# STOCHASTIC CONTROL PROBLEMS IN OPTIMAL CONSUMPTION AND OPTIMAL DIVIDENDS

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# Certificate of Originality

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___________________________(Signed)

Yue Yang (Name of student)

Dedicated to my family.

# Abstract

This thesis studies two applications of stochastic control in quantitative finance and insurance, namely one problem of optimal entry decision making and dynamic consumption with habit formation and one problem of optimal dividend payment for an insurance group in face of external default risk. By using dynamic programming argument and some delicate partial differential equation (PDE) analysis, we can characterize the value function of each control problem as the solution to the associated *Hamilton-Jacobi-Bellman* (HJB) variational inequality in a classical sense or in a viscosity sense.

In the first project, we consider a composite problem to choose an optimal entry time from complete market information to incomplete information bearing information costs. Starting from the chosen time, the investor no longer pays the fee for acquiring the extra market information and chooses dynamic investment and consumption strategies through partial observations of the public stock price. In addition, the habit formation preference is considered for the dynamic consumption problem. By employing the stochastic Perron's method, the value function of this composite problem is proved to be a viscosity solution of the HJB variational inequality. For the interior optimal investment-consumption problem, the feedback control policies are obtained. The numerical illustration of the continuation region and stopping region is also presented.

In the second project, a multi-dimensional optimal dividend problem for an insurance group is formulated and studied. The novelty of our work is to incorporate the systemic risk modelled by the contagious credit default risk among subsidiaries. That is, each subsidiary of the insurance group runs a product line and all subsidiaries suffer from the external credit risk from the financial market. The default contagion is considered in the sense that one default event may increase the default probabilities of all surviving subsidiaries. By studying the recursive system of the Hamilton-Jacobi-Bellman variational inequalities (HJBVIs), the optimal singular dividend of each subsidiary satisfies a barrier type and the optimal barrier is dynamically modulated by the current default state. In the case of two subsidiaries, the value function and optimal barriers are given in analytical forms, allowing us to conclude that the optimal barrier of one subsidiary decreases if the other subsidiary defaults.

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# List of Notations

| | |
|---|---|
| $(\Omega, \mathbb{F}, \mathbb{P})$ | probability space |
| $S^0$ | bond price process |
| $S_t$ | stock price process |
| $\mu_t$ | drift process |
| $\hat{\mu}_t$ | conditional expectation process |
| $\bar{\mu}, a_i$ | constant drift of the underlying process |
| $\sigma_S, \sigma_\mu, b_i$ | constant volatility of the underlying process |
| $\rho_{ij}$ | constant correlation coefficient |
| $p$ | constant risk aversion coefficient |
| $W_t, B_t, W_i(t)$ | Brownian motion |
| $\alpha(t)$ | persistence of the past level |
| $\delta(t)$ | intensity of consumption history |
| $(\pi_t, c_t)$ | investment and consumption policy process |
| $\mathcal{A}_t$ | time-modulated admissible set of the pair of investment and consumption process $(\pi_s, c_s)_{t \le s \le T}$ |
| $\tau$ | stopping time |
| $\tau_i$ | ruin time |
| $\sigma_i$ | default time |
| $\hat{X}_t$ | total wealth process |

| | |
|---|---|
| $\hat{X}_i(t)$ | pre-default surplus process |
| $\widetilde{X}_i(t)$ | actual surplus process considering external credit risk |
| $X_i(t)$ | surplus process in the presence of dividend payments |
| $\mathbf{X}(t) := (X_1(t), \ldots, X_N(t))$ | vector surplus process |
| $Z_t$ | habit formation process |
| $\mathbf{Z}(t) = (Z_1(t), \ldots, Z_N(t))$ | default indicator process |
| $D_i(\cdot)$ | dividend strategy |
| $\mathbf{D}(t) = (D_1(t), \ldots, D_N(t))$ | total dividend strategy |

# Chapter 1

# Introduction

The aim of this chapter is to briefly review some applications of stochastic control and dynamic programming approach based on HJB variational inequality. The existing literature is far-reaching, thus, we will only refer to a small portion of the abundant work that is closely related to our models and mathematical methods. The outline of this thesis is given at the end.

## 1.1 Optimal Control Problem and Variational Inequalities

Mathematical finance attaches great importance to optimal control problems as a consequence of the comfort and convenience that they bring in the real world, especially the optimal entry and consumption problem and the optimal dividend problem. Those problems address a basic model that can be described as HJB variational inequality to meet the increasing demands and requirements of detailed considerations, for example, optimal entry time, habit formation preference and contagious credit default risk.

Variational inequalities are important mathematical tools in vast research on stochastic singular control, impulse control and optimal stopping problems. We refer the comprehensive review of this approach in financial applications in [72]. HJB variational inequalities have been theoretically elaborated from those that relate to control problems in [10, 16]. Specifically, HJB variational inequalities have been used in optimal consumption and portfolio optimization in [32, 83] and optimal dividend control problems in [61, 84, 88]. The combined optimal stopping and stochastic control problem has been investigated in [26, 55]. The variational inequalities that formulate the optimal stopping problem was used in [16], and then developed in [50] in diffusion models for American options, and then further applied in [71] to the optimal stopping, free boundary, and American options in a jump-diffusion model. The optimal insurance demand problem with marked point processes shocks were analyzed through HJB variational inequality in [62, 63]. The relationship between optimal risk control and dividend distribution policies with the excess-of-loss reinsurance risk control method was explored in [64]. Although the same problem was studied in [82], they considered the diffusion model with a terminal value. The swing option has been discussed as another application in [15, 31].

The majority of research relating to the background of optimal control problems in this thesis has been carried out in previous studies. However, with reference to the difference of opinions regarding the impact of information costs on optimal investment [1, 51, 53, 74], we assume that investors need to afford higher information costs during the waiting time. Additionally, we employ the habit formation preference as the habit formation is illustrated to depict preferences on consumption rate, which has been investigated in both complete [35, 38, 66] and incomplete [86, 87] market models. A large body of work has considered various aspects of the optimal dividend payment, a necessary signal for both companies and shareholders [3, 6, 7, 8, 9, 27, 34, 43, 47, 56, 59, 68, 69, 70, 78]. It is also interesting to take default contagion into account, as studied by [4, 33, 80].

## 1.2 Outline of the Thesis

In this thesis, we discuss the optimal entry and consumption problem and the optimal dividend problem. Detailed introductions are given in each chapter to provide better background to the model, the mathematical challenges, and our contributions. Each value function of the corresponding control problem can be formulated as the solution to an HJB variational inequality, in a classical sense or in a viscosity sense using different mathematical arguments.

Chapter 2 analyzes a composite optimal control problem, that is, a portfolio and consumption optimization problem under habit formation, together with choosing an optimal entry time. First, the interior utility maximization problem with habit formation under partial observations are explicitly solved. The explicit interior value function then allows us to regard exterior optimal stopping as an optimization problem over the renewed input of the drift process using Kalman-Bucy filtering and a linear information cost function. Finally, using the stochastic Perron's method with a linear cost function for the exterior optimal entry problem, we prove that the value function of this composite control problem is the unique viscosity solution to some HJB variational inequality. The numerical example of the sensitivity analysis of the free boundary curve is also given.

Chapter 3 studies the optimal dividend strategy for a multi-line insurance group whose subsidiaries are exposed to some external credit default risk. As opposed to Chapter 2, a multi-dimensional control problem is considered, which creates some new mathematical hindrances. Firstly, an HJB variational inequality for two subsidiaries is derived and then a closed-form solution of this value function is solved completely. Moreover, the current default state modulates the optimal barrier of the dividend policy, which is a consequence of the recursive system of the HJBVIs and the smooth-fit principle. Furthermore, these results are generalized to a multi-line insurance group using mathematical induction. We also present the numerical example of two subsidiaries to show the change of the optimal barrier when default occurs.

Chapter 4 summarizes the main results and contributions of this thesis. In addition, it also gives a brief introduction to some future work.

# Chapter 2

# Optimal Entry and Consumption Under Habit Formation

In this chapter [1], we formulate a composite problem involving the decision making of the optimal entry time and dynamic consumption afterwards. In stage-1, the investor has access to full market information, subject to some information costs, and needs to choose an optimal stopping time to initiate stage-2. In stage-2, starting from the chosen stopping time, the investor terminates the costly full information acquisition and starts dynamic investment and consumption under partial observations of free public stock prices. The habit formation preference is employed, in which past consumption affects the investor's current decisions. The value function of the composite problem is proved to be the unique viscosity solution of some variational inequalities.

## 2.1 Introduction

We consider a basic model to incorporate information costs in a continuous-time finite horizon portfolio-consumption problem. In particular, we study a two-stage composite problem under complete and incomplete filtrations, sequentially. The drift process of the stock price is assumed to be of the Ornstein-Uhlenbeck type. In the first stage, from the

[1] A version of this chapter has been submitted to Advances in Applied Probability, which is currently under review.

initial time, the investor needs to pay information costs to access the full information filtration generated by both drift and stock price processes to update their dynamic distributions and decide the optimal time to enter the second stage. The information costs for full market information may refer to search cost and storage cost to obtain data generated by the stochastic drift process, as well as communication cost, investor's attention cost, and other service costs. We consider linear information costs in the present chapter, which have a constant cost rate per unit of time and are subtracted directly from the investor's initial wealth as time moves on. Therefore, the longer the first stage is, the higher information costs the investor needs to afford. Some previous work has addressed impacts of information costs to optimal investment from different perspectives (see [51], [74], [1] and [53]). In our first stage, the mathematical problem becomes an optimal stopping problem under the complete market information filtration. The second stage starts from the chosen entry time and the investor terminates the full observations of the drift process. Instead, the investor starts to choose the investment and consumption policy dynamically, based on the prior data inputs and the free partial observations of public stock prices, which corresponds to an optimal control problem under incomplete information filtration. As the value function of the interior control problem depends on the stopping time and data inputs of wealth and drift processes at the chosen stopping time, the exterior problem can be equivalently understood as to choose to wait in an optimal way, subjecting to some waiting costs for the input values to achieve certain levels in order to maximize the interior function.

Portfolio optimization under partial observations have been actively studied in past decades (see a few examples among [18, 21, 23, 57, 65, 81]) with different financial motivations. As illustrated in these works, the value function under incomplete information filtration is strictly lower than the counterpart under full information filtration, and this gap is usually regarded as the loss of information. The present chapter attempts to contribute to the study of partial observations from the perspective that the full market information is available, but costly, because more data, services and personal attentions are involved.

The information cost may significantly change the investor's attitude towards the usage of full observations because it is no longer true that the more information he observes, the higher profit he can attain. Moreover, from some previous work on partial observations, we know that the value function eventually depends on the given initial input of random factors, such as the drift process. As in [23, 57], it is conventionally assumed that the initial data of the unobservable drift is a Gaussian random variable so that the Kalman-Bucy filtering can be applied. We take this input into account and consider a model that the investor can wait on and dynamically update the distribution of inputs using the full market information, subjecting to information costs. We can show that starting sharp from the initial time to invest and consume under incomplete information is not necessarily the optimal decision. The optimal solution suggests that the investor can be better off if he delays his dynamic decisions and waits until the observed drift process hits a certain level.

On the other hand, the habit formation has become a new paradigm for modelling preferences on consumption rate in recent years, which can better match with some empirical observations (see [29, 60]). The literature suggests that the past consumption pattern may enforce a continuing impact on the individual's current consumption decisions and therefore the preference should depend on the consumption path. In particular, the linear habit formation preference has been widely accepted, in which there exists an index term that stands for the accumulative consumption history. This habit formation preference has been well studied in [35, 38, 66] in complete market models and in [86, 87] in incomplete market models. It is noted that the utility function is decreasing in the habit level. In the present chapter, we assume that there is no consumption during stage-1 and the investor starts to form consumption habits only in stage-2. Therefore, it may yield that an early entry time to stage-2 may not be the optimal decision because the investor has a longer time to develop a much higher habit level. This is our second motivation to investigate the exterior optimal entry time problem in order to see whether longer waiting and updating inputs can benefit the investor more as the resulting habit level can be much lower and lead to a higher interior value function.

We show that the value function of the composite problem is the unique viscosity solution to some variational inequalities. To this end, we can choose to apply either the classical Perron's method or the stochastic version of Perron's method introduced in [11]. For the classical Perron's method, in order to establish the equivalence between the value function and the viscosity solution, we must either prove the dynamic programming principle or upgrade the global regularity of the solution and prove the verification theorem. The convexity of the value function with respect to the state variable is usually crucial in some standard arguments to improve global regularity. However, the convexity is not clear in our composite problem. The global regularity of the value function along the free boundaries is not guaranteed, and the direct verification proof for the exterior problem becomes difficult. Instead, we choose the stochastic Perron's method, which allows us to show the equivalence between the value function and the viscosity solution without global regularity. For some related literature on optimal stopping using the viscosity solution, we refer to [76] and [71] (see also some recent work on stochastic control problems using the stochastic Perron's method, e.g. [11, 12, 13, 14, 58, 79]). One important step to complete the argument of the stochastic Perron's method is the comparison principle of the associated variational inequalities, which is also established in the present chapter.

The rest of the chapter is organized as follows. Section 2.2 introduces the market model and the habit formation preference and formulates the 2-stage optimization problem. Section 2.3 not only gives the main result of the interior utility maximization problem with habit formation and partial observations, but studies the exterior optimal entry problem with linear information costs as well. Using the stochastic Perron's method, we show that the value function of the composite problem is the unique viscosity solution of some variational inequalities. Some auxiliary results are reported in Appendix A.

## 2.2 Mathematical Model and Preliminaries

### 2.2.1 Market Model

Given the probability space $(\Omega, \mathbb{F}, \mathbb{P})$ with full information filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ that satisfies the usual conditions, we consider the market with one risk-free bond and one risky asset over a finite time horizon $[0, T]$. It is assumed that the bond process satisfies $S_t^0 \equiv 1$, for $t \in [0, T]$, which amounts to the standard change of numéraire.

The stock price $S_t$ satisfies

$$dS_t = \mu_t S_t dt + \sigma_S S_t dW_t, \quad 0 \leq t \leq T, \tag{2.1}$$

with $S_0 = s > 0$. Some empirical studies such as [24, 25, 40, 75] have observed that the drift process of many risky assets follows the so-called mean reverting diffusion. This structure has been widely used not only due to the financial evidence, but also in view of its advantage to make the mathematical problem tractable. We therefore consider that the drift process $\mu_t$ in (2.1) satisfies the Ornstein-Uhlenbeck SDE as

$$d\mu_t = -\lambda(\mu_t - \bar{\mu})dt + \sigma_\mu dB_t, \quad 0 \leq t \leq T. \tag{2.2}$$

Here, $(W_t)_{0\leq t\leq T}$ and $(B_t)_{0\leq t\leq T}$ are $\mathcal{F}_t$-adapted Brownian motions with correlation coefficient $\rho \in [-1, 1]$. For simplicity, the initial value $\mu_0$ of the drift is a given constant. We assume that market coefficients $\sigma_S$, $\lambda$, $\bar{\mu}$ and $\sigma_\mu$ are given non-negative constants based on calibrations from historical data.

It is assumed that the investor starts with initial wealth $x(0) = x_0 > 0$ at time $t = 0$. Also, starting from the initial time $t = 0$, the access to the full market information $\mathcal{F}_t$ generated by $W$ and $B$ incurs information costs $\kappa t$, where $\kappa > 0$ is the constant cost rate per unit time. The information costs may refer to storage cost, search cost, communication cost, investor's attention cost or other service costs to fully observe the market information $\mathcal{F}_t$. Moreover, to simplify the mathematical problem, it is assumed that starting from $t = 0$ to a chosen stopping time $\tau$, the investor purely waits and updates dynamic distributions of processes $\mu_t$ and $S_t$ and does not invest and consume at all. This assumption makes sense as long as the value of the optimal entry time $\tau$ is short in the model. The dynamic wealth process after the information costs at time $t$ is simply given by a deterministic function $x(t) = x_0 - \kappa t$ for any $t \leq \tau$.

As the full market information filtration is costly, the investor needs to choose optimally choose an $\mathcal{F}_t$-adapted stopping time $\tau$ to terminate the full information acquisition and enter the second stage. From the chosen stopping time $\tau$, he switches to the partial observations filtration $\mathcal{F}_t^S = \mathcal{F}_\tau \bigvee \sigma(S_u : \tau \leq u \leq t)$ for $\tau \leq t \leq T$, which is the union of the sigma algebra $\mathcal{F}_\tau$ and the natural filtration generated by the stock price $S$ up to time $t$. Moreover, for any time $\tau \leq t \leq T$, the investor chooses a dynamic consumption rate $c_t \geq 0$ and decides the amounts $\pi_t$ of his wealth to invest in the risky asset and the rest in the bond. Without paying information costs, the drift process $\mu_t$ and Brownian motions $W_t$ and $B_t$ are no longer observable for $t \geq \tau$. Therefore, the investment-consumption pair $(\pi_t, c_t)$ is only assumed to be adapted to the partial observation filtration $\mathcal{F}_t^S$ for $\tau \leq t \leq T$. Recall that at the entry time $\tau$, the investor only has wealth $x(\tau) = x_0 - \kappa\tau$ left. Under the incomplete filtration $\mathcal{F}_t^S$, the investor's total wealth process $\hat{X}_t$ can be written as

$$d\hat{X}_t = (\pi_t \mu_t - c_t)dt + \sigma_S \pi_t dW_t, \quad \tau \leq t \leq T, \tag{2.3}$$

with the initial value $\hat{X}_\tau = x(\tau) = x_0 - \kappa\tau > 0$. Note that $W_t$ is no longer a Brownian motion under the partial observations filtration $\mathcal{F}_t^S$, we have to apply the Kalman-Bucy filtering and consider the *Innovation Process* defined by

$$d\hat{W}_t := \frac{1}{\sigma_S}\Big[(\mu_t - \hat{\mu}_t)dt + \sigma_S dW_t\Big] = \frac{1}{\sigma_S}\Big(\frac{dS_t}{S_t} - \hat{\mu}_t dt\Big), \quad \tau \leq t \leq T,$$

which becomes a Brownian motion under $\mathcal{F}_t^S$. The best estimation of the unobservable drift process $\mu_t$ under $\mathcal{F}_t^S$ is the conditional expectation process $\hat{\mu}_t = \mathbb{E}\Big[\mu_t \Big| \mathcal{F}_t^S\Big]$, for $\tau \leq t \leq T$ with the initial input $\hat{\mu}_\tau = \mu_\tau$ at the stopping time $\tau$ where $\mu_\tau$ is determined via (2.2) by paying information costs up to $\tau$. By standard Kalman-Bucy filtering, $\hat{\mu}_t$

satisfies the SDE

$$d\hat{\mu}_t = -\lambda(\hat{\mu}_t - \bar{\mu})dt + \left(\frac{\hat{\Sigma}(t) + \sigma_S\sigma_\mu\rho}{\sigma_S}\right) d\hat{W}_t, \quad \tau \leq t \leq T,$$

with $\hat{\mu}_\tau = \mu_\tau$. The conditional variance $\hat{\Sigma}(t) = \mathbb{E}\Big[(\mu_t - \hat{\mu}_t)^2 \Big| \mathcal{F}_t^S\Big]$ satisfies the deterministic Riccati ODE

$$\frac{d\hat{\Sigma}(t)}{dt} = -\frac{1}{\sigma_S^2}\hat{\Sigma}^2(t) + \Big(-\frac{2\sigma_\mu\rho}{\sigma_S} - 2\lambda\Big)\hat{\Sigma}(t) + (1-\rho^2)\sigma_\mu^2, \quad \tau \leq t \leq T,$$

with the initial value $\hat{\Sigma}(\tau) = \mathbb{E}\Big[(\mu_\tau - \hat{\mu}_\tau)^2 \Big| \mathcal{F}_t^S\Big] = 0$ in view of $\hat{\mu}_\tau = \mu_\tau$, $\mathbb{P}$-a.s.. It can be solved explicitly as

$$\hat{\Sigma}(t) = \sqrt{k}\sigma_S \frac{k_1 \exp(2(\frac{\sqrt{k}}{\sigma_S})t) + k_2}{k_1 \exp(2(\frac{\sqrt{k}}{\sigma_S})t) - k_2} - \left(\lambda + \frac{\sigma_\mu\rho}{\sigma_S}\right)\sigma_S^2, \quad \tau \leq t \leq T,$$

where

$$k = \lambda^2\sigma_S^2 + 2\sigma_S\sigma_\mu\lambda\rho + \sigma_\mu^2,$$

$$k_1 = \sqrt{k}\sigma_S + (\lambda\sigma_S^2 + \sigma_S\sigma_\mu\rho),$$

$$k_2 = -\sqrt{k}\sigma_S + (\lambda\sigma_S^2 + \sigma_S\sigma_\mu\rho).$$

For the second stage dynamic control problem, we employ the habit formation preference. In particular, we denote $Z_t := Z(c_t)$ as *habit formation* process or *the standard of living* process, which describes the consumption habits level. It is assumed conventionally that the accumulative reference $Z_t$ satisfies the recursive equation (see [35]) that

$$dZ_t = (\delta(t)c_t - \alpha(t)Z_t)dt, \quad \tau \leq t \leq T,$$

where $Z_\tau = z_0 \geq 0$ is called the *initial consumption habit* of the investor. Equivalently, we have

$$Z_t = z_0 e^{-\int_\tau^t \alpha(u)du} + \int_\tau^t \delta(u)e^{-\int_u^t \alpha(s)ds} c_u du, \quad \tau \leq t \leq T,$$

which is the exponentially weighted average of the initial habit and the past consumption. Here, the deterministic discount factors $\alpha(t) \geq 0$ and $\delta(t) \geq 0$ measure, respectively, the persistence of the past level and the intensity of consumption history. We are interested in *addictive habits* in the present chapter, namely it is required that the investor's current consumption strategies shall never fall below the level of standard of living that $c_t \geq Z_t$ a.s., for $\tau \leq t \leq T$.

Under the partial observation filtration $(\mathcal{F}_t^S)_{\tau \leq t \leq T}$, the stock price dynamics (2.1) can be rewritten by

$$dS_t = \hat{\mu}_t S_t dt + \sigma_S S_t d\hat{W}_t,$$

and the wealth dynamics (2.3) can be rewritten as

$$d\hat{X}_t = (\pi_t \hat{\mu}_t - c_t)dt + \sigma_S \pi_t d\hat{W}_t, \quad \tau \leq t \leq T.$$

To facilitate the formulation of the stochastic control problem and the derivation of the dynamic programming equation, for any $t \in [0, T]$, we denote $\mathcal{A}_t(y)$ the time-modulated admissible set of the pair of investment and consumption process $(\pi_s, c_s)_{t \leq s \leq T}$ with the initial wealth $\hat{X}_t = y$, which is $\mathcal{F}_s^S$-progressively measurable and satisfies the integrability conditions

$$\int_t^T \pi_s^2 ds < +\infty, \quad \mathbb{P} - \text{a.s.},$$

$$\int_t^T c_s ds < +\infty, \quad \mathbb{P} - \text{a.s.},$$

with the addictive habit formation constraint that $c_s \geq Z_s$, $\mathbb{P}$-a.s., $t \leq s \leq T$. Moreover, no bankruptcy is allowed, i.e., the investor's wealth remains nonnegative, i.e. $\hat{X}_s \geq 0$, $\mathbb{P}$-a.s., $t \leq s \leq T$.

