MEAN FIELD GAME AND TEAM WITH STOCHASTIC LEADER-FOLLOWER INTERACTION

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Mean Field Game and Team with Stochastic Leader-Follower Interaction

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CERTIFICATE OF ORIGINALITY

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_____(Signed)

Tinghan XIE (Name of student)

Dedicate to my parents.

Abstract

The thesis is concerned with the application of mean field game (MFG) and mean field team (MFT) in the leader-follower (LF) interaction. We first introduce the LF game and MFT, separately, which can be treated as two preliminary chapters, then the LF game combine with MFT and MFG are investigated. More details about the four topics in this thesis are introduced as follows.

- The first topic studies a mixed linear quadratic (LQ) stochastic LF game with input constraint, where the model involves two agents with the same hierarchy in decision making and each agent has two controls which act as a leader and a follower, respectively. By solving a follower problem, we obtain a Nash equilibrium. Then a leader problem with constrained controls is tackled and the optimal controls are presented by projection mappings. Moreover, we consider the case that the control weights are singular. In this case, a sufficient condition for the uniform convexity of the cost functional is given and a minimizing sequence of solutions with non-degenerate control weights is constructed to investigate the weak convergence of the corresponding personal cost functionals.

- The second topic investigates the robust LQ MFT control under a direct approach, where a global uncertainty drift is involved for a large number of weakly-coupled interactive agents. All agents treat the uncertainty as an adversarial agent to obtain a "worst case" disturbance. Using variational analysis, we first obtain the centralized controls by a set of forward-backward stochastic

differential equations. Then the decentralized controls are designed by mean field heuristics. Finally, the proof of asymptotically social optimality is given.

- The third topic combines the LF problem and the MFT problem, which involves one leader and a large number of weakly-coupled interactive followers. All agents cooperate to optimize the social cost functional. Unlike the second topic, we apply the fixed point approach in this topic to solve the problem and obtain a set of decentralized social optimality strategies (the asymptotical Stackelberg equilibrium) through a consistency condition (CC) system.

- The fourth topic is a new game by combing three factors: hierarchical structure for iterative decision, model uncertainty with asymmetric information, and weak-coupling in a large population system. In particular, two classes of agents involved are denoted as leaders and followers, who sequentially make decisions with a hierarchical structure. As a consequence, the information structures between different hierarchies become asymmetric due to their iterative positions. Model uncertainty then arises in their decisions since the lacking of communication among non-cooperative leaders/followers. Moreover, all agents are framed within a weakly-coupled large population system with complex interrelations. Thus, leaders or followers play a Nash game with each other in their own hierarchy, while leaders and followers play a Stackelberg game between the two hierarchies. Applying the MFG theory, we obtain an asymptotic Stackelberg-Nash-Cournot equilibrium based on a CC system. The well-posedness of such consistency is derived by the fixed point analysis under mild conditions.

Keywords: Stackelberg game, weak-coupling, mean field game and team, input constraint, model uncertainty, forward-backward stochastic differential equation.

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Contents

Notation

$\mathbb{R}^n$	n-dimensional real Euclidean space.
$\mathbb{R}^{m\times n}$	the set of $(m \times n)$ real matrices.
$\mathbb{S}^n$	the set of all $(n \times n)$ symmetric matrices.
$\mathbb{1}_A$	the indicator function of the given set A.
$v^\top$	the transpose of vector (or matrix) v.
$\mathrm{tr}M$	the trace of the square matrix M.
M^{-1}	the inverse of matrix M.
$M \geq 0$ (≤ 0)	M is positive (negative) semi-definite.
$M > 0$ (< 0)	M is positive (negative) definite.
$\lvert v\rvert$	the standard Euclidean norm for vector v.
$\lvert M\rvert$	the Frobenius norm for matrix M and $\lvert M\rvert = \sqrt{\mathrm{tr}[MM^T]}$.
$\lVert M\rVert$	the norm for matrix function $M : [0, T] \to \mathbb{R}$ such that $\lVert M(\cdot)\rVert = \sup_{0\leq t\leq T} \lvert M(t)\rvert$.
$\langle\cdot,\cdot\rangle$	the standard Euclidean inner product and for matrices M, N, $\langle M, N\rangle = \mathrm{tr}[M^T N]$.
$:=$	Defined to be.
a.s.	almost surely
$\lvert v\rvert_M^2$	the quadratic form $v^T M v$, for given $M \geq 0$. It can also defined as $\langle Mv, v\rangle$.

$C(0,T;\mathbb{R}^n)$	the set of all continuous functions $\phi : [0,T] \to \mathbb{R}^n$.
$C^1(0,T;\mathbb{R}^n)$	the set of all continuously differentiable functions $\phi : [0,T] \to \mathbb{R}^n$.
$L^1_S(\Omega;\mathbb{R}^n)$	the set of Lebesgue measurable function $\phi : \Omega \to \mathbb{R}^n$ such that $\int_\Omega \sqrt{\phi^T(\omega)S\phi(\omega)}d\mathbb{P}(\omega) < \infty$, for $S > 0$, $\omega \in \Omega$.
$L^\infty(\Omega;\mathbb{R}^{n\times n})$	the set of essentially bounded and measurable in matrix norm $\lvert\cdot\rvert_{n\times n}$ a.s. such that $\phi : \Omega \to \mathbb{R}^{n\times n}$.
$L^p(0,T;\mathbb{R}^n)$	the set of Lebesgue measurable functions $\phi : [0,T] \to \mathbb{R}^n$ such that $\int_0^T \lvert\phi(t)\rvert^p dt < \infty$ $(1 \le p < \infty)$.
$L^\infty(0,T;\mathbb{R}^n)$	the set of essentially bounded and measurable functions $\phi : [0,T] \to \mathbb{R}^n$.
$(\Omega,\mathcal{F},\mathbb{P})$	probability space.
$\mathbb{F} = \{\mathcal{F}_t\}_{0\le t\le T}$	filtration.
$(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{0\le t\le T},\mathbb{P})$	filtered probability space.
$\mathbb{E}\ \xi$	the expectation of the random variable ξ.
$L^p_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$	the set of $\mathbb{R}^n$-valued $\mathcal{F}_T$-measurable random variables ξ such that $\mathbb{E}\lvert\xi\rvert^p < \infty$ $(1 \le p < \infty)$.
$L^p_{\mathbb{F}}(0,T;\mathbb{R}^n)$	the set of all $\mathbb{F}$-progressively measurable processes $x : [0,T]\times\Omega \to \mathbb{R}^n$ such that $\lvert x\rvert^p_{L^p} := \mathbb{E}\int_0^T \lvert x(t)\rvert^p dt < \infty$ $(1 \le p < \infty)$.
$L^p_{\mathbb{F}}(\Omega;C(0,T;\mathbb{R}^n))$	the set of $\mathbb{F}$-progressively measurable, continuous processes $x : [0,T]\times\Omega \to \mathbb{R}^n$ such that $\mathbb{E}\sup_{t\in[0,T]} \lvert x(t)\rvert^p < \infty$ $(1 \le p < \infty)$.
$\mathcal{M}(0,T)$	the set of $L^p_{\mathbb{F}}(\Omega;C(0,T;\mathbb{R}^n)) \times L^p_{\mathbb{F}}(\Omega;C(0,T;\mathbb{R}^m)) \times L^p_{\mathbb{F}}(0,T;\mathbb{R}^{m\times d})$ $(1 \le p < \infty)$.

Chapter 1

Introduction

1.1 The review of the leader-follower (or Stackelberg game) problem

The study of equilibrium problems has attracted extensive and consistent research attentions across the optimization or decision making community because of their important theoretical values and significant application potentials.

Along this research line, Nash equilibrium (NE) provides one important and fundamental notion for the solvability. In an NE, each agent is assumed to know the equilibrium strategies of the other agents and no one can increase its expected payoff by changing only its own strategy. The related NE problem has been well explored from a variety of viewpoints and considerable outcomes are thus generated. For example, see [16, 93, 103, 165, 166] for stochastic NE problem; [87, 141] for robust NE problem.

On the other hand, the leader-follower (LF) problem, which can be traced back to the work of von Stackelberg in 1934 (see [173]), provides another theoretical notion for equilibrium studies. It is a strategic game problem with at least two hierarchies of players. One hierarchy with a major position is defined as a leader and the other with a minor position is defined as a follower. The leader moves first, and then the follower will observe the leaders decision to move sequentially. Meanwhile, the

leaders anticipate the responses of the followers and then intake such responses when making their decisions. The optimal strategies for the leader and the follower form a Stackelberg equilibrium.

From the economic aspect, in the LF game, the firm as a leader strives towards a position of independence and dominates the market when the firm as a follower favours a position of dependence which is different from the NE that both the two firms want to be the position of dependence and do not become market dominance.

Form the aspect of structure, the Stackelberg game is hierarchical and sequential where the leader first announces a strategy as an anticipation and the follower find out his optimal strategy based on the leader's anticipation. Then, the leader minimizes his cost functional by taking his own optimal strategy as a realization after anticipating the followers best response. Thus, there is a looking forward and backward processes for the leader. By contrast, the Nash game is simultaneous where all the players optimize their strategies in single hierarchy at the same time. And the anticipation and realization of each player in the game are both uniform and simultaneous in Nash game.

Form the aspect of information structure, to achieve the LF equilibrium, each leader should know the complete information of all players in system (including the leaders and the followers) when anticipating the NE response of the followers, however, each follower is not necessary to know all the information of the leaders and only needs to know the strategies that the leaders announce and the information of all the followers. Thus, the information between the leaders and the followers are asymmetric. To achieve an NE, all players need to know the complete information and the information between the players are symmetric.

The LF problem can be categorized by the static and dynamic context. The dynamic context can be further categorized by the deterministic case and stochastic case. More details will be introduced as follows.

1.1.1 The review of the static leader-follower problem in mathematical field

In the static context, the Stackelberg problem is studied without time variable and the LF equilibrium will not be related to time. Then such problem can be viewed as a particular bilevel optimization problem (see [30, 38, 188]).

In mathematical field, [154] investigated a deterministic multiple leader-follower (multi-LF) game and [155] extended the work that the leader anticipates the response explicitly by the aggregate follower reaction curve and proved the existence and uniqueness of the equilibrium. If the follower's response is not unique, then it needs to find the best respose under the worst choices from the rational response set instead of the rational response curve [123]. Unlike the deterministic multi-LF games, [66] studied a stochastic multi-LF game and showed the existence and uniqueness of the stochastic multiple-leader Stackelberg-Nash-Cournot (SNC) equilibrium under some assumptions.

The Stackelberg game problem can be transformed to other forms of mathematical optimization problem. For example, a class of multi-LF game that can be formulated as a generalized NE problem with convexified strategy sets was considered in [144] and the similar original problem that can be constructed as variational inequalities was mentioned in [91]. By the special structure of Stackelberg game, the best response (or responses) of the leader (or leaders) are found by solving an optimization problem with constraints for the best response (or responses) of the follower (or followers), which leads to a mathematical program with equilibrium constraints (see [55, 67, 125, 132, 186]).

Sometimes both the leader and follower doubt the model specification, or the model, in real world, normally contains uncertain parameter. To solve such problems, [92] used the robust method that supposed the uncertain parameters belong to some

sets and minimized the cost function with respect to the worst-case scenario. Then it reformulated and solved the problem as a generalized variational inequality problem. While [66] applied the Bayesian method by assuming the uncertainty in the inverse demand function following a probability distributions of a random variable, and then solved it to obtain the stochastic multi-LF SNC equilirum. In [131], since the ambiguity of the true probability distribution, each player selected the optimal strategy with respect to the worst distribution rather than the worst scenario to hedge the risk.

1.1.2 The review of the static leader-follower problem in economy and management field

In economy and management field, as we mention before, von Stackelberg first put forward such game structure to model duopoly competition in his book [173]. After that, there are a lot of research literature about the LF competition (see [111]). [124] investigated the Stackelberg equilibrium on monopolistic competition and the Stackelberg game combined with the theory of price agreement on monopolistic competition was considered in [34]. [77, 139] studied the Stackelberg game model under the background of duopoly game and the existence and stability of such Stackelberg equilibrium were shown under some general conditions.

One of the interested points for the economists to the Stackelberg competition is its advantage and efficiency in economy and management. Stackelberg illustrated the advantage of moving first in a oligopolistic interaction and [152] made the concept "first-mover advantage" deeply rooted in people's mind. [63] showed that all sequential move structures are beneficial compared to the simultaneous move Cournot markets by investigating a class of Stackelberg markets. [110] designed an experimental market to compare the quantity between the Stackelberg and Cournot game. And it proved that, in any matching scheme, the Stackelberg market yield higher

output, higher consumer rents, and higher welfare levels than Cournot markets and, thus, higher efficiency.

However, the "first-mover advantage" does not always hold. [8] gave a striking result that the "first-mover advantage" is eliminated when there is even a slight imperfection in the observability of the leader's choice for the follower. At that moment, the set of pure strategy (If only one specific strategy can be selected under each given information in the complete information game, this strategy is a pure strategy. A mixed strategy is an assignment of a probability to each pure strategy.) NE obtained for the LF game with imperfect observation, if the follower's best response was single valued, coincided exactly with the set of pure strategy NE obtained for the associated simultaneous move game.

In recent year, the Stackelberg equilibrium has been applied in supply chain such as product line [68], inventory [149], retail [51], product remanufacturing [151]. In the financial market, [39, 153, 163] investigated the LF competition in the forward market and [67] studied the Stackelberg game in the commodity market. Also, the static LF game has been used in telecommunication industry [66, 170, 192] and electricity markets [55, 92, 144]. Especially, [55] showed that the largest producer can gain profits by withholding emission allowances and driving up the emission cost for rival followers which illustrated the "first-mover advantage" again. This result is quite important for the national strategy of "carbon neutrality" since, in some cases, the emission rights are equal to the right to development for the developing countries.

1.1.3 The review of the dynamic leader-follower problem (deterministic)

In the dynamic (deterministic) context, the Stackelberg problem is investigated with time variable. After the Stackelberg model had been put forward, most of the liter-

ature are related to the economy and management in static context.

Until 1970s, the dynamic (deterministic) Stackelberg game in the continuous time was considered primarily in the work of [50]. It was also the first to treat the feedback Stackelberg solutions for discrete time games. Then, [158] investigated the Stackelberg competition in static and dynamic nonzero-sum two-player games and gave a discussion of the linear-quadratic Stackelberg differential game. And some properties of the controls that are functions of the state variables of the LF game in addition to time was discussed in [159]. Another references that consider the open-loop Stackelberg solutions under discrete-time framework was investigated in [118] and the feedback Stackelberg solutions in discrete-time dynamic games was considered in [62]. [16, 127] gave a very comprehensive review about the dynamic non-cooperative LF game with discrete time and continuous time framework.

The open-loop information structure, where the players are committed to the strategy based on initial state and no measurement of state is available, is very important for tackling the LF optimization problem. [50] first studied the open-loop Stackelberg solution in dynamic games. Some other references [158, 159] that considered the necessary condition for the existence of an open-loop Stackelberg solution with two-player games and [160] for the multi-leader and multi-follower differential game. The necessary conditions of an open-loop Stackelberg solution and a Hamiltionian system, which was exploited to solve a two point boundary value problem, of a linear-quadratic Stackelberg games was given in [1]. [80] studied a new sufficient existence conditions for an open-loop LF equilibrium in terms of the solvability of a terminal-value problem of two symmetric Riccati differential equations and a coupled system of Riccati matrix differential equations, and [14] discussed the mixed Stackelberg open-loop solution in nonzero-sum differential games.

Unlike the open-loop information structure, the closed-loop information structure can be classified into three types. By [16, 26], if each player can access to current

state measurements, then it is called the feedback information structure. If each player can access to current state measurements and the initial state value, then it is called the closed-loop memoryless information structure. If each player can access to current state measurements and adapt his strategy to the evolution of the system, then it is called the closed-loop information structure. Compare to the open-loop information structure, the closed-loop information structure is more difficult to determine the Stackelberg solution in differential games. The main reason is the expression of the rational reaction set of the follower, the partial derivative of the leaders control with respect to the state measurement, and some attempts have been mentioned in [16, 62]. There are two main approaches to investigate the closed-loop dynamic Stackelberg game problem.

The first approach is the min-min LF strategies with a team-optimal method which first optimize the leader criterion as a team and then both controls are selected such that the follower's control react the leader's control in a rational reaction set (see [17, 18]). Especially, [112] studied both the min-max and min-min LF solutions with closed-loop information structure. Another approach is to define the follower's rational reaction set for a given control of the leader and turn the original control problem to be non-classical problem. [134] used a variational method to solve the non-classical problem with assuming that is a normal optimization problem, while [145] emphasized such technique does lead to a solution for all initial states.

Meanwhile, the dynamic (deterministic) LF game has been applied in pricing and production planning [76], traffic networks [82], time delay problem [187], and macroeconomics, such as policy making [84]. Since the government and private agent both doubt a common approximating model and have different preferences in [84], it defined the LF (or Ramsey plan) equilibrium with robust decision makers in which the government and private agent have different linear-quadratic worst-case model and the leader's current and future control setting was tracked past by a vector of

Lagrange multipliers.

1.1.4 The review of the dynamic leader-follower problem (stochastic)

In the dynamic (stochastic) context, the Stackelberg problem contains a noise or a stochastic process. The first literature to discuss the equilibrium in stochastic Stackelberg dynamic games was in [48].

The LF problem with additive noise is that the diffusion term only contains constants and the state and control do not appear in the diffusion term. The literature that related to the additive noise are given as follows. The LF equilibrium solution where players have access to noisy (but redundant) state information was considered in [11]. [12] investigated the linear-quadratic stochastic LF dynamic games with noisy observation and obtained the feedback Stackelberg solution. The existence of stochastic incentive problems with nested information and multiple levels of hierarchy was discussed in [13]. [156, 157] considered the stochastic Stackelberg differential game with asymmetric information, overlapping information, and their applications.

Sometimes, the diffusion term can contain state, or control, or both. If only the state appears in the diffusion term, we called it multiplicative noise. For example, The maximum principle for the global Stackelberg solution with multiplicative noise and adapting to closed-loop memoryless information structure was introduced in [26]. An application of such model in manufacturer-retailer cooperative advertising game was mentioned in [25]. Meanwhile, it can be used to study the real options games in complete and incomplete markets (see [27]) Note that The closed-loop information structure in stochastic case need to additionally adapted to the filtration generating by a Brownian motion. This is the same for the open-loop information structure in stochastic case comparing with deterministic case.

If the control appears in the diffusion term, we called it controlled diffusion. For

example, the open-loop LF problem with random coefficients and controlled diffusion, was first investigated in [190]. [126] explored the model of linear-quadratic generalized Stackelberg game with controlled diffusion and proved its unique solvability. The linear-quadratic stochastic LF differential games for jump-diffusion systems with controlled diffusion and random coefficients was discussed in [135] and a mixed linear quadratic Stackelberg game with input constraint was considered in [183]. These kinds of model have a wide application in financial market [71, 88], insurance industry [41, 52].

With the development of mean field game theory, the hierarchical structure under a large population system was investigated in [136, 178, 138]. The mean field Stackelberg game with aggregation of delayed instruction and state control delay were introduced in [22] and [23], respectively. The open-loop Stackelberg strategy for mean field type linear-quadratic stochastic differential game was given in [130] and the open-loop Stackelberg strategy for linear-quadratic stochastic mean field team problem was discussed in [101]. Also, the mean field stochastic LF game has been applied in mitigating epidemics, such as COVID-19 [5].

1.1.5 Another classification for the leader-follower problem

A general LF problem may also be classified into four types depending on N and M, the population of the leaders and followers, respectively.

- Type 1 is the single-leader and single-follower (SL/SF) problem ($N = M = 1$), which is the most basic type of LF problem endowed with the simplest but illustrative structure. For instance, interested readers may refer to [4, 173] for its static study and [50, 52, 53, 127, 130, 156, 190] for dynamic one.

- Type 2 and Type 3 are the multi-leader and single-follower (ML/SF) problem ($N \geq 2, M = 1$) and the single-leader and multi-follower (SL/MF) problem

($M \geq 2, N = 1$), respectively. These two types involve multiple leaders (or followers). Thus, there arises not only the LF problem between two hierarchical agents but also the NE problem among the agents within the same hierarchy. Therefore, the so-called Stackelberg-Nash-Cournot (SNC) equilibrium is designed where Cournot implies all leaders (or followers) are homogeneous. Readers may refer to [92] for robust ML/SF game, while [101, 138, 144, 178, 185] for SL/MF game.

- Type 4 is the multi-leader and multi-follower (ML/MF) problem ($N \geq 2, M \geq 2$) (see [66, 126, 160]) which is an extension of ML/SF and SL/MF problems in Type 2 and Type 3. The followers are non-cooperative and thus compete in a Nash game parameterized by the strategy profile of leaders. Likewise, all leaders are also competitive in a Nash game parameterized by the NE responses from all followers. Moreover, all leaders and followers compete in a Stackelberg game at an upper level. For more relevant studies, readers may refer to [67] for stochastic SNC equilibrium in the European gas market; [155] for ML/MF game in an oligopolistic market; [163] for ML/MF game in a forward market equilibrium model.

1.2 The review of the weak-coupling, mean field game (or team), and model uncertainty

1.2.1 The weak-coupling and mean field game (or team)

The main purpose of this thesis is to study the LF problem in the context of a large-population (or large-scaled) system ($N \gg 2, or M \gg 2$) where all (leaders/followers) agents are weakly-coupled with more realistic interactions. It is noteworthy that a large-population system arises often and naturally in various fields such as economics [32, 73, 121, 180], engineering [103, 115], medicine [19, 120] as well as management

science [64]. The most salient feature of a large-population system is the existence of weak-coupling interaction amongst all involved agents. Under the weak-coupling condition, the individual behaviors from a micro-scale can be negligible, whereas the overall mass effects of all agents cannot be ignored on a macro-scale. A weak-coupling system is strongly motivated by a variety of practical applications in reality, and we defer its more detailed illustrations in Chapter 6.

When N (or M) is sufficiently large in our multi-agent system, the interaction across all agents becomes rather complex and difficult to be handled. This is related to the so-called "curse of dimensionality," and more details are deferred in Chapter 6. The mean field game (MFG) theory (see [28, 40, 44, 43, 122]) or mean field team (MFT) theory (see [101, 105, 147, 176]) provide us a tractable approach to analyze such problems and compute the associated equilibrium or social optimality. As the trade-off, some approximated asymptotic equilibrium (or social optimality) can be designed with a more effective computation load. By this approach, we can reduce the complexity in computation and obtain an approximated solution via some consistency condition (CC) matching scheme (see more details deferred in Chapter 5 and 6). In the last decade, the MFG (or MFT) has been well studied in various research areas such as economics [41, 78, 175], engineering [61, 103, 115], medicine, and vaccination [81, 109], especially for the recent COVID-19 pandemic [5, 58, 70]. One of the particular applications of the MFG (or MFT) is the linear quadratic mean field (LQ-MF) game (team) problem, which can model various problems. Readers may refer to [10, 42, 99, 100, 105, 128, 169, 177, 179] and the reference therein for a comprehensive review.

Normally, there are two routes to solve the MFG and MFT problems. One is called the fixed point approach (see [101, 103, 105, 138]), which starts by applying the mean field approximation and constructing a fixed point problem. Then, the N-player game degenerates to an optimal control problem. By analyzing the optimal

response of the representative player, the decentralized strategies can be designed, which are proved to be asymptotically optimal. Another route is called the direct approach (see [69, 108, 176, 184]), which starts by solving the N-player game problem and a Riccati-like equation system formally under a large-population and high dimensional environment. Then, by letting N goes to infinity after obtaining the centralized optimal strategies, one can derive the decentralized optimal control laws. Thus, the difference between the two methods is the timing of using the mean field heuristics technique.

1.2.2 Model uncertainty

In general, mathematical models only describe and simulate the complicatedly real world in an approximated approach. Therefore, it is very meaningful to investigate a model with uncertainty parameters. Recall that there are two main methods to deal with the model uncertainty: one is the robust method that uses the minimax technique and considers the worst-case analysis (see [2, 87]); another one is the Bayesian method for which some subjective probability measure is introduced to average all possible realizations (see [85, 86] for details). The model uncertainty is also well documented in control theory literature. For example, in [98, 99, 174, 184], the LQ-MF control problem with a global uncertainty parameter is considered by researchers. More details, in [98], the so-called "hard constraint" approach was adopted to overcome difficulties after using the Lagrange multiplier. The "soft constraint" case (see [15, 172, 72]) was investigated in [99, 174, 184], which removed the bound of the disturbance and add a penalty for the disturbance in cost functional. The situation that a local disturbance appears between each agent was studied in [137, 168].

1.3 Problem Statement and Main Contributions of Each Topic in the Thesis

1.3.1 The first topic

In the first topic (Chapter 3) of this thesis, we investigate a mixed LF linear quadratic (LQ) control problem. Compare to the model in [14] or the classical Stackelberg differential games that have hierarchies between each player, Chapter 3 studies a stochastic differential game under the LQ framework with two players and there is no hierarchy of decision making between these two players. However, the hierarchy appears inside each player which means two players act as both leaders and followers. More specifically, the state equation and cost functional for each player contain two controls. On one hand, from a single player's aspect, for a given strategy, the player first chooses one control which acts as a follower, and then according to his former choice to seek another control which acts as a leader to minimize his own cost functional. If we consider each player as a system, then the system is very similar to the LF social optimization problem in that all the players inside the system work cooperatively and aim to minimize (maximize) the total cost functional (payoff) (see [101]). On the other hand, the two players play a non-cooperative differential game and seek the NE. Player 1 and player 2 always pick their own strategies simultaneously. In other words, for the mixed LF problem, we need to first solve a follower problem and look for an open-loop NE between player 1 and player 2. Then, with the former NE responses, we solve a leader problem and seek another open-loop NE. The latter optimal strategies and the former optimal strategies constitute the Stackelberg solution to our original mixed LF problem.

Compare to the previous work [14, 16, 25] that have no control constrained or mixed structure, our study is the first to investigate the mixed LF LQ differential game with input constrained and singular control weights. The constrained control

makes the classical approaches fail to apply (see [54, 97]) and brings some difficulties to our study here. First, the related forward-backward stochastic differential equation (FBSDE, see [133, 189]) system is no longer linear which becomes a nonlinear FBSDE with a projection operator. Second, since the related FBSDE is nonlinear, we cannot obtain the linear state feedback control by using the standard Riccati equation method. In our study, we only consider the case that the controls who act as leaders are constrained in a closed convex set of the whole space. The situation for constrained controls that act as followers is not investigated here since their corresponding solutions are non-smooth and the whole system becomes very difficult to be tackled. The LQ problem with input constraint has been studied from various aspects. For example, the relative topic under the LQ-MF games context was investigated in [93, 94]. A class of LQ problems in which the control process is constrained in a cone was discussed in [29, 97, 129].

Moreover, unlike [93] that the control weights are positive-definite, the weight coefficients of controls in Chapter 3 are allowed to be singular, which is realistic and full of challenges. In classical LQ control problems (like [16, 101, 138]), the control weights are required to be positive-definite or greater than δI (for some $\delta > 0$ and I is an identity matrix), such that the problem admits a unique solution in the deterministic case or additive-noise case (see [93, 101, 138, 191]). Even if the unconstrained control weights can be negative in the stochastic case, like [130, 164, 165, 166, 181], a sufficiently positive-definite condition (e.g., $D^\top P(t)D > 0, t \in [s, T], s \in [0, T)$ in [191]) will be added. In Chapter 3, the controls have constraints and the weights are degenerate. For this reason, we cannot use a projection operator to map the original control value from the whole space to its closed convex subset or apply the Riccati equation method. Instead, we decouple the states in leader problem which is a FBSDE system and estimate each part of the system to obtain a sufficient condition for the uniform convexity of the leader problem with respect to

the corresponding control.

The contributions of the first topic can be summarized as follows:

- We introduce and analyze a class of mixed stochastic differential LF games where two players with the same hierarchy play a non-cooperative game and find an open-loop NE in differently hierarchical Nash games. In our setting, the controls act as leaders are constrained and the control weights can be singular.

- For the non-singular case, the optimal pair is represented by a projection mapping, and a Hamiltonian system is obtained for the stochastic mixed LF games problem through the FBSDE system with projection operators.

- For the singular case, we discuss the uniform convexity of the cost functionals whose corresponding states are some fully coupled FBSDE systems and give out sufficient criteria. The near-optimal control sequence is obtained and the minimizing sequence method is applied to prove the weak convergence of its corresponding cost functionals.

1.3.2 The second topic

The second topic (Chapter 4) of the thesis studies the social optimality of the robust LQ-MF control model with a common uncertain drift by using a direct approach. The common uncertain term appears in both the state equation and the cost functional of each agent. Unlike [99] is related to MFG problem, Chapter 4 studies a MFT problem. Meanwhile, compare with the fixed point approach (like [93, 94, 101, 105, 174]), we use the direct approach method to solve the MFT problem in Chapter 4 (see Section 1.2.1). For detail, we first perturb all the agents and used duality procedures to tackle the large-scale problem with high dimensional FBSDE. After that, the centralized controls explicitly depending on x_i and the state average term $x^{(N)}$ are obtained first and then the decentralized controls are designed by mean

field heuristics. In reality, it is almost impossible for one agent to have all the information of other agents. Therefore, the decentralized controls which are based on the individual information sets will be used instead of the centralized controls which are based on the full information set and the information structure of each agent is different. Unlike [99, 174] that the weight coefficient Q of state in the cost functional is allowed to be indefinite, the coefficients in Chapter 4 are time-varying, which means the coefficients can be changed at different times. The time-variant systems could be applied in many areas such as the earth's thermodynamic and the human vocal tract (see [162, 167]).

Compared with the previous works [99, 174], the second topic mainly makes the following contributions:

- Instead of using the fixed-point method (see [101, 174]), the direct approach is applied to solve the robust LQ-MF social control problem, where the state weight Q is allowed to be indefinite.

- By the solvability of the low-dimensional Riccati-liked equation system, we obtain the condition for the uniform convexity of a high-dimensional control problem.

- Comparing to [174] whose CC system contains five coupled equations, we just have four coupled Riccati-liked equations, which are much easier to be tackled. The number of coupled equations can be even reduced to three under a specific condition. Moreover, in proving asymptotic optimality, we obtain the consistency of mean field approximations without setting an additional assumption.

1.3.3 The third topic

The third topic (Chapter 5) of this thesis investigates the social optimality of the LF LQ-MF control problem, which can be considered as a combination of the first topic

and the second topic. However, unlike the previous works [16, 190] and Chapter 3, the model in Chapter 5 contains one leader and N followers. The leader's state appears in both the state equation and the cost functional of each follower. It shows that the dynamics and the cost functionals of the N followers are directly influenced by the behavior of the leader. Meanwhile, compared to the model in [107, 138], the followers' state average term in Chapter 5 appears in all the state equations and the cost functionals, which implies that the state dynamics and the cost functionals are highly interactive and coupled, respectively. Different from the direct approach in previous works [69, 108, 176, 184] and Chapter 4, the third topic uses the fixed point approach to solve the LF MFT problem which starts by freezing the state average term and constructing an auxiliary control problem to obtain the decentralized controls.

Compared with the previous works [99, 105, 138], the third topic mainly makes the following contributions:

- A social optimum problem is studied for mean field models with hierarchical structure. Unlike [138] where the leader and the followers play a noncooperative game and try to minimize their cost functionals, all individuals in Chapter 5 aim to minimize the social cost functional which equals the summation of all individual cost functionals. Since the cost functional presents individual performance in the game problems, the order of the magnitude of its perturbation is $\frac{1}{N}$ which can be ignored (note that the N followers are coupled by the state average) and the state average may be approximated by a stochastic process directly (see [99]). However, in Chapter 5, the order of magnitude of the perturbation cannot be ignored after summing up all the cost functionals, which makes the problem very complicated. To overcome such difficulties, we approximate some terms as N goes to infinity and use a duality procedure

combined with auxiliary equations to transform the variation of the social cost functional into a standard LQ control form. Then, we construct an auxiliary control problem and a forward-backward consistency system to help us obtain the decentralized form of the optimal controls for the N followers.

- The decentralized controls of the LF problem are obtained and the solvability of a high-dimensional CC system is discussed. Since the leader's state equation and cost functional are fully coupled with the followers' state equations and cost functionals, it is more difficult to solve the leader's problem. Except constructing auxiliary problem by mean field approximation as in the former part, we need to construct six auxiliary equations and use duality relations to obtain the decentralized form of the optimal control for the leader. Unlike the followers' problem, the final CC system of the leader's problem contains ten equations which becomes a high-dimensional problem. To solve such equations directly is very difficult since they are fully coupled and have high-dimensional characteristics. We transform the CC system to a simple form of linear FBSDE and discuss the solvability of the FBSDE through the Ricatti equation method (see [133]).

- The decentralized strategies of the LF problem are proved to be asymptotic Stackelberg equilibrium by perturbation analysis. Different from [105, 107, 138], we, in Chapter 5, discuss the asymptotic Stackelberg equilibrium for the team problem. First, we need to prove the decentralized strategies for the followers have asymptotic social optimality. Since the LF problem contains two hierarchies, we need to consider both the leader's and the followers' cost functionals when using the standard method (see [105]). The asymptotic optimality is proved by decoupling the above cost functionals with two duality procedures and some arguments in error estimates. Second, we need to prove the decen-

tralized strategies for the LF problem are asymptotic Stackelberg equilibrium. Some error estimates very hard to be given directly since they are fully coupled. We decompose them by applying the Ricatti equation method and then estimate them in proper order.

1.3.4 The fourth topic

The fourth topic (Chapter 6) of this thesis discusses the application of the MFG approach in the weakly-coupling and model uncertainty ML/MF problem under a large-population system ($N \gg 2$, $M \gg 2$) with an incomplete and asymmetric information structure. Since all the agents are competitive, they will not share their information with others. Each leader and follower can only observe their objective functions, which depend on the strategy average. However, by analyzing the previous performance of the whole system, the agents can obtain the empirical distribution of the strategy average and uncertain parameters which can be seen as some common information in our weakly-coupled LF problem. Based on the common information, we apply the mean field method to the weakly-coupled LF model with model uncertainty and obtain an asymptotic equilibrium.

This topic contains many elements of previous chapters such as the LF equilibrium (see Chapters 3 and 5), the weakly-coupling with mean field heuristics (see Chapters 4 and 5), the model uncertainty (see Chapter 4), the input constraint (see Chapter 3), and the non-cooperative game (see Chapter 3). However, the whole study of Chapter 6 is under a *static* context (like [4, 66, 92, 144]), which is different from the previous chapters with a dynamic background (see [28, 43, 103, 122, 138]). For this reason, the general form game, LF game, and corresponding information structure under the static background are first introduced and then the motivation of weakly-coupling (see [106, 121, 146, 171]) and model uncertainty (see [2, 20, 21, 85, 86, 92, 141]) are provided as some explanation in Chapter 6. Meanwhile, it can also be considered

as some supplement for Chapters 4 and 5 although the LF MFG problem had been studied under the dynamic context (see [138, 178]).

The main contributions can be sketched as follows

- It is the first time that the weakly-coupled LF game problem in the large-population and static optimization setting is introduced.
- The ubiquitous model (parameter) uncertainty is naturally introduced in the weakly-coupled LF problem in the presence of empirical distribution from the large-population system.
- The MFG theory is applied to the above static weakly-coupled LF equilibrium problem, and some approximated decentralized Stackelberg-Nash-Cournot (SNC) strategy is designed based on a CC system.
- The well-posedness of the CC system is established under mild conditions for both the general nonlinear and quadratic cases.
- A numerical example is provided to simulate the mean field approximation.

1.4 Organization of the Thesis

In this thesis, we study the MFG and MFT problems with stochastic LF interaction. The chapters of this thesis are arranged carefully. We first introduce a mixed LF problem between two players to give out the principle of the Stackelberg game and then study an MFT control problem under a large-population system to provide some mean field techniques. After that, an MFT LF problem is investigated which can be considered as a combination of the previous two topics. Finally, an MFG LF problem under a static optimization context is given as the MFG topic of this thesis (since the MFG LF problem under dynamic context had been discussed in [138]).

All these topics mentioned above are original research articles. The first three topics have been published in *Applied Mathematics & Optimization* and *ESAIM: Control, Optimisation and Calculus of Variations*, respectively (see [101, 183, 184]) and the last topics have been finished and will be published soon. The following content is the organization of each chapter.

In Chapter 2, we present a brief introduction of the LQ stochastic control problem. Some inequalities which will be used in the following chapters are also given.

Chapter 3 investigates a mixed LF differential games problem, where the model involves two players with the same hierarchy in decision making and each player has two controls that act as a leader and a follower, respectively. We first formulate the mixed LF game problem and discuss the followers' problem by using variational analysis and obtain an NE with linear state feedback control forms. Then, the leaders' problem with constrained control is solved and a Hamiltonian system with non-degenerated controls weights are obtained. After that, we study the solvability of the leaders' problem under singular controls weights and obtain the near-optimal control sequence of the problem. Finally, two examples are provided.

Chapter 4 studies a robust LQ MFT control problem. The model involves a global uncertainty drift which is common for a large number of weakly-coupled interactive agents. The robust LQ-MF problem is formulated first, and then the worst disturbance based on the maximum principle is solved. Moreover, we seek the socially optimal solution under the "worst case" uncertainty and design asymptotically optimal decentralized controls by handling coupled FBSDEs. Finally, the asymptotically social optimality of decentralized controls is proved and a numerical example is provided to simulate the efficiency of decentralized control.

Chapter 5 deals with an LQ MFT LF problem, where the model involves one leader and a large number of weakly-coupled interactive followers. We first solve the optimal controls for followers based on person-by-person optimality and obtain the

CC system of the follower's problem. Then, we seek the socially optimal solution to the leader's problem and give the CC system of the leader's problem. Meanwhile, the CC system is transformed to a simple form of linear FBSDE, and its well-posedness is discussed. Finally, the details of proving the asymptotic Stackelberg equilibrium are given and a numerical example is provided to simulate the efficiency of decentralized control.

Chapter 6 introduces the ML/MF problem in the context of a large-population system where all agents are weakly-coupled with more realistic interactions. The definition of the general Nash game, the ML/MF game, symmetric game, and the corresponding information structure is introduced first. And then we formulate the weak-coupling LF problem and the motivation of weakly-coupling and model uncertainty. After that, a general case and a quadratic case of the weak-coupling LF problem are represented, respectively. The MFG approach is applied to tackle the problem in the above two cases and the existence and uniqueness of the fixed point system (or the CC system) are also discussed. Finally, a numerical example is provided.

Chapter 7 concludes the whole thesis and plans for future work.

Chapter 2

Preliminary

In this chapter, we introduce the linear quadratic (LQ) stochastic control model and present some lemmas which will be used in the following chapters.

2.1 Linear Quadratic Stochastic Control Model

In this thesis, we mainly focus on the LQ framework, therefore we first introduce the LQ stochastic control model. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space (see [191, Chapter 1]), $\xi \in \mathbb{R}^n$ be the values of the initial state, and $W(\cdot)$ be an one-dimensional standard Brownian motions. ξ and $W(\cdot)$ are defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{F}_t = \sigma(W(s), 0 \leq s \leq t)$ augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ is the natural filtration generated by the Brownian motions. Let $T > 0$ be given. For any $\xi \in \mathbb{R}^n$, consider the following linear stochastic differential equation (SDE):

$$\begin{cases} dx(t) =& [A(t)x(t) + B(t)u(t) + f(t)]dt \\ & \quad + [C(t)x(t) + D(t)u(t) + \sigma(t)]dW(t), \quad t \in [0, T], \\ x(0) =& \xi, \end{cases} \tag{2.1}$$

where $A(\cdot)$, $B(\cdot)$, $f(\cdot)$, $C(\cdot)$, $D(\cdot)$, $\sigma(\cdot)$ are deterministic matrix-valued functions of suitable sizes. In addition, the quadratic cost functional is given as

$$J(\xi; u(\cdot)) = \frac{1}{2}\mathbb{E}\int_0^T \left\{ |x(t)|^2_{Q(t)} + |u(t)|^2_{R(t)} \right\} dt + \frac{1}{2}\mathbb{E}|x(T)|^2_G, \tag{2.2}$$

where $Q(\cdot)$ and $R(\cdot)$ are $\mathbb{S}^n$- and $\mathbb{S}^m$-valued functions, respectively, and $G \in \mathbb{S}^n$. If

$$\begin{cases} A(\cdot), C(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n}), \quad B(\cdot), D(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times m}), \\ f(\cdot), \sigma(\cdot) \in L^2(0,T;\mathbb{R}^n), \end{cases}$$

and $u(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$, then, for any $\xi \in \mathbb{R}^n$, (2.1) admits a unique strong solution $x(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^n))$. Moreover, there exists a constant $c > 0$ such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |x(t)|^2\Big] \le c\mathbb{E}\Big[|\xi|^2 + \int_0^T |u(t)|^2 + |f(t)|^2 + |\sigma(t)|^2 dt\Big]. \tag{2.3}$$

The right hand side of (2.2) is well-defined under $u(\cdot)$.

In what follows, if v consists of several sub-vectors $v_1, \cdots, v_N$, it is sometimes written for simplicity as $(v_1, \cdots, v_N)$ or $(v_1^\top, \cdots, v_N^\top)^\top$ by abusing the matrix formation.

Next, we introduce a general LQ stochastic control model under a large-population system (or large-scale system) with N agents. Let $\xi_i \in \mathbb{R}^n$ are the values of the initial states and $W_i(\cdot)$ are one-dimensional standard Brownian motions, where $i = 1, \cdots, N$. ξ_i and $W_i(\cdot)$ are defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Unlike the above classic LQ stochastic control model, here we define σ-algebra $\mathcal{F}_t^i = \sigma(W_i(s), 0 \le s \le t)$, where $1 \le i \le N$ and $\mathcal{F}_t = \sigma(W_i(s), 0 \le s \le t, 1 \le i \le N)$. $\mathbb{F}^i = \{\mathcal{F}_t^i\}_{0\le t\le T}$, is the natural filtration generated by $W_i(\cdot)$ and $\mathbb{F} = \{\mathcal{F}_t\}_{0\le t\le T}$. We denote the N agents by $\{\mathcal{A}_i\}_{i\in\mathcal{I}}$, where $\mathcal{I} = \{1, \cdots, N\}$ denotes the index set of the agents. The aggregation of all agents is denoted by $\mathcal{A} := \{\mathcal{A}_i\}_{i\in\mathcal{I}}$. The state process of agent $\mathcal{A}_i$ is modeled by the following linear SDE:

$$\begin{cases} dx_i(t) = [A(t)x_i(t) + B(t)u_i(t) + F(t)x^{(N)}(t) + f(t)]dt + [C(t)x_i(t) \\ \qquad\qquad + D(t)u_i(t) + \bar{F}(t)x^{(N)}(t) + \sigma(t)(t)]dW_i(t), \quad t \in [0,T], \\ x_i(0) = \xi_i, \end{cases} \tag{2.4}$$

where $x^{(N)}(t) := \frac{1}{N}\sum_{i=1}^{N} x_i(t)$, $t \in [0,T]$, is the state average of the agents. $A(\cdot)$, $B(\cdot)$, $F(\cdot)$, $f(\cdot)$, $C(\cdot)$, $D(\cdot)$, $\bar{F}(\cdot)$, $\sigma(\cdot)$ are deterministic matrix-valued functions of suitable sizes. The cost functional of $\mathcal{A}_i$ is modeled by

$$\begin{aligned} J_i(\xi; u(\cdot)) =& \frac{1}{2}\mathbb{E}\int_0^T \left\{ |x_i(t) - \Gamma(t)x^{(N)}(t) - \eta(t)|^2_{Q(t)} + |u_i(t)|^2_{R(t)} \right\} dt \\ &+ \frac{1}{2}\mathbb{E}|x_i(T) - \hat{\Gamma}x^{(N)}(T) - \hat{\eta}|^2_G, \end{aligned} \tag{2.5}$$

where $\xi = \{\xi_1, \cdots, \xi_N\}$ and $u(\cdot) = \{u_1(\cdot), \cdots, u_N(\cdot)\}$. $\Gamma(\cdot)$ and $\eta(\cdot)$ are $\mathbb{R}^{n\times n}$- and $\mathbb{R}^n$-valued functions, and $\hat{\Gamma} \in \mathbb{R}^{n\times n}$, $\hat{\eta} \in \mathbb{R}^n$. By some mild conditions, (2.4) admits a unique strong solution with similar estimate as (2.3) and (2.5) is well-defined under $u(\cdot)$. The details will be given in following chapters and we omit the corresponding discussion here.

