# RELATIVE LIPSCHITZ-LIKE PROPERTY OF PARAMETRIC SYSTEMS VIA PROJECTIONAL CODERIVATIVES

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# Relative Lipschitz-like Property of Parametric Systems via Projectional Coderivatives

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# CERTIFICATE OF ORIGINALITY

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______________________________(Signed)

Yao Wenfang (Name of student)

# Abstract

In this thesis, fundamental properties of a newly introduced tool, projectional coderivatives, are illustrated. Some examples of calculation are also presented. When the set we refer to is a smooth manifold, the projectional coderivative can be simplified as a fixed-point expression. Therefore, we extend the generalized Mordukhovich criterion to such a setting. Moreover, chain rules and sum rules are developed for projectional coderivatives. Different levels of constraint qualifications are incorporated to generate upper estimates accordingly and all these upper estimates converge under the setting of smooth manifolds. By applying the sum rule to parametric systems, we obtain the upper estimate of the projectional coderivative of the solution mapping, which is also an implicit mapping, making it possible to analyse the relative Lipschitz-like property via projectional coderivatives. The difference between the approach of projectional coderivatives and directional normal cones is illustrated through an example. Under the framework of parametric systems, we analyse linear constraint systems, linear complementarity problems and affine variational inequalities. For linear constraint systems with a polyhedral setting, we show that by the generalized Mordukhovich criterion it enjoys the Lipschitz-like property relative to its domain automatically. Besides, we derive the corresponding graphical modulus. For linear complementarity problems with a $Q_0$-matrix, we investigate the sufficient and necessary condition for the Lipschitz-like property relative to its convex domain. For affine variational inequality, a generalized critical face condition is obtained to

characterize the Lipschitz-like property relative to a polyhedral convex set under a constraint qualification. By exploiting the structure of linear constraint systems, we investigate the Lipschitz-like property of such systems with an explicit set constraint under full perturbations (including the matrix perturbation) and derive some sufficient and necessary conditions. Some other approaches like outer-subdifferentials and error bounds are also taken into scope to characterize such property. The criterion is later applied on a linear portfolio selection optimization problem.

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# Contents

# List of Notations

| | |
|---|---|
| $\mathbb{R}$ | set of all real numbers |
| $\overline{\mathbb{R}}$ | set of extended real numbers |
| $\mathbb{R}^n$ | $n$-dimensional Euclidean vector space |
| $\mathbb{R}^n_+$ ($\mathbb{R}^n_-$) | non-negative (non-positive) orthant in $\mathbb{R}^n$ |
| $\mathbb{R}^n_{++}$ ($\mathbb{R}^n_{--}$) | positive (negative) orthant in $\mathbb{R}^n$ |
| $\|x\|$ | Euclidean norm of vector $x$ |
| $\langle x, y\rangle$ or $x^\top y$ | inner product |
| $x^\top$ | transpose of vector $x$ |
| $[x]$ | linear subspace generated by vector $x$ |
| $\mathbb{B}$ | closed unit ball |
| $\mathbb{S}$ | unit sphere |
| $\mathbb{B}_r(x)$ | closed unit ball with radius $r > 0$ centered at $x$ |
| $\mathcal{N}(x)$ | collection of neighborhoods of $x$ |
| $\mathbb{R}^{m\times n}$ | vector space of $m \times n$ real matrices |
| $E$ | the unit matrix |
| $A^*$ | transpose of matrix $A$ (also the adjoint operator of linear operator $A$) |
| $A^{-1}$ | inverse of matrix $A$ (also the inverse mapping of linear operator $A$) |
| $\det A$ | determinant of matrix $A$ |
| $\ker A$ | the kernel of matrix $A$ (also the null space of the linear operator corresponding to matrix $A$) |
| $\mathrm{rg}\, A$ | range of linear operator $A$ (also the column space of the matrix $A$) |
| $\mathrm{int}\, C$ | interior of set $C$ |
| $\mathrm{rint}\, C$ | relative interior of set $C$ |
| $\mathrm{cl}\, C$ | closure of set $C$ |
| $\mathrm{bdry}\, C$ | boundary of set $C$ |

| | |
|---|---|
| $\operatorname{conv} C$ | convex hull of set $C$ |
| $\operatorname{pos} C$ | positive hull of set $C$ |
| $C \setminus D$ | relative complement |
| $C^{\perp}$ | orthogonal complement of set $C$ |
| $C^{*}$ | polar cone of set $C$ |
| $\sigma_C$ | support function of set $C$ |
| $\delta_C$ | indicator function of set $C$ |
| $d(x, C)$ | distance from vector $x$ to set $C$ |
| $\operatorname{proj}_C(x)$ | projection of vector $x$ onto set $C$ |
| $\widehat{T}_C(x)$ | regular tangent cone of $C$ at $x$ |
| $T_C(x)$ | tangent/contingent cone of $C$ at $x$ |
| $N_C^{\text{prox}}(x)$ | proximal normal cone of $C$ at $x$, |
| $\widehat{N}_C(x)$ | regular(Fréchet) normal cone of $C$ at $x$ |
| $N_C(x)$ | (basic/limiting/Mordukhovich) normal cone of $C$ at $x$ |
| $S\vert_X$ | mapping $S$ restricted on set $X$ |
| $\operatorname{gph} S$ | graph of $S$ |
| $\operatorname{dom} S$ | domain of $S$ |
| $\operatorname{rg} S$ | range of $S$ |
| $\vert S\vert^{+}$ | outer norm of positively homogeneous mapping $S$ |
| $\text{g-limsup}_{\nu} S^{\nu}$ | graphical outer limit of a sequence of mappings $S^{\nu}$ |
| $\operatorname{lip} S(x \mid u)$ | graphical modulus of $S$ at $x$ for $u$ |
| $\operatorname{lip}_X S(x \mid u)$ | graphical modulus of $S$ relative to $X$ at $x$ for $u$ |
| $\widehat{D}^{*}S(x \mid u)$ | regular/Fréchet coderivative $S$ at $x$ for $u$ |
| $D^{*}S(x \mid u)$ | (normal) coderivative of $S$ at $x$ for $u$ |
| $\widehat{D}S(x \mid u)$ | regular derivative $S$ at $x$ for $u$ |
| $DS(x \mid u)$ | graphical derivative of $S$ at $x$ for $u$ |
| $\widehat{\partial} f(x)$ | regular/Fréchet subdifferential of $f$ at $x$ |
| $\partial f(x)$ | (limiting/Mordukhovich) subdifferential of $f$ at $x$ |
| $\partial^{>} f(x)$ | outer limiting subdifferential of $f$ at $x$ |
| $\nabla F(x)$ | Jacobian of a smooth mapping $F$ at $x$ |
| $\binom{n}{k}$ | combination of choosing $k$ elements from $n$ elements |

# Chapter 1

# Introduction

Real-world applications often involve input data which comes with fluctuations, like time delay, rapid change of prices, market frictions etc. Stability analysis aims at determining a verifiable condition, such that for a region of perturbation, the solution stays stable with an accuracy quantifiable by the size of the perturbation (Rockafellar and Wets [81]). Among various stability properties, the Lipschitz-like property (also known as the Aubin property) plays a central role and has deep consequences with error bounds and metric regularity, which are widely adopted in convergence analysis of iterative algorithms. The Lipschitz-like property arises from the Lipschitzian behavior of the set-valued mappings but focuses on localization around the reference point rather than all points in the domain of the mapping. By virtue of the Mordukhovich criterion, such property can be captured by coderivatives, along with describing the graphical modulus with an outer norm of coderivatives. The coderivative employed here, provides a geometric perspective on local behaviors of the graph of the set-valued mapping. The calculus rules of coderivatives offer a bridge to apply the Mordukhovich criterion to analyze the Lipschitz-like property of parametric optimization problems like generalized equations, parametric linear constraint systems, linear complementarity problems and variational inequalities.

However, one stringent assumption of the Lipschitz-like property is that the reference point lies in the interior of the domain of the set-valued mappings. Such an implicit assumption can be observed by both the definition of the property and the local boundedness requirement on coderivatives in Mordukhovich criterion. This assumption could hinder the study of this property when the point of interest lies on the boundary of the domain, or when one wish to study the property relative to certain directions only. Thus, the relative Lipschitz-like property comes into play. Although the Lipschitz-like property has been widely studied, the Lipschitz-like property relative to a set is greatly understudied. Not until recently, a new tool in variational analysis, projectional coderivative, has been introduced in Meng et al. [59] and employed to establish a generalized Mordukhovich criterion to characterize the property relative to a closed and convex set. This verifiable condition also pins down the associated graphical modulus of the set-valued mapping relative to the closed and convex set with the outer norm of projectional coderivatives.

For this newly acquainted tool, only a few properties and examples are presented due to its complicated nature. It involves interactions between normal cone of the set-valued mapping and projection onto the tangent cone of the set in the neighborhood. Thus, natural questions can be raised like, when does the projected normal cone enjoy outer semicontinuity like normal cone or under what conditions can the expression be simplified. More importantly, can we obtain calculus rules for projectional coderivatives similar to coderivatives and apply them to parametric systems to characterize the relative Lipschitz-like property? What kind of constraint qualifications should be imposed? These questions motivate our work.

## 1.1 Literature review

The early work on stability analysis can be traced back to Hoffman [35] where the error bound of linear inequality system under right-hand side perturbation was discussed. However, the study of stability did not proliferate until recent decades. Robinson played a leading role in studying the extension of the theorem (see Robinson [74]). Later works, Luo and Tseng [58], Azé and Corvellec [3] and Peña et al. [69] excelled in obtaining the error bound modulus of the linear systems and Van Ngai et al. [84], Kruger et al. [47, 48] worked on the semi-infinite ones. An outer subdifferential was used in error bounds: Cánovas et al. [14], Eberhard et al. [20], Fabian et al. [21], Kruger et al. [47], Ioffe [40], Ioffe and Outrata [43], Li et al. [56]. Related works on stability analysis are also introduced in monographs by Fiacco [24], Mordukhovich [66].

In 1984, Aubin [2] originated the definition of 'pseudo-Lipschitz' in the format of inverse function $F^{-1}$. Such property is later named after the author as the Aubin property (Rockafellar and Wets [81]), *a.k.a.* Lipschitz-like property, to capture it as an extension of Lipschitzian behavior of multifunction (see Mordukhovich [66]).

To characterize the Lipschitz-like property, Mordukhovich criterion was at first derived directly in Mordukhovich [61] with the estimates for the corresponding graphical modulus represented as the outer norm of the coderivative. The equivalence between the inverse Lipschitz-like property, openness at a linear rate and metric regularity is introduced thoroughly later in Mordukhovich [62]. The proof employing Ekeland variational principle and extremal principle (in Asplund spaces) can also be found in Mordukhovich [65] and a more direct one utilizing the essential variational analysis tools in Rockafellar and Wets [81]. The calculus rules of coderivatives widely adopted were largely initiated in Mordukhovich [63] and introduced in later monographs by Rockafellar and Wets [81] and Mordukhovich [65] under the assumptions of local boundedness and graph-convexity. These rules facilitate analyzing the Lipschitz-like property of parametric optimization problems. In Mordukhovich [64], the Lipschitz-like property was analyzed for a parametric system with the sum of two mappings, where the set-valued one did not involve the decision parameter. Later in

Levy and Mordukhovich [53], an extended form of the implicit set-valued mapping was considered, where both mappings involved the decision parameter. The upper estimate of the coderivative of the solution mapping is guaranteed under some constraint qualification and equality is also attainable. Such an estimate not only paves the way for studying stationary-point set mapping, but also facilitates the work of Huyen and Yen [38], which made full use of the condition for equality and studied the Lipschitz-like property and the Robinson metric regularity of a parametric linear constraint system under full perturbations:

$$S(A,b) = \{x \mid Ax + b \in K\} \tag{1.1.1}$$

with $K$ being a closed set only. Huyen and Yen [38] also applied the result to the solution mappings for linear complementarity problems and affine variational inequalities. A similar result on smooth function with regularity of the set is also mentioned in [81, Example 9.51] with graphical modulus obtained. Lee and Yen [50] employed the upper estimate in Levy and Mordukhovich [53] to analyze the Lipschitz-like property of the solution of the trust region-subproblem under full perturbations. For more introduction on parametric optimization problems, see monographs by Bonnans and Shapiro [6], Dontchev and Rockafellar [19], Ioffe [41] and Klatte and Kummer [46].

Among parametric systems, both linear and nonlinear systems have attracted a lot of attention on stability analysis due to its wide application. An example on portfolio selection using minimax rule can be found in Cai et al. [8] and Meng et al. [60]. Under the Robinson constraint qualification, Borwein [7] and Robinson [75] analyzed the Lipschitz property of the solution mapping of nonlinear inequality system with a closed and convex cone. Later in Jongen et al. [44], sufficiency for such stablity is provided with extended Mangasarian-Fromovitz constraint qualification at infinity. For linear semi-infinite systems, Goberna and López [31] gave some characterizations of the Lipschitz-like continuity. In Cánovas et al. [10], complete characterizations of the Lipschitz-like property of the solution mapping of both linear semi-infinite and infinite systems were obtained by developing a Mordukhovich criterion in an arbitrary Banach space based on coderivative. More recently, Li and Ng [54] used error bound results for approximate solutions to analyze the Lipschitz-like property of an abstract inequality system. For other stability results on these types of systems, see Cánovas et al. [12, 11] for calmness, Cánovas et al. [9] for metric regularity and Li and Li [55] and Gfrerer and Mordukhovich [26] for Robinson metric regularity.

As Huyen and Yen [38] treated the solution mappings of linear complementarity problems and affine variational inequalities under the framework of (1.1.1), $K$ becomes a union of polyhedral sets with special structures: graph of a normal cone mapping of a polyhedral set. In general, calculating the coderivative of this normal cone mapping requires some effort. Henrion and Outrata [33] considered the normal cone of finite union of polyhedral cones at the origin. Lee and Yen [50] considered the coderivative of a normal cone mapping of a Euclidean ball. In Henrion et al. [34], the coderivative formula was given for normal cone mappings of inequality systems with different assumptions, like full rank, linear independence constraint qualification (LICQ) and Mangasarian-Fromovitz constraint qualification (MFCQ). Gfrerer and Outrata [28] continued the work with a weaker constraint qualification. Qui [71] gave some estimates on the coderivative of right-hand side perturbed polyhedral normal cone mappings.

For a polyhedral normal cone mapping, Dontchev and Rockafellar [18] represented its coderivative by union of the difference (and its polar) between two critical faces of the set with a selection rule. In this way, a sufficient and necessary condition for the Lipschitz-like property of the solution mapping of affine variational inequality under linear perturbation,

$$L(q) = \{x \mid 0 \in q + Ax + N_C(x)\}\,, \tag{1.1.2}$$

was derived via a reduction lemma, named 'critical face condition'. In this work, the system (1.1.2) is closely related to the following one as the affine variational inequality can be treated as a linearization of $0 \in z + f(w, x) + N_C(x)$ under a proper setting

$$S(z, w) = \{x \mid 0 \in z + f(w, x) + N_C(x)\}\,. \tag{1.1.3}$$

Moreover, equivalences between single-valuedness along with Lipschitz continuity and the Lipschitz-like property, both for $L$ and $S$ also established, when differentiability and Lipschitz continuity of $f$ are assumed. Earlier work on such equivalence can also be found in Robinson [77] under strongly regularity condition and Robinson [79] under coherent orientation condition. See also Ralph [73], Scholtes [82] and Ioffe [42] for more related works.

For the case $C$ in (1.1.2) being a multifunction, the system can be taken as a quasi-variational inequality. New rules for coderivative calculus for this type of problems were introduced in Mordukhovich and Outrata [67] and thus efficient conditions for Lipschitzian stability were derived. Yen [85] considered $C$ as a polyhedral set with linear perturbation and established that the solution mapping of nonlinear variational inequality is Lipschitz continuous and single-valued under the Lipschitz and strong monotonicity assumptions. Lu and Robinson [57] deducted the determinantal condition for the existence of a single-valued, Lipschitz continuous, piecewise-affine solution. For the case that $C$ is a linear constraint system with full perturbations (both left-hand side and right-hand side), Robinson [76] and Cánovas et al. [13] characterized the calmness property for the stationary set mapping in terms of a Slater condition. Under a positively linear independence assumption, Qui [72] provided a sufficient condition for the Lipschitz-like property of the solution mapping of a fully perturbed affine variational inequality.

For linear complementarity problems, they can be seen as a special type of affine variational inequality with $C$ being the nonnegative orthant and therefore share the results above. Gowda and Pang [32] obtained a sufficient condition for calmness and existence of the mixed linear complementarity problem by degree theory and extended the result to the nonlinear one. For more applications of variational inequality problems and linear complementarity problems, see monographs by Cottle et al. [16], Kinderlehrer and Stampacchia [45], Facchinei and Pang [22] and Lee et al. [51]. More introduction on the solution mappings for variational problems and implicit mappings can also be found in the monograph by Dontchev and Rockafellar [19].

As mentioned before, the Lipschitz-like property has an implicit prerequisite that the reference point lies in the interior of the domain. In recent years, the relative stability has gained much attention, most of which employed the tool directional limiting coderivatives, introduced in Gfrerer [25] with directional limiting calculus initiated in Ginchev and Mordukhovich [30]. For directionally Lipschitzian single-valued mappings and generalized directional derivatives, see Clarke [15]. Gfrerer and Outrata [27] established sufficient conditions for the calmness and the Lipschitz-like property of implicit multifunctions by using a directional limiting coderivative along with graphical derivative. The formula for computing the directional limiting coderivative of the normal-cone map with a polyhedral set was also presented employing the critical face condition framework in Dontchev and Rockafellar [18]. Further extension of critical face condition to the Lipschitz-like property of solution mapping to a generalized equation can also be found in Gfrerer and Outrata [29] under the framework of Mordukhovich and Outrata [67]. In Benko et al. [5], sufficient conditions for the Lipschitz-like property relative to a closed set of the solution map for a class of parameterized variational systems were derived. These conditions require computation of directional limiting coderivatives of the normal-cone mapping for the so-called critical directions. However, these works only provided sufficiency for the relative Lipschitz-like property, which could still fail when the reference point lies on the boundary. In Meng et al. [59], a new tool, the projectional coderivative, was introduced and both sufficiency and necessity were provided for characterizing the Lipschitz-like property relative to a closed and convex set. For other stability properties relative to a set, see Van Ngai and Théra [83], Ioffe [39], Arutyunov and Izmailov [1] and Bonnans and Shapiro [6] for the relative metric regularity, Mordukhovich and Wang [68] for the restrictive metric regularity and Benko et al. [5] for the relative isolated calmness.

## 1.2 Organization of the thesis

In this dissertation, we restrict our scope of stability analysis mainly to the Lipschitz-like property and the relative Lipschitz-like property. The remaining of the current chapter introduces the tools widely adopted in variational analysis. Our work begins in Chapter 2 by introducing some properties of projectional coderivatives, mainly from the perspective of projection. Some connections between coderivatives and projectional ones are also stated. More specifically, when we consider a smooth manifold, the complicated expression of projectional coderivative can be fine-tuned as a fixed-point one. To take advantage of this structure, we extend the generalized Mordukhovich criterion from the convex setting to a smooth manifold. Similar to coderivatives, we obtain the chain rules for projectional coderivatives and establish an equation for smooth manifolds. Some special cases like when inner or outer layer of the function is single-valued are also discussed. Subsequently, a few sum rules are analyzed with different levels of constraint qualifications. The differences are mainly caused by: 1) the fact that a more stringent constraint qualification corresponds to a tighter upper estimate, and 2) how we deal with restricting the mappings onto the set. The first type of differences can be eliminated when the set is a smooth manifold.

As one useful objective of developing these rules is to analyze the relative Lipschitz-like property in a broader range, we apply these rules to a parametric system in Chapter 3. Inspired by Levy and Mordukhovich [53], we consider the parametric system which involves the sum of a $\mathcal{C}^1$ single-valued function and a set-valued one. Both of these two mappings are perturbed by the same parameter. When the system involves only one set-valued mapping, i.e., under the general implicit mapping setting, with a stricter constraint qualification we can represent the upper estimate of projectional coderivatives of the system by that of the mapping. We also compare our approach for analyzing relative Lipschitz-like property with that in Benko et al. [5] under the same setting. An example demonstrates that when a set or a direction involves part of the boundary of the domain, our approach works better as it characterizes the property in full. We also show that in some particular cases, constraint qualifications can be bypassed. Later we apply these results to specific problems with structures. For linear constraint systems under right-hand side perturbations, we first update some characterizations on the Lipschitz-like property by exploiting the structure of its domain. When the set is a union of polyhedrons, we give an expression of projectional coderivative of the system relative to its domain and when the set is one polyhedron, we obtain the graphical modulus relative to its domain. For linear complementarity problems, we consider a particular case where $M$ is a $Q_0$-matrix. In this way the domain is convex and we can apply the generalized Mordukhovich criterion and develop a fixed-point sufficient and necessary condition for the Lipschitz-like property relative to its domain. The corresponding graphical modulus is also presented. For affine variational inequalities, to characterize the Lipschitz-like property relative to a polyhedral convex set which lies within the domain of the solution mapping, we derive a generalized critical face condition in light of Dontchev and Rockafellar [18].

In Chapter 4, we take a step back to consider the Lipschitz-like property for the linear constraint system in Huyen and Yen [38]. We observe that for right-hand side and full perturbations, the Lipschitz-like property are equivalent and thereby give the relations of the Lipschitz-like property between different types of perturbations. Particularly for the linear constraint system (1.1.1) with a closed and convex set $K$, we establish the equivalences among the Lipschitz-like property of $S$, the normal cone and the tangent cone of its domain and the regularity at the candidate point. As an extension, we consider a linear constraint system with an explicit set constraint with full perturbations. Characterizations from perspectives of outer subdifferential and Robinson stability are also presented. Furthermore, we consider a variational inequality with linear approximation. With nonsingularity assumption of the matrix and some continuity assumptions on the approximated function $f$, we construct the equivalence between the original system and the linearized one. Lastly we consider a practical problem: linear portfolio selection. The stability analysis is performed both on feasible set mappings and the optimal solution mapping and some conditions that are easy to verify are obtained.

## 1.3 Preliminaries

In this section, we provide backgrounds on notations, tools and corresponding properties widely adopted in the study of variational analysis. Most of these can be found in monographs by Rockafellar and Wets [81] and Mordukhovich [65]. Readers who are familiar with these notations may safely skip this section.

For a nonempty set $C \subseteq \mathbb{R}^n$, the interior, the relative interior, the closure, the boundary, the convex hull and the positive hull of $C$ are denoted respectively by

$\operatorname{int} C$, $\operatorname{rint} C$, $\operatorname{cl} C$, $\operatorname{bdry} C$, $\operatorname{conv} C$ and $\operatorname{pos} C := \{0\} \cup \{\lambda x \mid x \in C \text{ and } \lambda > 0\}$. The orthogonal complement $C^{\perp}$, the polar cone $C^*$ and the horizon cone $C^{\infty}$ are defined respectively by

$$\begin{aligned}
&C^{\perp} := \{v \in \mathbb{R}^n \mid \langle v, x\rangle = 0,\ \forall x \in C\}, \\
&C^* := \{v \in \mathbb{R}^n \mid \langle v, x\rangle \le 0,\ \forall x \in C\},\ \text{and} \\
&C^{\infty} := \{x \in \mathbb{R}^n \mid \exists x_k \in C,\, \lambda_k \searrow 0, \text{ with } \lambda_k x_k \to x\}.
\end{aligned}$$

The support function $\sigma_C$ of $C$ is defined by

$$\sigma_C(x) := \sup_{v \in C} \langle v, x\rangle.$$

The indicator function $\delta_C$ of $C$ is defined by

$$\delta_C(x) := \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{otherwise.} \end{cases}$$

The distance from $x$ to $C$ is defined by

$$d(x, C) := \inf_{y \in C} ||y - x||.$$

The projection mapping $\operatorname{proj}_C$ is defined by

$$\operatorname{proj}_C(x) := \{y \in C \mid \|y - x\| = d(x, C)\}.$$

For a set $X \subset \mathbb{R}^n$, we denote the projection of $X$ onto $C$ by

$$\operatorname{proj}_C X := \{y \in C \mid \exists x \in X, \text{ with } \|y - x\| = d(x, C)\}.$$

If $C = \emptyset$, by convention we set that $d(x, C) := +\infty$, $\operatorname{proj}_C(x) := \emptyset$, and $\operatorname{proj}_C X := \emptyset$.

Let $x \in C$. We use $T_C(x)$ to denote the tangent/contingent cone to $C$ at $x$, i.e. $w \in T_C(x)$ if there exist sequences $t_k \searrow 0$ and $\{w_k\} \subset \mathbb{R}^n$ with $w_k \to w$ and $x + t_k w_k \in C,\ \forall k$. It can also be expressed as an outer limit:

$$T_C(x) = \limsup_{t \searrow 0} \frac{C - x}{t}.$$

The regular/Fréchet normal cone, $\widehat{N}_C(x)$, is the polar cone of $T_C(x)$, defined by

$$\widehat{N}_C(x) = \left\{ v \in \mathbb{R}^n \;\middle|\; \limsup_{\substack{x' \xrightarrow{C} x \\ \neq}} \frac{\langle v, x' - x \rangle}{\|x' - x\|} \leq 0 \right\}.$$

Here $x' \underset{\neq}{\xrightarrow{C}} x$ means $x' \to x$, $x' \in C$, $x' \neq x$. The (basic/limiting/Mordukhovich) normal cone to $C$ at $x$, $N_C(x)$, is defined via the outer limit of $\widehat{N}_C$ as

$$N_C(x) = \limsup_{x' \xrightarrow{C} x} \widehat{N}_C(x') := \left\{ v \in \mathbb{R}^n \mid \exists \text{ sequences } x_k \xrightarrow{C} x,\ v_k \to v,\ v_k \in \widehat{N}_C(x_k),\ \forall k \right\}.$$

If $C$ is a convex set,

$$\widehat{N}_C(x) = N_C(x) = \{ v \in \mathbb{R}^n \mid \langle v, x' - x \rangle \leq 0, \forall x' \in C \}.$$

We say that $C$ is locally closed at a point $x \in C$ if $C \cap U$ is closed for some closed neighborhood $U \in \mathcal{N}(x)$. $C$ is said to be regular at $x$ in the sense of Clarke if it is locally closed at $x$ and $\widehat{N}_C(x) = N_C(x)$. For any $x \notin C$, we set by convention $T_C(x) = \emptyset,\ N_C(x) = \emptyset,\ \widehat{N}_C(x) = \emptyset$.

Let $C \subset \mathbb{R}^n$ be a nonempty convex set. A face of $C$ is a convex subset $C'$ of $C$ such that every closed line segment in $C$ with a relative interior point in $C'$ has both endpoints in $C'$. An exposed face of $C$ is the intersection of $C$ and a non-trivial supporting hyperplane to $C$. In other words, $F$ is an exposed face of $C$ if and only if there is some $x \in \mathbb{R}^n$ such that $F = \arg\max_{v \in C} \langle x, v \rangle$. See the book [80] for more details. In this thesis, we also use the notion of semi-closed faces, which originated from Fang et al. [23] with 'semi-closed polyhedral. The concept 'semi-closed polyhedral' is an extension of polyhedron, defined as the intersection of finitely many closed or open half-spaces. For semi-closed faces $F'$, it can be expressed as the intersection of finitely many closed or open half-spaces and $\operatorname{cl} F'$ must correspond to some closed face $F$ of $C$.

Let $f : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ be an extended real-valued function and let $\bar{x}$ be a point with $f(\bar{x})$ finite. The vector $v \in \mathbb{R}^n$ is a regular/Fréchet subgradient of $f$ at $\bar{x}$, written $v \in \widehat{\partial} f(\bar{x})$, if

$$f(x) \geq f(\bar{x}) + \langle v, x - \bar{x} \rangle + o(||x - \bar{x}||).$$

The vector $v \in \mathbb{R}^n$ is a (general/basic) subgradient of $f$ at $\bar{x}$, written $v \in \partial f(\bar{x})$, if there exist sequences $x_k \to \bar{x}$ and $v_k \to v$ with $f(x_k) \to f(\bar{x})$ and $v_k \in \widehat{\partial} f(x_k)$. The subdifferential set $\partial f(\bar{x})$ is also referred to as limiting/Mordukhovich subdifferential. The vector $v \in \mathbb{R}^n$ is a horizon/singular subgradient of $f$ at $\bar{x}$, written $v \in \partial^\infty f(\bar{x})$, if there are sequences $x_k \to \bar{x}$ with $f(x_k) \to f(\bar{x})$, $\lambda_k \searrow 0$ and $v_k \in \widehat{\partial} f(x_k)$ such that $\lambda_k v_k \to v$.

The outer limiting subdifferential of $f$ at $\bar{x}$ introduced in [43] is denoted and defined as follows:

$$\partial^> f(\bar{x}) := \left\{ \lim_{k \to +\infty} v_k \mid \exists x_k \xrightarrow{f} \bar{x},\ \forall k :\ f(x_k) > f(\bar{x}) \text{ and } v_k \in \partial f(x_k) \right\}.$$

For a set-valued mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, we denote by

$$\operatorname{gph} S := \{(x, u) \mid u \in S(x)\} \quad \text{and} \quad \operatorname{dom} S := \{x \mid S(x) \neq \emptyset\}$$

the graph and the domain of $S$, respectively. $S$ is said to be positively homogeneous if

$$0 \in S(0) \quad \text{and} \quad S(\lambda x) = \lambda S(x) \quad \text{for all } \lambda > 0 \text{ and } x,$$

or in other words, $\operatorname{gph} S$ is a cone. If $S$ is a positively homogeneous mapping, the outer norm of $S$ is denoted and defined by

$$|S|^+ := \sup_{x \in \mathbb{B}} \sup_{u \in S(x)} \|u\|, \tag{1.3.1}$$

which is the infimum over all constants $\kappa \geq 0$ such that $\|u\| \leq \kappa \|x\|$ for all pairs $(x, u) \in \operatorname{gph} S$.

Consider a point $\bar{x} \in \operatorname{dom} S$. The outer limit of $S$ at $\bar{x}$ is defined by

$$\limsup_{x \to \bar{x}} S(x) := \{u \in \mathbb{R}^m \mid \exists x_k \to \bar{x}, \exists u_k \to u \text{ with } u_k \in S(x_k)\}.$$

$S$ is said to be outer semicontinuous at $\bar{x}$ if

$$\limsup_{x \to \bar{x}} S(x) \subset S(\bar{x}).$$

For a sequence of mappings $S^\nu : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, the graphical outer limit, is given via the graph:

$$\operatorname{gph}(\text{g-limsup}_\nu S^\nu) = \operatorname{limsup}_\nu(\operatorname{gph} S^\nu). \tag{1.3.2}$$

and therefore has $(\text{g-limsup}_\nu S^\nu)(x) = \bigcup_{\{x^\nu \to x\}} \limsup_{\nu \to \infty} S^\nu(x^\nu)$.

The (normal) coderivative and the regular/Fréchet coderivative of $S$ at $\bar{x}$ for any $\bar{u} \in S(\bar{x})$ are respectively the mapping $D^*S(\bar{x} \mid \bar{u}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ defined by

$$x^* \in D^*S(\bar{x} \mid \bar{u})(u^*) \Longleftrightarrow (x^*, -u^*) \in N_{\operatorname{gph} S}(\bar{x}, \bar{u}), \tag{1.3.3}$$

the mapping $\widehat{D}^*S(\bar{x} \mid \bar{u}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ defined by

$$x^* \in \widehat{D}^*S(\bar{x} \mid \bar{u})(u^*) \Longleftrightarrow (x^*, -u^*) \in \widehat{N}_{\operatorname{gph} S}(\bar{x}, \bar{u}).$$

Therefore with outer semicontinuity of normal cone mappings, by (1.3.2),

$$D^*S(\bar{x} \mid \bar{u}) = \underset{(x,u) \xrightarrow{\operatorname{gph} S} (\bar{x}, \bar{u})}{\text{g-limsup}} \hat{D}^*S(x \mid u) = \underset{(x,u) \xrightarrow{\operatorname{gph} S} (\bar{x}, \bar{u})}{\text{g-limsup}} D^*S(x \mid u).$$

For a set $X \subset \mathbb{R}^n$, we denote by

$$S|_X := \begin{cases} S(x) & \text{if } x \in X, \\ \emptyset & \text{if } x \notin X, \end{cases}$$

the restricted mapping of $S$ on $X$. It is clear to see that

$$\operatorname{gph} S|_X = \operatorname{gph} S \cap (X \times \mathbb{R}^m) \quad \text{and} \quad \operatorname{dom} S|_X = X \cap \operatorname{dom} S.$$

**Definition 1.3.1** (local boundedness relative to a set,[81, Page 162])**.** *For a mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, a closed set $X \subset \mathbb{R}^n$ and a given point $\bar{x} \in X$, if for some neighborhood $V \in \mathcal{N}(\bar{x})$, $S(V \cap X)$ is bounded, we say $S$ is locally bounded relative to $X$ at $\bar{x}$. Such definition is equivalent to local boundedness of $S|_X$ at $\bar{x}$, where $S|_X$ means the mapping $S$ restricted to $X$.*

**Remark 1.3.2.** *By definition of local boundedness, if $S$ is locally bounded at $\bar{x}$, then $S$ is also locally bounded in a certain neighborhood of $\bar{x}$. Similarly, if $S$ is locally bounded relative to $X$ at $\bar{x}$, then $S$ is also locally bounded relative to $X$ in a certain neighborhood of $\bar{x}$.*

Similar to definition of regularity in [81, Defnition 6.4], here we introduce a local version:

**Definition 1.3.3.** *For a set $C \subseteq \mathbb{R}^n$, we say $C$ is regular at around $\bar{x} \in C$ (in the sense of Clarke) if it is locally closed around $\bar{x}$ and there exists a neighborhood $X$ of $\bar{x}$, such that for any $x \in X \cap C$, $\widehat{N}_C(x) = N_C(x)$.*

**Definition 1.3.4** (Outer semicontinuity relative to a set, [81, Definition 5.4])**.** *A set-valued mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is outer semicontinuous (osc) at $\bar{x}$ relative to $X$ if $\bar{x} \in X$ and*

$$\limsup_{x \xrightarrow{X} \bar{x}} S(x) = S(\bar{x}).$$

*Such definition is equivalent to outer semicontinuity of $S|_X$ as*

$$\limsup_{x \xrightarrow{X} \bar{x}} S(x) = \limsup_{x \to \bar{x}} S|_X(x) = S|_X(\bar{x}).$$

Next we present some definitions and properties on stability of $S$, most of which are borrowed from [81].

**Definition 1.3.5** (Lipschitz-like property relative to a set, [81, Definition 9.36])**.** *A mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ has the Lipschitz-like property relative to $X$ at $\bar{x}$ for $\bar{u}$, where $\bar{x} \in X$ and $\bar{u} \in S(\bar{x})$, if* $\operatorname{gph} S$ *is locally closed at $(\bar{x}, \bar{u})$ and there are neighborhoods $V \in \mathcal{N}(\bar{x})$, $W \in \mathcal{N}(\bar{u})$, and a constant $\kappa \in \mathbb{R}_+$ such that*

$$S(x') \cap W \subset S(x) + \kappa \|x' - x\| \mathbb{B} \quad \forall x, x' \in X \cap V. \tag{1.3.4}$$

*The graphical modulus of $S$ relative to $X$ at $\bar{x}$ for $\bar{u}$ is then*

$$\begin{aligned} \operatorname{lip}_X S(\bar{x} \mid \bar{u}) := \inf \{ \ \kappa \geq 0 \ \ | \ \exists V \in \mathcal{N}(\bar{x}), W \in \mathcal{N}(\bar{u}), \ \textit{such that} \\ S(x') \cap W \subset S(x) + \kappa \|x' - x\| \mathbb{B}, \quad \forall x, x' \in X \cap V \ \}. \end{aligned}$$

The property with $V$ in place of $X \cap V$ in (1.3.4) is the Lipschitz-like property along with the graphical modulus $\operatorname{lip} S(\bar{x} \mid \bar{u})$. For the Lipschitz-like property, a useful test is provided, the Mordukhovich criterion.

**Theorem 1.3.6** (Mordukhovich criterion, see [81, Theorem 9.40], [65, Theorem 4.10])**.** *For a mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with* $\operatorname{gph} S$ *being locally closed at $(\bar{x}, \bar{u}) \in \operatorname{gph} S$, $S$ has the Lipschitz-like property at $\bar{x}$ for $\bar{u}$ if and only if*

$$D^* S(\bar{x} \mid \bar{u})(0) = \{0\}, \tag{1.3.5}$$

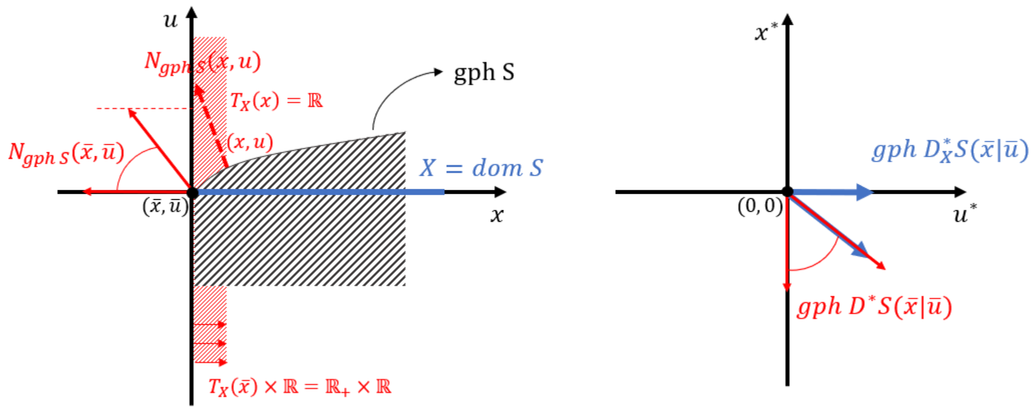
*or equivalently $|D^* S(\bar{x} \mid \bar{u})|^+ < \infty$. In this case $\operatorname{lip} S(\bar{x} \mid \bar{u}) = |D^* S(\bar{x} \mid \bar{u})|^+$.*

To characterize the relative Lipschitz-like property, in [59], a new tool, the projectional coderivative, is introduced.

**Definition 1.3.7** ([59, Definition 2.2])**.** *$D^*_X S(\bar{x} \mid \bar{u}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ of $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at $\bar{x} \in X$ for any $\bar{u} \in S(\bar{x})$ with respect to $X$ is defined as*

$$t^* \in D^*_X S(\bar{x} \mid \bar{u})(u^*) \Longleftrightarrow (t^*, -u^*) \in \limsup_{(x,u) \xrightarrow{\operatorname{gph} S|_X} (\bar{x}, \bar{u})} \operatorname{proj}_{T_X(x) \times \mathbb{R}^m} N_{\operatorname{gph} S|_X}(x, u). \tag{1.3.6}$$

Figure 1.1: An example of projectional coderivatives




Here we give an example of the calculation on projectional coderivatives.

Given this new tool, a handy test for the Lipschitz-like property relative to a closed and convex set is developed similarly to the Mordukhovich criterion, and is named as the generalized Mordukhovich criterion.

**Theorem 1.3.8** (generalized Mordukhovich criterion, [59, Theorem 2.4])**.** *Consider $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, $\bar{x} \in X \subset \mathbb{R}^n$ and $\bar{u} \in S(\bar{x})$. Suppose that* $\operatorname{gph} S$ *is locally closed at $(\bar{x}, \bar{u})$ and that $X$ is closed and convex. Then $S$ has the Lipschitz-like property relative to $X$ at $\bar{x}$ for $\bar{u}$ if and only if*

$$D_X^* S(\bar{x} \mid \bar{u})(0) = \{0\} \tag{1.3.7}$$

*or equivalently $|D_X^* S(\bar{x} \mid \bar{u})|^+ < +\infty$. In this case,*

$$\operatorname{lip}_X S(\bar{x} \mid \bar{u}) = |D_X^* S(\bar{x} \mid \bar{u})|^+. \tag{1.3.8}$$

For a single-valued mapping $F : D \to \mathbb{R}^m$ where $D \subseteq \mathbb{R}^n$, we say that $F$ is strictly continuous at $\bar{x}$ relative to $X \subseteq D$ if $\bar{x} \in X$ and

$$\operatorname{lip}_X F(\bar{x}) := \limsup_{\substack{x, x' \xrightarrow{X} \bar{x} \\ x \neq x'}} \frac{\|F(x') - F(x)\|}{\|x' - x\|}$$

is finite. $F$ is strictly continuous at $\bar{x}$ if $\bar{x} \in \operatorname{int} D$ and

$$\operatorname{lip} F(\bar{x}) := \limsup_{\substack{x, x' \longrightarrow \bar{x} \\ x \neq x'}} \frac{\|F(x') - F(x)\|}{\|x' - x\|}$$

is finite.

# Chapter 2

# Projectional Coderivatives and Chain Rules

The projectional coderivatives initiated by [59] (see Definition 1.3.6), was invented to capture the Lipschitz-like property of a multifunction relative to a closed and convex set in full. Such a definition is derived via the sufficient and necessary condition for the property (see [59, Theorem 2.3]) with existence of $\kappa > 0$:

$$\|\mathrm{proj}_{T_X(x)}(x^*)\| \le \kappa \|u^*\|,\ \forall x^* \in D^*S|_X(x \mid u)(u^*)$$

for all $(x, u)$ being close enough to $(\bar{x}, \bar{u})$ in $\mathrm{gph}\, S|_X$. As by definition,

$$t^* \in D^*_X S(\bar{x} \mid \bar{u})(u^*) \Longleftrightarrow (t^*, -u^*) \in \limsup_{(x,u) \xrightarrow{\mathrm{gph}\, S|_X} (\bar{x}, \bar{u})} \mathrm{proj}_{T_X(x) \times \mathbb{R}^m} N_{\mathrm{gph}\, S|_X}(x, u).$$

the projectional coderivative involves taking limsup of projected normal cones of points $(x, u)$ tending to $(\bar{x}, \bar{u})$ in $\mathrm{gph}\, S|_X$. When $\bar{x} \in \mathrm{int}\, X$, the projectional coderivative becomes coderivatives naturally. Given the osc of normal cone mappings, we start to explore under what condition will the projected normal cone be osc as well, i.e.,

$$\limsup_{(x,u) \xrightarrow{\mathrm{gph}\, S|_X} (\bar{x}, \bar{u})} \mathrm{proj}_{T_X(x) \times \mathbb{R}^m} N_{\mathrm{gph}\, S|_X}(x, u) = \mathrm{proj}_{T_X(\bar{x}) \times \mathbb{R}^m} N_{\mathrm{gph}\, S|_X}(\bar{x}, \bar{u}).$$

Based on the fact that the tangent cone of a smooth manifold varies from one point to another continuously, we are able to express the projectional coderivative in a fixed-point pattern and develop the equation for the chain rules.

## 2.1 Projectional coderivatives and properties of smooth manifolds

We first introduce some properties of projection and some natural observations of projectional coderivatives. Below we present an observation on the connection between projectional coderivatives and coderivatives.

**Lemma 2.1.1.** *For a set valued-mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, and a closed set $X \subseteq \mathbb{R}^n$, for any $(\bar{x}, \bar{w}) \in \operatorname{gph} S|_X$,*

$$D^*S|_X(\bar{x} \mid \bar{w})^{-1}(0) \subseteq D^*_X S(\bar{x} \mid \bar{w})^{-1}(0). \tag{2.1.1}$$

*Proof.* For $w' \in D^*S|_X(\bar{x} \mid \bar{w})^{-1}(0)$, it is equivalent that $(0, -w') \in N_{\operatorname{gph} S|_X}(\bar{x}, \bar{w})$. As $\operatorname{proj}_{T_X(\bar{x})}(0) = 0$, we have $(0, -w') \in \operatorname{proj}_{T_X(\bar{x}) \times \mathbb{R}^m} N_{\operatorname{gph} S|_X}(\bar{x}, \bar{w})$. Then by definition of projectional coderivatives, $w' \in D^*_X S(\bar{x} \mid \bar{w})^{-1}(0)$. □

Next we introduce some properties of projection onto cones.

**Lemma 2.1.2.** *For any nonempty closed convex cone $K \subseteq \mathbb{R}^n$ with its polar $K^* = K^\perp$, where $K^\perp = \{v | \langle v, x \rangle = 0, \ \forall x \in K\}$,*

$$\operatorname{proj}_K(v + y) = \operatorname{proj}_K(y), \ \textit{for any } y \in \mathbb{R}^n \textit{ and } v \in K^\perp. \tag{2.1.2}$$

*Proof.* Let $y \in \mathbb{R}^n$ and $x = \operatorname{proj}_K(y)$. As $K$ is a nonempty closed and convex set, [4, Theorem 3.16], it is equivalent that $x \in K$ and $y - x \in N_K(x)$. Then for any $x' \in K$, $\langle y - x, x' - x \rangle \leq 0$. For any choice of $v \in K^* = K^\perp$ we have $\langle v, x' - x \rangle = 0$. Therefore $\langle v + y - x, x' - x \rangle \leq 0$ and $v + y - x \in N_X(x)$. Thus $x = \operatorname{proj}_K(v + y)$. Then the equation (2.1.2) is proved along with uniqueness of projection. □

**Lemma 2.1.3.** *For a nonempty closed cone $C \subseteq \mathbb{R}^n$, for any $y \in \mathrm{proj}_C(x)$, $\lambda y \in \mathrm{proj}_C(\lambda x)$ for any $\lambda \geq 0$.*

*Proof.* As $\lambda \geq 0$, $y \in C$ and $C$ is a cone, $\lambda y \in C$. If $\lambda = 0$, $0 \in \mathrm{proj}_C(0)$. If $\lambda > 0$, suppose $\lambda y \notin \mathrm{proj}_C(\lambda x)$, i.e., $\exists w \in C$ such that $d(\lambda x, C) = d(\lambda x, w) < d(\lambda x, \lambda y) = \lambda d(x, y)$. Again as $C$ is a cone we can always find $y' \in C$ such that $w = \lambda y'$. Then $d(\lambda x, w) = d(\lambda x, \lambda y') = \lambda d(x, y') \geq \lambda d(x, C) = \lambda d(x, y)$, which contradicts our assumption. □

From the expression of (1.3.6) we can see that the calculation of projectional coderivative also involves neighboring points. When the set we refer to, $X$, has special structure, for example, a smooth manifold, such representation can be refined as a fixed-point expression. Before introducing the exact form, we present some basic properties of a smooth manifold in the following proposition. In what follows, let $X$ be a $d$-dimensional smooth manifold in $\mathbb{R}^n$ around the point $\bar{x} \in X$, in the sense that $X$ can be represented relative to an open neighborhood $O \in \mathcal{N}(\bar{x})$ as the set of solutions to $F(x) = 0$, where $F : O \to \mathbb{R}^{n-d}$ is a smooth (i.e., $\mathcal{C}^1$) mapping with $\nabla F(\bar{x})$ of full rank $n - d$. This definition is borrowed from [81, Example 6.8]. For more thorough details of smooth manifold, see the monograph by Lee [52].

**Proposition 2.1.4** (Basic properties of smooth manifolds)**.** *Let $X \in \mathbb{R}^n$ be a smooth manifold at $\bar{x}$. We have the following basic properties.*

**(a)** *$X$ is regular at around $\bar{x}$. The tangent and normal cones to $X$ at any $x$ being close enough to $\bar{x}$ are linear subspaces orthogonally complementary to each other, namely*

$$T_X(x) = \{w \in \mathbb{R}^n \mid \nabla F(x) w = 0\} \quad \textit{and} \quad N_X(x) = \{\nabla F(x)^* y \mid y \in \mathbb{R}^{n-d}\}.$$

*Moreover,*

$$\mathrm{proj}_{T_X(x)}(x^*) = \left[I - \nabla F(x)^* \left(\nabla F(x) \nabla F(x)^*\right)^{-1} \nabla F(x)\right] x^* \quad \forall x^*. \tag{2.1.3}$$

**(b)** *For any $x$ being close enough to $\bar{x}$ in $X$ and $x^* \in \mathbb{R}^n$, it holds that*

$$T_X(x_k) \to T_X(x) \quad \textit{and} \quad \mathrm{proj}_{T_X(x_k)}(x_k^*) \to \mathrm{proj}_{T_X(x)}(x^*),$$

*where $\{x_k\}$ and $\{x_k^*\}$ are two sequences such that $x_k \xrightarrow{X} x$ and $x_k^* \to x^*$.*

**(c)** *For any $\varepsilon > 0$, there exists some $\delta > 0$ such that*

$$\left\| \frac{y-x}{\|y-x\|} - \mathrm{proj}_{T_X(x)}\left(\frac{y-x}{\|y-x\|}\right) \right\| \leq \varepsilon$$

*holds for all $x, y \in X \cap \mathbb{B}_\delta(\bar{x})$ with $x \neq y$.*

*Proof.* As $F : O \to \mathbb{R}^{n-d}$ is smooth with $\nabla F(\bar{x})$ of full rank $n-d$, $\nabla F(x)$ is also of full rank $n-d$ for all $x$ close enough to $\bar{x}$ in $X$. The properties in (a) then follows readily from [81, Exercise 6.7, Example 6.8]. Therefore the projection onto $T_X(x)$ is equivalent to projection on the column space of $\nabla F(x)$. Given that $\nabla F(x)$ has full rank $n-d$, by [49, Page 365] we have (2.1.3).

For (b), with regularity, $T_X(x) = \widehat{T}_X(x) = \liminf_{x' \xrightarrow{X} x} T_X(x')$ by [81, Corollary 6.29 (b)]. To prove continuity relative to $X$, it remains to prove $\limsup_{x' \xrightarrow{X} x} T_X(x') \subseteq T_X(x)$. For $w \in \limsup_{x' \xrightarrow{X} x} T_X(x')$, it is equivalent that there exist sequences $x_k \xrightarrow{X} x$ and $w_k \in T_X(x_k) = \{w \in \mathbb{R}^n \mid \nabla F(x_k)w = 0\}$ such that $w_k \to w$. Given $\nabla F(x_k) \to \nabla F(x)$ when $x_k \to x$ and $\nabla F(x_k)w_k = 0$ , then $\nabla F(x_k)w_k \to \nabla F(x)w = 0$ when $k \to \infty$, which shows $w \in T_X(x)$ and thus $T_X(\cdot)$ is continuous at $x$ relative to $X$ and always convex-valued. By [81, Example 5.35], we have $\mathrm{proj}_{T_X(x_k)}(x_k^*) \to \mathrm{proj}_{T_X(x)}(x^*)$ for $x_k \xrightarrow{X} x$ and $x_k^* \to x^*$.

