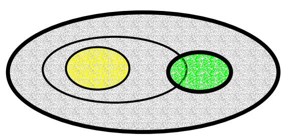
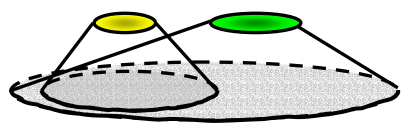
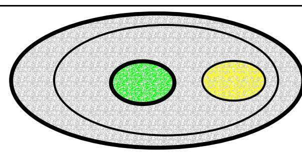
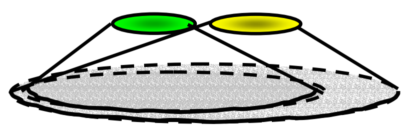
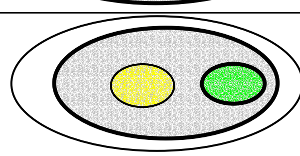
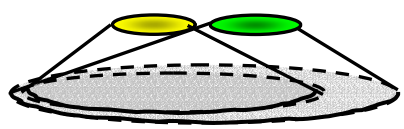
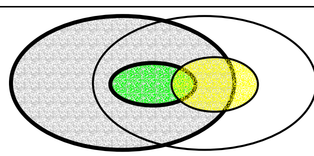
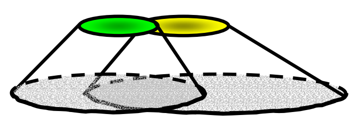
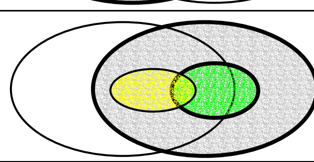
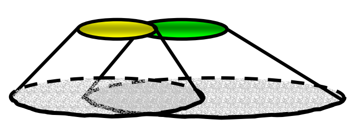
The topological relations among simple fuzzy regions to simple line segments, simple fuzzy regions to fuzzy points, simple fuzzy line segments to simple fuzzy line segments and simple fuzzy line segments to fuzzy points will be also investigated and reported in this chapter.

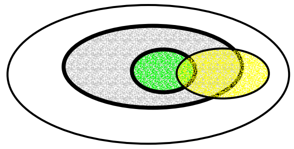
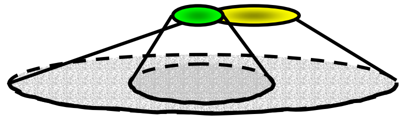
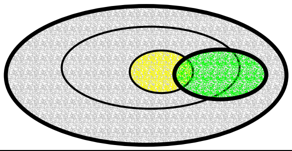
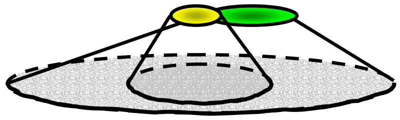
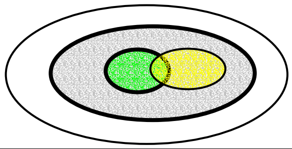
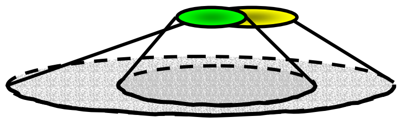
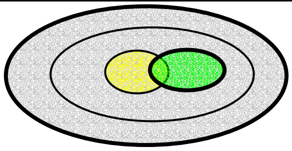
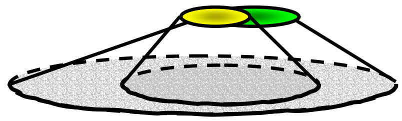
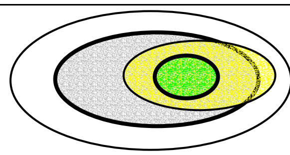
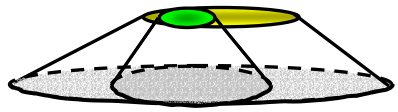
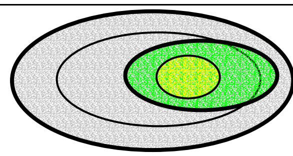
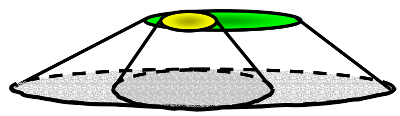
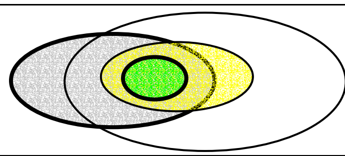
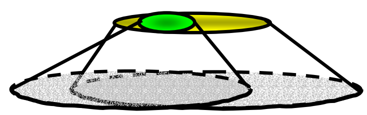
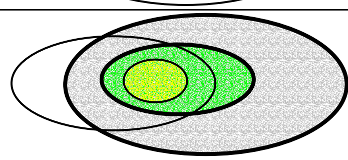
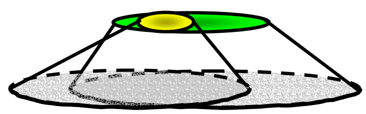
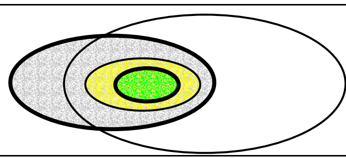
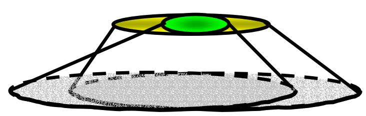
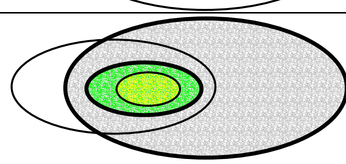
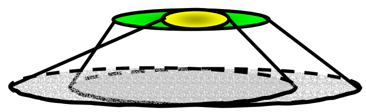
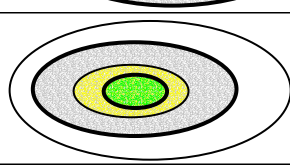
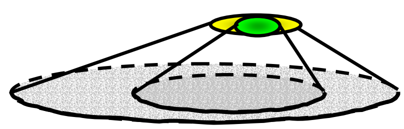
Table 9.1: The 44 relations between two simple fuzzy regions in $\mathbf{R}^2$