2.2 Some Lemmas

Lemma 2.1 (Gronwall's Inequality)**.** *Suppose the continuously real-valued function $g(t)$ satisfy*

$$0 \le g(t) \le \alpha(t) + \beta \int_0^t g(s)ds, \quad 0 \le t \le T,$$

with $\beta \ge 0$ and α integrable on $[0,T]$. Then

$$g(t) \le \alpha(t) + \beta \int_0^t \alpha(s)e^{\beta(t-s)}ds, \quad 0 \le t \le T.$$

Proof The proof is trivial and we omit here. □

Lemma 2.2 (Burkholder-Davis-Gundy Inequality)**.** *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$ be a complete filtered probability space augmented by all $\mathbb{P}$-null sets in $\mathcal{F}$ and let $W(t)$ be*

an m-dimensional standard Brownian motion. Let $X \in L^{2,loc}_{\mathcal{F}}(0,T;\mathbb{R}^{n\times m})$, where

$$L^{2,loc}_{\mathcal{F}}(0,T;\mathbb{R}^{n\times m}) = \{x : [0,T]\times\Omega \to \mathbb{R}^{n\times m} | x(\cdot) \text{ is } \{\mathcal{F}_t\}_{0\le t\le T} - \text{adapted and} \\ \int_0^T |x(t)|^2 dt < \infty, \quad \mathbf{P} - a.s.\}.$$

Then, for any $r > 0$, there exists a constant $C_r > 0$ such that for any stopping time τ,

$$\frac{1}{C_r}\mathbb{E}\Big\{\int_0^\tau |X(s)|^2 ds\Big\}^r \le \mathbb{E}\Big\{\sup_{0\le t\le\tau}\Big|\int_0^t X(s)dW(s)\Big|^{2r}\Big\} \le C_r\mathbb{E}\Big\{\int_0^\tau |X(s)|^2 ds\Big\}^r.$$

Proof See the Theorem 5.4 in Chapter 1 of [191]. □

Lemma 2.3. *Let Γ is a r-dimensional linear subspace in $\mathbb{R}^n$ and $r \le n$ with $(v_1, \cdots, v_r)$ as basis. $\mathbf{P}_\Gamma(\cdot) : \mathbb{R}^n \to \Gamma$ is a projection operator defined under $\langle\cdot,\cdot\rangle_M = \langle M^{\frac{1}{2}}\cdot, M^{\frac{1}{2}}\cdot\rangle$, $M > 0$. Denote $V = (v_1, \cdots, v_r)$, then the projection operators can be expressed as $\mathbf{P}_\Gamma = V(V^\top MV)^{-1}V^\top M$.*

Proof First, it is easy to verify that $\langle M^{\frac{1}{2}}\cdot, M^{\frac{1}{2}}\cdot\rangle$ is a well-defined inner product on $\mathbb{R}^n$. By [35, Chapter 5], there exists a unique projection $\mathbf{P}_\Gamma$ with respect to (w.r.t.) Γ and $\langle\cdot,\cdot\rangle_M$. Then, for any vector $v_1 \in \Gamma$, there exists a vector $\theta_1 \in \mathbb{R}^n$ such that $v_1 = V\theta_1$. Thus, for any vector $\theta_2 \in \mathbb{R}^n$, $\mathbf{P}_\Gamma\theta_2 \in \Gamma$ and there exists a vector $\theta_3 \in \mathbb{R}^n$ such that

$$\mathbf{P}_\Gamma\theta_2 = V\theta_3.$$

Hence,

$$\theta_2 - \mathbf{P}_\Gamma\theta_2 \perp V\theta_1.$$

Therefore,

$$\langle\theta_2 - \mathbf{P}_\Gamma\theta_2, V\theta_1\rangle_M = 0,$$

which derives that

$$V^\top M(\theta_2 - V\theta_3) = V^\top M(\theta_2 - \mathbf{P}_\Gamma \theta_2) = 0.$$

Then, it follows that

$$\mathbf{P}_\Gamma \theta_2 = V(V^\top MV)^{-1} V^\top M\theta_2,$$

which implies the lemma. $\square$

From now on, we may suppress the notation of time t in Chapter 3, 4, and 5 if necessary.

Chapter 3

Mixed Linear Quadratic Stochastic Differential Leader-Follower Game with Input Constraint

In this chapter, a mixed leader-follower (LF) differential games problem with input constraint is introduced. Two players play a Nash game with each other in the same hierarchy and each player plays an LF game with his two controls which act as a leader and a follower, respectively. Meanwhile, the controls act as followers are unconstrained and the controls act as leaders are constrained. In addition, this chapter discusses the case that the control weights are allowed to be singular.

3.1 Problem Formulation

Suppose that there are two players, 1 and 2, engaged in the game. The system state is described by the following SDE on $[0, T]$:

$$\begin{cases} dx_i = [A^i x_i + B_1^i u_i + B_2^i v_i + b^i]dt + [C_1^i x_i + D_1^i u_i + D_2^i v_i + \sigma_1^i]dW_1 \\ \qquad\quad + [C_2^i x_i + F_1^i u_i + F_2^i v_i + \sigma_2^i]dW_2, \\ x_i(0) = \xi_i, \quad i = 1, 2, \end{cases} \tag{3.1}$$

where $A^i, B_1^i, B_2^i, b^i, C_1^i, D_1^i, D_2^i, \sigma_1^i, C_2^i, F_1^i, F_2^i, \sigma_2^i$ are matrix-valued functions of suitable sizes and all these coefficients are satisfy following assumption:

$$(\mathbf{A3.1})\begin{cases} A^i(\cdot) \in L^2(0,T;\mathbb{R}^{n\times n}),\ B_1^i(\cdot) \in L^2(0,T;\mathbb{R}^{n\times m}),\ B_2^i(\cdot) \in L^2(0,T;\mathbb{R}^{n\times m}), \\ C_1^i(\cdot) \in L^2(0,T;\mathbb{R}^{n\times n}),\ D_1^i(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times m}),\ D_2^i(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times m}), \\ C_2^i(\cdot) \in L^2(0,T;\mathbb{R}^{n\times n}),\ F_1^i(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times m}),\ F_2^i(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times m}), \\ b^i(\cdot),\ \sigma_1^i(\cdot),\ \sigma_2^i(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n), \quad i=1,2. \end{cases}$$

The state equation of each player contains two controls. The controls u_1, u_2 act as leaders and the controls v_1, v_2 act as followers. The set of admissible controls u_i is defined as follows:

$$\mathcal{U}_i = \left\{u_i | u_i(t) \in L^2_{\mathbb{F}}(0,T;\Gamma)\right\}, \quad i=1,2,$$

where $\Gamma \subset \mathbb{R}^m$ is a closed convex set. Unlike [25], our controls u_1, u_2 are constrained in a subset of full space $\mathbb{R}^m$, which leads to some difficulties for solving the optimal controls in the following. The set of admissible controls v_i is defined as follows:

$$\mathcal{V}_i = \left\{v_i | v_i(t) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)\right\}, \quad i=1,2,$$

and we let $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$, $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$.

The cost functional J_i for player i is defined as follows:

$$\begin{aligned} J_i(u_1;v_1;u_2;v_2) =& \frac{1}{2}\mathbb{E}\int_0^T |x_i - k_i x_j|^2_{Q_i} + |u_i|^2_{R_1^i} + |v_i|^2_{R_2^i} dt + \frac{1}{2}\mathbb{E}\Big[|x_i(T) \\ & - k_i x_j(T)|^2_{G_i} + 2\langle g_i, x_i(T) - k_i x_j(T)\rangle\Big], \quad i=1,2, \quad j \neq i, \end{aligned} \tag{3.2}$$

where parameter $k_i \in [0,1]$ and Q_i, R_1^i, R_2^i, G_i are weight matrices. g_1, g_2 are random variables. The coefficients satisfy following assumption:

$$(\mathbf{A3.2})\begin{cases} Q_i(\cdot) \in L^2(0,T;\mathbb{S}^n),\ R_1^i(\cdot) \in L^\infty(0,T;\mathbb{S}^m), \\ R_2^i(\cdot) \in L^\infty(0,T;\mathbb{S}^m),\ G_i \in \mathbb{S}^n,\ g_i \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n), \quad i=1,2. \end{cases}$$

For each player, (3.2) could be essentially considered as a team optimization problem [89, 101] which is different from game problem [138, 156, 190]. Unlike [166], to avoid the independence of two players, we add coefficients k_1, k_2, such that the two players will concern about others' performance when they make decisions. We can see that (3.2) is equivalent to

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\int_0^T |(1-k_i)x_i + k_i(x_i - x_j)|^2_{Q_i} + |u_i|^2_{R_1^i} + |v_i|^2_{R_2^i} dt \\
&+ \frac{1}{2}\mathbb{E}\Big[|(1-k_i)x_i(T) + k_i(x_i(T) - x_j(T))|^2_{G_i} + 2\langle g_i, (1-k_i)x_i(T) \\
&+ k_i(x_i(T) - x_j(T))\rangle\Big].
\end{aligned}$$

By [73], this is called relative performance concerns. In the finance area, k_i determines the ith player's preference for absolute wealth versus relative wealth. When k_i is large, the ith player is more concerned with relative wealth than absolute wealth (see [65, 73]). However, this makes the system very complicated to be tackled since the players' states are coupled with each other.

Remark 3.1. *Note that we allow the weight coefficients R_1^i and R_2^i, $i = 1, 2$, to be singular. Thus, the classical open-loop solutions (see [166, 190]) cannot be obtained directly. By [195], we look for the feedback control form, however, the controls u_1, u_2 are constrained that the FBSDE system cannot be decoupled by using standard Riccati equation method. For these reasons, handling the problem becomes even more challenging.*

By [191, Section 4], under (**A3.1**)-(**A3.2**), for any $u_i \in \mathcal{U}_i$, $v_i \in \mathcal{V}_i$, (4.1) admits a unique strong solution for $x_i \in L^2_{\mathbb{F}}(\Omega; C(0,T;\mathbb{R}^n))$. Moreover, there exists a constant $c > 0$ such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |x_i|^2\Big] \leq c\mathbb{E}\Big[|\xi_i|^2 + \int_0^T (|u_i|^2 + |v_i|^2 + |b^i|^2 + |\sigma_1^i|^2 + |\sigma_2^i|^2)dt\Big].$$

Meanwhile, for any $(u_1, u_2, v_1, v_2) \in \mathcal{U} \times \mathcal{V}$, (3.2) is well defined.

In a mixed LF game, both the followers' and the leaders' problems are non-cooperative differential games between two players with the same hierarchy (see [14, 25]). Thus, our mixed LF game problem can divide into a follower part and a leader part:

Problem 3.1. (**FP**) *For any pair $(u_1, u_2) \in \mathcal{U}$, given $\xi_1, \xi_2 \in \mathbb{R}^n$, player 1 and player 2 seek for an Nash equilibrium, that is a pair $(\bar{v}_1, \bar{v}_2)$, where $\bar{v}_i(u_1; u_2)$: $\mathcal{U} \to \mathcal{V}_i$, $i = 1, 2$, is a mapping, such that*

$$J_1\Big(u_1; \bar{v}_1(u_1; u_2); u_2; \bar{v}_2(u_1; u_2)\Big) = \inf_{v_1 \in \mathcal{V}_1} J_1\Big(u_1; v_1(u_1; u_2); u_2; \bar{v}_2(u_1; u_2)\Big),$$

$$J_2\Big(u_1; \bar{v}_1(u_1; u_2); u_2; \bar{v}_2(u_1; u_2)\Big) = \inf_{v_2 \in \mathcal{V}_2} J_2\Big(u_1; \bar{v}_1(u_1; u_2); u_2; v_2(u_1; u_2)\Big).$$

Problem 3.2. (**LP**) *For given $\xi_1, \xi_2 \in \mathbb{R}^n$ and optimal pair $(\bar{v}_1, \bar{v}_2) \in \mathcal{V}$, player 1 and player 2 correspondingly seek for an Nash equilibrium again, that is a pair strategy $(\bar{u}_1, \bar{u}_2) \in \mathcal{U}$, such that*

$$J_1\Big(\bar{u}_1; \bar{v}_1(\bar{u}_1; \bar{u}_2); \bar{u}_2; \bar{v}_2(\bar{u}_1; \bar{u}_2)\Big) = \inf_{u_1 \in \mathcal{U}_1} J_1\Big(u_1; \bar{v}_1(u_1; \bar{u}_2); \bar{u}_2; \bar{v}_2(u_1; \bar{u}_2)\Big),$$

$$J_2\Big(\bar{u}_1; \bar{v}_1(\bar{u}_1; \bar{u}_2); \bar{u}_2; \bar{v}_2(\bar{u}_1; \bar{u}_2)\Big) = \inf_{u_2 \in \mathcal{U}_2} J_2\Big(\bar{u}_1; \bar{v}_1(\bar{u}_1; u_2); u_2; \bar{v}_2(\bar{u}_1; u_2)\Big).$$

For classical Stackelberg differential games problem, like [190], the two hierarchies of players have their own cost functionals. The procedure of solving these problems is

first solving the follower's problem to obtain an optimal control $\bar{u}_2(u_1)$, for any fixed u_1, and then solving the leader's problem to obtain $\bar{u}_1$. The optimal pair $(\bar{u}_1, \bar{u}_2(\bar{u}_1))$ constitutes the Stackelberg equilibrium of the classical Stackelberg differential game problem. However, in (**FP**) and (**LP**), we can see that player 1 and player 2 has the same hierarchy and make decisions simultaneously. From a *horizontal* view of (**FP**) and (**LP**), the two players play a non-cooperative game. Both of them will concern about others' performance and seek an NE such that no player can benefit from unilaterally changing its own strategy. More specifically, at the v_i, $i = 1, 2$, actions, player 1 and 2 play a Nash game, with the *additional* information from the u_i, $i = 1, 2$, actions and find a pair $(\bar{v}_1(u_1; u_2), \bar{v}_2(u_1; u_2))$ (depending on u_1, u_2) such that the equations in (**FP**) hold. Then, at the u_i, $i = 1, 2$, actions, player 1 and 2 play a Nash game again and find a pair $(\bar{u}_1, \bar{u}_2)$ such that the equations in (**LP**) hold. The optimal pair $(\bar{u}_1, \bar{v}_1(\bar{u}_1; \bar{u}_2), \bar{u}_2, \bar{v}_2(\bar{u}_1; \bar{u}_2))$ constitutes the solution of our original mixed LF game problem. From a *vertical* view of the two Nash games, they play with hierarchies (first obtain $(\bar{v}_1, \bar{v}_2)$, and then obtain $(\bar{u}_1, \bar{u}_2)$) and therefore constitute a Stackelberg game. According to the above reason, we call this game a mixed LF game.

3.2 The Follower Part

In this section, we first discuss the open-loop NE for the follower part. Since the states and cost functionals of player 1 and player 2 are symmetric, in what follows, we only consider one of them and the situation of another player is similar. For a given pair $(u_1, u_2) \in \mathcal{U}$, we give our first proposition of this section:

Proposition 3.1. *Suppose that* (**A3.1**)-(**A3.2**) *hold. Then,* $(\bar{v}_1, \bar{v}_2) \in \mathcal{V}$ *is an open-loop NE of* (**FP**) *if and only if the following two conditions hold:*

1. *The adapted solution* $(\bar{x}_i, p_i, \beta_i, \gamma_i)$ *to the FBSDE*

$$\begin{cases} d\bar{x}_i =[A^i\bar{x}_i + B_1^i u_i + B_2^i\bar{v}_i + b^i]dt + [C_1^i\bar{x}_i + D_1^i u_i + D_2^i\bar{v}_i \\ \qquad + \sigma_1^i]dW_1 + [C_2^i\bar{x}_i + F_1^i u_i + F_2^i\bar{v}_i + \sigma_2^i]dW_2, \\ dp_i = -[(A^i)^\top p_i + (C_1^i)^\top\beta_i + (C_2^i)^\top\gamma_i + Q(\bar{x}_i - k_i\bar{x}_j)]dt \\ \qquad + \beta_i dW_1 + \gamma_i dW_2, \\ \bar{x}_i(0) = \xi_i, \quad p_i(T) = G_i(\bar{x}_i(T) - k_i\bar{x}_j(T)) + g_i, \quad i = 1, 2, \end{cases} \tag{3.3}$$

satisfies the following stationary condition:

$$R_2^i\bar{v}_i + (B_2^i)^\top p_i + (D_2^i)^\top\beta_i + (F_2^i)^\top\gamma_i = 0. \tag{3.4}$$

2. *The following convexity condition holds:*

$$\mathbb{E}\Big\{\int_0^T \langle Q_i x_i, x_i\rangle + \langle R_2^i v_i, v_i\rangle dt + \langle G_i x_i(T), x_i(T)\rangle\Big\} \geq 0, \quad i = 1, 2,$$

where x_i *is the solution to the following SDE:*

$$\begin{cases} dx_i = (A^i x_i + B_2^i v_i)dt + (C_1^i x_i + D_2^i v_i)dW_1 + (C_2^i x_i + F_2^i v_i)dW_2, \\ x_i(0) = 0. \end{cases}$$

Proof By variational analysis (see [166, Theorem 4.1]), the result can be obtained. Thus, we omit it here. □

By the stationary condition (3.4) in Proposition 3.1, if R_2^i is invertible, then we have the optimal controls:

$$\bar{v}_i = -(R_2^i)^{-1}[(B_2^i)^\top p_i + (D_2^i)^\top\beta_i + (F_2^i)^\top\gamma_i], \quad i = 1, 2.$$

Since the controls v_1, v_2 have no constraint, we can consider their linear state feedback representations. We set the following nonhomogeneous relationships: $p_i(t) =$

$P_i(t)\bar{x}_i(t) + \varphi_i(t),\ t \in [0,T]$, $i = 1, 2$. Then, by [190, 191], we have following Riccati equations:

$$\begin{cases} \dot{P}_i + P_i A^i + (A^i)^\top P_i + \sum_{m=1}^{2} (C_m^i)^\top P_i (C_m^i) \\ \qquad\qquad + Q_i - (\widehat{B}_2^i)^\top (\widehat{R}_2^i)^{-1} \widehat{B}_2^i = 0, \\ P_i(T) = G_i, \quad \widehat{R}_2^i > 0, \quad i = 1, 2, \end{cases} \tag{3.5}$$

where $P_i(\cdot) \in C^1(0,T;\mathbb{R}^{n\times n})$ are matrix-value functions. Here

$$\begin{cases} \widehat{B}_2^i = (B_2^i)^\top P_i + (D_2^i)^\top P_i C_1^i + (F_2^i)^\top P_i C_2^i, \\ \widehat{R}_2^i = R_2^i + (D_2^i)^\top P_i D_2^i + (F_2^i)^\top P_i F_2^i, \quad i = 1, 2. \end{cases}$$

Now, we assume (3.5) admits adapted solutions, then we have following backward stochastic differential equations (BSDEs):

$$\begin{cases} d\varphi_i = -\,[(\widehat{A}^i)^\top \varphi_i + (\widehat{C}_1^i)^\top \theta_1^i + (\widehat{C}_2^i)^\top \theta_2^i + \Psi_i^\top u_i + (\widehat{C}_1^i)^\top P_i \sigma_1^i \\ \qquad + (\widehat{C}_2^i)^\top P_i \sigma_2^i - k_i Q_i \bar{x}_j + P_i b^i] dt + \theta_1^i dW_1 + \theta_2^i dW_2, \\ \varphi_i(T) = -k_i G_i \bar{x}_j(T) + g_i, \quad i = 1, 2, \end{cases} \tag{3.6}$$

where the unknown here are 3-tuple $(\varphi_i, \theta_1^i, \theta_2^i)$ of $\mathbb{F}$-progressively measurable $\mathbb{R}^n$-valued processes and

$$\begin{cases} \widehat{A}^i = A^i - B_2^i (\widehat{R}_2^i)^{-1} \widehat{B}_2^i, \quad \Phi_i = (D_2^i)^\top P_i D_1^i + (F_2^i)^\top P_i F_1^i, \\ \widehat{C}_1^i = C_1^i - D_2^i (\widehat{R}_2^i)^{-1} \widehat{B}_2^i, \quad \widehat{C}_2^i = C_2^i - F_2^i (\widehat{R}_2^i)^{-1} \widehat{B}_2^i, \\ \Psi_i = (B_1^i)^\top P_i + (D_1^i)^\top P_i C_1^i + (F_1^i)^\top P_i C_2^i - \Phi_i^\top (\widehat{R}_2^i)^{-1} \widehat{B}_2^i, \quad i = 1, 2. \end{cases}$$

By [164, Page 2285], the solutions of equation (3.5) are said to be strong regular if

$$\widehat{R}_2^i = R_2^i + (D_2^i)^\top P_i D_2^i + (F_2^i)^\top P_i F_2^i \geq \delta I, \quad \text{a.e. } t \in [0,T], \quad i = 1, 2,$$

for some $\delta > 0$. According to the Theorem 4.5 and Corollary 3.4 in [164], for any fixed u_1, u_2, $\widehat{R}_2^i \geq \delta I$ for some $\delta > 0$ is equivalent to $v_1(\cdot) \mapsto J_i(u_1, v_1(\cdot), u_2, \bar{v}_2)$ is uniformly convex, when $i = 1$ (the situation is similar for $i = 2$). Then, by the property of uniform convexity, (**FP**) is uniquely solvable. Thus, for our further analysis, we give the following assumption:

(**A3.3**) $\widehat{R}_2^i \geq \delta I$, $i = 1, 2$, for some $\delta > 0$.

Concerning (**A3.3**), we have following proposition:

Proposition 3.2. *Suppose that* (**A3.1**)-(**A3.3**) *hold. Then* (**FP**) *admits the unique optimal control pair* $(\bar{v}_1, \bar{v}_2) \in \mathcal{V}$ *of state feedback forms, where*

$$\begin{aligned} \bar{v}_i = &-(\widehat{R}_2^i)^{-1}[\widehat{B}_2^i \bar{x}_i + \Phi_i u_i + (B_2^i)^\top \varphi_i + (D_2^i)^\top \theta_1^i \\ &+ (F_2^i)^\top \theta_2^i + (D_2^i)^\top P_i \sigma_1^i + (F_2^i)^\top P_i \sigma_2^i], \quad i = 1, 2, \end{aligned} \tag{3.7}$$

and for any $\xi_i \in \mathbb{R}^n$, $i = 1, 2$, $j \neq i$, *the corresponding optimal costs are*

$$\begin{aligned} &\inf_{v_i \in \mathcal{V}_i} J_i(u_i; v_i; u_j; \bar{v}_j) \\ =&\frac{1}{2}\mathbb{E}\langle P_i(0)\xi_i, \xi_i\rangle + \langle \varphi_i(0), \xi_i - k_i \xi_j\rangle + \frac{1}{2}\mathbb{E}\int_0^T -|(\widehat{R}_2^i)^{-\frac{1}{2}}(\Phi_i u_i \\ &+ (B_2^i)^\top \varphi_i + (D_2^i)^\top \theta_1^i + (F_2^i)^\top \theta_2^i + (D_2^i)^\top P_i \sigma_1^i + (F_2^i)^\top P_i \sigma_2^i)|^2 \\ &+ \langle (R_1^i + (D_1^i)^\top P_i D_1^i + (F_1^i)^\top P_i F_1^i) u_i, u_i\rangle + \langle P_i \sigma_1^i, \sigma_1^i\rangle + \langle P_i \sigma_2^i, \sigma_2^i\rangle \\ &+ 2\langle (B_1^i)^\top \varphi_i + (D_1^i)^\top \theta_1^i + (F_1^i)^\top \theta_2^i + (D_1^i)^\top P_i \sigma_1^i + (F_1^i)^\top P_i \sigma_2^i, u_i\rangle \\ &+ 2k_i\langle [(\widehat{A}^i)^\top - I]\varphi_i + (\widehat{C}_1^i)^\top \theta_1^i + (\widehat{C}_2^i)^\top \theta_2^i + \Psi_i^\top u_i + (\widehat{C}_1^i)^\top P_i \sigma_1^i \\ &+ (\widehat{C}_2^i)^\top P_i \sigma_2^i + P_i b^i, \bar{x}_j\rangle + 3k_i^2\langle Q_i \bar{x}_j, \bar{x}_j\rangle + 2\langle \theta_1^i, \sigma_1^i\rangle + 2\langle \theta_2^i, \sigma_2^i\rangle dt. \end{aligned} \tag{3.8}$$

Proof The proofs of the optimal controls and the optimal costs are similar to the discussion in Page 313 to 317 of [191, Chapter 6] and theorem 2.3 of [190]. We omit the them here. □

Remark 3.2. *If v_1, v_2, are constrained, we cannot obtain the linear feedback form of controls by constructing Riccati equations. Then the problem becomes very difficult to be tackled and its corresponding solutions are non-smooth. Thus, we do not consider the constrained case here.*

3.3 The Leader Part

After solving (**FP**), we turn to the leaders part. Note that when the players take their optimal controls $\bar{v}_i(\cdot)$, $i = 1, 2$, given by (3.7), the leaders problem end up with the following state equation:

$$\begin{cases} dx_i =[\widehat{A}^i x_i + \widehat{B}^i_3\varphi_i + \widehat{D}^i_2\theta^i_1 + \widehat{F}^i_2\theta^i_2 + \widehat{B}^i_1 u_i + b^i + \widehat{D}^i_2 P_i\sigma^i_1 \\ \qquad + \widehat{F}^i_2 P_i\sigma^i_2]dt + [\widehat{C}^i_1 x_i + (\widehat{D}^i_2)^\top\varphi_i + \widehat{D}^i_3\theta^i_1 + \widehat{D}^i_4\theta^i_2 + \widehat{D}^i_1 u_i \\ \qquad + (\widehat{D}^i_3 P_i + I)\sigma^i_1 + \widehat{D}^i_4\sigma^i_2]dW_1 + [\widehat{C}^i_2 x_i + (\widehat{F}^i_2)^\top\varphi_i + (\widehat{D}^i_4)^\top\theta^i_1 \\ \qquad + \widehat{F}^i_3\theta^i_2 + \widehat{F}^i_1 u_i + (\widehat{D}^i_4)^\top\sigma^i_1 + (\widehat{F}^i_3 P_i + I)\sigma^i_2]dW_2, \\ d\varphi_i = -[(\widehat{A}^i)^\top\varphi_i + (\widehat{C}^i_1)^\top\theta^i_1 + (\widehat{C}^i_2)^\top\theta^i_2 + \Psi_i^\top u_i + (\widehat{C}^i_1)^\top P_i\sigma^i_1 \\ \qquad + (\widehat{C}^i_2)^\top P_i\sigma^i_2 - k_i Q_i x_j + P_i b^i]dt + \theta^i_1 dW_1 + \theta^i_2 dW_2, \\ x_i(0) =\xi_i, \quad \varphi_i(T) = -k_i G_i x_j(T) + g_i, \quad i = 1, 2, \quad j \neq i, \end{cases} \tag{3.9}$$

where

$$\begin{cases} \widehat{B}^i_3 = -B^i_2(\widehat{R}^i_2)^{-1}(B^i_2)^\top,\ \widehat{D}^i_2 = -B^i_2(\widehat{R}^i_2)^{-1}(D^i_2)^\top,\ \widehat{F}^i_2 = -B^i_2(\widehat{R}^i_2)^{-1}(F^i_2)^\top, \\ \widehat{B}^i_1 = B^i_1 - B^i_2(\widehat{R}^i_2)^{-1}\Phi_i,\ \widehat{D}^i_3 = -D^i_2(\widehat{R}^i_2)^{-1}(D^i_2)^\top,\ \widehat{D}^i_4 = -D^i_2(\widehat{R}^i_2)^{-1}(F^i_2)^\top, \\ \widehat{F}^i_3 = -F^i_2(\widehat{R}^i_2)^{-1}(F^i_2)^\top,\ \widehat{D}^i_1 = D^i_1 - D^i_2(\widehat{R}^i_2)^{-1}\Phi_i,\ \widehat{F}^i_1 = F^i_1 - F^i_2(\widehat{R}^i_2)^{-1}\Phi_i, \end{cases}$$

and the cost functional:

$$
\begin{aligned}
&J_i(u_1;\bar{v}_1(u_1;u_2);u_2;\bar{v}_2(u_1;u_2))\\
=&\frac{1}{2}\mathbb{E}\int_0^T |x_i-k_ix_j|^2_{Q_i}+|u_i|^2_{R_1^i}+\Big|(\widehat{R}_2^i)^{-1}[\widehat{B}_2^i\bar{x}_i+\Phi_iu_i+(B_2^i)^\top\varphi_i\\
&+(D_2^i)^\top\theta_1^i+(F_2^i)^\top\theta_2^i+(D_2^i)^\top P_i\sigma_1^i+(F_2^i)^\top P_i\sigma_2^i]\Big|^2_{R_2^i}dt\\
&+\frac{1}{2}\mathbb{E}(|x_i(T)-k_ix_j(T)|^2_{G_i}+\langle g_i,x_i(T)-k_ix_j(T)\rangle,\ i=1,2,\ j\neq i,
\end{aligned}
\tag{3.10}
$$

which are still quadratic forms.

Next, we discuss the open-loop NE for the leaders part. Let us rewrite (3.9) and (3.10) compactly such that

$$
\begin{cases}
dX=&[\widehat{A}X+\widehat{B}_3\varphi+\widehat{D}_2\theta_1+\widehat{F}_2\theta_2+\widehat{B}_1u+b_1]dt\\
&\quad+[\widehat{C}_1X+(\widehat{D}_2)^\top\varphi+\widehat{D}_3\theta_1+\widehat{D}_4\theta_2+\widehat{D}_1u+\sigma_1]dW_1\\
&\quad+[\widehat{C}_2X+(\widehat{F}_2)^\top\varphi+(\widehat{D}_4)^\top\theta_1+\widehat{F}_3\theta_2+\widehat{F}_1u+\sigma_2]dW_2,\\
d\varphi=&-[(\widehat{A})^\top\varphi+(\widehat{C}_1)^\top\theta_1+(\widehat{C}_2)^\top\theta_2+\Psi^\top u-QX+b_2]dt\\
&\quad+\theta_1dW_1+\theta_2dW_2,\\
X(0)=&(\xi_1^\top\ \xi_2^\top)^\top,\quad \varphi(T)=-GX(T)+g,
\end{cases}
\tag{3.11}
$$

where

$$
X=\begin{pmatrix}x_1\\x_2\end{pmatrix},\ \varphi=\begin{pmatrix}\varphi_1\\\varphi_2\end{pmatrix},\ u=\begin{pmatrix}u_1\\u_2\end{pmatrix},\ \theta_1=\begin{pmatrix}\theta_1^1\\\theta_1^2\end{pmatrix},\ \theta_2=\begin{pmatrix}\theta_2^1\\\theta_2^2\end{pmatrix},\ g=\begin{pmatrix}g_1\\g_2\end{pmatrix},
$$

$$
\widehat{A}=\begin{pmatrix}\widehat{A}^1&0\\0&\widehat{A}^2\end{pmatrix},\ \widehat{B}_3=\begin{pmatrix}\widehat{B}_3^1&0\\0&\widehat{B}_3^2\end{pmatrix},\ \widehat{D}_2=\begin{pmatrix}\widehat{D}_2^1&0\\0&\widehat{D}_2^2\end{pmatrix},\ \widehat{F}_2=\begin{pmatrix}\widehat{F}_2^1&0\\0&\widehat{F}_2^2\end{pmatrix},
$$

$$\widehat{B}_1=\begin{pmatrix}\widehat{B}_1^1 & 0\\ 0 & \widehat{B}_1^2\end{pmatrix},\ \widehat{C}_1=\begin{pmatrix}\widehat{C}_1^1 & 0\\ 0 & \widehat{C}_1^2\end{pmatrix},\ \widehat{D}_3=\begin{pmatrix}\widehat{D}_3^1 & 0\\ 0 & \widehat{D}_3^2\end{pmatrix},\ \widehat{D}_4=\begin{pmatrix}\widehat{D}_4^1 & 0\\ 0 & \widehat{D}_4^2\end{pmatrix},$$

$$\widehat{D}_1=\begin{pmatrix}\widehat{D}_1^1 & 0\\ 0 & \widehat{D}_1^2\end{pmatrix},\ \widehat{C}_2=\begin{pmatrix}\widehat{C}_2^1 & 0\\ 0 & \widehat{C}_2^2\end{pmatrix},\ \widehat{F}_3=\begin{pmatrix}\widehat{F}_3^1 & 0\\ 0 & \widehat{F}_3^2\end{pmatrix},\ \widehat{F}_1=\begin{pmatrix}\widehat{F}_1^1 & 0\\ 0 & \widehat{F}_1^2\end{pmatrix},$$

$$\Psi=\begin{pmatrix}\Psi_1 & 0\\ 0 & \Psi_2\end{pmatrix},\ Q=\begin{pmatrix}0 & k_1Q_1\\ k_2Q_2 & 0\end{pmatrix},\ G=\begin{pmatrix}0 & k_1G_1\\ k_2G_2 & 0\end{pmatrix}, v=\begin{pmatrix}v_1\\ v_2\end{pmatrix},$$

$$b_1=\begin{pmatrix}b^1+\widehat{D}_2^1P_1\sigma_1^1+\widehat{F}_2^1P_1\sigma_2^1\\ b^2+\widehat{D}_2^2P_2\sigma_1^2+\widehat{F}_2^2P_2\sigma_2^2\end{pmatrix},\ b_2=\begin{pmatrix}(\widehat{C}_1^1)^\top P_1\sigma_1^1+(\widehat{C}_2^1)^\top P_1\sigma_2^1+P_1b^1\\ (\widehat{C}_1^2)^\top P_2\sigma_1^2+(\widehat{C}_2^2)^\top P_2\sigma_2^2+P_2b^2\end{pmatrix},$$

$$\sigma_1=\begin{pmatrix}(\widehat{D}_3^1P_1+I)\sigma_1^1+\widehat{D}_4^1\sigma_2^1\\ (\widehat{D}_3^2P_2+I)\sigma_1^2+\widehat{D}_4^2\sigma_2^2\end{pmatrix},\ \sigma_2=\begin{pmatrix}(\widehat{D}_4^1)^\top\sigma_1^1+(\widehat{F}_3^1P_1+I)\sigma_2^1\\ (\widehat{D}_4^2)^\top\sigma_1^2+(\widehat{F}_3^2P_2+I)\sigma_2^2\end{pmatrix},$$

and

$$\begin{aligned}J_1(u;\bar{v}(u))=&\frac{1}{2}\mathbb{E}\Big\{\int_0^T\langle K_1Q_1K_1^\top X,X\rangle+\langle\widetilde{R}_1^1u,u\rangle+\langle\widetilde{R}_2^1\widehat{R}_2^{-1}[\widehat{B}_2X\\ &+\Phi u+B_2^\top\varphi+D_2^\top\theta_1+F_2^\top\theta_2+\sigma_3],\widehat{R}_2^{-1}[\widehat{B}_2X+\Phi u\\ &+B_2^\top\varphi+D_2^\top\theta_1+F_2^\top\theta_2+\sigma_3]\rangle dt\\ &+\langle K_1G_1K_1^\top X(T),X(T)\rangle+2\langle K_1e_1^\top g,X(T)\rangle\Big\},\end{aligned}\tag{3.12}$$

$$\begin{aligned}J_2(u;\bar{v}(u))=&\frac{1}{2}\mathbb{E}\Big\{\int_0^T\langle K_2Q_2K_2^\top X,X\rangle+\langle\widetilde{R}_1^2u,u\rangle+\langle\widetilde{R}_2^2\widehat{R}_2^{-1}[\widehat{B}_2X\\ &+\Phi u+B_2^\top\varphi+D_2^\top\theta_1+F_2^\top\theta_2+\sigma_3],\widehat{R}_2^{-1}[\widehat{B}_2X+\Phi u\\ &+B_2^\top\varphi+D_2^\top\theta_1+F_2^\top\theta_2+\sigma_3]\rangle dt\\ &+\langle K_2G_2K_2^\top X(T),X(T)\rangle+2\langle K_2e_2^\top g,X(T)\rangle\Big\},\end{aligned}\tag{3.13}$$

where

$$K_1=\begin{pmatrix}1\\-k_1\end{pmatrix},\ \widetilde{R}_1^1=\begin{pmatrix}R_1^1&0\\0&0\end{pmatrix},\ \widetilde{R}_2^1=\begin{pmatrix}R_2^1&0\\0&0\end{pmatrix},\ \widehat{B}_2=\begin{pmatrix}\widehat{B}_2^1&0\\0&\widehat{B}_2^2\end{pmatrix},$$

$$\widehat{R}_2=\begin{pmatrix}\widehat{R}_2^1&0\\0&\widehat{R}_2^2\end{pmatrix},\ B_2=\begin{pmatrix}B_2^1&0\\0&B_2^2\end{pmatrix},\ D_2=\begin{pmatrix}D_2^1&0\\0&D_2^2\end{pmatrix},\ F_2=\begin{pmatrix}F_2^1&0\\0&F_2^2\end{pmatrix},$$

$$\Phi=\begin{pmatrix}\Phi_1&0\\0&\Phi_2\end{pmatrix},\ \sigma_3=\begin{pmatrix}(D_2^1)^\top P_1\sigma_1^1+(F_2^1)^\top P_1\sigma_2^1\\(D_2^2)^\top P_2\sigma_1^2+(F_2^2)^\top P_2\sigma_2^2\end{pmatrix},\ K_2=\begin{pmatrix}1\\-k_2\end{pmatrix},$$

$$\widetilde{R}_1^2=\begin{pmatrix}0&0\\0&R_1^2\end{pmatrix},\ \widetilde{R}_2^2=\begin{pmatrix}0&0\\0&R_2^2\end{pmatrix},\ e_1=\begin{pmatrix}1\\0\end{pmatrix},\ e_2=\begin{pmatrix}0\\1\end{pmatrix}.$$

Then, we give out the proposition for (**LP**) with non-degenerate control weights.

Proposition 3.3. *Suppose that* (**A3.1**)-(**A3.3**) *and the following inequalities hold:*

$$\begin{aligned}\mathbb{E}\Big\{\int_0^T&\langle K_iQ_iK_i^\top X_i,X_i\rangle+\langle\widetilde{R}_1^i\delta u_i,\delta u_i\rangle+\langle\widetilde{R}_2^i\widehat{R}_2^{-1}[\widehat{B}_2X+\Phi\delta u_i\\&+B_2^\top\varphi+D_2^\top\theta_1+F_2^\top\theta_2],\widehat{R}_2^{-1}[\widehat{B}_2X+\Phi\delta u_i+B_2^\top\varphi+D_2^\top\theta_1\\&+F_2^\top\theta_2]\rangle dt+\langle K_iG_iK_i^\top X_i(T),X_i(T)\rangle\Big\}>0,\quad i=1,2,\end{aligned}\tag{3.14}$$

where $\delta u_1=(u_1^T\ 0^T)^T$, $\delta u_2=(0^T\ u_2^T)^T$ *and* $(X_i,\varphi_i,\theta_{1i},\theta_{2i})$ *are the solutions to the following SDE:*

$$\begin{cases}dX_i=&[\widehat{A}X_i+\widehat{B}_3\varphi_i+\widehat{D}_2\theta_{1i}+\widehat{F}_2\theta_{2i}+\widehat{B}_1\delta u_i]dt+[\widehat{C}_1X_i+\widehat{D}_2^\top\varphi_i\\&+\widehat{D}_3\theta_{1i}+\widehat{D}_4\theta_{2i}+\widehat{D}_1\delta u_i]dW_1+[\widehat{C}_2X_i+\widehat{F}_2^\top\varphi_i+\widehat{D}_4^\top\theta_{1i}\\&+\widehat{F}_3\theta_{2i}+\widehat{F}_1\delta u_i]dW_2,\quad X_i(0)=(0^\top\ 0^\top)^\top,\\d\varphi_i=&-(\widehat{A}^\top\varphi_i+\widehat{C}_1^\top\theta_{1i}+\widehat{C}_2^\top\theta_{2i}+\Psi^\top\delta u_i-QX_i)dt\\&+\theta_{1i}dW_1+\theta_{2i}dW_2,\quad\varphi_i(T)=-GX_i(T),\quad i=1,2.\end{cases}\tag{3.15}$$

Then, $(\bar{u}_1, \bar{u}_2) \in \mathcal{U}$ *is an open-loop NE of* (**LP**) *if and only if the adapted solution* $(\overline{X}, \overline{\varphi}, \overline{\theta}_1, \overline{\theta}_2, \phi_i, Y_i, \widehat{\beta}_i, \widehat{\gamma}_i)$ *to the FBSDE*

$$
\begin{cases}
d\overline{X} =[\widehat{A}\overline{X} + \widehat{B}_3\overline{\varphi} + \widehat{D}_2\overline{\theta}_1 + \widehat{F}_2\overline{\theta}_2 + \widehat{B}_1\bar{u} + b_1]dt + [\widehat{C}_1\overline{X} + (\widehat{D}_2)^\top\overline{\varphi} \\
\qquad + \widehat{D}_3\overline{\theta}_1 + \widehat{D}_4\overline{\theta}_2 + \widehat{D}_1\bar{u} + \sigma_1]dW_1 + [\widehat{C}_2\overline{X} + (\widehat{F}_2)^\top\overline{\varphi} \\
\qquad + (\widehat{D}_4)^\top\overline{\theta}_1 + \widehat{F}_3\overline{\theta}_2 + \widehat{F}_1\bar{u} + \sigma_2]dW_2, \quad \overline{X}(0) = (\xi_1^\top \ \xi_2^\top)^\top, \\
d\overline{\varphi} = -[(\widehat{A})^\top\overline{\varphi} + (\widehat{C}_1)^\top\overline{\theta}_1 + (\widehat{C}_2)^\top\overline{\theta}_2 + \Psi^\top\bar{u} - Q\overline{X} + b_2]dt \\
\qquad + \overline{\theta}_1 dW_1 + \overline{\theta}_2 dW_2, \quad \overline{\varphi}(T) = -G\overline{X}(T) + g, \\
dY_i = -[\widehat{A}^\top Y_i + \widehat{C}_1^\top\widehat{\beta}_i + \widehat{C}_2^\top\widehat{\gamma}_i - Q^\top\phi_i + K_iQ_iK_i^\top\overline{X} + \widehat{B}_2^\top\widehat{R}_2^{-1}\widetilde{R}_2^i \\
\qquad \cdot \widehat{R}_2^{-1}\Upsilon]dt + \widehat{\beta}_i dW_1 + \widehat{\gamma}_i dW_2, \\
d\phi_i =[\widehat{A}\phi_i + (\widehat{B}_3^\top Y_i + \widehat{D}_2\widehat{\beta}_i + \widehat{F}_2\widehat{\gamma}_i) + B_2\widehat{R}_2^{-1}\widetilde{R}_2^i\widehat{R}_2^{-1}\Upsilon]dt \\
\qquad + [\widehat{C}_1\phi_i + (\widehat{D}_2^\top Y_i + \widehat{D}_3^\top\widehat{\beta}_i + \widehat{D}_4\widehat{\gamma}_i) + D_2\widehat{R}_2^{-1}\widetilde{R}_2^i\widehat{R}_2^{-1}\Upsilon]dW_1 \\
\qquad + [\widehat{C}_2\phi_i + (\widehat{F}_2^\top Y_i + \widehat{D}_4^\top\widehat{\beta}_i + \widehat{F}_3^\top\widehat{\gamma}_i) + F_2\widehat{R}_2^{-1}\widetilde{R}_2^i\widehat{R}_2^{-1}\Upsilon]dW_2, \\
Y_i(T) = K_iG_iK_i^\top\overline{X}(T) + K_ie_i^\top g - G^\top\phi_i(T), \quad \phi_i(0) = 0, \quad i = 1, 2,
\end{cases}
\tag{3.16}
$$

satisfies the following condition:

$$
\begin{aligned}
&\langle(\widetilde{R}_1^i + \Phi^\top\widehat{R}_2^{-1}\widetilde{R}_2^i\widehat{R}_2^{-1}\Phi)\bar{u} + \Phi^\top\widehat{R}_2^{-1}\widetilde{R}_2^i\widehat{R}_2^{-1}[\widehat{B}_2X + B_2^\top\varphi + D_2^\top\theta_1 \\
&\qquad + F_2^\top\theta_2 + \sigma_3] + \widehat{B}_1^\top Y_i + \widehat{D}_1^\top\widehat{\beta}_i + \widehat{F}_1\widehat{\gamma}_i - \Psi\phi_i, u^i - \bar{u}\rangle \geq 0, \\
&\qquad\quad \textit{for all } u_i \in \Gamma, \ \textit{a.e. } t \in [0, T], \ \mathbb{P}\text{-}\textit{a.s}, \quad i = 1, 2,
\end{aligned}
\tag{3.17}
$$

where $u^1 = (u_1^\top \ \bar{u}_2^\top)^\top$, $u^2 = (\bar{u}_1^\top \ u_2^\top)^\top$, $\bar{u} = (\bar{u}_1^\top \ \bar{u}_2^\top)^\top$ *and* $\Upsilon = \widehat{B}_2\overline{X} + \Phi\bar{u} + B_2^\top\overline{\varphi} + D_2^\top\overline{\theta}_1 + F_2^\top\overline{\theta}_2 + \sigma_3$.

