LINEAR MAPS PRESERVING CERTAIN UNITARILY INVARIANT NORMS OF TENSOR PRODUCTS OF MATRICES

ZHENG RUN

MPhil

The Hong Kong Polytechnic University

2022

The Hong Kong Polytechnic University

Department of Applied Mathematics

Linear maps preserving certain unitarily invariant norms of tensor products of matrices

ZHENG Run

A thesis submitted in partial fulfilment of the requirements

for the degree of Master of Philosophy

August 2021

CERTIFICATE OF ORIGINALITY

I hereby declare that this thesis is my own work and that, to the best of my knowledge and belief, it reproduces no material previously published or written, nor material that has been accepted for the award of any other degree or diploma, except where due acknowledgement has been made in the text.

____________________________(Signed)

ZHENG Run (Name of student)

Abstract

Linear preserver problem is an active and popular research topic in matrix theory and functional analysis. The main goal of linear preserver problems is to characterise the structure of linear maps on matrix spaces or operator spaces that preserve certain functions, subsets or relations. Let M_n denote the $n \times n$ complex matrix space. The first linear preserver problem proposed by Frobenius in 1896 was to characterise linear maps $\phi : M_n \to M_n$ such that

$$\det(\phi(A)) = \det(A) \quad \text{for all } A \in M_n.$$

In recent years, partly due to the development of quantum science, much attention has been paid to the study of linear maps leaving invariant tensor products or certain propositions of tensor products.

Fošner et al. characterised linear preservers for Schatten p-norms and Ky Fan k-norms of tensor products of square matrices. In this thesis, we generalize their results by characterising the form of linear maps preserving the γ-norms or the (p, k)-norms with $2 < p < \infty$ of tensor products of square matrices. Let $m \geq 2$ and $n_1, \ldots, n_m$ be integers larger than or equal to 2. Suppose that $\|\cdot\|$ is the γ-norm or the (p, k)-norm with $2 < p < \infty$. We show in this thesis that a linear map $\phi : M_{n_1 \cdots n_m} \to M_{n_1 \cdots n_m}$ satisfies

$$\|\phi(A_1 \otimes \cdots \otimes A_m)\| = \|A_1 \otimes \cdots \otimes A_m\| \quad \text{for all } A_i \in M_{n_i}, i = 1, \ldots, m,$$

if and only if there exist unitary matrices $U, V \in M_{n_1 \cdots n_m}$ such that

$$\phi(A_1 \otimes \cdots \otimes A_m) = U(\varphi_1(A) \otimes \cdots \otimes \varphi_m(A))V \quad \text{for all } A_i \in M_{n_i}, i = 1, \ldots, m,$$

where φ_i is the identity map or the transposition map $A \mapsto A^T$ for $i = 1, \ldots, m$.

We develop some new techniques to show that $\phi(E_{ii} \otimes E_{jj})$ and $\phi(E_{rr} \otimes E_{ss})$ are orthogonal for any distinct $(i, j) \neq (r, s)$, which is a key step in our proof. Suppose that $\gamma = (\gamma_1, \ldots, \gamma_n)$ with $\gamma_1 \geq \cdots \geq \gamma_k > 0 = \gamma_{k+1} = \cdots = \gamma_n$. Our characterization of linear preservers for γ-norms mainly relies on the observation that if $\|E + F\|_\gamma = \|E\|_\gamma + \|F\|_\gamma$, then $UEV = E_1 \oplus E_2$ and $UFV = F_1 \oplus F_2$ for some unitary matrices U and V with $E_1, F_1 \in M_k$ and $E_2, F_2 \in M_{n-k}$. Some equalities have been applied to obtain our results on (p, k)-norms.

Keywords: linear preserver problems, matrix space, unitarily invariant norms, γ-norms, (p, k)-norms, singular values, tensor products

Acknowledgements

First and foremost, I would like to express my deepest gratitude to my supervisor Dr. Nung-Sing Sze for his detailed guidance, full support, strict regulation, and warm encouragement. His dedication and enthusiasm for research and life always motivate me to overcome the challenges and difficulties during my MPhil study.

I must express my sincerest gratitude to Dr. Zejun Huang of Shenzhen University for his patient guidance and useful advices. I will never forget those days I spent in Shenzhen under his support. Without his help, I can not complete this thesis.

Last but not least, I am grateful to my family and friends for their companies and supports.

Contents

List of Notations

LPP	Linear preserver problem
A^T	the transpose of A
A^*	the conjugate transpose of A
E_{ij}	a matrix which the (i, j)-th entry is equal to one and all the other entries are zeros.
M_n	the $n \times n$ complex matrix space
$M_{m,n}$	the $m \times n$ complex matrix space
H_n	the $n \times n$ Hermitian matrix space
GL_n	the set of $n \times n$ nonsingular matrices
$\mathscr{U}_n$	the set of $n \times n$ unitary matrices
I_n	the identity matrix of size n
$A \otimes B$	the tensor product of A and B
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{R}^+$	the set of positive real numbers
$\mathbb{R}^n$	the n-dimensional vector space over $\mathbb{R}$
$\mathbb{C}^n$	the n-dimensional vector space over $\mathbb{C}$
$\\| \cdot \\|_\gamma$	the γ-norm
$\\| \cdot \\|_{(p,k)}$	the (p, k)-norm
$\\| \cdot \\|_{tr}$	the trace norm
$x \succ y$	x majorizes y
$x \succ_w y$	x weakly majorizes y
$\text{rank}(A)$	the rank of A
$\det(A)$	the determinant of A
$\text{tr}(A)$	the trace of A

$A \perp B$	A and B are orthogonal
$A_1 \otimes \cdots \otimes A_k$	the tensor product of A_1 through A_k
$\bigotimes_{i=1}^{k} A_i$	the tensor product $A_1 \otimes \cdots \otimes A_k$
$\mathrm{diag}(a_1, \ldots, a_n)$	the $n \times n$ diagonal matrix with $a_1, \ldots, a_n$ as its diagonal entries

Chapter 1

Introduction

1.1 Linear preserver problems

Linear preserver problem is an active and popular research topic in matrix theory and functional analysis. The main goal of linear preserver problems (LPPs) is to characterise the structure of linear maps on matrix spaces or operator spaces that preserve certain functions, subsets or relations. Suppose $\mathbb{F}$ is a field. Let $M_{m,n}(\mathbb{F})$ denote the $m \times n$ matrix space over $\mathbb{F}$. For simplicity, we denote by $M_{m,n}$ the $m \times n$ complex matrix space, and in particular denote by M_n the $n \times n$ complex matrix space. Let $\mathscr{U}_n$, GL_n and H_n denote the sets of $n \times n$ unitary matrices, nonsingular matrices and Hermitian matrices, respectively. Denote by $\mathbb{C}$ and $\mathbb{R}$ the complex number field and real number field, respectively. In 1897, Frobenius [9] first initiated *linear preserver problem* by studying linear maps $\phi : M_n \to M_n$ such that

$$\det(\phi(A)) = \det(A) \quad \text{for all } A \in M_n, \tag{1.1}$$

where $\det(A)$ denotes the determinant of A. It was shown that such linear maps $\phi : M_n \to M_n$ have the form

$$\phi(A) = UAV \quad \text{or} \quad \phi(A) = UA^T V \quad \text{for all } A \in M_n, \tag{1.2}$$

where A^T denotes the transpose of A and $U, V \in GL_n$ satisfy $\det(UV) = 1$. In the past few decades, much effort has been devoted to this topic and there were many great works and results on LPPs. The following are some some typical problems.

(I) Suppose that $\mathscr{P}$ is a certain property of matrices. The first type is to determine the structure of linear maps ϕ leaving the property $\mathscr{P}$ invariant, i.e.,

$$\phi(A) \text{ satisfies } \mathscr{P} \quad \text{whenever } A \text{ satisfies } \mathscr{P}.$$

One example of this type is the *rank-one* LPP, that is, to characterise linear maps ϕ such that

$$\operatorname{rank}(\phi(A)) = 1 \quad \text{whenever } \operatorname{rank}(A) = 1, \tag{1.3}$$

where $\operatorname{rank}(A)$ denotes the rank of A. Marcus and Moyls characterised *rank-one* linear preservers on M_n [30]; Johonson and Pierce [16] characterised nonsingular *rank-one* linear preservers on the $n \times n$ Hermitian matrix space H_n; Chooi and Lim [3] characterised *rank-one* preservers on upper triangular matrix space; Li and Rodman et al. [19] characterised *rank-one* preservers from $M_{m\times n}(\mathbb{F})$ to $M_{p\times q}(\mathbb{F})$ for any given field $\mathbb{F}$ and integers m, n, p, q. The study of *rank-one* LPP is an important topic and many LPPs can be reduced to the characterisation of *rank-one* preservers. In fact, the above problem proposed by Frobenius can also be reduced *rank-one* LPP. It was shown that if ϕ satisfies (1.1), then it will send *rank-one* matrices to *rank-one* matrices.

(II) Suppose that $\mathcal{S}$ is a subset or a subgroup of a given matrix space. The second type is to characterise linear maps ϕ such that

$$\phi(\mathcal{S}) \subseteq \mathcal{S}.$$

Recall that GL_n and $\mathscr{U}_n$ denote the sets of $n \times n$ nonsingular matrices and $n \times n$ unitary matrices, respectively. Marcus and Purves [28, 31] characterised linear maps ϕ on M_n mapping GL_n or $\mathscr{U}_n$ into itself. It was shown that such linear maps also have the standard form in (1.2) with $U, V \in GL_n$ and $U, V \in \mathscr{U}_n$, respectively. Let I_n denote the identity matrix of size n. Cheung and Li [2] extended these results by showing that if $\phi : M_n \to M_m$ is a linear map such that $\phi(\mathscr{U}_n) \subseteq \mathscr{U}_m$, then m is a multiple of n and

$$\phi(A) = U[(A \otimes I_s) \oplus (A^T \otimes I_r)]V$$

for some matrices $U, V \in \mathscr{U}_m$. Note that the type (I) and type (II) might overlap. For example, suppose that $\mathcal{S}$ is the set of all *rank-one* matrices, the problem to characterise linear maps satisfying $\phi(\mathcal{S}) \subseteq \mathcal{S}$ falls in both type (I) and type (II).

(III) Suppose that f is a given (scalar-valued, vector-valued or set-valued) function of matrices. Problems of the third category aim at determining the structure of linear maps ϕ on a matrix space M preserving f i.e.,

$$f(\phi(A)) = f(A) \quad \text{for all } A \in M. \tag{1.4}$$

One active topic is the study of linear maps preserving functions of singular values. For example, let E_r be the r-th elementary symmetric function. Then a function f on $M_{m,n}$ can be defined as $f(A) = E_r(s_1(A), \ldots, s_n(A))$, where $r \leq \min\{m, n\}$ and $s_1(A), \ldots, s_n(A)$ are the singular values of A in decreasing order. Given a complex number $x \in \mathbb{C}$, we denote by $|x|$ the absolute value of x. Marcus and Gordon [29] proved that if a linear map ϕ on $M_{m,n}$ leaves the above function f invariant, then one of the following statements holds.

(a) if $r < m = n$, then ϕ has the form in (1.2) with $U \in \mathscr{U}_n, V \in \mathscr{U}_m$;

(b) if $r < \min\{m, n\}$ and $m \neq n$, then ϕ has the form $A \mapsto UAV$ with $U \in \mathscr{U}_n, V \in \mathscr{U}_m$;

(c) if $r = m < n$, then ϕ has the form $A \mapsto UAV$ with $|\det(U)| = 1$ and $V \in \mathscr{U}_n$;

(d) if $r = n < m$, then ϕ has the form $A \mapsto UAV$ with $|\det(V)| = 1$ and $U \in \mathscr{U}_m$;

(e) if $r = m = n$, then ϕ has the form in (1.2) with $|\det(UV)| = 1$.

Their proof mainly relies on some results on *rank-one* linear preservers.

(IV) Suppose that $\sim$ is a relation. The fourth type is to find all linear maps ϕ such that

$$\phi(A) \sim \phi(B) \quad \text{whenever } A \sim B$$

or

$$\phi(A) \sim \phi(B) \quad \text{if and only if } A \sim B.$$

For example, there are many works targeting on characterising linear maps preserving similarity. Two matrices $A, B \in M_n$ are said to be similar if $A = SBS^{-1}$ for some matrix $S \in GL_n$. Hiai [13] characterised linear maps ϕ on M_n such that $\phi(A)$ and $\phi(B)$ are similar whenever A and B are similar. Then the result was improved and extended by Lim, Li and Tsing [14, 25, 34]. Scholars also considerd linear maps ϕ such that $\phi(A)$ and $\phi(B)$ are commutative if A and B are commutative, i.e.,

$$\phi(A)\phi(B) = \phi(B)\phi(A) \quad \text{whenever } AB = BA. \tag{1.5}$$

Suppose that $n \geq 3$ and $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. Then a nonsingular linear map ϕ on $M_n(\mathbb{F})$ satisfies (1.5) if and only if there exist nonsingular matrix $S \in M_n(\mathbb{F})$, real number $\alpha \in \mathbb{R}$ and a linear function f on $M_n(\mathbb{F})$ such that

$$\phi(A) = \alpha S^{-1}AS + f(A)I_n$$

or

$$\phi(A) = \alpha S^{-1}A^TS + f(A)I_n$$

for all $A \in M_n(\mathbb{F})$; See [24, 35].

1.2 Unitarily invariant norms

For simplicity, we may assume that $m \leq n$ in this section. Recall that $\mathscr{U}_n$ denotes the set of $n \times n$ unitary matrices. A norm $\|\cdot\|$ on $M_{m,n}$ is called a unitarily invariant norm if

$$\|A\| = \|UAV\| \quad \text{for all } A \in M_{m,n}, U \in \mathscr{U}_n \text{ and } V \in \mathscr{U}_m.$$

Denote by $s_1(A) \geq s_2(A) \geq \cdots \geq s_m(A)$ the singular values of $A \in M_{m,n}$ in decreasing order. Common examples of unitarily invariant norms include

(i) the spectral norm defined by $\|A\|_{op} = s_1(A)$;

(ii) the trace norm defined by $\|A\|_{tr} = \sum_{i=1}^{m} s_i(A)$;

(ii) the Frobenius norm defined by $\|A\|_F = \{tr(AA^*)\}^{\frac{1}{2}}$.

One important class of unitarily invariant norms is the Ky Fan k-norms. Suppose that k is an integer with $1 \leq k \leq m$. The Ky Fan k-norm of $A \in M_{m,n}$ is defined as

$$\|A\|_{(k)} = \sum_{i=1}^{k} s_i(A).$$

Evidently, the spectral norm and the trace norm are also Ky Fan k-norms with $k = 1$ and $k = m$, respectively. The following theorem called *Fan Dominance Principle* is a beautiful and useful result on Ky Fan norms.

Theorem 1.1. (Fan Dominance Principle [5]) *Let $A, B \in M_n$. If $\|A\|_{(k)} \leq \|B\|_{(k)}$ for all $1 \leq k \leq n$, then $\|A\| \leq \|B\|$ for any unitarily invariant norm $\|\cdot\|$.*

Readers can also see Theorem 4.25 in [39] for the proof of the above theorem. Grone and Marcus proposed a further generalization of Ky Fan k-norm to the (p, k)-norm. Suppose that $1 \leq k \leq m$ is an integer and $1 \leq p \leq \infty$. The (p, k)-norm of $A \in M_{m,n}$ is defined by

$$\|A\|_{(p,k)} = \left[\sum_{i=1}^{k} s_i^p(A)\right]^{\frac{1}{p}}.$$

Obviously, the (p,k)-norm reduces to the Ky Fan k-norm when $p = 1$ and reduces to the Frobenius norm when $p = 2$ and $k = m$. Besides, the (p,k)-norm on $M_{m,n}$ with $k = m$ is also called the Schatten $p-$norm denoted by $\|A\|_p$ for $A \in M_{m,n}$, that is,

$$\|A\|_p = \left[\sum_{i=1}^{m} s_i^p(A)\right]^{\frac{1}{p}},$$

which corresponds to the l_p norm on $\mathbb{R}^n$, the n-dimensional vector space over the real number field $\mathbb{R}$. Clearly, all the above unitarily invariant norms are functions of singular values of matrices. In fact, one can conclude from the singular value decomposition that any unitarily invariant norm is a function of singular values of matrices, but not vice versa. In other words, not all functions of singular values could be a norm of matrices. So naturally, one may wonder what kind of functions can be unitarily invariant norms. Von Neumann answered this problem by giving Theorem 1.2. To show this interesting result, we first introduce some related definitions and notations.

Let $\mathbb{F}$ be a field. The set $\mathbb{F}^n$ of n-tuples with entries from $\mathbb{F}$ forms an n-dimensional vector space over $\mathbb{F}$. In particular, $\mathbb{R}^n$ and $\mathbb{C}^n$ denote the vector spaces over the real number field $\mathbb{R}$ and the complex number field $\mathbb{C}$, respectively. Let $\mathbb{R}^+$ denote the set of positive real numbers. For $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, denote $|x| = (|x_1|, \ldots, |x_n|)$. A norm $\|\cdot\|$ on $\mathbb{C}^n(\mathbb{R}^n)$ is called absolute if $\||x|\| = \|x\|$ for all $x \in \mathbb{C}^n(\mathbb{R}^n)$.

Definition 1.1. *A function $f : \mathbb{R}^n \to \mathbb{R}^+$ is said to be a symmetric Gauge function if f is an absolute norm on $\mathbb{R}^n$ and*

$$f(x_{j_1}, \ldots, x_{j_n}) = f(x_1, \ldots, x_n)$$

for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and permutation $(j_1, \ldots, j_n)$ of $(1, \ldots, n)$.

Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. One can easily verify that the l_∞ norm, defined by $\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$, is a symmetric *Gauge function*. Suppose that $|x_{j_1}| \geq \cdots \geq |x_{j_n}|$

for some permutation $(j_1, \ldots, j_n)$ of $(1, \ldots, n)$, we define $|x|_\downarrow = (|x_{j_1}|, \ldots, |x_{j_n}|)$. Denote $\mathbb{R}^n_{+,\downarrow} = \{(x_1, \ldots, x_n) \mid x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}$. A function f on $\mathbb{R}^n_{+,\downarrow}$ can be extended to a function $\widetilde{f}$ on $\mathbb{R}^n$ as

$$\widetilde{f}(x) = f(|x|_\downarrow) \quad \text{for all } x \in \mathbb{R}^n. \tag{1.6}$$

Theorem 1.2. [39, Theorem 4.23] *Let f be a function on $\mathbb{R}^n_{+,\downarrow}$ and $\|\cdot\|_f$ be defined by*

$$\|A\|_f = f(s_1(A), \ldots, s_m(A)) \quad \textit{for all } A \in M_{m,n}.$$

Then $\|\cdot\|_f$ is a unitarily invariant norm on $M_{m,n}$ if and only if $\widetilde{f}$ is a symmetric Gauge function.