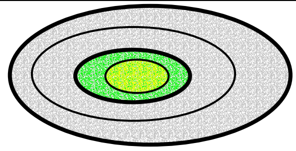
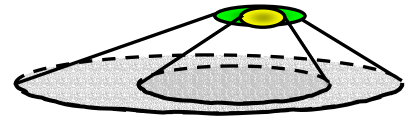
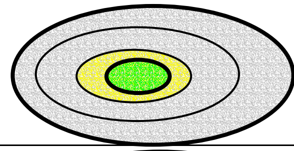
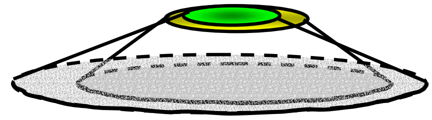
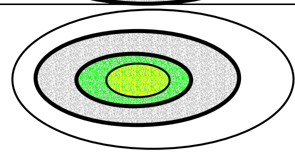
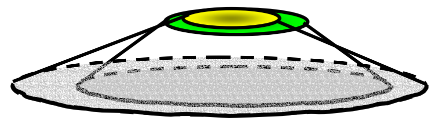
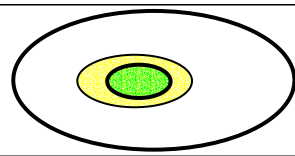
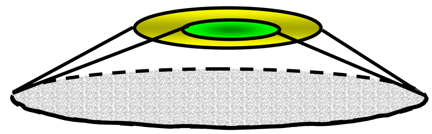
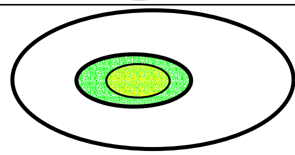
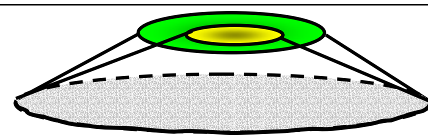
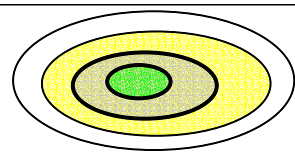
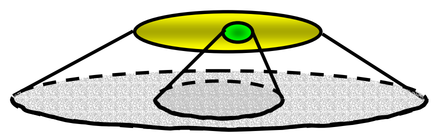
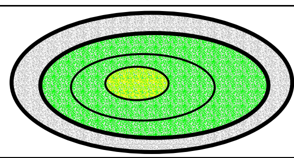
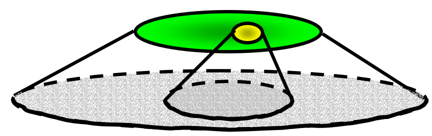
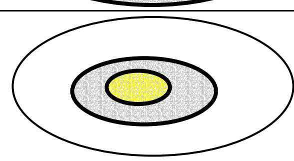
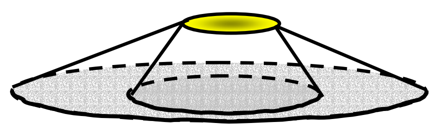
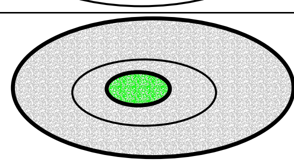
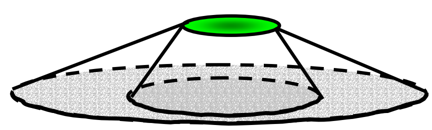


Matrix	Illustration Support view	Illustration in Membership value
1. $\begin{pmatrix} \phi & \phi & \sim\phi \\ \phi & \phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$		
2. $\begin{pmatrix} \phi & \phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$		

3. $\begin{pmatrix} \phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$		
4. $\begin{pmatrix} \phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$		
5. $\begin{pmatrix} \phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		
6. $\begin{pmatrix} \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$		
7. $\begin{pmatrix} \sim\phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$		
8. $\begin{pmatrix} \sim\phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		
9. $\begin{pmatrix} \sim\phi & \phi & \phi \\ \phi & \sim\phi & \phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		
10. $\begin{pmatrix} \sim\phi & \phi & \phi \\ \phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$		
11. $\begin{pmatrix} \phi & \phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$		
12. $\begin{pmatrix} \phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$		
13. $\begin{pmatrix} \phi & \phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$		

14. $\begin{pmatrix} \phi & \sim\phi & \phi \\ \phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$		
15. $\begin{pmatrix} \phi & \phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		
16. $\begin{pmatrix} \phi & \sim\phi & \phi \\ \phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$		
17. $\begin{pmatrix} \phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$		
18. $\begin{pmatrix} \phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$		
19. $\begin{pmatrix} \phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		
20. $\begin{pmatrix} \phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$		
21. $\begin{pmatrix} \phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$		
22. $\begin{pmatrix} \phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		
23. $\begin{pmatrix} \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$		
24. $\begin{pmatrix} \sim\phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$		

25. $\begin{pmatrix} \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		
26. $\begin{pmatrix} \sim\phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$		
27. $\begin{pmatrix} \sim\phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		
28. $\begin{pmatrix} \sim\phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$		
29. $\begin{pmatrix} \sim\phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		
30. $\begin{pmatrix} \sim\phi & \phi & \phi \\ \sim\phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$		
31. $\begin{pmatrix} \sim\phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$		
32. $\begin{pmatrix} \sim\phi & \phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$		
33. $\begin{pmatrix} \sim\phi & \sim\phi & \phi \\ \phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$		
34. $\begin{pmatrix} \sim\phi & \phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$		
35. $\begin{pmatrix} \sim\phi & \sim\phi & \phi \\ \phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		

36. $\begin{pmatrix} \sim\phi & \phi & \phi \\ \sim\phi & \sim\phi & \phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$		
37. $\begin{pmatrix} \sim\phi & \sim\phi & \phi \\ \phi & \sim\phi & \phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$		
38. $\begin{pmatrix} \sim\phi & \phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		
39. $\begin{pmatrix} \sim\phi & \sim\phi & \phi \\ \phi & \sim\phi & \phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		
40. $\begin{pmatrix} \sim\phi & \phi & \phi \\ \sim\phi & \sim\phi & \phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		
41. $\begin{pmatrix} \sim\phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		
42. $\begin{pmatrix} \sim\phi & \phi & \phi \\ \sim\phi & \phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$		
43. $\begin{pmatrix} \sim\phi & \phi & \phi \\ \phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		
44. $\begin{pmatrix} \sim\phi & \phi & \phi \\ \phi & \sim\phi & \phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$		

9.2 Multiple integrations

In fuzzy topology, because the memberships functions in $\mathbf{R}^2$ are involving two variables. In computing their integration ratios, we need to consider whether it is volume integration or line integration. In this section, we will review the concept of volume integration and line integration.

9.2.1 Surface Integrals in R^3

Let $\mu_A : \mathrm{R}^2 \to [0,1]$ be a fuzzy membership function of fuzzy set A. In GIS, point, line, region are three elementary elements. We first let $\mu_A(x, y)$ be a fuzzy region in $\mathbf{R}^2$, then fuzzy set A with membership function $\mu_A(x, y)$ can be thought as a graph of the surface *S* giving by graphing $\mu_A(x, y)$ in $\mathbf{R}^3$ (see Figure 9.3).

