A STUDY OF TWO BEHAVIORAL FINANCE MODELS

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A Study of Two Behavioral Finance Models

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CERTIFICATE OF ORIGINALITY

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Dedicated to love.

Abstract

This thesis is concerned with financial models that incorporate components from the burgeoning field of behavioral finance. The goal of behavioral finance is to describe illogical behaviors and anomalies seen in financial markets, as well as to investigate the patterns that emerge as people make decisions. I present two specific behavioral financial models: the first one is a portfolio selection problem in continuous time, and the second one is an optimal insurance design problem. These two models have one feature in common: they both deviate from the traditional paradigm that is built on mathematical assumptions like global convexity (concavity) and linear expectation, resulting in the failure of conventional methods. To tackle them, I reduce them to quantile optimization problems. Using the relaxation and calculus of variation methods, the optimal solutions to them are derived and explicit results are obtained under specific settings.

I begin this thesis by giving a brief historical overview of portfolio selection as well as a summary of the contributions and organization of the thesis. The storytelling of the two models is connected by a detailed illustration of decision-making theory under uncertainty that provides solid grounds and inspiration for modern behavioral economics. Some important prerequisites are presented in the last section of Chapter 1.

In Chapter 2, I present a return-oriented continuous-time portfolio selection model under the cumulative prospect theory. The model is considered in a stan-

dard complete and no-arbitrage market, and it also captures the heuristics and biases that occur during the agent's decision-making process. Benchmark and lower bound constraints are introduced to the model to measure performance and control the downside risk. The problem turns out to be a non-classical stochastic control problem, which can be addressed by solving a corresponding quantile optimization problem. The procedure heavily depends on the concept of quantile, which has long been used in nonlinear, nonadditive measures. The problem is converted to a locally concavified optimization problem using the relaxation method, and an optimal solution is derived. The last part of this chapter focuses on deriving the optimal portfolio, which boils down to solving a related partial differential equation (PDE). In particular, explicit expressions are obtained under the Black-Scholes setting.

In Chapter 3, I present an optimal insurance problem where the risk preference of the insured is characterized by the rank-dependent utility theory (RDUT) and the premium principle is based on Wang's class of premium principle. It is required that the insurance policy should not cause an issue of moral hazard, which means both the compensation and retention functions are non-decreasing with respect to the loss. The problem is converted to an equivalent quantile optimization problem. Using the calculus of variation method, the optimal solution is expressed via the solution of an ordinary integral-differential equation (OIDE). A numerical example is provided as well.

This thesis ends up with some concluding remarks and expectations for future work.

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List of Notations

$\mathbb{R}^n$	the n-dimensional real Euclidean space.
$\mathbb{R}^{n\times m}$	the set of all real $n \times m$ matrices.
M'	the transpose of a matrix or vector M.
$\\|M\\|$	the L^2-norm for a matrix or vector $M = (m_{ij})$, equal to $\sqrt{\sum_{i,j} m_{ij}^2}$.
$(\Omega, \mathcal{F}, \mathbb{P})$	the probability space $([0,1], \mathcal{B}[0,1], L)$, with L being the *Lebesgue measure*.
$W(\cdot)$	a standard n-dimensional Brownian motion.
$\{\mathcal{F}_t\}_{t\geqslant 0}$	the filtration generated by $W(\cdot)$.
$L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$	the set of $\{\mathcal{F}_t\}_{0\leqslant t\leqslant T}$-adapted, $\mathbb{R}^n$-valued progressively measurable processes $X(\cdot)$ such that $\mathbb{E}\left[\int_0^T \\|X(t)\\|^2 \, \mathrm{d}t\right] < +\infty$.
$L^2_{\mathcal{F}}$	the set of random variables X such that $\mathbb{E}[X^2] < \infty$.

Chapter 1

Introduction

In the last short but brilliant century, the world has witnessed a tremendous growth in the size of the financial market as well as a massive leap forward in the related academic research fields. The purpose of this chapter is to situate the thesis in the picture of the whole discipline. Some fundamental concepts and their backgrounds, including literature reviews, will be introduced. Such components altogether make up the motivation for this thesis.

1.1 Portfolio Selection: A View from Two Predominant Rules

It is critical to figure out what rules decision makers/agents follow in practice, as well as the underlying patterns of how they make judgements or evaluate outcomes during investment and other economic activities, before we use models to explore the interplay between them and the markets. In particular, dealing with risk or uncertainty is a common and central job throughout the entire evaluation process. Therefore, a basic question goes down to how people make decisions over choices that are associated with uncertainties, the so-called risk preference. Unfortunately, the vast number of normative and descriptive theories and models proposed by social economists and cognitive psychologists cannot provide a comprehensive picture of human behavior and preference for risky choices. However, fortunately, at least some of them can be treated as good approximations of reality, and indeed, these rules have been widely adopted in financial modeling. In this section, the skeleton will be expanded from the perspective of two predominant rules that appear in the context of portfolio selection.

The first rule traces back to Harry Markowitz's pioneering paper, "Portfolio Selection", published in 1952, *Journal of Finance*, which has been renowned as the start of modern financial economics. The Mean-Variance (MV) analysis framework, also known as the MV rule, proposed in the paper has been widely accepted and practiced as a persuasive philosophy in both the financial industry and the academic world. Within nearly seventy years, the original model has brought out hundreds of extensions and variations, making "portfolio selection" a prolonged but flourishing and fadeless topic in the fields of portfolio management, financial engineering, and financial mathematics. The essence of the MV rule is simple but extraordinary: risk cannot be fully eliminated by diversification[1], and a trade-off exists between risk and return. A "suboptimal" option is to find a portfolio that has the minimum risk given an expected level of return or the maximum expected return subject to a threshold of risk. At that time, such insightful assertions more or less reshaped the direction of how people analyzed the market. Readers can refer to Markowitz's monograph [92] for a systematic and detailed illustration of this research area.

On the basis of Markowitz's work, several cornerstones of pricing theories and models emerged, such as the famous Capital Assets Pricing Model (CAPM) proposed by Sharpe [111] and Linter [83], Arbitrage Pricing Theory (APT) introduced by Ross [105]. These models turn out to formulate and promote a specific prototype called the "Multi-factor" model, which aims at describing the cross-sectional difference between

[1] A popular analogy is "putting the eggs into plenty of baskets before we may break them all at once"

the average return on different assets. Such a target has been a central problem in empirical asset pricing. A notable work in this field is the Fama-French three factor model, introduced by Fama and French [51]. The regression techniques and results proposed in [51] are highly praised and applied in the real markets. It has directly led to the fashion of factors-mining in the financial industries. Nowadays, trading strategies and risk analysis frameworks surrounding the model have become one of the most prevalent in mutual funds, private equity, and hedge funds.

Despite the influences above, the appearance of the MV rule imposes a threat to another predominant rule for decision-making under uncertainty. Unlike the MV rule, the Expected Utility Maximization (EUM) rule, proposed by Daniel Bernoulli in 1738 and later axiomatized as the Expected Utility Theory (EUT) by Von Neumann and Morgenstern [124], is heavily rooted in the cradle of economics. It provides an interpretation of St. Petersburg's Paradox: a gamble with the observation that most people do not maximize the expected monetary profit or wealth. It is proposed that people evaluate a gamble using a hypothetical notion of "utility", which stands for a personal, subjective measure of satisfaction or well-being. A "utility function" describes a preference ordering over different risky choices, and the shape of the utility function (convex or concave) is a good determinant of the individual's altitude over risk[2]. Although the EUT has a comprehensible and appealing structure, how to choose a person's utility function is certainly another area of professional expertise and beyond the scope of this thesis. This limits the implementation of the EUT, and a debate between the EUM rule and the MV rule exists. In the literature, it is divided into two parties. On one hand, critiques of the MV rule argue that it is consistent with the axioms of the EUT only if the agent has a quadratic expected utility and the asset price follows a Gaussian distribution; see, e.g., [10], [22], [52].

[2] A concave utility function indicates a preference for risk aversion, implying that the agent always prefers a deterministic amount of x over a random variable with an expected outcome of x. The convex situation is simply the reverse, which means risk-seeking behavior

On the other hand, supporters of the MV rule believe that it is a good approximation of the EUM rule and is easy to apply due to its simplicity; see, e.g., [101], [102], [120], [75], [93].

Putting aside these controversial issues, both rules have been adopted in multi-period and continuous-time settings. Continuous-time financial models are usually distinguished by the use of Itô diffusion or other complex processes to characterize the randomness of asset prices over time, and most of them involve dedicated mathematical concepts and tools, such as stochastic analysis, stochastic control, and differential equation theories. Merton's seminal works, [95], [96], in which the rates of return were assumed to be normally distributed, is the foundation for continuous-time portfolio selection models under the EUM framework. He showed that finding an optimal policy can be reduced to finding a solution to the corresponding Hamilton-Jacobi-Bellman (HJB) equation, which bridges descriptive financial ideas with a rigorous mathematical formulation. However, the formulation of a continuous-time counterpart of Markowitz's original model using the MV rule encountered numerous obstacles. A major difficulty was that the stochastic dynamic programming principle could not be applied directly to solve the problem because of the variance term in the objective functional. After an enduring endeavor of thirty years, by means of an embedding technique introduced in Li and Ng [78], Zhou and Li [139] formulated the counterpart into an auxiliary stochastic linear quadratic problem and obtained a closed-form expression of the efficient frontier.

Surprisingly, the continuous-time reformulation of Markowitz's model has brought new possibilities to itself. In the past several decades, a considerable number of academic attempts have focused on incorporating realistic conditions or constraints into the continuous-time portfolio selection model, including but not limited to, the prohibition of short-selling [79], [34], [15], no-bankruptcy [18], transaction costs [89], [37], [84], [114], time-variant coefficients, [81], [85], regime-switching [140], robust and ambiguity environments [58], [141], [50], [56], [72], partial information [130], neural networks [53], reinforcement learning [126], behavioral preference [71], [68]. All the subdivision directions mentioned above have significantly enriched the world of portfolio selection, and inevitably, part of this thesis's work also lies in this category. Of course, most of the work mentioned is still done within the framework of the EUM rule[3].

A return-rate based portfolio selection model that incorporates behavioral preference will be presented in Chapter 2 of this thesis. Here, a (non-EU) behavioral preference literally means a departure from the expected utility preference and renders that investors are rational, which is a priori assumption of the EUT. At least two reasons exist for considering behavioral preference. Firstly, an acquiescence in daily life is that "rational" investors only take up a small proportion of participants in the markets, and people rarely achieve the maximization of their expected utility. The EUT has failed to capture the impact of psychological interference when people face large potential gains and losses. Few people could keep calm and behave rationally in a complicated and variable mood, not to mention whether anyone is capable of making perfect strategies at the perfect time with the correct expected utility calculation for every choice[4]. It is natural to wonder how "normal" or "emotional" investors choose their portfolios. Secondly, experimental evidence shows that people are not entirely risk averse. They are willing to take risks, especially when they are confronted with a sure loss, and this phenomenon violates the EUT. The EUM rule as an approximation to reality somehow captures the inner desire of investors, but it provides limited insight and unconvincing arguments (even wrong) for understanding a lot of basic facts (such as the Friedman and Savage puzzle [55], the tax evasion

[3] We would like to treat the MV criteria as a special case of the EUM criteria, so as to distinguished it from the behavioral criteria introduced later.

[4] There are other issues, including liquidation, short positions, transaction costs, etc

problem [45], the equity premium puzzle [94], and the stochastic volatility puzzle [27]) happening in the underlying markets.

In brief, the traditional finance paradigm built on the EUM rule, along with its implicit philosophy behind, has already received full investigation in academia and approached its descriptive and predictive limitations. Progress in portfolio selection has entered a new stage in which it is faced with challenges but also opportunities brought by the rise of a new branch of finance, behavioral finance. One of the building blocks of this field is the psychology of characterization of non-EU preferences, which we introduce in the next section.

1.2 Non-Expected Utility Theory

The dominated EUT, like the eldest son of a big "family", deals with a quantitative representation of a decision maker's preference over risky choices. Such a "family" has two basic tenets: a non-empty set X for decision and comparison, and an ordered preference $\succ$ (or binary relation) between elements in the set X. For a choice $a \in X$ which usually involves with uncertainty, we call $a = (y_1, p_1; \cdots y_n, p_n; \cdots)$ a *prospect* where y_i $(i = 1, 2, \cdots)$ denote all the possible outcomes with associated probabilities p_i $(i = 1, 2, \cdots)$ and $\sum_i p_i = 1$. The utility of a proposed by Von Neumann and Morgenstern [124] is a weighted sum of the utility of outcomes:

$$U(a) := \sum_{i=1} u(y_i) p_i \tag{1.1}$$

where u is the utility function defined on the set of outcomes. In this case[5]

$$a \succ b \quad \text{if and only if} \quad U(a) \geq U(b).$$

[5] For any $a, b \in X$, $a \succ b$ means the decision maker prefers a than b. We have either $a \succ b$ or $b \succ a$.

The existence of u and its cardinality is guaranteed and generalized by the four known axioms of preference in [124], namely completeness, transitivity, continuity, and independence.

(Completeness) For $a, b \in X$, either $a \succ b$ or $b \succ a$;

(Transitivity) For $a, b, c \in X$, if $a \succ b$ and $b \succ a$, then $a \succ c$;

(Continuity) For $a, b, c \in X$, if $a \succ b \succ c$, then there exist $\alpha, \beta \in (0, 1)$ such that $\alpha a + (1 - \alpha)c \succ b$ and $b \succ \beta a + (1 - \beta)c$;

(Independence) For any $a, b, c \in X$ and $\lambda \in (0, 1)$, if $a \succ b$, then $\lambda a + (1 - \lambda)c \succ \lambda b + (1 - \lambda)c$.

This axiomatization procedure lies at the heart of the EUT as well as other normative theories of choice introduced in this paragraph. A large amount of work which aims at generalizing the EUT can be reduced to a relaxation or even sacrifice of the above axioms. In [124], The axioms of completeness and transitivity together promise that choices can be ordered, but the latter has been a controversial property, bringing not only conveniences but also limitations. It may neglect the accumulation of small differences or vagueness between risky choices. A potential violation is that choices should be indifferent by applying transitivity, but it actually may not be indifferent in the decision maker's mind, and there may exist a cycle of preference[6]. A typical example of non-transitive preference can refer to the regret theory proposed by Bell [12], Fishburn [54], and Loomes and Sugden [86]. In addition, empirical observations of violations of the independence axiom have been collected since the 1950s. For instance, the well known common ratio effects and common sequence effects discovered in Allais [4] and the Ellsberg paradox [49], etc. These laboratory evidences have been the foothold of theories including but not restricted to the

[6] It means we may have $a \succ b$, $b \succ c$, and $c \succ a$

weighted utility theory proposed by Chew and MacCrimmon [30] and its variants such as Dekel [41] and Chew [31], the lottery-dependent utility of Becker and Sarin [11], the quadratic utility of Chew, Epstein and Segal [33], the disappointment theory of Loomes and Sugden [87], Bell [13]. Although the model settings and implicit motivations behind these theories are quite different, they share a similar structure for utility calculation, which is essentially a combination of the subjective utility of an outcome and its objective probability distribution through specific functions.

Another critical characteristic of the EUT that departs from reality is the hypothesis of risk aversion. In order to explain St. Petersburg's Paradox, diminishing marginal utility is assumed and leads to a concave utility function that indicates risk aversion behaviors. However, it is still insufficient to explain risk-taking behaviors observed in real-life scenarios, such as purchasing lottery tickets and gambling. Even though one may realize that the objective probability of death or winning is minimal, they still buy them in case of "lucky". A psychological explanation of these behaviors in the literature is called probability weighting. In other words, people sometimes do not make decisions based on the objective probability, especially when some extreme events with small probabilities happen. Emotional feelings such as fear, aspiration dominate the mind of decision maker, resulting in irrational behaviors that violate the EUT. The theory that considers probability weighting is called decision weighted utility theory. Usually the weight attached to $u(y_i)$ in (1.1) is no longer p_i, but mathematically replaced by a transformation of that. Examples are the rank-dependent expected utility developed by Quiggin [103], Wakker, Erev and Weber [125] and its variants, such as Chew and Epstein [32], Green and Jullien [64], Segal [110], Yaari [135].

So far, one common feature of axiomatic models we mentioned above is that there is always a priori restriction on individuals' preferences, such as rationality (the EUT), betweenness ([41], [31]), mixture symmetry ([33]), and comonotonic independence ([125]). Normally, these restrictions are the results of backward induction of a predefined target, which do not conform to the real decision-making progress which is more flexible and complex. As pointed out in Tversky and Kahneman [121], "people rarely obey some cast-iron rules in decision-making, they routinely violate dominance and invariance." Starmer [115] summarized this typical evidence as violations of procedure invariance and description invariance, which the axiomatic models would fail to explain. For example, in the so-called preference reversal phenomenon, first observed by Linderman [82], Lichtenstein and Slovic [80], participants exhibit opposite preferences over risky choices that differ slightly in description but are essentially the same from a normative perspective. In other words, how we frame choices may also influence individuals' preferences.

Another branch of the non-expected utility theory is represented by the prospect theory of Kahneman and Tversky [73], the cumulative prospect theory of Tversky and Kahneman [122]. These theories basically give up an axiomatization procedure. Instead, they focus primarily on the patterns of how people actually count in the decision-making process, considering secondarily which conditions need to be specified. The patterns mainly refer to those mental activities consisting of heuristics and biases, and they have a close intersection with the context of psychology. Barberis and Thaler [8] summarized a series of uniform beliefs that people appear to form in decision-making, such as overconfidence, conservatism, optimism, and wishful thinking, etc. Although these descriptive laws learned by psychologists have been well supported by designed experiments and field data, how to abstract them into mathematical structure for further quantitative analysis is an important and challenging task for financial economists. Next, we give a brief introduction to the cumulative prospect theory that will be used in our model.

1.2.1 Cumulative Prospect Theory

Among all the alternatives to the EUT, the prospect theory (PT) may be the most convincing model which captures experimental results as well as builds a general mathematical structure. By this theory, the process of making a choice has been simplified into two stages. The first stage is a valuation process of outcomes based on heuristics supported by a large scale of experimental data, and the second stage is a calculation over prospects which inherits the traditional form of decision weighted utility, namely it involves non-linear probability weighting. The PT has a disadvantage in that it may violate first-order stochastic dominance. To tackle it, Tversky and Kahneman [122] extended the PT to a revised version called the cumulative prospect theory (CPT). The model ends up with an analogy to (1.1):

$$V(x) := \sum_{i=1} v(y_i - b_i) w_i, ^{7} \tag{1.2}$$

where b_i is the reference point for state i,

$$w_i = \pi\Big(\sum_{j \geq i} p_i\Big) - \pi\Big(\sum_{j > i} p_i\Big)$$

and π is a probability weighting/distortion function. Besides, it proposed the following shape of the valuation function v:

[7] In [122], this expression has been divided into a gain part and a loss part, each of which uses its own probability weighting function. For the sake of simplicity, we assume they are the same in this thesis.

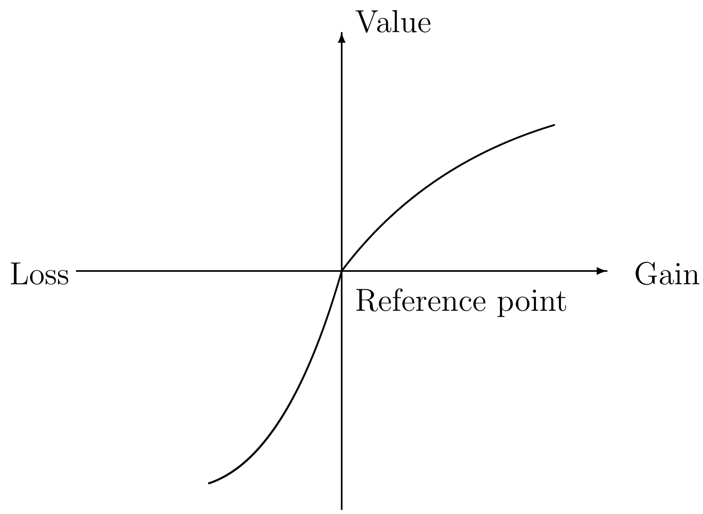

Figure 1.1: The function v.

As already mentioned, this S-shaped function is deduced by the heuristics observed in experiments instead of an axiomatization procedure. The first critical heuristic is a prevalent observation that people usually evaluate outcomes from a perspective of gain or loss instead of the final wealth. As pointed out by Barberis and Thaler [8], "it is consistent with the way people think, perceive attributes such as brightness, loudness, or temperature relative to earlier levels, rather than in absolute terms". It explains why the value function and corresponding probability weighting are divided into the gain part and the loss part. Gain and loss are relative concepts talked about on the basis of a notion called reference point. The reference point may be numerous and adaptive. An outcome above (or below) the level of reference point is regarded as a gain (or loss) situation.

The second heuristic is an observation of over-weighting of probabilities, no matter it is a gain situation or a loss situation. The over-weighting of sure gains contributes to risk aversion behavior; on the contrary, the over-weighting of sure losses contributes to risk-seeking behavior. These are the certainty effects and reflection effects identified in [73]. A psychological ground for the observations is called diminishing sensitivity, which means people become less sensitive to the marginal change as the outcome moves away from the reference point. This heuristic requires that the utility shape under prospect theory be an S-shape, namely concave above the reference point ($v''(x) < 0$) and convex below that ($v''(x) > 0$). The last important heuristic is called "loss aversion". It means a certain loss from the reference point causes a bigger marginal change than an equal gain does. In other words, we have the condition $v'(x) < v'(-x)$, for $x > 0$. So the loss part of the utility function is steeper than the gain part. All these heuristics explain the shape of v in Figure 1.1.

Apart from that, the curvature of probability weighting function has also been extensively studied in the literature. A widely accepted form is an inverted S-shaped π displayed in Figure 1.2. There are several reasons for that. Firstly, it is a consensus

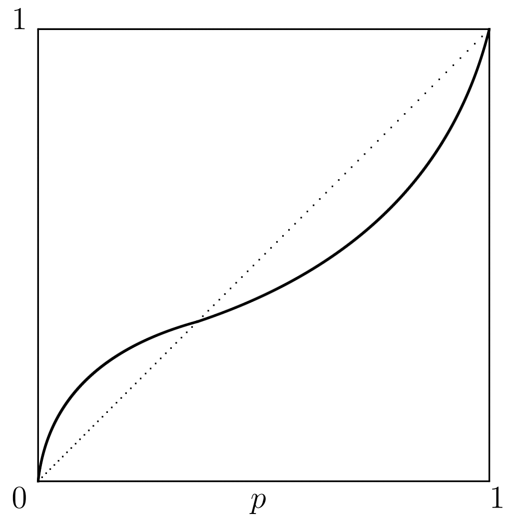

Figure 1.2: The probability weighting function π.

that π should be an increasing function. It should also satisfy $\pi(0) = 0$ and $\pi(1) = 1$, since people have no disagreement on "extreme" events: events with objective probability of zero or one. Besides, a critical observation in Kahneman and Tversky [73] is that people tend to over-weight small probabilities ($\pi(p) > p$) near zero and under-weight large probabilities ($\pi(p) < p$) near one. This means the shape of π on $[0,1]$ shall be concave at beginning and convex in the end. In Tversky and Kahneman [122], the inverted S-shaped probability weighting functions were found to fit the experiment results fairly well and later this finding was testified to be robust in other decision environments; see Abdellaoui [1], Bleichrodt and Pinto [20], Camerer and Ho [26], Gonzalez and Wu [63], Prelec [100], Wu and Gonzalez [128], Lattibmore, Baker, and Witte [76]. It is worth noting that such an inverted S-shape can also be explained by diminishing sensitivity if one treats the "extreme" events as the reference points. The shape shall be steeper near reference points and flatter on the intermediate probabilities.

Several parametric forms of the inverted S-shaped probability weighting function π were proposed in the aforementioned literature. We list them in the following table:

	π	parameter(s)
Lattibmore, et. al.[76]	$\pi(p_i) = \frac{\alpha p_i^\beta}{\alpha p_i^\beta + \sum_{k=1}^n p_i^\beta}$	$\alpha, \beta > 0, k \neq i$
Tversky and Kahneman[122]	$\pi(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}}$	$0 < \gamma < 1$
Gonzalez and Wu[63]	$\pi(p) = \frac{\delta p^\gamma}{\delta p^\gamma + (1-p)^\gamma}$	$\delta > 0, 0 < \gamma < 1$
Prelec[100]	$\pi(p) = e^{-\delta(-\ln p)^\gamma}$	$\delta > 0, 0 < \gamma < 1$

Table 1.1: Parametric forms of π

1.3 Behavioral Portfolio Selection

Incorporating the concept of behavioral preference into portfolio selection models is a burgeoning field that started at the beginning of this century. As same as the old trend, attempts are first conducted in the dimension of one-single period. Shefrin and Statman [112] proposed a behavioral portfolio theory (BPT) on the basis of Lopes' SP/A theory [88] and the mental accounting (MA) structure introduced by Thaler [118], [119]. Different from Markwoitz's model, the risk in BPT is measured by the probability of ruin (Roy [106]) instead of the variance of return. A number of studies focus on a comparison between the BPT optimal portfolio and the mean-variance efficient frontier. Normally, the former does not coincide with the latter. Das et al. [40] proposed an MA framework which integrates the appealing structure of the mean-variance model with BPT. The resulting optimal portfolio is also mean-variance efficient. However, Alexander and Baptista [3], Baptista [7] presented completely different results if additional conditions (such as background risk and delegation) are considered.

