QUALITY FUNCTION DEPLOYMENT OPTIMIZATION FROM GAME-THEORETIC AND FUZZY PERSPECTIVES

YUNWEN MIAO

PhD

The Hong Kong Polytechnic University

2021

The Hong Kong Polytechnic University
Department of Industrial and Systems Engineering

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Yunwen MIAO

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy

August 2020

CERTIFICATE OF ORIGINALITY

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_____Yunwen MIAO_____(Name of student)

Abstract

In regard to the widely applied customer-centric product design nowadays, quality function deployment (QFD) can be viewed as one of the effective and efficient tools to interpret customer requirements (CRs) of a certain product to engineering characteristics (ECs) in the manufacturing aspect. Generally, the competitiveness and customer satisfaction of the product are expected to be enhanced after implementing the QFD optimization procedure. Based on the traditional QFD optimization framework, this research attempts to conduct product development from the perspectives of cooperative games and fuzzy uncertainty, respectively.

For a certain manufacturing product, several CRs and ECs are selected in the QFD procedure for optimization. Commonly, the respective importance weights of CRs and ECs, and target values or target levels of ECs are significant research points, which aim at maximizing the overall customer satisfaction of the product under limited resources. Firstly, the angle of a two-stage cooperative game integrating a quantitative Kano's model in QFD is hardly considered in the previous literature. More specifically, Shapley value is utilized to obtain the CR relative importance weights, while Nash bargaining is applied to the objective function in a deterministic optimization model to attain target values of ECs.

Secondly, as far as the QFD optimization under the fuzzy perspective is concerned, some novel derivations on calculations for expected values of different fuzzy events expressed by α-optimistic values of fuzzy variables are given. Therefore, the fuzzy importance of ECs can be measured, and the expected return of

the fuzzy objective and several expected constraints of a fuzzy optimization model can be transformed into more simplified ones. On this basis, an improved hybrid intelligent algorithm (iHIA), which consists of a novel fuzzy simulation technique for the expected value of fuzzy events and a genetic algorithm, is proposed to solve the simplified model.

Thirdly, in order to accomplish the novel fuzzy simulation procedure in the iHIA, a series of improved fuzzy simulation techniques are generated. At the beginning, a new operational law regarding membership functions of continuous and strictly monotone functions of regular fuzzy numbers or intervals is set forth. As a consequence, several novel fuzzy simulation techniques for the possibility and expected value of fuzzy events are successively raised as a theoretical basis. Another enhancement on the expected value simulation is based on the analytical expressions of α-optimistic values of fuzzy variables.

On the whole, by applying the proposed QFD optimization methods from two perspectives, research outcomes of CRs and ECs provide useful guidelines, suggestions, and managerial implications for the decision-makers. The implementation of the methods to the case study of a notebook development in this thesis can also be extended to other manufacturing products. Meanwhile, the theoretical improvements on fuzzy simulation will also make contributions to the development of fuzzy theories.

Acknowledgements

It has been 10 years since I went to college, and now in this very special 2020, I expect to finish my Ph.D. study and want to give a satisfactory closure to my student days. Someday in the future, when I look back, I hope that I can still find the original intention and passion for academic researches.

During this three-year period, I have encountered hardship, confusion, and of course, happiness. It has been a great honor to have Prof. Ping Ji as my chief supervisor. Most of the time, he fulfills his duty as a mentor that guides me and encourages me with many valuable insights and suggestions on researches. And sometimes, he acts like a parent or a friend to help me solve problems in life. I deeply appreciate his academic guidance and great kindness, and meanwhile I will always treasure the memories and laughters we share when climbing mountains. At the same time, I express my sincere thanks to my previous supervisors in Shanghai University, Prof. Yizeng Chen, and Prof. Jian Zhou. They have laid a solid foundation for my progress on the academic path.

An unforgettable highlight in this journey is the half-year attachment in the Department of Management and Marketing of the University of Melbourne. There, my host supervisor, Dr. William Ho provides me with many new knowledge and enlightenment in the supply chain area, which I am not familiar with before. In addition, it is so fortunate for me to get acquainted with several like-minded friends, which makes my exchange life full of color and without loneliness. Certainly, all these gains should credit to the student stipend and the attachment subsidy offered by the research committee of The Hong Kong Polytechnic University.

The last but not the least, I would like to express heartful thanks for my parents and grandma's selfless dedication and support. I am also very grateful for my boyfriend, who gives me a strong psychological backup. Meanwhile, thanks should be conveyed to some important companions on this academic path, Dr. Jian Jin, Dr. Ying Cheng, Dr. Xin Ma, Dr. Dan Zhuge, Dr. Lei Chen, Dr. Jiage Huo, and Ph.D. candidates, Xiajie Yi, Tiantian Chen, Ruojue Lin, Ming Gao, Songman Wu, Tingting Han, Mingxuan Zhao, and all the colleagues in Room DE404.

It starts from DE404, and also ends here. May all have a bright future.

Acronyms

Acronyms for algorithms
SDS: stochastic discretization simulation
SDA: stochastic discretization algorithm
HIA: hybrid intelligent algorithm
UDA: uniform discretization algorithm
NIA-G: general numerical integration algorithm
UDS: uniform discretization simulation
UDS-Joint: extended version of the UDS for the possibility of joint fuzzy events
iSDA: improved stochastic discretization algorithm
SDA*: an intermediate algorithm between the SDA and iSDA
NIA-S: special numerical integration algorithm
TiSDA: extended version of the iSDA for regular fuzzy intervals
TNIA-S: extended version of the NIA-S for regular fuzzy intervals
SVCA: Shapley value calculation algorithm
VFCA: value function calculation algorithm
iHIA: improved hybrid intelligent algorithm
Acronyms for fuzzy variables
$\mathcal{T}(a,b,c)$: triangular fuzzy number (TFN)
$\mathcal{N}(c,\sigma)$: normal fuzzy number
$\mathcal{G}(c,b)$: Gaussian fuzzy number
$\mathcal{A}(a,b,c,d)$: trapezoidal fuzzy number (TpFN)
$\mathcal{B}(a,b,c,d), \boldsymbol{c}(a,b,c,d)$: two specified regular fuzzy intervals
Other acronyms in this thesis
QFD: quality function deployment
HoQ: house of quality
CR: customer requirement
EC: engineering characteristic
CS/DS: customer satisfaction/dissatisfaction
OCS: overall customer satisfaction
EVM: expected value model

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Chapter 1

Introduction

1.1 Background

Many consumer products have undergone the evolution from the handicraft age, to the commonly seen mass production nowadays and even a small batch of customization. It is not difficult to explore the underlying reason for this unavoidable tendency. That is, during these three stages, the center of manufacturing has shifted from the product to the customer gradually, and the participation of customers is also enhanced [Tao18]. It implies that at present manufacturers should focus more on customers' multiple and diversified needs on products [Hau88, Gri93]. The fulfillment degree of these expectations and perceptions will directly affect the final customer satisfaction, which in turn determines whether the product becomes a business success.

Therefore, for the sake of grasping targeted customers and occupying market share in the long term, under today's free market economies, manufacturers should generate rapid productions by following customer needs accurately and iteratively [Efe20]. Meanwhile, fierce competition emerges in the quality of products, and also in the supply chain and inventory management. High-level product development becomes imperative for companies to gain superiority and competitiveness. Under this circumstance, systematic approaches should be initiated to assist manufacturers in coping with these upcoming opportunities and challenges.

As far as the mainstream customer-centric product design is concerned, quality function deployment (QFD) proposed by Akao [Aka90] is regarded as a comprehensive and systematic method. It helps interpret "whats - customers' voices" to "hows - design attributes" to further improve the target product's performance. This useful tool was originated from Japan during the 1960s and later gained extensive support and acknowledgment worldwide. Many applications of QFD can be seen in the areas of product design, quality management, decision making, and team building, etc [Wol09]. QFD is also adopted by many renowned companies in automobile, electronics, appliances, and garment industries [Xio09], such as Haima Automobile, China Telecom, Kindee software, and Midea air conditioner.

A crucial step before implementing QFD is how to accurately identify and understand customers' voices, that is, customer requirements (CRs) [Gri93]. The method of dealing with CRs is not standardized, and Kano's model [Kan84] can be a solution to this essential obstacle to manufacturers. According to different features of customers' voices, these needs are categorized into the major five types, i.e., One-dimensional (O), Attractive (A), Must-be (M), Indifferent (I), and Reverse (R). The former three kinds are primarily identified in Kano *et al.* [Kan84]'s research, which were depicted by straight lines or curves. Currently, the tendency of qualitative analyses based on Kano's model has turned to quantitative analyses by forming linear or non-linear functions or regressions [Che04, Liu14]. And some quantitative Kano's models are aggregated with QFD so as to further modify customers' voices in a more analytical way [Ber93, Tin02, Wan10, Fin11]. A combination of Kano's model with cooperative game theory is another direction [Con04, Con05]. As a consequence, the underlying perceptions of customers can be observed and reached, even the pain points of a certain product can be identified.

Technically, during the course of the QFD optimization procedure, CRs of a certain product together with their relative importance weights are translated into several engineering characteristics (ECs) in the manufacturing aspect. To complete this transformation, a diagram named house of quality (HoQ) was put forward by Hauser and Clausing [Hau88], whose layout resembles a house with walls, rooms, and a roof. Critical matrices regarding CRs and ECs, along with the competitor information and EC technical values are recorded in the HoQ. CRs and their corresponding importance weights are displayed on the left wall of the HoQ, and these weights are usually obtained from external rating methods like analytic hierarchy process (AHP) or analytic network process (ANP) [Chu01, Zai14]. After the weights of CRs are derived, with the aid of all the data listed in the HoQ, deterministic, fuzzy or uncertain mathematical models can be built to optimize the current design of the target product. Then, target values of ECs are settled to realize the ultimate objective of obtaining a maximal degree of customer satisfaction [Che04, Che05, Jip14, Mia17].

Notably, Kano's model and QFD are two beneficial customer-driven quality tools, and in this research the former will be embedded in the latter. Both of them play significant roles when dealing with CRs and ECs of a certain manufacturing product. In order to get closer to real-life applications and achieve research improvements, the QFD optimization will be conducted from two different angles, i.e., game-theoretic and fuzzy perspectives, respectively. The detailed motivation of these two perspectives are elaborated in the forthcoming section.

1.2 Motivation

The first is the QFD optimization based on the perspective of cooperative game theory. Here, the research gap is clarified. Although Kano's model, QFD, and cooperative game theory are respectively mature in both theory and practice, the combination of these three approaches was less studied. On this basis, the research in this part attempts to imitate the collaboration relationship among several CRs by using a rigorous mathematical tool, cooperative game theory. Whereas in traditional researches, this collaboration relationship among CRs was not specified and clearly pointed out. And usually, these CRs were treated individually or were implemented by pairwise comparisons in the AHP and ANP related methods in the importance rating [Chu01, Zai14].

More specifically, Shapley value and Nash bargaining adopted in this part are familiar concepts in cooperative game theory. Nevertheless, the integration of them with the whole QFD procedure from the CR weighting to the EC target value determining is hardly considered. This novel cooperative game-theoretic angle will endow both Shapley value and Nash bargaining with practical significance in the QFD product planning.

The other is the QFD optimization based on the perspective of fuzzy theories. Such combinations were common in previous literature. Due to the reason that experts' evaluations in the matrices of the HoQ are usually expressed by linguistic variables, fuzzy variables seem better than crisp values to describe these subjective assessments [Che05, Zho14]. Subsequently, optimization models with fuzzy parameters are formulated to derive target levels of ECs. In order to solve the models, it is an alternative to transform the fuzzy objective function and constraints into deterministic ones, e.g., fuzzy expected value models (EVM) [Che05, Mia17]. In some simple cases, the optimal solution of the fuzzy EVM can be directly obtained via analytical calculations. In some intricate cases, the fuzzy EVM is needed to be solved by a heuristic algorithm, e.g., the *hybrid intelligent algorithm* (HIA) proposed by Liu [Liu02a]. The HIA consists of a fuzzy simulation process for expected values of functions of fuzzy variables and a genetic algorithm.

The original fuzzy simulation process in the HIA is discovered to have some deficiencies during the actual operation. Therefore, in this research, firstly, the fuzzy simulation of both the possibility and expected value of functions of fuzzy variables are improved based on a newly raised operational law, and a series of novel algorithms are put forward. In addition, an improved HIA equipped with a novel fuzzy simulation technique is used to solve the fuzzy EVM in QFD. It is noted that the improved HIA is also applicable to other optimization models in the fuzzy environment.

1.3 Research Objectives

Currently, customer-centric manufacturing products still occupy a dominant position in the market. Many products are upgraded based on their former versions, such as mobile phones and automobiles. The main factors that trigger these upgrades are those diversified and changeable customer needs towards the target product. Meanwhile, with the rapid growth of the Internet technology and E-commerce, a variety of online shopping websites or online communities emerge, which provide platforms for customers to express their views on products.

In order to adapt to the above situation, the main objective of the research in this dissertation is to understand and analyze customer needs and to enhance customer satisfaction degree of the target product by implementing the QFD optimization procedure. On this basis, the target product is expected to acquire considerable competitiveness in products of the same kind. Notably, the target product here can be either the end user products or industry products. To achieve this goal, the QFD optimization procedure will be studied from two different perspectives, i.e., game-theoretic and fuzzy perspectives. The three objectives of this research are described as follows:

(1) *To generate theoretical and practical improvements in the fuzzy simulation area, from the possibility to the expected value of fuzzy events.* At the theoretical level, the existing fuzzy simulation techniques for the possibility and expected

value of fuzzy events are proved to be defective in their stochastic sampling process. Therefore, novel fuzzy simulation techniques are proposed based on a newly raised operational law. At the practical level, the original HIA is further revised by employing a novel fuzzy simulation technique, which can be utilized to solve fuzzy optimization models of real-life applications.

(2) *To provide a systematic and effective guideline by Kano's model, QFD, and cooperative game theory for manufacturers to follow in improving the target product.* From the cooperative game-theoretic perspective, the marginal contributions and bargaining among several CRs are considered. Kano's model runs through the entire QFD process, which includes analyzing critical CRs together with their relative importance weights, and computing target values of ECs through a deterministic optimization model of maximizing the overall customer satisfaction.

(3) *To propose a general methodological framework in the integration researches of Kano's model with QFD in the fuzzy environment for manufacturers.* In this regard, fuzzy variables are utilized to describe subjective evaluations inside the HoQ of the QFD process. The fuzzy importance of ECs are ranked through their expected values. Afterwards, a fuzzy expected value model integrating Kano's model is formulated, whose objective function is also to maximize the overall customer satisfaction, and decision variables are target levels of ECs. The expected return of the objective function and some expected constraints are further derived, so as to be solved by an improved HIA.

1.4 Dissertation Overview

This dissertation is composed of two major parts, i.e., theoretical improvements in fuzzy simulation, and QFD optimizations with crisp and fuzzy parameters. The detailed structure is arranged as follows:

Chapter 2 firstly reviews some basic concepts of fuzzy arithmetic including fuzzy uncertainty, fuzzy variables and fuzzy measures. Then, the relevant literature regarding Kano's model, QFD, and the integration researches of these two quality tools with some methods are discussed, respectively.

Chapters 3 and 4 aim at theoretical improvements on fuzzy simulation techniques for the possibility and expected value of fuzzy events, respectively. In Chapter 3, originally, a *stochastic discretization simulation* (SDS) [Liu98a] was generated to deal with the possibility simulation. However, it is proved that the stochastic sampling process inside the SDS does not strictly follow Zadeh's extension principle, which unavoidably leads to inaccurate simulation results. To overcome this drawback, a new operational law is initiated to specify membership functions of continuous and strictly monotone functions of regular fuzzy intervals. On this basis, a *uniform discretization algorithm* (UDS) is proposed to approximate the possibility of individual fuzzy event. Furthermore, the UDS is extended to the UDS-Joint to simulate the possibility of joint fuzzy events. Several numerical examples are conducted to illustrate the deficiency of the SDS, and the effectiveness of the UDS and the UDS-Joint.

Chapter 4 focuses on the improvements on fuzzy simulation techniques for expected values of continuous and strictly monotone functions of fuzzy variables based on two existing fuzzy simulation techniques. The new algorithms raised in this chapter are called the *improved stochastic discretization algorithm* (iSDA), and the *special numerical integration algorithm* (NIA-S), respectively. The original SDA [Liu02b] shares the same stochastic sampling process with the SDS, and the drawback of this sampling method is explained detailedly in Chapter 3. The iSDA is also initiated in the light of the new operational law set forth in Chapter 3, which revises two deficiencies in the original SDA. Subsequently, the NIA-S is proposed to simplify the bisection procedure of the original NIA [Lix15]. Since the iSDA and NIA-S are designed for regular fuzzy numbers, after a series of theorems for regular fuzzy intervals are proved, both algorithms are extended to approximate expected values of functions of regular fuzzy intervals with new efforts, i.e., the TiSDA and TNIA-S. The feasibility and effectiveness of all the proposed algorithms are validated through numerical examples, from which the superiority of both the iSDA and NIA-S over others are conspicuously displayed in aspects of accuracy, stability, and efficiency.

Chapters 5 and 6 target on QFD optimizations with crisp and fuzzy parameters, respectively. In Chapter 5, cooperative games are incorporated in QFD in two sequential stages. The first stage is to obtain the CR relative importance weights by applying customer satisfaction and dissatisfaction values of Kano's model to the Shapley value calculations. This procedure is simplified by proposing two novel algorithms according to Conklin *et al.* [Con04, Con05]. The second stage is to formulate a mixed integer non-linear programming model, whose objective function is a Nash bargaining function which contains the CR weights and a quantitative Kano's model presented in [Wan10, Jip14]. The model is designed to obtain the maximal overall customer satisfaction and derive target values of ECs. Finally, the proposed two-stage cooperative game is implemented to an illustrative example of a notebook computer development to demonstrate its performance. Target values of ECs are settled according to customer perceptions, and some discussions and managerial implications are addressed for the decision-makers.

Chapter 6 conducts the QFD optimization procedure for a certain manufacturing product in the fuzzy environment. The CR importance weights and relationships between CRs and ECs are evaluated by linguistic variables, which are expressed by trapezoidal fuzzy numbers (TpFNs). The procedure starts from the ranking of the fuzzy importance of ECs, to the determination of target levels of ECs. The expected value of fuzzy variables and fuzzy events play an important role no matter in calculating the expected value of the fuzzy importance of ECs, or the establishment of a fuzzy expected value model (EVM). In order to solve the model, the objective function and constraints in the fuzzy EVM are further derived into simplified ones with the aid of the definitions and theorems on regular fuzzy intervals in Chapter 4. Afterwards, an *improved hybrid intelligent algorithm* (iHIA), which integrates the TNIA-S with a genetic algorithm is designed to attain optimal solutions. At last, the proposed method is applied to the same case study in Chapter 5. The ranking of ECs are obtained, and different combinations of target levels of ECs are computed with respect to different confidence levels.

Chapeter 7 concludes the whole dissertation and proposes a future research direction.

Chapter 2

Literature Review

In this chapter, firstly, some basic concepts of fuzzy theories including the fuzzy uncertainty, fuzzy variables, and fuzzy measures are introduced. Secondly, a series of qualitative and quantitative researches on Kano's model are reviewed. Subsequently, six research points in the HoQ of QFD are enumerated, and the relevant literature is recalled in detail. At last, the researches on integration of QFD with Kano's model, fuzzy theories, and other useful methods are summarized.

2.1 Basic Concepts of Fuzzy Arithmetic

2.1.1 Introduction to fuzzy uncertainty

In real life, some events are considered to be deterministic with clear boundaries. For example, "the sun rises from the east every day", and "like charges repel each other" are definite events. In the coin-tossing game, although appearing heads or tails is a stochastic event, these two kinds of results are deterministic. Commonly, the probability measure based on the probability theory is adopted to estimate the occurrence of this kind of stochastic events.

However, sometimes a situation may be encountered that the boundary of an event is unclear and is hard to be measured by the probability, like the division of regions or the transition of four seasons, as well as in some adjectives"many, high, young, pretty, far" and some adverbs "approximately, perhaps", etc. This kind of uncertainty is different from the stochastic uncertainty, but more based on humans' subjective assessments from their own knowledge and experience. Just as Shakespeare said, "there are a thousand Hamlets in a thousand people's eyes." Therefore, corresponding to the probability theory, Zadeh [Zad65] established the fuzzy set theory in 1965, and the uncertain judgment or the belief degree were utilized to describe vague and ambiguous events.

Ever since then, fuzzy theories have been gradually developed and improved, and they are still moving forward today. Inspired by the establishment of the probability theory, several fuzzy measures were put forward to enrich theoretical findings and facilitate practical applications. For example, the possibility measure [Zad78], the necessity measure (the dual of possibility) [Zad79], the λ-fuzzy measure [Lee95], and the regular fuzzy measure [Nar00] were proposed over the past three decades. Furthermore, Liu and Liu [Liu02b] defined the credibility of a fuzzy event as the average of its possibility and necessity, to overcome the absence of the self-duality of the possibility and necessity measures.

In order to derive the membership degrees for a real function f of fuzzy variables, Zadeh's extension principle was raised in 1975 [Zad75], which is now served as a basis in fuzzy theories. Even so, the computation of this principle towards real-life problems with many fuzzy parameters is usually difficult, due to its internal operations of repeatedly calculating the maximal and minimal membership degrees. Therefore, a variety of fuzzy simulation techniques were set forth to deal with this obstacle. For instance, Liu and Iwamura [Liu98a] estimated the possibility of a fuzzy event, $\mathrm{Pos}\{f(\boldsymbol{\xi}) \le 0\}$, by designing a Monte Carlo simulation-like technique, which is called the *stochastic discretization simulation* (SDS). Later, the similar idea was employed by Liu [Liu04] to approximate the credibility of a fuzzy event, $\mathrm{Cr}\{f(\boldsymbol{\xi}) \le 0\}$. In addition, regarded as a significant notion in mathematics, the expected value of a fuzzy event, $E[f(\boldsymbol{\xi})]$, was simulated by Liu and Liu [Liu02b], which is called the *stochastic discretization algorithm* (SDA).

Thereafter, researchers showed their continuous interests in exploring the simulation methods for the expected value. Liu [Liu06b] suggested a uniform sampling-based simulation technique for $E[f(\boldsymbol{\xi})]$, taking advantage of some newly proved convergent results of sequences of fuzzy numbers. The accuracy of his approach was raised to a higher level compared with that of the SDA. However, its computational complexity may lead to a lower deficiency when handling practical problems. In recent years, Li [Lix15] firstly utilized a bisection algorithm to attain α-optimistic values of $f(\boldsymbol{\xi})$, and then the optimistic values were incorporated in a *numerical integration algorithm* (NIA) to approach the exact value of $E[f(\boldsymbol{\xi})]$. The simulation method innovated by Li [Lix15] performed well in aspects of accuracy, stability, and operation time. As a parallel research, Zhou *et al.* [Zho16c] put forward a fuzzy operational law regarding the inverse credibility distribution of fuzzy variables, which shared an analogous underlying principle with [Lix15]'s theorems on α-optimistic values. By virtue of these fuzzy simulation techniques and fuzzy arithmetic, practical applications in some areas [Duj17, Yan19, Guy19] were settled.

The development and improvement of fuzzy simulation is a beneficial foundation for solving real-life fuzzy optimization problems. In traditional mathematical programming models, crisp decision vectors of optimal values are usually achieved for objectives. In contrast, fuzzy decisions are obtained from fuzzy optimization models. As introduced in Liu [Liu02a], three kinds of possibility/credibility measure-based fuzzy optimization models were proposed to cope with the case of fuzzy decisions. The first type is the expected value model (EVM), in which a maximal expected return is subject to several expected fuzzy constraints. The second type is the chance-constrained programming model (CCP), which is composed of a maximal return and possibilistic/credibilistic constraints at different confidence levels. And the last type is the dependent-chance programming model

(DCP) with a possibilistic/credibilistic return. Sometimes, these three kinds of models can be solved via analytical computations. For intricate models that are hard to obtain analytical results, Liu and Liu [Liu02b] suggested a *hybrid intelligent algorithm* (HIA), which integrates different fuzzy simulation procedures for fuzzy variables with genetic algorithms, neural network algorithms, or other heuristic algorithms. Some recent implementations of the HIA-based algorithms can be found in literature [Guo16, Wan18, Zho18].

2.1.2 Fuzzy variables and fuzzy measures

At first, the definitions of the possibility measure and a fuzzy variable together with its membership function are given based on the possibility space as follows:

Definition 2.1 (Nahmias [Nah78]) *Let* Θ *be a nonempty set, and* $\mathcal{P}(\Theta)$ *be the power set of* Θ*. For each* $A \in \mathcal{P}(\Theta)$*, there is a nonnegative number* $\mathrm{Pos}\{A\}$*, called its possibility, such that*

(i) $\mathrm{Pos}\{\emptyset\} = 0, \mathrm{Pos}\{\Theta\} = 1$*; and*

(ii) $\mathrm{Pos}\{\cup_k A_k\} = \sup_k \mathrm{Pos}\{A_k\}$ *for any arbitrary collection* $\{A_k\}$ *in* $\mathcal{P}(\Theta)$*.*

The triplet $(\Theta, \mathcal{P}(\Theta), \mathrm{Pos})$ *is called a possibility space, and the function* Pos *is referred to as a possibility measure.*

Definition 2.2 (Liu [Liu02a]) *A fuzzy variable is defined as a function from the possibility space* $(\Theta, \mathcal{P}(\Theta), \mathrm{Pos})$ *to the real line* $\mathbb{R}$*.*

Definition 2.3 (Liu [Liu02a]) *Let* ξ *be a fuzzy variable on the possibility space* $(\Theta, \mathcal{P}(\Theta), \mathrm{Pos})$*. Then its membership function is derived from the possibility measure* Pos *by*

$$\mu(x) = \mathrm{Pos}\{\theta \in \Theta \mid \xi(\theta) = x\}.$$

The necessity measure is the dual of the possibility measure, which is defined to evaluate the impossibility of the opposite set A^c as follows,

Definition 2.4 (Liu [Liu02a]) *Let* $(\Theta, \mathcal{P}(\Theta), \mathrm{Pos})$ *be a possibility space, and* A *be a set in* $\mathcal{P}(\Theta)$*. Then the necessity measure of* A *is defined by*

$$\mathrm{Nec}\{A\} = 1 - \mathrm{Pos}\{A^c\}.$$

The credibility measure takes the average of the possibility and necessity of a fuzzy event, which is presented as follows:

Definition 2.5 (Liu and Liu [Liu02b]) *Let* $(\Theta, \mathcal{P}(\Theta), \mathrm{Pos})$ *be a possibility space, and* A *be a set in* $\mathcal{P}(\Theta)$*. Then the credibility measure of* A *is defined by*

$$\mathrm{Cr}\{A\} = \frac{1}{2}\big(\mathrm{Pos}\{A\} + \mathrm{Nec}\{A\}\big).$$

It can be seen that the credibility measure is self-dual, i.e., $\mathrm{Cr}\{A\} + \mathrm{Cr}\{A^c\} = 1$ for any $A \in \mathcal{P}(\Theta)$.

As a simple illustration, suppose that ξ is a fuzzy variable, μ is the membership function of ξ, and r is a real number. Then the fuzzy event $\{\xi \leq r\}$ has the following possibility, necessity, and credibility,

$$\mathrm{Pos}\{\xi \leq r\} = \sup_{x \leq r} \mu(x),$$

$$\mathrm{Nec}\{\xi \leq r\} = 1 - \mathrm{Pos}\{\xi > r\} = 1 - \sup_{x > r} \mu(x).$$

$$\mathrm{Cr}\{\xi \leq r\} = \frac{1}{2}\big(\mathrm{Pos}\{\xi \leq r\} + \mathrm{Nec}\{\xi \leq r\}\big).$$

Notably, a specialized type of fuzzy variables, called LR fuzzy numbers were defined by Dubois and Prade [Dub87], which derives as follows:

Definition 2.6 (Dubois and Prade [Dub78]) *A shape function* L *(or* R*) is a decreasing function from* $\mathbb{R}^+ \to [0, 1]$ *such that*

(1) $L(0) = 1$;

(2) $L(x) < 1, \ \forall x > 0$;

(3) $L(x) > 0, \ \forall x < 1$;

(4) $L(1) = 0$ [*or* $L(x) > 0, \ \forall x$ *and* $L(+\infty) = 0$].

Definition 2.7 (Dubois and Prade [Dub87]) *A fuzzy number ξ is of LR-type if there exist shape functions L (for left) and R (for right), and scalers $\gamma > 0$, $\beta > 0$ with membership function*

$$\mu_\xi(x) = \begin{cases} L\left(\dfrac{c-x}{\gamma}\right), & \text{if } x \le c \\ \\ R\left(\dfrac{x-c}{\beta}\right), & \text{if } x > c, \end{cases} \tag{2.1}$$

where the real number c is called the mean value or peak of ξ, and γ and β are called the left and right spreads, respectively. Symbolically, ξ is denoted by $(c, \gamma, \beta)_{LR}$.

Meanwhile, a generalized definition for fuzzy intervals together with LR fuzzy intervals were proposed by Dubois and Prade [Dub88], which are reviewed as follows:

Definition 2.8 (Dubois and Prade [Dub88]) *A fuzzy interval $\widetilde{\xi}$ is a quantity with a quasi-concave membership function μ, i.e., a convex fuzzy subset of the real line $\mathbb{R}$ such that*

$$\mu(z) \ge \min\{\mu(x), \mu(y)\}, \quad \forall x, y \in \mathbb{R}, z \in [x, y]. \tag{2.2}$$

Definition 2.9 (Dubois and Prade [Dub88]) *A fuzzy interval $\widetilde{\xi}$ is of LR-type if there exist shape functions L (for left), R (for right) and four parameters $(\underline{c}, \overline{c}) \in \mathbb{R}^2 \bigcup\{-\infty, +\infty\}, \gamma > 0, \beta > 0$ with membership function*

$$\mu_{\widetilde{\xi}}(x) = \begin{cases} L\left(\dfrac{\underline{c}-x}{\gamma}\right), & \text{if } x \le \underline{c} \\ \\ \quad 1, & \text{if } \underline{c} < x \le \overline{c} \\ \\ R\left(\dfrac{x-\overline{c}}{\beta}\right), & \text{if } x > \overline{c}, \end{cases} \tag{2.3}$$

and the fuzzy interval is represented as $\widetilde{\xi} = (\underline{c}, \overline{c}, \gamma, \beta)_{LR}$.

It is obtained that when $\overline{c} = \underline{c}$, an LR fuzzy interval is turned to be an LR fuzzy number. This situation implies that an LR fuzzy number can be regarded as a degradation form of an LR fuzzy interval.

2.2 Kano's Model

Kano's model [Kan84] was generated to get a better understanding of CRs (customer requirements) as well as their influence on customer satisfaction, which analyzed satisfaction qualitatively at the very beginning. Three different types of CRs were defined in accordance with different fulfillment levels to customer satisfaction, i.e., Attractive, One-dimensional, and Must-be attributes, which are demonstrated in Figure 2.1. The horizontal axis represents the fulfillment level of CR, and the vertical axis is the fulfillment level of customer satisfaction. Originally, the Kano questionnaire designs both functional and dysfunctional questions for each customer need. And the Kano category is ascertained by the highest response frequency through an evaluation table [Kan84].

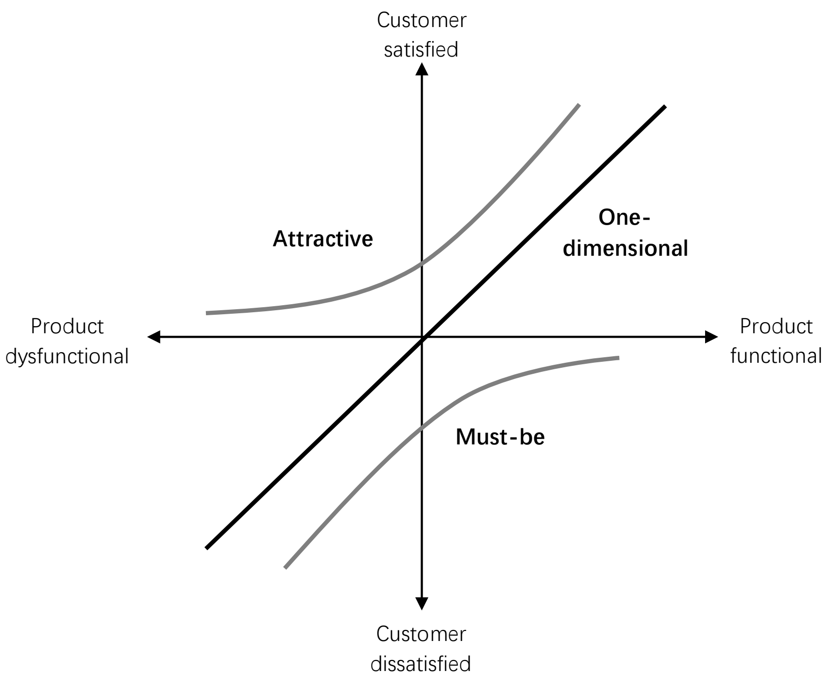


Figure 2.1: The diagram of traditional Kano's model.

From Figure 2.1, it can be seen that the fulfillment of a One-dimensional attribute (O) is positively related to the fulfillment level of customer satisfaction in a linear way, such as the camera function of a smart phone. And the fulfillment of an Attractive attribute (A) leads to higher level of satisfaction proportionally. Nevertheless, since Attractive attributes are not expected by customers, the lack of this kind of attribute will not result in high customer dissatisfaction, such as the appearance design of a smart phone. Lastly, as to a Must-be attribute (M), if not satisfied, the customers will be very displeased, such as the calling function of a phone. Apart from the aforementioned three types, there also exist three other types, Indifferent (I), Reverse (R), and Questionable (Q) attributes, which can be distinguished from their names. i.e., I, R, and Q indicate those attributes that customers do not care at all, dislike the requirements, and a contradiction may be caused with customers' expectations, respectively. In Yang [Yan05], a refined Kano's model was proposed, where quality attributes were divided into more distinct and precise categories by considering the importance of product attributes from the customers' point of view.

From the beginning of the 21st century, Kano's model has been applied to many areas and gains sufficient recognition from researchers. Matzler *et al.* [Mat04] tried to apply Kano's model to the employee satisfaction, and used a regression analysis with dummy variables to find an asymmetric relationship between the satisfaction with different factors and the overall employee satisfaction. Except for the employee satisfaction measurement of improving people management, Kano's model was also applied to other fields like web community service quality [Kuo04, Ilb17], customer knowledge discovery [Che06b], analysis of the attractive factors of regional characteristics [Che16b], and environmental correlates of residential satisfaction [Yin16]. These ideas mainly focused on the classification and qualitative analysis on different Kano categories of customer needs.

In terms of the two-dimensional quality model in Figure 2.1, some quantitative researches can be carried out, which was first suggested by Berger *et al.* [Ber93]. With respect to the asymmetric and non-linear relationship between different store quality attributes and customer satisfaction, Ting and Chen [Tin02] verified these relationships by a logarithm model. Wang and Ji [Wan10] employed an S-CR relationship function between the fulfillment level of customer satisfaction and the fulfillment level of CRs, which were plotted by linear and exponential functions. Finn [Fin11] incorporated the prospect theory to depict the quality attribute's straight line or curve shape of customer satisfaction.

For the past few years, some enhancements on the traditional Kano's model have also been conducted, for example, modifying questionnaires, improving classification methods, or integrating Kano's model with other tools. Lee and Huang [Lee09] applied a kind of subjective fuzzy questionnaire to modify the original Kano questionnaire, and calculated the fuzzy mode results as Kano's classifications. Florez-Lopez and Ramon-Jeronimo [Flo12] developed an integration framework of Kano's model, fuzzy distances, and fuzzy models to handle logistics customer service. Wang [Wan13a] employed a fuzzy Kano's model to elicit customer perception on product attributes and then used information entropy to derive their importance weights. Bu and Park [Buk16] utilized the DAQ (directly-asked-question) model to find the most accurate fuzzy Kano categories in sports lesson programs. Kuo *et al.* [Kuo12] proposed the IPA-Kano (importance-performance analysis) model to categorize and diagnose service quality attributes. This approach avoided the limitation of the original Kano's model of neglecting the attribute performance and importance, and also eliminated the weakness of the IPA model to concern One-dimensional attributes only. Further, a fuzzy Kano's model was combined with the IPA model to investigate the service quality in restaurant industry [Pai18]. Madzik [Mad18a] set forth a Type IV approach, which was based on a modification of the requirement categorization process, and intended to minimize the discrepancy zone between the calculated and the real position of a particular requirement. Kano's model was integrated with FMEA (failure mode and effects analysis) to determine the categories of requirements more precisely, in which the Kano parameter k was calculated with a novel risk priority number [Mad18b].

2.3 Quality Function Deployment (QFD)

As introduced in Section 1.1, the concept of QFD stems from Japan in the late 1960s [Aka90]. The house of quality (HoQ) [Hau88] is the core concept of QFD, which is a diagram that resembles a house. Illustrated in Figure 2.2, the HoQ includes several matrices, i.e., CRs and their relative importance weights are listed in the left wall, ECs (engineering characteristics) together with their correlations are enumerated in the ceiling and roof separately. The relationship room of the HoQ displays the relationships between CRs and ECs, and the data of competing products are listed in the strategic planning room. Besides, the technical specifications of ECs are arranged in the technical priorities room.

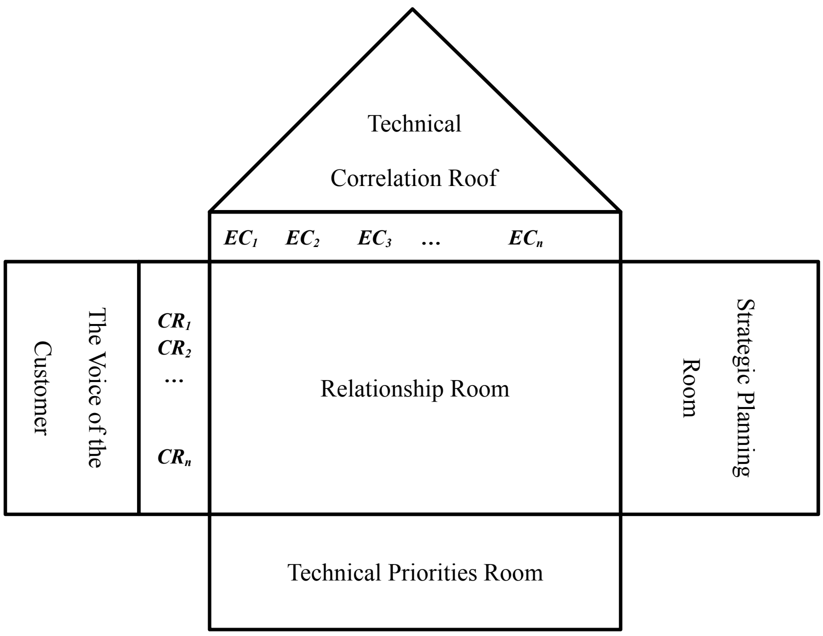


Figure 2.2: The diagram of house of quality (HoQ).

It is seen that both Kano's model and the left wall of the HoQ are utilized to describe customers' voices. However, it is hard to directly combine Kano's model into the HoQ to present a visual model, due to the reason that Kano's model is only a schematic diagram. Even so, the Kano category of each CR can be marked in the left wall of the HoQ.

Substantially, each part of the HoQ can be a research point. The mainstream researches regarding QFD lie on the following several aspects:

(a) The identification of customer requirements from customers' voices.

Traditionally, if manufacturers want to get feedback of a product, questionnaires may be sent out or interviews are organized to collect customers' voices, such as the conventional Kano questionnaire [Kan84]. Sometimes, it is time-consuming and only can cover a part of customers. The customer information acquired are regarded as a sample in statistics, which means the obtained CRs may be incomplete or even unilateral if the survey population and questionnaire lack a proper planning. Nowadays, with the prosperity of information technology and the spreading of E-commerce, it is very simple for customers to shop online. The transformation from shop offline to online leads to a large quantity of customer data, which contains customer search logs, purchasing behaviors, and customer reviews, etc. Customer reviews are usually appeared as subjective or objective linguistic statements in a positive or negative way. Negative comments usually express the urgent customer pain points, or potential customer requirements. Therefore, to some extent it is more beneficial to identify and analyze negative customer opinions to find out some critical CRs of a product.

The analysis towards big online customer data not only is an opportunity facing manufacturers, but also has raised great interests in academic researches. As one of the approaches, CRFs (conditional random fields) is widely utilized to summarize online reviews. Jakob and Gurevych [Jak10] defined features of online comments by taking advantage of tokens, POS (part of speech) tags, short dependency paths, and word distances, etc. Zhong *et al.* [Zho16a] reviewed some representative researches on big data in the service and manufacturing sectors detailedly, together with its challenges, opportunities and future perspectives. A big data mining method which incorporated neural network analysis was adopted by Chong *et al.* [Cho15] to build a platform for understanding and predicting online consumer demands. A quadruple Q = <F, S, A, R> was originated by Jin *et al.* [Jin16b] to describe online reviews, where F, S, A, and R stand for product features, sentiment polarity, aspects of product features, and detailed reasons, respectively. A framework constituted of several advanced data analytics was designed by [Ire18] to understand customer needs through transforming qualitative data to quantitative insights on products. The internal sentiment analysis was achieved via natural language processing, POS tags, machine learning, etc.

Some other useful data mining techniques are based on the concept of biclustering, which is capable of clustering the rows and columns simultaneously to find biclusters of analogous information. The idea of biclustering method was first put forward by Hartigan in 1972, but no one applied it for almost 30 years. Until the beginning of 21st century, the analysis of microarray data brought biclustering back into focus. Afterwards, many algorithms were proposed to find similarities between biological gene expressions under different conditions, like CC algorithm [Che00], Plaid algorithm [Laz00], Xmotifs algorithm [Mur03], Spectral algorithm [Klu03], BCBimax algorithm [Pre06]. Except for applications in biological data analysis, the biclustering method has gained extensive support in other areas, such as text-mining [Dec07], market data analysis [Dol11], recommendation systems [Inb11], financial forecasting and trading [Hua11], collaborative filtering [Sym08], and market segmentation [Wan16].

Actually, in the area of big data mining, a variety of technologies according to different theories were set forth, like genetic algorithm, inductive learning theory, Bayesian network, decision tree, pattern recognition, high-performance computing, and statistical analysis, the details of which can be found in Zhang *et al.* [Zha16] and Sang *et al.* [San16]. Quite probably, the aforementioned data-mining approaches can be employed and adapted in the future research to hear customers' voices by extracting online customer data of reviews or opinions of a certain product.

(b) The determination of CR relative importance weights in the left wall of the HoQ, which is the input of the QFD optimization procedure.

(c) The professional assessment of the relationship matrix between CRs and ECs in the relationship room of the HoQ.

(d) The generation of ECs in the product design, as well as the determination of their importance weights in the ceiling and roof of the HoQ.

The solutions of (b), (c), and (d) usually have overlapping areas due to the reason that comparisons are generated during these determination processes, and these evaluations are usually conducted by professionals or experts. As a consequence, the following literature recalled may also have overlapping among these three research points.

Before the optimization model is established, confirming the CR relative importance weights, the internal relations and correlations are of great significance. In the establishment of the house of quality, Hauser and Clausing [Hau88] described that the QFD team is responsible for the priorities/importance weights of CRs to balance the cost of fulfilling a need with the benefit to the customer. The principle to decide the weights is to consider the importance of each CR to customers. At beginning, these weights are determined through a direct market research with customers.