It remains to show (c). Suppose by contradiction that there exist some $\varepsilon_0 > 0$ and some sequences $x_k, y_k \xrightarrow{X} \bar{x}$ with $x_k \neq y_k$ such that

$$\left\| \frac{y_k - x_k}{\|y_k - x_k\|} - \mathrm{proj}_{T_X(x_k)}\left(\frac{y_k - x_k}{\|y_k - x_k\|}\right) \right\| > \varepsilon_0 \quad \forall k. \tag{2.1.4}$$

Without loss of generality, we can assume that $x_k, y_k \in O$ with $\nabla F(x_k)$ of full rank $n - d$ for all $k$, and that there is some $w \in \mathbb{S}$ such that

$$\frac{y_k - x_k}{\|y_k - x_k\|} \to w.$$

It then follows that

$$\int_0^1 \nabla F(\tau y_k + (1-\tau)x_k)\, d\tau \cdot \frac{y_k - x_k}{\|y_k - x_k\|} = \frac{F(y_k) - F(x_k)}{\|y_k - x_k\|} = 0 \quad \forall k, \tag{2.1.5}$$

where the integral of a matrix is to be understood componentwise. Applying componentwise the first mean value theorem for definite integrals, we have

$$\int_0^1 \nabla F(\tau y_k + (1-\tau)x_k)\, d\tau \to \nabla F(\bar{x}),$$

and hence

$$\int_0^1 \nabla F(\tau y_k + (1-\tau)x_k)\, d\tau \cdot \frac{y_k - x_k}{\|y_k - x_k\|} \to \nabla F(\bar{x})w.$$

In view of (2.1.5), we have

$$\nabla F(\bar{x})w = 0.$$

As $\nabla F(x_k)$ is of full row rank, we have

$$T_X(x_k) = \{w \mid \nabla F(x_k)w = 0\},$$

and hence

$$\begin{aligned} &\frac{y_k - x_k}{\|y_k - x_k\|} - \mathrm{proj}_{T_X(x_k)}\left(\frac{y_k - x_k}{\|y_k - x_k\|}\right) \\ &= \nabla F(x_k)^* \left(\nabla F(x_k)\nabla F(x_k)^*\right)^{-1} \nabla F(x_k)\frac{y_k - x_k}{\|y_k - x_k\|} \\ &\to \nabla F(\bar{x})^* \left(\nabla F(\bar{x})\nabla F(\bar{x})^*\right)^{-1} \nabla F(\bar{x})w = 0, \end{aligned}$$

contradicting to (2.1.4). This completes the proof. $\square$

**Proposition 2.1.5** (Coderivatives of a set-valued mapping restricted on a smooth manifold)**.** *Consider* $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *and* $\bar{u} \in S(\bar{x})$*. Suppose that* $\operatorname{gph} S$ *is locally closed at* $(\bar{x}, \bar{u})$ *and* $X$ *is a smooth manifold at around* $\bar{x}$ *with* $\bar{x} \in X$*. The following properties hold for all* $(x, u)$ *close enough to* $(\bar{x}, \bar{u})$ *in* $\operatorname{gph} S|_X$*:*

**(a)** $\operatorname{proj}_{T_X(x)\times\mathbb{R}^m} \widehat{N}_{\operatorname{gph} S|_X}(x, u) = \widehat{N}_{\operatorname{gph} S|_X}(x, u) \cap (T_X(x) \times \mathbb{R}^m)$.

**(b)** $\operatorname{proj}_{T_X(x)\times\mathbb{R}^m} N_{\operatorname{gph} S|_X}(x, u) = N_{\operatorname{gph} S|_X}(x, u) \cap (T_X(x) \times \mathbb{R}^m)$.

**(c)** $D^*_X S(x \mid u)(u^*) = \operatorname{proj}_{T_X(x)} D^* S|_X(x \mid u)(u^*) = D^* S|_X(x \mid u)(u^*) \cap T_X(x), \quad \forall u^*$.

*Proof.* In what follows, let $(x, u)$ be close enough to $(\bar{x}, \bar{u})$ in $\operatorname{gph} S|_X$ such that the properties in Proposition 2.1.4 (a) and (b) holds.

To prove (a), it suffices to show

$$\operatorname{proj}_{T_X(x)\times\mathbb{R}^m} \widehat{N}_{\operatorname{gph} S|_X}(x, u) \subset \widehat{N}_{\operatorname{gph} S|_X}(x, u) \cap (T_X(x) \times \mathbb{R}^m)\,. \tag{2.1.6}$$

Let $(y^*, u^*)$ belong to the left-hand side of (2.1.6). Then there exists some $x^*$ such that $y^* = \operatorname{proj}_{T_X(x)}(x^*)$ and $(x^*, u^*) \in \widehat{N}_{\operatorname{gph} S|_X}(x, u)$. Then by definition we have

$$\limsup_{\substack{(x',u') \xrightarrow{\operatorname{gph} S|_X} (x,u) \\ (x',u') \neq (x,u)}} \frac{\langle (x^*, u^*),\ (x' - x, u' - u)\rangle}{\|(x' - x, u' - u)\|} \leq 0. \tag{2.1.7}$$

Let $z^* := \operatorname{proj}_{N_X(x)}(x^*)$. As $N_X(x)$ is a linear subspace, we have $\pm z^* \in N_X(x)$. This implies that

$$\lim_{\substack{x' \xrightarrow{X} x \\ x' \neq x}} \frac{\langle z^*, x' - x\rangle}{\|x' - x\|} = 0,$$

and hence that

$$\lim_{\substack{(x',u') \xrightarrow{\operatorname{gph} S|_X} (x,u) \\ (x',u') \neq (x,u)}} \frac{\langle z^*, x' - x\rangle}{\|(x' - x, u' - u)\|} = 0. \tag{2.1.8}$$

Since $x^* = y^* + z^*$, it follows from (2.1.7) and (2.1.8) that

$$\limsup_{\substack{(x',u') \xrightarrow{\operatorname{gph} S|_X} (x,u) \\ (x',u') \neq (x,u)}} \frac{\langle (y^*, u^*),\ (x'-x, u'-u) \rangle}{\|(x'-x, u'-u)\|} \leq 0,$$

which amounts to that $(y^*, u^*) \in \widehat{N}_{\operatorname{gph} S|_X}(x, u)$. From the fact that $y^* \in T_X(x)$, it then follows that $(y^*, u^*)$ belongs to the right-hand side of (2.1.6).

To prove (b), it suffices to show

$$\operatorname{proj}_{T_X(x) \times \mathbb{R}^m} N_{\operatorname{gph} S|_X}(x, u) \subset N_{\operatorname{gph} S|_X}(x, u) \cap (T_X(x) \times \mathbb{R}^m). \tag{2.1.9}$$

Let $(y^*, u^*)$ belong to the left-hand side of (2.1.9). Then there exists $x^*$ such that $y^* = \operatorname{proj}_{T_X(x)}(x^*)$ and $(x^*, u^*) \in N_{\operatorname{gph} S|_X}(x, u)$. By definition there are sequences $(x_k, u_k) \xrightarrow{\operatorname{gph} S|_X} (x, u)$ and $(x_k^*, u_k^*) \in \widehat{N}_{\operatorname{gph} S|_X}(x_k, u_k)$ such that $(x_k^*, u_k^*) \to (x^*, u^*)$. In view of (a), we have for all sufficiently large $k$,

$$\left(\operatorname{proj}_{T_X(x_k)}(x_k^*),\ u_k^*\right) \in \widehat{N}_{\operatorname{gph} S|_X}(x_k, u_k).$$

By Proposition 2.1.4 (b), we have $\operatorname{proj}_{T_X(x_k)}(x_k^*) \to \operatorname{proj}_{T_X(x)}(x^*) = y^*$. Then by definition we have

$$(y^*, u^*) \in N_{\operatorname{gph} S|_X}(x, u).$$

From the fact that $y^* \in T_X(x)$, it then follows that $(y^*, u^*)$ belongs to the right-hand side of (2.1.9). Let $u^* \in \mathbb{R}^m$. From (b) and the definition of coderivatives (1.3.3), we get

$$D_X^* S(x \mid u)(u^*) \supset \operatorname{proj}_{T_X(x)} D^* S|_X(x \mid u)(u^*) = D^* S|_X(x \mid u)(u^*) \cap T_X(x).$$

To show (c), it suffices to show

$$D_X^* S(x \mid u)(u^*) \subset D^* S|_X(x \mid u)(u^*) \cap T_X(x). \tag{2.1.10}$$

Let $y^*$ belong to the left-hand side of (2.1.10). Then by definition there are some $(x_k, u_k) \xrightarrow{\operatorname{gph} S|_X} (x, u)$ and $x_k^* \in D^*S|_X(x_k \mid u_k)(u_k^*)$ such that $u_k^* \to u^*$ and $y_k^* := \operatorname{proj}_{T_X(x_k)}(x_k^*) \to y^*$. By (b), we have for all sufficiently large $k$,

$$y_k^* \in D^*S|_X(x_k \mid u_k)(u_k^*),$$

implying that $y^* \in D^*S|_X(x \mid u)(u^*)$. As $y_k^* \in T_X(x_k)$, we get from Proposition 2.1.4 (b) that $y^* \in T_X(x)$. That is, $y^*$ belongs to the right-hand side of (2.1.10). This completes the proof. □

Next we give a simple example to for geometric interpretation of the properties.

**Example 2.1.6.** *Consider a multifunction $S : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ defined as*

$$S\left((x_1, x_2)^\top\right) = \begin{cases} \{(0, -x_2)^\top\}, & x_1 \neq 0 \\ \mathbb{R}^2, & x_1 = 0 \end{cases}.$$

*For $X = \mathbb{R} \times \{1\} \subseteq \operatorname{dom} S = \mathbb{R}^2$ and $\bar{x} = (0, 1)^\top$, $\bar{u} = (0, 0)^\top$, $(\bar{x}, \bar{u}) \in \operatorname{gph} S|_X$. By calculation we have $N_{\operatorname{gph} S|_X}(\bar{x} \mid \bar{u}) = \mathbb{R}^2 \times \{0_2\}$, $T_X(\bar{x}) = \mathbb{R} \times \{0\}$ and*

$$D^*S|_X(\bar{x} \mid \bar{u})(u^*) = \begin{cases} \mathbb{R}^2, & u^* = (0, 0)^\top \\ \emptyset, & u^* \neq (0, 0)^\top \end{cases}.$$

*Then we can see that*

$$\begin{aligned} &\operatorname{proj}_{T_X(x) \times \mathbb{R}^2} N_{\operatorname{gph} S|_X}(\bar{x} \mid \bar{u}) = \operatorname{proj}_{\mathbb{R} \times \{0\} \times \mathbb{R}^2} \left(\mathbb{R}^2 \times \{0_2\}\right) \\ =&\mathbb{R} \times \{0_3\} = N_{\operatorname{gph} S|_X}(\bar{x} \mid \bar{u}) \cap (T_X(\bar{x}) \times \mathbb{R}^m) \end{aligned}$$

*and*

$$\begin{aligned} D_X^*S(\bar{x} \mid \bar{u})(u^*) = \operatorname{proj}_{T_X(\bar{x})} D^*S|_X(\bar{x} \mid \bar{u})(u^*) &= \begin{cases} \mathbb{R} \times \{0\}, & u^* = (0, 0)^\top \\ \emptyset, & u^* \neq (0, 0)^\top \end{cases} \\ &= D^*S|_X(\bar{x} \mid \bar{u}) \cap T_X(\bar{x}). \end{aligned}$$

The coming corollary is a natural observation from Propositions 2.1.4 and 2.1.5.

**Corollary 2.1.7.** *Consider* $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *and* $\bar{u} \in S(\bar{x})$*. Suppose that* $\operatorname{gph} S$ *is locally closed at* $(\bar{x}, \bar{u})$*,* $\bar{x} \in X$ *and* $X$ *is a smooth manifold at around* $\bar{x}$ *with* $X \subseteq \operatorname{dom} S$*. Then the mapping* $(x, u) \mapsto \operatorname{proj}_{T_X(x) \times \mathbb{R}^m} N_{\operatorname{gph} S|_X}(x, u)$ *is outer semicontinuous relative to* $\operatorname{gph} S|_X$ *at* $(\bar{x}, \bar{u})$ *and*

$$D_X^* S(\bar{x} \mid \bar{u}) = \underset{(x,u) \xrightarrow{\operatorname{gph} S|_X} (\bar{x},\bar{u})}{\text{g-limsup}} D_X^* S(x \mid u) = \operatorname{proj}_{T_X(\bar{x})} D^* S|_X(\bar{x} \mid \bar{u}). \tag{2.1.11}$$

*Proof.* By Proposition 2.1.4 (b) and outer semicontinuity of the normal cone mapping $N_{\operatorname{gph} S|_X}$ relative to $\operatorname{gph} S|_X$ at $(\bar{x}, \bar{u})$, we have

$$\limsup_{(x,u) \xrightarrow{\operatorname{gph} S|_X} (\bar{x},\bar{u})} \operatorname{proj}_{T_X(x) \times \mathbb{R}^m} N_{\operatorname{gph} S|_X}(x, u) = \operatorname{proj}_{T_X(\bar{x}) \times \mathbb{R}^m} N_{\operatorname{gph} S|_X}(\bar{x}, \bar{u}).$$

By definition of projectional coderivative (1.3.6) and the property (c) in Proposition 2.1.5, we have (2.1.11). □

Although the generalized Mordukhovich criterion (Theorem 1.3.8) provides a useful tool to examine the relative Lipschitz-like property, we next show by a smooth function that the calculation of projectional coderivative may not be as simple as the coderivatives as the projection of normal cone does not enjoy outer semicontinuity.

**Lemma 2.1.8** (Projectional coderivatives of a smooth function)**.** *For* $F : \mathbb{R}^n \to \mathbb{R}^m$ *being smooth and single-valued on* $\mathbb{R}^n$ *and a closed set* $X \subseteq \mathbb{R}^n$*, for any* $\bar{x} \in \operatorname{bdry} X$*,*

$$D_X^* F(\bar{x})(y) = \limsup_{x \xrightarrow{X} \bar{x}, y' \to y} \left\{ \operatorname{proj}_{T_X(x)} (\nabla F(x)^* y' + w) \mid w \in N_X(x) \right\}. \tag{2.1.12}$$

*If furthermore* $X$ *is regular at around* $\bar{x}$*,*

$$D_X^* F(\bar{x})(y) = \limsup_{x \xrightarrow{X} \bar{x}} \left\{ \operatorname{proj}_{T_X(x)} (\nabla F(x)^* y + w) \mid w \in N_X(x) \right\}. \tag{2.1.13}$$

*In particular when $X$ is a smooth manifold at around $\bar{x}$,*

$$D_X^* F(\bar{x})(y) = \mathrm{proj}_{T_X(\bar{x})} \left(\nabla F(\bar{x})^* y\right). \tag{2.1.14}$$

*Proof.* For smooth mapping $F$ defined on $\mathbb{R}^n$, we can always find an open set $O \supseteq X$ such that $F$ remains smooth on $O$. In this way, $\nabla F|_X(x) = \nabla F(x)$ for any $x \in X$. Then for any $x \in X$ and [81, Example 8.34],

$$N_{\mathrm{gph}\, F}(x, F(x)) = \left\{ (\nabla F(x)^* y, -y) \;\middle|\; y \in \mathbb{R}^m \right\}.$$

By expressing $F|_X = F + \delta_X$ and [81, Exercise 10.43],

$$D^* F|_X(x)(y) = N_X(x) + \nabla F(x)^* y, \ \forall y \in \mathbb{R}^m.$$

That is, for all $x \in X$,

$$N_{\mathrm{gph}\, F|_X}(x, F(x)) = \left\{ (\nabla F(x)^* y + w, -y) \;\middle|\; y \in \mathbb{R}^m, w \in N_X(x) \right\}.$$

From definition of projectional coderivative (1.3.6),

$$\begin{aligned}
& t \in D_X^* F(\bar{x})(y) \\
\iff & (t, -y) \in \limsup_{x \xrightarrow{X} \bar{x}} \mathrm{proj}_{T_X(x) \times \mathbb{R}^m} N_{\mathrm{gph}\, S|_X}(x, F(x)) \\
\iff & (t, -y) \in \limsup_{x \xrightarrow{X} \bar{x}} \mathrm{proj}_{T_X(x) \times \mathbb{R}^m} \left\{ (\nabla F(x)^* y' + w, -y') \;\middle|\; y' \in \mathbb{R}^m, \ w \in N_X(x) \right\}.
\end{aligned}$$

Therefore we have (2.1.12).

For $t \in \limsup_{x \xrightarrow{X} \bar{x}, y' \to y} \left\{ \mathrm{proj}_{T_X(x)} \left(\nabla F(x)^* y' + w\right), \ w \in N_X(x) \right\}$, there exist sequences $x_k \xrightarrow{X} \bar{x}$, $w_k \in N_X(x_k)$, $y_k \in \mathbb{R}^m$ and $t_k \in \mathrm{proj}_{T_X(x_k)} \left(\nabla F(x_k)^* y_k + w_k\right)$, such that $t_k \to t$ and $y_k \to y$. When $X$ is regular at around $\bar{x}$, $T_X(x)$ is convex for all $x \in X$ around $\bar{x}$. With nonexpansive property of $\mathrm{proj}_{T_X(x_k)}$ for sufficiently large $k$, we have

$$\left\| \mathrm{proj}_{T_X(x_k)} \left(\nabla F(x_k)^* y + w_k\right) - t \right\|$$

$$\le \left\|\mathrm{proj}_{T_X(x_k)}\left(\nabla F(x_k)^* y + w_k\right) - \mathrm{proj}_{T_X(x_k)}\left(\nabla F(x_k)^* y_k + w_k\right)\right\|$$
$$+ \left\|\mathrm{proj}_{T_X(x_k)}\left(\nabla F(x_k)^* y_k + w_k\right) - t\right\|$$
$$\le \|\nabla F(x_k)^* (y - y_k)\| + \left\|\mathrm{proj}_{T_X(x_k)}\left(\nabla F(x_k)^* y_k + w_k\right) - t\right\|.$$

As $\|\nabla F(x_k)^* (y - y_k)\|$ and $\left\|\mathrm{proj}_{T_X(x_k)}\left(\nabla F(x_k)^* y_k + w_k\right) - t\right\|$ both tend to 0 when $k \to \infty$, we have $\mathrm{proj}_{T_X(x_k)}\left(\nabla F(x_k)^* y + w_k\right) \to t$ as well. Therefore we have

$$\limsup_{x \xrightarrow{X} \bar{x}, y' \to y} \left\{\mathrm{proj}_{T_X(x)}\left(\nabla F(x)^* y' + w\right),\ w \in N_X(x)\right\}$$
$$\subseteq \limsup_{x \xrightarrow{X} \bar{x}} \left\{\mathrm{proj}_{T_X(x)}\left(\nabla F(x)^* y + w\right),\ w \in N_X(x)\right\}.$$

Given that the inclusion in reverse is obvious by taking $y_k := y$, we arrive at (2.1.13).

If furthermore $X$ is a smooth manifold, $T_X(x)$ and $N_X(x)$ are linear subspaces orthogonally complementary to each other for any $x \in X$ being sufficiently close to $\bar{x}$ ([81, Example 6.8]). By Lemma 2.1.2

$$\mathrm{proj}_{T_X(x)}\left(\nabla F(x)^* y + w\right) = \mathrm{proj}_{T_X(x)}\left(\nabla F(x)^* y\right),\ \forall w \in N_X(x).$$

Then the fixed-point expression (2.1.14) is obtained via Lemma 2.1.2. $\square$

**Example 2.1.9.** *Consider a smooth, single-valued mapping $F : \mathbb{R}^n \to \mathbb{R}^m$. By Lemma 2.1.8 we can obtain some formulas for the projectional coderivatives of $F$ with respect to sets having simple structures. In the case of an affine set*

$$X := \{x \in \mathbb{R}^n \mid Bx = b\},$$

*where $B$ is an $d \times n$ matrix and $b \in \mathbb{R}^d$, we have for all $\bar{x} \in \mathrm{bdry}\, X$,*

$$D_X^* F(\bar{x})(y) = \mathrm{proj}_{\ker B}(\nabla F(\bar{x})^* y),$$

*where* $\ker B := \{x \in \mathbb{R}^n \mid Bx = 0\}$. *While in the case of a closed half-space*

$$X := \{x \mid \langle a, x \rangle \le \beta\},$$

*we have for all* $\bar{x} \in \operatorname{bdry} X$*,*

$$D_X^* F(\bar{x})(y) = \begin{cases} \left[\nabla F(\bar{x})^* y,\ \ \operatorname{proj}_{[a]^\perp}(\nabla F(\bar{x})^* y)\right], & \text{if } \langle \nabla F(\bar{x})^* y, a\rangle \le 0, \\ \left\{\nabla F(\bar{x})^* y,\ \ \operatorname{proj}_{[a]^\perp}(\nabla F(\bar{x})^* y)\right\}, & \text{if } \langle \nabla F(\bar{x})^* y, a\rangle > 0. \end{cases}$$

## 2.2 Lipschitz-like property relative to a smooth manifold

As in the last section, we derived a fixed-point expression for projectional coderivative of a mapping relative to a smooth manifold (see Proposition 2.1.5). Considering that the generalized Mordukhovich criterion in [59] are for closed and convex sets, in this section, we extend the criterion to the setting of a smooth manifold.

First, we give the sufficient and necessary conditions respectively for $S$ to be Lipschitz-like relative to a smooth manifold. Recall that the Lipschitz-like property relative to a set is given in Definition 1.3.5.

**Lemma 2.2.1** (Necessity)**.** *Consider a mapping* $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$*,* $\bar{x} \in X \subset \mathbb{R}^n$ *where* $X$ *is a smooth manifold at around* $\bar{x}$*,* $\bar{u} \in S(\bar{x})$ *and* $\kappa \ge 0$*. If* $S$ *has the Lipschitz-like property relative to* $X$ *at* $\bar{x}$ *for* $\bar{u}$ *with constant* $\kappa$*, then the condition*

$$\|\operatorname{proj}_{T_X(x)}(x^*)\| \le \kappa \|u^*\| \quad \forall x^* \in \widehat{D}^* S|_X(x \mid u)(u^*)$$

*holds for all* $(x, u)$ *close enough to* $(\bar{x}, \bar{u})$ *in* $\operatorname{gph} S|_X$*.*

*Proof.* As $X$ is a smooth manifold around $\bar{x}$, $X$ is locally closed at $\bar{x}$. By [59, Theorem 2.1], we have

$$\max_{w \in T_X(x) \cap \mathbb{S}} \langle x^*, w\rangle \le \kappa \|u^*\| \quad \forall x^* \in \widehat{D}^* S|_X(x \mid u)(u^*)$$

holds for all $(x, u)$ close enough to $(\bar{x}, \bar{u})$ in $\operatorname{gph} S|_X$. As $T_X(x)$ is a linear subspace, we have

$$\|\operatorname{proj}_{T_X(x)}(x^*)\| = \max\left\{\max_{w \in T_X(x) \cap \mathbb{S}} \langle x^*, w\rangle, 0\right\}.$$

This completes the proof. $\square$

The necessary condition is a direct application of [59, Theorem 2.1]. For the sufficient condition, some efforts need to be made to change the set from $\operatorname{cl}\operatorname{pos}(X-x)$ to $T_X(x)$. We give the proof similar to the one in [59, Theorem 2.2] employing the property of smooth manifold, Proposition 2.1.4 (c).

**Lemma 2.2.2** (Sufficiency). *Consider a mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, $\bar{x} \in X \subset \mathbb{R}^n$ where $X$ is a smooth manifold at around $\bar{x}$, $\bar{u} \in S(\bar{x})$ and $\tilde{\kappa} > \kappa > 0$. Suppose that* $\operatorname{gph} S$ *is locally closed at $(\bar{x}, \bar{u})$. If the condition*

$$\|\operatorname{proj}_{T_X(x)}(x^*)\| \leq \kappa \|u^*\|, \quad \forall x^* \in D^*S|_X(x \mid u)(u^*) \tag{2.2.1}$$

*holds for all $(x, u)$ close enough to $(\bar{x}, \bar{u})$ in* $\operatorname{gph} S|_X$*, then $S$ has the Lipschitz-like property relative to $X$ at $\bar{x}$ for $\bar{u}$ with constant $\tilde{\kappa}$.*

*Proof.* Observing that all the properties involved depend on the nature of $\operatorname{gph} S$ in an arbitrary small neighborhood of $(\bar{x}, \bar{u})$, without loss of generality, from now on we assume that $\operatorname{gph} S$ is closed in its entirety.

Let $0 < \varepsilon' < \frac{\tilde{\kappa}-\kappa}{\kappa+\tilde{\kappa}}$. Then by Proposition 2.1.4 (c), there is some $\delta > 0$ such that the following holds for all $x', \tilde{x} \in X \cap \mathbb{B}_\delta(\bar{x})$ with $x' \neq \tilde{x}$:

$$\left\| \frac{x' - \tilde{x}}{\|x' - \tilde{x}\|} - \operatorname{proj}_{T_X(\tilde{x})}\left(\frac{x' - \tilde{x}}{\|x' - \tilde{x}\|}\right) \right\| \leq \varepsilon'. \tag{2.2.2}$$

Let $0 < \varepsilon < \min\left\{\dfrac{\tilde{\kappa} - \kappa - (\kappa + \tilde{\kappa})\varepsilon'}{4\tilde{\kappa}(1 + \varepsilon')}, \dfrac{1}{3}\delta\right\}$. Suppose by contradiction that $S$ does not have the Lipschitz-like property relative to $X$ at $\bar{x}$ for $\bar{u}$ with constant $\tilde{\kappa}$, meaning that there exist $x', x'' \in \mathbb{B}_\varepsilon(\bar{x}) \cap X$ with $x' \neq x''$, and $u'' \in S(x'') \cap \mathbb{B}_\varepsilon(\bar{u})$ such that

$$d(u'', S(x')) > \tilde{\kappa}\|x'' - x'\| := \beta. \tag{2.2.3}$$

Clearly, we have $0 < \beta \le 2\tilde{\kappa}\varepsilon$. Define $\varphi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ by

$$\varphi(x,u) := \|x - x'\| + \delta_{\mathrm{gph}\, S|_X}(x,u).$$

Clearly, $\varphi$ is lsc (due to closedness of gph $S$ and $X$) with $\inf \varphi$ being finite, and

$$\varphi(x'', u'') \le \inf \varphi + \frac{\beta}{\tilde{\kappa}}.$$

By equipping the product space $\mathbb{R}^n \times \mathbb{R}^m$ with a norm $p$ defined by

$$p(x,u) := \beta\|x\| + \|u\|,$$

we apply the Ekeland's variational principle to obtain some $(\tilde{x}, \tilde{u}) \in \mathbb{R}^n \times \mathbb{R}^m$ such that

$$p(\tilde{x} - x'', \tilde{u} - u'') \le \frac{\kappa + \tilde{\kappa}}{2} \frac{\beta}{\tilde{\kappa}}, \tag{2.2.4}$$

$$\varphi(\tilde{x}, \tilde{u}) \le \varphi(x'', u''), \tag{2.2.5}$$

$$\operatorname*{arg\,min}_{x,\, u} \left\{ \varphi(x,u) + \frac{2}{\kappa + \tilde{\kappa}} p(x - \tilde{x}, u - \tilde{u}) \right\} = \{(\tilde{x}, \tilde{u})\}. \tag{2.2.6}$$

From (2.2.4), it follows that

$$(\tilde{x}, \tilde{u}) \in \mathrm{gph}\, S \cap (X \times \mathbb{R}^m) = \mathrm{gph}\, S|_X \tag{2.2.7}$$

and hence that

$$\|\tilde{x} - x'\| \le \|x'' - x'\|.$$

Then by the triangle inequality, we have

$$\|\tilde{x} - \bar{x}\| \le \|\tilde{x} - x'\| + \|x' - \bar{x}\| \le \|x'' - x'\| + \|x' - \bar{x}\| \le \|x'' - \bar{x}\| + 2\|x' - \bar{x}\| \le 3\varepsilon. \tag{2.2.8}$$

From (2.2.4), it follows that

$$\|\tilde{u} - u''\| \le \frac{\kappa + \tilde{\kappa}}{2} \frac{\beta}{\tilde{\kappa}} < \beta \le 2\tilde{\kappa}\varepsilon$$

and hence by the triangle inequality that

$$\|\tilde{u} - \bar{u}\| \leq \|\tilde{u} - u''\| + \|u'' - \bar{u}\| \leq (2\tilde{\kappa} + 1)\varepsilon. \tag{2.2.9}$$

So we have $\tilde{x} \neq x'$, for otherwise we have

$$d(u'', S(x')) = d(u'', S(\tilde{x})) \leq \|\tilde{u} - u''\| < \beta,$$

contradicting to (2.2.3). From (2.2.6) and the generalized version of Fermat's rule [81, Theorem 10.1], it follows that

$$(0, 0) \in \partial(\psi + \delta_{\operatorname{gph} S|_X})(\tilde{x}, \tilde{u}), \tag{2.2.10}$$

where

$$\psi(x, u) := \|x - x'\| + \frac{2}{\kappa + \tilde{\kappa}} \left(\beta\|x - \tilde{x}\| + \|u - \tilde{u}\|\right).$$

Clearly, $\psi$ is convex and Lipschitz continuous and in terms of closed unit balls $\mathbb{B}_1$ in $\mathbb{R}^n$ and $\mathbb{B}_2$ in $\mathbb{R}^m$,

$$\partial\psi(\tilde{x}, \tilde{u}) = \left(\frac{\tilde{x} - x'}{\|\tilde{x} - x'\|} + \frac{2\beta}{\kappa + \tilde{\kappa}}\mathbb{B}_1\right) \times \frac{2}{\kappa + \tilde{\kappa}}\mathbb{B}_2. \tag{2.2.11}$$

Applying the calculus rule for subgradients of Lipschitzian sums [81, Exercise 10.10], we deduce from (2.2.10) that

$$(0, 0) \in \partial\psi(\tilde{x}, \tilde{u}) + N_{\operatorname{gph} S|_X}(\tilde{x}, \tilde{u}).$$

This, together with (2.2.11), implies the existence of $v_1 \in \mathbb{B}_1$, $v_2 \in \mathbb{B}_2$ and

$$(x^*, -u^*) \in N_{\operatorname{gph} S|_X}(\tilde{x}, \tilde{u}) \Longleftrightarrow x^* \in D^*S|_X(\tilde{x} \mid \tilde{u})(u^*) \tag{2.2.12}$$

such that

$$x^* = -\frac{\tilde{x} - x'}{\|\tilde{x} - x'\|} - \frac{2\beta}{\kappa + \tilde{\kappa}}v_1,$$

and

$$u^* = \frac{2}{\kappa + \tilde{\kappa}} v_2.$$

Since $\tilde{x}, x' \in X \cap \mathbb{B}_{3\varepsilon}(\bar{x}) \subset X \cap \mathbb{B}_{\delta}(\bar{x})$ with $\tilde{x} \neq x'$, it follows from (2.2.2) that

$$\|w^* - w\| \leq \varepsilon',$$

where

$$w^* := \frac{x' - \tilde{x}}{\|x' - \tilde{x}\|} \quad \text{and} \quad w := \mathrm{proj}_{T_X(\tilde{x})} \frac{x' - \tilde{x}}{\|x' - \tilde{x}\|} \in T_X(\tilde{x}).$$

Then we have

$$\begin{aligned}
\langle x^*, w \rangle - \kappa \|u^*\| &= \langle x^*, w^* \rangle - \kappa \|u^*\| + \langle x^*, w - w^* \rangle \\
&= 1 - \frac{2\beta}{\kappa + \tilde{\kappa}} \langle v_1, w^* \rangle - \frac{2\kappa}{\kappa + \tilde{\kappa}} \|v_2\| + \langle x^*, w - w^* \rangle \\
&\geq 1 - \frac{2\beta}{\kappa + \tilde{\kappa}} - \frac{2\kappa}{\kappa + \tilde{\kappa}} - (1 + \frac{2\beta}{\kappa + \tilde{\kappa}})\varepsilon' \\
&\geq 1 - \frac{4\tilde{\kappa}\varepsilon}{\kappa + \tilde{\kappa}} - \frac{2\kappa}{\kappa + \tilde{\kappa}} - (1 + \frac{4\tilde{\kappa}\varepsilon}{\kappa + \tilde{\kappa}})\varepsilon' \\
&> 0,
\end{aligned}$$

where the first inequality follows from the Cauchy-Schwarz inequality, the second one from the fact that $\beta \leq 2\tilde{\kappa}\varepsilon$, and the last one from our setting that $\varepsilon < \frac{\tilde{\kappa} - \kappa - (\kappa + \tilde{\kappa})\varepsilon'}{4\tilde{\kappa}(1 + \varepsilon')}$. Therefore, we have

$$\mathrm{proj}_{T_X(\tilde{x})}(x^*) = \max_{\tilde{w} \in T_X(\tilde{x}) \cap \mathbb{S}} \langle x^*, \tilde{w} \rangle \geq \langle x^*, w \rangle > \kappa \|u^*\|. \tag{2.2.13}$$

In view of (2.2.7) - (2.2.9), (2.2.12) and (2.2.13) and the fact that $\varepsilon$ could be any number such that

$$0 < \varepsilon < \min \left\{ \frac{\tilde{\kappa} - \kappa - (\kappa + \tilde{\kappa})\varepsilon'}{4\tilde{\kappa}(1 + \varepsilon')}, \ \frac{1}{3}\delta \right\},$$

we conclude that condition (2.2.1) cannot hold for all $(x, u)$ close enough to $(\bar{x}, \bar{u})$ in $\mathrm{gph}\, S|_X$, which forms a contradiction. This completes the proof. $\square$

Next we present the characterization of the Lipschitz-like property relative to a smooth manifold in full. In [17, Proposition 18], they showed that the condition (d) in the following theorem provided the sufficiency. We improve this result with necessity implemented. Recall that the notation $|\cdot|^+$ is the outer norm of a set-valued mapping (see (1.3.1)).

**Theorem 2.2.3** (Lipschitz-like property relative to a smooth manifold)**.** *Consider a mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, $\bar{x} \in X \subset \mathbb{R}^n$ where $X$ is a smooth manifold at around $\bar{x}$, and $\bar{u} \in S(\bar{x})$. Suppose that* gph $S$ *is locally closed at $(\bar{x}, \bar{u})$. The following properties are equivalent:*

**(a)** *$S$ has the Lipschitz-like property relative to $X$ at $\bar{x}$ for $\bar{u}$.*

**(b)** $\mathrm{proj}_{T_X(\bar{x})} D^*S|_X(\bar{x} \mid \bar{u})(0) = \{0\}$.

**(c)** $\left|\mathrm{proj}_{T_X(\bar{x})} D^*S|_X(\bar{x} \mid \bar{u})\right|^+ < +\infty$.

**(d)** $D^*S|_X(\bar{x} \mid \bar{u})(0) \cap T_X(\bar{x}) = 0$.

**(e)** $D^*S|_X(\bar{x} \mid \bar{u})(0) = N_X(\bar{x})$.

**(f)** $D^*_X S(\bar{x} \mid \bar{u})(0) = \{0\}$.

*Furthermore, we have*

$$\mathrm{lip}_X S(\bar{x} \mid \bar{u}) = \left|\mathrm{proj}_{T_X(\bar{x})} D^*S|_X(\bar{x} \mid \bar{u})\right|^+. \tag{2.2.14}$$

*Proof.* It is clear to see that

$$D^*S|_X(\bar{x} \mid \bar{u})(0) \supset \widehat{D}^*S|_X(\bar{x} \mid \bar{u})(0) \supset \widehat{D}^*S(\bar{x} \mid \bar{u})(0) + N_X(\bar{x}) \supset N_X(\bar{x}), \tag{2.2.15}$$

and that the mapping $\mathrm{proj}_{T_X(\bar{x})} D^*S|_X(\bar{x} \mid \bar{u})$ is outer semicontinuous and positively homogeneous. Then the equivalence of (b) and (c) follows immediately from [81, Proposition 9.23]. The equivalences among (b), (d) and (f) follows readily from

Proposition 2.1.4 (c). In view of (2.2.15), we get the equivalence of (b) and (e). It remains to prove the equivalence of (a) and (b).

[(a) $\Longrightarrow$ (c)] Assuming (a), we will show (c) by proving the inequality

$$\left|\,\mathrm{proj}_{T_X(\bar{x})} D^*S|_X(\bar{x}\mid\bar{u})\,\right|^+ \le \mathrm{lip}_X S(\bar{x}\mid\bar{u}). \tag{2.2.16}$$

Choose any $\kappa \in (\mathrm{lip}_X S(\bar{x}\mid\bar{u}), +\infty)$. Then $S$ has the Lipschitz-like property relative to $X$ at $\bar{x}$ for $\bar{u}$ with constant $\kappa$. Let $(u^*, v^*)$ be given arbitrarily such that $v^* \in \mathrm{proj}_{T_X(\bar{x})} D^*S|_X(\bar{x}\mid\bar{u})(u^*)$. Then there is some $x^* \in D^*S|_X(\bar{x}\mid\bar{u})(u^*)$ such that

$$v^* = \mathrm{proj}_{T_X(\bar{x})}(x^*). \tag{2.2.17}$$

By the definition of the limiting coderivatives, there are some $(x_k, u_k) \to (\bar{x}, \bar{u})$ with $(x_k, u_k) \in \mathrm{gph}\, S|_X$ and $x_k^* \in \widehat{D}^*S|_X(x_k \mid u_k)(u_k^*)$ such that $(x_k^*, -u_k^*) \to (x^*, -u^*)$. By Lemma 2.2.1, there exists some positive integer $k'$ such that

$$\|\mathrm{proj}_{T_X(x_k)}(x_k^*)\| \le \kappa \|u_k^*\| \quad \forall k \ge k'. \tag{2.2.18}$$

Since $X$ is a smooth manifold around $\bar{x}$, we have

$$\mathrm{proj}_{T_X(x_k)}(x_k^*) \to \mathrm{proj}_{T_X(\bar{x})}(x^*). \tag{2.2.19}$$

In view of (2.2.17-2.2.19), we have $\|v^*\| \le \kappa\|u^*\|$ and hence

$$\left|\,\mathrm{proj}_{T_X(\bar{x})} D^*S|_X(\bar{x}\mid\bar{u})\,\right|^+ \le \kappa.$$

Due to $\kappa \in (\mathrm{lip}_X S(\bar{x}\mid\bar{u}), +\infty)$ being chosen arbitrarily, we get (2.2.16) immediately.

[(c) $\Longrightarrow$ (a)] Assuming (c), we will show (a) by proving the inequality

$$\mathrm{lip}_X S(\bar{x}\mid\bar{u}) \le \left|\,\mathrm{proj}_{T_X(\bar{x})} D^*S|_X(\bar{x}\mid\bar{u})\,\right|^+,$$

from which the equality (2.2.14) follows as the inequality in the other direction has been proved earlier.

Suppose by contradiction that $\left|\operatorname{proj}_{T_X(\bar{x})} D^*S|_X(\bar{x} \mid \bar{u})\right|^+ < \operatorname{lip}_X S(\bar{x} \mid \bar{u})$. Choose any $\kappa', \kappa''$ as

$$\kappa' \in \left(\left|\operatorname{proj}_{T_X(\bar{x})} D^*S|_X(\bar{x} \mid \bar{u})\right|^+, \operatorname{lip}_X S(\bar{x} \mid \bar{u})\right), \; \kappa'' \in \left(\left|\operatorname{proj}_{T_X(\bar{x})} D^*S|_X(\bar{x} \mid \bar{u})\right|^+, \kappa'\right).$$

Clearly, $S$ fails to have the Lipschitz-like property relative to $X$ at $\bar{x}$ for $\bar{u}$ with constant $\kappa'$. By Lemma 2.2.2, there exist some sequences $(x_k, u_k) \to (\bar{x}, \bar{u})$ with $(x_k, u_k) \in \operatorname{gph} S|_X$ and some $x_k^* \in D^*S|_X(x_k \mid u_k)(u_k^*)$ such that $\|v_k^*\| > \kappa''\|u_k^*\|$, $\forall k$, where $v_k^* := \operatorname{proj}_{T_X(x_k)}(x_k^*)$. By Proposition 2.1.5, we have

$$v_k^* \in D^*S|_X(x_k \mid u_k)(u_k^*) \cap T_X(x_k), \; \forall k.$$

Clearly, we have $v_k^* \neq 0$ for all $k$. By taking a subsequence if necessary, we assume that there is some $v^* \in T_X(\bar{x})$ with $\|v^*\| = 1$ such that

$$\frac{v_k^*}{\|v_k^*\|} \to v^*.$$

As we have

$$\frac{\|u_k^*\|}{\|v_k^*\|} < \frac{1}{\kappa''} \quad \forall k,$$

by taking a subsequence if necessary again, we assume that there is some $u^*$ with $\kappa''\|u^*\| \le 1$ such that

$$\frac{u_k^*}{\|v_k^*\|} \to u^*.$$

Thus, we have $v^* \in D^*S|_X(\bar{x} \mid \bar{u})(u^*)$ and $\|v^*\| \ge \kappa''\|u^*\|$. So we have

$$\begin{aligned}
\left|\operatorname{proj}_{T_X(\bar{x})} D^*S|_X(\bar{x} \mid \bar{u})\right|^+ &:= \sup_{\tilde{u}^* \in \mathbb{B}} \sup_{\tilde{x}^* \in D^*S|_X(\bar{x}|\bar{u})(\tilde{u}^*)} \|\operatorname{proj}_{T_X(\bar{x})}(\tilde{x}^*)\| \\
&\ge \|\operatorname{proj}_{T_X(\bar{x})}(\kappa'' v^*)\| \\
&= \kappa''\|v^*\| \\
&= \kappa'',
\end{aligned}$$

contradicting to the setting that $\kappa'' \in \Big( \big| \operatorname{proj}_{T_X(\bar{x})} D^*S|_X(\bar{x} \mid \bar{u}) \big|^+ , \kappa' \Big)$. This completes the proof. $\square$

## 2.3 Chain rules for projectional coderivatives

To broaden the scope of application of the projectional coderivative onto various systems, one important thing would be developing the corresponding calculus rules for it, which is also the main goal of the coming two sections. Unlike the chain rule for coderivatives (see [81, Theorem 10.37]), the one for projectional coderivatives comes with stricter assumptions as it involves projection.

**Theorem 2.3.1** (Projectional coderivative chain rule)**.** *Suppose $S = S_2 \circ S_1$ for mappings $S_1 : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ and $S_2 : \mathbb{R}^p \rightrightarrows \mathbb{R}^m$. Let $X \subseteq \mathbb{R}^n$ be a closed set with $\bar{x} \in X$. Here $S_1$ is outer semicontinuous relative to $X$ and $S_2$ is outer semicontinuous. For a pair $(\bar{x}, \bar{u}) \in \operatorname{gph} S|_X = \operatorname{gph}(S_2 \circ S_1|_X)$, assume:*

*(a) the mapping $(x, u) \mapsto S_1|_X(x) \cap S_2^{-1}(u)$ is locally bounded at $(\bar{x}, \bar{u})$, or equivalently, the mapping $(x, u) \mapsto S_1(x) \cap S_2^{-1}(u)$ is locally bounded relative to $X \times \mathbb{R}^m$ at $(\bar{x}, \bar{u})$ (this being true in particular if either $S_1$ is locally bounded relative to $X$ at $\bar{x}$ or $S_2^{-1}$ is locally bounded at $\bar{u}$). In this way, $S_2 \circ S_1|_X$ is outer semicontinuous (see [81, Proposition 5.52 (b)]).*

*(b) $D^*S_2(\bar{w}|\bar{u})(0) \cap D^*_X S_1(\bar{x}|\bar{w})^{-1}(0) = \{0\}$ holds for any $\bar{w} \in S_1|_X(\bar{x}) \cap S_2^{-1}(\bar{u})$ (this being true in particular if $S_2$ has Lipschitz-like property at $\bar{w}$ for $\bar{u}$).*

*Then $\operatorname{gph} S|_X$ is locally closed around $(\bar{x}, \bar{u})$ and*

$$D^*_X S(\bar{x}|\bar{u}) \subset \bigcup_{\bar{w} \in S_1|_X(\bar{x}) \cap S_2^{-1}(\bar{u})} D^*_X S_1(\bar{x}|\bar{w}) \circ D^*S_2(\bar{w}|\bar{u}). \tag{2.3.1}$$

*Besides, if (a) and (b) hold, and $S_1|_X$ and $S_2$ are graph-convex, then $S|_X$ is graph-convex as well and*

$$\operatorname{proj}_{T_X(\bar{x})} D^*S_1|_X(\bar{x}|\bar{w}) \circ D^*S_2(\bar{w}|\bar{u}) \subseteq D^*_X S(\bar{x}|\bar{u}),\ \forall \bar{w} \in S_1|_X(\bar{x}) \cap S_2^{-1}(\bar{u}). \quad (2.3.2)$$

*If (a) and (b) hold, and $X$ is a smooth manifold at $\bar{x}$,*

$$D^*_X S_1(\bar{x}|\bar{w}) \circ D^*S_2(\bar{w}|\bar{u}) = \operatorname{proj}_{T_X(\bar{x})} D^*S_1|_X(\bar{x}|\bar{w}) \circ D^*S_2(\bar{w}|\bar{u}). \quad (2.3.3)$$

*Therefore, when assumptions (a) and (b) hold, $S_1|_X$ and $S_2$ are graph-convex, and $X$ is a smooth manifold at $\bar{x}$, we obtain an equation:*

$$D^*_X S(\bar{x}|\bar{u}) = D^*_X S_1(\bar{x}|\bar{w}) \circ D^*S_2(\bar{w}|\bar{u}), \forall \bar{w} \in S_1|_X(\bar{x}) \cap S_2^{-1}(\bar{u}). \quad (2.3.4)$$

*Proof.* By (2.1.1), we have that the constraint qualification (b) also indicates the constraint qualification below:

$$D^*S_2(\bar{w}|\bar{u})(0) \cap D^*S_1|_X(\bar{x}|\bar{w})^{-1}(0) = \{0\}. \quad (2.3.5)$$

Let $C = \{(x, w, u) | (x, w) \in \operatorname{gph} S_1|_X,\ (w, u) \in \operatorname{gph} S_2\}$ and $G : (x, w, u) \mapsto (x, u)$. Then $\operatorname{gph} S|_X = G(C)$. With assumption (a), we can obtain $\varepsilon > 0$ such that $G^{-1}(\mathcal{N}_\varepsilon(\bar{x}, \bar{u})) \cap C$ is bounded. Then by Theorem 6.43 on $\operatorname{gph} S|_X$ at $(\bar{x}, \bar{u})$, we have that $\operatorname{gph} S|_X$ is locally closed at $(\bar{x}, \bar{u})$ and

$$\begin{aligned} N_{\operatorname{gph} S|_X}(\bar{x}, \bar{u}) &\subset \bigcup_{(\bar{x}, \bar{w}, \bar{u}) \in G^{-1}(\bar{x}, \bar{u}) \cap C} \left\{ (v, -y) \;\middle|\; \nabla G(\bar{x}, \bar{w}, \bar{u})^*(v, -y) \in N_C(\bar{x}, \bar{w}, \bar{u}) \right\} \\ &= \bigcup_{\bar{w} \in S_1|_X(\bar{x}) \cap S_2^{-1}(\bar{u})} \left\{ (v, -y) \;\middle|\; (v, 0, -y) \in N_C(\bar{x}, \bar{w}, \bar{u}) \right\}. \end{aligned}$$

Next we try to obtain the expression for $N_C(\bar{x}, \bar{w}, \bar{u})$. Let $D = \operatorname{gph} S_1|_X \times \operatorname{gph} S_2$. For $F : (x, w, u) \mapsto (x, w, w, u)$, we have $C = F^{-1}(D)$. Here the definition of $F$ ensures the component $w$ of $(x, w, w, u)$ in $D$ belongs to $S_1|_X(x) \cap S_2^{-1}(u)$. Here we apply [81, Theorem 6.14] on $C = F^{-1}(D)$. The constraint qualification requires that:

$$\forall q \in N_D(F(\bar{x}, \bar{w}, \bar{u})) \text{ with } -\nabla F(\bar{x}, \bar{w}, \bar{u})^* q = 0 \Longrightarrow q = 0,$$

which is

$$\begin{cases}(q_1,q_2)\in N_{\operatorname{gph} S_1|_X}(\bar{x},\bar{w})\\ (q_3,q_4)\in N_{\operatorname{gph} S_2}(\bar{w},\bar{u}) & \Longrightarrow q_1,q_2,q_3,q_4=0\\ (q_1,q_2+q_3,q_4)=0\end{cases}$$

due to the product form of $D=\operatorname{gph} S_1|_X\times\operatorname{gph} S_2$ (see [81, Proposition 6.41]). By expressing in coderivatives, it becomes

$$0\in D^*S_1|_X(\bar{x}|\bar{w})(q_3),\ q_3\in D^*S_2(\bar{w}|\bar{u})(0)\Longrightarrow q_3=0 \text{ for all } \bar{w}\in S_1|_X(\bar{x})\cap S_2^{-1}(\bar{u}),$$

which can be reformulated as in (2.3.5). Then we can have the inclusion:

$$\begin{aligned}N_C(\bar{x},\bar{w},\bar{u})&\subset\left\{\nabla F(\bar{x},\bar{w},\bar{u})^*q\;\middle|\;q\in N_D(\bar{x},\bar{w},\bar{w},\bar{u})\right\}\\&=\left\{(q_1,q_2+q_3,q_4)\;\middle|\;(q_1,q_2)\in N_{\operatorname{gph} S_1|_X}(\bar{x},\bar{w}),(q_3,q_4)\in N_{\operatorname{gph} S_2}(\bar{w},\bar{u})\right\}.\end{aligned}$$

Next we prove that the constraint qualification (2.3.5) also holds for all $(x,u)$ in $\operatorname{gph} S|_X$ sufficiently near to $(\bar{x},\bar{u})$ by contradiction. Suppose there exist sequences $(x_k,u_k)\xrightarrow{\operatorname{gph} S|_X}(\bar{x},\bar{u})$, $w_k\in S_1|_X(x_k)\cap S_2^{-1}(u_k)$ , and $w_k^*\in D^*S_2(w_k|u_k)(0)\cap D^*S_1|_X(x_k|w_k)^{-1}(0)$ (which is a cone) such that $w_k^*\neq 0$. Without loss of generality we assume $\|w_k^*\|=1$. Note that under assumption (a), $w_k\to\bar{w}\in S_1|_X(\bar{x})\cap S_2^{-1}(\bar{u})$. By outer semicontinuity of normal cone mappings, $w_k^*$ must converge to some $w^*\in D^*S_2(\bar{w}|\bar{u})(0)\cap D^*S_1|_X(\bar{x}|\bar{w})^{-1}(0)$ with $\|w^*\|=1$, which contradicts (2.3.5). As the assumption (a) and (2.3.5) hold for all $(x,u)$ in $\operatorname{gph} S|_X$ around $(\bar{x},\bar{u})$, the inclusion can be obtained:

$$N_{\operatorname{gph} S|_X}(x,u)\subset\bigcup_{w\in S_1|_X(x)\cap S_2^{-1}(u)}\left\{(x^*,-u^*)\;\middle|\;\begin{array}{l}\exists w^* \text{ s.t. } x^*\in D^*S_1|_X(x|w)(w^*),\\ w^*\in D^*S_2(w|u)(u^*)\end{array}\right\}\tag{2.3.6}$$

for all $(x,u)$ in $\operatorname{gph} S|_X$ around $(\bar{x},\bar{u})$. Given the upper estimate of $N_{\operatorname{gph} S|_X}(x,u)$ in (2.3.6), we now proceed to exploring the estimate of projectional coderivative

$D^*_X S(\bar{x}, \bar{u})$. Let $t^* \in D^*_X S(\bar{x}|\bar{u})(u^*)$, then there are sequences $(x_k, u_k) \xrightarrow{\mathrm{gph}\, S|_X} (\bar{x}, \bar{u})$ and $(x_k^*, -u_k^*) \in N_{\mathrm{gph}\, S|_X}(x_k, u_k)$ such that $t_k^* \in \mathrm{proj}_{T_X(x_k)}(x_k^*)$, $t_k^* \to t^*$ and $u_k^* \to u^*$. By (2.3.6), $\exists w_k \in S_1|_X(x_k) \cap S_2^{-1}(u_k)$ and $w_k^*$ such that $(x_k^*, -w_k^*) \in N_{\mathrm{gph}\, S_1|_X}(x_k, w_k)$ and $(w_k^*, -u_k^*) \in N_{\mathrm{gph}\, S_2}(w_k, u_k)$.