Readers can refer to Chapter 4 of [39] for more results on unitarily invariant norms and functions of singular values of matrices. Below we focus on the study of LPPs about unitarily invariant norms. Schur [37] showed that an analytic map ϕ on $M_{m,n}$ satisfies

$$\|\phi(A)\|_{op} = \|A\|_{op} \quad \text{for all } A \in M_{m,n},$$

if and only if ϕ has the form in (1.2) when $m = n$, or the form $A \mapsto UAV$ when $m \neq n$ with $U \in \mathscr{U}_m$ and $V \in \mathscr{U}_n$. Later, Morita [33] and Sugawara [38] reproved this result based on Morita's result on *rank-one* preservers. Suppose that $\|\cdot\|$ is a norm on $M_{m,n}$. The unit sphere in $M_{m,n}$ with respect to $\|\cdot\|$ is the set $\{A : \|A\| = 1, A \in M_{m,n}\}$. Suppose that $\mathcal{S}$ is a set, then $x \in \mathcal{S}$ is said to be an extreme point of $\mathcal{S}$ if there do not exist $x_1, x_2 \in \mathcal{S}$ and $0 < t < 1$ such that $x_1 \neq x_2$ and $x = tx_1 + (1-t)x_2$, in other words, $x = tx_1 + (1-t)x_2$ for some $0 < t < 1$ implies that $x = x_1 = x_2$. Let $\mathscr{E}$ be the set of all the extreme points of the unit sphere $\{A : \|A\| = 1, A \in M_{m,n}\}$. One can easily check that a nonsingular linear map ϕ on $M_{m,n}$ that preserves $\|\cdot\|$ maps $\mathscr{E}$ into itself. This observation was applied to characterise many norm preservers. For instance, Russo [36] showed that the set of extreme points of the unit sphere with respect to the trace norm, $\{A : \|A\|_{tr}, A \in M_n\}$, is simply the set of those matrices of rank one and trace norm one. It follows that a linear map ϕ on M_n preserves the trace norm only if it preserves rank one, that is, ϕ satisfies (1.3). With this, he characterised unital linear maps on M_n that preserve the trace norm. Li and Tsing [22] applied a special property of unit sphere with respect to (p,k)-norms to characterize linear maps on $M_{m,n}$ preserving (p,k)-norms. Let $\mathscr{B} = \{A : \|A\|_{(p,k)} = 1, A \in M_{m,n}\}$ with $(p,k) \neq (2,m)$ and $1 < p < \infty$. It was shown by them that a matrix $A \in \mathscr{B}$ is of rank greater than $k-1$ if and only if there exists $B \in \mathscr{B}$ such that $B \neq A$ and

$$\alpha A + (1-\alpha)B \in \mathscr{B} \quad \text{for all } 0 \leq \alpha \leq 1.$$

With this result, they proved that a linear map ϕ on $M_{m,n}$ preserves (p,k)-norms if and only if ϕ maps the set of all matrices of rank greater than $k-1$ into itself. Grone and Marcus [11] showed that linear maps ϕ on M_n preserving Ky Fan $k-$norms have the form in (1.2) with matrices $U, V \in \mathscr{U}_n$. And then this result was extended to the space of rectangular matrices [10]. Suppose that $\gamma = (\gamma_1, \ldots, \gamma_m) \in \mathbb{R}^m_{+,\downarrow}$. Another generalization of the Ky Fan $k-$norm is the γ-norm defined by

$$\|A\|_\gamma = \sum_{i=1}^{m} s_i(A)\gamma_i \quad \text{for all } A \in M_{m,n}.$$

In [21], Li and Tsing proved that there exist linear maps ϕ on $M_{m,n}$ such that $\|\phi(A)\|_\gamma = \|A\|_{\hat{\gamma}}$, $A \in M_{m,n}$ for some given $\gamma = (\gamma_1, \ldots, \gamma_m)$, $\hat{\gamma} = (\hat{\gamma_1}, \ldots, \hat{\gamma_m}) \in \mathbb{R}^m_{+,\downarrow}$ only if γ is a scalar multiple of $\hat{\gamma}$ and in this case there exist matrices $U \in \mathscr{U}_m$ and $V \in \mathscr{U}_n$ such that

$$\phi(A) = \frac{\gamma_1}{\hat{\gamma_1}} UAV$$

or when $m = n$

$$\phi(A) = \frac{\gamma_1}{\hat{\gamma_1}} UA^TV$$

for all $A \in M_{m,n}$. Clearly, when $m = n$ and $\gamma = \hat{\gamma}$, ϕ reduces to the form in (1.2).

As we can see, all the above linear preservers on $M_{m,n}$ have the standard form $A \mapsto UAV$ or when $m = n$ $A \mapsto UA^TV$ with $U \in \mathscr{U}_m$ and $V \in \mathscr{U}_n$. Actually, linear preservers for any unitarily invariant norm have this structure. Suppose that $\|\cdot\|$ is a unitarily invariant norm, then a linear map $\phi : M_{m,n} \to M_{m,n}$ satisfies

$$\|\phi(A)\| = \|A\| \quad \text{for all } A \in M_{m,n}$$

if and only if ϕ has the form $A \mapsto UAV$ or when $m = n$ $A \mapsto UA^TV$ with matrices $U \in \mathscr{U}_m$ and $V \in \mathscr{U}_n$; See [23]. One might think that corresponding results on $M_{m,n}(\mathbb{R})$ could also be obtained. However, this is not true for the case when $m = n = 4$; See [1, 23] for details. Readers can also refer to [1] for an excellent survey of LLPs on unitarily invariant norms.

1.3 Linear preservers on tensor products

In recent years, partly due to the development of quantum science, much attention has been paid to the study of linear maps leaving invariant tensor products or certain propositions of tensor products. Let $A = [a_{ij}] \in M_{m,\ell}$ and $B \in M_{n,t}$. The tensor product of A and B, denoted by $A \otimes B$, is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1\ell}B \\ a_{21}B & a_{22}B & \cdots & a_{2\ell}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{m\ell}B \end{bmatrix} \in M_{mn,\ell t}.$$

The tensor product is also called the *Kronecker product*. Denote by A^* the conjugate transpose of $A \in M_n$. Then A is said to be a Hermitian matrix if $A = A^*$. Recall that H_n denotes the set of all $n \times n$ Hermitian matrices. In quantum science, the state of an n-physical-state quantum system is represented by a density matrix, which is a positive semidefinite matrix of trace one in H_n. Let $A \in H_m$ and $B \in H_n$ be density matrices describing two quantum systems. Then $A \otimes B \in H_{mn}$ is a quantum state of the composite system. A density matrix $C \in H_{mn}$ is said to be separable if

$$C = \sum_{i}^{k} p_i A_i \otimes B_i, \tag{1.7}$$

for some density matrices $A_i \in H_m, B_i \in H_n$ and real number $0 < p_i \leq 1$ with $\sum_{i=1}^{k} p_i = 1$. In particular, if $k = 1$, C is called a pure separable state. Otherwise, C is said to be an inseparable state or an entangled state. Generalizations of these definitions to multipartite systems $H_{n_1} \otimes \cdots \otimes H_{n_m}$ with $m \geq 3$ are obvious. Clearly, the set of separable states is the convex hull of the set of pure separable states.

Entangled states have many applications in quantum information and quantum computation. One significant problem in quantum science is to distinguish separable states from entangled states efficiently. Unfortunately, it was proved in [12] that this problem is NP hard. Nevertheless, it is well worth finding transformations which can simplify a given state so that it is easier to determine whether it is separable or not. Obviously, such a transformation should not change the separability of a state. This leads to the study of linear operators preserving the set of separable states. It was shown in [8] that a linear transformation ϕ on $H_{m_1 \cdots m_k}$ preserving the set of pure separable states $\{A_1 \otimes \cdots \otimes A_k \mid A_i \in H_{m_i}\}$ or its convex hull if and only if there exists a permutation $(j_1, \ldots, j_k)$ of $(1, \ldots, k)$ such that

$$\phi(A_1 \otimes \cdots \otimes A_k) = \psi_1(A_{j_1}) \otimes \cdots \otimes \psi_k(A_{j_k}) \quad \text{for all } A_i \in H_{m_i}, i = 1, \ldots, k,$$

where ψ_i has the form

$$A \mapsto U_i A U_i^* \quad \text{or} \quad A \mapsto U_i A^T U_i^*$$

with matrices $U_i \in \mathscr{U}_{m_i}$ and $m_{j_i} = m_i$ for $i = 1, \ldots, k$. The evolution of a closed quantum system is described by a unitary transformation. Moreover, let ρ_1, ρ_2 be the states of a system at time t_1 and t_2, respectively. There exists a unitary matrix U which only depends on time t_1 and t_2 such that $\rho_1 = U\rho_2 U^*$. Therefore, it is well worth studying the similarity orbits $\mathcal{U}(C)$, defined by $\mathcal{U}(C) = \{UCU^* \mid U \in \mathscr{U}_n\}$, of a matrix $C \in H_n$. Suppose that $C_i, D_i \in H_{m_i}$ for $i = 1, \ldots, k$. Let

$$S_1 = \{X_1 \otimes \cdots \otimes X_k \mid X_i \in \mathcal{U}(C_i),\ i = 1, \ldots, k\},$$

$$S_2 = \{X_1 \otimes \cdots \otimes X_k \mid X_i \in \mathcal{U}(D_i),\ i = 1, \ldots, k\}.$$

Authors in [18] characterised linear transformations on $H_{m_1 \cdots m_k}$ satisfying $\phi(S_1) = S_2$. In [7], authors characterised linear maps $\phi : H_{mn} \to H_{mn}$ that preserve the spectrum or the spectral radius of tensor products $A \otimes B$ for all $A \in H_m$ and $B \in H_n$. Another interesting topic is the study of preservers for rank of tensor products of matrices. In [26], Lim gave the structure of additive maps between tensor products of two real vector spaces of Hermitian matrices that preserve the rank of tensor products of *rank-one* matrices. Zheng et al. [40] showed that a linear map $\phi : M_{m_1 \cdots m_k} \to M_{m_1 \cdots m_k}$ satisfying

$$\operatorname{rank}(\phi(A_1 \otimes \cdots \otimes A_k)) = \operatorname{rank}(A_1 \otimes \cdots \otimes A_k) \quad \text{for all } A_i \in M_{m_i}, i = 1, \ldots, k,$$

if and only if

$$\phi(A_1 \otimes \cdots \otimes A_k) = U(\psi_1(A_1) \otimes \cdots \otimes \psi_k(A_k))V \quad \text{for all } A_i \in M_{m_i}, i = 1, \ldots, k, \quad (1.8)$$

where $U, V \in M_{m_1 \cdots m_k}$ are nonsingular matrices and ψ_i is the identity map or the transposition map $A \mapsto A^T$ for $i = 1, \ldots, k$. Next Lim [27] extended this result to arbitrary field $\mathbb{F}$ by showing that a linear map $\phi : M_{m_1 \cdots m_k}(\mathbb{F}) \to M_{p,q}(\mathbb{F})$ satisfying

$$\operatorname{rank}(\phi(A_1 \otimes \cdots \otimes A_k)) = 1 \quad \text{for all rank one matrix } A_i \in M_{m_i}(\mathbb{F}), \text{and}$$

$$\operatorname{rank}(\phi(A_1 \otimes \cdots \otimes A_k)) = \prod_{i=1}^{k} m_i \quad \text{for all rank } m_i \text{ marix } A_i \in M_{m_i}(\mathbb{F})$$

also have the structure in (1.8). Hang et al. [15] extended the above result by characterising linear maps sending tensor products of *rank-one* complex matrices to *rank-one* matrices. It is worth noting that such preservers might have a more complicated form. One challenging problem is to characterise linear maps preserving determinant of tensor products of matrices. The following theorem is one recent result on this problem obtained by Ding et al. in [4].

Theorem 1.3. *Let $\phi : H_{mn} \to H_{mn}$ be a linear map such that $\phi(R \otimes S)$ is a positive or negative definite matrix for some $R \in H_m, S \in H_n$. Then ϕ satisfies*

$$\det(\phi(A \otimes B)) = \det(A \otimes B) \quad \text{for all } A \in H_m \text{ and } B \in H_n \tag{1.9}$$

if and only if there exists $U \in H_{mn}$ such that $\det(UU^*) = 1$ and

$$\phi(A \otimes B) = \epsilon U(\psi_1(A) \otimes \psi_2(B))U^* \quad \text{for all } A \in H_m \text{ and } B \in H_n, \tag{1.10}$$

where ψ_i is the identity map or the transposition map $A \mapsto A^T$ for $i = 1, 2$, $\epsilon = 1$ when $\phi(R \otimes S)$ is positive definite, and $\epsilon = -1$ when $\phi(R \otimes S)$ is negative definite.

The assumption that $\phi(R \otimes S)$ is positive or negative definite is essential. In fact, one can check that a linear map $\phi : H_4 \to H_4$ defined by

$$\phi(A \otimes B) = \begin{bmatrix} 0 & AB \\ BA & 0 \end{bmatrix} \quad \text{for all } A, B \in H_2.$$

satisfies (1.9) but does not have the form in (1.10).

There are many results on linear maps preserving unitarily invariant norms of matrices (without the tensor structure). Naturally, one may want to extend these results to tensor products of matrices. For example, authors of [6] considered linear maps preserving Ky Fan k-norms and Schatten $p-$norms of tensor products of matrices.

In this thesis, we extend their results to another two classes of unitarily invariant norms by giving the structure of linear maps preserving γ-norms or (p, k)-norms of tensor products of matrices. Denote by E_{ij} the matrix which the (i, j)-th entry is equal to one and all the other entries are equal to zero, where the size of E_{ij} should be clear in the context. Let $A, B \in M_{m,n}$ be two matrices. Then A and B are said to be orthogonal, denoted by $A \perp B$, if $AB^* = 0$ and $A^*B = 0$. In Chapter 2, we focus on linear maps preserving γ-norms of tensor products of square matrices. Suppose that $\gamma = (\gamma_1, \ldots, \gamma_{mn})$ with $\gamma_1 \geq \cdots \geq \gamma_k > 0 = \gamma_{k+1} = \cdots = \gamma_{mn}$ for some integer $2 \leq k \leq mn$. Let $A = \phi(E_{ii} \otimes E_{jj})$ and $B = \phi(E_{ii} \otimes E_{ss})$ with $j \neq s$. We observe that a linear map ϕ on M_{mn} such that $\|\phi(C \otimes D)\|_\gamma = \|C \otimes D\|_\gamma$ for all $C \in M_m$ and $D \in M_n$ should satisfy that

$$\|2A + (x+1)e^{i\theta}B\|_\gamma = \|A + e^{i\theta}B\|_\gamma + \|A + xe^{i\theta}B\|_\gamma$$

for all $0 < x \leq 1$ and $\theta \in [0, 2\pi)$. With this observation, we develop some techniques to show that there exist some matrices $U, V \in \mathscr{U}_{mn}$ such that

$$A = U(A_1 \oplus A_2)V \quad \text{and} \quad B = U(B_1 \oplus B_2)V$$

with $A_1, B_1 \in M_k$ and $A_1 \perp B_1$. Then we use some methods to show that $A_2 \perp B_2$. It follows that $A \perp B$, which is a key step of our proof of the main result in Chapter 2. In Chapter 3, we apply some equalities about the eigenvalues of positive semidefinite matrices, which are crucial to our characterisation of linear maps preserving (p, k)-norms of tensor products of matrices.

Chapter 2

Linear maps preserving γ-norms of tensor products of matrices

2.1 Introduction

Let $m, n \geq 2$ be two integers. Given any nonzero $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{mn}) \in \mathbb{R}^{mn}_{+,\downarrow}$, in this chapter, we aim at characterising linear maps $\phi : M_{mn} \to M_{mn}$ satisfying

$$\|\phi(C \otimes D)\|_\gamma = \|C \otimes D\|_\gamma \quad \text{for all } C \in M_m \text{ and } D \in M_n. \tag{2.1}$$

Obviously, if $\gamma_2 = 0$, then the γ-norm reduces to a scalar multiple of the Ky Fan 1-norm, also called the spectral norm. In [6], Fošner et al. showed that linear maps ϕ on M_{mn} preserving spectral norms of tensor products of matrices have form

$$\phi(C \otimes D) = U(\varphi_1(C) \otimes \varphi_2(D))V \quad \text{for all } C \in M_m \text{ and } D \in M_n,$$

where $U, V \in \mathscr{U}_{mn}$ and ϕ_s is the identity map or the transposition map for $s = 1, 2$. It follows that if γ_2=0, then a linear map ϕ satisfying (2.1) also has the above form. So in the following sections, we only need consider the case when $\gamma_2 > 0$. Denote by I_n and 0_n the $n \times n$ identity matrix and zero matrix, respectively. Recall that two matrices $A, B \in M_n$ are said to be orthogonal, denote by $A \perp B$, if $A^*B = AB^* = 0$. It was shown in [20] that A and B are orthogonal if and only if there exist matrices $U, V \in \mathscr{U}_n$ such that $UAV = \text{diag}(a_1, \ldots, a_n)$ and $UBV = \text{diag}(b_1, \ldots, b_n)$ with $a_i b_i = 0$ for $i = 1, \ldots, n$.

Suppose that $\|\cdot\|$ is a norm on M_{mn}. Note that $C_1 \otimes D_1 + C_2 \otimes D_2$ may not be of the form $C \otimes D$ with $C \in M_m$ and $D \in M_n$. So even if a linear map ϕ satisfies that $\|\phi(C \otimes D)\| = \|C \otimes D\|$ for all $C \in M_m, D \in M_n$, we may not have $\|\phi(C_1 \otimes D_1 + C_2 \otimes D_2)\| = \|C_1 \otimes D_1 + C_2 \otimes D_2\|$. Thus, some techniques and methods applied to characterise linear maps $\phi : M_n \to M_n$ preserving a certain norm cannot be used to characterise linear preservers for norms of tensor products of matrices. One key step in the characterisation of linear maps ϕ on M_{mn} preserving Ky Fan k-norms of tensor products is to show that $\phi(E_{ii} \otimes E_{jj})$ and $\phi(E_{rr} \otimes E_{ss})$ are orthogonal for any distinct pairs (i, j) and (r, s); See [6]. Similar methods can also be seen in the characterisation of linear maps on H_{mn} preserving the spectrum or the spectral radius of $C \otimes D$ for all $C \in H_m$ and $D \in H_n$; See [6, 7].

However, approaches to complete the key step in these previous literatures do not work for our problem. So we have to develop some new techniques to solve this problem. Let $A = \phi(E_{ii} \otimes E_{jj})$ and $B = \phi(E_{ii} \otimes E_{ss})$ with $j \neq s$. Our proof mainly relies on the observation that

$$\|2A + (x + 1)e^{i\theta}B\|_\gamma = \|A + e^{i\theta}B\|_\gamma + \|A + xe^{i\theta}B\|_\gamma$$

and

$$\|A + xe^{i\theta}B\|_\gamma = \gamma_1 + x\gamma_2$$

for all $\theta \in [0, 2\pi)$ and $0 < x \leq 1$. In Section 2.2, with the above observation, we will prove in Assertion 2.1 that A and B are orthogonal. Notice that similar equations also hold for $G = \phi(E_{ii} \otimes (E_{jj} + E_{ss}))$ and $H = \phi(E_{tt} \otimes (E_{jj} + E_{ss}))$. Then with this, we will prove in Assertion 2.2 that G and H are orthogonal, too. The results in the first two assertions directly imply Assertion 2.3 that $\phi(E_{ii} \otimes E_{jj})$ and $\phi(E_{rr} \otimes E_{ss})$ are orthogonal for any distinct $(i, j) \neq (r, s)$. At last, we will complete the proof of our main result in Assertion 2.4. In Section 2.3, we will extend the result on bipartite system to multipartite system.

2.2 Bipartite system

Theorem 2.1. *Let $m, n \geq 2$ be integers. For any given $\gamma = (\gamma_1, \ldots, \gamma_{mn}) \in \mathbb{R}^{mn}_{+,\downarrow}$ with $\gamma_2 > 0$, a linear map $\phi : M_{mn} \to M_{mn}$ satisfies*

$$\|\phi(C \otimes D)\|_\gamma = \|C \otimes D\|_\gamma \quad \text{for all } C \in M_m \text{ and } D \in M_n, \tag{2.2}$$

if and only if there exist matrices $U, V \in \mathscr{U}_{mn}$ such that

$$\phi(C \otimes D) = U(\varphi_1(C) \otimes \varphi_2(D))V \quad \text{for all } C \in M_m \text{ and } D \in M_n,$$

where φ_s is the identity map or the transposition map $X \mapsto X^T$, for $s = 1, 2$.

To prove the Theorem, we need the following lemmas.

