LINEAR QUADRATIC SOCIAL OPTIMA AND MEAN FIELD GAMES

QIU ZHENGHONG

PhD

The Hong Kong Polytechnic University

2021

The Hong Kong Polytechnic University

Department of Applied Mathematics

Linear Quadratic Social Optima and Mean Field Games

QIU Zhenghong

A thesis submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy

December 2020

CERTIFICATE OF ORIGINALITY

I hereby declare that this thesis is my own work and that, to the best of my knowledge and belief, it reproduces no material previously published or written, nor material that has been accepted for the award of any other degree or diploma, except where due acknowledgement has been made in the text.

Name: QIU Zhenghong

Dated: 24/10/2021

Acknowledgements

The endeavor of carrying out research is a fascinatingly non-isolated activity. I am grateful to the several individuals who have supported me in various ways during the PhD program and would like to hereby acknowledge their assistance. First and foremost, I am greatly indebted to my supervisor, Prof. Huang Jianhui, James, for his enlightening guidance, invaluable discussions and insightful ideas throughout the years. What I have benefited most from him is the rigorous and diligent attitude to scientific research. Furthermore, I wish to express my deep thanks to Dr. Qingyuan Qi and Dr. Xinwei Feng, who once offered me valuable research experience, for their strong encouragement and help during my PhD programme. Last but not least, I owe much to my beloved parents for their loving considerations and great confidence in me all through these years. My gratitude also extends to my friends for their love, encouragement and support.

Abstract

The thesis explores linear quadratic (LQ) mean field (MF) large population (LP) systems. Three topics are considered:

1. The MF social optima (which is also called MF team (MFT)) problem for a LP system.

2. The MFT problem for a major-minor LP system.

3. The relation among the MF type control (MFC) problem, the MF game (MFG) problem and the MFT problem.

In the first topic, the MF approximation method is applied to the social optimal problem. In this problem, the agents' states and strategies access the diffusion terms. In addition, the control weight for the cost functional might be indefinite. Firstly, we consider the convexity of the social cost functional. We derive some low-dimensional criteria to determine this convexity via algebra analysis. Secondly, under the person-by-person optimality principle, we apply some stochastic variational techniques and MF approximation to obtain the decentralized auxiliary control. Thirdly, to resolve the solvability of the consistency condition, which is represented as a MF forward-backward stochastic differential equations (MF-FBSDEs) system, we apply the decentralizing method to convert it to a general FBSDEs system. Furthermore, we apply the decoupling method and obtain a Riccati equation. Lastly, because the agent states access the diffusion term, we should consider the convergence of the average of a se-

ries of weakly coupled BSDEs. Through the decoupling method, we obtain two Lyapunov equations, and their uniform boundness ensure the convergence.

In the second topic, the social optima of the major-minor LP system are considered. In our model, a considerable number of minor agents are cooperative to minimize the social cost as the sum of individual costs, while the major agent and minor agents competitively aim for Nash equilibrium. Moreover, as in topic one, the agents' states and strategies access the diffusion terms, and this brings essential difficulty to the proof of asymptotic optimality. In our research, we firstly study the decentralized control of the major agent. By freezing the minor state-average, we obtain the auxiliary control problem for the major agent. Furthermore, under the person-by-person optimality principle and applying some MF approximation, we obtain the auxiliary control problems for the minor agents. The consistency condition is also a MF-FBSDEs system. The well-posedness of the consistency condition system is obtained by the discounting method. The related asymptotic optimality is also verified.

In the third topic, we study the relation among the MFC, MFG and MFT problems. Notably, the individual admissible controls are constrained in a linear subset. By introducing a new type of Riccati equation, we obtain a uniform convexity condition which is weaker than the "standard condition" widely used in previous literature. Also, using this new type of Riccati equation, we obtain the constrained feedback form optimal control and MF strategies for MFC, MFG and MFT problems respectively. Moreover, through analysing the corresponding Hamiltonian systems of these three problems, it can be concluded that under some mild conditions, the MF strategies of the MFG and MFT problems are equivalent to the optimal control of the MFC problem. Lastly, we also find that the MFT strategy obtained via the direct approaching method is identical to that obtained by the fixed-point approaching method.

Table of Contents

Chapter 1 Introduction

1.1 Background

The purpose of this thesis is to study the mean field (MF) social optima problem which also known as the MF team (MFT) problem, and the connections among the MF type control (MFC) problem, the MF game (MFG) problem and the MFT problem. A common feature shared by these three types of problems is the existence of considerable interactive agents. Thus, such systems are also called large population (LP) systems.

1.1.1 LP system

LP systems have been widely applied in various areas, including economics, biology, engineering, and social science (see, [1, 2, 3]). The most salient feature of the LP system is the existence of considerable insignificant agents whose dynamics and cost functionals are interrelated via the state-average or control-average. Although the effect of an individual agent is negligible, the combined effects of their statistical behaviors cannot be ignored at the population scale. Recently, the dynamic decisions of LP systems have attracted more attention because of the rapid growth of practical models with large-scaled interactions.

However, when the number of agents is sufficiently large, we have to face two severe problems: (i) Complex coupling features arise with the growth of the population and it is unrealistic for a given agent to obtain all other agents' information (centralizing information) to solve this coupling problem. (ii) Because of the coupling structure, a large population leads to high dimensionality and is thus subjected to the so-called "curse of dimensionality" (see [4, 5]). As a consequence, the computational complexity increases exponentially in practical numerical analysis.

Consequently, the MF method has drawn increasing research attention because it provides an effective scheme to obtain an asymptotically optimal decentralized controls based on the limited information of individual agents with a much lower computational burden. In particular, via the MF method, each agent takes advantage of MF interaction to transform the analysis of the original LP problem (centralizing and high-dimensional) to an optimization problem of itself (decentralized and low-dimensional) with responding to aggregation effects of the other agents. Interested reader are referred to [6, 7] for application of the MF method in economics, [8, 9, 10] in engineering, [11, 12] in biology, as well as [13, 14] in management science and operational research.

1.1.2 MFG problem

The MF method has also been widely applied in games model framework (see [1, 2, 15, 14, 16]). The agents in these aforementioned works are competitive, and thus we call such models MFG. The derivation of MFG can be traced back to the parallel works of Lasry and Lions [16] and Huang, Caines, and Malhamé [14]. Specifically, in [14] an ε-Nash equilibrium is designed via the Nash certainty equivalence (NCE) approach which is also called the consistency condition (CC).

In [16], MFG is studied using two limiting coupled partial differential equation systems which are inspired by physical particle systems. The first one is the Hamilton-Jacobi-Bellman equation which describes the reaction of agents to the population aggregation. The second one is the Fokker-Planck-Kolmogorov equation which describes the behaviors of population aggregations. For more detailed discussion of MFG, interested readers are referred to [17, 18, 19] for linear-quadratic-Gaussian (LQG) MFG; [20] for probabilistic analysis in MFG; [21] for risk-sensitive MFG; [22] for discrete-time MFG; and [23, 24, 25] for robust MFG.

1.1.3 MFT problem

Apart from noncooperative MFG, cooperative multi-agent decision problems (social optima) are also a research hotspot due to interest in their theoretical analysis and real application potentials. In the MFT problems, all agents usually work cooperatively to minimize a social cost which is generally framed as the sum of the N individual costs containing MF coupling.

The recent research into MFT largely follows two routes. One is called the fixed-point approach. Usually each agent first establishes an auxiliary control problem through applying the variational method, duality and MF approximations to the social cost functional. Then by solving the auxiliary control problem, the agent can obtain an auxiliary optimal control involving some pre-frozen MF terms. Lastly the agent would determine the MF terms by formalizing a fixed-point problem and this is why it is called the fixed-point approach. For more details of the fixed-point approach method, readers are referred to [15, 26, 14, 27].

The other route is called the direct approach, where each agent starts by directly formally solving the MFT problem as a high dimensional control problem. Usually, an optimal control equation with feedback form representation would be obtained via some "Riccati-type" equations which would converge to some standard Riccati equations. The next step is to derive the limit of optimal control via the limiting Riccati equations as the population size tends to infinity. For more discussion of the direct approach method, readers are referred to [28, 29, 30].

Existing MFT studies are tremendously rich and for more comprehensive details, interested readers may refer to [27] for the centralized and decentralized controls in a linear-quadratic-Gaussian (LQG) MFT problem where the asymptotic optimality of the MF strategies is also illustrated; [31] for the MF social solution to consensus problems; [32] for social optima of MF LQG control models with Markov jump parameters; and [33] for an LQG MFT problem involving a major player.

1.1.4 MFC problem

In the aforementioned MFT problem, each agent has the free will to choose its own control. On the other hand, if one considers a system of interacting agents under centralized control, then it would lead to a MFC problem (see [34, 35, 36]). Such a MFC problem shares a similar form to that proposed in MFT problem, but the MF term is now uniformly determined by a centralizing system instead of being affected by the aggregation of the population. The MFC problem is aimed at assigning a strategy to all agents at once, such that the resulting population behavior is optimal with respect to the costs imposed on a central planner. In the MFC problem, the MF term is influenced by the agent individually, and the problem is thus a control problem. Indeed, the state equation also contains the probability distribution of the state and thus is called the McKean–Vlasov SDE, a kind of MF forward stochastic differential equation (MF-FSDE), which was suggested by Kac [37] in 1956 as a stochastic toy model for the Vlasov kinetic equation of plasma, and a study of this was initiated by McKean [38] in 1966.

1.2 Contributions and organization of the thesis

As for the novelty, this thesis mainly considers the MF method in LP systems. Details can be summarized as follows:

1.2.1 Contributions and organization of chapter 2

In Chapter 2, we consider the social optima problem in a weakly-coupled LP system with N individual agents. Our setup has the following features in its structure.

- All agents are highly interactive and coupled in their dynamics and cost functionals due to the presence of the state-average $x^{(N)}$. In particular, the individual cost functionals depend on $u = \{u_1, u_2, \cdots, u_n\}$, the control profile of all agents owning to such weak-coupling. Thus, all agents frame a LP system of MF type. Such a system arises in various fields, as seen in [1], [20], [39], [40], [16] and [41]. Because of such MF structure, the related dynamic optimization is subject to the curse of dimensionality and complexity in numerical computation. Consequently, decentralized controls which are based on the local information set will be used instead of centralized controls which are based on the full information set. Note that the decentralized control for multi-agent system are well documented in literature (see [11], [14], [26] [42]).

- Unlike the noncooperative game in [43], [44], [45], where agents try to minimize their own individual cost functionals, all agents in our model aim to minimize the social cost functional which is the summation of the cost functionals of all agents. Thus, the relevant analysis is also very different: all agents aim to reach the same criteria of social (Pareto) optimality. They are seeking social optimal points instead of the Nash equilibrium. Also, in the presence of weakly-coupled interactive agents, some variational analysis should be applied to the person-by-person optimality to obtain some necessary condition for such Pareto optimality criteria. Note that social optima is also well studied in literature, and readers can refer to [46], [47], [48],[49], [50], [51].

- Based on the state dynamics and cost functional, our work should be framed as a MF social optima problem with numerous cooperative agents. Note that such kind of problems have drawn increasing research attention in recent years, see [27], [52] on LQG social optima with constant noise, [32] for related analysis in Markov LQG setup, [53] for related analysis in economic social welfare problem and [52] for robust LQG social optima with drift uncertainty.

Note that, in our state dynamics, a generalized setting is considered. The state process $x_i(\cdot)$, control process $u_i(\cdot)$ and state-average $x^{(N)}$ enter the diffusion term when $C, D, \tilde{F} \neq 0$, while in the recent research of MF social optima (see [14], [52], [30]), only constant volatility situation is studied. In particular, when $D \neq 0$, the diffusion term is dependent on the control directly, and the related LQG problem can be referred to as a *multiplicative-noise* control problem. The inclusion of the control variable in diffusion is well motivated by various real applications. One such example comes from the well-known mean-variance portfolio problem ([54], [55], [56], [57]) where the control process (risky portfolio allocation) naturally enters the dynamics for given wealth process. There is various literature on the discussion of related LQG problems, and readers can refer to [58], [59], [60], [61], [62].

Our control-dependent setup is different from [63], [43], [14], [27], [52] in which $D = 0$ and only drift is directly control-dependent. Such problems can be referred as *additive-noise* LQG control problem, and has already been well investigated (see [43], [14], [41], [32]). In principle, the additive-noise problem has no essential distinction to deterministic or constant volatility LQG (see [43], [27], [52]). Actually, with the help of constant variation method and separable property for linear system, the state can be represented by linear functional on state and control separately.

Besides, in this chapter, the weight matrices of the cost functional are indefinite. Recently, the LQG frameworks with indefinite weight matrices have been studied extensively in [58] and [59]. This setting has some interesting applications in mathematical finance (see [62], [57]). However, in recent literature on social optima (e.g., [27], [52], [30]), only positive semi-definite weight is considered. Consequently, our research might be the first to formulate a social optima problem under such generalized setting with realistic significance. However, such extension also brings some practical difficulties to our research.

Compared with previous works, the difficulties appear in our research are as follows:

1. In contrast with the conditions in [63], [27], [44], [52], the weight coefficients in our model can be indefinite, and the convexity of the social cost

functional need to be discussed. Unlike the MFG problem, where each agent only aims to minimize its own low-dimensional personal cost functional, our social optima problem is essentially a high-dimensional control problem. Hence, due to the dimensionality, it is very difficult to verify its convexity directly. However, the convexity is crucial to the problem solvability (see [58]). In this chapter, we provide some low-dimensional criteria for the convexity of the social cost functional using some algebraic techniques.

2. In general, for classic MFG problem (see [1], [17], [64]), the auxiliary control problem can be obtained directly by freezing the state-average as some deterministic term. However, in searching for the social optima, this scheme will bring some ineffective strategy, which can not achieve the asymptotic optimality. Thus, variational techniques are used to distinguish the high-order infinitesimals after MF approximation. In particular, $N+1$ additional dual processes need to be introduced to deal with the cross-terms in the cost functional variation and a new type of auxiliary control problem is derived.

3. The proposed CC system in this chapter is a highly coupled MF forward-backward stochastic differential equations (MF-FBSDEs) system, instead of an ordinary differential equations (ODEs) system as in some other general cases. Because $C, \tilde{F} \neq 0$, the adjoint terms of the backward equations enter the drift term. Thus, the dynamics of the MF terms cannot be obtained by taking expectation. It is complicated to investigate its solvability directly by decoupling method. Thus, by applying decentralizing method,

we transform it to a linear FBSDE system, and study the well-posedness of the new system.

4. Unlike some previous works, such as [27], [52], [32], we use the linear operator method and Fréchet derivative to prove the asymptotic social optimality. Because $C, F, \tilde{F} \neq 0$, the error estimates are very hard to obtain directly, since some of them are coupled without explicit expression. To overcome such difficulty, we decouple them via Lyapunov equations and estimate them in proper order.

The main contributions of this chapter are summarized as follows:

- We setup a class of LQG control problem where both the drift and diffusion terms are dependent on state process, control process and state-average, as well as propose an approach for obtaining the social optimal solution.

- By discussing the weight coefficients, some low-dimensional criteria for the convexity of the social cost functional, which is a high-dimensional system, are obtained.

- A highly coupled CC system (MF-FBSDE) is transformed to an equivalent FBSDE by decentralizing transformation. The existence and uniqueness of the CC system solution is characterized by a Riccati equation.

- By using the perturbation method to analyse the Fréchet derivative of cost functional, the decentralized MF strategies we derived are proved to be asymptotically optimal. In addition, to prove its asymptotical optimality, we apply some classical estimates of SDEs, and investigate the optimality loss.

Chapter 2 is organized as follows: In Section 2.1, we formulate the social optimal LQG control problem. In Section 2.2, the convexity of the social cost functional is discussed. We construct an auxiliary optimal control problem based on person-by-person optimality and design the decentralized control in Section 2.3 and Section 2.4 respectively. In Section 2.5, some prior lemmas are given, and based on them the asymptotic social optimality is proved. A numerical example is provided to simulate the efficiency of decentralized control in Section 2.6. Section 2.7 conclude this chapter.

1.2.2 Contributions and organization of Chapter 3

In Chapter 3, within the MF modeling, we investigate a new class of stochastic LQG optimization problems involving a major agent and a large number of weakly-coupled minor agents. Specifically, the minor agents are cooperative to minimize the social cost as the sum of individual costs, while the major agent and minor agents are competitive, aiming for Nash equilibrium in a nonzero-sum game manner. To the best of our knowledge, this is the first time this kind of problem is being studied. In MFG problems, the major agent and all the minor agents are competitive so as to achieve a Nash equilibrium; while in social optimal problems, all the agents are cooperative to find the social optimal strategies. Besides the new framework, our study also offers other new features. For instance, in the state equations of the major agent and minor agents, the control process enters both the drift and diffusion coefficients.

Our setup is an extension of the well-studied two-player (non-cooperative) game in which two agents make competitive decisions based on *individual but centralized* information. In our setup's extension: one agent no longer engages in such centralized decision-making, instead, all its sub-units or branches would apply distributed information to jointly optimize the original cost (e.g., [65], [66] and [67]), which is reformulated as some team-cost form now. Thereby, all sub-units become "minor" agents and formalize a (cooperative) team, while another agent still applies centralized information, becoming a "major" and non-cooperative player, from the viewpoint of all "minor" agents described above.

Somewhat similarly, [68] studies the competition between a centralized firm and a decentralized firm. In this case the centralized firm is treated as a "major" agent, while all sub-units of decentralized firm are treated as "minor" agents of a same team. Moreover, in [69] the centralized and decentralized charging options for electric vehicles are studied.

In our study, the problem is solved in the following way. Firstly, for the major agent, we freeze the state-average as a process only depending on the major noise. Thus, the auxiliary control problem for the major agent can be obtained. By the result in [62], the auxiliary optimal control for the major agent can be derived, which depends on the frozen MF term. Secondly, for the minor agents, based on the person-by-person optimality principle, by applying variational techniques and introducing some MF terms, the original minor social optimization problem is also converted to an auxiliary LQG control problem which can be solved using some traditional scheme in [62] as well. Thirdly, to determine the frozen MF terms, we construct a CC system by some fixed-point analysis. The MF terms can be obtained by solving the CC system, while the solvability of the CC system can be determined through the discounting method. Lastly, by using some asymptotic analysis and standard estimation of (SDEs), we show that the MF strategies really bring us an efficient approximation (i.e., the optimal loss tend to 0 when the population N tends to infinity).

Chapter 3 is organized as follows: In Section 3.1, we give the formulation of the mixed LQG social optima problem. In Section 3.2 and Section 3.3, we use the MF approximation and person-by-person optimality to find the auxiliary control problem of the major agent and minor agents, respectively. The CC system is derived in Section 3.4. Meanwhile, the well-posedness of the CC system is also established. In Section 3.5, we obtain the asymptotic optimality of the decentralized strategies. In Section 3.6, a numerical example motivated by electric charging network model is computed to illustrate the theoretical results of this chapter.

1.2.3 Contributions and organization of Chapter 4

The main contributions of Chapter 4 can be summarized as follows:

1. We study the MFG, MFT and MFC problem under a unified mathematical form with input constrained on a linear subspace. By using some algebraic techniques and modified SMP, we obtain a new type of Hamiltonian system which is related to the characterization of the constrained optimal control and MF strategy. We also obtain a new type of Riccati equation which is related to the uniform convexity of cost functional and feedback form representation of the constrained control.

To the best of our knowledge, the current study is the first to introduce such Hamiltonian system and Riccati equation, and with the help of them, the optimal control and MF strategy can be represented explicitly. In contrast, in other relevant literature, the designed control can only be represented implicitly; that is, it will be embedded into an projection mapping coupled with the dual process (see equation 2.10 in [64], page 905 in [18] and equation 18 in [70]). Notably, our explicit representation of the designed control would provide great practical advantage especially in numerical computation.

2. We also study the uniform convexity of the MFC problem on the constrained admissible control set. Note that the uniform convexity of a cost functional on the whole space does imply the same on a linear subspace, but not the other way around. We futher provide a counter example to illustrate it. In this sense, some previous works (e.g., [61, 13, 1]) can be treated as special cases of the current study.

   Through the aforementioned new-type Riccati equation, we obtain the uniform convexity condition of the MFC problem. Our condition is much weaker than the so-called "standard assumption" which is widely applied in other relevant literature (e.g., [61, 13, 1]). Under standard assumption, the weight coefficients in the cost functional should be positive semidefinite. In contrast, in our condition such coefficients could be indefinite, and we also provide an example to illustrate this.

3. We study the relation of the uniform convexity between the MFC problem and the related augmented control problem which is mentioned in [61] but

has not been discussed. We find that generally, the uniform convexity of the related augmented control problem implies the same of MFC problem, but not the other way around.

4. We obtain the relation among the optimal control of MFC problem and the MF strategies of MFG and MFT problem. Through analyzing the Hamiltonian system of MFC problem and the CC systems of MFG and MFT problem, we find that the optimal control of MFC problem is equivalent to the MF strategy of MFT problem. Such result provides a shortcut for dealing with MFT problem in practical application. Each agent in the system could directly calculate its own related MFC problem instead of deriving an auxiliary control problem (fixed-point approach) or computing the limit of the centralized optimal control (direct approach).

   Moreover, we also find that in certain cases, the optimal control of the MFC problem is also the MF strategy of the MFG problem. Such result is consistent to that in [13], which can be treated as a special case of the current study. Lastly, we compare the MF strategies of MFT problem derived by fixed-point approach and direct approach and also find that they are identical. These two routes lead to a same MF strategy.

Chapter 4 is organized as follows: Section 4.1-4.3 discusses MFC, MFG and MFT problem with input constrained on a linear subspace respectively. Section 4.4 analyzes the relation among the optimal control of MFC and the MF strategies of MFG, MFT problem. Section 4.5 concludes this chapter.

1.3 Preliminaries and notations

The following basic notations will be used throughout this paper

- $\| \cdot \|$: standard Euclidean norm
- $\langle \cdot, \cdot \rangle$: standard Euclidean inner product
- $\mathrm{Tr}(M)$: the trace of matrix M
- $\|M\|$: the matrix norm of matrix M, where $\|M\| := \sqrt{\mathrm{Tr}(M^T M)}$
- $\|M\|_{\max}$: the max-norm of matrix M, which is equal to the maximum absolute value of all elements
- x^T: transpose of a vector (or matrix) x
- $\mathbb{S}^n$: the set of symmetric $n \times n$ matrices with real elements
- $\|v\|_S^2$: for any vector v and symmetric matrix S, $\|v\|_S^2 := \langle Sv, v \rangle = v^T S v$
- $M > (\geq) 0$: $M \in \mathbb{S}^n$ is positive (semi)definite
- $M \gg 0$: for some $\varepsilon > 0$, $M - \varepsilon I \geq 0$
- $\lambda_{\max}(M)$: the maximum eigenvalue of matrix M
- $\lambda_{\min}(M)$: the minimum eigenvalue of matrix M

- $v \perp \Lambda$: the vector v is vertical to the linear subspace Λ (i.e., for any vector $v' \in \Lambda$, $\langle v, v' \rangle = 0$).

Throughout this paper, we suppose that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a complete filtered probability space, and $W(\cdot) = (W_0(\cdot), W_1(\cdot), \cdots, W_N(\cdot))$ is a $(N+1)$-dimensional standard Brownian motion defined on it. Let σ-algebra $\mathcal{F}_t := \sigma\{W_i(s), 0 \le s \le t, 0 \le i \le N\}$, and $\mathcal{F}_t^i := \sigma\{W_i(s), 0 \le s \le t\}$, $0 \le i \le N$. $\mathbb{F} := \{\mathcal{F}_t\}_{t \ge 0}$ is the natural filtration generated by $W(\cdot)$ augmented by all $\mathbb{P}$-null sets in $\mathcal{F}$, and $\mathbb{F}^i := \{\mathcal{F}_t^i\}_{t \ge 0}$ is the natural filtration generated by $W_i(\cdot)$ augmented by all $\mathbb{P}$-null sets in $\mathcal{F}$. Next, for any given Euclidean space $\mathbb{H}$ and filtration $\mathbb{G}$, we introduce the following spaces:

- $L^2(0,T;\mathbb{H}) = \left\{ x : [0,T] \to \mathbb{H} \middle| \int_0^T \|x(t)\|^2 dt < \infty \right\}$

- $L^\infty(0,T;\mathbb{H}) = \left\{ x : [0,T] \to \mathbb{H} \middle| x(\cdot) \text{ is bounded and measurable} \right\}$

- $C([0,T];\mathbb{H}) = \left\{ x : [0,T] \to \mathbb{H} \middle| x(\cdot) \text{ is continuous} \right\}$

- $L^2_{\mathbb{G}}(\Omega;\mathbb{H}) = \left\{ x : \Omega \to \mathbb{H} \middle| x \text{ is } \mathbb{G}\text{-measurable, } \mathbb{E}\|\xi\|^2 < \infty \right\}$

- $L^2_{\mathbb{G}}(0,T;\mathbb{H}) = \Big\{ x : [0,T] \times \Omega \to \mathbb{H} \Big| \ x(\cdot)$ is $\mathbb{G}$-progressively measurable, $\|x(t)\|^2_{L^2} := \mathbb{E} \int_0^T \|x(t)\|^2 dt < \infty \Big\}$

- $L^2_{\mathbb{G}}(\Omega; C([0,T];\mathbb{H})) = \Big\{ x : [0,T] \times \Omega \to \mathbb{H} \Big| \ x(\cdot)$ is $\mathbb{G}$-progressively measurable, continuous, $\mathbb{E} \sup_{t \in [0,T]} \|x(t)\|^2 < \infty \Big\}$

Next, we introduce some inequalities which are commonly used in stochastic estimation

Theorem 1.1 (Burkholder-Davis-Gundy inequality). *In the complete filtered probability space* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *defined above, let*

$$L^{2,loc}_{\mathbb{F}}(0,T;\mathbb{H}) = \Big\{ x : [0,T] \times \Omega \to \mathbb{H} \Big| \; x(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable}, \int_0^T \|x(t)\|^2 dt < \infty, \quad \mathbb{P} - a.s. \Big\}.$$

Then for any process $x(\cdot) \in L^{2,loc}_{\mathbb{F}}(0,T;\mathbb{H})$*, and any real number* $p > 0$*, there exists a constant* $K(r) > 0$ *(only depends on* r*) such that for any stopping time* τ*,*

$$\frac{1}{K(r)} \mathbb{E} \left[\int_0^\tau \|x(s)\|^2 ds \right]^p \leq \mathbb{E} \left[\sup_{0 \leq t \leq \tau} \left\| \int_0^t x(s) dW \right\|^{2p} \right] \leq K(r) \mathbb{E} \left[\int_0^\tau \|x(s)\|^2 ds \right]^p .$$

For more details of Burkholder-Davis-Gundy inequality, please refer to [62, Chapter 1, Theorem 5.4]. Actually, in this thesis we only need the following corollary of Burkholder-Davis-Gundy inequality:

Corollary 1.1. *In the complete filtered probability space* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *defined above, for any process* $x(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{H})$*, there exists a constant* $K(r) > 0$ *(only depends on* r*) such that for any time* $\tau \in \mathbb{R}^+$*,*

$$\mathbb{E} \left[\sup_{0 \leq t \leq \tau} \left\| \int_0^t x(s) dW \right\|^2 \right] \leq K(r) \mathbb{E} \left[\int_0^\tau \|x(s)\|^2 ds \right] .$$

We also introduce the following version of Grönwall's inequality which may be applied in this thesis.

Theorem 1.2 (Grönwall's inequality)**.** *Let $x(\cdot)$ be a $[0,T] \to \mathbb{H}$ continuous function. $a(\cdot)$, $b(\cdot)$ are non-negtive increasing function. If $x(t) \leq a(t)+b(t)\int_0^t x(s)ds$, then it holds that*

$$x(t) \leq a(t)e^{b(t)\times t}.$$

Chapter 2 MF Strategy in Social Optima

2.1 Problem formulation

In this chapter, we consider a weakly-coupled LP system with N agents denoted by $\{\mathcal{A}_i\}_{1\le i\le N}$. The state process of the i^{th} agent $\mathcal{A}_i$ is modeled by the following controlled linear SDE on finite time horizon $[0,T]$: $\mathcal{A}_0$

$$\begin{cases} dx_i(t) = (A(t)x_i(t) + B(t)u_i(t) + F(t)x^{(N)}(t))dt \\ \qquad\qquad + (C(t)x_i(t) + D(t)u_i(t) + \tilde{F}(t)x^{(N)}(t))dW_i(t), \\ x_i(0) = \xi_0. \end{cases} \tag{2.1}$$

where $x^{(N)}(t) := \frac{1}{N}\sum_{i=1}^{N} x_i(t)$ denotes the state-average. $A(\cdot)$, $B(\cdot)$, $F(\cdot)$, $C(\cdot)$, $D(\cdot)$, $\tilde{F}(\cdot)$ are deterministic matrix-valued functions with appropriate dimensions. In addition, to evaluate the performance of the control laws, we also introduce the following individual cost functional for $\mathcal{A}_i$:

$$\begin{aligned} \mathcal{J}_i(\xi_0; u(\cdot)) =& \frac{1}{2}\mathbb{E}\bigg\{ \int_0^T \|x_i(t) - \Gamma(t)x^{(N)}(t) - \eta(t)\|^2_{Q(t)} + \|u_i(t)\|^2_{R(t)} dt \\ & + \|x_i(T) - \bar{\Gamma}x^{(N)}(T) - \bar{\eta}\|^2_G \bigg\} \end{aligned} \tag{2.2}$$

where $u(\cdot) = (u_1(\cdot), \cdots, u_N(\cdot))$ and $Q(\cdot)$, $R(\cdot)$ and $G(\cdot)$ are weight matrices. All agents are cooperative and aim to minimize the social cost functional, which is denoted as follows:

$$\mathcal{J}^{(N)}_{soc}(\xi_0; u(\cdot)) = \sum_{i=1}^{N} \mathcal{J}_i(\xi_0; u(\cdot)), \tag{2.3}$$

and in this chapter, for the sake of notation simplicity, we may suppress the time notation "(t)" and "$(\cdot)$" if necessary.

Further, based on the information structure, two types of admissible strategy sets are defined as follows. The centralized admissible strategy set is given by:

$$\mathcal{U}_c := \Big\{ u | u \text{ is adapted to } \mathbb{F}, \text{ and } \mathbb{E}\int_0^T \|u(t)\|^2 dt < \infty \Big\}.$$

Correspondingly, the decentralized admissible strategy set for the i^{th} agent is given by:

$$\mathcal{U}_i := \Big\{ u | u \text{ is adapted to } \big\{ \mathcal{F}_t^i \vee \sigma\{x_0(s), 0 \le s \le t\} \big\}_{t \ge 0}, \mathbb{E}\int_0^T \|u_i(t)\|^2 dt < \infty \Big\},$$

Now, we introduce the following two assumptions

(A2.1) The coefficients of the state equation satisfy the following:

$$A, C, F, \tilde{F} \in L^\infty(0,T;\mathbb{R}^{n\times n}), \quad B, D \in L^\infty(0,T;\mathbb{R}^{n\times m}).$$

(A2.2) The weighting coefficients in the cost functional satisfy the following:

$$\begin{cases} Q \in L^\infty(0,T;\mathbb{S}^n),\ \Gamma \in L^\infty(0,T;\mathbb{R}^{n\times n}),\ R \in L^\infty(0,T;\mathbb{S}^m), \\ \eta \in L^2(0,T;\mathbb{R}^n),\ \bar{\Gamma} \in \mathbb{R}^{n\times n},\ G \in \mathbb{S}^n,\ \bar{\eta} \in \mathbb{R}^n. \end{cases}$$

Under (A2.1), for any given $(u_1, \cdots, u_N) \in \mathcal{U}_c \times \cdots \times \mathcal{U}_c$, (2.1) admits a unique solution $(x_1, \cdots, x_N)$. Under (A2.2), the cost functional (2.2) is well-posed. We aim to work out the optimal strategy for each agent to minimize our social cost. Thus we propose the following social optimal problem:

Problem 2.1. *Minimize* $\mathcal{J}_{soc}^{(N)}(\xi_0; u)$ *over* $\{u = (u_1, \cdots, u_N) | u_i \in \mathcal{U}_c\}$.

Remark 2.1. *Problem 2.1 can be solved by the method in [58] in a high-dimensional approach. However, it will face some difficulties in the practical application. Firstly, an agent can only access its own information (i.e.,* $\sigma\{\mathcal{F}_t^i \cup \sigma(x_i(s), s \leq t)\}$*) most of the time and the information of the others may be unavailable for it in real world (see [71, 72, 73]). Secondly, by the large population structure, the dynamic optimization will be subjected to the curse of dimensionality and complexity in numerical analysis in practice.*

For these reasons, we aim to work out a so-called decentralized strategy, which only depends on the agent's own information. Let us introduce following problem:

Problem 2.2. *Minimize* $\mathcal{J}_{soc}^{(N)}(\xi_0; u)$ *over* $\{u = (u_1, \cdots, u_N) | u_i \in \mathcal{U}_i\}$.

2.2 Convexity

In this section, the convexity of the social cost functional will be studied. The weight coefficients Q, R and G profoundly influence the convexity of the cost functional. We start with positive semi-definite weight case, which is relatively simple, and then further consider indefinite weight case. Indefinite weight setting has some interesting mathematical finance background (see [62], [57]). Before that, the dynamics of the agent states should be rewritten as follows:

$$d\mathbf{x} = (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u})dt + \sum_{i=1}^{N}(\mathbf{C}_i\mathbf{x} + \mathbf{D}_i\mathbf{u})dW_i, \quad \mathbf{x}(0) = \Xi, \tag{2.4}$$

where

$$\mathbf{A}=\begin{pmatrix} A+\frac{F}{N} & \frac{F}{N} & \cdots & \frac{F}{N} \\ \frac{F}{N} & A+\frac{F}{N} & \cdots & \frac{F}{N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{F}{N} & \frac{F}{N} & \cdots & A+\frac{F}{N} \end{pmatrix}_{(Nn\times Nn)}, \mathbf{B}=\begin{pmatrix} B & 0 & \cdots & 0 \\ 0 & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B \end{pmatrix}_{(Nn\times Nm)}, \Xi=\begin{pmatrix} \xi_0 \\ \vdots \\ \xi_0 \end{pmatrix}_{(Nn\times 1)},$$

$$\mathbf{C}_i = \begin{array}{c} 1 \\ \vdots \\ i^{\text{th}} \\ \vdots \\ N \end{array} \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\tilde{F}}{N} & \cdots & \frac{\tilde{F}}{N} & \frac{\tilde{F}}{N}+C & \frac{\tilde{F}}{N} & \cdots & \frac{\tilde{F}}{N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{(Nn\times Nn)}, \quad \mathbf{D}_i = \begin{array}{c} 1 \\ \vdots \\ i^{\text{th}} \\ \vdots \\ N \end{array} \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & D & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}_{(Nn\times Nm)}, \tag{2.5}$$

and the cost functional can be rewritten as follows:

$$\begin{aligned} \mathcal{J}_{soc}^{(N)}(\xi_0; \mathbf{u}) = \frac{1}{2}\mathbb{E}\int_0^T \Big\{ & \mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{S}_1^T\mathbf{x} + N\eta^T Q\eta + \mathbf{u}^T\mathbf{R}\mathbf{u}dt \\ & + \mathbf{x}^T(T)\mathbf{G}\mathbf{x}(T) + 2\mathbf{S}_2^T\mathbf{x}(T) + N\bar{\eta}^T G\bar{\eta} \Big\}, \end{aligned} \tag{2.6}$$

and the weight coefficients are:

$$\begin{aligned} \mathbf{Q} &= \begin{pmatrix} Q+\frac{1}{N}(\Gamma^T Q\Gamma - Q\Gamma - \Gamma^T Q) & \frac{1}{N}(\Gamma^T Q\Gamma - Q\Gamma - \Gamma^T Q) & \cdots & \frac{1}{N}(\Gamma^T Q\Gamma - Q\Gamma - \Gamma^T Q) \\ \frac{1}{N}(\Gamma^T Q\Gamma - Q\Gamma - \Gamma^T Q) & Q+\frac{1}{N}(\Gamma^T Q\Gamma - Q\Gamma - \Gamma^T Q) & \cdots & \frac{1}{N}(\Gamma^T Q\Gamma - Q\Gamma - \Gamma^T Q) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N}(\Gamma^T Q\Gamma - Q\Gamma - \Gamma^T Q) & \frac{1}{N}(\Gamma^T Q\Gamma - Q\Gamma - \Gamma^T Q) & \cdots & Q+\frac{1}{N}(\Gamma^T Q\Gamma - Q\Gamma - \Gamma^T Q) \end{pmatrix}_{(Nn\times Nn)} \\ &= \begin{pmatrix} Q & 0 & \cdots & 0 \\ 0 & Q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q \end{pmatrix} + \frac{1}{N}\begin{pmatrix} \hat{Q} & \cdots & \hat{Q} \\ \vdots & \ddots & \vdots \\ \hat{Q} & \cdots & \hat{Q} \end{pmatrix} - \frac{1}{N}\begin{pmatrix} Q & \cdots & Q \\ \vdots & \ddots & \vdots \\ Q & \cdots & Q \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}_{(Nn\times 1)}, \\ \mathbf{G} &= \begin{pmatrix} G & 0 & \cdots & 0 \\ 0 & G & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G \end{pmatrix} + \frac{1}{N}\begin{pmatrix} \hat{G} & \cdots & \hat{G} \\ \vdots & \ddots & \vdots \\ \hat{G} & \cdots & \hat{G} \end{pmatrix} - \frac{1}{N}\begin{pmatrix} G & \cdots & G \\ \vdots & \ddots & \vdots \\ G & \cdots & G \end{pmatrix}, \quad \mathbf{S}_1 = -\begin{pmatrix} \Gamma^T Q\eta - Q\eta \\ \vdots \\ \Gamma^T Q\eta - Q\eta \end{pmatrix}_{(Nn\times 1)}, \\ \mathbf{S}_2 &= -\begin{pmatrix} \bar{\Gamma}^T G\bar{\eta} - G\bar{\eta} \\ \vdots \\ \bar{\Gamma}^T G\bar{\eta} - G\bar{\eta} \end{pmatrix}_{(Nn\times 1)}, \mathbf{R} = \begin{pmatrix} R & 0 & \cdots & 0 \\ 0 & R & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R \end{pmatrix}_{(Nn\times Nn)}, \quad \mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}_{(Nm\times 1)}, \end{aligned} \tag{2.7}$$

where $\hat{Q} := (\Gamma - I)^T Q(\Gamma - I)$, $\hat{G} := (\bar{\Gamma} - I)^T G(\bar{\Gamma} - I)$.

Through (2.4) and (2.6), one can see that Problem 2.1 is actually an $N \times n$-dimensional standard control problem. Using the method in [58], the solvability and the optimal control can be derived. However, usually the population N is large in practical application. This brings great computational complexity due to the "curse of dimensional". Thus, in what follows, some low-dimensional criteria for the convexity of the cost functional will be studied.

2.2.1 Case 1: For Q, G, R are positive semi-definite

We start with the simplest case that the weighting coefficients are all positive semi-definite. For the convexity of the cost functional, it follows that

Proposition 2.1. *Under* (A2.1)-(A2.2) *and* $Q, G, R \geq 0$, $\mathbf{u} \mapsto \mathcal{J}_{soc}^{(N)}(\xi_0; \mathbf{u})$ *is convex with respect to* $\mathbf{u}$. *Moreover, if* $R \gg 0$, *then* $\mathbf{u} \mapsto \mathcal{J}_{soc}^{(N)}(\xi_0; \mathbf{u})$ *is uniformly convex.*

Proof. Under the assumption $Q, G, R \geq 0$, we know $\begin{pmatrix} \hat{Q} & \cdots & \hat{Q} \\ \vdots & \ddots & \vdots \\ \hat{Q} & \cdots & \hat{Q} \end{pmatrix} \geq 0$, $\begin{pmatrix} \hat{G} & \cdots & \hat{G} \\ \vdots & \ddots & \vdots \\ \hat{G} & \cdots & \hat{G} \end{pmatrix} \geq 0$ and $\mathbf{R} \geq 0$. By the definition of positive semi-definiteness, we can obtain the following two inequalities:

$$\begin{pmatrix} Q & 0 & \cdots & 0 \\ 0 & Q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q \end{pmatrix} - \frac{1}{N} \begin{pmatrix} Q & \cdots & Q \\ \vdots & \ddots & \vdots \\ Q & \cdots & Q \end{pmatrix} \geq 0, \quad \begin{pmatrix} G & 0 & \cdots & 0 \\ 0 & G & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G \end{pmatrix} - \frac{1}{N} \begin{pmatrix} G & \cdots & G \\ \vdots & \ddots & \vdots \\ G & \cdots & G \end{pmatrix} \geq 0.$$

Thus, $\mathbf{Q}, \mathbf{G} \geq 0$ and the convexity of $\mathcal{J}_{soc}^{(N)}$ would follow. Moreover, if $R \gg 0$, then $\mathbf{R} \gg 0$ and $\mathcal{J}_{soc}^{(N)}$ is uniformly convex. $\square$

Next, we consider two more general situations.

2.2.2 Case 2: For $\mathbf{F} = \tilde{\mathbf{F}} = 0$, Q, R and G could be indefinite

In this case, the agent's state will not be influenced by others (see [27]). The agents' state dynamics are decoupled. For the weighting coefficients of cost functional, we have the following proposition.

Proposition 2.2. *Suppose that $Q - \hat{Q} \geq 0$ and $G - \hat{G} \geq 0$, then for any ΔQ, $\Delta G \in \mathbb{S}^n$ such that $\Delta Q \geq Q - \hat{Q}$ and $\Delta G \geq G - \hat{G}$, we have $\mathbf{Q} - \mathbf{Q}_2 \geq 0$ and $\mathbf{G} - \mathbf{G}_2 \geq 0$ where*

$$\mathbf{Q}_2 = \begin{pmatrix} Q-\Delta Q & 0 & \cdots & 0 \\ 0 & Q-\Delta Q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q-\Delta Q \end{pmatrix}, \quad \mathbf{G}_2 = \begin{pmatrix} G-\Delta G & 0 & \cdots & 0 \\ 0 & G-\Delta G & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G-\Delta G \end{pmatrix}. \tag{2.8}$$

Proof. Consider matrices $\mathbf{Q} - \mathbf{Q}_2$ and $\mathbf{G} - \mathbf{G}_2$ which are

$$\mathbf{Q} - \mathbf{Q}_2 = \begin{pmatrix} \Delta Q+\frac{1}{N}\hat{Q}-\frac{1}{N}Q & \frac{1}{N}\hat{Q}-\frac{1}{N}Q & \cdots & \frac{1}{N}\hat{Q}-\frac{1}{N}Q \\ \frac{1}{N}\hat{Q}-\frac{1}{N}Q & \Delta Q+\frac{1}{N}\hat{Q}-\frac{1}{N}Q & \cdots & \frac{1}{N}\hat{Q}-\frac{1}{N}Q \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N}\hat{Q}-\frac{1}{N}Q & \frac{1}{N}\hat{Q}-\frac{1}{N}Q & \cdots & \Delta Q+\frac{1}{N}\hat{Q}-\frac{1}{N}Q \end{pmatrix},$$

$$\mathbf{G} - \mathbf{G}_2 = \begin{pmatrix} \Delta G+\frac{1}{N}\hat{G}-\frac{1}{N}G & \frac{1}{N}\hat{G}-\frac{1}{N}G & \cdots & \frac{1}{N}\hat{G}-\frac{1}{N}G \\ \frac{1}{N}\hat{G}-\frac{1}{N}G & \Delta G+\frac{1}{N}\hat{G}-\frac{1}{N}G & \cdots & \frac{1}{N}\hat{G}-\frac{1}{N}G \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N}\hat{G}-\frac{1}{N}G & \frac{1}{N}\hat{G}-\frac{1}{N}G & \cdots & \Delta G+\frac{1}{N}\hat{G}-\frac{1}{N}G \end{pmatrix}.$$

If $Q-\hat{Q}\geq 0$ holds, then for any non-zeros vector $(x_1^T,\cdots,x_N^T)^T\in\mathbb{R}^{Nn\times 1}$ and any matrix $\Delta Q\geq Q-\hat{Q}$, it holds that

$$
\begin{aligned}
&\begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_N\end{pmatrix}^T
\begin{pmatrix}
\Delta Q+\frac{1}{N}\hat{Q}-\frac{1}{N}Q & \frac{1}{N}\hat{Q}-\frac{1}{N}Q & \cdots & \frac{1}{N}\hat{Q}-\frac{1}{N}Q\\
\frac{1}{N}\hat{Q}-\frac{1}{N}Q & \Delta Q+\frac{1}{N}\hat{Q}-\frac{1}{N}Q & \cdots & \frac{1}{N}\hat{Q}-\frac{1}{N}Q\\
\vdots & \vdots & \ddots & \vdots\\
\frac{1}{N}\hat{Q}-\frac{1}{N}Q & \frac{1}{N}\hat{Q}-\frac{1}{N}Q & \cdots & \Delta Q+\frac{1}{N}\hat{Q}-\frac{1}{N}Q
\end{pmatrix}
\begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_N\end{pmatrix}\\
=&(x_1^T\Delta Q x_1+\cdots+x_N^T\Delta Q x_N)+\frac{1}{N}(x_1+\cdots+x_N)^T\hat{Q}(x_1+\cdots+x_N)\\
&-\frac{1}{N}(x_1+\cdots+x_N)^TQ(x_1+\cdots+x_N)\geq 0\\
=&(x_1^T\Delta Q x_1+\cdots+x_N^T\Delta Q x_N)-\frac{1}{N}(x_1+\cdots+x_N)^T(Q-\hat{Q})(x_1+\cdots+x_N)\\
\geq&(x_1^T\Delta Q x_1+\cdots+x_N^T\Delta Q x_N)-(x_1^T(Q-\hat{Q})x_1+\cdots+x_N^T(Q-\hat{Q})x_N)\\
=&x_1^T(\Delta Q-(Q-\hat{Q}))x_1+\cdots+x_N^T(\Delta Q-(Q-\hat{Q}))x_N\geq 0.
\end{aligned}
$$

By similar argument, one can also obtain:

$$
\begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_N\end{pmatrix}^T
\begin{pmatrix}
\Delta G+\frac{1}{N}\hat{G}-\frac{1}{N}G & \frac{1}{N}\hat{G}-\frac{1}{N}G & \cdots & \frac{1}{N}\hat{G}-\frac{1}{N}G\\
\frac{1}{N}\hat{G}-\frac{1}{N}G & \Delta G+\frac{1}{N}\hat{G}-\frac{1}{N}G & \cdots & \frac{1}{N}\hat{G}-\frac{1}{N}G\\
\vdots & \vdots & \ddots & \vdots\\
\frac{1}{N}\hat{G}-\frac{1}{N}G & \frac{1}{N}\hat{G}-\frac{1}{N}G & \cdots & \Delta G+\frac{1}{N}\hat{G}-\frac{1}{N}G
\end{pmatrix}
\begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_N\end{pmatrix}\geq 0.
$$

Thus, $\mathbf{Q}-\mathbf{Q}_2\geq 0$ and $\mathbf{G}-\mathbf{G}_2\geq 0$ and the proposition is proved. $\square$

Consequently, Proposition 2.2 implies

$$
\frac{1}{2}\mathbb{E}\Big\{\int_0^T \mathbf{x}^T\mathbf{Q}\mathbf{x}+\mathbf{u}^T\mathbf{R}\mathbf{u}dt+\mathbf{x}^T(T)\mathbf{G}\mathbf{x}(T)\Big\}\geq\frac{1}{2}\mathbb{E}\Big\{\int_0^T \mathbf{x}^T\mathbf{Q}_2\mathbf{x}+\mathbf{u}^T\mathbf{R}\mathbf{u}dt+\mathbf{x}^T(T)\mathbf{G}_2\mathbf{x}(T)\Big\}. \tag{2.9}
$$

Moreover, by [58], we have the following result:

Proposition 2.3. *$\mathbf{u} \mapsto \mathcal{J}_{soc}^{(N)}(\xi_0;\mathbf{u})$ is (uniformly) convex if and only if $\check{\mathcal{J}}_{soc}^{(N)}(0;\mathbf{u}) \geq 0$ (or $\geq \varepsilon\|\mathbf{u}\|_{L^2}^2$) where*

$$
\begin{cases}
\check{\mathcal{J}}_{soc}^{(N)}(0;\mathbf{u}) = \dfrac{1}{2}\mathbb{E}\Big\{\displaystyle\int_0^T \mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}dt + \mathbf{x}(T)^T\mathbf{G}\mathbf{x}(T)\Big\}, \\
d\mathbf{x} = (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u})dt + \displaystyle\sum_{i=1}^N(\mathbf{C}_i\mathbf{x} + \mathbf{D}_i\mathbf{u})dW_i, \quad \mathbf{x}(0) = 0.
\end{cases}
\tag{2.10}
$$

Motivated by (2.9) and Proposition 2.3, we have the convexity of $\mathcal{J}_{soc}^{(N)}$ as follows:

Proposition 2.4. *Under* (A2.1)-(A2.2)*, $F = \tilde{F} = 0$, $Q - \hat{Q} \geq 0$ and $G - \hat{G} \geq 0$; if there exist some ΔQ, $\Delta G \in \mathbb{S}^n$ such that $\Delta Q \geq Q - \hat{Q}$, $\Delta G \geq G - \hat{G}$ and cost functional $u_i \mapsto \widetilde{J}(0;u_i)$ of the following low-dimensional control problem:*

$$
\begin{cases}
\widetilde{J}(0;u_i) = \dfrac{1}{2}\mathbb{E}\Big\{\displaystyle\int_0^T x_i^T(Q-\Delta Q)x_i + u_i^T R u_i dt + x_i^T(T)(G-\Delta G)x_i(T)\Big\}, \\
s.t. \quad dx_i = (Ax_i + Bu_i)dt + (Cx_i + Du_i)dW_i, \quad x_i(0) = 0,
\end{cases}
\tag{2.11}
$$

is (uniformly) convex, then $\mathbf{u} \mapsto \mathcal{J}_{soc}^{(N)}(\xi_0;\mathbf{u})$ is (uniformly) convex.

Proof. If $u_i \mapsto \widetilde{J}(0;u_i)$ is (uniformly) convex, then by noting that $F = \tilde{F} = 0$, we have $\widetilde{\mathcal{J}}_{soc}^{(N)}(0;\mathbf{u}) = \sum_{i\in\mathcal{I}} \widetilde{J}(0;u_i) \geq 0$ (or $\geq \varepsilon\sum_{i\in\mathcal{I}}\|u_i\|_{L^2}^2 = \varepsilon\|\mathbf{u}\|_{L^2}^2$, for some $\varepsilon > 0$) where

$$
\begin{cases}
\widetilde{\mathcal{J}}_{soc}^{(N)}(0;\mathbf{u}) = \dfrac{1}{2}\mathbb{E}\Big\{\displaystyle\int_0^T \mathbf{x}^T\mathbf{Q}_2\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}dt + \mathbf{x}(T)^T\mathbf{G}_2\mathbf{x}(T)\Big\}, \\
d\mathbf{x} = (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u})dt + \displaystyle\sum_{i=1}^N(\mathbf{C}_i\mathbf{x} + \mathbf{D}_i\mathbf{u})dW_i, \quad \mathbf{x}(0) = 0,
\end{cases}
$$

and $\mathbf{Q}_2$, $\mathbf{G}_2$ are given by (2.8) for any $\Delta Q \geq Q - \hat{Q}$ and $\Delta G \geq G - \hat{G}$. Then by relation (2.9), it follows that $\check{\mathcal{J}}_{soc}^{(N)}(0;\mathbf{u}) \geq \widetilde{\mathcal{J}}_{soc}^{(N)}(0;\mathbf{u}) \geq 0$ (or $\geq \varepsilon \|\mathbf{u}\|_{L^2}^2$) which ends the proof by Proposition 2.3. $\square$

To the best of our knowledge, it is complicated to verify the convexity of the low-dimensional control problem (2.11) generally. However, by [58], the uniform convexity of problem (2.11) is related to the Riccati equation, and then we have the following corollary.

Corollary 2.1. *Under* (A2.1)-(A2.2), $F = \tilde{F} = 0$, $Q - \hat{Q} \geq 0$ *and* $G - \hat{G} \geq 0$*; if there exist some* ΔQ, $\Delta G \in \mathbb{S}^n$ *such that* $\Delta Q \geq Q - \hat{Q}$, $\Delta G \geq G - \hat{G}$ *and the following Riccati equation*

$$\begin{cases} \dot{P}+PA+A^TP+C^TPC+Q-\Delta Q-(PB+C^TPD)(R+D^TPD)^{-1}(B^TP+D^TPC)=0, \\ P(T) = G-\Delta G, \end{cases} \tag{2.12}$$

admits a solution $P \in C([0,T];\mathbb{S}^n)$ *such that* $R+D^TPD \gg 0$*, then the low-dimensional control problem* (2.11) *is uniformly convex. Consequently,* $\mathbf{u} \mapsto \mathcal{J}_{soc}^{(N)}(\xi_0;\mathbf{u})$ *is uniformly convex.*

Next, we present a numerical example to illustrate Proposition 2.4.

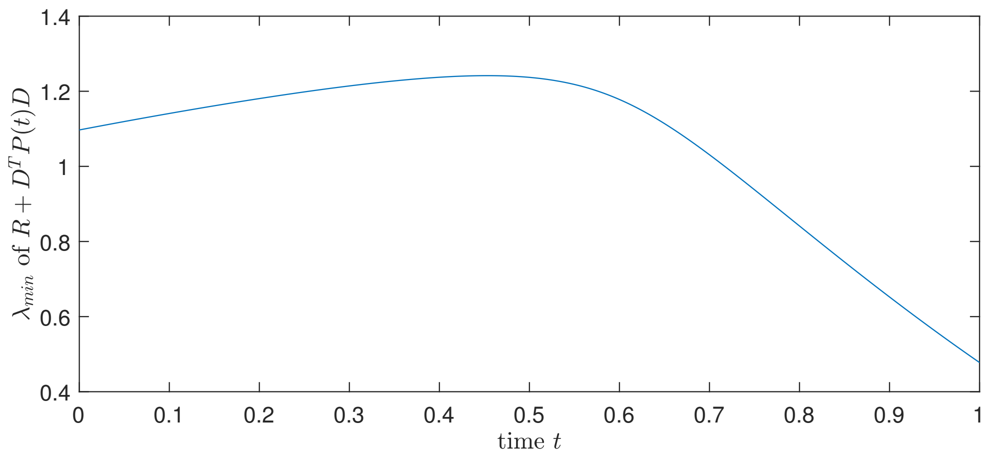
Example 2.1. *We let*

$$A = \left(\begin{smallmatrix} 0.2 & 0.5 \\ 0.4 & 0.1 \end{smallmatrix}\right), B = \left(\begin{smallmatrix} 0.6 & 0.4 \\ 0.2 & 0.6 \end{smallmatrix}\right), C = \left(\begin{smallmatrix} 0.3 & 0.6 \\ 0.3 & 0.3 \end{smallmatrix}\right), D = \left(\begin{smallmatrix} 0.8 & 0.7 \\ 1 & 0.3 \end{smallmatrix}\right), R = \left(\begin{smallmatrix} -0.3 & 0 \\ 0 & 1.7 \end{smallmatrix}\right), Q = \left(\begin{smallmatrix} -0.1 & 0 \\ 0 & 1.5 \end{smallmatrix}\right),$$

$$\hat{Q} = \left(\begin{smallmatrix} -0.3 & 0 \\ 0 & 1.1 \end{smallmatrix}\right), \Delta Q = \left(\begin{smallmatrix} 0.3 & 0 \\ 0 & 0.5 \end{smallmatrix}\right), G = \left(\begin{smallmatrix} -0.2 & 0 \\ 0 & 1.6 \end{smallmatrix}\right), \Delta G = \left(\begin{smallmatrix} 0.3 & 0 \\ 0 & 0.5 \end{smallmatrix}\right), \hat{G} = \left(\begin{smallmatrix} -0.4 & 0 \\ 0 & 1.2 \end{smallmatrix}\right),$$

and time interval is $[0,1]$. *Then,* Q, R *and* G *are all indefinite, and such coefficients satisfy Proposition 3.3, since* $\Delta Q = \left(\begin{smallmatrix} 0.3 & 0 \\ 0 & 0.5 \end{smallmatrix}\right) \geq Q - \hat{Q} = \left(\begin{smallmatrix} 0.2 & 0 \\ 0 & 0.4 \end{smallmatrix}\right) \geq 0$, $\Delta G = \left(\begin{smallmatrix} 0.3 & 0 \\ 0 & 0.5 \end{smallmatrix}\right) \geq G - \hat{G} = \left(\begin{smallmatrix} 0.2 & 0 \\ 0 & 0.4 \end{smallmatrix}\right) \geq 0$.

Thus, if the low-dimensional control problem (2.11) *is (uniformly) convex, then* $\mathcal{J}_{soc}^{(N)}$ *is (uniformly) convex. The related Riccati equation of* (2.11) *is* (2.12). *By solving* (2.12), *we have the trajectory of the minimum eigenvalue of* $R + D^T P(t) D$*:*



Thus, by letting $\lambda = 0.4$, *we have* $R + D^T P(t) D \geq \lambda I$, *and* (2.12) *admits a strongly regular solution. By [58], problem* (2.11) *is uniformly convex, and so is* $\mathcal{J}_{soc}^{(N)}(u)$.

Note that the convexity of a low-dimensional control problem has been well studied in [58] and [59]. Thus, we will not further discuss the convexity of (2.11) here.

Next, the situation: $F, \tilde{F} \neq 0$ will be studied. In this situation, to deal with the terminal term, we also assume that $G \geq 0$ for discussion simplicity.