Berkelaar, Kouwenberg, and Post [16] considered a continuous-time portfolio choice problem with a loss aversion investor characterized by the piece-wise power utility. Their solution shows that loss aversion may significantly influence an investor's weight on a stock in a short investment horizon. However, the paper does not involve probability weighting, which is a critical component of behavioral criteria. Jin and Zhou [71] first established a general continuous-time portfolio selection model under CPT theory, which captures both the S-shaped value function and probability distortion. They obtained an analytical result by a "divide and conquer" machinery: splitting the corresponding optimization problem into three subproblems and solving them separately. Later, Zhang, Jin and Zhou [138] considered a constrained version of [71] by imposing an upper bound on the loss. The quantile formulation used in [71] turns out to be a powerful tool and has been extended by He and Zhou [68] to a general model with law-invariant performance criteria. The law-invariant criteria captures the common features of a broad class of models, including the goal-reaching model of Browne [24], [25], Yaari's dual model [135], Lopes' SP/A model [88], and those involving VaR and CVaR. Shi, Cui and Li [113] constructed a multi-period CPT model which respectively extended the one-period versions of He and Zhou

[68], Barberis and Xiong [9], Pirvu and Schulze [99].

A main obstacle that appears in continuous-time behavioral portfolio choice problems is the failure of traditional approaches. For instance, once a nonlinear probability weighting is introduced, the target becomes a nonlinear expectation, for which the well-known conditional tower property in probability may no longer exist. Furthermore, Bellman's principle of optimality no longer exists either, and consequently, the dynamic programming principle (DPP), which was first utilized by Merton [95], [96] to derive a corresponding HJB equation, failed. This situation is also called dynamic inconsistency, tracing back to Strotz [116]. Under this circumstance, an optimal strategy in the long-term may have terrible performance in the short-term, and investors have the incentives to give up and change their objectives and preferences in the midway.

The martingale approach becomes inapplicable as well. It was developed by Harrison and Kreps [65], Harrison and Pliska [66], [67] and extended to the convex duality method in Cvitanić and Karatzas [36]. Usually, techniques under the traditional expected utility framework are premised on the global concavity or convexity assumption. But unfortunately, this desiring property also vanishes due to the non-concave or non-convex shape of the utility. Apart from that, it may also rise an issue of well-posedness, which is not common in the traditional paradigm.

One systematic approach developed recently to tackle those non-concave (non-convex), nonlinear optimization problems is to study their quantile optimization problems instead. Specifically, rewrite the functionals and constraints by replacing the decision variables with their quantiles (the inverse of distribution functions). This procedure is called quantile formulation, which was first utilized by Jin and Zhou [71] to tackle the intractability raised by the probability weighting function. He and Zhou [68] illustrated that this procedure is valid as long as the performance criteria are law-invariant, namely the payoff functional is only related to the distribution function of the decision variable. To further address the quantile optimization technique, Xia and Zhou [129] proposed a systematic calculus of variation method, while Xu [132] alternatively proposed a much simplified change-of-variable and relaxation technique. The latter has considerably relaxed the strong conditions imposed on weighting functions in [71], [68], [129] and cleverly avoided the feasibility, well-posedness, attainability, and uniqueness issues by embedding it into a traditional Merton's problem. This patterned technique will be used to simplify the situations investigated in Chapters 2 and 3. In exchange, the questions will pose new theoretical challenges and help to improve existing analysis methods.

1.4 Summary of Contributions of the Thesis

This thesis studies two separate models in the fields of portfolio selection and optimal insurance contract design. Both of them assume that agents have behavioral preferences, which leads to quite different theoretical problems compared with traditional ones. The original contributions are summarized as follows:

Return-Rate Based Portfolio Selection Model

1. To the best of our knowledge, it is the first model that considers a behavioral investor with the goal of maximizing prospective utility of log-return. Compared with the conventional expected utility of terminal wealth, the model is more realistic, both psychologically and practically. Under the utility maximization framework, the optimal terminal wealth presents a slightly different structure from that of conventional behavioral portfolio selection models. Both the investors' risk tolerance level and the benchmark chosen to measure performance are found to play a critical role in determining the loss scenarios and the state of the optimal terminal wealth. In a complete market with deterministic parameters, we derive an analytical result on the optimal proportional strategy

given the form of benchmark return.

2. Theoretically, the return-oriented objective involved with behavioral risk preference brings us a different mathematical structure, causing a failure of traditional approaches. We propose a relaxation method to solve a constrained quantile optimization problem. The problem has a non-concave objective functional (M-shaped), which is the major difficulty. It is proved that this non-concave problem is equivalent to a "locally concavified" problem which can be solved. A comparison between the return-based objective and a traditional case is also provided in this model.

Optimal Moral-Hazard-Free Insurance Model

1. We consider a moral-hazard-free insurance problem in which the insurer calculates the premium based on Wang's premium principle and the insured's risk preference is characterized by the rank-dependent utility theory. This model generalizes the model in which the insurer is risk-neutral. Moral-hazard-free means the compensation and retention must be non-decreasing on the loss; without this requirement, the insured has the incentive to report the loss falsely. This consideration is more realistic but causes a great theoretical challenge as the corresponding optimization problem is constrained with a bound on the derivatives of the compensation and the retention.

2. Theoretically, we solve a constrained quantile optimization problem where the derivative of the admissible quantile is globally bounded. By means of the calculation of variation, the solution is characterized by the solution of an ordinary differential equation (ODE) that can be solved numerically. A numerical example is presented as well.

1.5 Organization of the Thesis

Chapter 2 introduces the return-oriented portfolio selection model. The inspiration behind our model's distinctive features is explained in Section 2.1. Its theoretical significance stems from observation in practice. In Section 2.2, we define an optimal control problem. The underlying financial market on which the investor trades is specified, as well as the risk preferences of the agents in this model. There are a few assumptions and constraints that will be made. In Section 2.3, we reduce the optimal control problem to an equivalent solvable problem using tools like the quantile formulation and relaxation method. Finally, we derive a closed-form solution and compare it with the conventional result.

Chapter 3 presents the optimal moral-hazard-free insurance model. In Section 3.1, we provide background information, including a literature review and motivation. The formulation of the optimal control problem is the focus of Section 3.2, and the original problem is converted to a constrained quantile optimization problem. Section 3.3 and Section 3.4 present a characterization of the optimal solution and a numerical illustration, respectively.

Chapter 4 summarizes the contribution and discusses the potential future works.

1.6 Preliminaries

Before looking deeply into the models, we give a brief introduction to the fundamental concepts about quantile function and list some necessary results that will be used throughout the thesis.

We consider a canonical probability space $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}([0,1]), L)$, where L stands for the *Lebesgue measure*, and $\mathcal{B}([0,1])$ is the Borel set of $[0,1]$. In this probability space, $\xi : \Omega \to \mathbb{R}$ is called a *random variable* (r.v.) if its inverse map $\xi^{-1} : \mathcal{B}(\mathbb{R}) \to \mathcal{F}$, which is defined as

$$\xi^{-1}(B) := \{\omega \in \Omega \mid \xi(\omega) \in B\}, \quad \forall\, B \in \mathcal{B}(R),$$

is $\mathcal{F}$-measurable. Here $\mathcal{B}(\mathbb{R})$ is the Borel set of $\mathbb{R}$. The function

$$F_\xi(x) := \mathbb{P}\{\xi \le x\} = \mathbb{P} \circ \xi^{-1}\{(-\infty, x)\}$$

is called the *cumulative distribution function* (cdf) of ξ and it maps $\mathbb{R}$ to $[0,1]$. The induced probability measure $\mathbb{P} \circ \xi^{-1}$ is called the *law* of ξ. Clearly F_ξ is increasing[8] by definition. Because

$$F_\xi(z) = \mathbb{P}\{\xi \le z\} = \mathbb{E}[1_{\xi \le z}]\,,$$

by the Dominated Convergence Theorem, we see F_ξ is increasing and right-continuous with $\lim_{x\to\infty} F_\xi(x) = 1$ and $\lim_{x\to-\infty} F_\xi(x) = 0$. Conversely, we will show below any function with the above properties is a cdf. Hereafter we say r.v.s X and Y are equal, or $X = Y$ almost surely (a.s.) if $\mathbb{P}\{X \ne Y\} = 0$. We will not distinguish equal r.v.s.

Lemma 1.1. *Given any increasing and right-continuous function $F : \mathbb{R} \to [0,1]$ with $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$, let*

$$\xi(\omega) = \inf\{z \in \mathbb{R} \mid F(z) \ge \omega\}, \quad \omega \in [0,1].$$

Then ξ is a r.v. and its cdf is F.

[8] Throughout this thesis, "increasing" means "non-decreasing" and "decreasing" means "non-increasing".

Proof. For any $\omega \in [0,1]$, $z \in \mathbb{R}$ such that $F(z) \geq \omega$, by the definition of ξ, we have $\xi(\omega) \leq z$. On the other hand, if $F(z) < \omega$, then by the right-continuity of F, we have $F(z+\epsilon) < \omega$ for some sufficiently small $\epsilon > 0$. So

$$\xi(\omega) = \inf\{z \in \mathbb{R} \mid F(z) \geq \omega\} \geq \inf\{z \in \mathbb{R} \mid F(z) > F(z+\epsilon)\} \geq z+\epsilon > z,$$

by the increasing property of F. Hence, we proved that $\omega \leq F(z)$ if and only if $\xi(\omega) \leq z$. Consequently,

$$\mathbb{P}\{\xi(\omega) \leq z\} = \mathbb{P}\{\omega \leq F(z)\} = F(z).$$

Therefore, F is the cdf of ξ. □

We give the definition of the quantile function for r.v. which plays a critical role in our analysis. It is the (left-continuous) inverse function of cdf.

Definition 1.1. *For a r.v. ξ, the (lower) quantile function of ξ is defined as*

$$Q_\xi(p) := \inf\{z \in \mathbb{R} \mid F_\xi(z) \geq p\}, \quad p \in (0,1),$$

with the convention that $Q_\xi(0) = Q_\xi(0+)$, $Q_\xi(1) = Q_\xi(1-)$ and $\inf \emptyset = +\infty$.

Remark 1.1. *In the insurance literature, people also consider the upper quantile function of ξ, which is defined as*

$$Q_\xi^+(p) = \inf\{z \in \mathbb{R} \mid F_\xi(z) > p\}, \quad p \in (0,1).$$

This will not make any difference to our portfolio choice and insurance problems because the two definitions are different at a zero measure set and the integrals in the target assign the same value for the two definitions.

Lemma 1.2. *Given a r.v. ξ, $F_\xi(Q_\xi(p)) \geq p$ and $Q_\xi(F_\xi(p)) \leq p$ for any $p \in [0,1]$. Moreover,*

$$F_\xi(z) = \inf\big\{p \in [0,1] \mid Q_\xi(p) > z\big\}, \quad z \in \mathbb{R}\,.$$

Proof. By the increasing property of F_ξ and the definition of Q_ξ, $F_\xi(Q_\xi(p)+\epsilon) \geq p$ for any $\epsilon > 0$. Because F_ξ is right-continuous, we see $F_\xi(Q_\xi(p)) \geq p$. Similarly

$$Q_\xi(F_\xi(p)) = \inf\{z \in \mathbb{R} \mid F_\xi(z) \geq F_\xi(p)\} \leq p.$$

Let us show the second conclusion

$$F_\xi(z) = \inf\big\{p \in [0,1] \mid Q_\xi(p) > z\big\}.$$

Suppose there exists c such that

$$\inf\big\{p \in [0,1] \mid Q_\xi(p) > z\big\} > c > F_\xi(z).$$

Then $Q_\xi(c) \leq z$. And by our first conclusion $F_\xi(z) \geq F_\xi(Q_\xi(c)) \geq c > F_\xi(z)$, a contradiction. Therefore, we have

$$\inf\big\{p \in [0,1] \mid Q_\xi(p) > z\big\} \leq F_\xi(z).$$

Now suppose there exists c such that

$$\inf\big\{p \in [0,1] \mid Q_\xi(p) > z\big\} < c < F_\xi(z).$$

Then $Q_\xi(c) > z$. And by our first conclusion $Q_\xi(F_\xi(z)) \leq z < Q_\xi(c) \leq Q_\xi(F_\xi(z))$, a contradiction. This proves our claim. □

The set of all (lower) quantile functions is denoted by $\mathbb{Q}$. We have the following characterization.

Lemma 1.3. *We have*

$$\mathbb{Q} = \big\{Q : [0,1] \to \mathbb{R} \mid Q \textit{ is increasing and left-continuous}\big\}.$$

Proof. Clearly $\mathbb{Q}$ is contained in the set on the right hand side.

To show the reverse implication, suppose Q is increasing and left-continuous. Let $\xi(\omega) = Q(\omega)$ for $\omega \in (0,1)$ and $\xi(0) = \xi(1) = 0$. We want to show that the quantile function of ξ is Q. By definition, $F_\xi(z) = \mathbb{P}(\xi \leq x) = \mathbb{P}(Q(\omega) \leq z)$, so

$$Q_\xi(p) = \inf\{z \in \mathbb{R} \mid \mathbb{P}(Q(\omega) \leq z) \geq p\}, \quad p \in (0,1).$$

By the increasing property of Q, we have $\mathbb{P}(Q(\omega) \leq Q(p)) \geq \mathbb{P}(\omega \leq p) = p$, so $Q_\xi(p) \leq Q(p)$. On the other hand, because Q is left-continuous, for any $z < Q(p)$, there exists $\epsilon > 0$ such that $z < Q(p - \epsilon)$. Hence

$$\mathbb{P}(Q(\omega) \leq z) \leq \mathbb{P}(Q(\omega) < Q(p-\epsilon)) \leq \mathbb{P}(\omega < p - \epsilon) = p - \epsilon < p.$$

This indicates $Q_\xi(p) \geq Q(p)$. This completes the proof. □

By the proof we can see

Corollary 1.1. *We have* $\xi(\omega) = Q_\xi(\omega)$ *a.s.*

If a r.v. U satisfies $\mathbb{P}\{U \leq p\} = p$ for $p \in [0,1]$. Then we say U is uniformly distributed on $[0,1]$. Denote $\mathcal{U}$ the set of random variables which are uniformly distributed on $[0,1]$.

Corollary 1.2. *Given any r.v.* ξ*. If a r.v.* U *is uniformly distributed on* $[0,1]$*, then* $Q_\xi(U)$ *has the same cdf as* ξ*.*

Proof. Given $p \in [0,1]$, we see $\mathbb{P}\{\xi \leq p\} = \mathbb{P}\{Q_\xi(\omega) \leq p\}$ by the above Corollary 1.1. Because Q_ξ is left-continuous, there exists c such that $\{\omega \in [0,1] : Q_\xi(\omega) \leq p\} = \{\omega \in [0,1] : \omega \leq c\}$. Therefore, $\{Q_\xi(U(\omega)) \leq p\} = \{U(\omega) \leq c\}$ and

$$\mathbb{P}\{Q_\xi(U(\omega)) \leq p\} = \mathbb{P}\{U(\omega) \leq c\} = c = \mathbb{P}\{\omega \leq c\} = \mathbb{P}\{Q_\xi(\omega) \leq p\} = \mathbb{P}\{\xi \leq p\}.$$

It completes the proof. □

Corollary 1.3. *For any r.v. ξ, and any increasing continuous function $f : \mathbb{R} \to \mathbb{R}$, we have $f(Q_\xi(p)) = Q_{f(\xi)}(p)$ for any $p \in [0,1]$.*

Proof. For any $p_0 \in [0,1]$, let $z_0 = Q_{f(\xi)}(p_0)$, namely

$$z_0 = \inf\{z \in \mathbb{R} \mid \mathbb{P}\{f(\xi) \leq z\} \geq p_0\}.$$

By definition, we have $\mathbb{P}\{f(\xi) \leq z_0\} \geq p_0$, and for $h < z_0$, $\mathbb{P}\{f(\xi) \leq h\} < p_0$. Note that f is continuous and non-decreasing, so there exists an interval $[b, c]$ such that $\{x : f(x) = z_0\} = [b, c]$. Then $f(x_1) < f(b) = z_0$ for any $x_1 < b$. By the definition of z_0, it implies $\mathbb{P}\{\xi \leq x_1\} \leq \mathbb{P}\{f(\xi) \leq f(x_1)\} < p_0$. Hence

$$Q_\xi(p_0) = \inf\{z \in \mathbb{R} \mid \mathbb{P}\{\xi \leq z\} \geq p_0\} \geq b.$$

On the other hand, $f(x_2) > f(c) = z_0$ for any $x_2 > c$. So $f(\xi) \leq z_0$ implies $\xi \leq c$. Besides, $\{\xi \leq c\} \subseteq \{f(\xi) \leq z_0\}$. It means $\{f(\xi) \leq z_0\}$ is equivalent to $\{\xi \leq c\}$, so $\mathbb{P}\{\xi \leq c\} = \mathbb{P}\{f(\xi) \leq z_0\} \geq p_0$, and

$$Q_\xi(p_0) = \inf\{z \in \mathbb{R} \mid \mathbb{P}\{\xi \leq z\} \geq p_0\} \leq c.$$

Therefore, $Q_\xi(p_0) \in [b, c]$ and $f\big(Q_\xi(p_0)\big) = z_0 = Q_{f(\xi)}(p_0)$. The proof is complete. □

Next we introduce a very important concept called *comonotonicity* which characterizes the dependency between random variables. Dhaene, et al [44] gave several characterizations of comonotonicity. We give the following definition:

Definition 1.2. *We say two r.v.s X and Y are comonotonic, if the joint distribution of (X, Y) defined by $F_{X,Y}(x, y) = \mathbb{P}\{X \leq x,\ Y \leq y\}$ satisfies $F_{X,Y}\{x, y\} = \min\{F_X(x), F_Y(y)\}$ for any $x, y \in \mathbb{R}$. We say X and Y are anti-comonotonic if X and $-Y$ are comonotonic.*

We have following basic lemma.

Lemma 1.4. *Two r.v.s X and Y are equal if and only if they are comonotonic and have the same cdf.*

Proof. It is easy to check equal random variables are comonotonic and have the same cdf. Suppose X and Y are comonotonic and have the same cdf. Then by Corollary 1.1,

$$\mathbb{P}\{\omega : X(\omega) = Y(\omega)\} = \mathbb{P}\{\omega : Q_X(\omega) = Q_Y(\omega)\} = 1.$$

This completes the proof. □

Corollary 1.4. *Given r.v.s $U \in \mathcal{U}$ and X. Then $X = Q_X(U)$, a.s. if and only if X and U are comonotonic*

Proof. It is the immediate consequence of Lemma 1.4 and Corollary 1.2 □

Corollary 1.5. *Given r.v.s $U \in \mathcal{U}$ and X. Then X is anti-comonotonic with $Q_X(1-U)$ if and only if X and U are comonotonic.*

Proof. It is an immediate consequence of Corollary 1.4. □

To construct a specific $U \in \mathcal{U}$ such that U is comonotonic with a given random variable, we have following lemma. We call a r.v. is atomless if its cdf is a continuous function on $\mathbb{R}$.

Lemma 1.5. *If a r.v. X is atomless, then $F_X(X) \in \mathcal{U}$. Morover, $F_X(X)$ and X are comonotonic.*

Proof. Because F_X is continuous, we can see $\{X \leq Q_X(p)\}$ if and only if $\{F_X(X) \leq p\}$ for $p \in (0,1)$. Therefore,

$$\mathbb{P}\{F_X(X) \leq p\} = \mathbb{P}\{X \leq Q_X(p)\} = F_X(Q_X(p)) = p,$$

where the last equality is also ensured by continuity. □

Remark 1.2. *For atomic r.v.s, a construction can be found in Xu [131].*

Next we give a critical lemma that will be used in the subsequent section. It is called the *Hardy-Littlewood Inequality.*

Lemma 1.6 (Hardy-Littlewood Inequality)**.** *Suppose r.v.s* ξ_1 *and* η *are comonotonic,* ξ_2 *and* η *are anti-comonotonic, and* ξ_1 *and* ξ_2 *have the same cdf* Q*. Then for any r.v.* $\tilde{\xi}$ *having the cdf* Q*, we have*

$$\mathbb{E}[\xi_2 \cdot \eta] \leq \mathbb{E}\left[\tilde{\xi} \cdot \eta\right] \leq \mathbb{E}[\xi_1 \cdot \eta]\,,$$

provided that the first and last expectations exist and finite, where the first and second equalities hold if and only if $\tilde{\xi} = \xi_2$ *and* $\tilde{\xi} = \xi_1$*, respectively.*

Proof. Without losing of generality, we only consider non-negative random variables, as one can use the monotone convergence theorem to prove the general case. For non-negative variable ξ, we have the following expression

$$\xi(\omega) = \int_0^\infty 1_{\{\xi(\omega)\geq t\}}\,\mathrm{d}t.$$

By Fubini's Theorem, we have

$$\begin{aligned}
\mathbb{E}\left[\tilde{\xi} \cdot \eta\right] &= \mathbb{E}\int_0^\infty 1_{\{\tilde{\xi}(\omega)\geq t\}}\,\mathrm{d}t \int_0^\infty 1_{\{\eta(\omega)\geq s\}}\,\mathrm{d}s \\
&= \int_0^\infty \int_0^\infty \mathbb{E}\left[1_{\{\tilde{\xi}(\omega)\geq t\}} 1_{\{\eta(\omega)\geq s\}}\right] \mathrm{d}t\,\mathrm{d}s \\
&= \int_0^\infty \int_0^\infty \mathbb{P}\{\tilde{\xi}(\omega) \geq t, \eta(\omega) \geq s\}\,\mathrm{d}t\,\mathrm{d}s \\
&\leq \int_0^\infty \int_0^\infty \min\{\mathbb{P}\{\tilde{\xi}(\omega) \geq t\}, \mathbb{P}\{\eta(\omega) \geq s\}\}\,\mathrm{d}t\,\mathrm{d}s \\
&= \int_0^\infty \int_0^\infty \min\{\mathbb{P}\{\xi_1(\omega) \geq t\}, \mathbb{P}\{\eta(\omega) \geq s\}\}\,\mathrm{d}t\,\mathrm{d}s \\
&= \int_0^\infty \int_0^\infty \mathbb{P}\{\xi_1(\omega) \geq t, \eta(\omega) \geq s\}\,\mathrm{d}t\,\mathrm{d}s \\
&= \int_0^\infty \int_0^\infty \mathbb{E}\left[1_{\{\xi_1(\omega)\geq t\}} 1_{\{\eta(\omega)\geq s\}}\right] \mathrm{d}t\,\mathrm{d}s = \mathbb{E}[\xi_1 \cdot \eta]\,.
\end{aligned}$$

The inequality is due to that $\mathbb{P}(A \cap B) \leq \min(\mathbb{P}(A), \mathbb{P}(B))$, the last third and second equalities are due to that $\tilde{\xi}$ and ξ have the same cdf and Definition 1.2 respectively.

Therefore, $\mathbb{E}\left[\tilde{\xi} \cdot \eta\right] \le \mathbb{E}[\xi_1 \cdot \eta]$. Using the fact that ξ_2 is comonotonic with $-\eta$, we have $\mathbb{E}[\xi_2 \cdot -\eta] \ge \mathbb{E}\left[\tilde{\xi} \cdot -\eta\right]$, namely $\mathbb{E}[\xi_2 \cdot \eta] \le \mathbb{E}\left[\tilde{\xi} \cdot \eta\right]$. The proof is complete. $\square$

Chapter 2

A Return-Rate Based Portfolio Selection Model

In this chapter, we present a continuous-time portfolio optimization problem capturing behavioral preference agents with the goal of maximizing utility of log-return. The illustration can be split into three sections. Section 2.1 goes through the motivation and background, as well as the theoretical significance, behind this problem. In Section 2.2, we define a market driven by Brownian motion and a risk measure characterized by the CPT, and then formulate an optimal control problem. In Section 2.3, we first convert the problem into an equivalent quantile optimization problem and then derive the associated optimal solution using change of variable and a relaxation method.

2.1 Motivation

One of the tacit rules adopted in continuous-time portfolio optimization problems is the expected utility hypothesis on the psychology of people's choice-making. It goes back to the pioneering work of Merton [95], [96] in which the agent has a portfolio built upon a simplified market and seeks to maximize the expected utility from consumption. However, for problems that do not consider consumption behavior and other injections of income during the investment horizon, the targets are almost maximization of the expected utility from terminal wealth or/plus either exogenous or endogenous penalty terms. The idea is quite natural since the majority of investors are concerned about how much money they will obtain when they decide to liquidate their portfolios. But in practice, as one may notice, practitioners and institutions with long-term investment horizons do not always keep their eyes on the book value of their portfolios. Moreover, they care about those return-based indexes such as net return, periodic annual return, and return on investment (ROI), which are adopted as measurements of the portfolios' performances. Taking return as a criterion not only allows for a normalized comparison between wealth managers but also contributes to capital raising [1]for institutions like funds and banks by removing the impact of the initial endowment. For fund managers, achieving a higher return represents excellent investment skills and brings fame and fortune at the end of each year. In this light, it is reasonable to imagine an agent whose purpose is to maximize the expected utility from return rather than terminal wealth.

