STOCHASTIC CONTROL PROBLEMS AND RELATED FREE BOUNDARIES IN MATHEMATICAL FINANCE AND INSURANCE

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CERTIFICATE OF ORIGINALITY

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____________________________(Signed)

__________ZHOU Rui__________(Name of student)

Dedicate to my mom and dad

Abstract

This thesis deals with free boundary problems arising from stochastic control models in the context of investment portfolios and insurance. It consists of three parts. The first part is devoted to an optimal dynamic reinsurance and dividend-payout problem for an insurer who has to manage her risk exposure in order to maintain the solvency capital requirement and business viability. We aim to determine the optimal dividend and reinsurance policies with the objective of maximizing the expected cumulative discounted dividend payout until either bankruptcy or a given maturity time comes. Mathematically, it is a combined classical-singular control issue. The corresponding Hamilton-Jacobi-Bellman equation is a variational inequality with a fully nonlinear operator and a gradient constraint. Using a standard penalty approximation method and the comparison principle for its gradient function, we can prove the existence and uniqueness of a $C^{2,1}$ solution to the variational inequality. We find that a risk and time dependent reinsurance barrier and a time dependent dividend-payout barrier can partition the surplus-time space into three non-overlapping zones. The localities of the regions are also explicitly estimated.

The second part is concerned with an infinite-time optimal stopping problem where a mutual fund manager wants to maximize the expected utility of her capital with option compensation at the stopping time. We formally derive an explicit solution of the value function using the Legendre transformation approach and determine the optimal investing strategy as well as the ideal selling price. Furthermore, we give

numerical examples to illustrate our results.

The third part investigates an exit strategy problem in a finite-time horizon. We assume that a portfolio manager invests on behalf of small investors with an expected utility maximization investment schedule, while each investor has the option of redeeming her position before maturity. We seek to identify an appropriate exit point for an investor so as to minimize the expected relative error between her redeeming worth and the high-water mark over a given period. The problem can be also formulated as a stopping problem with a variational inequality. We employ partial differential equation techniques to solve it and show that an investor will either reject the manager's wealth management program from the start or hold the position until the end.

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$\mathbb{R}^n$	n-dimensional real Euclidean vector space.
$\mathbb{R}^n_+$	n-dimensional real vector with nonnegative components.
$\emptyset$	the empty set.
x^T	the transpose of the vector or (matrix) x.
a^+	the positive part of real number a.
a^-	the negative part of real number a.
a $\wedge$ b	the smaller one of a and b.
a $\vee$ b	the bigger one of a and b.
$\mathbb{E}(X)$	the expectation of a random variable X.
$\mathbb{E}(X\|\mathcal{G})$	conditional expectation of X given $\mathcal{G}$.
$L^p_{\mathcal{F}_T}(\Omega, A)$	the set of all A- valued, $\mathcal{F}_T$ - measurable random variables with finite $L^p_{\mathcal{F}_T}$ norm.
$(\Omega, \mathcal{F}, \mathbb{P})$	probability space.
$\{\mathcal{F}_t\}_{t\geqslant 0}$	filtration.
$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$	filtered probability space.
$C^k(U)$	the k-times continuously differentiable function on U. .

Chapter 1

Introduction

Stochastic control problems where there are no bounds on the rate of control can be reduced to free boundary problems in PDEs (partial differential equations). Examples of such stochastic control problems include the singular control and optimal stopping. This thesis studies stochastic control problems in applications of stochastic control to insurance and financial mathematics and characterizes the related free boundaries. Specifically, Chapter 2 studies optimal reinsurance and dividend strategies for an insurer, which can boil down to a combined classical and singular control problem. Chapter 3 and Chapter 4 focus on discretionary stopping problems of optimal timing in investment decisions, which end up with mixed classical and stopping problems.

1.1 Overview

1.1.1 Optimal dividend payments and reinsurance policy

A common goal of risk managers in insurance companies is to improve the solvency and stability of their companies. This goal can be reached through choices of dividend-payout and reinsurance strategies. Those are, how many of a company's surpluses to be paid out as dividends, and how many of claim risks to be ceded to a reinsurer. The former decision normally requires a tradeoff between the insurer and the shareholders on optimal dividend distribution. On one hand, shareholders would react positively when the dividend payments increase. On the other hand, paying out dividends would definitely reduce the future reserves that are essential for the survival of the company faced with financial distress. Meanwhile, due to the solvency capital requirements, especially in emergency incidents, such as the black swan event "COVID-19" since 2020, an insurance company can divert a proportion of risk exposure from the balance sheet to reinsurers at the expense of reinsurance premiums. Therefore, it comes to a tradeoff between the insurer (ceding company) and the reinsurer (ceded company) on transferring the insurance risk by reinsurance contracts.

A considerable amount of literature is devoted to optimal dividend payments s-ince the seminal work De Finetti (1957). Many of the existing results show that the optimal dividend strategy is a band type and can be reduced to a barrier strategy in some cases, see Albrecher and Thonhauser (2009) and Avanzi (2009) for a comprehensive exposition. To be more specific, different types of dividend strategies are associated with different control problems. In the first case, the dividend-payout rate is constrained in a bounded interval $[0, l]$. The optimal dividend-payout strategy turns out to be an "all or nothing" policy with respect to (w.r.t.) the dividend rate. That is, paying out dividends at the minimum rate 0 if the surplus is lower than a threshold, or paying out at the maximum rate ℓ otherwise. In this case, the optimal dividend problem is a classical control problem. The studies in this case can be found in Jin et al. (2015), Zhu (2015), and among others. In the second case, the dividend payout is not continuous and a dividend strategy consists of a pair of processes $(T_i, D_i)_{i=1,2,\cdots}$, where D_i is the amount of *i-th* dividend at T_i. The corresponding problem is an impulse control problem. We refer to Cadenillas et al. (2006), Cadenillas et al. (2007), and Yao et al. (2011) for related investigation in this case. In the third case, the dividend-payout rate is unbounded. The optimal strategy is to keep the surplus under a certain barrier, that is, either paying out all reserves in excess of the barrier as dividends, or doing nothing under the barrier, which leads to a singular control problem. See Azcue and Muler (2005), Sotomayor and Cadenillas (2011) for related studies in this case.

In particular, Karatzas and Shreve (1985) demonstrate the connections between the singular stochastic control and optimal stopping under suitable conditions on the cost functions. More recently, some optimal dividend problems related to singular control are transformed into optimal stopping problems where the value functions turn out to be the derivative of those in the original problems and the problems can be further addressed by probabilistic arguments. For example, De Angelis and Ekström (2017) reformulate a finite-time optimal dividend problem as an optimal stopping problem for a diffusion reflected at 0 where they establish regularities of the value function of the optimal stopping problem and show that the corresponding value function of the original optimal dividend problem is the unique classical solution of a suitable Hamilton-Jacobi-Bellman equation. Following the ideas in De Angelis and Ekström (2017), the work is extended later to the cases in which additional ingredients are involved, such as the partial information (De Angelis (2020)), capital injection (Ferrari and Schuhmann (2019)) and stochastic discount factor (Bandini et al. (2020)).

Besides the dividend payments, reinsurance is another important theme on an insurer's agenda. Reinsurance is a contract between the insurer and reinsurer, by which the cedent could transfer the extra insurance risk from the balance sheet to the reinsurer at the expense of premium income. Typically, the premium is calculated by the expected value premium principle and variance premium principle. In general, most insurance contracts can be categorized as life insurance or property and casualty insurance. For example, a life insurance company is mainly faced with morality risk. There are many types of life insurance policies providing coverage to policyholders, such as the whole life insurance, term life insurance, endowment life insurance, and group life insurance. In this regard, it is natural for an insurer to consider the total risk exposure as the combination of losses arising from different business lines. Mathematically, the total risk exposure can take any discrete or continuous, probability distributions. Consequently, the reinsurance strategies based on various types of risk result in non-classical complicated reinsurance policies. Although the optimal reinsurance problem has been extensively investigated, most of existing literature analyzes under the assumption of typical reinsurance policies such as proportional and excess of loss reinsurance (e.g., Schmidli (2002), Schmidli (2001), Taksar and Markussen (2003), Hipp and Vogt (2003), Hipp and Taksar (2010), Albrecher et al. (2017), Schmidli (2007), Asmussen et al. (2000), Højgaard and Taksar (1998)). In Chapter 2, we formulate a stochastic control problem for an insurer who controls both dividend scheme and reinsurance policy, meanwhile, we do not confine the reinsurance contracts to be particular ones such as proportional or excess of loss reinsurance.

1.1.2 Discretionary stopping time in investment decisions

The optimal stopping problem involving a continuous-time model is a well-developed class in the control theory, the objective of the problem is to determine the optimal stopping time when the underlying continuous time Markovian process should be stopped in order to minimize or maximize the value of an objective functional. The optimal stopping time is proved to be the first time that the associated Markovian process exits from a continuation region specified by an optimal stopping boundary. A comprehensive overview of the connection between the optimal stopping problem for a continuous Markovian process and the free boundary problem for a differential operator can be found in Peskir and Shiryaev (2006). By some arguments from stochastic analysis including Doob's optional sample theorem and It$\hat{o}$ formula, the analytical solution of a free boundary problem turns out to be the solution of the corresponding stopping problem. It is proved in Shiryaev (2007) as well as Peskir and Shiryaev (2006) that if the underlying process exits the continuation region continuously, the smooth pasting conditions for the value function at the stopping boundary should hold. In mathematical finance, optimal stopping problems have been extensively studied in pricing American type derivative, such as American option, mortgage prepayment option and convertible bond (e.g., Shreve (2004), Yong and Zhou (1999), Yan et al. (2015), Yagi and Sawaki (2005), De Rossi and Vargiolu (2010)). In other applied areas, problems could be more complicated with more general controlled diffusions (e.g., Ceci and Bassan (2004), Dayanik and Karatzas (2003), Chang et al. (2009)).

Our work contributes to the literature on the discretionary stopping problem in the context of investment decisions. Assessing profits and risks in financial industries are often related to find optimal choices in decision-making. In the equity-investment department, optimality considerations are crucial for fund managers, who have to address capital issues (how to distribute and allocate a company's financial resources in ways that will increase efficiency) and time issues (when to sell stocks or portfolios). Choi et al. (2004) investigate a model of optimal consumption-investment selection where an investor has an option to switch her active role of consumption and portfolio selection to passive management. Li and Zhou (2006) incorporate a stopping rule into a mean-variance efficient strategy. Henderson (2007) considers the investment timing of a risk averse entrepreneur in an incomplete market. Karatzas and Wang (2000) consider an agent is allowed freely to stop before or at a prespecified final time in order to maximize the expected utility of his wealth and consumption up to the stopping time. Li and Wu (2008) illustrate the principal-agent conflict between investors and managers regarding different risk-taking behaviours and managerial investment schemes in fund management. Li and Wu (2009) address corporate investment issues illustrated by target-beating in capital budgeting, and discuss applications in capital finance. Jian et al. (2014) study optimal investment and stopping strategies of a risk averse investor. Chen et al. (2020) analyze the optimal withdrawal time for an investor and give the solution of a piecewise linear payoff function for an optimal stopping problem on the fund's assets.

In previous studies, researchers normally use expected utility theory to make decisions. In such models, it is often assumed a concave utility function. Chapter 3 extends Carpenter (2000), where she solves a dynamic investment problem of a risk averse manager compensated with a call option on the assets, into an optimal stopping framework, which is equivalent to assuming a non-smooth utility function.

It is known that a significant portion of fund managers' compensation comes from the performance bonuses, which are usually based on a percentage of the money they made for their clients and the firm. Factors such as how long they have been working, how many clients they support and how successful their clients' investments become may also affect what type of bonus they receive. Therefore, it is natural to consider an optimal portfolio selection and investment stopping problem for a mutual fund manager ("she") with the objective of maximizing the expected utility of her compensation package. The package compensation is based on a management fee and an incentive fee, with the former determined by a fixed proportion of the asset value at the start of the investment period and the latter determined by the profit of a call option on the assets under management, i.e. the fund manager's net profit. Then, the so-called non-smooth utility function aforementioned is a composite of a convex function and a classical concave utility function. The decisions faced by the fund manager are selecting the optimal investment policy and best selling time.

In Chapter 4, we consider an investor's redeeming behaviour where the portfolio manager of an equity trust is able to actively manage the account based on power utility maximization. It would be ideal if she could redeem the assets exactly at the high-water mark of a given period. However, this mission is apparently impossible since the trading decision is based on the available information to date while the high-water mark over the entire period is not known yet until the terminal time T. It is therefore crucial for the investor to predict the global maximum price and choose the best redeeming time such that the transaction price is as close as possible to the ideal price.

The earliest study of the problem, called as the prediction problem in probabilistic literature, can date back to Graversen et al. (2001) who measure the closeness by a square error distance between a standard Brownian motion and its global maximum. Du Toit and Peskir (2007) extend their work to a drifted Brownian motion. More recently, Shiryaev (2007) apply the idea to model a stock trading behaviour. In our work, we are interested in the decision made by an investor to maximize the predicted relative error between the selling value and the maximum potential asset value by choosing an appropriate time to redeem. This model is consistent with the common investment notion that upon redeeming, an investor naturally attempts to get as close as possible to the highest price.

1.2 The Scope of the Thesis

In Chapter 2, we investigate an optimal reinsurance and dividend-payout problem where the insurer's surplus is modeled by a controlled diffusion, which is a good approximation of the classical Cramér-Lundberg process as well-justified by Grandell (2012). The target of the insurer is to maximize the expected cumulative discounted dividend payments until either bankruptcy or a given maturity time comes. It turns out to be a mixed classical-singular stochastic control problem. The closest model to ours is Guan et al. (2019), which constrains a proportional reinsurance contract such that the risk control model is relatively simple in comparison to ours. Tan et al. (2020) consider a similar surplus process, but their model is in an infinite time horizon, therefore the associated Hamilton-Jacobi-Bellman (HJB) equation is an ordinary differential equation (ODE), which is considerably easier to handle than that of a PDE in our case.

Different from the existing studies on the dividend and reinsurance problem, our model has the following features. First, the insurer chooses the reinsurance policy subject to the expected value premium principle, we do not confine the reinsurance contract to be particular ones such as proportional reinsurance or excess of loss reinsurance. Second, the reinsurance contract is dynamically chosen based on the surplus level, which leads to an optimal feedback reinsurance policy depending on surplus, time, and risk magnitude. Third, the claims from policyholders admit any probability distribution with a tail constraint. Finally, a finite time horizon condition makes the HJB equation be a complicated fully nonlinear one with a gradient constraint.

Similar to most existing optimal dividend models, the HJB equation of our problem turns out to be a variational inequality problem with a gradient constraint. However, since the reinsurance scheme can be time-dependently chosen, the HJB equation involves a complicated fully nonlinear operator due to a functional optimization problem appearing in the operator, which prevents us from connecting the gradient function to an optimal stopping problem. That means the ideas in De Angelis and Ekström (2017) do not work in our problem, we have to resort to a purely PDE approach to study the HJB equation. Using the method in Dai et al. (2010), Dai et al. (2009) and Dai and Yi (2009), we find that the gradient function satisfies an obstacle PDE problem. We then study it by a penalty approximation method and derive a comparison principle and prove necessary properties (such as $C^{2,1}$ smoothness and uniqueness of the value function) for the original full nonlinear HJB equation. In particular, our approach requires a number of profound PDE and functional analysis results, such as the Leray-Schauder fixed point theorem, the

Sobolev embedding theorem, the $C^{\alpha,\frac{\alpha}{2}}$ estimation, the Schauder estimation, and the comparison principle for nonlinear PDEs.

Our model provides numerous economic insights. We show that there is a smooth, time-dependent, dividend-payout barrier that divides the surplus-time space into a non-dividend-payout region and a dividend-payout region. The insurer ought to pay out all reserves in excess of the dividend-payout barrier, that is, all reserves in the dividend-payout region. Furthermore, we find a risk-magnitude and time-dependent smooth reinsurance barrier that divides the non-dividend-payout region into a reinsurance-covered region and a reinsurance-uncovered region, in an increasing order of surplus, the reinsurance is an excess of loss reinsurance policy. Economically, when faced with the same magnitude of risk, an insurer with a larger surplus will not cede the risk to a reinsurer. In other words, as the magnitude of the risk gets smaller, the reinsurance-covered region shrinks (i.e., less insurance companies with different surplus levels tend to cover the risk of this magnitude); whereas, the reinsurance-uncovered region expands. The former disappears when the magnitude of the risk is smaller than an explicitly given constant (namely, all the insurance companies choose to cover the risk by themself); by contrast, the latter never vanishes. The insurance company can be exposed to a higher risk as surplus increases in the reinsurance region; exposed to the risk once its surplus falls into the non-reinsurance region. We also provide accurate explicit estimations for the localities of these regions. Moreover, our results indicate that there is a uniform non-action region where the insurer should be exposed to all risk and not pay out dividends when the claims are bounded random variables.

Most works are done for solving the investment problem with a concave utility maximization objective, while the problems with a non-smooth criterion on wealth are quite limited. Chapter 3 roots in Carpenter (2000) where she presents solutions to a risk averse investor with a convex compensation function and interprets the risk taking behaviour. Distinct from the previous works, we consider a mixed optimal control and stopping problem in an infinite horizon. It can be reduced to a free boundary problem which satisfies smooth pasting conditions. We derive the solution in terms of variational inequalities with a nonlinear ordinary differential equation. By Legendre-Fenchel transformation, we transfer the nonlinear equation to a linear homogeneous ordinary differential equation and obtain its closed-form solution. We then verify that the solution to the free boundary problem is the value function and we give the optimal investment strategy and free boundary that represents the optimal selling price. The numerical examples illustrated by our results are quite intuitive and consistent with the common investment practice. We show that the optimal selling point drops as the commission ratio and basic salary increase. While the manager has to pursue a higher selling price in exchange for a higher incentive charge as the pre-specified reference level rises.

In Chapter 4, we see that the problem can be reformulated as an optimal stopping problem. Its PDE formulation can be described by a free boundary problem or an obstacle problem. We generalize Shiryaev et al. (2008) where they consider optimal stock trading using a purely probabilistic approach to an optimal redeeming problem on a trust asset, our main focus is on the characterization of the free boundary only at time $t = 0$. As opposed to Shiryaev et al. (2008), we make use of a PDE approach to conduct a thorough qualitative analysis. We show that the optimal exercise time is the first time that the underlying process exits from the region restricted by the boundary depending on a so-called drawdown process, which can be represented as the ratio of the running maximum to the current asset value. It turns out that the redeeming strategy of an individual investor heavily depends on the portfolio manager's risk aversion degree γ. When $0 < \gamma < \frac{1}{2}$, one should not redeem the trust until the maturity. However, in the case of $\frac{1}{2} \leqslant \gamma < 1$, one should not invest the trust initially.

Chapter 2

Dynamic Optimal Reinsurance and Dividend-payout in a Finite Time Horizon

This chapter addresses a stochastic control problem in the context of reinsurance and dividend from the perspective of an insurer. We are interested in how an insurance company chooses reinsurance and dividend strategies for its long term survival and financial viability.

We consider a dynamic reinsurance design and dividend problem for an insurer whose surplus is modeled by a diffusion approximation of the standard Cramér-Lundberg model with an unbounded dividend rate. As a result, the problem is reduced to a singular control problem. The target of the insurer is to maximize the expected cumulative discounted dividend payout until either bankruptcy or a given maturity time comes.

We find that the ceded risk and time dependently smooth reinsurance barrier partitions the non-dividend-payout region into a reinsurance region and a non-reinsurance region. As the ceded risk increases, the non-reinsurance region shrinks, while the reinsurance region expands. Besides, the insurance company can be exposed to a higher risk as the surplus increases in the reinsurance region and exposed to all risks once its surplus level falls into the non-reinsurance region.

2.1 Model

In the classical Cramér-Lundberg model, there are two different components that affect the cash reserves (also called surplus) dynamics. One is continuous premium inflow from policyholders at a constant rate p. The other is the outgoing payments for claims. We denote the total number of claims received until time t by $\mathbb{N}_t$ and the size of ith claim by Z_i, then the company's reserve/surplus R_t at time t is given by

$$R_t = R_0 + pt - \sum_{i=1}^{\mathbb{N}_t} Z_i, \tag{2.1.1}$$

where $\{\mathbb{N}_t\}_{t \geqslant 0}$ is a Poisson process with intensity 1, and all claims are independent and identically distributed random variables independent of $\mathbb{N}_t$.

The insurance company has to protect itself by sharing risk with a ceded company (the reinsurer). The insurer buys reinsurance contracts from the reinsurer. Given a reinsurance contract $I(\cdot)$, the reinsurer should compensate $I(z)$ to the insurer when a claim amount z for the ceded risk is received by the insurer. This function $I(\cdot)$ is known as the ceded loss function, and $H(z) := z - I(z)$ is known as the retained loss function. The reinsurance policy consists of a series of reinsurance contracts $\{I_t\}_{t \geqslant 0}$ over time, where I_t denotes the reinsurance contract signed at time t. Note that the reinsurance contracts which can be dynamically chosen by the insurer, are time and surplus dependent.

The presence of reinsurance modifies the risk exposure of the insurer. It distorts the incoming and outgoing cash flow of the insurer's surplus process (2.1.1). As well-justified by Grandell (2012), the surplus process R_t can be approximated by the following diffusion process

$$\mathrm{d}R_t = (p - p(I_t) - \mathbb{E}_{R_{t-}}[Z_1 - I_t(Z_1)])\mathrm{d}t + \sqrt{\mathbb{E}_{R_{t-}}[(Z_1 - I_t(Z_1))^2]}\,\mathrm{d}W_t, \tag{2.1.2}$$

where $\{W_t\}_{t\geqslant 0}$ is a standard Brownian motion independent of the random claim Z_1, $\mathbb{E}_{R_{t-}}[\,\cdot\,] = \mathbb{E}[\,\cdot \mid R_{t-}]$, and $p(I_t)$ denotes the reinsurance premium corresponding to the reinsurance contract I_t. It is worth mentioning that the contract I_t depends on the state of the surplus process up to t.

We consider the expected value premium principle for both insurance and reinsurance contracts, which is given by

$$p = (1+\delta)\mathbb{E}[Z_1], \quad p(I_t) = (1+\rho)\mathbb{E}_{R_{t-}}[I_t(Z_1)],$$

where δ, $\rho > 0$ are safety loadings of insurance premium and reinsurance premium. This is fundamental to the insurance pricing as it stipulates that the (re)insurer has a positive safety loading on the underwritten risk. In this case, we can rewrite (2.1.2) as

$$\mathrm{d}R_t = (-\gamma + \rho\mathbb{E}_{R_{t-}}[Z_1 - I_t(Z_1)])\mathrm{d}t + \sqrt{\mathbb{E}_{R_{t-}}[(Z_1 - I_t(Z_1))^2]}\,\mathrm{d}W_t,$$

with $\gamma = (\rho - \delta)\mathbb{E}[Z_1]$. We set $\rho > \delta$, i.e., $\gamma > 0$, to ensure reinsurance is non-cheap. Otherwise, the insurer can eliminate the risk exposure by ceding all arising claims to the reinsurer, receiving a guaranteed profit of $\delta - \rho > 0$ with no chance of ruin. For simplicity, we will assume $\rho = 1$ from now on.

We assume that the insurer pays out dividends to the shareholders. Let L_t be the cumulative dividend extracted from the surplus process until t, which is a non-decreasing càdlàg (right continuous with left limits) process. It is chosen by the insurer according to its surplus state. Then the new surplus process $\{R_s\}_{s\geqslant t-}$ begins at time $t-$ with an initial value x satisfying

$$\begin{cases} \mathrm{d}R_s = \Big(-\gamma + \mathbb{E}_{R_{s-}}[H_s(Z_1)]\Big)\mathrm{d}s + \sqrt{\mathbb{E}_{R_{s-}}[H_s^2(Z_1)]}\,\mathrm{d}W_s - \mathrm{d}L_s, \quad s \geqslant t, \\ R_{t-} = x > 0, \end{cases} \tag{2.1.3}$$

where $H(z) = z - I(z)$. The surplus process R_t is also a càdlàg process. It jumps at the same time as L does with the same jump size but with an opposite sign, namely $R_s - R_{s-} = -(L_s - L_{s-})$ for any $s \geqslant t$. Define the ruin time of the insurer as

$$\tau := \inf\{s \geqslant t \mid R_s \leqslant 0\}. \tag{2.1.4}$$

The amounts of dividends paid out by the insurer are not allowed to exceed the existing surplus, so $L_s - L_{s-} \leqslant R_{s-}$ at any time s. As a consequence, $R_s = R_{s-} - (L_s - L_{s-}) \geqslant 0$. Then the surplus of the insurance company is $R_\tau = 0$ at the ruin time.