Proof Using a similar argument in Proposition 3.1, we let $(\bar{u}_1, \bar{u}_2) \in \mathcal{U}$ be the optimal pair of (**LP**) and $(\overline{X}, \overline{\varphi}, \overline{\theta}_1, \overline{\theta}_2, \phi_1, Y_1, \widehat{\beta}_1, \widehat{\gamma}_1)$ be the adapted solution to (3.16) with

$i=1$. Since $\mathcal{U}_1$ is convex, for any $\bar{u}_1+u_1\in\mathcal{U}_1$ and $\varepsilon\in[0,1]$, we have

$$\bar{u}_1+\varepsilon u_1=\varepsilon(\bar{u}_1+u_1)+(1-\varepsilon)\bar{u}_1\in\mathcal{U}_1.$$

Let X be the perturbed state with following state equation:

$$\begin{cases} dX=&[\widehat{A}X+\widehat{B}_3\varphi+\widehat{D}_2\theta_1+\widehat{F}_2\theta_2+\widehat{B}_1(\bar{u}+\varepsilon\delta u_1)+b_1]dt\\ &+[\widehat{C}_1X+\widehat{D}_2^\top\varphi+\widehat{D}_3\theta_1+\widehat{D}_4\theta_2+\widehat{D}_1(\bar{u}+\varepsilon\delta u_1)+\sigma_1]dW_1\\ &+[\widehat{C}_2X+\widehat{F}_2^\top\varphi+\widehat{D}_4^\top\theta_1+\widehat{F}_3\theta_2+\widehat{F}_1(\bar{u}+\varepsilon\delta u_1)+\sigma_2]dW_2,\\ d\varphi=&-[\widehat{A}^\top\varphi+\widehat{C}_1^\top\theta_1+\widehat{C}_2^\top\theta_2+\Psi^\top(\bar{u}+\varepsilon\delta u_1)-QX+b_2]dt\\ &+\theta_1dW_1+\theta_2dW_2,\\ X(0)=&(\xi_1^\top\ \xi_2^\top)^\top,\quad \varphi(T)=-GX(T)+g. \end{cases}$$

We let $X=\overline{X}+\varepsilon X_1$, $\varphi=\overline{\varphi}+\varepsilon\varphi_1$, $\theta_1=\overline{\theta}_1+\varepsilon\theta_{11}$, $\theta_2=\overline{\theta}_2+\varepsilon\theta_{21}$ and X_1, φ_1, θ_{11}, θ_{21} are the solutions of (3.15) when $i=1$. Then, one can obtain

$$\begin{aligned} &J_1(\bar{u}_1+\varepsilon u_1;\bar{v}_1(\bar{u}_1+\varepsilon u_1;\bar{u}_2);\bar{u}_2;\bar{v}_2(\bar{u}_1+\varepsilon u_1;\bar{u}_2))\\ &-J_1(\bar{u}_1;\bar{v}_1(\bar{u}_1;\bar{u}_2);\bar{u}_2;\bar{v}_2(\bar{u}_1;\bar{u}_2))\\ =&\varepsilon\mathbb{E}\bigg\{\int_0^T\langle K_1Q_1K_1^\top\overline{X}+\widehat{B}_2^\top\widehat{R}_2^{-1}\widetilde{R}_2^1\widehat{R}_2^{-1}\Upsilon,X_1\rangle+\langle\widetilde{R}_1^1\bar{u}+\Phi^\top\widehat{R}_2^{-1}\widetilde{R}_2^1\widehat{R}_2^{-1}\Upsilon,\delta u_1\rangle\\ &+\langle B_2\widehat{R}_2^{-1}\widetilde{R}_2^1\widehat{R}_2^{-1}\Upsilon,\varphi_1\rangle+\langle D_2\widehat{R}_2^{-1}\widetilde{R}_2^1\widehat{R}_2^{-1}\Upsilon,\theta_{11}\rangle+\langle F_2\widehat{R}_2^{-1}\widetilde{R}_2^1\widehat{R}_2^{-1}\Upsilon,\theta_{21}\rangle dt\\ &+\langle K_1G_1K_1^\top\overline{X}(T)+K_1e_1^\top g,X_1(T)\rangle\bigg\}+\frac{\varepsilon^2}{2}\mathbb{E}\bigg\{\int_0^T\langle K_1Q_1K_1^\top X_1,X_1\rangle\\ &+\langle\widetilde{R}_1^1\delta u_1,\delta u_1\rangle+\langle\widetilde{R}_2^1\widehat{R}_2^{-1}[\widehat{B}_2X_1+\Phi\delta u_1+B_2^\top\varphi_1+D_2^\top\theta_{11}+F_2^\top\theta_{21}],\\ &\widehat{R}_2^{-1}[\widehat{B}_2X_1+\Phi\delta u_1+B_2^\top\varphi_1+D_2^\top\theta_{11}+F_2^\top\theta_{21}]\rangle dt+\langle K_1G_1K_1^\top X_1(T),X_1(T)\rangle\bigg\}. \end{aligned}$$

On the other hand, we introduce following auxiliary equations

$$\begin{cases} dY_1=\widehat{\alpha}_1dt+\widehat{\beta}_1dW_1+\widehat{\gamma}_1dW_2,\quad Y_1(T)=K_1G_1K_1^\top\overline{X}(T)+K_1e_1^\top g-G^\top\phi_1(T),\\ d\phi_1=\psi_1dt+\omega_1^1dW_1+\omega_2^1dW_2,\quad \phi_1(0)=0, \end{cases}$$

where $Y_1 = ((y_1^1)^\top \ (y_1^2)^\top)^\top$, $\widehat{\beta}_1 = ((\widehat{\beta}_1^1)^\top \ (\widehat{\beta}_1^2)^\top)^\top$, $\widehat{\gamma}_1 = ((\widehat{\gamma}_1^1)^\top \ (\widehat{\gamma}_1^2)^\top)^\top$, $\phi_1 = ((\phi_1^1)^\top \ (\phi_1^2)^\top)^\top$ and

$$\begin{cases} \widehat{\alpha}_1 = -\left[\widehat{A}^\top Y_1 + \widehat{C}_1^\top \widehat{\beta}_1 + \widehat{C}_2^\top \widehat{\gamma}_1 - Q^\top \phi_1 + K_1 Q_1 K_1^\top \overline{X} + \widehat{B}_2^\top \widehat{R}_2^{-1} \widetilde{R}_2^1 \widehat{R}_2^{-1} \Upsilon\right], \\ \psi_1 = \widehat{A}\phi_1 + [\widehat{B}_3^\top Y_1 + \widehat{D}_2 \widehat{\beta}_1 + \widehat{F}_2 \widehat{\gamma}_1] + B_2 \widehat{R}_2^{-1} \widetilde{R}_2^1 \widehat{R}_2^{-1} \Upsilon, \\ \omega_1^1 = \widehat{C}_1 \phi_1 + [\widehat{D}_2^\top Y_1 + \widehat{D}_3^\top \widehat{\beta}_1 + \widehat{D}_4 \widehat{\gamma}_1] + D_2 \widehat{R}_2^{-1} \widetilde{R}_2^1 \widehat{R}_2^{-1} \Upsilon, \\ \omega_2^1 = \widehat{C}_2 \phi_1 + [\widehat{F}_2^\top Y_1 + \widehat{D}_4^\top \widehat{\beta}_1 + \widehat{F}_3^\top \widehat{\gamma}_1] + F_2 \widehat{R}_2^{-1} \widetilde{R}_2^1 \widehat{R}_2^{-1} \Upsilon, \end{cases}$$

$\widehat{\beta}_1, \widehat{\gamma}_1 \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^{2n})$. Applying Itô formula to $\langle Y_1, X_1 \rangle$ and $\langle \phi_1, \varphi_1 \rangle$, we have

$$\begin{aligned} & \langle Y_1(T), X_1(T) \rangle - \langle Y_1(0), X_1(0) \rangle \\ =& \mathbb{E} \int_0^T \langle \widehat{\alpha}_1 + \widehat{A}^\top Y_1 + \widehat{C}_1^\top \widehat{\beta}_1 + \widehat{C}_2^\top \widehat{\gamma}_1, X_1 \rangle + \langle \widehat{B}_3^\top Y_1 + \widehat{D}_2 \widehat{\beta}_1 + \widehat{F}_2 \widehat{\gamma}_1, \varphi_1 \rangle \\ & + \langle \widehat{D}_2^\top Y_1 + \widehat{D}_3^\top \widehat{\beta}_1 + \widehat{D}_4 \widehat{\gamma}_1, \theta_{11} \rangle + \langle \widehat{F}_2^\top Y_1 + \widehat{D}_4^\top \widehat{\beta}_1 + \widehat{F}_3^\top \widehat{\gamma}_1, \theta_{21} \rangle \\ & + \langle \widehat{B}_1^\top Y_1 + \widehat{D}_1^\top \widehat{\beta}_1 + \widehat{F}_1 \widehat{\gamma}_1, \delta u_1 \rangle, \end{aligned}$$

and

$$\begin{aligned} \langle \phi_1(T), \varphi_1(T) \rangle - \langle \phi_1(0), \varphi_1(0) \rangle = \mathbb{E} \int_0^T & \langle \psi_1 - \widehat{A}\phi_1, \varphi_1 \rangle \\ & + \langle \omega_1^1 - \widehat{C}_1 \phi_1, \theta_{11} \rangle + \langle \omega_2^1 - \widehat{C}_2 \phi_1, \theta_{21} \rangle - \langle \Psi \phi_1, \delta u_1 \rangle - \langle Q^\top \phi_1, X_1 \rangle. \end{aligned}$$

Hence,

$$\begin{aligned} & J_1(\bar{u}_1 + \varepsilon u_1; \bar{v}_1(\bar{u}_1 + \varepsilon u_1; \bar{u}_2); \bar{u}_2; \bar{v}_2(\bar{u}_1 + \varepsilon u_1; \bar{u}_2)) \\ & - J_1(\bar{u}_1; \bar{v}_1(\bar{u}_1; \bar{u}_2); \bar{u}_2; \bar{v}_2(\bar{u}_1; \bar{u}_2)) \\ =& \varepsilon \mathbb{E} \Big\{ \int_0^T \langle \widetilde{R}_1^1 \bar{u} + \Phi^\top \widehat{R}_2^{-1} \widetilde{R}_2^1 \widehat{R}_2^{-1} \Upsilon + \widehat{B}_1^\top Y_1 + \widehat{D}_1^\top \widehat{\beta}_1 + \widehat{F}_1 \widehat{\gamma}_1 - \Psi \phi_1, \delta u_1 \rangle dt \Big\} \end{aligned}$$

$$
\begin{aligned}
&+\frac{\varepsilon^2}{2}\mathbb{E}\Big\{\int_0^T \langle K_1Q_1K_1^\top X_1, X_1\rangle + \langle \widetilde{R}_1^1\delta u_1, \delta u_1\rangle + \langle \widetilde{R}_2^1\widehat{R}_2^{-1}[\widehat{B}_2X_1 + \Phi\delta u_1 \\
&+ B_2^\top\varphi_1 + D_2^\top\theta_{11} + F_2^\top\theta_{21}], \widehat{R}_2^{-1}[\widehat{B}_2X_1 + \Phi\delta u_1 + B_2^\top\varphi_1 + D_2^\top\theta_{11} \\
&+ F_2^\top\theta_{21}]\rangle dt + \langle K_1G_1K_1^\top X_1(T), X_1(T)\rangle\Big\}.
\end{aligned}
$$

Therefore, under the condition (3.14) for $i = 1$,

$$
\begin{aligned}
&J_1(\bar{u}_1; \bar{v}_1(\bar{u}_1; \bar{u}_2); \bar{u}_2; \bar{v}_2(\bar{u}_1; \bar{u}_2)) \\
\leq &J_1(\bar{u}_1 + \varepsilon u_1; \bar{v}_1(\bar{u}_1 + \varepsilon u_1; \bar{u}_2); \bar{u}_2; \bar{v}_2(\bar{u}_1 + \varepsilon u_1; \bar{u}_2)),
\end{aligned}
$$

for any $\bar{u}_1 + u_1 \in \mathcal{U}_1$, $\varepsilon \in [0, 1]$ if and only if (3.17) holds for $i = 1$. Similarly, we have the result for $i = 2$. The proposition follows. □

By the discussion in [93], if $(R_1^i + \Phi_i^\top(\widehat{R}_2^1)^{-1}R_2^i(\widehat{R}_2^1)^{-1}\Phi_i) > 0$, there exists two projection mappings $\mathbf{P}_{\Gamma^1}(\cdot) : \mathbb{R}^m \to \Gamma$ and $\mathbf{P}_{\Gamma^2}(\cdot) : \mathbb{R}^m \to \Gamma$, where Γ is a closed convex subset of $\mathbb{R}^m$, under the norm $|\cdot|_{R_1^i+\Phi_i^\top(\widehat{R}_2^1)^{-1}R_2^i(\widehat{R}_2^1)^{-1}\Phi_i}$ such that

$$
\begin{aligned}
\bar{u}_1 = &\mathbf{P}_{\Gamma^1}\{-(R_1^1 + \Phi_1^\top(\widehat{R}_2^1)^{-1}R_2^1(\widehat{R}_2^1)^{-1}\Phi_1)^{-1}[\Phi_1^\top(\widehat{R}_2^1)^{-1}R_2^1(\widehat{R}_2^1)^{-1}\cdot \\
&(\widehat{B}_2^1x_1 + (B_2^1)^\top\varphi_1 + (D_2^1)^\top\theta_1^1 + (F_2^1)^\top\theta_2^1 + (D_2^1)^\top P_1\sigma_1^1 + (F_2^1)^\top P_1\sigma_2^1) \\
&+ (\widehat{B}_1^1)^\top y_1^1 + (\widehat{D}_1^1)^\top\widehat{\beta}_1^1 + \widehat{F}_1^1\widehat{\gamma}_1^1 - \Psi_1\phi_1^1]\}, \\
\bar{u}_2 = &\mathbf{P}_{\Gamma^2}\{-(R_1^2 + \Phi_2^\top(\widehat{R}_2^2)^{-1}R_2^2(\widehat{R}_2^2)^{-1}\Phi_2)^{-1}[\Phi_2^\top(\widehat{R}_2^2)^{-1}R_2^2(\widehat{R}_2^2)^{-1}\cdot \\
&(\widehat{B}_2^2x_2 + (B_2^2)^\top\varphi_2 + (D_2^2)^\top\theta_1^2 + (F_2^2)^\top\theta_2^2 + (D_2^2)^\top P_2\sigma_1^2 + (F_2^2)^\top P_2\sigma_2^2) \\
&+ (\widehat{B}_1^2)^\top y_2^2 + (\widehat{D}_1^2)^\top\widehat{\beta}_2^2 + \widehat{F}_1^2\widehat{\gamma}_2^2 - \Psi_2\phi_2^2]\}.
\end{aligned}
\tag{3.18}
$$

Let $\bar{u}^1 = (\bar{u}_1^\top \ 0^\top)^\top$ and $\bar{u}^2 = (0^\top \ \bar{u}_2^\top)^\top$, then we obtain the related Hamiltonian

system:

$$
\begin{cases}
d\overline{X} =[\widehat{A}\overline{X} + \widehat{B}_3\overline{\varphi} + \widehat{D}_2\overline{\theta}_1 + \widehat{F}_2\overline{\theta}_2 + \widehat{B}_1\bar{u}^i + b_1]dt + [\widehat{C}_1\overline{X} + (\widehat{D}_2)^\top\overline{\varphi} \\
\qquad + \widehat{D}_3\overline{\theta}_1 + \widehat{D}_4\overline{\theta}_2 + \widehat{D}_1\bar{u}^i + \sigma_1]dW_1 + [\widehat{C}_2\overline{X} + (\widehat{F}_2)^\top\overline{\varphi} + (\widehat{D}_4)^\top\overline{\theta}_1 \\
\qquad + \widehat{F}_3\overline{\theta}_2 + \widehat{F}_1\bar{u}^i + \sigma_2]dW_2, \quad \overline{X}(0) = (\xi_1^\top\ \xi_2^\top)^\top, \\
d\overline{\varphi} = -[(\widehat{A})^\top\overline{\varphi} + (\widehat{C}_1)^\top\overline{\theta}_1 + (\widehat{C}_2)^\top\overline{\theta}_2 + \Psi^\top\bar{u}^i - Q\overline{X} + b_2]dt + \overline{\theta}_1 dW_1 \\
\qquad + \overline{\theta}_2 dW_2, \quad \overline{\varphi}(T) = -G\overline{X}(T) + g, \\
dY_i = -[\widehat{A}^\top Y_i + \widehat{C}_1^\top\widehat{\beta}_i + \widehat{C}_2^\top\widehat{\gamma}_i - Q^\top\phi_i + K_iQ_iK_i^\top\overline{X} + \widehat{B}_2^\top\widehat{R}_2^{-1}\widetilde{R}_2^i\cdot \\
\qquad \widehat{R}_2^{-1}(\widehat{B}_2\overline{X} + \Phi\bar{u}^i + B_2^\top\overline{\varphi} + D_2^\top\overline{\theta}_1 + F_2^\top\overline{\theta}_2 + \sigma_3)]dt + \widehat{\beta}_i dW_1 \\
\qquad + \widehat{\gamma}_i dW_2, \quad Y_i(T) = K_iG_iK_i^\top\overline{X}(T) + K_ie_i^\top g - G^\top\phi_i(T), \\
d\phi_i =[\widehat{A}\phi_i + (\widehat{B}_3^\top Y_i + \widehat{D}_2\widehat{\beta}_i + \widehat{F}_2\widehat{\gamma}_i) + B_2\widehat{R}_2^{-1}\widetilde{R}_2^i\widehat{R}_2^{-1}(\widehat{B}_2\overline{X} + \Phi\bar{u}^i + B_2^\top\overline{\varphi} \\
\qquad + D_2^\top\overline{\theta}_1 + F_2^\top\overline{\theta}_2 + \sigma_3)]dt + [\widehat{C}_1\phi_i + (\widehat{D}_2^\top Y_i + \widehat{D}_3^\top\widehat{\beta}_i + \widehat{D}_4\widehat{\gamma}_i) + D_2\cdot \\
\qquad \widehat{R}_2^{-1}\widetilde{R}_2^i\widehat{R}_2^{-1}(\widehat{B}_2\overline{X} + \Phi\bar{u}^i + B_2^\top\overline{\varphi} + D_2^\top\overline{\theta}_1 + F_2^\top\overline{\theta}_2 + \sigma_3)]dW_1 \\
\qquad + [\widehat{C}_2\phi_i + (\widehat{F}_2^\top Y_i + \widehat{D}_4^\top\widehat{\beta}_i + \widehat{F}_3^\top\widehat{\gamma}_i) + F_2\widehat{R}_2^{-1}\widetilde{R}_2^i\widehat{R}_2^{-1}(\widehat{B}_2\overline{X} \\
\qquad + \Phi\bar{u}^i + B_2^\top\overline{\varphi} + D_2^\top\overline{\theta}_1 + F_2^\top\overline{\theta}_2 + \sigma_3)]dW_2, \quad \phi_i(0) = 0, \quad i = 1, 2.
\end{cases}
$$

In summary, the mixed LF game problem with constrained and non-degenerate control weights is solved. Next, we give some examples for the explicit expression of the projection operators $\mathbf{P}_{\Gamma^i}$, $i = 1, 2$.

3.3.1 Some examples for the projection operators

In general, the projection operator on a convex-closed set does not admit some explicit representation. In some sense, we can characterize it using the so-called *variational inequality* but some numerical algorithm should be invoked for real computation. On the other hand, in case the convex-closed set has more structure, it is

possible to construct more explicit representation, as addressed below. Before that, in equation (3.18), we let

$$
\begin{aligned}
&\vartheta^i(x_i;\varphi_i;\theta_1^i;\theta_2^i;y_i^i;\widehat{\beta}_i^i;\widehat{\gamma}_i^i;\phi_i^i)\\
=&-(R_1^i+\Phi_i^\top(\widehat{R}_2^i)^{-1}R_2^i(\widehat{R}_2^i)^{-1}\Phi_i)^{-1}[\Phi_i^\top(\widehat{R}_2^i)^{-1}R_2^i(\widehat{R}_2^i)^{-1}(\widehat{B}_2^i x_i\\
&+(B_2^i)^\top\varphi_i+(D_2^i)^\top\theta_1^i+(F_2^i)^\top\theta_2^i+(D_2^i)^\top P_i\sigma_1^i+(F_2^i)^\top P_i\sigma_2^i)\\
&+(\widehat{B}_1^i)^\top y_i^i+(\widehat{D}_1^i)^\top\widehat{\beta}_i^i+\widehat{F}_1^i\widehat{\gamma}_i^i-\Psi_i\phi_i^i],\quad i=1,2.
\end{aligned}
$$

and $\vartheta_1^i,\vartheta_2^i,\cdots,\vartheta_m^i$ are the components of ϑ^i.

Case 1: Convex-closed cone. First, we consider the case when Γ^i, $i=1,2$ are two *convex-closed cones*. Recall a set Γ is said to be a convex-closed cone if it is closed convex set, and closed under positive scalar operations (namely, $\vartheta^i\in\Gamma^i\subseteq\mathbb{R}^m$, then $\kappa^i\vartheta^i\in\Gamma^i$ for all $\kappa^i>0$). This definition may be revised as for all $\kappa^i\geq 0$. Indeed, since $\lim_{\kappa^i\to 0}\kappa^i\vartheta^i=0$ by noting the closeness of Γ^i, so a closed cone contains the original point $0\in\Gamma^i$.

In a finite-dimensional space such as $\mathbb{R}^m$, a closed convex cone can be characterized by m half-spaces. We may present this point when $m=2$: suppose $\mathbf{P}_{\Gamma^i}:\mathbb{R}^2\to\Gamma^i$, $i=1,2$ and $\Gamma^i\subset\mathbb{R}^2$ are closed convex cones, then $\Gamma^i=\{\vartheta^i\in\mathbb{R}^2|\langle\alpha_1^i,\vartheta^i\rangle\geq 0;\ \langle\alpha_2^i,\vartheta^i\rangle\geq 0\}$ for vectors $\alpha_1^i,\alpha_2^i\in\mathbb{R}^2$. Then, we can introduce the normal cones for Γ^i. Recall the normal cone for a set Γ (not necessary to be convex set) at point, say original 0, is:

$$N_\Gamma(0)=\{p\in\mathbb{R}^m|\ \langle p,\vartheta^i\rangle\leq 0,\ \forall\ \vartheta^i\in\Gamma\}.$$

In particular, suppose β_1^i,β_2^i respectively the normal vectors to α_1^i,α_2^i with obtuse angle arrangements: that is $\langle\alpha_1^i,\beta_2^i\rangle\leq 0$, $\langle\alpha_2^i,\beta_1^i\rangle\leq 0$. Then, the normal cone of Γ^i takes form: $N_{\Gamma^i}(0)=\{\vartheta^i\in\mathbb{R}^2|\langle\beta_1^i,\vartheta^i\rangle\leq 0;\ \langle\beta_2^i,\vartheta^i\rangle\leq 0\}$. In this case, the

projection operator admits more explicit expressions as:

$$\mathbf{P}_{\Gamma^i}\{\vartheta^i\} = \begin{cases} \vartheta^i & \vartheta^i \in \Gamma^i, \quad i = 1, 2, \\ 0 & \vartheta^i \in N_{\Gamma^i}(0), \\ \frac{\alpha_1^i \langle \alpha_1^i, \vartheta^i \rangle}{\langle \alpha_1^i, \alpha_1^i \rangle} & \vartheta^i \in \Gamma^\dagger = \{\vartheta^i \in \mathbb{R}^2 | \ \langle \alpha_1^i, \vartheta^i \rangle \geq 0; \ \langle \beta_1^i, \vartheta^i \rangle \geq 0\}, \\ \frac{\alpha_2^i \langle \alpha_2^i, \vartheta^i \rangle}{\langle \alpha_2^i, \alpha_2^i \rangle} & \vartheta^i \in \Gamma^\ddagger = \{\vartheta^i \in \mathbb{R}^2 | \ \langle \alpha_2^i, \vartheta^i \rangle \geq 0; \ \langle \beta_2^i, \vartheta^i \rangle \geq 0\}. \end{cases}$$

Case 2: Convex-closed orthant cone. Moreover, we have the following representation for more specific orthant cone. Recall a nonnegative (closed, convex) orthant cone in $\mathbb{R}^m$ space: $\Gamma^i = \{\vartheta^i = (\vartheta_1^i, \cdots, \vartheta_m^i) \in \mathbb{R}^m | \vartheta_1^i \geq 0, \cdots, \vartheta_m^i \geq 0\}$. Note that the positive orthant cone $(\Gamma^i)' = \{\vartheta^i = (\vartheta_1^i, \cdots, \vartheta_m^i) \in \mathbb{R}^m | \vartheta_1^i > 0, \cdots, \vartheta_m^i > 0\}$ is *not closed*. In this case, we have

$$\mathbf{P}_{\Gamma^i}\{\vartheta^i\} = ((\vartheta_1^i)^+, \cdots, (\vartheta_m^i)^+),$$

where $(\vartheta_k^i)^+ = \max\{\vartheta_k^i, 0\}$, $k = 1, 2, \cdots, m$.

Case 3: Subspace. A complete explicit form of projection may be the subspace which is a very special but still important closed-convex set (note that subspace constraint is well documented in literature such as [73, 95, 113]). Suppose Γ^i are r^i-dimensional subspaces in $\mathbb{R}^m$, $r^i \leq m$ with $h^1, \cdots, h^{r^i}$ as basis. Introduce $H_i := (h^1, \cdots, h^{r^i})$, then we have explicit expression (see [9, Chapter 8]):

$$\mathbf{P}_{\Gamma^i}\{\vartheta^i\} = \widehat{H}_i \vartheta^i,$$

with $\widehat{H}_i = H_i(H_i^\top R_1^i H_i)^{-1} H_i^\top R_1^i$, $i = 1, 2$ (see Chapter 2).

Besides, in case with standard *orthonormal basis* $e_1, \cdots, e_r$, we have

$$\mathbf{P}_{\Gamma^i}\{\vartheta^i\} = (e_1, \cdots, e_r)(e_1, \cdots, e_r)^\top \vartheta^i.$$

More financial linear constraints in subspace can refer to [73, Section 5].

3.4 The Solvability of Singular Case

In Section 4, we obtain the open-loop NE for (**LP**), and the related mixed LF strategy can be completely determined. A crucial assumption therein is the control weights are positive-definite that restated as follows:

$$R_1^i(t) > 0, \quad R_2^i(t) > 0, \quad a.e.\ t \in [0, T], \quad i = 1, 2. \tag{3.19}$$

Nevertheless, in reality, there arise various cases in which the control weight might be indefinite or degenerated (e.g., the mean-variance optimization where control weight becomes singular [129, 191, 195]). In the case that the control weight is not invertible, it is impossible to decouple the Hamiltonian system like classical LQ problems (see [101, 190, 191]). When handling (**LP**) under this case, the difficulties mainly arise from two sides: (i) due to the singular control weight, the classical FBSDE representation of Hamiltonian system via explicit stationary condition is no longer workable; (ii) due to the input constrained for u_i, $i = 1, 2$, it is also impossible to impose the Riccati equation to have some conditions (like $D^\top PD > 0$ in [191]).

We plan to attack this problem by some weak-convergence method to get some near-optimal control sequence. This provides some tractable solution, although near-optimality, for real application purposes. It also provides some new insight into the classical singular LQ control problem.

Unlike [164, 166], the state equation here becomes a fully-coupled FBSDE, thus the standard completion-square method (see [191, Chapter 6]) via Riccati equation may not be applicable directly. Meanwhile, the state equation with a fully-coupled FBSDE has been investigated in [182, 193], their control weights are positive-definite, thus the convexity of the corresponding cost functional is obvious. However, since R_1^i and R_2^i, $i = 1, 2$ here are degenerate, we first need to discuss the (uniform) convexity of the cost functional of (**LP**). By using a Riccati-type equation to decouple (3.11), (**LP**) can be decomposed into a forward problem and a backward problem. Then,

we can discuss the convexity of the above two problems, respectively. To simplify, we let $g_i = 0$. Since the states and cost functionals of player 1 and player 2 are symmetric, we only consider player 1. For player 2, the situation is similar.

First, we rewrite (3.11) as follows:

$$\begin{cases} dX =[\widehat{A}X + \widehat{B}_3\varphi + \widehat{D}_2\theta_1 + \widehat{F}_2\theta_2 + \acute{B}_1\tilde{u}_1]dt + [\widehat{C}_1X + \widehat{D}_2^\top\varphi \\ \qquad + \widehat{D}_3\theta_1 + \widehat{D}_4\theta_2 + \acute{D}_1\tilde{u}_1]dW_1 + [\widehat{C}_2X + \widehat{F}_2^\top\varphi + \widehat{D}_4^\top\theta_1 \\ \qquad + \widehat{F}_3\theta_2 + \acute{F}_1\tilde{u}_1]dW_2, \quad X(0) = (\xi_1^\top\ \xi_2^\top)^\top, \\ d\varphi = -[\widehat{A}^\top\varphi + \widehat{C}_1^\top\theta_1 + \widehat{C}_2^\top\theta_2 + \acute{\Psi}_1^\top\tilde{u}_1 - QX]dt + \theta_1 dW_1 \\ \qquad + \theta_2 dW_2, \quad \varphi(T) = -GX(T), \end{cases} \tag{3.20}$$

where $\tilde{u}_1 = (u_1^\top\ 0^\top)^\top$ and

$$\acute{B}_1 = \begin{pmatrix} \widehat{B}_1^1 & 0 \\ 0 & 0 \end{pmatrix},\ \acute{D}_1 = \begin{pmatrix} \widehat{D}_1^1 & 0 \\ 0 & 0 \end{pmatrix},\ \acute{F}_1 = \begin{pmatrix} \widehat{F}_1^1 & 0 \\ 0 & 0 \end{pmatrix},\ \acute{\Psi}_1 = \begin{pmatrix} \Psi_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $\varphi = \Lambda X + \lambda$ and

$$\begin{aligned} d\varphi =& \dot{\Lambda}X + \Lambda[\widehat{A}X + \widehat{B}_3\varphi + \widehat{D}_2\theta_1 + \widehat{F}_2\theta_2 + \acute{B}_1\tilde{u}_1]dt + \Lambda[\widehat{C}_1X + \widehat{D}_2^\top\varphi \\ &+ \widehat{D}_3\theta_1 + \widehat{D}_4\theta_2 + \acute{D}_1\tilde{u}_1]dW_1 + \Lambda[\widehat{C}_2X + \widehat{F}_2^\top\varphi + \widehat{D}_4^\top\theta_1 \\ &+ \widehat{F}_3\theta_2 + \acute{F}_1\tilde{u}_1]dW_2 + \dot{\lambda} + \iota_1 dW_1 + \iota_2 dW_2 \\ =& -[\widehat{A}^\top\varphi + \widehat{C}_1^\top\theta_1 + \widehat{C}_2^\top\theta_2 + \acute{\Psi}_1^\top\tilde{u}_1 - QX]dt + \theta_1 dW_1 + \theta_2 dW_2, \end{aligned}$$

where $\Lambda(\cdot) \in C^1(0,T;\mathbb{R}^{2n\times 2n})$ and $\lambda(\cdot) \in C^1(0,T;\mathbb{R}^{2n})$. By comparing the diffusion terms, we have

$$\begin{aligned} \theta_1 =& [\Lambda_3(\widehat{C}_1 + \widehat{D}_2^\top P) + \Lambda_1\widehat{D}_4\Lambda_2(\widehat{C}_2 + \widehat{F}_2^\top P)]X + (\Lambda_3\widehat{D}_2^\top + \Lambda_1\widehat{D}_4\Lambda_2\widehat{F}_2^\top)\lambda \\ &+ (\Lambda_3\acute{D}_1^\top + \Lambda_1\widehat{D}_4\Lambda_2\acute{F}_1)\tilde{u}_1 + \iota_1, \\ \theta_2 =& [\Lambda_2(\widehat{C}_2 + \widehat{F}_2^\top P) + \Lambda_2\widehat{D}_4^\top P_1(\widehat{C}_1 + \widehat{D}_2^\top P)]X + (\Lambda_2\widehat{D}_2^\top + \Lambda_2\widehat{D}_4\Lambda_1\widehat{D}_3^\top)\lambda \\ &+ (\Lambda_2\acute{F}_1 + \Lambda_2\widehat{D}_4^\top P_1\acute{D}_1)\tilde{u}_1 + \iota_2, \end{aligned}$$

where

$$\begin{cases}\Lambda_1 =(I-\Lambda\widehat{D}_3)^{-1}\Lambda, \quad \Lambda_2=(I-\Lambda\widehat{F}_3-\Lambda\widehat{D}_4^\top\Lambda_1\widehat{D}_4)^{-1},\\ \Lambda_3 =\Lambda_1+\Lambda_1^\top\widehat{D}_4\Lambda_2\widehat{D}_4^\top\Lambda_1.\end{cases}$$

By elementary calculation, one can obtain that Λ satisfies following process

$$-d\Lambda = H(\widehat{A};\widehat{B}_3;\widehat{C}_1;\widehat{C}_2;\widehat{D}_2;\widehat{D}_4;\widehat{F}_2)dt, \quad \Lambda(T)=G, \tag{3.21}$$

with

$$\begin{cases}H(\widehat{A};\widehat{B}_3;\widehat{C}_1;\widehat{C}_2;\widehat{D}_2;\widehat{D}_4;\widehat{F}_2)\\ \quad :=\Lambda\widehat{A}+\widehat{A}^\top\Lambda+\Lambda\widehat{B}_3\Lambda-Q+\widehat{C}_1^\top\Lambda\widehat{C}_1+\widehat{C}_2^\top\Lambda\widehat{C}_2+\Lambda\widehat{D}_2\Lambda_3\widehat{C}_1\\ \qquad +\widehat{C}_1^\top\Lambda_3\widehat{D}_2^\top\Lambda+\Lambda\widehat{F}_2\Lambda_2\widehat{C}_2+\widehat{C}_2^\top\Lambda_2\widehat{F}_2^\top\Lambda+\Lambda\widehat{D}_2\widehat{D}_2^\top\Lambda+\Lambda\widehat{F}_2\widehat{F}_2^\top\Lambda\\ \qquad +\Lambda\widehat{F}_2\Lambda_2\widehat{D}_4^\top\Lambda_1\widehat{C}_1+\widehat{C}_1^\top\Lambda_1\widehat{D}_4\Lambda_2\widehat{F}_2^\top\Lambda+\Lambda\widehat{D}_2\Lambda_1\widehat{D}_4\Lambda_2\widehat{C}_2\\ \qquad +\widehat{C}_2^\top\Lambda_2\widehat{D}_4^\top\Lambda_1\widehat{D}_2^\top\Lambda+\Lambda\widehat{D}_2\Lambda_1\widehat{D}_4\Lambda_2\widehat{F}_2^\top\Lambda+\Lambda\widehat{F}_2\Lambda_2\widehat{D}_4^\top\Lambda_1\widehat{D}_2^\top\Lambda\\ \qquad +\widehat{C}_1^\top\Lambda_1\widehat{D}_4\Lambda_2\widehat{C}_2+\widehat{C}_2^\top\Lambda_2\widehat{D}_4^\top\Lambda_1\widehat{C}_1.\end{cases}$$

If equation (3.21) is solvable, then the FBSDE (3.20) can be decoupled as follows

$$\begin{cases}dX =(\check{A}_1X+\check{C}_1\lambda+\widehat{D}_2\iota_1+\widehat{F}_2\iota_2+\check{B}_1\tilde{u}_1)dt+(\check{A}_2X+\check{C}_2\lambda\\ \qquad +\widehat{D}_3\iota_1+\widehat{D}_4\iota_2+\check{B}_2\tilde{u}_1)dW_1+(\check{A}_3X+\check{C}_3\lambda+(\widehat{D}_4)^\top\iota_1\\ \qquad +\widehat{F}_3\iota_2+\check{B}_3\tilde{u}_1)dW_2, \quad X(0)=(\xi_1^\top\ \xi_2^\top)^\top,\\ d\lambda =-\{\check{A}_4\lambda+(\Lambda\widehat{D}_2+\widehat{C}_1^\top)\iota_1+(\Lambda\widehat{F}_2+\widehat{C}_2^\top)\iota_2+\check{B}_4\tilde{u}_1\}dt\\ \qquad +\iota_1dW_1+\iota_2dW_2, \quad \lambda(T)=0,\end{cases} \tag{3.22}$$

where

$$
\begin{cases}
\check{A}_1 =\widehat{A} + \widehat{B}_3\Lambda + \widehat{D}_2(\Lambda_3(\widehat{C}_1 + \widehat{D}_2^\top\Lambda) + \Lambda_1\widehat{D}_4\Lambda_2(\widehat{C}_2 + \widehat{F}_2^\top\Lambda)) \\
\qquad + \widehat{F}_2(\Lambda_2(\widehat{C}_2 + \widehat{F}_2^\top P) + \Lambda_2\widehat{D}_4^\top P_1(\widehat{C}_1 + \widehat{D}_2^\top P)), \\
\check{A}_2 =\widehat{C}_1 + (\widehat{D}_2)^\top\Lambda + \widehat{D}_3(\Lambda_3(\widehat{C}_1 + \widehat{D}_2^\top P) + \Lambda_1\widehat{D}_4\Lambda_2(\widehat{C}_2 + \widehat{F}_2^\top P)) \\
\qquad + \widehat{D}_4(\Lambda_2(\widehat{C}_2 + \widehat{F}_2^\top P) + \Lambda_2\widehat{D}_4^\top P_1(\widehat{C}_1 + \widehat{D}_2^\top P)), \\
\check{A}_3 =\widehat{C}_2 + (\widehat{F}_2)^\top\Lambda + (\widehat{D}_4)^\top(\Lambda_3(\widehat{C}_1 + \widehat{D}_2^\top P) + \Lambda_1\widehat{D}_4\Lambda_2(\widehat{C}_2 + \widehat{F}_2^\top P)) \\
\qquad + \widehat{F}_3(\Lambda_2(\widehat{C}_2 + \widehat{F}_2^\top P) + \Lambda_2\widehat{D}_4^\top P_1(\widehat{C}_1 + \widehat{D}_2^\top P)), \\
\check{A}_4 =\widehat{A}^\top + \Lambda\widehat{B}_3 + (\Lambda\widehat{D}_2 + \widehat{C}_1^\top)(\Lambda_3\widehat{D}_2^\top + \Lambda_1\widehat{D}_4\Lambda_2\widehat{F}_2^\top) \\
\qquad + (\Lambda\widehat{F}_2 + \widehat{C}_2^\top)(\Lambda_2\widehat{D}_2^\top + \Lambda_2\widehat{D}_4\Lambda_1\widehat{D}_3^\top), \\
\check{C}_1 =\widehat{B}_3 + \widehat{D}_2(\Lambda_3\widehat{D}_2^\top + \Lambda_1\widehat{D}_4\Lambda_2\widehat{F}_2^\top) + \widehat{F}_2(\Lambda_2\widehat{D}_2^\top + \Lambda_2\widehat{D}_4\Lambda_1\widehat{D}_3^\top), \\
\check{C}_2 =(\widehat{D}_2)^\top + \widehat{D}_3(\Lambda_3\widehat{D}_2^\top + \Lambda_1\widehat{D}_4\Lambda_2\widehat{F}_2^\top) + \widehat{D}_4(\Lambda_2\widehat{D}_2^\top + \Lambda_2\widehat{D}_4\Lambda_1\widehat{D}_3^\top), \\
\check{C}_3 =(\widehat{F}_2)^\top + (\widehat{D}_4)^\top(\Lambda_3\widehat{D}_2^\top + \Lambda_1\widehat{D}_4\Lambda_2\widehat{F}_2^\top) + \widehat{F}_3(\Lambda_2\widehat{D}_2^\top + \Lambda_2\widehat{D}_4\Lambda_1\widehat{D}_3^\top), \\
\check{B}_1 =\acute{B}_1 + \widehat{D}_2(\Lambda_3\acute{D}_1^\top + \Lambda_1\widehat{D}_4\Lambda_2\acute{F}_1) + \widehat{F}_2(\Lambda_2\acute{F}_1 + \Lambda_2\widehat{D}_4^\top P_1\acute{D}_1), \\
\check{B}_2 =\acute{D}_1 + \widehat{D}_3(\Lambda_3\acute{D}_1^\top + \Lambda_1\widehat{D}_4\Lambda_2\acute{F}_1) + \widehat{D}_4(\Lambda_2\acute{F}_1 + \Lambda_2\widehat{D}_4^\top P_1\acute{D}_1), \\
\check{B}_3 =\acute{F}_1 + (\widehat{D}_4)^\top(\Lambda_3\acute{D}_1^\top + \Lambda_1\widehat{D}_4\Lambda_2\acute{F}_1) + \widehat{F}_3(\Lambda_2\acute{F}_1 + \Lambda_2\widehat{D}_4^\top P_1\acute{D}_1), \\
\check{B}_4 =(\Lambda\acute{B}_1 + \acute{\Psi}_1^\top) + (\Lambda\widehat{D}_2 + \widehat{C}_1^\top)(\Lambda_3\acute{D}_1^\top + \Lambda_1\widehat{D}_4\Lambda_2\acute{F}_1) \\
\qquad + (\Lambda\widehat{F}_2 + \widehat{C}_2^\top)(\Lambda_2\acute{F}_1 + \Lambda_2\widehat{D}_4^\top P_1\acute{D}_1), \\
\check{b}_1 =b_1 + \widehat{D}_2(\Lambda_3\sigma_1 + \Lambda_1\widehat{D}_4\Lambda_2\sigma_2) + \widehat{F}_2(\Lambda_2\sigma_2 + \Lambda_2\widehat{D}_4^\top P_1\sigma_1), \\
\check{b}_2 =\sigma_1 + \widehat{D}_3(\Lambda_3\sigma_1 + \Lambda_1\widehat{D}_4\Lambda_2\sigma_2) + \widehat{D}_4(\Lambda_2\sigma_2 + \Lambda_2\widehat{D}_4^\top P_1\sigma_1), \\
\check{b}_3 =\sigma_2 + (\widehat{D}_4)^\top(\Lambda_3\sigma_1 + \Lambda_1\widehat{D}_4\Lambda_2\sigma_2) + \widehat{F}_3(\Lambda_2\sigma_2 + \Lambda_2\widehat{D}_4^\top P_1\sigma_1), \\
\check{b}_4 =\Lambda b_1 + b_2 + (\Lambda\widehat{D}_2 + \widehat{C}_1^\top)(\Lambda_3\sigma_1 + \Lambda_1\widehat{D}_4\Lambda_2\sigma_2) \\
\qquad + (\Lambda\widehat{F}_2 + \widehat{C}_2^\top)(\Lambda_2\sigma_2 + \Lambda_2\widehat{D}_4^\top P_1\sigma_1).
\end{cases}
$$

Next, we give out a very important (sufficient) condition for the uniform convexity of the cost functional (3.12) in (**LP**).

Theorem 3.1. *Suppose that* (**A3.1**)-(**A3.3**) *hold,* (3.21) *is solvable. For some* $\delta_4 > 0$, $\gamma > 0$, *if*

$$\delta_4\gamma > \left|\frac{1}{2}L_1T + L_2 + 1 + (\frac{1}{\delta_1''} + \frac{1}{\delta_2''} + \frac{1}{\delta_3''})L_3T\left[L_2 + \frac{L_1(T+1)}{2} + 1\right]\right|,$$

where

$$L_1 = \exp\Big(2\int_0^T (|\check{A}_4| + \frac{1}{\delta_1} + \frac{1}{\delta_2} + \frac{1}{\delta_3})dt\Big),$$

$$L_2 = \Big(2\int_0^T (|\check{A}_4| + \frac{1}{\delta_1} + \frac{1}{\delta_2} + \frac{1}{\delta_3})L_1dt\Big),$$

$$L_3 = (1 + 27\delta_2' + 27\delta_3')\exp\Big(\int_0^T (2|\check{A}_1| + \frac{1}{\delta_1'}) + (1 + \frac{1}{\delta_2'})|\check{A}_2|^2 + (1 + \frac{1}{\delta_3'})|\check{A}_3|^2dt\Big),$$

and

$$\begin{cases} \delta_1 = \dfrac{1}{2|(\Lambda\widehat{D}_2 + \widehat{C}_1^\top)^\top(\Lambda\widehat{D}_2 + \widehat{C}_1^\top)|}, \quad \delta_1'' = \dfrac{1}{9\|P_3\|^2\max\{\|\check{C}_1\|^2, \|\widehat{D}_2\|^2, \|\widehat{F}_2\|^2\}}, \\ \delta_1' = \dfrac{1}{9\max\{\|\check{C}_1\|^2, \|\widehat{D}_2\|^2, \|\widehat{F}_2\|^2\}}, \quad \delta_2' = \dfrac{1}{9\max\{\|\check{C}_2\|^2, \|\widehat{D}_3\|^2, \|\widehat{D}_4\|^2\}}, \\ \delta_3' = \dfrac{1}{9\max\{\|\check{C}_3\|^2, \|\widehat{D}_4^\top\|^2, \|\widehat{F}_3\|^2\}}, \quad \delta_2 = \dfrac{1}{2|(\Lambda\widehat{F}_2 + \widehat{C}_2^\top)^\top(\Lambda\widehat{F}_2 + \widehat{C}_2^\top)|}, \\ \delta_2'' = \dfrac{2\|P_3\| + 1/\max\{\|\check{C}_2\|^2, \|\widehat{D}_3\|^2, \|\widehat{D}_4\|^2\}}{9\|\check{A}_2^\top P_3\|^2}, \quad \delta_3 = \dfrac{1}{|\check{B}_4^\top\check{B}_4|}, \\ \delta_3'' = \dfrac{2\|P_3\| + 1/\max\{\|\check{C}_3\|^2, \|\widehat{D}_4^\top\|^2, \|\widehat{F}_3\|^2\}}{9\|\check{A}_3^\top P_3\|^2}, \end{cases}$$

then cost functional (3.12) *is uniformly convex on* u.