Afterwards, the simplest and most intuitive way to prioritize CRs is based on the rating systems of 1-2-3-4-5 or 1-3-5-7-9 [Gri93], which is marked as the relative importance degrees. Otherwise, according to the pairwise comparison system - AHP (analytic hierarchy process) initiated by Saaty in 1980 [Saa80], the CR relative importance weights were calculated [Fun98]. In Ho *et al.* [How11], AHP was combined with QFD not only in the determination of relative importance weights of CRs, but also the functional relationships between CRs and ECs, which aimed at enhancing the effectiveness of sourcing decisions among diverse suppliers. Chuang [Chu01] attached AHP to QFD for a location decision from a requirement perspective, where the AHP method was applied to measure the weight for each location requirement. As an advanced version, ANP (analytic network process) was further proposed by Saaty in 1996 [Saa05], which is more complicated than AHP, in which the computation complexity is largely raised owing to the network design inside. Zaim *et al.* [Zai14] incorporated the ANP weighted relative importance weights to prioritize CRs.

Naturally, the usage of crisp values will help decrease the calculation difficulty in AHP or ANP related problems. However, researchers found that crisp values may be inappropriate since the internal evaluations in (b), (c), and (d) are subjective and ambiguous. Usually, these importance degrees or relationships are expressed by linguistic statements from experts, like "quite important" or "a little weak", which cannot be mapped into crisp values directly. Therefore, stochastic variables and fuzzy variables are more proper to represent them. As stated in Section 2.1.1, stochastic variables are inclined to describe natural events, like "the chance of raining tomorrow", which belongs to the probability theory. Here it seems quite suitable to adopt fuzzy variables to express subjective linguistic data. On this basis, the previous introduction of the fuzzy uncertainty and fuzzy arithmetic is essential.

Served as a crucial branch of the QFD research, an increasing number of fuzzy QFD researches have arisen in recent decades. Shen *et al.* [She01] used fuzzy variables to indicate the human perception and judgment in QFD as mentioned, together with fuzzy arithmetic and the defuzzification techniques. A fuzzy least-square regression approach to depict relationships in QFD was considered by Kwong *et al.* [Kwo10], taking both the fuzziness and randomness into account. Otherwise, fuzzy linear regressions or non-linear regressions were employed to link the weighting of CRs, and the functional relationships between CRs and ECs with the ultimate customer satisfaction. Notably, the h value is a vital parameter in fuzzy linear regression models, which guarantees the observed crisp outputs are included in the h intervals of the fuzzy outputs obtained from models. Other than setting the h value in fuzzy linear models arbitrarily by the decision-makers, Liu *et al.* [Liu15b] put forward an approach using the fuzzy linear regression models attached with optimized h values to identify the functional relationships in QFD. And the coefficients inside were assumed to be symmetric triangular fuzzy numbers. Soon after, another approach was proposed by Chen *et al.* [Che16a] to optimize the h value for fuzzy linear regression analysis. It took advantage of the minimum fuzziness criterion with symmetric triangular fuzzy coefficients to obtain the maximum reliability. As to fuzzy non-linear regressions, Liu *et al.* [Liu14] developed a fuzzy non-linear regression method to acquire the degree of compensation among CRs. The overall customer satisfaction was derived in accordance with a trade-off strategy which consisted of the CR relative importance weights and the degree of compensation among them. Even though the linear and non-linear regressions appear to be more objective, sometimes the sparse data obtained from them will result in less practicable solutions.

In regard to the determination of the priority of ECs in (d), due to the inherent fuzziness in matrices in the HoQ, it is not simple to derive the importance degrees of ECs straightforwardly. This situation generates great interests among researchers. Chen *et al.* [Che06a] originated a fuzzy weighted average method (the h-cut method) in a fuzzy expected value operator so as to rank technical attributes in fuzzy QFD. Kwong *et al.* [Kwo07] set up an aggregated importance of ECs, which considered both the conventional meaning of EC importance as well as the impacts of one EC on other ECs. In 2011, Kwong *et al.* [Kwo11] proposed a fuzzy group decision-making method which combined a fuzzy weighted average method with a consensus ordinal ranking technique by concerning two types of uncertainties, human perception and customer heterogeneity, simultaneously. Wang [Wan12] viewed the group decision-making QFD problem as a series of preferential combinations of customers and QFD team members, and prioritized ECs by comparing their normalized fuzzy technical importance ratings via the method of centroid defuzzification. In recent years, Liu *et al.* [Liu16] proposed an exact expected value-based method to prioritize technical attributes in fuzzy QFD, in which the expected values of the importance of ECs were obtained through the inverse credibility distribution of fuzzy numbers. Yu *et al.* [Yul18] scored technical attributes by virtue of interval-valued intuitionistic fuzzy sets and Choquet integral. Meanwhile, the CR relative importance weights were also derived through converting interval-valued intuitionistic fuzzy numbers.

(e) The determination of target values of ECs in the improved product, which is the output of the QFD optimization procedure.

As to a new or improved product, the product design procedure based on QFD aims at determining a series of $x_1, x_2, \cdots, x_n$ for ECs constrained to restricted resources. The ultimate overall consumer satisfaction is supposed to be equal to or larger than that of other potential competitors in the current market. During this complex operation course, definitely diverse variables, trade-offs and multiple contradictions will be involved.

Abundant fuzzy modelling studies regarding how to get a series of target values for ECs have been carried out. It is quite reasonable to absorb fuzziness to depict inner indeterminate factors in the HoQ by means of fuzzy concepts. For example, Chen *et al.* [Che05] brought up a fuzzy expected value model to determine target values of ECs, which was in consideration of the maximum consumer expectation or the minimum development expense, respectively. Erginel [Erg10] set forth a fuzzy multi-objective decision model by integrating the information from design failure and effect analysis. The means-end chain notion was incorporated by Chen and Ko [Che10] to establish a fuzzy linear modelling approach in calculating the amount of contribution of individual "how" to the whole consumer perception. Sener and Karsak [Sen10, Sen11] suggested some fuzzy mathematical programming, including a fuzzy non-linear regression and optimization method, and an integrated fuzzy linear regression with a fuzzy multiple objective programming approach to determine target levels of ECs. Liu *et al.* [Liu14] embedded the compensation degrees among CRs into QFD, which combined the minimized fuzziness benchmark with the aid of a non-linear regression to realize it. Zhong *et al.* [Zho14] set up a fuzzy chance-constrained programming model in setting target values of technical attributes, which was solved by the HIA.

(f) The benchmark management with rival companies in the strategic planning room of the HoQ.

The goal of the QFD optimization procedure is to give the decision-makers a guided map and assist them in competing with their rival companies in the current market. On this basis, considering the benchmark management or the strategic planning is indispensable during the optimization.

Generally, there are two ways of setting the benchmark of the preferred customer satisfaction degree towards a certain product. Firstly, as described in the notebook computer design case in Ji *et al.* [Jip14], there existed four competitor companies, and their customer satisfaction degrees of several CRs were outlined in the strategic planning room of the HoQ. Then, the average values of four companies were set as the lowest limits in the constraints of the proposed optimization model. Besides, another way of benchmark setting was introduced in the motorcar design case in Chen *et al.* [Che05]. The technical specifications of ECs for five competitor companies were accumulated in the HoQ. According to the overall customer satisfaction calculation formula, the overall customer satisfaction degree of each company can be readily obtained, which provided reference substances for the design team of the motorcar. More academically, traditional customer-competitive benchmarking based on customers' perceptions towards a set of products/service was analyzed [Fra18], and a new method was proposed to transfer subjective judgements of customers to a collective cardinal scaling. On the whole, compared with other research points in the HoQ, there are few articles specifically studying the benchmark setting in QFD. In practice, the benchmark management for enterprises is essential especially in a competitive market environment.

2.4 Integration of Kano's Model with QFD

Owing to the fact that Kano's model is concerned about the attributes of diversified CRs, it helps provide an effective tool for measuring the input of the QFD procedure. This attempt was first accomplished by Matzler and Hinterhuber [Mat98] in 1998, who categorized CRs in QFD by Kano's model according to their different impacts on customer satisfaction. Their research was the pioneer in the subsequent academic researches of integrating the Kano's model with the product development. Tan and Shen [Tan00] incorporated Kano's model into the planning matrix of QFD to help accurately and deeply understand the nature of customers' voices. In view of the Kano's model analysis, an approximate transformation function was proposed to adjust the improvement ratio of each CR. In contrast to the former literature, Tontini [Ton07] provided a method which adequately treated CRs according to their category in Kano's model. And the CR importance weights was modified with respect to the impacts on satisfaction or dissatisfaction that their presence or absence may cause to customers. Among the relevant integration studies, Kano's model was still viewed as a qualitative measure of sorting. For example, in Chang and Chen [Cha11], Kano's model was generated to explore brand contact elements from customers of a hot spring hotel, and QFD was employed to take the identified elements into design under the compromise of a service provider's technical considerations. Kuo *et al.* [Kuo16] implemented the analytical model of Kano's model and QFD to enhance the city hotel service quality.

To deal with the vague and ambiguous information of the product development procedure, fuzzy variables are introduced to enrich the integration researches, either in the fuzzy Kano's model or the fuzzy QFD framework. In terms of the Kano's two-dimensional quality classification, this qualitative relation is quantified by fuzzy logic in the QFD matrix [Che07] . Lee *et al.* [Lee08] presented a combination approach by integrating Kano's model with fuzzy mode into the QFD matrix and adjusted the weights of CRs, which appeared to be more objective during the course of weighting. Wang [Wan13b] incorporated Kano's model into QFD to recognize the degree of urgency with respect to the enhancement and priority of CRs, and the optimal aggregation weights were accomplished by a fuzzy linguistic quantifier with a soft majority concept. Chen and Ko [Che08] adopted fuzzy approaches to represent the importance weights of CRs, the relationships between CRs and ECs, and the correlations among ECs. In consideration of the traditional Kano category of CRs, they proposed a fuzzy non-linear model to determine target values of ECs to obtain the maximal customer satisfaction. Mu *et al.* [Mul08] suggested a fuzzy multi-objective model to reconcile the trade-off between customer satisfaction and cost, where a quantitative Kano's model of three general functions was used to illustrate the linear and non-linear influence between CRs and ECs. Yeh [Yeh10] integrated the refined Kano's model proposed by Yang [Yan05], with QFD and fuzzy integrals to ascertain the medical service improvement priority. Yeh and Chen [Yeh14] also adopted the refined Kano's model with QFD and grey relational analysis to improve the service quality of nursing homes. Kano's model was used to filter customer needs and transform the Attractive ones into a kind of parameters, which was served as the cornerstone in the subsequent QFD process [Hab18]. Avikal *et al.* [Avi20] utilized the two quality tools to classify the aesthetic attributes of SUV car and compared the performances of fuzzy Kano's model with traditional Kano's model.

Apart from some joint studies on Kano's model and QFD, there are also a small amount of literature were conducted on cooperative games with Kano's model or cooperative games with QFD. Conklin *et al.* [Con04] and Conklin and Lipovetsky [Con05] applied the notion of Shapley value from cooperative game theory to risk analysis and Kano theory, so as to identify the key drivers of CRs that led to customer satisfaction. Their calculation procedure is also adopted in this thesis. Meanwhile, the concept of Nash bargaining in cooperative game theory was taken into account to obtain target levels of ECs by means of a fuzzy optimization model [Yan14]. Lately, QFD was combined with other useful tools or theories, such as MCDM (multiple criteria decision making) methods including AHP [Abd18, Hab18], ANP [Asa17], and TOPSIS [Cho17]; DEA (data envelopment analysis) [Zha19], FMEA [Kum18], Markov models [Asa17, Got18], etc., which has brought a variety of new research and application opportunities.

2.5 Summary

The literature review in this chapter was divided into two major parts. The first part introduced the development of fuzzy uncertainty, including fuzzy measures, fuzzy simulation, and fuzzy programming. Some basic concepts in fuzzy arithmetic were outlined as preliminaries for novel fuzzy simulation techniques to be proposed in Chapters 3 and 4. The second part reviewed the researches on two quality tools, i.e., Kano's model and QFD. Firstly, the origin of Kano's model and meanings of six attributes in Kano categorization were explained. Then, both the qualitative and quantitative researches on Kano's model were recalled in detail. Subsequently, the origin of QFD and different matrices in the HoQ were described. More specifically, six research points in the HoQ were elaborated with a variety of references. Finally, relevant integration researches were displayed.

It is observed that the integration research of these two quality tools with other methods is popular at present and is also the tendency in the future. Corresponding to the motivation and research objectives of this thesis expounded in Sections 1.2 and 1.3, the integration researches will be conducted among Kano's model, QFD, cooperative game theory, and fuzzy theories. In particular, for the QFD optimization procedure, The research points (b), (d), and (e) will be treated as the main focus in Chapters 5 and 6.

The next chapter deals with the fuzzy simulation for the possibility of fuzzy events. The principle and limitation of a commonly used simulation technique for the possibility will be described. Then, two novel fuzzy simulation techniques are proposed to approximate possibilities of individual and joint fuzzy events based on a novel operational law.

Chapter 3

A Uniform Discretization Simulation for the Possibility of Fuzzy Events

The possibility theory and the possibility measure were set forth by Zadeh in 1978, which is an accepted theoretical foundation for many subsequent theories and measures in the fuzzy area. Later in 1998, Liu and Iwamura applied the possibility measure to fuzzy constraints in fuzzy programming. And a stochastic discretization simulation with a stochastic sampling process inside was firstly put forward to estimate the possibility of fuzzy events. Nevertheless, in this chapter their method is proved to have obvious flaws in the actual operation of sampling.

To overcome this drawback, two novel fuzzy simulation algorithms for individual and joint fuzzy events by virtue of a uniform sampling process are suggested, which are based on two newly raised and proved theorems for a specialized type of fuzzy variables - regular fuzzy intervals. In addition, from four progressive numerical experiments, it can be seen that the simulation accuracy is greatly enhanced and the computational time is also decreased when compared with Liu and Iwamura's method. The proposed uniform sampling process is also applicable to other fuzzy simulations, like the expected value and credibility for functions of fuzzy variables. Meanwhile, this innovative fuzzy simulation technique can also be embedded in heuristic algorithms to solve fuzzy programming models with individual or joint possibilistic constraints.

3.1 Fuzziness and Its Measure

As stated in [Zad78], human decisions are mainly based on possibilistic rather than probabilistic information in nature, since they focus more on the meaning of the information. Commonly, the information is imperfect with a vague, fuzzy, general or ambiguous content/value (called the imprecision), and its truth/confidence can be subjectively assessed (called the uncertainty) [Dub88]. To deal with such uncertainty, the possibility theory was established by Zadeh in 1978 [Zad78], in which the fuzzy set theory [Zad65] was served as a natural basis. Ever since then, the possibility theory has been gradually accepted to handle the fuzziness, and the possibility measure, Pos, is used to evaluate the degree of belief of fuzzy events.

Later, with the development of fuzzy programming and fuzzy optimization problems [Liu02a], the fuzzy constraints are usually assumed to satisfy at a possibility of α. More specifically, as described in Liu and Iwamura [Liu98a], the fuzzy constraints in a chance-constrained programming model with fuzzy parameters are frequently converted into the following crisp forms as individual and joint possibilistic constraints, i.e.,

$$\text{Pos}\{g(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0\} \geq \alpha, \tag{3.1}$$

or

$$\text{Pos}\{g_k(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0, k = 1, 2, \cdots, s\} \geq \beta \tag{3.2}$$

where $\boldsymbol{x}$ is a decision vector, $\boldsymbol{\xi}$ is a fuzzy vector, and α and β are predetermined confidence levels to corresponding constraints. To cope with these two kinds of possibilistic constraints, Liu and Iwamura first gave the respective crisp equivalents for some special cases, whose results are discussed detailedly in [Liu98a].

Simultaneously, for other intricate cases without known crisp forms, they further suggested a more generalized fuzzy simulation technique to handle them, called the *stochastic discretization simulation* (SDS). Subsequently, the SDS was attached to a genetic algorithm as a *hybrid intelligent algorithm* (HIA) to find feasible solutions to chance-constrained programming models in a fuzzy environment [Mai06, Mai07, Dai14].

It can be seen that the SDS is conducted as an important part of the HIA, whose outcomes have direct influence on the final solutions of fuzzy optimization models. After the internal structure of the SDS is investigated, it is discovered that its design principle is consistent with Zadeh's extension principle in [Zad75] while its actual operation is not. This discrepancy is caused by the stochastic sampling process inside, which shares some similarities with Monte Carlo simulation [Rub81, Rub98]. From both theory and practice, the stochastic sampling is proved to inevitably generate deviations on simulation results in this chapter. Analogously, another fuzzy simulation with a wide impact for the expected value of functions of fuzzy variables adopted this stochastic sampling procedure as well [Liu02b, Zha05, Keh10], and its performance was discussed by Li in [Lix15]. For the sake of enhancing the performance of the HIA, a novel *uniform discretization simulation* (UDS) for individual fuzzy events is initiated to substitute the SDS. And the UDS is designed based on a newly raised operational law of continuous and strictly monotone functions of regular fuzzy intervals (a specialized LR fuzzy intervals with continuous and strictly decreasing shape functions). Meanwhile, the UDS is also extended to the UDS-Joint to deal with joint fuzzy events as another new theorem is developed.

The remaining content of this chapter is arranged as follows. Firstly, Section 3.2 reviews the underlying philosophy of the SDS, and then clearly points out its inherent deficiency through an analytical analysis, which is validated by the simulation results from three progressive numerical examples. Subsequently, Section 3.3 elaborates the principle and algorithm design of the UDS. The distinctions between the SDS and UDS are summarized as well, and their performances are compared via the identical three numerical examples in Section 3.2. Afterwards, Section 3.4 expounds the details of the UDS-Joint, and another numerical example is used to demonstrate its efficiency. Finally, Section 3.5 concludes the whole chapter.

3.2 The Limitation of the SDS

3.2.1 Stochastic discretization simulation

The SDS has become a feasible way to approximate the possibility of fuzzy events since its design. To start with, assume that an individual fuzzy event $g(\boldsymbol{\xi}) \leq 0$ exists, in which $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$ and ξ_i are continuous fuzzy variables with membership functions μ_{ξ_i}, $i = 1, 2, \cdots, n$, respectively. Then, $g(\boldsymbol{\xi})$ is also a continuous fuzzy variable according to Liu [Liu02a]. Through employing the possibility theory established by Zadeh [Zad78] and Zadeh's extension principle [Zad75], the principle of the SDS to calculate the possibility, $\mathrm{Pos}\{g(\boldsymbol{\xi}) \leq 0\}$, is presented in the following theorem.

Theorem 3.1 (Liu [Liu02a]) *Let $\xi_1, \xi_2, \cdots, \xi_n$ be fuzzy variables, and $g : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Then the possibility of the fuzzy event $g(\xi_1, \xi_2, \cdots, \xi_n) \leq 0$ is*

$$\mathrm{Pos}\{g(\xi_1, \xi_2, \cdots, \xi_n) \leq 0\} = \sup_{x_1, x_2, \cdots, x_n \in \mathbb{R}} \Big\{ \min_{1 \leq i \leq n} \mu_{\xi_i}(x_i) \mid g(x_1, x_2, \cdots, x_n) \leq 0 \Big\}. \tag{3.3}$$

In order to achieve $\mathrm{Pos}\{g(\xi) \leq 0\}$ through Theorem 3.1, the SDS operates the simulation by transferring the continuous fuzzy variable $g(\boldsymbol{\xi})$ into a discrete fuzzy variable $g(\boldsymbol{\xi})^*$ through randomly generating sample points. The specific steps of the SDS are illustrated as follows:

Algorithm 1 (SDS of Liu & Iwamura [Liu98a])

Step 1. Initialize the number of sample points N, and set $M = \gamma$ ($\gamma > 0$) as a lower estimation.

Step 2. Randomly generate u_i from the γ-level sets of ξ_i, $i = 1, 2, \cdots, n$, respectively, and denote $\boldsymbol{u} = (u_1, u_2, \cdots, u_n)$.

Step 3. Set $\mu = \mu_1(u_1) \wedge \mu_2(u_2) \wedge \cdots \wedge \mu_n(u_n)$.

Step 4. If $g(\boldsymbol{u}) \le 0$, and $M < \mu$, then reset $M = \mu$.

Step 5. Repeat steps 2, 3, and 4 for N times.

Step 6. Return M as the simulation value of $\mathrm{Pos}\{g(\boldsymbol{\xi}) \le 0\}$.

It can be seen that the estimated membership degree $\mu(a)^*$ for $g(\boldsymbol{\xi})^*$ at a real number a in the SDS can be obtained through its stochastic sampling process in Steps $2 \sim 5$ and expressed as

$$\mu(a)^* = \max_{1 \le k \le N} \left\{ \min_{1 \le i \le n} \mu_i(u_i^k) \mid g(u_1^k, u_2^k, \cdots, u_n^k) = a \right\}. \tag{3.4}$$

Corresponding to the basic computing principle in Eq. (3.3), technically Eq. (3.4) seems practicable to attain a specified membership degree when the number of sample points N is large enough. However, when this stochastic sampling process is meticulously inspected from the aspect of actual operation, it is noticed that two unavoidable deficiencies emerge. Firstly, the general setting of N is relative small, e.g., 3000, 5000, or 10000, in practical applications, which does not strictly follow Zadeh's extension principle. As a direct consequence, the simulated membership degree $\mu(a)^*$ may be inaccurate. Secondly, it is intuitive that, with the growing of the size of fuzzy variables, n, inside the function g, the quantity of sample points needed increases remarkably. For instance, in order to get a relative precise membership degree when $n = 5$, the number of sample points N is required to set to be at least 10^{10} in the SDS. Nevertheless, it is found that whatever the size n is, N is generally set as 10000 for time-saving. As a natural result, either the membership degree or the membership function of the discrete fuzzy variable $g(\boldsymbol{\xi})^*$ simulated by the SDS is deviated from those of the original continuous fuzzy variable $g(\boldsymbol{\xi})$. To more clearly demonstrate this deviation, three numerical examples are conducted successively.

3.2.2 Three numerical examples for the SDS

The forthcoming three numerical examples contain two, four, and ten fuzzy variables, respectively. To observe the distance between the simulation value and the exact value, the following equation is utilized to calculate the error rate, i.e.,

$$\text{Error} = \frac{|\text{Simulation value} - \text{Exact value}|}{\text{Exact value}} \times 100\%. \tag{3.5}$$

Example 3.1 Suppose that η_1 and η_2 are two Gaussian fuzzy numbers $\mathcal{G}(2,1)$ and $\mathcal{G}(1,1)$ with membership functions $\mu_{\eta_1}(x) = \exp[-(x-2)^2]$ and $\mu_{\eta_2}(x) = \exp[-(x-1)^2]$, respectively. Let $g_1 = x_1 - x_2$, $\boldsymbol{\eta}_1 = (\eta_1, \eta_2)$, and calculate the possibility of $\eta_1 \le \eta_2$ or $\text{Pos}\{g_1(\boldsymbol{\eta}_1) \le 0\}$ by the fuzzy simulation (see [Liu98a]).

Notably, the analytical result of $\text{Pos}\{\eta_1 \le \eta_2\}$ can be easily figured out as 0.7788, according to Zadeh's extension principle. As a comparison, the SDS is applied to Example 3.1 to compute the simulation result simultaneously. Parallel to the setting of N in [Liu98a], the distribution of 3000 sample points of $g_1(\boldsymbol{\eta}_1)$ generated by the SDS is depicted in Figure 3.1. After rounding and employing Eq. (3.4), the membership function of $g_1(\boldsymbol{\eta}_1)$ can be obtained approximately in Figure 3.2. The horizontal axis is the value range of $g_1(\boldsymbol{\eta}_1)$, and the vertical axis is the membership degree μ. On this basis, it is simple to acquire the simulation result, i.e., $\text{Pos}\{g_1(\boldsymbol{\eta}_1) \le 0\} = 0.7739$. Through Eq. (3.5), the error rate is calculated as 0.63%, which shows a small difference with the exact value 0.7788.

Example 3.2 Let $g_2 = x_1^2 + x_2 * x_3 - x_4^{-1}$, $x_1 > 0$, and $\boldsymbol{\eta}_2 = (\eta_1, \eta_2, \eta_3, \eta_4)$, in which η_1 is a Gaussian fuzzy number $\mathcal{G}(0,1)$ with $\mu_{\eta_1}(x) = \exp[-x^2]$, the

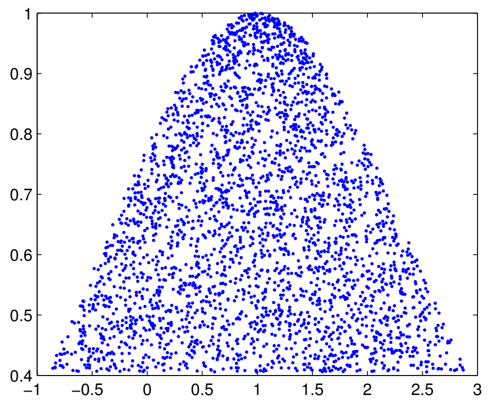


Figure 3.1: The distribution of 3000 sample points in Example 3.1 generated by the SDS.

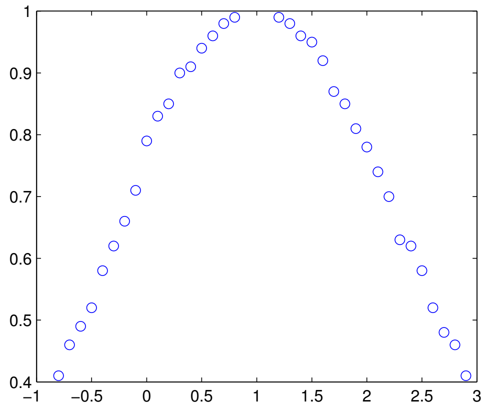


Figure 3.2: The simulated membership function of $g_1(\boldsymbol{\eta}_1)$ in Example 3.1 by the SDS.

membership function of η_2 is $\mu_{\eta_2}(x) = \exp[-|x-2|]$, η_3 is a trapezoidal fuzzy number $\mathcal{A}(-1,1,2,3)$, and η_4 is a triangular fuzzy number $\mathcal{T}(1,2,3)$. Calculate the value of $\text{Pos}\{g_2(\boldsymbol{\eta}_2) \geq 4\}$ by the fuzzy simulation (see [Liu98a]).

The exact value of $\text{Pos}\{g_2(\boldsymbol{\eta}_2) \geq 4\}$ is 0.9086, which can also be derived analytically. Similarly to Example 3.1, the distribution of 5000 sample points of $g_2(\boldsymbol{\eta}_2)$ generated by the SDS is shown in Figure 3.3, and the simulated membership function of $g_2(\boldsymbol{\eta}_2)$ is attained in Figure 3.4. In contrast to Example 3.1, this membership function is unclear and not coherent, and the simulation result of $\text{Pos}\{g_2(\boldsymbol{\eta}_2) \geq 4\}$ is extracted as 0.8752. With the increasing of both quantity and type of fuzzy variables in $g_2(\boldsymbol{\eta}_2)$, the error rate between the simulation result 0.8752 and the exact value 0.9086 is 3.68%, which becomes larger than that of Example 3.1.

From Examples 3.1 and 3.2, it is discovered that the accuracy of the possibility simulated by the SDS is not guaranteed, which verifies the theoretical inference of Eq. (3.4) in Section 3.2.1. To make this deviation more clearly, the quantity of fuzzy variables, n, is expanded to ten in the following example.

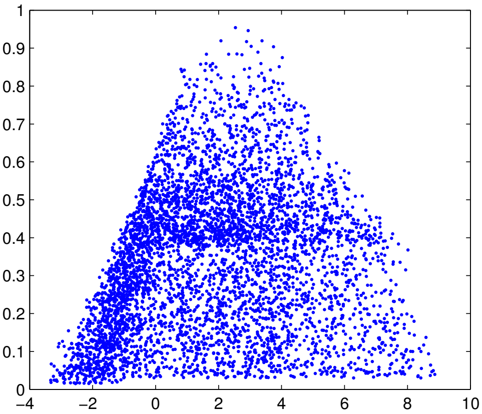


Figure 3.3: The distribution of 5000 sample points in Example 3.2 generated by the SDS.

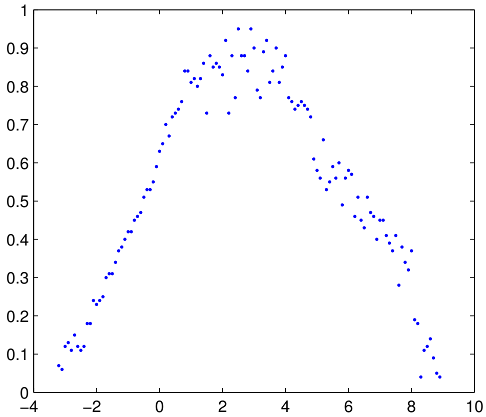


Figure 3.4: The simulated membership function of $g_2(\boldsymbol{\eta}_2)$ in Example 3.2 by the SDS.

Example 3.3 It is assumed that $\eta_i, i = 1, 2, \cdots, 10$, are ten independent triangular fuzzy numbers listed in Table 3.1, and denote $\boldsymbol{\eta}_3 = (\eta_1, \eta_2, \cdots, \eta_{10})$. They are incorporated in a continuous and increasing function $g_3 = x_1 + x_2 + \cdots + x_{10}$. Calculate the value of $\text{Pos}\{g_3(\boldsymbol{\eta}_3) \geq 40\}$ by the fuzzy simulation.

After plugging the values of $\eta_i, i = 1, 2, \cdots, 10$ in Table 3.1, it is easy to calculate that $g_3(\boldsymbol{\eta}_3)$ is also a triangular fuzzy number $\mathcal{T}(26, 38, 52)$. Then, the exact possibility is calculated as 0.8571 in this example. When it comes to the simulation aspect, among the outputs of 100 membership degrees generated by the SDS in Table 3.2, the largest is 0.5128 and the smallest is 0.0006 (marked in bold). As a further step, 10000 outputs of sample points are displayed in Figure 3.5 to more aptly and credibly view the distribution of membership degrees, in which the maximal and minimal values are 0.6276 and 0 accordingly. Then, by applying Eq. (3.4), the simulated membership function of $g_3(\boldsymbol{\eta}_3)$ in Example 3.3 is acquired and illustrated in Figure 3.6. Obviously, from Figures 3.5 and 3.6, it seems hard for the SDS to generate membership degrees with larger values in [0.7,1] under the circumstance of ten fuzzy variables in $g_3(\boldsymbol{\eta}_3)$. In other words, it is of a small

Table 3.1: Ten triangular fuzzy numbers in Example 3.3.

Index	Triangular fuzzy number
η_1	$\mathcal{T}(2,3,4)$
η_2	$\mathcal{T}(5,6,8)$
η_3	$\mathcal{T}(6,7,8)$
η_4	$\mathcal{T}(4,5,6)$
η_5	$\mathcal{T}(3,4,6)$
η_6	$\mathcal{T}(7,9,10)$
η_7	$\mathcal{T}(-5,-3,-2)$
η_8	$\mathcal{T}(5,6,8)$
η_9	$\mathcal{T}(0,1,2)$
η_{10}	$\mathcal{T}(-1,0,2)$

probability to compute a larger membership degree via the SDS. Consequently, the membership function obtained through the SDS is greatly deviated from the exact one. Back to Example 3.3, the simulation result of $\text{Pos}\{g_3(\boldsymbol{\eta}_3) \geq 40\}$ towards 10000 sample points is 0.4776, which shows a much greater deviation of 44.28% than those of Examples 3.1 and 3.2.

Table 3.2: The membership degrees in Example 3.3 via the SDS when $N = 100$.

0.0025	0.0225	0.0093	0.1711	0.0882	**0.0006**	0.0264	0.0763	0.2273	0.0150
0.0668	0.1841	0.2460	0.1053	0.0511	0.0081	0.2199	0.0884	0.0032	0.1108
0.0554	0.0060	0.1370	0.2178	0.1898	0.1984	0.0704	0.1190	0.2290	0.0614
0.0658	0.1600	0.0177	0.1328	0.0835	0.0440	0.2897	0.1323	0.1659	0.2046
0.0228	0.0881	0.0130	0.0143	0.0064	0.1946	0.0621	0.0364	0.0384	0.0111
0.0772	0.1181	0.1099	0.1406	0.0188	0.0744	0.2079	0.1333	0.1851	0.0556
0.0381	0.0450	0.0275	0.0041	0.0558	0.0949	0.0204	0.1753	0.0696	**0.5128**
0.1630	0.0099	0.0458	0.0185	0.0810	0.1749	0.1921	0.0026	0.0278	0.0421
0.0330	0.0019	0.0808	0.0222	0.1192	0.0620	0.0079	0.2497	0.0090	0.0858
0.0782	0.2367	0.0050	0.0035	0.1560	0.1351	0.0361	0.0013	0.0366	0.1230

To sum up briefly, the approximate possibilities obtained from the SDS in

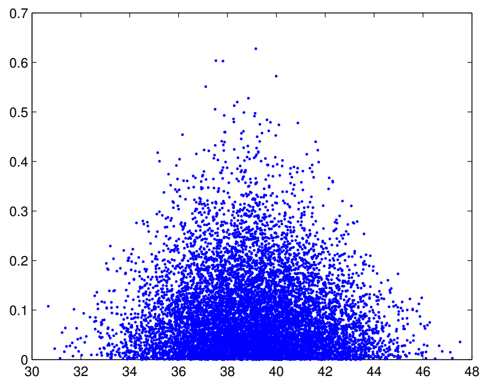


Figure 3.5: The distribution of 10000 sample points in Example 3.3 generated by the SDS.

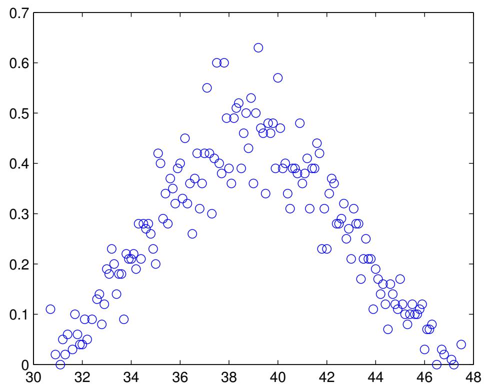


Figure 3.6: The simulated membership function of $g_3(\boldsymbol{\eta}_3)$ in Example 3.3 by the SDS.

Examples 3.1 $\sim$ 3.3 have different extents of deviation with their respective exact values. This deviation is more obviously seen on $g_3(\boldsymbol{\eta}_3)$ that contains a larger quantity of fuzzy variables. Meanwhile, the increase of sample points from 3000, 5000 to 10000 cannot aggrandize the accuracy of simulation, but instead bring up a more centralized distribution without larger membership degrees, e.g., in Figure 3.5 of Example 3.3. Therefore, the membership function of the discrete fuzzy variable $g_3(\boldsymbol{\eta}_3)^*$ has a distinct disparity with that of the original continuous fuzzy variable $g_3(\boldsymbol{\eta}_3)$ in Example 3.3.

In addition, two specific details in the algorithm design of the SDS are needed to be expounded. Firstly, the SDS starts from a lower estimation in Step 1 of Algorithm 1, and takes a hypercube containing the γ-cut set for fuzzy variables as an interval, by the intuition that people are usually not interested in the points with too low possibility [Liu02a]. In fact, this predetermined setting of interval is tested to have subtle influence on the outcome, which is further indicated by the numerical examples in the next section. Secondly, although the SDS is born with some defects, it is applicable to all kinds of fuzzy variables and general functions.

3.3 A Novel UDS

The fuzzy simulation discussed in this section is to approximate the value of $\text{Pos}\{g(\boldsymbol{\xi}) \leq 0\}$, where g is a continuous and strictly monotone function, and $\boldsymbol{\xi}$ is a fuzzy vector of regular fuzzy intervals. Firstly, a new operational law on the membership function of a continuous and strictly monotone function of regular fuzzy intervals is proposed and proved. Afterwards, based on the new operational law, a novel fuzzy simulation technique, called the *uniform discretization simulation* (UDS) is put forward to estimate the value of $\text{Pos}\{g(\boldsymbol{\xi}) \leq 0\}$.

3.3.1 Some basic concepts

Before the new operational law is proposed, the notions of regular fuzzy numbers, regular fuzzy intervals, and continuous and strictly monotone functions are introduced at first place.

Definition 3.1 (Zhou et al. [Zho16c]) *An LR fuzzy number is said to be regular if the shape functions L and R are continuous and strictly decreasing on the open intervals $\{0 < L(x) < 1\}$ and $\{0 < R(x) < 1\}$, respectively.*

Definition 3.2 (Regular Fuzzy Interval) *An LR fuzzy interval is said to be regular if the shape functions L and R are continuous and strictly decreasing on the open intervals $\{0 < L(x) < 1\}$ and $\{0 < R(x) < 1\}$, respectively.*

In accordance with the preliminaries of LR fuzzy numbers and LR fuzzy intervals in Section 2.1.2, the regular fuzzy number can be viewed as the degradation form of the regular fuzzy interval. Meanwhile, the definition of regular fuzzy numbers from the perspective of credibility distribution can also be found in [Zho16c]. Generally, triangular, normal, Gaussian, and trapezoidal fuzzy numbers are commonly used regular fuzzy intervals in practice. With the aid of the shape functions L and R in Eq. (2.3), for any $\alpha \in (0,1]$, the α-cuts of a regular fuzzy interval can be formulated via the inverse functions L^{-1} and R^{-1} of L and R as follows:

$$\begin{aligned} x_\alpha^L &= \underline{c} - \gamma L^{-1}(\alpha), \\ x_\alpha^R &= \bar{c} + \beta R^{-1}(\alpha). \end{aligned} \tag{3.6}$$

For some frequently used regular fuzzy intervals as mentioned, the analytical expressions of their L^{-1} and R^{-1} are usually not difficult to obtain. For instance, as to the four different kinds of regular fuzzy intervals that appear in three numerical examples in Section 3.2.2, their shape functions and α-cuts expressions are presented in the following two examples.

Example 3.4 The shape functions L and R of a triangular fuzzy number $\xi \sim (c, \gamma, \beta)_{LR}$ or $\mathcal{T}(c-\gamma, c, c+\beta)$ and a trapezoidal fuzzy number $\xi \sim (\underline{c}, \bar{c}, \gamma, \beta)_{LR}$ or $\mathcal{A}(\underline{c}-\gamma, \underline{c}, \bar{c}, \bar{c}+\beta)$ are written as

$$L(x) = R(x) = \max\{0, 1-x\}.$$

Then, their α-cuts for $\alpha \in (0,1]$, are formulated based on Eq. (3.6) as

$$\xi \sim (c, \gamma, \beta)_{LR} : \begin{cases} x_\alpha^L = c - \gamma(1-\alpha), \\ x_\alpha^R = c + \beta(1-\alpha). \end{cases} \tag{3.7}$$

$$\xi \sim (\underline{c}, \bar{c}, \gamma, \beta)_{LR} : \begin{cases} x_\alpha^L = \underline{c} - \gamma(1-\alpha), \\ x_\alpha^R = \bar{c} + \beta(1-\alpha). \end{cases} \tag{3.8}$$

Example 3.5 The shape functions L and R of a Gaussian fuzzy number $\xi \sim (c, w, w)_{LR}$ or $\mathcal{G}(c, w)$ is written as

$$L(x) = R(x) = e^{-x^2}.$$

And its α-cuts for $\alpha \in (0, 1]$ is expressed as

$$\xi \sim (c, w, w)_{LR} : \begin{cases} x_\alpha^L = c - w\sqrt{-\ln(\alpha)}, \\ \\ x_\alpha^R = c + w\sqrt{-\ln(\alpha)}. \end{cases} \tag{3.9}$$

For the regular fuzzy interval, $\eta_2 \sim (c, p, p)_{LR}$, in Example 3.2, its shape functions L and R are

$$L(x) = R(x) = e^{-|x|}.$$

The α-cuts for $\alpha \in (0, 1]$ is displayed as

$$\eta_2 \sim (c, p, p)_{LR} : \begin{cases} x_\alpha^L = c + p\ln(\alpha), \\ \\ x_\alpha^R = c - p\ln(\alpha). \end{cases} \tag{3.10}$$

Besides, another important notion, the continuous and strictly monotone function, is defined as follows:

Definition 3.3 (Liu [Liu15a]) *A real-valued function $f(x_1, x_2, \cdots, x_n)$ is said to be strictly monotone if it is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, that is*

$$f(x_1, \cdots, x_m, x_{m+1}, \cdots, x_n) \leq f(y_1, \cdots, y_m, y_{m+1}, \cdots, y_n)$$

whenever $x_i \leq y_i$, for $i = 1, 2, \cdots, m$ and $x_i \geq y_i$, for $i = m+1, m+2, \cdots, n$, and

$$f(x_1, \cdots, x_m, x_{m+1}, \cdots, x_n) < f(y_1, \cdots, y_m, y_{m+1}, \cdots, y_n)$$

whenever $x_i < y_i$, for $i = 1, 2, \cdots, m$ and $x_i > y_i$, for $i = m+1, m+2, \cdots, n$.

3.3.2 The new operational law

Based on the aforementioned basic concepts, a new operational law for the membership function of a fuzzy vector, $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$, incorporated in a continuous and strictly monotone function g is proposed and proved. The new operational law is initiated to substitute Theorem 3.1, and is given as follows:

Theorem 3.2 *Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent regular fuzzy intervals. If the continuous function $g(x_1, x_2, \cdots, x_n)$ is strictly increasing in regard to $x_1, x_2, \cdots, x_h$ and strictly decreasing in regard to $x_{h+1}, x_{h+2}, \cdots, x_n$, then the membership function of the fuzzy variable $g(\xi_1, \xi_2, \cdots, \xi_n)$ is*

$$\mu(x) = \mu_1(x_1)\Big| \quad x = g(x_1, x_2, \cdots, x_n), \ (x_1, x_2, \cdots, x_n) \in \mathcal{L} \cup \mathcal{R}, \tag{3.11}$$

where μ_1 is the membership function of ξ_1, and

$$\begin{aligned} \mathcal{L} &= \{(\xi_1^L(\alpha), \cdots, \xi_h^L(\alpha), \xi_{h+1}^R(\alpha), \cdots, \xi_n^R(\alpha)) : 0 < \alpha \le 1\}, \\ \mathcal{R} &= \{(\xi_1^R(\alpha), \cdots, \xi_h^R(\alpha), \xi_{h+1}^L(\alpha), \cdots, \xi_n^L(\alpha)) : 0 < \alpha \le 1\}, \end{aligned} \tag{3.12}$$

and $[\xi_i^L(\alpha), \xi_i^R(\alpha)]$ is the α-level set of ξ_i, $i = 1, 2, \cdots, n$, i.e.,

$$\xi_i^L(\alpha) = \inf\{r \mid \mathrm{Pos}\{\xi_i \le r\} \ge \alpha\},$$

$$\xi_i^R(\alpha) = \sup\{r \mid \mathrm{Pos}\{\xi_i \ge r\} \ge \alpha\}.$$

Proof: Here, only the case that $h = 1$ and $n = 2$ is proved. That is, $g(x_1, x_2)$ is continuous and strictly increasing with respect to x_1 and decreasing with respect to x_2. Suppose that there exist two different vectors, $(x_1, x_2) \in \mathcal{L} \cup \mathcal{R}$ and $(y_1, y_2) \in \mathcal{L} \cup \mathcal{R}$, such that $g(x_1, x_2) = g(y_1, y_2)$. Without loss of generality, assume that $x_1 < y_1$. Then it follows from the monotonicity of g that $x_2 < y_2$. Since the two vectors are both in $\mathcal{L} \cup \mathcal{R}$, according to the definition of $\mathcal{L}$ and $\mathcal{R}$, there exist α_1 and α_2 such that

$$\begin{cases} (x_1, x_2) = (\xi_1^L(\alpha_1), \xi_2^R(\alpha_1)) \ \ \textit{or} & (3.13) \\ \\ (x_1, x_2) = (\xi_1^R(\alpha_1), \xi_2^L(\alpha_1)) & (3.14) \end{cases}$$

$$\begin{cases} (y_1, y_2) = (\xi_1^L(\alpha_2), \xi_2^R(\alpha_2)) \ \ \textit{or} & (3.15) \\ \\ (y_1, y_2) = (\xi_1^R(\alpha_2), \xi_2^L(\alpha_2)) & (3.16) \end{cases}$$

There are four possible combinations, (3.13) + (3.15), (3.13) + (3.16), (3.14) + (3.15), (3.14) + (3.16). Following the strictly monotonicity of the shape functions of ξ_1 and ξ_2, it is easy to derive that any combination would lead to contradiction. Take (3.13) + (3.15) as an example. Since $x_1 < y_1$, i.e., $\xi_1^L(\alpha_1) < \xi_1^L(\alpha_2)$, it can be deduced that $\alpha_1 < \alpha_2$, which follows that $\xi_1^R(\alpha_1) > \xi_1^R(\alpha_2)$, equivalently, $x_2 > y_2$. The contradiction proves the uniqueness.