The objective of the optimal reinsurance and dividend-payout problem is to find a retained loss policy $\mathbb{H}^t = \{H_s\}_{s \geqslant t-}$ (or equivalently, a reinsurance policy $\mathbb{I}^t = \{I_s\}_{s \geqslant t-}$) and a dividend-payout policy $\mathbb{L}^t = \{L_s\}_{s \geqslant t-}$ to maximize the expectation of discounted cumulative dividend payouts until either bankruptcy or a given maturity $T > 0$ comes. The value function of our problem is defined as

$$V(x,t) = \sup_{\mathbb{H}^t, \mathbb{L}^t} \mathbb{E}\left[\int_t^{T \wedge \tau} e^{-c(s-t)} \mathrm{d}L_s \,\middle|\, R_{t-} = x \right], \quad x > 0, \quad 0 \leqslant t \leqslant T, \tag{2.1.5}$$

where c is a positive discount factor and the retained loss function (ceded function) is subject to the constraint

$$0 \leqslant H_s(Z) \leqslant Z, \quad s \in [t-, T].$$

This is a mixed singular-classical control problem.

In the rest of this chapter, we will investigate the value function and provide the optimal reinsurance and dividend-payout strategies.

2.2 Solution

In this section, we study Problem (2.1.5) by the dynamic programming principle.

2.2.1 HJB equation

We first introduce the following variational inequality

$$\begin{cases} \min\{v_t - \mathcal{L}v, v_x - 1\} = 0, \text{in} \quad \mathcal{Q}_T := (0, +\infty) \times (0, T], \\ v(0,t) = 0, \quad 0 < t \leqslant T, \\ v(x,0) = x, \quad x > 0, \end{cases} \tag{2.2.6}$$

where

$$\begin{aligned} \mathcal{L}v &:= \sup_{H \in \mathcal{H}} \left(\frac{v_{xx}}{2} \int_0^\infty H(z)^2 \mathrm{d}F(z) + v_x \int_0^\infty H(z) \mathrm{d}F(z) \right) - \gamma v_x - cv, \\ \mathcal{H} &:= \big\{ H : [0,\infty) \to [0,\infty) \mid 0 \leqslant H(z) \leqslant z \big\}. \end{aligned} \tag{2.2.7}$$

Here, $F(\cdot)$ denotes the common cumulative distribution function of the claims with $F(0-) = 0$ (due to the non-negativity of the claims). The variational inequality (2.2.6) is indeed the (time reversed) HJB equation for our problem (2.1.5).

The theory of viscosity solution (see Yong and Zhou (1999)) can link an HJB equation to a control problem (2.1.5) (see Yong and Zhou (1999)). However, the proof of uniqueness of the viscosity solution in the classical approach is an extremely challenging task for us. Moreover, this approach usually cannot provide a classical solution to the HJB equation. In what follows, we adopt a different approach. We first show that (2.2.6) has a classical solution by purely PDE techniques, and then demonstrate that the solution is essentially the value function of problem (2.1.5) by a verification theorem. The first result is shown by the penalty approximation method in PDE, one can compute the value function as well as the optimal strategies for problem (2.1.5) by numerically solving the approximation PDE. This is an advantage of our method compared with other probabilistic arguments or viscosity solution approaches.

Throughout this chapter we put the following technical assumption

$$z^3(1 - F(z)) \text{ is bounded on } [0, \infty), \tag{2.2.8}$$

which allows $F(\cdot)$ to be discontinuous.

Theorem 2.1. *Problem* (2.2.6) *has a unique solution* $v \in C^{2,1}\big(\overline{\mathcal{Q}_T}\backslash\{(0,0)\}\big)\bigcap C\big(\overline{\mathcal{Q}_T}\big)$ *that satisfies*

$$v_x \geqslant 1, \tag{2.2.9}$$

$$v_t \geqslant 0, \tag{2.2.10}$$

$$v_{xx} \leqslant 0, \tag{2.2.11}$$

$$v_{xxx} \geqslant 0 \text{ in a weak sense}, \tag{2.2.12}$$

$$v_{xt} \geqslant 0, \tag{2.2.13}$$

$$\lambda v_x + v_{xx} \geqslant 0, \tag{2.2.14}$$

where λ is the unique positive root of the function defined by (2.4.41).

Proof. The proof is based on purely PDE methods and we leave it in Section 2.4.1. □

Theorem 2.2 (Verification Theorem)**.** *Suppose that* $v \in C^{2,1}\big(\overline{\mathcal{Q}_T}\backslash\{(0,0)\}\big)\bigcap C\big(\overline{\mathcal{Q}_T}\big)$ *to Problem* (2.2.6) *is increasing and concave w.r.t. x. Then the value function of optimal reinsurance and dividend-payout problem (2.1.5) is given by*

$$V(x,t) = v(x, T-t). \tag{2.2.15}$$

Moreover, the optimal ceded loss policy I^* can be given by the feedback control of the loss and surplus

$$I_s^*(s, R_{s-}^*) = \max\left\{0, z + \frac{v_x(R_{s-}^*, T-s)}{v_{xx}(R_{s-}^*, T-s)}\right\},$$

and the dividend-payout strategy L^* for problem (2.1.5) is given by

$$\begin{cases} L_s^* - L_{s-}^* = R_{s-}^* - d^*(T-s), & \text{if} \quad R_{s-}^* > d^*(T-s); \\ L_s^* - L_{s-}^* = 0, & \text{if} \quad R_{s-}^* \leqslant d^*(T-s), \end{cases}$$

where the payout free boundary d^* *is given by*

$$d^*(s) = \inf\{x \geqslant 0 | v_x(x,s) = 1\}, s \in [0,T],$$

with the convention that $\inf \emptyset = \infty$.

Proof. The proof is standard and given in Section 2.4.2. □

In what follows, we fix v as that in Theorem 2.1. By Verification Theorem 2.2, v completely characterizes the value function of the optimal reinsurance and dividend-payout problem (2.1.5). Next, we study the properties of the optimal reinsurance and dividend payout strategies.

2.2.2 Optimal strategies

In the previous section, we have illustrated the existence and uniqueness for the time-reversed HJB equation (2.2.6). Next, we will investigate the optimal strategies for problem (2.1.5) where we use the notations in 2.4.1 such as parameters μ_1 and μ_2, functions $A(\cdot)$, $B(\cdot)$, and the operator $\mathcal{T}$, defined by (2.4.31), (2.4.35), (2.4.36) and (2.4.40).

If $\gamma \geqslant \mu_1$, it is easy to check that $v \equiv x$ is the solution to Problem (2.2.6). In this case, one can see that the drift of the surplus process in (3.1.1) is either negative or 0 depending on whether $\mathbb{E}[I_t(Z_1)] > 0$. Correspondingly, the optimal policy is to pay out all reserves as dividends such that the company goes bankrupt immediately. This gives the optimal value x for Problem (2.1.5). The model is trivial and unrealistic. Therefore, we assume $0 < \gamma < \mu_1$ hereafter.

We investigate the dividend-payout plan first, followed by the reinsurance strategy.

Optimal dividend-payout strategy

To investigate the optimal dividend-payout strategy, we divide the whole domain $\mathcal{Q}^T := (0, +\infty) \times [0, T)$ into a dividend-payout region

$$\mathcal{D} = \left\{ (x,t) \in \mathcal{Q}^T \;\middle|\; v_x(x, \vec{t}\,) = 1 \right\},$$

and a non-dividend-payout region

$$\mathcal{ND} = \left\{ (x,t) \in \mathcal{Q}^T \;\middle|\; v_x(x, \vec{t}\,) > 1 \right\}.$$

Here and hereafter we use $\vec{t} = T - t$. Since $v_{xx} \leqslant 0$, we can express them as

$$\mathcal{D} = \{x \geqslant d(\vec{t}\,)\}, \quad \mathcal{ND} = \{x < d(\vec{t}\,)\},$$

where $d(\cdot)$ is the dividend-payout boundary, defined by

$$d(\vec{t}\,) = \inf\{x \geqslant 0 \mid v_x(x, \vec{t}\,) = 1\}, \quad \vec{t} > 0.$$

In what follows, we come to show the boundary $d(\cdot)$ is uniformly upper bounded by an explicit given constant. To this end, we construct a function $\widehat{u}(x)$ such that $\widehat{u}(x) \geqslant v_x(x,t)$, then it is clear that $\inf\{x \geqslant 0 \mid \widehat{u}(x) = 1\}$ provides a uniformly upper bound for $d(\cdot)$.

First, we show that $v_x(0,t)$ is uniformly bounded. We construct a function

$$\widehat{v}(x) := \begin{cases} C_1(1 - e^{-\frac{x}{\gamma}}), & 0 < x \leqslant x_1, \\ C_2 + x - x_1, & x > x_1, \end{cases}$$

where

$$C_1 = \frac{\mu_1}{c} + \gamma, \quad C_2 = \frac{\mu_1}{c}, \quad x_1 = \gamma \ln \frac{C_1}{\gamma} > 0.$$

It indicates $\widehat{v}(x_1-) = \widehat{v}(x_1+)$ and $\widehat{v}_x(x_1-) = \widehat{v}_x(x_1+)$. Thus, we have $\widehat{v} \in C^1[0, \infty)$. We can obtain that $\widehat{v}_x$ is continuous and decreasing such that $\widehat{v}$ is a concave function.

When $0 < x \leqslant x_1$, it suggests that

$$\begin{aligned}\widehat{v}_t - \mathcal{L}\widehat{v} &= \widehat{v}_t - A\left(-\frac{\widehat{v}_{xx}}{\widehat{v}_x}\right)\widehat{v}_{xx} - B\left(-\frac{\widehat{v}_{xx}}{\widehat{v}_x}\right)\widehat{v}_x + \gamma\widehat{v}_x + c\widehat{v} \\ &= -\int_0^\infty \sup_{0\leqslant h\leqslant z}\Big(\frac{1}{2}h^2\widehat{v}_{xx} + h\widehat{v}_x\Big)\mathrm{d}F(z) + \gamma\widehat{v}_x + c\widehat{v} \\ &\geqslant -\sup_{0\leqslant h<\infty}\Big(\frac{1}{2}h^2\widehat{v}_{xx} + h\widehat{v}_x\Big) + \gamma\widehat{v}_x = \frac{\widehat{v}_x^2}{2\widehat{v}_{xx}} + \gamma\widehat{v}_x = \frac{C_1}{2}e^{-\frac{x}{\gamma}} > 0,\end{aligned}$$

and in the case of $x > x_1$, it gives

$$\widehat{v}_t - \mathcal{L}\widehat{v} = -\mu_1 + \gamma + c(C_2 + x - x_1) = \gamma + c(x - x_1) > 0.$$

Therefore, $\widehat{v} \in W_p^{2,1}(\mathcal{Q}_T)$ is a super solution to problem (2.2.6). Since $\widehat{v}(0) = v(0,t) = 0$, we obtain $v_x(0,t) \leqslant \widehat{v}_x(0) = \frac{C_1}{\gamma}$ by the comparison principle.

Remark 2.1. *We provide a probabilistic representation for $\hat{v}$. Consider*

$$\sup_{h_{s\geqslant t-},L_{s\geqslant t-}} \mathbb{E}\Big[\int_{t-}^{T\wedge\theta} 1\mathrm{d}L_s | R_{t-} = x\Big], \tag{2.2.16}$$

for the surplus

$$\begin{cases}\mathrm{d}R_s = (-\frac{\gamma}{2} + h_s)\mathrm{d}s + h_s\mathrm{d}W_s - \mathrm{d}L_s, \quad s \geqslant t, \\ R_{t-} = x > 0.\end{cases} \tag{2.2.17}$$

By establishing an analogue of verification theorem 2.2, one can show that $\hat{v}$ is the value function for this problem.

For any reinsurance and dividend-payout strategy $\{H_s\}_{s\geqslant t-}$ and $\{L_s\}_{s\geqslant t-}$ for the original surplus (2.1.3)*, we take the same dividend-payout strategy $\{L_s\}_{s\geqslant t-}$ and choose $\{h_s\}_{s\geqslant t-} = \sqrt{\mathbb{E}_{R_{s-}}[H_s^2(Z_1)]}_{s\geqslant t-}$ for the new surplus* (2.2.17)*, then one can easily show that the new surplus is greater than the surplus* (2.1.3) *by virtue of*

$$-\frac{\gamma}{2} + h_s = -\frac{\gamma}{2} + \sqrt{\mathbb{E}_{R_{s-}}[H_s^2(Z_1)]} > -\gamma + \mathbb{E}_{R_{s-}}[H_s(Z_1)].$$

Meanwhile, the cost functional (2.2.16) *is no less than* (2.1.5)*, which leads to $\hat{v} \geqslant V$.*

Now we are ready to construct an upper bound function for $v_x(x,t)$. To this end, let

$$\widehat{u}(x) := \begin{cases} C_3(x_2 - x)^2 + 1, & 0 < x \leqslant x_2, \\ 1, & x > x_2, \end{cases}$$

where

$$C_3 = \frac{c^2}{c\mu_2 + \gamma^2}, \quad x_2 := \sqrt{\frac{C_2}{C_3\gamma}} = \sqrt{\frac{1}{\gamma c^3}\left(\mu_1 + c\gamma\right)\left(c\mu_2 + \gamma^2\right)}.$$

Clearly, $\widehat{u}$ is convex and $\widehat{u} \in W_p^{2,1}(\mathcal{Q}_T)$. If $0 < x < x_2$, then $\widehat{u}_x \leqslant 0$. By exploiting (2.4.37) and the elementary inequality $x^2 - 2xy \geqslant -y^2$, we obtain

$$\begin{aligned} \widehat{u}_t - \mathcal{T}\widehat{u} &\geqslant -\frac{1}{2}\mu_2\widehat{u}_{xx} + \gamma\widehat{u}_x + c\widehat{u} \\ &= C_3\left(-\mu_2 - 2\gamma(x_2 - x) + c(x_2 - x)^2\right) + c \\ &\geqslant C_3\left(-\mu_2 - \gamma^2/c\right) + c = 0. \end{aligned}$$

If $x > x_2$, then $\widehat{u}_t - \mathcal{T}\widehat{u} = c > 0$. Since $\widehat{u}(0) = \frac{C_1}{\gamma} \geqslant v_x(0,t) = u(0,t)$, we can conclude that $\widehat{u} \geqslant u$ by the comparison principle. As a result, $x_2 = \inf\{x \geqslant 0 \mid \widehat{u}(x) = 1\}$ is an upper bound for $d(\cdot)$.

To summarize the above results, the dividend-payout boundary is completely characterized in the following theorem.

Theorem 2.3. *The dividend-payout boundary $d(\vec{t}\,)$ is continuous and increasing in $\vec{t}\,$, satisfying*

$$d(0+) = 0 < d(\vec{t}\,) \leqslant d(\infty) \leqslant x_2 = \sqrt{\frac{1}{\gamma c^3}\left(\mu_1 + c\gamma\right)\left(c\mu_2 + \gamma^2\right)}, \qquad (2.2.18)$$

where $d(0+) = \lim_{\vec{t} \to 0+} d(\vec{t}\,)$ and $d(\infty) := \lim_{\vec{t} \to +\infty} d(\vec{t}\,)$. Furthermore, if Z_1 is a bounded random variable, then $d(\vec{t}\,) \in C^\infty(0,T)$.

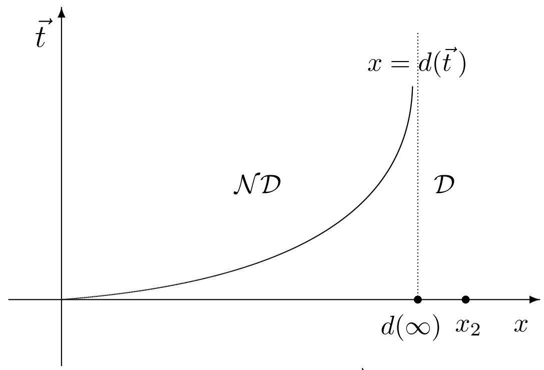

Figure 2.1: The dividend-payout barrier $x = d(\vec{t})$ divides the surplus-time space into a dividend-payout region $\mathcal{D}$ and a non-dividend-payout region $\mathcal{ND}$.

See Figure 2.1 for an illustration of the dividend-payout barrier $d(\vec{t}\,)$ as well as dividend-payout region $\mathcal{D}$ and non-dividend-payout region $\mathcal{ND}$.

Proof. The monotone property of $d(\cdot)$ results from (2.2.13). We are now able to prove the continuity. If $d(\cdot)$ is not continuous, there exists a $\vec{t}_0$ such that $d(\vec{t}_0-) < d(\vec{t}_0+)$. Then for any $x \in (d(\vec{t}_0-), d(\vec{t}_0+))$, the continuity and the monotonicity of v_x and $d(\cdot)$ yield

$$v_x(x,\vec{t}_0) = \lim_{\epsilon\to 0+} v_x(x,\vec{t}_0-\epsilon) \leqslant \lim_{\epsilon\to 0+} v_x(d(\vec{t}_0-\epsilon),\vec{t}_0-\epsilon) = 1.$$

Due to $v_x \geqslant 1$, we have $v_x(x,\vec{t}_0) = 1$. Therefore, for any $x \in (d(\vec{t}_0-), d(\vec{t}_0+))$, $v_{xx}(x,\vec{t}_0) = 0$ holds. Consequently, we infer from (2.1.5) that $v_t(x,\vec{t}_0) = -\mu_1 + \gamma - cv(x,\vec{t}_0)$ and thus $v_{xt}(x,\vec{t}_0) = -cv_x(x,\vec{t}_0) = -c < 0$, which is in contradiction with (2.2.13). So we conclude that $d(\cdot)$ is continuous. Similarly, we can prove $\lim\limits_{\tau\to 0+} d(\tau) = 0$.

Next, we prove $d(\cdot) > 0$. Suppose it is not true, by monotonicity, there exists a $\vec{t}_0 > 0$ such that $d(\vec{t}\,) = 0$ for all $0 < \vec{t} < \vec{t}_0$. This implies $v_x \equiv 1$ or $v \equiv x$ for $0 < \vec{t} < \vec{t}_0$. Denote $\widehat{v}(x,\vec{t}\,) = v(x,\vec{t}-\vec{t}_0)$, then both $\widehat{v}$ and v satisfy (2.2.6). By the uniqueness, we get $\widehat{v} \equiv v$, namely $v \equiv x$ for $0 < \vec{t} < 2\vec{t}_0$. By mathematical induction, we have $v \equiv x$ for all $\vec{t} > 0$. This leads to

$$v_t - \mathcal{L}v = -\mu_1 + \gamma + cx,$$

which contradicts (2.1.5) for a sufficiently small x when $\gamma < \mu_1$.

The only task left is to show the smoothness of $d(\cdot)$ when Z_1 is bounded. Suppose $F(\hat{z}) = 1$ for some $\hat{z} > 0$. Then by (2.4.35) and (2.4.36),

$$A(y) = \frac{1}{2}\mu_2, \quad B(y) = \mu_1, \quad y \leqslant \frac{1}{\hat{z}}. \tag{2.2.19}$$

For any $\vec{t}_0 \in (0, T]$, since v_x and v_{xx} are continuous, we have $v_x = 1$ and $v_{xx} = 0$ at $(d(\vec{t}_0), \vec{t}_0)$. This implies that $v_x + \hat{z}v_{xx} > 0$ is true in a neighborhood $\mathcal{B}$ of $(d(\vec{t}_0), \vec{t}_0)$. It follows from (2.4.45) and (2.2.19) that $u = v_x$ satisfies

$$\min\left\{u_t - \frac{\mu_2}{2}u_{xx} - (\mu_1 - \gamma)u_x + cu,\ u - 1\right\} = 0, \quad (x, \vec{t}\,) \in \mathcal{B}.$$

Since the coefficients are constants in the above equation, using the method in Friedman (2008), we can prove $d(\vec{t}\,) \in C^\infty$ at a neighborhood of $\vec{t}_0$. Since $\vec{t}_0$ is arbitrarily chosen, we conclude that $d(t) \in C^\infty(0, T)$. □

In view of the original optimal reinsurance and dividend-payout problem (2.1.5), by the proof of Theorem 2.2, the optimal dividend-payout policy L_s^* is the local time of corresponding reserve process R_s^* at the level $d(T - s)$, namely

$$\begin{cases} L_s^* - L_{s-}^* = R_{s-}^* - d(T-s), & \text{if} \quad R_{s-}^* > d(T-s), \\ \mathrm{d}L_s^* = 0, & \text{if} \quad R_{s-}^* \leqslant d(T-s). \end{cases} \tag{2.2.20}$$

Under this policy, the reserve process R_s^* is continuous and no more than $d(T - s)$, except for the initial time t. When the surplus R_{t-}^* exceeds the threshold $d(T - t)$, the insurance company would pay out the reserves of an amount of $R_{t-}^* - d(T - t)$ as dividends to its shareholders at the initial time t; otherwise, pay out nothing. The accumulated dividends increase with the local time at the boundary.

Optimal reinsurance strategy

In this section, we study the optimal reinsurance strategy and its behaviour. Recall that we have ascertained the value function $V(x,t) = v(x,\vec{t}\,)$ by Theorem 2.2.

For the insurer, if its current state (x,t), by (2.2.9), (2.2.11) and (2.4.33), the corresponding optimal retained function will become

$$z \mapsto \widehat{H}(z,x,\vec{t}\,) := \begin{cases} -\frac{v_x}{v_{xx}}(x,\vec{t}\,), & \text{if } -\frac{v_{xx}}{v_x}(x,\vec{t}\,) > \frac{1}{z}, \\ z, & \text{otherwise}, \end{cases} \tag{2.2.21}$$

and the optimal reinsurance function will become

$$z \mapsto \widehat{I}(z,x,\vec{t}\,) := z - \widehat{H}(z,x,t).$$

This reinsurance contract depends on the claim z, the current insurance surplus x and time t.

For each $z > 0$, depending on whether $\widehat{I}$ is zero, we divide the surplus-time space into a reinsurance region

$$\mathcal{R}_z = \left\{(x,t) \in \mathcal{Q}^T \;\middle|\; \widehat{I}(z,x,\vec{t}\,) > 0\right\}$$

and a reinsurance-uncovered (non-reinsurance) region

$$\mathcal{NR}_z = \left\{(x,t) \in \mathcal{Q}^T \;\middle|\; \widehat{I}(z,x,\vec{t}\,) = 0\right\}.$$

By (2.2.21), they can also be expressed as

$$\mathcal{R}_z = \left\{(x,t) \in \mathcal{Q}^T \;\middle|\; -\frac{v_{xx}}{v_x}(x,\vec{t}\,) > \frac{1}{z}\right\}, \quad \mathcal{NR}_z = \left\{(x,t) \in \mathcal{Q}^T \;\middle|\; -\frac{v_{xx}}{v_x}(x,\vec{t}\,) \leqslant \frac{1}{z}\right\}.$$

By the above discussions, we have $\mathcal{D} \subseteq \mathcal{NR}_z$ and $\mathcal{R}_z \subseteq \mathcal{ND}$.

Lemma 2.1. *Let v be the (time reversed) value function defined in Theorem (2.2), we have*

$$v_x,\ v_t \in C^{2,1}\big(\mathcal{ND}\big). \tag{2.2.22}$$

Furthermore, $v_{xx} < 0$ if $(x,t) \in \mathcal{ND}$. Consequently, $\widehat{I}(z,x,\vec{t}) = \max\left\{z + \frac{v_x}{v_{xx}}(x,\vec{t}),\ 0\right\}$ for $z > 0$ and $(x,t) \in \mathcal{ND}$.

Proof. The proof is given in Section 2.4.3. □

These results imply that $\widehat{I}$ is an increasing function w.r.t z, which has the perfect financial implication that the insurer will receive more compensation from the reinsurance business when a larger claim arises. Another result is that

$$\widehat{I}(z,x,\vec{t}\,) < z, \quad (x,t) \in \mathcal{ND}.$$

Except for the initial time, the optimal reserve process R^* is always continuous and never exceeds the dividend payout barrier, so we have $\widehat{I}(z, R^*_{t-}, \vec{t}\,) < z$. In this case, the insurer shall undertake part of risks instead of ceding all risks due to the non-cheap reinsurance assumption.

Lemma 2.2. *Let v be the time reversed value function defined in Theorem (2.2), it satisfies*

$$\left(-\frac{v_x}{v_{xx}}\right)\Bigg|_{x=0} = \frac{1}{\lambda}, \tag{2.2.23}$$

$$-\frac{v_x}{v_{xx}}(x,\vec{t}\,) > \frac{1}{\lambda} \quad \text{if} \quad (x,t) \in \mathcal{ND}, \tag{2.2.24}$$

$$\partial_x\left(-\frac{v_x}{v_{xx}}\right)(x,\vec{t}\,) \geqslant 2c \quad \text{if} \quad (x,t) \in \mathcal{ND}, \tag{2.2.25}$$

where λ is the unique positive root of the function f defined by (2.4.41).