In fact, the utility of return is not a fresh terminology and exsits in the literature. For example, Benartzi and Thaler [14] tried to use prospect theory to explain the famous equity premium puzzle (Campbell and Cochrane [28]). In their model, the prospective utility is calculated based on the changes in wealth, namely returns. Besides, if one treats the Markowitz's one-period mean-variance analysis as an investor with quadratic utility, then the utility function has already been imposed on the return right from the beginning. An important question is, will it be different if the target is to maximize the utility of return? Intuitively, one may believe that the optimal portfolio that maximizes the utility of terminal wealth must also maximize the utility of return. Theoretically, such equivalence does not exist in general. Markowitz [93] demonstrated that the equivalence between the utility of

[1] For example, advertising a financial product reputed to generate \$10,000 in a year is clearly no better than titling it with a corresponding 20% annual return. The former expression has a chance of being eclipsed by a billionaire who thinks \$10,000 is not attractive at all.

gross return and the utility of wealth disappears when the utility function is exponential. Apart from that, using different return indexes may result in different objection functionals, which makes the problems more challenging. For example, in this model, we consider log-return, a typical type of return used in financial time series, which means the return is continuously compounded. It is natural to consider it in a continuous-time setting, and it possesses an additive property on time, which is convenient for calculating multi-period returns and annualizing. In addition, those market anomalies, such as the equity premium puzzle and the volatility puzzle (Campbell [27]), have both found statistical support on the log-return. It conforms to reality. As we will see, using log-return results in a non-concave utility shape, which poses the major hurdle to the problem. Recently, Dai et al. [38] investigated a dynamic mean-variance portfolio choice problem based on log-return (Log-MV criteria). The optimal policies under specific settings are found to be consistent with several conventional investment wisdoms that are usually contradicted by models based on terminal wealth.

Furthermore, the situation becomes more complex for behavioral investors. As most non-expected utility theories (Friedman and Savage [55], Markowitz [91], A. Tversky and D. Kahneman [122], [73]) have been supported by empirical observation and experiments, the problem becomes challenging and quite different once we introduce non-expected utility functions and non-linear decision weights. Incorporating non-expected utility theory into continuous-time portfolio choice problems has received much attention in recent years, as already mentioned in the introduction chapter (See Jin and Zhou [71], He and Zhou [68], Xu [132], Berkelaar, Kouwenberg, and Post [16]). To our best knowledge, there are fewer than a handful of papers available in the literature that investigate return-oriented portfolio choice problems using behavioral performance criteria. We will present a behavioral portfolio choice model in this chapter to fill this gap. Anyway, it deserves to take a close look at return-oriented portfolio selection problems under non-EU preference, theoretically and practically.

Considering that institutions use prominent indices or portfolios to evaluate funds' performance in practice, we will add a benchmark and a performance constraint to our model. Apart from that, we will also consider weighted objective probability rather than itself, to explain those biases and errors appeared in the decision-making progress. Under this setting, the target of the portfolio choice problem becomes a non-linear expectation (Choquet expectation) of log-return. An immediate consequence is the failure of Bellman's optimality, which relies on the tower property of linear expectation. So, the classical dynamic programming principle is not applicable to our problem. One possible approach is to rewrite the target by replacing the decision variable with its quantile, which is the so-called "quantile formulation". This method is used in Jin and Zhou [71], He and Zhou [68], among many others. Xu [132] investigated a portfolio choice problem under the Rank-dependent utility theory (RDUT). He solved the corresponding quantile optimization problem by using a relaxation method. His optimization problem is a global optimization problem since there are no constraints on the domain of the decision variable other than the budget constraint. In our model, the problem becomes a local optimization problem due to the performance constraint. Besides, the utility function in Xu [132] is concave, but in this paper, we find the utility of the corresponding quantile optimization problem is of concave-convex-concave type (called M-shaped). This non-concave quantile optimization problem has rarely been considered in the literature. We prove that it is equivalent to studying its concavified problem. In the next section, we will give a detailed illustration of our model and formulate the corresponding optimal control problem.

2.2 Problem Formulation

Let $T > 0$ be a fixed known investment maturity throughout this chapter. Let us introduce our financial market. The underlying market is defined as the filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{0\leqslant t\leqslant T})$ on which a standard $\mathcal{F}_t$-adapted, n-dimensional Brownian motion $W(\cdot) \equiv (W^1(\cdot), \ldots, W^n(\cdot))'$ is defined. We assume the uncertainty of the market entirely comes from the Brownian motion and define the information filtration $\mathcal{F}_t = \sigma\{W(s), 0 \leq s \leq t\}$, which is augmented by all the $\mathbb{P}$-null sets. Also $\mathcal{F}_T = \mathcal{F}$.

2.2.1 Market and Portfolio

Suppose the financial market consists of $n+1$ assets which are traded continuously over the investment horizon $[0, T]$ without friction (there is no transaction costs, tax or any other restriction imposed on transaction). One of the assets is a *bond* (also called risk-less asset), whose price $S_0(\cdot)$ evolves according to the ordinary differential equation (ODE).

$$\begin{cases} \mathrm{d}S_0(t) = r(t)S_0(t)\,\mathrm{d}t, \quad t \in [0, T], \\ S_0(0) = s_0 > 0, \end{cases}$$

where $r(t)$ is the instantaneous interest rate of the bond at time t. The remaining n assets are *stocks* (also called risky assets), and their prices $S_i(\cdot)$, $i = 1, 2, \ldots, n$, are modeled by the system of stochastic differential equations (SDEs)

$$\begin{cases} \mathrm{d}S_i(t) = S_i(t)\{b_i(t)\,\mathrm{d}t + \sum_{j=1}^{m} \sigma_{ij}(t)\,\mathrm{d}W^j(t)\}, \quad t \in [0, T], \\ S_i(0) = s_i > 0, \end{cases} \tag{2.1}$$

where $b_i : [0, T] \times \Omega \to \mathbb{R}$ with $b_i(t) > 0$ is the *appreciation rate* of the stock i and $\sigma_{ij} : [0, T] \times \Omega \to \mathbb{R}$ is the *volatility* coefficient of stock i with respect to W^j at time t. Define the volatility matrix $\sigma(t) := (\sigma_{ij}(t))_{n\times m} : [0, T] \times \Omega \to \mathbb{R}^{n\times m}$ and the excess return vector process $\mu(t) = (b_1(t) - r(t), \ldots, b_n(t) - r(t))' : [0, T] \times \Omega \to \mathbb{R}^n$. We have the following basic technical assumption to ensure an existence and uniqueness of solution of the above ODE and SDEs

Assumption 2.1. *The processes $r(t)$, $b(t)$, $\sigma(t)$ are progressively measurable with respect to the filtration $\{\mathcal{F}_t\}$ and satisfy*

$$\int_0^T |r(s)|ds < +\infty, \ \ a.s.$$

and

$$\int_0^T \left[\sum_{i=1}^m |b_i(t)| + \sum_{i,j=1}^m |\sigma_{ij}(t)|^2 \right] \mathrm{d}t < +\infty, \ \ a.s.$$

Moreover, the SDEs (2.1) *admits a unique strong solution.*

Consider an agent with an initial wealth $x_0 > 0$. She invests in the assets in the market but her action cannot affect the market and assets price. Let $\pi_i(t)$ denote the proportion of her total wealth invested in the stock i at time t, $i = 1, \ldots, n$. Obviously, the proportion invested in the bond can be derived by $\pi_0(t) = 1 - \sum_{i=1}^n \pi_i(t)$. We call the vector process $\pi(t) := (\pi_1(t), \ldots, \pi_n(t))'$ a *portfolio* process and denote $X^\pi(t)$ the agent's related *wealth* process at time t with the implement of portfolio $\pi(t)$. We assume there is no transaction costs, or any other kinds of withdrawal (consumption) or income (dividend) during the investment horizon $[0, T]$, namely the change of wealth process only comes from the change of the assets price in the market. This is the so-called self-financing trading strategy. Mathematically, if $N_i^\pi(t)$ denotes the number of shares invested in the asset i under a portfolio π, then

$$X^\pi(t) = N_0^\pi(t) S_0(t) + \sum_{i=1}^n N_i^\pi(t) S_i(t).$$

Therefore, for a self-financing strategy π,

$$\mathrm{d}X^\pi(t) = N_0^\pi(t)\,\mathrm{d}S_0(t) + \sum_{i=1}^n N_i^\pi(t)\,\mathrm{d}S_i(t).$$

Because $N_i^\pi(t) = \frac{X^\pi(t)\pi_i(t)}{S_i(t)}$ and $\pi_0(t) = 1 - \sum_{i=1}^n \pi_i(t)$, we have the following proposition:

Proposition 2.1 (Self-financing)**.** *An $\{\mathcal{F}_t\}$-progressively measurable portfolio $\pi(t) : [0,T] \times \Omega \to \mathbb{R}^n$ is called self-financed if and only if:*

$$X^\pi(t) = x_0 + \int_0^t \frac{X^\pi(t)(1 - \sum_{i=1}^n \pi_i(t))}{S_0(t)}\,\mathrm{d}S_0(t) + \sum_{i=1}^n \int_0^t \frac{X^\pi(t)\pi_i(t)}{S_i(t)}\,\mathrm{d}S_i(t),\ \ a.s.$$

For a self-financing strategy π,

$$\begin{aligned}
\mathrm{d}X^\pi(t) &= X^\pi(t)(1 - \sum_{i=1}^n \pi_i(t))r(t)\,\mathrm{d}t + \sum_{i=1}^n X^\pi(t)\pi_i(t)\big[b_i(t)\,\mathrm{d}t + \sum_{j=1}^m \sigma_{ij}(t)\,\mathrm{d}W^j(t)\big] \\
&= X^\pi(t)\sum_{i=1}^n \pi_i(t)r(t)\,\mathrm{d}t + X^\pi(t)\sum_{i=1}^n \pi_i(t)\big[(b_i(t) - r(t))\,\mathrm{d}t + \sum_{j=1}^m \sigma_{ij}(t)\,\mathrm{d}W^j(t)\big] \\
&= X^\pi(t)r(t)\,\mathrm{d}t + X^\pi(t)\big[\pi(t)'\mu(t)\,\mathrm{d}t + \pi(t)'\sigma(t)\cdot\mathrm{d}W(t)\big].
\end{aligned}$$

Therefore, the wealth process $X^\pi(\cdot)$ evolves according to the following SDE:

$$\begin{cases} \mathrm{d}X^\pi(t) = X^\pi(t)\big[(r(t) + \pi(t)'\mu(t))\,\mathrm{d}t + \pi(t)'\sigma(t)\cdot\mathrm{d}W(t)\big], & t \geqslant 0, \\ X^\pi(0) = x_0 > 0. \end{cases} \tag{2.2}$$

We call a portfolio π feasible if it is self-financing and (2.2) has a unique strong solution $X^\pi(t)$. From now on, we only consider feasible portfolios.

Applying Itô's lemma to $\ln X^\pi(t)$,

$$\mathrm{d}\ln X^\pi(t) = \big(r(t) + \pi(t)'\mu(t) - \frac{1}{2}\|\pi(t)'\sigma(t)\|^2\big)\,\mathrm{d}t + \pi(t)'\sigma(t)\cdot\mathrm{d}W(t).$$

Integrating both sides yields

$$X^\pi(t) = x_0 \cdot \exp\left\{ \int_0^t \left(r(s) + \pi(s)'\mu(s) - \frac{1}{2}\|\pi(s)'\sigma(s)\|^2\right) \mathrm{d}s \right.$$
$$\left. + \int_0^t \pi(s)'\sigma(s) \cdot \mathrm{d}W(s) \right\} > 0.$$

Therefore, we have a no-bankruptcy condition inherent in the wealth process.

Another general acquiescence in the financial market is the assumption of no-arbitrage. Mathematically, an *arbitrage* means there exists a portfolio $\pi(t)$ such that $X^\pi(0) = 0$, $X^\pi(T) \geq 0$, and $\mathbb{P}(X^\pi(T) > 0) > 0$ or a slightly stronger condition that $X^\pi(0) < 0$, $X^\pi(T) \geq 0$. The existing of an arbitrage means that one can make a (potential positive) profit in the market without facing any risk of loss.

In the meanwhile, we assume the market is complete. This means any target can be perfectly hedged if one is provided with enough initial endowment. Technically, we assume

Assumption 2.2. *There exists a unique essentially bounded risk premium process* $\theta(t)$ *such that* $\sigma(t)\theta(t) = \mu(t)$, $t \in [0, T]$.

This assumption indicates that the matrix process $\sigma(t)$ is invertible.

Let

$$\rho(t) = \exp\left(-\int_0^t \left(r(s) + \tfrac{1}{2}\|\theta(s)\|^2\right) \mathrm{d}s - \int_0^t \theta(s) \cdot \mathrm{d}W(s) \right), \quad t \in [0, T].$$

Then it follows from Itô's lemma that

$$\mathrm{d}\rho(t) = -\rho(t)\big(r(t)\,\mathrm{d}t + \theta(t) \cdot \mathrm{d}W(t)\big).$$

Denote $\rho = \rho(T)$. It is called the *pricing kernel* or *stochastic discount factor* of the market in the literature.

For any feasible portfolio π, we have by Itô's lemma,

$$\mathrm{d}\left(\rho(t)X^\pi(t)\right) = \rho(t)X^\pi(t)\big(\pi(t)'\sigma(t) - \theta(t)'\big)\,\mathrm{d}W(t). \tag{2.3}$$

Hence $\rho(t)X^\pi(t)$ is a local martingale, but it is a positive process, so it is a supermartingale. Hence

$$\mathbb{E}[\rho X^\pi(T)] \le x_0. \tag{2.4}$$

This is often called the *budget constraint*. No matter how the investor trades in the market, the result must obey this constraint. It is easy to check that for any π with $X^\pi(T) \ge 0$ and $\mathbb{P}(X^\pi(T) > 0) > 0$, we have $x_0 \ge \mathbb{E}[\rho X^\pi(T)] > 0$, so the no-arbitrage condition is satisfied for any feasible strategy.

One feature of this model is the performance will be measured by return instead of terminal wealth. Let $\mathfrak{R}^\pi(t) = \ln(X^\pi(t)/x_0)$ be the total logarithmic return of an portfolio $\pi(\cdot)$ at time t. The (normalized) return rate over $[0,t]$ is then given by $t^{-1}\mathfrak{R}^\pi(t)$ for $t > 0$. By Itô's Lemma and (2.2), we have

$$\begin{cases} \mathrm{d}\mathfrak{R}^\pi(t) = \big(r(t) + \pi(t)'\mu(t) - \frac{1}{2}\pi(t)'\sigma(t)\sigma(t)'\pi(t)\big)\,\mathrm{d}t + \pi(t)'\sigma(t)\cdot \mathrm{d}W(t), \quad t \geqslant 0, \\ \mathfrak{R}^\pi(0) = 0. \end{cases} \tag{2.5}$$

Notice that $\mathfrak{R}^\pi(t)$ may be negative, meaning the position of wealth slides into a loss. Compared with (2.2), this expression is irrelevant to the investment initial value x_0, making the model universal (that is, the optimal strategy is for all investors). The budget constraint can also be written in terms of $\mathfrak{R}^\pi(T)$ as

$$\mathbb{E}\left[\rho e^{\mathfrak{R}^\pi(T)}\right] \le 1. \tag{2.6}$$

Furthermore, to measure the performance relatively, we introduce a benchmark process $\mathfrak{B}$ and denote by $\mathfrak{B}_t$ its value at time t. It is an $\{\mathcal{F}_t\}$-adapted stochastic or deterministic process. Apparently, $\mathfrak{B}$ must be chosen carefully or at least be set up not very high. In practice, $\mathfrak{B}$ can be a short-term return of a stock index or structured products, a benchmark one-year deposit rate, a personal target, etc. We assume that the investor will not use any strategies that lead to very poor performance in the end. Mathematically, we impose the following *lower bound* constraint

$$\mathfrak{R}^\pi(T) - \mathfrak{B}_T \ge -c, \tag{2.7}$$

where $c > 0$ is a given constant risk tolerance level. To exclude the trivial case, we assume there exists at least one portfolio π_0 such that

$$\mathfrak{R}^{\pi_0}(T) - \mathfrak{B}_T > -c.$$

Notice that (2.7) is equivalent to $X^{\pi}(T) \geq x_0 e^{\mathfrak{B}_T - c}$. Combining it with the budget constraint (2.4), we see $x_0 \geq \mathbb{E}[\rho X^{\pi}(T)] \geq \mathbb{E}\left[\rho x_0 e^{\mathfrak{B}_T - c}\right]$, namely $\ln(\mathbb{E}\left[\rho e^{\mathfrak{B}_T}\right]) \leq c$. We list it as a basic assumption to ensure its feasibility.

Assumption 2.3 (feasibility). *The trio of the pricing kernel ρ, benchmark $\mathfrak{B}$ and lower bound parameter $c > 0$ satisfies*

$$\ln(\mathbb{E}\left[\rho e^{\mathfrak{B}_T}\right]) \leq c.$$

If the above inequality is not satisfied, then there is no feasible strategy to satisfy both (2.4) and (2.7).

Remark 2.1. *If the benchmark is a constant and the interest rate process is a deterministic function, then the above assumption is equivalent to $\mathfrak{B}_T - c \leq \int_0^T r(s)\,\mathrm{d}s$. It means one can not set up an extremely high target of $\mathfrak{B}_T$ with a specified c, otherwise the problem may be ill-posed. One can also derive it from the lower bound constraint* (2.7) *by taking the trivial strategy: placing all the money in the* bond.

Remark 2.2. *An implicit fact underlying Assumption 2.3 is that it is impossible to set a higher benchmark return than $\mathfrak{B}_T$ without accepting the risk of a higher potential maximum loss than c, which corresponds to the saying "higher return comes higher risk".*

Finally, we define the set of admissible portfolios. Given a Hilbert space $\mathcal{H}$ with the norm $\|\cdot\|_{\mathcal{H}}$, we can define a Banach space

$$L^2_{\mathcal{F}}(a,b;\mathcal{H}) = \left\{ \varphi(\cdot) \;\middle|\; \begin{array}{l} \varphi(\cdot) \text{ is an } \{\mathcal{F}_t\}_{t \geqslant 0}\text{-adapted, } \mathcal{H}\text{-valued progressively measurable} \\ \text{process defined on } [a,b] \text{ and satisfies } \|\varphi(\cdot)\|_{\mathcal{F}} < +\infty \end{array} \right\}$$

with the norm

$$\|\varphi(\cdot)\|_{\mathcal{F}} = \left(\mathbb{E}\left[\int_a^b \|\varphi(t,\omega)\|_{\mathcal{H}}^2 \,\mathrm{d}t\right]\right)^{\frac{1}{2}}.$$

We call a feasible portfolio $\pi(t)$ is *admissible* if it satisfied the aforementioned constraints and belongs to the set of admissible portfolios given by

$$\mathcal{A} := \left\{\pi(\cdot) \,\middle|\, X^{\pi}(\cdot)\sigma'(\cdot)\pi(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n),\ X^{\pi}(T) \geq x_0 e^{\mathfrak{B}_T - c}\right\}.$$

We have following important hedging result.

Theorem 2.1. *Suppose $\rho e^{\xi} \in L^2_{\mathcal{F}}$ satisfies $\xi \geq \mathfrak{B}_T - c$ and $\mathbb{E}\big[\rho e^{\xi}\big] = 1$. Then there exists an admissible portfolio π such that $X^{\pi}(T) = x_0 e^{\xi}$ a.s..*

Proof. Let $Y(t) = \mathbb{E}\big[x_0 \rho e^{\xi} \mid \mathcal{F}_t\big]$, then $Y(t)$ is a square integrable martingale. By the martingale representation theorem (Yong and Zhou [136] Chapter 1. Theorem 5.7 PP 38), there exists a unique $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted $Z(t) : [0,T] \to \mathbb{R}^n$ such that $\mathbb{E}\Big[\int_0^T \|Z(t)\|^2 \,\mathrm{d}t\Big] < +\infty$ and $\mathrm{d}Y(t) = Z(t)\,\mathrm{d}W(t)$. Let $X(t) = \rho(t)^{-1}Y(t)$ and $\pi(t)' = \big(\theta(t) + Y(t)^{-1}Z(t)\big)\sigma(t)^{-1}$. Then $X(0) = \rho(0)^{-1}Y(0) = x_0$. By Itô's Lemma, we have

$$\mathrm{d}\left(\rho(t)^{-1}\right) = \rho(t)^{-1}\big((r(t) + \theta(t)^2)\,\mathrm{d}t + \theta(t)\,\mathrm{d}W(t)\big)$$

and

$$\begin{aligned}
\mathrm{d}\,X(t) &= \rho(t)^{-1}\,\mathrm{d}Y(t) + Y(t)\,\mathrm{d}\rho(t)^{-1} + \mathrm{d}\,\langle \rho(t)^{-1}, Y(t)\rangle \\
&= \rho(t)^{-1}Z(t)\,\mathrm{d}W(t) + Y(t)\rho(t)^{-1}\big((r(t) + \theta(t)^2)\,\mathrm{d}t + \theta(t)\,\mathrm{d}W(t)\big) + \rho(t)^{-1}\theta(t)Z(t)\,\mathrm{d}t \\
&= \rho(t)^{-1}Y(t)\Big(\big(r(t) + \big(\theta(t) + Y(t)^{-1}Z(t)\big)\theta(t)\big)\,\mathrm{d}t + \big(\theta(t) + Y(t)^{-1}Z(t)\big)\,\mathrm{d}W(t)\Big) \\
&= X(t)\Big(\big(r(t) + \pi(t)'\sigma(t)\theta(t)\big)\,\mathrm{d}t + \pi(t)'\sigma(t)\,\mathrm{d}W(t)\Big).
\end{aligned}$$

Therefore, $(X(t), \pi(t))$ is a solution to (2.2) such that $X(T) = \rho^{-1}Y(T) = x_0 e^{\xi}$. □

2.2.2 Risk Preference

In our model, we consider a CPT investor rather than an EUT investor. It is implemented by adding a process of probability weighting and using a non-concave value function.

Firstly, we call a function $w\colon [0,1] \mapsto [0,1]$ a probability distortion (or weighting) function if it is strictly increasing and continuously differentiable with $w(0) = 0$ and $w(1) = 1$. The probability weighting function w is arbitrary chosen but fixed throughout this paper. Particularly, we are interested in concave w and inverted S-shaped w. As we introduce the idea of probability weighting into our model, the mathematical expectation involved w for a random variable ξ becomes nonlinear Choquet expectation. Here we define the Choquet expectation of ξ as

$$\mathscr{E}[\xi] = \int_0^{\infty} w(1 - F_\xi(x))\,\mathrm{d}x + \int_{-\infty}^{0} (w(1 - F_\xi(x)) - 1)\,\mathrm{d}x, \tag{2.8}$$

provided that one of the integrals is finite.

Remark 2.3. *Tversky and Kahneman [122] used different probability weighting functions for the gain part (when $\xi > 0$) and loss part (when $\xi < 0$). For simplicity of the presentation, in this paper we use the same probability weighting function for both parts. It is possible to consider different probability weighting functions for the two cases using the "divide and conquer" machinery developed in Jin and Zhou [71] to study the corresponding model. We leave the case for the interested readers.*

Remark 2.4. *When the probability weighting function is the identical function ($w(x) = x$), i.e., there is no probability weighting, the Choquet expectation* (2.8) *reduces to the classical linear expectation. Therefore our model is a generalization of the classical case.*

Secondly, we adopt the piece-wise power value function in Tversky and Kahneman

[122]:

$$u(x) = \begin{cases} x^\alpha, & x \ge 0; \\ -\kappa(-x)^\beta, & x < 0, \end{cases} \tag{2.9}$$

where $0 < \alpha, \beta < 1$ are risk parameters, $\kappa > 0$ represents the degree of *loss aversion*.[2] Notice that it is convex on $(-\infty, 0]$ and concave on $[0, \infty)$ as well as continuous and strictly increasing on $\mathbb{R}$, and continuous differentiable except at the point $x = 0$.

Remark 2.5. *Apart from adopting a power-type piece-wise utility function, one can also consider general case*

$$u(x) = \begin{cases} \text{any increasing concave function,} & x \ge 0; \\ \text{any increasing function with } A_u \text{ having the following properties,} & x < 0, \end{cases}$$

where Arrow-Pratt's measure of absolute risk aversion

$$A_u(x) = -\frac{u''(x)}{u'(x)}$$

is large than -1 first and less than -1.

Finally we can now define the CPT risk preference for a r.v. ξ based on (2.8) and (2.9) as

$$\mathscr{E}[u(\xi)] .$$

The target of our model is given by

$$\max_{\pi(\cdot)\in\mathcal{A}} \quad \mathscr{E}[u(\mathfrak{R}^\pi(T) - \mathfrak{B}_T)] . \tag{2.10}$$

Altogether with (2.7), we can formulate an optimal portfolio choice problem in terms of logarithmic return variable:

$$\max_{\pi(\cdot)\in\mathcal{A}} \quad \mathscr{E}[u(\mathfrak{R}^\pi(T) - \mathfrak{B}_T)] \tag{2.11}$$

[2] In this utility function we used 0 as the reference point, it will make no difference to the subsequent argument if we use a different reference point as we may rename the benchmark of the investor. Furthermore, the specific parameters simulated in [122] are $\alpha = \beta = 0.88$ and $\kappa = 2.5$. We may consider different power indices α, β for the gain and loss situations in the model.

$$\text{s.t.} \quad \mathfrak{R}^{\pi}(T) - \mathfrak{B}_T \geq -c.$$

As the end of Section 2.2, I would like to distinguish the return-oriented objective of Problem (2.11) from those objective functionals which specify a logarithmic utility of wealth.[3] The portfolios induced by such targets are also called *growth optimal portfolio* (GOP), which traces back to Kelly [74] and have been investigated for more than half a century in the literature. The conventional objective $\max_{\pi(\cdot)} \mathbb{E}[u(X^{\pi}(T))]$ becomes maximizing the expected value of log-return by taking logarithmic utility, which looks similar to (2.11). But they are quite different problems.