Simple and similar proof can be provided to prove the contradiction of the other three combinations. The proof is complete.

$\square$

3.3.3 Uniform discretization simulation

In this part, the underlying principle and specific steps of the UDS to simulate $\text{Pos}\{g(\boldsymbol{\xi}) \leq 0\}$ are elaborated at first, and a comparison is also conducted between the SDS and UDS regarding their algorithm designs. To complete the procedure of UDS, a novel uniform sampling method taking advantage of the new operational law in Theorem 3.2 is set forth.

More specifically, the continuous fuzzy variable $g(\boldsymbol{\xi})$, $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$ is transformed into a discrete fuzzy variable $g^*(\boldsymbol{\xi})$ from sets $\mathcal{L}$ and $\mathcal{R}$ uniformly with respect to $\alpha \in (0, 1]$. Let $\xi_i \sim (\underline{c}_i, \bar{c}_i, \gamma_i, \beta_i)_{LR}$ be the ith regular fuzzy interval with shape functions L_i and R_i, $i = 1, 2, \cdots, n$. Uniformly split the interval $(0, 1]$ into N pieces and write $\theta_j = j/N, j = 1, 2, \cdots, N$, in which N is a sufficiently large integer. Afterwards, based on Eq. (3.6), the discrete points in terms of the left and right α-cuts, x_{ij}^L and x_{ij}^R, can be respectively formulated as

$$\begin{aligned} x_{ij}^L &= \underline{c}_i - \gamma_i L_i^{-1}(\theta_j), \quad j = 1, 2, \cdots, N-1, \\ x_{ij}^R &= \bar{c}_i + \beta_i R_i^{-1}(\theta_j), \quad j = 1, 2, \cdots, N-1. \end{aligned} \tag{3.17}$$

Suppose that the function g is continuous and strictly increases in regard to $\xi_1, \xi_2, \cdots, \xi_h$, and strictly decreases in regard to $\xi_{h+1}, \xi_{h+2}, \cdots, \xi_n$. Then, it is denoted that

$$\begin{aligned}
&\boldsymbol{x}_j^L = (x_{1j}^L, x_{2j}^L, \cdots, x_{hj}^L, x_{(h+1)j}^R, \cdots, x_{nj}^R),\ j = 1, 2, \cdots, N-1,\\
&\boldsymbol{x}_j^R = (x_{1j}^R, x_{2j}^R, \cdots, x_{hj}^R, x_{(h+1)j}^L, \cdots, x_{nj}^L),\ j = 1, 2, \cdots, N-1,\\
&\underline{\boldsymbol{c}} = (\underline{c}_1, \underline{c}_2, \cdots, \underline{c}_h, \bar{c}_{h+1}, \cdots, \bar{c}_n),\\
&\bar{\boldsymbol{c}} = (\bar{c}_1, \bar{c}_2, \cdots, \bar{c}_h, \underline{c}_{h+1}, \cdots, \underline{c}_n).
\end{aligned} \tag{3.18}$$

Apparently, $\boldsymbol{x}_j^L \subset \mathcal{L}$ and $\boldsymbol{x}_j^R \subset \mathcal{R}$ for $j = 1, 2, \cdots, N-1$, where $\mathcal{L}$ and $\mathcal{R}$ are described in Theorem 3.2. By virtue of the above uniform sampling procedure, a new discrete fuzzy variable $g^*(\boldsymbol{\xi})$ is thereby defined to approach the continuous fuzzy variable $g(\boldsymbol{\xi})$ as

$$g^*(\boldsymbol{\xi}) = \begin{cases} g(\boldsymbol{x}_j^L), & \text{with membership degree } \theta_j,\ j = 1, 2, \cdots, N-1\\ g(\boldsymbol{x}_j^R), & \text{with membership degree } \theta_j,\ j = 1, 2, \cdots, N-1\\ g(\underline{\boldsymbol{c}}), & \text{with membership degree } 1\\ g(\bar{\boldsymbol{c}}), & \text{with membership degree } 1. \end{cases} \tag{3.19}$$

Notably, when applying Eqs. (3.17)-(3.19) to the version of regular fuzzy numbers $\xi_i \sim (c_i, \gamma_i, \beta_i)_{LR}, i = 1, 2, \cdots, n$, both $\underline{c}_i$ and $\bar{c}_i$ in Eq. (3.17) will be replaced by c_i, while both $\underline{\boldsymbol{c}}$ and $\bar{\boldsymbol{c}}$ in Eq. (3.18) will be substituted by $\boldsymbol{c} = (c_1, c_2, \cdots, c_n)$, and both $f(\underline{\boldsymbol{c}})$ and $f(\bar{\boldsymbol{c}})$ in Eq. (3.19) will be reduced to $f(\boldsymbol{c})$, respectively. Take $h = 1$ and $n = 2$ in Eq. (3.18) as an example, then the continuous function $g(x_1, x_2)$ is strictly increasing with x_1, and strictly decreasing with x_2. Suppose that η_1 and η_2 are both triangular fuzzy numbers with membership functions $\mu_1(x)$ and $\mu_2(x)$, respectively. Hereafter, the continuous fuzzy variable $g(\boldsymbol{\eta})$, $\boldsymbol{\eta} = (\eta_1, \eta_2)$ is approached by the following discrete fuzzy variable $g^*(\boldsymbol{\eta})$ as

$$g^*(\boldsymbol{\eta}) = \begin{cases} g(x_{1j}^L, x_{2j}^R), & \text{with membership degree } \theta_j,\ j = 1, 2, \cdots, N-1\\ g(x_{1j}^R, x_{2j}^L), & \text{with membership degree } \theta_j,\ j = 1, 2, \cdots, N-1\\ g(\boldsymbol{c}), & \text{with membership degree } 1, \end{cases} \tag{3.20}$$

where $\boldsymbol{x}_j^L = (x_{1j}^L, x_{2j}^R)$ and $\boldsymbol{x}_j^R = (x_{1j}^R, x_{2j}^L)$, and they are depicted in Figures 3.7 and 3.8, respectively.

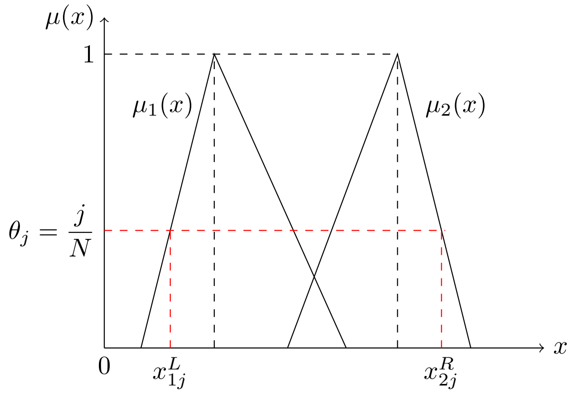


Figure 3.7: $\boldsymbol{x}_j^L = (x_{1j}^L, x_{2j}^R)$ in Eq. (3.20).

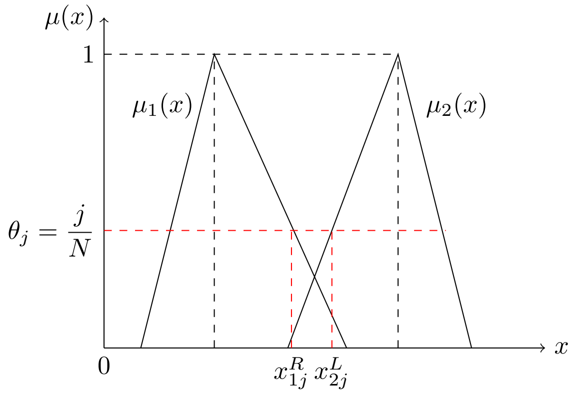


Figure 3.8: $\boldsymbol{x}_j^R = (x_{1j}^R, x_{2j}^L)$ in Eq. (3.20).

Back to the general situation, now $g^*(\boldsymbol{\xi})$ in Eq. (3.19) is a discrete fuzzy variable, and due to the monotonicity, the relationship of $f(\underline{\boldsymbol{c}}) < f(\bar{\boldsymbol{c}})$ can be easily obtained. Analogously to the SDS in Algorithm 1, here $\mathrm{Pos}\{g(\boldsymbol{\xi}) \le 0\}$ can be approximated by $M' = \mathrm{Pos}\{g(\boldsymbol{\xi})^* \le 0\}$ as

$$M' = \begin{cases} \max\limits_{1 \le j \le N-1} \{\theta_j \,|\, g(x_j^L) \le 0\}, & \text{if } g(\underline{\boldsymbol{c}}) > 0 \\ \\ 1, & \text{if } g(\underline{\boldsymbol{c}}) \le 0, \end{cases} \tag{3.21}$$

and $\mathrm{Pos}\{g(\boldsymbol{\xi}) \ge 0\}$ is estimated by $M'' = \mathrm{Pos}\{g^*(\boldsymbol{\xi}) \ge 0\}$ as

$$M'' = \begin{cases} \max\limits_{1 \le j \le N-1} \{\theta_j \,|\, g(x_j^R) \ge 0\}, & \text{if } g(\bar{\boldsymbol{c}}) < 0 \\ \\ 1, & \text{if } g(\bar{\boldsymbol{c}}) \ge 0. \end{cases} \tag{3.22}$$

In accordance with the above detailed elaboration of a novel uniform sampling method and the calculation formulae in Eqs. (3.21)-(3.22), the specific steps of the UDS to approximate $\mathrm{Pos}\{g(\boldsymbol{\xi}) \le 0\}$ and $\mathrm{Pos}\{g(\boldsymbol{\xi}) \ge 0\}$ are respectively arranged as follows:

Algorithm 2 (UDS for $\mathrm{Pos}\{g(\boldsymbol{\xi}) \le 0\}$)

Step 1. Initialize the number of sample points N. Set $M' = 0$ and $j = 1$.

Step 2. Calculate $g(\underline{\boldsymbol{c}})$, where $\underline{\boldsymbol{c}} = (\underline{c}_1, \underline{c}_2, \cdots, \underline{c}_h, \bar{c}_{h+1}, \bar{c}_{h+2}, \cdots, \bar{c}_n)$.

Step 3. If $g(\underline{\boldsymbol{c}}) \leq 0$, return $M' = \text{Pos}\{g(\boldsymbol{\xi}) \leq 0\} = 1$. Otherwise, go to Step 4.

Step 4. Denote $\theta_j = j/N$, calculate $g(\boldsymbol{x}_j^L)$ with respect to Eq. (3.19).

Step 5. If $g(\boldsymbol{x}_j^L) \leq 0$, and $M' < \theta_j$, then reset $M' = \theta_j$ and $j = j + 1$.

Step 6. If $j < N$, go to Step 4. Otherwise go to Step 7.

Step 7. Return M' as the simulation value of $\text{Pos}\{g(\boldsymbol{\xi}) \leq 0\}$.

Algorithm 3 (UDS for $\text{Pos}\{g(\boldsymbol{\xi}) \geq 0\}$)

Step 1. Initialize the number of sample points N. Set $M'' = 0$ and $j = 1$.

Step 2. Calculate $g(\bar{\boldsymbol{c}})$, where $\bar{\boldsymbol{c}} = (\bar{c}_1, \bar{c}_2, \cdots, \bar{c}_h, \underline{c}_{h+1}, \underline{c}_{h+2}, \cdots, \underline{c}_n)$.

Step 3. If $g(\bar{\boldsymbol{c}}) \geq 0$, return $M'' = \text{Pos}\{g(\boldsymbol{\xi}) \geq 0\} = 1$. Otherwise, go to Step 4.

Step 4. Denote $\theta_j = j/N$, calculate $g(\boldsymbol{x}_j^R)$ with respect to Eq. (3.19).

Step 5. If $g(\boldsymbol{x}_j^R) \geq 0$, and $M'' < \theta_j$, then reset $M'' = \theta_j$ and $j = j + 1$.

Step 6. If $j < N$, go to Step 4. Otherwise go to Step 7.

Step 7. Return M'' as the simulation value of $\text{Pos}\{g(\boldsymbol{\xi}) \geq 0\}$.

The distinctions between the SDS and UDS regarding their corresponding objects, philosophies, and additional outcomes are summarized in Table 3.3. Combining this comparison with the former three numerical examples conducted by the SDS in Section 3.2.2, it can be seen that although the SDS is equipped with a larger application range in objects, the internal stochastic sampling procedure may lead to imprecise possibility simulations, and further affect the accuracy of the membership function. The UDS is proposed to refine this defect and can also provide a relatively fitted membership function, which will be validated vividly by numerical examples in the forthcoming section.

Table 3.3: Distinctions between the SDS and UDS in algorithm design.

Algorithm	SDS Liu & Iwamura [Liu98a]	UDS proposed in this chapter
Object	all kinds fuzzy variables general functions	regular fuzzy intervals strictly monotone functions
Philosophy	Theorem 3.1 stochastic sampling	Theorem 3.2 uniform sampling by α-cuts
Attached outcome	membership function of a discrete fuzzy number $g(\boldsymbol{\xi})^*$	membership function of a discrete fuzzy number $g^*(\boldsymbol{\xi})$

3.3.4 Three numerical examples for the UDS

In order to test the performance of the UDS in contrast to that of the SDS, the identical three numerical examples in Section 3.2.2 are also adopted here to carry out comparisons on the simulation accuracy and time between these two algorithms. The simulation results are computed and displayed in Table 3.4.

From Table 3.4, it is observed that in the SDS, a pair of outputs with or without a hypercube are listed simultaneously. The reason of employing this hypercube can be found in [Liu02a]. That is, as known a lower estimation of γ ($\gamma > 0$) is given in Algorithm 1, then $u_1, u_2, \cdots, u_n$ are randomly generated from the γ-level sets of $\xi_1, \xi_2, \cdots, \xi_n$, respectively. However, if this γ-level is not easy for computers to identify, then a larger region like a hypercube that contains the γ-level set will be further defined. For the UDS part, Example 3.1 is simulated by Algorithm 2, while Examples 3.2 and 3.3 are attained via Algorithm 3, in which the calculations of α-cuts of the four regular fuzzy intervals incorporated in these three numerical examples are illustrated in Eqs. (3.7)-(3.10) in Section 3.3.2.

Some conclusions can be drawn from Table 3.4. First of all, from the aspect of accuracy, the preciseness of the UDS can be straightforwardly seen since its

Table 3.4: Simulation results of the SDS and UDS in Examples 3.1 $\sim$ 3.3.

Example 3.1: Exact value: $\text{Pos}\{g_1(\boldsymbol{\eta}_1) \le 0\} = 0.7788$, $g_1 = x_1 - x_2$.						
Sample points	**SDS** (Hypercube)	Error	Time(s)	**SDS** (None)	Error	Time(s)
3000	0.7646	1.82%	0.000	0.7739	0.63%	0.000
5000	0.7646	1.82%	0.000	0.7739	0.63%	0.000
Sample points	**UDS**	Error	Time(s)			
3000	0.7787	0.01%	0.000			
5000	0.7788	0.00%	0.000			
Example 3.2: Exact value: $\text{Pos}\{g_2(\boldsymbol{\eta}_2) \ge 4\} = 0.9086$, $g_2 = x_1^2 + x_2 * x_3 - x_4^{-1}$.						
Sample points	**SDS** (Hypercube)	Error	Time(s)	**SDS** (None)	Error	Time(s)
3000	0.8068	11.01%	0.001	0.7960	12.39%	0.001
5000	0.8752	3.68%	0.001	0.8581	5.56%	0.001
Sample points	**UDS**	Error	Time(s)			
3000	0.9087	0.01%	0.000			
5000	0.9086	0.00%	0.001			
Example 3.3: Exact value: $\text{Pos}\{g_3(\boldsymbol{\eta}_3) \ge 40\} = 0.8571$, $g_3 = x_1 + x_2 + \cdots + x_{10}$.						
Sample points	**SDS** (Hypercube)	Error	Time(s)	**SDS** (None)	Error	Time(s)
3000	0.3883	54.70%	0.001	0.3988	53.47%	0.001
5000	0.5699	33.51%	0.002	0.4776	44.28%	0.002
10000	0.5699	33.51%	0.003	0.4776	44.28%	0.003
Sample points	**UDS**	Error	Time(s)			
3000	0.8570	0.01%	0.000			
5000	0.8570	0.01%	0.000			
10000	0.8571	0.00%	0.001			

simulation results are nearly the same as the corresponding exact values, 0.7788, 0.9086, and 0.8571, respectively. When it comes to the performance of the SDS, neither the increase of sample points, or the presence or absence of a hypercube does not greatly help enhance the accuracy of the final simulation results. Meanwhile, the adoption of a hypercube is not stable and reliable, e.g., hypercubes involved in Examples 3.2 and 3.3 have a small positive impact while it performs adversely in Example 3.1. In addition, with the increasing of the number of fuzzy variables contained in the function g from two in g_1 to four in g_2, and ten in g_3, it is noticed that the error degree of the SDS becomes larger and larger accordingly. The last but not the least, the computational time of both the SDS and UDS are very short, and the UDS is slightly quicker when encoutering functions with more fuzzy variables.

Apart from the simulation results of possibilities and the computational time, the membership function of $g^*(\boldsymbol{\xi})$ can also be acquired from the UDS as attached outcomes. As an illustration, Figure 3.9 depicts the simulated membership function of $g_1(\boldsymbol{\eta}_1) = \eta_1 - \eta_2$, which is similar to the one generated by the SDS in Figure 3.2. Nonetheless, in terms of the membership function of $g_3(\boldsymbol{\eta}_3) = \eta_1 + \eta_2 + \cdots + \eta_{10} \sim \mathcal{T}(26, 38, 52)$, it is found that the simulated membership function of $g_3(\boldsymbol{\eta}_3)$ by the UDS in Figure 3.10 is of high accuracy. While another simulated one by the SDS in Figure 3.6 is not clear enough and also not capable of reaching larger membership degrees over 0.7.

3.4 The UDS-Joint for $\mathrm{Pos}\{g_k(\boldsymbol{\xi}) \leq 0, k = 1, 2, \cdots, s\}$

In the former section, the scenario of individual possibilistic constraint is discussed, and detailed comparisons are generated between the SDS and UDS. Not limited to this, in many fuzzy programming models, it is common to meet the

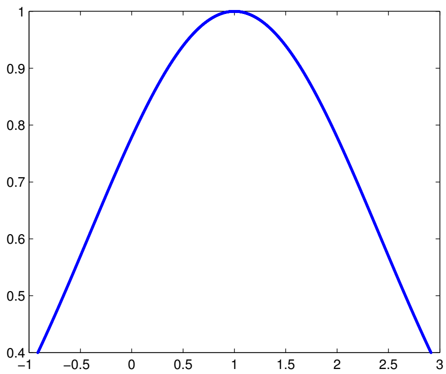


Figure 3.9: The simulated membership function of $g_1(\boldsymbol{\eta}_1)$ in Example 3.1 by the UDS.

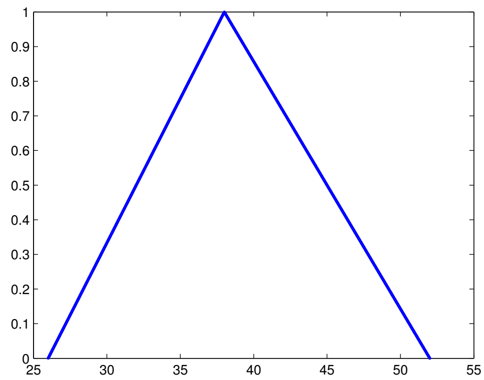


Figure 3.10: The simulated membership function of $g_3(\boldsymbol{\eta}_3)$ in Example 3.3 by the UDS.

scenario of joint possibilistic constraints. Therefore, another theorem is proposed as well as another UDS-based algorithm, UDS-Joint, is designed in this section to handle this situation. Finally, the UDS-joint is implemented to a numerical example to illustrate its effectiveness.

Theorem 3.3 *Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent regular fuzzy intervals, and denote $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$. If the functions $f_k(x_1, x_2, \cdots, x_n)$, $k = 1, 2, \cdots, s$, are continuous and strictly monotone, then*

$$\mathrm{Pos}\{f_k(\boldsymbol{\xi}) \leq 0, k = 1, 2, \cdots, s\} = \min_{1 \leq k \leq s} \mathrm{Pos}\{f_k(\boldsymbol{\xi}) \leq 0\}. \tag{3.23}$$

where the values of $\mathrm{Pos}\{f_k(\boldsymbol{\xi}) \leq 0\}, k = 1, 2, \cdots, s$ are obtained according to Theorem 3.2.

Proof: The proof is based on the condition that $n = 2$ and $s = 2$.

Case 1: f_1 and f_2 are both continuous and increasing functions. According to Theorem 3.2, suppose that a pair of $(x_1, x_2) = (\xi_1^L(\alpha_1), \xi_1^L(\alpha_1)) \in \mathcal{L} \cup \mathcal{R}$ will satisfy $f_1(x_1, x_2) \leq 0$, which does not satisfy $f_2(x_1, x_2) \leq 0$. Another pair of $(x_1', x_2') = (\xi_1^L(\alpha_2), \xi_1^L(\alpha_2)) \in \mathcal{L} \cup \mathcal{R}$ holds for $f_2(x_1', x_2') \leq 0$. Due to the monotonicity, it is easy to figure out that $x_1 > x_1'$, and $x_2 > x_2'$, which means $\alpha_2 < \alpha_1$, and under this condition, $f_1(x_1', x_2') \leq 0$ also holds.

Case 2: f_1 is continuous and increasing, while f_2 is a continuous and monotone function which increases with x_1 and decreases with x_2. In this case, if $(x_1, x_2) = (\xi_1^L(\alpha_1), \xi_1^L(\alpha_1))$ holds for $f_1 \leq 0$, and $(x_1', x_2') = (\xi_1^L(\alpha_2), \xi_1^R(\alpha_2))$ holds for $f_2 \leq 0$. Then $f_1(\xi_1^L(\alpha_2), \xi_1^L(\alpha_2)) \leq 0$ will only hold when $\alpha_2 < \alpha_1$.

Case 3: f_1 and f_2 are both continuous and monotone functions which increase with x_1 and decrease with x_2. In this case, $(x_1, x_2) = (\xi_1^L(\alpha_1), \xi_1^R(\alpha_1))$, and $(x_1', x_2') = (\xi_1^L(\alpha_2), \xi_1^R(\alpha_2))$. In accordance with the monotonicity, it is obtained that $x_1 > x_1'$, $x_2 < x_2'$ and $\alpha_2 < \alpha_1$, and meanwhile $f_1(x_1', x_2') \leq 0$ also holds.

From the above three cases, it is attained that

$$\begin{aligned}
\mathrm{Pos}\left\{ \begin{array}{c} f_1(\xi_1, \xi_2) \leq 0 \\ f_2(\xi_1, \xi_2) \leq 0 \end{array} \right\} &= \min\Big\{\mathrm{Pos}\{f_1(\xi_1, \xi_2) \leq 0\}, \mathrm{Pos}\{f_2(\xi_1, \xi_2) \leq 0\}\Big\} \\
&= \min\{\alpha_1, \alpha_2\} = \alpha_2.
\end{aligned} \tag{3.24}$$

The proof is complete.

□

It is known that the possibility of an individual fuzzy event can be easily solved by the UDS. Based on Theorem 3.3, an advanced algorithm called the UDS-Joint is set up to obtain the minimum possibility among s individual fuzzy events as the possibility of a joint fuzzy event, and the details are delivered as follows:

Algorithm 4 (UDS-Joint for $\mathrm{Pos}\{g_k(\boldsymbol{\xi}) \leq 0, k = 1, 2, \cdots, s\}$)

Step 1. Initialize the number of sample points N.

Step 2. Simulate $\mathrm{Pos}\{g_k(\boldsymbol{\xi}) \leq 0\}$ via the UDS in Algorithm 2 to attain the approximation values $M_k', k = 1, 2, \cdots, s$, respectively.

Step 3. Return the minimum value in $\{M'_1, M'_2, \cdots, M'_s\}$ as the simulation value for $\text{Pos}\{g_k(\boldsymbol{\xi}) \le 0, k = 1, 2, \cdots, s\}$.

Example 3.6 Denote that $\boldsymbol{\xi} = (\eta_1, \eta_2, \cdots, \eta_{10})$, where $\eta_i, i = 1, 2, \cdots, 10$ are ten independent triangular fuzzy numbers in Table 3.1. They are incorporated in three continuous and monotone functions $g_3 = x_1 + x_2 + \cdots + x_{10}$, $g_4 = x_1 + \cdots + x_5 - x_6 - \cdots - x_{10}$, and $g_5 = -(x_1 \wedge x_2 \wedge \cdots \wedge x_{10})$, respectively. Calculate the possibilities of joint fuzzy events: (1) $\text{Pos}\left\{\begin{array}{l} g_3(\boldsymbol{\xi}) \le 35 \\ g_4(\boldsymbol{\xi}) \le 6 \end{array}\right\}$, (2) $\text{Pos}\left\{\begin{array}{l} g_3(\boldsymbol{\xi}) \le 35 \\ g_5(\boldsymbol{\xi}) \le 2.6 \end{array}\right\}$, and (3) $\text{Pos}\left\{\begin{array}{l} g_3(\boldsymbol{\xi}) \le 35 \\ g_4(\boldsymbol{\xi}) \le 6 \\ g_5(\boldsymbol{\xi}) \le 2.6 \end{array}\right\}$, respectively.

First of all, it is simple to find $g_3(\boldsymbol{\xi}) \sim \mathcal{T}(26, 38, 52)$ and $g_4(\boldsymbol{\xi}) \sim \mathcal{T}(0, 12, 26)$ by means of a simple fuzzy computation. Further, the exact values can be figured out as $\text{Pos}\{g_3(\boldsymbol{\xi}) \le 35\} = 0.75$, $\text{Pos}\{g_4(\boldsymbol{\xi}) \le 6\} = 0.50$, and $\text{Pos}\{g_5(\boldsymbol{\xi}) \le 2.6\} = 0.60$, respectively. According to Theorem 3.3, the exact values for the joint events are 0.50, 0.60, and 0.50, respectively. The simulation results of the UDS-Joint are listed in Table 3.5, in which the number of sample points N are all set to be 10000. Under this condition, both the UDS and UDS-Joint can return the precise approximation results perfectly with no error. As to the computational time, it is intuitive that the UDS-Joint may need some time when dealing with a joint fuzzy event, especially when the expression of function g is complicated and the number of s is large.

3.5 Summary

As is known, the possibility measure, Pos, was the first attempt to evaluate the belief degree of fuzzy events, which plays a significant role in the fuzzy field. As a further application, Pos was employed to modify fuzzy constraints into possibilistic constraints in fuzzy chance-constrained programming models. This

Table 3.5: Simulation results of the UDS in Example 3.6.

Algorithm	Fuzzy event	Exact value	Simulation value	Error	Time(s)
UDS	$\text{Pos}\{g_3(\boldsymbol{\xi}) \leq 35\}$	0.75	0.75	0.00%	0.001
	$\text{Pos}\{g_4(\boldsymbol{\xi}) \leq 6\}$	0.50	0.50	0.00%	0.001
	$\text{Pos}\{g_5(\boldsymbol{\xi}) \leq 2.6\}$	0.60	0.60	0.00%	0.002
UDS-Joint	(1)	0.50	0.50	0.00%	0.001
	(2)	0.60	0.60	0.00%	0.003
	(3)	0.50	0.50	0.00%	0.006

chapter introduced a novel fuzzy simulation technique on possibilistic constraints, which can also be integrated in heuristic model-solving algorithms, like the HIA.

The main contributions of this chapter lie in the following three aspects. Firstly, although the SDS was widely acknowledged as a pioneer technique for simulating the possibility, its actual operation did not strictly follow Zadeh's extension principle. This inherent deficiency of the SDS was detailedly analyzed and verified from both theory and practice. Secondly, a new operational law for the membership function of continuous and strictly monotone functions of regular fuzzy intervals was raised and proved. On this basis, a novel simulation technique is initiated, namely, the UDS with a uniform sampling process inside, to compete with the SDS. It turns out that the UDS can return quite satisfactory simulation results with nearly no error in a very short time period, whereas the SDS deviates more from the exact values especially when encountering different kinds of fuzzy variables or complicated functions. Lastly, since the UDS was designed for individual fuzzy event, for the sake of handling joint fuzzy events in practice, the UDS is further extended to the UDS-Joint for a larger application region according to another new theorem.

It is noted that both the UDS and UDS-joint are applicable to continuous and strictly monotone functions of regular fuzzy intervals, as one of the future directions, these two algorithms may be stretched to more kinds of fuzzy variables and general functions. Another direction is, the UDS and UDS-Joint can be embedded in heuristic algorithms to solve fuzzy programming models with possibilistic constraints towards real-life optimization problems, which may help enhance the preciseness of the final solutions. In addition, the novel uniform sampling procedure inside the UDS may provide insights for the subsequent new fuzzy simulations of expected value, credibility, or other important notions in the fuzzy area.

The next chapter deals with the fuzzy simulation for the expected value of fuzzy events. Two improved fuzzy simulation techniques for the expected value are put forward to compete with two existing techniques. One is partly based on the new operational law in Chapter 3, and the other utilizes the analytical expression of α-optimistic values of fuzzy variables.

Chapter 4

On Fuzzy Simulations for Expected Values of Functions of Fuzzy Numbers and Intervals

In the previous chapter, several fuzzy simulation techniques on possibilities of individual and joint fuzzy events were discussed. Based on existing fuzzy simulation algorithms, this chapter presents two innovative techniques for approximating the expected values of fuzzy numbers' monotone functions, which is of utmost importance in fuzzy optimization literature. In this regard, the stochastic discretization algorithm presented by [Liu02b] is enhanced by updating the discretization process for the simulation of the membership function and the calculation formula for the expected values. This is achieved through initiating a novel uniform sampling process and employing a formula for discrete fuzzy numbers, respectively, as the generated membership function in the stochastic discretization algorithm would adversely affect its accuracy to some extent.

What is more, in consideration of the fact that the bisection procedure involved in the numerical integration algorithm in [Lix15] is time-consuming and also not necessary for the specified types of fuzzy numbers, a special numerical integration algorithm is proposed. It simplifies the simulation procedure by adopting the analytical expressions of α-optimistic values.

Subsequently, as for the extensive applications of regular fuzzy intervals, several theorems are introduced and proved as an extended effort to apply the improved stochastic discretization algorithm and the special numerical integration algorithm to the issues of fuzzy intervals. Throughout the chapter, a series of numerical experiments are conducted from which the superiority of both the two novel techniques over others are conspicuously displayed in aspects of accuracy, stability, and efficiency.

4.1 Introduction

Intuitively, the expected value is a well documented measurement of great importance in both academic literature and real-world applications. In particular, for the mathematical study of the mean value of fuzzy numbers, several definitions have been proposed by leading researchers in the relevant literature. In this direction, Dubois and Prade [Dub87] constructed the expected value on the foundations of possibility theory for a fuzzy number, and it was formulated as an interval bounded by expected values obtained using the upper and lower distribution functions. Further, Heilpern [Hei92] introduced the concepts of expected interval and expected value of fuzzy numbers, and the latter was calculated as the center of the former. Lower and upper possibilistic mean values were studied by Carlsson and Fullér [Car01] as well as the relation between the interval-valued possibilistic and probabilistic means. All the above definitions are framed upon the possibility measure. However, the possibility alongside with the necessity measure has been proved to have a lack of self-duality, which might unavoidably lead to counterintuitive results. Thus, in this regard, Liu and Liu [Liu02b] established the credibility measure by taking advantage of the average of the possibility and necessity measurements to compensate for this serious limitation. In addition, they proposed an expected value operator by utilizing the credibility measure and Choquet integral.

In real-life projects, it seems reasonable that measuring expected values for different functions that contain fuzzy parameters to obtain a general evaluation, like the expected value of the wind speed [Zho19] or the expected value of the lifetime of a certain product [Zha05, Zho18]. For a single fuzzy variable, based on the credibility measure, Xue *et al.* [Xue08] derived a direct formula for calculating the exact expected value of a monotone function of a fuzzy variable with a continuous membership function. However, the existence of a variety of structures for the fuzzy numbers, and particularly for their complex functions, derives further challenges on the analytical calculation of the expected value when it is compared with the single fuzzy number counterpart.

Alternatively, the use of fuzzy simulation techniques provides us with an effective method to approximate the expected value. In this regard, a *stochastic discretization algorithm* (SDA) was employed by Liu and Liu [Liu02b] to simulate the expected value. The basic idea of the SDA is firstly to transform continuous fuzzy numbers to discrete ones through a stochastical generation of sample points, and then to compute the mean values for functions of these discrete counterparts. Since its establishment, the SDA has not only gained extensive support in the fuzzy expected value simulation literature, but also played a critical role in solving fuzzy expected value models whose target is to optimize the expected objectives with respect to several expected constraints. The SDA along with the SDA-based heuristic algorithms has been widely employed in handling fuzzy expected value models in various areas like portfolio selection with fuzzy returns [Lix09, Zho16b], system reliability analysis [Lix18], project scheduling problem [Keh10], amongst others.

Analogous to the SDA, Liu [Liu06b] proposed a *uniform discretization algorithm* (UDA) from the perspective of uniformly generating sample points, whose guiding principle is the convergence concept of sequences for fuzzy numbers. In practice, the UDA appears to be far more complex both as a concept and calculation procedure. Li [Lix15] commented that both the SDA and UDA demonstrate good performance of accuracy and computational time when it comes to functions of fuzzy numbers with low dimensions, but they fail to return satisfactory approximation values as the dimension increases substantially. Therefore, Li [Lix15] introduced a *numerical integration algorithm* (NIA) to calculate expected values by means of α-optimistic values of strictly monotone functions of regular fuzzy numbers (i.e., a special type of LR fuzzy numbers with continuous and strictly decreasing shape functions, such as triangular, normal and Gaussian fuzzy numbers in [Lix15, Zho16c]), which was proved to be stable and reliable.

The fact that the SDA would return inaccurate results when high-dimensional functions occurred was reflected by the comparative results of numerical experiments between the SDA and NIA in Li [Lix15]'s work. Chapter 3 further explained the reasoning behind it, indicating that the membership degrees utilized in the SDA were not obtained by sticking strictly to Zadeh's extension principle. In addition, it is known that the SDA is not merely designed for singular use, but can also be served as a significant step in solving fuzzy expected value models where the SDA is incorporated in a *hybrid intelligent algorithm* (HIA). This sophisticated algorithm was first proposed by Liu [Liu02a] and later gained great popularity in applications ([Zha05, Lix09, Keh10]). However, as pointed out early, the computation of the SDA was proved to be not accurate both from the scenarios of theory and practice. In order to better facilitate the integration of expected value simulation to the HIA for more precise solutions of fuzzy expected value models, the inherent deficiencies of the SDA are rectified in this chapter.

Therefore, on the basis of the contents in Chapter 3, this chapter proposes an *improved stochastic discretization algorithm* (iSDA) to generate the expected value simulation for continuous and strictly monotone functions involving regular fuzzy numbers, in which not only the stochastic sampling process in the SDA is substituted by a novel uniform sampling process, but also the original calculation formula of the expected value is replaced by another discrete calculation formula. More specifically, a novel simulation method of sampling and fitting membership functions of continuous and strictly monotone functions that contain regular fuzzy numbers is proposed. Through this method, simulated membership functions of higher accuracy are obtained compared with those attained from the SDA. Afterwards, some analytical supplementaries for the NIA are carried out and a special NIA (NIA-S) is thereby proposed so as to further simplify the NIA when the analytical expressions of α-optimistic values of regular fuzzy numbers are not so complicated to derive. In addition, due to the vast number of real-world applications for regular fuzzy intervals (i.e., a special type of LR fuzzy intervals with continuous and strictly decreasing shape functions, such as the trapezoidal fuzzy numbers), some theorems about α-optimistic and α-pessimistic values, and expected values of continuous and strictly monotone functions of regular fuzzy intervals are proposed and proved. On this basis, for fuzzy intervals, the extension algorithms of the iSDA and NIA are introduced, respectively. It should be noted that the discussions in this chapter mainly focus on fuzzy numbers and fuzzy intervals, while fuzzy variables cover a larger range.

The rest of the chapter is organized as follows. In Section 4.2, the concepts of the SDA and iSDA are expounded, whose performances are demonstrated by three numerical experiments. Subsequently, in Section 4.3, the algorithm designs of the NIA and NIA-S, together with some connections and differences between the iSDA, NIA, and NIA-S, are elaborated through other three numerical examples. Section 4.4 introduces regular fuzzy intervals, related theorems, and algorithms along with the conduction of two illustrative examples of four kinds of functions. Finally, Section 4.5 presents some conclusions of the whole chapter.

4.2 Improved Stochastic Discretization Algorithm

In 1998, Liu and Iwamura [Liu98a, Liu98b] firstly proposed a fuzzy simulation technique, known as the *stochastic discretization simulation* (SDS), which aims at calculating the possibility of a fuzzy event (as discussed in Chapter 3). Later, the SDS was extended to the SDA to simulate the expected value, where the credibility measure [Liu02b] is employed.

In this section, the specific contents of the SDA including its basic principle and algorithm steps are reviewed first together with two derived deficiencies. Then, a novel uniform sampling method of generating membership functions of regular fuzzy numbers, and the expected value calculation formula for a discrete fuzzy number are successively elaborated. Based on them, the iSDA is put forward to handle the deficiencies derived by the SDA approach.

4.2.1 Stochastic discretization algorithm

Liu and Liu [Liu02b] defined the expected value of fuzzy variables in light of the credibility measure (see Definition 2.5) as follows.

Definition 4.1 (Liu and Liu [Liu02b], Liu [Liu06a]) *Let ξ be a fuzzy variable with membership function μ. Then the expected value of ξ is defined by*

$$E[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Cr}\{\xi \leq r\}\mathrm{d}r \tag{4.1}$$

provided that at least one of the two integrals is finite, in which Cr *is the credibility measure with*

$$\mathrm{Cr}\{\xi \geq r\} = \frac{1}{2}\Big(\sup_{x\geq r}\mu(x) + 1 - \sup_{x<r}\mu(x)\Big),$$

$$\mathrm{Cr}\{\xi \leq r\} = \frac{1}{2}\Big(\sup_{x\leq r}\mu(x) + 1 - \sup_{x>r}\mu(x)\Big).$$

Suppose that f is an n-ary real-valued function, and ξ_i are fuzzy numbers with respective membership functions μ_i, $i = 1, 2, \cdots, n$. Then $f(\boldsymbol{\xi})$, $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$, is also a fuzzy variable (Liu [Liu02a]), whose expected value is given by

$$E[f(\boldsymbol{\xi})] = \int_0^{+\infty} \mathrm{Cr}\{f(\boldsymbol{\xi}) \geq r\}\mathrm{d}r - \int_{-\infty}^0 \mathrm{Cr}\{f(\boldsymbol{\xi}) \leq r\}\mathrm{d}r. \tag{4.2}$$

For the purpose of estimating $E[f(\boldsymbol{\xi})]$ as well as for solving a fuzzy expected value model, the following process was proposed by Liu and Liu [Liu02b]. Randomly generate u_1^j, u_2^j, $\cdots, u_n^j (j = 1, 2, \cdots, m)$ from the ϵ-level sets of ξ_1, ξ_2, $\cdots, \xi_n$, respectively, in which m is a sufficiently large integer, while ϵ is a sufficiently small number. Denote $\boldsymbol{u}_j = (u_1^j, u_2^j, \cdots, u_n^j)$ and $v_j = \mu_1(u_1^j) \wedge \mu_2(u_2^j) \wedge \cdots \wedge \mu_n(u_n^j)$ for $j = 1, 2, \cdots, m$. Accordingly, for any $r \in \mathbb{R}$, the credibilities $\mathrm{Cr}\{f(\boldsymbol{\xi}) \geq r\}$ and $\mathrm{Cr}\{f(\boldsymbol{\xi}) \leq r\}$ can be respectively estimated by

$$\begin{aligned} E^R(r) &= \frac{1}{2}\Big(\max_{j=1,2,\cdots,m}\{v_j \mid f(\boldsymbol{u}_j) \geq r\} + 1 - \max_{j=1,2,\cdots,m}\{v_j \mid f(\boldsymbol{u}_j) < r\}\Big), \\ E^L(r) &= \frac{1}{2}\Big(\max_{j=1,2,\cdots,m}\{v_j \mid f(\boldsymbol{u}_j) \leq r\} + 1 - \max_{j=1,2,\cdots,m}\{v_j \mid f(\boldsymbol{u}_j) > r\}\Big). \end{aligned} \tag{4.3}$$

In Eq. (4.3), if one of the two sets is empty, then the maximal value is 0. Through applying the SDA, continuous fuzzy numbers are converted to discrete counterparts. Thus, the expected value of the function with respect to these discrete fuzzy numbers can be derived by Eq. (4.2). To summarize, the steps of the SDA are given in Algorithm 1.

Algorithm 1 (SDA of Liu and Liu [Liu02b])

Step 1. Initialize the numbers of sample points m and integration points N, and a sufficient small number ϵ. Set $E = 0$.

Step 2. Randomly generate $u_1^j, u_2^j, \cdots, u_n^j$ from the ϵ-level sets of $\xi_1, \xi_2, \cdots, \xi_n$, respectively, and denote $\boldsymbol{u}_j = (u_1^j, u_2^j, \cdots, u_n^j)$ for $j = 1, 2, \cdots, m$.

Step 3. Identify the minimal and maximal values $p = f(\boldsymbol{u}_1) \wedge f(\boldsymbol{u}_2) \wedge \cdots \wedge f(\boldsymbol{u}_m)$ and $q = f(\boldsymbol{u}_1) \vee f(\boldsymbol{u}_2) \vee \cdots \vee f(\boldsymbol{u}_m)$, respectively.

Step 4. Randomly generate a real number r from $[p, q]$.

Step 5. If $r \geq 0$, reset $E = E + E^R(r)$.

Step 6. If $r < 0$, reset $E = E - E^L(r)$.

Step 7. Repeat the Steps 4, 5, 6 for N times.