Case 3: For $G \geq 0,\ F, \tilde{F} \neq 0$

When $F, \tilde{F} \neq 0$, the state of each agent will be influenced by the others (see [30] where $F \neq 0$ is assumed). It is inaccessible to obtain a decoupled low-dimensional problem due to the coupling of the state dynamics. To analyse $\breve{\mathcal{J}}_{soc}^{(N)}(0;\mathbf{u})$, we firstly estimate the L^2 norm of $\mathbf{x}$ which is given by (2.10). Applying Itô's formula to $\mathbf{x}$ and taking expectation, one can obtain:

$$\mathbb{E}\|\mathbf{x}(t)\|^2 = \mathbb{E}\int_0^t \Big[\mathbf{x}^T\Big(\mathbf{A} + \mathbf{A}^T + \sum_{i=1}^N \mathbf{C}_i^T\mathbf{C}_i\Big)\mathbf{x} + \mathbf{x}^T\Big(2\mathbf{B} + 2\sum_{i=1}^N \mathbf{C}_i^T\mathbf{D}_i\Big)\mathbf{u} + \mathbf{u}^T\sum_{i=1}^N \mathbf{D}_i^T\mathbf{D}_i\mathbf{u}\Big]ds, \tag{2.13}$$

where

$$\sum_{i=1}^N \mathbf{C}_i^T\mathbf{C}_i = \begin{pmatrix} C^TC & 0 & \cdots & 0 \\ 0 & C^TC & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C^TC \end{pmatrix} + \frac{1}{N}\begin{pmatrix} (\tilde{F}+C)^T(\tilde{F}+C) & \cdots & (\tilde{F}+C)^T(\tilde{F}+C) \\ \vdots & \ddots & \vdots \\ (\tilde{F}+C)^T(\tilde{F}+C) & \cdots & (\tilde{F}+C)^T(\tilde{F}+C) \end{pmatrix} - \frac{1}{N}\begin{pmatrix} C^TC & \cdots & C^TC \\ \vdots & \ddots & \vdots \\ C^TC & \cdots & C^TC \end{pmatrix} \geq 0,$$

$$\begin{aligned}
\mathbf{A}^T + \mathbf{A} &= \begin{pmatrix} A^T+A & 0 & \cdots & 0 \\ 0 & A^T+A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A^T+A \end{pmatrix} + \frac{1}{N}\begin{pmatrix} F^T+F & \cdots & F^T+F \\ \vdots & \ddots & \vdots \\ F^T+F & \cdots & F^T+F \end{pmatrix} \\
&\leq \begin{pmatrix} A^T+A & 0 & \cdots & 0 \\ 0 & A^T+A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A^T+A \end{pmatrix} + \begin{pmatrix} \lambda_{\max}(F^T+F)I & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{\max}(F^T+F)I \end{pmatrix},
\end{aligned}$$

$$\sum_{i=1}^N \mathbf{D}_i^T\mathbf{D}_i = \begin{pmatrix} D^TD & 0 & \cdots & 0 \\ 0 & D^TD & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D^TD \end{pmatrix}, \quad \sum_{i=1}^N \mathbf{C}_i^T\mathbf{D}_i = \frac{1}{N}\begin{pmatrix} \tilde{F}^TD & \cdots & \tilde{F}^TD \\ \vdots & \ddots & \vdots \\ \tilde{F}^TD & \cdots & \tilde{F}^TD \end{pmatrix} + \begin{pmatrix} C^TD & 0 & \cdots & 0 \\ 0 & C^TD & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C^TD \end{pmatrix},$$

$$
\begin{aligned}
\sum_{i=1}^{N}\mathbf{D}_i^T\mathbf{C}_i\sum_{i=1}^{N}\mathbf{C}_i^T\mathbf{D}_i &= \frac{1}{N}\begin{pmatrix} D^T\tilde{F} & \cdots & D^T\tilde{F} \\ \vdots & \ddots & \vdots \\ D^T\tilde{F} & \cdots & D^T\tilde{F}\end{pmatrix}\frac{1}{N}\begin{pmatrix} \tilde{F}^T D & \cdots & \tilde{F}^T D \\ \vdots & \ddots & \vdots \\ \tilde{F}^T D & \cdots & \tilde{F}^T D\end{pmatrix} \\
&+ \begin{pmatrix} D^T C & 0 & \cdots & 0 \\ 0 & D^T C & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D^T C\end{pmatrix}\frac{1}{N}\begin{pmatrix} \tilde{F}^T D & \cdots & \tilde{F}^T D \\ \vdots & \ddots & \vdots \\ \tilde{F}^T D & \cdots & \tilde{F}^T D\end{pmatrix} \\
&+ \frac{1}{N}\begin{pmatrix} D^T\tilde{F} & \cdots & D^T\tilde{F} \\ \vdots & \ddots & \vdots \\ D^T\tilde{F} & \cdots & D^T\tilde{F}\end{pmatrix}\begin{pmatrix} C^T D & 0 & \cdots & 0 \\ 0 & C^T D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C^T D\end{pmatrix} \\
&+ \begin{pmatrix} D^T C & 0 & \cdots & 0 \\ 0 & D^T C & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D^T C\end{pmatrix}\begin{pmatrix} C^T D & 0 & \cdots & 0 \\ 0 & C^T D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C^T D\end{pmatrix} \\
&= \begin{pmatrix} D^T CC^T D & 0 & \cdots & 0 \\ 0 & D^T CC^T D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D^T CC^T D\end{pmatrix} \\
&+ \frac{1}{N}\begin{pmatrix} D^T(\tilde{F}\tilde{F}^T + C\tilde{F}^T + \tilde{F}C^T)D & \cdots & D^T(\tilde{F}\tilde{F}^T + C\tilde{F}^T + \tilde{F}C^T)D \\ \vdots & \ddots & \vdots \\ D^T(\tilde{F}\tilde{F}^T + C\tilde{F}^T + \tilde{F}C^T)D & \cdots & D^T(\tilde{F}\tilde{F}^T + C\tilde{F}^T + \tilde{F}C^T)D\end{pmatrix}.
\end{aligned}
$$

Moreover, to estimate $\mathbf{x}$, we still need the following result:

Proposition 2.5. *Suppose S is a real symmetric matrix, then for any real vector x, it holds that*

$$
\lambda_{\min}(S)\|x\|^2 \leq x^T Sx \leq \lambda_{\max}(S)\|x\|^2.
$$

Proof. $S \in \mathbb{S}^n$ is a real symmetric matrix. There exists an orthogonal matrix P such that $P^{-1}SP = \Lambda$, where Λ is diagonal and the diagonal elements are the eigenvalues $\lambda_1, \ldots, \lambda_n$ of S. Hence, by letting $P^T x = y = (y_1^T, \cdots, y_n^T)^T$, we have:

$$
x^T Sx = x^T P\Lambda P^T x = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2.
$$

This implies that $\lambda_{\min}(S)\|y\|^2 \leq x^T S x \leq \lambda_{\max}(S)\|y\|^2$. Noting the orthogonality of P, we have $\|y\|^2 = x^T P P^T x = \|x\|^2$ and the result of Proposition 2.5 follows. $\square$

Based on (2.13) and Proposition 2.5, the estimation of $\|\mathbf{x}\|^2_{L^2}$ and the convexity of $\mathcal{J}^{(N)}_{soc}(\xi_0; u)$ can be obtained as follows:

Proposition 2.6. *Under* (A2.1)-(A2.2), $G \geq 0$, $F, \tilde{F} \neq 0$ *and* $Q - \hat{Q} \geq 0$, *if there exist some* $\Delta Q \in \mathbb{S}^n$ *such that* $\Delta Q \geq Q - \hat{Q}$, $\lambda_{\min}(Q - \Delta Q) \leq 0$ *and* $Ke^{2KT}\lambda_{\min}(Q - \Delta Q) + \frac{1}{2}\lambda_{\min}(R) \geq 0$ *(or* $\geq \varepsilon I$ *for some* $\varepsilon > 0$*), then* $\mathbf{u} \mapsto \mathcal{J}^{(N)}_{soc}(\xi_0; \mathbf{u})$ *is convex (or uniformly convex).* K *is given by:*

$$K = \max\Big\{\big(\lambda_{\max}(A^T + A) + \lambda_{\max}(F^T + F)\big), \lambda_{\max}\big(C^T C + (\tilde{F} + C)^T(\tilde{F} + C)\big), \sqrt{\lambda_{\max}(B^T B)}, \sqrt{\lambda_{\max}\big(D^T(\tilde{F}\tilde{F}^T + C\tilde{F}^T + \tilde{F}C^T)D\big) + \lambda_{\max}\big(D^T C C^T D\big)}, \lambda_{\max}(D^T D)\Big\}.$$

Proof. By Proposition 2.5 and noting that the non-zero eigenvalues of $\frac{1}{N}\begin{pmatrix} A & \cdots & A \\ \vdots & \vdots & \vdots \\ A & \cdots & A \end{pmatrix}$, $\begin{pmatrix} A & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & A \end{pmatrix}$, A are the same, we have:

$$\begin{cases}
\mathbf{x}^T\left(\sum_{i=1}^N \mathbf{C}_i^T \mathbf{C}_i\right)\mathbf{x} \leq \lambda_{\max}\Big(C^T C + (\tilde{F} + C)^T(\tilde{F} + C)\Big)\|\mathbf{x}\|^2, \\
\mathbf{x}^T\left(\mathbf{A}^T + \mathbf{A}\right)\mathbf{x} \leq \left[\lambda_{\max}(A^T + A) + \lambda_{\max}(F^T + F)\right]\|\mathbf{x}\|^2, \\
2\mathbf{x}^T \mathbf{B}\mathbf{u} = \langle \mathbf{B}\mathbf{u}, \mathbf{x}\rangle + \langle \mathbf{u}, \mathbf{B}^T\mathbf{x}\rangle \leq \sqrt{\langle \mathbf{B}\mathbf{u}, \mathbf{B}\mathbf{u}\rangle\langle \mathbf{x}, \mathbf{x}\rangle} + \sqrt{\langle \mathbf{u}, \mathbf{u}\rangle\langle \mathbf{B}^T\mathbf{x}, \mathbf{B}^T\mathbf{x}\rangle} \\
\qquad\qquad \leq 2\sqrt{\lambda_{\max}(B^T B)}\|\mathbf{u}\|\|\mathbf{x}\|, \\
2\mathbf{x}^T \sum_{i=1}^N \mathbf{C}_i^T \mathbf{D}_i \mathbf{u} \leq 2\sqrt{\lambda_{\max}\left(\sum_{i=1}^N \mathbf{D}_i^T \mathbf{C}_i \sum_{i=1}^N \mathbf{C}_i^T \mathbf{D}_i\right)}\|\mathbf{u}\|\|\mathbf{x}\| \\
\qquad\qquad \leq 2\sqrt{\lambda_{\max}\big(D^T(\tilde{F}\tilde{F}^T + C\tilde{F}^T + \tilde{F}C^T)D\big) + \lambda_{\max}\big(D^T C C^T D\big)}\|\mathbf{u}\|\|\mathbf{x}\|, \\
\mathbf{u}^T \sum_{i=1}^N \mathbf{D}_i^T \mathbf{D}_i \mathbf{u} \leq \lambda_{\max}(D^T D)\|\mathbf{u}\|^2.
\end{cases}$$

Thus, the estimation below would follow:

$$
\begin{aligned}
\mathbb{E}\|\mathbf{x}(t)\|^2 &= \mathbb{E}\int_0^t \Big[\mathbf{x}^T\Big(\mathbf{A}+\mathbf{A}^T+\sum_{i=1}^N \mathbf{C}_i^T\mathbf{C}_i\Big)\mathbf{x}+\mathbf{x}^T\Big(2\mathbf{B}+2\sum_{i=1}^N \mathbf{C}_i^T\mathbf{D}_i\Big)\mathbf{u}+\mathbf{u}^T\sum_{i=1}^N \mathbf{D}_i^T\mathbf{D}_i\mathbf{u}\Big]ds \\
&\le K\mathbb{E}\int_0^t \Big[\|\mathbf{x}\|^2+2\|\mathbf{u}\|\|\mathbf{x}\|+\|\mathbf{u}\|^2\Big]ds \\
&\le 2K\int_0^t \Big[\mathbb{E}\|\mathbf{x}\|^2+\mathbb{E}\|\mathbf{u}\|^2\Big]ds,
\end{aligned}
$$

where

$$
\begin{aligned}
K = \max\Big\{&\big(\lambda_{\max}(A^T+A)+\lambda_{\max}(F^T+F)\big), \lambda_{\max}\big(C^TC+(\tilde{F}+C)^T(\tilde{F}+C)\big), \sqrt{\lambda_{\max}(B^TB)}, \\
&\sqrt{\lambda_{\max}\big(D^T(\tilde{F}\tilde{F}^T+C\tilde{F}^T+\tilde{F}C^T)D\big)+\lambda_{\max}\big(D^TCC^TD\big)}, \lambda_{\max}(D^TD)\Big\} \ge 0.
\end{aligned}
$$

Further, by Grönwall's inequality, $\mathbb{E}\|\mathbf{x}(t)\|^2 \le 2Ke^{2Kt}\mathbb{E}\|\mathbf{u}(t)\|^2$. Noting that $G \ge 0$ and $\lambda_{\min}(Q-\Delta Q) \le 0$, the estimation of $\check{\mathcal{J}}_{soc}^{(N)}(0;\mathbf{u})$ can be derived as:

$$
\begin{aligned}
&\frac{1}{2}\mathbb{E}\Big\{\int_0^T \mathbf{x}^T\mathbf{Q}\mathbf{x}+\mathbf{u}^T\mathbf{R}\mathbf{u}dt+\mathbf{x}^T(T)\mathbf{G}\mathbf{x}(T)\Big\} \\
\ge&\frac{1}{2}\mathbb{E}\Big\{\int_0^T \mathbf{x}^T\mathbf{Q}_2\mathbf{x}+\mathbf{u}^T\mathbf{R}\mathbf{u}dt\Big\} \ge Ke^{2KT}\lambda_{\min}(Q-\Delta Q)\|\mathbf{u}\|_{L^2}^2+\frac{1}{2}\lambda_{\min}(R)\|\mathbf{u}\|_{L^2}^2,
\end{aligned}
$$

and Proposition 2.6 can be obtained straightforwardly by Proposition 2.3.

Moreover, if we further assume that $Q \le 0$, then $\hat{Q} = (\Gamma - I)^T Q(\Gamma - I) \le 0$. Thus, it follows that $0 \ge \hat{Q} \ge Q - \Delta Q$ and condition $\lambda_{\min}(Q-\Delta Q) \le 0$ will follow as well. □

Remark 2.2. *There is a trade-off among the three uniform convexity conditions presented in Proposition 2.1,2.4,2.6. In Proposition 2.1, the condition $Q, G, R \ge 0$ is the strongest one but easiest to verify. In Proposition 2.4, $F = \tilde{F} = 0$ is assumed and the dynamics of agents' states should be decoupled,* *while in Proposition 2.6 the dynamics of agents' states could be coupled but the weighting matrices in the cost functional is restricted (i.e.,* $\lambda_{\min}(Q - \Delta Q) \leq 0$ *and* $Ke^{2KT}\lambda_{\min}(Q - \Delta Q) + \frac{1}{2}\lambda_{\min}(R) \geq 0$*).*

Through the discussion above, we have studied the (uniform) convexity of the social cost functional. For the sake of simplicity, we introduce the following uniform convexity assumption.

(A2.3) $\mathbf{u} \mapsto \mathcal{J}_{soc}^{(N)}(\xi_0; \mathbf{u})$ is uniformly convex.

Under (A2.1)-(A2.3), $\mathbf{u} \mapsto \mathcal{J}_{soc}^{(N)}(\xi_0; \mathbf{u})$ is uniformly convex and consequently Problem 2.1, 2.2 are unique solvable. In what follows, the problem will be discussed under the uniform convexity assumption. Next, we will apply variational method to the cost functional to derive a set of decentralized strategies based on person-by-person optimality principle.

2.3 Person-by-person optimality

Person-by-person optimality is a critical technique in MF social optima scheme. It has been applied in many recent social optima literature (e.g., [52], [30], [32]). Unlike MFG framework, here the auxiliary control problem can not be derived directly by freezing the state-average directly, since this would bring some ineffective control laws. Thus, in this section, under the person-by-person optimality principle, variation method will be used to analyze the MF approximation.

Consider the optimal centralized strategy $\bar{u} = (\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_N)$ of the LP system. $\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_N)$ denotes the associated optimal state. Perturb $\bar{u}_i$ and keep $\bar{u}_{-i} := (\bar{u}_1, \cdots, \bar{u}_{i-1}, \bar{u}_{i+1}, \cdots, \bar{u}_N)$ fixed. We denote the perturbed strategy as $\bar{u}_i + \delta u_i$ and the associated perturbed states are denoted by $(\bar{x}_1 + \delta x_1, \cdots, \bar{x}_N + \delta x_N)$. The perturbation of the cost functional is denoted by $\delta \mathcal{J}_j := \mathcal{J}_j(\xi_0; u_j, \bar{u}_{-i}) - \mathcal{J}_j(\xi_0; \bar{u}_j, \bar{u}_{-i})$, for $j = 1, \cdots, N$. Thus, the dynamic of the state variation of $\mathcal{A}_i$ is

$$d\delta x_i = (A\delta x_i + B\delta u_i + F\delta x^{(N)})dt + (C\delta x_i + D\delta u_i + \tilde{F}\delta x^{(N)})dW_i, \quad \delta x_i(0) = 0,$$

and the dynamics of the other state variations are

$$d\delta x_j = (A\delta x_j + F\delta x^{(N)})dt + (C\delta x_j + \tilde{F}\delta x^{(N)})dW_j, \quad \delta x_j(0) = 0. \tag{2.14}$$

Sum up (2.14), and we have:

$$\begin{cases} d\delta x^{(-i)} = \Big[A\delta x^{(-i)} + F\dfrac{N-1}{N}(\delta x^{(-i)} + \delta x_i)\Big]dt + \displaystyle\sum_{j\neq i}\Big[C\delta x_j + \dfrac{\tilde{F}}{N}(\delta x^{(-i)} + \delta x_i)\Big]dW_j, \\ \delta x^{(-i)}(0) = 0, \end{cases} \tag{2.15}$$

where $\delta x^{(-i)} := \sum_{j\neq i} \delta x_j$. By some elementary calculations, one can further obtain the variation of $\mathcal{A}_i$ cost functional:

$$\begin{aligned} \delta \mathcal{J}_i =& \mathbb{E}\Big\{ \int_0^T \langle Q(\bar{x}_i - \Gamma\bar{x}^{(N)} - \eta), \delta x_i - \Gamma\delta x^{(N)}\rangle + \langle R\bar{u}_i, \delta u_i\rangle dt \\ &+ \mathbb{E}\langle G(\bar{x}_i(T) - \bar{\Gamma}\bar{x}^{(N)}(T) - \bar{\eta}), \delta x_i(T) - \bar{\Gamma}\delta x^{(N)}(T)\rangle \Big\}, \end{aligned} \tag{2.16}$$

and correspondingly, for $j \neq i$:

$$\begin{aligned} \delta \mathcal{J}_j =& \mathbb{E}\Big\{ \int_0^T \langle Q(\bar{x}_j - \Gamma\bar{x}^{(N)} - \eta), \delta x_j - \Gamma\delta x^{(N)}\rangle dt \\ &+ \mathbb{E}\langle G(\bar{x}_i(T) - \bar{\Gamma}\bar{x}^{(N)}(T) - \bar{\eta}), \delta x_j(T) - \bar{\Gamma}\delta x^{(N)}(T)\rangle \Big\}. \end{aligned} \tag{2.17}$$

Hence, by combining (2.16) and (2.17), the variation of the social cost satisfies:

$$
\begin{aligned}
\delta \mathcal{J}_{soc}^{(N)} :=& \delta \mathcal{J}_i + \sum_{j\neq i} \delta \mathcal{J}_j \\
=& \mathbb{E}\bigg\{ \int_0^T \langle Q(\bar{x}_i - \Gamma \bar{x}^{(N)} - \eta), \delta x_i \rangle - \langle \Gamma^T Q(\bar{x}_i - \Gamma \bar{x}^{(N)} - \eta), \delta x^{(N)} \rangle \\
& + \sum_{j\neq i} \langle Q(\bar{x}_j - \Gamma \bar{x}^{(N)} - \eta), \delta x_j \rangle - \sum_{j\neq i} \langle \Gamma^T Q(\bar{x}_j - \Gamma \bar{x}^{(N)} - \eta), \delta x^{(N)} \rangle \\
& + \langle R\bar{u}_i, \delta u_i \rangle dt + \langle G(\bar{x}_i(T) - \bar{\Gamma} \bar{x}^{(N)}(T) - \bar{\eta}), \delta x_i(T) \rangle \\
& - \langle \bar{\Gamma}^T G(\bar{x}_i(T) - \bar{\Gamma} \bar{x}^{(N)}(T) - \bar{\eta}), \delta x^{(N)}(T) \rangle \\
& + \sum_{i\neq j} \langle G(\bar{x}_j(T) - \bar{\Gamma} \bar{x}^{(N)}(T) - \bar{\eta}), \delta x_j(T) \rangle \\
& - \sum_{i\neq j} \langle \bar{\Gamma}^T G(\bar{x}_j(T) - \bar{\Gamma} \bar{x}^{(N)}(T) - \bar{\eta}), \delta x^{(N)}(T) \rangle \bigg\}.
\end{aligned}
$$

We use MF term $\hat{x}$ to replace $\bar{x}^{(N)}$. Then, we have

$$
\begin{aligned}
\delta \mathcal{J}_{soc}^{(N)} = \mathbb{E}\bigg\{ & \int_0^T \langle Q(\bar{x}_i - \Gamma \hat{x} - \eta), \delta x_i \rangle + \sum_{j\neq i} \frac{1}{N} \langle Q(\bar{x}_j - \Gamma \hat{x} - \eta), N\delta x_j \rangle \\
& - \langle \frac{1}{N} \sum_{j\neq i} \Gamma^T Q(\bar{x}_j - \Gamma \hat{x} - \eta), \delta x_i + \delta x^{(-i)} \rangle + \langle R\bar{u}_i, \delta u_i \rangle dt \\
& + \langle G(\bar{x}_i(T) - \bar{\Gamma} \hat{x}(T) - \bar{\eta}), \delta x_i(T) \rangle \\
& + \sum_{i\neq j} \frac{1}{N} \langle G(\bar{x}_j(T) - \bar{\Gamma} \hat{x}(T) - \bar{\eta}), N\delta x_j(T) \rangle \\
& - \sum_{i\neq j} \frac{1}{N} \langle \bar{\Gamma}^T G(\bar{x}_j(T) - \bar{\Gamma} \hat{x}(T) - \bar{\eta}), \delta x_i(T) + \delta x^{(-i)}(T) \rangle + \varepsilon_1 + \varepsilon_2 \bigg\},
\end{aligned}
$$

where

$$\begin{cases}\varepsilon_1 = \mathbb{E}\Big\{ \displaystyle\int_0^T \langle (\Gamma^T Q\Gamma - Q\Gamma)(\bar{x}^{(N)} - \hat{x}), N\delta x^{(N)}\rangle dt \\ \qquad\qquad + \langle (\bar{\Gamma}^T Q\bar{\Gamma} - Q\bar{\Gamma})(\bar{x}^{(N)}(T) - \hat{x}(T)), N\delta x^{(N)}(T)\rangle \Big\}, \\ \varepsilon_2 = \mathbb{E}\Big\{ \displaystyle\int_0^T -\langle \Gamma^T Q(\bar{x}_i - \Gamma\bar{x}^{(N)} - \eta), \delta x^{(N)}\rangle dt \\ \qquad\qquad - \langle \bar{\Gamma}^T G(\bar{x}_i(T) - \bar{\Gamma}\hat{x}(T) - \bar{\eta}), \delta x^{(N)}(T)\rangle \Big\}. \end{cases} \tag{2.18}$$

For the next step, one can introduce limiting processes x^{**} and x_j^* satisfying:

$$\begin{cases} dx^{**} = (Ax^{**} + F\delta x_i + Fx^{**})dt, \quad x^{**}(0) = 0, \\ dx_j^* = (Ax_j^* + F\delta x_i + Fx^{**})dt + (Cx_j^* + \tilde{F}\delta x_i + \tilde{F}x^{**})dW_j, \quad x_j^*(0) = 0, \end{cases} \tag{2.19}$$

to substitute $\delta x^{(-i)}$ and $N\delta x_j$. This implies

$$\begin{aligned}\delta\mathcal{J}_{soc}^{(N)} = \mathbb{E}\Big\{ \int_0^T &\langle Q(\bar{x}_i - \Gamma\hat{x} - \eta), \delta x_i\rangle + \frac{1}{N}\sum_{j\neq i}\langle Q(\bar{x}_j - \Gamma\hat{x} - \eta), x_j^*\rangle \\ &- \langle \Gamma^T Q((I-\Gamma)\hat{x} - \eta), \delta x_i\rangle - \langle \Gamma^T Q(\hat{x} - \Gamma\hat{x} - \eta), x^{**}\rangle + \langle R\bar{u}_i, \delta u_i\rangle dt \\ &+ \langle G(\bar{x}_i(T) - \bar{\Gamma}\hat{x}(T) - \bar{\eta}), \delta x_i(T)\rangle + \frac{1}{N}\sum_{j\neq i}\langle G(\bar{x}_j(T) - \bar{\Gamma}\hat{x}(T) - \bar{\eta}), x_j^*(T)\rangle \\ &- \langle \bar{\Gamma}^T G((I-\bar{\Gamma})\hat{x}(T) - \bar{\eta}), \delta x_i(T)\rangle - \langle \bar{\Gamma}^T G((I-\bar{\Gamma})\hat{x}(T) - \bar{\eta}), x^{**}(T)\rangle + \sum_{i=1}^4 \varepsilon_i \Big\},\end{aligned} \tag{2.20}$$

where

$$
\begin{cases}
\varepsilon_3 = \mathbb{E}\Big\{ \displaystyle\int_0^T \frac{1}{N}\sum_{j\neq i}\langle Q(\bar{x}_j - \Gamma\hat{x} - \eta), N\delta x_j - x_j^*\rangle - \frac{1}{N}\sum_{j\neq i}\langle \Gamma^T Q(\bar{x}_j - \Gamma\hat{x} - \eta), \delta x^{(-i)} - x^{**}\rangle dt \\
\qquad\qquad + \displaystyle\frac{1}{N}\sum_{j\neq i}\langle G(\bar{x}_j(T) - \bar{\Gamma}\hat{x}(T) - \bar{\eta}), N\delta x_j(T) - x_j^*(T)\rangle \\
\qquad\qquad - \displaystyle\frac{1}{N}\sum_{j\neq i}\langle \bar{\Gamma}^T G(\bar{x}_j(T) - \bar{\Gamma}\hat{x}(T) - \bar{\eta}), \delta x^{(-i)}(T) - x^{**}(T)\rangle \Big\}, \\
\varepsilon_4 = \mathbb{E}\Big\{ \displaystyle\int_0^T -\Big\langle \Gamma^T Q\Big(\frac{\sum_{j\neq i}\bar{x}_j}{N} - \hat{x}\Big), \sum_{i=1}^N \delta x_i \Big\rangle dt - \Big\langle \bar{\Gamma}^T G\Big(\frac{\sum_{j\neq i}\bar{x}_j(T)}{N} - \hat{x}(T)\Big), \sum_{i=1}^N \delta x_i(T)\Big\rangle \Big\}.
\end{cases}
\tag{2.21}
$$

ε_1-ε_4 are actually $o(1)$ order and the rigorous proof will show in Section 2.5.

Remark 2.3. *In previous social optima literature, the limiting processes x^{**}, x_j^* are usually deterministic or even unnecessary. Here,* (2.19) *is a SDEs system, and this difficulty comes from $F, \tilde{F}, C \neq 0$. If $\tilde{F} = C = 0$ (e.g., [52]), then the dynamics of $\delta x^{(-i)}$* (2.15) *and δx_j* (2.14) *become*

$$
\begin{cases}
d\delta x_j = (A\delta x_j + \dfrac{F}{N}\delta x^i + \dfrac{F}{N}\delta x^{(-i)})dt, \quad \delta x_j(0) = 0, \\
d\delta x^{(-i)} = \Big[A\delta x^{(-i)} + F\dfrac{N-1}{N}(\delta x^{(-i)} + \delta x_i)\Big]dt, \quad \delta x^{(-i)}(0) = 0.
\end{cases}
$$

Clearly, δx_j and $\delta x^{(-i)}$ are all deterministic and so are x^{**}, x_j^*.

Further, if $F = \tilde{F} = 0$ (e.g., [27]), then $\delta x^{(-i)}$ and δx_j becomes

$$
\begin{cases}
d\delta x_j = A\delta x_j dt + C\delta x_j dW_j, \quad \delta x_j(0) = 0, \\
d\delta x^{(-i)} = A\delta x^{(-i)}dt + \displaystyle\sum_{j\neq i} C\delta x_j dW_j, \quad \delta x^{(-i)}(0) = 0.
\end{cases}
$$

By the homogeneity, $\delta x_j = \delta x^{(-i)} = 0$ and no limiting approximation is needed. It will bring many technical difficulties that the limiting processes are stochastic.

Mainly, when using dual method in what follows to substitute x^{**} and x_j^*, the processes average will enter the drift term of the dual process, and to substitute it, the residual between the processes average and processes expectation should be estimated in Section 2.5.

Observing (2.20), the direct variation decomposition of $\delta\mathcal{J}_{soc}^{(N)}$ gives rise to terms like $\langle Q(\bar{x}_j - \Gamma\hat{x} - \eta), x_j^*\rangle$, $\left\langle \Gamma^T Q(\hat{x} - \Gamma\hat{x} - \eta), x^{**}\right\rangle$ containing x^{**}, x_j^* which are some intermediate variation terms related to the basic variation term δx_i, δu_i indirectly. This is not what we desire, since this will prevent our construction of an auxiliary LQG control problem. Thus, it is necessary to apply some duality procedure (see [74, 52] for similar duality argument) to break the dependence of $\delta\mathcal{J}_{soc}^{(N)}$ on x^{**}, x_j^*, which enable us to reformulate $\delta\mathcal{J}_{soc}^{(N)}$ being dependent on basic variation δx_i and δu_i only. As a consequence, some auxiliary problem can thus be constructed. We introduce the dual processes y_1^j, y_2 satisfying:

$$\begin{cases} dy_1^j = -[A^T y_1^j + C^T \beta_1^j + Q(\bar{x}_j - \Gamma\hat{x} - \eta)]dt + \beta_1^j dW_j + \displaystyle\sum_{j' \neq j} \beta_1^{j'} dW_{j'}, \\ dy_2 = -[(A+F)^T y_2 + F^T \mathbb{E} y_1^j + \tilde{F}^T \mathbb{E}\beta_1^j - \Gamma^T Q(\hat{x} - \Gamma\hat{x} - \eta)]dt, \\ y_1^j(T) = G(\bar{x}_j(T) - \bar{\Gamma}\hat{x}(T) - \bar{\eta}), \quad y_2(T) = -\bar{\Gamma}^T G((I - \bar{\Gamma})\hat{x}(T) - \bar{\eta}). \end{cases} \tag{2.22}$$

By applying Itô formula, we have the following duality relations:

$$\begin{aligned} &\mathbb{E}\langle G(\bar{x}_j(T) - \bar{\Gamma}\hat{x}(T) - \bar{\eta}), x_j^*(T)\rangle \\ =&\mathbb{E}\langle y_1^j(T), x_j^*(T)\rangle - \mathbb{E}\langle y_1^j(0), x_j^*(0)\rangle \\ =&\mathbb{E}\int_0^T \langle -Q(\bar{x}_j - \Gamma\hat{x} - \eta), x_j^*\rangle + \langle \tilde{F}^T\beta_1^j + F^T y_1^j, x^{**}\rangle + \langle \tilde{F}^T\beta_1^j + F^T y_1^j, \delta x_i\rangle dt, \end{aligned} \tag{2.23}$$

and

$$
\begin{aligned}
&\mathbb{E}\langle -\bar{\Gamma}^T G((I-\bar{\Gamma})\hat{x}(T)-\bar{\eta}), x^{**}(T)\rangle \\
=&\mathbb{E}\langle y_2(T), x^{**}(T)\rangle - \mathbb{E}\langle y_2(0), x^{**}(0)\rangle \\
=&\mathbb{E}\int_0^T \langle \Gamma^T Q(\hat{x}-\Gamma\hat{x}-\eta) - F^T\mathbb{E}y_1^j - \tilde{F}^T\mathbb{E}\beta_1^j, x^{**}\rangle + \langle F^T y_2, \delta x_i\rangle dt.
\end{aligned}
\tag{2.24}
$$

Combining (2.20), (2.23) and (2.24), it holds that

$$
\begin{aligned}
\delta\mathcal{J}_{soc}^{(N)} = \mathbb{E}\int_0^T &\langle Q\bar{x}_i, \delta x_i\rangle + \langle R\bar{u}_i, \delta u_i\rangle - \langle Q(\Gamma\hat{x}+\eta) + \Gamma^T Q((I-\Gamma)\hat{x}-\eta) \\
&- F^T y_2 - \tilde{F}^T\mathbb{E}\beta_1^j - F^T\mathbb{E}y_1^j, \delta x_i\rangle dt + \langle G\bar{x}_i(T), \delta x_i(T)\rangle \\
&- \langle G(\bar{\Gamma}\hat{x}(T)+\bar{\eta}) + \bar{\Gamma}^T G((I-\bar{\Gamma})\hat{x}(T)-\bar{\eta}), \delta x_i(T)\rangle + \sum_{i=1}^5 \varepsilon_i,
\end{aligned}
\tag{2.25}
$$

where

$$
\varepsilon_5 = \mathbb{E}\int_0^T \left\langle \tilde{F}^T\left(\mathbb{E}\beta_1^j - \frac{\sum_{j\neq i}^N \beta_1^j}{N}\right) + F^T\left(\mathbb{E}y_1^j - \frac{\sum_{j\neq i}^N y_1^j}{N}\right), \delta x_i\right\rangle dt. \tag{2.26}
$$

Remark 2.4. *Here, we introduce $N+1$ dual processes to break away $\delta\mathcal{J}_{soc}^{(N)}$ from the dependence on x^{**} and x_j^*. This difficulty is brought by $F, \tilde{F} \neq 0$. By contrast, if $F = \tilde{F} = 0$ (e.g., [27]), then (2.25) becomes*

$$
\begin{aligned}
\delta\mathcal{J}_{soc}^{(N)} = \mathbb{E}\int_0^T &\langle Q\bar{x}_i, \delta x_i\rangle + \langle R\bar{u}_i, \delta u_i\rangle - \langle Q(\Gamma\hat{x}+\eta) + \Gamma^T Q((I-\Gamma)\hat{x}-\eta), \delta x_i\rangle dt \\
&+ \langle G\bar{x}_i(T), \delta x_i(T)\rangle - \langle G(\bar{\Gamma}\hat{x}(T)+\bar{\eta}) + \bar{\Gamma}^T G((I-\bar{\Gamma})\hat{x}(T)-\bar{\eta}), \delta x_i(T)\rangle + \varepsilon.
\end{aligned}
$$

Clearly, in this case, no additional dual process is needed to derive the auxiliary problem.

Moreover, due to $F, \tilde{F} \neq 0$, ε_5 is the residual between the average and the expectation of y_1^j, β_1^j. Generally speaking, to obtain the dynamics of adjoint terms β_1^j is inaccessible. To estimate it, some decoupling method is applied through two Lyapunov equations in Section 2.5.

Therefore, by using MF term $\hat{y}_2$, $\hat{y}_1$, $\hat{\beta}_1$ to replace y_2, $\mathbb{E}y_1^j$, $\mathbb{E}\beta_1^j$ respectively, we can introduce the decentralized auxiliary cost functional variation δJ_i as follows:

$$\begin{aligned}\delta J_i = \mathbb{E}\int_0^T & \langle Q\bar{\alpha}_i, \delta\alpha_i\rangle + \langle R\bar{v}_i, \delta v_i\rangle \\ & -\langle Q(\Gamma\hat{x}+\eta)+\Gamma^T Q[(I-\Gamma)\hat{x}-\eta]-F^T\hat{y}_2-\tilde{F}^T\hat{\beta}_1-F^T\hat{y}_1, \delta\alpha_i\rangle dt \\ & +\langle G\bar{\alpha}_i(T), \delta\alpha_i(T)\rangle - \langle G(\bar{\Gamma}\hat{x}(T)+\bar{\eta})+\bar{\Gamma}^T G[(I-\bar{\Gamma})\hat{x}(T)-\bar{\eta}], \delta\alpha_i(T)\rangle,\end{aligned} \tag{2.27}$$

and

$$\delta\mathcal{J}_{soc}^{(N)} = \delta J_i + \sum_{i=1}^{6}\varepsilon_i,$$

where

$$\begin{aligned}\varepsilon_6 =\mathbb{E}\int_0^T & \langle Q(\bar{x}_i-\bar{\alpha}_i), \delta x_i\rangle + \langle Q\bar{\alpha}_i, \delta x_i - \delta\alpha_i\rangle + \langle R(\bar{u}_i-\bar{v}_i), \delta u_i\rangle + \langle R\bar{v}_i, \delta u_i - \delta v_i\rangle \\ & +\langle F^T(y_2-\hat{y}_2)+\tilde{F}^T(\mathbb{E}\beta_1^j-\hat{\beta}_1)+F^T(\mathbb{E}y_1^j-\hat{y}_i), \delta x_i\rangle \\ & +\langle F^T\hat{y}_2+\tilde{F}^T\hat{\beta}_1+F^T\hat{y}_1-Q(\Gamma\hat{x}+\eta)-\Gamma^T Q[(I-\Gamma)\hat{x}-\eta], \delta x_i-\delta\alpha_i\rangle dt \\ & +\langle G(\bar{x}_i(T)-\bar{\alpha}_i(T)), \delta x_i(T)\rangle + \langle G\bar{\alpha}_i(T), \delta x_i(T)-\delta\alpha_i(T)\rangle \\ & -\langle G(\bar{\Gamma}\hat{x}(T)+\bar{\eta})+\bar{\Gamma}^T G[(I-\bar{\Gamma})\hat{x}(T)-\bar{\eta}], \delta x_i(T)-\delta\alpha_i(T)\rangle.\end{aligned} \tag{2.28}$$

Similarly to ε_1-ε_4, it actually follows that ε_5, ε_6 are also $o(1)$ order. Thus if $\delta J_i = 0$, we have $\delta\mathcal{J}_{soc}^{(N)} \longrightarrow 0$.

2.4 Decentralized strategy design

2.4.1 Auxiliary problem

Motivated by equation (2.27), to achieve $\delta J_i = 0$, one can introduce the following auxiliary control problem:

Problem 2.3. *Minimize* $J_i(v_i)$ *over* $v_i \in \mathcal{U}_i$, *where*

$$\begin{cases} J_i := \dfrac{1}{2}\mathbb{E}\Big\{ \displaystyle\int_0^T \|\alpha_i\|_Q^2 + \|v_i\|_R^2 + 2\langle q_1, \alpha_i\rangle dt + \|\alpha_i(T)\|_G^2 + 2\langle q_2, \alpha_i(T)\rangle \Big\}, \\ d\alpha_i = (A\alpha_i + Bv_i + F\hat{x})dt + (C\alpha_i + Dv_i + \tilde{F}\hat{x})dW_i, \quad \alpha_i(0) = \xi_0, \end{cases}$$

and

$$\begin{cases} q_1 := -Q(\Gamma\hat{x} + \eta) - \Gamma^T Q[(I-\Gamma)\hat{x} - \eta] + F^T\hat{y}_2 + F^T\hat{y}_1 + \tilde{F}^T\hat{\beta}_1, \\ q_2 := -G(\bar{\Gamma}\hat{x}(T) + \bar{\eta}) - \bar{\Gamma}^T G[(I-\bar{\Gamma})\hat{x}(T) - \bar{\eta}]. \end{cases}$$

This is a standard LQ control problem, where the MF terms $\hat{x}$, $\hat{y}_2$, $\hat{y}_1$, $\hat{\beta}_1$ will be determined by the CC system in Section 2.4.2-2.4.3. For the solvability of Problem 2.3, by referring [62], we have the following result.

Lemma 2.1. *Under the* (A2.1)-(A2.3), *if the following Riccati equation:*

$$\begin{cases} \dot{P} + PA + A^T P + C^T PC + Q - (PB + C^T PD)(R + D^T PD)^{-1}(B^T P + D^T PC) = 0, \\ P(T) = G, \end{cases} \tag{2.29}$$

admits a strongly regular solution (i.e., $P \in C([0,T]; \mathbb{S}^n)$ *such that* $R(t) + D^T(t)P(t)D(t) \gg 0$ *a.e.* $t \in [0,T]$*), then for any given* $\hat{x}$, $\hat{y}_2$, $\hat{y}_1$, $\hat{\beta}_1$ $\in$

$L^1(s,T;\mathbb{R}^n)$, the auxiliary control problem (Problem 2.3) admits a unique feedback form optimal control $\bar{v}_i = \Theta_1\bar{\alpha}_i + \Theta_2$, where

$$\Theta_1 := -(R+D^TPD)^{-1}(B^TP+D^TPC), \quad \Theta_2 := -(R+D^TPD)^{-1}(B^T\varphi+D^TP\tilde{F}\hat{x}), \tag{2.30}$$

and φ is the unique solution of

$$\begin{cases} \dot{\varphi}+[A^T-(PB+C^TPD)(R+D^TPD)^{-1}B^T]\varphi \\ \quad -[(PB+C^TPD)(R+D^TPD)^{-1}D^T-C^T]P\tilde{F}\hat{x}+PF\hat{x}+q_1=0, \\ \varphi(T)=q_2. \end{cases} \tag{2.31}$$

$\bar{\alpha}_i$ is the corresponding optimal auxiliary state satisfying:

$$\begin{cases} d\bar{\alpha}_i = \big[(A+B\Theta_1)\bar{\alpha}_i+B\Theta_2+F\hat{x}\big]dt+\big[(C+D\Theta_1)\bar{\alpha}_i+D\Theta_2+\tilde{F}\hat{x}\big]dW_i, \\ \bar{\alpha}_i(0)=\xi_0. \end{cases} \tag{2.32}$$

2.4.2 CC system

Applying the decentralized control law (2.30) to $\mathcal{A}_i$, let $\tilde{x}_i$ be the realized state. By the limiting approximation, the conditions $\mathbb{E}\tilde{x}_i=\hat{x}$, $\mathbb{E}y_1^j=\hat{y}_1$, $\mathbb{E}\beta_1^j=\hat{\beta}_1$ should holds. Then $\tilde{x}_i$ satisfies

$$\begin{cases} d\tilde{x}_i = (A\tilde{x}_i+B(\Theta_1\tilde{x}_i+\Theta_2)+F\tilde{x}^{(N)})dt+(C\tilde{x}_i+D(\Theta_1\tilde{x}_i+\Theta_2)+\tilde{F}\tilde{x}^{(N)})dW_i, \\ \tilde{x}_i(0)=\xi_0, \end{cases} \tag{2.33}$$

and

$$\Theta_1 = -(R+D^TPD)^{-1}(B^TP+D^TPC), \quad \Theta_2 = -(R+D^TPD)^{-1}(B^T\varphi+D^TP\tilde{F}\mathbb{E}\tilde{x}_i),$$

where

$$
\begin{cases}
\dot{\varphi} + [A^T - (PB + C^T PD)(R + D^T PD)^{-1} B^T]\varphi \\
\quad - [(PB + C^T PD)(R + D^T PD)^{-1} D^T - C^T] P\tilde{F}\mathbb{E}\tilde{x}_i + PF\mathbb{E}\tilde{x}_i \\
\quad - Q(\Gamma\mathbb{E}\tilde{x}_i + \eta) - \Gamma^T Q[(I - \Gamma)\mathbb{E}\tilde{x}_i - \eta] + F^T \hat{y}_2 + F^T \hat{y}_1 + \tilde{F}^T \hat{\beta}_1 = 0, \\
\varphi(T) = -G(\bar{\Gamma}\mathbb{E}\tilde{x}_i(T) + \bar{\eta}) - \bar{\Gamma}^T G[(I - \bar{\Gamma})\mathbb{E}\tilde{x}_i(T) - \bar{\eta}].
\end{cases}
$$

The dual processes (2.22) become

$$
\begin{cases}
dy_1^j = -[A^T y_1^j + C^T \beta_1^j + Q(\tilde{x}_j - \Gamma\mathbb{E}\tilde{x}_i - \eta)]dt + \beta_1^j dW_j + \sum\limits_{j' \neq j} \beta_1^{j'} dW_{j'}, \\
dy_2 = -[(A + F)^T y_2 + F^T \mathbb{E}y_1^j + \tilde{F}^T \mathbb{E}\beta_1^j - \Gamma^T Q(\mathbb{E}\tilde{x}_i - \Gamma\mathbb{E}\tilde{x}_i - \eta)]dt, \\
y_1^j(T) = G(\tilde{x}_j(T) - \bar{\Gamma}\mathbb{E}\tilde{x}_i(T) - \bar{\eta}), \quad y_2(T) = -\bar{\Gamma}^T G((I - \bar{\Gamma})\mathbb{E}\tilde{x}_i(T) - \bar{\eta}).
\end{cases}
$$

By taking summation and expectation to (2.33), we have

$$
\begin{cases}
d\tilde{x}^{(N)} = (A\tilde{x}^{(N)} + B(\Theta_1 \tilde{x}^{(N)} + \Theta_2) + F\tilde{x}^{(N)})dt \\
\qquad\qquad + \dfrac{1}{N}\sum\limits_{i=1}^{N} (C\tilde{x}_i + D(\Theta_1 \tilde{x}_i + \Theta_2) + \tilde{F}\tilde{x}^{(N)})dW_i, \quad \tilde{x}^{(N)}(0) = \xi_0, \\
d\mathbb{E}\tilde{x}_i = (A\mathbb{E}\tilde{x}_i + B(\Theta_1 \mathbb{E}\tilde{x} + \Theta_2) + F\mathbb{E}\tilde{x}_i)dt, \quad \mathbb{E}\tilde{x}_i(0) = \xi_0.
\end{cases}
$$

On the other hand, by basic property of Itô integral, for $i \neq j$, we have

$$
\mathbb{E}\left\langle \int_0^t (C\tilde{x}_i + D(\Theta_1 \tilde{x}_i + \Theta_2) + \tilde{F}\tilde{x}^{(N)})dW_i, \int_0^t (C\tilde{x}_j + D(\Theta_1 \tilde{x}_j + \Theta_2) + \tilde{F}\tilde{x}^{(N)})dW_j \right\rangle = 0.
$$

Thus, it follows that

$$
\lim_{N \to +\infty} \mathbb{E}\left\| \frac{1}{N}\sum_{i=1}^{N} \int_0^t (C\tilde{x}_i + D(\Theta_1 \tilde{x}_i + \Theta_2) + \tilde{F}\tilde{x}^{(N)})dW_i \right\|^2 = 0, \text{ in } L^2(0, T; \mathbb{R}^n).
$$

This implies that $\tilde{x}^{(N)} \to \mathbb{E}\tilde{x}_i$ when $N \to \infty$, and correspondingly we denotes $\tilde{x}_i \to \check{x}$, $y_1^j \to \check{y}_1$, $y_2 \to \check{y}_2$. By letting

$$\begin{cases} \Pi_1 = A - B(R + D^T PD)^{-1}(B^T P + D^T PC), \quad \Pi_2 = F - B(R + D^T PD)^{-1} D^T P\tilde{F}, \\ \Pi_3 = -B(R + D^T PD)^{-1} B^T, \\ \Pi_4 = (PB + C^T PD)(R + D^T PD)^{-1} D^T P\tilde{F} - C^T P\tilde{F} + PF + Q\Gamma + \Gamma^T Q(I - \Gamma), \\ \Pi_1' = C - D(R + D^T PD)^{-1}(B^T P + D^T PC), \\ \Pi_2' = \tilde{F} - D(R + D^T PD)^{-1} D^T P\tilde{F}, \quad \Pi_3' = -D(R + D^T PD)^{-1} B^T, \end{cases} \tag{2.34}$$

we have the following CC system

$$\begin{cases} d\check{x} = (\Pi_1 \check{x} + \Pi_2 \mathbb{E}\check{x} + \Pi_3 \check{\varphi})dt + (\Pi_1' \check{x} + \Pi_2' \mathbb{E}\check{x} + \Pi_3' \check{\varphi})dB(t), \\ d\check{\varphi} = (-\Pi_1^T \check{\varphi} + \Pi_4 \mathbb{E}\check{x} - F^T \check{y}_2 - F^T \mathbb{E}\check{y}_1 - \tilde{F}^T \mathbb{E}\check{\beta}_1 + Q\eta - \Gamma^T Q\eta)dt, \\ d\check{y}_1 = (-Q\check{x} + Q\Gamma\mathbb{E}\check{x} - A^T \check{y}_1 - C^T \check{\beta}_1 + Q\eta)dt + \check{\beta}_1 dB(t), \\ d\check{y}_2 = [\Gamma^T Q(I - \Gamma)\mathbb{E}\check{x} - F^T \mathbb{E}\check{y}_1 - \tilde{F}^T \mathbb{E}\check{\beta}_1 - (A + F)^T \check{y}_2 - \Gamma^T Q\eta]dt, \\ \check{x}(0) = \xi_0, \quad \check{\varphi}(T) = -G(\bar{\Gamma}\mathbb{E}\check{x}(T) + \bar{\eta}) - \bar{\Gamma}^T G[(I - \bar{\Gamma})\mathbb{E}\check{x}(T) - \bar{\eta}], \\ \check{y}_1(T) = G(\check{x}(T) - \bar{\Gamma}\mathbb{E}\check{x}(T) - \bar{\eta}), \quad \check{y}_2(T) = -\bar{\Gamma}^T G((I - \bar{\Gamma})\mathbb{E}\check{x}(T) - \bar{\eta}), \end{cases} \tag{2.35}$$

where $B(t)$ is a generic Brownian motion. Due to the symmetry and decentralization, only one generic Brownian motion $B(t)$ is needed to characterize the CC system here. Moreover, the MF terms $\hat{x}$, $\hat{y}_1$, $\hat{\beta}_1$ can be determined by $\hat{x} = \mathbb{E}\check{x}$, $\hat{y}_1 = \mathbb{E}\check{y}_1$, $\hat{\beta}_1 = \mathbb{E}\check{\beta}_1$, $\hat{y}_2 = \mathbb{E}\check{y}_2 = \check{y}_2$.

Remark 2.5. *This CC system is a highly coupled MF-FBSDEs system. It is different to a general CC system (e.g., [17], [63], [43]), since the adjoint terms (i.e., $C^T \check{\beta}_1$, $\tilde{F}^T \mathbb{E}\check{\beta}_1$) here enter the drift term. In general situation, by taking expectation on the realized state, an ODEs-type CC system would be obtained.*

However, in our system here, $C^T\check{\beta}_1$, $\tilde{F}^T\mathbb{E}\check{\beta}_1$ can not be determined. Thus, the dynamics of $\hat{x}$, $\hat{y}_1$ and $\hat{\beta}_1$ cannot be obtained directly, and we can only represent them in an embedded way via (2.35).

2.4.3 Solvability of CC system

CC system (2.35) is a highly coupled MF-FBSDEs system. To study its global solvability, in what follows, we apply some decentralizing method (see [61]) to simplify it. We start with rewriting the CC system as follows:

$$
\begin{cases}
dX = \big(A_1X + \bar{A}_1\mathbb{E}X + B_1Y\big)\,dt + \big(A_1'X + \bar{A}_1'\mathbb{E}X + B_1'Y\big)\,dB(t),\\
dY = \big(A_2X + \bar{A}_2\mathbb{E}X + B_2Y + \bar{B}_2\mathbb{E}Y + C_2Z + \bar{C}_2\mathbb{E}Z + f\big)\,dt + ZdB(t),\\
X(0) = \bar{\xi}_0, \quad Y(T) = \bar{G}X(T) + \bar{G}'\mathbb{E}X(T) + g,
\end{cases}
\tag{2.36}
$$

where

$$
\begin{cases}
X := \begin{pmatrix} \check{x} \\ 0 \\ 0 \end{pmatrix}, Y := \begin{pmatrix} \check{\varphi} \\ \check{y}_1 \\ \check{y}_2 \end{pmatrix}, Z := \begin{pmatrix} 0 \\ \beta_1 \\ 0 \end{pmatrix}, \bar{\xi}_0 := \begin{pmatrix} \xi_0 \\ 0 \\ 0 \end{pmatrix},\\
A_1 := \begin{pmatrix} \Pi_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \bar{A}_1 := \begin{pmatrix} \Pi_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B_1 := \begin{pmatrix} \Pi_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_1' := \begin{pmatrix} \Pi_1' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \bar{A}_1' := \begin{pmatrix} \Pi_2' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\\
B_1' := \begin{pmatrix} \Pi_3' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_2 := \begin{pmatrix} 0 & 0 & 0 \\ -Q & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \bar{A}_2 := \begin{pmatrix} \Pi_4 & 0 & 0 \\ Q\Gamma & 0 & 0 \\ -\Gamma^TQ(I-\Gamma) & 0 & 0 \end{pmatrix},\\
B_2 := \begin{pmatrix} -\Pi_1^T & 0 & F^T \\ 0 & -A^T & 0 \\ 0 & 0 & -(A+F)^T \end{pmatrix}, \bar{B}_2 := \begin{pmatrix} 0 & -F^T & 0 \\ 0 & 0 & 0 \\ 0 & -F^T & 0 \end{pmatrix}, C_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & -C^T & 0 \\ 0 & 0 & 0 \end{pmatrix}, \bar{C}_2 := \begin{pmatrix} 0 & -\tilde{F}^T & 0 \\ 0 & 0 & 0 \\ 0 & -\tilde{F}^T & 0 \end{pmatrix},\\
f := \begin{pmatrix} Q\eta-\Gamma^TQ\eta \\ Q\eta \\ \Gamma^TQ\eta \end{pmatrix}, \bar{G} := \begin{pmatrix} 0 & 0 & 0 \\ G & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \bar{G}' := \begin{pmatrix} [-G\bar{\Gamma}-\bar{\Gamma}^TG(I-\bar{\Gamma})] & 0 & 0 \\ -G\bar{\Gamma} & 0 & 0 \\ -\bar{\Gamma}^TG(I-\bar{\Gamma}) & 0 & 0 \end{pmatrix}, g := \begin{pmatrix} \bar{\Gamma}^TG\bar{\eta}-G\bar{\eta} \\ -G\bar{\eta} \\ \bar{\Gamma}^TG\bar{\eta} \end{pmatrix}.
\end{cases}
$$

By taking expectation of (2.36), one can obtain the following decentralized system

$$\begin{cases} d\mathbb{E}X = \big[(A_1 + \bar{A}_1)\mathbb{E}X + B_1\mathbb{E}Y\big]\,dt, \quad \mathbb{E}X(0) = \bar{\xi}_0, \\ d(X - \mathbb{E}X) = [A_1(X - \mathbb{E}X) + B_1(Y - \mathbb{E}Y)]\,dt \\ \qquad\qquad + \big[A_1'(X - \mathbb{E}X) + (A_1' + \bar{A}_1')\mathbb{E}X + B_1'(Y - \mathbb{E}Y) + B_1'\mathbb{E}Y\big]\,dB(t), \\ d\mathbb{E}Y = \big[(A_2 + \bar{A}_2)\mathbb{E}X + (B_2 + \bar{B}_2)\mathbb{E}Y + (C_2 + \bar{C}_2)\mathbb{E}Z + f\big]\,dt, \\ d(Y - \mathbb{E}Y) = [A_2(X - \mathbb{E}X) + B_2(Y - \mathbb{E}Y) + C_2(Z - \mathbb{E}Z)]\,dt + ZdB(t), \\ (X - \mathbb{E}X)(0) = 0, \quad \mathbb{E}Y(T) = (\bar{G} + \bar{G}')\mathbb{E}X(T) + g, \quad Y(T) - \mathbb{E}Y(T) = \bar{G}(X - \mathbb{E}X)(T). \end{cases} \tag{2.37}$$

Motivated by (2.37), we introduce the following FBSDEs system

$$\begin{cases} dX_1 = \big[(A_1 + \bar{A}_1)X_1 + B_1Y_1\big]\,dt, \\ dX_2 = [A_1X_2 + B_1Y_2]\,dt + \big[A_1'X_2 + (A_1' + \bar{A}_1')X_1 + B_1'Y_2 + B_1'Y_1\big]\,dB(t), \\ dY_1 = \big[(A_2 + \bar{A}_2)X_1 + (B_2 + \bar{B}_2)Y_1 + (C_2 + \bar{C}_2)\mathbb{E}Z + f\big]\,dt, \\ dY_2 = [A_2X_2 + B_2Y_2 + C_2(Z - \mathbb{E}Z)]\,dt + ZdB(t), \\ X_1(0) = \bar{\xi}_0, \; X_2(0) = 0, \; Y_1(T) = (\bar{G} + \bar{G}')X_1(T) + g, \; Y_2(T) = \bar{G}X_2(T). \end{cases} \tag{2.38}$$

By comparing (2.36) and (2.38), we have the following result.

Proposition 2.7. *If*

$$\det\left((0, I)\Phi(T, 0)\begin{pmatrix} 0 \\ I \end{pmatrix}\right) \neq 0, \tag{2.39}$$

where Φ is the transmission matrix w.r.t. $\left(\begin{smallmatrix} A_1 & B_1 \\ A_2 - \bar{G}A_1 + (B_2 - \bar{G}B_1)\bar{G} & B_2 - \bar{G}B_1 \end{smallmatrix}\right)$, then the MF-FBSDEs system (2.36) *is equivalent to the FBSDEs system* (2.38).

Proof. Suppose that (X, Y, Z) is the adapted solution of system (2.36). Let $X_1 = \mathbb{E}X$, $Y_1 = \mathbb{E}Y$, $X_2 = X - \mathbb{E}X$, $Y_2 = Y - \mathbb{E}Y$, and it is easy to verify that X_1, Y_1, X_2, Y_2 satisfy the system (2.38). Conversely, if (X_1, Y_1, X_2, Y_2) is the solutions of system (2.38), then let $X = X_1 + X_2$, $Y = Y_1 + Y_2$, and we have

$$\begin{cases} dX = \big(A_1 X + \bar{A}_1 X_1 + B_1 Y\big)\,dt + \big(A_1' X + \bar{A}_1' X_1 + B_1' Y\big)\,dB(t), \\ dY = \big(A_2 X + \bar{A}_2 X_1 + B_2 Y + \bar{B}_2 Y_1 + C_2 Z + \bar{C}_2 \mathbb{E}Z + f\big)\,dt + Z dB(t), \\ X(0) = \bar{\xi}_0, \quad Y(T) = \bar{G}X(T) + \bar{G}' X_1(T) + g. \end{cases}$$

Thus, we just need to verify that $X_1 = \mathbb{E}(X_1 + X_2)$ and $Y_1 = \mathbb{E}(Y_1 + Y_2)$. Considering the expectation of X_2 and Y_2, it follows that

$$\begin{cases} d\mathbb{E}X_2 = [A_1 \mathbb{E}X_2 + B_1 \mathbb{E}Y_2]\,dt, \\ d(\mathbb{E}Y_2 - \bar{G}\mathbb{E}X_2) = \big[(A_2 - \bar{G}A_1 + (B_2 - \bar{G}B_1)\bar{G})\mathbb{E}X_2 + (B_2 - \bar{G}B_1)(\mathbb{E}Y_2 - \bar{G}\mathbb{E}X_2)\big]\,dt, \\ \mathbb{E}X_2(0) = 0, \quad (\mathbb{E}Y_2 - \bar{G}\mathbb{E}X_2)(T) = 0. \end{cases}$$

Clearly, if

$$\det\left((0, I)\Phi(T, 0)\begin{pmatrix} 0 \\ I \end{pmatrix}\right) \neq 0,$$

then $\mathbb{E}X_2 = \mathbb{E}Y_2 \equiv 0$. Besides, X_1 and Y_1 are deterministic, which implies $X_1 = \mathbb{E}(X_1 + X_2)$ and $Y_1 = \mathbb{E}(Y_1 + Y_2)$. $\square$

To study the solvability of (2.38), we firstly rewrite it as following compact form:

$$\begin{cases} d\tilde{X} = (\tilde{A}_1 \tilde{X} + \tilde{B}_1 \tilde{Y})dt + (\tilde{A}_1' \tilde{X} + \tilde{B}_1' \tilde{Y})dB(t), \quad \tilde{X}(0) = \tilde{\xi}_0, \\ d\tilde{Y} = (\tilde{A}_2 \tilde{X} + \tilde{B}_2 \tilde{Y} + \tilde{C}_2 \tilde{Z} + \tilde{\bar{C}}_2 \mathbb{E}\tilde{Z} + \tilde{f})dt + \tilde{Z} dB(t), \quad \tilde{Y}(T) = \tilde{G}\tilde{X} + \tilde{g}, \end{cases} \tag{2.40}$$

where $\tilde{X} = \binom{X_1}{X_2}$, $\tilde{Y} = \binom{Y_1}{Y_2}$, $\tilde{A}_1 = \begin{pmatrix} A_1+\bar{A}_1 & 0 \\ 0 & A_1 \end{pmatrix}$, $\tilde{B}_1 = \begin{pmatrix} B_1 & 0 \\ 0 & B_1 \end{pmatrix}$, $\tilde{A}_1' = \begin{pmatrix} 0 & 0 \\ A_1'+\bar{A}_1' & A_1' \end{pmatrix}$, $\tilde{B}_1' = \begin{pmatrix} 0 & 0 \\ B_1' & B_1' \end{pmatrix}$, $\tilde{A}_2 = \begin{pmatrix} A_2+\bar{A}_2 & 0 \\ 0 & A_2 \end{pmatrix}$, $\tilde{B}_2 = \begin{pmatrix} B_2+\bar{B}_2 & 0 \\ 0 & B_2 \end{pmatrix}$, $\tilde{C}_2 = \begin{pmatrix} 0 & 0 \\ 0 & C_2 \end{pmatrix}$, $\tilde{\bar{C}}_2 = \begin{pmatrix} 0 & C_2+\bar{C}_2 \\ 0 & -C_2 \end{pmatrix}$, $\tilde{G} = \begin{pmatrix} \bar{G}+\bar{G}' & 0 \\ 0 & \bar{G} \end{pmatrix}$, $\tilde{f} = \binom{f}{0}$, $\tilde{\xi}_0 = \binom{\bar{\xi}_0}{0}$, $\tilde{g} = \binom{g}{0}$. (2.40) is a FBSDEs system and its solvability can be obtained by decoupling method. Through introducing a Riccati equation, we have the following lemma.

Lemma 2.2. *Under the* (A2.1)-(A2.2), *if the following Riccati equation*

$$\begin{cases} -\dot{K} + \tilde{B}_2 K + (\tilde{C}_2 + \tilde{\bar{C}}_2) K \tilde{A}_1' + (\tilde{C}_2 + \tilde{\bar{C}}_2) K \tilde{B}_1' K - K\tilde{A}_1 - K\tilde{B}_1 + \tilde{A}_2 = 0, \\ K(T) = \tilde{G}, \end{cases} \tag{2.41}$$

admits a unique solution and condition (2.39) *holds, then the system* (2.40) *admits a unique solution, and equivalently, the CC system* (2.35) *admits a unique solution.*

Proof. Let $\tilde{Y} = K\tilde{X} + \kappa$. By Itô formula, we have

$$\begin{aligned} &(\tilde{A}_2\tilde{X} + \tilde{B}_2(K\tilde{X} + \kappa) + \tilde{C}_2\tilde{Z} + \tilde{\bar{C}}_2\mathbb{E}\tilde{Z} + \tilde{f})dt + \tilde{Z}dB(t) = d\tilde{Y} \\ =& dK \times \tilde{X} + K \times d\tilde{X} + d\kappa \\ =& dK \times \tilde{X} + (K\tilde{A}_1\tilde{X} + K\tilde{B}_1K\tilde{X} + K\tilde{B}_1\kappa)dt + (K\tilde{A}_1'\tilde{X} + K\tilde{B}_1'K\tilde{X} + K\tilde{B}_1'\kappa)dB(t) + d\kappa. \end{aligned}$$

Comparing the diffusion term, one can obtain

$$\tilde{Z} = K\tilde{A}_1'\tilde{X} + K\tilde{B}_1'K\tilde{X} + K\tilde{B}_1'\kappa,$$

which implies

$$\tilde{\bar{C}}_2\mathbb{E}\tilde{Z} = \tilde{\bar{C}}_2K\tilde{A}_1'\mathbb{E}\tilde{X} + \tilde{\bar{C}}_2K\tilde{B}_1'K\mathbb{E}\tilde{X} + \tilde{\bar{C}}_2K\tilde{B}_1'\kappa.$$

Comparing the drift terms, we obtain

$$
\begin{aligned}
&\tilde{A}_2\mathbb{E}\tilde{X} + \tilde{B}_2K\mathbb{E}\tilde{X} + \tilde{B}_2\kappa + (\tilde{C}_2 + \tilde{\tilde{C}}_2)K\tilde{A}_1'\mathbb{E}\tilde{X} + (\tilde{C}_2 + \tilde{\tilde{C}}_2)K\tilde{B}_1'K\mathbb{E}\tilde{X} + (\tilde{C}_2 + \tilde{\tilde{C}}_2)K\tilde{B}_1'\kappa + \tilde{f} \\
=&\dot{K}\mathbb{E}\tilde{X} + K\tilde{A}_1\mathbb{E}\tilde{X} + K\tilde{B}_1K\mathbb{E}\tilde{X} + K\tilde{B}_1\kappa + \dot{\kappa}.
\end{aligned}
\tag{2.42}
$$

By comparing the coefficients of (2.42), K should be the solution of (2.41) and κ satisfies

$$
-\dot{\kappa} + \big[\tilde{B}_2 + (\tilde{C}_2 + \tilde{\tilde{C}}_2)K\tilde{B}_1' - K\tilde{B}_1\big]\kappa + \tilde{f} = 0, \quad \kappa(T) = \tilde{g}. \tag{2.43}
$$

Under (A2.1)-(A2.2), by [59], equation (2.43) always admits a unique solution. Thus, if the Riccati equation (2.41) admits a unique solution, then the system (2.40) also admits a unique solution. By the equivalence, the CC system (2.35) admits a unique solution as well. □

The Riccati equation (2.41) can be rewritten as follows:

$$
\dot{K} = \tilde{A}_2 + \tilde{B}_2K - K(\tilde{A}_1 + \tilde{B}_1K) + (\tilde{C}_2 + \tilde{\tilde{C}}_2)K(\tilde{A}_1' + \tilde{B}_1'K), \quad K(T) = \tilde{G}, \tag{2.44}
$$

which is not a general symmetric Riccati equation. Thus, it is not always solvable on $[0, T]$. However, explicit solutions can still be obtained in some reduced but non-trivial cases. For example, by [60, Theorem 5.3], we have the following proposition:

Proposition 2.8. *If $\tilde{C}_2 + \tilde{\tilde{C}}_2 = 0$ (i.e., $C = \tilde{F} = 0$) and the following condition holds:*

$$
\left[(0, I)\Psi(T, t)\begin{pmatrix} 0 \\ I \end{pmatrix}\right]^{-1} \in L^1(0, T; \mathbb{R}^{n\times n}), \tag{2.45}
$$

then the Riccati equation (2.44) *admits a unique solution* K *which is given by:*

$$K = -\left[(0, I)\Psi(T, t)\begin{pmatrix} 0 \\ I \end{pmatrix}\right]^{-1}(0, I)\Psi(T, t)\begin{pmatrix} I \\ 0 \end{pmatrix}, \quad t \in [0, T],$$

where $\Psi(t, s)$ *is the fundamental matrix w.r.t.* $\begin{pmatrix} \tilde{A}_1 & \tilde{B}_1 \\ \tilde{A}_2 & \tilde{B}_2 \end{pmatrix}$.