On one hand, the financial background and motivation behind the targets are different. The GOP is commonly interpreted as an investor whose purpose is to maximize the geometric mean value of gross return. Accidentally, it can be explained as a special example of maximizing the expected log-utility of terminal wealth, rather than the expected utility of return. Due to the logarithmic utility function, they become related to our objective. As already mentioned, in Markowitz [93] (see PP. 3), the author gave a detailed discussion of the relationship between utility of gross return and utility of wealth when the agent has an EU-preference. Although when the utility function is logarithmic, maximizing the former is equivalent to maximizing the latter, such an equivalence has not been verified when the agent is characterized by behavioral risk preference, not to mention that we use log-return rather than gross return. Intuitively, maximizing the utility of gross return within the framework of the EUT does not lead to a very different mathematical structure, and perhaps this is one of the reasons why there was less attention paid to return-oriented objectives in continuous-time portfolio choice problems before.

On the other hand, portfolio choice problems that adopt the EU-preference are naturally concave optimization problems. Traditional approaches, including the con-

[3] Examples can be found in Akian, Sulem, Taksar[2], Goll, Kallsen [59],[60]

vex duality method and dynamic programming principle, may be applicable directly. By contrast, Problem (2.11) is a non-concave optimization problem. The non-EU preference and a different decision variable make the problem challenging and deserves further study.

2.3 Quantile Optimization Problem and its Solution

This section focuses on solving Problem (2.11). In the definition of Choquet expectation (2.8), there are two elements: the weighting function $w(\cdot)$ and the survival function $\mathbb{P}(\xi > x)$. The latter represents the distribution of the decision variable, while the former represents a subjective distortion of the distribution. The explicit form of $w(\cdot)$ proposed in literature (see A. Tversky and D. Kahneman [122]) imposes big complexity if we try to calculate the nonlinear expectation directly. Writing the objective in terms of its quantile of the decision variable will enable us to treat the two elements separately. This is one of the motivations and advantages of the quantile formulation. This technique first appeared in Jin and Zhou [71], and a simplified version was given by Xu [132]. Here we apply this method to solving Problem (2.11).

2.3.1 Quantile Formulation

Firstly, we still write Problem (2.11) in terms of $X^\pi(T)$, which is

$$\begin{aligned} \max_{\pi(\cdot)\in\mathcal{A}} \quad & \mathscr{E}[u(\ln(X^\pi(T)/x_0) - \mathfrak{B}_T)] \\ \text{s.t.} \quad & \ln(X^\pi(T)/x_0) - \mathfrak{B}_T \geq -c. \end{aligned}$$

For any admissible portfolio π, it must obey the budget constraint (2.4), that is, $\mathbb{E}[\rho X^\pi(T)] \leq x_0$. Therefore, this problem is equivalent to

$$\max_{\pi(\cdot)\in\mathcal{A}} \quad \mathscr{E}[u(\ln(X^\pi(T)/x_0) - \mathfrak{B}_T)] \tag{2.12}$$

$$\text{s.t.} \quad \mathbb{E}[\rho X^\pi(T)] \le x_0, \quad \ln(X^\pi(T)/x_0) - \mathfrak{B}_T \ge -c.$$

Let $\zeta^\pi = X^\pi(T)/(x_0 e^{\mathfrak{B}_T})$ and $v(x) = u(\ln x)$. The objective of (2.12) becomes

$$\max_{\pi(\cdot)} \quad \mathscr{E}[v(\zeta^\pi)]\,. \tag{2.13}$$

Note that (2.8) can be rewritten in terms of the quantile of ξ. By partial integration,

$$\begin{aligned}
\mathscr{E}[\xi] &= \int_{F_\xi(0)}^1 w(1-p)\,\mathrm{d}Q_\xi(p) + \int_0^{F_\xi(0)} (w(1-p)-1)\,\mathrm{d}Q_\xi(p) \\
&= Q_\xi(p)w(1-p)\Big|_{F_\xi(0)}^1 + \int_{F_\xi(0)}^1 Q_\xi(p)w'(1-p)\,\mathrm{d}p \\
&\quad + Q_\xi(p)(w(1-p)-1)\Big|_0^{F_\xi(0)} + \int_0^{F_\xi(0)} Q_\xi(p)w'(1-p)\,\mathrm{d}p \\
&= \int_0^1 Q_\xi(p)w'(1-p)\,\mathrm{d}p.
\end{aligned}$$

Therefore, we obtain the identity

$$\mathscr{E}[\xi] = \int_0^1 Q_\xi(p)w'(1-p)\,\mathrm{d}p.$$

It will be very useful in the subsequent analysis.

By virtue of the above expression and Corollary 1.3, we have

$$\mathscr{E}[v(\zeta^\pi)] = \int_0^1 Q_{v(\zeta^\pi)}(p)w'(1-p)\,\mathrm{d}p = \int_0^1 v(Q_{\zeta^\pi}(p))w'(1-p)\,\mathrm{d}p. \tag{2.14}$$

The budget constraint (2.4) in terms of ζ^π reads

$$x_0 \ge \mathbb{E}[\rho X^\pi(T)] = \mathbb{E}\left[\rho\zeta^\pi x_0 e^{\mathfrak{B}_T}\right]$$

or

$$\mathbb{E}\left[\rho\zeta^\pi e^{\mathfrak{B}_T}\right] \le 1.$$

Note that ρ represents the market pricing kernel. We call $\eta = \rho e^{\mathfrak{B}_T}$ the *adjusted pricing kernel* under target $\mathfrak{B}_T$. Then we can rewrite the budget constraint as

$$\mathbb{E}[\zeta^\pi \eta] \le 1.$$

The following lemma is critical for us.

Lemma 2.1. *For any admissible $\pi \in \mathcal{A}$, any optimal candidate ζ^π for* (2.13) *must be anti-comonotonic with the adjusted pricing kernel η, and*

$$\mathbb{E}[\zeta^\pi \eta] = 1. \tag{2.15}$$

Proof. Suppose ζ^{π^*} is an optimal solution of (2.13). According to Lemma 1.5, we can find a r.v. $U \in \mathcal{U}$ such that U and η are comonotonic, then $Q_{\zeta^{\pi^*}}(1-U)$ is anti-comonotonic with η and has the same cdf as ζ^{π^*}. By the Hardy-Littlewood Inequality of Lemma 1.6, we have

$$\mathbb{E}\left[Q_{\zeta^{\pi^*}}(1-U)\cdot\eta\right] \le \mathbb{E}\left[\zeta^{\pi^*}\cdot\eta\right] \le 1.$$

If ζ^{π^*} is not anti-comonotonic with η, then the first inequality is strict; or if the second inequality is strict, then we always have

$$\mathbb{E}\left[Q_{\zeta^{\pi^*}}(1-U)\cdot\eta\right] < 1.$$

Let $\delta = \frac{1-\mathbb{E}\left[Q_{\zeta^{\pi^*}}(1-U)\cdot\eta\right]}{\mathbb{E}[\eta]} > 0$ and $\hat{\zeta}^{\pi^*} = Q_{\zeta^{\pi^*}}(1-U)+\delta$. One can check that we have

$$\mathbb{E}\left[\hat{\zeta}^{\pi^*}\eta\right] = 1$$

and

$$\mathscr{E}\left[v(\zeta^{\pi^*})\right] = \mathscr{E}\left[v(Q_{\zeta^{\pi^*}}(1-U))\right] < \mathscr{E}\left[v(Q_{\zeta^{\pi^*}}(1-U)+\delta)\right] = \mathscr{E}\left[v(\hat{\zeta}^{\pi^*})\right],$$

where the two equalities are due to the fact that the objective functional is law-invariant. Thanks to Theorem 2.1, we can find an admissible portfolio $\hat{\pi}$ such that $\hat{\zeta}^{\pi^*} = X^{\hat{\pi}}(T)/(x_0 e^{\mathfrak{B}_T})$. So $\hat{\zeta}^{\pi^*}$ is better than ζ^{π^*}, a contradiction. $\square$

Remark 2.6. *The similar result can be also found in Theorem 7, Xu [131]; B.1, Jin and Zhou [71]; Lemma 2.5, He and Zhou [68]. Note that the wealth process in our model is slightly different from theirs.*

In the classical case, i.e., there is no benchmark, then the optimal candidate ζ^π must be anti-comonotonic with the pricing kernel ρ. In our model, by contrast, ζ^π may not be anti-comonotonic with the pricing kernel ρ, since ρ and η may not be comonotonic. For example, when the benchmark is $\mathfrak{B}_T = \frac{1}{\rho^2}$, $\eta = \rho e^{\mathfrak{B}_T} = \rho\frac{1}{\rho^2} = \frac{1}{\rho}$; since ζ^π is anti-comonotonic with the adjusted pricing kernel η, it is indeed comonotonic with ρ. Economically speaking, the benchmark may influence how we judge, the traditional optimal solution which maximizes the utility of terminal wealth may not be the one that maximizes the utility of relative return. Therefore, the reference plays a significant role in deciding the optimal strategy.

Let $U \in \mathcal{U}$ be comonotonic with η. According to Corollary 1.4, we then have $Q_\eta(U) = \eta$, and $Q_{\zeta^\pi}(1-U)$ is anti-comonotonic with η. So the constraint (2.15) can be written in a quantile form below:

$$\mathbb{E}[\zeta^\pi\eta] = \mathbb{E}[Q_\eta(U)Q_{\zeta^\pi}(1-U)] = \int_0^1 Q_{\zeta^\pi}(p)Q_\eta(1-p)\,\mathrm{d}p = 1. \tag{2.16}$$

Furthermore, the lower bound constraint (2.7) reads

$$\zeta^\pi = X^\pi(T)/(x_0 e^{\mathfrak{B}_T}) = e^{\mathfrak{R}^\pi(T)-\mathfrak{B}_T} \geq e^{-c} := \hat{c}.$$

In terms of quantile, it becomes

$$Q_{\zeta^\pi}(0+) \geq \hat{c}. \tag{2.17}$$

We note that $0 < \hat{c} < 1$.

Putting the objective (2.14), the budget constraint (2.16) and the lower bound constraint (2.17) together, we arrive at a quantile optimization problem

$$\max_{Q\in\mathbb{Q}} \quad \int_0^1 v(Q(p))w'(1-p)\,\mathrm{d}p \tag{2.18}$$

$$\text{s.t.} \quad \int_0^1 Q(p)Q_\eta(1-p)\,\mathrm{d}p = 1, \quad Q(0+) \geq \hat{c}.$$

As the end of this section, we present the relationship between the optimal solutions of Problem (2.18) and Problem (2.12) as well as Problem (2.11).

Theorem 2.2. *If Q^* is an optimal solution of Problem* (2.18)*, then*

$$X^*(T) = x_0 e^{\mathfrak{B}_T} Q^*(1 - F_\eta(\eta))$$

and

$$R^*(T) = \mathfrak{B}_T + \ln(Q^*(1 - F_\eta(\eta)))$$

are optimal outcomes of Problem (2.12) *and Problem* (2.11)*, respectively.*

Proof. Given an optimal solution Q^* which is the quantile function of $\zeta^{\pi^*} = X^*(T)/x_0 e^{\mathfrak{B}_T}$. Based on Corollary 1.4, finding the optimal candidate $X^*(T)$ of Problem (2.12) is reduced to finding a $U \in \mathcal{U}$ which is comonotonic with ζ^{π^*} such that $\zeta^{\pi^*} = Q^*(U)$. Lemma 2.1 requires that ζ^{π^*} must be anti-comonotonic with η, namely $Q^*(U)$ is anti-comonotonic with η. Base on Corollary 1.5 and Lemma 1.5, we have $1 - F_\eta(\eta) \in \mathcal{U}$ which is anti-comonotonic with η, then we have $Q^*(1 - F_\eta(\eta))$ is anti-comonotonic with η. So, we have $\zeta^{\pi^*} = Q^*(1 - F_\eta(\eta))$, the same goes for $R^*(T)$. □

2.3.2 Change of Variable

To simplify Problem (2.18), we first remove the distortion function w from its objective functional. Introduce

$$\nu(p) = 1 - w^{-1}(1-p), \quad p \in [0, 1].$$

It is also a distortion function, sometimes called the dual distortion of w. Let

$$G(p) = Q(\nu(p)), \quad p \in (0, 1).$$

The map between Q and G is one-to-one as ν is strictly increasing. Furthermore, G is a quantile if and only if so is Q.

We rewrite

$$\begin{aligned}\int_0^1 v(Q(p))w'(1-p)\,\mathrm{d}p &= \int_0^1 v(Q(p))\,\mathrm{d}\,(1-w(1-p)) \\ &= \int_0^1 v(Q(\nu(p)))\,\mathrm{d}\,(1-w(1-\nu(p))) \\ &= \int_0^1 v(G(p))\,\mathrm{d}p\end{aligned}$$

and

$$\int_0^1 Q(p)Q_\eta(1-p)\,\mathrm{d}p = \int_0^1 Q(\nu(p))Q_\eta(1-\nu(p))\,\mathrm{d}\nu(p) = \int_0^1 G(p)\varphi'(p)\,\mathrm{d}p, \quad (2.19)$$

where

$$\varphi(p) = -\int_p^1 Q_\eta(1-\nu(s))\nu'(s)\,\mathrm{d}s, \quad p \in [0,1] \quad (2.20)$$

is a differentiable increasing function. Notice $Q(0+) \geq \hat{c}$ if and only if $G(0+) \geq \hat{c}$. Therefore, Problem (2.18) is, in terms of G, equivalent to

$$\begin{aligned}\max_{G\in\mathbb{Q}} \quad & \int_0^1 v(G(p))\,\mathrm{d}p \quad (2.21)\\ \text{s.t.} \quad & \int_0^1 G(p)\varphi'(p)\,\mathrm{d}p = 1, \quad G(0+) \geq \hat{c}.\end{aligned}$$

Unfortunately, this problem is not a concave optimization problem as we will show v is not concave, hence the Lagrange method cannot be applied directly to tackle it. This is different from the problem in Xu [132] wherein the constraint $G(0+) \geq \hat{c}$ is missing and v is concave. For the above non-concave optimization problem, Xu's [132] idea cannot be applied directly.

We now focus on Problem (2.21). Similar to Theorem 2.2, we see if G^* is an optimal solution of the Problem (2.21), then

$$X^*(T) = x_0 e^{\mathfrak{B}_T} G^*\big(\nu^{-1}\left(1 - F_\eta(\eta)\right)\big)$$

and

$$R^*(T) = \mathfrak{B}_T + \ln\Big(G^*\big(\nu^{-1}\left(1 - F_\eta(\eta)\right)\big)\Big)$$

are the optimal outcomes of Problem (2.12) and Problem (2.11).

2.3.3 Utility of Relative Return

To solve Problem (2.21), the shape of v plays a key role. Observed from (2.13), it can be regarded as a utility function of the random variable ζ^π, hereafter we label it the utility of relative return.

Notice

$$v(x) = u(\ln x) = \begin{cases} (\ln x)^\alpha, & x \geq 1; \\ -\kappa(-\ln x)^\beta, & 0 < x < 1. \end{cases}$$

Hence we have

$$v'(x) = u'(\ln x)x^{-1} > 0$$

and

$$v''(x) = [u''(\ln x) - u'(\ln x)]x^{-2}, \quad x > 0.$$

We see from the above relations that v would be increasing and global concave if so was u. That happends in EUT and RDUT. By contrast, the shape of v in our model is essentially different from them. Indeed,

- When $x > 1$, we have $u''(\ln x) < 0$ and $u'(\ln x) > 0$, so $v''(x) < 0$ and v is strictly concave.

- When $0 < x < 1$, we have

$$\begin{aligned} v''(x) &= [u''(\ln x) - u'(\ln x)]x^{-2} \\ &= -\kappa\beta\big((\beta - 1) + (-\ln x)\big)(-\ln x)^{\beta-2}x^{-2} \\ &= \begin{cases} > 0, & e^{\beta-1} < x < 1; \\ < 0, & 0 < x < e^{\beta-1}. \end{cases} \end{aligned}$$

So v is strictly concave on $(0, e^{\beta-1}]$ and strictly convex on $[e^{\beta-1}, 1)$.

To summarize, the function v is strictly concave on $(0, e^{\beta-1}]$ and $[1, \infty)$, respectively, and strictly convex on $[e^{\beta-1}, 1]$. Furthermore, we want to point out that v' is continuous at $x = 1$ but v'' not.

Since $\hat{c}$ and β are entirely decided by the investor, both the cases $\hat{c} > e^{\beta-1}$ and $\hat{c} \leq e^{\beta-1}$ can happen. The shape of v is demonstrated in Figure 2.1 and Figure 2.2.

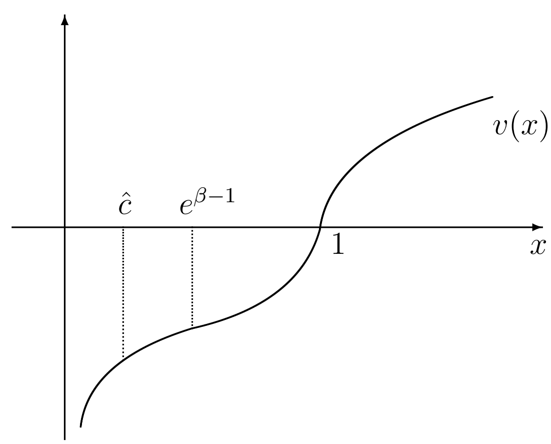

Figure 2.1: The function v when $\hat{c} < e^{\beta-1}$.

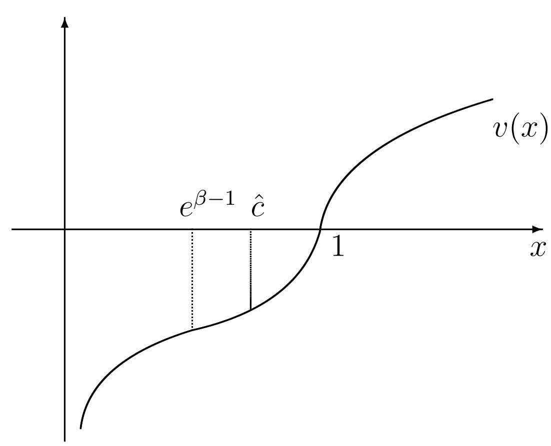

Figure 2.2: The function v when $\hat{c} > e^{\beta-1}$.

Due to the special shape of v, Problem (2.21) is not a concave or S-shaped utility optimization problem as in Jin and Zhou [71] and Xu [132]. It is an M-shaped function. The M-shaped utility has already been proposed in the theory of decision-making under uncertainty, dating back to Friedman and Savage [55]. The authors initially proposed such a shape of utility ("F-S" hypothesis) to explain the purchasing of both insurance and lottery tickets. As demonstrated in Friedman and Savage [55] (Fig 1 in PP.290), a concave segment indicated a preference for certainty (insurance) while the convex segment indicated a preference for risk (gambling). In their paper, the M-shaped utility was endowed with a reasonable interpretation that a lower socioeconomic class consumer whose income was placed corresponding to the first concave segment wished to shift himself to a higher socioeconomic class whose income was placed corresponding to the last concave segment. The convex segment with increasing marginal utility represented a transitional stage between two classes.

Although the shape of utility v in this paper is also M-shaped, there are some essential differences we would like to highlight. Firstly, speaking of the meaning of utility, in the context of [55], it refers to the utility of "income", a general concept which may vary subjectively. Whereas the "utility" investigated in our model typically refers to the utility of the relative return variable $\zeta^\pi = X^\pi(T)/x_0 e^{\mathfrak{B}_T}$. As one may note, the form of the utility function relies heavily on the hypothesis of the piece-wise utility of wealth introduced by Tversky and Kahneman [122]. The shape of the utility is determined once the parameters α, β, κ are fixed. Besides, the variable $\zeta^\pi = X^\pi(T)/x_0 e^{\mathfrak{B}_T}$ has standardized the influence of initial wealth x_0 and the benchmark return $\mathfrak{B}_T$. These parameters do not affect the curvature of the utility in our setting. We may interpret them as a reflection of investor's economic status and aspiration level.

More interestingly, as self-explained in [55], the M-shaped utility is only plausible for consumer units whose level of income is corresponding to the first concave segment of the utility shape. If that were so, suppose an investor's "income" entirely comes from the return of investment in the market. Then in turn, the "F-S" hypothesis implies that the investor's ability to invest, measured by the relative return variable $\zeta^\pi = X^\pi(T)/x_0 e^{\mathfrak{B}_T}$, should take values in the first concave segment, namely the region $(0, e^{\beta-1})$. That is to say, the gross return of investor should be subject to $0 < \frac{X^\pi(T)}{x_0} < e^{\mathfrak{B}_T+\beta-1}$. If we take $\beta = 0.88$, $\mathfrak{B}_T$ to be the common value of risk-free interest rate (normally less than 0.05), then these investors typically refer to those who suffer losses in the market as $e^{\mathfrak{B}_T+\beta-1} < 1$.

After all, we have to admit that so far there lacks empirical experiments or evidences to support that a utility of return or any other decision variables associated with wealth could be simply induced by a corresponding transformation from the utility of wealth, and certainly it is beyond the scope of this thesis. The model proposed by Tversky and Kahneman [122] is one of the most persuasive suggestions on risk preference we can count on. What we focus on here is how to deal with this different non-concave quantile optimization problem when considering a different decision variable.

2.3.4 Concavified Problems

One naive way to solve Problem (2.21) is to consider its concavified problem. For this, let us introduce the concave envelope function of v on $(0, \infty)$, that is, the small concave function dominating v on $(0, \infty)$, denoted by $\hat{v}_0$. Mathematically it is given by

$$\hat{v}_0(x) = \sup_{\substack{0<y\leq x\leq z \\ y\neq z}} \left(\frac{z-x}{z-y} v(y) + \frac{x-y}{z-y} v(z) \right), \quad x > 0.$$

We have the following lemma:

Lemma 2.2. *There are two scalars $0 < a < 1 < b$ such that the function $\hat{v}_0$ is concave* *on* $(0,\infty)$*, coincides with* v *on* $(-\infty,a]\cup[b,\infty)$*, and is affine on* $[a,b]$*. Moreover,* a *and* b *are determined by the equations*

$$v'(a)=v'(b)=\frac{v(b)-v(a)}{b-a}. \tag{2.22}$$

Proof. Let

$$f_1(\beta)=\min\{\beta x-v(x):x\geq 1\}$$

and

$$f_2(\beta)=\min\{\beta x-v(x):0<x\leq 1\}$$

Then both f_1 and f_2 are continuous function in its domain in $(0,\infty)$.

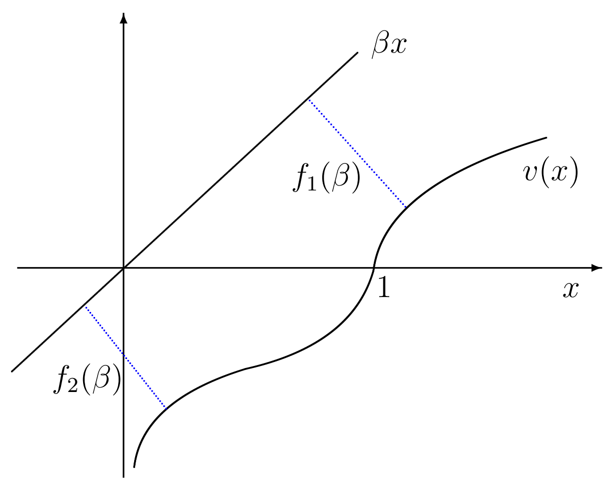

Figure 2.3: The functions f_1 and f_2 in blue dot line.

One can show $f_1(\beta)>f_2(\beta)$ when β is sufficiently large, and $f_1(\beta)<f_2(\beta)$ when β is small. So for some $\beta>0$, $f_1(\beta)=f_2(\beta)$. Also there exists $b>1$ and $0<a<1$ such that $f_1(\beta)=\beta b-v(b)$ and $f_2(\beta)=\beta a-v(a)$. It follows $\beta a-v(a)=\beta b-v(b)$. Since a and b minimize $\beta x-v(x)$, respectively, on $(0,1)$ and $(1,\infty)$, the first order condition gives $v'(b)=\beta$ and $v'(a)=\beta$. So we conclude (2.22). □

The positions of a and b are demonstrated in Figure 2.4. The global concave envelope function $\hat{v}_0$ is different from v on the interval (a, b), in red dot line.

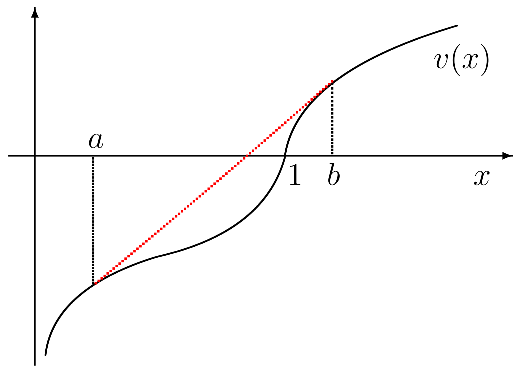

Figure 2.4: The function $\hat{v}_0$ in red dot line.

A naive way to solve the problem (2.21) is described as follows. One first tries to solve the following *globally concavified* problem

$$\begin{aligned} \max_{G \in \mathbb{Q}} \quad & \int_0^1 \hat{v}_0(G(p))\,\mathrm{d}p \\ \text{s.t.} \quad & \int_0^1 G(p)\varphi'(p)\,\mathrm{d}p = 1, \quad G(0+) \geq \hat{c}. \end{aligned} \tag{2.23}$$

And then tries to show that the optimal solution of Problem (2.23) is also an optimal solution of Problem (2.21). But unfortunately this approach turns out to fail due to the low bound constraint $G(0+) \geq \hat{c}$ in the problem. In fact, one can easily see from the formulation of Problem (2.21) that its optimal value as well as its solution are only related to the utility v on $[\hat{c}, \infty)$. Therefore, it makes sense to use the local concave envelope of v on $[\hat{c}, \infty)$ rather than the global one $\hat{v}_0$ on $(0, \infty)$.

