CALCULUS OF KURDYKA-ŁOJASIEWICZ EXPONENTS AND ITS APPLICATIONS IN THE ANALYSIS OF FIRST-ORDER METHODS

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Calculus of Kurdyka-Łojasiewicz exponents and its applications in the analysis of first-order methods

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____________________________(Signed)

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Abstract

In this thesis, we study calculus rules of the Kurdyka-Łojasiewicz (KL) exponents and show how KL exponents are applied in analyzing first-order methods for widely used optimization problems.

First, we focus on calculus rules that derive the KL exponents of new functions from functions with known KL exponents. These include deriving the KL exponent of the inf-projection of a function from that of its original function, the KL exponent of the sum of a continuous function and the indicator function defined by a set of constraints from that of its Lagrangian and the KL exponent of a fractional function from the difference between the numerator and (a suitable scaling of) the denominator. Using these rules, we derive explicit KL exponents of some concrete optimization models such as the fractional model in [115, 116], the model of minimizing ℓ_1 subject to logistic/Poisson loss, some semidefinite-programming-representable functions and some functions with C^2-cone reducible structures.

Second, we show how KL exponents are employed in analyzing an existing first-order method, the sequential convex programming method with monotone line search (SCP_{ls}) in [83] for difference-of-convex (DC) optimization problem with multiple smooth inequality constraints. By imposing suitable KL assumptions, we analyze the convergence rate of the sequence generated by SCP_{ls} in both nonconvex and convex settings. We also discuss how the various conditions required in our analysis can be verified for minimizing ℓ_{1-2} [123] subject to residual error measured by ℓ_2

norm/Lorentzian norm [36].

To further illustrate the applications of KL exponents, finally, we focus on the minimization of the quotient of ℓ_1 and ℓ_2 (denoted as ℓ_1/ℓ_2) subject to one possibly nonsmooth constraint [97]. We show that the sum of ℓ_1/ℓ_2 and the indicator function of an affine constraint set satisfies the KL property with exponent 1/2; this allows us to establish linear convergence of the algorithm proposed in [116, Eq. 11] under mild assumptions. We next extend the ℓ_1/ℓ_2 model to handle compressed sensing problems with noise. We establish the solution existence for some of these models under the spherical section property [114, 128], and extend the algorithm in [116, Eq. 11] for solving these problems. We prove the subsequential convergence of our algorithm under mild conditions, and establish global convergence of the whole sequence generated by our algorithm by imposing additional KL and differentiability assumptions on a specially constructed potential function. Finally, we perform numerical experiments on robust compressed sensing and basis pursuit denoising with residual error measured by ℓ_2 norm or Lorentzian norm via solving the corresponding ℓ_1/ℓ_2 models by our algorithm. Our numerical simulations show that our algorithm is able to recover the original sparse vectors with reasonable accuracy.

Publications Arising from the Thesis

1. P. Yu, G. Li and T. K. Pong.
   Kurdyka-Łojasiewicz exponent via inf-projection.
   To appear in *Found. Comput. Math.*
   Available at https://arxiv.org/abs/1902.03635.

2. P. Yu, T. K. Pong and Z. Lu.
   Convergence rate analysis of a sequential convex programming method with line search for a class of constrained difference-of-convex optimization problems
   To appear in *SIAM J. Optim.*
   Available at https://arxiv.org/abs/2001.06998.

3. L. Zeng, P. Yu and T. K. Pong.
   Analysis and algorithms for some compressed sensing models based on L1/L2 minimization.
   SIAM J. Optim. 31:1576–1603, 2021.
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Contents

List of Notations

$\mathbb{R}$	the set of real numbers
$\mathbb{R}_+$	the set of nonnegative real numbers
$\mathcal{N}_+$	the set of nonnegative integers including 0
$\mathcal{N}$	the set of natural numbers
$\mathbb{R}^n$	the n-dimensional Euclidean space with inner product $\langle \cdot, \cdot \rangle$
$\|x\|_0$	the number of nonzero entries of a vector x
$\|x\|_1$	the ℓ_1 norm of a vector x
$\|x\|$	the ℓ_2 norm of a vector x
$B(x, r)$	the closed ball centered at x with radius $r \geq 0$
$\mathbb{R}^{m \times n}$	the set of $m \times n$ real matrices
$\langle A, B \rangle$	$\mathrm{tr}(A^T B)$ with tr denotes the trace of a matrix
$\|X\|_F$	$\sqrt{X^T X}$
$\mathcal{S}^n$	The space of $n \times n$ symmetric matrices
$\mathcal{S}^n_+$	The cone of positive semidefinite matrices
$\mathcal{A}$	A linear map
$\mathcal{A}^*$	The adjoint of a linear map
$\arg\min f$	the minimizer of f that has a unique minimizer
$\mathrm{Arg}\min f$	the set of minimizers of f

Chapter 1

Introduction

Many problems in machine learning, signal processing and data analysis involve large-scale nonsmooth nonconvex optimization problems. These problems are typically solved using first-order methods, which are noted for their scalability and ease of implementation. Commonly used first-order methods include the proximal gradient method and its variants, and splitting methods such as Douglas-Rachford splitting method and its variants; see the recent expositions [30, 92] and references therein for more detail. In the general nonconvex nonsmooth setting, convergence properties of the sequences generated by these algorithms are typically analyzed by assuming a certain potential function to have the so-called Kurdyka-Łojasiewicz (KL) property. Moreover, when it comes to estimating *local convergence rate*, the so-called KL exponent plays a key role; see, for example, [6, Theorem 2], [54, Theorem 3.4] and [74, Theorem 3]. We now give a more detailed introduction about the KL property and KL exponent in Section 1.1. In Sections 1.2 and 1.3, we introduce the applications of KL properties in the analysis of algorithms for optimization models.

1.1 KL property and KL exponent

The KL property originates from the seminal Łojasiewicz inequality that bounds the function value deviation of a real-analytic function in terms of its gradient; see [80].

This inequality was extended to the case of C^1 subanalytic functions by Kurdyka in [66] using the notion of desingularizing function. An important breakthrough was made in [20, 21], where the Łojasiewicz inequality was further generalized to nonsmooth cases by using tools of modern variational analysis and semialgebraic geometry. This generalization significantly broadened the applicability of the aforementioned KL inequality to nonconvex settings, and it allowed us to perform convergence rate analysis for various important algorithms in nonsmooth optimization and subgradient dynamical systems.

The KL property[1] is satisfied by a large class of functions such as proper closed semi-algebraic functions; see, for example, [7]. It has been the main workhorse for establishing convergence of sequences generated by various first-order methods, especially in nonconvex settings [6–8, 24]. Moreover, when it comes to estimating *local convergence rate*, the so-called KL exponent plays a key role; see, for example, [6, Theorem 2], [54, Theorem 3.4] and [74, Theorem 3]. Roughly speaking, an exponent of $\alpha \in (0, \frac{1}{2}]$ of a suitable potential function corresponds to a linear convergence rate, while an exponent of $\alpha \in (\frac{1}{2}, 1)$ corresponds to a sublinear convergence rate, see for example [6, 54, 74]. However, as noted in [85, Page 63, Section 2.1], explicit estimation of KL exponent for a given function is difficult in general. Nevertheless, due to its significance in convergence rate analysis, KL exponent computation has become an important research topic in recent years and some positive results have been obtained. For instance, we now know the KL exponent of the maximum of finitely many polynomials [73, Theorem 3.3] and the KL exponent of a class of quadratic optimization problems with matrix variables satisfying orthogonality constraints [77]. In addition, it has been shown that the KL exponent is closely related to several existing and widely-studied error bound concepts such as the Hölder

[1] See Definition 2.1 for the precise definition.

growth condition and the first-order error bound mentioned in [22, 86, 110];[2] see for example, [22, Theorem 5], [48, Theorem 3.7], [48, Proposition 3.8], [49, Corollary 3.6] and [75, Theorem 4.1]. Taking advantage of these connections, we now also know that convex models that satisfy the second-order growth condition have KL exponent $\frac{1}{2}$, so do models that satisfy the first-order error bound condition together with a mild assumption on the separation of stationary values; see the recent work [43, 75, 129] for concrete examples. This sets the stage for developing calculus rules for KL exponent in [75] to deduce the KL exponent of a function from functions with known KL exponents. For example, it was shown in [75, Corollary 3.1] that under mild conditions, if f_i is a KL function with exponent $\alpha_i \in [0,1)$, $1 \le i \le m$, then the KL exponent of $\min_{1\le i\le m} f_i$ is given by $\max_{1\le i\le m} \alpha_i$. This was then used in [75, Section 5.2] for showing that the least squares loss with smoothly clipped absolute deviation (SCAD) [53] or minimax concave penalty (MCP) regularization [127] has KL exponent $\frac{1}{2}$.

In Chapter 3 of this thesis, we will further explore this line of research and study three types of calculus of KL exponent:

1. **Lagrangian of functions:**

   - **For equality constraints** We determine the KL exponent of $F + \delta_{G^{-1}\{0\}}$ from its Lagrangian relaxation, where F and G are continuously differentiable functions with ∇G being injective, $\delta_{G^{-1}\{0\}}$ is the indicator function of the set $G^{-1}\{0\} := \{x : G(x) = 0\}$ with $G^{-1}\{0\} \neq \emptyset$; see Theorem 3.1.
   - **For inequality constraints** We determine the KL exponent of $P_1(x) + \delta_{g(\cdot)\le 0}(x)$ from its Lagrangian (see Theorem 3.2) under suitable assumptions, where P_1 is convex continuous, the function $g(x)$ is of the form $(l_1(A_1x), \ldots, l_m(A_mx))$ with each $A_i \in \mathbb{R}^{q_i\times n}$ and $l_i : \mathbb{R}^{q_i} \to \mathbb{R}$ being

[2] This type of first-order error bound is sometimes called the Luo-Tseng error bound; see [75, 126].

strictly convex, and $\{x: \; g(x) \leq 0\} \neq \emptyset$. This enables us to deduce that the function F corresponding to minimizing ℓ_1 subject to logistic/Poisson loss is a KL function with exponent $\frac{1}{2}$ under mild conditions, see Remark 3.2.

2. **Fractional functions** We establish a calculus rule for deducing the KL exponent of a fractional objective from the difference between the numerator and (a suitable scaling of) the denominator, see Theorem 3.3. As we can see in Section 5.2, this can be used in deducing the explicit convergence rate of the sequence generated by the algorithm proposed in [116, Eq. 11].

3. **Inf-projection** This is a generalization of the operation of taking the minimum of finitely many functions. Precisely speaking, let $\mathbb{X}$ and $\mathbb{Y}$ be two finite dimensional Hilbert spaces and let $F : \mathbb{X} \times \mathbb{Y} \to \mathbb{R} \cup \{\infty\}$ be a proper closed function,[3] we call the function $f(x) := \inf_{y \in \mathbb{Y}} F(x, y)$ for $x \in \mathbb{X}$ an inf-projection of F. The name comes from the fact that the strict epigraph of f, defined as $\{(x, r) \in \mathbb{X} \times \mathbb{R} : \; f(x) < r\}$, is equal to the projection of the strict epigraph of F onto $\mathbb{X} \times \mathbb{R}$. Functions represented in terms of inf-projections arise naturally in sensitivity analysis as *value functions*; see, for example, [25, Chapter 3.2]. Inf-projection also appears when representing functions as optimal values of linear programming problems, or more generally, semidefinite programming (SDP) problems; see [58] for SDP-representable functions. It is known that inf-projection preserves nice properties of F such as convexity [100, Proposition 2.22(a)]. In this thesis, we show that, under mild assumptions, the KL exponent is also preserved under inf-projection. Based on this result and the ubiquity of inf-projection, we are then able to obtain KL exponents of various important convex and nonconvex models that were out of

[3] We refer the readers to Chapter 2 for relevant definitions.

reach in the previous study. These include convex models such as a large class of SDP-representable functions, and some functions with C^2-cone reducible structures, as well as nonconvex models such as difference-of-convex functions and Bregman envelopes. These models are discussed in details in Section 3.3.1 with the general strategy for deducing their KL exponents outlined.

1.2 KL property in the convergence analysis of a sequential convex programming method with line search (SCP$_{ls}$)

Constrained optimization problems naturally arise when one attempts to find a solution that minimizes a certain objective under some restrictions, see [10,17,31,36,59]. In this section and Chapter 4, we consider the following specific type of difference-of-convex (DC) constrained optimization problem:

$$\min_{x\in\mathbb{R}^n} F(x) := f(x) + P_1(x) - P_2(x) + \delta_{g(\cdot)\le 0}(x), \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is smooth, $P_1 : \mathbb{R}^n \to \mathbb{R}$ and $P_2 : \mathbb{R}^n \to \mathbb{R}$ are convex continuous, the function $g(x) = (g_1(x), \ldots, g_m(x))$ with each $g_i : \mathbb{R}^n \to \mathbb{R}$ is continuous and $\{x : \ g(x) \le 0\} \neq \emptyset$. In typically applications, the f in (1.1) arises as measures for data fidelity, g is used for modeling restrictions on the decision variable x, and $P_1 - P_2$ is a regularizer for inducing desirable structures; see [57, Table 1] for examples of such regularizers. In our subsequent algorithmic development for (1.1), we also consider the following additional assumption.

Assumption 1.1. *Let f, g and F be as in* (1.1).

(i) $f : \mathbb{R}^n \to \mathbb{R}$ *has Lipschitz continuous gradient with Lipschitz modulus L_f.*

(ii) *Each g_i has Lipschitz continuous gradient with Lipschitz modulus L_{g_i}.*

(iii) *The function F is level-bounded.*

Under Assumption 1.1, the solution set of (1.1) is nonempty and $\inf F > -\infty$.

To design algorithms for solving (1.1) under Assumption 1.1, one common approach is to resort to the majorization-minimization (MM) procedure: in this procedure, one iteratively constructs and minimizes a surrogate function that locally majorizes F; see [23, 48, 49, 51, 107] for related models and discussions. For (1.1) under Assumption 1.1, one natural way to construct surrogate function is to make use of the 2nd-order Taylor's expansions of f and g: the resulting algorithms are the moving balls approximation method (MBA) proposed in [10] (for $P_1 = P_2 = 0$) and its variants [19, 23]. In each iteration, these algorithms approximate the constraint $g(x) \leq 0$ in (1.1) by

$$\bar{G}(x, y, w) := \begin{pmatrix} g_1(y) + \langle \nabla g_1(y), x - y \rangle + \frac{w_1}{2}\|x - y\|^2 \\ \vdots \\ g_m(y) + \langle \nabla g_m(y), x - y \rangle + \frac{w_m}{2}\|x - y\|^2 \end{pmatrix} \leq 0 \tag{1.2}$$

for some fixed (y, w): the feasible region of the resulting subproblem is an intersection of m balls. For the sequence generated by MBA, global convergence to a minimizer was established in [10] when $\{f, g_1, \ldots, g_m\}$ are in addition convex and the Slater condition holds. The linear convergence of the sequence generated by MBA was also proved in [10] when f in (1.1) is additionally strongly convex. In [23], when $\{f, g_1, \ldots, g_m\}$ are semi-algebraic and $P_1 = P_2 = 0$ in (1.1), the whole sequence generated by an MBA variant was shown to converge to a critical point and its convergence rate was also established, under the Mangasarian-Fromovitz constraint qualification (MFCQ).

When the possibly nonsmooth DC function $P_1 - P_2$ in (1.1) is nonzero (these nonsmooth functions arise naturally as regularizers in applications such as sparse recovery [36, 57, 123]), the aforementioned MBA-type methods such as the multiprox method in [19] cannot be directly applied to (1.1). Fortunately, under Assumption 1.1, problem (1.1) has DC objective and DC constraints: indeed, one can write f and each g_i in (1.1) as the difference of two convex functions as follows:

$$f(x) = \frac{L_f}{2}\|x\|^2 - \left(\frac{L_f}{2}\|x\|^2 - f(x)\right) \text{ and } g_i(x) = \frac{L_{g_i}}{2}\|x\|^2 - \left(\frac{L_{g_i}}{2}\|x\|^2 - g_i(x)\right).$$

DC algorithms (DCA) (see, for example, [69, 71]) can thus be applied. A variant that specializes in functional constraints is the sequential convex programming (SCP) method proposed in [83] [4]; see also [96, Remark 5]. When applied to (1.1) under Assumption 1.1, this method maintains feasibility at each iteration[5] and each subproblem is constrained over an intersection of balls: thus, this method can also be viewed as a variant of MBA. It was shown that any accumulation point of the sequence generated by SCP is a stationary point under Slater's condition. However, convergence and convergence rate of the whole sequence generated remain unknown.[6]

For empirical acceleration, a variant of MBA that involves a line search scheme was proposed in [19], which is called the Multiproximal method with backtracking step sizes (Multiprox$_{bt}$). When applied to (1.1) under Assumption 1.1, the sequence generated by Multiprox$_{bt}$ converges to a minimizer when $\{f, g_1, \ldots, g_m\}$ are additionally convex, $P_1 = P_2 = 0$ and the Slater condition holds. However, Multiprox$_{bt}$ uses monotone initial step sizes, i.e., $\widetilde{\alpha}$ in [19, Eq. (37)] is nondecreasing as the algorithm progresses, which rules out widely used choices such as the truncated Barzilai-Borwein step sizes [12, 18]. On the other hand, the line search variant of SCP proposed in [83] can incorporate flexible line search schemes like the truncated Barzilai-Borwein step

[4] We would like to point out that the methods proposed in [83] (including SCP and its variant) were designed to solve more general models than (1.1). In particular, they can deal with problems with nonsmooth constraints, and allow for nonmonotone line search.

[5] There are some DCA variants for solving (1.1) under Assumption 1.1 that do not maintain feasibility throughout. We refer the interested readers to [70, 71, 76, 105, 109] for more discussions.

[6] We point out that convergence of the whole sequence and the convergence rate generated by some DCA variants were considered in [5, 69] under suitable Kurdyka-Łojasiewicz (KL) assumptions; however, their problem formulations do not explicitly involve functional constraints as in (1.1).

size and is general enough to be applied to (1.1) under Assumption 1.1 with possibly nonsmooth $P_1 - P_2$. In [83], the well-definedness of the proposed algorithm was established, and it was also shown that any accumulation point of it is a stationary point under Slater's condition. However, convergence of the whole sequence generated and the corresponding convergence rate is still open.

In Chapter 4, we further study the line search variant of the SCP method proposed in [83] with its line search being monotone, i.e., M in [83, Eq. (22)] being 0. We call this variant SCP_{ls}; see Algorithm 2.1 below. We analyze the convergence properties of the sequence generated by SCP_{ls} for solving (1.1) under Assumption 1.1. The main convergence rate analysis of SCP_{ls} is presented in Section 4.1. We derive global convergence rate of the sequence generated by SCP_{ls} in the following two scenarios:

- *F in* (1.1) *is possibly nonconvex with each g_i being twice continuously differentiable and P_2 being Lipschitz continuously differentiable on an open set Γ that contains the set of stationary points of F.*

  Our analysis is based on the following specially constructed potential function:
  $$\bar{F}(x, y, w) = f(x) + P_1(x) - P_2(x) + \delta_{\bar{G}(\cdot)\leq 0}(x, y, w), \tag{1.3}$$
  where $\bar{G}$ is defined as in (1.2). Under MFCQ, we characterize the local convergence rate of the sequence generated by SCP_{ls} according to the Kurdyka-Łojasiewicz (KL) exponent of $\bar{F}$. Note the mapping $(x, y) \mapsto \bar{F}(x, y, L)$ with $P_1 = P_2 = 0$ and L being a constant positive vector (related to the step size) was used previously in [23] for establishing the convergence of an MBA variant when $P_1 = P_2 = 0$ and $\{f, g_1, \ldots, g_m\}$ in (1.1) are semi-algebraic. This kind of potential functions was called "value function" in [94] and was used there for deducing the global convergence properties of the composite Gauss-Newton method for composite optimization problems. Our potential function $\bar{F}$ allows us to deal with more flexible stepsize rules than those studied in [23, 94].

- *$\{f, g_1 \ldots, g_m\}$ in (1.1) are convex and $P_2 = 0$.*

  This same convex setting was considered in [19, Section 3.2.3]. In this setting, we impose KL assumptions directly on F in (1.1) (instead of on $\bar{F}$). In particular, a local *linear* convergence rate is established when F is a KL function with exponent $\frac{1}{2}$, under MFCQ. This is different from many existing analysis (see, for example, [7, 23, 74, 90]), which typically make use of the KL property of a potential function constructed out of F instead of F itself.

In Section 4.2, we study a relationship between the KL property of $\bar{F}$ in (1.3) and that of F in (1.1).

In Section 4.3, we discuss some concrete models to which SCP_{ls} can be applied. Specifically, we consider models of the following form:

$$\begin{array}{ll} \min\limits_{x} & \|x\|_1 - \mu\|x\| \\ \text{s.t.} & \ell(Ax - b) \leq \delta, \end{array} \tag{1.4}$$

where $\mu \in [0, 1]$, $A \in \mathbb{R}^{q \times n}$ has *full row rank*, $b \in \mathbb{R}^q$, $\ell : \mathbb{R}^q \to \mathbb{R}_+$ is analytic with Lipschitz continuous gradient and satisfies $\ell(0) = 0$, and $\delta \in (0, \ell(-b))$. This model arises in compressed sensing where the measurements b may be corrupted by different types of noise; see [35]. We focus on two concrete choices of ℓ: the square of norm (for noise following Gaussian distribution) and the Lorentzian norm (for noise following Cauchy distribution). For these two choices, we provide suitable conditions on the problem data so that the assumptions in our convergence results are satisfied. Then we perform numerical tests on solving (1.4) with ℓ being either the square of norm or the Lorentzian norm via two methods: SCP_{ls} and SCP [83]. We observe that SCP_{ls} appears to converge linearly and is much faster.

1.3 KL property in ℓ_1/ℓ_2 minimization

In compressed sensing (CS), a high-dimensional sparse or approximately sparse signal $x_0 \in \mathbb{R}^n$ is compressed (linearly) as Ax_0 for transmission, where $A \in \mathbb{R}^{m\times n}$ is the sensing matrix. The CS problem seeks to recover the original signal x_0 from the possibly noisy low-dimensional measurement $b \in \mathbb{R}^m$. This problem is NP-hard in general; see [88].

When there is no noise in the transmission, i.e., $Ax_0 = b$, one can recover x_0 exactly by minimizing the ℓ_1 norm over $A^{-1}\{b\}$ if x_0 is sufficiently sparse and the matrix A satisfies certain assumptions [34, 39]. To empirically enhance the recovery ability, various nonconvex models like ℓ_p $(0 < p < 1)$ minimization model [37] and ℓ_{1-2} minimization model [81] have been proposed, in which the ℓ_p quasi-norm and the difference of ℓ_1 and ℓ_2 norms are minimized over $A^{-1}\{b\}$, respectively. Recently, a new nonconvex model based on minimizing the quotient of the ℓ_1 and ℓ_2 norms was introduced in [97, 122] and further studied in [115, 116]:

$$\nu_{cs}^* := \min_{x\in\mathbb{R}^n} \frac{\|x\|_1}{\|x\|} \qquad \text{s.t.} \quad Ax = b, \tag{1.5}$$

where $A \in \mathbb{R}^{m\times n}$ has full row rank and $b \in \mathbb{R}^m\backslash\{0\}$. As discussed in [97], the above ℓ_1/ℓ_2 model has the advantage of being *scale-invariant* when reconstructing signals and images with high dynamic range. An efficient algorithm was proposed for solving (1.5) in [116, Eq. 11] and subsequential convergence was established under mild assumptions.

In practice, however, there is noise in the measurement, i.e., $b = Ax_0 + \epsilon$ for some noise vector ϵ, and (1.5) is not applicable for (approximately) recovering x_0. To deal with noisy situations, it is customary to relax the equality constraint in (1.5) to an inequality constraint [33]. In this section and Chapter 5, we consider the following model that minimizes the ℓ_1/ℓ_2 objective over an inequality constraint:

$$\nu_{ncs}^* = \min_{x\in\mathbb{R}^n} \frac{\|x\|_1}{\|x\|} \qquad \text{s.t. } q(x) \le 0, \tag{1.6}$$

where $q(x) = P_1(x) - P_2(x)$ with $P_1 : \mathbb{R}^n \to \mathbb{R}$ being continuously differentiable with globally Lipschitz continuous gradient and $P_2 : \mathbb{R}^n \to \mathbb{R}$ being convex continuous, and we assume that $\{x : \ q(x) \le 0\} \neq \emptyset$ and $q(0) > 0$. Our assumptions on q are general enough to cover commonly used loss functions for modeling noise in various scenarios:

1. **Gaussian noise**: When the noise in the measurement follows the Gaussian distribution, the least squares loss function $y \mapsto \|y - b\|^2$ is typically employed [33, 39]. One may consider the following ℓ_1/ℓ_2 minimization problem:

$$\min_{x\in\mathbb{R}^n} \frac{\|x\|_1}{\|x\|} \qquad \text{s.t. } \|Ax - b\|^2 - \sigma^2 \le 0, \tag{1.7}$$

   where $\sigma > 0$, $A \in \mathbb{R}^{m\times n}$ has full row rank and $b \in \mathbb{R}^m$ satisfies $\|b\| > \sigma$. Problem (1.7) corresponds to (1.6) with $q(x) = P_1(x) = \|Ax - b\|^2 - \sigma^2$ and $P_2 = 0$.

2. **Cauchy noise**: When the noise in the measurement follows the Cauchy distribution (a heavy-tailed distribution), the Lorentzian norm[7] $\|y\|_{LL_2,\gamma} := \sum_{i=1}^m \log\left(1 + \gamma^{-2} y_i^2\right)$ is used as the loss function [35,36], where $\gamma > 0$. Note that the Lorentzian norm is continuously differentiable with Lipschitz continuous gradient. One may then consider the following ℓ_1/ℓ_2 minimization problem:

$$\min_{x\in\mathbb{R}^n} \frac{\|x\|_1}{\|x\|} \qquad \text{s.t. } \|Ax - b\|_{LL_2,\gamma} - \sigma \le 0, \tag{1.8}$$

[7] We refer the readers to [36, Equation (12)] for the definition and notation of Lorentzian norm.

where $\sigma > 0$, $A \in \mathbb{R}^{m\times n}$ has full row rank, and $b \in \mathbb{R}^m$ with $\|b\|_{LL_{2,\gamma}} > \sigma$. Problem (1.8) corresponds to (1.6) with $q(x) = P_1(x) = \|Ax - b\|_{LL_{2,\gamma}} - \sigma$ and $P_2 = 0$.

3. **Robust compressed sensing**: In this scenario, the measurement is corrupted by both Gaussian noise and electromyographic noise [36, 95]: the latter is sparse and may have large magnitude (outliers). Following [79, Section 5.1.1], one may make use of the loss function $y \mapsto \mathrm{dist}^2(y, S)$, where $S := \{z \in \mathbb{R}^m : \|z\|_0 \le r\}$, $\|z\|_0$ is the number of nonzero entries in z and r is an estimate of the number of outliers. One may then consider the following ℓ_1/ℓ_2 minimization problem:

$$\min_{x\in\mathbb{R}^n} \frac{\|x\|_1}{\|x\|} \qquad \text{s.t. } \mathrm{dist}^2(Ax - b, S) - \sigma^2 \le 0, \tag{1.9}$$

where $\sigma > 0$, $S = \{z \in \mathbb{R}^m : \|z\|_0 \le r\}$ with $r \ge 0$, $A \in \mathbb{R}^{m\times n}$ has full row rank and $b \in \mathbb{R}^m$ satisfies $\mathrm{dist}(b, S) > \sigma$. Notice that

$$\mathrm{dist}^2(Ax - b, S) - \sigma^2 = \underbrace{\|Ax - b\|^2 - \sigma^2}_{P_1(x)} - \underbrace{\max_{z\in S}\{\langle 2z, Ax - b\rangle - \|z\|^2\}}_{P_2(x)}, \tag{1.10}$$

with P_1 being continuously differentiable with Lipschitz continuous gradient and P_2 being convex continuous. So this problem corresponds to (1.6) with P_1 and P_2 as in (1.10) and $q = P_1 - P_2$.

In the literature, algorithms for solving (1.7) with ℓ_1 norm or ℓ_p quasi-norm in place of the quotient of the ℓ_1 and ℓ_2 norms have been discussed in [17, 40, 103], and [125] discussed an algorithm for solving (1.8) with ℓ_1 norm in place of the quotient of the ℓ_1 and ℓ_2 norms. These existing algorithms, however, are not directly applicable for solving (1.6) due to the fractional objective and the possibly nonsmooth continuous function q in the constraint.

In Chapter 5, we further study properties of the ℓ_1/ℓ_2 models (1.5) and (1.6), and propose an algorithm for solving (1.6). In particular, we first argue that an optimal solution of (1.5) exists by making connections with the s-spherical section property [114, 128] of $\ker A$: a property which is known to hold with high probability when n is much greater than m for Gaussian matrices. We then revisit the algorithm proposed in [116, Eq. 11] (see Algorithm 5.1 below) for solving (1.5). Specifically, we consider the following function

$$F(x) := \frac{\|x\|_1}{\|x\|} + \delta_{A^{-1}\{b\}}(x), \tag{1.11}$$

where $A \in \mathbb{R}^{m\times n}$ has full row rank and $b \in \mathbb{R}^m\backslash\{0\}$. We show in Section 5.2.1 that F is a Kurdyka-Łojasiewicz (KL) function with exponent $\frac{1}{2}$. This together with standard convergence analysis based on KL property [6–8] allows us to deduce local linear convergence of the sequence $\{x^t\}$ generated by Algorithm 5.1 when $\{x^t\}$ is bounded. The KL exponent of F is obtained based on the calculus rule deduced in Section .

Next, for the model (1.6), we also relate existence of solutions to the s-spherical section property of $\ker A$ when q takes the form in (1.7) and (1.8). We then propose an algorithm, which we call $\mathrm{MBA}_{\ell_1/\ell_2}$ (see Algorithm 5.2), for solving (1.6), which can be seen as an extension of Algorithm 5.1 by incorporating *moving-balls-approximation* (MBA) techniques. The MBA algorithm was first proposed in [10] for minimizing a smooth objective function subject to multiple smooth constraints, and was further studied in [19, 23, 125] for more general objective functions. However, the existing convergence results of these algorithms cannot be applied to $\mathrm{MBA}_{\ell_1/\ell_2}$ because of the possibly nonsmooth continuous function q and the fractional objective in (1.6). Our convergence analysis of $\mathrm{MBA}_{\ell_1/\ell_2}$ relies on a specially constructed potential function, which involves the indicator function of the lower level set of a *proper closed* function related to q (see (5.22)). We prove that any accumulation point of the sequence generated by $\mathrm{MBA}_{\ell_1/\ell_2}$ is a so-called Clarke critical point (see 5.3 for explicit definition), under mild assumptions; Clarke criticality reduces to the usual notion of stationarity when q is regular. Moreover, by imposing additional KL assumptions on this potential function and assuming P_1 is twice continuously differentiable, we show that the sequence generated by $\mathrm{MBA}_{\ell_1/\ell_2}$ is globally convergent, and the convergence rate is related to the KL exponent of the potential function. Finally, we perform numerical experiments to illustrate the performance of our algorithm on solving (1.7), (1.8) and (1.9).

Chapter 2

Notation and Preliminaries

In this chapter, we first present the notation and preliminary results used throughout this thesis in Section 2.1. Sections 2.2, 2.3 and 2.4 give the notation and preliminaries that are only used in Chapters 3, 4 and 5 respectively.

2.1 Basic notation and preliminaries

Throughout this thesis, we use $\mathbb{X}$ and $\mathbb{Y}$ to denote two finite dimensional Hilbert spaces. We use $\langle \cdot, \cdot \rangle$ to denote the inner product of the underlying Hilbert space and use $\|\cdot\|$ to denote the associated norm. We let $\mathbb{R}$ denote the set of real numbers and $\mathcal{N}_+$ denote the set of positive integers. The n-dimensional Euclidean space is denoted by $\mathbb{R}^n$, and the nonnegative orthant is denoted by $\mathbb{R}^n_+$. For two vectors x and $y \in \mathbb{R}^n$, we write $x \geq y$ if $x_i \geq y_i$ for all i. The ℓ_0 norm (the number of nonzero entries) of x by $\|x\|_0$ and the ℓ_1 norm of x is denoted by $\|x\|_1$. For $x \in \mathbb{R}^n$ and $r \geq 0$, we let $B(x, r)$ denote the closed ball centered at x with radius r, i.e., $B(x, r) = \{y : \ \|x - y\| \leq r\}$.

We say that an extended-real-valued function $f : \mathbb{R}^n \to (-\infty, \infty]$ is proper if its domain $\mathrm{dom} f := \{x : \ f(x) < \infty\} \neq \emptyset$. A proper function f is said to be closed if it is lower semicontinuous. For a proper function f, the regular subdifferential of f at $x \in \mathrm{dom} f$ is defined by

$$\widehat{\partial} f(x) := \left\{ \zeta : \liminf_{z \to x, z \neq x} \frac{f(z) - f(x) - \langle \zeta, z - x \rangle}{\|z - x\|} \geq 0 \right\}.$$

The (limiting) subdifferential of f at $x \in \text{dom} f$ is defined by

$$\partial f(x) := \left\{ \zeta : \exists x^k \xrightarrow{f} x, \zeta^k \to \zeta \text{ with } \zeta^k \in \widehat{\partial} f(x^k) \text{ for each } k \right\},$$

where $x^k \xrightarrow{f} x$ means both $x^k \to x$ and $f(x^k) \to f(x)$. Moreover, we set $\partial f(x) = \widehat{\partial} f(x) = \emptyset$ for $x \notin \text{dom}\, f$ by convention, and we write $\text{dom}\, \partial f := \{x : \partial f(x) \neq \emptyset\}$. It is known that $\partial h(x) = \{\nabla h(x)\}$ if h is continuously differentiable at x [100, Exercise 8.8(b)]. When f is proper convex, thanks to [83, Proposition 8.12], the limiting subdifferential and regular subdifferential of f at an $x \in \text{dom}\, f$ reduce to the classical subdifferential, which is given by

$$\partial f(x) = \{\zeta : \langle \zeta, y - x \rangle \leq f(y) - f(x) \text{ for all } y\}.$$

The convex conjugate of a proper closed convex function h is defined as

$$h^*(y) = \sup_{x \in \mathbb{R}^n} \{\langle x, y \rangle - h(x)\}.$$

We recall the following relationship concerning convex conjugate and subdifferential of a proper closed convex function h; see [100, Proposition 11.3]:

$$y \in \partial h(x) \iff x \in \partial h^*(y) \iff h(x) + h^*(y) \leq \langle x, y \rangle \iff h(x) + h^*(y) = \langle x, y \rangle. \quad (2.1)$$

For a proper closed convex function f, its asymptotic (or recession) function f^∞ is defined by $f^\infty(d) := \liminf_{t \to \infty, d' \to d} \frac{f(td')}{t}$; see [9, Theorem 2.5.1]. Finally, for a proper function f, we say that it is level-bounded if, for each $\alpha \in \mathbb{R}$, the set $\{x : f(x) \leq \alpha\}$ is bounded.

For a locally Lipschitz function h, its Clarke subdifferential at $\bar{x} \in \mathbb{R}^n$ is defined by

$$\partial^\circ h(\bar{x}) := \left\{ v : \limsup_{x \to \bar{x}, t \downarrow 0} \frac{h(x + tw) - h(x)}{t} \geq \langle v, w \rangle \text{ for all } w \in \mathbb{R}^n \right\};$$

it holds that $\partial h(\bar{x}) \subseteq \partial^{\circ} h(\bar{x})$; see [27, Theorem 5.2.22].

For a nonempty set C, the indicator function δ_C is defined as

$$\delta_C(x) := \begin{cases} 0 & x \in C, \\ \infty & x \notin C. \end{cases}$$

The normal cone (resp., regular normal cone) of C at an $x \in C$ is defined as $N_C(x) := \partial \delta_C(x)$ (resp., $\widehat{N}_C(x) := \widehat{\partial} \delta_C(x)$), and the distance from a point $x \in \mathbb{R}^n$ to C is denoted by $\operatorname{dist}(x, C)$. If $\mathfrak{D}$ is in addition convex, we define its tangent cone at $x \in \mathfrak{D}$ by $T_{\mathfrak{D}}(x) := [N_{\mathfrak{D}}(x)]^{\circ}$.

We next recall the Kurdyka-Łojasiewicz (KL) property and the notion of KL exponent; see [6–8, 66, 75, 80]. This property has been used extensively in analyzing convergence of first-order methods; see, for example, [6–8, 24, 119].

Definition 2.1 (**Kurdyka-Łojasiewicz property and exponent**). *We say that a proper closed function $h : \mathbb{X} \to \mathbb{R} \cup \{\infty\}$ satisfies the Kurdyka-Łojasiewicz (KL) property at $\widehat{x} \in \operatorname{dom} \partial h$ if there are $a \in (0, \infty]$, a neighborhood V of $\widehat{x}$ and a continuous concave function $\varphi : [0, a) \to [0, \infty)$ with $\varphi(0) = 0$ such that*

(i) *φ is continuously differentiable on $(0, a)$ with $\varphi' > 0$ on $(0, a)$;*

(ii) *For any $x \in V$ with $h(\widehat{x}) < h(x) < h(\widehat{x}) + a$, it holds that*

$$\varphi'(h(x) - h(\widehat{x}))\operatorname{dist}(0, \partial h(x)) \geq 1. \tag{2.2}$$

If h satisfies the KL property at $\widehat{x} \in \operatorname{dom} \partial h$ and the $\varphi(s)$ in (2.2) can be chosen as $\bar{c}\, s^{1-\alpha}$ for some $\bar{c} > 0$ and $\alpha \in [0, 1)$, then we say that h satisfies the KL property at $\widehat{x}$ with exponent α.

A proper closed function h satisfying the KL property at every point in $\operatorname{dom} \partial h$ is said to be a KL function, and a proper closed function h satisfying the KL property with exponent $\alpha \in [0, 1)$ at every point in $\operatorname{dom} \partial h$ is said to be a KL function with exponent α.

KL functions is a broad class of functions which arise naturally in many applications. For instance, it is known that proper closed semi-algebraic functions are KL functions with exponent $\alpha \in [0,1)$; see, for example, [7]. KL property is a key ingredient in many contemporary convergence analysis for first-order methods, and the KL exponent plays an important role in identifying *local convergence rate*; see, for example, [6, Theorem 2], [54, Theorem 3.4] and [74, Theorem 3]. In this thesis, we will study how the KL exponent behaves under inf-projection, and use the rules developed to compute the KL exponents of various functions and to derive new calculus rules for KL exponent.

2.2 Notation and preliminaries in Chapter 3

For a linear map $\mathcal{A} : \mathbb{X} \to \mathbb{Y}$, we use $\mathcal{A}^*$ to denote its adjoint. We also let $\mathrm{I\!R}^{m\times n}$ denote the set of all $m \times n$ matrices. The (trace) inner product of two matrices A and $B \in \mathrm{I\!R}^{m\times n}$ is defined as $\langle A, B\rangle := \mathrm{tr}(A^T B)$, where tr denotes the trace of a square matrix. The Fröbenius norm of a matrix $A \in \mathrm{I\!R}^{m\times n}$ is denoted by $\|A\|_F$, which is defined as $\|A\|_F := \sqrt{\mathrm{tr}(A^T A)}$. Finally, the space of $n \times n$ symmetric matrices is denoted by $\mathcal{S}^n$, the cone of $n \times n$ positive semidefinite matrices is denoted by $\mathcal{S}^n_+$, and we write $X \succeq 0$ (resp., $X \succ 0$) to mean $X \in \mathcal{S}^n_+$ (resp., $X \in \mathrm{int}\,\mathcal{S}^n_+$, where $\mathrm{int}\,\mathcal{S}^n_+$ is the interior of $\mathcal{S}^n_+$).

The closure (resp., interior) of $\mathfrak{D}$ is denoted by $\mathrm{cl}\,\mathfrak{D}$ (resp., $\mathrm{int}\,\mathfrak{D}$). For a convex set $\mathfrak{C} \subseteq \mathbb{X}$, we denote its relative interior by $\mathrm{ri}\,\mathfrak{C}$, and use $\mathfrak{C}^\circ$ to denote its polar, which is defined as

$$\mathfrak{C}^\circ := \{z \in \mathbb{X} : \langle x, z\rangle \le 1 \text{ for all } x \in \mathfrak{C}\}.$$

We use $\sigma_{\mathfrak{D}}$ to denote its support function, which is defined as $\sigma_{\mathfrak{D}}(x) := \sup_{z\in\mathfrak{D}}\langle x, z\rangle$ for $x \in \mathbb{X}$.

For a mapping $\Theta : \mathbb{X} \to \mathbb{Y}$ that is continuously differentiable on $\mathbb{X}$, we use $D\Theta(x)$

to denote the derivative mapping of Θ at $x \in \mathbb{X}$: this is the linear map defined by

$$[D\Theta(x)]h := \lim_{t \to 0} \frac{\Theta(x+th) - \Theta(x)}{t} \quad \text{for all } h \in \mathbb{X}.$$

We denote the adjoint of the derivative mapping by $\nabla\Theta(x)$. This latter mapping is referred to as the gradient mapping of Θ at x. Then, following [101, Definition 3.1], we say that a closed set $\mathfrak{D} \subseteq \mathbb{X}$ is C^2-cone reducible at $\bar{w} \in \mathfrak{D}$ if there exist a closed convex pointed cone $K \subseteq \mathbb{Y}$, $\rho > 0$ and a mapping $\Theta : \mathbb{X} \to \mathbb{Y}$ that maps $\bar{w}$ to 0 and is twice continuously differentiable in $B(\bar{w}, \rho)$ with $D\Theta(\bar{w})$ being onto, such that

$$\mathfrak{D} \cap B(\bar{w}, \rho) = \{w : \Theta(w) \in K\} \cap B(\bar{w}, \rho).$$

We say that the set $\mathfrak{D}$ is C^2-cone reducible if, for all $\bar{w} \in \mathfrak{D}$, $\mathfrak{D}$ is C^2-cone reducible at $\bar{w}$. It is known that convex polyhedral sets, the positive semidefinite cone and the second-order cone are all C^2-cone reducible; see, for example, the discussion following [101, Definition 3.1]. Finally, following the discussion right after [43, Definition 6], we say that an extended-real-valued function is C^2-cone reducible if its epigraph is a C^2-cone reducible set, where the epigraph of an extended-real-valued function $f : \mathbb{X} \to [-\infty, \infty]$ is defined as $\operatorname{epi} f := \{(x, t) \in \mathbb{X} \times \mathbb{R} : \ f(x) \le t\}$.

For a proper function $F : \mathbb{X} \times \mathbb{Y} \to \mathbb{R} \cup \{\infty\}$, following [100, definition 1.16], we say that F is level-bounded in y locally uniformly in x if for each $\bar{x} \in \mathbb{X}$ and $\alpha \in \mathbb{R}$ there is a neighborhood V of $\bar{x}$ such that the set $\{(x, y) \in \mathbb{X} \times \mathbb{Y} : \ x \in V \text{ and } F(x, y) \le \alpha\}$ is bounded. When a function F is level-bounded in y locally uniformly in x, its inf-projection $f(x) := \inf_y F(x, y)$ has the following properties, which can be found in [100]. We include the proof here.

Lemma 2.1. *Let $F : \mathbb{X} \times \mathbb{Y} \to \mathbb{R} \cup \{\infty\}$ be a proper closed function and define $f(x) := \inf_{y \in \mathbb{Y}} F(x, y)$ and $Y(x) := \operatorname{Arg\,min}_{y \in \mathbb{Y}} F(x, y)$ for $x \in \mathbb{X}$. Suppose F is level-bounded in y locally uniformly in x. Then the following statements hold:*

(i) *The function f is proper and closed, and the set $Y(x)$ is nonempty and compact for any $x \in \operatorname{dom} \partial f$.*

(ii) *For any $x \in \operatorname{dom} \partial f$, it holds that*

$$\partial f(x) \subseteq \bigcup_{y \in Y(x)} \{\xi \in \mathbb{X} \ : \ (\xi, 0) \in \partial F(x, y)\}. \tag{2.3}$$

(iii) *For any $\bar{x} \in \operatorname{dom} \partial f$, it holds that*

$$\limsup_{\operatorname{dom} \partial f \ni x \xrightarrow{f} \bar{x}} Y(x) \subseteq Y(\bar{x}); \tag{2.4}$$

(iv) *For any $\bar{x} \in \operatorname{dom} \partial f$ and any $\nu > 0$, there exists $\epsilon > 0$ such that*

$$\operatorname{dist}(y, Y(\bar{x})) \leq \frac{\nu}{2}$$

whenever $y \in Y(x)$ with $x \in B(\bar{x}, \epsilon) \cap \operatorname{dom} \partial f$ and $|f(x) - f(\bar{x})| < \epsilon$.

Proof. Since F is proper, closed and level-bounded in y locally uniformly in x, we have from [100, Theorem 1.17] that f is proper and closed, and $Y(x)$ is a nonempty compact set whenever $x \in \operatorname{dom} \partial f$. Applying [100, Theorem 10.13], we conclude that (2.3) holds for any $x \in \operatorname{dom} \partial f$.

We now prove (iii) and (iv) respectively. For (iii), fix any $\bar{x} \in \operatorname{dom} \partial f$ and any y^* satisfying $y^* \in \limsup_{\operatorname{dom} \partial f \ni x \xrightarrow{f} \bar{x}} Y(x)$ and recall from [100, Section 5B] that $\limsup_{\operatorname{dom} \partial f \ni x \xrightarrow{f} \bar{x}} Y(x)$ equals to

$$\left\{ y : \ \exists\, x^k \xrightarrow{f} \bar{x}, \ y^k \to y \text{ with } y^k \in Y(x^k) \text{ and } x^k \in \operatorname{dom} \partial f \text{ for each } k \right\}.$$

So, there exist $x^k \xrightarrow{f} \bar{x}$ with $x^k \in \operatorname{dom} \partial f$ and $y^k \to y^*$ such that $y^k \in Y(x^k)$ for all k. Then we have

$$F(\bar{x}, y^*) \overset{\text{(a)}}{\leq} \liminf_k F(x^k, y^k) \overset{\text{(b)}}{=} \liminf_k f(x^k) \overset{\text{(c)}}{=} f(\bar{x}),$$

where (a) is due to the closedness of F, (b) holds because $y^k \in Y(x^k)$, and (c) holds because $x^k \xrightarrow{f} \bar{x}$. The above relation implies that $y^* \in Y(\bar{x})$. This proves (2.4).

Finally, for (iv), fix any $\bar{x} \in \operatorname{dom} \partial f$ and any $\nu > 0$. Since F is level-bounded in y locally uniformly in x, there exist $\widetilde{\epsilon} > 0$ and a bounded set D so that whenever $x \in B(\bar{x}, \widetilde{\epsilon}) \cap \operatorname{dom} \partial f$, we have $\{y : F(x,y) \leq f(\bar{x}) + 1\} \subseteq D$. Thus, for any x satisfying $x \in B(\bar{x}, \widetilde{\epsilon}) \cap \operatorname{dom} \partial f$ and $f(x) < f(\bar{x}) + 1$, we obtain

$$Y(x) = \{y : F(x,y) \leq f(x)\} \subseteq \{y : F(x,y) \leq f(\bar{x}) + 1\} \subseteq D. \tag{2.5}$$

Since (2.4) holds, by picking $\eta > 0$ so that $D \subseteq B(0, \eta)$ and following the proof of [100, Proposition 5.12(a)], we see that for this η, there exists $\epsilon \in (0, \min\{\widetilde{\epsilon}, 1\})$ such that

$$Y(x) = Y(x) \cap D \subseteq Y(x) \cap B(0, \eta) \subseteq Y(\bar{x}) + B(0, \nu/2),$$

whenever $x \in B(\bar{x}, \epsilon) \cap \operatorname{dom} \partial f$ and $|f(x) - f(\bar{x})| < \epsilon$, where the first equality follows from (2.5) and the facts that $\epsilon < \widetilde{\epsilon}$ and $\epsilon < 1$. This further implies that

$$\operatorname{dist}(y, Y(\bar{x})) \leq \frac{\nu}{2}.$$

for any $y \in Y(x)$ with $x \in B(\bar{x}, \epsilon) \cap \operatorname{dom} \partial f$ and $|f(x) - f(\bar{x})| < \epsilon$. $\square$

Before ending this section, we present one auxiliary lemma that concerns the uniformized KL property and will be used in Chapter 3. It is a specialization of [24, Lemma 6] and explicitly involves the KL exponent.

Lemma 2.2 (Uniformized KL property with exponent). *Suppose that $h : \mathbb{X} \to \mathbb{R} \cup \{\infty\}$ is a proper closed function and let Ω be a nonempty compact set with $\Omega \subseteq \operatorname{dom} \partial h$. If h takes a constant value on Ω and satisfies the KL property at each point of Ω with exponent α, then there exist $\epsilon, a, c > 0$ such that*

$$\operatorname{dist}(0, \partial h(x)) \geq c\,(h(x) - h(\bar{x}))^{\alpha}$$

for any $\bar{x} \in \Omega$ *and any* x *satisfying* $h(\bar{x}) < h(x) < h(\bar{x}) + a$ *and* $\mathrm{dist}(x, \Omega) < \epsilon$.

Proof. Replace the $\varphi_i(t)$ in the proof of [24, Lemma 6] by $c_i t^{1-\alpha}$ for some $c_i > 0$. The desired conclusion can then be proved analogously as in [24, Lemma 6]. $\square$

The next lemma is a direct consequence of results in [106]; see [106, Theorem 3.3] and the discussion following [106, Eq. (1.4)] concerning the degree of singularity for semidefinite feasibility system.