Lemma 2.1. *Let $A, B \in M_{m,n}$. Then $A \perp B$ if and only if there exist some matrices $U \in \mathscr{U}_m$, $V \in \mathscr{U}_n$, $\hat{A} \in M_r$ and $\hat{B} \in M_{m-r,n-r}$ such that*

$$UAV = \begin{bmatrix} \hat{A} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad UBV = \begin{bmatrix} 0_r & 0 \\ 0 & \hat{B} \end{bmatrix}$$

Proof. The sufficiency part is clear and we only need to prove the necessity part. If $A = 0$, then there is nothing to prove. Suppose that A is nonzero, then by the singular value decomposition, we have

$$UAV = \begin{bmatrix} \hat{A} & 0 \\ 0 & 0 \end{bmatrix}$$

for some matrices $U \in \mathscr{U}_m$, $V \in \mathscr{U}_n$ and nonsingular matrix $\hat{A} \in M_r$ with $1 \leq r \leq \min\{m, n\}$. Let UBV be partitioned as

$$UBV = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

with $B_{11} \in M_r$ and $B_{22} \in M_{m-r,n-r}$. We conclude from $A \perp B$ that

$$(UAV)^*(UBV) = VA^*BV = 0 \quad \text{and} \quad (UAV)(UBV)^* = UAB^*U^* = 0,$$

that is,

$$\begin{bmatrix} \hat{A}^* B_{11} & \hat{A}^* B_{12} \\ 0 & 0 \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} \hat{A} B_{11}^* & \hat{A} B_{21}^* \\ 0 & 0 \end{bmatrix} = 0.$$

Since $\hat{A}$ is nonsingular, it follows that $B_{11} = 0, B_{12} = 0$ and $B_{21} = 0$. Let $\hat{B} = B_{11}$. Then we have

$$UBV = \begin{bmatrix} 0_r & 0 \\ 0 & \hat{B} \end{bmatrix}.$$

This completes the the proof. □

Lemma 2.2. *Let $A, B, C \in M_{m,n}$. If $(A + B) \perp C$ and $A \perp B$, then*

$$A \perp C \quad \textit{and} \quad B \perp C.$$

Proof. Since $A \perp B$, we apply Lemma 2.1 to conclude that there exist some matrices $U \in \mathscr{U}_m$, $V \in \mathscr{U}_n$, $\hat{A} \in M_r$ and $\hat{B} \in M_{m-r,n-r}$ such that

$$UAV = \begin{bmatrix} \hat{A} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad UBV = \begin{bmatrix} 0_r & 0 \\ 0 & \hat{B} \end{bmatrix}.$$

Then we have

$$U(A + B)V = \begin{bmatrix} \hat{A} & 0 \\ 0 & \hat{B} \end{bmatrix}.$$

Let UCV be partitioned as

$$UCV = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

with $C_{11} \in M_r$ and $C_{22} \in M_{m-r,n-r}$. We conclude from $(A + B) \perp C$ that

$$(U(A + B)V)^*(UCV) = V^*(A + B)^*CV = 0, \textit{and}$$

$$(U(A + B)V)(UCV)^* = U(A + B)C^*U^* = 0,$$

that is,

$$\begin{bmatrix} \hat{A}^* C_{11} & \hat{A}^* C_{12} \\ \hat{B}^* C_{21} & \hat{B}^* C_{22} \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} \hat{A} C_{11}^* & \hat{A} C_{21}^* \\ \hat{B} C_{12}^* & \hat{B} C_{22}^* \end{bmatrix} = 0.$$

This implies that

$$(UAV)^*(UCV) = \begin{bmatrix} \hat{A}^*C_{11} & \hat{A}^*C_{12} \\ 0 & 0 \end{bmatrix} = 0 \quad \text{and} \quad (UAV)(UCV)^* = \begin{bmatrix} \hat{A}C_{11}^* & \hat{A}C_{21}^* \\ 0 & 0 \end{bmatrix} = 0,$$

and therefore $A^*C = 0$ and $AC^* = 0$, that is, $A \perp C$. Similarly, we can also conclude that $B \perp C$. □

Lemma 2.3. *Let $E, F \in M_n$. Given $\gamma = (\gamma_1, \ldots, \gamma_n)$ with $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_k > 0 = \gamma_{k+1} = \cdots = \gamma_n$ for some integer $2 \leq k \leq n$. Suppose $\|E+F\|_\gamma = \|E\|_\gamma + \|F\|_\gamma$ and there exist matrices $U, V \in \mathscr{U}_n$ such that*

$$U(E+F)V = \operatorname{diag}\left(s_{\ell_1}(E+F), \ldots, s_{\ell_n}(E+F)\right)$$

for some permutation $(\ell_1, \ell_2, \ldots, \ell_n)$ of $(1, 2, \ldots, n)$. Let $L = \{j : \ell_j \leq k\}$ and $\bar{L} = \{j : \ell_j > k\}$ be the index sets, and let a_{ij} and b_{ij} be the (i,j)-th entries of UEV and UFV, respectively. Then

1. $\|E\|_\gamma = \sum_{j \in L} a_{jj}\gamma_{\ell_j}$ *and* $\|F\|_\gamma = \sum_{j \in L} b_{jj}\gamma_{\ell_j}$,
2. *the (i,j)-th entries of UEV and UFV are zero for all $(i,j) \in (L \times \bar{L}) \bigcup (\bar{L} \times L)$, and*
3. *the two $k \times k$ submatrices of UEV and UFV obtained the columns and rows from the index L are positive semidefinite with $s_1(E), \ldots, s_k(E)$ and $s_1(F), \ldots, s_k(F)$ as their eigenvalues, respectively.*

Proof. By replacing (U, V) with (PU, VP^T) for some permutation P, if necessary, we may assume that $(\ell_1, \ldots, \ell_n) = (1, \ldots, n)$, i.e.,

$$U(E+F)V = \operatorname{diag}\left(s_1(E+F), \ldots, s_n(E+F)\right). \tag{2.3}$$

In this case, $L = \{1, \ldots, k\}$ and $\bar{L} = \{k+1, \ldots, n\}$. Then we aim to show that

$$\|E\|_\gamma = \sum_{j=1}^{k} a_{jj}\gamma_j \quad \text{and} \quad \|F\|_\gamma = \sum_{j=1}^{k} b_{jj}\gamma_j$$

and

$$UEV = E_1 \oplus E_2 \quad \text{and} \quad UFV = F_1 \oplus F_2,$$

where $E_1, F_1 \in M_k$ are positive semidefinite with $s_1(E), \ldots, s_k(E)$ and $s_1(F), \ldots, s_k(F)$ as their eigenvalues, respectively.

Notice that

$$\sum_{j=1}^{r} |a_{jj}| \leq \sum_{j=1}^{r} s_j(UEV) = \sum_{j=1}^{r} s_j(E) \quad \text{for } r = 1, \ldots, n.$$

Recall that $\gamma_1 \geq \cdots \geq \gamma_k > \gamma_{k+1} = 0$. Thus,

$$\begin{aligned}\sum_{j=1}^{k} |a_{jj}|\gamma_j = \sum_{r=1}^{k} \left[(\gamma_r - \gamma_{r+1}) \sum_{j=1}^{r} |a_{jj}| \right] &\leq \sum_{r=1}^{k} \left[(\gamma_r - \gamma_{r+1}) \sum_{j=1}^{r} s_j(E) \right] \\ &= \sum_{j=1}^{k} s_j(E)\gamma_j.\end{aligned} \tag{2.4}$$

Furthermore, the equality holds if and only if

$$(\gamma_r - \gamma_{r+1}) \sum_{j=1}^{r} |a_{jj}| = (\gamma_r - \gamma_{r+1}) \sum_{j=1}^{r} s_j(E)$$

for $r = 1, \ldots, k$. In particular, $(\gamma_k - \gamma_{k+1}) \sum_{j=1}^{k} |a_{jj}| = (\gamma_k - \gamma_{k+1}) \sum_{j=1}^{k} s_j(E)$ implies $\sum_{j=1}^{k} |a_{jj}| = \sum_{j=1}^{k} s_j(E)$. By the same argument, these observations also hold for F.

Now by our assumption in (2.3) and the above observations, we have

$$\begin{aligned}\|E+F\|_\gamma = \sum_{j=1}^{k} s_j(E+F)\gamma_j = \sum_{j=1}^{k}(a_{jj}+b_{jj})\gamma_j &\le \sum_{j=1}^{k}|a_{jj}|\gamma_j + \sum_{j=1}^{k}|b_{jj}|\gamma_j \\ &\le \sum_{j=1}^{k} s_j(E)\gamma_j + \sum_{j=1}^{k} s_j(F)\gamma_j \qquad (2.5)\\ &= \|E\|_\gamma + \|F\|_\gamma.\end{aligned}$$

The assumption that $\|E+F\|_\gamma = \|E\|_\gamma + \|F\|_\gamma$ implies that the two equalities in (2.5) both hold. It follows that

$$\|E\|_\gamma = \sum_{j=1}^{k} s_j(E)\gamma_j = \sum_{j=1}^{k}|a_{jj}|\gamma_j \quad \text{and} \quad \|F\|_\gamma = \sum_{j=1}^{k} s_j(F)\gamma_j = \sum_{j=1}^{k}|b_{jj}|\gamma_j,$$

and $a_{jj}, b_{jj} \ge 0$ for all $j = 1 \ldots k$. With the inequality (2.4) and the discussion after that, we can further conclude that $\sum_{j=1}^{k} a_{jj} = \sum_{j=1}^{k} s_j(E)$ and $\sum_{j=1}^{k} b_{jj} = \sum_{j=1}^{k} s_j(F)$. Then applying Corollary 3.2 in [17], we have

$$UEV = E_1 \oplus E_2 \quad \text{and} \quad UFV = F_1 \oplus F_2,$$

where $E_1, F_1 \in M_k$ are positive semidefinite with eigenvalues $s_1(E), \ldots, s_k(E)$ and $s_1(F), \ldots, s_k(F)$, respectively. □

Lemma 2.4. *Let $A \in M_n$ be a nonzero matrix and $U \in \mathscr{U}_n$. Suppose that $V, W \in \mathscr{U}_n$ are matrices such that*

$$A = V(A_1 \oplus 0_{n-r})W^*,$$

where $1 \le r \le n$ and $A_1 \in M_r$ is positive definite. Then UA is positive semidefinite if and only if

$$U = W(I_r \oplus \hat{U})V^*$$

for some matrix $\hat{U} \in \mathscr{U}_{n-r}$.

Proof. The sufficiency part is obvious, here we only need prove the necessity part. By replacing (U, A) with (W^*UV, V^*AW), we may assume that $V = W = I_n$, i.e.,

$$A = A_1 \oplus 0_{n-r}.$$

We aim to show that $U = I_r \oplus \hat{U}$ for some matrix $\hat{U} \in \mathscr{U}_{n-r}$.

Let U be partitioned as

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$$

with $U_{11} \in M_r$ and $U_{22} \in M_{n-r}$. Then

$$UA = \begin{bmatrix} U_{11}A_1 & 0 \\ U_{21}A_1 & 0 \end{bmatrix}$$

is positive semidefinite. It follows that $U_{11}A_1$ is positive semidefinite and $U_{21}A_1 = 0$. Recall that A_1 is positive definite. Therefore, $U_{21}A_1 = 0$ implies that $U_{21} = 0$. With the assumption that U is unitary, we have

$$I_n = UU^* = \begin{bmatrix} U_{11}U_{11}^* + U_{12}U_{12}^* & U_{12}U_{22}^* \\ U_{22}U_{12}^* & U_{22}U_{22}^* \end{bmatrix},$$

and therefore $U_{11}U_{11}^* + U_{12}U_{12}^* = I_r, U_{22}U_{22}^* = I_{n-r}$ and $U_{12}U_{22}^* = 0$. It follows that $U_{12} = 0$ and U_{11} and U_{22} are unitary, i.e., $U = U_{11} \oplus U_{22}$ with unitary matrices $U_{11} \in M_r$ and $U_{22} \in M_{n-r}$. Recall that $U_{11}A_1$ is positive semidefinite and A_1 is positive definite. Let $P = U_{11}A_1$. Then we can conclude that

$$P^2 = P^*P = (U_{11}A_1)^*(U_{11}A_1) = A_1^*A_1 = A_1^2.$$

It follows that $P = A_1$. Hence we can conclude from $P = U_{11}A_1$ that $U_{11} = I_r$. It follows that $U = I_r \oplus U_{22}$. Let $\hat{U} = U_{22}$. This completes our proof. □

Proof of Theorem 2.1. With the assumption stated in Theorem 2.1, we can conclude that there exists an integer $2 \leq k \leq mn$ such that $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_k >$

$0 = \gamma_{k+1} = \cdots = \gamma_{mn}$. Since the sufficiency part is clear, we consider only the necessity part. Suppose the linear map $\phi : M_{mn} \to M_{mn}$ satisfies (2.2). We will prove the necessity part through the following assertions.

Assertion 2.1. *For any matrices $X \in \mathscr{U}_m$ and $Y \in \mathscr{U}_n$,*

$$\phi(XE_{ii}X^* \otimes YE_{jj}Y^*) \perp \phi(XE_{ii}X^* \otimes YE_{ss}Y^*) \quad \text{whenever } j \neq s. \tag{2.6}$$

And similarly,

$$\phi(XE_{rr}X^* \otimes YE_{jj}Y^*) \perp \phi(XE_{tt}X^* \otimes YE_{jj}Y^*) \quad \text{whenever } t \neq r.$$

Also $\operatorname{rank}(\phi(XE_{ii}X^* \otimes YE_{jj}Y^*)) < k$ *for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.*

Proof. Without loss of generality, we need only prove the claim in (2.6) holds. For simplicity, we denote $\phi(XE_{ii}X^* \otimes YE_{jj}Y^*)$ and $\phi(XE_{ii}X^* \otimes YE_{ss}Y^*)$ by A and B, respectively. Let $h = \operatorname{rank}(A)$ and $x_0 = \min\left\{\frac{s_h(A)}{2s_1(B)}, \frac{1}{2}\right\}$. We divide the proof into the following steps.

Step 1. We claim that there exist an integer T and matrices $U, V \in \mathscr{U}_{mn}$ such that

$$UAV = \bigoplus_{j=1}^{T} \tilde{A}_j \quad \text{and} \quad UBV = \bigoplus_{j=1}^{T} \tilde{B}_j \tag{2.7}$$

and for each $\theta \in [0, 2\pi)$, there exists a nonzero subset $J(\theta) \subseteq \{1, \ldots, T\}$ satisfying

(1.a) $A_{J(\theta)} = \bigoplus_{j \in J(\theta)} \tilde{A}_j \in M_k$ and $B_{J(\theta)} = \bigoplus_{j \in J(\theta)} \tilde{B}_j \in M_k$, and

(1.b) $s_j\left(2A_{J(\theta)} + (x_0+1)e^{i\theta}B_{J(\theta)}\right) = s_j\left(2A + (x_0+1)e^{i\theta}B\right)$ for $j = 1, \ldots, k$.

We prove the above claims by showing that for some integer T and matrices $U, V \in \mathscr{U}_{mn}$, the direct sum decomposition (2.7) satisfies (1.a) and (1.b). First of all, the decomposition clearly exists when $T = 1$ with $U = V = I_{mn}$ i.e., $\tilde{A}_1 = A_1$ and

$\tilde{B}_1 = B_1$. If (1.a) and (1.b) both hold for such $\tilde{A}_1$ and $\tilde{B}_1$, then Step 1 is correct. Otherwise, we can consider the following decomposition

$$UAV = \bigoplus_{j=1}^{T} \tilde{A}_j \quad \text{and} \quad UBV = \bigoplus_{j=1}^{T} \tilde{B}_j,$$

where $T \geq 1$ is an integer and $\tilde{A}_j, \tilde{B}_j \in M_{n_j}$ with $\sum_{j=1}^{T} n_j = mn$. Clearly,

$$U(2A + (x_0+1)e^{i\theta}B)V = \bigoplus_{j=1}^{T} 2\tilde{A}_j + (x_0+1)e^{i\theta}\tilde{B}_j.$$

Then for each $\theta \in [0, 2\pi)$, there exist $k_1, \dots, k_T$ with $0 \leq k_j \leq n_j$ and $\sum_{j=1}^{T} k_j = k$ such that the largest k singular values of $U(2A + (x_0+1)e^{i\theta}B)V$, as well as the largest k singular values of $2A + (x_0+1)e^{i\theta}B$, come from the largest k_j singular values of $2\tilde{A}_j + (x_0+1)e^{i\theta}\tilde{B}_j$. That is,

$$\Big(s_1(2\tilde{A}_1 + (x_0+1)e^{i\theta}\tilde{B}_1), \dots, s_{k_1}(2\tilde{A}_1 + (x_0+1)e^{i\theta}\tilde{B}_1),$$
$$s_1(2\tilde{A}_2 + (x_0+1)e^{i\theta}\tilde{B}_2), \dots, s_{k_2}(2\tilde{A}_2 + (x_0+1)e^{i\theta}\tilde{B}_2),$$
$$\cdots\cdots$$
$$s_1(2\tilde{A}_T + (x_0+1)e^{i\theta}\tilde{B}_T), \dots, s_{k_T}(2\tilde{A}_T + (x_0+1)e^{i\theta}\tilde{B}_T)\Big) \quad (2.8)$$

is equal to

$$\big(s_{j_1}(2A + (x_0+1)e^{i\theta}B), \dots, s_{j_k}(2A + (x_0+1)e^{i\theta}B)\big)$$

for some permutation $(j_1, \dots, j_k)$ of $(1, \dots, k)$. Here the integers $k_1, \dots, k_T$ depend on θ. Suppose for some $\theta \in [0, 2\pi)$, there exists $1 \leq j \leq T$ such that $0 < k_j < n_j$. Without loss of generality, we may assume $j = 1$, i.e., $0 < k_1 < n_1$. By the singular value decomposition, there exist matrices $\hat{U}_j, \hat{V}_j \in \mathscr{U}_{n_j}$ such that

$$\hat{U}_j(2\tilde{A}_j + (x_0+1)e^{i\theta}\tilde{B}_j)\hat{V}_j = \operatorname{diag}\left(s_1(2\tilde{A}_j + (x_0+1)e^{i\theta}\tilde{B}_j), \ldots, s_{n_j}(2\tilde{A}_j + (x_0+1)e^{i\theta}\tilde{B}_j)\right).$$

Let $\hat{U} = \left(\bigoplus_{j=1}^{T} \hat{U}_j\right) U$ and $\hat{V} = V\left(\bigoplus_{j=1}^{T} \hat{V}_j\right)$. Then

$$\hat{U}(2A + (x_0+1)e^{i\theta}B)\hat{V} = \bigoplus_{j=1}^{T} \operatorname{diag}\left(s_1(2\tilde{A}_j + (x_0+1)e^{i\theta}\tilde{B}_j), \ldots, s_{n_j}(2\tilde{A}_j + (x_0+1)e^{i\theta}\tilde{B}_j)\right)$$

is an $mn \times mn$ diagonal matrix. From (2.8), the first $k_1 \times k_1$ block diagonal submatrix of $\hat{U}(2A + (x_0+1)e^{i\theta}B)\hat{V}$ is

$$\begin{aligned}\operatorname{diag}&\left(s_1(2\tilde{A}_1 + (x_0+1)e^{i\theta}\tilde{B}_1), \ldots, s_{k_1}(2\tilde{A}_1 + (x_0+1)e^{i\theta}\tilde{B}_1)\right) \\ &= \operatorname{diag}\left(s_{j_1}(2A + (x_0+1)e^{i\theta}B), \ldots, s_{j_{k_1}}(2A + (x_0+1)e^{i\theta}B)\right). \qquad (2.9)\end{aligned}$$

By the assumption in (2.2), we have

$$\|2A + (x+1)e^{i\theta}B\|_\gamma = \|XE_{ii}X^* \otimes Y(2E_{jj} + (x+1)E_{ss})Y^*)\|_\gamma = 2\gamma_1 + (x+1)\gamma_2,$$

$$\|A + xe^{i\theta}B\|_\gamma = \|XE_{ii}X^* \otimes Y(E_{jj} + xE_{ss})Y^*)\|_\gamma = \gamma_1 + x\gamma_2, \text{ and}$$

$$\|A + e^{i\theta}B\|_\gamma = \|XE_{ii}X^* \otimes Y(E_{jj} + E_{ss})Y^*)\|_\gamma = \gamma_1 + \gamma_2$$

for all $0 < x \le 1$. It follows from the above equations that

$$\|2A + (x_0+1)e^{i\theta}B\|_\gamma = \|A + e^{i\theta}B\|_\gamma + \|A + x_0e^{i\theta}B\|_\gamma. \qquad (2.10)$$

Applying Lemma 2.3 with $(E, F) = (A + e^{i\theta}B, A + x_0e^{i\theta}B)$, we can conclude that the (i,j)-th entries of $\hat{U}(A + e^{i\theta}B)\hat{V}$ and $\hat{U}(A + x_0e^{i\theta}B)\hat{V}$ are zero for all $(i,j) \in$

$\big(\{1,\ldots,k_1\}\times\{k_1+1,\ldots,n_1\}\big)\bigcup\big(\{k_1+1,\ldots,n_1\}\times\{1,\ldots,k_1\}\big)$, so as $\hat{U}A\hat{V}$ and $\hat{U}B\hat{V}$. Notice that

$$\hat{U}A\hat{V}=\bigoplus_{j=1}^{T}\hat{U}_j\tilde{A}_j\hat{V}_j \quad \text{and} \quad \hat{U}B\hat{V}=\bigoplus_{j=1}^{T}\hat{U}_j\tilde{B}_j\hat{V}_j.$$

Then we can conclude from the above observation that the (i,j)-th entries of $\hat{U}_1\tilde{A}_1\hat{V}_1$ and $\hat{U}_1\tilde{B}_1\hat{V}_1$ are zero for all $(i,j)\in\big(\{1,\ldots,k_1\}\times\{k_1+1,\ldots,n_1\}\big)\bigcup\big(\{k_1+1,\ldots,n_1\}\times\{1,\ldots,k_1\}\big)$. With the assumption that $0<k_1<n_1$, we can write

$$\hat{U}_1\tilde{A}_1\hat{V}_1=\hat{A}_1\oplus\hat{A}_2 \quad \text{and} \quad \hat{U}_1\tilde{B}_1\hat{V}_1=\hat{B}_1\oplus\hat{B}_2$$

with $\hat{A}_1,\hat{B}_1\in M_{k_1}$ and $\hat{A}_2,\hat{B}_2\in M_{n_1-k_1}$. Let $\hat{A}_{j+1}=\hat{U}_j\tilde{A}_j\hat{V}_j$ and $\hat{B}_{j+1}=\hat{U}_j\tilde{B}_j\hat{V}_j$ for $j=2,\ldots,T$. Then we can conclude that

$$\hat{U}A\hat{V}=\bigoplus_{j=1}^{T+1}\hat{A}_j \quad \text{and} \quad \hat{U}B\hat{V}=\bigoplus_{j=1}^{T+1}\hat{B}_j.$$

With the new unitary matrices $\hat{U}$ and $\hat{V}$, we can re-define $n_1,\ldots,n_{T+1}$, and $k_1,\ldots,k_{T+1}$ accordingly. If there still exists some $\theta\in[0,2\pi)$ such that $0<k_j<n_j$ for some $1\le j\le T+1$, we can repeat the above argument again so that for some matrices $U,V\in\mathscr{U}_{mn}$,

$$UAV=\bigoplus_{j=1}^{T+2}\tilde{A}_j \quad \text{and} \quad UBV=\bigoplus_{j=1}^{T+2}\tilde{B}_j.$$

Since the number of diagonal blocks is at most mn, the above argument can be repeated for finitely many times only. Therefore, we may conclude that, after finitely many times, for all $\theta\in[0,2\pi)$, either $k_j=0$ or $k_j=n_j$ for all $j=1,\ldots,T$, where $n_1,\ldots,n_T$, and $k_1,\ldots,k_T$ are the quantities defined with respect to the diagonal block decomposition,

$$UAV=\bigoplus_{j=1}^{T}\tilde{A}_j \quad \text{and} \quad UBV=\bigoplus_{j=1}^{T}\tilde{B}_j.$$

With (2.8), this is equivalent to say, for each $\theta \in [0, 2\pi)$, there exists an index set $J(\theta) \subseteq \{1, \ldots, T\}$ such that $\sum\limits_{j \in J(\theta)} n_j = \sum\limits_{j \in J(\theta)} k_j = k$ and

$$s_j \left(\bigoplus_{j \in J(\theta)} 2\tilde{A}_j + (x_0 + 1)e^{i\theta} \tilde{B}_j \right) = s_j \left(2A + (x_0 + 1)e^{i\theta} B \right) \quad \text{for } j = 1, \ldots, k.$$

Now we have completed the proof of Step 1.