Step 8. Return $E[f(\boldsymbol{\xi})] = p \vee 0 + q \wedge 0 + E \cdot (q - p)/N$.

Except for the initialization in Step 1 of Algorithm 1, the SDA mainly contains two parts. The first part (Step 2) targets on transforming continuous fuzzy numbers to discrete counterparts through random generation of sample points, while the second part (Steps 3 to 8) intends to attain the mean value based on Eqs. (4.2)-(4.3) via the integration simulation. As discussed in Section 3.2, the limitation of the SDS was found and explicitly proved. And meanwhile, the SDS and SDA share the same stochastic sampling process, whose membership degree $\mu(a)^*$ for $f(\boldsymbol{\xi})$ at a real number a is expressed as

$$\mu(a)^* = \max_{1 \leq j \leq m} \{ \min_{1 \leq i \leq n} \mu_i(u_i^j) \mid f(u_1^j, u_2^j, \cdots, u_n^j) = a \}. \tag{4.4}$$

Technically, Eq. (4.4) is capable of obtaining a satisfactory membership degree when the number of sample points m is large enough. However, from the aspect of actual operation of this stochastic sampling process, the general setting of m is a relatively small quantity of 10^3 or 10^4 level regardless of the dimension n, which does not strictly follow Zadeh's extension principle [Zad75].

With respect to the new operational law in Chapter 3, a novel uniform sampling method and a novel simulation technique, namely iSDA, which concerns LR fuzzy numbers are proposed here to improve the SDA. Additionally, in the iSDA, the original calculation formula of the expected value, as illustrated in Step 8 of Algorithm 1, is also substituted. Both the basic principle and the iSDA are explained in detail in the following section.

4.2.2 Improved stochastic discretization algorithm

For this part, a specialized type of LR fuzzy numbers is considered (see Definition 2.7) with continuous and strictly decreasing shape functions L and R on the open intervals $\{x \,|\, 0 < L(x) < 1\}$ and $\{x \,|\, 0 < R(x) < 1\}$ respectively, which are called regular fuzzy numbers in [Zho16c] and utilized in [Lix15, Zho16c].

Three commonly used regular fuzzy numbers are given in Examples 4.1-4.3 as follows, including the triangular, normal, and Gaussian fuzzy numbers.

Example 4.1 *When the shape functions L and R are written by the following form,*

$$L(x) = R(x) = \max\{0, 1 - x\},$$

the corresponding LR fuzzy number is a triangular fuzzy number, whose membership function is determined by the triplet (a, c, b) with $a < c < b$ as

$$\mu_{\mathcal{T}}(x) = \begin{cases} \dfrac{x-a}{c-a}, & \text{if } a \le x \le c \\[2ex] \dfrac{x-b}{c-b}, & \text{if } c < x \le b \\[2ex] 0, & \text{otherwise,} \end{cases}$$

which can also be denoted as $\xi = (c, c-a, b-c)_{LR}$ or $\xi \sim \mathcal{T}(a, c, b)$.

Example 4.2 *When the shape functions L and R are written by the following form,*

$$L(x) = R(x) = 2\left(1 + \exp(\pi x/\sqrt{6})\right)^{-1},$$

the corresponding LR fuzzy number is a normal fuzzy number, whose membership function is known as

$$\mu_{\mathcal{N}}(x) = 2\left(1 + \exp\left(\pi|x-c|/\sqrt{6}\sigma\right)\right)^{-1}, \quad x \in \mathbb{R}, \sigma > 0,$$

which can also be expressed by $\xi = (c, \sigma, \sigma)_{LR}$ or $\xi \sim \mathcal{N}(c, \sigma)$.

Example 4.3 *When the shape functions L and R are written by the following form,*

$$L(x) = R(x) = e^{-x^2},$$

the LR fuzzy number is a Gaussian fuzzy number, whose membership function is expressed as

$$\mu_{\mathcal{G}}(x) = e^{-(\frac{x-c}{b})^2}, \quad x \in \mathbb{R}, b > 0,$$

and can also be represented by $\xi = (c, b, b)_{LR}$ or $\xi \sim \mathcal{G}(c, b)$.

As for regular fuzzy numbers $\xi_i (i = 1, 2, \cdots, n)$ and a continuous and strictly monotone function f defined in Definition 3.3 [Liu15a], the operational law for the membership function of a fuzzy number $f(\boldsymbol{\xi}), \boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$, is given in Theorem 4.1 in accordance with Theorem 3.2 in Chapter 3.

Theorem 4.1 *Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent regular fuzzy numbers. If the continuous function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing in regard to $x_1, x_2, \cdots$, x_h and strictly decreasing in regard to $x_{h+1}, x_{h+2}, \cdots$, x_n, then the membership function of the fuzzy number $f(\xi_1, \xi_2, \cdots, \xi_n)$ is*

$$\mu(x) = \mu_1(x_1)\Big| \quad x = f(x_1, x_2, \cdots, x_n), \ (x_1, x_2, \cdots, x_n) \in \mathcal{L} \cup \mathcal{R},$$

where μ_1 is the membership function of ξ_1,

$$\mathcal{L} = \{(\xi_1^L(\alpha), \cdots, \xi_h^L(\alpha), \xi_{h+1}^R(\alpha), \cdots, \xi_n^R(\alpha)) : 0 < \alpha \leq 1\},$$

$$\mathcal{R} = \{(\xi_1^R(\alpha), \cdots, \xi_h^R(\alpha), \xi_{h+1}^L(\alpha), \cdots, \xi_n^L(\alpha)) : 0 < \alpha \leq 1\},$$

and $[\xi_i^L(\alpha), \xi_i^R(\alpha)]$ is the α-level set of ξ_i, $i = 1, 2, \cdots, n$, i.e.,

$$\xi_i^L(\alpha) = \inf\{r \mid \mathrm{Cr}\{\xi_i \leq r\} \geq \alpha\},$$

$$\xi_i^R(\alpha) = \sup\{r \mid \mathrm{Cr}\{\xi_i \geq r\} \geq \alpha\}.$$

Based on Theorem 4.1, a novel uniform sampling method is initiated to approximate the continuous fuzzy number, $f(\boldsymbol{\xi})$, by using a discrete counterpart, $f^*(\boldsymbol{\xi})$, $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$. First, denote the closure of the support of ξ_i by $S_i = [a_i, b_i]$ for $i = 1, 2, \cdots, n$ (the support of ξ_i contains all x with $\mu_{\xi_i}(x) > 0$). When the range of S_i is not finite, a set including the most values is utilized to substitute S_i as an alternative. Since ξ_i is regular, it is easy to know that there exists one and only one value $c_i \in S_i$ such that $\mu_{\xi_i}(c_i) = 1$ and $a_i < c_i < b_i$. Second, define

$$\begin{aligned} x_{ij}^L &= a_i + (c_i - a_i) \times \tfrac{j}{k},\ j = 0, 1, \cdots, k-1, \\ x_{ij}^R &= b_i - (b_i - c_i) \times \tfrac{j}{k},\ j = 0, 1, \cdots, k-1, \end{aligned} \tag{4.5}$$

and write

$$\begin{aligned} &\boldsymbol{X}_j^L = (x_{1j}^L, \cdots, x_{hj}^L, x_{h+1j}^R, \cdots, x_{nj}^R),\ j = 0, 1, \cdots, k-1, \\ &\boldsymbol{X}_j^R = (x_{1j}^R, \cdots, x_{hj}^R, x_{h+1j}^L, \cdots, x_{nj}^L),\ j = 0, 1, \cdots, k-1, \\ &\boldsymbol{c} = (c_1, c_2, \cdots, c_n). \end{aligned} \tag{4.6}$$

Afterwards, a new discrete fuzzy number, $f^*(\boldsymbol{\xi})$, is defined as follows:

$$f^*(\boldsymbol{\xi}) = \begin{cases} f(\boldsymbol{X}_j^L), & \text{with membership degree } \mu_1(x_{1j}^L), \\ & \quad j = 0, 1, \cdots, k-1 \\ f(\boldsymbol{X}_j^R), & \text{with membership degree } \mu_1(x_{1j}^R), \\ & \quad j = 0, 1, \cdots, k-1 \\ f(\boldsymbol{c}), & \text{with membership degree 1.} \end{cases} \tag{4.7}$$

Denote $\mathcal{L}' = \{\boldsymbol{X}_0^L, \boldsymbol{X}_1^L, \cdots, \boldsymbol{X}_{k-1}^L\}$ and $\mathcal{R}' = \{\boldsymbol{X}_0^R, \boldsymbol{X}_1^R, \cdots, \boldsymbol{X}_{k-1}^R\}$. Obviously $\mathcal{L}'$ and $\mathcal{R}'$ are respectively subsets of $\mathcal{L}$ and $\mathcal{R}$ defined in Theorem 4.1.

It is easy to derive that the discrete fuzzy number $f^*(\boldsymbol{\xi})$ is in close proximity to the continuous fuzzy number $f(\boldsymbol{\xi})$, when k is large enough. As a consequence, the mean value of $f^*(\boldsymbol{\xi})$ can be reasonably viewed to be an approximation of the expected value of $f(\boldsymbol{\xi})$. Subsequently, by taking advantage of the calculation formula of the expected value of discrete fuzzy numbers presented in both [Liu02a] and [Liu02b], the expected value of $f^*(\boldsymbol{\xi})$ is calculated by

$$E[f^*(\boldsymbol{\xi})] = \sum_{j=0}^{k-1} w_j f(\boldsymbol{X}_j^L) + w_k f(\boldsymbol{c}) + \sum_{j=0}^{k-1} w_{m-j} f(\boldsymbol{X}_j^R), m = 2k, \tag{4.8}$$

where w_j, $j = 0, 1, \cdots, 2k$, are ascertained by

$$\begin{aligned} w_j &= \frac{1}{2}\Big(\max_{t \le j} \mu(f(\boldsymbol{X}_t^L)) - \max_{t<j} \mu(f(\boldsymbol{X}_t^L)) + \max_{t \ge j} \mu(f(\boldsymbol{X}_t^L)) - \max_{t>j} \mu(f(\boldsymbol{X}_t^L)) \Big), \\ &\quad j = 0, 1, \cdots, k-1, \\ w_k &= \frac{1}{2}\left(2 - \mu(f(\boldsymbol{X}_{k-1}^L)) - \mu(f(\boldsymbol{X}_{k-1}^R))\right), \\ w_{m-j} &= \frac{1}{2}\Big(\max_{t \le j} \mu(f(\boldsymbol{X}_t^R)) - \max_{t<j} \mu(f(\boldsymbol{X}_t^R)) + \max_{t \ge j} \mu(f(\boldsymbol{X}_t^R)) - \max_{t>j} \mu(f(\boldsymbol{X}_t^R)) \Big), \\ &\quad j = 0, 1, \cdots, k-1, \end{aligned} \tag{4.9}$$

and μ represents the membership function of $f^*(\boldsymbol{\xi})$ in Eq. (4.7). Further, utilizing the strict monotonicity of the shape functions of ξ_1 (that is, $\mu_1(x_{1i}^L) < \mu_1(x_{1j}^L)$ and $\mu_1(x_{1i}^R) > \mu_1(x_{1j}^R)$ hold for all $i < j$), Eq. (4.9) can be simplified as follows:

$$\begin{aligned} w_0 &= \frac{1}{2}\mu_1(x_{10}^L), \quad w_m = \frac{1}{2}\mu_1(x_{10}^R), \\ w_j &= \frac{1}{2}\big(\mu_1(x_{1j}^L) - \mu_1(x_{1(j-1)}^L)\big), j = 1, 2, \cdots, k-1, \\ w_k &= 1 - \frac{1}{2}\big(\mu_1(x_{1(k-1)}^L) + \mu_1(x_{1(k-1)}^R)\big), \\ w_{m-j} &= \frac{1}{2}\big(\mu_1(x_{1j}^R) - \mu_1(x_{1(j-1)}^R)\big), j = 1, 2, \cdots, k-1. \end{aligned} \tag{4.10}$$

Therefore, a novel simulation technique, namely iSDA, to simulate the expected value $E[f(\boldsymbol{\xi})]$ is proposed by combining the uniform sampling process in Eqs. (4.5)-(4.7) and the expected value calculation formula for discrete fuzzy numbers in Eqs. (4.8)-(4.10). And the detailed procedure of the iSDA is described as follows:

Algorithm 2 (iSDA)

Step 1. Initialize the number of sample points m. Set $k = m/2$, $E = 0$ and $j = 0$.

Step 2. Calculate $f(\boldsymbol{X}_j^L)$ with Eqs. (4.5)-(4.6).

Step 3. Calculate w_j with Eq. (4.10). Reset $E = E + w_j f(\boldsymbol{X}_j^L)$ and $j = j + 1$.

Step 4. If $j < k$, go to Step 2. Otherwise, reset $j = 0$ and go to Step 5.

Step 5. Calculate $f(\boldsymbol{X}_j^R)$ with Eqs. (4.5)-(4.6).

Step 6. Calculate w_{m-j} with Eq. (4.10). Reset $E = E + w_{m-j} f(\boldsymbol{X}_j^R)$ and $j = j + 1$.

Step 7. If $j < k$, go to Step 5. Otherwise, go to Step 8.

Step 8. Calculate $f(\boldsymbol{c})$ and w_k. Reset $E = E + w_k f(\boldsymbol{c})$.

Step 9. Return E as the simulation value of the expected value $E[f(\boldsymbol{\xi})]$.

Similarly to the SDA, the calculation procedure of the iSDA basically consists of two parts. Steps 2 and 5 indicate the uniform sampling process, and Steps 3, 6 and 8 represent the expected value calculation procedure for the discrete fuzzy number, $f^*(\boldsymbol{\xi})$.

In order to clearly demonstrate the feasibility and effectiveness of the iSDA, a series of contrast outcomes of the SDA and iSDA considering different fuzzy variables and functions are presented in the following two subsections. Furthermore, since the calculation formula in Step 8 of Algorithm 1 is not easy to be understood, and to observe the efficiency of this formula, an intermediate simulation algorithm, SDA*, is specifically designed. It employs the same uniform sampling process with the iSDA in Steps 2 and 5 of Algorithm 2 and utilizes the same calculation procedure of the expected value with the SDA from Steps 3 to 8 of Algorithm 1.

4.2.3 Comparative study between the SDA and iSDA: the case of triangular fuzzy numbers

The comparative results between the SDA and iSDA as well as for the SDA* facilitating a numerical example are presented in this section, including the simulation accuracy, computational time, and complexity analysis of each algorithm.

Example 4.4 *Suppose that* $\eta_i, i = 1, 2, \cdots, 10,$ *are independent triangular fuzzy numbers listed in Table 4.1, incorporated in a continuous and strictly increasing function* $f_1(x_1, x_2, \cdots, x_{10}) = x_1 + x_2 + \cdots + x_{10}$*. This example [Lix15] aims at calculating the expected value,* $E[\xi]$*, of the fuzzy number,* $\xi = f_1(\eta_1, \eta_2, \cdots, \eta_{10})$*.*

Table 4.1: Different kinds of regular fuzzy numbers utilized in examples.

Index	Triangular Fuzzy Number	Normal Fuzzy Number	Gaussian Fuzzy Number
η_1	$\mathcal{T}(2,3,4)$	$\mathcal{N}(0,1)$	$\mathcal{G}(0,1)$
η_2	$\mathcal{T}(5,6,8)$	$\mathcal{N}(0,2)$	$\mathcal{G}(0,2)$
η_3	$\mathcal{T}(6,7,8)$	$\mathcal{N}(1,2)$	$\mathcal{G}(1,2)$
η_4	$\mathcal{T}(4,5,6)$	$\mathcal{N}(2,4)$	$\mathcal{G}(2,4)$
η_5	$\mathcal{T}(3,4,6)$	$\mathcal{N}(4,6)$	$\mathcal{G}(4,6)$
η_6	$\mathcal{T}(7,9,10)$	$\mathcal{N}(5,8)$	$\mathcal{G}(5,8)$
η_7	$\mathcal{T}(-5,-3,-2)$	$\mathcal{N}(-1,2)$	$\mathcal{G}(-1,2)$
η_8	$\mathcal{T}(5,6,8)$	$\mathcal{N}(-3,6)$	$\mathcal{G}(-3,6)$
η_9	$\mathcal{T}(0,1,2)$	$\mathcal{N}(-5,2)$	$\mathcal{G}(-5,2)$
η_{10}	$\mathcal{T}(-1,0,2)$	$\mathcal{N}(-7,7)$	$\mathcal{G}(-7,7)$

Before the simulation is conducted, in terms of the linearity towards the expected value operator for independent fuzzy numbers proved by [Liu02b], the exact value of $E[\xi]$ can be calculated in a straightforward manner, that is,

$$E[\xi] = E[\eta_1] + E[\eta_2] + \cdots + E[\eta_{10}] = 38.5.$$

With regard to Example 4.4, the SDA, SDA*, and iSDA are operated ten times for each and their simulation results are recorded in Table 4.2, accordingly. The quantity of integration points in the SDA is settled as 10000, while those of sample points in the three algorithms are all set as 1000. In order to express the relative error degree of all the results via the three algorithms, the same index called "*Error*" is displayed on the fourth column of Table 4.2, which is defined in Eq. (3.5) in Chapter 3. Note that the simulation value utilized in the calculation of the Error in Table 4.2 is the average value of the ten times simulation results. From Table 4.2, it is clear seen that both the stability and accuracy of the iSDA are superior to those of the SDA and SDA*.

Table 4.2: Ten comparative results among the SDA, SDA*, and iSDA for Example 4.4.

Algorithm	Simulation Value of $E[\xi]$ for Time 1-10				Deviation	Error
	38.8528	38.8765	38.8738			
SDA	38.8918	38.8617	38.8764		0.02	1.00%
	38.9195	38.8955	38.8751	38.9079		
	38.4900	38.6885	38.2088			
SDA*	38.4986	38.0610	38.4801		0.21	0.26%
	38.2785	38.2741	38.7293	38.3066		
	38.4990	38.4990	38.4990			
iSDA	38.4990	38.4990	38.4990		0.00	0.00%
	38.4990	38.4990	38.4990	38.4990		

In order to further demonstrate the performance of the SDA, SDA*, and iSDA, their simulation values, deviation, and computational time are obtained through the variation of the quantity of sample points m as well as that of integration points N. Particularly, m in the SDA is changed when N is set to be 10000 or 20000 respectively to test whether the increasing of sample points will positively affect the accuracy of the final results. Accordingly, the detailed results towards the above-mentioned experiment are displayed in Table 4.3 and visualized in Figure 4.1. It is noted that the simulation value and the computational time listed in this table as well as in any subsequent tables are all the average values of running the corresponding algorithm for ten times.

Table 4.3: Comparative results among the SDA, SDA*, and iSDA for Example 4.4.

Algorithm	Number of Sample Points m	Number of Integration Points N	Simulation Value of $E[\xi]$	Error	CPU Time(s)
SDA	1000	10000	38.8831	1.00%	0.180
	3000	10000	38.7269	0.59%	0.527
	5000	10000	38.8252	0.84%	0.843
	10000	10000	38.8618	0.94%	1.907
	15000	10000	38.7456	0.64%	2.866
	20000	10000	38.7887	0.75%	3.898
	1000	20000	38.8902	1.01%	0.328
	3000	20000	38.7277	0.59%	0.845
	5000	20000	38.8238	0.84%	1.950
	10000	20000	38.8492	0.91%	3.643
	15000	20000	38.7456	0.64%	5.938
	20000	20000	38.8043	0.79%	7.800
SDA*	1000	1000	38.4015	0.26%	0.022
	3000	3000	38.4164	0.22%	0.169
	5000	5000	38.4400	0.16%	0.393
	10000	10000	38.4612	0.10%	1.777
	15000	15000	38.4802	0.05%	4.053
	20000	20000	38.4889	0.03%	7.294
iSDA	1000	none	38.4990	0.00%	0.001
	3000	none	38.4997	0.00%	0.002
	5000	none	38.4998	0.00%	0.003
	10000	none	38.4999	0.00%	0.004
	15000	none	38.4999	0.00%	0.005
	20000	none	38.5000	0.00%	0.006

The detailed analyses of Table 4.3 and Figure 4.1 are presented here. First, from the point of view of the derived accuracy, as shown in Figure 4.1, the results of the iSDA are steadily converged and almost coincide with the exact value 38.5, which nearly provide no error even when the number of integration points m is small (e.g., $m = 1000$ in Table 4.3). It can also be seen that under

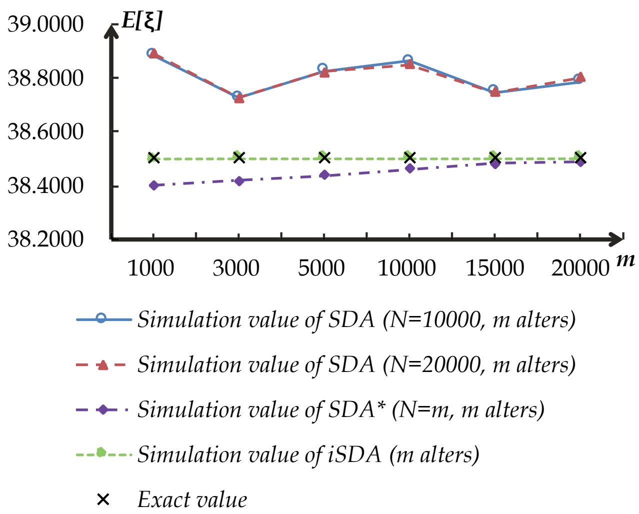


Figure 4.1: The visualisation of comparative results in Table 4.3.

different combinations of m and N, the biggest error of the SDA is 1.01%, and the changes of m in the SDA do not affect positively the final results. Second, from the stability point of view, the experimental results of the SDA have a larger deviation, while the iSDA is quite stable in returning the simulation values. Third, the computational time of iSDA is hundreds of times faster than SDA. More precisely, in Table 4.3 it is noticed that the longest time of the SDA is 7.800s and that of the iSDA is only 0.006s. It is known that the computational time has a strong relationship with the algorithm complexity, whose expressions of the SDA and iSDA are O(mN) and O(N), respectively. To summarize, the iSDA is equipped with prominent advantages speaking of accuracy, stability, and operation speed in contrast to the SDA.

As an intermediate algorithm, the results of the SDA* are listed in Table 4.3 and depicted in Figure 4.1 as well. It can be seen that both the accuracy and the computational time of the SDA* are not comparable to those of the iSDA. On the one hand, through comparing the SDA* with the SDA, it reveals that the uniform sampling method is reasonable and effective, especially reflected on the convergence of the simulation results in the SDA*. On the other hand, through comparing the SDA* with the iSDA, the effectiveness of the calculation formula of the expected value of discrete fuzzy numbers in Eqs. (4.8)-(4.10) utilized in the iSDA can also be validated. These two comparisons demonstrate the feasibility and reliability of the two improvements for the iSDA.

4.2.4 Comparative study between different functions and fuzzy numbers

Two more examples are given in this section to further demonstrate the superiority of the iSDA among the other two algorithms.

Example 4.5 *This example targets on simulating the expected values of the same function f_1 of Example 4.4 using the SDA, SDA*, and iSDA, but the fuzzy variables included are triangular, normal, and Gaussian fuzzy numbers, see Table 4.1, respectively.*

The simulation results are illustrated in Table 4.4, in which (m/N) represents the number of sample points m, or integration points N, adopted in each algorithm. Since the support S_i of a normal fuzzy number $\mathcal{N}(c_i, \sigma_i)$ or a Gaussian fuzzy number $\mathcal{G}(c_i, b_i)$ is infinite, $S_i = [c_i - g\sigma_i, c_i + g\sigma_i]$ or $S_i = [c_i - gb_i, c_i + gb_i]$ is obtained, where g is a positive integer. The range of the support S_i in a triangular fuzzy number is finite. It is known that $\pm 6\sigma$ can cover a relative large range of values to 99.99966%, then three values, 1, 3, and 6 are assigned to g to observe the differences.

From Table 4.4, it is obvious that the iSDA is rather accurate, reliable, and fast on the outputs. In addition, the iSDA performs better when $g = 6$ in contrast to $g = 1$, which reflects its sensitivity to the support S_i. Compared with the SDA, the effect of the SDA* is enhanced due to the replacement of the stochastic sampling process, but still is far from obtaining accurate values, especially for normal and Gaussian fuzzy numbers.

Table 4.4: Comparative results among the SDA, SDA*, and iSDA for the case that $f_1 = x_1 + x_2 + \cdots + x_{10}$.

Algorithm (m/N)	Triangular	Normal			Gaussian		
SDA (3000/10000)		$g = 1$	$g = 3$	$g = 6$	$g = 1$	$g = 3$	$g = 6$
Exact Value	38.5000	-4.0000	-4.0000	-4.0000	-4.0000	-4.0000	-4.0000
Simulation Value	38.7269	-3.5069	-4.2920	-7.5761	-4.3014	-5.6030	-9.2436
Error	0.59%	12.33%	7.30%	89.40%	7.54%	40.08%	131.09%
CPU Time (s)	0.527	0.476	0.479	0.498	0.468	0.498	0.495
SDA* (10000/10000)		$g = 1$	$g = 3$	$g = 6$	$g = 1$	$g = 3$	$g = 6$
Exact Value	38.5000	-4.0000	-4.0000	-4.0000	-4.0000	-4.0000	-4.0000
Simulation Value	38.4612	-3.8402	-3.6853	-3.7063	-3.8285	-3.7282	-3.8002
Error	0.10%	4.00%	7.87%	7.34%	4.29%	6.80%	5.00%
CPU Time (s)	1.777	1.188	1.413	1.816	1.411	1.583	1.807
iSDA (10000/none)		$g = 1$	$g = 3$	$g = 6$	$g = 1$	$g = 3$	$g = 6$
Exact Value	38.5000	-4.0000	-4.0000	-4.0000	-4.0000	-4.0000	-4.0000
Simulation Value	38.4999	-3.9983	-3.9995	-4.0000	-3.9985	-4.0000	-4.0000
Error	0.00%	0.04%	0.01%	0.00%	0.04%	0.00%	0.00%
CPU Time (s)	0.004	0.006	0.006	0.006	0.005	0.006	0.006

Example 4.6 *A more complex function $f_2 = -(x_1 \wedge x_2 \wedge \cdots \wedge x_{10})$ is employed in this example. Calculate the expected values of f_2 of triangular, normal, and Gaussian fuzzy numbers by the SDA, SDA*, and iSDA, respectively.*

The simulation results of Example 4.6 are recorded in Table 4.5, which share some similar conclusions with those of Example 4.5. From Tables 4.4 and 4.5, several remarks on the three algorithms used are outlined. First, the results of Examples 4.5 and 4.6 are similar with those derived from Example 4.4, as there still exist great differences in accuracy and time between the SDA and iSDA. Second, in terms of the parameter g, generally the performance of the SDA is barely acceptable for $g = 1$, but when g gets larger, the results become worse. Whereas the iSDA returns the simulation results of the highest accuracy at $g = 6$, and the biggest error is 0.03%. Third, the SDA* reduces the error rate due to the incorporated uniform sampling process compared with the SDA. In summary, the results of three examples demonstrate that the iSDA works better regardless of different functions or kinds of fuzzy variables.

Table 4.5: Comparative results among the SDA, SDA*, and iSDA for the case that $f_2 = -(x_1 \wedge x_2 \wedge \cdots \wedge x_{10})$.

Algorithm (m/N)	Triangular	Normal			Gaussian		
SDA (3000/10000)		$g = 1$	$g = 3$	$g = 6$	$g = 1$	$g = 3$	$g = 6$
Exact Value	3.2500	8.8300	8.8300	8.8300	8.2664	8.2664	8.2664
Simulation Value	3.4669	8.1734	12.9329	22.9128	8.0238	13.1042	23.4327
Error	6.67%	7.44%	46.47%	159.49%	2.93%	58.52%	183.47%
CPU Time (s)	0.783	0.576	0.565	0.608	0.534	0.591	0.600
SDA* (10000/10000)		$g = 1$	$g = 3$	$g = 6$	$g = 1$	$g = 3$	$g = 6$
Exact Value	3.2500	8.8300	8.8300	8.8300	8.2664	8.2664	8.2664
Simulation Value	3.2458	7.8663	8.7163	8.7785	7.9114	8.2307	8.2065
Error	0.13%	10.91%	1.29%	0.58%	4.29%	0.43%	0.72%
CPU Time (s)	1.753	1.565	1.688	1.844	1.682	1.713	1.810
iSDA (10000/none)		$g = 1$	$g = 3$	$g = 6$	$g = 1$	$g = 3$	$g = 6$
Exact Value	3.2500	8.8300	8.8300	8.8300	8.2664	8.2664	8.2664
Simulation Value	3.2500	7.8745	8.7462	8.8271	7.9176	8.2657	8.2652
Error	0.00%	10.82%	0.95%	0.03%	4.22%	0.01%	0.01%
CPU Time (s)	0.004	0.006	0.006	0.006	0.006	0.006	0.006

4.3 Special Numerical Integration Algorithm

With respect to the particular case of continuous and strictly monotone functions of regular fuzzy numbers (also called ordinary fuzzy variables in [Lix15]), Li [Lix15] proposed a *numerical integration algorithm* (NIA) to approximate expected values by means of the concept of α-optimistic values. In this section, the NIA and its related principles and concepts are primarily recalled. Subsequently, after the analytical expressions of α-optimistic values for regular fuzzy numbers are derived, owing to the specific features of regular fuzzy numbers, a *special numerical integration algorithm* (NIA-S) is further proposed to simplify the calculation procedure of the original NIA set forth by Li [Lix15] (renamed as a general NIA, NIA-G for short, in this chapter for being distinguishable).

4.3.1 General numerical integration algorithm

Before the calculation procedure of the NIA-G proposed in [Lix15] is introduced, the relevant definitions and theorems are brought in.

Definition 4.2 (Liu [Liu04]) *The credibility distribution of a fuzzy variable ξ is defined as*

$$\Phi(x) = \mathrm{Cr}\{\xi \le x\}, \quad \forall x \in \mathbb{R}. \tag{4.11}$$

Analogously, $\Psi(x) = \mathrm{Cr}\{\xi \ge x\}$ is denoted, and $\Psi + \Phi \equiv 1$ if ξ is a continuous fuzzy variable, which implies that

$$\Psi(x) = 1 - \Phi(x). \tag{4.12}$$

Definition 4.3 (Liu [Liu04]) *For any $\alpha \in (0,1]$, the α-optimistic value of a fuzzy variable ξ is*

$$\xi_{\sup}(\alpha) = \sup\{r \mid \mathrm{Cr}\{\xi \ge r\} \ge \alpha\}. \tag{4.13}$$

Theorem 4.2 (Li [Lix15]) *If ξ is a regular fuzzy number, for any $\alpha \in (0,1]$, we have that*

$$\xi_{\sup}(\alpha) = \Psi^{-1}(\alpha). \tag{4.14}$$

Assuming that the membership function, μ_ξ, of a regular fuzzy number ξ is known, Ψ can be deduced via μ_ξ as follows,

$$\Psi(x) = \begin{cases} \mu_\xi(x)/2, & \text{if } x \ge c \\ 1 - \mu_\xi(x)/2, & \text{if } x < c, \end{cases} \tag{4.15}$$

in which $\mu_\xi(c) = 1$.

According to the mathematical property of μ_ξ, it is found that Ψ is continuous and strictly decreasing. Then in terms of Eqs. (4.14)-(4.15), Li [Lix15] designed a bisection algorithm to simulate $\xi_{\sup}(\alpha)$ for any given $\alpha \in (0, 1]$.

Algorithm 3 (Bisection Algorithm of Li [Lix15])

Step 1. Initialize a small enough number $\epsilon > 0$, and $[a, b]$ such that $\Psi(a) > \alpha > \Psi(b)$.

Step 2. Denote $d = (a + b)/2$.

Step 3. If $\Psi(d) > \alpha$, reset $a = d$. If $\Psi(d) < \alpha$, reset $b = d$. Otherwise, stop and return d.

Step 4. If $|\Psi(b) - \Psi(a)| \le \epsilon$, return $(a + b)/2$. Otherwise, go to Step 2.

On this basis, α-optimistic values are derived, which can be further utilized to obtain mean values for continuous and strictly monotone functions of regular fuzzy numbers by the following theorem.

Theorem 4.3 (Li [Lix15]) *Assume that $\xi_1, \xi_2, \cdots, \xi_n$ are independent regular fuzzy numbers. If the function $f(x_1, x_2, \cdots, x_n)$ is continuous and strictly increases in regard to $x_1, x_2, \cdots, x_h$ and strictly decreases in regard to $x_{h+1}, x_{h+2}, \cdots, x_n$, for any $\alpha \in (0, 1]$, the expected value of $f(\boldsymbol{\xi}) = f(\xi_1, \xi_2, \cdots, \xi_n)$ is given by*

$$E[f(\boldsymbol{\xi})] = \int_0^1 f\Big((\xi_1)_{\sup}(\alpha), \cdots, (\xi_h)_{\sup}(\alpha), (\xi_{h+1})_{\sup}(1-\alpha), \cdots, (\xi_n)_{\sup}(1-\alpha)\Big) \mathrm{d}\alpha. \tag{4.16}$$

According to Eq. (4.16), Li [Lix15] designed an integration simulation algorithm NIA-G to calculate $E[f(\boldsymbol{\xi})]$ by utilizing $\xi_{\sup}(\alpha)$ obtained from the bisection algorithm.

Algorithm 4 (NIA-G of Li [Lix15])

Step 1. Initialize the number of integration points N. Set $E = 0$ and $k = 1$.

Step 2. Let $\alpha = k/N$. Using Algorithm 3, for each $1 \le i \le n$, calculate

$$x_i = \begin{cases} (\xi_i)_{\sup}(\alpha), & \text{if } 1 \le i \le h \\ (\xi_i)_{\sup}(1-\alpha), & \text{if } h < i \le n. \end{cases}$$

Step 3. Reset $E = E + f(x_1, x_2, \cdots, x_n)/N$ and $k = k + 1$.

Step 4. If $k \le N$, go to Step 2. Otherwise return E as the simulation value of the expected value $E[f(\boldsymbol{\xi})]$.

4.3.2 Special numerical integration algorithm

As a matter of fact, for the commonly used regular fuzzy numbers, deriving the analytical expressions of their α-optimistic values is not difficult. Then, the bisection procedure in the NIA-G could be replaced by the clear calculation formula of $\xi_{\sup}(\alpha)$. Based upon this concept, the NIA-S is thus put forward to improve NIA-G as follows:

As to a regular fuzzy number ξ, which is of LR-type with continuous and strictly decreasing shape functions L and R, in regard to Eqs. (2.1) and (4.15), it is obtained that

$$\Psi(x) = \begin{cases} \dfrac{1}{2} R\left(\dfrac{x-c}{\beta}\right), & \text{if } x \ge c \\ 1 - \dfrac{1}{2} L\left(\dfrac{c-x}{\gamma}\right), & \text{if } x < c. \end{cases} \tag{4.17}$$

Due to the strict monotonicity of L and R, their inverse functions exist and are denoted by L^{-1} and R^{-1}, respectively. Then the α-optimistic value of ξ, $\xi_{\sup}(\alpha)$, can be derived from Eqs. (4.14) and (4.17) as follows:

$$\xi_{\sup}(\alpha) = \Psi^{-1}(\alpha) = \begin{cases} c + \beta R^{-1}(2\alpha), & \text{if } 0 < \alpha \le 0.5 \\ c - \gamma L^{-1}(2 - 2\alpha), & \text{if } 0.5 < \alpha \le 1. \end{cases} \tag{4.18}$$

According to Eqs. (4.17)-(4.18), the α-optimistic values of a triangular, normal, and Gaussian fuzzy number enumerated in Examples 4.1-4.3 can be respectively obtained as

$$\xi_{\mathcal{T}\,\sup}(\alpha) = \begin{cases} 2\alpha c + (1-2\alpha)b, & \text{if } 0 < \alpha \le 0.5 \\ \\ (2\alpha-1)a + (2-2\alpha)c, & \text{if } 0.5 < \alpha \le 1. \end{cases} \tag{4.19}$$

$$\xi_{\mathcal{N}\,\sup}(\alpha) = c + (\ln(1-\alpha) - \ln\alpha)\sqrt{6}\sigma/\pi, \quad \alpha \in (0,1). \tag{4.20}$$

$$\xi_{\mathcal{G}\,\sup}(\alpha) = \begin{cases} c + b\sqrt{-\ln(2\alpha)}, & \text{if } 0 < \alpha \le 0.5 \\ \\ c - b\sqrt{-\ln(2-2\alpha)}, & \text{if } 0.5 < \alpha < 1. \end{cases} \tag{4.21}$$

Consequently, based on Theorem 4.3 and Eqs. (4.18)-(4.21), a special NIA is then set forth by using the analytical expressions of α-optimistic values of regular fuzzy numbers to substitute the bisection algorithm in the NIA-G. The steps of the NIA-S are described as follows:

Algorithm 5 (NIA-S)

Step 1. Initialize the number of integration points N. Let $E = 0$ and $k = 1$.

Step 2. Set $\alpha = k/N$. For each $1 \le i \le n$, according to the calculation formula of α-optimistic values in Eq. (4.18), calculate

$$x_i = \begin{cases} (\xi_i)_{\sup}(\alpha), & \text{if } 1 \le i \le h, \\ (\xi_i)_{\sup}(1-\alpha), & \text{if } h < i \le n. \end{cases}$$

Step 3. Reset $E = E + f(x_1, x_2, \cdots, x_n)/N$ and $k = k + 1$.

Step 4. If $k \le N$, go to Step 2. Otherwise, return E as the simulation value of the expected value $E[f(\boldsymbol{\xi})]$.

As a general rule, the clear analytical expressions of the inverse functions of L and R are not difficult to obtain. Under this case, the NIA-S is more suitable to be chosen for the simulation of the expected value. But when it comes to a situation that the inverse functions are too complex to figure out, then the bisection algorithm is preferred to calculate the value of $\Psi^{-1}(\alpha)$ (see Algorithm 3) or the "polyfit" function of Matlab to generate approximate functions for $\Psi^{-1}(\alpha)$.

4.3.3 Comparative study with different functions of different fuzzy numbers

In this section, three numerical examples considering expected values of continuous and strictly monotone functions of regular fuzzy numbers are conducted to compare the performances of the iSDA, NIA-G, and NIA-S based on the accuracy, stability, and operation speed measurements.

Example 4.7 *According to the data and function given in Example 4.4, accomplish the expected value* $E[\xi]$ *of the fuzzy number* $\xi = f_1(\eta_1, \eta_2, \cdots, \eta_{10})$ *by means of the iSDA, NIA-G, and NIA-S, respectively, in which* η_i, $i = 1, 2, \cdots, 10$, *are triangular fuzzy numbers.*

The final simulation results of the iSDA, NIA-G, and NIA-S are obtained through altering the numbers of sample points or integration points and reported in Table 4.6. Here the small enough number ϵ in the bisection part of the NIA-G is set to be 10^{-3} on account of the trade-off between the accuracy and time. Meanwhile, the analytical expression of the α-optimistic value of a triangular fuzzy number in the NIA-S is Eq. (4.19).

From Table 4.6, it is seen that along with the increasing number of integration points N, the accuracy degrees for the NIA-G and NIA-S are both greatly enhanced, and this point is not obviously reflected on the iSDA. On the whole, regardless of the accuracy, stability, or operation speed, the performance of the iSDA in Example 4.7 is clearly superior among all the three algorithms compared.

Table 4.6: Simulation results for the iSDA, NIA-G, and NIA-S in Example 4.7.

Number of Sample Points or Integration Points N	iSDA Simulation Value	iSDA CPU Time (s)	NIA-G Simulation Value	NIA-G CPU Time (s)	NIA-S Simulation Value	NIA-S CPU Time (s)
1000	38.4990	0.000	38.4870	0.010	38.4870	0.000
3000	38.4997	0.000	38.4957	0.046	38.4957	0.005
5000	38.4998	0.000	38.4974	0.070	38.4974	0.010
10000	38.4999	0.000	38.4987	0.140	38.4987	0.015
15000	38.4999	0.010	38.4991	0.202	38.4991	0.020
20000	38.5000	0.010	38.4994	0.265	38.4994	0.030
10^6	38.5000	0.330	38.5000	10.256	38.5000	0.883

Example 4.8 *This example is designed to find the expected value* $E[\xi]$ *of* $\xi = f_1(\eta_1, \eta_2, \cdots, \eta_{10})$ *using the iSDA, NIA-G, and NIA-S, in which the function* $f_1 = x_1 + x_2 + \cdots + x_{10}$*, and the fuzzy variables* $\eta_i, i = 1, 2, \cdots, 10$*, included are respective triangular, normal, and Gaussian fuzzy numbers in Table 4.1.*

The simulation outcomes are summarized in Table 4.7, in which the figure in () after each algorithm represents the number of sample points or integration points involved. It is observed that there is no g in the NIA-S since it utilizes the inverse functions directly rather than the range of the support S_i. Not surprisingly, the iSDA with the setting $g = 6$ still performs better either on the accuracy or with respect to the time.

Example 4.9 *Other conditions stay unchanged, only the function* f_1 *in Example 4.8 is replaced by* $f_2 = -(x_1 \wedge x_2 \wedge \cdots \wedge x_{10})$*, and the mean value* $E[\xi]$ *of* $\xi = f_2(\eta_1, \eta_2, \cdots, \eta_{10})$ *is obtained by using the iSDA, NIA-G, and NIA-S.*

The simulation results of Example 4.9 are enumerated in Table 4.8 for comparison purposes. Analogously to Section 4.2.4, the computational time does not change so much for the iSDA, but the performance is quite well when $g = 6$. The

Table 4.7: Comparative results among the iSDA, NIA-G, and NIA-S for the case that $f_1 = x_1 + x_2 + \cdots + x_{10}$.

	Triangular		Normal			Gaussian	
iSDA (10000)		$g=1$	$g=3$	$g=6$	$g=1$	$g=3$	$g=6$
Exact Value	38.5000	-4.0000	-4.0000	-4.0000	-4.0000	-4.0000	-4.0000
Simulation Value	38.4999	-3.9983	-3.9995	-4.0000	-3.9985	-4.0000	-4.0000
Error	0.00%	0.04%	0.01%	0.00%	0.04%	0.00%	0.00%
CPU Time (s)	0.004	0.006	0.006	0.006	0.005	0.006	0.006
NIA-G (10000)		$g=1$	$g=3$	$g=6$	$g=1$	$g=3$	$g=6$
Exact Value	38.5000	-4.0000	-4.0000	-4.0000	-4.0000	-4.0000	-4.0000
Simulation Value	38.4987	-3.9996	-3.9996	-3.9996	-3.9996	-3.9996	-3.9996
Error	0.00%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%
CPU Time (s)	0.140	0.402	0.419	0.411	0.333	0.348	0.353
NIA-S (10000)							
Exact Value	38.5000		-4.0000			-4.0000	
Simulation Value	38.4987		-3.9996			-3.9996	
Error	0.00%		0.01%			0.01%	
CPU Time (s)	0.008		0.036			0.017	

NIA-S is also effective except for its computational time which is several times longer than that of the iSDA. Certainly, the NIA-G is able to achieve a satisfactory result when $g = 6$, nevertheless the time needed is hundreds times greater than the iSDA.

Notably, for the iSDA or NIA-G, when it comes to the function f_1, whatever the value g is, the simulation results are already good enough. However, as for the function f_2, only when $g = 6$, a result of high accuracy can be obtained. The main cause of this difference may come from the features of these two functions, that is, f_1 focuses on the overall sum while f_2 aims at the minimum value only.