Proof The FBSDE (3.22) can be further divided into:

$$\begin{cases} dX_1 =(\check{A}_1X_1+\check{B}_1\tilde{u}_1)dt+(\check{A}_2X_1+\check{B}_2\tilde{u}_1)dW_1+(\check{A}_3X_1+\check{B}_3\tilde{u}_1)dW_2, \\ dX_2 =(\check{A}_1X_2+\check{C}_1\lambda+\widehat{D}_2\iota_1+\widehat{F}_2\iota_2)dt+(\check{A}_2X_2+\check{C}_2\lambda+\widehat{D}_3\iota_1 \\ \qquad +\widehat{D}_4\iota_2)dW_1+(\check{A}_3X_2+\check{C}_3\lambda+(\widehat{D}_4)^\top\iota_1+\widehat{F}_3\iota_2)dW_2, \\ d\lambda =-\{\check{A}_4\lambda+(\Lambda\widehat{D}_2+\widehat{C}_1^\top)\iota_1+(\Lambda\widehat{F}_2+\widehat{C}_2^\top)\iota_2+\check{B}_4\tilde{u}_1\}dt \\ \qquad +\iota_1dW_1+\iota_2dW_2, \\ X_1(0)=(\xi_1^\top\ \xi_2^\top)^\top, \quad X_2(0)=(0^\top\ 0^\top)^\top, \quad \lambda(T)=0, \end{cases} \tag{3.23}$$

and the cost functional (3.12) can be rewritten as

$$\begin{aligned} J_1(u_1) =&\frac{1}{2}\mathbb{E}\Big\{\int_0^T\langle K_1Q_1K_1^\top(X_1+X_2),(X_1+X_2)\rangle dt+\langle K_1G_1K_1^\top(X_1 \\ &+X_2)(T),(X_1+X_2)(T)\rangle\Big\} \\ \geq&\frac{1}{2}\mathbb{E}\Big\{\int_0^T\langle(1-\varepsilon)K_1Q_1K_1^\top X_1,X_1\rangle+\langle(1-\frac{1}{\varepsilon})K_1Q_1K_1^\top X_2,X_2\rangle dt \\ &+\langle(1-\varepsilon)K_1G_1K_1^\top X_1(T),X_1(T)\rangle+\langle(1-\frac{1}{\varepsilon})K_1G_1K_1^\top X_2(T),X_2(T)\rangle\Big\}, \end{aligned}$$

where $\varepsilon>0$. Then, according to the decoupled system (3.23), we can construct two auxiliary systems:

$$\begin{cases} dX_1=(\check{A}_1X_1+\check{B}_1\tilde{u}_1)dt+(\check{A}_2X_1+\check{B}_2\tilde{u}_1)dW_1+(\check{A}_3X_1+\check{B}_3\tilde{u}_1)dW_2, \\ X_1(0)=(\xi_1^\top\ \xi_2^\top)^\top, \\ J_1^1(u_1)=\dfrac{1}{2}\mathbb{E}\Big\{\displaystyle\int_0^T\langle(1-\varepsilon)K_1Q_1K_1^\top X_1,X_1\rangle dt \\ \qquad\qquad +\langle(1-\varepsilon)K_1G_1K_1^\top X_1(T),X_1(T)\rangle\Big\}, \end{cases} \tag{3.24}$$

and

$$
\begin{cases}
dX_2 = (\check{A}_1 X_2 + \check{C}_1 \lambda + \widehat{D}_2 \iota_1 + \widehat{F}_2 \iota_2)dt + (\check{A}_2 X_2 + \check{C}_2 \lambda + \widehat{D}_3 \iota_1 \\
\qquad + \widehat{D}_4 \iota_2)dW_1 + (\check{A}_3 X_2 + \check{C}_3 \lambda + (\widehat{D}_4)^\top \iota_1 + \widehat{F}_3 \iota_2)dW_2, \\
X_2(0) = (0^\top \ 0^\top)^\top, \\
J_1^2(\lambda, \iota_1, \iota_2) = \dfrac{1}{2}\mathbb{E}\Big\{ \displaystyle\int_0^T \langle (1 - \dfrac{1}{\varepsilon}) K_1 Q_1 K_1^\top X_2, X_2 \rangle dt \\
\qquad\qquad + \langle (1 - \dfrac{1}{\varepsilon}) K_1 G_1 K_1^\top X_2(T), X_2(T) \rangle \Big\}.
\end{cases}
\tag{3.25}
$$

We first consider the BSDE in (3.23). By the similar argument in [194, Chapter 4], [191, Chapter 7] and using Itô formula,

$$
\begin{aligned}
&\mathbb{E}\Big[|\lambda(t)|^2 + \int_t^T |\iota_1|^2 ds + \int_t^T |\iota_2|^2 ds\Big] \\
=&\mathbb{E}\int_t^T \langle 2\lambda, (\check{A}_4 \lambda + (\Lambda \widehat{D}_2 + \widehat{C}_1^\top)\iota_1 + (\Lambda \widehat{F}_2 + \widehat{C}_2^\top)\iota_2 + \check{B}_4 \tilde{u}_1) \rangle ds \\
\leq&\mathbb{E}\int_t^T 2|\check{A}_4| \langle \lambda, \lambda \rangle + \frac{1}{\delta_1}\langle \lambda, \lambda \rangle + \delta_1 \langle (\Lambda \widehat{D}_2 + \widehat{C}_1^\top)^\top (\Lambda \widehat{D}_2 + \widehat{C}_1^\top)\iota_1, \iota_1 \rangle + \frac{1}{\delta_2}\langle \lambda, \lambda \rangle \\
&+ \delta_2 \langle (\Lambda \widehat{F}_2 + \widehat{C}_2^\top)^\top (\Lambda \widehat{F}_2 + \widehat{C}_2^\top)\iota_2, \iota_2 \rangle + \frac{1}{\delta_3}\langle \lambda, \lambda \rangle + \delta_3 \langle \check{B}_4^\top \check{B}_4 \tilde{u}_1, \tilde{u}_1 \rangle ds \\
\leq&\mathbb{E}\int_t^T 2(|\check{A}_4| + \frac{1}{\delta_1} + \frac{1}{\delta_2} + \frac{1}{\delta_3})|\lambda|^2 + \frac{1}{2}|\iota_1|^2 + \frac{1}{2}|\iota_2|^2 ds + \mathbb{E}\int_0^T |\tilde{u}_1|^2 ds,
\end{aligned}
$$

where $\delta_1 = \frac{1}{2|(\Lambda \widehat{D}_2 + \widehat{C}_1^\top)^\top (\Lambda \widehat{D}_2 + \widehat{C}_1^\top)|}$, $\delta_2 = \frac{1}{2|(\Lambda \widehat{F}_2 + \widehat{C}_2^\top)^\top (\Lambda \widehat{F}_2 + \widehat{C}_2^\top)|}$, $\delta_3 = \frac{1}{|\check{B}_4^\top \check{B}_4|}$. Then

$$
\begin{aligned}
&\mathbb{E}\Big[|\lambda(t)|^2 + \frac{1}{2}\int_t^T |\iota_1|^2 ds + \frac{1}{2}\int_t^T |\iota_2|^2 ds\Big] \\
\leq&\mathbb{E}\int_t^T 2(|\check{A}_4| + \frac{1}{\delta_1} + \frac{1}{\delta_2} + \frac{1}{\delta_3})|\lambda|^2 ds + \mathbb{E}\int_0^T |\tilde{u}_1|^2 ds, \ \forall\ t \in [0, T],
\end{aligned}
\tag{3.26}
$$

which, together with Fubini's theorem, implies that

$$\mathbb{E}\Big[|\lambda(t)|^2\Big] \leq \int_t^T 2(|\check{A}_4| + \frac{1}{\delta_1} + \frac{1}{\delta_2} + \frac{1}{\delta_3})\mathbb{E}|\lambda|^2 ds + \mathbb{E}\int_t^T |\tilde{u}_1|^2 ds.$$

By Gronwall's inequality, we have

$$\mathbb{E}\Big[|\lambda(t)|^2\Big] \leq L_1 \mathbb{E}\int_0^T |\tilde{u}_1|^2 ds, \quad \forall\, t \in [0,T], \tag{3.27}$$

where $L_1 = \exp\Big(2\int_0^T (|\check{A}_4| + \frac{1}{\delta_1} + \frac{1}{\delta_2} + \frac{1}{\delta_3})dt\Big)$. Then, letting $t = 0$ and plugging (3.27) into (3.26), we have

$$\mathbb{E}\Big[\int_0^T |\iota_1|^2 dt + \int_0^T |\iota_2|^2 dt\Big] \leq (2L_2 + 2)\mathbb{E}\int_0^T |\tilde{u}_1|^2 dt, \tag{3.28}$$

where $L_2 = \Big(2\int_0^T (|\check{A}_4| + \frac{1}{\delta_1} + \frac{1}{\delta_2} + \frac{1}{\delta_3})L_1 dt\Big)$.

Secondly, we discuss the second system (3.25). Let $\delta_1' = \frac{1}{9\max\{\|\check{C}_1\|^2, \|\widehat{D}_2\|^2, \|\widehat{F}_2\|^2\}}$, $\delta_2' = \frac{1}{9\max\{\|\check{C}_2\|^2, \|\widehat{D}_3\|^2, \|\widehat{D}_4\|^2\}}$, $\delta_3' = \frac{1}{9\max\{\|\check{C}_3\|^2, \|\widehat{D}_4^\top\|^2, \|\widehat{F}_3\|^2\}}$. Applying Itô's formula and combining with (3.27) and (3.28), it follows that

$$\begin{aligned}
\mathbb{E}|X_2(t)|^2 &\leq \mathbb{E}\int_0^T \Big[(2|\check{A}_1| + \frac{1}{\delta_1'}) + (1 + \frac{1}{\delta_2'})|\check{A}_2|^2 + (1 + \frac{1}{\delta_3'})|\check{A}_3|^2\Big]|X_2|^2 \\
&\quad + (1 + 27\delta_2' + 27\delta_3')(|\lambda|^2 + |\iota_1|^2 + |\iota_2|^2)ds \\
&\leq \mathbb{E}\int_0^T \Big[(2|\check{A}_1| + \frac{1}{\delta_1'}) + (1 + \frac{1}{\delta_2'})|\check{A}_2|^2 + (1 + \frac{1}{\delta_3'})|\check{A}_3|^2\Big]|X_2|^2 ds \\
&\quad + (1 + 27\delta_2' + 27\delta_3')(2L_2 + L_1(T+1) + 2)\mathbb{E}\int_0^T |\tilde{u}_1|^2 ds, \quad \forall\, t \in [0,T].
\end{aligned}$$

Applying Gronwall's inequality, we have

$$\mathbb{E}|X_2(t)|^2 \leq L_3(2L_2 + L_1(T+1) + 2)\mathbb{E}\int_0^T |\tilde{u}_1|^2 ds, \quad \forall\, t \in [0,T]. \tag{3.29}$$

where $L_3 = (1+27\delta_2'+27\delta_3')\exp\Big(\int_0^T(2|\check{A}_1|+\frac{1}{\delta_1'})+(1+\frac{1}{\delta_2'})|\check{A}_2|^2+(1+\frac{1}{\delta_3'})|\check{A}_3|^2ds\Big)$.

Let $P_3 \in C^1(0,T;\mathbb{S}^{2n})$ satisfies the following Lyapunov equation:

$$\begin{cases}\dot{P}_3+P_3\check{A}_1+\check{A}_1^\top P_3+\check{A}_2^\top P_3\check{A}_2+\check{A}_3^\top P_3\check{A}_3+(1-\dfrac{1}{\varepsilon})K_1Q_1K_1^\top=0,\\ P_3(T)=(1-\dfrac{1}{\varepsilon})K_1G_1K_1^\top.\end{cases}$$

Then, combining (3.27)-(3.29) and using Itô's formula to $\langle P_3X_2,X_2\rangle$, we have

$$\begin{aligned}
&J_1^2(\lambda;\iota_1;\iota_2)\\
=&\frac{1}{2}\mathbb{E}\int_0^T 2\langle P_3(\check{C}_1\lambda+\widehat{D}_2\iota_1+\widehat{F}_2\iota_2),X_2\rangle+2\langle\check{A}_2^\top P_3(\check{C}_2\lambda+\widehat{D}_3\iota_1+\widehat{D}_4\iota_2),X_2\rangle\\
&+2\langle\check{A}_3^\top P_3(\check{C}_3\lambda+(\widehat{D}_4)^\top\iota_1+\widehat{F}_3\iota_2),X_2\rangle+2\langle P_3(\check{C}_2\lambda+\widehat{D}_3\iota_1+\widehat{D}_4\iota_2),(\check{C}_2\lambda\\
&+\widehat{D}_3\iota_1+\widehat{D}_4\iota_2)\rangle+2\langle P_3(\check{C}_3\lambda+(\widehat{D}_4)^\top\iota_1+\widehat{F}_3\iota_2),(\check{C}_3\lambda+(\widehat{D}_4)^\top\iota_1+\widehat{F}_3\iota_2)\rangle dt\\
\geq&\frac{1}{2}\mathbb{E}\int_0^T -\delta_1''|P_3|^2|\check{C}_1\lambda+\widehat{D}_2\iota_1+\widehat{F}_2\iota_2|^2-(\frac{1}{\delta_1''}+\frac{1}{\delta_2''}+\frac{1}{\delta_3''})|X_2|^2-(\delta_2''|\check{A}_2^\top P_3|^2\\
&-2|P_3|)|\check{C}_2\lambda+\widehat{D}_3\iota_1+\widehat{D}_4\iota_2|^2-(\delta_3''|\check{A}_3^\top P_3|^2-2|P_3|)|\check{C}_3\lambda+\widehat{D}_4^\top\iota_1+\widehat{F}_3\iota_2|^2dt\\
\geq&\frac{1}{2}\mathbb{E}\int_0^T -(|\lambda|^2+|\iota_1|^2+|\iota_2|^2)-(\frac{1}{\delta_1''}+\frac{1}{\delta_2''}+\frac{1}{\delta_3''})|X_2|^2dt\\
\geq&-\left\{\frac{1}{2}L_1T+L_2+1+(\frac{1}{\delta_1''}+\frac{1}{\delta_2''}+\frac{1}{\delta_3''})L_3T\left[L_2+\frac{L_1(T+1)}{2}+1\right]\right\}\mathbb{E}\int_0^T|\tilde{u}_1|^2dt,
\end{aligned}\tag{3.30}$$

where $\delta_1''=\frac{1}{9\|P_3\|^2\max\{\|\check{C}_1\|^2,\|\widehat{D}_2\|^2,\|\widehat{F}_2\|^2\}}$, $\delta_2''=\frac{2\|P_3\|+1/\max\{\|\check{C}_2\|^2,\|\widehat{D}_3\|^2,\|\widehat{D}_4\|^2\}}{9\|\check{A}_2^\top P_3\|^2}$, $\delta_3''=\frac{2\|P_3\|+1/\max\{\|\check{C}_3\|^2,\|\widehat{D}_4^\top\|^2,\|\widehat{F}_3\|^2\}}{9\|\check{A}_3^\top P_3\|^2}$.

Moreover, since the first system (3.24) is a standard control system, by the dis-

cussion in [164, Theorem 4.5], there exists a Riccati equation

$$\begin{cases} \dot{\Pi} + \Pi\check{A}_1 + \check{A}_1^\top\Pi + \sum_{m=1}^{2}\check{A}_m^\top\Pi\check{A}_m + (1-\varepsilon)K_1Q_1K_1^\top - (\check{B}_1^\top\Pi \\ \quad + \sum_{m=2}^{3}\check{B}_m^\top\Pi\check{A}_m)^\top(\check{A}_2^\top\Pi\check{A}_2 + \check{A}_3^\top\Pi\check{A}_3)^{-1}(\check{B}_1^\top\Pi + \sum_{m=2}^{3}\check{B}_m^\top\Pi\check{A}_m) = 0, \\ \Pi(T) = (1-\varepsilon)K_1G_1K_1^\top, \quad \check{A}_2^\top\Pi\check{A}_2 + \check{A}_3^\top\Pi\check{A}_3 \geq \lambda I, \end{cases}$$

for some $\lambda > 0$ and $\Pi(\cdot) \in C^1(0,T;\mathbb{R}^{2n\times 2n})$ is matrix-value functions. Denoting

$$\Theta = (\check{A}_2^\top\Pi\check{A}_2 + \check{A}_3^\top\Pi\check{A}_3)^{-1}(\check{B}_1^\top\Pi + \sum_{m=2}^{3}\check{B}_m^\top\Pi\check{A}_m),$$

and applying Itô's formula to $\langle \Pi X_1, X_1\rangle$, it follows that

$$\begin{aligned} J_1^1(u_1) =& \frac{1}{2}\mathbb{E}\int_0^T \langle(\dot{\Pi} + \Pi\check{A}_1 + \check{A}_1^\top\Pi + \sum_{m=1}^{2}\check{A}_m^\top\Pi\check{A}_m + (1-\varepsilon)K_1Q_1K_1^\top)X_1, X_1\rangle \\ &+ 2\langle\Theta X_1, \tilde{u}_1\rangle + \langle(\check{A}_2^\top\Pi\check{A}_2 + \check{A}_3^\top\Pi\check{A}_3)\tilde{u}_1, \tilde{u}_1\rangle dt + \langle\Pi(0)X_1(0), X_1(0)\rangle \\ =& \frac{1}{2}\mathbb{E}\int_0^T \langle(\check{A}_2^\top\Pi\check{A}_2 + \check{A}_3^\top\Pi\check{A}_3)(\tilde{u}_1 - \Theta X_1), (\tilde{u}_1 - \Theta X_1)\rangle dt + \langle\Pi(0)X_1(0), X_1(0)\rangle. \end{aligned}$$

Letting $\check{A}_2^\top\Pi\check{A}_2 + \check{A}_3^\top\Pi\check{A}_3 \geq \delta_4 I$, for some $\delta_4 > 0$. By Lemma 2.3 in [164], we have

$$J_1^1(u_1) \geq \delta_4\gamma\mathbb{E}\int_0^T |\tilde{u}_1|^2 dt, \quad \forall\ \tilde{u}_1 \in \mathcal{U}_1, \tag{3.31}$$

for some $\gamma > 0$ (e.g. $\delta_4\gamma = \frac{1}{2}(\check{A}_2^\top\Pi\check{A}_2 + \check{A}_3^\top\Pi\check{A}_3)$). Then, $J_1^1(u_1)$ is uniformly convex.

Finally, combining (3.30) and (3.31), one can obtain

$$\begin{aligned} J_1(u_1) =& J_1^1(u_1) + J_1^2(\lambda;\iota_1;\iota_2) \geq \Big\{\delta_4\gamma - \frac{1}{2}L_1T + L_2 + 1 + (\frac{1}{\delta_1''} \\ &+ \frac{1}{\delta_2''} + \frac{1}{\delta_3''})L_3T\Big[L_2 + \frac{L_1(T+1)}{2} + 1\Big]\Big\}\mathbb{E}\int_0^T |\tilde{u}_1|^2 dt, \quad \forall\ \tilde{u}_1 \in \mathcal{U}_1, \end{aligned}$$

is uniformly convex, if

$$\delta_4\gamma > \left|\frac{1}{2}L_1T + L_2 + 1 + (\frac{1}{\delta_1''} + \frac{1}{\delta_2''} + \frac{1}{\delta_3''})L_3T\left[L_2 + \frac{L_1(T+1)}{2} + 1\right]\right|.$$

The theorem follows. □

According to above discussion, we study the solvability of (**LP**) under degenerate control weight case with the following assumption:

(**A3.4**) Assume that (3.21) admits unique solution and for some $\delta_4 > 0$ and $\gamma > 0$,

$$\delta_4\gamma > \left|\frac{1}{2}L_1T + L_2 + 1 + (\frac{1}{\delta_1''} + \frac{1}{\delta_2''} + \frac{1}{\delta_3''})L_3T\left[L_2 + \frac{L_1(T+1)}{2} + 1\right]\right|.$$

If (**A3.4**) hold, by the property of uniform convexity, **(LP)** admits a unique equilibrium pair $(\bar{u}_1, \bar{u}_2)$. However, as we discussed above, it is intractable to characterize it. Thus, we may rest upon the classical weak-convergence method to have some near-optimal sequence to approximate it. Then we consider a minimizing sequence for (**LP**). Let $R_1^{i,(\varepsilon)} = \varepsilon I$, for some $\varepsilon > 0$, and

$$\begin{aligned}
&J_i^{(\varepsilon)}(u;\bar{v}(u)) = J_i^{(\varepsilon)}(u_i;\bar{v}_i(u_i;u_j);u_j;\bar{v}_j(u_i;u_j))\\
=&\frac{1}{2}\mathbb{E}\Big\{\int_0^T \langle K_iQ_iK_i^\top X, X\rangle + \langle \widetilde{R}_1^{i,(\varepsilon)}u, u\rangle dt + \langle K_iG_iK_i^\top X(T), X(T)\rangle\\
&+ 2\langle K_ie_i^\top g, X(T)\rangle\Big\}\\
:=&J_i(u_i;\bar{v}_i(u_i;\bar{u}_j);\bar{u}_j;\bar{v}_j(u_i;\bar{u}_j)) + \frac{1}{2}\mathbb{E}\int_0^T |u_i|^2_{R_1^{i,(\varepsilon)}}dt,\ i=1,2,\ j\neq i,
\end{aligned} \tag{3.32}$$

where

$$\widetilde{R}_1^{1,(\varepsilon)} = \begin{pmatrix} R_1^{1,(\varepsilon)} & 0\\ 0 & 0\end{pmatrix},\quad \widetilde{R}_1^{2,(\varepsilon)} = \begin{pmatrix} 0 & 0\\ 0 & R_1^{2,(\varepsilon)}\end{pmatrix}.$$

Then, for fixed $\bar{u}_2$, by equation (3.16), (3.18) and (3.20), we let

$$\bar{u}_1^{(\varepsilon)} = \mathbf{P}_{\Gamma^1}\Big\{-(R_1^{1,(\varepsilon)})^{-1}[(\widehat{B}_1^1)^\top y_1^1 + (\widehat{D}_1^1)^\top\widehat{\beta}_1^1 + \widehat{F}_1^1\widehat{\gamma}_1^1 - \Psi_1\phi_1^1]\Big\}.$$

Similarly, for fixed $\bar{u}_1$,

$$\bar{u}_2^{(\varepsilon)} = \mathbf{P}_{\Gamma^2}\Big\{ -(R_1^{2,(\varepsilon)})^{-1}[(\widehat{B}_1^2)^\top y_2^2 + (\widehat{D}_1^2)^\top \widehat{\beta}_2^2 + \widehat{F}_1^2\widehat{\gamma}_2^2 - \Psi_2\phi_2^2]\Big\}.$$

Then, by [164, Corollary 4.7] and the uniform convexity of (3.12), for fixed $\bar{u}_j$, $j \neq i$, $i = 1, 2$, $\bar{u}_i^{(\varepsilon)}$ is the unique optimal control of (3.32). Using the minimizing sequence, we have following proposition:

Proposition 3.4. *Suppose that* (**A3.1**)-(**A3.4**) *hold. For* $i = 1, 2$ *and* $j \neq i$, $\bar{u}_i^{(\varepsilon)}$ *is a minimizing sequence of*

$$u_i \to J_i(u_i; \bar{v}_i(u_i; \bar{u}_j); \bar{u}_j; \bar{v}_j(u_i; \bar{u}_j)),$$

for some $\varepsilon > 0$, *i.e.,*

$$\lim_{\varepsilon\to 0} J_i(\bar{u}_i^{(\varepsilon)}; \bar{v}_i(\bar{u}_i^{(\varepsilon)}; \bar{u}_j); \bar{u}_j; \bar{v}_j(\bar{u}_i^{(\varepsilon)}; \bar{u}_j)) = \inf_{u_i\in\mathcal{U}_i} J_i(u_i; \bar{v}_i(u_i; \bar{u}_j); \bar{u}_j; \bar{v}_j(u_i; \bar{u}_j)).$$

Moreover, the sequence $\{\bar{u}_i^{(\varepsilon)}\}_{\varepsilon>0}$ *admits a weakly convergent subsequence.*

Proof By (3.32), for fixed $\bar{u}_j$, $j \neq i$, $i = 1, 2$,

$$\begin{aligned} &J_i^{(\varepsilon)}(u_i; \bar{v}_i(u_i; \bar{u}_j); \bar{u}_j; \bar{v}_j(u_i; \bar{u}_j)) \\ =&J_i(u_i; \bar{v}_i(u_i; \bar{u}_j); \bar{u}_j; \bar{v}_j(u_i; \bar{u}_j)) + \frac{1}{2}\mathbb{E}\int_0^T |u_i|^2_{R_1^{i,(\varepsilon)}} dt, \end{aligned}$$

then we have

$$\inf_{u_i\in\mathcal{U}_i} J_i^{(\varepsilon)}(u_i; \bar{v}_i(u_i; \bar{u}_j); \bar{u}_j; \bar{v}_j(u_i; \bar{u}_j)) \geq \inf_{u_i\in\mathcal{U}_i} J_i(u_i; \bar{v}_i(u_i; \bar{u}_j); \bar{u}_j; \bar{v}_j(u_i; \bar{u}_j)). \tag{3.33}$$

On the other hand, since J_i is uniformly convex, for any $\delta > 0$, there exists $u_i^{(\delta)} \in \mathcal{U}_i$ such that

$$J_i(u_i^{(\delta)}; \bar{v}_i(u_i^{(\delta)}; \bar{u}_j); \bar{u}_j; \bar{v}_j(u_i^{(\delta)}; \bar{u}_j)) \leq \inf_{u_i\in\mathcal{U}_i} J_i(u_i; \bar{v}_i(u_i; \bar{u}_j); \bar{u}_j; \bar{v}_j(u_i; \bar{u}_j)) + \delta.$$

Hence,

$$
\begin{aligned}
&\inf_{u_i\in\mathcal{U}_i} J_i^{(\varepsilon)}(u_i;\bar{v}_i(u_i;\bar{u}_j);\bar{u}_j;\bar{v}_j(u_i;\bar{u}_j))\\
\leq& J_i(u_i^{(\delta)};\bar{v}_i(u_i^{(\delta)};\bar{u}_j);\bar{u}_j;\bar{v}_j(u_i^{(\delta)};\bar{u}_j))+\frac{1}{2}\mathbb{E}\int_0^T |u_i^{(\delta)}|^2_{R_1^{i,(\varepsilon)}}dt\\
\leq& \inf_{u_i\in\mathcal{U}_i} J_i(u_i;\bar{v}_i(u_i;\bar{u}_j);\bar{u}_j;\bar{v}_j(u_i;\bar{u}_j))+\delta+\frac{1}{2}\mathbb{E}\int_0^T |u_i^{(\delta)}|^2_{R_1^{i,(\varepsilon)}}dt.
\end{aligned}
$$

Since $\delta>0$ is arbitrary, it follows that

$$
\lim_{\varepsilon\to 0}\inf_{u_i\in\mathcal{U}_i} J_i^{(\varepsilon)}(u_i;\bar{v}_i(u_i;\bar{u}_j);\bar{u}_j;\bar{v}_j(u_i;\bar{u}_j))\leq \inf_{u_i\in\mathcal{U}_i} J_i(u_i;\bar{v}_i(u_i;\bar{u}_j);\bar{u}_j;\bar{v}_j(u_i;\bar{u}_j)). \tag{3.34}
$$

Combining (3.33) and (3.34), we have

$$
\lim_{\varepsilon\to 0}\inf_{u_i\in\mathcal{U}_i} J_i^{(\varepsilon)}(u_i;\bar{v}_i(u_i;\bar{u}_j);\bar{u}_j;\bar{v}_j(u_i;\bar{u}_j))= \inf_{u_i\in\mathcal{U}_i} J_i(u_i;\bar{v}_i(u_i;\bar{u}_j);\bar{u}_j;\bar{v}_j(u_i;\bar{u}_j)).
$$

Note that $\bar{u}_i^{(\varepsilon)}$ is the unique optimal control of (3.32), then one can obtain

$$
\begin{aligned}
&\frac{1}{2}\mathbb{E}\int_0^T |\bar{u}_i^{(\varepsilon)}|^2_{R_1^{i,(\varepsilon)}}dt\\
=& J_i^{(\varepsilon)}(\bar{u}_i^{(\varepsilon)};\bar{v}_i(\bar{u}_i^{(\varepsilon)};\bar{u}_j);\bar{u}_j;\bar{v}_j(\bar{u}_i^{(\varepsilon)};\bar{u}_j))-J_i(\bar{u}_i^{(\varepsilon)};\bar{v}_i(\bar{u}_i^{(\varepsilon)};\bar{u}_j);\bar{u}_j;\bar{v}_j(\bar{u}_i^{(\varepsilon)};\bar{u}_j))\\
=& \inf_{u_i\in\mathcal{U}_i} J_i^{(\varepsilon)}(u_i;\bar{v}_i(u_i;\bar{u}_j);\bar{u}_j;\bar{v}_j(u_i;\bar{u}_j))-J_i(\bar{u}_i^{(\varepsilon)};\bar{v}_i(\bar{u}_i^{(\varepsilon)};\bar{u}_j);\bar{u}_j;\bar{v}_j(\bar{u}_i^{(\varepsilon)};\bar{u}_j))\\
\leq& \inf_{u_i\in\mathcal{U}_i} J_i^{(\varepsilon)}(u_i;\bar{v}_i(u_i;\bar{u}_j);\bar{u}_j;\bar{v}_j(u_i;\bar{u}_j))-\inf_{u_i\in\mathcal{U}_i} J_i(u_i;\bar{v}_i(u_i;\bar{u}_j);\bar{u}_j;\bar{v}_j(u_i;\bar{u}_j)),
\end{aligned}
$$

hence,

$$
\lim_{\varepsilon\to 0}\frac{1}{2}\mathbb{E}\int_0^T |\bar{u}_i^{(\varepsilon)}|^2_{R_1^{i,(\varepsilon)}}dt=0,\quad \forall\ \varepsilon>0.
$$

Therefore,

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} J_i(\bar{u}_i^{(\varepsilon)};\bar{v}_i(\bar{u}_i^{(\varepsilon)};\bar{u}_j);\bar{u}_j;\bar{v}_j(\bar{u}_i^{(\varepsilon)};\bar{u}_j))\\
=&\lim_{\varepsilon\to 0}\Big[J_i^{(\varepsilon)}(\bar{u}_i^{(\varepsilon)};\bar{v}_i(\bar{u}_i^{(\varepsilon)};\bar{u}_j);\bar{u}_j;\bar{v}_j(\bar{u}_i^{(\varepsilon)};\bar{u}_j))-\frac{1}{2}\mathbb{E}\int_0^T|\bar{u}_i^{(\varepsilon)}|^2_{R_1^{i,(\varepsilon)}}dt\Big]\\
=&\lim_{\varepsilon\to 0} J_i^{(\varepsilon)}(\bar{u}_i^{(\varepsilon)};\bar{v}_i(\bar{u}_i^{(\varepsilon)};\bar{u}_j);\bar{u}_j;\bar{v}_j(\bar{u}_i^{(\varepsilon)};\bar{u}_j))=\inf_{u_i\in\mathcal{U}_i} J_i(u_i;\bar{v}_i(u_i;\bar{u}_j);\bar{u}_j;\bar{v}_j(u_i;\bar{u}_j)).
\end{aligned}
$$

Furthermore, since (**A3.4**) hold, by the property of uniform convexity, **(LP)** is uniquely solvable. Let $\bar{u}_i$ be the optimal control of $J_i(u_i,\bar{v}_i(u_i,\bar{u}_j),\bar{u}_j,\bar{v}_j(u_i,\bar{u}_j))$, we have

$$
\begin{aligned}
&J_i(\bar{u}_i;\bar{v}_i(\bar{u}_i;\bar{u}_j);\bar{u}_j;\bar{v}_j(\bar{u}_i;\bar{u}_j))+\frac{1}{2}\mathbb{E}\int_0^T|\bar{u}_i^{(\varepsilon)}|^2_{R_1^{i,(\varepsilon)}}dt\\
\le&J_i^{(\varepsilon)}(\bar{u}_i^{(\varepsilon)};\bar{v}_i(\bar{u}_i^{(\varepsilon)};\bar{u}_j);\bar{u}_j;\bar{v}_j(\bar{u}_i^{(\varepsilon)};\bar{u}_j))\le J_i^{(\varepsilon)}(\bar{u}_i;\bar{v}_i(\bar{u}_i;\bar{u}_j);\bar{u}_j;\bar{v}_j(\bar{u}_i;\bar{u}_j))\\
=&J_i(\bar{u}_i;\bar{v}_i(\bar{u}_i;\bar{u}_j);\bar{u}_j;\bar{v}_j(\bar{u}_i;\bar{u}_j))+\frac{1}{2}\mathbb{E}\int_0^T|\bar{u}_i|^2_{R_1^{i,(\varepsilon)}}dt,
\end{aligned}
$$

i.e., $\{\bar{u}_i^{(\varepsilon)}\}_{\varepsilon>0}$ is bounded in the Hilbert space $\mathcal{U}_i$ and hence admits a weak-convergence subsequence. □

3.5 Two Examples

We give an example with non-singular control weight and obtain a related Hamiltonian system. Then, another example with singular control weight is provided.

3.5.1 Example 1

We now look at a one-dimensional case and assume the following:

$$
\begin{cases}
A^i=0,\quad B_1^i=1,\quad B_2^i=1,\quad b^i=0,\quad \xi_i=0,\quad G_i=2,\quad T=1,\\
C_1^i=0,\quad D_1^i=1,\quad D_2^i=1,\quad \sigma_1^i=0,\quad R_1^i=R_2^i=1,\quad Q_i=-2,\\
C_2^i=0,\quad F_1^i=0,\quad F_2^i=0,\quad \sigma_2^i=0,\quad g_i=0,\quad k_i=0.5,\quad i=1,2.
\end{cases}
\tag{3.35}
$$

Then, the state equation can be show as follows:

$$dx_i = (u_i + v_i)dt + (u_i + v_i)dW, \quad x_i(0) = 0, \quad i = 1, 2,$$

and the cost functional is

$$\begin{aligned} J_i(u_1; v_1; u_2; v_2) =& \frac{1}{2}\mathbb{E}\Big\{ \int_0^1 -2|x_i - 0.5x_j|^2 + |u_i|^2 + |v_i|^2 dt \Big\} \\ &+ \frac{1}{2}\mathbb{E}\Big[2|x_i(T) - 0.5x_j(T)|^2 \Big], \quad i = 1, 2. \end{aligned}$$

The corresponding Riccati equation $P_i(\cdot)$ satisfy

$$\dot{P}_i - 2 - \frac{P_i^2}{1 + P_i} = 0, \quad P_i(1) = 2, \quad i = 1, 2. \tag{3.36}$$

By computation, the solutions of (3.36) are shown as follows

$$P_i(t) = (2e^{2t-2} - 1)^{\frac{1}{2}} - 1, \text{ or } P_i(t) = -(2e^{2t-2} - 1)^{\frac{1}{2}} - 1,$$

where $t \in [0, 1]$. According to our assumption (3.35), the state for (**LP**) is

$$\begin{cases} dX =[\widehat{A}X + \widehat{B}_3\varphi + \widehat{D}_2\theta_1 + \widehat{B}_1 u]dt + [\widehat{C}_1 X + \widehat{D}_2^\top \varphi + \widehat{D}_3\theta_1 + \widehat{D}_1 u]dW, \\ d\varphi = -[\widehat{A}^\top \varphi + (\widehat{C}_1)^\top \theta_1 + \Psi^\top u - QX]dt + \theta_1 dW, \\ X(0) = (0^\top \ 0^\top)^\top, \quad \varphi(T) = -GX(T), \end{cases} \tag{3.37}$$

where

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \ \widehat{B}_1 = \widehat{D}_1 = \begin{pmatrix} 1 - \frac{1}{1+P_1} & 0 \\ 0 & 1 - \frac{1}{1+P_2} \end{pmatrix},$$

$$\widehat{A} = \widehat{C}_1 = \begin{pmatrix} -\frac{P_1}{1+P_1} & 0 \\ 0 & -\frac{P_2}{1+P_2} \end{pmatrix}, \ \widehat{B}_3 = \widehat{D}_2 = \widehat{D}_3 = \begin{pmatrix} -\frac{1}{1+P_1} & 0 \\ 0 & -\frac{1}{1+P_2} \end{pmatrix},$$

$$Q = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \ \Psi = \begin{pmatrix} \frac{P_1}{1+P_1} & 0 \\ 0 & \frac{P_2}{1+P_2} \end{pmatrix}, \ u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \ G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and the cost functional is

$$
\begin{aligned}
J_i(u;\bar{v}(u)) =& \frac{1}{2}\mathbb{E}\Big\{\int_0^1 -2\langle K_iK_i^\top X, X\rangle + \langle \widetilde{R}_1^i u, u\rangle + \langle \widetilde{R}_2^i \widehat{R}_2^{-1}[\widehat{B}_2 X \\
&+ \Phi u + B_2^\top \varphi + D_2^\top \theta_1], \widehat{R}_2^{-1}[\widehat{B}_2 X + \Phi u + B_2^\top \varphi \\
&+ D_2^\top \theta_1]\rangle dt + 2\langle K_iK_i^\top X(T), X(T)\rangle\Big\}, \quad i = 1, 2,
\end{aligned}
$$

where

$$
\begin{aligned}
K_1 =& K_2 = \begin{pmatrix} 1 \\ -0.5 \end{pmatrix}, \ \widetilde{R}_1^1 = \widetilde{R}_2^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \widehat{B}_2 = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}, \\
\widehat{R}_2 =& B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \widetilde{R}_1^2 = \widetilde{R}_2^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}
$$

Since $R_1^1 = R_1^2 = 1 > 0$ and $R_2^1 = R_2^2 = 1 > 0$, then let

$$
\vartheta_i(P_i; x_i; \varphi_i; \theta_1^i; y_i^i; \widehat{\beta}_i^i) := P_i(P_i x_i + \varphi_i + \theta_1^i) + (1 + P_i)(y_i^i + \widehat{\beta}_i^i) - P_i^3 - P_i,
$$

there exists two projection maps $\mathbf{P}_{\Gamma^1}(\cdot)$ and $\mathbf{P}_{\Gamma^2}(\cdot)$ such that

$$
\bar{u}_1 = \mathbf{P}_{\Gamma^1}\Big\{-\frac{\vartheta_1}{1 + 2P_1 + 2P_1^2}\Big\}, \quad \bar{u}_2 = \mathbf{P}_{\Gamma^2}\Big\{-\frac{\vartheta_2}{1 + 2P_2 + 2P_2^2}\Big\}.
$$

We let $\bar{u}^1 = (\bar{u}_1^\top \ 0^\top)^\top$ and $\bar{u}^2 = (0^\top \ \bar{u}_2^\top)^\top$ and the related Hamiltonian system is

$$
\begin{cases}
d\overline{X} =& [\widehat{A}\overline{X} + \widehat{B}_3\overline{\varphi} + \widehat{D}_2\overline{\theta}_1 + \widehat{B}_1\bar{u}^i]dt + [\widehat{C}_1\overline{X} + \widehat{D}_2^\top\overline{\varphi} + \widehat{D}_3\overline{\theta}_1 + \widehat{D}_1\bar{u}^i]dW_1, \\
d\overline{\varphi} =& -[(\widehat{A})^\top\overline{\varphi} + (\widehat{C}_1)^\top\overline{\theta}_1 + \Psi^\top\bar{u}^i - Q\overline{X}]dt + \overline{\theta}_1 dW_1, \quad \overline{\varphi}(T) = -G\overline{X}(T), \\
dY_i =& -[\widehat{A}^\top Y_i + \widehat{C}_1^\top\widehat{\beta}_i - Q^\top\phi_i + K_iQ_iK_i^\top\overline{X} + \widehat{B}_2^\top\widehat{R}_2^{-1}\widetilde{R}_2^i\widehat{R}_2^{-1}(\widehat{B}_2\overline{X} + \Phi\bar{u}^i \\
&\quad + B_2^\top\overline{\varphi} + D_2^\top\overline{\theta}_1)]dt + \widehat{\beta}_i dW_1, \quad Y_i(T) = K_iG_iK_i^\top\overline{X}(T) - G^\top\phi_i(T), \\
d\phi_i =& [\widehat{A}\phi_i + (\widehat{B}_3^\top Y_i + \widehat{D}_2\widehat{\beta}_i) + B_2\widehat{R}_2^{-1}\widetilde{R}_2^i\widehat{R}_2^{-1}(\widehat{B}_2\overline{X} + \Phi\bar{u}^i + B_2^\top\overline{\varphi} + D_2^\top\overline{\theta}_1)]dt \\
&\quad + [\widehat{C}_1\phi_i + (\widehat{D}_2^\top Y_i + \widehat{D}_3^\top\widehat{\beta}_i) + D_2\widehat{R}_2^{-1}\widetilde{R}_2^i\widehat{R}_2^{-1}(\widehat{B}_2\overline{X} + \Phi\bar{u}^i + B_2^\top\overline{\varphi} \\
&\quad + D_2^\top\overline{\theta}_1)]dW_1, \quad \phi_i(0) = 0, \quad \overline{X}(0) = (0^\top \ 0^\top)^\top, \quad i = 1, 2.
\end{cases}
$$

Suppose that $\mathbf{P}_{\Gamma^i}:\mathbb{R}\to\Gamma^i$, $\Gamma^i=[a^i,b^i]\subset\mathbb{R}$, for some $a^i\le b^i$, $i=1,2$, and we have

$$\bar{u}_1=\mathbf{P}_{\Gamma^1}\Big\{-\frac{\vartheta_1}{1+2P_1+2P_1^2}\Big\}=a^1\vee\Big(-\frac{\vartheta_1}{1+2P_1+2P_1^2}\Big)\wedge b^1,$$

$$\bar{u}_2=\mathbf{P}_{\Gamma^2}\Big\{-\frac{\vartheta_2}{1+2P_2+2P_2^2}\Big\}=a^2\vee\Big(-\frac{\vartheta_2}{1+2P_2+2P_2^2}\Big)\wedge b^2.$$

If $a^i=0$ and $b^i\to+\infty$, then $\Gamma^i=\mathbb{R}^+$. Under this case, $-\frac{\vartheta_i}{1+2P_i+2P_i^2}$ only take its positive part and

$$\begin{aligned}\bar{u}^1=&\mathbf{P}_{\Gamma^1}\Big\{-\frac{\vartheta_1}{1+2P_1+2P_1^2}\Big\}=\Big\{-\frac{\vartheta_1}{1+2P_1+2P_1^2}\Big\}^+\\&=\begin{cases}-\dfrac{\vartheta_1}{1+2P_1+2P_1^2}, & \text{if } -\dfrac{\vartheta_1}{1+2P_1+2P_1^2}>0,\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

$$\begin{aligned}\bar{u}^2=&\mathbf{P}_{\Gamma^2}\Big\{-\frac{\vartheta_2}{1+2P_2+2P_2^2}\Big\}=\Big\{-\frac{\vartheta_2}{1+2P_2+2P_2^2}\Big\}^+\\&=\begin{cases}-\dfrac{\vartheta_2}{1+2P_2+2P_2^2}, & \text{if } -\dfrac{\vartheta_2}{1+2P_2+2P_2^2}>0,\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

In portfolio selection, letting $\Gamma=\mathbb{R}^+$ usually presents the constraint for short-selling prohibition (see [97, 129]).