Lemma 2.3 (**Error bound for standard SDP problems under strict complementarity**). *Let* $C \in \mathcal{S}^d$, $\mathcal{A} : \mathcal{S}^d \to \mathbb{R}^m$ *be a linear map,* $b \in \mathrm{Range}\,(\mathcal{A})$ *and define the function* $G : \mathcal{S}^d \to \mathbb{R} \cup \{\infty\}$ *by*

$$G(X) := \langle C, X \rangle + \delta_{\mathfrak{L}}(X),$$

where $\mathfrak{L} = \mathcal{A}^{-1}\{b\} \cap \mathcal{S}^d_+$. *Suppose that* $\mathcal{A}^{-1}\{b\} \cap \mathrm{int}\,\mathcal{S}^d_+ \neq \emptyset$ *and there exists* $\bar{X} \in \mathfrak{L}$ *satisfying* $0 \in \mathrm{ri}\,\partial G(\bar{X})$. *Then for any bounded neighborhood* $\mathfrak{U}$ *of* $\bar{X}$, *there exists* $c > 0$ *such that for any* $X \in \mathfrak{U} \cap \mathfrak{L}$,

$$\mathrm{dist}\,(X, \mathrm{Arg\,min}\, G) \leq c \left(G(X) - G(\bar{X})\right)^{\frac{1}{2}}.$$

Proof. Observe that

$$\begin{aligned} 0 \in \mathrm{ri}\,\partial G(\bar{X}) &\overset{\text{(a)}}{=} C + \mathrm{ri}\,N_{\mathfrak{L}}(\bar{X}) \overset{\text{(b)}}{=} C + \mathrm{ri}\,\left(N_{\mathcal{A}^{-1}\{b\}}(\bar{X}) + N_{\mathcal{S}^d_+}(\bar{X})\right) \\ &\overset{\text{(c)}}{=} C + \mathrm{ri}\,N_{\mathcal{A}^{-1}\{b\}}(\bar{X}) + \mathrm{ri}\,N_{\mathcal{S}^d_+}(\bar{X}), \end{aligned} \tag{2.6}$$

where (a) follows from [100, Exercise 8.8], (b) follows from [99, Theorem 23.8] and the assumption $\mathcal{A}^{-1}\{b\} \cap \mathrm{int}\,\mathcal{S}^d_+ \neq \emptyset$, and (c) follows from [99, Corollary 6.6.2]. Since $N_{\mathcal{A}^{-1}\{b\}}(\bar{X}) = \mathrm{Range}\,(\mathcal{A}^*)$, we deduce further from (2.6) the existence of $\bar{y}$ satisfying

$$\mathcal{A}^*\bar{y} - C \in \mathrm{ri}\,N_{\mathcal{S}^d_+}(\bar{X}). \tag{2.7}$$

Next, since $0 \in \partial G(\bar{X})$, we have that $\bar{X} \in \operatorname{Arg\,min} G$ and thus

$$\operatorname{Arg\,min} G = \{W:\ \mathcal{A}W = b\} \cap \{W:\ \langle C, W\rangle = \inf G\} \cap \mathcal{S}^d_+ \neq \emptyset.$$

This together with (2.7) implies that the singularity degree of the semidefinite feasibility system $\left(\{W:\ \mathcal{A}W = b\} \cap \{W:\ \langle C, W\rangle = \inf G\}, \mathcal{S}^d_+\right)$ is one. Combining this with [50, Theorem 2.3], we conclude that for any bounded neighborhood $\mathfrak{U}$ of $\bar{X}$, there exists $c_1 > 0$ such that for any $X \in \mathfrak{U} \cap \mathfrak{L}$,

$$\begin{aligned}\operatorname{dist}(X, \operatorname{Arg\,min} G) &\le c_1 \sqrt{\operatorname{dist}(X, \{W:\ \mathcal{A}W = b\} \cap \{W: \langle C, W\rangle = \inf G\})} \\ &\le c\left(\langle C, X\rangle - \inf G\right)^{\frac{1}{2}} = c\left(G(X) - G(\bar{X})\right)^{\frac{1}{2}},\end{aligned}$$

where the second inequality holds for some $c > 0$ thanks to the Hoffman error bound [52, Lemma 3.2.3]. This completes the proof. □

Remark 2.1. *In the above lemma, the Slater's condition $\mathcal{A}^{-1}\{b\} \cap \operatorname{int} \mathcal{S}^d_+ \neq \emptyset$ together with the relative interior (ri) condition $0 \in \operatorname{ri} \partial G(\bar{X})$ implies that (2.7) holds. The condition (2.7) is widely used in the SDP literature and is often referred to as the strict complementarity condition; see [93, 102, 111] for detailed discussions. In particular, it is known that if strict complementarity condition (2.7) holds , then the singular degree of the associated semidefinite feasibility system is one (see [82, Proposition 7] or the discussion following [106, Eq. (1.4)]).*

As we shall see in Section 3.4, this strict complementarity condition is crucial for deriving a KL exponent of $\frac{1}{2}$ for some SDP representable functions.

2.3 Notation and preliminaries in Chapter 4

Now we recall the definition of stationary points of (1.1) when g_i are smooth.

Definition 2.2 (Stationary point). *Consider* (1.1) *and assume that each* g_i *is smooth. We say that an* $x \in \mathbb{R}^n$ *is a stationary point of* (1.1) *if there exists* $\lambda \in \mathbb{R}^m_+$ *such that* (x, λ) *satisfies*

$$g(x) \le 0, \ \lambda_i g_i(x) = 0 \text{ for all } i, \text{ and } 0 \in \nabla f(x) + \partial P_1(x) - \partial P_2(x) + \sum_{i=1}^m \lambda_i \nabla g_i(x).$$

The following assumption will be used repeatedly in Chapter 4.

Assumption 2.1. *Each* g_i *in* (1.1) *is smooth and the Mangasarian-Fromovitz constraint qualification (MFCQ) holds in the whole domain of* F *in* (1.1), *i.e., for every* x *satisfying* $g(x) \le 0$, *there exists* $d \in \mathbb{R}^n$ *such that*

$$\langle \nabla g_i(x), d \rangle < 0 \text{ for each } i \in I(x) := \{j : \ \ g_j(x) = 0\}.$$

Under Assumptions 1.1 and 2.1, it is routine to show that any local minimizer of (1.1) is a stationary point in the sense of Definition 2.2. In fact, let $\widehat{x}$ be a local minimizer of (1.1). Using [100, Theorem 10.1], we have

$$\begin{aligned} 0 \in \partial F(\widehat{x}) &\overset{\text{(a)}}{\subseteq} \nabla f(\widehat{x}) + \partial P_1(\widehat{x}) + \partial(-P_2)(\widehat{x}) + \partial \delta_{g(\cdot)\le 0}(\widehat{x}) \\ &\overset{\text{(b)}}{\subseteq} \nabla f(\widehat{x}) + \partial P_1(\widehat{x}) + \partial^\circ(-P_2)(\widehat{x}) + \partial \delta_{g(\cdot)\le 0}(\widehat{x}) \\ &\overset{\text{(c)}}{=} \nabla f(\widehat{x}) + \partial P_1(\widehat{x}) - \partial^\circ P_2(\widehat{x}) + \partial \delta_{g(\cdot)\le 0}(\widehat{x}) \\ &= \nabla f(\widehat{x}) + \partial P_1(\widehat{x}) - \partial P_2(\widehat{x}) + \partial \delta_{g(\cdot)\le 0}(\widehat{x}), \end{aligned} \tag{2.8}$$

where (a) follows from [100, Exercise 10.10], the inclusion (b) uses [27, Theorem 5.2.22], where $\partial^\circ(-P_2)$ is the Clarke subdifferential of $-P_2$, the equality (c) uses [41, Proposition 2.3.1] and the last equality holds because of the convexity of P_2 and [25, Theorem 6.2.2]. In addition, we can deduce that

$$\begin{aligned} \partial \delta_{g(\cdot)\le 0}(\widehat{x}) = N_{g(\cdot)\le 0}(\widehat{x}) &= \left\{ \sum_{i=1}^m \lambda_i \nabla g_i(\widehat{x}) : \lambda \in N_{-\mathbb{R}^m_+}(g(\widehat{x})) \right\} \\ = \left\{ \sum_{i=1}^m \lambda_i \nabla g_i(\widehat{x}) : \lambda \in \mathbb{R}^m_+, \ \lambda_i g_i(\widehat{x}) = 0 \text{ for } i = 1, \ldots, m \right\}, \end{aligned}$$

where the second equality follows from MFCQ and [100, Theorem 6.14] and the last equality follows from the definition of normal cone. The above display together with (2.8) shows that $\widehat{x}$ is a stationary point of (1.1). In passing, we would like to point out that x^* is a stationary point of (1.1) in the sense of Definition 2.2 if and only if there exists ξ^* such that $0 \in \partial \widetilde{F}(x^*, \xi^*)$, where $\widetilde{F}(x, \xi) := f(x) + P_1(x) - \langle \xi, x \rangle + P_2^*(\xi) + \delta_{g(\cdot)\leq 0}(x)$, with $\{P_1, P_2\}$ given in (1.1) and P_2^* being the Fenchel conjugate of P_2. This type of stationary points is widely used in the DC literature; see, for example, [108, 109, 119]. Note that there are other concepts of stationarity used in the literature, such as the Clarke stationarity, d-stationarity and B-stationarity; we refer to [1, 64, 91] for more discussions. The notion of stationarity defined in Definition 2.2 is in general weaker than these aforementioned notions.

Before ending this section, we introduce the algorithm we consider here and in Chapter 4 and present some auxiliary results for our subsequent analysis. The algorithm, SCP_{ls} proposed in [83], is presented in Algorithm 2.1, where $\bar{G}$ is defined as in (1.2). Notice that by rearranging terms of the constraint functions of the subproblem (2.10), we can see that the constraint there is equivalent to

$$x \in \bigcap_{i=1}^{m} B\left(\widetilde{s}_i, \sqrt{\widetilde{R}_i}\right), \tag{2.9}$$

where $\widetilde{s}_i := x^t - \frac{1}{\left(\widetilde{L}_g\right)_i} \nabla g_i(x^t)$ and $\widetilde{R}_i := \left\| \frac{\nabla g_i(x^t)}{\left(\widetilde{L}_g\right)_i} \right\|^2 - \frac{2}{\left(\widetilde{L}_g\right)_i} g_i(x^t)$. Thus, when $m = 1$, the constraint reduces to a *single* ball constraint and a simple root-finding scheme was discussed in [103] for exactly and efficiently solving the subproblem (2.10) with $m = 1$, $P_2 = 0$ and P_1 being the ℓ_1 norm or the nuclear norm, etc. However, solving subproblem (2.10) in general requires an iterative solver; see [10, Section 6] for the case when $P_1 = P_2 = 0$.

Algorithm 2.1. Sequential convex programming method with monotone line search (SCP_{ls}) for (1.1) under Assumption 1.1

Step 0. *Choose parameters* $c > 0$, $0 < \underline{L} < \bar{\mathrm{L}}$, $\tau > 1$ *and an* x^0 *with* $g(x^0) \le 0$. *Set* $t = 0$.

Step 1. *Pick any* $\xi^t \in \partial P_2(x^t)$.

Step 2. *Choose* $L_f^{t,0} \in [\underline{L}, \bar{\mathrm{L}}]$ *and* $L_g^{t,0} \in [\underline{L}, \bar{\mathrm{L}}]^m$ *arbitrarily. Set* $\widetilde{L}_f = L_f^{t,0}$ *and* $\widetilde{L}_g = L_g^{t,0}$.

Step 3. *Compute*

$$\begin{aligned} \widetilde{x} = \arg\min_x & \left\{ \langle \nabla f(x^t) - \xi^t, x - x^t \rangle + \frac{\widetilde{L}_f}{2}\|x - x^t\|^2 + P_1(x) \right\} \\ \text{s.t.} & \quad \bar{G}(x, x^t, \widetilde{L}_g) \le 0. \end{aligned} \tag{2.10}$$

Step 3a) *If* $g(\widetilde{x}) \le 0$ *and*

$$F(\widetilde{x}) \le F(x^t) - \frac{c}{2}\|\widetilde{x} - x^t\|^2 \tag{2.11}$$

holds, go to step 4.

Step 3b) *If* $g(\widetilde{x}) \not\le 0$, *let* $\widetilde{L}_g \leftarrow \tau \widetilde{L}_g$ *and go to step 3.*

Step 3c) *If* (2.11) *does not hold, let* $\widetilde{L}_f \leftarrow \tau \widetilde{L}_f$ *and go to step 3.*

Step 4. *If a termination criterion is not met, set* $L_g^t = \widetilde{L}_g$, $L_f^t = \widetilde{L}_f$ *and* $x^{t+1} = \widetilde{x}$. *Update* $t \leftarrow t + 1$ *and go to* **Step 1.**

In the next lemma, we discuss the well-definedness of SCP_{ls} and also establish some inequalities needed in our analysis below. Note that the well-definedness of SCP_{ls} was already proved in [83, Theorem 3.6] in a more general setting. Here we include its proof for completeness.

Lemma 2.4. *Consider* (1.1) *and suppose that Assumptions 1.1 and 2.1 hold. Then the following statements hold:*

(i) *SCP_{ls} is well defined, i.e., the subproblems* (2.10) *are well defined and there exists a* $k_0 \in \mathcal{N}_+$ *(independent of* t*) such that in any iteration* $t \ge 0$, *the inner loop stops after at most* k_0 *iterations.*

(ii) *The sequence* $\{(L_f^t, L_g^t)\}$ *generated by SCP_{ls} is bounded.*

(iii) *For each* $i \in \{1, \ldots, m\}$, *each* $t \geq 0$ *and each* $(\widetilde{L}_f, \widetilde{L}_g)$, *the* $\widetilde{R}_i$ *in* (2.9) *is positive.*

(iv) *For each* $t \geq 0$ *and each* $(\widetilde{L}_f, \widetilde{L}_g)$, *the problem* (2.10) *has a Lagrange multiplier* $\widetilde{\lambda}$. *Let* $\widetilde{L}_{fg} := \widetilde{L}_f + \langle \widetilde{\lambda}, \widetilde{L}_g \rangle$ *and let* $\widetilde{x}$ *be as in* (2.10). *Then*

$$\widetilde{\lambda}_i \left(g_i(x^t) + \langle \nabla g_i(x^t), \widetilde{x} - x^t \rangle + \frac{(\widetilde{L}_g)_i}{2} \|\widetilde{x} - x^t\|^2 \right) = 0 \text{ for all } i, \tag{2.12}$$

and

$$0 \in \nabla f(x^t) - \xi^t + \widetilde{L}_{fg}(\widetilde{x} - x^t) + \partial P_1(\widetilde{x}) + \sum_{i=1}^{m} \widetilde{\lambda}_i \nabla g_i(x^t), \tag{2.13}$$

where $\{x^t\}$ *and* $\{\xi^t\}$ *are generated by* SCP_{ls}. *Moreover, if* $g(\widetilde{x}) \leq 0$, *then for any* $x \in \mathbb{R}^n$ *we have*

$$\begin{aligned} F(\widetilde{x}) \leq\, & f(x^t) + \left\langle \nabla f(x^t) - \xi^t, x - x^t \right\rangle + \frac{\widetilde{L}_{fg}}{2}\|x - x^t\|^2 + P_1(x) - P_2(x^t) \\ & + \sum_{i=1}^{m} \widetilde{\lambda}_i \left(g_i(x^t) + \langle \nabla g_i(x^t), x - x^t \rangle \right) - \frac{\widetilde{L}_{fg}}{2}\|x - \widetilde{x}\|^2 - \frac{\widetilde{L}_f - L_f}{2}\|\widetilde{x} - x^t\|^2. \end{aligned} \tag{2.14}$$

Proof. Let an x^t satisfying $g(x^t) \leq 0$ be given for some $t \geq 0$. We will first show that the corresponding subproblems (2.10) are well defined (for any $(\widetilde{L}_f, \widetilde{L}_g)$) and the conclusions of items (iii) and (iv) hold for this t. Using these, we will then show that there exists k_0 (independent of t) so that the inner loop in Step 3 terminates after k_0 iterations and returns an x^{t+1} that satisfies $g(x^{t+1}) \leq 0$. This together with $g(x^0) \leq 0$ and an induction argument will show that SCP$_{ls}$ is well defined and that items (iii) and (iv) hold for all $t \geq 0$. Finally, we show that $\{(L_f^t, L_g^t)\}$ is bounded.

Suppose that an x^t satisfying $g(x^t) \leq 0$ is given for some $t \geq 0$. Notice that for any $(\widetilde{L}_f, \widetilde{L}_g)$, the feasible region of (2.10) is nonempty (it contains x^t) and the subproblem is to minimize a strongly convex continuous function over a nonempty closed convex set. Thus, $\widetilde{x}$ exists and is unique. Now, fix any $i \in \{1, \ldots, m\}$. Since $g(x^t) \leq 0$ and $(\widetilde{L}_g)_i > 0$, we have $-\frac{2}{(\widetilde{L}_g)_i} g_i(x^t) \geq 0$ and thus $\widetilde{R}_i \geq 0$. Suppose to the contrary that $\widetilde{R}_i = 0$. Then we have $\nabla g_i(x^t) = 0$ and $g_i(x^t) = 0$, contradicting Assumption 2.1. Thus, we must have $\widetilde{R}_i > 0$ at the t^{th} iteration.

Next, using a similar proof of [10, Proposition 2.1(iii)], we deduce using MFCQ that the Slater condition holds for (2.10) for this t. Therefore, using [99, Corollary 28.2.1, Theorem 28.3], for problem (2.10), there exists a Lagrange multiplier $\widetilde{\lambda} \in \mathbb{R}^m_+$ such that (2.12) holds at the t^{th} iteration and $\widetilde{x}$ is a minimizer of the following function:

$$
\begin{aligned}
L_t(x, \widetilde{\lambda}) :=& f(x^t) + \left\langle \nabla f(x^t), x - x^t \right\rangle + \frac{\widetilde{L}_f}{2}\|x - x^t\|^2 + P_1(x) - P_2(x^t) \\
& - \left\langle \xi^t, x - x^t \right\rangle + \langle \widetilde{\lambda}, \bar{G}(x, x^t, \widetilde{L}_g) \rangle.
\end{aligned}
$$

This together with [100, Theorem 10.1, Exercise 8.8] shows that (2.13) holds at the t^{th} iteration.

In addition, note that $x \mapsto L_t(x, \widetilde{\lambda})$ is strongly convex with modulus $\widetilde{L}_{fg}$. Then we see that for any $x \in \mathbb{R}^n$,

$$
\begin{aligned}
& f(x^t) + \left\langle \nabla f(x^t), \widetilde{x} - x^t \right\rangle + \frac{\widetilde{L}_f}{2}\|\widetilde{x} - x^t\|^2 + P_1(\widetilde{x}) - P_2(x^t) - \left\langle \xi^t, \widetilde{x} - x^t \right\rangle \\
&= L_t(\widetilde{x}, \widetilde{\lambda}) \leq L_t(x, \widetilde{\lambda}) - \frac{\widetilde{L}_{fg}}{2}\|x - \widetilde{x}\|^2 \\
&= f(x^t) + \left\langle \nabla f(x^t), x - x^t \right\rangle + \frac{\widetilde{L}_{fg}}{2}\|x - x^t\|^2 + P_1(x) - P_2(x^t) - \left\langle \xi^t, x - x^t \right\rangle \\
&\quad + \sum_{i=1}^m \widetilde{\lambda}_i \left(g_i(x^t) + \langle \nabla g_i(x^t), x - x^t \rangle \right) - \frac{\widetilde{L}_{fg}}{2}\|x - \widetilde{x}\|^2,
\end{aligned}
\tag{2.15}
$$

where the first equality makes use of (2.12). On the other hand, since f has Lipschitz continuous gradient (with modulus L_f), if $g(\widetilde{x}) \leq 0$, then we have for any $x \in \mathbb{R}^n$

that

$$
\begin{aligned}
&F(\widetilde{x}) = f(\widetilde{x}) + P_1(\widetilde{x}) - P_2(\widetilde{x})\\
&\le f(x^t) + \left\langle \nabla f(x^t), \widetilde{x} - x^t \right\rangle + \frac{L_f}{2}\|\widetilde{x} - x^t\|^2 + P_1(\widetilde{x}) - P_2(\widetilde{x})\\
&= f(x^t) + \left\langle \nabla f(x^t), \widetilde{x} - x^t \right\rangle + \frac{\widetilde{L}_f}{2}\|\widetilde{x} - x^t\|^2 + P_1(\widetilde{x}) - P_2(\widetilde{x}) - \frac{\widetilde{L}_f - L_f}{2}\|\widetilde{x} - x^t\|^2\\
&\overset{\text{(a)}}{\le} f(x^t) + \left\langle \nabla f(x^t), \widetilde{x} - x^t \right\rangle + \frac{\widetilde{L}_f}{2}\|\widetilde{x} - x^t\|^2 + P_1(\widetilde{x})\\
&\quad - P_2(x^t) - \left\langle \xi^t, \widetilde{x} - x^t \right\rangle - \frac{\widetilde{L}_f - L_f}{2}\|\widetilde{x} - x^t\|^2\\
&\le f(x^t) + \left\langle \nabla f(x^t), x - x^t \right\rangle + \frac{\widetilde{L}_{fg}}{2}\|x - x^t\|^2 + P_1(x) - P_2(x^t) - \left\langle \xi^t, x - x^t \right\rangle\\
&\quad + \sum_{i=1}^m \widetilde{\lambda}_i \left(g_i(x^t) + \langle \nabla g_i(x^t), x - x^t \rangle \right) - \frac{\widetilde{L}_{fg}}{2}\|x - \widetilde{x}\|^2 - \frac{\widetilde{L}_f - L_f}{2}\|\widetilde{x} - x^t\|^2,
\end{aligned}
$$

where (a) uses the convexity of P_2 and the fact that $\xi^t \in \partial P_2(x^t)$, while the last inequality holds due to (2.15). This shows that (2.14) holds at the t^{th} iteration.

Now we show that there exists k_0 (independent of t) so that the inner loop in Step 3 terminates after finitely many iterations at the t^{th} iteration and returns an x^{t+1} satisfying $g(x^{t+1}) \le 0$. To this end, let $k_1 \in \mathcal{N}_+$ be such that $\underline{L}\tau^{k_1} > \max\{\frac{1}{2}(c + L_f), L_{g_1}, \ldots, L_{g_m}\}$. Then k_1 does not depend on t and we have

$$
L_f^{t,0}\tau^{k_1} - \frac{L_f}{2} \ge \underline{L}\tau^{k_1} - \frac{L_f}{2} > \frac{c}{2} \text{ and } (L_g^{t,0})_i\tau^{k_1} \ge \underline{L}\tau^{k_1} \ge L_{g_i} \text{ for } i = 1, \ldots, m. \tag{2.16}
$$

Note that for each i, since g_i has Lipschitz gradient with Lipschitz modulus L_{g_i}, we have for any $(\widetilde{L}_g)_i > 0$ that

$$
\begin{aligned}
g_i(\widetilde{x}) &\le g_i(x^t) + \langle \nabla g_i(x^t), \widetilde{x} - x^t \rangle + \frac{L_{g_i}}{2}\|\widetilde{x} - x^t\|^2\\
&= \bar{G}(\widetilde{x}, x^t, \widetilde{L}_g) + \frac{L_{g_i} - (\widetilde{L}_g)_i}{2}\|\widetilde{x} - x^t\|^2.
\end{aligned}
$$

This together with (2.16) and the update rule of $\widetilde{L}_g$ in Step 3b) shows that after at most k_1 calls of Step 3b), we have $g(\widetilde{x}) \leq 0$. Whenever $\widetilde{x}$ satisfies $g(\widetilde{x}) \leq 0$, we can apply (2.14) with x being x^t to conclude that

$$
\begin{aligned}
F(\widetilde{x}) &\leq f(x^t)+P_1(x^t)-P_2(x^t)+\langle\widetilde{\lambda}, g(x^t)\rangle-\frac{\widetilde{L}_{fg}}{2}\|x^t-\widetilde{x}\|^2-\left[\frac{\widetilde{L}_f-L_f}{2}\right]\|x^t-\widetilde{x}\|^2 \\
&\leq f(x^t)+P_1(x^t)-P_2(x^t)-\frac{\langle\widetilde{\lambda}, \widetilde{L}_g\rangle}{2}\|x^t-\widetilde{x}\|^2-\left[\widetilde{L}_f-\frac{L_f}{2}\right]\|x^t-\widetilde{x}\|^2 \\
&\leq F(x^t)-\left[\widetilde{L}_f-\frac{L_f}{2}\right]\|x^t-\widetilde{x}\|^2,
\end{aligned}
$$

where the second inequality holds because $\widetilde{\lambda} \in \mathbb{R}^n_+$ and $g(x^t) \leq 0$; we also used the fact that $\widetilde{L}_{fg} = \widetilde{L}_f + \langle\widetilde{\lambda}, \widetilde{L}_g\rangle$. Thus, in view of the above two displays, the conditions in Step 3a) must hold when $(\widetilde{L}_g)_i \geq L_{g_i}$ for all i and $\widetilde{L}_f \geq \frac{L_f}{2}$; according to the update rules of $\widetilde{L}_f$ and $\widetilde{L}_g$, this happens after at most k_1 calls of Step 3b) and k_1 calls of Step 3c). Thus, at iteration t, the inner loop stops after at most $k_0 := 2k_1$ iterations and outputs an x^{t+1} satisfying $g(x^{t+1}) \leq 0$.

Finally, since $g(x^0) \leq 0$ to start with, by induction, we know that for any $t \geq 0$, the inner loop stops after at most k_0 iterations. This together with the fact that $\{(L_f^{t,0}, L_g^{t,0})\} \subseteq [\underline{\mathrm{L}}, \bar{\mathrm{L}}]^{m+1}$ implies that $\{(L_f^t, L_g^t)\}$ is bounded. Therefore, SCP$_{ls}$ is well defined and items (ii), (iii) and (iv) hold. This completes the proof. □

2.4 Notation and preliminaries in Chapter 5

We give the definition of (subdifferential) regularity which will be needed in our discussion later ; see [100, Definition 6.4] and [100, Definition 7.25].

Definition 2.3. *A nonempty closed set C is regular at $x \in C$ if $N_C(x) = \widehat{N}_C(x)$, and a proper closed function h is (subdifferentially) regular at $x \in \operatorname{dom} h$ if its epigraph* $\operatorname{epi} h := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \ h(x) \leq t\}$ *is regular at $(x, h(x))$.*

According to [100, Example 7.28], continuously differentiable functions are regular everywhere. Thus, the constraint functions in (1.7) and (1.8) are regular everywhere. In addition, a nonsmooth regular function particularly relevant to our discussion is the objective function of (1.5). Indeed, in view of [87, Corollary 1.111(i)], it holds that:

$$\text{At any } \bar{x} \neq 0,\ \frac{\|\cdot\|_1}{\|\cdot\|} \text{ is regular and } \partial\frac{\|\bar{x}\|_1}{\|\bar{x}\|} = \frac{1}{\|\bar{x}\|}\partial\|\bar{x}\|_1 - \frac{\|\bar{x}\|_1}{\|\bar{x}\|^3}\bar{x}. \tag{2.17}$$

We will also need the following auxiliary lemma concerning the subdifferential of a particular class of functions in our analysis in Section 5.4.

Lemma 2.5. *Let $q = P_1 - P_2$ with $P_1 : \mathbb{R}^n \to \mathbb{R}$ being continuously differentiable and $P_2 : \mathbb{R}^n \to \mathbb{R}$ being convex continuous. Then for any $x \in \mathbb{R}^n$, we have*

$$\partial^\circ q(x) = \nabla P_1(x) - \partial P_2(x). \tag{2.18}$$

Proof. Note that for any $x \in \mathbb{R}^n$, we have

$$\partial^\circ q(x) \overset{\text{(a)}}{=} \nabla P_1(x) + \partial^\circ(-P_2)(x) \overset{\text{(b)}}{=} \nabla P_1(x) - \partial^\circ P_2(x) \overset{\text{(c)}}{=} \nabla P_1(x) - \partial P_2(x),$$

where (a) follows from Corollary 1 of [41, Proposition 2.3.3], (b) holds because of [41, Proposition 2.3.1] and (c) follows from [41, Proposition 2.2.7]. $\square$

Chapter 3

KL Exponents and KL Calculus Rules

In this chapter, we provide a collection of KL calculus rules and provide some examples showing how these rules can be applied to obtain explicit KL exponents.

3.1 KL exponents concerning Lagrangian function

3.1.1 Equality constrained problems

In this section, we consider the following model:

$$g(x) := h(x) + \delta_{G^{-1}\{0\}}(x), \tag{3.1}$$

where $h : \mathbb{X} \to \mathrm{I\!R}$ and $G : \mathbb{X} \to \mathbb{Y}$ be continuously differentiable with $G^{-1}\{0\} \neq \emptyset$ and the linear map $\nabla G(\bar{x}) : \mathbb{Y} \to \mathbb{X}$ being injective.

Theorem 3.1. *Let g be defined as in* (3.1)*. Define the function g_1 by*

$$g_1(x, \lambda) := h(x) + \langle \lambda, G(x) \rangle.$$

Let $\bar{x} \in \operatorname{dom} \partial g$. Then the following statements hold:

(i) *There exists $\epsilon > 0$ so that for each $x \in B(\bar{x}, \epsilon)$, the function $\lambda \mapsto \|\nabla h(x) + \nabla G(x)\lambda\|$ has a unique minimizer.*

(ii) *If* g_1 *satisfies the KL property at* $(\bar{x}, \lambda(\bar{x}))$ *with exponent* α*, then* g *satisfies the KL property at* $\bar{x}$ *with exponent* α*, where* $\lambda(\bar{x})$ *is the unique minimizer of* $\lambda \mapsto \|\nabla h(\bar{x}) + \nabla G(\bar{x})\lambda\|$.

Proof. We first prove (i). Since $\nabla G(\bar{x})$ is an injective linear map and $x \mapsto \nabla G(x)$ is continuous, there exists an $\epsilon > 0$ so that $\nabla G(x)$ is an injective linear map whenever $x \in B(\bar{x}, \epsilon)$. Then statement (i) follows immediately because the function $\lambda \mapsto \|\nabla h(x) + \nabla G(x)\lambda\|$ is minimized if and only if the quantity $\|\nabla h(x) + \nabla G(x)\lambda\|^2$ is minimized, and this latter function is a strongly convex function in λ whenever $x \in B(\bar{x}, \epsilon)$, thanks to the fact that $\nabla G(x)$ is an injective linear map from $\mathbb{Y}$ to $\mathbb{X}$.

We now prove (ii). Let $x \in B(\bar{x}, \epsilon)$ and $\lambda(x)$ denote the unique minimizer of $\lambda \mapsto \|\nabla h(x) + \nabla G(x)\lambda\|$. Then $\lambda(x)$ is also the unique minimizer of $\lambda \mapsto \|\nabla h(x) + \nabla G(x)\lambda\|^2$. Using the first-order optimality condition, we see that $\lambda(x)$ has to satisfy the relation $\nabla G(x)^* \left(\nabla h(x) + \nabla G(x)\lambda(x)\right) = 0$, which gives

$$\lambda(x) = -(\nabla G(x)^*\nabla G(x))^{-1}(\nabla G(x)^*\nabla h(x));$$

here the inverse exists because $\nabla G(x)$ is injective. Since h and G are continuously differentiable, we conclude that λ is a continuous function on $B(\bar{x}, \epsilon)$.

Since g_1 satisfies the KL property at $(\bar{x}, \lambda(\bar{x}))$ with exponent α, there exist $a, \nu, c > 0$ such that whenever $(x, \lambda) \in B\left((\bar{x}, \lambda(\bar{x})), \nu\right)$ and $g_1(\bar{x}, \lambda(\bar{x})) < g_1(x, \lambda) < g_1(\bar{x}, \lambda(\bar{x})) + a$, it holds that

$$\|\nabla g_1(x, \lambda)\| \geq c\left(g_1(x, \lambda) - g_1(\bar{x}, \lambda(\bar{x}))\right)^{\alpha}. \tag{3.2}$$

Next, using [100, Exercise 8.8], for any $x \in B(\bar{x}, \epsilon) \cap \operatorname{dom} \partial g$, we have

$$\partial g(x) = \nabla h(x) + N_{G^{-1}\{0\}}(x) \subseteq \nabla h(x) + \left\{\nabla G(x)\lambda : \ \lambda \in \mathbb{Y}\right\},$$

where the inclusion follows from [100, Corollary 10.50] and the injectivity of $\nabla G(x)$. This implies that for any $x \in B(\bar{x}, \epsilon) \cap \operatorname{dom} \partial g$,

$$\operatorname{dist}\left(0, \partial g(x)\right) \geq \inf_{\lambda} \|\nabla h(x) + \nabla G(x)\lambda\| = \|\nabla h(x) + \nabla G(x)\lambda(x)\|, \tag{3.3}$$

where the equality follows from the definition of $\lambda(x)$ as the unique minimizer.

On the other hand, we have for any $x \in \operatorname{dom} \partial g$ and any λ that

$$\nabla g_1(x, \lambda) = \begin{bmatrix} \nabla h(x) + \nabla G(x)\lambda \\ G(x) \end{bmatrix} = \begin{bmatrix} \nabla h(x) + \nabla G(x)\lambda \\ 0 \end{bmatrix}, \tag{3.4}$$

where the second equality holds because $G(x) = 0$ whenever $x \in \operatorname{dom} \partial g$. Combining (3.4) with (3.3), we then obtain for any $x \in B(\bar{x}, \epsilon) \cap \operatorname{dom} \partial g$ that

$$\operatorname{dist}(0, \partial g(x)) \geq \|\nabla g_1(x, \lambda(x))\|. \tag{3.5}$$

Now, choose $0 < \epsilon' < \min\{\epsilon, \frac{\nu}{\sqrt{2}}\}$ small enough so that when $x \in B(\bar{x}, \epsilon') \cap \operatorname{dom} \partial g$, we have $\|\lambda(x) - \lambda(\bar{x})\| \leq \frac{\nu}{\sqrt{2}}$; such an ϵ' exists thanks to the continuity of $\lambda(\cdot)$. This implies that $(x, \lambda(x)) \in B((\bar{x}, \lambda(\bar{x})), \nu)$ whenever $x \in B(\bar{x}, \epsilon') \cap \operatorname{dom} \partial g$. Therefore, for $x \in B(\bar{x}, \epsilon') \cap \operatorname{dom} \partial g$ with $g(\bar{x}) < g(x) < g(\bar{x}) + a$, we have $(x, \lambda(x)) \in B((\bar{x}, \lambda(\bar{x})), \nu)$ and

$$g_1(\bar{x}, \lambda(\bar{x})) = g(\bar{x}) < g(x) = g_1(x, \lambda(x)) < g(\bar{x}) + a = g_1(\bar{x}, \lambda(\bar{x})) + a.$$

For these x, combining (3.2) with (3.5), we have

$$\operatorname{dist}(0, \partial g(x)) \geq c\left(g_1(x, \lambda(x)) - g_1(\bar{x}, \lambda(\bar{x}))\right)^{\alpha} = c\,(g(x) - g(\bar{x}))^{\alpha},$$

where the equality holds because $G(x) = 0$ whenever $x \in \operatorname{dom} \partial g$. This completes the proof. $\square$

Remark 3.1. *As we shall see in Section 3.5.3, the above relation can be used in deducing the explicit KL exponent of the sum of least squares and the indicator function of a rank constraint.*

3.1.2 Inequality constrained problems

In this subsection, we consider the following multiply constrained optimization problem:

$$\min_{x \in \mathbb{R}^n} P_1(x) + \delta_{G(\cdot) \leq 0}(x), \tag{3.6}$$

where P_1 is convex continuous, the function $G(x) = (g_1(A_1x), \ldots, g_m(A_mx))$ with each $A_i \in \mathbb{R}^{q_i \times n}$ and $g_i : \mathbb{R}^{q_i} \to \mathbb{R}$ being *strictly* convex, and $\{x : \; G(x) \leq 0\} \neq \emptyset$. Define

$$F(x) := P_1(x) + \delta_{G(\cdot)\leq 0}(x) = P_1(x) + \sum_{i=1}^{m} \delta_{g_i(\cdot)\leq 0}(A_ix). \tag{3.7}$$

We will derive rules to deduce the KL exponent of F in (3.7) from its Lagrangian. Similar rules were introduced in [75] and [124], which studied the KL exponent of F in (3.7) respectively when $m = 1$ and when the constraint set is defined by equality constraints, under suitable assumptions. Here, we look at (3.7) that involves multiple inequality constraints.

Theorem 3.2 (**KL exponent of** (3.7) **from its Lagrangian**). *Let F be as in* (3.7) *and $\bar{x} \in \operatorname{Arg\,min} F$. Suppose the following conditions hold:*

(i) *There exists a Lagrange multiplier $\bar{\lambda} \in \mathbb{R}^m_+$ for* (3.6) *and $x \mapsto P_1(x) + \langle \bar{\lambda}, G(x)\rangle$ is a KL function with exponent $\alpha \in (0, 1)$.*

(ii) *The strict complementarity condition holds at $(\bar{x}, \bar{\lambda})$, i.e., for every i satisfying $\bar{\lambda}_i = 0$, it holds that $g_i(A_i\bar{x}) < 0$.*

Then F satisfies the KL property with exponent α at $\bar{x}$.

Proof. Let $F_{\bar{\lambda}}(x) := P_1(x) + \langle \bar{\lambda}, G(x)\rangle$. By the definition of Lagrange multiplier, we have

$$F(\bar{x}) = \inf F = P_1(\bar{x}) = \inf F_{\bar{\lambda}} \leq F_{\bar{\lambda}}(\bar{x}) \leq F(\bar{x}), \tag{3.8}$$

where the second inequality holds because $G(\bar{x}) \leq 0$ and $\bar{\lambda} \in \mathbb{R}^m_+$. On the other hand, thanks to (ii), it holds that $\{i : \bar{\lambda}_i > 0\} = I(\bar{x})$. This together with [99, Theorem 28.1] gives

$$\bar{x} \in \operatorname{Arg\,min} F = \bigcap_{i\in I(\bar{x})} \{x \colon \; g_i(A_ix) = 0\} \cap \bigcap_{i\notin I(\bar{x})} \{x : \; g_i(A_ix) \leq 0\} \cap \operatorname{Arg\,min} F_{\bar{\lambda}}. \tag{3.9}$$

Since g_i is strictly convex and $\bar{\lambda}_i > 0$ for $i \in I(\bar{x})$, we see that $A_i x$ is constant over $\operatorname{Arg\,min} F_{\bar{\lambda}}$ for each $i \in I(\bar{x})$. This together with the fact that $g_i(A_i\bar{x}) = 0$ for $i \in I(\bar{x})$ and (3.9) implies that

$$\bar{x} \in \operatorname{Arg\,min} F = \bigcap_{i \notin I(\bar{x})} \{x :\ g_i(A_i x) \le 0\} \cap \operatorname{Arg\,min} F_{\bar{\lambda}}. \tag{3.10}$$

Next, since $g_i(A_i\bar{x}) < 0$ for each $i \notin I(\bar{x})$, there exists $\epsilon_0 > 0$ such that

$$g_i(A_i x) < 0,\ \forall x \in B(\bar{x}, \epsilon_0),\ \forall\ i \notin I(\bar{x}).$$

This together with (3.10) implies that

$$\bar{x} \in \operatorname{Arg\,min} F \cap B(\bar{x}, \epsilon_0) = \operatorname{Arg\,min} F_{\bar{\lambda}} \cap B(\bar{x}, \epsilon_0). \tag{3.11}$$

Now, using (i) and [22, Theorem 5(i)] together with the fact that $\bar{x} \in \operatorname{Arg\,min} F_{\bar{\lambda}}$, we see that there exist $\bar{a} > 0$, $\bar{c} > 0$ and $0 < \epsilon < \epsilon_0$ such that

$$\operatorname{dist}(x, \operatorname{Arg\,min} F_{\bar{\lambda}}) \le \bar{c}(F_{\bar{\lambda}}(x) - F_{\bar{\lambda}}(\bar{x}))^{1-\alpha} \tag{3.12}$$

whenever $\|x - \bar{x}\| \le \epsilon$ and $F_{\bar{\lambda}}(\bar{x}) \le F_{\bar{\lambda}}(x) < F_{\bar{\lambda}}(\bar{x}) + \bar{a}$. Note that for any x satisfying $F(\bar{x}) < F(x) < F(\bar{x}) + \bar{a}$, we have $g_i(A_i x) \le 0$ for each i and

$$F(\bar{x}) = F_{\bar{\lambda}}(\bar{x}) \le F_{\bar{\lambda}}(x) \le F(x) < F(\bar{x}) + \bar{a} = F_{\bar{\lambda}}(\bar{x}) + \bar{a}, \tag{3.13}$$

where the first and the last equalities follow from (3.8) and the second inequality holds because $\bar{\lambda}_i \ge 0$ and $g_i(A_i x) \le 0$ for each $i = 1, \ldots, m$. Therefore, for any x satisfying $F(\bar{x}) < F(x) < F(\bar{x}) + \bar{a}$ and $\|x - \bar{x}\| \le \epsilon$, we have

$$\begin{aligned}
&\operatorname{dist}(x, \operatorname{Arg\,min} F) \le \operatorname{dist}(x, \operatorname{Arg\,min} F \cap B(\bar{x}, \epsilon_0)) \overset{\text{(a)}}{=} \operatorname{dist}(x, \operatorname{Arg\,min} F_{\bar{\lambda}} \cap B(\bar{x}, \epsilon_0)) \\
&\overset{\text{(b)}}{\le} 4\max\{\operatorname{dist}(x, \operatorname{Arg\,min} F_{\bar{\lambda}}), \operatorname{dist}(x, B(\bar{x}, \epsilon_0))\} \overset{\text{(c)}}{=} 4\operatorname{dist}(x, \operatorname{Arg\,min} F_{\bar{\lambda}}) \\
&\overset{\text{(d)}}{\le} 4\bar{c}(F_{\bar{\lambda}}(x) - F_{\bar{\lambda}}(\bar{x}))^{1-\alpha} \le 4\bar{c}(F(x) - F(\bar{x}))^{1-\alpha},
\end{aligned}$$

where (a) follows from (3.11), (b) follows from [72, Lemma 4.10], (c) holds because $\epsilon < \epsilon_0$, (d) follows from (3.12) and (3.13) and the last inequality holds because of (3.8) (so that $F_{\bar{\lambda}}(\bar{x}) = F(\bar{x})$), $g_i(A_i x) \le 0$ for each i and $\bar{\lambda} \in \mathbb{R}^m_+$. The desired conclusion now follows immediately from this and [22, Theorem 5(ii)]. $\square$

Now, we give a corollary that deals with (3.6) with $m = 1$. This result is different from [75, Theorem 3.5] because, here, it is the constraint function that is a composition of strictly convex function and a linear map, but not the objective function.

Corollary 3.1. *Let F be defined as in (3.7) with $m = 1$. Suppose the following conditions hold:*

(i) *It holds that* $\inf P_1 < \inf F$.

(ii) *There exists a Lagrange multiplier*[1] $\bar{\lambda} \ge 0$ *for* (3.6) *and* $x \mapsto P_1(x) + \bar{\lambda} g_1(A_1 x)$ *is a KL function with exponent* $\alpha \in (0, 1)$.

Then F is KL function with exponent α.

Proof. Let $F_{\bar{\lambda}}(x) := P_1(x) + \bar{\lambda} g_1(A_1 x)$. In view of [75, Lemma 2.1] and the convexity of F, it suffices to show that F has KL property at every point in $\{x : \ 0 \in \partial F(x)\} = \operatorname{Arg\,min} F$ with exponent α. Fix any $\bar{x}$ with $0 \in \partial F(\bar{x})$. Then one can see from condition (i) and the definition of Lagrange multiplier that $\bar{\lambda} > 0$ and thus $g_1(A_1 \bar{x}) = 0$. Therefore, Assumption (ii) of Theorem 3.2 is satisfied. This together with (ii) and Theorem 3.2 shows that F satisfies the KL property at $\bar{x}$ with exponent α. $\square$

Remark 3.2. *When $P_1(\cdot) = \|\cdot\|_1$ in (3.6), we deduce from [75, Corollary 5.1] and Corollary 3.1 that the KL exponent of F in (3.7) is $\frac{1}{2}$ if $m = 1$ and g_1 takes one of the following forms with $b \in \mathbb{R}^q$ and $\delta > 0$ chosen so that the Slater condition holds and the origin is not feasible:*

---

[1] Following [99, Page 274], we say that $\bar{\lambda}$ is a Lagrange multiplier for (3.7) if $\bar{\lambda} \ge 0$ and $\inf_{x \in \mathbb{R}^n} \{P_1(x) + \bar{\lambda} g_1(Ax)\} = \inf_{x \in \mathbb{R}^n} \{P_1(x) + \delta_{g_1(\cdot) \le 0}(A_1 x)\} > -\infty$.

(i) *(Basis pursuit denoising [32])* $g_1(z) = \frac{1}{2}\|z - b\|^2 - \delta$.

(ii) *(Logistic loss [60, 65])* $g_1(z) = \sum_{i=1}^q \log(1 + \exp(b_i z_i)) - \delta$ *for some* $b \in \mathbb{R}^q$.

(iii) *(Poisson loss [56, 67, 130])* $g_1(z) = \sum_{i=1}^q (-b_i z_i + \exp(z_i)) - \delta$ *for some* $b \in \mathbb{R}^q$.

3.2 KL exponent of fractional functions

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be proper closed and $g : \mathbb{R}^n \to \mathbb{R}$ be a continuous nonnegative function that is continuously differentiable on an open set containing $\operatorname{dom} f$. Suppose that $\inf f \geq 0$ and $\inf_{\operatorname{dom} f} g > 0$. We consider the following fractional programming problem:

$$\min_x G(x) := \frac{f(x)}{g(x)}. \tag{3.14}$$

In algorithmic developments for solving (3.14) (see, for example, [42, 46]), it is customary to consider functions of the following form

$$H_u(x) := f(x) - \frac{f(u)}{g(u)} g(x), \tag{3.15}$$

where u typically carries information from the previous iterate. In the literature, KL-type assumptions are usually imposed on G or H_u for establishing the global convergence of the sequence generated by first-order methods for solving (3.14); see, for example, the discussions in [28, Theorem 16] and [29, Theorem 5.5]. Here, we study a relationship between the KL exponent of G in (3.14) and that of $H_{\bar{x}}$ in (3.15) when $\bar{x}$ is a stationary point of G.

Theorem 3.3 (KL exponent of fractional functions)**.** *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ *be a proper closed function with* $\inf f \geq 0$ *and* $g : \mathbb{R}^n \to \mathbb{R}$ *be a continuous nonnegative function that is continuously differentiable on an open set containing* $\operatorname{dom} f$ *with* $\inf_{\operatorname{dom} f} g > 0$. *Assume that one of the following conditions hold:*

(i) *f is locally Lipschitz.*

(ii) *$f = h + \delta_D$ for some continuously differentiable function h and nonempty closed set D.*

(iii) *$f = h + \delta_D$ for some locally Lipschitz function h and nonempty closed set D, and h and D are regular at every point in D.*

Let $\bar{x}$ be such that $0 \in \partial G(\bar{x})$, where G is defined as in (3.14). Then $\bar{x} \in \operatorname{dom} \partial H_{\bar{x}}$. If $H_{\bar{x}}$ defined as in (3.15) satisfies the KL property with exponent $\theta \in [0, 1)$ at $\bar{x}$, then so does G.

Proof. It is clear that $\operatorname{dom} H_{\bar{x}} = \operatorname{dom} f = \operatorname{dom} G$. We first argue that under the assumptions on f and g, we have for any $x \in \operatorname{dom} G$ that

$$\partial H_{\bar{x}}(x) = \partial f(x) - G(\bar{x})\nabla g(x) \quad \text{and} \quad \partial G(x) = \frac{1}{g(x)}\left(\partial f(x) - G(x)\nabla g(x)\right). \tag{3.16}$$

Indeed, in all cases, the first relation in (3.16) follows from [100, Exercise 8.8(c)]. When f is locally Lipschitz, the second relation in (3.16) follows from [87, Corollary 1.111(i)]. When $f = h + \delta_D$ for some continuously differentiable function h and nonempty closed set D, the second relation in (3.16) follows by first applying [100, Exercise 8.8(c)] to $G = \frac{h}{g} + \delta_D$, then applying the usual quotient rule to the differentiable function $\frac{h}{g}$, and subsequently using $\partial f = \nabla h + \partial \delta_D$ (thanks to [100, Exercise 8.8(c)]). Finally, when $f = h + \delta_D$ for some locally Lipschitz function h and nonempty closed set D with h and D being regular at every point in D, we have that the function $\frac{h}{g}$ is regular for all $x \in D$ in view of [87, Corollary 1.111(i)]. This together with the regularity of

D gives

$$\begin{aligned}\partial G(x) &= \partial\left(\frac{h}{g}\right)(x) + \partial\delta_D(x)\\ &= \frac{g(x)\partial h(x) - h(x)\nabla g(x)}{g(x)^2} + \partial\delta_D(x)\\ &= \frac{g(x)\partial f(x) - f(x)\nabla g(x)}{g(x)^2},\end{aligned}$$

where the first and the last equalities follow from [100, Corollary 10.9] and [100, Exercise 8.14], and the second equality follows from [87, Corollary 1.111(i)].

Now, in view of (3.16), we have $\operatorname{dom}\partial H_{\bar{x}} = \operatorname{dom}\partial f = \operatorname{dom}\partial G$. In addition, in all three cases, it holds that $\operatorname{dom} f = \operatorname{dom}\partial f$. Indeed, when f is locally Lipschitz, this claim follows from Exercise 8(c) of [25, Section 6.4]. When $f = h + \delta_D$ as in (ii), the claim follows from [100, Exercise 8.8(c)], while for case (iii), we have $\operatorname{dom} f = \operatorname{dom}\partial f = D$ in view of [100, Corollary 10.9], [100, Exercise 8.14] and Exercise 8(c) of [25, Section 6.4]. Consequently, in all three cases, we have

$$\Xi := \operatorname{dom} G = \operatorname{dom}\partial G = \operatorname{dom} H_{\bar{x}} = \operatorname{dom}\partial H_{\bar{x}} = \operatorname{dom} f = \operatorname{dom}\partial f,$$

and $H_{\bar{x}}$ is continuous relative to Ξ. In particular, $\bar{x} \in \operatorname{dom}\partial G = \operatorname{dom}\partial H_{\bar{x}}$.

Let U be the open set containing $\operatorname{dom} f$ on which g is continuously differentiable. Since $H_{\bar{x}}$ satisfies the KL property with exponent θ at $\bar{x}$ and is continuous relative to Ξ, there exist $\epsilon > 0$ and $c > 0$ so that $B(\bar{x}, 2\epsilon) \subseteq U$ and

$$\operatorname{dist}(0, \partial H_{\bar{x}}(x)) \geq c(H_{\bar{x}}(x) - H_{\bar{x}}(\bar{x}))^{\theta} = c(H_{\bar{x}}(x))^{\theta} \tag{3.17}$$

whenever $x \in \Xi$, $H_{\bar{x}}(x) > 0$ and $\|x - \bar{x}\| \leq \epsilon$. Let $M := \sup_{\|x-\bar{x}\|\leq\epsilon} \max\{g(x), \|\nabla g(x)\|\}$, which is finite as g is continuously differentiable on $U \supseteq B(\bar{x}, 2\epsilon)$. Using the facts that $\theta \in [0, 1)$, $H_{\bar{x}}$ is continuous relative to Ξ, $H_{\bar{x}}(\bar{x}) = 0$ and $\inf_{\operatorname{dom} f} g > 0$, we deduce that there exists $\epsilon' \in (0, \epsilon)$ such that

$$|H_{\bar{x}}(x)|^{1-\theta} \leq \frac{c \inf_{\operatorname{dom} f} g}{2M} \quad \text{whenever } \|x - \bar{x}\| \leq \epsilon' \text{ and } x \in \Xi, \tag{3.18}$$

where c is given in (3.17).