Through the discussion above, we have studied the solvability of the CC system. Besides the decoupling method, the solvability of a FBSDEs system can still be studied through many other techniques, like monotone condition (see [75]) and contraction mapping method (see [76]). However, it is not the main topic we study in this chapter, and we would not discuss it further. Thus, for the sake of discussion simplicity, in what follows, we introduce the assumption:

(A2.4) The CC system (2.35) admit a unique solution $(\check{x}, \check{y}_1, \check{y}_2, \check{\beta}_1)$.

By Lemma 2.1, we have the following theorem.

Theorem 2.1. *Under* (A2.1)-(A2.4)*, if the Riccati equation* (2.1) *admits a strongly regular solution* $P \in C([0, T]; \mathbb{S}^n)$ *such that* $R(t) + D^T(t)P(t)D(t) \gg 0$ *a.e.* $t \in [0, T]$*, then Problem 2.2 admits a feedback form MF decentralized strategy* $\tilde{u}_i = \Theta_1 \tilde{x}_i + \Theta_2$*, where*

$$\Theta_1 := -(R+D^T PD)^{-1}(B^T P+D^T PC), \ \Theta_2 := -(R+D^T PD)^{-1}(B^T \varphi+D^T P\tilde{F}\hat{x}),$$

and φ is the unique solution of

$$\begin{cases}\dot{\varphi}+[A^T-(PB+C^TPD)(R+D^TPD)^{-1}B^T]\varphi\\ \quad -[(PB+C^TPD)(R+D^TPD)^{-1}D^T-C^T]P\tilde{F}\hat{x}+PF\hat{x}\\ \quad -Q(\Gamma\hat{x}+\eta)-\Gamma^TQ[(I-\Gamma)\hat{x}-\eta]+(F^T\hat{y}_2+F^T\hat{y}_1+\tilde{F}^T\hat{\beta}_1)=0,\\ \varphi(T)=-G(\bar{\Gamma}\hat{x}(T)+\bar{\eta})-\bar{\Gamma}^TG[(I-\bar{\Gamma})\hat{x}(T)-\bar{\eta}].\end{cases}$$

MF terms $\hat{x}$, $\hat{y}_2$, $\hat{y}_1$, $\hat{\beta}_1$ are determined by $\hat{x}=\mathbb{E}\check{x}$, $\hat{y}_2=\check{y}_2$, $\hat{y}_1=\mathbb{E}\check{y}_1$, $\hat{\beta}_1=\mathbb{E}\check{\beta}_1$ and $(\check{x},\check{y}_2,\check{y}_1,\check{\beta}_1)$ is the solution of CC system (2.35)*. $\tilde{x}_i$ is realized state satisfying*

$$\begin{cases}d\tilde{x}_i=(A\tilde{x}_i+B(\Theta_1\tilde{x}_i+\Theta_2)+F\tilde{x}^{(N)})dt+(C\tilde{x}_i+D(\Theta_1\tilde{x}_i+\Theta_2)+\tilde{F}\tilde{x}^{(N)})dW_i,\\ \tilde{x}_i(0)=\xi_0,\end{cases} \tag{2.46}$$

where $\tilde{x}^{(N)}:=\frac{1}{N}\sum_{i=1}^N\tilde{x}_i$.

Through the discussion above, the MF decentralized strategy has been characterized in Theorem 2.1. In what follows, the performance of such MF decentralized strategy will be studied. Specifically, the asymptotic optimality will be proved.

2.5 Asymptotic social optimality

In this section, we will prove that the MF decentralized strategy given by Theorem 2.1 is asymptotically optimal. We introduce a new generic approach, which is different from traditional MFG scheme (e.g., [17], [64], [43]), where the authors usually used the auxiliary cost functional as a bridge to obtain the asymptotic optimality. Also, our method is different from some reduced social optima models (e.g., [27], [52], [30]), where the optimality loss could be calculated directly with completing square method. Specifically, in this section we estimate the social optimality loss by studying the Fréchet differential of the social cost functional.

Firstly, we distinguish the realized state of original problem and the optimal state of auxiliary problem. Let $\tilde{u} := (\tilde{u}_1, \cdots, \tilde{u}_N)$ be the MF decentralized strategy given by Theorem 2.1, where $\tilde{u}_i = \Theta_1 \tilde{x}_i + \Theta_2$. The realized decentralized state $\tilde{x}_i$ satisfies (2.46) which depends on the state-average $\tilde{x}^{(N)}$, while the optimal auxiliary state $\bar{\alpha}_i$ satisfies (2.32) which depends on the MF term $\hat{x}$, and the optimal auxiliary control is $\bar{v}_i = \Theta_1 \bar{\alpha}_i + \Theta_2$. Correspondingly, the original cost functional $\mathcal{J}_i$ w.r.t the MF decentralized strategy $\tilde{u}$ is

$$\mathcal{J}_i(\tilde{u}_i, \tilde{u}_{-i}) = \frac{1}{2}\mathbb{E}\bigg\{ \int_0^T \|\tilde{x}_i - \Gamma \tilde{x}^{(N)} - \eta\|_Q^2 + \|\tilde{u}_i\|_R^2 dt + \|\tilde{x}_i(T) - \bar{\Gamma}\tilde{x}^{(N)}(T) - \bar{\eta}\|_G^2 \bigg\},$$

and the auxiliary cost functional J_i w.r.t the optimal auxiliary control $\bar{v}_i$ is

$$J_i(\bar{v}_i) = \frac{1}{2}\mathbb{E}\bigg\{ \int_0^T \|\bar{\alpha}_i - \Gamma \hat{x} - \eta\|_Q^2 + \|\bar{v}_i\|_R^2 dt + \|\bar{\alpha}_i(T) - \bar{\Gamma}\hat{x}(T) - \bar{\eta}\|_G^2 \bigg\}.$$

Next, we present the definition of asymptotic optimality.

Definition 2.1. *A decentralized strategy set* $u^\varepsilon := (u_1^\varepsilon, \cdots, u_N^\varepsilon) \in \mathcal{U}_1 \times \cdots \times \mathcal{U}_N$ *has asymptotic social optimality if*

$$\frac{1}{N}\Big| \inf_{u \in \mathcal{U}_c} \mathcal{J}_{soc}^{(N)}(u) - \mathcal{J}_{soc}^{(N)}(u^\varepsilon) \Big| = \varepsilon(N), \quad \varepsilon(N) \to 0, \text{ when } N \to \infty.$$

2.5.1 Preliminary estimations

To study the asymptotic optimality, we first provide several prior lemmas which will play a significant role in future analysis. In the discussion below, for the sake of notation simplicity, we use L to denote a generic constant whose value may change from line to line and only depend on the coefficients (i.e., A, B, C, D, F, $\tilde{F}$, Γ, η, Q, R, ξ_0).

Lemma 2.3. *Under* (A2.1)-(A2.4)*, there exists some constant L such that*

$$\sup_{1\le i\le N}\mathbb{E}\sup_{0\le t\le T}\|\tilde{x}_i(t)\|^2 \le L, \quad \sup_{0\le t\le T}\mathbb{E}\|\tilde{x}^{(N)}(t)\|^2 \le L, \quad \mathbb{E}\int_0^T \|\tilde{u}_i\|^2 dt \le L,$$

where $\tilde{u}_i = \Theta_1\tilde{x}_i + \Theta_2$.

Proof. By referring [64, Section 5], we have the first and the second inequality. Based on the boundness of $\tilde{x}_i$ and φ, the third inequality could be obtained easily. The detailed proof is omitted here. □

Next, the estimation of the difference between the MF term $\hat{x}$ and the realized state-average $\tilde{x}^{(N)}$ is given as follows:

Lemma 2.4. *Under* (A2.1)-(A2.4)*, it holds that*

$$\mathbb{E}\sup_{0\le t\le T}\|\tilde{x}^{(N)} - \hat{x}\|^2 = O\left(\frac{1}{N}\right).$$

Proof. See Appendix A.1. □

Based on Lemma 2.4 we have the following estimation:

Lemma 2.5. *Under* (A2.1)-(A2.4)*, for some constant* L*, it holds that*

$$\sup_{1\leq i\leq N} \mathbb{E} \sup_{0\leq t\leq T} \|\tilde{x}_i - \bar{\alpha}_i\|^2 < L, \quad \sup_{1\leq i\leq N} \mathbb{E} \sup_{0\leq t\leq T} \|\tilde{x}_i - \hat{x}\|^2 < L.$$

Proof. See Appendix A.2. □

Moreover, based on Lemmas 2.3, 2.4, 2.5, we have the following estimation of the social cost functional.

Lemma 2.6. *Under* (A2.1)-(A2.4)*, for some constant* L *such that*

$$\mathcal{J}_{soc}^{(N)}(\tilde{u}_1, \cdots, \tilde{u}_N) \leq NL.$$

Proof. See Appendix A.3. □

Since we are studying the asymptotic optimality of $\tilde{u}$, it is sufficient only to consider those admissible strategies perform better then $\tilde{u}$. Specifically, we only consider those admissible strategies $\acute{u}$ satisfying

$$\mathcal{J}_{soc}^{(N)}(\acute{u}) \leq \mathcal{J}_{soc}^{N}(\tilde{u}) \leq NL. \tag{2.47}$$

For these admissible strategies satisfying (2.47), we have the following estimation.

Proposition 2.9. *Under* (A2.1)-(A2.4)*, for any admissible strategy* $\acute{u} \in \mathcal{U}_c$ *satisfying* (2.47)*, there exists a constant* L *such that*

$$\sum_{i=1}^{N} \mathbb{E} \int_0^T \|\acute{u}_i\|^2 dt \leq NL.$$

Proof. By (2.47), this result can be obtained forthrightly. □

Next, we introduce the last lemma for single agent perturbation. Consider an admissible strategy $(\tilde{u}_1, \cdots, \tilde{u}_{i-1}, \acute{u}_i, \tilde{u}_{i+1}, \cdots, \tilde{u}_N)$. Note that here all the agents apply the MF decentralized control law given by Theorem 2.1 except $\mathcal{A}_i$. Correspondingly the agent state is denoted by $(\acute{x}_1, \cdots, \acute{x}_N)$. We denote $\delta u_i := \acute{u}_i - \tilde{u}_i$ and correspondingly $\delta x_j := \acute{x}_j - \tilde{x}_j$, $\acute{x}^{(N)} := \frac{1}{N}\sum_{j=1}^{N} \acute{x}_j$, $\delta x^{(N)} := \acute{x}^{(N)} - \tilde{x}^{(N)}$ for $j = 1, \cdots, N$. Similarly, $\delta\alpha_i := \acute{\alpha}_i - \bar{\alpha}_i$, where $\bar{\alpha}_i$ is the optimal auxiliary state (2.32) and $\acute{\alpha}_i$ is the δu_i-perturbed auxiliary state satisfying

$$d\acute{\alpha}_i = (A\acute{\alpha}_i + B\acute{v}_i + F\hat{x})dt + (C\acute{\alpha}_i + D\acute{v}_i + \tilde{F}\hat{x})dW_i, \quad \acute{\alpha}_i(0) = \xi_0.$$

The δu_i-perturbed auxiliary control $\acute{v}_i = \Theta_1\bar{\alpha}_i + \Theta_2 + \delta u_i$.

Lemma 2.7. *Under* (A2.1)-(A2.4)*, for some constant* L *and any admissible strategy with form* $(\tilde{u}_1, \cdots, \tilde{u}_{i-1}, \acute{u}_i, \tilde{u}_{i+1}, \cdots, \tilde{u}_N)$ *satisfying* $\mathbb{E}\int_0^T \|\acute{u}_i\|^2 dt < L$*, it follows that*

$$\mathbb{E} \sup_{0 \leq t \leq T} \|\delta x_i - \delta a_i\|^2 = O\Big(\frac{1}{N^2}\Big).$$

Proof. See Appendix A.4. □

2.5.2 Asymptotic optimality

Next, we begin to estimate the optimality loss. Recalling Section 2.2, the original LP system (2.1)-(2.2) can be rewritten as follows:

$$\begin{cases} \text{minimize: } \mathcal{J}_{soc}^{(N)}(\mathbf{u}) = \frac{1}{2}\mathbb{E}\Big\{ \int_0^T \mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{S}_1^T\mathbf{x} + N\eta^T Q\eta + \mathbf{u}^T\mathbf{R}\mathbf{u}dt \\ \qquad\qquad + \mathbf{x}(T)^T\mathbf{G}\mathbf{x}(T) + 2\mathbf{S}_2^T\mathbf{x}(T) + N\bar{\eta}^T G\bar{\eta}\Big\}, \\ \text{subject to: } d\mathbf{x} = (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u})dt + \sum_{i=1}^N (\mathbf{C}_i\mathbf{x} + \mathbf{D}_i\mathbf{u})dW_i, \quad \mathbf{x}(0) = \Xi, \end{cases} \tag{2.48}$$

where $\mathbf{x}$, $\mathbf{u}$, $\mathbf{Q}$, $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{R}$, $\mathbf{G}$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, Ξ follow (2.5) and (2.7). For any given admissible $\mathbf{u}$, the state $\mathbf{x}$ can be determined by

$$\mathbf{x}(t) = \Phi(t)\Xi + \Phi(t)\int_0^t \Phi(s)^{-1}\big[(\mathbf{B} - \sum_{i=1}^N \mathbf{C}_i\mathbf{D}_i)\mathbf{u}(s)\big]ds + \sum_{i=1}^N \Phi(t)\int_0^t \Phi(s)^{-1}\mathbf{D}_i\mathbf{u}dW_i(s),$$

where

$$d\Phi(t) = \mathbf{A}\Phi(t)dt + \sum_{i=1}^m \mathbf{C}_i\Phi(t)dW_i, \quad \Phi(0) = I.$$

Define the following operators:

$$\begin{cases} (L\mathbf{u}(\cdot))(\cdot) := \Phi(\cdot)\Big\{\int_0^\cdot \Phi(s)^{-1}\big[(\mathbf{B} - \sum_{i=1}^N \mathbf{C}_i\mathbf{D}_i)\mathbf{u}(s)\big]ds + \sum_{i=1}^N \int_0^\cdot \Phi(s)^{-1}\mathbf{D}_i\mathbf{u}dW_i(s)\Big\}, \\ \tilde{L}\mathbf{u}(\cdot) = (L\mathbf{u}(\cdot))(T), \quad \Gamma\Xi(\cdot) = \Phi(\cdot)\Phi^{-1}(0)\Xi, \quad \tilde{\Gamma}\Xi = (\Gamma\Xi)(T). \end{cases}$$

Correspondingly, L^* is defined as the adjoint operator of L (see [62]). Given any admissible $\mathbf{u}$, $\mathbf{x}$ can be represented as follows:

$$\mathbf{x}(\cdot) = (L\mathbf{u}(\cdot))(\cdot) + \Gamma\Xi(\cdot), \quad \mathbf{x}(T) = \tilde{L}\mathbf{u}(\cdot) + \tilde{\Gamma}\Xi,$$

and the cost functional can be rewritten as

$$
\begin{aligned}
2\mathcal{J}_{soc}^{(N)}(\mathbf{u}) =& \mathbb{E}\bigg\{ \int_0^T \mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{S}_1^T\mathbf{x} + N\eta^T Q\eta + \mathbf{u}^T\mathbf{R}\mathbf{u}dt + \mathbf{x}(T)^T\mathbf{G}\mathbf{x}(T) \\
&\qquad + 2\mathbf{S}_2^T\mathbf{x}(T) + N\bar{\eta}^T G\bar{\eta}\bigg\} \\
=& \langle (L^*\mathbf{Q}L + \tilde{L}^*\mathbf{G}\tilde{L} + \mathbf{R})\mathbf{u}(\cdot), \mathbf{u}(\cdot)\rangle + 2\langle L^*(\mathbf{Q}\Gamma\Xi(\cdot) + \mathbf{S}_1) \\
&+ \tilde{L}^*(\mathbf{G}\tilde{\Gamma}\Xi(\cdot) + \mathbf{S}_2), \mathbf{u}(\cdot)\rangle + \langle \mathbf{Q}\Gamma\Xi(\cdot), \Gamma\Xi(\cdot)\rangle + 2\langle \mathbf{S}_1, \Gamma\Xi(\cdot)\rangle \\
&+ 2\langle \mathbf{S}_2, \tilde{\Gamma}\Xi(\cdot)\rangle + TN\eta^T Q\eta + N\bar{\eta}^T G\bar{\eta} \\
:=& \langle M_2\mathbf{u}(\cdot), \mathbf{u}(\cdot)\rangle + 2\langle M_1, \mathbf{u}(\cdot)\rangle + M_0,
\end{aligned}
$$

where

$$
\begin{cases}
M_2 := L^*\mathbf{Q}L + \tilde{L}^*\mathbf{G}\tilde{L} + \mathbf{R}, \quad M_1 := L^*(\mathbf{Q}\Gamma\Xi(\cdot) + \mathbf{S}_1) + \tilde{L}^*(\mathbf{G}\tilde{\Gamma}\Xi(\cdot) + \mathbf{S}_2), \\
M_0 := \langle \mathbf{Q}\Gamma\Xi(\cdot), \Gamma\Xi(\cdot)\rangle + 2\langle \mathbf{S}_1, \Gamma\Xi(\cdot)\rangle + 2\langle \mathbf{S}_2, \tilde{\Gamma}\Xi(\cdot)\rangle + TN\eta^T Q\eta + N\bar{\eta}^T G\bar{\eta}.
\end{cases}
$$

Note that, M_2 is a bounded self-adjoint linear operator. Let $\tilde{\mathbf{u}} = (\tilde{u}_1^T, \cdots, \tilde{u}_n^T)^T$ be the decentralized strategy given by Theorem 2.1. Consider a perturbation: $\mathbf{u} = \tilde{\mathbf{u}} + \delta\mathbf{u}$. Then

$$
\begin{aligned}
2\mathcal{J}_{soc}^{(N)}(\tilde{\mathbf{u}} + \delta\mathbf{u}) &= \langle M_2(\tilde{\mathbf{u}} + \delta\mathbf{u}), \tilde{\mathbf{u}} + \delta\mathbf{u}\rangle + 2\langle M_1, \tilde{\mathbf{u}} + \delta\mathbf{u}\rangle + M_0 \\
&= 2\mathcal{J}_{soc}^{(N)}(\tilde{\mathbf{u}}) + 2\langle M_2\tilde{\mathbf{u}} + M_1, \delta\mathbf{u}\rangle + o(\delta\mathbf{u}).
\end{aligned} \tag{2.49}
$$

Here, $\langle M_2\tilde{\mathbf{u}} + M_1, \cdot\rangle$ is the Fréchet differential of $J_{soc}^{(N)}$ on $\tilde{\mathbf{u}}$. By the linearity, we also have

$$
\mathcal{J}_{soc}^{(N)}(\tilde{\mathbf{u}} + \delta\mathbf{u}) = \mathcal{J}_{soc}^{(N)}(\tilde{\mathbf{u}}) + \sum_{i=1}^{N} \langle M_2\tilde{\mathbf{u}} + M_1, \delta\mathbf{u}_i\rangle + o(\delta\mathbf{u}),
$$

where $\delta\mathbf{u}_i := (0^T, \cdots, 0^T, \delta u_i^T, 0^T, \cdots, 0^T)^T$. Specifically,

$$\mathcal{J}_{soc}^{(N)}(\tilde{\mathbf{u}} + \delta\mathbf{u}_i) = \mathcal{J}_{soc}^{(N)}(\tilde{\mathbf{u}}) + \langle M_2\tilde{\mathbf{u}} + M_1, \delta\mathbf{u}_i\rangle + o(\delta\mathbf{u}_i). \tag{2.50}$$

For the estimation of $M_2\tilde{\mathbf{u}} + M_1$, we have the following Lemma.

Lemma 2.8. *Under* (A2.1)-(A2.4)*, for the MF decentralized strategy* $(\tilde{u}_1, \cdots, \tilde{u}_N)$ *given by Theorem 2.1, we have*

$$\|M_2\tilde{\mathbf{u}} + M_1\| = O\Big(\frac{1}{\sqrt{N}}\Big).$$

Proof. See Appendix A.5. □

Based on the discussion above, we can introduce the following result of the asymptotic optimality.

Theorem 2.2. *Under* (A2.1)-(A2.4)*, the MF decentralized strategy* $\tilde{u}$ *given by Theorem 2.1 has asymptotic social optimality such that*

$$\frac{1}{N}\Big\| \inf_{u\in\mathcal{U}_c} \mathcal{J}_{soc}^{(N)}(u) - \mathcal{J}_{soc}^{(N)}(\tilde{u})\Big\| = O\Big(\frac{1}{\sqrt{N}}\Big).$$

Proof. By the representation of (2.48), $\mathcal{J}_{soc}^{(N)}(\tilde{\mathbf{u}}) - \mathcal{J}_{soc}^{(N)}(\tilde{\mathbf{u}} + \delta\mathbf{u}) = o(N)$ is aimed to prove, for any admissible strategy $\tilde{\mathbf{u}} + \delta\mathbf{u}$ satisfying condition (2.47).

By (2.49), the following relation can be obtained

$$
\begin{aligned}
0 \leq & \mathcal{J}_{soc}^{(N)}(\tilde{\mathbf{u}}) - \mathcal{J}_{soc}^{(N)}(\tilde{\mathbf{u}}+\delta\mathbf{u}) = -\left(\sum_{i=1}^{N}\langle M_2\tilde{\mathbf{u}}+M_1,\delta\mathbf{u}_i\rangle\right) - \frac{1}{2}\,\langle M_2\delta\mathbf{u},\delta\mathbf{u}\rangle \\
\leq & \sqrt{\sum_{i=1}^{N}\|M_2\tilde{\mathbf{u}}+M_1\|^2\sum_{i=1}^{N}\|\delta\mathbf{u}_i\|^2} - \frac{1}{2}\,\langle M_2\delta\mathbf{u},\delta\mathbf{u}\rangle \\
= & \sqrt{N\|M_2\tilde{\mathbf{u}}+M_1\|^2}\times O(\sqrt{N}) - \frac{1}{2}\langle M_2\delta\mathbf{u},\delta\mathbf{u}\rangle.
\end{aligned}
\tag{2.51}
$$

Thus, by Lemma 2.8 and noting the convexity (i.e., $\frac{1}{2}\langle M_2\delta\mathbf{u},\delta\mathbf{u}\rangle \geq 0$), we have

$$
\mathcal{J}_{soc}^{(N)}(\tilde{\mathbf{u}}) - \mathcal{J}_{soc}^{(N)}(\tilde{\mathbf{u}}+\delta\mathbf{u}) = O\big(\sqrt{N}\big).
$$

Theorem 2.2 follows. $\square$

2.6 A numerical example based on navigation application

In this section, we present a numerical example with navigation application background. A company decides to deploy a group of robots to explore an unknown terrain. The population of the robots is $N = 1000$. The states of the robots are driven by the linear equations (2.1) with the following coefficients:

$$
\begin{cases}
A = \begin{pmatrix} 0.9723 & 0.9707 \\ 0.7409 & 0.0118 \end{pmatrix}, B = \begin{pmatrix} 0.7310 & 0.7980 \\ 0.2814 & 0.6108 \end{pmatrix}, F = \begin{pmatrix} 0.2077 & 0.4383 \\ 0.5265 & 0.2515 \end{pmatrix}, C = \begin{pmatrix} 0.5469 & 0.9669 \\ 0.3363 & 0.8207 \end{pmatrix}, \\
D = \begin{pmatrix} 0.9051 & 0.8551 \\ 0.8856 & 0.4914 \end{pmatrix}, \tilde{F} = \begin{pmatrix} 0.4969 & 0.5103 \\ 0.4094 & 0.2017 \end{pmatrix}, \xi_0 = \begin{pmatrix} 0.1627 \\ 0.6570 \end{pmatrix}.
\end{cases}
$$

The exploring time duration can be normalized by $T = 1$, and the individual running cost of each robot is given by (2.2) with the following coefficients:

$$
\begin{aligned}
&\Gamma = \begin{pmatrix} 0.7420 & 0.9669 \\ 0.2016 & 0.1553 \end{pmatrix},\ \eta = \begin{pmatrix} 0.8740 \\ 0.7733 \end{pmatrix},\ Q = \begin{pmatrix} 0.1845 & 0 \\ 0 & 0.1785 \end{pmatrix},\ R = \begin{pmatrix} 0.6587 & 0 \\ 0 & 0.8763 \end{pmatrix}, \\
&\bar{\Gamma} = \begin{pmatrix} 0.0683 & 0.9089 \\ 0.2087 & 0.1938 \end{pmatrix},\ \bar{\eta} = \begin{pmatrix} 0.8829 \\ 0.9076 \end{pmatrix},\ G = \begin{pmatrix} 0.7313 & 0 \\ 0 & 0.4721 \end{pmatrix}.
\end{aligned}
$$

Recall (2.2):

$$\mathcal{J}_i = \frac{1}{2}\mathbb{E}\bigg\{ \int_0^T \|x_i - \Gamma x^{(N)} - \eta\|_Q^2 + \|u_i\|_R^2 dt + \|x_i(T) - \bar{\Gamma} x^{(N)}(T) - \bar{\eta}\|_G^2 \bigg\}.$$

The cost of each robot is based on a tradeoff between moving toward a certain reference signal: η, and staying near the average of the members' states: $x^{(N)}$. The individual cost functions penalize the deviation from the reference signal and the state average. Thus, we introduce the first part of the cost functional:

$$\frac{1}{2}\mathbb{E}\bigg\{ \int_0^T \|x_i - \Gamma x^{(N)} - \eta\|_Q^2 dt \bigg\}.$$

Moreover, the running cost of the adopted strategy u_i should be considered as well, and we introduce the second part of the cost functional:

$$\frac{1}{2}\mathbb{E}\bigg\{ \int_0^T \|u_i\|_R^2 dt \bigg\}.$$

The company also focuses the terminal states of the robots which are based on a tradeoff between reaching a certain destination signal: $\bar{\eta}$, and staying near the average of the members' terminal states: $x^{(N)}(T)$. Thus, we introduce the third part (terminal part) of the cost functional:

$$\frac{1}{2}\mathbb{E}\bigg\{ \|x_i(T) - \bar{\Gamma} x^{(N)}(T) - \bar{\eta}\|_G^2 \bigg\},$$

to penalize the deviation from the destination signal and the terminal state average. For more similar applications of such quadratic cost functionals, interested readers are referred to [77, 78]. Note that $\mathcal{J}_i$ is designed to characterize the individual cost of each robot, and the total cost of the whole company is

$$\mathcal{J}_{soc}^{(N)} = \sum_{i=1}^{N} \mathcal{J}_i.$$

The target of the company is to assign a decentralized strategy to each robot which is much more cost-effective compared with centralized strategy. Actually, to calculate the centralized strategy, the central server should collect all the real-time information uploaded by all the robots (e.g., $W_1, \cdots, W_N$ and $x_1, \cdots, x_N$). This brings great computational burden to the central server, and such data stream also requires much more channels and bandwidths.

Thus, by applying the theoretical results we obtained above, such decentralized strategy can be derived. Specifically, the feedback coefficients (Θ_1, Θ_2) can be calculated beforehand and stored in each robot. Then each robot can calculate its individual decentralized strategy offline according to its individual real-time state: x_i with very low computational burden (i.e., $u_i = \Theta_1 x_i + \Theta_2$). Moreover, it is noteworthy that in this case, the communication between the robots and the central server is also unnecessary. Each robot can complete such simple calculation by its individual microcomputer. Then the company can save the cost of supporting the bandwidth of the channel and the high frequency communication.

To calculate the decentralized strategy, firstly, by applying Runge-Kutta methods to (2.41) and (2.43), we can obtain K and κ respectively. Then by using the relation $\tilde{Y} = K\tilde{X} + \kappa$ and $\tilde{Z} = K\tilde{A}_1'\tilde{X} + K\tilde{B}_1'K\tilde{X} + K\tilde{B}_1'\kappa$, (2.40) can be solved and we can determine $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$. Correspondingly, $\hat{x}$, $\hat{y}_1$, $\hat{\beta}_1$ can also be obtained. By calculating (2.29), (2.30) and (2.31), we have P, φ, Θ_1, Θ_2, and then the realized state $\tilde{x}_i$ would follow by (2.46). Through the calculation above, we can obtain the two coordinates of the trajectories of the realized states of the 1000 robots given by Figure 2.1.

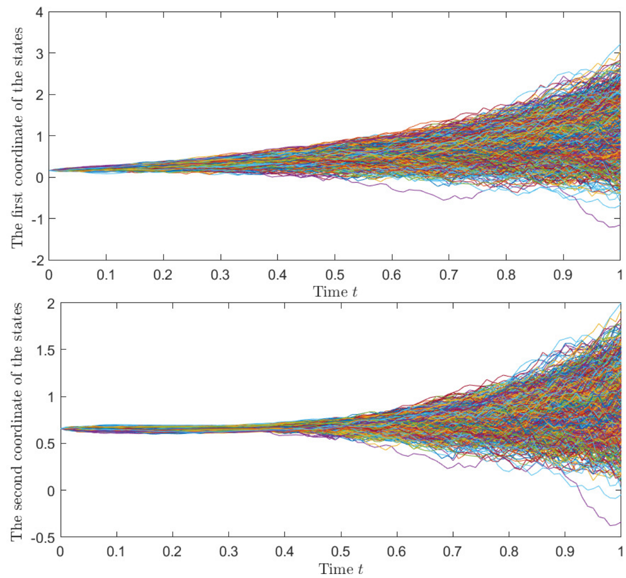


Figure 2.1: the trajectories of the realized states of the 1000 robots

Moreover, the realized state-average $\tilde{x}^{(N)}$ and MF term $\hat{x}$ are given in Figure 2.2.

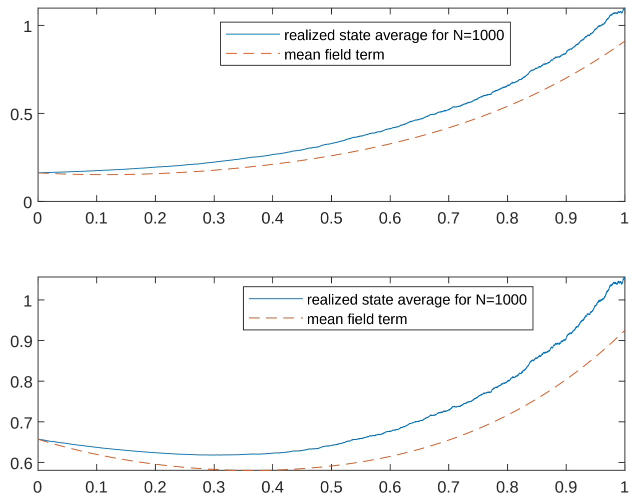


Figure 2.2: the trajectories of realized state-average $\tilde{x}^{(N)}$ and MF term $\hat{x}$

By Figure 2.2, we see that $\hat{x}$ is approximate to $\tilde{x}^{(N)}$, which is consistent to our theoretical result in section 2.5. To illustrate such asymptotic approximation between $\hat{x}$ and $\tilde{x}^{(N)}$ better, we also calculate the change of the value of $\mathbb{E}\sup_{0\leq t\leq T}\|\tilde{x}^{(N)}(t)-\hat{x}(t)\|^2$ with the number of the robots N. The trajectory is given by Figure 2.3 below.

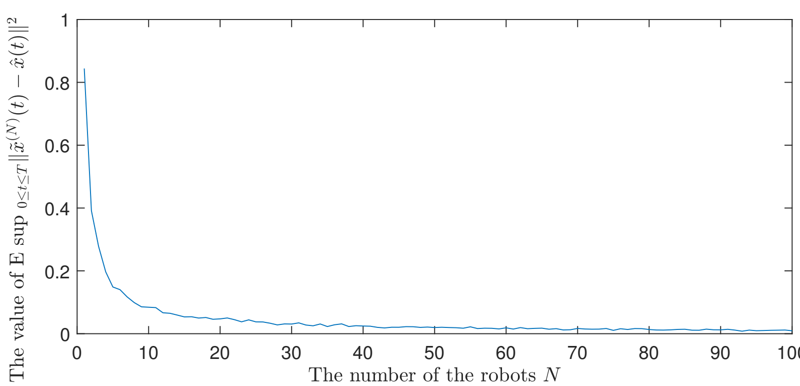


Figure 2.3: the change of the value of $\mathbb{E}\sup_{0\leq t\leq T}\|\tilde{x}^{(N)}(t)-\hat{x}(t)\|^2$ along with N

We see that the residual $\mathbb{E}\sup_{0\le t\le T}\|\tilde{x}^{(N)}(t)-\hat{x}(t)\|^2$ tends to 0 as the population of the robots N tends to ∞. More exactly, the residual $\mathbb{E}\sup_{0\le t\le T}\|\tilde{x}^{(N)}(t)-\hat{x}(t)\|^2$ can be fitted by the following relation:

$$\mathbb{E}\sup_{0\le t\le T}\|\tilde{x}^{(N)}(t)-\hat{x}(t)\|^2=\frac{0.8316}{N},$$

with SSE $=0.002642$ and $R^2=0.9972$. The fitting performance is given by the following figure 2.4.

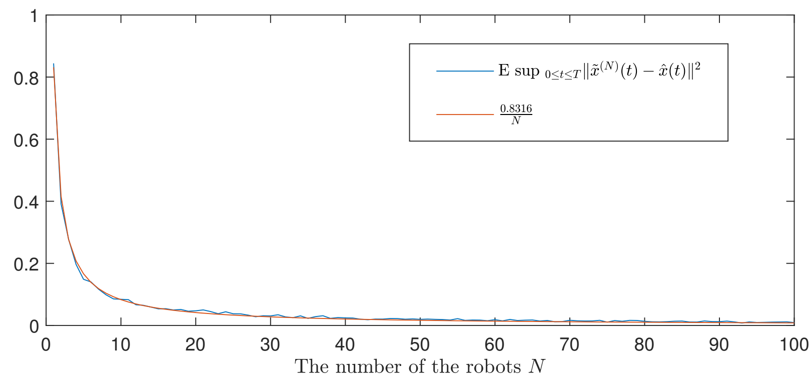


Figure 2.4: the fitting performance of $\frac{0.8316}{N}$

In this sense, the result in Lemma 2.4 can be verified in our numerical example.

Next, we aim to verify the result in Proposition 2.8. The corresponding components of $\left[(0,I)\Psi(T,t)\binom{0}{I}\right]^{-1}$ are shown in Figure 2.5, respectively.

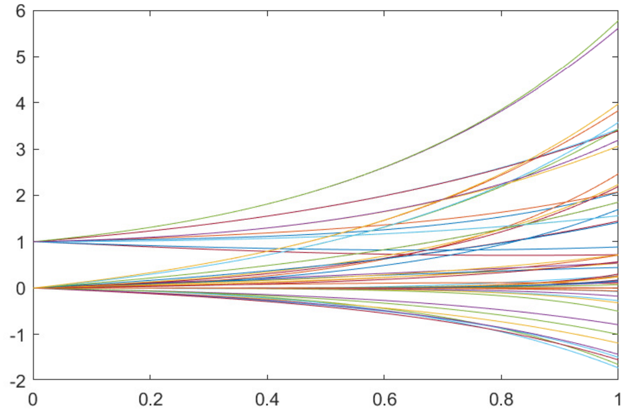


Figure 2.5: the trajectories of the components in equation (2.45), when $C, F \neq 0$

Further, if we let $F = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $C = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, then the components of $\left[(0, I)\Psi(T,t)\binom{0}{I}\right]^{-1}$ are shown in Figure 2.6 respectively,

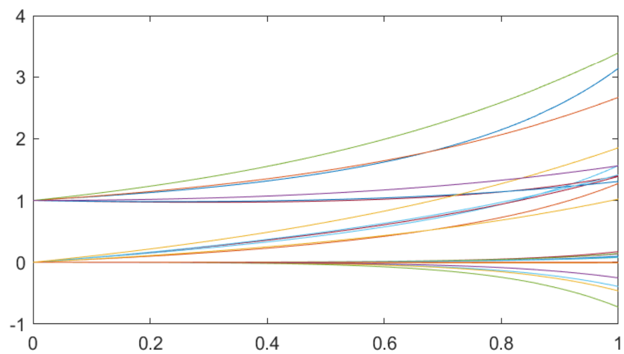


Figure 2.6: the trajectories of the components in equation (2.45), when $C = F = 0$

and we have

$$\left[(0, I)\Psi(T,t)\begin{pmatrix} 0 \\ I \end{pmatrix}\right]^{-1} \in L^1(0, T; \mathbb{R}^{n\times n}).$$

Then, in this case by the result of Proposition 2.8, we know that the Riccati equation (2.44) admits a unique solution K which is given by the following:

$$K = -\left[(0, I)\Psi(T, t)\begin{pmatrix} 0 \\ I \end{pmatrix}\right]^{-1}(0, I)\Psi(T, t)\begin{pmatrix} I \\ 0 \end{pmatrix}, \quad t \in [0, T],$$

where $\Psi(t, s)$ is the fundamental matrix w.r.t. $\begin{pmatrix} \tilde{A}_1 & \tilde{B}_1 \\ \tilde{A}_2 & \tilde{B}_2 \end{pmatrix}$.

2.7 Conclusion

In this chapter, we investigate the social optima of a MF LQG control problem with multiplicative noise. First, we discuss the convexity of the social cost functional and summarize some conditions for some cases of indefinite weight coefficients. Then based on person by person optimality and duality procedures, a set of decentralized strategies is designed by optimizing the social cost functional subject to a MF-FBSDEs system (CC system) in MF approximations. We study the well-posedness of the MF-FBSDEs system and obtain the conditions for its solvability. Finally, the corresponding decentralized strategies is proved to has asymptotic social optimality.

Chapter 3 MF Strategy in Mixed Social Optima

3.1 Problem formulation

In this chapter, we consider a weakly coupled LP system with a major agent $\mathcal{A}_0$ and N individual minor agents denoted by $\{\mathcal{A}_i : 1 \leq i \leq N\}$. The dynamics of the $N+1$ agents are given by a system of linear SDEs with MF coupling:

$$\begin{cases} dx_0(t) = \Big[A_0(t)x_0(t) + B_0(t)u_0(t) + F_0(t)x^{(N)}(t)\Big]dt \\ \qquad\qquad + \Big[C_0(t)x_0(t) + D_0(t)u_0(t) + \widetilde{F}_0(t)x^{(N)}(t)\Big]dW_0(t), \\ x_0(0) = \xi_0 \in \mathbb{R}^n, \end{cases} \tag{3.1}$$

and for $1 \leq i \leq N$,

$$\begin{cases} dx_i(t) = \Big[A(t)x_i(t) + B(t)u_i(t) + F(t)x^{(N)}(t)\Big]dt \\ \qquad\qquad + \Big[C(t)x_i(t) + D(t)u_i(t) + \widetilde{F}(t)x^{(N)}(t) + \widetilde{G}(t)x_0(t)\Big]dW_i(t), \\ x_i(0) = \xi \in \mathbb{R}^n, \end{cases}$$

where $x^{(N)}(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t)$ is the average state of the minor agents.

Remark 3.1. *We remark that the control process and state-average enter both the drift and diffusion terms. This makes our paper different to standard MFG (e.g., [52]) or social optima (e.g., [27]) literature in which only drift terms are control-dependent.*

Let $u(\cdot) := (u_0(\cdot), u_1(\cdot), \cdots, u_N(\cdot))$ be the set of strategies of all $N+1$ agents, $u_{-0}(\cdot) := (u_1(\cdot), \cdots, u_N(\cdot))$ and $u_{-i}(\cdot) := (u_0(\cdot), \cdots, u_{i-1}(\cdot), u_{i+1}(\cdot), \cdots, u_N(\cdot))$, $0 \leq i \leq N$. The centralized admissible strategy set is given by

$$\mathcal{U}_c := \Big\{ u(\cdot) | u(\cdot) \text{ is adapted to } \mathbb{F}, \text{ and } \mathbb{E}\int_0^T \|u(t)\|^2 dt < \infty \Big\}.$$

Correspondingly, the feedback decentralized admissible strategy set for the major agent is given by

$$\begin{aligned}\mathcal{U}_0 := \Big\{ & u_0(\cdot) | u_0(\cdot) \text{ is adapted to } \big\{\mathcal{F}_t^0 \vee \sigma\{x_0(s), 0 \leq s \leq t\}\big\}_{t\geq 0}, \\ & \mathbb{E}\int_0^T \|u_i(t)\|^2 dt < \infty \Big\},\end{aligned}$$

and the feedback decentralized admissible strategy set for the i^{th} minor agent is given by

$$\begin{aligned}\mathcal{U}_i := \Big\{ & u_i(\cdot) | u_i(\cdot) \text{ is adapted to } \big\{\mathcal{F}_t^i \vee \sigma\{x_i(s), 0 \leq s \leq t\}\big\}_{t\geq 0}, \\ & \mathbb{E}\int_0^T \|u_i(t)\|^2 dt < \infty \Big\}.\end{aligned}$$

For simplicity, define

$$\mathcal{U}_{-0} := \{(u_1(\cdot), \cdots, u_N(\cdot)) | u_i(\cdot) \in \mathcal{U}_i, \ i = 1, \cdots, N\}.$$

The cost functional for $\mathcal{A}_0$ is given by

$$\begin{aligned}&\mathcal{J}_0(u_0(\cdot), u_{-0}(\cdot)) \\ =& \frac{1}{2}\mathbb{E}\int_0^T \Big[\Big\langle Q_0(t)(x_0(t) - H_0(t)x^{(N)}(t)), x_0(t) - H_0(t)x^{(N)}(t)\Big\rangle \\ & + \Big\langle R_0(t)u_0(t), u_0(t)\Big\rangle\Big] dt,\end{aligned} \tag{3.2}$$

and the cost functional for $\mathcal{A}_i$, $1 \le i \le N$, is given by

$$
\begin{aligned}
&\mathcal{J}_i(u_i(\cdot), u_{-i}(\cdot)) \\
&= \frac{1}{2}\mathbb{E}\int_0^T \Big[\Big\langle Q(t)(x_i(t) - H(t)x_0(t) - \hat{H}(t)x^{(N)}(t)), \\
&\qquad\qquad x_i(t) - H(t)x_0(t) - \hat{H}(t)x^{(N)}(t)\Big\rangle + \Big\langle R(t)u_i(t), u_i(t)\Big\rangle\Big]dt.
\end{aligned}
\tag{3.3}
$$

Remark 3.2. *It is worth pointing out that it brings no essential difficulty to introduce a terminal cost term in* (3.2) *and* (3.3). *This will only change the terminal value of the associated Riccati equations. Thus, for simplicity, we only consider Lagrange type cost functional here.*

The aggregate team cost of N minor agents is

$$
\mathcal{J}_{soc}^{(N)}(u(\cdot)) = \sum_{i=1}^{N} \mathcal{J}_i(u_i(\cdot), u_{-i}(\cdot)). \tag{3.4}
$$

We impose the following general assumptions, which are commonly used in LQG models, on the coefficients:

(A3.1) $A_0(\cdot), F_0(\cdot), C_0(\cdot), \widetilde{F}_0(\cdot), A(\cdot), F(\cdot), C(\cdot), \widetilde{F}(\cdot),\ \widetilde{G}(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n})$, $B_0(\cdot), D_0(\cdot), B(\cdot), D(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times m})$.

(A3.2) $Q_0(\cdot), H_0(\cdot), Q(\cdot), H(\cdot), \hat{H}(\cdot) \in L^\infty(0,T;\mathbb{S}^n)$, $R_0(\cdot), R(\cdot) \in L^\infty(0,T;\mathbb{S}^m)$.

Remark 3.3. *Under* (A3.1), *the system* (3.1) *and* (3.1) *admits a unique strong solution* $(x_0, \cdots, x_N) \in L^2_{\mathcal{F}_t}(\Omega; C(0,T;\mathbb{R}^n)) \times \cdots \times L^2_{\mathcal{F}_t}(\Omega; C(0,T;\mathbb{R}^n))$ *for any* *given admissible control* $(u_0, \cdots, u_N) \in \mathcal{U}_c \times \cdots \times \mathcal{U}_c$. *Under* (A3.2)*, the cost functionals* (3.2) *and* (3.3) *are well defined.*

Note that while the coefficients are dependent on the time variable t, in this chapter, the variable (t), $(\cdot)$ will usually be suppressed if no confusion would occur. We propose the following social optimization problem:

Problem 3.1. *Find a strategy set* $\bar{u}$=($\bar{u}_0$, $\bar{u}_1$, $\cdots$, $\bar{u}_N$) *where* $\bar{u}_i \in \mathcal{U}_i$, $0 \le i \le N$*, such that*

$$\begin{cases} \mathcal{J}_0(\bar{u}_0, \bar{u}_{-0}) = \inf\limits_{u_0 \in \mathcal{U}_0} \mathcal{J}_0(u_0, \bar{u}_{-0}), \\ \mathcal{J}_{soc}^{(N)}(\bar{u}_0, \bar{u}_{-0}) = \inf\limits_{u_{-0} \in \mathcal{U}_{-0}} \mathcal{J}_{soc}^{(N)}(\bar{u}_0, u_{-0}). \end{cases} \tag{3.5}$$

3.2 Stochastic optimal control problem of the major agent

Replacing $x^{(N)}$ of (3.1) and (3.2) by $\hat{x}$ which will be determined in Section 3.4, the limiting major agent's state is given by:

$$dz_0 = \big(A_0 z_0 + B_0 u_0 + F_0 \hat{x}\big)dt + \big(C_0 z_0 + D_0 u_0 + \widetilde{F}_0 \hat{x}\big)dW_0, \quad z_0(0) = \xi_0, \tag{3.6}$$

and correspondingly the limiting cost functional is

$$J_0(u_0) = \frac{1}{2}\mathbb{E}\int_0^T \Big[\langle Q_0(z_0 - H_0\hat{x}), z_0 - H_0\hat{x}\rangle + \langle R_0 u_0, u_0\rangle\Big]dt. \tag{3.7}$$

We define the following auxiliary control problem for major agent:

Problem 3.2. *For major agent* $\mathcal{A}_0$*, minimize* $J_0(u_0)$ *over* $\mathcal{U}_0$.

This is a standard LQ stochastic control problem. For its solvability, one can introduce the following standard assumption:

(SA) $Q_0 \geq 0$, $Q \geq 0$, $R_0 \gg 0$, $R \gg 0$.

By [58, pp 2285, Theorem 4.3], we have the following result:

Proposition 3.1. *Under* (A3.1)-(A3.2) *and* (SA), *the following Riccati equation:*

$$\begin{cases} -(P_0B_0 + C_0^T P_0 D_0)(R_0 + D_0^T P_0 D_0)^{-1}(B_0^T P_0 + D_0^T P_0 C_0) + \dot{P}_0 + P_0 A_0 \\ + A_0^T P_0 + C_0^T P_0 C_0 + Q_0 = 0, \quad P_0(T) = 0, \end{cases} \tag{3.8}$$

is strongly regularly solvable, and Problem 3.2 admits a feedback optimal control $\bar{u}_0 = \Theta_1 \bar{z}_0 + \Theta_2$ *where*

$$\begin{cases} \Theta_1 = -(R_0 + D_0^T P_0 D_0)^{-1}(B_0^T P_0 + D_0^T P_0 C_0), \\ \Theta_2 = -(R_0 + D_0^T P_0 D_0)^{-1}(B_0^T \phi + D_0^T \zeta + D_0^T P_0 \widetilde{F}_0 \hat{x}), \end{cases} \tag{3.9}$$

and ϕ *satisfies*

$$\begin{cases} d\phi = -\Big\{ \Big[A_0^T - (P_0B_0 + C_0^T P_0 D_0)(R_0 + D_0^T P_0 D_0)^{-1} B_0^T \Big] \phi \\ \qquad + \Big[C_0^T - (P_0 B_0 + C_0^T P_0 D_0)(R_0 + D_0^T P_0 D_0)^{-1} D_0^T \Big] \zeta \\ \qquad + \Big[C_0^T - (P_0 B_0 + C_0^T P_0 D_0)(R_0 + D_0^T P_0 D_0)^{-1} D_0^T \Big] P_0 \widetilde{F}_0 \hat{x} \\ \qquad + P_0 F_0 \hat{x} - Q_0 H_0 \hat{x} \Big\} dt + \zeta dW_0, \quad \phi(T) = 0. \end{cases} \tag{3.10}$$

The corresponding optimal state is

$$
\begin{cases}
d\bar{z}_0 = \Big[(A_0+B_0\Theta_1)\bar{z}_0+B_0\Theta_2+F_0\hat{x}\Big]dt+\Big[(C_0+D_0\Theta_1)\bar{z}_0+D_0\Theta_2+\widetilde{F}_0\hat{x}\Big]dW_0(t),\\
z_0(0)=\xi_0.
\end{cases}
\tag{3.11}
$$

3.3 Stochastic optimal control problem for minor agents

In this section, we aim to derive the MF strategy set for the minor agents. The methodology (e.g., variational analysis and duality method) is similar to that in Chapter 2 Section 2.3-2.4. However, due to the existence of the major agent, the corresponding variation function, limit process and dual process of the major agent should be introduced additionally. Thus, in what follows, we would still sketch some key steps.

3.3.1 Person-by-person optimality

Let $(\bar{u}_1,\cdots,\bar{u}_n)$ be centralized optimal strategies of the minor agents. We now perturb u_i and keep $\bar{u}_{-i}=(\bar{u}_0,\ \bar{u}_1,\ \cdots,\ \bar{u}_{i-1},\ \bar{u}_{i+1},\ \cdots,\ \bar{u}_N)$ fixed. For $j=1,\cdots,N,\ j\neq i$, denote the perturbation $\delta u_i=u_i-\bar{u}_i$, $\delta x_i=x_i-\bar{x}_i$, $\delta x_j=x_j-\bar{x}_j$, $\delta x^{(N)}=\frac{1}{N}\sum_{j=1}^N\delta x_j$, and $\delta\mathcal{J}_j$ is the first variation (Fréchet differential) of $\mathcal{J}_j$ w.r.t. δu_j. Therefore, δx_i, δx_j, δx_0 and $\delta x_{-(0,i)}:=\sum_{j=1,j\neq i}^N\delta x_j$ are given by

$$
\begin{cases}
d\delta x_i=(A\delta x_i+B\delta u_i+F\delta x^{(N)})dt+(C\delta x_i+D\delta u_i+\widetilde{F}\delta x^{(N)}+\widetilde{G}\delta x_0)dW_i,\\
d\delta x_j=(A\delta x_j+F\delta x^{(N)})dt+(C\delta x_j+\widetilde{F}\delta x^{(N)}+\widetilde{G}\delta x_0)dW_j,\\
d\delta x_0=(A_0\delta x_0+F_0\delta x^{(N)})dt+(C_0\delta x_0+\widetilde{F}_0\delta x^{(N)})dW_0,\\
d\delta x_{-(0,i)}=[A\delta x_{-(0,i)}+F(N-1)\delta x^{(N)}]dt+\sum_{j\neq i}(C\delta x_j+\widetilde{F}\delta x^{(N)}+\widetilde{G}\delta x_0)dW_j,\\
\delta x_i(0)=0,\ \delta x_j(0)=0,\ \delta x_0(0)=0,\ \delta x_{-(0,i)}(0)=0.
\end{cases}
$$

By some elementary calculations, we can further obtain $\delta\mathcal{J}_i$ of the cost functional of $\mathcal{A}_i$ as follows

$$\delta\mathcal{J}_i=\mathbb{E}\int_0^T\langle Q(\bar{x}_i-\hat{H}\bar{x}^{(N)}-H\bar{x}_0),\delta x_i-\hat{H}\delta x^{(N)}-H\delta x_0\rangle+\langle R\bar{u}_i,\delta u_i\rangle dt. \tag{3.12}$$

For $j\neq i$, $\delta\mathcal{J}_j$ of the cost functional of $\mathcal{A}_j$ is given by

$$\delta\mathcal{J}_j=\mathbb{E}\int_0^T\langle Q(\bar{x}_j-\hat{H}\bar{x}^{(N)}-H\bar{x}_0),\delta x_j-H\delta x_0-\hat{H}\delta x^{(N)}\rangle dt. \tag{3.13}$$

We can further obtain $\delta\mathcal{J}_{soc}^{(N)}$, the first variation of the social cost, satisfying

$$\begin{aligned}\delta\mathcal{J}_{soc}^{(N)}=\mathbb{E}\int_0^T\Big[&\langle Q(\bar{x}_i-\hat{H}\bar{x}^{(N)}-H\bar{x}_0),\delta x_i-H\delta x_0-\hat{H}\delta x^{(N)}\rangle\\&+\sum_{j\neq i}\langle Q(\bar{x}_j-\hat{H}\bar{x}^{(N)}-H\bar{x}_0),\delta x_j-H\delta x_0-\hat{H}\delta x^{(N)}\rangle+\langle R\bar{u}_i,\delta u_i\rangle\Big]dt.\end{aligned} \tag{3.14}$$

Replacing $\bar{x}^{(N)}$ in (3.14) by $(\bar{x}^{(N)}-\hat{x})+\hat{x}$, we have

$$\begin{aligned}\delta\mathcal{J}_{soc}^{(N)}=\mathbb{E}\int_0^T\Big[&\langle Q\bar{x}_i,\delta x_i\rangle-\Big\langle Q(\hat{H}\hat{x}+H\bar{x}_0)+\hat{H}Q(\hat{x}-\hat{H}\hat{x}-H\bar{x}_0),\delta x_i\Big\rangle\\&-\big\langle\hat{H}Q(\hat{x}-\hat{H}\hat{x}-H\bar{x}_0),\delta x_{-(0,i)}\big\rangle-\big\langle HQ(\hat{x}-\hat{H}\hat{x}-H\bar{x}_0),N\delta x_0\big\rangle\\&+\frac{1}{N}\sum_{j\neq i}\big\langle Q(\bar{x}_j-\hat{H}\hat{x}-H\bar{x}_0),N\delta x_j\big\rangle+\langle R\bar{u}_i,\delta u_i\rangle\Big]dt+\sum_{l=1}^{4}\varepsilon_l,\end{aligned}$$

where

$$\begin{cases}\varepsilon_1=E\int_0^T\langle(Q\hat{H}-\hat{H}Q\hat{H})(\hat{x}-\bar{x}^{(N)}),N\delta x^{(N)}\rangle dt,\\ \varepsilon_2=-E\int_0^T\langle HQ\hat{H}(\hat{x}-\bar{x}^{(N)}),NH\delta x_0\rangle dt,\\ \varepsilon_3=E\int_0^T\langle HQ(\hat{x}-\bar{x}^{(N)}),N\delta x_0\rangle dt,\\ \varepsilon_4=E\int_0^T\langle\hat{H}Q(\hat{x}-\bar{x}^{(N)}),N\delta x^{(N)}\rangle.\end{cases} \tag{3.15}$$

Introduce the limit processes (x_0^*, x_j^*, x^{**}) to replace $(N\delta x_0,\ N\delta x_j,\ \delta x_{-(0,i)})$ by $((N\delta x_0 - x_0^*) + x_0^*,\ (N\delta x_j - x_j^*) + x_j^*,\ (\delta x_{-(0,i)} - x^{**}) + x^{**})$ where

$$\begin{cases} dx_0^* = \Big(A_0 x_0^* + F_0 \delta x_i + F_0 x^{**}\Big)dt + \Big(C_0 x_0^* + \widetilde{F}_0 \delta x_i + \widetilde{F}_0 x^{**}\Big)dW_0, \quad x_0^*(0) = 0, \\ dx_j^* = \Big(A x_j^* + F\delta x_i + F x^{**}\Big)dt + \Big(C x_j^* + \widetilde{F}\delta x_i + \widetilde{F} x^{**} + \widetilde{G} x_0^*\Big)dW_j, \quad x_j^*(0) = 0, \\ dx^{**} = \Big(A x^{**} + F\delta x_i + F x^{**}\Big)dt, \quad x^{**}(0) = 0. \end{cases} \tag{3.16}$$

Therefore,

$$\begin{aligned} \delta \mathcal{J}_{soc}^{(N)} = \mathbb{E}\int_0^T \Big[& \langle Q\bar{x}_i, \delta x_i\rangle - \langle Q(\hat{H}\hat{x} + H\bar{x}_0) + \hat{H}Q(\hat{x} - \hat{H}\hat{x} - H\bar{x}_0), \delta x_i\rangle \\ & - \langle \hat{H}Q(\hat{x} - \hat{H}\hat{x} - H\bar{x}_0), x^{**}\rangle - \langle HQ(\hat{x} - \hat{H}\hat{x} - H\bar{x}_0), x_0^*\rangle \\ & + \frac{1}{N}\sum_{j\neq i}\langle Q(\bar{x}_j - \hat{H}\hat{x} - H\bar{x}_0), x_j^*\rangle + \langle R\bar{u}_i, \delta u_i\rangle\Big]dt + \sum_{l=1}^{7}\varepsilon_l, \end{aligned}$$

where

$$\begin{cases} \varepsilon_5 = E\int_0^T \langle \hat{H}Q(\hat{x} - \hat{H}\hat{x} - H\bar{x}_0), x^{**} - \delta x_{-(0,i)}\rangle dt, \\ \varepsilon_6 = E\int_0^T \langle HQ(\hat{x} - \hat{H}\hat{x} - H\bar{x}_0), x_0^* - N\delta x_0\rangle dt, \\ \varepsilon_7 = E\int_0^T \frac{1}{N}\sum_{j\neq i}\langle Q(\bar{x}_j - \hat{H}\hat{x} - H\bar{x}_0), N\delta x_j - x_j^*\rangle dt. \end{cases} \tag{3.17}$$

Replacing $\bar{x}_0$ by $(\bar{x}_0 - z_0) + z_0$, we have

$$\begin{aligned} \delta \mathcal{J}_{soc}^{(N)} = \mathbb{E}\int_0^T \Big[& \langle Q\bar{x}_i, \delta x_i\rangle - \langle Q(\hat{H}\hat{x} + Hz_0) + \hat{H}Q(\hat{x} - \hat{H}\hat{x} - Hz_0), \delta x_i\rangle \\ & - \langle \hat{H}Q(\hat{x} - \hat{H}\hat{x} - Hz_0), x^{**}\rangle - \langle HQ(\hat{x} - \hat{H}\hat{x} - Hz_0), x_0^*\rangle \\ & + \frac{1}{N}\sum_{j\neq i}\langle Q(\bar{x}_j - \hat{H}\hat{x} - Hz_0), x_j^*\rangle + \langle R\bar{u}_i, \delta u_i\rangle\Big]dt + \sum_{l=1}^{10}\varepsilon_l, \end{aligned} \tag{3.18}$$

where

$$\begin{cases} \varepsilon_8 = E\int_0^T \langle QH(z_0-\bar{x}_0)+\hat{H}Q(z_0-\bar{x}_0)+\hat{H}QH(\bar{x}_0-z_0), x^{**}\rangle dt, \\ \varepsilon_9 = E\int_0^T \langle HQH(\bar{x}_0-z_0), x_0^*\rangle dt, \\ \varepsilon_{10} = E\int_0^T \frac{1}{N}\sum_{j\neq i}\langle QH(z_0-\bar{x}_0), x_j^*\rangle dt. \end{cases} \tag{3.19}$$