3.5.2 Example 2

The following example shows the singular case and gives out the criteria for the uniformly convexity in Theorem 5.1. Consider the following one-dimensional case and assume the following:

$$\begin{cases}A^i=1,\quad B_1^i=-1,\quad B_2^i=1,\quad b^i=0,\quad \xi_i=0,\quad G_i=1,\quad T=1,\\ C_1^i=0,\quad D_1^i=1,\quad D_2^i=1,\quad \sigma_1^i=0,\quad R_1^i=R_2^i=0,\quad Q_i=0,\\ C_2^i=0,\quad F_1^i=0,\quad F_2^i=0,\quad \sigma_2^i=0,\quad g_i=0,\quad k_i=0.5,\quad i=1,2.\end{cases}\tag{3.38}$$

Then, the state equation can be show as follows:

$$dx_i = [x_1 - u_i + v_i]dt + [u_i + v_i]dW, \quad x_i(0) = 0, \quad i = 1, 2,$$

and the cost functional is

$$J_i(u_1; v_1; u_2; v_2) = \frac{1}{2}\mathbb{E}|x_i(1) - 0.5x_j(1)|^2, \quad i = 1, 2.$$

The corresponding Riccati equation $P_i(\cdot)$ satisfy

$$\dot{P}_i + P_i = 0, \quad P_i(1) = 1, \quad i = 1, 2, \tag{3.39}$$

then (3.39) has a unique solution that

$$P_i(t) = e^{t-1}, \quad t \in [0, 1].$$

By our assumption (3.38), some corresponding coefficients becomes

$$\begin{cases} \widehat{B}_2^i = (B_2^i)^\top P_i = P_i, \quad \widehat{R}_2^i = (D_2^i)^\top P_i D_2^i = P_i, \quad \widehat{A}^i = 0, \quad \widehat{C}_1^i = -1, \\ \Phi_i = (D_2^i)^\top P_i D_1^i = P_i, \quad \Psi_i = (B_1^i)^\top P_i - \Phi_i^\top (\widehat{R}_2^i)^{-1} \widehat{B}_2^i = -2P_i, \\ \widehat{B}_3^i = -B_2^i (\widehat{R}_2^i)^{-1} (B_2^i)^\top = -P_i^{-1}, \quad \widehat{D}_2^i = -B_2^i (\widehat{R}_2^i)^{-1} (D_2^i)^\top = -P_i^{-1}, \\ \widehat{B}_1^i = B_1^i - B_2^i (\widehat{R}_2^i)^{-1} \Phi_i = -2, \quad \widehat{D}_3^i = -D_2^i (\widehat{R}_2^i)^{-1} (D_2^i)^\top = -P_i^{-1}, \\ \widehat{D}_1^i = D_1^i - D_2^i (\widehat{R}_2^i)^{-1} \Phi_i = 0, \quad i = 1, 2. \end{cases}$$

We consider the case that $i = 1$ (it is similar when $i = 2$), and the state equation (3.20) is

$$\begin{cases} dX = [\widehat{B}_3 \varphi + \widehat{D}_2 \theta_1 + \acute{B}_1 \tilde{u}_1]dt, \\ \qquad + [\widehat{C}_1 X + \widehat{D}_2^\top \varphi + \widehat{D}_3 \theta_1]dW, \quad X(0) = (0^\top \ 0^\top)^\top, \\ d\varphi = -[\widehat{C}_1^\top \theta_1 + \acute{\Psi}_1^\top \tilde{u}_1]dt + \theta_1 dW, \quad \varphi(T) = -GX(T), \end{cases} \tag{3.40}$$

where $\tilde{u}_1 = (u_1^\top, 0^\top)^\top$ and

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix},\ u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},\ \theta_1 = \begin{pmatrix} \theta_1^1 \\ \theta_1^2 \end{pmatrix},$$

$$\widehat{B}_3 = \widehat{D}_2 = \widehat{D}_3 = \begin{pmatrix} -P_1^{-1} & 0 \\ 0 & -P_2^{-1} \end{pmatrix} = -e^{1-t} I,\ \widehat{C}_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I,$$

$$G = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix},\ \acute{B}_1 = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix},\ \acute{\Psi}_1 = \begin{pmatrix} -2P_1 & 0 \\ 0 & 0 \end{pmatrix} = -e^{t-1} \acute{B}_1.$$

The cost functional is

$$J_1(u; \bar{v}(u)) = \frac{1}{2}\, \mathbb{E} \left\langle \begin{pmatrix} 1 \\ -0.5 \end{pmatrix} \begin{pmatrix} 1 \\ -0.5 \end{pmatrix}^\top X(T), X(T) \right\rangle. \tag{3.41}$$

Meanwhile, considering the Riccati equation

$$-d\Lambda = H(\widehat{B}_3; \widehat{C}_1; \widehat{D}_2; \widehat{D}_3) dt, \quad \Lambda(1) = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix},$$

with

$$\begin{aligned} H(\widehat{B}_3; \widehat{C}_1; \widehat{D}_2; \widehat{D}_3) :=& \Lambda \widehat{B}_3 \Lambda + \widehat{C}_1^\top \Lambda \widehat{C}_1 + \Lambda \widehat{D}_2 (I - \Lambda \widehat{D}_3)^{-1} \Lambda \widehat{C}_1 \\ &+ \widehat{C}_1^\top (I - \Lambda \widehat{D}_3)^{-1} \Lambda \widehat{D}_2^\top \Lambda + \Lambda \widehat{D}_2 \widehat{D}_2^\top \Lambda, \end{aligned} \tag{3.42}$$

if we let

$$\Lambda = \begin{pmatrix} \Lambda^1 & \Lambda^2 \\ \Lambda^2 & \Lambda^3 \end{pmatrix}, \quad \Lambda' = e^{t-1} + \Lambda^2,$$

then

$$\begin{aligned} (I - \Lambda \widehat{D}_3)^{-1} =& (I + e^{1-t} \Lambda)^{-1} = e^{t-1} \begin{pmatrix} \Lambda' & \Lambda^1 \\ \Lambda^3 & \Lambda' \end{pmatrix}^{-1} \\ =& \frac{e^{t-1}}{(\Lambda')^2 - \Lambda^1 \Lambda^3} \begin{pmatrix} \Lambda' & \Lambda^1 \\ \Lambda^3 & \Lambda' \end{pmatrix}. \end{aligned}$$

Thus,

$$
\begin{aligned}
(I-\Lambda\widehat{D}_3)^{-1}\Lambda =& \frac{e^{t-1}}{(\Lambda')^2-\Lambda^1\Lambda^3}\begin{pmatrix} \Lambda' & \Lambda^1 \\ \Lambda^3 & \Lambda' \end{pmatrix}\begin{pmatrix} \Lambda^1 & \Lambda^2 \\ \Lambda^2 & \Lambda^3 \end{pmatrix} \\
=& \frac{e^{t-1}}{(\Lambda')^2-\Lambda^1\Lambda^3}\begin{pmatrix} e^{t-1}\Lambda^1 & \Lambda'\Lambda^2-\Lambda^1\Lambda^3 \\ \Lambda'\Lambda^2-\Lambda^1\Lambda^3 & e^{t-1}\Lambda^3 \end{pmatrix},
\end{aligned}
$$

and

$$
\begin{aligned}
&\Lambda(I-\Lambda\widehat{D}_3)^{-1}\Lambda = \frac{e^{t-1}}{(\Lambda')^2-\Lambda^1\Lambda^3}\Lambda\begin{pmatrix} e^{t-1}\Lambda^1 & \Lambda'\Lambda^2-\Lambda^1\Lambda^3 \\ \Lambda'\Lambda^2-\Lambda^1\Lambda^3 & e^{t-1}\Lambda^3 \end{pmatrix} \\
=& \frac{e^{t-1}}{(\Lambda')^2-\Lambda^1\Lambda^3} \\
&\cdot\begin{pmatrix} e^{t-1}(\Lambda^1)^2+\Lambda'(\Lambda^2)^2-\Lambda^1\Lambda^2\Lambda^3 & \Lambda'\Lambda^1\Lambda^2-(\Lambda^1)^2\Lambda^3+e^{t-1}\Lambda^2\Lambda^3 \\ e^{t-1}\Lambda^1\Lambda^2+\Lambda'\Lambda^2\Lambda^3-\Lambda^1(\Lambda^3)^2 & e^{t-1}(\Lambda^3)^2+\Lambda'(\Lambda^2)^2-\Lambda^1\Lambda^2\Lambda^3 \end{pmatrix}.
\end{aligned}
$$

Hence, by (3.42), one can obtain that

$$
\begin{aligned}
&H(\widehat{B}_3;\widehat{C}_1;\widehat{D}_2;\widehat{D}_3) \\
=&(e^{2-2t}-e^{1-t})\Lambda\Lambda-\Lambda+\frac{1}{(\Lambda')^2-\Lambda^1\Lambda^3} \\
&\cdot\begin{pmatrix} e^{t-1}(\Lambda^1)^2+\Lambda'(\Lambda^2)^2-\Lambda^1\Lambda^2\Lambda^3 & \Lambda'\Lambda^1\Lambda^2-(\Lambda^1)^2\Lambda^3+e^{t-1}\Lambda^2\Lambda^3 \\ e^{t-1}\Lambda^1\Lambda^2+\Lambda'\Lambda^2\Lambda^3-\Lambda^1(\Lambda^3)^2 & e^{t-1}(\Lambda^3)^2+\Lambda'(\Lambda^2)^2-\Lambda^1\Lambda^2\Lambda^3 \end{pmatrix}.
\end{aligned}
$$

According to above discussion, we see that in Riccati equation

$$
\begin{cases}
-d\Lambda = H(\widehat{B}_3;\widehat{C}_1;\widehat{D}_2;\widehat{D}_3)dt, \quad \Lambda(1)=\begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix}, \\
(e^{t-1}+\Lambda^2)^2-\Lambda^1\Lambda^3 \neq 0,
\end{cases}
\tag{3.43}
$$

the components Λ^1, Λ^2 and Λ^3 are heavily coupled. Therefore, an explicit solution of (3.43) is difficult to be obtained. We assume that (3.43) is solvable and the FBSDE (3.40) can be decoupled as follows

$$
\begin{cases}
dX = (\check{A}_1X+\check{C}_1\lambda+\check{B}_1\tilde{u}_1)dt+(\check{A}_2X+\check{C}_2\lambda)dW, \quad X(0)=0, \\
d\lambda = -\{\check{A}_4\lambda+(\Lambda\widehat{D}_2+\widehat{C}_1^\top)\iota_1+\check{B}_4\tilde{u}_1\}dt+\iota_1dW, \quad \lambda(1)=0,
\end{cases}
$$

where

$$\begin{cases} \check{A}_1 =\widehat{B}_3\Lambda + \widehat{D}_2(I-\Lambda\widehat{D}_3)^{-1}\Lambda(\widehat{C}_1+\widehat{D}_2^\top\Lambda), \\ \check{A}_2 =\widehat{C}_1 + (\widehat{D}_2)^\top\Lambda + \widehat{D}_3(I-\Lambda\widehat{D}_3)^{-1}\Lambda(\widehat{C}_1+\widehat{D}_2^\top P), \\ \check{A}_4 =\Lambda\widehat{B}_3 + (\Lambda\widehat{D}_2+\widehat{C}_1^\top)(I-\Lambda\widehat{D}_3)^{-1}\Lambda\widehat{D}_2^\top, \\ \check{C}_1 =\widehat{B}_3 + \widehat{D}_2(I-\Lambda\widehat{D}_3)^{-1}\Lambda\widehat{D}_2^\top, \quad \check{B}_1 = \acute{B}_1, \\ \check{C}_2 =(\widehat{D}_2)^\top + \widehat{D}_3(I-\Lambda\widehat{D}_3)^{-1}\Lambda\widehat{D}_2^\top, \quad \check{B}_4 = \Lambda\acute{B}_1 + \acute{\Psi}_1^\top. \end{cases}$$

Furthermore, it can be divided into:

$$\begin{cases} dX_1 =(\check{A}_1X_1 + \check{B}_1\tilde{u}_1)dt + \check{A}_2X_1dW, \quad X_1(0)=0, \\ dX_2 =(\check{A}_1X_2 + \check{C}_1\lambda)dt + (\check{A}_2X_2 + \check{C}_2\lambda)dW, \quad X_2(0)=0, \\ d\lambda = -\{\check{A}_4\lambda + (\Lambda\widehat{D}_2+\widehat{C}_1^\top)\iota_1 + \check{B}_4\tilde{u}_1\}dt + \iota_1 dW, \quad \lambda(1)=0, \end{cases}$$

and the criteria for the uniformly convexity of the cost functional (3.41) is: for some $\delta_4 > 0$ and $\gamma > 0$,

$$\begin{aligned} \delta_4\gamma > \Bigg| \frac{1}{2}L_1 + L_2 + 1 + \Big(9\|P_3\|^2 \max\{\|\check{C}_1\|^2, \|\widehat{D}_2\|^2\} \\ + \frac{9\|\check{A}_2^\top P_3\|^2}{2\|P_3\| + 1/\max\{\|\check{C}_2\|^2, \|\widehat{D}_3\|^2\}} \Big) L_3(L_2+L_1+1) \Bigg|, \end{aligned}$$

where

$$\begin{cases} L_1 = \exp\Big(2(|\check{A}_4| + 2|(\Lambda\widehat{D}_2+\widehat{C}_1^\top)^\top(\Lambda\widehat{D}_2+\widehat{C}_1^\top)| + |\check{B}_4^\top\check{B}_4|)\Big), \\ L_2 =2\Big(|\check{A}_4| + 2|(\Lambda\widehat{D}_2+\widehat{C}_1^\top)^\top(\Lambda\widehat{D}_2+\widehat{C}_1^\top)| + |\check{B}_4^\top\check{B}_4|\Big)L_1, \\ L_3 =(1+\frac{3}{\max\{\|\check{C}_2\|^2, \|\widehat{D}_3\|^2\}})\exp\Big(2|\check{A}_1| + 9\max\{\|\check{C}_1\|^2, \|\widehat{D}_2\|^2\} \\ \qquad + (1+9\max\{\|\check{C}_2\|^2, \|\widehat{D}_3\|^2\})|\check{A}_2|^2\Big), \end{cases}$$

and

$$\dot{P}_3 + P_3\check{A}_1 + \check{A}_1^\top P_3 + \check{A}_2^\top P_3\check{A}_2 = 0, \quad P_3(1) = (1 - \frac{1}{\varepsilon})\begin{pmatrix} 1 \\ -0.5 \end{pmatrix}\begin{pmatrix} 1 \\ -0.5 \end{pmatrix}^\top.$$

3.6 Conclusion

In this chapter, we study a mixed Stackelberg game problem that two players have the same hierarchy and each of them contains an unconstrained control and a constrained control. The unconstrained controls act as followers and the constrained controls act as leaders. We first solve the problem under the case that the control weight coefficients are non-degenerate and obtain the corresponding NE. Then, we discuss the problem under the case in which the control weights are singular. Finally, a minimizing sequence of the solutions is obtained and the weak convergence of the corresponding cost functionals is proved. For future work, one can extend the results of this paper and further investigate the limit solutions when the control weights are singular.

Chapter 4

Robust Linear Quadratic Mean Field Social Control: A Direct Approach

In this chapter, an LQ mean field team (MFT) problem with model uncertainty is solved by using a direct approach. Unlike the person-by-person optimality, which will be introduced in the next chapter, all the agents here are perturbed and the duality procedures are used to tackle the large-scale problem with high-dimensional FBSDEs. After that, the centralized controls explicitly depending on x_i and the state average $x^{(N)}$ are obtained first and then the decentralized controls are designed by mean field heuristics.

4.1 Problem Formulation

We consider a large-population system with N weakly-coupled agents. By the discussion in Section 2.1 of Chapter 2, we define σ-algebra $\mathcal{G}_t^i = \mathcal{F}_t^i \bigvee \sigma\{\xi_i\}$, where $1 \leq i \leq N$, and $\mathcal{G}_t = \mathcal{F}_t \bigvee \sigma\{\xi_i, 1 \leq i \leq N\}$. $\mathbb{G}^i = \{\mathcal{G}_t^i\}_{0 \leq t \leq T}$, where $1 \leq i \leq N$, and $\mathbb{G} = \{\mathcal{G}_t\}_{0 \leq t \leq T}$. The state processes of the agent $\mathcal{A}_i$, $i = 1, 2, \cdots, N$, is modelled by

the following linear SDE on a finite time horizon $[0,T]$:

$$\begin{cases} dx_i =[Ax_i + Bu_i + Fx^{(N)} + f]dt + \sigma dW_i, \\ x_i(0) = \xi_i, \end{cases} \tag{4.1}$$

where $x^{(N)} := \frac{1}{N}\sum_{i=1}^{N} x_i$ is the state average of the agents. A, B, F, σ are deterministic matrix-valued functions of suitable sizes. $f(\cdot) \in L^2_{\mathbb{G}}(0,T;\mathbb{R}^n)$ is an unknown disturbance that agents are imposed by the environment. The coefficients appearing in (4.1) satisfy

$$(\mathbf{A4.1}) \qquad A(\cdot), F(\cdot), \sigma(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n}), \quad B(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times m}).$$

The cost functional of $\mathcal{A}_i$ is given by

$$\begin{aligned} \mathcal{J}_i^F(u;f) =& \frac{1}{2}\mathbb{E}\int_0^T \left\{ |x_i - \Gamma x^{(N)} - \eta|_Q^2 + |u_i|_{R_1}^2 - |f|_{R_2}^2 \right\} dt \\ &+ \frac{1}{2}\mathbb{E}|x_i(T) - \hat{\Gamma} x^{(N)}(T) - \hat{\eta}|_G^2, \end{aligned} \tag{4.2}$$

where $u = \{u_1, \cdots, u_N\}$. Q, R_1, R_2 and G are weight matrices and the coefficients appearing in (4.2) satisfy

$$(\mathbf{A4.2}) \begin{cases} Q(\cdot) \in L^\infty(0,T;\mathbb{S}^n), \quad R_1(\cdot), R_2(\cdot) \in L^\infty(0,T;\mathbb{S}^m), \quad \Gamma(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n}), \\ G \in \mathbb{S}^n, \quad \hat{\Gamma} \in \mathbb{R}^{n\times n}, \quad \eta(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n), \quad \hat{\eta} \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n). \end{cases}$$

All the agents in the system work cooperatively to optimize the social cost functional

$$\mathcal{J}_{soc}^F(u;f) = \sum_{i=1}^{N} \mathcal{J}_i^F(u;f). \tag{4.3}$$

The decentralized control set is defined as follows:

$$\mathcal{U}_i^F = \left\{ u_i | u_i(t) \in L^2_{\mathbb{G}^i}(0,T;\mathbb{R}^m), 1 \le i \le N \right\},$$

and the decentralized control set of all agents is defined as $\mathcal{U}^F = \mathcal{U}_1^F \times \mathcal{U}_2^F \times \cdots \times \mathcal{U}_N^F$. For comparison, the centralized control set is given by

$$\mathcal{U}_c^F = \Big\{(u_1, \cdots, u_N) | u_i(t) \in L^2_{\mathbb{G}}(0, T; \mathbb{R}^m), 1 \leq i \leq N\Big\}.$$

According to the minimax control problem, we need to consider the possible of worst case scenario. Thus, the social cost under the worst-case disturbance as

$$\mathcal{J}_{soc}^{wo}(u) = \sup_{f \in \mathcal{U}_c^F} \mathcal{J}_{soc}^F(u; f). \tag{4.4}$$

For further analysis, we introduce the following assumptions.

(**A4.3**) $\{x_i(0)\}$ are independent with the same expectation. $\mathbb{E}x_i(0) = \hat{\xi}$, $1 \leq i \leq N$. There exists a constant c_0 such that $\sup_{1 \leq i \leq N} \mathbb{E}|x_i(0)|^2 \leq c_0$, where c_0 is independent of N. Furthermore, $\{x_i(0)\}$ and $W_i(t), i = 1, 2, \cdots, N$ are mutually independent.

(**A4.4**) $R_1(\cdot) > 0$, $R_2(\cdot) > 0$ and $G \geq 0$.

Now, we introduce our robust LQ-MF problem:

Problem 4.1. (**P4.1**) *Seek a set of decentralized control laws* $\bar{u} = \{\bar{u}_1, \cdots, \bar{u}_N\} \in \mathcal{U}^F$ *such that for* $\varepsilon > 0$,

$$\mathcal{J}_{soc}^{wo}(\bar{u}) - \varepsilon \leq \inf_{u \in \mathcal{U}_c^F} \mathcal{J}_{soc}^{wo}(u) \leq \mathcal{J}_{soc}^{wo}(\bar{u}).$$

For the sake of notation simplicity, we will use c to denote a generic constant in following discussion. The value of c may be different at different places and it only depends on the coefficients and initial values.

4.2 The LQ-MF Control Problem for the Disturbance

In this section, we seek the worst-case disturbance f. First, we fix $u_i = \check{u}_i \in \mathcal{U}_c^F, i = 1, \cdots, N$ and consider the optimal control problem for the disturbance:

$$(\mathbf{P4.2})\ \text{maximize}_{f \in \mathcal{U}_c^F} \mathcal{J}_{soc}^F(\check{u}; f).$$

Then, (**P4.2**) can be rewritten as an equivalent problem

$$(\mathbf{P4.2a})\ \text{minimize}_{f\in\mathcal{U}_c^F}\check{\mathcal{J}}_{soc}^F(f),$$

where

$$\begin{aligned}\check{\mathcal{J}}_{soc}^F(f) =&\frac{1}{2}\sum_{i=1}^{N}\mathbb{E}\int_0^T\Big\{-|x_i-\Gamma x^{(N)}-\eta|_Q^2+|f|_{R_2}^2\Big\}dt\\ &-\frac{1}{2}\sum_{i=1}^{N}\mathbb{E}|x_i(T)-\hat{\Gamma}x^{(N)}(T)-\hat{\eta}|_G^2.\end{aligned}\tag{4.5}$$

Here x_i are the solution to corresponding $\check{u}_i$, $i=1,2,\cdots,N$. To obtain the worst disturbance, we need to discuss the convexity of (4.5).

Let $\mathbf{x}$=$(x_1^\top,\cdots,x_N^\top)^\top$, $\mathbf{u}$=$(u_1^\top,\cdots,u_N^\top)^\top$, $\mathbf{W}$=$(W_1^\top,\cdots,W_N^\top)^\top$, $\mathbf{A}$=$diag(A,\cdots,A)$, $\mathbf{B}$=$diag(B,\cdots,B)$ and $\check{\sigma}=diag(\sigma,\cdots,\sigma)$. Then our state equation can be rewritten as

$$d\mathbf{x}=(\check{\mathbf{A}}\mathbf{x}+\mathbf{B}\mathbf{u}+\mathbf{1}\otimes f)dt+\check{\sigma}d\mathbf{W}(t),$$

where $\check{\mathbf{A}}=\mathbf{A}+\frac{1}{N}(\mathbf{1}\mathbf{1}^\top\otimes F)$, $\mathbf{1}$=$(1,\cdots,1)^\top$. Correspondingly, (**P4.2a**) can be rewritten as

$$\min_{f\in\mathcal{U}_c^F}\Big\{\frac{1}{2}\mathbb{E}\int_0^T(-\mathbf{x}^\top\mathbf{Q}\mathbf{x}+2\check{\eta}\mathbf{x}+Nf^\top R_2f)dt-\frac{1}{2}\mathbb{E}[\mathbf{x}^\top(T)\mathbf{G}\mathbf{x}(T)+2\grave{\eta}\mathbf{x}(T)]\Big\},$$

where $\mathbf{Q}$=$diag(Q,\cdots,Q)-\frac{1}{N}\mathbf{1}\mathbf{1}^\top\otimes Q_\Gamma$, $\mathbf{G}$=$diag(G,\cdots,G)-\frac{1}{N}\mathbf{1}\mathbf{1}^\top\otimes G_{\hat{\Gamma}}$, $\check{\eta}=\mathbf{1}\otimes\eta_\Gamma$ and $\grave{\eta}=\mathbf{1}\otimes\hat{\eta}_{\hat{\Gamma}}$, $Q_\Gamma\triangleq\Gamma^\top Q+Q\Gamma-\Gamma^\top Q\Gamma$, $G_{\hat{\Gamma}}\triangleq\hat{\Gamma}^\top G+G\hat{\Gamma}-\hat{\Gamma}^\top G\hat{\Gamma}$, $\eta_\Gamma=Q\eta-\Gamma^\top Q\eta$ and $\hat{\eta}_{\hat{\Gamma}}=G\hat{\eta}-\hat{\Gamma}^\top G\hat{\eta}$.

For our further analysis, we have the following assumption:

(**A4.5**) The map $f\mapsto\check{\mathcal{J}}_{soc}^F(f)$ is uniformly convex.

Next, we give a necessary and sufficient condition which is useful in future discussion.

Proposition 4.1. *The following statements are equivalent: (i)* (**A4.5**) *holds true. (ii) The following equation*

$$\dot{\boldsymbol{P}} + \check{\boldsymbol{A}}^\top \boldsymbol{P} + \boldsymbol{P}\check{\boldsymbol{A}} - \boldsymbol{P}(\boldsymbol{1} \otimes I)(NR_2)^{-1}(\boldsymbol{1}^\top \otimes I)\boldsymbol{P} - \boldsymbol{Q} = \boldsymbol{0}, \quad \boldsymbol{P}(T) = -\boldsymbol{G},$$

admits a solution in $C^1(0,T;\mathbb{S}^{nN})$. *(iii) The equation*

$$\begin{cases} \dot{P} + P(A+F) + (A+F)^\top P - PR_2^{-1}P - (Q - Q_\Gamma) = 0, \\ P(T) = -(G - G_{\hat{\Gamma}}), \end{cases} \tag{4.6}$$

admits a solution in $C^1(0,T;\mathbb{S}^n)$.

(iv) $\det\left\{(0,I)e^{\mathbb{A}t}\left(\begin{smallmatrix}0\\ I\end{smallmatrix}\right)\right\} > 0$, $\forall\, t \in [0,T]$, *holds, where*

$$\mathbb{A} = \begin{pmatrix} A + F + R_2^{-1}G & -R_2^{-1} \\ \breve{Q} & -(A + F + R_2^{-1}G)^\top \end{pmatrix}, \tag{4.7}$$

and $\breve{Q} = GR_2^{-1}G + (I-\Gamma)^\top Q(I-\Gamma) + (A+F)^\top G + G(A+F)$.

Proof (i) $\Longleftrightarrow$ (ii) is proved in [164, Theorem 4.5]. By [164, Theorem 4.5], we obtain (i) $\Longleftrightarrow$ (iii). Moreover, we construct an auxiliary control problem

$$\begin{cases} dy = ((A+F)y + g)dt, \quad y(0) = 0, \\ \check{\mathcal{J}}'^F_{soc}(g) = \displaystyle\sum_{i=1}^N \mathbb{E}\Big\{ \int_0^T (-y^\top \hat{Q} y + g^\top R_2 g)dt - [y^\top(T)\hat{G}y(T)] \Big\}, \end{cases}$$

where $\hat{Q} = (I-\Gamma)^\top Q(I-\Gamma)$ and $\hat{G} = (I-\hat{\Gamma})^\top G(I-\hat{\Gamma})$. Let p be the adjoint equation of state y and

$$dp = -[(A+F)^\top p - \hat{Q}y]dt, \quad p(T) = -\hat{G}y.$$

By Itô formula to $\langle p, y\rangle$, we have

$$\sum_{i=1}^N \mathbb{E}(\langle p(T), y(T)\rangle - \mathbb{E}\langle p(0), y(0)\rangle) = \sum_{i=1}^N \mathbb{E}\int_0^T \langle -\hat{Q}y, y\rangle + \langle p, g\rangle dt.$$

Thus, $\check{\mathcal{J}}_{soc}'^{F}(g) = 0$ is equivalent to $g = -R_2^{-1}p$. Considering system

$$\begin{cases} dy = [(A+F)y - R_2^{-1}p]dt, \quad y(0) = 0, \\ dp = -[(A+F)^\top p - \hat{Q}y]dt, \quad p(T) = -\hat{G}y, \end{cases} \tag{4.8}$$

and letting $p = Py + \kappa$, one can obtain that

$$dp = \dot{P}ydt + P((A+F)y - R_2^{-1}p)dt + d\kappa = -[(A+F)^\top p - \hat{Q}y]dt.$$

Hence, P and κ should be the solution to

$$\begin{cases} \dot{P} + P(A+F) + (A+F)^\top P - PR_2^{-1}P - \hat{Q} = 0, \quad P(T) = -\hat{G}, \\ \dot{\kappa} + [(A+F)^\top - PR_2^{-1}]\kappa = 0, \quad \kappa(T) = 0. \end{cases}$$

For (iv) $\Longrightarrow$ (iii) is proved in [133, Theorem 4.3]. On the other hand, we suppose (iii) holds. By Proposition 5.5 and Theorem 6.1 of [191, Chapter 6], linear forward-backward ordinary differential equation (4.8) is solvable. Set $\tilde{p} = p + \hat{G}y$, then (4.8) can be rewritten as

$$\begin{cases} dy = [(A+F+R_2^{-1}\hat{G})y - R_2^{-1}\tilde{p}]dt, \\ d\tilde{p} = [(\hat{Q} + (A+F)^\top\hat{G} + \hat{G}(A+F) + GR_2^{-1}G)y - (A+F+R_2^{-1}\hat{G})^\top\tilde{p}]dt, \\ y(0) = 0, \quad \tilde{p}(T) = 0, \end{cases}$$

which implies

$$\begin{cases} d\begin{pmatrix} y \\ \tilde{p} \end{pmatrix} = \left\{\mathbb{A}\begin{pmatrix} y \\ \tilde{p} \end{pmatrix} + \mathbb{C}\beta\right\}dt + \left\{\mathbb{A}_1\begin{pmatrix} y \\ \tilde{p} \end{pmatrix} + \mathbb{C}_1\beta\right\}dW(t), \\ y(0) = 0, \quad \tilde{p}(T) = 0, \end{cases}$$

where $\mathbb{A}$ satisfies (4.7),

$$\mathbb{C} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbb{A}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbb{C}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By [133, Theorem 3.7], it follows that (iii) $\Longrightarrow$ (iv). The proposition follows. $\square$

Example 4.1. *Consider Proposition 4.1 with parameters* $A = B = R_2 = 1$, $F = -2$, $\Gamma = 0.5$, $Q = 4$, $G = 0$, $T = 1$. *Then, by* (4.6) *we have*

$$P(t) = -\frac{1}{t-2} - 1, \quad t \in [0,1]. \tag{4.9}$$

$P(t)$ *is well defined on* $[0,1]$. *And by the local Lipschitz continuity property of* (4.6), (4.9) *is unique. Furthermore, one can obtain that*

$$\mathbb{A} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \quad e^{\mathbb{A}t} = \begin{pmatrix} 1-t & -t \\ t & t+1 \end{pmatrix},$$

and

$$\det\left\{(0,1)e^{\mathbb{A}t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} = t+1 > 0, \quad \forall\ t \in [0,1],$$

which implies $(iii) \Longleftrightarrow (iv)$.

According to above discussion, we have following theorem.

Theorem 4.1. *Suppose that* (**A4.1**)-(**A4.4**) *hold. Then* (**P4.2a**) *has a unique minimizer if and only if* (**A4.5**) *hold and the following FBSDE admits a unique solution,*

$$\begin{cases} d\check{x}_i = (A\check{x}_i + B\check{u}_i + F\check{x}^{(N)} + \check{f})dt + \sigma dW_i, \\ \\ d\check{p}_i = -[A^\top \check{p}_i + F^\top \check{p}^{(N)} - (Q - Q_\Gamma)\check{x}^{(N)} + \eta_\Gamma]dt + \displaystyle\sum_{j=1}^{N} \beta_i^j dW_j, \\ \\ \check{x}_i(0) = \xi_i, \quad \check{p}_i(T) = -G\check{x}_i(T) + G_{\hat{\Gamma}}\check{x}^{(N)}(T) + \hat{\eta}_{\hat{\Gamma}}, \end{cases} \tag{4.10}$$

where $p^{(N)} = \frac{1}{N}\sum_{i=1}^{N} p_i$ *and* $\check{f} = -R_2^{-1}p^{(N)}$.

Proof By the variational analysis in [176, Theorem 3.1], the theorem follows. □

By taking average of (4.10) and letting $u^{(N)} = \frac{1}{N}\sum_{i=1}^{N} u_i$, we have following

equations:

$$\begin{cases} d\check{x}^{(N)} = ((A+F)\check{x}^{(N)} + B\check{u}^{(N)} - R_2^{-1}\check{p}^{(N)})dt + \dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\sigma dW_i, \\ d\check{p}^{(N)} = -[(A+F)^\top\check{p}^{(N)} - (Q-Q_\Gamma)\check{x}^{(N)} + \eta_\Gamma]dt + \dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\sum_{j=1}^{N}\beta_i^j dW_j, \\ \check{x}^{(N)}(0) = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\xi_i, \quad \check{p}^{(N)}(T) = -(G-G_{\hat{\Gamma}})\check{x}^{(N)}(T) + \hat{\eta}_{\hat{\Gamma}}. \end{cases} \tag{4.11}$$

Now we discuss the feedback form of disturbance in (**P4.2a**). We make the ansatz $\check{p}^{(N)}(t) = \bar{P}(t)\check{x}^{(N)}(t) + \check{s}(t)$, $t \in [0,T]$, where $\bar{P}(\cdot) \in C^1(0,T;\mathbb{S}^n)$ is a matrix-value function and $\check{s}(\cdot) \in C^1(0,T;\mathbb{R}^n)$. Combining this equation and (4.11), one can obtain that

$$\begin{aligned} d\check{p}^{(N)} =& \dot{\bar{P}}\check{x}^{(N)}dt + \bar{P}((A+F-R_2^{-1}P)\check{x}^{(N)} + B\check{u}^{(N)} - R_2^{-1}\check{s})dt + \frac{\bar{P}}{N}\sum_{i=1}^{N}\sigma dW_i + d\check{s} \\ =& -[(A+F)^\top(\bar{P}\check{x}^{(N)} + \check{s}) - (Q-Q_\Gamma)\check{x}^{(N)} + \eta_\Gamma]dt + \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\beta_i^j dW_j. \end{aligned} \tag{4.12}$$

Hence, $\bar{P}(\cdot)$ is a solution of

$$\begin{cases} \dot{\bar{P}} + \bar{P}(A+F) + (A+F)^\top\bar{P} - \bar{P}R_2^{-1}\bar{P} - (Q-Q_\Gamma) = 0, \\ \bar{P}(T) = -(G-G_{\hat{\Gamma}}), \end{cases} \tag{4.13}$$

and $\check{s}(\cdot)$ is the solution of the following BSDE:

$$\begin{cases} d\check{s} + [(A+\bar{F})^\top\check{s} + \bar{P}B\check{u}^{(N)} + \eta_\Gamma]dt + \dfrac{1}{N}\displaystyle\sum_{j=1}^{N}\sum_{i=1}^{N}(\frac{\bar{P}\sigma}{N} - \beta_i^j)dW_j = 0, \\ \check{s}(T) = \hat{\eta}_{\hat{\Gamma}}, \end{cases} \tag{4.14}$$

where $\bar{F} = F - R_2^{-1}\bar{P}$. Thus, $\check{f} = -R_2^{-1}(\bar{P}\check{x}^{(N)} + \check{s})$. We can easily see that P in (4.6) is equal to $\bar{P}$ here. In what follows, $\bar{P}$ will be substituted by P.

4.3 Distributed Strategy Design

After applying the worst disturbance $\check{f}$, one can obtain the following optimal control problem.

(**P4.3**): Minimize $\mathcal{J}_{soc}^{wo}(u; \check{f}(u))$ over $\{u = (u_1, \cdots, u_N) \in \mathcal{U}_c^F\}$, where

$$\begin{cases} dx_i = [Ax_i + Bu_i + \bar{F}x^{(N)} - R_2^{-1}s]dt + \sigma dW_i, \\ ds = -[(A+\bar{F})^\top s + PBu^{(N)} + \eta_\Gamma]dt + \dfrac{1}{N}\displaystyle\sum_{j=1}^{N}\sum_{i=1}^{N}(\beta_i^j - \dfrac{P\sigma}{N})dW_j, \\ x_i(0) = \xi_i, \quad s(T) = \hat{\eta}_{\hat{\Gamma}}, \end{cases} \tag{4.15}$$

and

$$\begin{aligned} \mathcal{J}_{soc}^{wo}(u) =& \frac{1}{2}\sum_{i=1}^{N}\mathbb{E}\int_0^T \left\{|x_i - \Gamma x^{(N)} - \eta|_{Q(t)}^2 + |u_i|_{R_1}^2 - |P(t)x^{(N)} + s|_{R_2^{-1}}^2\right\}dt \\ &+ \frac{1}{2}\sum_{i=1}^{N}\mathbb{E}|x_i(T) - \hat{\Gamma}x^{(N)}(T) - \hat{\eta}|_G^2. \end{aligned} \tag{4.16}$$

To solve (**P4.3**), we first give out a proposition.

Proposition 4.2. *Suppose that* (**A4.1**)-(**A4.5**) *hold. If* (**P4.3**) *is uniformly convex in u, then* (**P4.3**) *has a set of optimal controls and the following FBSDE admits a*

set of solutions

$$\begin{cases} dx_i = [Ax_i + Bu_i + \bar{F}x^{(N)} - R_2^{-1}s]dt + \sigma dW_i, \quad x_i(0) = \xi_i, \\ ds = -[(A+\bar{F})^\top s + PBu^{(N)} + \eta_\Gamma]dt + \dfrac{1}{N}\displaystyle\sum_{j=1}^N\sum_{i=1}^N(\beta_i^j - \dfrac{P\sigma}{N})dW_j, \\ dk_i = [-A^\top k_i - \bar{F}^\top k^{(N)} - Qx_i + Q_\Gamma x^{(N)} + \eta_\Gamma + PR_2^{-1}(Px^{(N)} + s)]dt \\ \qquad + \zeta_i^i dW_i + \displaystyle\sum_{j\neq i}\zeta_i^j dW_j, \quad k_i(T) = G\bar{x}_i(T) - G_{\hat{\Gamma}}\bar{x}^{(N)}(T) - \hat{\eta}_{\hat{\Gamma}}, \\ dl = [(A+\bar{F})l + R_2^{-1}(k^{(N)} + Px^{(N)} + s)]dt, \quad l(0) = 0, \quad s(T) = \hat{\eta}_{\hat{\Gamma}}, \end{cases} \tag{4.17}$$

where $k^{(N)} = \frac{1}{N}\sum_{i=1}^N k_i$ *and* $R_1u_i + B^Tk_i - B^TPl = 0$, $i = 1, 2, \cdots, N$.

Proof Let $u^* = \{u_1^*, u_2^*, \cdots, u_N^*\} \in \mathcal{U}_c^F$ be the unique centralized optimal control of the N agents and $x^* = \{x_1^*, x_2^*, \cdots, x_N^*\}$ be their unique corresponding states. We perturb u^* and denote $\delta u = u - u^*$, $\delta u^{(N)} = u^{(N)} - (u^*)^{(N)}$, $\delta x_i = x_i - x_i^*$, $\delta x^{(N)} = \frac{1}{N}\sum_{i=1}^N \delta x_i = \frac{1}{N}\sum_{i=1}^N(x_i - x_i^*) = \frac{1}{N}\sum_{i=1}^N x_i - \frac{1}{N}\sum_{i=1}^N x_i^* = x^{(N)} - (x^*)^{(N)}$, $\delta s = s - s^*$. Then we have

$$\begin{cases} d\delta x_i = [A\delta x_i + B\delta u_i + \bar{F}\delta x^{(N)} - R_2^{-1}\delta s]dt, \quad \delta x_i(0) = 0, \\ d\delta s = -[(A+\bar{F})^\top \delta s + PB\delta u^{(N)}]dt + \dfrac{1}{N}\displaystyle\sum_{j=1}^N\sum_{i=1}^N \delta\beta_i^j dW_j, \quad \delta s(T) = 0. \end{cases}$$

The Fréchet differential of the corresponding social cost functional is

$$\delta\mathcal{J}_{soc}^{wo}(\delta u) = \mathcal{J}_{soc}^{wo}(u) - \mathcal{J}_{soc}^{wo}(u^*) + o(|\delta u|_{L^2}^2) = \Lambda_1 + \frac{1}{2}\Lambda_2,$$

where

$$\begin{aligned} \Lambda_1 := &\sum_{i=1}^N \mathbb{E}\int_0^T \langle Q(x_i^* - \Gamma(x^*)^{(N)} - \eta), \delta x_i - \Gamma\delta x^{(N)}\rangle + \langle R_1u_i^*, \delta u_i\rangle \\ &- \langle R_2^{-1}(P(x^*)^{(N)} + \bar{s}), (P\delta x^{(N)} + \delta s)\rangle dt \end{aligned}$$

$$+\sum_{i=1}^{N}\mathbb{E}\langle G(x_i^*(T)-\hat{\Gamma}(x^*)^{(N)}(T)-\hat{\eta}),\delta x_i(T)-\hat{\Gamma}\delta x^{(N)}(T)\rangle,$$

$$\Lambda_2:=\sum_{i=1}^{N}\mathbb{E}\int_0^T\left\{|\delta x_i-\Gamma\delta x^{(N)}|_Q^2+|\delta u_i|_{R_1}^2-|P\delta x^{(N)}+\delta s|_{R_2^{-1}}^2\right\}dt$$

$$+\sum_{i=1}^{N}\mathbb{E}|\delta x_i(T)-\hat{\Gamma}\delta x^{(N)}(T)|_G^2.$$

Note that

$$\begin{aligned}&\sum_{i=1}^{N}\mathbb{E}\int_0^T\langle -Q(x_i^*-\Gamma(x^*)^{(N)}-\eta),\delta\Gamma x^{(N)}\rangle dt\\&=\sum_{i=1}^{N}\mathbb{E}\int_0^T\langle\Gamma^\top Q((I-\Gamma)(x^*)^{(N)}-\eta),\delta x_i\rangle dt.\end{aligned}\tag{4.18}$$

and

$$\begin{aligned}&\sum_{i=1}^{N}\mathbb{E}\int_0^T-\langle R_2^{-1}(P(x^*)^{(N)}+\bar{s}),(P\delta x^{(N)}+\delta s)\rangle dt\\&=\sum_{i=1}^{N}\mathbb{E}\int_0^T-\langle P^\top R_2^{-1}(P(x^*)^{(N)}+\bar{s}),\delta x_i\rangle-\langle R_2^{-1}(P(x^*)^{(N)}+\bar{s}),\delta s\rangle dt.\end{aligned}\tag{4.19}$$

Let

$$\begin{cases}dk_i=\alpha_i dt+\zeta_i^i dW_i+\sum_{j\neq i}\zeta_i^j dW_j,\quad k_i(T)=Gx_i^*(T)-G_{\hat{\Gamma}}(x^*)^{(N)}(T)-\hat{\eta}_{\hat{\Gamma}},\\ dl=\gamma dt,+\nu_i dW_i+\sum_{j\neq i}\nu_j dW_j,\quad l(0)=0.\end{cases}\tag{4.20}$$

where

$$\begin{cases}\alpha_i=-\left[A^\top k_i+\bar{F}^\top k^{(N)}+Qx_i^*-Q_\Gamma(x^*)^{(N)}-\eta_\Gamma-PR_2^{-1}(P(x^*)^{(N)}+\bar{s})\right],\\ \gamma=(A+\bar{F})l+R_2^{-1}(k^{(N)}+P(x^*)^{(N)}+\bar{s}),\quad \sum_{j=1}^{N}\nu_j=0.\end{cases}\tag{4.21}$$

By Itô formula,

$$
\begin{aligned}
&\sum_{i=1}^{N}\mathbb{E}\langle k_i(T),\delta x_i(T)\rangle-\sum_{i=1}^{N}\mathbb{E}\langle k_i(0),\delta x_i(0)\rangle\\
=&\sum_{i=1}^{N}\mathbb{E}\int_0^T\langle\alpha_i+A^\top k_i+\bar{F}^\top k^{(N)},\delta x_i\rangle+\langle B^\top k_i,\delta u_i\rangle-\langle R_2^{-1}k^{(N)},\delta s\rangle dt,
\end{aligned}
\tag{4.22}
$$

and

$$
\begin{aligned}
&\sum_{i=1}^{N}\mathbb{E}\langle l(T),\delta s(T)\rangle-\mathbb{E}\langle l(0),\delta s(0)\rangle=0\\
=&\sum_{i=1}^{N}\mathbb{E}\int_0^T\langle\gamma-(A+\bar{F})l,\delta s\rangle-\langle B^\top Pl,\delta u_i\rangle+\frac{1}{N}\langle\sum_{j=1}^{N}\nu_j,\sum_{i=1}^{N}\delta\beta_i^j\rangle dt.
\end{aligned}
\tag{4.23}
$$

Consequently by (4.18), (4.19), (4.21), (4.22) and (4.23), we have

$$
\Lambda_1=\sum_{i=1}^{N}\mathbb{E}\int_0^T\langle R_1u_i^*+B^\top k_i-B^\top Pl,\delta u_i\rangle dt.
$$

Thus, $\Lambda_1=0$ is equivalent to $R_1u_i^*+B^\top k_i-B^\top Pl=0$, $i=1,2,\cdots,N$. Then, considering (4.15), (4.20) and (4.21), we have (4.17). The proposition follows. □

It follows from (4.17) that

$$
\begin{cases}
dx^{(N)}=[(A+\bar{F})x^{(N)}+Bu^{(N)}-R_2^{-1}s]dt+\dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\sigma dW_i,\quad x^{(N)}(0)=\dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\xi_i,\\
dk^{(N)}=[-(A+\bar{F})^\top k^{(N)}-(Q-Q_\Gamma)\bar{x}^{(N)}+\eta_\Gamma+PR_2^{-1}(Px^{(N)}+s)]dt\\
\qquad+\dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\sum_{j=1}^{N}\zeta_i^jdW_j,\quad k^{(N)}(T)=(G-G_{\hat{\Gamma}})x^{(N)}(T)-\hat{\eta}_{\hat{\Gamma}}.
\end{cases}
\tag{4.24}
$$

To discuss the state feedback form of the optimal controls we solved in (**P4.3**), we consider the following nonhomogeneous relationships:

$$\begin{cases} k_i(t) = K(t)x_i(t) + L(t)x^{(N)}(t) + M(t)l(t) + \varphi(t), \\ s(t) = \bar{M}(t)l(t) + \bar{L}(t)x^{(N)}(t) + \phi(t), \\ k^{(N)}(t) = (K(t) + L(t))x^{(N)}(t) + M(t)l(t) + \varphi(t), \quad t \in [0, T], \end{cases} \tag{4.25}$$

where $K(\cdot)$, $L(\cdot)$, $M(\cdot)$, $\bar{M}(\cdot)$, $\bar{L}(\cdot) \in C^1(0, T; \mathbb{R}^{n\times n})$ and $\varphi(\cdot)$, $\phi(\cdot) \in C^1(0, T; \mathbb{R}^n)$. By (4.25), (4.17) and (4.24), we have

$$\begin{aligned} dk_i =& \dot{K}x_i dt + K[Ax_i + Bu_i + \bar{F}x^{(N)} - R_2^{-1}s]dt + K\sigma dW_i + \dot{L}x^{(N)}dt \\ &+ L[(A + \bar{F})x^{(N)} + Bu^{(N)} - R_2^{-1}s]dt + \frac{L}{N}\sum_{i=1}^{N}\sigma dW_i \\ &+ \dot{M}ldt + M((A + \bar{F})l + R_2^{-1}(k^{(N)} + Px^{(N)} + s))dt + d\varphi \\ =& -[A^\top k_i + \bar{F}^\top k^{(N)} + Q(x_i - \Gamma x^{(N)} - \eta) - \Gamma^\top Q((I - \Gamma)x^{(N)} - \eta) \\ &- PR_2^{-1}(Px^{(N)} + s)]dt + \sum_{j=1}^{N}\zeta_i^j dW_j, \end{aligned} \tag{4.26}$$

$$\begin{aligned} ds =& \dot{\bar{M}}ldt + \bar{M}[(A + \bar{F})l + R_2^{-1}(k^{(N)} + Px^{(N)} + s)]dt \\ &+ \dot{\bar{L}}x^{(N)}dt + \bar{L}((A + \bar{F})x^{(N)} + Bu^{(N)} - R_2^{-1}s)dt + \frac{\bar{L}}{N}\sum_{i=1}^{N}\sigma dW_i + d\phi \\ =& -[(A + \bar{F})^\top(\bar{M}l + \bar{L}x^{(N)} + \phi) + PBu^{(N)} + (I - \Gamma)^\top Q\eta]dt \\ &+ \frac{1}{N}\sum_{j=1}^{N}\sum_{i=1}^{N}(\beta_i^j - \frac{P\sigma}{N})dW_j. \end{aligned} \tag{4.27}$$

Comparing the diffusion terms in (4.26), we have the following results: $(K+\frac{L}{N})\sigma = \zeta_i^i$, $\frac{L}{N}\sigma = \zeta_i^j$, $j \neq i$, $\frac{\bar{L}}{N}\sum_{i=1}^{N}\sigma = \frac{1}{N}\sum_{j=1}^{N}\sum_{i=1}^{N}(\beta_i^j - \frac{P\sigma}{N})$. Hence, combining (4.25) and

the equation $R_1 u_i^* + B^\top k_i - B^\top P l = 0$, we have

$$\begin{cases} u_i^* = -R_1^{-1}\big\{B^\top K x_i + B^\top L x^{(N)} + B^\top \varphi - B^\top (P-M) l\big\}, \\ (u^*)^{(N)} = -R_1^{-1}\big\{B^\top (K+L) x^{(N)} + B^\top \varphi - B^\top (P-M) l\big\}. \end{cases} \tag{4.28}$$

Thus, using the same argument from (4.12) to (4.14), it follows that K is a solution of

$$\dot{K} + KA + A^\top K - (B^\top K)^\top R_1^{-1} B^\top K + Q = 0, \quad K(T) = G, \tag{4.29}$$

and L, M, $\bar{L}$, $\bar{M}$ satisfy

$$\begin{cases} \dot{L} + L(A+\bar{F}) + (A+\bar{F})^\top L + K\bar{F} + \bar{F}^\top K - PR_2^{-1}(P+\bar{L}) + MR_2^{-1}(K \\ \quad + L + P + \bar{L}) - (K+L)R_2^{-1}\bar{L} - (B^\top (K+L))^\top R_1^{-1} B^\top (K+L) \\ \quad + (B^\top K)^\top R_1^{-1} B^\top K - Q_\Gamma = 0, \quad L(T) = -G_{\hat{\Gamma}}, \\ \dot{\bar{L}} + \bar{L}(A+\bar{F}) + (A+\bar{F})^\top \bar{L} - \bar{L}R_2^{-1}\bar{L} + \bar{M}R_2^{-1}(K+L+P+\bar{L}) \\ \quad - (B^\top (P+\bar{L}))^\top R_1^{-1} B^\top (K+L) = 0, \quad \bar{L}(T) = 0, \\ \dot{M} + M(A+\bar{F}) + (A+\bar{F})^\top M - (K+L+P)R_2^{-1}\bar{M} + MR_2^{-1}(M+\bar{M}) \\ \quad + (B^\top (K+L))^\top R_1^{-1} B^\top (P-M) = 0, \quad M(T) = 0, \\ \dot{\bar{M}} + \bar{M}(A+\bar{F}) + (A+\bar{F})^\top \bar{M} + \bar{M}R_2^{-1}(M+\bar{M}) - \bar{L}R_2^{-1}\bar{M} \\ \quad + (B^\top (P+\bar{L}))^\top R_1^{-1} B^\top (P-M) = 0, \quad \bar{M}(T) = 0, \end{cases} \tag{4.30}$$

and φ, ϕ satisfy

$$\begin{cases} d\varphi + (MR_2^{-1} + A + \bar{F})^\top \varphi - (B^\top (K+L))^\top R_1^{-1} B^\top \varphi - (K+L+P \\ \quad - M)R_2^{-1}\phi - \eta_\Gamma = 0, \\ d\phi + ((\bar{M} - \bar{L})R_2^{-1} + A + \bar{F})^\top \phi - PBR_2^{-1}B^\top \varphi + \bar{M}R_2^{-1}\varphi + \eta_\Gamma = 0, \\ \varphi(T) = -\hat{\eta}_{\hat{\Gamma}}, \quad \phi(T) = \hat{\eta}_{\hat{\Gamma}}. \end{cases} \tag{4.31}$$

Remark 4.1. *Equation* (4.30) *are the non-symmetric Riccati equations (or the Riccati-like equations). For more details about their property and solvabibility, readers could refer to [79, 108, 116].*

If $L(\cdot)$, $M(\cdot)$, $\bar{M}(\cdot) \in C^1(0,T;\mathbb{S}^n)$ are the unique solutions in (4.30), we have the following result:

Proposition 4.3. *Suppose that* (**A4.1**)-(**A4.5**) *hold. If* L, M, $\bar{M}$ *in* (4.30) *satisfy* $L(\cdot)$, $M(\cdot)$, $\bar{M}(\cdot) \in C^1(0,T;\mathbb{S}^n)$, *then* $\bar{L} = -M$ *and the original four coupled Riccati-like equations can be simplified to three coupled equations.*

Proof According to equation (4.29), K is symmetric. It follows from taking transpose on M and multiply -1 on both sides in (4.30) that,

$$\begin{aligned}
&-\dot{M}^\top + (A+\bar{F})^\top(-M)^\top + (-M)^\top(A+\bar{F}) + \bar{M}^\top R_2^{-1}(K+L+P-M)^\top \\
&\quad - M^\top R_2^{-1} M^\top - (B^\top(P-M^\top))^\top R_1^{-1} B^\top(K+L) = 0, \quad -M^\top(T) = 0,
\end{aligned}$$

Since L, M, $\bar{M}$ are symmetric, we have $\bar{L} = -M^\top = -M$. Putting this result into system (4.30), it could be simplified as

$$\begin{cases}
\dot{L} + L(A+\bar{F}) + (A+\bar{F})^\top L + K\bar{F} + \bar{F}^\top K - PR_2^{-1}P - MR_2^{-1}M \\
\quad + MR_2^{-1}(K+L+P) + (K+L+P)R_2^{-1}M - (B^\top(K+L))^\top R_1^{-1}B^\top(K \\
\quad + L) + (B^\top K)^\top R_1^{-1} B^\top K - Q_\Gamma = 0, \quad L(T) = -G_{\hat{\Gamma}}, \\
\dot{M} + M(A+\bar{F}) + (A+\bar{F})^\top M - (K+L+P-M)R_2^{-1}\bar{M} + MR_2^{-1}M \\
\quad + (B^\top(K+L))^\top R_1^{-1} B^\top(P-M) = 0, \quad M(T) = 0, \\
\dot{\bar{M}} + \bar{M}(A+\bar{F}) + (A+\bar{F})^\top \bar{M} + \bar{M}R_2^{-1}M + MR_2^{-1}\bar{M} + \bar{M}R_2^{-1}\bar{M} \\
\quad + (B^\top(P-M))^\top R_1^{-1} B^\top(P-M) = 0, \quad \bar{M}(T) = 0.
\end{cases}$$

The proposition follows. □

By a similar argument in [174, Lemma 2.1], if (**A4.1**)-(**A4.5**) hold, there exists a constant $\delta > 0$ such that $R_1(t) > \delta I$ and $R_2(t) > \delta I$, *a.e.*, $t \in [0, T]$. Then, for $Q \geq 0$, (**P4.3**) is uniformly convex. However, when Q is indefinite, we have following result.