In summary, the three algorithms, the iSDA, NIA-S, and NIA-G, are much better than the SDA in terms of the accuracy, stability, or computational time, and their individual outputs are steady, unlike those of the SDA. Generally, the

Table 4.8: Comparative results among the iSDA, NIA-G, and NIA-S for the case that $f_2 = -(x_1 \wedge x_2 \wedge \cdots \wedge x_{10})$.

	Triangular		Normal			Gaussian	
iSDA (10000)		$g = 1$	$g = 3$	$g = 6$	$g = 1$	$g = 3$	$g = 6$
Exact Value	3.2500	8.8300	8.8300	8.8300	8.2664	8.2664	8.2664
Simulation Value	3.2500	7.8745	8.7462	8.8271	7.9176	8.2657	8.2652
Error	0.00%	10.82%	0.95%	0.03%	4.22%	0.01%	0.01%
CPU Time (s)	0.004	0.006	0.006	0.006	0.006	0.006	0.006
NIA-G (10000)		$g = 1$	$g = 3$	$g = 6$	$g = 1$	$g = 3$	$g = 6$
Exact Value	3.2500	8.8300	8.8300	8.8300	8.2664	8.2664	8.2664
Simulation Value	3.2497	7.8740	8.7455	8.8252	7.9170	8.2650	8.2650
Error	0.01%	10.83%	0.96%	0.05%	4.23%	0.02%	0.02%
CPU Time (s)	0.136	0.385	0.395	0.411	0.284	0.321	0.337
NIA-S (10000)							
Exact Value	3.2500		8.8300			8.2664	
Simulation Value	3.2499		8.8274			8.2650	
Error	0.00%		0.03%			0.02%	
CPU Time (s)	0.010		0.046			0.025	

iSDA outperforms all the other algorithms in all aspects, i.e., it is highly efficient and time-saving. The NIA-S is slightly inferior to the iSDA from the aspect of time, but the good point is that its calculating procedure is not related to the range of the support (no change of g). The main disadvantage of the NIA-G lies on the computational time, due to the reason that there exists a bisection circulation in its algorithm design.

As to the application of both the iSDA and NIA-S, firstly, they can be utilized to calculate the expected value of fuzzy events. Secondly, they can be served as the internal simulation procedure of heuristic algorithms for solving fuzzy expected value models with expected returns of fuzzy objectives. Nevertheless, the application scenarios of the two algorithms are not the same, and which one to choose should depend on actual needs. Certainly, the accuracy of the iSDA and NIA-S are comparable to each other. The iSDA computes a little faster than the NIA-S, which is more suitable for users have time requirements. For some commonly used fuzzy variables, α-optimistic values are easy to obtain, the NIA-S seems more appropriate. Meanwhile, the iSDA need to consider the range of $\pm 6\sigma$ when the closure of the support of a fuzzy variable is not finite, while there is no such consideration in the NIA-S.

4.4 Extensions to Regular Fuzzy Intervals

It is clear that regular fuzzy intervals are also of great importance no matter in theoretical developments like its variance research [Guy19] and the entropy calculation and simulation [She19], or in practical applications like the portfolio optimization [Liu15c]. One of the representative forms of regular fuzzy intervals is the commonly used trapezoidal fuzzy number. Researchers have continued interests in updating the fuzzy simulation of the expected value of functions that contain trapezoidal fuzzy numbers ([Zha05],[Zho18]). In this section, to calculate the expected value of a strictly monotone function f of regular fuzzy intervals $\widetilde{\xi}_1, \widetilde{\xi}_2, \cdots, \widetilde{\xi}_n$, the α-optimistic value $\widetilde{\xi}_{\sup}(\alpha)$ and α-pessimistic value $\widetilde{\xi}_{\inf}(\alpha)$ of regular fuzzy intervals are deduced. Then, Theorem 4.3 is further extended for the case of regular fuzzy intervals. On this basis, two extension algorithms called the TiSDA and TNIA-S are proposed to simulate the expected value $E[f(\widetilde{\xi}_1, \widetilde{\xi}_2, \cdots, \widetilde{\xi}_n)]$, respectively.

4.4.1 Regular fuzzy interval

The definition of regular fuzzy intervals based on LR fuzzy intervals (see Definition 2.9) is in accordance with that of regular fuzzy numbers, which is described as follows:

Definition 4.1 *An LR fuzzy interval is said to be regular if the shape functions L and R are continuous and strictly decreasing functions on the open intervals*

$\{0 < L(x) < 1\}$ *and* $\{0 < R(x) < 1\}$*, respectively.*

In this part, expected values of continuous and strictly monotone functions of regular fuzzy intervals are considered. Before that, several properties of α-optimistic values of regular fuzzy intervals are elaborated as follows:

Theorem 4.4 *Let $\widetilde{\xi}$ be a regular fuzzy interval. For any $\alpha \in (0, 1]$, it is obtained that*

$$\mathrm{Cr}\{\widetilde{\xi} \geq \widetilde{\xi}_{\sup}(\alpha)\} = \alpha. \tag{4.22}$$

Proof: For any $\alpha \in (0, 1]$ and $\alpha \neq 0.5$, it follows from the continuity of the distribution function and the definition of optimistic value in Eq. (4.13) that

$$\mathrm{Cr}\{\widetilde{\xi} \geq \widetilde{\xi}_{\sup}(\alpha)\} = \lim_{n\to\infty} \mathrm{Cr}\{\widetilde{\xi} \geq \widetilde{\xi}_{\sup}(\alpha) - \frac{1}{n}\} \geq \alpha, \tag{4.23}$$

$$\mathrm{Cr}\{\widetilde{\xi} \geq \widetilde{\xi}_{\sup}(\alpha)\} = \lim_{n\to\infty} \mathrm{Cr}\{\widetilde{\xi} \geq \widetilde{\xi}_{\sup}(\alpha) + \frac{1}{n}\} \leq \alpha. \tag{4.24}$$

If $\alpha = 0.5$, it is calculated that

$$\begin{aligned}
&\mathrm{Cr}\{\widetilde{\xi} \geq \widetilde{\xi}_{\sup}(0.5)\} \\
&= \frac{1}{2}\Big(\mathrm{Pos}\{\widetilde{\xi} \geq \widetilde{\xi}_{\sup}(0.5)\} + 1 - \mathrm{Pos}\{\widetilde{\xi} < \widetilde{\xi}_{\sup}(0.5)\}\Big) \\
&= \frac{1}{2}(0.5 + 1 - 0.5) = 0.5.
\end{aligned} \tag{4.25}$$

With Eqs. (4.23)-(4.25), the proof is complete. □

Theorem 4.5 *If $\widetilde{\xi}$ is a regular fuzzy interval, for any $\alpha \in (0, 1]$, it is obtained that*

$$\widetilde{\xi}_{\sup}(\alpha) = \begin{cases} \Psi^{-1}(\alpha), & \text{if } \alpha \neq 0.5 \\ \\ \overline{c}, & \text{if } \alpha = 0.5. \end{cases} \tag{4.26}$$

Proof: For any given $\alpha \in (0, 1]$, denote $\Psi(x) = \mathrm{Cr}\{\widetilde{\xi} \geq x\} = \alpha$. Since $\Psi(x)$ is strictly decreasing in $\{x \leq \underline{c}\}$ and $\{x \geq \overline{c}\}$, for any $\alpha \in (0, 1]$ and $\alpha \neq 0.5$,

$x = \Psi^{-1}(\alpha)$. Combining the above with Eq. (4.22) in Theorem 4.4, it is obtained that $\widetilde{\xi}_{\sup}(\alpha) = \Psi^{-1}(\alpha)$. In addition, it follows immediately from the definition of $\widetilde{\xi}_{\sup}$ that $\widetilde{\xi}_{\sup}(\alpha) = \overline{c}$ when $\alpha = 0.5$.

□

According to Theorem 4.5, in order to obtain $\widetilde{\xi}_{\sup}(\alpha)$, $\Psi(x)$ of regular fuzzy interval $\tilde{\xi}$ is needed to be calculated first. If the membership function $\mu_{\tilde{\xi}}$ of a regular fuzzy interval $\widetilde{\xi}$ is attained, $\Psi(x)$ can be deduced via $\mu_{\tilde{\xi}}$ as follows:

$$\Psi(x) = \begin{cases} 1 - \mu_{\tilde{\xi}}(x)/2, & \text{if } x < \underline{c}, \\ \dfrac{1}{2}, & \text{if } \underline{c} \le x \le \overline{c} \\ \mu_{\tilde{\xi}}(x)/2, & \text{if } x > \overline{c}. \end{cases} \tag{4.27}$$

Based on Eqs. (2.3) and (4.27), it is attained that

$$\Psi(x) = \begin{cases} 1 - \dfrac{1}{2} L\left(\dfrac{\underline{c} - x}{\gamma}\right), & \text{if } x < \underline{c} \\ \dfrac{1}{2}, & \text{if } \underline{c} < x \le \overline{c} \\ \dfrac{1}{2} R\left(\dfrac{x - \overline{c}}{\beta}\right), & \text{if } x > \overline{c}. \end{cases} \tag{4.28}$$

Since the shape functions L and R are both continuous and strictly decreasing, the inverse functions L^{-1} and R^{-1} exist. Consequently, the analytical expression of $\widetilde{\xi}_{\sup}(\alpha)$ of a regular fuzzy interval in Eq. (4.26) is obtained as

$$\tilde{\xi}_{\sup}(\alpha) = \begin{cases} \beta R^{-1}(2\alpha) + \overline{c}, & \text{if } 0 < \alpha < 0.5 \\ \overline{c}, & \text{if } \alpha = 0.5 \\ \underline{c} - \gamma L^{-1}(2 - 2\alpha), & \text{if } 0.5 < \alpha \le 1. \end{cases} \tag{4.29}$$

Further, in order to better understand the following theorems, the concept of the α-pessimistic value is also introduced in this section.

Definition 4.4 (Liu [Liu04]) *For any $\alpha \in (0,1]$, the α-pessimistic value of a fuzzy variable ξ is*

$$\xi_{\inf}(\alpha) = \inf\{r \bigm| \mathrm{Cr}\{\xi \le r\} \ge \alpha\}. \tag{4.30}$$

Theorem 4.6 *Let $\widetilde{\xi}$ be a regular fuzzy interval. For any $\alpha \in (0,1]$, it is obtained that*

$$\mathrm{Cr}\{\widetilde{\xi} \le \widetilde{\xi}_{\inf}(\alpha)\} = \alpha. \tag{4.31}$$

Proof: Analogous to the proof of Theorem 4.4, it is calculated that

$$\mathrm{Cr}\{\widetilde{\xi} \le \widetilde{\xi}_{\inf}(\alpha)\} = \lim_{n\to\infty} \mathrm{Cr}\{\widetilde{\xi} \le \widetilde{\xi}_{\inf}(\alpha) + \frac{1}{n}\} \ge \alpha, \tag{4.32}$$

$$\mathrm{Cr}\{\widetilde{\xi} \le \widetilde{\xi}_{\inf}(\alpha)\} = \lim_{n\to\infty} \mathrm{Cr}\{\widetilde{\xi} \le \widetilde{\xi}_{\inf}(\alpha) - \frac{1}{n}\} \le \alpha, \tag{4.33}$$

and if $\alpha = 0.5$, it yields that

$$\begin{aligned} &\mathrm{Cr}\{\widetilde{\xi} \le \widetilde{\xi}_{\inf}(0.5)\} \\ &= \frac{1}{2}\Big(\mathrm{Pos}\{\widetilde{\xi} \le \widetilde{\xi}_{\inf}(0.5)\} + 1 - \mathrm{Pos}\{\widetilde{\xi} > \widetilde{\xi}_{\inf}(0.5)\}\Big) \\ &= \frac{1}{2}(0.5 + 1 - 0.5) = 0.5. \end{aligned} \tag{4.34}$$

With Eqs. (4.32)-(4.34), the proof is complete.

□

Theorem 4.7 *If $\widetilde{\xi}$ is a regular fuzzy interval, for any $\alpha \in (0,1]$, it is obtained that*

$$\widetilde{\xi}_{\inf}(\alpha) = \begin{cases} \Psi^{-1}(1-\alpha), & \text{if } \alpha \ne 0.5 \\ \underline{c}, & \text{if } \alpha = 0.5. \end{cases} \tag{4.35}$$

Proof: For any given $\alpha \in (0,1]$, denote $\Phi(x) = \mathrm{Cr}\{\widetilde{\xi} \le x\} = \alpha$. Since $\Phi(x) = \mathrm{Cr}\{\xi \le x\}$ is strictly increasing in $\{x \le \underline{c}\}$ and $\{x \ge \overline{c}\}$, for $\alpha \in (0,1]$ and $\alpha \ne 0.5$, $x = \Phi^{-1}(\alpha)$. With Eq. (4.31) in Theorem 4.6, it is calculated that $\widetilde{\xi}_{\inf}(\alpha) = \Phi^{-1}(\alpha)$. In terms of Eq. (4.12), $\Phi(x) = 1 - \Psi(x) = \alpha$ is attained, which follows that $\widetilde{\xi}_{\inf}(\alpha) = \Psi^{-1}(1-\alpha)$. Besides, according to the definition of $\widetilde{\xi}_{\inf}$ that when $\alpha = 0.5$, $\widetilde{\xi}_{\inf}(\alpha) = \underline{c}$ can be obtained.

$\square$

Analogously, the analytical expression of $\widetilde{\xi}_{\inf}(\alpha)$ in Eq. (4.35) is derived based on Eq. (4.29) as

$$\widetilde{\xi}_{\inf}(\alpha) = \begin{cases} \underline{c} - \gamma L^{-1}(2\alpha), & \text{if } 0 < \alpha < 0.5 \\ \\ \underline{c}, & \text{if } \alpha = 0.5 \\ \\ \beta R^{-1}(2-2\alpha) + \overline{c}, & \text{if } 0.5 < \alpha \leq 1. \end{cases} \tag{4.36}$$

On the basis of the above analytical analyses, some regular fuzzy intervals are illustrated in Examples 4.10 $\sim$ 4.12, and their corresponding α-optimistic and α-pessimistic values are deduced in light of Eqs. (4.29) and (4.36), respectively.

Example 4.10 *When the shape functions L and R are*

$$L(x) = R(x) = \max\{0, 1-x\},$$

the corresponding LR fuzzy interval $\widetilde{\xi}$ is a trapezoidal fuzzy number. The membership function of a trapezoidal fuzzy number $\widetilde{\xi}$ with $a < b < c < d$ is

$$\mu_{\mathcal{A}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & \textit{if } a \leq x < b \\ \\ 1, & \textit{if } b \leq x \leq c \\ \\ \dfrac{d-x}{d-c}, & \textit{if } c < x \leq d \\ \\ 0, & \textit{otherwise}, \end{cases} \tag{4.37}$$

which is denoted by $\widetilde{\xi} \sim \mathcal{A}(a,b,c,d)$, and is illustrated in Figure 4.2.

Further, in light of Eqs. (4.29) and (4.36), the α-optimistic and α-pessimistic values of a trapezoidal fuzzy number $\widetilde{\xi} \sim \mathcal{A}(a,b,c,d)$ are derived as follows:

$$\widetilde{\xi}_{\mathcal{A}\sup}(\alpha) = \begin{cases} d - 2(d-c)\alpha, & \textit{if } 0 < \alpha \leq 0.5 \\ \\ 2b - a - 2(b-a)\alpha, & \textit{if } 0.5 < \alpha \leq 1 \end{cases} \tag{4.38}$$

$$
\widetilde{\xi}_{\mathcal{A}\,\inf}(\alpha) = \begin{cases} a + 2(b-a)\alpha, & \textit{if } 0 < \alpha \le 0.5 \\ \\ 2c - d + 2(d-c)\alpha, & \textit{if } 0.5 < \alpha \le 1, \end{cases} \tag{4.39}
$$

which are depicted in Figures 4.3 and 4.4, respectively.

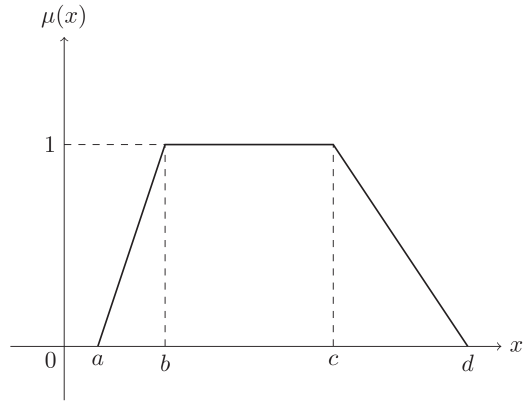


Figure 4.2: The membership function of $\mathcal{A}(a, b, c, d)$ in Eq. (4.37).

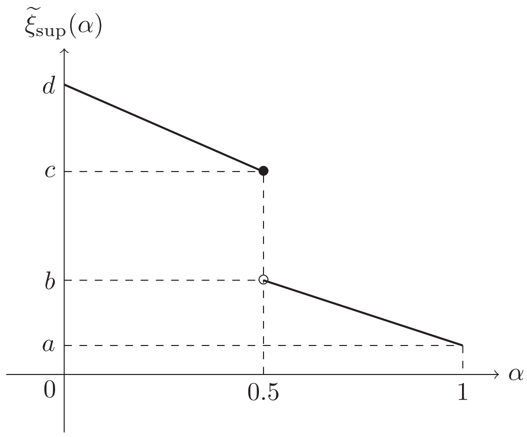


Figure 4.3: The $\widetilde{\xi}_{\sup}(\alpha)$ value of $\mathcal{A}(a, b, c, d)$ in Eq. (4.38).

Example 4.11 *When the shape functions L and R are*

$$
L(x) = \max\{0, 1-x\}, R(x) = \max\{0, 1-x^2\},
$$

a new LR fuzzy interval $\widetilde{\xi}$ is established, whose membership function with $a <$

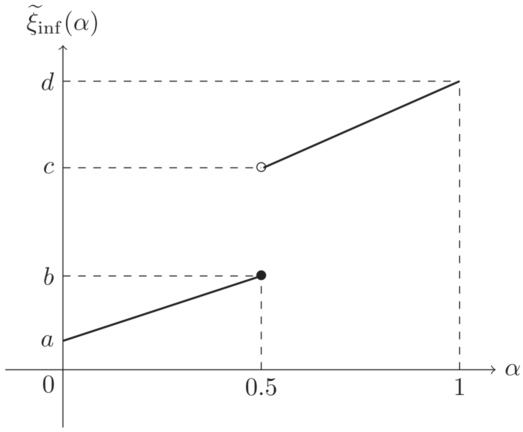


Figure 4.4: The $\widetilde{\xi}_{\inf}(\alpha)$ value of $\mathcal{A}(a,b,c,d)$ in Eq. (4.39).

$b < c < d$ is

$$\mu_{\mathcal{B}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & \text{if } a \le x < b \\ 1, & \text{if } b \le x \le c \\ \dfrac{(x+d-2c)(d-x)}{(d-c)^2}, & \text{if } c < x \le d \\ 0, & \text{otherwise,} \end{cases} \tag{4.40}$$

which is denoted by $\widetilde{\xi} \sim \mathcal{B}(a,b,c,d)$. Similarly, its α-optimistic and α-pessimistic values are respectively derived as

$$\widetilde{\xi}_{\mathcal{B}\sup}(\alpha) = \begin{cases} c + (d-c)\sqrt{1-2\alpha}, & \text{if } 0 < \alpha \le 0.5 \\ 2b - a - 2(b-a)\alpha, & \text{if } 0.5 < \alpha \le 1 \end{cases} \tag{4.41}$$

$$\widetilde{\xi}_{\mathcal{B}\inf}(\alpha) = \begin{cases} a + 2(b-a)\alpha, & \text{if } 0 < \alpha \le 0.5 \\ c + (d-c)\sqrt{2\alpha-1}, & \text{if } 0.5 < \alpha \le 1. \end{cases} \tag{4.42}$$

Example 4.12 *When the shape functions L and R are*

$$L(x) = \max\{0, 1-x^2\}, R(x) = e^{-x},$$

another new LR fuzzy interval $\widetilde{\xi}$ is built, whose membership function with $a <$

$b < c < d$ is

$$\mu_{\boldsymbol{c}}(x) = \begin{cases} \dfrac{(2b-a-x)(x-a)}{(b-a)^2}, & \text{if } a \le x < b \\ 1, & \text{if } b \le x \le c \\ e^{\frac{c-x}{d-c}}, & \text{if } c < x \le d \\ 0, & \text{otherwise,} \end{cases} \tag{4.43}$$

which is written as $\widetilde{\xi} \sim \boldsymbol{c}(a,b,c,d)$. Correspondingly, it is obtained that

$$\widetilde{\xi}_{\boldsymbol{c}\,\sup}(\alpha) = \begin{cases} c-(d-c)\ln(2\alpha), & \text{if } 0 < \alpha \le 0.5 \\ b-(b-a)\sqrt{2\alpha-1}, & \text{if } 0.5 < \alpha \le 1 \end{cases} \tag{4.44}$$

$$\widetilde{\xi}_{\boldsymbol{c}\,\inf}(\alpha) = \begin{cases} b-(b-a)\sqrt{1-2\alpha}, & \text{if } 0 \le \alpha \le 0.5 \\ c-(d-c)\ln(2-2\alpha), & \text{if } 0.5 < \alpha < 1. \end{cases} \tag{4.45}$$

Theorem 4.8 *Let $\widetilde{\xi}$ be a regular fuzzy interval. Then*

$$\widetilde{\xi}_{\inf}(\alpha) = \widetilde{\xi}_{\sup}(1-\alpha) \tag{4.46}$$

holds for $\alpha \in (0,1]$ except $\alpha = 0.5$. Especially, if $\widetilde{\xi}$ is a regular fuzzy number, Eq. (4.46) holds for $\alpha \in (0,1]$.

Proof: It follows immediately from Definitions 4.3 and 4.4, and Eqs. (4.26) and (4.35). The proof is complete.

□

Theorem 4.9 *Assume that $\widetilde{\xi}_1, \widetilde{\xi}_2, \cdots, \widetilde{\xi}_n$ are independent regular fuzzy intervals. Denote $\widetilde{\boldsymbol{\xi}} = (\widetilde{\xi}_1, \widetilde{\xi}_2, \cdots, \widetilde{\xi}_n)$. If the function $f(x_1, x_2, \cdots, x_n)$ is continuous and strictly increases in regard to $x_1, x_2, \cdots, x_h$ and strictly decreases in regard to $x_{h+1}, x_{h+2}, \cdots, x_n$, for any $\alpha \in (0,1]$, it is obtained that*

$$f(\widetilde{\boldsymbol{\xi}})_{\sup}(\alpha) = f\Big((\widetilde{\xi}_1)_{\sup}(\alpha), \cdots, (\widetilde{\xi}_h)_{\sup}(\alpha), (\widetilde{\xi}_{h+1})_{\inf}(\alpha), \cdots, (\widetilde{\xi}_n)_{\inf}(\alpha)\Big).$$

Proof: Without loss of generality, only the case of $h = 1$ and $n = 2$ will be proved. On the basis that $\widetilde{\xi}_1$ and $\widetilde{\xi}_2$ are independent regular fuzzy intervals, for any $\alpha \in (0, 1]$, it is calculated that

$$\begin{aligned}
&\mathrm{Cr}\{f(\widetilde{\xi}_1, \widetilde{\xi}_2) \geq f((\widetilde{\xi}_1)_{\sup}(\alpha), (\widetilde{\xi}_2)_{\inf}(\alpha))\} \\
&\geq \mathrm{Cr}\{\{\widetilde{\xi}_1 \geq (\widetilde{\xi}_1)_{\sup}(\alpha)\} \cap \{\widetilde{\xi}_2 \leq (\widetilde{\xi}_2)_{\inf}(\alpha)\}\} \\
&= \mathrm{Cr}\{\widetilde{\xi}_1 \geq (\widetilde{\xi}_1)_{\sup}(\alpha)\} \wedge \mathrm{Cr}\{\widetilde{\xi}_2 \leq (\widetilde{\xi}_2)_{\inf}(\alpha)\} \\
&= \alpha \wedge \alpha \\
&= \alpha.
\end{aligned}$$

Then again, since the function f is continuous, for any $\epsilon > 0$, there exists a real number $\delta > 0$ such that if $|x_1 - (\widetilde{\xi}_1)_{\sup}(\alpha)| + |x_2 - (\widetilde{\xi}_2)_{\inf}(\alpha)| \leq \delta$, $|f(x_1, x_2) - f((\widetilde{\xi}_1)_{\sup}(\alpha), (\widetilde{\xi}_2)_{\inf}(\alpha))| < \epsilon$ holds. By taking advantage of the independence,

$$\begin{aligned}
&\mathrm{Cr}\{f(\widetilde{\xi}_1, \widetilde{\xi}_2) \geq f((\widetilde{\xi}_1)_{\sup}(\alpha), (\widetilde{\xi}_2)_{\inf}(\alpha)) + \epsilon\} \\
&\leq \mathrm{Cr}\{\{\widetilde{\xi}_1 \geq (\widetilde{\xi}_1)_{\sup}(\alpha) + \delta\} \cup \{\widetilde{\xi}_2 \leq (\widetilde{\xi}_2)_{\inf}(\alpha) - \delta\}\} \\
&= \mathrm{Cr}\{\widetilde{\xi}_1 \geq (\widetilde{\xi}_1)_{\sup}(\alpha) + \delta\} \vee \mathrm{Cr}\{\widetilde{\xi}_2 \leq (\widetilde{\xi}_2)_{\inf}(\alpha) - \delta\} \\
&< \alpha.
\end{aligned}$$

Eventually, it is attained that

$$f(\widetilde{\boldsymbol{\xi}})_{\sup}(\alpha) = f((\widetilde{\xi}_1)_{\sup}(\alpha), (\widetilde{\xi}_2)_{\inf}(\alpha)).$$

The proof is complete.

□

Theorem 4.10 *Let $\widetilde{\xi}$ be a regular fuzzy interval. If its expected value exists, then*

$$E[\widetilde{\xi}] = \int_0^1 \widetilde{\xi}_{\inf}(\alpha)\mathrm{d}\alpha = \int_0^1 \widetilde{\xi}_{\sup}(\alpha)\mathrm{d}\alpha. \tag{4.47}$$

Proof: Denote $\widetilde{\xi} = (\underline{c}, \overline{c}, \gamma, \beta)_{LR}$. Provided that $\underline{c} \geq 0$, it follows from the definition of the expected value operator in Eq. (4.1) and the credibility distribution in Eq. (4.11) that

$$
\begin{aligned}
E[\widetilde{\xi}] &= \int_0^{+\infty} \mathrm{Cr}\{\widetilde{\xi} \geq x\}\mathrm{d}x - \int_{-\infty}^{0} \mathrm{Cr}\{\widetilde{\xi} \leq x\}\mathrm{d}x \\
&= \int_0^{\underline{c}} (1-\Phi(x))\mathrm{d}x + \int_{\underline{c}}^{\overline{c}} (1-\Phi(x))\mathrm{d}x + \int_{\overline{c}}^{+\infty} (1-\Phi(x))\mathrm{d}x - \int_{-\infty}^{0} \Phi(x)\mathrm{d}x \\
&= \int_0^{\underline{c}} x\mathrm{d}\Phi(x) + \int_{\overline{c}}^{+\infty} x\mathrm{d}\Phi(x) + \int_{-\infty}^{0} x\mathrm{d}\Phi(x) \\
&= \int_{\Phi(0)}^{0.5} \Phi^{-1}(\alpha)\mathrm{d}\alpha + \int_{\alpha\downarrow 0.5}^{1} \Phi^{-1}(\alpha)\mathrm{d}\alpha + \int_0^{\Phi(0)} \Phi^{-1}(\alpha)\mathrm{d}\alpha \\
&= \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha = \int_0^1 \widetilde{\xi}_{\inf}(\alpha)\mathrm{d}\alpha.
\end{aligned}
\tag{4.48}
$$

With Theorem 4.8, Eq. (4.48) can be further written as

$$
\begin{aligned}
E[\widetilde{\xi}] &= \int_0^1 \widetilde{\xi}_{\inf}(\alpha)\mathrm{d}\alpha = \int_0^1 \widetilde{\xi}_{\sup}(1-\alpha)\mathrm{d}\alpha \\
&= -\int_1^0 \widetilde{\xi}_{\sup}(\alpha)\mathrm{d}\alpha = \int_0^1 \widetilde{\xi}_{\sup}(\alpha)\mathrm{d}\alpha.
\end{aligned}
$$

Similar proof procedure can be achieved to derive Eq. (4.47) if $\underline{c} \leq 0$. The proof is complete.

□

By the results presented in Theorems 4.9 and 4.10, the calculation formula on expected values of continuous and strictly monotone functions of regular fuzzy intervals is provided in the following theorem as

Theorem 4.11 *Suppose that $\widetilde{\xi}_1, \widetilde{\xi}_2, \cdots, \widetilde{\xi}_n$ are independent regular fuzzy intervals. If the function $f(x_1, x_2, \cdots, x_n)$ is continuous and strictly increases in regard to $x_1, x_2, \cdots, x_h$ and strictly decreases in regard to $x_{h+1}, x_{h+2}, \cdots, x_n$, for any $\alpha \in (0,1]$, the expected value of $f(\widetilde{\boldsymbol{\xi}}) = f(\widetilde{\xi}_1, \widetilde{\xi}_2, \cdots, \widetilde{\xi}_n)$ is*

$$
E[f(\widetilde{\boldsymbol{\xi}})] = \int_0^1 f\Big((\widetilde{\xi}_1)_{\sup}(\alpha), \cdots, (\widetilde{\xi}_h)_{\sup}(\alpha), (\widetilde{\xi}_{h+1})_{\inf}(\alpha), \cdots, (\widetilde{\xi}_n)_{\inf}(\alpha)\Big)\mathrm{d}\alpha.
$$

Proof: The proof derives directly from Theorems 4.9 and 4.10, thus it is omitted.

□

Notably, on the basis of Theorem 4.8, Theorem 4.11 will be directly transformed into Theorem 4.3 for obtaining expected values of functions of a series of independent regular fuzzy numbers.

As illustrated in this chapter, regular fuzzy numbers can be observed as a special case of regular fuzzy intervals. This means all the definitions and theorems raised in this section for regular fuzzy intervals also hold for regular fuzzy numbers, which is consistent with the results presented by Li [Lix15].

4.4.2 Simulation algorithms

For the purpose of carrying out expected values of continuous and strictly monotone functions of regular fuzzy intervals, the iSDA and NIA-S are extended from regular fuzzy numbers to their relevant interval versions, called the TiSDA and TNIA-S, respectively.

The basic concept of the TiSDA resembles that of the iSDA except that the interval range where the membership degree equals to 1 in regular fuzzy intervals is not considered. Analogously, a continuous regular fuzzy interval is discretized according to the extended version of Theorem 4.1 at the beginning. Without loss of generality, as to a regular fuzzy interval $\widetilde{\xi}_i$, its closure of the support is denoted by $S_i = [a_i, b_i]$. And there exists an interval $[\underline{c}_i, \overline{c}_i] \in S_i$ such that its membership degree corresponds to 1 and $a_i < \underline{c}_i < \overline{c}_i < b_i$. Then, the left part of $\underline{c}_i$ and the right part of $\overline{c}_i$ in S_i (i.e., $[a_i, \underline{c}_i]$ and $[\overline{c}_i, b_i]$) are equally divided into k pieces, respectively. The jth point of the left part is set as x_{ij}^L and the $(k-j)$th point of the right part is set as x_{ij}^R for $i = 1, 2, \cdots, n$, i.e.,

$$\begin{aligned} x_{ij}^L &= a_i + (\underline{c}_i - a_i) \times \tfrac{j}{k},\ j = 0, 1, \cdots, k-1, \\ x_{ij}^R &= b_i - (b_i - \overline{c}_i) \times \tfrac{j}{k},\ j = 0, 1, \cdots, k-1. \end{aligned} \tag{4.49}$$

The forms of $\boldsymbol{X}_j^L$ and $\boldsymbol{X}_j^R$ of regular fuzzy intervals are identical to those in Eq. (4.6), in which x_{ij}^L and x_{ij}^R are listed in Eq. (4.49). Additionally, $\underline{\boldsymbol{c}} = (\underline{c}_1, \cdots, \underline{c}_h, \overline{c}_{h+1}, \cdots, \overline{c}_n)$ and $\overline{\boldsymbol{c}} = (\overline{c}_1, \cdots, \overline{c}_h, \underline{c}_{h+1}, \cdots, \underline{c}_n)$ are included in the discretization procedure. Similarly to Eq. (4.7), the discrete fuzzy interval $f^*(\widetilde{\boldsymbol{\xi}})$ is defined, where $f(\boldsymbol{X}_j^L)$ and $f(\boldsymbol{X}_j^R)$ are with membership degrees $\mu_1(x_{1j}^L)$ and $\mu_1(x_{1j}^R)$ for $j = 0, 1, \cdots, k-1$, respectively, and $f(\underline{\boldsymbol{c}})$ and $f(\overline{\boldsymbol{c}})$ are with the membership degree 1. Next, the mean value for $f^*(\widetilde{\boldsymbol{\xi}})$ is calculated, in which $w_k f(\boldsymbol{c})$ in Eq. (4.8) is needed to be replaced by $w_{k_1} f(\underline{\boldsymbol{c}}) + w_{k_2} f(\overline{\boldsymbol{c}})$, where $w_{k_1} = \dfrac{1}{2}\big(1 - \mu_1(x_{1(k-1)}^L)\big), w_{k_2} = \dfrac{1}{2}\big(1 - \mu_1(x_{1(k-1)}^R)\big)$. The calculation of other w_j for $j = 1, 2, \cdots, m$ are based on Eq. (4.10).

As a result, after Steps 2 and 5 of the iSDA in Algorithm 2 are substituted by the above discretization procedure of regular fuzzy intervals and "Reset $E = E + w_k f(\boldsymbol{c})$" in Step 8 is replaced by "Reset $E = E + (w_{k_1} f(\underline{\boldsymbol{c}}) + w_{k_2} f(\overline{\boldsymbol{c}}))$", a new simulation algorithm, the TiSDA is constituted for regular fuzzy intervals. It is noted that there are some differences between the iSDA and TiSDA. The peak value $\boldsymbol{c}$ in the iSDA is extended to $\underline{\boldsymbol{c}}$ and $\overline{\boldsymbol{c}}$ in the TiSDA. Meanwhile, the number of discrete points in the iSDA is $2k+1$, while that of the TiSDA is $2k+2$.

Further, on the basis of Theorem 4.11 and the analytical expressions of $\widetilde{\xi}_{\sup}(\alpha)$ in Eq. (4.26) and $\widetilde{\xi}_{\inf}(\alpha)$ in Eq. (4.35), the TNIA-S is proposed to approximate expect values for continuous and strictly monotone functions of regular fuzzy intervals, which shares a similar concept with the NIA-S. Likewise, when the inverse functions of L and R are not easy to derive in some situations, with the aid of the "polyfit" function in Matlab or taking advantage of the bisection algorithm (see Algorithm 3), the value of $\Psi^{-1}(\alpha)$ can be obtained directly.

4.4.3 Comparative study among the SDA, TiSDA, and TNIA-S

Two numerical examples regarding the widely used trapezoidal fuzzy number and other two regular fuzzy intervals are implemented in this section to indicate the efficiencies of the TiSDA and TNIA-S. Since the SDA is suitable for simulating expected values of general functions that contain all kinds of fuzzy variables, here the simulation results of the SDA are also taken into account for the purpose of comparison.

Example 4.13 *Assume that* $\widetilde{\eta}_i, i = 1, 2, \cdots, 10,$ *are independent trapezoidal fuzzy numbers summarized in Table 4.9 involved in two continuous and strictly monotone functions* $f_2 = -(x_1 \wedge x_2 \wedge \cdots \wedge x_{10})$, *and* $f_3 = x_1 + \cdots + x_5 - x_6 - \cdots - x_{10}$. *The expected value* $E[\widetilde{\xi}]$ *of the fuzzy number* $\widetilde{\xi}_j = f_j(\widetilde{\eta}_1, \widetilde{\eta}_2, \cdots, \widetilde{\eta}_{10}), j = 2, 3,$ *is needed to be accomplished.*

Table 4.9: Different kinds of regular fuzzy intervals utilized in examples.

Index	Trapezoidal fuzzy number	Two regular fuzzy intervals	
$\widetilde{\eta}_1$	$\mathcal{A}(2,3,5,8)$	$\mathcal{B}(2,3,5,8)$	$\mathcal{C}(2,3,5,8)$
$\widetilde{\eta}_2$	$\mathcal{A}(4,6,7,9)$	$\mathcal{B}(4,6,7,9)$	$\mathcal{C}(4,6,7,9)$
$\widetilde{\eta}_3$	$\mathcal{A}(5,6,7,8)$	$\mathcal{B}(5,6,7,8)$	$\mathcal{C}(5,6,7,8)$
$\widetilde{\eta}_4$	$\mathcal{A}(2,4,5,6)$	$\mathcal{B}(2,4,5,6)$	$\mathcal{C}(2,4,5,6)$
$\widetilde{\eta}_5$	$\mathcal{A}(3,5,6,9)$	$\mathcal{B}(3,5,6,9)$	$\mathcal{C}(3,5,6,9)$
$\widetilde{\eta}_6$	$\mathcal{A}(6,7,9,10)$	$\mathcal{B}(6,7,9,10)$	$\mathcal{C}(6,7,9,10)$
$\widetilde{\eta}_7$	$\mathcal{A}(-5,-3,-2,-1)$		
$\widetilde{\eta}_8$	$\mathcal{A}(2,6,8,9)$		
$\widetilde{\eta}_9$	$\mathcal{A}(0,1,2,4)$		
$\widetilde{\eta}_{10}$	$\mathcal{A}(-1,0,2,5)$		

Initially, the exact value of $E[\widetilde{\xi}_3]$ is obtained on the basis of the linearity of the expected value operator, i.e.,

$$E[\widetilde{\xi}_3] = E[\widetilde{\eta}_1] + \cdots + E[\widetilde{\eta}_5] - E[\widetilde{\eta}_6] - \cdots - E[\widetilde{\eta}_{10}] = 12.75.$$

While the exact values of $E[\widetilde{\xi}_2]$ is a little more challenging to derive, which are calculated with Matlab and recorded in Table 4.10.

Table 4.10: Comparative results among the SDA, TiSDA, and TNIA-S for the case of f_2 and f_3.

Algorithm	f_2 $-(x_1 \wedge x_2 \wedge \cdots \wedge x_{10})$	f_3 $x_1 + x_2 + \cdots - x_{10}$
SDA (3000/10000)		
Exact Value	2.7500	12.7500
Simulation Value	2.8448	12.6033
Error	3.45%	1.15%
CPU Time (s)	0.551	0.585
TiSDA (10000/none)		
Exact Value	2.7500	12.7500
Simulation Value	2.7500	12.7500
Error	0.00%	0.00%
CPU Time (s)	0.001	0.000
TNIA-S (none/10000)		
Exact Value	2.7500	12.7500
Simulation Value	2.7499	12.7477
Error	0.00%	0.02%
CPU Time (s)	0.007	0.008

The approximation results of the SDA, TiSDA, and TNIA-S are also listed in Table 4.10, in which (m/N) indicates the numbers of sample points m, or integration points N, involved in the experiment. Similarly to above, the outputs of the SDA are unsteady, and thus the average value of ten times outputs is employed in the table, while the simulation results of the TiSDA and TNIA-S are identical every time. As to the two types of functions in Example 4.13, it is explicit that the TiSDA and TNIA-S are reliable and stable regardless of the accuracy or the operation speed. In contrast, the largest error degree of the SDA is 3.45% when it comes to the function f_2 and the time consumed is hundreds of times larger than the other two algorithms. Although both the TiSDA and TNIA-S return satisfactory simulation results at the end, the overall performance of the TiSDA is still better than the TNIA-S. It is expected that the accuracy of the TNIA-S will be further enhanced as the number of integration points N rises, but the time needed will grow as well which surely decreases its competitiveness. Overall, the TiSDA outperforms the SDA in every aspects, and it is able to compute a precise enough value in a relatively short time period.

Example 4.14 *Three types of regular fuzzy intervals of Examples 4.10 ~ 4.12 are listed in Table 4.9, which are respectively incorporated in a continuous and strictly increasing function* $f_4 = \sqrt{x_1^2 + x_2^2 + \cdots + x_6^2}, x_i \geq 0, i = 1, 2, \cdots, 6$*, and another continuous and strictly monotone function* $f_5 = x_1x_2x_3/(x_4x_5x_6)$*. Calculate the corresponding expected value of* $E[\widetilde{\xi}]$ *of the fuzzy number* $\widetilde{\xi} = f_4(\widetilde{\eta}_1, \widetilde{\eta}_2, \cdots, \widetilde{\eta}_6)$ *or* $\widetilde{\xi} = f_5(\widetilde{\eta}_1, \widetilde{\eta}_2, \cdots, \widetilde{\eta}_6)$ *for three types of regular fuzzy intervals.*

Six kinds of outputs of three regular fuzzy intervals under two continuous and strictly monotone functions are clearly illustrated in Table 4.11. Firstly, as to different functions, the SDA returns better computations in f_4 than f_5 for the former two regular fuzzy intervals. Along with other two functions f_2 and f_3 in Example 4.13, it is observed that the SDA is not reliable when encountering different functions. In contrast, the TiSDA and TNIA-S are more dependable, flexible, and adaptable to changeable functions, and can both return perfect simulation results. Secondly, as to different regular fuzzy intervals, it is explicit that the error degree in the SDA becomes larger as the form of the membership function gets complicated, especially in $\mathcal{C}(a, b, c, d)$. This situation also happens in the TiSDA and TNIA-S. However, their simulation outcomes are still quite satisfactory. The performances of the TiSDA and TNIA-S in this example are comparable to each other as well, whereas the TiSDA is more time-saving. These

Table 4.11: Comparative results among the SDA, TiSDA, and TNIA-S for the case of f_4 and f_5 of three regular fuzzy intervals.

Algorithm	Trapezoidal fuzzy number		$\mathcal{B}(a,b,c,d)$		$\mathcal{C}(a,b,c,d)$	
SDA (3000/10000)	f_4	f_5	f_4	f_5	f_4	f_5
Exact Value	14.8960	0.9464	15.2597	0.9596	15.7620	1.0014
Simulation Value	14.9497	1.8682	15.2901	2.0975	14.9121	2.3484
Error	0.36%	97.40%	0.20%	118.58%	5.39%	134.51%
CPU Time (s)	0.176	0.137	0.233	0.162	0.228	0.159
TiSDA (10000/none)	f_4	f_5	f_4	f_5	f_4	f_5
Exact Value	14.8960	0.9464	15.2597	0.9596	15.7620	1.0014
Simulation Value	14.8956	0.9464	15.2597	0.9596	15.7566	1.0013
Error	0.00%	0.00%	0.00%	0.00%	0.03%	0.01%
CPU Time (s)	0.001	0.000	0.002	0.001	0.002	0.001
TNIA-S (none/10000)	f_4	f_5	f_4	f_5	f_4	f_5
Exact Value	14.8960	0.9464	15.2597	0.9596	15.7620	1.0014
Simulation Value	14.8956	0.9465	15.2593	0.9597	15.7594	1.0015
Error	0.00%	0.01%	0.00%	0.01%	0.02%	0.01%
CPU Time (s)	0.004	0.002	0.007	0.002	0.009	0.003

results are consistent with previous analyses in numerical examples.