Now we introduce the following dual processes of y_1^0, y_1^j and y_2:

$$\begin{cases} dy_1^0 = \Big[HQ(\hat{x}-\hat{H}\hat{x}-Hz_0)-A_0^Ty_1^0-C_0^T\beta_1^0-\widetilde{G}\mathbb{E}[\beta_1^{jj}|\mathcal{F}_t^{W_0}]\Big]dt+\beta_1^0dW_0, \quad y_1^0(T)=0, \\ dy_1^j = \Big[-Q(\bar{x}_j-\hat{H}\hat{x}-Hz_0)-A^Ty_1^j-C^T\beta_1^{jj}\Big]dt+\beta_1^{jj}dW_j+\sum_{k\neq j}\beta_1^{jk}dW_k, \quad y_1^j(T)=0, \\ dy_2 = \Big[\hat{H}Q(\hat{x}-\hat{H}\hat{x}-Hz_0)-F^T\mathbb{E}[y_1^j|\mathcal{F}_t^{W_0}]-\widetilde{F}^T\mathbb{E}[\beta_1^{jj}|\mathcal{F}_t^{W_0}]-\widetilde{F}^T\mathbb{E}[\beta_1^{jj}|\mathcal{F}_t^{W_0}] \\ \qquad -(A+F)^Ty_2-F_0^Ty_1^0-\widetilde{F}_0^T\beta_1^0\Big]dt+\beta_2^0dW_0, \quad y_2(T)=0, \quad j=1,\cdots,N, \end{cases}$$

to replace the intermediate variation terms x_0^*, x_j^* and x^{**}, respectively. Applying Itô's formula to $\langle y_1^j, x_j^*\rangle$, $\langle y_2, x^{**}\rangle$ and $\langle y_1^0, x_0^*\rangle$, we have

$$\begin{aligned} 0 =&\mathbb{E}\langle y_1^j(T), x_j^*(T)\rangle - \mathbb{E}\langle y_1^j(0), x_j^*(0)\rangle \\ =&\mathbb{E}\int_0^T \Big[\langle -Q(\bar{x}_j-\hat{H}\hat{x}-Hz_0), x_j^*\rangle + \langle F^Ty_1^j+\widetilde{F}\beta_1^{jj}, x^{**}\rangle \\ &+\langle \widetilde{G}^T\beta_1^{jj}, x_0^*\rangle + \langle F^Ty_1^j+\widetilde{F}^T\beta_1^{jj}, \delta x_i\rangle\Big]dt, \end{aligned} \tag{3.20}$$

$$\begin{aligned} 0 =&\mathbb{E}\langle y_2(T), x^{**}(T)\rangle - \mathbb{E}\langle y_2(0), x^{**}(0)\rangle \\ =&\mathbb{E}\int_0^T \Big[\langle \hat{H}Q(\hat{x}-\hat{H}\hat{x}-Hz_0)-F^T\mathbb{E}[y_1^j|\mathcal{F}_t^{W_0}] \\ &-\widetilde{F}^T\mathbb{E}[\beta_1^{jj}|\mathcal{F}_t^{W_0}]-F_0^Ty_1^0-\widetilde{F}_0^T\beta_1^0, x^{**}\rangle + \langle F^Ty_2, \delta x_i\rangle\Big]dt, \end{aligned} \tag{3.21}$$

$$\begin{aligned} 0 =&\mathbb{E}\langle y_1^0(T), x_0^*(T)\rangle - \mathbb{E}\langle y_1^0(0), x_0^*(0)\rangle \\ =&\mathbb{E}\int_0^T \Big[\langle HQ(\hat{x}-\hat{H}\hat{x}-Hz_0)-\widetilde{G}^T\mathbb{E}[\beta_1^{jj}|\mathcal{F}_t^{W_0}], x_0^*\rangle \\ &+\langle F_0^Ty_1^0+\widetilde{F}_0^T\beta_1^0, x^{**}\rangle + \langle F_0^Ty_1^0+\widetilde{F}_0^T\beta_1^0, \delta x_i\rangle\Big]dt. \end{aligned} \tag{3.22}$$

Adding to (3.18), we have

$$\delta\mathcal{J}_{soc}^{(N)}=\mathbb{E}\int_0^T\Big[\langle Q\bar{x}_i,\delta x_i\rangle-\langle Q(\hat{H}\hat{x}+Hz_0)+\hat{H}Q(\hat{x}-\hat{H}\hat{x}-Hz_0)-F^Ty_2-F^T\mathbb{E}[y_1^j|\mathcal{F}_t^{W_0}]$$
$$-\widetilde{F}^T\mathbb{E}[\beta_1^{jj}|\mathcal{F}_t^{W_0}]-F_0^Ty_1^0-\widetilde{F}_0^T\beta_1^0,\delta x_i\rangle+\langle R\bar{u}_i,\delta u_i\rangle\Big]dt+\sum_{l=1}^{12}\varepsilon_l, \tag{3.23}$$

where

$$\begin{cases}\varepsilon_{11}=\mathbb{E}\int_0^T\Big\langle F^T\Big(\frac{1}{N}\sum_{j\neq i}y_1^j-\mathbb{E}[y_1^j|\mathcal{F}_t^{W_0}]\Big)+\widetilde{F}^T\Big(\frac{1}{N}\sum_{j\neq i}\beta_1^{jj}-\mathbb{E}[\beta_1^{jj}|\mathcal{F}_t^{W_0}]\Big),x^{**}\Big\rangle dt,\\ \varepsilon_{12}=\mathbb{E}\int_0^T\widetilde{G}\Big(\frac{1}{N}\sum_{j\neq i}\beta_1^{jj}-\mathbb{E}[\beta_1^{jj}|\mathcal{F}_t^{W_0}]\Big),x_0^*\rangle dt,\\ \varepsilon_{13}=\mathbb{E}\int_0^T\Big\langle F^T\Big(\frac{1}{N}\sum_{j\neq i}y_1^j-\mathbb{E}[y_1^j|\mathcal{F}_t^{W_0}]\Big)+\widetilde{F}^T\Big(\frac{1}{N}\sum_{j\neq i}\beta_1^{jj}-\mathbb{E}[\beta_1^{jj}|\mathcal{F}_t^{W_0}]\Big),\delta x_i\Big\rangle dt.\end{cases} \tag{3.24}$$

Therefore, considering the case when $N\longrightarrow\infty$, we introduce the first variation of the decentralized auxiliary cost functional δJ_i as follows:

$$\delta J_i=\mathbb{E}\int_0^T\Big[-\langle Q(\hat{H}\hat{x}+Hz_0)+\hat{H}Q(\hat{x}-\hat{H}\hat{x}-Hz_0)-F^Ty_2-F^T\hat{y}_1-\widetilde{F}^T\hat{\beta}_1$$
$$-F_0^Ty_1^0-\widetilde{F}_0^T\beta_1^0,\delta x_i\rangle+\langle Q\bar{x}_i,\delta x_i\rangle+\langle R\bar{u}_i,\delta u_i\rangle\Big]dt. \tag{3.25}$$

Remark 3.4. *In* (3.25)*, we ignore* $\varepsilon_1,\cdots,\varepsilon_{13}$ *and introduce the first variation of the auxiliary cost functional* δJ_i*. Actually,* $\varepsilon_1,\cdots,\varepsilon_{13}$ *have some order as* $\bar{x}^{(N)}-\hat{x}$*, and it is sufficient to conjecture* $\|\bar{x}^{(N)}-\hat{x}\|_{L^2}^2\longrightarrow 0$ *when* $N\longrightarrow\infty$ *by considering the weakly coupled structure of our problem. The rigorous proof will be given in Section 3.5.*

3.3.2 Decentralized strategy

Motivated by (3.25), one can introduce the following auxiliary problem:

Problem 3.3. *Minimize $J_i(u_i)$ over $u_i \in \mathcal{U}_i$ where*

$$\begin{cases} dx_i = \big(Ax_i + Bu_i + F\hat{x}\big)dt + \big(Cx_i + Du_i + \widetilde{F}\hat{x} + \widetilde{G}z_0\big)dW_i, \;\; x_i(0) = x, \\ J_i(u_i) = \frac{1}{2}\mathbb{E}\int_0^T [\langle Qx_i, x_i\rangle - 2\langle S, x_i\rangle + \langle Ru_i, u_i\rangle]dt, \\ S = Q(\hat{H}\hat{x} + Hz_0) + \hat{H}Q(\hat{x} - \hat{H}\hat{x} - Hz_0) - F^T y_2 - F^T\hat{y}_1 - \widetilde{F}^T\hat{\beta}_1 - F_0^T y_1^0 - \widetilde{F}_0^T\beta_1^0. \end{cases} \tag{3.26}$$

The MF terms $\hat{x}$, z_0, y_2, $\hat{y}_1$, $\hat{\beta}_1$, y_1^0, β_1^0 will be determined by the CC system in Section 3.4. From [58], we have the following result:

Proposition 3.2. *Under the assumptions* (A3.1)-(A3.2) *and* (SA)*, the following Riccati equation*

$$\begin{cases} \dot{P} + PA + A^T P + C^T PC + Q - (PB + C^T PD)^{-1}(B^T P + D^T PC) = 0, \\ P(T) = 0, \end{cases} \tag{3.27}$$

is strongly regularly solvable, and Problem 3.3 admits a feedback optimal control $\bar{u}_i = \Lambda_1\bar{x}_i + \Lambda_2$ where

$$\begin{cases} \Lambda_1 = -(R + D^T PD)^{-1}(B^T P + D^T PC), \\ \Lambda_2 = -(R + D^T PD)^{-1}(B^T\varphi + D^T\eta + D^T P(\widetilde{F}\hat{x} + \widetilde{G}z_0)), \end{cases} \tag{3.28}$$

and (φ, η) satisfies

$$\begin{cases} d\varphi(t) = -\Big\{[A^T - (PB + C^T PD)(R + D^T PD)^{-1}B^T]\varphi \\ \qquad\qquad + [C^T - (PB + C^T PD)(R + D^T PD)^{-1}D^T]\eta \\ \qquad\qquad + [(PB + C^T PD)(R + D^T PD)^{-1}D^T - C^T]P(\widetilde{F}\hat{x} + \widetilde{G}z_0) \\ \qquad\qquad + PF\hat{x} - S\Big\}dt + \eta dW_0(t), \quad \varphi(T) = 0. \end{cases} \tag{3.29}$$

3.4 CC system

Because of the symmetric and decentralized character, we only need a generic Brownian motion (still denoted by W_1), which is independent of W_0, to characterize the CC system.

Proposition 3.3. *The undetermined quantities in Problem 3.2, 3.3 can be determined by* $(\hat{x},\ z_0,\ y_1^0,\ \beta_1^0,\ \hat{y}_1,\ \hat{\beta}_1,\ y_2)=(\mathbb{E}[z|\mathcal{F}_t^{W_0}],\ z_0,\ \check{y}_0,\ \check{\beta}_0,\ \mathbb{E}[\check{y}_1|\mathcal{F}_t^{W_0}],\ \mathbb{E}[\check{\beta}_1^1|\mathcal{F}_t^{W_0}],\ \check{y}_2)$*, where* $(z,\ z_0,\ \check{y}_0,\ \check{\beta}_0,\ \check{y}_1,\ \check{\beta}_1^1,\ \check{y}_2)$ *is the solution of the following MF-FBSDEs system:*

$$
\begin{cases}
dz_0 = [(A_0-B_0\mathcal{R}_0^{-1}\mathcal{P}_0)z_0-B_0\mathcal{R}_0^{-1}B_0^T\check{\phi}-B_0\mathcal{R}_0^{-1}D_0^T\check{\zeta} \\
\qquad + (F_0-B_0\mathcal{R}_0^{-1}D_0^TP_0\widetilde{F}_0)\mathbb{E}[z|\mathcal{F}_t^{W_0}]]dt+[(C_0-D_0\mathcal{R}_0^{-1}\mathcal{P}_0)z_0 \\
\qquad -D_0\mathcal{R}_0^{-1}B_0^T\check{\phi}-D_0\mathcal{R}_0^{-1}D_0^T\check{\zeta}+(\widetilde{F}_0-D_0\mathcal{R}_0^{-1}D_0^TP_0\widetilde{F}_0)\mathbb{E}[z|\mathcal{F}_t^{W_0}]]dW_0, \\
dz = [(A-B\mathcal{R}^{-1}\mathcal{P})z-B\mathcal{R}^{-1}D^TP\widetilde{G}z_0-B\mathcal{R}^{-1}B^T\check{\varphi}-B\mathcal{R}^{-1}D^T\check{\eta} \\
\qquad +(F-B\mathcal{R}^{-1}D^TP\widetilde{F})\mathbb{E}[z|\mathcal{F}_t^{W_0}]]dt+[(C-D\mathcal{R}^{-1}\mathcal{P})z \\
\qquad +(\widetilde{G}-D\mathcal{R}^{-1}D^T\widetilde{G})z_0-D\mathcal{R}^{-1}B^T\check{\varphi}-D\mathcal{R}^{-1}D^T\check{\eta}+(\widetilde{F}-D\mathcal{R}^{-1}D^TP\widetilde{F})\mathbb{E}[z|\mathcal{F}_t^{W_0}]]dW_1, \\
d\check{y}_0=[-HQHz_0+HQ(I-\hat{H})\mathbb{E}[z|\mathcal{F}_t^{W_0}]-A_0^T\check{y}_0-C_0^T\check{\beta}_0-\widetilde{G}^T\mathbb{E}[\check{\beta}_1^1|\mathcal{F}_t^{W_0}]]dt+\check{\beta}_0dW_0, \\
d\check{y}_1 = [QHz_0-Qz+Q\hat{H}\mathbb{E}[z|\mathcal{F}_t^{W_0}]-A^T\check{y}_1-C^T\check{\beta}_1^1]dt+\check{\beta}_1^0dW_0+\check{\beta}_1^1dW_1, \\
d\check{y}_2 = [-\hat{H}QHz_0+\hat{H}Q(I-\hat{H})\mathbb{E}[z|\mathcal{F}_t^{W_0}]-F^T\mathbb{E}[\check{y}_1|\mathcal{F}_t^{W_0}] \\
\qquad -\widetilde{F}^T\mathbb{E}[\check{\beta}_1^1|\mathcal{F}_t^{W_0}]-(A+F)^T\check{y}_2-F_0^T\check{y}_0-\widetilde{F}_0^T\check{\beta}_0]dt+\check{\beta}_2dW_0, \\
d\check{\phi} =-[(A_0^T-\mathcal{P}_0^T\mathcal{R}_0^{-1}B_0^T)\check{\phi}+(C_0^T-\mathcal{P}_0^T\mathcal{R}_0^{-1}D_0^T)\check{\zeta}-(\mathcal{P}_0^T\mathcal{R}_0^{-1}D_0^T-C_0^T) \\
\qquad \times P_0\widetilde{F}_0\mathbb{E}[z|\mathcal{F}_t^{W_0}]+P_0F_0\mathbb{E}[z|\mathcal{F}_t^{W_0}]-Q_0H_0\mathbb{E}[z|\mathcal{F}_t^{W_0}]]dt+\check{\zeta}dW_0, \\
d\check{\varphi} =-[(A^T-\mathcal{P}^T\mathcal{R}^{-1}B^T)\check{\varphi}+(C^T-\mathcal{P}^T\mathcal{R}^{-1}D^T)\check{\eta}+(\mathcal{P}^T\mathcal{R}^{-1}D^T-C^T)P\widetilde{G}z_0 \\
\qquad +(PF+(\mathcal{P}^T\mathcal{R}^{-1}D^T-C^T)P\widetilde{F})\mathbb{E}[z|\mathcal{F}_t^{W_0}]-S]dt+\check{\eta}dW_0, \\
z_0(0) = \xi_0,\ z(0) = \xi,\ \check{y}_0(T) = 0,\ \check{y}_1(T) = 0,\ \check{y}_2(T) = 0,\ \check{\phi}(T) = 0,\ \check{\varphi}(T) = 0,
\end{cases}
\tag{3.30}
$$

with

$$\begin{cases} \mathcal{P} := B^T P + D^T P C, \quad \mathcal{P}_0 := B_0^T P_0 + D_0^T P_0 C_0 \\ \mathcal{R} := R + D^T P D, \quad \mathcal{R}_0 := R_0 + D_0^T P_0 D_0. \end{cases}$$

By letting $X = (z_0^T, z^T)^T$, $Y = (\check{y}_0^T, \check{y}_1^T, \check{y}_2^T, \check{\phi}^T, \check{\varphi}^T)^T$, $Z_1 = (\check{\beta}_0^T,\ \check{\beta}_1^{0T},\ \check{\beta}_2^T,\ \check{\zeta}^T,\ \check{\eta}^T)^T$, $Z_2 = (0^T,\ \check{\beta}_1^{1T},\ 0^T,\ 0^T,\ 0^T)^T$, $Z = (Z_1, Z_2)$ and $W = (W_0^T, W_1^T)^T$, (3.30) takes the following form:

$$\begin{cases} dX = \Big[\mathbb{A}_1 X + \bar{\mathbb{A}}_1 \mathbb{E}[X|\mathcal{F}_t^{W_0}] + \mathbb{B}_1 Y + \mathbb{F}_1 Z_1\Big] dt \\ \qquad + \Big[\mathbb{C}_1^0 X + \bar{\mathbb{C}}_1^0 \mathbb{E}[X|\mathcal{F}_t^{W_0}] + \mathbb{D}_1^0 Y + \mathbb{F}_1^0 Z_1\Big] dW_0 \\ \qquad + \Big[\mathbb{C}_1^1 X + \bar{\mathbb{C}}_1^1 \mathbb{E}[X|\mathcal{F}_t^{W_0}] + \mathbb{D}_1^1 Y + \mathbb{F}_1^1 Z_1\Big] dW_1, \\ dY = \Big[\mathbb{A}_2 X + \bar{\mathbb{A}}_2 \mathbb{E}[X|\mathcal{F}_t^{W_0}] + \mathbb{B}_2 Y + \bar{\mathbb{B}}_2 \mathbb{E}[Y|\mathcal{F}_t^{W_0}] \\ \qquad + \mathbb{C}_2 Z_1 + \tilde{\mathbb{C}}_2 Z_2 + \bar{\mathbb{C}}_2 \mathbb{E}[Z_2|\mathcal{F}_t^{W_0}]\Big] dt + Z_1 dW_0 + Z_2 dW_1, \\ X(0) = (\xi_0^T, \xi^T)^T, \quad Y(T) = (0^T, 0^T, 0^T, 0^T, 0^T)^T, \end{cases} \tag{3.31}$$

where

$$\mathbb{A}_1 = \begin{pmatrix} A_0 - B_0 \mathcal{R}_0^{-1} \mathcal{P}_0 & 0 \\ -B\mathcal{R}^{-1} D^T P \widetilde{G} & A - B\mathcal{R}^{-1}\mathcal{P} \end{pmatrix}, \bar{\mathbb{A}}_1 = \begin{pmatrix} 0 & F_0 - B_0 \mathcal{R}_0^{-1} D_0^T P_0 \widetilde{F}_0 \\ 0 & F - B\mathcal{R} D^T P \widetilde{F} \end{pmatrix}, \mathbb{B}_1 = \begin{pmatrix} 0 & 0 & 0 & -B_0 \mathcal{R}_0^{-1} B_0^T & 0 \\ 0 & 0 & 0 & 0 & -B\mathcal{R}B^T \end{pmatrix},$$

$$\mathbb{F}_1 = \begin{pmatrix} 0 & 0 & 0 & -B_0 \mathcal{R}_0^{-1} D_0^T & 0 \\ 0 & 0 & 0 & 0 & -B_0 \mathcal{R}_0^{-1} D^T \end{pmatrix}, \mathbb{F}_1^0 = \begin{pmatrix} 0 & 0 & 0 & -D_0 \mathcal{R}_0^{-1} D_0^T & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \bar{\mathbb{C}}_1^0 = \begin{pmatrix} 0 & \widetilde{F}_0 - D_0 \mathcal{R}_0^{-1} D_0^T P_0 \widetilde{F}_0 \\ 0 & 0 \end{pmatrix},$$

$$\mathbb{C}_1^0 = \begin{pmatrix} C_0 - D_0 \mathcal{R}_0^{-1} \mathcal{P}_0 & 0 \\ 0 & 0 \end{pmatrix}, \mathbb{C}_1^1 = \begin{pmatrix} 0 & 0 \\ \widetilde{G} - D\mathcal{R}^{-1} D^T \widetilde{G} & C - D\mathcal{R}^{-1}\mathcal{P} \end{pmatrix}, \bar{\mathbb{C}}_1^1 = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{F} - D\mathcal{R}^{-1} D^T P \widetilde{F} \end{pmatrix},$$

$$\mathbb{D}_1^0 = \begin{pmatrix} 0 & 0 & 0 & -D_0 \mathcal{R}_0^{-1} B_0^T & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathbb{D}_1^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -D\mathcal{R}^{-1} B^T \end{pmatrix}, \mathbb{F}_1^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -D\mathcal{R}^{-1} D^T \end{pmatrix},$$

$$\mathbb{A}_2 = \begin{pmatrix} -HQH & 0 \\ QH & -Q \\ -\hat{H}QH & 0 \\ 0 & 0 \\ -(\mathcal{P}^T \mathcal{R}^{-1} D^T - C^T) P \widetilde{G} + QH - \hat{H}QH & 0 \end{pmatrix}, \mathbb{B}_2 = \begin{pmatrix} -A_0^T & 0 & 0 & 0 & 0 \\ 0 & -A^T & 0 & 0 & 0 \\ -F_0^T & 0 & -(A+F)^T & 0 & 0 \\ 0 & 0 & 0 & \mathbb{B}_2' & 0 \\ -F_0^T & 0 & -F^T & 0 & \mathbb{B}_2'' \end{pmatrix},$$

$$\bar{\mathbb{A}}_2 = \begin{pmatrix} 0 & HQ(I - \hat{H}) \\ 0 & Q\hat{H} \\ 0 & \hat{H}Q(I - \hat{H}) \\ 0 & \bar{\mathbb{A}}_2' \\ 0 & \bar{\mathbb{A}}_2'' \end{pmatrix}, \bar{\mathbb{B}}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -F^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -F^T & 0 & 0 & 0 \end{pmatrix}, \bar{\mathbb{C}}_2 = \begin{pmatrix} 0 & -\widetilde{G}^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\widetilde{F}^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\widetilde{F}^T & 0 & 0 & 0 \end{pmatrix},$$

$$\bar{\mathbb{A}}_2' = [\mathcal{P}_0^T\mathcal{R}_0^{-1}D_0^T - C_0^T]P_0\widetilde{F}_0 - P_0F_0 + Q_0H_0, \quad \bar{\mathbb{A}}_2'' = -F + [\mathcal{P}^T\mathcal{R}^{-1}D^T - C^T]P\widetilde{F} + \hat{H}Q + Q\hat{H} - \hat{H}Q\hat{H},$$

$$\mathbb{B}_2' = -A_0^T + \mathcal{P}_0^T\mathcal{R}_0^{-1}B_0^T, \quad \mathbb{B}_2'' = -A^T + \mathcal{P}^T\mathcal{R}^{-1}B^T, \quad \mathbb{C}_2' = -(C_0^T - \mathcal{P}_0^T\mathcal{R}_0^{-1}D_0^T), \quad \mathbb{C}_2'' = -(C^T - \mathcal{P}^T\mathcal{R}^{-1}D^T).$$

Next we use discounting method to study the global solvability of the FBSDEs system (3.31). To start, we first give some results for general nonlinear forward-backward system:

$$\begin{cases} dX(t) = b\Big(t, X(t), \mathbb{E}[X(t)|\mathcal{F}_t^{W_0}], Y(t), Z(t)\Big)dt \\ \qquad\qquad + \sigma\Big(t, X(t), \mathbb{E}[X(t)|\mathcal{F}_t^{W_0}], Y(t), Z(t)\Big)dW(t), \qquad X(0) = x, \\ dY(t) = -f\Big(t, X(t), \mathbb{E}[X(t)|\mathcal{F}_t^{W_0}], Y(t), \mathbb{E}[Y(t)|\mathcal{F}_t^{W_t}], Z(t), \mathbb{E}[Z(t)|\mathcal{F}_t^{W_0}]\Big)dt \\ \qquad\qquad + Z(t)dW(t), \qquad Y(T) = 0, \end{cases} \tag{3.32}$$

where $W = \binom{W_0}{W_1}$, and the coefficients satisfy the following conditions:

(H3.1) There exist $\rho_1, \rho_2 \in \mathbb{R}$ and positive constants $k_i, i = 1, \cdots, 12$ such that for all $t, x, \bar{x}, y, \bar{y}, z, \bar{z}$, a.s.,

1. $\langle b(t, x_1, \bar{x}, y, z) - b(t, x_2, \bar{x}, y, z), x_1 - x_2\rangle \le \rho_1\|x_1 - x_2\|^2$,

2. $\|b(t, x, \bar{x}_1, y_1, z_1) - b(t, x, \bar{x}_2, y_2, z_2)\| \le k_1\|\bar{x}_1 - \bar{x}_2\| + k_2\|y_1 - y_2\| + k_3\|z_1 - z_2\|$,

3. $\langle f(t, x, \bar{x}, y_1, \bar{y}, z, \bar{z}) - f(t, x, \bar{x}, y_2, \bar{y}, z, \bar{z}), y_1 - y_2\rangle \le \rho_2\|y_1 - y_2\|^2$,

4. $\|f(t, x_1, \bar{x}_1, y, \bar{y}_1, z_1, \bar{z}_1) - f(t, x_2, \bar{x}_2, y, \bar{y}_2, z_2, \bar{z}_2)\| \le k_4\|x_1 - x_2\| + k_5\|\bar{x}_1 - \bar{x}_2\| + k_6\|\bar{y}_1 - \bar{y}_2\| + k_7\|z_1 - z_2\| + k_8\|\bar{z}_1 - \bar{z}_2\|$,

5. $\|\sigma(t,x_1,\bar{x}_1,y_1,z_1) - \sigma(t,x_2,\bar{x}_2,y_2,z_2)\|^2 \leq k_9^2\|x_1 - x_2\|^2 + k_{10}^2\|\bar{x}_1 - \bar{x}_2\|^2 + k_{11}^2\|y_1 - y_2\|^2 + k_{12}^2\|z_1 - z_2\|^2$.

(H3.2)

$$\mathbb{E}\int_0^T \Big[\|b(t,0,0,0,0)\|^2 + \|\sigma(t,0,0,0,0)\|^2 + \|f(t,0,0,0,0,0,0,0)\|^2\Big]dt < \infty.$$

Similar to [64] and [76], we have the following result of solvability of (3.32). For the readers' convenience, we give the proof in Appendix B.

Theorem 3.1. *Suppose the assumptions* (H3.1) *and* (H3.2) *hold. There exists a constant* $\delta_1 > 0$ *depending on* $\rho_1, \rho_2, T, k_i, i = 1,6,7,8,9,10$ *such that if* $k_i \in [0,\delta_1)$, $i = 2,3,4,5,11,12$*, the FBSDEs system* (3.32) *admits a unique adapted solution* $(X,Y,Z) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)\times L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)\times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{m\times d})$. *Furthermore, if* $2\rho_1 + 2\rho_2 < -2k_1 - 2k_6 - 2k_7^2 - 2k_8^2 - k_9^2 - k_{10}^2$*, there exists a constant* $\delta_2 > 0$ *depending on* $\rho_1, \rho_2, k_i, i = 1,6,7,8,9,10$ *such that if* $k_i \in [0,\delta_2)$, $i = 2,3,4,5,11,12$*, the FBSDEs system* (3.32) *admits a unique adapted solution* $(X,Y,Z) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^m) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{m\times d})$.

Let ρ_1^* and ρ_2^* be the largest eigenvalue of $\frac{1}{2}(\mathbb{A}_1+\mathbb{A}_1^T)$ and $\frac{1}{2}(\mathbb{B}_2+\mathbb{B}_2^T)$, respectively. Comparing (3.32) with (3.31), we can check that the parameters of (H3.1) can be chosen as follows:

$$k_1 = \|\bar{\mathbb{A}}_1\|, \quad k_2 = \|\mathbb{B}_1\|, \quad k_3 = \|F_1\|, \quad k_4 = \|\mathbb{A}_2\|, \quad k_5 = \|\bar{\mathbb{A}}_2\|, \quad k_6 = \|\bar{\mathbb{B}}_2\|,$$
$$k_7 = \|\mathbb{C}_2\| + \|\tilde{\mathbb{C}}_2\|, \quad k_8 = \|\bar{\mathbb{C}}_2\|, \quad k_9 = \|\mathbb{C}_1^0\| + \|\mathbb{C}_1^1\|, \quad k_{10} = \|\bar{\mathbb{C}}_1^0\| + \|\bar{\mathbb{C}}_1^1\|,$$
$$k_{11} = \|\mathbb{D}_1^0\| + \|\mathbb{D}_1^1\|, \quad k_{12} = \|F_1^0\| + \|F_1^1\|.$$

Now we introduce the following assumption:

(A3.3) $2\rho_1^* + 2\rho_2^* < -2\|\bar{\mathbb{A}}_1\| - 2\|\bar{\mathbb{B}}_2\| - 2\|\mathbb{C}_2\|^2 - 2\|\bar{\mathbb{C}}_2\|^2 - \|\mathbb{C}_1^0\|^2 - \|\bar{\mathbb{C}}_1^0\|^2.$

We have the following result:

Proposition 3.4. *Under the assumptions* (A3.1)-(A3.3)*, there exists a constant* $\delta_3 > 0$ *depending on* ρ_1^**,* ρ_2^**,* k_i*,* $i = 1, 6, 7, 8, 9, 10$*, such that if* $k_i \in [0, \delta_3)$*,* $i = 2, 3, 4, 5, 11, 12$*, the FBSDEs system* (3.31) *admits a unique adapted solution* $(X, Y, Z) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^m) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m\times d})$.

In what follows, we give an example to show how exactly such conditions can be applied.

Remark 3.5. *For* $\varepsilon > 0$*, let* $\rho_1 = \frac{k_2}{\varepsilon}$*,* $\rho_2 = \frac{k_3}{\varepsilon}$*,* $\rho_3 = \frac{k_4}{\varepsilon}$*,* $\rho_4 = \frac{k_5}{\varepsilon}$*,* $\rho_5 = \frac{k_7}{2k_7^2+2\varepsilon}$*,* $\rho_6 = \frac{k_8}{2k_8^2+2\varepsilon}$*,* $d = -2k_1 - 2k_6 - 2k_7^2 - 2k_8^2 - k_9^2 - k_10^2 - 2\rho_1^* - 2\rho_2^* - 4\varepsilon$*,* $\bar{\rho}_1 = \bar{\rho}_2 = \frac{d}{2}$*,* $\theta = \big(\frac{1}{\bar{\rho}_2} + \frac{1}{1-k_7\rho_5-k_8\rho_{10}}\big)\big(\frac{1}{\bar{\rho}_1}\big) = \big(\frac{2}{d} + \frac{(k_7^2+\varepsilon)(k_8^2+\varepsilon)}{\varepsilon^2}\big)\big(\frac{2}{d}\big)$*. If* $d > 0$*,* $\theta\big(\frac{k_4^2}{\varepsilon} + \frac{k_5^2}{\varepsilon}\big) < 1$*,* $\theta\big(\frac{k_2^2}{\varepsilon} + k_{11}^2\big) < 1$*,* $\theta\big(\frac{k_3^2}{\varepsilon} + k_{12}^2\big) < 1$*. Then* (3.31) *admits a unique solution.*

Thus, via Proposition 3.1, 3.2, 3.4, we can conclude the following procedure to calculate the MF strategy.

- Under (A3.1)-(A3.3) and (SA), each agent can calculate CC system (3.30) and obtain (z, z_0, $\check{y}_0$, $\check{\beta}_0$, $\check{y}_1$, $\check{\beta}_1^1$, $\check{y}_2$). Then by taking conditional expectation, the MF terms can be obtained by ($\hat{x}$, z_0, y_1^0, β_1^0, $\hat{y}_1$, $\hat{\beta}_1$, y_2)=($\mathbb{E}[z|\mathcal{F}_t^{W_0}]$, z_0, $\check{y}_0$, $\check{\beta}_0$, $\mathbb{E}[\check{y}_1|\mathcal{F}_t^{W_0}]$, $\mathbb{E}[\check{\beta}_1^1|\mathcal{F}_t^{W_0}]$, $\check{y}_2$).

- With $(\hat{x},\ z_0,\ y_1^0,\ \beta_1^0,\ \hat{y}_1,\ \hat{\beta}_1,\ y_2)$, the agents can solve Riccati equations (3.8), (3.27) and BSDEs (3.10), (3.29) to obtain P_0, P, ϕ, φ.

- With P_0, P, ϕ, φ, the agents can obtain (Θ_1, Θ_2) and (Λ_1, Λ_2) by (3.9) and (3.28) respectively. Then the feedback form MF decentralized strategies are given by $\tilde{u}_0 = \Theta_1 \tilde{x}_0 + \Theta_2$, $\tilde{u}_i = \Lambda_1 \tilde{x}_i + \Lambda_2$, for $i = 1, \cdots, N$, where $\tilde{x}_0$ and $\tilde{x}_i$ are the realized states satisfying

$$\begin{cases} d\widetilde{x}_0 = \Big[(A_0 + B_0\Theta_1)\widetilde{x}_0 + B_0\Theta_2 + F_0\widetilde{x}^{(N)}\Big]dt + \Big[(C_0 + D_0\Theta_1)\widetilde{x}_0 + D_0\Theta_2 + \widetilde{F}_0\widetilde{x}^{(N)}\Big]dW_0, \\ d\widetilde{x}_i = \Big[(A + B\Lambda_1)\widetilde{x}_i + B\Lambda_2 + F\widetilde{x}^{(N)}\Big]dt + \Big[(C + D\Lambda_1)\widetilde{x}_i + D\Lambda_2 + \widetilde{F}\widetilde{x}^{(N)} + \widetilde{G}\widetilde{x}_0\Big]dW_i, \\ \widetilde{x}_0(0) = \xi_0, \qquad \widetilde{x}_i(0) = \xi, \qquad 1 \le i \le N. \end{cases} \tag{3.33}$$

Through the discussion above, the MF decentralized strategies have been characterized. For the next part, we will study the performance of the MF strategy. Specifically, we will prove its asymptotic optimality.

3.5 Asymptotic ε optimality

Definition 3.1. *A mixed strategy set $\{u_i^\varepsilon \in \mathcal{U}_i\}_{i=0}^N$ is called asymptotically ε-optimal if there exists $\varepsilon = \varepsilon(N) > 0$, $\lim_{N\to\infty} \varepsilon(N) = 0$ such that*

$$\begin{cases} \mathcal{J}_0(u_0^\varepsilon, u_{-0}^\varepsilon) \le \inf\limits_{u_0 \in \mathcal{U}_0} \mathcal{J}_0(u_0, u_{-0}^\varepsilon) + \varepsilon, \\ \dfrac{1}{N}\big(\mathcal{J}_{soc}^{(N)}(u_0^\varepsilon, u_{-0}^\varepsilon) - \inf\limits_{u_{-0} \in \mathcal{U}_{-0}} \mathcal{J}_{soc}^{(N)}(u_0^\varepsilon, u_{-0})\big) \le \varepsilon, \end{cases}$$

where $u_{-0}^\varepsilon := \{u_1^\varepsilon, \cdots, u_N^\varepsilon\}$. In this case, $u_0^\varepsilon, u_{-0}^\varepsilon$ achieve an asymptotic ε-equilibrium, and $u_1^\varepsilon, \cdots, u_N^\varepsilon$ achieve an asymptotic ε-social optima.

Let $\widetilde{u}$ be the MF strategy given in Section 3.4 and the realized decentralized states $(\widetilde{x}_0, \widetilde{x}_1, \cdots, \widetilde{x}_N)$ satisfy (3.33) and $\widetilde{x}^{(N)} = \frac{1}{N}\sum_{i=1}^N \widetilde{x}_i$. For the optimal controls of the auxiliary problems $\bar{u} = (\bar{u}_0,\ \bar{u}_1,\ \cdots,\ \bar{u}_N)$, the corresponding optimal states are:

$$\begin{cases} d\bar{x}_0 = \Big[(A_0+B_0\Theta_1)\bar{x}_0+B_0\Theta_2+F_0\mathbb{E}[z|\mathcal{F}_t^{W_0}]\Big]dt \\ \qquad\qquad +\Big[(C_0+D_0\Theta_1)\bar{x}_0+D_0\Theta_2+\widetilde{F}_0\mathbb{E}[z|\mathcal{F}_t^{W_0}]\Big]dW_0(t), \\ d\bar{x}_i = \Big[(A+B\Lambda_1)\bar{x}_i+B\Lambda_2+F\mathbb{E}[z|\mathcal{F}_t^{W_0}]\Big]dt \\ \qquad\qquad +\Big[(C+D\Lambda_1)\bar{x}_i+D\Lambda_2+\widetilde{F}\mathbb{E}[z|\mathcal{F}_t^{W_0}]+\widetilde{G}\bar{x}_0\Big]dW_i(t), \\ \bar{x}_0(0) = \xi_0, \qquad \bar{x}_i(0) = \xi, \qquad 1 \le i \le N, \end{cases}$$

where z is the solution of (3.30). Therefore, the optimal control of Problem 3.2 is $\bar{u}_0 = \Theta_1\bar{x}_0 + \Theta_2$. Before further discussion, we need some estimations. In the proofs below, we will use K to denote a generic constant whose value may change from line to line.

Lemma 3.1. *[64, Lemma 5.1] Under* (A3.1)-(A3.3) *and* (SA)*, there exists a constant K_1 independent of N such that*

$$\sup_{0\le i\le N} \mathbb{E} \sup_{0\le t\le T} \|\widetilde{x}_i(t)\|^2 \le K_1.$$

Lemma 3.2. *Under* (A3.1)-(A3.3) *and* (SA)*, there exists a constant K_2 independent of N such that*

$$\mathbb{E} \sup_{0\le t\le T} \left\|\widetilde{x}^{(N)}(t) - \mathbb{E}[z|\mathcal{F}_t^{W_0}]\right\|^2 \le \frac{K_2}{N}.$$

Proof. It is easy to get that

$$d\Big(\widetilde{x}^{(N)} - \mathbb{E}[z|\mathcal{F}_t^{W_0}]\Big) = (A + B\Lambda_1 + F)\Big(\widetilde{x}^{(N)} - \mathbb{E}[z|\mathcal{F}_t^{W_0}]\Big)dt + \frac{1}{N}\sum_{i=1}^{N}\Big[(C + D\Lambda_1)\widetilde{x}_i + D\Lambda_2 + \widetilde{F}\widetilde{x}^{(N)} + \widetilde{G}\widetilde{x}_0\Big]dW_i.$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\sup_{0\le s\le t}\Big\|\widetilde{x}^{(N)} - \mathbb{E}[z|\mathcal{F}_t^{W_0}]\Big\|^2 \\
\le& K\mathbb{E}\int_0^t \Big\|\widetilde{x}^{(N)} - \mathbb{E}[z|\mathcal{F}_t^{W_0}]\Big\|^2 ds \\
&+ \frac{K}{N^2}\mathbb{E}\sup_{0\le s\le t}\Big\|\int_0^s\sum_{i=1}^{N}\Big[(C + D\Lambda_1)\widetilde{x}_i + D\Lambda_2 + \widetilde{F}\widetilde{x}^{(N)} + \widetilde{G}\widetilde{x}_0\Big]dW_i\Big\|^2.
\end{aligned}$$

Note that $\widetilde{x}_i$, $\widetilde{x}_0$ and $\widetilde{x}^{(N)}$ are all continuous, and the coefficients $(C + D\Lambda_1)$, $D\Lambda_2$, $\tilde{F}$ and $\tilde{G}$ are all bounded. Then the integrand $(C+D\Lambda_1)\widetilde{x}_i+D\Lambda_2+\widetilde{F}\widetilde{x}^{(N)}+\widetilde{G}\widetilde{x}_0 \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$. Thus, we can apply Burkholder-Davis-Gundy inequality (see Corollary 1.1) and obtain

$$\begin{aligned}
&\mathbb{E}\sup_{0\le s\le t}\Big\|\widetilde{x}^{(N)} - \mathbb{E}[z|\mathcal{F}_t^{W_0}]\Big\|^2 \\
\le& K\mathbb{E}\int_0^t \Big\|\widetilde{x}^{(N)} - \mathbb{E}[z|\mathcal{F}_t^{W_0}]\Big\|^2 ds + \frac{K}{N^2}\mathbb{E}\int_0^t\sum_{i=1}^{N}\Big\|(C+D\Lambda_1)\widetilde{x}_i+D\Lambda_2+\widetilde{F}\widetilde{x}^{(N)}+\widetilde{G}\widetilde{x}_0\Big\|^2 ds \\
\le& K\mathbb{E}\int_0^t \Big\|\widetilde{x}^{(N)} - \mathbb{E}[z|\mathcal{F}_t^{W_0}]\Big\|^2 ds + \frac{K}{N}\Big(1+\sup_{0\le i\le N}\mathbb{E}\sup_{0\le t\le T}\|\widetilde{x}_i(t)\|^2\Big).
\end{aligned}$$

Finally, it follows from Grönwall inequality, and Lemma 3.1 that there exists a constant K_2 independent of N such that

$$\mathbb{E}\sup_{0\le t\le T}\Big\|\widetilde{x}^{(N)}(t) - \mathbb{E}[z|\mathcal{F}_t^{W_0}]\Big\|^2 \le \frac{K_2}{N}.$$

$\square$

Lemma 3.3. *Under* (A3.1)-(A3.3) *and* (SA), *there exists a constant* K_3 *independent of* N *such that*

$$\sup_{0\le i\le N}\mathbb{E}\sup_{0\le t\le T}\left\|\widetilde{x}_i(t)-\bar{x}_i(t)\right\|^2\le\frac{K_3}{N}.$$

Proof. It is easy to check that

$$\begin{aligned}d(\widetilde{x}_i-\bar{x}_i)=&\Big[(A+B\Lambda_1)(\widetilde{x}_i-\bar{x}_i)+F(\widetilde{x}^{(N)}-\mathbb{E}[z|\mathcal{F}_t^{W_0}])\Big]dt\\&+\Big[(C+D\Lambda_1)(\widetilde{x}_i-\bar{x}_i)+\widetilde{F}(\widetilde{x}^{(N)}-\mathbb{E}[z|\mathcal{F}_t^{W_0}])+\widetilde{G}(\widetilde{x}_0-\bar{x}_0)\Big]dW_i(t),\end{aligned}$$

and

$$\begin{aligned}d(\widetilde{x}_0-\bar{x}_0)=&\Big[(A_0+B_0\Theta_1)(\widetilde{x}_0-\bar{x}_0)+F_0(\widetilde{x}^{(N)}-\mathbb{E}[z|\mathcal{F}_t^{W_0}])\Big]dt\\&+\Big[(C_0+D_0\Theta_1)(\widetilde{x}_0-\bar{x}_0)+\widetilde{F}_0(\widetilde{x}^{(N)}-\mathbb{E}[z|\mathcal{F}_t^{W_0}])\Big]dW_0(t).\end{aligned}$$

Therefore, it follows from Burkholder-Davis-Gundy inequality that

$$\begin{aligned}&\mathbb{E}\sup_{0\le s\le t}\|\widetilde{x}_i-\bar{x}_i\|^2\\\le&K\mathbb{E}\int_0^t\Big[\|\widetilde{x}_i-\bar{x}_i\|^2+\|\widetilde{x}^{(N)}-\mathbb{E}[z|\mathcal{F}_t^{W_0}]\|^2\Big]ds\\&+2\mathbb{E}\sup_{0\le s\le t}\Big\|\int_0^s\Big[(C+D\Lambda_1)(\widetilde{x}_i-\bar{x}_i)+\widetilde{F}(\widetilde{x}^{(N)}-\mathbb{E}[z|\mathcal{F}_t^{W_0}])+\widetilde{G}(\widetilde{x}_0-\bar{x}_0)\Big]dW_i(t)\Big\|^2\\\le&C\mathbb{E}\int_0^t\|\widetilde{x}_i-\bar{x}_i\|^2ds+C\mathbb{E}\int_0^t\Big[\|\widetilde{x}^{(N)}-\mathbb{E}[z|\mathcal{F}_t^{W_0}]\|^2+\|\widetilde{x}_0-\bar{x}_0\|^2\Big]ds,\end{aligned}$$

and

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le s\le t}\|\widetilde{x}_0(s)-\bar{x}_0(s)\|^2\\
\le& C\mathbb{E}\int_0^t\Big[\|\widetilde{x}_0-\bar{x}_0\|^2+\|\widetilde{x}^{(N)}-\mathbb{E}[z|\mathcal{F}_t^{W_0}]\|^2\Big]ds\\
&+2\mathbb{E}\sup_{0\le s\le t}\Big\|\int_0^s\Big[(C_0+D_0\Theta_1)(\widetilde{x}_0-\bar{x}_0)+\widetilde{F}_0(\widetilde{x}^{(N)}-\mathbb{E}[z|\mathcal{F}_t^{W_0}])\Big]dW_0(s)\Big\|^2\\
\le& K\mathbb{E}\int_0^t\|\widetilde{x}_0-\bar{x}_0\|^2ds+K\mathbb{E}\int_0^t\Big\|\widetilde{x}^{(N)}-\mathbb{E}[z|\mathcal{F}_s^{W_0}]\Big\|^2ds\\
\le& K\int_0^t\mathbb{E}\sup_{0\le r\le s}\|\widetilde{x}_0(r)-\bar{x}_0(r)\|^2ds+K\mathbb{E}\int_0^t\Big\|\widetilde{x}^{(N)}-\mathbb{E}[z|\mathcal{F}_s^{W_0}]\Big\|^2ds.
\end{aligned}
$$

Note that K is a constant and is obviously non-decreasing with time t. Moreover, by $\Big\|\widetilde{x}^{(N)}-\mathbb{E}[z|\mathcal{F}_s^{W_0}]\Big\|^2\ge 0$, we also know $K\mathbb{E}\int_0^t\Big\|\widetilde{x}^{(N)}-\mathbb{E}[z|\mathcal{F}_s^{W_0}]\Big\|^2ds$ is non-decreasing with time t. Therefore, it follows from Grönwall inequality (Theorem 1.2) and Lemma 3.2 that

$$
\sup_{0\le i\le N}\mathbb{E}\sup_{0\le t\le T}\|\widetilde{x}_i(t)-\bar{x}_i(t)\|^2\le\frac{K_3}{N}.
$$

□

Note that

$$
\begin{aligned}
d(\widetilde{x}_0-z_0)=&\Big[(A_0+B_0\Theta_1)(\widetilde{x}_0-z_0)+F_0(\widetilde{x}^{(N)}-\mathbb{E}[z|\mathcal{F}_t^{W_0}])\Big]dt\\
&+\Big[(C_0+D_0\Theta_2)(\widetilde{x}_0-z_0)+\widetilde{F}_0(\widetilde{x}^{(N)}-\mathbb{E}[z|\mathcal{F}_t^{W_0}])\Big]dW_0.
\end{aligned}
$$

Similar to the proof of Lemma 3.3, we have the following result:

Lemma 3.4. *Under* (A3.1)-(A3.3) *and* (SA)*, there exists a constant K_5 independent of N such that*

$$\mathbb{E}\sup_{0\le t\le T}\|\widetilde{x}_0(t)-z_0(t)\|^2\le\frac{K_5}{N}.$$

3.5.1 Major agent

Lemma 3.5. *Under* (A3.1)-(A3.3) *and* (SA),

$$\Big|\mathcal{J}_0(\widetilde{u}_0,\widetilde{u}_{(-0)})-J_0(\bar{u}_0)\Big|=O(\frac{1}{\sqrt{N}}).$$

Proof. Recall (3.2) and (3.7), it follows from

$$\begin{aligned}
&\mathcal{J}_0(\widetilde{u}_0,\widetilde{u}_{(-0)})-J_0(\bar{u}_0)\\
=&\frac{1}{2}\mathbb{E}\int_0^T\Big\{\Big\langle Q_0(\widetilde{x}_0-H_0\widetilde{x}^{(N)}),\widetilde{x}_0-H_0\widetilde{x}^{(N)}\Big\rangle\\
&-\Big\langle Q_0(\bar{x}_0-H_0\mathbb{E}[z|\mathcal{F}_t^{W_0}]),\bar{x}_0-H_0\mathbb{E}[z|\mathcal{F}_t^{W_0}]\Big\rangle\Big\}dt\\
=&\frac{1}{2}\mathbb{E}\int_0^T\Big\langle Q_0(\widetilde{x}_0-\bar{x}_0-H_0(\widetilde{x}^{(N)}-\mathbb{E}[z|\mathcal{F}_t^{W_0}])),\widetilde{x}_0-\bar{x}_0-H_0(\widetilde{x}^{(N)}-\mathbb{E}[z|\mathcal{F}_t^{W_0}])\Big\rangle dt\\
&+\mathbb{E}\int_0^T\Big\langle Q_0(\widetilde{x}_0-\bar{x}_0-H_0(\widetilde{x}^{(N)}-\mathbb{E}[z|\mathcal{F}_t^{W_0}])),\bar{x}_0-H_0\mathbb{E}[z|\mathcal{F}_t^{W_0}]\Big\rangle dt\\
\le&K\mathbb{E}\int_0^T\Big[\|\widetilde{x}_0-\bar{x}_0\|^2+\|\widetilde{x}^{(N)}-\mathbb{E}[z|\mathcal{F}_t^{W_0}]\|^2\Big]dt\\
&+K\int_0^T\Big[(\mathbb{E}\|\widetilde{x}_0-\bar{x}_0\|^2)^{\frac{1}{2}}+(\mathbb{E}\|\widetilde{x}^{(N)}-\mathbb{E}[z|\mathcal{F}_t^{W_0}]\|^2)^{\frac{1}{2}}\Big]dt\\
=&O\Big(\frac{1}{\sqrt{N}}\Big),
\end{aligned}$$

where the last equality follows from Lemmas 3.1-3.4. □

Consider the case when the major agent $\mathcal{A}_0$ uses an alternative strategy u_0 and the minor agent $\mathcal{A}_i$ still uses the strategy $\widetilde{u}_i$. The realized states with major agent's perturbation are

$$\begin{cases} d\alpha_0 = \Big(A_0\alpha_0 + B_0u_0 + F_0\alpha^{(N)}\Big)dt + \Big(C_0\alpha_0 + D_0u_0 + \widetilde{F}_0\alpha^{(N)}\Big)dW_0, \\ d\alpha_i = \Big[(A + B\Lambda_1)\alpha_i + B\Lambda_2 + F\alpha^{(N)}\Big]dt + \Big[(C + D\Lambda_1)\alpha_i + D\Lambda_2 + \widetilde{F}\alpha^{(N)} + \widetilde{G}\alpha_0\Big]dW_i, \\ \alpha_0(0) = \xi_0, \qquad \alpha_i(0) = \xi, \qquad 1 \le i \le N, \end{cases}$$

where $\alpha^{(N)} = \frac{1}{N}\sum_{i=1}^{N}\alpha_i$. The decentralized limiting states with major agent's perturbation are

$$\begin{cases} d\bar{\alpha}_0 = \Big[A_0\bar{\alpha}_0 + B_0u_0 + F_0\mathbb{E}[z|\mathcal{F}_t^{W_0}]\Big]dt + \Big[C_0\bar{\alpha}_0 + D_0u_0 + \widetilde{F}_0\mathbb{E}[z|\mathcal{F}_t^{W_0}]\Big]dW_0, \\ d\bar{\alpha}_i = \Big[(A + B\Lambda_1)\bar{\alpha}_i + B\Lambda_2 + F\mathbb{E}[z|\mathcal{F}_t^{W_0}]\Big]dt \\ \qquad\qquad + \Big[(C + D\Lambda_1)\bar{\alpha}_i + D\Lambda_2 + \bar{F}\mathbb{E}[z|\mathcal{F}_t^{W_0}] + \widetilde{G}\bar{\alpha}_0\Big]dW_i, \\ \bar{\alpha}_0(0) = \xi_0, \qquad \bar{\alpha}_i(0) = \xi, \qquad 1 \le i \le N. \end{cases}$$

Similar to Lemma 3.2 and 3.3, we have

Lemma 3.6. *Under* (A3.1)-(A3.3) *and* (SA), *there exists a constant* K_6 *independent of* N *such that*

$$\mathbb{E}\sup_{0\le t\le T}\Big\|\alpha^{(N)}(t) - \mathbb{E}[z|\mathcal{F}_t^{W_0}]\Big\|^2 \le \frac{K_6}{N}.$$

Lemma 3.7. *Under* (A3.1)-(A3.3) *and* (SA)*, there exists a constant K_7 independent of N such that*

$$\sup_{0\le i\le N}\mathbb{E}\sup_{0\le t\le T}\Big\|\alpha_i(t)-\bar{\alpha}_i(t)\Big\|^2\le\frac{K_7}{N}.$$

Thus, we have the following result.

Lemma 3.8. *Under* (A3.1)-(A3.3) *and* (SA)*, we have*

$$\Big\|\mathcal{J}_0(u_0,\widetilde{u}_{(-0)})-J_0(u_0)\Big\|=O(\frac{1}{\sqrt{N}}).$$

Proof.

$$\begin{aligned}
&\mathcal{J}_0(u_0,\widetilde{u}_{(-0)})-J_0(u_0)\\
=&\frac{1}{2}\mathbb{E}\int_0^T\bigg\{\Big\langle Q_0(\alpha_0-H_0\alpha^{(N)}),\alpha_0-H_0\alpha^{(N)}\Big\rangle\\
&\qquad-\Big\langle Q_0(\bar{\alpha}_0-H_0\mathbb{E}[z|\mathcal{F}_t^{W_0}]),\bar{\alpha}_0-H_0\mathbb{E}[z|\mathcal{F}_t^{W_0}]\Big\rangle\bigg\}dt\\
=&\frac{1}{2}\mathbb{E}\int_0^T\Big\langle Q_0(\alpha_0-\bar{\alpha}_0-H_0(\alpha^{(N)}-\mathbb{E}[z|\mathcal{F}_t^{W_0}])),\alpha_0-\bar{\alpha}_0-H_0(\alpha^{(N)}-\mathbb{E}[z|\mathcal{F}_t^{W_0}])\Big\rangle dt\\
&+\mathbb{E}\int_0^T\Big\langle Q_0(\alpha_0-\bar{\alpha}_0-H_0(\alpha^{(N)}-\mathbb{E}[z|\mathcal{F}_t^{W_0}])),\bar{\alpha}_0-H_0\mathbb{E}[z|\mathcal{F}_t^{W_0}]\Big\rangle dt\\
\le&K\mathbb{E}\int_0^T\Big[\|\alpha_0-\bar{\alpha}_0\|^2+\|\alpha^{(N)}-\mathbb{E}[z|\mathcal{F}_t^{W_0}]\|^2\Big]dt\\
&\qquad+K\int_0^T\Big[(\mathbb{E}\|\alpha_0-\bar{\alpha}_0\|^2)^{\frac{1}{2}}+(\mathbb{E}\|\alpha^{(N)}-\mathbb{E}[z|\mathcal{F}_t^{W_0}]\|^2)^{\frac{1}{2}}\Big]dt\\
=&O\Big(\frac{1}{\sqrt{N}}\Big).
\end{aligned}$$

□

Theorem 3.2. *Under* (A3.1)-(A3.3) *and* (SA), $\tilde{u}_0$ *is an asymptotically* ε-*optimal strategy for the major agent.*

Proof. It follows from Lemma 3.5 and Lemma 3.8 that

$$\mathcal{J}_0(\widetilde{u}_0,\widetilde{u}_{(-0)})\leq J_0(\bar{u}_0)+O(\frac{1}{\sqrt{N}})\leq J_0(u_0)+O(\frac{1}{\sqrt{N}})\leq \mathcal{J}_0(u_0,\widetilde{u}_{(-0)})+O(\frac{1}{\sqrt{N}}).$$

$\square$

3.5.2 Minor agent

The proof of the asymptotical ε-optimality of the minor agents' MF strategy set is similar to that in Chapter 2, and we just sketch some key points in what follows.

Representation of social cost

Rewrite the LP system (3.1) and (3.1) as follows:

$$d\mathbf{x} = (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u})dt + \sum_{i=0}^{N}(\mathbf{C}_i\mathbf{x} + \mathbf{D}_i\mathbf{u})dW_i, \quad \mathbf{x}(0) = \Xi, \tag{3.34}$$

where

$$\mathbf{A} = \begin{pmatrix} A_0 & \frac{F_0}{N} & \frac{F_0}{N} & \cdots & \frac{F_0}{N} \\ 0 & A+\frac{F}{N} & \frac{F}{N} & \cdots & \frac{F}{N} \\ 0 & \frac{F}{N} & A+\frac{F}{N} & \cdots & \frac{F}{N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{F}{N} & \frac{F}{N} & \cdots & A+\frac{F}{N} \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{pmatrix}, \mathbf{B} = \begin{pmatrix} B_0 & 0 & 0 & \cdots & 0 \\ 0 & B & 0 & \cdots & 0 \\ 0 & 0 & B & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B \end{pmatrix}, \mathbf{u} = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \end{pmatrix},$$

$$\mathbf{C}_0 = \begin{pmatrix} C_0 & \frac{\widetilde{F}_0}{N} & \frac{\widetilde{F}_0}{N} & \cdots & \frac{\widetilde{F}_0}{N} \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \mathbf{D}_0 = \begin{pmatrix} D_0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \mathbf{C}_i = \begin{array}{c} 1 \\ \vdots \\ i+1 \\ \vdots \\ N+1 \end{array} \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots & \vdots & \vdots & \vdots \\ \frac{\widetilde{F}}{N} & \cdots & \frac{\widetilde{F}}{N} & \frac{\widetilde{F}}{N}+C & \frac{\widetilde{F}}{N} & \cdots & \frac{\widetilde{F}}{N} \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$\mathbf{D}_i = \begin{array}{c} 1 \\ \vdots \\ i+1 \\ \vdots \\ N+1 \end{array} \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & D & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \Xi = \begin{pmatrix} \xi_0 \\ \xi \\ \vdots \\ \xi \end{pmatrix}.$$

Similarly, the social cost takes the following form:

$$\begin{aligned}\mathcal{J}_{soc}^{(N)}(\mathbf{u})=&\frac{1}{2}\sum_{i=1}^{n}\mathbb{E}\int_0^T\Big[\Big\langle Q(x_i-Hx_0-\hat{H}x^{(N)}),(x_i-Hx_0-\hat{H}x^{(N)})\Big\rangle+\langle Ru_i,u_i\rangle\Big]dt\\ =&\frac{1}{2}\mathbb{E}\int_0^T\Big[\langle\mathbf{Q}\mathbf{x},\mathbf{x}\rangle+\langle\mathbf{R}\mathbf{u},\mathbf{u}\rangle\Big]dt,\end{aligned}$$

where

$$\mathbf{Q}=\begin{pmatrix}\mathbf{Q}_{00}&\mathbf{Q}_{01}&\mathbf{Q}_{02}&\cdots&\mathbf{Q}_{0N}\\ \mathbf{Q}_{10}&\mathbf{Q}_{11}&\mathbf{Q}_{12}&\cdots&\mathbf{Q}_{1N}\\ \mathbf{Q}_{20}&\mathbf{Q}_{21}&\mathbf{Q}_{22}&\cdots&\mathbf{Q}_{2N}\\ \vdots&\vdots&\vdots&\ddots&\vdots\\ \mathbf{Q}_{N0}&\mathbf{Q}_{N1}&\mathbf{Q}_{N2}&\cdots&\mathbf{Q}_{NN}\end{pmatrix},\mathbf{R}=\begin{pmatrix}0&0&0&\cdots&0\\ 0&R&0&\cdots&0\\ 0&0&R&\cdots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\ 0&0&0&\cdots&R\end{pmatrix},$$

and for $i=1,\cdots,N, j\neq i$,

$$\begin{cases}\mathbf{Q}_{00}=NQ+\hat{H}^TQ\hat{H}-Q\hat{H}-\hat{H}^TQ,\ \mathbf{Q}_{0i}=-\hat{H}^TQH+QH,\\ \mathbf{Q}_{i0}=-HQ\hat{H}+HQ,\ \mathbf{Q}_{ii}=Q+\frac{1}{N}(\hat{H}^TQ\hat{H}-Q\hat{H}-\hat{H}^TQ),\\ \mathbf{Q}_{ij}=\frac{1}{N}(\hat{H}^TQ\hat{H}-Q\hat{H}-\hat{H}^TQ).\end{cases}$$

Next, by the variation of constant formula, we know that the strong solution of (3.34) admits the following representation:

$$\mathbf{x}(t)=\Phi(t)\Xi+\Phi(t)\int_0^t\Phi(s)^{-1}\Big[(\mathbf{B}-\sum_{i=0}^{N}\mathbf{C}_i\mathbf{D}_i)\mathbf{u}(s)\Big]ds+\sum_{i=0}^{N}\Phi(t)\int_0^t\Phi(s)^{-1}\mathbf{D}_iu(s)dW_i(s),$$

where

$$d\Phi(t)=\mathbf{A}\Phi(t)dt+\sum_{i=0}^{N}\mathbf{C}_i\Phi(t)dW_i,\quad \Phi(0)=I.$$

Define the following operators

$$\begin{cases}(L\mathbf{u}(\cdot))(\cdot):=\Phi(\cdot)\Big\{\int_0^\cdot\Phi(s)^{-1}\big[(\mathbf{B}-\sum_{i=0}^{N}\mathbf{C}_i\mathbf{D}_i)\mathbf{u}(s)\big]ds+\sum_{i=0}^{N}\int_0^\cdot\Phi(s)^{-1}\mathbf{D}_i\mathbf{u}dW_i(s)\Big\},\\ \widetilde{L}\mathbf{u}(\cdot):=(L\mathbf{u}(\cdot))(T),\quad \Gamma\Xi(\cdot):=\Phi(\cdot)\Phi^{-1}(0)\Xi,\quad \widetilde{\Gamma}\Xi:=(\Gamma\Xi)(T).\end{cases}$$

Correspondingly, L^* is defined as the adjoint operator of L (seeing [62]). Given any admissible $\mathbf{u}$, we can express $\mathbf{x}$ as follows:

$$\mathbf{x}(\cdot) = (L\mathbf{u}(\cdot))(\cdot) + \Gamma\Xi(\cdot), \quad \mathbf{x}(T) = \widetilde{L}\mathbf{u}(\cdot) + \widetilde{\Gamma}\Xi.$$

Hence, we can rewrite the cost functional as follows:

$$\begin{aligned}
2\mathcal{J}_{soc}^{(N)}(\mathbf{u}) = \mathbb{E}\int_0^T & \Big[\langle \mathbf{Q}\mathbf{x}, \mathbf{x}\rangle + \langle \mathbf{R}\mathbf{u}, \mathbf{u}\rangle\Big] dt \\
&= \langle L^*\mathbf{Q}L\mathbf{u}(\cdot), \mathbf{u}(\cdot)\rangle + 2\langle L^*\mathbf{Q}\Gamma\Xi(\cdot), \mathbf{u}(\cdot)\rangle + \langle \mathbf{Q}\Gamma\Xi(\cdot), \Gamma\Xi(\cdot)\rangle + \langle \mathbf{R}\mathbf{u}, \mathbf{u}\rangle \\
&= \langle (L^*\mathbf{Q}L + \mathbf{R})\mathbf{u}(\cdot), \mathbf{u}(\cdot)\rangle + 2\langle L^*Q\Gamma\Xi(\cdot), \mathbf{u}(\cdot)\rangle + \langle Q\Gamma\Xi(\cdot), \Gamma\Xi(\cdot)\rangle \\
&:= \langle M_2\mathbf{u}(\cdot), \mathbf{u}(\cdot)\rangle + 2\langle M_1, \mathbf{u}(\cdot)\rangle + M_0.
\end{aligned}$$

Note that, M_2 is a self-adjoint positive semidefinite bounded linear operator.