Now, consider any $x \in \Xi$ satisfying $\|x - \bar{x}\| \leq \epsilon'$ and $G(\bar{x}) < G(x) < G(\bar{x}) + \epsilon'$. Then we have from (3.16) that

$$\begin{aligned}
&\operatorname{dist}(0, \partial G(x)) = \frac{1}{g(x)} \inf_{\xi \in \partial f(x)} \|\xi - G(x)\nabla g(x)\| \overset{\text{(a)}}{\geq} \frac{1}{M} \inf_{\xi \in \partial f(x)} \|\xi - G(x)\nabla g(x)\| \\
&\overset{\text{(b)}}{\geq} \frac{1}{M} \inf_{\xi \in \partial f(x)} \|\xi - G(\bar{x})\nabla g(x)\| - \frac{1}{M}|G(x) - G(\bar{x})|\|\nabla g(x)\| \\
&\overset{\text{(c)}}{\geq} \frac{1}{M} \inf_{\xi \in \partial f(x)} \|\xi - G(\bar{x})\nabla g(x)\| - (G(x) - G(\bar{x})) \\
&= \frac{1}{M}\operatorname{dist}(0, \partial H_{\bar{x}}(x)) - \frac{1}{g(x)} H_{\bar{x}}(x) \overset{\text{(d)}}{\geq} \frac{1}{M}\operatorname{dist}(0, \partial H_{\bar{x}}(x)) - \frac{1}{\inf_{\operatorname{dom} f} g} H_{\bar{x}}(x) \\
&\overset{\text{(e)}}{\geq} \frac{c}{M}(H_{\bar{x}}(x))^\theta - \frac{1}{\inf_{\operatorname{dom} f} g} H_{\bar{x}}(x) \overset{\text{(f)}}{\geq} \frac{c}{2M}(H_{\bar{x}}(x))^\theta \\
&= \frac{c(g(x))^\theta}{2M}(G(x) - G(\bar{x}))^\theta \overset{\text{(g)}}{\geq} \frac{c(\inf_{\operatorname{dom} f} g)^\theta}{2M}(G(x) - G(\bar{x}))^\theta,
\end{aligned}$$

where (a) holds because $g(x) \leq M$, (b) follows from the triangle inequality, (c) holds because $\|\nabla g(x)\| \leq M$ and $G(x) > G(\bar{x})$, (d) holds because $H_{\bar{x}}(x) > 0$ (thanks to $G(x) > G(\bar{x})$), (e) then follows from (3.17) and (f) follows from (3.18) and the fact that $H_{\bar{x}}(x) > 0$. Finally, (g) holds because $G(x) > G(\bar{x})$. This completes the proof. □

Remark 3.3. *As we shall see in Section 5.2, the above fractional rule plays a key role in deducing the linear convergence of the sequence generated by the algorithm proposed in [116, Eq. 11] for ℓ_1/ℓ_2 minimization problem.*

3.3 KL exponent via inf-projection

In this section, we study how the KL exponent behaves under inf-projection. Specifically, given a proper closed function $F : \mathbb{X} \times \mathbb{Y} \to \mathbb{R} \cup \{\infty\}$ with known KL exponent, we would like to deduce the KL exponent of $\inf_{y \in \mathbb{Y}} F(\cdot, y)$ under suitable assumptions.

Theorem 3.4 (**KL exponent via inf-projection**). *Let $F : \mathbb{X} \times \mathbb{Y} \to \mathbb{R} \cup \{\infty\}$ be a proper closed function and define $f(x) := \inf_{y \in \mathbb{Y}} F(x, y)$ and $Y(x) := \operatorname{Arg\,min}_{y \in \mathbb{Y}} F(x, y)$ for $x \in \mathbb{X}$. Suppose that the function F is level-bounded in y locally uniformly in x. Let $\alpha \in [0, 1)$ and $\bar{x} \in \operatorname{dom} \partial f$.*[2] *Suppose in addition the following conditions hold:*

(i) *It holds that $\partial F(\bar{x}, \bar{y}) \neq \emptyset$ for all $\bar{y} \in Y(\bar{x})$.*

(ii) *The function F satisfies the KL property with exponent α at every point in $\{\bar{x}\} \times Y(\bar{x})$.*

Then f satisfies the KL property at $\bar{x}$ with exponent α.

Proof. Using the nonemptiness and compactness of $Y(\bar{x})$ given by Lemma 2.1(i), and the facts that $F(x, y) \equiv f(\bar{x})$ on $\Omega := \{\bar{x}\} \times Y(\bar{x}) \subseteq \operatorname{dom} \partial F$ and F satisfies the KL property with exponent α at every point in Ω, we deduce from Lemma 2.2 that there exist $\nu, a, c > 0$ such that

$$\operatorname{dist}\,(0, \partial F(x, y)) \geq c\,(F(x, y) - f(\bar{x}))^{\alpha} \tag{3.19}$$

for any (x, y) satisfying

$$f(\bar{x}) < F(x, y) < f(\bar{x}) + a \quad \text{and} \quad \operatorname{dist}((x, y), \Omega) < \nu. \tag{3.20}$$

By decreasing a if necessary, without loss of generality, we may assume $a \in (0, 1)$.

Next, using Lemma 2.1(iv), we see that there exists $\epsilon \in (0, \min\{\nu/2, a\})$ such that

$$\operatorname{dist}(y, Y(\bar{x})) \leq \frac{\nu}{2}$$

whenever $y \in Y(x)$ with $x \in B(\bar{x}, \epsilon) \cap \operatorname{dom} \partial f$ and $f(\bar{x}) < f(x) < f(\bar{x}) + \epsilon$. Hence, for any $x \in B(\bar{x}, \epsilon) \cap \operatorname{dom} \partial f$ with $f(\bar{x}) < f(x) < f(\bar{x}) + \epsilon$ and any $y \in Y(x)$, we have

$$\operatorname{dist}((x, y), \Omega) \leq \|x - \bar{x}\| + \operatorname{dist}(y, Y(\bar{x})) \leq \epsilon + \frac{\nu}{2} < \nu,$$

[2] Here, f is a proper closed function, thanks to Lemma 2.1(i).

where the last inequality follows from the choice of ϵ. The above relation together with the fact that $\epsilon < a$ shows that the relation (3.20) holds for any such x and any $y \in Y(x)$. Thus, using (3.19) we conclude that for any such x and any $y \in Y(x)$,

$$
\begin{aligned}
&\operatorname{dist}(0, \partial f(x)) = \operatorname{dist}\left(0, \begin{bmatrix} \partial f(x) \\ 0 \end{bmatrix}\right) \geq \inf_{y \in Y(x)} \operatorname{dist}(0, \partial F(x,y)) \\
&\geq \inf_{y \in Y(x)} c\,(F(x,y) - f(\bar{x}))^{\alpha} = c\,(f(x) - f(\bar{x}))^{\alpha},
\end{aligned}
$$

where the first inequality follows from (2.3) and the last equality follows from the definition of $Y(x)$. This completes the proof. $\square$

Theorem 3.4 can be viewed as a generalization of [75, Theorem 3.1], which studies the KL exponent of the minimum of finitely many proper closed functions with known KL exponents. Indeed, let f_i, $1 \leq i \leq m$, be proper closed functions. If we let $\mathbb{Y} = \mathbb{R}$ and define $F : \mathbb{X} \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ by

$$
F(x,y) = \begin{cases} f_y(x) & \text{if } y = 1, 2, \ldots, m, \\ \infty & \text{otherwise,} \end{cases} \tag{3.21}
$$

then it is not hard to see that this F is a proper closed function, and $\inf_{y \in \mathbb{R}} F(x,y) = \min_{1 \leq i \leq m} f_i(x)$ for all $x \in \mathbb{X}$. Moreover, one can check directly from the definition that

$$
\partial F(x,y) = \begin{cases} \partial f_y(x) \times \mathbb{R} & \text{if } y = 1, 2, \ldots, m, \\ \emptyset & \text{otherwise.} \end{cases} \tag{3.22}
$$

Thus, we have the following immediate corollary of Theorem 3.4, which is a slight generalization of [75, Theorem 3.1] by dropping the continuity assumption on $\min_{1 \leq i \leq m} f_i$.

Corollary 3.2 (**KL exponent for minimum of finitely many functions**). *Let f_i, $1 \leq i \leq m$, be proper closed functions, and define $f := \min_{1 \leq i \leq m} f_i$. Let $\bar{x} \in \operatorname{dom} \partial f \cap \bigcap_{i \in I(\bar{x})} \operatorname{dom} \partial f_i$, where $I(\bar{x}) := \{i : f_i(\bar{x}) = f(\bar{x})\}$. Suppose that for each*

$i \in I(\bar{x})$, the function f_i satisfies the KL property at $\bar{x}$ with exponent $\alpha_i \in [0,1)$. Then f satisfies the KL property at $\bar{x}$ with exponent $\alpha = \max\{\alpha_i : i \in I(\bar{x})\}$.

Proof. Define F as in (3.21). Then F is proper and closed, and $f(x) = \inf_{y \in \mathbb{R}} F(x,y)$. Moreover, $I(x) = Y(x) := \operatorname{Arg\,min}_{y\in\mathbb{R}} F(x,y)$. It is clear that this F is level-bounded in y locally uniformly in x. Moreover, in view of (3.22) and the assumption that $\bar{x} \in \bigcap_{i\in I(\bar{x})} \operatorname{dom} \partial f_i$, we see that $\partial F(\bar{x}, \bar{y}) \neq \emptyset$ whenever $\bar{y} \in Y(\bar{x})$. Finally, it is routine to show that F satisfies the KL property with exponent α_i at $(\bar{x}, i)$ for $i \in I(\bar{x})$. Thus, F satisfies the KL property with exponent $\alpha = \max\{\alpha_i : i \in I(\bar{x})\}$ on $\{\bar{x}\} \times I(\bar{x})$. The desired conclusion now follows from Theorem 3.4. $\square$

The next corollary can be proved similarly as [75, Corollary 3.1] by using Corollary 3.2 in place of [75, Theorem 3.1].

Corollary 3.3. *Let f_i, $1 \leq i \leq m$, be proper closed functions with $\operatorname{dom} f_i = \operatorname{dom} \partial f_i$ for all i, and define $f := \min_{1\leq i\leq m} f_i$. Suppose that for each i, the function f_i is a KL function with exponent $\alpha_i \in [0,1)$. Then f is a KL function with exponent $\alpha = \max\{\alpha_i : 1 \leq i \leq m\}$.*

Finally, we show in the next corollary that one can relax some conditions of Theorem 3.4 when F is in addition convex.

Corollary 3.4 (KL exponent via inf-projections under convexity). *Let $F : \mathbb{X} \times \mathbb{Y} \to \mathbb{R} \cup \{\infty\}$ be a proper closed convex function and define $f(x) := \inf_{y\in\mathbb{Y}} F(x,y)$ and $Y(x) := \operatorname{Arg\,min}_{y\in\mathbb{Y}} F(x,y)$ for $x \in \mathbb{X}$. Suppose there exists $\bar{u}$ such that $f(\bar{u}) \in \mathbb{R}$ and $Y(\bar{u})$ is nonempty and compact. Then the following statements hold:*

(i) *The function f is proper and closed, and $Y(x)$ is nonempty and compact for any $x \in \operatorname{dom} \partial f$.*

(ii) *It holds that $\partial F(x,y) \neq \emptyset$ for all $x \in \operatorname{dom} \partial f$ and $y \in Y(x)$.*

(iii) *If* $\bar{x} \in \operatorname{dom} \partial f$, $\alpha \in [0, 1)$ *and the function* F *satisfies the KL property with exponent* α *at every point in* $\{\bar{x}\} \times Y(\bar{x})$*, then* f *satisfies the KL property at* $\bar{x}$ *with exponent* α*.*

Proof. For (i), we first show that F is level-bounded in y locally uniformly in x. Suppose to the contrary that there exist $x_0 \in \mathbb{X}$ and $\beta \in \mathbb{R}$ so that $\mathfrak{C} := \{(x, y) : x \in B(x_0, 1) \text{ and } F(x, y) \leq \beta\}$ is unbounded. Then there exists $\{(x^k, y^k)\} \subset \mathfrak{C}$ with $\|y^k\| \to \infty$. By passing to a subsequence if necessary, we may assume $\lim_{k\to\infty} \frac{y^k}{\|y^k\|} = d$ for some d with $\|d\| = 1$. Since $F(x^k, y^k) \leq \beta$ and $\{x^k\} \subset B(x_0, 1)$ is bounded, we have

$$F^\infty(0, d) \leq \liminf_{k\to\infty} \frac{F(x^k, y^k)}{\|(x^k, y^k)\|} \leq \liminf_{k\to\infty} \frac{\beta}{\|(x^k, y^k)\|} = 0,$$

where F^∞ is the asymptotic function of F and the first inequality follows from [9, Theorem 2.5.1]. This together with the convexity of F and [9, Proposition 2.5.2] shows that

$$F(x, y + td) \leq F(x, y) \quad \text{for all } t > 0 \text{ and for all } (x, y) \in \operatorname{dom} F.$$

Since $Y(\bar{u}) \neq \emptyset$ and $f(\bar{u}) \in \mathbb{R}$, we have $\{\bar{u}\} \times Y(\bar{u}) \subseteq \operatorname{dom} F$. Hence, we can take $\bar{v} \in Y(\bar{u})$ and set $x = \bar{u}$ and $y = \bar{v}$ in the above display to conclude that $F(\bar{u}, \bar{v} + td) \leq F(\bar{u}, \bar{v})$ for all $t > 0$. This further implies that $\bar{v} + td \in Y(\bar{u})$ for all $t > 0$, which contradicts the compactness of $Y(\bar{u})$. Thus, for any $x_0 \in \mathbb{X}$ and $\beta \in \mathbb{R}$, the set $\{(x, y) : x \in B(x_0, 1) \text{ and } F(x, y) \leq \beta\}$ is bounded. Using Lemma 2.1(i), we see that (i) holds.

Next, we prove (ii). To this end, fix any $u \in \operatorname{dom} \partial f$ and $v \in Y(u)$. Note that the function f is convex as inf-projection of the convex function F; see [100, Proposition 2.22(a)]. Now, for the proper convex function f, we have from the definition that $f^*(w) = \sup_x\{\langle w, x\rangle - f(x)\} = \sup_{x,y}\{\langle w, x\rangle - F(x, y)\} = F^*(w, 0)$

for any $w \in \mathbb{X}$. Taking a $\bar{w} \in \partial f(u)$ and using (2.1), we see further that for any $v \in Y(u)$,

$$F(u,v) + F^*(\bar{w},0) = f(u) + f^*(\bar{w}) = \langle u, \bar{w}\rangle,$$

where the equality $F(u,v) = f(u)$ holds because $v \in Y(u)$. In view of (2.1), the above relation further implies that $(\bar{w},0) \in \partial F(u,v)$. This proves (ii).

Now, suppose in addition that $\bar{x} \in \operatorname{dom} \partial f$, $\alpha \in [0,1)$ and the function F satisfies the KL property with exponent α at every point in $\{\bar{x}\} \times Y(\bar{x})$. Recall that we have shown that F is level-bounded in y locally uniformly in x in the proof of item (i) and we have $\{\bar{x}\} \times Y(\bar{x}) \subseteq \operatorname{dom} \partial F$ from item (ii). The conclusion (iii) now follows by applying Theorem 3.4. □

Remark 3.4. *In addition to the inf-projection, another closely related operation, which appears frequently in optimization, would be taking the supremum over a family of functions. However, we would like to point out that, as opposed to the inf-projection, the supremum operation may not preserve KL exponents. For example, consider* $F : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $F = \max\{f_1, f_2\}$ *with* $f_1(x) = x_1^2$ *and* $f_2(x) = (x_1+1)^2 + x_2^2 - 1$. *Clearly,* f_1 *and* f_2 *are both quadratic and are KL functions with exponent* $\frac{1}{2}$. *On the other hand, it was shown in [63, Page 1617] that* F *has an optimal solution at* $(0,0)$ *and the KL exponent of* F *at* $(0,0)$ *is* $\frac{3}{4}$ *and cannot be* $\frac{1}{2}$. *It would be of interest to see, under what additional conditions, the supremum operation can preserve the KL exponents. This could be one interesting future research direction.*

3.3.1 Optimization models that can be represented as inf-projections

Inf-projection is ubiquitous in optimization. In this section, we present some commonly encountered models that can be written as inf-projections. This includes a large class of semidefinite-programming-representable (SDP-representable) functions, rank constrained least squares problems, and Bregman envelopes. These are important convex and nonconvex models whose explicit KL exponents were out of reach in previous studies. In Sections 3.4 and 3.5, we will study their KL exponents based on their inf-projection representations, Theorem 3.4 and Corollary 3.4.

Convex models that can be written as inf-projections

(i) SDP-representable functions Following [58, Eq. (1.3)], we say that a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, is semidefinite-programming-representable (SDP-representable) if its epigraph can be expressed as the feasible region of some SDP problems, i.e.,

$$\operatorname{epi} f = \left\{(x,t)\in\mathbb{R}^n\times\mathbb{R} :\ \exists u\in\mathbb{R}^N \ \text{ s.t. } A_{00}+A_0t+\sum_{i=1}^n A_ix_i+\sum_{j=1}^N B_ju_j \succeq 0\right\} \tag{3.23}$$

for some $\{A_{00}, A_0, A_1, \ldots, A_n, B_1, \ldots, B_N\} \subset \mathcal{S}^d$, $d \geq 1$ and $N \geq 1$. These functions arise in various applications and include important examples such as least squares loss functions, ℓ_1 norm, and nuclear norm, etc; see, for example, [16, Section 4.2] for more discussions. Using the symmetric matrices in (3.23), we define a linear map $\mathcal{A} : \mathcal{S}^d \to \mathbb{R}^{n+N+1}$ as

$$\mathcal{A}(W) := [\langle A_1, W\rangle \ \cdots \ \langle A_n, W\rangle \ \langle B_1, W\rangle \ \cdots \ \langle B_N, W\rangle \ \langle A_0, W\rangle]^T . \tag{3.24}$$

Then it is routine to show that $\mathcal{A}^* : \mathbb{R}^{n+N+1} \to \mathcal{S}^d$ is given by $\mathcal{A}^*(x,u,t) = A_0t + \sum_{i=1}^n A_ix_i + \sum_{j=1}^N B_ju_j$ for $(x,u,t) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}$. Now, if we define

$$F(x,u,t) := t + \delta_{\mathfrak{D}}(x,u,t) \ \text{ with } \ \mathfrak{D} = \{(x,u,t) :\ A_{00} + \mathcal{A}^*(x,u,t) \succeq 0\}\,, \tag{3.25}$$

then it holds that $f(x) = \inf_{u,t} F(x,u,t)$ for all $x \in \mathbb{R}^n$. We will show in Theorem 3.5 (using Corollary 3.4) that a proper closed SDP-representable function has KL property with exponent $\frac{1}{2}$ at points satisfying suitable assumptions on the SDP representation of F in (3.25).

(ii) Sum of LMI-representable functions We say that a function $h : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, is LMI-representable (see [58, Eq. (1.1)]) if there exist symmetric matrices A_{00}, A_j, $j = 0, \ldots, n$, such that

$$\operatorname{epi} h = \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R} : \ A_{00} + \sum_{j=1}^{n} A_j x_j + A_0 t \succeq 0 \right\}.$$

It is clear that LMI-representable functions form a special class of SDP-representable functions. Many commonly used functions are LMI-representable such as the least squares loss function, the ℓ_1, ℓ_2, ℓ_∞ norm functions, the indicator functions of their corresponding norm balls, and the indicator function of the matrix operator norm ball, etc.

Let $f = \sum_{i=1}^m f_i$ be the sum of m proper closed LMI-representable functions. In Theorem 3.6, we show that f has KL property with exponent $\frac{1}{2}$ at points under suitable assumptions. Different from Theorem 3.5, which imposes the "strict complementarity condition" on the corresponding F in (3.25), Theorem 3.6 *directly* imposes such kind of condition on the original function f. Explicit optimization models which can be written as sum of LMI-representable functions include (non-overlapping) group Lasso and group fused Lasso, and are discussed in Example 3.1.

(iii) Sum of LMI-representable functions and the nuclear norm In various applications, the nuclear norm has been used for inducing low rank of solutions; see, for example, [98] for more discussions. Noticing that the nuclear norm is a special SDP-representable function, we further consider the sum of LMI-representable functions and the nuclear norm:

$$f(X) := \sum_{k=1}^{p} f_k(X) + \|X\|_*, \tag{3.26}$$

where $X \in \mathbb{R}^{m\times n}$, $\|X\|_*$ denotes the nuclear norm of X (the sum of all singular values of X) and each $f_k : \mathbb{R}^{m\times n} \to \mathbb{R} \cup \{\infty\}$ is a proper closed LMI-representable function. Define a function $F : \mathcal{S}^{n+m} \to \mathbb{R} \cup \{\infty\}$ by

$$F(Z) := \sum_{k=1}^{p} f_k(X) + \frac{1}{2}(\mathrm{tr}(U) + \mathrm{tr}(V)) + \delta_{\mathcal{S}_+^{m+n}}(Z); \tag{3.27}$$

here, we partition the matrix variable $Z \in \mathcal{S}^{n+m}$ as follows:

$$Z = \begin{bmatrix} U & X \\ X^T & V \end{bmatrix}, \tag{3.28}$$

where $U \in \mathcal{S}^m$, $V \in \mathcal{S}^n$ and $X \in \mathbb{R}^{m\times n}$. Then one can show that $f(X) = \inf_{U,V} F(Z)$; see (3.62) below. In Theorem 3.7, we will show that f in (3.26) satisfies KL property with exponent $\frac{1}{2}$ at points $\bar{X}$ such that $0 \in \mathrm{ri}\,\partial f(\bar{X})$, under mild conditions. Explicit optimization models of the form (3.26) are introduced in Remark 3.7.

(iv) Convex models with C^2-cone reducible structure SDP representable functions are all semi-algebraic. As an attempt to go beyond semi-algebraicity, we analyze functions involving C^2-cone reducible structure. Specifically, we consider the following function $f : \mathbb{X} \to \mathbb{R} \cup \{\infty\}$:

$$f(x) := \ell(\mathcal{A}x) + \langle v, x\rangle + \gamma(x), \tag{3.29}$$

where γ is a closed gauge[3] whose polar gauge[4] is C^2-cone reducible, the function $\ell : \mathbb{Y} \to \mathbb{R}$ is strongly convex on any compact convex set and has locally Lipschitz gradient, $\mathcal{A} : \mathbb{X} \to \mathbb{Y}$ is a linear map, and $v \in \mathbb{X}$.

Notice that $f(x) = \inf_t F(x,t)$, where

$$F(x,t) := \ell(\mathcal{A}x) + \langle v, x\rangle + t + \delta_{\mathfrak{D}}(x,t), \tag{3.30}$$

[3] A gauge is a nonnegative positively homogeneous convex function that vanishes at the origin.

[4] See [55, Proposition 2.1(iii)].

with $\mathfrak{D} = \{(x,t) \in \mathbb{X} \times \mathbb{R} : \gamma(x) \le t\}$. In Section 3.4.4, we will deduce that f in (3.29) has KL property with exponent $\frac{1}{2}$ at points satisfying assumptions involving relative interior of some subdifferential sets; see Corollary 3.5. Optimization models in the form of (3.29) are presented in Example 3.2.

Nonconvex optimization models that can be written as inf-projections

(i) Difference-of-convex functions We consider difference-of-convex (DC) functions of the following form:

$$f(x) = P_1(x) - P_2(\mathcal{A}x), \tag{3.31}$$

where $P_1 : \mathbb{X} \to \mathbb{R} \cup \{\infty\}$ is a proper closed convex function, $P_2 : \mathbb{Y} \to \mathbb{R}$ is a continuous convex function and $\mathcal{A} : \mathbb{X} \to \mathbb{Y}$ is a linear map. These functions arise in many contemporary applications including compressed sensing; see, for example, [2, 112, 119, 123] and references therein. In the literature, the following function is a typically used majorant for designing and analyzing algorithms for minimizing DC functions. It is obtained from (3.31) by majorizing the concave function $-P_2$ using the Fenchel conjugate P_2^* of P_2:

$$F(x,y) = P_1(x) - \langle \mathcal{A}x, y \rangle + P_2^*(y). \tag{3.32}$$

Note that $f(x) = \inf_y F(x,y)$ thanks to the definition of Fenchel conjugate and [99, Theorem 12.2]. In Theorem 3.9, we will deduce the KL exponent of f in (3.31) from that of F in (3.32).

(ii) Bregman envelope The Bregman envelope of a proper closed function $f : \mathbb{X} \to \mathbb{R} \cup \{\infty\}$, is defined in [15] as follows:

$$F_\phi(x) := \inf_y \{f(y) + \mathbb{B}_\phi(y,x)\} \tag{3.33}$$

where $\phi : \mathbb{X} \to \mathbb{R}$ is a differentiable convex function and

$$\mathbb{B}_\phi(y,x) = \phi(y) - \phi(x) - \langle \nabla\phi(x), y - x \rangle \tag{3.34}$$

is the Bregman distance. Note that F_ϕ is an inf-projection by definition. In Section 3.5.2, we will show that if ϕ satisfies Assumption 3.1 and f is a KL function with exponent $\alpha \in (0,1]$ and satisfies $\inf f > -\infty$, then F_ϕ in (3.33) is also a KL function with exponent $\alpha \in (0,1]$. As we shall see in Remark 3.9, the F_ϕ with ϕ satisfying Assumption 3.1 covers the widely studied Moreau envelope (see, for example, [100, Section 1G]) and the recently proposed forward-backward envelope [104].

(iii) Least squares loss function with rank constraint Consider the following least squares loss function with rank constraint:

$$f(X) := \frac{1}{2}\|\mathcal{A}X - b\|^2 + \delta_{\mathrm{rank}(\cdot)\le k}(X), \tag{3.35}$$

where $X \in \mathbb{R}^{m\times n}$, $\mathcal{A} : \mathbb{R}^{m\times n} \to \mathbb{R}^p$ is a linear map, $b \in \mathbb{R}^p$ and k is an integer between 1 and $\min\{m,n\} - 1$. The model above is considered in many applications such as principal components analysis (PCA); see [113] for more details. Notice that f in (3.35) is an inf-projection in the following form:

$$f(X) = \inf_U \left\{\frac{1}{2}\|\mathcal{A}X - b\|^2 + \delta_{\widehat{\mathfrak{D}}}(X,U)\right\}, \tag{3.36}$$

where

$$\widehat{\mathfrak{D}} := \{(X,U) \in \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times(m-k)} :\ U^TX = 0 \text{ and } U^TU = I_{m-k}\},$$

and I_{m-k} is the identity matrix of size $m-k$. In Section 3.5.3, we first establish an auxiliary KL calculus rule concerning Lagrangian in Theorem 3.1. Then, using this result together with Theorem 3.4, we give an explicit KL exponent (dependent on n, m and k) of f in (3.35) in Theorem 3.11.

3.4 KL exponents via inf-projection for some convex models

3.4.1 Convex models with SDP-representable structure

In this section, we explore the KL exponent of SDP-representable functions introduced in Section 3.3.1(i). More specifically, we will deduce the KL exponent of a proper closed function f with its epigraph represented as in (3.23), under suitable conditions on F in (3.25). To this end, we collect the u components in $\mathfrak{D}$ in (3.25) for each fixed $x \in \operatorname{dom} \partial f$ and define the following set:

$$\mathfrak{D}_x = \left\{ u \in \mathbb{R}^N : \ (x, u, f(x)) \in \mathfrak{D} \right\}. \tag{3.37}$$

Roughly speaking, these are extra variables that correspond to the "x-slice" in the "lifted" SDP representation. As we shall see in the proof of Theorem 3.6, when f is the sum of LMI-representable functions (which is SDP-representable), one can have $\mathfrak{D}_x = \{(f_1(x), \ldots, f_m(x))\}$.

We begin with three auxiliary lemmas. The first one relates the KL exponent of f, whose epigraph is represented as in (3.23), to that of F in (3.25).

Lemma 3.1. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a proper closed SDP-representable function with its epigraph represented as in* (3.23)*. Then the function F defined in* (3.25) *is proper, closed and convex.*

Next, suppose in addition that $\bar{x} \in \operatorname{dom} \partial f$, $\alpha \in [0, 1)$, and that the following conditions hold:

(i) *The set $\mathfrak{D}_{\bar{x}}$ defined as in* (3.37) *is nonempty and compact.*

(ii) *The function F defined in* (3.25) *satisfies the KL property with exponent α at every point in $\{\bar{x}\} \times \mathfrak{D}_{\bar{x}} \times \{f(\bar{x})\}$.*

Then f satisfies the KL property at $\bar{x}$ with exponent α.

Proof. Observe from the definition that

$$f(x) = \inf_{u,t} F(x,u,t).$$

First, note that $\mathfrak{D} \neq \emptyset$ because f is proper. Since $\mathfrak{D}$ is clearly closed and convex, we conclude that F is proper, closed and convex. We will now check the conditions in Corollary 3.4 and apply the corollary to deduce the KL property of f from that of F.

To this end, by assumption, we see that F satisfies the KL property with exponent α on $\{\bar{x}\} \times \mathfrak{D}_{\bar{x}} \times \{f(\bar{x})\} = \{\bar{x}\} \times \operatorname{Arg\,min}_{u,t} F(\bar{x},u,t)$ and that $\mathfrak{D}_{\bar{x}}$ is nonempty and compact. The desired conclusion now follows from a direct application of Corollary 3.4. This completes the proof. $\square$

The second lemma relates the KL exponent of F in (3.25) to that of another SDP-representable function with carefully constructed matrices involved in its representation.

Lemma 3.2. *Let f be a proper closed function and $\bar{x} \in \operatorname{dom} f$. Suppose that f is SDP-representable with its epigraph represented as in* (3.23)*, and that there exists (x^s,u^s,t^s) such that $A_{00} + \mathcal{A}^*(x^s,u^s,t^s) \succ 0$, where A_{00} and $\mathcal{A}$ are given in* (3.23) *and* (3.24) *respectively. Let F be defined as in* (3.25) *and $\mathfrak{D}_{\bar{x}}$ be defined as in* (3.37).[5] *Let $\bar{u} \in \mathfrak{D}_{\bar{x}}$ and suppose that $0 \in \partial F(\bar{x},\bar{u},f(\bar{x}))$. Then the following statements hold:*

(i) *It holds that $A_0 \neq 0$. Moreover, the set* $\operatorname{span}\{A_1,\ldots,A_n,B_1,\ldots,B_N,A_0\}$ *has an orthogonal basis $\{\widehat{A}_0,\ldots,\widehat{A}_p\}$, where $p \geq 0$ and $\widehat{A}_0 \neq 0$, such that*

$$\begin{bmatrix}\mathbf{a}_1 \ \ldots \ \mathbf{a}_n \ \mathbf{b}_1 \ \ldots \ \mathbf{b}_N \ \mathbf{a}_0\end{bmatrix} = \begin{bmatrix}\widehat{\mathbf{a}}_1 \ldots \widehat{\mathbf{a}}_p \ \widehat{\mathbf{a}}_0\end{bmatrix} U$$

for some $U \in \mathbb{R}^{(p+1)\times(n+N+1)}$ having full row rank and the entries of the $(p+1)^{\text{th}}$ row of U are 0 except for $U_{p+1,n+N+1} = 1$; here, $\mathbf{a}_i$, $\mathbf{b}_j$ and $\widehat{\mathbf{a}}_k \in \mathbb{R}^{d^2}$ are the columnwise vectorization of the matrices A_i, B_j and $\widehat{A}_k$, respectively.

[5] Notice that F is proper and closed thanks to the existence of the Slater point (x^s,u^s,t^s).

(ii) *Define* $F_1 : \mathbb{R}^{p+1} \to \mathbb{R} \cup \{\infty\}$ *by*

$$F_1(z,t) := t + \delta_{\mathfrak{D}_1}(z,t) \text{ with } \mathfrak{D}_1 = \left\{(z,t): \ A_{00} + \widehat{A}_0 t + \sum_{w=1}^{p} \widehat{A}_w z_w \succeq 0\right\}, \tag{3.38}$$

where $p \geq 0$ *and* $\{\widehat{A}_0, \ldots, \widehat{A}_p\}$ *is the orthogonal basis constructed in* (i).[6] *Suppose that* $U(\bar{x}, \bar{u}, f(\bar{x})) \in \operatorname{dom} \partial F_1$ *and* F_1 *satisfies the KL property at* $U(\bar{x}, \bar{u}, f(\bar{x}))$ *with exponent* $\alpha \in [0,1)$, *where* U *is the same as in* (i).[7] *Then* F *satisfies the KL property at* $(\bar{x}, \bar{u}, f(\bar{x}))$ *with exponent* α.

Proof. Since $0 \in \partial F(\bar{x}, \bar{u}, f(\bar{x}))$, we have in view of [100, Exercise 8.8] that

$$0_{n+N+1} \in (0_n, 0_N, 1) + N_{\mathfrak{D}}(\bar{x}, \bar{u}, f(\bar{x})), \tag{3.39}$$

where $\mathfrak{D}$ is defined as in (3.25), and 0_k is the zero vector of dimension k. Next, since $\delta_{\mathfrak{D}}(x,u,t) = [\delta_{\mathcal{S}^d_+ - A_{00}} \circ \mathcal{A}^*](x,u,t)$ and we have $\mathcal{A}^*(x^s, u^s, t^s) \succ -A_{00}$ by assumption, using [99, Theorem 23.9], we deduce that

$$N_{\mathfrak{D}}(\bar{x}, \bar{u}, f(\bar{x})) = \partial\Big[\delta_{\mathcal{S}^d_+ - A_{00}} \circ \mathcal{A}^*\Big](\bar{x}, \bar{u}, f(\bar{x})) = \mathcal{A} N_{\mathcal{S}^d_+ - A_{00}}(\mathcal{A}^*(\bar{x}, \bar{u}, f(\bar{x}))).$$

This together with (3.39) implies that there exists $Y \in N_{\mathcal{S}^d_+ - A_{00}}(\mathcal{A}^*(\bar{x}, \bar{u}, f(\bar{x})))$ such that

$$\langle A_1, Y\rangle = \cdots = \langle A_n, Y\rangle = \langle B_1, Y\rangle = \cdots = \langle B_N, Y\rangle = 0 \ \text{ but } \ \langle A_0, Y\rangle = -1;$$

in particular, $A_0 \notin \operatorname{span}\{A_1, \ldots, A_n, B_1, \ldots, B_N\}$ and hence $A_0 \neq 0$.

If $\operatorname{span}\{A_1, \ldots, A_n, B_1, \ldots, B_N\} = \{0\}$, then $A_i = B_j = 0$ for $i = 1, \ldots, n$ and $j = 1, \ldots, N$. In this case, set $\widehat{A}_0 = A_0$. We see that $\{\widehat{A}_0\}$ is an orthogonal set and we have

$$\begin{bmatrix}\mathbf{a}_1 & \ldots & \mathbf{a}_n & \mathbf{b}_1 & \ldots & \mathbf{b}_N & \mathbf{a}_0\end{bmatrix} = \widehat{\mathbf{a}}_0 \begin{bmatrix}0_{n+N}^T & 1\end{bmatrix},$$

[6] Note that F_1 is proper and closed thanks to the existence of the Slater point (x^s, u^s, t^s).

[7] Here and henceforth, $U(\bar{x}, \bar{u}, f(\bar{x}))$ is a short-hand notation for the matrix vector product $U\begin{bmatrix}\bar{x}\\ \bar{u}\\ f(\bar{x})\end{bmatrix}$.

where 0_{n+N} is the zero vector of dimension $n+N$. Thus, the conclusion in (i) holds in this case.

Otherwise, $\operatorname{span}\{A_1,\ldots,A_n,B_1,\ldots,B_N\}\neq\{0\}$ and we let $\{\bar{A}_1,\ldots,\bar{A}_p\}$ be a maximal linearly independent subset of $\{A_1,\ldots,A_n,B_1,\ldots,B_N\}$. Then there exists $M_0\in\mathbb{R}^{p\times(n+N)}$ with full row rank such that $[\mathbf{a}_1\ \ldots\ \mathbf{a}_n\ \mathbf{b}_1\ \ldots\ \mathbf{b}_N]=[\bar{\mathbf{a}}_1\ \ldots\ \bar{\mathbf{a}}_p]\,M_0$, where $\bar{\mathbf{a}}_i\in\mathbb{R}^{d^2}$ is the columnwise vectorization of $\bar{A}_i$. Thus

$$[\mathbf{a}_1\ \ldots\ \mathbf{a}_n\ \mathbf{b}_1\ \ldots\ \mathbf{b}_N\ \mathbf{a}_0]=[\bar{\mathbf{a}}_1\ \ldots\ \bar{\mathbf{a}}_p\ \mathbf{a}_0]\begin{bmatrix}M_0 & 0\\ 0 & 1\end{bmatrix}. \tag{3.40}$$

Using Gram-Schmidt process followed by a suitable scaling to $\{\bar{A}_1,\ldots,\bar{A}_p,A_0\}$, there exists an invertible upper triangle matrix $U_0\in\mathbb{R}^{(p+1)\times(p+1)}$ with the $(U_0)_{p+1,p+1}=1$ and an orthogonal basis $\{\widehat{A}_1,\ldots,\widehat{A}_p,\widehat{A}_0\}$ of $\operatorname{span}\{\bar{A}_1,\ldots,\bar{A}_p,A_0\}$ such that

$$[\bar{\mathbf{a}}_1\ \ldots\ \bar{\mathbf{a}}_p\ \mathbf{a}_0]=\left[\widehat{\mathbf{a}}_1\ \ldots\ \widehat{\mathbf{a}}_p\ \widehat{\mathbf{a}}_0\right]U_0,$$

where $\widehat{\mathbf{a}}_i$ is the columnwise vectorization of $\widehat{A}_i$. This together with (3.40) shows that

$$\left[\mathbf{a}_1\ \ldots\ \mathbf{a}_n\ \mathbf{b}_1\ \ldots\ \mathbf{b}_N\ \mathbf{a}_0\right]=[\bar{\mathbf{a}}_1\ \ldots\ \bar{\mathbf{a}}_p\ \mathbf{a}_0]\begin{bmatrix}M_0 & 0\\ 0 & 1\end{bmatrix}=\left[\widehat{\mathbf{a}}_1\ \ldots\ \widehat{\mathbf{a}}_p\ \widehat{\mathbf{a}}_0\right]U,$$

where $U:=U_0\begin{bmatrix}M_0 & 0\\ 0 & 1\end{bmatrix}$ has full row rank and the entries of the $(p+1)^{\text{th}}$ row of U are 0 except for $U_{p+1,n+N+1}=1$. This proves (i).

Now, using the definition of F_1 in (3.38), we have $F(x,u,t)=F_1(U(x,u,t))$. Since U is surjective and the KL exponent of F_1 is α at $U(\bar{x},\bar{u},f(\bar{x}))$, using a similar argument as in [75, Theorem 3.2], the KL exponent of F at $(\bar{x},\bar{u},f(\bar{x}))$ equals α. This completes the proof. $\square$

Finally, we rewrite F_1 in (3.38) suitably as a function on $\mathcal{S}^d$ that satisfies a certain "strict complementarity" condition so that Lemma 2.3 can be readily applied to deducing the KL exponent of F_1 explicitly.

Lemma 3.3. *Let f be a proper closed function and $\bar{x} \in \operatorname{dom} f$. Suppose in addition that f is SDP-representable with its epigraph represented as in* (3.23)*. Let F be defined as in* (3.25)*, $\mathfrak{D}_{\bar{x}}$ be defined as in* (3.37)*, and $\bar{u} \in \mathfrak{D}_{\bar{x}}$. Suppose that the following conditions hold:*

(i) **(Slater's condition)** *There exists (x^s, u^s, t^s) such that $A_{00} + \mathcal{A}^*(x^s, u^s, t^s) \succ 0$, where A_{00} and $\mathcal{A}$ are given in* (3.23) *and* (3.24) *respectively.*[8]

(ii) **(Strict complementarity)** *It holds that $0 \in \operatorname{ri} \partial F(\bar{x}, \bar{u}, f(\bar{x}))$.*

Let F_1 be defined as in (3.38)*. Then $U(\bar{x}, \bar{u}, f(\bar{x})) \in \operatorname{dom} \partial F_1$ and F_1 satisfies the KL property at $U(\bar{x}, \bar{u}, f(\bar{x}))$ with exponent $\frac{1}{2}$, where U is given in Lemma 3.2*(i)*.*

Proof. Define $\bar{\mathcal{A}} : \mathcal{S}^d \to \mathbb{R}^{p+1}$ by

$$\bar{\mathcal{A}}(W) := \left[\langle \widehat{A}_1, W \rangle \ \ldots \ \langle \widehat{A}_p, W \rangle \ \langle \widehat{A}_0, W \rangle\right]^T,$$

where $\{\widehat{A}_0, \ldots, \widehat{A}_p\}$ is given by Lemma 3.2(i). Since $\{\widehat{A}_0, \ldots, \widehat{A}_p\}$ is orthogonal, we see that $\bar{\mathcal{A}}$ is surjective and $\bar{\mathcal{A}}^* : \mathbb{R}^{p+1} \to \mathcal{S}^d$ with $\bar{\mathcal{A}}^*(z,t) := \widehat{A}_0 t + \sum_{w=1}^p \widehat{A}_w z_w$ is injective. Also, for any $(z,t) \in \mathbb{R}^{p+1}$, by orthogonality,

$$\bar{\mathcal{A}}\bar{\mathcal{A}}^*(z,t) = \bar{\mathcal{A}}\left(\widehat{A}_0 t + \sum_{w=1}^{p} \widehat{A}_w z_w\right) = \left(\|\widehat{A}_1\|_F^2 z_1, \ldots, \|\widehat{A}_p\|_F^2 z_p, \|\widehat{A}_0\|_F^2 t\right).$$

Choose a basis $\{H_1, H_2, \ldots, H_r\}$ of $\ker \bar{\mathcal{A}}$ and define a linear map $\mathcal{H} : \mathcal{S}^d \to \mathbb{R}^r$ by[9]

$$\mathcal{H}(W) := [\langle H_1, W \rangle \ \cdots \ \langle H_r, W \rangle]^T. \tag{3.41}$$

Define a proper closed function $F_2 : \mathcal{S}^d \to \mathbb{R} \cup \{\infty\}$ by

$$F_2(X) := \|\widehat{A}_0\|_F^{-2} \langle \widehat{A}_0, X \rangle + \delta_{\mathfrak{D}_2}(X) \text{ with } \mathfrak{D}_2 := \left\{X \in \mathcal{S}^d_+ : \ \mathcal{H}X = \mathcal{H}A_{00}\right\}. \tag{3.42}$$

[8] Note that this condition implies that both F in (3.25) and F_1 in (3.38) are proper and closed.

[9] In the case when $\ker \bar{\mathcal{A}} = \{0\}$ so that the basis is empty (i.e., $r = 0$), we define $\mathcal{H}$ to be the *unique* linear map that maps $\mathcal{S}^d$ onto the *zero vector space.*

Thanks to the identity $(\ker \bar{\mathcal{A}})^\perp = \mathrm{Range}\,(\bar{\mathcal{A}}^*)$ and the fact that $\mathcal{H}X = \mathcal{H}A_{00}$ if and only if $X - A_{00} \in (\ker \bar{\mathcal{A}})^\perp$, we have the following relations concerning $\mathfrak{D}_2$ and the $\mathfrak{D}_1$ defined in (3.38):

$$\begin{aligned} (z,t) \in \mathfrak{D}_1 &\Longrightarrow A_{00} + \bar{\mathcal{A}}^*(z,t) \in \mathfrak{D}_2, \\ X \in \mathfrak{D}_2 \Rightarrow \exists \text{ unique } (z,t) &\text{ s.t. } A_{00} + \bar{\mathcal{A}}^*(z,t) = X, \text{ and } (z,t) \in \mathfrak{D}_1, \end{aligned} \tag{3.43}$$

where the second implication also makes use of the injectivity of $\bar{\mathcal{A}}^*$. We then deduce further that for any $(z,t) \in \mathbb{R}^{p+1}$,

$$\begin{aligned} &F_2(A_{00} + \bar{\mathcal{A}}^*(z,t)) - \|\widehat{A}_0\|_F^{-2}\langle \widehat{A}_0, A_{00}\rangle \\ &= \langle \bar{\mathcal{A}}\left(\|\widehat{A}_0\|_F^{-2}\widehat{A}_0\right), (z,t)\rangle + \delta_{\mathfrak{D}_2}(A_{00} + \bar{\mathcal{A}}^*(z,t)) \\ &= t + \delta_{\mathfrak{D}_2}(A_{00} + \bar{\mathcal{A}}^*(z,t)) = F_1(z,t), \end{aligned} \tag{3.44}$$

where the last equality follows from (3.43).

Next, let U be as in Lemma 3.2(i). Since the entries in the $(p+1)^{\text{th}}$ row of U are 0 except for $U_{p+1,n+N+1} = 1$, there exists $\bar{z} \in \mathbb{R}^p$ such that[10]

$$U(\bar{x}, \bar{u}, f(\bar{x})) = (\bar{z}, f(\bar{x})). \tag{3.45}$$

Now, define

$$\bar{X} := A_{00} + \bar{\mathcal{A}}^*(\bar{z}, f(\bar{x})). \tag{3.46}$$

We claim that $0 \in \mathrm{ri}\,\partial F_2(\bar{X})$. We first show that

$$0 \in \mathrm{ri}\,\partial F_1(\bar{z}, f(\bar{x})). \tag{3.47}$$

In fact, using [99, Theorem 23.9] (note that $U(x^s, u^s, t^s) \in \mathrm{int}\,\mathfrak{D}_1$ thanks to assumption (i)) together with the assumption (ii), we have

$$0 \in \mathrm{ri}\,\partial F(\bar{x}, \bar{u}, f(\bar{x})) = \mathrm{ri}\,\left[U^T \partial F_1\left(U(\bar{x}, \bar{u}, f(\bar{x}))\right)\right] = U^T \mathrm{ri}\,\partial F_1\left(U(\bar{x}, \bar{u}, f(\bar{x}))\right),$$

[10] Recall that $p \geq 0$. When $p = 0$, we interpret $\bar{z}$ as a null vector so that $U(\bar{x}, \bar{u}, f(\bar{x})) = f(\bar{x})$.

where the second equality follows from [99, Theorem 6.6]. Since U has full row rank and thus U^T is injective, recalling the definition of $\bar{z}$ in (3.45), we deduce further that (3.47) holds. Now, using this and [100, Exercise 8.8], we have

$$0 \in \mathrm{ri}\,\partial F_1(\bar{z}, f(\bar{x})) = (\underbrace{0, \ldots, 0}_{p \text{ entries}}, 1) + \mathrm{ri}\, N_{\mathfrak{D}_1}(\bar{z}, f(\bar{x})). \tag{3.48}$$

Now, notice that $\delta_{\mathfrak{D}_1}(z,t) = \left[\delta_{\mathcal{S}^d_+ - A_{00}} \circ \bar{\mathcal{A}}^*\right](z,t)$ and

$$\mathfrak{D}_2 \ni X^s := A_{00} + \bar{\mathcal{A}}^*(z^s, t^s) = A_{00} + \mathcal{A}^*(x^s, u^s, t^s) \succ 0 \tag{3.49}$$

with $(z^s, t^s) = U(x^s, u^s, t^s)$, where the inclusion holds thanks to (3.43). Using these and [99, Theorem 23.9], we see that

$$\mathrm{ri}\, N_{\mathfrak{D}_1}(\bar{z}, f(\bar{x})) = \mathrm{ri}\,\partial \left[\delta_{\mathcal{S}^d_+ - A_{00}} \circ \bar{\mathcal{A}}^*\right](\bar{z}, f(\bar{x})) = \mathrm{ri}\,\bar{\mathcal{A}} N_{\mathcal{S}^d_+}(\bar{X}) = \bar{\mathcal{A}}\,\mathrm{ri}\, N_{\mathcal{S}^d_+}(\bar{X}),$$

where the last equality follows from [99, Theorem 6.6]. This together with (3.48) implies that there exists $\widetilde{Y} \in \mathrm{ri}\, N_{\mathcal{S}^d_+}(\bar{X})$ such that

$$\langle \widehat{A}_1, \widetilde{Y} \rangle = \cdots = \langle \widehat{A}_p, \widetilde{Y} \rangle = 0 \text{ and } \langle \widehat{A}_0, \widetilde{Y} \rangle = -1. \tag{3.50}$$

The second relation in (3.50) gives $\langle \widehat{A}_0, \widetilde{Y} + \|\widehat{A}_0\|_F^{-2} \widehat{A}_0 \rangle = \langle \widehat{A}_0, \widetilde{Y} \rangle + 1 = 0$. In addition, in view of the first relation in (3.50) and the orthogonality of $\{\widehat{A}_0, \ldots, \widehat{A}_p\}$, we have $\langle \widehat{A}_i, \widetilde{Y} + \|\widehat{A}_0\|_F^{-2} \widehat{A}_0 \rangle = \langle \widehat{A}_i, \widetilde{Y} \rangle + \langle \widehat{A}_i, \|\widehat{A}_0\|_F^{-2} \widehat{A}_0 \rangle = 0$ for all $i = 1, \ldots, p$. Thus, it holds that $\widetilde{Y} + \|\widehat{A}_0\|_F^{-2} \widehat{A}_0 \in \ker \bar{\mathcal{A}}$. Hence, there exists $\omega \in \mathbb{R}^r$ such that

$$\widetilde{Y} + \|\widehat{A}_0\|_F^{-2} \widehat{A}_0 = \sum_{i=1}^{r} H_i \omega_i \tag{3.51}$$

with r and H_i defined as in (3.41).[11] Using (3.51) and the definition of $\widetilde{Y}$, we have further that

$$0 = \widetilde{Y} + \|\widehat{A}_0\|_F^{-2} \widehat{A}_0 - \sum_{i=1}^{r} H_i \omega_i \in \mathrm{ri}\, N_{\mathcal{S}^d_+}(\bar{X}) + \|\widehat{A}_0\|_F^{-2} \widehat{A}_0 + \mathrm{Range}\, \mathcal{H}^*. \tag{3.52}$$

[11] In the case when $\ker \bar{\mathcal{A}} = \{0\}$ (i.e., $r = 0$), we have $\widetilde{Y} + \|\widehat{A}_0\|_F^{-2} \widehat{A}_0 = 0$. In this case, we interpret ω as a null vector.