Step 2. There exist matrices $U, V \in \mathscr{U}_{mn}$ and an infinite subset $\Theta \subseteq [0, 2\pi)$ such that

$$UAV = A_1 \oplus A_2 \quad \text{and} \quad UBV = B_1 \oplus B_2 \tag{2.11}$$

with $A_1, B_1 \in M_k$ and $A_2, B_2 \in M_{mn-k}$, and for any $\theta \in \Theta$,

$$s_j(2A_1 + (x_0 + 1)e^{i\theta} B_1) = s_j(2A + (x_0 + 1)e^{i\theta} B) \quad \text{for } j = 1, \ldots, k. \tag{2.12}$$

From Step 1, A and B have the decomposition (2.7) and satisfy (1.a) and (1.b). Since $[0, 2\pi)$ is an infinite set and the number of subsets of $\{1, 2, \ldots, T\}$ is finite, we can conclude that $J(\theta)$ are the same for infinitely many $\theta \in [0, 2\pi)$. Denote by Θ and J the set of these infinitely many θ and the common subset $J(\theta)$, respectively. Then we have for any $\theta \in \Theta$,

$$s_j \left(2A_J + (x_0 + 1)e^{i\theta} B_J \right) = s_j \left(2A + (x_0 + 1)e^{i\theta} B \right) \quad \text{for } j = 1, \ldots, k,$$

where $A_J = \bigoplus\limits_{j \in J} \tilde{A}_j \in M_k$ and $B_J = \bigoplus\limits_{j \in J} \tilde{B}_j \in M_k$. By replacing (U, V) with (PU, VP^T) for some permutation P, if necessary, we may assume that $J = \{1, \ldots, \hat{T}\}$ for some $1 \leq \hat{T} \leq T$. Let

$$A_1 = \bigoplus_{j=1}^{\hat{T}} \tilde{A}_j, A_2 = \bigoplus_{j=\hat{T}+1}^{T} \tilde{A}_j, B_1 = \bigoplus_{j=1}^{\hat{T}} \tilde{B}_j \text{ and } B_2 = \bigoplus_{j=\hat{T}+1}^{T} \tilde{B}_j.$$

Then we have $UAV = A_1 \oplus A_2$ and $UBV = B_1 \oplus B_2$. This completes the proof of Step 2.

Step 3. The matrices A_1 and B_1 obtained in Step 2 are orthogonal, and hence there exist matrices $W, \hat{W} \in \mathscr{U}_k$ and some integer $0 \le r \le k$ such that

$$W A_1 \hat{W} = \operatorname{diag}(a_1, \ldots, a_r) \oplus 0_{k-r} \quad \text{and} \quad W B_1 \hat{W} = 0_r \oplus \operatorname{diag}(b_{r+1}, \ldots, b_k) \quad (2.13)$$

with $a_j > 0$ for $j = 1, \ldots, r$ and $b_j \ge 0$ for $j = r+1, \ldots, k$.

If $A_1 = 0$ or $B_1 = 0$, then there is nothing to prove. So we may suppose that A_1 and B_1 are both nonzero matrices. For simplicity, we may assume that $U = V = I_{mn}$ in the equation (2.11). Then (2.11) and (2.12) imply that for any $\theta \in \Theta$, there exist matrices $X_\theta, Y_\theta \in \mathscr{U}_k$ and $\hat{X}_\theta, \hat{Y}_\theta \in \mathscr{U}_{mn-k}$ such that

$$\begin{aligned}\big(X_\theta \oplus \hat{X}_\theta\big)\big(2A + (x_0+1)e^{i\theta}B\big)\big(Y_\theta \oplus \hat{Y}_\theta\big) &= \\ X_\theta\big(2A_1 + (x_0+1)e^{i\theta}B_1)Y_\theta \oplus \hat{X}_\theta(2A_2 + (x_0+1)e^{i\theta}B_2)\hat{Y}_\theta &= \\ \operatorname{diag}\big(s_1(2A + (x_0+1)e^{i\theta}B), \ldots, s_{mn}(2A + (x_0+1)e^{i\theta}B)\big). & \quad (2.14)\end{aligned}$$

Recall that

$$\|2A + (x_0+1)e^{i\theta}B\|_\gamma = \|A + e^{i\theta}B\|_\gamma + \|A + x_0 e^{i\theta}B\|_\gamma.$$

Applying Lemma 2.3 again with $(E, F) = (A + e^{i\theta}B, A + x_0e^{i\theta}B)$, we conclude from the above two equations that $X_\theta(A_1 + e^{i\theta}B_1)Y_\theta$ and $X_\theta(A_1 + x_0e^{i\theta}B_1)Y_\theta$ are both positive semidefinite with eigenvalues $s_1(A + e^{i\theta}B), \ldots, s_k(A + e^{i\theta}B)$ and $s_1(A + x_0e^{i\theta}B), \ldots, s_k(A + x_0e^{i\theta}B)$, respectively. It follows that $Y_\theta X_\theta(A_1 + e^{i\theta}B_1)$ and $Y_\theta X_\theta(A_1 + x_0e^{i\theta}B_1)$ are positive semidefinite. For simplicity, we denoted $Y_\theta X_\theta$ by U_θ. Clearly, U_θ is unitary. By now, we have showed that for any $\theta \in \Theta$, there exists a matrix $U_\theta \in \mathscr{U}_k$ such that

$$U_\theta(A_1 + e^{i\theta}B_1) \quad \text{and} \quad U_\theta(A_1 + x_0e^{i\theta}B_1) \text{ are both positive semidefinite.} \quad (2.15)$$

Thus, $U_\theta A_1$ is Hermitian. We claim that $U_\theta A_1$ is also positive semidefinite. Otherwise, since $U_\theta A_1$ is Hermitian, there exists an eigenvalue λ of $U_\theta A_1$ such that $\lambda = -s(U_\theta A_1)$ for some nonzero singular value $s(U_\theta A_1)$ of $U_\theta A_1$. Let y be a unit eigenvector corresponding to λ, that is, $U_\theta A_1 y = \lambda y$. Then we have

$$y^* U_\theta A_1 y = \lambda = -s(U_\theta A_1). \tag{2.16}$$

Furthermore, U_θ is unitary implies that $s(U_\theta A_1)$ is also a nonzero singular value of A_1, therefore, as well as a nonzero singular value of A. It follows that $s(U_\theta A_1) \geq s_h(A)$. Then with the assumption that $x_0 = \min\left\{\frac{s_h(A)}{2s_1(B)}, \frac{1}{2}\right\}$, we have

$$y^* U_\theta(A_1 + x_0 e^{i\theta} B_1)y \leq -s(U_\theta A_1) + x_0 s_1(B_1) \leq -s_h(A) + x_0 s_1(B) < 0,$$

contrary to (2.15). Thus, our claim is correct, i.e., $U_\theta A_1$ is positive semidefinite. Let $G, W \in \mathscr{U}_k$ be matrices such that

$$G^* A_1 W = A_{11} \oplus 0_{k-r} \quad \text{and} \quad G^* B_1 W = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \tag{2.17}$$

where $B_{11}, A_{11} \in M_r$ for some $1 \leq r \leq k$ and A_{11} is positive definite. Applying Lemma 2.4, we have $U_\theta = W(I_r \oplus X_\theta)G^*$ with $X_\theta \in \mathscr{U}_{k-r}$. Recall our assumption that B_1 is nonzero. We claim that B_{22} is nonzero. Otherwise, $B_{22} = 0$, then we have

$$U_\theta(A_1 + e^{i\theta} B_1) = W \begin{bmatrix} A_{11} + e^{i\theta} B_{11} & e^{i\theta} B_{12} \\ X_\theta e^{i\theta} B_{21} & 0 \end{bmatrix} W^*.$$

Recall that $U_\theta(A_1 + e^{i\theta} B_1)$ is positive semidefinite and A_{11} is positive definite. It follows that $B_{12} = 0$, $B_{21} = 0$ and $e^{i\theta} B_{11} = -e^{i\theta} B_{11}^*$. Since this is true for all $\theta \in \Theta$ and Θ is an infinite set, it follows that $B_{11} = 0$ and therefore $B_1 = 0$, which is contrary to our assumption that B_1 is nonzero. Thus, our claim is correct, that is, B_{22} is nonzero. So by replacing G^* and W with $(I_r \oplus G_1^*)G^*$, and $W(I_r \oplus W_1)$

respectively, for some $G_1, W_1 \in \mathscr{U}_{k-r}$, we can further rewrite equations in (2.17) as

$$G^* A_1 W = A_{11} \oplus 0_{k-r} \quad \text{and} \quad G^* B_1 W = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & 0 \\ B_{31} & 0 & 0 \end{bmatrix},$$

where $A_{11} \in M_r$ and $B_{22} \in M_\ell$ are positive definite and $B_{11} \in M_r$ for some $1 \le r \le k$ and $1 \le \ell \le k-r$. And we still have $U_\theta = W(I_r \oplus X_\theta)G^*$ with $X_\theta \in \mathscr{U}_{k-r}$. Recall that $U_\theta(A_1 + e^{i\theta}B_1)$ is positive semidefinite. This implies that $e^{i\theta}X_\theta(B_{22} \oplus 0_{k-r-\ell})$ is positive semdefinite. Clearly, $e^{i\theta}X_\theta$ is also unitary. We use Lemma 2.4 again to conclude that $e^{i\theta}X_\theta = I_\ell \oplus Y_\theta$ with $Y_\theta \in \mathscr{U}_{k-r-\ell}$, or equivalently, $X_\theta = e^{-i\theta}I_\ell \oplus e^{-i\theta}Y_\theta$. Then we have $U_\theta = W\big(I_r \oplus e^{-i\theta}I_\ell \oplus e^{-i\theta}Y_\theta\big)G^*$. It follows that

$$U_\theta(A_1 + e^{i\theta}B_1) = W \begin{bmatrix} A_{11} + e^{i\theta}B_{11} & e^{i\theta}B_{12} & e^{i\theta}B_{13} \\ B_{21} & B_{22} & 0 \\ Y_\theta B_{31} & 0 & 0 \end{bmatrix} W^*$$

is positive semidefinite. Recall that $A_{11} \in M_r$ is positive definite. Therefore, $B_{31} = 0$, $B_{13} = 0$, $e^{i\theta}B_{12} = B_{21}^*$ and $e^{i\theta}B_{11} = e^{-i\theta}B_{11}^*$. Since this is true for all $\theta \in \Theta$ and Θ is an infinite set, it follows that B_{12}, B_{21} and B_{11} are all zero matrices. Then we have

$$G^* B_1 W = 0_r \oplus B_{22} \oplus 0_{k-r-\ell}. \tag{2.18}$$

Clearly, it follows that A_1 and B_1 are orthogonal. This confirms the Step 3.

Step 4. The matrices A and B are orthogonal and $\operatorname{rank}(A) < k$. Thus, Assertion 2.1 holds.

For simplicity, we may assume that $U = V = I_{mn}$ in (2.11) and $W = \hat{W} = I_k$ in (2.13). It follows that for some $0 \le r \le k$,

$$A = \operatorname{diag}(a_1, \ldots, a_r) \oplus 0_{k-r} \oplus A_2 \quad \text{and} \quad B = 0_r \oplus \operatorname{diag}(b_{r+1}, \ldots, b_k) \oplus B_2 \tag{2.19}$$

with $a_j > 0$ for $j = 1, \ldots, r$ and $b_j \ge 0$ for $j = r+1, \ldots, k$. Choose a certain θ_0 from Θ. Denote the singular values of $2A + (x_0 + 1)e^{i\theta_0}B$ by $s_1 \ge s_2 \ge \cdots \ge s_{mn}$. Then

(2.12) implies that

$$\big(s_{\ell_1},\dots,s_{\ell_k}\big)=\big(2a_1,\dots,2a_r,(x_0+1)b_{r+1},\dots,(x_0+1)b_k\big)$$

for some permutation $(\ell_1,\dots,\ell_k)$ of $(1,\dots,k)$. Let $\hat{X},\hat{Y}\in\mathscr{U}_{mn-k}$ such that

$$\hat{X}\big(2A_2+(x_0+1)e^{i\theta_0}B_2\big)\hat{Y}=\operatorname{diag}\big(s_{k+1},\dots,s_{mn}\big).$$

Let $U=I_r\oplus e^{-i\theta_0}I_{k-r}\oplus\hat{X}$ and $V=I_r\oplus I_{k-r}\oplus\hat{Y}$. Then we have

$$U(2A+(x_0+1)e^{i\theta_0}B)V=\operatorname{diag}(s_{\ell_1},\dots,s_{\ell_k},s_{k+1},\dots,s_{mn}),$$
$$U(A+e^{i\theta_0}B)V=\operatorname{diag}(a_1,\dots,a_r,b_{r+1},\dots,b_k)\oplus\hat{X}(A_2+e^{i\theta_0}B_2)\hat{Y},$$
$$\text{and } U(A+x_0e^{i\theta_0}B)V=\operatorname{diag}(a_1,\dots,a_r,x_0b_{r+1},\dots,x_0b_k)\oplus\hat{X}(A_2+x_0e^{i\theta_0}B_2)\hat{Y}.$$

We apply Lemma 2.3 with $(E,F)=(A+e^{i\theta_0}B,A+x_0e^{i\theta_0}B)$ to conclude that

$$\|A+e^{i\theta_0}B\|_\gamma=\sum_{j=1}^{r}a_j\gamma_{\ell_j}+\sum_{j=r+1}^{k}b_j\gamma_{\ell_j},\ \text{and}$$

$$\|A+x_0e^{i\theta_0}B\|_\gamma=\sum_{j=1}^{r}a_j\gamma_{\ell_j}+x_0\sum_{j=r+1}^{k}b_j\gamma_{\ell_j}.$$

With the assumption that $\phi:M_{mn}\to M_{mn}$ satisfies (2.2), we have

$$\|A+xe^{i\theta_0}B\|_\gamma=\|(XE_{ii}X^*\otimes YE_{jj}Y^*)+xe^{i\theta_0}(XE_{ii}X^*\otimes YE_{ss}Y^*)\|_\gamma=\gamma_1+x\gamma_2$$

for all $0<x\le 1$. The above three equations imply that

$$\|A+xe^{i\theta_0}B\|_\gamma=\gamma_1+x\gamma_2=\sum_{j=1}^{r}a_j\gamma_{\ell_j}+x\sum_{j=r+1}^{k}b_j\gamma_{\ell_j}\quad\text{for all } 0<x\le 1. \tag{2.20}$$

Notice that $\|A\|_\gamma=\|XE_{ii}X^*\otimes YE_{jj}Y^*\|_\gamma=\gamma_1$ and $\gamma_1\ge\cdots\ge\gamma_k>0$ with $k\ge 2$. Then we conclude from (2.20) that

$$\sum_{j=1}^{r}a_j\gamma_{\ell_j}=\gamma_1=\|A\|_\gamma\quad\text{and}\quad\sum_{j=r+1}^{k}b_j\gamma_{\ell_j}=\gamma_2. \tag{2.21}$$

It follows from the right equation in (2.21) that $r < k$. We claim that $A_2 = 0$. Otherwise, since $\text{rank}(A_1) \le r < k$, we must have $\|A\|_\gamma > \|A_1\|_\gamma \ge \sum_{j=1}^{r} a_j\gamma_{\ell_j}$, contrary to (2.21). Therefore, $A = A_1 \oplus 0_{n-k}$ and $B = B_1 \oplus B_2$. Since A_1 and B_1 are orthogonal, so as A and B. Furthermore, $\text{rank}(A) = \text{rank}(A_1) \le r < k$. This completes the proof. □

Assertion 2.2. *For any matrices $X \in \mathscr{U}_m$ and $Y \in \mathscr{U}_n$,*

$$\phi(XE_{ii}X^* \otimes Y(E_{jj} + E_{ss})Y^*) \perp \phi(XE_{tt}X^* \otimes Y(E_{jj} + E_{ss})Y^*),$$

whenever $i \ne t$ and $j \ne s$.