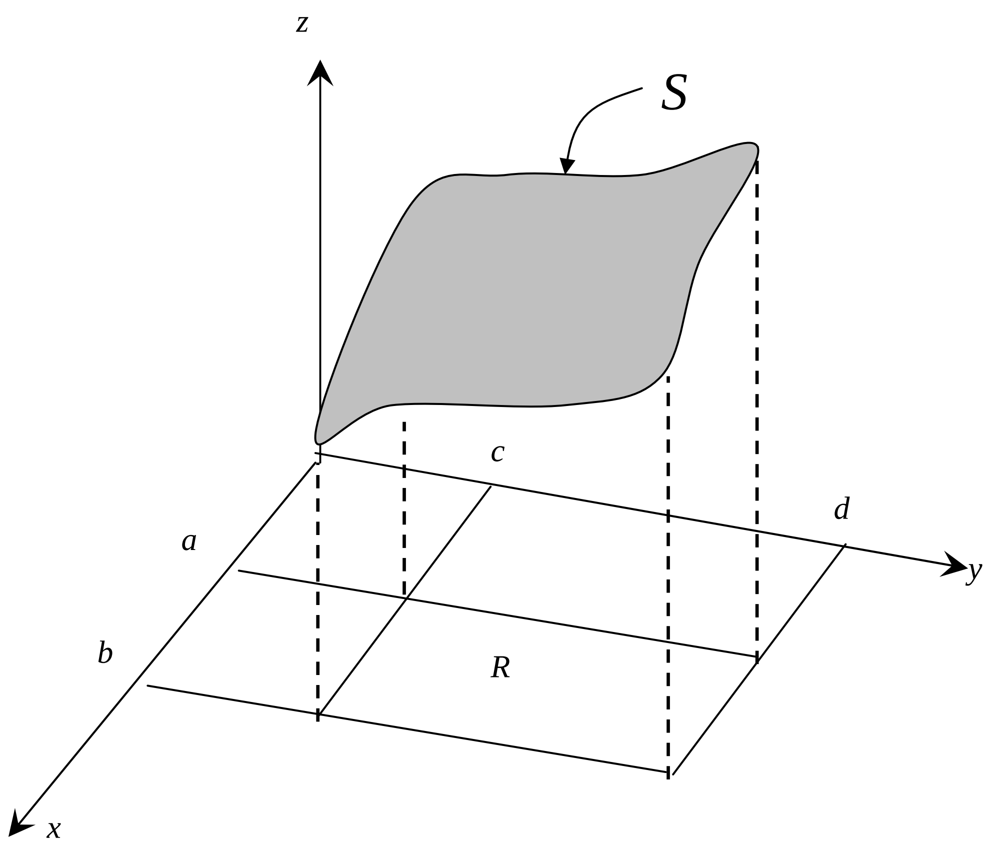


Figure 9.3: The concept of surface integration

To calculate the volume of a fuzzy set in $\mathbf{R}^2$, we actually need to calculate the volume of the region under *S* (and above the *xy*-plane).

The double integral of a function of two variables over a rectangular region R is defined as:

$$\iint_R \mu_A(x, y)\, dA = \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} \mu_A(x_i, y_j)\, \Delta A,$$

where $\mu_A(x_i, y_j)$ is the membership height of each rectangular region.

9.2.2 Line Integrals

Let $\mu_L(x, y)$ be a fuzzy line in $\mathbf{R}^2$, then fuzzy set L with membership function $\mu_L(x, y)$ can be thought as a graph of a smooth curve C giving by graphing $\mu_L(x, y)$ in $\mathbf{R}^3$ (see Figure 9.4).

Then the **line integral** of $\mu_L(x, y)$ along a smooth curve C is denoted by, $\int_C \mu_L(x, y)\, ds$,

where ds is the line integral of μ_L with respect to arc length. Let the parametric equation of the smooth curve C be $\mu_L(x(t), y(t))$, then the **line integral** of $\mu_L(x, y)$ along a smooth curve C can be written as $\int_a^b \mu_L(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ or $\int_a^b \mu_L(x(t), y(t))\,\|\, \gamma'(t) \,\|\, dt$, where $P = (x(a), y(a))$ and $Q = (x(b), y(b))$ are the two end point of a smooth curve C and $\|\, \gamma'(t) \,\|$ is the magnitude or norm of $\gamma'(t)$ (see Figure 9.4).

An basic property of line integrals is that $\int_C \mu_L(x, y)\, ds = \int_{-C} \mu_L(x, y)\, ds$, which means we can change the direction of a line integral with respect to arc length and will not change the value of the integral.

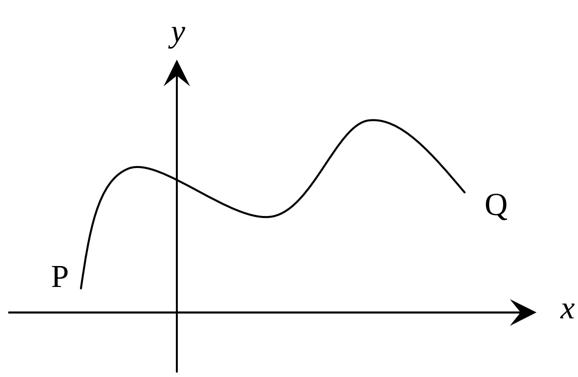


Figure 9.4: A smooth curve C

A piecewise smooth curve $\mu_L(x, y)$ is a curve that can be written as the union of a finite number of smooth curves, C_1, …, C_n where the end point of C_i is the starting point of C_{i+1}. Figure 9.5 is an illustration of a piecewise smooth curve.

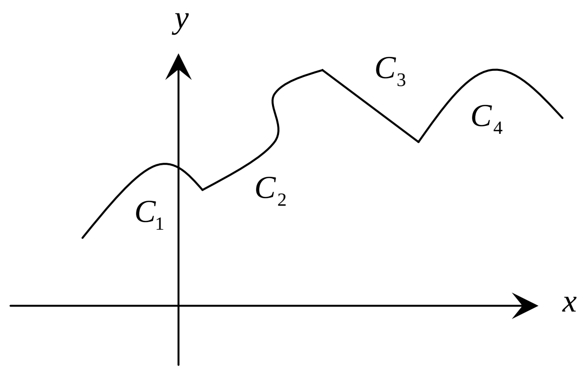


Figure 9.5: A piecewise smooth curve C

The line integral over a **piecewise smooth curves** can be evaluated by the formula $\int_C \mu_L(x, y)\, ds = \int_{C_1} \mu_L(x, y)\, ds + \int_{C_2} \mu_L(x, y)\, ds + \ldots + \int_{C_n} \mu_L(x, y)\, ds$.