Asymptotic optimality

In order to prove asymptotic optimality for the minor agents, it suffices to consider the perturbations $u_{-0} \in \mathcal{U}_{-0}$ such that $\mathcal{J}_{soc}^{(N)}(\widetilde{u}_0, u_{-0}) \leq \mathcal{J}_{soc}^{(N)}(\widetilde{u}_0, \widetilde{u}_{-0})$. It is easy to check that $\mathcal{J}_{soc}^{(N)}(\widetilde{u}_0, \widetilde{u}_{-0}) \leq KN$, where K is a constant independent of N. Therefore, in what follows, we only consider the perturbations $u_{-0} \in \mathcal{U}_{-0}$ satisfying $\sum_{j=1}^N \mathbb{E}\int_0^T \|u_j\|^2 dt \leq KN$. Let $\delta u_i = u_i - \widetilde{u}_i$, and consider a perturbation $u = \widetilde{u} + (0, \delta u_1, \cdots, \delta u_N) := \widetilde{u} + \delta u$. We have

$$\begin{aligned}
2\mathcal{J}_{soc}^{(N)}(\widetilde{u} + \delta u) =& \langle M_2(\widetilde{u} + \delta u), \widetilde{u} + \delta u\rangle + 2\langle M_1, \widetilde{u} + \delta u\rangle + M_0 \\
=& 2\mathcal{J}_{soc}^{(N)}(\widetilde{u}) + 2\sum_{i=1}^N \langle M_2\widetilde{u} + M_1, \delta u_i\rangle + \langle M_2\delta u, \delta u\rangle,
\end{aligned}$$

where $\langle M_2\widetilde{u} + M_1, \delta u_i\rangle$ is the Fréchet differential of $\mathcal{J}_{soc}^{(N)}$ on $\widetilde{u}$ with variation δu_i. By Cauchy-Schwarz inequality,

$$\begin{aligned}
&\mathcal{J}_{soc}^{(N)}(\widetilde{u}+\delta u) - \mathcal{J}_{soc}^{(N)}(\widetilde{u}) \\
\geq& -\sqrt{\sum_{i=1}^{N}\|M_2\widetilde{u}+M_1\|^2\sum_{i=1}^{N}\|\delta u_i\|^2} + \frac{1}{2}\langle M_2\delta u, \delta u\rangle \\
\geq& -\|M_2\widetilde{u}+M_1\|O(N).
\end{aligned}$$

Therefore, in order to prove asymptotic optimality for the minor agents, we only need to show that $\|M_2\widetilde{u}+M_1\| = o(1)$. To this end, we introduce another assumption:

(A3.4) There exists constants $L_1, L_2 > 0$ independent of N such that

$$E\int_0^T \left\|\mathbb{E}[y_1^1|\mathcal{F}_t^{W_0}] - \frac{1}{N}\sum_{j\neq i} y_1^j\right\|^2 dt \leq \frac{L_1}{N}, \tag{3.35}$$

and

$$E\int_0^T \left\|\mathbb{E}[\beta_1^{11}|\mathcal{F}_t^{W_0}] - \frac{1}{N}\sum_{j\neq i} \beta_1^{jj}\right\|^2 dt \leq \frac{L_2}{N}. \tag{3.36}$$

Theorem 3.3. *Under* (A3.1)-(A3.4) *and* (SA), $(\tilde{u}_1, \cdots, \tilde{u}_N)$ *is an asymptotically ε-optimal strategy set for the minor agents*

Proof. We have

$$\langle M_2\widetilde{u}+M_1, \delta u_i\rangle = \mathbb{E}\int_0^T \Big[\langle Q\widetilde{l}_i, \delta l_i\rangle - \langle S, \delta l_i\rangle + \langle R\widetilde{u}_i, \delta u_i\rangle\Big] dt + \sum_{l=1}^{15}\varepsilon_l.$$

From the optimality of $\widetilde{u}$, we have

$$\mathbb{E}\int_0^T \Big[\langle Q\widetilde{l}_i, \delta l_i\rangle - \langle S, \delta l_i\rangle + \langle R\widetilde{u}_i, \delta u_i\rangle\Big]dt = 0.$$

Moreover, similar to Chapter 2, we have

$$\sum_{l=1}^{10} \varepsilon_l + \varepsilon_{14} + \varepsilon_{15} = O\Big(\frac{1}{\sqrt{N}}\Big).$$

Therefore,

$$\|M_2\widetilde{u} + M_1\| = O\Big(\frac{1}{\sqrt{N}}\Big).$$

$\square$

Remark 3.6. *Note that*

$$\begin{aligned}
&d\Big(\frac{1}{N}\sum_{j\neq i}\widetilde{x}_j - \mathbb{E}[\widetilde{x}_1|\mathcal{F}_t^{W_0}]\Big)\\
=&\Big[(A+B\Lambda_1)\Big(\frac{1}{N}\sum_{j\neq i}\widetilde{x}_j - \mathbb{E}[\widetilde{x}_1|\mathcal{F}_t^{W_0}]\Big) + F\widetilde{x}^{(N)} - F\mathbb{E}[\widetilde{x}^{(N)}|\mathcal{F}_t^{W_0}]\Big]dt\\
&+\frac{1}{N}\sum_{j\neq i}\Big[(C+D\Lambda_1)\widetilde{x}_j + D\Lambda_2 + \widetilde{F}x^{(N)} + \widetilde{G}\widetilde{x}_0\Big]dW_j.
\end{aligned}$$

Therefore, it follows from Burkholder-Davis-Gundy inequality and Grönwall inequality that

$$\mathbb{E}\sup_{0\leq s\leq t}\Big\|\frac{1}{N}\sum_{j\neq i}\widetilde{x}_j - \mathbb{E}[\widetilde{x}_1|\mathcal{F}_t^{W_0}]\Big\|^2 \leq \frac{K}{N}.$$

If $C=0$*, applying Itô's formula to* $\big\|\frac{1}{N}\sum_{j\neq i} y_1^j - \mathbb{E}[y_1^1|\mathcal{F}_t^{W_0}]\big\|^2$*, it is easy to check that* (3.35) *in* (A3.4) *holds.*

Remark 3.7. *If the states has the following form*

$$dx_0 = \Big(A_0x_0 + B_0u_0 + F_0x^{(N)}\Big)dt + \Big(C_0x_0 + D_0u_0 + \widetilde{F}_0x^{(N)}\Big)dW_0, \quad x_0(0) = \xi_0,$$

and for $1 \leq i \leq N$,

$$dx_i = \Big(Ax_i + Bu_i + Fx^{(N)} + Gx_0\Big)dt + DdW_i, \quad x_i(0) = \xi \in \mathbb{R}^n,$$

then assumption (A3.4) *is not needed to obtain the asymptotic optimality of the minor agents. However, if the state equations of the minor agents take the form* (3.1), *we need to suppose the assumption* (A3.4) *to hold and we will continue to study this in the future work.*

Lastly, by combining Theorem 3.2, 3.3 and considering the Definition 3.1, we have the following result

Theorem 3.4. *The MF strategies* $\tilde{u}_0$, $\tilde{u}_{-0}$ *achieve an asymptotic* ε*-equilibrium between the major agent and the aggregation of minor agents, where* $\tilde{u}_0 = \Theta_1\tilde{x}_0 + \Theta_2$, $\tilde{u}_i = \Lambda_1\tilde{x}_i + \Lambda_2$ *and* $\tilde{u}_{-0} = (\tilde{u}_1, \cdots, \tilde{u}_N)$. *Moreover* $\tilde{u}_1, \cdots, \tilde{u}_N$ *achieve an asymptotic* ε*-social optima among the aggregation of minor agents. Thus,* $(\tilde{u}_0, \tilde{u}_{-0})$ *is asymptotically* ε*-optimal.*

Proof. By Theorem 3.2, we have

$$\mathcal{J}_0(\tilde{u}_0, \tilde{u}_{-0}) \leq \inf_{u_0 \in \mathcal{U}_0} \mathcal{J}_0(u_0, \tilde{u}_{-0}) + O\Big(\frac{1}{\sqrt{N}}\Big).$$

Moreover, by Theorem 3.3, we have

$$\frac{1}{N}\Big(\mathcal{J}_{soc}^{(N)}(\tilde{u}_0, \tilde{u}_{-0}) - \inf_{u_{-0}\in\mathcal{U}_{-0}} \mathcal{J}_{soc}^{(N)}(\tilde{u}_0, u_{-0})\Big) \leq O\Big(\frac{1}{\sqrt{N}}\Big).$$

Thus, by Definition 3.1, Theorem 3.4 holds straightforwardly. □

3.6 Numerical analysis

This section presents some numerical example to illustrate our theoretical results. Our example is motivated by an electric charging control problem in presence of distributed information network. Relevant literature include e.g., [79] and [8].

Consider two competitive charging providers in a power-grid network for given city. One provider (namely, the major agent in our model) still retains the traditional charging scheme upon centralized information, whereby its charging strategy is determined by a central controller. In this case, we do not need to differentiate all its sub-units on charging nodes since they formalize one decision entity with consistency actions.

On the other hand, another provider, taking account the well-recognized distributed datum, prefers to adopt some decentralized charging scheme, where the strategy is determined by each distributed charging unit on grid-node, only upon their decentralized information *on or around that node.* In this case, such distributed provider is actually formalized into a cooperative team wherein all its sub-units or nodes acts as the minor agents as in our model.

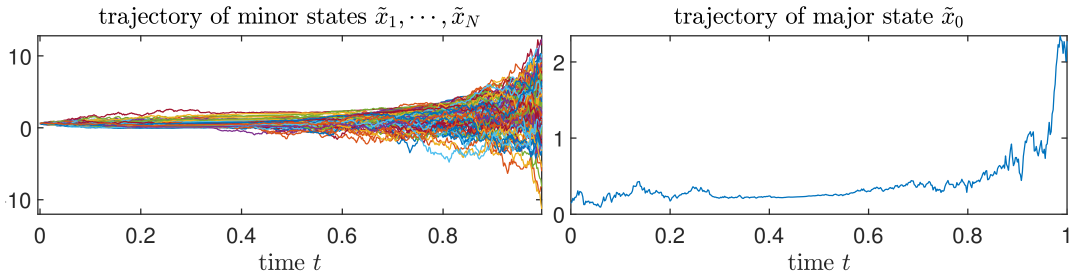
Moreover, under some mild conditions on (linear) demand-supply curve, above competitive problem may be fit into a linear-quadratic setup whenever a quadratic deviation or tracing criteria is applied, as in [80] and [13]. Thus, we cook one example in our theoretical framework.

We now specify such example in details, by randomly generate its coefficients: $A_0 = \begin{pmatrix} 0.6423 & 0.2057 \\ 0.0287 & 0.7907 \end{pmatrix}$, $F_0 = \begin{pmatrix} 0.9225 & 0.3780 \\ 0.6933 & 0.8048 \end{pmatrix}$, $C_0 = \begin{pmatrix} 0.0125 & 0.4517 \\ 0.4720 & 0.1117 \end{pmatrix}$, $\widetilde{F}_0 = \begin{pmatrix} 0.8084 & 0.4284 \\ 0.7032 & 0.1955 \end{pmatrix}$, $A = \begin{pmatrix} 0.1023 & 0.2995 \\ 0.1027 & 0.9415 \end{pmatrix}$, $F = \begin{pmatrix} 0.0377 & 0.8910 \\ 0.2866 & 0.1003 \end{pmatrix}$, $C = \begin{pmatrix} 0.1641 & 0.0360 \\ 0.3271 & 0.8063 \end{pmatrix}$, $\widetilde{F} = \begin{pmatrix} 0.3751 & 0.6241 \\ 0.4491 & 0.5093 \end{pmatrix}$, $\widetilde{G}(\cdot) = \begin{pmatrix} 0.5018 & 0.7881 \\ 0.6989 & 0.1633 \end{pmatrix}$, $B_0 = \begin{pmatrix} 0.4514 & 0.2916 \\ 0.4309 & 0.9989 \end{pmatrix}$, $D_0(\cdot) = \begin{pmatrix} 0.4514 & 0.2916 \\ 0.4309 & 0.9989 \end{pmatrix}$, $B = \begin{pmatrix} 0.4389 & 0.4766 \\ 0.2411 & 0.3539 \end{pmatrix}$, $D = \begin{pmatrix} 0.8756 & 0.9451 \\ 0.7493 & 0.8354 \end{pmatrix}$, $Q_0 = \begin{pmatrix} 0.6210 & 0 \\ 0 & 0.8691 \end{pmatrix}$, $H_0 = \begin{pmatrix} 0.3250 & 0 \\ 0 & 0.5957 \end{pmatrix}$, $Q = \begin{pmatrix} 0.8701 & 0 \\ 0 & 0.1925 \end{pmatrix}$, $H = \begin{pmatrix} 0.3865 & 0 \\ 0 & 0.2957 \end{pmatrix}$, $\hat{H} = \begin{pmatrix} 0.7027 & 0 \\ 0 & 0.0354 \end{pmatrix}$, $R_0 = \begin{pmatrix} 0.7160 & 0 \\ 0 & 0.5594 \end{pmatrix}$, $R = \begin{pmatrix} 0.3885 & 0 \\ 0 & 0.4182 \end{pmatrix}$, $Q_0 = \begin{pmatrix} 0.6210 & 0 \\ 0 & 0.8691 \end{pmatrix}$, $Q = \begin{pmatrix} 0.8701 & 0 \\ 0 & 0.1925 \end{pmatrix}$, $R_0 = \begin{pmatrix} 0.7160 & 0 \\ 0 & 0.5594 \end{pmatrix}$, $R = \begin{pmatrix} 0.3885 & 0 \\ 0 & 0.4182 \end{pmatrix}$. It is easy to see that such generated coefficients are constants and surely L^∞ and Lipschitz continuous. Thus, assumption (A3.1)-(A3.2) and (A1)-(A2) hold.

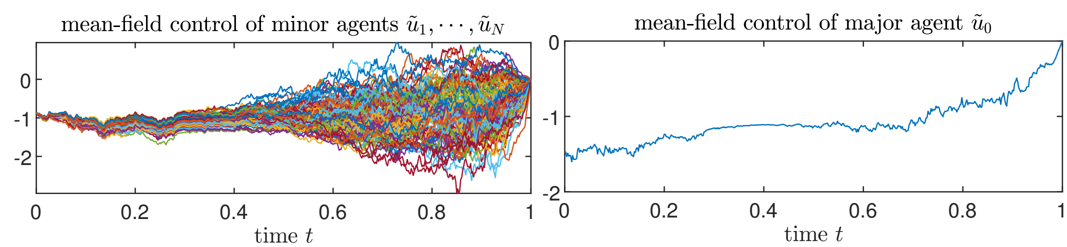
In the following simulation, we will calculate the feedback form MF strategies and also the corresponding state trajectories of major and minor agents. The convergence of the population average will be also simulated.

Firstly, we solve (3.30) by decentralizing method and decoupling method, and ($\hat{x}$, z_0, y_1^0, β_1^0, $\hat{y}_1$, $\hat{\beta}_1$, y_2) can be obtained. Further, by (3.9) and (3.28), we can calculate Θ_1, Θ_2, Λ_1, Λ_2. Then, the realized states can be obtained by (3.1) and

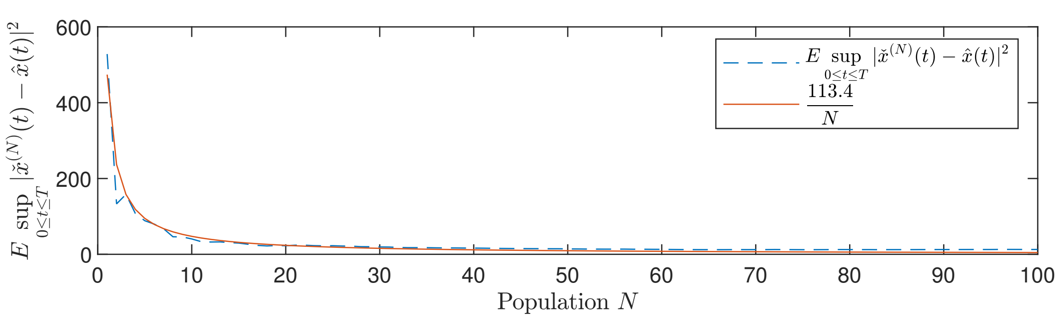
(3.1). The following graphs are the first coordinate of the realized states.



Then, the corresponding feedback form MF strategies can be obtained as well. The following graphs are the first coordinate of the MF strategies.



Next, we simulate the convergence of the population state-average $\check{x}^{(N)}(t)$ to the MF $\hat{x}$. Specifically, we will calculate $\mathbb{E}\sup_{0\le t\le T}\|\check{x}^{(N)}(t)-\hat{x}\|^2$. First, $\sup_{0\le t\le T}\|\check{x}^{(N)}(t)-\hat{x}\|^2$ can be calculated directly. Second, for the expectation, we repeat such process enough times (200 times) and take the average to simulate it.



The relation between $\mathbb{E}\sup_{0\le t\le T}\|\check{x}^{(N)}(t)-\hat{x}(t)\|^2$and N can be fitted by $\mathbb{E}\sup_{0\le t\le T}\|\check{x}^{(N)}(t)-\hat{x}(t)\|^2=\frac{113.4}{N}$with R-square 0.9944. In this sense, $\mathbb{E}\ \sup_{0\le t\le T}$ $\|\check{x}^{(N)}(t)-\hat{x}(t)\|^2=O\big(\frac{1}{N}\big)$.

By the simulation above, we can see that the MF strategy is asymptotically optimal.

Chapter 4 Relation among MFC, MFG, MFT Constrained on a Linear Subspace

In this section, we study the relation among MFC, MFG, MFT problem constrained on a linear subspace. For a given linear subspace $\Lambda \subseteq \mathbb{R}^m$, we can introduce the following centralized admissible control set constrained on Λ:

$$\mathcal{U}_c^\Lambda = \Big\{ u(\cdot) | u(\cdot) \in L^2_{\mathbb{F}}(0,T;\Lambda) \Big\},$$

and decentralized admissible control set constrained on Λ:

$$\mathcal{U}_i^\Lambda = \left\{ u_i(\cdot) | u_i(\cdot) \in L^2_{\mathbb{F}^i}(0,T;\Lambda) \right\}, \quad 1 \le i \le N.$$

Without loss of generality, we assume $\dim(\Lambda) = m' < m$. We let $\Lambda = \operatorname{span}(v_1, \cdots, v_{m'})$, and the basis vectors $v_1, \cdots, v_{m'}$ are linear independent. Denote $V := (v_1, \cdots, v_{m'})$ being the $m \times m'$ corresponding matrix. Then for any positive definite matrix M, we have the following result which will be used frequently throughout this chapter.

Lemma 4.1. *If Λ is a linear subspace of $\mathbb{R}^m$ spanned by $v_1, \cdots, v_{m'}$ and M is positive definite matrix, then $\langle M\cdot, \cdot\rangle$ is a well defined inner product on $\mathbb{R}^m$ and $\|\cdot\|_M$ is the corresponding norm. Moreover, $V^TMV > 0$ as well. For every $v' \in \mathbb{R}^m$, there exists a unique $v^* \in \Lambda$, such that:*

$$\|v' - v^*\|_M = \inf_{v \in \Lambda} \|v' - v\|_M.$$

Moreover, v^* is characterized by the property:

$$v^* \in \Lambda, \quad \langle M(v^* - v'), v - v^* \rangle \geq 0, \quad \forall v \in \Lambda. \tag{4.1}$$

The above element v^* is called the projection of v' onto Λ and is denoted by $\mathbf{P}_\Lambda^M(v')$. Moreover, the projection $\mathbf{P}_\Lambda^M$ can be represented as:

$$\mathbf{P}_\Lambda^M = V(V^T M V)^{-1} V^T M. \tag{4.2}$$

Proof. Firstly, it can be verified directly that $\langle M\cdot, \cdot\rangle$ is a well defined inner product on $\mathbb{R}^m$ if $M > 0$. Moreover, for any vector $0 \neq x := (x_1, \cdots, x_{m'})^T \in \mathbb{R}^{m'}$, it holds that

$$x^T \left(V^T M V\right) x = \left(\sum_{i=1}^{m'} x_i v_i\right)^T M \left(\sum_{i=1}^{m'} x_i v_i\right) > 0,$$

since $v_1, \cdots, v_{m'}$ are linear independent and $\sum_{i=1}^{m'} x_i v_i \neq 0$.

Secondly, by Chapter 5 of [81], there exits a unique projection $\mathbf{P}_\Lambda^M$ w.r.t. the linear subspace Λ and inner product $\langle M\cdot, \cdot\rangle$, which is characterized by (4.1).

Lastly, we desire to prove the representation (4.2) of $\mathbf{P}_\Lambda^M$. The range (image) of V is Λ, and any vector $v \in \Lambda$ can be written as $v = Vb$ for some $b \in \mathbb{R}^{m'}$. For any vector $c \in \mathbb{R}^m$, we have $c - \mathbf{P}_\Lambda c \perp \Lambda$. Equivalently, for any $b \in \mathbb{R}^{m'}$, $c - \mathbf{P}_\Lambda c \perp Vb$. Thus, $\langle M(c - \mathbf{P}_\Lambda c), Vb\rangle = 0$ for any $(b, c) \in \mathbb{R}^{m'} \times \mathbb{R}^m$. Hence, $V^T M(c - Vb') = V^T M(c - \mathbf{P}_\Lambda c) = 0$ for some $b' \in \mathbb{R}^m$. Consequently, $b' = (V^T M V)^{-1} V^T M c$ and $\mathbf{P}_\Lambda c = Vb' = V(V^T M V)^{-1} V^T M c$ which completes the proof. $\square$

4.1 MFC problem constrained on a linear subspace

We firstly study the LQG MFC problem constrained on a linear subspace (for short, (**MFC-c**)). The problem can be represented as follows:

(**MFC-c**): For given initial value $\xi_0 \in \mathbb{R}^n$, find a $\bar{u}_i(\cdot) \in \mathcal{U}_i^\Lambda$ such that $\mathcal{J}_i(\xi_0; \bar{u}_i(\cdot)) = \inf_{u_i(\cdot)\in\mathcal{U}_i^\Lambda} \mathcal{J}_i(\xi_0; u_i(\cdot))$, where

$$\mathcal{J}_i(\xi_0; u_i(\cdot)) = \frac{1}{2}\mathbb{E}\left\{ \int_0^T \|x_i - \Gamma_1 \mathbb{E}x_i\|_Q^2 + \|u_i\|_R^2 dt + \|x_i(T) - \Gamma_2 \mathbb{E}x_i(T)\|_G^2 \right\}, \tag{4.3}$$

$$\text{s.t.} \quad \begin{cases} dx_i = (Ax_i + \bar{A}\mathbb{E}x_i + Bu_i + \bar{B}\mathbb{E}u_i)dt + (Cx_i + \bar{C}\mathbb{E}x_i + Du_i + \bar{D}\mathbb{E}u_i)dW_i, \\ x_i(0) = \xi_0. \end{cases} \tag{4.4}$$

Note that all agents are homogeneous here, and we suppress the subscript i in case when no confusion occurs hereafter in this section. For the coefficients, we apply assumptions (A4.1) and (A4.2) as follows:

(**A4.1**) : A, $\bar{A}$, C, $\bar{C} \in L^\infty(0, T; \mathbb{R}^{n\times n})$, B, $\bar{B}$, D, $\bar{D} \in L^\infty(0, T; \mathbb{R}^{n\times m})$.

(**A4.2**) : $Q, \in L^\infty(0, T; \mathbb{S}^{n\times n})$, $G \in \mathbb{S}^{n\times n}$, $R \in L^\infty(0, T; \mathbb{S}^{m\times m})$, Γ_1, $\Gamma_2 \in \mathbb{R}^{n\times n}$.

Under (A4.1), for any given $u(\cdot) \in \mathcal{U}^\Lambda$, (4.3) admits a unique strong solution $x(\cdot) \equiv x(\cdot; \xi_0, u(\cdot))$ by Proposition 2.6 in [61]. Furthermore, under (A4.2), $\mathcal{J}(\xi_0; u(\cdot))$ is well-defined for all $u(\cdot) \in \mathcal{U}^\Lambda$.

4.1.1 Convexity

In this section, we will introduce some basic conditions ensuring the convexity of the cost functional, since convexity plays a crucial role in the study of finiteness and solvability of (**MFC-c**). Firstly, we introduce the following definitions of uniform convexity and positive definiteness:

Definition 4.1. *For any given admissible control set $\mathcal{V} \subseteq \mathcal{U}$, cost functional $u(\cdot) \mapsto \mathcal{J}(\xi_0; u(\cdot))$ is said to be uniformly convex on $\mathcal{V}$ if*

$$\mathcal{J}(0; u(\cdot)) \geq \varepsilon \|u(\cdot)\|_{L^2}^2, \quad \forall u(\cdot) \in \mathcal{V},$$

for some constant $\varepsilon > 0$.

Definition 4.2. *For any given linear subspace Λ of $\mathbb{R}^m$, a matrix M is said to be positive (semi-)definite on Λ if*

$$\langle Mv, v\rangle > (\geq)\ 0, \quad \forall v \in \Lambda,\ v \neq 0.$$

Next, we rewrite the system (4.3)-(4.4) as follows:

$$\begin{cases} \mathcal{J}(\xi_0; u(\cdot)) = \mathbb{E}\displaystyle\int_0^T \left\langle \begin{pmatrix} Q & 0 \\ 0 & \widehat{Q} \end{pmatrix} \begin{pmatrix} x-\mathbb{E}x \\ \mathbb{E}x \end{pmatrix}, \begin{pmatrix} x-\mathbb{E}x \\ \mathbb{E}x \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} R & 0 \\ 0 & \widehat{R} \end{pmatrix} \begin{pmatrix} u-\mathbb{E}u \\ \mathbb{E}u \end{pmatrix}, \begin{pmatrix} u-\mathbb{E}u \\ \mathbb{E}u \end{pmatrix} \right\rangle dt \\ \qquad\qquad + \left\langle \begin{pmatrix} G & 0 \\ 0 & \widehat{G} \end{pmatrix} \begin{pmatrix} x-\mathbb{E}x \\ \mathbb{E}x \end{pmatrix}(T), \begin{pmatrix} x-\mathbb{E}x \\ \mathbb{E}x \end{pmatrix}(T) \right\rangle, \\ d\begin{pmatrix} x-\mathbb{E}x \\ \mathbb{E}x \end{pmatrix} = \left[\begin{pmatrix} A & 0 \\ 0 & \mathcal{A} \end{pmatrix} \begin{pmatrix} x-\mathbb{E}x \\ \mathbb{E}x \end{pmatrix} + \begin{pmatrix} B & 0 \\ 0 & \mathcal{B} \end{pmatrix} \begin{pmatrix} u-\mathbb{E}u \\ \mathbb{E}u \end{pmatrix} \right] dt \\ \qquad\qquad + \left[\begin{pmatrix} C & \mathcal{C} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x-\mathbb{E}x \\ \mathbb{E}x \end{pmatrix} + \begin{pmatrix} D & \mathcal{D} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u-\mathbb{E}u \\ \mathbb{E}u \end{pmatrix} \right] dW, \\ \begin{pmatrix} x-\mathbb{E}x \\ \mathbb{E}x \end{pmatrix}(0) = \begin{pmatrix} 0 \\ \xi_0 \end{pmatrix}, \quad u(\cdot) \in \mathcal{U}^\Lambda, \end{cases} \tag{4.5}$$

where $\widehat{Q} = (I-\Gamma_1)^T Q(I-\Gamma_1)$, $\widehat{G} = (I-\Gamma_2)^T G(I-\Gamma_2)$, $\mathcal{A} = A+\bar{A}$, $\mathcal{B} = B+\bar{B}$, $\mathcal{C} = C+\bar{C}$, $\mathcal{D} = D+\bar{D}$. By letting

$$\mathbf{Q} = \begin{pmatrix} Q & 0 \\ 0 & \widehat{Q} \end{pmatrix}, \mathbf{R} = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}, \mathbf{G} = \begin{pmatrix} G & 0 \\ 0 & \widehat{G} \end{pmatrix}, \mathbf{A} = \begin{pmatrix} A & 0 \\ 0 & \mathcal{A} \end{pmatrix}, \mathbf{B} = \begin{pmatrix} B & 0 \\ 0 & \mathcal{B} \end{pmatrix}, \mathbf{C} = \begin{pmatrix} C & \mathcal{C} \\ 0 & 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} D & \mathcal{D} \\ 0 & 0 \end{pmatrix},$$

$$\mathbf{x}_0 = \begin{pmatrix} 0 \\ \xi_0 \end{pmatrix}, \mathcal{V} = \left\{ \mathbf{u}(\cdot) \middle| \mathbf{u}(\cdot) = \begin{pmatrix} u_1(\cdot) \\ u_2(\cdot) \end{pmatrix}; u_1(\cdot) \in L^2_{\mathbb{F}^i}(0,T;\Lambda), \mathbb{E}(u_1) = 0, u_2(\cdot) \in L^2(0,T;\Lambda) \right\},$$

system (4.5) can be rewritten as

$$\begin{cases} \mathcal{J}(\mathbf{x}_0, \mathbf{u}(\cdot)) = \mathbb{E}\displaystyle\int_0^T \langle \mathbf{Q}\mathbf{x}, \mathbf{x}\rangle + \langle \mathbf{R}\mathbf{u}, \mathbf{u}\rangle\, dt + \langle \mathbf{G}\mathbf{x}(T), \mathbf{x}(T)\rangle, \\ d\mathbf{x} = (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u})\, dt + (\mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u})\, dW, \quad \mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{u}(\cdot) \in \mathcal{V}. \end{cases} \tag{4.6}$$

Moreover, we can also denote

$$\mathbf{\Lambda} = \Lambda \times \Lambda := \left\{ v \middle| v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}; v_1, v_1 \in \Lambda \right\},$$

and

$$\mathcal{U}^{\mathbf{\Lambda}} = \mathcal{U}^{\Lambda} \times \mathcal{U}^{\Lambda} := \left\{ u(\cdot) \middle| u(\cdot) = \begin{pmatrix} u_1(\cdot) \\ u_2(\cdot) \end{pmatrix}; u_1(\cdot), u_2(\cdot) \in L^2_{\mathbb{F}^i}(0,T;\Lambda) \right\} = \left\{ u(\cdot) \middle| u(\cdot) \in L^2_{\mathbb{F}^i}(0,T;\mathbf{\Lambda}) \right\}.$$

Then triggered by (4.6), we can also introduce the following related augmented system:

$$\begin{cases} \mathcal{J}'(\mathbf{x}_0, \mathbf{u}(\cdot)) = \mathbb{E}\displaystyle\int_0^T \langle \mathbf{Q}\mathbf{x}, \mathbf{x}\rangle + \langle \mathbf{R}\mathbf{u}, \mathbf{u}\rangle\, dt + \langle \mathbf{G}\mathbf{x}(T), \mathbf{x}(T)\rangle, \\ d\mathbf{x} = (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u})\, dt + (\mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u})\, dW, \quad \mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{u}(\cdot) \in \mathcal{U}^{\mathbf{\Lambda}}. \end{cases} \tag{4.7}$$

Here we note that $\mathbf{\Lambda}$ is also a linear subspace of $\mathbb{R}^{2m}$ satisfying $\mathbf{\Lambda} = \operatorname{span}\left(\begin{pmatrix} v_1 \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} v_{m'} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ v_{m'} \end{pmatrix}\right)$. Correspondingly, we denote $\mathbf{V} = \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}$.

Remark 4.1. *Although system* (4.5) *(or equivalently* (4.6)*) forms a standard-looking stochastic LQ problem, system* (4.5) *(or equivalently* (4.6)*) is not equivalent to* (4.7) *since the control has to be of the form* $\binom{u-\mathbb{E}u}{\mathbb{E}u}$ *and the collection of all such processes is* $\mathcal{V}$ *instead of* $L^2_{\mathbb{F}}(0,T;\mathbf{\Lambda})$, *which should be the set of all admissible controls of* (4.7). *Hence, the above reduction does not lead to a direct application of standard stochastic LQ theory. However, in what follows, we can still study the relation of the uniform convexity between* (4.5) *(or equivalently* (4.6)*) and* (4.7).

We introduce the following Riccati equations (RE1) and (RE2) related to system (4.3)+(4.4) (or (4.5), (4.6) equivalently):

$$\text{(RE1)}\begin{cases}\dot{P}_1+P_1A+Q+C^TP_1C+A^TP_1-(P_1B+C^TP_1D)\mathcal{R}_1(B^TP_1+D^TP_1C)=0,\\ P_1(T)=G,\end{cases}$$

$$\text{(RE2)}\begin{cases}\dot{P}_2+P_2\mathcal{A}+\widehat{Q}+\mathcal{C}^TP_1\mathcal{C}+\mathcal{A}^TP_2-(P_2\mathcal{B}+\mathcal{C}P_1\mathcal{D})\mathcal{R}_2(\mathcal{B}^TP_2+\mathcal{D}^TP_1\mathcal{C})=0,\\ P_2(T)=\widehat{G},\end{cases}$$

where $\mathcal{R}_1=V(V^T(D^TP_1D+R)V)^{-1}V^T$ and $\mathcal{R}_2=V(V^T(\mathcal{D}^TP_1\mathcal{D}+R)V)^{-1}V^T$, and Riccati equation (RE0) related to system (4.7):

$$\text{(RE0)}\begin{cases}\dot{\mathbf{P}}+\mathbf{PA}+\mathbf{A}^T\mathbf{P}+\mathbf{C}^T\mathbf{PC}+\mathbf{Q}-\left(\mathbf{PB}+\mathbf{C}^T\mathbf{PD}\right)\mathcal{R}_0\left(\mathbf{B}^T\mathbf{P}+\mathbf{D}^T\mathbf{PC}\right)=0,\\ \mathbf{P}(T)=\mathbf{G},\end{cases}$$

where $\mathcal{R}_0=\mathbf{V}\left[\mathbf{V}^T\left(\mathbf{R}+\mathbf{D}^T\mathbf{PD}\right)\mathbf{V}\right]^{-1}\mathbf{V}^T$. Then we have the following result.

Lemma 4.2. *Let* (A4.1)–(A4.2) *hold. Among the following statements:*

(i) *$u(\cdot) \mapsto \mathcal{J}(\xi_0; u(\cdot))$ is uniformly convex on $\mathcal{U}^{\Lambda}$,*

(i)' *$\mathbf{u}(\cdot) \mapsto \mathcal{J}'(\mathbf{x}_0; \mathbf{u}(\cdot))$ is uniformly convex on $\mathcal{U}^{\boldsymbol{\Lambda}}$,*

(ii) (RE1), (RE2) *admit solutions $P_1(\cdot), P_2(\cdot) \in C([0,T]; \mathbb{S}^n)$ such that $\mathcal{D}^T(t)P_1(t)\mathcal{D}(t)+R(t),\ D^T(t)P_1(t)D(t)+R(t) \gg 0$ on Λ, a.e. $t \in [0,T]$,*

(ii)' (RE0) *admits a solution $\mathbf{P}(\cdot) \in C([0,T]; \mathbb{S}^{2n})$ such that $\mathbf{R}(t)+\mathbf{D}^T(t)\mathbf{P}(t)\mathbf{D}(t) \gg 0$ on $\boldsymbol{\Lambda}$, a.e. $t \in [0,T]$,*

the following implications hold:

$$\begin{array}{ccc} \text{(i)'} & \substack{\Longrightarrow \\ \not\Longleftarrow} & \text{(i)} \\ \Updownarrow & & \Uparrow \\ \text{(ii)'} & \not\Longleftarrow & \text{(ii)} \end{array}$$

Proof. See Appendix C.1. □

In the discussion above, we have studied the uniform convexity on the linear subspace, and it is actually weaker than that on the whole space. We cook up the following example to illustrate it.

Example 4.1. *Consider a system with form of* (4.7) *and we let:*

$$\mathbf{A} = \begin{pmatrix} -0.1 & -0.2 \\ -0.1 & -0.1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0.5 & 0.6 \\ -0.4 & -0.1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -0.9 & 0.4 \\ -0.6 & -0.1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} -0.7 & 0.2 \\ -0.3 & -0.6 \end{pmatrix},$$

$$\mathbf{Q} = \begin{pmatrix} 0.3 & -0.1 \\ -0.1 & -0.1 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} -0.2 & -0.3 \\ -0.3 & 0.6 \end{pmatrix}, \mathbf{G} = \begin{pmatrix} -0.1 & -0.4 \\ -0.4 & 0.2 \end{pmatrix}, \mathbf{V} = \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix},$$

and time interval is $[0,1]$. *Then* $\mathbf{\Lambda} = \operatorname{span}\left(\left(\begin{smallmatrix}0.5\\-0.5\end{smallmatrix}\right)\right)$ *and for any vector* $v \in \mathbf{\Lambda}$, *v is with form of* $\left(\begin{smallmatrix}a\\-a\end{smallmatrix}\right)$, $a \in \mathbb{R}$. *By Lemma* (4.2), $\mathbf{u}(\cdot) \mapsto \mathcal{J}'(\xi_0, \mathbf{u}(\cdot))$ *is uniform convex on* $\mathcal{U}^{\mathbf{\Lambda}}$ *if* (RE0) *admits a solution* $\mathbf{P}(\cdot) \in C([0,T];\mathbb{S}^{2n})$ *such that* $\mathbf{R}(t)+\mathbf{D}^T(t)\mathbf{P}(t)\mathbf{D}(t) \gg 0$ *on* $\mathbf{\Lambda}$, *a.e.* $t \in [0,T]$ *which is equivalent to* $\mathbf{V}^T\left(\mathbf{R}(t)+\mathbf{D}^T(t)\mathbf{P}(t)\mathbf{D}(t)\right)\mathbf{V} \gg 0$, *a.e.* $t \in [0,T]$. *By solving* (RE0), *we have the following trajectory of* $\mathbf{V}^T\left(\mathbf{R}+\mathbf{D}^T\mathbf{P}\mathbf{D}\right)\mathbf{V}$ *(which is actually 1-dimensional in this case):*

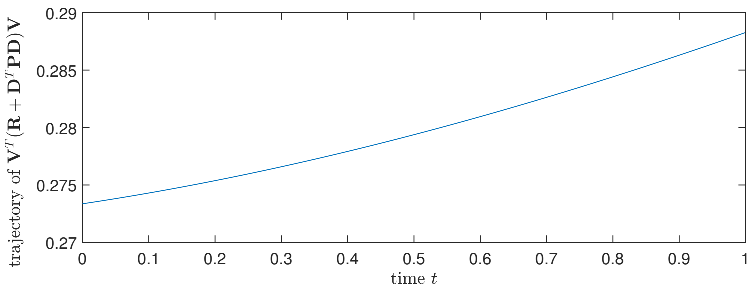


Figure 4.1: the trajectory of $\mathbf{V}^T\left(\mathbf{R}+\mathbf{D}^T\mathbf{P}\mathbf{D}\right)\mathbf{V}$

It holds that $\mathbf{V}^T\left(\mathbf{R}+\mathbf{D}^T\mathbf{P}\mathbf{D}\right)\mathbf{V} > 0.2$ *and hence* $\mathbf{u}(\cdot) \mapsto \mathcal{J}'(\xi_0, \mathbf{u}(\cdot))$ *is uniform convex on* $\mathcal{U}^{\mathbf{\Lambda}}$. *However when we consider the following classic Riccati equation:*

$$\begin{cases}\dot{\mathbf{P}}+\mathbf{P}\mathbf{A}+\mathbf{A}^T\mathbf{P}+\mathbf{C}^T\mathbf{P}\mathbf{C}+\mathbf{Q}-\left(\mathbf{P}\mathbf{B}+\mathbf{C}^T\mathbf{P}\mathbf{D}\right)\left(\mathbf{R}+\mathbf{D}^T\mathbf{P}\mathbf{D}\right)^{-1}\left(\mathbf{B}^T\mathbf{P}+\mathbf{D}^T\mathbf{P}\mathbf{C}\right)=0,\\ \mathbf{P}(T)=\mathbf{G},\end{cases}$$

we have the following trajectories of the eigenvalues of $\mathbf{R}+\mathbf{D}^T\mathbf{P}\mathbf{D}$*:*

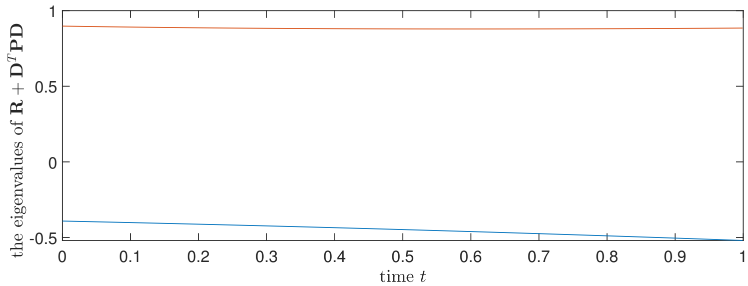


Figure 4.2: the trajectories of the eigenvalues of $\mathbf{R}+\mathbf{D}^T\mathbf{PD}$

$\mathbf{R}+\mathbf{D}^T\mathbf{PD}$ *is indefinite on the whole time interval* $[0,1]$ *and hence* $\mathbf{u}(\cdot) \mapsto \mathcal{J}'(\xi_0, \mathbf{u}(\cdot))$ *is not uniform convex on the whole space.*

Remark 4.2. *Our extension study of the uniform convexity on the linear subspace has practical potential in various areas such as finance. The linear subspace represents that each manager has access to the whole market except some fixed firm who has private information. In this case, the unique solvability of the problem can be guaranteed even if the cost functional is not uniform convex on the whole space, which is usually assumed in the previous literature. We only require the uniform convexity on the linear subspace. For more examples of linear constraints and their economic meaning, interested readers are referred to [82].*

To conclude our results, we introduce the following assumption:

(A4.3) (RE1), (RE2) admit solutions $P_1(\cdot), P_2(\cdot) \in C([0,T];\mathbb{S}^n)$ such that $\mathcal{D}^T(t)P_1(t)\mathcal{D}(t)+R(t)$, $D^T(t)P_1(t)D(t)+R(t) \gg 0$ on Λ, a.e. $t \in [0,T]$.

Then we can obtain the main result of the convexity of $\mathcal{J}$ directly as follows:

Theorem 4.1. *Under* (A4.1)-(A4.3), *the functional* $u(\cdot) \mapsto \mathcal{J}(\xi_0; u(\cdot))$ *is uniformly convex on* $\mathcal{U}^\Lambda$, *and hence* (**MFC-c**) *admits a unique minimizer on* $\mathcal{U}^\Lambda$.

We have established the relation between the Riccati equations (RE1), (RE2) and the uniform convexity of the cost functional. The convexity condition given in Theorem 4.1 is much weaker than the *standard assumption* represented as follows in terms of our notation:

$$(SA) : Q, G \geq 0, R \gg 0.$$

The (SA) is widely used in other relevant literature, e.g., [13, 61] where the MFC problem is studied and [64] where the constrained LQG MFG problem is studied. The following proposition will show the relation between (SA) and (A4.3) more illustratively.

Proposition 4.1. *If* (A4.1)-(A4.2) *and* (SA) *are assumed, then* (A4.3) *holds. By contrast, if* (A4.1)-(A4.2) *and* (A4.3) *are assumed,* (SA) *does NOT necessarily hold.*

Proof. By Theorem 4.1 in [83], under (A4.1)-(A4.2) and (SA), we consider the following iterative scheme with index $\alpha \geq 0$:

$$\begin{cases} K_0 = I, \\ \Theta_0 = \left[V^T(D^T K_0 D + R)V\right]^{-1}(V^T B^T K_0 + V^T D^T K_0 C), \\ K_{\alpha+1} : \dot{K}_{\alpha+1} + K_{\alpha+1}(A - BV\Theta_\alpha) + (A - BV\Theta_\alpha)^T K_{\alpha+1} + (C - DV\Theta_\alpha)^T K_{\alpha+1}(C - DV\Theta_\alpha) \\ \qquad + \Theta_\alpha^T V^T R V \Theta_\alpha + Q = 0, \quad K_{\alpha+1}(T) = G, \\ \Theta_{\alpha+1} = \left[V^T(D^T K_{\alpha+1} D + R)V\right]^{-1}(V^T B^T K_{\alpha+1} + V^T D^T K_{\alpha+1} C), \quad \alpha \geq 0. \end{cases}$$

By (SA) and Lemma 4.1, we have $K_{\alpha+1}$, $V^T(D^TK_{\alpha+1}D+R)V \gg 0$. Then $\lim_{\alpha\to\infty} K_\alpha \longrightarrow P_1$ and hence (RE1) admits a unique solution $P_1 \geq 0$. Thus, by $R \gg 0$ we have D^TP_1D+R, $\mathcal{D}^TP_1\mathcal{D}+R \gg 0$.

Next, we consider (RE2) which can be rewritten as:

$$\text{(RE2)} \begin{cases} \dot{P}_2+P_2\left(\mathcal{A}-\mathcal{B}\mathcal{R}_2\mathcal{D}^TP_1\mathcal{C}\right)+\left(\mathcal{A}^T-\mathcal{C}^TP_1\mathcal{D}\mathcal{R}_2\mathcal{B}^T\right)P_2+\widetilde{Q} \\ -P_2\left(\mathcal{B}V\right)\left[V^T(\mathcal{D}^TP_1\mathcal{D}+R)V\right]^{-1}(V\mathcal{B})^TP_2=0, \quad P_2(T)=\widehat{G}, \end{cases}$$

where $\widetilde{Q} := \mathcal{C}^TP_1\mathcal{C}+\widehat{Q}-\mathcal{C}^TP_1\mathcal{D}\mathcal{R}_2\mathcal{D}^TP_1\mathcal{C}$. Since $Q, G \geq 0, R \gg 0$, then $\widehat{Q}, \widehat{G} \geq 0$ and $V^T(\mathcal{D}^TP_1\mathcal{D}+R)V \gg 0$. Thus we desire to prove $\widetilde{Q} \geq 0$. Since $P_1 \geq 0$ and $R \gg 0$, then there exist two unique matrix-value functions $P_1^{\frac{1}{2}}(t) \geq 0$ and $R^{\frac{1}{2}}(t) \gg 0$ such that $P_1^{\frac{1}{2}}P_1^{\frac{1}{2}} = P_1$ and $R^{\frac{1}{2}}R^{\frac{1}{2}} = R$. We consider the related matrix $\mathbb{M} := \begin{pmatrix} V^T(\mathcal{D}^TP_1\mathcal{D}+R)V & (\mathcal{D}V)^TP_1 \\ P_1\mathcal{D}V & P_1 \end{pmatrix}$. Noting that $V^T(\mathcal{D}^TP_1\mathcal{D}+R)V \gg 0$, by Schur complement lemma, $\mathbb{M} \geq 0$ if and only if $P_1-P_1\mathcal{D}\mathcal{R}_2\mathcal{D}^TP_1 \geq 0$. Using $P_1^{\frac{1}{2}}$ and $R^{\frac{1}{2}}$, $\mathbb{M}$ can be represented as:

$$\mathbb{M} = \begin{pmatrix} V^T(\mathcal{D}^TP_1\mathcal{D}+R)V & (\mathcal{D}V)^TP_1 \\ P_1\mathcal{D}V & P_1 \end{pmatrix} = \begin{pmatrix} (\mathcal{D}V)^TP_1^{\frac{1}{2}} & V^TR^{\frac{1}{2}} \\ P_1^{\frac{1}{2}} & 0 \end{pmatrix} \begin{pmatrix} P_1^{\frac{1}{2}}(\mathcal{D}V) & P_1^{\frac{1}{2}} \\ R^{\frac{1}{2}}V & 0 \end{pmatrix} \geq 0.$$

Thus, $P_1-P_1\mathcal{D}\mathcal{R}_2\mathcal{D}^TP_1 \geq 0$ and $\widetilde{Q}=\mathcal{C}^TP_1\mathcal{C}+\widehat{Q}-\mathcal{C}^TP_1\mathcal{D}\mathcal{R}_2\mathcal{D}^TP_1\mathcal{C} = \mathcal{C}^T(P_1-P_1\mathcal{D}\mathcal{R}_2\mathcal{D}^TP_1)\mathcal{C}+\widehat{Q} \geq 0$, and (A4.3) holds.

Moreover, in Example C.1, we see that Q, R, G are all indefinite, which results that (SA) fails to hold. However, $\mathcal{J}$ is still uniformly convex and (**MFC-c**) is uniquely solvable. This ends the proof. $\square$

4.1.2 Optimality condition

In this section, we derive the characterization of the optimal pair of (**MFC-c**).

Proposition 4.2. *Let* (A4.1)-(A4.3) *hold.* (**MFC-c**) *admits a unique optimal pair* $(\bar{x}, \bar{u})$ *on* $\mathcal{U}^\Lambda$*. The following Hamiltonian system* (H1)*:*

$$
(\mathbf{H1})\begin{cases}
d\bar{x} = (A\bar{x}+\bar{A}\mathbb{E}\bar{x}+B\bar{u}+\bar{B}\mathbb{E}\bar{u})dt+(C\bar{x}+\bar{C}\mathbb{E}\bar{x}+D\bar{u}+\bar{D}\mathbb{E}\bar{u})dW,\\
dk = -\left(Q\bar{x}-(Q\Gamma_1+\Gamma_1^TQ-\Gamma_1^TQ\Gamma_1)\mathbb{E}\bar{x}+A^Tk+\bar{A}^T\mathbb{E}k+C^T\zeta+\bar{C}^T\mathbb{E}\zeta\right)dt+\zeta dW,\\
\bar{x}(0)=\xi_0, \quad k(T)=G\bar{x}(T)-(G\Gamma_2+\Gamma_2^TG-\Gamma_2^TG\Gamma_2)\mathbb{E}\bar{x}(T),\\
V^T\left(B^Tk+\bar{B}^T\mathbb{E}k+D^T\zeta+\bar{D}^T\mathbb{E}\zeta+R\bar{u}\right)=0, \ \ a.e. \ t\in[0,T], \ \mathbb{P}-a.s.,
\end{cases} \tag{4.8}
$$

admits a unique adapted solution $(\bar{x}, \bar{u}, k, \zeta)$ *where* $(\bar{x}, \bar{u})$ *is the optimal pair of* (**MFC-c**).

Proof. See Appendix C.2. □

Remark 4.3. *If we further assume that* $R > 0$*, then by Lemma 4.1,* $\bar{u}$ *can be represented explicitly as follows:*

$$
\bar{u} = -V(V^TRV)^{-1}V^T\left(B^Tk+\bar{B}^T\mathbb{E}k+D^T\zeta+\bar{D}^T\mathbb{E}\zeta\right), \quad a.e. \ t\in[0,T], \ \mathbb{P}-a.s..
$$

Moreover, in what follows we will also introduce another feedback form representation of $\bar{u}$*, and in that case* R *could be indefinite.*

Proposition 4.3. *Under* (A4.1)–(A4.3)*, the following closed-loop system:*

$$
\begin{cases}
d\bar{x} = (\left[A-B\mathcal{R}_1(B^TP_1+D^TP_1C)\right](\bar{x}-\mathbb{E}\bar{x})+\left[\mathcal{A}-\mathcal{B}\mathcal{R}_2(\mathcal{B}^TP_2+\mathcal{D}^TP_1\mathcal{C})\right]\mathbb{E}\bar{x})dt\\
\qquad +(\left[C-D\mathcal{R}_1(B^TP_1+D^TP_1C)\right](\bar{x}-\mathbb{E}\bar{x})+\left[\mathcal{C}-\mathcal{D}\mathcal{R}_2(\mathcal{B}^TP_2+\mathcal{D}^TP_1\mathcal{C})\right]\mathbb{E}\bar{x})dW,\\
\bar{x}(0)=\xi_0,
\end{cases} \tag{4.9}
$$

admits a unique solution $\bar{x}$, and by defining:

$$\begin{cases} \bar{u} = -\mathcal{R}_2(\mathcal{B}^T P_2 + \mathcal{D}^T P_1 \mathcal{C})\mathbb{E}\bar{x} - \mathcal{R}_1(B^T P_1 + D^T P_1 C)(\bar{x} - \mathbb{E}\bar{x}), \\ k = P_1(\bar{x} - \mathbb{E}\bar{x}) + P_2(\mathbb{E}\bar{x}), \\ \zeta = \left[P_1 C - P_1 D \mathcal{R}_1(B^T P_1 + D^T P_1 C)\right](\bar{x} - \mathbb{E}\bar{x}) + \left[P_1 \mathcal{C} - P_1 \mathcal{D}\mathcal{R}_2(\mathcal{B}^T P_2 + \mathcal{D}^T P_1 \mathcal{C})\right]\mathbb{E}\bar{x}, \end{cases} \tag{4.10}$$

the quadruple $(\bar{x}, \bar{u}, k, \zeta)$ is the unique adapted solution to (H1)*, and $(\bar{x}, \bar{u})$ is the unique optimal pair of* (**MFC-c**)*. Moreover,*

$$\inf_{u \in \mathcal{U}^\Lambda} \mathcal{J}(\xi_0; u) = \mathcal{J}(\xi_0; \bar{u}) = \frac{1}{2}\langle P_2(0)\xi_0, \xi_0 \rangle. \tag{4.11}$$

Proof. It can be verified directly that $(\bar{x}, \bar{u}, k, \zeta)$ defined by (4.9)-(4.10) is the adapted solution of (H1). The uniqueness and optimality of $(\bar{x}, \bar{u})$ follow by Proposition 4.2. Thus, what remains to prove is (4.11). Noting that $\langle P_1(x(0) - \mathbb{E}x(0)), (x(0) - \mathbb{E}x(0)) \rangle = \langle P_1(\xi_0 - \xi_0), (\xi_0 - \xi_0) \rangle = 0$, similar to the proof of Lemma (4.2), we have:

$$\begin{aligned} & 2\mathcal{J}(\xi_0; u(\cdot)) \\ =& \mathbb{E}\int_0^T \left\langle (\mathcal{D}^T P_1 \mathcal{D} + R)\left[\mathbb{E}u + \mathcal{R}_2(\mathcal{B}^T P_2 + \mathcal{D}^T P_1 \mathcal{C})\mathbb{E}x\right], \mathbb{E}u + \mathcal{R}_2(\mathcal{B}^T P_2 + \mathcal{D}^T P_1 \mathcal{C})\mathbb{E}x \right\rangle dt \\ & + \mathbb{E}\int_0^T \Big\langle (D^T P_1 D + R)\left[(u - \mathbb{E}u) + \mathcal{R}_1(B^T P_2 + D^T P_1 C)(x - \mathbb{E}x)\right], \\ & \qquad (u - \mathbb{E}u) + \mathcal{R}_1(B^T P_2 + D^T P_1 C)(x - \mathbb{E}x) \Big\rangle dt + \langle P_2(0)\xi_0, \xi_0 \rangle. \end{aligned}$$

Since $(\mathcal{D}^T P_1 \mathcal{D} + R)$, $(D^T P_1 D + R) \gg 0$ on Λ, then $\inf_{u \in \mathcal{U}^\Lambda} \mathcal{J}(\xi_0; u) = \frac{1}{2}\langle P_2(0)\xi_0, \xi_0 \rangle$. This ends the proof. □

Through the discussion above, we can conclude the following contributions:

- We consider the MFC problem constrained on a linear subspace. To our best knowledge, this thesis is the first research to tackle such model.

- We study the relation of the uniform convexity between systems (4.5) and (4.7) which is mentioned but has not been discussed in [61].

- We obtain a weaker condition of the uniform convexity of (**MFC-c**) compared with (SA) which has been widely used in other relevant studies (e.g., [13, 64, 61]). In our condition, the cost functional weight coefficients could be indefinite.

- We establish the relation between the optimal pair of (**MFC-c**) and the solution of Hamiltonian system (H1). We also derive a feedback form representation of the optimal control $\bar{u}$.

In next section we will analyze the MFG problem constrained on a linear subspace with similar scheme.

4.2 MFG problem constrained on a linear subspace

In this section we study the LQG MFG problem constrained on a linear subspace (for short, (**MFG-c**)). Then the problem can be represented as follows:

(**MFG-c**): For given initial value ξ_0, each agent $\mathcal{A}_i$ find a control $\bar{u}_i(\cdot) \in \mathcal{U}_c^{\Lambda}$

such that $\mathcal{J}_i(\xi_0;\bar{u}_i(\cdot),\bar{u}_{-i}(\cdot)) = \inf_{u_i(\cdot)\in\mathcal{U}_c^\Lambda} \mathcal{J}_i(\xi_0;u_i(\cdot),\bar{u}_{-i}(\cdot))$, where

$$\mathcal{J}_i(\xi_0;u_i(\cdot),u_{-i}(\cdot))=\frac{1}{2}\mathbb{E}\left\{\int_0^T \|x_i-\Gamma_1 x^{(N)}\|_Q^2+\|u_i\|_R^2 dt+\|x_i(T)-\Gamma_2 x^{(N)}(T)\|_G^2\right\},\tag{4.12}$$

$$\text{s.t.}\quad\begin{cases} dx_i=(Ax_i+\bar{A}x^{(N)}+Bu_i+\bar{B}u^{(N)})dt+(Cx_i+\bar{C}x^{(N)}+Du_i+\bar{D}u^{(N)})dW_i,\\ x_i(0)=\xi_0.\end{cases}\tag{4.13}$$

We call $\bar{\mathbf{u}} := (\bar{u}_1,\cdots,\bar{u}_N)$ a centralized Nash equilibrium for (**MFG-c**). For comparison, we also present the definition of ε-Nash equilibrium.

Definition 4.3. *A control set $\mathbf{u}^\varepsilon := (u_1^\varepsilon,\cdots,u_N^\varepsilon)\in\prod_{i\in\mathcal{I}}\mathcal{U}_c^\Lambda$ is called an ε-Nash equilibrium if*

$$\left|\mathcal{J}_i(\xi_0;u_i^\varepsilon(\cdot),u_{-i}^\varepsilon(\cdot))-\inf_{u_i(\cdot)\in\mathcal{U}_c^\Lambda}\mathcal{J}_i(\xi_0;u_i(\cdot),u_{-i}^\varepsilon(\cdot))\right|=\varepsilon(N),\quad \varepsilon(N)\to 0,\ \textit{when}\ N\to\infty.$$

Remark 4.4. *If $\varepsilon=0$, Definition 4.3 reduces to the usual exact Nash equilibrium.*

Note that in (**MFG-c**) each agent chooses its control in centralized admissible control set $\mathcal{U}_c^\Lambda$ and this will face some difficulties in the practical application. Firstly, an agent may be only able to access its own information (i.e., $\mathbb{F}^i$) most of the time, and the information of the others may be unavailable for it in real world (see [71, 72, 73]).

Secondly, by the coupling structure, the dynamic optimization will be subjected to the curse of dimensionality and complexity in numerical analysis in practice (see [4, 5]). Thus, to some extent, decentralized control would be more practicable in real application than centralized control (see [14, 40]).

Thus, in what follows, we aim to derive some decentralized control set for (**MFG-c**) satisfying some asymptotic optimality (e.g., ε-Nash equilibrium) and with less computational burden in the practical application. Before we begin further discussion, we should introduce some basic assumption in this section. We still apply assumptions (A4.1)-(A4.2) to the coefficients. Similar to (**MFC-c**), under (A4.1), for any given $\mathbf{u} := (u_1, \cdots, u_N) \in \prod_{i\in\mathcal{I}} \mathcal{U}_c^{\Lambda}$, (4.13) admits a unique strong solution $\mathbf{x}(\cdot) \equiv \mathbf{x}(\cdot;\xi_0,\mathbf{u}(\cdot)) := (x_1(\cdot;\xi_0,\mathbf{u}(\cdot)), \cdots, x_N(\cdot;\xi_0,\mathbf{u}(\cdot)))$, and under (A4.2), each $\mathcal{J}_i$, $i \in \mathcal{I}$ is well-defined.

4.2.1 MFG scheme

Next we will apply MFG method to analyze (**MFG-c**), which would bring us a decentralized ε-Nash equilibrium. Initially, we introduce the classical procedure of MFG method. Here we just briefly sketch some key points and interested readers are referred to [20, 15] for more details.