Proof. The proof is given in Section 2.4.4. □

From (2.2.24) we see that, if $(x,t) \in \mathcal{ND}$, then

$$\widehat{I}(z,x,\vec{t}\,) = \max\left\{z + \frac{v_x}{v_{xx}}(x,\vec{t}\,),\ 0\right\} \leqslant \max\left\{z - \frac{1}{\lambda},\ 0\right\}. \tag{2.2.26}$$

So $\hat{I}(z,x,\vec{t}\,) = 0$ when $z \leqslant \frac{1}{\lambda}$, which means the insurance company should bear all arising claim below $\frac{1}{\lambda}$ by itself. In other words, any arising claim below $\frac{1}{\lambda}$ should not be shared with a ceded company.

Together with (2.2.24) and (2.2.25), we have $-\frac{v_{xx}}{v_x}$ is a strictly decreasing function in x in $\mathcal{ND}$. Since $\mathcal{R}_z \subseteq \mathcal{ND}$, we have

$$\mathcal{R}_z = \left\{(x,t) \in \mathcal{Q}^T \;\middle|\; x < K(z,\vec{t}\,)\right\}, \quad \mathcal{NR}_z = \left\{(x,t) \in \mathcal{Q}^T \;\middle|\; x \geqslant K(z,\vec{t}\,)\right\},$$

where $K(z,\vec{t}\,)$ is the reinsurance boundary given by

$$K(z,\vec{t}\,) := \inf\left\{x > 0 \;\middle|\; -\frac{v_{xx}}{v_x}(x,\vec{t}\,) \leqslant \frac{1}{z}\right\}.$$

Since $\mathcal{R}_z \subseteq \mathcal{ND}$, we have $K(z,\vec{t}\,) \leqslant d(\vec{t}\,)$. Moreover, when $z \leqslant \frac{1}{\lambda}$, (2.2.26) leads to $\mathcal{R}_z = \emptyset$ and $K(z,\vec{t}\,) = 0$.

For each $z > 0$, define the overlapping of the non-reinsurance and non-dividend regions as the non-action region

$$\mathcal{NA}_z = \mathcal{NR}_z \bigcap \mathcal{ND} = \left\{(x,t) \in \mathcal{Q}^{\mathrm{T}} \middle| K(z,\vec{t}\,) \leqslant x < d(\vec{t}\,)\right\}.$$

In this region, the insurer should not cede risks or pay out dividends. We turn to show that this region is always non-empty. Recall $K(z,\vec{t}\,) \leqslant d(\vec{t}\,)$, it is equivalent to showing $K(z,\vec{t}\,) \neq d(\vec{t}\,)$. Note that $d(\vec{t}\,) = \inf\{x \geqslant 0 \mid v_x(x,\vec{t}\,) = 1\}$ and $v \in C^{2,1}\big(\overline{\mathcal{Q}_T} \setminus \{(0,0)\}\big) \bigcap C\big(\overline{\mathcal{Q}_T}\big)$, so $(v_x + zv_{xx})\big|_{(d(\vec{t}\,),\vec{t}\,)} = 1$ and $(v_x + zv_{xx})\big|_{(K(z,\vec{t}\,),\vec{t}\,)} = 0$. As a result, $K(z,\vec{t}\,) \neq d(\vec{t}\,)$.

Combining the above results, we find that, for each magnitude of risk $z > 0$, the reinsurance and dividend-payout barriers divide the surplus-time space into three non-overlapping regions: a (possibly empty) reinsurance region, a non-action region and a dividend-payout region, in an increasing order of the surplus. This is depicted in Figure 2.2.

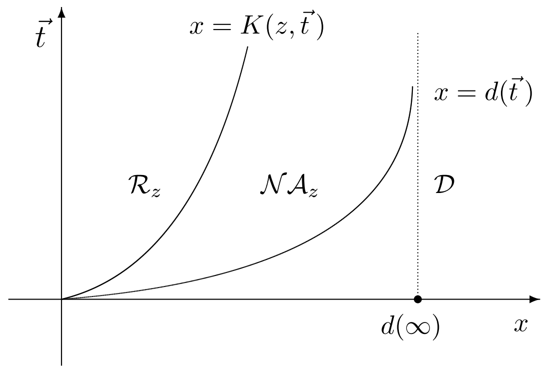

Figure 2.2: The surplus-time space is divided into three non-overlapping regions: $\mathcal{R}_z$, $\mathcal{NA}_z$ and $\mathcal{D}$ by the reinsurance barrier $x = K(z, \vec{t})$ and dividend-payout barrier $x = d(\vec{t})$.

Economically speaking, if the surplus level of an insurance company is relatively low ($x < K(z, \vec{t})$), all risk z should be ceded to a reinsurer to avoid bankruptcy. If the surplus level is medium ($K(z, \vec{t}) \leqslant x \leqslant d(\vec{t})$), the company can cover the claim z on its own, but the reserves are insufficient to pay out dividends, thus no actions are required. If the surplus is large enough ($x > d(\vec{t})$), the company should distribute extra reserves to shareholders as dividends . It is never a good idea for an insurer to obtain reinsurance contracts and pay out dividends at the same time.

By Lemma 2.2, as the cede risk z increases, $K(z, \vec{t})$ extends and the reinsurance-covered region $\mathcal{R}_z$ expands. Economically speaking, if a risk is covered by a reinsurance contract, then any higher risk should be covered as well.

Define a uniform non-action region

$$\mathcal{NA}_0 = \bigcap_{z>0} \mathcal{NA}_z.$$

In this region the insurer would not cede any risk or pay out dividends.

If $\hat{z} = \operatorname{ess\,sup} Z_1 < \infty$, then $K(z, \vec{t}\,) \leqslant K(\hat{z}, \vec{t}\,)$. Therefore,

$$\bigcup_{z>0} \mathcal{R}_z = \Big\{ (x,t) \in \mathcal{Q} \;\Big|\; x < K(\hat{z}, \vec{t}\,) \Big\} = \mathcal{R}_{\hat{z}}.$$

As $K(\hat{z}, \vec{t}\,) < d(\vec{t}\,)$, we can see that

$$\mathcal{NA}_0 \supseteq \Big\{ (x,t) \in \mathcal{Q} \;\Big|\; K(\hat{z}, \vec{t}\,) \leqslant x < d(\vec{t}\,) \Big\} = \mathcal{NA}_{\hat{\ddagger}} \neq \emptyset.$$

This is illustrated in Figure 2.3.

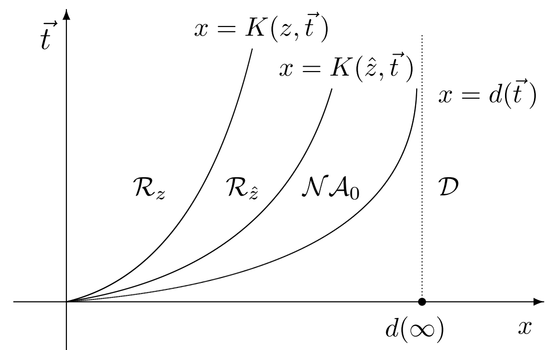

Figure 2.3: The reinsurance region $\mathcal{R}_z$ expands to $\mathcal{R}_{\hat{z}}$ when z increases to $\hat{z}$.

Economically speaking, when the underlying claims are bounded ($\leqslant \hat{z}$) and the surplus level is quite large ($x \geqslant K(\hat{z}, \vec{t}\,)$), the insurance company can cover the total risk without sharing with a ceded company.

By (2.2.25), we further have

$$\partial_x \widehat{I}(z, x, \vec{t}\,) \leqslant -2c \quad \text{in} \quad \mathcal{R}_z.$$

This means that the insurance company should significantly reduce its purchase of reinsurance contracts as the surplus increases. If its surplus level is very high, i.e., $x \geqslant \frac{z}{2c} - \frac{1}{2c\lambda}$, then we claim that there is no need to buy reinsurance for the claim z, that is, $(x,t) \notin \mathcal{R}_z$. In fact, if $z \leqslant 1/\lambda$, then there is nothing to prove since $\mathcal{R}_z = \emptyset$.

Otherwise, suppose $(x,t) \in \mathcal{R}_z$, by (2.2.25), (2.2.26) and the mean value theorem, we have

$$\widehat{I}(z,x,\vec{t}\,) \leqslant \widehat{I}(z,0+,\vec{t}\,) - 2cx \leqslant z - 1/\lambda - 2cx \leqslant 0,$$

which contradicts $(x,t) \in \mathcal{R}_z$. Therefore, we conclude that

$$\mathcal{R}_z \subseteq \left\{ (x,t) \in \mathcal{Q}^T \;\middle|\; x < \frac{z}{2c} - \frac{1}{2c\lambda} \right\},$$

and consequently,

$$K(z,\vec{t}\,) \leqslant \frac{z}{2c} - \frac{1}{2c\lambda}.$$

This is illustrated in Figure 2.4.

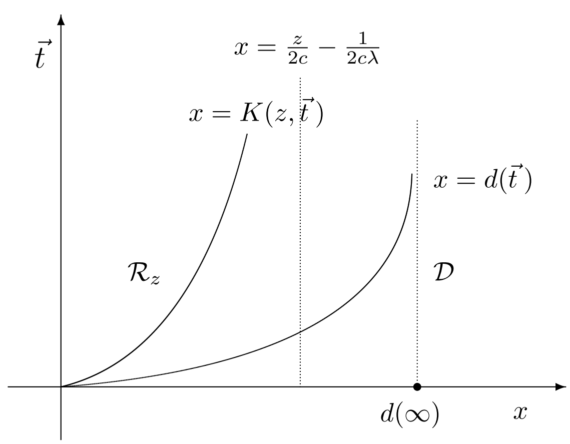

Figure 2.4: The reinsurance barrier $K(z,\vec{t}\,)$ with an upper bound $x = \frac{z}{2c} - \frac{1}{2c\lambda}$ and the reinsurance region $\mathcal{R}_z$ when $z > 1/\lambda$.

Economically speaking, an insurance company only shares large claims with a ceded company.

2.3 An example

In this section we assume that claims Z follow a discrete probability distribution given by

$$\mathbb{P}(Z_1 = z_j) = p_j > 0, \quad j = 1, 2, ..., N,$$

with

$$0 < z_1 < z_2 < ... < z_{i_0-1} \leqslant \frac{1}{\lambda} < z_{i_0} < ... < z_N \quad \text{and} \quad \sum_{j=1}^{N} p_j = 1.$$

Define the j^{th} reinsurance boundary as

$$K_j(\vec{t}\,) := \inf \left\{ x > 0 \,\middle|\, -\frac{v_{xx}}{v_x}(x, \vec{t}\,) \leqslant \frac{1}{z_j} \right\}, \quad j = 1, 2, ..., N.$$

Also, define the j^{th} reinsurance region as

$$\mathcal{R}_z^j = \left\{ (x, t) \in \mathcal{Q}^T \,\middle|\, \widehat{I}(z_j, x, \vec{t}\,) > 0 \right\}.$$

Thanks to Lemma 2.2, the properties of these reinsurance boundaries are given in the following results.

Theorem 2.4. *The reinsurance boundaries $K_j(\cdot)$, $j = 1, 2, \cdots, N$, are all continuously differentiable in time. Moreover,*

$$K_j(\vec{t}\,) = 0 \quad \textit{for each} \quad j = 1, 2, \cdots, i_0 - 1; \tag{2.3.27}$$

$$0 < K_{i_0}(\vec{t}\,) < \frac{z_{i_0} - \frac{1}{\lambda}}{2c}; \tag{2.3.28}$$

$$0 < K_j(\vec{t}\,) - K_{j-1}(\vec{t}\,) < \frac{z_j - z_{j-1}}{2c} \quad \textit{for each} \quad j = 2, ..., N; \tag{2.3.29}$$

$$\lim_{\vec{t} \to 0+} K_j(\vec{t}\,) = 0. \tag{2.3.30}$$

Proof. By (2.2.23), (2.2.25) and $0 < z_1 < z_2 < \cdots < z_{i_0-1} \leqslant \frac{1}{\lambda}$, we have (2.3.27). Note that $\left(-\frac{v_x}{v_{xx}}\right)(K_{i_0}(t), t) = z_{i_0} > \frac{1}{\lambda}$. Since $-\frac{v_x}{v_{xx}}$ is continuous in $\mathcal{ND}$, we obtain

$K_{i_0}(t) > 0$. The mean value theorem and (2.2.25) imply that

$$z_1 - \frac{1}{\lambda} = \big(-\frac{v_x}{v_{xx}}\big)(K_1(t), t) - \big(-\frac{v_x}{v_{xx}}\big)(0, t) > 2cK_1(t).$$

which gives (2.3.28). The proof of (2.3.29) is similar. The fact of $K_1(t) < \cdots < K_N(t) < d(t)$ and $\lim\limits_{t \to 0^+} d(t) = 0$ give (2.3.30).

Now we prove $K_j(t) \in C^1((0, T])$ for $j \geqslant i_0$. By Lemma 2.1 and v_{xx} in $\mathcal{ND}$, we see that $-\frac{v_x}{v_{xx}} \in C^{1,2}(\mathcal{ND})$. Since $\partial_x(-\frac{v_x}{v_{xx}}) > 2c$ in $\mathcal{ND}$, the implicit function theorem implies $K_j(t) \in C^1((0, T])$. $\square$

Figure 2.5 illustrates this result.

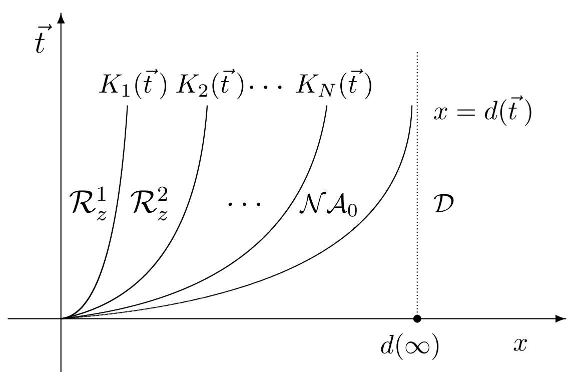

Figure 2.5: The reinsurance boundaries K_j and reinsurance regions $\mathcal{R}_z^j$ and the uniform non-action region $\mathcal{NA}_{z_N}$

Economically speaking, the insurer should cede the claims $z_j, z_{j+1}, \cdots z_N$, if the surplus level falls in $\mathcal{R}_z^j$.

2.4 Proofs

2.4.1 Proof of Theorem 2.1.

Based on (2.2.8), $z(1-F(z))$ is dominated by z^{-2} near infinity, together with Fubini's theorem and the dominated convergence theorem, it is certain that

$$\int_0^\infty z^2 \mathrm{d}F(z) = 2\int_0^\infty z(1-F(z))\mathrm{d}z < \infty.$$

Therefore, the first and second moments of the claims are finite, that is

$$\mu_1 := \int_0^\infty z\mathrm{d}F(z) < \infty, \quad \mu_2 := \int_0^\infty z^2\mathrm{d}F(z) < \infty. \tag{2.4.31}$$

To solve Problem (2.2.6), we first solve the supremum in the operator $\mathcal{L}$, namely

$$\sup_{H\in\mathcal{H}} \Big(\frac{v_{xx}}{2} \int_0^\infty H(z)^2 \mathrm{d}F(z) + v_x \int_0^\infty H(z)\mathrm{d}F(z) \Big),$$

or

$$\sup_{H\in\mathcal{H}} \int_0^\infty \Big(\frac{1}{2} H(z)^2 v_{xx} + H(z) v_x \Big) \mathrm{d}F(z). \tag{2.4.32}$$

Define

$$h^*(z,y) = \operatorname*{argmax}_{0\leqslant h\leqslant z} \left(-\frac{1}{2}h^2 y + h \right) = \begin{cases} \min\{z, y^{-1}\}, & \text{if} \quad 0 < y < \infty, \\ z, & \text{otherwise.} \end{cases} \tag{2.4.33}$$

Notice that $v_x \geqslant 1$ in (2.2.6), so (2.4.32) is equal to

$$v_{xx} \int_0^\infty \frac{1}{2} \left(h^*\Big(z, -\frac{v_{xx}}{v_x}\Big) \right)^2 \mathrm{d}F(z) + v_x \int_0^\infty h^*\Big(z, -\frac{v_{xx}}{v_x}\Big) \mathrm{d}F(z). \tag{2.4.34}$$

For $y > 0$, we have

$$\int_0^{\infty} \frac{1}{2}(h^*(z,y))^2 \mathrm{d}F(z) = \int_0^{\frac{1}{y}} \frac{1}{2} z^2 \mathrm{d}F(z) + \int_{\frac{1}{y}}^{\infty} \frac{1}{2} y^{-2} \mathrm{d}F(z)$$

$$= -\frac{1}{2} z^2 (1 - F(z))|_0^{\frac{1}{y}} + \int_0^{\frac{1}{y}} z(1 - F(z))\mathrm{d}z + \frac{1}{2} y^{-2}(1 - F(y^{-1}))$$

$$= \int_0^{\frac{1}{y}} z(1 - F(z))\mathrm{d}z,$$

and

$$\int_0^{\infty} h^*(z,y)\mathrm{d}F(z) = \int_0^{\frac{1}{y}} z\mathrm{d}F(z) + \int_{\frac{1}{y}}^{\infty} y^{-1}\mathrm{d}F(z)$$

$$= -z(1 - F(z))|_0^{\frac{1}{y}} + \frac{0}{\frac{1}{y}}(1 - F(z))\mathrm{d}z + y^{-1}(1 - F(y^{-1}))$$

$$= \int_0^{\frac{1}{y}} (1 - F(z))\mathrm{d}z.$$

Similarly, for $y \leqslant 0$, we have

$$\int_0^{\infty} \frac{1}{2}(h^*(z,y))^2 \mathrm{d}F(z) = \frac{1}{2}\int_0^{\infty} z^2 \mathrm{d}F(z) = \frac{1}{2}\mu_2,$$

and

$$\int_0^{\infty} h^*(z,y)\mathrm{d}F(z) = \int_0^{\infty} z\mathrm{d}F(z) = \mu_1.$$

Hence, (2.4.34) is equal to

$$A\left(-\frac{v_{xx}}{v_x}\right) v_{xx} + B\left(-\frac{v_{xx}}{v_x}\right) v_x,$$

where functions $A(y)$ and $B(y)$ are defined by

$$A(y) = \begin{cases} \int_0^{1/y} z(1 - F(z))\mathrm{d}z, & \text{if } 0 < y < +\infty, \\ \int_0^{\infty} z(1 - F(z))\mathrm{d}z = \frac{1}{2}\mu_2, & \text{if } y \leqslant 0, \end{cases} \tag{2.4.35}$$

and

$$B(y)=\begin{cases}\int_0^{1/y}(1-F(z))\mathrm{d}z, & \text{if } 0<y<+\infty,\\ \\ \int_0^{\infty}(1-F(z))\mathrm{dz}=\mu_1, & \text{if } y\leqslant 0.\end{cases} \tag{2.4.36}$$

It suffices to show $A(y)$ and $B(y)$ are both decreasing functions in $C^1(\mathbb{R})$ and satisfy

$$0<A(y)\leqslant\frac{1}{2}\min\{y^{-2},\ \mu_2\},\quad 0<B(y)\leqslant\min\{y^{-1},\ \mu_1\},\quad \text{for } y>0, \tag{2.4.37}$$

$$y^3A'(y)=y^2B'(y)=F(y^{-1})-1\leqslant 0,\quad \text{for } y>0, \tag{2.4.38}$$

$$A'(y)=B'(y)=0,\quad \text{for } y\leqslant 0. \tag{2.4.39}$$

We obtain

$$\mathcal{L}v:=A\left(-\frac{v_{xx}}{v_x}\right)v_{xx}+B\left(-\frac{v_{xx}}{v_x}\right)v_x-\gamma v_x-cv.$$

Hence, we can see that problem (2.2.6) is a variational inequality problem for a nonlinear elliptic operator with a gradient constraint. A usual approach to study this kind of problem is to transform it into a variational inequality problem for its gradient. Then the gradient constraint becomes a value constraint and the new variational inequality becomes a well-studied obstacle problem, see (Dai et al. (2009), Dai and Yi (2009)). Hence, we adopt a similar approach.

In what follows, we want to find a variational inequality for v_x. Notice that

$$\begin{aligned}&\partial_x\left[A\left(-\frac{v_{xx}}{v_x}\right)v_{xx}+B\left(-\frac{v_{xx}}{v_x}\right)v_x\right]\\ =&A\left(-\frac{v_{xx}}{v_x}\right)v_{xxx}+B\left(-\frac{v_{xx}}{v_x}\right)v_{xx}+\left[A'\left(-\frac{v_{xx}}{v_x}\right)v_{xx}+B'\left(-\frac{v_{xx}}{v_x}\right)v_x\right]\partial_x\left(-\frac{v_{xx}}{v_x}\right)\\ =&A\left(-\frac{v_{xx}}{v_x}\right)v_{xxx}+B\left(-\frac{v_{xx}}{v_x}\right)v_{xx},\end{aligned}$$

where we combine (2.4.38) and (2.4.39) to get the last equation. By this, one can easily deduce that

$$\partial_x(\mathcal{L}v)=\mathcal{T}v_x,$$

where the operator $\mathcal{T}$ is defined as

$$\mathcal{T}u := A\left(-\frac{u_x}{u}\right)u_{xx} + B\left(-\frac{u_x}{u}\right)u_x - \gamma u_x - cu. \tag{2.4.40}$$

Next, we deduce a boundary condition for v_x. Define a continuous function

$$f(y) := -yA(y) + B(y) - \gamma. \tag{2.4.41}$$

By (2.4.37)-(2.4.39), we have

$$f'(y) = -A(y) < 0. \tag{2.4.42}$$

Hence f is strictly decreasing. Also, note that

$$f(0+) = \mu_1 - \gamma > 0, \quad f(+\infty) = -\gamma < 0,$$

so f has a unique and positive root denoted by λ . See Figure 2.6 for an illustration of the function f.

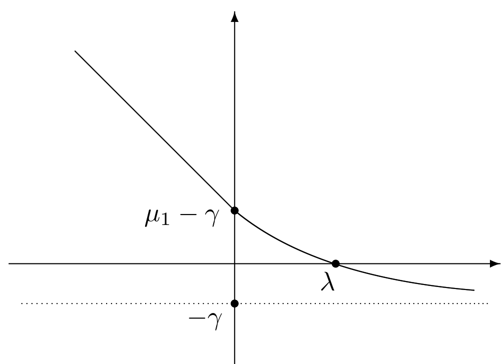

Figure 2.6: The function $f \in C^1(\mathbb{R})$ is strictly decreasing and admits a unique positive root λ.

On the other hand, owing to the boundary condition $v(0,t) = 0$ and $v_t - \mathcal{L}v = 0$ near $\{x = 0\}$, we have

$$\left(A\left(-\frac{v_{xx}}{v_x}\right)v_{xx} + B\left(-\frac{v_{xx}}{v_x}\right)v_x - \gamma v_x\right)\Bigg|_{x=0} = 0. \tag{2.4.43}$$

Dividing both sides by v_x yields $f\left(-\frac{v_{xx}}{v_x}\right)\Big|_{x=0} = 0$. Then $-\frac{v_{xx}}{v_x}\Big|_{x=0} = \lambda$ by the strictly monotonicity of f, which leads to a boundary condition

$$\left(\lambda v_x + v_{xx}\right)\Big|_{x=0} = 0. \tag{2.4.44}$$

To sum up, we present the following variational inequality for $u = v_x$

$$\begin{cases} \min\{u_t - \mathcal{T}u,\ u-1\} = 0 \quad \text{in} \quad \mathcal{Q}_T, \\ \left(\lambda u + u_x\right)(0,t) = 0, \quad 0 < t \leqslant T, \\ u(x,0) = 1, \quad x > 0. \end{cases} \tag{2.4.45}$$

This is an obstacle problem for a quasilinear elliptic operator with mixed boundary conditions. We address it by using a penalty approximation method, see Yi (2008); Dai and Yi (2009); Guan et al. (2019). We first prove the existence of a solution to Problem (2.4.45), and then construct a solution to problem (2.2.6).

For a sufficiently small $\varepsilon > 0$, let $\beta_\varepsilon(\cdot)$ be a penalty function that satisfies

$$\beta_\varepsilon(\cdot) \in C^2(-\infty,+\infty), \quad \beta_\varepsilon(0) = -c, \quad \beta_\varepsilon(x) = 0 \ \text{ for } \ x \geqslant \varepsilon > 0,$$

$$\beta_\varepsilon(\cdot) \leqslant 0, \quad \beta_\varepsilon'(\cdot) \geqslant 0, \quad \beta_\varepsilon''(\cdot) \leqslant 0, \quad \lim_{\varepsilon\to 0+} \beta_\varepsilon(x) = \begin{cases} 0, & \text{if } \ x > 0, \\ -\infty, & \text{if } \ x < 0. \end{cases}$$

See Figure 2.7 for an illustration of the penalty function β_ε.