Lemma 4.1. (**P4.3**) *has uniform convexity if equations* (4.29)-(4.31) *has a solution, respectively.*

Proof By [176, Proposition 3.1] and [99, Section 3], we first let $\acute{u}_i \in \mathcal{U}_c^F$, $\acute{s}(\cdot) \in C^1(0, T; \mathbb{R}^n)$ and consider the system

$$\begin{cases} dy_i = Ay_i + B\acute{u}_i + Fy^{(N)} - R_2^{-1}(Py^{(N)} + \acute{s})dt, \quad y_i(0) = 0, \\ dy^{(N)} = Ay^{(N)} + B\acute{u}^{(N)} + Fy^{(N)} - R_2^{-1}(Py^{(N)} + \acute{s})dt, \quad y^{(N)}(0) = 0, \\ d(y_i - y^{(N)}) = A(y_i - y^{(N)}) + B(\acute{u}_i - \acute{u}^{(N)})dt, \quad (y_i - y^{(N)})(0) = 0. \end{cases}$$

By (4.13), (4.29) and using Itô formula to $|y_i - y^{(N)}|_K^2$ and $|y_i - y^{(N)}|_P^2$, we have

$$\begin{aligned} &\mathbb{E}|y_i(T) - y^{(N)}(T)|_G^2 = \mathbb{E}|y_i(T) - y^{(N)}(T)|_{K(T)}^2 - \mathbb{E}|y_i(0) - y^{(N)}(0)|_{K(0)}^2 \\ =&\mathbb{E}\int_0^T \langle (K^\top BR_1^{-1}B^\top K - Q)(y_i - y^{(N)}), y_i - y^{(N)} \rangle \\ &+ 2\langle \acute{u}_i - \acute{u}^{(N)}, B^\top K(y_i - y^{(N)}) \rangle, \end{aligned}$$

and

$$\begin{aligned} &-\mathbb{E}|y^{(N)}(T)|_{G-G_{\hat{\Gamma}}}^2 = \mathbb{E}|y^{(N)}(T)|_{P(T)}^2 - \mathbb{E}|y^{(N)}(0)|_{P(0)}^2 \\ =&\mathbb{E}\int_0^T \langle (PR_2^{-1}P + Q - Q_\Gamma)y^{(N)}, y^{(N)} \rangle + 2\langle \acute{u}^{(N)}, B^\top Py^{(N)} \rangle - 2\langle \acute{s}, R_2^{-1}Py^{(N)} \rangle. \end{aligned}$$

By Lemma 2.1 in [174], we know that $\sum_{i=1}^N \mathbb{E}\int_0^T |y_i|^2 dt \leq \frac{c}{N}\sum_{i=1}^N \mathbb{E}\int_0^T |\acute{u}_i|^2 dt$,

$\sum_{i=1}^N \mathbb{E}\int_0^T |\acute{s}|^2 dt \leq \frac{c}{N}\sum_{i=1}^N \mathbb{E}\int_0^T |\acute{u}_i|^2 dt$. By Lemma 2.3 in [164] and Proposition 4.1,

$$\sum_{i=1}^N \mathbb{E}\int_0^T \Big(|y_i - \Gamma y^{(N)}|_Q^2 + |\acute{u}_i|_{R_1}^2 - |Py^{(N)} + \acute{s}|_{R_2^{-1}}^2\Big)dt + \sum_{i=1}^N \mathbb{E}|y_i(T) - \hat{\Gamma} y^{(N)}(T)|_G^2$$

$$= \sum_{i=1}^N \mathbb{E}\int_0^T \Big(|y_i - \Gamma y^{(N)}|_Q^2 + |y^{(N)}|_{Q-Q_\Gamma}^2 + |\acute{u}_i - \acute{u}^{(N)}|_{R_1}^2 + |\acute{u}^{(N)}|_{R_1}^2 - |Py^{(N)}|_{R_2^{-1}}^2$$

$$- 2\langle \acute{s}, R_2^{-1}Py^{(N)}\rangle - |\acute{s}|_{R_2^{-1}}^2\Big)dt + \sum_{i=1}^N \mathbb{E}|y_i(T) - y^{(N)}(T)|_G^2 + |y^{(N)}(T)|_{G-G_{\hat{\Gamma}}}^2$$

$$= \sum_{i=1}^N \mathbb{E}\int_0^T \Big(|\acute{u}_i - \acute{u}^{(N)} + R_1^{-1}B^\top K(y_i - y^{(N)})|_{R_1}^2 + |\acute{u}^{(N)} - R_1^{-1}BPy^{(N)}|_{R_1}^2$$

$$- |Py^{(N)}|_{B^\top R_1^{-1}B + 2R_2^{-1}}^2 - |\acute{s}|_{R_2^{-1}}^2\Big)dt$$

$$\geq \sum_{i=1}^N \mathbb{E}\int_0^T \Big(|\acute{u}_i + R_1^{-1}B^\top Ky_i - R_1^{-1}B(P+K)y^{(N)}|_{R_1}^2 - \frac{c}{N}|\acute{u}_i|^2\Big)dt$$

$$\geq \delta \sum_{i=1}^N \mathbb{E}\int_0^T |\acute{u}_i|^2 dt.$$

The lemma follows. □

For further proofs, we have the following assumption:

(**A4.6**) Assume that (4.29)-(4.30) admit unique solutions.

Then, by above discussion, we have following theorem.

Theorem 4.2. *Suppose that* (**A4.1**)-(**A4.6**) *hold. Then* (**P4.3**) *is uniquely solvable with the optimal control* u_i *in* (4.28).

Proof Since (4.29)-(4.31) has a solution, respectively, the system (4.17) is decoupled and solvable (see the Theorem 3.7 and Theorem 4.3 in [133, Chapter 2]). By Lemma 4.1, (**P4.3**) has uniform convexity and can achieves an optimal control, where $u_i = -R_1^{-1}B^\top\{Kx_i + Lx^{(N)} + \varphi - (P-M)l\}$. □

We use $\hat{x}$, $\hat{l}$ to approximate $x^{(N)}$, l in (4.17) and (4.24), respectively.

$$\begin{cases} d\hat{x} = \Big\{ [A + \bar{F} - BR_1^{-1}B^\top(K+L) - R_2^{-1}\bar{L}]\hat{x} - BR_1^{-1}B^\top\varphi \\ \qquad + [BR_1^{-1}B^\top(P-M) - R_2^{-1}\bar{M}]\hat{l} - R_2^{-1}\phi \Big\} dt, \\ d\hat{l} = [(A + \bar{F} + R_2^{-1}(M+\bar{M}))\hat{l} + R_2^{-1}((K+L+P+\bar{L})\hat{x} + \varphi + \phi)]dt, \\ \hat{x}(0) = \hat{\xi}, \quad \hat{l}(0) = 0, \end{cases} \tag{4.32}$$

where K, L, $\bar{L}$, M, $\bar{M}$, φ and ϕ are determined by (4.29)-(4.31). Then, according to Theorem 4.2, one can obtain the decentralized control law for the agent $\mathcal{A}_i$

$$\bar{u}_i = -R_1^{-1}B^\top[K\bar{x}_i + L\hat{x} + \varphi - (P-M)\hat{l}]. \tag{4.33}$$

Meanwhile, we have the decentralized terms $\bar{k}_i = K\bar{x}_i + L\hat{x} + M\hat{l} + \varphi$, $\bar{s} = \bar{M}\bar{l} + \bar{L}\bar{x}^{(N)} + \phi$, and $\hat{s} = \bar{M}\hat{l} + \bar{L}\hat{x} + \phi$. By applying (4.33), we have the following closed-loop system

$$\begin{cases} d\bar{x}_i = \Big\{ [A - BR_1^{-1}B^\top K]\bar{x}_i + (\bar{F} - R_2^{-1}\bar{L})\bar{x}^{(N)} - BR_1^{-1}B^\top L\hat{x} - BR_1^{-1}B^\top\varphi \\ \qquad + BR_1^{-1}B^\top(P-M)\hat{l} - R_2^{-1}(\bar{M}\bar{l} + \phi) \Big\} dt + \sigma dW_i, \quad \bar{x}_i(0) = \xi_i \\ d\bar{s} = -\Big\{ (A+\bar{F})^\top\bar{s} + (B^\top P)^\top R_1^{-1}B^\top K\bar{x}^{(N)} + (B^\top P)^\top R_1^{-1}B^\top L\hat{x} \\ \qquad + (B^\top P)^\top R_1^{-1}B^\top\varphi - (B^\top P)^\top R_1^{-1}B^\top(P-M)\hat{l} - (I-\Gamma)^\top Q\eta \Big\} dt \\ \qquad + \dfrac{1}{N}\sum_{j=1}^{N}\sum_{i=1}^{N}(\dfrac{P\sigma}{N} - \beta_i^j)dW_j, \quad \bar{s}(T) = \hat{\eta}_{\hat{\Gamma}}, \quad \bar{l}(0) = 0, \\ d\bar{l} = [(A + \bar{F} + R_2^{-1}\bar{M})\bar{l} + R_2^{-1}((K+P+\bar{L})\bar{x}^{(N)} + L\hat{x} + M\hat{l} + \varphi + \phi)]dt. \end{cases} \tag{4.34}$$

Remark 4.2. *We use the direct approach here and first obtain a set of centralized optimal controls, then the decentralized controls are designed. Note that* L, M, $\bar{M}$, $\bar{L}$, φ, ϕ *are not coupled with* $\hat{x}$ *and it is simpler to solve four Riccati-liked equations than solving the consistency condition (CC) system in [174] who contains five highly coupled FBSDEs. Thus, the fixed-point equation system is not necessary here. In addition, if* L, M, $\bar{M}$ *are symmetric, the original four equations can further degenerate to three Riccati-liked equations.*

Remark 4.3. *Note that here the weight* Q *is allowed to be indefinite. If* Q *is negative semi-definite, then* (4.13) *admits a solution necessarily. If* Q *is positive semi-definite, then* (4.29) *also admits a solution necessarily. However, to ensure that both* (4.13) *and* (4.29) *admit solutions, the selection of* Q *should reach a compromise and the magnitude of* Q *cannot be too "positive" or "negative".*

4.4 Asymptotic Optimality

Definition 4.1. *A set of control laws* $\bar{u} = \{\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_N\} \in \mathcal{U}^F$ *has robust asymptotic social optimality if*

$$\left| \frac{1}{N} \mathcal{J}_{soc}^{wo}(\bar{u}) - \frac{1}{N} \inf_{u \in \mathcal{U}_c^F} \mathcal{J}_{soc}^{wo}(u) \right| = O(\frac{1}{\sqrt{N}}),$$

where $\mathcal{U}_c^F$ *is a set of centralized information-based control.*

Before proving asymptotically social optimality, we need to introduce some estimations first.

Lemma 4.2. *Suppose that* (**A4.1**)-(**A4.6**) *hold. Then*

$$\mathbb{E} \int_0^T |\bar{x}^{(N)} - \hat{x}|^2 dt + \mathbb{E} \int_0^T |\bar{l} - \hat{l}|^2 dt = O(\frac{1}{N}).$$

Proof By (4.34), we have

$$\begin{cases} d\bar{x}^{(N)} = \Big\{[A - BR^{-1}B^\top K + \bar{F} - R_2^{-1}\bar{L}]\bar{x}^{(N)} - M)\hat{l} - BR_1^{-1}B^\top L\hat{x} \\ \quad - BR_1^{-1}B^\top\varphi + BR_1^{-1}B^\top(P - R_2^{-1}(\bar{M}\bar{l} + \phi)\Big\}dt + \dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\sigma dW_i, \\ \bar{x}^{(N)}(0) = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\xi_i. \end{cases} \tag{4.35}$$

Combining (4.24), (4.34), (4.32) and (4.35), one can obtain

$$\begin{cases} d\tilde{x} = [(A - BR_1^{-1}B^\top K + \bar{F} - R_2^{-1}\bar{L})\tilde{x} - R_2^{-1}\bar{M}\tilde{l}]dt + \dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\sigma dW_i, \\ d\tilde{l} = [(A + \bar{F} - R_2^{-1}\bar{M})\tilde{l} + R_2^{-1}(K + P + \bar{L})\tilde{x}]dt, \\ \tilde{x}(0) = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\xi_i - \hat{\xi}, \quad \tilde{l}(0) = 0, \end{cases} \tag{4.36}$$

where $\tilde{x} = \bar{x}^{(N)} - \hat{x}$, $\tilde{l} = \bar{l} - \hat{l}$. By the Cauchy-Schwarz inequality and the Burkholder-Davis-Gundy inequality, we have

$$\mathbb{E}\sup_{0\le s\le t}|\mathbb{X}|^2 = \mathbb{E}\sup_{0\le s\le t}\left|\mathbb{X}(0) + \int_0^s \mathbb{A}\mathbb{X}dr + \int_0^s \frac{1}{N}\sum_{i=1}^{N}\begin{pmatrix}\sigma\\0\end{pmatrix}dW_i\right|^2$$

$$\le c\,\mathbb{E}\sup_{0\le s\le t}\int_0^s |\mathbb{X}|^2dr + \frac{3}{N^2}\sum_{i=1}^{N}\mathbb{E}\left|\int_0^t\begin{pmatrix}\sigma\\0\end{pmatrix}\right|^2 dr \le c\,\mathbb{E}\int_0^t|\mathbb{X}|^2dr + O\Big(\frac{1}{N}\Big),$$

where $\mathbb{X} = (\tilde{x}^\top, \tilde{l}^\top)^\top$,

$$\mathbb{A} = \begin{pmatrix} A - BR_1^{-1}B^\top K + \bar{F} - R_2^{-1}\bar{L} & -R_2^{-1}\bar{M} \\ R_2^{-1}(K + P + \bar{L}) & A + \bar{F} - R_2^{-1}\bar{M} \end{pmatrix},$$

and constant c is independent of N. Then, by Gronwall's inequality, one can obtain that

$$\mathbb{E}\sup_{0\le t\le T}|\mathbb{X}|^2 = O(\frac{1}{N}).$$

The lemma follows. □

Remark 4.4. *In [174], an additional Riccati equation is needed for proving*

$$\sup_{0\le t\le T}\mathbb{E}(|\bar{x}^{(N)}-\hat{x}|^2+|\bar{s}-\hat{s}|^2)=O(\frac{1}{N}),$$

since $(\bar{x}^{(N)}-\hat{x})$ *and* $(\bar{s}-\hat{s})$ *satisfy an FBSDE system and they need to be decoupled by using the Riccati equation method. However, in our model,* $\tilde{x}$ *and* $\tilde{l}$ *evolve by forward SDEs and we can estimate them directly without setting such an assumption.*

Theorem 4.3. *Suppose that* (**A4.1**)-(**A4.6**) *hold. The set of decentralized control laws* $\bar{u}=\{\bar{u}_1,\bar{u}_2,\cdots,\bar{u}_N\}\in\mathcal{U}^F$ *given by* (4.33) *has robust asymptotic social optimality.*

Proof Let $\grave{x}_i=x_i-\bar{x}_i$, $\grave{u}_i=u_i-\bar{u}_i$, $\grave{x}^{(N)}=x^{(N)}-\bar{x}^{(N)}$ and $\grave{s}=s-\bar{s}$, where $i=1,2,\cdots,N$. Then by (4.17),

$$\begin{cases} d\grave{x}_i=[A\grave{x}_i+B\grave{u}_i+\bar{F}\grave{x}^{(N)}-R_2^{-1}\grave{s})]dt, \quad \grave{x}_i(0)=0,\\ d\grave{s}=-[(A+\bar{F})^\top\grave{s}-PB\grave{u}^{(N)}]dt+\dfrac{1}{N}\displaystyle\sum_{j=1}^N\sum_{i=1}^N(-\grave{\beta}_i^j)dW_j, \quad \grave{s}(T)=0. \end{cases} \tag{4.37}$$

By Lemma 5.4 in [101], if (**A4.1**)-(**A4.6**) hold, for all $u_i\in\mathcal{U}_c^F$, $i=1,2,\cdots,N$, we have $\frac{1}{N}\mathcal{J}_{soc}^{wo}(u)\le\frac{1}{N}\mathcal{J}_{soc}^{wo}(\bar{u})\le c$, where c is independent of N. That implies $\mathbb{E}\int_0^T|u_i|^2dt<c$. Then, by (4.37), we have $\mathbb{E}\int_0^T(|\grave{x}_i|^2+|\grave{u}_i|^2+|\grave{s}|^2)dt<c$. Next, considering (4.2) and (4.16), we denote

$$\begin{aligned}\mathcal{J}_{soc}^{wo}(u)=&\frac{1}{2}\sum_{i=1}^N\mathbb{E}\int_0^T\Big\{|x_i-\Gamma x^{(N)}-\eta|_Q^2+|u_i|_{R_1}^2-|Px^{(N)}+s|_{R_2^{-1}}^2\Big\}dt\\ &+\frac{1}{2}\sum_{i=1}^N\mathbb{E}|x_i(T)-\hat{\Gamma}x^{(N)}(T)-\hat{\eta}|_G^2=\sum_{i=1}^N(\mathcal{J}_i^F(\bar{u})+\grave{\mathcal{J}}_i^F(\grave{u})+I_i)\end{aligned} \tag{4.38}$$

where

$$\grave{\mathcal{J}}_i^{wo}(\grave{u}) = \frac{1}{2}\mathbb{E}\int_0^T \left\{ |\grave{x}_i - \Gamma \grave{x}^{(N)}|_Q^2 + |\grave{u}_i|_{R_1}^2 - |P\grave{x}^{(N)} + \grave{s}|_{R_2^{-1}}^2 \right\} dt$$
$$+ \frac{1}{2}\mathbb{E}|\grave{x}_i(T) - \hat{\Gamma}\grave{x}^{(N)}(T)|_G^2,$$

and

$$\begin{aligned}
\sum_{i=1}^N I_i = &\sum_{i=1}^N \mathbb{E}\int_0^T \langle Q\bar{x}_i - Q_\Gamma \bar{x}^{(N)} - \eta_\Gamma - PR_2^{-1}(P\hat{x} + \hat{s}), \grave{x}_i\rangle \\
&- \langle PR_2^{-1}[P(\bar{x}^{(N)} - \hat{x}) + (\bar{s} - \hat{s})], \grave{x}_i\rangle - \langle R_2^{-1}(P\hat{x} + \hat{s}), \grave{s}\rangle \\
&- \langle R_2^{-1}[P(\bar{x}^{(N)} - \hat{x}) + (\bar{s} - \hat{s})], \grave{s}\rangle + \langle R_1\bar{u}_i, \grave{u}_i\rangle dt \\
&+ \sum_{i=1}^N \mathbb{E}\langle G\bar{x}_i(T) - G_{\hat{\Gamma}}\bar{x}^{(N)}(T) - \hat{\eta}_{\hat{\Gamma}}, \grave{x}_i(T)\rangle.
\end{aligned} \tag{4.39}$$

We now prove $\sum_{i=1}^N I_i = O(\frac{1}{\sqrt{N}})$. By (4.25), (4.34), (4.37), and the similar techniques from (4.18) to (4.23),

$$\begin{aligned}
\sum_{i=1}^N \mathbb{E}\langle \bar{k}_i(T), \grave{x}_i(T)\rangle = &\sum_{i=1}^N \mathbb{E}\int_0^T \langle -Q\bar{x}_i + Q_\Gamma \bar{x}^{(N)} + \eta_\Gamma - KR_2^{-1}(\bar{s} - \hat{s}) \\
&+ PR_2^{-1}(P + \bar{L})\hat{x} + PR_2^{-1}\bar{M}\hat{l} + PR_2^{-1}\phi + (K\bar{F} + \bar{F}^\top K)(\bar{x}^{(N)} - \hat{x}), \grave{x}_i\rangle \\
&- \langle R_2^{-1}(K\bar{x}_i + L\hat{x} + M\hat{l} + \varphi), \grave{s}\rangle + \langle B^\top(K\bar{x}_i + L\hat{x} + M\hat{l} + \varphi), \grave{u}_i\rangle dt,
\end{aligned} \tag{4.40}$$

and

$$\begin{aligned}
0 = \sum_{i=1}^N \langle \bar{l}(T), \grave{s}(T)\rangle - \langle \bar{l}(0), \grave{s}(0)\rangle = &\sum_{i=1}^N \mathbb{E}\int_0^T \langle R_2^{-1}\bar{k}_i, \grave{s}\rangle + \langle R_2^{-1}(P\hat{x} + \hat{s}), \grave{s}\rangle \\
&- \langle B^\top P\bar{l}, \grave{u}_i\rangle + \langle R_2^{-1}(P + \bar{L})(\bar{x}^{(N)} - \hat{x}) + R_2^{-1}\bar{M}(\bar{l} - \hat{l}), \grave{s}\rangle.
\end{aligned} \tag{4.41}$$

Combining (4.39)-(4.41) and Lemma 5.1, note that $\bar{s} - \hat{s} = \bar{M}\tilde{l} + \bar{L}\tilde{x}$ and $R_1\bar{u}_i +$

$B^\top[K\bar{x}_i + L\hat{x} - B^\top(P-M)\hat{l} + \varphi] = 0$, one can obtain

$$\frac{1}{N}\sum_{i=1}^{N} I_i = O(\frac{1}{\sqrt{N}}).$$

Then, we put this into (4.38), the theorem follows. □

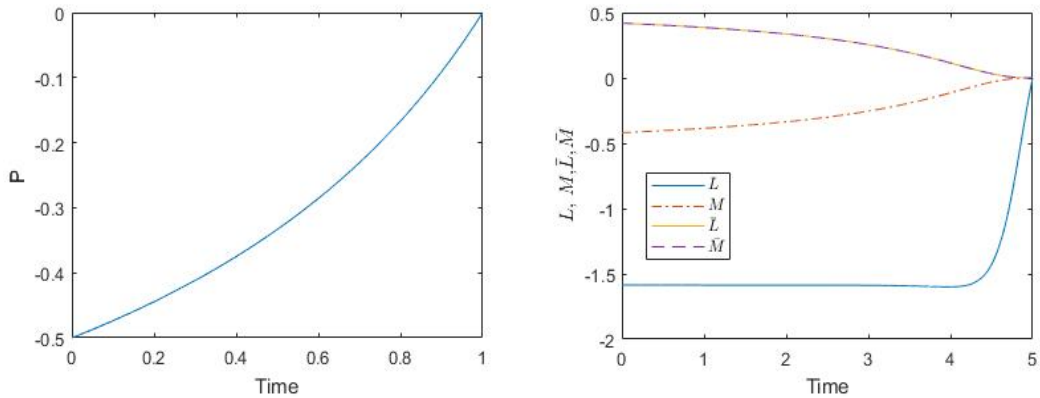
4.5 Numerical Examples

We continue to use the parameters in Example 4.1. First, we give the figure of $P(t) = -\frac{1}{t-2} - 1$, $t \in [0,1]$. Since $P(t)$ in (4.6) is the same as it in (4.13), $P(t) = -\frac{1}{t-2} - 1$ could also be solution for (4.13) and its trajectory is shown in Figure 1(a). Let the population $N = 100$, $R_1 = 0.5$, $\sigma = 5$, $\eta = \hat{\eta} = 0$, $\hat{\Gamma} = 0.5$ and the time interval is [0,5]. Using Matlab computation and by (4.13), (4.29), P and K can be easily computed. After that, we simulate the BSDEs from (4.30) to (4.31) and obtain their figures in Figure 1(b)[1]. Taking the initial values independently from a uniform distribution $U(-30, 60)$ and by equations (4.36), the curve of $\tilde{x}$, $\tilde{l}$ and $\tilde{s}$ is described in Figure 2(a). Denote that $\varepsilon_1^2 = \mathbb{E}\int_0^1 |\bar{x}^{(N)} - \hat{x}|^2 dt$, $\varepsilon_2^2 = \mathbb{E}\int_0^1 |\bar{l} - \hat{l}|^2 dt$. We let N increase from 1 to 100 and the curves of ε_1^2 and ε_2^2 are shown in Figure 2(b). It shows that they are getting close to zero when N is becoming larger and larger.

4.6 Conclusion

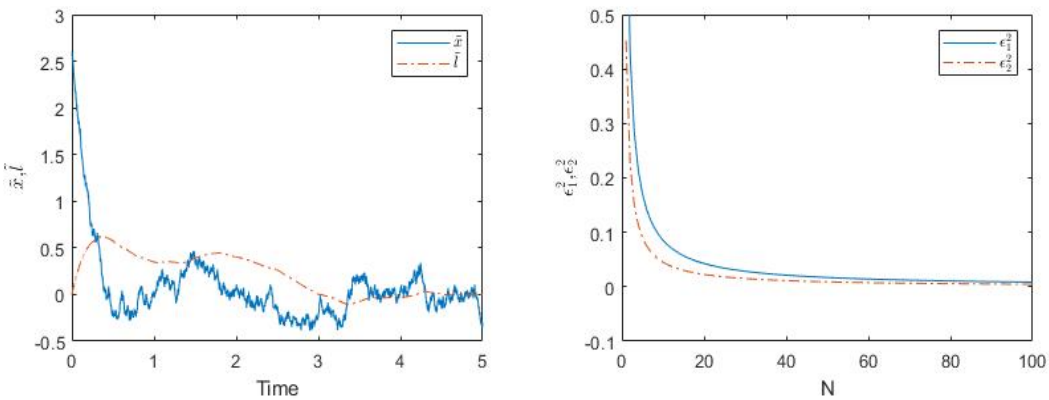
In this chapter, we study a class of social optimality for robust LQ-MF problems with a common uncertain drift. By the robust optimization approach, we obtain a "worst case" disturbance for all agents. Using variational analysis and decoupling

[1] By observing figure 1(b), we could find that the curve of $\bar{L}$ and $\bar{M}$ is overlapping. In fact, the situation does not change even though we try many sets of numbers for the parameters. Therefore, we have a hypothesis that $\bar{L}$ may be equal to $\bar{M}$. If so, the system (4.30) and (4.31) are decoupled and solved directly, which may be useful in other more complicated models. Unfortunately, we cannot prove the hypothesis rigorously in mathematics.



(a) (b)

Figure 4.1: (a) is the curve of $P(t)$ and (b) is the curves of L, M, $\bar{L}$ and $\bar{M}$.



(a) (b)

Figure 4.2: (a) is the curves of $\tilde{x}$, $\tilde{l}$ and (b) is the curves of ε_1^2, ε_2^2 when time interval is $[0, 5]$.

the FBSDEs with mean field approximation, we construct the decentralized controls, which are further proved to be an asymptotically social optimum. For further work, it is interesting to investigate the social optimality for robust LQ-MF problems with uncertainty in common noise by the direct approach.

Chapter 5

Social Optima in Leader-Follower Mean Field Linear Quadratic Control

After introducing the LF problem and the MFT problem in Chapter 3 and Chapter 4, which can be considered as two preliminary chapters, respectively, we are going to investigate the social optimality of the LF LQ-MF control problem. The model in this chapter involves one leader and a large number of weakly-coupled interactive followers and all the agents (including the leaders and the followers) cooperate to optimize the social cost. Unlike the previous chapter that using the direct approach, we apply the person-by-person optimality here and construct two auxiliary control problems (the fixed point approach). By solving these two auxiliary problems sequentially with consistent mean field approximations, a set of decentralized control can be obtained with the help of a consistency condition (CC) system. By some proper conditions, the asymptotic Stackelberg equilibrium is proved.

5.1 Problem Formulation

Consider a large-population system which contains one leader and N followers. Since it contains a leader, by the discussion in Section 2.1 of Chapter 2, there are

$N+1$ agents in the system. We define the leader as $\mathcal{A}_0$ and let σ-algebra $\mathcal{G}_t^i = \mathcal{F}_t^i \bigvee \sigma\{\xi_i, \xi_0, W_0(s), 0 \le s \le t\}$, where $0 \le i \le N$, and $\mathcal{G}_t = \mathcal{F}_t \bigvee \sigma\{\xi_i, 0 \le i \le N\}$. $\mathbb{G}^i = \{\mathcal{G}_t^i\}_{0 \le t \le T}$, where $0 \le i \le N$, and $\mathbb{G} = \{\mathcal{G}_t\}_{0 \le t \le T}$. The state processes of the leader $\mathcal{A}_0$ and the follower $\mathcal{A}_i$, $i = 1, 2, \cdots, N$, are modeled by the following linear SDE on a finite time horizon $[0, T]$:

$$\begin{cases} dx_0 = [A_0 x_0 + B_0 u_0 + C_0 x^{(N)}]dt + D_0 dW_0, \\ dx_i = [A x_i + B u_i + C x^{(N)} + F x_0]dt + D dW_i, \\ x_0(0) = \xi_0, \quad x_i(0) = \xi_i, \end{cases} \tag{5.1}$$

where $x^{(N)} := \frac{1}{N}\sum_{i=1}^N x_i$ is the state average of the followers and the coefficients are satisfy the following assumption:

$$(\mathbf{A5.1}) \begin{cases} A_0(\cdot), C_0(\cdot), A(\cdot), C(\cdot), F(\cdot) \in L^\infty(0, T; \mathbb{R}^{n\times n}), \\ B_0(\cdot), B(\cdot) \in L^\infty(0, T; \mathbb{R}^{n\times m}), \quad D_0(\cdot), D(\cdot) \in L^\infty(0, T; \mathbb{R}^{n\times d}). \end{cases}$$

The set of admissible controls for $\mathcal{A}_0$ is defined as follows:

$$\mathcal{U}_0 = \left\{u_0 | u_0(t) \in L^2_{\mathbb{G}^0}(0, T; \mathbb{R}^m)\right\},$$

and the set of admissible controls for the follower $\mathcal{A}_i$ is defined as follows:

$$\mathcal{U}_i = \left\{u_i | u_i(t) \in L^2_{\mathbb{G}^i}(0, T; \mathbb{R}^m)\right\}, \quad 1 \le i \le N.$$

These are the decentralized control sets and we let $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_N$. For comparison, the centralized control set is given by

$$\mathcal{U}_c = \Big\{(u_0, u_1, \cdots, u_N) | u_i(t) \in L^2_{\mathbb{G}}(0, T; \mathbb{R}^m), 0 \le i \le N\Big\}.$$

Now we introduce the cost functionals of the leader $\mathcal{A}_0$ and the follower $\mathcal{A}_i$,

$1 \leq i \leq N$. For the leader, the cost functional is defined as follows:

$$\begin{aligned}\mathcal{J}_0(u_0;u) =&\mathbb{E}\Big\{\int_0^T \big[|x_0-\Theta_0 x^{(N)}-\eta_0|^2_{Q_0}+|u_0(t)|^2_{R_0}\big]dt \\ &+|x_0(T)-\hat{\Theta}_0 x^{(N)}(T)-\hat{\eta}_0|^2_{G_0}\Big\},\end{aligned} \tag{5.2}$$

where $u=(u_1,\cdots,u_N)\in\mathcal{U}_c$. Q_0, R_0 and G_0 are weight matrices. Q_0 and Θ_0 represent the coupling between the leader and the state average term. This implies that the states of the followers can influence the cost functional of the leader. For the follower $\mathcal{A}_i$, his individual cost functional is defined as follows:

$$\begin{aligned}\mathcal{J}_i(u_0;u) =&\mathbb{E}\Big\{\int_0^T \big[|x_i-\Theta x^{(N)}-\Theta_1 x_0-\eta|^2_{Q}+|u_i|^2_{R}\big]dt \\ &+|x_i(T)-\hat{\Theta} x^{(N)}(T)-\hat{\Theta}_1 x_0(T)-\hat{\eta}|^2_{G}\Big\},\end{aligned} \tag{5.3}$$

where Q, R and G are weight matrices. Q, Θ and Θ_1 represent the coupling between the follower $\mathcal{A}_i$, the state average term and the leader $\mathcal{A}_0$. This implies that the cost functional of the follower $\mathcal{A}_i$ will be affected by the behavior of both the leader and the other followers. All the individuals in the system, including the leader and the followers, aim to minimize the social cost functional, which is denoted by

$$\mathcal{J}^{(N)}_{soc}(u_0;u)=\alpha N\mathcal{J}_0(u_0;u)+\sum_{i=1}^N \mathcal{J}_i(u_0;u), \quad \alpha>0. \tag{5.4}$$

Similar to [107] and [140], we have a scaling factor αN before $\mathcal{J}_0(u_0;u)$ such that $\mathcal{J}_0(u_0;u)$ and $\mathcal{J}_i(u_0;u)$ have the same order of magnitude. Otherwise, if $\alpha N=1$, then the performance of the leader will be insensitive when N becomes larger. Now we give some assumptions that will be applied in the further analysis.

(**A5.2**) The coefficients of (5.2) and (5.3) satisfy

$$\begin{cases} Q_0(\cdot), Q(\cdot) \in L^\infty(0,T;\mathbb{S}^n), \quad R_0(\cdot), R(\cdot) \in L^\infty(0,T;\mathbb{S}^m), \\ \Theta_0(\cdot), \Theta_1(\cdot), \Theta(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n}), \quad \eta_0(\cdot), \eta(\cdot) \in L^2(0,T;\mathbb{R}^n), \\ \hat{\Theta}_0, \hat{\Theta}_1, \hat{\Theta} \in \mathbb{R}^{n\times n}, \quad G_0, G \in \mathbb{S}^n, \quad \hat{\eta}_0, \hat{\eta} \in \mathbb{R}^n. \end{cases}$$

(**A5.3**) $x_0(0)$ and $W_0(\cdot)$ are mutually independent. $\{x_i(0), 1 \le i \le N\}$ and $\{W_i(\cdot), 1 \le i \le N\}$ are independent of each other. $\mathbb{E}x_i(0) = \hat{\xi}$, $1 \le i \le N$. There exists a constant K independent of N such that $\sup_{1\le i\le N} \mathbb{E}|x_i(0)|^2 \le K$. Furthermore, $x_0(0)$, $W_0(\cdot)$ and $\{x_i(0), 1 \le i \le N\}$, $\{W_i(t), 1 \le i \le N\}$ are independent of each other.
(**A5.4**) $Q_0(\cdot) \ge 0$, $R_0(\cdot) > \delta I$, $G_0 \ge 0$ and $Q(\cdot) \ge 0$, $R(\cdot) > \delta I$, $G \ge 0$, for some $\delta > 0$.

Next, we introduce our LF MFT problem:

Problem 5.1. (**P5.1**) *For any $u_0 \in \mathcal{U}_0$, to find a mapping $\mathcal{M}$: $\mathcal{U}_0 \to \mathcal{U}$, and a control $\bar{u}_0 \in \mathcal{U}_0$ such that*

$$\begin{cases} \mathcal{J}_{soc}^{(N)}(u_0; \mathcal{M}(u_0)) = \inf\limits_{u\in\mathcal{U}_c} \mathcal{J}_{soc}^{(N)}(u_0; u), \\ \mathcal{J}_{soc}^{(N)}(\bar{u}_0; \mathcal{M}(\bar{u}_0)) = \inf\limits_{u_0\in\mathcal{U}_0} \mathcal{J}_{soc}^{(N)}(u_0; \mathcal{M}(u_0)). \end{cases}$$

Note that the $\mathcal{M}$ here is a mapping, which is different from the notation $\mathcal{M}[0,T]$.

5.2 The LQ-MF Control Problem for the N Followers

5.2.1 The person-by-person optimality

Fix $u_0 \in \mathcal{U}_0$. The leader firstly announces his own open-loop strategy. Let $\bar{u} = \{\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_N\}$ be the centralized optimal control of the followers and $\bar{x} = \{\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_N\}$ be the corresponding states. Now we perturb $\bar{u}_i$ and fix other $\bar{u}_j$, where $j \ne i$. Then

we denote $\delta u_i = u_i - \bar{u}_i$, $\delta x_i = x_i - \bar{x}_i$, where u_i is the control after perturbing and x_i is its corresponding state. The Fréchet differential $\delta\mathcal{J}_0(\delta u_i) = \mathcal{J}_0(u_0; u) - \mathcal{J}_0(u_0; \bar{u}) + o(|\delta u_i|^2_{L^2})$ and $\delta\mathcal{J}_i(\delta u_i) = \mathcal{J}_i(u_0; u) - \mathcal{J}_i(u_0; \bar{u}) + o(|\delta u_i|^2_{L^2})$, where $i = 1, \cdots, N$. Therefore, the variations of the state equations for the leader, the ith follower and the jth follower, where $j \neq i$, are

$$\begin{cases} d\delta x_0 = (A_0\delta x_0 + C_0\delta x^{(N)})dt, \quad \delta x_0(0) = 0, \\ d\delta x_i = (A\delta x_i + B\delta u_i + C\delta x^{(N)} + F\delta x_0)dt, \quad \delta x_i(0) = 0, \\ d\delta x_j = (A\delta x_j + C\delta x^{(N)} + F\delta x_0)dt, \quad \delta x_j(0) = 0, \quad j \neq i, \end{cases}$$

and the variations of their corresponding cost functionals are

$$\begin{aligned} \frac{1}{2}\delta\mathcal{J}_0(\delta u_i) =& \mathbb{E}\Big\{ \int_0^T \langle Q_0(\bar{x}_0 - \Theta_0\bar{x}^{(N)} - \eta_0), \delta x_0 - \Theta_0\delta x^{(N)}\rangle dt \\ &+ \langle G_0(\bar{x}_0(T) - \hat{\Theta}_0\bar{x}^{(N)}(T) - \hat{\eta}_0), \delta x_0(T) - \hat{\Theta}_0\delta x^{(N)}(T)\rangle \Big\}, \end{aligned}$$

$$\begin{aligned} \frac{1}{2}\delta\mathcal{J}_i(\delta u_i) =& \mathbb{E}\Big\{ \int_0^T \langle Q(\bar{x}_i - \Theta\bar{x}^{(N)} - \Theta_1\bar{x}_0 - \eta), \delta x_i - \Theta\delta x^{(N)} \\ &- \Theta_1\delta x_0\rangle + \langle R\bar{u}_i, \delta u_i\rangle dt + \langle G(\bar{x}_i(T) - \hat{\Theta}\bar{x}^{(N)}(T) \\ &- \hat{\Theta}_1\bar{x}_0(T) - \hat{\eta}), \delta x_i(T) - \hat{\Theta}\delta x^{(N)}(T) - \hat{\Theta}_1\delta x_0(T)\rangle \Big\}, \end{aligned}$$

$$\begin{aligned} \frac{1}{2}\delta\mathcal{J}_j(\delta u_i) =& \mathbb{E}\Big\{ \int_0^T \langle Q(\bar{x}_j - \Theta\bar{x}^{(N)} - \Theta_1\bar{x}_0 - \eta), \delta x_j - \Theta\delta x^{(N)} \\ &- \Theta_1\delta x_0\rangle dt + \langle G(\bar{x}_j(T) - \hat{\Theta}\bar{x}^{(N)}(T) - \hat{\Theta}_1\bar{x}_0(T) - \hat{\eta}), \\ &\delta x_j(T) - \hat{\Theta}\delta x^{(N)}(T) - \hat{\Theta}_1\delta x_0(T)\rangle \Big\}, \end{aligned}$$

respectively. Consequently, we have the variation of the social cost functional as:

$$\frac{1}{2}\delta\mathcal{J}_{soc}^{(N)}(\delta u_i) = \frac{1}{2}\Big[\alpha N\delta\mathcal{J}_0(\delta u_i) + \sum_{j\neq i}\delta\mathcal{J}_j(\delta u_i) + \delta\mathcal{J}_i(\delta u_i)\Big]$$

$$= \mathbb{E}\Big\{\int_0^T \alpha N\langle Q_0(\bar{x}_0 - \Theta_0\bar{x}^{(N)} - \eta_0), \delta x_0\rangle - \alpha N\langle\Theta_0^\top Q_0(\bar{x}_0 - \Theta_0\bar{x}^{(N)} - \eta_0), \delta x^{(N)}\rangle$$

$$+ \langle Q(\bar{x}_i - \Theta\bar{x}^{(N)} - \Theta_1\bar{x}_0 - \eta), \delta x_i\rangle - \langle\Theta^\top Q(\bar{x}_i - \Theta\bar{x}^{(N)} - \Theta_1\bar{x}_0 - \eta), \delta x^{(N)}\rangle$$

$$- \langle\Theta_1^\top Q(\bar{x}_i - \Theta\bar{x}^{(N)} - \Theta_1\bar{x}_0 - \eta), \delta x_0\rangle + \langle R\bar{u}_i, \delta u_i\rangle + \sum_{j\neq i}\langle Q(\bar{x}_j - \Theta\bar{x}^{(N)} - \Theta_1\bar{x}_0$$

$$- \eta), \delta x_j\rangle - \sum_{j\neq i}\langle\Theta^\top Q(\bar{x}_j - \Theta\bar{x}^{(N)} - \Theta_1\bar{x}_0 - \eta), \delta x^{(N)}\rangle - \sum_{j\neq i}\langle\Theta_1^\top Q(\bar{x}_j - \Theta\bar{x}^{(N)}$$

$$- \Theta_1\bar{x}_0 - \eta), \delta x_0\rangle dt + \alpha N\langle G_0(\bar{x}_0(T) - \hat{\Theta}_0\bar{x}^{(N)}(T) - \hat{\eta}_0), \delta x_0(T)\rangle$$

$$- \alpha N\langle\hat{\Theta}_0^\top G_0(\bar{x}_0(T) - \hat{\Theta}_0\bar{x}^{(N)}(T) - \hat{\eta}_0), \delta x^{(N)}(T)\rangle + \langle G(\bar{x}_i(T) - \hat{\Theta}\bar{x}^{(N)}(T)$$

$$- \hat{\Theta}_1\bar{x}_0(T) - \hat{\eta}), \delta x_i(T)\rangle - \langle\hat{\Theta}^\top G(\bar{x}_i(T) - \hat{\Theta}\bar{x}^{(N)}(T) - \hat{\Theta}_1\bar{x}_0(T) - \hat{\eta}), \delta x^{(N)}(T)\rangle$$

$$- \langle\hat{\Theta}_1^\top G(\bar{x}_i(T) - \hat{\Theta}\bar{x}^{(N)}(T) - \hat{\Theta}_1\bar{x}_0(T) - \hat{\eta}), \delta x_0(T)\rangle + \sum_{j\neq i}\langle G(\bar{x}_j(T) - \hat{\Theta}\bar{x}^{(N)}(T)$$

$$- \hat{\Theta}_1\bar{x}_0(T) - \hat{\eta}), \delta x_j(T)\rangle - \sum_{j\neq i}\langle\hat{\Theta}^\top G(\bar{x}_j(T) - \hat{\Theta}\bar{x}^{(N)}(T) - \hat{\Theta}_1\bar{x}_0(T)$$

$$- \hat{\eta}), \delta x^{(N)}(T)\rangle - \sum_{j\neq i}\langle\hat{\Theta}_1^\top G(\bar{x}_j(T) - \hat{\Theta}\bar{x}^{(N)}(T) - \hat{\Theta}_1\bar{x}_0(T) - \hat{\eta}), \delta x_0(T)\rangle\Big\}. \tag{5.5}$$