(MG1) Freeze $x^{(N)}$, $u^{(N)}$ by some deterministic terms $\bar{m}$, $\bar{w}$ respectively, and obtain the auxiliary problem:

$(\mathbf{MFG\text{-}c})^*$: For given initial value ξ_0, each agent $\mathcal{A}_i$ find a control $\check{u}_i \in \mathcal{U}_i^\Lambda$ such that $J_i(\xi_0;\check{u}_i(\cdot)) = \inf_{u_i \in \mathcal{U}_i^\Lambda} J_i(\xi_0;u_i(\cdot))$, where

$$J_i(\xi_0;u_i(\cdot))=\frac{1}{2}\mathbb{E}\left\{\int_0^T \|x_i-\Gamma_1\bar{m}\|_Q^2+\|u_i\|_R^2 dt + \|x_i(T)-\Gamma_2\bar{m}(T)\|_G^2\right\}, \tag{4.14}$$

$$\text{s.t.}\quad \begin{cases} dx_i = (Ax_i+\bar{A}\bar{m}+Bu_i+\bar{B}\bar{w})dt+(Cx_i+\bar{C}\bar{m}+Du_i+\bar{D}\bar{w})dW_i, \\ x_i(0) = \xi_0. \end{cases} \tag{4.15}$$

Here, $\bar{m}$, $\bar{w}$ are undetermined at this moment, thus they should be treated as some exogenous terms.

(MG2) Solve the auxiliary control $(\mathbf{MFG\text{-}c})^*$ for each agent, and obtain the auxiliary optimal control set $\check{\mathbf{u}}(\cdot;\xi_0,\bar{m}(\cdot),\bar{w}(\cdot)) := (\check{u}_1(\cdot;\xi_0,\bar{m}(\cdot),\bar{w}(\cdot)), \cdots, \check{u}_N(\cdot;\xi_0,\bar{m}(\cdot),\bar{w}(\cdot)))$ and the corresponding optimal trajectories $\check{\mathbf{x}}(\cdot;\xi_0,\bar{m}(\cdot),\bar{w}(\cdot)) := (\check{x}_1(\cdot;\xi_0,\bar{m}(\cdot),\bar{w}(\cdot)), \cdots, \check{x}_N(\cdot;\xi_0,\bar{m}(\cdot),\bar{w}(\cdot)))$.

(MG3) Determine the pre-frozen terms $\bar{m}(\cdot)$, $\bar{w}(\cdot)$ by the following CC system:

$$\bar{m}(\cdot) = \mathbb{E}\check{x}_i(\cdot;\xi_0,\bar{m}(\cdot),\bar{w}(\cdot)), \quad \bar{w}(\cdot) = \mathbb{E}\check{u}_i(\cdot;\xi_0,\bar{m}(\cdot),\bar{w}(\cdot)). \tag{4.16}$$

Then by plugging the determined $\bar{m}(\cdot)$, $\bar{w}(\cdot)$ into $\check{\mathbf{u}}(\cdot;\xi_0,\bar{m}(\cdot),\bar{w}(\cdot))$, we obtain the MFG strategy set of $(\mathbf{MFG\text{-}c})$.

Remark 4.5. *Note that in* $(\mathbf{MFG\text{-}c})^*$ *we restrict the admissible control in the decentralized set* $\mathcal{U}_i^\Lambda$*, since* $(\mathbf{MFG\text{-}c})^*$ *is decoupled, and each agent* $\mathcal{A}_i$

only need the decentralized information to minimize the cost functional J_i (i.e., $\inf_{u_i(\cdot)\in\mathcal{U}_c^\Lambda} J_i(\xi_0;u_i(\cdot)) \equiv \inf_{u_i(\cdot)\in\mathcal{U}_i^\Lambda} J_i(\xi_0;u_i(\cdot))$). Hence the MFG method could bring us a decentralized strategy set $\check{\mathbf{u}}$.

Note that all agents are homogeneous in (**MFG-c**)*, and we suppress the subscript i in case when no confusion occurs hereafter in this chapter.

4.2.2 MFG strategy set

Firstly, we start with procedures (MG1)-(MG2) to derive the MFG strategy set. By observing Riccati equation (RE1), we obtain the following result whose proof is similar to Proposition 4.2.

Proposition 4.4. *Let* (A4.1)-(A4.3) *hold.* (**MFG-c**)* *is uniformly convex and thus admits a unique optimal pair $(\check{x},\check{u})$ on $\mathcal{U}^\Lambda$. The following Hamiltonian system:*

$$(\mathbf{H2})\begin{cases} d\check{x}=(A\check{x}+\bar{A}\bar{m}+B\check{u}+\bar{B}\bar{w})dt+(C\check{x}+\bar{C}\bar{m}+D\check{u}+\bar{D}\bar{w})dW, \\ dl=-\left(A^Tl+C^T\varsigma+Q\check{x}-Q\Gamma_1\bar{m}\right)dt+\varsigma dW, \\ \check{x}(0)=\xi_0, \quad l(T)=G\check{x}(T)-G\Gamma_2\bar{m}(T), \\ V^T\left(B^Tl+D^T\varsigma+R\check{u}\right)=0, \ a.e.\ t\in[0,T],\ \mathbb{P}-a.s.. \end{cases}$$

admits a unique adapted solution $(\check{x},\check{u},l,\varsigma)$ where $(\check{x},\check{u})$ is the optimal pair of (**MFG-c**)**. Moreover, if $R>0$ is assumed, the optimal control can be represented explicitly as $\check{u}=-V(V^TRV)^{-1}V^T\big(B^Tl+D^T\varsigma\big)$.*

Secondly, we tackle procedure (MG3) by plugging (4.16) into (H2), and we have the following CC system:

$$
(\mathbf{CC\text{-}1})\begin{cases} d\check{x} = (A\check{x}+\bar{A}\mathbb{E}\check{x}+B\check{u}+\bar{B}\mathbb{E}\check{u})dt+(C\check{x}+\bar{C}\mathbb{E}\check{x}+D\check{u}+\bar{D}\mathbb{E}\check{u})dW, \\ dl = -\left(A^T l+C^T\varsigma + Q\check{x} - Q\Gamma_1\mathbb{E}\check{x}\right)dt+\varsigma dW, \\ \check{x}(0) = \xi_0, \quad l(T) = G\check{x}(T)-G\Gamma_2\mathbb{E}\check{x}(T), \\ V^T\left(B^T l+D^T\varsigma+R\check{u}\right)=0. \end{cases}
$$

Although (CC-1) only takes an indirect embedding representation, it is still rather tractable. Actually, by using the discounting method (see [64, 76]) or reduction decoupling method (see [60]), the solvability condition of (28) can be set up. By decentralizing method and decoupling method, which can be found in [84, 61], we can even obtain an explicit solution of $\bar{m}$ and $\bar{w}$ via Riccati equation. We introduce the following asymmetric Riccati equations:

$$
(\mathrm{RE3})\begin{cases} \dot{P}_3+P_3\mathcal{A}+A^TP_3+C^TP_1\mathcal{C}-(P_3\mathcal{B}+C^TP_1\mathcal{D})\mathcal{R}_3(B^TP_3+D^TP_1\mathcal{C})+(Q-Q\Gamma_1), \\ P_3(T) = G(I-\Gamma_2), \end{cases} \tag{4.17}
$$

where $\mathcal{R}_3 = V(V^T(D^TP_1\mathcal{D}+R)V)^{-1}V^T$. Then we have the following result:

Proposition 4.5. *Under* (A4.1)–(A4.2)*, if Riccati equations* (RE1)*,* (RE3) *admit solutions $P_1 \in C([0,T],\mathbb{S}^n)$, $P_3 \in C([0,T],\mathbb{R}^n)$ such that $(D^T(t)P_1(t)D(t)+R(t)), (D^T(t)P_1(t)\mathcal{D}(t)+R(t)) \gg 0$ on Λ, a.e. $t \in [0,T]$, then the following* *closed-loop system:*

$$\begin{cases} d\check{x} = \left[\left(A - B\mathcal{R}_1(B^T P_1 + D^T P_1 C)\right)(\check{x} - \mathbb{E}\check{x}) + \left(\mathcal{A} - \mathcal{B}\mathcal{R}_3(B^T P_3 + D^T P_1 \mathcal{C})\right)\mathbb{E}\check{x}\right] dt \\ \qquad + \left[\left(C - D\mathcal{R}_1(B^T P_1 + D^T P_1 C)\right)(\check{x} - \mathbb{E}\check{x}) + \left(\mathcal{C} - \mathcal{D}\mathcal{R}_3(B^T P_3 + D^T P_1 \mathcal{C})\right)\mathbb{E}\check{x}\right] dW, \\ \check{x}(0) = \xi_0, \end{cases} \tag{4.18}$$

admits a solution $\check{x}$*, and by defining:*

$$\begin{cases} \check{u} = -\mathcal{R}_1(B^T P_1 + D^T P_1 C)(\check{x} - \mathbb{E}\check{x}) - \mathcal{R}_3(B^T P_3 + D^T P_1 \mathcal{C})\mathbb{E}\check{x}, \\ l = P_1(\check{x} - \mathbb{E}\check{x}) + P_3(\mathbb{E}\check{x}), \\ \varsigma = P_1\left(C - D\mathcal{R}_1(B^T P_1 + D^T P_1 C)\right)(\check{x} - \mathbb{E}\check{x}) + P_1\left(\mathcal{C} - \mathcal{D}\mathcal{R}_3(B^T P_3 + D^T P_1 \mathcal{C})\right)\mathbb{E}\check{x}, \end{cases} \tag{4.19}$$

the 4-tuple $(\check{x}, \check{u}, l, \varsigma)$ *is the adapted solution to* (CC-1) *and* $\bar{m} = \mathbb{E}\check{x}$*,* $\bar{w} = -\mathcal{R}_3(B^T P_3 + D^T P_1 \mathcal{C})\bar{m}$*.*

Proof. This result can be verified directly by plugging (4.19) into (CC-1). □

From the above, we derive an explicit representation of the solution of (CC-1). As for the uniqueness of (CC-1), we would also prove that (CC-1) is equivalent (H1) under some conditions in Section 4.4, which would lead to the equivalence of the MFC control and MFG strategy. Thus, through the discussion above, for each agent $\mathcal{A}_i$, (**MFG-c**) admits a unique feedback form MFG strategy:

$$\tilde{u}_i = -\mathcal{R}_3(B^T P_3 + D^T P_3 C)(\tilde{x}_i - \mathbb{E}\tilde{x}_i) - \mathcal{R}_5(B^T P_5 + D^T P_3 \mathcal{C})\mathbb{E}\tilde{x}_i, \tag{4.20}$$

where $\tilde{x}_i$ is the realized state satisfying the following dynamic:

$$\begin{cases} d\tilde{x}_i = (A\tilde{x}_i + \bar{A}\tilde{x}^{(N)} + B\tilde{u}_i + \bar{B}\tilde{u}^{(N)})dt + (C\tilde{x}_i + \bar{C}\tilde{x}^{(N)} + D\tilde{u}_i + \bar{D}\tilde{u}^{(N)})dW_i, \\ \tilde{x}_i(0) = \xi_0, \end{cases}$$

and $\tilde{x}^{(N)} = \frac{\sum_{j\in\mathcal{I}} \tilde{x}_j}{N}$, $\tilde{u}^{(N)} = \frac{\sum_{j\in\mathcal{I}} \tilde{u}_j}{N}$.

Lastly, for the performance of MFG strategy set $\tilde{\mathbf{u}}$ determined by (MG1)-(MG3) we have the following result whose proof is similar to that in [64, 18].

Proposition 4.6. *Under* (A4.1)–(A4.2)*, if Riccati equations* (RE1)*,* (RE3) *admit unique solutions* $P_1 \in C([0,T], \mathbb{S}^n)$*,* $P_3 \in C([0,T], \mathbb{R}^n)$ *such that* $(D^T(t)P_1(t)D(t) + R(t)), (D^T(t)P_1(t)\mathcal{D}(t) + R(t)) \gg 0$ *on* Λ*, a.e.* $t \in [0,T]$*, and* (CC-1) *is uniquely solvable, then the MFG strategy set* $\tilde{\mathbf{u}}$ *determined by* (MG1)-(MG3) *(i.e.,* (4.20)*) is an* ε*-Nash equilibrium.*

In this section, we establish the relation between the MF strategy of (**MFG-c**) and the solution of Hamiltonian system (H2) and CC system (CC-1). In next section we will analyze the MFT problem constrained on a linear subspace with similar scheme.

4.3 MFT problem constrained on a linear subspace

In what follows, we study LQG MFT problem constrained on a linear subspace (for short, (**MFT-c**)). We denote $\mathcal{U}_c^{\mathbf{\Lambda}} = \prod_{i\in\mathcal{I}} \mathcal{U}_c^{\Lambda}$ where $\mathbf{\Lambda} := \prod_{i\in\mathcal{I}} \Lambda$ which is also a linear subspace of $\mathbb{R}^{Nm}$ satisfying $\mathbf{\Lambda} = \text{span}\big(\binom{v_1}{0}, \cdots, \binom{v_{m'}}{0}, \cdots, \binom{0}{v_1}, \cdots, \binom{0}{v_{m'}}\big)$. Correspondingly, we denote $\mathbf{V} := \text{diag}(V, \cdots, V)$. Note that for the sake of notation simplicity, we still use $\mathbf{\Lambda}$, $\mathbf{V}$ to represent the augmented linear subspace and matrix, whose meanings are different to those in Section 4.1. Then the problem can be represented as follows:

(**MFT-c**): For given initial value ξ_0, find a control set $\bar{\mathbf{u}} = (\bar{u}_1, \cdots, \bar{u}_N) \in \mathcal{U}_c^{\mathbf{\Lambda}}$ such that $\mathcal{J}_{soc}^{(N)}(\xi_0; \bar{\mathbf{u}}(\cdot)) = \inf_{\mathbf{u}(\cdot)\in\mathcal{U}_c^{\mathbf{\Lambda}}} \mathcal{J}_{soc}^{(N)}(\xi_0; \mathbf{u}(\cdot))$, where

$$\mathcal{J}_{soc}^{(N)}(\xi_0; \mathbf{u}(\cdot)) := \sum_{i\in\mathcal{I}} \mathcal{J}_i(\xi_0; \mathbf{u}(\cdot)), \tag{4.21}$$

$$\mathcal{J}_i(\xi_0; \mathbf{u}(\cdot)) = \mathcal{J}_i(\xi_0; u_i(\cdot), u_{-i}(\cdot)) \tag{4.22}$$

$$= \frac{1}{2}\mathbb{E}\left\{\int_0^T \|x_i - \Gamma_1 x^{(N)}\|_Q^2 + \|u_i\|_R^2 dt + \|x_i(T) - \Gamma_2 x^{(N)}(T)\|_G^2\right\}, \tag{4.23}$$

$$\text{s.t.} \quad \begin{cases} dx_i = (Ax_i + \bar{A}x^{(N)} + Bu_i + \bar{B}u^{(N)})dt + (Cx_i + \bar{C}x^{(N)} + Du_i + \bar{D}u^{(N)})dW_i, \\ x_i(0) = \xi_0. \end{cases} \tag{4.24}$$

We call $\bar{\mathbf{u}} = (\bar{u}_1, \cdots, \bar{u}_N)$ a centralized optimal control set for (**MFT-c**). For comparison, we also present the definition of ε-asymptotically optimal control set.

Definition 4.4. *A control set $\mathbf{u}^\varepsilon := (u_1^\varepsilon, \cdots, u_N^\varepsilon) \in \mathcal{U}_c^{\mathbf{\Lambda}}$ is ε-asymptotically optimal if*

$$\frac{1}{N}\left|\mathcal{J}_{soc}^{(N)}(\xi_0; \mathbf{u}^\varepsilon(\cdot)) - \inf_{\mathbf{u}(\cdot)\in\mathcal{U}_c^{\mathbf{\Lambda}}} \mathcal{J}_{soc}^{(N)}(\xi_0; \mathbf{u}(\cdot))\right| = \varepsilon(N), \quad \varepsilon(N) \to 0, \textit{ when } N \to \infty.$$

We also apply assumptions (A4.1)–(A4.2) to the coefficients in (**MFT-c**). Under (A4.1), for any given $\mathbf{u}(\cdot) \in \mathcal{U}_c^{\mathbf{\Lambda}}$, (4.24) admits a unique strong solution $\mathbf{x}(\cdot) \equiv$

$\mathbf{x}(\cdot;\xi_0,\mathbf{u}(\cdot)) := (x_1(\cdot;\xi_0,\mathbf{u}(\cdot)),\ \cdots,\ x_N(\cdot;\xi_0,\mathbf{u}(\cdot)))$ by Proposition 2.6 in [61]. Furthermore, under (A4.2), each $\mathcal{J}_i$, $i \in \mathcal{I}$ is well-defined.

4.3.1 Convexity and MF strategy design

The state dynamics (4.24) and the social cost functional (4.22) could be rewritten in a high dimensional form as follows:

$$d\mathbf{x} = (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u})dt + \sum_{i=1}^{N}(\mathbf{C}_i\mathbf{x} + \mathbf{D}_i\mathbf{u})dW_i, \quad \mathbf{x}(0) = \Xi, \tag{4.25}$$

$$\mathcal{J}_{soc}^{(N)}(\xi_0;\mathbf{u}(\cdot)) = \frac{1}{2}\mathbb{E}\Big\{\int_0^T \mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}dt + \mathbf{x}^T(T)\mathbf{G}\mathbf{x}(T)\Big\}, \tag{4.26}$$

where

$$\begin{gathered}
\mathbf{A} = \begin{pmatrix} A+\frac{\bar{A}}{N} & \frac{\bar{A}}{N} & \cdots & \frac{\bar{A}}{N} \\ \frac{\bar{A}}{N} & A+\frac{\bar{A}}{N} & \cdots & \frac{\bar{A}}{N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\bar{A}}{N} & \frac{\bar{A}}{N} & \cdots & A+\frac{\bar{A}}{N} \end{pmatrix}_{(Nn\times Nn)}, \mathbf{B} = \begin{pmatrix} B+\frac{\bar{B}}{N} & \frac{\bar{B}}{N} & \cdots & \frac{\bar{B}}{N} \\ \frac{\bar{B}}{N} & B+\frac{\bar{B}}{N} & \cdots & \frac{\bar{B}}{N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\bar{B}}{N} & \frac{\bar{B}}{N} & \cdots & B+\frac{\bar{B}}{N} \end{pmatrix}_{(Nn\times Nm)}, \\
\mathbf{C}_i = \begin{array}{c} 1 \\ \vdots \\ i^{\text{th}} \\ \vdots \\ N \end{array} \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ \frac{\bar{C}}{N} & \cdots & \frac{\bar{C}}{N}+C & \cdots & \frac{\bar{C}}{N} \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}_{(Nn\times Nn)}, \ \mathbf{D}_i = \begin{array}{c} 1 \\ \vdots \\ i^{\text{th}} \\ \vdots \\ N \end{array} \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ \frac{\bar{D}}{N} & \cdots & \frac{\bar{D}}{N}+D & \cdots & \frac{\bar{D}}{N} \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}_{(Nn\times Nm)}, \\
\Xi = \begin{pmatrix} \xi_0 \\ \vdots \\ \xi_0 \end{pmatrix}_{(Nn\times 1)}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}_{(Nn\times 1)}, \quad \mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}_{(Nm\times 1)},
\end{gathered} \tag{4.27}$$

and

$$\begin{gathered}
\mathbf{Q} = \begin{pmatrix} Q & 0 & \cdots & 0 \\ 0 & Q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q \end{pmatrix} + \frac{1}{N}\begin{pmatrix} \widehat{Q} & \cdots & \widehat{Q} \\ \vdots & \ddots & \vdots \\ \widehat{Q} & \cdots & \widehat{Q} \end{pmatrix} - \frac{1}{N}\begin{pmatrix} Q & \cdots & Q \\ \vdots & \ddots & \vdots \\ Q & \cdots & Q \end{pmatrix}, \mathbf{R} = \begin{pmatrix} R & 0 & \cdots & 0 \\ 0 & R & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R \end{pmatrix}, \\
\mathbf{G} = \begin{pmatrix} G & 0 & \cdots & 0 \\ 0 & G & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G \end{pmatrix} + \frac{1}{N}\begin{pmatrix} \widehat{G} & \cdots & \widehat{G} \\ \vdots & \ddots & \vdots \\ \widehat{G} & \cdots & \widehat{G} \end{pmatrix} - \frac{1}{N}\begin{pmatrix} G & \cdots & G \\ \vdots & \ddots & \vdots \\ G & \cdots & G \end{pmatrix},
\end{gathered} \tag{4.28}$$

where $\widehat{Q} := (\Gamma_1 - I)^T Q(\Gamma_1 - I)$ and $\widehat{G} := (\Gamma_2 - I)^T G(\Gamma_2 - I)$. For the sake of notation simplicity, we still use $\mathbf{A}$, $\mathbf{B}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{G}$, $\mathbf{x}$, $\mathbf{u}$ to represent the augmented matrices and vectors, whose meanings are different to those in Section 4.1.

We have studied the *unconstrained* MFT problem in Chapter 2. In like manner, we can obtain the corresponding results of (**MFT-c**). In what follows, we would present these results directly and omit the detailed proofs. Firstly, we list the results of the convexity:

Proposition 4.7. *Under* (A4.1)–(A4.2), $\mathbf{u}(\cdot) \mapsto \mathcal{J}_{soc}^{(N)}(\xi_0; \mathbf{u}(\cdot))$ *is uniformly convex on* $\mathcal{U}_c^{\mathbf{\Lambda}}$ *if and only if the following Riccati equation:*

$$\begin{cases} \dot{\mathbf{P}}+\mathbf{A}^T\mathbf{P}+\mathbf{P}\mathbf{A}+\sum\limits_{i\in\mathcal{I}}\mathbf{C}_i^T\mathbf{P}\mathbf{C}_i+\mathbf{Q}-\left(\mathbf{B}^T\mathbf{P}+\sum\limits_{i\in\mathcal{I}}\mathbf{D}_i^T\mathbf{P}\mathbf{C}_i\right)^T\mathcal{R}_N\left(\mathbf{B}^T\mathbf{P}+\sum\limits_{i\in\mathcal{I}}\mathbf{D}_i^T\mathbf{P}\mathbf{C}_i\right)=0, \\ \mathbf{P}(T)=\mathbf{G}, \end{cases} \tag{4.29}$$

admits a solution $\mathbf{P} \in C([0,T];\mathbb{R}^{Nn})$ *such that* $\mathbf{R}(t)+\sum_{i\in\mathcal{I}}\mathbf{D}_i^T(t)\mathbf{P}(t)\mathbf{D}_i(t) \gg 0$ *on* $\mathbf{\Lambda}$, *a.e.* $t\in[0,T]$, *where* $\mathcal{R}_N=\mathbf{V}\left[\mathbf{V}^T\left(\mathbf{R}+\sum_{i\in\mathcal{I}}\mathbf{D}_i^T\mathbf{P}\mathbf{D}_i\right)\mathbf{V}\right]^{-1}\mathbf{V}^T$.

Proposition 4.8. *Assume that* (A4.1)–(A4.2) *hold and* $\mathbf{u}(\cdot) \mapsto \mathcal{J}_{soc}^{(N)}(\xi_0; \mathbf{u}(\cdot))$ *is uniformly convex on* $\mathcal{U}_c^{\mathbf{\Lambda}}$. *Then* (**MFT-c**) *admits a unique optimal pair* $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ *on* $\mathcal{U}_c^{\mathbf{\Lambda}}$. *Moreover, the Hamiltonian system:*

$$\begin{cases} d\bar{\mathbf{x}} = (\mathbf{A}\bar{\mathbf{x}}+\mathbf{B}\bar{\mathbf{u}})dt+\sum\limits_{i=1}^{N}(\mathbf{C}_i\bar{\mathbf{x}}+\mathbf{D}_i\bar{\mathbf{u}})dW_i, \quad \bar{\mathbf{x}}(0)=\xi_0, \\ d\mathbf{p} = -\left(\mathbf{Q}\bar{\mathbf{x}}+\mathbf{A}^T\mathbf{p}+\sum\limits_{i\in\mathcal{I}}\mathbf{C}_i^T\mathbf{q}_i\right)dt+\sum\limits_{i\in\mathcal{I}}\mathbf{q}_i dW_i, \quad \mathbf{p}(T)=\mathbf{G}\bar{\mathbf{x}}(T), \\ \mathbf{V}^T\left(\mathbf{B}^T\mathbf{p}+\sum\limits_{i\in\mathcal{I}}\mathbf{D}_i^T\mathbf{q}_i+\mathbf{R}\bar{\mathbf{u}}\right)=0, \quad a.e.\ t\in[0,T],\ \mathbb{P}-a.s., \end{cases} \tag{4.30}$$

admits a unique adapted solution $(\bar{\mathbf{x}}, \bar{\mathbf{u}}, \mathbf{p}, \mathbf{q}_1, \cdots, \mathbf{q}_N)$ *where* $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ *is the optimal pair of* (**MFC-c**). *In this case, we also call* $\bar{\mathbf{u}}$ *centralized optimal control,* *since $\bar{\mathbf{u}}$ accesses the centralized information. Moreover, the optimal cost satisfies*

$$\inf_{\mathbf{u}(\cdot)\in\mathcal{U}_c^{\mathbf{\Lambda}}} \mathcal{J}_{soc}^{(N)}(\xi_0;\mathbf{u}(\cdot)) = \mathcal{J}_{soc}^{(N)}(\xi_0;\bar{\mathbf{u}}(\cdot)) = \langle \mathbf{P}(0)\xi_0, \xi_0\rangle\,,$$

where $\mathbf{P}(\cdot)$ is the solution of (4.29).

Proposition 4.9. *Under* (A4.1)-(A4.2)*, if $Q, G \geq 0$ and $R \gg 0$, then $\mathbf{u}(\cdot) \mapsto \mathcal{J}_{soc}^{(N)}(\xi_0;\mathbf{u}(\cdot))$ is uniformly convex on $\mathcal{U}_c^{\mathbf{\Lambda}}$.*

Proposition 4.10. *Under* (A4.1)-(A4.2)*, if $\bar{A} = \bar{B} = \bar{C} = \bar{D} = 0$, $Q - \widehat{Q} \geq 0$, $G - \widehat{G} \geq 0$, and there exist some ΔQ, $\Delta G \in \mathbb{S}^n$ such that $\Delta Q \geq Q - \widehat{Q}$, $\Delta G \geq G - \widehat{G}$ and the following Riccati equation:*

$$\begin{aligned}&\dot{P}+PA+A^TP+C^TPC+(Q-\Delta Q)-(PB+C^TPD)V\left[V^T(R+D^TPD)V\right]^{-1}\\&\times V^T(B^TP+D^TPC)=0, \quad P(T)=(G-\Delta G),\end{aligned} \tag{4.31}$$

admits a solution $P \in C([0,T];\mathbb{S}^n)$ such that $R(t)+D^T(t)P(t)D(t) \gg 0$ on Λ, a.e. $t \in [0,T]$, then $\mathbf{u}(\cdot) \mapsto \mathcal{J}_{soc}^{(N)}(\xi_0;\mathbf{u}(\cdot))$ is uniformly convex on $\mathcal{U}_c^{\mathbf{\Lambda}}$.

Proposition 4.11. *Under* (A4.1)-(A4.2)*, let $Q - \widehat{Q} \geq 0$, and $G \geq 0$. If there exists some $\Delta Q \in \mathbb{S}^n$ such that $\Delta Q \geq Q - \widehat{Q}$, $\lambda_{\min}(Q - \Delta Q) \leq 0$ and $Ke^{2KT}\lambda_{\min}(Q-\Delta Q) + \frac{1}{2}\lambda_{\min}(R) \geq \varepsilon I$, then $\mathbf{u}(\cdot) \mapsto \mathcal{J}_{soc}^{(N)}(\xi_0;\mathbf{u}(\cdot))$ is uniformly* *convex on* $\mathcal{U}_c^{\mathbf{\Lambda}}$*. Constant* K *is given by:*

$$K = \max\Big\{ \left[\lambda_{\max}(A^T + A) + \lambda_{\max}(\bar{A}^T + \bar{A})\right], \left[\lambda_{\max}\left(C^T C\right) + \lambda_{\max}\left(\mathcal{C}^T\mathcal{C} - C^T C\right)\right],$$
$$\sqrt{\lambda_{\max}(B^T B) + \lambda_{\max}(\mathcal{B}^T\mathcal{B} - B^T B)},$$
$$\sqrt{\lambda_{\max}\left(\mathcal{D}^T\mathcal{C}\mathcal{C}^T\mathcal{D} - D^T C C^T D\right) + \lambda_{\max}\left(D^T C C^T D\right)},$$
$$\left[\lambda_{\max}\left(D^T D\right) + \lambda_{\max}\left(\mathcal{D}^T\mathcal{D} - D^T D\right)\right]\Big\} \geq 0. \tag{4.32}$$

Secondly, we list the results of the characterization of the MF strategy set.

For agent $\mathcal{A}_i$, its auxiliary control problem is given by:

(**MFT-c**)*: For given initial value ξ_0, agent $\mathcal{A}_i$ find a strategy $\check{u}_i \in \mathcal{U}_i^{\Lambda}$ such that $J_i(\xi_0; \check{u}_i) = \inf_{u_i \in \mathcal{U}_i^{\Lambda}} J_i(\xi_0; u_i)$, where

$$\begin{cases} J_i(\xi_0; u_i) = \dfrac{1}{2}\mathbb{E}\Big\{ \displaystyle\int_0^T \|x_i\|_Q^2 + 2\langle S_1, x_i\rangle + 2\langle S_2, u_i\rangle + \|u_i\|_R^2 dt + \|x_i(T)\|_G^2 + 2\langle S_3, x_i(T)\rangle \Big\}, \\ \text{s.t. } dx_i = (Ax_i + Bu_i + \bar{A}\bar{m} + \bar{B}\bar{w})dt + (Cx_i + Du_i + \bar{C}\bar{m} + \bar{D}\bar{w})dW_i, \quad x_i(0) = \xi_0, \\ \text{where} \quad S_1 = \left(\Gamma_1^T Q\Gamma_1 - \Gamma_1^T Q - Q\Gamma_1\right)\bar{m} + \bar{A}^T\mathbb{E}p_i^1 + \bar{C}^T\mathbb{E}q_i^1 + \bar{A}^T p^2, \\ \qquad\qquad S_2 = \bar{B}^T\mathbb{E}p_i^1 + \bar{D}^T\mathbb{E}q_i^1 + \bar{B}^T p^2, \quad S_3 = (\Gamma_2^T G\Gamma_2 - \Gamma_2^T G - G\Gamma_2)\bar{m}(T), \end{cases} \tag{4.33}$$

and ($\bar{m}$, $\bar{w}$, p_i^1, q_i^1, p^2) are the prefrozen MF terms.

Proposition 4.12. *Let* (A4.1)-(A4.3) *hold.* (**MFT-c**)* *is uniformly convex on* $\mathcal{U}_i^{\Lambda}$ *and thus admits a unique optimal pair* $(\check{x}_i, \check{u}_i)$ *on* $\mathcal{U}_i^{\Lambda}$*. The following*

Hamiltonian system :

$$
(\mathbf{H3})\begin{cases}
d\check{x}_i=(A\check{x}_i+B\check{u}_i+\bar{A}\bar{m}+\bar{B}\bar{w})dt+(C\check{x}_i+D\check{u}_i+\bar{C}\bar{m}+\bar{D}\bar{w})dW_i\\
dp_i=\big(-Q\check{x}_i+Q\Gamma_1\bar{m}-A^Tp_i-C^Tq_i+\Gamma_1^TQ(I-\Gamma)\bar{m}-\bar{A}^Tp^2\\
\qquad\qquad -\bar{C}^T\mathbb{E}q_i^1-\bar{A}^T\mathbb{E}p_i^1\big)dt+q_idW_i,\\
\check{x}_i(0)=\xi_0,\quad p_i(T)=G\check{x}_i(T)+S_3,\\
V^T\left(R\check{u}_i+B^Tp_i+D^Tq_i+\bar{B}^T\mathbb{E}p_i^1+\bar{D}^T\mathbb{E}q_i^1+\bar{B}^Tp^2\right)=0,\quad a.e.\ t\in[0,T],\ \mathbb{P}-a.s.,
\end{cases}
$$

admits a unique adapted solution $(\check{x}_i,\check{u}_i,p_i,q_i)$ *where* $(\check{x}_i,\check{u}_i)$ *is the optimal pair of* (**MFG-c**)**. Moreover, if* $R>0$ *is assumed, the optimal control can be represented explicitly as* $\check{u}=-V(V^TRV)^{-1}V^T\big(B^Tp_i+D^Tq_i+\bar{B}^T\mathbb{E}p_i^1+\bar{D}^T\mathbb{E}q_i^1+\bar{B}^Tp^2\big)$.

The prefrozen MF terms ($\bar{m}$, $\bar{w}$, p_i^1, q_i^1, p^2) can be determined by the following CC system:

$$
(\mathbf{CC\text{-}2})\begin{cases}
d\check{x}_i=(A\check{x}_i+B\check{u}_i+\bar{A}\mathbb{E}\check{x}_i+\bar{B}\mathbb{E}\check{u}_i)dt+(C\check{x}_i+D\check{u}_i+\bar{C}\mathbb{E}\check{x}_i+\bar{D}\mathbb{E}\check{u}_i)dW_i,\\
dp_i=\big[-Q\check{x}_i+(Q\Gamma_1+\Gamma_1^TQ-\Gamma_1^TQ\Gamma_1)\mathbb{E}\check{x}_i-A^Tp_i-C^Tq_i-\bar{A}^Tp^2\\
\qquad\qquad -\bar{C}^T\mathbb{E}q_i^1-\bar{A}^T\mathbb{E}p_i^1\big]dt+q_idW_i,\\
dp_i^1=-\left(Q\check{x}_i+A^Tp_i^1+C^Tq_i^1\right)dt+q_i^1dW_i,\\
dp^2=-\left[(\Gamma_1^TQ\Gamma_1-\Gamma_1^TQ-Q\Gamma_1)\mathbb{E}\check{x}_i+\bar{A}^T\mathbb{E}p_i^1+\bar{C}^T\mathbb{E}q_i^1+\bar{A}^Tp^2+A^Tp^2\right]dt,\\
\check{x}_i(0)=\xi_0,\quad p_i(T)=G\check{x}_i(T)+(\Gamma_2^TG\Gamma_2-\Gamma_2^TG-G\Gamma_2)\mathbb{E}\check{x}_i(T),\\
p_i^1(T)=G\check{x}_i(T),\quad p^2(T)=(\Gamma_2^TG\Gamma_2-\Gamma_2^TG-G\Gamma_2)\mathbb{E}\check{x}_i(T),\\
V^T\left(R\check{u}_i+B^Tp_i+D^Tq_i+\bar{B}^T\mathbb{E}p_j^1+\bar{D}^T\mathbb{E}q_j^1+\bar{B}^Tp^2\right)=0.
\end{cases}
\tag{4.34}
$$

Similar to (CC-1), (CC-2) is also a fully coupled MF-FBSDE, the exogenous terms $(\bar{m},\bar{w},p_j^1,p^2,q_j^1)$ are characterized through some embedding representation. We would prove that (CC-2) is equivalent (H1) under some conditions in

Section 4.4, which would lead to the equivalence of the MFC control and MFT strategy.

4.4 The relation among MFC, MFG and MFT

Through the discussion in Section 4.1-4.3, we have characterized the optimal control and MF strategy of (**MFC-c**), (**MFG-c**) and (**MFT-c**) through the Hamiltonian system and CC system. In this section, we will analyze their relation further.

4.4.1 Relation of uniform convexity

Firstly, we study the relation of the uniform convexity. Noting that by Proposition 4.4, we know that (A4.3) leads to the uniform convexity of both (**MFC-c**) and (**MFG-c**)*. Thus, we mainly focus on the relation between (**MFC-c**) and (**MFT-c**) and we have the following result.

Proposition 4.13.

(i) Under (A4.1)-(A4.2)*, if $Q, G \geq 0$ and $R \gg 0$, then* (**MFC-c**) *cost functional $u_i(\cdot) \mapsto \mathcal{J}_i(\xi_0; u_i(\cdot))$ is uniformly convex on $\mathcal{U}_i^{\Lambda}$, and* (**MFT-c**) *social cost functional $\mathbf{u}(\cdot) \mapsto \mathcal{J}_{soc}^{(N)}(\xi_0; \mathbf{u}(\cdot))$ is also uniformly convex on $\mathcal{U}_c^{\mathbf{\Lambda}}$.*

(ii) Under (A4.1)-(A4.3)*, if $\bar{A} = \bar{B} = \bar{C} = \bar{D} = 0$, $Q = \widehat{Q}$, $G = \widehat{G}$, then $u_i(\cdot) \mapsto \mathcal{J}_i(\xi_0; u_i(\cdot))$ is uniformly convex on $\mathcal{U}_i^{\Lambda}$, and $\mathbf{u}(\cdot) \mapsto \mathcal{J}_{soc}^{(N)}(\xi_0; \mathbf{u}(\cdot))$ is uniformly convex on $\mathcal{U}_c^{\mathbf{\Lambda}}$.*

Proof. For item (i), the result can be obtained directly by Proposition 4.1 and Proposition 4.9. For item (ii), by Theorem 4.1, $u_i(\cdot) \mapsto \mathcal{J}_i(\xi_0; u_i(\cdot))$ is uniformly convex under (A4.1)-(A4.3). Moreover by letting $\Delta Q = \Delta G = 0$, we also have

$0 = \Delta Q \geq Q - \widehat{Q} = 0$, $\Delta G \geq G - \widehat{G} = 0$ and Riccati equation (4.31) is equivalent to (RE1) in this case. Thus, by Proposition 4.10, $\mathbf{u}(\cdot) \mapsto \mathcal{J}_{soc}^{(N)}(\xi_0; \mathbf{u}(\cdot))$ is also uniformly convex. $\square$

4.4.2 Relation of the designed control

Next, we study the relation among the optimal control of (**MFC-c**) and the MF strategies of (**MFG-c**) and (**MFT-c**). Firstly, we focus on (**MFC-c**) and (**MFG-c**). We recall (H1) and (CC-1):

$$
(\mathbf{H1})\begin{cases}
d\bar{x} = (A\bar{x}+\bar{A}\mathbb{E}\bar{x}+B\bar{u}+\bar{B}\mathbb{E}\bar{u})dt+(C\bar{x}+\bar{C}\mathbb{E}\bar{x}+D\bar{u}+\bar{D}\mathbb{E}\bar{u})dW, \\
dk = -\left(Q\bar{x}-(Q\Gamma_1 + \Gamma_1^T Q-\Gamma_1^T Q\Gamma_1)\mathbb{E}\bar{x}+A^T k+\bar{A}^T\mathbb{E}k+C^T\zeta+\bar{C}^T\mathbb{E}\zeta\right) dt+\zeta dW, \\
\bar{x}(0) = \xi_0, \quad k(T) = G\bar{x}(T)-(G\Gamma_2 + \Gamma_2^T G-\Gamma_2^T G\Gamma_2)\mathbb{E}\bar{x}(T), \\
V^T\left(B^T k+\bar{B}^T\mathbb{E}k+D^T\zeta + \bar{D}^T\mathbb{E}\zeta+R\bar{u}\right)=0, \text{ a.e. } t \in [0,T], \ \mathbb{P}-a.s..
\end{cases}
$$

$$
(\mathbf{CC\text{-}1})\begin{cases}
d\check{x} = (A\check{x}+\bar{A}\mathbb{E}\check{x}+B\check{u}+\bar{B}\mathbb{E}\check{u})dt+(C\check{x}+\bar{C}\mathbb{E}\check{x}+D\check{u}+\bar{D}\mathbb{E}\check{u})dW, \\
dl = -\left(A^T l+C^T\varsigma + Q\check{x} - Q\Gamma_1\mathbb{E}\check{x}\right) dt+\varsigma dW, \\
\check{x}(0) = \xi_0, \quad l(T) = G\check{x}(T)-G\Gamma_2\mathbb{E}\check{x}(T), \\
V^T\left(B^T l+D^T\varsigma+R\check{u}\right)=0.
\end{cases}
$$

By comparing (CC-1) and (H1), we have the following result:

Lemma 4.3. *Under* (A4.1)–(A4.3)*, if* $\bar{A} = \bar{B} = \bar{C} = \bar{D} = 0$ *and* $\Gamma_1^T Q – \Gamma_1^T Q\Gamma_1 = 0$ *(e.g.,* $\Gamma_1 = \Gamma_2 = I$ *or* $\Gamma_1 = \Gamma_2 = 0$*), then* (CC-1) *and* (H1) *are identical.*

Proof. The result of Lemma 4.3 can be obtained directly by comparing the coefficients of (CC-1) and (H1). $\square$

By Lemma 4.3, we can obtain the relation between (**MFC-c**) and (**MFG-c**) as follows:

Theorem 4.2. *Let* (A4.1)–(A4.3) *hold and* $\bar{A} = \bar{C} = \bar{B} = \bar{D} = 0$, $\Gamma_1^T Q - \Gamma_1^T Q \Gamma_1 = 0$ *(e.g.,* $\Gamma_1 = \Gamma_2 = I$ *or* $\Gamma_1 = \Gamma_2 = 0$*). Then for each agent* $\mathcal{A}_i$*,* (**MFC-c**) *admits a unique optimal control denoted by* $\bar{u}_i^{MFC} \in \mathcal{U}_i^{\Lambda}$*, and* (**MFG-c**) *admits a unique MFG strategy denoted by* $\bar{u}_i^{MFG} \in \mathcal{U}_i^{\Lambda}$*. Moreover, in this case* $\bar{u}_i^{MFC} = \bar{u}_i^{MFG}$*.*

Proof. Under (A4.1)–(A4.3), by Proposition 4.2, (**MFC-c**) admits a unique optimal control $\bar{u}_i^{MFC}$ which is determined by (H1), and (H1) admits a unique adapted solution. Then what we desire to prove next is the existence and uniqueness of the MFG strategy. By Lemma 4.3, (CC-1) is also uniquely solvable. Thus, the MF terms $\bar{m}$ and $\bar{w}$ can be uniquely determined by (CC-1). Consequently, by Proposition 4.4 and procedure (MG1)-(MG3), there exists a unique MFG strategy $\bar{u}_i^{MFG}$ for $\mathcal{A}_i$, which is determined by (H2). Lastly, the relation $\bar{u}_i^{MFC} = \bar{u}_i^{MFG}$ follows the equivalence of (CC-1) and (H1). □

Secondly, we focus on (**MFC-c**) and (**MFT-c**). By comparing (CC-2) and (H1), we have the following result:

Lemma 4.4. *Under* (A4.1)–(A4.2)*,* (CC-2) *and* (H1) *are identical.*

Proof. See Appendix C.3. □

By the discussion above, we can conclude the following result of the equivalence between the MFC optimal control and MF strategy of (**MFT-c**), whose proof is similar to Theorem 4.2, and we omit it.

Theorem 4.3. *Under* (A4.1)–(A4.3)*, for each agent* $\mathcal{A}_i$*,* (**MFC-c**) *admits a unique optimal control denoted by* $\bar{u}_i^{MFC} \in \mathcal{U}_i^{\Lambda}$*, and* (**MFT-c**) *admits a unique MF strategy denoted by* $\bar{u}_i^{MFT} \in \mathcal{U}_i^{\Lambda}$*. Moreover* $\bar{u}_i^{MFC} = \bar{u}_i^{MFT}$*.*

By using Theorem 4.3, we can obtain a feedback form MF strategy of (**MFT-c**) by the representation of the (**MFC-c**) optimal control given in Proposition 4.3.

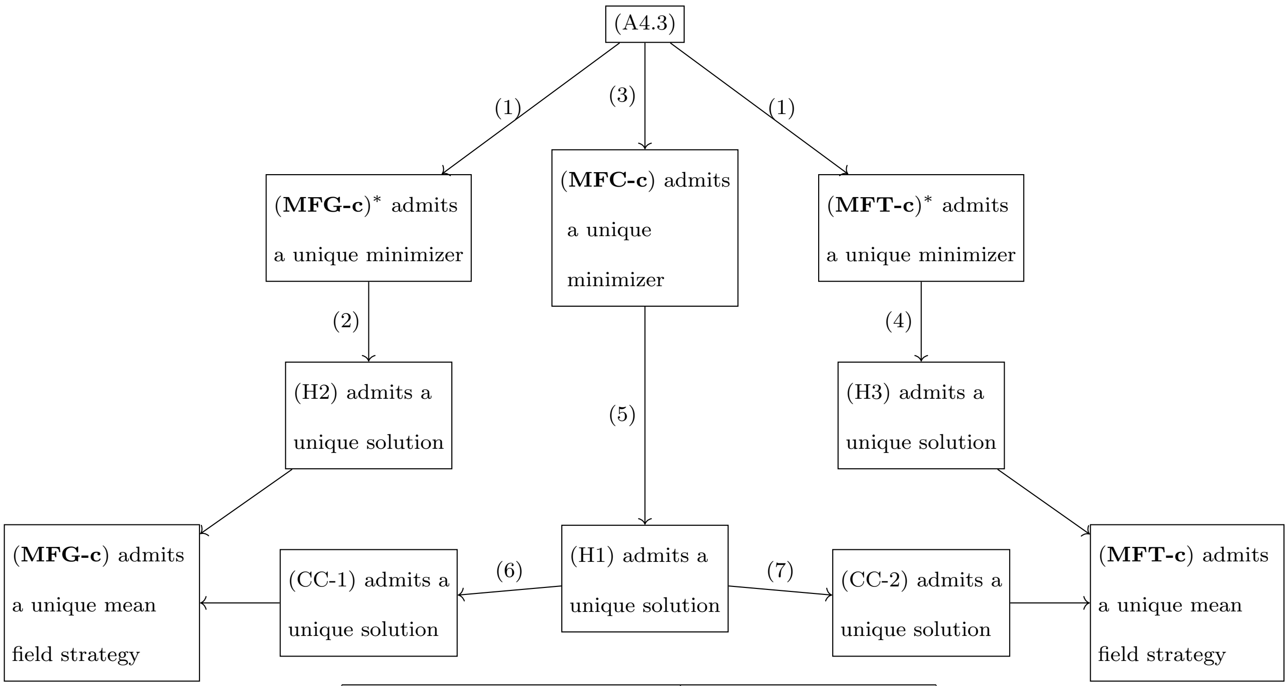
Corollary 4.1. *Under* (A4.1)–(A4.3)*, in* (**MFT-c**) *each agent* $\mathcal{A}_i$ *has a unique feedback form MF strategy:*

$$\tilde{u}_i = -\mathcal{R}_2(\mathcal{B}^T P_2 + \mathcal{D}^T P_1 \mathcal{C})\mathbb{E}\tilde{x}_i - \mathcal{R}_1(B^T P_1 + D^T P_1 C)(\tilde{x}_i - \mathbb{E}\tilde{x}_i), \tag{4.35}$$

where $\tilde{x}_i$ *is the realized state satisfying the following dynamic:*

$$\begin{cases} d\tilde{x}_i = (A\tilde{x}_i + \bar{A}\tilde{x}^{(N)} + B\tilde{u}_i + \bar{B}\tilde{u}^{(N)})dt + (C\tilde{x}_i + \bar{C}\tilde{x}^{(N)} + D\tilde{u}_i + \bar{D}\tilde{u}^{(N)})dW_i, \\ \tilde{x}_i(0) = \xi_0. \end{cases}$$

Lastly, we use the following diagram to conclude the relation among (**MFC-c**), (**MFG-c**) and (**MFT-c**):



(1) Riccati equation (RE1)	(5) Proposition 4.2
(2) Proposition 4.4	(6) Lemma 4.3
(3) Theorem 4.1	(7) Lemma 4.4
(4) Proposition 4.12	

4.4.3 Relation of fixed-point analysis and direct method in (MFT-c)

The method we apply in Section 4.3 is so-called fixed-point approach, since for each single agent an auxiliary control is constructed based on consistent mean field approximations and formalize a fixed-point problem (CC system) to determine the MF terms. For more details of such fixed-point approach method, readers are referred to [15, 26, 14, 27]. Another route to deal with MFT problem is direct approach method which starts by formally solving the high dimensional problem directly to obtain a large coupled solution equation system and the next step is to derive a limit for the solution by taking $N \to \infty$. For more discussion of direct approach method, readers are referred to [28, 29, 30].

In what follows we briefly sketch some key points of direct approach method, and compare the MF strategy obtained by direct approach method and fixed-point approach. Firstly system (4.30) can be rewritten as follows:

$$\begin{cases} d\bar{x}_i = (A\bar{x}_i + \bar{A}\bar{x}^{(N)} + B\bar{u}_i + \bar{B}\bar{u}^{(N)})dt + (C\bar{x}_i + \bar{C}x^{(N)} + D\bar{u}_i + \bar{D}\bar{u}^{(N)})dW_i, \\ dp_i = -\left[Q\bar{x}_i + (\widehat{Q} - Q)\bar{x}^{(N)} + A^T p_i + \bar{A}^T p^{(N)} + C^T q_i^i + \frac{\bar{C}^T}{N}\sum_{j\in\mathcal{I}} q_j^j\right]dt + \sum_{j\in\mathcal{I}} q_i^j dW_j, \\ \bar{x}_i(0) = \xi_0, \quad p_i(T) = G\bar{x}_i(T) + (\widehat{G} - G)\bar{x}^{(N)}(T), \\ V^T\left(B^T p_i + \bar{B}^T p^{(N)} + D^T q_i^i + \frac{\bar{D}^T}{N}\sum_{j\in\mathcal{I}} q_j^j + R\bar{u}_i\right) = 0. \end{cases} \tag{4.36}$$

Then we also have:

$$\begin{cases} d\bar{x}^{(N)} = (\mathcal{A}\bar{x}^{(N)} + \mathcal{B}\bar{u}^{(N)})dt + \frac{1}{N}\sum_{i\in\mathcal{I}}(C\bar{x}_i + \bar{C}x^{(N)} + D\bar{u}_i + \bar{D}\bar{u}^{(N)})dW_i, \\ dp^{(N)} = -\left[\widehat{Q}\bar{x}^{(N)} + \mathcal{A}^T p^{(N)} + \frac{C^T}{N}\sum_{i\in\mathcal{I}} q_i^i + \frac{\bar{C}^T}{N}\sum_{j\in\mathcal{I}} q_j^j\right]dt + \frac{1}{N}\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{I}} q_i^j dW_j, \\ \bar{x}^{(N)}(0) = \xi_0, \quad p^{(N)}(T) = \widehat{G}\bar{x}^{(N)}(T), \\ V^T\left(\mathcal{B}^T p^{(N)} + \frac{D^T}{N}\sum_{i\in\mathcal{I}} q_i^i + \frac{\bar{D}^T}{N}\sum_{j\in\mathcal{I}} q_j^j + R\bar{u}^{(N)}\right) = 0. \end{cases}$$

Remark 4.6. *Here if we apply fixed-point approach method by replacing* $\left(\bar{x}^{(N)}, \bar{u}^{(N)}, p^{(N)}, \frac{\sum_{j\in\mathcal{I}} q_j^j}{N}\right)$ *with* $(\mathbb{E}\bar{x}_i, \mathbb{E}\bar{u}_i, \mathbb{E}p_i, \mathbb{E}q_i^i)$*, then* (4.36) *becomes*

$$\begin{cases} d\bar{x}_i = (A\bar{x}_i+\bar{A}\mathbb{E}\bar{x}_i+B\bar{u}_i+\bar{B}\mathbb{E}\bar{u}_i)dt+(C\bar{x}_i+\bar{C}\mathbb{E}\bar{x}_i+D\bar{u}_i+\bar{D}\mathbb{E}\bar{u}_i)dW_i, \\ dp_i = -\left[Q\bar{x}_i+(\widehat{Q}-Q)\mathbb{E}\bar{x}_i+A^Tp_i+\bar{A}^T\mathbb{E}p_i+C^Tq_i^i+\bar{C}^T\mathbb{E}q_i^i\right]dt+q_i^idW_i, \\ \bar{x}_i(0)=\xi_0, \quad p_i(T) = G\bar{x}_i(T)+(\widehat{G}-G)\mathbb{E}\bar{x}_i(T), \\ V^T\left(B^Tp_i+\bar{B}^T\mathbb{E}p_i+D^Tq_i^i+\bar{D}^T\mathbb{E}q_i^i+R\bar{u}_i\right)=0, \end{cases} \tag{4.37}$$

which is identical to (H1) *and by Lemma 4.4 also identical to* (CC-2)*. Thus, it also leads to a MF strategy set identical to the one we derived in Section 4.3.*

Let $p_i = P_N\bar{x}_i+K_N\bar{x}^{(N)}$, then by applying Itô formulat to p_i, we can obtain the equations w.r.t P_N and K_N as follows:

$$\begin{cases} Q+\dot{P}_N+P_NA+A^TP_N+C^T\left(P_N+\dfrac{K_N}{N}\right)C-\left[C^T\left(P_N+\dfrac{K_N}{N}\right)D+P_NB\right] \\ \times V\left(V^T\left[D^T\left(P_N+\dfrac{K_N}{N}\right)D+R\right]V\right)^{-1}V^T\left[B^TP_N+D^T\left(P_N+\dfrac{K_N}{N}\right)C\right]=0, \\ P_N(T)=G, \end{cases} \tag{4.38}$$

$$
\begin{cases}
\widehat{Q}-Q+P_N\bar{A}+\dot{K}_N+K_N\mathcal{A}+A^TK_N+\bar{A}^TP_N+\bar{A}^TK_N+C^T\left(P_N+\frac{K_N}{N}\right)\bar{C} \\
+\bar{C}^T\left(P_N+\frac{K_N}{N}\right)\mathcal{C}-\left[C^T\left(P_N+\frac{K_N}{N}\right)D+P_NB\right]V \\
\times\left(V^T\left[D^T\left(P_N+\frac{K_N}{N}\right)D+R\right]V\right)^{-1}V^T \\
\times\Big\{B^TK_N+\bar{B}^T\left(P_N+K_N\right)+D^T\left(P_N+\frac{K_N}{N}\right)\bar{C}+\bar{D}^T\left(P_N+\frac{K_N}{N}\right)\mathcal{C} \\
\quad-\left(\mathcal{D}^T\left(P_N+\frac{K_N}{N}\right)\mathcal{D}+R\right)V\left[V^T\left(\mathcal{D}^T\left(P_N+\frac{K_N}{N}\right)\mathcal{D}+R\right)V\right]^{-1}V^T \\
\times\left[\mathcal{B}^T\left(P_N+K_N\right)+\mathcal{D}^T\left(P_N+\frac{K_N}{N}\right)\mathcal{C}\right]\Big\} \\
-\left[C^T\left(P_N+\frac{K_N}{N}\right)\bar{D}+\bar{C}^T\left(P_N+\frac{K_N}{N}\right)\mathcal{D}+\left(P_N\bar{B}+K_N\mathcal{B}\right)\right] \\
\times V\left[V^T\left(\mathcal{D}^T\left(P_N+\frac{K_N}{N}\right)\mathcal{D}+R\right)V\right]^{-1}V^T\times\left[\mathcal{B}^T(P_N+K_N)+\mathcal{D}^T\left(P_N+\frac{K_N}{N}\right)\mathcal{C}\right]=0, \\
K_N(T)=(\widehat{G}-G),
\end{cases}
\tag{4.39}
$$

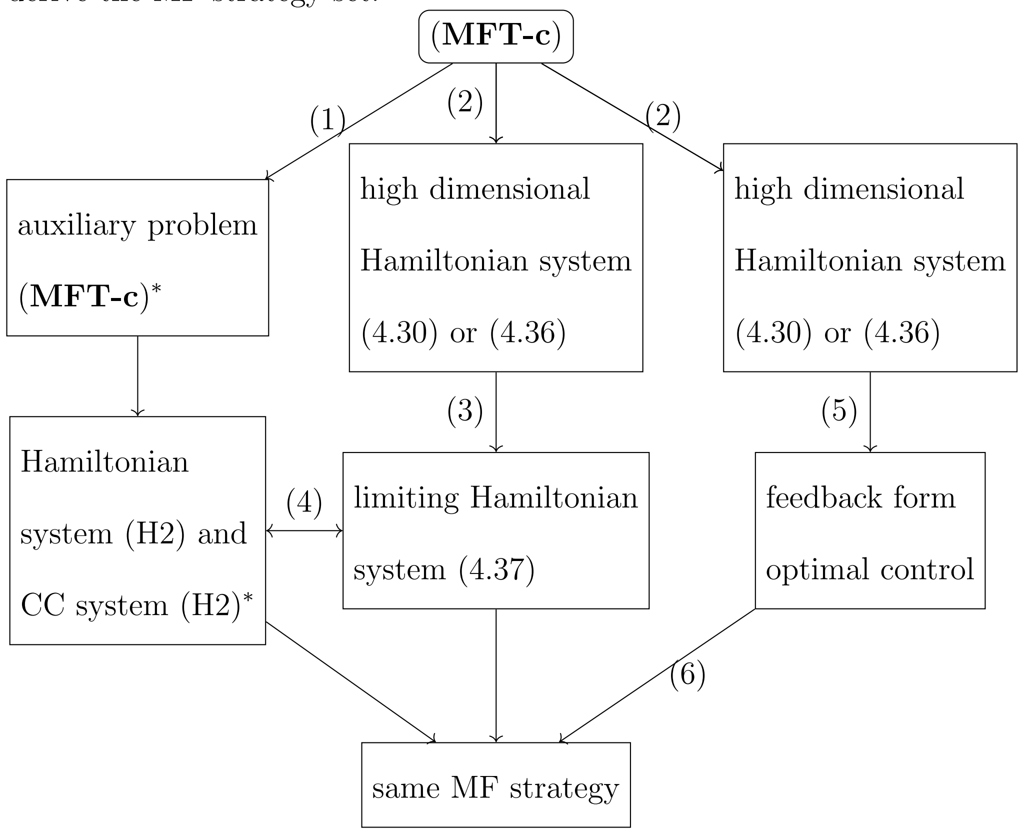
and the optimal control takes the following form:

$$
\begin{aligned}
\bar{u}_i=&-V\left[V^T\left(D^T\left(P_N+\frac{K_N}{N}\right)D+R\right)V\right]^{-1}V^T\times\left[B^TP_N+D^T\left(P_N+\frac{K_N}{N}\right)C\right]\bar{x}_i \\
&-V\left[V^T\left(D^T\left(P_N+\frac{K_N}{N}\right)D+R\right)V\right]^{-1}V^T \\
&\quad\times\Big\{B^TK_N+\bar{B}^T\left(P_N+K_N\right)+D^T\left(P_N+\frac{K_N}{N}\right)\bar{C}+\bar{D}^T\left(P_N+\frac{K_N}{N}\right)\mathcal{C} \\
&\qquad-\left(\mathcal{D}^T\left(P_N+\frac{K_N}{N}\right)\mathcal{D}+R\right)V\left[V^T\left(\mathcal{D}^T\left(P_N+\frac{K_N}{N}\right)\mathcal{D}+R\right)V\right]^{-1}V^T \\
&\qquad\times\left[\mathcal{B}^T\left(P_N+K_N\right)+\mathcal{D}^T\left(P_N+\frac{K_N}{N}\right)\mathcal{C}\right]\Big\}\bar{x}^{(N)}
\end{aligned}
$$

By following the argument in Theorem 4 of [28], $\lim_{N\to\infty}P_N = P_1$ and $\lim_{N\to\infty}(P_N+K_N) = P_2$ and the limiting optimal control can be represented as follows:

$$
\bar{u}_i=-\mathcal{R}_2(\mathcal{B}^TP_2+\mathcal{D}^TP_1\mathcal{C})\mathbb{E}\bar{x}_i-\mathcal{R}_1(B^TP_1+D^TP_1C)(\bar{x}_i-\mathbb{E}\bar{x}_i),
$$

which is identical to (4.35), the MF strategy designed by fixed-point approach method. Thus, combined with Remark 4.6, we have the following three routes to derive the MF strategy set:



(1) Variation method and mean field approximations	(5) Riccati equations (4.38) and (4.39)
(2) Variation method and duality	(6) Take the limitation as $N \to \infty$
(3) mean field approximations	
(4) Remark 4.6	

4.5 Concluding remarks

This chapter investigates MFC, MFG and MFT problem constrained on a linear subspace under a unified mathematical framework involving both the state and control MF term. Firstly we extend the result in [13, 61] of MFC problem to the case constrained on a linear subspace. The relation of the uniform convexity between the MFC problem and the related augmented control problem has been studied. Based on some algebraic analysis, a new type of Riccati equation is introduced and subsequently a uniform convexity condition is introduced which is weaker than (SA). We also derive the explicit feedback form representations of the optimal control and MF strategies of MFC, MFG and MFT problem respectively, while in some relevant literature [64, 18, 70], the designed control can only be represented in an embedded form coupled with the dual process through a projection mapping. Lastly, we compare the optimal control and MF strategy of MFC, MFG and MFT problem, and some equivalent relations have been found.

Chapter 5 Ongoing Work and Publications

- (**Conditionally accepted by IEEE TAC**) X. Feng, J. Huang, and Z. Qiu, “Mixed Social Optima and Nash equilibrium in Linear-Quadratic-Gaussian Mean-field System,” arXiv [math.OC], Nov. 05, 2019.

- (**Accepted by MCRF**) Z. Qiu, J. Huang, and T. Xie, “Linear Quadratic Gaussian Mean-Field Controls of Social Optima,” arXiv [math.OC], May 14, 2020.

- (**Accepted by AMC**) Asymmetric Information Control for Stochastic Systems with Different Intermittent Observations

- (**Published**) Q. Qi, Z. Qiu, X. Liang, and C. Tan, “Control for Multiplicative Noise Systems With Intermittent Noise and Input Delay,” IEEE Access, vol. 8, pp. 17713–17721, 2020.