### 2.2.2 Problem Formulation

The two-stage optimal decision making problem is formulated as the composite problem involving the optimal stopping and the stochastic control afterwards, which is defined by

$$\widetilde{V}(0,\mu_0;x_0,z_0) := \sup_{\tau\geq 0}\mathbb{E}\left[\operatorname*{esssup}_{(\pi,c)\in\mathcal{A}_\tau(x_0-\kappa\tau)}\mathbb{E}\left[\int_\tau^T \frac{(c_s-Z_s)^p}{p}ds\Big|\mathcal{F}_\tau^S\right]\right]. \tag{2.4}$$

In particular, starting from the chosen stopping time $\tau$, we are interested in the utility maximization on consumption with habit formation, in which the power utility function $U(x) = x^p/p$ is defined on the difference $c_t - Z_t$. To simplify the presentation, we only consider in the present paper that the risk aversion coefficient $p < 0$. The indirect utility process of the interior control problem is denoted by

$$\widehat{V}(t, x_0-\kappa t, z_0, \mu_t; 0) := \operatorname*{esssup}_{(\pi,c)\in\mathcal{A}_t(x_0-\kappa t)}\mathbb{E}\left[\int_t^T \frac{(c_s-Z_s)^p}{p}ds\Big|\mathcal{F}_t^S\right]$$

$$= \operatorname*{esssup}_{(\pi,c)\in\mathcal{A}_t(x_0-\kappa t)}\mathbb{E}\left[\int_t^T \frac{(c_s-Z_s)^p}{p}ds\Big|\hat{X}_t = x_0-\kappa t, \hat{\mu}_t=\mu_t, Z_t=z_0; \hat{\Sigma}(t)=0\right].$$

To determine the exterior optimal stopping time, we need to maximize over the inputs of values $\tau$, $\hat{X}_\tau$, $Z_\tau$ and $\hat{\mu}_\tau$. Recall that the investor does not manage his investment and consumption before $\tau$, it follows that $\hat{X}_\tau = x_0 - \kappa\tau$, $Z_\tau = z_0$ and $\hat{\Sigma}(\tau) = 0$ can all be taken as parameters instead of variables. That is, $\mu_\tau = \hat{\mu}_\tau$ is the only random input and we can regard $\mu_t$ as the only underlying state process. Therefore, the dynamic counterpart of (2.4) is defined by

$$\widetilde{V}(t,\eta;x_0-\kappa t,z_0) := \operatorname*{esssup}_{\tau\geq t}\mathbb{E}\left[\operatorname*{esssup}_{(\pi,c)\in\mathcal{A}_\tau(x_0-\kappa\tau)}\mathbb{E}\left[\int_\tau^T \frac{(c_s-Z_s)^p}{p})ds\Big|\mathcal{F}_\tau^S\right]\Big|\mu_t=\eta\right]. \tag{2.5}$$

**Remark 2.1.** *We focus on the case $p < 0$ in the present chapter because functions $A(t,s)$, $B(t,s)$ and $C(t,s)$ as solutions to some future ODEs (2.13), (2.14) and (2.15) are all bounded and the utility $U(x)$ is also bounded from above, which can significantly simplify the proof of the verification result Theorem 2.3 and the proof of comparison results Proposition 2.1. The other case $0 < p < 1$ can essentially be handled in a similar way.*

*However, as the process $\hat{\mu}_t$ is unbounded and functions $A(t,s)$, $B(t,s)$ and $C(t,s)$ may explode at some $t \in [0,T]$, one needs some additional parameter assumptions to guarantee integrability conditions and martingale properties in the proofs of some main results.*

**Assumption 2.1.** *According to Remark 2.3 for the interior control problem, it is assumed from this point onwards that $x_0 - \kappa t > z_0 m(t)$ for any $0 \leq t \leq T$, i.e. the initial wealth is sufficiently large to support that the interior control problem is always well defined for any $0 \leq t \leq T$, where $m(t)$ is defined by*

$$m(t) := \int_t^T \exp\left( \int_t^s (\delta(v) - \alpha(v)) dv \right) ds, \quad 0 \leq t \leq T. \tag{2.6}$$

*Here $m(t)$ in (2.6) represents the cost of subsistence consumption per unit of standard of living at time t because the interior control problem is solvable if and only if $\hat{X}_t^* \geq m(t) Z_t$, $0 \leq t \leq T$, see Proposition 3.4.1 in [85].*

The function $\widehat{V}$ can be solved in the explicit form given in (2.12) later. The process $\widetilde{V}(t, \mu_t; x_0 - \kappa t, z_0)$ with the function $\widetilde{V}$ defined in (2.5) is the Snell envelope of the process $\widehat{V}(t, x_0 - \kappa t, z_0, \mu_t)$ above. The function $\widetilde{V}$ in (2.5) can therefore be written as

$$\widetilde{V}(t, \eta; x_0 - \kappa t, z_0) = \operatorname*{esssup}_{\tau \geq t} \mathbb{E}\Big[ \widehat{V}(\tau, x_0 - \kappa \tau, z_0, \mu_\tau) \Big| \mu_t = \eta \Big].$$

The continuation region, interpreted as the continuation of full information observations to update the input value, is denoted by

$$\mathcal{C} = \{(t,\eta) \in [0,T) \times \mathbb{R} : \widetilde{V}(t,\eta; x_0 - \kappa t, z_0) > \widehat{V}(t, x_0 - \kappa t, z_0, \eta)\},$$

and the free boundary is

$$\partial\mathcal{C} = \{(t,\eta) \in [0,T) \times \mathbb{R} : \widetilde{V}(t,\eta; x_0 - \kappa t, z_0) = \widehat{V}(t, x_0 - \kappa t, z_0, \eta)\}.$$

Let us denote $\widetilde{V}(t,\eta; x_0 - \kappa t, z_0)$ by $\widetilde{V}(t,\eta)$ for short when there is no confusion. By some heuristic arguments, we can write the HJB variational inequalities with the terminal condition $\widetilde{V}(T,\eta)=0$, $\eta\in\mathbb{R}$, by

$$\min\left\{\widetilde{V}(t,\eta)-\widehat{V}(t,x_0-\kappa t,z_0,\eta),\ -\frac{\partial\widetilde{V}(t,\eta)}{\partial t}-\mathcal{L}\widetilde{V}(t,\eta)\right\}=0, \tag{2.7}$$

where $\mathcal{L}\widetilde{V}(t,\eta)=-\lambda(\eta-\bar{\mu})\frac{\partial\widetilde{V}}{\partial\eta}(t,\eta)+\frac{1}{2}\sigma_\mu^2\frac{\partial^2\widetilde{V}}{\partial\eta^2}(t,\eta)$. To simplify notations in the following sections, we shall rewrite (2.7) by

$$\begin{cases} F(t,\eta,\widetilde{V},\widetilde{V}_t,\widetilde{V}_\eta,\widetilde{V}_{\eta\eta})=0, & \text{on } [0,T)\times\mathbb{R}, \\ v(T,\eta)=0, & \text{for } \eta\in\mathbb{R}, \end{cases} \tag{2.8}$$

where $F(t,\eta,v,v_t,v_\eta,v_{\eta\eta}):=\min\left\{v-\widehat{V},\ -\frac{\partial v}{\partial t}-\mathcal{L}v\right\}$.

**Remark 2.2.** *The second term $-\frac{\partial\widetilde{V}}{\partial t}-\mathcal{L}\widetilde{V}=0$ in (2.7) is a linear parabolic PDE and does not depend on the interior control $(\pi,c)$. The comparison part $\widetilde{V}-\widehat{V}$ in (2.7) depends on the optimal control $(\pi,c)$ as the $\widehat{V}$ is the value function of the interior control problem provided the input $\hat{X}_t=x_0-\kappa t$, $Z_t=z_0$ and $\hat{\mu}_t=\mu_t=\eta$.*

The next theorem is the main result of this chapter.

**Theorem 2.1.** *$\widetilde{V}(t,\eta)$ defined in (2.5) is the unique bounded and continuous viscosity solution to variational inequalities (2.7). In addition, the optimal entry time for the composite problem (2.5) is given by the $\mathcal{F}_t$-adapted stopping time*

$$\tau^*:=T\wedge\inf\left\{t\geq 0:\widetilde{V}(t,\mu_t;x_0-\kappa t,z_0)=\widehat{V}(t,x_0-\kappa t,z_0,\mu_t)\right\}. \tag{2.9}$$

*We also have that the process $\widetilde{V}(t,\mu_t;x_0-\kappa t,z_0)$ is a martingale with respect to the full information filtration $\mathcal{F}_t$, $0\leq t\leq\tau^*$.*

The proof will be provided in Section 2.3.

### 2.2.3 Numerical Example

We present here some numerical results of sensitivity analysis of the free boundary curve, i.e. the shape of the continuation region and stopping region, with respect to changes of the parameter $\delta$. In particular, we want to illustrate that waiting in the full information filtration can benefit the investor more and it is optimal for the drift process to achieve certain thresholds that gives the optimal entry time for the interior control problem under habit formation and partial observations. We choose parameters $T = 12.5$, $p = -1$, $\rho = 0.2$, $\sigma_S = 0.5$, $x_0 = 1000000$, $z_0 = 0.5$, $\sigma_\mu = 0.4$, $\lambda = 0.1$, $\alpha = 0.04$, $\bar{\mu} = 0.25$ and the information cost rate $\kappa = 5000$, and plot free boundary curves with respect to the parameter $\delta = 0.05, 0.25, 0.45, 0.55, 0.75$ respectively. The shaded regions correspond to the continuation regions, which should be understood as the region to purely update the input by observing the costly full information generated by both $\mu_t$ and $S_t$.

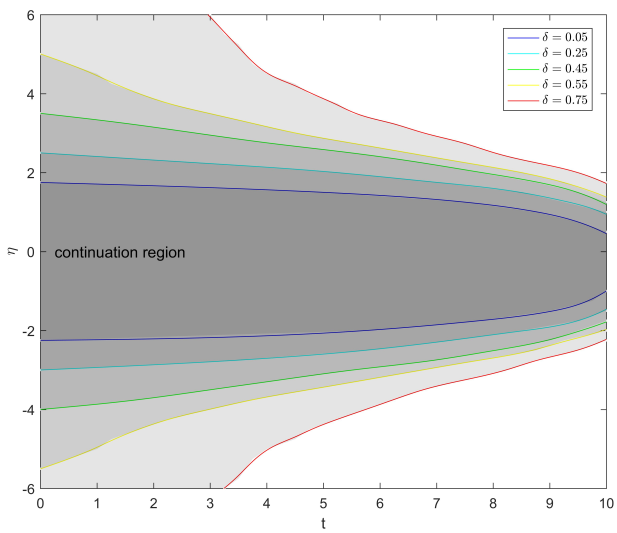


Figure 2.1: Sensitivity analysis of the free boundary curve

For each time $t$, we can first identify the barrier level for the input of the drift process $\mu_t$ such that it is optimal to terminate the full observations of the drift process and initiate the investment and consumption under partial observations only when the observed $|\mu_t|$ is large enough. From Figure 1, we can see that if the discount factor $\delta$ increases in the habit formation preference, the free boundary barrier also increases so that the optimal stopping time increases. This can be understood that if the weighting intensity of the past consumption is larger, the trigger level (absolute value of $\eta$) to start consumption is lifted up and the investor would prefer to wait longer in the first stage and delay his consumption in order to maximize his total profit.

Moreover, we can also easily verify the following sensitivity results of the composite value function.

**Lemma 2.1.** *If the information cost rate $\kappa$ increases, the value function $\widetilde{V}(t, \eta)$ decreases for any $t < T$.*

*Proof.* By the definition of $\widetilde{V}(t, \eta)$ and the explicit form of $\widehat{V}(t, x_0 - \kappa t, z_0, \eta)$ in (2.12) and explicit form of $m(t)$ in (2.6), $\widehat{V}(t, x_0 - \kappa t, z_0, \eta)$ decreases if $x_0 - \kappa t$ decrease, then it clearly follows that $\widetilde{V}(t, \eta)$ is decreasing in $\kappa$. □

**Lemma 2.2.** *We have the following sensitivity properties of the value function $\widetilde{V}(t, \eta)$:*

*(i) Suppose that $\alpha >$ and $\delta > 0$ are both constants in the definition of habit formation process such that $\delta > \alpha$. We have that $\widetilde{V}(t, \eta; \alpha, \delta)$ is decreasing in $\delta$ and increasing in $\alpha$.*

*(ii) If the initial habit $z_0$ increases, the value function $\widetilde{V}(t, \eta)$ decreases.*

*Proof.* By the definition of $\widetilde{V}(t, \eta)$ and the explicit form of $\widehat{V}(t, x_0 - \kappa t, z_0, \eta)$ in (2.12) and explicit form of $m(t)$ in (2.6), for constants $\delta > \alpha$, it is clear that $\widehat{V}(t, x_0 - \kappa t, z_0, \eta)$ is decreasing in $\delta$ and increasing in $\alpha$, which implies that $\widetilde{V}(t, \eta)$ has the same sensitivity property. Similarly, it is clear that $\widehat{V}(t, x_0 - \kappa t, z_0, \eta)$ decreases while $z_0$ increases, and hence $\widetilde{V}(t, \eta)$ is decreasing in $z_0$. □

## 2.3 Exterior Optimal Stopping Problem

This section mainly aims to solve the exterior optimal entry problem. To determine the optimal stopping time, we need to maximize over the inputs of values $\tau$, $\hat{X}_\tau$, $Z_\tau$ and $\hat{\mu}_\tau$.

We recall that the investor does not manage his investment and consumption before $\tau$, it follows that $\hat{X}_\tau = x_0 - \kappa\tau$, $Z_\tau = z_0$ and $\hat{\Sigma}(\tau) = 0$ can all be taken as parameters. The mathematical problem corresponds to an optimal stopping problem in which $\mu_t$ becomes the only underlying state process. To this end, we choose to apply the stochastic Perron's method to verify that the value function of the composite problem corresponds to the unique viscosity solution of some variational inequality.

The proof can be summarized as follows: we first introduce sets of stochastic semi-solutions $\mathcal{V}^+$ and $\mathcal{V}^-$ and prove that $v^- \leq \widetilde{V} \leq v^+$, where $v^-$ and $v^+$ are defined later in (2.17) and (2.18). By using the stochastic Perron's method, we can get that $v^+$ is a bounded and upper semi-continuous (u.s.c.) viscosity subsolution and $v^-$ is a bounded and lower semi-continuous (l.s.c.) viscosity supersolution. At last, we prove the comparison principle, namely if we have any bounded and u.s.c. viscosity subsolution $u$ and bounded and l.s.c. viscosity supersolution $v$ of (2.8), we must have $u \leq v$. It follows that $v^+ \leq v^-$, which leads to the desired conclusion that $v^- = \widetilde{V} = v^+$ and the value function is the unique viscosity solution.

First, the similar result in [85] with respect to the interior utility maximization under partial observations will be showed. For some fixed time $0 \leq k \leq T$, the dynamic interior stochastic control problem under habit formation is defined by

$$\begin{aligned} &\widehat{V}(k, x, z, \eta; \theta) \\ :=& \sup_{(\pi,c)\in\mathcal{A}_k(x)} \mathbb{E}\left[\int_k^T \frac{(c_s - Z_s)^p}{p} ds \Big| \mathcal{F}_k^S\right] \\ =& \sup_{(\pi,c)\in\mathcal{A}_k(x)} \mathbb{E}\left[\int_k^T \frac{(c_s - Z_s)^p}{p} ds \Big| \hat{X}_k = x, Z_k = z, \hat{\mu}_k = \eta; \hat{\Sigma}(k) = \theta\right], \end{aligned} \tag{2.10}$$

where $\mathcal{A}_k(x)$ denotes the admissible control space starting from time $k$. Here, as the conditional variance $\hat{\Sigma}(t)$ is a deterministic function of time, we set $\theta$ as a parameter instead of a state variable. We only consider in the present chapter that the risk aversion coefficient $p < 0$.

By using the optimality principle and Itô's formula, we can heuristically obtain the HJB equation as

$$
\begin{aligned}
& V_t - \alpha(t) z V_z - \lambda(\eta - \bar{\mu}) V_\eta + \frac{\left(\hat{\Sigma}(t) + \sigma_S \sigma_\mu \rho\right)^2}{2\sigma_S^2} V_{\eta\eta} \\
& + \max_{(\pi,c)\in\mathcal{A}} \left[ -cV_x + c\delta(t) V_z + \frac{(c-z)^p}{p} \right] \\
& + \max_{(\pi,c)\in\mathcal{A}} \left[ \pi\eta V_x + \frac{1}{2}\sigma_S^2 \pi^2 V_{xx} + V_{x\eta} \left(\hat{\Sigma}(t) + \sigma_S \sigma_\mu \rho\right) \pi \right] = 0, \quad k \le t \le T,
\end{aligned} \tag{2.11}
$$

with the terminal condition $V(T, x, z, \eta) = 0$.

The following results are part of discussions in section 3.3.2 by [85]. We present them for the completeness.

**Theorem 2.2.** *For fixed $t \in [k, T]$, we can define the* effective domain *for the pair $(x, z)$ by $\mathbb{D}_t := \{(x', z') \in (0, +\infty) \times [0, +\infty);\ x' \ge m(t) z'\}$, where $k \le t \le T$. The HJB equation (2.11) admits a classical solution on $[k, T] \times \mathbb{D}_t \times \mathbb{R}$ that*

$$
\begin{aligned}
V(t, x, z, \eta) = & \Big[ \int_t^T \Big(1 + \delta(s) m(s)\Big)^{\frac{p}{p-1}} \exp\Big(A(t,s)\eta^2 + B(t,s)\eta + C(t,s)\Big) ds \Big]^{1-p} \\
& \times \frac{[(x - m(t) z)]^p}{p},
\end{aligned} \tag{2.12}
$$

*where $A(t,s)$, $B(t,s)$ and $C(t,s)$ satisfy the following ODEs:*

$$
\begin{aligned}
& A_t(t,s) + \frac{p}{2(1-p)^2 \sigma_S^2} + 2\left[ -\lambda + \frac{p(\hat{\Sigma}(t) + \sigma_S \sigma_\mu \rho)}{\sigma_S^2 (1-p)} \right] A(t,s) \\
& + \frac{2(\hat{\Sigma}(t) + \sigma_S \sigma_\mu \rho)^2}{\sigma_S^2} A^2(t,s) = 0,
\end{aligned} \tag{2.13}
$$

$$
\begin{aligned}
& B_t(t,s) + \left[ -\lambda + \frac{p(\hat{\Sigma}(t) + \sigma_S \sigma_\mu \rho)}{\sigma_S^2 (1-p)} \right] B(t,s) + 2\lambda\bar{\mu} A(t,s) \\
& + \frac{2(\hat{\Sigma}(t) + \sigma_S \sigma_\mu \rho)^2}{\sigma_S^2} A(t,s) B(t,s) = 0,
\end{aligned} \tag{2.14}
$$

$$C_t(t,s) + \lambda\bar{\mu}B(t,s) + \frac{\left(\hat{\Sigma}(t) + \sigma_S\sigma_\mu\rho\right)^2}{2\sigma_S^2}\left(B^2(t,s) + 2A(t,s)\right) = 0, \qquad (2.15)$$

*with terminal conditions $A(s,s) = B(s,s) = C(s,s) = 0$. The explicit solutions of ODEs (2.13), (2.14), (2.15) are reported in Appendix A.*

**Remark 2.3.** *The effective domain of $V(t,x,z,\eta)$ mandates some constraints on the optimal wealth process $\hat{X}_t^*$ and habit formation process $Z_t^*$ such that $\hat{X}_t^* \geq m(t)Z_t^*$ for $t \in [k,T]$. In particular, we have to enforce the initial wealth-habit budget constraint that $\hat{X}_k \geq m(k)Z_k$ at time $k$.*

The following results is Theorem 3.3.3. in [85].

**Theorem 2.3.** *(The Verification Theorem) If the initial budget constraint $\hat{X}_k \geq m(k)Z_k$ holds at time $k$, the unique solution (2.12) of HJB equation equals the value function defined in (2.10), i.e., $V(k,x,z,\eta) = \widehat{V}(k,x,z,\eta)$. Moreover, the optimal investment policy $\pi_t^*$ and optimal consumption policy $c_t^*$ are given in the feedback form by $\pi_t^* = \pi^*(t,\hat{X}_t^*,Z_t^*,\hat{\mu}_t)$ and $c_t^* = c^*(t,\hat{X}_t^*,Z_t^*,\hat{\mu}_t)$, $k \leq t \leq T$. The function $\pi^*(t,x,z,\eta) : [k,T] \times \mathbb{D}_t \times \mathbb{R} \to \mathbb{R}$ is given by*

$$\pi^*(t,x,z,\eta) = \left[\frac{\eta}{(1-p)\sigma_S^2} + \frac{\left(\hat{\Sigma}(t) + \sigma_S\sigma_\mu\rho\right)}{\sigma_S^2}\frac{N_\eta(t,\eta)}{N(t,\eta)}\right](x - m(t)z),$$

*and the function $c^*(t,x,z,\eta) : [k,T] \times \mathbb{D}_t \times \mathbb{R} \to \mathbb{R}^+$ is given by*

$$c^*(t,x,z,\eta) = z + \frac{(x - m(t)z)}{\left(1 + \delta(t)m(t)\right)^{\frac{1}{1-p}} N(t,\eta)}.$$

*The optimal wealth process $\hat{X}_t^*$, $k \leq t \leq T$, is given by*

$$\hat{X}_t^* = (x - m(k)z)\frac{N(t,\hat{\mu}_t)}{N(k,\eta)}\exp\left(\int_k^t \frac{(\hat{\mu}_u)^2}{2(1-p)\sigma_S^2}du + \int_k^t \frac{\hat{\mu}_u}{(1-p)\sigma_S}d\hat{W}_u\right) + m(t)Z_t^*.$$

**Remark 2.4.** *Recall that the interior value function $\widehat{V}$ is of the form in (2.12). Moreover, by Remark A.1, functions $A(t,s) \leq 0$ and $B(t,s) \leq 0$ in (2.12) due to $p < 0$. That is, if we take $\widehat{V}(\tau, \hat{\mu}_\tau)$ as a functional of the input $\hat{\mu}_\tau$, it is not globally convex or concave in $\hat{\mu}_\tau \in \mathbb{R}$ because the function* $\exp\left(A(t,s)\eta^2 + B(t,s)\eta + C(t,s)\right)$ *is not globally convex or concave in the variable $\eta \in \mathbb{R}$, which depends on values of $A(t,s)$ and $B(t,s)$. Therefore, the composite value function $\widetilde{V}(t,\eta)$ in (2.5) is not globally convex or concave in $\eta \in \mathbb{R}$, which depends on all model parameters.*

The proof of the above theorem can be found in [85]. Let us then give the following definitions similar to [11, 13].

**Definition 2.1.** *The set of stochastic super-solutions for the PDE (2.8), denoted by $\mathcal{V}^+$, is the set of functions $v : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ which have the following properties:*
*(i) $v$ is u.s.c. and bounded on $[0,T] \times \mathbb{R}$ and $v(t,\eta) \geq \widehat{V}(t, x_0 - \kappa t, z_0, \eta)$ for any $(t,\eta) \in [0,T] \times \mathbb{R}$.*
*(ii) for each $(t,\eta) \in [0,T] \times \mathbb{R}$ and any stopping time $t \leq \tau_1 \in \mathcal{T}$, we have $v(\tau_1, \mu_{\tau_1}) \geq \mathbb{E}[v(\tau_2, \mu_{\tau_2}) | \mathcal{F}_{\tau_1}]$, $\mathbb{P} -$ a.s. for any $\tau_2 \in \mathcal{T}$ and $\tau_2 \geq \tau_1$. That is to say, the function $v$ along the solution of the SDE (2.2) is a super-martingale with respect to full information filtration $(\mathcal{F}_t)_{t\in[0,T]}$ between $\tau_1$ and $T$.*

**Definition 2.2.** *The set of stochastic sub-solutions for the PDE (2.8), denoted by $\mathcal{V}^-$, is the set of functions $v : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ which have the following properties:*
*(i) $v$ is l.s.c. and bounded on $[0,T] \times \mathbb{R}$ and $v(T,\eta) \leq 0$ for any $\eta \in \mathbb{R}$.*
*(ii) for each $(t,\eta) \in [0,T] \times \mathbb{R}$ and any stopping time $t \leq \tau_1 \in \mathcal{T}$, we have $v(\tau_1, \mu_{\tau_1}) \leq \mathbb{E}[v(\tau_2 \wedge \zeta, \mu_{\tau_2 \wedge \zeta}) | \mathcal{F}_{\tau_1}]$, $\mathbb{P} -$ a.s. for any $\tau_2 \in \mathcal{T}$ and $\tau_2 \geq \tau_1$. Hence, the function $v$ along the solution to (2.2) is a sub-martingale with respect to full information filtration $(\mathcal{F}_t)_{t\in[0,T]}$ between $\tau_1$ and $\zeta$, where*

$$\zeta := \inf\{t \in [\tau_1, T] : v(t, \mu_t; x_0 - \kappa t, z_0) \geq \widehat{V}(t, x_0 - \kappa t, z_0, \mu_t)\}. \tag{2.16}$$

**Remark 2.5.** *We note that the definitions of stochastic super-solutions and stochastic sub-solutions for the optimal stopping problem are not symmetric, which are consistent with the similar definitions in [13]. The main reason for these differences comes from the natural supermartingale property of the Snell envelop process and its martingale property between the initial time and the first hitting time* $\zeta$ *in (2.16). That is, we naturally need* $v(t,\eta) \geq \widehat{V}(t, x_0 - \kappa t, z_0, \eta)$ *for all* $(t,\eta) \in [0,T] \times \mathbb{R}$ *including the terminal time* $T$ *in item* $(i)$ *of Definition 2.1 of stochastic super-solution, but we only require* $v(T,\eta) \leq \widehat{V}(T, x_0 - \kappa t, z_0, \eta) = 0$ *at the terminal time* $T$ *in item* $(i)$ *of Definition 2.2 for stochastic sub-solution. These comparison results and the supermartingale and submartingale properties will play important roles to establish the desired sandwich result* $v^- \leq \widetilde{V} \leq v^+$ *in Lemma 2.6.*

**Lemma 2.3.** $\widehat{V}(t, x_0 - \kappa t, z_0, \eta; 0)$ *is bounded and continuous for* $(t,\eta) \in [0,T] \times \mathbb{R}$.

*Proof.* For fixed $x_0$ and $z_0$, it is clear that $\widehat{V}(t, x_0 - \kappa t, z_0, \eta)$ in the explicit form in Theorem 2.3 is continuous and $\widehat{V}(t, x_0 - \kappa t, z_0, \eta) \leq 0$. So we only show that $\widehat{V}$ is lower bounded. By Appendix A, we know that $A(u) \leq 0$, $B(u) \leq 0$ and $C(u) \leq K$ for some $K \geq 0$ by using $p < 0$. We deduce that $\Big(A(u)\eta^2 + B(u)\eta + C(u)\Big) \leq K_1$ for some $K_1 > 0$ and it follows that $\widehat{V}(t, x_0 - \kappa t, z_0, \eta)$ is lower bounded by some constant for $(t,\eta) \in [0,T] \times \mathbb{R}$ as $p < 0$. ☐

As it is trivial to see that $0 \in \mathcal{V}^-$ and $0 \in \mathcal{V}^+$, we have the following result.

**Lemma 2.4.** $\mathcal{V}^+$ *and* $\mathcal{V}^-$ *are nonempty.*

**Definition 2.3.** *We define*

$$v^- := \sup_{p \in \mathcal{V}^-} p; \tag{2.17}$$

$$v^+ := \inf_{q \in \mathcal{V}^+} q. \tag{2.18}$$

Similar to Lemma 2.2. of [11], the next result holds.

**Lemma 2.5.** *We have $v^- \in \mathcal{V}^-$ and $v^+ \in \mathcal{V}^+$.*

Next, we have the following comparison result.

**Lemma 2.6.** *We have $v^- \leq \widetilde{V} \leq v^+$.*

*Proof.* For each $v \in \mathcal{V}^+$, let us consider $\tau_1 = t \geq 0$ in Definition 2.1. For any $\tau \geq t$, we have

$$v(t, \eta) \geq \mathbb{E}[v(\tau, \mu_\tau)|\mathcal{F}_t] \geq \mathbb{E}[\widehat{V}(\tau, x_0 - \kappa\tau, z_0, \mu_\tau)|\mathcal{F}_t],$$

because of the sup-martingale property in Definition 2.1. It readily follows that

$$v(t, \eta) \geq \text{esssup}_{t \leq \tau} \mathbb{E}[\widehat{V}(\tau, x_0 - \kappa\tau, z_0, \mu_\tau)|\mathcal{F}_t].$$

This implies that $v(t, \eta) \geq \widetilde{V}(t, \eta)$ in view of the definition of $\widetilde{V}(t, \eta)$ and hence $\widetilde{V} \leq v^+$ by the Definition (2.18). On the other hand, for each $v \in \mathcal{V}^-$, by taking $\tau_1 = t \geq 0$ in the Definition 2.2, we have $v(t, \eta) \leq \mathbb{E}[v(\tau \wedge \zeta, \mu_{\tau\wedge\zeta})|\mathcal{F}_t]$ for any $\tau \geq t$ because of the sub-martingale property in Definition 2.2. In particular, using the definition of $\zeta$, we further have

$$\begin{aligned} v(t, \eta) &\leq \mathbb{E}[v(\tau \wedge \zeta, \mu_{\tau\wedge\zeta})|\mathcal{F}_t] \\ &\leq \mathbb{E}[\widehat{V}(\tau \wedge \zeta, x_0 - f(\tau \wedge \zeta), z_0, \mu_{\tau\wedge\zeta})|\mathcal{F}_t] \\ &\leq \text{esssup}_{\tau \geq t} \mathbb{E}[\widehat{V}(\tau, x_0 - \kappa\tau, z_0, \mu_\tau)|\mathcal{F}_t] = \widetilde{V}(t, \eta). \end{aligned}$$

Thus, it follows that $\widetilde{V} \geq v^-$ because of (2.17). In conclusion, we have the inequality $v^- \leq \widetilde{V} \leq v^+$. □

**Theorem 2.4.** *(Stochastic Perron's Method) $v^-$ in Definition 2.3 is a bounded and l.s.c. viscosity super-solution of*

$$\begin{cases} F(t, \eta, v, v_t, v_\eta, v_{\eta\eta}) \geq 0, & \textit{on } [0, T) \times \mathbb{R}, \\ v(T, \eta) \geq 0, & \textit{for any } \eta \in \mathbb{R}, \end{cases}$$

*and $v^+$ in Definition 2.3 is a bounded and u.s.c. viscosity sub-solution of*

$$\begin{cases} F(t, \eta, v, v_t, v_\eta, v_{\eta\eta}) \leq 0, & \textit{on } [0, T) \times \mathbb{R}, \\ v(T, \eta) \leq 0, & \textit{for any } \eta \in \mathbb{R}. \end{cases} \tag{2.19}$$

*Proof.* We follow similar arguments as in [11, 13].