On the other hand, using the definition of F_2 in (3.42), we have

$$\mathrm{ri}\,\partial F_2(\bar{X}) = \|\widehat{A}_0\|_F^{-2}\widehat{A}_0 + \mathrm{ri}\,\partial\delta_{\mathfrak{D}_2}(\bar{X}) = \|\widehat{A}_0\|_F^{-2}\widehat{A}_0 + \mathrm{ri}\,\left(N_{\mathcal{H}^{-1}\{\mathcal{H}A_{00}\}}(\bar{X}) + N_{\mathcal{S}_+^d}(\bar{X})\right)$$

$$= \|\widehat{A}_0\|_F^{-2}\widehat{A}_0 + \mathrm{ri}\,N_{\mathcal{H}^{-1}\{\mathcal{H}A_{00}\}}(\bar{X}) + \mathrm{ri}\,N_{\mathcal{S}_+^d}(\bar{X}) = \|\widehat{A}_0\|_F^{-2}\widehat{A}_0 + \mathrm{Range}\,\mathcal{H}^* + \mathrm{ri}\,N_{\mathcal{S}_+^d}(\bar{X}),$$

where the second equality follows from [99, Theorem 23.8] and (3.49), and the third equality follows from [99, Corollary 6.6.2]. This together with (3.52) shows

$$0 \in \mathrm{ri}\,\partial F_2(\bar{X}). \tag{3.53}$$

In view of (3.49) and (3.53), we can now apply Lemma 2.3 and deduce that, for a given compact neighborhood $\mathfrak{U}$ of $\bar{X}$, there exists $c > 0$ such that for any $X \in \mathfrak{U} \cap \mathfrak{D}_2$,

$$\mathrm{dist}\,(X, \mathrm{Arg\,min}\,F_2) \le c\left(F_2(X) - F_2(\bar{X})\right)^{\frac{1}{2}}. \tag{3.54}$$

Thus, fix an $\epsilon > 0$ so that $A_{00} + \bar{\mathcal{A}}^*(z,t) \in \mathfrak{U}$ whenever $(z,t) \in B((\bar{z}, f(\bar{x})), \epsilon)$; such an ϵ exists thanks to the definitions of $\bar{z}$ in (3.45) and $\bar{X}$ in (3.46). Now, consider any (z,t) satisfying $(z,t) \in B((\bar{z}, f(\bar{x})), \epsilon)$ and $F_1(\bar{z}, f(\bar{x})) < F_1(z,t) < F_1(\bar{z}, f(\bar{x})) + \epsilon$. Then $(z,t) \in \mathrm{dom}\,F_1$, which means $A_{00} + \bar{\mathcal{A}}^*(z,t) \in \mathfrak{D}_2$ according to (3.43). Hence, using (3.54), we have

$$\begin{aligned}
&\mathrm{dist}^2((z,t), \mathrm{Arg\,min}\,F_1) \le \|(z,t) - (z^*,t^*)\|^2 \overset{\text{(a)}}{\le} c_1\left\|\bar{\mathcal{A}}^*(z,t) - \bar{\mathcal{A}}^*(z^*,t^*)\right\|_F^2\\
&= c_1\|A_{00} + \bar{\mathcal{A}}^*(z,t) - X^*\|_F^2 = c_1 \mathrm{dist}^2(A_{00} + \bar{\mathcal{A}}^*(z,t), \mathrm{Arg\,min}\,F_2)\\
&\le c^2 c_1\left(F_2(A_{00} + \bar{\mathcal{A}}^*(z,t)) - F_2(\bar{X})\right) \overset{\text{(b)}}{=} c^2 c_1\left(F_1(z,t) - F_1(\bar{z}, f(\bar{x}))\right),
\end{aligned}$$

where X^* denotes the projection of $A_{00} + \bar{\mathcal{A}}^*(z,t)$ on $\mathrm{Arg\,min}\,F_2$ and (z^*,t^*) is the corresponding element in $\mathrm{Arg\,min}\,F_1$ such that $X^* = A_{00} + \bar{\mathcal{A}}^*(z^*,t^*)$ (the existence of (z^*,t^*) follows from (3.43) and (3.44)), (a) holds for some $c_1 > 0$ because $\bar{\mathcal{A}}^*$ is injective, and (b) follows from (3.44). Combining this with [22, Theorem 5], we conclude that F_1 satisfies the KL property with exponent $\frac{1}{2}$ at $(\bar{z}, f(\bar{x})) = U(\bar{x}, \bar{u}, f(\bar{x}))$. $\square$

We are now ready to state and prove our main result in this section.

Theorem 3.5 (**KL exponent of SDP-representable functions**). *Let f be a proper closed function and $\bar{x} \in \operatorname{dom} \partial f$. Suppose in addition that f is SDP-representable with its epigraph represented as in* (3.23) *and that the following conditions hold:*

(i) **(Slater's condition)** *There exists (x^s, u^s, t^s) such that $A_{00} + \mathcal{A}^*(x^s, u^s, t^s) \succ 0$, where A_{00} and $\mathcal{A}$ are given in* (3.23) *and* (3.24) *respectively.*

(ii) **(Compactness)** *The set $\mathfrak{D}_{\bar{x}}$ defined as in* (3.37) *is nonempty and compact.*

(iii) **(Strict complementarity)** *It holds that $0 \in \operatorname{ri} \partial F(\bar{x}, u, f(\bar{x}))$ for all $u \in \mathfrak{D}_{\bar{x}}$, where F is defined as in* (3.25) *and $\mathfrak{D}_{\bar{x}}$ is defined as in* (3.37).[12]

Then f satisfies the KL property at $\bar{x}$ with exponent $\frac{1}{2}$.

Remark 3.5. *In Theorem 3.5, we require $0 \in \operatorname{ri} \partial F(\bar{x}, u, f(\bar{x}))$ for all $u \in \mathfrak{D}_{\bar{x}}$ with $\mathfrak{D}_{\bar{x}}$ defined as in* (3.37)*. This can be hard to check in practice. In Sections 3.4.2 and 3.4.3, we will impose additional assumptions on f so that this condition can be replaced by $0 \in \operatorname{ri} \partial f(\bar{x})$, which is a form of strict complementarity condition imposed on the original function f (rather than the representation F in the lifted space).*

Proof. In view of Lemma 3.1, it suffices to show that F satisfies the KL property with exponent $\frac{1}{2}$ at every point in $\{\bar{x}\} \times \mathfrak{D}_{\bar{x}} \times \{f(\bar{x})\}$. Fix any $\bar{u} \in \mathfrak{D}_{\bar{x}}$. From Lemma 3.3, we know that F_1 defined as in (3.38) has KL property with exponent $\frac{1}{2}$ at $U(\bar{x}, \bar{u}, f(\bar{x})) \in \operatorname{dom} \partial F_1$, where U is given in Lemma 3.2(i). Using this together with Lemma 3.2, we know that F satisfies the KL property with exponent $\frac{1}{2}$ at $(\bar{x}, \bar{u}, f(\bar{x}))$. This completes the proof. $\square$

[12] We note that because of the Slater's condition, the function F in (3.25) is proper and closed.

We would like to point out that the third condition in Theorem 3.5 cannot be replaced by "$0 \in \operatorname{ri} \partial f(\bar{x})$" in general. One concrete counter-example is $f(x) = x^4$. Indeed, for this function, the global minimizer is 0 and we have $\partial f(0) = \{\nabla f(0)\} = \{0\}$, which implies that $0 \in \operatorname{ri} \partial f(0)$. Moreover, this function is SDP-representable:

$$\operatorname{epi} f = \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R} : \begin{bmatrix} 1 & y & 0 & 0 \\ y & t & 0 & 0 \\ 0 & 0 & 1 & x \\ 0 & 0 & x & y \end{bmatrix} \succeq 0 \text{ for some } y \right\}.$$

It is easy to check that the first two conditions of Theorem 3.5 are satisfied for $\bar{x} = 0$. However, it can be directly verified that this f does not have KL property with exponent $\frac{1}{2}$ at 0. This concrete example suggests that the third condition in Theorem 3.5 cannot be replaced by $0 \in \operatorname{ri} \partial f(\bar{x})$ in general.

Next, in Sections 3.4.2 and 3.4.3, we will look at special SDP-representable functions and show that the third condition in Theorem 3.5 can indeed be replaced by $0 \in \operatorname{ri} \partial f(\bar{x})$ in those cases.

3.4.2 Sum of LMI-representable functions

In this section, we discuss how the KL exponent of the sum of finitely many proper closed LMI-representable functions as defined in Section 3.3.1(ii) can be deduced through Theorem 3.5. Compared with Theorem 3.5, the strict complementarity condition in this section is now imposed *directly* on the original function.

Theorem 3.6 (KL exponent of sum of LMI-representable functions). *Let $f = \sum_{i=1}^m f_i$, where each $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is proper and closed. Suppose that each f_i is LMI-representable, i.e., there exist $d_i \geq 1$ and matrices $\{A_{00}^i, A_0^i, A_1^i, \ldots, A_n^i\} \subset \mathcal{S}^{d_i}$ such that*

$$\operatorname{epi} f_i = \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R} : \; A_{00}^i + \sum_{j=1}^n A_j^i x_j + A_0^i t \succeq 0 \right\}.$$

Suppose in addition that there exist $x^s \in \mathbb{R}^n$ and $s^s \in \mathbb{R}^m$ such that for $i = 1, \dots, m$,

$$A_{00}^i + \sum_{j=1}^n A_j^i x_j^s + A_0^i s_i^s \succ 0.$$

If $\bar{x} \in \operatorname{dom} \partial f$ satisfies $0 \in \operatorname{ri} \partial f(\bar{x})$, then f satisfies the KL property at $\bar{x}$ with exponent $\frac{1}{2}$.

Proof. We first derive an SDP representation of $\operatorname{epi} f$. To this end, define

$$\widehat{\mathfrak{D}} := \left\{ (x, s, t) : \ t \geq \sum_{i=1}^m s_i \text{ and } s_i \geq f_i(x), \ \forall i = 1, \dots, m \right\}.$$

Then it holds that $(x, s, t) \in \widehat{\mathfrak{D}}$ if and only if

$$\begin{bmatrix} t - \sum_{i=1}^m s_i & 0 & \cdots & 0 \\ 0 & A_{00}^1 + \sum_{j=1}^n A_j^1 x_j + A_0^1 s_1 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & A_{00}^m + \sum_{j=1}^n A_j^m x_j + A_0^m s_m \end{bmatrix} \succeq 0. \tag{3.55}$$

Since

$$(x, t) \in \operatorname{epi} f \iff t \geq \sum_{i=1}^m f_i(x) \iff \exists s \in \mathbb{R}^m \text{ s.t. } (x, s, t) \in \widehat{\mathfrak{D}}, \tag{3.56}$$

we see that f is SDP-representable. Moreover, if we define

$$F(x, s, t) := t + \delta_{\widehat{\mathfrak{D}}}(x, s, t), \tag{3.57}$$

then it holds that $f(x) = \inf_{s,t} F(x, s, t)$ for all $x \in \mathbb{R}^n$. We next show that f and the F defined in (3.57) satisfy the conditions required in Theorem 3.5.

First, from the definition of $x^s \in \mathbb{R}^n$ and $s^s \in \mathbb{R}^m$, we have

$$\begin{bmatrix} t^s - \sum_{i=1}^m s_i^s & 0 & \cdots & 0 \\ 0 & A_{00}^1 + \sum_{j=1}^n A_j^1 x_j^s + A_0^1 s_1^s & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & A_{00}^m + \sum_{j=1}^n A_j^m x_j^s + A_0^m s_m^s \end{bmatrix} \succ 0,$$

where $t^s := \sum_{i=1}^m s_i^s + 1$. This together with (3.55) and (3.56) shows that condition (i) in Theorem 3.5 holds.

Next, note that the set $\{s : \ (\bar{x}, s, f(\bar{x})) \in \widehat{\mathfrak{D}}\} = \{(f_1(\bar{x}), \ldots, f_m(\bar{x}))\}$, which is clearly nonempty and compact. In view of this and (3.57), we conclude that condition (ii) in Theorem 3.5 is satisfied.

Finally, we look at the strict complementarity condition, i.e., condition (iii) in Theorem 3.5. Notice that the definition of $x^s \in \mathbb{R}^n$ implies

$$x^s \in \bigcap_{i=1}^m \operatorname{int}\operatorname{dom} f_i. \tag{3.58}$$

Write $\bar{s} := (f_1(\bar{x}), \cdots, f_m(\bar{x}))$ for notational simplicity. Define

$$\mathfrak{C}_0 = \left\{(x, s, t) : \ t \geq \sum_{i=1}^m s_i\right\} \text{ and } \mathfrak{C}_i = \{(x, s, t) : \ s_i \geq f_i(x)\}, \ \forall i = 1, \ldots, m.$$

Then $\widehat{\mathfrak{D}} = \bigcap_{i=0}^m \mathfrak{C}_i$. Moreover, using [99, Theorem 7.6], we have for $i = 1, \ldots, m$ that

$$\begin{aligned} &\operatorname{ri}\mathfrak{C}_i = \operatorname{ri}\{(x, s, t) : \ g_i(x, s, t) \leq 0\} = \{(x, s, t) \in \operatorname{ri}\operatorname{dom} g_i : \ g_i(x, s, t) < 0\} \\ &= \{(x, s, t) \in \operatorname{ri}\operatorname{dom} f_i \times \mathbb{R}^m \times \mathbb{R} : \ g_i(x, s, t) < 0\}, \end{aligned}$$

where $g_i(x, s, t) = f_i(x) - s_i$ for each i. This together with (3.58) shows that $\bigcap_{i=0}^m \operatorname{ri}\mathfrak{C}_i \neq \emptyset$. Using this, [99, Theorem 23.8] and the definition of F in (3.57), we have

$$\partial F(\bar{x}, \bar{s}, f(\bar{x})) = (0_{n+m}, 1) + \sum_{i=0}^m N_{\mathfrak{C}_i}(\bar{x}, \bar{s}, f(\bar{x})), \tag{3.59}$$

where 0_p is the zero vector of dimension p, and recall that $\bar{s} = (f_1(\bar{x}), \cdots, f_m(\bar{x}))$.

We claim that $0 \in \operatorname{ri}\partial F(\bar{x}, \bar{s}, f(\bar{x}))$. To this end, note first that the assumption $0 \in \operatorname{ri}\partial f(\bar{x})$ and (3.58) together with [99, Theorem 23.8] imply that $\bar{x} \in \bigcap_i \operatorname{dom}\partial f_i$.

Hence, we have from [99, Theorem 23.7] that for each $i = 1, \ldots, m$,

$$N_{\mathfrak{C}_i}(\bar{x}, \bar{s}, f(\bar{x})) = \mathrm{cl}\ [\mathrm{cone}\,\partial g_i(\bar{x}, \bar{s}, f(\bar{x}))] = \mathrm{cl} \bigcup_{\lambda_i \geq 0} (\lambda_i \partial f_i(\bar{x}), 0_{i-1}, -\lambda_i, 0_{m+1-i}) \tag{3.60}$$

where the second equality follows from [100, Proposition 10.5] and cone $\mathfrak{B}$ denotes the convex conical hull of $\mathfrak{B}$. Similarly, we also have

$$N_{\mathfrak{C}_0}(\bar{x}, \bar{s}, f(\bar{x})) = \mathrm{cl} \bigcup_{\lambda_0 \geq 0} (0_n, \lambda_0 \cdot 1_m, -\lambda_0)\,, \tag{3.61}$$

where 1_m is the m-dimensional vector of all ones. Using (3.59), (3.60) and (3.61), we have

$$\begin{aligned}
&\mathrm{ri}\,\partial F(\bar{x}, \bar{s}, f(\bar{x})) \overset{\text{(a)}}{=} (0_{n+m}, 1) + \sum_{i=0}^{m} \mathrm{ri}\, N_{\mathfrak{C}_i}(\bar{x}, \bar{s}, f(\bar{x})) \\
&\overset{\text{(b)}}{=} (0_{n+m}, 1) + \sum_{i=1}^{m} \mathrm{ri} \left[\mathrm{cl} \bigcup_{\lambda_i \geq 0} (\lambda_i \partial f_i(\bar{x}), 0_{i-1}, -\lambda_i, 0_{m+1-i}) \right] + \mathrm{ri} \left[\mathrm{cl} \bigcup_{\lambda_0 \geq 0} (0_n, \lambda_0 \cdot 1_m, -\lambda_0) \right] \\
&\overset{\text{(c)}}{=} (0_{n+m}, 1) + \sum_{i=1}^{m} \bigcup_{\lambda_i > 0} (\lambda_i\, \mathrm{ri}\,\partial f_i(\bar{x}), 0_{i-1}, -\lambda_i, 0_{m+1-i}) + \bigcup_{\lambda_0 > 0} (0_n, \lambda_0 \cdot 1_m, -\lambda_0)
\end{aligned}$$

where (a) follows from (3.59) and [99, Corollary 6.6.2], (b) follows from (3.60) and (3.61), and (c) follows from [99, Theorem 6.3] and [99, Corollary 6.8.1]. This together with $0 \in \mathrm{ri}\,\partial f(\bar{x})$ yields

$$\begin{aligned}
0 &\in (\mathrm{ri}\,\partial f(\bar{x}), 0_m, 0) = (0_n, 0_m, 1) + (\mathrm{ri}\,\partial f(\bar{x}), -1_m, 0) + (0_n, 1_m, -1) \\
&= (0_n, 0_m, 1) + \left(\sum_{i=1}^{m} \mathrm{ri}\,\partial f_i(\bar{x}), -1_m, 0 \right) + (0_n, 1_m, -1) \subseteq \mathrm{ri}\,\partial F(\bar{x}, \bar{s}, f(\bar{x})),
\end{aligned}$$

where the second equality follows from [99, Theorem 23.8] and [99, Corollary 6.6.2], thanks to (3.58). Thus, condition (iii) in Theorem 3.5 is also satisfied. The desired conclusion now follows from Theorem 3.5. $\square$

Example 3.1. *Note that ℓ_1-norm, ℓ_2-norm, convex quadratic functions and indicator functions of second-order cones are all LMI-representable. Using these, we can infer from Theorem 3.6 that the following functions f satisfy the KL property with exponent $\frac{1}{2}$ at any $\bar{x}$ that verifies $0 \in \operatorname{ri} \partial f(\bar{x})$:*

(i) **Group Lasso with overlapping blocks of variables:**

$$f(x) = \frac{1}{2}\|Ax - b\|^2 + \sum_{i=1}^{s} w_i \|x_{J_i}\|,$$

where $b \in \mathbb{R}^p$, $A \in \mathbb{R}^{p\times n}$, $J_i \subseteq \{1, \dots, n\}$ with $\bigcup_{i=1}^{s} J_i = \{1, \dots, n\}$, x_{J_i} is the subvector of x indexed by J_i, and $w_i \geq 0$, $i = 1, \dots, s$. We emphasize here that $J_i \cap J_j$ can be nonempty when $i \neq j$.

(ii) **Least squares with products of second-order cone constraints:**

$$f(x) = \frac{1}{2}\|Ax - b\|^2 + \delta_{\prod_{i=1}^{s} \mathrm{SOC}_{n_i}}(x),$$

where $b \in \mathbb{R}^p$, $A \in \mathbb{R}^{p\times n}$, $x = (x_1, \dots, x_s) \in \prod_{i=1}^{s} \mathbb{R}^{n_i}$ with $x_i \in \mathbb{R}^{n_i}$, $i = 1, \dots, s$, and SOC_{n_i} is the second-order cone in $\mathbb{R}^{n_i}$.

(iii) **Group fused Lasso [3]:**

$$f(x) = \frac{1}{2}\|Ax - b\|^2 + \sum_{i=1}^{s} w_i \|x_{J_i}\| + \sum_{i=2}^{s} \nu_i \|x_{J_i} - x_{J_{i-1}}\|,$$

where $b \in \mathbb{R}^p$, $A \in \mathbb{R}^{p\times n}$ with $n = rs$ for some $r \in \mathcal{N}$, J_i is an equi-partition of $\{1, \dots, n\}$ in the sense that $\bigcup_{i=1}^{s} J_i = \{1, \dots, n\}$, $J_i \cap J_j = \emptyset$ and $|J_i| = |J_j| = r$ for $i \neq j$, w_i, $\nu_i \geq 0$, $i = 1, \dots, s$.

3.4.3 Sum of LMI-representable functions and the nuclear norm

In this section, we apply Theorem 3.6 and Corollary 3.4 to derive the KL exponent of the function in (3.26) under suitable assumptions. It is known (see, for example [98]) that the nuclear norm can be expressed as

$$\|X\|_* = \frac{1}{2}\inf_{U,V}\left\{\mathrm{tr}(U)+\mathrm{tr}(V):\ \begin{bmatrix} U & X \\ X^T & V\end{bmatrix}\succeq 0,\, U\in\mathcal{S}^m, V\in\mathcal{S}^n\right\} \tag{3.62}$$

for any $X \in \mathbb{R}^{m\times n}$. This fact plays an important role for our analysis later on, and shows that the nuclear norm is an SDP representable function. To the best of our knowledge, it is not known that whether the nuclear norm is LMI representable. Our analysis is an attempt to generalize our results on the sum of LMI representable functions (with strict complementarity assumption on the original function) to a large subclass of SDP representable functions that arises in many important areas such as matrix completion [98].

Theorem 3.7 (**KL exponent of sum of LMI-representable functions and the nuclear norm**). *Let f be defined as in* (3.26) *and let symmetric matrices A^k_{00}, A^k_0, A^k_{ij}, $i = 1,\ldots,m$ and $j = 1,\ldots,n$, be such that*

$$\mathrm{epi}\, f_k = \left\{(X,t):\ A^k_{00} + \sum_{i=1}^m\sum_{j=1}^n A^k_{ij}X_{ij} + A^k_0 t \succeq 0\right\}.$$

Suppose in addition that there exist $X^s \in \mathbb{R}^{m\times n}$ and $s^s \in \mathbb{R}^p$ such that for $k = 1,\ldots,p$,

$$A^k_{00} + \sum_{i=1}^m\sum_{j=1}^n A^k_{ij}X^s_{ij} + A^k_0 s^s_k \succ 0.$$

If $\bar X \in \mathrm{dom}\,\partial f$ satisfies $0 \in \mathrm{ri}\,\partial f(\bar X)$, then f satisfies the KL property at $\bar X$ with exponent $\frac{1}{2}$.

Remark 3.6. *Similar to Theorem 3.6, the "ri-condition" here is also imposed on f itself, while such a condition is imposed on the F in* (3.25) *in Theorem 3.5.*

Proof. Let F be defined as in (3.27) with the matrix variable $Z \in \mathcal{S}^{n+m}$ partitioned as in (3.28). Then $f(X) = \inf_{U,V} F(Z)$, thanks to (3.62). Let $r = \operatorname{rank}(\bar{X})$ and

$$\bar{X} = [P_+ \; P_0] \begin{bmatrix} \Sigma_+ & 0 \\ 0 & 0 \end{bmatrix} [Q_+ \; Q_0]^T = P_+ \Sigma_+ Q_+^T,$$

be a singular value decomposition of $\bar{X}$, where $\Sigma_+ \in \mathbb{R}^{r \times r}$ is a diagonal matrix whose diagonal entries are the r positive singular values of $\bar{X}$, $[P_+ \; P_0]$ is orthogonal with $P_+ \in \mathbb{R}^{m \times r}$ and $P_0 \in \mathbb{R}^{m \times (m-r)}$, $[Q_+ \; Q_0]$ is orthogonal with $Q_+ \in \mathbb{R}^{n \times r}$ and $Q_0 \in \mathbb{R}^{n \times (n-r)}$. Define[13]

$$\bar{Z} := \begin{bmatrix} P_+ \Sigma_+ P_+^T & \bar{X} \\ \bar{X}^T & Q_+ \Sigma_+ Q_+^T \end{bmatrix}.$$

Then $\bar{Z} \succeq 0$. Now, using [99, Theorem 23.8], the definition of F and [99, Corollary 6.6.2], we have

$$\operatorname{ri} \partial F(\bar{Z}) = \left\{ \frac{1}{2} \begin{bmatrix} I_m & \Lambda \\ \Lambda^T & I_n \end{bmatrix} + Y : \Lambda \in \operatorname{ri} \partial \left(\sum_{k=1}^{p} f_k \right)(\bar{X}) \text{ and } Y \in \operatorname{ri} N_{\mathcal{S}_+^{m+n}}(\bar{Z}) \right\}. \tag{3.63}$$

Next, since $0 \in \operatorname{ri} \partial f(\bar{X})$ and the nuclear norm is continuous, we see from [99, Theorem 23.8] and [99, Corollary 6.6.2] that

$$0 \in \operatorname{ri} \partial f(\bar{X}) = \operatorname{ri} \partial \left(\sum_{k=1}^{p} f_k \right)(\bar{X}) + \operatorname{ri} \partial \|\bar{X}\|_*. \tag{3.64}$$

Moreover, recall from [118, Example 2] and [99, Corollary 7.6.1] that

$$\operatorname{ri} \partial \|\bar{X}\|_* = \left\{ [P_+ \; P_0] \begin{bmatrix} I_r & 0 \\ 0 & W \end{bmatrix} [Q_+ \; Q_0]^T : \; W \in \mathbb{R}^{(m-r)\times(n-r)}, \|W\|_2 < 1 \right\}, \tag{3.65}$$

where $\|W\|_2$ is the operator norm of W, that is, the largest singular value of W. Combining (3.64) and (3.65), we conclude that there exist $C \in \operatorname{ri} \partial \left(\sum_{k=1}^{p} f_k\right)(\bar{X})$ and

[13] When $r = 0$, we set $\bar{Z} = 0 \in \mathcal{S}^{m+n}$.

W_0 with $\|W_0\|_2 < 1$ such that

$$0 = C + [P_+ \ P_0] \begin{bmatrix} I_r & 0 \\ 0 & W_0 \end{bmatrix} [Q_+ \ Q_0]^T = C + P_0 W_0 Q_0^T + P_+ Q_+^T. \tag{3.66}$$

On the other hand, using the definition of $\bar{Z}$ and a direct computation, we have

$$\bar{Z} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} P_+ & P_0 & 0 & \frac{1}{\sqrt{2}} P_+ \\ \frac{1}{\sqrt{2}} Q_+ & 0 & Q_0 & -\frac{1}{\sqrt{2}} Q_+ \end{bmatrix}}_{\widehat{P}} \begin{bmatrix} 2\Sigma_+ & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} P_+ & P_0 & 0 & \frac{1}{\sqrt{2}} P_+ \\ \frac{1}{\sqrt{2}} Q_+ & 0 & Q_0 & -\frac{1}{\sqrt{2}} Q_+ \end{bmatrix}^T . \tag{3.67}$$

Note that $\widehat{P}^T \widehat{P} = \widehat{P} \widehat{P}^T = I_{m+n}$, meaning that (3.67) is an eigenvalue decomposition of $\bar{Z}$. Thus, we can compute that

$$\begin{aligned} \operatorname{ri} N_{\mathcal{S}_+^{m+n}}(\bar{Z}) &= \operatorname{ri} \left[(-\mathcal{S}_+^{m+n}) \cap \{\bar{Z}\}^\perp \right] = \widehat{P} \begin{bmatrix} 0 & 0 \\ 0 & -\operatorname{int} \mathcal{S}_+^{m+n-r} \end{bmatrix} \widehat{P}^T \\ &\ni \begin{bmatrix} \frac{1}{\sqrt{2}} P_+ & P_0 & 0 & \frac{1}{\sqrt{2}} P_+ \\ \frac{1}{\sqrt{2}} Q_+ & 0 & Q_0 & -\frac{1}{\sqrt{2}} Q_+ \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} I_{m-r} & \frac{1}{2} W_0 & 0 \\ 0 & \frac{1}{2} W_0^T & -\frac{1}{2} I_{n-r} & 0 \\ 0 & 0 & 0 & -I_r \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} P_+^T & \frac{1}{\sqrt{2}} Q_+^T \\ P_0^T & 0 \\ 0 & Q_0^T \\ \frac{1}{\sqrt{2}} P_+^T & -\frac{1}{\sqrt{2}} Q_+^T \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} -I_m & -C \\ -C^T & -I_n \end{bmatrix}, \end{aligned}$$

where the inclusion holds because $\|W_0\|_2 < 1$, and the last equality follows from (3.66) and a direct computation. This together with (3.63) and the definition of C implies that $0 \in \operatorname{ri} \partial F(\bar{Z})$. Moreover, one can see that F is the sum of $p+1$ proper closed LMI-representable functions and the Slater's condition required in Theorem 3.6 holds. Thus, we conclude from Theorem 3.6 that F in (3.27) has KL property at $\bar{Z}$ with exponent $\frac{1}{2}$.

Finally, recall that for the F defined in (3.27), we have

$$\inf_{U,V} F(Z) = f(X) \text{ and } \operatorname*{Arg\,min}_{U,V} F\left(\begin{bmatrix} U & \bar{X} \\ \bar{X}^T & V \end{bmatrix} \right) = \left\{ (P_+ \Sigma_+ P_+^T, Q_+ \Sigma_+ Q_+^T) \right\}.[14]$$

[14] When $r = 0$, this set is $\{(0,0)\}$ and $\bar{Z} = 0$.

These together with Corollary 3.4 and the fact that the KL exponent of F at $\bar{Z}$ is $\frac{1}{2}$ shows that f satisfies the KL property at $\bar{X}$ with exponent $\frac{1}{2}$. $\square$

Remark 3.7. *In [129, Proposition 12], it was shown that if $\ell : \mathbb{R}^p \to \mathbb{R}$ is strongly convex on any compact convex set with locally Lipschitz gradient and $\mathcal{A} : \mathbb{R}^{m\times n} \to \mathbb{R}^p$ is a linear map, then the function*

$$f(X) = \ell(\mathcal{A}X) + \|X\|_*$$

satisfies the KL property with exponent $\frac{1}{2}$ at any $\bar{X}$ that verifies $0 \in \operatorname{ri} \partial f(\bar{X})$. In particular, the loss function $X \mapsto \ell(\mathcal{A}X)$ is smooth. The more general case where the nuclear norm is replaced by a general spectral function was considered in [43, Theorem 3.12], and a sufficient condition involving the relative interior of the subdifferential of the conjugate of the spectral function *was proposed in [43, Proposition 3.13], which, in general, is different from the regularity condition $0 \in \operatorname{ri} \partial f(\bar{X})$.*

On the other hand, using our Theorem 3.7, we can deduce the KL exponent of functions in the form of (3.26) *at points $\bar{X}$ satisfying the condition $0 \in \operatorname{ri} \partial f(\bar{X})$, but with a different set of conditions on the loss function. For instance, one can prove using Theorem 3.7 that the following functions f satisfy the KL property with exponent $\frac{1}{2}$ at a point $\bar{X}$ verifying $0 \in \operatorname{ri} \partial f(\bar{X})$:*

(i) $f(X) = \frac{1}{2}\|\mathcal{A}X - b\|^2 + \mu \sum_{i,j} |X_{ij}| + \nu \|X\|_*$, *where $\mu > 0$ and $\nu > 0$, $b \in \mathbb{R}^p$ and $\mathcal{A} : \mathbb{R}^{m\times n} \to \mathbb{R}^p$ is a linear map.*

(ii) $f(X) = \|\mathcal{A}X - b\| + \mu \sum_{i,j} |X_{ij}| + \nu \|X\|_*$, *where $\mu > 0$ and $\nu > 0$, $b \in \mathbb{R}^p$ and $\mathcal{A} : \mathbb{R}^{m\times n} \to \mathbb{R}^p$ is a linear map.*

In view of [43, Theorem 3.12], it would be of interest to extend Theorem 3.7 to cover more general spectral functions. However, since our analysis in this subsection *is based on LMI or SDP representability, it is not clear how this can be achieved at this moment. This would be a potential important future research direction.*

Remark 3.8 (**Discussion of the relative interior conditions**)**.** *In Theorems 3.5, 3.6 and 3.7, the conclusions of KL exponent being* $1/2$ *were derived under relative interior conditions. If these relative interior conditions were dropped, then the corresponding conclusions could fail, in general. For example, in [129, equation (53)], the authors provided an example of* $\widetilde{f}(X) := f_1(X) + \|X\|_*$ *for* $X \in \mathbb{R}^{2\times 2}$*, where* f_1 *is a convex quadratic function on* $\mathbb{R}^{2\times 2}$*, and showed that* $0 \notin \mathrm{ri}\, \partial \widetilde{f}(\overline{X})$ *for some* $\overline{X} \in \mathbb{R}^{2\times 2}$ *and the first-order error bound is not satisfied at* $\overline{X}$*. Recalling [22, Theorem 5] and [49, Corollary 3.6], this means that* $\widetilde{f}$ *cannot have a KL exponent of* $\frac{1}{2}$ *at* $\overline{X}$*.*

We also would like to point out that, when the relative interior condition fails, one can follow the approach in Section 3.4.1 and the general error bound result for ill-posed semidefinite programs [50, 106] to derive a KL exponent that depends on the degree of singularity of a certain semidefinite system in the lifted representation. In general, this KL exponent will approach 1 *quickly as the dimension grows, which can be of less interest. For simplicity, we do not discuss this in detail.*

3.4.4 Convex models with C^2-cone reducible structure

In this section, we explore the KL exponent of functions that involve C^2-cone reducible structures. Our first theorem concerns the sum of the *support* function of a C^2-cone reducible closed convex set and a specially structured smooth convex function. In the theorem, we will also make use of the so-called bounded linear regularity condition [13, Definition 5.6]. Recall that $\{\mathfrak{D}_1, \mathfrak{D}_2\}$ is said to be boundedly linearly regular at $\bar{x} \in \mathfrak{D}_1 \cap \mathfrak{D}_2$ if for any bounded neighborhood $\mathfrak{U}$ of $\bar{x}$, there exists $c > 0$ such that

$$\mathrm{dist}\,(x, \mathfrak{D}_1 \cap \mathfrak{D}_2) \le c[\mathrm{dist}\,(x, \mathfrak{D}_1) + \mathrm{dist}\,(x, \mathfrak{D}_2)] \text{ for all } x \in \mathfrak{U}.$$

It is known that if $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are both polyhedral, then $\{\mathfrak{D}_1, \mathfrak{D}_2\}$ is boundedly linearly regular at any $\bar{x} \in \mathfrak{D}_1 \cap \mathfrak{D}_2$; moreover, if $\mathfrak{D}_1$ is polyhedral and $\mathfrak{D}_1 \cap \operatorname{ri} \mathfrak{D}_2 \neq \emptyset$, then $\{\mathfrak{D}_1, \mathfrak{D}_2\}$ is also boundedly linearly regular at any $\bar{x} \in \mathfrak{D}_1 \cap \mathfrak{D}_2$; see [14, Corollary 3].

Theorem 3.8 (**Composite convex models with C^2-cone reducible structure**). *Let $\ell : \mathbb{Y} \to \mathbb{R}$ be a function that is strongly convex on any compact convex set and has locally Lipschitz gradient, $\mathcal{A} : \mathbb{X} \to \mathbb{Y}$ be a linear map, and $v \in \mathbb{X}$. Consider the function*

$$h(x) := \ell(\mathcal{A}x) + \langle v, x\rangle + \sigma_{\mathfrak{D}}(x)$$

with $\mathfrak{D}$ being a nonempty C^2-cone reducible closed convex set. Suppose $0 \in \partial h(\bar{x})$. Then, one has

$$\bar{x} \in N_{\mathfrak{D}}(-\mathcal{A}^*\nabla \ell(\mathcal{A}\bar{x}) - v).$$

If we assume in addition that $\{\mathcal{A}^{-1}\{\mathcal{A}\bar{x}\},\ N_{\mathfrak{D}}(-\mathcal{A}^*\nabla\ell(\mathcal{A}\bar{x}) - v)\}$ is boundedly linearly regular at $\bar{x}$, then h satisfies the KL property at $\bar{x}$ with exponent $\frac{1}{2}$.

Proof. Since $0 \in \partial h(\bar{x})$, we see from [100, Exercise 8.8] that

$$\bar{w} := -\mathcal{A}^*\nabla \ell(\mathcal{A}\bar{x}) - v \in \partial \sigma_{\mathfrak{D}}(\bar{x}) = \partial \delta^*_{\mathfrak{D}}(\bar{x}) = (\partial \delta_{\mathfrak{D}})^{-1}(\bar{x}),$$

where the last equality follows from [100, Proposition 11.3]. This implies $\bar{x} \in \partial \delta_{\mathfrak{D}}(\bar{w}) = N_{\mathfrak{D}}(\bar{w})$.

We now assume in addition the bounded linear regularity condition and prove the alleged KL property. First, since $\mathfrak{D}$ is a C^2-cone reducible closed convex set, there exists $\widetilde{\rho} > 0$ and a mapping $\Theta : \mathbb{X} \to \mathbb{V}$ which is twice continuously differentiable on $B(\bar{w}, \widetilde{\rho})$ and a closed convex pointed cone $K \subseteq \mathbb{V}$ such that $\Theta(\bar{w}) = 0$, $D\Theta(\bar{w})$ is onto and $\mathfrak{D} \cap B(\bar{w}, \widetilde{\rho}) = \{w : \Theta(w) \in K\} \cap B(\bar{w}, \widetilde{\rho})$.

Fix any $\rho \in (0, \widetilde{\rho})$ so that $D\Theta(w)$ is onto whenever $w \in B(\bar{w}, \rho)$. Then, we have from [100, Exercise 10.7] that

$$N_{\mathfrak{D}}(w) = D\Theta(w)^* N_K(\Theta(w)) \quad \text{for all } w \in B(\bar{w}, \rho). \tag{3.68}$$

Now, fix any $\delta > 0$. Take $w \in \mathfrak{D} \cap B(\bar{w}, \rho)$ and $x \in N_{\mathfrak{D}}(w) \cap B(\bar{x}, \delta)$. Then $x = D\Theta(w)^* u_x$ for some $u_x \in N_K(\Theta(w))$ according to (3.68). For such a u_x, one can observe that

$$D\Theta(\bar{w})^* u_x \in D\Theta(\bar{w})^* N_K(\Theta(w)) \subseteq D\Theta(\bar{w})^* K^\circ = D\Theta(\bar{w})^* N_K(\Theta(\bar{w})) = N_{\mathfrak{D}}(\bar{w}),$$

where K° is the polar of K, the set inclusion follows from the definition of normal cone and the fact that K is a closed convex cone, the first equality holds because $\Theta(\bar{w}) = 0$ and the last equality follows from (3.68). Thus, for any $w \in \mathfrak{D} \cap B(\bar{w}, \rho)$ and $x \in N_{\mathfrak{D}}(w) \cap B(\bar{x}, \delta)$, we have

$$\operatorname{dist}(x, N_{\mathfrak{D}}(\bar{w})) \leq \|x - D\Theta(\bar{w})^* u_x\| = \|D\Theta(w)^* u_x - D\Theta(\bar{w})^* u_x\| \leq L\|u_x\|\|w - \bar{w}\|, \tag{3.69}$$

where L is the Lipschitz continuity modulus of $D\Theta$ over the set $B(\bar{w}, \rho)$, which is finite because Θ is twice continuously differentiable.

Next, for each $z \in B(\bar{w}, \rho)$, define the linear map

$$\mathcal{W}(z) = \left(D\Theta(z) D\Theta(z)^*\right)^{-1} D\Theta(z).$$

Then $\mathcal{W}$ is continuously differentiable on $B(\bar{w}, \rho)$ because Θ is twice continuously differentiable on $B(\bar{w}, \rho)$ with surjective gradient map. Moreover, for any $w \in \mathfrak{D} \cap B(\bar{w}, \rho)$ and $x \in N_{\mathfrak{D}}(w) \cap B(\bar{x}, \delta)$, it follows from the definition of u_x that $[\mathcal{W}(w)](x) = u_x$. Let M be the Lipschitz continuity modulus of $w \mapsto \mathcal{W}(w)$ on $B(\bar{w}, \rho)$, which is finite because $\mathcal{W}$ is continuously differentiable on $B(\bar{w}, \rho)$. Then we have for any $w \in \mathfrak{D} \cap B(\bar{w}, \rho)$ and $x \in N_{\mathfrak{D}}(w) \cap B(\bar{x}, \delta)$ that

$$\begin{aligned}
&\|u_x - u_{\bar{x}}\| = \|[\mathcal{W}(w)](x) - [\mathcal{W}(\bar{w})](\bar{x})\| \\
&\leq \|[\mathcal{W}(w)](x) - [\mathcal{W}(\bar{w})](x)\| + \|[\mathcal{W}(\bar{w})](x) - [\mathcal{W}(\bar{w})](\bar{x})\| \\
&\leq M\|x\|\,\|w - \bar{w}\| + \|\mathcal{W}(\bar{w})\|\|x - \bar{x}\| \\
&\leq M\rho(\|\bar{x}\| + \|x - \bar{x}\|) + \|\mathcal{W}(\bar{w})\|\|x - \bar{x}\|,
\end{aligned}$$

where the last inequality follows from triangle inequality and the fact that $w \in B(\bar{w}, \rho)$. In particular, $\|u_x\| \leq \|u_{\bar{x}}\| + M\rho(\|\bar{x}\| + \delta) + \|\mathcal{W}(\bar{w})\|\delta =: \kappa$. This together with (3.69) implies that

$$N_{\mathfrak{D}}(w) \cap B(\bar{x}, \delta) \subseteq N_{\mathfrak{D}}(\bar{w}) + \kappa L \|w - \bar{w}\| B(0,1) \quad \text{for all } w \in B(\bar{w}, \rho).$$

This means that the mapping $w \mapsto N_{\mathfrak{D}}(w)$ is calm at $\bar{w}$ with respect to $\bar{x}$; see [47, Page 182]. Thus, according to [47, Theorem 3H.3], the mapping $x \mapsto (N_{\mathfrak{D}})^{-1}(x)$ is metrically subregular at $\bar{x}$ with respect to $\bar{w}$; see [47, Page 183] for the definition. Noting also that $\partial\sigma_{\mathfrak{D}} = (N_{\mathfrak{D}})^{-1}$ according to [100, Example 11.4], we then deduce from [4, Theorem 3.3] that there exist $\delta' \in (0, \delta)$ and $c_0 > 0$ such that

$$\sigma_{\mathfrak{D}}(x) - \sigma_{\mathfrak{D}}(\bar{x}) - \langle \bar{w}, x - \bar{x} \rangle \geq c_0 \, \mathrm{dist}\,(x, (\partial\sigma_{\mathfrak{D}})^{-1}(\bar{w}))^2 = c_0 \, \mathrm{dist}\,(x, N_{\mathfrak{D}}(\bar{w}))^2 \quad (3.70)$$

whenever $\|x - \bar{x}\| \leq \delta'$. We now follow a similar line of argument used in [129, Theorem 2] and [49, Theorem 4.2] to show the desired conclusion. Observe that

$$\begin{aligned} &\operatorname{Arg\,min} h = \{z : 0 \in \partial h(z)\} \\ &= \{z : \mathcal{A}z = \mathcal{A}\bar{x} \text{ and } -\mathcal{A}^*\nabla\ell(\mathcal{A}z) - v \in (N_{\mathfrak{D}})^{-1}(z)\} \\ &= \{z : \mathcal{A}z = \mathcal{A}\bar{x} \text{ and } z \in N_{\mathfrak{D}}(-\mathcal{A}^*\nabla\ell(\mathcal{A}\bar{x}) - v)\}. \end{aligned}$$

Then it follows that for any bounded convex neighborhood $\mathfrak{U}$ of $\bar{x}$ with $\mathfrak{U} \subseteq B(\bar{x}, \delta')$, there exists $c_1 > 0$ such that for any $z \in \mathfrak{U}$,

$$\begin{aligned} &\mathrm{dist}\,(z, \operatorname{Arg\,min} h) = \mathrm{dist}\,(z, \mathcal{A}^{-1}\{\mathcal{A}\bar{x}\} \cap N_{\mathfrak{D}}(\bar{w})) \\ &\overset{(a)}{\leq} \alpha[\mathrm{dist}\,(z, \mathcal{A}^{-1}\{\mathcal{A}\bar{x}\}) + \mathrm{dist}\,(z, N_{\mathfrak{D}}(\bar{w}))] \\ &\overset{(b)}{\leq} \alpha[c_1 \|\mathcal{A}\bar{x} - \mathcal{A}z\| + \mathrm{dist}\,(z, N_{\mathfrak{D}}(\bar{w}))] \\ &\overset{(c)}{\leq} \alpha\left[c_1 \|\mathcal{A}\bar{x} - \mathcal{A}z\| + c_0^{-\frac{1}{2}} \sqrt{\sigma_{\mathfrak{D}}(z) - \sigma_{\mathfrak{D}}(\bar{x}) - \langle \bar{w}, z - \bar{x} \rangle}\right]; \end{aligned} \quad (3.71)$$

here, (a) holds for some $\alpha > 0$ because of the bounded linear regularity assumption, (b) holds for some $c_1 > 0$ thanks to the Hoffman error bound, and (c) follows from (3.70). Now, as ℓ is strongly convex on compact convex sets, there exists $\beta > 0$ such that for all $z \in \mathfrak{U}$, we have

$$\beta\|\mathcal{A}\bar{x} - \mathcal{A}z\|^2 \leq \ell(\mathcal{A}z) - \ell(\mathcal{A}\bar{x}) - \langle \mathcal{A}^*\nabla\ell(\mathcal{A}\bar{x}), z - \bar{x}\rangle.$$

Combining this with (3.71), we have for any $z \in \mathfrak{U}$ that

$$\operatorname{dist}(z, \operatorname{Arg\,min} h) \leq \alpha\left(c_1 \|\mathcal{A}\bar{x} - \mathcal{A}z\| + c_0^{-\frac{1}{2}} \sqrt{\sigma_{\mathfrak{D}}(z) - \sigma_{\mathfrak{D}}(\bar{x}) - \langle \bar{w}, z - \bar{x}\rangle}\right)$$

$$\leq \alpha\left(c_1\beta^{-\frac{1}{2}}\sqrt{\ell(\mathcal{A}z) - \ell(\mathcal{A}\bar{x}) - \langle \mathcal{A}^*\nabla\ell(\mathcal{A}\bar{x}), z - \bar{x}\rangle} + c_0^{-\frac{1}{2}} \sqrt{\sigma_{\mathfrak{D}}(z) - \sigma_{\mathfrak{D}}(\bar{x}) - \langle \bar{w}, z - \bar{x}\rangle}\right)$$

Note that $\sqrt{a} + \sqrt{b} \leq \sqrt{2}\sqrt{a+b}$ for $a, b \geq 0$, and

$$h(z) - h(\bar{x}) = \ell(\mathcal{A}z) - \ell(\mathcal{A}\bar{x}) - \langle \mathcal{A}^*\nabla\ell(\mathcal{A}\bar{x}), z - \bar{x}\rangle + \sigma_{\mathfrak{D}}(z) - \sigma_{\mathfrak{D}}(\bar{x}) - \langle \bar{w}, z - \bar{x}\rangle.$$

Thus, there exists $c > 0$ such that for all $z \in \mathfrak{U}$, $\operatorname{dist}(z, \operatorname{Arg\,min} h) \leq c\sqrt{h(z) - h(\bar{x})}$. Combining this with [22, Theorem 5], we conclude that h satisfies the KL property at $\bar{x}$ with exponent $\frac{1}{2}$. $\square$

As a corollary of the preceding theorem, we consider the KL exponent of a class of gauge regularized optimization problems. Recall that a convex function $\gamma : \mathbb{X} \to \mathbb{R} \cup \{\infty\}$ is called a gauge if it is nonnegative, positively homogeneous, and vanishes at the origin. It is clear that any norm is a gauge. In the next corollary, we make explicit use of the gauge structure and replace the relative interior condition in Theorem 3.8 by one involving the so-called polar gauge. Recall from [55, Proposition 2.1(iii)] that for a gauge γ, its polar can be given by $\gamma^\circ(x) = \sup_z\{\langle x, z\rangle : \gamma(z) \leq 1\}$; moreover, polar of norms are their corresponding dual norms.

Corollary 3.5. *Let f be defined as in* (3.29). *Suppose that $0 \in \partial f(\bar{x})$ and $\gamma(\bar{x}) > 0$. Then $\gamma^\circ(-\mathcal{A}^*\nabla\ell(\mathcal{A}\bar{x}) - v) = 1$. Suppose in addition that $-\mathcal{A}^*\nabla\ell(\mathcal{A}\bar{x}) - v \in \operatorname{dom}\partial\gamma^\circ$ and the following relative interior condition holds:*

$$\mathcal{A}^{-1}\{\mathcal{A}\bar{x}\} \cap \left(\bigcup_{\lambda>0} \lambda\left(\operatorname{ri}\partial\gamma^\circ(-\mathcal{A}^*\nabla\ell(\mathcal{A}\bar{x}) - v)\right)\right) \neq \emptyset. \tag{3.72}$$

Then f satisfies the KL property at $\bar{x}$ with exponent $\frac{1}{2}$.

Proof. Since $0 \in \partial f(\bar{x})$, we see from [100, Exercise 8.8] that

$$\bar{w} := -\mathcal{A}^*\nabla\ell(\mathcal{A}\bar{x}) - v \in \partial\gamma(\bar{x}).$$

Since we have from [55, Proposition 2.1(iv)] that $\gamma^* = \delta_{\mathfrak{C}}$ with $\mathfrak{C} = \{x :\ \gamma^\circ(x) \leq 1\}$, we conclude from (2.1) that $\gamma^\circ(\bar{w}) \leq 1$ and $\gamma(\bar{x}) = \langle \bar{x}, \bar{w}\rangle$. Since $\gamma(\bar{x}) > 0$, we also have from $\gamma(\bar{x}) = \langle \bar{x}, \bar{w}\rangle$ and [55, Proposition 2.1(iii)] that

$$1 = \frac{\langle \bar{x}, \bar{w}\rangle}{\gamma(\bar{x})} \leq \sup_z\{\langle \bar{w}, z\rangle :\ \gamma(z) \leq 1\} = \gamma^\circ(\bar{w}).$$

Thus, it holds that $\gamma^\circ(\bar{w}) = 1$.