Since the left boundary condition and initial condition in (2.4.45) are not consistent at $(0,0)$, in order to show the existence of a solution, we choose $f_\varepsilon(t) \in C^2([0,+\infty))$ that satisfies

$$f_\varepsilon(t) = \begin{cases} \lambda, & t = 0, \\ \text{decreasing}, & 0 \leqslant t < \varepsilon, \\ 0, & t \geqslant \varepsilon, \end{cases}$$

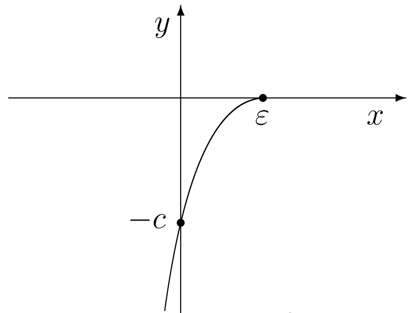

Figure 2.7: The penalty function β_ε.

and consider the following penalty approximation problem

$$\begin{cases} u^\varepsilon_t - \mathcal{T}u^\varepsilon + \beta_\varepsilon(u^\varepsilon - 1) = 0 \quad \text{in} \quad \mathcal{Q}_{L,T} := (0,L)\times(0,T], \\ \big(\lambda u^\varepsilon + u^\varepsilon_x\big)(0,t) = f_\varepsilon(t), \quad 0 < t \leqslant T, \\ u^\varepsilon(L,t) = 1, \quad 0 < t \leqslant T, \\ u^\varepsilon(x,0) = 1, \quad 0 < x < L, \end{cases} \tag{2.4.46}$$

where L is a fixed positive constant and $\mathcal{T}$ is defined by (2.4.40).

The following result is useful to solve the above problem.

Lemma 2.3. *Let A and B follow* (2.4.35) *and* (2.4.36) *respectively, we define*

$$F(y,z) := A\left(-\frac{y}{z}\right), \qquad G(y,z) := B\left(-\frac{y}{z}\right).$$

Then F and G are uniformly Lipschitz continuous in $(-\infty,\infty)\times[1,\infty)$.

Proof. By (2.4.38), F is continuous in $(-\infty,\infty)\times[1,\infty)$ and for any $z \geqslant 1$,

$$\partial_y F(y,z) = A'\left(-\frac{y}{z}\right)\frac{-1}{z} = \begin{cases} \left(-\frac{z}{y}\right)^3\left(1 - F\left(-\frac{z}{y}\right)\right)\frac{1}{z}, & \text{if} \quad y < 0; \\ 0, & \text{if} \quad y \geqslant 0, \end{cases}$$

and

$$\partial_z F(y,z) = A'\left(-\frac{y}{z}\right)\frac{y}{z^2} = \begin{cases} \left(-\frac{z}{y}\right)^2\left(1 - F\left(-\frac{z}{y}\right)\right)\frac{1}{z}, & \text{if} \quad y < 0; \\ 0, & \text{if} \quad y \geqslant 0. \end{cases}$$

Due to (2.2.8) and $z \geqslant 1$, we see that $\partial_y F$ and $\partial_z F$ are bounded, so F is uniformly Lipschitz continuous in $(-\infty, \infty) \times [1, \infty)$. In the same way, we can show that G is also uniformly Lipschitz continuous in $(-\infty, \infty) \times [1, \infty)$. □

We first establish a comparison principle for Problem (2.4.46).

Lemma 2.4. *Suppose that* $u_1,\ u_2 \in C^{2,1}\big(\mathcal{Q}_{L,T}\big) \bigcap C\big(\overline{\mathcal{Q}_{L,T}}\big)$ *satisfy*

$$\begin{cases} \partial_t u_1 - \mathcal{T} u_1 + \beta_\varepsilon(u_1 - 1) \leqslant \partial_t u_2 - \mathcal{T} u_2 + \beta_\varepsilon(u_2 - 1) \quad \text{in} \quad \mathcal{Q}_{L,T}, \\ \big(\lambda u_1 + \partial_x u_1\big)(0,t) \geqslant \big(\lambda u_2 + \partial_x u_2\big)(0,t), \quad 0 < t \leqslant T, \\ u_1(L,t) \leqslant u_2(L,t), \quad 0 < t \leqslant T, \\ u_1(x,0) \leqslant u_2(x,0), \quad 0 < x < L. \end{cases}$$

If $u_1,\ u_2 \geqslant 1$, *then*

$$u_1(x,t) \leqslant u_2(x,t), \quad (x,t) \in \mathcal{Q}_{L,T}. \tag{2.4.47}$$

Proof. Let

$$w_1(x,t) = e^{x/\lambda} u_1(x,t), \quad w_2(x,t) = e^{x/\lambda} u_2(x,t).$$

Then

$$\begin{cases} \partial_t w_1 - e^{x/\lambda}\mathcal{T}(e^{-x/\lambda} w_1) + e^{x/\lambda}\beta_\varepsilon(e^{-x/\lambda} w_1 - 1) \leqslant \partial_t w_2 - e^{x/\lambda}\mathcal{T}(e^{-x/\lambda} w_2) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + e^{x/\lambda}\beta_\varepsilon(e^{-x/\lambda} w_2 - 1), \\ \partial_x w_1(0,t) \geqslant \partial_x w_2(0,t), \quad 0 < t \leqslant T, \\ w_1(L,t) \leqslant w_2(L,t), \quad 0 < t \leqslant T, \\ w_1(x,0) \leqslant w_2(x,0), \quad 0 < x < L. \end{cases}$$

By Lemma 2.3, the assumption of u_1, $u_2 \geqslant 1$ is sufficient to guarantee that $F(\cdot,\cdot)$ and $G(\cdot,\cdot)$ in $\mathcal{T}(e^{-x/\lambda} w_i)$ are Lipschitz continuous on w_i, w_{ix}, $i = 1, 2$. By the comparison principle for nonlinear equations (see Lieberman (1996) Theorem 14.3), we obtain $w_1 \leqslant w_2$ in $\mathcal{Q}_{L,T}$. This confirms the claim. □

Lemma 2.5. *There exists a solution* $u^\varepsilon \in C^{2,1}\big(\overline{\mathcal{Q}_{L,T}}\big)$ *to Problem (2.4.46). Moreover, for a sufficiently small* $\varepsilon > 0$, u^ε *satisfies*

$$1 \leqslant u^\varepsilon \leqslant \frac{Ke^{\Lambda t}}{x+1/\lambda}, \quad \text{in} \quad \mathcal{Q}_{L,T}, \tag{2.4.48}$$

where

$$K = L + 1/\lambda + 1, \quad \Lambda = \frac{\mu_2}{\gamma^2} + \gamma\lambda > 0.$$

Proof. Using the Leray-Schauder fixed point theorem (see Gilbarg and Trudinger (2015)) and the embedding theorem (see Lieberman (1996) Theorem 6.8), we obtain the existence of a $C^{1+\alpha,\frac{1+\alpha}{2}}\big(\overline{\mathcal{Q}_{L,T}}\big)(0 < \alpha < 1)$ solution u^ε to Problem (2.4.46). By the Schauder estimation (see Lieberman (1996) Theorem 4.23), we also have $u^\varepsilon \in C^{2+\alpha,1+\frac{\alpha}{2}}\big(\overline{\mathcal{Q}_{L,T}}\big)$.

We are now to prove (2.4.48). Let $\phi \equiv 1$, then

$$\begin{cases} \phi_t - A\left(-\frac{u^\varepsilon_x}{u^\varepsilon}\right)\phi_{xx} - B\left(-\frac{u^\varepsilon_x}{u^\varepsilon}\right)\phi_x + \gamma\phi_x + c\phi + \beta_\varepsilon(\phi-1) = 0, \\ \big(\lambda\phi + \phi_x\big)(0,t) = \lambda \geqslant f_\varepsilon(t), \quad 0 < t \leqslant T, \\ \phi(L,t) = 1, \quad 0 < t \leqslant T, \\ \phi(x,0) = 1, \quad 0 < x < L. \end{cases}$$

Assume that $A\left(-\frac{u^\varepsilon_x}{u^\varepsilon}\right)$ and $B\left(-\frac{u^\varepsilon_x}{u^\varepsilon}\right)$ are known coefficients, we can establish a comparison principle for the above PDE by the same method in Lemma 2.4, which yields $u^\varepsilon \geqslant \phi = 1$.

Let $\Phi = Ke^{\Lambda t}/(x+1/\lambda)$. Then, in $\mathcal{Q}_{L,T}$, for a sufficiently small $\varepsilon > 0$, we have $\beta_\varepsilon(\Phi - 1) = 0$. Note that

$$\Phi_t = \Lambda\Phi, \quad \Phi_x = -\frac{\Phi}{x+1/\lambda} < 0, \quad \Phi_{xx} = \frac{2\Phi}{(x+1/\lambda)^2} > 0,$$

by (2.4.37),

$$\Phi_t - A\left(-\frac{\Phi}{\Phi_x}\right)\Phi_{xx} - B\left(-\frac{\Phi}{\Phi_x}\right)\Phi_x + \gamma\Phi_x + c\Phi + \beta_\varepsilon(\Phi - 1)$$
$$\geqslant \Phi_t - \frac{1}{2}\mu_2\Phi_{xx} + \gamma\Phi_x + c\Phi = \left(\Lambda - \frac{\mu_2}{(x+1/\lambda)^2} - \frac{\gamma}{x+1/\lambda} + c\right)\Phi \ \geqslant \ 0.$$

Together with boundary conditions

$$\begin{cases} \big(\lambda\Phi + \Phi_x\big)(0,t) = 0 \leqslant f_\varepsilon(t), & 0 < t \leqslant T, \\ \Phi(L,t) \geqslant 1, & 0 < t \leqslant T, \\ \Phi(x,0) \geqslant 1, & 0 < x \leqslant L, \end{cases}$$

we can obtain $u^\varepsilon \leqslant \Phi$ by applying Lemma 2.4. □

Before passing ϵ to the limit, we give some properties of u^ε.

Lemma 2.6. *Let $u^\varepsilon \in C^{2,1}\big(\overline{\mathcal{Q}_{L,T}}\big)$ be the solution to Problem (2.4.46), we have*

$$u_t^\varepsilon \geqslant 0, \tag{2.4.49}$$

$$u_x^\varepsilon \leqslant 0. \tag{2.4.50}$$

Proof. We first prove (2.4.49). For any $0 < \Delta < T$, let $\widetilde{u}^\varepsilon(x,t) = u^\varepsilon(x,t+\Delta)$, then both $\widetilde{u}^\varepsilon$ and u^ε satisfy the equation in (2.4.46) in the domain of $(0,L) \times (0, T-\Delta]$. Moreover,

$$\begin{cases} \Big(\lambda\widetilde{u}^\varepsilon + \widetilde{u}_x^\varepsilon\Big)(0,t) = f_\varepsilon(t+\Delta t) \leqslant f_\varepsilon(t) = \Big(\lambda u^\varepsilon + u_x^\varepsilon\Big)(0,t) \quad \text{in} & (0,L)\times(0,T-\Delta], \\ \widetilde{u}^\varepsilon(L,t) = u^\varepsilon(L,t) = 1, & 0 < t \leqslant T-\Delta, \\ \widetilde{u}^\varepsilon(x,0) = u^\varepsilon(x,\Delta) \geqslant 1 = u^\varepsilon(x,0), & 0 < x < L. \end{cases}$$

According to Lemma 2.4 we have $\widetilde{u}^\varepsilon \geqslant u^\varepsilon$ in $(0,L)\times(0,T-\Delta]$, which implies (2.4.49).

To prove (2.4.50), we differentiate the equation in (2.4.46) w.r.t. x such that

$$\partial_t u_x^\varepsilon - \partial_x \left[A\left(-\frac{u_x^\varepsilon}{u^\varepsilon}\right)\partial_x u_x^\varepsilon \right] - B\left(-\frac{u_x^\varepsilon}{u^\varepsilon}\right)\partial_x u_x^\varepsilon$$
$$- B'\left(-\frac{u_x^\varepsilon}{u^\varepsilon}\right)\partial_x\left(-\frac{u_x^\varepsilon}{u^\varepsilon}\right)u_x^\varepsilon + cu_x^\varepsilon + \beta_\varepsilon'(u^\varepsilon - 1)u_x^\varepsilon = 0. \quad (2.4.51)$$

Note that

$$\partial_x\left(-\frac{u_x^\varepsilon}{u^\varepsilon}\right) = -\frac{u_{xx}^\varepsilon}{u^\varepsilon} + \left(-\frac{u_x^\varepsilon}{u^\varepsilon}\right)^2,$$

so (2.4.51) can be written as

$$\partial_t u_x^\varepsilon - \partial_x \left[\left\{ A\left(-\frac{u_x^\varepsilon}{u^\varepsilon}\right)\right\}\partial_x u_x^\varepsilon \right] - \left\{ B\left(-\frac{u_x^\varepsilon}{u^\varepsilon}\right) + B'\left(-\frac{u_x^\varepsilon}{u^\varepsilon}\right)\frac{-u_x^\varepsilon}{u^\varepsilon}\right\}\partial_x u_x^\varepsilon$$
$$+ \left\{ -B'\left(-\frac{u_x^\varepsilon}{u^\varepsilon}\right)\left(-\frac{u_x^\varepsilon}{u^\varepsilon}\right)^2 + c + \beta_\varepsilon'(u^\varepsilon - 1)\right\}u_x^\varepsilon = 0. \quad (2.4.52)$$

It is a linear equation about u_x^ε in a divergence form if we regard the terms in $\{\cdots\}$ of (2.4.52) as known coefficients. (2.4.37)-(2.4.39) indicate that,

$$|A(y)| \leqslant \mu_2, \quad |B(y)| \leqslant \mu_1, \quad |B'(y)\,y| \leqslant |y^{-1}(1 - F(y^{-1}))| \leqslant \mu_1,$$
$$\left|B'(y)\,y^2\right| \leqslant |1 - F(y^{-1})| \leqslant 1, \quad |\beta_\varepsilon'(u^\varepsilon - 1)| \geqslant 0,$$

which imply that all the coefficients of (2.4.52) are bounded, except the last one has a lower bound. Moreover, since

$$u_x^\varepsilon(0,t) = (f_\varepsilon(t) - \lambda u^\varepsilon(0,t)),$$

together with $f_\varepsilon \leqslant \lambda \leqslant \lambda u^\varepsilon$, we have

$$u_x^\varepsilon(0,t) \leqslant 0.$$

From $u^\varepsilon \geqslant 1$ and $u^\varepsilon(L,t) = 1$, we have

$$u_x^\varepsilon(L,t) \leqslant 0.$$

Moreover, with the knowledge of

$$u_x^\varepsilon(x,0)=0,$$

we can deduce $u_x^\varepsilon \leqslant 0$ by the maximum principle for the weak solution (see Lieberman (1996) Corollary 6.16). □

Lemma 2.7. *Let $u^\varepsilon \in C^{2,1}\left(\overline{Q_{L,T}}\right)$ be the solution to Problem (2.4.46), we have*

$$\lambda u^\varepsilon + u_x^\varepsilon \geqslant 0. \tag{2.4.53}$$

Proof. If $u^\varepsilon(0,t)=1$, it follows from $u^\varepsilon \geqslant 1$ and $u_x^\varepsilon \leqslant 0$ that $u^\varepsilon(x,t)\equiv 1$ and $u_x^\varepsilon(x,t)\equiv 0$, for all $x \geqslant 0$, so $(\lambda u^\varepsilon + u_x^\varepsilon)(x,t)\equiv \lambda > 0$ and then (2.4.53) holds.

Otherwise, if $u^\varepsilon(0,t)>1$, we have

$$(cu^\varepsilon + \beta_\varepsilon(u^\varepsilon-1))(0,t) > c+\beta_\varepsilon(0)=0. \tag{2.4.54}$$

Consider the equation in (2.4.46), i.e.

$$u_t^\varepsilon - A\left(-\frac{u_x^\varepsilon}{u^\varepsilon}\right)u_{xx}^\varepsilon - B\left(-\frac{u_x^\varepsilon}{u^\varepsilon}\right)u_x^\varepsilon + \gamma u_x^\varepsilon + cu^\varepsilon + \beta_\varepsilon(u^\varepsilon-1)=0.$$

If $u_x^\epsilon = 0$, then it reaches its global minimum value 0, so $u_{xx}^\epsilon = 0$. Together with $u_t^\epsilon \geqslant 0$, the above equation gives $cu^\epsilon + \beta_\epsilon(u^\epsilon - 1) \leqslant 0$, and the definition of β_ϵ implies $u^\epsilon = 1$. Therefore, $\lambda u^\epsilon + u_x^\epsilon = \lambda > 0$, and (2.4.53) follows. Otherwise, we have $u_x^\epsilon < 0$. Dividing both sides by u_x^ϵ and using the identity

$$-\frac{u_{xx}^\varepsilon}{u_x^\varepsilon} = \left[\partial_x\left(-\frac{u^\varepsilon}{u_x^\varepsilon}\right)+1\right]\left(\frac{-u_x^\varepsilon}{u^\varepsilon}\right),$$

lead to

$$\frac{u_t^\varepsilon}{u_x^\varepsilon} + A\left(-\frac{u_x^\varepsilon}{u^\varepsilon}\right)\left[\partial_x\left(-\frac{u^\varepsilon}{u_x^\varepsilon}\right)+1\right]\left(\frac{-u_x^\varepsilon}{u^\varepsilon}\right) - B\left(-\frac{u_x^\varepsilon}{u^\varepsilon}\right) + \gamma + \frac{cu^\varepsilon + \beta_\varepsilon(u^\varepsilon-1)}{u_x^\varepsilon} = 0.$$

Denote $\nu^\varepsilon = -u^\varepsilon/u_x^\varepsilon$, we have

$$A\left(\frac{1}{\nu^\varepsilon}\right)\frac{\nu_x^\varepsilon+1}{\nu^\varepsilon} - B\left(\frac{1}{\nu^\varepsilon}\right) + \gamma - \frac{cu^\varepsilon + \beta_\varepsilon(u^\varepsilon - 1)}{u^\varepsilon}\nu^\varepsilon = -\frac{u_t^\varepsilon}{u_x^\varepsilon} \geqslant 0.$$

By Lemma 2.6, we get

$$\nu_x^\varepsilon \geqslant \frac{1}{A\left(\frac{1}{\nu^\varepsilon}\right)}\left[f\left(\frac{1}{\nu^\varepsilon}\right)\nu^\varepsilon + \frac{cu^\varepsilon + \beta_\varepsilon(u^\varepsilon-1)}{u^\varepsilon}(\nu^\varepsilon)^2\right],$$

where $f(\cdot)$ is defined in (2.4.41). Notice that

$$cu^\varepsilon + \beta_\varepsilon(u^\varepsilon - 1) \geqslant c + \beta_\varepsilon(0) = 0,$$

and $f(\frac{1}{z}) \geqslant 0$ when $z \geqslant 1/\lambda$, so

$$\nu_x^\varepsilon \geqslant 0 \quad \text{if} \quad \nu^\varepsilon \geqslant 1/\lambda. \tag{2.4.55}$$

Moreover, by $f(\lambda) = 0$, (2.4.54) and

$$\nu^\varepsilon(0,t) = 1/\lambda, \tag{2.4.56}$$

we arrive at

$$\nu_x^\varepsilon(0,t) > 0. \tag{2.4.57}$$

Taking (2.4.56)-(2.4.55) into account, we get $\nu^\varepsilon \geqslant 1/\lambda$, which implies (2.4.53). □

By Lemma 2.5, Lemma 2.6 and Lemma 2.7, we see that $|u_x^\varepsilon| \leqslant \lambda u^\varepsilon \leqslant K\lambda^2 e^{\Lambda T}$ in $\mathcal{Q}_{L,T}$. This provides an upper bound for $|u_x^\varepsilon|$, independent of ε, so u^ε is uniformly Lipschitz continuous in x. Moreover, by Lemma 2.6 and Lemma 2.7 and the monotonicity of A,

$$A\left(-\frac{u_x^\varepsilon}{u^\varepsilon}\right) \geqslant A(\lambda) > 0. \tag{2.4.58}$$

This confirms the uniform parabolic condition in (2.4.46) about the equation of u^ε (where we regard $A\left(-u_x^\varepsilon/u^\varepsilon\right)$ and $B\left(-u_x^\varepsilon/u^\varepsilon\right)$ as known coefficients in the operator $\mathcal{T}$).

Lemma 2.8. *Let $u^\varepsilon \in C^{2,1}\big(\overline{\mathcal{Q}_{L,T}}\big)$ be the solution to Problem (2.4.46), we have*

$$u^\varepsilon_{xx} \geqslant 0. \tag{2.4.59}$$

Proof. By the equation in (2.4.46), Lemma 2.5, Lemma 2.6, $f(\lambda)=0$, and (2.4.58), we have

$$\begin{aligned} A\left(-\frac{u^\varepsilon_x}{u^\varepsilon}\right)u^\varepsilon_{xx} &= u^\varepsilon_t - B\left(-\frac{u^\varepsilon_x}{u^\varepsilon}\right)u^\varepsilon_x + \gamma u^\varepsilon_x + cu^\varepsilon + \beta_\varepsilon(u^\varepsilon-1) \\ &\geqslant \left[-B\left(-\frac{u^\varepsilon_x}{u^\varepsilon}\right)+\gamma\right]u^\varepsilon_x \geqslant \left[-B(\lambda)+\gamma\right]u^\varepsilon_x = -\lambda A(\lambda)u^\varepsilon_x \geqslant 0. \end{aligned}$$

This gives (2.4.59). □

Now we give some uniform norm estimates for u^ε. First, we rewrite the equation of u^ϵ in the divergence form. By (2.4.46),(2.4.38) and (2.4.39), we have

$$u^\epsilon_t - \partial_x\big(A\big(-\frac{u^\epsilon_x}{u^\epsilon}\big)u^\epsilon_x - B\big(-\frac{u^\epsilon_x}{u^\epsilon}\big)u^\epsilon\big) + \gamma u^\epsilon_x + cu^\epsilon + \beta_\epsilon(u^\epsilon - 1) = 0.$$

Since $A(\cdot)$ and $B(\cdot)$ are bounded, by applying $C^{\alpha,\frac{\alpha}{2}}$ estimate (see Lieberman (1996) Theorem 6.33 for the interior estimate and boundary estimate), we have

$$|u^\varepsilon|_{\alpha,\mathcal{Q}_{L,T}} \leqslant C(|u^\varepsilon|_{0,\mathcal{Q}_{L,T}} + |\beta_\varepsilon(\cdot)|_{L_p(\mathcal{Q}_{L,T})} + 1) \leqslant C.$$

We rewrite (2.4.52) as

$$\begin{aligned} \partial_t u^\varepsilon_x - \partial_x\left[A\left(-\frac{u^\varepsilon_x}{u^\varepsilon}\right)\partial_x u^\varepsilon_x\right] - \left\{B\left(-\frac{u^\varepsilon_x}{u^\varepsilon}\right) + B'\left(-\frac{u^\varepsilon_x}{u^\varepsilon}\right)\frac{-u^\varepsilon_x}{u^\varepsilon}\right\}\partial_x u^\varepsilon_x \\ + \left\{-B'\left(-\frac{u^\varepsilon_x}{u^\varepsilon}\right)\left(-\frac{u^\varepsilon_x}{u^\varepsilon}\right)^2 + c\right\}u^\varepsilon_x = -\partial_x\bigg[\beta_\varepsilon(u^\varepsilon-1)\bigg]. \end{aligned}$$

Applying $C^{\alpha,\frac{\alpha}{2}}$ interior (with partial boundary) estimate, we obtain

$$|u^\varepsilon_x|_{\alpha,\mathcal{Q}^{r/2}} \leqslant C(|u^\varepsilon_x|_{0,\mathcal{Q}^{r/4}} + |\beta_\varepsilon(\cdot)|_{L_p(\mathcal{Q}^{r/4})} + 1) \leqslant C,$$

where

$$\mathcal{Q}^r := \mathcal{Q}_{L,T} \setminus \{(x,t) \mid x^2 + t^2 \leqslant r\}.$$

According to Lemma 2.3, $A\left(-u_x^\varepsilon/u^\varepsilon\right)$ and $B\left(-u_x^\varepsilon/u^\varepsilon\right)$ are uniformly $C^{\alpha,\frac{\alpha}{2}}$ in $\mathcal{Q}^{r/2}$. So we can apply the $W_p^{2,1}$ interior estimate (see Lieberman (1996) Theorem 7.13 for the interior estimate and Theorem 7.17 for the boundary estimate) to (2.4.46) in $\mathcal{Q}_{L,T}$ to obtain

$$|u^\varepsilon|_{W_p^{2,1}(\mathcal{Q}^r)} \leqslant C(|u^\varepsilon|_{L_p(\mathcal{Q}^{r/2})} + |\beta_\varepsilon(u^\varepsilon - 1)|_{L_p(\mathcal{Q}^{r/2})} + 1) \leqslant C.$$

We emphasize that Cs in the above estimates are independent of ε, so there exist a $u \in W_{p,\,\mathrm{loc}}^{2,1}(\mathcal{Q}_{L,T}) \bigcap C(\overline{\mathcal{Q}_{L,T}})$ and a subsequence of u^ε (still denoted by u^ε) such that

$$u^\varepsilon \longrightarrow u \quad \text{weakly in} \quad W_p^{2,1}(\mathcal{Q}^r) \quad \text{and uniformly in} \quad C(\overline{\mathcal{Q}_{L,T}}).$$

Now, set

$$v(x,t) = \int_0^x u(y,t)\mathrm{d}y,$$

we come to prove that v is a solution to Problem (2.2.6) in $\mathcal{Q}_{L,T}$. The initial and boundary conditions are clearly satisfied. Owing to $v(0,t) = 0$ and $v_t(0,t) = 0$, together with the boundary condition in (2.4.45) that $(\lambda u + u_x)(0,t) = 0$, we have $(v_t - \mathcal{L}v)(0,t) = 0$. Therefore,

$$\begin{aligned}(v_t - \mathcal{L}v)(x,t) &= (v_t - \mathcal{L}v)(0,t) + \int_0^x \partial_x(v_t - \mathcal{L}v)(y,t)\mathrm{d}t \\ &= \int_0^x (u_t - \mathcal{T}u)(y,t)\mathrm{d}t \geqslant 0.\end{aligned} \tag{2.4.60}$$

On the other hand, if $v_x(x,t) = u(x,t) > 1$, then, by (2.4.50), $u(y,t) > 1$ for all $y \in [0,x]$, which implies $(u_t - \mathcal{T}u)(y,t) = 0$ for $y \in [0,x]$, thus the inequality in (2.4.60) becomes an equality. Hence v satisfies the variational inequality in problem (2.2.6) in $\mathcal{Q}_{L,T}$.