Proof. For simplicity, we denote $\phi(XE_{ii}X^* \otimes Y(E_{jj} + E_{ss})Y^*)$ and $\phi(XE_{tt}X^* \otimes Y(E_{jj}+E_{ss})Y^*)$ by G and H, respectively. Let $h = \text{rank}(G)$ and $x_0 = \min\{\frac{s_h(G)}{2s_1(H)}, \frac{1}{2}\}$. By the assumption in (2.2), we can use a similar argument as used in (2.10) to show that

$$\|2G + (x_0 + 1)e^{i\theta}H\|_\gamma = \|G + e^{i\theta}H\|_\gamma + \|G + x_0e^{i\theta}H\|_\gamma$$

for all $\theta \in [0, 2\pi)$. We can use the same argument in Assertion 2.1 to conclude that there exist matrices $U, V \in \mathscr{U}_{mn}$ such that for some $0 \le r \le k$,

$$\begin{aligned} UGV &= \text{diag}(a_1, \dots, a_r) \oplus 0_{k-r} \oplus G_2 \\ UHV &= 0_r \oplus \text{diag}(b_{r+1}, \dots, b_k) \oplus H_2 \end{aligned} \tag{2.22}$$

with $a_j > 0$ for $j = 1, \dots, r$ and $b_j \ge 0$ for $j = r + 1, \dots, k$, and

$$\begin{aligned} \|G + x_0e^{i\theta_0}H\|_\gamma &= \sum_{j=1}^{r} a_j\gamma_{\ell_j} + x_0 \sum_{j=r+1}^{k} b_j\gamma_{\ell_j} \\ \|G + e^{i\theta_0}H\|_\gamma &= \sum_{j=1}^{r} a_j\gamma_{\ell_j} + \sum_{j=r+1}^{k} b_j\gamma_{\ell_j} \end{aligned} \tag{2.23}$$

for some $\theta_0 \in [0, 2\pi)$ and permutation $(\ell_1, \dots, \ell_k)$ of $(1, \dots, k)$. With the assumption that $\phi : M_{mn} \to M_{mn}$ satisfies (2.2), we have

$$\|G + xe^{i\theta} H\|_\gamma = \gamma_1 + \gamma_2 + x(\gamma_3 + \gamma_4) \quad \text{for all } 0 < x \le 1 \text{ and } \theta \in [0, 2\pi). \tag{2.24}$$

It follows from (2.23) and (2.24) that

$$\sum_{j=1}^{r} a_j \gamma_{\ell_j} = \gamma_1 + \gamma_2 = \|G\|_\gamma \quad \text{and} \quad \sum_{j=r+1}^{k} b_j \gamma_{\ell_j} = \gamma_3 + \gamma_4.$$

If $k \ge 3$, then $\displaystyle\sum_{j=r+1}^{k} b_j \gamma_{\ell_j} = \gamma_3 + \gamma_4 > 0$, and hence $r < k$. Then we can use the same argument in Step 4 of Assertion 2.1 to show that $G_2 = 0$, and therefore $G \perp H$. We now turn to the case when $k = 2$. In this case, by the result of Assertion 2.1, we have $\operatorname{rank}\big(\varphi(XE_{ii}X^* \otimes YE_{jj}Y^*)\big) = 1$ for $i = 1, \dots, m$ and $j = 1, \dots, n$. It follows that

$$\|\varphi(XE_{ii}X^* \otimes YE_{jj}Y^*)\|_\gamma = s_1(\varphi(XE_{ii}X^* \otimes YE_{jj}Y^*))\gamma_1. \tag{2.25}$$

Besides, by the assumption in (2.2), we have

$$\|\varphi(XE_{ii}X^* \otimes YE_{jj}Y^*)\|_\gamma = \|XE_{ii}X^* \otimes YE_{jj}Y^*\|_\gamma = \gamma_1. \tag{2.26}$$

The above equations imply that $s_1(\varphi(XE_{ii}X^* \otimes YE_{jj}Y^*)) = 1$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$. Notice that $G = \phi(XE_{ii}X^* \otimes YE_{jj}Y^*) + \phi(XE_{ii}X^* \otimes YE_{ss}Y^*)$. Then with the result in Assertion 2.1, we have

$$\operatorname{rank}(G) = 2 \quad \text{and} \quad s_1(G) = s_2(G) = 1.$$

The same observations also hold for H. It follows that

$$a_j = 1 \text{ for } j = 1, \dots, r \quad \text{and} \quad b_j \in \{0, 1\} \text{ for } j = r+1, \dots, 2.$$

However, with the equations in (2.23), $b_j = 0$ for some $r + 1 \le j \le 2$ leads to

$$\|G + e^{i\theta} H\|_\gamma = \sum_{j=1}^{r} a_j \gamma_{\ell_j} + \sum_{j=r+1}^{2} b_j \gamma_{\ell_j} < \gamma_1 + \gamma_2,$$

contrary to (2.2). Thus, $b_j = 1$ for $j = r+1, \ldots, 2$. Next we show that G_2 and H_2 are orthogonal. If $G_2 = 0$, then there is nothing to prove. If $G_2 \neq 0$, we may assume that

$$G_2 = \begin{bmatrix} I_\ell & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad H_2 = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$$

with $H_{11} \in M_\ell$ for some $1 \leq \ell \leq 2$. Then

$$G_2 + e^{i\theta} H_2 = \begin{bmatrix} I_\ell + e^{i\theta} H_{11} & e^{i\theta} H_{12} \\ e^{i\theta} H_{21} & e^{i\theta} H_{22} \end{bmatrix}.$$

We claim that $H_{11} = 0, H_{12} = 0, H_{21} = 0$, and hence $H = 0 \oplus H_{22}$. Otherwise, $s_1(G_2 + e^{i\theta_0} H_2) > 1$, and therefore $s_1(G + e^{i\theta_0} H) > 1$ for some $\theta_0 \in [0, 2\pi)$. It follows that $\|G + e^{i\theta_0} H\|_\gamma > \gamma_1 + \gamma_2$, contrary to (2.2). Thus, our claim is correct, that is, $H = 0 \oplus H_{22}$. It follows that $G \perp H$. This completes our proof. □

Assertion 2.3. *For any matrices $X \in \mathscr{U}_m$ and $Y \in \mathscr{U}_n$,*

$$\phi(XE_{ii}X^* \otimes YE_{jj}Y^*) \perp \phi(XE_{rr}X^* \otimes YE_{ss}Y^*) \text{ whenever } (i,j) \neq (r,s).$$

Proof. If $i = r$ or $j = s$, then the result in Assertion 2.1 directly implies that

$$\phi(XE_{ii}X^* \otimes YE_{jj}Y^*) \perp \phi(XE_{rr}X^* \otimes YE_{ss}Y^*).$$

Next, we suppose that $i \neq r$ and $j \neq s$. With Assertion 2.1, we have

$$\phi(XE_{ii}X^* \otimes YE_{jj}Y^*) \perp \phi(XE_{ii}X^* \otimes YE_{ss}Y^*) \tag{2.27}$$

and

$$\phi(XE_{rr}X^* \otimes YE_{jj}Y^*) \perp \phi(XE_{rr}X^* \otimes YE_{ss}Y^*). \tag{2.28}$$

By Assertion 2.2, we have

$$\phi(XE_{ii}X^* \otimes Y(E_{jj} + E_{ss})Y^*) \perp \phi(XE_{rr}X^* \otimes Y(E_{jj} + E_{ss})Y^*). \tag{2.29}$$

Applying Lemma 2.2, we conclude from (2.27) and (2.29) that

$$\phi(XE_{ii}X^* \otimes YE_{jj}Y^*) \perp \phi(XE_{rr}X^* \otimes Y(E_{jj} + E_{ss})Y^*). \tag{2.30}$$

Then we apply Lemma 2.2 again to conclude from (2.28) and (2.30) that

$$\phi(XE_{ii}X^* \otimes YE_{jj}Y^*) \perp \phi(XE_{rr}X^* \otimes YE_{ss}Y^*).$$

This completes the proof. □

Assertion 2.4. *There exist matrices U and V in $\mathscr{U}_{mn}$ such that*

$$\phi(C \otimes D) = U(\varphi_1(C) \otimes \varphi_2(D))V \quad \textit{for all } C \in M_m \textit{ and } D \in M_n,$$

where φ_s is the identity map or the transposition map for $s = 1, 2$.

Proof. For any $Y \in \mathscr{U}_n$, by Assertion 2.3,

$$\{\phi(E_{ii} \otimes YE_{jj}Y^*) : i = 1, \ldots, m \text{ and } j = 1 \ldots, n\}$$

is a set of mn orthogonal matrices in M_{mn}. It follows that all of the matrices in this set are of rank one. Then there exist matrices $U_Y, V_Y \in \mathscr{U}_{mn}$ such that

$$\phi(E_{ii} \otimes YE_{jj}Y^*) = U_Y(E_{ii} \otimes E_{jj})V_Y^* \quad \text{for all } i = 1, \ldots, m \text{ and } j = 1, \ldots, n. \tag{2.31}$$

Without loss of generality, we may assume that $U_I = V_I = I_{mn}$, i.e.,

$$\phi(E_{ii} \otimes E_{jj}) = E_{ii} \otimes E_{jj} \quad \text{for all } i = 1, \ldots, m \text{ and } j = 1, \ldots, n. \tag{2.32}$$

With (2.31) and (2.32), we have

(i) $I_{mn} = \phi(I_m \otimes I_n) = U_Y(I_m \otimes I_n)V_Y^*$;

(ii) $E_{ii} \otimes I_n = \phi(E_{ii} \otimes I_n) = U_Y(E_{ii} \otimes I_n)V_Y^*$ for all $i = 1, \ldots, m$.

It follows that $U_Y = V_Y$ and U_Y commutes with $E_{ii} \otimes I_n$ for all $i = 1, \ldots, m$. Therefore, we have U_Y commuting with $E_{11} \otimes I_n + 2E_{22} \otimes I_n + \cdots + mE_{mm} \otimes I_n$, which implies that $U_Y = \bigoplus_{i=1}^{m} U_i$ with $U_i \in \mathscr{U}_n$ for all $i = 1, \ldots, m$. It follows that $\phi(E_{ii} \otimes YE_{jj}Y^*) = E_{ii} \otimes U_iE_{jj}U_i^*$. Now, we have showed that for any $Y \in \mathscr{U}_n$, there exists $U_i \in \mathscr{U}_n$ depending on i and Y such that

$$\phi(E_{ii} \otimes YE_{jj}Y^*) = E_{ii} \otimes U_iE_{jj}U_i^* \quad \text{for } j = 1, \ldots, n.$$

By the linearity of ϕ, we conclude from the above equation that for any $i = 1, \ldots, m$, there is a linear map ψ_i such that

$$\phi(E_{ii} \otimes B) = E_{ii} \otimes \psi_i(B) \quad \text{for all } B \in M_n.$$

Let $\hat{\gamma} = (\gamma_1, \ldots, \gamma_n)$. Then it is easy to check that

$$\|\psi_i(B)\|_{\hat{\gamma}} = \|E_{ii} \otimes \psi_i(B)\|_{\gamma} = \|E_{ii} \otimes B\|_{\gamma} = \|B\|_{\hat{\gamma}} \quad \text{for all } B \in M_n.$$

That is, ψ_i is a linear map on M_n preserving $\hat{\gamma}$-norm. Thus, by Theorem 4 in [21], ψ_i has form $B \mapsto W_iB\widetilde{W}_i$ or $B \mapsto W_iB^T\widetilde{W}_i$ for some matrices $W_i, \widetilde{W}_i \in \mathscr{U}_n$. Let $W = \bigoplus_{i=1}^{m} W_i$ and $\widetilde{W} = \bigoplus_{i=1}^{m} \widetilde{W}_i$. It follows that for any $i = 1, \ldots, m$,

$$\phi(E_{ii} \otimes B) = W(E_{ii} \otimes \varphi_i(B))\widetilde{W} \quad \text{for all } B \in M_n,$$

where φ_i is the identity map or the transposition map. Recall that $I_{mn} = \phi(I_m \otimes I_n)$. Thus, we have $\widetilde{W} = W^*$. Applying Assertion 2.3 again, we can repeat the same argument above to show that for any unitary matrix $X \in M_m$ and $1 \le i \le n$, there exists unitary matrix W_X such that

$$\phi(XE_{ii}X^* \otimes B) = W_X(E_{ii} \otimes \varphi_{i,X}(B))W_X^* \quad \text{for all } B \in M_n, \tag{2.33}$$

where $\varphi_{i,X}$ is the identity map or the transposition map. For simplicity, we may further assume that $W_I = I_{mn}$, i.e.,

$$\phi(E_{ii} \otimes B) = E_{ii} \otimes \varphi_{i,I_n}(B) \quad \text{for all } B \in M_n, \tag{2.34}$$

where φ_{i,I_n} is the identity map or the transposition map. Next, we use the same arguments in the last two paragraphs of the proof of Theorem 2.1 in [6] to show that $\varphi_{i,X}$ are the same for all $i = 1, \ldots, m$ and $X \in \mathscr{U}_m$. With (2.33) and (2.34), we have for any real symmetric $S \in M_n$ and $X \in \mathscr{U}_m$,

$$I_m \otimes S = \phi(I_m \otimes S) = \sum_{i=1}^{m} \phi(XE_{ii}X^* \otimes S) = W_X(I_m \otimes S)W_X^*.$$

It follows that W_X commutes with $I_m \otimes S$ for all real symmetric $S \in M_n$. This yields that $W_X = Z_X \otimes I_n$ for some $Z_X \in \mathscr{U}_n$, and hence

$$\phi(XE_{ii}X^* \otimes B) = (Z_X E_{ii} Z_X^*) \otimes \varphi_{i,X}(B) \quad \text{for all } i = 1, \ldots, m \text{ and } B \in M_n.$$

Define linear maps $\mathrm{tr}_1 : M_{mn} \to M_n$ and $\mathrm{Tr}_1 : M_{mn} \to M_n$ as

$$\mathrm{tr}_1(A \otimes B) = (\mathrm{trA})B \quad \text{and} \quad \mathrm{Tr}_1(A \otimes B) = \mathrm{tr}_1(\phi(A \otimes B))$$

for all $A \in M_m$ and $B \in M_n$. The map tr_1 is also called the partial trace function in quantum science. Then

$$\mathrm{Tr}_1(XE_{ii}X^* \otimes B) = \varphi_{i,X}(B),$$

where $\varphi_{i,X}$ is the identity map or the transposition map. Note that Tr_1 is linear and therefore continuous and the set

$$\{XE_{ii}X^* \mid 1 \le i \le m, X \in \mathscr{U}_n\} = \{xx^* \in M_m \mid x^*x = 1\}$$

is connected. So, all the maps $\varphi_{i,X}$ are the same. By replacing ϕ with the map $A \otimes B \mapsto \phi(A \otimes B^T)$, if necessary, we may assume that $\varphi_{i,X}$ is the identity map for all $i = 1, \ldots, m$ and unitary $X \in M_n$. It follows that

$$\phi(A \otimes B) = \varphi_1(A) \otimes B \quad \text{for all } A \in M_m,$$

where φ_1 is a linear map on M_m. Let $\widetilde{\gamma} = (\gamma_1, \ldots, \gamma_m)$. It is easy to verify that φ_1 is a linear map on M_m preserving $\widetilde{\gamma}$-norm. Hence, φ_1 also has the form $A \mapsto UAV$ or $A \mapsto UA^TV$ for some matrices $U, V \in \mathscr{U}_m$. This completes our proof. □

2.3 Multipartite system

We now consider the multipartite case.

Theorem 2.2. *Given an integer $m \geq 2$. Let $n_i \geq 2$ be integers for $i = 1, \ldots, m$ and $N = \prod_{i=1}^{m} n_i$. For any $\gamma = (\gamma_1, \ldots, \gamma_N) \in \mathbb{R}^N_{+,\downarrow}$ with $\gamma_2 > 0$, a linear map $\phi : M_N \to M_N$ satisfies*

$$\|\phi(A_1 \otimes \cdots \otimes A_m)\|_\gamma = \|A_1 \otimes \cdots \otimes A_m\|_\gamma \quad \textit{for all } A_i \in M_{n_i}, i = 1, \ldots, m, \tag{2.35}$$

if and only if there are unitary matrices $U, V \in M_N$ such that

$$\phi(A_1 \otimes \cdots \otimes A_m) = U(\varphi_1(A_1) \otimes \cdots \otimes \varphi_m(A_m))V \quad \textit{for all } A_i \in M_{n_i}, i = 1, \ldots, m,$$

where φ_i is the identity map or the transposition map $A \mapsto A^T$, for $i = 1, \ldots, m$.

Proof. The sufficiency part is clear. To prove the necessity part, we use induction on m. By Theorem 2.1, we already know that the statement of Theorem 2.2 holds for $m = 2$. So, we assume that $m \geq 3$ and the result holds for any $(m-1)$-partite system. We need to prove that the same is true for any m-partite system.

With the assumption for γ, we can conclude that there exists an integer $2 \leq k \leq N$ such that $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_k > 0 = \gamma_{k+1} = \cdots = \gamma_N$. Given any matrices $X_i \in \mathscr{U}_{n_i}, i = 1, \ldots, m$, we first claim that

$$\phi(X_1 E_{i_1 i_1} X_1^* \otimes \cdots \otimes X_m E_{i_m i_m} X_m^*) \perp \phi(X_1 E_{j_1 j_1} X_1^* \otimes \cdots \otimes X_m E_{j_m j_m} X_m^*) \tag{2.36}$$

for any distinct $(i_1, \ldots, i_m) \neq (j_1, \ldots, j_m)$.

Without loss of generality, we may assume that X_i are identity matrices for $i = 1 \ldots, m$. Then it is sufficient to show that for all $s = 1, \ldots, m$,

$$\phi\left(\bigotimes_{u=1}^{s-1}(E_{i_u i_u} + E_{j_u j_u}) \otimes E_{i_s i_s} \otimes \bigotimes_{u=s+1}^{m} E_{i_u i_u}\right) \perp \phi\left(\bigotimes_{u=1}^{s-1}(E_{i_u i_u} + E_{j_u j_u}) \otimes E_{j_s j_s} \otimes \bigotimes_{u=s+1}^{m} E_{i_u i_u}\right)$$

for any $\mathrm{i} = (i_1, \ldots, i_m)$ and $\mathrm{j} = (j_1, \ldots, j_m)$ with $i_u \neq j_u$, $1 \leq u \leq s$. We denote by $A_s(\mathrm{i}, \mathrm{j})$ and $B_s(\mathrm{i}, \mathrm{j})$ the above matrices accordingly. It is easy to check that

$$\|2A_s(\mathrm{i},\mathrm{j}) + (x+1)e^{i\theta}B_s(\mathrm{i},\mathrm{j})\|_\gamma = \|A_s(\mathrm{i},\mathrm{j}) + e^{i\theta}B_s(\mathrm{i},\mathrm{j})\|_\gamma + \|A_s(\mathrm{i},\mathrm{j}) + xe^{i\theta}B_s(\mathrm{i},\mathrm{j})\|_\gamma \tag{2.37}$$

for all $s = 1, \ldots, m$, $0 < x \leq 1$ and $\theta \in [0, 2\pi)$.

Case 1. Suppose that $k > 2^{m-1}$. For simplicity, denote $A_s = A_s(\mathrm{i}, \mathrm{j})$ and $B_s = B_s(\mathrm{i}, \mathrm{j})$. Let $h = \mathrm{rank}(A_s)$ and $x_0 = \min\left\{\frac{s_h(A_s)}{2s_1(B_s)}, \frac{1}{2}\right\}$. With (2.37), we apply the same argument in the proof of Assertion 2.1 to conclude that there exist matrices $U, V \in \mathscr{U}_N$ such that for some integer $0 \leq r \leq k$,

$$UA_sV = \mathrm{diag}(a_1, \ldots, a_r) \oplus 0_{k-r} \oplus \widetilde{A}_s \quad \text{and} \quad UB_sV = 0_r \oplus \mathrm{diag}(b_{r+1}, \ldots, b_k) \oplus \widetilde{B}_s, \tag{2.38}$$

with $a_j > 0$ for $j = 1, \ldots, r$ and $b_j \geq 0$ for $j = r+1, \ldots, k$, and

$$\|A_s + x_0 e^{i\theta_0} B_s\|_\gamma = \sum_{j=1}^{r} a_j \gamma_{\ell_j} + x_0 \sum_{j=r+1}^{k} b_j \gamma_{\ell_j}$$

$$\|A_s + e^{i\theta_0} B_s\|_\gamma = \sum_{j=1}^{r} a_j \gamma_{\ell_j} + \sum_{j=r+1}^{k} b_j \gamma_{\ell_j}$$

for some $\theta_0 \in [0, 2\pi)$ and permutation $(\ell_1, \ldots, \ell_k)$ of $(1, \ldots, k)$. With the assumption that $\phi : M_N \to M_N$ satisfies (2.35), we have

$$\|A_s + xB_s\|_\gamma = \sum_{j=1}^{2^{s-1}} \gamma_j + x \sum_{j=2^{s-1}+1}^{2^s} \gamma_j \quad \text{for all } 0 < x \le 1.$$

Notice that $\|A_s\|_\gamma = \sum_{j=1}^{2^{s-1}} \gamma_j$. It follows from the above three equations that

$$\sum_{j=1}^{r} a_j \gamma_{\ell_j} = \sum_{j=1}^{2^{s-1}} \gamma_j = \|A_s\|_\gamma \quad \text{and} \quad \sum_{j=r+1}^{k} b_j \gamma_{\ell_j} = \sum_{j=2^{s-1}+1}^{2^s} \gamma_j.$$

Since $k > 2^{m-1}$, we have $k \ge 2^{s-1}+1$, that is, $\gamma_{2^{s-1}+1} > 0$, for $s = 1, \dots, m$. Therefore, $\sum_{j=r+1}^{k} b_j \gamma_{\ell_j} = \sum_{j=2^{s-1}+1}^{2^s} \gamma_j > 0$. Then we can use the same argument in Assertion 2.1 to show that $\widetilde{A}_s = 0$, and therefore $A_s \perp B_s$ for all $s = 1, \dots, m$. Furthermore, by replacing U and V with $(I_k \oplus \hat{U})U$ and $V(I_k \oplus \hat{V})$ for some unitary matrices $\hat{U}, \hat{V} \in M_{N-k}$, the equations in (2.38) can be rewritten as

$$UA_sV = \mathrm{diag}(a_1, \dots, a_r) \oplus 0_{N-r} \quad \text{and} \quad UB_sV = 0_r \oplus \mathrm{diag}(b_{r+1}, \dots, b_N).$$

Case 2. Suppose that $k \le 2^{m-1}$. Then there exists an integer $1 \le s_0 \le m-1$ such that $2^{s_0-1} < k \le 2^{s_0}$. We can use the same argument in the *Case 1* to conclude that for any $\mathrm{i} = (i_1, \dots, i_m)$ and $\mathrm{j} = (j_1, \dots, j_m)$ with $i_u \ne j_u$, $1 \le u \le s$,

(2.a) $A_s(\mathrm{i},\mathrm{j}) \perp B_s(\mathrm{i},\mathrm{j})$ for all $s = 1, \dots, s_0$;

(2.b) There exist unitary matrices $U, V \in M_N$ such that

$UA_{s_0}(\mathrm{i},\mathrm{j})V = \mathrm{diag}(a_1, \dots, a_r) \oplus 0_{N-r}$ and $UB_{s_0}(\mathrm{i},\mathrm{j})V = 0_r \oplus \mathrm{diag}(b_{r+1}, \dots, b_N)$,

with $\|A_{s_0}(\mathrm{i},\mathrm{j}) + e^{i\theta_0} B_{s_0}(\mathrm{i},\mathrm{j})\|_\gamma = \sum_{t=1}^{r} a_t \gamma_{\ell_t} + \sum_{t=r+1}^{k} b_t \gamma_{\ell_t}$ for some $\theta_0 \in [0, 2\pi)$ and permutation $(\ell_1, \dots, \ell_k)$ of $(1, \dots, k)$.