Next, suppose in addition that $\bar{w} \in \operatorname{dom}\partial\gamma^\circ$ and (3.72) holds. Let $F(x,t)$ be defined as in (3.30). Observe that

$$F(x,t) = \ell(\widetilde{\mathcal{A}}(x,t)) + \langle (v,1), (x,t)\rangle + \sigma_{\mathfrak{D}^\circ}(x,t)$$

where $\widetilde{\mathcal{A}}(x,t) := \mathcal{A}x$ and $\mathfrak{D}^\circ$ is the polar of $\mathfrak{D}$, which is given by $\mathfrak{D}^\circ = \{(x,t) :\ \gamma^\circ(x) + t \leq 0\}$ according to the proof of [99, Theorem 15.4]. From our assumption, the set $\{(x,t) :\ \gamma^\circ(x) \leq t\}$ is a C^2-cone reducible closed convex set, which implies that $\mathfrak{D}^\circ$ is also C^2-cone reducible. Now, observe from [99, Theorem 23.7] that for any $(u,s) \in \operatorname{dom}\partial\gamma^\circ \times \mathrm{I\!R}$ satisfying $\gamma^\circ(u) + s = 0$, we have

$$N_{\mathfrak{D}^\circ}(u,s) = \operatorname{cl}\left(\bigcup_{\lambda\geq 0} \lambda\big(\partial\gamma^\circ(u), 1\big)\right),$$

which together with [99, Theorem 6.3] and [99, Corollary 6.8.1] gives

$$\operatorname{ri} N_{\mathfrak{D}^\circ}(u,s) = \bigcup_{\lambda>0} \lambda\big(\operatorname{ri}\partial\gamma^\circ(u), 1\big).$$

Applying this relation with $(u,s) = (\bar{w}, -\gamma^\circ(\bar{w})) = (\bar{w}, -1)$ together with the relative interior condition (3.72) shows that

$$\big(\mathcal{A}^{-1}\{\mathcal{A}\bar{x}\} \times \mathbb{R}\big) \cap \operatorname{ri} N_{\mathfrak{D}^\circ}(\bar{w}, -1) \neq \emptyset.$$

In view of this and [14, Corollary 3], we obtain that $\{\big(\mathcal{A}^{-1}\{\mathcal{A}\bar{x}\} \times \mathbb{R}\big), N_{\mathfrak{D}^\circ}(\bar{w}, -1)\}$ is boundedly linearly regular. It follows from Theorem 3.8 that F satisfies the KL property at $(\bar{x}, \gamma(\bar{x}))$ with exponent $\frac{1}{2}$. Since $f(x) = \inf_{t\in\mathbb{R}} F(x,t)$, we see from Corollary 3.4 that f satisfies the KL property at $\bar{x}$ with exponent $\frac{1}{2}$. $\square$

While checking C^2-cone reducibility directly using the definition can be difficult, a sufficient condition related to standard constraint qualifications was given in [101, Proposition 3.2].[15] Specifically, let $K \subseteq \mathbb{Y}$ be a C^2-cone reducible closed convex set and $G : \mathbb{X} \to \mathbb{Y}$ be a twice continuously differentiable function. If $G(\bar{x}) \in K$ and G is nondegenerate at $\bar{x}$ in the sense that

$$DG(\bar{x})\mathbb{X} + \big(T_K(G(\bar{x})) \cap \big[-T_K(G(\bar{x}))\big]\big) = \mathbb{Y}, \tag{3.73}$$

then $G^{-1}(K)$ is a C^2-cone reducible set. In particular, if $g_1, \ldots, g_m$ are C^2 functions with $\{\nabla g_i(\bar{x}) : i \in I(\bar{x})\}$ being linearly independent, where $I(\bar{x}) := \{i : \; g_i(\bar{x}) = 0\}$, then the set $\{x : g_i(x) \leq 0, i = 1, \ldots, m\}$ is C^2-cone reducible at $\bar{x}$.

We will now present a few concrete examples of functions to which Theorem 3.8 and Corollary 3.5 can be applied, taking advantage of the aforementioned sufficient condition (3.73) for checking C^2-cone reducibility.

[15] The quoted result is for C^1-cone reducibility. However, it is apparent from the proof how to adapt the result for C^2-cone reducibility.

Example 3.2. *Let $\ell : \mathbb{Y} \to \mathbb{R}$ be a function that is strongly convex on any compact convex set and has locally Lipschitz gradient, $\mathcal{A} : \mathbb{X} \to \mathbb{Y}$ be a linear map, and $v \in \mathbb{X}$.*

(i) **(Entropy-like regularization)** *Let $\mathbb{X} = \mathbb{R}^n$ and $\mathbb{Y} = \mathbb{R}^m$. Denote*

$$p(x) = \begin{cases} \sum_{i=1}^n x_i \log(x_i) - (\sum_{i=1}^n x_i) \log(\sum_{i=1}^n x_i) & \text{if } x \in \mathbb{R}^n_+, \\ \infty & \text{else,} \end{cases}$$

with the convention that $0 \log 0 = 0$. This function is proper closed convex and arises in the study of maximum entropy optimization [100, Example 11.12]. We claim that $f(x) = \ell(\mathcal{A}x) + \langle v, x\rangle + p(x)$ satisfies the KL property with exponent $\frac{1}{2}$ at any stationary point $\bar{x}$. To see this, recall from [100, Example 11.12] that

$$p(x) = \sigma_{\mathfrak{D}}(x), \quad \text{where} \quad \mathfrak{D} = \{x \in \mathbb{R}^n : g(x) \leq 0\},$$

and $g(x) = \log(\sum_{i=1}^n e^{x_i})$. Then we have from Theorem 3.8 that $-\mathcal{A}^*\nabla\ell(\mathcal{A}\bar{x}) - v \in \mathfrak{D}$. Moreover, for all $x \in \mathfrak{D}$, $\nabla g(x) = (\frac{e^{x_1}}{\sum_{i=1}^n e^{x_i}}, \ldots, \frac{e^{x_n}}{\sum_{i=1}^n e^{x_i}}) \neq 0$. Thus, in view of the discussion preceding this example, $\mathfrak{D}$ is C^2-cone reducible. Finally, notice that for any $x \in \mathfrak{D}$, the set

$$N_{\mathfrak{D}}(x) = \begin{cases} \bigcup_{\lambda \geq 0} \lambda\{\nabla g(x)\} & \text{if } g(x) = 0, \\ \{0\} & \text{if } g(x) < 0, \end{cases}$$

is polyhedral, and hence, $\{\mathcal{A}^{-1}\{\mathcal{A}\bar{x}\}, N_{\mathfrak{D}}(-\mathcal{A}^*\nabla\ell(\mathcal{A}\bar{x}) - v)\}$ is boundedly linearly regular [13, Corollary 5.26]. So, Theorem 3.8 implies that f satisfies the KL property with exponent $\frac{1}{2}$ at any stationary point $\bar{x}$.

(ii) **(Positive semidefinite cone constraints)** *Let $\mathbb{X} = \mathcal{S}^n$ and $\mathbb{Y} = \mathbb{R}^m$. Using the C^2-cone reducibility of $\mathcal{S}^n_+$, one can see that $f(X) = \ell(\mathcal{A}X) + \langle V, X\rangle + \delta_{\mathcal{S}^n_+}(X)$ satisfies the KL property with exponent $\frac{1}{2}$ at any stationary point $\bar{X}$ under the* *relative interior condition* $\mathcal{A}^{-1}\{\mathcal{A}\bar{X}\} \cap \mathrm{ri}\left(N_{-\mathcal{S}^n_+}(-\mathcal{A}^*\nabla\ell(\mathcal{A}\bar{X}) - V)\right) \neq \emptyset$. *We note that this result has also been derived in [44] via a different approach.*

(iii) **(Schatten p-norm regularization)** *Let* $\mathbb{X} = \mathcal{S}^n$ *and* $\mathbb{Y} = \mathbb{R}^m$. *Let* $p \in [1, 2] \cup \{\infty\}$ *and consider the following optimization model with Schatten p-norm regularization:*

$$f(X) = \ell(\mathcal{A}X) + \langle V, X\rangle + \tau\|X\|_p \quad \text{for all } X \in \mathcal{S}^n,$$

where $\|X\|_p = \left(\sum_{i=1}^n |\lambda_i(X)|^p\right)^{\frac{1}{p}}$ *and* $\lambda_n(X) \geq \lambda_{n-1}(X) \geq \cdots \geq \lambda_1(X)$ *are eigenvalues of* X. *The dual norm of* $\|\cdot\|_p$ *is the Schatten q-norm with* $\frac{1}{p} + \frac{1}{q} = 1$ *where* $q \in \{1\} \cup [2, \infty]$. *Let* $g(\lambda_1, \ldots, \lambda_n) = \left(\sum_{i=1}^n |\lambda_i|^q\right)^{\frac{1}{q}}$. *It can be directly verified that g is convex, symmetric and C^2-cone reducible. So,* $\|X\|_q = g(\lambda(X))$ *is also C^2-cone reducible [43, Proposition 3.2]. Thus, from Corollary 3.5, f satisfies the KL property with exponent $\frac{1}{2}$ at any nonzero stationary point $\bar{X}$ under the relative interior condition* (3.72) *with* $\gamma(X) = \|X\|_p$.

3.5 KL exponents via inf-projection for some nonconvex models

3.5.1 Difference-of-convex functions

In this section, we study a relationship between the KL exponents of the difference-of-convex (DC) function f in (3.31) and the auxiliary function F in (3.32). In [79, Theorem 4.1], it was shown that if f in (3.31) satisfies the KL property at $\bar{x} \in \mathrm{dom}\,\partial f$ with exponent $\frac{1}{2}$ and P_2 has globally Lipschitz gradient, then F in (3.32) satisfies the KL property at $(\bar{x}, \nabla P_2(\mathcal{A}\bar{x})) \in \mathrm{dom}\,\partial F$ with exponent $\frac{1}{2}$. Here we study the converse implication as a corollary to Theorem 3.4.

Theorem 3.9 (**KL exponent of DC functions**)**.** *Suppose that f and F are defined in* (3.31) *and* (3.32) *respectively. If F is a KL function with exponent $\alpha \in [0,1)$, then f is a KL function with exponent α.*

Proof. Let $\bar{x} \in \operatorname{dom} \partial f$. We will show that f satisfies the KL property at $\bar{x}$ with exponent α.

Note that we have $\operatorname{dom} \partial f = \operatorname{dom} \partial P_1$ thanks to [100, Corollary 10.9] and the fact that continuous convex functions are locally Lipschitz continuous. Hence, we actually have $\bar{x} \in \operatorname{dom} \partial P_1$.

Now, using [100, Exercise 8.8] and [100, Proposition 10.5], we have for any $\bar{\xi} \in \partial P_2(\mathcal{A}\bar{x})$ that

$$\partial F(\bar{x}, \bar{\xi}) = \begin{bmatrix} \partial P_1(\bar{x}) - \mathcal{A}^*\bar{\xi} \\ \partial P_2^*(\bar{\xi}) - \mathcal{A}\bar{x} \end{bmatrix} \supseteq \begin{bmatrix} \partial P_1(\bar{x}) - \mathcal{A}^*\bar{\xi} \\ 0 \end{bmatrix}. \tag{3.74}$$

where the inclusion follows from the fact that $\partial P_2^* = \partial P_2^{-1}$ (see [100, Proposition 11.3]). Since $\bar{x} \in \operatorname{dom} \partial P_1$, we see further from (3.74) that $\{\bar{x}\} \times \partial P_2(\mathcal{A}\bar{x}) \subseteq \operatorname{dom} \partial F$. Then condition (i) of Theorem 3.4 holds because one can show using (2.1) that $\operatorname{Arg\,min}_y F(\bar{x}, y) = \partial P_2(\mathcal{A}\bar{x})$. On the other hand, the assumption on KL property of F shows that condition (ii) of Theorem 3.4 holds. Now, it remains to prove that F is level-bounded in y locally uniformly in x before we can apply Theorem 3.4 to establish the desired KL property.

To this end, we will show that for any $x^* \in \mathbb{X}$ and $\beta \in \mathbb{R}$, the following set is bounded:

$$\{(x, y) : \ \|x - x^*\| \le 1, \ F(x, y) \le \beta\}. \tag{3.75}$$

Suppose to the contrary that the above set is unbounded for some x^* and β. Then there exists a sequence

$$\{(x^k, y^k)\} \subseteq \{(x, y) : \ \|x - x^*\| \le 1, \ F(x, y) \le \beta\} \tag{3.76}$$

with $\|y^k\| \to \infty$. Passing to a subsequence if necessary, we may assume without loss of generality that $x^k \to \widetilde{x}$ for some $\widetilde{x} \in B(x^*, 1)$ and that $\lim_k \frac{y^k}{\|y^k\|}$ exists. Denote this latter limit by d. Then $\|d\| = 1$. Next, using the definition of $\{(x^k, y^k)\}$ in (3.76) and the definition of F, we have for all sufficiently large k that

$$\beta \geq F(x^k, y^k) = P_1(x^k) - \langle \mathcal{A}x^k, y^k \rangle + P_2^*(y^k) \geq f(x^k) \tag{3.77}$$

$$\Rightarrow \frac{\beta}{\|y^k\|} \geq \frac{P_1(x^k)}{\|y^k\|} - \left\langle \mathcal{A}x^k, \frac{y^k}{\|y^k\|} \right\rangle + \frac{P_2^*(y^k)}{\|y^k\|}, \tag{3.78}$$

where the second inequality in (3.77) follows from the definition of Fenchel conjugate. Then we see in particular from (3.77) and the closedness of f that $\widetilde{x} \in \operatorname{dom} f = \operatorname{dom} P_1$. Using this, the closedness of P_1 and the definition of d, we have upon passing to limit inferior in (3.78) that

$$\begin{aligned}
0 &\geq -\langle \mathcal{A}\widetilde{x}, d \rangle + \liminf_{k\to\infty} \frac{P_2^*(y^k)}{\|y^k\|} \overset{\text{(a)}}{\geq} -\langle \mathcal{A}\widetilde{x}, d \rangle + (P_2^*)^\infty(d) \\
&\overset{\text{(b)}}{=} -\langle \mathcal{A}\widetilde{x}, d \rangle + \sigma_{\operatorname{dom} P_2}(d) = -\langle \mathcal{A}\widetilde{x}, d \rangle + \sup_{x \in \operatorname{dom} P_2} \{\langle x, d \rangle\},
\end{aligned}$$

where (a) follows from [9, Theorem 2.5.1] and (b) follows from [9, Theorem 2.5.4]. Since $\operatorname{dom} P_2 = \mathbb{Y}$, we deduce from the above inequality that $d = 0$, which contradicts the fact that $\|d\| = 1$. Thus, we have shown that (3.75) is bounded for any $x^* \in \mathbb{X}$ and any $\beta \in \mathbb{R}$, which implies that F is level-bounded in y locally uniformly in x. This completes the proof. □

3.5.2 Bregman envelope

In this section, we discuss the KL exponent of the Bregman envelope (3.33) of a proper closed function. We consider the following assumption on ϕ in (3.34), which is general enough for the corresponding (3.33) to include the celebrated Moreau envelope and the forward-backward envelope introduced in [104] as special cases. Further comments on this assumption will be given in Remark 3.9 below.

Assumption 3.1. *The function ϕ in* (3.34) *is twice continuously differentiable and there exists $a_1 > 0$ such that for all $x \in \mathbb{X}$,*

$$\nabla^2\phi(x) - a_1\mathcal{I} \succeq 0; \tag{3.79}$$

here $\mathcal{I}$ is the identity map, and for a linear map $\mathcal{A} : \mathbb{X} \to \mathbb{X}$, $\mathcal{A} \succeq 0$ means it is positive semidefinite, i.e., $\mathcal{A} = \mathcal{A}^*$ and $\langle h, \mathcal{A}h\rangle \geq 0$ for all $h \in \mathbb{X}$.

Given a proper closed function f and a function ϕ satisfying Assumption 3.1, we first analyze the KL property of the following auxiliary function:

$$F(x, y) := f(y) + \mathbb{B}_\phi(y, x) \tag{3.80}$$

with $\mathbb{B}_\phi$ defined in (3.34). For this function, applying [100, Proposition 8.8] and [100, Proposition 10.5], we have the following formula for ∂F at any $x \in \mathbb{X}$ and $y \in \operatorname{dom} f$,

$$\partial F(x, y) = \begin{bmatrix} -\nabla^2\phi(x)(y - x) \\ \partial f(y) + \nabla\phi(y) - \nabla\phi(x) \end{bmatrix}. \tag{3.81}$$

This formula will be used repeatedly in our discussion below.

Lemma 3.4. *Let $f : \mathbb{X} \to \mathbb{R} \cup \{\infty\}$ be a KL function with exponent $\alpha \in [\frac{1}{2}, 1)$. Let F be defined in* (3.80) *with ϕ satisfying Assumption 3.1. Then F is a KL function with exponent α.*

Proof. Thanks to [75, Lemma 2.1], it suffices to show that F satisfies the KL property at any point (x, y) with $0 \in \partial F(x, y)$. Let $(\bar{x}, \bar{y})$ be such that $0 \in \partial F(\bar{x}, \bar{y})$. Then in view of (3.81), we see that $0 \in \partial F(\bar{x}, \bar{y})$ implies that $\nabla^2\phi(\bar{x})(\bar{y} - \bar{x}) = 0$. Combining this with (3.79) we deduce that $\bar{y} = \bar{x}$.

Next, since f is a KL function with exponent α, there exist $c, \eta, \epsilon > 0$ such that

$$\frac{1}{c}\operatorname{dist}^{\frac{1}{\alpha}}(0, \partial f(y)) \geq f(y) - f(\bar{x}) \tag{3.82}$$

whenever $y \in B(\bar{x}, \epsilon) \cap \operatorname{dom}\partial f$ and $f(y) < f(\bar{x}) + \eta$. Since ϕ is twice continuously differentiable, by shrinking ϵ further if necessary, we see that there exists $b_1 > a_1$ with

a_1 being as in (3.79) such that for any $(x,y) \in B((\bar{x},\bar{x}),\epsilon)$, there exists $x_0 \in B(\bar{x},\epsilon)$ so that

$$\|\nabla\phi(y)-\nabla\phi(x)\| \le b_1\|y-x\| \text{ and } \langle y-x, \nabla\phi(y)-\nabla\phi(x)\rangle = \langle y-x, [\nabla^2\phi(x_0)](y-x)\rangle.$$

To the second relation in the above display, apply Cauchy-Schwartz inequality to the left hand side and apply (3.79) to the right hand side to obtain $\|y-x\|\|\nabla\phi(x) - \nabla\phi(y)\| \ge a_1\|y-x\|^2$. Combining this with the first relation in the above display, we obtain that

$$b_1\|y-x\| \ge \|\nabla\phi(y) - \nabla\phi(x)\| \ge a_1\|y-x\|. \tag{3.83}$$

Now, combining (3.81) with [75, Lemma 2.2], we deduce that there exists $C_0 > 0$ such that for $(x,y) \in B((\bar{x},\bar{x}),\epsilon)$ with $y \in \operatorname{dom}\partial f$,

$$\begin{aligned}
&\operatorname{dist}^{\frac{1}{\alpha}}(0, \partial F(x,y)) \ge C_0\left(\|\nabla^2\phi(x)(y-x)\|^{\frac{1}{\alpha}} + \inf_{\xi\in\partial f(y)}\|\xi + \nabla\phi(y) - \nabla\phi(x)\|^{\frac{1}{\alpha}}\right)\\
&\overset{\text{(a)}}{\ge} C_0\left(a_1^{\frac{1}{\alpha}}\|y-x\|^{\frac{1}{\alpha}} + (a_1b_1^{-1})^{\frac{1}{\alpha}}\inf_{\xi\in\partial f(y)}\|\xi + \nabla\phi(y) - \nabla\phi(x)\|^{\frac{1}{\alpha}}\right)\\
&\overset{\text{(b)}}{\ge} C_0\left(a_1^{\frac{1}{\alpha}}\|y-x\|^{\frac{1}{\alpha}} + (a_1b_1^{-1})^{\frac{1}{\alpha}}\inf_{\xi\in\partial f(y)}\eta_1\|\xi\|^{\frac{1}{\alpha}} - (a_1b_1^{-1})^{\frac{1}{\alpha}}\eta_2\|\nabla\phi(y) - \nabla\phi(x)\|^{\frac{1}{\alpha}}\right)\\
&\overset{\text{(c)}}{\ge} C_0\left(a_1^{\frac{1}{\alpha}}\|y-x\|^{\frac{1}{\alpha}} + (a_1b_1^{-1})^{\frac{1}{\alpha}}\inf_{\xi\in\partial f(y)}\eta_1\|\xi\|^{\frac{1}{\alpha}} - a_1^{\frac{1}{\alpha}}\eta_2\|y-x\|^{\frac{1}{\alpha}}\right)\\
&\ge C_1\left(\inf_{\xi\in\partial f(y)}\|\xi\|^{\frac{1}{\alpha}} + \|y-x\|^{\frac{1}{\alpha}}\right),
\end{aligned} \tag{3.84}$$

where (a) follows from (3.79) and the fact that $\left(\frac{a_1}{b_1}\right)^{\frac{1}{\alpha}} < 1$, (b) follows from [75, Lemma 3.1] for some $\eta_1 > 0$ and $\eta_2 \in (0,1)$, (c) follows from the first inequality in (3.83), and the last inequality holds with $C_1 := C_0\min\{(1-\eta_2)a_1^{\frac{1}{\alpha}}, \eta_1(a_1b_1^{-1})^{\frac{1}{\alpha}}\} > 0$.

Next, since $\nabla\phi$ is Lipschitz continuous on $B(\bar{x},\epsilon/2)$ with Lipschitz constant b_1 in view of (3.83), by shrinking ϵ further, we may assume $2b_1\epsilon^2 < 1$ and that for any

$(x, y) \in B((\bar{x}, \bar{x}), \epsilon)$,

$$0 \leq I\!B_\phi(y, x) = \phi(y) - \phi(x) - \langle \nabla\phi(x), y - x\rangle \leq \frac{b_1}{2}\|y - x\|^2 \leq \frac{b_1}{2}(2\epsilon)^2 < 1, \quad (3.85)$$

where the first inequality follows from the convexity of ϕ. Combining this with (3.84), we deduce further that for $(x, y) \in B((\bar{x}, \bar{x}), \epsilon)$ with $y \in \operatorname{dom}\partial f$ and $F(x, y) < F(\bar{x}, \bar{x}) + \eta$,

$$\begin{aligned}
&\operatorname{dist}^{\frac{1}{\alpha}}(0, \partial F(x, y)) \geq C_1 \left(\inf_{\xi \in \partial f(y)} \|\xi\|^{\frac{1}{\alpha}} + \left(2b_1^{-1} I\!B_\phi(y, x)\right)^{\frac{1}{2\alpha}} \right) \\
&\overset{\text{(a)}}{\geq} C_1 \left(\inf_{\xi \in \partial f(y)} \|\xi\|^{\frac{1}{\alpha}} + (2b_1^{-1})^{\frac{1}{2\alpha}} I\!B_\phi(y, x) \right) \\
&\overset{\text{(b)}}{=} C_1 c \left(\inf_{\xi \in \partial f(y)} c^{-1}\|\xi\|^{\frac{1}{\alpha}} + (2b_1^{-1})^{\frac{1}{2\alpha}} c^{-1} I\!B_\phi(y, x) \right) \\
&\overset{\text{(c)}}{\geq} C_2 \left(\inf_{\xi \in \partial f(y)} c^{-1}\|\xi\|^{\frac{1}{\alpha}} + I\!B_\phi(y, x) \right) \overset{\text{(d)}}{\geq} C_2 \left(f(y) - f(\bar{x}) + I\!B_\phi(y, x) \right) \\
&= C_2 \left(F(x, y) - F(\bar{x}, \bar{x}) \right)
\end{aligned}$$

where (a) holds because $\frac{1}{2\alpha} \leq 1$ and $I\!B_\phi(y, x) < 1$, thanks to (3.85), the constant c for (b) comes from (3.82), (c) holds with $C_2 := C_1 c \min\{1, (2b_1^{-1})^{\frac{1}{2\alpha}} c^{-1}\}$, (d) follows from (3.82) because $(x, y) \in B((\bar{x}, \bar{x}), \epsilon)$, $y \in \operatorname{dom}\partial f$ and $f(y) \leq F(x, y) < F(\bar{x}, \bar{x}) + \eta = f(\bar{x}) + \eta$, and the last equality holds because $f(\bar{x}) = F(\bar{x}, \bar{x})$. This completes the proof. $\square$

We are now ready to analyze the KL property of the Bregman envelope F_ϕ in (3.33).

Theorem 3.10 (KL exponent of Bregman envelope). *Let $f : \mathbb{X} \to \mathbb{R} \cup \{\infty\}$ be a proper closed function with* $\inf f > -\infty$. *Suppose that ϕ satisfies Assumption 3.1 and that f is a KL function with exponent $\alpha \in [\frac{1}{2}, 1)$. Then F_ϕ defined in* (3.33) *is a KL function with exponent α.*

Proof. Let F be defined as in (3.80). We will use Theorem 3.4 to deduce the KL exponent of F_ϕ from that of F. To this end, we need to check all the conditions required by Theorem 3.4.

First, we claim that F is level-bounded in y locally uniformly in x. To prove this, fix any $x_0 \in \mathbb{X}$ and $t \in \mathbb{R}$. Define

$$U_{x_0} := \{(x,y) : \ \|x - x_0\| \leq 1, \ F(x,y) \leq t\}.$$

Thus, it suffices to show that U_{x_0} is bounded. To this end, note that ϕ is strongly convex with modulus a_1 according to Assumption 3.1. We have from this and the definition of Bregman distance that for any $(x,y) \in U_{x_0}$,

$$\frac{a_1}{2}\|x-y\|^2 \leq \mathbb{B}_\phi(y,x).$$

Since $\inf f > -\infty$ by assumption, we deduce further that for any $(x,y) \in U_{x_0}$,

$$\inf f + \frac{a_1}{2}\|x-y\|^2 \leq \inf f + \mathbb{B}_\phi(y,x) \leq f(y) + \mathbb{B}_\phi(y,x) = F(x,y) \leq t.$$

Since $x \in B(x_0, 1)$, we deduce from the above inequality that U_{x_0} is bounded. Thus, we have shown that F is level-bounded in y locally uniformly in x.

Next, using [100, Exercise 8.8], we have for any $x \in \operatorname{dom}\partial F_\phi$ and any $\bar{y} \in \operatorname{Arg\,min}_y F(x,y)$ that

$$0 \in \partial f(\bar{y}) + \nabla \mathbb{B}_\phi(\cdot, x)(\bar{y}),$$

which implies that $\partial f(\bar{y}) \neq \emptyset$. This together with (3.81) implies that $\partial F(x,\bar{y}) \neq \emptyset$ for any such x and $\bar{y}$. In particular, condition (i) in Theorem 3.4 is satisfied.

Finally, note that condition (ii) in Theorem 3.4 is also satisfied thanks to Lemma 3.4. Thus, we deduce from Theorem 3.4 that F_ϕ satisfies the KL property with exponent α at any $x \in \operatorname{dom}\partial F_\phi$. $\square$

Remark 3.9. *The Bregman envelope* (3.33) *with* ϕ *satisfying Assumption 3.1 covers several envelopes studied in the literature.*

(i) *When* $\phi(\cdot) = \frac{1}{2\lambda}\|\cdot\|^2$ *with some* $\lambda > 0$, *the function* F_ϕ *in* (3.33) *becomes*

$$F_\phi(x) = \inf_y \left\{ f(y) + \frac{1}{2\lambda}\|x-y\|^2 \right\} =: e_\lambda f(x).$$

This function is known as the Moreau envelope of f. *In [75, Theorem 3.4], it was proved that if* f *is a convex KL function with exponent* $\alpha \in (0, \frac{2}{3})$ *that is continuous on* $\operatorname{dom}\partial f$, *then* $e_\lambda f$ *is a KL function with exponent* $\max\left\{\frac{1}{2}, \frac{\alpha}{2-2\alpha}\right\}$. *Here, without the convexity and continuity assumptions, we can obtain a tighter estimate on the KL exponent of* $e_\lambda f$ *via Theorem 3.10: if* f *is a KL function with exponent* $\alpha \in [\frac{1}{2}, 1)$ *and* $\inf f > -\infty$, *then* $e_\lambda f$ *is a KL function with exponent* α.

(ii) *If the function* f *in* (3.33) *takes the form* $h+g$, *where* g *is a proper closed function, and* h *is twice continuously differentiable with Lipschitz gradient whose modulus is less than* $\frac{1}{\gamma}$, *then the function* $\phi(x) := \frac{1}{2\gamma}\|x\|^2 - h(x)$ *is convex and satisfies Assumption 3.1. The forward-backward envelope* ψ_γ *of the function* $f = h + g$ *was defined in [104] as follows (see also the discussion in [78, Section 2]):*

$$\psi_\gamma(x) = \inf_y \{h(y) + g(y) + \mathbb{B}_\phi(y, x)\}.$$

In [78, Theorem 3.2], it was shown that if the first-order error bound condition (or error bound condition in the sense of Luo-Tseng) holds for $h+g$, *with* h *being in addition analytic and* g *being in addition convex, continuous on* $\operatorname{dom}\partial g$, *subanalytic and bounded below, then* ψ_γ *is a KL function with exponent* $\frac{1}{2}$. *Here, in view of Theorem 3.10, we can deduce the KL exponent of* ψ_γ *without the convexity and (sub)analyticity assumptions: if* $f = h + g$ *is a KL function with* *exponent* $\alpha \in [\frac{1}{2}, 1)$ *and* $\inf f > -\infty$, g *is a proper closed function, and* h *is twice continuously differentiable with Lipschitz gradient whose modulus is less than* $\frac{1}{\gamma}$, *then* ψ_γ *is a KL function with exponent* α.

(iii) *The* $\phi(x)$ *satisfying Assumption 3.1 can also be chosen as* $\frac{1}{4}\|x\|^2 + \frac{1}{2}\|x\|^2$, *which was proposed in [84, Section 2.1].*

3.5.3 Least squares loss function with rank constraint

In this section, we compute an explicit KL exponent of the function f in (3.35), which can be rewritten as an inf-projection as in (3.36). Now, observe further that one can relax the orthogonality constraint and introduce a penalty function without changing the optimal value in (3.36), i.e.,

$$f(X) = \inf_U \left\{ \underbrace{\frac{1}{2}\|\mathcal{A}X - b\|^2 + \frac{1}{2}\|U^TU - I_{m-k}\|_F^2 + \delta_{\widetilde{\mathfrak{D}}}(X,U)}_{\widetilde{f}(X,U)} + \delta_{\widetilde{\mathfrak{B}}}(X,U) \right\}, \quad (3.86)$$

where

$$\widetilde{\mathfrak{D}} := \{(X,U) \in \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times(m-k)} : \; U^TX = 0\},$$

$$\widetilde{\mathfrak{B}} := \{(X,U) \in \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times(m-k)} : 0.5I_{m-k} \preceq U^TU \preceq 2I_{m-k}\},$$

where $A \preceq B$ means the matrix $B - A$ is positive semidefinite. In view of (3.86), as another application of Theorem 3.4, we will deduce the KL exponent of f via that of $\widetilde{f} + \delta_{\widetilde{\mathfrak{B}}}$.

We now make use of Theorem 3.1 to deduce the KL exponent of $\widetilde{f} + \delta_{\widetilde{\mathfrak{B}}}$ in (3.86) at points $(\bar{X}, \bar{U}) \in \operatorname{dom}\partial(\widetilde{f} + \delta_{\widetilde{\mathfrak{B}}})$ with $\bar{U}^T\bar{U} = I_{m-k}$. For notational simplicity, we write

$$\tau := mn + m(m-k) + n(m-k) - 1. \quad (3.87)$$

Lemma 3.5. *The function* $\widetilde{f} + \delta_{\widetilde{\mathfrak{B}}}$ *given in* (3.86) *satisfies the KL property with exponent* $1 - \frac{1}{4\cdot 9^\tau}$ *at points* $(\bar{X}, \bar{U}) \in \operatorname{dom}\partial(\widetilde{f} + \delta_{\widetilde{\mathfrak{B}}})$ *with* $\bar{U}^T\bar{U} = I_{m-k}$*, where* τ *is given in* (3.87)*.*

Proof. Define the function $G : \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times(m-k)} \to \mathbb{R}^{(m-k)\times n}$ by $G(X, U) := U^T X$, one can rewrite $\widetilde{f}$ as

$$\widetilde{f}(X, U) = \frac{1}{2}\|\mathcal{A}X - b\|^2 + \frac{1}{2}\|U^T U - I_{m-k}\|_F^2 + \delta_{G^{-1}\{0\}}(X, U).$$

Now, for $X \in \mathbb{R}^{m\times n}$, $U \in \mathbb{R}^{m\times(m-k)}$ and $\Lambda \in \mathbb{R}^{(m-k)\times n}$, define

$$\widetilde{f}_1(X, U, \Lambda) := \frac{1}{2}\|\mathcal{A}X - b\|^2 + \frac{1}{2}\|U^T U - I_{m-k}\|_F^2 + \operatorname{tr}(\Lambda^T U^T X).$$

Note that $\widetilde{f}_1$ is a polynomial of degree 4 on $\mathbb{R}^\tau$ where τ is given in (3.87). We deduce from [45, Theorem 4.2] that $\widetilde{f}_1$ is a KL function with exponent $1 - \frac{1}{4\cdot 9^\tau}$.

Next, since $(\bar{X}, \bar{U}) \in \operatorname{dom}\partial(\widetilde{f} + \delta_{\widetilde{\mathfrak{B}}})$ with $\bar{U}^T\bar{U} = I_{m-k}$, we see that $(\bar{X}, \bar{U})$ lies in the interior of $\widetilde{\mathfrak{B}}$. Thus, we have $(\bar{X}, \bar{U}) \in \operatorname{dom}\partial\widetilde{f}$. We will now check the conditions in Theorem 3.1 for the functions $\widetilde{f}_1$ and $\widetilde{f}$ (in place of g_1 and g, respectively) at $(\bar{X}, \bar{U})$. Notice first that the functions $(X, U) \mapsto \frac{1}{2}\|\mathcal{A}X - b\|^2 + \frac{1}{2}\|U^T U - I_{m-k}\|_F^2$ and G are continuously differentiable, and $G^{-1}\{0\}$ is clearly nonempty. We next claim that the linear map $\nabla G(\bar{X}, \bar{U})$ is injective. To this end, let $Y \in \ker\nabla G(\bar{X}, \bar{U})$. Then, using the definition of the derivative mapping of G, for any $(H, K) \in \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times(m-k)}$, we have

$$\begin{aligned} 0 &= \langle (H, K), [\nabla G(\bar{X}, \bar{U})](Y)\rangle = \langle [DG(\bar{X}, \bar{U})](H, K), Y\rangle \\ &= \langle \bar{U}^T H + K^T\bar{X}, Y\rangle = \langle H, \bar{U}Y\rangle + \langle \bar{X}Y^T, K\rangle. \end{aligned}$$

Since H and K are arbitrary, we deduce that

$$\bar{U}Y = 0 \ \text{ and } \ \bar{X}Y^T = 0.$$

These together with $\bar{U}^T\bar{U} = I_{m-k}$ imply that $Y = 0$. Thus, we have $\ker(\nabla G(\bar{X}, \bar{U})) = \{0\}$, i.e., $\nabla G(\bar{X}, \bar{U})$ is an injective linear map. Now, using Theorem 3.1, we conclude that $\widetilde{f}$ satisfies the KL property at $(\bar{X}, \bar{U})$ with exponent $1 - \frac{1}{4 \cdot 9^\tau}$.

Finally, since $(\bar{X}, \bar{U}) \in \operatorname{int} \widetilde{\mathfrak{B}}$, one can verify directly from the definition that, at $(\bar{X}, \bar{U})$, the KL exponent of $\widetilde{f} + \delta_{\widetilde{\mathfrak{B}}}$ is the same as that of $\widetilde{f}$. This completes the proof. $\square$

Now we are ready to compute the KL exponent of f in (3.35). Interestingly, the derived KL exponent can be determined explicitly in terms of the number of rows/columns of the matrix involved and the upper bound constant in the rank constraint.

Theorem 3.11. *The function f given in* (3.35) *is a KL function with exponent $1 - \frac{1}{4 \cdot 9^\tau}$, where τ is given in* (3.87).

Proof. Notice that $f(X) = \inf_U(\widetilde{f} + \delta_{\widetilde{\mathfrak{B}}})(X, U)$ and that for any $X \in \operatorname{dom} \partial f$,

$$\operatorname*{Arg\,min}_U(\widetilde{f} + \delta_{\widetilde{\mathfrak{B}}})(X, U) = \{U : \ U^T X = 0 \text{ and } U^T U = I_{m-k}\}, \tag{3.88}$$

where $\widetilde{f} + \delta_{\widetilde{\mathfrak{B}}}$ is given in (3.86). We will check the conditions in Theorem 3.4 and apply the theorem to deducing the KL exponent of f.

First, the function $\widetilde{f} + \delta_{\widetilde{\mathfrak{B}}}$ is clearly proper and closed. Next, for any fixed X, the U with $(X, U) \in \widetilde{\mathfrak{D}} \cap \widetilde{\mathfrak{B}}$ satisfies $0.5 I_{m-k} \preceq U^T U \preceq 2 I_{m-k}$. This shows that $\widetilde{f} + \delta_{\widetilde{\mathfrak{B}}}$ is bounded in U locally uniformly in X. Furthermore, for any $X \in \operatorname{dom} \partial f$ and any $U \in \operatorname{Arg\,min}_U(\widetilde{f} + \delta_{\widetilde{\mathfrak{B}}})(X, U)$, we have using (3.88) and [100, Exercise 8.8] that

$$\partial(\widetilde{f} + \delta_{\widetilde{\mathfrak{B}}})(X, U) = (\mathcal{A}^*(\mathcal{A}X - b), 0) + N_{\widetilde{\mathfrak{D}} \cap \widetilde{\mathfrak{B}}}(X, U) \neq \emptyset.$$

These together with (3.88) and Lemma 3.5 implies that the conditions required by Theorem 3.4 are satisfied. Applying Theorem 3.4, we conclude that f is a KL function of exponent $1 - \frac{1}{4 \cdot 9^\tau}$. $\square$

Chapter 4

KL property in the study of SCP$_{ls}$

In this chapter, we show how KL property can be applied in the convergence analysis of the SCP$_{ls}$ method for multiply constrained difference-of-convex model introduced in Section 1.2.

4.1 Convergence properties of SCP$_{ls}$

4.1.1 Convergence analysis in nonconvex settings

In this section, we analyze SCP$_{ls}$ when F in (1.1) is possibly nonconvex. We first prove some basic properties of the sequence generated by SCP$_{ls}$. Item (iii) in the following theorem was already proved in [83, Theorem 3.7]; we include its proof here.

Theorem 4.1. *Consider* (1.1) *and suppose that Assumptions 1.1 and 2.1 hold. Let* $\{(x^t, L_g^t)\}$ *be generated by SCP$_{ls}$. Then the following statements hold:*

(i) *The sequence* $\{x^t\}$ *is bounded.*

(ii) *The sequence* $\{\bar{F}(x^{t+1}, x^t, L_g^t)\}$ *is nonincreasing and convergent to some real number* $\bar{F}^*$*, where* $\bar{F}$ *is defined as in* (1.3)*. Moreover, for any* $t \geq 1$*, we have*

$$\bar{F}(x^{t+1}, x^t, L_g^t) \leq \bar{F}(x^t, x^{t-1}, L_g^{t-1}) - \frac{c}{2}\|x^{t+1} - x^t\|^2. \tag{4.1}$$

(iii) *It holds that* $\lim\limits_{t\to\infty} \|x^{t+1} - x^t\| = 0$.

Proof. Let F be defined as in (1.1). Then for any $t \geq 0$, we have

$$F(x^{t+1}) - F(x^0) = \sum_{i=0}^{t}[F(x^{i+1}) - F(x^i)] \leq -\sum_{i=0}^{t}\frac{c}{2}\|x^{i+1} - x^i\|^2 \leq 0, \tag{4.2}$$

where the first inequality follows from (2.11). Since F is level-bounded by Assumption 1.1(iii), we deduce that $\{x^t\}$ is bounded and the conclusion in item (i) holds.

We now prove (ii). Since for any $t \geq 0$, the x^{t+1} belongs to $\operatorname{dom} F$ and is feasible for (2.10) with $(\widetilde{L}_f, \widetilde{L}_g) = (L_f^t, L_g^t)$, it holds that

$$\bar{F}(x^{t+1}, x^t, L_g^t) = F(x^{t+1}) \;\text{ for } t \geq 0. \tag{4.3}$$

This together with (2.11) shows that $\{\bar{F}(x^{t+1}, x^t, L_g^t)\}$ is nonincreasing and (4.1) holds for all $t \geq 1$. Also, thanks to (4.3) and Assumption 1.1, we have

$$\inf_t \bar{F}(x^{t+1}, x^t, L_g^t) = \inf_t F(x^t) \geq \inf F > -\infty,$$

implying that $\{\bar{F}(x^{t+1}, x^t, L_g^t)\}$ is bounded from below. Thus, we conclude that the sequence $\{\bar{F}(x^{t+1}, x^t, L_g^t)\}$ is convergent. We denote this limit by $\bar{F}^*$.

Finally, we prove (iii). Since $\{\bar{F}(x^{t+1}, x^t, L_g^t)\}$ converges to $\bar{F}^*$, passing to the limit as t goes to infinity in (4.2) and invoking (4.3), we have

$$\sum_{i=0}^{\infty}\frac{c}{2}\|x^{i+1} - x^i\|^2 \leq F(x^0) - \lim_{t\to\infty}\bar{F}(x^{t+1}, x^t, L_g^t) = F(x^0) - \bar{F}^* < \infty.$$

Therefore, item (iii) holds. This completes the proof. $\square$

Next, we show that $\{\lambda^t\}$ with each λ^t being a Lagrange multiplier[1] of (2.10) with $(\widetilde{L}_f, \widetilde{L}_g) = (L_f^t, L_g^t)$ is bounded and any cluster point of the sequence $\{x^t\}$ generated by SCP_{ls} is a stationary point of (1.1) in the sense of Definition 2.2. The latter conclusion was also proved in [83, Theorem 3.7]. We include its proof for completeness.

[1] The existence of λ^t follows from Lemma 2.4(iv).

Theorem 4.2. *Consider* (1.1) *and suppose that Assumptions 1.1 and 2.1 hold. Let* $\{x^t\}$ *be the sequence generated by* SCP_{ls} *and* λ^t *be a Lagrange multiplier of* (2.10) *with* $(\widetilde{L}_f, \widetilde{L}_g) = (L_f^t, L_g^t)$*. Then the sequence* $\{\lambda^t\}$ *is bounded and any accumulation point of* $\{x^t\}$ *is a stationary point of* (1.1)*.*

Proof. Suppose to the contrary that $\{\lambda^t\}$ is unbounded and let $\{\lambda^{t_j}\}$ be a subsequence of $\{\lambda^t\}$ such that $\|\lambda^{t_j}\| \xrightarrow{j} \infty$. Passing to a further subsequence if necessary, we may assume that there exist $\lambda^* \in \mathbb{R}^m_+$ and x^* such that $\lim\limits_{j\to\infty} \frac{\lambda^{t_j}}{\|\lambda^{t_j}\|} = \lambda^*$ and $\lim\limits_{j\to\infty} x^{t_j} = x^*$, where the existence of x^* is due to Theorem 4.1(i).

Using (2.13), the definition of $\widetilde{L}_{fg}$ there and the fact $(\widetilde{L}_f, \widetilde{L}_g) = (L_f^t, L_g^t)$, we have

$$\eta^t := \sum_{i=1}^m \lambda_i^t \left[\nabla g_i(x^t) + (L_g^t)_i(x^{t+1} - x^t)\right] \in -\nabla f(x^t) - L_f^t(x^{t+1} - x^t) - \partial P_1(x^{t+1}) + \xi^t.$$

Since the functions ∇f, P_1 and P_2 are continuous, and $\{(x^t, L_f^t)\}$ is bounded thanks to Theorem 4.1(i) and Lemma 2.4(ii), we deduce from the above display that $\{\eta^t\}$ is bounded. Then, dividing η^{t_j} by $\|\lambda^{t_j}\|$ and letting $j \to \infty$, using the continuity of ∇g and Theorem 4.1(iii) together with Lemma 2.4(ii), we deduce further that

$$\sum_{i=1}^m \lambda_i^* \nabla g_i(x^*) = 0. \tag{4.4}$$

On the other hand, using (2.12) with $(\widetilde{x}, \widetilde{\lambda}, \widetilde{L}_g) = (x^{t+1}, \lambda^t, L_g^t)$, the continuity of ∇g_i for each i, Lemma 2.4(ii) and Theorem 4.1(iii), we see that $\lambda_i^* g_i(x^*) = 0$ for all $i = 1, \ldots, m$. This further implies that

$$\lambda_i^* = 0 \text{ for } i \notin I(x^*).$$

The above display and (4.4) imply that

$$\sum_{i\in I(x^*)} \lambda_i^* \nabla g_i(x^*) = 0.$$

Combining this with MFCQ (Assumption 2.1) and recalling that $\lambda^* \in \mathbb{R}^m_+$, we conclude that $\lambda^*_i = 0$ for $i \in I(x^*)$. Therefore, we have $\lambda^* = 0$, contradicting the fact that $\|\lambda^*\| = 1$. Thus, the sequence $\{\lambda^t\}$ is bounded.

For the second conclusion of this theorem, let $\bar{x}$ be an accumulation point of $\{x^t\}$ with $\lim_{k\to\infty} x^{t_k} = \bar{x}$. Since $\{\lambda^t\}$ is bounded, passing to a further subsequence if necessary, we assume without loss of generality that $\lim_{k\to\infty} \lambda^{t_k} = \bar{\lambda}$ for some $\bar{\lambda}$. Since the sequence $\{(L^t_f, L^t_g, \lambda^t)\}$ is bounded thanks to Lemma 2.4(ii) and the boundedness of $\{\lambda^t\}$, using Theorem 4.1(iii), we have that $\lim_{k\to\infty} \left(L^{t_k}_f + \langle \lambda^{t_k}, L^{t_k}_g \rangle\right)(x^{t_k+1} - x^{t_k}) = 0$. Using this fact together with the closedness of ∂P_1 and ∂P_2, the Lipschitz continuity of ∇f and ∇g and Theorem 4.1(iii), we have upon passing to the limit as k goes to infinity in (2.13) with $(\widetilde{x}, \widetilde{\lambda}, \widetilde{L}_f, \widetilde{L}_g) = (x^{t_k+1}, \lambda^{t_k}, L^{t_k}_f, L^{t_k}_g)$ and $t = t_k$ that

$$0 \in \nabla f(\bar{x}) + \partial P_1(\bar{x}) - \partial P_2(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i \nabla g_i(\bar{x}). \tag{4.5}$$

On the other hand, using (2.12) with $(\widetilde{x}, \widetilde{\lambda}, \widetilde{L}_g) = (x^{t_k+1}, \lambda^{t_k}, L^{t_k}_g)$ and $t = t_k$, letting $k \to \infty$, we have upon using the continuity of ∇g, Theorem 4.1(iii) and Lemma 2.4(ii) that

$$\bar{\lambda}_i g_i(\bar{x}) = 0 \text{ for all } i = 1, \ldots, m. \tag{4.6}$$

Finally, since $\lambda^t \geq 0$ for any $t \geq 0$, we have $\bar{\lambda} \geq 0$. Also, since g_i is continuous for each i and $g(x^t) \leq 0$ thanks to Step 3a) of SCP$_{ls}$, we have $g(\bar{x}) \leq 0$. These together with (4.5) and (4.6) imply that $\bar{x}$ is a stationary point of (1.1). □

Lemma 4.1. *Consider* (1.1) *and suppose that Assumptions 1.1 and 2.1 hold. Let* $\{(x^t, L^t_g)\}$ *be the sequence generated by SCP$_{ls}$ and let* Ω *be the set of accumulation points of the sequence* $\{(x^{t+1}, x^t, L^t_g)\}$. *Then* $\Omega \neq \emptyset$ *and* $\bar{F} \equiv \bar{F}^*$ *on* Ω, *where* $\bar{F}$ *is defined as in* (1.3) *and* $\bar{F}^*$ *is given in Theorem 4.1(ii).*

Proof. From Theorem 4.1(i) and Lemma 2.4(ii) we know that $\Omega \neq \emptyset$. Fix any $(x^\Omega, y^\Omega, L^\Omega) \in \Omega$ and let $\{(x^{t_j+1}, x^{t_j}, L_g^{t_j})\}$ be a subsequence that converges with $\lim_{j\to\infty}(x^{t_j+1}, x^{t_j}, L_g^{t_j}) = (x^\Omega, y^\Omega, L^\Omega)$. Since each ∇g_i is continuous and x^{t_j+1} belongs to $\operatorname{dom} F$ and is feasible for (2.10) with $t = t_j$ and $(\widetilde{L}_f, \widetilde{L}_g) = (L_f^{t_j}, L_g^{t_j})$, we have

$$g(x^\Omega) = \lim_{j\to\infty} g(x^{t_j+1}) \le 0, \quad \bar{G}(x^\Omega, y^\Omega, L^\Omega) = \lim_{j\to\infty} \bar{G}(x^{t_j+1}, x^{t_j}, L_g^{t_j}) \le 0 \tag{4.7}$$

and $F(x^{t_j+1}) = \bar{F}(x^{t_j+1}, x^{t_j}, L_g^{t_j})$ for all j. Then, using the continuity of F on its closed domain, we have

$$F(x^\Omega) = \lim_{j\to\infty} F(x^{t_j+1}) = \lim_{j\to\infty} \bar{F}(x^{t_j+1}, x^{t_j}, L_g^{t_j}) = \bar{F}^*,$$

where the last equality follows from Theorem 4.1(ii). Thus, we deduce that

$$\bar{F}(x^\Omega, y^\Omega, L^\Omega) = F(x^\Omega) = \bar{F}^*,$$

where the first equality follows from (4.7). Since $(x^\Omega, y^\Omega, L^\Omega) \in \Omega$ is arbitrary, we conclude that $\bar{F} \equiv \bar{F}^*$ on Ω. $\square$

To analyze the global convergence properties of SCP$_{ls}$, we need a bound on the subdifferential of $\bar{F}$ in (1.3). To this end, we consider the following additional differentiability assumption on g_i.

Assumption 4.1. *Each g_i in* (1.1) *is twice continuously differentiable.*

Lemma 4.2. *Consider* (1.1) *and suppose that Assumption 4.1 holds. Let $(x, y, w) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ and assume that P_2 is continuously differentiable around x. Then*

$$\partial \bar{F}(x, y, w) \supseteq \begin{pmatrix} \nabla f(x) - \nabla P_2(x) + \partial P_1(x) + \sum_{i=1}^m \lambda_i [\nabla g_i(y) + w_i(x - y)] \\ \sum_{i=1}^m \lambda_i [\nabla^2 g_i(y)(x - y) - w_i(x - y)] \\ \frac{1}{2}\|x - y\|^2 \lambda \end{pmatrix} \tag{4.8}$$

whenever $\lambda \in N_{-\mathbb{R}_+^m}(\bar{G}(x, y, w))$, where $\bar{F}$ and $\bar{G}$ are defined as in (1.3).

Proof. We only consider the case where $(x,y,w) \in \mathrm{dom}\bar{F}$, since (4.8) holds trivially otherwise. Using [100, Exercise 8.8, Corollary 10.9, Proposition 10.5], we have

$$\partial \bar{F}(x,y,w) \supseteq \widehat{\partial}\bar{F}(x,y,w) \supseteq \begin{pmatrix} \nabla f(x) - \nabla P_2(x) + \widehat{\partial} P_1(x) \\ 0 \\ 0 \end{pmatrix} + \widehat{\partial}\delta_{\bar{G}(\cdot)\le 0}(x,y,w)$$

$$\overset{\text{(a)}}{=} \begin{pmatrix} \nabla f(x) - \nabla P_2(x) + \partial P_1(x) \\ 0 \\ 0 \end{pmatrix} + \widehat{N}_{\bar{G}(\cdot)\le 0}(x,y,w)$$

$$\overset{\text{(b)}}{\supseteq} \begin{pmatrix} \nabla f(x) - \nabla P_2(x) + \partial P_1(x) \\ 0 \\ 0 \end{pmatrix} + \sum_{i=1}^{m} \lambda_i \begin{pmatrix} \nabla g_i(y) + w_i(x-y) \\ \nabla^2 g_i(y)(x-y) - w_i(x-y) \\ \frac{1}{2}\|x-y\|^2 e_i \end{pmatrix},$$

where (a) uses the convexity of P_1 and [100, Proposition 8.12], $e_i \in \mathbb{R}^m$ is the i^{th} standard basis vector and (b) holds for any $\lambda \in \widehat{N}_{-\mathbb{R}^m_+}(\bar{G}(x,y,w)) = N_{-\mathbb{R}^m_+}(\bar{G}(x,y,w))$, thanks to [100, Theorem 6.14]. $\square$

We also need the following assumption to derive the desired bound on $\partial\bar{F}$. This assumption was also used in [119] for analyzing the global convergence property of the sequence generated by the proximal DCA with extrapolation (pDCA$_e$).