Given $w_k \in S_1|_X(x_k) \cap S_2^{-1}(u_k)$, the outer semicontinuity of $S_1|_X$ and $S_2^{-1}$ and local boundedness of the mapping $(x, u) \mapsto S_1|_X(x) \cap S_2^{-1}(u)$ around $(\bar{x}, \bar{u})$, $\{w_k\}$ must converge to some $\bar{w} \in S_1|_X(\bar{x}) \cap S_2^{-1}(\bar{u})$ (taking a subsequence if necessary). For $(w_k^*, -u_k^*) \in N_{\mathrm{gph}\, S_2}(w_k, u_k)$ and outer semicontinuity of $(w, u) \to N_{\mathrm{gph}\, S_2}(w, u)$ at $(\bar{w}, \bar{u})$, we have either $w_k^* \to w^*$ or $\lambda_k w_k^* \to w^*$ with $\lambda_k \searrow 0$. For the first case we have $w^* \in D^* S_2(\bar{w}|\bar{u})(u^*)$. Given $(x_k^*, -w_k^*) \in N_{\mathrm{gph}\, S_1|_X}(x_k, w_k)$ and $t_k^* \in \mathrm{proj}_{T_X(x_k)}(x_k^*)$ with $t_k^* \to t^*$, then $\bar{t} \in D^*_X S_1(\bar{x}|\bar{w})(\bar{z})$. Thus $\bar{t} \in D^*_X S_1(\bar{x}|\bar{w}) \circ D^* S_2(\bar{w}|\bar{u})(\bar{y})$ with $\bar{w} \in S_1|_X(\bar{x}) \cap S_2^{-1}(\bar{u})$.

For the second case, without loss of generality we can assume $\|w^*\| = 1$. Under the conic nature, $\lambda_k w_k^* \in D^* S_2(w_k|u_k)(\lambda_k u_k^*)$. Given $\{u_k^*\}$ is bounded with $u_k^* \to u^*$, then $\lambda_k u_k^* \to 0$ and we have $w^* \in D^* S_2(\bar{w}|\bar{u})(0)$. Similarly we have $(\lambda_k x_k^*, -\lambda_k w_k^*) \in N_{\mathrm{gph}\, S_1|_X}(x_k, w_k)$. As $T_X(x)$ is a nonempty closed cone for any $x \in X$ around $\bar{x}$, $\lambda_k t_k^* \in \mathrm{proj}_{T_X(x_k)}(\lambda_k x_k^*)$ and $\lambda_k t_k^* \to 0$. That is, $0 \in D^*_X S_1(\bar{x}|\bar{w})(w^*)$. Thus we have $w^* \in D^* S_2(\bar{w}|\bar{u})(0) \cap D^*_X S_1(\bar{x}|\bar{w})^{-1}(0) = \{0\}$ with $\|w^*\| = 1$, which contradicts the assumption (b). Therefore the case $\{\lambda_k w_k^*\} \to w^*$ can be abandoned and the inclusion (2.3.1) is thus proved.

Note that by definition, $\mathrm{proj}_{T_X(\bar{x})} D^* S|_X(\bar{x}|\bar{u}) \subseteq D^*_X S(\bar{x}|\bar{u})$. When $\mathrm{gph}\, S_1|_X$ and $\mathrm{gph}\, S_2$ are convex, the inclusion (2.3.6) becomes an equation for every $w \in S_1|_X(x) \cap S_2^{-1}(u)$ and the union becomes superfluous. Then the inclusion (2.3.2) is obtained and $\mathrm{gph}\, S|_X$ is convex as well. Besides, when $X$ is a smooth manifold around $\bar{x}$, by Proposition 2.1.5, we obtain (2.3.3) and further the equation (2.3.4). □

Based on Theorem 2.3.1 above, similar to [81, Exercise 10.39, Theorem 10.40], we give the following two corollaries when one of the mapping in the composition is a single-valued one. When the outer layer involves a single-valued one, we can apply Theorem 2.3.1 directly.

**Corollary 2.3.2** (Outer composition with a single-valued function)**.** *Let $X$ be a closed set in $\mathbb{R}^n$ and $S = F \circ S_0$ for a mapping $S_0 : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ being outer semicontinuous relative to $X$ and a single-valued function $F : \mathbb{R}^p \to \mathbb{R}^m$. Let $\bar{u} \in S|_X(\bar{x})$ and suppose $F$ is strictly continuous at every $\bar{w} \in S_0(\bar{x})$. Suppose also that the mapping $(x, u) \mapsto S_0|_X(x) \cap F^{-1}(u)$ is locally bounded at $(\bar{x}, \bar{u})$. Then*

$$D_X^* S(\bar{x}|\bar{u}) \subseteq \bigcup_{\bar{w} \in S_0(\bar{x}) \cap F^{-1}(\bar{u})} D_X^* S_0(\bar{x}|\bar{w}) \circ D^* F(\bar{w}).$$

*If in addition $S_0|_X$ is graph-convex, $X$ is a smooth manifold, and $F$ is linear, then*

$$D_X^* S(\bar{x}|\bar{u}) = D_X^* S_0(\bar{x}|\bar{w}) \circ \nabla F(\bar{w})^*.$$

*Proof.* This result is obtained directly as a special case of Theorem 2.3.1. □

Next we give a simple example for illustration.

**Example 2.3.3.** *Let $S_0(x) = \left[-\sqrt{|x|}, \sqrt{|x|}\right],\ F(w) = 2w,\ X = \mathbb{R}_+$. Then $S|_X(x) = F \circ S_0|_X(x) = \left[-2\sqrt{|x|}, 2\sqrt{|x|}\right]$. For $\bar{x} = 0$ and $\bar{u} = 0 \in S|_X(\bar{x}) = F \circ S_0|_X(\bar{x})$. Then we have $\bar{w} \in S_0|_X(\bar{x}) \cap F^{-1}(\bar{u}) = \{0\}$ and*

$$D_X^* S_0(\bar{x} \mid \bar{w})(z) = \begin{cases} \mathbb{R}_- & z = 0 \\ \emptyset & z \neq 0 \end{cases}, \quad \nabla F(\bar{w})^* y = 2y.$$

*Therefore*

$$D_X^* S(\bar{x} \mid \bar{u})(y) = D_X^* S_0(\bar{x} \mid \bar{w}) \circ \nabla F(\bar{w})^* y = \begin{cases} \mathbb{R}_- & y = 0 \\ \emptyset & y \neq 0 \end{cases}.$$

However, when the inner layer of the composition involves a single-valued function, it varies from direct application of Theorem 2.3.1 in that $F$ is restricted on $X$ rather than defined on the whole space when we try to derive an equation for the projectional coderivative. Before that, we illustrate the expression of the coderivative of $F$ restricted on $X$.

**Lemma 2.3.4.** *Let $X$ be a closed set in $\mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^m$ be strictly continuous at $\bar{x}$ relative to $X$. Then for all $z \in \mathbb{R}^m$ we have*

$$\widehat{D}^* F|_X(\bar{x})(z) = \widehat{\partial}(zF|_X)(\bar{x}), \tag{2.3.7}$$

$$D^* F|_X(\bar{x})(z) = \partial(zF|_X)(\bar{x}). \tag{2.3.8}$$

*Proof.* For $v \in \widehat{D}^* F|_X(\bar{x})(z)$, it is equivalent that

$$\langle v, x - \bar{x}\rangle - \langle z, F|_X(x) - F|_X(\bar{x})\rangle \leq o(\|(x, F|_X(x)) - (\bar{x}, F|_X(\bar{x}))\|).$$

As $F$ is strictly continuous at $\bar{x}$ relative to $X$, we can replace $o(\|(x, F|_X(x)) - (\bar{x}, F|_X(\bar{x}))\|)$ with $o(\|x - \bar{x}\|)$, i.e.,

$$(zF|_X)(x) \geq (zF|_X)(\bar{x}) + \langle v, x - \bar{x}\rangle + o(\|x - \bar{x}\|).$$

Thus it is equivalent that $v \in \widehat{\partial}(zF|_X)(\bar{x})$. Given that $F$ is also strictly continuous at $x$ relative to $X$ for $x$ being sufficiently close to $\bar{x}$, such equation (2.3.7) also holds for any $x \in X \cap O$ for some $O \in \mathcal{N}(\bar{x})$. Let $v \in D^* F|_X(\bar{x})(z)$, then there exist sequences $x_k \xrightarrow{X} \bar{x}$ and $v_k \in \widehat{D}^* F|_X(x_k)(z_k)$ with $v_k \to v$, $z_k \to z$. By (2.3.7), $v_k \in \widehat{\partial}(z_k F|_X)(x_k) \subseteq \partial(z_k F|_X)(x_k) = \partial[zF|_X + (z_k - z)F|_X]((x_k) \subseteq \partial(zF|_X)(x_k) + \partial[(z_k - z)F|_X]((x_k)$. When $k \to \infty$, $\partial[(z_k - z)F|_X](x_k) \to \{0\}$ as $z_k \to z$. Given $F$ is strictly continuous at $\bar{x}$ relative to $X$, $zF|_X(x_k) \to zF|_X(\bar{x})$ when $x \xrightarrow{X} \bar{x}$. Then we have $v_k \to v \in \partial(zF|_X)(\bar{x})$. For the inclusion in reverse for (2.3.8), let $v \in \partial(zF|_X)(\bar{x})$. Then there exist sequences $x_k \xrightarrow{zF|_X} \bar{x}$ and $v_k \in \widehat{\partial}(zF|_X)(x_k)$ such that $v_k \to v$. Then it is equivalent that $x \xrightarrow{X} \bar{x}$ and $v_k \in \widehat{D}^* F|_X(x_k)(z)$ and by definition of normal cone mappings we have $v_k \to v \in D^* F|_X(\bar{x})(z)$. $\square$

**Theorem 2.3.5** (Inner composition with a single-valued function)**.** *Let $X$ be a closed set in $\mathbb{R}^n$, and $S = S_0 \circ F$ for an outer semicontinuous mapping $S_0 : \mathbb{R}^p \rightrightarrows \mathbb{R}^m$ and a single-valued mapping $F : \mathbb{R}^n \to \mathbb{R}^p$ that is strictly continuous at $\bar{x}$ relative to $X$. Let $\bar{u} \in S|_X(\bar{x})$. If*

$$D^* S_0(F(\bar{x}) \mid \bar{u})(0) \cap D^*_X F(\bar{x})^{-1}(0) = \{0\}, \tag{2.3.9}$$

*then*

$$D^*_X S(\bar{x} \mid \bar{u}) \subseteq D^*_X F(\bar{x}) \circ D^* S_0(F(\bar{x}) \mid \bar{u}). \tag{2.3.10}$$

*Still under (2.3.9), suppose that $S_0$ is graphically regular at $F(\bar{x})$ for $\bar{u}$, and the function $zF|_X$ is regular at $\bar{x}$ for all $z \in \operatorname{rg} D^* S_0\,(F(\bar{x}) \mid \bar{u})$, and $X$ is a smooth manifold in $\mathbb{R}^n$. Then $S|_X$ is graphically regular at $\bar{x}$ for $\bar{u}$, and*

$$D^*_X S(\bar{x} \mid \bar{u}) = D^*_X F(\bar{x}) \circ D^* S_0(F(\bar{x}) \mid \bar{u}). \tag{2.3.11}$$

*Proof.* First we prove

$$\widehat{D}^* F|_X(\bar{x}) \circ \widehat{D}^* S_0(F(\bar{x}) \mid \bar{u}) \subseteq \widehat{D}^* S|_X(\bar{x} \mid \bar{u}). \tag{2.3.12}$$

Let $v \in \widehat{D}^* F|_X(\bar{x})(z)$ and $z \in \widehat{D}^* S_0(F(\bar{x}) \mid \bar{u})(y)$. By definition we have:

$$\limsup_{\substack{(x,u) \xrightarrow{\operatorname{gph} S|_X} (\bar{x},\bar{u}) \\ \neq}} \frac{\langle v, x - \bar{x}\rangle - \langle z, F|_X(x) - F|_X(\bar{x})\rangle}{\|(x - \bar{x}, F|_X(x) - F|_X(\bar{x}))\|}$$

$$\leq \limsup_{\substack{x \xrightarrow{X} \bar{x} \\ \neq}} \frac{\langle v, x - \bar{x}\rangle - \langle z, F|_X(x) - F|_X(\bar{x})\rangle}{\|(x - \bar{x}, F|_X(x) - F|_X(\bar{x}))\|} \leq 0 \tag{2.3.13}$$

and

$$\limsup_{\substack{(F|_X(x),u) \xrightarrow{\operatorname{gph} S_0} (F|_X(\bar{x}),\bar{u}) \\ \neq}} \frac{\langle z, F|_X(x) - F|_X(\bar{x})\rangle - \langle y, u - \bar{u}\rangle}{\|(F|_X(x), u) - (F|_X(\bar{x}), \bar{u})\|}$$

$$\leq \limsup_{\substack{(w,u)\xrightarrow{\operatorname{gph} S_0}(F|_X(\bar{x}),\bar{u})\\ \neq}} \frac{\langle z, w - F|_X(\bar{x})\rangle - \langle y, u-\bar{u}\rangle}{\|(w,u)-(F|_X(\bar{x}),\bar{u})\|} \leq 0. \tag{2.3.14}$$

As $F$ is strictly continuous at $\bar{x}$ relative to $X$, $F|_X(x) \to F|_X(\bar{x})$ when $x \xrightarrow{X} \bar{x}$ and

$$\limsup_{\substack{x\xrightarrow{X}\bar{x}\\ \neq}} \frac{\|F|_X(x)-F|_X(\bar{x})\|}{\|x-\bar{x}\|} < \infty.$$

Therefore by (2.3.13) and (2.3.14) we have:

$$\limsup_{\substack{(x,u)\xrightarrow{\operatorname{gph} S|_X}(\bar{x},\bar{u})\\ \neq}} \frac{\langle v, x-\bar{x}\rangle - \langle z, F|_X(x)-F|_X(\bar{x})\rangle}{\|x-\bar{x}\|+\|u-\bar{u}\|}$$

$$= \limsup_{\substack{(x,u)\xrightarrow{\operatorname{gph} S|_X}(\bar{x},\bar{u})\\ \neq}} \left(\frac{\langle v, x-\bar{x}\rangle - \langle z, F|_X(x)-F|_X(\bar{x})\rangle}{\|(x-\bar{x}, F|_X(x)-F|_X(\bar{x}))\|} \cdot \frac{\|(x-\bar{x}, F|_X(x)-F|_X(\bar{x}))\|}{\|x-\bar{x}\|+\|u-\bar{u}\|}\right) \leq 0 \tag{2.3.15}$$

and

$$\limsup_{\substack{(F|_X(x),u)\xrightarrow{\operatorname{gph} S_0}(F|_X(\bar{x}),\bar{u})\\ \neq}} \frac{\langle z, F|_X(x)-F|_X(\bar{x})\rangle - \langle y, u-\bar{u}\rangle}{\|x-\bar{x}\|+\|u-\bar{u}\|}$$

$$= \limsup_{\substack{(F|_X(x),u)\xrightarrow{\operatorname{gph} S_0}(F|_X(\bar{x}),\bar{u})\\ \neq}} \frac{\langle z, F|_X(x)-F|_X(\bar{x})\rangle - \langle y, u-\bar{u}\rangle}{\|(F|_X(x),u)-(F|_X(\bar{x}),\bar{u})\|} \cdot \frac{\|(F|_X(x),u)-(F|_X(\bar{x}),\bar{u})\|}{\|x-\bar{x}\|+\|u-\bar{u}\|} \leq 0. \tag{2.3.16}$$

Given that $x \in X$ and $(F|_X(x), u) \in \operatorname{gph} S_0$ is equivalently to $(x,u) \in \operatorname{gph} S|_X$, combining (2.3.15) and (2.3.16) we have

$$\limsup_{\substack{(x,u)\xrightarrow{\operatorname{gph} S|_X}(\bar{x},\bar{u})\\ \neq}} \frac{\langle v, x-\bar{x}\rangle - \langle y, u-\bar{u}\rangle}{\|(x,u)-(\bar{x},\bar{u})\|}$$

$$\leq \limsup_{\substack{(x,u)\xrightarrow{\operatorname{gph} S|_X}(\bar{x},\bar{u})\\ \neq}} \frac{\langle v, x-\bar{x}\rangle - \langle z, F|_X(x)-F|_X(\bar{x})\rangle}{\|x-\bar{x}\|+\|u-\bar{u}\|}$$

$$+ \limsup_{\substack{(x,u)\xrightarrow{\operatorname{gph} S|_X}(\bar{x},\bar{u})\\ \neq}} \frac{\langle z, F|_X(x)-F|_X(\bar{x})\rangle - \langle y, u-\bar{u}\rangle}{\|x-\bar{x}\|+\|u-\bar{u}\|} \leq 0,$$

which means $v \in \widehat{D}^* S|_X(\bar{x} \mid \bar{u})(y)$. The inclusion (2.3.10) comes from directly applying the chain rule for projectional coderivatives, as $F|_X(\cdot)$ is locally bounded and single-valued at $\bar{x}$. For the equation part, note that (2.3.9) also indicates

$$z \in D^* S_0(F(\bar{x}) \mid \bar{u})(0),\ 0 \in D^* F|_X(\bar{x})(z) = \partial(zF|_X)(\bar{x}) \Longrightarrow z = 0.$$

By [81, Theorem 10.37], we have

$$D^* S|_X(\bar{x} \mid \bar{u}) \subseteq D^* F|_X(\bar{x}) \circ D^* S_0(F(\bar{x}) \mid \bar{u}). \tag{2.3.17}$$

With (2.3.12) and (2.3.17), if we assume that $\operatorname{gph} S_0$ is regular at $(F(\bar{x}), \bar{u})$ and the function $zF|_X$ is regular at $\bar{x}$ for all $z \in \operatorname{rg} D^* S_0\,(F(\bar{x}) \mid \bar{u})$, we have

$$D^* S|_X(\bar{x} \mid \bar{u}) = D^* F|_X(\bar{x}) \circ D^* S_0(F(\bar{x}) \mid \bar{u})) \tag{2.3.18}$$

and also that $S|_X$ is graphically regular at $\bar{x}$ for $\bar{u}$. Therefore

$$\operatorname{proj}_{T_X(\bar{x})} D^* F|_X(\bar{x}) \circ D^* S_0(F(\bar{x}) \mid \bar{u})) = \operatorname{proj}_{T_X(\bar{x})} D^* S|_X(\bar{x} \mid \bar{u}) \subseteq D^*_X S(\bar{x} \mid \bar{u}).$$

Besides, when $X$ is a smooth manifold,

$$\begin{aligned} D^*_X S(\bar{x}|\bar{u}) = \operatorname{proj}_{T_X(\bar{x})} D^* S|_X(\bar{x}|\bar{u}) &\subseteq \operatorname{proj}_{T_X(\bar{x})} D^* F|_X(\bar{x}) \circ D^* S_0(F(\bar{x})|\bar{u})) \\ &= D^*_X F(\bar{x}) \circ D^* S_0(F(\bar{x})|\bar{u})), \end{aligned} \tag{2.3.19}$$

where the two equations come from applying Proposition 2.1.5 to $S$ and $F$ and the inclusion comes from (2.3.17). Combining these two inclusions (2.3.18) and (2.3.19), we can obtain (2.3.11). □

## 2.4 Sum rules for projectional coderivatives

Next we present four sum rules on projectional coderivatives obtained by different methods. The differences are mainly caused by restricting $X$ onto different functions: $F(x) = (x, \ldots, x)$ or $S_i(x)$ and accordingly, different levels of constraint qualifications are taken into consideration. When restricting $F$ onto $X$ (Sum rule-1 and Sum rule-2), we separate $X$-related expressions from $S_i$ and when restricting $S_i$ onto $X$ (Sum rule-3 and Sum rule-4), the calculation is performed on $S_i|_X$. Within each type, two sum rules are given using different methods: via the corollaries we obtain above or directly via the sum rule of coderivative, [81, Theorem 10.41]. First, we introduce a sum rule obtained via Corollary 2.3.2 and Corollary 2.3.5.

**Theorem 2.4.1** (Sum rule-1)**.** *Let $S = S_1 + \cdots + S_p$ for $S_i : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ being outer semicontinuous relative to $X$ and let $\bar{x} \in \operatorname{dom} S \cap X$, $\bar{u} \in S|_X(\bar{x})$. Assume the following conditions are satisfied:*

- *(boundedness condition): the mapping*

$$(x, u) \mapsto \left\{ (u_1, \cdots, u_p) \;\middle|\; u_i \in S_i|_X(x),\ \forall i = 1, \ldots, p,\ \sum_{i=1}^{p} u_i = u \right\} \tag{2.4.1}$$

  *is locally bounded at $(\bar{x}, \bar{u})$.*

- *(constraint qualification):*

$$\left.\begin{array}{c} v_i \in D^* S_i(\bar{x} \mid u_i)(0),\ u_i \in S_i(\bar{x}),\ \sum_{i=1}^{p} u_i = \bar{u} \\ 0 \in D^*_X F(\bar{x})(v_1, \ldots, v_p) \end{array}\right\} \Longrightarrow v_i = 0 \text{ for } i = 1, \ldots, p \tag{2.4.2}$$

  *holds for $(\bar{x}, \bar{u}) \in \operatorname{gph} S|_X$, where $F : \mathbb{R}^n \to \mathbb{R}^{np}$ is defined as $F(x) = (x, \ldots, x)$.*

*Then $\operatorname{gph} S|_X$ is locally closed at $(\bar{x}, \bar{u})$ and one has*

$$D^*_X S(\bar{x} \mid \bar{u}) \subseteq \bigcup_{\substack{u_i \in S_i(\bar{x}) \\ \sum_{i=1}^{p} u_i = \bar{u}}} D^*_X F(\bar{x}) \circ \prod_{i=1}^{p} D^* S_i(\bar{x}|u_i). \tag{2.4.3}$$

*If in addition $X$ is a smooth manifold, the constraint qualification (2.4.2) can be simplified as*

$$\left.\begin{array}{c} v_i \in D^* S_i(\bar{x} \mid u_i)(0),\ u_i \in S_i(\bar{x}),\ \sum_{i=1}^{p} u_i = \bar{u} \\ \sum_{i=1}^{p} v_i \in N_X(\bar{x}) \end{array}\right\} \Longrightarrow v_i = 0 \text{ for } i = 1, \ldots, p \tag{2.4.4}$$

*and the inclusion becomes a fixed-point expression as*

$$D_X^* S(\bar{x} \mid \bar{u}) \subseteq \bigcup_{\substack{u_i \in S_i(\bar{x}) \\ \sum_{i=1}^p u_i = \bar{u}}} \mathrm{proj}_{T_X(\bar{x})} \left( \sum_{i=1}^p D^* S_i(\bar{x}|u_i) \right). \tag{2.4.5}$$

*Moreover, when every $S_i|_X$ is graph-convex, the union is superfluous and the inclusion becomes equation.*

*Proof.* First let $S' = S_0 \circ F$ where $F(x) = (x, \cdots, x)$ ($p$ copies), $S_0(x_1, \cdots, x_p) = S_1(x_1) \times \cdots \times S_p(x_p)$. Then by [81, Proposition 6.43, Example 8.34, Exercise 10.43], we have

$$\begin{aligned} D^* S_0 \left( F(\bar{x}) \mid u_1, \cdots, u_p \right) (y_1, \cdots, y_p) &= \left( \prod_{i=1}^p D^* S_i(\bar{x} \mid u_i) \right) (y_1, \cdots, y_p) \\ &= \prod_{i=1}^p D^* S_i(\bar{x} \mid u_i)(y_i). \end{aligned}$$

For $v = (v_1, \ldots, v_p)$ and $v' = (v_1', \ldots, v_p')$ in $\mathbb{R}^{np}$,

$$\begin{aligned} D_X^* F(\bar{x})(v) &= \limsup_{x \xrightarrow{X} \bar{x}, v' \to v} \mathrm{proj}_{T_X(x)} \left( \partial (v' F|_X)(x) \right) \\ &= \limsup_{x \xrightarrow{X} \bar{x}, v' \to v} \mathrm{proj}_{T_X(x)} \left( \nabla F(x)^* v' + N_X(x) \right) \\ &= \limsup_{x \xrightarrow{X} \bar{x}, v_i' \to v_i} \mathrm{proj}_{T_X(x)} \left( \sum_{i=1}^p v_i' + N_X(x) \right). \end{aligned} \tag{2.4.6}$$

Then the constraint qualification (2.3.9): $D^* S_0(F(\bar{x}) \mid u_1, \cdots, u_p)(0, \cdots, 0) \cap D_X^* F(\bar{x})^{-1}(0) = \{0\}$ can be written as (2.4.2). Without loss of generality, we can relax the requirement that $S_0$ being outer semicontinuous to being outer semicontinuous relative to the set $X$ as we restrict our scope only to $X$ here. That is, $S_i$ is outer semicontinuous relative to $X$ for each $i$. Then by applying Corollary 2.3.5 we have for any $u_i \in S_i(\bar{x})$,

$$
\begin{aligned}
D_X^* S'(\bar{x}\,|u_1,\cdots,u_p) \subseteq & D_X^* F(\bar{x}) \circ D^* S_0\left(F(\bar{x})\,|u_1,\cdots,u_p\right) \\
= & D_X^* F(\bar{x}) \circ \left(\prod_{i=1}^{p} D^* S_i(\bar{x} \mid u_i)\right).
\end{aligned}
\tag{2.4.7}
$$

Secondly we write $S = F_2 \circ S'$ with $F_2(u_1,\cdots,u_p) = \sum_{i=1}^{p} u_i$, then

$$
S'|_X(x) \cap F_2^{-1}(u) = \left\{ (u_1,\cdots,u_p) \;\middle|\; u_i \in S_i|_X(x),\ \forall i = 1,\ldots,p,\ \sum_{i=1}^{p} u_i = u \right\}.
$$

Therefore the boundedness assumption can be put as (2.4.1) and we have

$$
D_X^* S(\bar{x} \mid \bar{u}) \subseteq \bigcup_{\substack{u_i \in S_i(\bar{x}) \\ \sum_{i=1}^{p} u_i = \bar{u}}} D_X^* S'(\bar{x}\,|u_1,\cdots,u_p) \circ \nabla F_2(u_1,\cdots,u_p)^*, \tag{2.4.8}
$$

according to Corollary 2.3.2 with the boundedness condition satisfied. Combining (2.1.12), (2.4.7) and (2.4.8), we arrive at

$$
\begin{aligned}
D_X^* S(\bar{x} \mid \bar{u})(y) \subseteq & \bigcup_{\substack{u_i \in S_i(\bar{x}) \\ \sum_{i=1}^{p} u_i = \bar{u}}} D_X^* F(\bar{x}) \circ \prod_{i=1}^{p} D^* S_i(\bar{x} \mid u_i)(y) \\
= & \bigcup_{\substack{u_i \in S_i(\bar{x}) \\ \sum_{i=1}^{p} u_i = \bar{u}}} \limsup_{x \xrightarrow{X} \bar{x}, v_i' \to v_i} \left\{ \operatorname{proj}_{T_X(x)} \left( \sum_{i=1}^{p} v_i' + w \right) \;\middle|\; v_i \in D^* S_i(\bar{x} \mid u_i)(y), \right. \\
& \left. w \in N_X(x) \right\}.
\end{aligned}
\tag{2.4.9}
$$

When $X$ is a smooth manifold, by Proposition 2.1.5(c) and Lemma 2.1.8, we have

$$
D_X^* F(\bar{x})(v) = \operatorname{proj}_{T_X(\bar{x})} \left( \sum_{i=1}^{p} v_i \right)
$$

and further the inclusion (2.4.5) and the constraint qualification expressed as in (2.4.4). With each $S_i|_X$ being graph-convex (or each $S_i$ being graph-convex together with $X$ being convex) the union becomes superfluous and the equation is obtained as

$$D_X^* S(\bar{x} \mid \bar{u}) = \mathrm{proj}_{T_X(\bar{x})} \left( \sum_{i=1}^{p} D^* S_i(\bar{x} \mid u_i) \right).$$

□

Next we present a sum rule that can be seen as taking an intermediate step as it is obtained by applying [81, Theorem 10.41] directly and involves the process of taking limsup.

**Theorem 2.4.2** (Sum rule-2)**.** *Let $S = S_1 + \cdots + S_p$ for $S_i : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ being outer semicontinuous relative to $X$ and let $\bar{x} \in \mathrm{dom}\, S \cap X$, $\bar{u} \in S|_X(\bar{x})$. Assume the boundedness condition (2.4.1) holds. If the following constraint qualification is satisfied:*

$$\left.\begin{array}{c} v_i \in D^* S_i(\bar{x} \mid u_i)(0),\ u_i \in S_i(\bar{x}),\ \sum_{i=1}^p u_i = \bar{u} \\ 0 \in \sum_{i=1}^p v_i + N_X(\bar{x}) \end{array}\right\} \Longrightarrow v_i = 0 \text{ for } i = 1, \ldots, p \tag{2.4.10}$$

*Then* $\mathrm{gph}\, S|_X$ *is locally closed at $(\bar{x}, \bar{u})$ and one has*

$$D_X^* S(\bar{x} \mid \bar{u})(y) \subseteq \limsup_{\substack{(x,u) \xrightarrow{\mathrm{gph}\, S|_X} (\bar{x},\bar{u}) \\ y' \to y}} \bigcup_{\substack{u_i' \in S_i(x) \\ \sum_{i=1}^p u_i' = u}} \mathrm{proj}_{T_X(x)} \left( \sum_{i=1}^{p} D^* S_i(x \mid u_i')(y') + N_X(x) \right). \tag{2.4.11}$$

*When the constraint qualification is strengthened into (2.4.2), the right-hand side of (2.4.11) is included by that of (2.4.9) (or (2.4.3)).*

*Proof.* Given (2.4.1) and (2.4.10) and that (2.4.10) holds for points $(x, u) \in \mathrm{gph}\, S|_X$ around $(\bar{x}, \bar{u})$ (as similar proof is given in Theorem 2.3.1), by [81, Theorem 6.42 and

Theorem 10.41], we have for all $(x,u) \in \operatorname{gph} S|_X$ around $(\bar{x},\bar{u})$,

$$N_{\operatorname{gph} S|_X}(x,u) \subseteq \bigcup_{\substack{u_i' \in S_i(x) \\ \sum_{i=1}^p u_i' = u}} \left\{ \left( \sum_{i=1}^p v_i + w, -y' \right) \middle| \, v_i \in D^* S_i(x \mid u_i')(y'), w \in N_X(x) \right\}$$

and therefore,

$$\begin{aligned} &\limsup_{(x,u) \xrightarrow{\operatorname{gph} S|_X} (\bar{x},\bar{u})} \operatorname{proj}_{T_X(x) \times \mathbb{R}^m} N_{\operatorname{gph} S|_X}(x,u) \\ &\subseteq \limsup_{(x,u) \xrightarrow{\operatorname{gph} S|_X} (\bar{x},\bar{u})} \bigcup_{\substack{u_i' \in S_i(x) \\ \sum_{i=1}^p u_i' = u}} \left\{ \left( \operatorname{proj}_{T_X(x)} \left( \sum_{i=1}^p v_i + w \right), -y' \right) \middle| \, v_i \in D^* S_i(x \mid u_i')(y'), \right. \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. w \in N_X(x) \right\}. \end{aligned} \tag{2.4.12}$$

Given the definition of projectional coderivative and the upper estimate (2.4.12) we have (2.4.11). Combining the expression of $D_X^* F(\bar{x})(v)$ in (2.4.6) and Lemma 2.1.1, we can see that the constraint qualification (2.4.13) is indicated by (2.4.2).

If a stronger constraint qualification (2.4.2) is given, we next prove the set in right-hand side of (2.4.12) is included by that of (2.4.9), i.e.,

$$\begin{aligned} D_X^* S(\bar{x} \mid \bar{u})(y) &\subseteq \limsup_{\substack{(x,u) \xrightarrow{\operatorname{gph} S|_X} (\bar{x},\bar{u}) \\ y' \to y}} \bigcup_{\substack{u_i' \in S_i(x) \\ \sum_{i=1}^p u_i' = u}} \operatorname{proj}_{T_X(x)} \left( \sum_{i=1}^p D^* S_i(x \mid u_i')(y') + N_X(x) \right) \\ &\subseteq \bigcup_{\substack{u_i \in S_i(\bar{x}) \\ \sum_{i=1}^p u_i = \bar{u}}} \limsup_{x \xrightarrow{X} \bar{x}, v_i' \to v_i} \left\{ \operatorname{proj}_{T_X(x)} \left( \sum_{i=1}^p v_i' + w \right) \middle| \, v_i \in D^* S_i(\bar{x} \mid u_i)(y), \right. \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. w \in N_X(x) \right\}. \end{aligned}$$

For $t \in D_X^* S(\bar{x} \mid \bar{u})(y)$, there exist sequences $(x_k,u_k) \xrightarrow{\operatorname{gph} S|_X} (\bar{x},\bar{u})$, $u_{ik}' \in S_i(x_k)$ with $\sum_{i=1}^p u_{ik}' = u_k$, $(v_{ik}, -y_k) \in N_{\operatorname{gph} S_i}(x_k, u_{ik}')$, $w_k \in N_X(x_k)$ such that $y_k \to y$

and $t_k \in \operatorname{proj}_{T_X(x_k)}(\sum_{i=1}^p v_{ik} + w_k) \to t$. As $S_i$ are outer semicontinuous relative to $X$ and the boundedness condition (2.4.1) holds, $\{u'_{ik}\}$ must converge to some $u_i \in S_i(\bar{x})$. Given $\sum_{i=1}^p u'_{ik} = u_k \to \bar{u}$, $\sum_{i=1}^p u_i = \bar{u}$. For $(v_{ik}, -y_k) \in N_{\operatorname{gph} S_i}(x_k, u'_{ik})$, when $\{v_{ik}\}$ are all bounded for $i = 1, \ldots, p$, then $v_{ik} \to v_i \in D^*S_i(\bar{x} \mid u_i)(y)$ and accordingly $t$ belongs to the right-hand side of (2.4.9). For the case that there exists at least one $j \in \{1, \ldots, p\}$ such that $\{v_{jk}\}$ is unbounded, then $\{\lambda_k v_{jk}\}$ must converge to some $v_j \in D^*S_j(\bar{x} \mid u_j)(0)$ with $v_j \neq 0$ for $\lambda_k \searrow 0$. For $i \neq j$, $\lambda_k v_{ik} \to 0$. Also, $0 \leftarrow \lambda_k t_k \in \operatorname{proj}_{T_X(x_k)}(\sum_{i=1}^p \lambda_k v_{ik} + \lambda_k w_k)$. Then $v_j \neq 0$ contradicts the constraint qualification (2.4.2) and therefore this case is abandoned and the inclusion is proved. □

**Remark 2.4.3.** *When $X$ is a smooth manifold at $\bar{x}$, the constraint qualification (2.4.2) turns into (2.4.4), and coincides with (2.4.10) as $N_X(\bar{x}) = -N_X(\bar{x})$. In this case, the upper estimate of $D^*_X S(\bar{x}, \bar{u})$, (2.4.11) is the same as (2.4.5).*

Comparing the constraint qualification (2.4.10) with the one in [81, Theorem 10.41], we can see that (2.4.10) also serves as a constraint qualification to express $N_{\operatorname{gph} S|_X}$ via $N_{\operatorname{gph} S}$ and $N_X$. Next we present a sum rule where each $S_i$ is restricted onto $X$. Unlike in Theorem 2.4.1 where we restrict $F$ onto $X$, in this sum rule we restrict each $S_i$ onto $X$ and therefore only Corollary 2.3.2 is employed rather than both Corollaries 2.3.2 and 2.3.5.

**Theorem 2.4.4** (Sum rule-3)**.** *Let $S = S_1 + \cdots + S_p$ for $S_i : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ being outer semicontinuous relative to $X$ and let $\bar{x} \in \operatorname{dom} S \cap X$, $\bar{u} \in S|_X(\bar{x})$. Assume the boundedness condition (2.4.1) is satisfied and the constraint qualification holds:*

$$\left.\begin{array}{c} v_i \in D^*S_i|_X(\bar{x} \mid u_i)(0),\ u_i \in S_i(\bar{x}),\ \sum_{i=1}^p u_i = \bar{u} \\ \sum_{i=1}^p v_i = 0 \end{array}\right\} \Longrightarrow v_i = 0 \text{ for } i = 1, \ldots, p. \tag{2.4.13}$$

*Then* $\operatorname{gph} S|_X$ *is locally closed at* $(\bar{x}, \bar{u})$ *and one has*

$$D_X^* S(\bar{x} \mid \bar{u})(y) \subseteq \limsup_{\substack{(x,u) \xrightarrow{\operatorname{gph} S|_X} (\bar{x},\bar{u}) \\ y_i' \to y, i=1,\ldots,p}} \bigcup_{\substack{u_i' \in S_i(x) \\ \sum_{i=1}^p u_i' = u}} \operatorname{proj}_{T_X(x)} \left( \sum_{i=1}^p D^* S_i|_X (x \mid u_i')(y_i') \right). \tag{2.4.14}$$

*If in addition* $X$ *is a smooth manifold, the inclusion becomes a fixed-point expression as*

$$D_X^* S(\bar{x} \mid \bar{u}) \subseteq \bigcup_{\substack{u_i \in S_i|_X(\bar{x}) \\ \sum_{i=1}^p u_i = \bar{u}}} \operatorname{proj}_{T_X(\bar{x})} \left( \sum_{i=1}^p D^* S_i|_X (\bar{x}|u_i) \right). \tag{2.4.15}$$

*Moreover, when every* $S_i|_X$ *is graph-convex, the union is superfluous and the inclusion becomes equation.*

*Proof.* First let $S' = S_0 \circ F$ where $F(x) = (x, \ldots, x)$ ($p$ copies), $S_0(x_1, \cdots, x_p) = S_1(x_1) \times \cdots \times S_p(x_p)$. By restricting each $S_i$ onto $X$, we have

$$S'|_X(x) = S_0|_{X \times \cdots \times X} (F(\bar{x})) = S_1|_X(x) \times \cdots \times S_p|_X(x).$$

By [81, Proposition 6.41], we have

$$\begin{aligned} {}^* S_0|_{X \times \cdots \times X} (F(\bar{x}) \mid u_1, \ldots, u_p)(y_1, \ldots, y_p) &= \left( \prod_{i=1}^p D^* S_i|_X (\bar{x} \mid u_i) \right) (y_1, \ldots, y_p) \\ &= \prod_{i=1}^p D^* S_i|_X (\bar{x} \mid u_i)(y_i). \end{aligned} \tag{2.4.16}$$

Note that $D^* F(\bar{x})(v_1, \ldots, v_p) = \sum_{i=1}^p v_i$. Then the constraint qualification in [81, Theorem 10.40]: $D^* S_0|_{X \times \cdots \times X}(F(\bar{x}) \mid u_1, \ldots, u_p)(0, \ldots, 0) \cap D^* F(\bar{x})^{-1}(0) = \{0\}$ can be written as (2.4.13). Note that this constraint qualification also indicates the one for any point $(x, u_1', \ldots, u_p') \in \operatorname{gph} S'|_X$ being sufficiently close to $(\bar{x}, u_1, \ldots, u_p)$

(similar to the previous proof of Theorem 2.3.1), therefore by [81, Theorem 10.40] we have

$$D^*S'|_X\left(x\left|u_1',\dots,u_p'\right.\right)\subseteq D^*F(x)\circ D^*S_0|_{X\times\cdots\times X}\left(F(x)\left|u_1',\dots,u_p'\right.\right)\tag{2.4.17}$$

for any $(x,u_1',\dots,u_p')\in\operatorname{gph}S'|_X$ being sufficiently close to $(\bar{x},u_1,\dots,u_p)$. Then by definition of projectional coderivative and (2.4.17),

$$\begin{aligned}&D_X^*S'\left(\bar{x}\left|u_1,\dots,u_p\right.\right)(y_1,\dots,y_p)\\ \subseteq&\limsup_{\substack{(x,u_1',\dots,u_p')\xrightarrow{\operatorname{gph}S'|_X}(\bar{x},u_1,\dots,u_p)\\(y_1',\dots,y_p')\to(y_1,\dots,y_p)}}\operatorname{proj}_{T_X(x)}D^*F(x)\circ D^*S_0|_{X\times\cdots\times X}\left(F(x)\left|u_1',\dots,u_p'\right.\right)\left(y_1',\dots,y_p'\right).\end{aligned}\tag{2.4.18}$$

Secondly we write $S=F_2\circ S'$ with $F_2(u_1,\dots,u_p)=\sum_{i=1}^p u_i$, then

$$S'|_X(x)\cap F_2^{-1}(u)=\left\{(u_1,\dots,u_p)\;\middle|\;u_i\in S_i|_X(x),\ \forall i=1,\dots,p,\ \sum_{i=1}^p u_i=u\right\}.$$

Therefore according to Corollary 2.3.2 with the boundedness assumption satisfied, we have

$$D_X^*S(\bar{x}|\bar{u})\subseteq\bigcup_{\substack{u_i\in S_i(\bar{x})\\\sum_{i=1}^p u_i=\bar{u}}}D_X^*S'\left(\bar{x}\left|u_1,\cdots,u_p\right.\right)\circ\nabla F_2(u_1,\cdots,u_p)^*.\tag{2.4.19}$$

Combining (2.4.16), (2.4.18) and (2.4.19), we arrive at

$$D_X^*S(\bar{x}\mid\bar{u})(y)\subseteq\bigcup_{\substack{u_i\in S_i(\bar{x})\\\sum_{i=1}^p u_i=\bar{u}}}\limsup_{\substack{(x,u_1',\dots,u_p')\xrightarrow{\operatorname{gph}S'|_X}(\bar{x},u_1,\dots,u_p)\\y_i'\to y,\ i=1,\dots,p}}\operatorname{proj}_{T_X(x)}\left(\sum_{i=1}^p D^*S_i|_X(x\mid u_i')(y_i')\right).$$

For $(x,u_1',\dots,u_p')\xrightarrow{\operatorname{gph}S'|_X}(\bar{x},u_1,\dots,u_p)$, it is equivalent that $(x,u_i')\xrightarrow{\operatorname{gph}S_i|_X}(\bar{x},u_i)$ for $i=1,\dots,p$. Besides, $u:=\sum_{i=1}^p u_i'\to\sum_{i=1}^p u_i=\bar{u}$, which means $(x,u)\in$

$\operatorname{gph} S|_X$. Therefore we have

$$\bigcup_{\substack{u_i \in S_i(\bar{x}) \\ \sum_{i=1}^p u_i = \bar{u}}} \limsup_{\substack{(x, u'_1, \ldots, u'_p) \xrightarrow{\operatorname{gph} S'|_X} (\bar{x}, u_1, \ldots, u_p) \\ y'_i \to y,\ i=1,\ldots,p}} \operatorname{proj}_{T_X(x)} \left( \sum_{i=1}^p D^* S_i|_X(x \mid u'_i)(y'_i) \right)$$

$$\subseteq \limsup_{\substack{(x,u) \xrightarrow{\operatorname{gph} S|_X} (\bar{x}, \bar{u}) \\ y'_i \to y,\ i=1,\ldots,p}} \bigcup_{\substack{u'_i \in S_i|_X(x) \\ \sum_{i=1}^p u'_i = u}} \operatorname{proj}_{T_X(x)} \left( \sum_{i=1}^p D^* S_i|_X(x \mid u'_i)(y'_i) \right)$$

and accordingly (2.4.14) holds. Besides, if $X$ is a smooth manifold, by Proposition 2.1.5, we have

$$D^*_X S(\bar{x} \mid \bar{u})(y) \subseteq \bigcup_{\substack{u_i \in S_i(\bar{x}) \\ \sum_{i=1}^p u_i = \bar{u}}} \operatorname{proj}_{T_X(\bar{x})} \left( \sum_{i=1}^p D^* S_i|_X(\bar{x} \mid u_i)(y) \right).$$

With each $S_i|_X$ being graph-convex, both the inclusions (2.4.17) and (2.4.19) become equations respectively and the union in (2.4.15) becomes superfluous and we have

$$\operatorname{proj}_{T_X(\bar{x})} \left( \sum_{i=1}^p D^* S_i|_X(\bar{x} \mid u_i)(y) \right) \subseteq D^*_X S(\bar{x} \mid \bar{u})(y),\ \forall u_i \in S_i(\bar{x}),\ \sum_{i=1}^p u_i = \bar{u}.$$

□

In next sum rule, a tighter upper estimate is given as we apply directly, [81, Theorem 10.41].

**Theorem 2.4.5** (Sum rule-4)**.** *Still under the setting of Theorem 2.4.4, we can also have*

$$D^*_X S(\bar{x} \mid \bar{u})(y) \subseteq \limsup_{\substack{(x,u) \xrightarrow{\operatorname{gph} S|_X} (\bar{x}, \bar{u}) \\ y' \to y}} \bigcup_{\substack{u'_i \in S_i(x) \\ \sum_{i=1}^p u'_i = u}} \operatorname{proj}_{T_X(x)} \left( \sum_{i=1}^p D^* S_i|_X(x \mid u'_i)(y') \right), \tag{2.4.20}$$

*where the right-hand side of (2.4.20) is included in that of (2.4.14).*

*Proof.* By Theorem 10.41, for any $(x,u) \in \operatorname{gph} S|_X$ being close enough to $(\bar{x},\bar{u})$:

$$N_{\operatorname{gph} S|_X}(x,u) \subseteq \bigcup_{\substack{u_i' \in S_i(x) \\ \sum_{i=1}^p u_i' = u}} \left\{ \left( \sum_{i=1}^p v_i, -y' \right) \middle| \, v_i \in D^* S_i|_X(x \mid u_i')(y') \right\}$$

and therefore

$$\limsup_{(x,u) \xrightarrow{\operatorname{gph} S|_X} (\bar{x},\bar{u})} \operatorname{proj}_{T_X(x) \times \mathbb{R}^m} N_{\operatorname{gph} S|_X}(x,u)$$

$$\subseteq \limsup_{(x,u) \xrightarrow{\operatorname{gph} S|_X} (\bar{x},\bar{u})} \bigcup_{\substack{u_i' \in S_i(x) \\ \sum_{i=1}^p u_i' = u}} \left\{ \left( \operatorname{proj}_{T_X(x)} \left( \sum_{i=1}^p v_i \right), -y' \right) \middle| \, v_i \in D^* S_i|_X(x \mid u_i')(y') \right\}.$$

By comparing the terms in the right-hand side of (2.4.20) and that of (2.4.14) we can see that the former is a special case of the latter where each $y_i'$ is taken as $y'$ and therefore included by the latter. $\square$

As both Theorem 2.4.5 and Theorem 2.4.2 are obtained directly from applying, [81, Theorem 10.41], the difference mainly exists in using different constraint qualifications. The one in Theorem 2.4.5 carries $X$ in each $S_i|_X$ in the calculation while the one in Theorem 2.4.2 separates $X$ from $S_i$. Besides, the difference between Theorem 2.4.5 and Theorem 2.4.4 comes from using a larger estimate in Corollary 2.3.2 with the form of projectional coderivatives.

# Chapter 3

# Relative Lipschitz-like Property for Parametric Systems

In this chapter, we consider an extended form of parametric system under the framework of [53]:

$$S(w) := \{x \in \mathbb{R}^n \mid 0 \in G(w,x) + M(w,x)\} \tag{3.0.1}$$

where $G : \mathbb{R}^{m+n} \to \mathbb{R}^d$ is a $\mathcal{C}^1$ mapping and $M : \mathbb{R}^{m+n} \rightrightarrows \mathbb{R}^d$ is a multifunction with a closed graph. We first develop the upper estimates of projectional coderivative of $S$ under different levels of constraint qualifications accordingly and illustrate how adaptive these constraint qualifications can be when applying them to different systems via some simple examples. As this type of system (3.0.1) also includes the one with $0 \in M(w,x)$, we treat the latter as a special case and compare our result with the one in [5] for sufficiency on the relative Lipschitz-like property. Structural differences of these two approaches are demonstrated via an example. When $M(w,x)$ is a multifunction of $x$ only, i.e., $M(w,x) = M(x)$, this framework can be applied to various types of problems. For example, when $M(x)$ is a normal cone mapping, then linear complementarity problems and affine variational inequalities fit in. The discussion on non-emptiness of $S(w)$ for these cases can be found in monographs [22, 51]. For the remaining sections, we apply this upper estimate to a wide range of problems covered by the framework (3.0.1): linear constraint systems, linear complementarity problems and affine variational inequalities. For the first two types of problems, we give exact expression of projectional coderivatives for the solution mappings relative to their domains with the structure of the union of polyhedral sets and the sufficient and necessary conditions for their Lipschitz-like property relative to their domain. For the affine variational inequalities, we consider in general a set within its domain and obtain an upper estimate of the projectional coderivatives under some constraint qualification and formulate a generalized critical face condition in view of [18].

## 3.1 Projectional coderivatives for parametric systems

First we present the result in [53] on coderivatives of solution maps of (3.0.1). We slightly tune the statement by switching the position of $w$ and $x$.