*(i) The sub-solution property of $v^+$.* First, definition in (2.18) and Lemma 2.5 imply that $v^+$ is bounded and upper semi-continuous. Suppose $v^+$ is not a viscosity subsolution, there exists some interior point $(\bar{t},\bar{\eta}) \in (0,T)\times\mathbb{R}$ and a $C^{1,2}$-test function $\varphi : [0,T]\times\mathbb{R}\to\mathbb{R}$ such that $v^+-\varphi$ attains a strict local maximum that is equal to zero and $F(\bar{t},\bar{\eta},v,v_{\bar{t}},v_{\bar{\eta}},v_{\bar{\eta}\bar{\eta}})>0$. It follows that

$$\begin{cases} v^+(\bar{t},\bar{\eta}) - \widehat{V}(\bar{t},x_0-f(\bar{t}),z_0,\bar{\eta}) > 0, \\ -\frac{\partial\varphi}{\partial t}(\bar{t},\bar{\eta}) - \mathcal{L}\varphi(\bar{t},\bar{\eta}) > 0. \end{cases}$$

As coefficients of the variational inequality are continuous, there exists a ball $B(\bar{t},\bar{\eta},\varepsilon)$ small enough that

$$\begin{cases} -\frac{\partial\varphi}{\partial t} - \mathcal{L}\varphi > 0 \ \text{ on } \ \overline{B(\bar{t},\bar{\eta},\varepsilon)}, \\ \varphi > v^+ \ \text{ on } \ \overline{B(\bar{t},\bar{\eta},\varepsilon)}\backslash(\bar{t},\bar{\eta}). \end{cases}$$

In addition, as $\varphi(\bar{t},\bar{\eta}) = v^+(\bar{t},\bar{\eta}) > \widehat{V}(\bar{t},x_0-f(\bar{t}),z_0,\bar{\eta})$, $\varphi$ is continuous and $\widehat{V}$ is continuous, we can derive that for some $\varepsilon$ small enough, we have $\varphi-\varepsilon \geq \widehat{V}$ on $\overline{B(\bar{t},\bar{\eta},\varepsilon)}$. Because $v^+-\varphi$ is upper semi-continuous and $\overline{B(\bar{t},\bar{\eta},\varepsilon)}\backslash B(\bar{t},\bar{\eta},\frac{\varepsilon}{2})$ is compact, it then follows that there exists a $\delta>0$ such that $\varphi-\delta \geq v^+$ on $\overline{B(\bar{t},\bar{\eta},\varepsilon)}\backslash B(\bar{t},\bar{\eta},\frac{\varepsilon}{2})$.

If we choose $0<\xi<\delta\wedge\varepsilon$, the function $\varphi^\xi = \varphi-\xi$ will satisfy the following properties:

$$\begin{cases} -\frac{\partial\varphi^\xi}{\partial t} - \mathcal{L}\varphi^\xi > 0 \ \text{ on } \ \overline{B(\bar{t},\bar{\eta},\varepsilon)}, \\ \varphi^\xi > v^+ \ \text{ on } \ \overline{B(\bar{t},\bar{\eta},\varepsilon)}\backslash B(\bar{t},\bar{\eta},\frac{\varepsilon}{2}), \\ \varphi^\xi \geq \widehat{V} \ \text{ on } \ \overline{B(\bar{t},\bar{\eta},\varepsilon)}, \end{cases}$$

and $\varphi^\xi(\bar{t},\bar{\eta}) = v^+(\bar{t},\bar{\eta})-\xi$.

Let us define an auxiliary function by

$$v^\xi := \begin{cases} v^+\wedge\varphi^\xi \ \text{ on } \ \overline{B(\bar{t},\bar{\eta},\varepsilon)}, \\ v^+ \ \text{ outside } \ \overline{B(\bar{t},\bar{\eta},\varepsilon)}. \end{cases}$$

It is easy to check that $v^\xi$ is upper semi-continuous and $v^\xi(\bar{t},\bar{\eta}) = \varphi^\xi(\bar{t},\bar{\eta}) < v^+(\bar{t},\bar{\eta})$. We claim that $v^\xi$ satisfies the terminal condition. To this end, we pick some $\varepsilon>0$ that satisfies

$T > \bar{t} + \varepsilon$ and recall that $v^+$ satisfies the terminal condition. We then continue to show that $v^\xi \in \mathcal{V}^+$ to obtain a contradiction.

Let us fix $(t, \eta)$ and recall that $((\mu_s)_{t \le s \le T}, (W_s, B_s)_{t \le s \le T}, \Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_s)_{t \le s \le T}) \in \chi$, where $\chi$ is the nonempty set of all weak solutions. We need to show that the process $(v^\xi(s, \mu_s))_{t \le s \le T}$ is a super-martingale on $(\Omega, \mathbb{P})$ with respect to $(\mathcal{F}_s)_{t \le s \le T}$. We first assume that $(v^+(s, \mu_s))_{t \le s \le T}$ has right continuous paths. In this case, $v^\xi$ is a super-martingale locally in the region $[t, T] \times \mathbb{R} \backslash B(\bar{t}, \bar{\eta}, \frac{\varepsilon}{2})$ because it equals the right continuous super-martingale $(v^+(s, \mu_s))_{t \le s \le T}$. As the process $(v^\xi(s, \mu_s))_{t \le s \le T}$ is the minimum between two local super-martingales in the region $B(\bar{t}, \bar{\eta}, \varepsilon)$, it is a local super-martingale. As two regions $[t, T] \times \mathbb{R} \backslash B(\bar{t}, \bar{\eta}, \frac{\varepsilon}{2})$ and $B(\bar{t}, \bar{\eta}, \varepsilon)$ overlap over an open region, $(v^\xi(s, \mu_s))_{t \le s \le T}$ is actually a super-martingale.

If the process $(v^+(s, \mu_s))_{t \le s \le T}$ is not right continuous, we can consider its right continuous limit over rational times to transform it to the special case discussed above. In particular, for a given rational number $r$ and fixed $0 \le t \le r \le s \le T$ and $\eta \in \mathbb{R}$, it remains to show the process $(Y_u)_{t \le u \le T} := (v^\xi(u, \mu_u))_{t \le u \le T}$ between $r$ and $s$ is a super-martingale, which is equivalent to show $Y_r \ge \mathbb{E}[Y_s | \mathcal{F}_r]$.

Let us denote $G_u := v^+(u, \mu_u),\ r \le u \le s$ and stop the process $G$ after time s, i.e. $G_u := v^+(s, \mu_s),\ s \le u \le T$. As $(G_u)_{r \le u \le T}$ may not be right continuous, by Proposition 1.3.14 in [52], we can define its right continuous modification as

$$G_u^+(\omega) := \lim_{u' \to u,\ u' > u,\ u' \in \mathbb{Q}} G_{u'}(\omega), \quad r \le u \le T.$$

Note that $G^+$ is a right continuous super-martingale with respect to $\mathcal{F}$ which satisfies the usual conditions. Because $v^+$ is upper semi-continuous and the process remains the same after $s$, we conclude that $G_r \ge G_r^+,\ G_s = G_s^+$. Recall that $v^+ < \varphi - \delta$ in the open region $B(\bar{t}, \bar{\eta}, \varepsilon) \backslash \overline{B(\bar{t}, \bar{\eta}, \frac{\varepsilon}{2})}$, if we take right limits inside this region and use continuous function $\varphi$, we have

$$G_u^+ < \varphi^\xi(u, \mu_u), \text{ if } (u, \mu_u) \in B(\bar{t}, \bar{\eta}, \varepsilon) \backslash \overline{B(\bar{t}, \bar{\eta}, \frac{\varepsilon}{2})}.$$

Thus, if we consider the process

$$Y_u^+ := \begin{cases} G_u^+, \ (u, \mu_u) \notin \overline{B(\bar{t}, \bar{\eta}, \frac{\varepsilon}{2})}, \\ G_u^+ \wedge \varphi^\xi(u, \mu_u), \ (u, \mu_u) \in B(\bar{t}, \bar{\eta}, \varepsilon), \end{cases}$$

we also have $Y_r \geq Y_r^+, \ Y_s = Y_s^+$.

Because $G^+$ has right continuous paths, we can conclude that $Y$ is a super-martingale such that

$$Y_r \geq Y_r^+ \geq \mathbb{E}[Y_s^+ | \mathcal{F}_r] = \mathbb{E}[Y_s | \mathcal{F}_r].$$

*(ii) The terminal condition of* $v^+$.

For some $\eta_0 \in \mathbb{R}$, we assume that $v^+(T, \eta_0) > 0$ and will show a contradiction. As $\widehat{V}$ is continuous on $\mathbb{R}$, we can choose an $\varepsilon > 0$ such that $0 \leq v^+(T, \eta_0) - \varepsilon$ and $|\eta - \eta_0| \leq \varepsilon$. On the compact set $(\overline{B(T, \eta_0, \varepsilon)} \backslash B(T, \eta_0, \frac{\varepsilon}{2})) \cap ([0, T] \times \mathbb{R})$, $v^+$ is bounded above by the definition of $\mathcal{V}^+$ and that $v^+ \in \mathcal{V}^+$. Moreover, as $v^+$ is upper semi-continuous on this compact set, we can find $\delta > 0$ small enough such that

$$v^+(T, \eta_0) + \frac{\varepsilon^2}{4\delta} \geq \varepsilon + \sup_{(t,\eta) \in (\overline{B(T,\eta_0,\varepsilon)} \backslash B(T,\eta_0,\frac{\varepsilon}{2})) \cap ([0,T] \times \mathbb{R})} v^+(t, \eta). \tag{2.20}$$

Next, for $k > 0$, we define the function $\varphi^{\delta,\varepsilon,k}(t, \eta) := v^+(T, \eta_0) + \frac{|\eta - \eta_0|^2}{\delta} + k(T - t)$. For $k$ large enough, we derive that $-\varphi_t^{\delta,\varepsilon,k} - \mathcal{L}\varphi^{\delta,\varepsilon,k} > 0$ on $\overline{B(T, \eta_0, \varepsilon)}$. Moreover, we have the following result in view of (2.20)

$$\varphi^{\delta,\varepsilon,k} \geq \varepsilon + v^+ \text{ on } (\overline{B(T, \eta_0, \varepsilon)} \backslash B(T, \eta_0, \frac{\varepsilon}{2})) \cap ([0, T] \times \mathbb{R}),$$

and $\varphi^{\delta,\varepsilon,k}(T, \eta) \geq v^+(T, \eta_0) \geq 0 + \varepsilon$ for $|\eta - \eta_0| \leq \varepsilon$.

Now, we can find $\xi < \varepsilon$ and define the function as follows,

$$v^{\delta,\varepsilon,k,\xi} := \begin{cases} v^+ \wedge (\varphi^{\delta,\varepsilon,k} - \xi) \ \text{ on } \ \overline{B(T, \eta_0, \varepsilon)}, \\ v^+ \ \text{ outside } \ \overline{B(T, \eta_0, \varepsilon)}. \end{cases}$$

By following similar argument in Step $(i)$, one can obtain that $v^{\delta,\varepsilon,k,\xi} \in \mathcal{V}^+$, but $v^{\delta,\varepsilon,k,\xi}(T, \eta_0) = v^+(T, \eta_0) - \xi$, which leads to a contradiction.

*(iii) The super-solution property of $v^-$.*

Let us only provide a sketch of the proof as it is essentially similar to Step $(i)$. Suppose that $v^-$ is not a viscosity super-solution, then there exist some interior point $(\bar{t}, \bar{\eta}) \in (0, T) \times \mathbb{R}$ and a $C^{1,2}$-test function $\psi : [0, T] \times \mathbb{R} \to \mathbb{R}$ such that $v^- - \psi$ attains a strict local minimum that is equal to zero. As $F(\bar{t}, \bar{\eta}, v, v_{\bar{t}}, v_{\bar{\eta}}, v_{\bar{\eta}\bar{\eta}}) < 0$, there are two separate cases to check.

case(i) $v^-(\bar{t}, \bar{\eta}) - \widehat{V}(\bar{t}, x_0 - f(\bar{t}), z_0, \bar{\eta}) < 0$. This already leads to a contradiction with $v^-(\bar{t}, \bar{\eta}) \geq \widehat{V}(\bar{t}, x_0 - f(\bar{t}), z_0, \bar{\eta})$ by the definition of $v^-$.

case(ii) $-\frac{\partial \psi}{\partial t}(\bar{t}, \bar{\eta}) - \mathcal{L}\psi(\bar{t}, \bar{\eta}) < 0$. We can find a small enough ball $B(\bar{t}, \bar{\eta}, \varepsilon)$ such that $-\frac{\partial \psi}{\partial t} - \mathcal{L}\psi < 0$ on $\overline{B(\bar{t}, \bar{\eta}, \varepsilon)}$. Moreover, as $v^- - \psi$ is lower semi-continuous and $\overline{B(\bar{t}, \bar{\eta}, \varepsilon)} \backslash B(\bar{t}, \bar{\eta}, \frac{\varepsilon}{2})$ is compact, there exists a $\delta > 0$ such that $\psi + \delta \leq v^-$ on $\overline{B(\bar{t}, \bar{\eta}, \varepsilon)} \backslash B(\bar{t}, \bar{\eta}, \frac{\varepsilon}{2})$. We can then choose $\xi \in (0, \frac{\delta}{2})$ small such that $\psi^\xi = \psi + \xi$ satisfies the following three properties: (i) $-\frac{\partial \psi^\xi}{\partial t} - \mathcal{L}\psi^\xi < 0$ on $\overline{B(\bar{t}, \bar{\eta}, \varepsilon)}$; (ii) we have $v^- \geq \psi + \delta > \psi + \xi = \psi^\xi$ on $\overline{B(\bar{t}, \bar{\eta}, \varepsilon)} \backslash B(\bar{t}, \bar{\eta}, \frac{\varepsilon}{2})$; (iii) $\psi^\xi(\bar{t}, \bar{\eta}) = \psi(\bar{t}, \bar{\eta}) + \xi = v^-(\bar{t}, \bar{\eta}) + \xi > v^-(\bar{t}, \bar{\eta})$. Thus, we can define an auxiliary function as

$$v^\xi := \begin{cases} v^- \vee \psi^\xi & \text{on } \overline{B(\bar{t}, \bar{\eta}, \varepsilon)}, \\ v^- & \text{outside } \overline{B(\bar{t}, \bar{\eta}, \varepsilon)}. \end{cases}$$

By repeating similar argument in Step $(i)$, we have that $v^\xi \in \mathcal{V}^-$ by showing that $(v^\xi(s, \mu_s))_{t \leq s \leq T}$ is a sub-martingale. If $v^-$ has right continuous paths, then the proof is trivial. In general, by Proposition 1.3.14 in [52], we can define the right continuous sub-martingale $G_u^+(\omega) := \lim_{u' \to u,\ u' > u,\ u' \in \mathbb{Q}} G_{u'}(\omega),\ \omega \in \Omega^*,\ r \leq u \leq T$, where $G_u := v^-(u, \mu_u),\ r \leq u \leq s$ and we stop it at time $t$, that is to say, $G_u := v^-(s, \mu_s),\ s \leq u \leq T$, given fixed $0 \leq t \leq r \leq s \leq T$ and $\eta \in \mathbb{R}$. Similar to Step $(i)$, we note that $G^+$ is the right continuous sub-martingale and therefore $G_r \leq G_r^+,\ G_s = G_s^+$. As $G_u^+ > \psi^\xi(u, \mu_u),$ if $(u, \mu_u) \in B(\bar{t}, \bar{\eta}, \varepsilon) \backslash \overline{B(\bar{t}, \bar{\eta}, \frac{\varepsilon}{2})}$, we can define the process

$$Y_u^+ := \begin{cases} G_u^+,\ (u, \mu_u) \notin \overline{B(\bar{t}, \bar{\eta}, \frac{\varepsilon}{2})}, \\ G_u^+ \vee \psi^\xi(u, \mu_u),\ (u, \mu_u) \in \overline{B(\bar{t}, \bar{\eta}, \frac{\varepsilon}{2})}. \end{cases}$$

We can conclude that $Y_r \leq Y_r^+$, $Y_s = Y_s^+$ and $Y$ is a sub-martingale that $Y_r \leq Y_r^+ \leq \mathbb{E}[Y_s^+|\mathcal{F}_r] = \mathbb{E}[Y_s|\mathcal{F}_r]$, which completes the proof.

*(iv) The terminal condition of $v^-$.*

For some $\eta_0 \in \mathbb{R}$, suppose that $v^-(T,\eta_0) < 0$ and we will show a contradiction. As $\widehat{V}$ is continuous on $\mathbb{R}$, we can choose an $\varepsilon > 0$ such that $0 \geq v^-(T,\eta_0) + \varepsilon$ and $|\eta - \eta_0| \leq \varepsilon$. Similar to Step $(ii)$, we can find $\delta > 0$ small enough so that

$$v^-(T,\eta_0) - \frac{\varepsilon^2}{4\delta} \leq \inf_{(t,\eta)\in(\overline{B(T,\eta_0,\varepsilon)}\backslash B(T,\eta_0,\frac{\varepsilon}{2}))\cap([0,T]\times\mathbb{R})} v^-(t,\eta) - \varepsilon. \tag{2.21}$$

Then, for $k > 0$, we consider $\psi^{\delta,\varepsilon,k}(t,\eta) := v^-(T,\eta_0) - \frac{|\eta-\eta_0|^2}{\delta} - k(T-t)$. For $k$ large enough, we have that $-\psi_t^{\delta,\varepsilon,k} - \mathcal{L}\psi^{\delta,\varepsilon,k} < 0$ on $\overline{B(T,\eta_0,\varepsilon)}$. Furthermore, in view of (2.21), we have

$$\psi^{\delta,\varepsilon,k} \leq v^- - \varepsilon \text{ on } (\overline{B(T,\eta_0,\varepsilon)}\backslash B(T,\eta_0,\frac{\varepsilon}{2})) \cap ([0,T]\times\mathbb{R}),$$

and $\psi^{\delta,\varepsilon,k}(T,\eta) \leq v^-(T,\eta_0) \leq -\varepsilon$ for $|\eta - \eta_0| \leq \varepsilon$.

Next, we can find $\xi < \varepsilon$ and define the function by

$$v^{\delta,\varepsilon,k,\xi} := \begin{cases} v^- \vee (\psi^{\delta,\varepsilon,k} + \xi) & \text{on } \overline{B(T,\eta_0,\varepsilon)}, \\ v^- & \text{outside } \overline{B(T,\eta_0,\varepsilon)}. \end{cases}$$

Similar to Step $(iii)$, we obtain that $v^{\delta,\varepsilon,k,\xi} \in \mathcal{V}^-$, but $v^{\delta,\varepsilon,k,\xi}(T,\eta_0) = v^-(T,\eta_0) + \xi$, which gives a contradiction. $\square$

Let us then reverse the time and consider $s := T - t$. However, for the simplicity of presentation, let us continue to use $t$ in the place of $s$ if there is no confusion. The variational inequalities can be rewritten as

$$\min\Bigg\{\widetilde{V}(t,\eta;x_0 - f(T-t), z_0) - \widehat{V}(t, x_0 - f(T-t), z_0, \eta),$$
$$\frac{\partial \widetilde{V}(t,\eta)}{\partial t} - \mathcal{L}\widetilde{V}(t,\eta)\Bigg\} = 0, \tag{2.22}$$

where $\mathcal{L}\widetilde{V}(t,\eta) = -\lambda(\eta-\bar{\mu})\frac{\partial \widetilde{V}}{\partial \eta}(t,\eta) + \frac{1}{2}\sigma_\mu^2 \frac{\partial^2 \widetilde{V}}{\partial \eta^2}(t,\eta)$ and also $\widetilde{V}(0,\eta) = 0$.

Let us denote it equivalently as

$$\begin{cases} F(t,\eta,v,v_t,v_\eta,v_{\eta\eta}) = 0, \text{on } (0,T]\times\mathbb{R}, \\ v(0,\eta) = \widehat{V}(0,x_0-f(0),z_0,\eta), \text{for any } \eta\in\mathbb{R}, \end{cases} \tag{2.23}$$

where $F(t,\eta,v,v_t,v_\eta,v_{\eta\eta}) := \min\left\{v-\widehat{V},\ \frac{\partial v}{\partial t} - \mathcal{L}v\right\}$. We also have the continuation region as $\mathcal{C} = \{(t,\eta)\in(0,T]\times\mathbb{R} : \widetilde{V}(t,\eta;x_0-f(T-t),z_0) > \widehat{V}(t,x_0-f(T-t),z_0,\eta)\}$.

**Proposition 2.1.** *(Comparison Principle) Let $u, v$ be u.s.c viscosity subsolution and l.s.c. viscosity supersolution of (2.23), respectively. If $u(0,\eta)\le v(0,\eta)$ on $\mathbb{R}$, then we have $u\le v$ on $(0,T]\times\mathbb{R}$.*

*Proof.* We will follow similar arguments in [14, 72] with modifications to fit into our framework. We suppose that $u(0,\eta)\le v(0,\eta)$ on $\mathbb{R}$, then, we try to prove that $u\le v$ on $[0,T]\times\mathbb{R}$. We first construct the strict supersolution to the system (2.23) with suitable perturbations of $v$. Let us recall that $A\le 0$, $B\le 0$ and $C$ is bounded above by some constant, which are shown in Appendix A. Moreover, $\widehat{V}(t,x_0-\kappa t,z_0,\eta)\le 0$. Let us fix a constant $C_2>0$ small enough such that $\lambda > C_2\sigma_\mu^2$ and set $\psi(t,\eta) = C_0e^t + e^{C_2\eta^2}$ with some $C_0>1$. Thus, we have the following inequality:

$$\begin{aligned}\frac{\partial\psi}{\partial t} - \mathcal{L}\psi =& C_0e^t + C_2\Big[2(\lambda-C_2\sigma_\mu^2)\eta^2 - 2\lambda\bar{\mu}\eta - \sigma_\mu^2\Big]e^{C_2\eta^2} \\ \ge& C_0e^t + C_2\frac{-2(\lambda-C_2\sigma_\mu^2)\sigma_\mu^2 - \lambda^2\bar{\mu}^2}{2(\lambda-C_2\sigma_\mu^2)} \\ >& C_0 + C_2\frac{-2(\lambda-C_2\sigma_\mu^2)\sigma_\mu^2 - \lambda^2\bar{\mu}^2}{2(\lambda-C_2\sigma_\mu^2)}.\end{aligned}$$

We can then choose $C_0>1$ large enough such that $C_0 + C_2\frac{-2(\lambda-C_2\sigma_\mu^2)\sigma_\mu^2-\lambda^2\bar{\mu}^2}{2(\lambda-C_2\sigma_\mu^2)} > 1$, which guarantees that

$$\frac{\partial\psi}{\partial t} - \mathcal{L}\psi > 1. \tag{2.24}$$

We define $v^\Lambda := (1-\Lambda)v + \Lambda\psi$ on $[0,T]\times\mathbb{R}$ for any $\Lambda \in (0,1)$. It follows that

$$\begin{aligned} v^\Lambda - \widehat{V} &= (1-\Lambda)v + \Lambda\psi - \widehat{V} = (1-\Lambda)v + \Lambda(C_0 e^t + e^{C_2\eta^2}) - \widehat{V} \\ &\geq (1-\Lambda)v + \Lambda(C_0 e^t + e^{C_2\eta^2}) + \Lambda\widehat{V} - \widehat{V} \\ &> (1-\Lambda)(v - \widehat{V}) + \Lambda C_0 > \Lambda, \end{aligned} \tag{2.25}$$

where we used $v - \widehat{V} \geq 0$ in the last inequality. From (2.24) and (2.25), we can deduce that for $\Lambda \in (0,1)$, $v^\Lambda$ is a supersolution to

$$\min\left\{ v^\Lambda - \widehat{V},\ \frac{\partial v^\Lambda}{\partial t} - \mathcal{L}v^\Lambda \right\} \geq \Lambda. \tag{2.26}$$

In order to prove the comparison principle, it suffices to show the claim that $\sup(u - v^\Lambda) \leq 0$ for all $\Lambda \in (0,1)$, as the required result is obtained by letting $\Lambda$ go to $0$. To this end, we will prove the claim by showing a contradiction and suppose that there exists some $\Lambda \in (0,1)$ such that $M := \sup(u - v^\Lambda) > 0$.

It is clear that $u$, $v$ and $\widehat{V}$ have the same growth conditions: in view of the explicit forms of $A, B, C$ and $\widehat{V}$, it follows that $\widehat{V}$ has growth condition in $t$ as $e^{e^{K_1 t}}$ for some $K_1 < 0$ and has growth condition in $\eta$ as $e^{K_2\eta^2}$ for some $K_2 < 0$; on the other hand, $\psi$ has growth condition in $t$ as $e^t$ and has growth condition in $\eta$ as $e^{C_2\eta^2}$. Thus, we have that $u(t,\eta) - v^\Lambda(t,\eta) = (u - (1-\Lambda)v - \Lambda\psi)(t,\eta)$ goes to $-\infty$ as $t \to T, \eta \to \infty$. Consequently, the u.s.c. function $(u - v^\Lambda)$ attains its maximum $M$.