Assumption 4.2. *Each g_i in* (1.1) *is smooth, and the P_2 in* (1.1) *is continuously differentiable on an open set Γ that contains all stationary points of* (1.1). *Moreover, the function ∇P_2 is locally Lipschitz continuous on Γ.*

Using this assumption and Lemma 4.2, we can prove the following property of $\partial\bar{F}$.

Lemma 4.3. *Consider* (1.1) *and suppose that Assumptions 1.1, 2.1, 4.1 and 4.2 hold. Let $\{(x^t, L_g^t)\}$ be the sequence generated by SCP$_{ls}$ and let $\bar{F}$ be defined as in* (1.3). *Then there exist $\kappa > 0$ and $\underline{t} \in \mathcal{N}_+$ such that*

$$\mathrm{dist}(0, \partial\bar{F}(x^{t+1}, x^t, L_g^t)) \le \kappa\|x^{t+1} - x^t\| \quad \text{for all } t > \underline{t}. \tag{4.9}$$

Proof. From Theorem 4.1(i), we know that $\{x^t\}$ is bounded. Thus, denoting the set of accumulation points of $\{x^t\}$ as Ω_x, we have that Ω_x is compact and $\Omega_x \subseteq \Gamma$ thanks to Theorem 4.2, where Γ is the open set give in Assumption 4.2. Choose an $\epsilon > 0$ so that $\Gamma_\epsilon := \{x : \operatorname{dist}(x, \Omega_x) < \epsilon\} \subseteq \Gamma$ and ∇P_2 is Lipschitz continuous with modulus L_{P_2} on Γ_ϵ, which exists thanks to the compactness of Ω_x and Assumption 4.2. Moreover, since Ω_x is compact, from the definition of cluster points, we see that there exists $t_0 \in \mathcal{N}_+$ such that $\operatorname{dist}(x^t, \Omega_x) < \epsilon$ whenever $t > t_0$. In particular, P_2 is continuously differentiable around each x^t whenever $t > t_0$. In addition, thanks to Theorem 4.1(iii), we can further choose $\underline{t} > t_0 + 1$ such that for $t > \underline{t}$, we have

$$\|x^{t+1} - x^t\|^2 \leq \|x^{t+1} - x^t\|. \tag{4.10}$$

Now, let λ^t be a Lagrange multiplier of (2.10) with $(\widetilde{L}_f, \widetilde{L}_g) = (L_f^t, L_g^t)$, which exists thanks to Lemma 2.4(iv). Then it holds that $\lambda^t \in N_{-\mathbb{R}_+^m}(\bar{G}(x^{t+1}, x^t, L_g^t))$. Therefore, using (4.8) with $\lambda = \lambda^t$ for any $t > \underline{t}$, we have that

$$\partial \bar{F}(x^{t+1}, x^t, L_g^t) \supseteq \begin{pmatrix} J^t \\ \sum_{i=1}^m \lambda_i^t \left(\nabla^2 g_i(x^t)(x^{t+1} - x^t) - (L_g^t)_i(x^{t+1} - x^t)\right) \\ \frac{1}{2}\|x^{t+1} - x^t\|^2 \lambda^t \end{pmatrix} \tag{4.11}$$

with $J^t := \nabla f(x^{t+1}) + \partial P_1(x^{t+1}) - \nabla P_2(x^{t+1}) + \sum_{i=1}^m \lambda_i^t \left(\nabla g_i(x^t) + (L_g^t)_i(x^{t+1} - x^t)\right)$. For this J^t, using (2.13) with $\widetilde{x} = x^{t+1}$ and recalling the definition of ξ^t, we have that

$$\begin{aligned} J^t \ni & \nabla f(x^{t+1}) - \nabla P_2(x^{t+1}) + \sum_{i=1}^m \lambda_i^t \left(\nabla g_i(x^t) + (L_g^t)_i(x^{t+1} - x^t)\right) \\ & + \left(-\nabla f(x^t) - L_f^t(x^{t+1} - x^t) + \nabla P_2(x^t) - \sum_{i=1}^m \lambda_i^t \left(\nabla g_i(x^t) + (L_g^t)_i(x^{t+1} - x^t)\right)\right) \\ = & \nabla f(x^{t+1}) - \nabla f(x^t) + \nabla P_2(x^t) - \nabla P_2(x^{t+1}) - L_f^t(x^{t+1} - x^t). \end{aligned}$$

Using this together with Cauchy-Schwarz inequality, for $t > \underline{t}$, it holds that

$$\begin{aligned}
\|J^t\|^2 &\leq 3\Big(\|\nabla f(x^{t+1}) - \nabla f(x^t)\|^2 + \|\nabla P_2(x^{t+1}) - \nabla P_2(x^t)\|^2 + \|L_f^t(x^{t+1} - x^t)\|^2\Big) \\
&\overset{(a)}{\leq} 3L_f^2\|x^{t+1} - x^t\|^2 + 3L_{P_2}^2\|x^{t+1} - x^t\|^2 + 3(L_f^t)^2\|x^{t+1} - x^t\|^2 \\
&= \Big(3L_f^2 + 3(L_f^t)^2 + 3L_{P_2}^2\Big)\|x^{t+1} - x^t\|^2,
\end{aligned} \tag{4.12}$$

where (a) makes use of the fact that $t > \underline{t}$ (so that $x^t \in \Gamma_\epsilon$) and the Lipschitz continuity of ∇f and ∇P_2.

On the other hand, since $\{(x^t, L_g^t, \lambda^t)\}$ is bounded thanks to Theorem 4.1(i), Lemma 2.4(ii) and Theorem 4.2, using the continuity of $\nabla^2 g_i$ for each i, there exists $D_1 > 0$ such that

$$\begin{aligned}
&\left\|\sum_{i=1}^m \lambda_i^t\Big(\nabla^2 g_i(x^t)(x^{t+1} - x^t) - (L_g^t)_i(x^{t+1} - x^t)\Big)\right\|^2 \\
&\leq m\sum_{i=1}^m (\lambda_i^t)^2\|\nabla^2 g_i(x^t)(x^{t+1} - x^t) - (L_g^t)_i(x^{t+1} - x^t)\|^2 \leq D_1\|x^{t+1} - x^t\|^2,
\end{aligned} \tag{4.13}$$

where the first inequality uses the Cauchy-Schwarz inequality.

Therefore, since $\{(L_f^t, \lambda^t)\}$ is bounded thanks to Lemma 2.4(ii) and Theorem 4.2, combining (4.10), (4.11), (4.12) and (4.13), we conclude that there exists $\kappa > 0$ such that (4.9) holds. This completes the proof. $\square$

Now, if we suppose in addition that $\bar{F}$ is a KL function with exponent $\alpha \in [0, 1)$, then using the results above and following the analysis in [6–8, 24, 79, 119], we can deduce the convergence of the sequence $\{x^t\}$ generated by SCP_{ls} to a stationary point of (1.1) and estimate its local convergence rate. Specifically, using similar proofs as in [79, 119], we have the following results. The lines of arguments are standard and we omit its proof for brevity.

Theorem 4.3 (**Convergence rate of SCP$_{ls}$ in nonconvex settings**). *Consider* (1.1). *Suppose that Assumptions 1.1, 2.1, 4.1 and 4.2 hold, and $\bar{F}$ in* (1.3) *is a KL function. Let $\{(x^t, L_g^t)\}$ be the sequence generated by SCP$_{ls}$ and let Ω be the set of accumulation points of the sequence $\{(x^{t+1}, x^t, L_g^t)\}$. Then $\{x^t\}$ converges to a stationary point x^* of* (1.1). *Moreover, if $\bar{F}$ satisfies the KL property with exponent $\alpha \in [0, 1)$ at every point in Ω, then there exists $\underline{t} \in \mathcal{N}_+$ such that the following statements hold:*

(i) *If $\alpha = 0$, then $\{x^t\}$ converges finitely, i.e., $x^t \equiv x^*$ for $t > \underline{t}$.*

(ii) *If $\alpha \in (0, \frac{1}{2}]$, then there exist $a_0 \in (0, 1)$ and $a_1 > 0$ such that*

$$\|x^t - x^*\| \le a_1 a_0^t \quad \text{for } t > \underline{t}.$$

(iii) *If $\alpha \in (\frac{1}{2}, 1)$, then there exists $a_2 > 0$ such that*

$$\|x^t - x^*\| \le a_2 t^{-\frac{1-\alpha}{2\alpha-1}} \quad \text{for } t > \underline{t}.$$

4.1.2 Convergence analysis in convex settings

In this section, we study the convergence properties of SCP$_{ls}$ under the following convex settings:

Assumption 4.3. *Suppose that in* (1.1), *$P_2 = 0$ and $\{f, g_1, \ldots, g_m\}$ are convex.*

Assumption 4.3 was also considered in [19, Section 3.2.3] for analyzing MBA, and in [19, Section 4] for its line search variant Multiprox$_{bt}$ [19, Eq. (37)]. Here, we would like to point out that the line search criterion in Multiprox$_{bt}$ [19, Eq. (37)] is different from the criterion (2.11) used in SCP$_{ls}$. The criterion in Multiprox$_{bt}$ relies on a local majorant of the objective function, while (2.11) uses the objective function directly, and is originated from SpaRSA; see [120, Eq. (22)]. We will establish global convergence of the whole sequence generated by SCP$_{ls}$ in the above convex settings, under suitable assumptions. Unlike the analysis in the previous subsection, our analysis here is based on KL property of F in (1.1) instead of that of $\bar{F}$, and we will *not* assume g to be twice continuously differentiable (i.e., we do not require Assumption 4.1). We start with two auxiliary lemmas. The first lemma is an analogue of [24, Lemma 6] and follows immediately from an application of [22, Theorem 5] and standard compactness argument. We omit the proof for brevity.

Lemma 4.4. *Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be a level-bounded proper closed convex function with $\Lambda := \operatorname{Arg\,min} f \neq \emptyset$. Let $\underline{f} := \inf f$. Suppose that f satisfies the KL property at each point in Λ with exponent $\alpha \in [0, 1)$. Then there exist $\epsilon > 0$, $r_0 > 0$ and $c_0 > 0$ such that*

$$\operatorname{dist}(x, \Lambda) \le c_0(f(x) - \underline{f})^{1-\alpha}$$

for any $x \in \operatorname{dom}\partial f$ satisfying $\operatorname{dist}(x, \Lambda) \le \epsilon$ and $\underline{f} \le f(x) < \underline{f} + r_0$.

The next lemma is an analogue of Lemma 4.1 for F in (1.1).

Lemma 4.5. *Consider (1.1) and suppose that Assumptions 1.1 and 2.1 hold. Let $\{x^t\}$ be the sequence generated by SCP$_{ls}$ for (1.1) and let Ω_x be the set of accumulation point of $\{x^t\}$. Then the following statements hold:*

(i) *It holds that $\Omega_x \neq \emptyset$ and $F \equiv \bar{F}^*$ on Ω_x, where F is defined as in (1.1) and $\bar{F}^*$ is given in Theorem 4.1(ii).*

(ii) *The sequence $\{F(x^t)\}$ is nonincreasing and convergent to $\bar{F}^*$.*

Proof. We note first from Theorem 4.1(i) that $\Omega_x \neq \emptyset$. In addition, since $x^t \in \operatorname{dom} F$ and is feasible for (2.10) (with $(t-1, L_f^{t-1}, L_g^{t-1})$ in place of $(t, \widetilde{L}_f, \widetilde{L}_g)$), we have

$$F(x^t) = f(x^t) + P_1(x^t) - P_2(x^t) = \bar{F}(x^t, x^{t-1}, L_g^{t-1}), \text{ for all } t \ge 1. \tag{4.14}$$

Fix any $x^* \in \Omega_x$ and let $\lim_{j\to\infty} x^{t_j} = x^*$. Using the continuity of F on its closed domain and (4.14), we see that

$$F(x^*) = \lim_{j\to\infty} f(x^{t_j}) + P_1(x^{t_j}) - P_2(x^{t_j}) = \lim_{j\to\infty} \bar{F}(x^{t_j}, x^{t_j-1}, L_g^{t_j-1}) = \bar{F}^*,$$

where the last equality makes use of Theorem 4.1(ii). This proves (i). The conclusion in (ii) now follows immediately upon combining the above display and (4.14) with Theorem 4.1(ii). This completes the proof. □

Now we present our main result in this subsection.

Theorem 4.4 (**Convergence rate of SCP$_{ls}$ in convex settings**). *Consider* (1.1) *and suppose that Assumptions 1.1, 2.1 and 4.3 hold. Let $\{x^t\}$ be the sequence generated by SCP$_{ls}$. Then $\{x^t\}$ converges to a minimizer x^* of* (1.1)*. If in addition F in* (1.1) *is a KL function with exponent $\alpha \in [0, 1)$, then the following statements hold:*

(i) *If $\alpha \in [0, \frac{1}{2}]$, then there exist $c_0 > 0$, $Q_1 \in (0, 1)$ and $\underline{t} \in \mathcal{N}_+$, such that*

$$\|x^t - x^*\| \le c_0\, Q_1^t \quad \text{for } t > \underline{t}.$$

(ii) *If $\alpha \in (\frac{1}{2}, 1)$, then there exist $c_0 > 0$ and $\underline{t} \in \mathcal{N}_+$ such that*

$$\|x^t - x^*\| \le c_0\, t^{-\frac{1-\alpha}{2\alpha-1}} \quad \text{for } t > \underline{t}.$$

Proof. Let $S := \operatorname{Arg\,min} F$ for notational simplicity. Note that $S \neq \emptyset$ thanks to Assumption 1.1. Since $P_2 = 0$ and $\{f, g_1, \ldots, g_m\}$ are convex by Assumption 4.3, using Theorem 4.2 and [99, Theorem 28.3], we see that

$$\emptyset \neq \Omega_x \subseteq S, \tag{4.15}$$

where Ω_x is as in Lemma 4.5. This together with Lemma 4.5 implies that $\bar{F}^* = \inf F$.

Next, let λ^t be a Lagrange multiplier of (2.10) with $(\widetilde{L}_f, \widetilde{L}_g) = (L_f^t, L_g^t)$, which exists thanks to Lemma 2.4(iv). Since $P_2 = 0$ and $g(x^t) \le 0$ for all t, for any $\bar{x} \in S$, using (2.14) with $x = \bar{x}$, $\widetilde{x} = x^{t+1}$, $\widetilde{\lambda} = \lambda^t$, $\widetilde{L}_f = L_f^t$ and $\widetilde{L}_{fg} = L_{fg}^t := L_f^t + \langle \lambda^t, L_g^t \rangle$, we deduce that

$$
\begin{aligned}
F(x^{t+1}) &\le f(x^t) + \left\langle \nabla f(x^t), \bar{x} - x^t \right\rangle + P_1(\bar{x}) + \frac{L_{fg}^t}{2}\|\bar{x} - x^t\|^2 - \frac{L_{fg}^t}{2}\|\bar{x} - x^{t+1}\|^2 \\
&\qquad + \sum_{i=1}^m \lambda_i^t \left(g_i(x^t) + \langle \nabla g_i(x^t), \bar{x} - x^t \rangle \right) - \frac{L_f^t - L_f}{2}\|x^{t+1} - x^t\|^2 \\
&\overset{\text{(a)}}{\le} f(\bar{x}) + P_1(\bar{x}) + \frac{L_{fg}^t}{2}\|\bar{x} - x^t\|^2 - \frac{L_{fg}^t}{2}\|\bar{x} - x^{t+1}\|^2 - \frac{L_f^t - L_f}{2}\|x^{t+1} - x^t\|^2 \\
&\overset{\text{(b)}}{\le} f(\bar{x}) + P_1(\bar{x}) + \frac{L_{fg}^t}{2}\|\bar{x} - x^t\|^2 - \frac{L_{fg}^t}{2}\|x^{t+1} - \bar{x}\|^2 + \frac{(L_f - L_f^t)_+}{c}(F(x^t) - F(x^{t+1})) \\
&\le f(\bar{x}) + P_1(\bar{x}) + \frac{L_{fg}^t}{2}\|\bar{x} - x^t\|^2 - \frac{L_{fg}^t}{2}\|x^{t+1} - \bar{x}\|^2 + \frac{M_0}{c}(F(x^t) - F(x^{t+1})),
\end{aligned}
$$

where (a) holds because $\{f, g_1, \ldots, g_m\}$ are convex, and $\lambda_i^t \ge 0$ and $g_i(\bar{x}) \le 0$ for all i, (b) follows from (2.11), and the M_0 in the last inequality is an upper bound of $\{(L_f - L_f^t)_+\}$, which exists thanks to Lemma 2.4(ii). Rearranging terms in the above inequality and noting $\bar{F}^* = \inf F = f(\bar{x}) + P_1(\bar{x})$ whenever $\bar{x} \in S$, we have for any $\bar{x} \in S$ that

$$
\frac{F(x^{t+1}) - \bar{F}^*}{L_{fg}^t} \le \frac{1}{2}\|\bar{x} - x^t\|^2 - \frac{1}{2}\|x^{t+1} - \bar{x}\|^2 + \frac{M_0}{cL_{fg}^t}\left(F(x^t) - F(x^{t+1})\right).
$$

Let $L_{\max}$ be the upper bound of $\{L_{fg}^t\}$ (which exists according to Lemma 2.4(ii) and Theorem 4.2) and recall that $L_{fg}^t \ge L_f^t \ge \underline{\mathrm{L}} > 0$ for all t, where $\underline{\mathrm{L}}$ is the one used in Step 2 of SCP$_{ls}$. Then we have from the above display that for any $\bar{x} \in S$,

$$
\gamma\left(F(x^{t+1}) - \bar{F}^*\right) \le \frac{1}{2}\|\bar{x} - x^t\|^2 - \frac{1}{2}\|x^{t+1} - \bar{x}\|^2 + \theta\left(F(x^t) - F(x^{t+1})\right),
$$

where $\gamma := \frac{1}{L_{\max}}$ and $\theta := \frac{M_0}{c\underline{\mathrm{L}}}$. Rearranging terms in the above inequality, we have

$$
(\gamma + \theta)\left(F(x^{t+1}) - \bar{F}^*\right) \le \frac{1}{2}\|\bar{x} - x^t\|^2 - \frac{1}{2}\|x^{t+1} - \bar{x}\|^2 + \theta\left(F(x^t) - \bar{F}^*\right). \tag{4.16}
$$

The inequality above in particular implies that for any $\bar{x} \in S$,

$$
\begin{aligned}
&\frac{1}{2}\|x^{t+1} - \bar{x}\|^2 \le \frac{1}{2}\|\bar{x} - x^t\|^2 + \theta\left(F(x^t) - \bar{F}^*\right) - (\gamma + \theta)\left(F(x^{t+1}) - \bar{F}^*\right) \\
&\le \frac{1}{2}\|\bar{x} - x^t\|^2 + (\gamma + \theta)\left(F(x^t) - F(x^{t+1})\right),
\end{aligned}
\tag{4.17}
$$

where the last inequality holds because $\bar{F}^* = \inf F \le F(x^t)$. Since $\{F(x^t) - F(x^{t+1})\}$ is nonnegative and summable thanks to Lemma 4.5(ii), using (4.15), (4.17) and [62, Proposition 1], we conclude that $\{x^t\}$ converges to a minimizer x^* of (1.1).

Now, we suppose in addition that F is a KL function with exponent $\alpha \in [0, 1)$. Let $\bar{x}^t \in S$ satisfy $\|x^t - \bar{x}^t\| = \mathrm{dist}(x^t, S)$. Since $\bar{x}^t \in S$, it holds that $-\|x^{t+1} - \bar{x}^t\|^2 \le -\mathrm{dist}^2(x^{t+1}, S)$. Using this and applying (4.16) with $\bar{x}^t$ in place of $\bar{x}$ gives

$$
(\gamma + \theta)\left(F(x^{t+1}) - \bar{F}^*\right) \le \frac{1}{2}\mathrm{dist}^2(x^t, S) - \frac{1}{2}\mathrm{dist}^2(x^{t+1}, S) + \theta\left(F(x^t) - \bar{F}^*\right). \tag{4.18}
$$

For notational simplicity, let

$$
\beta_t := F(x^t) - \bar{F}^* + \frac{1}{2(\gamma + \theta)}\mathrm{dist}^2(x^t, S). \tag{4.19}
$$

Using this, rearranging terms and dividing $\gamma + \theta$ from both sides of (4.18), we have

$$
\beta_{t+1} \le \frac{\theta}{\gamma + \theta}\left(F(x^t) - \bar{F}^*\right) + \frac{1}{2(\gamma + \theta)}\mathrm{dist}^2(x^t, S). \tag{4.20}
$$

Since F is a proper closed convex level-bounded KL function with exponent $\alpha \in [0, 1)$, using Lemma 4.4, there exist $0 < \bar{a} < 1$, $\bar{c} > 0$ and $0 < \epsilon < 1$ such that

$$
\mathrm{dist}(x, S)^{\frac{1}{1-\alpha}} \le \bar{c}\left(F(x) - \bar{F}^*\right) \tag{4.21}
$$

for any $x \in \mathrm{dom}\partial F$ satisfying $\mathrm{dist}(x, S) \le \epsilon$ and $\bar{F}^* \le F(x) < \bar{F}^* + \bar{a}$.

Clearly, $\{x^t\} \subset \mathrm{dom}\partial F = \{x : \ g(x) \le 0\}$. Next, since $\{x^t\}$ is bounded thanks to Theorem 4.1(i), using (4.15), there exists t_1 such that

$$
\mathrm{dist}(x^t, S) \le \mathrm{dist}(x^t, \Omega_x) < \epsilon, \text{ for } t > t_1. \tag{4.22}
$$

On the other hand, using Lemma 4.5(ii), we see that there exists t_2 such that

$$\bar{F}^* \leq F(x^t) < \bar{F}^* + \bar{a}, \text{ for } t > t_2. \tag{4.23}$$

We now consider the cases when $\alpha \in [0, \frac{1}{2}]$ and $\alpha \in (\frac{1}{2}, 1)$ separately.

Case (i) $\alpha \in [0, \frac{1}{2}]$. Combining (4.21), (4.22) and (4.23), we conclude that for any $t > t_3 := \max\{t_1, t_2\}$,

$$\operatorname{dist}^2(x^t, S) \leq \operatorname{dist}^{\frac{1}{1-\alpha}}(x^t, S) \leq \bar{c}\left(F(x^t) - \bar{F}^*\right), \tag{4.24}$$

where the first inequality holds because $\frac{1}{1-\alpha} \leq 2$ and $\operatorname{dist}(x^t, S) < \epsilon < 1$. Next, let $\zeta := \frac{2\theta+\bar{c}}{2(\gamma+\theta)+\bar{c}} \in (0,1)$. Then one can show that

$$\frac{\theta}{\gamma+\theta} + \frac{(1-\zeta)\bar{c}}{2(\gamma+\theta)} = \zeta. \tag{4.25}$$

Using this and (4.20), we have for all $t > t_3$ that

$$\begin{aligned}
\beta_{t+1} &\leq \frac{\theta}{\gamma+\theta}(F(x^t) - \bar{F}^*) + \frac{1-\zeta}{2(\gamma+\theta)}\operatorname{dist}^2(x^t, S) + \frac{\zeta}{2(\gamma+\theta)}\operatorname{dist}^2(x^t, S) \\
&\overset{\text{(a)}}{\leq} \left(\frac{\theta}{\gamma+\theta} + \frac{(1-\zeta)\bar{c}}{2(\gamma+\theta)}\right)(F(x^t) - \bar{F}^*) + \frac{\zeta}{2(\gamma+\theta)}\operatorname{dist}^2(x^t, S) \\
&\overset{\text{(b)}}{=} \zeta\left(F(x^t) - \bar{F}^* + \frac{1}{2(\gamma+\theta)}\operatorname{dist}^2(x^t, S)\right) = \zeta\beta_t,
\end{aligned}$$

where (a) follows from (4.24) and (b) follows from (4.25). Combining the above inequality with the definition of β_t in (4.19) gives

$$F(x^t) - \bar{F}^* \leq \beta_t \leq \zeta^{t-t_3-1}\beta_{t_3+1} \quad \text{for } t > t_3. \tag{4.26}$$

Then, for $t > t_3$, we have

$$\begin{aligned}
\|x^* - x^t\| &\leq \sum_{j=t+1}^{\infty} \|x^j - x^{j-1}\| \leq \sum_{j=t+1}^{\infty} \sqrt{\frac{2}{c}}\sqrt{F(x^{j-1}) - F(x^j)} \\
&\leq \sum_{j=t+1}^{\infty} \sqrt{\frac{2}{c}}\sqrt{F(x^{j-1}) - \bar{F}^*} \leq \sum_{j=t+1}^{\infty} \sqrt{\frac{2}{c}}\sqrt{\zeta^{j-t_3-2}\beta_{t_3+1}} = \sqrt{\frac{2\beta_{t_3+1}}{c\zeta^{t_3+1}}}\frac{(\sqrt{\zeta})^t}{1-\sqrt{\zeta}},
\end{aligned}$$

where the second inequality follows from (2.11), the third inequality follows from Lemma 4.5(ii) and the last inequality follows from (4.26). This proves (i).

Case (ii) $\alpha \in (\frac{1}{2}, 1)$. Using (4.20) and the definition of β_t in (4.19), for any $t > t_3 = \max\{t_1, t_2\}$, we have

$$\begin{aligned}
\beta_{t+1} &\leq \beta_t - \frac{\gamma}{\gamma+\theta}\left(F(x^t) - \bar{F}^*\right) \\
&= \beta_t - \frac{1}{2}c_3\left[F(x^t) - \bar{F}^* + \bar{c}\left(\frac{1}{2(\gamma+\theta)}\right)^{\frac{1}{2(1-\alpha)}}\left(F(x^t) - \bar{F}^*\right)\right] \\
&\overset{\text{(a)}}{\leq} \beta_t - \frac{1}{2}c_3\left[F(x^t) - \bar{F}^* + \left(\frac{1}{2(\gamma+\theta)}\right)^{\frac{1}{2(1-\alpha)}}\operatorname{dist}(x^t, S)^{\frac{1}{1-\alpha}}\right] \\
&\overset{\text{(b)}}{\leq} \beta_t - \frac{1}{2}c_3\left[\left(F(x^t) - \bar{F}^*\right)^{\frac{1}{2(1-\alpha)}} + \left(\frac{1}{2(\gamma+\theta)}\operatorname{dist}^2(x^t, S)\right)^{\frac{1}{2(1-\alpha)}}\right],
\end{aligned}$$

where $c_3 = 2\frac{\frac{\gamma}{\gamma+\theta}}{1+\bar{c}\left(\frac{1}{2(\gamma+\theta)}\right)^{\frac{1}{2(1-\alpha)}}}$, (a) follows from (4.21), (4.22), (4.23) and the fact that $\{x^t\} \subset \operatorname{dom}\partial F = \{x : \; g(x) \leq 0\}$, and (b) holds because $0 \leq F(x^t) - \bar{F}^* < \bar{a} < 1$ (thanks to (4.23)) and $\frac{1}{2(1-\alpha)} > 1$. Since the mapping $w \mapsto w^{\frac{1}{2(1-\alpha)}}$ is convex, for $t > t_3$, we obtain further that

$$\begin{aligned}
\beta_{t+1} &\leq \beta_t - c_3c_4\left(F(x^t) - \bar{F}^* + \frac{1}{2(\gamma+\theta)}\operatorname{dist}^2(x^t, S)\right)^{\frac{1}{2(1-\alpha)}} \\
&= \beta_t - c_3c_4\beta_t^{\frac{1}{2(1-\alpha)}} = \beta_t\left(1 - c_3c_4\beta_t^{\frac{1}{2(1-\alpha)}-1}\right),
\end{aligned}$$

where $c_4 := 2^{-\frac{1}{2(1-\alpha)}}$. Since $\frac{1}{2(1-\alpha)} - 1 = \frac{2\alpha-1}{2(1-\alpha)} > 0$, using the above inequality and [26, Lemma 4.1], we have

$$\beta_t \leq \left(\beta_{t_3+1}^{-\frac{2\alpha-1}{2(1-\alpha)}} + \frac{2\alpha-1}{2(1-\alpha)}c_3c_4(t - t_3 - 1)\right)^{-\frac{2(1-\alpha)}{2\alpha-1}} \quad \text{for } t > t_3. \tag{4.27}$$

Then, for any $t > t_3$ and $t' \geq 0$, we have

$$
\begin{aligned}
&\|x^t - x^{t+t'}\|^2 \leq 2\left(\|x^t - \bar{x}^t\|^2 + \|\bar{x}^t - x^{t+t'}\|^2\right) \\
&\overset{\text{(a)}}{\leq} 2\left(\|x^t - \bar{x}^t\|^2 + \|\bar{x}^t - x^t\|^2 + 2(\gamma+\theta)\left(F(x^t) - F(x^{t+t'})\right)\right) \\
&= 2\left(2\text{dist}^2(x^t, S) + 2(\gamma+\theta)\left(F(x^t) - F(x^{t+t'})\right)\right) \\
&\overset{\text{(b)}}{\leq} 2\left(2\text{dist}^2(x^t, S) + 4(\gamma+\theta)\left(F(x^t) - \bar{F}^*\right)\right) \\
&\overset{\text{(c)}}{=} 8(\gamma+\theta)\beta_t \leq 8(\gamma+\theta)\left(\beta_{t_3+1}^{-\frac{2\alpha-1}{2(1-\alpha)}} + \frac{2\alpha-1}{2(1-\alpha)}c_3c_4(t - t_3 - 1)\right)^{-\frac{2(1-\alpha)}{2\alpha-1}},
\end{aligned}
$$

where (a) follows from (4.17) and the first equality uses the definition of $\bar{x}^t$ (i.e., the projection of x^t onto S), (b) follows from Lemma 4.5(ii), (c) uses the definition of β_t and the last inequality follows from (4.27). Letting $t' \to \infty$ and recalling that $x^t \to x^*$, we see that the conclusion in (ii) holds. This completes the proof. $\square$

Remark 4.1. *From the proof of the above theorem, we can actually deduce that the sequence* $\left\{F(x^t) - \bar{F}^* + c_0\text{dist}^2(x^t, S)\right\}$ *(with some suitable* $c_0 > 0$*) is Q-linearly convergent when* F *is a KL function with exponent* $\alpha \in [0, \frac{1}{2}]$*, and is sublinearly convergent when* F *is a KL function with exponent* $\alpha \in (\frac{1}{2}, 1)$*; see* (4.26) *and* (4.27).

4.2 KL properties of $\bar{F}$ and F

In Section 4.1, we deduced the rate of convergence of the sequence $\{x^t\}$ generated by SCP_{ls} under nonconvex and convex settings by imposing KL assumptions on $\bar{F}$ in (1.3) and F in (1.1), respectively; see Theorem 4.3 and Theorem 4.4. Note that the assumptions in Theorem 4.3 and Theorem 4.4 for (1.1) are different as follows:

- Assumptions 1.1, 2.1, 4.1 and 4.2 are used in Theorem 4.3.
- Assumptions 1.1, 2.1, 4.3 are used in Theorem 4.4.

Thus, it is interesting to find a relationship between KL exponent of $\bar{F}$ and that of F when all the above assumptions hold. In this regard, we have the following theorem.

Theorem 4.5 (**Relation between the KL exponents of** $\bar{F}$ **and** F)**.** *Let F be defined as in* (1.1) *and suppose that Assumptions 1.1, 2.1, 4.1 and 4.3 hold. If $\bar{F}$ defined in* (1.3) *is a KL function with exponent $\alpha \in [0, 1)$, then F is also a KL function with exponent α.*

Proof. Fix any $x_0 \in \mathrm{dom}\partial F$ and $w_0 \in \mathbb{R}$. Using (4.8) and noting that $P_2 = 0$ (Assumption 4.3), we have for any $x \in \mathrm{dom}\partial F$ that

$$\begin{aligned}
&\partial \bar{F}(x, x, w_0) \\
&\supseteq \left\{ \begin{pmatrix} \nabla f(x) + \partial P_1(x) + \sum_{i=1}^m \lambda_i \nabla g_i(x) \\ 0 \\ 0 \end{pmatrix} : \ \lambda \in N_{-\mathbb{R}^m_+}(\bar{G}(x, x, w_0)) \right\} \\
&\overset{\text{(a)}}{=} \left\{ \begin{pmatrix} \nabla f(x) + \partial P_1(x) + \sum_{i=1}^m \lambda_i \nabla g_i(x) \\ 0 \\ 0 \end{pmatrix} : \ \lambda \in N_{-\mathbb{R}^m_+}(g(x)) \right\} \\
&\overset{\text{(b)}}{=} \begin{pmatrix} \nabla f(x) + \partial P_1(x) + N_{g(\cdot)\le 0}(x) \\ 0 \\ 0 \end{pmatrix} \overset{\text{(c)}}{=} \begin{pmatrix} \partial F(x) \\ 0 \\ 0 \end{pmatrix},
\end{aligned} \tag{4.28}$$

where (a) follows from the fact that $g(x) = \bar{G}(x, x, w_0)$, (b) follows from Assumption 2.1 and [100, Theorem 6.14], and (c) holds due to [100, Exercise 8.8] and [99, Theorem 23.8] together with the convexity of P_1 and g and the continuity of P_1. Using this together with the assumption that $x_0 \in \mathrm{dom}\partial F$, we have $(x_0, x_0, w_0) \in \mathrm{dom}\partial \bar{F}$. Then, from the KL assumption on $\bar{F}$, we see that there exist $a > 0$, $\epsilon > 0$ and $c_0 > 0$ such that

$$\mathrm{dist}(0, \partial \bar{F}(x, y, w)) \ge a(\bar{F}(x, y, w) - \bar{F}(x_0, x_0, w_0))^{\alpha} \tag{4.29}$$

whenever $0 < \bar{F}(x, y, w) - \bar{F}(x_0, x_0, w_0) < c_0$ and $\|(x, y, w) - (x_0, x_0, w_0)\| \le \epsilon$.

In addition, thanks to the fact that $g(x) = \bar{G}(x, x, w_0)$, for any $x \in \operatorname{dom}\partial F$ satisfying $F(x_0) < F(x) < F(x_0) + c_0$, we have

$$\bar{F}(x_0, x_0, w_0) < \bar{F}(x, x, w_0) < \bar{F}(x_0, x_0, w_0) + c_0. \tag{4.30}$$

On the other hand, for x such that $\|x - x_0\| \leq \frac{1}{2}\epsilon$, we have $\|(x, x, w_0) - (x_0, x_0, w_0)\| \leq \epsilon$. Using this and (4.30), for $x \in \operatorname{dom}\partial F$ satisfying $\|x - x_0\| \leq \frac{1}{2}\epsilon$ and $F(x_0) < F(x) < F(x_0) + c_0$, we have

$$\begin{aligned}\operatorname{dist}(0, \partial F(x)) &\overset{\text{(a)}}{\geq} \operatorname{dist}(0, \partial \bar{F}(x, x, w_0)) \overset{\text{(b)}}{\geq} a(\bar{F}(x, x, w_0) - \bar{F}(x_0, x_0, w_0))^\alpha \\ &\overset{\text{(c)}}{=} a(F(x) - F(x_0))^\alpha,\end{aligned}$$

where (a) follows from (4.28), (b) uses (4.29) and (c) holds thanks to $g(x) = \bar{G}(x, x, w_0)$. This completes the proof. □

4.3 Applications in compressed sensing

In this section, we consider applications of (1.1) and discuss how the various assumptions required in our analysis of SCP_{ls} can be verified. We focus on the problem of compressed sensing, which attempts to reconstruct sparse signals from possibly noisy low-dimensional measurements; see [36] for a recent review. We specifically look at the following model:

$$\begin{array}{cl} \min\limits_{x} & \|x\|_1 - \mu\|x\| \\ \text{s.t.} & \ell(Ax - b) \leq \delta, \end{array} \tag{4.31}$$

where $\mu \in [0, 1]$, $A \in \mathbb{R}^{q \times n}$ has *full row rank*, $b \in \mathbb{R}^q$, $\ell : \mathbb{R}^q \to \mathbb{R}_+$ is an analytic function whose gradient is Lipschitz continuous with modulus L_ℓ and satisfies $\ell(0) = 0$, and $\delta \in (0, \ell(-b))$. The ℓ in (4.31) is typically chosen according to different types of noise. We will look at two specific choices in Section 4.3.1 and Section 4.3.2, respectively.

Problem (4.31) is a special case of (1.1) with $f = 0$, $P_1(x) = \|x\|_1$, $P_2(x) = \mu\|x\|$ and $g(x) = \ell(Ax - b) - \delta$.[2] Then the F from (1.1) corresponding to (4.31) is

$$F(x) = \|x\|_1 - \mu\|x\| + \delta_{\ell(A\cdot - b)\le\delta}(x), \tag{4.32}$$

and the $\bar{F}$ from (1.3) corresponding to (4.31) is

$$\bar{F}(x, y, w) = \|x\|_1 - \mu\|x\| + \delta_{\bar{G}(\cdot)\le 0}(x, y, w) \tag{4.33}$$

with

$$\bar{G}(x, y, w) = \ell(Ay - b) + \langle A^T\nabla\ell(Ay - b), x - y\rangle + \frac{w}{2}\|x - y\|^2 - \delta. \tag{4.34}$$

Our next theorem concerns the KL conditions needed in Theorems 4.3 and 4.4.

Theorem 4.6. *Let F and $\bar{F}$ be defined as in (4.32) and (4.33), respectively, and let Ξ and Υ be compact subsets of $\operatorname{dom} F$ and $\operatorname{dom} \bar{F}$, respectively. Then there exists $\alpha \in [0, 1)$ so that F (resp., $\bar{F}$) satisfies the KL property with exponent α at every point in Ξ (resp., in Υ).*

Proof. Let $\mathfrak{D}_0 := \{x : \ \ell(Ax - b) \le \delta\}$ and $\mathfrak{D}_1 = \{(x, y, w) : \ \bar{G}(x, y, w) \le 0\}$, where $\bar{G}$ is as in (4.34). Since ℓ and $\bar{G}$ are analytic, we have that $\mathfrak{D}_0$ and $\mathfrak{D}_1$ are semianalytic; see [52, Page 596] for the definition.

On the other hand, since $x \mapsto \|x\|_1 - \mu\|x\|$ is semialgebraic, it holds that $\mathfrak{F}_0 := \{(x, z) : \ z = \|x\|_1 - \mu\|x\|\}$ and $\mathfrak{F}_1 := \{(x, y, w, z) : \ z = \|x\|_1 - \mu\|x\|\}$ are subanalytic (see [52, Page 597(p2)] for the subanalyticity of $\mathfrak{F}_1$). Therefore,

$$\operatorname{gph}(F) = \mathfrak{F}_0 \cap (\mathfrak{D}_0 \times \mathbb{R}) \quad \text{and} \quad \operatorname{gph}(\bar{F}) = \mathfrak{F}_1 \cap (\mathfrak{D}_1 \times \mathbb{R})$$

are subanalytic, thanks to [52, Page 597(p1)&(p2)]. Also, the functions F and $\bar{F}$ have closed domains and are continuous on their respective domains. Thus, the desired conclusion follows from [20, Theorem 3.1] and a standard compactness argument as in the proof of [6, Lemma 1]. $\square$

[2] Note that $\{x : \ g(x) \le 0\} \neq \emptyset$ because A has full row rank and $\ell(0) = 0 < \delta$.

We next focus on two common choices of ℓ in (4.31): $\ell(\cdot) = \frac{1}{2}\|\cdot\|^2$ (for Gaussian noise [17]) and $\ell(\cdot) = \|\cdot\|_{LL_{2,\gamma}}$ being the Lorentzian norm (for Cauchy noise [35]) for some $\gamma > 0$. We will discuss how to verify the other assumptions necessary for the applications of Theorem 4.3 or Theorem 4.4 to (4.31) with these two choices of ℓ.

4.3.1 When $\ell(\cdot) = \frac{1}{2}\|\cdot\|^2$

In this case, the model (4.31) becomes

$$\begin{array}{ll} \min\limits_{x} & \|x\|_1 - \mu\|x\| \\ \text{s.t.} & \frac{1}{2}\|Ax - b\|^2 \le \delta, \end{array} \tag{4.35}$$

and the corresponding F in (1.1) becomes:

$$F(x) = \|x\|_1 - \mu\|x\| + \delta_{g(\cdot)\le 0}(x), \tag{4.36}$$

with $f = 0$, $P_1(x) = \|x\|_1$, $P_2(x) = \mu\|x\|$ and $g(x) = \frac{1}{2}\|Ax - b\|^2 - \delta$ for A, b, δ and μ as in (4.31). Then, for (4.35), P_1 and P_2 are convex continuous, and Assumption 1.1(i) and (ii) and Assumption 4.1 are satisfied. Moreover, A having full row rank and $\delta \in (0, \frac{1}{2}\|b\|^2)$ imply that Slater condition holds for (4.35). Hence, it holds that $\{x : \ g(x) \le 0\} \ne \emptyset$, and we also have Assumption 2.1 hold, thanks to [25, Section 3.2, Exercise 10]. In addition, this P_2 satisfies Assumption 4.2 since its only possible point of nondifferentiability (the origin) is not feasible thanks to the fact that $\delta < \frac{1}{2}\|b\|^2$. Furthermore, the required KL conditions follow from Theorem 4.6.[3] In order to apply Theorem 4.3 (or Theorem 4.4), we now demonstrate how conditions can be imposed so that Assumption 1.1(iii) (level-boundedness) is satisfied.

Proposition 4.1. *Let F be defined as in* (4.36)*. The following statements hold:*

(i) *If $\mu \in [0, 1)$, then F is level-bounded.*

[3] Specifically, if $\mu = 0$, then F is convex and level-bounded, and the set of stationary points (minimizers) is compact. We can then deduce from Theorem 4.6 that F is a KL function with some exponent $\alpha \in [0, 1)$. On the other hand, the KL property required in the nonconvex case (see Theorem 4.3) follows directly from Theorem 4.6.

(ii) *If* $\mu = 1$ *and* A *does not have zero columns, then* F *is level-bounded.*

Proof. Note first that if $0 \le \mu < 1$, then $x \mapsto \|x\|_1 - \mu\|x\|$ is level-bounded and hence (i) holds trivially. We next focus on the case where $\mu = 1$.

Suppose to the contrary that there exists σ and $\{x^t\} \subseteq \{x : \; F(x) \le \sigma\}$ such that $\|x^t\| \to \infty$. By passing to a further subsequence if necessary, we may assume that there exists d with $\|d\| = 1$ and $\lim\limits_{t\to\infty} \frac{x^t}{\|x^t\|} = d$. Since $\frac{1}{2}\|Ax^t - b\|^2 \le \delta$ thanks to $F(x^t) \le \sigma$ for each t, we have

$$\frac{1}{2}\|Ad\|^2 = \lim_{t\to\infty} \frac{1}{2}\frac{\|Ax^t - b\|^2}{\|x^t\|^2} \le \lim_{t\to\infty} \frac{\delta}{\|x^t\|^2} = 0. \tag{4.37}$$

On the other hand, since $F(x^t) \le \sigma$, it holds that

$$0 \le \|x^t\|_1 - \|x^t\| \le \sigma \Longrightarrow 0 \le \lim_{t\to\infty} \frac{\|x^t\|_1 - \|x^t\|}{\|x^t\|} = \|d\|_1 - 1 \le 0.$$

This together with $\|d\| = 1$ implies that exactly one coordinate of d is nonzero. Since A does not have zero columns, we obtain that $\|Ad\| \ne 0$, which contradicts (4.37). Thus, the statement in (ii) holds. $\square$

Therefore, if the assumptions in the above proposition hold, one can apply Theorem 4.3 or Theorem 4.4 to deducing the convergence rate of the sequence generated by SCP$_{ls}$ when applied to solving (4.35). When $\mu = 0$ in (4.35), since we assumed $\delta \in (0, \frac{1}{2}\|b\|^2)$ and A has full row rank, we know from Remark 3.2 that $x \mapsto \|x\|_1 + \delta_{\frac{1}{2}\|A(\cdot)-b\|^2 \le \delta}(x)$ is a KL function with exponent $\frac{1}{2}$. Consequently, the sequence $\{x^t\}$ generated by SCP$_{ls}$ for (4.35) converges locally linearly. When $\mu \in (0, 1]$, although no explicit KL exponent is known for the corresponding $\bar{F}$, we still observe in our numerical experiments below that the sequence $\{x^t\}$ generated by SCP$_{ls}$ for (4.35) appears to converge linearly.

4.3.2 When ℓ is the Lorentzian norm

Recall that, given $\gamma > 0$, the Lorentzian norm of a vector $y \in \mathbb{R}^q$ is defined as

$$\|y\|_{LL_2,\gamma} := \sum_{i=1}^{q} \log\left(1 + \frac{y_i^2}{\gamma^2}\right).$$

In this case, the model (4.31) becomes

$$\begin{aligned} &\min_x \; \|x\|_1 - \mu\|x\| \\ &\text{s.t. } \|Ax - b\|_{LL_2,\gamma} \le \delta, \end{aligned} \tag{4.38}$$

and the corresponding F in (1.1) now takes the following form:

$$F(x) = \|x\|_1 - \mu\|x\| + \delta_{g(\cdot)\le 0}(x), \tag{4.39}$$

with $f = 0$, $P_1(x) = \|x\|_1$, $P_2(x) = \mu\|x\|$ and $g(x) = \|Ax - b\|_{LL_2,\gamma} - \delta$ for A, b, δ and μ defined as in (4.31). One can show that the mapping $z \mapsto \|z\|_{LL_2,\gamma} - \delta$ has Lipschitz gradient with modulus $L_\ell = \frac{2}{\gamma^2}$ and is twice continuously differentiable. From these one can readily see that P_1 and P_2 are convex continuous, and Assumption 1.1(i) and (ii) and Assumption 4.1 are satisfied. Also, since A has full row rank and $\delta \in (0, \|b\|_{LL_2,\gamma})$, we see that $\{x : \; g(x) \le 0\} \neq \emptyset$. In addition, this P_2 satisfies Assumption 4.2 since its only possible point of nondifferentiability is not feasible, thanks to $\delta \in (0, \|b\|_{LL_2,\gamma})$. Furthermore, the required KL conditions follow from Theorem 4.6. In order to apply Theorem 4.3, we show below that Assumption 2.1 holds and impose conditions so that Assumption 1.1(iii) is satisfied.

Proposition 4.2. *Let F be defined as in* (4.39). *The following statements hold:*

(i) *The MFCQ holds in the whole feasible set of* (4.38).

(ii) *If $\mu \in [0, 1)$, then F is level-bounded.*

(iii) *If $\mu = 1$ and A does not have zero columns, then F is level-bounded.*

Proof. For (i), using the definition of MFCQ, it suffices to show that for every feasible point x with $g(x)=0$, it holds that $\nabla g(x)\neq 0$. Suppose to the contrary that there exists $\widehat{x}$ such that $g(\widehat{x})=0$ and

$$\nabla g(\widehat{x}) = A^T\left(\frac{2(A\widehat{x}-b)_1}{\gamma^2+(A\widehat{x}-b)_1^2},\ldots,\frac{2(A\widehat{x}-b)_q}{\gamma^2+(A\widehat{x}-b)_q^2}\right)^T = 0.$$

Since A is surjective, we deduce that $\left(\frac{2(A\widehat{x}-b)_1}{\gamma^2+(A\widehat{x}-b)_1^2},\ldots,\frac{2(A\widehat{x}-b)_q}{\gamma^2+(A\widehat{x}-b)_q^2}\right)=0$. This shows that $A\widehat{x}-b=0$ and thus $g(\widehat{x})=\|A\widehat{x}-b\|_{LL_2,\gamma}-\delta=-\delta\neq 0$, a contradiction. Therefore, the MFCQ holds in the whole feasible set of (4.38).

The assertion in (ii) holds trivially. We now prove (iii). Suppose to the contrary that there exist σ and $\{x^t\}\subseteq\{x:\ F(x)\leq\sigma\}$ such that $\|x^t\|\to\infty$. By passing to a further subsequence if necessary, we may assume that there exists d with $\|d\|=1$ and $d=\lim\limits_{t\to\infty}\frac{x^t}{\|x^t\|}$. Since $\ell(Ax^t-b)\leq 0$ thanks to $F(x^t)\leq\sigma$ for each t, and the Lorentzian norm is level-bounded, we see that there exists ξ such that $\|Ax^t-b\|\leq\xi$ for all t. The rest of the proof is then the same as that of Proposition 4.1(ii). $\square$

Therefore, if the assumptions in the above proposition hold, one can apply Theorem 4.3 to deducing the convergence rate of the sequence $\{x^t\}$ generated by SCP$_{ls}$ when applied to solving (4.38). Although no explicit KL exponent is known for the corresponding $\bar{F}$, in our numerical experiments below, we observe empirically that the sequence $\{x^t\}$ generated by SCP$_{ls}$ for (4.38) appears to converge linearly.

4.3.3 Numerical experiments

In this subsection, we perform numerical experiments to illustrate the convergence results of SCP$_{ls}$ established in Section 4.1. We apply SCP$_{ls}$ to (4.31) with ℓ being either $\frac{1}{2}\|\cdot\|^2$ (as in (4.35)) or the Lorentzian norm (as in (4.38)). We also consider the SCP in [83] in our experiments below.

Algorithms and their parameters We consider the following algorithms:

(i) **SCP**$_{ls}$: We solve the corresponding subproblem (2.10) through a root-finding scheme outlined in Section 4.3.4. Moreover, we let $\tau = 2$, $c = 10^{-4}$, $\underline{\mathrm{L}} = 10^{-8}$, $\bar{\mathrm{L}} = 10^{8}$. For $t = 0$, we choose $L_f^{t,0} = 1$ and $L_g^{t,0} = 1$. For $t \geq 1$, we choose:

$$L_f^{t,0} = 1,\ L_g^{t,0} = \begin{cases} \max\left\{10^{-8}, \min\left\{\frac{\langle \Delta x, \Delta g\rangle}{\|\Delta x\|^2}, 10^8\right\}\right\} & \text{if } \langle \Delta x, \Delta g\rangle \geq 10^{-12}, \\ \max\left\{10^{-8}, \min\left\{L_g^{t-1}/\tau, 10^8\right\}\right\} & \text{else,} \end{cases}$$

where $\Delta x = x^t - x^{t-1}$ and $\Delta g = \nabla g(x^t) - \nabla g(x^{t-1})$. We initialize SCP$_{ls}$ at $A^\dagger b$ and terminate it when $\|x^{t+1} - x^t\| < 10^{-8}\max\{1, \|x^{t+1}\|\}$.