Moreover, the estimates (2.2.9)-(2.2.14) follow from (2.4.48), (2.4.49), (2.4.50), (2.4.53) and (2.4.59).

We are now ready to ascertain the order of smoothness of v. Based on $v_x = u \in W_p^{2,1}(Q^r) \bigcap C(\overline{\mathcal{Q}_{L,T}})$, the Sobolev embedding theorem implies that $v_{xx} = u_x \in C(\overline{\mathcal{Q}^r})$. Moreover, using the method in Friedman (1975), we can prove that $v_{xt} = u_t$ continuously passes through the free boundary.

Next, we prove the uniqueness. Suppose v_1, v_2 are two solutions to (2.2.6). Set $\mathcal{N} = \{\partial_x v_1 > \partial_x v_2\}$, then

$$\begin{cases} \partial_t v_1 - \mathcal{L} v_1 = 0, \quad \partial_t v_2 - \mathcal{L} v_2 \geqslant 0, \quad (x,t) \in \mathcal{N}, \\ v_1 = v_2 = 0, \quad (x,t) \in \partial\mathcal{N} \cap \{x = 0\}, \\ \partial_x v_1 = \partial_x v_2, \quad (x,t) \in \partial\mathcal{N} \setminus (\{x=0\} \cup \{t=0\} \cup \{t=T\}), \\ v_1 = v_2 = x, \quad (x,t) \in \partial\mathcal{N} \cap \{t=0\}. \end{cases}$$

Applying the comparison principle to the fully nonlinear equation (see Lieberman (1996) Theorem 14.3), we have $v_2 \geqslant v_1$ in $\mathcal{N}$, which implies

$$\{\partial_x v_1 > \partial_x v_2\} \subset \{v_2 \geqslant v_1\},$$

i.e.

$$\mathcal{C} := \{v_2 < v_1\} \subset \{\partial_x v_1 \leqslant \partial_x v_2\}.$$

If $\mathcal{C}$ is non-empty, using the fact that $v_2 = v_1$ on the left boundary of $\mathcal{C}$ and $\partial_x v_1 \leqslant \partial_x v_2$ in $\mathcal{C}$, we get $v_1 \leqslant v_2$ in $\mathcal{C}$, which is impossible. This completes the proof of the uniqueness.

Let x_2 be defined in (2.2.18) and choose $L > x_2$. Using a similar argument in Section 2.2.2, we have $v_x(x,t) = 1$ for $x \in [x_2, L]$, then we can extend our solution to the unbounded domain $\mathcal{Q}_T$ by setting $v(t,x) = v(L,t) + (x - L)$ for $x > L$. Then, we can infer that $v \in C^{2,1}(\overline{\mathcal{Q}_T}\ (0,0)) \bigcap C(\overline{\mathcal{Q}_T})$ is a unique solution to (2.2.6) in $\mathcal{Q}_T$. Moreover, the properties (2.2.9)- (2.2.14) remain true in $\mathcal{Q}_T$.

Furthermore, Lemma 2.3 and the equation in $\{v_x > 1\}$ imply $v_{xt} = u_t \in C\big(\{v_x > 1\} \setminus \{(0,0)\}\big)$, so $v_{xt} \in C\big(\overline{\mathcal{Q}_{L,T}} \setminus \{(0,0)\}\big)$. Hence we have

$$v,\ v_x \in C\big(\overline{\mathcal{Q}_T}\big), \quad v_{xx},\ v_t,\ v_{xt} \in C\big(\overline{\mathcal{Q}_T} \setminus \{(0,0)\}\big).$$

This completes the proof.

2.4.2 Proof of Theorem 2.2

Denote $\widehat{V}(x,t) = v(x, T-t)$. We first prove $\widehat{V}(x,t) \geqslant V(x,t)$. For any admissible retained loss policy $\mathbb{H}^t = \{H_s(z)\}_{s\geqslant t}$ and a dividend-payout policy $\mathbb{L}^t = \{L_s\}_{s\geqslant t}$, assume that R_s is the solution to (2.1.3) with the control variables $(\mathbb{H}^t, \mathbb{L}^t)$, and τ is the ruin time of R_s defined by (2.1.4). Then by Itô's formula, it follows that

$$\begin{aligned}\widehat{V}(x,t) = {} & \mathbb{E}\left[e^{-c(T\wedge\tau - t)}\widehat{V}(R_{T\wedge\tau}, T\wedge\tau)\right] \\ & + \mathbb{E}\bigg[\int_t^{T\wedge\tau} e^{-c(s-t)}\bigg(-\widehat{V}_t - \frac{\widehat{V}_{xx}}{2}\int_0^\infty H_s(z)^2 \mathrm{d}F(z) \\ & \qquad - \widehat{V}_x \int_0^\infty H_s(z)\mathrm{d}F(z) + \gamma\widehat{V}_x + c\widehat{V}\bigg)(R_{s-}, s)\mathrm{d}s\bigg] \\ & + \mathbb{E}\bigg[\int_t^{T\wedge\tau} e^{-c(s-t)}\widehat{V}_x(R_{s-}, s)\mathrm{d}L_s^c\bigg] - \mathbb{E}\sum_{t\leqslant s\leqslant T\wedge\tau} e^{-c(s-t)}\big(\widehat{V}(R_s, s) - \widehat{V}(R_{s-}, s)\big),\end{aligned}$$

where L_s^c is the continuous part of L_s. The former two expectation terms on the right hand side are non-negative since $\widehat{V} \geqslant 0$ and $-\widehat{V}_t - \mathcal{L}\widehat{V} \geqslant 0$ by (2.2.6). Meanwhile, since $\widehat{V}_x \geqslant 1$ and $R_s \leqslant R_{s-}$, we have

$$\mathbb{E}\bigg[\int_t^{T\wedge\tau} e^{-c(s-t)}\widehat{V}_x(R_{s-}, s)\mathrm{d}L_s^c\bigg] \geqslant \mathbb{E}\bigg[\int_t^{T\wedge\tau} e^{-c(s-t)}\mathrm{d}L_s^c\bigg],$$

and

$$\widehat{V}(R_s, s) - \widehat{V}(R_{s-}, s) \leqslant R_s - R_{s-} = L_{s-} - L_s.$$

Thus

$$\widehat{V}(x,t) \geqslant \mathbb{E}\left[\int_t^{T\wedge\tau} e^{-c(s-t)}\mathrm{d}L_s^c + \sum_{t\leqslant s\leqslant T\wedge\tau} e^{-c(s-t)}(L_s - L_{s-})\right] = \mathbb{E}\left[\int_t^{T\wedge\tau} e^{-c(s-t)}\mathrm{d}L_s\right].$$

Since the policy pair $(\mathbb{H}^t, \mathbb{L}^t)$ is arbitrarily chosen, it implies $\widehat{V}(x,t) \geqslant V(x,t)$.

We now prove that the reverse inequality $\widehat{V}(x,t) \leqslant V(x,t)$ holds. Define a boundary

$$d^*(s) = \inf\{x \geqslant 0 \mid \widehat{V}_x(x,s) = 1\}, \quad s \in [0,T].$$

Since $\widehat{V}$ is concave in x by hypothesis, it yields

$$\widehat{V}_x(x,s) > 1, \quad \text{if} \quad x < d^*(T-s).$$

By the equation in (2.2.6), it follows that

$$-\widehat{V}_t - \mathcal{L}\widehat{V} = 0, \quad \text{if} \quad x \leqslant d^*(T-s), \tag{2.4.61}$$

where the $C^{2,1}$ continuity of $\widehat{V}$ ensures that the above equation holds at $(d^*(T-s), s)$. Recalling the definitions of I^* and L^* in the statement, we have

$$H_s^*(z, R_{s-}^*) = z - I_s^*(z, R_{s-}^*) = h^*\left(z, -\frac{\widehat{V}_{xx}(R_{s-}^*, s)}{\widehat{V}_x(R_{s-}^*, s)}\right),$$

where h^* is defined by (2.4.33). By combining the feedback controls H^* and L^* with the property of $R_s^* \leqslant d^*(T-s)$ for $s > t$, one can show that (2.1.3) admits a strong solution R_s^*. Moreover R_s^* is continuous for $s > t$. Denote τ^* as the corresponding ruin time.

Now we show that the controls defined above are indeed the optimal controls. By Itô's formula,

$$\widehat{V}(x,t) = \mathbb{E}\left[e^{-c(T\wedge\tau^*-t)}\widehat{V}(R_{T\wedge\tau^*}^*, T\wedge\tau^*)\right]$$

$$
+\mathbb{E}\Bigg[\int_t^{T\wedge\tau^*} e^{-c(s-t)}\Bigg(-\widehat{V}_t-\frac{\widehat{V}_{xx}}{2}\int_0^\infty H_s^*(z,R_{s-}^*)^2\mathrm{d}F(z)
$$

$$
-\widehat{V}_x\int_0^\infty H_s^*(z,R_{s-}^*)\mathrm{d}F(z)+\gamma\widehat{V}_x+c\widehat{V}\Bigg)(R_{s-}^*,s)\mathrm{d}s\Bigg]
$$

$$
+\mathbb{E}\Bigg[\int_t^{T\wedge\tau^*} e^{-c(s-t)}\widehat{V}_x(R_{s-}^*,s)\mathrm{d}L_s^{*c}\Bigg]
$$

$$
-\mathbb{E}\Bigg[\sum_{t\leqslant s\leqslant T\wedge\tau^*} e^{-c(s-t)}(\widehat{V}(R_s^*,s)-\widehat{V}(R_{s-}^*,s))\Bigg]. \tag{2.4.62}
$$

According to the boundary condition in (2.2.6), $R_{T\wedge\tau^*}^*=R_{\tau^*}^*=0$ when $\tau^*<T$ and $\widehat{V}(R_{T\wedge\tau^*}^*,T\wedge\tau^*)=\widehat{V}(R_T^*,T)=0$ otherwise. Therefore, the first expectation in (2.4.62) is zero.

By our choice of H^*, we see that at (R_{s-}^*,s)

$$
-\widehat{V}_t-\frac{\widehat{V}_{xx}}{2}\int_0^\infty H_s^*(z,R_{s-}^*)^2\mathrm{d}F(z)-\widehat{V}_x\int_0^\infty H_s^*(z,R_{s-}^*)\mathrm{d}F(z)+\gamma\widehat{V}_x+c\widehat{V}
$$

$$
=-\widehat{V}_t-\sup_{H\in\mathcal{H}}\left(\frac{\widehat{V}_{xx}}{2}\int_0^\infty H(z)^2\mathrm{d}F(z)+\widehat{V}_x\int_0^\infty H(z)\mathrm{d}F(z)\right)+\gamma\widehat{V}_x+c\widehat{V}=-\widehat{V}_t-\mathcal{L}\widehat{V}.
$$

Since $R_{s-}^*\leqslant d^*(T-s)$ for $s>t$ and by (2.4.61), we conclude that the second expectation in (2.4.62) is zero.

Now we consider

$$
\widehat{V}(x,t)=\mathbb{E}\Bigg[\int_t^{T\wedge\tau^*} e^{-c(s-t)}\widehat{V}_x(R_{s-}^*,s)\mathrm{d}L_s^{*c}\Bigg]-\mathbb{E}\Bigg[\sum_{t\leqslant s\leqslant T\wedge\tau^*} e^{-c(s-t)}(\widehat{V}(R_s^*,s)-\widehat{V}(R_{s-}^*,s))\Bigg].
$$

Note that $\mathrm{d}L_s^{*c}=0$ unless $R_{s-}^*=d^*(T-s)$, thus

$$
\widehat{V}_x(R_{s-}^*,s)\mathrm{d}L_s^{*c}=\widehat{V}_x(d^*(T-s),s)\mathrm{d}L_s^{*c}=\mathrm{d}L_s^{*c}.
$$

Since R_s^* is continuous for $s>t$, we have

$$
\mathbb{E}\Big[\sum_{t\leqslant s\leqslant T\wedge\tau^*} e^{-c(s-t)}(\widehat{V}(R_s^*,s)-\widehat{V}(R_{s-}^*,s))\Big]=\mathbb{E}\Big[\widehat{V}(R_t^*,t)-\widehat{V}(R_{t-}^*,t)\Big],
$$

which indicates

$$\widehat{V}(x,t) = \mathbb{E}\left[\int_t^{T\wedge\tau^*} e^{-c(s-t)}\mathrm{d}L_s^{*c}\right] - \mathbb{E}\Big[\widehat{V}(R_t^*,t) - \widehat{V}(R_{t-}^*,t)\Big]$$

$$= \mathbb{E}\left[\int_t^{T\wedge\tau^*} e^{-c(s-t)}\mathrm{d}L_s\right] - \mathbb{E}\Big[(L_t^* - L_{t-}^*) + \widehat{V}(R_t^*,t) - \widehat{V}(R_{t-}^*,t)\Big].$$

If $R_{t-}^* \leqslant d^*(T-t)$, then $L_t^* - L_{t-}^* = 0$ and $R_t^* = R_{t-}^*$. Consequently,

$$(L_t - L_{t-}) + \widehat{V}(R_t^*,t) - \widehat{V}(R_{t-}^*,t) = 0.$$

If $R_{t-}^* > d^*(T-t)$, then $R_t^* = d^*(T-t)$ holds. Since $\widehat{V}_x(y,t) = 1$ for $y \geqslant d^*(T-t)$, we also obtain

$$(L_t^* - L_{t-}^*) + \widehat{V}(R_t^*,t) - \widehat{V}(R_{t-}^*,t) = (L_t^* - L_{t-}^*) + R_t^* - R_{t-}^* = 0.$$

Now we conclude that

$$\widehat{V}(x,t) = \mathbb{E}\left[\int_t^{T\wedge\tau^*} e^{-c(s-t)}\mathrm{d}L_s^*\right],$$

of which the right hand side is no more than $V(x,t)$ by definition. This completes the proof.

2.4.3 Proof of Lemma 2.1.

The equation (2.2.6) gives

$$v_t - A\left(-\frac{v_{xx}}{v_x}\right)v_{xx} - B\left(-\frac{v_{xx}}{v_x}\right)v_x + \gamma v_x + cv = 0 \quad \text{in} \quad \mathcal{ND}. \tag{2.4.63}$$

Differentiating (2.4.63) w.r.t. x and t respectively and combining (2.4.38), we obtain

$$v_{tx} - \left[A\left(-\frac{v_{xx}}{v_x}\right)v_{xxx} + B\left(-\frac{v_{xx}}{v_x}\right)v_{xx}\right] + \gamma v_{xx} + cv_x = 0 \quad \text{in} \quad \mathcal{ND}, \tag{2.4.64}$$

and

$$v_{tt} - \left[A\left(-\frac{v_{xx}}{v_x}\right)v_{xxt} + B\left(-\frac{v_{xx}}{v_x}\right)v_{xt}\right] + \gamma v_{xt} + cv_t = 0 \quad \text{in} \quad \mathcal{ND}. \quad (2.4.65)$$

Since

$$0 < A(\lambda) \leqslant A\left(-\frac{v_{xx}}{v_x}\right) \leqslant \mu_2, \quad B\left(-\frac{v_{xx}}{v_x}\right) \leqslant \mu_2$$

and

$$A\left(-\frac{v_{xx}}{v_x}\right),\ B\left(-\frac{v_{xx}}{v_x}\right) \in C^{\alpha,\alpha/2}(\mathcal{Q}_T)$$

owing to v_x, $v_{xx} \in C^{\alpha,\alpha/2}(\mathcal{Q}_T)$ and Remark 2.3, we can apply the Schauder estimate (see Lieberman (1996) Theorem 4.23) to (2.4.64) and (2.4.65), respectively, to obtain

$$v_x,\ v_t \in C^{2+\alpha,1+\alpha/2}(\mathcal{ND}).$$

Suppose $v_{xx}(x_0, t_0) = 0$ for some $(x_0, t_0) \in \mathcal{ND}$, since $v_{xx} \leqslant 0$, (x_0, t_0) is a maximizer point for v_{xx}, the first order condition gives $v_{xxx}(x_0, t_0) = 0$. By (2.4.64),

$$v_{tx}(x_0, t_0) + cv_x(x_0, t_0) = 0,$$

which contradicts (2.2.9) and (2.2.13). Therefore $v_{xx} < 0$ on $\mathcal{ND}$.

2.4.4 Proof of Lemma 2.2.

Equation (2.2.23) is derived from the boundary condition in (2.4.45). Then (2.2.24) is an immediate consequence of (2.2.23) and (2.2.25). So it is only left to prove (2.2.25). In $\mathcal{ND}$, the equation in (2.4.45) holds, i.e.,

$$v_{xt} - A\left(-\frac{v_{xx}}{v_x}\right)v_{xxx} - B\left(-\frac{v_{xx}}{v_x}\right)v_{xx} + \gamma v_{xx} + cv_x = 0.$$

Since

$$v_{xxx} = \left[\partial_x\left(-\frac{v_x}{v_{xx}}\right) + 1\right]\frac{v_{xx}^2}{v_x},$$

it follows that

$$v_{xt} - A\left(-\frac{v_{xx}}{v_x}\right)\left[\partial_x\left(-\frac{v_x}{v_{xx}}\right)+1\right]\frac{v_{xx}^2}{v_x} - B\left(-\frac{v_{xx}}{v_x}\right)v_{xx} + \gamma v_{xx} + cv_x = 0.$$

Dividing v_x on both sides yields

$$-A\left(-\frac{v_{xx}}{v_x}\right)\left[\partial_x\left(-\frac{v_x}{v_{xx}}\right)+1\right]\frac{v_{xx}^2}{v_x^2} - B\left(-\frac{v_{xx}}{v_x}\right)\frac{v_{xx}}{v_x} + \gamma\frac{v_{xx}}{v_x} + c = -\frac{v_{xt}}{v_x} \leqslant 0,$$

by (2.2.9) and (2.2.13). Denote $\nu = -\frac{v_x}{v_{xx}}$, the above inequality can be expressed as

$$\frac{A\left(\frac{1}{\nu}\right)(\nu_x+1)}{2\nu^2} - \frac{B\left(\frac{1}{\nu}\right)}{\nu} + \frac{\gamma}{\nu} - c \geqslant 0.$$

Hence,

$$\begin{aligned}\nu_x &\geqslant \frac{1}{A(1/\nu)}\left[\left(-\frac{1}{\nu}A\left(\frac{1}{\nu}\right) + B\left(\frac{1}{\nu}\right) - \gamma\right)\nu + c\nu^2\right] \\ &= \frac{1}{A(1/\nu)}\left[f\left(\frac{1}{\nu}\right)\nu + c\nu^2\right].\end{aligned} \tag{2.4.66}$$

By (2.2.14), we have $\nu \geqslant 1/\lambda > 0$. In addition, according to (2.4.42), f is decreasing, so $f(1/\nu) \geqslant f(\lambda) = 0$. Together with (2.4.37), we obtain $\nu_x \geqslant 2c$. This completes the proof.

2.5 Concluding Remarks

We formulate a reinsurance and dividend-payout problem as a mixed classical-singular control problem. The dynamic programming method and free boundary PDE approach are employed to solve the problem and we derive the optimal reinsurance and dividend payout strategies. The model studied in this chapter can be explored further in different directions. For example, it would be challenging to generalize our results to a general premium principle, such as the Wang's premium principle (Wang et al. (1997)) or the Mean-CVaR premium principle (Tan et al. (2020)). Besides, we can consider other objectives such as the rank-dependent utility objective as in Xu et al. (2019); Xu (2021).

Chapter 3

Discretionary Stopping in an Investment Problem with Option Compensation

In this chapter, we solve a dynamic investment and stopping problem for a mutual fund manager in a complete continuous-time market. In particular, the risk averse manager can manage her account by asset allocation and decide when to exit the portfolio in her own right. She is compensated with a call option on the assets and aims to maximize the utility of her compensation at the stopping time. Therefore, her risk preference is measured by the composite function of a standard concave utility function and a non-smooth and non-concave function, which is firstly proposed by Carpenter (2000) to study an investor's investment risk appetite.

We apply the dynamic programming method to solve the optimization problem. The technical contribution of this chapter is a detailed analysis of the existence of the optimal stopping time with an exponential (constant absolute risk aversion) utility. In addition to mathematical treatment with the problem, we also present economic implications to illustrate the impact of market coefficients on the optimal selling (stopping) point.

3.1 Model

3.1.1 Market

A mutual fund manager allocates assets in a complete and arbitrage-free financial market where $n+1$ assets are continuously traded. One of the assets is a risk-free bond with instantaneously interest rate r. The price $S_0(\cdot)$ evolves according to the ordinary differential equation

$$\begin{cases} \mathrm{d}S_0(t) = rS_0(t)\mathrm{d}t, \ t \geqslant 0 \\ S_0(0) = s_0 > 0. \end{cases}$$

The remaining n assets are stocks, and the prices $S_i(\cdot), i = 1, 2, \cdots, n$ are modeled by the system of stochastic differential equations

$$\begin{cases} \mathrm{d}S_i(t) = S_i(t)\{(r+\mu_i)dt + \sum_{j=1}^{n} \sigma_{ij} dW^j(t)\}, \quad t \geqslant 0, \\ S_i(0) = s_i > 0, \end{cases}$$

where $\mu := (\mu_1, \mu_2, \cdots, \mu_n)^{\mathrm{T}}$ is the risk premium rate, $\sigma := (\sigma_{ij})_{n\times n}$ is the volatility coefficient and $W(t) := (W^1(t), W^2(t), \cdots, W^n(t))^{\mathrm{T}}$ is a standard n-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We assume that $\sigma'\sigma > \epsilon I_{n\times n}$, where $\epsilon > 0$.

We refer to an n-dimensional process $\pi(t) := (\pi_1(t), \cdots, \pi_n(t))^{\mathrm{T}}$ as a portfolio of the mutual fund manager, where i-th component is the dollar amount allocated in i-th stock at time t. An admissible strategy $\pi(t)$ is progressively measurable with respect to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ such that $X(t) \geqslant 0$. Note that $X(t) = \pi_0(t) + \sum_{i=1}^{n} \pi_i(t)$, $\pi_0(t)$ is the dollar amount traded in a risk-free bond. Thus, the wealth process X_t satisfies:

$$\mathrm{d}X(t) = (rX(t) + \mu^{\mathrm{T}}\pi(t))\mathrm{d}t + \pi(t)^{\mathrm{T}}\sigma\mathrm{d}W(t), \qquad X(0) = x_0, \tag{3.1.1}$$

where π_t belongs to the admissible strategy set Π

$$\Pi := \{\pi(t) \in \mathcal{L}^2_{\mathcal{F}}([0,\infty); \mathbb{R}^n) | X(t) \geqslant 0\}.$$

The aim of the manager is to determine an optimal asset allocation strategy $\pi(t)$ and find the optimal stopping time τ to maximize the expected discounted utility of her compensation package which includes base payment and incentive salary. The incentive salary is analogous to a call option on the total assets

$$\sup_{\tau,\pi} \mathbb{E}\Big[e^{-c\tau} U(\alpha(X(\tau) - R)^+ + B)\Big],$$

where $\alpha > 0$ represents the commission ratio, $B > 0$ stands for the base payment, the discount factor is denoted by $c > 0$ and $U(\cdot) : [0, \infty) \to \mathbb{R}$ is a strictly increasing and concave function defined by

$$U(x) = 1 - \exp(-\beta x),$$

with a constant $\beta > 0$ representing the degree of risk aversion.