Let us consider the u.s.c. function $\Phi_\varepsilon(t,t',\eta,\eta') = u(t,\eta) - v^\Lambda(t',\eta') - \phi_\varepsilon(t,t',\eta,\eta')$, where $\phi_\varepsilon(t,t',\eta,\eta') = \frac{1}{2\varepsilon}((t-t')^2 + (\eta-\eta')^2)$, $\varepsilon > 0$ and $(t_\varepsilon, t'_\varepsilon, \eta_\varepsilon, \eta'_\varepsilon)$ attains the maximum of $\Phi_\varepsilon$. We have

$$M_\varepsilon = \max \Phi_\varepsilon = \Phi_\varepsilon(t_\varepsilon, t'_\varepsilon, \eta_\varepsilon, \eta'_\varepsilon) \to M \text{ and } \phi_\varepsilon(t_\varepsilon, t'_\varepsilon, \eta_\varepsilon, \eta'_\varepsilon) \to 0 \text{ when } \varepsilon \to 0. \tag{2.27}$$

We give an equivalent definition of viscosity solutions in terms of superjets and subjets. In particular, we define $\bar{\mathcal{P}}^{2,+}u(\bar{t},\bar{\eta})$ as the set of elements $(\bar{q},\bar{k},\bar{M}) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}$ satisfying $u(t,\eta) \leq u(\bar{t},\bar{\eta}) + \bar{q}(t-\bar{t}) + \bar{k}(\eta-\bar{\eta}) + \frac{1}{2}\bar{M}(\eta-\bar{\eta})^2 + o((t-\bar{t}) + (\eta-\bar{\eta})^2)$. We define $\bar{\mathcal{P}}^{2,-}v^\Lambda(\bar{t},\bar{\eta})$ similarly.

Thanks to Crandall-Ishii's lemma, we can find $A_\varepsilon, B_\varepsilon \in \mathbb{R}$ such that

$$(\frac{t_\varepsilon - t'_\varepsilon}{\varepsilon}, \frac{\eta_\varepsilon - \eta'_\varepsilon}{\varepsilon}, A_\varepsilon) \in \bar{\mathcal{P}}^{2,+}u(t_\varepsilon, \eta_\varepsilon),$$

$$(\frac{t_\varepsilon - t'_\varepsilon}{\varepsilon}, \frac{\eta_\varepsilon - \eta'_\varepsilon}{\varepsilon}, B_\varepsilon) \in \bar{\mathcal{P}}^{2,-}v^\Lambda(t'_\varepsilon, \eta'_\varepsilon),$$

$$\sigma^2(\eta_\varepsilon)A_\varepsilon - \sigma^2(\eta'_\varepsilon)B_\varepsilon \leq \frac{3}{\varepsilon}(\sigma(\eta_\varepsilon) - \sigma(\eta'_\varepsilon))^2.$$

By combining the viscosity subsolution property (2.19) of $u$ and the viscosity strict supersolution property (2.26) of $v^\Lambda$, we have the following inequalities

$$\begin{aligned}\min\Big\{&u(t_\varepsilon, \eta_\varepsilon) - \widehat{V}(t_\varepsilon, x_0 - f(t_\varepsilon), z_0, \eta_\varepsilon),\\ &\frac{t_\varepsilon - t'_\varepsilon}{\varepsilon} - \frac{\eta_\varepsilon - \eta'_\varepsilon}{\varepsilon} b(t_\varepsilon, \eta_\varepsilon) - \frac{1}{2}\sigma^2(\eta_\varepsilon)A_\varepsilon\Big\} \leq 0, \end{aligned} \tag{2.28}$$

$$\begin{aligned}\min\Big\{&v^\Lambda(t'_\varepsilon, \eta'_\varepsilon) - \widehat{V}(t'_\varepsilon, x_0 - f(t'_\varepsilon), z_0, \eta'_\varepsilon),\\ &\frac{t_\varepsilon - t'_\varepsilon}{\varepsilon} - \frac{\eta_\varepsilon - \eta'_\varepsilon}{\varepsilon} b(t'_\varepsilon, \eta'_\varepsilon) - \frac{1}{2}\sigma^2(\eta'_\varepsilon)B_\varepsilon\Big\} \geq \Lambda, \end{aligned} \tag{2.29}$$

where $b(t_\varepsilon, \eta_\varepsilon) = -\lambda(\eta_\varepsilon - \bar{\mu})$, $\sigma^2(\eta_\varepsilon) = \sigma_\mu^2$, $b(t'_\varepsilon, \eta'_\varepsilon) = -\lambda(\eta'_\varepsilon - \bar{\mu})$ and $\sigma^2(\eta'_\varepsilon) = \sigma_\mu^2$.

If $u - \widehat{V} \leq 0$ in (2.28), then because $v^\Lambda - \widehat{V} \geq \Lambda$ in (2.29), we obtain that $u - v^\Lambda \leq -\Lambda < 0$ by contradiction with $\sup(u - v^\Lambda) = M > 0$. On the other hand, if $u - \widehat{V} > 0$ in (2.28), then we have

$$\begin{cases} \frac{t_\varepsilon - t'_\varepsilon}{\varepsilon} - \frac{\eta_\varepsilon - \eta'_\varepsilon}{\varepsilon} b(t_\varepsilon, \eta_\varepsilon) - \frac{1}{2}\sigma^2(\eta_\varepsilon)A_\varepsilon \leq 0, \\ \frac{t_\varepsilon - t'_\varepsilon}{\varepsilon} - \frac{\eta_\varepsilon - \eta'_\varepsilon}{\varepsilon} b(t'_\varepsilon, \eta'_\varepsilon) - \frac{1}{2}\sigma^2(\eta'_\varepsilon)B_\varepsilon \geq \Lambda. \end{cases}$$

Furthermore, after mixing these two inequalities above, we derive that

$$\begin{aligned}&\frac{\eta_\varepsilon - \eta'_\varepsilon}{\varepsilon}(b(t_\varepsilon, \eta_\varepsilon) - b(t'_\varepsilon, \eta'_\varepsilon)) + \frac{3}{2\varepsilon}(\sigma(\eta_\varepsilon) - \sigma(\eta'_\varepsilon))^2\\ \geq &\frac{\eta_\varepsilon - \eta'_\varepsilon}{\varepsilon}(b(t_\varepsilon, \eta_\varepsilon) - b(t'_\varepsilon, \eta'_\varepsilon)) + \frac{1}{2}(\sigma^2(\eta_\varepsilon)A_\varepsilon - \sigma^2(\eta'_\varepsilon)B_\varepsilon) \geq \Lambda.\end{aligned}$$

The first inequality holds due to the Crandall-Ishii's lemma. Moreover, by letting $\varepsilon \to 0$, we get $\frac{\eta_\varepsilon - \eta'_\varepsilon}{\varepsilon}(b(t_\varepsilon, \eta_\varepsilon) - b(t'_\varepsilon, \eta'_\varepsilon)) + \frac{3}{2\varepsilon}(\sigma(\eta_\varepsilon) - \sigma(\eta'_\varepsilon))^2 = 0$ thanks to (2.27). It follows that we have $0 \geq \Lambda > 0$, which leads to a contradiction and therefore our claim holds. $\square$

**Lemma 2.7.** *For all $(t, \eta) \in \mathcal{C}$ in the continuation region, $\widetilde{V}$ in (2.5) has Hölder continuous derivatives.*

*Proof.* The proof follows closely the argument in Section 6.3 of [41]. First, let us recall that

$$\frac{\partial \widetilde{V}}{\partial t}(t, \eta) + \lambda(\eta - \bar{\mu}) \frac{\partial \widetilde{V}}{\partial \eta}(t, \eta) - \frac{1}{2}\sigma_\mu^2 \frac{\partial^2 \widetilde{V}}{\partial \eta^2}(t, \eta) = 0 \text{ on } \mathcal{C}. \tag{2.30}$$

The definition of viscosity solution of $\widetilde{V}$ to (2.22) gives that $\widetilde{V}$ is a supersolution to (2.30). On the other hand, for any $(\bar{t}, \bar{\eta}) \in \mathcal{C}$, let $\varphi$ be a $C^2$ test function such that $(\bar{t}, \bar{\eta})$ is a maximum of $\widetilde{V} - \varphi$ with $\widetilde{V}(\bar{t}, \bar{\eta}) = \varphi(\bar{t}, \bar{\eta})$. By definition of $\mathcal{C}$, we have $\widetilde{V}(\bar{t}, \bar{\eta}) > \widehat{V}(\bar{t}, x_0 - f(\bar{t}), z_0, \bar{\eta})$, so that

$$\frac{\partial \varphi}{\partial t}(\bar{t}, \bar{\eta}) + \lambda(\eta - \bar{\mu}) \frac{\partial \varphi}{\partial \eta}(\bar{t}, \bar{\eta}) - \frac{1}{2}\sigma_\mu^2 \frac{\partial^2 \varphi}{\partial \eta^2}(\bar{t}, \bar{\eta}) \leq 0,$$

due to the viscosity sub-solution property of $\widetilde{V}$ to (2.22). It follows that $\widetilde{V}$ is a viscosity subsolution and therefore viscosity solution to (2.30).

Let us consider an initial boundary value problem:

$$\begin{aligned} -\frac{\partial w}{\partial t}(t, \eta) - \lambda(\eta - \bar{\mu}) \frac{\partial w}{\partial \eta}(t, \eta) + \frac{1}{2}\sigma_\mu^2 \frac{\partial^2 w}{\partial \eta^2}(t, \eta) = 0 \text{ on } Q \cup B_T, \\ w(0, \eta) = 0 \text{ on } B, \\ w(t, \eta) = \widehat{V}(t, x_0 - \kappa t, z_0, \eta) \text{ on } S. \end{aligned} \tag{2.31}$$

Here, $Q$ is an arbitrary bounded open region in $\mathcal{C}$, $Q$ lies in the strip $0 < t < T$. $\tilde{B} = \bar{Q} \cap \{t = 0\}$, $\tilde{B}_T = \bar{Q} \cap \{t = T\}$, $B_T$ denotes the interior of $\tilde{B}_T$, $B$ denotes the interior of $\tilde{B}$, $S_0$ denotes the boundary of $Q$ lying in the strip $0 \leq t \leq T$ and $S = S_0 \backslash B_T$.

Theorem 3.6 in [41] provides the existence and uniqueness of a solution $w$ on $Q \cup B_T$ to (2.31), and the solution $w$ has Hölder continuous derivatives $w_t$, $w_\eta$ and $w_{\eta\eta}$. Because the solution $w$ is a viscosity solution to (2.30) on $Q \cup B_T$, from standard uniqueness results on viscosity solution, we know that $\widetilde{V} = w$ on $Q \cup B_T$. As $Q \subset \mathcal{C}$ is arbitrary, it follows that $\widetilde{V}$ has the same property in the continuation region $\mathcal{C}$. Therefore, $\widetilde{V}$ has Hölder continuous derivatives $\widetilde{V}_t$, $\widetilde{V}_\eta$ and $\widetilde{V}_{\eta\eta}$. □

We can finally prove our main result Theorem 2.1.

*Proof.* We have shown the inequality $v^- = \sup_{p \in \mathcal{V}^-} p \le \widetilde{V} \le v^+ = \inf_{q \in \mathcal{V}^+} q$ in Lemma 2.6. By using the comparison result in Proposition 2.1, we also have $v^+ \le v^-$. Putting all pieces together, we conclude that $v^+ = \widetilde{V}(t,\eta) = v^-$ and therefore the value function $\widetilde{V}(t,\eta)$ is the unique viscosity solution of the HJB variational inequality (2.7). By following similar argument for Theorem 1 in [36], fix the $\mathcal{F}_t$-adapted stopping time $\tau^*$ defined in (2.9), Itô-Tanaka's formula (see Theorem IV.1.5, Corollary IV.1.6 of [77]) can be applied to $\widetilde{V}(t,\mu_t)$ in view of Hölder continuous derivatives of $\widetilde{V}(t,\eta)$ and we get that

$$
\begin{aligned}
&\widehat{V}(\tau^* \wedge \tau_n, x_0 - \kappa\tau^* \wedge \tau_n, z_0, \mu_{\tau^* \wedge \tau_n}) \\
=&\widetilde{V}(t,\mu_t) + \left[\widehat{V}(\tau^* \wedge \tau_n, x_0 - \kappa\tau^* \wedge \tau_n, z_0, \mu_{\tau^* \wedge \tau_n}) - \widetilde{V}(\tau^* \wedge \tau_n, \mu_{\tau^* \wedge \tau_n})\right] \\
+&\int_t^{\tau^* \wedge \tau_n} \sigma_\mu \frac{\partial \widetilde{V}}{\partial \eta}(s,\mu_s) dB_s + \int_t^{\tau^* \wedge \tau_n} \left[\frac{\partial \widetilde{V}(s,\mu_s)}{\partial t} + \mathcal{L}\widetilde{V}(s,\mu_s)\right] ds,
\end{aligned}
$$

where $\tau_n \uparrow T$ is the localizing sequence. As $\widetilde{V}(t,\eta)$ satisfies HJB variational inequality (2.7), by taking conditional expectations and the definition of $\tau^*$ in (2.9), we obtain that

$$
\mathbb{E}_t\left[\widehat{V}(\tau^* \wedge \tau_n, x_0 - \kappa\tau^* \wedge \tau_n, z_0, \mu_{\tau^* \wedge \tau_n}) \mathbf{1}_{\{\tau^* \le \tau_n\}}\right] + \mathbb{E}_t\left[\widetilde{V}(\tau_n, \mu_{\tau_n}) \mathbf{1}_{\{\tau^* > \tau_n\}}\right] = \widetilde{V}(t,\mu_t).
$$

By taking the limit of $\tau_n$ and dominated convergence theorem, we can verify that

$$
\mathbb{E}_t\left[\widehat{V}(\tau^*, x_0 - \kappa\tau^*, z_0, \mu_{\tau^*})\right] = \widetilde{V}(t,\mu_t),
$$

and therefore $\tau^*$ is the optimal entry time.

At last, the martingale property between $t = 0$ and $\tau^*$ follows from the definition of stochastic subsolution and stochastic supersolution. $\square$

# Chapter 3

# Optimal Dividend Strategy for an Insurance Group with Contagious Default Risk

This chapter [1] studies the optimal dividend for a multi-line insurance group, in which each subsidiary runs a product line and is exposed to some external credit risk. The default contagion is considered such that one default event may increase the default probabilities of all surviving subsidiaries. The total dividend problem for the insurance group is investigated and we find that the optimal dividend strategy is still of the barrier type. Furthermore, we show that the optimal barrier of each subsidiary is modulated by the default state. That is, how many and which subsidiaries have defaulted will determine the dividend threshold of each surviving subsidiary. These conclusions are based on the analysis of the associated recursive system of *Hamilton-Jacobi-Bellman* variational inequalities (HJBVIs). The existence of the classical solution is established and the verification theorem is proved. In the case of two subsidiaries, the value function and optimal barriers are given in analytical forms, allowing us to conclude that the optimal barrier of one subsidiary decreases if the other subsidiary defaults.

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[1] A version of this chapter has been accepted by Scandinavian Actuarial Journal, which is forthcoming.

## 3.1 Introduction

Dividend payment is always a focused issue in insurance and corporate finance, which is regarded as an important signal of the company's future growth opportunities and has direct impact on the wealth of shareholders. Meanwhile, insurance companies also dynamically invest money in the financial market in order to pay future claims. The pioneer work [34] solves the optimal dividend problem up to the financial ruin time when the surplus process follows a simple random walk. Later, vast research has been devoted to finding optimal dividend strategies in various discrete and continuous time risk models, see a short list of related work in [6, 8, 9, 27, 43, 47, 56, 59, 68, 69, 70, 78] and references therein. We refer to [3] and [7] for some comprehensive surveys on the topic of dividend optimization.

The present chapter has a particular interest in a multi-line insurance group, which is a parent insurer consisting of multiple subsidiaries in the market where each subsidiary runs a product line such as life insurance, auto insurance, income protection insurance, housing insurance and etc. Each product line is subject to bankruptcy separately and has its own premiums and losses with very distinctive claim frequency, which motivates some recent academic studies on multi-line insurance business. In a multi-line insurance group framework, the insurance pricing model by line is studied in [73]. The capital allocation strategy for a multi-line insurance company is investigated in [67], which reveals that allocations depend on the uncertainty of each line's losses and the marginal contribution of each line. Under the assumption that losses from all product lines follow a sharing rule, some premiums problems are examined in [48].

What is missing in the literature is the investigation of external systemic risk for the insurance group. Our work enriches the study of the insurance group by considering the group dividend optimization problem in which each subsidiary may go default due to some contagious default risk. In practice, many subsidiaries share the same reserves pool from the parent group company. It is reasonable to assume that all subsidiaries are exposed to some common credit risk. Our model can depict some real life situations that the group manager collects cash reserves from different subsidiaries and invests them into some financial credit instruments such as defaultable Bonds, CDS, equity default swaps and etc. The insolvency and termination of one subsidiary business caused by the market credit risk may quickly spread to all other subsidiaries if they share the same underlying credit assets. Some empirical studies find that defaults are indeed contagious in certain cases and exhibit the so-called default-clustering phenomenon, see [33]. In particular, a dependent credit risk model is studied in [80], which analyzes the contagious defaults affected by a common macroeconomic factor. A financial network model is later developed in [4], in which the contagious defaults are caused by a macroeconomic shock. In the context of insurance, it is also reasonable to consider the investment of net-reserves in some credit assets and the default risk in the financial market may lead to some massive domino effects in surplus management and subsidiaries operations.

It is worth noting that some recent work such as [2], [46] and [45] consider the collaborating dividend problem between multiple insurance companies, in which the credit default and default contagion are again not concerned. Instead, they consider some independent insurance companies and assume that one insurance company can inject capital into other companies whenever their financial ruins occur. The optimal dividend for two collaborating insurance companies in compound Poisson and diffusion models are studied in [2] and [46] respectively. The extension to different solvency criteria is considered later in [45]. Although these work differ substantially from the present chapter, we confront similar challenges from the multi-dimensional singular control problem and some new mathematical methods are required.

To ensure the tractability, we work in the interacting intensity framework to model default contagion, which allows sequential defaults and assumes that the credit default of one subsidiary can affect other surviving names by increasing their default intensities.

This type of default contagion has been actively studied recently in the context of portfolio management, see among [17, 19, 20, 21, 22] and many others. The key observation in these work is that the system of HJB partial differential equations (PDEs) is recursive and the depth of the recursion equals the number of risky assets. The system of PDEs can therefore be analyzed using a backward recursion from the state in which all assets are defaulted towards the state that all assets are alive. As opposed to portfolio optimization, we confront a singular control problem that stems from the dividend payment, and we consequently need to handle variational inequalities instead of PDE problems. To the best of our knowledge, our work appears as the first one attempting to introduce the default contagion to the insurance group dividend control framework. In particular, we distinguish the ruin caused by insurance claims (i.e. the surplus process diffuses to zero) and the termination caused by credit default jump. It is observed in this chapter that the optimal group dividend is of the barrier type and the optimal barrier for each subsidiary is default-state-modulated, i.e., the optimal barrier of each surviving subsidiary will be adjusted whenever some subsidiaries go default. In the simple case of two subsidiaries, we can rigorously prove that the group manager lowers the dividend barrier of the surviving subsidiary and forces it to pay dividend soon, see Corollary 3.1.

Our mathematical contribution is the study of the recursive system of HJBVIs (3.36), which differs from some conventional PDE problems in portfolio optimization. We adopt the core idea in [17, 19, 20, 22] and follow the backward recursion based on the number of defaulted subsidiaries. In addition, we take the full advantage of the risk neutral valuation of the group control and simplify the multi-dimensional value function into a separation form. Our arguments can be outlined as follows. Firstly, we start from the case when there is only one surviving subsidiary and work inductively to the case when all subsidiaries are alive. The classical solution in the step with $k$ surviving subsidiaries will appear as variable coefficients in the step with $k + 1$ surviving subsidiaries, and we can continue to show the existence of classical solution with $k + 1$ names. Secondly, to show the existence of classical solution in each step with a fixed number of subsidiaries, we conjecture a separation form of the value function, and split the variational inequality from the group control into a subsystem of auxiliary variational inequalities. To tackle each auxiliary variational inequality, we first obtain the existence of a classical solution to the ODE problem. By applying the smooth-fit principle, we deduce the existence of a free boundary point depending on the default state and construct the desired classical solution to the auxiliary variational inequality. The rigorous proof of the verification theorem is provided to show that the value function coincides with the classical solution to the recursive system of HJB-VIs (3.36). As a byproduct, the optimal dividend is proved to be a reflection strategy with the barrier depending on the default state indicator process, see (3.6) in Theorem 3.1.

The rest of the chapter is organized as follows. Section 3.2 introduces the model of the multi-line insurance group with external credit default contagion. The optimal group dividend problem for all subsidiaries is formulated and the main theorem is presented therein. In Section 3.3, we derive the HJBVI (3.9) for two subsidiaries and solve the value function in an explicit manner. The optimal barriers of the dividend are constructed using the smooth-fit principle. Section 3.4 generalizes the results to a multi-line insurance group. The proof of the verification theorem is given in Section 3.5. The derivation of the HJBVI (3.9) for two subsidiaries is reported in Appendix B.

## 3.2 Model Formulation

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space where $\mathbb{F} := \{\mathcal{F}_t\}$ is a right-continuous, $\mathbb{P}$-completed filtration. We consider an insurance group consisting of $N$ subsidiary business units and each business unit is managed independently within the group. In particular, the decision maker in the present chapter is the insurance group manager, who collects the premiums and contributes shares of the dividend for the whole group of subsidiaries.

After the pioneer work [49], the diffusion-approximation of the classical Cramér-Lundberg model has been popular in the study of optimal dividend and reinsurance thanks to its tractability and allowance of explicit control strategies, see among [37], [44], [5], [28], [42] and many others. Following their setting, it is assumed in this chapter that all subsidiaries have the same form of surplus processes with different drifts and insurance claim distributions and the pre-default surplus process $\hat{X}_i(t)$ for each subsidiary satisfies the diffusion model that

$$d\hat{X}_i(t) = a_i dt - b_i dW_i(t),$$

where constants $a_i > 0$ and $b_i > 0$ represent the mean and the volatility of the surplus process respectively, and each $W_i(t)$ is a standard $\mathbb{P}$-Brownian motion. For $1 \leq i, j \leq N$, the correlation coefficient between $W_i$ and $W_j$ is denoted by the constant $-1 \leq \rho_{ij} \leq 1$ and the correlation coefficient matrix is denoted by $\Sigma = (\rho_{ij})_{N \times N}$. The model covers correlated insurance claims from different subsidiaries including possible scenarios that some subsidiaries are running product lines that depend on other product lines and some subsidiaries serve certain overlapping customers.

We consider in this chapter that each subsidiary allocates a large proportion of its net-reserves in some credit assets. Each subsidiary is exposed to some external credit risk in the financial market, and a wave of defaults in these credit assets may lead to large loss of net-reserves in all subsidiaries. One example is the collapse of AIG, which is exposed to substantial credit risk in its balance sheet in the 2008 financial crisis. To make our multi-dimensional dividend control problem tractable and facilitate the backward induction method, we consider the extreme case in the present chapter that the external default will terminate the operation of the subsidiary and no salvage value can be paid as dividend at the moment of default. To model these extreme and irreparable default events, we choose the so-called default indicator process that is described by an N-dimensional $\mathbb{F}$-adapted process $\mathbf{Z}(t) = (Z_1(t), \ldots, Z_N(t))$ taking values on $\{0,1\}^N$. For each $i$,

$Z_i(t) = 1$ indicates that the $i$-th subsidiary has defaulted up to time $t$, while $Z_i(t) = 0$ indicates that the $i$-th subsidiary is still alive at time $t$. The process $\mathbf{Z}(t)$ is assumed to be independent of all Brownian motions $W_i(t)$, $i = 1, \ldots, N$, to reflect that these external default events stem from the credit assets and they do not depend on the claims of each subsidiary's insurance products.

For each $i = 1, \ldots, N$, the default time $\sigma_i$ for the $i$-th subsidiary is given by

$$\sigma_i := \inf \{t \geq 0; Z_i(t) = 1\}.$$

The stochastic intensity of $\sigma_i$ is modeled by $(1 - Z_i(\cdot)) \lambda_i (\mathbf{Z}(\cdot))$, where $\lambda_i$ maps $\{0,1\}^N$ to $(0, +\infty)$ and the process

$$M_i(t) := Z_i(t) - \int_0^{t \wedge \sigma_i} \lambda_i (\mathbf{Z}(s)) \, ds, \tag{3.1}$$

is a martingale with respect to the filtration generated by $\mathbf{Z}$. Note that this process $Z_i(t)$ can also be viewed as a Cox process truncated above by constant 1, whose intensity process is $(1 - Z_i(t))\lambda_i(\mathbf{Z}(t)) + Z_i(t)$.

Let us take $N = 2$ as an example and consider the default state $\mathbf{Z}(t) = (0, 0)$ at time $t$. The values $\lambda_1(0, 0)$ and $\lambda_2(0, 0)$ give the default intensity of subsidiary 1 and subsidiary 2 at time $t$ respectively. Suppose that subsidiary 1 has already defaulted before time $t$ and only subsidiary 2 is alive, then $\lambda_2(1, 0)$ represents the default intensity of subsidiary 2 at time $t$. Similarly, if the subsidiary 2 has already defaulted before time $t$ and only subsidiary 1 is alive, then $\lambda_1(0, 1)$ represents the default intensity of subsidiary 1 at time $t$. Moreover, we consider the default contagion in the sense that $\lambda_1(0, 0) \leq \lambda_1(0, 1)$ and $\lambda_2(0, 0) \leq \lambda_2(1, 0)$ such that the default intensity of one subsidiary increases after the other subsidiary defaults.

For the general case with $N$ subsidiaries, the default indicator process at time $t$ may jump from a state

$$\mathbf{Z}(t) = (Z_1(t), \ldots, Z_{i-1}(t), Z_i(t), Z_{i+1}(t), \ldots, Z_N),$$

in which the subsidiary $i$ is alive ($Z_i(t) = 0$) to the neighbour state

$$(Z_1(t), \ldots, Z_{i-1}(t), 1 - Z_i(t), Z_{i+1}(t), \ldots, Z_N),$$

in which the subsidiary $i$ has defaulted with the stochastic rate $\lambda_i(\mathbf{Z}(t))$. It is assumed from this point on that $Z_i$, $i = 1, \ldots, N$, will not jump simultaneously in the sense that

$$\Delta Z_i(t)\Delta Z_j(t) = 0, \quad 1 \leq i < j \leq N, \quad t \geq 0. \tag{3.2}$$

Note that the default intensity of the $i$-th subsidiary $\lambda_i(\mathbf{Z}(t))$ depends on the whole vector process $\mathbf{Z}(t)$, and it is assumed that $\lambda_i(\mathbf{Z}(t))$ increases if any other subsidiary defaults. This is what we mean by default contagion for multiple subsidiaries. Let us denote the vector $\lambda(\mathbf{z}) = (\lambda_i(\mathbf{z}); i = 1, \ldots, N)^T$, for the given default vector $\mathbf{z} \in \{0, 1\}^N$.