**Lemma 3.1.1** ([53, Theorem 2.1])**.** *Consider the implicit mapping $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ of the form (3.0.1) with $G : \mathbb{R}^{m+n} \to \mathbb{R}^d$ a $\mathcal{C}^1$ mapping, and $M : \mathbb{R}^{m+n} \rightrightarrows \mathbb{R}^d$ a multifunction with a closed graph. Consider a pair $(\bar{w}, \bar{x}) \in \operatorname{gph} S$. If the constraint qualification holds:*

$$(0_m, 0_n) \in \nabla G(\bar{w}, \bar{x})^* y + D^* M \left( (\bar{w}, \bar{x}) \,|{-G(\bar{w}, \bar{x})} \right) (y) \Longrightarrow y = 0_d \tag{3.1.1}$$

*then*

$$D^* S \left( \bar{w} \mid \bar{x} \right) (r) \subseteq \bigcup_{y \in \mathbb{R}^d} \left\{ v \in \mathbb{R}^m \;\middle|\; (v, -r) \in \left( \nabla G(\bar{w}, \bar{x})^* y + D^* M \left( (\bar{w}, \bar{x}) \mid -G(\bar{w}, \bar{x}) \right) (y) \right) \right\}. \tag{3.1.2}$$

*The inclusion becomes an equation if one of the following conditions is satisfied:*

**(a)** *either $M$ is graphically regular at $(\bar{w}, \bar{x}, -G(\bar{w}, \bar{x}))$, or*

**(b)** *$M = M(x)$ and $\nabla_w G(\bar{w}, \bar{x})$ has full rank. In this case the constraint qualification*

*(3.1.1) holds automatically and one has*

$$D^*S(\bar{w} \mid \bar{x})(r) = \bigcup_{y \in \mathbb{R}^d} \left\{ \nabla_w G(\bar{w}, \bar{x})^* y \,\middle|\, -r \in \left(\nabla_x G(\bar{w}, \bar{x})^* y + D^*M(\bar{x} \mid -G(\bar{w}, \bar{x}))(y)\right) \right\}.$$

To ease the notations, we omit the dimension $m$, $n$ and $d$ in the subscripts of the zero vector 0 whenever it can be obviously derived. Most of the results we obtain for projectional coderivatives are based on the proposition above with $M$ additionally restricted on a set $W$.

**Theorem 3.1.2.** *Consider the implicit mapping $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ of the form (3.0.1) with $G : \mathbb{R}^{m+n} \to \mathbb{R}^d$ a $\mathcal{C}^1$ mapping, and $M : \mathbb{R}^{m+n} \rightrightarrows \mathbb{R}^d$ a multifunction with closed graph. Consider a pair $(\bar{w}, \bar{x}) \in \operatorname{gph} S|_W$ where $W \subseteq \operatorname{dom} S$ is a closed set. If the following constraint qualification holds:*

$$(0,0) \in \nabla G(\bar{w}, \bar{x})^* y + D^*M|_{W \times \mathbb{R}^n}\left((\bar{w}, \bar{x}) \,|{-G(\bar{w}, \bar{x})}\right)(y) \Longrightarrow y = 0, \tag{3.1.3}$$

*then we have*

$$D^*_W S(\bar{w} \mid \bar{x})(r) \subseteq \limsup_{\substack{(w,x) \xrightarrow{\operatorname{gph} S|_W} (\bar{w}, \bar{x}) \\ r' \to r}} \bigcup_{y \in \mathbb{R}^d} \left\{ \operatorname{proj}_{T_W(w)}(v) \,\middle|\, (v, -r') \in \left(\nabla G(w,x)^* y + D^*M|_{W \times \mathbb{R}^n}((w,x) \mid -G(w,x))(y)\right) \right\}. \tag{3.1.4}$$

*If we strengthen the constraint qualification (3.1.3) to*

$$(0,0) \in \limsup_{\substack{(w,x) \xrightarrow{\operatorname{gph} S|_W} (\bar{w}, \bar{x}) \\ y' \to y}} \operatorname{proj}_{T_W(w) \times \mathbb{R}^n}\left(\nabla G(w,x)^* y' + D^*M|_{W \times \mathbb{R}^n}\left((w,x) \,|{-G(w,x)}\right)(y')\right) \Longrightarrow y = 0 \tag{3.1.5}$$

*then the limsup in (3.1.4) can be put into the bracket as*

$$D_W^* S\,(\bar{w}\mid\bar{x})(r) \subseteq \left\{ t\in\mathbb{R}^m \,\middle|\, \exists y\in\mathbb{R}^d \text{ with } (t,-r)\in \limsup_{\substack{(w,x,-G(w,x))\xrightarrow{\text{gph } M|_{W\times\mathbb{R}^n}}(\bar{w},\bar{x},-G(\bar{w},\bar{x}))\\ y'\to y}} \operatorname{proj}_{T_W(w)\times\mathbb{R}^n}\big(\nabla G(w,x)^* y' + D^* M|_{W\times\mathbb{R}^n}\left((w,x)\mid -G(w,x)\right)(y')\big) \right\}. \tag{3.1.6}$$

*If in addition, $M|_{W\times\mathbb{R}^n}$ is graphically regular at $(\bar{w},\bar{x},-G(\bar{w},\bar{x}))$ and $W$ is a smooth manifold at $\bar{w}$, then*

$$D_W^* S\,(\bar{w}\mid\bar{x})\,(r) = \left\{ t\in\mathbb{R}^m \,\middle|\, \exists y\in\mathbb{R}^d \text{ with } (t,-r)\in \operatorname{proj}_{T_W(\bar{w})\times\mathbb{R}^n}\Big(\nabla G(\bar{w},\bar{x})^* y + D^* M|_{W\times\mathbb{R}^n}\left((\bar{w},\bar{x})\mid -G(\bar{w},\bar{x})\right)(y)\Big) \right\}. \tag{3.1.7}$$

*Proof.* Similar to the proof of Theorem 2.3.1, (3.1.3) also indicates that

$$(0,0)\in\nabla G(w,x)^* y + D^* M|_{W\times\mathbb{R}^n}\left((w,x)\,|{-G(w,x)}\right)(y) \Longrightarrow y=0$$

for any $(w,x)\in\operatorname{gph} S|_W$ sufficiently close to $(\bar{w},\bar{x})$. According to [53, Theorem 2.1], for any pair $(w,x)\in\operatorname{gph} S|_W$ sufficiently near $(\bar{w},\bar{x})$, we have

$$N_{\operatorname{gph} S|_W}(w,x)\subseteq \bigcup_{y\in\mathbb{R}^d}\left(\nabla G(w,x)^* y + D^* M|_{W\times\mathbb{R}^n}((w,x)\mid -G(w,x))(y)\right). \tag{3.1.8}$$

Therefore we have

$$\begin{aligned}
&\limsup_{(w,x)\xrightarrow{\text{gph } S|_W}(\bar{w},\bar{x})} \operatorname{proj}_{T_W(w)\times\mathbb{R}^n} N_{\operatorname{gph} S|_W}(w,x)\\
\subseteq &\limsup_{(w,x)\xrightarrow{\text{gph } S|_W}(\bar{w},\bar{x})} \bigcup_{y\in\mathbb{R}^d} \operatorname{proj}_{T_W(w)\times\mathbb{R}^n}\left(\nabla G(w,x)^* y + D^* M|_{W\times\mathbb{R}^n}((w,x)\mid -G(w,x))(y)\right)
\end{aligned} \tag{3.1.9}$$

and accordingly the inclusion (3.1.4) holds.

Now, assume that the constraint qualification (3.1.5) holds. By the definition of projectional coderivative (1.3.6) and (3.1.9), for $t \in D^*_W S(\bar{w} \mid \bar{x})(r)$, there exist sequences $(w_k, x_k) \xrightarrow{\operatorname{gph} S|_W} (\bar{w}, \bar{x})$, $y_k \in \mathbb{R}^d$ and

$$(v_k, -r_k) \in \nabla G(w_k, x_k)^* y_k + D^* M|_{W \times \mathbb{R}^n} \left((w_k, x_k) \mid -G(w_k, x_k)\right)(y_k),$$

such that $t_k \in \operatorname{proj}_{T_W(w_k)}(v_k) \to t$ and $r_k \to r$. Taking a subsequence if necessary, we have either $y_k \to y \in \mathbb{R}^d$ or $\lambda_k y_k \to y \in \mathbb{R}^d$ with $\lambda_k \searrow 0$. For the first case, we directly have that $t$ belongs to the right-hand side of (3.1.6). For the second case, without loss of generality we assume $\|y\| = 1$. With the conic structure we have $\lambda_k(v_k, -r_k) \in \nabla G(w_k, x_k)^*(\lambda_k y_k) + D^* M|_{W \times \mathbb{R}^n} \left((w_k, x_k) \mid -G(w_k, x_k)\right)(\lambda_k y_k)$ and accordingly $\lambda_k t_k \in \lambda_k \operatorname{proj}_{T_W(w_k)}(v_k) \to 0$, $\lambda_k r_k \to 0$, which contradicts the constraint qualification (3.1.5) with $\|y\| = 1$. Given that $(w, x) \xrightarrow{\operatorname{gph} S|_W} (\bar{w}, \bar{x})$ is equivalent to $(w, x, -G(w, x)) \xrightarrow{\operatorname{gph} M|_{W \times \mathbb{R}^n}} (\bar{w}, \bar{x}, -G(\bar{w}, \bar{x}))$ and therefore we have that $t$ also belongs to the set on the right-hand side of (3.1.6).

If furthermore $M|_{W \times \mathbb{R}^n}$ is graphically regular at $(\bar{w}, \bar{x}, -G(\bar{w}, \bar{x}))$, again by Proposition 3.1.1, we have (3.1.8) as an equation at the reference point $(\bar{w}, \bar{x})$ and therefore

$$\begin{aligned}
&\operatorname{proj}_{T_W(\bar{w})} D^* S|_W(\bar{w} \mid \bar{x})(r) \\
&= \Big\{ t \in \mathbb{R}^m \,\Big|\, \exists y \in \mathbb{R}^d \text{ with } (t, -r) \in \operatorname{proj}_{T_W(\bar{w}) \times \mathbb{R}^n} \big(\nabla G(\bar{w}, \bar{x})^* y \\
&\qquad\qquad + D^* M|_{W \times \mathbb{R}^n} \left((\bar{w}, \bar{x}) \mid -G(\bar{w}, \bar{x})\right)(y)\big) \Big\} \\
&\subseteq D^*_W S(\bar{w} \mid \bar{x})(r).
\end{aligned} \tag{3.1.10}$$

Besides, when $W$ is a smooth manifold at $\bar{w}$, by Proposition 2.1.5(c) and (3.1.8),

$$\begin{aligned} & D^*_W S\left(\bar{w} \mid \bar{x}\right)(r) = \operatorname{proj}_{T_W(\bar{w})} D^* S|_W \left(\bar{w} \mid \bar{x}\right)(r) \\ \subseteq & \left\{ t \in \mathbb{R}^m \middle| \exists y \in \mathbb{R}^d \text{ with } (t, -r) \in \operatorname{proj}_{T_W(\bar{w}) \times \mathbb{R}^n} \big( \nabla G(\bar{w}, \bar{x})^* y \right. \\ & \qquad\qquad \left. + D^* M|_{W \times \mathbb{R}^n} \left( (\bar{w}, \bar{x}) \mid -G(\bar{w}, \bar{x}) \right)(y) \big) \right\}. \end{aligned} \tag{3.1.11}$$

Combining the conditions that $M|_{W\times\mathbb{R}^n}$ is graphically regular at $(\bar{w}, \bar{x}, -G(\bar{w}, \bar{x}))$ and that $W$ is a smooth manifold at $\bar{w}$, (3.1.10) and (3.1.11) turn into equation (3.1.7). $\square$

**Remark 3.1.3.** *The constraint qualification (3.1.3) involves only coderivatives of $G(w, x) + M|_{W\times\mathbb{R}^n}(w, x)$ at the reference point while the stronger constraint qualification (3.1.5) involves the projected coderivatives. By Lemma 2.1.1 we can see that (3.1.5) indicates (3.1.3). When the stronger constraint qualification is satisfied, we can have a tighter estimate, as RHS of (3.1.6) is included by that of (3.1.4). The connection between the basic constraint qualification and the stronger one is better revealed in the Corollary 3.1.5 as it involves $M(w, x)$ only.*

Here we use a simple example to illustrate how the stronger constraint qualification can be applied in calculating the projectional coderivatives (3.1.7). Note that in a later example (Example 3.1.9) we will show how that the basic constraint qualification can be adopted but the stronger one fails.

**Example 3.1.4.** *For $S(w) := \{x \in \mathbb{R}^n \mid Ax + w \in K\}$ where $K \subseteq \mathbb{R}^m$ is a closed set. Let $G(w, x) = -Ax - w$ and $M(w, x) = K$. For $W \subseteq \operatorname{dom} S$ we can write $\operatorname{gph} M|_{W\times\mathbb{R}^n} = W \times \mathbb{R}^n \times K$ and accordingly for any $(w, x, u) \in \operatorname{gph} M|_{W\times\mathbb{R}^n}$,*

$$D^* M|_{W\times\mathbb{R}^n}\left((w, x) \mid u\right)(y) = \begin{cases} N_W(w) \times \{0\}, & \text{if } y \in -N_K(u) \\ \emptyset, & \text{if } y \notin -N_K(u) \end{cases}, \tag{3.1.12}$$

*and*

$$\nabla G(w,x)^* y = (-y, -A^* y). \tag{3.1.13}$$

*Let* $n = m = 2$, $K = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}$, $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. *Then we can also write the closed form of* $S$ *as*

$$S(w) = \begin{cases} \mathbb{R}^2, & w_1 = 0, \\ \mathbb{R} \times \{-w_2\}, & w_1 \neq 0. \end{cases}$$

*Then* $\operatorname{dom} S = K + \operatorname{rg} A = \mathbb{R}^2$. *Consider the particular pair* $(\bar{w}, \bar{x}) \in \operatorname{gph} S|_W$ *where* $\bar{w} = (0,1)^\top$, $\bar{x} = (0,0)^\top$ *and a smooth manifold* $W = \mathbb{R} \times \{1\} \subseteq \operatorname{dom} S$. *Then* $S|_W(w)$ *with* $w = (w_1, 1)^\top \in W$ *becomes*

$$S|_W(w) = \begin{cases} \mathbb{R}^2, & w_1 = 0, \\ \mathbb{R} \times \{-1\}, & w_1 \neq 0. \end{cases}$$

*Then the constraint qualification (3.1.5) becomes the one at the reference pair* $(\bar{w}, \bar{x})$ *(as* $W$ *is a smooth manifold):*

$$\begin{aligned} (0,0) &\in \operatorname{proj}_{T_W(\bar{w}) \times \mathbb{R}^n} \left( \nabla G(\bar{w}, \bar{x})^* y + D^* M|_{W \times \mathbb{R}^n} \left( (\bar{w}, \bar{x}) \mid -G(\bar{w}, \bar{x}) \right)(y) \right) \\ &= \left\{ (\operatorname{proj}_{T_W(\bar{w})}(v - y), -A^* y) \mid v \in N_W(\bar{w}), y \in -N_K(A\bar{x} + \bar{w}) \right\} \Longrightarrow y = 0, \end{aligned}$$

*where the equation is a direct result from (3.1.12) and (3.1.13). As* $-G(\bar{w}, \bar{x}) = A\bar{x} + \bar{w} = (0,1)^\top$, $K$ *is regular at* $-G(\bar{w}, \bar{x})$ *and*

$$N_K(-G(\bar{w}, \bar{x})) = N_K\left((0,1)^\top\right) = \mathbb{R} \times \{0\}.$$

*Thus* $M|_{W \times \mathbb{R}^n}$ *is graphically regular at* $(\bar{w}, \bar{x}, -G(\bar{w}, \bar{x}))$. *Besides, in view of the fact that* $T_W(\bar{w}) = \mathbb{R} \times \{0\}$ *and* $y \in -N_K(A\bar{x} + \bar{w}) = \mathbb{R} \times \{0\}$ *and by Lemma 2.1.2,*

$$0 = \operatorname{proj}_{T_W(\bar{w})}(v - y) = \operatorname{proj}_{T_W(\bar{w})}(-y) = -y \Longrightarrow y = 0.$$

*Thus the constraint qualification is satisfied. Applying (3.1.7) , we obtain*

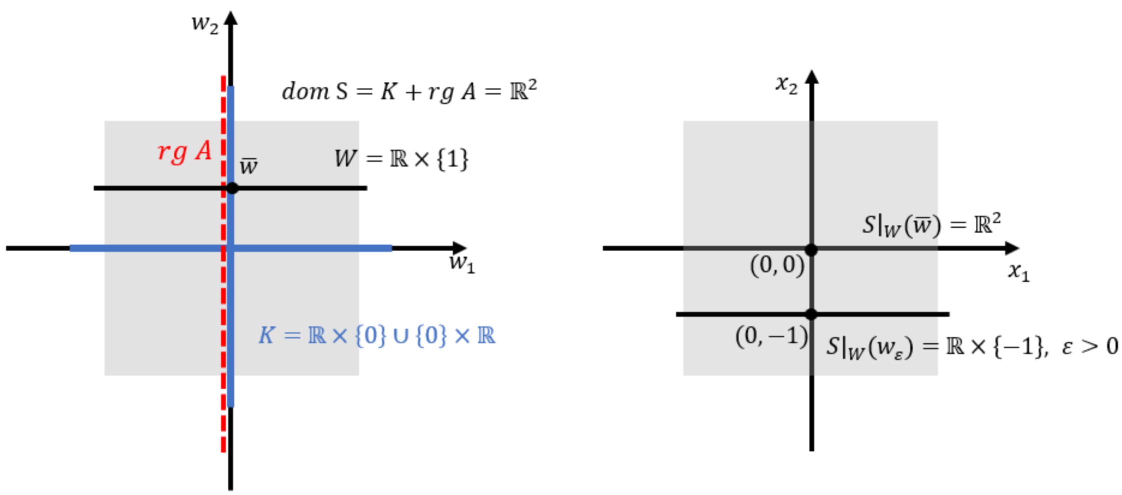
$$D_W^* S(\bar{w}, \bar{x})(r) = \{ y \mid y \in N_K(A\bar{w} + \bar{x}) \text{ with } A^* y = r \} = \begin{cases} \mathbb{R} \times \{0\} & \text{if } r = (0,0)^\top, \\ \emptyset & \text{if } r \neq (0,0)^\top. \end{cases}$$

*Thus,* $D^*_W S(\bar{w},\bar{x})((0,0)^\top) = \mathbb{R}\times\{0\} \neq \{(0,0)^\top\}$*,* $S$ *does not enjoy the Lipschitz-like property relative to* $W$ *at* $\bar{w}$ *for* $\bar{x}$ *according to Theorem 2.2.3. Also, for* $w_\varepsilon = (\varepsilon, 1)^\top \in W$ *with* $\varepsilon > 0$*, choose* $\rho > 0$ *being small enough. Then the inclusion*

$$S(\bar{w}) \cap \mathbb{B}_\rho(\bar{x}) = \mathbb{B}_\rho \subseteq S(w_\varepsilon) + \kappa\varepsilon\mathbb{B} = \mathbb{R}\times\{-1\} + \kappa\varepsilon\mathbb{B}$$

*does not hold unless* $\varepsilon \geq \frac{1+\rho}{\kappa}$*. Then by definition we can also draw the same conclusion. Below we give the figure of* $\operatorname{dom} S$ *and* $S|_W(\bar{w}), S|_W(w_\varepsilon)$*.*

Figure 3.1: $\operatorname{dom} S$ and $S|_W(w)$ of Example 3.1.4.




By observing the right-hand side of the expression (3.1.6), we can see that $(t, -r)$ actually belongs to the projectional coderivative of the multifunction $G(w,x) + M(w,x)$ relative to the set $W \times \mathbb{R}^n$. Next we present a simpler model by taking $G(w,x) = 0$ so that the relation of projectional coderivatives between $S$ and $M$ can be revealed more clearly. After that, we give a parallel comparison with a related result in [5], which uses directional coderivatives to characterize the Lipschitz-like property of $S$ relative to certain directions.

**Corollary 3.1.5.** *For an implicit mapping* $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ *as* $S(w) = \{x \mid 0 \in M(w,x)\}$ *where* $M : \mathbb{R}^m \times \mathbb{R}^n \rightrightarrows \mathbb{R}^d$ *is an outer semicontinuous mapping, consider a closed set*

*$W \subseteq \operatorname{dom} S$ and let $\bar{x} \in S|_W(\bar{w})$, if*

$$(0,0) \in D^*M|_{W\times\mathbb{R}^n}\left((\bar{w},\bar{x}) \mid 0\right)(y) \Longrightarrow y = 0, \tag{3.1.14}$$

*then*

$$D^*_W S\left(\bar{w} \mid \bar{x}\right)(r) \subseteq \limsup_{\substack{(w,x)\xrightarrow{\operatorname{gph} S|_W}(\bar{w},\bar{x})\\ r'\to r}} \bigcup_{y\in\mathbb{R}^d} \left\{ \operatorname{proj}_{T_W(w)}(v) \;\middle|\; (v,-r') \in D^*M|_{W\times\mathbb{R}^n}\left((w,x) \mid 0\right)(y) \right\}. \tag{3.1.15}$$

*If furthermore*

$$(0,0) \in D^*_{W\times\mathbb{R}^n} M\left((\bar{w},\bar{x}) \mid 0\right)(y) \Longrightarrow y = 0 \tag{3.1.16}$$

*then we have*

$$D^*_W S(\bar{w} \mid \bar{x})(r) \subseteq \left\{ t \mid \exists y \text{ such that } (t,-r) \in D^*_{W\times\mathbb{R}^n} M((\bar{w},\bar{x}) \mid 0)(y) \right\}. \tag{3.1.17}$$

*When in addition $M|_{W\times\mathbb{R}^n}$ is graphically regular at $(\bar{w},\bar{x},0)$ and $W$ is a smooth manifold at $\bar{w}$,*

$$D^*_W S(\bar{w} \mid \bar{x})(r) = \left\{ t \,\middle|\, \exists y \text{ s.t } (t,-r) \in D^*_{W\times\mathbb{R}^n} M((\bar{w},\bar{x}) \mid 0)(y) \right\}.$$

*Proof.* This corollary comes from direct application of Theorem 3.1.2 by taking $G(w,x) = 0$. As $(w,x) \xrightarrow{\operatorname{gph} S|_W} (\bar{w},\bar{x})$ is equivalent to $(w,x,0) \xrightarrow{\operatorname{gph} M|_{W\times\mathbb{R}^n}} (\bar{w},\bar{x},0)$,

$$\limsup_{\substack{(w,x)\xrightarrow{\operatorname{gph} S|_W}(\bar{w},\bar{x})\\ y'\to y}} \operatorname{proj}_{T_W(w)\times\mathbb{R}^n} D^*M|_{W\times\mathbb{R}^n}\left((w,x) \mid 0\right)(y') \subseteq D^*_{W\times\mathbb{R}^n} M\left((\bar{w},\bar{x}) \mid 0\right)(y).$$

Therefore we can rewrite the inclusions in Theorem 3.1.2 as (3.1.15) and (3.1.17) respectively. □

Observing (3.1.16) and (3.1.17), we can have a sufficient condition for the relative Lipschitz-like property of $S$.

**Corollary 3.1.6.** *For the set-valued mapping $S$ defined as in Corollary 3.1.5 with $(\bar{w}, \bar{x}) \in \operatorname{gph} S|_W$, where $W$ is a smooth manifold at around $\bar{w}$ or a closed and convex set, $S$ has the Lipschitz-like property relative to $W$ at $\bar{w}$ for $\bar{x}$ if*

$$(t, 0) \in D^*_{W \times \mathbb{R}^n} M\left((\bar{w}, \bar{x}) \mid 0\right)(y) \Longrightarrow t = 0,\ \ y = 0.$$

*Proof.* This is a simple result from (3.1.16) and (3.1.17) and application of Theorems 1.3.8 and 2.2.3. □

In [5, Theorem 3.5], a sufficient condition is given to examine the Lipschitz-like property of the parametric system relative to a set. The condition mainly involves directional limiting coderivatives. Next we illustrate how our upper estimate (3.1.15) can be applied to verify the property for comparison. Here we put down some necessary notations and the theorem for reference.

First we introduce the definitions of the directional limiting normal cone and the directional limiting coderivatives.

**Definition 3.1.7** ([30, Definition 2.3])**.** *For a closed set $\Omega \subset \mathbb{R}^n$ with $\bar{x} \in \Omega$ and a direction $u \in \mathbb{R}^n$, the directional limiting normal cone to $\Omega$ in direction $u$ at $\bar{x}$ is defined by*

$$N_\Omega(\bar{x}; u) := \limsup_{t \downarrow 0,\, u' \to u} \widehat{N}_\Omega(\bar{x} + tu'), \tag{3.1.18}$$

*while for a set-valued mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ having locally closed graph around $(\bar{w}, \bar{x}) \in \operatorname{gph} S$ and a pair of directions $(u, v) \in \mathbb{R}^n \times \mathbb{R}^m$, the set-valued mapping $D^*S((\bar{w}, \bar{x}); (u, v)) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, defined by*

$$D^*S((\bar{w}, \bar{x}); (u, v))(v^*) := \{u^* \in \mathbb{R}^n \mid (u^*, -v^*) \in N_{\operatorname{gph} S}((\bar{w}, \bar{x}); (u, v))\},\ \ \forall v^* \in \mathbb{R}^m \tag{3.1.19}$$

*is called the directional limiting coderivative of $S$ in the direction $(u, v)$ at $(\bar{w}, \bar{x})$. See [5] for more details and some basic properties of these notions.*

**Proposition 3.1.8** ([5, Theorem 3.5]). *For an implicit mapping $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ as*

$$S(w) = \{x \mid 0 \in M(w,x)\}$$

*where $M : \mathbb{R}^m \times \mathbb{R}^n \rightrightarrows \mathbb{R}^d$ is an outer semicontinuous mapping, consider a closed set $W \subseteq \mathbb{R}^m$ and let $\bar{x} \in S|_W(\bar{w})$, further assume that*

**(i)** *for every $w \in T_W(\bar{w})$ and every sequence $t_k \searrow 0$ there exists some $x \in \mathbb{R}^n$ satisfying*

$$\liminf_{k\to\infty} d\left((\bar{w} + t_k w, \bar{x} + t_k x, 0), \operatorname{gph} M\right)/t_k = 0 \tag{3.1.20}$$

**(ii)** *for every nonzero $(w,x) \in T_W(\bar{w}) \times \mathbb{R}^n$ verifying $(w,x,0) \in T_{\operatorname{gph} M}(\bar{w},\bar{x},0)$ one has the implication*

$$(v,0) \in D^*M\left((\bar{w},\bar{x},0);(w,x,0)\right)(y) \Longrightarrow y = 0. \tag{3.1.21}$$

*Then $S$ has the Lipschitz-like property relative to $W$ at $\bar{w}$ for $\bar{x}$.*

Next we give an example to illustrate how the upper estimate in Corollary 3.1.5 can be applied in verifying the Lipschitz-like property relative to a set with the basic constraint qualification (3.1.14).

**Example 3.1.9.** *For $S(w) := \{x \in \mathbb{R}^n \mid Ax + w \in K\}$ where $K \subseteq \mathbb{R}^m$ is a closed set. Then we have $M(w,x) = -Ax - w + K$. By writing*

$$\operatorname{gph} M = \{(w,x,u) \mid u + Ax + w \in K\}, \tag{3.1.22}$$

$$\operatorname{gph} M|_W = \left\{ (w,x,u) \;\middle|\; \begin{pmatrix} I & 0 & 0 \\ I & A & I \end{pmatrix} \begin{pmatrix} w \\ x \\ u \end{pmatrix} \in W \times K \right\},$$

*we can see that $\begin{pmatrix} I & 0 & 0 \\ I & A & I \end{pmatrix}$ has full rank $2m$ and $\begin{pmatrix} I & A & I \end{pmatrix}$ has full rank $m$. Thus we can apply [81, Exercise 6.7] to obtain, for any $(w,x,u) \in \operatorname{gph} M|_{W\times\mathbb{R}^n}$,*

$$N_{\operatorname{gph} M|_{W\times\mathbb{R}^n}}(w,x,u) = \{(v+y, A^*y, y) \mid v \in N_W(w), y \in N_K(u + Ax + w)\},$$

$$N_{\operatorname{gph} M}(w,x,u) = \{(y, A^*y, y) \mid y \in N_K(u+Ax+w)\}\,.$$

*By definition of coderivatives,*

$$D^*M|_{W\times\mathbb{R}^n}\left((w,x)\mid u\right)(y) = \{(-y,-A^*y) \mid -y \in N_K(u+Ax+w)\} + N_W(w)\times\{0\} \tag{3.1.23}$$

$$= D^*M\left((w,x)\mid u\right)(y) + N_W(w)\times\{0\}.$$

*Then the constraint qualification (3.1.14), together with the expression (3.1.23) becomes*

$$N_K(A\bar{x}+\bar{w}) \cap \ker A^* \cap (-N_W(\bar{w})) = \{0\} \tag{3.1.24}$$

*at the reference pair* $(\bar{w},\bar{x}) \in \operatorname{gph} S|_W$*.*

*Next we consider a particular case. Let* $n = m = 2$*,* $K = \mathbb{R}_+ \times \{0\} \cup \{0\} \times \mathbb{R}_+$ *and* $A = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$*. For the reference pair* $\bar{w} = (0,0)^\top$*,* $\bar{x} = (0,0)^\top$ *and* $W = \{(w_1,w_2)\in\mathbb{R}^2 \mid w_1+w_2 \geq 0, w_1 \geq 0\}$*, we have* $\bar{w} \in W$ *and* $W \subseteq \operatorname{dom} S = K + \operatorname{rg} A = \{(w_1,w_2)\in\mathbb{R}^2 \mid w_1+w_2\geq 0\}$*. By some calculation we can see that*

$$N_K(A\bar{x}+\bar{w}) = N_K(0) = (\mathbb{R}\times\{0\})\cup(\{0\}\times\mathbb{R})\cup\mathbb{R}^2_-,\ \ker A^* = \mathbb{R}(1,1)^\top$$

*and*

$$N_W(\bar{w}) = N_W(0) = W^* = \left\{(w_1,w_2)\in\mathbb{R}^2 \mid w_1 - w_2 \leq 0, w_2 \leq 0\right\}$$

*and therefore the constraint qualification (3.1.24) is satisfied. As* $\operatorname{gph} S|_W$ *is a union of polyhedral cones, we have only finite combinations of* $N_K(Ax+w)$ *and* $N_W(w)$*. By (3.1.23), we have for sufficiently small* $\varepsilon > 0$*,*

$$\limsup_{(w,x)\xrightarrow{\operatorname{gph} S|_W}(\bar{w},\bar{x})} \bigcup_{y\in\mathbb{R}^d} \operatorname{proj}_{T_W(w)\times\mathbb{R}^n} D^*M|_{W\times\mathbb{R}^n}((w,x)\mid 0)(y)$$

$$= \bigcup_{\substack{(w,x)\in\mathbb{B}_\varepsilon(\bar{w},\bar{x})\cap\operatorname{gph} S|_W \\ y\in -N_K(Ax+w)}} \left\{\left(\operatorname{proj}_{T_W(w)}(v-y), -A^*y\right) \;\middle|\; v\in N_W(w)\right\}$$

$$= \Bigg( \bigcup_{y\in(\mathbb{R}\times\{0\})\cup(\{0\}\times\mathbb{R})\cup\mathbb{R}^2_+} \Big( \left\{\left(\operatorname{proj}_{W_2}(v-y), -A^*y\right) \;\middle|\; v\in W_2^*\right\}$$

$$\cup \left\{ \left( \operatorname{proj}_W(v-y), -A^*y \right) \ \middle|\ v \in W^* \right\} \Big) \Big) \bigcup$$

$$\left( \bigcup_{y \in (\mathbb{R}\times\{0\})\cup(\{0\}\times\mathbb{R})} \left( \left\{ \left( \operatorname{proj}_{W_1}(v-y), -A^*y \right) \ \middle|\ v \in W_1^* \right\} \cup \{(-y, -A^*y)\} \right) \right),$$

*where* $W_1 = \mathbb{R}_+ \times \mathbb{R}$, $W_2 = \{(w_1, w_2) \mid w_1 + w_2 \geq 0\}$. *Together with (3.1.15) we have*

$$\begin{aligned} \operatorname{gph} D_W^* S(\bar{w} \mid \bar{x}) \subseteq & \Bigg( \bigcup_{y \in (\mathbb{R}\times\{0\})\cup(\{0\}\times\mathbb{R})\cup\mathbb{R}_+^2} \left\{ \left( A^*y, \operatorname{proj}_{W_2}(v-y) \right) \ \middle|\ v \in W_2^* \right\} \\ & \cup \left\{ \left( A^*y, \operatorname{proj}_W(v-y) \right) \ \middle|\ v \in W^* \right\} \Bigg) \bigcup \\ & \left( \bigcup_{y \in (\mathbb{R}\times\{0\})\cup(\{0\}\times\mathbb{R})} \left( \left\{ \left( A^*y, \operatorname{proj}_{W_1}(v-y) \right) \ \middle|\ v \in W_1^* \right\} \cup \{(A^*y, -y)\} \right) \right). \end{aligned} \tag{3.1.25}$$

*By the generalized Mordukhovich criterion, it is sufficient to examine the criterion on each of the subsets in right-hand side of (3.1.25) to obtain* $D_W^* S(\bar{w} \mid \bar{x})(0) = \{0\}$*:*

1. $y \in \left( (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R}) \cup \mathbb{R}_+^2 \right) \cap \ker A^* = \mathbb{R}_+ (1,1)^\top$. *By calculation we have* $W_2^* = \mathbb{R}_-(1,1)^\top$ *and* $W^* = \{(w_1, w_2) \mid w_1 - w_2 \leq 0, w_2 \leq 0\}$. *Then we can see that* $-y \in W_2^* \cap W^*$ *and thus*

$$\operatorname{proj}_{W_2}(v-y) = 0 \quad \text{for } v \in W_2^*,$$
$$\operatorname{proj}_W(v-y) = 0 \quad \text{for } v \in W^*.$$

2. $y \in ((\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R})) \cap \ker A^* = \{0\}$.

*Therefore we have* $D_W^* S(\bar{w}, \bar{x})(0) = \{0\}$ *and* $S$ *has the Lipschitz-like property relative to* $W$ *at* $\bar{w}$ *for* $\bar{x}$. *However, in this case the stronger constraint qualification does not hold, which can be verified via*

$$(0,0) \in \operatorname{proj}_{T_W(\bar{w})\times\mathbb{R}^n} D^* M|_{W\times\mathbb{R}^n} \left( (\bar{w}, \bar{x}) \mid 0 \right)(y)$$

$$= \left\{ (\text{proj}_W(v-y), -A^*y) \ \middle| v \in W^*, y \in \mathbb{R}\times\{0\}\cup\{0\}\times\mathbb{R}\cup\mathbb{R}^2_+ \right\}$$
$$\subseteq D^*_{W\times\mathbb{R}^n}\left((\bar{w},\bar{x}) \mid 0\right)(y) \Longrightarrow y = \mathbb{R}_+(1,1)^\top \neq \{(0,0)^\top\}.$$

*Next we will show how Proposition 3.1.8 fails on examining the property as for* $w = (1,-1)^\top \in T_W(\bar{w})$ *and* $x = (0,1)^\top$*, condition (ii) can't be satisfied. Given the conic and polyhedral structure of* $\operatorname{gph} M$ *(see representation(3.1.22)), again by [81, Exercise 6.7] we have*

$$(w,x,0) \in T_{\operatorname{gph} M}(\bar{w},\bar{x},0) = \{(w,x,u) \mid Ax+w+u \in T_K(A\bar{x}+\bar{w})\} = \operatorname{gph} M.$$

*By definition of directional limiting normal cone (3.1.18),*

$$\begin{aligned}
& N_{\operatorname{gph} M}\left((\bar{w},\bar{x},0);(w,x,0)\right) \\
:= \quad & \limsup\nolimits_{t\downarrow 0,\ (w',x',u')\to(w,x,0)} \widehat{N}_{\operatorname{gph} M}\left((\bar{w},\bar{x},0)+t(w',x',u')\right) \\
= \quad & \limsup\nolimits_{(w',x',u')\to(w,x,0)} \widehat{N}_{\operatorname{gph} M}\left(w',x',u'\right) \\
= \quad & N_{\operatorname{gph} M}\left(w,x,0\right) \\
= \quad & \left\{ (y,\ A^*y,\ y) \,\middle|\, y \in N_K(Ax+w) = N_K(0) = \mathbb{R}\times\{0\}\cup\{0\}\times\mathbb{R}\cup\mathbb{R}^2_- \right\},
\end{aligned}$$

*where the second equality follows from the conic structure of* $\operatorname{gph} M$*, the third one from the definition of the normal cone mappings, and the last one from (3.1.22). Then we have by definition of directional limiting coderivatives (3.1.19),*

$$(-y,-A^*y) \in D^*M((\bar{w},\bar{x},0);(w,x,0))(y) \ \textit{with} \ -A^*y = 0$$
$$\Longrightarrow y \in \mathbb{R}_+(1,1)^\top \neq \{(0,0)^\top\},$$

*suggesting that the Lipschitz-like property of* $S$ *relative to* $W$ *at* $\bar{w}$ *for* $\bar{x}$ *cannot be obtained via Proposition 3.1.8.*

**Remark 3.1.10.** *The reason why the sufficient condition in Proposition 3.1.8 fails in verifying the property in the above example is structural: the tool, directional limiting* *coderivatives, are intrinsically the coderivatives at the neighboring points in specifically given directions. By Mordukhovich criterion (Theorem 1.3.6), it is required that the coderivative of $S$ at $0$ includes $0$ only, which means that the condition (ii) in Proposition 3.1.8 requires the set $W$ does not involve any points on the boundary of $\operatorname{dom} S$ other than the reference point. On the other hand, projectional coderivatives come from taking the limsup of projected normal cone and therefore work efficiently when characterizing the property when on the boundary.*

**Remark 3.1.11.** *In Theorem 3.1.2, two different constraint qualifications are mentioned. We can see that the basic one (3.1.3) is ensuring the upper estimate in two ways: (i) restricting $S$ to $W$; (ii) expressing the normal cone of $\operatorname{gph} S$ via those of $\operatorname{gph} G$ and $\operatorname{gph} M$. In Corollary 3.1.5, the stronger constraint qualification (3.1.16) ensures that $D^*_W S(\bar{w} \mid \bar{x})$ can be expressed via $D^*_{W\times\mathbb{R}^n} M\left((\bar{w},\bar{x}) \mid 0\right)$. Note that in the proof of Corollary 3.1.5, we used a larger set. The difference between Theorem 3.1.2 and combining application of Corollary 3.1.5 to $M'(w,x) := G(w,x) + M(w,x)$ and that of the sum rule (Theorem 2.4.5) to $G(w,x) + M(w,x)$ is caused by this larger upper estimate.*

In the next theorem, we give a setting where the constraint qualification (3.1.3) could be bypassed. It is a result from Lemma 3.1.1.

**Theorem 3.1.12.** *For $S$ defined as in (3.0.1), if $M = M(x)$ and $\nabla_w G(w,x)$ has full rank, then*

$$D^*_{\operatorname{dom} S} S\left(\bar{w} \mid \bar{x}\right)(r) = \limsup_{\substack{(w,x)\xrightarrow{\operatorname{gph} S}(\bar{w},\bar{x}) \\ r'\to r}} \bigcup_{y\in\mathbb{R}^d} \Big\{ \operatorname{proj}_{T_{domS(w)}}(\nabla_w G(w,x)^* y) \;\Big|\; -r' \in \left(\nabla_x G(w,x)^* y + D^* M\left(x \mid -G(w,x)\right)(y)\right) \Big\}. \tag{3.1.26}$$

*Proof.* When the set $W := \operatorname{dom} S$, $S|_W = S$. By condition (b) in Lemma 3.1.1, we have for any $(w, x) \in \operatorname{gph} S$,

$$N_{\operatorname{gph} S}(w, x) = \bigcup_{y \in \mathbb{R}^d} \left( \nabla G(w, x)^* y + \{0\} \times D^* M \left( x \mid -G(w, x) \right) (y) \right).$$

Therefore we have

$$\limsup_{(w,x) \xrightarrow{\operatorname{gph} S|_W} (\bar{w}, \bar{x})} \operatorname{proj}_{T_W(w) \times \mathbb{R}^n} N_{\operatorname{gph} S|_W}(w, x) = \limsup_{(w,x) \xrightarrow{\operatorname{gph} S} (\bar{w}, \bar{x})} \operatorname{proj}_{T_{\operatorname{dom} S}(w) \times \mathbb{R}^n} N_{\operatorname{gph} S}(w, x)$$

$$= \limsup_{(w,x) \xrightarrow{\operatorname{gph} S} (\bar{w}, \bar{x})} \bigcup_{y \in \mathbb{R}^d} \operatorname{proj}_{T_{\operatorname{dom} S}(w) \times \mathbb{R}^n} \left( \nabla G(w, x)^* y + \{0\} \times D^* M(x \mid -G(w, x))(y) \right)$$

and thus the equation (3.1.26). □

Here the set we refer to, $W$, becomes the largest possible set $\operatorname{dom} S$. In this case, $S$ does not carry the set constraint along for calculation and thus the constraint qualification (3.1.3) goes back to the one in Lemma 3.1.1 and is satisfied automatically with the above setting. In the coming sections, we introduce some models under specific settings.

## 3.2 Linear constraint systems

Consider the solution mapping of a linear constraint system $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$:

$$S(b) = \{x \in \mathbb{R}^n \mid Ax + b \in K\} \tag{3.2.1}$$

where $K$ is a closed set in $\mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$. By calculation in [38], for any given pair $(\bar{b}, \bar{x}) \in \operatorname{gph} S$,

$$N_{\operatorname{gph} S}(\bar{b}, \bar{x}) = \left\{ (u, A^* u) \;\middle|\; u \in N_K(\bar{v}) \right\} \text{ where } \bar{v} = A\bar{x} + \bar{b}. \tag{3.2.2}$$

Here we denote the column space generated by $A$ (i.e., linear combinations generated by columns of $A$) as $\operatorname{rg} A$ and the orthogonal complement of $\operatorname{rg} A$ (also the null space of $A^*$) as $\ker A^*$. In [38] a sufficient and necessary condition of Lipschitz-like property of $S$ at $\bar{b}$ for $\bar{x}$ is given via Mordukhovich criterion:

$$\ker A^* \cap N_K(\bar{v}) = \{0\}. \tag{3.2.3}$$

Note that this condition can also be applied when both the matrix $A$ and the vector $b$ undergo perturbations. We know that the Lipschitz-like property suggests implicitly that the referred point $\bar{b}$ should lie in the interior of the domain of $S$. Therefore the criterion also fails when $\bar{b}$ falls on the boundary of $\operatorname{dom} S$. Next we focus on the Lipschitz-like property relative to $\operatorname{dom} S$.

### 3.2.1 Relative Lipschitz-like property and the graphical modulus

From expression (3.2.1) we can see that $\operatorname{gph} S$ can be taken as a linear transformation of $K$. For the domain $\operatorname{dom} S = K + \operatorname{rg} A$ where $\operatorname{rg} A = A(\mathbb{R}^n) = \{Ax \mid x \in \mathbb{R}^n\}$. Given that $\operatorname{rg} A$ is a subspace in $\mathbb{R}^m$, the domain of $S$ can be interpreted as a set generated by moving the set $K$ along the subspace $\operatorname{rg} A$. Therefore $\operatorname{gph} S$, $\operatorname{dom} S$ share something common in structure with the set $K$ like convexity and polyhedrality. In this section we first assume $K$ to be union of polyhedral sets and derive the expression for projectional coderivatives of $S$ relative to its domain. Later we consider the case that $K$ is a convex polyhedral set. Under such an assumption we can give an explicit form of its tangent cone and normal cone.

First we present a result of simply applying Theorem 3.1.12 to $S$ (3.2.1) with $K$ being a union a polyhedral sets. Under such a setting, the limsup in definition of projectional coderivative can be substituted by a union as the number of possible combinations of $N_{\operatorname{gph} S}(b, x)$ and $T_{\operatorname{dom} S}(b)$ are finite.

**Corollary 3.2.1.** *For the set mapping $S$ defined as in (3.2.1) with $K$ being a union* *of polyhedral sets and a pair* $(\bar{b},\bar{x}) \in \operatorname{gph} S$*, for sufficiently small* $\varepsilon > 0$*,*

$$D^*_{\operatorname{dom} S} S(\bar{b} \mid \bar{x})(y) = \bigcup_{(b,x)\in \operatorname{gph} S \cap \mathbb{B}_\varepsilon(\bar{b},\bar{x})} \Big\{ \operatorname{proj}_{T_{\operatorname{dom} S}(b)}(u) \;\Big|\; \exists u \in N_K(Ax+b) \\ s.t. - A^*u = y \Big\}. \tag{3.2.4}$$

*Proof.* Let $G(b,x) = -Ax - b$, $M(x) = K$, we can directly apply Theorem 3.1.12 to $S$ with

$$D^*M\,(x \mid Ax+b)\,(u) = \begin{cases} \{0\}, & \text{if } u \in -N_K(Ax+b) \\ \emptyset, & \text{if } u \notin -N_K(Ax+b) \end{cases}$$

and

$$\nabla G(b,x)^* u = (-u, -A^*u).$$

In view of the fact that $\operatorname{gph} S$ is also a union of polyhedral sets, we have for sufficiently small $\varepsilon > 0$,

$$\begin{aligned} & D^*_{\operatorname{dom} S} S\left(\bar{b} \mid \bar{x}\right)(y) \\ = & \limsup_{\substack{(b,x)\xrightarrow{\operatorname{gph} S}(\bar{b},\bar{x}) \\ y' \to y}} \bigcup_{u\in\mathbb{R}^d} \left\{ \operatorname{proj}_{T_{domS}(b)}(v) \;\middle|\; (v,-y') \in (-u,-A^*u)\,, u \in -N_K(Ax+b) \right\} \\ = & \bigcup_{(b,x)\in \operatorname{gph} S \cap \mathbb{B}_\varepsilon(\bar{b},\bar{x})} \left\{ \operatorname{proj}_{T_{domS}(b)}(-u) \;\middle|\; y = A^*u, u \in -N_K(Ax+b) \right\} \end{aligned}$$

and finally (3.2.4) by tuning the direction of $u$. $\square$

From now on we focus on the case where $K$ is a convex polyhedral set. In this case, $\operatorname{dom} S$ is also convex and we can apply the generalized Mordukhovich criterion (Theorem 1.3.8) to $S$ relative to $\operatorname{dom} S$. Besides, $\operatorname{gph} S$ is also convex polyhedral and $S$ enjoys the Lipschitz continuity on $\operatorname{dom} S$ automatically (see [81, Example 9.35]). Thus $S$ should enjoy the Lipschitz-like property relative to $\operatorname{dom} S$ as well and we next verify it by employing the generalized Mordukhovich criterion.

**Proposition 3.2.2.** *For the set mapping $S$ defined as in (3.2.1) with $K$ being convex, we have that* $\operatorname{dom} S = K + \operatorname{rg} A$ *is convex as well. For a given pair* $(\bar{b}, \bar{x}) \in \operatorname{gph} S$ *and* $\bar{v} = A\bar{x} + \bar{b}$*, we have*

$$T_{\operatorname{dom} S}(\bar{b}) = \operatorname{cl}(T_K(\bar{v}) + \operatorname{rg} A) = T_K(\bar{v}) + \operatorname{rg} A, \tag{3.2.5}$$

$$N_{\operatorname{dom} S}(\bar{b}) = \left(T_{\operatorname{dom} S}(\bar{b})\right)^* = N_K(\bar{v}) \cap \ker A^*. \tag{3.2.6}$$

*Proof.* As we know that $\bar{v} \in K$, $\bar{b} = \bar{v} - A\bar{x}$, $\operatorname{dom} S = K + \operatorname{rg} A$, and both $\operatorname{rg} A$ and $K$ are closed and convex sets, by direct application of [81, Exercise 6.44] we obtain (3.2.5) and (3.2.6). □

**Remark 3.2.3.** *For* $b \in \operatorname{bdry} \operatorname{dom} S$*, there exists at least one $x$ such that* $Ax + b \in \operatorname{bdry} K$*, but not vice versa.*

**Corollary 3.2.4.** *For the set mapping $S$ defined as in (3.2.1) with $K$ being convex polyhedral and a pair* $(\bar{b}, \bar{x}) \in \operatorname{gph} S$*, $S$ always has the Lipschitz-like property relative to its domain at $\bar{b}$ for $\bar{x}$.*

*Proof.* Here we use the generalized Mordukhovich criterion as $\operatorname{dom} S$ is also convex:

$$D^*_{\operatorname{dom} S} S(\bar{b}|\bar{x})(0) = \{0\}. \tag{3.2.7}$$

Given the expression of $D^*_{\operatorname{dom} S}(\bar{b} \mid \bar{x})$ as (3.2.4) and the polyhedrality of $K$, the criterion is equivalent to checking for all $(b, x) \xrightarrow{\operatorname{gph} S} (\bar{b}, \bar{x})$, if

$$A^* u = 0, u \in N_K(Ax + b) \Longrightarrow \operatorname{proj}_{T_{\operatorname{dom} S}(b)}(u) = 0,$$

which is equivalent to

$$u \in N_K(Ax + b) \cap \ker A^* \Longrightarrow \operatorname{proj}_{T_{\operatorname{dom} S}(b)}(u) = 0.$$

By convexity of $T_{\operatorname{dom} S}(b)$, we have

$$\operatorname{proj}_{T_{\operatorname{dom} S}(b)}(u) = 0 \Longleftrightarrow u \in N_{\operatorname{dom} S}(b) = N_K(Ax + b) \cap \ker A^*.$$

Therefore we have the property naturally. □

Given that the Lipschitz-like property relative to $\operatorname{dom} S$ holds automatically, next we explore the form of the modulus $\operatorname{lip}_{\operatorname{dom} S} S$. For $\operatorname{lip} S$, by calculation or [81, Example 9.44] we have

$$\operatorname{lip} S = \max_{u \in N_K(A\bar{x}+\bar{b}) \cap \mathbb{S}} \frac{1}{\|A^* u\|}. \tag{3.2.8}$$

For related results, see [9, Corollary 3.2] and [12, Remark 9]. First we give some results regarding the calculation of $\operatorname{proj}_{T_{\operatorname{dom} S(b)}}(u)$ with $u \in N_K(Ax+b)$.