Next, we use induction on s to show that for all $s = s_0, \ldots, m$, $\operatorname{rank}(A_s(\mathrm{i},\mathrm{j})) = 2^{s-1}$,

$$A_s(\mathrm{i},\mathrm{j}) \perp B_s(\mathrm{i},\mathrm{j}) \quad \text{and} \quad s_t(A_s(\mathrm{i},\mathrm{j})) = 1 \text{ for } t = 1, \ldots, 2^{s-1} \tag{2.39}$$

for all $\mathrm{j} = (j_1, \ldots, j_m)$, $\mathrm{i} = (i_1, \ldots, i_m)$ with $i_u \neq j_u$, $1 \leq u \leq s$. First, we claim that a_t and b_t obtained in (2.b) are not larger than one for $t = 1, \ldots, N$. Otherwise, we can conclude from (2.a) that the largest singular value of $\phi\Big(\bigotimes_{u=1}^{m} E_{i_u i_u}\Big)$ is larger than one for some $(i_1, \ldots, i_m)$, and thus $\Big\|\phi\Big(\bigotimes_{u=1}^{m} E_{i_u i_u}\Big)\Big\|_\gamma > \gamma_1$, contrary to (2.35). Therefore, our claim is correct. It follows that

$$\|A_{s_0}(\mathrm{i},\mathrm{j}) + e^{i\theta_0} B_{s_0}(\mathrm{i},\mathrm{j})\|_\gamma = \sum_{t=1}^{r} a_t \gamma_{\ell_t} + \sum_{t=r+1}^{k} b_t \gamma_{\ell_t} \leq \sum_{t=1}^{k} \gamma_t.$$

On the other hand, with (2.35), we have $\|A_{s_0}(\mathrm{i},\mathrm{j}) + e^{i\theta_0} B_{s_0}(\mathrm{i},\mathrm{j})\|_\gamma = \sum_{t=1}^{k} \gamma_t$, in other words, the above equality holds. This implies that $a_t = 1$ for $t = 1, \ldots, r$. Notice that

$$\|A_{s_0}(\mathrm{i},\mathrm{j})\|_\gamma = \sum_{t=1}^{2^{s_0-1}} \gamma_t. \tag{2.40}$$

If $r < 2^{s_0-1}$, then we have $\|A_{s_0}(\mathrm{i},\mathrm{j})\|_\gamma = \sum_{t=1}^{r} \gamma_t < \sum_{t=1}^{2^{s_0-1}} \gamma_t$; If $r > 2^{s_0-1}$, then we have $\|A_{s_0}(\mathrm{i},\mathrm{j})\|_\gamma = \sum_{t=1}^{r} \gamma_t > \sum_{t=1}^{2^{s_0-1}} \gamma_t$. Both of them are contrary to (2.35). Thus, we have $r = 2^{s_0-1}$. By now, we have showed that (2.39) holds for $s = s_0$.

Suppose that (2.39) holds for $s-1$ with $s - 1 \geq s_0$. With (2.37), we apply the same argument in Assertion 2.1 again to conclude that there exist unitary matrices $U, V \in M_N$ such that for some integer $0 \leq r \leq k$,

$$UA_s(\mathrm{i},\mathrm{j})V = \operatorname{diag}(a_1, \ldots, a_r) \oplus 0_{k-r} \oplus \widetilde{A}$$

$$UB_s(\mathrm{i},\mathrm{j})V = 0_r \oplus \operatorname{diag}(b_{r+1}, \ldots, b_k) \oplus \widetilde{B}$$

with $a_1 \geq \cdots \geq a_r > 0$ and $b_{r+1} \geq \cdots \geq b_k \geq 0$, and

$$\|A_s(\mathrm{i},\mathrm{j}) + e^{i\theta_1} B_s(\mathrm{i},\mathrm{j})\|_\gamma = \sum_{t=1}^{r} a_t \gamma_{\ell_t} + \sum_{t=r+1}^{k} b_t \gamma_{\ell_t} \tag{2.41}$$

for some $\theta_1 \in [0, 2\pi)$ and permutation $(\ell_1, \ldots, \ell_k)$ of $(1, \ldots, k)$. Notice that $A_s(\mathrm{i},\mathrm{j}) = A_{s-1}(\hat{\mathrm{i}},\hat{\mathrm{j}}) + B_{s-1}(\hat{\mathrm{i}},\hat{\mathrm{j}})$ for some $\hat{\mathrm{j}} = (\hat{j}_1, \ldots, \hat{j}_m)$, $\hat{\mathrm{i}} = (\hat{i}_1, \ldots, \hat{i}_m)$. Thus, with our assumption, we have

$$\mathrm{rank}(A_s(\mathrm{i},\mathrm{j})) = 2^{s-1} \quad \text{and} \quad s_t(A_s(\mathrm{i},\mathrm{j})) = 1 \text{ for } t = 1, \ldots, 2^{s-1}.$$

The same observation also holds for $B_s(\mathrm{i},\mathrm{j})$. It follows that $a_t = 1$ for $t = 1, \ldots r$ and $b_t = 1$ or 0 for $t = r+1, \ldots, k$. Then with (2.41), we use the same argument in last part of the proof of Assertion 2.2 to conclude that $\widetilde{A} \perp \widetilde{B}$, and therefore $A_s(\mathrm{i},\mathrm{j}) \perp B_s(\mathrm{i},\mathrm{j})$. By now, we can conclude that (2.39) holds for all $s = s_0, \ldots, m$. Therefore, $A_s(\mathrm{i},\mathrm{j}) \perp B_s(\mathrm{i},\mathrm{j})$ for all $s = 1 \ldots, m$. This proves our claim in (2.36).

It follows that for any unitary matrix $X_m \in M_{n_m}$, there exist unitary matrices U_{X_m} and V_{X_m} such that

$$\phi\left(\bigotimes_{i=1}^{m-1} E_{j_i j_i} \otimes X_m E_{j_m j_m} X_m^*\right) = U_{X_m}(E_{j_1 j_1} \otimes \cdots \otimes E_{j_m j_m}) V_{X_m}^* \tag{2.42}$$

for all $j_i = 1, \ldots, n_i$ with $1 \leq i \leq m$. For simplicity, we may assume that $U_I = V_I = I$, i.e.,

$$\phi(E_{j_1 j_1} \otimes \cdots \otimes E_{j_m j_m}) = E_{j_1 j_1} \otimes \cdots \otimes E_{j_m j_m} \tag{2.43}$$

It follows that $\phi(I_N) = I_N$. Applying a similar argument in Assertion 2.4, one can conclude from (2.42) and (2.43) that there are unitary matrices $W, \widetilde{W} \in M_N$ such that for any $1 \leq j_i \leq n_i$ with $1 \leq i \leq m-1$,

$$\phi\left(\bigotimes_{i=1}^{m-1} E_{j_i j_i} \otimes B\right) = W\left(\bigotimes_{i=1}^{m-1} E_{j_i j_i} \otimes \varphi_{j_1, \ldots, j_{m-1}}(B)\right)\widetilde{W},$$

where $\varphi_{j_1,\ldots,j_{m-1}}$ is the identity map or the transposition map. It follows that $\phi(I_N) = W\widetilde{W}$. Recall that $\phi(I_N) = I_N$ and W and $\widetilde{W}$ are both unitary matrices. Thus, we have $\widetilde{W} = W^*$. For any unitary matrices $X_i \in M_{n_i}$, $i = 1, \ldots, m-1$, denote $(X_1, \ldots, X_{m-1})$ by X, i.e., $X = (X_1, \ldots, X_{m-1})$. In particular, let $\mathrm{I} = (I_{n_1}, \ldots, I_{n_{m-1}})$. Following a similar argument as above, one can show that for any $X = (X_1, \ldots, X_{m-1})$ and $1 \leq j_i \leq n_i$ with $1 \leq i \leq m-1$, there exists a unitary matrix $W_X \in M_N$ such that

$$\phi\left(\bigotimes_{i=1}^{m-1} X_i E_{j_i j_i} X_i^* \otimes B\right) = W_X \left(\bigotimes_{i=1}^{m-1} E_{j_1 j_1} \otimes \varphi_{j_1,\ldots,j_{m-1},X}(B)\right) W_X^* \qquad (2.44)$$

for all $B \in M_{n_m}$, where $\varphi_{j_1,\ldots,j_{m-1},X}$ is the identity map or transposition map. For simplicity, we may further assume that $W_X = I_N$ when $X = (I_{n_1}, \ldots, I_{n_{m-1}})$, i.e., for any $1 \leq j_i \leq n_i$ with $1 \leq i \leq m-1$,

$$\phi\left(\bigotimes_{i=1}^{m-1} E_{j_i j_i} \otimes B\right) = \bigotimes_{i=1}^{m-1} E_{j_1 j_1} \otimes \varphi_{j_1,\ldots,j_{m-1},\mathrm{I}}(B) \quad \text{for all } B \in M_{n_m}, \qquad (2.45)$$

where $\varphi_{j_1,\ldots,j_{m-1},\mathrm{I}}$ is the identity map or the transposition map. Next we show that $\varphi_{j_1,\ldots,j_{m-1},\mathrm{I}}$ are the same. Considering all symmetric real matrix as in the proof of Assertion 2.4, one can conclude that there exists some unitary matrix $Z_X \in M_{n_1 \cdots n_{m-1}}$ such that

$$\phi\left(\bigotimes_{i=1}^{m-1} X_i E_{j_i j_i} X_i^* \otimes B\right) = Z_X \left(\bigotimes_{i=1}^{m-1} E_{j_1 j_1}\right) Z_X^* \otimes \varphi_{j_1,\ldots,j_{m-1},X}(B)$$

for all $B \in M_{n_m}$ and $1 \leq j_i \leq n_i$ with $1 \leq i \leq m-1$. Define linear maps $\mathrm{tr}_1 : M_N \to M_{n_m}$ and $\mathrm{Tr}_1 : M_N \to M_{n_m}$ by

$$\mathrm{tr}_1(A \otimes B) = \mathrm{tr}(A)B \quad \text{and} \quad \mathrm{Tr}_1(A \otimes B) = \mathrm{tr}_1(\phi(A \otimes B))$$

for all $A \in M_{n_1 \cdots n_{m-1}}$ and $B \in M_{n_m}$. Then

$$\mathrm{Tr}_1\left(\bigotimes_{i=1}^{m-1} X_i E_{j_i j_i} X_i^* \otimes B\right) = \varphi_{j_1,\dots,j_{m-1},X}(B).$$

Notice that Tr_1 is a linear and therefore continuous. Besides, the set

$$\left\{\bigotimes_{i=1}^{m-1} X_i E_{j_i j_i} X_i^* \mid 1 \le j_i \le n_i \text{ and } X_i \in \mathscr{U}_{n_i} \text{ for } i = 1, \dots, m-1\right\}$$
$$= \left\{\bigotimes_{i=1}^{m-1} x_i x_i^* \mid x_i \in \mathbb{C}^{n_i} \text{ with } x_i^* x_i = 1 \text{ for } i = 1, \dots, m-1\right\}$$

is connected. So, all the maps $\varphi_{j_1,\dots,j_{m-1},X}$ are the same. Denote the common map by φ_m, which is either the identity map or the transposition map. With the linearity of ϕ, we can conclude that for all $B \in M_{n_m}$ and $A_i \in M_{n_i}$ with $1 \le i \le m-1$,

$$\phi(A_1 \otimes \cdots \otimes A_{m-1} \otimes B) = \psi(A_1 \otimes \cdots \otimes A_{m-1}) \otimes \varphi_m(B),$$

where ψ is a linear map on $M_{n_1 \cdots n_{m-1}}$. Let $\hat{\gamma} = (\gamma_1, \dots, \gamma_{n_1 \cdots n_{m-1}})$. It is easy to check that

$$\|\psi(A_1 \otimes \cdots \otimes A_{m-1})\|_{\hat{\gamma}} = \|A_1 \otimes \cdots \otimes A_{m-1}\|_{\hat{\gamma}} \quad \text{for all } A_i \in M_{n_i},\ i = 1, \dots, m-1.$$

Hence, by the induction hypothesis, we conclude that there exist unitary matrices $\widetilde{U}, \widetilde{V}$ such that

$$\psi(A_1 \otimes \cdots \otimes A_{m-1}) = \widetilde{U}(\varphi_1(A_1) \otimes \cdots \otimes \varphi_{m-1}(A_{m-1}))\widetilde{V},$$

where φ_i is the identity map or the transposition map for $i = 1, \dots, m-1$. Then ϕ has the desired form and the proof is completed. □

Chapter 3

Linear maps preserving (p, k)-norms of tensor products of matrices

3.1 Introduction

In this chapter, we turn to the characterization of linear preservers for (p, k)-norms of tensor products of matrices. Recall that H_n denotes the set of $n \times n$ Hermitian matrices. For $A, B \in H_n$, we denote by $A \geq B$, or equivalently $B \leq A$, to mean that $A - B$ is positive semidefinite. In particular, $A \geq 0$ means that A is positive semidefinite. Let $1 \leq k \leq \min\{m, n\}$ be an integer and $1 \leq p \leq \infty$. Recall that the (p, k)-norm of $A \in M_{m,n}$ is defined by

$$\|A\|_{(p,k)} = \left[\sum_{i=1}^{k} s_i^p(A) \right]^{\frac{1}{p}}.$$

Clearly, the (p, k)-norm reduces to the spectral norm when $p = \infty$. In [22], Li and Tsing determined the form of linear preservers for (p, k)-norms on $M_{m,n}$. It was shown that such linear maps have the form

$$A \mapsto UAV \quad \text{or} \quad \text{when } m = n \ A \mapsto UA^T V$$

for some matrices $U \in \mathscr{U}_m$ and $V \in \mathscr{U}_n$. Notice that if $k = \min\{m, n\}$, then the (p, k)-norm reduces to the Schatten p-norm. In [6], the authors characterised the form of linear preservers for Schatten p-norms of tensor products of square matrices. We will extend this result to (p, k)-norms for $2 < p < \infty$. Our proof relies on some equalities, which do not hold for the case when $1 < p \leq 2$. So some other methods and techniques may be needed to tackle this case.

In the following sections, we first characterise linear preservers on bipartite system and then use induction on m to characterise corresponding linear preservers on m-partite system. Suppose that $A \in M_n$ is a positive semidefinite matrix. We denote the eigenvalues of A by $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$. Rearrange $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ in decreasing order as $x_{[1]} \geq \cdots \geq x_{[n]}$. Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Then x is said to weakly majorize y, denote by $x \succ_w y$, if

$$\sum_{i=1}^{k} x_{[i]} \geq \sum_{i=1}^{k} y_{[i]} \quad \text{for all } k = 1, \ldots, n.$$

Futhermore, if $x \succ_w y$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, then x is said to majorize y, denoted by $x \succ y$.

3.2 Bipartite system

Theorem 3.1. *Let $m, n, k \geq 2$ be integers with $k \leq mn$. Given a real number $2 < p < \infty$, a linear map $\phi : M_{mn} \to M_{mn}$ satisfies*

$$\|\phi(C \otimes D)\|_{(p,k)} = \|C \otimes D\|_{(p,k)} \quad \text{for all } C \in M_m \text{ and } D \in M_n, \tag{3.1}$$

if and only if there exist matrices $U, V \in \mathscr{U}_{mn}$ such that

$$\phi(C \otimes D) = U(\varphi_1(C) \otimes \varphi_2(D))V \quad \text{for all } C \in M_m \text{ and } D \in M_n, \tag{3.2}$$

where φ_s is the identity map or the transposition map $X \mapsto X^T$, for $s = 1, 2$.

To prove the theorem, we need some preliminary results. Notice that $x \mapsto x^\gamma$ $(x \geq 0)$ is a convex function for any real number $1 \leq \gamma < \infty$. With this, one can easily conclude the following lemma.

Lemma 3.1. *Let $a, b \in \mathbb{R}$. If $-a \leq b \leq a$. then for any real number $1 \leq \gamma < \infty$,*

$$(a+b)^\gamma + (a-b)^\gamma \geq 2a^\gamma.$$

We also need the following lemmas from [32, 39]

Lemma 3.2. [32, Lemma 2.1] *Let $A \in M_n$ be a positive semidefinite matrix. Then*

$$x^* A^\gamma x \geq (x^* A x)^\gamma \|x\|^{2(1-\gamma)} \quad \textit{for all } x \in \mathbb{C}^n \textit{ and } 1 \leq \gamma < \infty.$$

Lemma 3.3. [39, Lemma 3.7] *Let $A \in H_n$. Then*

$$\sum_{i=1}^{k} \lambda_i(A) = \max_{U^*U=I_k} \operatorname{tr}(U^*AU) \quad \textit{and} \quad \sum_{i=1}^{k} \lambda_{n-i+1}(A) = \min_{U^*U=I_k} \operatorname{tr}(U^*AU),$$

where I_k is the identity matrix of order k and $U \in M_{n,k}$.

Lemma 3.4. *Let $C, D \in H_n$ such that $-C \leq D \leq C$. Then for any real number $1 \leq \gamma < \infty$,*

$$\sum_{i=1}^{k} \lambda_i^\gamma(C+D) + \sum_{i=1}^{k} \lambda_i^\gamma(C-D) \geq 2 \sum_{i=1}^{k} \lambda_i^\gamma(C).$$

Proof. Let $U \in \mathscr{U}_n$ such that

$$UCU^* = \operatorname{diag}(\lambda_1(C), \lambda_2(C), \ldots, \lambda_n(C)).$$

Denote by u_i the i-th column of U for $i = 1, \ldots, n$. Let $\hat{U} = [u_1, u_2 \ldots, u_k]$. Then applying Lemma 3.3, we have

$$\sum_{i=1}^{k} \lambda_i^\gamma(C+D) \geq \operatorname{tr}(\hat{U}^*(C+D)^\gamma \hat{U}) \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i^\gamma(C-D) \geq \operatorname{tr}(\hat{U}^*(C-D)^\gamma \hat{U}).$$

Since $-C \leq D \leq C$, we have

$$C + D \geq 0, \quad C - D \geq 0,$$

and

$$-x^*Cx \leq x^*Dx \leq x^*Cx \quad \text{for all } x \in \mathbb{C}^n.$$

By Lemma 3.2, we have

$$u_i^*(C+D)^\gamma u_i \geq (u_i^*(C+D)u_i)^\gamma \quad \text{and} \quad u_i^*(C-D)^\gamma u_i \geq (u_i^*(C-D)u_i)^\gamma$$

for $i = 1, \ldots, n$. Applying Lemma 3.1 with $a = u_i^*Cu_i$ and $b = u_i^*Du_i$, we get

$$(u_i^*(C+D)u_i)^\gamma + (u_i^*(C-D)u_i)^\gamma \geq 2(u_i^*Cu_i)^\gamma \quad \text{for all } i = 1, \ldots, n.$$

It follows from the above inequalities that

$$\begin{aligned}
\sum_{i=1}^k \lambda_i^\gamma(C+D) + \sum_{i=1}^k \lambda_i^\gamma(C-D) \geq & \operatorname{tr}(\hat{U}^*(C+D)^\gamma \hat{U}) + \operatorname{tr}(\hat{U}^*(C-D)^\gamma \hat{U}) \\
= & \sum_{i=1}^k u_i^*(C+D)^\gamma u_i + \sum_{i=1}^k u_i^*(C-D)^\gamma u_i \\
\geq & \sum_{i=1}^k (u_i^*(C+D)u_i)^\gamma + \sum_{i=1}^k (u_i^*(C-D)u_i)^\gamma \\
\geq & 2\sum_{i=1}^k (u_i^*Cu_i)^\gamma = 2\sum_{i=1}^k \lambda_i^\gamma(C).
\end{aligned}$$

□

Corollary 3.1. *Let $2 < p < \infty$ be a real number and $A, B \in M_n$. Then*

$$\|A+B\|_{(p,k)}^p + \|A-B\|_{(p,k)}^p \geq 2\sum_{i=1}^k \lambda_i^{\frac{p}{2}}(A^*A + B^*B). \tag{3.3}$$

Proof. Notice that

$$\|A+B\|_{(p,k)}^p = \sum_{i=1}^{k} s_i^p(A+B) = \sum_{i=1}^{k} \lambda_i^{\frac{p}{2}}\big((A^*A+B^*B)+(A^*B+B^*A)\big)$$

and

$$\|A-B\|_{(p,k)}^p = \sum_{i=1}^{k} s_i^p(A-B) = \sum_{i=1}^{k} \lambda_i^{\frac{p}{2}}\big((A^*A+B^*B)-(A^*B+B^*A)\big).$$

Let $C = A^*A + B^*B$ and $D = A^*B + B^*A$. Then $C + D = (A+B)^*(A+B)$ and $C - D = (A-B)^*(A-B)$ are positive semidefinite. Applying Lemma 3.4, we get (3.3). □

Lemma 3.5. *Let $A, B \in M_n$ be nonzero matrices and $k \geq 2$ be an integer. Given a real number $0 < p < \infty$, if*

$$\|A+B\|_{(p,k)}^p = \|A\|_{(p,k)}^p + \|B\|_{(p,k)}^p \quad \textit{and} \quad A \perp B,$$

then $\operatorname{rank}(A+B) \leq k$.