The actual surplus process of subsidiary $i$ after the incorporation of external credit risk is denoted by $\widetilde{X}_i(t)$, where $i = 1, 2, \ldots, N$, and it is defined as

$$\widetilde{X}_i(t) := (1 - Z_i(t))\, \hat{X}_i(t).$$

Given the surplus process $\widetilde{X}_i(t)$, for each subsidiary $i$, we can then introduce the dividend policy. A dividend strategy $D_i(\cdot)$ is an $\mathcal{F}_t$-adapted process representing the accumulated amount of dividend paid up to time $t$. That is, $D_i(t)$ is a nonnegative and nondecreasing stochastic process that is right continuous and have left limits with $D_i(0^-) = 0$. The jump size of $D_i$ at time $t \geq 0$ is denoted by $\Delta D_i(t) := D_i(t) - D_i(t^-)$, and $D_i^c(t) := D_i(t) - \sum_{0 \leq s \leq t} \Delta D_i(s)$ denotes the continuous part of $D_i(t)$.

For the $i$-th subsidiary, the resulting surplus process in the presence of dividend payments can be written as

$$X_i(t) := \big(1 - Z_i(t)\big)\big(\widetilde{X}_i(t) - D_i(t)\big),\ X_i(0) = x_i \geq 0,$$

where $x_i$ stands for the initial surplus of the $i$-th subsidiary. We denote the vector process

$$\mathbf{X}(t) := (X_1(t), \ldots, X_N(t)).$$

The objective function for the insurance group is formulated as a corporative singular control of total dividend strategy

$$\mathbf{D}(t) = (D_1(t), \ldots, D_N(t)),$$

under the expected value of discounted future dividend payments up to the ruin time

$$J(\mathbf{x}, \mathbf{z}, \mathbf{D}(\cdot)) := \mathbb{E}\left(\sum_{i=1}^{N} \alpha_i \int_0^{\tau_i} e^{-rt} dD_i(t)\right),$$

where the weight parameter satisfies $\alpha_1 + \alpha_2 + \ldots + \alpha_N = 1$. The parameter $\alpha_i$ represents the relative weight of the subsidiary in the insurance group, and they add up to $1$ after scaling. $r > 0$ is a given discount rate. Recall that the insurance group manager is the decision maker, the surplus process of each subsidiary is therefore completely observable to the decision maker. The ruin time $\tau_i$ of the subsidiary $i$ is defined by

$$\tau_i := \inf\{t \geq 0 : X_i(t) = 0\}, \;\; i = 1, \ldots, N.$$

The initial surplus level is denoted by $X_i(0) = x_i$ and the initial default state is denoted by $Z_i(0) = z_i$, $i = 1, \ldots, N$. We also denote

$$\mathbf{X}(0) = \mathbf{x} := (x_1, \ldots, x_N),$$
$$\mathbf{Z}(0) = \mathbf{z} := (z_1, \ldots, z_N).$$

It is assumed henceforth that each admissible control process $D_i(t)$ can not jump simultaneously with $Z_i(t)$ in the sense that, for $t \geq 0$,

$$\Delta D_i(t) \Delta Z_i(t) = 0, \; 1 \leq i \leq N. \tag{3.3}$$

That is, the dividend for the subsidiary $i$ can not be paid right at the moment when the subsidiary $i$ goes default due to external credit risk. The assumption (3.3) is by no means restrictive because the process $D_i(t)$ is càdlàg and the default time $\sigma_i$ is totally inaccessible due to the existence of default intensity $\lambda_i$. In Appendix B, assumptions (3.2) and (3.3) are needed to derive the associated HJBVI. Moreover, it is assumed throughout the chapter that

$$\Delta D_i(t) \leq X_i(t-),$$
$$D_i(t) = D_i(t \wedge \tau_i),$$

where the first condition dictates that the subsidiary $i$ can not pay dividend more than its currently available fund and the second condition means that the subsidiary $i$ won't pay any dividend after its ruin time.

**Remark 3.1.** *In most cases, we simply choose* $\alpha_i = \frac{1}{N}$*, that is, every subsidiary shares the same percentage of the insurance group. In special cases, the product line of some subsidiary* $i$ *is the key business of the insurance group, in which weight* $\alpha_i$ *is a little bit bigger than other subsidiaries.*

Our goal is to find the optimal dividend strategy $\mathbf{D}^*$ such that the value function can be attained that

$$f(\mathbf{x}, \mathbf{z}) := \sup_D J(\mathbf{x}, \mathbf{z}, \mathbf{D}) = J(\mathbf{x}, \mathbf{z}, \mathbf{D}^*). \tag{3.4}$$

In particular, we are interested in the case that all subsidiaries are alive at the initial time, i.e., the value function $f(\mathbf{x}, \mathbf{0})$ can be characterized, where $\mathbf{0} = (0, \dots, 0)$ is the zero vector.

A barrier dividend strategy is to pay dividend whenever the surplus process excesses over the barrier. The optimal dividend for a single insurance company has been shown to fit this type of barrier control in various risk models. In our setting with default contagion, the optimal dividend for the insurance group also fits this barrier control. Nevertheless, the optimal barrier for each subsidiary is no longer a fixed level as in the model of a single insurance company. Instead, we identify that the optimal barrier is dynamically modulated by the defaulted subsidiaries and surviving ones. The dependence on the default state leads to some distinctive phenomena that the dividend barrier will be adjusted in the observation of sequential defaults. Furthermore, the change of the barrier for subsidiary $i$, i.e. the change of $m_i(\mathbf{Z}(t))$ in (3.6), is complicated and depends on all market parameters. In the case of two subsidiaries, we can prove in Corollary 3.1 that the default event of one subsidiary will stimulate the surviving one to pay dividend, albeit with less amount, because the dividend threshold decreases.

For any vectors $\mathbf{x} \in [0, +\infty)^N$ and $\mathbf{z} \in \{0, 1\}^N$, let us denote

$$\mathbf{x}^{(l)} := (x_1, \ldots, x_{l-1}, 0, x_{l+1}, \ldots, x_N),$$

$$\mathbf{z}^l := (z_1, \ldots, z_{l-1}, 1, z_{l+1}, \ldots, z_N).$$

The next theorem is the main result of this chapter.

**Theorem 3.1.** *Let us consider the initial surplus level* $\mathbf{X}(0) = \mathbf{x} \in [0, +\infty)^N$ *and the initial default state* $\mathbf{Z}(0) = \mathbf{z} := (z_1, \ldots, z_N) = \mathbf{0}$ *that all subsidiaries are alive at the initial time. The value function* $f(\mathbf{x}, \mathbf{0})$ *defined in* (3.4) *is the unique classical solution to the variational inequalities*

$$\max_{1 \le i \le N} \left\{ \mathcal{L}f(\mathbf{x}, \mathbf{z}) + \sum_{l=1}^{N} \lambda_l(\mathbf{z}) f(\mathbf{x}^{(l)}, \mathbf{z}^l), \alpha_i - \partial_i f(\mathbf{x}, \mathbf{z}) \right\} = 0, \tag{3.5}$$

*in which the operator is defined by*

$$\mathcal{L}f(\mathbf{x}, \mathbf{z}) := -\left( r + \sum_{k=1}^{N} \lambda_k(\mathbf{z}) \right) f(\mathbf{x}, \mathbf{z}) + \sum_{k=1}^{N} \left( a_k \partial_k f(\mathbf{x}, \mathbf{z}) + \frac{1}{2} b_k^2 \partial_{kk} f(\mathbf{x}, \mathbf{z}) \right)$$
$$+ \sum_{\substack{i,j=1 \\ i>j}}^{N} b_i b_j \rho_{ij} \partial_{ij}^2 f(\mathbf{x}, \mathbf{z}),$$

*where* $\partial_k f := \frac{\partial f}{\partial x_k}$ *and* $\partial_{kk} f := \frac{\partial^2 f}{\partial x_k^2}$.

*Moreover, for each* $i = 1, \ldots, N$*, there exists a mapping* $m_i : \{0, 1\}^N \mapsto (0, +\infty)$ *such that the optimal dividend* $\mathbf{D}^*$ *for the* $i$*-th subsidiary is given by the reflection strategy*

$$D_i^*(t) := \max \left\{ 0, \sup_{0 \le s \le t} \left\{ \widetilde{X}_i(s) - m_i(\mathbf{Z}(s)) \right\} \right\}, \quad i = 1, \ldots, N, \tag{3.6}$$

*and $m_i(\mathbf{Z}(t))$ represents the optimal barrier for the $i$-th subsidiary modulated by the N-dimensional default state indicator $\mathbf{Z}(t)$ at time $t$.*

From the form of HJBVI (3.5), we can see that the solution $f(\mathbf{x}, \mathbf{z})$ actually depends on the value function $f(\mathbf{x}, \mathbf{z}^l)$ with the initial default state $\mathbf{z}^l$ indicating that one subsidiary has already defaulted. Therefore, to show the existence of classical solution to HJBVI (3.5) with $\mathbf{z} = \mathbf{0}$, we have to analyze the existence of the classical solution of the entire system of HJBVIs with all different values of $\mathbf{z} \in \{0, 1\}^N$. To this end, we follow a recursive scheme that is based on default states of subsidiaries. The proof of Theorem 3.1 is postponed to Section 3.5.

## 3.3 Analysis of HJBVIs: Two Subsidiaries

To make our recursive arguments more readable, we first present the main result for only 2 subsidiaries. As one can see, the associated HJB variational inequalities can be solved explicitly for 2 initial subsidiaries and the optimal barriers of dividend for each subsidiary at time $t$ can be derived that depends on the default state $\mathbf{Z}(t)$. The recursive scheme to analyze the variational inequalities has a hierarchy feature, which is operated in a backward manner. To be more precise, we first solve a standard optimal dividend problem when only one subsidiary survives initially, and the associated value function appears as variable coefficients in the top level of HJBVI when both subsidiaries are initially alive. We can then continue to tackle the top level HJBVI with two subsidiaries by employing a separation form of its solution and the smooth-fit principle.

### 3.3.1 *One Surviving Subsidiary*

In this subsection, it is assumed that there is only one subsidiary at the initial time. That is, we need to consider default states $\mathbf{z}_1 := (0, 1)$ and $\mathbf{z}_2 := (1, 0)$. Here, the default state $\mathbf{z}_i$, $i = 1, 2$, indicates that subsidiary $i$ is alive initially while the other subsidiary has already defaulted due to the external credit risk.

For each $i$, let us consider the default state $\mathbf{z}_i$, and let $x_i \geq 0$ be the initial surplus level for the subsidiary $i$. The associated HJBVI for the default state $(0,1)$ and $(1,0)$ can be derived as

$$\max\left\{\mathcal{L}^{\mathbf{z}_i} f(x_i, \mathbf{z}_i), \alpha_i - \frac{\partial f}{\partial x_i}(x_i, \mathbf{z}_i)\right\} = 0, \quad i = 1, 2, \tag{3.7}$$

where the operator is defined by

$$\mathcal{L}^{\mathbf{z}_i} f := -\left(r + \lambda_i(\mathbf{z}_i)\right) f + \left(a_i \frac{\partial f}{\partial x_i} + \frac{1}{2} b_i^2 \frac{\partial^2 f}{\partial x_i^2}\right).$$

Here, we recall that $\lambda_i(\mathbf{z}_i)$ stands for the default intensity for subsidiary $i$ given that the other subsidiary has already defaulted.

We can follow some standard results in [6], which solves the stochastic control problem for a single insurance company. The positive discount rate $r > 0$ ensures that

$$\frac{1}{2} b_i^2 s^2 + a_i s - (r + \lambda_i(\mathbf{z}_i)) = 0,$$

admits two real roots. Let $\hat{\theta}_{i1}$, $-\hat{\theta}_{i2}$ denote the positive and negative root respectively that

$$\hat{\theta}_{i1} := \frac{-a_i + \sqrt{a_i^2 + 2b_i^2(r + \lambda_i(\mathbf{z}_i))}}{b_i^2}, \quad i = 1, 2,$$

$$-\hat{\theta}_{i2} := \frac{-a_i - \sqrt{a_i^2 + 2b_i^2(r + \lambda_i(\mathbf{z}_i))}}{b_i^2}, \quad i = 1, 2.$$

According to results in [6], for $i = 1, 2$, the solution to the HJBVI (3.7) is

$$f(x_i, \mathbf{z}_i) = \begin{cases} \alpha_i C_i(\mathbf{z}_i)(e^{\hat{\theta}_{i1} x_i} - e^{-\hat{\theta}_{i2} x_i}), & 0 \leq x_i \leq m_i(\mathbf{z}_i), \\ \alpha_i C_i(\mathbf{z}_i)(e^{\hat{\theta}_{i1} m_i(\mathbf{z}_i)} - e^{-\hat{\theta}_{i2} m_i(\mathbf{z}_i)}) + \alpha_i(x_i - m_i(\mathbf{z}_i)), & x_i \geq m_i(\mathbf{z}_i), \end{cases} \tag{3.8}$$

where

$$
\begin{aligned}
m_i(\mathbf{z}_i) &:= \frac{2}{\hat{\theta}_{i1}+\hat{\theta}_{i2}}\log\left(\frac{\hat{\theta}_{i2}}{\hat{\theta}_{i1}}\right) \\
&= \frac{b_i^2}{\sqrt{a_i^2+2b_i^2(r+\lambda_i(\mathbf{z}_i))}}\log\left(\frac{\sqrt{a_i^2+2b_i^2(r+\lambda_i(\mathbf{z}_i))}+a_i}{\sqrt{a_i^2+2b_i^2(r+\lambda_i(\mathbf{z}_i))}-a_i}\right), \\
C_i(\mathbf{z}_i) &:= \frac{1}{\hat{\theta}_{i1}e^{\hat{\theta}_{i1}m_i(\mathbf{z}_i)}+\hat{\theta}_{i2}e^{-\hat{\theta}_{i2}m_i(\mathbf{z}_i)}}, \quad i=1,2.
\end{aligned}
$$

### 3.3.2 *Auxiliary Results for Two Subsidiaries*

We continue to consider the case that both subsidiaries are alive at time $t=0$ with the initial surplus $\mathbf{x}=(x_1,x_2)$ and initial default state $\mathbf{z}=(0,0)$. Using heuristic arguments in Appendix B, the associated HJBVI for the value function can be written by

$$
\max\left\{\mathcal{L}^{(0,0)}f(\mathbf{x},(0,0)),\alpha_1-\partial_1 f(\mathbf{x},(0,0)),\alpha_2-\partial_2 f(\mathbf{x},(0,0))\right\}=0, \tag{3.9}
$$

with the operator

$$
\begin{aligned}
\mathcal{L}^{(0,0)}f(\mathbf{x},(0,0)) :=& -(r+\lambda_1(0,0)+\lambda_2(0,0))f(\mathbf{x},(0,0))+b_1b_2\rho_{12}\partial_{12}f(\mathbf{x},(0,0)) \\
&+\left(a_1\partial_1 f(\mathbf{x},(0,0))+\frac{1}{2}b_1^2\partial_{11}^2 f(\mathbf{x},(0,0))\right) \\
&+\left(a_2\partial_2 f(\mathbf{x},(0,0))+\frac{1}{2}b_2^2\partial_{22}^2 f(\mathbf{x},(0,0))\right) \\
&+\lambda_1(0,0)f(x_2,(1,0))+\lambda_2(0,0)f(x_1,(0,1)),
\end{aligned} \tag{3.10}
$$

where functions $f(x_1,(0,1))$ and $f(x_2,(1,0))$ are given explicitly in (3.8), and

$$
\partial_i f(\mathbf{x},(0,0)) := \frac{\partial f(\mathbf{x},(0,0))}{\partial x_i}, \text{ and } \partial_{ij} f(\mathbf{x},(0,0)) := \frac{\partial^2 f(\mathbf{x},(0,0))}{\partial x_i x_j}, \ \ i,j=1,2.
$$

To show the existence of a classical solution to HJBVI (3.9), we first conjecture that the solution $f(\mathbf{x},(0,0))$ with $\mathbf{x}=(x_1,x_2)\in[0,+\infty)^2$ admits a key separation form that

$$
f(\mathbf{x},(0,0)) = f_1(x_1,(0,0))+f_2(x_2,(0,0)), \quad x_1,x_2\geq 0, \tag{3.11}
$$

for some smooth functions $f_1$ and $f_2$, i.e., functions of $x_1$ and $x_2$ can be decoupled. The rigorous proof of this separation form will be given in the next subsection.

With the aid of the separation form (3.11), to solve HJBVI (3.9) is equivalent to solve two auxiliary variational inequalities with one dimensional variable $x \in [0, +\infty)$ defined by

$$\max\left\{\mathcal{A}_i f_i(x,(0,0)) + \frac{\lambda_1(0,0)\lambda_2(0,0)}{\lambda_i(0,0)} f(x,\mathbf{z}_i), \right. \tag{3.12}$$

$$\left. \alpha_i - f_i'(x,(0,0))\right\} = 0, \quad i = 1,2, \quad x \geq 0, \tag{3.13}$$

where the operators are defined as

$$\mathcal{A}_i f(x,(0,0)) := \frac{1}{2} b_i^2 f''(x,(0,0)) + a_i f'(x,(0,0))$$
$$-(r + \lambda_1(0,0) + \lambda_2(0,0)) f(x,(0,0)), \quad i = 1,2,$$

and the boundary condition $f_i(0,(0,0)) = 0$, $i = 1,2$.

**Remark 3.2.** *When two subsidiaries are alive, the function $f_1(x_1,(0,0))$ from the decomposition relationship* (3.33) *satisfies variational inequalities* (3.12)*. It is worth noting that this function $f_1(x_1,(0,0))$ can not be simplify interpreted as the value function of the optimal dividend problem for the single subsidiary 1 without considering all other subsidiaries. As one can observe from* (3.12)*, $f_1(x_1,(0,0))$ depends on the coefficient $\lambda_2(0,0)$ that is the default intensity of the subsidiary 2 and also depends on the value function $f_1(x,(0,1))$. However, as pointed out later in Remark 3.4, our mathematical approach can eventually verify that $f_1(x_1,(0,0))$ equals the expected value of the discounted dividend using the dividend control policy $D_1^*(t)$ for subsidiary 1, where $\mathbf{D}^*(t) = (D_1^*(t), D_2^*(t))$ is the optimal dividend for the whole group.*

By symmetry, for the existence of classical solution to the auxiliary variational inequality (3.12), for $i = 1,2$, it is sufficient to study the general form of variational inequality with one dimensional variable $x \in [0, +\infty)$ defined by

$$\max\{\mathcal{A}f(x) + h(x), \gamma - f'(x)\} = 0, \tag{3.14}$$

where $\gamma > 0$,

$$\mathcal{A}f(x) := -\mu f(x) + \nu f'(x) + \frac{1}{2}\sigma^2 f''(x), \quad \mu, \nu, \sigma > 0, \tag{3.15}$$

and the function $h$ is a $C^2$ function satisfying $h(0) = 0$, $\lim_{u \to +\infty} h(u) = +\infty$, $h(x) \geq 0$, $h'(x) > 0$, and $h''(x) \leq 0$, for $x \geq 0$.

To tackle the general variational inequality (3.14), we propose to examine the solution to the ODE part at first in the next lemma.

**Lemma 3.1.** *Let us consider the ODE problem*

$$\mathcal{A}g(x) + h(x) = 0, \quad x \geq 0, \tag{3.16}$$

*with the boundary condition* $g(0) = 0$ *and the operator* $\mathcal{A}$ *is defined in* (3.15)*,* $h$ *is the same as that in* (3.14)*. The classical solution* $g$ *to* (3.16) *admits the form*

$$g(x) = \phi_1(x) + C\phi_2(x),$$

*where* $C$ *is a parameter in* $\mathbb{R}$*, and*

$$\phi_1(x) := -\frac{2}{\sigma^2(\theta_1 + \theta_2)} \int_0^x h(u)(e^{\theta_1(x-u)} - e^{-\theta_2(x-u)})du, \quad x \geq 0, \tag{3.17}$$

$$\phi_2(x) := e^{\theta_1 x} - e^{-\theta_2 x}, \quad x \geq 0. \tag{3.18}$$

*Here* $\theta_1$*,* $-\theta_2$ *are the roots of the equation* $\frac{1}{2}\sigma^2\theta^2 + \nu\theta - \mu = 0$*.*

*Proof.* We first rewrite the ODE (3.16) in a vector form as

$$\frac{d}{dx}\begin{pmatrix} g(x) \\ g'(x) \end{pmatrix} = A \begin{pmatrix} g(x) \\ g'(x) \end{pmatrix} + \beta(x),$$

where

$$A := \begin{pmatrix} 0 & 1 \\ 2\sigma^{-2}\mu & -2\sigma^{-2}\nu \end{pmatrix}, \quad \beta(x) := \begin{pmatrix} 0 \\ -2\sigma^{-2}h(x) \end{pmatrix}.$$

One can solve it as

$$\begin{pmatrix} g(x) \\ g'(x) \end{pmatrix} = e^{Ax} \int_0^x e^{-Au} \beta(u) du + e^{Ax} \beta_0.$$

The boundary condition $g(0) = 0$ then yields that $\beta_0 = (0, g'(0))^\top$ and

$$e^{Ax}\beta_0 = \left( C(e^{\theta_1 x} - e^{-\theta_2 x}), C(\theta_1 e^{\theta_1 x} + \theta_2 e^{-\theta_2 x}) \right)^\top,$$

for some constant $C$. Note also that $\beta(x) = \left(0, -2\sigma^{-2} h(x)\right)$, hence it follows that

$$\begin{aligned} e^{Ax} \int_0^x e^{-Au} \beta(u) du &= -2\sigma^{-2} \int_0^x e^{A(x-u)} \begin{pmatrix} 0 \\ h(u) \end{pmatrix} du \\ &= -2\sigma^{-2} \int_0^x h(u) e^{A(x-u)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} du. \end{aligned}$$

Let $\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = e^{At} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we get that $\frac{d}{dt} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = A \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$, $y_1(0) = 0, y_2(0) = 1$. Then $y_1'(t) = y_2(t)$ implies that $y_1(t) = C_1 e^{\theta_1 t} + C_2 e^{-\theta_2 t}$, $y_1(0) = 0$, $y_1'(0) = 1$. We then deduce that $C_1 = -C_2 = \frac{1}{\theta_1 + \theta_2}$. Therefore, we have

$$\begin{aligned} e^{Ax} \int_0^x e^{-Au} \beta(u) du &= -2\sigma^{-2} \int_0^x h(u) e^{A(x-u)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} du \\ &= -\frac{2}{\sigma^2(\theta_1 + \theta_2)} \int_0^x \begin{pmatrix} h(u)(e^{\theta_1(x-u)} - e^{-\theta_2(x-u)}) \\ h(u)(\theta_1 e^{\theta_1(x-u)} + \theta_2 e^{-\theta_2(x-u)}) \end{pmatrix} du, \end{aligned}$$

and also

$$\begin{aligned} g(x, (0,0)) &= -\frac{2}{\sigma^2(\theta_1 + \theta_2)} \int_0^x h(u)(e^{\theta_1(x-u)} - e^{-\theta_2(x-u)}) du + C(e^{\theta_1 x} - e^{-\theta_2 x}) \\ &= \phi_1(x) + C\phi_2(x), \end{aligned}$$

where $C$ is a parameter, and $\phi_1(x)$ and $\phi_2(x)$ satisfy (3.17) and (3.18) respectively. □

Back to the variational inequality (3.14), we plan to apply the smooth-fit principle to mandate the solution to be smooth at the free boundary point. The next technical result becomes an important step to prove the main theorem.

**Lemma 3.2.** *Under the conditions in Lemma 3.1, we have* $\zeta > 0$ *and there exist positive constants* $(C, m)$ *such that*

$$\begin{cases} \phi_1'(m) + C\phi_2'(m) = \gamma, \\ \phi_1''(m) + C\phi_2''(m) = 0. \end{cases}$$

*Proof.* Let us start with some identities of derivatives by direct calculations that

$$\phi_1'(x) = -\frac{2}{\sigma^2(\theta_1+\theta_2)} \int_0^x h(u)(\theta_1 e^{\theta_1(x-u)} + \theta_2 e^{-\theta_2(x-u)})du \le 0, \tag{3.19}$$

$$\phi_1''(x) = -\frac{2}{\sigma^2(\theta_1+\theta_2)} \int_0^x h'(u)(\theta_1 e^{\theta_1(x-u)} + \theta_2 e^{-\theta_2(x-u)})du \le 0, \tag{3.20}$$

where the second inequality holds thanks to $h(0) = 0$, and

$$\begin{aligned} \phi_1''(x) &= -\frac{2}{\sigma^2(\theta_1+\theta_2)} h(x)\phi_2'(0) - \frac{2}{\sigma^2(\theta_1+\theta_2)} \int_0^x h(u)\phi_2''(x-u)du \\ &= -\frac{2}{\sigma^2(\theta_1+\theta_2)} \left(h(x)\phi_2'(0) - h(0)\phi_2'(x)\right) - \frac{2}{\sigma^2(\theta_1+\theta_2)} \int_0^x h(u)\phi_2''(x-u)du \\ &= \frac{2}{\sigma^2(\theta_1+\theta_2)} \int_0^x h(u)\phi_2''(x-u)du - \frac{2}{\sigma^2(\theta_1+\theta_2)} \int_0^x h'(u)\phi_2'(x-u)du \\ &\quad - \frac{2}{\sigma^2(\theta_1+\theta_2)} \int_0^x h(u)\phi_2''(x-u)du \\ &= -\frac{2}{\sigma^2(\theta_1+\theta_2)} \int_0^x h'(u)\phi_2'(x-u)du. \end{aligned}$$

Note that $\phi_2''(0) = \theta_1^2 - \theta_2^2 < 0$. As $\phi_2'(x) > 0$, the existence of $m \in (0, +\infty)$ boils down to the existence of root $x \in (0, +\infty)$, to the following equation

$$q(x) := \phi_1''(x) + \frac{\gamma - \phi_1'(x)}{\phi_2'(x)}\phi_2''(x) = 0.$$

As $\phi_1'(0) = \phi_1''(0) = 0$ by (3.19) and (3.20), we obtain that $q(0) = \frac{\gamma\phi_2''(0)}{\phi_2'(0)} < 0$.