9.3 Quantitative fuzzy topological relations between other simple fuzzy objects

Based on the definitions of simple fuzzy line segment in the previous section, and a further definition of quantitative method in this study, we now provide a method for determining fuzzy topological relations between simple fuzzy region to simple fuzzy region, simple fuzzy region to simple fuzzy line segment, simple fuzzy region to simple fuzzy point, simple fuzzy line segment to simple fuzzy line segment, simple fuzzy line segment to simple fuzzy points and simple fuzzy point to simple fuzzy point respectively. The case of simple fuzzy region to simple fuzzy region has been studied previously. Here we concentrate on the others and actually there are several technique problems need to overcome due to the problem of intersection integrals. The 3×3 integration model $\begin{pmatrix} \int_X A_\alpha \wedge B_\alpha \, dV & \int_X \partial A \wedge B_\alpha \, dV & \int_X (A^c)_\alpha \wedge B_\alpha \, dV \\ \int_X A_\alpha \wedge \partial B \, dV & \int_X \partial A \wedge \partial B \, dV & \int_X (A^c)_\alpha \wedge \partial B \, dV \\ \int_X A_\alpha \wedge (B^c)_\alpha \, dV & \int_X \partial A \wedge (B^c)_\alpha \, dV & \int_X (A^c)_\alpha \wedge (B^c)_\alpha \, dV \end{pmatrix}$ between spatial fuzzy objects is designed for fuzzy region to region. Where for any two fuzzy spatial objects (fuzzy sets) A, $B \in I^X$, $\int_X A \wedge B \, dV = \dfrac{\int_X (A \wedge B)(x)\, dx}{\int_X (A \vee B)(x)\, dx}$.

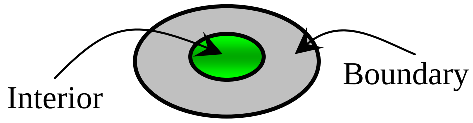
But in the case of fuzzy region to fuzzy line, the integration format should be changed. Indeed, for example, if L is a fuzzy line, B is a fuzzy region. Then $L \vee B$ is a fuzzy region, therefore, the integration in $\mathbf{R}^2$ should be surface integral. On the other hand, $L \wedge B$ is a fuzzy line, therefore, the integration should be line integral. In this case, the integration ration should be $\dfrac{\int_X (L \wedge B)(x)\, ds}{\int_X (L \vee B)(x)\, dV}$, where ds is the line integral of $L \wedge B(x)$ with respect to arc length and dV is the surface integral of $L \vee B(x)$ with respect to the surface. With no confusion, we still use the same notation to denote the model, that is

$$\begin{pmatrix} \int_X A_\alpha \wedge B_\alpha \, dV & \int_X \partial A \wedge B_\alpha \, dV & \int_X (A^c)_\alpha \wedge B_\alpha \, dV \\ \int_X A_\alpha \wedge \partial B \, dV & \int_X \partial A \wedge \partial B \, dV & \int_X (A^c)_\alpha \wedge \partial B \, dV \\ \int_X A_\alpha \wedge (B^c)_\alpha \, dV & \int_X \partial A \wedge (B^c)_\alpha \, dV & \int_X (A^c)_\alpha \wedge (B^c)_\alpha \, dV \end{pmatrix}.$$

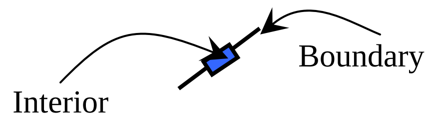
By using the 3×3 integration model, topological relations between simple fuzzy region to simple fuzzy line segment, simple fuzzy region to simple fuzzy point, simple fuzzy line segment to simple fuzzy line segment, simple fuzzy line segment to simple fuzzy points and simple fuzzy point to simple fuzzy point in $\mathbf{R}^2$ are identified (see Table 9.2, Table 9.3 and Table 9.4).

Table 9.2: The 16 relations between simple fuzzy region and simple fuzzy line segment in $\mathbf{R}^2$

Legend:



Fuzzy region



Fuzzy line segment

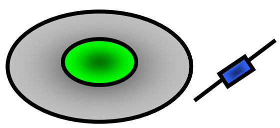
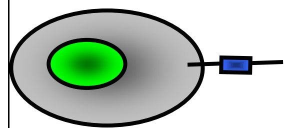
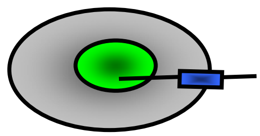
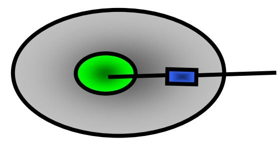
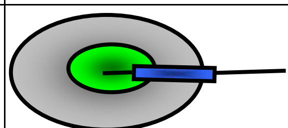
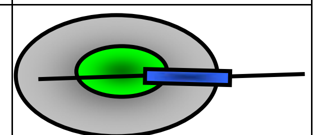
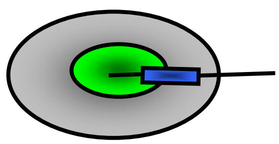
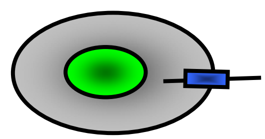
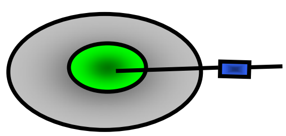
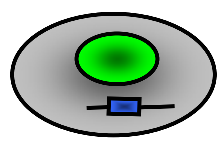
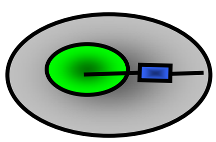
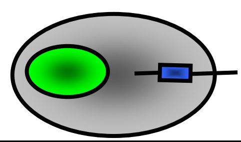
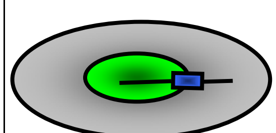
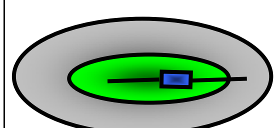
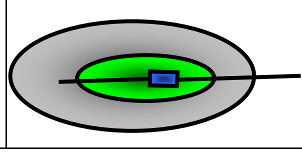
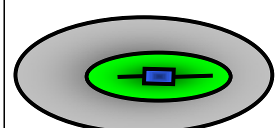
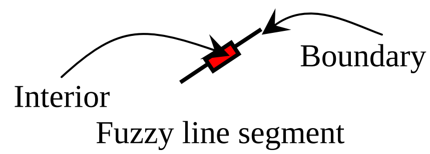
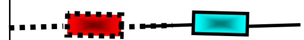
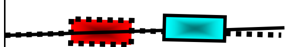
Matrix	Illustration	Matrix	Illustration
1. $\begin{pmatrix} \phi & \phi & \sim\phi \\ \phi & \phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$		2. $\begin{pmatrix} \phi & \phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$	
3. $\begin{pmatrix} \phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$		4. $\begin{pmatrix} \phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$	
5. $\begin{pmatrix} \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$		6. $\begin{pmatrix} \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$	
7. $\begin{pmatrix} \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$		8. $\begin{pmatrix} \phi & \phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$	
9. $\begin{pmatrix} \phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$		10. $\begin{pmatrix} \phi & \phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$	
11. $\begin{pmatrix} \phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		12. $\begin{pmatrix} \phi & \phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$	
13. $\begin{pmatrix} \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		14. $\begin{pmatrix} \sim\phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$	
15. $\begin{pmatrix} \sim\phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$		16. $\begin{pmatrix} \sim\phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$	

Table 9.3: The 46 topological relations between two simple fuzzy line segments.