Proof. With the assumption that $A \perp B$, we can suppose that the largest k singular values of $A + B$ are $s_1(A), \ldots, s_\ell(A), s_1(B), \ldots, s_{k-\ell}(B)$ for some $0 \leq \ell \leq k$. Then

$$\|A+B\|_{(p,k)}^p = \sum_{i=1}^{\ell} s_i^p(A) + \sum_{i=1}^{k-\ell} s_i^p(B) \leq \sum_{i=1}^{k} s_i^p(A) + \sum_{i=1}^{k} s_i^p(B). \tag{3.4}$$

On the other hand, $\|A+B\|_{(p,k)}^p = \|A\|_{(p,k)}^p + \|B\|_{(p,k)}^p = \sum_{i=1}^{k} s_i^p(A) + \sum_{i=1}^{k} s_i^p(B)$. Thus, the equality in (3.4) holds, which implies

$$\sum_{i=1}^{\ell} s_i^p(A) = \sum_{i=1}^{k} s_i^p(A) \quad \text{and} \quad \sum_{i=1}^{k-\ell} s_i^p(B) = \sum_{i=1}^{k} s_i^p(B).$$

Since A and B are both nonzero, we have $\sum_{i=1}^{k} s_i^p(A) > 0$ and $\sum_{i=1}^{k} s_i^p(B) > 0$. It follows that $\ell \geq 1$ and $k - \ell \geq 1$, i.e., $1 \leq \ell \leq k-1$, and

$$\sum_{i=\ell+1}^{k} s_i^p(A) = 0, \qquad \sum_{i=k-\ell+1}^{k} s_k^p(B) = 0.$$

This implies that $s_{\ell+1}(A) = 0$ and $s_{k-\ell+1}(B) = 0$. Then we can conclude that $\operatorname{rank}(A) \leq \ell$ and $\operatorname{rank}(B) \leq k - \ell$. Since $A \perp B$, this implies that $\operatorname{rank}(A+B) = \operatorname{rank}(A) + \operatorname{rank}(B) \leq \ell + k - \ell = k$. $\square$

Lemma 3.6. *Let $A, B \in M_n$ be positive semidefinite matrices and $1 < \gamma < \infty$ be a real number. Suppose*

$$\sum_{i=1}^{k} \lambda_i^\gamma(A + \alpha B) \leq \sum_{i=1}^{k} \lambda_i^\gamma(A) + \sum_{i=1}^{k} \lambda_i^\gamma(\alpha B) \quad \text{for all} \quad 0 < \alpha < 1, \tag{3.5}$$

and $U^*AU = \operatorname{diag}(\lambda_1(A), \ldots, \lambda_n(A))$ for some matrix $U \in \mathscr{U}_n$.

(a) *If $\lambda_k(A) = 0$, then $A \perp B$.*

(b) *If $\lambda_k(A) > 0$, then $U^*BU = 0_{k+\ell} \oplus \hat{B}$ with $\hat{B} \in M_{n-k-\ell}$, where ℓ is the largest integer such that $\lambda_{k+\ell}(A) = \lambda_k(A)$.*

Proof. Denote the i-th diagonal entry of U^*BU by b_i. Then $\lambda_i(A) + \alpha b_i$ is the i-th diagonal entry of $U^*(A + \alpha B)U$. It follows that

$$\big(\lambda_1(A + \alpha B), \ldots, \lambda_k(A + \alpha B)\big) \succ_w \big(\lambda_1(A) + \alpha b_1, \ldots, \lambda_k(A) + \alpha b_k\big).$$

Notice that $g(x) = x^\gamma$ $(x > 0)$ is an increasing convex function when $1 < \gamma < \infty$. We can apply the Theorem 3.26 in [39] to obtain

$$\big(\lambda_1^\gamma(A + \alpha B), \ldots, \lambda_k^\gamma(A + \alpha B)\big) \succ_w \big((\lambda_1(A) + \alpha b_1)^\gamma, \ldots, (\lambda_k(A) + \alpha b_k)^\gamma\big).$$

Thus, $\sum_{i=1}^{k} \lambda_i^{\gamma}(A+\alpha B) \geq \sum_{i=1}^{k} (\lambda_i(A)+\alpha b_i)^{\gamma}$. With the assumption in (3.5), we can conclude that

$$\sum_{i=1}^{k} (\lambda_i(A)+\alpha b_i)^{\gamma} \leq \sum_{i=1}^{k} \lambda_i^{\gamma}(A) + \sum_{i=1}^{k} \lambda_i^{\gamma}(\alpha B) \quad \text{for all} \quad 0<\alpha<1. \tag{3.6}$$

Let $f(\alpha) = \sum_{i=1}^{k} (\lambda_i(A)+\alpha b_i)^{\gamma} - \sum_{i=1}^{k} \lambda_i^{\gamma}(A) - \sum_{i=1}^{k} \lambda_i^{\gamma}(\alpha B)$ be a function on α. Then we have

$$f(\alpha) = f(0) + f^{'}(0)\alpha + o(\alpha) = \left[\sum_{i=1}^{k} \lambda_i^{\gamma-1}(A) b_i \gamma \right] \alpha + o(\alpha) \tag{3.7}$$

when α is sufficiently small. Since A and B are both positive semidefinite, we have $\lambda_i(A) \geq 0$ and $b_i \geq 0$ for all $i = 1, \ldots, n$. It follows that $\sum_{i=1}^{k} \lambda_i^{\gamma-1}(A) b_i \gamma \geq 0$. We claim that $\sum_{i=1}^{k} \lambda_i^{\gamma-1}(A) b_i \gamma = 0$. Otherwise, $\sum_{i=1}^{k} \lambda_i^{\gamma-1}(A) b_i \gamma > 0$ leads to $f(\alpha) > 0$ when $\alpha > 0$ is sufficiently small, which contradicts (3.6). It follows that

$$\lambda_i(A) b_i = 0 \quad \text{for } i = 1, \ldots, k.$$

For the case $\lambda_k(A) = 0$, we may assume that t is the largest integer such that $\lambda_t(A) > 0$. Then $U^*AU = \operatorname{diag}(\lambda_1(A), \ldots, \lambda_t(A)) \oplus 0_{n-t}$ and $b_i = 0$ for $i = 1, \ldots, t$. Recall that B is positive semidefinite. Thus, $U^*BU = 0_t \oplus \hat{B}$ with $\hat{B} \in M_{n-t}$. It follows that $A \perp B$.

For the case $\lambda_k(A) > 0$, we first have $b_i = 0$ for all $i = 1, \ldots, k$. Since B is positive semidefinite, it follows that $B = 0_k \oplus C$ with $C \in M_{n-k}$. Recall that ℓ is the largest integer such that $\lambda_{k+\ell}(A) = \lambda_k(A)$. If $\ell = 0$, then the proof is completed. If $\ell > 0$, then for any $i = k+1, \ldots, k+\ell$, replacing the role of $\lambda_k(A) + \alpha b_k$ with

$\lambda_i(A) + \alpha b_i$ in the above argument, we can conclude $b_i = 0$. Thus, we have $b_i = 0$ for $i = 1, \ldots, k + \ell$. It follows that $B = 0_{k+\ell} \oplus \hat{B}$ with $\hat{B} \in M_{n-k-\ell}$. □

Proof of Theorem 3.1. Since the sufficiency part is clear, we consider only the necessity part. So, suppose the linear map $\phi : M_{mn} \to M_{mn}$ satisfies (3.1), we will prove that ϕ has the form in (3.2) through the following 3 steps.

Step 1. For any matrices $X \in \mathscr{U}_m$ and $Y \in \mathscr{U}_n$,

$$\phi(XE_{ii}X^* \otimes YE_{jj}Y^*) \perp \phi(XE_{ii}X^* \otimes YE_{ss}Y^*)$$

and $\operatorname{rank}(\phi(XE_{ii}X^* \otimes Y(E_{jj} + E_{ss})Y^*)) \leq k$ for all $i = 1, \ldots, m$ and $j \neq s$.

For simplicity, we denote $\phi(XE_{ii}X^* \otimes YE_{jj}Y^*)$ and $\phi(XE_{ii}X^* \otimes YE_{ss}Y^*)$ by T and S, respectively. We aim to show that $T \perp S$ and $\operatorname{rank}(T + S) \leq k$. With the assumption in (3.1), we have

$$\|T + xS\|^p_{(p,k)} + \|T - xS\|^p_{(p,k)} = 2\|T\|^p_{(p,k)} + 2\|xS\|^p_{(p,k)} \quad \text{for all} \quad 0 < x < 1. \quad (3.8)$$

Applying Corollary 3.1 with $A = T$ and $B = xS$, we get

$$\|T + xS\|^p_{(p,k)} + \|T - xS\|^p_{(p,k)} \geq 2\sum_{i=1}^{k} \lambda_i^{\frac{p}{2}}(T^*T + x^2S^*S) \quad \text{for all} \quad 0 < x < 1. \quad (3.9)$$

Since $\|T\|^p_{(p,k)} = \sum_{i=1}^{k} \lambda_i^{\frac{p}{2}}(T^*T)$ and $\|xS\|^p_{(p,k)} = \sum_{i=1}^{k} \lambda_i^{\frac{p}{2}}(x^2S^*S)$, It follows from (3.8) and (3.9) that

$$\sum_{i=1}^{k} \lambda_i^{\frac{p}{2}}(T^*T + x^2S^*S) \leq \sum_{i=1}^{k} \lambda_i^{\frac{p}{2}}(T^*T) + \sum_{i=1}^{k} \lambda_i^{\frac{p}{2}}(x^2S^*S) \quad \text{for all} \quad 0 < x < 1. \quad (3.10)$$

Note the above observations also hold if (T, S) is replaced by (T^*, S^*), that is

$$\sum_{i=1}^{k} \lambda_i^{\frac{p}{2}}(TT^* + x^2SS^*) \leq \sum_{i=1}^{k} \lambda_i^{\frac{p}{2}}(TT^*) + \sum_{i=1}^{k} \lambda_i^{\frac{p}{2}}(x^2SS^*) \quad \text{for all} \quad 0 < x < 1. \quad (3.11)$$

Let $U, V \in \mathscr{U}_{mn}$ be matrices such that $V^*TU = \mathrm{diag}(s_1(T), \ldots, s_{mn}(T))$. Then we have

$$U^*T^*TU = \mathrm{diag}(s_1^2(T), \ldots, s_{mn}^2(T)) \quad \text{and} \quad V^*TT^*V = \mathrm{diag}(s_1^2(T), \ldots, s_{mn}^2(T)).$$

We claim that $s_k(T) = 0$. Otherwise, $s_k(T) > 0$. Then let ℓ be the largest integer such that $s_{k+\ell}(T) = s_k(T)$. Since $\lambda_i(T^*T) = \lambda_i(TT^*) = s_i^2(T)$ for all $i = 1, \ldots, mn$, we have $\lambda_k(TT^*) = \lambda_k(T^*T) > 0$ and ℓ is the largest integer such that $\lambda_{k+\ell}(T^*T) = \lambda_k(T^*T)$ and $\lambda_{k+\ell}(TT^*) = \lambda_k(TT^*)$. With (3.10) and (3.11), we can apply Lemma 3.6 twice to obtain

$$U^*S^*SU = 0_{k+\ell} \oplus C \quad \text{and} \quad V^*SS^*V = 0_{k+\ell} \oplus D$$

with $C, D \in M_{mn-k-\ell}$. It follows that

$$V^*SU = 0_{k+\ell} \oplus \hat{S}$$

with $\hat{S} \in M_{mn-k-\ell}$. Thus, there exists sufficiently small $x > 0$ such that the largest k singular values of $T + xS$ are $s_1(T), \ldots, s_k(T)$. Since $\|T\|_{(p,k)}^p = \|E_{ii} \otimes E_{jj}\|_{(p,k)}^p = 1$, this implies that

$$\|T + xS\|_{(p,k)}^p = \sum_{i=1}^{k} s_i^p(T + xS) = \sum_{i=1}^{k} s_i^p(T) = \|T\|_{(p,k)}^p = 1,$$

which contradicts the fact that

$$\|T + xS\|_{(p,k)}^p = \|E_{ii} \otimes (E_{jj} + xE_{ss})\|_{(p,k)}^p = 1 + x^p \quad \text{for all } 0 < x < 1.$$

So, our claim is correct, that is, $s_k(T) = 0$. Then we have $\lambda_k(T^*T) = \lambda_k(TT^*) = 0$. We can apply Lemma 3.6 twice to obtain $T^*T \perp S^*S$ and $TT^* \perp SS^*$. It follows that $T \perp S$. Notice that $\|T + S\|_{(p,k)}^p = \|T\|_{(p,k)}^p + \|S\|_{(p,k)}^p$. Then we can apply Lemma 3.5 to conclude that $\mathrm{rank}(T + S) \leq k$.

Step 2. For any matrices $X \in \mathscr{U}_m$ and $Y \in \mathscr{U}_n$,

$$\phi(XE_{ii}X^* \otimes Y(E_{jj} + E_{ss})Y^*) \perp \phi(XE_{tt}X^* \otimes (YE_{jj} + E_{ss})Y^*) \quad \text{whenever } i \neq t.$$

For simplicity, we denote $T = \phi(XE_{ii}X^* \otimes Y(E_{jj} + E_{ss})Y^*)$ and $S = \phi(XE_{tt}X^* \otimes Y(E_{jj} + E_{ss})Y^*)$. We aim to show that $T \perp S$. Applying Corollary 3.1 with $A = T$ and $B = xS$, we get

$$\|T + xS\|_{(p,k)}^p + \|T - xS\|_{(p,k)}^p \geq 2\sum_{i=1}^{k} \lambda_i^{\frac{p}{2}}(T^*T + x^2S^*S) \quad \text{for all } 0 < x < 1. \tag{3.12}$$

With the assumption in (3.1), we have

(i) $\|T + xS\|_{(p,k)}^p + \|T - xS\|_{(p,k)}^p = 2\|T\|_{(p,k)}^p + 2\|xS\|_{(p,k)}^p$ for the case $k \geq 4$;

(ii) $\|T + xS\|_{(p,k)}^p + \|T - xS\|_{(p,k)}^p = 2\|T\|_{(p,k)}^p + \|xS\|_{(p,k)}^p$ for the case $k = 3$;

(iii) $\|T + xS\|_{(p,k)}^p + \|T - xS\|_{(p,k)}^p = 2\|T\|_{(p,k)}^p$ for the case $k = 2$.

So we can conclude that for any integer $k \geq 2$,

$$\begin{aligned}\|T + xS\|_{(p,k)}^p + \|T - xS\|_{(p,k)}^p &\leq 2\|T\|_{(p,k)}^p + 2\|xS\|_{(p,k)}^p \\ &= 2\sum_{i=1}^{k} \lambda_i^{\frac{p}{2}}(T^*T) + 2\sum_{i=1}^{k} \lambda_i^{\frac{p}{2}}(x^2S^*S).\end{aligned} \tag{3.13}$$

It follows that

$$\sum_{i=1}^{k} \lambda_i^{\frac{p}{2}}(T^*T + x^2S^*S) \leq \sum_{i=1}^{k} \lambda_i^{\frac{p}{2}}(T^*T) + \sum_{i=1}^{k} \lambda_i^{\frac{p}{2}}(x^2S^*S) \quad \text{for all } 0 < x < 1. \tag{3.14}$$

The above observations also hold if (T, S) is replaced by (T^*, S^*), that is,

$$\sum_{i=1}^{k} \lambda_i^{\frac{p}{2}}(TT^* + x^2SS^*) \leq \sum_{i=1}^{k} \lambda_i^{\frac{p}{2}}(TT^*) + \sum_{i=1}^{k} \lambda_i^{\frac{p}{2}}(x^2SS^*) \quad \text{for all } 0 < x < 1. \tag{3.15}$$

If $s_k(T) = 0$, then we can use the same argument in Step 1 to conclude that $T^*T \perp S^*S$ and $TT^* \perp SS^*$, and hence $T \perp S$. Next, we consider the case when $s_k(T) > 0$. Notice that the result in Step 1 implies that $\text{rank}(T) \leq k$. Thus, there exist some matrices $U, V \in \mathscr{U}_{mn}$ such that

$$V^*TU = \text{diag}(s_1(T), \ldots, s_k(T)) \oplus 0_{mn-k}.$$

With (3.14) and (3.15), we can use the same argument in Step 1 to conclude that

$$V^*SU = 0_k \oplus \hat{S}$$

with $\hat{S} \in M_{mn-k}$. It follows that $T \perp S$.

Step 3. With the results in the first two steps, we have for any matrices $X \in \mathscr{U}_m$ and $Y \in \mathscr{U}_n$,

$$\phi(XE_{ii}X^* \otimes YE_{jj}Y^*) \perp \phi(XE_{rr}X^* \otimes YE_{ss}Y^*) \quad \text{for any } (i, j) \neq (r, s).$$

Then we can use the same argument in Assertion 2.4 of Chapter 2 to conclude that ϕ has the form in (3.2).

3.3 Multipartite system

Theorem 3.2. *Given $m \geq 2$. Let $n_i \geq 2$ be integers for $i = 1, \ldots, m$ and $N = \prod_{i=1}^{m} n_i$. Then for any given $2 < p < \infty$ and $k \geq 2$, a linear map $\phi : M_N \to M_N$ satisfies*

$$\|\phi(A_1 \otimes \cdots \otimes A_m)\|_{(p,k)} = \|A_1 \otimes \cdots \otimes A_m\|_{(p,k)} \quad \text{for all } A_i \in M_{n_i}, i = 1, \ldots, m, \tag{3.16}$$

if and only if there exist $U, V \in \mathscr{U}_N$ such that

$$\phi(A_1 \otimes \cdots \otimes A_m) = U(\varphi_1(A_1) \otimes \cdots \otimes \varphi_m(A_m))V \quad \text{for all } A_i \in M_{n_i}, i = 1, \ldots, m, \tag{3.17}$$

where φ_i is the identity map or the transposition map $A \mapsto A^T$, for $i = 1, \ldots, m$.

Proof. We use induction on m to prove Theorem 3.2. By Theorem 3.1, Theorem 3.2 obviously holds for $m = 2$. Thus, we may suppose that $m \geq 3$ and Theorem 3.2 holds for any $(m-1)$-partite system. Then we aim to show that Theorem 3.2 holds for any m-partite system.