**Lemma 3.2.5.** *For a given pair $(b,x) \in \operatorname{gph} S$ and $u \in N_K(Ax+b)$,*

$$\|\operatorname{proj}_{T_{\operatorname{dom} S(b)}}(u)\| = \begin{cases} \|AA^+u\|, & \text{if } b \in \operatorname{bdry} \operatorname{dom} S \\ \|u\|, & \text{if } b \in \operatorname{int} \operatorname{dom} S \end{cases} \tag{3.2.9}$$

*where $A^+$ is the pseudo-inverse of $A$ given by*

$$A^+ = \begin{cases} (A^*A)^{-1}A^*, & \text{if } A \text{ is tall and thin} \\ A^*(AA^*)^{-1}, & \text{if } A \text{ short and fat, or is tall but does not have full column rank,} \end{cases}$$

*and when $A$ has neither full row rank nor full column rank, $A^+$ is given by $AA^+ = UU^*$. Note that in this case $U$ is an orthonormal basis for $\operatorname{rg} A$, generated from the truncated SVD of $A$, and therefore is tall and thin with rank(A) columns (and full column rank).*

*Proof.* For any $b \in \operatorname{int} \operatorname{dom} S$, $T_{\operatorname{dom} S}(b) = \mathbb{R}^m$ and therefore the projection of $u$ is itself. For any $b \in \operatorname{bdry} \operatorname{dom} S$, $Ax+b$ must be lying on the $\operatorname{bdry} K$ as well for $\operatorname{bdry} \operatorname{dom} S = \operatorname{bdry}(K + \operatorname{rg} A) \subseteq \operatorname{bdry} K + \operatorname{rg} A$ (see Remark 3.2.3). By [81, Exercise 12.22], $u$ can be represented uniquely as $u = w + y$ with $w = \operatorname{proj}_{T_{\operatorname{dom} S(b)}}(u)$, $y = \operatorname{proj}_{N_{\operatorname{dom} S(b)}}(u)$ and $w \perp y$. As when $b \in \operatorname{bdry} \operatorname{dom} S$, $u \in N_K(Ax+b)$ and $N_{\operatorname{dom} S}(b) = N_K(Ax+b) \cap \ker A^*$. Therefore we have

$$\begin{aligned} &\|\operatorname{proj}_{T_{\operatorname{dom} S(b)}}(u)\| = \|u - \operatorname{proj}_{N_{\operatorname{dom} S(b)}}(u)\| \\ =&d(u, N_{\operatorname{dom} S}(b)) = d(u, \ker A^*) = \|\operatorname{proj}_{\operatorname{rg} A}(u)\| \end{aligned} \tag{3.2.10}$$

Therefore the projection on $T_{\operatorname{dom} S}(b)$ here is equivalent to projection on the column space of $A$, where the results can be found in [49, Page 365].

$\square$

For any $(b, x) \in \operatorname{gph} S$ with $v = Ax + b \in \operatorname{bdry} K$, we denote the set of all non-overlapping semi-closed faces $F$ of $K$ with $v \in \operatorname{bdry} F$ as $\mathcal{F}(v)$. Accordingly, the normal cones on each $F$ are the same and we denote them as $N_F$. Among these faces, some form the boundary of $\operatorname{dom} S$, and therefore we use $\mathcal{F}_e(v)$ to denote such collection, i.e.,

$$\mathcal{F}_e(v) := \{F \in \mathcal{F}(v) \mid F + \operatorname{rg} A \subseteq \operatorname{bdry} \operatorname{dom} S\}.$$

**Theorem 3.2.6.** *For any* $(\bar{b}, \bar{x}) \in \operatorname{gph} S$ *with* $\bar{b} \in \operatorname{bdry} \operatorname{dom} S$ *and* $\bar{v} = A\bar{x} + \bar{b}$, *we have*

$$\operatorname{lip}_{\operatorname{dom} S} S(\bar{b}|\bar{x}) = \max \left\{ \max_{F \in \mathcal{F}_e(\bar{v})} \sup_{u \in N_F \cap \mathbb{S}} \frac{\|AA^+ u\|}{\|A^* u\|}, \max_{F \in \mathcal{F}(\bar{v}) \setminus \mathcal{F}_e(\bar{v})} \sup_{u \in N_F \cap \mathbb{S}} \frac{\|u\|}{\|A^* u\|} \right\}. \tag{3.2.11}$$

*Proof.* According to [59, Theorem 2.3],

$$\operatorname{lip}_{\operatorname{dom} S} S(\bar{b}|\bar{x}) = \limsup_{(b,x) \xrightarrow{\operatorname{gph} S} (\bar{b}, \bar{x})} \sup_{A^* u \in \mathbb{B}} \sup_{u \in N_K(Ax+b)} \frac{\|\operatorname{proj}_{T_{\operatorname{dom} S}(b)}(u)\|}{\|A^* u\|}.$$

For $(b, x) \xrightarrow{\operatorname{gph} S} (\bar{b}, \bar{x})$, it is equivalent for $v := Ax + b \xrightarrow{\mathcal{F}(\bar{v})} \bar{v}$. Besides, as $\mathcal{F}_e(\bar{v})$ contains all faces that form the set $\operatorname{bdry} \operatorname{dom} S$, then for any $b \in \operatorname{bdry} \operatorname{dom} S$ such that $Ax + b \in F \in \mathcal{F}_e(\bar{v})$ and $u \in N_F$,

$$\|\operatorname{proj}_{T_{\operatorname{dom} S}(b)}(u)\| = \|\operatorname{proj}_{\operatorname{rg} A}(u)\| = \|AA^+ u\|.$$

For other semi-closed faces, i,e., $b \in F \in \mathcal{F}(\bar{v}) \setminus \mathcal{F}_e(\bar{v})$, $b \in \operatorname{int} \operatorname{dom} S$ and therefore $\operatorname{proj}_{T_{\operatorname{dom} S}(b)}(u) = u$ for any $u \in N_F$. Therefore we have

$$\operatorname{lip}_{\operatorname{dom} S} S(\bar{b}|\bar{x}) = \max_{F \in \mathcal{F}(\bar{v})} \max_{v \in F} \sup_{A^* u \in \mathbb{S}} \sup_{u \in N_K(v)} \frac{\|\operatorname{proj}_{T_{\operatorname{dom} S}(b)}(u)\|}{\|A^* u\|}$$

$$= \max_{F\in\mathcal{F}(\bar v)} \max_{v\in F} \sup_{u\in N_F\cap\mathbb{S}} \frac{\|\mathrm{proj}_{T_{\mathrm{dom}\,S}(b)}(u)\|}{\|A^*u\|},$$

and further (3.2.11) with possibilities of $F \in \mathcal{F}(\bar v)$ exhausted. $\square$

**Remark 3.2.7.** *When* $\mathrm{dom}\, S = \mathbb{R}^n$*, the set* $\mathcal{F}_e(\bar v) = \emptyset$ *and* $\cup_{F\in\mathcal{F}(\bar v)} N_F = N_K(\bar v)$*. In this way, the modulus (3.2.11) becomes identical to (3.2.8) as*

$$\mathrm{lip}_{\mathrm{dom}\,S} S(\bar b|\bar x) = \max_{F\in\mathcal{F}(\bar v)} \sup_{u\in N_F\cap\mathbb{S}} \frac{\|u\|}{\|A^*u\|} = \max_{u\in N_K(\bar v)\cap\mathbb{S}} \frac{1}{\|A^*u\|} = \mathrm{lip}\, S.$$

### 3.2.2 Some examples

Next we give some examples on modulus calculations with $K = \mathbb{R}^n_+$. Note that in the following examples, for the case $F = \mathbb{R}^n_{++}$, $u \in N_F = \{0_m\}$, which is trivial and thus is omitted here. It is worth mentioning that graphical modulus $\mathrm{lip}_{\mathrm{dom}\,S} S$, i.e., the largest $\kappa$ in (1.3.4), can be obtained in the interior of the domain (see Example 3.2.8) or on the boundary (see Example 3.2.9).

**Example 3.2.8.** $A = \begin{pmatrix} -2 & -3 \\ 2 & 3 \end{pmatrix}$, $\bar b = 0_2$, $\bar x = 0_2$. $\mathrm{dom}\, S = \{(y_1, y_2) \mid y_2 \ge -y_1\}$.

$\mathrm{proj}_{\mathrm{rg}\,A}(u) = UU^*u = \begin{pmatrix} \frac12 & -\frac12 \\ -\frac12 & \frac12 \end{pmatrix} u$ *with* $U = (-\frac{1}{\sqrt2}, \frac{1}{\sqrt2})^\top$.

$$\begin{aligned}\mathcal{F}(\bar v) &= \left\{\{0_2\},\ \{0\}\times\mathbb{R}_{++},\ \mathbb{R}_{++}\times\{0\},\ \mathbb{R}^2_{++}\right\},\\ \mathcal{F}_e(\bar v) &= \{\{0_2\}\}.\end{aligned}$$

1. $v = (0, 0)$, $u \in N_K(v) = \mathbb{R}^2_-$, $T_{\mathrm{dom}\,S}(b) = \{(y_1, y_2) \mid y_2 \ge -y_1\}$.

$$\kappa = \max_{u\in\mathbb{R}^2_-} \frac{\|\mathrm{proj}_{\mathrm{rg}\,A}(u)\|}{\|A^*u\|} = \max_{u\in\mathbb{R}^2_-} \frac{\|UU^*u\|}{\|A^*u\|} = \frac{1}{\sqrt{26}}.$$

2. $v \in \{0\}\times\mathbb{R}_{++}$, $u \in N_K(v) = \mathbb{R}_-\times\{0\}$, $T_{\mathrm{dom}\,S}(b) = \mathbb{R}^2$.

$$\kappa = \max_{u\in\mathbb{R}_-\times\{0\}} \frac{\|\mathrm{proj}_{\mathbb{R}^2}(u)\|}{\|A^*u\|} = \max_{u_1\in\mathbb{R}_-} \frac{|u_1|}{\left\|\begin{pmatrix}-2\\-3\end{pmatrix} u_1\right\|} = \frac{1}{\sqrt{13}}.$$

*3.* $v \in \mathbb{R}_{++} \times \{0\}, u \in N_K(v) = \{0\} \times \mathbb{R}_-,\ T_{\operatorname{dom} S}(b) = \mathbb{R}^2$.

$$\kappa = \max_{u \in \{0\} \times \mathbb{R}_-} \frac{\|\operatorname{proj}_{\mathbb{R}^2}(u)\|}{\|A^* u\|} = \max_{u \in \{0\} \times \mathbb{R}_-} \frac{\|u\|}{\|A^* u\|} = \max_{u_2 \in \mathbb{R}_-} \frac{|u_2|}{\left\| \begin{pmatrix} 2 \\ 3 \end{pmatrix} u_2 \right\|} = \frac{1}{\sqrt{13}}.$$

*In all,* $\operatorname{lip}_{\operatorname{dom} S} S(0_2 \mid 0_2) = \dfrac{1}{\sqrt{13}}$.

**Example 3.2.9.** $A = \begin{pmatrix} 1 & -4 \\ 0 & 0 \\ 2 & 2 \end{pmatrix},\ \bar{b} = 0_3,\ \bar{x} = 0_2.\ \operatorname{dom} S = \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}.$

$$\operatorname{proj}_{\operatorname{rg} A}(u) = A(A^*A)^{-1}A^* u = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} u = (u_1, 0, u_3)^\top.$$

$$\begin{aligned} \mathcal{F}(\bar{v}) = \big\{ &\{0_3\},\ \mathbb{R}_{++} \times \{0_2\},\ \{0_2\} \times \mathbb{R}_{++},\ \mathbb{R}_{++} \times \{0\} \times \mathbb{R}_{++}, \\ &\{0\} \times \mathbb{R}_{++} \times \{0\},\ \mathbb{R}^2_{++} \times \{0\},\ \{0\} \times \mathbb{R}^2_{++},\ \mathbb{R}^3_{++} \big\}, \\ \mathcal{F}_e(\bar{v}) = \{ &\{0_3\},\ \mathbb{R}_{++} \times \{0_2\},\ \{0_2\} \times \mathbb{R}_{++},\ \mathbb{R}_{++} \times \{0\} \times \mathbb{R}_{++} \}. \end{aligned}$$

*1.* $v = 0_3,\ u \in N_K(v) = \mathbb{R}^3_-,\ T_{\operatorname{dom} S}(b) = \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$.

$$\kappa = \max_{u \in \mathbb{R}^3_-} \frac{\|\operatorname{proj}_{\operatorname{rg} A}(u)\|}{\|A^* u\|} = \max_{u \in \mathbb{R}^3_-} \frac{\|(u_1, 0, u_3)^\top)\|}{\left\| \begin{pmatrix} 1 \\ -4 \end{pmatrix} u_1 + \begin{pmatrix} 2 \\ 2 \end{pmatrix} u_3 \right\|} = \frac{1}{\sqrt{5}}.$$

*2.* $v \in \mathbb{R}_{++} \times \{0_2\},\ u \in N_K(v) = \{0\} \times \mathbb{R}^2_-,\ T_{\operatorname{dom} S}(b) = \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$.

$$\kappa = \max_{u \in \{0\} \times \mathbb{R}^2_-} \frac{\|\operatorname{proj}_{\operatorname{rg} A}(u)\|}{\|A^* u\|} = \max_{u_3 \in \mathbb{R}_-} \frac{\|u_3\|}{\left\| \begin{pmatrix} 2 \\ 2 \end{pmatrix} u_3 \right\|} = \frac{1}{2\sqrt{2}}.$$

*3.* $v \in \{0_2\} \times \mathbb{R}_{++}, u \in N_K(v) = \mathbb{R}^2_- \times \{0\},\ T_{\operatorname{dom} S}(b) = \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$.

$$\kappa = \max_{u \in \mathbb{R}^2_- \times \{0\}} \frac{\|\operatorname{proj}_{\operatorname{rg} A}(u)\|}{\|A^* u\|} = \max_{u_1 \in \mathbb{R}_-} \frac{\|u_1\|}{\left\| \begin{pmatrix} 1 \\ -4 \end{pmatrix} u_1 \right\|} = \frac{1}{\sqrt{17}}.$$

*4.* $v \in \mathbb{R}_{++} \times \{0\} \times \mathbb{R}_{++}, u \in N_K(v) = \{0\} \times \mathbb{R}_- \times \{0\},\ T_{\operatorname{dom} S}(b) = \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$.

$$\kappa = \max_{u \in \{0\} \times \mathbb{R}_- \times \{0\}} \frac{\|\operatorname{proj}_{\operatorname{rg} A}(u)\|}{\|A^* u\|} = 0.$$

*5.* $v \in \{0\} \times \mathbb{R}_{++} \times \{0\}, u \in N_K(v) = \mathbb{R}_- \times \{0\} \times \mathbb{R}_-,\ T_{\operatorname{dom} S}(b) = \mathbb{R}^3$.

$$\kappa = \max_{u \in \mathbb{R}_- \times \{0\} \times \mathbb{R}_-} \frac{\|\operatorname{proj}_{\mathbb{R}^3}(u)\|}{\|A^* u\|} = \max_{u \in \mathbb{R}_- \times \{0\} \times \mathbb{R}_-} \frac{\left\|(u_1, 0, u_3)^\top)\right\|}{\left\| \begin{pmatrix} 1 \\ -4 \end{pmatrix} u_1 + \begin{pmatrix} 2 \\ 2 \end{pmatrix} u_3 \right\|} = \frac{1}{\sqrt{5}}.$$

*6.* $v \in \mathbb{R}^2_{++} \times \{0\}, u \in N_K(v) = \{0_2\} \times \mathbb{R}_-,\ T_{\operatorname{dom} S}(b) = \mathbb{R}^3$.

$$\kappa = \max_{u \in \{0\} \times \mathbb{R}^2_-} \frac{\|\operatorname{proj}_{\mathbb{R}^3}(u)\|}{\|A^* u\|} = \max_{u_3 \in \mathbb{R}_-} \frac{\|u_3\|}{\left\| \begin{pmatrix} 2 \\ 2 \end{pmatrix} u_3 \right\|} = \frac{1}{2\sqrt{2}}.$$

*7.* $v \in \{0\} \times \mathbb{R}^2_{++}, u \in N_K(v) = \mathbb{R}_- \times \{0_2\},\ T_{\operatorname{dom} S}(b) = \mathbb{R}^3$.

$$\kappa = \max_{u \in \mathbb{R}_- \times \{0_2\}} \frac{\|\operatorname{proj}_{\mathbb{R}^3}(u)\|}{\|A^* u\|} = \max_{u_1 \in \mathbb{R}_-} \frac{\|u_1\|}{\left\| \begin{pmatrix} 1 \\ -4 \end{pmatrix} u_1 \right\|} = \frac{1}{\sqrt{17}}.$$

In all, $\operatorname{lip}_{\operatorname{dom} S} S(0_3 \mid 0_2) = \dfrac{1}{\sqrt{5}}$.

## 3.3 Linear complementarity problems

In this section we consider the linear complementarity problem LCP$(q, M)$:

$$x \geq 0,\ Mx + q \geq 0,\ x^\top (Mx + q) = 0 \tag{3.3.1}$$

where $M \in \mathbb{R}^{n \times n}, q \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$. For the Lipschitz-like property of the solution mapping of this problem, see [38]. Here we consider only $q$ is changing, and we denote the solution mapping as $S(\cdot) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. Thus we have the domain and graph of $S$ written as follow:

$$\operatorname{dom} S = \{q \in \mathbb{R}^n \mid \exists x \geq 0,\ Mx + q \geq 0,\ \langle x,\, Mx + q\rangle = 0\}, \tag{3.3.2}$$

$$\operatorname{gph} S = \{(q, x) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \geq 0,\ Mx + q \geq 0,\ \langle x,\, Mx + q\rangle = 0\}. \tag{3.3.3}$$

Here we introduce some properties of the set $\operatorname{gph} S$ and $\operatorname{dom} S$.

### 3.3.1 Properties of graph and domain

To analyze $\operatorname{gph} S$, we first define a category of index combination $I_1, I_2, I_3 \subset I := \{1, \ldots, n\}$ with

**(a)** $I_1 \cup I_2 \cup I_3 = I$,

**(b)** $I_i \cap I_j = \emptyset, \forall i \neq j$.

Here we use $\mathcal{I}$ to denote the set of all such possible combinations:

$$\mathcal{I} := \{(I_1, I_2, I_3) \mid I_1 \cup I_2 \cup I_3 = I,\ I_i \cap I_j = \emptyset,\ \forall i \neq j\},\ (|\mathcal{I}| = 3^n).$$

In terms of each combination $(I_1, I_2, I_3) \in \mathcal{I}$, we denote:

$$(\operatorname{gph} S)_{(I_1,I_2,I_3)} := \left\{ (q, x) \in \mathbb{R}^n \times \mathbb{R}^n \;\middle|\; \begin{array}{lll} x_i = 0 & (Mx + q)_i > 0 & \text{if } i \in I_1 \\ x_i > 0 & (Mx + q)_i = 0 & \text{if } i \in I_2 \\ x_i = 0 & (Mx + q)_i = 0 & \text{if } i \in I_3 \end{array} \right\}. \tag{3.3.4}$$

We call $(\operatorname{gph} S)_{(I_1,I_2,I_3)}$ a slice of $\operatorname{gph} S$.

**Theorem 3.3.1.** $(\operatorname{gph} S)_{(\cdot,\cdot,\cdot)}$ *has the following properties:*

**(a)** *each index combination is unique, therefore the corresponding* $(\operatorname{gph} S)_{(\cdot,\cdot,\cdot)}$ *is mutually exclusive (non-overlapping) for different index combinations, that is,*

$$(\operatorname{gph} S)_{(I_1,I_2,I_3)} \cap (\operatorname{gph} S)_{(I_1',I_2',I_3')} = \emptyset,\ \forall (I_1, I_2, I_3) \neq (I_1', I_2', I_3');$$

**(b)** gph $S$ *is a union of slices* $(\operatorname{gph} S)_{(\cdot,\cdot,\cdot)}$ *where the index combination runs through all elements in* $\mathcal{I}$*:*

$$\operatorname{gph} S = \bigcup_{(I_1,I_2,I_3)\in\mathcal{I}} (\operatorname{gph} S)_{(I_1,I_2,I_3)}; \tag{3.3.5}$$

**(c)** $(\operatorname{gph} S)_{(I_1,I_2,I_3)}$ *is a nonempty convex semi-closed polyhedral cone for all* $(I_1, I_2, I_3) \in \mathcal{I}$*. Here we adopt the definition of cone* $C$ *as follows:*

$$\forall x \in C \Longrightarrow \lambda x \in C, \forall \lambda \in \mathbb{R}_{++}.$$

*Note: such cone may not contain* $0$*.*

*Proof.* The first two properties and the polyhedrality in the third property can be observed by checking the definition of gph $S$, (3.3.3), and gph $S$ with the index combination (3.3.4). For the conic structure and convexity mentioned in the third property, $\forall (q_1, x_1), (q_2, x_2) \in (\operatorname{gph} S)_{(I_1,I_2,I_3)}$, and $\lambda_1, \lambda_2 > 0$, $\lambda_1(q_1, x_1) + \lambda_2(q_2, x_2) \in (\operatorname{gph} S)_{(I_1,I_2,I_3)}$ as elements $(\lambda_1 x_1 + \lambda_2 x_2)_i$, $\lambda_1(Mx_1 + q_1)_i + \lambda_2(Mx_2 + q_2)_i$, for $i \in \{1, \ldots, n\}$ remain their status of being 0 or positive. $\square$

Depending on the combination above, the following proposition gives the interpretation of the index sets $I_1, I_2, I_3$.

**Proposition 3.3.2** ([16, Proposition 1.4.4])**.** *For any given* $q \in \operatorname{dom} S$*, it can be expressed as a nonnegative linear combination of columns in* $E$ *(the unit matrix) and* $-M$ *as*

$$q = c^+ - Mc^- = \sum_{i:c_i>0} E_{(\cdot,i)} c_i^+ - \sum_{i:c_i<0} M_{(\cdot,i)} c_i^-. \tag{3.3.6}$$

*Here* $c^+ = \max\{0, c\}$ *and* $c^- = \max\{0, -c\}$*. Then the corresponding solution under such expression is* $x = c^-$ *with* $Mx + q = c^+$*. In this case,* $I_1 = \{i \mid c_i > 0\}$*,* $I_2 = \{i \mid c_i < 0\}$*,* $I_3 = \{i \mid c_i = 0\}$*.*

From the proposition above and mutual exclusiveness between slices $(\operatorname{gph} S)_{(\cdot,\cdot,\cdot)}$, we can see that for every $(q,x) \in \operatorname{gph} S$, there is a unique index combination $(I_1, I_2, I_3) \in \mathcal{I}$ such that $(q,x) \in (\operatorname{gph} S)_{(I_1,I_2,I_3)}$. To avoid abuse of notations, accordingly we specify the unique index combination decided by $(q,x)$ as

$$\begin{aligned} I_1(q,x) :=&\{i \in I | x_i = 0, (Mx+q)_i > 0\}, \\ I_2(q,x) :=&\{i \in I | x_i > 0, (Mx+q)_i = 0\}, \\ I_3(q,x) :=&\{i \in I | x_i = 0, (Mx+q)_i = 0\}. \end{aligned} \tag{3.3.7}$$

Thus by the representation above we can see that when $(q,x)$ is given, for any pair $(q',x') \in (\operatorname{gph} S)_{(I_1(q,x),I_2(q,x),I_3(q,x))}$, the index combination $(I_1(q',x'),\ I_2(q',x'),\ I_3(q',x'))$ remains the same as the one of $(q,x)$.

**Remark 3.3.3.** *When we put the form of $S$ as $S(q) = \left\{x \mid 0 \in Mx + q + N_{\mathbb{R}^n_+}(x)\right\}$, the domain writes $\operatorname{dom} S = \bigcup_{x\in\mathbb{R}^n_+} \left\{-Mx - N_{\mathbb{R}^n_+}(x)\right\}$. Given that*

$$-N_{\mathbb{R}^n_+}(x) = \left\{x' \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} x'_i \in \mathbb{R}_+, & \textit{if } x_i = 0 \\ x'_i = 0, & \textit{if } x_i > 0 \end{array}\right\},$$

*we can have $-N_{\mathbb{R}^n_+}(x) = \sum_{i:c_i>0} E_{(\cdot,i)} c_i^+$ with $x = c^-$ taken as in (3.3.6).*

With the pair $(q,x) \in \operatorname{gph} S$ fixed, we may now proceed to the representation of $N_{\operatorname{gph} S}(q,x)$. To better illustrate the structure of $N_{\operatorname{gph} S}(q,x)$, we introduce a set defined by index combinations:

$$W(I_1,I_2,I_3) := \left\{(u^*,v^*) \in \mathbb{R}^n \times \mathbb{R}^n \;\middle|\; \begin{array}{ll} (u_i^*,v_i^*) \in \{0\} \times \mathbb{R} & \text{if } i \in I_1 \\ (u_i^*,v_i^*) \in \mathbb{R} \times \{0\} & \text{if } i \in I_2 \\ (u_i^*,v_i^*) \in \Omega & \text{if } i \in I_3 \end{array}\right\} \tag{3.3.8}$$

where $\Omega := (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R}) \cup \mathbb{R}^2_-$. Note that for $(q,x) \in \operatorname{gph} S$,

$$(u^*,v^*) \in W(I_1(q,x), I_2(q,x), I_3(q,x)) \Longleftrightarrow (v^*, -u^*) \in N_{\operatorname{gph} N_{\mathbb{R}^m_+}}(x, -Mx-q). \tag{3.3.9}$$

By calculation in [37] we have

$$N_{\operatorname{gph} S}(q,x) = \{(u^*,\ M^*u^* + v^*) \mid (u^*, v^*) \in W(I_1(q,x), I_2(q,x), I_3(q,x))\} \quad (3.3.10)$$

From (3.3.10) we can see that the normal cone of $\operatorname{gph} S$ at a given pair is decided by the associated index combination. Given the discussion above that the index combination remains unchanged for all elements in the slice $(\operatorname{gph} S)_{(I_1,I_2,I_3)}$, in accordance $N_{\operatorname{gph} S}(q,x)$ stays the same for all $(q,x) \in (\operatorname{gph} S)_{(I_1,I_2,I_3)}$ as well. In this way, we can use the index combination to recognize the behavior of neighboring points $(q,x) \xrightarrow{\operatorname{gph} S} (\bar{q},\bar{x})$ and the related $N_{\operatorname{gph} S}(q,x)$. Next theorem shows how we can group the elements by index combination.

**Theorem 3.3.4.** *For $(q^k,x^k) \xrightarrow{\operatorname{gph} S} (\bar{q},\bar{x})$, there is a subsequence $(q^{ki},x^{ki})$ such that the index combination $I_1(q^{ki},x^{ki})$, $I_2(q^{ki},x^{ki})$, $I_3(q^{ki},x^{ki})$ categorized as in (3.3.7) remain the same for all $i$.*

*Proof.* Let sequence $(q^k,x^k) \longrightarrow (\bar{q},\bar{x})$ in $\operatorname{gph} S$. For each $k$, $(q^k,x^k)$ has a corresponding combination of index set $I_1(q^k,x^k), I_2(q^k,x^k), I_3(q^k,x^k)$. As such combinations of index set is finite, there is a subsequence $(q^{ki},x^{ki})$ such that the corresponding combination of index sets $I_1(q^{ki},x^{ki})$, $I_2(q^{ki},x^{ki})$, $I_3(q^{ki},x^{ki})$ remains the same. Then the normal cone $N_{\operatorname{gph} S}(q^{ki},x^{ki})$ remains the same as well. □

As mentioned in the proof above, there are finite combinations of index sets around $(\bar{q},\bar{x})$. Next we illustrate what those combinations and neighboring slices are.

**Theorem 3.3.5.** *Given the pair $(\bar{q},\bar{x}) \in \operatorname{gph} S$, we denote the collection of all possible combinations of index sets as:*

$$\mathcal{I}(\bar{q},\bar{x}) = \{(I_1,I_2,I_3) \in \mathcal{I} \mid I_1 \supseteq I_1(\bar{q},\bar{x}), I_2 \supseteq I_2(\bar{q},\bar{x}), I_3 \subseteq I_3(\bar{q},\bar{x})\} \quad (3.3.11)$$

*The corresponding neighboring slices in* $\operatorname{gph} S$ *are finite as:*

$$(\operatorname{gph} S)_{(I_1,I_2,I_3)}, \ \forall (I_1, I_2, I_3) \in \mathcal{I}(\bar{q}, \bar{x}).$$

*Proof.* For $(\bar{q}, \bar{x}) \in \operatorname{gph} S$, specifically it lies on the slice $(\operatorname{gph} S)_{(I_1(\bar{q},\bar{x}),I_2(\bar{q},\bar{x}),I_3(\bar{q},\bar{x}))}$. Consider $(q, x)$ around $(\bar{q}, \bar{x})$ in $\operatorname{gph} S$. By definition (3.3.7), we can see that for $I_1(q, x), I_2(q, x)$ there are open constraints $(Mx + q)_i > 0$ and $x_i > 0$ respectively and for $I_3(q, x)$ there are equations: $(Mx + q)_i = 0$ and $x_i = 0$. Therefore for any element $(q, x) \xrightarrow{\operatorname{gph} S} (\bar{q}, \bar{x})$, $I_1(q, x)$ and $I_2(q, x)$ should include $I_1(\bar{q}, \bar{x})$ and $I_2(\bar{q}, \bar{x})$ as subsets respectively. For equality constraints, it can be tended through open constraints, either $(Mx + q)_i > 0$ or $x_i > 0$. Then we have the indices in $I_3(\bar{q}, \bar{x})$ being distributed into either $I_1(q, x)$, $I_2(q, x)$ or remained in $I_3(q, x)$. Thus $I_3(q, x) \subseteq I_3(\bar{q}, \bar{x})$. Combining all these finite possible status, we arrive at (3.3.11). With possible combinations decided, we can accordingly give neighboring slices as stated. □

With index combination of neighboring slices given, we can see that for a given pair $(\bar{q}, \bar{x})$, there are $\|\mathcal{I}(\bar{q}, \bar{x})\| = 3^{\|I_3(\bar{q},\bar{x})\|}$ neighboring slices in $\operatorname{gph} S$ (including the slice where the pair lies on). For example, for $(\bar{q}, \bar{x}) = (0, 0)$, $I_1(\bar{q}, \bar{x}) = \emptyset$, $I_2(\bar{q}, \bar{x}) = \emptyset$, $I_3(\bar{q}, \bar{x}) = I$. Then $\mathcal{I}(\bar{q}, \bar{x})$ gives all possible combinations of $I_1, I_2, I_3$: $\mathcal{I}(\bar{q}, \bar{x}) = \mathcal{I}$ and $\|\mathcal{I}(\bar{q}, \bar{x})\| = \|\mathcal{I}\| = 3^n$. The neighboring slices are all slices of $\operatorname{gph} S$.

Other than the index notation we introduced above, there is another set of index notation using $\alpha$, which is introduced in [16]. Before introducing $\operatorname{dom} S$, we first introduce complementary cones related to complementary matrices of $M$ and the index notation with $\alpha$.

**Definition 3.3.6** ([16, Definition 1.3.2])**.** *Given* $M \in \mathbb{R}^{n\times n}$ *and* $\alpha \subseteq I = \{1, \dots, n\}$, *we define a complementary matrix of $M$, $C_M(\alpha) \in \mathbb{R}^{n\times n}$ as*

$$C_M(\alpha)_{(\cdot,i)} = \begin{cases} -M_{(\cdot,i)}, & \text{if } i \in \alpha, \\ E_{(\cdot,i)}, & \text{if } i \notin \alpha. \end{cases} \tag{3.3.12}$$

*The associated cone,* $\operatorname{conv}\operatorname{pos}(C_M(\alpha))$ *is called the complementary cone (relative to $M$) and* $\operatorname{conv}\operatorname{pos}(C_M(\alpha))$ *stands for the set of all nonnegative linear combinations of columns of the matrix $C_M(\alpha)$.*

For an $n \times n$ matrix $M$, there are $2^n$ complementary cones (convex polyhedral) and the union of these cones is the domain of $S$, see (3.3.2). For $\alpha = \emptyset$, $C_M(\alpha) = E$. For $\alpha = I$, $C_M(\alpha) = -M$. Therefore we can see that $\operatorname{dom} S$ must contain $\operatorname{conv}\operatorname{pos}(E) = \mathbb{R}^n_+$ (i.e. the nonnegative orthant in $\mathbb{R}^n$) and $\operatorname{conv}\operatorname{pos}(-M)$, and moreover is contained in $\operatorname{conv}\operatorname{pos}(E, -M)$ where $\mathrm{LCP}(q, M)$ is feasible [16, Page 18]. Then we have:

$$\operatorname{dom} S = \bigcup_{\alpha \subseteq I} \operatorname{conv}\operatorname{pos} C_M(\alpha) \tag{3.3.13}$$

and

$$(\operatorname{conv}\operatorname{pos}(E) \cup \operatorname{conv}\operatorname{pos}(-M)) \subseteq \operatorname{dom} S \subseteq \operatorname{conv}\operatorname{pos}(E, -M). \tag{3.3.14}$$

Note that in general, $\operatorname{dom} S$ may not be convex and its convex hull is $\operatorname{conv}\operatorname{pos}(E, -M)$. Besides, the following notation for the index set $\alpha$ is also widely adopted on describing $\operatorname{gph} S$ [16, Page 646]:

$$(\operatorname{gph} S)_\alpha := \left\{ (q, x) \in \mathbb{R}^n \times \mathbb{R}^n \;\middle|\; \begin{array}{ll} (Mx + q)_i = 0, x_i \geq 0 & i \in \alpha \\ (Mx + q)_i \geq 0, x_i = 0 & i \notin \alpha \end{array} \right\}. \tag{3.3.15}$$

In this way, $\operatorname{gph} S$ is a union of $2^n$ closed convex polyhedral cones $(\operatorname{gph} S)_\alpha$ with $\alpha$ running over all subsets of $I$. Under such representation, for a given index set $\alpha$, $(\operatorname{gph} S)_\alpha$ has one-to-one correspondence with $\operatorname{conv}\operatorname{pos}(C_M(\alpha))$ in $\operatorname{dom} S$. That is, for $(q, x) \in (\operatorname{gph} S)_\alpha$, $q \in \operatorname{conv} \operatorname{pos} C_M(\alpha)$. And in reverse, for $q \in \operatorname{conv} \operatorname{pos} C_M(\alpha)$ there always exists $x$ such that $(q, x) \in (\operatorname{gph} S)_\alpha$.

The difference between using $\alpha$ and $(I_1, I_2, I_3)$ is that for the case $(Mx + q)_i = 0, x_i = 0$, it is specifically categorized as $I_3$ in the latter combination while in the former, such $i$ could fall in either $\alpha$ or $\overline{\alpha}$ (the complement of $\alpha$ in $I$) by definition. However, for such index notation, we can see that $N_{\operatorname{gph} S}$ has different values for $(q, x) \in (\operatorname{gph} S)_\alpha$ for a given $\alpha$. More specifically, $I_1 \subset \overline{\alpha}$, $I_2 \subset \alpha$.

As the assumption of the generalized Mordukhovich criterion requires the set to be closed and convex, to employ such criterion on $S$ relative to $\operatorname{dom} S$, it is natural to ask under what condition $\operatorname{dom} S$ is closed and convex. The following proposition provides the rationality behind such an assumption.

**Proposition 3.3.7** ([16, Proposition 3.2.1])**.** *For an LCP$(q, M)$, the following statements are equivalent:*

**(a)** *$M$ is a $Q_0$-matrix.*

**(b)** $\operatorname{dom} S$ *is convex;*

**(c)** $\operatorname{dom} S = \operatorname{conv} \operatorname{pos}(E, -M)$.

*Here $Q_0$-matrix means the type of matrices with LCP(3.3.1) being solvable whenever feasible.*

We can see that when $\operatorname{dom} S$ is closed and convex, it is also a convex polyhedral cone in $\mathbb{R}^n$, generated by the columns of $E$ and $-M$, which provides further simplification to calculate its normal cone and tangent cone.

**Theorem 3.3.8.** *Assume that* $\operatorname{dom} S$ *is closed and convex. For a given combination of index set $(I_1, I_2, I_3)$, any $q$ with $(q, x) \in \operatorname{gph} S_{(I_1, I_2, I_3)}$ for some $x$ has the following* *properties:*

$$N_{\operatorname{dom} S}(q) = \left\{ \; w \mid \langle w, v \rangle = 0, v \in \left(E_{(\cdot,I_1)}, -M_{(\cdot,I_2)}\right); \langle w, u \rangle \leq 0, u \in \left(E_{(\cdot,\overline{I_1})}, -M_{(\cdot,\overline{I_2})}\right) \right\}, \tag{3.3.16}$$

$$T_{\operatorname{dom} S}(q) = (N_{\operatorname{dom} S}(q))^* = \operatorname{conv} \operatorname{pos} \left(E_{(\cdot,\overline{I_1})},\; -M_{(\cdot,\overline{I_2})},\; \pm E_{(\cdot,I_1)},\; \pm M_{(\cdot,I_2)}\right). \tag{3.3.17}$$

*Proof.* Since $\operatorname{dom} S$ is a polyhedral convex cone, so are $T_{\operatorname{dom} S}(q)$ and $N_{\operatorname{dom} S}(q)$. By (10) in [18],

$$w \in N_{\operatorname{dom} S}(q) \Longleftrightarrow q \in \operatorname{dom} S,\; w \in (\operatorname{dom} S)^*, w \perp q.$$

Here $(\operatorname{dom} S)^*$ means the polar of $\operatorname{dom} S$. As $\operatorname{dom} S = \operatorname{conv} \operatorname{pos}(E, -M)$, by [81, Lemma 6.45], we have

$$(\operatorname{dom} S)^* = \left\{ w \mid \langle E_{(\cdot,i)}, w \rangle \leq 0, \langle -M_{(\cdot,i)}, w \rangle \leq 0, i = 1, \ldots, n \right\}.$$

Note that $(q, x) \in \operatorname{gph} S_{(I_1,I_2,I_3)}$, by representation (3.3.6) of $q$, $q$ can be expressed as a positive linear combination of $E_{(\cdot,i)}, i \in I_1$ and $-M_{(\cdot,j)}, j \in I_2$. Therefore we have

$$N_{\operatorname{dom} S}(q) = (\operatorname{dom} S)^* \cap [q]^{\perp} = \left\{ w \;\middle|\; \begin{array}{ll} \langle E_{(\cdot,i)}, w \rangle \leq 0, i \in \overline{I_1}, & \langle -M_{(\cdot,j)}, w \rangle \leq 0, j \in \overline{I_2} \\ \langle E_{(\cdot,i)}, w \rangle = 0, i \in I_1, & \langle -M_{(\cdot,j)}, w \rangle = 0, j \in I_2 \end{array} \right\}.$$

From the polar relation between a normal cone and a tangent cone of a convex set, we can derive:

$$T_{\operatorname{dom} S}(q) = (N_{\operatorname{dom} S}(q))^* = \operatorname{conv} \operatorname{pos} \left(E_{(\cdot,\overline{I_1})},\; -M_{(\cdot,\overline{I_2})},\; \pm E_{(\cdot,I_1)},\; \pm M_{(\cdot,I_2)}\right).$$

□

From the theorem above we can see that the tangent cone and the normal cone stay the same for all $(q, x)$ on the same slice of $\operatorname{gph} S$ as long as the index combination is fixed. Note that when only $q$ is given, without index combination, the linear combination in (3.3.6) is not unique.

### 3.3.2 Lipschitz-like property relative to domain under convexity

For linear complementarity problems, the constraint qualification (3.1.3) can also be avoided when the set we refer to is $\operatorname{dom} S$. The proposition below provides another application of Theorem 3.1.12.

**Proposition 3.3.9.** *For LCP(3.3.1) and the corresponding solution mapping $S$, let $(\bar{q}, \bar{x}) \in \operatorname{gph} S$. Then*

$$D^*_{\operatorname{dom} S} S(\bar{q} \mid \bar{x})(y^*) = \bigcup_{(q,x)\in \operatorname{gph} S \cap \mathbb{B}_\varepsilon(\bar{q},\bar{x})} \Big\{ \ \operatorname{proj}_{T_{\operatorname{dom} S(q)}}(u^*) \ \Big| \ \exists u^* \ s.t. \ (u^*, -y^* - M^* u^*) \in W\left(I_1(q,x), I_2(q,x), I_3(q,x)\right) \Big\} \tag{3.3.18}$$

*for sufficiently small $\varepsilon > 0$.*

*Proof.* Note that in Remark 3.3.3 we expressed $S$ as

$$S(q) = \left\{ x \in \mathbb{R}^n \mid 0 \in Mx + q + N_{\mathbb{R}^m_+}(x) \right\}.$$

Thus we can directly apply Theorem 3.1.12 to $S$ with

$$D^* N_{\mathbb{R}^m_+}(x \mid -Mx - q)(u^*) = \left\{ v^* \ \middle| \ (v^*, -u^*) \in N_{\operatorname{gph} N_{\mathbb{R}^m_+}}(x, -Mx - q) \right\}.$$

In view of the fact that $\operatorname{gph} S$ is also polyhedral, we have for sufficiently small $\varepsilon > 0$,

$$\begin{aligned}
&D^*_{\operatorname{dom} S} S(\bar{q} \mid \bar{x})(y^*) \\
&= \limsup_{\substack{(q,x) \xrightarrow{\operatorname{gph} S} (\bar{q},\bar{x}) \\ y'^* \to y^*}} \bigcup_{u^* \in \mathbb{R}^n} \left\{ \operatorname{proj}_{T_{domS(q)}}(u^*) \ \middle| \ y'^* = -M^* u^* - v^*, \right. \\
&\qquad\qquad \left. (v^*, -u^*) \in N_{\operatorname{gph} N_{\mathbb{R}^m_+}}(x, -Mx - q) \right\} \\
&= \bigcup_{(q,x)\in \operatorname{gph} S \cap \mathbb{B}_\varepsilon(\bar{q},\bar{x})} \left\{ \operatorname{proj}_{T_{domS(q)}}(u^*) \ \middle| \ y^* = -M^* u^* - v^*, \right.
\end{aligned}$$

$$
\left. (v^*, -u^*) \in N_{\mathrm{gph}\, N_{\mathbb{R}^m_+}}(x, -Mx - q) \right\}
$$

$$
= \bigcup_{(q,x)\in \mathrm{gph}\, S \cap \mathbb{B}_\varepsilon(\bar{q},\bar{x})} \left\{ \mathrm{proj}_{T_{domS(q)}}(u^*) \;\middle|\; y^* = -M^*u^* - v^*, \right.
$$

$$
\left. (u^*, v^*) \in W\left(I_1(q,x), I_2(q,x), I_3(q,x)\right) \right\}
$$

and finally (3.3.18) by tuning the expression of $y^*$. Here the first equation is obtained directly from the application of Theorem 3.1.12 and the second from the polyhedrality of $\mathrm{gph}\, S$. The third one can be derived via (3.3.9). $\square$

Although the expression of the projectional coderivative involves employing information of neighboring points, in next theorem we prove that under some specific setting we can use only the information at the given point to obtain a sufficient and necessary condition for relative Lipschitz-like property. Before presenting the condition, we introduce another set defined by index combination similar to $W(I_1, I_2, I_3)$:

$$
W'(I_1, I_2, I_3) := \left\{ (u^*, v^*) \in \mathbb{R}^n \times \mathbb{R}^n \;\middle|\; \begin{array}{ll} (u_i^*, v_i^*) \in \{0\} \times \mathbb{R}_- & \text{if } i \in I_1 \\ (u_i^*, v_i^*) \in \mathbb{R}_- \times \{0\} & \text{if } i \in I_2 \\ (u_i^*, v_i^*) \in \mathbb{R}^2_- & \text{if } i \in I_3 \end{array} \right\}. \tag{3.3.19}
$$

**Theorem 3.3.10.** *For LCP(3.3.1) with $M$ being a $Q_0$-matrix, let $\bar{x} \in S(\bar{q})$. The solution mapping $S$ has the Lipschitz-like property relative to its domain at $\bar{q}$ for $\bar{x}$ if and only if*

$$
\begin{aligned}
\forall (u^*, -M^*u^*) \in\, & W(I_1(\bar{q},\bar{x}), I_2(\bar{q},\bar{x}), I_3(\bar{q},\bar{x})) \\
& \Longrightarrow (u^*, -M^*u^*) \in W'(I_1(\bar{q},\bar{x}), I_2(\bar{q},\bar{x}), I_3(\bar{q},\bar{x})).
\end{aligned} \tag{3.3.20}
$$

*Equivalently, that is*

$$
\begin{aligned}
&\forall (u^*, -M^*u^*) \in W(I_1(\bar{q},\bar{x}), I_2(\bar{q},\bar{x}), I_3(\bar{q},\bar{x})) \\
&\qquad \Longrightarrow \begin{cases} \langle u^*, E_{(\cdot,i)}\rangle = 0, i \in I_1(\bar{q},\bar{x}), & \langle u^*, M_{(\cdot,j)}\rangle = 0, j \in I_2(\bar{q},\bar{x}) \\ \langle u^*, E_{(\cdot,i)}\rangle \le 0, i \in \overline{I_1(\bar{q},\bar{x})}, & \langle u^*, M_{(\cdot,j)}\rangle \ge 0, j \in \overline{I_2(\bar{q},\bar{x})}. \end{cases}
\end{aligned} \tag{3.3.21}
$$

*Proof.* Given Theorem 3.3.7, $\operatorname{dom} S$ here is a convex polyhedral cone as $M$ is a $Q_0$-matrix. By applying the criterion $D^*_{\operatorname{dom} S} S(\bar{q} \mid \bar{x})(0) = \{0\}$ of relative Lipschitz-like property, it remains to examine the behavior of $N_{\operatorname{gph} S}(q, x)$ around $(\bar{q}, \bar{x})$ with first element projected onto the tangent cones $T_{\operatorname{dom} S}(q)$.

We begin our proof by showing the equivalence between $D^*_{\operatorname{dom} S} S(\bar{q} \mid \bar{x})(0) = \{0\}$ and

$$\forall (I_1, I_2, I_3) \in \mathcal{I}(\bar{q}, \bar{x}),\ (u^*, -M^* u^*) \in W(I_1, I_2, I_3) \Longrightarrow (u^*, -M^* u^*) \in W'(I_1, I_2, I_3). \tag{3.3.22}$$

For a given pair $(\bar{q}, \bar{x}) \in \operatorname{gph} S$, the number of neighboring slices are finite (see Theorem 3.3.5) and on each slice, $N_{\operatorname{gph} S}$, $N_{\operatorname{dom} S}$ and $T_{\operatorname{dom} S}$ remain the same. First, given $(I_1, I_2, I_3) \in \mathcal{I}$, we introduce two notations as:

$$N(I_1, I_2, I_3) := \left\{ w \;\middle|\; \begin{array}{ll} \langle w, v \rangle = 0, & v \in \left(E_{(\cdot, I_1)}, -M_{(\cdot, I_2)}\right) \\ \langle w, u \rangle \le 0, & u \in \left(E_{(\cdot, \overline{I_1})}, -M_{(\cdot, \overline{I_2})}\right) \end{array} \right\}, \tag{3.3.23}$$

$$T(I_1, I_2, I_3) := \operatorname{conv} \operatorname{pos} \left( E_{(\cdot, \overline{I_1})}, -M_{(\cdot, \overline{I_2})}, \pm E_{(\cdot, I_1)}, \pm M_{(\cdot, I_2)} \right). \tag{3.3.24}$$

From expressions (3.3.16) and (3.3.17), we can see that

$$\forall (q, x) \in (\operatorname{gph} S)_{(I_1, I_2, I_3)} :\ N_{\operatorname{dom} S}(q) = N(I_1, I_2, I_3),\ T_{\operatorname{dom} S}(q) = T(I_1, I_2, I_3).$$

Therefore by Proposition 3.3.9 and Theorem 3.3.5 we have

$$\limsup_{(q,x) \xrightarrow{\operatorname{gph} S} (\bar{q}, \bar{x})} \operatorname{proj}_{T_{\operatorname{dom} S}(q) \times \mathbb{R}^m} N_{\operatorname{gph} S}(q, x)$$

$$= \bigcup_{\substack{(q,x) \in (\operatorname{gph} S)_{(I_1, I_2, I_3)} \\ (I_1, I_2, I_3) \in \mathcal{I}(\bar{q}, \bar{x})}} \operatorname{proj}_{T_{\operatorname{dom} S}(q) \times \mathbb{R}^m} N_{\operatorname{gph} S}(q, x) \tag{3.3.25}$$

$$= \bigcup_{(I_1, I_2, I_3) \in \mathcal{I}(\bar{q}, \bar{x})} \left\{ (w^*, M^* u^* + v^*) \;\middle|\; w^* = \operatorname{proj}_{T(I_1, I_2, I_3)}(u^*),\ (u^*, v^*) \in W(I_1, I_2, I_3) \right\} \tag{3.3.26}$$

and therefore

$$D^*_{\mathrm{dom}\,S}S(\bar{q} \mid \bar{x})(y^*) = \bigcup_{(I_1,I_2,I_3)\in\mathcal{I}(\bar{q},\bar{x})} \big\{\mathrm{proj}_{T(I_1,I_2,I_3)}(u^*) \mid \exists u^* \text{ s.t.}$$

$$(u^*, -y^* - M^*u^*) \in W(I_1, I_2, I_3)\big\}.$$

In this way, the criterion is equivalent to checking if $y^* = 0$ generates $\mathrm{proj}_{T(I_1,I_2,I_3)}(u^*) = 0$ for every index combination $(I_1, I_2, I_3) \in \mathcal{I}(\bar{q}, \bar{x})$. For $\mathrm{proj}_{T(I_1,I_2,I_3)}(u^*) = 0$, it is equivalent that $u^* \in (T(I_1, I_2, I_3))^* = N(I_1, I_2, I_3)$ considering the convexity of $T(I_1, I_2, I_3)$. Thus it becomes:

$$\forall (I_1, I_2, I_3) \in \mathcal{I}(\bar{q}, \bar{x}) : (u^*, -M^*u^*) \in W(I_1, I_2, I_3) \Longrightarrow u^* \in N(I_1, I_2, I_3).$$

As $\langle u^*, E_{(\cdot,i)}\rangle = u^*_i$ and $\langle u^*, -M_{(\cdot,i)}\rangle = -(M^*u^*)_i$ for $i = 1, \dots, n$ , we can see that

$$\begin{cases} \langle u^*, E_{(\cdot,i)}\rangle = 0, i \in I_1, \quad \langle u^*, M_{(\cdot,j)}\rangle = 0, j \in I_2 \\ \langle u^*, E_{(\cdot,i)}\rangle \leq 0, i \in \overline{I_1}, \quad \langle u^*, M_{(\cdot,j)}\rangle \geq 0, j \in \overline{I_2} \end{cases}$$
$$\Longleftrightarrow \begin{cases} (u^*_i, -(M^*u^*)_i) \in \{0\} \times \mathbb{R}_-, & i \in I_1 \\ (u^*_i, -(M^*u^*)_i) \in \mathbb{R}_- \times \{0\}, & i \in I_2 \\ (u^*_i, -(M^*u^*)_i) \in \mathbb{R}^2_-, & i \in I_3 \end{cases}, \tag{3.3.27}$$

which means that $u^* \in N(I_1, I_2, I_3)$ is equivalent to $(u^*, -M^*u^*) \in W'(I_1, I_2, I_3)$ defined as (3.3.19). Then we have proved (3.3.22).