(ii) **SCP**: This was proposed in [83]. The subproblem of SCP is solved using a root-finding scheme outlined in Section 4.3.4. We initialize SCP at $A^\dagger b$ and terminate it when $\|x^{t+1} - x^t\| < 10^{-8}\max\{1, \|x^{t+1}\|\}$.

Numerical results All codes are written in Matlab, and the experiments are performed in Matlab 2019b on a 64-bit PC with an Intel(R) Core(TM) i7-4790 CPU (3.60GHz) and 32GB of RAM.

For both models (4.35) and (4.38), we consider either $\mu = 0$ or 1. In our tests, we let $q = 720i$ and $n = 2560i$ with $i = 5$. We generate an $A \in \mathbb{R}^{q\times n}$ with i.i.d standard Gaussian entries, and then normalize this matrix so that each column of A has unit norm. Then we choose a subset T of size $s_0 = [\frac{q}{9}]$ uniformly at random from $\{1, 2, \ldots, n\}$ and an s_0-sparse vector $x_{\rm orig}$ having i.i.d. standard Gaussian entries on T is generated.

For (4.35), we let $b = Ax_{\rm orig} + 0.01\cdot\widehat{n}$ with $\widehat{n} \in \mathbb{R}^q$ being a random vector with i.i.d. standard Gaussian entries. We then set the δ in (4.35) to be $\frac{1}{2}\sigma^2$ with $\sigma = 1.1\|0.01\cdot\widehat{n}\|$.

For (4.38), we let $b = Ax_{\rm orig} + 0.01\cdot\bar{n}$ with $\bar{n}_i \sim$ Cauchy$(0,1)$, i.e., $\bar{n}_i :=$ $\tan(\pi(\widetilde{n}_i - 1/2))$ with $\widetilde{n} \in \mathbb{R}^m$ being a random vector with i.i.d. entries uniformly chosen in $[0,1]$. We set the δ in (4.38) to be $1.1\|0.01\bar{n}\|_{LL_{2,\gamma}}$ with $\gamma = 0.02$.

We compare the approximate solution obtained by SCP_{ls} and the original sparse solution in Figures 4.1 and 4.2 to illustrate the recovery ability of SCP_{ls}. In Figures 4.3 and 4.4, we plot $\|x^t - x^{out}\|$ (in logarithmic scale) against the number of iterations, where x^t and x^{out} are respectively the t^{th} iterate and the approximate solution obtained by the algorithm under study. As we can see, SCP_{ls} always appears to converge linearly and is also faster than SCP.

Figure 4.1: Recovery results by solving model (4.35) with $\mu = 0$ (left) and $\mu = 1$ (right) via SCP_{ls}. The approximate solution obtained by SCP_{ls} is marked by asterisk, and x_{orig} is marked by circle.

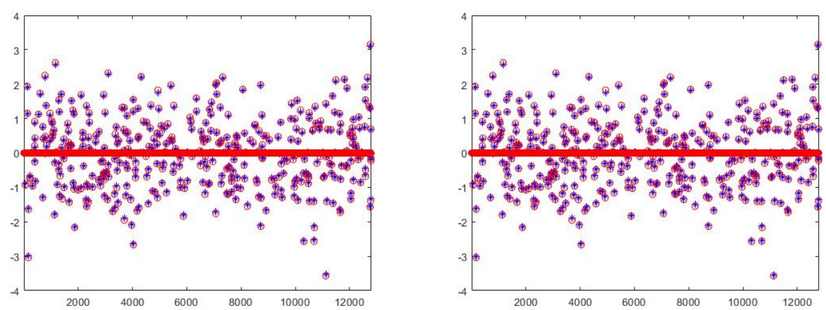


Figure 4.2: Recovery results by solving model (4.38) with $\mu = 0$ (left) and $\mu = 1$ (right) via SCP_{ls}. The approximate solution obtained by SCP_{ls} is marked by asterisk, and x_{orig} is marked by circle.

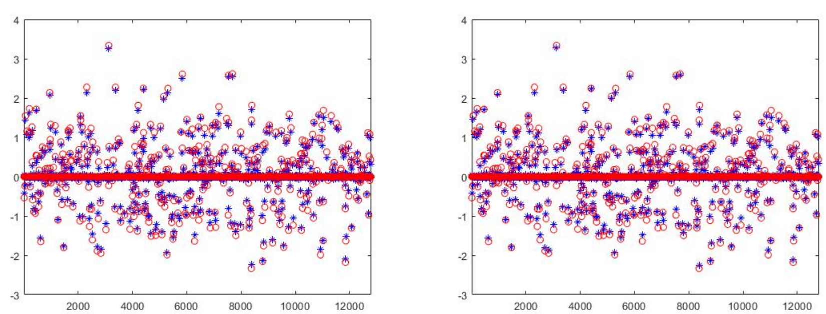


Figure 4.3: Plot of $\|x^t - x^{\text{out}}\|$ (in log scale) for model (4.35) with $\mu = 0$ (left) and $\mu = 1$ (right). The number in the parenthesis is the CPU time taken.

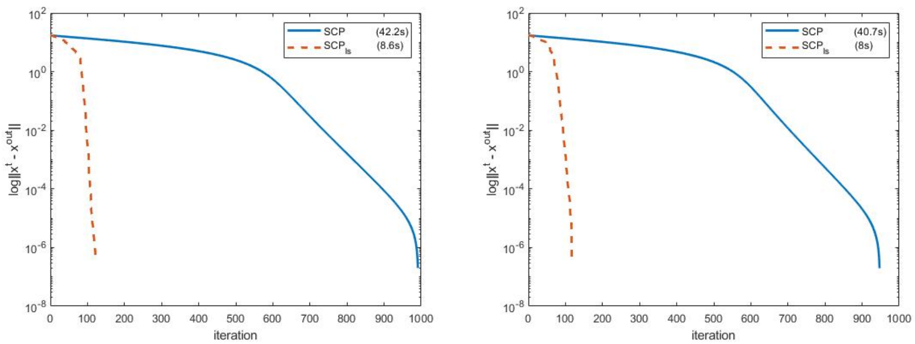
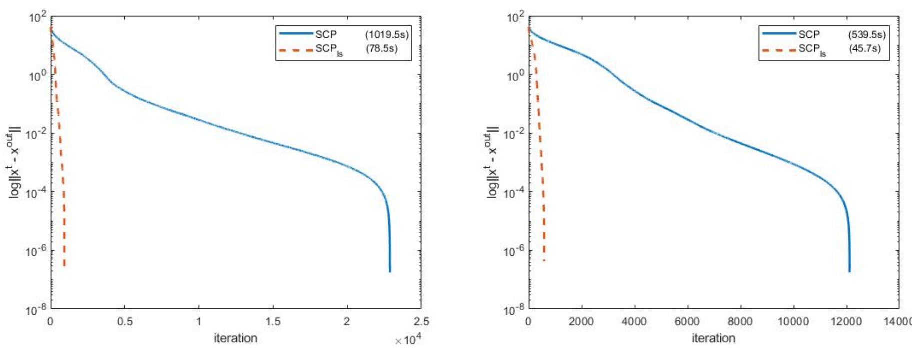


Figure 4.4: Plot of $\|x^t - x^{\text{out}}\|$ (in log scale) for model (4.38) with $\mu = 0$ (left) and $\mu = 1$ (right). The number in the parenthesis is the CPU time taken.



4.3.4 Exactly solving the subproblem of SCP$_{ls}$ with P_1 being the ℓ_1 norm, $P_2 = 0$ and $m = 1$

We discuss how the subproblem (2.10) that arises in our numerical tests when SCP$_{ls}$ is applied to (4.31) can be solved efficiently. Our approach is based on a root-finding strategy for solving the dual, which was also adopted in [103] for solving the subproblem that arises in the MBA variant there. Comparing with the subproblem considered in [103], our subproblem has an additional quadratic term, which slightly complicates the derivation and implementation.

At the t^{th} iteration, the corresponding subproblem (2.10) that arises when SCP$_{ls}$

is applied to (4.31) takes the following form:

$$\begin{aligned} \min_x \quad & \|x\|_1 + \tfrac{\alpha}{2}\|x-y\|^2 \\ \text{s.t.} \quad & \|x-s\|^2 \le r, \end{aligned} \tag{4.40}$$

where y, $s \in \mathbb{R}^n$, $\alpha > 0$ and $r > 0$.[4]

Recall that the Lagrangian function for (4.40) is given by

$$\widetilde{L}(x,\lambda) = \|x\|_1 + \frac{\alpha}{2}\|x-y\|^2 + \lambda(\|x-s\|^2 - r).$$

Using [99, Corollary 28.2.1, Theorem 28.3], we know that there exists (x^*, λ^*) with $\lambda^* \ge 0$ such that x^* is optimal for (4.40) and

$$\min_{x\in\mathbb{R}^n} \widetilde{L}(x,\lambda^*) = \min_{x\in\mathbb{R}^n} \|x\|_1 + \frac{\alpha}{2}\|x-y\|^2 + \delta_{\|(\cdot)-s\|^2\le r}(x).$$

If $\lambda^* = 0$, then the solution $\check{x}$ of $\min\limits_{x\in\mathbb{R}^n} \|x\|_1 + \frac{\alpha}{2}\|x-y\|^2$ lies in $\{x:\ \|x-s\|^2 \le r\}$ and $\check{x}$ solves (4.40). Moreover, $\check{x}$ is given explicitly as $\operatorname{sign}(y)\circ\max\{|y|-\frac{1}{\alpha},0\}$, where $\circ$ denotes the entrywise product, and the sign function, absolute value and maximum are taken componentwise.

If $\lambda^* > 0$, using [99, Theorem 28.3], we obtain that

$$0 \in \partial\|x^*\|_1 + \alpha(x^*-y) + 2\lambda^*(x^*-s) \text{ and } \|x^*-s\|^2 = r. \tag{4.41}$$

Using the first relation in (4.41), we have

$$x^* = \operatorname{Prox}_{\frac{1}{\alpha+2\lambda^*}\|\cdot\|_1}\left(\frac{\alpha}{\alpha+2\lambda^*}y + \frac{2\lambda^*}{\alpha+2\lambda^*}s\right), \tag{4.42}$$

where $\operatorname{Prox}_h(u) := \operatorname*{arg\,min}\limits_{v\in\mathbb{R}^n}\left\{h(v) + \frac{1}{2}\|u-v\|^2\right\}$ for a proper closed convex function h. Plugging this into the second relation in (4.41), we see that λ^* can be obtained by

[4] The fact that $r > 0$ follows from Lemma 2.4(iii).

solving the following one-dimensional nonsmooth equation and the solution x^* can then be recovered via (4.42):

$$\left\|\operatorname{Prox}_{\frac{1}{\alpha+2\lambda^*}\|\cdot\|_1}\left(\frac{\alpha}{\alpha+2\lambda^*}y+\frac{2\lambda^*}{\alpha+2\lambda^*}s\right)-s\right\|^2=r.$$

Upon the transformation $t^* = \frac{\alpha}{\alpha+2\lambda^*}$, the above equation becomes piecewise linear quadratic and can be solved efficiently by a standard root-finding procedure.

In passing, we note that a solution procedure for the subproblem that arises when SCP is applied to (4.31) can be derived similarly, where the subproblem takes the form

$$\begin{array}{ll}\min\limits_{x} & \|x\|_1 - \langle \xi, x\rangle \\ \text{s.t.} & \|x-s\|^2 \le r,\end{array}$$

for some ξ, $s \in \mathbb{R}^n$ and $r > 0$. We omit the details for brevity.

Chapter 5

KL property in ℓ_1/ℓ_2 Minimization

In this chapter, we will focus on the ℓ_1/ℓ_2 minimization problems (1.5) and (1.6), and show how the KL property and KL exponent are applied to analyze the convergence properties of the algorithms for solving (1.5) and (1.6) respectively.

5.1 Solution existence of model (1.5)

In this section, we establish the existence of optimal solutions to problem (1.5) under suitable assumptions. A similar discussion was made in [97, Theorem 2.2], where the existence of *local minimizers* was established under the strong null space property (see [97, Definition 2.1]) of the sensing matrix A. It was indeed shown that any sufficiently sparse solution of $Ax = b$ is a local minimizer for problem (1.5), under the strong null space property. Here, our discussion focuses on the existence of *globally optimal solutions*, and our analysis is based on the spherical section property (SSP) [114, 128].

Definition 5.1 (Spherical section property [114, 128])**.** *Let m, n be two positive integers such that $m < n$. Let V be an $(n-m)$-dimensional subspace of $\mathbb{R}^n$ and s be a positive integer. We say that V has the s-spherical section property if*

$$\inf_{v \in V \setminus \{0\}} \frac{\|v\|_1}{\|v\|} \geq \sqrt{\frac{m}{s}}.$$

Remark 5.1. *According to [128, Theorem 3.1], if $A \in \mathbb{R}^{m\times n}$ $(m < n)$ is a random matrix with i.i.d. standard Gaussian entries, then its $(n-m)$-dimensional nullspace has the s-spherical section property for $s = c_1(\log(n/m) + 1)$ with probability at least $1 - e^{-c_0(n-m)}$, where c_0 and c_1 are positive constants independent of m and n.*

We now present our analysis. We first characterize the existence of unbounded minimizing sequences of (1.5): recall that $\{x^t\}$ is called a minimizing sequence of (1.5) if $Ax^t = b$ for all t and $\lim_{t\to\infty} \frac{\|x^t\|_1}{\|x^t\|} = \nu_{cs}^*$. Our characterization is related to the following auxiliary problem, where A is as in (1.5):

$$\nu_d^* := \inf\left\{\frac{\|d\|_1}{\|d\|} :\ Ad = 0, d \neq 0\right\}. \tag{5.1}$$

Lemma 5.1. *Consider* (1.5) *and* (5.1)*. Then $\nu_{cs}^* = \nu_d^*$ if and only if there exists a minimizing sequence of* (1.5) *that is unbounded.*

Proof. We first suppose that there exists an unbounded minimizing sequence $\{x^t\}$ of (1.5). By passing to a subsequence if necessary, we may assume without loss of generality that $\|x^t\| \to \infty$ and that $\lim_{t\to\infty} \frac{x^t}{\|x^t\|} = x^*$ for some x^* with $\|x^*\| = 1$. Then we have $\|x^*\|_1 = \nu_{cs}^*$ using the definition of minimizing sequence, and

$$Ax^* = \lim_{t\to\infty} \frac{Ax^t}{\|x^t\|} = \lim_{t\to\infty} \frac{b}{\|x^t\|} = 0. \tag{5.2}$$

One can then see that

$$\nu_d^* \leq \frac{\|x^*\|_1}{\|x^*\|} = \|x^*\|_1 = \nu_{cs}^* < \infty. \tag{5.3}$$

Next, fix any x such that $Ax = b$ and choose any $d \neq 0$ satisfying $Ad = 0$ (these exist thanks to $\nu_d^* \leq \nu_{cs}^* < \infty$). Then it holds that

$$\nu_{cs}^* \leq \frac{\|x + sd\|_1}{\|x + sd\|}$$

for any $s \in \mathbb{R}$. It follows from the above display that

$$\nu_{cs}^* \leq \lim_{s\to\infty} \frac{\|x+sd\|_1}{\|x+sd\|} = \frac{\|d\|_1}{\|d\|}.$$

Then we have $\nu_{cs}^* \leq \nu_d^*$ by the arbitrariness of d. This together with (5.3) shows that $\nu_{cs}^* = \nu_d^*$.

We next suppose that $\nu_{cs}^* = \nu_d^*$. Since $\nu_{cs}^* < \infty$ (thanks to $A^{-1}\{b\} \neq \emptyset$), there exists a sequence $\{d^k\}$ satisfying $Ad^k = 0$ and $d^k \neq 0$ such that $\lim_{k\to\infty} \frac{\|d^k\|_1}{\|d^k\|} = \nu_d^*$. Passing to a further subsequence if necessary, we may assume without loss of generality that $\lim_{k\to\infty} \frac{d^k}{\|d^k\|} = d^*$ for some d^* with $\|d^*\| = 1$. It then follows that

$$Ad^* = \lim_{k\to\infty} \frac{Ad^k}{\|d^k\|} = 0 \quad \text{and} \quad \|d^*\|_1 = \lim_{k\to\infty} \left\| \frac{d^k}{\|d^k\|} \right\|_1 = \nu_d^*.$$

Now, choose any x^0 such that $Ax^0 = b$ and define $x^t = x^0 + td^*$ for each $t = 1, 2, \ldots$ Then we have $Ax^t = b$ for all t. Moreover $\|x^t\| \to \infty$ as $t \to \infty$ and

$$\lim_{t\to\infty} \frac{\|x^t\|_1}{\|x^t\|} = \frac{\|d^*\|_1}{\|d^*\|} = \nu_d^* = \nu_{cs}^*.$$

Thus, $\{x^t\}$ is an unbounded minimizing sequence for (1.5). This completes the proof. $\square$

We are now ready to present the theorem on solution existence for (1.5).

Theorem 5.1 (Solution existence for (1.5))**.** *Consider* (1.5)*. Suppose that* $\ker A$ *has the* s*-spherical section property for some* $s > 0$ *and there exists* $\widetilde{x} \in \mathbb{R}^n$ *such that* $\|\widetilde{x}\|_0 < m/s$ *and* $A\widetilde{x} = b$*. Then the optimal value* ν_{cs}^* *of* (1.5) *is attainable, i.e., the set of optimal solutions of* (1.5) *is nonempty.*

Proof. According to the s-spherical property of $\ker A$ and the definition of ν_d^* in (5.1), we see that $\nu_d^* \geq \sqrt{\frac{m}{s}}$. It then follows that

$$\nu_{cs}^* \overset{\text{(a)}}{\leq} \frac{\|\widetilde{x}\|_1}{\|\widetilde{x}\|} \overset{\text{(b)}}{\leq} \sqrt{\|\widetilde{x}\|_0} \overset{\text{(c)}}{<} \sqrt{\frac{m}{s}} \leq \nu_d^*,$$

where (a) follows from the definition of ν_{cs}^* and the fact that $A\widetilde{x} = b$, (b) follows from Cauchy-Schwarz inequality and (c) holds by our assumption. Invoking Lemma 5.1 and noting $\nu_{cs}^* < \infty$, we see that there is a bounded minimizing sequence $\{x^t\}$ for (1.5). We can then pass to a convergent subsequence $\{x^{t_j}\}$ so that $\lim_{j\to\infty} x^{t_j} = x^*$ for some x^* satisfying $Ax^* = b$. Since $b \neq 0$, this means in particular that $x^* \neq 0$. We then have upon using the continuity of $\frac{\|\cdot\|_1}{\|\cdot\|}$ at x^* and the definition of minimizing sequence that

$$\frac{\|x^*\|_1}{\|x^*\|} = \lim_{j\to\infty} \frac{\|x^{t_j}\|_1}{\|x^{t_j}\|} = \nu_{cs}^*.$$

This shows that x^* is an optimal solution of (1.5). This completes the proof. $\square$

5.2 KL exponent of F in (1.11) and global convergence of Algorithm 5.1

In this section, we discuss the KL exponent of (1.11) and its implication on the convergence rate of the algorithm proposed in [116, Eq. 11] for solving (1.5). For ease of reference, this algorithm is presented as Algorithm 5.1 below. It was shown in [116] that if the sequence $\{x^t\}$ generated by this algorithm is bounded, then any accumulation point is a stationary point of F in (1.11).

Algorithm 5.1. The algorithm proposed in [116, Eq. 11] for (1.5)

Step 0. *Choose x^0 with $Ax^0 = b$ and $\alpha > 0$. Set $\omega_0 = \|x^0\|_1/\|x^0\|$ and $t = 0$.*

Step 1. *Solve the subproblem*

$$\begin{aligned} x^{t+1} = \mathop{\arg\min}_{x\in\mathbb{R}^n} \quad & \|x\|_1 - \frac{\omega_t}{\|x^t\|}\langle x, x^t\rangle + \frac{\alpha}{2}\|x - x^t\|^2 \\ \text{s.t.} \quad & Ax = b. \end{aligned} \tag{5.4}$$

Step 2. *Compute $\omega_{t+1} = \|x^{t+1}\|_1/\|x^{t+1}\|$. Update $t \leftarrow t+1$ and go to* **Step 1.**

Here, we first remark that if the sequence $\{x^t\}$ generated by Algorithm 5.1 is bounded, then it converges to a stationary point x^* of F in (1.11). The argument is standard (see [7, 8, 24]), making use of **H1**, **H2**, **H3** in [8, Section 2.3]. We include the proof here.

Proposition 5.1 (Global convergence of Algorithm 5.1)**.** *Consider* (1.5)*. Let* $\{x^t\}$ *be the sequence generated by Algorithm 5.1 and suppose that* $\{x^t\}$ *is bounded. Then* $\{x^t\}$ *converges to a stationary point of* F *in* (1.11)*.*

Proof. First, according to [116, Lemma 1], the sequence $\{\omega_t\}$ generated by Algorithm 5.1 enjoys the following sufficient descent property:

$$\omega_t - \omega_{t+1} \geq \frac{\alpha}{2\|x^{t+1}\|}\|x^{t+1} - x^t\|^2. \tag{5.5}$$

Now, if we let λ^t denote a Lagrange multiplier of the subproblem (5.4) at iteration t, one then see from the first-order optimality condition that

$$-A^T\lambda^t + \frac{\|x^t\|_1}{\|x^t\|^2}x^t - \alpha(x^{t+1} - x^t) \in \partial\|x^{t+1}\|_1. \tag{5.6}$$

On the other hand, using (2.17) and noting that $x^t \neq 0$ for all t, we have

$$\begin{aligned}
&\frac{1}{\|x^{t+1}\|}\partial\|x^{t+1}\|_1 - \frac{\|x^{t+1}\|_1}{\|x^{t+1}\|^3}x^{t+1} + \frac{A^T\lambda^t}{\|x^{t+1}\|} \\
&= \partial\frac{\|x^{t+1}\|_1}{\|x^{t+1}\|} + \frac{A^T\lambda^t}{\|x^{t+1}\|} \subset \partial\frac{\|x^{t+1}\|_1}{\|x^{t+1}\|} + N_{A^{-1}\{b\}}(x^{t+1}) = \partial F(x^{t+1}),
\end{aligned}$$

where the last equality follows from [100, Corollary 10.9], the regularity at x^{t+1} of $\frac{\|\cdot\|_1}{\|\cdot\|}$ (see (2.17)) and $\delta_{A^{-1}\{b\}}(\cdot)$ (see [100, Theorem 6.9]), and the definition of F in (1.11). Combining (5.6) and the above display, we obtain that

$$\frac{1}{\|x^{t+1}\|}\left(\frac{\|x^t\|_1}{\|x^t\|^2}x^t - \frac{\|x^{t+1}\|_1}{\|x^{t+1}\|^2}x^{t+1}\right) - \frac{\alpha}{\|x^{t+1}\|}(x^{t+1} - x^t) \in \partial F(x^{t+1}).$$

On the other hand, since $\|x^t\| \geq \inf_{y\in A^{-1}\{b\}} \|y\| > 0$ for all t (thanks to $Ax^t = b$ and $b \neq 0$) and $\{x^t\}$ is bounded, we see that there exists $C_0 > 0$ so that

$$\left\| \frac{\|x^t\|_1}{\|x^t\|^2} x^t - \frac{\|x^{t+1}\|_1}{\|x^{t+1}\|^2} x^{t+1} \right\| \leq C_0 \|x^{t+1} - x^t\| \text{ for all } t.$$

Thus, in view of the above two displays, we conclude that

$$\operatorname{dist}(0, \partial F(x^{t+1})) \leq \frac{C_0 + \alpha}{\inf_{y\in A^{-1}\{b\}} \|y\|} \|x^{t+1} - x^t\| \text{ for all } t.$$

Using the boundedness of $\{x^t\}$, (5.5), the above display and the continuity of F on its domain, we see that the conditions **H1**, **H2**, **H3** in [8, Section 2.3] are satisfied. Since F is clearly proper closed semi-algebraic and hence a KL function, we can then invoke [8, Theorem 2.9] to conclude that $\{x^t\}$ converges to a stationary point of F. □

While it is routine to show that the sequence $\{x^t\}$ generated by Algorithm 5.1 is convergent when it is bounded, it is more challenging to deduce the asymptotic convergence rate: the latter typically requires an estimate of the KL exponent of F in (1.11), which was used in the above analysis. In what follows, we will show that the KL exponent of F is $\frac{1}{2}$. To do this, we will first establish a calculus rule for deducing the KL exponent of a fractional objective from the difference between the numerator and (a suitable scaling of) the denominator: this is along the line of the calculus rules for KL exponents developed in [75, 79, 124], and can be of independent interest.

5.2.1 KL exponent of F in (1.11)

Before proving our main result concerning the KL exponent of F in (1.11), we also need the following simple proposition.

Proposition 5.2. *Let p be a proper closed function, and let $\bar{x} \in \operatorname{dom} p$ be such that $p(\bar{x}) > 0$. Then the following statements hold.*

(i) *We have* $\partial(p^2)(x) = 2p(x)\partial p(x)$ *for all* x *sufficiently close to* $\bar{x}$.

(ii) *Suppose in addition that* $\bar{x} \in \operatorname{dom} \partial(p^2)$ *and* p^2 *satisfies the KL property at* $\bar{x}$ *with exponent* $\theta \in [0,1)$. *Then* p *satisfies the KL property at* $\bar{x}$ *with exponent* $\theta \in [0,1)$.

Proof. Since $p(\bar{x}) > 0$ and p is closed, there exists $\epsilon > 0$ so that

$$0 < p(x) < \infty$$

whenever $\|x - \bar{x}\| \le \epsilon$ and $x \in \operatorname{dom} p$. Then we deduce from [89, Lemma 1] that

$$\widehat{\partial}(p^2)(x) = 2p(x)\widehat{\partial}p(x) \quad \text{whenever } x \in \operatorname{dom} p \text{ and } \|x - \bar{x}\| \le \epsilon. \tag{5.7}$$

Using (5.7), and invoking the definition of limiting subdifferential and by shrinking ϵ if necessary, we deduce that

$$\partial(p^2)(x) = 2p(x)\partial p(x) \quad \text{whenever } x \in \operatorname{dom} p \text{ and } \|x - \bar{x}\| \le \epsilon. \tag{5.8}$$

In particular, if $\bar{x} \in \operatorname{dom} \partial(p^2)$, then $\bar{x} \in \operatorname{dom} \partial p$.

When p^2 also satisfies the KL property at $\bar{x}$ with exponent θ, by shrinking ϵ further if necessary, we see that there exists $c > 0$ so that

$$\operatorname{dist}(0, \partial(p^2)(x)) \ge c(p^2(x) - p^2(\bar{x}))^{\theta}, \tag{5.9}$$

whenever $p^2(\bar{x}) < p^2(x) < p^2(\bar{x}) + \epsilon(2p(\bar{x}) + \epsilon)$ and $\|x - \bar{x}\| \le \epsilon$. Thus, for $x \in \operatorname{dom} \partial p$ satisfying $\|x - \bar{x}\| \le \epsilon$ and $p(\bar{x}) < p(x) < p(\bar{x}) + \epsilon$, we have from (5.8) that

$$\begin{aligned}
&\operatorname{dist}(0, \partial p(x)) = \frac{1}{2p(x)}\operatorname{dist}(0, \partial(p^2)(x)) \ge \frac{1}{2p(\bar{x}) + 2\epsilon}\operatorname{dist}(0, \partial(p^2)(x)) \\
&\overset{\text{(a)}}{\ge} \frac{c}{2p(\bar{x}) + 2\epsilon}(p^2(x) - p^2(\bar{x}))^{\theta} = \frac{c}{2p(\bar{x}) + 2\epsilon}(p(x) + p(\bar{x}))^{\theta}(p(x) - p(\bar{x}))^{\theta} \\
&\ge \frac{c[p(\bar{x})]^{\theta}}{2^{1-\theta}(p(\bar{x}) + \epsilon)}(p(x) - p(\bar{x}))^{\theta},
\end{aligned}$$

where (a) follows from (5.9). This completes the proof. $\square$

We are now ready to show that the KL exponent of F in (1.11) is $\frac{1}{2}$. We remark that if the set $\mathcal{X} := \{x :\ 0 \in \partial F(x)\}$ is empty, then this claim holds trivially in view of [75, Lemma 2.1]. However, in general, one can have $\mathcal{X} \neq \emptyset$. Indeed, according to Theorem 5.1 and [100, Theorem 10.1], we have $\mathcal{X} \neq \emptyset$ with high probability when A is generated in a certain way.

Theorem 5.2. *The function F in* (1.11) *is a KL function with exponent $\frac{1}{2}$.*

Proof. In view of [75, Lemma 2.1], it suffices to look at the KL exponent at a stationary point $\bar{x}$ of F. For any $\bar{x}$ satisfying $0 \in \partial F(\bar{x})$, we have $F(\bar{x}) > 0$ since $b \neq 0$. Moreover, we have $0 \in \partial(F^2)(\bar{x})$ in view of Proposition 5.2(i). Next, note that the function

$$F_1(x) := \|x\|_1^2 - \frac{\|\bar{x}\|_1^2}{\|\bar{x}\|^2}\|x\|^2 + \delta_{A^{-1}\{b\}}(x)$$

can be written as $\min_{\sigma\in\mathfrak{R}}\{Q_\sigma(x) + P_\sigma(x)\}$, where $\mathfrak{R} = \{u \in \mathbb{R}^n :\ u_i \in \{1, -1\}\ \ \forall i\}$, and Q_σ are quadratic functions (nonconvex) and P_σ are polyhedral functions indexed by σ: indeed, for each $\sigma \in \mathfrak{R}$, one can define P_σ as the indicator function of the set $\{x :\ \ Ax = b, \sigma \circ x \geq 0\}$, where $\circ$ denotes the Hadamard product, and $Q_\sigma(x) := (\langle\sigma, x\rangle)^2 - \frac{\|\bar{x}\|_1^2}{\|\bar{x}\|^2}\|x\|^2$. Then, in view of [75, Corollary 5.2], F_1 is a KL function with exponent $\frac{1}{2}$. Since the convex function $\|\cdot\|_1^2$ is regular everywhere and the convex set $A^{-1}\{b\}$ is regular at every $x \in A^{-1}\{b\}$ (thanks to [100, Theorem 6.9]), we deduce using Theorem 3.3 that the function

$$x \mapsto \frac{\|x\|_1^2}{\|x\|^2} + \delta_{A^{-1}\{b\}}(x)$$

satisfies the KL property at $\bar{x}$ with exponent $\frac{1}{2}$. The desired conclusion now follows from this and Proposition 5.2(ii). $\square$

Equipped with the result above, by following the line of arguments in [6, Theorem 2], one can conclude further that the sequence $\{x^t\}$ generated by Algorithm 5.1

converges locally linearly to a stationary point of F in (1.11) if the sequence is bounded. The proof is standard and we omit it here for brevity.

Theorem 5.3 (Convergence rate of Algorithm 5.1)**.** *Consider* (1.5)*. Let* $\{x^t\}$ *be the sequence generated by Algorithm 5.1 and suppose that* $\{x^t\}$ *is bounded. Then* $\{x^t\}$ *converges to a stationary point* x^* *of* F *in* (1.11) *and there exist* $\underline{t} \in \mathcal{N}_+$*,* $a_0 \in (0,1)$ *and* $a_1 > 0$ *such that*

$$\|x^t - x^*\| \leq a_1 a_0^t \quad \textit{whenever } t > \underline{t}.$$

5.3 Compressed sensing with noise based on ℓ_1/ℓ_2 minimization

In the previous sections, we have been focusing on the model (1.5), which corresponds to noiseless compressed sensing problems. In this section and the next, we will be looking at (1.6). We will discuss conditions for existence of solutions and derive some first-order optimality conditions for (1.6) in this section. An algorithm for solving (1.6) will be proposed in the next section and will be shown to generate sequences that cluster at "critical" points in the sense defined in this section, under suitable assumptions.

5.3.1 Solution existence

Clearly, if q in (1.6) is in addition level-bounded, then the feasible set is compact and hence the set of optimal solutions is nonempty. However, in applications such as (1.7), (1.8) and (1.9), the corresponding q is not level-bounded. Here, we discuss solution existence for (1.7) and (1.8). Our arguments are along the same line as those in Section 5.1. We first present a lemma that establishes a relationship between the problems (1.7), (1.8) and (5.1).

Lemma 5.2. *Consider* (5.1) *and* (1.6) *with* q *given as in* (1.7) *or* (1.8). *Then* $\nu^*_{ncs} = \nu^*_d$ *if and only if there exists a minimizing sequence of* (1.6) *that is unbounded.*

The proof of this lemma is almost identical to that of Lemma 5.1. Here we omit the details and only point out a slight difference concerning the derivation of (5.2). Take (1.7) as an example and let $\{x^t\}$ be an unbounded minimizing sequence of it with $\lim_{t\to\infty} \frac{x^t}{\|x^t\|} = x^*$ for some x^* satisfying $\|x^*\| = 1$. Then one can prove $Ax^* = 0$ by using the facts that $\|Ax^t - b\| \leq \sigma$ for all t and $\|x^t\| \to \infty$. Similar deductions can be done for (1.8).

Using Lemma 5.2, we can deduce solution existence based on the SSP of $\ker A$ and the existence of a sparse feasible solution to (1.7) (or (1.8)). The corresponding arguments are the same as those in Theorem 5.1 and we omit the proof for brevity.

Theorem 5.4 (Solution existence for (1.7) and (1.8))**.** *Consider* (1.6) *with* q *given as in* (1.7) *or* (1.8). *Suppose that* $\ker A$ *has the* s*-spherical section property and there exists* $\widetilde{x} \in \mathbb{R}^n$ *such that* $\|\widetilde{x}\|_0 < m/s$ *and* $q(\widetilde{x}) \leq 0$. *Then the optimal value of* (1.6) *is attainable.*

5.3.2 Optimality conditions

We discuss first-order necessary optimality conditions for local minimizers. Our analysis is based on the following standard constraint qualifications.

Definition 5.2 (Generalized Mangasarian-Fromovitz constraint qualifications)**.** *Consider* (1.6). *We say that the general Mangasarian-Fromovitz constraint qualifications (GMFCQ) holds at an* x^* *satisfying* $q(x^*) \leq 0$ *if the following statement holds:*

- *If* $q(x^*) = 0$, *then* $0 \notin \partial^\circ q(x^*)$.

The GMFCQ reduces to the standard MFCQ when q is smooth. One can then see from Sections 5.1 and 5.2 of [125] that the GMFCQ holds at every x feasible for

(1.7) and (1.8) for all positive σ and γ, because A is surjective. We next study the GMFCQ for (1.9), in which A is also surjective.

Proposition 5.3. *The GMFCQ holds in the whole feasible set of* (1.9)*.*

Proof. It is straightforward to see that the GMFCQ holds for $x \in \{x : q(x) < 0\}$. Then it remains to consider those x satisfying $q(x) = 0$. Let q be as in (1.9) and $\bar{x}$ satisfy $q(\bar{x}) = 0$. Notice that a $\xi \in \mathrm{Proj}_S(A\bar{x} - b)$ takes the following form:

$$\xi_j = \begin{cases} [A\bar{x} - b]_j & \text{if } j \in I^*, \\ 0 & \text{otherwise,} \end{cases}$$

where I^* is an index set corresponding to the r-largest entries (in magnitude). Then for any $\xi \in \mathrm{Proj}_S(A\bar{x} - b)$, we have

$$\begin{aligned} \langle A\bar{x} - b, \xi \rangle &= \|\xi\|^2, \\ \|A\bar{x} - b\|^2 &= \|\xi\|^2 + \|A\bar{x} - b - \xi\|^2 \\ &= \|\xi\|^2 + \mathrm{dist}^2(A\bar{x} - b, S) \overset{\text{(a)}}{=} \|\xi\|^2 + \sigma^2, \end{aligned} \tag{5.10}$$

where (a) holds because $0 = q(\bar{x}) = \mathrm{dist}^2(A\bar{x} - b, S) - \sigma^2$. Furthermore, since A is surjective, we can deduce from [100, Example 8.53], [100, Exercise 10.7] and [100, Theorem 8.49] that

$$\partial^\circ q(\bar{x}) = \mathrm{conv}\{2A^T(A\bar{x} - b - \xi) : \xi \in \mathrm{Proj}_S(A\bar{x} - b)\}.$$

Now, suppose to the contrary that $0 \in \partial^\circ q(\bar{x})$. Using Carathéodory's theorem, we see that there exist $\lambda_i \geq 0$ and $\xi_i \in \mathrm{Proj}_S(A\bar{x} - b)$, $i = 1, \cdots, m+1$ such that $\sum_{i=1}^{m+1} \lambda_i = 1$ and $\sum_{i=1}^{m+1} \lambda_i A^T(A\bar{x} - b - \xi_i) = 0$. Since A is surjective, we then have

$$\sum_{i=1}^{m+1} \lambda_i (A\bar{x} - b - \xi_i) = 0.$$

Multiplying both sides of the above equality by $(A\bar{x}-b)^T$, we obtain further that

$$
\begin{aligned}
0 &= \sum_{i=1}^{m+1} \lambda_i \langle A\bar{x}-b, A\bar{x}-b-\xi_i\rangle = \sum_{i=1}^{m+1} \lambda_i [\|A\bar{x}-b\|^2 - \langle A\bar{x}-b, \xi_i\rangle] \\
&\overset{\text{(a)}}{=} \sum_{i=1}^{m+1} \lambda_i [\|\xi_i\|^2 + \sigma^2 - \|\xi_i\|^2] = \sigma^2 > 0,
\end{aligned}
$$

where (a) follows from (5.10) and the fact that $\xi_i \in \mathrm{Proj}_S(A\bar{x}-b)$ for each i, and the last equality holds because $\sum_{i=1}^{m+1} \lambda_i = 1$. This is a contradiction and thus we must have $0 \notin \partial^\circ q(\bar{x})$. This completes the proof. $\square$

In the next definition, we consider some notions of criticality. The first one is the standard notion of stationarity while the second one involves the Clarke subdifferential.

Definition 5.3. *Consider* (1.6)*. We say that an* $\bar{x} \in \mathbb{R}^n$ *satisfying* $q(\bar{x}) \le 0$ *is*

(i) *a stationary point of* (1.6) *if*

$$
0 \in \partial \left(\tfrac{\|\cdot\|_1}{\|\cdot\|} + \delta_{[q \le 0]}(\cdot) \right)(\bar{x}); \tag{5.11}
$$

(ii) *a Clarke critical point of* (1.6) *if there exists* $\bar{\lambda} \ge 0$ *such that*

$$
0 \in \partial \tfrac{\|\bar{x}\|_1}{\|\bar{x}\|} + \bar{\lambda}\partial^\circ q(\bar{x}) \quad \text{and} \quad \bar{\lambda} q(\bar{x}) = 0. \tag{5.12}
$$

As mentioned above, Definition 5.3(i) is standard and it is known that every local minimizer of (1.6) is a stationary point; see [100, Theorem 10.1]. We next study some relationships between these notions of criticality, and show in particular that every local minimizer is Clarke critical when the GMFCQ holds.

Proposition 5.4 (Stationarity vs Clarke criticality)**.** *Consider* (1.6) *and let* $\bar{x}$ *be such that* $q(\bar{x}) \le 0$*. Then the following statements hold.*

(i) *If $\bar{x}$ is a stationary point of* (1.6) *and the GMFCQ holds at $\bar{x}$, then $\bar{x}$ is a Clarke critical point.*

(ii) *If $\bar{x}$ is a Clarke critical point of* (1.6) *and q is regular at $\bar{x}$, then $\bar{x}$ is stationary.*

Remark 5.2. *Since local minimizers of* (1.6) *are stationary points, we see from Proposition 5.4(i) that when the GMFCQ holds in the whole feasible set, local minimizers are also Clarke critical.*

Proof. Suppose that $\bar{x}$ is a stationary point of (1.6) at which the GMFCQ holds. Then (5.11) holds and we consider two cases.

Case 1: $q(\bar{x}) < 0$. Since q is continuous, (5.11) implies $0 \in \partial \frac{\|\bar{x}\|_1}{\|\bar{x}\|}$ and hence (5.12) holds with $\bar{\lambda} = 0$. Thus, $\bar{x}$ is a Clarke critical point.

Case 2: $q(\bar{x}) = 0$. Since the GMFCQ holds for (1.6) at $\bar{x}$, we see that $0 \notin \partial^\circ q(\bar{x})$. Then we can deduce from (5.11) and [100, Exercise 10.10] that

$$0 \in \partial \frac{\|\bar{x}\|_1}{\|\bar{x}\|} + N_{[q\leq 0]}(\bar{x}) \overset{\text{(a)}}{\subseteq} \partial \frac{\|\bar{x}\|_1}{\|\bar{x}\|} + \bigcup_{\lambda \geq 0} \lambda \partial^\circ q(\bar{x}),$$

where (a) follows from [27, Theorem 5.2.22], the first corollary to [41, Theorem 2.4.7] and the fact that $0 \notin \partial^\circ q(\bar{x})$. Thus, (5.12) holds with some $\bar{\lambda} \geq 0$ (recall that $q(\bar{x}) = 0$), showing that $\bar{x}$ is a Clarke critical point. This proves item (i).

We now prove item (ii). Suppose that $\bar{x}$ is a Clarke critical point and that q is regular at $\bar{x}$. Then there exists $\bar{\lambda} \geq 0$ so that (5.12) holds. We again consider two cases.

Case 1: $\bar{\lambda} = 0$. In this case, we see from (5.12) that $0 \in \partial \frac{\|\bar{x}\|_1}{\|\bar{x}\|}$, which implies

$$\begin{aligned} 0 \in \partial \frac{\|\bar{x}\|_1}{\|\bar{x}\|} &\overset{\text{(a)}}{=} \widehat{\partial} \frac{\|\bar{x}\|_1}{\|\bar{x}\|} \subseteq \widehat{\partial} \frac{\|\bar{x}\|_1}{\|\bar{x}\|} + \widehat{N}_{[q\leq 0]}(\bar{x}) \\ &\overset{\text{(b)}}{\subseteq} \widehat{\partial} \left(\frac{\|\cdot\|_1}{\|\cdot\|} + \delta_{[q\leq 0]}(\cdot) \right)(\bar{x}) \overset{\text{(c)}}{\subseteq} \partial \left(\frac{\|\cdot\|_1}{\|\cdot\|} + \delta_{[q\leq 0]}(\cdot) \right)(\bar{x}), \end{aligned}$$

where (a) follows from (2.17) and [100, Corollary 8.11], (b) holds thanks to [100, Corollary 10.9], and (c) follows from [100, Theorem 8.6]. Thus, $\bar{x}$ is a stationary point.

Case 2: $\bar{\lambda} > 0$. In this case, we have from (5.12) that $q(\bar{x}) = 0$. Since q is regular at $\bar{x}$, we see from [100, Corollary 8.11] and the discussion right after [100, Theorem 8.49] that

$$\widehat{\partial} q(\bar{x}) = \partial q(\bar{x}) = \partial^\circ q(\bar{x}). \tag{5.13}$$

Now, in view of (5.13), $q(\bar{x}) = 0$ and [100, Proposition 10.3], we have

$$\widehat{N}_{[q\leq 0]}(\bar{x}) \supseteq \bigcup_{\lambda\geq 0} \lambda \widehat{\partial} q(\bar{x}) = \bigcup_{\lambda\geq 0} \lambda \partial^\circ q(\bar{x}). \tag{5.14}$$

We then deduce that

$$\partial\left(\frac{\|\cdot\|_1}{\|\cdot\|} + \delta_{[q\leq 0]}(\cdot)\right)(\bar{x}) \overset{\text{(a)}}{\supseteq} \widehat{\partial}\frac{\|\bar{x}\|_1}{\|\bar{x}\|} + \widehat{N}_{[q\leq 0]}(\bar{x}) \overset{\text{(b)}}{\supseteq} \partial\frac{\|\bar{x}\|_1}{\|\bar{x}\|} + \bigcup_{\lambda\geq 0} \lambda \partial^\circ q(\bar{x}),$$

where (a) follows from [100, Theorem 8.6] and [100, Corollary 10.9], and (b) follows from (5.14), (2.17) and [100, Corollary 8.11]. This together with the definition of Clarke criticality shows that (5.11) holds. This completes the proof. □

5.4 A moving-balls-approximation based algorithm for solving (1.6)

In this section, we propose and analyze an algorithm for solving (1.6), which is an extension of Algorithm 5.1 by incorporating *moving-balls-approximation* (MBA) techniques [10]. Our algorithm, which we call $\mathrm{MBA}_{\ell_1/\ell_2}$, is presented as Algorithm 5.2 below.

Algorithm 5.2. $\mathbf{MBA}_{\ell_1/\ell_2}$**: Moving-balls-approximation based algorithm for** (1.6)

Step 0. *Choose* x^0 *with* $q(x^0) \leq 0$, $\alpha > 0$ *and* $0 < l_{\min} < l_{\max}$. *Set* $\omega_0 = \|x^0\|_1/\|x^0\|$ *and* $t = 0$.

Step 1. *Choose* $l_t^0 \in [l_{\min}, l_{\max}]$ *arbitrarily and set* $l_t = l_t^0$. *Choose* $\zeta^t \in \partial P_2(x^t)$.

(1a) Solve the subproblem

$$\begin{aligned} \widetilde{x} = \underset{x \in \mathbb{R}^n}{\arg\min} \quad & \|x\|_1 - \frac{\omega_t}{\|x^t\|}\langle x, x^t\rangle + \frac{\alpha}{2}\|x - x^t\|^2 \\ \text{s.t.} \quad & q(x^t) + \langle \nabla P_1(x^t) - \zeta^t, x - x^t\rangle + \frac{l_t}{2}\|x - x^t\|^2 \leq 0. \end{aligned} \tag{5.15}$$

(1b) If $q(\widetilde{x}) \leq 0$, *go to* **Step 2**. *Else, update* $l_t \leftarrow 2l_t$ *and go to Step (1a).*

Step 2. *Set* $x^{t+1} = \widetilde{x}$ *and compute* $\omega_{t+1} = \|x^{t+1}\|_1/\|x^{t+1}\|$. *Set* $\bar{l}_t := l_t$. *Update* $t \leftarrow t + 1$ *and go to* **Step 1.**

Unlike previous works [19, 23, 125] that made use of MBA techniques, our algorithm deals with a *fractional* objective and a possibly *nonsmooth* continuous constraint function. Thus, the convergence results in [19, 23, 125] cannot be directly applied to analyze our algorithm. Indeed, as we shall see later in Section 5.4.2, we need to introduce a new potential function for our analysis to deal with the possibly nonsmooth q in the constraint.

We will show that Algorithm 5.2 is well defined later, i.e., for each $t \in \mathcal{N}_+$, the subproblem (5.15) has a unique solution for every l_t and the inner loop in **Step 1** terminates finitely. Here, it is worth noting that (5.15) can be efficiently solved using a root-finding procedure outlined in [125, Appendix A] since (5.15) takes the form of

$$\min_x \ \|x\|_1 + \frac{\alpha}{2}\|x - c^t\|^2 \quad \text{s.t.}\ \|x - s^t\|^2 \leq R_t$$

for some $c^t \in \mathbb{R}^n$, $s^t \in \mathbb{R}^n$ and $R_t \geq 0$.

5.4.1 Convergence analysis

In this subsection, we establish subsequential convergence of $\mathrm{MBA}_{\ell_1/\ell_2}$ under suitable assumptions. We start with the following auxiliary lemma that concerns well-definedness and sufficient descent. The proof of the sufficient descent property in item (iii) below is essentially the same as [116, Lemma 1]. We include it here for completeness.

Lemma 5.3 (Well-definedness and sufficient descent)**.** *Consider* (1.6)*. Then the following statements hold:*

(i) *MBA_{ℓ_1/ℓ_2} is well defined, i.e., for each $t \in \mathcal{N}_+$, the subproblem* (5.15) *has a unique solution for every l_t and the inner loop in* **Step 1** *terminates finitely.*

(ii) *The sequence $\{\bar{l}_t\}$ is bounded.*

(iii) *Let $\{(x^t, \omega_t)\}$ be the sequence generated by MBA_{ℓ_1/ℓ_2}. Then there exists $\delta > 0$ such that $\|x^t\| \geq \delta$ for every $t \in \mathcal{N}_+$, and the sequence $\{\omega_t\}$ satisfies*

$$\omega_t - \omega_{t+1} \geq \frac{\alpha}{2\|x^{t+1}\|}\|x^t - x^{t+1}\|^2, \quad t \in \mathcal{N}_+. \tag{5.16}$$

Proof. Suppose that an x^t satisfying $q(x^t) \leq 0$ is given for some $t \in \mathcal{N}_+$. Then $x^t \neq 0$ since $q(0) > 0$. Moreover, for any $l_t > 0$, x^t is feasible for (5.15) and the feasible set is thus nonempty. Since (5.15) minimizes a strongly convex continuous function over a nonempty closed convex set, it has a unique optimal solution, i.e., $\widetilde{x}$ exists.

Let L_p be the Lipschitz continuity modulus of ∇P_1. Then we have

$$\begin{aligned} q(\widetilde{x}) = P_1(\widetilde{x}) - P_2(\widetilde{x}) &\leq P_1(x^t) + \langle \nabla P_1(x^t), \widetilde{x} - x^t\rangle + \tfrac{L_p}{2}\|\widetilde{x} - x^t\|^2 - P_2(\widetilde{x}) \\ &\overset{(a)}{\leq} P_1(x^t) - P_2(x^t) + \langle \nabla P_1(x^t) - \zeta^t, \widetilde{x} - x^t\rangle + \tfrac{L_p}{2}\|\widetilde{x} - x^t\|^2 \overset{(b)}{\leq} \tfrac{L_p - l_t}{2}\|\widetilde{x} - x^t\|^2, \end{aligned} \tag{5.17}$$

where (a) holds because of the convexity of P_2 and the definition of ζ^t, and (b) follows from the feasibility of $\widetilde{x}$ for (5.15). Let $k_0 \in \mathcal{N}_+$ be such that $L_p - 2^{k_0} l_{\min} \leq 0$. Then by (5.17) and the definition of l_t we see that $q(\widetilde{x}) \leq 0$ after at most k_0 calls of Step (1b). Moreover, it holds that $\bar{l}_t \leq 2^{k_0} l_{\max}$. Therefore, if $q(x^t) \leq 0$, then the inner loop of Step 1 stops after at most k_0 iterations and outputs an x^{t+1} satisfying $q(x^{t+1}) \leq 0$ (in particular, $x^{t+1} \neq 0$) with $\bar{l}_t \leq 2^{k_0} l_{\max}$. Since we initialize our algorithm at an x^0 satisfying $q(x^0) \leq 0$, the conclusions in items (i) and (ii) now follow from an induction argument.