4.5 Summary

The regular fuzzy numbers which include triangular, normal and Gaussian fuzzy numbers, and the regular fuzzy intervals which contain trapezoidal fuzzy numbers are appeared in many real-world applications. In the corresponding literature, there exist two mainstream fuzzy simulation algorithms in approximating expected values for fuzzy variables. The first one, namely the SDA, was proposed by Liu and Liu [Liu02b], and it follows the concept that stochastically discretize continuous fuzzy numbers. The SDA is capable of simulating the expected value for general functions containing different fuzzy variables. The second algorithm, namely the NIA-G, was formulated by Li [Lix15], and it is based on the integration simulation and the bisection procedure.

In this chapter, two novel simulation techniques of calculating expected values for continuous and strictly monotone functions of regular fuzzy numbers were put forward. Firstly, the iSDA was proposed to revise the stochastic discretization procedure and the calculation formula of the expected value of the SDA. These two parts were substituted by a novel uniform sampling process and another calculation formula applied to discrete fuzzy numbers, respectively. Secondly, the NIA-S took advantage of the analytical expressions of α-optimistic values of regular fuzzy numbers directly in its algorithm design to replace the bisection procedure in the NIA-G. From the results obtained for regular fuzzy numbers, although the iSDA and NIA-S were based on distinct simulation concepts, they both performed better in the accuracy, stability, and computational time compared with the SDA. In addition, as to regular fuzzy intervals, the iSDA and NIA-S were extended to the TiSDA and TNIA-S according to a series of regular fuzzy interval related theorems, respectively. The simulation results demonstrated that either the TiSDA or TNIA-S outperforms the SDA. Besides, it is noted that the continuous and strictly monotone functions f in the examples of this chapter are not difficult. For every f, whether it is challenging to write the expression of f, it is possible to conduct the expected value simulation by using the proposed novel techniques.

So far, the improvements on fuzzy theories and fuzzy simulation in this dissertation have been accomplished. In summary, the identical stochastic sampling process in the original SDS for the possibility and the SDA for the expected value of fuzzy events was substituted by two kinds of uniform sampling processes based on the new operational law. In other words, both the UDS for the possibility of fuzzy events in Chapter 3 and the iSDA/TiSDA for the expected value of fuzzy events in Chapter 4 were initiated according to the new operational law. Meanwhile, the UDS, the iSDA/TiSDA, and the NIA-S/TNIA-S were applied to continuous and strictly monotone functions of regular fuzzy numbers or regular fuzzy intervals. Moreover, the novel theorems raised and proved on regular fuzzy intervals in Chapter 4 will also be implemented to the fuzzy expected value model formulation in Chapter 6. And the TNIA-S will be incorporated in a genetic algorithm to solve this model.

The next chapter deals with the quality function deployment optimization from the perspective of cooperative game theory. A two-stage cooperative game which integrates a quantitative Kano's model is proposed to determine the relative importance weights of customer requirements, and target levels of engineering characteristics of a manufacturing product.

Chapter 5

A Two-stage Cooperative Game for Integrating Kano's Model to QFD

The last two chapters elaborated the improvements on fuzzy simulation techniques for the possibility and expected value of fuzzy events. This chapter focuses on the determination of the relative importance weights of customer requirements (CRs) and target values of engineering characteristics (ECs) in quality function deployment (QFD) from a novel cooperative game-theoretic angle. QFD is a systematic and effective quality tool, which aims at mapping diversified CRs into several ECs.

To accomplish the aforementioned two goals, a two-stage cooperative game in QFD is initiated. Shapley value and Kano's model are involved in the first stage to complete the CR weighting, and two algorithms are further designed to facilitate its application. Subsequently, a mixed integer non-linear programming model is formulated in the second stage to derive target values of ECs. The objective function in this model reflects the bargaining among different fulfillment levels of CRs by integrating the CR weights and a quantitative Kano's model, so as to maximize the overall customer satisfaction.

Finally, the proposed method is implemented to a notebook computer design case study for quality development, and target values of ECs are settled according to customer perceptions. Additionally, some comparisons, discussions, and managerial implications are also raised in terms of the whole methodology and the case study.

5.1 Introduction

As a general rule, a comprehensive QFD optimization procedure contains two critical sequential stages, the determination of relative importance weights of CRs, and the determination of target values of ECs. It is investigated that many researches only focus on one of the stages and weaken the other, and rarely view these two stages together from the perspective of allocation in cooperative games. More details of the relevant literature review can be found in Section 2.3 of Chapter 2.

Therefore, this chapter attempts to observe these two stages from a novel cooperative game-theoretic angle. Stage 1 of the CR weighting is a marginal contribution based allocation using Shapley value, and Stage 2 of the EC target value setting is a resource based allocation handled by a mixed integer non-linear programming model. The model is signified by a Nash bargaining objective which involves the CR weights obtained from Stage 1 to optimize the satisfaction degree of consumers. And a quantitative Kano's model in [Wan10] is adopted and applied to both stages to characterize the relationship between the fulfillment level of each CR and its customer satisfaction. The literature review of Kano's model can be found in Section 2.2 of Chapter 2. These integrations will attach practical significance to Shapley value and Nash bargaining in cooperative games in the industrial product development.

Given a certain manufacturing product like computers, automobiles, or mobile phones, several CRs have been accumulated and categorized according to the routine of Kano's model. Since the selected CRs focus on the current product, with the aid of house of quality (HoQ), the next generation product can be improved to fulfill diversified customer perceptions. During this optimization procedure, two major problems should be solved. (1) The relative importance weights of crucial CRs should be determined, which are listed in the HoQ and used in the optimization model formulation later; (2) The objective function and constraints of the model should be discreetly formulated to adapt to the practical production scenario. Conventionally, maximizing the overall customer satisfaction (OCS) is a mainstream objective, and decision variables are target values of ECs. The optimization model is usually established based on the information displayed in the HoQ.

In order to solve the above two problems, both the quantitative Kano's model and cooperative game theory related knowledge are adopted. As mentioned, the importance weights of CRs are supposed to be settled prior to the optimization of the existing design. These weights are usually acquired by utilizing the average or normalization method based on the data in the Kano questionnaire. Unlike these methods, marginal contributions of CRs are considered in this chapter through employing a quantitative Kano's model and Shapley value in cooperative games, so as to derive the CR weighting in Stage 1. Then, this importance weight vector is involved in a Nash bargaining function in Stage 2, which will be served as the objective function of a mixed integer non-linear programming model in QFD.

Notably, there are commonality and distinctions between the game setups of these two stages regarding the setting of players, objectives, strategy sets, value functions, and calculation methods, which are explained detailedly in Table 5.1. The involved notation in this chapter is summarized in Table 5.2. As illustrated, there are n CRs, q ECs in the product design, and s competitor companies in the current market.

Table 5.1: The two-stage cooperative game setup in QFD.

	Stage 1	Stage 2
Player	Several specified CRs, which are indexed by CR_i, $i = 1, 2, \cdots, n$	
Objective	Determine relative importance weights of CRs	Maximize the overall customer satisfaction
Strategy set	Cooperate or not to cooperate in a coalition	Target values set of ECs $(x_1, x_2, \cdots, x_n)$ An agreement is reached among all players
Value function	customer satisfaction value in Kano's model	$d_i(y_i)$ relationship functions in Kano's model
Calculation	A simplified calculation formula of Shapley value and a normalization formula of Shapley value	A mixed integer non-linear programming model, whose objective function is a Nash bargaining function

The remaining contents are arranged as follows. Firstly, the complete methodology of the proposed two-stage cooperative game in QFD is elaborated in Sections 5.2 and 5.3. Afterwards, an illustrative example regarding a notebook computer development is conducted in Section 5.4. Several model solutions, comparisons, discussions, and managerial implications are figured out to manifest both the performance and effectiveness of the proposed method. Lastly, Section 5.6 illustrates some conclusions of this study.

5.2 Stage 1: The CR Weighting in QFD

5.2.1 Shapley value in the CR weighting

To consider whether customers are pleased or not in terms of one CR, the values of customer satisfaction (CS) and dissatisfaction (DS) which raised by [Mat98], are applied to this research as follows:

$$CS_i = \frac{f_A + f_O}{f_A + f_O + f_M + f_I} \tag{5.1}$$

$$DS_i = -\frac{f_O + f_M}{f_A + f_O + f_M + f_I}, \tag{5.2}$$

Table 5.2: Notation in this chapter.

Stage 1: the CR weighting in QFD	
CS_i	customer satisfaction value of CR_i, $i = 1, 2, \cdots, n$
DS_i	customer dissatisfaction value of CR_i, $i = 1, 2, \cdots, n$
S_i	Shapley value of CR_i, $i = 1, 2, \cdots, n$
w_i	the relative importance weight of CR_i, $i = 1, 2, \cdots, n$
Stage 2: the EC target value setting in QFD	
l_j	the target value of a continuous EC_j, $j = 1, 2, \cdots, q$
x_j	the fulfillment level of EC_j, $j = 1, 2, \cdots, q$
$x_{jk} = \begin{cases} 1 \\ 0 \end{cases}$	If the value k of a discrete EC_j is chosen in the product design, $k = 1, 2, \cdots, m$ Otherwise
h_{kj}	the fulfillment rating of the value k of a discrete EC_j ($h_{kj} \neq 0$), $k = 1, 2, \cdots, m$
d_i	the customer satisfaction degree of CR_i, $i = 1, 2, \cdots, n$
y_i	the fulfillment level of CR_i, $i = 1, 2, \cdots, n$
p_i^t	the performance of CR_i in company t, $t = 1, 2, \cdots, s$
r_{ij}	the relationship between CR_i and EC_j
γ_{jg}	the correlation between EC_j and EC_g
r_{ij}^{norm}	the normalized relationship between CR_i and EC_j in matrix R
c_j	the unit improvement cost for EC_j, $j = 1, 2, \cdots, q$
B	the budget during the entire product design process
ECL_j	the lower bound of the technical constraints for EC_j, $j = 1, 2, \cdots, q$
ECH_j	the upper bound of the technical constraints for EC_j, $j = 1, 2, \cdots, q$

where i denotes the ith CR, and f_A, f_O, f_M, and f_I represent the total fraction numbers of customers' preferences on CR_i through the categorization of A, O, M, and I attributes in Kano's model, respectively. The CS values will be utilized in the Shapley value calculations later.

Since all chosen CRs are significant and regarded as consumers' key needs towards a certain product, the joining of any CR would have an impact on the overall customer satisfaction. Thus, the angle can be switched to measure the marginal contributions of CRs in this procedure as a cooperative game. Certainly, CRs' contributions to the cooperation are different from each other, and will be expressed by their Shapley values in this section accordingly.

Shapley [Sha53] proposed the efficacious tool of Shapley value to handle arbitration between a pile of players, giving the definition of adding the player k into the coalition M as follows:

$$S_k = \sum_{all\ M} \gamma_n(M)[v(M \cup \{k\}) - v(M)], \tag{5.3}$$

where the weights $\gamma_n(M)$ is formulated by

$$\gamma_n(M) = \frac{m!(n-m-1)!}{n!}. \tag{5.4}$$

In Eqs. (5.3)-(5.4), n and m indicate the quantities of players inside the game and the coalition M, respectively, and $v(.)$ is the value function which represents the utility of every combination. As defined, Shapley value computes a unique solution that satisfies the basic requirements of the Nash equilibrium, and on the basis of three axioms including symmetry of players, effectiveness, and additivity.

Corresponding to the problem in this chapter, the players in the first-stage cooperative game are those CRs gathered and selected from the Kano questionnaire. During the computation of Shapley value, CS values for CRs in Eqs. (5.1) are specified as the value function of each participant, which means $v(k) = CS_k$ for CR$_k$. As the difficulty of calculation grows with the number of players in the original formula in Eq. (5.3), Conklin *et al.* [Con04] and [Con05] offered a solution in consideration of the mean values of coalitions at different levels with or without player k as follows:

$$\begin{aligned} S_k = {} & \frac{1}{n-1}(v(k) - Avg_1) + \frac{1}{n-2}(V_2(k) - Avg_2) + \frac{1}{n-3}(V_3(k) - Avg_3) \\ & + \cdots + \frac{1}{n-(n-1)}\left(V_{n-1}(k) - Avg_{n-1}\right) + \frac{1}{n}(v(all)). \end{aligned} \tag{5.5}$$

In Eq. (5.5), Avg_1 stands for the mean value of all combinations which only contain one player at first level, and under this circumstance $v(k)$ shows the mean value of each player itself. Continually, Avg_2 represents the average of all combinations at second level, and $V_2(k)$ is interpreted as the mean value of these combinations include player k, and so on until the coalition of $(n-1)$ players. The last one element equally allocates the utility value of all players together into Shapley value of each participant. Further, Eq. (5.5) can be simplified into

$$S_k = \frac{1}{n} v(M_{all}) + \sum_{j=1}^{n-1} \frac{1}{n-j} (\bar{v}(M_{kj}) - \bar{v}(M_j)). \tag{5.6}$$

5.2.2 Algorithm design in the CR weighting

In reality, sometimes customers may have various needs against the product (over 10 CRs or even more), then the number of coalitions will grow tremendously because of the factorial calculation. To handle this, following the idea of Eqs. (5.5)-(5.6), a simple algorithm, called the *Shapley Value Calculation Algorithm* (short for SVCA), is designed as follows:

Algorithm 1 (Shapley Value Calculation Algorithm, SVCA)

Step 1. Initialize the number n of players (CRs). Input the value function $v(k)$ of the player k, $k = 1, 2, \cdots, n$, and let $z = 1$.

Step 2. Calculate the number of combinations that contain player k at the zth level as $N_z = C_{n-1}^{z-1}$.

Step 3. Find combinations with z element(s) and also contain player k, $k = 1, 2, \cdots, n$, respectively.

Step 4. Calculate the corresponding value functions of these combinations and sum them as $Sum_z(k)$, $k = 1, 2, \cdots, n$, and then get the corresponding mean value $V_z(k) = \dfrac{Sum_z(k)}{N_z}$, $k = 1, 2, \cdots, n$, respectively.

Step 5. Calculate the mean value of all the combinations at the zth level as $Avg_z = \dfrac{\sum_{k=1}^{n} V_z(k)}{n}$.

Step 6. If $z < n-1$, $z = z+1$, and go to Step 2. Otherwise, go to Step 7.

Step 7. Calculate the value function of all the players together as $v(all)$.

Step 8. Return S_k via Eq. (5.5) as the Shapley value of player k, $k = 1, 2, \cdots, n$.

It is observed that the description of the summation value, $Sum_z(k)$, is not so clear in Step 4 of the SVCA. To be more specific, the value functions of different combinations are calculated according to the following disciplines. First of all, the CS value of each CR is considered as the corresponding value function, i.e., $v(k) = CS_k$ for CR_k, $k = 1, 2, \cdots, n$. Meanwhile, for other zth level coalitions, the maximum value function is served as the payoff of one coalition, e.g., in one combination {CR_1, CR2, CR_3} at third level,

$$v(1 \cup 2 \cup 3) = v(1) \vee v(2) \vee v(3) = CS_1 \vee CS_2 \vee CS_3. \tag{5.7}$$

On this basis, another *Value Function Calculation Algorithm* (short for VFCA) is developed as a complementary algorithm for Step 4 of the SVCA. Notably, the VFCA is just applied to the case of similar value function derivations as in Eq. (5.7).

Algorithm 2 (Value Function Calculation Algorithm, VFCA)

Step 1. Initialize the number n of players (CRs). Input the value function $v(k) = CS_k$ for CR_k, $k = 1, 2, \cdots, n$.

Step 2. Find all non-repeating combinations at the zth level such that $\{\mathrm{CR}_k, \mathrm{CR}_{x_1}, \mathrm{CR}_{x_2}, \cdots, \mathrm{CR}_{x_{z-1}}\} \subsetneqq \{\mathrm{CR}_1, \mathrm{CR}_2, \cdots, \mathrm{CR}_n\}$ and label them from $Comb_1$ to $Comb_{N_z}$. And let $Sum_z(k) = 0$, and $t = 1$.

Step 3. If $z > 1$, calculate $v_z(Comb_t) = \max\{v(k), v(x_1), v(x_2), \cdots, v(x_{z-1})\}$ and $Sum_z(k) = Sum_z(k) + v_z(Comb_t)$.

Step 4. If $t < N_z$, $t = t + 1$, and go to Step 3. Otherwise, return $Sum_z(k)$.

At last, according to the Shapley value calculation procedure expounded above, the following normalization equation is utilized to figure out the relative importance weight w_i of CR_i, i.e.,

$$w_i = \frac{S_i}{\sum_{i=1}^{n} S_i} \times 100\%. \tag{5.8}$$

5.3 Stage 2: The EC Target Value Setting in QFD

5.3.1 A Nash bargaining objective function

From the former literature, the overall customer satisfaction, OCS, was attained through a linearly additive operator of the fulfillment levels, y_i, of all CRs in [Poe07] as

$$OCS = \sum_{i=1}^{n} w_i y_i.$$

Similar to this idea, Ji *et al.* [Jip14] aggregated the OCS as a weighted summation of relationship functions between CR and CS, $d_i(y_i)$, i.e.,

$$OCS(y_1, y_2, \cdots, y_n) = \sum_{i=1}^{n} w_i d_i. \tag{5.9}$$

In regard to the CS and DS values introduced in Section 5.2.1, the analytical expressions, $d_i(y_i)$, of the lines and curves of distinct attributes can be further approximately derived through linear and exponential functions. Corresponding to the Kano diagram in Figure 2.1, more detailed relationship functions of A, O, and M attributes are obtained in Table 5.3.

Table 5.3: The $d_i(y_i)$ relationship functions employed from [Jip14].

KC[a]	$f(y_i)$	$d_i = a_i f(y_i) + b_i$
A	e^{y_i}	$d_i = \dfrac{CS_i - DS_i}{e-1} e^{y_i} - \dfrac{CS_i - eDS_i}{e-1}$
O	y_i	$d_i = (CS_i - DS_i) y_i + DS_i$
M	$-e^{-y_i}$	$d_i = -\dfrac{e(CS_i - DS_i)}{e-1} e^{-y_i} + \dfrac{eCS_i - DS_i}{e-1}$

[a]KC is short for Kano classification, which is indicated by the attribute of most replies.

In this section, different from the viewpoint in Eq. (5.9), the problem of maximizing the OCS can be observed as a second-stage cooperative game in QFD, and the bargaining among the fulfillment levels of CRs is considered. As a start, the basic notion of Nash bargaining is reviewed as follows. Assume that $f_i(x)$ is the payoff function for player i, and each player aims to maximize his/her payoff during the bargaining. As a basic principle, the bargaining function $B(.)$ must fulfill the following requirements,

$$\min(f_1, \cdots, f_n) < B(f_1, \cdots, f_n) < \max(f_1, \cdots, f_n).$$

And the generalized bargaining function is written as

$$B(x) = \prod_{i=1}^{n} (f_i(x) - f_i(x_w))^{w_i}, \tag{5.10}$$

where $f_i(x_w)$ represents the minimum value player i would pay in the payoff function $f_i(x)$, and w_i signifies the weight of the payoff function such that $\sum_{i=1}^{n} w_i = 1$, $w_i \in [0, 1]$ for $i = 1, 2, \cdots, n$ [Yan14].

Here in this research, the relative importance weight w_i for CR$_i$ is obtained from the first-stage cooperative game, and the relationship functions $d_i(y_i)$ are adopted as payoff functions of CRs in the Nash bargaining function. Therefore, Eq. (5.9) can be reconsidered to be a second-stage cooperative game version via the application of Eq. (5.10) as

$$B(y_i) = \prod_{i=1}^{n} (d_i(y_i) - DS_i)^{w_i}, \tag{5.11}$$

where DS_i is the customer dissatisfaction value of CR$_i$ in Eq. (5.2).

5.3.2 Normalization

For the sake of more accurately depicting the relationship between CRs and ECs, a normalization procedure is used for the relationship element r_{ij} and the correlation element γ_{jg} in the HoQ. According to [Was93], the normalized relationship between CR$_i$ and EC$_j$, r_{ij}^{norm} in matrix R, can be attained as follows:

$$r_{ij}^{norm} = \frac{\sum_{g=1}^{n} r_{ig} \cdot \gamma_{gj}}{\sum_{j=1}^{n}\sum_{g=1}^{n} r_{ij} \cdot \gamma_{jg}}, \quad i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, q. \tag{5.12}$$

Commonly, there exist two kinds of ECs, the discrete ones and continuous ones. In order to get rid of the impact coming from different measures, two normalization formulae will be used to convert target values of ECs to their fulfillment levels, x_j, accordingly. For discrete ECs, several options with fulfillment ratings are provided for selection. On this basis, the fulfillment level of a discrete EC can be expressed as follows:

$$x_j = \sum_{k=1}^{m} x_{jk} h_{kj}, \quad j = 1, 2, \cdots, q, \tag{5.13}$$

where x_{jk} is a binary variable and h_{kj} stands for the fulfillment rating. For continuous ECs, they are classified into two groups at the beginning, i.e., the cost type (*C-type*) and the benefit type (*B-type*). Afterwards, the target value l_j to its fulfillment level x_j can be scaled in line with [Che05] as follows:

$$x_j = \begin{cases} \dfrac{l_j^{max} - l_j}{l_j^{max} - l_j^{min}} & (\textit{C-type}) \\[2ex] \dfrac{l_j - l_j^{min}}{l_j^{max} - l_j^{min}} & (\textit{B-type}) \end{cases} \tag{5.14}$$

where $0 \leq x_j \leq 1$. For those *C-type* ECs, $l_j^{\max}$ is the maximal target value that matches the performance of competitors, while $l_j^{\min}$ indicates the lowest limit. When it comes to *B-type* ECs, $l_j^{\min}$ denotes the minimal target value that matches the performance of competitors, whereas $l_j^{\max}$ represents the largest limit.

5.3.3 A mixed integer non-linear programming model

Based on the contents above, for the purpose of maximizing the payoff function of individual CR, a mixed integer non-linear programming model is established through an optimal trade, which shows as follows:

$$
\begin{cases}
\max \prod_{i=1}^{n} (d_i(y_i) - DS_i)^{w_i} \\
\text{s.t.} \\
\qquad x_j = \sum_{k=1}^{m} x_{jk} h_{kj} \quad (for\ discrete\ EC_j, \ \ j = 1, 2, \cdots, q) \\
\qquad \sum_{k=1}^{m} x_{jk} = 1 \qquad\quad (for\ discrete\ EC_j, \ \ j = 1, 2, \cdots, q) \\
\qquad \sum_{j=1}^{q} c_j x_j \leq B \\
\qquad y_i = \sum_{j=1}^{q} r_{ij}^{norm} x_j, \quad i = 1, 2, \cdots, n \\
\qquad y_i \geq \frac{\sum_{t=1}^{s} p_i^t}{s}, \quad i = 1, 2, \cdots, n \\
\qquad 0 \leq x_j \leq 1, \quad j = 1, 2, \cdots, q \\
\qquad ECL_j \leq x_j \leq ECH_j, \quad j = 1, 2, \cdots, q \\
\qquad x_{jk} \in (0, 1),
\end{cases}
\tag{5.15}
$$

where the expressions of $d_i(y_i)$ are shown in Table 5.3. The first constraint indicates the normalization of discrete ECs and the second constraint guarantees that a unique k of a discrete EC is selected. The third constraint demonstrates the budget control through the process, and the fourth constraint transfers the fulfillment levels of ECs into those of CRs by means of the normalized relationship matrix R. The average value among the competitor companies is used as a benchmark for the fulfillment level of CS with respect to individual CR in the fifth constraint. Since target values of both continuous and discrete ECs are normalized into x_j in Section 5.3.2, then x_j is defined in the domain $[0, 1]$. Besides, some ECs are equipped with lower or upper bounds for technical concerns.

For a simple mathematical derivation, suppose that there are two players (two continuous CRs) in this bargaining now, $i = 1, 2.$, and CR_1 is an Attractive attribute while CR_2 is a One-dimensional attribute. Then, according to model

(5.15), the following model can be formulated as

$$\begin{cases} \max\ (d_1(y_1) - DS_1)^{w_1}(d_2(y_2) - DS_2)^{w_2} \\ \qquad d_1(y_1) = \dfrac{CS_1 - DS_1}{e-1}e^{y_1} - \dfrac{CS_1 - eDS_1}{e-1} \\ \qquad d_2(y_2) = (CS_2 - DS_2)y_2 + DS_2 \\ \text{s.t.} \\ \qquad c_1x_1 + c_2x_2 + \cdots + c_qx_q \leq B \\ \qquad y_1 = r_{11}^{norm}x_1 + r_{12}^{norm}x_2 + \cdots + r_{1q}^{norm}x_q \\ \qquad y_2 = r_{21}^{norm}x_1 + r_{22}^{norm}x_2 + \cdots + r_{2q}^{norm}x_q \\ \qquad y_1 \geq \dfrac{\sum_{t=1}^{s} p_1^t}{s},\ y_2 \geq \dfrac{\sum_{t=1}^{s} p_2^t}{s} \\ \qquad 0 \leq x_j \leq 1, \quad j = 1, 2, \cdots, q, \end{cases} \tag{5.16}$$

where x_j are decision variables of fulfillment levels of q ECs. The unit improvement cost c_j is positive as well as the relationship r_{ij}^{norm} between CR$_i$ and EC$_j$. After plugging the expressions of $d_i(y_i)$ into the objective function in model (5.16), it is obtained that

$$f(y_1, y_2) = \Big(\frac{CS_1 - DS_1}{e-1}(e^{y_1} - 1)\Big)^{w_1}\Big((CS_2 - DS_2)y_2\Big)^{w_2}. \tag{5.17}$$

For the purpose of better analyzing the above equation, we let $E = (CS_1 - DS_1)/(e-1)$ and $F = CS_2 - DS_2$. Since CS_i is positive and DS_i is negative, the values of E and F are both positive. After taking the first order derivative of Eq. (5.17) with respect to y_1, it is attained that

$$\frac{\partial f}{\partial y_1} = w_1E(e^{y_1} - 1)^{(w_1-1)}e^{y_1} \cdot (Fy_2)^{w_2}. \tag{5.18}$$

Next, by taking the second order partial derivative towards y_2, it is calculated that

$$\frac{\partial^2 f}{\partial y_1 \partial y_2} = w_1E(e^{y_1} - 1)^{(w_1-1)}e^{y_1} \cdot w_2(Fy_2)^{(w_2-1)}F. \tag{5.19}$$

In Eq. (5.19), since y_1 and y_2 are both positive, the second order partial derivative is larger than 0, which means Eq. (5.17) is a convex function.

On the other hand, in Eq. (5.17), $\left(\frac{CS_1 - DS_1}{e-1}(e^{y_1}-1)\right)^{w_1}$ is increasing with respect to $y_1 \in (0, +\infty)$ and $\left((CS_2 - DS_2)y_2\right)^{w_2}$ is increasing with respect to $y_2 \in (0, +\infty)$ as well. This indicates that the objective function $f(y_1, y_2)$ in model (5.16) is increasing both to y_1 and y_2 in the positive horizontal axis. If the value of $f(y_1, y_2)$ is expected to be maximized, the larger the values of y_1 and y_2, the larger that of $f(y_1, y_2)$. As a consequence, the cost constraint in model (5.16) should be equal to the budget to get the largest y_i, i.e., $c_1x_1+c_2x_2+\cdots+c_qx_q = B$.

When it comes to the objective function in model (5.15), two aspects should be considered in practical computations. Firstly, as a product, the value of the objective function is negatively related to the number of players, n. Secondly, the existence of parameter w_i will increase the difficulty of solving the model. To evade these two concerns, a simple mathematical conversion of the objective function is needed. Thereby, the objective function can be modified by applying a logarithmic transformation as follows:

$$\ln \prod_{i=1}^{n} (d_i(y_i) - DS_i)^{w_i} = \sum_{i=1}^{n} w_i \ln(d_i(y_i) - DS_i). \tag{5.20}$$

5.4 Case Study: A Notebook Computer Development

At first, seven major CRs of a notebook computer are extracted out of 125 valid feedback, including Stylish design (CR_1), Mobility (CR_2), High computing speed (CR_3), Powerful graphics solution (CR_4), Solid audio capability (CR_5), Large storage (CR_6), and High network performance (CR_7), and the details are summarized in Table 5.4. It is noted that the original Kano questionnaire data and Kano's model analysis of this case can be found in [Jip14].

Table 5.4: Kano questionnaire results in [Jip14].

Seven major CRs	A	O	M	I	R	Q	Sum	KC	CS	DS
a: Stylish design	**70**	25	14	15	0	1	125	A	0.7661	−0.3145
b: Mobility	15	**75**	27	7	1	0	125	O	0.7258	−0.8226
c: High computing speed	24	**64**	23	11	2	1	125	O	0.7213	−0.7131
d: Powerful graphics solution	**69**	31	15	9	1	0	125	A	0.8065	−0.3710
e: Solid audio capability	**76**	25	12	10	2	0	125	A	0.8211	−0.3008
f: Large storage	24	20	**70**	9	2	0	125	M	0.3577	−0.7317
High network performance	16	27	22	**57**	1	2	125	**I**	0.3525	−0.4016

The CS and DS values in Table 5.4 are acquired by means of Eqs. (5.1)-(5.2). In terms of the calculation of CR_1, the values of CS_1 and DS_1 can be obtained as

$$CS_1 = \frac{\frac{70}{125}+\frac{25}{125}}{\frac{70}{125}+\frac{25}{125}+\frac{14}{125}+\frac{15}{125}} = \frac{95}{124} = 0.7661,$$

$$DS_1 = -\frac{\frac{25}{125}+\frac{14}{125}}{\frac{70}{125}+\frac{25}{125}+\frac{14}{125}+\frac{15}{125}} = -\frac{39}{124} = -0.3145.$$

Analogously, the results of remaining CRs are attained, which will be served as value functions in the following Shapley value calculations. Notably, the KC of High network performance (CR_7) is remarked by I in Table 5.4, which implies that consumers feel indifferent to this function. As a consequence, the assignment of relative importance weights will only focus on the other six CRs (labeled by a-f for easier instructions later), and the sum of total weights is 1.

5.4.1 Results of Stage 1: the CR weighting

As introduced in Section 5.2, Shapley value is utilized to distribute the weights of CRs in accordance with their marginal contributions. In this empirical study, there exist six CRs in the current product planning, which means six levels of coalitions in the cooperative game.

All the value functions and mean values of different combinations can be derived with the aid of the SVCA in Algorithm 1 and the VFCA in Algorithm 2 in Section 5.2.2, and the results are summarized in Table 5.5. In the "Coalition"

Table 5.5: The calculation procedure of Shapley value in Eq. (5.5).

zth Level	Coalition		CR$_1$ a	CR$_2$ b	CR$_3$ c	CR$_4$ d	CR$_5$ e	CR$_6$ f
1^{st} level	a, b, c, d, e, f	Avg_1= 0.6998	$v(1)$= 0.7661	$v(2)$= 0.7258	$v(3)$= 0.7213	$v(4)$= 0.8065	$v(5)$= 0.8211	$v(6)$= 0.3577
2^{nd} level	$ab, ac, ad, ae, af, bc, bd, be, bf, cd, ce, cf, de, df, ef$	Avg_2= 0.7868	$V_2(1)$= 0.7852	$V_2(2)$= 0.7691	$V_2(3)$= 0.7682	$V_2(4)$= 0.8094	$V_2(5)$= 0.8211	$V_2(6)$= 0.7682
3^{rd} level	$abc, abd, abe, abf, acd, ace, acf, ade, adf, aef, bcd, bce, bcf, bde, bdf, bef, cde, cdf, cef, def$	Avg_3= 0.8037	$V_3(1)$= 0.8002	$V_3(2)$= 0.7962	$V_3(3)$= 0.7962	$V_3(4)$= 0.8123	$V_3(5)$= 0.8211	$V_3(6)$= 0.7962
4^{th} level	$abcd, abce, abcf, abde, abdf, abef, acde, acdf, acef, adef, bcde, bcdf, bcef, bdef, cdef$	Avg_4= 0.8135	$V_4(1)$= 0.8112	$V_4(2)$= 0.8112	$V_4(3)$= 0.8112	$V_4(4)$= 0.8153	$V_4(5)$= 0.8211	$V_4(6)$= 0.8112
5^{th} level	$abcde, abcdf, abcef, abdef, acdef, bcdef$	Avg_5= 0.8187	$V_5(1)$= 0.8182	$V_5(2)$= 0.8182	$V_5(3)$= 0.8182	$V_5(4)$= 0.8182	$V_5(5)$= 0.8211	$V_5(6)$= 0.8182
6^{th} level	$abcdef$	$v(all)$= **0.8211**						

column, all the combinations at the zth level for $z = 1, 2, \cdots, 6$ are listed, and the detailed meanings of Avg_k, $v(k)$, $V_z(k)$, and $v(all)$ can be found in Section 5.2. Based on the primary results in Table 5.5, Shapley value S_1 of CR$_1$ is computed by applying Eq. (5.5) as follows:

$$\begin{aligned} S_1 = &\frac{1}{5} * (0.7661 - 0.6998) + \frac{1}{4} * (0.7852 - 0.7868) + \frac{1}{3} * (0.8002 - 0.8037) \\ &+\frac{1}{2} * (0.8112 - 0.8135) + \frac{1}{1} * (0.8182 - 0.8187) + \frac{1}{6} * 0.8211 = 0.1469. \end{aligned} \tag{5.21}$$

Similarly to Eq. (5.21), the values of $S_2 \sim S_6$ can be easily figured out by the SVCA and are displayed in Table 5.6. It is observed that CR$_5$ possesses the most marginal contribution whereas CR$_6$ shows the least, which belong to Attractive and Must-be attributes, respectively. The sum $\sum_{i=1}^{6} S_i = 0.8211$ in Shapley value equals to the utility value when all the players collaborate in the sixth level in Table 5.5. This consistency validates the effectiveness axiom of Shapley value. Apart from that, it is noticed that CS_5 (0.8211) is two times larger than CS_6 (0.3577), and DS_6 (-0.7317) is two times larger than DS_5 (-0.3008) in Table 5.4. Consequently, it seems quite convincing that Shapley value of CR$_5$, S_5, is three times greater than that of CR$_6$, S_6.

Table 5.6: Shapley values and relative importance weights of CRs.

Shapley Value	S_1	S_2	S_3	S_4	S_5	S_6
	0.1469	0.1335	0.1323	0.1671	0.1817	0.0596
Relative Importance	w_1	w_2	w_3	w_4	w_5	w_6
Weight (%)	17.89	16.25	16.12	20.35	22.13	7.26

Then, in regard to Eq. (5.8), the relative importance weight w_i of each CR can be further generated. For example, the relative importance weight for CR$_1$ is $w_1 = (0.1469/0.8211) \times 100\% = 17.89\%$. In correspondence with each Shapley value, the relative importance weights of the remaining CRs are calculated in Table 5.6, in which CR$_5$ shows the most significance while CR$_6$ is the least.

5.4.2 Results of Stage 2: the EC target value setting

In this section, the second-stage cooperative game regarding the notebook computer design is conducted, which aims at determining target values of ECs in the new generation product by using a mixed integer non-linear programming model. The first step is to accomplish all the needed information in model (5.15).

Primarily, according to the quantitative Kano's model describing A, O, and M attributes in Table 5.3, the specific relationship functions between the fulfillment level of CR and CS, i.e., the $d_i(y_i)$ functions of different attributes in this case study are enumerated and calculated in Table 5.7.

Table 5.7: The $d_i(y_i)$ functions for CRs of the notebook computer design.

Six crucial CRs	KC	a_i	b_i	$f(y_i)$	$d_i = a_i f(y_i) + b_i$
CR_1: Stylish design	A	0.8529	−0.9434	e^{y_1}	$d_1 = 0.6289e^{y_1} - 0.9434$
CR_2: Mobility	O	1.5484	−0.8226	y_2	$d_2 = 1.5484y_2 - 0.8226$
CR_3: High computing speed	O	1.4344	−0.7131	y_3	$d_3 = 1.4344y_3 - 0.7131$
CR_4: Powerful graphics solution	A	0.6852	−1.0562	e^{y_4}	$d_4 = 0.6852e^{y_4} - 1.0562$
CR_5: Solid audio capability	A	0.6529	−0.9538	e^{y_5}	$d_5 = 0.6529e^{y_5} - 0.9538$
CR_6: Large storage	M	1.7235	0.9917	$-e^{-y_6}$	$d_6 = -1.7235e^{-y_6} + 0.9917$

Secondly, before the QFD analysis is conducted, the information of four matrices are gathered in the HoQ of the notebook computer design in Table 5.8. Notably, CRs together with their relative importance weights w_i are obtained in Section 5.4.1. Seven ECs are outlined by engineers to map into six crucial CRs, i.e., CPU, RAM, hard disk, sound card, graphic card, LCD display, and battery. Then, the normalized relationship matrix R with respect to Eq. (5.12) is assessed and given by experts as follows:

$$R = \left(r_{ij}^{norm}\right)_{n\times q} = \begin{bmatrix} 0.1172 & 0.1172 & 0.1176 & 0.0415 & 0.1176 & 0.2725 & 0.2163 \\ 0.1816 & 0.1683 & 0.1604 & 0.0923 & 0.1654 & 0.0283 & 0.2036 \\ 0.1876 & 0.1895 & 0.1686 & 0.1356 & 0.1628 & 0.0078 & 0.1481 \\ 0.1753 & 0.1702 & 0.1445 & 0.1101 & 0.1685 & 0.0829 & 0.1485 \\ 0.1909 & 0.1956 & 0.1603 & 0.1807 & 0.1458 & 0.0038 & 0.1230 \\ 0.1664 & 0.2081 & 0.2280 & 0.1119 & 0.1256 & 0.0060 & 0.1540 \end{bmatrix}. \tag{5.22}$$

The benchmark information of $Comp_i^t$ is listed in the right-side strategic room containing four competitor companies. Lastly, the cost coefficient c_j towards ECs is summarized on the floor as well as some technical constraints with the lower bound ECL_j or the upper bound ECH_j.

Thirdly, the detailed information on two kinds of ECs can be found in Table 5.9. Among them, EC_1 to EC_6 are discrete and each one is provided with several options for customers to choose. And their normalization is based on

Table 5.8: The HoQ of the notebook computer design.

CRs \ ECs	Relative Importance Weights w_i (%)	1. CPU	2. RAM	3. Hard disk	4. Sound card	5. Graphic card	6. LCD display	7. Battery	$Comp^1$	$Comp^2$	$Comp^3$	$Comp^4$
1. Stylish design (A)	0.1789	0.1172	0.1172	0.1176	0.0415	0.1176	0.2725	0.2163	0.85	0.74	0.70	0.76
2. Mobility (O)	0.1625	0.1816	0.1683	0.1604	0.0923	0.1654	0.0283	0.2036	0.60	0.71	0.45	0.60
3. High computing speed (O)	0.1612	0.1876	0.1895	0.1686	0.1356	0.1628	0.0078	0.1481	0.72	0.55	0.40	0.47
4. Powerful graphics solution (A)	0.2035	0.1753	0.1702	0.1445	0.1101	0.1685	0.0829	0.1485	0.90	0.84	0.75	0.71
5. Solid audio capability (A)	0.2213	0.1909	0.1956	0.1603	0.1807	0.1458	0.0038	0.1230	0.54	0.55	0.61	0.60
6. Large storage (M)	0.0726	0.1664	0.2081	0.2280	0.1119	0.1256	0.0060	0.1540	0.83	0.75	0.50	0.66
Cost Index (units)		19.8	17.5	15.5	14	16.5	14.5	13				
Technical Constraints (ECL)		0.65	-	-	-	-	-	-				
Technical Constraints (ECH)		-	-	0.85	-	-	-	-				

Table 5.9: Seven discrete and continuous ECs.

Discrete ECs	Option 1	Option 2	Option 3	Option 4	Option 5
EC_1: CPU	1.8Ghz ($h_{11} = 0.25$)	2.4Ghz ($h_{12} = 0.5$)	2.9Ghz ($h_{13} = 0.8$)	3.6Ghz ($h_{14} = 1$)	–
EC_2: RAM	1G ($h_{21} = 0.25$)	2G ($h_{22} = 0.5$)	4GB ($h_{23} = 0.75$)	8GB ($h_{24} = 1$)	–
EC_3: Hard disk	64G ($h_{31} = 0.2$)	128G ($h_{32} = 0.4$)	256G ($h_{33} = 0.6$)	512G ($h_{34} = 0.85$)	1TB ($h_{35} = 1$)
EC_4: Sound card	Level I ($h_{41} = 0.33$)	Level II ($h_{42} = 0.66$)	Level III ($h_{43} = 1$)	–	–
EC_5: Graphic card	Level I ($h_{51} = 0.33$)	Level II ($h_{52} = 0.66$)	Level III ($h_{53} = 1$)	–	–
EC_6: LCD display	11.1” ($h_{61} = 0.2$)	12.2” ($h_{62} = 0.4$)	13.1” ($h_{63} = 0.6$)	15.1” ($h_{64} = 0.8$)	17.1” ($h_{65} = 1$)
Continuous ECs	Category	Min EC value		Max EC value	
EC_7: Battery	*B-type*	2hr		8hr	

Eq. (5.13), in which h_{kj} is the fulfillment rating. When it comes to the continuous EC_7 of the B-type, the normalization formula is Eq. (5.14).

After all necessary information is gathered, the mixed integer non-linear programming model can be built to optimize the current design of the notebook computer. Denote $X = [x_1, x_2, \cdots, x_7]^T$ as the fulfillment level vector for ECs, and $Y = [y_1, y_2, \cdots, y_6]^T$ as the fulfillment level vector for CRs. Meanwhile, it is evaluated by the decision-makers that the total financial investment during the QFD procedure is 100 units. Then, the optimization model can be established according to all the case study information (see next page).

The proposed model (5.23) is solved by the Lingo software, and the results of decision variables x_j, EC options together with their technical values, and the relevant resource allocation are displayed in Table 5.10. It is easily observed that the fulfillment level of EC_6 (LCD display), 0.400, is barely satisfactory in contrast to those of the remaining ECs. And the design cost calculated indicates that the

$$
\left\{
\begin{array}{l}
\max \quad \displaystyle\sum_{i=1}^{6} w_i \ln(d_i(y_i) - DS_i) \\
\qquad d_1(y_1) - DS_1 = 0.6289(e^{y_1} - 1) \\
\qquad d_2(y_2) - DS_2 = 1.5484 y_2 \\
\qquad d_3(y_3) - DS_3 = 1.4344 y_3 \\
\qquad d_4(y_4) - DS_4 = 0.6852(e^{y_4} - 1) \\
\qquad d_5(y_5) - DS_5 = 0.6529(e^{y_5} - 1) \\
\qquad d_6(y_6) - DS_6 = -1.7235(e^{y_6} - 1) \\
\text{s.t.} \\
\quad x_1 = 0.2x_{11} + 0.5x_{12} + 0.8x_{13} + x_{14} \\
\quad x_2 = 0.25x_{21} + 0.5x_{22} + 0.75x_{23} + x_{24} \\
\quad x_3 = 0.2x_{31} + 0.4x_{32} + 0.6x_{33} + 0.85x_{34} + x_{35} \\
\quad x_4 = 0.33x_{41} + 0.66x_{42} + x_{43} \\
\quad x_5 = 0.33x_{51} + 0.66x_{52} + x_{53} \\
\quad x_6 = 0.2x_{61} + 0.4x_{62} + 0.6x_{63} + 0.8x_{64} + x_{65} \\
\quad x_{11} + x_{12} + x_{13} + x_{14} = 1 \\
\quad x_{21} + x_{22} + x_{23} + x_{24} = 1 \\
\quad x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = 1 \\
\quad x_{41} + x_{42} + x_{43} = 1 \\
\quad x_{51} + x_{52} + x_{53} = 1 \\
\quad x_{61} + x_{62} + x_{63} + x_{64} + x_{65} = 1 \\
\quad Y = RX \\
\quad y_1 \geq (0.85 + 0.74 + 0.7 + 0.76)/4 \\
\quad y_2 \geq (0.6 + 0.71 + 0.45 + 0.6)/4 \\
\quad y_3 \geq (0.72 + 0.55 + 0.4 + 0.47)/4 \\
\quad y_4 \geq (0.9 + 0.84 + 0.75 + 0.71)/4 \\
\quad y_5 \geq (0.54 + 0.55 + 0.61 + 0.6)/4 \\
\quad y_6 \geq (0.83 + 0.75 + 0.5 + 0.66)/4 \\
\quad 19.8x_1 + 17.5x_2 + 14.5x_3 + 14x_4 + 16.5x_5 + 15.5x_6 + 13x_7 \leq 100 \\
\quad 0 \leq x_j \leq 1, \quad j = 1, 2, \cdots, 7 \\
\quad x_{jk} \in (0, 1) \\
\quad x_1 \geq 0.65, x_3 \leq 0.85
\end{array}
\right.
\tag{5.23}
$$

total investment is distributed unevenly and inappropriately. Furthermore, this circumstance leads to the result that the fulfillment level of CR_1 (Stylish design), 0.8188, is the lowest as demonstrated in Table 5.11, while most of the remaining CR fulfillment levels are larger than 0.95.