It remains to prove that (3.3.22) can be replaced by (3.3.20). It can be easily observed that $(I_1(\bar{q}, \bar{x}), I_2(\bar{q}, \bar{x}), I_3(\bar{q}, \bar{x})) \in \mathcal{I}(\bar{q}, \bar{x})$ and therefore (3.3.22) $\Longrightarrow$ (3.3.20) naturally. Next we prove (3.3.20) $\Longrightarrow$ (3.3.22). For $\forall (I_1, I_2, I_3) \in \mathcal{I}(\bar{q}, \bar{x})$, by (3.3.8) and (3.3.11) we have

$$W(I_1, I_2, I_3) \subseteq W\left(I_1(\bar{q}, \bar{x}), I_2(\bar{q}, \bar{x}), I_3(\bar{q}, \bar{x})\right).$$

Therefore when (3.3.20) holds, we have

$$(u^*, -M^*u^*) \in W(I_1, I_2, I_3) \Longrightarrow (u^*, -M^*u^*) \in W'(I_1(\bar{q}, \bar{x}), I_2(\bar{q}, \bar{x}), I_3(\bar{q}, \bar{x}).$$

Given $I_3 \subseteq I_3(\bar{q},\bar{x})$, we have $(u_i^*, -(M^*u^*)_i) \in \mathbb{R}^2_-$ for $i \in I_3$ naturally. Besides, $(u^*, -M^*u^*) \in W(I_1, I_2, I_3)$ gives $(u_i^*, -(M^*u^*)_i) \in \{0\} \times \mathbb{R}_-$ for $i \in I_1$ and $(u_i^*, -(M^*u^*)_i) \in \mathbb{R}_- \times \{0\}$ for $i \in I_2$. As the last condition (3.3.21) can be derived via (3.3.27), the proof is completed. $\square$

**Remark 3.3.11.** *When $\bar{q} \in \operatorname{int}\operatorname{dom} S$, $N_{\operatorname{dom} S}(\bar{q}) = \{0\}$, the criterion above can be reduced to*

$$(u^*, -M^*u^*) \in W\left(I_1(\bar{q},\bar{x}), I_2(\bar{q},\bar{x}), I_3(\bar{q},\bar{x})\right) \Longrightarrow u^* = 0,$$

*which is equivalent to the sufficient and necessary condition for Lipschitz-like property of $S$ in [38] when $\bar{q} \in \operatorname{int}\operatorname{dom} S$.*

### 3.3.3 The graphical modulus

By (3.3.10), for a given pair $(\bar{q},\bar{x}) \in \operatorname{gph} S$,

$$\operatorname{lip} S(\bar{q},\bar{x}) = \sup_{(u^*,v^*) \in W(I_1(\bar{q},\bar{x}), I_2(\bar{q},\bar{x}), I_3(\bar{q},\bar{x}))} \frac{\|u^*\|}{\|M^*u^* + v^*\|}. \tag{3.3.28}$$

From expression (3.3.28) we can see when $M^*u^* + v^* = 0$ it is required that $u^* = 0$ as otherwise $\operatorname{lip} S$ becomes infinite, which is equivalent to the criterion of Lipschitz-like property of $S$ at $\bar{q}$ for $\bar{x}$ given by [38]:

$$(u^*, -M^*u^*) \in W(I_1(\bar{q},\bar{x}), I_2(\bar{q},\bar{x}), I_3(\bar{q},\bar{x})) \Longrightarrow u^* = 0. \tag{3.3.29}$$

Here we further simplify the modulus by getting rid of $v^*$.

**Theorem 3.3.12.** *For $(\bar{q},\bar{x}) \in \operatorname{gph} S$ defined as in (3.3.3),*

$$\operatorname{lip} S(\bar{q},\bar{x}) = \sup_{(I_1,I_2,I_3) \in \mathcal{I}(\bar{q},\bar{x})} \; \sup_{u^* \in U(I_1,I_2,I_3)} \frac{\|u^*\|}{\left\| \begin{matrix} M^*_{(\cdot,I_2)} u^* \\ \left(-M^*_{(\cdot,I_3)} u^*\right)_+ \end{matrix} \right\|}. \tag{3.3.30}$$

*where for a vector $y$, $(y)_+$ means $((y)_+)_i := (y_i)_+ = \max\{0, y_i\}$ and*

$$U(I_1, I_2, I_3) = \left\{ u^* \in \mathbb{R}^n \,\middle|\, \begin{array}{ll} u_i^* = 0, & i \in I_1 \\ u_i^* \in \mathbb{R}, & i \in I_2 \\ u_i^* \in \mathbb{R}_-, & i \in I_3 \end{array} \right\}. \tag{3.3.31}$$

*Proof.* The supremum in (3.3.28) is equivalent to first maximizing the fractional in (3.3.28) relative to $v^*$ and then relative to $u^*$. Therefore we begin by minimizing $\|M^*u^* + v^*\|$ element-wisely, i.e., $\min_{v_i^*} |(M^*u^*)_i + v_i^*|$ for each $i \in I$. Note that $(M^*u^*)_i = M^*_{(\cdot,i)}u^*$. Moreover, we have:

1. $i \in I_1(\bar{q}, \bar{x}) :\ (u_i^*, v_i^*) \in \{0\} \times \mathbb{R}$. $\min_{v_i^* \in \mathbb{R}} |M^*_{(\cdot,i)}u^* + v_i^*| = 0$ by taking $v_i^* = -M^*_{(\cdot,i)}u^*$.

2. $i \in I_2(\bar{q}, \bar{x}) :\ (u_i^*, v_i^*) \in \mathbb{R} \times \{0\}$. $\min_{v_i^* = 0} |M^*_{(\cdot,i)}u^* + v_i^*| = |M^*_{(\cdot,i)}u^*|$ .

3. $i \in I_3(\bar{q}, \bar{x}) :\ (u_i^*, v_i^*) \in \Omega$.

   Next we define the following index subset of $I_3$ considering the possibilities of the values of $u^*$,$v^*$ and $M^*u^*$ :

$$\begin{aligned} I_{31} &:= \{i \in I_3(\bar{q}, \bar{x}) \mid (u_i^*, v_i^*) \in \{0\} \times \mathbb{R}\}; \\ I_{32} &:= \{i \in I_3(\bar{q}, \bar{x}) \mid (u_i^*, v_i^*) \in \mathbb{R} \times \{0\}\}; \\ I_{33} &:= \left\{i \in I_3(\bar{q}, \bar{x}) \mid (u_i^*, v_i^*) \in \mathbb{R}^2_-\right\}; \\ I_{331} &:= \left\{i \in I_{33} \mid M^*_{(\cdot,i)}u^* \in \mathbb{R}_+\right\}; \\ I_{332} &:= \left\{i \in I_{33} \mid M^*_{(\cdot,i)}u^* \in \mathbb{R}_-\right\}. \end{aligned}$$

   Similarly it can be divided non-overlappingly as

   (a) $i \in I_{31}$ where $(u_i^*, v_i^*) \in \{0\} \times \mathbb{R}$, same as in $i \in I_1$.

   (b) $i \in I_{32}$ where $(u_i^*, v_i^*) \in \mathbb{R} \times \{0\}$, same as in $i \in I_2$.

(c) $i \in I_{33}$ where $(u_i^*, v_i^*) \in \mathbb{R}_-^2$. Here further dividing is required depending on the value of $M_{(\cdot,i)}^* u^*$.

i. $i \in I_{331}$: $M_{(\cdot,i)}^* u^* \in \mathbb{R}_+$. Then $\min_{v_i^* \in \mathbb{R}_-} |M_{(\cdot,i)}^* u^* + v_i^*| = 0$ by taking $v_i^* = -M_{(\cdot,i)}^* u^*$.

ii. $i \in I_{332}$: $M_{(\cdot,i)}^* u^* \in \mathbb{R}_-$. Then by taking $v_i^* = 0$,

$$\min_{v_i^* \in \mathbb{R}_-} |M_{(\cdot,i)}^* u^* + v_i^*| = |M_{(\cdot,i)}^* u^*| = -M_{(\cdot,i)}^* u^*.$$

Combining these two cases, we can see that for $i \in I_{33}$,

$$\min_{v_i^* \in \mathbb{R}_-} |M_{(\cdot,i)}^* u^* + v_i^*| = \left(-M_{(\cdot,i)}^* u^*\right)_+ .$$

For the first two cases (a) and (b), they are covered in $I_1$ and $I_2$ respectively when we consider all possible combinations $(I_1, I_2, I_3) \in I(\bar{q}, \bar{x})$ defined in (3.3.11). To avoid abuse of index notation, we take supremum over all these combinations.

□

**Remark 3.3.13.** *From the form of graphical modulus (3.3.30) we can also see that the fixed point condition (3.3.20) is equivalent to neighboring point condition (3.3.22).*

Next we consider the graphical modulus $\operatorname{lip}_{\operatorname{dom} S} S$ based on the sufficient and necessary condition (3.3.21). Under the setting that $\operatorname{dom} S$ is convex, by (3.3.26) we can see that

$$\begin{aligned} \operatorname{lip}_{\operatorname{dom} S}(\bar{q}, \bar{x}) &= \sup_{(u^*,v^*) \in W(I_1(\bar{q},\bar{x}), I_2(\bar{q},\bar{x}), I_3(\bar{q},\bar{x}))} \frac{d(u^*, N_{\operatorname{dom} S}(\bar{q}))}{\|M^* u^* + v^*\|} \\ &= \sup_{(I_1,I_2,I_3) \in \mathcal{I}(\bar{q},\bar{x})} \sup_{(u^*,v^*) \in W(I_1,I_2,I_3)} \frac{d(u^*, N(I_1, I_2, I_3))}{\|M^* u^* + v^*\|}. \end{aligned} \tag{3.3.32}$$

Next we give further simplifications according to Theorem 3.3.12.

**Corollary 3.3.14.** *For* $(\bar{q}, \bar{x}) \in \operatorname{gph} S$ *defined as in (3.3.3),*

$$\operatorname{lip}_{\operatorname{dom} S}(\bar{q}, \bar{x}) = \sup_{(I_1,I_2,I_3)\in\mathcal{I}(\bar{q},\bar{x})} \sup_{u^*\in U(I_1,I_2,I_3)} \frac{d(u^*, N(I_1, I_2, I_3))}{\left\| \begin{matrix} M^*_{(\cdot,I_2)}u^* \\ \left(-M^*_{(\cdot,I_3)}u^*\right)_+ \end{matrix} \right\|}, \tag{3.3.33}$$

*where* $U(I_1, I_2, I_3)$ *and* $N(I_1, I_2, I_3)$ *are defined as in (3.3.31) and (3.3.23) respectively.*

*Proof.* The step minimizing $\|M^*u^* + v^*\|$ relative to $v^*$ is already given in Theorem 3.3.12 for calculating $\operatorname{lip} S$ above. □

## 3.4 Affine variational inequalities

Next we mainly focus on the affine variational inequality:

$$0 \in q + Mx + N_C(x)$$

where $C$ is a polyhedral convex set and $M \in \mathbb{R}^{n\times n}$. The solution mapping writes:

$$S(q) = \{x \mid 0 \in q + Mx + N_C(x)\}. \tag{3.4.1}$$

For a closed set $Q \subseteq \operatorname{dom} S$, the graph of the multifunction $S$ restricted on $Q$ is

$$\operatorname{gph} S|_Q = \operatorname{gph} S \cap (Q \times \mathbb{R}^n) = \{(q, x) \in Q \times \mathbb{R}^n \mid 0 \in q + Mx + N_C(x)\}. \tag{3.4.2}$$

From now on, we consider the case where $Q$ is a polyhedral set. In this way, we may express $D^*_Q S$ in the form of union rather than limsup.

### 3.4.1 The upper estimate of the projectional coderivative

**Proposition 3.4.1.** *For AVI(3.4.1) and the corresponding solution mapping* $S$*, consider a union of polyhedral sets* $Q \subseteq \operatorname{dom} S$ *which is also closed. Let* $(\bar{q}, \bar{x}) \in \operatorname{gph} S|_Q$*. If the following constraint qualification holds:*

$$\forall u^* \in N_Q(\bar{q}),\ (M^*u^*, u^*) \in N_{\operatorname{gph} N_C}(\bar{x}, -M\bar{x} - \bar{q}) \Longrightarrow u^* = 0, \tag{3.4.3}$$

*then*

$$D_Q^* S(\bar{q} \mid \bar{x})(y^*) \subseteq \bigcup_{(q,x) \in \operatorname{gph} S|_Q \cap \mathbb{B}_\varepsilon(\bar{q},\bar{x})} \Big\{ \operatorname{proj}_{T_Q(q)}(-u^* + w^*) \ \Big| \ w^* \in N_Q(q), \quad \exists u^* \ s.t. \ (M^* u^* - y^*, u^*) \in N_{\operatorname{gph} N_C}(x, -Mx - q) \Big\} \tag{3.4.4}$$

*for sufficiently small* $\varepsilon > 0$.

*Proof.* Apply Theorem 3.1.2 to $S$. Then the constraint qualification (3.1.3) becomes

$$(0,0) = (u^*, M^* u^*) + (w^*, v^*) \text{ with } w^* \in N_Q(\bar{q}), \ v^* \in D^* N_C(\bar{x} \mid -M\bar{x} - \bar{q})(u^*) \Longrightarrow u^* = 0,$$

which is equivalent to

$$-u^* \in N_Q(\bar{q}), \ -M^* u^* \in D^* N_C(\bar{x} \mid -M\bar{x} - \bar{q})(u^*) \Longrightarrow u^* = 0.$$

By tuning the direction of $u^*$, we arrive at (3.4.3). And the upper estimate (3.1.4) can be put as

$$\begin{aligned} D_Q^* S(\bar{q} \mid \bar{x})(y^*) &\subseteq \limsup_{\substack{(q,x) \xrightarrow{\operatorname{gph} S} (\bar{q},\bar{x}) \\ y'^* \to y^*}} \bigcup_{u^* \in \mathbb{R}^n} \Big\{ \operatorname{proj}_{T_Q(q)}(u^* + w^*) \ \Big| \ y'^* = -M^* u^* - v^*, \\ &\qquad v^* \in D^* N_C(x \mid -Mx - q)(u^*), \ w^* \in N_Q(q) \Big\} \\ &= \bigcup_{(q,x) \in \operatorname{gph} S|_Q \cap \mathbb{B}_\varepsilon(\bar{q},\bar{x})} \Big\{ \operatorname{proj}_{T_Q(q)}(u^* + w^*) \ \Big| \ y^* = -M^* u^* - v^*, \\ &\qquad v^* \in D^* N_C(x \mid -Mx - q)(u^*), \ w^* \in N_Q(q) \Big\} \end{aligned}$$

for sufficiently small $\varepsilon > 0$. Here the second equation comes from the polyhedrality of both $\operatorname{gph} S|_Q$ and $Q$. By tuning the expression of $y^*$ and direction of $u^*$, we finally arrive at (3.4.4). $\square$

**Remark 3.4.2.** *When $Q$ is taken as* $\operatorname{dom} S$, *we can apply Theorem 3.1.12 instead and the constraint qualification (3.4.3) can be avoided and the inclusion (3.4.4) becomes an equation.*

Given the upper estimate (3.4.4), we can give a simple sufficient condition for the Lipschitz-like property of $S$ relative to $Q$ when $Q$ is in further convex.

**Corollary 3.4.3.** *For AVI(3.4.1) and the corresponding solution mapping $S$, consider a closed set $Q \subseteq \operatorname{dom} S$ that is also convex polyhedral. Let $(\bar{q}, \bar{x}) \in \operatorname{gph} S|_Q$ with the constraint qualification (3.4.3) being satisfied. If for every $(q, x) \in \operatorname{gph} S|_Q \cap \mathbb{B}_\varepsilon(\bar{q}, \bar{x})$ with sufficiently small $\varepsilon > 0$,*

$$(M^* u^*, u^*) \in N_{\operatorname{gph} N_C}(x, -Mx - q) \Longrightarrow u^* \in -N_Q(q), \tag{3.4.5}$$

*then $S$ has the Lipschitz-like property relative to $Q$ at $\bar{q}$ for $\bar{x}$.*

*Proof.* When $Q$ is in further convex, we may apply the generalized Mordukhovich criterion $D_Q^* S(\bar{q} \mid \bar{x})(0) = \{0\}$. It becomes sufficient to verify such criterion on the upper estimate (3.4.4) and

$$D_Q^* S(\bar{q} \mid \bar{x})(0) \subseteq \bigcup_{(q,x) \in \operatorname{gph} S|_Q \cap \mathbb{B}_\varepsilon(\bar{q}, \bar{x})} \left\{ \operatorname{proj}_{T_Q(q)}(-u^* + w^*) \;\middle|\; w^* \in N_Q(q), \right.$$

$$\left. \exists u^* \text{ s.t. } (M^* u^*, u^*) \in N_{\operatorname{gph} N_C}(x, -Mx - q) \right\}.$$

It is equivalent to examining if $\operatorname{proj}_{T_Q(q)}(-u^* + w^*) = 0$ for all $(q, x) \in \operatorname{gph} S|_Q \cap \mathbb{B}_\varepsilon(\bar{q}, \bar{x})$. As $T_Q(q)$ is convex, $\operatorname{proj}_{T_Q(q)}(-u^* + w^*) = 0$ is equivalent to $-u^* \in N_Q(q)$ when $w^* \in N_Q(q)$ is already given. Thus we arrive at the sufficient condition (3.4.5).

□

In this sufficient condition, we can see that $N_{\operatorname{gph} N_C}(x, -Mx - q)$ is used. In the coming subsection, we further exploit the structure of this normal cone and will give a more detailed form of the sufficient condition based on the critical face condition introduced in [18].

### 3.4.2 Generalized critical face condition

Before giving the generalized critical face condition, we first introduce some widely adopted tools in study of polyhedral convex sets.

**Lemma 3.4.4** ([80, Corollaries 16.4.2, 19.2.2, 19.3.2])**.** *For two closed polyhedral convex cones $K_1, K_2 \in \mathbb{R}^n$, $K_1^*, K_2^*, K_1+K_2$ are all polyhedral convex cones. Besides,*

$$(K_1 \cap K_2)^* = K_1^* + K_2^*,\ (K_1 + K_2)^* = K_1^* \cap K_2^*.$$

**Proposition 3.4.5.** *Let $C$ be a polyhedral set in $\mathbb{R}^n$. For $x \in C$ and $v \in N_C(x)$ consider the critical cone defined as $K(x,v) := T_C(x) \cap [v]^\perp$. Then $K(x,v)$ is a polyhedral convex cone and the polar of it is $(K(x,v))^* = N_C(x) + [v]$.*

*Proof.* As $C$ is a polyhedral set, $N_C(x)$ and $T_C(x)$ are polyhedral convex cones and $N_C(x) = (T_C(x))^*$, vice versa. By polar relation of polyhedral convex cones, we can see that $(K(x,v))^* = \mathrm{conv}((T_C(x))^* \cup [v]) = \mathrm{conv}(N_C(x) \cup [v])$. Next we will prove $\mathrm{conv}(N_C(x) \cup [v]) = N_C(x) + [v]$. It is easy to see that both of these two sets are cones.

As $N_C(x)$ is a polyhedral convex cone, suppose it is generated by vectors $a_1, \ldots, a_m$, i.e., $N_C(x) = \mathrm{conv}\,\mathrm{pos}\{a_1, \ldots, a_m\}$. With $v \in N_C(x)$, there are $\lambda_i \geq 0,\ i = 1, \ldots, m$ such that $v = \sum_{i=1}^m \lambda_i a_i$. Let $w \in \mathrm{conv}(N_C(x) \cup [v])$. Then there exist $\lambda_i' \geq 0, i = 1, \ldots, m$ and $\lambda_{m+1}' \in \mathbb{R},\ \tau \in [0,1]$ such that $w = \tau \sum_{i=1}^m \lambda_i' a_i + (1-\tau)\lambda_{m+1}' v$. As $\tau \sum_{i=1}^m \lambda_i' a_i \in N_C(x)$, $(1-\tau)\lambda_{m+1}' v \in [v]$, we have $w \in N_C(x) + [v]$.

Let $w \in N_C(x) + [v]$. Then there exist $u \in N_C(x)$ and $\tau \in R$ such that $w = u + \tau v$. Suppose $u = \sum_{i=1}^m \lambda_i' a_i$ with $\lambda_i' \geq 0, i = 1, \ldots, m$. Let $\tau' = \sum_{i=1}^m \lambda_i' + |\tau|$. By conic structure of $N_C(x) + [v]$, we have $\tau'^{-1} w \in N_C(x) + [v]$ as well. We can write

$$\frac{w}{\tau'} = \sum_{i=1}^m \frac{\lambda_i'}{\tau'} a_i + \frac{\tau}{\tau'} v,$$

which is a convex combination of elements in $N_C(x)$ and $[v]$. Then the proof is completed. $\square$

Below is an important result for introducing the face condition, which is given in [18]. The proof can be found in [19].

**Lemma 3.4.6** ([18, Reduction Lemma]). *For any $(x, v) \in \operatorname{gph} N_C$, there is a neighborhood $U$ of $(0, 0)$ in $\mathbb{R}^n \times \mathbb{R}^n$ such that for $(x', v') \in U$ one has*

$$v + v' \in N_C(x + x') \Longleftrightarrow v' \in N_{K(x,v)}(x'). \tag{3.4.6}$$

*Here the critical cone $K(x, v) = T_C(x) \cap [v]^\perp$ with $v \in N_C(x)$. In particular, $T_{\operatorname{gph} N_C}(x, v) = \operatorname{gph} N_{K(x,v)}$.*

**Lemma 3.4.7.** *From the proof of [18, Theorem 2] we can see that for any pair $(x, v) \in \operatorname{gph} N_C$,*

$$N_{\operatorname{gph} N_C}(x, v) = \{(F_1 - F_2)^* \times (F_1 - F_2) \mid F_2 \subset F_1 \in \mathcal{F}(K(x, v))\} \tag{3.4.7}$$

*where $\mathcal{F}(K(x, v))$ is the collection of all closed faces of the polyhedral convex cone $K(x, v)$ in the form of $F = K(x, v) \cap [v^*]^\perp$, where $v^* \in (K(x, v))^*$.*

In [27] they give an expression of directional limiting normal cone of $\operatorname{gph} N_C$ (see also (3.1.18) for its definition).

**Lemma 3.4.8** ([27, Theorem 2.12]). *Given $(x, v) \in \operatorname{gph} N_C$ and $(x', v') \in T_{\operatorname{gph} N_C}(x, v)$,*

$$\begin{aligned} N_{\operatorname{gph} N_C}((x, v); (x', v')) :=& \limsup_{\substack{t \searrow 0 \\ (\tilde{x}, \tilde{v}) \to (x', v')}} \widehat{N}_{\operatorname{gph} N_C}((x, v) + t(\tilde{x}, \tilde{v})) \\ =& \left\{(F_1 - F_2)^* \times (F_1 - F_2) \mid x' \in F_2 \subset F_1 \subset [v']^\perp,\ F_1, F_2 \in \mathcal{F}(K(x, v))\right\} \end{aligned}$$

Considering the conic structure of $\operatorname{gph} N_C$, we have the following result.

**Corollary 3.4.9.** *For a given pair* $(x,v) \in \operatorname{gph} N_C$ *and* $(x',v') \in T_{\operatorname{gph} N_C}(x,v)$ *sufficiently near to* $(0,0)$,

$$N_{\operatorname{gph} N_C}(x+x', v+v') = N_{\operatorname{gph} N_C}\left((x,v);(x',v')\right). \tag{3.4.8}$$

*Proof.* As $(x',v') \in T_{\operatorname{gph} N_C}(x,v)$ is sufficiently near to $(0,0)$ and $\operatorname{gph} N_C$ has conic structure,

$$\begin{aligned} N_{\operatorname{gph} N_C}(x+x', v+v') &= \limsup_{(x'',v'') \longrightarrow (x',v')} \widehat{N}_{\operatorname{gph} N_C}\left((x,v)+(x'',v'')\right) \\ &= \limsup_{\substack{t \searrow 0 \\ (\tilde{x},\tilde{v}) \to (x',v')}} \widehat{N}_{\operatorname{gph} N_C}\left((x,v)+t(\tilde{x},\tilde{v})\right). \end{aligned}$$

□

Then we can give another version of sufficient condition of Lipschitz-like property relative to the set $Q$.

**Proposition 3.4.10.** *For* $(\bar{q},\bar{x}) \in \operatorname{gph} S|_Q$ *defined as (3.4.2) , suppose the constraint qualification (3.4.3) holds for all* $(q,x) \in \operatorname{gph} S|_Q$ *sufficiently near to* $(\bar{q},\bar{x})$*. If in addition for* $q' = q - \bar{q},\ x' = x - \bar{x}$,

$$\forall (M^* u^*, u^*) \in (F_1 - F_2)^* \times (F_1 - F_2) \Longrightarrow u^* \in -N_Q(\bar{q}) \cap [q']^{\perp}, \tag{3.4.9}$$

*holds for all closed faces* $F_1, F_2 \in \mathcal{F}(K(\bar{x},\bar{v}))$ *with* $x' \in F_2 \subset F_1 \subset [-Mx' - q']^{\perp}$, *then* $S$ *has Lipschitz-like property relative to* $Q$ *at* $\bar{q}$ *for* $\bar{x}$*. Here* $\bar{v} = -M\bar{x} - \bar{q}$*. The condition becomes necessary when* $\operatorname{gph} N_C$ *is regular at* $(\bar{x},\bar{v})$.

*Proof.* For any $(q,x) \in \operatorname{gph} S|_Q$ sufficiently near to $(\bar{q},\bar{x})$, $(q-\bar{q}, x-\bar{x}) = (q',x') \in T_{\operatorname{gph} S|_Q}(\bar{q},\bar{x})$ is sufficiently near to $(0,0)$. By [81, Theorems 6.42, Exercise 6.7],

$$\begin{aligned} T_{\operatorname{gph} S|_Q}(\bar{q},\bar{x}) &\subseteq (T_Q(\bar{q}) \times \mathbb{R}^n) \cap T_{\operatorname{gph} S}(\bar{q},\bar{x}) \\ &= \left\{(q',x') \mid q' \in T_Q(\bar{q}),\ (x', -Mx' - q') \in T_{\operatorname{gph} N_C}(\bar{x},\bar{v})\right\}. \end{aligned}$$

The inclusion becomes an equation when $\operatorname{gph} N_C$ is regular at $(\bar{x}, \bar{v})$. Therefore $q' \in T_Q(\bar{q})$ and $(x', -Mx' - q') \in T_{\operatorname{gph} N_C}(\bar{x}, \bar{v})$. By polyhedrality of $Q$,

$$N_Q(\bar{q} + q') = N_Q(\bar{q}) \cap [q']^{\perp}.$$

By Corollary 3.4.9,

$$N_{\operatorname{gph} N_C}(x, -Mx - q) = \big\{ (F_1 - F_2)^* \times (F_1 - F_2) \mid x' \in F_2 \subset F_1 \subset [v']^{\perp}, \\ F_1, F_2 \in \mathcal{F}(K(\bar{x}, \bar{v}))\big\}$$

With $N_{\operatorname{gph} N_C}(x, v)$ specified in this way, (3.4.9) can be derived. Again, when $\operatorname{gph} N_C$ is regular at $(\bar{x}, \bar{v})$, the condition becomes necessary. □

By further simplifying the condition, we arrive at the following condition involving only the reference point.

**Theorem 3.4.11.** *For $(\bar{q}, \bar{x}) \in \operatorname{gph} S|_Q$ defined as (3.4.2), suppose the constraint qualification holds:*

$$u^* \in N_Q(\bar{q}),\ (M^* u^*, u^*) \in (F_1 - F_2)^* \times (F_1 - F_2) \Longrightarrow u^* = 0, \tag{3.4.10}$$

*where $F_1, F_2$ are closed faces with $F_2 \subset F_1 \in \mathcal{F}(K(\bar{x}, \bar{v}))$ and $\bar{v} = -M\bar{x} - \bar{q}$. If for such $F_1, F_2$,*

$$\forall (M^* u^*, u^*) \in (F_1 - F_2)^* \times (F_1 - F_2) \Longrightarrow u^* \in -N_Q(\bar{q}), \tag{3.4.11}$$

*then $S$ has Lipschitz-like property relative to $Q$ at $\bar{q}$ for $\bar{x}$. The condition becomes necessary when $\operatorname{gph} N_C$ is regular at $(\bar{x}, \bar{v})$.*

*Proof.* For this statement it remains to show that the constraint qualification and the sufficient condition holding at the reference point indicates the conditions for any $(q, x) \in \operatorname{gph} S|_Q$ sufficiently near to $(\bar{q}, \bar{x})$. Suppose (3.4.10) and (3.4.11) holds. Let

$q', x'$ denote $q - \bar{q}, x - \bar{x}$ respectively. Then $(q', x') \in T_{\operatorname{gph} S|_Q}(\bar{q}, \bar{x})$ is sufficiently near $(0,0)$. For $\tilde{F}_1, \tilde{F}_2 \in K(\bar{x}, \bar{v})$ with $x' \in \tilde{F}_2 \subset \tilde{F}_1 \subset [-Mx' - q']^{\perp}$,

$$tx' \in \tilde{F}_2 \subset \tilde{F}_1, \ \forall t \geq 0$$

by conic structure of $\tilde{F}_1, \tilde{F}_2$. Thus we have

$$[x'] \subset \tilde{F}_1 - \tilde{F}_2 \subset [-Mx' - q']^{\perp}$$

Note that $\left([-Mx' - q']^{\perp}\right)^* = [-Mx' - q']$ and $[x']^* = [x']^{\perp}$. Besides, $\tilde{F}_1 - \tilde{F}_2$ is still a convex polyhedral cone (see Lemma 3.4.4) and by polar relation we have

$$[-Mx' - q'] \subset \left(\tilde{F}_1 - \tilde{F}_2\right)^* \subset [x']^{\perp}.$$

Together that is

$$(M^* u^*, u^*) \in \left(\tilde{F}_1 - \tilde{F}_2\right)^* \times \left(\tilde{F}_1 - \tilde{F}_2\right) \subset [x']^{\perp} \times [-Mx' - q']^{\perp}.$$

From $M^* u^* \in (\tilde{F}_1 - \tilde{F}_2)^* \subset [x']^{\perp}$, we have $\langle u^*, Mx' \rangle = \langle M^* u^*, x' \rangle = 0$. From $u^* \in \tilde{F}_1 - \tilde{F}_2 \subset [-Mx' - q']^{\perp}$, $\langle u^*, -q' \rangle = \langle u^*, -Mx' - q' \rangle = 0$. Therefore, $u^* \in [q']^{\perp}$ and (3.4.9) holds automatically when (3.4.11) is satisfied. Moreover, for $u^* \in N_Q(\bar{q} + q') = N_Q(\bar{q}) \cap [q']^{\perp}$ and the choice of $\tilde{F}_1, \tilde{F}_2$ is also covered in $F_1, F_2$, the constraint qualification can be replaced by (3.4.10). □

Consider the polar relation in the expressions (3.4.7) and (3.4.11), we give the following lemma to further simplify the condition.

**Lemma 3.4.12** ([80, Corollary 16.3.2]). *Let $A$ be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. For any convex set $D$ in $\mathbb{R}^n$, one has*

$$(AD)^* = (A^*)^{-1} D^*. \tag{3.4.12}$$

Next we give another illustration of the generalized criterion:

**Theorem 3.4.13.** *For* $(\bar{q}, \bar{x}) \in \operatorname{gph} S|_Q$ *defined as (3.4.2), assume for all closed faces* $F_2 \subset F_1$ *in* $\mathcal{F}(K(\bar{x}, \bar{v}))$ *(where* $\bar{v} = -M\bar{x} - \bar{q}$*) the constraint qualification holds:*

$$N_Q(\bar{q}) \cap (F_1 - F_2) \cap (M(F_1 - F_2))^* = \{0\}. \tag{3.4.13}$$

*If*

$$(F_1 - F_2) \cap (M(F_1 - F_2))^* \subseteq -N_Q(\bar{q}), \tag{3.4.14}$$

*then* $S$ *has the Lipschitz-like property relative to* $Q$ *at* $\bar{q}$ *for* $\bar{x}$*. The condition becomes necessary when* $\operatorname{gph} N_C$ *is regular at* $(\bar{x}, \bar{v})$*.*

*Proof.* For any possible combinations of $F_1 - F_2$ with $F_2 \subset F_1 \in \mathcal{F}(K(\bar{x}, \bar{v}))$, $F_1, F_2$ are both convex cones and so is $F_1 - F_2$. By (3.4.12), $M^* u^* \in (F_1 - F_2)^*$ is equivalent to $u^* \in (M(F_1 - F_2))^*$. Then (3.4.13) and (3.4.14) can be derived from (3.4.10) and (3.4.11) respectively. $\square$

# Chapter 4

# Lipschitz-like Property for Linear Constraint Systems

In this chapter, we go back to the Lipschitz-like property and mainly focus on the linear constraint system with an explicit set constraint.

## 4.1 Linear constraint systems

### 4.1.1 Lipschitz-like property of linear constraint systems

In [38], they considered a linear constraint system under full perturbation:

$$S(A, b) = \{x \in \mathbb{R}^n \mid Ax + b \in K\} \tag{4.1.1}$$

where $K \subseteq \mathbb{R}^m$ is a closed set, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Here we present their result for reference:

**Lemma 4.1.1** ([38, Theorem 3.3])**.** *For the mapping $S$ defined in (4.1.1) and $(\bar{A}, \bar{b}, \bar{x}) \in \operatorname{gph} S$,*

$$D^*S\left((\bar{A}, \bar{b}) \mid \bar{x}\right)(x^*) = \left\{ \left(\left\{(v_i^*\bar{x}_j)_{i,j}\right\}, v^*\right) \mid \ v^* \in N_K(\bar{A}\bar{x} + \bar{b}),\ \textit{with } x^* = -\bar{A}^*v^* \right\}. \tag{4.1.2}$$

*Here $\{(v_i^*\bar{x}_j)_{i,j}\}$ stands for the matrix whose $(i, j)$-th entry is $(v_i^*\bar{x}_j)$. $S$ has the Lipschitz-like property at $(\bar{A}, \bar{b})$ for $\bar{x}$ if and only if*

$$\ker \bar{A}^* \cap N_K(\bar{A}\bar{x} + \bar{b}) = \{0\}. \tag{4.1.3}$$

From the representation of coderivative of $S$ (4.1.2), we can also have the expressions of coderivatives for several similar set mappings with different parameters: one with right-hand side perturbation and the other with left-hand side perturbation.

$$\begin{aligned} S'(b) =& \{x \in \mathbb{R}^n \mid \bar{A}x + b \in K\} \\ S''(A) =& \{x \in \mathbb{R}^n \mid Ax + \bar{b} \in K\}. \end{aligned} \tag{4.1.4}$$

Although a full perturbation is considered in [38], we will prove in the next theorem that from the perspective of the Lipschitz-like property, right-hand side perturbation is actually equivalent to full perturbation. And both of these two types of perturbation can indicate the Lipschitz-like property of the system under left-hand side perturbation only. Such result can be obtained both by definition and coderivatives. We use the latter method in the following proof.

**Theorem 4.1.2.** *For* $(\bar{A}, \bar{b}, \bar{x}) \in \operatorname{gph} S$ *and the following statements,*

**(a)** $\ker \bar{A}^* \cap N_K(\bar{A}\bar{x} + \bar{b}) = \{0\}$.

**(b)** *$S$ is Lipschitz-like at $(\bar{A}, \bar{b})$ for $\bar{x}$.*

**(c)** *$S'$ is Lipschitz-like at $\bar{b}$ for $\bar{x}$.*

**(d)** *$S''$ is Lipschitz-like at $\bar{A}$ for $\bar{x}$.*

*(a) $\Longleftrightarrow$ (b) $\Longleftrightarrow$ (c) $\Longrightarrow$ (d). If in addition $K$ is regular at $\bar{A}\bar{x} + \bar{b}$ and $\bar{x} \neq 0$, (d) $\Longrightarrow$ (a) and all the statements are equivalent.*

*Proof.* The first equivalence (a) $\Longleftrightarrow$ (b) comes from Lemma 4.1.1 . Similar to the proof of [38, Theorem 3.3], let $G(A, b, x) = -Ax - b$ and $M(x) = K$. We can see that $\nabla_b G(A, b, x) = -E$ has full rank $m$ and therefore we can write

$$D^*S'\left(\bar{b} \mid \bar{x}\right)(x^*) = \left\{v^* \middle|\ v^* \in N_K(\bar{A}\bar{x} + \bar{b}),\ \text{with } x^* = -\bar{A}^* v^*\right\}$$

directly according to (4.1.2) and Lemma 3.1.1. Thus the condition (4.1.3) is also a sufficient and necessary condition for the Lipschitz-like property of $S'$ at $\bar{b}$ for $\bar{x}$. Therefore we have the second equivalence.

Next we fix $b = \bar{b}$ in $G(A, b, x)$, by calculation in [38], $\nabla_A G(A, \bar{b}, x)^* v^* = \{-(v_i^* x_j)_{i,j}\}$. For $M(x) = K$ and any $v \in K$,

$$D^* M(x \mid v)(v^*) = \begin{cases} \{0\}, & \text{if } v^* \in -N_K(v) \\ \emptyset, & \text{if } v^* \notin -N_K(v) \end{cases}.$$

Together we have

$$\begin{aligned} &\left(\nabla_A G(\bar{A}, \bar{b}, \bar{x}), \nabla_x G(\bar{A}, \bar{b}, \bar{x})\right)^* v^* + D^* M(\bar{x} \mid A\bar{x} + \bar{b})(v^*) \\ &= \left\{ \left( -\left\{ (v_i^* \bar{x}_j)_{i,j} \right\}, -\bar{A}^* v^* \right) \;\middle|\; v^* \in -N_K(\bar{A}\bar{x} + \bar{b}) \right\} \\ &= \left\{ \left( \left\{ (v_i^* \bar{x}_j)_{i,j} \right\}, \bar{A}^* v^* \right) \;\middle|\; v^* \in N_K(\bar{A}\bar{x} + \bar{b}) \right\}. \end{aligned}$$

Then by Lemma 3.1.1, if the constraint qualification holds:

$$(0,0) \in \left\{ \left( \left\{ (v_i^* \bar{x}_j)_{i,j} \right\}, \bar{A}^* v^* \right) \;\middle|\; v^* \in N_K(\bar{A}\bar{x} + \bar{b}) \right\} \Longrightarrow v^* = 0, \tag{4.1.5}$$

then

$$D^* S''(\bar{A} \mid \bar{x})(x^*) \subseteq \left\{ \left\{ (v_i^* \bar{x}_j)_{i,j} \right\} \;\middle|\; v^* \in N_K(\bar{A}\bar{x} + \bar{b}), \text{ with } x^* = -\bar{A}^* v^* \right\}. \tag{4.1.6}$$

When the condition (4.1.3) holds, the constraint qualification (4.1.5) holds automatically and $D^* S''(\bar{A} \mid \bar{x})(0) \subseteq \{0\}$ as $v^* = 0$ indicates $(v_i^* \bar{x}_j)_{i,j} = 0, \ \forall i, j$. Therefore, the direction (a) $\Longrightarrow$ (d) is completed by the Mordukhovich criterion. When $\bar{x} \neq 0$, $(v_i^* \bar{x}_j)_{i,j} = 0$ for any $i, j$ is equivalent to $v^* = 0$ and therefore the constraint qualification (4.1.5) holds automatically. If in addition $K$ is regular at $\bar{A}\bar{x} + \bar{b}$, the inclusion (4.1.6) turns into an equation and by the Mordukhovich criterion, $S''$ is Lipschitz-like at $\bar{A}$ for $\bar{x}$ if and only if

$$\bar{A}^* v^* = 0, \ v^* \in N_K(\bar{A}\bar{x} + \bar{b}) \Longrightarrow (v_i^* \bar{x}_j)_{i,j} = 0, \forall i, j.$$

Therefore (d) $\Longrightarrow$ (a). □

Given the equivalence on the Lipschitz-like property of the linear constraint system under right-hand side perturbation and full perturbation introduced in Theorem 4.1.2, to better analyze the problem we next consider the one with right-hand side perturbation only, (4.1.4) with $\bar{A}$ given. We know that the Lipschitz-like property suggests implicitly that the referred parameter $\bar{b}$ should lie in the interior of the domain of $S'$. Therefore the criterion fails when $\bar{b}$ falls on the boundary of $\operatorname{dom} S'$. Next we give more illustrations of this criterion and further the Lipschitz-like stability of $S$ under different settings.

In this section we assume $K$ to be a convex set. From Proposition 3.2.2 in the last chapter, we notice that $N_{\operatorname{dom} S'}(\bar{b})$ is exactly the set employed in the criterion Theorem 4.1.2, (a). Then we can formulate the characterizations of Lipschitz-like property of $S'$ as follows.

**Theorem 4.1.3.** *For the set mapping $S'$ defined as in (4.1.4) with $K$ being convex and a pair $(\bar{b}, \bar{x}) \in \operatorname{gph} S'$, the followings are equivalent:*

**(a)** *$S'$ is Lipschitz-like at $\bar{b}$ for $\bar{x}$;*

**(b)** $\ker \bar{A}^* \cap N_K(\bar{v}) = \{0\}$*;*

**(c)** $N_{\operatorname{dom} S'}(\bar{b}) = \{0\}$*;*

**(d)** $T_{\operatorname{dom} S'}(\bar{b}) = \mathbb{R}^m$*;*

**(e)** $0 \in \operatorname{int}\big(T_K(\bar{v}) + \operatorname{rg} \bar{A}\big)$*;*

**(f)** $\bar{b} \in \operatorname{int} \operatorname{dom} S' = \operatorname{int}\big(K + \operatorname{rg} \bar{A}\big)$*, i.e., $\bar{b}$ is regular (see [76, Lemma 3]).*

*Proof.* The first equivalence (a) $\Longleftrightarrow$ (b) comes from the criterion in [38, Theorem 3.3] and subsequently the characterization (c) from (3.2.6) and (d) from polar relations between tangent cones and normal cones. (e) is an equivalent description of

$T_{\mathrm{dom}\, S'}(\bar{b}) = \mathbb{R}^m$ and the last one (f) is equivalent to (c) by characterization of interior points via normal cones. $\square$

**Remark 4.1.4.** *One shall compare the conditions here with [38, Theorem 4.1], where $K$ is assumed to be a closed convex cone. In their condition $(f)$: $\mathrm{rg}\, \bar{A} + cone(K - \bar{v}) = \mathbb{R}^m$ is a particular expression of condition (d): $T_{\mathrm{dom}\, S'}(\bar{b}) = \mathbb{R}^m$ here. A similar statement of (f) is also given in [64, Corollary 4.2] which requires $0 \in \mathrm{int\, dom}\, S'$ but $K$ being a multifunction with convex graph.*

**Remark 4.1.5.** *In [76] with $K$ being a convex polyhedral cone, Robinson shows that when the solution set $S'(b)$ is bounded, it has the upper Lipschitz continuity, i.e., calmness, involving both left-hand side and right-hand side perturbation. See [76, Lemma 2, Lemma 3]. Later in [78], Robinson gave the result that any polyhedral multifunction is locally upper Lipschitzian.*

### 4.1.2 Linear constraint system with a set constraint

Next we consider an extension of (4.1.1): adding a set constraint to the linear constraint system: $x \in X \subseteq \mathbb{R}^n$. The model becomes:

$$S(A, b) = \{x \in X \mid Ax + b \in K\} \tag{4.1.7}$$

where $K \subseteq \mathbb{R}^m$, $X \in \mathbb{R}^n$ are two closed sets, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Similar to Lemma 4.1.1, we first give the expression of the coderivative $D^*S$ and the sufficient and necessary condition for the Lipschitz-like property of $S$.

**Theorem 4.1.6.** *For the solution mapping $S$ of the linear constraint system introduced in (4.1.7) and the triplet $(\bar{A}, \bar{b}, \bar{x}) \in \mathrm{gph}\, S$,*

$$\begin{aligned} D^*S\left((\bar{A}, \bar{b}) \;\middle|\; \bar{x}\right)(x^*) = \big\{ ((v_i^* \bar{x}_j)_{i,j}, v^*) \mid -x^* = \bar{A}^* v^* + w^*, \\ v^* \in N_K(\bar{A}\bar{x} + \bar{b}),\ w^* \in N_X(\bar{x}) \big\}. \end{aligned} \tag{4.1.8}$$

*The mapping $S$ is Lipschitz-like at $\left(\bar{A},\bar{b}\right)$ for $\bar{x}$ if and only if*

$$-\,(\bar{A}^*)^{-1}N_X(\bar{x})\cap N_K(\bar{A}\bar{x}+\bar{b})=\{0\}. \tag{4.1.9}$$

*Proof.* Let $G(A,b,x)=-Ax-b$ and $M(x)=\begin{cases}K, & \text{if } x\in X\\ \emptyset, & \text{if } x\notin X\end{cases}$. Then for any $(x,v)\in$ gph $M$,

$$D^*M(x\mid v)(v')=\begin{cases}N_X(x), & \text{if } v'\in -N_K(v),\\ \emptyset, & \text{if } v'\notin -N_K(v).\end{cases}$$

As $\nabla_b G(A,b,x)=-E$ has full rank $m$ we can directly apply Lemma 3.1.1 with case (b) to obtain the equation for coderivative of $S$. Note that

$$\nabla G(\bar{A},\bar{b},\bar{x})^*(v')=\left(-(v_i^*\bar{x}_j)_{i,j},-v^*,-\bar{A}^*v^*\right).$$

Therefore we have

$$\begin{aligned}D^*S\left((\bar{A},\bar{b})\;\middle|\;\bar{x}\right)(x^*)=\big\{\,(-(v_i^*\bar{x}_j)_{i,j},-v^*)\mid -x^*=-\bar{A}^*v^*+w^*,\\ v^*\in -N_K(\bar{A}\bar{x}+\bar{b}),\; w^*\in N_X(\bar{x})\big\}\end{aligned}$$

and accordingly (4.1.8) by changing the direction of $v^*$. By the Mordukhovich criterion, when $x^*=-\bar{A}^*v^*-w^*=0$, i.e., $-\bar{A}^*v^*\in N_X(\bar{x})$, it is required $v^*=0$ and $(v_i^*\bar{x}_j)_{i,j}=0$ for any $i,j$. Therefore the sufficient and necessary condition (4.1.9) can be given. □

For this linear constraint system, we will next show that the Lipschitz-like property under right-hand side perturbation is also equivalent to the one under full perturbation. Besides, there are other characterizations using different tools. Before that, we give some results on error bounds.

Let $\bar{x}\in X,\bar{A}\in\mathbb{R}^{m\times n},\bar{b}\in\mathbb{R}^m$ and $\bar{v}:=\bar{A}\bar{x}+\bar{b}\in K$. Define the following mapping

$$g(x,A,b):=d(x,X)+d(Ax+b,K). \tag{4.1.10}$$

Then we can write (4.1.7) as

$$S(A,b) = \{x \in \mathbb{R}^n \mid g(x,A,b) = 0\}.$$

**Lemma 4.1.7.** *If the following condition holds*

$$0 \notin [\bar{A}^*(N_K(\bar{v}) \cap \mathbb{S}) + N_X(\bar{x}) \cap \mathbb{B}] \bigcup [\bar{A}^*(N_K(\bar{v}) \cap \mathbb{B}) + N_X(\bar{x}) \cap \mathbb{S}], \tag{4.1.11}$$

*then there exist some constants $\tau > 0, \delta > 0$ such that*

$$\tau d(x, S(A,b)) \leq g(x,A,b)$$

*for all $x \in \mathbb{B}_\delta(\bar{x})$ and $(A,b) \in \mathbb{B}_\delta(\bar{A},\bar{b})$.*

*Proof.* By (4.1.11), there exist constants $\delta > 0$ and $\tau > 0$ such that

$$d(0, A^*(N_K(v) \cap \mathbb{S}) + N_X(\bar{x}) \cap \mathbb{B}) > \tau \tag{4.1.12}$$

and

$$d(0, A^*(N_K(v) \cap \mathbb{B}) + N_X(\bar{x}) \cap \mathbb{S}) > \tau \tag{4.1.13}$$

for $(A,b) \in \mathbb{B}_\delta(\bar{A},\bar{b})$ and $v = A\bar{x} + b$. Consider the outer subdifferential of $g(\cdot, A, b)$ (see [43] for reference) for any $(A,b) \in \mathbb{B}_\delta(\bar{A},\bar{b})$,

$$\partial_x^{>} g(\bar{x},A,b) = \limsup_{\substack{x \xrightarrow[g(x,A,b)>g(\bar{x},A,b)]{g(\cdot,A,b)} \bar{x}}} \partial_x g(x,A,b). \tag{4.1.14}$$

From [81, Example 8.53, Exercise 10.10] we know that $\partial^\infty g(x,A,b) = \partial^\infty d(x,X) = \{0\}$. Let $F(x,A,b) := Ax + b$ and $v$ be the value $v = Ax + b$. Together with [81, Corollary 10.11],

$$\partial_x g(x,A,b) \subseteq \{x^* \mid \exists v^* \text{ s.t. } (x^*,v^*) \in \partial g(x,A,b)\} \subseteq \partial_x d(F(x,A,b),K) + \partial d(x,X). \tag{4.1.15}$$

Again, as $\partial^{\infty} d(v, K) = \{0\}$, we have by [81, Theorem 10.6],

$$\partial_x d(F(x, A, b), K) \subseteq \nabla_x F(x, A, b)^* \partial d(v, K). \tag{4.1.16}$$

Given $g(x, A, b) > g(\bar{x}, A, b)$, we have $v \notin K$ and/or $x \notin X$. In this case, by [81, Example 8.53] it is either

$$\partial d(v, K) = N_K(v) \cap \mathbb{B},\ \partial d(x, X) = N_X(x) \cap \mathbb{S} \tag{4.1.17}$$

or

$$\partial d(v, K) = N_K(v) \cap \mathbb{S},\ \partial d(x, X) = N_X(x) \cap \mathbb{B}. \tag{4.1.18}$$

Note that $\partial d(v, K) = N_K(v) \cap \mathbb{S}$, $\partial d(x, X) = N_X(x) \cap \mathbb{S}$ is already included in both equations. Combining (4.1.14) - (4.1.18), we arrive at

$$\partial_x^{>} g(\bar{x}, A, b) \subset [A^*(N_K(v) \cap \mathbb{S}) + N_X(\bar{x}) \cap \mathbb{B}] \bigcup [A^*(N_K(v) \cap \mathbb{B}) + N_X(\bar{x}) \cap \mathbb{S}]$$

for any $(A, b) \in \mathbb{B}_\delta(\bar{A}, \bar{b})$. This combining (4.1.12), (4.1.13) and [43, Theorem 2.1] ((d) $\Rightarrow$ (a)) yields the desired result. $\square$

**Lemma 4.1.8.** *If there exist some constants $\tau > 0, \delta > 0$ such that*

$$\tau d(x, S(A, b)) \leq g(x, A, b) \tag{4.1.19}$$

*for all $x \in N_\delta(\bar{x})$ and $(A, b) \in N_\delta(\bar{A}, \bar{b})$, then $S$ is Lipschitz-like at $(\bar{A}, \bar{b})$ for $\bar{x}$.*

*Proof.* It is easy to verify that

$$|g(x, A, b) - g(x, A', b')| \leq (||\bar{x}|| + \delta)||A - A'|| + ||b - b'|| \tag{4.1.20}$$

for all $x \in N_\delta(\bar{x})$ and $(A, b) \in N_\delta(\bar{A}, \bar{b})$. Suppose that $S$ does not enjoy the Lipschitz-like property at $(\bar{A}, \bar{b})$ for $\bar{x}$. Then for any $\kappa > 0$, there exist sequences $(A_k, b_k), (A'_k, b'_k) \to (\bar{A}, \bar{b})$ and $\{x_k\} \subset S(A_k, b_k)$ with $x_k \to \bar{x}$ such that

$$d(x_k, S(A'_k, b'_k)) > \kappa(||A_k - A'_k|| + ||b_k - b'_k||).$$

This together with (4.1.19) and (4.1.20) yields a contradiction. $\square$

To show that the Lipschitz-like property of the linear constraint system (4.1.7) under full perturbation is equivalent to that under right-hand side perturbation, we introduce the set-valued mapping similar to (4.1.7) but with $A = \bar{A}$ fixed.

$$S'(b) = \left\{x \in X \mid \bar{A}x + b \ \in K\right\}. \tag{4.1.21}$$

**Theorem 4.1.9.** *For the set-valued mappings $S$, $S'$ defined in (4.1.7) and (4.1.21) respectively, and the function $g(x, A, b)$ defined in (4.1.10), let $(\bar{A}, \bar{b}, \bar{x}) \in \operatorname{gph} S$. Then we also have $(\bar{b}, \bar{x}) \in \operatorname{gph} S'$ and the following statements are equivalent:*

**(a)** *$S$ is Lipschitz-like at $(\bar{A}, \bar{b})$ for $\bar{x}$.*

**(b)** *$S'$ is Lipschitz-like at $\bar{b}$ for $\bar{x}$.*

**(c)** $-(\bar{A}^*)^{-1}N_X(\bar{x}) \cap N_K(\bar{A}\bar{x} + \bar{b}) = \{0\}$.

**(d)** $0 \notin [\bar{A}^*(N_K(\bar{v}) \cap \mathbb{S}) + N_X(\bar{x}) \cap \mathbb{B}] \bigcup [\bar{A}^*(N_K(\bar{v}) \cap \mathbb{B}) + N_X(\bar{x}) \cap \mathbb{S}]$.