The above thinking motives us to consider the concave envelope function of v on $[\hat{c}, \infty)$, denoted by $\hat{v}_{\hat{c}}$. It is the smallest concave function dominating v on $[\hat{c}, \infty)$, called the *local concave envelope* of v and defined by

$$\hat{v}_{\hat{c}}(x) = \sup_{\substack{\hat{c} \le y \le x \le z \\ y \neq z}} \left(\frac{z-x}{z-y} v(y) + \frac{x-y}{z-y} v(z) \right), \quad x \ge \hat{c}. \tag{2.24}$$

We now introduce the *locally concavified* problem

$$\begin{aligned} \max_{G \in \mathbb{Q}} \quad & \int_0^1 \hat{v}_{\hat{c}}(G(p))\,\mathrm{d}p \\ \text{s.t.} \quad & \int_0^1 G(p)\varphi'(p)\,\mathrm{d}p = 1, \quad G(0+) \ge \hat{c}. \end{aligned} \tag{2.25}$$

Because the constraints are linear in the decision variable G in this problem, and the objective functional is concave in it, this is a concave optimization problem. By contrast, Problem (2.21) is not a concave optimization problem. Generally speaking, solving a concave optimization problem is easier than solving a non-concave one. Although the two problems seem different, it however turns out that the optimal solution of Problem (2.25) is also an optimal solution of Problem (2.21).

Before proving the above claim, let us first study the properties of the local concave envelope function $\hat{v}_{\hat{c}}$. Clearly $\hat{v}_{\hat{c}} \le \hat{v}_0$ on $[\hat{c}, \infty)$. One can easily show that they are identical on $[\hat{c}, \infty)$ if and only if $\hat{c} \le a$. If $a < \hat{c}$, then the function $\hat{v}_{\hat{c}}$ coincides with v on $[d, \infty)$ and is affine on $[\hat{c}, d]$ for some $1 < d < b$; moreover, $\hat{v}_{\hat{c}} < \hat{v}_0$ on $[\hat{c}, b]$ and $\hat{v}_{\hat{c}} > v$ on $(\hat{c}, d)$. Similar as before, we can find the value of d via an algebraic equation

$$v'(d) = \frac{v(d) - v(\hat{c})}{d - \hat{c}}.$$

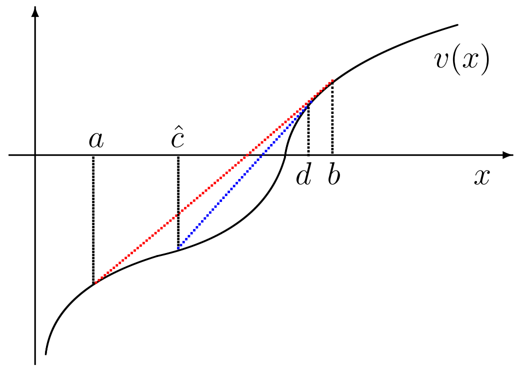

Figure 2.5: The function $\hat{v}_0$ in red dot line and $\hat{v}_{\hat{c}}$ in blue dot line when $\hat{c} > a$.

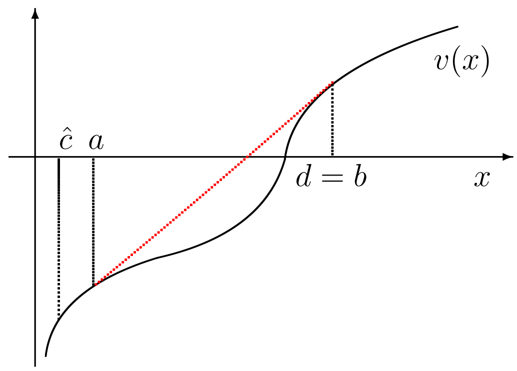

Figure 2.6: The function $\hat{v}_0$ and $\hat{v}_{\hat{c}}$ coincide in red dot line when $\hat{c} \leq a$.

Economically speaking, the bigger the value of c (equivalently, the smaller the value of $\hat{c}$), the smaller the local concave envelope function $\hat{v}_{\hat{c}}$ as well as the optimal value of Problem (2.21).

Because $\hat{v}'_{\hat{c}}$ is continuous and decreasing on $[\hat{c}, \infty)$, we may define its left-continuous inverse function as

$$I(x) = \inf\left\{y \geq \hat{c} \mid \hat{v}'_{\hat{c}}(y) \leq x\right\}, \quad x > 0. \tag{2.26}$$

We have the following important lemma:

Lemma 2.3. *The function I is decreasing and left-continuous, and satisfies the following properties.*

1. *If $a \le \hat{c}$, then*

$$I(x) = \begin{cases} (v')^{-1}(x), & \text{if } 0 < x < v'(d); \\ \hat{c}, & \text{if } x \ge v'(d). \end{cases} \tag{2.27}$$

   where v only takes the positive part, i.e. $v(x) = (\ln x)^\alpha,\ x > 1$.

2. *If $a > \hat{c}$, then*

$$I(x) = \begin{cases} (v')^{-1}(x), & \text{if } 0 < x < v'(b); \\ a, & \text{if } x = v'(b); \\ (v')^{-1}(x), & \text{if } v'(b) < x < v'(\hat{c}); \\ \hat{c}, & \text{if } x \ge v'(\hat{c}). \end{cases} \tag{2.28}$$

3. *For any $x > 0$,*

$$\hat{v}_{\hat{c}}(I(x)) = v(I(x)).$$

4. *For any $x > 0$,*

$$\max_{y \ge \hat{c}}(v(y) - xy) = \max_{y \ge \hat{c}}(\hat{v}_{\hat{c}}(y) - xy) = \hat{v}_{\hat{c}}(I(x)) - xI(x).$$

Proof. The first three properties are easy to prove, let us prove the last property. In fact $\hat{v}_{\hat{c}}$ is concave, so for any $x > 0$, we have

$$\sup_{y \ge \hat{c}}(\hat{v}_{\hat{c}}(y) - xy) = \hat{v}_{\hat{c}}(I(x)) - xI(x) = v(I(x)) - xI(x),$$

where the last equation is due to the third property. Hence

$$\sup_{y \ge \hat{c}}(\hat{v}_{\hat{c}}(y) - xy) = v(I(x)) - xI(x) \le \sup_{y \ge \hat{c}}(v(y) - xy).$$

But the reverse inequality

$$\sup_{y \ge \hat{c}}(\hat{v}_{\hat{c}}(y) - xy) \ge \sup_{y \ge \hat{c}}(v(y) - xy)$$

is trivial as $\hat{v}_{\hat{c}} \ge v$, so all the inequalities become identities. □

From the first three properties, we see that I does not take values in $(a \vee \hat{c}, d)$.

2.3.5 Optimal Solution

We are now ready to use the relaxation method to solve Problem (2.25). This method was introduced by Xu [132]. The idea is to replace $\varphi(p)$ defined by (2.20) with its concave envelop.

Let δ be the concave envelope of φ on $[0,1]$, that is,

$$\delta(x) = \sup_{\substack{0\le y\le x\le z\le 1\\ y\neq z}} \left(\frac{z-x}{z-y}\varphi(y) + \frac{x-y}{z-y}\varphi(z)\right), \quad x\in[0,1]. \tag{2.29}$$

Then it satisfies an ODE

$$\min\{-\delta''(p),\ \delta(p)-\varphi(p)\} = 0, \quad \text{for almost every } p\in[0,1], \tag{2.30}$$

with $\delta(0)=\varphi(0)$ and $\delta(1)=\varphi(1)$.

Remark 2.7. *When w is concave (including the case that there is no probability distortion), φ is concave, so $\delta=\varphi$.*

We give the main conclusion.

Theorem 2.3 (Verification Theorem)**.** *Suppose*

$$\int_0^1 I(\lambda\delta'(p))\varphi'(p)\,\mathrm{d}p = 1 \tag{2.31}$$

for some $\lambda>0$. Then $G^*(p)=I(\lambda\delta'(p))$ is an optimal solution of Problem (2.25) *as well as Problem* (2.21).

Proof. By the definition of I, we have $G^*\ge\hat{c}$, together with (2.31), we infer that G^* is a feasible solution of Problem (2.21) and Problem (2.25). Moreover, it is an optimal solution of the latter; see Xu [132] or Xia and Zhou [129]. Note

$$\hat{v}_{\hat{c}}(I(x)) = v(I(x))$$

for any $x > 0$, so

$$\hat{v}_{\hat{c}}(G^*(p)) = \hat{v}_{\hat{c}}(I(\lambda\delta'(p))) = v(I(\lambda\delta'(p))) = v(G^*(p)).$$

Therefore for any feasible solution G of Problem (2.21), we have

$$\int_0^1 v(G(p))\,\mathrm{d}p \le \int_0^1 \hat{v}_{\hat{c}}(G(p))\,\mathrm{d}p \le \int_0^1 \hat{v}_{\hat{c}}(G^*(p))\,\mathrm{d}p = \int_0^1 v(G^*(p))\,\mathrm{d}p,$$

and hence G^* is also optimal to Problem (2.21). □

Remark 2.8. *The verification theorem requires the existence of* λ. *Let*

$$f(\lambda) = \int_0^1 I(\lambda\delta'(p))\varphi'(p)\,\mathrm{d}p.$$

Since $I(x)$ *is decreasing and left-continuous, we can show by the monotone convergence theorem that* $f(\lambda)$ *is continuous and decreasing. Moreover,*

$$\lim_{\lambda\to 0} f(\lambda) = +\infty, \qquad \lim_{\lambda\to\infty} f(\lambda) = \hat{c}\int_0^1 \varphi'(p)\,\mathrm{d}p = \hat{c}\,\mathbb{E}[\eta]\,.$$

Thanks to Assumption 2.3, the existence of λ *in* (2.31) *is ensured since* $\hat{c}\,\mathbb{E}[\eta] \le 1$.

Finally, given an optimal solution of Problem (2.25)

$$G^*(p) = Q^*(\nu(p)) = I(\lambda^*\delta'(p)).$$

Therefore

$$G^*(\nu^{-1}(p)) = Q^*(p) = I(\lambda^*\delta'(\nu^{-1}(p))).$$

Recall Theorem 2.2, the corresponding optimal terminal wealth and optimal returns of Problem (2.12) and Problem (2.11) are

$$X^*(T) = x_0 e^{\mathfrak{B}_T} G^*(\nu^{-1}(1 - F_\eta(\eta))) = x_0 e^{\mathfrak{B}_T} I(\lambda^*\delta'(1 - \omega(F_\eta(\eta))))$$

and

$$\mathfrak{R}^*(T) - \mathfrak{B}_T = \ln(I(\lambda^* \delta^{'}(1 - \omega(F_\eta(\eta))))),$$

which depend on the function I in (2.27) and (2.28). Note that if $I \geq 1$, it is regarded as a gain situation, otherwise it is regarded as a loss situation.

In Berkelaar, Kouwenberg, and Post [16] and Jin and Zhou [71], the authors both obtained a two-case phenomenon (either a gain or maximum loss) for the optimal terminal wealth. In our model, one can also observe this phenomenon from (2.27) when the agent's risk tolerance level $c \leq -\ln a$ ($a \leq \hat{c}$). A slightly different observation from (2.28) is that when $c > -\ln a$ ($a > \hat{c}$), the loss will continuously decrease from a to $\hat{c}$ according to the state of the market, not necessarily a sure maximum loss.

2.3.6 A Comparison under Piece-Wise Power Utility

It is worth noting that Zhang, Jin and Zhou [138] also considered a continuous-time portfolio optimization problem under the CPT with lower bound constraint and obtained similar results. Their model is highly related to our model except that they maximize the utility of terminal wealth and use separated probability weighting functions for the gain and loss parts. Their results provide an opportunity for us to compare the differences between using returns and using terminal wealth as the investment targets. To illustrate it, we consider a parallel model of Problem (2.11), which is

$$\begin{aligned} \max_{\pi(\cdot)\in\mathcal{A}} \quad & \mathscr{E}\left[u(X^\pi(T) - \bar{\mathfrak{B}}_T)\right] \qquad (2.32) \\ \text{s.t.} \quad & \mathbb{E}[\rho X^\pi(T)] = x_0, \quad X^\pi(T) - \bar{\mathfrak{B}}_T \geq -\bar{c}. \end{aligned}$$

Namely, replace the terminal return $R^\pi(T)$ in (2.11) with the conventional terminal wealth $X^\pi(T)$. Let the other settings remain unchanged. Then, the formulation is exactly the problem studied in Zhang, Jin and Zhou [138], in which $\bar{\mathfrak{B}}_T$ and $\bar{c}$ was interpreted as the reference point at time T and the upper bound to control the loss.

By letting $\bar{X}^\pi(T) = X^\pi(T) - \bar{\mathfrak{B}}_T$, $\bar{x}_0 = x_0 - \mathbb{E}\left[\rho\bar{\mathfrak{B}}_T\right]$, the authors studied the equivalent problem

$$\begin{aligned} \max_{\pi(\cdot)\in\mathcal{A}} \quad & \mathscr{E}\left[u(\bar{X}^\pi(T))\right] \\ \text{s.t.} \quad & \mathbb{E}\left[\rho\bar{X}^\pi(T)\right] = \bar{x}_0, \quad \bar{X}^\pi(T) \geq -\bar{c}. \end{aligned} \tag{2.33}$$

By means of the "divide and conquer" machinery introduced in Jin and Zhou [71], Problem (2.33) reduced to a related three-dimensional mathematical programme problem. For details of it, readers can refer to Zhang, Jin and Zhou [138] (Theorem 5.1). By solving the three-dimension optimization problem and given a corresponding solution denoted as (d_1, d_2, x_+), the optimal wealth profile is classified to three cases with respect to the state of pricing kernel ρ

$$\bar{X}^\pi(T) = \begin{cases} (u_+')^{-1}(\lambda\frac{\rho}{T_+'(F_\rho(\rho))}), & \text{if } \rho \leq d_1; \\ -\frac{x_+ - \bar{x}_0 - \bar{c}\,\mathbb{E}\left[\rho 1_{\{\rho > d_2\}}\right]}{\mathbb{E}\left[\rho 1_{\{d_1 < \rho \leq d_2\}}\right]}, & \text{if } d_1 < \rho \leq d_2; \\ -\bar{c}, & \text{if } \rho > d_2, \end{cases} \tag{2.34}$$

where $u_+ = \max\{u, 0\}$, $\underline{\rho} \leq d_1 \leq d_2 \leq \overline{\rho}$, $\bar{x}_0^+ \leq x_+ \leq \bar{x}_0 + \bar{c}\,\mathbb{E}[\rho]$. $\underline{\rho}$, $\overline{\rho}$ are the essential infimum and supremum of ρ respectively. T_+, T_- is the probability weighting function for the positive outcome and negative outcome. The Largrange multiplier λ satisfys $\mathbb{E}\left[\rho(u_+')^{-1}(\lambda\frac{\rho}{T_+'(F_\rho(\rho))})1_{\{\rho\leq d_1\}}\right] = x_+$.

Note that the loss scenario has two cases: either a constant moderate loss $\frac{x_+ - \bar{x}_0 - c\,\mathbb{E}\left[\rho 1_{\{\rho > d_2\}}\right]}{\mathbb{E}\left[\rho 1_{\{d_1 < \rho \leq d_2\}}\right]}$ or a constant maximum loss $-\bar{c}$. Zhang, Jin and Zhou [138] (Section 6) demonstrated two possible analytical solutions when the utility function is piece-wise power type. It shows that (2.34) may be further reduced to a two-cases (either the moderate loss or the maximum loss disappear) which depends on the explicit form of T_- taken in the model.

To make a comparison, we first need to unify the benchmark and lower bound performance parameter taken in both targets. We assume $\bar{\mathfrak{B}}_T = x_0 e^{\mathfrak{B}_T}$, namely the benchmark used in both targets should be the same value in terms of terminal wealth and log-return, $\bar{\mathfrak{B}}_T$ should generate a log-return $\mathfrak{B}_T$ in turn. Similarly, we assume $\bar{c} = x_0 e^{-c} = x_0 \hat{c}$, then the solution of Problem (2.32) is

$$X^{\pi}(T) = \begin{cases} x_0 e^{\mathfrak{B}_T} + (u'_+)^{-1}(\lambda \frac{\rho}{T'_+(F_\rho(\rho))}), & \text{if } \rho \leq d_1; \\ x_0 e^{\mathfrak{B}_T} - \frac{x_+ - (x_0 - \mathbb{E}[\rho \bar{\mathfrak{B}}_T]) - x_0 \hat{c}\, \mathbb{E}[\rho 1_{\{\rho > d_2\}}]}{\mathbb{E}[\rho 1_{\{d_1 < \rho \leq d_2\}}]}, & \text{if } d_1 < \rho \leq d_2; \\ x_0 e^{\mathfrak{B}_T} - x_0 \hat{c}, & \text{if } \rho > d_1. \end{cases} \tag{2.35}$$

But in our result, because $w(\cdot)$, $F_\eta(\cdot)$ are non-decreasing, $\delta^{'}(\cdot)$ is non-increasing. then $h(\cdot) := \delta^{'}(1 - w(F_\eta(\cdot)))$ is non-decreasing with respect to η. According to (2.27) and (2.28), the optimal terminal wealth reduces to two scenarios:

If $\hat{c} \geq a$, then

$$X^{\pi}(T) = \begin{cases} x_0 e^{\mathfrak{B}_T} (v')^{-1}(\lambda^* \delta^{'}(1 - w(F_\eta(\eta)))), & \text{if } \eta \leq \bar{d}_2; \\ x_0 e^{\mathfrak{B}_T} \hat{c}, & \text{if } \eta > \bar{d}_2, \end{cases} \tag{2.36}$$

If $\hat{c} < a$, then

$$X^{\pi}(T) = \begin{cases} x_0 e^{\mathfrak{B}_T} (v')^{-1}(\lambda^* \delta^{'}(1 - w(F_\eta(\eta)))), & \text{if } \eta \leq \bar{d}_3; \\ x_0 e^{\mathfrak{B}_T} a, & \text{if } \eta = \bar{d}_3; \\ x_0 e^{\mathfrak{B}_T} (v')^{-1}(\lambda^* \delta^{'}(1 - w(F_\eta(\eta)))), & \text{if } \bar{d}_3 < \eta \leq \bar{d}_1; \\ x_0 e^{\mathfrak{B}_T} \hat{c}, & \text{if } \eta > \bar{d}_1, \end{cases} \tag{2.37}$$

where $\lambda^* h(\bar{d}_2) = v'(d)$, $\lambda^* h(\bar{d}_3) = v'(b)$, $\lambda^* h(\bar{d}_1) = v'(\hat{c})$.

Comparing it with (2.35), one can see that maximizing the utility of return leads to a completely different optimal solution. Especially, when we consider environment coefficients including benchmark and impose an upper bound on the loss. The level of risk tolerance c plays a critical role in determining the loss scenarios (either a sure maximum loss or a continuous increase to the maximum loss). Furthermore, whether the terminal wealth is a gain or a loss is determined by the adjusted pricing kernel η

rather than ρ being lower or higher above a threshold, implying that the impact of benchmark $\mathfrak{B}_T$ in our result is more complicated.

Remark 2.9. *Note that if $\mathfrak{B}_T \leq -c$, namely the benchmark return goes down below the risk tolerance level, which stands for a "bad" market. Then $x_0 e^{\mathfrak{B}_T} - x_0\hat{c} \leq 0$, which means the strategy under target* (2.32) *may go bankruptcy or even in debt. Since we use proportional strategy under a return-oriented target, the portfolio will never go bankruptcy.*

Remark 2.10. *In Jin and Zhou [71] and Zhang, Jin and Zhou [138], to ensure that $\frac{\rho}{T'_+(F_\rho(\rho))}$ is non-decreasing with respect to ρ. An assumption that $\frac{F_\rho^{-1}(\cdot)}{T'_+(\cdot)}$ should be non-decreasing has been made. As pointed out in Xu [132], one can find*

$$\varphi'(1 - w(F_\eta(\eta))) = Q_\eta(1 - \nu(1 - w(F_\eta(\eta))))\nu'(1 - w(F_\eta(\eta))) = \frac{\eta}{w'(F_\eta(\eta))},$$

which means $\varphi'(1 - w(F_\eta(\eta)))$ should non-decreasing with respect to η, namely φ' should be decreasing. An important truth which is proved in Xu [132] and our model is that such an assumption is not necessary if we replace φ with its concave envelope δ. Furthermore, by the definition of δ, we have

$$\delta'(1 - w(F_\eta(\eta))) = \frac{\eta}{w'(F_\eta(\eta))}.$$

Since $\delta'' \leq 0$, we have $\frac{\eta}{w'(F_\eta(\eta))}$ is non-decreasing with respect to η.

2.3.7 Optimal Controls under Deterministic Parameters

In this section, inspired by Bielecki, Jin, Pliska and Zhou [18], we give an explicit form of the optimal control under the condition that the parameters and coefficients are deterministic. Having

$$X^*(T) = x_0 e^{\mathfrak{B}_T} G^*(\nu^{-1}(1 - \omega(F_\eta(\eta)))) = x_0 e^{\mathfrak{B}_T} I(\lambda^* \delta'(1 - \omega(F_\eta(\eta))))$$

and

$$\mathfrak{R}^*(T) = \mathfrak{B}_T + \ln(I(\lambda^*\delta'(1-\omega(F_\eta(\eta))))),$$

where η, $\mathfrak{B}_T$ are $\mathcal{F}_T$-measurable r.v.s, $X^*(T)$ and $R^*(T)$ could be treated as functions of η.

To derive the replicating portfolio, it suffices to find a strategy π^* that satisfies the following *Backward Stochastic Differential Equation* (BSDE):

$$\begin{cases} \mathrm{d}X^*(t) = X^*(t)\big[(r(t)+\pi^*(t)'\mu(t))\,\mathrm{d}t + \pi^*(t)'\sigma(t)\,\mathrm{d}W(t)\big], \quad t \geqslant 0, \\ X^*(T) = x_0 e^{\mathfrak{B}_T} I(\lambda^*\delta'(1-\omega(F_\eta(\eta)))), \end{cases} \tag{2.38}$$

where I is given by (2.27) and (2.28).

Although the existence and uniqueness of such pair $(X^*(t), \pi^*(t))$ is promised by Theorem 2.1. But to find an analytical solution is not so easy. Inspired by Bielecki, Jin, Pliska and Zhou [18], we present an explicit optimal control when the parameters are all deterministic and the target is a function of $\rho = \rho(T)$. In particular, we assume the benchmark $\mathfrak{B}_T$ is a function of ρ, so $\eta = \rho e^{\mathfrak{B}_T}$ is also a function of ρ, then $X^*(T) = x_0 e^{\mathfrak{B}} I(\lambda^*\delta'(1-\omega(F_\eta(\eta))))$ can be regarded as a function of ρ as well.

Let $X^*(T) = g(\rho)$ for some deterministic function g. Then (2.38) becomes

$$\begin{cases} \mathrm{d}X^*(t) = X^*(t)\big[(r(t)+\pi^*(t)'\mu(t))\,\mathrm{d}t + \pi^*(t)'\sigma(t)\,\mathrm{d}W(t)\big], \quad t \geqslant 0, \\ X^*(T) = g(\rho). \end{cases} \tag{2.39}$$

Notice that $\rho(t)X^*(t)$ is a martingale, so we have

$$\rho(t)X^*(t) = \mathbb{E}[\rho(T)X^*(T) \mid \mathcal{F}_t] = \mathbb{E}[\rho(T)g(\rho(T)) \mid \mathcal{F}_t].$$

The last term can be expressed as a function of t and $\rho(t)$, so is $X^*(t)$.