Legend:



Matrix	Illustration	Matrix	Illustration
1. $\begin{pmatrix} \phi & \phi & \sim\phi \\ \phi & \phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$		2. $\begin{pmatrix} \phi & \phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$	
3. $\begin{pmatrix} \phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$		4. $\begin{pmatrix} \phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$	
5. $\begin{pmatrix} \phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		6. $\begin{pmatrix} \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$	
7. $\begin{pmatrix} \sim\phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$		8. $\begin{pmatrix} \sim\phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \phi \\ \phi & \phi & \sim\phi \end{pmatrix}$	
9. $\begin{pmatrix} \sim\phi & \phi & \phi \\ \phi & \sim\phi & \phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		10. $\begin{pmatrix} \sim\phi & \phi & \phi \\ \phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$	
11. $\begin{pmatrix} \phi & \phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$		12. $\begin{pmatrix} \phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$	
13. $\begin{pmatrix} \phi & \phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$		14. $\begin{pmatrix} \phi & \sim\phi & \phi \\ \phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$	
15. $\begin{pmatrix} \phi & \phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		16. $\begin{pmatrix} \phi & \sim\phi & \phi \\ \phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$	

17. $\begin{pmatrix} \phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$		18. $\begin{pmatrix} \phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$	
19. $\begin{pmatrix} \phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		20. $\begin{pmatrix} \phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$	
21. $\begin{pmatrix} \phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$		22. $\begin{pmatrix} \phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$	
23. $\begin{pmatrix} \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$		24. $\begin{pmatrix} \sim\phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$	
25. $\begin{pmatrix} \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		26. $\begin{pmatrix} \sim\phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$	
27. $\begin{pmatrix} \sim\phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		28. $\begin{pmatrix} \sim\phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$	
29. $\begin{pmatrix} \sim\phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		30. $\begin{pmatrix} \sim\phi & \phi & \phi \\ \sim\phi & \sim\phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$	
31. $\begin{pmatrix} \sim\phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$		32. $\begin{pmatrix} \sim\phi & \phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$	
33. $\begin{pmatrix} \sim\phi & \sim\phi & \phi \\ \phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$		34. $\begin{pmatrix} \sim\phi & \phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$	
35. $\begin{pmatrix} \sim\phi & \sim\phi & \phi \\ \phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		36. $\begin{pmatrix} \sim\phi & \phi & \phi \\ \sim\phi & \sim\phi & \phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$	
37. $\begin{pmatrix} \sim\phi & \sim\phi & \phi \\ \phi & \sim\phi & \phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$		38. $\begin{pmatrix} \sim\phi & \phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$	

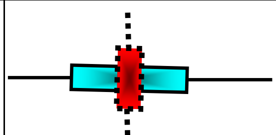
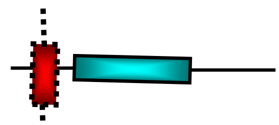
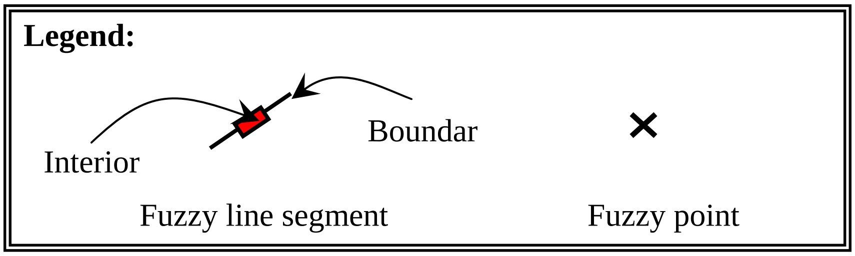
39. $\begin{pmatrix} \sim\phi & \sim\phi & \phi \\ \phi & \sim\phi & \phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		40. $\begin{pmatrix} \sim\phi & \phi & \phi \\ \sim\phi & \sim\phi & \phi \\ \phi & \phi & \sim\phi \end{pmatrix}$	
41. $\begin{pmatrix} \sim\phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		42. $\begin{pmatrix} \sim\phi & \phi & \phi \\ \sim\phi & \phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$	
43. $\begin{pmatrix} \sim\phi & \phi & \phi \\ \phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		44. $\begin{pmatrix} \sim\phi & \phi & \phi \\ \phi & \sim\phi & \phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$	
45. $\begin{pmatrix} \sim\phi & \phi & \sim\phi \\ \phi & \phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$		46. $\begin{pmatrix} \phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \\ \sim\phi & \sim\phi & \sim\phi \end{pmatrix}$	

Table 9.4: 3 relations between simple fuzzy region to fuzzy point in $\mathbf{R}^2$
and 3 relations between simple fuzzy line segment to fuzzy point in $\mathbf{R}^2$.



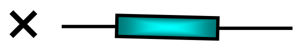
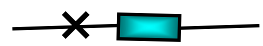
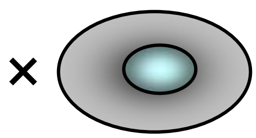
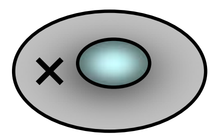
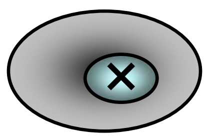
1. $\begin{pmatrix} \sim\phi & \sim\phi & \phi \\ \phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$		2. $\begin{pmatrix} \sim\phi & \phi & \phi \\ \sim\phi & \sim\phi & \sim\phi \\ \phi & \sim\phi & \sim\phi \end{pmatrix}$	
3. $\begin{pmatrix} \sim\phi & \sim\phi & \phi \\ \phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$			
1. $\begin{pmatrix} \sim\phi & \sim\phi & \phi \\ \phi & \sim\phi & \phi \\ \phi & \phi & \sim\phi \end{pmatrix}$		2. $\begin{pmatrix} \sim\phi & \phi & \phi \\ \sim\phi & \sim\phi & \phi \\ \phi & \phi & \sim\phi \end{pmatrix}$	
3. $\begin{pmatrix} \sim\phi & \sim\phi & \sim\phi \\ \phi & \phi & \sim\phi \\ \phi & \phi & \sim\phi \end{pmatrix}$			

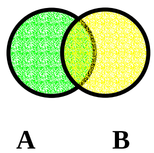
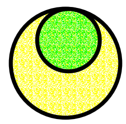
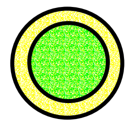
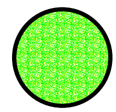
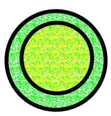
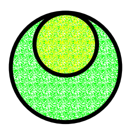
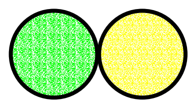
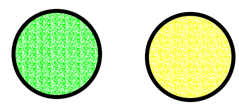
Table 9.2 shows that there are 16 relations between simple fuzzy region and simple fuzzy line segment in $\mathbf{R}^2$. Table 9.3 shows that there are 46 topological relations between two simple fuzzy line segments and Table 9.4 shows that there are 3 relations between simple fuzzy region to fuzzy point in $\mathbf{R}^2$ and 3 relations between simple fuzzy line segment to fuzzy point in $\mathbf{R}^2$.