**(e)** *there exist some constants $\tau > 0, \delta > 0$ such that*

$$\tau d(x, S(A, b)) \leq g(x, A, b)$$

*for all $x \in \mathbb{B}_\delta(\bar{x})$ and $(A, b) \in \mathbb{B}_\delta(\bar{A}, \bar{b})$.*

*Proof.* The equivalence between (a) and (c) has already been established in Theorem 4.1.6. The equation of $D^*S$ (4.1.8) is obtained via Lemma 3.1.1 with the full rank property of $\nabla_b G(A, b, x)$. Therefore, similar steps can be performed on $S'$ to obtain

$$D^*S'\left(\bar{b} \mid \bar{x}\right)(x^*) = \left\{v^* \ \middle| -x^* = \bar{A}^*v^* + w^*,\ v^* \in N_K(\bar{A}\bar{x} + \bar{b}),\ w^* \in N_X(\bar{x})\right\}.$$

Then the Mordukhovich criterion on $S'$ finally turns into (c) and therefore the equivalence between (b) and (c) is established. As (d) $\Rightarrow$ (e) and (e) $\Rightarrow$ (a) have been proved in previous lemmas, it remains to prove (c) $\Rightarrow$ (d). Suppose $0 \in [\bar{A}^*(N_K(\bar{v}) \cap \mathbb{S}) + N_X(\bar{x}) \cap \mathbb{B}] \bigcup [\bar{A}^*(N_K(\bar{v}) \cap \mathbb{B}) + N_X(\bar{x}) \cap \mathbb{S}]$. That means there exists

$v^* \in N_K(\bar{v})$, $w^* \in N_X(\bar{x})$ such that $\bar{A}^* v^* + w^* = 0$ with either $\|v^*\| = 1, \|w^*\| \leq 1$ or $\|v^*\| \leq 1, \|w^*\| = 1$. Both of these two cases contradict (c) and therefore the proof is completed. $\square$

**Remark 4.1.10.** *The sufficient and necessary condition (c) can also be taken as a linearized version of [81, Example 9.51] but improved on necessity without regularity presented.*

## 4.2 Linearization of nonlinear variational inequalities

In this section, we discuss a set-valued mapping rising from the optimality condition of the problem. Consider the following parametric optimization problem in which the parameters are $z, w, A, b$.

$$\begin{aligned} \min_{x} \quad & F(w, x) + z^\top x \\ \text{s.t.} \quad & Ax - b \in C \\ & x \in \mathbb{R}^n, A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, z \in \mathbb{R}^n \end{aligned} \tag{4.2.1}$$

where $F : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}$ is differentiable with respect to $x$ with $\nabla_x F(w, x) = f(w, x)$ and $C \subseteq \mathbb{R}^m$ is a convex polyhedral set. If the matrix $A$ satisfies the constraint qualification

$$A^* y = 0, y \in N_C(Ax - b) \Longrightarrow y = 0 \tag{4.2.2}$$

then the stationary point set-mapping of this problem under perturbation on parameters $(A, b, z, w)$ can be expressed as

$$S(A, b, z, w) = \Big\{ x \in \mathbb{R}^n \mid 0 \in z + f(w, x) + A^* N_{\mathbb{R}^m_-}(Ax - b) \Big\}. \tag{4.2.3}$$

For the coming content, we analyze how this set-valued mapping can be approximated when $A$ is a square matrix with full rank and the equivalence in terms of the Lipschitz-like property. The main task is to establish the equivalence of the Lipschitz-like property between (4.2.3) and the set-valued mapping below for a given point $(\bar{A}, \bar{b}, \bar{z}, \bar{w}, \bar{x}) \in \operatorname{gph} S$ when $\bar{A}$ is a square matrix with full rank:

$$L(A, b, q) = \{x \mid 0 \in q + Qx + N_C(Ax - b)\} \tag{4.2.4}$$

where $Q = (\bar{A}^*)^{-1}\nabla_x f(\bar{w}, \bar{x})$, $\bar{q} = (\bar{A}^*)^{-1}\bar{z} + (\bar{A}^*)^{-1} f(\bar{w}, \bar{x}) - (\bar{A}^*)^{-1}\nabla_x f(\bar{w}, \bar{x})\bar{x}$. Here we introduce another set-valued mapping for the purpose of bridging in the condition that $(A^*)^{-1}$ exists:

$$S'(A, b, z, w) = \{x \mid 0 \in (A^*)^{-1}z + (A^*)^{-1}f(w, x) + N_C(Ax - b)\}. \tag{4.2.5}$$

Before introducing the equivalence, we give some illustrations on the properties of the inverse of a matrix and $f$. Throughout this section, we use $\|\cdot\|$ to denote any matrix norm that satisfies $\|A^*\| = \|A\|$ with $A$ being a square matrix and $\|I\| = 1$.

**Proposition 4.2.1.** *Suppose $A$ is a nonsingular square matrix, and $A_\varepsilon = A + \Delta A$. If $\|A_\varepsilon - A\| < \|A^{-1}\|^{-1}$, then $A_\varepsilon$ is nonsingular.*

*Proof.* $A_\varepsilon = A(I + A^{-1}\Delta A)$. Given $A$ is nonsingular, $A_\varepsilon$ is nonsingular if and only if $I + A^{-1}\Delta A$ is nonsingular. By [36, Observation 1.1.7], this is equivalent to $0 \notin \sigma(I + A^{-1}\Delta A)$, where $\sigma(A)$ denotes the set of all eigenvalues of the matrix $A$. The condition is then passed with equivalence to $-1 \notin \sigma(A^{-1}\Delta A)$ by [36, Observation 1.1.8]. It is known that $\rho(A^{-1}\Delta A) \le \|A^{-1}\Delta A\| \le \|A^{-1}\|\|\Delta A\|$ with $\rho(A)$ denoting the spectral radius of $A$, i.e., the largest absolute value of all possible eigenvalues. $\|\Delta A\| = \|A_\varepsilon - A\| < \|A^{-1}\|^{-1}$ ensures $\rho(A^{-1}\Delta A) < 1$ and therefore $A_\varepsilon$ is nonsingular. $\square$

**Lemma 4.2.2.** *For a nonsingular square matrix $\bar{A}$, we denote $d := \dfrac{\|\bar{A}^{-1}\|}{1 - a\|\bar{A}^{-1}\|}$. Then for any $A', A'' \in \mathbb{B}_a(\bar{A})$ with $a < \|\bar{A}^{-1}\|^{-1}$ we have the following properties:*

**(a)** $\|(A')^{-1}\| \le d$.

**(b)** $\|(A'^*)^{-1} - (\bar{A}^*)^{-1}\| \leq ad\|\bar{A}^{-1}\|$.

**(c)** $\|(A'^*)^{-1} - (A''^*)^{-1}\| \leq d^2\|A' - A''\|$.

*Proof.* The first property comes directly from

$$\|(A')^{-1}\| \leq \frac{\|\bar{A}^{-1}\|}{1 - \|\bar{A}^{-1}(A' - \bar{A})\|} \leq \frac{\|\bar{A}^{-1}\|}{1 - a\|\bar{A}^{-1}\|} = d.$$

For the second,

$$\begin{aligned} \|(A'^*)^{-1} - (\bar{A}^*)^{-1}\| &= \|(A'^*)^{-1}(\bar{A}^* - A'^*)(\bar{A}^*)^{-1}\| \\ &\leq \|(\bar{A})^{-1}\| \; \|(A')^{-1}\| \; \|A' - \bar{A}\| \leq ad\|\bar{A}^{-1}\|. \end{aligned}$$

For the third,

$$\begin{aligned} \|(A'^*)^{-1} - (A''^*)^{-1}\| &= \|(A'^*)^{-1}(A''^* - A'^*)(A''^*)^{-1}\| \\ &\leq \|(A'^*)^{-1}\| \; \|A''^* - A'^*\| \; \|(A''^*)^{-1}\| \leq \; d^2\|A' - A''\|. \end{aligned}$$

□

For the coming proof in this section, we continue to use $d$ as

$$d := \frac{\|\bar{A}^{-1}\|}{1 - a\|\bar{A}^{-1}\|} \tag{4.2.6}$$

Similar to [18], we have the following assumptions on $f$:

(A) $f$ is differentiable with respect to $x$ with Jacobian matrix $\nabla_x f(w, x)$ depending continuously on $(w, x)$ in a neighborhood of $(\bar{w}, \bar{x})$;

(B) $f$ is Lipschitz continuous in $w$ uniformly in $x$ around $(\bar{w}, \bar{x})$; that is, there exist neighborhoods $U$ of $\bar{x}$ and $V$ of $\bar{w}$ and a number $l > 0$ such that for all $x \in U$ and $w_1, w_2 \in V$:

$$\|f(w_1, x) - f(w_2, x)\| \leq l\|w_1 - w_2\|. \tag{4.2.7}$$

**Lemma 4.2.3** ([18, Strict Differentiability Lemma]). *Under assumption (A), for any $\varepsilon > 0$, there exist neighborhoods $U$ of $\bar{x}$ and $V$ of $\bar{w}$ such that for all $x_1, x_2 \in U$ and $w \in V$,*

$$\|f(w, x_1) - f(w, x_2) - \nabla_x f(\bar{w}, \bar{x})(x_1 - x_2)\| \le \varepsilon \|x_1 - x_2\|. \tag{4.2.8}$$

*Readers are recommended to refer to [18] for the proof.*

Let $f_1(A, w, x) := (A^*)^{-1} f(w, x)$. Then we have $\nabla_x f_1(A, w, x) = (A^*)^{-1} \nabla_x f(w, x)$. Next we will prove that the Strict Differentiability Lemma holds as well with $f_1$:

**Proposition 4.2.4.** *Under assumption (A) for $f$, for any $\varepsilon' > 0$ there exist neighborhoods $U$ of $\bar{x}$, $V$ of $\bar{w}$ and $W$ of $\bar{A}$, such that for all $x_1, x_2 \in U$ and $(A, w) \in W \times V$,*

$$\|f_1(A, w, x_1) - f_1(A, w, x_2) - \nabla_x f_1(\bar{A}, \bar{w}, \bar{x})(x_1 - x_2)\| \le \varepsilon' \|x_1 - x_2\|. \tag{4.2.9}$$

*Proof.* Choose $0 < a < \|\bar{A}^{-1}\|^{-1}$ to ensure existence of $(A^*)^{-1}$ for all $A \in \mathbb{B}_a(\bar{A})$, and $b, t > 0$ as in Strict Differentiability Lemma for $f(w, x)$ with $w \in \mathbb{B}_b(\bar{w})$, $x_1, x_2 \in \mathbb{B}_t(\bar{x})$ and $\varepsilon = \dfrac{\varepsilon'}{2d}$ with $d$ defined as in (4.2.6) and $\varepsilon$ as in (4.2.8). Without loss of generality, suppose $\varepsilon' < 1$. Take

$$a < \min\left\{\frac{1}{\|\bar{A}^{-1}\|}, \frac{\varepsilon'}{(2c\|\bar{A}^{-1}\| + 1)\|\bar{A}^{-1}\|}\right\} \tag{4.2.10}$$

where $c := \|\nabla_x f(\bar{w}, \bar{x})\|$. First we have

$$\|(A^*)^{-1}\| \|f(w, x_1) - f(w, x_2) - \nabla_x f(\bar{w}, \bar{x})(x_1 - x_2)\| \le \frac{\varepsilon'}{2} \|x_1 - x_2\| \tag{4.2.11}$$

as $\|(A^*)^{-1}\| \le d$ by Lemma 4.2.2 (a) and $\|f(w, x_1) - f(w, x_2) - \nabla_x f(\bar{w}, \bar{x})(x_1 - x_2)\| \le \varepsilon \|x_1 - x_2\|$ by (4.2.8). Besides, by Lemma 4.2.2 (b) and (4.2.10)

$$\|(A^*)^{-1} - (\bar{A}^*)^{-1}\| \|\nabla_x f(\bar{w}, \bar{x})\| \le \frac{ac\|\bar{A}^{-1}\|^2}{1 - a\|\bar{A}^{-1}\|} < \frac{\varepsilon' c \|\bar{A}^{-1}\|}{2c\|\bar{A}^{-1}\| + 1 - \varepsilon'} < \frac{\varepsilon'}{2}. \tag{4.2.12}$$

Combining (4.2.11) and (4.2.12) we can derive

$$\begin{aligned}
&\|f_1(A,w,x_1) - f_1(A,w,x_2) - \nabla_x f_1(\bar{A},\bar{w},\bar{x})(x_1-x_2)\| \\
&= \|(A^*)^{-1}f(w,x_1) - (A^*)^{-1}f(w,x_2) - (\bar{A}^*)^{-1}\nabla_x f(\bar{w},\bar{x})(x_1-x_2)\| \\
&\le \|(A^*)^{-1}\|\ \|f(w,x_1) - f(w,x_2) - \nabla_x f(\bar{w},\bar{x})(x_1-x_2)\| \\
&\qquad\qquad + \|(A^*)^{-1} - (\bar{A}^*)^{-1}\|\ \|\nabla_x f(\bar{w},\bar{x})\|\ \|x_1-x_2\| \\
&\le \varepsilon'\|x_1-x_2\|.
\end{aligned}$$

□

**Proposition 4.2.5.** *Under assumptions (A) and (B) of $f$, $f_1$ is Lipschitz continuous in $(A,w)$ uniformly in $x$ around $(\bar{w},\bar{x})$, i.e. there exist neighborhoods $U$ of $\bar{x}$, $V$ of $\bar{w}$ and $W$ of $\bar{A}$ and a number $l'>0$ such that for all $x\in U$, $(A_1,w_1),(A_2,w_2)\in W\times V$,*

$$\|f_1(A_1,w_1,x) - f_1(A_2,w_2,x)\| \le l'(\|A_1-A_2\| + \|w_1-w_2\|). \tag{4.2.13}$$

*Proof.* Take $0 < a < \|\bar{A}^{-1}\|^{-1}$ and $w_1,w_2\in\mathbb{B}_r(\bar{w}), x\in\mathbb{B}_t(\bar{x})$ with $r,t>0$ chosen as in Lipschitz continuity of $f$ illustrated as (4.2.7) with constant $l>0$ and (4.2.8) satisfied for a certain $\varepsilon>0$. Consider any $A_1,A_2\in\mathbb{B}_a(\bar{A})$,

$$\begin{aligned}
&\|f_1(A_1,w_1,x) - f_1(A_2,w_2,x)\| = \|(A_1^*)^{-1}f(w_1,x) - (A_2^*)^{-1}f(w_2,x)\| \\
&\le \|(A_1^*)^{-1} - (A_2^*)^{-1}\|\ \|f(w_1,x)\| + \|(A_2^*)^{-1}\|\ \|f(w_1,x) - f(w_2,x)\|.
\end{aligned} \tag{4.2.14}$$

Given

$$\begin{aligned}
&\|f(w_1,x)\| \le \|f(w_1,x) - f(\bar{w},x)\| + \|f(\bar{w},x)\| \\
&\le lr + \|f(\bar{w},x) - f(\bar{w},\bar{x}) - \nabla_x f(\bar{w},\bar{x})(x-\bar{x})\| + \|f(\bar{w},\bar{x})\| + \|\nabla_x f(\bar{w},\bar{x})\|\ \|x-\bar{x}\| \\
&\le lr + (\varepsilon + \|\nabla_x f(\bar{w},\bar{x})\|)t + \|f(\bar{w},\bar{x})\| =: l_1,
\end{aligned}$$

we can say that $\|f(w_1,x)\|$ is bounded by $l_1$. Besides, $\|(A_1^*)^{-1} - (A_2^*)^{-1}\| \le d^2\|A_1 - A_2\|$ by Lemma 4.2.2 (c). Then we have:

$$\|(A_1^*)^{-1} - (A_2^*)^{-1}\|\ \|f(w_1,x)\| \le l_1 d^2\|A_1-A_2\|. \tag{4.2.15}$$

By (4.2.7) and Lemma 4.2.2 (a), we have that

$$\|(A_2^*)^{-1}\| \, \|f(w_1,x)-f(w_2,x)\| \leq dl\|w_1-w_2\|. \tag{4.2.16}$$

Take $l' = \max\{l_1 d^2, dl\}$ and combine (4.2.14), (4.2.15) and (4.2.16), we can see that (4.2.13) is proved. □

**Theorem 4.2.6.** *The following are equivalent for the mappings $S, S', L$:*

*(i) $S$ is Lipschitz-like at $(\bar{A}, \bar{b}, \bar{z}, \bar{w}, \bar{x})$;*

*(ii) $S'$ is Lipschitz-like at $(\bar{A}, \bar{b}, \bar{z}, \bar{w}, \bar{x})$;*

*(iii) $L$ is Lipschitz-like at $(\bar{A}, \bar{b}, \bar{q}, \bar{x})$ with*

$$\bar{q} = (\bar{A}^*)^{-1}\bar{z} + (\bar{A}^*)^{-1} f(\bar{w}, \bar{x}) - (\bar{A}^*)^{-1}\nabla_x f(\bar{w}, \bar{x})\bar{x}.$$

*Proof.* The equivalence (i) ⇔ (ii) can be guaranteed by choosing a neighborhood of $\bar{A}$ with radius $a < \|\bar{A}^{-1}\|^{-1}$, where the existence of $(A^*)^{-1}$ of any $B \in \mathbb{B}_a(\bar{A})$ is ensured. To prove (ii) ⇔ (iii), we first start with (iii) ⇒ (ii).

Let $L$ have the Lipschitz-like property at $(\bar{A}, \bar{b}, \bar{q}, \bar{x})$ with a constant $M$; that is, for some $r_1, r_2, r_3 > 0$ and $t > 0$ with $r_1 \leq a$, and for every $(A', b', q'), (A'', b'', q'') \in \mathbb{B}_{r_1}(\bar{A}) \times \mathbb{B}_{r_2}(\bar{b}) \times \mathbb{B}_{r_3}(\bar{q})$ we have

$$L(A', b', q') \cap \mathbb{B}_t(\bar{x}) \subset L(A'', b'', q'') + M(\|A' - A''\| + \|b' - b''\| + \|q' - q''\|)\mathbb{B}. \tag{4.2.17}$$

Let $\varepsilon' > 0$ be such that $M\varepsilon' < 1$ as specified in the Strict Differentiability Lemma. Choose $\alpha > 0$, $r_1' > 0$ and $r_5' > 0$ as radius for $x, B, w$ respectively, with

$$r_1' < \min\left\{a, \frac{r_3}{4(l' + d^2\|\bar{z}\|)}\right\}, \quad \alpha \leq \min\left\{t, \frac{r_3}{4\varepsilon'}\right\}$$

such that Strict Differentiability Lemma and Lipschitz continuity both hold for $f_1$. Let $r_1'', r_2', r_4, r_5 > 0$ be such that

$$r_1'' \leq \min\left\{r_1', \frac{\alpha(1-\varepsilon' M)}{16M[1 + l' + (r_4 + \|\bar{z}\|)d^2]}\right\}, \quad r_2' \leq \min\left\{r_2, \frac{\alpha(1-\varepsilon' M)}{16M}\right\},$$

$$r_4 \le \min\left\{\frac{r_3}{4d}, \frac{\alpha(1-\varepsilon' M)}{16Md}\right\}, \quad \text{and } r_5 \le \min\left\{r_5', \frac{r_3}{4l'}, \frac{\alpha(1-\varepsilon' M)}{16Ml'}\right\}.$$

Here $d = \dfrac{\|(\bar{A})^{-1}\|}{1 - a\|(\bar{A})^{-1}\|}$ and $l' > 0$ is the Lipschitz constant for $f_1$. It will be demonstrated that $S_2$ has the Lipschitz-like property at $(\bar{A}, \bar{b}, \bar{z}, \bar{w}, \bar{x})$ with constant

$$M' = \frac{M}{1-\varepsilon' M} \cdot \ \max\left\{1, d, l', 1 + l' + (r_4 + \|\bar{z}\|)d^2\right\}.$$

Fix $(A', b', z', w'), (A'', b'', z'', w'') \in \mathbb{B}_{r_1''}(\bar{A}) \times \mathbb{B}_{r_2'}(\bar{b}) \times \mathbb{B}_{r_4}(\bar{z}) \times \mathbb{B}_{r_5}(\bar{w})$, and consider any $x' \in S'(A', b', z', w') \cap \mathbb{B}_{\alpha/2}(\bar{x})$. Then

$$\begin{aligned}
0 &\in (A'^*)^{-1}z' + (A'^*)^{-1}f(w', x') + N_C(A'x' - b') \\
&= (A'^*)^{-1}z' + (A'^*)^{-1}f(w', x') - (\bar{A}^*)^{-1}\nabla_x f(\bar{w}, \bar{x})x' + Ax' + N_C(A'x' - b') \\
&= q' + Ax' + N_C(A'x' - b'),
\end{aligned}$$

Then $x' \in L(A', b', q')$ for $q' = (A'^*)^{-1}z' + (A'^*)^{-1}f(w', x') - (\bar{A}^*)^{-1}\nabla_x f(\bar{w}, \bar{x})x'$. We can write

$$q' - \bar{q} = (A'^*)^{-1}z' - (\bar{A}^*)^{-1}\bar{z} + (A'^*)^{-1}f(w', x') - (\bar{A}^*)^{-1}f(\bar{w}, \bar{x}) - (\bar{A}^*)^{-1}\nabla_x f(\bar{w}, \bar{x})(x' - \bar{x}).$$

Then

$$\begin{aligned}
\|q' - \bar{q}\| \le & \|(A'^*)^{-1}z' - (\bar{A}^*)^{-1}\bar{z}\| + \|(A'^*)^{-1}f(w', \bar{x}) - (\bar{A}^*)^{-1}f(\bar{w}, \bar{x})\| \\
& + \|(A'^*)^{-1}f(w', x') - (A'^*)^{-1}f(w', \bar{x}) - (\bar{A}^*)^{-1}\nabla_x f(\bar{w}, \bar{x})(x' - \bar{x})\|.
\end{aligned}$$

By Strict Differentiability Lemma stated before, $\|q' - \bar{q}\| \le \varepsilon' \|x' - \bar{x}\|$. By Lipschitz continuity of $f_1$,

$$\|(A'^*)^{-1}f(w', \bar{x}) - (\bar{A}^*)^{-1}f(\bar{w}, \bar{x})\| \le l'(\|A' - \bar{A}\| + \|w' - \bar{w}\|).$$

Moreover,

$$\|(A'^*)^{-1}z' - (\bar{A}^*)^{-1}\bar{z}\| \le \|(A'^*)^{-1}\| \ \|z' - \bar{z}\| + \|\bar{z}\| \ \|(A'^*)^{-1} - (\bar{A}^*)^{-1}\|$$

$$\leq d\|z' - \bar{z}\| + d^2\|\bar{z}\| \ \|A' - \bar{A}\|.$$

In all, we have

$$\|q' - \bar{q}\| \leq (d^2\|\bar{z}\| + l')\|A' - \bar{A}\| + d\|z' - \bar{z}\| + l'\|w' - \bar{w}\| + \varepsilon'\|x' - \bar{x}\|$$
$$\leq (d^2\|\bar{z}\| + l')r_1'' + dr_4 + l'r_5 + \varepsilon'\alpha/2 \leq \frac{r_3}{4} + \frac{r_3}{4} + \frac{r_3}{4} + \frac{r_3}{8} < r_3,$$

and therefore $q' \in \mathbb{B}_{r_3}(\bar{q})$. Analogously, for the vector

$$q'' = (A''^*)^{-1}z'' + (A''^*)^{-1}f(w'', x') - (\bar{A}^*)^{-1}\nabla_x f(\bar{w}, \bar{x})x',$$
$$q'' - \bar{q} = (A''^*)^{-1}z'' - (\bar{A}^*)^{-1}\bar{z} + (A''^*)^{-1}f(w'', x') - (\bar{A}^*)^{-1}f(\bar{w}, \bar{x})$$
$$- (\bar{A}^*)^{-1}\nabla_x f(\bar{w}, \bar{x})(x' - \bar{x}).$$

Therefore $q'' \in \mathbb{B}_{r_3}(\bar{q})$ as well.

Let $x_1 = x'$. By the Lipschitz-like property of $L$, there exists $x_2 \in L(A'', b'', q'')$ such that

$$0 \in (A''^*)^{-1}z'' + (A''^*)^{-1}f(w'', x_1) + (\bar{A}^*)^{-1}\nabla_x f(\bar{w}, \bar{x})(x_2 - x_1) + N_C(A''x_2 - b'')$$

and

$$\|x_2 - x_1\| \leq M(\|A' - A''\| + \|b' - b''\| + \|q' - q''\|).$$

As $q' - q'' = (A'^*)^{-1}z' - (A''^*)^{-1}z'' + (A'^*)^{-1}f(w', x_1) - (A''^*)^{-1}f(w'', x_1)$,

$$\|q' - q''\| \leq \|(A'^*)^{-1}\| \ \|z' - z''\| + \|z''\| \ \|(A'^*)^{-1} - (A''^*)^{-1}\|$$
$$+ l'(\|A' - A''\| + \|w' - w''\|)$$
$$\leq [(r_4 + \|\bar{z}\|)d^2 + l'] \ \|A' - A''\| + d\|z' - z''\| + l'\|w' - w''\|.$$

Then we have

$$\|x_2 - x_1\| \leq M\left([1 + l' + (r_4 + \|\bar{z}\|)d^2]\|A' - A''\| + \|b' - b''\| + d\|z' - z''\| + l'\|w' - w''\|\right). \tag{4.2.18}$$

By denoting the right-hand side part of (4.2.18) as $s$, we can have

$$s \le M \left\{2[1 + l' + (r_4 + \|\bar{z}\|)d^2]r_1'' + 2r_2' + 2dr_4 + 2l'r_5\right\} \le \frac{\alpha(1-\varepsilon' M)}{2}.$$

Suppose that there exist points $x_2, x_3, \ldots, x_{n-1}$ with

$$0 \in (A''^*)^{-1}z'' + (A''^*)^{-1}f(w'', x_{i-1}) + (\bar{A}^*)^{-1}\nabla_x f(\bar{w}, \bar{x})(x_i - x_{i-1}) + N_C(A''x_i - b'')$$

and

$$\|x_i - x_{i-1}\| \le (M\varepsilon')^{i-2}s \text{ for } i = 2, \ldots, n-1.$$

Then for every $i$ we have

$$\|x_i - \bar{x}\| \le \|x_1 - \bar{x}\| + \sum_{j=2}^{i} \|x_j - x_{j-1}\| \le \frac{\alpha}{2} + s\sum_{j=2}^{i}(M\varepsilon')^{j-2} \le \frac{\alpha}{2} + \frac{s}{1-\varepsilon' M} \le \alpha.$$

By setting

$$\begin{aligned} q_i &:= (A''^*)^{-1}z'' + (A''^*)^{-1}f(w'', x_i) - (\bar{A}^*)^{-1}\nabla_x f(\bar{w}, \bar{x})x_i \\ &= \bar{q} + (A''^*)^{-1}z'' - (\bar{A}^*)^{-1}\bar{z} - (\bar{A}^*)^{-1}f(\bar{w}, \bar{x}) + (A''^*)^{-1}f(w'', x_i) \\ &\qquad - (\bar{A}^*)^{-1}\nabla_x f(\bar{w}, \bar{x})(x_i - \bar{x}) \end{aligned}$$

for $i = 2, 3, \ldots, n-1$ we get

$$\begin{aligned} \|q_i - \bar{q}\| &\le \|(A''^*)^{-1}z'' - (\bar{A}^*)^{-1}\bar{z}\| + \|(A''^*)^{-1}f(w'', \bar{x}) - (\bar{A}^*)^{-1}f(\bar{w}, \bar{x})\| \\ &\qquad + \|(A''^*)^{-1}f(w'', x_i) - (A''^*)^{-1}f(w'', \bar{x}) - (\bar{A}^*)^{-1}\nabla_x f(\bar{w}, \bar{x})(x_i - \bar{x})\| \\ &\le (l' + d^2\|\bar{z}\|)\ \|A'' - \bar{A}\| + d\|z'' - \bar{z}\| + l'\|w'' - \bar{w}\| + \varepsilon'\|x_i - \bar{x}\| \\ &\le (l' + d^2\|\bar{z}\|)\ r_1'' + dr_4 + l'r_5 + \varepsilon'\alpha \le r_3, \end{aligned}$$

so that $q_i \in \mathbb{B}_{r_3}(\bar{q})$. As $x_{n-1} \in L(A'', b'', q_{n-2}) \cap \mathbb{B}_\alpha(\bar{x})$, by the Lipschitz-like property of $L$, there exists $x_n$ with

$$0 \in (A''^*)^{-1}z'' + (A''^*)^{-1}f(w'', x_{n-1}) + (\bar{A}^*)^{-1}\nabla_x f(\bar{w}, \bar{x})(x_n - x_{n-1}) + N_C(A''x_n - b'') \tag{4.2.19}$$

and

$$\begin{aligned}
&\|x_n - x_{n-1}\| \leq \ M\|q_{n-1} - q_{n-2}\| \\
&\leq M\|(A''^*)^{-1}f(w'', x_{n-1}) - (A''^*)^{-1}f(w'', x_{n-2}) - (\bar{A}^*)^{-1}\nabla_x f(\bar{w}, \bar{x})(x_{n-1} - x_{n-2})\| \\
&\leq M\varepsilon'\|x_{n-1} - x_{n-2}\| \leq (M\varepsilon')^{n-2}s.
\end{aligned}$$

The induction step is thereby joined. We obtain an infinite Cauchy sequence of points $x_1, x_2, \ldots, x_n, \ldots$ in $\mathbb{B}_\alpha(\bar{x})$ and therefore converges to some $x'' \in \mathbb{B}_\alpha(\bar{x})$. Since $(A''^*)^{-1}f(w'', \cdot)$ is continuous in $\mathbb{B}_\alpha(\bar{x})$ and the normal cone map $N_C(A''x - b'')$ has a closed graph, by (4.2.19), $x'' \in S'(A'', b'', z'', w'')$. Moreover, since

$$\begin{aligned}
\|x_n - x'\| &\leq \sum_{i=2}^{n} \|x_i - x_{i-1}\| \leq s\sum_{i=2}^{n}(M\varepsilon')^{i-2} \\
&\leq \frac{M}{1-\varepsilon' M}\big([1 + l' + (r_4 + \|\bar{z}\|)d^2]\|A' - A''\| + \|b' - b''\| \\
&\qquad\qquad + d\|z' - z''\| + l'\|w' - w''\|\big).
\end{aligned}$$

By passing to the limit, we have

$$\begin{aligned}
\|x'' - x'\| &\leq \frac{M}{1-\varepsilon' M}\big([1 + l' + (r_4 + \|\bar{z}\|)d^2]\ \|A' - A''\| + \|b' - b''\| \\
&\qquad\qquad + d\|z' - z''\| + l'\|w' - w''\|\big) \\
&\leq M'\left(\|A' - A''\| + \|b' - b''\| + \|z' - z''\| + \|w' - w''\|\right).
\end{aligned}$$

(iii) $\Rightarrow$ (ii) is established.

To prove the implication (ii) $\Rightarrow$ (iii), suppose $S'$ has the Lipschitz-like property at $(\bar{A}, \bar{b}, \bar{z}, \bar{w}, \bar{x})$ with constant $M$, i.e., for some $r_1, r_2, r_4, r_5, t > 0$, for every $(A', b', z', w'), (A'', b'', z'', w'') \in \mathbb{B}_{r_1}(\bar{A}) \times \mathbb{B}_{r_2}(\bar{b}) \times \mathbb{B}_{r_4}(\bar{z}) \times \mathbb{B}_{r_5}(\bar{w})$,

$$\begin{aligned}
S'(A', b', z', w') \cap \mathbb{B}_t(\bar{x}) \subset{}& S'(A'', b'', z'', w'') \\
&+ M(\|A' - A''\| + \|b' - b''\| + \|z' - z''\| + \|w' - w''\|)\mathbb{B}.
\end{aligned} \tag{4.2.20}$$

Let $\varepsilon'$ be such that $M\varepsilon' < 1$ and choose $\alpha', r_1', r_5' > 0$ as radius for $x, B, w$ respectively, with $\alpha' \le t,\ r_1' \le r_1$ as specified in the Strict Differentiability Lemma for $f_1$. Here we denote

$$u := \max\{1, \|\bar{A}\| + r_1'\},\ v := \|(\bar{A}^*)^{-1}\nabla_x f(\bar{w}, \bar{x})\|.$$

Let $r_1'', r_2', r_3, \alpha > 0$ be chosen as:

$$\alpha \le \min\left\{\alpha', \frac{r_4}{3\varepsilon' u}\right\},\ r_1'' \le \min\left\{r_1', \frac{r_4}{3(\|\bar{q}\| + v\|\bar{x}\|)}, \frac{\alpha(1-\varepsilon' M)}{12M[1 + \|\bar{q}\| + v(\|\bar{x}\| + \alpha/2)]}\right\},$$

$$r_2' \le \min\left\{r_2, \frac{\alpha(1-\varepsilon' M)}{12M}\right\},\ r_3 \le \min\left\{\frac{r_4}{3u}, \frac{\alpha(1-\varepsilon' M)}{24Mu}\right\}.$$

It will be demonstrated that $L$ has the Lipschitz-like property at $(\bar{A}, \bar{b}, \bar{q}, \bar{x})$ with constant

$$M' = \frac{Mu}{1-\varepsilon' M} \cdot \max\{u,\ 1 + \|\bar{q}\| + r_3 + v(\|\bar{x}\| + \alpha/2)\}. \tag{4.2.21}$$

Here we express the form of $\bar{z} = \bar{A}^*\bar{q} - f(\bar{w}, \bar{x}) + \nabla_x f(\bar{w}, \bar{x})\bar{x}$. Fix $(A', b', q), (A'', b'', q'') \in \mathbb{B}_{r_1''}(\bar{A}) \times \mathbb{B}_{r_2'}(\bar{b}) \times \mathbb{B}_{r_3}(\bar{q})$ and consider $x' \in L(A', b', q') \cap \mathbb{B}_{\alpha/2}(\bar{x})$. Then

$$0 \in q' + Ax' + N_C(A'x' - b') = q' + Ax' - (A'^*)^{-1}f(\bar{w}, x') + (A'^*)^{-1}f(\bar{w}, x') + N_C(A'x' - b').$$

Let $z' := (A'^*)q' + (A'^*)(\bar{A}^*)^{-1}\nabla_x f(\bar{w}, \bar{x})x' - f(\bar{w}, x')$, then we have $x' \in S'(A', b', z', \bar{w})$. Moreover,

$$z' - \bar{z} = A'^*q' - \bar{A}^*\bar{q} + (A'^*)(\bar{A}^*)^{-1}\nabla_x f(\bar{w}, \bar{x})x' - \nabla_x f(\bar{w}, \bar{x})\bar{x} - [f(\bar{w}, x') - f(\bar{w}, \bar{x})].$$

For the first part:

$$\|A'^*q' - \bar{A}^*\bar{q}\| = \|A'^*q' - A'^*\bar{q} + A'^*\bar{q} - \bar{A}^*\bar{q}\| \le \|A'^*\|\ \|q' - \bar{q}\| + \|A'^* - \bar{A}^*\|\ \|\bar{q}\|.$$

For the second part:

$$\|(A'^*)(\bar{A}^*)^{-1}\nabla_x f(\bar{w}, \bar{x})x' - \nabla_x f(\bar{w}, \bar{x})\bar{x} - [f(\bar{w}, x') - f(\bar{w}, \bar{x})]\|$$

$$= \| -(A'^*)[(A'^*)^{-1}f(\bar{w},x') - (A'^*)^{-1}f(\bar{w},\bar{x}) - (\bar{A}^*)^{-1}\nabla_x f(\bar{w},\bar{x})(x'-\bar{x})]$$
$$+ (A'^*)(\bar{A}^*)^{-1}\nabla_x f(\bar{w},\bar{x})\bar{x} - (\bar{A}^*)(\bar{A}^*)^{-1}\nabla_x f(\bar{w},\bar{x})\bar{x}\|$$
$$\leq \varepsilon'\|x'-\bar{x}\|\ \|A'^*\| + \|(\bar{A}^*)^{-1}\nabla_x f(\bar{w},\bar{x})\bar{x}\|\ \|A'^* - \bar{A}^*\|.$$

To sum up,

$$\begin{aligned}\|z'-\bar{z}\| &\leq \|A'^*\|\ \|q'-\bar{q}\| + \|A'^*-\bar{A}^*\|\ \|\bar{q}\| + \varepsilon'\|x'-\bar{x}\|\ \|A'^*\| \\ &\qquad + \|(\bar{A}^*)^{-1}\nabla_x f(\bar{w},\bar{x})\bar{x}\|\ \|A'^*-\bar{A}^*\| \\ &\leq (\|\bar{A}\| + r_1'')r_3 + (\|\bar{q}\| + \|(\bar{A}^*)^{-1}\nabla_x f(\bar{w},\bar{x})\bar{x}\|)r_1'' + \varepsilon'(\|\bar{A}\| + r_1'')\alpha/2 \\ &\leq ur_3 + (\|\bar{q}\| + v\|\bar{x}\|)r_1'' + u\varepsilon'\alpha/2 \leq \frac{r_4}{3} + \frac{r_4}{3} + \frac{r_4}{6} < r_4.\end{aligned}$$

Then $z' \in \mathbb{B}_{r_4}(\bar{z})$. Analogously, for $z'' := (A''^*)q'' + (A''^*)(\bar{A}^*)^{-1}\nabla_x f(\bar{w},\bar{x})x' - f(\bar{w},x')$, $z'' \in \mathbb{B}_{r_4}(\bar{z})$ as well. By (4.2.20), let $x_1 = x'$, there exists $x_2 \in S'(A'',b'',z'',\bar{w})$, i.e.,

$$\begin{aligned}0 &\in (A''^*)^{-1}z'' + (A''^*)^{-1}f(\bar{w},x_2) + N_C(A''x_2 - b'') \\ &= q'' + (A''^*)^{-1}[f(\bar{w},x_2) - f(\bar{w},x_1)] + (\bar{A}^*)^{-1}\nabla_x f(\bar{w},\bar{x})x_1 + N_C(A''x_2 - b''),\end{aligned}$$

and

$$\|x_2 - x_1\| \leq M\left(\|A'-A''\| + \|b'-b''\| + \|z'-z''\|\right).$$

As

$$\begin{aligned}\|z'-z''\| &= \|(A'^*)q' - (A''^*)q'' + (A'^*-A''^*)(\bar{A}^*)^{-1}\nabla_x f(\bar{w},\bar{x})x_1)\| \\ &\leq \|A'^*\|\ \|q'-q''\| + \|A'^*-A''^*\|\ \|q''\| + \|(\bar{A}^*)^{-1}\nabla_x f(\bar{w},\bar{x})x_1\|\ \|A'^*-A''^*\| \\ &\leq (\|\bar{A}\| + r_1'')\ \|q'-q''\| + [\|\bar{q}\| + r_3 + v(\|\bar{x}\| + \alpha/2)]\,\|A'-A''\|,\end{aligned}$$

we have

$$\begin{aligned}\|x_2 - x_1\| &\leq M\left\{[1 + \|\bar{q}\| + r_3 + v(\|\bar{x}\| + \alpha/2)]\,\|A'-A''\|\right. \\ &\qquad \left. + \|b'-b''\| + (\|\bar{A}\| + r_1'')\|q'-q''\|\right\} =: s \leq su.\end{aligned}$$

Here

$$s \leq M(2\,[1 + \|\bar{q}\| + r_3 + v(\|\bar{x}\| + \alpha/2)]\,r_1'' + 2r_2' + 2(\|\bar{A}\| + r_1'')r_3$$

$$\leq M(2\,[1+\|\bar{q}\|+v(\|\bar{x}\|+\alpha/2)]\,r_1''+2r_2'+2(\|\bar{A}\|+2r_1')r_3$$

$$\leq M(2\,[1+\|\bar{q}\|+v(\|\bar{x}\|+\alpha/2)]\,r_1''+2r_2'+4ur_3\leq\frac{\alpha(1-\varepsilon'M)}{2}.$$

Suppose that there exist points $x_2, x_3, \ldots, x_{n-1}$ with

$$0\in q''+(A''^{*})^{-1}[f(\bar{w},x_i)-f(\bar{w},x_{i-1})]+(\bar{A}^{*})^{-1}\nabla_x f(\bar{w},\bar{x})x_{i-1}+N_C(A''x_i-b'')$$

and

$$\|x_i-x_{i-1}\|\leq(M\varepsilon')^{i-2}su$$

for $i=2,3,\ldots,n-1$. Then for every $i$ we have

$$\|x_i-\bar{x}\|\leq\|x_1-\bar{x}\|+\sum_{j=2}^{i}\|x_j-x_{j-1}\|\leq\frac{\alpha}{2}+s\sum_{j=2}^{i}(M\varepsilon')^{j-2}\leq\frac{\alpha}{2}+\frac{s}{1-M\varepsilon'}\leq\alpha.$$

By setting $z_i:=(A''^{*})q''+(A''^{*})(\bar{A}^{*})^{-1}\nabla_x f(\bar{w},\bar{x})x_i-f(\bar{w},x_i)$ for $i=2,3,\ldots,n-1$, we get

$$z_i-\bar{z}=A''^{*}q''-\bar{A}^{*}\bar{q}+(A''^{*})(\bar{A}^{*})^{-1}\nabla_x f(\bar{w},\bar{x})x_i-\nabla_x f(\bar{w},\bar{x})\bar{x}-[f(\bar{w},x_i)-f(\bar{w},\bar{x})]$$

and therefore

$$\begin{aligned}\|z_i-\bar{z}\|\leq{}&\|A''^{*}\|\;\|q''-\bar{q}\|+\|A''^{*}-\bar{A}^{*}\|\;\|\bar{q}\|+\varepsilon'\|x_i-\bar{x}\|\;\|A''^{*}\|\\&+\|(\bar{A}^{*})^{-1}\nabla_x f(\bar{w},\bar{x})\bar{x}\|\;\|A''^{*}-\bar{A}^{*}\|\\\leq{}&(\|\bar{A}\|+r_1'')r_3+(\|\bar{q}\|+\|(\bar{A}^{*})^{-1}\nabla_x f(\bar{w},\bar{x})\bar{x}\|)r_1''+\varepsilon'(\|\bar{A}\|+r_1'')\alpha\\\leq{}&ur_3+(\|\bar{q}\|+v\|\bar{x}\|)r_1''+u\varepsilon'\alpha\leq r_4.\end{aligned}$$

So $z_i\in\mathbb{B}_{r_4}(\bar{z})$. Since $x_{n-1}\in S'(A'',b'',z_{n-2},\bar{w})\cap\mathbb{B}_\alpha(\bar{x})$, by the Lipschitz-like property of $S'$ (4.2.20), there exists $x_n\in S'(A'',b'',z_{n-1},\bar{w})$, i.e.,

$$0\in q''+(A''^{*})^{-1}[f(\bar{w},x_n)-f(\bar{w},x_{n-1})]+(\bar{A}^{*})^{-1}\nabla_x f(\bar{w},\bar{x})x_{n-1}+N_C(A''x_n-b'')\tag{4.2.22}$$

and

$$
\begin{aligned}
&\|x_n - x_{n-1}\| \leq \ M\|z_{n-1} - z_{n-2}\| \\
&= M\|(A''^*)(\bar{A}^*)^{-1}\nabla_x f(\bar{w}, \bar{x})(x_{n-1} - x_{n-2}) - (f(\bar{w}, x_{n-1}) - f(\bar{w}, x_{n-2}))\| \\
&\leq M\|A''\| \ \|(A''^*)^{-1}(f(\bar{w}, x_{n-1}) - f(\bar{w}, x_{n-2})) - (\bar{A}^*)^{-1}\nabla_x f(\bar{w}, \bar{x})(x_{n-1} - x_{n-2})\| \\
&\leq M\varepsilon'\|A''\| \ \|x_{n-1} - x_{n-2}\| \leq (M\varepsilon')^{n-2} su.
\end{aligned}
$$

The induction step is joined. We obtain an infinite sequence of points $x_1, x_2, \ldots, x_n, \ldots$ in $\mathbb{B}_\alpha(\bar{x})$ and therefore converges to some $x'' \in \mathbb{B}_\alpha(\bar{x})$. As $(A''^*)^{-1} f(w'', \cdot)$ is continuous in $\mathbb{B}_\alpha(\bar{x})$ and the normal cone map $N_C$ has a closed graph, by (4.2.22), taking $n \to \infty$, we have $x'' \in L(A'', b'', q'')$. Moreover, since

$$
\begin{aligned}
&\|x_n - x'\| \leq \ \sum_{i=2}^{n} \|x_i - x_{i-1}\| \leq \sum_{i=2}^{n} (M\varepsilon')^{i-2} su \leq \frac{su}{1 - \varepsilon' M} \\
&= \frac{Mu}{1 - \varepsilon' M} \left\{ [1 + \|\bar{q}\| + r_3 + v(\|\bar{x}\| + \alpha/2)] \, \|A' - A''\| + \|b' - b''\| + u\|q' - q''\| \right\},
\end{aligned}
$$

with definition of $M'$ (4.2.21) we can have

$$
\|x'' - x'\| \leq \ M'(\|A' - A''\| + \|b' - b''\| + \|q' - q''\|).
$$

$\square$

## 4.3 Application to a linear portfolio selection

In this section, we consider a linear portfolio selection problems with different settings. For the first model, the conservative strategy requires minimizing the largest invested risk with some specific constraint on asset allocation like number of invested stocks and no-shorting circumstance. Some easy-to-hold conditions are obtained to guarantee the Lipschitz-like property of the feasible set of such a problem. Later we focus on the stationary set of a portfolio selection problem under minimax rule in [8]. Sufficient conditions for this stationary point set mapping are also given.

### 4.3.1 Examples of normal cones of a set constraint

From the criterion (4.1.9) we can see that to verify the Lipschitz-like property for the system with a set constraint, it is essential to calculate $N_X(\bar{x})$ when $X$ is given. For example, when it comes to portfolio selection, we use $x$ to represent the weighting of selection of stocks. Therefore the summation constraint is a must: $\sum_{i=1}^n x_i = 1$. Besides, when short-selling is forbidden, a nonnegative constraint is added: $x_i \geq 0,\ i = 1, \ldots, n$. Among the universe of stocks we would like to limit the number of stocks to invest in and therefore a sparsity constraint can be imposed: $\|x\|_0 \leq p,\ p \leq n$. Note that the zero norm $\|x\|_0$ is defined as the number of nonzero entries of the vector $x$. We next give some calculation results on these set constraints.

For an arbitrary $x \in X$, we denote $i \in I(x)$ if $x_i \neq 0$ and $k = |I(x)| = \|x\|_0$, the size of $I(x)$ and also the number of nonzero entries of $x$. Besides, let $I := \{1, \ldots, n\}$ and $\overline{I(x)} := I \setminus I(x)$.

1. $X = \{x \in \mathbb{R}^n \mid \sum_{j=1}^n x_j = 1\}$.

   By [81, Example 6.8],
$$N_X(\bar{x}) = \mathbb{R}(1, \ldots, 1)^\top. \tag{4.3.1}$$
   Here $\mathbb{R}(1, \ldots, 1)^\top$ stands for $\{(x, \ldots, x)^\top \mid x \in \mathbb{R}\}$.

2. $X = \{x \in \mathbb{R}^n \mid \sum_{j=1}^n x_j = 1, x_j \geq 0\}$.

   In this case, $X$ is an $(n-1)$-simplex and therefore convex. Then $\widehat{N}_X(\bar{x}) = N_X(\bar{x})$. Let
$$X_1 = \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{j=1}^n x_j = 1 \right\},\ X_2 = \mathbb{R}^n_+.$$
   Then $X = X_1 \cap X_2$. For any $\bar{x} \in X$,

   $N_{X_1}(\bar{x}) = \mathbb{R}(1, \ldots, 1)^\top,$

$$N_{X_2}(\bar{x}) = \left\{x^* \in \mathbb{R}^n_- \mid \langle x^*, \bar{x}\rangle = 0\right\} = \left\{x^* \in \mathbb{R}^n_- \mid x_i^* = 0 \text{ for } i \in I(\bar{x})\right\},$$

where $N_{X_1}(\bar{x})$ is obtained from [81, Exercise 6.7] and $N_{X_2}(\bar{x})$ is a direct calculation. As both sets are convex and therefore regular and that

$$(-N_{X_1}(\bar{x})) \cap N_{X_2}(\bar{x}) = \begin{cases} \{0\}, & \text{if } \bar{x} \neq 0 \\ \mathbb{R}_-(1, \dots, 1)^\top, & \text{if } \bar{x} = 0 \end{cases},$$

where the latter case is forbidden due to $\bar{x} \in X$, then we can apply [81, Theorem 6.42] and obtain

$$N_X(\bar{x}) = N_{X_1}(\bar{x}) + N_{X_2}(\bar{x}) = \left\{ x^* \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} x_i^* = x_j^*, & \forall i, j \in I(\bar{x}) \\ x_i^* \leq x_j^*, & \forall i \in \overline{I(\bar{x})}, j \in I(\bar{x}) \end{array} \right\}. \tag{4.3.2}$$

Note that when $\bar{x} \in \operatorname{rint} X$, i.e., $x_i > 0$ for all $i = 1, \dots, n$,

$$N_X(\bar{x}) = N_{X_1}(\bar{x}) = \mathbb{R}(1, \dots, 1)^\top.$$

3. $X = \{x \in \mathbb{R}^n | \sum_{j=1}^n x_j = 1, x_j \geq 0, ||x||_0 \leq p, p < n\}$.