Theorem 2.4. *Suppose all the parameters are deterministic and the benchmark $\mathfrak{B}_T$ is a function of the pricing kernel ρ. Under Assumption 2.3 and assume $\int_0^T \|\theta(s)\|^2\,\mathrm{d}s >$*

0, *then there exists a unique optimal portfolio strategy for Problem* (2.11)*. Moreover, the optimal portfolio strategy and the associated wealth process and log-return process are respectively given as*

$$\pi^*(t)' = -\frac{\frac{\partial c}{\partial x}(t,\rho(t))}{c(t,\rho(t))}\rho(t)(\sigma(t)\sigma(t)')^{-1}\mu(t) \tag{2.40}$$

and

$$X^*(t) = c(t,\rho(t)), \qquad \mathfrak{R}^*(t) = \ln(X^*(t)/x_0), \tag{2.41}$$

where

$$c(t,x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\int_t^T r(s)\,\mathrm{d}s} g\big(x\cdot e^{-\int_t^T (r(s)-\frac{1}{2}\|\theta(s)\|^2)\,\mathrm{d}s - y\sqrt{\int_t^T \|\theta(s)\|^2\,\mathrm{d}s}}\big)e^{-\frac{1}{2}y^2}\,\mathrm{d}y, \tag{2.42}$$

$$\begin{aligned}&\frac{\partial c}{\partial x}(t,x)\\ =&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\int_t^T (2r(s)-\|\theta(s)\|^2)\,\mathrm{d}s} g'\big(x\cdot e^{-\int_t^T (r(s)-\frac{3}{2}\|\theta(s)\|^2)\,\mathrm{d}s - y\sqrt{\int_t^T \|\theta(s)\|^2\,\mathrm{d}s}}\big)e^{-\frac{1}{2}y^2}\,\mathrm{d}y,\end{aligned} \tag{2.43}$$

and $c(t,x)$ *is the solution of following second-order parabolic type partial differential equation:*

$$\begin{cases}\frac{\partial c}{\partial t}(t,x) + (\|\theta(t)\|^2 - r(t))x\frac{\partial c}{\partial x}(t,x) + \frac{1}{2}\frac{\partial^2 c}{\partial x^2}(t,x)x^2\|\theta(t)\|^2 = r(t)c(t,x),\\ c(T,x) = g(x).\end{cases} \tag{2.44}$$

Proof. Consider the PDE (2.44), according to Feynman-Kac Formula in Yong and Zhou [136] (Chapter 7. Theorem 4.1.), the solution $c(t,x)$, $(t,x)\in[0,T]\times\mathbb{R}$ can be expressed as the conditional expectation of a stochastic process:

$$\begin{aligned}c(t,x) &= \mathbb{E}\Big[e^{-\int_t^T r(s)\,\mathrm{d}s} g(y(T)) \mid y(t) = x\Big] &(2.45)\\ &= \mathbb{E}\Big[e^{-\int_t^T r(s)\,\mathrm{d}s} c(T, y(T)) \mid y(t) = x\Big],\end{aligned}$$

where $y(\cdot)$ is the solution of the following SDE

$$\begin{cases} \mathrm{d}y(s) = y(s)\big[(\|\theta(s)\|^2 - r(s))\,\mathrm{d}s - \theta(s)\,\mathrm{d}W(s)\big], & s \in [t,T], \\ y(t) = x. \end{cases} \tag{2.46}$$

One can check that

$$c(T,x) = \mathbb{E}\Big[e^{-\int_T^T r(s)\,\mathrm{d}s} g(y(T)) \mid y(T) = x\Big] = \mathbb{E}[g(y(T)) \mid y(T) = x] = g(x).$$

Since $y(T) = y(t)e^{-[\int_t^T (r(s)-\frac{1}{2}\|\theta(s)\|^2)\,\mathrm{d}s - \int_t^T \theta(s)\,\mathrm{d}W(s)]}$, thus

$$\begin{aligned} c(t,x) =& \mathbb{E}\Big[e^{-\int_t^T r(s)\,\mathrm{d}s} g(y(t)e^{-[\int_t^T (r(s)-\frac{1}{2}\|\theta(s)\|^2)\,\mathrm{d}s - \int_t^T \theta(s)\,\mathrm{d}W(s)]}) \mid y(t) = x\Big] \\ =& \mathbb{E}\Big[e^{-\int_t^T r(s)\,\mathrm{d}s} g(y(t)e^{-[\int_t^T (r(s)-\frac{1}{2}\|\theta(s)\|^2)\,\mathrm{d}s - Y\cdot\sqrt{\int_t^T \theta(s)\,\mathrm{d}s}]}) \mid y(t) = x\Big], \end{aligned}$$

where $Y = \frac{\int_t^T \theta(s)\,\mathrm{d}W(s)}{\sqrt{\int_t^T \theta(s)\,\mathrm{d}s}}$ is a centralized normal random variable.

Notice that Y, $e^{-\int_t^T r(s)\,\mathrm{d}s}$ and $e^{-\int_t^T (r(s)-\frac{1}{2}\|\theta(s)\|^2)\,\mathrm{d}s}$ are independent of $y(t)$. So we have

$$c(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\int_t^T r(s)\,\mathrm{d}s} g\big(x \cdot e^{-\int_t^T (r(s)-\frac{1}{2}\|\theta(s)\|^2)\,\mathrm{d}s - y\sqrt{\int_t^T \|\theta(s)\|^2\,\mathrm{d}s}}\big) e^{-\frac{1}{2}y^2}\,\mathrm{d}y.$$

Differentiating it with respect to x, we obtain

$$\begin{aligned} &\frac{\partial c}{\partial x}(t,x) \\ =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\int_t^T (2r(s)-\|\theta(s)\|^2)\,\mathrm{d}s} g'\big(x \cdot e^{-\int_t^T (r(s)-\frac{1}{2}\|\theta(s)\|^2)\,\mathrm{d}s - y\sqrt{\int_t^T \|\theta(s)\|^2\,\mathrm{d}s}}\big) e^{-\frac{1}{2}(y+\sqrt{\int_t^T \|\theta(s)\|^2\,\mathrm{d}s})^2}\,\mathrm{d}y \\ =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\int_t^T (2r(s)-\|\theta(s)\|^2)\,\mathrm{d}s} g'\big(x \cdot e^{-\int_t^T (r(s)-\frac{3}{2}\|\theta(s)\|^2)\,\mathrm{d}s - y\sqrt{\int_t^T \|\theta(s)\|^2\,\mathrm{d}s}}\big) e^{-\frac{1}{2}y^2}\,\mathrm{d}y. \end{aligned}$$

Furthermore, Let $X^*(t) = c(t,\rho(t))$ and $\pi^*(t)' = -\frac{\frac{\partial c}{\partial x}(t,\rho(t))}{c(t,\rho(t))}\rho(t)\theta(t)\sigma(t)^{-1}$. Applying Itô's Lemma to $X^*(t)$, one has

$$\mathrm{d}\,X^*(t) = \frac{\partial c}{\partial t}(t,\rho(t))\,\mathrm{d}t + \frac{\partial c}{\partial x}(t,\rho(t))\,\mathrm{d}\,\rho(t) + \frac{1}{2}\frac{\partial^2 c}{\partial x^2}(t,\rho(t))\,\mathrm{d}\,\langle\rho(t),\rho(t)\rangle$$

$$= \Big(\frac{\partial c}{\partial t}(t,\rho(t)) - \frac{\partial c}{\partial x}(t,\rho(t))\rho(t)r(t) + \frac{1}{2}\frac{\partial^2 c}{\partial x^2}(t,\rho(t))\rho(t)^2\|\theta(t)\|^2\Big)\,\mathrm{d}t$$
$$- \Big(\frac{\partial c}{\partial x}(t,\rho(t))\rho(t)\theta(t)\Big)\,\mathrm{d}W(t).$$

Based on (2.44), we have the drift term

$$\frac{\partial c}{\partial t}(t,\rho(t)) - \frac{\partial c}{\partial x}(t,\rho(t))\rho(t)r(t) + \frac{1}{2}\frac{\partial^2 c}{\partial x^2}(t,\rho(t))\rho(t)^2\|\theta(t)\|^2$$
$$= r(t)c(t,\rho(t)) - \frac{\partial c}{\partial x}(t,\rho(t))\rho(t)\|\theta(t)\|^2.$$

Notice that $X^*(t)\pi^*(t)'\sigma(t) = -\frac{\partial c}{\partial x}(t,\rho(t))\rho(t)\theta(t)$, thus

$$\begin{aligned}
\mathrm{d}\,X^*(t) &= \big(r(t)c(t,\rho(t)) - \frac{\partial c}{\partial x}(t,\rho(t))\rho(t)\|\theta(t)\|^2\big)\,\mathrm{d}t + \big(-\frac{\partial c}{\partial x}(t,\rho(t))\rho(t)\theta(t)\big)\,\mathrm{d}W(t)\\
&= \big(r(t)X^*(t) + X^*(t)\pi^*(t)'\mu(t)\big)\,\mathrm{d}t + X^*(t)\pi^*(t)'\sigma(t)\,\mathrm{d}W(t)\\
&= X^*(t)\big[(r(t) + \pi^*(t)'\mu(t))\,\mathrm{d}t + \pi^*(t)'\sigma(t)\,\mathrm{d}W(t)\big].
\end{aligned}$$

Combining it with $X^*(T) = c(T,\rho(T)) = g(\rho)$, we prove that $X^*(T)$ and $\pi^*(t)'$ are respectively the optimal wealth and portfolio. □

Remark 2.11. *When there is no probability weighting, the terminal condition in* (2.38) *reduces to* $X^*(T) = x_0 e^{\mathfrak{B}_T} I(\lambda^*\eta)$.

Remark 2.12. *Based on* (2.27) *and* (2.28), *one can check that*

$$\begin{aligned}
X^*(T) =& x_0 e^{\mathfrak{B}_T} I(\lambda^*\delta'(1-\omega(F_\eta(\eta))))\\
=& x_0 e^{\mathfrak{B}_T}\big(e^{-c} + [(v')^{-1}(\lambda^*\delta'(1-\omega(F_\eta(\eta)))) - e^{-c}]^+\big)\\
=& x_0 e^{\mathfrak{B}_T - c} + \big(x_0 e^{\mathfrak{B}_T}(v')^{-1}(\lambda^*\delta'(1-\omega(F_\eta(\eta)))) - x_0 e^{\mathfrak{B}_T - c}\big)^+.
\end{aligned}$$

Let $\bar{g}(\rho) = x_0 e^{\mathfrak{B}_T}(v')^{-1}(\lambda^*\delta'(1-\omega(F_\eta(\eta))))$. *Then the optimal terminal wealth is divided into two parts. The first part* $x_0 e^{\mathfrak{B}_T - c}$ *guarantees a return restricted by the lower bound constraint* $\mathfrak{R}^*(T) \geq \mathfrak{B}_T - c$ *at maturity, while the second part* $(\bar{g}(\rho) - x_0 e^{\mathfrak{B}_T - c})^+$

could be regarded as the payoff of a European call option striking at $x_0 e^{\mathfrak{B}_T - c}$, the asset it attached has value $\bar{g}(\rho)$ at time T. Practically, one could interpret the first part as allocating proportional initial capital $x_0 e^{-c}$ on the assets replicate the chosen benchmark. The potential financial instruments could be the constituent securities when the benchmark is a prominent index, and a bond when it is an expected constant target or a fund has a return $\mathfrak{B}_T$, etc. The strategy implied by the second part is the purchase of European call options. It could be any combination of options as long as it replicates the payoff $(\bar{g}(\rho) - x_0 e^{\mathfrak{B}_T - c})^+$ at time T.

Remark 2.13. *A random $\mathfrak{B}_T$ was considered in Berkelaar, Kouwenberg, and Post [16], but $\mathfrak{B}_T$ in their model has been endowed with another motivation: stochastic reference point. It means people will change their reference point according to the fluctuation of wealth and market status. The evolution of $\mathfrak{B}_T$ in [16] was defined by a dynamic updating rule for reference points, which is partly proportional to the change in wealth. So, in essence the randomness of $\mathfrak{B}_T$ is a replication of the terminal wealth $X^\pi(T)$, which is why the problem can be reduced to a static case with a fixed reference point and a shift in loss aversion degree. However, the setting cannot be directly adopted in our model, since, in practice, no one would use a benchmark that fluctuates almost synchronously with the portfolio itself. Normally, we expect the randomness of $\mathfrak{B}_T$ to be highly correlated to market status.*

2.4 Conclusion

Portfolio selection, as one of the basic topics in finance, has fascinated academics and practitioners for almost seventy years. The model presented in this chapter is an attempt to combine the features of behavioral finance with a return-oriented portfolio selection problem. Using the cumulative prospect theory to characterize the agent's risk preference makes the model psychologically realistic, but brings us new theoretical challenges. The results obtained are distinguished from those conventional objectives that aimed at maximizing the utility of terminal wealth. Shifting the carrier of utility from terminal wealth to return has found its roots in practice and academia, but somehow its difference may have been underestimated. Especially when one tries to explore more than just deriving the optimal trading policy but also the potential relationship within the market coefficients and the possibility to explain the observations and facts behind individual behaviors and the aggregate market. The generally observed two-case phenomenon (either a gain or a maximum loss) on the optimal solution in literature has verified the old saying, "There is no free lunch, higher return comes higher risk." In our demonstration, the factor that determines the gain or loss has changed from the state of pricing kernel ρ to the state of η. That is to say, the benchmark we choose to measure performance may be more critical in our investment decisions and influence the final optimal wealth at the same time. This point will not be observed if we just consider a CPT-investor with the purpose to maximize the utility of terminal wealth. In Zhang et. al. [138], the role of benchmark has been wiped out technically, but captured by the return-oriented objective in this model.

By means of "quantile formulation" and a relaxation method, we solved a difficult, non-concave quantile optimization problem, which cannot be tackled by traditional approaches and is rarely seen in the literature. Honestly, the relaxation method adopted to address the M-shaped utility may no longer be effective for other shapes of utility. One cannot expect there to be no gap between general non-concave optimization problems and their locally concavified problems. Finally, we derived the explicit replicating portfolio under deterministic coefficients, provided that the benchmark can be expressed as a function of the pricing kernel. In particular, it covers the case when the benchmark is a specified constant target for individuals.

Chapter 3

Optimal Moral-Hazard-Free Insurance Model

Non-EU preference also has its application in the insurance market. In Chapter 3, we introduce an insurance contract design problem in which the insurer and insured are both characterized by non-EU preferences. In Section 3.1, we go through the background and motivation behind our model, including a literature review. In Section 3.2, we formulate our optimal control problem and convert the problem into a constrained quantile optimization problem. In Section 3.3, we subsequently characterize the optimal solution by means of calculus of variation, which boils down to solving an ordinary-integrated differential equation. The equation can be solved numerically, and we present an example in Section 3.4.

3.1 Background and Motivation

There are at least two ways to eliminate one's risk exposure in the financial market. The first one is to hedge the risk by continuous trading of financial instruments in a standard liquid market. The other one is called "risk-sharing" which seeks risk reallocation across individuals and firms. In the literature, risk sharing models generally explore an equilibrium between two parties, which is significantly different from portfolio selection problems. A mechanism (contract) is designed to allow the beneficiary party to reduce risk by receiving payments from the other when a loss occurs. The basic risk-sharing problem in the insurance industry is to establish an insurance contract between the insurer and the insured that achieves Pareto optimality/efficiency (PO/PE). Pareto optimality means one party cannot increase its utility without affecting the other one's.

Mossin [97] showed that the optimal policy for a risk-averse insured is full coverage when the premium is sold at a fair price. A weakness of Mossin's model is that the premium is given. Schlesinger [109] provided a Mossin's theorem when the insurance policy is not given but upper-bounded. A more general framework was proposed by Borch [21], who first demonstrated that the Pareto optimum risk allocation is possible in a reinsurance market when agents' preferences are described by the EUT. Inspired by Borch's model, Arrow [5] imagined a risk-neutral insurer who calculates the premium based on the actuarial value of the policy and a proportionate loading. Given a non-negative reimbursement restriction and the assumption that the insured and the insurer share the same probabilistic belief about the random loss, the best policy for a risk-averse insured is a deductible scheme, that is, complete coverage of the loss above a threshold. Arrow's classical model has been used as a framework for optimal insurance design, and it has brought out numerous extensions and variations. For example, Arrow [6] extended the results of [5] to state-independent utilities. Raviv [104] expanded on Arrow's work by taking into account mild assumptions and constraints such as an upper limit of coverage and multiple risks. The insurance cost was shown to have a significant impact on the optimality of a deductible policy. In [104], the Pareto optimal policy for a risk-averse insured is a combination of deductible and coinsurance. Namely, the optimal coverage function is proportional to the amount of money lost over a deductible. Schlesinger [107] particularly studied the connection between aversion degree and the optimal level of coverage for deductible insurance. It was shown that the greater the insured's aversion, the lower the level they purchased. However, their result no longer holds if the indemnity scheme has an upper bound limit, as pointed out in Cummins and Mahul [35].

In most cases, the risk covered by an insurance contract cannot be covered by trading (portfolios) in liquid markets. But it does not mean the risk in liquid markets can not be hedged by insurance contracts. Brennan and Solanki [23] considered a portfolio insurance contract wherein the coverage is based on the investment performance of one's portfolio. Considering that there may not exist an ideal option to hedge the reference portfolio, this insurance contract can be regarded as complementary to the option market. Leland [77] identified those who will benefit from purchasing portfolio insurance, such as safety-first investors and those who have their wealth managed by institutions and are optimistic about a higher return. Black and Robert [19] provided a simplified and comprehensive strategy for how to design a portfolio insurance policy and illustrated it with an example.

On the other hand, the insurance market can not hedge all the social risks in a contract. Most of the works mentioned above isolate the insurance loss from those uninsurable risks such as natural disasters and wars. A special example of an uninsurable background risk is a setting of random initial wealth, which has been investigated by Doherty and Schlesinger [47], [48], Hong et al. [69], Mahul [90]. Doherty and Schlesinger [47] reexamined the optimality of full coverage in an incomplete market when there are two risks (one of which is uninsurable) with a two-state marginal distribution. They found Arrow's deductible optimal policy is invalid if the insurable loss is not independent of the uninsurable loss. Gollier [61] investigated an additively separable dependence between the coverage loss and the background risk. If the insured is prudent and the increase in insurable losses results in a more risky distribution of background risk, then the optimal policy is a deductible type. A non-separable case was explored by Vercammen [123] which observed the opposite result to Gorllier's. The optimal policy entails coinsurance above the deductible. Dana and

Scarsini [39] studied a more general circumstance of stochastic dependence ("stochastic increasing") between the insured and uninsurable loss, whose results hold for all risk-averse expected utility maximizers (EU-maximizers).

Apart from using the EUT to characterize the risk preference, it looks more realistic to consider non-EU preferences in risk-sharing scenarios, as one of the earliest observations of violations of the EUT is the purchase of insurance and lottery tickets. In general, the insurer and insured have different probabilistic beliefs to the potential risk. Doherty and Eeckhoudt [46] reexamined the standard results for optimal policy when the policy holder's risk preference follows Yarri's dual theory. Schlesinger [108] illustrated the robustness of Arrow's results and Mossin's theorem when the insured's preference is expressed as a function of mean and standard deviation, not necessarily an EU-maximizer. Dana et al. [29] established a general equilibrium set-up in which the agents have a non-additive measure, which involves Chqouet expectation (CEU-maximizer). The Pareto optimal allocation turns out to be identical with the classical results for the EU-maximizer if the capacity function is convex. Sung et al. [117] solved an insurance problem in which the agent's preference is modeled by the cumulative prospect theory (CPT). Gollier [62] considered a case in which the policy holder has an ambiguity aversion and the distribution of loss is also ambiguous. The author showed that if the ambiguity is concentrated on the realization of small losses, then it will diminish the demand for insurance. Bernard et al. [17] and Xu [133] investigated an insurance contract design problem in which the insured evaluated insurance contracts using the RDU risk measure and the insurer utilizes the expected premium principle.

In this chapter, we also consider a similar optimal insurance design model with an insured characterized by the RDU risk measure. But we consider a more general premium principle, namely Wang's premium principle. In the conventional paradigm, the premium is proportional to the expected coverage, which is linear and additive.

Non-additive risk measures have also been applied to premium calculation. Deprez and Gerber [43] generalized the commonly seen premium principle into a convex premium principle. The authors established the corresponding properties and applied the principle to optimal reinsurance problems as well as optimal cooperation under fairly general assumptions. Wang [127] proposed a general premium principle that is convex and involves distortion of probabilities, related to Yarri's dual theory. Yong [137] utilized Wang's premium principle to study the optimal insurance policy for a risk-aversion EU-maximizer. Because Wang's class of premium principles are non-linear, it becomes a huge technological challenge to overcome. The optimal policy was shown to be deductible when the distortion function is piece-wise linear, and to be deductible with coinsurance above when the distortion is power.

Another characteristic of our model is that we use Huberman, Mayers, Smith Jr [70] and Picard [98]'s incentive compatibility constraint. This constraint requires that, in an insurance problem, the compensation and retention functions must be non-decreasing in relation to the loss. Bernard et al. [17] abandoned it, resulting in a moral hazard contract in which the insured is encouraged to falsely disclose actual losses. Xu et al. [134] and Xu [133] took the constraint into consideration in Bernard et al.'s model, and their optimal contracts eliminate the moral hazard issue. While Ghossoub [57] imposed a state-verification cost to rule out moral hazard contracts.

This study will follow the technological procedure introduced by Xu [133]. The problem turns out to be a non-concave optimization problem caused by the nonlinear Choquet expectation. To tackle it, we transform it into a corresponding quantile optimization problem, which is a tractable concave optimization problem. We then apply the calculus of variations approach to obtain the corresponding optimality condition from an ordinary integral-differential equation (OIDE) and reduce the OIDE to a numerically solvable ordinary differential equation (ODE). The equivalency between the PO contract and the ODE is provided. The rest of this chapter is organized as follows. In Section 3.2, we introduce a Pareto optimality insurance problem where the insured uses the RDU risk measure to evaluate insurance contracts and the insurer uses Wang's class of premium principles. A quantile formulation of the insurance problem has been obtained. We will provide its optimal solution in Section 3.3.

3.2 Problem Formulation

When talking about an insurance contract in optimal insurance problems, we mean two things. The first is a *premium* $\mathscr{P} \in \mathbb{R}$ that the insured ("she") pays to the insurer (an insurance business) for acquiring the contract, and the second is a *compensation* (also known as an indemnity) scheme I in which the insurer reimburses the insured $I(x)$ in the event of an actual loss x. Throughout this model, we use the same notation as in Xu [133].

Let X be the random loss that the insured wants to share with the insurer. A well-known fact in practice is that X is bounded and its probability has a mass at 0. In most cases, it is impossible for the insured to foresee the occurrence and magnitude of a prospective loss x, and it is also unable to hedge it in the financial market. However, based on sufficient samples, the insurer may occasionally acquire the empirical distribution function of the random loss X, which is a good approximation of the real one. We denote F_X the cumulative distribution function of the random loss X and impose the following assumption:

Assumption 3.1. *The random loss X is bounded with the support $[0, M]$, and $F_X(x)$ is a continuous, strictly increasing function on $[0, M]$ with $F_X(0) = m_0 > 0$.*

Let $I(x)$ and $R(x)$ be the loss borne of the insurer and insured after claim of an actual loss x, which are both assumed to be functions of x. We must have

$$I(x) + R(x) = x,$$

meaning that the true loss x is shared by the two parties. $I(x)$ and $R(x)$ are called the compensation and retention functions, respectively, in the literature. It is an acquiescence to assume both compensation and retention cannot exceed the true loss x, implying that we have

$$I(0) = 0, \quad 0 \le I(x) \le x, \quad \forall\, x \in \mathbb{R}, \tag{3.1}$$
$$R(0) = 0, \quad 0 \le R(x) \le x, \quad \forall\, x \in \mathbb{R}\,.$$

When $I(x) \equiv x$, we say the insurance is *full coverage*. When $I(x) \equiv \max\{x - d, 0\}$, the insurance is called *deductible*.

Furthermore, both $I(x)$ and $R(x)$ should be non-decreasing functions with respect to x; otherwise, the insured is willing to report a smaller loss in exchange for a larger compensation $I(x)$ or a larger loss in exchange for a smaller retention $R(x)$, which are examples of the so-called *moral-hazard* behaviors. Mathematically, we should impose the following constraint to prevent such moral-hazard issues:

$$0 \le I(x) \le I(y), \quad 0 \le R(x) \le R(y), \quad \forall\, 0 \le x \le y,\ x,\ y \in \mathbb{R}\,. \tag{3.2}$$

Combining with (3.1) together, we give the set of compensation function by

$$\mathscr{C} := \Big\{ I : [0, \infty) \to [0, \infty) \;\Big|\; I \text{ is absolutely continuous}$$
$$\text{with } I(0) = 0 \text{ and } 0 \le I' \le 1 \text{ almost everywhere (a.e.).} \Big\}.$$

Note that the set of retention function $\mathscr{R}$ is the same as the set of compensation function $\mathscr{C}$.

In a traditional fashion, the premium $\mathscr{P}$ charged by the insurer should not be less than the expected value of compensation $I(x)$; otherwise the insurer will face insolvency due to the law of large numbers. It should also not exceed the maximum loss; otherwise, no one would buy it. Normally, one has

$$(1 + \theta)\, \mathbb{E}[I(X)] \le \mathscr{P} < M,^{1}$$

[1] In this model, we assume $(1 + \theta)H(I(X))) \le \mathscr{P} < M$, where $H(\cdot)$ stands for Wang's premium principle.

which is called the participation constraint of the insurer. Here, $\theta > 0$ is called the *safety loading* coefficient, which can be regarded as an additional cost built into the contract to cover unexpected potential loss. For example, if an insured person has already experienced five car accidents this year, the insurer would charge a higher θ next year than those who have not, since the person himself becomes a kind of risk which needs to be added in the insurer's eyes. In this model, we assume θ to be a fixed constant.

In addition, we assume the insurer calculates the premium using Wang's class of premium principles. In Wang [127], a premium principle H is defined to be a functional $X \to [0, \infty]$ as

$$H(X) = \int_0^\infty \mathfrak{g}(1 - F_{I(X)}(x))\,\mathrm{d}x,$$

where $F_{I(X)}$ is the cumulative distribution function of $I(X)$, $\mathfrak{g}$ is an increasing concave function with $\mathfrak{g}(0) = 0$ and $\mathfrak{g}(1) = 1$. Note that if $\mathfrak{g}(x) = x$, $H(X) = E(I(X))$ reduces to the classical case.

Based on Wang's premium principle, we define the value of an insurance contract $(\mathscr{P}, I)$ from the insurer's perspective:

$$\begin{aligned}\mathcal{U}_{\text{insurer}}(\mathscr{P}, I) =& \mathscr{P} - (1+\theta)H(X) \\ =& \mathscr{P} - (1+\theta)\int_0^\infty \mathfrak{g}(1 - F_{I(X)}(x))\,\mathrm{d}x. \end{aligned} \tag{3.3}$$

On the other hand, we characterize the insured's risk preference according to the rank-dependent utility theory of Quiggin [103]. Let $u : \mathbb{R} \to \mathbb{R}$ be the utility function of the insured that is differentiable, strictly concave, and strictly increasing, and let $w : [0, 1] \to [0, 1]$ be a probability weighting function that is continuous and strictly increasing with $w(0) = 0, w(1) = 1$. The RDU risk measure $\mathscr{E}$ of an random variable Y then is defined as follows:

$$\mathscr{E}[Y] = \int_0^1 u\left(F_Y^{-1}(p)\right)\mathrm{d}\left(1 - w(1-p)\right). \tag{3.4}$$

If there is no probability weighting, namely $w(x) = x$, then the RDU risk measure reduces to the traditional expected utility form.