Plugging (3.19) and (3.20) into the definition of $q$ above, we obtain that

$$q(x) = \gamma\frac{\phi_2''(x)}{\phi_2'(x)} + \frac{2}{\sigma^2(\theta_1+\theta_2)} \int_0^x \left[\frac{\phi_2''(x)}{\phi_2'(x)}h(u) - h'(u)\right](\theta_1 e^{\theta_1(x-u)} + \theta_2 e^{-\theta_2(x-u)})du. \tag{3.21}$$

As $h'' \leq 0$, $h' > 0$, it follows that $h'$ is bounded. Noting that $\lim_{x \to +\infty} \frac{\phi_2''(x)}{\phi_2'(x)} = \theta_1 > 0$, as well as that $\lim_{u \to +\infty} h(u) = +\infty$, we deduce from (3.21) that $\lim_{x \to +\infty} q(x) = +\infty$. Therefore $q$ admits at least one root $x \in (0, +\infty)$. We then define

$$m := \inf \{u : q(u) = 0\} \in (0, +\infty), \tag{3.22}$$

and choose

$$C := \frac{\gamma - \phi_1'(m)}{\phi_2'(m)} \geq \frac{\gamma}{\phi_2'(m)} > 0. \tag{3.23}$$

□

With the parameters $(C, m)$ obtained in (3.23) and (3.22) in the proof of Lemma 3.2, we can turn to the construction of a classical solution to the general variational inequality.

**Proposition 3.1.** *The variational inequality*

$$\max \{\mathcal{A}f(x) + h(x), \gamma - f'(x)\} = 0, \quad x \geq 0, \tag{3.24}$$

*with the boundary condition* $f(0) = 0$ *admits a* $C^2$ *solution, which has the form of*

$$f(x) = \begin{cases} \phi_1(x) + C\phi_2(x), & x \in [0, m], \\ \phi_1(m) + C\phi_2(m) + \gamma(x - m), & x \in [m, +\infty). \end{cases} \tag{3.25}$$

*Here* $\phi_1(x)$ *and* $\phi_2(x)$, $x \geq 0$, *are defined in* (3.17) *and* (3.18) *respectively and parameters* $C$ *and* $m$ *are determined in* (3.23) *and* (3.22).

*In particular, we have*

$$\begin{cases} \mathcal{A}f(x) + h(x) = 0, & x \in [0, m], \\ \gamma - f'(x) = 0, & x \in [m, +\infty), \end{cases} \tag{3.26}$$

*and* $f(0) = 0$, $f' > 0$, $f'' \leq 0$, $\lim_{x \to +\infty} f(x) = +\infty$.

*Proof of Proposition 3.1.* Let $g(x)$ be the classical solution to the ODE (3.16). We have that $f(x)$ coincides with $g(x)$ in Lemma 3.1, for $x \leq m$ and the function is a linear function, for $x > m$. We aim to prove that the function $f$ is the desired $C^2$ solution to the variational inequality (3.24). Thanks to Lemma 3.2, we deduce that $f'(m) = \gamma$, $f''(m) = 0$. In view of its definition, it is straightforward to see that $f$ belongs to $C^2$. On the other hand, Lemma 3.1 and (3.25) give the validity of (3.26). Therefore (3.24) holds once we show that

$$f'(x) = \phi_1'(x) + C\phi_2'(x) \geq \gamma, \text{ for } x \in [0, m],$$

as well as

$$\mathcal{A}f(x) + h(x) \leq 0, \text{ for } x \geq m.$$

Define the elliptic operator

$$Lf := -\frac{1}{2}\sigma^2 f'' - \nu f' + \mu f,$$

and consider $g(x)$ in Lemma 3.1 with $C$ in (3.23). Then we have

$$Lg(x) = h(x), \quad x \in (0, m).$$

Note that $h$ is twice differentiable, and that $h'' \leq 0$. It therefore follows that

$$Lg''(x) = h''(x) \leq 0, \quad x \in (0, m).$$

Since $\mu > 0$, according to the weak maximum principle (see Theorem 2 in §6.4 of [39]), we have

$$\max_{x \in [0,m]} g''(x) \leq \max\left\{ \left[g''(0)\right]^+, \left[g''(m)\right]^+ \right\} = 0.$$

Therefore, we have

$$\phi_1'(x) + C\phi_2'(x) \geq \phi_1'(m) + C\phi_2'(m) = \gamma, \text{ for } x \in [0, m].$$

In other words,

$$\phi_1''(x) + C\phi_2''(x) \leq 0, \quad x \in [0, m]. \tag{3.27}$$

We next show that $\mathcal{A}f'(x)+h'(x)\leq 0$, for $x\geq m$. In our previous argument, we have shown that $\phi_1''(x)+C\phi_2''(x)\leq 0$, $x\in[0,m]$, i.e., $f''(x)\leq 0$, $x\in[0,m]$. It follows that

$$f'''(m-)=\lim_{x\to m-}\frac{f''(m)-f''(x)}{m-x}=-\lim_{x\to m-}\frac{f''(x)}{m-x}\geq 0. \tag{3.28}$$

Thanks to the definition of $f$, we have that $\mathcal{A}f'(x)+h'(x)=0$ on $x\in[0,m)$. By sending $x\to m-$, we get

$$\mathcal{A}f'(m-)+h'(m)=0.$$

That is,

$$-\mu\gamma+h'(m)=-\frac{1}{2}\sigma^2 f'''(m-)\leq 0.$$

For $x>m$, we have $f''(x)=0$, $f'(x)=\gamma$, and $h'(x)\leq h'(m)$ as $h''\leq 0$. Hence, we have

$$\mathcal{A}f'(x)+h'(x)=-\mu f'(x)+h'(x)\leq -\mu\gamma+h'(m)\leq 0.$$

Then for $x\geq m$, we arrive at

$$\mathcal{A}f(x)+h(x)\leq \mathcal{A}f(m)+h(m)=0.$$

Putting all the pieces together, we can conclude that $f$ is the desired $C^2$ solution to the variational inequality (3.24).

To complete the proof, it remains to show that

$$f(0)=0,\ \ f'(x)>0,\ \ f''(x)\leq 0,\ \ x\geq 0.$$

In view of (3.17), (3.18) and (3.25), it holds that $f(0)=0$. Note that the variational inequality (3.24) gives $f'(x)>0$, $x\geq 0$. Moreover, in view of (3.27) and the fact that $f(x)$ is linear on $x\in[m,+\infty)$, we obtain that $f''(x)\leq 0$, $x\geq 0$, $\lim_{x\to+\infty}f(x)=+\infty$. $\square$

### 3.3.3 *Main Results for Two Subsidiaries*

In view of the explicit solution of the auxiliary variational inequality (3.24), for $i = 1, 2$, we can derive the explicit solution $f_i(x_i, (0,0))$ to the variational inequality (3.12) by setting $\mathcal{A} = \mathcal{A}_i$, $h(x_i) = \frac{\lambda_1(0,0)\lambda_2(0,0)}{\lambda_i(0,0)} f_i(x_i, \mathbf{z}_i)$ and $\gamma = \alpha_i$.

Moreover, for $i = 1, 2$, let us denote the constant $m$ and $C$ for variational inequality (3.12) by $m_i(0,0)$ and $C_i(0,0)$, because we can verify later that the constant $m_i(0,0)$ is the optimal barrier of the dividend strategy for the subsidiary $i$.

Let us define

$$K_i := \alpha_i C_i(\mathbf{z}_i)(e^{\hat{\theta}_{i1} m_i(\mathbf{z}_i)} - e^{-\hat{\theta}_{i2} m_i(\mathbf{z}_i)}) - \alpha_i m_i(\mathbf{z}_i), \quad i = 1, 2,$$

and we will construct the explicit solution of the variational inequality (3.12) in the following steps.

For $i = 1, 2$, let us denote $\theta_{i1}$, $-\theta_{i2}$ as the positive and negative roots of the equation $\frac{1}{2} b_i^2 \theta^2 + a_i \theta - (r + \lambda_1(0,0) + \lambda_2(0,0)) = 0$ respectively that

$$\theta_{i1} := \frac{-a_i + \sqrt{a_i^2 + 2b_i^2(r + \lambda_1(0,0) + \lambda_2(0,0))}}{b_i^2},$$

$$-\theta_{i2} := \frac{-a_i - \sqrt{a_i^2 + 2b_i^2(r + \lambda_1(0,0) + \lambda_2(0,0))}}{b_i^2}.$$

Let us first define for $i = 1, 2$ and the variable $x \geq 0$ that

$$f_{i1}(x,(0,0)) := \begin{cases} f_{i11}(x) := -\frac{2}{\sigma^2}\frac{\alpha_i\lambda_1(0,0)\lambda_2(0,0)C_i(\mathbf{z}_i)}{\lambda_i(0,0)(\theta_{i1}+\theta_{i2})} \\ \times\left[\frac{(\theta_{i1}+\theta_{i2})e^{\hat{\theta}_{i1}x}}{(\hat{\theta}_{i1}-\theta_{i1})(\hat{\theta}_{i1}+\theta_{i2})} + \frac{(\theta_{i1}+\theta_{i2})e^{-\hat{\theta}_{i2}x}}{(\hat{\theta}_{i2}+\theta_{i1})(-\hat{\theta}_{i2}+\theta_{i2})}\right. \\ \left. - \frac{(\hat{\theta}_{i1}+\hat{\theta}_{i2})e^{\theta_{i1}x}}{(\hat{\theta}_{i1}-\theta_{i1})(\hat{\theta}_{i2}+\theta_{i1})} - \frac{(\hat{\theta}_{i1}+\hat{\theta}_{i2})e^{-\theta_{i2}x}}{(\hat{\theta}_{i1}+\theta_{i2})(-\hat{\theta}_{i2}+\theta_{i2})}\right], \\ 0 \leq x \leq m_i(\mathbf{z}_i), \\ f_{i12}(x) := -\frac{2}{\sigma^2}\frac{\alpha_i\lambda_1(0,0)\lambda_2(0,0)C_i(\mathbf{z}_i)}{\lambda_i(0,0)(\theta_{i1}+\theta_{i2})} \\ \times\left[\frac{e^{\theta_{i1}x}}{\hat{\theta}_{i1}-\theta_{i1}}\Big(e^{(\hat{\theta}_{i1}-\theta_{i1})m_i(\mathbf{z}_i)}-1\Big) + \frac{e^{-\theta_{i2}x}}{\hat{\theta}_{i1}+\theta_{i2}}\Big(-e^{(\hat{\theta}_{i1}+\theta_{i2})m_i(\mathbf{z}_i)}+1\Big)\right. \\ + \frac{e^{\theta_{i1}x}}{\hat{\theta}_{i2}+\theta_{i1}}\Big(e^{-(\hat{\theta}_{i2}+\theta_{i1})m_i(\mathbf{z}_i)}-1\Big) \\ \left. + \frac{e^{-\theta_{i2}x}}{-\hat{\theta}_{i2}+\theta_{i2}}\Big(e^{(-\hat{\theta}_{i2}+\theta_{i2})m_i(\mathbf{z}_i)}-1\Big)\right] \\ - \frac{2}{\sigma^2}\frac{K_i\lambda_1(0,0)\lambda_2(0,0)}{\lambda_i(0,0)(\theta_{i1}+\theta_{i2})} \\ \times\left[\frac{1}{\theta_{i1}}\Big(e^{\theta_{i1}x-\theta_{i1}m_i(\mathbf{z}_i)}-1\Big) + \frac{1}{\theta_{i2}}\Big(e^{-\theta_{i2}x+\theta_{i2}m_i(\mathbf{z}_i)}-1\Big)\right] \\ - \frac{2}{\sigma^2}\frac{\alpha_i\lambda_1(0,0)\lambda_2(0,0)}{\lambda_i(0,0)(\theta_{i1}+\theta_{i2})} \\ \times\left[\frac{1}{(\theta_{i1})^2}\Big(-\theta_{i1}x-1+(\theta_{i1}m_i(\mathbf{z}_i)+1)e^{\theta_{i1}x-\theta_{i1}m_i(\mathbf{z}_i)}\Big)\right. \\ \left. + \frac{1}{(\theta_{i2})^2}\Big(-\theta_{i2}x+1+(\theta_{i2}m_i(\mathbf{z}_i)-1)e^{-\theta_{i2}x+\theta_{i2}m_i(\mathbf{z}_i)}\Big)\right], \\ m_i(\mathbf{z}_i) \leq x, \end{cases} \tag{3.29}$$

$$f_{i2}(x,(0,0)) = e^{\theta_{i1}x} - e^{-\theta_{i2}x}, x \geq 0. \tag{3.30}$$

In view of Lemma 3.2 and Proposition 3.1, we can define the constant

$$m_i(0,0) := \inf\{s : q_i(s) = 0\}, \quad i = 1, 2,$$

where

$$q_i(x) := f''_{i1}(x,(0,0)) + \frac{\alpha_i - f'_{i1}(x,(0,0))}{f'_{i2}(x,(0,0))} f''_{i2}(x,(0,0)), \quad i = 1, 2.$$

We also define $C_i(0,0) := \frac{\alpha_i - f'_{i1}(m_i(0,0))}{f'_{i2}(m_i(0,0))}$, $i = 1, 2$.

To illustrate the change of the optimal barrier when one subsidiary defaults, let us choose the model parameters: $a_1 = 0.1$, $b_1 = 0.07$, $a_2 = 0.15$, $b_2 = 0.06$, $\lambda_1(0,0) = 0.02$, $\lambda_1(0,1) = 0.04$, $\lambda_2(0,0) = 0.01$, $\lambda_2(1,0) = 0.04$, $r = 0.05$ and $\alpha_1 = 0.4$. We can see from Figure $1$ that the comparison results $m_1(0,0) > m_1(0,1)$ and $m_2(0,0) > m_2(1,0)$ hold. That is, both subsidiaries decrease the optimal barriers for dividend payment after the other subsidiary defaults. These observations are consistent with our intuition that the default contagion effect forces the surviving subsidiary to take into account that itself will go default very soon because of the increased default intensity. Therefore the surviving one prefers to pay dividend as soon as possible by setting a lower dividend threshold before the unexpected default happens.

We actually have the next theoretical result on the change of the optimal barrier when one subsidiary defaults.

**Corollary 3.1.** *For the case of two subsidiaries, as we have* $\lambda_1(0,1) \geq \lambda_1(0,0)$ *and* $\lambda_2(1,0) \geq \lambda_2(0,0)$*, we always have the orders that* $m_1(0,0) \geq m_1(0,1)$ *and* $m_2(0,0) \geq m_2(1,0)$.

*Proof.* It suffices to show that $m_1(0,0) \geq m_1(0,1)$. We first show that $f_1(x,(0,0)) \geq f_1(x,(0,1))$, $x \geq 0$. Define $f_\delta(x) := e^{-\delta x} f_1(x,(0,0))$, $\hat{f}_\delta(x) := e^{-\delta x} f_1(x,(0,1))$. Here,

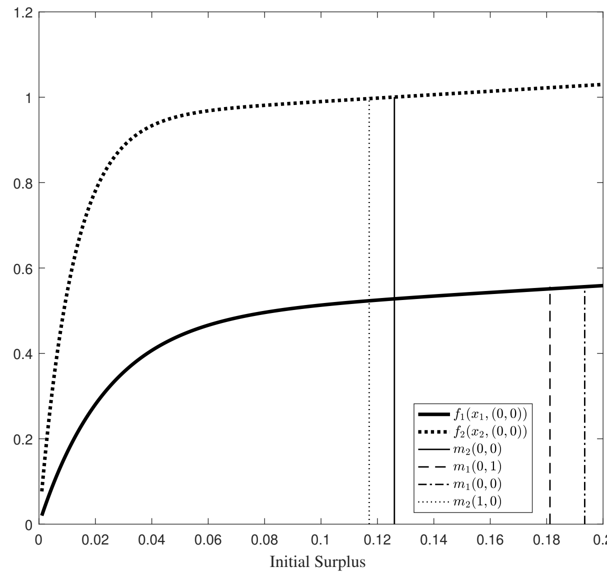


Figure 3.1: The change of the optimal barrier when default occurs

we choose the constant $\delta > 0$ small enough such that $r+\lambda_1(0,0)+\lambda_2(0,0)-\delta a_1-\frac{1}{2}\delta^2 b_1^2 > 0$. We can verify by direct calculation that $f_\delta(x)$ satisfies

$$\max\left\{\mathcal{A}_1^\delta f_\delta(x) + \lambda_2(0,0) f_1(x,(0,1)), \alpha_1 - \left(e^{\delta x} f_\delta(x)\right)'\right\} = 0, \quad x \geq 0,$$

with $f_\delta(0) = 0$ and the operator $\mathcal{A}_1^\delta$ defined by

$$\mathcal{A}_1^\delta f := \frac{1}{2} b_1^2 \big(e^{\delta x} f(x)\big)'' + a_1 \big(e^{\delta x} f(x)\big)' - \big(r + \lambda_1(0,0) + \lambda_2(0,0)\big) e^{\delta x} f(x).$$

On the other hand, we have that

$$\max\left\{\hat{\mathcal{A}}_1^\delta \hat{f}_\delta(x) + \lambda_2(0,0) f_1(x,(0,1)), \alpha_1 - \left(e^{\delta x} \hat{f}_\delta(x)\right)'\right\} = 0, \quad x \geq 0,$$

with $\hat{f}_\delta(0) = 0$ and the operator $\hat{\mathcal{A}}_1^\delta$ defined by

$$\hat{\mathcal{A}}_1^\delta f := \frac{1}{2} b_1^2 \big(e^{\delta x} f(x)\big)'' + a_1 \big(e^{\delta x} f(x)\big)' - \big(r + \hat{\lambda}_1(0,0) + \lambda_2(0,0)\big) e^{\delta x} f(x),$$

and $\hat{\lambda}_1(0,0) := \lambda_1(0,1)$. Noting that $\hat{\lambda}_1(0,0) \geq \lambda_1(0,0)$ and $\hat{f}_\delta \geq 0$, we thus have that

$$\max\left\{\mathcal{A}_1^\delta \hat{f}_\delta(x) + \lambda_2(0,0) f_1(x,(0,1)), \alpha_1 - \left(e^{\delta x} \hat{f}_\delta(x)\right)'\right\} \geq 0, \quad x \geq 0.$$

The comparison result of viscosity solutions (see e.g. Section 5B in [30]) yields that, for each $M > 0$,

$$\hat{f}_\delta(x) - f_\delta(x) \leq \max\left\{0, \hat{f}_\delta(M) - f_\delta(M)\right\}, \quad x \in [0, M].$$

Note that $M > 0$ is arbitrary and $\lim_{M\to+\infty} |\hat{f}_\delta(M) - f_\delta(M)| = 0$. Letting $M \to +\infty$ in the inequality above, we obtain that

$$f_\delta(x) - \hat{f}_\delta(x) \geq 0, \quad x \geq 0.$$

This gives that $f_1(x, (0, 0)) \geq f_1(x, (0, 1))$, $x \geq 0$.

Next, let us define $g(x_1) := f_1'(x_1, (0, 0))$. We claim that $g$ is the viscosity solution of

$$\max\{\mathcal{A}_1 g(x_1) + \lambda_2(0, 0) f_1'(x_1, (0, 1)), \alpha_1 - g(x_1)\} = 0, \tag{3.31}$$

with $g(0) = f_1'(0, (0, 0))$ and $g(M) = \alpha_1$, where the constant $M$ is sufficiently large that $M > m_1(0, 1) \vee m_1(0, 0)$. Indeed, on $(0, +\infty) \setminus \{m_1(0, 1)\}$, $g$ is $C^2$ and satisfies (3.31). On the other hand, similar to (3.28), we can derive that

$$\lim_{x\uparrow m_1(0,1)} g''(x) = \lim_{x\uparrow m_1(0,1)} f_1'''(x, (0, 1)) \geq 0,$$

as well as that $\lim_{x\downarrow m_1(0,1)} g''(x) = 0$. Hence

$$D^{+(2)} g(m_1(0, 1)) = \left\{(0, p) : p \geq \lim_{x\uparrow m_1(0,1)} f_1'''(x, (0, 1))\right\},$$
$$D^{-(2)} g(m_1(0, 1)) = \{(0, p) : p \leq 0\}.$$

Here, we denote $D^{+(2)}$ and $D^{-(2)}$ the second order Super-Jet and Sub-Jet respectively. For $(0, p) \in D^{+(2)} g(m_1(0, 1))$, we have that

$$\max\left\{\frac{1}{2} b_1^2 \cdot p + a_1 \cdot 0 - (r + \lambda_1(0, 0) + \lambda_2(0, 0)) g(m_1(0, 1)), \alpha_1 - g(m_1(0, 1))\right\} \geq 0,$$

while for $(0, p) \in D^{-(2)} g(m_1(0, 1))$, we have

$$\max\left\{\frac{1}{2} b_1^2 \cdot p + a_1 \cdot 0 - (r + \lambda_1(0, 0) + \lambda_2(0, 0)) g(m_1(0, 1)), \alpha_1 - g(m_1(0, 1))\right\} \leq 0.$$

Therefore $g$ is the viscosity solution of (3.31).

Let us define $\hat{g}(x) := f_1'(x,(0,1))$. Following the same arguments above, we have that $\hat{g}$ is the viscosity supersolution to (3.31), or equivalently, the viscosity solution to

$$\max\{\mathcal{A}_1\hat{g}(x_1) + \lambda_2(0,0)f_1'(x_1,(0,1)), \alpha_1 - \hat{g}(x_1)\} \geq 0,$$

with $\hat{g}(0) = f_1'(0,(0,1))$ and $\hat{g}(M) = \alpha_1$.

Because we have shown that

$$f_1(x,(0,0)) \geq f_1(x,(0,1)), \quad f_1(0,(0,0)) = f_1(0,(0,1)) = 0,$$

it follows that $f_1'(0,(0,0)) \geq f_1'(0,(0,1))$, i.e., $g(0) \geq \hat{g}(0)$. Moreover, $g(M) = \hat{g}(M) = \alpha_1$. The comparison result of viscosity solutions gives that $g(x) \geq \hat{g}(x)$, $x \in [0,M]$. That is, $f_1'(x,(0,0)) \geq f_1'(x,(0,1))$. We thus deduce that

$$\alpha_1 = f_1'(m_1(0,0),(0,0)) \geq f_1'(m_1(0,0),(0,1)) \geq \alpha_1,$$

which implies that $f_1'(m_1(0,0),(0,1)) = \alpha_1$. As $f_1'(x,(0,1)) > \alpha_1$, for $x \in (0, m_1(0,1))$, we can obtain the desired order that $m_1(0,1) \leq m_1(0,0)$. □

Based on solution forms in (3.29) and (3.30) and Corollary 3.1, we have $m_i(0,0) \geq m_i(\mathbf{z}_i)$, $i = 1,2$, and the solution of the auxiliary variational inequality (3.12) satisfies the piecewise form that

$$f_i(x_i,(0,0)) = \begin{cases} f_{i11}(x_i) + C_i(0,0)f_{i2}(x_i,(0,0)), & 0 \leq x_i < m_i(\mathbf{z}_i), \\ f_{i12}(x_i) + C_i(0,0)f_{i2}(x_i,(0,0)), & m_i(\mathbf{z}_i) \leq x_i \leq m_i(0,0), \\ f_{i12}(m_i(0,0)) + C_i(0,0)f_{i2}(m_i(0,0),(0,0)) & \\ \qquad +\alpha_i(x_i - m_i(0,0)), & x_i > m_i(0,0). \end{cases} \tag{3.32}$$

We can continue to verify the important conjecture $f(\mathbf{x},(0,0)) = f_1(x_1,(0,0)) + f_2(x_2,(0,0))$ in (3.11) and prove the existence of a classical solution to HJBVI (3.9) in the next theorem.

**Theorem 3.2.** *There exists a $C^2$ solution to HJBVI* (3.9) *that admits the form*

$$f(\mathbf{x},(0,0)) := f_1(x_1,(0,0)) + f_2(x_2,(0,0)), \tag{3.33}$$

*where $f_i(x_i,(0,0))$ given in* (3.32) *is the $C^2$ solution to the auxiliary variational inequality* (3.12)*, $i = 1, 2$.*

*Proof.* Thanks to Proposition 3.1, the auxiliary variational inequality (3.12) admits $C^2$ solution, for $i = 1, 2$. Let $f_i$ be the solution to (3.12), $i = 1, 2$. By setting $f(\mathbf{x},(0,0)) := f_1(x_1,(0,0)) + f_2(x_2,(0,0))$ and plugging into (3.10), we have

$$\begin{aligned}
\mathcal{L}^{(0,0)} f(\mathbf{x},(0,0)) = & - r f_1(x_1,(0,0)) - r f_2(x_2,(0,0)) \\
& + \left( a_1 \partial_1 f_1(x_1,(0,0)) + \frac{1}{2} b_1^2 \partial_{11}^2 f_1(x_1,(0,0)) \right) \\
& - \big(\lambda_1(0,0) + \lambda_2(0,0)\big) f_1(x_1,(0,0)) + \lambda_2(0,0) f(x_1,(0,1)) \\
& + \left( a_2 \partial_2 f_2(x_2,(0,0)) + \frac{1}{2} b_2^2 \partial_{22}^2 f_2(x_2,(0,0)) \right) \\
& - \big(\lambda_1(0,0) + \lambda_2(0,0)\big) f_2(x_2,(0,0)) + \lambda_2(0,0) f(x_1,(0,1)).
\end{aligned}$$

It readily yields that

$$\begin{aligned}
\mathcal{L}^{(0,0)} f(\mathbf{x},(0,0)) &= \mathcal{A}_1 f_1(x_1,(0,0)) + \lambda_2(0,0) f_1(x_1,(0,1)) \\
& \quad + \mathcal{A}_2 f_2(x_2,(0,0)) + \lambda_1(0,0) f_2(x_2,(1,0)), \\
\alpha_1 - \partial_1 f(\mathbf{x},(0,0)) &= \alpha_1 - f_1'(x_1,(0,0)), \\
\alpha_2 - \partial_2 f(\mathbf{x},(0,0)) &= \alpha_2 - f_2'(x_2,(0,0)).
\end{aligned}$$

As $f_i$ solves the variational inequality (3.12), $i = 1, 2$, we have that

$$\max\left\{\mathcal{L}^{(0,0)} f(\mathbf{x},(0,0)), \alpha_1 - \partial_1 f(\mathbf{x},(0,0)), \alpha_2 - \partial_2 f(\mathbf{x},(0,0))\right\} \le 0.$$

Moreover, if $\mathcal{L}^{(0,0)} f(\mathbf{x},(0,0)) < 0$, we get that

$$\mathcal{A}_1 f_1(x_1,(0,0)) + \lambda_2(0,0) f(x_1,(0,1)) < 0,$$

or

$$\mathcal{A}_2 f_2(x_2, (0,0)) + \lambda_1(0,0) f(x_2, (1,0)) < 0.$$

Without loss of generality, we assume that $\mathcal{A}_1 f_1(x_1, (0,0)) + \lambda_2(0,0) f(x_1, (0,1)) < 0$. By (3.12), we have that $\alpha - \partial_1 f(\mathbf{x}, (0,0)) = \alpha - f_1'(x_1, (0,0)) = 0$, and hence

$$\max\left\{\mathcal{L}^{(0,0)} f(\mathbf{x}, (0,0)), \alpha - \partial_1 f(\mathbf{x}, (0,0)), 1 - \alpha - \partial_2 f(\mathbf{x}, (0,0))\right\} = 0.$$

This shows that $f(\mathbf{x}, (0,0))$ in (3.33) is the solution of the HJBVI (3.9). □

## 3.4 Analysis of HJBVIs: Multiple Subsidiaries

This section generalizes the previous results to the case with $N \geq 3$ subsidiaries by employing mathematical induction. To this end, let us start to focus on the case that there are $k \leq N$ subsidiaries defaulted at the initial time and show the existence of classical solution to the associated variational inequality. The final verification proof of the optimal reflection dividend strategy for $N$ initial subsidiaries is given in the next section.