3.1.2 HJB equation

The value function is defined as

$$v(x) = \sup_{\tau,\pi} \mathbb{E}\Big[e^{-c\tau} U(\alpha(X(\tau) - R)^+ + B)\Big]. \tag{3.1.2}$$

Applying the dynamic programming principle, we obtain that the value function satisfies the variational inequality

$$\begin{cases} \min\{-\max_{\pi}[\frac{1}{2}\pi^T \sigma\sigma^T \pi v''(x) + \mu^T \pi v'(x)] - rxv'(x) + cv(x), \\ \qquad\qquad v(x) - U(\alpha(x - R)^+ + B)\} = 0, \quad x > 0, \\ v(0) = U(B). \end{cases} \tag{3.1.3}$$

It follows from the definition of (3.1.2) that v is increasing in x and the Hamilton operator

$$-\max_{\pi}[\frac{1}{2}\pi^T \sigma\sigma^T \pi v''(x) + \mu^T \pi v'(x)] - rxv'(x),$$

is singular if $v''(x) \geqslant 0$. Intuitively, we try to find a solution such that

$$v''(x) < 0, \quad v'(x) > 0, \quad x > 0. \tag{3.1.4}$$

Note that the gradient of $\pi^T \sigma \sigma^T \pi$ with respect to π is

$$\nabla_\pi (\pi^T \sigma \sigma^T \pi) = 2\sigma\sigma^T \pi,$$

so the optimal trading strategy is

$$\pi^* = -(\sigma\sigma^T)^{-1} \mu \frac{v'(x)}{v''(x)}.$$

Define $\|\theta\|^2 = \mu^T (\sigma\sigma^T)^{-1} \mu$, the variational inequalities can be formulated as follows

$$\begin{cases} \min\{\frac{1}{2}\frac{v'(x)^2}{v''(x)}\|\theta\|^2 - rxv'(x) + cv(x), v(x) - U(\alpha(x-R)^+ + B)\} = 0, \quad x > 0, \\ v(0) = U(B). \end{cases}$$

Suppose that there exists a free boundary point $\hat{x} > B$ such that

$$\begin{cases} v(x) > U(\alpha(x-R)^+ + B), & 0 < x < \hat{x}, \\ v(x) = U(\alpha(x-R) + B), & B < \hat{x} \leqslant x. \end{cases}$$

Then the problem can be rewritten as

$$\begin{cases} \frac{1}{2}\frac{(v'(x))^2}{v''(x)}\|\theta\|^2 - rxv'(x) + cv(x) = 0, & 0 < x < \hat{x}, \\ v(x) = U(\alpha(x-R) + B), & R < \hat{x} \leqslant x, \\ v(0) = U(B). \end{cases} \tag{3.1.5}$$

We will show that problem (3.1.5) has a free boundary point $\hat{x}$ under the exponential utility function and a solution $v(x)$ satisfies (3.1.4). We also present a verification theorem to prove the solution is $v(x)$ defined in (3.1.2).

3.2 Dual Problem

In this section, we formulate the dual problem first and then use inverse dual transformation to obtain a solution to the original problem.

Define a dual transformation of $v(x)$ by

$$w(y) := \max_{x \geqslant 0}(v(x) - xy), \quad y > 0.$$

For any fixed y, $v''(x) < 0$ leads to an optimal x at $x(y) := v_x^{-1}(y)$. Denote $I(y) \equiv v_x^{-1}(y)$ such that

$$w(y) = v(I(y)) - I(y)y, \quad y \in [\lim_{x \to +\infty} v'(x), v'(0)]. \tag{3.2.6}$$

Taking derivative on both sides yields

$$\begin{aligned} w'(y) &= v'(I(y))I'(y) - I(y) - yI'(y) = -I(y) = -x \leqslant 0, \\ w''(y) &= -I'(y) = \frac{-1}{v''(I(y))} > 0, \end{aligned} \tag{3.2.7}$$

which implies that $w(y)$ is strictly convex and decreasing on $y \in [\lim_{x \to +\infty} v'(x), v'(0)]$.

For any $x > 0$, there exists $y(x) = v'(x)$. Recall that (3.1.5), we deduce the ODE for w as follows:

$$\begin{cases} \frac{1}{2}\|\theta\|^2 y^2 w''(y) + (c - r)yw'(y) - cv(y) = 0, & \hat{y} < y < y(0), \\ w(y) - yw'(y) = U(\alpha(-w'(y) - R) + B), & \lim_{x \to +\infty} v'(x) < y \leqslant \hat{y}, \end{cases}$$

where $y(0) = v'(0)$ and $\hat{y} = v'(\hat{x})$. Combining (3.2.6) with the first expression in (3.2.7), we have

$$\begin{cases} w(y(0)) = v(0), \\ w'(y(0)) = 0. \end{cases}$$

Suppose $v'(x)$ is continuous at $x = \hat{x}$, by smooth pasting condition, we get

$$\begin{cases} \hat{y} = U_x(\alpha(x(\hat{y}) - R) + B), \\ w(\hat{y}) - yw'(\hat{y}) = U(\alpha(-v'(\hat{y}) - R) + B). \end{cases}$$

Based on the above analysis, we have the following lemmas.

Lemma 3.1. *The function $w(y)$ satisfies the equation*

$$\frac{1}{2}\|\theta\|^2 y^2 w''(y) + (c - r)yw'(y) - cw(y) = 0, \quad \hat{y} < y < y(0), \tag{3.2.8}$$

with boundary conditions

$$w(y(0)) = U(B), \tag{3.2.8a}$$

$$w'(y(0)) = 0, \tag{3.2.8b}$$

$$w(\hat{y}) - \hat{y}w'(\hat{y}) = U(\alpha(-v'(\hat{y}) - R) + B), \tag{3.2.8c}$$

$$\hat{y} = \beta\alpha \exp(-\beta(\alpha(-w'(\hat{y}) - R) + B)). \tag{3.2.8d}$$

Lemma 3.2. *If $B - \alpha R < 0$, the solution of $w(y)$ is unique.*

Proof. The general solution of (3.2.8) has the form as follows:

$$w(y) = C_1 y^{k_1} + C_2 y^{k_2}, \quad y \in [\hat{y}, y(0)],$$

where

$$\begin{cases} k_1 = \dfrac{-(c - r - \frac{1}{2}\|\theta\|^2) + \sqrt{(c - r - \frac{1}{2}\|\theta\|^2)^2 + 2\|\theta\|^2 c}}{\|\theta\|^2}, \\ k_2 = \dfrac{-(c - r - \frac{1}{2}\|\theta\|^2) - \sqrt{(c - r - \frac{1}{2}\|\theta\|^2)^2 + 2\|\theta\|^2 c}}{\|\theta\|^2}. \end{cases}$$

Following from the fact that $(c - r + \frac{1}{2}\|\theta\|^2)^2 \leqslant (c - r - \frac{1}{2}\|\theta\|^2)^2 + 2\|\theta\|^2 c$, we obtain $k_1 > 1$ and $k_2 < 0$.

Substituting $w(y)$ to (3.2.8a) and (3.2.8b) yields

$$C_1 = \frac{-k_2 U(B)}{(k_1 - k_2) y(0)^{k_1}}, \quad C_2 = \frac{k_1 U(B)}{(k_1 - k_2) y(0)^{k_2}}.$$

Hence,

$$w(y) = \frac{k_2(\exp(-\beta B) - 1)}{(k_1 - k_2)} \Big(\frac{y}{y(0)}\Big)^{k_1} - \frac{k_1(\exp(-\beta B) - 1)}{(k_1 - k_2)} \Big(\frac{y}{y(0)}\Big)^{k_2}, \quad y \in [\hat{y}, y(0)],$$

Substituting $w(y)$ to (3.2.8c), we have

$$\frac{(k_1 - 1)k_2 U(B)}{(k_1 - k_2)} \Big(\frac{\hat{y}}{y(0)}\Big)^{k_1} + \frac{(1 - n_2)n_1 U(B)}{(k_1 - k_2)} \Big(\frac{\hat{y}}{y(0)}\Big)^{k_2} = U(\alpha(-w'(\hat{y}) - R) + B). \tag{3.2.10}$$

Following from the definition of $U(x)$, we have

$$\begin{aligned} &\frac{(k_1 - 1)k_2 U(B)}{(k_1 - k_2)} \Big(\frac{\hat{y}}{y(0)}\Big)^{k_1} + \frac{(1 - k_2)k_1 U(B)}{(k_1 - k_2)} \Big(\frac{\hat{y}}{y(0)}\Big)^{k_2} \\ =&1 - e^{-\beta\left\{\alpha\left[\frac{k_1 k_2 U(B)}{(k_1 - k_2)y(0)}\left(\frac{\hat{y}}{y(0)}\right)^{k_1 - 1} - \frac{k_1 k_2 U(B)}{(k_1 - k_2)y(0))}\left(\frac{\hat{y}}{y(0)}\right)^{k_2 - 1} - R\right] + B\right\}}. \end{aligned}$$

Rearranging the above equation yields

$$\begin{aligned} &1 - \frac{(k_1 - 1)k_2 U(B)}{(k_1 - k_2)} \Big(\frac{\hat{y}}{y(0)}\Big)^{k_1} - \frac{(1 - k_2)k_1 U(B)}{(k_1 - k_2)} \Big(\frac{\hat{y}}{y(0)}\Big)^{k_2} \\ =&e^{-\beta\left\{\alpha\left[\frac{k_1 k_2 U(B)}{(k_1 - k_2)y(0)}\left(\frac{\hat{y}}{y(0)}\right)^{k_1 - 1} - \frac{k_1 k_2 U(B)}{(k_1 - k_2)y(0)}\left(\frac{\hat{y}}{y(0)}\right)^{k_2 - 1} - R\right] + B\right\}}. \end{aligned}$$

Note that $U(x(y)) < 1$, together with (3.2.10), we have

$$1 - \frac{(k_1 - 1)k_2 U(B)}{(k_1 - k_2)} \Big(\frac{\hat{y}}{y(0)}\Big)^{k_1} - \frac{(1 - k_2)k_1 U(B)}{(k_1 - k_2)} \Big(\frac{\hat{y}}{y(0)}\Big)^{k_2} > 0. \tag{3.2.11}$$

Taking the logarithm on both sides, $y(0)$ can be expressed by $\frac{\hat{y}}{y(0)}$ such that

$$y(0)=\frac{-\alpha\beta\big[\frac{k_1k_2U(B)}{(k_1-k_2)}\big(\frac{\hat{y}}{y(0)}\big)^{k_1-1}-\frac{k_1k_2U(B)}{(k_1-k_2)}\big(\frac{\hat{y}}{y(0)}\big)^{k_2-1}\big]}{\ln\Big(1-\frac{(k_1-1)k_2(1-e^{-\beta B})}{(k_1-k_2)}\big(\frac{\hat{y}}{y(0)}\big)^{k_1}-\frac{(1-k_2)k_1(1-e^{-\beta B})}{(k_1-k_2)}\big(\frac{\hat{y}}{y(0)}\big)^{k_2}\Big)-\beta\alpha R+\beta B}. \tag{3.2.12}$$

Combining (3.2.8c) with (3.2.8d), we have

$$\begin{aligned}\frac{w(\hat{y})-yw'(\hat{y})}{\hat{y}} &= \frac{1-e^{-\beta(\alpha(-w'(\hat{y})-R)+B)}}{\beta\alpha e^{-\beta(\alpha(-w'(\hat{y})-R)+B)}}\\ &= \frac{1}{\beta\alpha}e^{\beta(\alpha(-w'(\hat{y})-R)+B)}-\frac{1}{\beta\alpha},\end{aligned}$$

which implies

$$\begin{aligned}\hat{y}=&\Big[\beta\alpha\Big(\frac{(k_1-1)k_2U(B)}{(k_1-k_2)}\big(\frac{\hat{y}}{y(0)}\big)^{k_1}+\frac{(1-k_2)k_1U(B)}{(k_1-k_2)}\big(\frac{\hat{y}}{y(0)}\big)^{k_2}\Big)\Big]\\ &\cdot\Big[\frac{1}{e^{\beta(\alpha(-w'(\hat{y})-R)+B))}-1}\Big].\end{aligned} \tag{3.2.13}$$

Define $\eta=\frac{\hat{y}}{y(0)}$ and $Q(\eta):=(k_1-1)k_2\eta^{k_1}+(1-k_2)k_1\eta^{k_2}$. It follows from (3.2.11) that

$$Q(\eta)<\frac{k_1-k_2}{U(B)},$$

it is easy to verify

$$Q(0)=+\infty,\quad Q'(\eta)<0,\quad \eta>0.$$

Then we define

$$\hat{\eta}:=\big\{\eta|Q(\eta)=\frac{k_1-k_2}{U(B)}\big\}.$$

Note that $0<\hat{y}<y(0)$ and $\hat{\eta}<\eta<1$. Since $k_1>1$ and $k_2<0$, we have

$0 < \eta^{k_1-1} < \eta^{k_2-1}$. Divided (3.2.12) by (3.2.13), we show that

$$\begin{aligned} & k_1k_2(\eta^{k_1} - \eta^{k_2}) \\ = & -\Big[\ln\Big(1 - \frac{(k_1-1)k_2U(B)}{(k_1-k_2)}\Big(\frac{\hat{y}}{y(0)}\Big)^{k_1} - \frac{(1-k_2)k_1U(B)}{(k_1-k_2)}\Big(\frac{\hat{y}}{y(0)}\Big)^{k_2}\Big) - \beta\alpha R + \beta B\Big] \\ & \cdot\Big[(k_1-1)k_2\eta^{k_1} + (1-k_2)k_1\eta^{k_2}\Big] \times \frac{1}{e^{\beta(\alpha(-w'(\hat{y})-R)+B)} - 1}, \end{aligned}$$

where $w'(\hat{y}) = -\frac{U(B)k_1k_2}{k_1-k_2}(\eta^{k_1-1} - \eta^{k_2-1})$.

Let

$$\begin{aligned} P(\eta) = & k_1k_2\big[\eta^{k_1} - \eta^{k_2}\big] + \Big(\ln(1 - \frac{U(B)}{(k_1-k_2)}Q(\eta)) - \beta\alpha R + \beta B\Big) \times Q(\eta) \\ & \times \frac{1}{\exp\big\{\beta B - \beta\alpha R\big\}\exp\big\{\beta\alpha\frac{U(B)}{k_1-k_2}k_1k_2[\eta^{k_1} - \eta^{k_2}]\big\} - 1}. \end{aligned}$$

When $\beta B - \beta\alpha R < 0$, we have $P(\hat{\eta}+) = -\infty$, $P(1) = \beta\alpha B(k_1-k_2)\frac{1}{1-e^{\beta B - \beta\alpha R}} > 0$ and $P'(\eta) > 0$, which imply $P(\eta)$ admits a unique solution $\bar{\eta}$ such that

$$P(\bar{\eta}) = 0.$$

By (3.2.12) and (3.2.13), we then have

$$y(0) = \frac{-\alpha\beta\big[\frac{k_1k_2U(B)}{(k_1-k_2)}\eta^{*k_1-1} - \frac{k_1k_2U(B)}{(k_1-k_2)}\eta^{*n_2-1}\big]}{\ln\Big(1 - \frac{(n_1-1)n_2(1-e^{-\beta K})}{(k_1-k_2)}\bar{\eta}^{k_1} - \frac{(1-n_2)n_1(1-e^{-\beta K})}{(n_1-n_2)}\hat{\eta}^{k_2}\Big) - \beta\alpha R + \beta B},$$

and

$$\begin{aligned} \hat{y} = & \Big[\beta\alpha\Big(\frac{(k_1-1)k_2U(K)}{(n_1-n_2)}\hat{\eta}^{k_1} + \frac{(1-k_2)k_1U(B)}{(k_1-k_2)}\bar{\eta}^{k_2}\Big)\Big] \\ & \cdot\Big[\frac{1}{e^{\beta(\alpha(\frac{U(B)k_1k_2}{k_1-k_2}(\bar{\eta}^{k_1-1}-\bar{\eta}^{k_2-1})-R)+B))} - 1}\Big]. \end{aligned}$$

Thus, we have $w(y)$ over $[\hat{y}, y(0)]$. $\square$

3.3 Solution

In this section, we continue to find the free boundary of the original problem.

3.3.1 Free boundary

We proceed to derive $v(x)$ by employing $w(x)$.

$$v(x) = \min_{y \in [\hat{y}, y(0)]} [C_1 y^{k_1} + C_2 y^{k_2} + xy], \quad x \in (0, \hat{x}), \tag{3.3.14}$$

where

$$\hat{x} = -w'(\hat{y}) = \frac{k_1 k_2 (1 - e^{-\beta B})}{(k_2 - k_1) y(0)} [\bar{\eta}^{k_1 - 1} - \bar{\eta}^{k_2 - 1}]. \tag{3.3.15}$$

Denote $v'(x) = y_x$, the minimum y_x satisfies

$$C_1 k_1 y_x^{k_1 - 1} + C_2 k_2 y_x^{k_2 - 1} + x = 0, \quad x \in (0, \hat{x}).$$

Hence, we can obtain y_x for any $x \in (0, \hat{x})$. Then substituting it into (3.3.14) such that

$$v(x) = \begin{cases} C_1 y_x^{k_1} + C_2 y_x^{k_2} + x y_x, & x \in (0, \hat{x}), \\ 1 - e^{-\beta(\alpha(x - R) + B)}, & x \in [\hat{x}, \infty). \end{cases}$$

It is easy to check that

$$v'(x) = y_x > 0 \quad v''(x) < 0,$$

for any $x \in (0, \infty)$, which implies that $v(x)$ is increasing and concave and the optimal investment strategy π^* can be expressed accordingly. Moreover, the free boundary point $\hat{x}$ is given by $-w'(\hat{y})$ in (3.3.15). Thus, we can obtain $(\pi^*, \hat{x})$ in the original problem.

3.3.2 Verification theorem

Theorem 3.1. *Suppose $v(x)$ is the solution to HJB (3.1.5), then for any admissible π and τ, we have*

$$v(x) \geqslant \mathbb{E}[e^{-c\tau}U(\alpha(X(\tau)-R)^+ + B)],$$

the optimal strategy pair (π^*, τ^*) for problem (3.1.2) is

$$\begin{cases} \tau^* = \inf\{t > 0 : X(t) \geqslant \hat{x}\}, \\ \pi^* = -(\sigma\sigma^T)^{-1}\mu\frac{v'(x)}{v''(x)}, \end{cases}$$

and

$$v(x) = J_{\tau^*,\pi^*}(x).$$

Proof. For any admissible π and stopping time $\tau \geqslant 0$, with the corresponding state trajectory $X(\cdot)$, it follows from Itô formula that

$$\begin{aligned}
v(x) &= \mathbb{E}\big[e^{-c(\tau\wedge T)}v(X_{\tau\wedge T})\big] - \mathbb{E}\int_0^{\tau\wedge T} e^{-ct}\big[-cv(X_t) + (rX_t + \mu^T\pi)v'(X_t) \\
&\quad + \frac{1}{2}\pi^T(\sigma\sigma^T)\pi v''(X_t)\big]dt \\
&\geqslant \mathbb{E}[e^{-c(\tau\wedge T)}v(X_{\tau\wedge T})] + \mathbb{E}\int_0^{\tau\wedge T} e^{-ct}[cv(X_t) - rX_tv'(X_t) - \sup_{\pi}\big(\mu^T\pi v'(X_t)\big) \\
&\quad + \frac{1}{2}\pi^T(\sigma\sigma^T)\pi v''(X_t)]dt \\
&\geqslant \mathbb{E}\big[e^{-c(\tau\wedge T)}U\big(\alpha(X_{\tau\wedge T} - R)^+ + B\big)\big],
\end{aligned}$$

where the inequality is derived by applying HJB equation (3.1.3). Taking T to infinity and applying Fatou's Lemma give

$$v(x) \geqslant \lim_{T\to\infty}\mathbb{E}[e^{-c(\tau\wedge T)}U(\alpha(X_{\tau\wedge T} - R)^+ + B)] \geqslant J_{\pi,\tau}(x). \tag{3.3.16}$$

Next, applying the equality of (τ^*, π^*), we get

$$
\begin{aligned}
v(x) &= \mathbb{E}\big[e^{-c(\tau^*\wedge T)}v(X^*_{\tau^*\wedge T})\big] \\
&\quad + \mathbb{E}\int_0^{\tau^*\wedge T} e^{-ct}\big[-cv(X_t^*) + (rX_t^* + \mu^T\pi^*)v'(X_t^*) + \frac{1}{2}\pi^{*T}(\sigma\sigma^T)\pi^* v''(X_t^*)\big]dt \\
&= \mathbb{E}\big[e^{-c(\tau^*\wedge T)}v(X^*_{\tau^*\wedge T})\big] \\
&= \mathbb{E}\big[e^{-c(\tau^*\wedge T)}v(X^*_{\tau^*\wedge T})\mathbb{1}_{\{\tau^*\leqslant T\}} + e^{-c(\tau^*\wedge T)}v(X^*_{\tau^*\wedge T})\mathbb{1}_{\{\tau^*>T\}}\big] \\
&= \mathbb{E}\big[e^{-c\tau^*}U(\alpha(X^*_{\tau^*} - R)^+ + B)\mathbb{1}_{\{\tau^*>T\}}\big] + \mathbb{E}\big[e^{-cT}v(X_T^*)\mathbb{1}_{\{\tau^*>T\}}\big] \\
&\leqslant \mathbb{E}\big[e^{-c\tau^*}U(\alpha(X^*_{\tau^*} - R)^+ + B)\big] + \mathbb{E}\big[e^{-cT}v(X_T^*)\mathbb{1}_{\{\tau^*>T\}}\big].
\end{aligned}
$$

Consider that $X_T^* \leqslant \hat{x}$ when $\tau^* > T$. Due to $v'(x) > 0$, we have $v(X_T^*) < v(\hat{x})$, thus

$$v(x) \leqslant J_{\tau^*,\pi^*}(x) + e^{-cT}v(\hat{x}).$$

Taking T to infinity and combining (3.3.16) yield

$$v(x) = J_{\tau^*,\pi^*}(x).$$

□

3.4 Examples

In this section, we study how the optimal selling point $\hat{x}$ varies as α, R and K change. Let $r = 0.025$, $\mu = (0.08, 0.125, 0.15)^{\mathrm{T}}$, $\beta = 0.5$ and the volatility matrix

$$\sigma = \begin{pmatrix} 0.15 & 0 & 0 \\ 0.18 & 0.25 & 0 \\ 0.24 & -0.265 & 0.32 \end{pmatrix}$$

Case 1: Let $0.05 \leqslant \alpha \leqslant 0.1$, $R = 600$ and $B = 20$.

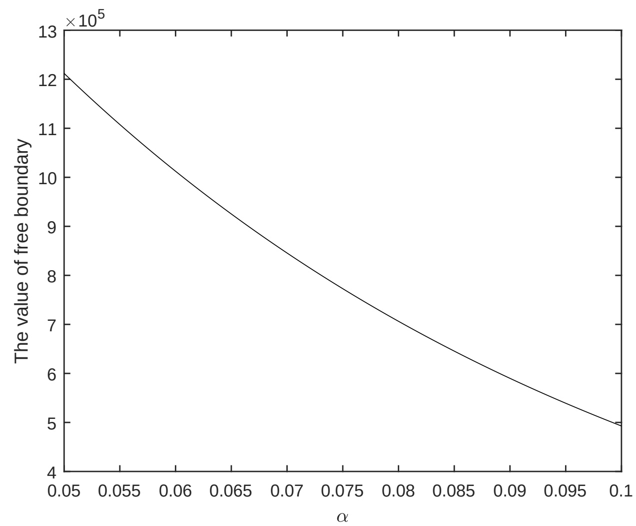

Figure 3.1: The free boundary $\hat{x}$ with commission ratio α

Case 2: Let $\alpha = 0.06$, $500 \leqslant R \leqslant 650$ and $B = 20$.

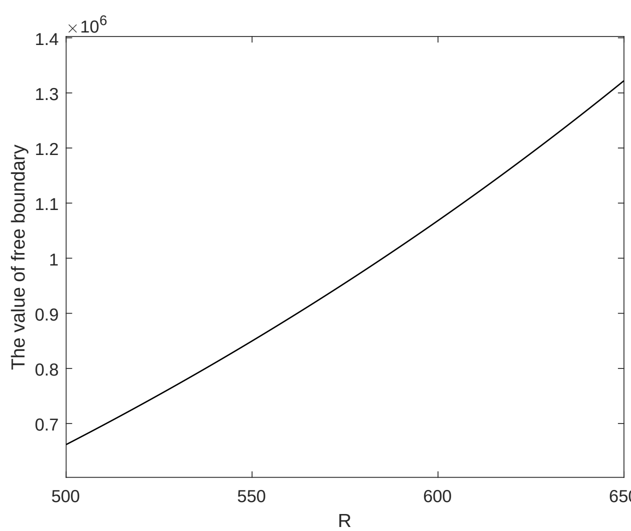

Figure 3.2: The free boundary $\hat{x}$ with reference level R

Case 3: Let $\alpha = 0.06$, $R = 650$ and $15 \leqslant B \leqslant 25$.