In the utility maximization framework, the insured seeks to maximize the utility of expected terminal wealth. Denote $\beta_{\text{insured}} > 0$ the initial economic status of the insured. Then the terminal position is expressed by

$$\beta_{\text{insured}} - \mathscr{P} - X + I(X) = \beta_{\text{insured}} - \mathscr{P} - R(X).$$

In our model, we do not consider insurance contracts $(\mathscr{P}, I)$ that lead to the bankruptcy of the terminal position, namely $\beta_{\text{insured}} - \mathscr{P} - X + I(X) \geq 0$, otherwise we need to consider a RDU measure with a negative part. To ensure it, we assume $\beta_{\text{insured}} > 2M$. Thus, the value of an insurance contract $(\mathscr{P}, I)$ under RDU insured is

$$\mathcal{U}_{\text{insured}}(\mathscr{P}, I) = \mathscr{E}\left[\left(\beta_{\text{insured}} - \mathscr{P} - X + I(X)\right)\right].$$

One can discover that $\mathcal{U}_{\text{insurer}}(\mathscr{P}, I)$ is increasing with respect to $\mathscr{P}$ and decreasing with respect to I, while $\mathcal{U}_{\text{insured}}(\mathscr{P}, I)$ is decreasing with respect to $\mathscr{P}$ and increasing with respect to I on the contrary. A trade-off exists between $\mathcal{U}_{\text{insurer}}(\mathscr{P}, I)$ and $\mathcal{U}_{\text{insured}}(\mathscr{P}, I)$.

In practice, the equilibrium between two parties is achieved by negotiation, which can be simplified into two steps. Firstly, the insurer calculates the base value of the insurance contract after examining the market quotation. They offer the potential choice of an insurance contract $(\mathscr{P}, I)$ to the insured. For any $(\mathscr{P}, I)$, one has

$$\mathcal{U}_{\text{insurer}}(\mathscr{P}, I) \geq \gamma,$$

where $\gamma > 0$.

The problem boils down to the following

$$\begin{aligned} \sup_{\mathscr{P}\in\mathbb{R},\, I\in\mathscr{C}} \quad & \mathcal{U}_{\text{insured}}(\mathscr{P}, I) = \mathscr{E}\left[\left(\beta_{\text{insured}} - \mathscr{P} - X + I(X)\right)\right] \\ \text{s.t.} \quad & \mathcal{U}_{\text{insurer}}(\mathscr{P}, I) \geq \gamma. \end{aligned} \tag{3.5}$$

The optimal insurance contract of Problem (3.5) is also called *Pareto optimal/efficient* (PO) which means one cannot improve the benefit of one party without hurting the other's or improve the benefits of both. Any PO contract $(\mathscr{P}^*, I^*)$ must satisfy

$$\mathcal{U}_{\text{insurer}}(\mathscr{P}^*, I^*) = \gamma,$$

namely

$$\mathscr{P}^* = \gamma + (1+\theta)\int_0^\infty \mathfrak{g}(1 - F_{I^*(X)}(x))\,\mathrm{d}x.$$

Because otherwise there exists $\mathscr{P}' < \mathscr{P}^*$ such that $\mathcal{U}_{\text{insurer}}(\mathscr{P}', I^*) = \gamma$, but

$$\mathcal{U}_{\text{insured}}(\mathscr{P}', I^*) > \mathcal{U}_{\text{insured}}(\mathscr{P}^*, I^*),$$

due to the decreasing property of $\mathcal{U}_{\text{insured}}$ on $\mathscr{P}$.

Consequently, Problem (3.5) can be reduced to a one-dimensional optimization problem:

$$\sup_{I \in \mathscr{C}} \quad \mathscr{E}\big(\beta_{\text{insured}} - \gamma - (1+\theta)\int_0^\infty \mathfrak{g}(1 - F_{I(X)}(x))\,\mathrm{d}x - X + I(X)\big). \tag{3.6}$$

Note that if I^*_γ is an optimal solution to the above problem, then a PO contract is expressed by

$$\Big(\gamma + (1+\theta)\int_0^\infty \mathfrak{g}(1 - F_{I^*_\gamma(X)}(x))\,\mathrm{d}x,\ I^*_\gamma\Big).$$

Since Problem (3.6) is highly involved with integral and probability distortion, making the objective functional a nonlinear Choquet expectation and is difficult to tackle directly. We use the same method of quantile formulation introduced in the previous chapters to study its equivalent quantile optimization problem. Change of variable is adopted to obtain a simplified version.

We use F_X^{-1} to denote the quantile function of F_X (or its left-continuous inverse function). Recall its definition as below

Definition 3.1. *The (lower) quantile function* F_X^{-1} *is defined as*

$$F_X^{-1}(p) = \inf\left\{z \in [0, M] \mid F_X(z) \geq p\right\}, \quad p \in (0, 1),$$

with the convention that $F_X^{-1}(0) = F_X^{-1}(0+)$, $F_X^{-1}(1) = F_X^{-1}(1-)$ *and* $\inf \emptyset = +\infty$.

By this definition and Assumption 3.1, $F_X^{-1}(p) = 0$ for $p \in [0, m_0]$ and $F_X^{-1}(1) = M$. It is non-decreasing and left-continuous.

Following [133], we have the following technical assumptions on F_X^{-1}.

Assumption 3.2. F_X^{-1} *is absolutely continuous on* $[0, 1]$ *and* $\left(F_X^{-1}\right)'(p) > 0$ *for a.e.* $p \in (m_0, 1)$.

As demonstrated in Corollary 1.4 and Lemma 1.5, one can find a random variable U uniformly distributed on $[0, 1]$, such that U is comonotonic with X, and $F_X^{-1}(U) = X$. Both $I(X)$ and $R(X)$ are non-decreasing functions with respect to X. As a result, U is also comonotonic with $I(X)$ and $R(X)$. It follows

$$I(X) = I(F_X^{-1}(U)) = F_{I(X)}^{-1}(U),$$

$$R(X) = R(F_X^{-1}(U)) = F_{R(X)}^{-1}(U).$$

Since $I(X) + R(X) = X$, we have

$$F_{I(X)}^{-1}(U) + F_{R(X)}^{-1}(U) = F_X^{-1}(U). \tag{3.7}$$

Furthermore, since $I(X)$ and $R(X)$ are comonotonic, their quantiles are additive (See Denneberg [42]).

Let

$$G(p) = F_{R(X)}^{-1}(p), \quad p \in [0, 1],$$

be the quantile function of $R(X)$, then from above

$$G(U) = F_X^{-1}(U) - F_{I(X)}^{-1}(U) = X - I(X).$$

Let

$$
\begin{aligned}
Y &= \beta_{\text{insured}} - \gamma - (1+\theta)\int_0^{\infty} \mathfrak{g}(1 - F_{I(X)}(x))\,\mathrm{d}x - X + I(X) \\
&= \beta_{\text{insured}} - \gamma - (1+\theta)\int_0^{1} \mathfrak{g}(1-q)\,\mathrm{d}F_{I(X)}^{-1}(q) - G(U) \\
&= \beta_{\text{insured}} - \gamma - (1+\theta)\big(\mathfrak{g}(1-q)F_{I(X)}^{-1}(q)|_0^1 - \int_0^1 F_{I(X)}^{-1}(q)\,\mathrm{d}\mathfrak{g}(1-q)\big) - G(U) \\
&= \beta_{\text{insured}} - \gamma + (1+\theta)\int_0^1 F_{I(X)}^{-1}(q)\,\mathrm{d}\mathfrak{g}(1-q) - G(U) \\
&= \beta_{\text{insured}} - \gamma + (1+\theta)\int_0^1 (F_X^{-1}(q) - G(q))\,\mathrm{d}\mathfrak{g}(1-q) - G(U) \\
&= \beta - (1+\theta)\int_0^1 G(q)\,\mathrm{d}\mathfrak{g}(1-q) - G(U),
\end{aligned}
$$

where

$$
\beta = \beta_{\text{insured}} - \gamma + (1+\theta)\int_0^1 F_X^{-1}(q)\,\mathrm{d}\mathfrak{g}(1-q).
$$

Note that Y is anti-comonotonic with U, so the quantile function of Y is given by

$$
F_Y^{-1}(p) = \beta - (1+\theta)\int_0^1 G(q)\,\mathrm{d}\mathfrak{g}(1-q) - G(1-p), \quad \text{a.e. } p \in [0,1].
$$

Since w is strictly increasing and continuous, its inverse w^{-1} exists. Let $\chi(p) = w^{-1}(p)$, which is also a probability weighting function. Then

$$
\begin{aligned}
\mathscr{E}[Y] &= \int_0^1 u\left(F_Y^{-1}(p)\right)\mathrm{d}\left(1 - w(1-p)\right) \\
&= \int_0^1 u\left(\beta - (1+\theta)\int_0^1 G(q)\,\mathrm{d}\mathfrak{g}(1-q) - G(1-p)\right)\mathrm{d}\left(1 - w(1-p)\right) \\
&= \int_0^1 u\left(\beta - (1+\theta)\int_0^1 G(q)\,\mathrm{d}\mathfrak{g}(1-q) - G(t)\right)\mathrm{d}w(t) \\
&= \int_0^1 u\left(\beta - (1+\theta)\int_0^1 G(\chi(t))\,\mathrm{d}\mathfrak{g}(1-\chi(t)) - G(\chi(s))\right)\mathrm{d}s
\end{aligned}
$$

$$= \int_0^1 u\left(\beta + (1+\theta)\int_0^1 Q(t)\,\mathrm{d}\nu(t) - Q(s)\right)\mathrm{d}s,$$

where

$$\nu(t) = 1 - \mathfrak{g}(1-\chi(t)),\ t \in [0,1]. \tag{3.8}$$

and

$$Q(s) = G(\chi(s)) = R(F_X^{-1}(\chi(s))),\ s \in [0,1]. \tag{3.9}$$

Note that ν is an increasing function with $\nu(0) = 0$ and $\nu(1) = 1$.

Let

$$h(p) := \big(F_X^{-1}\big)(\chi(p))', \quad p \in [0,1].$$

For $R \in \mathscr{R}$, we have $0 \le R' \le 1$, so

$$Q'(p) = R'(F_X^{-1}(\chi(p)))h(p) \in [0, h(p)], \quad p \in [0,1].$$

Clearly,

$$\int_0^p h(t)\,\mathrm{d}t = \big(F_X^{-1}\big)(\chi(p)) \le \big(F_X^{-1}\big)(1) = M,$$

according to Assumption 3.2.

Thus, we reduce Problem (3.6) to the following problem:

$$\sup_{Q\in\mathscr{Q}} \int_0^1 u\Big(\beta + (1+\theta)\int_0^1 Q(t)\,\mathrm{d}\nu(t) - Q(p)\Big)\,\mathrm{d}p, \tag{3.10}$$

where

$$\mathscr{Q} := \Big\{Q : [0,1] \to [0,\infty) \;\Big|\; Q \text{ is absolutely}$$

$$\text{continuous with } Q(0) = 0 \text{ and } 0 \le Q' \le h \text{ a.e.}\Big\},$$

and ν is given by (3.8),

$$h(p) = \big(F_X^{-1}\big)(\chi(p))', \quad \chi(p) = w^{-1}(p), \quad p \in [0,1].$$

Because the set $\mathscr{Q}$ is convex and the utility function is concave with respect to $Q \in \mathscr{Q}$, we have converted a non-concave optimization problem to a concave one. The derivatives of the admissible quantiles $Q \in \mathscr{Q}$ are both lower and upper bounded; such quantile optimization problems are of the second-type defined in Xu [133].[2]

We present a relationship between the optimal solutions of Problem (3.10) and Problem (3.6) as a lemma below.

Lemma 3.1. *A quantile function $Q^* \in \mathscr{Q}$ is an optimal solution to Problem* (3.10) *if and only if*

$$I^*(x) = x - Q^*(w(F_X(x))), \quad x \in [0, M],$$

is an optimal solution to Problem (3.6)*. And the optimal insurance contract is given by*

$$\Big(\gamma + (1+\theta)\int_0^\infty \mathfrak{g}(1 - F_{I^*(X)}(x))\,\mathrm{d}x,\ I^*\Big),$$

where γ is decided by the insurer.

Proof. By backward deduction of (3.9), one has

$$R^*(x) = G^*(F_X(x)) = Q^*(w(F_X(x))).$$

Since $I^*(x) = x - R^*(x)$, this completes the proof. □

3.3 Characterization of Optimal Solution

The relaxation method introduced in Chapter 3 cannot directly solve the quantile optimization problem (3.10) in last section. We use calculus of variation to derive an optimal condition. Problem (3.10) is expressed as:

$$\sup_{Q\in\mathscr{Q}} \int_0^1 u\Big(\beta + (1+\theta)\int_0^1 Q(t)\,\mathrm{d}\nu(t) - Q(p)\Big)\,\mathrm{d}p,$$

[2] The first-type problems only put one side (upper or lower) constraint on the derivatives of the admissible quantiles.

where

$$\nu(p) = 1 - \mathfrak{g}(1 - \chi(p)),\ \chi(p) = w^{-1}(p), \quad p \in [0,1], \tag{3.11}$$

$$\beta = \beta_{\text{insured}} - \gamma + (1+\theta)\int_0^1 F_X^{-1}(q)\,\mathrm{d}\mathfrak{g}(1-q), \tag{3.12}$$

and

$$\begin{aligned} \mathscr{Q} := \Big\{ Q : [0,1] \to [0,\infty) \;\Big|\; & Q \text{ is absolutely} \\ & \text{continuous with } Q(0) = 0 \text{ and } 0 \le Q' \le h \text{ a.e.} \Big\}, \end{aligned}$$

and u, $\mathfrak{g}$, w are respectively the utility function, the distortion function in the premium principle, the probability weighting function. Also β_{insured}, γ, θ are respectively the insured's initial wealth position, the lower bound of the insurer's preference, the safety loading.

Let

$$f_Q(p) = \beta + (1+\theta)\int_0^1 Q(t)\,\mathrm{d}\nu(t) - Q(p), \quad p \in [0,1].$$

Suppose Q^* is an optimal solution to Problem (3.10). For any $Q \in \mathscr{Q}$ and $0 < \varepsilon < 1$, define $Q_\varepsilon = Q^* + \varepsilon(Q - Q^*)$. Since $\mathscr{Q}$ is convex, we have $Q_\varepsilon \in \mathscr{Q}$ and $\lim_{\varepsilon\to 0} Q_\varepsilon = Q^*$. By the optimality of Q^* and Fatou's Lemma, we have

$$\begin{aligned} 0 &\ge \liminf_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\Big[\int_0^1 u\Big(f_{Q_\varepsilon}(p)\Big)\,\mathrm{d}p - \int_0^1 u\Big(f_{Q^*}(p)\Big)\,\mathrm{d}p\Big] \\ &\ge \int_0^1 \liminf_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\Big(u\big(f_{Q_\varepsilon}(p)\big) - u\big(f_{Q^*}(p)\big)\Big)\,\mathrm{d}p \\ &= \int_0^1 \lim_{\varepsilon\to 0} u'\Big(f_{Q^*}(p)\Big)\frac{1}{\varepsilon}\big(f_{Q_\varepsilon}(p) - f_{Q^*}(p)\big)\,\mathrm{d}p. \end{aligned}$$

Based on (3.11), one has

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\big(f_{Q_\varepsilon}(p) - f_{Q^*}(p)\big)$$

$$
\begin{aligned}
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\Big((1+\theta)\int_0^1 (Q_\varepsilon(t)-Q^*(t))\,\mathrm{d}\nu(t) - (Q_\varepsilon(t)-Q^*(t))\Big)\\
&=(1+\theta)\int_0^1 (Q(t)-Q^*(t))\,\mathrm{d}\nu(t) - (Q(t)-Q^*(t))\\
&=f_Q(p)-f_{Q^*}(p).
\end{aligned}
$$

So for $Q \in \mathscr{Q}$,

$$
\int_0^1 u'\Big(f_{Q^*}(p)\Big)\big(f_Q(p)-f_{Q^*}(p)\big)\,\mathrm{d}p \le 0. \tag{3.13}
$$

Reversely, suppose (3.13) holds. Since $u(x)$ is concave, we have $u(y)-u(x) \le u'(x)(y-x)$, for $y, x \in \mathbb{R}$. Hence for $Q \in \mathscr{Q}$

$$
u\Big(f_Q(p)\Big) - u\Big(f_{Q^*}(p)\Big) \le u'\Big(f_{Q^*}(p)\Big)\big(f_Q(p)-f_{Q^*}(p)\big).
$$

Then

$$
\int_0^1 u\Big(f_Q(p)\Big)\,\mathrm{d}p - \int_0^1 u\Big(f_{Q^*}(p)\Big)\,\mathrm{d}p \le \int_0^1 u'\Big(f_{Q^*}(p)\Big)\big(f_Q(p)-f_{Q^*}(p)\big)\,\mathrm{d}p \le 0,
$$

which proves the optimality of Q^*. Thus (3.13) is an equivalent characterization of the optimal solution.

Furthermore, let

$$
\Phi(p) = -\int_p^1 u'\Big(f_{Q^*}(t)\Big)\,\mathrm{d}t,
$$

then $\Phi(1)=0$ and $\Phi'(p)=u'\Big(f_{Q^*}(p)\Big)$. By partial integration and the fact $Q(0)=0$, the left-hand side of (3.13) can be written as

$$
\begin{aligned}
&\Phi(p)\big(f_Q(p)-f_{Q^*}(p)\big)\Big|_0^1 - \int_0^1 \Phi(p)\,\mathrm{d}\big(f_Q(p)-f_{Q^*}(p)\big)\\
=&-\Phi(0)\big(f_Q(0)-f_{Q^*}(0)\big) - \int_0^1 \Phi(p)\,\mathrm{d}\big(f_Q(p)-f_{Q^*}(p)\big).
\end{aligned}
$$

By (3.11), it reduces to

$$
\begin{aligned}
&-\Phi(0)(1+\theta)\int_0^1 (Q(t)-Q^*(t))\,\mathrm{d}\nu(t) + \int_0^1 \Phi(p)\big(Q'(p)-(Q^*)'(p)\big)\,\mathrm{d}p \\
=&-\Phi(0)(1+\theta)\Big[\nu(t)(Q(t)-Q^*(t))|_0^1 - \int_0^1 \nu(t)\big(Q'(t)-(Q^*)'(t)\big)\,\mathrm{d}t\Big] + \\
&\int_0^1 \Phi(p)\big(Q'(p)-(Q^*)'(p)\big)\,\mathrm{d}p \\
=&-\Phi(0)(1+\theta)\Big[(Q(1)-Q^*(1)) - \int_0^1 \nu(t)\big(Q'(t)-(Q^*)'(t)\big)\,\mathrm{d}t\Big] + \\
&\int_0^1 \Phi(p)\big(Q'(p)-(Q^*)'(p)\big)\,\mathrm{d}p \\
=&-\Phi(0)(1+\theta)\Big[\int_0^1 \big(Q'(t)-(Q^*)'(t)\big)\,\mathrm{d}t - \int_0^1 \nu(t)\big(Q'(t)-(Q^*)'(t)\big)\,\mathrm{d}t\Big] + \\
&\int_0^1 \Phi(p)\big(Q'(p)-(Q^*)'(p)\big)\,\mathrm{d}p \\
=&\int_0^1 \big(\Phi(p)-\Phi(0)(1+\theta)(1-\nu(p))\big)\big(Q'(p)-(Q^*)'(p)\big)\,\mathrm{d}p,
\end{aligned}
$$

where the first equation is by applying partial integration to $\int_0^1 (Q(t)-Q^*(t))\,\mathrm{d}\nu(t)$; the second equation is due to $\nu(0)=0, \nu(1)=1$; and the third equation is due to $Q(0)=0$.

So an equivalent expression of (3.13) is that for $Q \in \mathscr{Q}$,

$$
\int_0^1 \big(\Phi(p)-\Phi(0)(1+\theta)(1-\nu(p))\big)\big(Q'(p)-(Q^*)'(p)\big)\,\mathrm{d}p \le 0. \tag{3.14}
$$

Based on the equivalence between (3.10) and (3.13), one can realize that Q^* is also the optimal solution to the following problem

$$
\max_{Q\in\mathscr{Q}} \quad \int_0^1 \big(\Phi(p)-\Phi(0)(1+\theta)(1-\nu(p))\big)Q'(p)\,\mathrm{d}p, \tag{3.15}
$$

where

$$\Phi(p) = -\int_p^1 u'\Big(f_{Q^*}(t)\Big)\,\mathrm{d}t, \quad p\in[0,1].$$

For any admissible $Q\in\mathscr{Q}$, we have $0\le Q'\le h(p),\ p\in[0,1]$. In order to maximize (3.15), clearly Q^* must satisfy

$$\begin{cases}(Q^*)'(p)=h(p), & \text{if } \Phi(p)>\Phi(0)(1+\theta)(1-\nu(p));\\ (Q^*)'(p)\in[0,h(p)], & \text{if } \Phi(p)=\Phi(0)(1+\theta)(1-\nu(p));\\ (Q^*)'(p)=0, & \text{if } \Phi(p)<\Phi(0)(1+\theta)(1-\nu(p)),\end{cases} \quad \text{for a.e. } p\in[0,1]. \tag{3.16}$$

We assemble the three cases in (3.16) into the following equation, which is an ordinary integral-differential equation (OIDE).

Lemma 3.2. *For $Q\in\mathscr{Q}$, Q is an optimal solution to problem* (3.6) *if and only if Q is an optimal solution to the following OIDE:*

$$\begin{cases}\min\Big\{\max\big\{Q'(p)-h(p),\ \Phi(0)(1+\theta)(1-\nu(p))-\Phi(p)\big\},\ Q'(p)\Big\}=0, \quad a.e.\ p\in[0,1],\\ Q(0)=0,\end{cases} \tag{3.17}$$

where

$$\Phi(p) = -\int_p^1 u'\Big(f_Q(t)\Big)\,\mathrm{d}t, \quad p\in[0,1] \tag{3.18}$$

and

$$f_Q(p)=\beta+(1+\theta)\int_0^1 Q(t)\,\mathrm{d}\nu(t)-Q(p), \quad p\in[0,1]. \tag{3.19}$$

Moreover, it turns out OIDE (3.17) can be reduced to an ODE by writing it in terms of Φ. In subsequent analysis, we show Q' and $Q(0)$ in (3.17) can be expressed in terms of Φ. Before that, we prove a lemma.

Lemma 3.3. $\min\{\max\{a,b\},c\}=0$ *is equivalent to* $\min\{\max\{ma,nb\},hc\}=0$ *for any $m,n,h>0$.*

Proof. Given $\min\{\max\{a,b\},c\}=0$, it follows two possible cases

$$c=0, \quad \max\{a,b\}\geq 0; \quad \text{or} \quad c>0, \quad \max\{a,b\}=0.$$

In the former case, $hc=0$. Since $\max\{a,b\}\geq 0$, we have $a\geq 0$ or $b\geq 0$, then $ma\geq 0$ or $nb\geq 0$, namely $\max\{ma,nb\}\geq 0$. Thus, $\min\{\max\{ma,nb\},hc\}=0$. In the latter case, $hc>0$. Since $\max\{a,b\}=0$, we have $b\leq a=0$ or $a\leq b=0$, then $nb\leq na=0=ma$ or $ma\leq mb=0=nb$, namely $\max\{ma,nb\}=0$. Thus, $\min\{\max\{ma,nb\},hc\}=0$.

The proof of the reverse is by letting $m=n=h=1$. □

Next we turn (3.17) into an ODE. By differentiating (3.18) and taking the inverse, we have

$$f_Q(p)=(u')^{-1}(\Phi'(p)).$$

Putting it into (3.19)

$$Q(p)=\beta+(1+\theta)\int_0^1 Q(t)\,\mathrm{d}\nu(t)-(u')^{-1}(\Phi'(p)). \tag{3.20}$$

Differentiating (3.18) twice, we obtain

$$\Phi''(p)=-u''(f_Q(p))Q'(p).$$

Combining it with $f_Q(p)=(u')^{-1}(\Phi'(p))$, we see

$$Q'(p)=\frac{\Phi''(p)}{-u''\Big((u')^{-1}(\Phi'(p))\Big)}. \tag{3.21}$$

Plugging it into (3.17),

$$\min\Big\{\max\Big\{\frac{\Phi''(p)}{-u''\Big((u')^{-1}(\Phi'(p))\Big)}-h(p),\ \Phi(0)(1+\theta)(1-\nu(p))-\Phi(p)\Big\},$$
$$\frac{\Phi''(p)}{-u''\Big((u')^{-1}(\Phi'(p))\Big)}\Big\}=0,$$

and by Lemma 3.3, one can show it is equivalent to

$$\min\Big\{\max\big\{\Phi''(p)+u''\Big((u')^{-1}(\Phi'(p))\Big)h(p),\ \Phi(0)(1+\theta)(1-\nu(p))-\Phi(p)\big\},\ \Phi''(p)\Big\}=0.$$

Besides, integrating with respect to $\nu(p)$ on both sides of (3.20) gives that

$$\begin{aligned}\int_0^1 Q(p)\,\mathrm{d}\nu(p) &= \int_0^1\Big[\beta+(1+\theta)\int_0^1 Q(t)\,\mathrm{d}\nu(t)-(u')^{-1}(\Phi'(p))\Big]\,\mathrm{d}\nu(p)\\ &= \beta+(1+\theta)\int_0^1 Q(t)\,\mathrm{d}\nu(t)-\int_0^1 (u')^{-1}(\Phi'(p))\,\mathrm{d}\nu(p),\end{aligned}$$

namely

$$0=\beta+\theta\int_0^1 Q(t)\,\mathrm{d}\nu(t)-\int_0^1 (u')^{-1}(\Phi'(t))\,\mathrm{d}\nu(t).$$

Based on (3.20) and $Q(0)=0$, one has

$$Q(0)=\beta+(1+\theta)\int_0^1 Q(t)\,\mathrm{d}\nu(t)-(u')^{-1}(\Phi'(0))=0.$$

Comparing the above two equations, we obtain

$$\int_0^1 Q(t)\,\mathrm{d}\nu(t)=-\int_0^1 (u')^{-1}(\Phi'(t))\,\mathrm{d}\nu(t)+(u')^{-1}(\Phi'(0)) \tag{3.22}$$

and

$$\beta=(1+\theta)\int_0^1 (u')^{-1}(\Phi'(t))\,\mathrm{d}\nu(t)-\theta(u')^{-1}(\Phi'(0)). \tag{3.23}$$

Putting (3.22) and (3.23) back into (3.20), we have

$$Q(p)=(u')^{-1}\big(\Phi'(0)\big)-(u')^{-1}\big(\Phi'(p)\big).$$

We denote by $C^{2-}([0,1])$ the set of functions $f:[0,1]\to\mathbb{R}$ which are differentiable and their derivative functions f' are absolutely continuous on $[0,1]$. The optimal solution to Problem (3.10) is completely characterized in the following result.