- (**Published**) Q. Qi, Z. Qiu, and Z. Ji, “Optimal Continuous/Impulsive LQ Control With Quadratic Constraints,” IEEE Access, vol. 7, pp. 52955–52963, 2019.

- (**Ongoing work**) Relation among MFC, MFG, MFT Constrained on a Linear Subspace

- (**Ongoing work**) Optimal Stabilization Control

Publications

- (**Accepted by IEEE TAC**) X. Feng, J. Huang, and Z. Qiu, "Mixed Social Optima and Nash equilibrium in Linear-Quadratic-Gaussian Mean-field System," arXiv [math.OC], Nov. 05, 2019.

- (**Accepted by MCRF**) Z. Qiu, J. Huang, and T. Xie, "Linear Quadratic Gaussian Mean-Field Controls of Social Optima," arXiv [math.OC], May 14, 2020.

- (**Accepted by AMC**) Asymmetric Information Control for Stochastic Systems with Different Intermittent Observations

- (**Published**) Q. Qi, Z. Qiu, X. Liang, and C. Tan, "Control for Multiplicative Noise Systems With Intermittent Noise and Input Delay," IEEE Access, vol. 8, pp. 17713–17721, 2020.

- (**Published**) Q. Qi, Z. Qiu, and Z. Ji, "Optimal Continuous/Impulsive LQ Control With Quadratic Constraints," IEEE Access, vol. 7, pp. 52955–52963, 2019.

References

[1] A. Bensoussan, J. Frehse, and P. Yam, *Mean field games and mean field type control theory*, vol. 101. Springer, 2013.

[2] P. E. Caines, M. Huang, and R. P. Malhamé, "Mean field games," 2015.

[3] G. Y. Weintraub, C. L. Benkard, and B. Van Roy, "Markov perfect industry dynamics with many firms," *Econometrica*, vol. 76, no. 6, pp. 1375–1411, 2008.

[4] J. Rust, "Using randomization to break the curse of dimensionality," *Econometrica*, vol. 65, no. 3, pp. 487–516, 1997.

[5] F. Y. Kuo and I. H. Sloan, "Lifting the curse of dimensionality," *Notices of the AMS*, vol. 52, no. 11, pp. 1320–1328, 2005.

[6] P. Cardaliaguet and C.-A. Lehalle, "Mean field game of controls and an application to trade crowding," *Mathematics and Financial Economics*, vol. 12, no. 3, pp. 335–363, 2018.

[7] F. Slanina, "Mean-field approximation for a limit order driven market model," *Physical Review E*, vol. 64, no. 5, p. 056136, 2001.

[8] Z. Ma, D. S. Callaway, and I. A. Hiskens, "Decentralized charging control of large populations of plug-in electric vehicles," *IEEE Transactions on Control Systems Technology*, vol. 21, no. 1, pp. 67–78, 2011.

[9] M. Huang, P. E. Caines, and R. P. Malhamé, "Individual and mass behaviour in large population stochastic wireless power control problems: centralized and nash equilibrium solutions," in *42nd IEEE International Conference on Decision and Control (IEEE Cat. No. 03CH37475)*, vol. 1,

pp. 98–103, IEEE, 2003.

[10] R. Couillet, S. M. Perlaza, H. Tembine, and M. Debbah, "Electrical vehicles in the smart grid: A mean field game analysis," *IEEE Journal on Selected Areas in Communications*, vol. 30, no. 6, pp. 1086–1096, 2012.

[11] C. T. Bauch and D. J. Earn, "Vaccination and the theory of games," *Proceedings of the National Academy of Sciences*, vol. 101, no. 36, pp. 13391–13394, 2004.

[12] R. Breban, R. Vardavas, and S. Blower, "Mean-field analysis of an inductive reasoning game: application to influenza vaccination," *Physical Review E*, vol. 76, no. 3, p. 031127, 2007.

[13] P. J. Graber, "Linear quadratic mean field type control and mean field games with common noise, with application to production of an exhaustible resource," *Applied Mathematics and Optimization*, vol. 74, no. 3, pp. 459–486, 2016.

[14] M. Huang, P. E. Caines, and R. P. Malhamé, "Large-population cost-coupled lqg problems with nonuniform agents: individual-mass behavior and decentralized ε-nash equilibria," *IEEE Transactions on Automatic Control*, vol. 52, no. 9, pp. 1560–1571, 2007.

[15] R. Carmona and F. Delarue, *Probabilistic Theory of Mean Field Games with Applications I-II.* Springer, 2018.

[16] J.-M. Lasry and P.-L. Lions, "Mean field games," *Japanese Journal of Mathematics*, vol. 2, no. 1, pp. 229–260, 2007.

[17] A. Bensoussan, K. Sung, S. C. P. Yam, and S.-P. Yung, "Linear-quadratic mean field games," *Journal of Optimization Theory and Applications*, vol. 169, no. 2, pp. 496–529, 2016.

[18] Y. Hu, J. Huang, and X. Li, "Linear quadratic mean field game with control input constraint," *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 24, no. 2, pp. 901–919, 2018.

[19] J. Huang, S. Wang, and Z. Wu, "Backward mean-field linear-quadratic-gaussian (lqg) games: full and partial information," *IEEE Transactions on Automatic Control*, vol. 61, no. 12, pp. 3784–3796, 2016.

[20] R. Carmona and F. Delarue, "Probabilistic analysis of mean-field games," *SIAM Journal on Control and Optimization*, vol. 51, no. 4, pp. 2705–2734, 2013.

[21] J. Moon and T. Başar, "Risk-sensitive mean field games via the stochastic maximum principle," *Dynamic Games and Applications*, vol. 9, no. 4, pp. 1100–1125, 2019.

[22] J. Moon and T. Başar, "Discrete-time lqg mean field games with unreliable communication," in *53rd IEEE Conference on Decision and Control*, pp. 2697–2702, IEEE, 2014.

[23] D. Bauso, H. Tembine, and T. Başar, "Robust mean field games with application to production of an exhaustible resource," *IFAC Proceedings Volumes*, vol. 45, no. 13, pp. 454–459, 2012.

[24] D. Bauso, H. Tembine, and T. Başar, "Robust mean field games," *Dynamic Bames and Applications*, vol. 6, no. 3, pp. 277–303, 2016.

[25] J. Moon and T. Başar, "Robust mean field games for coupled markov jump linear systems," *International Journal of Control*, vol. 89, no. 7, pp. 1367–1381, 2016.

[26] T. Li and J.-F. Zhang, "Asymptotically optimal decentralized control for large population stochastic multiagent systems," *IEEE Transactions on Automatic Control*, vol. 53, no. 7, pp. 1643–1660, 2008.

[27] M. Huang, P. E. Caines, and R. P. Malhamé, “Social optima in mean field lqg control: centralized and decentralized strategies,” *IEEE Transactions on Automatic Control*, vol. 57, no. 7, pp. 1736–1751, 2012.

[28] M. Huang and M. Zhou, “Linear quadratic mean field games: Asymptotic solvability and relation to the fixed point approach,” *IEEE Transactions on Automatic Control*, vol. 65, no. 4, pp. 1397–1412, 2019.

[29] B.-C. Wang and H. Zhang, “Indefinite linear quadratic mean field social control problems with multiplicative noise,” *IEEE Transactions on Automatic Control*, 2020.

[30] B.-C. Wang, J. Huang, and J.-F. Zhang, “Social optima in robust mean field lqg control: From finite to infinite horizon,” *IEEE Transactions on Automatic Control*, 2020.

[31] M. Nourian, P. E. Caines, R. P. Malhame, and M. Huang, “Nash, social and centralized solutions to consensus problems via mean field control theory,” *IEEE Transactions on Automatic Control*, vol. 58, no. 3, pp. 639–653, 2012.

[32] B.-C. Wang and J.-F. Zhang, “Social optima in mean field linear-quadratic-gaussian models with markov jump parameters,” *SIAM Journal on Control and Optimization*, vol. 55, no. 1, pp. 429–456, 2017.

[33] M. Huang and S. L. Nguyen, “Linear-quadratic mean field social optimization with a major player,” *arXiv preprint arXiv:1904.03346*, 2019.

[34] M. Bardi and F. S. Priuli, “Linear-quadratic n-person and mean-field games with ergodic cost,” *SIAM Journal on Control and Optimization*, vol. 52, no. 5, pp. 3022–3052, 2014.

[35] D. Andersson and B. Djehiche, “A maximum principle for sdes of mean-field type,” *Applied Mathematics and Optimization*, vol. 63, no. 3, pp. 341–356, 2011.

[36] F. S. Priuli, "Linear-quadratic n-person and mean-field games: Infinite horizon games with discounted cost and singular limits," *Dynamic Games and Applications*, vol. 5, no. 3, pp. 397–419, 2015.

[37] M. Kac, "Foundations of kinetic theory," in *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, vol. 3, pp. 171–197, 1956.

[38] H. P. McKean Jr, "A class of markov processes associated with nonlinear parabolic equations," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 56, no. 6, p. 1907, 1966.

[39] T. Hamidou, Q. Zhu, and T. Başar, "Risk-sensitive mean-field games," *IEEE Transactions on Automatic Control*, vol. 59, no. 4, pp. 835–850, 2014.

[40] A. C. Kizilkale and R. P. Malhame, "Collective target tracking mean field control for markovian jump-driven models of electric water heating loads," *Proceedings of the 19th IFAC World Congress, Cape Town, South Africa*, pp. 1867–1972, 2014.

[41] J. Moon and T. Başar, "Linear quadratic risk-sensitive and robust mean field games," *IEEE Transactions on Automatic Control*, vol. 62, no. 3, pp. 1062–1077, 2017.

[42] B.-C. Wang and J.-F. Zhang, "Hierarchical mean field games for multiagent systems with tracking-type costs: Distributed ε-stackelberg equilibria," *IEEE Transactions on Automatic Control*, vol. 59, no. 8, pp. 2241–2247, 2014.

[43] M. Huang, "Large-population lqg games involving a major player: the nash certainty equivalence principle," *SIAM Journal on Control and Optimization*, vol. 48, no. 5, pp. 3318–3353, 2010.

[44] J. Moon and T. Başar, “Linear quadratic mean field stackelberg differential games,” *Automatica*, vol. 97, pp. 200–213, 2018.

[45] M. Nourian and P. E. Caines, “ϵ-nash mean field game theory for nonlinear stochastic dynamical systems with major and minor agents,” *SIAM Journal on Control and Optimization*, vol. 51, no. 4, pp. 3302–3331, 2013.

[46] D. Bauso and R. Pesenti, “Team theory and person-by-person optimization with binary decisions,” *SIAM Journal on Control and Optimization*, vol. 50, no. 5, pp. 3011–3028, 2012.

[47] G. Gnecco, M. Sanguineti, and M. Gaggero, “Suboptimal solutions to team optimization problems with stochastic information structure,” *SIAM Journal on Optimization*, vol. 22, no. 1, pp. 212–243, 2012.

[48] T. Groves, “Incentives in teams,” *Econometrica*, vol. 41, no. 4, pp. 617–631, 1973.

[49] Y.-C. Ho, “Team decision theory and information structures in optimal control problems–part i,” *IEEE Transactions on Automatic control*, vol. 17, no. 1, pp. 15–22, 1972.

[50] J. Marschak, “Elements for a theory of teams,” *Management Science*, vol. 1, no. 2, pp. 127–137, 1955.

[51] R. Radner, “Team decision problems,” *The Annals of Mathematical Statistics*, vol. 33, no. 3, pp. 857–881, 1962.

[52] B.-C. Wang and J. Huang, “Social optima in robust mean field lqg control,” in *2017 11th Asian Control Conference (ASCC)*, pp. 2089–2094, IEEE, 2017.

[53] J. Huang, G. Wang, and Z. Wu, “Optimal premium policy of an insurance firm: full and partial information,” *Insurance: Mathematics and Eco-*

nomics, vol. 47, no. 2, pp. 208–215, 2010.

[54] D. Duffie and H. R. Richardson, "Mean-variance hedging in continuous time," *The Annals of Applied Probability*, vol. 1, no. 1, pp. 1–15, 1991.

[55] R. C. Merton, "Optimum consumption and portfolio rules in a continuous-time model," *Journal of Economic Theory*, vol. 3, no. 4, pp. 373–413, 1971.

[56] J. Xiong and X. Y. Zhou, "Mean-variance portfolio selection under partial information," *SIAM Journal on Control and Optimization*, vol. 46, no. 1, pp. 156–175, 2007.

[57] X. Y. Zhou and D. Li, "Continuous-time mean-variance portfolio selection: A stochastic lq framework," *Applied Mathematics and Optimization*, vol. 42, no. 1, pp. 19–33, 2000.

[58] J. Sun, X. Li, and J. Yong, "Open-loop and closed-loop solvabilities for stochastic linear quadratic optimal control problems," *SIAM Journal on Control and Optimization*, vol. 54, no. 5, pp. 2274–2308, 2016.

[59] J. Sun and J. Yong, "Linear quadratic stochastic differential games: open-loop and closed-loop saddle points," *SIAM Journal on Control and Optimization*, vol. 52, no. 6, pp. 4082–4121, 2014.

[60] J. Yong, "Linear forward-backward stochastic differential equations with random coefficients," *Probability Theory and Related Fields*, vol. 135, no. 1, pp. 53–83, 2006.

[61] J. Yong, "Linear-quadratic optimal control problems for mean-field stochastic differential equations," *SIAM journal on Control and Optimization*, vol. 51, no. 4, pp. 2809–2838, 2013.

[62] J. Yong and X. Y. Zhou, *Stochastic controls: Hamiltonian systems and HJB equations*, vol. 43. Springer Science and Business Media, 1999.

[63] J. Huang and M. Huang, "Robust mean field linear-quadratic-gaussian games with unknown l^2-disturbance," *SIAM Journal on Control and Optimization*, vol. 55, no. 5, pp. 2811–2840, 2017.

[64] Y. Hu, J. Huang, and T. Nie, "Linear-quadratic-gaussian mixed mean-field games with heterogeneous input constraints," *SIAM Journal on Control and Optimization*, vol. 56, no. 4, pp. 2835–2877, 2018.

[65] F. Bloch, "Sequential formation of coalitions in games with externalities and fixed payoff division," *Games and Economic Behavior*, vol. 14, no. 1, pp. 90–123, 1996.

[66] S. Hart and M. Kurz, "Endogenous formation of coalitions," *Econometrica: Journal of the Econometric Society*, pp. 1047–1064, 1983.

[67] D. Ray and R. Vohra, "A theory of endogenous coalition structures," *Games and Economic Behavior*, vol. 26, no. 2, pp. 286–336, 1999.

[68] P. Pekgün, P. M. Griffin, and P. Keskinocak, "Centralized versus decentralized competition for price and lead-time sensitive demand," *Decision Sciences*, vol. 48, no. 6, pp. 1198–1227, 2017.

[69] R. Wang, G. Xiao, and P. Wang, "Hybrid centralized-decentralized (hcd) charging control of electric vehicles," *IEEE Transactions on Vehicular Technology*, vol. 66, no. 8, pp. 6728–6741, 2017.

[70] T. Xie, X. Feng, and J. Huang, "Mixed linear quadratic stochastic differential leader-follower game with input constraint," *Applied Mathematics and Optimization*, pp. 1–37, 2021.

[71] M. Aoki, "On feedback stabilizability of decentralized dynamic systems," *Automatica*, vol. 8, no. 2, pp. 163–173, 1972.

[72] R. Lau, R. Persiano, and P. Varaiya, "Decentralized information and con-

trol: A network flow example," *IEEE Transactions on Automatic Control*, vol. 17, no. 4, pp. 466–473, 1972.

[73] S.-H. Wang and E. Davison, "On the stabilization of decentralized control systems," *IEEE Transactions on Automatic Control*, vol. 18, no. 5, pp. 473–478, 1973.

[74] J. Huang, B.-C. Wang, and J. Yong, "Social optima in mean field linear-quadratic-gaussian control with volatility uncertainty," *SIAM Journal on Control and Optimization*, vol. 59, no. 2, pp. 825–856, 2021.

[75] S. Peng and Z. Wu, "Fully coupled forward-backward stochastic differential equations and applications to optimal control," *SIAM Journal on Control and Optimization*, vol. 37, no. 3, pp. 825–843, 1999.

[76] E. Pardoux and S. Tang, "Forward-backward stochastic differential equations and quasilinear parabolic pdes," *Probability Theory and Related Fields*, vol. 114, no. 2, pp. 123–150, 1999.

[77] R. Salhab, J. Le Ny, and R. P. Malhamé, "Dynamic collective choice: Social optima," *IEEE Transactions on Automatic Control*, vol. 63, no. 10, pp. 3487–3494, 2018.

[78] M. Nourian, R. P. Malhame, M. Huang, and P. E. Caines, "Mean field (nce) formulation of estimation based leader-follower collective dynamics," *International Journal of Robotics and Automation*, vol. 26, no. 1, p. 120, 2011.

[79] L. Gan, U. Topcu, and S. H. Low, "Optimal decentralized protocol for electric vehicle charging," *IEEE Transactions on Power Systems*, vol. 28, no. 2, pp. 940–951, 2012.

[80] H. Fang, Y. Wang, and J. Chen, "Health-aware and user-involved battery charging management for electric vehicles: Linear quadratic strategies,"

IEEE Transactions on Control Systems Technology, vol. 25, no. 3, pp. 911–923, 2016.

[81] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*. Springer Science and Business Media, 2010.

[82] G.-E. Espinosa and N. Touzi, "Optimal investment under relative performance concerns," *Mathematical Finance*, vol. 25, no. 2, pp. 221–257, 2015.

[83] S. Chen and X. Y. Zhou, "Stochastic linear quadratic regulators with indefinite control weight costs. ii," *SIAM Journal on Control and Optimization*, vol. 39, no. 4, pp. 1065–1081, 2000.

[84] Z. Qiu, J. Huang, and T. Xie, "Linear quadratic gaussian mean-field controls of social optima," *arXiv preprint arXiv:2005.06792*, 2020.

[85] R. Buckdahn, J. Li, and S. Peng, "Mean-field backward stochastic differential equations and related partial differential equations," *Stochastic Processes and Their Applications*, vol. 119, no. 10, pp. 3133–3154, 2009.

[86] R. Buckdahn, B. Djehiche, J. Li, and S. Peng, "Mean-field backward stochastic differential equations: a limit approach," *The Annals of Probability*, vol. 37, no. 4, pp. 1524–1565, 2009.

Appendix A: Chapter 2

A.1 Proof of Lemma 2.4

Proof. By (2.34), (2.35) and (2.46), the dynamic of $\tilde{x}^{(N)} - \hat{x}$ satisfies

$$\begin{cases} d(\tilde{x}^{(N)} - \hat{x}) = (\Pi_1 + F)(\tilde{x}^{(N)} - \hat{x})dt + \dfrac{1}{N}\displaystyle\sum_{i=1}^{N}[(C + D\Theta_1 + \tilde{F})\tilde{x}_i + D\Theta_2]dW_i, \\ (\tilde{x}^{(N)} - \hat{x})(0) = 0. \end{cases}$$

By applying Cauchy-Schwartz inequality, Burkholder-Davis-Gundy's inequality and Lemma 2.3, there exist some constant L such that

$$\begin{aligned} &\mathbb{E}\sup_{0\le s\le t}\|\tilde{x}^{(N)}(s) - \hat{x}(s)\|^2 \\ \le&2\mathbb{E}\sup_{0\le s\le t}\left\|\int_0^s (\Pi_1 + F)(\tilde{x}^{(N)} - \hat{x})dr\right\|^2 + \frac{2}{N^2}\mathbb{E}\sup_{0\le s\le t}\left\|\sum_{i=1}^{N}\int_0^s[(C + D\Theta_1 + \tilde{F})\tilde{x}_i + D\Theta_2]dW_i\right\|^2 \\ \le&2\mathbb{E}\int_0^t \left\|(\Pi_1 + F)(\tilde{x}^{(N)} - \hat{x})\right\|^2 dr + \frac{L}{N^2}\mathbb{E}\sum_{i=1}^{N}\int_0^t \left\|(C + D\Theta_1 + \tilde{F})\tilde{x}_i + D\Theta_2\right\|^2 dr \\ =&L\mathbb{E}\int_0^t \left\|\tilde{x}^{(N)} - \hat{x}\right\|^2 dr + O\Big(\frac{1}{N}\Big). \end{aligned}$$

Then, by Grönwall's inequality,

$$\mathbb{E}\sup_{0\le t\le T}\|\tilde{x}^{(N)}(t) - \hat{x}(t)\|^2 = O\bigg(\frac{1}{N}\bigg).$$

□

A.2 Proof of Lemma 2.5

Proof. By (2.46) and (2.35), we have

$$\begin{cases} d(\tilde{x}_i - \hat{x}) = \Big\{ \big[A - B(R + D^T PD)^{-1}(B^T P + D^T PC)\big](\tilde{x}_i - \hat{x}) + F(\tilde{x}^{(N)} - \hat{x}) \Big\} dt \\ \qquad\qquad\qquad + \big[(C + D\Theta_1)\tilde{x}_i + D\Theta_2 + \tilde{F}\tilde{x}^{(N)}\big] dW_i, \\ \tilde{x}_i(0) - \hat{x}(0) = 0. \end{cases}$$

By Cauchy-Schwarz inequality, Burkholder-Davis-Gundy's inequality and Lemma 2.4, there exist some constant L such that

$$\begin{aligned} & \mathbb{E} \sup_{0 \le s \le t} \|\tilde{x}_i(s) - \hat{x}(s)\|^2 \\ \le & 4\mathbb{E} \sup_{0 \le s \le t} \int_0^s \Big\| \big[A - B(R + D^T PD)^{-1}(B^T P + D^T PC)\big](\tilde{x}_i - \hat{x}) \Big\|^2 + \|F(\tilde{x}^{(N)} - \hat{x})\|^2 dr \\ & + 2\mathbb{E} \sup_{0 \le s \le t} \Big\| \int_0^s \big[(C + D\Theta_1)\tilde{x}_i + D\Theta_2 + \tilde{F}\tilde{x}^{(N)}\big] dW_i \Big\|^2 \\ \le & L \Big\{ \mathbb{E} \int_0^t \|\tilde{x}_i - \hat{x}\|^2 dr + O\Big(\frac{1}{N}\Big) + \mathbb{E} \int_0^t \Big\| (C + D\Theta_1)\tilde{x}_i + D\Theta_2 + \tilde{F}\tilde{x}^{(N)} \Big\|^2 dr \Big\} \\ \le & L \Big\{ \mathbb{E} \int_0^t \|\tilde{x}_i - \hat{x}\|^2 dr + O\Big(\frac{1}{N}\Big) + L \Big\}. \end{aligned}$$

Then, by Grönwall's inequality, one can obtain

$$\mathbb{E} \sup_{0 \le t \le T} \|\tilde{x}_i(t) - \hat{x}(t)\|^2 \le L,$$

where L is not related to i and the lemma follows. Similarly, by (2.32) and (2.46), The dynamic of $\tilde{x}_i - \bar{\alpha}_i$ satisfies

$$\begin{cases} d(\tilde{x}_i - \bar{\alpha}_i) = [(A + B\Theta_1)(\tilde{x}_i - \bar{\alpha}_i) + F(\tilde{x}^{(N)} - \hat{x})]dt \\ \qquad\qquad\qquad + [(C + D\Theta_1)(\tilde{x}_i - \bar{\alpha}_i) + \tilde{F}(\tilde{x}^{(N)} - \hat{x})]dW_i, \\ (\tilde{x}_i - \bar{\alpha}_i)(0) = 0. \end{cases}$$

Applying Burkholder-Davis-Gundy's inequality, Grönwall's inequality, and Lemma 2.4, the result can be obtained. □

A.3 Proof of Lemma 2.6

Proof. By (2.2) and Lemma 2.3, 2.4, 2.5, for some constant L we have

$$\begin{aligned} 2\mathcal{J}_{soc}^{(N)}(\tilde{u}_1, \cdots, \tilde{u}_N) =& \sum_{i=1}^{N} \mathbb{E}\bigg\{ \int_0^T \|\tilde{x}_i - \Gamma\hat{x} + \Gamma\hat{x} - \Gamma\tilde{x}^{(N)} - \eta\|_Q^2 + \|\tilde{u}_i\|_R^2 \, dt \\ & + \|\tilde{x}_i(T) - \bar{\Gamma}\hat{x}(T) + \bar{\Gamma}\hat{x}(T) - \bar{\Gamma}\tilde{x}^{(N)}(T) - \bar{\eta}\|_G^2 \bigg\} \\ \leq& L\sum_{i=1}^{N} \mathbb{E}\bigg\{ \int_0^T \|\tilde{x}_i - \hat{x}\|^2 + \|\hat{x} - \tilde{x}^{(N)}\|^2 + \|\eta\|^2 + \|\tilde{u}_i\|^2 dt \\ & + \|\tilde{x}_i(T) - \hat{x}(T)\|^2 + \|\hat{x}(T) - \tilde{x}^{(N)}(T)\|^2 + \|\bar{\eta}\|^2 \bigg\} \\ \leq& L\sum_{i=1}^{N} \mathbb{E} \int_0^T L + O\Big(\frac{1}{N}\Big) dt \leq NL. \end{aligned}$$

□

A.4 Proof of Lemma 2.7

To prove Lemma 2.7, we need the following lemmas.

Lemma A.1. *Under* (A2.1)-(A2.4)*, for some constant L and any admissible control with form $(\tilde{u}_1, \cdots, \tilde{u}_{i-1}, \acute{u}_i, \tilde{u}_{i+1}, \cdots, \tilde{u}_N) \in \mathcal{U}_c$ satisfying*

$\mathbb{E}\int_0^T \|\acute{u}_i\|^2 dt < L$, *it holds that*

$$\mathbb{E}\sup_{0\le t\le T}\|\delta x^{(N)}\|^2 = O\Big(\frac{1}{N^2}\Big).$$

Proof of Lemma A.1. The dynamic of $\delta x^{(N)}$ follows

$$\begin{cases} d\delta x^{(N)} = \Big[(A+F)\delta x^{(N)} + \dfrac{B}{N}\delta u_i\Big]dt + \dfrac{1}{N}\displaystyle\sum_{j=1}^{N}(C\delta x_j + \tilde{F}\delta x^{(N)})dW_j + \dfrac{1}{N}D\delta u_i dW_i, \\ \delta x^{(N)}(0) = 0. \end{cases}$$

By Lemma 2.4, we have $\mathbb{E}\sup_{0\le t\le T}\|\tilde{x}^{(N)}(t) - \hat{x}(t)\|^2 = O\big(\frac{1}{N}\big)$. By Burkholder-Davis-Gundy's inequality and the boundness of $\acute{u}_i$, there exist some constant L such that

$$\begin{aligned}\mathbb{E}\sup_{0\le s\le t}\|\delta x^{(N)}(s)\|^2 = &\mathbb{E}\sup_{0\le s\le t}\Bigg\|\int_0^s\Big[(A+F)\delta x^{(N)} + \frac{B}{N}\delta u_i\Big]dr + \frac{1}{N}D\int_0^s \delta u_i dW_i \\ &+ \frac{1}{N}\sum_{i=1}^{N}\int_0^s (C\delta x_j + \tilde{F}\delta x^{(N)})dW_j\Bigg\|^2 \\ \le& L\mathbb{E}\int_0^t \|\delta x^{(N)}\|^2 dr + \frac{L}{N^2}\sum_{i=1}^{N}\mathbb{E}\int_0^t \|\delta x_i\|^2 + \|\delta x^{(N)}\|^2 dr + O\Big(\frac{1}{N^2}\Big).\end{aligned}$$

Applying Grönwall's inequality, for some constant L, one can obtain

$$\begin{aligned}\mathbb{E}\sup_{0\le s\le t}\|\delta x_j(s)\|^2 \le& L\mathbb{E}\int_0^t \|\delta x^{(N)}\|^2 ds + \int_0^t L\mathbb{E}\int_0^s \|\delta x^{(N)}\|^2 dr ds \\ \le& L\mathbb{E}\int_0^t \|\delta x^{(N)}\|^2 ds, \quad j \neq i.\end{aligned}$$

Thus

$$\mathbb{E}\sup_{0\le s\le t}\|\delta x^{(N)}(s)\|^2 \le L\mathbb{E}\int_0^t \|\delta x^{(N)}\|^2 ds + O\Big(\frac{1}{N^2}\Big).$$

Again, by Grönwall's inequality

$$\mathbb{E}\sup_{0\le t\le T}\|\delta x^{(N)}(t)\|^2 = O\Big(\frac{1}{N^2}\Big).$$

$\square$

Lemma A.2. *Under* (A2.1)-(A2.4)*, for some constant L and any admissible control with form $(\tilde{u}_1,\ \cdots,\ \tilde{u}_{i-1},\ \acute{u}_i,\ \tilde{u}_{i+1},\ \cdots,\ \tilde{u}_N)\in\mathcal{U}_c$ satisfying $\mathbb{E}\int_0^T\|\acute{u}_i\|^2dt<L$, it follows that*

$$\sup_{1\le j\le N}\mathbb{E}\sup_{0\le t\le T}\|\delta x_j\|^2 = O\Big(\frac{1}{N^2}\Big). \tag{A.1}$$

Proof. By the proof of Lemma A.1, for some constant L, we have

$$\mathbb{E}\sup_{0\le s\le t}\|\delta x_j\|^2 \le L\mathbb{E}\int_0^t\|\delta x^{(N)}\|^2ds = O\Big(\frac{1}{N^2}\Big).$$

Note that L is independent of j and Proposition A.2 holds. $\square$

Based on Lemma A.1, we can also obtain the following estimation of $\delta x_i-\delta a_i$.

Proof of Lemma 2.7. The dynamics of $\delta x_i-\delta a_i$ follows

$$\begin{cases} d(\delta x_i-\delta a_i)=[A(\delta x_i-\delta a_i)+F\delta x^{(N)}]dt+[C(\delta x_i-\delta a_i)+\tilde{F}\delta x^{(N)}]dW_i,\\ (\delta x_i-\delta a_i)(0)=0.\end{cases}$$

By Lemma A.1, we have $\mathbb{E}\sup_{0\le t\le T}\|\delta x^{(N)}(t)\|^2 = O\big(\frac{1}{N^2}\big)$. Thus, applying Burkholder-Davis-Gundy's inequality and Grönwall's inequality, the lemma follows. $\square$

A.5 Proof of Lemma 2.8

Proof. Motivated by (2.50), we consider $\langle M_2\tilde{\mathbf{u}} + M_1, \delta\mathbf{u}_i\rangle$ for some single-agent bounded perturbation $\delta\mathbf{u}_i$ satisfying $\mathbb{E}\int_0^T \|\delta\mathbf{u}_i\|^2 dt < L$ for some constant L. By the calculation in Section 3, when we only perturb $\mathcal{A}_i$, the variation of the cost functional is

$$\begin{aligned}
&\mathcal{J}_{soc}^{(N)}(\tilde{\mathbf{u}}+\delta\mathbf{u}_i) = \mathcal{J}_{soc}^{(N)}(\tilde{\mathbf{u}}) + \mathbb{E}\bigg\{\int_0^T \langle Q(\tilde{x}_i - \Gamma\tilde{x}^{(N)} - \eta), \delta x_i\rangle - \langle \Gamma^T Q(\tilde{x}_i - \Gamma\tilde{x}^{(N)} - \eta), \delta x^{(N)}\rangle \\
&+ \sum_{j\neq i}\langle Q(\tilde{x}_j - \Gamma\tilde{x}^{(N)} - \eta), \delta x_j\rangle - \sum_{j\neq i}\langle \Gamma^T Q(\tilde{x}_j - \Gamma\tilde{x}^{(N)} - \eta), \delta x^{(N)}\rangle + \langle R\tilde{u}_i, \delta u_i\rangle dt + \langle G(\tilde{x}_i(T) \\
&- \bar{\Gamma}\tilde{x}^{(N)}(T) - \bar{\eta}), \delta x_i(T)\rangle - \langle \bar{\Gamma}^T G(\tilde{x}_i(T) - \bar{\Gamma}\tilde{x}^{(N)}(T) - \bar{\eta}), \delta x^{(N)}(T)\rangle + \sum_{j\neq i}\langle G(\tilde{x}_j(T) \\
&- \bar{\Gamma}\tilde{x}^{(N)}(T) - \bar{\eta}), \delta x_j(T)\rangle - \sum_{j\neq i}\langle \bar{\Gamma}^T G(\tilde{x}_j(T) - \bar{\Gamma}\tilde{x}^{(N)}(T) - \bar{\eta}), \delta x^{(N)}(T)\rangle\bigg\} + o(\delta\mathbf{u}_i).
\end{aligned}$$

Thus, for the Fréchet derivative of $\mathcal{A}_i$ on $\tilde{\mathbf{u}}$, we have

$$\begin{aligned}
&\langle M_2\tilde{\mathbf{u}} + M_1, \delta\mathbf{u}_i\rangle = \mathbb{E}\bigg\{\int_0^T \langle Q(\tilde{x}_i - \Gamma\tilde{x}^{(N)} - \eta), \delta x_i\rangle - \langle \Gamma^T Q(\tilde{x}_i - \Gamma\tilde{x}^{(N)} - \eta), \delta x^{(N)}\rangle \\
&+ \sum_{j\neq i}\langle Q(\tilde{x}_j - \Gamma\tilde{x}^{(N)} - \eta), \delta x_j\rangle - \sum_{j\neq i}\langle \Gamma^T Q(\tilde{x}_j - \Gamma\tilde{x}^{(N)} - \eta), \delta x^{(N)}\rangle + \langle R\tilde{u}_i, \delta u_i\rangle dt \\
&+ \langle G(\tilde{x}_i(T) - \bar{\Gamma}\tilde{x}^{(N)}(T) - \bar{\eta}), \delta x_i(T)\rangle - \langle \bar{\Gamma}^T G(\tilde{x}_i(T) - \bar{\Gamma}\tilde{x}^{(N)}(T) - \bar{\eta}), \delta x^{(N)}(T)\rangle \\
&+ \sum_{j\neq i}\langle G(\tilde{x}_j(T) - \bar{\Gamma}\tilde{x}^{(N)}(T) - \bar{\eta}), \delta x_j(T)\rangle - \sum_{j\neq i}\langle \bar{\Gamma}^T G(\tilde{x}_j(T) - \bar{\Gamma}\tilde{x}^{(N)}(T) - \bar{\eta}), \delta x^{(N)}(T)\rangle\bigg\}.
\end{aligned} \tag{A.2}$$

Next, we will verify $\varepsilon_1, \cdots, \varepsilon_6 = o(1)$, since

$$\langle M_2\tilde{\mathbf{u}} + M_1, \delta\mathbf{u}_i\rangle - \sum_{i=1}^{6}\varepsilon_i = \delta J_i = 0. \tag{A.3}$$

Equation (A.3) follows by (2.25), (A.2) and the optimality of the auxiliary cost functional. Firstly, we consider ε_1 which is given by (2.18). By Lemma 2.3, Lemma 2.4 and A.1, for some constant L, we have

$$\begin{aligned}
\varepsilon_1 =& \mathbb{E}\Big\{ \int_0^T \langle(\Gamma^T Q\Gamma - Q\Gamma)(\tilde{x}^{(N)} - \hat{x}), N\delta x^{(N)}\rangle dt \\
&\quad + \langle(\bar{\Gamma}^T Q\bar{\Gamma} - Q\bar{\Gamma})(\bar{x}^{(N)}(T) - \hat{x}(T)), N\delta x^{(N)}(T)\rangle\Big\} \\
&\le NL\sqrt{\mathbb{E}\int_0^T \|\tilde{x}^{(N)} - \hat{x}\|^2 dt \mathbb{E}\int_0^T \|\delta x^{(N)}\|^2 dt} + O\Big(\frac{1}{\sqrt{N}}\Big) = O\Big(\frac{1}{\sqrt{N}}\Big), \\
\varepsilon_2 =& \mathbb{E}\Big\{ \int_0^T -\langle\Gamma^T Q(\bar{x}_i - \Gamma\bar{x}^{(N)} - \eta), \delta x^{(N)}\rangle dt - \langle\bar{\Gamma}^T G(\bar{x}_i(T) - \bar{\Gamma}\hat{x}(T) - \bar{\eta}), \delta x^{(N)}(T)\rangle\Big\} \\
&= 2L \times O\Big(\frac{1}{N^2}\Big) = O\Big(\frac{1}{N^2}\Big).
\end{aligned}$$

Next, we will estimate ε_3 which is given by (2.21). We consider $\delta x^{(-i)} - x^{**}$ firstly, where

$$\begin{cases}
d(\delta x^{(-i)} - x^{**}) = \Big[(A+F)(\delta x^{(-i)} - x^{**}) - \dfrac{F}{N}\delta x^{(N)}\Big]dt + \displaystyle\sum_{j\ne i}\Big[C\delta x_j + \dfrac{\tilde{F}}{N}(\delta x^{(-i)} + \delta x_i)\Big]dW_j, \\
\delta x^{(-i)}(0) - x^{**}(0) = 0.
\end{cases}$$

By Lemma 2.4 and Corollary A.2, for some constant L such that

$$\begin{aligned}
&\mathbb{E}\sup_{0\le s\le t}\|\delta x^{(-i)}(s) - x^{**}(s)\|^2 \\
\le& L\mathbb{E}\int_0^t \Big[\|\delta x^{(-i)} - x^{**}\|^2 + \frac{K}{N^2}\|\delta x^{(N)}\|^2\Big]ds + \mathbb{E}\sup_{0\le s\le t}\Big\|\sum_{j\ne i}\int_0^t [C\delta x_j + \frac{\tilde{F}}{N}(\delta x^{(-i)} + \delta x_i)]dW_j\Big\|^2 \\
\le& L\mathbb{E}\int_0^t \|\delta x^{(-i)} - x^{**}\|^2 ds + O\Big(\frac{1}{N^4}\Big) + L\mathbb{E}\sum_{j\ne i}\int_0^t \|\delta x_j\|^2 + \|\delta x^{(N)}\|^2 ds \\
=& L\mathbb{E}\int_0^t \|\delta x^{(-i)} - x^{**}\|^2 ds + O\Big(\frac{1}{N^4}\Big) + O\Big(\frac{1}{N}\Big) + O\Big(\frac{1}{N}\Big).
\end{aligned}$$

Thus,

$$\mathbb{E}\sup_{0\le t\le T}\|\delta x^{(-i)}(t) - x^{**}(t)\|^2 = O\Big(\frac{1}{N}\Big).$$

Secondly, the dynamics of $N\delta x_j - x_j^*$ is given by

$$\begin{cases} d(N\delta x_j - x_j^*) = \Big[A(N\delta x_j - x_j^*) + F(\delta x^{(-i)} - x^{**})\Big]dt + \Big[C(N\delta x_j - x_j^*) + \tilde{F}(\delta x^{(-i)} - x^{**})\Big]dW_j, \\ N\delta x_j(0) - x_j^*(0) = 0. \end{cases}$$

For some constant L, one can obtain

$$\begin{aligned} \mathbb{E}\sup_{0\le s\le t}\|N\delta x_j - x_j^*\|^2 =& L\mathbb{E}\int_0^t \|N\delta x_j - x_j^*\|^2 + \|\delta x^{(-i)} - x^{**}\|^2 ds \\ =& L\mathbb{E}\int_0^t \|N\delta x_j - x_j^*\|^2 ds + O\Big(\frac{1}{N^2}\Big). \end{aligned}$$

Hence, by Grönwall's inequality,

$$\mathbb{E}\sup_{0\le t\le T}\|N\delta x_j - x_j^*\|^2 = O\Big(\frac{1}{N^2}\Big).$$

For ε_3, there exist some constants L such that

$$\begin{aligned} \varepsilon_3 =& \mathbb{E}\int_0^T \frac{1}{N}\sum_{j\neq i}\langle Q(\tilde{x}_j - \Gamma\hat{x} - \eta), N\delta x_j - x_j^*\rangle - \frac{1}{N}\sum_{j\neq i}\langle \Gamma^T Q(\tilde{x}_j - \Gamma\hat{x} - \eta), \delta x^{(-i)} \\ & - x^{**}\rangle dt + \frac{1}{N}\sum_{j\neq i}\langle G(\bar{x}_j(T) - \bar{\Gamma}\hat{x}(T) - \bar{\eta}), N\delta x_j(T) - x_j^*(T)\rangle - \frac{1}{N}\sum_{j\neq i}\langle \bar{\Gamma}^T G(\bar{x}_j(T) \\ & - \bar{\Gamma}\hat{x}(T) - \bar{\eta}), \delta x^{(-i)}(T) - x^{**}(T)\rangle \\ \le& \frac{1}{N}\sum_{j\neq i}\sqrt{\mathbb{E}\int_0^T \|Q(\tilde{x}_j - \Gamma\hat{x} - \eta)\|^2 dt\,\mathbb{E}\int_0^T \|N\delta x_j - x_j^*\|^2 dt} \\ & + \frac{1}{N}\sum_{j\neq i}\sqrt{\mathbb{E}\int_0^T \|\Gamma^T Q(\tilde{x}_j - \Gamma\hat{x} - \eta)\|^2 dt\,\mathbb{E}\int_0^T \|\delta x^{(-i)} - x^{**}\|^2 dt} + O\Big(\frac{1}{\sqrt{N}}\Big) \\ =& \frac{1}{N}\sum_{j\neq i}\sqrt{L\times O\Big(\frac{1}{N^2}\Big)} + \sqrt{L\times O\Big(\frac{1}{N}\Big)} = O\Big(\frac{1}{\sqrt{N}}\Big). \end{aligned}$$

In what follows, we aim prove that $\varepsilon_4 = o(1)$. By Lemma 2.4 and A.1 we have

$$
\begin{aligned}
\varepsilon_4 =& \mathbb{E}\int_0^T -\Big\langle \Gamma^T Q\Big(\frac{\sum_{j\neq i}\tilde{x}_j}{N} - \hat{x}\Big), \sum_{i=1}^N \delta x_i \Big\rangle dt - \Big\langle \bar{\Gamma}^T G\Big(\frac{\sum_{j\neq i}\bar{x}_j(T)}{N} - \hat{x}(T)\Big), \sum_{i=1}^N \delta x_i(T)\Big\rangle \\
=& 2 \times O\Big(\frac{1}{\sqrt{N}}\Big) \times O\Big(\frac{1}{N}\Big).
\end{aligned}
$$

For ε_5, which is given by (2.26), we need to estimate $\mathbb{E}\beta_1^j - \frac{\sum_{j\neq i}^N \beta_1^j}{N}$ and $\mathbb{E}y_1^j - \frac{\sum_{j\neq i}^N y_1^j}{N}$. Recall that in Section 2.3, (2.22) can be rewritten as follows:

$$
\begin{cases}
d\mathbf{y_1} = [-\bar{\mathbf{Q}}\tilde{\mathbf{x}} + \mathbf{q} - \bar{\mathbf{A}}^T\mathbf{y_1} + \displaystyle\sum_{j=1}^N \bar{\mathbf{C}}_\mathbf{j}^T \mathbf{z_1^j}]dt + \sum_{j=1}^N \mathbf{z_1^j} dW_j, \quad \mathbf{y_1}(T) = \mathbf{G}\tilde{\mathbf{x}} + \mathbf{g}, \\
d\tilde{\mathbf{x}} = (\mathbf{A}\tilde{\mathbf{x}} + \mathbf{b})dt + \displaystyle\sum_{i=1}^N (\mathbf{C}_i\tilde{\mathbf{x}} + \mathbf{h_i})dW_i, \quad \tilde{\mathbf{x}}(0) = \Xi,
\end{cases}
$$

where

$$
\bar{\mathbf{A}} = \begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \end{pmatrix}, \quad
\bar{\mathbf{C}}_\mathbf{j} = \begin{array}{c} 1 \\ \vdots \\ j^{\text{th}} \\ \vdots \\ N \end{array}
\begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & C & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}, \quad
\bar{\mathbf{G}} = \begin{pmatrix} G & 0 & \cdots & 0 \\ 0 & G & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G \end{pmatrix}, \quad
\bar{\mathbf{Q}} = \begin{pmatrix} Q & 0 & \cdots & 0 \\ 0 & Q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q \end{pmatrix},
$$

$$
\mathbf{A} = \begin{pmatrix} A + \frac{F}{N} + B\Theta_1 & \frac{F}{N} & \cdots & \frac{F}{N} \\ \frac{F}{N} & A + \frac{F}{N} + B\Theta_1 & \cdots & \frac{F}{N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{F}{N} & \frac{F}{N} & \cdots & A + \frac{F}{N} + B\Theta_1 \end{pmatrix}, \quad
\mathbf{b} = \begin{pmatrix} B\Theta_2 \\ \vdots \\ B\Theta_2 \end{pmatrix}, \quad
\Xi = \begin{pmatrix} \xi_0 \\ \vdots \\ \xi_0 \end{pmatrix}, \quad
\mathbf{y_1} = \begin{pmatrix} y_1^1 \\ \vdots \\ y_1^N \end{pmatrix}, \quad
\tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_N \end{pmatrix},
$$

$$
\mathbf{C}_i = \begin{array}{c} 1 \\ \vdots \\ i^{\text{th}} \\ \vdots \\ N \end{array}
\begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots \\ \frac{\tilde{F}}{N} & \cdots & \frac{\tilde{F}}{N} & \frac{\tilde{F}}{N} + C + D\Theta_1 & \frac{\tilde{F}}{N} & \cdots & \frac{\tilde{F}}{N} \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad
\mathbf{h_i} = \begin{array}{c} 1 \\ \vdots \\ i^{\text{th}} \\ \vdots \\ N \end{array}
\begin{pmatrix} 0 \\ \vdots \\ D\Theta_2 \\ \vdots \\ 0 \end{pmatrix}, \quad
\mathbf{g} = \begin{pmatrix} G\bar{\Gamma}\hat{x} + G\bar{\eta} \\ \vdots \\ G\bar{\Gamma}\hat{x} + G\bar{\eta} \end{pmatrix}, \quad
\mathbf{q} = \begin{pmatrix} Q\Gamma\hat{x} + Q\eta \\ \vdots \\ Q\Gamma\hat{x} + Q\eta \end{pmatrix}, \quad
\mathbf{z_1^j} = \begin{pmatrix} \beta_1^{1j} \\ \vdots \\ \beta_1^{Nj} \end{pmatrix}.
$$

Let $\mathbf{y_1} = \mathbf{\Lambda}\tilde{\mathbf{x}} + \lambda$ and the following equations can be derived

$$\begin{cases} \mathbf{z_1^j} = \mathbf{\Lambda}\mathbf{C}_j\tilde{\mathbf{x}} + \mathbf{\Lambda}\mathbf{h_j}, \\ \dot{\mathbf{\Lambda}} + \mathbf{\Lambda}\mathbf{A} + \bar{\mathbf{Q}} + \bar{\mathbf{A}}^T\mathbf{\Lambda} - \sum_{j=1}^{N} \bar{\mathbf{C}}_{\mathbf{j}}^T\mathbf{\Lambda}\mathbf{C_j} = 0,\ \mathbf{\Lambda}(T) = \mathbf{G}, \\ \dot{\lambda} + \bar{\mathbf{A}}^T\lambda - \mathbf{q} - \sum_{j=1}^{N} \bar{\mathbf{C}}_{\mathbf{j}}^T\mathbf{\Lambda}\mathbf{h_j} + \mathbf{\Lambda}\mathbf{b} = 0,\ \lambda(T) = \mathbf{g}, \end{cases} \tag{A.4}$$

where $\mathbf{\Lambda} = \begin{pmatrix} \Lambda_{11} & \cdots & \Lambda_{1N} \\ \vdots & \ddots & \vdots \\ \Lambda_{N1} & \cdots & \Lambda_{NN} \end{pmatrix}$. Thus, $\mathbb{E}y_1^j - \frac{\sum_{j=1}^N y_1^j}{N} = \frac{1}{N}(I, \cdots, I)\mathbf{\Lambda}\begin{pmatrix} \tilde{x}_1 - \mathbb{E}\tilde{x}_1 \\ \vdots \\ \tilde{x}_N - \mathbb{E}\tilde{x}_N \end{pmatrix}$. Clearly, if there exist some constants L such that $\sup_{1\le i\le N}\sup_{0\le t\le T} \|\sum_{k=1}^{N}\Lambda_{ki}(t)\|_{\max} < L$ holds, then by letting $E = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$ we have

$$\Big\|\mathbb{E}y_1^j - \frac{\sum_{j=1}^N y_1^j}{N}\Big\|^2 = \Big\|\frac{1}{N}\sum_{i=1}^{N}\Big(\sum_{k=1}^{N}\Lambda_{ki}\Big)(\tilde{x}_i - \mathbb{E}\tilde{x}_i)\Big\|^2 \le L\Big\|\frac{1}{N}\sum_{i=1}^{N}E(\tilde{x}_i - \mathbb{E}\tilde{x}_i)\Big\|^2$$
$$= L\Big\|E(\tilde{x}^{(N)} - \hat{x})\Big\|^2 = O\Big(\frac{1}{N}\Big).$$

Thus, next step is to investigate $\mathbf{\Lambda}$.

We can verify that $\Lambda_{ii} = \Lambda_1$ for $i = 1, \cdots, N$ and $\Lambda_{ij} = \Lambda_2$ for $i \neq j$, where Λ_1 and Λ_2 satisfy

$$\begin{cases} \dot{\Lambda}_1 + \Lambda_1(A + B\Theta_1) + \frac{(N-1)\Lambda_2 + \Lambda_1}{N}F + Q + A^T\Lambda_1 - \frac{1}{N}C^T\Lambda_1\tilde{F} - C^T\Lambda_1(C + D\Theta_1) = 0, \\ \dot{\Lambda}_2 + \Lambda_2(A + B\Theta_1) + \frac{(N-1)\Lambda_2 + \Lambda_1}{N}F + A^T\Lambda_2 - \frac{1}{N}C^T\Lambda_1\tilde{F} = 0, \\ \Lambda_1(T) = G, \quad \Lambda_2(T) = 0. \end{cases}$$

Hence, $\sum_{k=1}^{N}\Lambda_{ki} = \Lambda_1 + (N-1)\Lambda_2$ and we study the uniform boundness of Λ_1 and $\Lambda_2^* := (N-1)\Lambda_2$ w.r.t N and t, where

$$\begin{cases}\dot{\Lambda}_1+\Lambda_1\Big(A+B\Theta_1+\dfrac{F}{N}\Big)+A^T\Lambda_1 - C^T\Lambda_1(C+D\Theta_1+\dfrac{\tilde{F}}{N})+\dfrac{\Lambda_2^*}{N}F+Q=0,\\ \dot{\Lambda}_2^*+\Lambda_2^*\Big(A+B\Theta_1+\dfrac{N-1}{N}F\Big)+A^T\Lambda_2^*+\dfrac{N-1}{N}(\Lambda_1F-C^T\Lambda_1\tilde{F})=0,\\ \Lambda_1(T)=G,\quad \Lambda_2^*(T)=0.\end{cases} \tag{A.5}$$

Firstly, by the linearity, for any given N, Λ_1 and Λ_2^* are solvable on $[0,T]$. Let L $= \sup_{0\le t\le T}\{\|A(t)\|_{\max}, \|B(t)\Theta_1(t)\|_{\max}, \|F(t)\|_{\max}, \|C(t)\|_{\max}, \|D(t)\Theta_1(t)\|_{\max},$ $\|\tilde{F}(t)\|_{\max}, \|Q(t)\|_{\max}\}$. Introduce $\bar{\Lambda}_1$ and $\bar{\Lambda}_2^*$ satisfying

$$\begin{cases}\dot{\bar{\Lambda}}_1+3L\bar{\Lambda}_1E+LE\bar{\Lambda}_1+3LE\bar{\Lambda}_1E+LE\bar{\Lambda}_2^*+LE=0,\quad \bar{\Lambda}_1(T)=G,\\ \dot{\bar{\Lambda}}_2^*+2L\bar{\Lambda}_2^*E+LE\bar{\Lambda}_2^*+L(\bar{\Lambda}_1E+E\bar{\Lambda}_1E)=0,\quad \bar{\Lambda}_2^*(T)=0.\end{cases}$$

By the linearity, $\bar{\Lambda}_1$ and $\bar{\Lambda}_2^*$ are solvable on $[0,T]$ and surely, bounded. For any N and $t\in[0,T]$, $|\Lambda_1(t)|\le\bar{\Lambda}_1(t)$ and $|\Lambda_2^*(t)|\le\bar{\Lambda}_2^*(t)$. Thus, there exist some constants L such that $\sup\limits_{1\le i\le N}\sup\limits_{0\le t\le T}\|\sum\limits_{k=1}^{N}\Lambda_{ki}(t)\|_{\max}<L$ holds, and $\|\mathbb{E}y_1^j - \frac{\sum_{j=1}^{N}y_1^j}{N}\|^2 = o(1)$. Similarly, $\|\mathbb{E}\beta_1^j - \frac{\sum_{j\neq i}^{N}\beta_1^j}{N}\|^2=o(1)$ and consequently $\varepsilon_5=o(1)$.

The estimation of ε_6, which is given by (2.28), follows by Lemma 2.7 straightforwardly and $\varepsilon_6 = O\big(\frac{1}{N^2}\big)$. Thus,

$$\langle M_2\tilde{\mathbf{u}}+M_1,\delta\mathbf{u}_i\rangle=\delta J_i+\sum_{i=1}^{6}\varepsilon_i=O\Big(\frac{1}{\sqrt{N}}\Big).$$

Using $\mathbb{E}\int_0^T\|\delta\mathbf{u}_i\|^2dt<L$ and the Cauchy-Schwartz inequality, $\|M_2\tilde{\mathbf{u}}+M_1\| = O\big(\frac{1}{\sqrt{N}}\big)$. □

Appendix B: Chapter 3

For any given $(Y,Z)\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)\times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{m\times(d+1)})$ and $0\le t\le T$, the following SDE has a unique solution:

$$\begin{aligned} X(t) =& x + \int_0^t b(s,X(s),\mathbb{E}[X(s)|\mathcal{F}_s^{W_0}],Y(s),Z(s))ds \\ &+ \int_0^t \sigma(s,X(s),\mathbb{E}[X(s)|\mathcal{F}_s^{W_0}],Y(s),Z(s))dW(s). \end{aligned} \tag{B.1}$$

Therefore, we can introduce a map $\mathcal{M}_1:(Y,Z)\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)\times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{m\times(d+1)})\to X\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$ by (B.1). Moreover, we have the following result:

Lemma B.1. *Let X_i be the solution of* (B.1) *corresponding to (Y_i,Z_i), $i=1,2$ respectively. Then for all $\rho\in\mathbb{R}$ and some constant $l_1>0$, we have*

$$\begin{aligned} &\mathbb{E}e^{-\rho t}\|\hat{X}(t)\|^2+\bar{\rho}_1\mathbb{E}\int_0^t e^{-\rho s}\|\hat{X}(s)\|^2ds \\ \le&(k_2l_1+k_{11}^2)\mathbb{E}\int_0^t e^{-\rho s}\|\hat{Y}(s)\|^2ds+(k_3l_2+k_{12}^2)\mathbb{E}\int_0^t e^{-\rho s}\|\hat{Z}(s)\|^2ds, \end{aligned} \tag{B.2}$$

and

$$\begin{aligned} \mathbb{E}e^{-\rho t}\|\hat{X}(t)\|^2 \le& (k_2l_1+k_{11}^2)\mathbb{E}\int_0^t e^{-\bar{\rho}_1(t-s)-\rho s}\|\hat{Y}(s)\|^2ds \\ &+ (k_2l_2+k_{12}^2)\mathbb{E}\int_0^t e^{-\bar{\rho}_1(t-s)-\rho s}\|\hat{Z}(s)\|^2ds, \end{aligned} \tag{B.3}$$

where $\bar{\rho}_1 = \rho - 2\rho_1 - 2k_1 - k_2 l_1^{-1} - k_3 l_2^{-1} - k_9^2 - k_{10}^2$ *and* $\hat{\Phi} = \Phi_1 - \Phi_2$, $\Phi = X, Y, Z$. *Moreover,*

$$\begin{aligned}
&\mathbb{E}\int_0^T e^{-\rho t}\|\hat{X}(t)\|^2 dt \\
\leq &\frac{1 - e^{-\bar{\rho}_1 T}}{\bar{\rho}_1}\Big[(k_2 l_1 + k_{11}^2)\mathbb{E}\int_0^t e^{-\rho t}\|\hat{Y}(t)\|^2 dt \\
&+ (k_3 l_2 + k_{12}^2)\mathbb{E}\int_0^t e^{-\rho t}\|\hat{Z}(t)\|^2 dt\Big],
\end{aligned} \tag{B.4}$$

and

$$\begin{aligned}
e^{-\rho T}\mathbb{E}\|\hat{X}(T)\|^2 \leq (1 \vee e^{-\bar{\rho}_1 T})\Big[&(k_2 l_1 + k_{11}^2)\mathbb{E}\int_0^T e^{-\rho t}\|\hat{Y}(t)\|^2 dt \\
&+ (k_3 l_2 + k_{12}^2)\mathbb{E}\int_0^T e^{-\rho t}\|\hat{Z}(t)\|^2 dt\Big].
\end{aligned} \tag{B.5}$$

Specifically, if $\bar{\rho}_1 > 0$,

$$\begin{aligned}
e^{-\rho T}\mathbb{E}\|\hat{X}(T)\|^2 \leq &(k_2 l_1 + k_{11}^2)\mathbb{E}\int_0^T e^{-\rho t}\|\hat{Y}(t)\|^2 dt \\
&+ (k_3 l_2 + k_{12}^2)\mathbb{E}\int_0^T e^{-\rho t}\|\hat{Z}(t)\|^2 dt.
\end{aligned} \tag{B.6}$$

Proof. For any $\rho > 0$, applying Itô's formula to $e^{-\rho t}\|\hat{X}(t)\|^2$,

$$\begin{aligned}
&\mathbb{E}e^{-\rho t}\|\hat{X}(t)\|^2 + \rho\mathbb{E}\int_0^t e^{-\rho s}\|\hat{X}(s)\|^2 ds \\
=&2\mathbb{E}\int_0^t e^{-\rho s}\hat{X}(s)(b(s, X_1(s), \mathbb{E}[X_1(s)|\mathcal{F}_s^{W_0}], Y_1(s), Z_1(s)) \\
&\qquad - b(s, X_2(s), \mathbb{E}[X_2(s)|\mathcal{F}_s^{W_0}], Y_2(s), Z_2(s)))ds \\
&+ \mathbb{E}\int_0^t e^{-\rho s}(\sigma(s, X_1(s), \mathbb{E}[X_1(s)|\mathcal{F}_s^{W_0}], Y_1(s), Z_1(s)) \\
&\qquad - \sigma(s, X_2(s), \mathbb{E}[X_2(s)|\mathcal{F}_s^{W_0}], Y_2(s), Z_2(s)))^2 ds \\
\leq&\mathbb{E}\int_0^t e^{-\rho s}\Big[(2\rho_1 + 2k_1 + k_2 l_1^{-1} + k_3 l_2^{-1} + k_9^2 + k_{10}^2)\|\hat{X}(s)\|^2 \\
&\qquad + (k_2 l_1 + k_{11}^2)\|\hat{Y}(s)\|^2 + (k_3 l_2 + k_{12}^2)\|\hat{Z}(s)\|^2\Big] ds.
\end{aligned}$$

Similarly, applying Itô's formula to $e^{-\bar{\rho}_1(t-s)-\rho s}\|\hat{X}(s)\|^2$, we have (B.3). Integrating from 0 to T on both sides of (B.3) and noting that $\frac{1-e^{-\bar{\rho}_1(t-s)}}{\bar{\rho}_1} \le \frac{1-e^{-\bar{\rho}_1 T}}{\bar{\rho}_1}$, we have

$$\begin{aligned}
&\mathbb{E}\int_0^T e^{-\rho t}\|\hat{X}(t)\|^2 dt \\
\le&(k_2 l_1 + k_{11}^2)\frac{1-e^{-\bar{\rho}_1 T}}{\bar{\rho}_1}\mathbb{E}\int_0^T e^{-\rho s}\|\hat{Y}(s)\|^2 ds \\
&+ (k_3 l_2 + k_{12}^2)\frac{1-e^{-\bar{\rho}_1 T}}{\bar{\rho}_1}\mathbb{E}\int_0^T e^{-\rho s}\|\hat{Z}(s)\|^2 ds.
\end{aligned}$$

Letting $t = T$ in (B.3), we have (B.5). □

For any given $X \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$, the following BSDE has a unique solution:

$$\begin{aligned}
Y(t) = \int_t^T & f(s, X(s), \mathbb{E}[X(s)|\mathcal{F}_s^{W_0}], Y(s), \mathbb{E}[Y(s)|\mathcal{F}_s^{W_0}], \\
& Z(s), \mathbb{E}[Z(s)|\mathcal{F}_s^{W_0}])ds - \int_t^T Z(s)dW(s).
\end{aligned} \tag{B.7}$$

Thus, we can introduce another map $\mathcal{M}_2 : X \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n) \to (Y,Z) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{m\times(d+1)})$ by (B.7). Similarly, we have the following result:

Lemma B.2. *Let (Y_i, Z_i) be the solution of* (B.7) *corresponding to $X_i, i = 1,2$, respectively. Then for all $\rho \in \mathbb{R}$ and some constants $l_3, l_4, l_5, l_6 > 0$ such that*

$$\begin{aligned}
&\mathbb{E}e^{-\rho t}\|\hat{Y}(t)\|^2 + \bar{\rho}_2\mathbb{E}\int_t^T e^{-\rho s}\|\hat{Y}(s)\|^2 ds \\
&+ (1 - k_7 l_5 - k_8 l_6)\mathbb{E}\int_t^T e^{-\rho s}\|\hat{Z}(s)\|^2 ds \\
\le&(k_4 l_3 + k_5 l_4)\mathbb{E}\int_t^T e^{-\rho s}\|\hat{X}(s)\|^2 ds,
\end{aligned} \tag{B.8}$$

and

$$\begin{aligned}
&\mathbb{E}e^{-\rho t}\|\hat{Y}(t)\|^2 + (1 - k_7 l_5 - k_8 l_6)\mathbb{E}\int_t^T e^{-\rho s}\|\hat{Z}(s)\|^2 ds \\
\leq &(k_4 l_3 + k_5 l_4)\mathbb{E}\int_t^T e^{-\bar{\rho}_2(s-t)-\rho s}\|\hat{X}(s)\|^2 ds,
\end{aligned} \tag{B.9}$$

where $\bar{\rho}_2 = -\rho - 2\rho_2 - 2k_6 - k_4 l_3^{-1} - k_5 l_4^{-1} - k_7 l_5^{-1} - k_8 l_6^{-1}$*, and* $\hat{\Phi} = \Phi_1 - \Phi_2$, $\Phi = X, Y, Z$*. Moreover,*

$$\mathbb{E}\int_0^T e^{-\rho t}\|\hat{Y}(t)\|^2 dt \leq \frac{1 - e^{-\bar{\rho}_2 T}}{\bar{\rho}_2}(k_4 l_3 + k_5 l_4)\mathbb{E}\int_0^T e^{-\rho s}\|\hat{X}(s)\|^2 ds, \tag{B.10}$$

and

$$\begin{aligned}
&\mathbb{E}\int_0^T e^{-\rho t}\|\hat{Z}(t)\|^2 dt \\
\leq &\frac{(k_4 l_3 + k_5 l_4)(1 \vee e^{-\bar{\rho}_2 T})}{(1 - k_7 l_5 - k_8 l_6)(1 \wedge e^{-\bar{\rho}_2 T})}\mathbb{E}\int_0^T e^{-\rho s}\|\hat{X}(s)\|^2 ds.
\end{aligned} \tag{B.11}$$

Specifically, if $\bar{\rho}_2 > 0$*,*

$$\mathbb{E}\int_0^T e^{-\rho t}\|\hat{Z}(t)\|^2 dt \leq \frac{k_4 l_3 + k_5 l_4}{1 - k_7 l_5 - k_8 l_6}\mathbb{E}\int_0^T e^{-\rho s}\|\hat{X}(s)\|^2 ds. \tag{B.12}$$

Proof of Theorem 3.1: Define $\mathcal{M} := \mathcal{M}_2 \circ \mathcal{M}_1$, where $\mathcal{M}_1$ is defined by (B.1) and $\mathcal{M}_2$ is defined by (B.7). Thus, $\mathcal{M}$ is a mapping from $L^2_{\mathcal{F}}(0, T; \mathbb{R}^m) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m\times(d+1)})$ into itself. For $(U_i, V_i) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m\times(d+1)})$, let $X_i := \mathcal{M}_1(U_i, V_i)$ and $(Y_i, Z_i) := \mathcal{M}(U_i, V_i)$. Therefore,

$$
\begin{aligned}
&\mathbb{E}\int_0^T e^{-\rho t}\|Y_1(t)-Y_2(t)\|^2dt+\mathbb{E}\int_0^T e^{-\rho t}\|Z_1(t)-Z_2(t)\|^2dt\\
\leq&\Big[\frac{1-e^{-\bar{\rho}_2 T}}{\bar{\rho}_2}+\frac{1\vee e^{-\bar{\rho}_2 T}}{(1-k_7l_5-k_8l_6)(1\wedge e^{-\bar{\rho}_2 T})}\Big](k_4l_3+k_5l_4)\\
&\times\mathbb{E}\int_0^T e^{-\rho t}\|X_1(t)-X_2(t)\|^2dt\\
\leq&\Big[\frac{1-e^{-\bar{\rho}_2 T}}{\bar{\rho}_2}+\frac{1\vee e^{-\bar{\rho}_2 T}}{(1-k_7l_5-k_8l_6)(1\wedge e^{-\bar{\rho}_2 T})}\Big]\frac{1-e^{-\bar{\rho}_1 T}}{\bar{\rho}_1}\\
&\times(k_4l_3+k_5l_4)\Big[(k_2l_1+k_{11}^2)\mathbb{E}\int_0^T e^{-\rho t}\|U_1(t)-U_2(t)\|^2dt\\
&+(k_3l_2+k_{12}^2)\mathbb{E}\int_0^T e^{-\rho t}\|V_1(t)-V_2(t)\|^2dt\Big].
\end{aligned}
$$

Choosing suitable ρ, we get that $\mathcal{M}$ is a contraction mapping.