9.4 A comparison with the existing models

In dealing with fuzzy spatial objects, Cohn and Gotts (1996) proposed the egg-yolk model and suggested using two concentric sub-regions, indicating the degree of "membership" in a vague / fuzzy region, where "yolk" represents the precise part and "egg" represents the vague/ fuzzy part of the region. Based on the Region Connection

Calculus (RCC) theory (Randell, 1992), eight basic relations can be defined. They are: DC (Disconnected), EC (Externally Connected), PO (Partially Overlapping), TPP (Tangential Proper Part), NTPP (Non-tangential Proper Part), EQ (Equal), PPI (Proper Part Inverse), and TPPI (Tangential Proper Part Inverse), respectively (see Table 9.5).

Table 9.5: RCC relations between two regions

PO(A, B)	TPP(A, B)	NTTP(A, B)	EQ(A, B)
 **A** **B**			
NTPPI(A, B)	TPPI(A, B)	EC(A, B)	DC(A, B)
			

The egg-yolk model is an extension of the RCC theory into the vague / fuzzy region. A total of 46 relations can be identified (Cohn and Gotts, 1996).

In dealing with spatial objects with indeterminate boundaries, based on Egenhofer's nine-intersection model

$$\begin{pmatrix} A^o \cap B^o & \partial A \cap B^o & A^c \cap B^o \\ A^o \cap \partial B & \partial A \cap \partial B & A^c \cap \partial B \\ A^o \cap B^c & \partial A \cap B^c & A^c \cap B^c \end{pmatrix},$$

Clementini and Di Felice (1996) defined a region with a broad boundary, by using two simple regions. This broad boundary is denoted by ΔA. More precisely, the broad boundary is a simple connected subset of R^2 with a hole. The shaded region in Figure 9.6 is region A with a broad boundary. Based on the empty and non-empty invariance, Clementini and Di Felice's

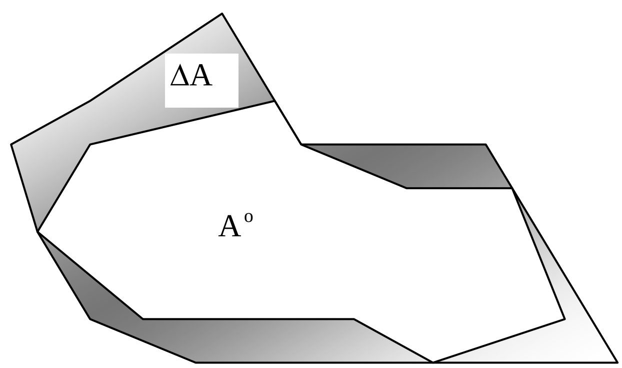


Figure 9.6: Region A with a broad boundary

Algebraic model, $\begin{pmatrix} A^{o} \cap B^{o} & \Delta A \cap B^{o} & A^{c} \cap B^{o} \\ A^{o} \cap \Delta B & \Delta A \cap \Delta B & A^{c} \cap \Delta B \\ A^{o} \cap B^{c} & \Delta A \cap B^{c} & A^{c} \cap B^{c} \end{pmatrix}$, gave a total of 44 relations between two spatial regions with a broad boundary.

To investigate the topological relations between fuzzy regions, Tang *et al* (2002, 2003a and 2003b) decomposed a fuzzy set A into several topological parts, as follows:

(i) the core, $A^{\oplus}$, which is the subset of the closure fuzzy set A with $\left(A^{-} \wedge A^{c-}\right)(x) = 0$, for all $x \in X$;

(ii) the fringe, ℓA, which is the subset of the closure fuzzy set A with $\left(A^{-} \wedge A^{c-}\right)(x) > 0$, for all $x \in X$;

(iii) the outer, $A^{=}$, the complement of the support of the closure of fuzzy set A.

By using the nine-intersection matrix, $\begin{pmatrix} A^{\oplus} \wedge B^{\oplus} & A^{\oplus} \wedge \ell B & A^{\oplus} \wedge B^{=} \\ \ell A \wedge B^{\oplus} & \ell A \wedge \ell B & \ell A \wedge B^{=} \\ A^{=} \wedge B^{\oplus} & A^{=} \wedge \ell B & A^{=} \wedge B^{=} \end{pmatrix}$, there are a total 44 relations between two simply fuzzy regions.

Different from the general topology, when decomposing a fuzzy set into interior, boundary, and exterior, the intersecting of two (interior and boundary or boundary and exterior or interior and exterior) may not be empty. Actually, Tang (http://www.itc.nl/library/Papers_2004/phd/xinming.pdf) considered this by introducing more topological invariants. Here, we created a computational fuzzy topological space and calculate this intersecting values and obtain a bounded which are $\left(A_{\alpha} \wedge \partial A\right)(x) < 1 - \alpha$ and $\left(\left(A^{c}\right)_{\alpha} \wedge \partial A\right)(x) < 1 - \alpha$. The existing models did not take this fact into consideration, which may lead to unexpected effects in modeling topological relations between spatial objects. In our research, we not only considered this factor, but also gave a significant bound on the overlapping parts that actually can be controlled by varying the level cutting.

Furthermore, the relations between objects have been quantified based on the ratio between integrations, which varies between 0 and 1. The advantage of this method not only provides the existence of the intersection between two parts of objects, but also provides a quantitative value for this intersection. This is a step further to the existing methods, which can only provide topological relations by giving a value of 1 (with intersection) or 0 (without intersection).

With a different method and based on the 3×3 integration matrix, there are 44 relations between the simple fuzzy regions in $\mathbf{R}^2$. This result agrees with two previous results (Clementini and Di Felice, 1996) and (Tang *et al*, 2002, 2003a and 2003b). Other new findings are on the number of topological relations, there are 16 relations between simple fuzzy region and simple fuzzy line segment in $\mathbf{R}^2$; 46 topological relations between two simple fuzzy line segments; three relations between simple fuzzy region to fuzzy point in $\mathbf{R}^2$ and three relations between simple fuzzy line segment to fuzzy point in $\mathbf{R}^2$.