Theorem 3.1 (Optimal solution)**.** *(1). If Q^* is the optimal solution to Problem* (3.10)*. Then*

$$\Phi(p) := -\int_p^1 u'\Big(\beta + (1+\theta)\int_0^1 Q^*(t)\,\mathrm{d}\nu(t) - Q^*(s)\Big)\,\mathrm{d}s \tag{3.24}$$

is a solution in $C^{2-}([0,1])$ to the following ODE:

$$\begin{cases} \min\Big\{ \max\big\{\Phi''(p) + h(p)u''\big((u')^{-1}\big(\Phi'(p)\big)\big),\ \Phi(0)(1+\theta)(1-\nu(p)) - \Phi(p)\big\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \Phi''(p)\Big\} = 0, \quad a.e.\ p \in [0,1], \\ \Phi(1) = 0, \quad \beta = (1+\theta)\int_0^1 (u')^{-1}\big(\Phi'(t)\big)\,\mathrm{d}\nu(t) - (u')^{-1}\big(\Phi'(0)\big)\theta. \end{cases} \tag{3.25}$$

(2). If Φ is a solution to (3.25) *in $C^{2-}([0,1])$. Then*

$$Q^*(p) := (u')^{-1}\big(\Phi'(0)\big) - (u')^{-1}\big(\Phi'(p)\big)$$

and

$$I^*(x) := x - (u')^{-1}\big(\Phi'(0)\big) + (u')^{-1}\big(\Phi'\big(w(F_X(x))\big)\big) \tag{3.26}$$

are the optimal solution to Problem (3.10) *and the optimal compensation function to Problem* (3.6)*, respectively.*

Proof. For (1), we have illustrated it in the deduction. For (2), given a $\Phi \in C^{2-}([0,1])$ which is the solution of (3.25), Let

$$Q(p) := (u')^{-1}\big(\Phi'(0)\big) - (u')^{-1}\big(\Phi'(p)\big).$$

Differentiate it we have

$$Q'(p) = \frac{\Phi''(p)}{-u''\Big((u')^{-1}(\Phi'(p))\Big)} \quad a.e.\ p \in [0,1].$$

Plugging it into (3.25), since $Q(0) = 0$, and by Lemma 3.3, we obtain OIDE (3.17).

Since

$$\beta = (1+\theta)\int_0^1 (u')^{-1}(\Phi'(t))\,\mathrm{d}\nu(t) - \theta(u')^{-1}(\Phi'(0))$$

$$
\begin{aligned}
&=(1+\theta)\int_0^1 (u')^{-1}(\Phi'(t))\,\mathrm{d}\nu(t)-(\theta+1)(u')^{-1}(\Phi'(0))+(u')^{-1}(\Phi'(0))\\
&=(1+\theta)\int_0^1 (u')^{-1}(\Phi'(t))\,\mathrm{d}\nu(t)-(\theta+1)\int_0^1 (u')^{-1}(\Phi'(0))\,\mathrm{d}\nu(t)+(u')^{-1}(\Phi'(0))\\
&=(1+\theta)\int_0^1 \big((u')^{-1}(\Phi'(t))-(u')^{-1}(\Phi'(0))\big)\,\mathrm{d}\nu(t)+(u')^{-1}(\Phi'(0))\\
&=-(1+\theta)\int_0^1 Q(p)\,\mathrm{d}\nu(t)+(u')^{-1}(\Phi'(0)).
\end{aligned}
$$

where the second equation is due to $\nu(1)=1$ and $\nu(0)=0$; the last equation is by the definition of Q. So

$$
(u')^{-1}(\Phi'(0))=\beta+(1+\theta)\int_0^1 Q(p)\,\mathrm{d}\nu(t).
$$

Putting it into $Q(p)=(u')^{-1}\big(\Phi'(0)\big)-(u')^{-1}\big(\Phi'(p)\big)$, we derive

$$
Q(p)=\beta+(1+\theta)\int_0^1 Q(p)\,\mathrm{d}\nu(t)-(u')^{-1}\big(\Phi'(p)\big),
$$

namely

$$
\Phi(p)=-\int_p^1 u'\Big(\beta+(1+\theta)\int_0^1 Q(t)\,\mathrm{d}\nu(t)-Q(s)\Big)\,\mathrm{d}s=-\int_p^1 u'(f_Q(s))\,\mathrm{d}s.
$$

This completes the proof $\square$

3.4 Numerical Example

So far, we have demonstrated that solving the quantile optimization problem (3.10) reduces to solving ODE (3.25). But this ODE is difficult to solve directly, since the boundary condition is on the endpoint and β has a complex form.

In this section, we present a numerical example by assuming $\mathfrak{g}(x)=x$, then

$$
\nu(p)=1-\mathfrak{g}(1-\chi(p))=\chi(p)=w^{-1}(p).
$$

It is the inverse function of probability weighting function w.

The problem reduces to solve the following ODE

$$\begin{cases} \min\Big\{ \max\big\{\Phi''(p) + h(p)u''\big((u')^{-1}\big(\Phi'(p)\big)\big),\ \Phi(0)(1+\theta)(1-\nu(p)) - \Phi(p)\big\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Phi''(p)\Big\} = 0, \quad \text{a.e. } p \in [0,1], \\ \Phi(1) = 0, \quad \beta = (1+\theta)\int_0^1 (u')^{-1}\big(\Phi'(t)\big)\,\mathrm{d}\nu(t) - (u')^{-1}\big(\Phi'(0)\big)\theta. \end{cases} \tag{3.27}$$

To solve it, we turn to considering the following problem first:

Fix $\sigma < 0$, for each $\varpi > 0$, we find a numerical solution $\Phi_{\sigma,\varpi}$ to the following ODE

$$\begin{cases} \min\Big\{ \max\big\{\Phi''(p) + h(p)u''\big((u')^{-1}\big(\Phi'(p)\big)\big),\ \Phi(0)(1+\theta)(1-\nu(p)) - \Phi(p)\big\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Phi''(p)\Big\} = 0, \quad \text{a.e. } p \in [0,1], \\ \Phi(0) = \sigma, \quad \Phi'(0) = \varpi. \end{cases} \tag{3.28}$$

Based on the comparison theorem for nonlinear ODE, one can check $\Phi_{\sigma,\varpi}$ is non-decreasing in ϖ, and we can find a ϖ^* such that $\Phi_{\sigma,\varpi^*}(1) = 0$. In particular, this trajectory Φ_{σ,ϖ^*} is the solution of (3.27) if

$$\beta = (1+\theta)\int_0^1 (u')^{-1}\big(\Phi'_{\sigma,\varpi^*}(t)\big)\,\mathrm{d}\nu(t) - (u')^{-1}\big(\varpi^*\big)\theta. \tag{3.29}$$

Namely, different σ will lead to different β via (3.29). For fixed β, ODE (3.27) admits at most one solution. As a result, if we go through all of the $\sigma < 0$, we can find all the optimal insurance contracts. Assume h is a continuous function, then $\Phi'_{\sigma,\varpi}$ is continuous in σ and ϖ. The map $\sigma \longmapsto \beta$ then is injective and continuous. So β in (3.29) is a monotone function of σ. Hence, given any feasible β, we can solve (3.27) by searching for the corresponding σ and Φ_{σ,ϖ^*}.

Hereafter, we present a numerical solution of (3.28). In this example, we consider a power utility function $u(x) = \frac{x^\alpha}{\alpha}$ with $\alpha = 0.5$, that is, $u(x) = 2\sqrt{x}$. We construct a specific S-shaped (first convex then concave) weighting function w with its inverse ν defined as below:

$$\nu(p) = \begin{cases} \gamma p + \frac{\theta}{a^2(1+\theta)}(2ap - p^2), & \text{if } p \in [0, a]; \\ \gamma \left(p + \frac{c_1}{3(b-a)}(p-a)^3\right) + \frac{\theta}{1+\theta}, & \text{if } p \in [a, b]; \\ \gamma \left(p + c_1(p-a)(p-b) + \frac{c_1}{3}(b-a)^2\right) + \frac{\theta}{1+\theta}, & \text{if } p \in [b, 1]. \end{cases}$$

where $\gamma = -\frac{\varpi}{\sigma(1+\theta)}$, $0 < a < b < 1$, and

$$c_1 = \frac{3(1-\gamma(1+\theta))}{\gamma(1+\theta)\left((b-a)^2 + 3(a-1)(b-1)\right)}.$$

We set parameters as follows:

$$a = 0.2, \quad b = 0.4, \quad \theta = 0.2, \quad \varpi = 0.5, \quad \sigma = -1.$$

Then $\gamma = \frac{5}{12}$ and $c_1 = \frac{75}{37}$.

The pictures of w (S-shaped) and its inverse ν (Inverted S-shaped) are drawn in Figure 3.1 and Figure 3.2

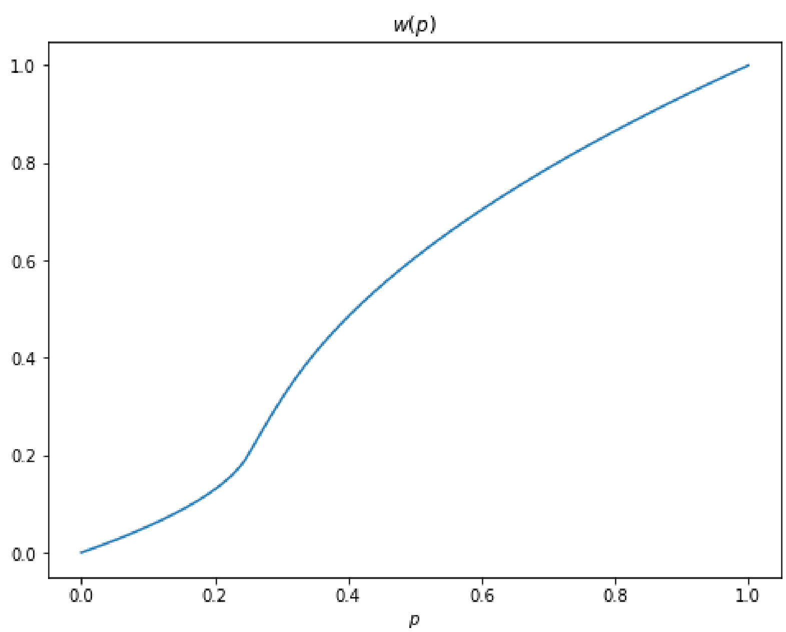

Figure 3.1: The probability weighting function w.

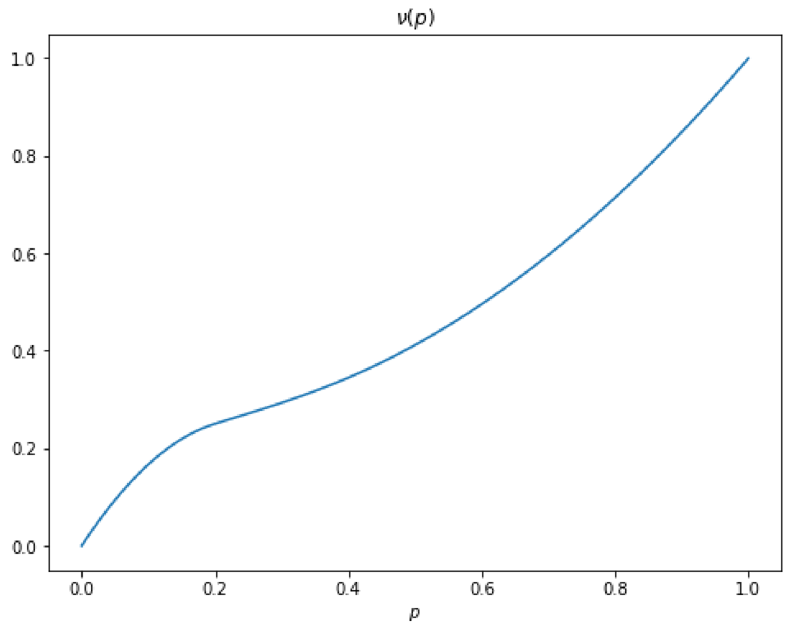

Figure 3.2: The inverse of the probability weighting $\nu = w^{-1}$.

We define:

$$h(p) = \begin{cases} 0, & \text{if } p \in [0,a]; \\ -\Big[(u')^{-1}\big(-\sigma(1+\theta)\nu'(p)\big)\Big]' + \int_a^p c_2(p-a)(p-b)^2\,\mathrm{d}p, & \text{if } p \in [a,b]; \\ -\Big[(u')^{-1}\big(-\sigma(1+\theta)\nu'(p) + c_3(p-b)(b+2-3p)\big)\Big]' & \text{if } p \in [b,1]. \end{cases} \tag{3.30}$$

where $c_3 = c_1 \frac{\varpi}{3(1-b)^2}$ and $c_2 = 5000$. The picture of $h(p)$ is given in Figure 3.3. One can observe that it is continuous.

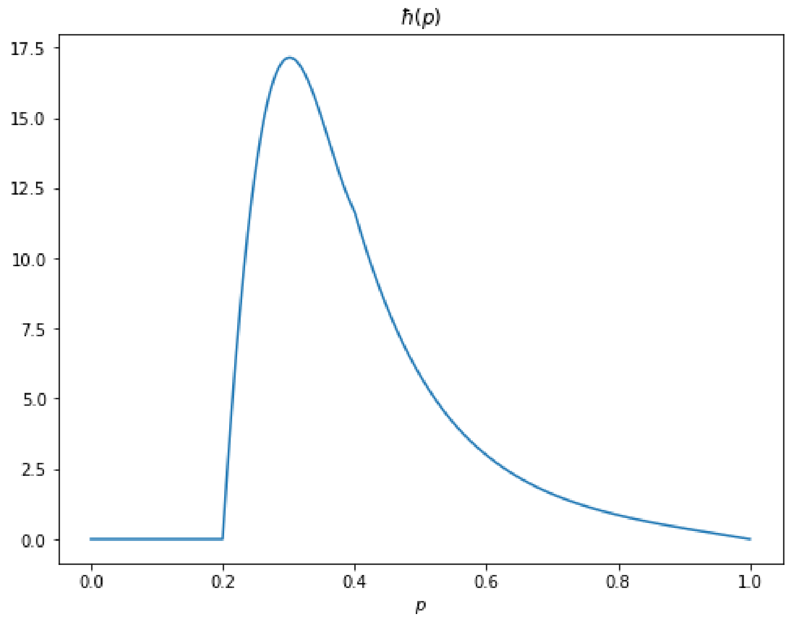

Figure 3.3: The upper bound $h(p)$.

Note that $h(p) = \left(F_X^{-1}\right)(\nu(p))'$. After integration, we have

$$F_X^{-1}(\nu(p)) = \begin{cases} 0, & \text{if } p \in [0, a]; \\ \varpi^{-2} - \left(\varpi(1 + c_1 \frac{(p-a)^2}{b-a})\right)^{-2} + \int_a^p c_2(p-a)(p-b)^2 \, \mathrm{d}p, & \text{if } p \in [a, b]; \\ \varpi^{-2} - \left(\varpi(1 + c_1(2p - a - b)) + c_3(p-b)^2(b + 3 - 4p)\right)^{-2} & \\ + \int_a^b c_2(p-a)(p-b)^2 \, \mathrm{d}p & \text{if } p \in [b, 1]. \end{cases} \tag{3.31}$$

where $c_3 = c_1 \frac{\varpi}{3(1-b)^2}$ and $c_2 = 5000$. The picture of $F_X^{-1}(p)$ is given in Figure 3.4. We can observe from the picture $F_X^{-1}(p) = 0$ for some $[0, m_0]$ which matchs the assumption we set in the beginning.

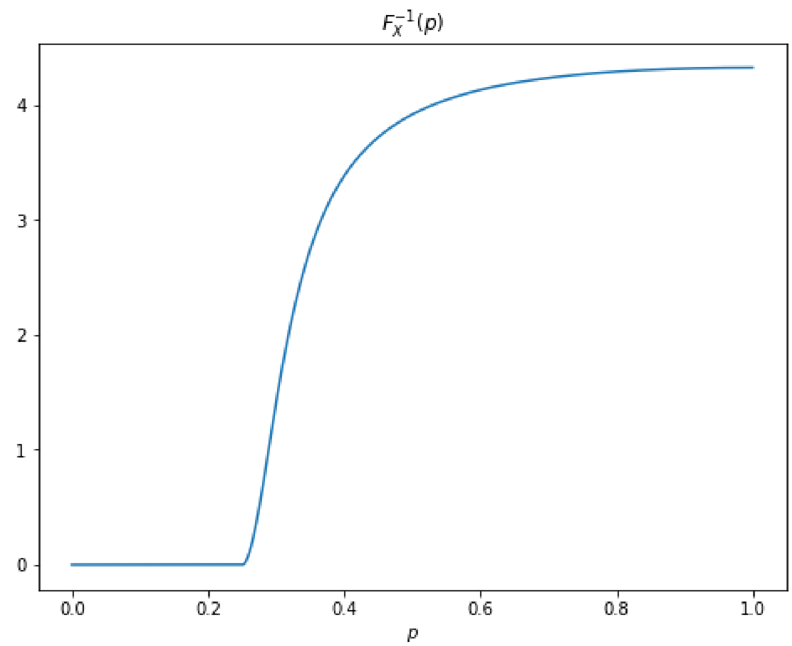

Figure 3.4: The quantile function of loss X.

Under the above settings, the solution $\Phi(p)$ turns out to be a three-step shape:

$$\begin{cases} \Phi(p) > \sigma(1+\theta)(1-\nu^*(p)), & \text{if } p \in [0,a]; \\ \Phi(p) = \sigma(1+\theta)(1-\nu^*(p)), & \text{if } p \in [a,b]; \\ \Phi(p) > \sigma(1+\theta)(1-\nu^*(p)), & \text{if } p \in [b,1]. \end{cases}$$

Furthermore, based on (3.16) and the value of $h(p)$, we have

$$\Phi(p) = \begin{cases} \sigma + \varpi p, & \text{if } p \in [0,a]; \\ \sigma(1+\theta)(1-\nu(p)), & \text{if } p \in [a,b]; \\ \sigma(1+\theta)(1-\nu(p)) + c_3(p-b)^3(1-p), & \text{if } p \in [b,1]. \end{cases} \tag{3.32}$$

The picture is given in Figure 3.5.

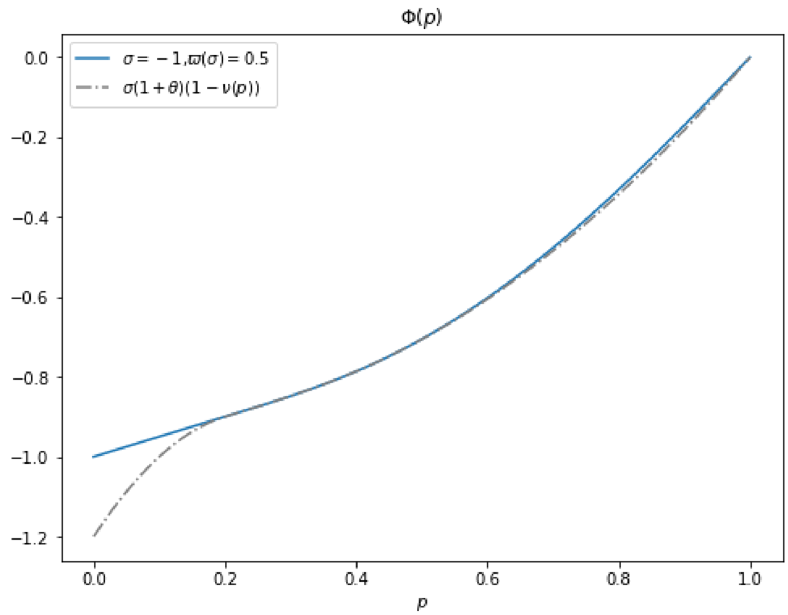

Figure 3.5: The optimal solution $\Phi_{\sigma,\varpi}$.

The associated optimal retention function is given by

$$R^*(x) = (u')^{-1}\big(\Phi'(0)\big) + (u')^{-1}\big(\Phi'\big(w(F_X(x))\big)\big).$$

We plot it in Figure 3.6.

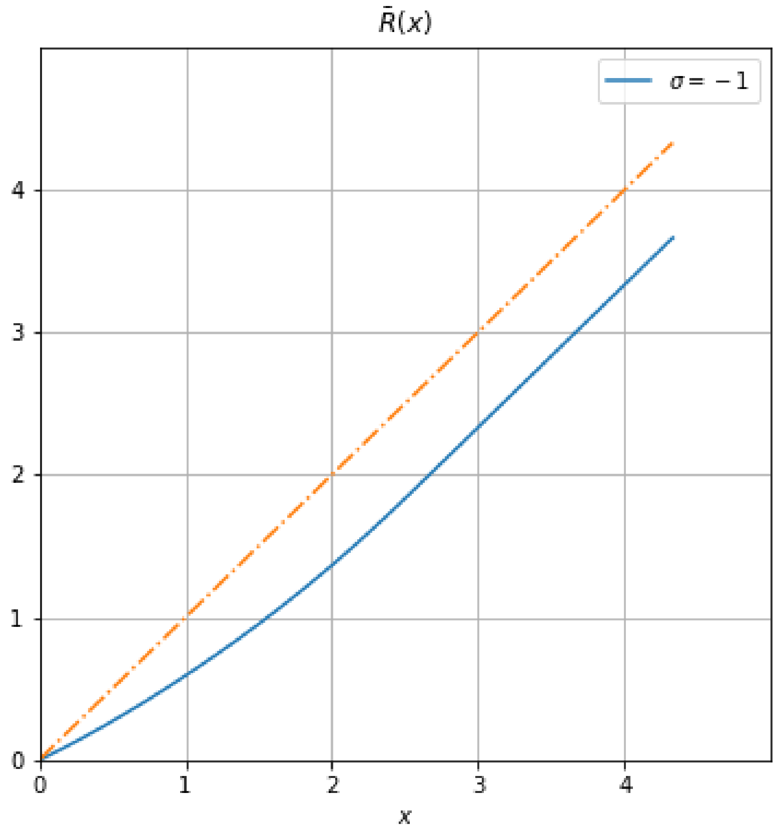

Figure 3.6: The optimal retention function $R(x)$.

3.5 Conclusion

In this model, we derive the pareto-efficiency insurance policy, which avoids the issue of moral hazard. The problem boils down to solving a corresponding ordinary differential equation numerically. Both the insurer and the policy holder in this model allow for a probability distortion, which generalizes the conventional expected utility maximization models. The difficulty of this model is that there is a global bounded constraint on the derivatives of compensation and retention due to the moral-hazard-free requirement. Technically, the method can be adopted to solve other general problems with this type of globally bounded constraint. We can also consider other behavioral risk preferences, such as non-concave utility, separated probability weighting functions for the gain and loss. In those cases, the problem becomes more challenging.

Chapter 4

Concluding Remarks

This thesis studies two behavioral finance models: a continuous-time portfolio selection model and a static optimal insurance design model. Mathematically, the involvement of behavioral preference leads to complicated targets involving time-inconsistency and non-standard stochastic control problems, in which Bellman's optimality and global concavity no longer exist, resulting in the failure of standard techniques.

In both models, problems are initially turned into quantile optimization problems to handle the nonlinear expectations caused by probability weighting. We have applied alternative ways to deal with the related quantile optimization difficulties. To tackle the M-shaped utility in the portfolio selection problem, we have applied a relaxation method to link the non-concave quantile optimization problem to a concave one and obtained the optimal solution, while the optimal insurance problem has been connected to a numerically solved ODE via calculus of variation.

In terms of the portfolio selection model, there are several possible directions for further improvements and generalization. First, interested readers can investigate more general utility forms, as stated in Remark 2.5. In that way, the induced utility of relative return may take on a more complicated shape and challenge the relaxation method we used in our formulation. Besides, in our findings, the benchmark $\mathfrak{B}$ plays an important role in determining the state of the optimal solution. We can further conduct an investigation into the regularity of how it would influence the gain and loss numerically. In addition, we can introduce more elements and constraints into the models, such as no-short positions and ambiguity environments. Another potential direction is to apply the return-oriented target to other portfolio selection models, especially those that have targets that maximize the utility of terminal wealth.

Regarding the optimal insurance design problem, we considered Wang's premium principle for the insurer, interested readers can conduct an empirical study on the optimal policy according to different classes of distortion function $\mathfrak{g}$, such as VaR, CVaR, etc. We can also consider other premium calculation principles for the insurer and behavioral risk preferences for the policy holder under the moral-hazard-free constraint.

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