For $0 \leq k \leq N$, let us consider the initial default state that $k$ subsidiaries have defaulted and denote $\mathbf{z} = 0^{j_1,\ldots,j_k}$ as the $N$ dimensional vector that $j_1$, ..., $j_k$ components are $1$ and all other components are $0$ if $k \geq 1$ and denote $\mathbf{z} = 0^{j_1,\ldots,j_k}$ as the N-dimensional zero vector **0** if $k = 0$. We also denote by $\{j_{k+1}, \ldots, j_N\} := \{1, 2, \ldots, N\} \setminus \{j_1, \ldots, j_k\}$. For example, if $(j_1, \ldots, j_k) = (1, 2, \ldots, k)$, then $(j_{k+1}, \cdots, j_N) = (k+1, \ldots, N)$.

Consider $\mathbf{z} = 0^{1,\ldots,k}$, $\mathbf{x} = (0, \ldots, 0, x_{k+1}, \ldots, x_N)$, and define the operator

$$\mathcal{L}^{\mathbf{z}} f(\mathbf{x}, \mathbf{z}) := -\left(r + \sum_{i=k+1}^{N} \lambda_i(\mathbf{z})\right) f(\mathbf{x}, \mathbf{z}) + \sum_{i=k+1}^{N} \left(a_i \partial_i f(\mathbf{x}, \mathbf{z}) + \frac{1}{2} b_i^2 \partial_{ii}^2 f(\mathbf{x}, \mathbf{z})\right) \tag{3.34}$$

$$+ \sum_{\substack{i,l=k+1 \\ i<l}}^{N} b_i b_l \rho_{il} \partial_{il}^2 f(\mathbf{x}, \mathbf{z}).$$

With the notation above, we introduce the recursive system of HJBVIs

$$\max_{k+1\le i\le N}\left\{\mathcal{L}^{\mathbf{z}}f(\mathbf{x},\mathbf{z})+\sum_{l=k+1}^{N}\lambda_l(\mathbf{z})f(\mathbf{x}^{(l)},\mathbf{z}^l),\alpha_i-\partial_i f(\mathbf{x},\mathbf{z})\right\}=0.$$

Similar to the previous section, we seek for the solution in the separation form

$$f(\mathbf{x},\mathbf{z})=\sum_{i=k+1}^{N}f_i(x_i,\mathbf{z}),$$

so that $x_{k+1}$, …, $x_N$ are decoupled, where we define, for any $x\ge 0$, that

$$f_i(x,\mathbf{z})=\begin{cases} f_{i,1}(x,\mathbf{z})+C_i(\mathbf{z})f_{i,2}(x,\mathbf{z}), & 0\le x\le m_i(\mathbf{z}),\\ f_{i,1}(m_i(\mathbf{z}),\mathbf{z})+C_i(\mathbf{z})f_{i,2}(m_i(\mathbf{z}),\mathbf{z})+\alpha_i(x-m_i(\mathbf{z})), & x\ge m_i(\mathbf{z}).\end{cases}$$

In particular, for $k+1\le i\le N$,

$$\begin{cases} \alpha_i-\partial_i f(\mathbf{x},\mathbf{z})=0, & \mathbf{x}\in U_i(\mathbf{z}),\\ \mathcal{L}^{\mathbf{z}}f(\mathbf{x},\mathbf{z})+\displaystyle\sum_{l=k+1}^{N}\lambda_l(\mathbf{z})f(\mathbf{x}^{(l)},\mathbf{z}^l)=0, & \mathbf{x}\in U(\mathbf{z}),\end{cases}$$

where we have introduced

$$U_i(\mathbf{z}):=\big\{x_i\ge m_i(\mathbf{z})\big\},\quad \text{and}\quad U(\mathbf{z}):=\bigcap_{i=k+1}^{N}U_i^c(\mathbf{z}). \tag{3.35}$$

For $\mathbf{z}=0^{j_1,\dots,j_k}$ and $\mathbf{x}=(x_1,\dots,x_N)$ with $x_{j_i}=0$, $1\le i\le k$, we can define $U_i(\mathbf{z})$, $U(\mathbf{z})$ and the operator $\mathcal{L}^{\mathbf{z}}$ in the same manner as (3.35) and (3.34), except that the notation $i$ and $l$ in the expression, satisfying $k+1\le i,l\le N$, is replaced with $j_i$ and $j_l$, satisfying $k+1\le i,l\le N$.

With the discussion and notations above, we now proceed to prove by induction that the following statement **(S$_n$)** holds, for $1\le n\le N$:

**($\mathbf{S}_n$)** For $N-n \le k \le N$ and $\mathbf{z} = 0^{j_1,\dots,j_k}$, there exists a solution $f$ to HJBVI

$$\max_{k+1\le i\le N}\left\{\mathcal{L}^{\mathbf{z}} f(\mathbf{x},\mathbf{z}) + \sum_{l=k+1}^{N} \lambda_{j_l}(\mathbf{z}) f(\mathbf{x}^{(j_l)}, \mathbf{z}^{j_l}), \alpha_{j_i} - \partial_{j_i} f(\mathbf{x},\mathbf{z})\right\} = 0, \quad (3.36)$$

where $f$ admits the form $f(\mathbf{x},\mathbf{z}) = \sum_{i=k+1}^{N} f_{j_i}(x_{j_i}, \mathbf{z})$, satisfying

$$f_{j_i}(x,\mathbf{z}) = \begin{cases} f_{j_i,1}(x,\mathbf{z}) + C_{j_i}(\mathbf{z}) f_{j_i,2}(x,\mathbf{z}), & 0 \le x \le m_{j_i}(\mathbf{z}), \\ f_{j_i,1}(m_{j_i}(\mathbf{z}),\mathbf{z}) + C_{j_i}(\mathbf{z}) f_{j_i,2}(m_{j_i}(\mathbf{z}),\mathbf{z}) & \\ + \alpha_{j_i}(x - m_{j_i}(\mathbf{z})), & x \ge m_{j_i}(\mathbf{z}). \end{cases} \quad (3.37)$$

In particular, for $k+1 \le i \le N$,

$$\begin{cases} \alpha_{j_i} - \partial_{j_i} f(\mathbf{x},\mathbf{z}) = 0, & \mathbf{x} \in U_i(\mathbf{z}), \\ \mathcal{L}^{\mathbf{z}} f(\mathbf{x},\mathbf{z}) + \sum_{l=k+1}^{N} \lambda_{j_l}(\mathbf{z}) f(\mathbf{x}^{(j_l)}, \mathbf{z}^{j_l}) = 0, & \mathbf{x} \in U(\mathbf{z}), \end{cases} \quad (3.38)$$

and $f_{j_i}(0,\mathbf{z}) = 0$, $f_{j_i} \ge 0$, $f'_{j_i} > 0$, $f''_{j_i} \le 0$, $\lim_{x\to+\infty} f_{j_i}(x,\mathbf{z}) = +\infty$.

The expressions of (3.8) and (3.25), Proposition 3.1 and Theorem 3.2 in the previous section imply that **($\mathbf{S}_n$)** holds when $n = 1, 2$.

Let $n$ be any fixed integer satisfying $1 \le n < N$. Assuming that statement **($\mathbf{S}_n$)** holds true, we continue to show by induction that statement **($\mathbf{S}_{n+1}$)** is also true. Due to symmetry, it suffices to show that HJBVI (3.36) admits a solution $f(\mathbf{x},\mathbf{z})$, for $\mathbf{z} = 0^{1,\dots,k}$ when $k = N-n-1$, as well as that $f(\mathbf{x},\mathbf{z})$ should admit the form specified in (3.37) and (3.38). In the case where $\mathbf{z} = 0^{1,\dots,k}$ and $k = N-n-1$, the previous HJBVI (3.36) turns out to be

$$\max_{N-n\le i\le N}\left\{\mathcal{L}^{\mathbf{z}} f(\mathbf{x},\mathbf{z}) + \sum_{l=N-n}^{N}\left(\sum_{j\ne l} \lambda_l(\mathbf{z}) f_j(x_j, \mathbf{z}^l)\right), \alpha_i - \partial_i f(\mathbf{x},\mathbf{z})\right\} = 0. \quad (3.39)$$

In the same fashion of the previous section with two subsidiaries, it is sufficient to study the auxiliary variational inequality, for $N-n \le i \le N$, with one dimensional variable $x \geq 0$ that

$$\max\left\{\mathcal{A}^{\mathbf{z},i} f_i(x,\mathbf{z}) + \left(\sum_{\substack{l=N-n \\ l\neq i}}^{N} \lambda_l(\mathbf{z}) f_i(x,\mathbf{z}^l)\right), \alpha_i - f_i'(x,\mathbf{z})\right\} = 0. \tag{3.40}$$

Here, we define the operator

$$\mathcal{A}^{\mathbf{z},i} f := -\left(r + \tilde{\lambda}(\mathbf{z})\right) f + a_i f' + \frac{1}{2} b_i^2 f'',$$

where $\tilde{\lambda}(\mathbf{z}) := \sum_{l=N-n}^{N} \lambda_l(\mathbf{z})$.

**Lemma 3.3.** *Suppose that statement* ($\mathbf{S}_n$) *is true, then the auxiliary variational inequality* (3.40) *with the boundary condition $f(\mathbf{0},\mathbf{z}) = 0$ admits a $C^2$ solution $f_i(x,\mathbf{z})$, $N-n \leq i \leq N$, where $\mathbf{z} = 0^{1,\dots,N-n-1}$, and*

$$f_i(x,\mathbf{z}) = \begin{cases} f_{i,1}(x,\mathbf{z}) + C_i(\mathbf{z}) f_{i,2}(x,\mathbf{z}), & 0 \leq x \leq m_i(\mathbf{z}), \\ f_{i,1}(m_i(\mathbf{z}),\mathbf{z}) + C_i(\mathbf{z}) f_{i,2}(m_i(\mathbf{z}),\mathbf{z}) + \alpha_i(x - m_i(\mathbf{z})), & x > m_i(\mathbf{z}). \end{cases} \tag{3.41}$$

*Moreover, for $x \geq 0$ and $N-n \leq i \leq N$, it holds that*

$$\begin{cases} \mathcal{A}^{\mathbf{z},i} f_i(x,\mathbf{z}) + \left(\sum_{\substack{l=N-n \\ l\neq i}}^{N} \lambda_l(\mathbf{z}) f_i(x,\mathbf{z}^l)\right) = 0, & x \in [0, m_i(\mathbf{z})], \\ \alpha_i - f_i'(x,\mathbf{z}) = 0, & x \in [m_i(\mathbf{z}), +\infty), \end{cases} \tag{3.42}$$

*as well as that $f_i(0,\mathbf{z}) = 0$, $f_i'(x,\mathbf{z}) > 0$, $f_i''(x,\mathbf{z}) \leq 0$, and* $\lim_{x\to+\infty} f_i(\mathbf{x},\mathbf{z}) = +\infty$.

*Proof.* Note that for any $N-n \leq l \leq N$, $\mathbf{z}^l = 0^{1,\dots,N-n-1,l}$. Our induction assumption ($\mathbf{S}_n$) gives the boundary condition $\sum_{l\neq i} \lambda_l(\mathbf{z}) f_i(0,\mathbf{z}^l) = 0$ as well as the results

$$\sum_{l\neq i} \lambda_l(\mathbf{z}) f_i(x,\mathbf{z}^l) \geq 0, \quad \left(\sum_{l\neq i} \lambda_l(\mathbf{z}) f_i(x,\mathbf{z}^l)\right)' > 0, \quad \left(\sum_{l\neq i} \lambda_l(\mathbf{z}) f_i(x,\mathbf{z}^l)\right)'' \leq 0,$$

for $N-n \leq i \leq N$. Therefore, for $N-n \leq i \leq N$, we can conclude the existence of $C^2$ solution $f_i(x, \mathbf{z})$ by using the same argument in the proof of Proposition 3.1 and obtain the existence of free boundary points $m_i(\mathbf{z})$ with $\mathbf{z} = 0^{1,\ldots,N-n-1}$ such that (3.42) holds. Moreover, we have $f_i(0, \mathbf{z}) = 0$, $f_i'(x, \mathbf{z}) > 0$, $f_i''(x, \mathbf{z}) \leq 0$, $x \geq 0$. In view of (3.41), we also have $\lim_{x \to +\infty} f_i(x, \mathbf{z}) = +\infty$. □

**Lemma 3.4.** *Suppose that statement* **($\mathbf{S}_n$)** *is true, then the variational inequality* (3.39) *admits a $C^2$ solution, which is in the separation form of*

$$f(\mathbf{x}, \mathbf{z}) = \sum_{i=N-n}^{N} f_i(x_i, \mathbf{z}), \tag{3.43}$$

*where each $f_i(x, \mathbf{z})$ defined in* (3.41) *is the solution to the auxiliary variational inequality* (3.40). *In particular, for $x \geq 0$, $f_i(x, \mathbf{z})$ satisfies* (3.42)*, $f_i(0, \mathbf{z}) = 0$, $f_i'(x, \mathbf{z}) > 0$, $f_i''(x, \mathbf{z}) \leq 0$, and $\lim_{x \to +\infty} f_i(\mathbf{x}, \mathbf{z}) = +\infty$. Therefore statement* **($\mathbf{S}_{n+1}$)** *is also true.*

*Proof.* It suffices to investigate the $C^2$ solution of the variational inequality (3.39). Let $f$ be the function defined in (3.43). It is then obvious that $f$ is $C^2$. In view of (3.40), we have

$$\begin{aligned}
&\mathcal{L}^{\mathbf{z}} f(\mathbf{x}, \mathbf{z}) + \sum_{i=N-n}^{N} \left( \sum_{l \neq i} \lambda_l(\mathbf{z}) f_i(x_i, \mathbf{z}^l) \right) \\
&= \sum_{i=N-n}^{N} \mathcal{A}^{\mathbf{z},i} f_i(x_i, \mathbf{z}) + \sum_{i=N-n}^{N} \left( \sum_{l \neq i} \lambda_l(\mathbf{z}) f_i(x_i, \mathbf{z}^l) \right) \\
&= \sum_{i=N-n}^{N} \left( \mathcal{A}^{\mathbf{z},i} f_i(x_i, \mathbf{z}) + \left( \sum_{l \neq i} \lambda_l(\mathbf{z}) f_i(x_i, \mathbf{z}^l) \right) \right) \leq 0.
\end{aligned}$$

Furthermore, $\alpha_i - \partial_i f(\mathbf{x}, \mathbf{z}) = \alpha_i - f_i'(x_i, \mathbf{z}) \leq 0$, $i = N-n, \ldots, N$. It follows that

$$\max_{N-n \leq i \leq N} \left\{ \mathcal{L}^{\mathbf{z}} f(\mathbf{x}, \mathbf{z}) + \sum_{i=N-n}^{N} \left( \sum_{l \neq i} \lambda_l(\mathbf{z}) f_i(x_i, \mathbf{z}^l) \right), \alpha_i - \partial_i f(\mathbf{x}, \mathbf{z}) \right\} \leq 0.$$

Now we claim that

$$\max_{N-n \leq i \leq N} \left\{ \mathcal{L}^{\mathbf{z}} f(\mathbf{x}, \mathbf{z}) + \sum_{i=N-n}^{N} \left( \sum_{l \neq i} \lambda_l(\mathbf{z}) f_i(x_i, \mathbf{z}^l) \right), \alpha_i - \partial_i f(\mathbf{x}, \mathbf{z}) \right\} = 0.$$

Fix $x_i \geq 0$, $N-n \leq i \leq N$ and $\mathbf{z} = 0^{1,\ldots,N-n-1}$. If

$$\mathcal{L}^{\mathbf{z}} f(\mathbf{x},\mathbf{z}) + \sum_{i=N-n}^{N} \left( \sum_{l \neq i} \lambda_l(\mathbf{z}) f_i(x_i, \mathbf{z}^l) \right) = 0,$$

then the equality trivially holds. If $\mathcal{L}^{\mathbf{z}} f(\mathbf{x},\mathbf{z}) + \sum_{i=N-n}^{N} \left( \sum_{l\neq i} \lambda_l(\mathbf{z}) f_i(x_i,\mathbf{z}^l) \right) < 0$ , it follows that $\mathcal{A}^{\mathbf{z},i} f_i(x_i,\mathbf{z}) + \left( \sum_{l\neq i} \lambda_l(\mathbf{z}) f_i(x_i,\mathbf{z}^l) \right) < 0$, for some $i$. As $f_i$ is chosen to solve (3.40), it holds that $\alpha_i - \partial_i f(\mathbf{x},\mathbf{z}) = \alpha_i - f_i'(x_i,\mathbf{z}) = 0$. Therefore, our claim holds that $f(\mathbf{x},\mathbf{z})$ is the $C^2$ solution to the variational inequality (3.39). Moreover, for $x \geq 0$, we have by Lemma 3.3 that $f_i(x,\mathbf{z})$ defined in (3.41) satisfies $f_i(0,\mathbf{z}) = 0$, $f_i'(x,\mathbf{z}) > 0$, $f_i''(x,\mathbf{z}) \leq 0$ and $\lim_{x\to+\infty} f_i(x,\mathbf{z}) = +\infty$. Meanwhile, (3.42) in Lemma 3.3 yields the desired property in (3.38).

Given the results above, we conclude that, for $\mathbf{z} = 0^{1,\ldots,N-n-1}$, HJBVI (3.36) has a solution $f(\mathbf{x},\mathbf{z})$, which admits the form in (3.37) and (3.38). This completes the proof of the statement (**S**$_{n+1}$). □

By mathematical induction, we can present the following main result.

**Theorem 3.3.** *Statement* (**S**$_N$) *is true. In particular, for* $0 \leq k \leq N$ *and* $\mathbf{z} = 0^{1,\ldots,k}$*, the recursive system of HJBVI* (3.36) *admits a* $C^2$ *solution in the separation form of*

$$f(\mathbf{x},\mathbf{z}) = \sum_{i=k+1}^{N} f_i(x_i,\mathbf{z}), \tag{3.44}$$

*where each* $f_i(x,\mathbf{z})$ *is defined in* (3.41)*, with* $n = N-1$*, i.e.,* $f_i(x,\mathbf{z})$ *is the solution to the auxiliary variational inequality* (3.40) *and satisfies* (3.42)*,* $k+1 \leq i \leq N$*.*

**Remark 3.3.** *It can be observed from* (3.40) *that each function* $f_i(x_i,\mathbf{z})$ *in the separation form* (3.43) *is actually independent of the correlation coefficient matrix* $\Sigma$*. Therefore, the solution* $f(\mathbf{x},\mathbf{z})$ *to the recursive system of HJBVI* (3.36)*, for* $0 \leq k \leq N$*, is also independent of the correlation coefficient matrix* $\Sigma = (\rho_{ij})_{N\times N}$*.*

## 3.5 Proof of Verification Theorem

In this section, we construct the optimal dividend strategy using the $C^2$ solution of the recursive system HJBVI (3.36) and complete the proof of the main theorem.

*Proof of Theorem 3.1.*

Thanks to Theorem 3.3, we can readily conclude that variational inequality (3.5) for the case $k = 0$ (i.e. $\mathbf{z} = \mathbf{0}$ and $N$ subsidiaries are alive) also admits the $C^2$ solution in the separation form (3.44). Moreover, as statement **($\mathbf{S}_N$)** holds, the existence of mapping $m_{j_i}(\mathbf{z}) : \{0,1\}^N \mapsto (0,+\infty)$ is also guaranteed, for any $\mathbf{z} = 0^{j_1,\dots,j_k}$, $1 \le i \le k$ as well as $\mathbf{z} = \mathbf{0}$.

Let $\tau$ be an arbitrary stopping time, and $\mathbf{D}(t) = (D_1(t),\dots,D_N(t))$ be an arbitrary admissible strategy. By using Itô's formula, we first get

$$
\begin{aligned}
&\sum_{i=1}^{N}\alpha_i\int_0^{\tau}e^{-rs}dD_i(s) + e^{-r\tau}f\left(\mathbf{X}(\tau),\mathbf{Z}(\tau)\right) - f(\mathbf{x},\mathbf{z})\\
&=\int_0^{\tau}e^{-rs}\left[\mathcal{L}^{\mathbf{Z}(s)}f(\mathbf{X}(s),\mathbf{Z}(s)) + \sum_{l=k+1}^{N}\lambda_l(\mathbf{Z}(s))f(\mathbf{X}^{(l)}(s),\mathbf{Z}^{l}(s))\right]ds\\
&\quad+\sum_{i=1}^{N}\int_0^{\tau}e^{-rs}\left[\alpha_i - \partial_i f(\mathbf{X}(s),\mathbf{Z}(s))\right]dD_i^c(s)\\
&\quad+\sum_{0<s\le\tau,\Delta Z(s)\neq 0}e^{-rs}\sum_{j=1}^{N}\Delta Z_j(s)\Big[f\left(\mathbf{X}^{(j)}(s-) - \Delta\mathbf{D}^{j}(s),\mathbf{Z}^{j}(s-)\right)\\
&\quad-f\left(\mathbf{X}^{(j)}(s-),\mathbf{Z}^{j}(s-)\right) + \sum_{\substack{i=1\\ i\neq j}}^{N}\alpha_i\Delta D_i(s)\Big]\\
&\quad+\sum_{0<s\le\tau,\Delta Z(s)=0}e^{-rs}\Big[f\left(\mathbf{X}(s)-\Delta\mathbf{D}(s),\mathbf{Z}(s-)\right) - f\left(\mathbf{X}(s-),\mathbf{Z}(s-)\right)\\
&\quad+\sum_{i=1}^{N}\alpha_i\Delta D_i(s)\Big] + \mathcal{M}_\tau
\end{aligned}
$$

$$=:I + II + III + IV + \mathcal{M}_\tau. \tag{3.45}$$

As $f$ solves (3.36), we have that $I, II, IV \leq 0$. Moreover, by noting that $f(\mathbf{x}, \mathbf{z}^j)$ also solves (3.36), we deduce that $III \leq 0$. Note that $\mathcal{M}_{t\wedge\tau}$ is a local martingale. There exists a sequence of stopping times $\{T_n\}_{n=1}^\infty$ satisfying $T_n \uparrow \infty$, and

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^N \alpha_i \int_0^\tau e^{-rs} dD_i(s)\right] \\
\leq &\lim_{n\to\infty} \mathbb{E}\left[\sum_{i=1}^N \alpha_i \int_0^{\tau\wedge T_n} e^{-rs} dD_i(s) + e^{-r(\tau\wedge T_n)} f(\mathbf{X}(\tau\wedge T_n), \mathbf{Z}(\tau\wedge T_n))\right] \\
\leq &f(\mathbf{x}, \mathbf{z}) + \lim_{n\to\infty} \mathbb{E}[\mathcal{M}_{\tau\wedge T_n}] = f(\mathbf{x}, \mathbf{z}).
\end{aligned} \tag{3.46}$$

In view that $\mathbf{D}(t)$ is arbitrary, we obtain by sending $\tau$ in (3.46) to $+\infty$ that

$$\sup_D J(\mathbf{x}, \mathbf{z}, \mathbf{D}) \leq f(\mathbf{x}, \mathbf{z}). \tag{3.47}$$

Let us continue to prove that "=" holds in (3.47). Consider the càdlàg strategy

$$\begin{aligned}
D_i^*(t) &:= \max\left\{0, \sup_{0\leq s\leq t}\left\{\widetilde{X}_i(s) - m_i\left(\mathbf{Z}(s)\right)\right\}\right\}, \\
X_i^*(t) &= \widetilde{X}_i(t) - D_i^*(t).
\end{aligned}$$

We set $A_i(t) := \mathbf{1}_{\left\{D_i^*(t) = \widetilde{X}_i(t) - m_i(\mathbf{Z}(t))\right\}}$. It follows that

$$\begin{aligned}
X_i^*(t) &= \widetilde{X}_i(t) - D_i^*(t) \leq m_i\left(\mathbf{Z}(t)\right), \\
dD_i^*(t) &= A_i(t) dD_i^*(t).
\end{aligned} \tag{3.48}$$

On $\left\{D_i^*(t) = \widetilde{X}_i(t) - m_i\left(\mathbf{Z}(t)\right)\right\}$, we have that

$$X_i^*(t) = \tilde{X}_i(t) - D_i^*(t) = m_i\left(\mathbf{Z}(t)\right),$$

and vise versa. It then follows that

$$dD_i^*(t) = A_i(t) dD_i^*(t) = \mathbf{1}_{\left\{X_i^*(t) = m_i(\mathbf{Z}(t))\right\}} dD_i^*(t).$$

Furthermore, we have on $\{X_i^*(t) = m_i(\mathbf{Z}(t))\}$ that

$$X_i^*(t-) = X_i^*(t) + \Delta D_i^*(t) \geq X_i^*(t) = m_i(\mathbf{Z}(t)). \tag{3.49}$$

In view of (3.48), (3.38), we have that

$$\mathcal{L}^{\mathbf{Z}(s)} f(\mathbf{X}^*(s), \mathbf{Z}(s)) + \sum_{l=k+1}^{N} \lambda_l(\mathbf{Z}(s)) f((\mathbf{X}^*)^{(l)}(s), \mathbf{Z}^l(s)) = 0. \tag{3.50}$$

Note that for $x_i \geq m_i(\mathbf{z})$, $\partial_i f(\mathbf{x}, \mathbf{z}) = f_i'(x_i, \mathbf{z}) = \alpha_i$. Hence, it holds that $\partial_i f(\mathbf{X}^*(s), \mathbf{Z}(s)) = \alpha_i$ on $\{X_i^*(t) = m_i(\mathbf{Z}(t))\}$, which then entails that

$$\begin{aligned}
&\sum_{i=1}^{N} \int_0^{\tau} e^{-rs} \left[\alpha_i - \partial_i f(\mathbf{X}^*(s), \mathbf{Z}(s))\right] (D_i^*)^c(s) \\
=&\sum_{i=1}^{N} \int_0^{\tau} e^{-rs} \left[\alpha_i - \partial_i f(\mathbf{X}^*(s), \mathbf{Z}(s))\right] \mathbf{1}_{\left\{X_i^*(t) = m_i(\mathbf{Z}(t))\right\}} d(D_i^*)^c(s) = 0.
\end{aligned} \tag{3.51}$$