When $N \to \infty$, it follows that

$$\frac{1}{2}\delta\mathcal{J}_{soc}^{(N)}(\delta u_i) = \mathbb{E}\Big\{\int_0^T \alpha\langle Q_0(\bar{x}_0 - \Theta_0\bar{x}^{(N)} - \eta_0), N\delta x_0\rangle - \alpha\langle\Theta_0^\top Q_0(\bar{x}_0 - \Theta_0\bar{x}^{(N)}$$

$$- \eta_0), N\delta x^{(N)}\rangle + \langle Q(\bar{x}_i - \Theta\bar{x}^{(N)} - \Theta_1\bar{x}_0 - \eta), \delta x_i\rangle + \langle R\bar{u}_i, \delta u_i\rangle + \langle\frac{1}{N}\sum_{j\neq i}Q(\bar{x}_j$$

$$- \Theta\bar{x}^{(N)} - \Theta_1\bar{x}_0 - \eta), N\delta x_j\rangle - \langle\frac{1}{N}\sum_{j\neq i}\Theta^\top Q(\bar{x}_j - \Theta\bar{x}^{(N)} - \Theta_1\bar{x}_0 - \eta), N\delta x^{(N)}\rangle$$

$$- \langle \frac{1}{N} \sum_{j \neq i} \Theta_1^\top Q(\bar{x}_j - \Theta \bar{x}^{(N)} - \Theta_1 \bar{x}_0 - \eta), N\delta x_0 \rangle dt + \alpha \langle G_0(\bar{x}_0(T) - \hat{\Theta}_0 \bar{x}^{(N)}(T)$$

$$- \hat{\eta}_0), N\delta x_0(T) \rangle - \alpha \langle \hat{\Theta}_0^\top G_0(\bar{x}_0(T) - \hat{\Theta}_0 \bar{x}^{(N)}(T) - \hat{\eta}_0), N\delta x^{(N)}(T) \rangle + \langle G(\bar{x}_i(T)$$

$$- \hat{\Theta} \bar{x}^{(N)}(T) - \hat{\Theta}_1 \bar{x}_0(T) - \hat{\eta}), \delta x_i(T) \rangle + \langle \frac{1}{N} \sum_{j \neq i} G(\bar{x}_j(T) - \hat{\Theta} \bar{x}^{(N)}(T) - \hat{\Theta}_1 \bar{x}_0(T)$$

$$- \hat{\eta}), N\delta x_j(T) \rangle - \langle \frac{1}{N} \sum_{j \neq i} \hat{\Theta}^\top G(\bar{x}_j(T) - \hat{\Theta} \bar{x}^{(N)}(T) - \hat{\Theta}_1 \bar{x}_0(T) - \hat{\eta}), N\delta x^{(N)}(T) \rangle$$

$$\left. - \langle \frac{1}{N} \sum_{j \neq i} \hat{\Theta}_1^\top G(\bar{x}_j(T) - \hat{\Theta} \bar{x}^{(N)}(T) - \hat{\Theta}_1 \bar{x}_0(T) - \hat{\eta}), N\delta x_0(T) \rangle \right\} + o(1).$$

Note that $\mathbb{E} \sup_{0 \leq t \leq T} |\delta x_0|^2 = O(\frac{1}{N^2})$, $\mathbb{E} \sup_{0 \leq t \leq T} |\delta x^{(N)}|^2 = O(\frac{1}{N^2})$ and $\langle \Theta^\top Q(\bar{x}_i - \Theta \bar{x}^{(N)} - \Theta_1 \bar{x}_0 - \eta), \delta x^{(N)} \rangle + \langle \Theta_1^\top Q(\bar{x}_i - \Theta \bar{x}^{(N)} - \Theta_1 \bar{x}_0 - \eta), \delta x_0 \rangle + \langle \hat{\Theta}^\top G(\bar{x}_i(T) - \hat{\Theta} \bar{x}^{(N)}(T) - \hat{\Theta}_1 \bar{x}_0(T) - \hat{\eta}), \delta x^{(N)}(T) \rangle + \langle \hat{\Theta}_1^\top G(\bar{x}_i(T) - \hat{\Theta} \bar{x}^{(N)}(T) - \hat{\Theta}_1 \bar{x}_0(T) - \hat{\eta}), \delta x_0(T) \rangle = o(1)$ (the rigorous proof will be shown in Section 5.5). Let

$$\begin{cases} \delta x_0^\dagger = \lim_{N \to +\infty} (N\delta x_0), \\ \delta x^\dagger = \lim_{N \to +\infty} (N\delta x_j) = \lim_{N \to +\infty} (\sum_{j \neq i} \delta x_j), \ j \neq i. \end{cases} \tag{5.6}$$

Here $N\delta x_0$ converges to $\delta x_0^\dagger$ such that $\mathbb{E} \int_0^T |N\delta x_0 - \delta x_0^\dagger|^2 = O(\frac{1}{N^2})$. Similarly, $\sum_{j \neq i} \delta x_j$ and $N\delta x_j$ converge to $\delta x^\dagger$ (see Section 5.5 for the detailed proof). Then one can obtain

$$\begin{cases} d\delta x_0^\dagger = (A_0 \delta x_0^\dagger + C_0 \delta x_i + C_0 \delta x^\dagger) dt, & \delta x_0^\dagger(0) = 0, \\ d\delta x^\dagger = (A\delta x^\dagger + C\delta x_i + C\delta x^\dagger + F\delta x_0^\dagger) dt, & \delta x^\dagger(0) = 0. \end{cases} \tag{5.7}$$

When $N \to \infty$, by mean field approximation, we use $\hat{x}$ to approximate $\bar{x}^{(N)}$. Note that $\hat{x}$ will be affected by u_0 which is given by the leader. Moreover, the influence of individual follower on $\hat{x}$ may be negligible. Hence, by straightforward computation,

we simplified the social cost functional as follows:

$$
\begin{aligned}
&\frac{1}{2}\delta\mathcal{J}_{soc}^{(N)}(\delta u_i)\\
=&\mathbb{E}\Big\{\int_0^T \langle \alpha Q_0\Psi_1 - \Theta_1^\top Q\Psi_3, \delta x_0^\dagger\rangle + \langle Q\Psi_2^i - \Theta^\top Q\Psi_3 - \alpha\Theta_0^\top Q_0\Psi_1, \delta x_i\rangle\\
&+ \langle R\bar{u}_i, \delta u_i\rangle + \langle Q\Psi_3 - \Theta^\top Q\Psi_3 - \alpha\Theta_0^\top Q_0\Psi_1, \delta x^\dagger\rangle dt + \langle \alpha G_0\Psi_4(T)\\
&- \hat{\Theta}_1^\top G\Psi_6(T), \delta x_0^\dagger(T)\rangle + \langle G\Psi_6(T) - \hat{\Theta}^\top G\Psi_6(T) - \alpha\hat{\Theta}_0^\top G_0\Psi_4(T),\\
&\delta x^\dagger(T)\rangle + \langle G\Psi_5^i(T) - \hat{\Theta}^\top G\Psi_6(T) - \alpha\hat{\Theta}_0^\top G_0\Psi_4(T), \delta x_i(T)\rangle\Big\},
\end{aligned}
\tag{5.8}
$$

where

$$
\begin{cases}
\Psi_1(\cdot) := \bar{x}_0 - \Theta_0\hat{x} - \eta_0, \quad \Psi_2^i(\cdot) := \bar{x}_i - \Theta\hat{x} - \Theta_1\bar{x}_0 - \eta,\\
\Psi_3(\cdot) := (I-\Theta)\hat{x} - \Theta_1\bar{x}_0 - \eta,
\end{cases}
$$

are related to time t, and

$$
\begin{cases}
\Psi_4(T) := \bar{x}_0(T) - \hat{\Theta}_0\hat{x}(T) - \hat{\eta}_0, \quad \Psi_5^i(T) := \bar{x}_i(T) - \hat{\Theta}\hat{x}(T) - \hat{\Theta}_1\bar{x}_0(T) - \hat{\eta},\\
\Psi_6(T) := (I-\hat{\Theta})\hat{x}(T) - \hat{\Theta}_1\bar{x}_0(T) - \hat{\eta},
\end{cases}
$$

are related to time T which are terminal terms.

Remark 5.1. *Note that, unlike the direct approach in Chapter 4 that perturbs all the agents' controls, the person-by-person optimality only perturbs the ith follower's control here. For this reason, it will derive three additional processes $N\delta x_0$, $N\delta x_j$, and $\sum_{j\neq i}\delta x_j$ and two limit process $\delta x_0^\dagger$ and $\delta x^\dagger$, which makes the problem more complicated. However, if $C_0 = 0$ and $C = 0$, then all these additional process can be vanished and the problem will be simplified (see [105, 138]).*

It is very important to formulate an auxiliary control problem to obtain the decentralized optimal control for analyzing the problem of social optimality. Usually,

an auxiliary control problem is a standard LQ control problem (see [105, 174]). However, (5.8) contains $\delta x_0^\dagger$ and $\delta x^\dagger$, which are the terms we do not want them appear in the social cost functional. Therefore, we need to use a duality procedure (see [191, Chapter 3]) to get off the dependence of $\delta\mathcal{J}_{soc}^{(N)}(\delta u_i)$ on $\delta x_0^\dagger$ and $\delta x^\dagger$. To this end, we introduce two auxiliary equations

$$\begin{cases} dk_1 = -(\alpha Q_0\Psi_1 - \Theta_1^\top Q\Psi_3 + F^\top k_2 + A_0^\top k_1)dt + \beta_1 dW_0, \\ dk_2 = -(Q\Psi_3 - \Theta^\top Q\Psi_3 - \alpha\Theta_0^\top Q_0\Psi_1 + C_0^\top k_1 + C^\top k_2 + A^\top k_2)dt + \beta_2 dW_0, \\ k_1(T) = \alpha G_0\Psi_4(T) - \hat{\Theta}_1^\top G\Psi_6(T),\ k_2(T) = (I - \hat{\Theta}^\top)G\Psi_6(T) - \alpha\hat{\Theta}_0^\top G_0\Psi_4(T), \end{cases}$$

and, by Itô formula (see Proposition 3.3 in Chapter 3 or Proposition 4.2 in Chapter 4), the variation of social cost functional is equivalent to

$$\begin{aligned} \frac{1}{2}\delta\mathcal{J}_{soc}^{(N)}(\delta u_i) = \mathbb{E}\Big\{ \int_0^T & \langle Q\bar{x}_i, \delta x_i\rangle + \langle R\bar{u}_i, \delta u_i\rangle + \langle -Q(\Theta\hat{x} + \Theta_1\bar{x}_0 + \eta) \\ & - \Theta^\top Q\Psi_3 - \alpha\Theta_0^\top Q_0\Psi_1 + C_0^\top k_1 + C^\top k_2, \delta x_i\rangle dt + \langle G\bar{x}_i(T), \delta x_i(T)\rangle \\ & + \langle -G(\hat{\Theta}\hat{x}(T) + \hat{\Theta}_1\bar{x}_0(T) + \hat{\eta}) - \hat{\Theta}^\top G\Psi_6(T) - \alpha\hat{\Theta}_0^\top G_0\Psi_4(T), \delta x_i(T)\rangle \Big\}. \end{aligned} \tag{5.9}$$

5.2.2 Decentralized strategy design for followers

As discussed in the previous subsection, when N is sufficiently large, a stochastic process $\hat{x}$ can be used to approximate $x^{(N)}$. Now, we can introduce the following auxiliary control problem for the ith follower.

Problem 5.2. (**P5.2**) *Minimize* $\hat{\mathcal{J}}_i((u_0, \hat{x}); u_i)$ *over* $u_i \in \mathcal{U}_i$, *where*

$$dx_i = [Ax_i + Bu_i + C\hat{x} + F\bar{x}_0(u_0)]dt + DdW_i,\ x_i(0) = \xi_i,\ i = 1, \cdots, N, \tag{5.10}$$

$$\hat{\mathcal{J}}_i((u_0, \hat{x}); u_i) = \mathbb{E}\Big\{ \int_0^T |x_i|_Q^2 + |u_i|_R^2 + 2\langle \chi_1, x_i\rangle dt + |x_i(T)|_G^2 + 2\langle \chi_2, x_i(T)\rangle \Big\}, \tag{5.11}$$

with

$$\begin{cases} \chi_1 = -Q(\Theta\hat{x} + \Theta_1\bar{x}_0(u_0) + \eta) - \Theta^\top Q\Psi_3 - \alpha\Theta_0^\top Q_0\Psi_1 + C_0^\top k_1 + C^\top k_2, \\ \chi_2 = -G(\hat{\Theta}\hat{x}(T) + \hat{\Theta}_1\bar{x}_0(u_0)(T) + \hat{\eta}) - \hat{\Theta}^\top G\Psi_6(T) - \alpha\hat{\Theta}_0^\top G_0\Psi_4(T). \end{cases}$$

Here, $\bar{x}_0(u_0)$ means $\bar{x}_0$ is related to u_0. $\bar{x}_0$, $\hat{x}$, k_1 and k_2 are determined by

$$\begin{cases} d\bar{x}_0 = [A_0\bar{x}_0 + B_0u_0 + C_0\hat{x}]dt + D_0dW_0, \quad \bar{x}_0(0) = \xi_0, \\ d\hat{x} = [A\hat{x} + B\hat{u} + C\hat{x} + F\bar{x}_0(u_0)]dt, \quad \hat{x}(0) = \hat{\xi}, \\ dk_1 = -(\alpha Q_0\Psi_1 - \Theta_1^\top Q\Psi_3 + F^\top k_2 + A_0^\top k_1)dt + \beta_1 dW_0, \\ dk_2 = -(Q\Psi_3 - \Theta^\top Q\Psi_3 - \alpha\Theta_0^\top Q_0\Psi_1 + C_0^\top k_1 + C^\top k_2 + A^\top k_2)dt + \beta_2 dW_0, \\ k_1(T) = \alpha G_0\Psi_4(T) - \hat{\Theta}_1^\top G\Psi_6(T), \ k_2(T) = (I - \hat{\Theta}^\top)G\Psi_6(T) - \alpha\hat{\Theta}_0^\top G_0\Psi_4(T), \end{cases} \tag{5.12}$$

where $\hat{x}$ and $\hat{u}$ are the approximations of $x^{(N)}$ and $\frac{1}{N}\sum_{i=1}^N u_i$, respectively.

In what follows, we let $\bar{u} = \mathcal{M}(u_0) = \{\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_N\} \in \mathcal{U}$. Note that $\bar{u}$ here represents the decentraliezd optimal control, which is different from the same notation in the beginning of this section.

Proposition 5.1. *Assume that* (**A5.1**)-(**A5.4**) *hold. For given $u_0 \in \mathcal{U}_0$,* (**P5.2**) *has a unique optimal control*

$$\bar{u}_i = -R^{-1}B^\top p_i, \tag{5.13}$$

where p_i is an adaptive solution to the following BSDE

$$dp_i = -(A^\top p_i + Q\bar{x}_i + \chi_1)dt + \zeta_0 dW_0 + \zeta_i dW_i, \ p_i(T) = G\bar{x}_i(T) + \chi_2. \tag{5.14}$$

Proof By variational analysis (see Proposition 3.1 in Chapter 3), the result can be obtained. □

Substituting (5.13) into (5.10) and combining (5.14), we have the following FBSDE

$$\begin{cases} d\bar{x}_i = [A\bar{x}_i - BR^{-1}B^\top p_i + C\hat{x} + F\bar{x}_0]dt + DdW_i, \\ dp_i = -(A^\top p_i + Q\bar{x}_i + \chi_1)dt + \zeta_0 dW_0 + \zeta_i dW_i, \\ x_i(0) = \xi_i, \quad p_i(T) = Gx_i(T) + \chi_2, \quad i = 1, 2, \cdots, N. \end{cases} \tag{5.15}$$

By taking limits, the above FBSDE can be rewritten as:

$$\begin{cases} d\hat{x} = [(A+C)\hat{x} + F\bar{x}_0 - BR^{-1}B^\top \hat{p}]dt, \ \hat{x}(0) = \hat{\xi}, \\ d\hat{p} = -(A^\top \hat{p} + Q\hat{x} + \chi_1)dt + \zeta_0 dW_0, \ \hat{p}(T) = G\hat{x}(T) + \chi_2. \end{cases} \tag{5.16}$$

5.2.3 The consistency condition of the follower problem

Let

$$\begin{cases} \Xi_1 := (I - \Theta^\top)Q(I - \Theta) + \alpha\Theta_0^\top Q_0\Theta_0, \ \Xi_1^G := (I - \hat{\Theta}^\top)G(I - \hat{\Theta}) + \alpha\hat{\Theta}_0^\top G_0\hat{\Theta}_0, \\ \Xi_2 := (I - \Theta^\top)Q\Theta_1 + \alpha\Theta_0^\top Q_0, \ \Xi_2^G := (I - \hat{\Theta}^\top)G\hat{\Theta}_1 + \alpha\hat{\Theta}_0^\top G_0, \\ \Xi_3 := (I - \Theta^\top)Q\eta - \alpha\Theta_0^\top Q_0\eta_0, \ \Xi_3^G := (I - \hat{\Theta}^\top)G\hat{\eta} - \alpha\hat{\Theta}_0^\top G_0\hat{\eta}_0, \\ \Xi_4 := \Theta_1^\top Q\Theta_1 + \alpha Q_0, \ \Xi_4^G := \hat{\Theta}_1^\top G\hat{\Theta}_1 + \alpha G_0, \\ \Xi_5 := \Theta_1^\top Q\eta - \alpha Q_0\eta_0, \ \Xi_5^G := \hat{\Theta}_1^\top G\hat{\eta} - \alpha G_0\hat{\eta}_0. \end{cases}$$

Combining (5.12) and (5.16), we can obtain the CC system

$$\begin{cases} d\hat{x} = [(A+C)\hat{x} + F\bar{x}_0 - BR^{-1}B^\top k_2]dt, \ \hat{x}(0) = \hat{\xi}, \\ d\bar{x}_0 = [A_0\bar{x}_0 + B_0u_0 + C_0\hat{x}]dt + D_0dW_0, \ \bar{x}_0(0) = \xi_0, \\ dk_1 = -[\Xi_4\bar{x}_0 - \Xi_2^\top\hat{x} + A_0^\top k_1 + F^\top k_2 + \Xi_5]dt + \beta_1 dW_0, \\ dk_2 = -[\Xi_1\hat{x} - \Xi_2\bar{x}_0 + C_0^\top k_1 + (A+C)^\top k_2 - \Xi_3]dt + \beta_2 dW_0, \\ k_1(T) = \Xi_4^G\bar{x}_0(T) - (\Xi_2^G)^\top\hat{x}(T) + \Xi_5^G, \ k_2(T) = \Xi_1^G\hat{x}(T) - \Xi_2^G\bar{x}_0(T) - \Xi_3^G, \end{cases} \tag{5.17}$$

where $\hat{p} = k_2$ can be easily verified.

5.3 The Optimal Control Problem for the Leader

Now, let (**P5.2**) have a unique solution. Then, for $u_0 \in \mathcal{U}_0$ given by the leader, the followers choose their optimal control $\bar{u} = \mathcal{M}(u_0) = \{\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_N\} \in \mathcal{U}$, where $\bar{u}_i$ is shown in (5.13). Now we consider the optimal control of the leader to further minimize the social cost functional. In the infinite population system, $x^{(N)}$ may be approximated by $\hat{x}$. Hence, we can construct the following auxiliary optimal control problem for the leader.

Problem 5.3. (**P5.3**) *Minimize* $\hat{\mathcal{J}}_{soc}^{(N)}(u_0; \bar{u})$ *over* $u_0 \in \mathcal{U}_0$, *where*

$$\begin{aligned} &dx_0 = [A_0 x_0 + B_0 u_0 + C_0 \hat{x}]dt + D_0 dW_0, \quad x_0(0) = \xi_0, \\ &\hat{\mathcal{J}}_{soc}^{(N)}(u_0; \bar{u}) = \alpha N \hat{\mathcal{J}}_0(u_0; \bar{u}) + \sum_{i=1}^{N} \hat{\mathcal{J}}_i(u_0; \bar{u}). \end{aligned} \tag{5.18}$$

(**P5.3**) *is based on* (**P5.2**). *Therefore, combining* (5.13), (5.3), *and* (5.4) *with mean field approximations, the cost functionals of the leader* $\mathcal{A}_0$ *and the follower* $\mathcal{A}_i$ *are*

$$\begin{aligned} \hat{\mathcal{J}}_0(u_0; \bar{u}) =& \mathbb{E}\Big\{ \int_0^T \langle Q_0 \Psi_1, \Psi_1 \rangle + \langle R_0 u_0, u_0 \rangle dt + \langle G_0 \Psi_4, \Psi_4 \rangle \Big\}, \\ \hat{\mathcal{J}}_i(u_0; \bar{u}) =& \mathbb{E}\Big\{ \int_0^T \langle Q \Psi_2^i, \Psi_2^i \rangle + \langle B^\top p_i, R^{-1} B^\top p_i \rangle dt + \langle G \Psi_5^i, \Psi_5^i \rangle \Big\}, \end{aligned}$$

where $\hat{x}$, $\bar{x}_0$, k_1, k_2, $\bar{x}_i$, p_i *are determined by* (5.17) *and* (5.15).

We let $\bar{u}_0$ be the optimal strategy of the leader and perturb u_0 in (5.18), where $\delta u_0 = u_0 - \bar{u}_0$. Since $\bar{x}_0$, $\hat{x}$, $\bar{x}_i$ and p_i are determined by u_0, we denote their corresponding perturbations as: $\delta \bar{x}_0 = \bar{x}_0(u_0) - \bar{x}_0(\bar{u}_0)$, $\delta \hat{x} = \hat{x}(u_0) - \hat{x}(\bar{u}_0)$, $\delta \bar{x}_i = \bar{x}_i(u_0) - \bar{x}_i(\bar{u}_0)$ and $\delta p_i = p_i(u_0) - p_i(\bar{u}_0)$. For sake of notation simplicity, we drop $(\bar{u}_0)$ in the following $\bar{x}_0(\bar{u}_0)$, $\hat{x}(\bar{u}_0)$, $\bar{x}_i(\bar{u}_0)$ and $p_i(\bar{u}_0)$, etc. Then, one can obtain

$$d\delta \bar{x}_0 = [A_0 \delta \bar{x}_0 + B_0 \delta u_0 + C_0 \delta \hat{x}]dt, \ \delta \bar{x}_0(0) = 0,$$

and the variations of corresponding cost functionals

$$\frac{1}{2}\delta\hat{\mathcal{J}}_0(\delta u_0) =\mathbb{E}\bigg\{ \int_0^T \langle Q_0\Psi_1, \delta\bar{x}_0 - \Theta_0\delta\hat{x}\rangle + \langle R_0\bar{u}_0, \delta u_0\rangle dt$$

$$+ \langle G_0\Psi_4, \delta\bar{x}_0(T) - \hat{\Theta}_0\delta\hat{x}(T)\rangle \bigg\},$$

and

$$\frac{1}{2}\sum_{i=1}^N \delta\hat{\mathcal{J}}_i(\delta u_0) = \sum_{i=1}^N \mathbb{E}\bigg\{ \int_0^T \langle Q\Psi_2^i, \delta\bar{x}_i - \Theta\delta\hat{x} - \Theta_1\delta\bar{x}_0\rangle + \langle R^{-1}B^\top p_i, B^\top\delta p_i\rangle dt$$

$$+ \langle G\Psi_5^i, \delta\bar{x}_i(T) - \hat{\Theta}\delta\hat{x}(T) - \hat{\Theta}_1\delta\bar{x}_0(T)\rangle \bigg\}.$$

Here Ψ_1, Ψ_2^i, $\Psi_4(T)$, $\Psi_5^i(T)$, are related to $\bar{u}_0$, and, in what follows, Ψ_1, Ψ_2^i, Ψ_3, $\Psi_4(T)$, $\Psi_5^i(T)$, $\Psi_6(T)$ will be related to $\bar{u}_0$. Then, the variation of the social cost functional is

$$\begin{aligned}\frac{1}{2}\delta\hat{\mathcal{J}}_{soc}^{(N)}(\delta u_0) =&\alpha N\mathbb{E}\int_0^T \langle Q_0\Psi_1, \delta\bar{x}_0\rangle - \langle \Theta_0^\top Q_0\Psi_1, \delta\hat{x}\rangle + \langle R_0\bar{u}_0, \delta u_0\rangle dt \\ &+ \sum_{i=1}^N \mathbb{E}\int_0^T \langle Q\Psi_2^i, \delta\bar{x}_i\rangle - \langle \Theta^\top Q\Psi_2^i, \delta\hat{x}\rangle - \langle \Theta_1^\top Q\Psi_2^i, \delta\bar{x}_0\rangle \\ &+ \langle BR^{-1}B^\top p_i, \delta p_i\rangle dt + \alpha N\langle G_0\Psi_4(T), \delta\bar{x}_0(T)\rangle \\ &- \alpha N\langle \hat{\Theta}_0^\top G_0\Psi_4(T), \delta\hat{x}(T)\rangle - \sum_{i=1}^N \langle \hat{\Theta}_1^\top G\Psi_5^i(T), \delta\bar{x}_0(T)\rangle \\ &- \sum_{i=1}^N \langle \hat{\Theta}^\top G\Psi_5^i(T), \delta\hat{x}(T)\rangle + \sum_{i=1}^N \langle G\Psi_5^i(T), \delta\bar{x}_i(T)\rangle.\end{aligned} \tag{5.19}$$

Similarly, the variations of those equations in (5.15) and (5.17) are given by

$$\begin{cases} d\delta\bar{x}_i =[A\delta\bar{x}_i - BR^{-1}B^\top\delta p_i + C\delta\hat{x} + F\delta\bar{x}_0]dt, \quad \delta\bar{x}_i(0) = 0, \quad i = 1, 2, \cdots, N, \\ d\delta p_i = -(A^\top\delta p_i + Q\delta x_i + [\Xi_1 - Q]\delta\hat{x} - \Xi_2\delta\bar{x}_0 + C_0^\top\delta k_1 + C^\top\delta k_2)dt \\ \qquad + \delta\zeta_0 dW_0 + \delta\zeta_i dW_i, \quad \delta p_i(T) = G\delta x_i(T) + [\Xi_1^G - G]\delta\hat{x}(T) - \Xi_2^G\delta\bar{x}_0(T), \end{cases}$$

and

$$\begin{cases} d\delta\hat{x} = [(A+C)\delta\hat{x} + F\delta\bar{x}_0 - BR^{-1}B^\top\delta k_2]dt, \quad \delta\hat{x}(0) = 0, \\ d\delta k_1 = -[\Xi_4\delta\bar{x}_0 - \Xi_2^\top\delta\hat{x} + A_0^\top\delta k_1 + F^\top\delta k_2]dt + \delta\beta_1 dW_0, \\ d\delta k_2 = -[\Xi_1\delta\hat{x} - \Xi_2\delta\bar{x}_0 + C_0^\top\delta k_1 + (A+C)^\top\delta k_2]dt + \delta\beta_2 dW_0, \\ \delta k_1(T) = \Xi_4^G\delta\bar{x}_0(T) - (\Xi_2^G)^\top\delta\hat{x}(T), \quad \delta k_2(T) = \Xi_1^G\delta\hat{x}(T) - \Xi_2^G\delta\bar{x}_0(T). \end{cases}$$

Since (5.19) contains many terms that we do not want them to appear in the social cost functional, we will use a similar argument in the last section to get off the dependence of $\delta\hat{\mathcal{J}}_{soc}^{(N)}(\delta u_0)$ on those terms. Therefore, we need the following auxiliary equations

$$\begin{cases} dq_i = m_i dt + n_i^0 dW_0 + n_i dW_i, \quad q_i(0) = 0, \quad i = 1, 2, \cdots, N, \\ dl_1 = s_1 dt + r_1 dW_0, \quad l_1(0) = 0, \\ dl_2 = s_2 dt + r_2 dW_0, \quad l_2(0) = 0. \end{cases}$$

where

$$\begin{cases} m_i = -(BR^{-1}B^\top p_i - BR^{-1}B^\top y_i - Aq_i), \\ s_1 = C_0 l_2 + A_0 l_1 - C_0 q_i, \quad s_2 = (A+C)l_2 - BR^{-1}B^\top\hat{y}^i + Fl_1 - Cq_i, \\ n_i = 0, \quad n_i^0 = 0, \quad r_1 = 0, \quad r_2 = 0, \end{cases}$$

and

$$\begin{cases} d\hat{y}^i = \hat{\alpha}dt + \hat{\beta}dW_0 + \sum_{i=1}^N \hat{\beta}_i dW_i, \\ \hat{y}^i(T) = \alpha\hat{\Theta}_0^\top G_0\Psi_4(T) + \hat{\Theta}^\top G\Psi_5^i(T) - (\Xi_2^G)^\top l_1(T) + (\Xi_1^G)^\top l_2(T) + (\Xi_1^G - G)^\top q_i(T), \\ dy_0^i = \hat{\alpha}_0 dt + \hat{\beta}_0 dW_0 + \sum_{i=1}^N \hat{\beta}_i^0 dW_i, \\ y_0^i(T) = \alpha G_0\Psi_4(T) - \hat{\Theta}_1^\top G\Psi_5^i(T) - (\Xi_4^G)^\top l_1(T) + (\Xi_2^G)^\top l_2(T) + (\Xi_2^G)^\top q_i(T), \\ dy_i = \alpha_i dt + \beta_0 dW_0 + \beta_i dW_i, \quad y_i(T) = G\Psi_5^i(T) - Gq_i(T), \end{cases}$$

where

$$\begin{cases} \hat{\alpha}_0 = -\left(\alpha Q_0 \Psi_1 - \Theta_1^\top Q \Psi_2^i + F^\top y_i - F^\top \hat{y}^i + A_0^\top y_0^i - \Xi_4^\top l_1 + \Xi_2^\top l_2 + \Xi_2^\top q_i\right), \\ \hat{\alpha} = -\alpha \Theta_0^\top Q_0 \Psi_1 - \Theta^\top Q \Psi_2^i + C^\top y_i - (A+C)^\top \hat{y}^i + C_0^\top y_0^i + \Xi_2 l_1 \\ \qquad - \Xi_1^\top l_2 - (\Xi_1 - Q)^\top q_i, \\ \alpha_i = -\left(Q \Psi_2^i + A^\top y_i - Q^\top q_i\right), \end{cases}$$

to help us obtain the optimal control of the leader. Here q_i, l_1, l_2, $\hat{y}^i$, y_0^i, and y_i are used to free $\delta \hat{\mathcal{J}}_{soc}^{(N)}(\delta u_0)$ from the dependence on p_i, k_1, k_2, $\delta \hat{x}$, $\delta \bar{x}_0$, and $\delta \bar{x}_i$, respectively.

Similarly, by Itô formula and the duality relations, the variation of the social cost functional can be derived as follows:

$$\frac{1}{2} \delta \hat{\mathcal{J}}_{soc}^{(N)}(\delta u_0) = \mathbb{E} \int_0^T \langle \alpha N R_0 \bar{u}_0 + \sum_{i=1}^N B_0^\top y_0^i, \delta u_0 \rangle dt.$$

Thus, letting $\frac{1}{2} \delta \hat{\mathcal{J}}_{soc}^{(N)}(\delta u_0) = 0$ is equivlant to

$$\alpha N R_0 \bar{u}_0 + \sum_{i=1}^N B_0^\top y_0^i = 0.$$

Then, we have the centralized form of the optimal control for the leader

$$\bar{u}_0 = -\frac{1}{\alpha N} R_0^{-1} B_0^\top \sum_{i=1}^N y_0^i := u_0^{(N)}, \tag{5.20}$$

where $\bar{u}_0$ relies on N and the following FBSDE

$$\begin{cases} dy_i = -(A^\top y_i - Q^\top q_i + Q\Psi_2^i)dt + \beta_0 dW_0 + \beta_i dW_i, \quad y_i(T) = G\Psi_5^i(T) - Gq_i(T), \\ dq_i = (BR^{-1}B^\top y_i + Aq_i - BR^{-1}B^\top p_i)dt, \quad q_i(0) = 0, \quad i = 1, 2, \cdots, N, \\ d\hat{y}^i = (-\alpha\Theta_0^\top Q_0\Psi_1 - \Theta^\top Q\Psi_2^i + C^\top y_i - (A+C)^\top \hat{y}^i + C_0^\top y_0^i + \Xi_2 l_1 - \Xi_1^\top l_2 \\ \qquad - (\Xi_1 - Q)^\top q_i)dt + \hat{\beta} dW_0 + \sum_{i=1}^N \hat{\beta}_i dW_i, \\ \hat{y}^i(T) = \alpha\hat{\Theta}_0^\top G_0\Psi_4(T) + \hat{\Theta}^\top G\Psi_5^i(T) - (\Xi_2^G)^\top l_1(T) + (\Xi_1^G)^\top l_2(T) \\ \qquad + (\Xi_1^G - G)^\top q_i(T), \\ dy_0^i = -(\alpha Q_0\Psi_1 - \Theta_1^\top Q\Psi_2^i + F^\top y_i - F^\top \hat{y}^i + A_0^\top y_0^i - \Xi_4^\top l_1 + \Xi_2^\top l_2 + \Xi_2^\top q_i)dt \\ \qquad + \hat{\beta}_0 dW_0 + \sum_{i=1}^N \hat{\beta}_i^0 dW_i, \\ y_0^i(T) = \alpha G_0\Psi_4(T) - \hat{\Theta}_1^\top G\Psi_5^i(T) - (\Xi_4^G)^\top l_1(T) + (\Xi_2^G)^\top l_2(T) + (\Xi_2^G)^\top q_i(T), \\ dl_1 = (A_0 l_1 + C_0 l_2 - C_0 q_i)dt, \quad l_1(0) = 0, \\ dl_2 = [Fl_1 + (A+C)l_2 - BR^{-1}B^\top \hat{y}^i - Cq_i]dt, \quad l_2(0) = 0. \end{cases} \tag{5.21}$$

Denote

$$\begin{cases} y^* = \lim_{N\to+\infty} \frac{1}{N}\sum_{i=1}^N y_i, \quad \hat{y}^* = \lim_{N\to+\infty} \frac{1}{N}\sum_{i=1}^N \hat{y}^i, \quad y_0^* = \lim_{N\to+\infty} \frac{1}{N}\sum_{i=1}^N y_0^i, \\ q^* = \lim_{N\to+\infty} \frac{1}{N}\sum_{i=1}^N q_i, \quad l_1^* = \lim_{N\to+\infty} \frac{1}{N}\sum_{i=1}^N l_1, \quad l_2^* = \lim_{N\to+\infty} \frac{1}{N}\sum_{i=1}^N l_2. \end{cases}$$

Here, using a similar argument of (5.6), we can easily prove that $\frac{1}{N}\sum_{i=1}^N y_i$, $\frac{1}{N}\sum_{i=1}^N \hat{y}^i$, $\frac{1}{N}\sum_{i=1}^N y_0^i$, $\frac{1}{N}\sum_{i=1}^N q_i$, $\frac{1}{N}\sum_{i=1}^N l_1$ and $\frac{1}{N}\sum_{i=1}^N l_2$ converge to y^*, $\hat{y}^*$, y_0^*, q^*, l_1^* and l_2^*, respectively. Thus, combining (5.17) and (5.21), when $N \to \infty$, we can obtain the

CC system for the LF MFT problem

$$
\begin{cases}
d\hat{x} = [(A+C)\hat{x} + F\bar{x}_0 - BR^{-1}B^\top k_2]dt, \quad \hat{x}(0) = \hat{\xi}, \\
d\bar{x}_0 = [A_0\bar{x}_0 + C_0\hat{x} - B_0(\alpha R_0)^{-1}B_0^\top y_0^*]dt + D_0 dW_0, \quad \bar{x}_0(0) = \xi_0, \\
dk_1 = -[\Xi_4\bar{x}_0 - \Xi_2^\top\hat{x} + A_0^\top k_1 + F^\top k_2 + \Xi_5]dt + \beta_1 dW_0, \\
dk_2 = -[\Xi_1\hat{x} - \Xi_2\bar{x}_0 + C_0^\top k_1 + (A+C)^\top k_2 - \Xi_3]dt + \beta_2 dW_0, \\
k_1(T) = \Xi_4^G\bar{x}_0(T) - (\Xi_2^G)^\top\hat{x}(T) + \Xi_5^G, \quad k_2(T) = \Xi_1^G\hat{x}(T) - \Xi_2^G\bar{x}_0(T) - \Xi_3^G, \\
dy^* = -(A^\top y^* - Q^\top q^* + Q\Psi_3)dt + \beta^* dW_0, \quad y^*(T) = G\Psi_6(T) - Gq^*(T), \\
dq^* = (BR^{-1}B^\top y^* + Aq^* - BR^{-1}B^\top k_2)dt, \quad q^*(0) = 0, \\
d\hat{y}^* = [-\alpha\Theta_0^\top Q_0\Psi_1 - \Theta^\top Q\Psi_3 + C^\top y^* - (A+C)^\top\hat{y}^* + C_0^\top y_0^* + \Xi_2 l_1^* - \Xi_1^\top l_2^* \\
\qquad - (\Xi_1 - Q)^\top q^*]dt + \hat{\beta}^* dW_0, \\
\hat{y}^*(T) = \alpha\hat{\Theta}_0^\top G_0\Psi_4(T) + \hat{\Theta}^\top G\Psi_6(T) - (\Xi_2^G)^\top l_1^*(T) + (\Xi_1^G)^\top l_2^*(T) \\
\qquad + (\Xi_1^G - G)^\top q^*(T), \\
dy_0^* = -(\alpha Q_0\Psi_1 - \Theta_1^\top Q\Psi_3 + F^\top y^* - F^\top\hat{y}^* + A_0^\top y_0^* - \Xi_4^\top l_1^* + \Xi_2^\top l_2^* + \Xi_2^\top q^*)dt \\
\qquad + \hat{\beta}_0^* dW_0, \\
y_0^*(T) = \alpha G_0\Psi_4(T) - \hat{\Theta}_1^\top G\Psi_6(T) - (\Xi_4^G)^\top l_1^*(T) + (\Xi_2^G)^\top l_2^*(T) + (\Xi_2^G)^\top q^*(T), \\
dl_1^* = (A_0 l_1^* + C_0 l_2^* - C_0 q^*)dt, \quad l_1^*(0) = 0, \\
dl_2^* = [Fl_1^* + (A+C)l_2^* - BR^{-1}B^\top\hat{y}^* - Cq^*]dt, \quad l_2^*(0) = 0,
\end{cases}
\tag{5.22}
$$

and the decentralized optimal control for the leader

$$
u_0^* = -(\alpha R_0)^{-1}B_0^\top y_0^*. \tag{5.23}
$$

The final CC system is highly coupled with five forward equations and five backward equations. The existence and uniqueness of (5.22) are very important for obtaining the optimal control, however, it is very difficult to solve such a high-dimensional

system. We need to simplify the CC system to an FBSDE using block matrices and these will be discussed in the next section.

5.4 Well-Posedness of the CC System

Note that in (5.22), the equations of $(\hat{x}, \bar{x}_0, k_1, k_2)$ form a coupled FBSDE and $(y^*, q^*, \hat{y}^*, y_0^*, l_1^*, l_2^*)$ form another coupled FBSDE. The two FBSDEs are also fully coupled with each other. Therefore, we try to look at the above FBSDEs differently. To this end, we set

$$\mathbb{X} = \begin{pmatrix} \hat{x} \\ \bar{x}_0 \\ q^* \\ l_1^* \\ l_2^* \end{pmatrix}, \ \mathbb{Y} = \begin{pmatrix} y^* \\ \hat{y}^* \\ y_0^* \\ k_1 \\ k_2 \end{pmatrix}, \ \mathbb{X}(0) = \begin{pmatrix} \hat{\xi} \\ \xi_0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbb{Y}(T) = \begin{pmatrix} G\Psi_6 - Gq^*(T) \\ \alpha\hat{\Theta}_0^\top G_0\Psi_4 - \hat{\Theta}^\top G\Psi_6 - (\Xi_2^G)^\top l_1^*(T) + (\Xi_1^G)^\top l_2^*(T) + (\Xi_1^G - G)^\top q^*(T) \\ \alpha G_0\Psi_4 - \hat{\Theta}_1^\top G\Psi_6 - (\Xi_4^G)^\top l_1^*(T) + (\Xi_2^G)^\top l_2^*(T) + (\Xi_2^G)^\top q^*(T) \\ \Xi_4^G\bar{x}_0(T) - (\Xi_2^G)^\top\hat{x}(T) + \Xi_5^G \\ \Xi_1^G\hat{x}(T) - \Xi_2^G\bar{x}_0(T) - \Xi_3^G \end{pmatrix}.$$

Then (5.22) is equivalent to

$$\begin{cases} d\mathbb{X} = [\mathbb{A}\mathbb{X} + \mathbb{B}\mathbb{Y} + b]dt + \mathbb{D}dW_0, \quad \mathbb{X}(0) = (\hat{\xi}^\top \quad \xi_0^\top \quad 0 \quad 0 \quad 0)^\top, \\ d\mathbb{Y} = [\hat{\mathbb{A}}\mathbb{X} + \hat{\mathbb{B}}\mathbb{Y} + \hat{b}]dt + \hat{\mathbb{D}}dW_0, \quad \mathbb{Y}(T) = \mathbb{G}\mathbb{X}(T) + g, \end{cases} \tag{5.24}$$

with

$$\mathbb{A} = \begin{pmatrix} A+C & F & 0 & 0 & 0 \\ C_0 & A_0 & 0 & 0 & 0 \\ 0 & 0 & A & 0 & 0 \\ 0 & 0 & -C_0 & A_0 & C_0 \\ 0 & 0 & -C & F & (A+C) \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbb{D} = \begin{pmatrix} 0 \\ D_0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbb{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & -BR^{-1}B^\top \\ 0 & 0 & -B_0(\alpha R_0)^{-1}B_0^\top & 0 & 0 \\ BR^{-1}B^\top & 0 & 0 & 0 & -BR^{-1}B^\top \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -BR^{-1}B^\top & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{\mathbb{A}} = \begin{pmatrix} -Q(I-\Theta) & Q\Theta_1 & Q^\top & 0 & 0 \\ \Xi_1 - Q(I-\Theta) & -\Xi_2 + Q\Theta_1 & -(\Xi_1 - Q)^\top & \Xi_2 & -\Xi_1^\top \\ \Xi_2^\top & -\Xi_4 & -\Xi_2^\top & \Xi_4^\top & -\Xi_2^\top \\ \Xi_2^\top & -\Xi_4 & 0 & 0 & 0 \\ -\Xi_1 & \Xi_2 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{b} = \begin{pmatrix} Q\eta \\ -\Xi_3 + Q\eta \\ -\Xi_5 \\ -\Xi_5 \\ \Xi_3 \end{pmatrix}, \quad \hat{\mathbb{D}} = \begin{pmatrix} \beta^* \\ \hat{\beta}^* \\ \hat{\beta}_0^* \\ \beta_1 \\ \beta_2 \end{pmatrix}, \quad g = \begin{pmatrix} -G\hat{\eta} \\ \Xi_3^G - G\hat{\eta} \\ \Xi_5^G \\ \Xi_5^G \\ -\Xi_3^G \end{pmatrix},$$

$$\hat{\mathbb{B}} = \begin{pmatrix} -A^\top & 0 & 0 & 0 & 0 \\ C^\top & -(A+C)^\top & C_0^\top & 0 & 0 \\ -F^\top & F^\top & -A_0^\top & 0 & 0 \\ 0 & 0 & 0 & -A_0^\top & -F^\top \\ 0 & 0 & 0 & -C_0^\top & -(A+C)^\top \end{pmatrix},$$

$$\mathbb{G} = \begin{pmatrix} G(I-\hat{\Theta}) & -G\hat{\Theta}_1 & -G & 0 & 0 \\ -\Xi_1^G + G(I-\hat{\Theta}) & \Xi_2^G - G\hat{\Theta}_1 & (\Xi_1^G - G)^\top & -(\Xi_2^G)^\top & (\Xi_1^G)^\top \\ -(\Xi_2^G)^\top & \Xi_4^G & (\Xi_2^G)^\top & -(\Xi_4^G)^\top & (\Xi_2^G)^\top \\ -(\Xi_2^G)^\top & \Xi_4^G & 0 & 0 & 0 \\ \Xi_1^G & -\Xi_2^G & 0 & 0 & 0 \end{pmatrix}.$$

Let

$$\bar{\mathbb{A}} = \begin{pmatrix} \mathbb{A} + \mathbb{B}\mathbb{G} & \mathbb{B} \\ \hat{\mathbb{A}} - \mathbb{G}\mathbb{A} + \hat{\mathbb{B}}\mathbb{G} - \mathbb{G}\hat{\mathbb{B}}\mathbb{G} & \hat{\mathbb{B}} - \mathbb{G}\mathbb{B} \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} b \\ \hat{b} - \mathbb{G}b \end{pmatrix},$$

$$\bar{\mathbb{D}} = \begin{pmatrix} \mathbb{D} \\ \hat{\mathbb{D}} - \mathbb{G}\mathbb{D} \end{pmatrix}, \quad \bar{\mathbb{Y}} = \mathbb{Y} - \mathbb{G}\mathbb{X},$$

then (5.24) can be rewritten as:

$$\begin{cases} d\begin{pmatrix} \mathbb{X} \\ \bar{\mathbb{Y}} \end{pmatrix} = \left\{ \bar{\mathbb{A}} \begin{pmatrix} \mathbb{X} \\ \bar{\mathbb{Y}} \end{pmatrix} + \bar{b} \right\} dt + \bar{\mathbb{D}} dW_0, \\ \mathbb{X}(0) = (\hat{\xi}^\top \quad \xi_0^\top \quad 0 \quad 0 \quad 0)^\top, \quad \bar{\mathbb{Y}}(T) = g. \end{cases} \tag{5.25}$$

This is a fully coupled FBSDE. By the Theorem 3.7 in [133, Chapter 2], the FBSDE (5.25) is solvable for all $g \in L^2_{\mathbb{F}}(\Omega;\mathbb{R}^{5n})$ if and only if the following condition holds:

$$\det\left\{(0,I)e^{\bar{\mathbb{A}}t}\begin{pmatrix}0\\ I\end{pmatrix}\right\} > 0, \quad \forall\, t \in [0,T]. \tag{5.26}$$

In the case, (5.24) admits an unique solution for any given $g \in L^2_{\mathbb{F}}(\Omega;\mathbb{R}^{5n})$.