Table 5.10: The optimal solution of model (5.23).

Discrete ECs	EC option	EC fulfillment level (x_j)	EC technical value	Resource allocation
EC_1: CPU	x_{14}	1.000	3.6GHz	19.8
EC_2: RAM	x_{24}	1.000	8GB	17.5
EC_3: Hard disk	x_{34}	0.850	512G	12.3
EC_4: Sound card	x_{43}	1.000	Level III	14
EC_5: Graphic card	x_{53}	1.000	Level III	16.5
EC_6: LCD display	x_{62}	0.400	12.2"	6.2
Continuous ECs		EC fulfillment level (x_j)	EC technical value	Resource allocation
EC_7: Battery		1.000	8hr	13

Table 5.11: Results of CR fulfillment levels with regard to model (5.23).

CRs	CR fulfillment level (y_i)
CR_1: Stylish design	**0.8188**
CR_2: Mobility	0.9589
CR_3: High computing speed	0.9700
CR_4: Powerful graphics solution	0.9286
CR_5: Solid audio capability	0.9738
CR_6: Large storage	0.9622

The cause of this low fulfillment level is that CR_1 is greater connected with EC_6 than other CRs in the normalized relationship matrix R. In addition to the aforementioned imbalance in CR fulfillment levels, the technical value of EC_6 is 12.2", which seems not so acceptable for customers out of the five choices, i.e., 11.1", 12.2", 13.1", 15.1", and 17.1". The size of LCD display not only has an effect on the appearance of the notebook computer, but also on customers' visual experience.

In order to enhance customers' perceptions on the stylish design and consider the engineers' desire for a more balanced design, model (5.23) is revised through substituting "$y_1 \geq (0.85+0.74+0.7+0.76)/4$" by "$y_1 \geq 0.9$" directly. Then, after the revised model is solved by the Lingo software, more reasonable results can be obtained and are listed in Table 5.12. The data in bold are different solutions in contrast to those in Table 5.10, including fulfillment levels and technical values of EC_4, EC_6, and EC_7.

Table 5.12: The solution of the revised model (5.23).

Discrete ECs	EC option	EC fulfillment level (x_j)	EC technical value	Resource allocation
EC_1: CPU	x_{14}	1.000	3.6GHz	19.8
EC_2: RAM	x_{24}	1.000	8GB	17.5
EC_3: Hard disk	x_{34}	0.850	512G	12.3
EC_4: Sound card	$\boldsymbol{x_{42}}$	**0.660**	**Level II**	9.2
EC_5: Graphic card	x_{53}	1.000	Level III	16.5
EC_6: LCD display	$\boldsymbol{x_{64}}$	**0.800**	**15.1"**	12.4
Continuous ECs		EC fulfillment level (x_j)	EC technical value	Resource allocation
EC_7: Battery		**0.941**	**7.65hr**	12.2

It can be seen that in the revised solution in Table 5.12, the size of LCD display of the notebook computer becomes larger, which straightly leads to a shorter battery endurance time. Meanwhile, the performance of sound card is a bit lower compared with the former solution. Although the fulfillment level of EC_6 is improved by sacrificing those of EC_4 and EC_7, the customer satisfaction of CR_1 is explicitly enhanced to 0.9010 in Table 5.13. Now, all CR fulfillment levels are above 0.9 and more evenly distributed. The customer satisfaction degree d_i of each CR is also calculated in Table 5.13, and the OCS calculation in Eq. (5.9) and Table 5.6, i.e., $\sum_{i=1}^{6} w_i \cdot d_i = 0.6073$. Since d_i is proportional to CS_i in the quantitative Kano's model, the full customer satisfaction degree is calculated as $\sum_{i=1}^{6} w_i \cdot CS_i = 0.7341$. After the optimization, the product achieves an overall satisfaction level of 0.6073 out of 0.7431, i.e.,

$$Satisfaction\ level = \frac{\sum_{i=1}^{6} w_i \cdot d_i}{\sum_{i=1}^{6} w_i \cdot CS_i} = \frac{0.6073}{0.7431} = 81.73\%, \tag{5.24}$$

which is relatively satisfactory to the decision-makers.

Table 5.13: Results of CR fulfillment levels with regard to the revised model (5.23).

CRs	CR fulfillment level (y_i)	Customer satisfaction (d_i)	CS_i	Satisfaction level (%)
CR_1: Stylish design	**0.9010**	0.6050	0.7661	78.97
CR_2: Mobility	0.9268	0.6125	0.7258	84.38
CR_3: High computing speed	0.9183	0.6041	0.7213	83.75
CR_4: Powerful graphics solution	0.9156	0.6556	0.8065	81.29
CR_5: Solid audio capability	0.9066	0.6628	0.8211	80.72
CR_6: Large storage	0.9175	0.3032	0.3577	84.77
Overall customer satisfaction (OCS)		0.6073	0.7431	81.73

Besides, for the sake of observing the effect of the budget setting on the final results, the budget constraint in model (5.23) is modified to be "$19.8x + 17.5x + 14.5x + 14x + 16.5x + 15.5x + 13x \leq 90$", while other conditions stay unchanged. By solving this new model, the same results of x_j as in Table 5.10 are obtained except that $x_4 = 0.33$. It means under a smaller budget limit of 90 units, EC_4 is the first to be affected and sacrificed, and the practitioners should choose Level I sound card in the new notebook computer design. As a direct consequence, the lowest and largest CR fulfillment level are $y_1 = 0.7910$ and $y_2 = 0.8970$, respectively, which indicates all CR fulfillment levels are under 0.9. On this basis, the satisfaction level will drop to 70.11% compared with 81.73%.

In this section, with the steps in the methodology part, the notebook computer development in the CR weighting stage and the product improvement stage are successively conducted. The computational results are very reasonable in the practical scenario by considering diverse variables and trade-offs. The sensitivity of EC_4 towards the overall customer satisfaction and the budget setting is also analyzed. After the whole QFD procedure with a budget of 100 units, target values of discrete and continuous ECs in a revised solution are settled and selected according to customer perceptions.

5.5 Discussions

The proposed two-stage cooperative game-theoretic approach integrating Kano's model in QFD is a generalized and systematic method, which can be implemented to the vast majority of manufacturing products in real life. As a close contrast, in this part the performances between this research and Ji *et al.* [Jip14]'s method in both stages in terms of the same case study will be respectively expounded.

As to the contents in Stage 1, the distinctions on relative importance weights of both researches are extracted and outlined in Table 5.14 for further discussion. Since Ji *et al.* [Jip14] put the focus on the derivation of a novel quantitative

Table 5.14: The comparison on relative importance weights of CRs.

CRs	CR_1	CR_2	CR_3	CR_4	CR_5	CR_6
Kano category	A	O	O	A	A	M
The method in this chapter	w_1	w_2	w_3	w_4	w_5	w_6
Weights	0.1789	0.1625	0.1612	0.2035	0.2213	0.0726
Ranking	3	4	5	2	1	6
Ji *et al.* [Jip14]	w'_1	w'_2	w'_3	w'_4	w'_5	w'_6
Weights	0.1460	0.1562	0.1857	0.1624	0.1792	0.1705
Ranking	6	5	1	4	2	3

Kano's model in that paper, the weights $w'_i, i = 1, 2, \cdots, 6$ are given directly without a detailed elaboration. Notably, the weights w'_i are relatively even distributed and CR_3 (O), CR_5 (A), and CR_6 (M) are concerned about with larger degrees. However, the outcome of the method raised in this chapter puts more efforts on the three customer needs CR_5, CR_4, and CR_1 of Attractive attribute, which is intuitive and reasonable since this is the developing trend of the new product for the sake of grasping current and potential customers. In addition, CR_6 is entitled with the least importance w_6 and the lowest priority in this method owing to its Must-be attribute. It means CR_6 is just a basic requirement nowadays in the new product design that should be well achieved without doubt. Through the comparison, it can be seen that the determination of CR weights based on Shapley value is of CR weights is of more rationality and effectiveness. Meanwhile, this structured method is also accompanied by two calculation algorithms to simply facilitate the actual operation for the decision-makers.

When it comes to the contents in Stage 2, optimization models are built in both researches by considering separate objective functions. The identical solution of decision variables is computed, which validates the feasibility of the proposed Nash bargaining based model. Even so, there exist different analyses on the final results, and the efficiencies of two models are not the same as well. Firstly, the final solution adopted is a revised one and whose reason is clearly explained in Section 5.4.2. Analogously, such a situation also happened in [Jip14], however they omitted the relevant solution process description. Secondly, it is noted that in the calculation formula of satisfaction level in Eq. (5.24), w_i plays an important role and has a direct effect on the outcome. This close connection indicates that the determination of CR weights is of great significance in more precisely evaluating the satisfaction level of the upgraded product after optimization. Lastly, the budget analysis in Section 5.4.2 reflects the importance of the capital investment on CR fulfillment levels and consumers' satisfaction level towards the product development.

On the whole, three pieces of managerial implications can be figured out in this case study. Firstly, the rational ascertaining of CR weights is the first and vital step before formulating optimization models in the QFD procedure so as to guarantee the feasibility and effectiveness of subsequent computations. Based on this rule, this chapter helps provide a systematic and useful approach for the decision-makers to obtain the CR weights for most manufacturing products. Secondly, a revised solution sometimes should be considered when the original solution seems not so reasonable in practice. And this research elaborates the detailed reasoning behind it as a demonstration for the decision-makers. The last but not the least, it is also vital to put a sufficient investment in new product design. It is seen that the fulfillment level of each CR largely exceeds that of other companies $Comp^1$-$Comp^4$ in Table 5.8, due to a budget of 100 units.

5.6 Summary

The research in this chapter combined another benefit or resource distribution tool - cooperative game theory into product design process in two stages. Different from traditional QFD which considers CRs individually or comparatively, CRs were treated as a group with cooperation or bargaining. On this basis, the whole QFD process was observed from a novel cooperative game-theoretic angle.

In summary, the major contributions lay in several aspects as follows. Firstly, Shapley value was employed as each CR's marginal contribution to the whole customer satisfaction, and a Nash bargaining function among all customer requirements was served as the objective function of an optimization model. As a consequence, two critical issues in QFD were solved, i.e., the CR weighting and the EC target value determination. Secondly, a quantitative Kano's model was also adopted through the whole QFD process, and the selected figures and functions in Kano's model were defined as corresponding value functions in the Shapley value and Nash bargaining calculations. Thirdly, two simple algorithms to obtain the CR weighting were designed, which targeted on improving the feasibility and execution of the proposed method in real-life applications. Finally, the proposed method was implemented to a notebook computer design case study for quality development in defeating rivals in the market, and target values of ECs were determined according to customer perceptions. Meanwhile, comparisons and discussions with other researches were also conducted. The example demonstrated that the proposed approach was able to model the practical product planning process effectively and efficiently.

The next chapter deals with quality function deployment optimization from the perspective of fuzzy theories. The analytical derivations on expected values of different fuzzy events play a significant role. At first, the fuzzy importance of ECs are ranked through expected values. Then, target levels of ECs are determined through an expected value model with fuzzy parameters.

Chapter 6

Determination of Fuzzy Importance and Target Levels of ECs in Fuzzy QFD

The previous chapter elaborated a two-stage cooperative game, which dealt with the determination of CR relative importance weights and target values of ECs. All parameters in the optimization procedure were settled to be crisp numbers. In this chapter, the HoQ is evaluated by linguistic statements which correspond to a group of trapezoidal fuzzy numbers, based on which two research points regarding ECs in fuzzy QFD are conducted.

Primarily, the rating of the fuzzy importance of ECs is obtained with the aid of a newly derived calculation formula, making use of the fuzzy weights of CRs and fuzzy relationships between CRs and ECs. Afterwards, an expected value model with fuzzy parameters integrating a quantitative Kano's model is established to determine target levels of ECs in accordance with all the information listed in the fuzzy HoQ. An improved hybrid intelligent algorithm (iHIA), which consists of a fuzzy simulation procedure for the expected value of fuzzy events, the TNIA-S, and a genetic algorithm, is utilized to solve the proposed fuzzy optimization model. Finally, the whole method is implemented to the notebook computer development case study to demonstrate its feasibility and effectiveness. The priority of the fuzzy importance of ECs in the case study is attained. Meanwhile, by applying the iHIA, different combinations of target levels of ECs are computed in regard to varied confidence levels. In consideration of the preference and trade-off of the decision-makers, a set of target levels at a certain confidence level are ascertained as the ultimate solution.

6.1 Introduction

Since the HoQ in QFD is usually evaluated according to experts' knowledge and experience, the fuzzy set theory of describing vague and ambiguous events seems appropriate to be incorporated in this subjective evaluation. The process is, the judgments expressed by linguistic variables are firstly utilized to depict the fuzzy elements in the HoQ. Then, the corresponding fuzzy variables of these linguistic variables are defined, which will be listed in different matrices of the HoQ. Commonly, the fuzzy elements include the CR importance weights, the relationships between CRs and ECs, and the correlations among ECs, etc.

The fuzzy variables considered in this chapter are trapezoidal fuzzy numbers (TpFNs), which is a kind of widely used fuzzy variables in applications like [Zha05, Liu15c, Zho18]. Suppose that n CRs and q ECs are outlined from a manufacturing product. Then, different TpFNs will be predefined to assess the fuzzy weights of CRs, w_i, $i = 1, 2, \cdots, n$ in matrix W, and the fuzzy relationships between CRs and ECs, r_{ij}, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, q$ in matrix R.

Notably, the basic research points in the traditional QFD are also applicable to the fuzzy QFD. Relevant literature review can be found in Section 2.3. In particular, this chapter focuses on two of the research points regarding ECs, i.e., the determination of the priority of the fuzzy importance of ECs, and fulfillment levels of ECs through a fuzzy optimization model. Generally, the fuzzy importance of ECs is obtained by calculating the expected value of the multiplication of two fuzzy matrices, W and R. To simplify this calculation procedure, a direct formula for the expected value for the product of two TpFNs is given in this chapter. When it comes to the setting of target levels of ECs in fuzzy QFD, expected value models (EVM) with fuzzy parameters are commonly built [Che05]. Then, the proposed fuzzy EVM is transformed into a deterministic and simplified optimization model by virtue of several definitions and theorems on the expected value of fuzzy variables and fuzzy events. In addition, an *improved hybrid intelligent algorithm* (iHIA) which integrates the TNIA-S in Chapter 4 and a genetic algorithm is combined to solve the proposed fuzzy EVM, through which a series of target levels of ECs can be obtained.

The remaining contents of this chapter are arranged as follows. Firstly, the detailed calculation procedure on the expected value for the fuzzy importance of ECs is elaborated in Section 6.2. Subsequently, the model formulation and solution of a fuzzy EVM in QFD are set forth in Section 6.3, respectively. In Section 6.4, the fuzzy HoQ with six CRs and seven ECs of the same case study of a notebook computer development is given at first. On this basis, the priority of the fuzzy importance of ECs is addressed and different solutions of the fuzzy EVM are discussed. Finally, Section 6.5 concludes the whole chapter.

6.2 The Fuzzy Importance of ECs

Corresponding to the CR importance weights, the acquisition of the EC fuzzy importance or the prioritization of ECs is also an important research point in QFD. As introduced above, the fuzzy importance of ECs can be obtained via the multiplication of two matrices W and R in the HoQ. According to Chen *et al.* [Che05], the EC importance, v_j, for EC_j can be formulated as

$$v_j = \sum_{i=1}^{n} w_i r_{ij}, \ j = 1, 2, \cdots, q, \tag{6.1}$$

in which w_i in matrix W represents the relative importance weight of CR_i, $i = 1, 2, \cdots, n$, and r_{ij} in matrix R stands for the relationship between CR_i and EC_j,

$i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, q$. Since both w_i and r_{ij} are assumed to be fuzzy variables in this chapter, the product of them is also fuzzy, which seems hard to be measured in an intuitive way. On this basis, an expected value operator for v_j, $E[v_j]$, is utilized as a defuzzification approach [Che05], i.e.,

$$E[v_j] = E\Big[\sum_{i=1}^{n} w_i r_{ij}\Big], \ j = 1, 2, \cdots, q \tag{6.2}$$

Nevertheless, the expected value for the product of two fuzzy variables is also not simple to obtain. Chen *et al.* [Che06a] took advantage of a fuzzy weighted average method through h-cuts in [0,1] to approximate the expected value. Compared with the research in [Che06a], Liu *et al.* [Liu16] directly put forward a calculation formula to derive the exact expected value for the product of two triangular fuzzy numbers (TFNs). Notably, the fuzzy variables to describe the ambiguous linguistic statements in matrices W and R in this chapter are TpFNs, which belong to the area of regular fuzzy intervals. Thereby, in this section, by means of the computation principle in [Liu16] and several newly proved theorems on regular fuzzy intervals in Chapter 4, the calculation formula on the exact expected value for the product of two TpFNs will be obtained.

First of all, according to Theorem 4.10, the expected value of a TpFN $\xi \sim \mathcal{A}(a, b, c, d)$ can be derived as

$$\begin{aligned} E[\xi] &= \int_0^1 \xi_{\mathcal{A}\sup}(\alpha) \\ &= \int_0^{0.5} \Big(d - 2(d-c)\alpha\Big)\mathrm{d}\alpha + \int_{0.5}^1 \Big(2b - a - 2(b-a)\alpha\Big)\mathrm{d}\alpha \\ &= \frac{a+b+c+d}{4}, \end{aligned} \tag{6.3}$$

in which the analytical expression of the α-optimistic value of a TpFN, $\xi_{\mathcal{A}\sup}(\alpha)$, is in Eq. (4.38). It is easily verified that, the calculation result in Eq. (6.3) based on the α-optimistic value is identical to the one acquired by virtue of the original credibility measure-based definition on the expected value operator for fuzzy variables in [Liu02a, Liu02b]. A simple example regarding the expected value of a TpFN is given as follows:

Example 6.1 *Suppose that ξ is a TpFN, and is written as $\mathcal{A}(1,2,3,5)$. Then, in line with Eq. (6.3), the expected value of ξ is calculated as*

$$E[\xi] = \frac{1+2+3+5}{4} = \frac{11}{4}.$$

Subsequently, assume that $\xi_1 \sim \mathcal{A}(a_1, b_1, c_1, d_1)$ and $\xi_2 \sim \mathcal{A}(a_2, b_2, c_2, d_2)$ are two TpFNs with α-optimistic values $\xi_{1\,\sup}(\alpha)$ and $\xi_{2\,\sup}(\alpha)$, respectively. If both ξ_1 and ξ_2 are nonnegative TpFNs (i.e., $a_1 \geq 0$ and $a_2 \geq 0$), then the product $\xi = \xi_1\xi_2$ is strictly increasing with respect to ξ_1 and ξ_2, respectively. Therefore, based on Theorems 4.10 and 4.11, the expected value for the product of two TpFNs $\xi = \xi_1\xi_2$ can be formulated as

$$E[\xi] = E[\xi_1\xi_2] = \int_0^1 \xi_{1\,\sup}(\alpha)\xi_{2\,\sup}(\alpha)\mathrm{d}\alpha. \tag{6.4}$$

Afterwards, by plugging the analytical expression of the α-optimistic value of a TpFN in Eq. (4.38) into Eq. (6.4), it is attained that

$$\begin{aligned} E[\xi] = E[\xi_1\xi_2] &= \int_0^1 \xi_{1\,\sup}(\alpha)\xi_{2\,\sup}(\alpha)\mathrm{d}\alpha \\ &= \int_0^{0.5} \Big(d_1 - 2(d_1 - c_1)\alpha\Big) \times \Big(d_2 - 2(d_2 - c_2)\alpha\Big)\mathrm{d}\alpha \\ &\quad + \int_{0.5}^1 \Big(2b_1 - a_1 - 2(b_1 - a_1)\alpha\Big) \times \Big(2b_2 - a_2 + 2(b_2 - a_2)\alpha\Big)\mathrm{d}\alpha \\ &= \frac{1}{6}\Big(a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2\Big) + \frac{1}{12}\Big(a_1b_2 + a_2b_1 + c_1d_2 + c_2d_1\Big). \end{aligned} \tag{6.5}$$

The following numerical example is served as a simple display of the above calculation formula.

Example 6.2 *Let $\xi_1 \sim \mathcal{A}(1,2,3,5)$ and $\xi_2 \sim \mathcal{A}(3,5,7,10)$. Then, in accordance with Eq. (6.5), the expected value for the product of these two TpFNs is computed* *as*

$$E[\xi_1\xi_2] = \frac{1}{6}\Big(1 \times 3 + 2 \times 5 + 3 \times 7 + 5 \times 10\Big) + \frac{1}{12}\Big(1 \times 5 + 3 \times 2$$
$$+3 \times 10 + 7 \times 5\Big) = \frac{61}{3}.$$

Afterwards, in order to obtain the expected value of the fuzzy importance of ECs, a simplified expression of Eq. (6.2) can be further generated. That is, due to the independence of $w_1 r_{1q}, w_2 r_{2q}, \cdots, w_n r_{nq}$ in the QFD process, and the linearity of the expected value operator proved in [Liu02b], Eq. (6.2) is transformed into

$$E[v_j] = \sum_{i=1}^{n} E[w_i r_{ij}] \ j = 1, 2, \cdots, q. \tag{6.6}$$

For instance, $E[v_1]$ can be represented as

$$E[v_1] = E[w_1 r_{11}] + E[w_2 r_{21}] + \cdots + E[w_n r_{n1}], \tag{6.7}$$

in which each part can be achieved by applying Eq. (6.5). As a consequence, the fuzzy importance of ECs can be correspondingly prioritized according to their exact expected values, $E[v_j]$.

6.3 A Fuzzy Expected Value Model in QFD

Different from the mixed integer non-linear programming model with crisp parameters in Chapter 5, an expected value model (EVM) with fuzzy parameters is established in this section. It is clearly that the fuzzy elements are the CR importance weights and the relationships between CRs and ECs.

The fuzzy EVM is composed of an expected return of a fuzzy objective function and several constraints including constraints on the expected fulfillment levels of CRs. The model will be formulated with a detailed elaboration of the objective function and all the constraints. In particular, the analytical expressions of the original objective function and some constraints will be further derived into simplified ones, so as to be incorporated in the iHIA to get solutions. The TNIA-S for regular fuzzy intervals in Chapter 4 will be embedded in the iHIA to replace the original fuzzy simulation procedure for the expected value of fuzzy events in the HIA - the SDA. The specific algorithm steps of the iHIA for the fuzzy EVM will also be provided.

6.3.1 Model formulation with fuzzy parameters

The objective function and constraints of the proposed fuzzy EVM are mainly based on the optimization model from Ji *et al.* [Jip14], which are formulated as follows:

$$
\begin{cases}
\max \sum_{i=1}^{n} E[w_i \cdot d_i(y_i)] & (6.8) \\
\text{s.t.} & \\
\qquad \sum_{j=1}^{q} c_j x_j \le B & (6.9) \\
\qquad y_i = \dfrac{1}{D_i'} \sum_{j=1}^{q} r_{ij} x_j, \quad i = 1, 2, \cdots, n & (6.10) \\
\qquad \dfrac{E[y_i]}{E[Y_i]} \ge \alpha, \quad i = 1, 2, \cdots, n & (6.11) \\
\qquad 0 \le x_j \le 1, \quad j = 1, 2, \cdots, q & (6.12) \\
\qquad L_j \le x_j \le H_j, \quad j = 1, 2, \cdots, q & (6.13)
\end{cases}
$$

In the above fuzzy EVM, decision variables are fulfillment levels of ECs, x_j, $j = 1, 2, \cdots, q$. To make it easier to observe the changes of decision variables, all ECs are supposed to be continuous ones of either the cost type (*C-type*) or the benefit type (*B-type*) in Eq. (5.14). Normally, all $x_j \in [0, 1]$, $j = 1, 2, \cdots, q$, as written in (6.12), and under some circumstances, x_j is expected to have a lower bound L_j or an upper bound H_j as written in (6.13).

More specifically, the objective function (6.8) indicates the calculation on the expected value for the overall customer satisfaction (OCS), which is consistent with Eq. (5.9) in Chapter 5. Nevertheless, the differences lie in two aspects, i.e., w_i, $i = 1, 2, \cdots, n$, are known fuzzy parameters predetermined by experts, and $d_i(y_i)$, $i = 1, 2, \cdots, n$, contain three fuzzy relationship functions between the fulfillment levels of CRs, y_i, and their respective customer satisfaction in Kano's model. Analogously to the derivation process of Eq. (6.6), the expected return of this fuzzy objective function undergoes the following transformation in line with Eq. (5.9),

$$E[OCS] = E\big[\sum_{i=1}^{n} w_i \cdot d_i(y_i)\big] = \sum_{i=1}^{n} E[w_i \cdot d_i(y_i)]. \tag{6.14}$$

As to all the constraints, constraint (6.9) represents a budget limit, B, of the whole product development with crisp cost coefficients, c_j, for EC$_j$, $j = 1, 2, \cdots, q$. Apart from this, it can be seen that two particular constraints are related to fulfillment levels of CRs, y_i. Firstly, constraint (6.10) demonstrates the fulfillment level of each CR, through the multiplication between the fuzzy relationship matrix of CRs and ECs, $(r_{ij})_{n\times q}$, and the decision variable vector, $X = [x_1, x_2, \cdots, x_q]^T$. Due to the reason that r_{ij} is evaluated by TpFNs in this chapter, y_i is calculated to be a new TpFN via basic fuzzy computations, i.e., $y_i \sim \mathcal{A}(\frac{A_i}{D'_i}, \frac{B_i}{D'_i}, \frac{C_i}{D'_i}, \frac{D_i}{D'_i})$ with $0 < \frac{A_i}{D'_i} < \frac{B_i}{D'_i} < \frac{C_i}{D'_i} < \frac{D_i}{D'_i}$. Secondly, in constraint (6.11), the expected value for y_i over the expected value for Y_i is preferred to be larger than a confidence level of α estimated by the decision-makers, in which $\alpha \in (0, 1)$. The value of Y_i is obtained based on (6.10) as follows:

$$Y_i = \frac{1}{D'_i} \sum_{j=1}^{q} r_{ij}, \quad i = 1, 2, \cdots, n, \tag{6.15}$$

where all x_j, $j = 1, 2, \cdots, q$ are supposed to satisfy the maximum fulfillment level of 1 in (6.10). Similarly, it is attained that $Y_i \sim \mathcal{A}(\frac{A'_i}{D'_i}, \frac{B'_i}{D'_i}, \frac{C'_i}{D'_i}, \frac{D'_i}{D'_i})$ or $\mathcal{A}(\frac{A'_i}{D'_i}, \frac{B'_i}{D'_i}, \frac{C'_i}{D'_i}, 1)$ with $0 < \frac{A'_i}{D'_i} < \frac{B'_i}{D'_i} < \frac{C'_i}{D'_i} < 1$. And it is easily found that $D_i \leq D'_i$. Therefore, the values of $E[y_i]$ and $E[Y_i]$ can be formulated according to Eq. (6.3), and are respectively expressed as follows:

$$E[y_i] = \frac{A_i + B_i + C_i + D_i}{4D_i'}, \quad i = 1, 2, \cdots, n \tag{6.16}$$

$$E[Y_i] = \frac{A_i' + B_i' + C_i' + D_i'}{4D_i'}, \quad i = 1, 2, \cdots, n. \tag{6.17}$$

Notably, it can be further derived that if the fuzzy relationships, r_{ij}, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, q$ are settled, the value of $E[Y_i]$ is known. Besides, constraint (6.11) can be rewritten as

$$\frac{E[y_i]}{E[Y_i]} = \frac{A_i + B_i + C_i + D_i}{A_i' + B_i' + C_i' + D_i'} \geq \alpha. \tag{6.18}$$

6.3.2 Model solution with an improved HIA

The biggest obstacle of solving the proposed optimization model is the computation of the objective function. Based on Theorem 4.11, the expression of $E[w_i \cdot d_i(y_i)]$ in the objective function (6.8) can be equivalently transformed into:

$$E[w_i \cdot d_i(y_i)] = \int_0^1 w_{i\,\sup}(\alpha) \cdot d_i(y_{i\,\sup}(\alpha))\mathrm{d}\alpha, \tag{6.19}$$

in which w_i and y_i are TpFNs, and $w_{i\,\sup}(\alpha)$ and $y_{i\,\sup}(\alpha)$ are their α-optimistic values, respectively. The analytical expression of the α-optimistic value of a TpFN can be found in Eq. (4.38). The detailed relationship functions, $d_i(y_i)$, are exponential or linear functions displayed in Table 5.3 in Chapter 5. It is figured out that fulfillment levels of CRs, y_i, are fuzzy now, then the results of fuzzy functions $d_i(y_i)$ are also fuzzy, for $i = 1, 2, \cdots, n$.

According to Liu [Liu02a], the HIA can be employed to solve the proposed optimization model. However, as proved by several numerical examples in Chapter 4, its internal fuzzy simulation procedure, the SDA, for expected values of fuzzy events lacks accuracy and stability. Since the α-optimistic value of a TpFN is easily obtained, the TNIA-S is adopted here to substitute the original SDA in the HIA to help solve the model. In order to adapt to the model formulation, the steps of TNIA-S especially designed for the objective function are described as follows:

Algorithm 1 (TNIA-S for $\sum_{i=1}^{n} E[w_i \cdot d_i(y_i)]$)

Step 1. Initialize the number of integration points N. Let $E = 0$ and $k = 1$.

Step 2. Set $\alpha = k/N$. For each $1 \leq i \leq n$, according to the calculation formula of α-optimistic values in Eq. (6.19), calculate the value of $E[w_i \cdot d_i(y_i)]$.

Step 3. Reset $E = E + \sum_{i=1}^{n} E[w_i \cdot d_i(y_i)]/N$ and $k = k + 1$.

Step 4. If $k \leq N$, go to Step 2. Otherwise, return E as the simulation value of the expected value $\sum_{i=1}^{n} E[w_i \cdot d_i(y_i)]$.

On this basis, an improved HIA (iHIA) which incorporates the fuzzy simulation for the objective function in Algorithm 1 and a genetic algorithm is provided for solving the proposed fuzzy EVM. The detailed steps are listed as follows:

Algorithm 2 (iHIA for solving fuzzy EVM)

Step 1. Initialize *pop-size* chromosomes that satisfy constraints.

Step 2. Calculate the values of $\sum_{i=1}^{n} E[w_i \cdot d_i(y_i)]$ for all chromosomes with the aid of Algorithm 1.

Step 3. Compute the fitness of each chromosome by the rank-based evaluation function based on the values obtained in Step 2.

Step 4. Select the chromosomes by spinning the roulette wheel.

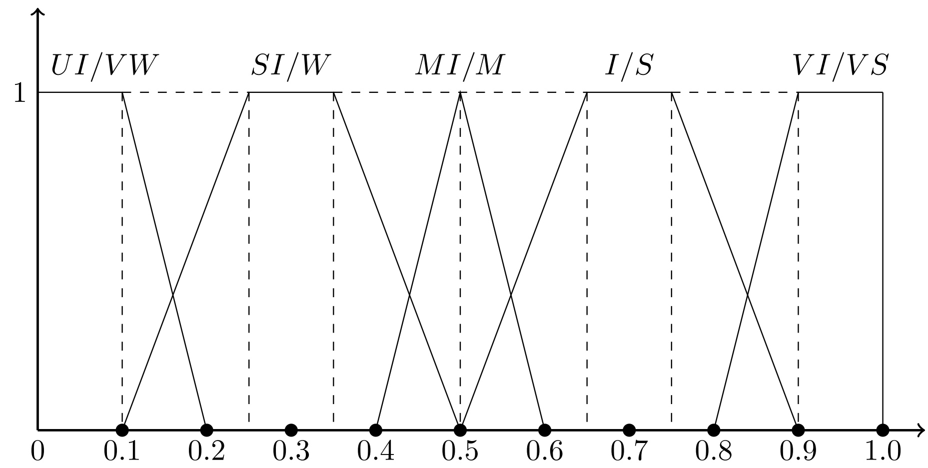
Step 5. Update the chromosomes by crossover and mutation operations, and check the feasibility of offsprings by constraints.

Step 6. Repeat Steps 2, 3, 4, and 5 for a given number of cycles.

Step 7. Report the best chromosome as the optimal solution.

6.4 Case Study: A Notebook Computer Development

In this section, the proposed methods of determining the fuzzy importance of ECs and target levels of ECs will be implemented to the same case study of the notebook computer development as in Chapter 5. The identical six major CRs of the notebook computer are mapped into seven specific ECs. Above all, the original crisp HoQ is reevaluated by some predefined linguistic variables, and are expressed by a series of fuzzy variables accordingly, which are enumerated and illustrated in Figure 6.1 as follows:



CR importance weight	CR and EC relationship	TFN/TpFN
UI : *unimportant*	VW : *very weak*	$\mathcal{A}(0, 0, 0.1, 0.2)$
SI : *some important*	W : *weak*	$\mathcal{A}(0.1, 0.25, 0.35, 0.5)$
MI : *moderately important*	M : *moderate*	$\mathcal{T}(0.4, 0.5, 0.6)$
I : *important*	S : *strong*	$\mathcal{A}(0.5, 0.65, 0.75, 0.9)$
VI : *very important*	VS : *very strong*	$\mathcal{A}(0.8, 0.9, 1, 1)$

Figure 6.1: The linguistic variables and fuzzy variables in matrices W and R.

It can be seen that the CR importance weights are categorized into five levels, that is, unimportant (UI), some important (SI), moderately important (MI), important (I), and very important (VI). Meanwhile, five levels of strength are utilized to describe relationships between CRs and ECs, that is, very weak (VW), weak (W), moderate (M), strong (S), and very strong (VS). The TpFNs $\mathcal{A}(0, 0, 0.1, 0.2)$, $\mathcal{A}(0.1, 0.25, 0.35, 0.5)$, $\mathcal{A}(0.4, 0.5, 0.5, 0.6)$, $\mathcal{A}(0.5, 0.65, 0.75, 0.9)$, and $\mathcal{A}(0.8, 0.9, 1, 1)$ are used to quantify the five levels of linguistic variables. In fact, as explained in Chapter 4, a TFN can be viewed as a degradation form of a TpFN. Thus, the third TpFN $\mathcal{A}(0.4, 0.5, 0.5, 0.6)$ will be replaced by the TFN $\mathcal{T}(0.4, 0.5, 0.6)$ directly. The membership functions of all the fuzzy variables are also illustrated in Figure 6.1. Notably, this scoring method is adopted and slightly simplified from [Che06a].

Subsequently, the fuzzy HoQ of the notebook computer design can be attained in Table 6.1. The evaluation of all the fuzzy elements in matrices W and R is mutually agreed via the Delphi method of 20 relevant experts and professionals. It is observed that four minus marks appear in the matrix R, which implies a potential reverse relationship. For example, as to the relationship between CR_1 (Stylish design) and EC_7 (Battery), "S(-)" means the design of the battery, such as its shape, size, and weight, may have a strong but negative effect on the stylish design. Besides, the cost coefficients and technical measures with minimal and maximal values for ECs in the current market are also listed on the bottom of the fuzzy HoQ.

6.4.1 Ranking of the EC importance

In the previous chapter, half of the focus was put on the determination of the CR importance weights, which is an in-depth understanding of customer needs. Whereas, the determination of the EC importance is also a vital basis for the decision-makers to allocate enterprise resources properly. According to the contents in Section 6.1 and the fuzzy HoQ in Table 6.1, the fuzzy importance of ECs can be obtained and prioritized through their expected values.

Table 6.1: The fuzzy HoQ of the notebook computer design.

CRs \ ECs	Relative Importance Weights w_i (%)	1. CPU	2. RAM	3. Hard disk	4. Sound card	5. Graphic card	6. LCD display	7. Battery
1. Stylish design (A)	**MI**	**W**	**W**	**M(-)**	**VW**	**VW**	**VS**	**S(-)**
2. Mobility (O)	**MI**	**W**	**W**	**M(-)**	**VW**	**VW**	**S**	**VS(-)**
3. High computing speed (O)	**VI**	**VS**	**VS**	**M**	**W**	**VS**	**VW**	**S**
4. Powerful graphics solution (A)	**I**	**VS**	**VS**	**W**	**VW**	**VS**	**S**	**S**
5. Solid audio capability (A)	**I**	**M**	**M**	**VW**	**VS**	**VW**	**VW**	**W**
6. Large storage (M)	**SI**	**W**	**W**	**VS**	**VW**	**VW**	**VW**	**VW**
Cost Index (units)		19.8	17.5	15.5	14	16.5	14.5	13
Technical Measures		GHz	GB	G	KHz	MHz	Inch	Hour
Technical Constraints (ECL)		1.8	1	64	44.1	500	11.1	2
Technical Constraints (ECH)		3.6	8	1024	192	1500	17.1	8

Firstly, the importance of EC_1, v_1, is calculated as a demonstration. In this case study, the quantity of CRs, $n = 6$, in Eq. (6.7), i.e.,

$$E[v_1] = \sum_{i=1}^{6} E[w_i r_{i1}] = E[w_1 r_{11}] + E[w_2 r_{21}] + \cdots + E[w_6 r_{61}], \tag{6.20}$$

where $w_1, w_2, \cdots, w_6$, and $r_{11}, r_{21}, \cdots, r_{61}$ are all independent and nonnegative TFNs and TpFNs listed in Table 6.1. It is seen that $w_1 \sim \mathcal{T}(0.4, 0.5, 0.6)$, and $r_{11} \sim \mathcal{A}(0.1, 0.25, 0.35, 0.5)$, and after the fuzzy variables are plugged into Eq. (6.20) via the simplified expression in Eq. (6.5), the values of $E[w_1 r_{11}]$ can be attained as

$$\begin{aligned} E[w_1 r_{11}] &= \frac{1}{6} \times (0.4 \times 0.1 + 0.5 \times 0.25 + 0.5 \times 0.35 + 0.6 \times 0.5) \\ &\quad + \frac{1}{12} \times (0.4 \times 0.25 + 0.1 \times 0.5 + 0.5 \times 0.5 + 0.35 \times 0.6) \\ &= 0.1575. \end{aligned} \tag{6.21}$$

Following the calculation steps in Eq. (6.21), the values of $E[w_2 r_{21}] \sim E[w_6 r_{61}]$ are easily acquired. Then, the value of $E[v_1]$ can be computed as follows:

$$
\begin{aligned}
E[v_1] &= \sum_{i=1}^{6} E[w_i r_{i1}] = E[w_1 r_{11}] + E[w_2 r_{21}] + \cdots + E[w_6 r_{61}] \\
&= 0.1575 + 0.1575 + 0.8617 + 0.6575 + 0.3575 + 0.1075 \\
&= 2.2992.
\end{aligned}
\tag{6.22}
$$

Analogous to the calculation procedure for the expected value of the fuzzy importance of EC_1, $E[v_1]$, the expected values of the fuzzy importance of all the remaining ECs from EC_2 to EC_7 are figured out and summarized in Table 6.2. As a general rule, the larger the value of $E[v_j]$ is, the larger importance and the higher priority EC_j can get. Thereby, the ranking of the fuzzy importance of ECs are shown in Table 6.2 in accordance with their corresponding values of $E[v_j]$, $j = 1, 2, \cdots, 7$. Apparently, EC_1 (CPU) and EC_2 (RAM) both score the highest

Table 6.2: The ranking of the fuzzy importance of ECs in the case study.

Seven ECs	EC_1 CPU	EC_2 RAM	EC_3 Hard disk	EC_4 Sound card	EC_5 Graphic card	EC_6 LCD display	EC_7 Battery
$E[v_j]$	2.2992	2.2992	1.5508	1.1233	1.6975	1.5017	2.2492
Ranking	1	1	5	7	4	6	3

and are equipped with a greater superiority compared with other ECs, while EC_7 (Battery) is close behind at the third place. Then, EC_5 (Graphic card) has a slight advantage to win the fourth place, and is followed by EC_3 (Hard disk) and EC_6 (LCD display) successively. Finally, EC_4 (Sound card) is distinguished as the least important. It is remarked that this ranking is calculated based on the subjective evaluation on two matrices of fuzzy variables, and the expected value is served as a defuzzification method.

6.4.2 Results of the EC fulfillment levels

In order to establish the fuzzy EVM for the case study, the expressions of both the objective function and constraints written by the case data are needed to be derived. Firstly, based on the information listed in the fuzzy HoQ in Table 6.1, all the constraints of the fuzzy EVM can be primarily obtained as follows:

Above all, the budget constraint is the same as the one in the case study in Chapter 5, where the cost coefficient for each EC, c_j, is enumerated in the fuzzy HoQ, and the budget, B, is also 100 units. Then, since the relationship matrix, $(r_{ij})_{n\times q}$, is predetermined, according to Eq. (6.15), it is acquired that $Y_1 \sim \mathcal{A}(\frac{1.9}{3.9}, \frac{2.55}{3.9}, \frac{3.15}{3.9}, 1)$, $Y_2 \sim \mathcal{A}(\frac{1.9}{3.9}, \frac{2.55}{3.9}, \frac{3.15}{3.9}, 1)$, $Y_3 \sim \mathcal{A}(\frac{3.4}{5.2}, \frac{4.1}{5.2}, \frac{4.7}{5.2}, 1)$, $Y_4 \sim \mathcal{A}(\frac{3.5}{5.5}, \frac{4.25}{5.5}, \frac{4.95}{5.5}, 1)$, $Y_5 \sim \mathcal{A}(\frac{1.7}{3.3}, \frac{2.15}{3.3}, \frac{2.65}{3.3}, 1)$, and $Y_6 \sim \mathcal{A}(\frac{1}{2.8}, \frac{1.4}{2.8}, \frac{2.1}{2.8}, 1)$. It can be found that, the values of $D'_1 \sim D'_6$ are 3.9, 3.9, 5.2, 5.5, 3.3, and 2.8, respectively. Thereby, by virtue of the formula in (6.10), the TpFNs for $y_i \sim \mathcal{A}(\frac{A_1}{D'_1}, \frac{B_1}{D'_1}, \frac{C_1}{D'_1}, \frac{D_1}{D'_1})$ can also be computed, which are represented in Table 6.3.