Furthermore, if $2\rho_1+2\rho_2<-2k_1-2k_6-2k_7^2-2k_8^2-k_9^2-k_{10}^2$, we can choose $\rho\in\mathbb{R}$, $0<k_7l_5<\frac{1}{2}$ and $0<k_8l_6<\frac{1}{2}$ and sufficient large l_1,l_2,l_3,l_4 such that

$$
\bar{\rho}_1>0,\qquad \bar{\rho}_2>0,\qquad 1-k_7l_5-k_8l_6>0.
$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}\int_0^T e^{-\rho t}\|Y_1(t)-Y_2(t)\|^2dt+\mathbb{E}\int_0^T e^{-\rho t}\|Z_1(t)-Z_2(t)\|^2dt\\
\leq&\Big[\frac{1}{\bar{\rho}_2}+\frac{1}{1-k_7l_5-k_8l_6}\Big]\frac{1}{\bar{\rho}_1}(k_4l_3+k_5l_4)\\
&\times\Big[(k_2l_1+k_{11}^2)\mathbb{E}\int_0^T e^{-\rho t}\|U_1(t)-U_2(t)\|^2dt\\
&\quad+(k_3l_2+k_{12}^2)\mathbb{E}\int_0^T e^{-\rho t}\|V_1(t)-V_2(t)\|^2dt\Big].
\end{aligned}
$$

The proof is complete. $\square$

Appendix C: Chapter 4

C.1 Proof of Lemma 4.2

Proof. We divide our proof into three parts:

- the equivalent relation: (i)' $\Longleftrightarrow$ (ii)',
- the inclusion relation: (i)' $\Longrightarrow$ (i); (ii) $\Longrightarrow$ (i),
- a counter example: (i) $\not\Longrightarrow$ (i)'; (ii) $\not\Longrightarrow$ (ii)',

We start with part one: (i)' $\Longleftrightarrow$ (ii)', and firstly we consider (ii)' $\Longrightarrow$ (i)'. By applying Itô formula to $\langle \mathbf{Px}, \mathbf{x} \rangle$ we have:

$$\begin{aligned}\langle \mathbf{Gx}(T), \mathbf{x}(T) \rangle = \int_0^T & \left\langle \left(\dot{\mathbf{P}} + \mathbf{PA} + \mathbf{A}^T\mathbf{P} + \mathbf{C}^T\mathbf{PC} \right) \mathbf{x}, \mathbf{x} \right\rangle \\ & + 2 \left\langle \left(\mathbf{B}^T\mathbf{P} + \mathbf{D}^T\mathbf{PC} \right) \mathbf{x}, \mathbf{u} \right\rangle + \left\langle \mathbf{D}^T\mathbf{PDu}, \mathbf{u} \right\rangle dt.\end{aligned}$$

Combined with (4.7), we have:

$$\begin{aligned}\mathcal{J}'(0, \mathbf{u}(\cdot)) = \mathbb{E} \int_0^T & \left\langle \left(\dot{\mathbf{P}} + \mathbf{PA} + \mathbf{A}^T\mathbf{P} + \mathbf{C}^T\mathbf{PC} + \mathbf{Q} \right) \mathbf{x}, \mathbf{x} \right\rangle \\ & + \left\langle \left(\mathbf{R} + \mathbf{D}^T\mathbf{PD} \right) \mathbf{u}, \mathbf{u} \right\rangle + 2 \left\langle \left(\mathbf{B}^T\mathbf{P} + \mathbf{D}^T\mathbf{PC} \right) \mathbf{x}, \mathbf{u} \right\rangle dt.\end{aligned}$$

By (RE0), it holds that

$$\mathcal{J}'(0,\mathbf{u}(\cdot))=\mathbb{E}\int_0^T \left\langle \left(\mathbf{PB}+\mathbf{C}^T\mathbf{PD}\right)\mathcal{R}_0\left(\mathbf{B}^T\mathbf{P}+\mathbf{D}^T\mathbf{PC}\right)\mathbf{x},\mathbf{x}\right\rangle$$
$$+\left\langle\left(\mathbf{R}+\mathbf{D}^T\mathbf{PD}\right)\mathbf{u},\mathbf{u}\right\rangle+2\left\langle\left(\mathbf{PB}+\mathbf{C}^T\mathbf{PD}\right)\mathbf{u},\mathbf{x}\right\rangle dt.$$

Moreover, we also have:

$$\begin{aligned}
&\mathcal{R}_0-\mathcal{R}_0\left(\mathbf{R}+\mathbf{D}^T\mathbf{PD}\right)\mathcal{R}_0\\
&=\mathbf{V}\Big[(\mathbf{V}^T\left(\mathbf{R}+\mathbf{D}^T\mathbf{PD}\right)\mathbf{V})^{-1}-(\mathbf{V}^T\left(\mathbf{R}+\mathbf{D}^T\mathbf{PD}\right)\mathbf{V})^{-1}\mathbf{V}^T\left(\mathbf{R}+\mathbf{D}^T\mathbf{PD}\right)\mathbf{V}\\
&\qquad\times(\mathbf{V}^T\left(\mathbf{R}+\mathbf{D}^T\mathbf{PD}\right)\mathbf{V})^{-1}\Big]\mathbf{V}^T\\
&=\mathbf{V}\left[\left(\mathbf{R}+\mathbf{D}^T\mathbf{PD}\right)^{-1}(I-\mathbf{V}^T\left(\mathbf{R}+\mathbf{D}^T\mathbf{PD}\right)\mathbf{V}\left(\mathbf{R}+\mathbf{D}^T\mathbf{PD}\right)^{-1})\right]\mathbf{V}^T\\
&=\mathbf{V}\left[(\mathbf{V}^T\left(\mathbf{R}+\mathbf{D}^T\mathbf{PD}\right)\mathbf{V})^{-1}(I-I)\right]\mathbf{V}^T=0.
\end{aligned}$$

Then we have:

$$\mathcal{J}'(0,\mathbf{u}(\cdot))=\mathbb{E}\int_0^T \left\langle \left(\mathbf{PB}+\mathbf{C}^T\mathbf{PD}\right)\mathcal{R}_0\left(\mathbf{R}+\mathbf{D}^T\mathbf{PD}\right)\mathcal{R}_0\left(\mathbf{B}^T\mathbf{P}+\mathbf{D}^T\mathbf{PC}\right)\mathbf{x},\mathbf{x}\right\rangle$$
$$+\left\langle\left(\mathbf{R}+\mathbf{D}^T\mathbf{PD}\right)\mathbf{u},\mathbf{u}\right\rangle+2\left\langle\left(\mathbf{B}^T\mathbf{P}+\mathbf{D}^T\mathbf{PC}\right)\mathbf{x},\mathbf{u}\right\rangle dt.$$

For any given $(t,\omega)\in[0,T]\times\Omega$ we have $\mathbf{u}(t,\omega)\in\mathbf{\Lambda}\subseteq\mathbb{R}^{2m}$. Then there exists some vectors $\mathbf{v}'\in\mathbb{R}^{2m'}$ such that $\mathbf{V}\mathbf{v}'=\mathbf{u}(t,\omega)$. Thus, it follows that

$$\begin{aligned}
&\left\langle\left(\mathbf{PB}+\mathbf{C}^T\mathbf{PD}\right)\mathbf{u},\mathbf{x}\right\rangle=\left\langle\left(\mathbf{PB}+\mathbf{C}^T\mathbf{PD}\right)\mathbf{V}\mathbf{v}',\mathbf{x}\right\rangle\\
&=\left\langle\left(\mathbf{PB}+\mathbf{C}^T\mathbf{PD}\right)\mathbf{V}(\mathbf{V}^T\left(\mathbf{R}+\mathbf{D}^T\mathbf{PD}\right)\mathbf{V})^{-1}(\mathbf{V}^T\left(\mathbf{R}+\mathbf{D}^T\mathbf{PD}\right)\mathbf{V})\mathbf{v}',\mathbf{x}\right\rangle\\
&=\left\langle\left(\mathbf{PB}+\mathbf{C}^T\mathbf{PD}\right)\mathcal{R}_0\left(\mathbf{R}+\mathbf{D}^T\mathbf{PD}\right)\mathbf{V}\mathbf{v}',\mathbf{x}\right\rangle=\left\langle\left(\mathbf{PB}+\mathbf{C}^T\mathbf{PD}\right)\mathcal{R}_0\left(\mathbf{R}+\mathbf{D}^T\mathbf{PD}\right)\mathbf{u},\mathbf{x}\right\rangle.
\end{aligned}$$

Then we have:

$$\mathcal{J}'(0,\mathbf{u}(\cdot))=\mathbb{E}\int_0^T \left\langle\left(\mathbf{R}+\mathbf{D}^T\mathbf{PD}\right)\left[\mathbf{u}+\mathcal{R}_0\left(\mathbf{B}^T\mathbf{P}+\mathbf{D}^T\mathbf{PC}\right)\mathbf{x}\right],\mathbf{u}+\mathcal{R}_0\left(\mathbf{B}^T\mathbf{P}+\mathbf{D}^T\mathbf{PC}\right)\mathbf{x}\right\rangle dt.$$

Noting that $\mathcal{R}_0 = \mathbf{V}\left[\mathbf{V}^T\left(\mathbf{R}+\mathbf{D}^T\mathbf{P}\mathbf{D}\right)\mathbf{V}\right]^{-1}\mathbf{V}^T$, then $\mathcal{R}_0(t)\big(\mathbf{B}^T(t)\mathbf{P}(t) + \mathbf{D}^T(t)\mathbf{P}(t)\mathbf{C}(t)\big)\mathbf{x}(t,\omega) \in \mathbf{\Lambda}$ for any given $(t,\omega) \in [0,T]\times\Omega$. If $\mathbf{R}(t) + \mathbf{D}^T(t)\mathbf{P}(t)\mathbf{D}(t) \gg 0$ on $\mathbf{\Lambda}$, a.e. $t\in[0,T]$, then by Lemma 2.3 in [58], it holds that $\mathcal{J}'(0,\mathbf{u}(\cdot)) \geq \varepsilon\|\mathbf{u}\|_{L^2}^2$ for some constant $\varepsilon > 0$, and $\mathbf{u}(\cdot)\mapsto\mathcal{J}'(0;\mathbf{u}(\cdot))$ is uniformly convex on $\mathcal{U}^{\mathbf{\Lambda}}$.

Secondly we consider (i)' $\Longrightarrow$ (ii)':
This part of the proof is similar to that of Theorem 4.5 in [58]. Here we just briefly sketch some key points. Consider the following Lyapunov equation:

$$\begin{cases}\dot{\mathbf{K}}+\mathbf{K}(\mathbf{A}+\mathbf{B}\mathbf{V}\Theta)+(\mathbf{A}+\mathbf{B}\mathbf{V}\Theta)^T\mathbf{K}+(\mathbf{C}+\mathbf{D}\mathbf{V}\Theta)^T\mathbf{K}(\mathbf{C}+\mathbf{D}\mathbf{V}\Theta)+\Theta^T\mathbf{V}^T\mathbf{R}\mathbf{V}\Theta+\mathbf{Q}=0,\\ \mathbf{K}(T)=\mathbf{G},\end{cases}$$

where $\Theta(\cdot)\in L^2(0,T;\mathbb{R}^{2m\times 2n})$. Then we have:

$$\begin{aligned}\mathcal{J}'(0,\mathbf{V}\Theta\tilde{\mathbf{x}}+\mathbf{u}(\cdot))=\mathbb{E}\int_0^T &\left\langle\left(\mathbf{R}+\mathbf{D}^T\mathbf{K}\mathbf{D}\right)\mathbf{u},\mathbf{u}\right\rangle\\ &+2\left\langle\left[\mathbf{B}^T\mathbf{K}+\mathbf{D}^T\mathbf{K}\mathbf{C}+\left(\mathbf{R}+\mathbf{D}^T\mathbf{K}\mathbf{D}\right)\mathbf{V}\Theta\right]\tilde{\mathbf{x}},\mathbf{u}\right\rangle dt,\end{aligned}$$

for any $\mathbf{u}(\cdot)\in\mathcal{U}^{\mathbf{\Lambda}}$. $\tilde{\mathbf{x}}$ is the solution of

$$d\tilde{\mathbf{x}}=[(\mathbf{A}+\mathbf{B}\mathbf{V}\Theta)\tilde{\mathbf{x}}+\mathbf{B}\mathbf{u}]\,dt+[(\mathbf{C}+\mathbf{D}\mathbf{V}\Theta)\tilde{\mathbf{x}}+\mathbf{B}\mathbf{u}]\,dW,\quad \tilde{\mathbf{x}}(0)=0.$$

Noting that $(\mathbf{V}\Theta\tilde{\mathbf{x}}+\mathbf{u})(\cdot)\in\mathcal{U}^{\mathbf{\Lambda}}$ and by the uniform convexity of $\mathbf{u}(\cdot)\mapsto\mathcal{J}'(0;\mathbf{u}(\cdot))$ on $\mathcal{U}^{\mathbf{\Lambda}}$, it holds that

$$\mathbb{E}\int_0^T\left\langle\left(\mathbf{R}+\mathbf{D}^T\mathbf{K}\mathbf{D}-\varepsilon I\right)\mathbf{u},\mathbf{u}\right\rangle+2\left\langle\left[\mathbf{B}^T\mathbf{K}+\mathbf{D}^T\mathbf{K}\mathbf{C}+\left(\mathbf{R}+\mathbf{D}^T\mathbf{K}\mathbf{D}-\varepsilon I\right)\mathbf{V}\Theta\right]\tilde{\mathbf{x}},\mathbf{u}\right\rangle dt\geq 0.$$

Let $\mathbf{u}_0 \in \mathbf{\Lambda} \subseteq \mathbb{R}^{2m}$ and take $\mathbf{u} = \mathbf{u}_0 \mathbb{1}_{[s,s+h]}$ with $0 \le s < s+h \le T$. Then by Lebesgue differentiation theorem, for some constant $\varepsilon > 0$, it holds that

$$\left\langle \left(\mathbf{R}(t) + \mathbf{D}^T(t)\mathbf{K}(t)\mathbf{D}(t) - \varepsilon I\right) \mathbf{u}_0, \mathbf{u}_0 \right\rangle \ge 0, \quad \text{a.e. } t \in [0,T], \quad \forall \mathbf{u}_0 \in \mathbf{\Lambda}.$$

Thus, $\mathbf{R}(t) + \mathbf{D}^T(t)\mathbf{K}(t)\mathbf{D}(t) \gg 0$ on $\mathbf{\Lambda}$ a.e. $t \in [0,T]$. By Proposition 4.1 in [58], we also have $\langle \mathbf{K}(t)\mathbf{x}_0, \mathbf{x}_0 \rangle \ge \varepsilon \|\mathbf{x}_0\|^2$ for some constant $\varepsilon > 0$ and $\forall (t, \mathbf{x}_0) \in [0,T] \times \mathbb{R}^{2m}$. Then we consider the following Lyapunov equation sequence with index $\alpha \ge 0$:

$$\begin{cases} \dot{\mathbf{K}}_0 + \dot{\mathbf{K}}_0 \mathbf{A} + \mathbf{A}^T + \mathbf{C}^T \mathbf{K}_0 \mathbf{C} + \mathbf{Q} = 0, \quad \mathbf{K}_0(T) = \mathbf{G}, \\ \dot{\mathbf{K}}_{\alpha+1} + \mathbf{K}_{\alpha+1}(\mathbf{A} + \mathbf{B}\mathbf{V}\Theta_\alpha) + (\mathbf{A} + \mathbf{B}\mathbf{V}\Theta_\alpha)^T \mathbf{K}_{\alpha+1} + (\mathbf{C} + \mathbf{D}\mathbf{V}\Theta_\alpha)^T \mathbf{K}_{\alpha+1}(\mathbf{C} + \mathbf{D}\mathbf{V}\Theta_\alpha) \\ + \Theta_\alpha^T \mathbf{V}^T \mathbf{R} \mathbf{V} \Theta_\alpha + \mathbf{Q} = 0, \quad \mathbf{K}_{\alpha+1}(T) = \mathbf{G}, \\ \Theta_\alpha = -\left[\mathbf{V}^T \left(\mathbf{R} + \mathbf{D}^T \mathbf{K}_\alpha \mathbf{D}\right) \mathbf{V}\right] \mathbf{V}^T \left(\mathbf{B}^T \mathbf{K}_\alpha + \mathbf{D}^T \mathbf{K}_\alpha \mathbf{C}\right), \\ \mathcal{R}_0^\alpha = \mathbf{V} \left[\mathbf{V}^T \left(\mathbf{R} + \mathbf{D}^T \mathbf{K}_\alpha \mathbf{D}\right) \mathbf{V}\right]^{-1} \mathbf{V}^T. \end{cases}$$

By noting that $\mathcal{R}_0^\alpha = \mathcal{R}_0^\alpha \left(\mathbf{R} + \mathbf{D}^T \mathbf{K}_\alpha \mathbf{D}\right) \mathcal{R}_0^\alpha$, we have:

$$\begin{cases} \lim_{\alpha \to \infty} \mathbf{K}_\alpha = \mathbf{P}, \quad \lim_{\alpha \to \infty} \mathbf{R} + \mathbf{D}^T \mathbf{K}_\alpha \mathbf{D} = \mathbf{R} + \mathbf{D}^T \mathbf{P} \mathbf{D}, \quad \lim_{\alpha \to \infty} \mathcal{R}_0^\alpha = \mathcal{R}_0, \\ \lim_{\alpha \to \infty} \Theta_\alpha = -\left[\mathbf{V}^T \left(\mathbf{R} + \mathbf{D}^T \mathbf{P} \mathbf{D}\right) \mathbf{V}\right] \left(\mathbf{B}^T \mathbf{P} + \mathbf{D}^T \mathbf{P} \mathbf{C}\right), \\ \mathbf{R}(t) + \mathbf{D}^T(t) \mathbf{P}(t) \mathbf{D}(t) \gg 0 \text{ on } \mathbf{\Lambda} \text{ a.e. } t \in [0,T], \end{cases}$$

and this leads to (ii)'.

For part two, we consider (i)' $\Longrightarrow$ (i) first: Apparently, $\binom{u - \mathbb{E}u}{\mathbb{E}u} \in \mathcal{U}^{\mathbf{\Lambda}}$ for $\forall u \in \mathcal{U}^{\Lambda}$. Then for $\forall u \in \mathcal{U}^{\Lambda}$, if (i)' holds, it holds that $\mathcal{J}'(0, \binom{u - \mathbb{E}u}{\mathbb{E}u}) \ge \varepsilon \left\| \binom{u - \mathbb{E}u}{\mathbb{E}u} \right\|_{L^2}^2 =$

$\varepsilon\|u\|_{L^2}^2$. Note that system (4.5) is of the form as (4.7). Thus, $\mathcal{J}(0,u) = \mathcal{J}'(0, \binom{u-\mathbb{E}u}{\mathbb{E}u}) \geq \varepsilon\|u\|_{L^2}^2$ and (i) holds.

Secondly, we consider (ii) $\Longrightarrow$ (i): By letting $\xi_0 = 0$, the dynamics of $x-\mathbb{E}x$ is given by:

$$\begin{cases} d(x-\mathbb{E}x) = \Big[A(x-\mathbb{E}x)+B(u-\mathbb{E}u)\Big]dt+\Big[C(x-\mathbb{E}x)+D(u-\mathbb{E}u)+\mathcal{C}\mathbb{E}x+\mathcal{D}\mathbb{E}u\Big]dW, \\ (x-\mathbb{E}x)(0) = 0. \end{cases}$$

The dynamics of $\mathbb{E}x$ is given by:

$$d\mathbb{E}x = (\mathcal{A}\mathbb{E}x+\mathcal{B}\mathbb{E}u)\,dt, \quad \mathbb{E}x(0) = 0.$$

Apply the Itô formula and chain rule to $\langle P_1(x-\mathbb{E}x),(x-\mathbb{E}x)\rangle$ and $\langle P_2\mathbb{E}x, \mathbb{E}x\rangle$, and we have:

$$\begin{aligned}
&\mathbb{E}\left\{\|(x-\mathbb{E}x)(T)\|_G^2\right\} \\
=&\mathbb{E}\int_0^T \Big\langle\Big(\dot{P}_1+P_1A+A^TP_1+C^TP_1C\Big)(x-\mathbb{E}x),(x-\mathbb{E}x)\Big\rangle\,dt \\
&+2\mathbb{E}\int_0^T \big\langle\big(P_1B+C^TP_1D\big)(u-\mathbb{E}u),(x-\mathbb{E}x)\big\rangle\,dt+\mathbb{E}\int_0^T \big\langle\big(D^TP_1D\big)(u-\mathbb{E}u),(u-\mathbb{E}u)\big\rangle\,dt, \\
&+\mathbb{E}\int_0^T\big\langle\mathcal{C}^TP_1\mathcal{C}\mathbb{E}x,\mathbb{E}x\big\rangle\,dt+\mathbb{E}\int_0^T\big\langle\mathcal{D}^TP_1\mathcal{D}\mathbb{E}u,\mathbb{E}u\big\rangle\,dt+2\mathbb{E}\int_0^T\big\langle\mathcal{C}^TP_1\mathcal{D}\mathbb{E}u,\mathbb{E}x\big\rangle\,dt, \\
&\mathbb{E}\left\{\|\mathbb{E}x(T)\|_{\widehat{G}}^2\right\} = \mathbb{E}\int_0^T\Big\langle\Big(\dot{P}_2+P_2\mathcal{A}+\mathcal{A}^TP_2\Big)\mathbb{E}x,\mathbb{E}x\Big\rangle\,dt+2\mathbb{E}\int_0^T\langle(P_2\mathcal{B})\,\mathbb{E}u,\mathbb{E}x\rangle\,dt.
\end{aligned} \tag{C.1}$$

Plugging (C.1) into $\mathcal{J}(0;u(\cdot))$ results that

$$\begin{aligned}
\mathcal{J}(0;u(\cdot))=&\frac{1}{2}\mathbb{E}\int_0^T\left\langle\left(\dot{P}_1+P_1A+A^TP_1+C^TP_1C+Q\right)(x-\mathbb{E}x),(x-\mathbb{E}x)\right\rangle dt\\
&+\frac{1}{2}\mathbb{E}\int_0^T\left\langle\left(\dot{P}_2+P_2\mathcal{A}+\mathcal{A}^TP_2+\widehat{Q}+\mathcal{C}^TP_1\mathcal{C}\right)\mathbb{E}x,\mathbb{E}x\right\rangle dt\\
&+\mathbb{E}\int_0^T\left\langle\left(P_1B+C^TP_1D\right)(u-\mathbb{E}u),(x-\mathbb{E}x)\right\rangle dt+\mathbb{E}\int_0^T\left\langle\left(P_2\mathcal{B}+\mathcal{C}^TP_1\mathcal{D}\right)\mathbb{E}u,\mathbb{E}x\right\rangle dt\\
&+\frac{1}{2}\mathbb{E}\int_0^T\left\langle\left(D^TP_1D+R\right)(u-\mathbb{E}u),(u-\mathbb{E}u)\right\rangle dt+\frac{1}{2}\mathbb{E}\int_0^T\left\langle(R+\mathcal{D}^TP_1\mathcal{D})\mathbb{E}u,\mathbb{E}u\right\rangle dt.
\end{aligned}$$

By (RE1) and (RE2) we have:

$$\begin{aligned}
&2\mathcal{J}(0;u(\cdot))\\
=&\mathbb{E}\int_0^T\left\langle(P_2\mathcal{B}+\mathcal{C}^TP_1\mathcal{D})\mathcal{R}_2(\mathcal{B}^TP_2+\mathcal{D}^TP_1\mathcal{C})\mathbb{E}x,\mathbb{E}x\right\rangle dt+2\mathbb{E}\int_0^T\left\langle\left(P_2\mathcal{B}+\mathcal{C}^TP_1\mathcal{D}\right)\mathbb{E}u,\mathbb{E}x\right\rangle dt\\
&+\mathbb{E}\int_0^T\left\langle(\mathcal{D}^TP_1\mathcal{D}+R)\mathbb{E}u,\mathbb{E}u\right\rangle dt+2\mathbb{E}\int_0^T\left\langle\left(P_1B+C^TP_1D\right)(u-\mathbb{E}u),(x-\mathbb{E}x)\right\rangle dt\\
&+\mathbb{E}\int_0^T\left\langle(P_1B+C^TP_1D)\mathcal{R}_1(B^TP_1+D^TP_1C)(x-\mathbb{E}x),(x-\mathbb{E}x)\right\rangle dt\\
&+\mathbb{E}\int_0^T\left\langle(D^TP_1D+R)(u-\mathbb{E}u),(u-\mathbb{E}u)\right\rangle dt.
\end{aligned}\tag{C.2}$$

Moreover, similar to part (ii)' $\Longrightarrow$ (i)' , we have:

$$\begin{cases}
\mathcal{R}_2-\mathcal{R}_2(\mathcal{D}^TP_1\mathcal{D}+R)\mathcal{R}_2=\mathcal{R}_1-\mathcal{R}_1(D^TP_1D+R)\mathcal{R}_1=0,\\
\left\langle\left(P_2\mathcal{B}+\mathcal{C}^TP_1\mathcal{D}\right)\mathbb{E}u,\mathbb{E}x\right\rangle=\left\langle\left(P_2\mathcal{B}+\mathcal{C}^TP_1\mathcal{D}\right)\mathcal{R}_2(\mathcal{D}^TP_1\mathcal{D}+R)\mathbb{E}u,\mathbb{E}x\right\rangle,\\
\left\langle\left(P_1B+C^TP_1D\right)(u-\mathbb{E}u),(x-\mathbb{E}x)\right\rangle=\left\langle\left(P_1B+C^TP_1D\right)\mathcal{R}_1(D^TP_1D+R)\mathbb{E}u,\mathbb{E}x\right\rangle.
\end{cases}$$

Combined with (C.2), it holds that

$$
\begin{aligned}
&2\mathcal{J}(0;u(\cdot))\\
=&\mathbb{E}\int_0^T \left\langle(\mathcal{D}^T P_1\mathcal{D}+R)\left[\mathbb{E}u+\mathcal{R}_2(\mathcal{B}^T P_2+\mathcal{D}^T P_1\mathcal{C})\mathbb{E}x\right], \mathbb{E}u+\mathcal{R}_2(\mathcal{B}^T P_2+\mathcal{D}^T P_1\mathcal{C})\mathbb{E}x\right\rangle dt\\
&+\mathbb{E}\int_0^T \Big\langle(D^T P_1 D+R)\left[(u-\mathbb{E}u)+\mathcal{R}_1(B^T P_2+D^T P_1 C)(x-\mathbb{E}x)\right],\\
&\qquad\qquad (u-\mathbb{E}u)+\mathcal{R}_1(B^T P_2+D^T P_1 C)(x-\mathbb{E}x)\Big\rangle dt.
\end{aligned}
\tag{C.3}
$$

Next we can put the dynamics of $x-\mathbb{E}x$, $\mathbb{E}x$ together as follows:

$$
\begin{cases}
d\begin{pmatrix} x-\mathbb{E}x\\ \mathbb{E}x\end{pmatrix}=\left[\begin{pmatrix} A & 0\\ 0 & \mathcal{A}\end{pmatrix}\begin{pmatrix} x-\mathbb{E}x\\ \mathbb{E}x\end{pmatrix}+\begin{pmatrix} B & 0\\ 0 & \mathcal{B}\end{pmatrix}\begin{pmatrix} u-\mathbb{E}u\\ \mathbb{E}u\end{pmatrix}\right]dt+\left[\begin{pmatrix} C & \mathcal{C}\\ 0 & 0\end{pmatrix}\begin{pmatrix} x-\mathbb{E}x\\ \mathbb{E}x\end{pmatrix}+\begin{pmatrix} D & \mathcal{D}\\ 0 & 0\end{pmatrix}\begin{pmatrix} u-\mathbb{E}u\\ \mathbb{E}u\end{pmatrix}\right]dW,\\
\begin{pmatrix} x-\mathbb{E}x\\ \mathbb{E}x\end{pmatrix}(0)=\begin{pmatrix} 0\\ 0\end{pmatrix}.
\end{cases}
$$

Then for some constant $\varepsilon>0$ and any $\begin{pmatrix} \Theta_1 & 0\\ 0 & \Theta_2\end{pmatrix}$, where $\Theta_1,\Theta_2\in L^2(0,T;\mathbb{R}^{m\times n})$, by Lemma 2.3 in [58], it holds that

$$
\|(u-\mathbb{E}u)-\Theta_1(x-\mathbb{E}x)\|_{L^2}^2+\|\mathbb{E}u-\Theta_1\mathbb{E}x\|_{L^2}^2=\left\|\begin{pmatrix} u-\mathbb{E}u\\ \mathbb{E}u\end{pmatrix}-\begin{pmatrix} \Theta_1 & 0\\ 0 & \Theta_2\end{pmatrix}\begin{pmatrix} x-\mathbb{E}x\\ \mathbb{E}x\end{pmatrix}\right\|_{L^2}^2\geq\varepsilon\left\|\begin{pmatrix} u-\mathbb{E}u\\ \mathbb{E}u\end{pmatrix}\right\|_{L^2}^2.
$$

Moreover, we also have:

$$
\begin{aligned}
\left\|\begin{pmatrix} u-\mathbb{E}u\\ \mathbb{E}u\end{pmatrix}\right\|_{L^2}^2=&\mathbb{E}\int_0^T\langle u-\mathbb{E}u,u-\mathbb{E}u\rangle\,dt+\mathbb{E}\int_0^T\langle\mathbb{E}u,\mathbb{E}u\rangle\,dt\\
=&\mathbb{E}\int_0^T\langle u,u\rangle-2\langle u,\mathbb{E}u\rangle+\langle\mathbb{E}u,\mathbb{E}u\rangle+\langle\mathbb{E}u,\mathbb{E}u\rangle\,dt\\
=&\mathbb{E}\int_0^T\langle u,u\rangle\,dt=\|u\|_{L^2}^2.
\end{aligned}
$$

Note that

$$\begin{cases} \mathcal{R}_2(t)\left[\mathcal{B}^T(t)P_2(t)+\mathcal{D}^T(t)P_1(t)\mathcal{C}(t)\right]\mathbb{E}x(t,w)\in\Lambda \text{ for } \forall(t,w)\in[0,T]\times\Omega, \\ \mathcal{R}_1(t)\left[B^T(t)P_2(t)+D^T(t)P_1(t)C(t)\right](x-\mathbb{E}x)(t,w)\in\Lambda \text{ for } \forall(t,w)\in[0,T]\times\Omega, \\ \mathcal{D}^T(t)P_1(t)\mathcal{D}(t)+R(t),\ D^T(t)P_1(t)D(t)+R(t)\gg 0 \text{ on } \Lambda, \text{ a.e. } t\in[0,T]. \end{cases}$$

Thus, it follows that $\mathcal{J}(0;u) \geq \varepsilon(\|\mathbb{E}u\|_{L^2}^2+\|u-\mathbb{E}u\|_{L^2}^2) = \varepsilon\|u\|_{L^2}^2$ for some constant $\varepsilon>0$ and $\forall u\in\mathcal{U}^\Lambda$. This leads to (i).

For part three, we cook up the following example to prove (ii) $\not\Longrightarrow$ (ii)':

Example C.1. *We let* $n=m=2$ *and* $\Lambda=\mathbb{R}^2$, $\mathbf{\Lambda}=\mathbb{R}^4$. *Thus* $V=\mathrm{diag}(1,1)$ *and* $\mathbf{V}=\mathrm{diag}(1,1,1,1)$. *Moreover, the coefficients are given as follows:*

$$A=\begin{pmatrix}-0.7 & 0.2\\ -0.2 & -0.4\end{pmatrix}, \bar{A}=\begin{pmatrix}-0.4 & -0.3\\ -0.9 & 0.1\end{pmatrix}, B=\begin{pmatrix}0.1 & 0.9\\ -0.5 & 0.4\end{pmatrix}, \bar{B}=\begin{pmatrix}-0.8 & -0.3\\ 0.8 & -0.6\end{pmatrix}, C=\begin{pmatrix}0.4 & -0.1\\ 0.2 & 0.4\end{pmatrix},$$

$$\bar{C}=\begin{pmatrix}-0.3 & 0.3\\ -0.2 & 0\end{pmatrix}, D=\begin{pmatrix}-0.7 & -0.9\\ 0.8 & 0.3\end{pmatrix}, \bar{D}=\begin{pmatrix}0.6 & 0\\ 0.4 & 0.6\end{pmatrix}, R=\begin{pmatrix}0.2 & -0.3\\ -0.3 & 0.3\end{pmatrix}, Q=\begin{pmatrix}-0.3 & -0.4\\ -0.4 & 0\end{pmatrix},$$

$$G=\begin{pmatrix}-0.1 & -0.4\\ -0.4 & 0.2\end{pmatrix}, \Gamma_1=\begin{pmatrix}0.5 & 0.3\\ -0.6 & -0.9\end{pmatrix}, \Gamma_2=\begin{pmatrix}0.1 & 0.8\\ 0 & 0.4\end{pmatrix},$$

and time interval is $[0,1]$. *Then such coefficients satisfy* (A4.1)-(A4.2). *Moreover*

$$\widehat{Q}=(I-\Gamma_1)^TQ(I-\Gamma_1)=\begin{pmatrix}-0.315 & -0.263\\ -0.263 & 0.429\end{pmatrix},$$

$$\widehat{G}=(I-\Gamma_2)^TG(I-\Gamma_2)=\begin{pmatrix}-0.082 & -0.144\\ -0.144 & 0.392\end{pmatrix}.$$

Note that all the coefficients here are constants. By [62], if (RE1), (RE2) *and* (RE0) *are solvable, then their solutions are all unique. Thus, by solving* (RE1), (RE2) *and* (RE0) *we have the trajectories of all components of* $P_1(t)$*,* $P_2(t)$ *and* $\mathbf{P}(t)$*.*

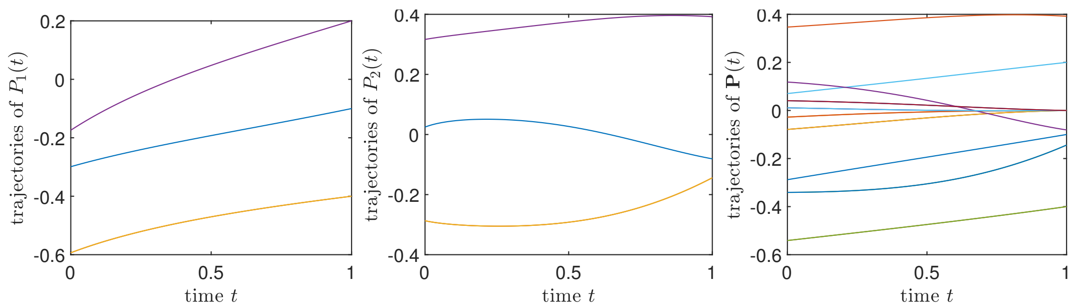


Figure C.1: the trajectories of all components of P_1, P_2 and $\mathbf{P}$

(RE1)*,* (RE2) *and* (RE0) *are all uniquely solvable on* $[0, 1]$*. Moreover, we also have the trajectories of the eigenvalues of* $\mathcal{D}^T P_1 \mathcal{D}+R$ *and* $D^T P_1 D+R$*.*

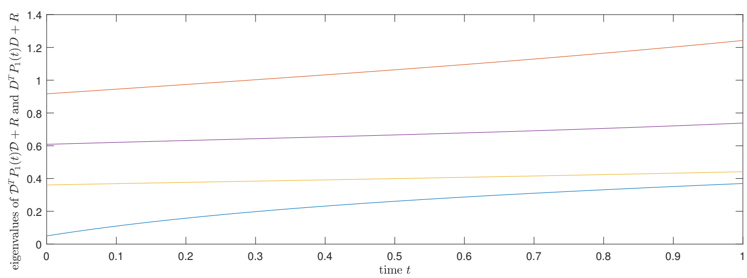


Figure C.2: the trajectories of the eigenvalues of $\mathcal{D}^T P_1 \mathcal{D}+R$ and $D^T P_1 D+R$

Thus, by letting $\varepsilon = 0.04$*, we have* $\mathcal{D}^T P_1(t)\mathcal{D}+R$*,* $D^T P_1(t)D+R \geq \varepsilon I$*,* $\forall t \in [0, 1]$*, and (ii) holds. However for* (RE0)*,* $\mathbf{R}+\mathbf{D}^T\mathbf{P}(t)\mathbf{D} \gg 0$ *does not always hold. Through the following graph, we can see that the minimum eigenvalue of* $\mathbf{R}+\mathbf{D}^T\mathbf{P}(t)\mathbf{D}$ *is always negative on* $[0, 1]$ *and* $\mathbf{R}+\mathbf{D}^T\mathbf{P}(t)\mathbf{D}$ *is indefinite.*

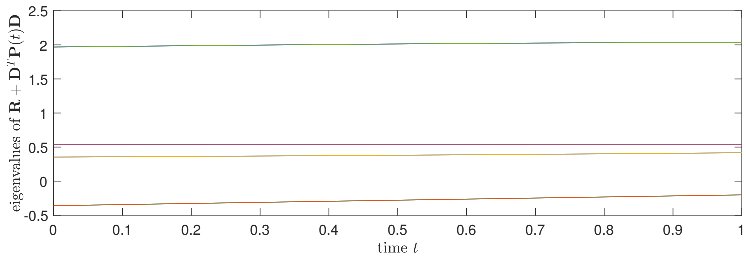


Figure C.3: the trajectories of the eigenvalues of $\mathbf{R}+\mathbf{D}^T\mathbf{P}(t)\mathbf{D}$

Thus, through the counter example above, we see that (ii) $\not\Longrightarrow$ (ii)'.

(i) $\not\Longrightarrow$ (i)': If (i) $\Longrightarrow$ (i)' holds, then by (ii) $\Longrightarrow$ (i) $\Longrightarrow$ (i)' $\Longrightarrow$ (ii)', we have (ii) $\Longrightarrow$ (ii)' which leads to a contradiction. $\square$

C.2 Proof of Proposition 4.2

Proof. We divide our proof into 3 parts: (i) The optimal pair of (**MFC-c**) is the adapted solution of (H1). (ii) The adapted solution of (H1) is the optimal pair of (**MFC-c**). (iii) (H1) is unique solvable.

(i) Under (A4.1)-(A4.3), (**MFC-c**) admits a unique optimal pair $(\bar{x}, \bar{u})$ by Theorem 4.1. Then by [85, 86], the following MF-BSDE:

$$\begin{cases} dk = -\left(Q\bar{x} - (Q\Gamma_1 + \Gamma_1^T Q - \Gamma_1^T Q\Gamma_1)\mathbb{E}\bar{x} + A^T k + \bar{A}^T\mathbb{E}k + C^T\zeta + \bar{C}^T\mathbb{E}\zeta\right) dt + \zeta dW, \\ k(T) = G\bar{x}(T) - (G\Gamma_2 + \Gamma_2^T G - \Gamma_2^T G\Gamma_2)\mathbb{E}\bar{x}(T), \end{cases} \tag{C.4}$$

admits a unique adapted solution (k, ζ). Thus, we mainly desire to verify:

$$V^T\left(B^T k + \bar{B}^T\mathbb{E}k + D^T\zeta + \bar{D}^T\mathbb{E}\zeta + R\bar{u}\right) = 0, \text{ a.e. } t \in [0, T], \ \mathbb{P} - a.s.. \tag{C.5}$$

Since Λ is a linear subspace, for any $u \in \mathcal{U}^\Lambda$ then maximum principle reads as the following form:

$$\left\langle R\bar{u}+B^T k+\bar{B}^T\mathbb{E}k+D^T\zeta+\bar{D}^T\mathbb{E}\zeta, u-\bar{u}\right\rangle = 0, \text{ a.e. } t \in [0,T], \ \mathbb{P}-a.s.. \quad \text{(C.6)}$$

(C.6) implies that $\left(R\bar{u}+B^T k+\bar{B}^T\mathbb{E}k+D^T\zeta+\bar{D}^T\mathbb{E}\zeta\right) \perp \Lambda$ a.e. $t \in [0,T], \ \mathbb{P}-a.s.$, which leads to (C.5).

(ii) Suppose $(\bar{x}, \bar{u}, k, \zeta)$ is an adapted solution to (H1). For any admissible control $u \in \mathcal{U}^\Lambda$, denote the differences: $\Delta u = u - \bar{u} \in \mathcal{U}^\Lambda$, $\Delta x = x(\xi_0; u) - \bar{x}(\xi_0; \bar{u})$ and $\Delta\mathcal{J} = \mathcal{J}(\xi_0; u) - \mathcal{J}(\xi_0; \bar{u})$. Then Δx satisfies that:

$$\begin{cases} d\Delta x = (A\Delta x+\bar{A}\mathbb{E}\Delta x+B\Delta u+\bar{B}\mathbb{E}\Delta u)dt+(C\Delta x+\bar{C}\mathbb{E}\Delta x+D\Delta u+\bar{D}\mathbb{E}\Delta u)dW, \\ \Delta x(0) = 0, \end{cases}$$

and

$$\begin{aligned} \Delta\mathcal{J} = \mathbb{E}\int_0^T & \left\langle Q\bar{x}+(\Gamma_1^T Q\Gamma_1-\Gamma_1^T Q-Q\Gamma_1)\mathbb{E}\bar{x}, \Delta x\right\rangle+\langle R\bar{u}, \Delta u\rangle \\ & +\frac{1}{2}\left\langle Q\Delta x+(\Gamma_1^T Q\Gamma_1-\Gamma_1^T Q-Q\Gamma_1)\mathbb{E}\Delta x, \Delta x\right\rangle+\frac{1}{2}\langle R\Delta u, \Delta u\rangle \, dt \\ & +\left\langle G\bar{x}(T)+(\Gamma_2^T G\Gamma_2-\Gamma_2^T G-G\Gamma_2)\mathbb{E}\bar{x}(T), \Delta x(T)\right\rangle \\ & +\frac{1}{2}\left\langle G\Delta x(T)+(\Gamma_2^T G\Gamma_2-\Gamma_2^T G-G\Gamma_2)\mathbb{E}\Delta x(T), \Delta x(T)\right\rangle. \end{aligned}$$

What we desire to prove is $\Delta\mathcal{J} \geq 0$ which is equivalent to the optimality of $(\bar{x}, \bar{u})$. Applying Itô formula to $d\langle k, \Delta x\rangle$, it holds that:

$$
\begin{aligned}
&\mathbb{E}\left\langle G\bar{x} - (G\Gamma_2 + \Gamma_2^T G(T) - \Gamma_2^T G\Gamma_2)\mathbb{E}\bar{x}(T), \Delta x(T)\right\rangle = \mathbb{E}\int_0^T d\langle k, \Delta x\rangle \\
=&\mathbb{E}\int_0^T \left\langle -\left(Q\bar{x} - (Q\Gamma_1 + \Gamma_1^T Q - \Gamma_1^T Q\Gamma_1)\mathbb{E}\bar{x} + A^T k + \bar{A}^T\mathbb{E}k + C^T\zeta + \bar{C}^T\mathbb{E}\zeta\right), \Delta x\right\rangle \\
&+ \left\langle k, (A\Delta x + \bar{A}\mathbb{E}\Delta x + B\Delta u + \bar{B}\mathbb{E}\Delta u)\right\rangle + \left\langle \zeta, (C\Delta x + \bar{C}\mathbb{E}\Delta x + D\Delta u + \bar{D}\mathbb{E}\Delta u)\right\rangle dt.
\end{aligned}
$$

Plugging into to $\Delta\mathcal{J}$, then we can obtain:

$$
\begin{aligned}
\Delta\mathcal{J} = \mathbb{E}\int_0^T &\left\langle R\bar{u} + B^T k + \bar{B}^T\mathbb{E}k + D^T\zeta + \bar{D}^T\mathbb{E}\zeta, \Delta u\right\rangle \\
&+ \frac{1}{2}\left\langle Q\Delta x + (\Gamma_1^T Q\Gamma_1 - \Gamma_1^T Q - Q\Gamma_1)\mathbb{E}\Delta x, \Delta x\right\rangle + \frac{1}{2}\langle R\Delta u, \Delta u\rangle\, dt \\
&+ \frac{1}{2}\left\langle G\Delta x(T) + (\Gamma_2^T G\Gamma_2 - \Gamma_2^T G - G\Gamma_2)\mathbb{E}\Delta x(T), \Delta x(T)\right\rangle.
\end{aligned}
$$

Since $V^T\left(B^T k + \bar{B}^T\mathbb{E}k + D^T\zeta + \bar{D}^T\mathbb{E}\zeta + R\bar{u}\right) = 0$, a.e. $t \in [0, T]$, $\mathbb{P} - a.s.$, then for any vector $v \in \mathbb{R}^{m'}$, it holds that:

$$
\left\langle \left(B^T k + \bar{B}^T\mathbb{E}k + D^T\zeta + \bar{D}^T\mathbb{E}\zeta + R\bar{u}\right), Vv\right\rangle = 0, \text{ a.e. } t \in [0, T],\ \mathbb{P} - a.s.,
$$

and $\Delta u \in \mathcal{U}^\Lambda$. Thus,

$$
\int_0^T \mathbb{E}\left\langle R\bar{u} + B^T k + \bar{B}^T\mathbb{E}k + D^T\zeta + \bar{D}^T\mathbb{E}\zeta, \Delta u\right\rangle dt = 0.
$$

Note that under (A4.1)-(A4.3) $\mathcal{J}$ is uniformly convex by Theorem 4.1. Then by observing the dynamics of Δx, we have $\Delta\mathcal{J} = \mathcal{J}(0; \Delta u) \geq 0$.

(iii) Assume that $(\bar{x}', \bar{u}', k', \zeta')$ is another adapted solution of (H1). Then by part (ii) of this proof, $(\bar{x}', \bar{u}')$ should be the optimal pair of (**MFC-c**).

Thus $(\bar{x}', \bar{u}') \equiv (\bar{x}, \bar{u})$, and hence by the unique solvability of (C.4), it follows that $(\bar{x}', \bar{u}', k', \zeta') \equiv (\bar{x}, \bar{u}, k, \zeta)$ proving this part.

□

C.3 Proof of Lemma 4.4

Proof. Firstly, we prove (CC-2)$\Longrightarrow$(H1) part.

Let $(\check{x}_i, \check{u}_i, p_i, p_i^1, p^2, q_i, q_i^1)$ be the adapted solution of (CC-2). The dynamic of $p^2 + p_i^1 - p_i$ satisfies that:

$$\begin{cases} d(p^2 + p_i^1 - p_i) = -\left[A^T(p^2+p_i^1-p_i)+C^T(q_i^1-q_i)\right]dt + (q_i^1 - q_i)dW_i, \\ (p^2 + p_i^1 - p_i)(T) = 0. \end{cases} \tag{C.7}$$

Under (A4.1)–(A4.2), (C.7) admits a unique solution, and we have $p^2+p_i^1-p_i \equiv 0$ and $q_i^1 - q_i \equiv 0$. Thus, we obtain $\mathbb{E}q_i^1 \equiv \mathbb{E}q_i$. Next, we have the dynamic of $p^2 + \mathbb{E}p_i^1$ satisfying

$$\begin{cases} d(p^2 + \mathbb{E}p_i^1) = \left[(\Gamma_1^T - I)Q(I-\Gamma_1)\mathbb{E}\check{x}_i - \bar{A}^T(\mathbb{E}p_i^1+p^2) - \bar{C}^T\mathbb{E}q_i^1 - A^T(p^2+\mathbb{E}p_i^1) - C^T\mathbb{E}q_i^1\right]dt, \\ (p^2 + \mathbb{E}p_i^1)(T) = (I-\Gamma_2^T)G(I-\Gamma_2)\mathbb{E}\check{x}_i(T), \end{cases}$$

and the dynamic of $\mathbb{E}p_i$ satisfying

$$\begin{cases} d\mathbb{E}p_i = \left[(\Gamma_1^T - I)Q(I-\Gamma_1)\mathbb{E}\check{x}_i - A^T\mathbb{E}p_i - C^T\mathbb{E}q_i - \bar{A}^T(\mathbb{E}p_i^1+p^2) - \bar{C}^T\mathbb{E}q_i^1\right]dt, \\ \mathbb{E}p_i(T) = (I-\Gamma_2^T)G(I-\Gamma_2)\mathbb{E}\check{x}_i(T). \end{cases}$$

By taking difference, we have:

$$\begin{cases} d(p^2+\mathbb{E}p_i^1-\mathbb{E}p_i)=\left[-A^T(p^2+\mathbb{E}p_i^1-\mathbb{E}p_i)-C^T(\mathbb{E}q_i^1-\mathbb{E}q_i)\right]dt, \\ (p^2+\mathbb{E}p_i^1-\mathbb{E}p_i)(T)=0. \end{cases}$$

By $q_i^1-q_i\equiv 0$, we have:

$$\begin{cases} d(p^2+\mathbb{E}p_i^1-\mathbb{E}p_i)=\left[-A^T(p^2+\mathbb{E}p_i^1-\mathbb{E}p_i)\right]dt, \\ (p^2+\mathbb{E}p_i^1-\mathbb{E}p_i)(T)=0, \end{cases}$$

and $p^2+\mathbb{E}p_i^1-\mathbb{E}p_i\equiv 0$. By plugging $\mathbb{E}q_i^1\equiv\mathbb{E}q_i$ and $p^2+\mathbb{E}p_i^1-\mathbb{E}p_i\equiv 0$ into (CC-2), we have:

$$\begin{cases} d\check{x}_i=(A\check{x}_i+B\check{u}_i+\bar{A}\mathbb{E}\check{x}_i+\bar{B}\mathbb{E}\check{u}_i)dt+(C\check{x}_i+D\check{u}_i+\bar{C}\mathbb{E}\check{x}_i+\bar{D}\mathbb{E}\check{u}_i)dW_i, \\ dp_i=\left[-Q\check{x}_i+(Q\Gamma_1+\Gamma_1^TQ-\Gamma_1^TQ\Gamma_1)\mathbb{E}\check{x}_i-A^Tp_i-C^Tq_i-\bar{A}^T\mathbb{E}p_i-\bar{C}^T\mathbb{E}q_i\right]dt+q_idW_i, \\ \check{x}_i(0)=\xi_0, \quad p_i(T)=G\check{x}_i(T)+(\Gamma_2^TG\Gamma_2-\Gamma_2^TG-G\Gamma_2)\mathbb{E}\check{x}_i(T), \\ V^T\left(R\check{u}_i+B^Tp_i+D^Tq_i+\bar{B}^T\mathbb{E}p_i+\bar{D}^T\mathbb{E}q_i\right)=0, \end{cases}$$

which is identical to (H1) and $(\check{x}_i,\check{u}_i,p_i,q_i)$ is a solution of (H1).

Secondly, we prove (H1)$\Longrightarrow$(CC-2) part.

Under (A4.1)–(A4.2), for any adapted solution $(\bar{x},\bar{u},k,\zeta)$ of (H1), the following BSDE admits a unique solution:

$$\begin{cases} dk^1=-\left(Q\bar{x}+A^Tk^1+C^T\zeta^1\right)dt+\zeta^1dW, \\ k^1(T)=G\bar{x}(T), \end{cases} \tag{C.8}$$

and thus so does the following backward ordinary differential equation (BODE):

$$\begin{cases} dk^2 = -\left[(\Gamma_1^T Q\Gamma_1 - \Gamma_1^T Q - Q\Gamma_1)\mathbb{E}\bar{x} + \bar{A}^T \mathbb{E}k^1 + \bar{C}^T \mathbb{E}\zeta^1 + \bar{A}^T k^2 + A^T k^2\right] dt, \\ k^2(T) = (\Gamma_2^T G\Gamma_2 - \Gamma_2^T G - G\Gamma_2)\mathbb{E}\bar{x}(T). \end{cases} \tag{C.9}$$

Then by combining (H1), (C.8) and (C.9), we have:

$$\begin{cases} d\bar{x} = (A\bar{x} + B\bar{u} + \bar{A}\mathbb{E}\bar{x} + \bar{B}\mathbb{E}\bar{u})dt + (C\bar{x} + D\bar{u} + \bar{C}\mathbb{E}\bar{x} + \bar{D}\mathbb{E}\bar{u})dW, \\ dk = -\left(Q\bar{x} - (Q\Gamma_1 + \Gamma_1^T Q - \Gamma_1^T Q\Gamma_1)\mathbb{E}\bar{x} + A^T k + \bar{A}^T \mathbb{E}k + C^T\zeta + \bar{C}^T \mathbb{E}\zeta\right) dt + \zeta dW, \\ dk^1 = -\left(Q\bar{x} + A^T k^1 + C^T \zeta^1\right) dt + \zeta^1 dW, \\ dk^2 = -\left[(\Gamma_1^T Q\Gamma_1 - \Gamma_1^T Q - Q\Gamma_1)\mathbb{E}\bar{x} + \bar{A}^T \mathbb{E}k^1 + \bar{C}^T \mathbb{E}\zeta^1 + \bar{A}^T k^2 + A^T k^2\right] dt, \\ \bar{x}(0) = \xi_0, \quad k(T) = G\bar{x}(T) + (\Gamma_2^T G\Gamma_2 - \Gamma_2^T G - G\Gamma_2)\mathbb{E}\bar{x}(T), \\ k^1(T) = G\bar{x}(T), \quad k^2(T) = (\Gamma_2^T G\Gamma_2 - \Gamma_2^T G - G\Gamma_2)\mathbb{E}\bar{x}(T), \\ V^T\left(B^T k + \bar{B}^T \mathbb{E}k + D^T\zeta + \bar{D}^T \mathbb{E}\zeta + R\bar{u}\right) = 0. \end{cases} \tag{C.10}$$

Comparing (C.10) with (CC-2), what remains to prove is that $k^2 + \mathbb{E}k^1 \equiv \mathbb{E}k$ and $\mathbb{E}\zeta^1 \equiv \mathbb{E}\zeta$. Then we consider the dynamic of $k^2 + k^1 - k$:

$$\begin{cases} \quad d(k^2 + k^1 - k) \\ = -\left(A^T(k^1 + k^2 - k) + C^T(\zeta^1 - \zeta) + \bar{A}^T(\mathbb{E}k^1 + k^2 - \mathbb{E}k) + \bar{C}^T(\mathbb{E}\zeta^1 - \mathbb{E}\zeta)\right) dt + (\zeta^1 - \zeta)dW, \\ = -\left(A^T(k^1 + k^2 - k) + C^T(\zeta^1 - \zeta) + \bar{A}^T\mathbb{E}(k^1 + k^2 - k) + \bar{C}^T\mathbb{E}(\zeta^1 - \zeta)\right) dt + (\zeta^1 - \zeta)dW, \\ \quad (k^2 + k^1 - k)(T) = 0. \end{cases} \tag{C.11}$$

Under (A4.1)–(A4.2), the MF-BSDE (C.11) admits a unique adapted solution, and obviously $(k^2 + k^1 - k, \zeta^1 - \zeta) = (0, 0)$ is an adapted solution of (C.11). Thus, $k^2 + k^1 \equiv k$, $\zeta^1 \equiv \zeta$, $k^2 + \mathbb{E}k^1 \equiv \mathbb{E}k$ and $\mathbb{E}\zeta^1 \equiv \mathbb{E}\zeta$. Plugging such relations into

(C.10) and we have:

$$
\begin{cases}
d\bar{x} = (A\bar{x}+B\bar{u}+\bar{A}\mathbb{E}\bar{x}+\bar{B}\mathbb{E}\bar{u})dt+(C\bar{x}+D\bar{u}+\bar{C}\mathbb{E}\bar{x}+\bar{D}\mathbb{E}\bar{u})dW,\\
dk=-\left(Q\bar{x}-(Q\Gamma_1+\Gamma_1^TQ-\Gamma_1^TQ\Gamma_1)\mathbb{E}\bar{x}+A^Tk+\bar{A}^Tk^2+\bar{A}^T\mathbb{E}k^1+C^T\zeta+\bar{C}^T\mathbb{E}\zeta^1\right)dt+\zeta dW,\\
dk^1 = -\left(Q\bar{x}+A^Tk^1+C^T\zeta^1\right)dt+\zeta^1 dW,\\
dk^2 = -\left[(\Gamma_1^TQ\Gamma_1-\Gamma_1^TQ-Q\Gamma_1)\mathbb{E}\bar{x}+\bar{A}^T\mathbb{E}k^1+\bar{C}^T\mathbb{E}\zeta^1+\bar{A}^Tk^2+A^Tk^2\right]dt,\\
\bar{x}(0)=\xi_0, \quad k(T)=G\bar{x}(T)+(\Gamma_2^TG\Gamma_2-\Gamma_2^TG-G\Gamma_2)\mathbb{E}\bar{x}(T),\\
k^1(T)=G\bar{x}(T), \quad k^2(T)=(\Gamma_2^TG\Gamma_2-\Gamma_2^TG-G\Gamma_2)\mathbb{E}\bar{x}(T),\\
\bar{u}=V^T\left(B^Tk+\bar{B}^Tk^2+\bar{B}^T\mathbb{E}k^1+D^T\zeta+\bar{D}^T\mathbb{E}\zeta^1+R\bar{u}\right),
\end{cases}
$$

which is identical to (CC-2) and $(\bar{x},\bar{u},k,k_1,k_2,\zeta,\zeta_1)$ is a solution of (CC-2), where (k_1,ζ_1) and k_2 are the unique solutions of (C.8) and (C.9) respectively. This completes the proof. □