The following table is the summary number of relations identified based on the existing models and our newly developed models in this study

Table 2: A comparison of number of topological relations based on several models

Authors	Region-Region	Region-Line	Region-Point	Line-Line	Line-Point
Our Model	44	16	3	46	3
Tang and Kainz's Model	44	30	3	97	3
Cohn and Gotts's Model	46	Nil	Nil	Nil	Nil
Clementini and Di Felice' Model	44	Nil	Nil	Nil	Nil

From the above table, we can see the number of topological relations between simple fuzzy regions from this study is similar to the results from previous studies (Clementini and Di Felice, 1996) and (Tang and Kainz, 2002). However, the method used in this study is different from the early studies. Furthermore, the findings on regions to line and line to line relations are different from Tang's model. This is due to the different definition on fuzzy line.

9.5 Summary

In this chapter, we presented a research outcome on modeling topological relations between simple fuzzy regions, simple fuzzy region to simple fuzzy line segment, simple fuzzy region to simple fuzzy point, simple fuzzy line segment to simple fuzzy line segment, simple fuzzy line segment to simple fuzzy points and simple fuzzy point to simple fuzzy point in $\mathbf{R}^2$, which the intersection concepts and the integration method are applied.

Since the intersection of the interior and boundary or boundary and exterior or interior and exterior may not be empty, there is double counting of the integration. In this chapter, we have carefully studied the relations among the interior, boundary and exterior and hence have achieved an approximation of these overlapping parts. That is, for any α the values of the intersection of the interior and boundary or boundary and exterior are always bounded by $1-\alpha$, where α is the value of the level cutting. Based on a previous finding that the intersection of interior and exterior is empty when the value of the level cutting is greater than 0.5, we can control the uncertainty (the size of the overlap) between the interior and exterior to the boundary by controlling the value of α. This is a new finding, and never has been mentioned in previous studies on topological relations. Moreover, the value of α provides an accurate control of the modeling of topological relations between spatial objects in GIS.

For computing the topological relations between spatial objects, the intersection concepts and the integration method are applied, and a computational nine-intersection model is thus developed. The computational topological relations between spatial objects are defined based on the ratio of the area/ volume of the *meet* to the *join* of two fuzzy spatial objects. This is a step ahead of the existing topological relation models: from the conceptual definition of topological relations to the computable definition of topological relations. As a result, the quantitative value of topological relations can be calculated. With a different method and based on the 3×3 integration matrix, there area 44 relations between the simple fuzzy regions in $\mathbf{R}^2$. This result agrees with two previous results (Clementini and Di Felice, 1996) and (Tang and Kainz, 2002). Another new findings are on the number of topological relations, there are 16 relations between simple fuzzy region and simple fuzzy line segment in $\mathbf{R}^2$; 46 topological relations between two simple fuzzy line segments; 3 relations between simple fuzzy region to fuzzy point in $\mathbf{R}^2$ and 3 relations between simple fuzzy line segment to fuzzy point in $\mathbf{R}^2$.

CHAPTER TEN
CONCLUSIONS AND DISCUSSIONS

10.1 Summary

Boundaries of certain objects may be vague or fuzzy and the classical set theory is based on a crisp boundary. Therefore, it may lead to information loss and inaccuracy in GIS analysis by using the classical set theory. Fuzzy theory provided an alternative solution and gives us many information in fuzzy related representation in GIS.

The main topic of this thesis is concentrated on modeling topological relations between spatial objects in GIS. Several issues of theory in modeling topological relations between spatial objects are discussed which including (a) given the definition of topological relations and the definition of fuzzy GIS elements; (b) proved that topological relations between spatial objects are shape dependent; (c) modeled topological relations between spatial objects by using the concepts of quasi-coincidence and quasi-difference in fuzzy topological theory; (d) created computable fuzzy topological space in order to practically implement these conceptual topological relations in a computer environment.

The first issue is on giving a new definition of the topological relations between two spatial objects which actually is an extended model for topological relations between two spatial objects. For this, we have found that the number of topological relations between the two sets is not as simple as finite; actually, it is infinite and can be approximated by a sequence of matrices. Moreover, as point, line and region (polygon) are the basic elements in GIS, we define them based on a fuzzy set.

Topology is normally considered as independent of shape of spatial objects. This may not necessarily be true in describing relations between spatial objects in GIS. In relating to this, we presented a proof that the topological relations between spatial objects are dependent on the shape of spatial objects. That is, that the topological relations of non-convex sets cannot be deformed to the topological relations of convex sets. The significant theoretical value of this finding that topology of spatial objects are shape dependent. This indicates that when we want to describe topological relations between spatial objects in GIS, both topology and the shape of objects need to be considered.

There are two theoretical issues on modeling topological relations between spatial objects. The first one is using the concepts of quasi-coincidence and quasi-difference to distinguish the topological relations between fuzzy objects and to indicate the effect of one fuzzy object on another in a fuzzy topological space. Secondly, based on the developed computational fuzzy topological space, methods for computing the fuzzy topological relations of spatial objects are proposed in this issue. For modeling the topological relations between spatial objects, the concepts of a bound on the intersection of the boundary and interior, and the boundary and exterior are defined based on the computational fuzzy topological space. Furthermore, the qualitative measures for the intersections are specified based on the α-cut induced fuzzy topological space, which are $(A_\alpha \wedge \partial A)(x) < 1-\alpha$ and $((A^c)_\alpha \wedge \partial A)(x) < 1-\alpha$. For computing the topological relations between spatial objects, the intersection concept and the integration method are applied, and a computational 9-intersection model is thus developed. The computational topological relations between spatial objects are defined based on the ratio of the area/volume of the *meet* of two fuzzy spatial objects to the *join* of two fuzzy spatial objects. This is a step ahead of the existing topological relations models: from a conceptual definition of topological relations to the computable definition of topological relations. As a result, the quantitative values of topological relations can be calculated.

10.2 Conclusions and Discussions

The following are the general conclusions and discussions of this research:

(1) Many researches have discussed the topological relations between crisp spatial objects. However, there exist two common insufficiencies in the existing models. Firstly, most of the existing intersection models mention that a line segment in two-dimensional (2D) space have non-empty interior. But actually, a line should have an empty interior in 2D space, while it has non-empty interior in one-

dimensional (1D) space. Therefore, when talking about the intersection relations, we should consider what space it belongs to. Secondly, there are many topological properties and it is insufficient to simply consider the empty and non-empty invariants.

(2) We have given a mathematical proof to show that the number of components in the intersection of the interior of two convex spatial regions in two-dimensional space is at most two, while the number of components can be more than one if they are not convex. Therefore, the topological relations between spatial objects cannot be modeled by convex sets only, since the number of components in spatial objects is an invariant property of topological relations.