In this case, $X$ is a intersection of simplex and a set with non-trivial sparsity constraint. Therefore $X$ is a union of $\binom{n}{p}$ $(p-1)$-simplices. Here we discuss the possible cases depending on the sparsity level $k = |I(\bar{x})| = \|\bar{x}\|_0$ at the given point $\bar{x}$.

**Case (i)** : $k = p$. In this case, $\bar{x}$ must be an relative interior point of some $(p-1)$-simplex in $\mathbb{R}^n$ and $X$ is regular at $\bar{x}$. We denote such simplex as

$$C(\bar{x}) = \left\{ x \in \mathbb{R}^n_+ \;\middle|\; x_i = 0 \text{ for } i \notin I(\bar{x}), \sum_{i \in I(\bar{x})} x_i = 1 \right\}.$$

Referring to (4.3.2), we have

$$N_X(\bar{x}) = N_{C(\bar{x})}(\bar{x}) = \left\{ x^* \in \mathbb{R}^n \;\middle|\; x_i^* = x_j^*, \ \forall i, j \in I(\bar{x}) \right\}. \tag{4.3.3}$$

**Case (ii)** : $k < p$. In this case, $\bar{x}$ must lie in the intersection of $\binom{n-k}{p-k}$ $(p-1)$-simplexes in $\mathbb{R}^n$. Since we already know that $k$ elements in $x$ should not equal 0 and there are $n-k$ elements left to select $p-k$ nonzero entries. We denote each neighboring $(p-1)$-simplex as $C_t$ with $t$ representing a unique selection of the position where $p-k$ nonzero entries lie, $t = 1, ..., \binom{n-k}{p-k}$. Here we use $I(C_t)$ to denote the index set where for $x \in C_t$, $x_i \geq 0, i \in I(C_t)$ and $x_i = 0, i \in \overline{I(C_t)}$. As $C_t$ is a simplex, by (4.3.2) we have

$$N_{C_t}(\bar{x}) = \left\{ x^* \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} x_i^* = x_j^*, & \forall i, j \in I(\bar{x}) \\ x_i^* \leq x_j^*, & \forall i \in I(C_t) \backslash I(\bar{x}), j \in I(\bar{x}) \end{array} \right\}.$$

Given the sparsity constraint in $X$: $\|x\|_0 \leq p$, as $\bar{x} \in \bigcap\limits_t C_t$, $x \xrightarrow{X} \bar{x}$ is equivalent to $x \xrightarrow{\bigcup_t C_t} \bar{x}$. Thus we have

$$N_X(\bar{x}) = \limsup_{x \xrightarrow{X} \bar{x}} \widehat{N}_X(x) = \limsup_{x \xrightarrow{\bigcup_t C_t} \bar{x}} \widehat{N}_X(x) \tag{4.3.4}$$

For $x \xrightarrow{\bigcup_t C_t} \bar{x}$, when $x = \bar{x}$,

$$\begin{aligned} \widehat{N}_X(\bar{x}) &= \widehat{N}_{\cup C_t}(\bar{x}) = \bigcap_t \widehat{N}_{C_t}(\bar{x}) = \bigcap_t N_{C_t}(\bar{x}) \\ &= \bigcap_t \left\{ x^* \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} x_i^* = x_j^*, & \forall i, j \in I(\bar{x}) \\ x_i^* \leq x_j^*, & \forall i \in I(C_t) \backslash I(\bar{x}), j \in I(\bar{x}) \end{array} \right\} \\ &= \left\{ x^* \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} x_i^* = x_j^*, & \forall i, j \in I(\bar{x}) \\ x_i^* \leq x_j^*, & \forall i \in \overline{I(\bar{x})}, j \in I(\bar{x}) \end{array} \right\}. \end{aligned} \tag{4.3.5}$$

The last equation comes from $t$ running through all the possibilities and accordingly,

$$\bigcup_t (I(C_t) \backslash I(\bar{x})) = \overline{I(\bar{x})}.$$

When $x \underset{\neq}{\xrightarrow{\bigcup_t C_t}} \bar{x}$, they can approach $\bar{x}$ through relative interior of simplexes in dimensions ranging from $(k-1)$ to $(p-1)$ and for any $x$ being in the relative interior of the same simplex, $\widehat{N}_X(x)$ stays the same as well.

For the $(p-1)$-simplex, the simplex is $C_t$, $t = 1, \dots, \binom{n-k}{p-k}$. Then for $x \in \operatorname{rint} C_t$,

$$\widehat{N}_X(x) = \widehat{N}_{C_t}(x) = \left\{x^* \in \mathbb{R}^n \mid x_i^* = x_j^*,\ i, j \in I(C_t)\right\}. \tag{4.3.6}$$

For the $(p-2)$-simplex, the simplex is an intersection of $\binom{n-p+1}{1}$ $(p-1)$-simplexes. Suppose the related simplixes are $C_s$, $s = 1, \dots, \binom{n-p+1}{1}$. For $x \in \operatorname{rint}(\bigcap_s C_s)$,

$$\widehat{N}_X(x) = \widehat{N}_{\bigcup_s C_s}(x) = \bigcap_s \widehat{N}_{C_s}(x)$$

$$= \left\{ x^* \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} x_i^* = x_j^*, & \forall i, j \in \bigcap_s I(C_s) \\ x_i^* \leq x_j^*, & \forall i \in \left(\bigcup_s I(C_s)\right) \setminus \left(\bigcap_s I(C_s)\right), j \in \bigcap_s I(C_s) \end{array} \right\}.$$

Note that in this case, $I(\bar{x}) \subseteq \bigcap_s I(C_s)$ and

$$\left(\bigcup_s I(C_s)\right) \setminus \left(\bigcap_s I(C_s)\right) = \overline{\bigcap_s I(C_s)} \subseteq \overline{I(\bar{x})}.$$

Therefore we have $\widehat{N}_X(x) \subseteq \widehat{N}_X(\bar{x})$. Similarly, we can have for $r$-simplex with $k-1 \leq r \leq p-2$ and $x$ approximates $\bar{x}$ via the relative interior of the simplex, $\widehat{N}_X(x) \subseteq \widehat{N}_X(\bar{x})$.

In conclusion, for the case $k < p$, combining (4.3.2), (4.3.5) and (4.3.6), we have

$$\begin{aligned} N_X(\bar{x}) = &\left( \bigcup_{t=1}^{\binom{n-k}{p-k}} \left\{x^* \in \mathbb{R}^n \;\middle|\; x_i^* = x_j^*,\ i, j \in I(C_t)\right\} \right) \\ &\bigcup \left\{ x^* \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} x_i^* = x_j^*, & \forall i, j \in I(\bar{x}) \\ x_i^* \leq x_j^*, & \forall i \in \overline{I(\bar{x})}, j \in I(\bar{x}) \end{array} \right\}. \end{aligned} \tag{4.3.7}$$

4. $X = \{x \in \mathbb{R}^n | \sum_{j=1}^n x_j = 1, ||x||_0 \le p, p < n\}$.

   This set differs from the previous one as there is no restriction on $x_i$. When put in the scenario of portfolio selection, it means shorting is allowed. Once the non-negative constraint is abandoned, the set $X$ is composed of $\binom{n}{p}$ $(p-1)$-dimensional affine subspaces.

   **Case (i)** : $k = p$. In this case, $\bar{x}$ must lie in the specific $(p-1)$-dimensional affine subspace

$$C(\bar{x}) = \left\{ x \in \mathbb{R}^n \middle| sgn(x_i) = sgn(\bar{x}_i), \text{ for } i = 1, \ldots, n, \sum_{i \in I(\bar{x})} x_i = 1 \right\}$$

   only and is not adjacent to any other $(p-1)$-dim affine subspaces. Therefore, we have

$$\begin{aligned} N_X(\bar{x}) = N_{C(\bar{x})}(\bar{x}) = \widehat{N}_{C(\bar{x})}(\bar{x}) &= \{x^* \in \mathbb{R}^n \mid \langle x^*, x - \bar{x} \rangle \le 0 \text{ for all } x \in C(\bar{x})\} \\ &= \left\{ x^* \in \mathbb{R}^n \mid x_i^* = x_j^*, \ \forall i, j \in I(\bar{x}) \right\}. \quad (4.3.8) \end{aligned}$$

   **Case (ii)** : $k < p$. In this case, $\bar{x}$ lie in the intersection of $\binom{n-k}{p-k}$ $(p-1)$-dimensional affine subspace in $\mathbb{R}^n$. We denote each neighboring $(p-1)$-dim affine subspace as $C_t$ with $t$ representing a unique selection of the position where $p-k$ nonzero entries lie, $t = 1, ..., \binom{n-k}{p-k}$. Here we use $I(C_t)$ to denote the index set where for $x \in C_t$, $x_i = 0, i \in \overline{I(C_t)}$. Given $C_t$ is an affine subspace, for any $x \in C_t = \left\{ x \in \mathbb{R}^n \mid \sum_{i \in I(C_t)} x_i = 1, x_i = 0 \text{ for } i \notin I(C_t) \right\}$,

$$\begin{aligned} N_{C_t}(x) &= \{x^* \in \mathbb{R}^n \mid \langle x^*, x' - x \rangle \le 0, \forall x' \in C_t\} \\ &= \left\{ x^* \in \mathbb{R}^n \,\middle|\, x_i^* = x_j^*, \ \forall i, j \in I(C_t) \right\}. \end{aligned}$$

   Given the sparsity constraint in $X$: $\|x\|_0 \le p$, as $\bar{x} \in \bigcap_t C_t, x \xrightarrow{X} \bar{x}$ is equivalent to $x \xrightarrow{\bigcup_t C_t} \bar{x}$. Thus we have

$$N_X(\bar{x}) = \limsup_{x \xrightarrow{X} \bar{x}} \widehat{N}_X(x) = \limsup_{x \xrightarrow{\bigcup_t C_t} \bar{x}} \widehat{N}_X(x). \tag{4.3.9}$$

For $x \xrightarrow{\bigcup_t C_t} \bar{x}$, when $x = \bar{x}$,

$$\begin{aligned} \widehat{N}_X(\bar{x}) &= \widehat{N}_{\cup C_t}(\bar{x}) = \bigcap_t \widehat{N}_{C_t}(\bar{x}) = \bigcap_t N_{C_t}(\bar{x}) \\ &= \bigcap_t \left\{ x^* \in \mathbb{R}^n \;\middle|\; x_i^* = x_j^*,\ \forall i,j \in I(C_t) \right\} = \mathbb{R}(1,\ldots,1)^\top. \end{aligned} \tag{4.3.10}$$

The last equation comes from the fact that $\bigcup_t I(C_t) = \{1,\ldots,n\}$.

For $x \underset{\neq}{\xrightarrow{\bigcup_t C_t}} \bar{x}$, they can approach $\bar{x}$ through affine subspaces in dimensions ranging from $(k-1)$ to $(p-1)$.

For the $(p-1)$-dim affine subspace $C_t$, $t = 1,\ldots,\binom{n-k}{p-k}$ and $x \in C_t$ with $\|x\|_0 = p$,

$$\widehat{N}_X(x) = \widehat{N}_{C_t}(x) = N_{C_t}(x) = \left\{ x^* \in \mathbb{R}^n \mid x_i^* = x_j^*,\ i,j \in I(C_t) \right\}. \tag{4.3.11}$$

For the $(p-2)$-dim affine subspace, it is an intersection of $\binom{n-p+1}{1}$ $(p-1)$-dim affine subspace. Suppose the related $(p-1)$-dim affine subspaces are $C_s$, $s = 1,\ldots,\binom{n-p+1}{1}$. For $x \in (\bigcap_s C_s)$ with $\|x\|_0 = p-1$ ,

$$\begin{aligned} \widehat{N}_X(x) = \widehat{N}_{\bigcup_s C_s}(x) = \bigcap_s \widehat{N}_{C_s}(x) = &\left\{ x^* \in \mathbb{R}^n \;\middle|\; x_i^* = x_j^*,\ \forall i,j \in \bigcup_s I(C_s) \right\} \\ =&\mathbb{R}(1,\ldots,1)^\top. \end{aligned}$$

Similarly, we can have for $r$-dim affine subspaces with $k-1 \le r \le p-2$ and $x$ approximates $\bar{x}$ with $k \le \|x\|_0 \le p-1$ accordingly, $\widehat{N}_X(x) = \widehat{N}_X(\bar{x})$. In conclusion, for the case $k < p$, combining (4.3.9), (4.3.10) and (4.3.11) and the fact that $\mathbb{R}(1, \ldots, 1)^\top \subseteq \{x^* \in \mathbb{R}^n \mid x_i^* = x_j^*,\ i, j \in I(C_t)\}$ for any $t$, we have

$$N_X(\bar{x}) = \left( \bigcup_{t=1}^{\binom{n-k}{p-k}} \left\{x^* \in \mathbb{R}^n \,\middle|\, x_i^* = x_j^*,\ i, j \in I(C_t)\right\} \right) \bigcup \mathbb{R}(1, \ldots, 1)^\top$$

$$= \left( \bigcup_{t=1}^{\binom{n-k}{p-k}} \left\{x^* \in \mathbb{R}^n \,\middle|\, x_i^* = x_j^*,\ i, j \in I(C_t)\right\} \right). \tag{4.3.12}$$

### 4.3.2 Stability of feasible sets

Here we consider a conservative strategy of portfolio selection using the minimax risk measure, which is a variant from the the framework of [8]. We minimize the largest invested risk $q_j x_j$ on an individual stock when given a desired return level in all, $\bar{r}$. Note that here we impose the assumption that all assets are risky, i.e., $q_j > 0$.

$$\begin{aligned} \min_{x,y} \quad & y \\ \text{s.t.} \quad & q_j x_j \le y, q_j > 0, \forall j = 1, ..., n, \\ & \sum_{j=1}^{n} \bar{r}_j x_j \ge \bar{r} \\ & x \in X \end{aligned} \tag{4.3.13}$$

Here $x_j \ge 0,\ j = 1, \ldots, n$ stands for the allocation of investments on the $j$-th asset, $\bar{r}_j, q_j$ denotes the expected rate of return and expected absolute deviation of the $j$-th asset respectively. Therefore $\bar{r}_j x_j$ gives the expected return of the investment on $j$-th asset. Therefore any investors who adopt such selection rule can best avoid high risks in any invested assets. The constraint $x \in X$ are mainly some possible restrictions on the investment, like no-shorting, number of investing stocks, etc. For this model, $X$ must contain the no-shorting requirements. The individual data in the given model, risk level $q_j$ and return $\bar{r}_j$, often subject to perturbation. It is essential to ensure the solution is stable under minor perturbations. We can write the feasible set as a set-valued mapping $S(A,b) = \{z \in Z \mid Az + b \in K\}$ and a given pair $(\bar{A}, \bar{b})$ is

$$\bar{A} = \begin{pmatrix} q_1 & 0 & \cdots & 0 & -1 \\ 0 & q_2 & \cdots & 0 & -1 \\ & & \ddots & & \\ 0 & 0 & \cdots & q_n & -1 \\ -\bar{r}_1 & -\bar{r}_2 & \cdots & -\bar{r}_n & 0 \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)},\ \bar{b} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \bar{r} \end{pmatrix}$$

and $K = \mathbb{R}^{n+1}_{-}$, $Z = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; x \in X, y \in \mathbb{R} \right\}$. Next we explain how condition (4.1.9) works when given a reference point. We next show that the feasible set mapping $S$, with two different set constraints, is Lipschitz-like at $(\bar{A}, \bar{b})$ for $(\bar{x}, \bar{y})$ when some natural conditions are satisfied.

**Theorem 4.3.1.** *For the portfolio selection problem (4.3.13) with $X$ being one of the following sets:*

*1)* $X = \left\{ x \in \mathbb{R}^n \mid \sum_{j=1}^n x_j = 1, x_j \geq 0 \right\}$

*2)* $X = \left\{ x \in \mathbb{R}^n \mid \sum_{j=1}^n x_j = 1, x_j \geq 0, \|x\|_0 < p \right\}$

*let $\bar{z} = (\bar{x}, \bar{y}) \in S(\bar{A}, \bar{b})$. If one of the following conditions is satisfied, $S$ is Lipschitz-like at $(\bar{A}, \bar{b})$ for $\bar{z}$.*

*(a)* $\sum_{i=1}^n \bar{r}_i \bar{x}_i > \bar{r}$.

*(b) the number of invested stocks is greater than 1, i.e., $\|\bar{x}\|_0 \geq 2$ and for the invested stocks, there exists at least two stocks with different returns, i.e.,$\exists i, j \in I(\bar{x})$ and $i \neq j$ such that $\bar{r}_i \neq \bar{r}_j$, where $I(\bar{x}) = \{i \in I \mid \bar{x}_i \neq 0\}$.*

*Note that condition (b) is easy to be satisfied when the stock pool is big enough.*

*Proof.* By (4.1.9), $S$ is Lipschitz-like at $(\bar{A},\bar{b})$ for $\bar{z}$ if and only if $-(\bar{A}^*)^{-1}N_Z(\bar{z})\cap N_K(\bar{A}\bar{z}+\bar{b})=\{0\}$, i.e.,

$$z^*\in N_K(\bar{A}\bar{z}+\bar{b}),\ -\bar{A}^*z^*\in N_Z(\bar{z})\Longrightarrow z^*=0. \tag{4.3.14}$$

For

$$\begin{aligned}\bar{A}\bar{z}+\bar{b}&=\begin{pmatrix} q_1 & 0 & \cdots & 0 & -1\\ 0 & q_2 & \cdots & 0 & -1\\ & & \ddots & & \\ 0 & 0 & \cdots & q_n & -1\\ -\bar{r}_1 & -\bar{r}_2 & \cdots & -\bar{r}_n & 0\end{pmatrix}\begin{pmatrix}\bar{x}_1\\ \vdots\\ \bar{x}_n\\ \bar{y}\end{pmatrix}+\begin{pmatrix}0\\ \vdots\\ 0\\ \bar{r}\end{pmatrix}\\ &=\begin{pmatrix} q_1\bar{x}_1-\bar{y}\\ \vdots\\ q_n\bar{x}_n-\bar{y}\\ \sum_{i=1}^n -\bar{r}_i\bar{x}_i+\bar{r}\end{pmatrix}\in K=\mathbb{R}^{n+1}_-,\end{aligned}$$

and $z^*\in N_K(\bar{A}\bar{z}+\bar{b})=N_{\mathbb{R}^{n+1}_-}(\bar{A}\bar{z}+\bar{b})$, we have

$$\begin{cases} z_i^*\geq 0, & \text{if } q_i\bar{x}_i-\bar{y}=0\\ z_i^*=0, & \text{if } q_i\bar{x}_i-\bar{y}<0\end{cases}\ \text{for } i=1,\ldots,n,\quad \begin{cases} z_{n+1}^*\geq 0, & \text{if } \sum_{i=1}^n -\bar{r}_i\bar{x}_i+\bar{r}=0\\ z_{n+1}^*=0, & \text{if } \sum_{i=1}^n -\bar{r}_i\bar{x}_i+\bar{r}<0\end{cases}. \tag{4.3.15}$$

For

$$-\bar{A}^*z^*=-\begin{pmatrix} q_1 & 0 & \cdots & 0 & -\bar{r}_1\\ 0 & q_2 & \cdots & 0 & -\bar{r}_2\\ & & & & \\ 0 & 0 & \cdots & q_n & -\bar{r}_n\\ -1 & -1 & \cdots & -1 & 0\end{pmatrix}\begin{pmatrix} z_1^*\\ \vdots\\ z_n^*\\ z_{n+1}^*\end{pmatrix}=\begin{pmatrix}\bar{r}_1z_{n+1}^*-q_1z_1^*\\ \vdots\\ \bar{r}_nz_{n+1}^*-q_nz_n^*\\ \sum_{i=1}^n z_i^*\end{pmatrix}\in N_Z(\bar{z}),$$

since $X$ and $\mathbb{R}$ are closed sets, by [81, Proposition 6.41], we have $N_Z(\bar{z})=N_{X\times\mathbb{R}}(\bar{x},\bar{y})=N_X(\bar{x})\times N_{\mathbb{R}}(\bar{y})=N_X(\bar{x})\times\{0\}$. Thus we first have $\sum_{i=1}^n z_i^*=0$. Combining (4.3.15), we have $z_i^*=0$ for all $i=1,\ldots,n$. When condition (a) holds, we have $z_{n+1}^*=0$ and (4.3.14) is satisfied. If not, it remains to obtain $z_{n+1}^*=0$ given $(\bar{r}_1z_{n+1}^*,\ldots,\bar{r}_nz_{n+1}^*)^\top\in N_X(\bar{x})$ and $z_{n+1}^*\geq 0$ by (4.3.14). By comparing between

(4.3.2), (4.3.3) and (4.3.7) we can see that for the two choices of $X$ mentioned in the theorem, we can derive something in common:

$$\bar{r}_i z^*_{n+1} = \bar{r}_j z^*_{n+1}, \ i, j \in I(\bar{x}).$$

Then by condition (b) we have $z^*_{n+1} = 0$ as there exists $\bar{r}_i \neq \bar{r}_j$ for $i \neq j, \ i, j \in I(\bar{x})$. $\square$

Next we present an example when both condition (a) and (b) fail, $S$ does not enjoy the Lipschitz-like property at the reference point.

**Example 4.3.2.** *Consider the portfolio selection problem with two stocks:*

$$\begin{aligned}
\min \quad & y \\
s.t. \quad & 0.1x_1 \leq y, \\
& 0.5x_2 \leq y, \\
& 0.1x_1 + 0.1x_2 \geq 0.1 \textit{(trivial)} \\
& x_1 + x_2 = 1, x_1, x_2 \geq 0.
\end{aligned}$$

*That is, we have the value:*

$$\bar{A} = \begin{pmatrix} 0.1 & 0 & -1 \\ 0 & 0.5 & -1 \\ -0.1 & -0.1 & 0 \end{pmatrix} \in \mathbb{R}^{3\times 3}, \bar{b} = \begin{pmatrix} 0 \\ 0 \\ 0.1 \end{pmatrix}$$

*with the set constraint* $X = \{x \in \mathbb{R}^2 \mid x_1 + x_2 = 1, x_1, x_2 \geq 0\}$. *For this problem, the optimal solution is* $\bar{z} = (\bar{x}, \bar{y}) = (\frac{5}{6}, \frac{1}{6}, \frac{1}{12})^\top$ *where* $\bar{z} \in S(\bar{A}, \bar{b})$. *Both condition (a) and (b) are not satisfied as* $\bar{r}_1\bar{x}_1 + \bar{r}_2\bar{x}_2 = \bar{r} = 0.1$ *and* $\bar{r}_1 = \bar{r}_2$.

*We will show that $S$ does not enjoy the Lipschitz-like property at the optimal solution point by the Mordukhovich criterion and by definition. To check by the Mordukhovich criterion, we have* $N_K(\bar{A}\bar{z}+\bar{b}) = N_{\mathbb{R}^3_-}(\bar{A}\bar{z}+\bar{b}) = N_{\mathbb{R}^3_-}\left((0,0,0)^\top\right) = \mathbb{R}^3_+$

*and*

$$-(\bar{A}^{*})^{-1}\left(N_Z(\bar{z})\right) = -(\bar{A}^{*})^{-1}(N_X(\bar{x}) \times \{0\}) = \frac{1}{6}\begin{pmatrix} -10 & 10 & 5 \\ 10 & -10 & 1 \\ 50 & 10 & 5 \end{pmatrix}\begin{pmatrix} t \\ t \\ 0 \end{pmatrix}(t \in \mathbb{R})$$

$$= \mathbb{R}(0,0,1)^{\top}.$$

*Then* $-(\bar{A}^{*})^{-1}\left(N_Z(\bar{z})\right) \cap N_K(\bar{v}) = \{0_2\} \times \mathbb{R}_+ \neq \{0_3\}$ *and the Mordukhovich criterion is not satisfied here.*

*To verify by definition, suppose there exists* $l > 0, U \in \mathcal{N}(\bar{A}, \bar{b}), V \in \mathcal{N}(\bar{z})$*, such that*

$$S(A', b') \cap V \subset S(A, b) + l\left(\|A' - A\| + \|b' - b\|\right)\mathbb{B}_{\mathbb{R}^3}, \forall (A', b'),\ (A, b) \in U. \quad (4.3.16)$$

*By taking*

$$A_{\varepsilon} = \begin{bmatrix} 0.1 & 0 & -1 \\ 0 & 0.5 & -1 \\ -0.1+\varepsilon & -0.1 & 0 \end{bmatrix}$$

*and* $\varepsilon > 0$ *small enough and choosing* $\rho > 0$ *small enough such that* $\mathbb{B}(\bar{z}, \rho) \subset V$*, let* $A' = \bar{A}$*,* $A = A_{\varepsilon}$ *and* $b' = b = \bar{b}$ *in (4.3.16), we should have*

$$S(\bar{A}, \bar{b}) \cap \mathbb{B}(\bar{z}, \rho) \subset S(A_{\varepsilon}, \bar{b}) + l\varepsilon\mathbb{B}_{\mathbb{R}^2}.$$

*However, given*

$$S(\bar{A}, \bar{b}) = \left\{(x_1, x_2, y) \in Z \mid x_1 \leq 10y, x_2 \leq 2y, x_1 + x_2 = 1, x_1, x_2 \geq 0\right\},$$

$$S(A_{\varepsilon}, \bar{b}) = \left\{(x_1, x_2, y) \in Z \ \mid x_1 \leq 10y, x_2 \leq 2y, x_1 + x_2 \geq 1 + 10\varepsilon x_1\right\}$$

$$= \left\{(0, 1, y) \mid y \geq 1/2\right\},$$

*the inclusion does not hold for any* $\varepsilon \in (0, \frac{\frac{5}{6}+\rho}{l})$*. Thus* $S$ *does not enjoy the Lipschitz-like property at* $\left(\bar{A}, \bar{b}\right)$ *for* $\bar{z}$*, which conforms to what we derived from the Mordukhovich criterion.*

### 4.3.3 Stability of a stationary point set

Next we consider a selection model which balances the return and risk by a parameter $\lambda$. In [8] a parametric portfolio optimization problem is considered:

$$\begin{aligned} \min_{x,y} \quad & \lambda y + (1-\lambda)\left(-\sum_{i=1}^{n} r_i x_i\right) \\ \text{s.t.} \quad & q_j x_j \le y, \forall j = 1, \dots, n, \\ & \sum_{i=1}^{n} x_i = 1, x_i \ge 0, \forall i = 1, \dots, n \end{aligned} \tag{4.3.17}$$

where $\lambda \in (0,1)$ is an investor's risk tolerance parameter. Similar to the setting of (4.3.13), in this problem we are both minimizing the highest risk of each individual asset and maximizing the expected rate of return of the portfolio. The balance between these two goals is achieved via the parameter $\lambda$. In other words, investors can adjust the strategy according to their preferences between the overall return($\sum_{i=1}^{n} r_i x_i$) and the largest individual asset risk ($y$) by controlling $\lambda$. For this model, three assumptions are considered:

**(A1)** $r_1 \le r_2 \le \cdots \le r_n$,

**(A2)** no two identical assets exist, i.e., $\nexists i \neq j$ with $r_i = r_j,\ q_i = q_j$,

**(A3)** all assets are risky, i.e., $q_j > 0$ for $i = 1, \dots, n$.

**Lemma 4.3.3** ([8, Theorem 3.1]). *With the assumptions (A1-A3) introduced above, the optimal solution to (4.3.17) is given as*

$$\bar{x}_i = \begin{cases} \dfrac{1}{q_i}\left(\displaystyle\sum_{l \in \mathcal{I}^*(\lambda)} \frac{1}{q_l}\right)^{-1}, & i \in \mathcal{I}^*(\lambda) \\ 0, & i \notin \mathcal{I}^*(\lambda) \end{cases}, \qquad \bar{y} = \left(\sum_{l \in \mathcal{I}^*(\lambda)} \frac{1}{q_l}\right)^{-1} \tag{4.3.18}$$

*where $\mathcal{I}^*(\lambda)$ is the set of assets to be invested decided by the following rule:*

**(i)** *if there exists an integer $k \in [0, n-2]$ such that*

$$\sum_{i=0}^{j-1} \frac{r_{n-i} - r_{n-j}}{q_{n-i}} < \frac{\lambda}{1-\lambda}, \ \textit{for } j = 1, \ldots, k \tag{4.3.19}$$

$$\sum_{i=0}^{k} \frac{r_{n-i} - r_{n-k-1}}{q_{n-i}} \geq \frac{\lambda}{1-\lambda}, \tag{4.3.20}$$

*then $\mathcal{I}^*(\lambda) = \{n, n-1, \ldots, n-k\}$.*

**(ii)** *otherwise, $\mathcal{I}^*(\lambda) = \{n, n-1, \ldots, 1\}$.*

Next we will show that this optimal solution enjoys stability when the parameters $r_i$, $i = 1, \ldots, n$ undergo perturbations under some assumptions. First we reformulate the optimization problem (4.3.17) as follows,

$$\begin{aligned} \min \quad & \bar{c}^\top z \\ \text{s.t.} \quad & Az + \bar{b} \in \mathbb{R}^{2n}_{-} \times \{1\} \end{aligned} \tag{4.3.21}$$

where

$$A = \begin{pmatrix} q_1 & 0 & \cdots & 0 & -1 \\ 0 & q_2 & \cdots & 0 & -1 \\ & & \ddots & & \\ 0 & 0 & \cdots & q_n & -1 \\ -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & -1 & 0 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}, \ z = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ y \end{pmatrix}, \ \bar{c} = \begin{pmatrix} (\lambda-1)r_1 \\ \vdots \\ (\lambda-1)r_n \\ \lambda \end{pmatrix}, \ \bar{b} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4.3.22}$$

Let

$$A^*\begin{pmatrix}\mu\\ \gamma\\ \tau\end{pmatrix}=\begin{pmatrix} q_1 & \cdots & 0 & -1 & \cdots & 0 & 1\\ & \ddots & & & \ddots & & \vdots\\ 0 & \cdots & q_n & 0 & \cdots & -1 & 1\\ -1 & \cdots & -1 & 0 & \cdots & 0 & 0\end{pmatrix}\begin{pmatrix}\mu_1\\ \vdots\\ \mu_n\\ \gamma_1\\ \vdots\\ \gamma_n\\ \tau\end{pmatrix}=\begin{pmatrix} q_1\mu_1-\gamma_1+\tau\\ \vdots\\ q_n\mu_n-\gamma_n+\tau\\ -\sum_{i=1}^n\mu_i\end{pmatrix}=0$$

with

$$\begin{pmatrix}\mu\\ \gamma\\ \tau\end{pmatrix}\in N_{\mathbb{R}_-^{2n}\times\{1\}}(Az+b)=\prod_{i=1}^n N_{\mathbb{R}_-}(q_ix_i-y)\times\prod_{i=1}^n N_{\mathbb{R}_-}(-x_i)\times N_{\{1\}}\left(\sum_{i=1}^n x_i\right)\subseteq\mathbb{R}_+^{2n}\times\mathbb{R}.$$

By $\sum_{i=1}^n\mu_i=0$ with $\mu_i\geq 0$, we have $\mu_i=0$ for $i=1,\ldots,n$. As there is at least one $x_i>0$ due to the constraint $\sum_{i=1}^n x_i=1$, the relative $\gamma_i=0$ and therefore $\tau=0$ and $\gamma_i=0$, for $i=1,\ldots,n$. We can naturally obtain $\mu=0,\ \gamma=0,\ \tau=0$ for any $z$ being in the feasible set. Considering the set $\mathbb{R}_-^{2n}\times\{1\}$ is convex, the optimal solution set-mapping can be put as

$$S(c,b)=\left\{z\in\mathbb{R}^{n+1}\mid 0\in c+A^*N_{\mathbb{R}_-^{2n}\times\{1\}}(Az+b)\right\}. \tag{4.3.23}$$

To analyze the stability of optimal solution of (4.3.17) subject to changes on $r_i$, it would be sufficient to study the Lipschitz-like property of the solution mapping $S$ at the given pair $(\bar{c},\bar{b})$ in (4.3.22) for $\bar{z}$ in (4.3.18) . In [8, Appendix A], when $\mathcal{I}^*(\lambda)$ is given, the related Lagrangian multiplier $(\bar{\mu},\bar{\gamma},\bar{\tau})\in N_{\mathbb{R}_-^{2n}\times\{1\}}(A\bar{z}+\bar{b})$ is unique as

$$\bar{\mu}_i=\begin{cases}\dfrac{1}{q_i}\left[(1-\lambda)r_i-\left(\displaystyle\sum_{l\in\mathcal{I}^*(\lambda)}\frac{1}{q_l}\right)^{-1}\left((1-\lambda)\displaystyle\sum_{l\in\mathcal{I}^*(\lambda)}\frac{r_l}{q_l}\right)\right]>0, & i\in\mathcal{I}^*(\lambda)\\ 0, & i\notin\mathcal{I}^*(\lambda)\end{cases}, \tag{4.3.24}$$

$$\bar{\gamma}_i = \begin{cases} 0, & i \in \mathcal{I}^*(\lambda) \\ -(1-\lambda)r_i + \left(\sum_{l\in\mathcal{I}^*(\lambda)} \frac{1}{q_l}\right)^{-1} \left((1-\lambda)\sum_{l\in\mathcal{I}^*(\lambda)} \frac{r_l}{q_l} - \lambda\right) \geq 0, & i \notin \mathcal{I}^*(\lambda) \end{cases}, \tag{4.3.25}$$

and

$$\bar{\tau} = \left(\sum_{l\in\mathcal{I}^*(\lambda)} \frac{1}{q_l}\right)^{-1} \left((1-\lambda)\sum_{l\in\mathcal{I}^*(\lambda)} \frac{r_l}{q_l} - \lambda\right). \tag{4.3.26}$$

Next we give the upper estimate of $N_{\operatorname{gph} S}(\bar{c}, \bar{b}, \bar{z})$.

**Proposition 4.3.4.** *Let $(\bar{c}, \bar{b}, \bar{z})$ be given as in (4.3.18) and (4.3.22) . Then*

$$N_{\operatorname{gph} S}(\bar{c}, \bar{b}, \bar{z}) \subseteq \left\{(-t^*, v^*, A^*v^*) \;\middle|\; (v^*, At^*) \in N_{\operatorname{gph} N_{\mathbb{R}^{2n}_- \times \{1\}}}(A\bar{z} + \bar{b}, \bar{\eta})\right\} \tag{4.3.27}$$

*where $\bar{\eta} := (\bar{\mu}, \bar{\gamma}, \bar{\tau})$ is given as in (4.3.24), (4.3.25) and (4.3.26).*

*Proof.* Here we denote $K := \mathbb{R}^{2n}_- \times \{1\}$ and $D := F(\operatorname{gph} N_K)$ where $F : \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1} \times \mathbb{R}^{n+1}$ is defined as $F(v, t) = \begin{pmatrix} E & 0 \\ 0 & A^* \end{pmatrix} \begin{pmatrix} v \\ t \end{pmatrix} = \begin{pmatrix} v \\ A^*t \end{pmatrix}$. Then we can rewrite (4.3.23) as

$$\operatorname{gph} S = \{(c, b, z) \mid (Az + b, -c) \in D\}.$$

For $U \in \mathcal{N}(A\bar{z} + \bar{b}, -\bar{c})$, $F^{-1}(U) \cap \operatorname{gph} N_K$ is either a single-point set or an empty set and therefore is bounded. Besides, as $\bar{\eta}$ is unique in $-\bar{c} = A^*\bar{\eta}$, by [81, Theorem 6.43], we have

$$\begin{aligned} N_D(A\bar{z} + \bar{b}, -\bar{c}) \subseteq & \left\{(v^*, t^*) \;\middle|\; \begin{pmatrix} E & 0 \\ 0 & A \end{pmatrix} (v^*, t^*) \in N_{\operatorname{gph} N_K}(A\bar{z} + \bar{b}, \bar{\eta})\right\} \\ & \left\{(v^*, t^*) \mid (v^*, At^*) \in N_{\operatorname{gph} N_K}(A\bar{z} + \bar{b}, \bar{\eta})\right\} \end{aligned} \tag{4.3.28}$$

The inclusion (4.3.28) becomes an equation when $\operatorname{gph} N_K$ is convex at around $(A\bar{z} + \bar{b}, \bar{\eta})$.

Let $G(c,b,z) := (Az+b,-c)$. Then we have $\operatorname{gph} S = G^{-1}(D)$ and that $\nabla G(\bar{c},\bar{b},\bar{z}) = \begin{pmatrix} 0 & E & A \\ -E & 0 & 0 \end{pmatrix}$ has full rank $3n+2$. By [81, Exercise 6.7],

$$\begin{aligned} N_{\operatorname{gph} S}(\bar{c},\bar{b},\bar{z}) =& \nabla G(\bar{c},\bar{b},\bar{z})^* N_D(A\bar{z}+\bar{b},-\bar{c}) \\ & \subseteq \left\{ (-t^*, v^*, A^*v^*) \;\middle|\; (v^*, At^*) \in N_{\operatorname{gph} N_K}(A\bar{z}+\bar{b},\bar{\eta}) \right\}. \end{aligned}$$

□

**Remark 4.3.5.** *If we consider $S$ as a mapping of $c$ only, i.e.,*

$$S(c) = \left\{ z \in \mathbb{R}^{n+1} \mid 0 \in c + A^* N_{\mathbb{R}^{2n}_{-}\times\{1\}}(Az) \right\},$$

*we can also obtain the upper estimate*

$$N_{\operatorname{gph} S}(\bar{c},\bar{z}) \subseteq \left\{ (-t^*, A^*v^*) \;\middle|\; (v^*, At^*) \in N_{\operatorname{gph} N_{\mathbb{R}^{2n}_{-}\times\{1\}}}(A\bar{z},\bar{\eta}) \right\}$$

*via [81, Theorem 6.14] as the constraint qualification*

$$\begin{cases} (v^*, At^*) \in N_{\operatorname{gph} N_K}(A\bar{z},\bar{\eta}) \\ (-t^*, A^*v^*) = 0 \end{cases} \Longrightarrow (v^*, t^*) = 0$$

*holds automatically.*

**Theorem 4.3.6.** *The solution to the portfolio optimization problem (4.3.17) enjoys the Lipschitz-like property at $(\bar{x},\bar{y})$ for $(r_1,\ldots,r_n)$ if one of the following conditions is satisfied:*

**(a)** $\mathcal{I}^*(\lambda) = \{n, n-1, \ldots, 1\}$,

**(b)** $\mathcal{I}^*(\lambda) = \{n, n-1, \ldots, n-k\}$ *for* $k \in [0, n-2]$ *and* $\displaystyle\sum_{i=0}^{k} \frac{r_{n-i} - r_{n-k-1}}{q_{n-i}} > \frac{\lambda}{1-\lambda}$, *Proof.* The solution to the problem (4.3.17) is Lipschitz-like at the reference point if $S$ is Lipschitz-like at $(\bar{c}, \bar{b})$ for $\bar{z}$ by representation (4.3.23). With upper estimate (4.3.27), it is sufficient to have

$$\begin{cases}(v^*, At^*) \in N_{\operatorname{gph} N_{\mathbb{R}^{2n}_- \times \{1\}}}(A\bar{z} + \bar{b}, \bar{\eta}) \\ A^*v^* = 0\end{cases} \Longrightarrow v^* = 0, t^* = 0,$$

where $\bar{\eta} := (\bar{\mu}, \bar{\gamma}, \bar{\tau})$ is given as in (4.3.24), (4.3.25) and (4.3.26). Note that

$$A^*v^* = \begin{pmatrix} q_1 v_1^* - v_{n+1}^* + v_{2n+1}^* \\ \vdots \\ q_n v_n^* - v_{2n}^* + v_{2n+1}^* \\ -\sum_{i=1}^n v_i^* \end{pmatrix}, \quad At^* = \begin{pmatrix} q_1 t_1^* - t_{n+1}^* \\ \vdots \\ q_n t_n^* - t_{n+1}^* \\ -t_1^* \\ \vdots \\ -t_n^* \\ \sum_{i=1}^n t_i^* \end{pmatrix}.$$

For $(v^*, At^*) \in N_{\operatorname{gph} N_{\mathbb{R}^{2n}_- \times \{1\}}}(A\bar{z}+\bar{b}, \bar{\eta})$, by [81, Proposition 6.41], it is equivalent that

$$\begin{cases}(v_i^*, q_i t_i^* - t_{n+1}^*) \in N_{\operatorname{gph} N_{\mathbb{R}_-}}(q_i\bar{x}_i - \bar{y}, \bar{\mu}_i), & i = 1, \dots, n \\ (v_{n+j}^*, -t_j^*) \in N_{\operatorname{gph} N_{\mathbb{R}_-}}(-\bar{x}_j, \bar{\gamma}_j), & j = 1, \dots, n \\ (v_{2n+1}^*, \sum_{i=1}^n t_i^*) \in N_{\operatorname{gph} N_{\{1\}}}(\sum_{i=1}^n \bar{x}_i, \bar{\tau}). \end{cases} \tag{4.3.29}$$

We know that $\operatorname{gph} N_{\mathbb{R}_-} = \mathbb{R}_- \times \{0\} \cup \{0\} \times \mathbb{R}_+$. Thus for any $(u_1, u_2) \in \operatorname{gph} N_{R_-}$,

$$N_{\operatorname{gph} N_{\mathbb{R}_-}}(u_1, u_2) = \begin{cases}\{0\} \times \mathbb{R}, & (u_1, u_2) \in \mathbb{R}_{--} \times \{0\} \\ \mathbb{R} \times \{0\}, & (u_1, u_2) \in \{0\} \times \mathbb{R}_{++} \\ \{0\} \times \mathbb{R} \cup \mathbb{R} \times \{0\} \cup \mathbb{R}_+ \times \mathbb{R}_-, & (u_1, u_2) = (0, 0)\end{cases}. \tag{4.3.30}$$

Besides, for $\operatorname{gph} N_{\{1\}} = \{1\} \times \mathbb{R}$, $N_{\operatorname{gph} N_{\{1\}}}(u_1, u_2) = \mathbb{R} \times \{0\}$ if $u_1 = 1$. Together with (4.3.24)-(4.3.26) and (4.3.30), we can further simplify (4.3.29) as

1. $i \in \mathcal{I}^*(\lambda)$: $q_i\bar{x}_i = \bar{y}$, $\bar{\mu}_i > 0$, $\bar{x}_i > 0$, $\bar{\gamma}_i = 0$.

$$(v_i^*, q_i t_i^* - t_{n+1}^*) \in N_{\operatorname{gph} N_{\mathbb{R}_-}}(q_i\bar{x}_i - \bar{y}, \bar{\mu}_i) = \mathbb{R} \times \{0\}. \tag{4.3.31}$$

$$(v_{n+i}^*, -t_i^*) \in N_{\operatorname{gph} N_{\mathbb{R}_-}}(-\bar{x}_i, \bar{\gamma}_i) = \{0\} \times \mathbb{R}. \tag{4.3.32}$$

2. $i \notin \mathcal{I}^*(\lambda)$: $q_i\bar{x}_i = 0 < \bar{y}$, $\bar{\mu}_i = 0$, $\bar{x}_i = 0$, $\bar{\gamma}_i \geq 0$.

$$(v_i^*, q_i t_i^* - t_{n+1}^*) \in N_{\operatorname{gph} N_{\mathbb{R}_-}}(q_i\bar{x}_i - \bar{y}, \bar{\mu}_i) = \{0\} \times \mathbb{R}. \tag{4.3.33}$$

$$(v_{n+i}^*, -t_i^*) \in N_{\operatorname{gph} N_{\mathbb{R}_-}}(-\bar{x}_i, \bar{\gamma}_i) = \begin{cases} \mathbb{R} \times \{0\}, & \text{if } \bar{\gamma}_i > 0 \\ \{0\} \times \mathbb{R} \cup \mathbb{R} \times \{0\} \cup \mathbb{R}_+ \times \mathbb{R}_-, & \text{if } \bar{\gamma}_i = 0 \end{cases}. \tag{4.3.34}$$

Besides, $A^*v^* = 0$ and $(v_{2n+1}^*, \sum_{i=1}^n t_i^*) \in N_{\operatorname{gph} N_{\{1\}}}(\sum_{i=1}^n \bar{x}_i, \bar{\tau})$ generate

$$q_i v_i^* - v_{n+i}^* + v_{2n+1}^* = 0, i = 1, \ldots, n \tag{4.3.35}$$

$$\sum_{i=1}^n v_i^* = 0 \tag{4.3.36}$$

$$\sum_{i=1}^n t_i^* = 0. \tag{4.3.37}$$

Combining (4.3.32), (4.3.33), (4.3.35) and (4.3.36) we can obtain $v^* = 0$. It remains to verify $t^* = 0$.

When $\mathcal{I}^*(\lambda) = \{1, \ldots, n\}$, by (4.3.31) and (4.3.37) we directly have $t^* = 0$. When $\mathcal{I}^*(\lambda) = \{n, n-1, \ldots, n-k\}$ where the integer $k \in [0, n-2]$ is decided by (4.3.19) and (4.3.20). By (4.3.25), for $i = 1, \ldots, n-k-1$,

$$\begin{aligned} \bar{\gamma}_i =& -(1-\lambda) r_i + \left( \sum_{l \in \mathcal{I}^*(\lambda)} \frac{1}{q_l} \right)^{-1} \left( (1-\lambda) \sum_{l \in \mathcal{I}^*(\lambda)} \frac{r_l}{q_l} - \lambda \right) \\ =& (1-\lambda) \left( \sum_{l \in \mathcal{I}^*(\lambda)} \frac{1}{q_l} \right)^{-1} \left( \sum_{l \in \mathcal{I}^*(\lambda)} \frac{r_l - r_i}{q_l} - \frac{\lambda}{1-\lambda} \right). \end{aligned}$$

When condition (b) holds, with assumption (A1), $\bar{\gamma}_1 \geq \bar{\gamma}_2 \geq \cdots \geq \bar{\gamma}_{n-k-1} > 0$. Therefore we can update (4.3.34) as

$$(v_{n+i}^*, -t_i^*) \in N_{\operatorname{gph} N_{\mathbb{R}_-}}(-\bar{x}_i, \bar{\gamma}_i) = \mathbb{R} \times \{0\}, i \notin \mathcal{I}^*(\lambda).$$

Together with (4.3.31) and (4.3.37) we can obtain $t^* = 0$. $\square$

**Remark 4.3.7.** *For both of the conditions (a) and (b), strictly complementarity between $A\bar{z}+\bar{b}$ and $(\bar{\mu}, \bar{\gamma}, \bar{\tau})$ is achieved.*

# Chapter 5

# Conclusions and Future Research

## 5.1 Summary of the thesis

The Lipschitz-like property relative to a set is an important task to study the stability of parametric systems under perturbations within a certain set as it puts no assumption on where the reference point lies in. Thanks to the projectional coderivative and the generalized Mordukhovich criterion developed in Meng et al. [59], we were able to characterize such a property relative to a closed and convex set.

In this thesis, we focused on introducing more properties of this newly introduced tool, projectional coderivatives, and deriving corresponding calculus rules. By exploiting the structure of smooth manifolds, we simplified the expression of projectional coderivatives of any set-valued mapping relative to a smooth manifold to a fixed-point one and extended the generalized Mordukhovich criterion under such a setting. For a closed set in general, we introduced the chain rule of this tool for composition of two set-valued mappings with outer semicontinuity. We also particularly developed chain rules when any one of these two mappings is single-valued. Based on these results, sum rules were presented with different types of constraint qualifications.

Subsequently we considered the parametric system under the framework in Levy and Mordukhovich [53] and gave the upper estimates of the projectional coderivatives under different settings. Several examples were given to illustrate how the upper estimate can be applied to analyze the stability. The comparison with the other tool, directional coderivatives, was also carried out. With wide applicability of such a system, we studied linear constraint systems, linear complementarity problems and affine variational inequalities. For the first two problems, we gave expression of projectional coderivatives relative to their domains under polyhedrality and convexity and derive the corresponding graphical modulus. For the third one, we provided an upper estimate of the projectional coderivative relative to a polyhedral set within its domain with a constraint qualification assumed. In particular we developed the sufficient condition of the relative Lipschitz-like property as a generalized critical face condition under the framework of Dontchev and Rockafellar [18].

For the Lipschitz-like property, we focused on the linear constraint systems and showed that the relations between different types of perturbation in terms of the property. We also extended to result to the linear constraint system with an implicit set constraint and characterized the property using various tools. Besides, we showed the equivalence on this property between a variational inequality and its linear approximation under full perturbation. Additionally, we considered a practical problem, a linear portfolio selection problem with two different models and drew some easy-to-hold condition for the Lipschitz-like property for the feasible set mapping and the stationary point set mapping respectively.

## 5.2 Future work

With results obtained as stated above, we plan to investigate in the following directions.

1. For chain rules we derived, it requires the set to be a smooth manifold to obtain the equation. Can we relax this setting and obtain an equation with

other conditions?

2. The calculation of projectional coderivative comes with the representation of the normal cone, $N_{\mathrm{gph}\,S|_X}$. To express it in terms of $N_{\mathrm{gph}\,S}$ and $N_X$, a constraint qualification is almost a must. As Penot [70] showed that the linear estimate can also replace the constraint qualification, can we use this to simplify the expression of projectional coderivatives for some specific problems?

3. We know that the Lipschitz-like property of $S$ corresponds to the metric regularity of $S^{-1}$. For the projectional coderivative defined as in (1.3.6), it is asymmetrical and can be used to verify the relative Lipschitz-like property according to the generalized Mordukhovich criterion. Is it possible to make some modifications on the projectional coderivative and extend the criterion for other relative stability properties, like relative metric regularity?

4. The equivalence on the Lipschitz-like property between the generalized equation and its linear approximation has been illustrated in Dontchev and Rockafellar [19]. As it is relatively more convenient to deal with the linearized system, can we obtain similar results for the relative Lipschitz-like property?

5. In Section 4.3.1 we calculate the normal cone of a simplex with sparsity constraint. Is it possible to apply the method to calculating the normal cone of the solution mapping of linear complementarity problems with sparsity constraints?

6. For affine variational inequalities, we derived an upper estimate of the projectional coderivative relative to a polyhedral set in the domain of the solution mapping under a constraint qualification. For a set that is closely tied to the critical face and the normal cone on the face, can we bypass the constraint

qualification and give a sufficient condition for the relative Lipschitz-like property?

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