By virtue of (3.49), we can see that whenever $\Delta D_i^*(s) \neq 0$, it holds that $X_i^*(s-) > X_i^*(s-) - \Delta D_i^*(s) = X_i^*(s) = m_i(\mathbf{Z}(s))$. By using the fact that $\partial_i f(\mathbf{x}, \mathbf{z}) = f_i'(x_i, \mathbf{z}) = \alpha_i$, for $x_i \geq m_i(\mathbf{z})$, again, we obtain that

$$\begin{aligned}
&\sum_{j=1}^{N} \Delta Z_j(s) \Big[ f\left((\mathbf{X}^*)^{(j)}(s-) - \Delta(\mathbf{D}^*)^{(j)}(s), \mathbf{Z}^j(s-)\right) \\
&\quad - f\left((\mathbf{X}^*)^{(j)}(s-), \mathbf{Z}^j(s-)\right) + \sum_{\substack{i=1 \\ i \neq j}}^{N} \alpha_i \Delta D_i^*(s) \Big] \\
=&\sum_{j=1}^{N} \Delta Z_j(s) \Big[ f\left((\mathbf{X}^*)^{(j)}(s-) - \Delta(\mathbf{D}^*)^{(j)}(s), \mathbf{Z}(s)\right) \\
&\quad - f\left((\mathbf{X}^*)^{(j)}(s-), \mathbf{Z}(s)\right) + \sum_{\substack{i=1 \\ i \neq j}}^{N} \alpha_i \Delta D_i^*(s) \Big] \\
=&0.
\end{aligned} \tag{3.52}$$

Similarly, we obtain the equality that

$$\begin{aligned}
&\sum_{0<s\leq\tau,\Delta Z(s)=0} e^{-rs}\Bigg[f\left(\mathbf{X}^*(s-)-\Delta\mathbf{D}^*(s),\mathbf{Z}(s-)\right)\\
&-f\left(\mathbf{X}^*(s-),\mathbf{Z}(s-)\right)+\sum_{i=1}^N \alpha_i\Delta D_i^*(s)\Bigg]\\
=&\sum_{0<s\leq\tau,\Delta Z(s)=0} e^{-rs}\Bigg[f\left(\mathbf{X}^*(s-)-\Delta\mathbf{D}^*(s),\mathbf{Z}(s)\right)\\
&-f\left(\mathbf{X}^*(s-),\mathbf{Z}(s)\right)+\sum_{i=1}^N \alpha_i\Delta D_i^*(s)\Bigg]\\
=&0.
\end{aligned} \tag{3.53}$$

Putting all the pieces together, we conclude from (3.45) and (3.50)-(3.53) that

$$\sum_{i=1}^N \alpha_i\int_0^\tau e^{-rs}dD_i^*(s)+e^{-r\tau}f\left(\mathbf{X}^*(\tau),\mathbf{Z}(\tau)\right)-f(\mathbf{x},\mathbf{z})=\mathcal{M}_\tau,\quad \tau\geq 0, \tag{3.54}$$

where $\mathcal{M}_\tau$ is a local martingale. Hence, there exists a sequence of stopping times $\{T_n\}_{n=1}^\infty$ satisfying $T_n\uparrow\infty$, and

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^N \alpha_i\int_0^{\tau\wedge T_n} e^{-rs}dD_i^*(s)+e^{-r(\tau\wedge T_n)}f\left(\mathbf{X}^*(\tau\wedge T_n),\mathbf{Z}(\tau\wedge T_n)\right)\right]-f(\mathbf{x},\mathbf{z})\\
=&\mathbb{E}\left[\mathcal{M}_{\tau\wedge T_n}\right]=0.
\end{aligned} \tag{3.55}$$

In view of (3.48), we have $0\leq X_i^*(\tau)\leq m_i\left(\mathbf{Z}(\tau)\right)$, $\tau\geq 0$, which entails that $X_i^*(\tau)$ is a bounded process. It follows that $f\left(\mathbf{X}^*(\tau),\mathbf{Z}(\tau)\right)$ is also bounded. Note that

$$\lim_{n\to\infty} e^{-r(\tau\wedge T_n)}f\left(\mathbf{X}^*(\tau\wedge T_n),\mathbf{Z}(\tau\wedge T_n)\right)=e^{-r\tau}f\left(\mathbf{X}^*(\tau),\mathbf{Z}(\tau)\right)\quad \text{a.s..}$$

By passing the limit in (3.55), we arrive at

$$\mathbb{E}\left[\sum_{i=1}^N \alpha_i\int_0^\tau e^{-rs}dD_i^*(s)+e^{-r\tau}f\left(\mathbf{X}^*(\tau),\mathbf{Z}(\tau)\right)\right]-f(\mathbf{x},\mathbf{z})=0. \tag{3.56}$$

Note that $\lim_{\tau\to+\infty} e^{-r\tau} f(\mathbf{X}^*(\tau), \mathbf{Z}(\tau)) = 0$ a.s.. Sending $\tau$ to $+\infty$ in (3.56) yields that

$$\mathbb{E}\left[\sum_{i=1}^{N} \alpha_i \int_0^{\tau_i} e^{-rs} dD_i^*(s)\right] - f(\mathbf{x}, \mathbf{z}) = 0, \tag{3.57}$$

which completes the proof. □

**Remark 3.4.** *Similar to the derivation of* (3.54)*, for* $i = 1, \ldots, N$*, if we extend the definition of* $f_i$ *in such a way that* $f_i(x_i, \mathbf{z}) = 0$ *whenever the* $i$*-th component of* $\mathbf{z}$ *is* $1$*, then, following the proof of Theorem 3.1 and using* (3.40)*, we can show*

$$\alpha_i \int_0^{\tau} e^{-rs} dD_i^*(s) + e^{-r\tau} f_i\left(X_i^*(\tau), \mathbf{Z}(\tau)\right) - f_i(x_i, \mathbf{z}) = \widetilde{\mathcal{M}}_\tau^{(i)}, \quad i = 1, \ldots, N,$$

*where* $\widetilde{\mathcal{M}}_\tau^{(i)}$ *are local martingales, for* $x_i \in [0, +\infty)$*,* $i = 1, \ldots, N$*, and* $\mathbf{z} = \mathbf{0}$*. In the same fashion to obtain* (3.57)*, one can also get*

$$\mathbb{E}\left[\alpha_i \int_0^{\tau_i} e^{-rs} dD_i^*(s)\right] - f_i(x_i, \mathbf{z}) = 0, \quad i = 1, \ldots, N.$$

*This equality implies a natural linear separation form of* $f(\mathbf{x}, \mathbf{z})$ *in* (3.43) *because we can see that*

$$f(\mathbf{x}, \mathbf{z}) = \mathbb{E}\left(\sum_{i=1}^{N} \alpha_i \int_0^{\tau_i} e^{-rt} dD_i^*(t)\right) = \sum_{i=1}^{N} \mathbb{E}\left[\alpha_i \int_0^{\tau_i} e^{-rt} dD_i^*(t)\right],$$

*and each* $f_i(x_i, \mathbf{z})$ *stands for the expected value that* $f_i(x_i, \mathbf{z}) = \mathbb{E}\left[\alpha_i \int_0^{\tau_i} e^{-rt} dD_i^*(t)\right]$ *given the optimal dividend policy* $D_i^*$ *for the subsidiary i. However, we also point out that* $D_i^*$ *is the i-th component of the optimal control* $\mathbf{D}^*$ *which solves the group dividend problem. One can not simply interpret that* $f_i(x_i, \mathbf{z})$ *is the value function or* $D_i^*$ *is the optimal control when we purely solve a dividend optimization problem for the single subsidiary i without taking account all other subsidiaries. The vector process* $\mathbf{D}^*$ *is the solution that is optimal for a whole group and it has a coupled nature because the variational inequality* (3.40) *or the solution form* (3.41) *for each* $f_i(x_i, \mathbf{z})$ *depends on the default intensities of all surviving subsidiaries and the value functions given that one more subsidiary has defaulted.*

# Chapter 4

# Conclusion

This thesis aims to investigate the applications of stochastic control in one optimal entry and investment-consumption problem and one optimal dividend problem. Employing dynamic programming arguments and PDE analysis, the value function of each problem is related to the HJB variational inequality. In this Chapter, we provide a summary of the main contributions stemming from this thesis and related future work.

## 4.1 Main Contributions

The first project on investment and consumption extends existing work by considering the optimal entry time and consumption with habit formation preference. This composite problem can be analyzed in two stages. In stage-1, the exterior problem is an optimal stopping problem under the complete market information filtration, where the investor needs to pay information costs and decides the initial time of the interior control problem. In stage-2, the interior problem is an optimal control problem under incomplete information filtration, where the investor selects a dynamic optimal investment and consumption strategy. Mathematically, with the help of the stochastic Perron's method and comparison principle of variational inequality, the value function of this composite problem is proved to be the unique viscosity solution to some HJB variational inequality. A numerical example is given to illustrate the free boundary curve.

The second project formulates and investigates an optimal dividend problem for a multi-line insurance group. Each subsidiary within the group runs a product line and all subsidiaries are exposed to some external contagious default risk. Using the backward recursive scheme and the smooth-fit principle, the associated recursive system of HJBVIs is studied and the value function of the expected total dividend is proved to be its classical solution that has a separation form. We verify that the optimal dividend fits the type of barrier control and the barrier for each surviving subsidiary is dynamically modulated by the default state. The numerical analysis for the change of the optimal barrier when one subsidiary defaults is also provided.

## 4.2 Future Work

As this thesis focuses on two different applications of optimal control problems in the background of mathematical finance, there remain some interesting open problems that deserve further studies.

For the first problem, information acquisition can be analyzed from a different perspective for future research. From the beginning time 0, it is assumed that the investor can only observe stock prices instead of underlying Brownian motions. Then, the investor is allowed to choose the unknown drift from a confidence interval estimated using Kalman-Bucy filtering to solve the path-dependent optimal investment and consumption control problem with partial observations. Information acquisition comes into play when the investor can update the confidence interval dynamically by using new updated data so that the confidence interval shrinks over time. This leads to a non-standard robust control problem when the uncertainty of the model can be improved dynamically. One can try to study the corresponding Hamilton-Jacobi-Bellman-Isaacs equation and prove that the value function is the unique viscosity solution of this equation.

For the second problem, some future research can be conducted along different directions. Firstly, one can consider the more general model of $\hat{X}_i$ with jumps such as the classical Cramér-Lundberg model or other jump-diffusion models. Secondly, we note that the real life default events from credit assets can hurt the surplus management but may not lead to domino bankruptcies of subsidiaries due to strict regulations of the whole insurance sector. It is more realistic to consider the problem when $Z_i(t)$ can take values in $[0, 1]$ so that the default event only leads to a large size downward jump of the surplus process and certain recovery rate can be incorporated. Moreover, the default intensity $\lambda_i\left(\mathbf{Z}(t), X_i(t)\right)$ of $Z_i(t)$ may also depend on the surplus level $X_i(t)$ of the $i$-th subsidiary to depict the situation that a larger surplus level guarantees a smaller default probability. The inclusion of these factors will complicate the analysis of HJBVIs significantly because the backward induction can not be applied in a simple way and it is an open problem whether the optimal dividend of each subsidiary is still of the barrier type. It will be interesting to study these model extensions by applying some distinctive PDE arguments. Another appealing future work is to accommodate the collaborating bail-out dividend (see [2], [46] and [45]) in the present setting with contagious default risk so that each subsidiary can perform capital injection to other subsidiaries within the group whenever their financial ruins or credit default events happen. Finally, we can consider other types of ruin in the future research such as the ruin time when the total surplus of the insurance group hits zero or the first ruin time among different subsidiaries.

# Appendix A

# Fully Explicit Solutions to The Auxiliary ODEs in Chapter 2

**Lemma A.1.** *For $k \leq t \leq s \leq T$, consider the following auxiliary ODEs for $a(t,s)$, $b(t,s)$, $l(t,s)$, $w(t,s)$ and $g(t,s)$:*

$$a_t = -\frac{2(1-p+p\rho^2)}{1-p}\sigma_\mu^2 a^2 + \left(2\lambda - \frac{2p\rho\sigma_\mu}{(1-p)\sigma_S}\right)a - \frac{p}{2(1-p)\sigma_S^2}, \tag{A.1}$$

$$b_t = -\frac{2(1-p+p\rho^2)}{1-p}\sigma_\mu^2 ab - 2\lambda\bar{\mu}a + \left(\lambda - \frac{p\rho\sigma_\mu}{(1-p)\sigma_S}\right)b, \tag{A.2}$$

$$l_t = -\sigma_\mu^2 a - \frac{(1-p+p\rho^2)\sigma_\mu^2}{2(1-p)}b^2 - \lambda\bar{\mu}b, \tag{A.3}$$

$$w_t = -2(1-\rho^2)\sigma_\mu^2 w^2 + 2\frac{\lambda\sigma_S + \rho\sigma_\mu}{\sigma_S}w + \frac{1}{2\sigma_S^2}, \tag{A.4}$$

$$g_t = \sigma_\mu^2(1-\rho^2)(w-a), \tag{A.5}$$

*with the terminal conditions $a(s,s) = b(s,s) = l(s,s) = w(s,s) = g(s,s) = 0$. If we adopt the convention $\frac{0}{0} = 0$, the solutions of ODEs (2.13), (2.14), (2.15) are given by:*

$$A(t,s) := \frac{a(t,s)}{(1-p)\left(1 - 2a(t,s)\hat{\Sigma}(t)\right)},$$

$$B(t,s) := \frac{b(t,s)}{(1-p)\left(1 - 2a(t,s)\hat{\Sigma}(t)\right)},$$

$$C(t,s) := \frac{1}{1-p}\Big[l(t,s) + \frac{\hat{\Sigma}(t)}{\Big(1-2a(t,s)\hat{\Sigma}(t)\Big)}b^2(t,s) - \frac{1-p}{2}\log\Big(1-2a(t,s)\hat{\Sigma}(t)\Big)$$

$$-\frac{p}{2}\log\Big(1-2w(t,s)\hat{\Sigma}(t)\Big) - pg(t,s)\Big].$$

By [54], we can solve auxiliary ODEs (A.1), (A.2), (A.3), (A.4) and (A.5) explicitly by

$$a(t,s) = \frac{p(1-e^{2\xi(t-s)})}{2(1-p)\sigma_S^2\Big[2\xi - (\xi+\gamma_2)(1-e^{2\xi(t-s)})\Big]},$$

$$b(t,s) = \frac{p\lambda\bar{\mu}(1-e^{\xi(t-s)})^2}{(1-p)\sigma_S^2\xi\Big[2\xi - (\xi+\gamma_2)(1-e^{2\xi(t-s)})\Big]},$$

$$l(t,s) = \frac{p}{2(1-p)\sigma_S^2}\left(\frac{\lambda^2\bar{\mu}^2}{\xi^2} - \frac{\sigma_\mu^2\gamma_2}{\gamma_2^2-\xi^2}\right)(s-t)$$

$$+ \frac{p\lambda^2\bar{\mu}^2\Big[(\xi+2\gamma_2)e^{2\xi(t-s)} - 4\gamma_2 e^{\xi(t-s)} + 2\gamma_2 - \xi\Big]}{2(1-p)\sigma_S^2\xi^3\left[2\xi - (\xi+\gamma_2)(1-e^{2\xi(t-s)})\right]}$$

$$+ \frac{p\sigma_\mu^2}{2(1-p)\sigma_S^2(\xi^2-\gamma_2^2)}\log\left|\frac{2\xi - (\xi+\gamma_2)(1-e^{2\xi(t-s)})}{2\xi e^{\xi(t-s)}}\right|,$$

$$w(t,s) = -\frac{1}{2\sigma_S}\frac{1-e^{2\xi_1(t-s)}}{(\sigma_S\xi_1 + \lambda\sigma_S + \rho\sigma_\mu) + (\sigma_S\xi_1 - \lambda\sigma_S - \rho\sigma_\mu)e^{2\xi_1(t-s)}},$$

$$g(t,s) = \frac{1}{2}\log\left(\frac{(\sigma_S\xi_1 + \lambda\sigma_S + \rho\sigma_\mu) + (\sigma_S\xi_1 - \lambda\sigma_S - \rho\sigma_\mu)e^{2\xi_1(t-s)}}{2\sigma_S\xi_1 e^{\xi_1(t-s)}}\right)$$

$$- \frac{(1-p)(1-\rho^2)}{2(1-p+p\rho^2)}\log\left(\frac{(\sigma_S\xi + \lambda\sigma_S - \frac{\rho\sigma_\mu p}{1-p}) + (\sigma_S\xi - \lambda\sigma_S + \frac{\rho\sigma_\mu p}{1-p})e^{2\xi(t-s)}}{2\sigma_S\xi e^{\xi(t-s)}}\right)$$

$$- \frac{\rho^2\lambda(s-t)}{2(1-p+p\rho^2)} - \frac{\rho\sigma_\mu(s-t)}{2(1-p+p\rho^2)\sigma_S},$$

where

$$\Delta := \lambda^2 - \frac{2\lambda p\rho\sigma_\mu}{(1-p)\sigma_S} - \frac{p\sigma_\mu^2}{(1-p)\sigma_S^2} > 0, \tag{A.6}$$

and

$$\xi := \sqrt{\Delta} = \sqrt{\gamma_2^2 - \gamma_1\gamma_3}, \;\; \xi_1 := \frac{\sqrt{(1-\rho^2)\sigma_\mu^2 + (\lambda\sigma_S + \rho\sigma_\mu)^2}}{\sigma_S},$$

$$\gamma_1 := \frac{(1-p+p\rho^2)}{1-p}\sigma_\mu^2, \;\; \gamma_2 := -\lambda + \frac{p\rho\sigma_\mu}{(1-p)\sigma_S}, \;\; \gamma_3 := \frac{p}{(1-p)\sigma_S^2}.$$

The condition for the bounded Normal solution is $\gamma_3 > 0$, or $\gamma_1 > 0$, or $\gamma_2 < 0$.

**Remark A.1.** *If $p < 0$, (A.6) clearly holds and we have $\gamma_2 < 0$, therefore $a(t,s) \leq 0$ is a bounded solution as well as $1 - 2a(t,s)\hat{\Sigma}(t) > 1$ and $1 - w(t,s)\hat{\Sigma}(t) > 1$. Hence solutions of ODEs (2.13), (2.14), (2.15) are bounded on $k \leq t \leq s \leq T$. We also note that $A(t,s) = \frac{a(t,s)}{(1-p)(1-2a(t,s)\hat{\Sigma}(t))} \leq 0$, on $k \leq t \leq s \leq T$.*

# Appendix B

# Derivation of (3.9) in Chapter 3

For the default process starting from $\mathbf{Z}(0) = \mathbf{z} = (0,0)$, we present here the argument to derive the associated HJBVI using Itô's lemma. For a given function $\psi(\cdot, \mathbf{z}) \in C^2(\mathbb{R}^2)$, let us rewrite

$$\alpha_1 \int_0^\tau e^{-rs} dD_1(s) + \alpha_2 \int_0^\tau e^{-rs} dD_2(s) + e^{-r\tau}\psi\left(\mathbf{X}(\tau), \mathbf{Z}(\tau)\right) - \psi(\mathbf{x}, \mathbf{z})$$

$$= \int_0^\tau e^{-rs} \tilde{\mathcal{L}}^{(0,0)} \psi(s) ds + \int_0^\tau e^{-rs} \left[\alpha_1 - \partial_1 \psi(s)\right] dD_1^c(s)$$

$$+ \int_0^\tau e^{-rs} \left[\alpha_2 - \partial_2 \psi(s)\right] dD_2^c(s)$$

$$+ \alpha_1 \int_0^\tau e^{-rs} dD_1(s) + \alpha_2 \int_0^\tau e^{-rs} dD_2(s)$$

$$+ \sum_{0<s\leq\tau} e^{-rs} \left[\psi\left(\mathbf{X}(s), \mathbf{Z}(s)\right) - \psi\left(\mathbf{X}(s-), \mathbf{Z}(s-)\right)\right] + \mathcal{M}_\tau$$

$$= \int_0^\tau e^{-rs} \tilde{\mathcal{L}}^{(0,0)} \psi(s) ds + \int_0^\tau e^{-rs} \left[\alpha_1 - \partial_1 \psi(s)\right] dD_1^c(s)$$

$$+ \int_0^\tau e^{-rs} \left[\alpha_2 - \partial_2 \psi(s)\right] dD_2^c(s)$$

$$+ \alpha_1 \int_0^\tau e^{-rs} dD_1(s) + \alpha_2 \int_0^\tau e^{-rs} dD_2(s)$$

$$+ \sum_{0<s\leq\tau, \Delta Z(s)\neq 0} e^{-rs} \left[\psi\left(\mathbf{X}(s), \mathbf{Z}(s)\right) - \psi\left(\mathbf{X}(s-), \mathbf{Z}(s-)\right)\right]$$

$$+\sum_{0<s\le\tau,\Delta Z(s)=0} e^{-rs}\left[\psi\left(\mathbf{X}(s)+\Delta D(s),\mathbf{Z}(s-)\right)-\psi\left(\mathbf{X}(s-),\mathbf{Z}(s-)\right)\right]+\mathcal{M}_\tau$$

$$=\int_0^\tau e^{-rs}\mathcal{L}^{(0,0)}\psi(s)ds+\int_0^\tau e^{-rs}\left[\alpha_1-\partial_1\psi(s)\right]dD_1^c(s)$$

$$+\int_0^\tau e^{-rs}\left[\alpha_2-\partial_2\psi(s)\right]dD_2^c(s)$$

$$+\sum_{0<s\le\tau,\Delta Z(s)\neq 0} e^{-rs}\Delta Z_1(s)$$

$$\times\left[\psi\left(0,X_2(s-)-\Delta D_2(s),(1,0)\right)-\psi\left(\mathbf{X}(s-),(0,0)\right)+\alpha_2\Delta D_2(s)\right]$$

$$+\sum_{0<s\le\tau,\Delta Z(s)\neq 0} e^{-rs}\Delta Z_2(s)$$

$$\times\left[\psi\left(X_1(s-)-\Delta D_1(s),0,(0,1)\right)-\psi\left(\mathbf{X}(s-),(0,0)\right)+\alpha_1\Delta D_1(s)\right]$$

$$+\sum_{0<s\le\tau,\Delta Z(s)=0} e^{-rs}\Big[\psi\left(\mathbf{X}(s)-\Delta D(s),\mathbf{Z}(s-)\right)-\psi\left(\mathbf{X}(s-),\mathbf{Z}(s-)\right)$$

$$+\alpha_1\Delta D_1(s)+\alpha_2\Delta D_2(s)\Big]+\mathcal{M}_\tau,$$

where $\mathcal{M}_\tau$ is a local martingale.

Let us turn to the jump terms. According to assumptions that no simultaneous jumps can occur in the sense of (3.2) and (3.3), it follows that

$$\Delta Z_1(s)\Delta D_1(s)=\Delta Z_2(s)\Delta D_2(s)=\Delta Z_1(s)\Delta Z_2(s)=0.$$

On $\{\Delta Z(s)\neq 0\}$, let us consider $\mathbf{Z}(s-)=(0,0)$. We have

$$\begin{aligned}&e^{-rs}\left[\psi\left(\mathbf{X}(s),\mathbf{Z}(s)\right)-\psi\left(\mathbf{X}(s-),\mathbf{Z}(s-)\right)\right]\\ =&e^{-rs}\Delta Z_1(s)\left[\psi\left((0,X_2(s-)-\Delta D_2(s)),(1,0)\right)-\psi\left(\mathbf{X}(s-),(0,0)\right)\right]\\ &+e^{-rs}\Delta Z_2(s)\left[\psi\left((X_1(s-)-\Delta D_1(s),0),(0,1)\right)-\psi\left(\mathbf{X}(s-),(0,0)\right)\right],\end{aligned}$$

as well as

$$\begin{aligned}&e^{-rs}\Delta Z_1(s)\left[\psi\left((0,X_2(s-)-\Delta D_2(s)),(1,0)\right)-\psi\left(\mathbf{X}(s-),(0,0)\right)\right]\\ =&e^{-rs}\Delta Z_1(s)\left[\psi\left((0,X_2(s-)-\Delta D_2(s)),(1,0)\right)-\psi\left(0,X_2(s-),(1,0)\right)\right]\end{aligned}$$

$$+ e^{-rs}\Delta Z_1(s)\left[\psi\left((0, X_2(s-)), (1,0)\right) - \psi\left(\mathbf{X}(s-), (0,0)\right)\right].$$

Similarly, one can get

$$\begin{aligned}
&e^{-rs}\Delta Z_2(s)\left[\psi\left((X_1(s-) - \Delta D_1(s), 0), (0,1)\right) - \psi\left(\mathbf{X}(s-), (0,0)\right)\right] \\
=&e^{-rs}\Delta Z_2(s)\left[\psi\left((X_1(s-) - \Delta D_1(s), 0), (0,1)\right) - \psi\left(X_1(s-), 0, (0,1)\right)\right] \\
&+ e^{-rs}\Delta Z_2(s)\left[\psi\left((X_1(s-), 0), (0,1)\right) - \psi\left(\mathbf{X}(s-), (0,0)\right)\right].
\end{aligned}$$

On $\{\Delta Z(s) = 0\}$, we have

$$\begin{aligned}
&e^{-rs}\left[\psi\left(\mathbf{X}(s), \mathbf{Z}(s)\right) - \psi\left(\mathbf{X}(s-), \mathbf{Z}(s-)\right)\right] \\
=&e^{-rs}\left[\psi\left(\mathbf{X}(s-) - \Delta D(s), \mathbf{Z}(s-)\right) - \psi\left(\mathbf{X}(s-), \mathbf{Z}(s-)\right)\right],
\end{aligned}$$

and also

$$\alpha_i \int_0^\tau e^{-rs} dD_i(s) = \sum_{0<s\le\tau, \Delta Z_2(s)\neq 0} \alpha_i e^{-rs}\Delta D_i(s) + \sum_{0<s\le\tau, \Delta Z_2(s)=0} \alpha_i e^{-rs}\Delta D_i(s).$$

Thanks to the martingale property in (3.1) and the fact that, for any $h \in C^1(\mathbb{R})$ and $y \in \mathbb{R}$,

$$h(y - \Delta D_i(s)) - h(y) = -\int_0^{\Delta D_i(s)} h'(y-u)du,$$

we obtain the desired HJBVI (3.9).

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