(3) A framework for describing the topological relations between two fuzzy objects was then presented. This framework was based on quasi-coincidence and quasi-difference. By applying these two concepts, we can obtain a seven-tupled topological relation and this seven-tupled relation can be immediately used in GIS to (a) describe fuzzy topological relations between two objects and, (b) to quantify the effect of one fuzzy object to the other fuzzy objects, which is a step further from the traditional fuzzy topological models, which only provide descriptions. The proposed solution can describe the topological relations between any two fuzzy objects without any constraints.

(4) The computational fuzzy topological model introduced which provides a solution to quantitatively compute the topological relations between spatial objects. This is a step ahead of the topological models developed in the past. This model not only provides conceptual definitions, but also quantitative descriptions of the topological relations between spatial objects.

(5) The former approaches (Cohn and Gotts, 1996; Smith, 1996; Tang and Kainz, 2002) have introduced the concept of fuzzy topology into GIS. Moreover, Tang (http://www.itc.nl/library/Papers_2004/phd/xinming.pdf) has contributed on modeling the topological relations between fuzzy objects. We can thus apply the computational fuzzy topological space to calculate the interior, boundary, and exterior of fuzzy spatial objects. An example of classifying the fuzzy interior, boundary, and exterior of the areas affected by *Mikania micrantha* is provided.

Moreover, this topology is a computable fuzzy topology and the fuzzy theory can be applied directly. As a result, the developed theory and the practical operation are linked closely. Therefore, the potential applications of fuzzy topology, presented in this chapter, are very wide.

(6) Fuzzy topology is dependant on the α used in leveling cuts. Different values of α generate different fuzzy topologies and may have different topological structures. Therefore, we can generate a suitable fuzzy topological space by adjusting the value of α; the generated fuzzy topological space can thus match the cases of the application concerned. Moreover, different values of α can provide a multi-directional spatial analysis in GIS. For example, different values of α provide different values of interior, boundary, and exterior. An optimal value of α can be obtained by investigating these fuzzy topologies. Thus, more information can be generated for spatial queries by applying fuzzy topology.

(7) As points, lines and polygons are fundamental elements for spatial objects in GIS, we have developed a method for computing fuzzy topological relations between simple fuzzy regions, between simple fuzzy regions and simple fuzzy line, and so forth, by applying the computational fuzzy topology that has been developed here. Thus, we have discovered that (a) the topological relations between simple fuzzy regions is 44, (b) the topological relations between simple fuzzy region to line segment are 16, (c) the topological relations between simple fuzzy line segments are 46, and (d) 3 topological relations between simple fuzzy region to fuzzy point and simple fuzzy line segment to fuzzy point.

(8) Object in an image can be firstly treated as a fuzzy set in a fuzzy space. The fuzzy set (object) is then decomposed into three parts, the interior, boundary and exterior, which is based on the optimal threshold value and the distribution of the object. The interior represents the core of objects, the boundary represents the overlapping part of object and its background, the exterior represents the unwanted part. Finally, the parts boundary and interior of the object is combined by using the property of spatial connectivity which is developed in this thesis.

10.3 Contributions

The main contributions of the thesis are follows:

(1) Actually, many other topological properties, such as connectivity, compactness, first fundamental group and subspace topology, can help to distinguish the topological relations in the use of GIS. In this aspect, we have extended the topological relations between GIS objects by considering more topological properties that included connectivity and first fundamental group. Moreover, by considered such invariants we have obtained a model that the topological relations can be described by a sequence of matrices and this is an infinite sequence of matrices. The proposed model can be immediately applied on the design and implementation of a GIS, the forest-grass topological relations is an example. The proposed solution is able to represent the topological relations between any two arbitrary objects without holes and connected sets.

(2) We present a proof that the topological relations between spatial objects depends on the shape of spatial objects. The significant theoretical value of this issue is on its findings that topology of spatial objects are shaped dependent. This indicates that when we want to describe topological relations between spatial objects in GIS, both topology and the shape of objects need to be considered. As a result, spatial data modeling, query and analysis based on the existing understanding of topology of spatial objects may need re-assessed.

(3) In modeling topological relations between spatial object with concept of quasi-coincidence and quasi-difference, (a) we have described the information on the overlapped part of two uncertain objects, (b) we have shown that the information on one fuzzy object is not affected by another fuzzy object by the concept of quasi-difference.

(4) When applying fuzzy topology in GIS, it is inappropriate to assume that the membership values of the interior is equal to one and the membership values of boundary is between 0 and 1 exclusively. Indeed, when $\alpha > \frac{1}{2}$, the interior value may not be one and the boundary value may be one.

(5) For modeling the topological relations between spatial objects, the concepts of a bound on the intersection of the boundary and interior, and the boundary and exterior are defined in this paper based on the computational fuzzy topology. Furthermore, the qualitative measures for the intersections are specified based on the α-cut induced fuzzy topological space, which are $(A_\alpha \wedge \partial A)(x) < 1 - \alpha$ and $((A^c)_\alpha \wedge \partial A)(x) < 1 - \alpha$. Actually, Tang has considered these intersecting bounded by introducing more topological invariants (http://www.itc.nl/library/Papers_2004/phd/xinming.pdf).

(6) Fuzzy topological relations between uncertain objects can be used for fuzzy spatial queries, fuzzy spatial analyses, and other questions. The existing topological models, such as the 9-intersection model and other models on topological relations, can thus be practically implemented in a GIS and used for computing fuzzy topologies, based on the computational fuzzy topology solution developed in this study.

10.4 Further Research Work

This thesis has contributed a theoretic framework for modeling fuzzy spatial objects by using fuzzy topological theory. There are several aspects that have been identified for future development.

(1) In the aspect of GIS theory, we actually can do more. We may ask the question that how much of this preserved properties can be used in GIS? How can we apply these theories in GIS? Moreover, what is the relations between the topological relations between spatial objects and the preserved properties? The fuzzy topology is a useful tool to develop theories in GIS.

(2) In order to demonstrate the applicability of the proposed solutions for modeling fuzzy topological relations to practical GIS problems. The following are two practical examples, for which the developed theory can be applied.

(i) Define town center of Hong Kong geographically

A GIS is used to model each of the key factors for a particular town's urban area, based on the collecting (population, number of retail, road network density and etc) data in each grid and each grid was assigned a fuzzy value. These values are then used to generate a surface that represented the fuzzy space of the study area. The modules were then overlaid to produce a so called “Town center” for the study area. This new urban classification will be developed based on the new developed fuzzy theory.

(ii) Environmental modeling by using fuzzy topology

In practice, since we cannot easily obtain the exact distribution of uncertainties of sensor observations, the uncertainty representation using fuzzy values is more flexible and intuitive in the light of the engineer's senses for many application areas. In the example, we try to first represent the uncertainties of geometric objects (trees, grasses, lake, the movement of birds and etc), then study the fuzzy relation among them so that we can understand the effect and relation of their activities.

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