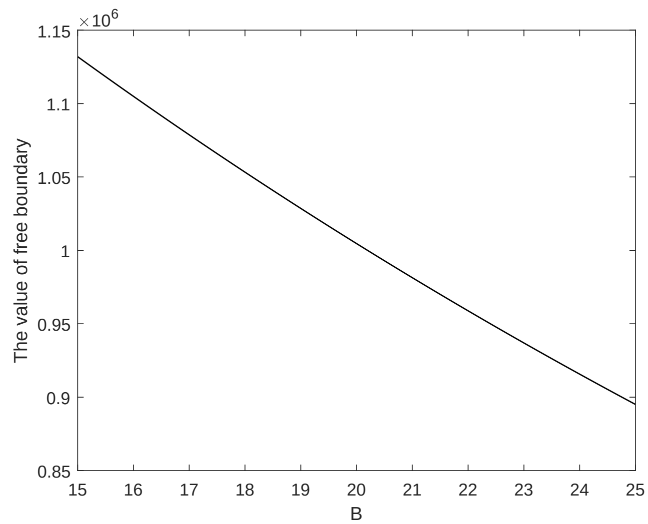

Figure 3.3: The free boundary $\hat{x}$ with base salary B

We can see that the free boundary $\hat{x}$ grows exponentially as the commission ratio α and the base payment B increase. Besides, it exhibits a declining trend with respect to the reference performance level R. It is consistent with the economic intuition. As α and B increase, even if the mutual fund manager lowers her relative performance, she is still able to obtain a satisfactory incentive pay or management fee due to the fact that she has a relatively high commission ratio or base payment. However, with the benchmark performance level rising, the manager has to seek a higher selling price for a better profit.

3.5 Concluding Remarks

We formulate a fund management problem as an optimal stopping problem. We apply the dual approach to obtain closed-form solutions. To maintain the analytical tractability, we consider an infinite-time horizon case and our setting allows us to calculate the results explicitly. It is interesting to generalize our model into a finite-time horizon case to match the industry practice. In addition, one can adopt other preferences such as the recursive utility in Epstein and Zin (1991).

Chapter 4

Optimal Redeeming Strategy from an Active Portfolio Management

This chapter continues the line of research on an optimal stopping problem. We provide a continuous-time investment and exit decision model in a principal-agent setting. Suppose that the portfolio manager of an equity trust as well as agent has an active management scheme with a power utility maximization objective. From the perspective of the investor as well as principal, we mainly focus on determining the optimal redeeming time of the portfolio to minimize the relative error between the transfer price and the high-water mark over the given period for the sake of liquidity requirements.

We explore a non-standard stopping problem in the sense that the high-water mark is not adapted to the current information and convert it to a standard problem by stochastic analysis techniques. Our results indicate that the optimal redeeming strategy is of a bang-bang type which is similar to that in Shiryaev et al. (2008). In particular, we apply the PDE approach to characterize the problem and find that the redeeming strategy heavily depends on the risk aversion degree of the portfolio manager.

4.1 Model

Assume that a portfolio manager manages her investment portfolio which consists of a risk free bond and n risky assets in a complete and arbitrage-free financial market over the period $[0, T]$.

The price process of the risk-free bond evolves as an ordinary differential equation

$$\begin{cases} \mathrm{d}S_0(u) &= r(u)S_0(u)du, \quad u \geqslant 0, \\ S_0(0) &= s_0 > 0, \end{cases} \tag{4.1.1}$$

where $r(t)$ represents the interest rate.

The price process $S_i(u)$ of i-th risky asset, $i = 1, 2, \cdots n$, follows the dynamics

$$\begin{cases} \mathrm{d}S_i(u) &= S_i(u)\Big\{b_i(u)\mathrm{d}u + \sum_{j=1}^{n} \sigma_{ij}(u)\mathrm{d}B^j(u)\Big\}, \quad u \geqslant 0, \\ S_i(0) &= s_i > 0, \end{cases} \tag{4.1.2}$$

where $b(u) := (b_1(u), b_2(u), \cdots, b_n(u))^{\mathrm{T}}$ is the appreciation rate, $\sigma(u) := (\sigma_{ij}(u))_{n\times n}$ is the volatility matrix, $B(u) \equiv (B^1(u), B^2(u), \cdots, B^n(u))$ is a standard $\{\mathcal{F}_u\}_{u\geqslant 0}$-adapted n-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_u\}_{u\geqslant 0})$. We assume that the market parameters $r(u)$, $b(u)$ $(> r(u))$, $\sigma(u)$ are deterministic, Borel measurable and bounded on $[0, T]$. In addition, the nondegeneracy condition satisfies

$$\sigma(u)'\sigma(u) > \delta I, \quad \forall u \in [0, T],$$

where $\delta > 0$ is a constant.

Suppose that the manager has a self-financing trading strategy with an initial wealth x_0. Then the wealth process is driven by a stochastic differential equation

$$\begin{cases} \mathrm{d}X(u) = \Big\{ r(u)X(u) + \sum_{i=1}^{n}(b_i(u) - r(u))\pi_i(u) \Big\}\mathrm{d}t + \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij}(u)\pi_i(t)\mathrm{d}B^j(u), \\ \hfill 0 \leqslant u \leqslant T, \\ X(0) = x_0, \end{cases} \tag{4.1.3}$$

where π_i is the dollar amount invested in i-th risky asset.

Denote $\pi(u) := (\pi_1(u), \pi_2(u), \cdots, \pi_n(u))^{\mathrm{T}}$. It is required that $\Pi = \{\pi(u)\}_{u \geqslant 0}$ is $\{\mathcal{F}_u\}_{u\geqslant 0}$-adapted and satisfies the integrability condition $\mathbb{E}[\int_0^T \|\pi(u)\|^2 du] < \infty$ such that $X(u)$ has a pathwise unique solution.

Define the running maximum of wealth process over the fixed time horizon as

$$M(u) = \max_{0 \leqslant s \leqslant u} X(s), \;\; s \geqslant 0.$$

Suppose that an individual investor buys a T -year equity trust managed by the portfolio manager with the above trading strategy. On one hand, the portfolio manager's aim is utility maximization of her managed account at time T. On the other hand, the investor has the option to transfer her position, equivalently, "sell the product" before maturity. Hence the investor attempts to have good timing τ to transfer the trust product at a relative high water mark to minimize the relative error between transfer value and the highest possible wealth value

$$\min_{\tau \in \mathcal{T}} \;\; \mathbb{E}\Big[\frac{M(T) - X(\tau)}{M(T)}\Big], \tag{4.1.4}$$

$$\text{s.t.} \quad \begin{cases} \max\limits_{\pi(\cdot)} \;\; \mathbb{E}\Big[\frac{(X(T))^\gamma}{\gamma}\Big], \\ (X(\cdot), \pi(\cdot)) \;\; \text{satisfy (4.1.3)}, \end{cases} \tag{4.1.5}$$

where $0 < \gamma < 1$, $\tau \in [0, T]$ and $\mathcal{T}$ is the set of all $\mathcal{F}_u$- stopping time.

4.2 Free Boundary Problem

4.2.1 The optimal investment strategy of the agent

Before investigating the optimal redeeming time, we first derive the optimal wealth management strategy of the portfolio manager in (4.1.5) via the dynamic programming method, that is,

$$\pi(u) = (\pi_1(u), \pi_2(u), \cdots, \pi_n(u))' = \tfrac{1}{1-\gamma}(\sigma(u)\sigma(u)')^{-1}(b(u) - r(u)\mathbf{1})X(u), \qquad (4.2.6)$$

where $\mathbf{1} = (1, 1, \cdots, 1)'$ is an n-dimensional column vector.

Substituting the above strategy (4.2.6) into (4.1.5) yields

$$\begin{cases} \mathrm{d}X(u) &= X(u)\big\{(r(u) + \tfrac{1}{1-\gamma}\|\theta(u)\|^2)\mathrm{d}u + \tfrac{1}{1-\gamma}\theta(u)'\mathrm{d}B(u)\big\}, \\ X(0) &= x_0, \end{cases} \qquad (4.2.7)$$

where $\theta(u) = \sigma(u)^{-1}(b(u) - r(u)\mathbf{1})$.

In particular, we change the num$\acute{e}$raire and denominate the wealth value in terms of the money market account. Define $D(u) = e^{-\int_0^u r(s)ds}$, the asset value becomes $D(u)X(u)$. In other words, at time t, the asset under management is worth $D(u)X(u)$ shares of the money market account. Then the dynamics of the asset can be expressed as

$$\begin{cases} \mathrm{d}X(u) &= X(u)\big\{\tfrac{1}{1-\gamma}\|\theta(u)\|^2)\mathrm{d}u + \tfrac{1}{1-\gamma}\theta(u)'\mathrm{d}B(u)\big\}, \\ X(0) &= x_0, \end{cases}$$

By virtue of the time change technique, there exists a one-dimensional Brownian motion $W(u), u \geqslant 0$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\frac{1}{1-\gamma}\int_0^u \theta(s)'dB(s) = W(\beta(u)),$$

where $\beta(u) \equiv \frac{1}{(1-\gamma)^2}\int_0^u \|\theta(s)\|^2 ds$.

Let $t := \frac{1}{(1-\gamma)^2}\int_0^u \|\theta(s)\|^2 ds$ and $\overline{T} := \frac{1}{(1-\gamma)^2}\int_0^T \|\theta(s)\|^2 ds$, the wealth process $X(\cdot)$ is equivalent to

$$\begin{cases} \mathrm{d}X(t) = X(t)\big\{(1-\gamma)\mathrm{d}t + \mathrm{d}W(t)\big\}, \quad 0 \leqslant t \leqslant \overline{T}, \\ X(0) = x_0, \end{cases}$$

and thus $X(t) = x_0 \exp\big\{(\frac{1}{2}-\gamma)t + W(t)\big\}$.

4.2.2 HJB equation

In this section, we consider the optimal redeeming strategy of an individual investor.

We reformulate the stopping problem as follows

$$\min_{\tau \in \mathcal{T}} \ \mathbb{E}\Big[\frac{M(\overline{T}) - X(\tau)}{M(\overline{T})}\Big],$$

where $\mathcal{T}$ is the set of all $\mathcal{F}_t$- stopping time, $\tau \in [0, \overline{T}]$. It is equivalent to

$$\max_{\tau \in \mathcal{T}} \ \mathbb{E}\Big[\frac{X(\tau)}{M(\overline{T})}\Big], \tag{4.2.8}$$

where $M(\overline{T})$ is the global maximum over the period $[0, \overline{T}]$, i.e.

$$M(\overline{T}) = \max_{0 \leqslant s \leqslant \overline{T}} X(s).$$

The objective of this is to find the optimal time $\tau^* \in \mathcal{T}$ such that

$$\mathbb{E}\Big[\frac{X(\tau^*)}{M(\overline{T})}\Big] = \max_{\tau \in \mathcal{T}} \ \mathbb{E}\Big[\frac{X(\tau)}{M(\overline{T})}\Big].$$

Then for any optimal stopping time $\tau \in \mathcal{T}$, we have

$$\begin{aligned}\mathbb{E}\Big[\frac{X(\tau)}{M(\overline{T})}\Big] &= \mathbb{E}\Big[\frac{X(\tau)}{\max\{M(\tau), \max\limits_{\tau\leqslant s\leqslant\overline{T}} X(s)\}}\Big] = \mathbb{E}\Big[\mathbb{E}\big[\max\{\frac{M(\tau)}{X(\tau)}, \max_{\tau\leqslant s\leqslant\overline{T}}\frac{X(s)}{X(\tau)}\}\big]^{-1}\big|\mathcal{F}_\tau\Big] \\ &= \mathbb{E}\Big[\mathbb{E}\big[\min\big\{e^{-y}, e^{-\max\limits_{t\leqslant s\leqslant\overline{T}}(\frac{1}{2}-\gamma)(s-t)+W(s)-W(t)}\big\}\big|\tau = t, \ln\frac{M(\tau)}{X(\tau)} = y\big]\Big] \\ &= \mathbb{E}\big[F\big(\tau, \ln\frac{M(\tau)}{X(\tau)}\big)\big].\end{aligned} \tag{4.2.9}$$

A direct computation shows that

$$\begin{aligned}F(t,y) :&= \mathbb{E}\Big[\min\Big\{e^{-y}, e^{-\max\limits_{t\leqslant s\leqslant\overline{T}}(\frac{1}{2}-\gamma)(s-t)+W(s)-W(t)}\Big\}\Big], \quad \forall (t,y)\in[0,\overline{T})\times(0,\infty) \\ &= \frac{1+2\gamma}{2\gamma}e^{\gamma(\overline{T}-t)}\Phi(d_1) + e^{-y}\Phi(d_2) - \frac{1}{2\gamma}e^{-2\gamma y}\Phi(d_3),\end{aligned}$$

where $d_1 = \frac{-y-(\gamma+\frac{1}{2})(\overline{T}-t)}{\sqrt{\overline{T}-t}}$, $d_2 = \frac{y-(\frac{1}{2}-\gamma)(\overline{T}-t)}{\sqrt{\overline{T}-t}}$, $d_3 = \frac{-y-(\frac{1}{2}-\gamma)(\overline{T}-t)}{\sqrt{\overline{T}-t}}$ and $\Phi(\cdot)$ is the cumulative distribution function of a standard normal distribution.

The problem (4.2.8) is not a standard stopping problem since $M(\overline{T})$ is not $\mathcal{F}_t$-adapted. In view of equation (4.2.9), we turn it to a standard one with terminal payoff F and an adapted state process

$$Y_t = \ln\big(\frac{M(t)}{X(t)}\big), \quad Y_0 = 0.$$

By the dynamic programming method, we introduce the value function

$$V(t,y) = \min_{0\leqslant\tau\leqslant\overline{T}-t} \mathbb{E}_{t,y}\big[F(t+\tau, Y_{t+\tau})\big], \tag{4.2.10}$$

where $Y_t = y$ is given and fixed under $\mathbb{P}_{t,y}$ with $(t,y)\in[0,\overline{T})\times(0,+\infty)$. It is clear that the original problem is $V(0,0) = \min\limits_{0\leqslant\tau\leqslant\overline{T}} \mathbb{E}\big[\frac{X(\tau)}{M(\overline{T})}\big]$.

The value function (4.2.10) satisfies the variational inequalities

$$\begin{cases} \min\{-\partial_t V - (\gamma - \frac{1}{2})\partial_y V - \frac{1}{2}\partial_{yy} V, V - F\} = 0, \ in \ \ \mathcal{Q} := [0, \overline{T}) \times (0, +\infty), \\ V_y(0,t) = 0, \quad V(y, \overline{T}) = F(y, \overline{T}). \end{cases} \tag{4.2.11}$$

It is known as an obstacle problem in physics. We can see that the optimal stopping strategy depends on the drawdown process Y_t as well as time t.

Define two set

$$\mathcal{HR} = \{(t, y) \in [0, \overline{T}) \times [0, +\infty) : V(t, y) > F(t, y)\},$$

and

$$\mathcal{TR} = \{(t, y) \in [0, \overline{T}] \times [0, +\infty) : V(t, y) = F(t, y)\},$$

which represent the holding region and redeeming region respectively. Therefore, by the general optimal stopping theory, optimal stopping time is defined by

$$\tau^* = \inf\{t \in [0, \overline{T}] | V(t, y) = F(t, y)\}.$$

4.3 Solution

In this section, we use purely PDE method to study the optimal stopping strategy.

4.3.1 PDE formulation

First, we present some transformation for further analysis

$$\Psi(t, y) = \ln(F(t, y)), \quad U(t, y) = \ln\left(\frac{V(t, y)}{F(t, y)}\right). \tag{4.3.12}$$

Since $F(t, y)$ satisfies[1]

$$\begin{cases} \mathcal{L}_0 F = F_y + \gamma F, \\ F_y(0, t) = 0, F(y, T) = e^{-y}. \end{cases}$$

[1] See Shiryaev et al. (2008) Appendix B for details.

where $\mathcal{L}_0 = -\partial_t - (\gamma - \frac{1}{2})\partial_y - \frac{1}{2}\partial_{yy}$. Then $\Psi(t, y)$ satisfies

$$\begin{cases} -\Psi_t - \frac{1}{2}(\Psi_{xx} + \Phi_x^2) - (\frac{1}{2} + \gamma)\Psi_x - \gamma = 0, \quad in \quad \mathcal{Q}, \\ \Psi_y(t, 0) = 0, \Psi(T, y) = -y. \end{cases} \tag{4.3.13}$$

The obstacle problem (4.2.11) can be reduced to

$$\begin{cases} \min\{\mathcal{L}_1 U + \Phi_y + \gamma, U\} = 0, \quad in \quad \mathcal{Q}, \\ U_y(t, 0) = 0, U(T, y) = 0, \end{cases} \tag{4.3.14}$$

where $\mathcal{L}_1 = -\partial_t - \frac{1}{2}[\partial_{yy} + (\partial_y)^2 + 2\Psi_y \partial_y] + (\frac{1}{2} - \gamma)\partial_y$. Therefore, we introduce $\hat{\mathcal{Q}} := [0, T) \times [0, +\infty)$ and rewrite the redeeming region as

$$\mathcal{TR} = \{(\tau, y) \in \hat{\mathcal{Q}} : U(\tau, y) = 0\}. \tag{4.3.15}$$

Let us introduce a lemma that is useful for the following analysis.

Lemma 4.1. *Let $\Psi(t, y)$ be the solution to (4.3.13). Then it has the following properties:*

1. $-1 < \Psi_y(t, y) < 0$, $\forall (t, y) \in \mathcal{Q} := [0, \overline{T}) \times (0, +\infty)$;
2. $\Psi_{yy}(t, y) \leqslant 0$, $\forall (t, y) \in \mathcal{Q}$;
3. $\Psi_{ty}(t, y) \leqslant 0$, $\forall (t, y) \in \mathcal{Q}$;
4. $\Psi_y(t, y, 1 - \gamma + \epsilon) \leqslant \Psi_y(t, y; 1 - \gamma) + \epsilon$, $\forall \epsilon > 0, (t, y) \in \mathcal{Q}$.

Proof. To prove property 1 and 2, denote $\bar{\Psi}(t, y) := \Psi_y(t, y)$ and $\tilde{\Psi}(t, y) := \Psi_{yy}(t, y)$. It follows from (4.3.13) that

$$\begin{cases} -\bar{\Psi}_t - \frac{1}{2}(\bar{\Psi}_{yy} + 2\bar{\Psi}\bar{\Psi}_y) - (\gamma + \frac{1}{2})\bar{\Psi}_y = 0, \quad in \quad \mathcal{Q}, \\ \bar{\Psi}(t, 0) = 0, \quad \bar{\Psi}(T, y) = -1, \end{cases}$$

and

$$\begin{cases} -\tilde{\Psi}_t - \frac{1}{2}(\tilde{\Psi}_{yy} + 2\Psi_x\tilde{\Psi}_x + 2(\tilde{\Psi})^2) - (\gamma + \frac{1}{2})\tilde{\Psi}_y = 0, \quad in \quad \mathcal{Q}, \\ \tilde{\Psi}(t,0) \leqslant 0, \quad \tilde{\Psi}(T,x) = 0. \end{cases}$$

By the strong maximum principle, we obtain property 1 and 2.

Denote $\bar{\Psi}^{-\epsilon}(t,y) = \Psi_y(t-\epsilon,y)$, property 3 holds if $\psi(t,y) = \bar{\Psi}^{-\epsilon}(t,y) - \bar{\Psi}(t,y) \geqslant 0, \forall \epsilon \geqslant 0$. It is easy to check that $\psi(t,y)$ satisfies

$$\begin{cases} -\psi_t - \frac{1}{2}(\psi_{yy} + 2\bar{\Psi}^{-\epsilon}\psi_y + 2\tilde{\Psi}\psi) - (\frac{1}{2} + \gamma)\psi_y = 0, \quad in \quad \mathcal{Q}, \\ \psi(t,0) = 0, \quad \psi(T,y) = \bar{\Psi}(T-\epsilon,y) + 1 \geqslant 0. \end{cases}$$

Due to the minimum principle, we have $\psi(t,y) \geqslant 0$, which leads to $\Psi_{ty} \leqslant 0$.

To prove property 4, denote $\bar{\Psi}^{\epsilon}(t,y) = \Psi_y(t,y;1-\gamma+\epsilon)$ and $\hat{\Psi}^{\epsilon}(t,y) = \Psi_y(t,y;1-\gamma)+\epsilon$. Let $\varphi = \bar{\Psi}^{\epsilon} - \hat{\Psi}^{\epsilon}$, it is easy to verify $\bar{\Psi}^{\epsilon}_1(t,y)$ and $\bar{\Psi}^{\epsilon}_2(t,y)$ satisfy

$$\begin{cases} -\bar{\Psi}^{\epsilon}_t - \frac{1}{2}(\bar{\Psi}^{\epsilon}_{yy} + 2\bar{\Psi}^{\epsilon}\bar{\Psi}^{\epsilon}_x) + (\epsilon - \gamma - \frac{1}{2})\bar{\Psi}^{\epsilon}_x = 0, \quad in \quad \mathcal{Q}, \\ \bar{\Psi}^{\epsilon}(0,t) = 0, \quad \bar{\Psi}^{\epsilon}(T,y) = -1, \end{cases} \tag{4.3.16}$$

and

$$\begin{cases} -\hat{\Psi}^{\epsilon}_t - \frac{1}{2}(\hat{\Psi}^{\epsilon}_{yy} + 2\hat{\Psi}^{\epsilon}\hat{\Psi}^{\epsilon}_y) + (\epsilon - \gamma - \frac{1}{2})\hat{\Psi}^{\epsilon}_y = 0, \quad in \quad \mathcal{Q}, \\ \hat{\Psi}^{\epsilon}(0,t) = \epsilon, \hat{\Psi}^{\epsilon}(T,y) = -1 + \epsilon. \end{cases} \tag{4.3.17}$$

Subtracting (4.3.17) from (4.3.16) yields

$$\begin{cases} -\varphi_t - \frac{1}{2}(\varphi_{yy} + \hat{\Psi}^{\epsilon}\varphi_y + \bar{\Psi}^{\epsilon}_y\varphi) + (\epsilon - \gamma - \frac{1}{2})\varphi_y = 0, \quad in \quad \mathcal{Q}, \\ \varphi(t,0) = -\epsilon, \varphi(T,x) = -\epsilon. \end{cases}$$

By virtue of the maximum principle, we obtain the desired result. □

4.3.2 Optimal strategies

We first introduce an auxiliary problem

$$\begin{cases} \mathcal{L}_1 U + \Psi_y + \gamma = 0, \quad in \quad \mathcal{Q} := [0,T) \times (0,+\infty), \\ U_y(t,0) = 0, \quad U(T,y) = 0. \end{cases} \tag{4.3.18}$$

Lemma 4.2. *Let $\hat{U}(t,y)$ be the solution to (4.3.18). For any $t \in [0,T)$,*

$$\hat{U}(t,0) > 0 \text{ if } \quad 0 < \gamma < \frac{1}{2},$$

$$\hat{U}(t,0) = 0 \text{ if } \quad \gamma = \frac{1}{2},$$

$$\hat{U}(t,0) < 0 \text{ if } \quad \frac{1}{2} < \gamma < 1.$$

Proof. Let $\hat{V}(t,y) = \mathbb{E}\big[\frac{X_{\overline{T}}}{M_{\overline{T}}} | \ln \frac{M_t}{X_t} = y\big]$ represent the value function with the trivial strategy, holding the position until terminal time T. It follows from transformation (4.3.12) that

$$\hat{U}(t,y) = \ln\big(\frac{\hat{V}(t,y)}{F(t,y)}\big).$$

By the definition of $\hat{U}$, we turn to prove

$$\hat{V}(t,0) > F(t,0) \text{ if } \quad 0 < \gamma < \frac{1}{2},$$

$$\hat{V}(t,0) = F(t,0) \text{ if } \quad \gamma = \frac{1}{2},$$

$$\hat{V}(t,0) < F(t,0) \text{ if } \quad \frac{1}{2} < \gamma < 1.$$

We first consider the case of $\gamma = \frac{1}{2}$ and prove $\mathbb{E}[F(\overline{T}, Y^0_{\overline{T}})] = F(0,0)$. The general expression of $\mathbb{E}[F(T, Y^y_{\overline{T}})] = \mathbb{E}[e^{-Y^y_{\overline{T}}}]$ leads to

$$\mathbb{E}[F(\overline{T}, Y^y_{\overline{T}})] = e^{-y+\frac{\overline{T}}{2}}\Phi(\frac{y-\overline{T}}{\sqrt{\overline{T}}}) + e^{y+\frac{\overline{T}}{2}}\Phi(\frac{-y-\overline{T}}{\sqrt{\overline{T}}}).$$

The definition of F in (4.2.9) leads to the expression of $F(0,y)$ with $\gamma = \frac{1}{2}$

$$F(0,y) = 2e^{\frac{\overline{T}}{2}}\Phi(\frac{-y-\overline{T}}{\sqrt{\overline{T}}}) - e^{-y}\Phi(\frac{-y}{\sqrt{\overline{T}}}) + e^{-y}\Phi(\frac{y}{\sqrt{\overline{T}}}).$$

Subsequently, we can prove $\mathbb{E}[F(\overline{T}, Y^0_{\overline{T}})] = F(0,0)$.