Afterwards, by applying Eq. (6.18), the expressions of $\dfrac{E[y_i]}{E[Y_i]}$ in constraint (6.11) can be further derived. For instance, the simplified expression of $\dfrac{E[y_1]}{E[Y_1]}$ can be obtained as follows:

$$
\begin{aligned}
A_1 + B_1 + C_1 + D_1 &= (0.1x_1 + 0.1x_2 + 0.4x_3 + 0.8x_6 + 0.5x_7) \\
&\quad +(0.25x_1 + 0.25x_2 + 0.5x_3 + 0.9x_6 + 0.65x_7) \\
&\quad +(0.35x_1 + 0.35x_2 + 0.5x_3 + 0.1x_4 + 0.1x_5 + x_6 + 0.75x_7) \\
&\quad +(0.5x_1 + 0.5x_2 + 0.6x_3 + 0.2x_4 + 0.2x_5 + x_6 + 0.9x_7) \\
&= 1.2x_1 + 1.2x_2 + 2x_3 + 0.3x_4 + 0.3x_5 + 3.7x_6 + 2.8x_7 \\
A'_1 + B'_1 + C'_1 + D'_1 &= 1.9 + 2.55 + 3.15 + 3.9 \\
\frac{E[y_1]}{E[Y_1]} &= \frac{A_1 + B_1 + C_1 + D_1}{A'_1 + B'_1 + C'_1 + D'_1} \\
&= \frac{1.2x_1 + 1.2x_2 + 2x_3 + 0.3x_4 + 0.3x_5 + 3.7x_6 + 2.8x_7}{1.9 + 2.55 + 3.15 + 3.9}
\end{aligned}
\tag{6.23}
$$

Table 6.3: The expressions of y_i by TpFNs in the case study.

y_i	$\frac{A_i}{D'_i}$	$\frac{B_i}{D'_i}$	$\frac{C_i}{D'_i}$	$\frac{D_i}{D'_i}$
y_1	$\frac{1}{3.9} * (0.1x_1 + 0.1x_2 + 0.4x_3 + 0.8x_6 + 0.5x_7)$	$\frac{1}{3.9} * (0.25x_1 + 0.25x_2 + 0.5x_3 + 0.9x_6 + 0.65x_7)$	$\frac{1}{3.9} * (0.35x_1 + 0.35x_2 + 0.5x_3 + 0.1x_4 + 0.1x_5 + x_6 + 0.75x_7)$	$\frac{1}{3.9} * (0.5x_1 + 0.5x_2 + 0.6x_3 + 0.2x_4 + 0.2x_5 + x_6 + 0.9x_7)$
y_2	$\frac{1}{3.9} * (0.1x_1 + 0.1x_2 + 0.4x_3 + 0.5x_6 + 0.8x_7)$	$\frac{1}{3.9} * (0.25x_1 + 0.25x_2 + 0.5x_3 + 0.65x_6 + 0.9x_7)$	$\frac{1}{3.9} * (0.35x_1 + 0.35x_2 + 0.5x_3 + 0.1x_4 + 0.1x_5 + 0.75x_6 + x_7)$	$\frac{1}{3.9} * (0.5x_1 + 0.5x_2 + 0.6x_3 + 0.2x_4 + 0.2x_5 + 0.9x_6 + x_7)$
y_3	$\frac{1}{5.2} * (0.8x_1 + 0.8x_2 + 0.4x_3 + 0.1x_4 + 0.8x_5 + 0.5x_7)$	$\frac{1}{5.2} * (0.9x_1 + 0.9x_2 + 0.5x_3 + 0.25x_4 + 0.9x_5 + 0.65x_7)$	$\frac{1}{5.2} * (x_1 + x_2 + 0.5x_3 + 0.35x_4 + x_5 + 0.1x_6 + 0.75x_7)$	$\frac{1}{5.2} * (x_1 + x_2 + 0.6x_3 + 0.5x_4 + x_5 + 0.2x_6 + 0.9x_7)$
y_4	$\frac{1}{5.5} * (0.8x_1 + 0.8x_2 + 0.1x_3 + 0.8x_5 + 0.5x_6 + 0.5x_7)$	$\frac{1}{5.5} * (0.9x_1 + 0.9x_2 + 0.25x_3 + 0.9x_5 + 0.65x_6 + 0.65x_7)$	$\frac{1}{5.5} * (x_1 + x_2 + 0.35x_3 + 0.1x_4 + x_5 + 0.75x_6 + 0.75x_7)$	$\frac{1}{5.5} * (x_1 + x_2 + 0.5x_3 + 0.2x_4 + x_5 + 0.9x_6 + 0.9x_7)$
y_5	$\frac{1}{3.3} * (0.4x_1 + 0.4x_2 + 0.8x_4 + 0.1x_7)$	$\frac{1}{3.3} * (0.5x_1 + 0.5x_2 + 0.9x_4 + 0.25x_7)$	$\frac{1}{3.3} * (0.5x_1 + 0.5x_2 + 0.1x_3 + x_4 + 0.1x_5 + 0.1x_6 + 0.35x_7)$	$\frac{1}{3.3} * (0.6x_1 + 0.6x_2 + 0.2x_3 + x_4 + 0.2x_5 + 0.2x_6 + 0.5x_7)$
y_6	$\frac{1}{2.8} * (0.1x_1 + 0.1x_2 + 0.8x_3)$	$\frac{1}{2.8} * (0.25x_1 + 0.25x_2 + 0.9x_3)$	$\frac{1}{2.8} * (0.35x_1 + 0.35x_2 + x_3 + 0.1x_4 + 0.1x_5 + 0.1x_6 + 0.1x_7)$	$\frac{1}{2.8} * (0.5x_1 + 0.5x_2 + x_3 + 0.2x_4 + 0.2x_5 + 0.2x_6 + 0.2x_7)$

Analogously to the above procedure, the simplified expressions for $\dfrac{E[y_i]}{E[Y_i]}$, $i = 2, 3, \cdots, 6$ can be easily derived. Consequently, the fuzzy EVM of the case study is formulated as follows:

$$
\begin{cases}
\max \quad \displaystyle\sum_{i=1}^{6} \int_0^1 w_{i\,\sup}(\alpha) \cdot d_i(y_{i\,\sup}(\alpha))\mathrm{d}\alpha \\
\qquad d_1(y_{1\,\sup}(\alpha)) = 0.6289e^{y_{1\,\sup}(\alpha)} - 0.9434 \\
\qquad d_2(y_{2\,\sup}(\alpha)) = 1.5484y_{2\,\sup}(\alpha) - 0.8226 \\
\qquad d_3(y_{3\,\sup}(\alpha)) = 1.4344y_{3\,\sup}(\alpha) - 0.7131 \\
\qquad d_4(y_{4\,\sup}(\alpha)) = 0.6852e^{y_{4\,\sup}(\alpha)} - 1.0562 \\
\qquad d_5(y_{5\,\sup}(\alpha)) = 0.6529e^{y_{5\,\sup}(\alpha)} - 0.9538 \\
\qquad d_6(y_{i6sup}(\alpha)) = -1.7235e^{-y_{6\,\sup}(\alpha)} + 0.9917 \\
\text{s.t.} \\
\qquad 19.8x_1 + 17.5x_2 + 14.5x_3 + 14x_4 + 16.5x_5 + 15.5x_6 + 13x_7 \le 100 \\
\qquad \dfrac{1.2x_1 + 1.2x_2 + 2x_3 + 0.3x_4 + 0.3x_5 + 3.7x_6 + 2.8x_7}{1.9 + 2.55 + 3.15 + 3.9} \ge \alpha \\
\qquad \dfrac{1.2x_1 + 1.2x_2 + 2x_3 + 0.3x_4 + 0.3x_5 + 2.8x_6 + 3.7x_7}{1.9 + 2.55 + 3.15 + 3.9} \ge \alpha \\
\qquad \dfrac{3.7x_1 + 3.7x_2 + 2x_3 + 1.2x_4 + 3.7x_5 + 0.3x_6 + 2.8x_7}{3.4 + 4.1 + 4.7 + 5.2} \ge \alpha \\
\qquad \dfrac{3.7x_1 + 3.7x_2 + 1.2x_3 + 0.3x_4 + 3.7x_5 + 2.8x_6 + 2.8x_7}{3.5 + 4.25 + 4.95 + 5.5} \ge \alpha \\
\qquad \dfrac{2x_1 + 2x_2 + 0.3x_3 + 3.7x_4 + 0.3x_5 + 0.3x_6 + 1.2x_7}{1.7 + 2.15 + 2.65 + 3.3} \ge \alpha \\
\qquad \dfrac{1.2x_1 + 1.2x_2 + 3.7x_3 + 0.3x_4 + 0.3x_5 + 0.3x_6 + 0.3x_7}{1 + 1.4 + 2.1 + 2.8} \ge \alpha \\
\qquad 0 \le x_j \le 1, \quad j = 1, 2, \cdots, 7
\end{cases}
\tag{6.24}
$$

As far as the objective function is concerned, the calculations on $E[w_i \cdot d_i(y_i)]$ for $i = 1, 2, \cdots, 6$ in the objective function are respectively expounded here. As is known, $w_1 \sim \mathcal{T}(0.4, 0.5, 0.6)$ in Table 6.1, and $y_1 \sim \mathcal{A}(\frac{A_1}{3.9}, \frac{B_1}{3.9}, \frac{C_1}{3.9}, \frac{D_1}{3.9})$ in Table 6.3.

Then, according to Eq. (6.19), the detailed expression for $E[w_1 \cdot d_1(y_1)]$ written by the α-optimistic value is obtained as

$$\begin{aligned} E[w_1 \cdot d_1(y_1)] &= \int_0^1 w_{1\sup}(\alpha) \cdot d_1(y_{1\sup}(\alpha))\mathrm{d}\alpha \\ &= \int_0^{0.5} (0.6 - 0.2\alpha)\Big(0.6289e^{\left(\frac{D_1}{3.9} - 2(\frac{D_1}{3.9} - \frac{C_1}{3.9})\alpha\right)} - 0.9434\Big)\mathrm{d}\alpha \\ &+ \int_{0.5}^1 (0.6 - 0.2\alpha)\Big(0.6289e^{\left(\frac{2B_1}{3.9} - \frac{A_1}{3.9} - 2(\frac{B_1}{3.9} - \frac{A_1}{3.9})\alpha\right)} - 0.9434\Big)\mathrm{d}\alpha, \end{aligned} \tag{6.25}$$

where the derivations of $w_{1\sup}(\alpha)$ and $y_{1\sup}(\alpha)$ are based on Eq. (4.38), and the exponential function $d_1(y_{1\sup}(\alpha))$ for CR_1 of Attractive attribute in Kano's model appears in model (6.24). Similarly, as to the calculation on $E[w_2 \cdot d_2(y_2)]$, $w_2 \sim \mathcal{T}(0.4, 0.5, 0.6)$, and $y_2 \sim \mathcal{A}(\frac{A_2}{3.9}, \frac{B_1}{3.9}, \frac{C_1}{3.9}, \frac{D_1}{3.9})$, so it is calculated that

$$\begin{aligned} E[w_2 \cdot d_2(y_2)] &= \int_0^1 w_{2\sup}(\alpha) \cdot d_2(y_{2\sup}(\alpha))\mathrm{d}\alpha \\ &= \int_0^{0.5} (0.6 - 0.2\alpha)\Big(1.5484(\frac{D_2}{3.9} - 2(\frac{D_2}{3.9} - \frac{C_2}{D3.9})\alpha) - 0.8226\Big)\mathrm{d}\alpha \\ &+ \int_{0.5}^1 (0.6 - 0.2\alpha)\Big(1.5484(\frac{2B_2}{3.9} - \frac{A_2}{3.9} - 2(\frac{B_2}{3.9} - \frac{A_2}{3.9})\alpha) - 0.8226\Big)\mathrm{d}\alpha. \end{aligned} \tag{6.26}$$

Then, $w_3 \sim \mathcal{A}(0.8, 0.9, 1, 1)$, and $y_3 \sim \mathcal{A}(\frac{A_3}{5.2}, \frac{B_3}{5.2}, \frac{C_3}{5.2}, \frac{D_3}{5.2})$, it is attained that

$$\begin{aligned} E[w_3 \cdot d_3(y_3)] &= \int_0^1 w_{3\sup}(\alpha) \cdot d_3(y_{3\sup}(\alpha))\mathrm{d}\alpha \\ &= \int_0^{0.5} 1 * \Big(1.4344(\frac{D_3}{5.2} - 2(\frac{D_3}{5.2} - \frac{C_3}{5.2})\alpha) - 0.7131\Big)\mathrm{d}\alpha \\ &+ \int_{0.5}^1 (1 - 0.2\alpha)\Big(1.4344(\frac{2B_3}{5.2} - \frac{A_3}{5.2} - 2(\frac{B_3}{5.2} - \frac{A_3}{5.2})\alpha) - 0.7131\Big)\mathrm{d}\alpha. \end{aligned} \tag{6.27}$$

Both $d_2(y_{2\sup}(\alpha))$ and $d_3(y_{3\sup}(\alpha))$ are linear functions for CR_2 and CR_3 of One-dimensional attribute in Kano's model. Afterwards, w_4 and $w_5 \sim$

$\mathcal{A}(0.5, 0.65, 0.75, 0.9)$, and $y_4 \sim \mathcal{A}(\frac{A_4}{5.5}, \frac{B_4}{5.5}, \frac{C_4}{5.5}, \frac{D_4}{5.5})$, $y_5 \sim \mathcal{A}(\frac{A_5}{3.3}, \frac{B_5}{3.3}, \frac{C_5}{3.3}, \frac{D_5}{3.3})$, then it is respectively obtained that

$$\begin{aligned} E[w_4 \cdot d_4(y_4)] &= \int_0^1 w_{4\,\sup}(\alpha) \cdot d_4(y_{4\,\sup}(\alpha))\mathrm{d}\alpha \\ &= \int_0^{0.5} (0.9 - 0.3\alpha)\Big(0.6852e^{\left(\frac{D_4}{5.5} - 2(\frac{D_4}{5.5} - \frac{C_4}{5.5})\alpha\right)} - 1.0562\Big)\mathrm{d}\alpha \\ &+ \int_{0.5}^1 (0.8 - 0.3\alpha)\Big(0.6852e^{\left(\frac{2B_4}{5.5} - \frac{A_4}{5.5} - 2(\frac{B_4}{5.5} - \frac{A_4}{5.5})\alpha\right)} - 1.0562\Big)\mathrm{d}\alpha. \end{aligned} \tag{6.28}$$

$$\begin{aligned} E[w_5 \cdot d_5(y_5)] &= \int_0^1 w_{5\,\sup}(\alpha) \cdot d_5(y_{5\,\sup}(\alpha))\mathrm{d}\alpha \\ &= \int_0^{0.5} (0.9 - 0.3\alpha)\Big(0.6529e^{\left(\frac{D_5}{3.3} - 2(\frac{D_5}{3.3} - \frac{C_5}{3.3})\alpha\right)} - 0.9538\Big)\mathrm{d}\alpha \\ &+ \int_{0.5}^1 (0.8 - 0.3\alpha)\Big(0.6529e^{\left(\frac{2B_5}{3.3} - \frac{A_5}{3.3} - 2(\frac{B_5}{3.3} - \frac{A_5}{3.3})\alpha\right)} - 0.9538\Big)\mathrm{d}\alpha. \end{aligned} \tag{6.29}$$

Both $d_4(y_{4\,\sup}(\alpha))$ and $d_5(y_{5\,\sup}(\alpha))$ are exponential functions, since CR_4 and CR_5 are of attractive attributes in Kano's model. At last, $w_6 \sim \mathcal{A}(0.1, 0.25, 0.35, 0.5)$, and $y_6 \sim \mathcal{A}(\frac{A_6}{2.8}, \frac{B_6}{2.8}, \frac{C_6}{2.8}, \frac{D_6}{2.8})$, then it can be obtained that

$$\begin{aligned} E[w_6 \cdot d_6(y_6)] &= \int_0^1 w_{6\,\sup}(\alpha) \cdot d_6(y_{6\,\sup}(\alpha))\mathrm{d}\alpha \\ &= \int_0^{0.5} (0.5 - 0.3\alpha)\Big(- 1.7235e^{-\left(\frac{D_6}{2.8} - 2(\frac{D_6}{2.8} - \frac{C_6}{2.8})\alpha\right)} + 0.9917\Big)\mathrm{d}\alpha \\ &+ \int_{0.5}^1 (0.4 - 0.3\alpha)\Big(- 1.7235e^{-\left(\frac{2B_6}{2.8} - \frac{A_6}{2.8} - 2(\frac{B_5}{2.8} - \frac{A_6}{2.8})\alpha\right)} + 0.9917\Big)\mathrm{d}\alpha, \end{aligned} \tag{6.30}$$

in which $d_6(y_{6\,\sup}(\alpha))$ is an exponential function for CR_6 of Must-be attribute in Kano's model.

Although the analytical results of the integrals in Eqs. (6.26) and (6.27) are not difficult to derive, it is not simple to obtain the analytical results of the integrals in Eqs. (6.25), (6.28), (6.29), and (6.30), especially when A_i, B_i, C_i, D_i,

$i = 1, 2, \cdots, 6$, all contain decision variables. On this basis, the iHIA is applied to the fuzzy EVM for the case study in model (6.24) so as to figure out the optimal solution. Before the fuzzy EVM is solved, some essential parameters in the iHIA are predetermined. The population size (*pop-size*) is 30, the probability of crossover P_c is 0.3, the probability of mutation P_m is 0.2, and the parameter a in the rank-based evaluation function is set as 0.05. All the parameters setting is adopted from [Liu02a]. Meanwhile, the number of integration points N in Algorithm 1 is 5000 in consideration of both the accuracy and computational time.

The confidence level α is scaled from 0.1 to 0.9 to observe the corresponding changes of the fulfillment levels of ECs, the expected return, and the design cost, whose results are illustrated in Table 6.4. Several conclusions can be drawn here.

Table 6.4: The fulfillment levels of ECs, the expected return, and the design cost with respect to different confidence levels of α.

α	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Expected return	Design cost
0.1	0.8960	0.8228	0.7466	0.1741	0.8589	0.7105	0.5135	0.4353	77.2630
0.2	0.8960	0.8228	0.7466	0.1741	0.8589	0.7105	0.5135	0.4353	77.2630
0.3	0.8677	0.9116	0.6147	0.7277	0.0432	0.6678	0.9765	0.5346	75.9926
0.4	0.8677	0.9116	0.6147	0.7277	0.0432	0.6678	0.9765	0.5346	75.9926
0.5	0.5463	0.9117	0.5226	0.7306	0.9993	0.9693	0.8674	0.7771	87.3664
0.6	0.7645	0.9379	0.7320	0.7466	0.9936	0.4348	0.9020	0.8264	87.4766
0.7	0.8787	0.8560	0.6006	05575	0.9544	0.9425	0.9958	0.9990	92.1937
0.8	0.9462	0.7902	0.8359	0.8949	0.8417	0.9926	0.9775	1.2066	99.1933
0.9	0.9587	0.9716	0.9232	0.8744	0.7721	0.8162	0.9979	1.2737	99.9767

Firstly, it is observed that the solutions for $\alpha = 0.1$ and $\alpha = 0.2$ are identical, which may be caused by the relative low setting of the confidence level α. It also brings the lowest result of the fulfillment level of EC_4, x_4. Such a situation also happens to $\alpha = 0.3$ and 0.4, where the fulfillment level of EC_5, x_5 is also severely low. In contrast to them, the solutions for $\alpha = 0.8$ and 0.9 are much better, in which the fulfillment levels for all ECs are more evenly distributed. Most fulfillment levels of ECs achieve the largest at $\alpha = 0.8$ or 0.9, except for the fulfillment level of EC_5. Besides, it is noted that the expected value of the fuzzy return is increasing with the increasing of the confidence levels, and this tendency is also depicted in Figure 6.2. It is intuitive that the decision-makers may have more preference for the results at a relative high confidence level. Meanwhile, as another natural result, the design cost also raises when the confidence level α gets larger, and the biggest expense, 99.9767, occurs at $\alpha = 0.9$. Whereas the smallest expense, 75.9926, occurs at $\alpha = 0.3$ and 0.4, which may be the reason for generating the lowest fulfillment level of EC_5, 0.0432. The changes of the design cost are shown in Figure 6.3.

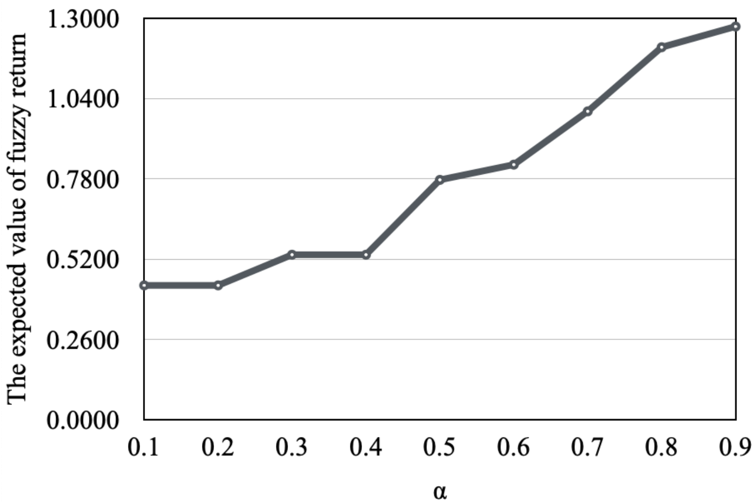


Figure 6.2: The expected return with respect to different confidence levels of α.

The changes of all the fulfillment levels of ECs are also visualized in Figure 6.4. It is seen that there are no stable trends for individual EC, however, the results at confidence levels of $\alpha \geq 0.7$ are more acceptable and satisfactory for the decision-makers. Even so, in the solutions for $\alpha \geq 0.8$, not all $x_j \geq 0.8$. The fulfillment level for EC_2 is 0.7902 when $\alpha = 0.8$, while the fulfillment level for EC_5 is 0.7721 when $\alpha = 0.9$. To view the difference, an additional setting of a lower bound

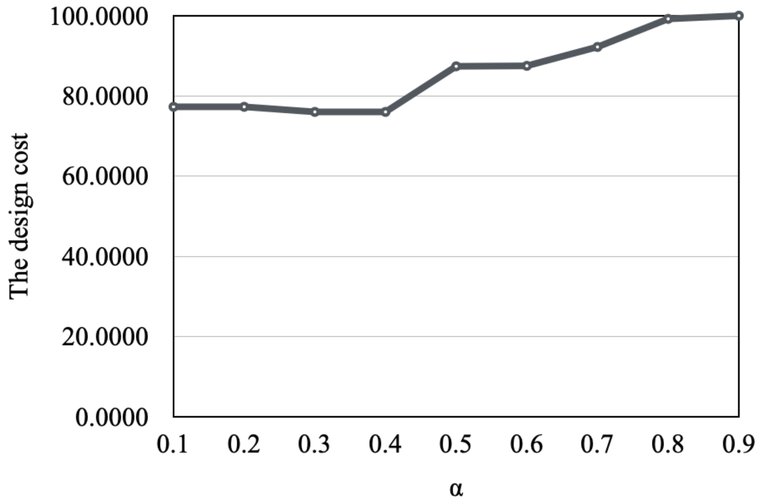


Figure 6.3: The design cost with respect to different confidence levels of α.

$L_j = 0.8$ for x_j, $j = 1, 2, \cdots, 7$, is incorporated into model (6.24). However, this compulsory setting of $x_j \in [0.8, 1]$ may bring a side effect. That is, when $\alpha \leq 0.8$, the constraint (6.11) always holds. So it is not surprising the solutions for α from 0.1 to 0.7 are the same as that for $\alpha = 0.8$. Meanwhile, due to the budget limit of 100 units and the lower bound setting of x_j, the confidence level cannot reach values in (0.9,1] in this case. As a consequence, only the solutions for $\alpha = 0.8$ and $\alpha = 0.9$ that expressed by x_j^*, $j = 1, 2, \cdots, 7$, are listed in Table 6.5 as follows:

Table 6.5: The fulfillment levels of ECs with respect to $\alpha = 0.8$ and 0.9, where $x_j \in [0.8, 1]$, $j = 1, 2, \cdots, 7$.

α	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Expected return	Design cost
0.8	0.9462	0.7902	0.8359	0.8949	0.8417	0.9926	0.9775	1.2066	99.1933
0.9	0.9587	0.9716	0.9232	0.8744	0.7721	0.8162	0.9979	1.2737	99.9767
α	x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	x_6^*	x_7^*	Expected return*	Design cost*
0.8	0.9162	0.9635	0.9077	0.8808	0.8523	0.8412	0.9378	1.2398	99.7878
0.9	0.9128	0.9993	0.8722	0.8248	0.8478	0.8506	0.9839	1.2588	99.7190

From Table 6.5, it seems that the modified x_j^*, the expected return* at $\alpha = 0.8$

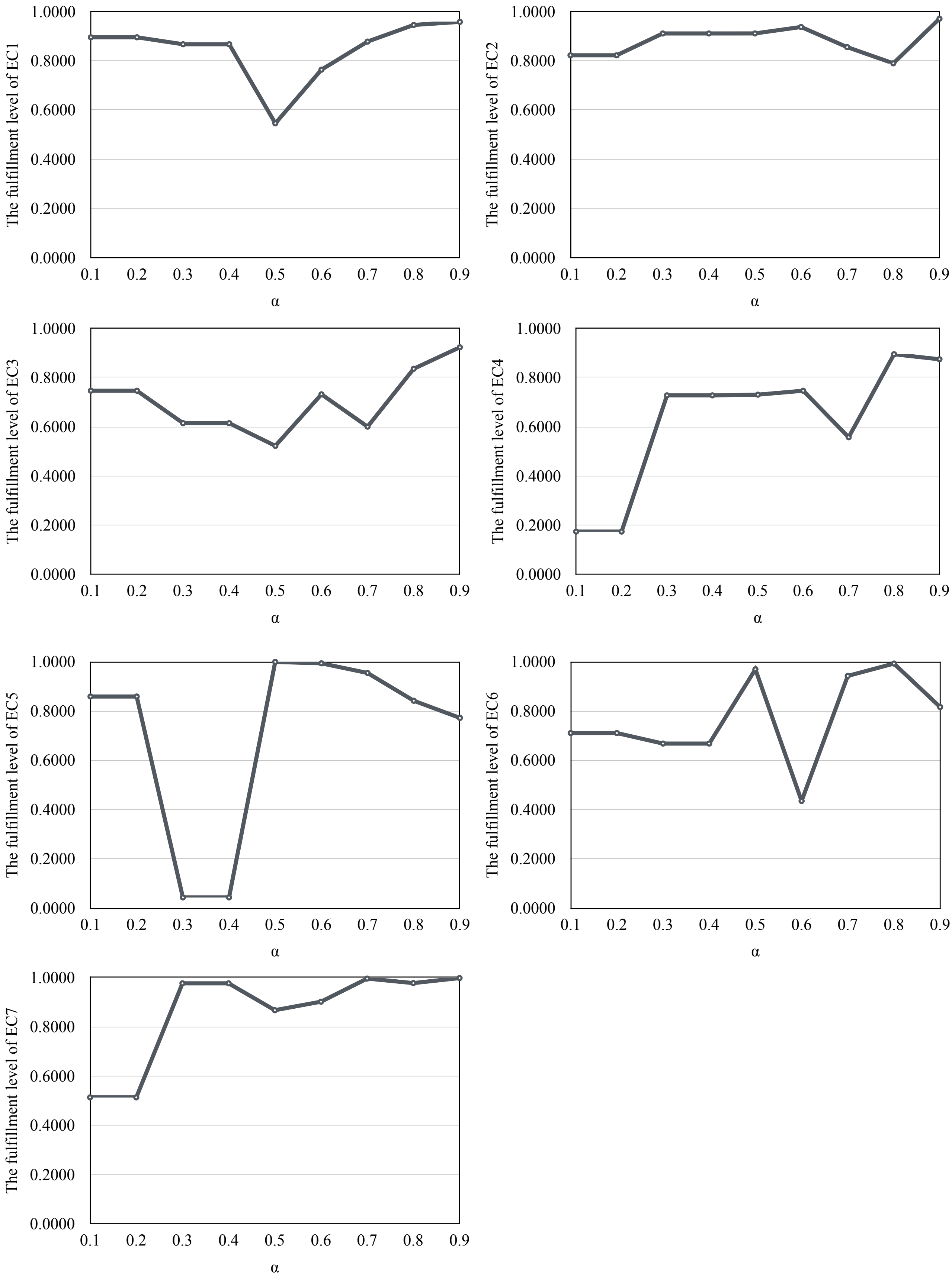


Figure 6.4: The fulfillment levels of $\text{EC}_1 \sim \text{EC}_7$ with respect to different confidence levels of α.

are better than the original ones. Nonetheless, as explained above, under this circumstance the constraint (6.11) is completely ineffective. When it comes to $\alpha = 0.9$, the modified solution is comparable to the original solution. Firstly, x_j^*, $j = 1, 2, \cdots, 7$, are distributed with more balance by sacrificing a part of the expected return, i.e., from 1.2737 to 1.2588. At the same time, a lower design cost of 99.7190 is consumed in contrast to the original one, 99.9767. Secondly, it is obviously observed that the expected return of the modified solution at $\alpha = 0.8$, 1.2398, is smaller than that at $\alpha = 0.9$, 1.2588, while the design cost* needed, 99.7878, is larger than 99.7190. This counter-intuitive comparison will rule out the modified solution at $\alpha = 0.8$, and determine the modified solution at $\alpha = 0.9$ as the final fulfillment levels of ECs after the fuzzy optimization procedure. For the decision-makers, this determination takes both the balance of fulfillment levels of ECs and the expected return into account, and is marked in blue in Table 6.5. Notably, if the number of CRs and ECs are of large-scale, for the fuzzy optimization model, it may be hard for the heuristic algorithm to obtain an optimal solution, and the computation may be more time-consuming.

6.5 Summary

In this chapter, a generalized methodology for two specific research points of QFD in the fuzzy environment was proposed. One was the ranking of fuzzy importance of ECs, and the other one was the setting of target levels of ECs through a fuzzy expected value model.

The major contributions can be summarized in the following three aspects. Firstly, the analytical expression of the expected value for the product of two trapezoidal fuzzy numbers (TpFNs) was derived, which was utilized to measure and prioritize the fuzzy importance of ECs. Secondly, a fuzzy expected value model was established to determine target levels of ECs. The objective function was transformed by taking advantage of α-optimistic values of TpFNs, and the constraints were rewritten by plugging the expected value of TpFNs. The fuzzy optimization model was solved by an improved HIA, iHIA, which integrated the TNIA-S in Chapter 4 with a genetic algorithm. Lastly, the proposed methodology was applied to the same case study of a notebook computer development as in Chapter 5, except that all ECs were assumed to be continuous ones in this chapter. More specifically, the priority of seven ECs was ascertained, and discussions were conducted towards the changes of fulfillment levels of ECs with respect to different confidence levels in the model. According to the decision-makers' preference and trade-off, target levels of ECs, x_j, $j = 1, 2, \cdots, 7$, at the confidence level $\alpha = 0.9$ were chosen, where $x_j \in [0.8, 1]$.

Up to this chapter, the main contents of this dissertation is completed. The next chapter elaborates the concluding remarks, major contributions, and future research directions of the current research.

Chapter 7

Conclusions and Future Work

7.1 Conclusions

Both quality function deployment (QFD) and Kano's model are effective quality tools that have been popular for decades. Various successful applications were generated with the aid of these two tools no matter in the traditional manufacturing or the emerging service quality areas. The QFD optimization procedure in this research was conducted from the cooperative game-theoretic and fuzzy perspectives, respectively. The main purpose was to provide guidelines, suggestions, and managerial implications for the decision-makers of manufacturing products. In conclusion, the three specific research objectives raised in Chapter 1 were realized through the contents in the four chapters.

Chapters 3 and 4 achieved the first research objective, where several novel fuzzy simulation techniques for the possibility and expected value of fuzzy events were put forward, respectively. The original simulation techniques for the possibility, the SDS, and the expected value, the SDA, shared the same stochastic sampling process. Nevertheless, the lack of accuracy and efficiency of this process were proved from the analytical inference and numerical examples. To substitute the SDS and the SDA, a new operational law to generate a uniform sampling process was initiated. On this basis, the UDS and UDS-Joint were respectively proposed for possibilities of individual and joint fuzzy events, and the iSDA/TiSDA was designed for expected values of fuzzy events. Besides, another fuzzy simulation technique for the expected value, the NIA-S/TNIA-S, was designed with respect to the analytical expressions of α-optimistic values of some commonly used fuzzy variables, like triangular, normal, Gaussian, and trapezoidal fuzzy numbers. It should be noted that the contents in Chapters 3 and 4 not only presented the improvements in fuzzy theories and fuzzy simulation, but also can be served as a vital basis in the iHIA in Chapter 6.

Chapter 5 achieved the second research objective, where the relative importance weights of customer requirements (CRs) and target values of engineering characteristics (ECs) were successively obtained. A quantitative Kano's model and two significant concepts in cooperative games, i.e., Shapley value and Nash bargaining were taken advantage in two sequential stages. The first stage was to ascertain the relative importance weights of CRs based on Shapley value, which considered the marginal contribution of individual CR to the whole customer satisfaction. During this course, the quantitative results of Kano's model helped provide the value functions in calculations. The second stage was to establish a mixed integer non-linear programming model to determine target values of ECs so as to maximize the overall customer satisfaction. The objective of this model was a Nash bargaining function of the importance weights and the fulfillment levels of CRs, which was different from conventional objectives in the literature.

Chapter 6 achieved the third research objective, where the fuzzy importance of ECs and target levels of ECs were determined by means of the identical quantitative Kano's model with fuzzy parameters and the expected values for different fuzzy events. In particular, expected values for individual trapezoidal fuzzy number (TpFN), the product of two TpFNs, and continuous and strictly monotone functions of TpFNs were calculated based on the definitions and theorems on regular fuzzy intervals in Chapter 4. These outcomes played important roles in accomplishing the aforementioned two tasks of ECs. More specifically, the fuzzy importance of ECs was defuzzified and prioritized through expected values, and target levels of ECs were achieved through a fuzzy expected value model. The proposed model was solved by an improved HIA, the iHIA, which integrated the TNIA-S in Chapter 4 with a genetic algorithm.

Last but not the least, the QFD optimization procedures proposed from the perspectives of cooperative game theory and fuzzy theories were implemented into a case study of notebook computer development. Detailed elaborations were addressed towards the determinations of the relative importance weights of CRs via Shapley value, the fuzzy importance of ECs via the expected value of the product of two TpFNs, target values of ECs via a deterministic mixed integer non-linear programming model, and target levels of ECs via a fuzzy expected value model, respectively. Corresponding suggestions were also outlined for the decision-makers, taking the competitiveness in the market, the trade-off between design cost and the overall customer satisfaction, and the decision-makers' preference into account. Not limited to the case study chosen in this research, the proposed QFD optimization framework is also applicable to other manufacturing products.

7.2 Contributions

The contributions of this research are summarized as follows:

Firstly, for the research work in Chapter 3, the drawback of the SDS was clearly pointed out and proved, and meanwhile a new operational law regarding continuous and strictly monotone functions of regular fuzzy intervals was given and proved. On this basis, the UDS and UDS-Joint were put forward to deal with individual and joint fuzzy events, respectively, whose accuracy and computational time were demonstrated by several numerical examples.

Secondly, for the research work in Chapter 4, according to the new operational law in Chapter 3 and another calculation formula on the expected value of discrete fuzzy variables, the iSDA was set forth to improve the original SDA. Simultaneously, the NIA-S which utilized the analytical expressions of α-optimistic values of fuzzy variables was proposed to get rid of the original bisection algorithm in the NIA-G. In addition, both the iSDA and NIA-S were extended to their separate versions of regular fuzzy intervals, i.e., the TiSDA and TNIA-S. Analogously, the accuracy, stability, and computational time of the proposed simulation techniques were illustrated by a series of numerical examples.

Thirdly, for the research work in Chapter 5, Shapley value and Nash bargaining were incorporated in two stages from the determinations of the relative importance weights of CRs to target values of ECs. The customer satisfaction and dissatisfaction values, and the detailed mathematical functions obtained in the quantitative Kano's model were utilized. Moreover, two algorithms, the SVCA and VFCA were put forward to calculate Shapley values of CRs, which were intended to decrease the calculation complexity when over 10 CRs were identified for the target product. Combined the proposed deterministic mixed integer non-linear programming model with the case study, some practical discussions on target values of ECs, the development budget, and competitors were expounded.

Finally, for the research work in Chapter 6, simplified calculation formulae on expected values for a TpFN and the product of two TpFNs were derived, which were utilized to attain the expected value of the fuzzy importance of ECs. As a consequence, the ECs were ranked according to these expected values. The expected return of the objective function and some expected constraints in the proposed fuzzy expected value model were transformed into more simplified ones by virtue of the relevant definitions and theorems on α-optimistic values of TpFNs in Chapter 4. The model was solved by an improved HIA which combined the TNIA-S for the transformed objective function with a genetic algorithm. Discussions were also conducted on different combinations of target levels of ECs at different confidence levels, and an ultimate solution was settled for the consideration of the decision-makers.

7.3 Future Research Work

The future research directions are divided into two parts. The first part elaborates the potential improvements for the limitations of the current research. And the second part sets forth further researches that based on current contents. As to the first part, the following two major limitations and their future directions are listed, which focus on the applications in the proposed QFD framework.

- At present, the critical CRs are extracted from a specially designed Kano questionnaire. The Kano questionnaire can be further improved or the acquisition of CRs can be reconsidered by some big data-based mining methods.

Firstly, in the Kano questionnaire, the questions on the preferred cost of customers towards each CR can be added to observe customers' perceptions and willingness on the payment for different needs and functions. On this basis, the real-life application of Kano's model can be further strengthened. Secondly, in order to identify CRs timely and dynamically under today's big data environment, online reviews of a manufacturing product can be paid more attention to.

Compared with the traditional methods of acquiring customers' voices by questionnaires, interviews, or feedback, nowadays it is intuitive to extract and capture these needs from relevant online websites. Customers are able to post product reviews on many websites, like JD.com and Amazon.com, based on which a series of researches were conducted to inject new vitality to the classical QFD [Jin15, Jin16a]. Additionally, some smart phone makers even have their own online communities. Viewed as one of the upcoming stars in mobile phone industry in recent years, Huawei's online community is shared by different groups of users, like programmers, engineers, Huawei fans, and common users. A market segmentation study was carried out with respect to another smart phone - Xiaomi's online community by using customer pain points [Wan16].

As stated in the research point (a) of QFD in Chapter 2, the extraction of CRs can be viewed as a front end work and the input of the whole QFD process. Due to the reason that the inaccurate identification of customer needs will definitely lead to improper resource allocation of ECs in the subsequent research and development process, the extraction method selection is vital and crucial. In this context, some advanced web crawling and analyzing methods can be incorporated in the future studies to accomplish the extraction of real-time CRs. Apart from the extraction of CRs, other directions based on big online reviews are listed for researchers to take. For instance, the lead user identification in terms of the customer participation, and data-driven precision marketing. The objectives of these researches target on developing new generation products that fulfill customer needs to the largest extent.

- The quantitative Kano's model between the the fulfillment level of CR and fulfillment level of customer satisfaction (CS) is directly adopted from [Jip14] without modification. Other kinds of functions except exponential functions can also be reasonably formulated to simulate the Attractive and Must-be attributes in the quantitative Kano's model.

As reviewed in Chapter 2, Kano's model is usually used to determine the Kano category of CRs qualitatively. The quantification researches on the curves in the Kano diagram is limited. The proposed QFD method does not attempt to enhance the quantitative Kano's model proposed in [Jip14] but just adopts it. This innovation adoption strengthens the application of quantitative Kano's model and make the results more reasonable to convince users.

As mentioned above, the quantitative Kano's model expressed by mathematical functions plays an important role in formulating either the deterministic or the fuzzy optimization model. It is noted that, as described in Ji *et al.* [Jip14], the exponential functions were figured out based on two endpoints $(0, DS_i)$ and $(1, CS_i)$, $i = 1, 2, \cdots, n$ for n CRs. If possible, a third coordinate can be determined and used to obtain a more complicated analytical expressions of the functions. Basically, the proposed methodology in this research is completed. If other kinds of functions, like power functions and logarithmic functions for the quantitative Kano's model are generated, they can be embedded in the models directly to substitute the current exponential functions.

As to the second part, the further researches mainly focus on the fuzzy optimization models and fuzzy simulation, which are displayed as follows:

- The fuzzy parameters, the fuzzy objective function and constraints, and the model type of the fuzzy optimization model proposed in Chapter 6 can be further modified according to different considerations of the decision-makers.

Firstly, except for the fuzzy importance weights of CRs and the fuzzy relationship functions between CRs and ECs, more fuzzy parameters can be predetermined. For example, no matter in the the deterministic model or the fuzzy model, the cost coefficients of ECs, c_j, $j = 1, 2, \cdots, q$, are assumed to be crisp values, which are listed in the cost index row in Tables 5.8 and 6.1. As a matter of fact, the market prices of raw materials are fluctuated and usually have a lower or an upper quotation. Thereby, interval fuzzy numbers can also be employed to represent this unit improvement cost c_j. Meanwhile, these fuzzy parameters can be evaluated by some multi-criteria decision making (MCDM) methods. It is found that, the objective functions in the proposed two optimization models aim at maximizing the overall customer satisfaction degree while the total design cost is restricted to a settled budget. However, for some small-size enterprises, the decision-makers may expect to minimize the total design cost while the overall customer satisfaction degree only need to reach an acceptable level. Therefore, different objective functions and constraints can be considered during the model formulation. Meanwhile, the fuzzy expected value model proposed in Chapter 6 can be modified to other model types, like fuzzy chance-constrained programming models with credibilistic constraints and dependent chance programming models, etc. Additionally, in contrast to tangible manufacturing products, intangible services can also be optimized in an analogous way, which may need a detailed investigation together with a practical case study in the future.

- The fuzzy simulation discussed in the current research are intended for the possibility and expected value of fuzzy events, respectively. On this basis, researches to enrich fuzzy arithmetic, fuzzy theories, and fuzzy simulation can be generated with new efforts.

For instance, the simulation algorithms for the continuous and strictly monotone functions can be extended to more generalized functions. And the successful applications of regular fuzzy numbers and regular fuzzy intervals can be extended to more generalized fuzzy variables. What is more, the new operational law raised in Chapter 3 can also provide insights for other important notions in the fuzzy area, like new simulation techniques on the credibility, the variance, the skewness, and the entropy of fuzzy events.

In summary, this research project has developed two novel quality function deployment optimization frameworks integrating a quantitative Kano's model from the game-theoretic and fuzzy perspectives, respectively. The relative importance weights of customer requirements and the fuzzy importance of engineering characteristics of manufacturing products are correspondingly studied. Meanwhile, the overall customer satisfaction is maximized through deterministic and fuzzy optimization models, respectively. In particular, a series of improved fuzzy simulation techniques for the possibility and expected value of fuzzy events based on the existing techniques are put forward. The improvements on the expected value simulation is further served as a basis for solving the fuzzy optimization model in QFD by a heuristic-based algorithm. These two methods in QFD help provide guided maps for the decision-makers to follow in improving the target product. It is noted that the detailed elaborations of the case study under two perspectives can be promoted to other manufacturing products. Future researches can be directed to the extraction of customer requirements, other forms of quantitive Kano's models, fuzzy simulation for other concepts in the fuzzy area, and different type of fuzzy optimization models towards the QFD optimization procedure.

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