We first show that for any $X_i \in \mathscr{U}_{n_i}$, $i = 1, \ldots, m$,

$$\phi(X_1 E_{i_1 i_1} X_1^* \otimes \cdots \otimes X_m E_{i_m i_m} X_m^*) \perp \phi(X_1 E_{j_1 j_1} X_1^* \otimes \cdots \otimes X_m E_{j_m j_m} X_m^*) \quad (3.18)$$

for any distinct $(i_1, \ldots, i_m) \neq (j_1, \ldots, j_m)$. Without loss of generality, we need only prove that (3.18) holds when X_i are identity matrices for $i = 1 \ldots, m$. It is sufficient to show that for all $s = 1, \ldots, m$,

$$\phi\left(\bigotimes_{u=1}^{s-1}(E_{i_u i_u} + E_{j_u j_u}) \otimes E_{i_s i_s} \otimes \bigotimes_{u=s+1}^{m} E_{i_u i_u}\right) \perp \phi\left(\bigotimes_{u=1}^{s-1}(E_{i_u i_u} + E_{j_u j_u}) \otimes E_{j_s j_s} \otimes \bigotimes_{u=s+1}^{m} E_{i_u i_u}\right) \quad (3.19)$$

for $\mathrm{i} = (i_1, \ldots, i_m)$ and $\mathrm{j} = (j_1, \ldots, j_m)$ with $i_u \neq j_u$, $1 \leq u \leq s$. Denote by $A_s(\mathrm{i},\mathrm{j})$ and $B_s(\mathrm{i},\mathrm{j})$ the two matrices in (3.19) accordingly. It is easy to check that for all $s = 1, \ldots, m$,

$$\|A_s(\mathrm{i},\mathrm{j}) + xB_s(\mathrm{i},\mathrm{j})\|_{(p,k)}^p + \|A_s(\mathrm{i},\mathrm{j}) - xB_s(\mathrm{i},\mathrm{j})\|_{(p,k)}^p \leq 2\|A_s(\mathrm{i},\mathrm{j})\|_{(p,k)}^p + 2\|xB_s(\mathrm{i},\mathrm{j})\|_{(p,k)}^p.$$

Then apply the same argument in the proof of Theorem 3.1, we have

$$\sum_{i=1}^{k} \lambda_i^{\frac{p}{2}}\left(A_s^*(\mathrm{i},\mathrm{j})A_s(\mathrm{i},\mathrm{j}) + B_s^*(\mathrm{i},\mathrm{j})B_s(\mathrm{i},\mathrm{j})\right) \leq \sum_{i=1}^{k} \lambda_i^{\frac{p}{2}}\left(A_s^*(\mathrm{i},\mathrm{j})A_s(\mathrm{i},\mathrm{j})\right) + \sum_{i=1}^{k} \lambda_i^{\frac{p}{2}}\left(x^2 B_s^*(\mathrm{i},\mathrm{j})B_s(\mathrm{i},\mathrm{j})\right) \quad (3.20)$$

and

$$\sum_{i=1}^{k} \lambda_i^{\frac{p}{2}}\big(A_s(\mathrm{i},\mathrm{j})A_s^*(\mathrm{i},\mathrm{j}) + B_s(\mathrm{i},\mathrm{j})B_s^*(\mathrm{i},\mathrm{j})\big)$$

$$\le \sum_{i=1}^{k} \lambda_i^{\frac{p}{2}}\big(A_s(\mathrm{i},\mathrm{j})A_s^*(\mathrm{i},\mathrm{j})\big) + \sum_{i=1}^{k} \lambda_i^{\frac{p}{2}}\big(x^2 B_s(\mathrm{i},\mathrm{j})B_s^*(\mathrm{i},\mathrm{j})\big) \quad (3.21)$$

for all $s = 1, \ldots, m$ and $\mathrm{i} = (i_1, \ldots, i_m)$ and $\mathrm{j} = (j_1, \ldots, j_m)$ with $i_u \neq j_u$, $1 \le u \le s$.

Case 1. Suppose that $k > 2^{m-1}$. For simplicity, we denote $A_s = A_s(\mathrm{i},\mathrm{j})$ and $B_s = B_s(\mathrm{i},\mathrm{j})$. Then

$$\|A_s + xB_s\|_{(p,k)}^p = 2^{s-1} + a_s x^p \quad \text{for } 0 < x < 1, \quad (3.22)$$

where $a_s = 2^{s-1}$ for $s = 1, \ldots, m-1$ and $a_m = \min\{k - 2^{m-1}, 2^{m-1}\}$. We claim that $s_k(A_s) = 0$ for all $s = 1, \ldots, m$. Otherwise, $s_k(A_s) > 0$ for some $1 \le s \le m$. Then with (3.20) and (3.21), we use the same argument in Step 1 to conclude that there exists sufficiently small $x > 0$ such that

$$\|A_s + xB_s\|_{(p,k)}^p = \|A_s\|_{(p,k)}^p = 2^{s-1},$$

which is contrary to (3.22). Thus, our claim is right, that is, $s_k(A_s) = 0$ for $s = 1, \ldots, m$. Then we can apply Lemma 3.6 to conclude that $A_sA_s^* \perp B_sB_s^*$ and $A_s^*A_s \perp B_sB_s^*$, and therefore $A_s \perp B_s$ for all $s = 1, \ldots, m$.

Case 2. Suppose that $k \le 2^{m-1}$. Let s_0 be the integer such that $2^{s_0-1} < k \le 2^{s_0}$. We can use the same argument in *Case 1* to show that

$$A_s(\mathrm{i},\mathrm{j}) \perp B_s(\mathrm{i},\mathrm{j}) \quad \text{and} \quad s_k(A_s(\mathrm{i},\mathrm{j})) = 0 \quad (3.23)$$

for all $s = 1, \ldots, s_0$, $\mathrm{i} = (i_1, \ldots, i_m)$ and $\mathrm{j} = (j_1, \ldots, j_m)$ with $i_u \neq j_u$, $1 \le u \le s$.

Next, we use induction on s to prove that for any $s = s_0+1, \ldots, m$, $\mathrm{i} = (i_1, \ldots, i_m)$ and $\mathrm{j} = (j_1, \ldots, j_m)$ with $i_u \neq j_u$, $1 \le u \le s$. There exist matrices $U, V \in \mathscr{U}_N$

depending on s and (i,j) such that

$$UA_s(\mathrm{i},\mathrm{j})V = I_{2^{s-1}} \oplus 0_{N-2^{s-1}} \quad \text{and} \quad A_s(\mathrm{i},\mathrm{j}) \perp B_s(\mathrm{i},\mathrm{j}). \tag{3.24}$$

First with (3.23), we have $A_{s_0}(\mathrm{i},\mathrm{j}) \perp B_{s_0}(\mathrm{i},\mathrm{j})$ and there exist matrices $U, V \in \mathscr{U}_N$ and integer $0 \le r < k$ such that

$$UA_{s_0}(\mathrm{i},\mathrm{j})V = \mathrm{diag}(a_1, \dots, a_r) \oplus 0 \quad \text{and} \quad UB_{s_0}(\mathrm{i},\mathrm{j})V = 0_r \oplus \mathrm{diag}(b_{r+1}, \dots, b_N),$$

where $a_1 \ge \cdots \ge a_r > 0$ and $b_{r+1} \ge \cdots \ge b_N \ge 0$. If $a_1 > 1$, then with (3.23), we have $s_1\Big(\phi\big(\bigotimes_{u=1}^{m} E_{i_u i_u}\big)\Big) > 1$ for some $(i_1, \dots, i_m)$. It follows that $\Big\|\phi\Big(\bigotimes_{u=1}^{m} E_{i_u i_u}\Big)\Big\|_{(p,k)} > 1$, contrary to (3.16). Thus, $a_1 \le 1$, and similarly $b_{r+1} \le 1$. It follows that

$$\sum_{j=1}^{r} a_j^p + \sum_{j=r+1}^{k} b_j^p \le k. \tag{3.25}$$

Clearly $a_1 \ge \cdots \ge a_r \ge xb_{r+1} \ge \cdots \ge xb_k$ are the largest k singular values of $A_{s_0}(\mathrm{i},\mathrm{j}) + xB_{s_0}(\mathrm{i},\mathrm{j})$ for all $0 < x \le \frac{a_r}{b_{r+1}}$. Thus, we have

$$\|A_{s_0}(\mathrm{i},\mathrm{j}) + xB_{s_0}(\mathrm{i},\mathrm{j})\|_{(p,k)}^p = \sum_{j=1}^{r} a_j^p + x^p \sum_{j=r+1}^{k} b_j^p \quad \text{for all } 0 < x \le \frac{a_r}{b_{r+1}}.$$

On the other hand, with (3.16), we have

$$\|A_{s_0}(\mathrm{i},\mathrm{j}) + xB_{s_0}(\mathrm{i},\mathrm{j})\|_{(p,k)}^p = 2^{s_0-1} + x^p(k - 2^{s_0-1}) \quad \text{for all } 0 < x \le 1.$$

It follows from the above two equations that $\sum_{j=1}^{r} a_j^p = 2^{s_0-1}$ and $\sum_{j=r+1}^{k} b_j^p = k - 2^{s_0-1}$.

Therefore, $\sum_{j=1}^{r} a_j^p + \sum_{j=r+1}^{k} b_j^p = k$, in other words, the equality in (3.25) holds, which implies that $a_j = 1$ for $j = 1, \dots, r$. Notice that $\|A_{s_0}(\mathrm{i},\mathrm{j})\|_{(p,k)}^p = 2^{s_0-1}$. Thus,

$r = 2^{s_0-1}$, that is, $UA_{s_0}(\mathrm{i},\mathrm{j})V = I_{2^{s_0-1}} \oplus 0_{N-2^{s_0-1}}$. By now, we have showed that (3.24) holds for s_0.

Suppose that (3.24) holds for $s-1$ with $s_0 < s \le m$. Then we will show that this also holds for s. Notice that $A_s(\mathrm{i},\mathrm{j}) = A_{s-1}(\hat{\mathrm{i}},\hat{\mathrm{j}}) + B_{s-1}(\hat{\mathrm{i}},\hat{\mathrm{j}})$ for some $\hat{\mathrm{i}} = (\hat{i}_1,\dots,\hat{i}_m)$ and $\hat{\mathrm{j}} = (\hat{j}_1,\dots,\hat{j}_m)$ with $\hat{i}_u \ne \hat{j}_u$, $1 \le u \le s-1$. Then with our assumption, we have

$$UA_s(\hat{\mathrm{i}},\hat{\mathrm{j}})V = I_{2^{s-1}} \oplus 0_{N-2^{s-1}} \quad \text{and} \quad UB_s(\hat{\mathrm{i}},\hat{\mathrm{j}})V = 0_{2^{s-1}} \oplus I_{2^{s-1}} \oplus 0_{N-2^s}$$

for some matices $U, V \in \mathscr{U}_N$. It follows that

$$UA_s(\mathrm{i},\mathrm{j})V = I_{2^s} \oplus 0.$$

Then with (3.20) and (3.21), we apply Lemma 3.6 twice to conclude that

$$UB_s(\mathrm{i},\mathrm{j})V = 0_{2^s} \oplus \hat{B}$$

for some $\hat{B} \in M_{N-2^s}$. It follows that $A_s(\mathrm{i},\mathrm{j}) \perp B_s(\mathrm{i},\mathrm{j})$. Now we have proved that (3.24) holds for s. Then we can conclude from the above discussion that for any $s = 1,\dots,m$,

$$A_s(\mathrm{i},\mathrm{j}) \perp B_s(\mathrm{i},\mathrm{j})$$

for all $\mathrm{i} = (i_1,\dots,i_m)$ and $\mathrm{j} = (j_1,\dots,j_m)$ with $i_u \ne j_u$, $1 \le u \le s$, that is, (3.18) holds. At last we can use the same argument in the last paragraph of the proof of Theorem 2.2 to conclude that ϕ has the form in (3.17) . This completes our proof. □

3.4 Rectangular case

We have characterised linear maps preserving (p,k)-norms of tensor products of square matrices with $2 < p < \infty$ in the above two sections. It is expected that our main results, Theorem 3.1 and Theorem 3.2, can be extended to the space of rectangular matrices. In fact, one can use the same argument in Step 1 and Step 2 of the proof of Theorem 3.1 to conclude the following result.

Lemma 3.7. *Let $m, n, \ell, t \geq 2$ be integers. If $\phi : M_{mn,\ell t} \to M_{mn,\ell t}$ is a linear map satisfying*

$$\phi(A \otimes B) = U(\varphi_1(A) \otimes \varphi_2(B))V \quad \text{for all } A \in M_{m,\ell} \text{ and } B \in M_{n,t},$$

then for any unitary matrices $X_1 \in M_m, X_2 \in M_\ell, Y_1 \in M_n$ and $Y_2 \in M_t$,

$$\phi(X_1 E_{ii} X_2 \otimes Y_1 E_{jj} Y_2) \perp \phi(X_1 E_{rr} X_2 \otimes Y_1 E_{ss} Y_2) \quad \text{for any } (i,j) \neq (r,s).$$

Chapter 4

Conclusion and future work

In this thesis, we have characterised linear maps preserving γ-norms or (p, k)-norms with $2 < p < \infty$ of tensor products of square matrices. It has been shown that such linear maps have the form

$$A_1 \otimes \cdots \otimes A_m \mapsto U \left(\psi_1(A_1) \otimes \cdots \otimes \psi_m(A_m)\right) V$$

where U and V are unitary matrices and ψ_s is either the identity map or the transpose map for $s = 1, \ldots, m$. It is expected that our main result on the (p, k)-norm can be extended to the space of rectangular matrices. However, our techniques used in the characterization of linear preservers for (p, k)-norms with $2 < p < \infty$ can not be applied to tackle the case when $1 < p < 2$. Notice that the (p, k)-norm and the γ-norms are both unitarily invariant norms. It is naturally to expect that linear maps preserving any unitarily invariant norm would have the above form. In the future, we will devote to the following problems.

- We will try to extend our results to the tensor products space of rectangular matrices, that is, to characterise linear maps preserving (p, k)-norms or γ-norms of tensor products of rectangular matrices.

- We will consider the linear preservers for (p, k)-norms of tensor products of matrices for $1 < p < 2$.

- Given any unitarily invariant norm $\| \cdot \|$. We will consider linear maps $\phi : M_{mn} \to M_{mn}$ such that $\|\phi(A \otimes B)\| = \|A \otimes B\|$ for all $A \in M_m$ and $B \in M_n$, and its extension to multipartite system.

Bibliography

[1] J. T. Chan, C. K. Li, and N. S. Sze. Isometries for unitarily invariant norms. *Linear Algebra and its Applications*, 399:53–70, 2005.

[2] W. S. Cheung and C. K. Li. Linear maps transforming the unitary group. *Canadian Mathematical Bulletin*, 46(1):54–58, 2003.

[3] W. L. Chooi and M. H. Lim. Linear preservers on triangular matrices. *Linear Algebra and its Applications*, 269(1-3):241–255, 1998.

[4] Y. Ding, A. Fošner, J. Xu, and B. Zheng. Linear maps preserving determinant of tensor products of hermitian matrices. *Journal of Mathematical Analysis and Applications*, 446(2):1139–1153, 2017.

[5] K. Fan. Maximum properties and inequalities for the eigenvalues of completely continuous operators. *Proceedings of the National Academy of Sciences of the United States of America*, 37(11):760, 1951.

[6] A. Fošner, Z. Huang, C. K. Li, and N. S. Sze. Linear maps preserving Ky Fan norms and schatten norms of tensor products of matrices. *SIAM Journal on Matrix Analysis and Applications*, 34(2):673–685, 2013.

[7] A. Fošner, Z. Huang, C. K. Li, and N. S. Sze. Linear preservers and quantum information science. *Linear and Multilinear Algebra*, 61(10):1377–1390, 2013.

[8] S. Friedland, C. K. Li, Y. T. Poon, and N. S. Sze. The automorphism group of separable states in quantum information theory. *Journal of mathematical physics*, 52(4):042203, 2011.

[9] G. Frobenius. Über die Darstellung der endlichen Gruppen durch lineare Substitutionen. S. B. *Deutsch Akad. Wiss. Berlin,* 994-1015, 1897.

[10] R. Grone. The invariance of partial isometries. *Indiana University Mathematics Journal*, 28(3):445–449, 1979.

[11] R. Grone and M. Marcus. Isometries of matrix algebras. *Journal of Algebra*, 47(1):180–189, 1977.

[12] L. Gurvits. Classical deterministic complexity of edmonds' problem and quantum entanglement. In *Proceedings of the thirty-fifth annual ACM symposium on Theory of computing*, pages 10–19, 2003.

[13] F. Hiai. Similarity preserving linear maps on matrices. *Linear Algebra and its Applications*, 97:127–139, 1987.

[14] R. A. Horn, C. K. Li, and N. K. Tsing. Linear operators preserving certain equivalence relations on matrices. *SIAM Journal on Matrix Analysis and Applications*, 12(2):195–204, 1991.

[15] Z. Huang, S. Shi, and N. S. Sze. Linear rank preservers of tensor products of rank one matrices. *Linear Algebra and its Applications*, 508:255–271, 2016.

[16] C. R. Johnson and S. Pierce. Linear maps on hermitian matrices: The stabilizer of an inertia class, ii. *Linear and Multilinear Algebra*, 19(1):21–31, 1986.

[17] C. K. Li. Matrices with some extremal properties. *Linear Algebra and its Applications*, 101:255–267, 1988.

[18] C. K. Li, Y. T. Poon, and N. S. Sze. Linear preservers of tensor product of unitary orbits, and product numerical range. *Linear Algebra and its Applications*, 438(10):3797–3803, 2013.

[19] C. K. Li, L. Rodman, and P. Šemrl. Linear transformations between matrix spaces that map one rank specific set into another. *Linear Algebra and its Applications*, 357(1-3):197–208, 2002.

[20] C. K. Li, P. Šemrl, and A. R. Sourour. Isometries for ky-fan norms on block triangular matrix algebras. *Archiv der Mathematik*, 81(2):175–181, 2003.

[21] C. K. Li and N. K. Tsing. Duality between some linear preserver problems. ii. isometries with respect to c-special norms and matrices with fixed singular values. *Linear Algebra and its Applications*, 110:181–212, 1988.

[22] C. K. Li and N. K. Tsing. Some isometries of rectangular complex matrices. *Linear and Multilinear Algebra*, 23(1):47–53, 1988.

[23] C. K. Li and N.-K. Tsing. Linear operators preserving unitarily invariant norms of matrices. *Linear and Multilinear Algebra*, 26(1-2):119–132, 1990.

[24] C. K. Li and N. K. Tsing. Linear preserver problems: A brief introduction and some special techniques. *Linear Algebra and its Applications*, 162-164:217–235, 1992.

[25] M. H. Lim. A note on similarity preserving linear maps on matrices. *Linear Algebra and its Applications*, 190:229–233, 1993.

[26] M. H. Lim. Additive preservers of tensor product of rank one Hermitian matrices. *The Electronic Journal of Linear Algebra*, 23:356-374, 2012.

[27] M. H. Lim. A note on linear preservers of certain ranks of tensor products of matrices. *Linear and Multilinear Algebra*, 63(7):1442–1447, 2015.

[28] M. Marcus. All linear operators leaving the unitary group invariant. *Duke Mathematical Journal*, 26(1):155–163, 1959.

[29] M. Marcus and W. R. Gordon. A generalization of the unitary group. *Linear Algebra and its Applications*, 3(2):225–247, 1970.

[30] M. Marcus and B. N. Moyls. Linear transformations on algebras of matrices. *Canadian Journal of Mathematics*, 11:61–66, 1959.

[31] M. Marcus and R. Purves. Linear transformations on algebras of matrices: The invariance of the elementary symmetric functions. *Canadian Journal of Mathematics*, 11:383–396, 1959.

[32] C. A. McCarthy. C_p. *Israel Journal of Mathematics*, 5(4):249–271, 1967.

[33] K. Morita. Analytical characterization of displacements in general poincare space. *Proceedings of the Imperial Academy*, 17(10):489–494, 1941.

[34] T. Petek. A note on unitary similarity preserving linear mappings on $B(H)$. *Linear Algebra and its Applications*, 394:217–224, 2005.

[35] S. Pierce and W. Watkins. Invariants of linear maps on matrix algebras. *Linear and Multilinear Algebra*, 6(3):185–200, 1978.

[36] B. Russo. Trace preserving mappings of matrix algebras. *Duke Mathematical Journal*, 36(2):297–300, 1969.

[37] I. Schur. Einige Bemerkungen zur Determinantentheorie. *Sitzungsber. Preuss. Akad. Wiss. Berlin,* 25:454-463, 1925.

[38] M. Sugawara. On the general schwarzian lemma. *Proceedings of the Imperial Academy*, 17(10):483–488, 1941.

[39] X. Zhan. *Matrix theory*, Volume 147. American Mathematical Soc., 2013.

[40] B. Zheng, J. Xu, and A. Fošner. Linear maps preserving rank of tensor products of matrices. *Linear and Multilinear Algebra*, 63(2):366–376, 2015.