Define $\overline{W}_t^{\upsilon} = \upsilon t + W(t)$ and $\widetilde{W}_t^{\upsilon} = \max_{0 \leqslant s \leqslant t} \big(\upsilon s + W(s)\big)$, where $\upsilon = \frac{1}{2} - \gamma$. Accordingly, we have

$$\mathbb{E}[F(T, Y_T)] - F(0, y) = \mathbb{E}\big[e^{-(y \vee \widetilde{W}_T^{\upsilon} - \overline{W}_T^{\upsilon})}\big], \ F(0, y) = \mathbb{E}\big[e^{-y \vee \widetilde{W}_T^{\upsilon}}\big], \ \forall y \geqslant 0.$$

By Girsanov theorem, we obtain

$$\begin{aligned} \mathbb{E}[F(\overline{T}, Y_{\overline{T}})] - F(0, y) &= \mathbb{E}\big[e^{-y \vee \widetilde{W}_T^{\upsilon}}(e^{\overline{W}_T^{\upsilon}} - 1)\big] \\ &= \mathbb{E}^Q\big[e^{-y \vee \widetilde{W}_T^{Q}}(e^{\overline{W}_T^{Q}} - 1)e^{-\frac{\upsilon^2 T}{2} + \upsilon \overline{W}_T^{Q}}\big] \\ &= \mathbb{E}\big[e^{-y \vee \widetilde{W}_T}(e^{\overline{W}_T} - 1)e^{-\frac{\upsilon^2 T}{2} + \upsilon \overline{W}_T}\big], \end{aligned}$$

where $\overline{W}_t^{Q} = \upsilon t + W(t)$ is a standard Brownian motion under probability Q.

In the case of $\upsilon \neq 0$, i.e. $\gamma \neq \frac{1}{2}$, we have

$$e^{\frac{\upsilon^2 T}{2}}(\mathbb{E}[F(T, Y_T)] - F(0, y)) = \mathbb{E}\big[e^{-y \vee \widetilde{W}_T}(e^{\overline{W}_T} - 1)e^{\upsilon \overline{W}_T}\big]$$

and

$$\frac{\partial}{\partial \upsilon}\Big(e^{\frac{\upsilon^2 T}{2}}(\mathbb{E}[F(T, Y_T)] - F(0, y))\Big) = \mathbb{E}^Q\big[e^{-y \vee \widetilde{W}_T}(e^{\overline{W}_T} - 1)\overline{W}_T e^{\upsilon \overline{W}_T}\big] > 0, \quad \forall y \geqslant 0. \tag{4.3.19}$$

It follows from (4.3.19) that $\mathbb{E}[F(\overline{T}, Y_{\overline{T}}^0)] > F(0, 0)$ when $\upsilon > 0$ $(0 < \gamma < \frac{1}{2})$ and $\mathbb{E}[F(\overline{T}, Y_{\overline{T}}^0)] < F(0, 0)$ when $\upsilon < 0$ $(1 > \gamma > \frac{1}{2})$.

By the arbitrariness of T, we can prove in the similar way that

$$\begin{aligned} &\mathbb{E}_{t,0}[F(\overline{T}, Y_{\overline{T}})] > F(t, 0) \text{ if } \quad 0 < \gamma < \frac{1}{2}, \\ &\mathbb{E}_{t,0}[F(\overline{T}, Y_{\overline{T}})] = F(t, 0) \text{ if } \quad \gamma = \frac{1}{2}, \\ &\mathbb{E}_{t,0}[F(\overline{T}, Y_{\overline{T}})] < F(t, 0) \text{ if } \quad \frac{1}{2} < \gamma < 1. \end{aligned} \tag{4.3.20}$$

The desired results are implied by (4.3.20).

□

Lemma 4.3. *The variational inequality (4.3.14) has a unique solution* $U(t,y) \in W_p^{1,2}(\bar{\mathcal{Q}})$*,* $1 < p < +\infty$*, where* $\bar{\mathcal{Q}}$ *is any bounded set in* $\mathcal{Q}$*. For any* $(t,y) \in \bar{\mathcal{Q}}$*.*

(i) $0 \leqslant U_y \leqslant 1$*;*

(ii) $U(t,y;1-\gamma) \leqslant U(t,y;1-\gamma+\epsilon)$ *for* $\epsilon > 0$*;*

(iii) $U(t,y) = \hat{U}(t,y) > 0$ *for* $0 < \gamma < \frac{1}{2}$*. For any* $t \in [0,T)$

$$U(t,0) > 0 \ \textit{if} \quad 0 < \gamma < \frac{1}{2}; \tag{4.3.21}$$

$$U(t,0) = 0 \ \textit{if} \quad \gamma = \frac{1}{2}; \tag{4.3.22}$$

$$U(t,0) = 0 \ \textit{if} \quad \frac{1}{2} < \gamma < 1. \tag{4.3.23}$$

Proof. (i). It is easy to show that (4.3.14) has a unique solution $U(t,y) \in W_p^{1,2}(\bar{\mathcal{Q}})$, $1 < p < +\infty$ by penalty approach. To prove (i), we only need to show $0 \leqslant U_y \leqslant 1$ in the set $\Omega = \{(t,y) \in \mathcal{Q} : U < 0\}$. Denote $u = U_y$ and $w = U_y - 1$, then u and w satisfy

$$\begin{cases} -u_t - \frac{1}{2}(u_{yy} + 2uu_y + 2\Psi_{yy}u + 2\Psi_y u_y) + (\frac{1}{2} - \gamma)u_y = -\Psi_{yy}, & \textit{in } \Omega \\ u|_{\partial\Omega} = 0, \end{cases}$$

and

$$\begin{cases} -w_t - \frac{1}{2}(w_{yy} + 2ww_y + 2\Psi_{yy}w + 2\Psi_y w_y) + -(\frac{1}{2} + p)u_y = 0, & \textit{in } \Omega \\ w|_{\partial\Omega} = -1, \end{cases}$$

Since $\Psi_{yy} \leqslant 0$, by the maximum principle, we obtain $u \geqslant 0$ and $w \leqslant 0$ in Ω, which imply the desired result (i).

To show (ii), we denote $G(t,y) = U(t,y;1-\gamma+\epsilon) - U(t,y)$ and $\bar{\Psi}^\epsilon(t,y) = \Psi_y(t,y;1-\gamma+\epsilon)$. If (ii) fails, then

$$\Lambda = \{(t,y) \in \mathcal{Q} : G(t,y) < 0\} \neq \emptyset,$$

It can be verified that

$$\begin{cases} -G_t - \frac{1}{2}(G_{yy} + G_y^2 + 2U_yG_y + 2\bar{\Psi}^\epsilon G_y) + (\frac{1}{2} + \epsilon - \gamma)G_y \geqslant -(\bar{\Psi}^\epsilon - \Psi_y - \epsilon)(1 - U_y) \\ \hfill in \ \ \Lambda, \\ G|_{\partial\Lambda} = 0. \end{cases}$$

Property 4 in Lemma 4.1 implies $\bar{\Psi}^\epsilon < \Psi_y + \epsilon$, together with (i), we can infer

$$-(\bar{\Psi}^\epsilon - \Psi_y - \epsilon)(1 - U_y) \geqslant 0.$$

Using again the maximum principle, we get $G \geqslant 0$ in Λ, which contradicts the definition of Λ.

(iii). It can be seen from (4.3.18) that $\hat{U}(t, y) > 0$ in $\mathcal{Q}$ by $\Psi_{yy} \leqslant 0$ and the maximum principle. Together with Lemma (4.2), we have $\hat{U}(t, y) > 0$ in $\mathcal{Q}$ when $0 < \gamma \leqslant \frac{1}{2}$, therefore $\hat{U}(t, y) > 0$ is the solution to Problem (4.3.18) and $U(t, y) = \hat{U}(t, y)$ for $0 < \gamma \leqslant \frac{1}{2}$, which implies (4.3.21) and (4.3.22). To prove (4.3.23), it is clear that $U(t, 0) \geqslant 0$, by virtue of (ii) and (4.3.22), we get $U(t, 0) \leqslant 0$ for $\frac{1}{2} < \gamma < 1$, which leads to (4.3.23). This completes the proof.

□

We then study the optimal stopping strategy of the investor.

Theorem 4.1. *Let $\mathcal{TR}$ be the optimal redeeming region defined as (4.3.15), then the following statements hold:*

(1) *If $0 < \gamma < \frac{1}{2}$, then $\mathcal{TR} = \emptyset$;*

(2) *If $\gamma = \frac{1}{2}$, then $\mathcal{TR} = \{y = 0\}$;*

(3) *If $\frac{1}{2} < \gamma < 1$, then $\{y = 0\} \in \mathcal{TR}$.*

Proof. Part (1) and (2) follow from (iii) in Lemma 4.3. Due to (4.3.23), we can get $\{y = 0\} \subset TR$, which yields part (3). Combining $U_y > 0$, we can further define

$$y(t) = \sup\{y \in [0, +\infty) : U(t, y) = 0\}, \forall t \in [0, T).$$

□

Corollary 4.1. *The optimal strategy for an investor should be either rejecting the trust product immediately at time 0 if* $\frac{1}{2} \leqslant \gamma < 1$ *or holding the position until time* T *if* $0 < \gamma < \frac{1}{2}$.

4.4 Concluding Remarks

We determine the optimal decisions of an individual investor to exit from an investment schedule dependent on portfolio manager's management performance based on power utility maximization in a given period. This is an optimal stopping problem with a variational inequality. We prove the optimal exit strategy at the initial time via a PDE approach. The results can be further explored to give a thorough characterization of the free boundary (investigate optimal decisions at any time t).

Chapter 5

Conclusion

This thesis focuses on two types of free boundary problems: a singular control problem and optimal stopping problems in the context of the continuous financial market. The first one considers the reinsurance and dividend-payout strategy of an insurer, while the latter two are concerned with the assets selling behaviours of the fund manager and individual investor.

In chapter 2, we consider an optimal reinsurance and dividend problem by maximizing the cumulative discounted dividend distribution, which boils down to a combined classical-singular control problem. We obtain a variational inequality with a gradient constraint by the dynamic programming principle.

To tackle the corresponding variational inequality and free boundary problem, we take the partial derivative of the original variational inequality to obtain a standard one. We also adopt the similar method in Yi (2008) to prove the existence and s-moothness of the value function. Although the main technique has been investigated, it still requires some trivial analysis to obtain the regularities of the value function by PDE techniques. Furthermore, we characterize the smoothness and the monotonicity of the free boundary, which represent the reinsurance scheme and dividend strategies, respectively. We show that the whole domain is divided by the reinsurance barrier and dividend-payout barrier. The non-reinsurance region expands as the surplus grows while the reinsurance region disappears when a claim incurred is relatively small.

We then study a discretionary stopping problem in Chapter 3 to find the optimal portfolio selection and market timing for a mutual fund manager to sell her assets under management. Mathematically, it is formulated as a combined classical control and stopping problem, the objective is to maximize the utility of an American call option type function of assets at stopping time. Then we obtain a variational inequality on the value function, which can be reduced to a so-called free boundary problem and give rise to a free boundary. In the infinite time horizon case, a free boundary degenerates to a point.

Since it is not straightforward to solve the variational inequality with a nonlinear ordinary differential equation(ODE), we alternatively apply a dual approach to convert it to a dual problem with a linear ODE. By a delicate analysis of the ODE, we arrive at the unique solution of the value function by smooth pasting conditions and find the optimal investment strategy as well as the free boundary (a point) of the original problem. We find that the optimal selling point rises as the benchmark pay rises while it falls as the commission ratio and basic salary rises. Our findings are compatible with economic intuition that the manager must wait for the portfolio to reach a higher point in order to make a profit. When the manager receives a decent base income, she does not require a large compensation package to match her salary expectation.

Finally, we study a finite-time horizon optimal stopping problem in Chapter 4. More specifically, the optimal wealth management strategy of a portfolio manager with the power utility maximization objective leads to the GBM wealth value of an investor. We formulate the optimal position transfer (selling) as an optimal stopping problem by minimizing the relative error between the transfer value and the high-water mark over a given period from the perspective of the investor. We employ the PDE approach to find the optimal strategy, which turns out to be relatively simple and heavily dependent on the portfolio manager's risk aversion degree γ. More precisely, in the case of $\frac{1}{2} \leqslant \gamma < 1$, the investor ought to reject the manager's investment management scheme at the very beginning. However, in the case of $0 < \gamma < \frac{1}{2}$, the investor ought to hold the position until the expiration date.

Appendix A

Appendix to Chapter 4

A.1 Change of Time for Martingale

This theorem can be found in Karatzas and Shreve (2012).

Theorem A.1. *(Dambis-Dubins-Schwarz(DDS)): Let $M(t)$ be a continuous martingale, null at zero, such that $[M, M](t)$ is non-decreasing to ∞, and τ_t defined by*

$$\tau_t = \inf\{s : [M, M](s) > t\}.$$

Then the process $B(t) = M(\tau_t)$ is a Brownian motion with respect to the filtration, and the martingale M can be obtained from the Brownian motion B by the change of time $M(t) = B([M, M])(t)$.

A.2 The Expression of F

The expression of function $F(t, y)$ defined by

$$\begin{aligned}
F(t, y) &= \mathbb{E}\big[\min\{e^{-y}, e^{-\max\limits_{t \leqslant s \leqslant T}(\frac{1}{2}-\gamma)(s-t)+W(s)-W(t)}\}\big] \\
&= \mathbb{E}\big[\min\{e^{-y}, e^{-\max\limits_{0 \leqslant s \leqslant T-t}(\frac{1}{2}-\gamma)s+W(s)}\}\big] \\
&= \int_0^\infty e^{-z} \mathrm{d}\mathbb{P}(\max_{0 \leqslant s \leqslant T-t}(\frac{1}{2} - \gamma)s + W(s) \leqslant z) + e^{-y} \mathrm{d}\mathbb{P}(\max_{0 \leqslant s \leqslant T-t}(\frac{1}{2} - \gamma)s + W(s) \leqslant y).
\end{aligned}$$

Note that

$$\mathbb{P}(\max_{0\leqslant s\leqslant T-t}(\frac{1}{2}-\gamma)s+W(s)\leqslant z)= \quad \Phi\big[\frac{z-(\frac{1}{2}-\gamma)(T-t)}{\sqrt{T-t}}\big]-e^{(1-2\gamma)z}\Phi\big[\frac{z-(\frac{1}{2}-\gamma)(T-t)}{\sqrt{T-t}}\big],$$

we have

$$\begin{aligned}\int_y^\infty e^{-z}\mathrm{d}\Phi\big[\frac{z-(\frac{1}{2}-\gamma)(T-t)}{\sqrt{T-t}}\big] &= \int_y^\infty e^{-z}\frac{1}{\sqrt{2\pi(T-t)}}e^{-\frac{(z-(\frac{1}{2}-\gamma)(T-t))^2}{2(T-t)}}\mathrm{d}z \\ &= e^{\gamma(T-t)}\Phi\big[\frac{-y-(\frac{1}{2}+\gamma)(T-t)}{\sqrt{T-t}}\big].\end{aligned}$$

Applying integration by parts

$$\begin{aligned}&\int_y^\infty e^{-z}\mathrm{d}\big[e^{(1-2\gamma)z}\Phi\big(\frac{-z-(\frac{1}{2}-\gamma)(T-t)}{\sqrt{T-t}}\big)\big] \\ =&\frac{1-2\gamma}{2\gamma}e^{-2\gamma y}\Phi\big[\frac{-y-(\frac{1}{2}-\gamma)(T-t)}{\sqrt{T-t}}\big]+-\frac{1}{2\gamma}e^{\gamma(T-t)}\Phi\big[\frac{-y-(\frac{1}{2}+\gamma)(T-t)}{\sqrt{T-t}}\big].\end{aligned}$$

Hence

$$\begin{aligned}F(t,y) &= \mathbb{E}\big[\min\{e^{-y},e^{-\max_{t\leqslant s\leqslant T}(\frac{1}{2}-\gamma)(s-t)+W(s)-W(t)}\}\big] \\ &= \int_0^\infty e^{-z}\mathrm{d}\mathbb{P}(\max_{0\leqslant s\leqslant T-t}(\frac{1}{2}-\gamma)s+W(s)\leqslant z)+e^{-y}\mathrm{d}\mathbb{P}(\max_{0\leqslant s\leqslant T-t}\big(\frac{1}{2}-\gamma)s+W(s)\leqslant y\big) \\ &= \frac{1+2\gamma}{2\gamma}e^{\gamma(T-t)}\Phi(\frac{-y-(\gamma+\frac{1}{2})(T-t)}{\sqrt{T-t}})+e^{-y}\Phi(\frac{y-(\frac{1}{2}-\gamma)(T-t)}{\sqrt{T-t}}) \\ &- \frac{1}{2\gamma}e^{-2\gamma y}\Phi(\frac{-y-(\frac{1}{2}-\gamma)(T-t)}{\sqrt{T-t}}).\end{aligned}$$

Appendix B

Notations and General Theory of PDEs

The notations and results stated in this appendix can be found in Evans (2010).

Assume $U \subset \mathbb{R}^n$ is open and $0 < \alpha < 1$. If

$$|u(x) - u(y)| \leqslant C|x - y|^{\alpha} \quad (x, y \in U)$$

for some $0 < \alpha \leqslant 1$ and a constant C. Then u is said to be Hölder continuous with exponent α.

If $u : U \to \mathbb{R}$ is bounded and continuous, we write

$$\|u\|_{C(\bar{U})} := \sup_{x \in U} |u(x)|.$$

The α^{th}- Hölder seminorm of $u : U \to \mathbb{R}$ is

$$[u]_{C^{\alpha}(\bar{U})} := \sup_{x,y \in U, x \neq y} \left\{ \frac{u(x) - u(y)}{|x - y|^{\alpha}} \right\},$$

the α^{th}- Hölder norm is

$$\|u\|_{C^{\alpha}(\bar{U})} := \|u\|_{C(\bar{U})} + [u]_{C^{\alpha}(\bar{U})}.$$

Let k be a positive integer. The Hölder space

$$C^{k+\alpha}(\bar{U})$$

consists of all functions $u \in C^k(\bar{U})$ for which the norm

$$\|u\|_{C^{k+\alpha}(\bar{U})} := \sum_{|\beta| \leqslant k} \|D^\beta u\|_{C(\bar{U})} + \sum_{|\beta|=k} [D^\beta u]_{C^\alpha(\bar{U})}$$

is finite.

Define

$$\overline{[u]}_{C^{i,\alpha}(\bar{U})} = \sup_{x,y \in \bar{U}} d_{x,y}^{i+\alpha} \frac{|u(x)-u(y)|}{|x-y|^\alpha}, \quad \overline{[u]}_{C^\alpha(\bar{U})} = \overline{[u]}_{C^{0,\alpha}(\bar{U})},$$

where $d_x = dist(x, \partial\Omega)$, $d_{x,y} = \min\{d_x, d_y\}$. Define the norms

$$\overline{\|u\|}_{C^\alpha(\bar{U})} := \|u\|_{C(\bar{U})} + \overline{[u]}_{C^\alpha(\bar{U})},$$

$$\overline{\|u\|}_{C^{k+\alpha}(\bar{U})} := \sum_{|\beta| \leqslant k} |d_x^{|\beta|} D^\beta u(x)| + \sum_{|\beta|=k} \overline{[u]}_{C^{i,\alpha}(\bar{U})}.$$

We say $u \in \bar{C}^{k+\alpha}(\bar{U})$ if $\overline{\|u\|}_{C^{k+\alpha}(\bar{U})} < \infty$. If $u \in \bar{C}^{k+\alpha}(\bar{U}_0)$ for any open set U_0 with $\bar{U}_0 \subset U$, then we write $u \in \bar{C}^{k+\alpha}(\bar{U})$.

Let $p \in [1, +\infty]$ be a constant. For $u \in L^p(U)$, we define the α^{th}- weak partial derivative $D^\alpha u : U \to \mathbb{R}$ by

$$\int_U D^\alpha u \cdot \varphi(x) dx = (-1)^{|\alpha|} \int_U u(x) \cdot D^\alpha \varphi(x) dx, \quad \forall \varphi \in C_c^\infty(U),$$

where

$$\alpha = (\alpha_1, \cdots, \alpha_n),$$

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n},$$

$$D_i = \frac{\partial}{\partial x_i},$$

$$|\alpha| = \alpha_1 + \cdots + \alpha_n,$$

and C_c^∞ denotes the space of infinitely differentiable function with compact support in U.

$W^{k,p}(U)$ stands for the sobolev space consists of all locally summable functions $u : U \to \mathbb{R}$ such that for each multiindex α with $|\alpha| \leqslant k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$. $W^{k,p}(U)$ is a Banach space with norm

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \Big(\sum_{|\alpha|\leqslant k} \int_U |D^\alpha u|^p dx \Big)^{\frac{1}{p}}, & (1 \leqslant p < \infty), \\ \sum_{|\alpha|\leqslant k} ess \sup_U |D^\alpha u|, & (p = \infty). \end{cases}$$

If $p = 2$, we usually write $H^k(U) = W^{k,2}(U) \quad (k = 0, 1, \cdots)$.

Theorem B.1. *(General Sobolev inequalities). Let U be a bounded open subset of $\mathbb{R}^n$, with a C^1 boundary. Assume $u \in W^{k,p}$. If $k < \frac{n}{p}$, then $u \in L^q(\bar{U})$, where $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$.If $k > \frac{n}{p}$, then $u \in C^{k-[\frac{n}{p}]-1}(\bar{U})$, where*

$$\gamma = \begin{cases} [\frac{n}{p}] + 1 - \frac{n}{p}, & if \ \frac{n}{p} \text{ is not an integer}, \\ \text{any positive number } < 1, & if \ \frac{n}{p} \text{ is an integer}. \end{cases}$$

Consider the operator

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} b^i(x) u_{x_i} + c(x)u.$$

Theorem B.2. *(Schauder's Interior Estimates). Suppose that $U \in \mathbb{R}^n$ is a bounded domain with diameter $\leqslant D$, $f \in \bar{C}^\alpha(U)$, $\alpha < 1$ and*

$$\sum_{i,j=1}^{n} \overline{\|a_{ij}\|}_{C^\alpha(U)} + \sum_{i=1}^{n} \overline{\|b_i\|}_{C^\alpha(U)} + \overline{\|c\|}_{C^\alpha(U)} \leqslant K,$$

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant \lambda|\xi|^2, \quad \forall x \in U, \xi \in \mathbb{R}^n,$$

where $\lambda > 0$.

If $u \in C^2(U) \cap L^\infty(U)$ *and*

$$\mathcal{A}u = f, \text{ in } U.$$

Then

$$\overline{\|u\|}_{C^{2+\alpha}(U)} \leqslant C(\overline{\|f\|}_{C^\alpha(U)} + \overline{\|u\|}_{C(U)}),$$

where C *is a constant depending only on* λ, K *and* U.

Theorem B.3. *(Schauder's Boundary Estimates). Let* U *be an open set in* $\mathbb{R}^n$. *Suppose that* $\partial\Omega$ *is locally in* $C^{2+\alpha}$, $f \in C^\alpha(\bar{U})$, $\varphi \in C^{2+\alpha}(\bar{U})$, *and*

$$\sum_{i,j=1}^{n} \overline{\|a_{ij}\|}_{C^\alpha(U)} + \sum_{i=1}^{n} \overline{\|b_i\|}_{C^\alpha(U)} + \overline{\|c\|}_{C^\alpha(U)} \leqslant K,$$

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant \lambda|\xi|^2, \quad \forall x \in U, \xi \in \mathbb{R}^n,$$

where $\lambda > 0$. *If* $u \in C^2(\bar{U})$ *and*

$$\mathcal{A}u = f, \text{ in } U.$$

Then

$$\|u\|_{C^{2+\alpha}(U)} \leqslant C(\|f\|_{C^\alpha(U)} + \|u\|_{C(U)} + \|\varphi\|_{C^{2+\alpha}(U)}),$$

where C *is a constant depending only on* λ, K *and* U.

Theorem B.4. *(Schauder's Fixed Point Theorem).* X *denotes a Banach space. Suppose* $K \subset X$ *is compact and convex, and assume also* $A : K \to K$ *is continuous. Then* A *has a fixed point in* K.

Theorem B.5. *(Strong maximum principle with* $c \geqslant 0$*). Assume* $u \in C^2(U) \bigcap C(\bar{U})$ *and* $c \geqslant 0$ *in* U,

(i) *If* $Lu \leqslant 0$ *in* U *and* u *attains a nonnegative maximum over* $\bar{U}$ *at an interior point, then* u *is constant within* U*.*

(ii) *If* $Lu \geqslant 0$ *in* U *and* u *attains a nonpositive minimum over* $\bar{U}$ *at an interior point, then* u *is constant within* U*.*

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