Next, we prove item (iii). Since $q(0) > 0$, we see immediately from the continuity of q that there exists some $\delta > 0$ such that $\|x\| \geq \delta$ whenever $q(x) \leq 0$. Thus, $\|x^t\| \geq \delta$ for all $t \in \mathcal{N}_+$, thanks to $q(x^t) \leq 0$. Now consider (5.15) with $l_t = \bar{l}_t$. Then x^t is feasible and x^{t+1} is optimal. This together with the definition of ω_t yields

$$\|x^{t+1}\|_1 - \frac{\|x^t\|_1}{\|x^t\|^2} \left\langle x^{t+1}, x^t \right\rangle + \frac{\alpha}{2}\|x^{t+1} - x^t\|^2 \leq \|x^t\|_1 - \frac{\|x^t\|_1}{\|x^t\|^2} \left\langle x^t, x^t \right\rangle + \frac{\alpha}{2}\|x^t - x^t\|^2 = 0.$$

Dividing both sides of the above inequality by $\|x^{t+1}\|$ and rearranging terms, we have

$$\frac{\|x^{t+1}\|_1}{\|x^{t+1}\|} + \frac{\alpha}{2\|x^{t+1}\|}\|x^t - x^{t+1}\|^2 \leq \frac{\|x^t\|_1}{\|x^t\|^2}\frac{\langle x^{t+1}, x^t\rangle}{\|x^{t+1}\|} \leq \frac{\|x^t\|_1}{\|x^t\|^2}\frac{\|x^{t+1}\|\|x^t\|}{\|x^{t+1}\|} = \frac{\|x^t\|_1}{\|x^t\|}.$$

This proves (iii) and completes the proof. □

We next introduce the following assumption.

Assumption 5.1. *The GMFCQ for* (1.6) *holds at every point in* $[q \leq 0]$.

Recall from Proposition 5.3 and the discussions preceding it that Assumption 5.1 holds for (1.7), (1.8) and (1.9) since A is surjective. We next derive the Karush-Kuhn-Tucker (KKT) conditions for (5.15) at every iteration t under Assumption 5.1, which will be used in our subsequent analysis.

Lemma 5.4 (KKT conditions for (5.15))**.** *Consider* (1.6) *and suppose that Assumption 5.1 holds. Let* $\{x^t\}$ *be the sequence generated by* MBA_{ℓ_1/ℓ_2}*. Then the following statements hold:*

(i) *The Slater's condition holds for the constraint of* (5.15) *at each* $t \in \mathcal{N}_+$.

(ii) *For each* $t \in \mathcal{N}_+$, $\zeta^t \in \partial P_2(x^t)$ *and* $l_t > 0$, *the subproblem* (5.15) *has a Lagrange multiplier* $\lambda_t \geq 0$. *Moreover, if* $\widetilde{x}$ *is as in* (5.15), *then it holds that*

$$\lambda_t \left(q(x^t) + \left\langle \nabla P_1(x^t) - \zeta^t, \widetilde{x} - x^t \right\rangle + \frac{l_t}{2} \|\widetilde{x} - x^t\|^2 \right) = 0, \tag{5.18}$$

$$0 \in \partial \|\widetilde{x}\|_1 - \frac{\omega_t x^t}{\|x^t\|} + \lambda_t (\nabla P_1(x^t) - \zeta^t) + (\alpha + \lambda_t l_t)(\widetilde{x} - x^t). \tag{5.19}$$

Proof. Notice that we can rewrite the feasible set of (5.15) as $B\left(s^t, \sqrt{R_t}\right)$ with $s^t := x^t - \frac{1}{l_t}(\nabla P_1(x^t) - \zeta^t)$ and $R_t := \frac{1}{l_t^2}\|\nabla P_1(x^t) - \zeta^t\|^2 - \frac{2}{l_t} q(x^t)$, where $R_t \geq 0$ because $q(x^t) \leq 0$. Suppose to the contrary that $R_t = 0$. Then we have $q(x^t) = 0$ and $\nabla P_1(x^t) - \zeta^t = 0$. The latter relation together with (2.18) implies $0 \in \partial^\circ q(x^t)$, contradicting the GMFCQ assumption at x^t. Thus, we must have $R_t > 0$ and hence the Slater's condition holds for (5.15) at the t^{th} iteration.

Since the Slater's condition holds for (5.15), we can apply [99, Corollary 28.2.1] and [99, Theorem 28.3] to conclude that there exists a Lagrange multiplier λ_t such that the relation (5.18) holds at the t^{th} iteration and $\widetilde{x}$ minimizes the following function:

$$\begin{aligned} \mathfrak{L}_t(x) :=& \|x\|_1 - \frac{\omega_t}{\|x^t\|}\langle x, x^t \rangle + \frac{\alpha}{2}\|x - x^t\|^2 \\ & + \lambda_t \left(q(x^t) + \left\langle \nabla P_1(x^t) - \zeta^t, x - x^t \right\rangle + \frac{l_t}{2}\|x - x^t\|^2 \right). \end{aligned}$$

This fact together with [100, Exercise 8.8] and [100, Theorem 10.1] implies that (5.19) holds at the t^{th} iteration. This completes the proof. □

Now we are ready to establish the subsequential convergence of Algorithm 5.2. In our analysis, we assume that the GMFCQ holds and that the $\{x^t\}$ generated by $\text{MBA}_{\ell_1/\ell_2}$ is bounded. The latter boundedness assumption was also used in [116] for analyzing the convergence of Algorithm 5.1. We remark that this assumption is not too restrictive. Indeed, for the sequence $\{x^t\}$ generated by $\mathrm{MBA}_{\ell_1/\ell_2}$, in view of Lemma 5.3(i), we know that $q(x^t) \leq 0$ for all t. Thus, if q is level-bounded, then $\{x^t\}$ is bounded. On the other hand, if q is only known to be bounded from below (as in (1.7), (1.8) and (1.9)) but the corresponding (1.6) is known to have an optimal solution, then one may replace $q(x)$ by the level-bounded function $q_M(x) := q(x) + (\|x\| - M)_+^2$ for a sufficiently large M. As long as $M > \|x^*\|$ for some optimal solution x^* of (1.6), replacing q by q_M in (1.6) will not change the optimal value.

Theorem 5.5 (Subsequential convergence of $\mathrm{MBA}_{\ell_1/\ell_2}$). *Consider* (1.6) *and suppose that Assumption 5.1 holds. Let* $\{(x^t, \zeta^t, \bar{l}_t)\}$ *be the sequence generated by* $\mathit{MBA}_{\ell_1/\ell_2}$ *and* λ_t *be a Lagrange multiplier of* (5.15) *with* $l_t = \bar{l}_t$. *Suppose in addition that* $\{x^t\}$ *is bounded. Then the following statements hold:*

(i) $\lim_{t\to\infty} \|x^{t+1} - x^t\| = 0$*;*

(ii) *The sequences* $\{\lambda_t\}$ *and* $\{\zeta^t\}$ *are bounded;*

(iii) *Let* $\bar{x}$ *be an accumulation point of* $\{x^t\}$*. Then* $\bar{x}$ *is a Clarke critical point of* (1.6)*. If* q *is also regular at* $\bar{x}$*, then* $\bar{x}$ *is a stationary point.*

Proof. Since $\{x^t\}$ is bounded, there exists $M > 0$ such that $\|x^t\| \leq M$ for all $t \in \mathcal{N}_+$. Using (5.16), we obtain

$$\sum_{t=0}^{\infty} \frac{\alpha}{2M} \|x^t - x^{t+1}\|^2 \leq \omega_0 - \liminf_{t\to\infty} \omega_t \leq \omega_0,$$

which proves item (i).

Now we turn to item (ii). The boundedness of $\{\zeta^t\}$ follows from the boundedness of $\{x^t\}$ and [112, Theorem 2.6]. We next prove the boundedness of $\{\lambda_t\}$. Suppose to the contrary that $\{\lambda_t\}$ is unbounded. Then there exists a subsequence $\{\lambda_{t_k}\}$ such that $\lim_{k\to\infty} \lambda_{t_k} = \infty$. Passing to a subsequence if necessary, we can find subsequences

$\{x^{t_k}\}$ and $\{\lambda_{t_k}\}$ such that $\lim_{k\to\infty} x^{t_k} = x^*$ and $\lambda_{t_k} > 0$ for all $k \in \mathcal{N}_+$, where the existence of x^* is due to the boundedness of $\{x^t\}$. According to (5.18) and the definition of x^{t_k+1}, we obtain

$$q(x^{t_k}) + \langle \nabla P_1(x^{t_k}) - \zeta^{t_k}, x^{t_k+1} - x^{t_k} \rangle + \frac{\bar{l}_{t_k}}{2} \left\| x^{t_k+1} - x^{t_k} \right\|^2 = 0.$$

Since $\{x^t\}$ is bounded and ∇P_1 is Lipschitz continuous, we then see that $\{\nabla P_1(x^t)\}$ is bounded. Moreover, $\{\bar{l}_{t_k}\}$ is bounded thanks to Lemma 5.3(ii) and we also know that $\{\zeta^t\}$ is bounded. Using these facts, item (i) and the continuity of q, we have upon passing to the limit in the above display that $q(x^*) = 0$. Since the GMFCQ holds for (1.6) at x^*, we then have $0 \notin \partial^\circ q(x^*)$.

Let $t = t_k$, $l_t = \bar{l}_{t_k}$, $\widetilde{x} = x^{t_k+1}$ in (5.19), and divide both sides of (5.19) by λ_{t_k}. Then

$$\nabla P_1(x^{t_k}) - \zeta^{t_k} \in -\frac{1}{\lambda_{t_k}} \partial \|x^{t_k+1}\|_1 + \frac{\omega_{t_k} x^{t_k}}{\lambda_{t_k} \|x^{t_k}\|} - \left(\bar{l}_{t_k} + \frac{\alpha}{\lambda_{t_k}} \right) (x^{t_k+1} - x^{t_k}).$$

Thus, there exists a sequence $\{\eta^k\}$ satisfying $\eta^k \in \partial \|x^{t_k+1}\|_1$ and

$$\nabla P_1(x^{t_k}) - \zeta^{t_k} = -\frac{1}{\lambda_{t_k}} \eta^k + \frac{\omega_{t_k} x^{t_k}}{\lambda_{t_k} \|x^{t_k}\|} - \left(\bar{l}_{t_k} + \frac{\alpha}{\lambda_{t_k}} \right) (x^{t_k+1} - x^{t_k}).$$

Note that $\{\eta^k\}$ is bounded since $\partial \|x\|_1 \subseteq [-1, 1]^n$ for any $x \in \mathbb{R}^n$. Moreover, $\{\omega_{t_k}\}$ is bounded since $\|x\| \leq \|x\|_1 \leq \sqrt{n}\|x\|$ for any $x \in \mathbb{R}^n$. Furthermore, we have the boundedness of $\{\bar{l}_{t_k}\}$ from Lemma 5.3(ii). Also recall that $\lim_{k\to\infty} \lambda_{t_k} = \infty$ and $\zeta^t \in \partial P_2(x^t)$. Using these together with item (i), we have upon passing to the limit in the above display and invoking the closedness of ∂P_2 (see Exercise 8 of [25, Section 4.2]) that $\nabla P_1(x^*) \in \partial P_2(x^*)$. This together with (2.18) further implies $0 \in \partial^\circ q(x^*)$, leading to a contradiction. Thus, the sequence $\{\lambda_t\}$ is bounded.

We now turn to item (iii). Suppose $\bar{x}$ is an accumulation point of $\{x^t\}$ with $\lim_{j\to\infty} x^{t_j} = \bar{x}$ for some convergent subsequence $\{x^{t_j}\}$. Since $\{\lambda_t, \bar{l}_t\}$ and $\{\zeta^t\}$ are bounded (thanks to Lemma 5.3(ii) and item (ii)), passing to a further subsequence if necessary, we may assume without loss of generality that

$$\lim_{j\to\infty}(\lambda_{t_j},\bar{l}_{t_j})=(\bar{\lambda},\bar{l})\text{ for some }\bar{\lambda},\ \bar{l}\geq 0,\quad \lim_{j\to\infty}\zeta^{t_j}=\bar{\zeta}\text{ for some }\bar{\zeta}\in\partial P_2(\bar{x});\quad (5.20)$$

here, $\bar{\zeta}\in\partial P_2(\bar{x})$ because of the closedness of ∂P_2 (see Exercise 8 of [25, Section 4.2]). On the other hand, according to Lemma 5.3(iii), we have $\|x^t\|\geq\delta>0$ for all $t\in\mathcal{N}_+$. This together with the definition of $\bar{x}$ yields $\|\bar{x}\|\neq 0$. It then follows that $\frac{\|\cdot\|_1}{\|\cdot\|}$ is continuous at $\bar{x}$. Thus, we have, upon using this fact, the definition of ω_t, the continuity of ∇P_1, the closedness of $\partial\|\cdot\|_1$, item (i), (5.20), and passing to the limit as $j\to\infty$ in (5.19) with $(\widetilde{x},\lambda_t,l_t)=(x^{t_j+1},\lambda_{t_j},\bar{l}_{t_j})$ and $t=t_j$ that

$$0\in\partial\|\bar{x}\|_1-\frac{\|\bar{x}\|_1}{\|\bar{x}\|^2}\bar{x}+\bar{\lambda}(\nabla P_1(\bar{x})-\bar{\zeta}).$$

We then divide both sides of the above inclusion by $\|\bar{x}\|$ and obtain

$$0\in\frac{1}{\|\bar{x}\|}\partial\|\bar{x}\|_1-\frac{\|\bar{x}\|_1}{\|\bar{x}\|^3}\bar{x}+\frac{\bar{\lambda}}{\|\bar{x}\|}(\nabla P_1(\bar{x})-\bar{\zeta})=\partial\frac{\|\bar{x}\|_1}{\|\bar{x}\|}+\frac{\bar{\lambda}}{\|\bar{x}\|}(\nabla P_1(\bar{x})-\bar{\zeta}),\quad (5.21)$$

where the equality holds due to (2.17). In addition, using (5.18) with $(\widetilde{x},\lambda_t,l_t)=(x^{t_j+1},\lambda_{t_j},\bar{l}_{t_j})$ and $t=t_j$, we have

$$\lim_{j\to\infty}\lambda_{t_j}\left[q(x^{t_j})+\langle\nabla P_1(x^{t_j})-\zeta^{t_j},x^{t_j+1}-x^{t_j}\rangle+\frac{\bar{l}_{t_j}}{2}\|x^{t_j+1}-x^{t_j}\|^2\right]=0.$$

This together with item (i) and (5.20) shows that $\bar{\lambda}q(\bar{x})=0$. Combining this with (5.21), $\bar{\zeta}\in\partial P_2(\bar{x})$ (see (5.20)), (2.18) and the fact that $q(\bar{x})\leq 0$ (because $q(x^t)\leq 0$ for all t) shows that $\bar{x}$ is a Clarke critical point. Finally, the claim concerning stationarity follows immediately from Proposition 5.4. This completes the proof. □

5.4.2 Global convergence under KL assumption

We now discuss global convergence of the sequence $\{x^t\}$ generated by Algorithm 5.2. Our analysis follows the line of analysis in [6–8, 19, 23, 125] and is based on the following auxiliary function:

$$\widetilde{F}(x,y,\zeta,w) := \frac{\|x\|_1}{\|x\|} + \delta_{[\widetilde{q}\leq 0]}(x,y,\zeta,w) + \delta_{\|\cdot\|\geq\rho}(x), \tag{5.22}$$

with

$$\widetilde{q}(x,y,\zeta,w) := P_1(y) + \langle\nabla P_1(y), x-y\rangle + P_2^*(\zeta) - \langle\zeta, x\rangle + \frac{w}{2}\|x-y\|^2, \tag{5.23}$$

where P_1 and P_2 are as in (1.6), and $\rho > 0$ is chosen such that $\{x:\ q(x)\leq 0\}\subset\{x:\ \|x\|>\rho\}$. Some comments on $\widetilde{F}$ are in place. First, recall that in the potential function used in [23] for analyzing their MBA variant, the authors replaced $P_1(x)$ by a quadratic majorant $P_1(y)+\langle\nabla P_1(y), x-y\rangle + \frac{L_p}{2}\|x-y\|^2$, where L_p is the Lipschitz continuity modulus of ∇P_1. In this section, we will also assume P_1 to be twice continuously differentiable. Here, as in [125], we further introduce the variable w to handle the varying $\bar{l}_t$. Finally, to deal with the possibly nonsmooth $-P_2$, we replaced $-P_2(x)$ by its majorant $P_2^*(\zeta)-\langle\zeta,x\rangle$ as in [79].

The next proposition concerns the subdifferential of $\widetilde{F}$ and will be used for deriving global convergence of the sequence generated by $\mathrm{MBA}_{\ell_1/\ell_2}$.

Proposition 5.5. *Consider* (1.6) *and assume that P_1 is twice continuously differentiable. Suppose that Assumption 5.1 holds. Let $\{(x^t,\zeta^t,\bar{l}_t)\}$ be the sequence generated by MBA_{ℓ_1/ℓ_2} and suppose that $\{x^t\}$ is bounded. Let $\widetilde{F}$ and $\widetilde{q}$ be given in* (5.22) *and* (5.23) *respectively. Then the following statements hold:*

(i) *For any $t\in\mathcal{N}_+$, we have $\widetilde{q}(x^{t+1},x^t,\zeta^t,\bar{l}_t)\leq 0$.*

(ii) *There exist* $\kappa > 0$ *and* $\underline{t} \in \mathcal{N}_+$ *such that*

$$\operatorname{dist}(0, \partial\widetilde{F}(x^{t+1}, x^t, \zeta^t, \bar{l}_t)) \le \kappa\|x^{t+1} - x^t\| \quad \text{for all } t > \underline{t}.$$

Proof. We first observe that

$$\begin{aligned}
&\widetilde{q}(x^{t+1}, x^t, \zeta^t, \bar{l}_t) \\
&= P_1(x^t) + \langle \nabla P_1(x^t), x^{t+1} - x^t\rangle + P_2^*(\zeta^t) - \langle \zeta^t, x^{t+1}\rangle + \frac{\bar{l}_t}{2}\|x^{t+1} - x^t\|^2 \\
&= P_1(x^t) + \langle \nabla P_1(x^t) - \zeta^t, x^{t+1} - x^t\rangle + P_2^*(\zeta^t) - \langle \zeta^t, x^t\rangle + \frac{\bar{l}_t}{2}\|x^{t+1} - x^t\|^2 \qquad (5.24)\\
&\overset{\text{(a)}}{=} P_1(x^t) - P_2(x^t) + \langle \nabla P_1(x^t) - \zeta^t, x^{t+1} - x^t\rangle + \frac{\bar{l}_t}{2}\|x^{t+1} - x^t\|^2 \\
&= q(x^t) + \langle \nabla P_1(x^t) - \zeta^t, x^{t+1} - x^t\rangle + \frac{\bar{l}_t}{2}\|x^{t+1} - x^t\|^2 \le 0,
\end{aligned}$$

where (a) follows from (2.1) because $\zeta^t \in \partial P_2(x^t)$, and the last inequality holds because x^{t+1} is feasible for (5.15) with $l_t = \bar{l}_t$. This proves item (i).

Now, note that $N_{\mathbb{R}_-}(\widetilde{q}(x^{t+1}, x^t, \zeta^t, \bar{l}_t)) = \{0\}$ if $\widetilde{q}(x^{t+1}, x^t, \zeta^t, \bar{l}_t) < 0$. Using this together with [100, Proposition 10.3], we conclude that at any $(x^{t+1}, x^t, \zeta^t, \bar{l}_t)$ (regardless of whether $\widetilde{q}(x^{t+1}, x^t, \zeta^t, \bar{l}_t) < 0$ or $\widetilde{q}(x^{t+1}, x^t, \zeta^t, \bar{l}_t) = 0$), the relation

$$\widehat{N}_{[\widetilde{q}\le 0]}(x^{t+1}, x^t, \zeta^t, \bar{l}_t) \supseteq \lambda\widehat{\partial}\widetilde{q}(x^{t+1}, x^t, \zeta^t, \bar{l}_t)$$

holds for any $\lambda \in N_{\mathbb{R}_-}(\widetilde{q}(x^{t+1}, x^t, \zeta^t, \bar{l}_t))$. Thus, for any $\lambda \in N_{\mathbb{R}_-}(\widetilde{q}(x^{t+1}, x^t, \zeta^t, \bar{l}_t))$, we have that

$$\begin{aligned}
&\widehat{N}_{[\widetilde{q}\le 0]}(x^{t+1}, x^t, \zeta^t, \bar{l}_t) \supseteq \lambda\widehat{\partial}\widetilde{q}(x^{t+1}, x^t, \zeta^t, \bar{l}_t) \\
&\overset{\text{(a)}}{=} \begin{bmatrix} \lambda[\nabla P_1(x^t) - \zeta^t + \bar{l}_t(x^{t+1} - x^t)] \\ \lambda[\nabla^2 P_1(x^t)(x^{t+1} - x^t) - \bar{l}_t(x^{t+1} - x^t)] \\ \lambda\partial P_2^*(\zeta^t) - \lambda x^{t+1} \\ \frac{\lambda}{2}\|x^{t+1} - x^t\|^2 \end{bmatrix} \overset{\text{(b)}}{\ni} \begin{bmatrix} \lambda V_1^t \\ \lambda V_2^t \\ \lambda(x^t - x^{t+1}) \\ \frac{\lambda}{2}\|x^{t+1} - x^t\|^2 \end{bmatrix},
\end{aligned} \qquad (5.25)$$

with

$$\begin{aligned}
V_1^t &:= \nabla P_1(x^t) - \zeta^t + \bar{l}_t(x^{t+1} - x^t), \\
V_2^t &:= \nabla^2 P_1(x^t)(x^{t+1} - x^t) - \bar{l}_t(x^{t+1} - x^t),
\end{aligned} \qquad (5.26)$$

where (a) uses the definition of $\widetilde{q}$, [100, Exercise 8.8(c)], [100, Proposition 10.5] and [100, Proposition 8.12] (so that $\partial P_2^*(\zeta^t) = \widehat{\partial} P_2^*(\zeta^t)$), and (b) uses (2.1) and the fact that $\zeta^t \in \partial P_2(x^t)$. On the other hand, we have from [100, Theorem 8.6] that

$$\begin{aligned}
&\partial \widetilde{F}(x^{t+1}, x^t, \zeta^t, \bar{l}_t) \supseteq \widehat{\partial} \widetilde{F}(x^{t+1}, x^t, \zeta^t, \bar{l}_t) \\
&\overset{\text{(a)}}{\supseteq} \begin{bmatrix} \frac{1}{\|x^{t+1}\|} \partial \left\|x^{t+1}\right\|_1 - \frac{\|x^{t+1}\|_1}{\|x^{t+1}\|^3} x^{t+1} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \widehat{N}_{[\widetilde{q} \le 0]}(x^{t+1}, x^t, \zeta^t, \bar{l}_t),
\end{aligned} \tag{5.27}$$

where (a) uses [100, Corollary 10.9], (2.17) and [100, Corollary 8.11], and the facts that $\widehat{\partial} \delta_{[\widetilde{q} \le 0]}(x^{t+1}, x^t, \zeta^t, \bar{l}_t) = \widehat{N}_{[\widetilde{q} \le 0]}(x^{t+1}, x^t, \zeta^t, \bar{l}_t)$ and $\widehat{N}_{\|\cdot\| \ge \rho}(x^{t+1}) = \{0\}$.

Let $\lambda_t \ge 0$ be a Lagrange multiplier of (5.15) with $l_t = \bar{l}_t$, which exists thanks to Lemma 5.4. In view of the inequality and the last equality in (5.24) and using (5.18) with $(\widetilde{x}, l_t) = (x^{t+1}, \bar{l}_t)$, we deduce that $\lambda_t \in N_{\mathbb{R}_-}(\widetilde{q}(x^{t+1}, x^t, \zeta^t, \bar{l}_t))$, which in turn implies that $\frac{\lambda_t}{\|x^{t+1}\|} \in N_{\mathbb{R}_-}(\widetilde{q}(x^{t+1}, x^t, \zeta^t, \bar{l}_t))$. We can hence let $\lambda = \frac{\lambda_t}{\|x^{t+1}\|}$ in (5.25) to obtain an element in $\widehat{N}_{[\widetilde{q} \le 0]}(x^{t+1}, x^t, \zeta^t, \bar{l}_t)$. Plugging this particular element into (5.27) yields

$$\partial \widetilde{F}(x^{t+1}, x^t, \zeta^t, \bar{l}_t) \supseteq \begin{bmatrix} \frac{1}{\|x^{t+1}\|} \partial \left\|x^{t+1}\right\|_1 - \frac{\|x^{t+1}\|_1}{\|x^{t+1}\|^3} x^{t+1} + \frac{\lambda_t}{\|x^{t+1}\|} V_1^t \\ \frac{\lambda_t}{\|x^{t+1}\|} V_2^t \\ \frac{\lambda_t}{\|x^{t+1}\|} (x^t - x^{t+1}) \\ \frac{\lambda_t}{2\|x^{t+1}\|} \left\|x^{t+1} - x^t\right\|^2 \end{bmatrix}, \tag{5.28}$$

where V_1^t and V_2^t are given in (5.26). On the other hand, applying (5.19) with $(\widetilde{x}, l_t) = (x^{t+1}, \bar{l}_t)$ and recalling that $\omega_t = \|x^t\|_1 / \|x^t\|$, we obtain

$$\begin{aligned}
\partial \|x^{t+1}\|_1 &\ni \frac{\|x^t\|_1}{\|x^t\|^2} x^t - \lambda_t (\nabla P_1(x^t) - \zeta^t) - (\alpha + \lambda_t \bar{l}_t)(x^{t+1} - x^t) \\
&= \frac{\|x^t\|_1}{\|x^t\|^2} x^t - \lambda_t V_1^t - \alpha (x^{t+1} - x^t).
\end{aligned} \tag{5.29}$$

Combining (5.28) and (5.29), we see further that

$$\partial\widetilde{F}(x^{t+1},x^t,\zeta^t,\bar{l}_t) \ni \begin{bmatrix} J_1^t \\ \frac{\lambda_t}{\|x^{t+1}\|}V_2^t \\ \frac{\lambda_t}{\|x^{t+1}\|}(x^t - x^{t+1}) \\ \frac{\lambda_t}{2\|x^{t+1}\|}\|x^{t+1}-x^t\|^2 \end{bmatrix}, \tag{5.30}$$

where

$$J_1^t := \frac{1}{\|x^{t+1}\|}\left(\frac{\|x^t\|_1}{\|x^t\|^2}x^t - \frac{\|x^{t+1}\|_1}{\|x^{t+1}\|^2}x^{t+1}\right) - \frac{\alpha}{\|x^{t+1}\|}(x^{t+1}-x^t).$$

Next, recall from Lemma 5.3(iii) that

$$\|x^{t+1}\| \geq \delta, \text{ for all } t \in \mathcal{N}_+. \tag{5.31}$$

Using this together with our assumption that $\{x^t\}$ is bounded, we see that there exists $L_1 > 0$ such that

$$\left\|\frac{\|x^t\|_1}{\|x^t\|^2}x^t - \frac{\|x^{t+1}\|_1}{\|x^{t+1}\|^2}x^{t+1}\right\| \leq L_1\|x^{t+1}-x^t\| \text{ for all } t.$$

Combining the above three displays, we deduce that

$$\left\|J_1^t\right\| \leq \frac{L_1+\alpha}{\delta}\|x^{t+1}-x^t\|. \tag{5.32}$$

On the other hand, one can see from (5.31), the definition of V_2^t (see (5.26)), the boundedness of $\{\lambda_t, \bar{l}_t\}$ (see Theorem 5.5(ii) and Lemma 5.3(ii)), the continuity of $\nabla^2 P_1$ and the boundedness of $\{x^t\}$ that there exist $L_2 > 0$ and $L_3 > 0$ such that

$$\frac{\lambda_t}{\|x^{t+1}\|} \leq \frac{L_2}{\delta} \text{ and } \left\|\frac{\lambda_t}{\|x^{t+1}\|}V_2^t\right\| \leq L_3\|x^{t+1}-x^t\|. \tag{5.33}$$

Moreover, we can see from Theorem 5.5(i) that there exists $\underline{t} \in \mathcal{N}_+$ such that

$$\|x^{t+1}-x^t\|^2 \leq \|x^{t+1}-x^t\|$$

whenever $t \geq \underline{t}$. Now we can conclude from (5.30), (5.32), (5.33) and the above display that there exists $\kappa > 0$ such that

$$\operatorname{dist}(0, \partial \widetilde{F}(x^{t+1}, x^t, \zeta^t, \bar{l}_t)) \leq \kappa \|x^{t+1} - x^t\|$$

for all $t \geq \underline{t}$. This completes the proof. $\square$

When the sequence $\{x^t\}$ generated by $\mathrm{MBA}_{\ell_1/\ell_2}$ is bounded, one can show that the set of accumulation points Ω of $\{(x^{t+1}, x^t, \zeta^t, \bar{l}_t)\}$ is compact. This together with Lemma 5.3(iii) and the continuity of $\widetilde{F}$ on its domain shows that $\widetilde{F}$ is constant on $\Omega \subseteq \operatorname{dom} \partial \widetilde{F}$. Using this together with Proposition 5.5 and Lemma 5.3(iii), one can prove the following convergence result by imposing additional KL assumptions on $\widetilde{F}$. The proof is standard and follows the line of arguments as in [6–8, 24, 79, 119]. We omit the proof here for brevity.

Theorem 5.6 (Global convergence and convergence rate of $\mathrm{MBA}_{\ell_1/\ell_2}$)**.** *Consider* (1.6) *and assume that P_1 is twice continuously differentiable. Suppose that Assumption 5.1 holds. Let $\{x^t\}$ be the sequence generated by MBA_{ℓ_1/ℓ_2} and assume that $\{x^t\}$ is bounded. If $\widetilde{F}$ in* (5.22) *is a KL function, then $\{x^t\}$ converges to a Clarke critical point x^* of* (1.6) *(x^* is stationary if q is in addition regular at x^*). Moreover, if $\widetilde{F}$ is a KL function with exponent $\theta \in [0, 1)$, then the following statements hold:*

(i) *If $\theta = 0$, then $\{x^t\}$ converges finitely.*

(ii) *If $\theta \in (0, \frac{1}{2}]$, then there exist $c_0 > 0$, $Q_1 \in (0,1)$ and $\underline{t} \in \mathcal{N}_+$ such that*

$$\|x^t - x^*\| \leq c_0 Q_1^t \ \ \textit{for} \ \ t > \underline{t}.$$

(iii) *If $\theta \in (\frac{1}{2}, 1)$, then there exist $c_0 > 0$ and $\underline{t} \in \mathcal{N}_+$ such that*

$$\|x^t - x^*\| \leq c_0 t^{-\frac{1-\theta}{2\theta-1}} \ \ \textit{for} \ \ t > \underline{t}.$$

Remark 5.3 (KL property of $\widetilde{F}$ corresponding to (1.7), (1.8) and (1.9)). (i) *In both* (1.7) *and* (1.8), *we have* $q = P_1$ *being analytic and* $P_2^* = \delta_{\{0\}}$. *Hence* $\widetilde{F}$ *becomes* $\widetilde{F}(x,y,\zeta,w) = \frac{\|x\|_1}{\|x\|} + \delta_\Delta(x,y,\zeta,w)$ *with* $\Delta = \{(x,y,\zeta,w) :\ P_1(y) + \langle \nabla P_1(y), x - y\rangle + \frac{w}{2}\|x-y\|^2 \le 0, \zeta = 0, \|x\| \ge \rho\}$. *Hence, the graph of* $\widetilde{F}$ *is*

$$\left\{ (x,y,\zeta,w,z) : \begin{array}{l} \|x\|_1 = z\|x\|, \quad \|x\| \ge \rho, \quad \zeta = 0, \\ P_1(y) + \langle \nabla P_1(y), x-y\rangle + \frac{w}{2}\|x-y\|^2 \le 0. \end{array} \right\},$$

which is semianalytic [52, Page 596]. This means that $\widetilde{F}$ *is subanalytic [52, Definition 6.6.1]. Moreover, the domain of* $\widetilde{F}$ *is closed and* $\widetilde{F}|_{\operatorname{dom}\widetilde{F}}$ *is continuous. Therefore,* $\widetilde{F}$ *satisfies the KL property according to [20, Theorem 3.1].*

(ii) *For* (1.9), *first note that* P_2 *is a convex piecewise linear-quadratic function (see, for example, the proof of [79, Theorem 5.1]). Then* P_2^* *is also piecewise linear-quadratic function thanks to [100, Theorem 11.14]. Thus, one can see that* $\widetilde{q}$ *corresponding to* (1.9) *is semialgebraic and so is the set* $\Theta = \{(x,y,\zeta,w) : \widetilde{q}(x,y,\zeta,w) \le 0\}$. *Therefore* $\widetilde{F}$ *is semialgebraic as the sum of the semialgebraic functions* $x \mapsto \frac{\|x\|_1}{\|x\|} + \delta_{\|\cdot\|\ge\rho}(x)$ *and* δ_Θ, *and is hence a KL function [7].*

Using Theorem 5.6, Remark 5.3, Proposition 5.3 and the discussions preceding it, and recalling that continuously differentiable functions are regular, we have the following immediately corollary.

Corollary 5.1 (Global convergence of $\text{MBA}_{\ell_1/\ell_2}$ for problems (1.7), (1.8) and (1.9)). *The following conclusions hold:*

1. *If we apply* MBA_{ℓ_1/ℓ_2} *to* (1.7) *or* (1.8), *then the sequence generated converges to a stationary point of the problem if the sequence is bounded.*

2. *If we apply* MBA_{ℓ_1/ℓ_2} *to* (1.9), *then the sequence generated converges to a Clarke critical point of the problem if the sequence is bounded.*

5.5 Numerical simulations

In this section, we perform numerical experiments on solving random instances of (1.7), (1.8) and (1.9) by $\mathrm{MBA}_{\ell_1/\ell_2}$. All numerical experiments are performed in MATLAB 2019b on a 64-bit PC with an Intel(R) Core(TM) i7-6700 CPU (3.40GHz) and 32GB of RAM.

We set $l_{\min} = 10^{-8}$, $l_{\max} = 10^8$ and $\alpha = 1$ in $\mathrm{MBA}_{\ell_1/\ell_2}$. We let $l_0^0 = 1$ and compute, for each $t \geq 1$,

$$l_t^0 = \begin{cases} \max\left\{l_{\min}, \min\left\{\frac{\langle d_x^t, d_g^t\rangle}{\|d_x^t\|^2}, l_{\max}\right\}\right\} & \text{if } \langle d_x^t, d_g^t\rangle \geq 10^{-12}, \\ \max\left\{l_{\min}, \min\left\{\frac{l_{t-1}}{2}, l_{\max}\right\}\right\} & \text{otherwise,} \end{cases}$$

where $d_x^t = x^t - x^{t-1}$ and $d_g^t = \xi^t - \xi^{t-1}$ with $\xi^t = \nabla P_1(x^t) - \zeta^t$: specifically, $\zeta^t = 0$ when solving (1.7) and (1.8), while for (1.9), we pick any $\zeta^t \in \mathrm{Proj}_S(Ax^t - b)$, which can be obtained by finding the largest r entries of $Ax^t - b$.

We initialize $\mathrm{MBA}_{\ell_1/\ell_2}$ at some feasible point x_{feas} and terminate $\mathrm{MBA}_{\ell_1/\ell_2}$ when

$$\|x^t - x^{t-1}\| \leq tol \cdot \max\{\|x^t\|, 1\}; \tag{5.34}$$

we will specify the choices of x_{feas} and tol in each of the subsections below.

5.5.1 Robust compressed sensing problems (1.9)

We generate a sensing matrix $A \in \mathbb{R}^{(p+\iota)\times n}$ with i.i.d standard Gaussian entries and then normalize each column of A. Next, we generate the original signal $x_{\mathrm{orig}} \in \mathbb{R}^n$ as a k-sparse vector with k i.i.d standard Gaussian entries at random (uniformly chosen) positions. We then generate a vector $z_\iota \in \mathbb{R}^\iota$ with i.i.d. standard Gaussian entries, and set $z \in \mathbb{R}^{p+\iota}$ to be a vector with the first p entries being zero and the last ι entries being $2\,\mathrm{sign}(z_\iota)$. The vector b in (1.9) is then generated as $b = Ax_{\mathrm{orig}} - z + 0.01\varepsilon$, where $\varepsilon \in \mathbb{R}^{p+\iota}$ has i.i.d. standard Gaussian entries. Finally, we set $\sigma = 1.2\|0.01\varepsilon\|$

and $r = 2\iota$. In MBA$_{\ell_1/\ell_2}$, we set $x_{\text{feas}} = A^\dagger b$,[1] and $tol = 10^{-6}$ in (5.34).

In our numerical tests, we consider $(n, p, k, \iota) = (2560i, 720i, 80i, 10i)$ with $i \in \{2, 4, 6, 8, 10\}$. For each i, we generate 20 random instances as described above. The computational results are shown in Table 5.1. We present the time t_{qr} for the (reduced) QR decomposition when generating x_{feas}, the CPU times t_{mba} and t_{sum},[2] the recovery error RecErr $= \frac{\|x_{\text{out}} - x_{\text{orig}}\|}{\max\{1, \|x_{\text{orig}}\|\}}$, and the Residual $= \text{dist}^2(Ax_{\text{out}} - b, S) - \sigma^2$, averaged over the 20 random instances, where x_{out} is the approximate solution returned by MBA$_{\ell_1/\ell_2}$. We see that x_{orig} are approximately recovered in a reasonable period of time.

Table 5.1: Random tests on robust compressed sensing

i	t_{qr}	$t_{\text{mba}}(t_{\text{sum}})$	RecErr	Residual
2	0.5	1.2 (1.7)	3.3e-02	-3e-11
4	3.1	4.1 (7.2)	3.3e-02	-5e-11
6	9.8	8.3 (18.1)	3.3e-02	-9e-11
8	24.0	14.3 (38.4)	3.3e-02	-1e-10
10	43.6	21.5 (65.3)	3.3e-02	-2e-10

5.5.2 CS problems with Cauchy noise (1.8)

Similar to the previous subsection, we generate the sensing matrix $A \in \mathbb{R}^{m \times n}$ with i.i.d standard Gaussian entries and then normalize each column of A. We then generate the original signal $x_{\text{orig}} \in \mathbb{R}^n$ as a k-sparse vector with k i.i.d standard Gaussian entries at random (uniformly chosen) positions. However, we generate b as $b = Ax_{\text{orig}} + 0.01\varepsilon$ with $\varepsilon_i \sim \text{Cauchy}(0, 1)$, i.e., $\varepsilon_i = \tan(\pi(\widetilde{\varepsilon}_i - 1/2))$ for some random vector $\widetilde{\varepsilon} \in \mathbb{R}^m$ with i.i.d. entries uniformly chosen in $[0, 1]$. Finally, we set $\gamma = 0.02$ and $\sigma = 1.2\|0.01\varepsilon\|_{LL_2,\gamma}$.

[1] We compute $A^\dagger b$ via the MATLAB commands [Q,R] = qr(A',0); xfeas = Q*(R'\b).

[2] t_{mba} is the run time of MBA$_{\ell_1/\ell_2}$, while t_{sum} includes the run time of MBA$_{\ell_1/\ell_2}$, the time for performing (reduced) QR factorization on A^T and the time for computing $Q(R^{-1})^T b$.

We compare the ℓ_1 minimization model (which minimizes ℓ_1 norm in place of ℓ_1/ℓ_2 in (1.8); see [125, Eq. (5.8)] with $\mu = 0$) with our ℓ_1/ℓ_2 model. We use $\mathrm{SCP_{ls}}$ in [125] for solving the ℓ_1 minimization model. We use the same parameter settings for $\mathrm{SCP_{ls}}$ as in [125, Section 5], except that we terminate $\mathrm{SCP_{ls}}$ when (5.34) is satisfied with $tol = 10^{-6}$ in Table 5.2. We initialize $\mathrm{MBA}_{\ell_1/\ell_2}$ at the approximate solution $x_{\rm scp}$ given by $\mathrm{SCP_{ls}}$, and terminate $\mathrm{MBA}_{\ell_1/\ell_2}$ when (5.34) is satisfied with $tol = 10^{-6}$.

In our numerical experiments, we consider $(n, m, k) = (2560i, 720i, 80i)$ with $i \in \{2, 4, 6, 8, 10\}$. For each i, we generate 20 random instances as described above. Our computational results are presented in Table 5.2, which are averaged over the 20 random instances. Here we show the CPU time $t_{\rm qr}$ for performing (reduced) QR decomposition on A^T, the CPU time,[3] the recovery error $\mathrm{RecErr} = \frac{\|x_{\rm out} - x_{\rm orig}\|}{\max\{1, \|x_{\rm orig}\|\}}$ and the residual $\mathrm{Residual} = \|Ax_{\rm out} - b\|_{LL_2,\gamma} - \sigma$ of both $\mathrm{SCP_{ls}}$ and $\mathrm{MBA}_{\ell_1/\ell_2}$, where $x_{\rm out}$ is the approximate solution returned by the respective algorithm. We see that the recovery error is significantly improved by solving the nonconvex model.

Finally, as suggested by one reviewer, we investigate the effect of initialization on the performance of $\mathrm{MBA}_{\ell_1/\ell_2}$. Specifically, we test $\mathrm{SCP_{ls}}$ and $\mathrm{MBA}_{\ell_1/\ell_2}$ on the same set of instances used in Table 5.2, but terminate $\mathrm{SCP_{ls}}$ when (5.34) is satisfied with $tol = 10^{-3}$. We then initialize $\mathrm{MBA}_{\ell_1/\ell_2}$ at the approximate solution returned by $\mathrm{SCP_{ls}}$, and terminate $\mathrm{MBA}_{\ell_1/\ell_2}$ when (5.34) is satisfied with $tol = 10^{-6}$. The computational results are presented in Table 5.3. Not too surprisingly, we can see that $\mathrm{MBA}_{\ell_1/\ell_2}$ can result in large recovery errors with this initialization, though the recovery errors may still be small sometimes (see $i = 6$). Thus, the performance of $\mathrm{MBA}_{\ell_1/\ell_2}$ is quite sensitive to its initialization.

[3] For $\mathrm{MBA}_{\ell_1/\ell_2}$, the time in parenthesis is the total run time including the time for computing the initial point $A^\dagger b$ for $\mathrm{SCP_{ls}}$ and the run times of $\mathrm{SCP_{ls}}$ and $\mathrm{MBA}_{\ell_1/\ell_2}$, the time without parenthesis is the actual run time of $\mathrm{MBA}_{\ell_1/\ell_2}$ starting from $x_{\rm feas} = x_{\rm scp}$.

Table 5.2: Random tests on CS problems with Cauchy noise ($tol = 10^{-6}$ for $\mathrm{SCP_{ls}}$)

i	t_{qr}	CPU		RecErr		Residual	
		$\mathrm{SCP_{ls}}$	$\mathrm{MBA}_{\ell_1/\ell_2}$	$\mathrm{SCP_{ls}}$	$\mathrm{MBA}_{\ell_1/\ell_2}$	$\mathrm{SCP_{ls}}$	$\mathrm{MBA}_{\ell_1/\ell_2}$
2	0.5	10.0	0.6 (11.1)	1.3e-01	6.5e-02	-2e-07	-8e-08
4	3.0	52.4	2.0 (57.5)	1.3e-01	6.6e-02	-6e-07	-2e-07
6	9.4	87.3	4.1 (100.9)	1.3e-01	6.6e-02	-9e-07	-2e-07
8	23.4	281.6	7.0 (312.1)	1.3e-01	6.5e-02	-1e-06	-3e-07
10	42.4	285.5	11.4 (339.5)	1.3e-01	6.5e-02	-2e-06	-4e-07

Table 5.3: Random tests on CS problems with Cauchy noise ($tol = 10^{-3}$ for $\mathrm{SCP_{ls}}$)

i	t_{qr}	CPU		RecErr		Residual	
		$\mathrm{SCP_{ls}}$	$\mathrm{MBA}_{\ell_1/\ell_2}$	$\mathrm{SCP_{ls}}$	$\mathrm{MBA}_{\ell_1/\ell_2}$	$\mathrm{SCP_{ls}}$	$\mathrm{MBA}_{\ell_1/\ell_2}$
2	0.5	3.0	50.8 (54.3)	1.8e+00	1.6e+00	-3e+01	-6e-05
4	3.0	11.8	457.6 (472.5)	4.3e+00	4.2e+00	-1e+02	-5e-04
6	9.5	30.5	4.9 (44.9)	2.1e-01	6.6e-02	-9e-01	-2e-07
8	22.9	37.7	78.5 (139.2)	9.7e+00	9.6e+00	-6e+01	-9e-03
10	41.5	71.9	3164.0 (3277.6)	2.1e+00	1.7e+00	-1e+02	-2e-04

5.5.3 Badly scaled CS problems with Gaussian noise (1.7)

In this section, we generate test instances similar to those in [116]. Specifically, we first generate $A = [a_1, \cdots, a_n] \in \mathbb{R}^{m\times n}$ with

$$a_j = \frac{1}{\sqrt{m}} \cos\left(\frac{2\pi w j}{F}\right), \quad j = 1, \cdots, n,$$

where $w \in \mathbb{R}^m$ have i.i.d. entries uniformly chosen in $[0, 1]$. Next, we generate the original signal $x_{\mathrm{orig}} \in \mathbb{R}^n$ using the following MATLAB command:

```
I = randperm(n); J = I(1:k); xorig = zeros(n,1);
xorig(J) = sign(randn(k,1)).*10.^(D*rand(k,1));
```

We then set $b = Ax_{\mathrm{orig}} + 0.01\varepsilon$, where $\varepsilon \in \mathbb{R}^m$ has i.i.d standard Gaussian entries. Finally, we set $\sigma = 1.2\|0.01\varepsilon\|$.

We compare the ℓ_1 minimization model (which minimizes ℓ_1 norm in place of ℓ_1/ℓ_2 in (1.7); see [125, Eq. (5.5)] with $\mu = 0$) with our ℓ_1/ℓ_2 model. The ℓ_1 minimization model is solved via SPGL1 [17] (version 2.1) using default settings. The initial point for $\mathrm{MBA}_{\ell_1/\ell_2}$ is generated from the approximate solution $x_{\rm spgl1}$ of SPGL1 as follows: Specifically, since $x_{\rm spgl1}$ may violate the constraint slightly, we set the initial point of $\mathrm{MBA}_{\ell_1/\ell_2}$ as

$$x_{\rm feas} = \begin{cases} A^\dagger b + \sigma \frac{x_{\rm spgl1} - A^\dagger b}{\|Ax_{\rm spgl1} - b\|} & \text{if } \|Ax_{\rm spgl1} - b\| > \sigma, \\ x_{\rm spgl1} & \text{otherwise.} \end{cases}$$

We terminate $\mathrm{MBA}_{\ell_1/\ell_2}$ when (5.34) is satisfied with $tol = 10^{-8}$.

In our numerical tests, we set $n = 1024$, $m = 64$ and consider $k \in \{8, 12\}$, $F \in \{5, 15\}$ and $D \in \{2, 3\}$. For each (k, F, D), we generate 20 random instances as described above. We present the computational results (averaged over the 20 random instances) in Table 5.4. Here we show the CPU time,[4] the recovery error $\mathrm{RecErr} = \frac{\|x_{\rm out} - x_{\rm orig}\|}{\max\{1, \|x_{\rm orig}\|\}}$, the Residual $= \|Ax_{\rm out} - b\|^2 - \sigma^2$ of both SPGL1 and $\mathrm{MBA}_{\ell_1/\ell_2}$, where $x_{\rm out}$ is the approximate solution returned by the respective algorithm. We again observe that the recovery error is significantly improved (on average) by solving the nonconvex model in most instances, except when $(k, F, D) = (12, 15, 3)$. In this case, we see that the $x_{\rm spgl1}$ can be highly infeasible and thus the starting point $x_{\rm feas}$ provided to $\mathrm{MBA}_{\ell_1/\ell_2}$ may not be a good starting point. This might explain the relatively poor performance of $\mathrm{MBA}_{\ell_1/\ell_2}$ in this case.

[4] For $\mathrm{MBA}_{\ell_1/\ell_2}$, the time in parenthesis is the total run time including the time for computing the feasible point $A^\dagger b$ and the run times of SPGL1 and $\mathrm{MBA}_{\ell_1/\ell_2}$, the time without parenthesis is the actual run time of $\mathrm{MBA}_{\ell_1/\ell_2}$ starting from $x_{\rm feas}$.

Table 5.4: Random tests on badly scaled CS problems with Gaussian noise

k	F	D	CPU		RecErr		Residual	
			SPGL1	MBA$_{\ell_1/\ell_2}$	SPGL1	MBA$_{\ell_1/\ell_2}$	SPGL1	MBA$_{\ell_1/\ell_2}$
8	5	2	0.07	0.13 (0.20)	3.2e-02	2.3e-03	-4e-05	-1e-13
8	5	3	0.06	0.14 (0.20)	3.2e-03	6.8e-04	-4e-05	-2e-11
8	15	2	0.08	3.92 (4.01)	4.7e-01	1.5e-01	-9e-05	-7e-13
8	15	3	0.11	31.46 (31.58)	3.8e-01	5.3e-02	2e-02	-5e-11
12	5	2	0.06	2.26 (2.32)	1.4e-01	3.6e-02	-3e-04	-8e-13
12	5	3	0.08	4.05 (4.14)	6.0e-02	3.8e-03	1e-04	-7e-11
12	15	2	0.09	8.32 (8.41)	5.2e-01	2.0e-01	-1e-04	-1e-12
12	15	3	0.11	403.80 (403.91)	5.2e-01	1.5e+00	6e-02	-3e-10

Chapter 6

Conclusion

In this thesis, we develop a collection of KL calculus rules and provide some examples showing how these rules can be applied to obtain explicit KL exponents. In the second part, we show how KL property and KL exponent are applied in deducing the convergence rate of the sequence generated by SCP_{ls}. In the last part, we consider an ℓ_1/ℓ_2-based constrained optimization problem.

In the future, we will explore more KL calculus rules as well as the explicit KL exponents of functions. For example, the explicit KL exponent for

$$F(x) := \left\|Y^T x\right\|_1 + \delta_{\|\cdot\|=1}(x) \tag{6.1}$$

is still unknown, where $Y \in \mathbb{R}^{n\times p}$ is a given matrix with full row rank. Minimizing the above function has applications in dual principal component pursuit and orthogonal dictionary learning; see [11, 38] for more introduction. Next, we plan to study the convergence properties of first-order algorithms whose convergence properties have not been fully unraveled. For example, the explicit local convergence rates of the whole sequences generated by the manifold proximal point algorithm and the alternating direction method of multipliers studied in [38, 121] respectively are still unknown. We will also investigate efficient methods to solve optimization models in different applications such as sparse clustering [117], robust subspace recovery (RSR) [68] and robust low-rank matrix completion [61].

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