Under the condition (5.26), we may decouple the FBSDE (5.25) by

$$\bar{\mathbb{Y}}(t) = \mathbb{K}(t)\mathbb{X}(t) + \kappa(t), \qquad t \in [0,T],$$

where $\mathbb{K} \in C^1(0,T;\mathbb{S}^{5n})$ is a solution of the following Ricatti equation

$$\begin{cases}\dot{\mathbb{K}} + \mathbb{K}(\mathbb{A}+\mathbb{B}\mathbb{G}) + \mathbb{K}\mathbb{B}\mathbb{K} - (\hat{\mathbb{B}} - \mathbb{G}\mathbb{B})\mathbb{K} - (\hat{\mathbb{A}} - \mathbb{G}\mathbb{A} + \hat{\mathbb{B}}\mathbb{G} - \mathbb{G}\hat{\mathbb{B}}\mathbb{G}) = 0,\\ \mathbb{K}(T) = 0, \quad t \in [0,T],\end{cases}$$

and $\kappa \in C^1(0,T;\mathbb{R}^{5n})$ satisfies

$$\dot{\kappa} + (\mathbb{K}\mathbb{B} - (\hat{\mathbb{B}} - \mathbb{G}\mathbb{B}))\kappa + \mathbb{K}b - (\hat{b} - \mathbb{G}b) = 0,\ t \in [0,T], \quad \kappa(T) = g. \tag{5.27}$$

By the Theorem 3.7 and Theorem 4.3 in [133, Chapter 2], if (5.26) hold, then the Ricatti equation admits a unique solution $\mathbb{K}(\cdot)$ with the following representation:

$$\mathbb{K}(t) = -\left[(0,I)e^{\bar{\mathbb{A}}(T-t)}\begin{pmatrix}0\\ I\end{pmatrix}\right]^{-1}\left[(0,I)e^{\bar{\mathbb{A}}(T-t)}\begin{pmatrix}I\\ 0\end{pmatrix}\right], \quad t \in [0,T]. \tag{5.28}$$

Example 5.1. *Consider the system* (5.25) *with parameters* $A_0 = 0.1$, $B_0 = 1$, $C_0 = 0.01$, $D_0 = 1$, $A = 0.05$, $B = 1$, $C = 0.05$, $D = 1$, $F = 0.3$, $\Theta_0 = 1$, $Q_0 = 1$, $R_0 = 10$, $G_0 = 0$, $\Theta = 0.1$, $\Theta_1 = 1$, $Q = 0.9$, $R = 15$, $G = 0$, $\alpha = 1.02$, $T = 12$,

$\eta_0 = \eta = 0$. *Then, we have*

$$\mathbb{A} = \begin{pmatrix} 0.10 & 0.30 & 0 & 0 & 0 \\ 0.01 & 0.10 & 0 & 0 & 0 \\ 0 & 0 & 0.05 & 0 & 0 \\ 0 & 0 & -0.01 & 0.10 & 0.01 \\ 0 & 0 & -0.05 & 0.30 & 0.10 \end{pmatrix}, \mathbb{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & -0.0667 \\ 0 & 0 & -0.0980 & 0 & 0 \\ 0.0667 & 0 & 0 & 0 & -0.0667 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -0.0667 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{\mathbb{A}} = \begin{pmatrix} -0.81 & 0.90 & 0.90 & 0 & 0 \\ 0.939 & -0.93 & -2.649 & 1.83 & -1.749 \\ 1.83 & -1.92 & -1.83 & 1.92 & -1.83 \\ 1.83 & -1.92 & 0 & 0 & 0 \\ -1.749 & 1.83 & 0 & 0 & 0 \end{pmatrix}, \hat{\mathbb{B}} = \begin{pmatrix} -0.05 & 0 & 0 & 0 & 0 \\ 0.05 & -0.10 & 0.01 & 0 & 0 \\ -0.30 & 0.30 & -0.10 & 0 & 0 \\ 0 & 0 & 0 & -0.10 & -0.30 \\ 0 & 0 & 0 & -0.01 & -0.10 \end{pmatrix}.$$

Hence, according to the simulation through Matlab software, for any $t \in [0, T]$, *we obtain*

$$\bar{\mathbb{A}} = \begin{pmatrix} \mathbb{A} + \mathbb{B}\mathbb{G} & \mathbb{B} \\ \hat{\mathbb{A}} - \mathbb{G}\mathbb{A} + \hat{\mathbb{B}}\mathbb{G} - \mathbb{G}\hat{\mathbb{B}}\mathbb{G} & \hat{\mathbb{B}} - \mathbb{G}\mathbb{B} \end{pmatrix}, \qquad \det\left\{(0, I)e^{\bar{\mathbb{A}}t}\begin{pmatrix} 0 \\ I \end{pmatrix}\right\} > 0,$$

(e.g. for $t = 6$, $\det\left\{(0, I)e^{\bar{\mathbb{A}}t}\begin{pmatrix} 0 \\ I \end{pmatrix}\right\} = 12.7053 > 0$*). By the argument above,* (5.24) *is solvable.*

For further analysis, we make the following assumption:

(**A5.5**) The equation (5.25) has a unique solution and the solution $(\mathbb{X}, \bar{\mathbb{Y}}, \bar{\mathbb{D}})$ belongs to $\mathcal{M}[0, T]$.

For the following equation

$$\begin{cases} d\bar{x}_i = [A\bar{x}_i - BR^{-1}B^\top p_i + C\hat{x} + F\bar{x}_0]dt + DdW_i, \ x_i(0) = \xi_i, \ i = 1, 2, \cdots, N, \\ dp_i = -[A^\top p_i + Q\bar{x}_i + \chi_1]dt + \zeta_0 dW_0 + \zeta_i dW_i, \ p_i(T) = Gx_i(T) + \chi_2, \end{cases} \tag{5.29}$$

where χ_1 and χ_2 are related to $\bar{u}_0$. We let $p_i = \bar{P}\bar{x}_i + \bar{\varphi}$, $t \in [0, T]$, where $\bar{P} \in C^1(0, T; \mathbb{S}^n)$ is a solution of the following Ricatti equation and $\bar{\varphi} \in C^1(0, T; \mathbb{R}^n)$ satisfies

$$\begin{cases} \dot{\bar{P}} + \bar{P}A - \bar{P}BR^{-1}B^\top\bar{P} + A^\top\bar{P} + Q = 0, \ t \in [0, T], \ \bar{P}(T) = G, \\ \dot{\bar{\varphi}} + (A^\top - \bar{P}BR^{-1}B^\top)\bar{\varphi} + \chi_1 + \bar{P}C\hat{x} + \bar{P}F\bar{x}_0 = 0, \ t \in [0, T], \ \bar{\varphi}(T) = \chi_2. \end{cases}$$

Since the Ricatti equation is standard, it has a unique solution. Hence, (5.29) is uniquely solvable and the solution belongs to $\mathcal{M}[0,T]$.

5.5 Asymptotically Social Optimality

In this section, we discuss that if the leader announces u_0^* obtained in (5.23) to the N followers, then the set of the optimal decentralized controls for the leader and the followers will constitute an approximated Stackelberg equilibrium. First, for the open-loop decentralized strategy (u_0^*, u^*) in (5.23) and (5.13), we have the realized decentralized state x_0^* and x_i^*, satisfies

$$\begin{cases} dx_0^*(t) = [A_0 x_0^*(t) - B_0(\alpha R_0)^{-1} B_0^\top y_0^*(t) + C_0 (x^*)^{(N)}(t)]dt + D_0 dW_0(t), \\ dx_i^*(t) = [A x_i^*(t) - BR^{-1}B^\top p_i(t) + C(x^*)^{(N)}(t) + F x_0^*(t)]dt + D dW_i(t), \\ x_0^*(0) = \xi_0, \qquad x_i^*(0) = \xi_i, \quad i = 1, 2, \cdots, N, \end{cases} \tag{5.30}$$

where y_0^*, p_i satisfy (5.22) and (5.29), respectively. Then, by [16] and [138], we give the definition of the asymptotic Stackelberg equilibrium.

Definition 5.1. *A set of control laws $\mathcal{M}(\check{u}_0) \in \mathcal{U}$ has asymptotic social optimality if*

$$\left| \frac{1}{N} \mathcal{J}_{soc}^{(N)}(\check{u}_0; \mathcal{M}(\check{u}_0)) - \frac{1}{N} \inf_{(\check{u}_0, \check{u}) \in \mathcal{U}_c} \mathcal{J}_{soc}^{(N)}(\check{u}_0; \check{u}) \right| = O(\frac{1}{\sqrt{N}}),$$

where $\mathcal{M}$ is a mapping and $\mathcal{M} : \mathcal{U}_0 \to \mathcal{U}$. $\mathcal{U}_c$ is a set of centralized information-based control.

Definition 5.2. *A set of control laws $(u_0^*, u^*) \in \mathcal{U}_0 \times \mathcal{U}$, where $u^* = \mathcal{M}(u_0^*)$, is an asymptotic Stackelberg equilibrium with respect to $\mathcal{J}_{soc}^{(N)}(u_0, u)$ if the following two properties hold:*

1. *$\mathcal{M}(\check{u}_0)$ has an asymptotic social optimality under $\check{u}_0$.*

2. The following equation is satisfied

$$\left|\frac{1}{N}\mathcal{J}_{soc}^{(N)}(u_0^*;\mathcal{M}(u_0^*)) - \frac{1}{N}\inf_{\check{u}_0\in\mathcal{U}_c}\mathcal{J}_{soc}^{(N)}(\check{u}_0;\mathcal{M}(\check{u}_0))\right| = O(\frac{1}{\sqrt{N}}).$$

We first need to introduce some lemmas before proving the asymptotic Stackelberg equilibrium. In what follows, the value of K may be different at different places and it only depends on the coefficients and initial values.

Lemma 5.1. *Assume that* (**A5.1**)-(**A5.5**) *hold. Then*

$$\mathbb{E}\int_0^T |(x^*)^{(N)} - \hat{x}|^2 dt + \mathbb{E}\int_0^T |p^{(N)} - \hat{p}|^2 dt + \mathbb{E}\int_0^T |x_0^* - \bar{x}_0|^2 dt = O(\frac{1}{N}).$$

Proof The proof is similar to Lemma 4.2 in Chapter 4. For the detail proof, readers may refer to [101, Appendix A]. □

Lemma 5.2. *Assume that* (**A5.1**)-(**A5.5**) *hold. There exists a constant K, which is independent of N, such that*

$$\mathcal{J}_{soc}^{(N)}(u_0^*;u^*) \leq NK.$$

Proof See [101, Appendix B]. □

Proposition 5.2. *Assume that* (**A5.1**)-(**A5.5**) *hold. For all $(\check{u}_0;\check{u}) \in \mathcal{U}_c$, there exists a constant K, which is independent of N, such that*

$$\alpha N|\check{u}_0|_{L^2}^2 + |\check{u}|_{L^2}^2 \leq NK.$$

Proof The proof is trivial, we omit it here. □

The following two propositions will give the rigorous proofs for the approximations in Section 5.2.

Proposition 5.3. *Assume that* (**A5.1**)-(**A5.5**) *hold. Then, for* (5.5), $\mathbb{E}\sup_{0\le t\le T}|\delta x_0|^2 = O(\frac{1}{N^2})$, $\mathbb{E}\sup_{0\le t\le T}|\delta x^{(N)}|^2 = O(\frac{1}{N^2})$ *and* $\langle \Theta^\top Q(\bar{x}_i - \Theta\bar{x}^{(N)} - \Theta_1\bar{x}_0 - \eta), \delta x^{(N)}\rangle + \langle \Theta_1^\top Q(\bar{x}_i - \Theta\bar{x}^{(N)} - \Theta_1\bar{x}_0 - \eta), \delta x_0\rangle + \langle \hat{\Theta}^\top G(\bar{x}_i(T) - \hat{\Theta}\bar{x}^{(N)}(T) - \hat{\Theta}_1\bar{x}_0(T) - \hat{\eta}), \delta x^{(N)}(T)\rangle + \langle \hat{\Theta}_1^\top G(\bar{x}_i(T) - \hat{\Theta}\bar{x}^{(N)}(T) - \hat{\Theta}_1\bar{x}_0(T) - \hat{\eta}), \delta x_0(T)\rangle = o(1)$.

Proof See [101, Appendix C]. □

Proposition 5.4. *Assume that* (**A5.1**)-(**A5.5**) *hold. Then,* $N\delta x_j$, $N\delta x_0$, $N\delta x_j$ *converge to* $\sum_{j\neq i}\delta x_j$, $\delta x_0^\dagger$, $\delta x^\dagger$ *such that*

$$\begin{cases} \mathbb{E}\displaystyle\int_0^T |N\delta x_j - \sum_{j\neq i}\delta x_j|^2 = O(\frac{1}{N^2}), \quad \mathbb{E}\int_0^T |N\delta x_0 - \delta x_0^\dagger|^2 = O(\frac{1}{N^2}), \\ \mathbb{E}\displaystyle\int_0^T |N\delta x_j - \delta x^\dagger|^2 = O(\frac{1}{N^2}). \end{cases}$$

Proof See [101, Appendix C]. □

By the lemmas and propositions we discussed above, we give the main result.

Theorem 5.1. *Assume that* (**A5.1**)-(**A5.5**) *hold. Then the pair* (u_0^*, u^*) *given in* (5.23) *and* (5.13) *is an asymptotic Stackelberg equilibrium with respect to the social cost functional.*

Proof For $(\check{u}_0; \check{u}) \in \mathcal{U}_c$, let

$$\frac{1}{N}\mathcal{J}_{soc}^{(N)}(u_0^*; u^*) - \frac{1}{N}\mathcal{J}_{soc}^{(N)}(\check{u}_0; \check{u}) = \frac{1}{N}\mathcal{J}_{soc}^{(N)}(u_0^*; \mathcal{M}(u_0^*)) - \frac{1}{N}\mathcal{J}_{soc}^{(N)}(\check{u}_0; \mathcal{M}(\check{u}_0))$$
$$+ \frac{1}{N}\mathcal{J}_{soc}^{(N)}(\check{u}_0; \mathcal{M}(\check{u}_0)) - \frac{1}{N}\mathcal{J}_{soc}^{(N)}(\check{u}_0; \check{u}) := \Delta_1 + \Delta_2,$$

where $\Delta_1 = \frac{1}{N}\mathcal{J}_{soc}^{(N)}(u_0^*; \mathcal{M}(u_0^*)) - \frac{1}{N}\mathcal{J}_{soc}^{(N)}(\check{u}_0; \mathcal{M}(\check{u}_0))$, $\Delta_2 = \frac{1}{N}\mathcal{J}_{soc}^{(N)}(\check{u}_0; \mathcal{M}(\check{u}_0)) - \frac{1}{N}\mathcal{J}_{soc}^{(N)}(\check{u}_0; \check{u})$. Since $\check{u}_0$ is fixed, by following the standard method in [105], we obtain

$|\Delta_2|^2 \leq c(|\check{u}_0|^2_{L^2})\frac{1}{N}$. Specifically, we denote $\grave{x}_i$ as the state of the ith follower when its control is $\mathcal{M}_i(\check{u}_0)$, thus $\grave{x}_i$ is equivalent to $\bar{x}_i$ in Section 5.2. Let

$$\begin{cases} \tilde{u}_0 = \check{u}_0 - \check{u}_0 = 0, \quad \tilde{u} = \check{u} - \mathcal{M}(\check{u}_0), \quad \tilde{u}_i = \check{u}_i - \mathcal{M}_i(\check{u}_0), \\ \tilde{x}_0 = \check{x}_0 - \grave{x}_0, \quad \tilde{x}_i = \check{x}_i - \grave{x}_i. \end{cases}$$

Then we have

$$\begin{aligned} \mathcal{J}^{(N)}_{soc}(\check{u}_0;\check{u}) &= \alpha N \mathcal{J}_0(\check{u}_0;\check{u}) + \sum_{i=1}^{N} \mathcal{J}_i(\check{u}_0;\check{u}) \\ =&\alpha N \mathcal{J}_0(\check{u}_0;\mathcal{M}(\check{u}_0)) + \alpha N H_0 + \alpha N I_0 + \sum_{i=1}^{N} \mathcal{J}_i(\check{u}_0;\mathcal{M}(\check{u}_0)) + \sum_{i=1}^{N} H_i + \sum_{i=1}^{N} I_i, \end{aligned}$$

where

$$\begin{aligned} \mathcal{J}_0(\check{u}_0;\mathcal{M}(\check{u}_0)) = \mathbb{E}\Big\{ &\int_0^T |\grave{x}_0 - \Theta_0 \grave{x}^{(N)} - \eta_0|^2_{Q_0} + |\check{u}_0|^2_{R_0} dt \\ &+ |\grave{x}_0(T) - \hat{\Theta}_0 \grave{x}^{(N)}(T) - \hat{\eta}_0|^2_{G_0} \Big\}, \end{aligned}$$

$$H_0 = \mathbb{E}\Big\{ \int_0^T |\tilde{x}_0 - \Theta_0 \tilde{x}^{(N)}|^2_{Q_0} dt + |\tilde{x}_0(T) - \hat{\Theta}_0 \tilde{x}^{(N)}(T)|^2_{G_0} \Big\},$$

$$\begin{aligned} \mathcal{J}_i(\check{u}_0;\mathcal{M}(\check{u}_0)) = \mathbb{E}\Big\{ &\int_0^T |\grave{x}_i - \Theta \grave{x}^{(N)} - \Theta_1 \grave{x}_0 - \eta|^2_{Q} + |\mathcal{M}_i(\check{u}_0)|^2_{R} dt \\ &+ |\grave{x}_i(T) - \hat{\Theta} \grave{x}^{(N)}(T) - \hat{\Theta}_1 \grave{x}_0(T) - \hat{\eta}|^2_{G} \Big\}, \end{aligned}$$

$$H_i = \mathbb{E}\Big\{ \int_0^T |\tilde{x}_i - \Theta \tilde{x}^{(N)} - \Theta_1 \tilde{x}_0|^2_{Q} + |\tilde{u}_i|^2_{R} dt + |\tilde{x}_i(T) - \hat{\Theta} \tilde{x}^{(N)}(T) - \hat{\Theta}_1 \tilde{x}_0(T)|^2_{G} \Big\},$$

$$\begin{aligned} I_0 = \mathbb{E}\Big\{ &\int_0^T (\grave{x}_0 - \Theta_0 \grave{x}^{(N)} - \eta_0)^\top Q_0 (\tilde{x}_0 - \Theta_0 \tilde{x}^{(N)}) dt \\ &+ (\grave{x}_0(T) - \hat{\Theta}_0 \grave{x}^{(N)}(T) - \hat{\eta}_0)^\top G_0 (\tilde{x}_0(T) - \hat{\Theta}_0 \tilde{x}^{(N)}(T)) \Big\}, \end{aligned}$$

$$I_i = \mathbb{E}\Big\{ \int_0^T (\grave{x}_i - \Theta\grave{x}^{(N)} - \Theta_1\grave{x}_0 - \eta)^\top Q(\tilde{x}_i - \Theta\tilde{x}^{(N)} - \Theta_1\tilde{x}_0) + \mathcal{M}_i^\top(\check{u})R\tilde{u}_i dt$$

$$+ (\grave{x}_i(T) - \hat{\Theta}\grave{x}^{(N)}(T) - \hat{\Theta}_1\grave{x}_0(T) - \hat{\eta})^\top G(\tilde{x}_i(T) - \hat{\Theta}\tilde{x}^{(N)}(T) - \hat{\Theta}_1\tilde{x}_0(T))\Big\}.$$

By straightforward computation

$$\begin{aligned}
\alpha N I_0 = \mathbb{E}\Big\{ \int_0^T & \alpha N[\Psi_1^\top Q_0 - (\Theta_0\upsilon_1)^\top Q_0]\tilde{x}_0 - \alpha[\Psi_1^\top Q_0\Theta_0 \\
& - (\Theta_0\upsilon_1)^\top Q_0\Theta_0]\sum_{i=1}^N \tilde{x}_i dt + \alpha N[\Psi_4(T)^\top G_0 - (\hat{\Theta}_0\upsilon_1(T))^\top G_0]\tilde{x}_0(T) \\
& - \alpha[\Psi_4(T)^\top G_0\hat{\Theta}_0 - (\hat{\Theta}_0\upsilon_1(T))^\top G_0\hat{\Theta}_0]\sum_{i=1}^N \tilde{x}_i(T)\Big\},
\end{aligned} \tag{5.31}$$

$$\begin{aligned}
\sum_{i=1}^N I_i = \mathbb{E}\Big\{ \int_0^T & \sum_{i=1}^N (\Psi_2^i)^\top Q\tilde{x}_i - [(\Theta\upsilon_1)^\top Q + \Psi_3^\top Q\Theta - ((I-\Theta)\upsilon_1)^\top Q\Theta]\sum_{i=1}^N \tilde{x}_i \\
& - N[\Psi_3^\top Q\Theta_1 - [(I-\Theta)\upsilon_1]^\top Q\Theta_1]\tilde{x}_0 + \mathcal{M}_i^\top(\check{u})R\tilde{u}_i dt + \sum_{i=1}^N (\Psi_5^i(T))^\top G\tilde{x}_i(T) \\
& - [(\hat{\Theta}\upsilon_1(T))^\top G + \Psi_6(T)^\top G\hat{\Theta} - ((I-\hat{\Theta})\upsilon_1(T))^\top G\hat{\Theta}]\sum_{i=1}^N \tilde{x}_i(T) \\
& - N[\Psi_6(T)^\top G\hat{\Theta}_1 - [(I-\hat{\Theta})\upsilon_1(T)]^\top G\hat{\Theta}_1]\tilde{x}_0(T)\Big\}.
\end{aligned} \tag{5.32}$$

where $\upsilon_1 = \grave{x}^{(N)} - \hat{x}$. By (5.22), (5.29) and Itô formula, we obtain following relations:

$$\begin{aligned}
& N\langle k_1(T), \tilde{x}_0(T)\rangle = \langle \alpha N G_0\Psi_4, \tilde{x}_0(T)\rangle - \langle N\hat{\Theta}_1^\top G\Psi_6, \tilde{x}_0(T)\rangle \\
= & \mathbb{E}\int_0^T -\langle \alpha N Q_0\Psi_1, \tilde{x}_0\rangle + \langle N\Theta_1^\top Q\Psi_3, \tilde{x}_0\rangle - \langle k_2, NF\tilde{x}_0\rangle + \langle C_0^\top k_1, \sum_{i=1}^N \tilde{x}_i\rangle dt,
\end{aligned} \tag{5.33}$$

and

$$\sum_{i=1}^{N}\langle p_i(T),\tilde{x}_i(T)\rangle = \mathbb{E}\int_0^T \langle \Theta^\top Q\Psi_3, \sum_{i=1}^{N}\tilde{x}_i\rangle - \langle Q\Psi_2^i, \sum_{i=1}^{N}\tilde{x}_i\rangle + \langle \alpha\Theta_0^\top Q_0\Psi_1, \sum_{i=1}^{N}\tilde{x}_i\rangle$$

$$- \langle C_0^\top k_1, \sum_{i=1}^{N}\tilde{x}_i\rangle - \langle p^{(N)} - k_2, C\sum_{i=1}^{N}\tilde{x}_i\rangle + \sum_{i=1}^{N}\langle p_i, B\tilde{u}_i\rangle + \langle p^{(N)}, NF\tilde{x}_0\rangle dt. \tag{5.34}$$

Meanwhile, by (5.13), we have

$$\begin{aligned}&\sum_{i=1}^{N}\langle \mathcal{M}_i(\check{u}), R\tilde{u}_i\rangle + \sum_{i=1}^{N}\langle p_i, B\tilde{u}_i\rangle \\ =&\sum_{i=1}^{N}\langle R\mathcal{M}_i(\check{u}) + B^\top p_i, \tilde{u}_i\rangle = \sum_{i=1}^{N}\langle R(-R^{-1}B^\top p_i) + B^\top p_i, \tilde{u}_i\rangle = 0.\end{aligned} \tag{5.35}$$

Combining (5.31)-(5.35), Lemma 5.1 and Lemma 5.2, it follows that

$$\frac{1}{N}(\alpha N I_0 + \sum_{i=1}^{N} I_i) = O(\frac{1}{\sqrt{N}}).$$

Moreover, $\frac{1}{N}(\alpha N H_0 + \sum_{i=1}^{N} H_i) \geq 0$. Thus, we have

$$\Delta_2 = \frac{1}{N}\mathcal{J}_{soc}^{(N)}(\check{u}_0; \mathcal{M}(\check{u}_0)) - \frac{1}{N}\mathcal{J}_{soc}^{(N)}(\check{u}_0; \check{u}) \leq c(|\check{u}_0|_{L^2}^2)\frac{1}{\sqrt{N}}. \tag{5.36}$$

For Δ_1, we decompose it as follows:

$$\Delta_1 = \frac{1}{N}\mathcal{J}_{soc}^{(N)}(u_0^*; \mathcal{M}(u_0^*)) - \frac{1}{N}\mathcal{J}_{soc}^{(N)}(\check{u}_0; \mathcal{M}(\check{u}_0)) = \frac{1}{N}\mathcal{J}_{soc}^{(N)}(u_0^*; \mathcal{M}(u_0^*))$$

$$- \frac{1}{N}\mathcal{J}_{soc}^{(N)}(u_0^{(N)}; \mathcal{M}(u_0^{(N)})) + \frac{1}{N}\mathcal{J}_{soc}^{(N)}(u_0^{(N)}; \mathcal{M}(u_0^{(N)})) - \frac{1}{N}\mathcal{J}_{soc}^{(N)}(\check{u}_0; \mathcal{M}(\check{u}_0)).$$

Note that $u_0^{(N)}$ is the centralized social optimal control in (5.20), thus one can easily obtain that

$$\frac{1}{N}\mathcal{J}_{soc}^{(N)}(u_0^{(N)}; \mathcal{M}(u_0^{(N)})) \leq \frac{1}{N}\mathcal{J}_{soc}^{(N)}(\check{u}_0; \mathcal{M}(\check{u}_0)). \tag{5.37}$$

We know that $\mathcal{J}_{soc}^{(N)}(u_0;\mathcal{M}(u_0))$ continuously depends on u_0. Since $\mathcal{M}(u_0)$ is the solution of (5.29) which continuously depends on parameters, we have $\mathcal{M}(u_0)$ is continuous in u_0. Note that $\mathcal{J}_{soc}^{(N)}(u_0;\mathcal{M}(u_0))$ is a quadratic functional and u_0^* is fixed. Let $\check{x}_0^{(N)}$ and $\check{x}_i^{(N)}$ be the state of the leader and the ith follower when the control of the leader is $u_0^{(N)}$. Denote

$$\begin{cases} \acute{u}_0 = u_0^{(N)} - u_0^*, \quad \delta\mathcal{M}(u_0) = \mathcal{M}(u_0^{(N)}) - \mathcal{M}(u_0^*), \\ \delta\mathcal{M}_i(u_0) = \mathcal{M}_i(u_0^{(N)}) - \mathcal{M}_i(u_0^*), \quad \acute{x}_0 = \check{x}_0^{(N)} - x_0^*, \quad \acute{x}_i = \check{x}_i^{(N)} - x_i^*. \end{cases}$$

Then we have

$$\begin{aligned} &\left|\mathcal{J}_{soc}^{(N)}(u_0^{(N)};\mathcal{M}(u_0^{(N)})) - \mathcal{J}_{soc}^{(N)}(u_0^*;\mathcal{M}(u_0^*))\right| \\ =&\left|\mathcal{J}_{soc}^{(N)}(u_0^{(N)} - u_0^* + u_0^*;\mathcal{M}(u_0^{(N)}) - \mathcal{M}(u_0^*) + \mathcal{M}(u_0^*)) - \mathcal{J}_{soc}^{(N)}(u_0^*;\mathcal{M}(u_0^*))\right|, \end{aligned}$$

and

$$\mathcal{J}_{soc}^{(N)}(u_0^{(N)};\mathcal{M}(u_0^{(N)})) = \alpha N[\mathcal{J}_0(u_0^*;\mathcal{M}(u_0^*)) + H_0' + I_0'] + \sum_{i=1}^{N}[\mathcal{J}_i(u_0^*;\mathcal{M}(u_0^*)) + H_i' + I_i'],$$

where

$$\begin{aligned} \mathcal{J}_0(u_0^*;\mathcal{M}(u_0^*)) = \mathbb{E}\Big\{ &\int_0^T |x_0^* - \Theta_0(x^*)^{(N)} - \eta_0|_{Q_0}^2 + |u_0^*|_{R_0}^2 dt \\ &+ |x_0^*(T) - \hat{\Theta}_0(x^*)^{(N)}(T) - \hat{\eta}_0|_{G_0}^2\Big\}, \end{aligned}$$

$$H_0' = \mathbb{E}\Big\{\int_0^T |\acute{x}_0 - \Theta_0\acute{x}^{(N)}|_{Q_0}^2 + |\acute{u}_0|_{R_0}^2 dt + |\acute{x}_0(T) - \hat{\Theta}_0\acute{x}^{(N)}(T)|_{G_0}^2\Big\},$$

$$\mathcal{J}_i(u_0^*;\mathcal{M}(u_0^*)) = \mathbb{E}\Big\{ \int_0^T |x_i^* - \Theta(x^*)^{(N)} - \Theta_1 x_0^* - \eta|_Q^2 + |\mathcal{M}_i(u_0^*)|_R^2 dt$$

$$+ |x_i^*(T) - \hat{\Theta}(x^*)^{(N)}(T) - \hat{\Theta}_1 x_0^*(T) - \hat{\eta}|_G^2 \Big\},$$

$$H_i' = \mathbb{E}\Big\{ \int_0^T |\acute{x}_i - \Theta \acute{x}^{(N)} - \Theta_1 \acute{x}_0|_Q^2 + |\delta\mathcal{M}_i(u_0)|_R^2 dt$$

$$+ |\acute{x}_i(T) - \hat{\Theta}\acute{x}^{(N)}(T) - \hat{\Theta}_1 \acute{x}_0(T)|_G^2 \Big\},$$

$$I_0' = \mathbb{E}\Big\{ \int_0^T (x_0^* - \Theta_0(x^*)^{(N)} - \eta_0)^\top Q_0(\acute{x}_0 - \Theta_0 \acute{x}^{(N)}) dt$$

$$+ (x_0^*(T) - \hat{\Theta}_0(x^*)^{(N)}(T) - \hat{\eta}_0)^\top G_0(\acute{x}_0(T) - \hat{\Theta}_0 \acute{x}^{(N)}(T)) \Big\},$$

$$I_i' = \mathbb{E}\Big\{ \int_0^T (x_i^* - \Theta(x^*)^{(N)} - \Theta_1 x_0^* - \eta)^\top Q(\acute{x}_i - \Theta\acute{x}^{(N)} - \Theta_1 \acute{x}_0) + \mathcal{M}_i^\top(u_0^*) R \delta\mathcal{M}_i(u_0) dt$$

$$+ (x_i^*(T) - \hat{\Theta}(x^*)^{(N)}(T) - \hat{\Theta}_1 x_0^*(T) - \hat{\eta})^\top G(\acute{x}_i(T) - \hat{\Theta}\acute{x}^{(N)}(T) - \hat{\Theta}_1 \acute{x}_0(T)) \Big\}.$$

By the similar arguments in Lemma 5.1 to Lemma 5.2 and $|\Delta_2|^2 \leq c(|\check{u}_0|_{L^2}^2)\frac{1}{N}$, we obtain

$$\frac{1}{N}H_0' + \frac{1}{N}H_i' + \alpha I_0' + \frac{1}{N}\sum_{i=1}^N I_i' = O(\frac{1}{\sqrt{N}}).$$

Hence, we have

$$-\frac{1}{N}\mathcal{J}_{soc}^{(N)}(u_0^{(N)};\mathcal{M}(u_0^{(N)})) + \frac{1}{N}\mathcal{J}_{soc}^{(N)}(u_0^*;\mathcal{M}(u_0^*)) \leq K(\frac{1}{\sqrt{N}}) = O(\frac{1}{\sqrt{N}}), \qquad (5.38)$$

where K is independent of N. By (5.38) and (5.37), it follows that

$$\frac{1}{N}\mathcal{J}_{soc}^{(N)}(u_0^*;\mathcal{M}(u_0^*)) - \frac{1}{N}\mathcal{J}_{soc}^{(N)}(u_0^{(N)};\mathcal{M}(u_0^{(N)})) = O(\frac{1}{\sqrt{N}}),$$

and

$$\frac{1}{N}\mathcal{J}_{soc}^{(N)}(u_0^{(N)};\mathcal{M}(u_0^{(N)})) - \frac{1}{N}\mathcal{J}_{soc}^{(N)}(\check{u}_0;\mathcal{M}(\check{u}_0)) \leq 0,$$

respectively. Thus, we have

$$\Delta_1 = \frac{1}{N}\mathcal{J}_{soc}^{(N)}(u_0^*; \mathcal{M}(u_0^*)) - \frac{1}{N}\mathcal{J}_{soc}^{(N)}(\check{u}_0; \mathcal{M}(\check{u}_0)) \leq O(\frac{1}{\sqrt{N}}). \tag{5.39}$$

By Proposition 5.2, there exists K independent of N such that $|\check{u}_0|_{L^2}^2 \leq K$. Then, combining (5.36), (5.39), we can obtain:

$$\Delta_1 + \Delta_2 \leq O(\frac{1}{\sqrt{N}}) + c(|\check{u}_0|_{L^2}^2)\frac{1}{\sqrt{N}}) \leq K \cdot O(\frac{1}{\sqrt{N}}) = O(\frac{1}{\sqrt{N}}),$$

where K is independent of N. The theorem follows. □

5.6 Numerical Examples

We now give a numerical example for Lemma 5.1. By (5.28) and (5.27), $\mathbb{K}$ and κ can be easily computed. Consider $\mathbb{Y} = \mathbb{K}\mathbb{X} + \kappa$, we can obtain that

$$d\mathbb{X} = [(\mathbb{A} + \mathbb{B}\mathbb{K})\mathbb{X} + \mathbb{B}\kappa + b]dt + \mathbb{D}dW_0, \quad \mathbb{Y} = \mathbb{K}\mathbb{X} + \kappa,$$

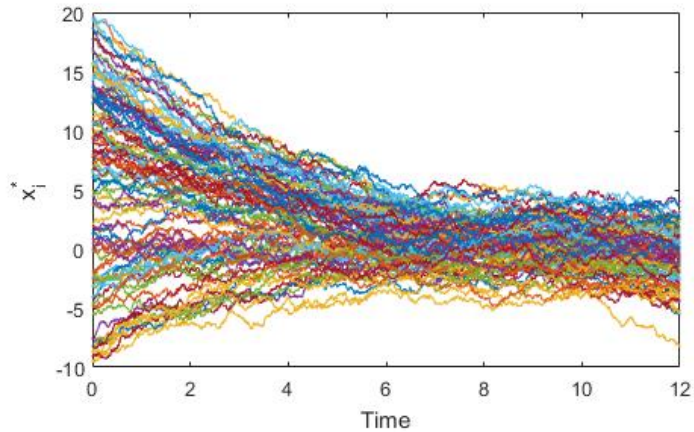
where $\mathbb{X} = ((\hat{x})^\top\ (\bar{x}_0)^\top\ (q^*)^\top\ (l_1^*)^\top\ (l_2^*)^\top)^\top$, $\mathbb{Y} = ((y^*)^\top\ (\hat{y}^*)^\top\ (y_0^*)^\top\ (k_1)^\top\ (k_2)^\top)^\top$. Since $p_i = \bar{P}\bar{x}_i + \bar{\varphi}$, by the following equations below (5.29), we have

$$d\bar{x}_i = [(A - BR^{-1}B^\top\bar{P})\bar{x}_i - BR^{-1}B^\top\bar{\varphi} + C\hat{x} + F\bar{x}_0]dt + DdW_i.$$

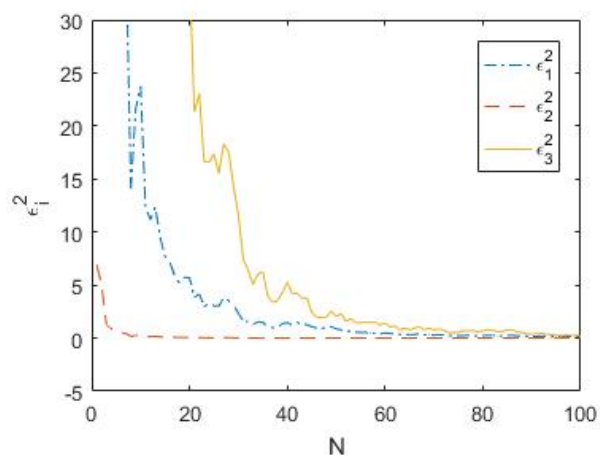
The realized decentralized state x_0^* and $(x^*)^{(N)}$, can be derived by (5.30). Combining them with (5.22), one can obtain

$$\begin{cases} d\begin{pmatrix} x_0^* - \bar{x}_0 \\ (x^*)^{(N)} - \hat{x} \end{pmatrix} = \left[\begin{pmatrix} A_0 & C_0 \\ F & A+C \end{pmatrix}\begin{pmatrix} x_0^* - \bar{x}_0 \\ (x^*)^{(N)} - \hat{x} \end{pmatrix} - \begin{pmatrix} 0 \\ BR^{-1}B^T \end{pmatrix}(p^{(N)} - \hat{p})\right]dt + \frac{1}{N}\begin{pmatrix} 0 \\ \sum_1^N D \end{pmatrix}dW_i, \\ \begin{pmatrix} x_0^* - \bar{x}_0 \\ (x^*)^{(N)} - \hat{x} \end{pmatrix}(0) = \begin{pmatrix} 0 \\ \frac{1}{N}\sum_1^N \xi_i - \hat{\xi} \end{pmatrix}, \end{cases}$$

where $\hat{p} = k_2$.



(a)



(b)

Figure 5.1: (a) is the trajectories of x_i^*, $i = 1, \cdots, 100$ and (b) is the curves of ε_i^2, $i = 1, 2, 3$ when time interval is $[0, 12]$.

We continuously use the parameters in Example 5.1. The population $N = 100$ and the time interval is $[0, 12]$. By Matlab computation, the trajectories of the realized state x_i^* are shown in Figure 1(a).

We defined $\varepsilon_1^2 = \mathbb{E}\int_0^{12} |(x*)^{(N)} - \hat{x}|^2 dt$, $\varepsilon_2^2 = \mathbb{E}\int_0^{12} |x_0^* - \bar{x}_0|^2 dt$, $\varepsilon_3^2 = \mathbb{E}\int_0^{12} |p^{(N)} - \hat{p}|^2 dt$. When N increases from 1 to 100, the curves of ε_1^2, ε_2^2 and ε_3^2 are shown in Figure 1(b). The X axis indicates N and the Y axis indicates $\varepsilon_i^2, i = 1, 2, 3$. It can be seen that they are approaching to zero when N is growing larger and larger.

5.7 Conclusion

This chapter has analyzed a class of LQ MFT control problems. We obtain the decentralized form of optimal controls for the leader and N followers. By the Riccati equation method, we discuss the solvability of the FBSDE. Finally, an asymptotic Stackelberg equilibrium theorem is established. For future work, one can extend the results of this paper to the hierarchical control with many leaders cases.

Chapter 6

Stackelberg-Nash-Cournot Equilibrium with Model Uncertainty and Weak-coupling: a Mean Field Consistency Approach

In this chapter, a multi-leader and multi-follower (ML/MF) game equilibrium problem with a hierarchical structure, model uncertainty, and weak-coupling is introduced. The leaders or followers play a Nash game with each other in their hierarchy, while leaders and followers play a Stackelberg game between the two hierarchies. The information structures between leaders and followers are asymmetric and model uncertainty appears since the lacking of communication among the agents. Moreover, all agents are framed under a weakly-coupled large population system with complex interrelations. According to the mean field game (MFG) theory, it can obtain an asymptotic Stackelberg-Nash-Cournot (SNC) equilibrium for leaders and followers based on a CC system.

This chapter can be considered as a supplement for the thesis. It is different from the previous chapters since it is focusing on discovering the LF MFG under the *static* model. Meanwhile, to our best knowledge, this is the first time to investigate the weakly-coupled LF problem (**w-LF**) in a *static* optimization context.

6.1 General Nash and Stackelberg Game

6.1.1 General Nash game

We recall the multiple-agent game in its general normal form with related notions.

Definition 6.1 (General form of game)**.** *A multiple-agent game is a quadruple* $G := (\mathcal{A},\ \mathbf{J},\ \Gamma,\ \Theta)$, *where* $\mathcal{A} = \{\mathcal{A}_i\}_{i\in\mathcal{I}}$ *denotes the set of agents involved.* $\mathcal{I}$ *is the agent index set and its cardinality is assumed to be finite with* $|\mathcal{I}| < +\infty$. $\mathbf{J} = (\mathcal{J}_1, \cdots \mathcal{J}_i \cdots, \mathcal{J}_{|\mathcal{I}|})$ *denotes the cost functional profile of* $\{\mathcal{A}_i\}_{i\in\mathcal{I}}$ *with* $\mathcal{J}_i = \mathcal{J}_i(x_i; x_{-i}; \theta_i) : \Gamma_i \times \prod_{j\neq i, j\in\mathcal{I}} \Gamma_j \times \Theta_i \subseteq \mathbb{R}^{n_i} \times \prod_{j\neq i, j\in\mathcal{I}} \mathbb{R}^{n_j} \times \mathbb{R}^{m_i} \to \mathbb{R}$ *the individual functional of agent* $\mathcal{A}_i$. x_i *is the individual strategy (decision) taken by agent* $\mathcal{A}_i$ *while* $x_{-i} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{|\mathcal{I}|})$ *is the collective strategies of the other agents.* θ_i *is the individual parameter of* $\mathcal{A}_i$. $\Gamma_i \subseteq \mathbb{R}^{n_i}$ *is the individual admissibility of* $\mathcal{A}_i$ *while* $\Gamma = \prod_{i\in\mathcal{I}} \Gamma_i$ *the admissible set of all strategies.* $\mathcal{P} = (\theta_1, \cdots \theta_i \cdots, \theta_{|\mathcal{I}|}) \in \Theta := \prod_{i\in\mathcal{I}} \Theta_i \subseteq \prod_{i\in\mathcal{I}} \mathbb{R}^{m_i}$ *denotes the parameter sets amongst all agents while* $\Theta_i \subseteq \mathbb{R}^{m_i}$ *the individual parameter support for* $\mathcal{A}_i$.

Remark 6.1. *In general, we assume* $\forall i \neq j \in \mathcal{I}, \mathcal{J}_i \neq \mathcal{J}_j$ *so that all agents in Definition 6.1 are completely competitive with fully conflictive costs. Another extreme case is* $\mathcal{J}_i \equiv \mathcal{J}_j \equiv \mathcal{J}$ *for all* $i, j \in \mathcal{I}$, *and in this case, all agents formalize a vector team optimization (see Chapter 4 and 5). We may denote* $\mathbf{x} = (x_1, \cdots x_i \cdots, x_{|\mathcal{I}|}) = (x_i; x_{-i}) \in \Gamma_i \times \prod_{j\neq i, j\in\mathcal{I}} \Gamma_j = \Gamma$ *to emphasize the particular role of decision* x_i *for agent* $\mathcal{A}_i$.

For multiple-agent game problem G, the following solvability notion is meaningful.

Definition 6.2. *An* NE *for a general multiple-agent game is an* $|\mathcal{I}|$*-tuple* $\bar{\mathbf{x}} = (\bar{x}_i; \bar{x}_{-i})$ *satisfying*

$$\mathcal{J}_i(\bar{x}_i; \bar{x}_{-i}; \theta_i) \leq \mathcal{J}_i(x_i; \bar{x}_{-i}; \theta_i), \quad \forall x_i \in \Gamma_i, \quad \forall i \in \mathcal{I}.$$

Cournot game (or Cournot duopoly model) is one of the applications of NE. Cournot model is a kind of non-cooperative game model in which two firms (it could be generalized to N firms) with identical cost functions compete with homogeneous products (see [59, 75, 143]). For example, suppose $P(x_1 + x_2)$ be the price function (or inverse demand function) for the firms and $C_i(x_i)$ be the cost function of firm $\mathcal{A}_i$, where $x_i \in \mathbb{R}$ is the quantity of product of firm $\mathcal{A}_i$. Then the profit is $\mathcal{J}_i(x_1; x_2) = P(x_1 + x_2)x_i - C_i(x_i)$, $i = 1, 2$. To calculate the Cournot equilibrium (the NE), it should first take the derivative of $\mathcal{J}_i$, $i = 1, 2$, and set this to zero for maximization (the firms want to maximize their profits, thus the inequality in Definition 6.2 should has an opposite direction here):

$$\frac{\partial \mathcal{J}_i}{\partial x_i} = \frac{\partial P(x_1 + x_2)}{\partial x_i} x_i + P(x_1 + x_2) - \frac{\partial C_i(x_i)}{\partial x_i} = 0.$$

The best responses of the firm $\mathcal{A}_i$ is the values of $\bar{x}_i$, $i = 1, 2$, that satisfy above equation and the NE is the pair $(\bar{x}_1, \bar{x}_2)$. More specifically, if the price function is linear $P(x_1 + x_2) = a - b(x_1 + x_2)$ and the cost is quadratic $C_i(x_i) = cx_i^2$, where a, b, $c > 0$. Without loss of generality, consider $\mathcal{A}_1$'s problem, then

$$\begin{aligned} &\frac{\partial \mathcal{J}_1}{\partial x_1} = \frac{\partial P(x_1 + x_2)}{\partial x_1} x_1 + P(x_1 + x_2) - \frac{\partial C_1(x_1)}{\partial x_1} = 0 \\ \Longrightarrow &- bx_1 + a - b(x_1 + x_2) - cx_1 = 0. \end{aligned}$$

The best responses of $\mathcal{A}_1$ and $\mathcal{A}_2$ (by symmetry) are

$$\bar{x}_1 = \bar{x}_2 = \frac{a}{3b + c},$$

and the profits of $\mathcal{A}_1$ and $\mathcal{A}_2$ under $(\bar{x}_1, \bar{x}_2)$ are

$$\mathcal{J}_1(\bar{x}_1; \bar{x}_2) = \mathcal{J}_2(\bar{x}_1; \bar{x}_2) = \frac{a^2 b}{(3b + c)^2}.$$

If the number of firms generalized to N, then by some elementary calculation (see [155]), the best responses of $\mathcal{A}_i$, $i = 1, \cdots, N$, is

$$\bar{x}_i = \frac{a}{(N+1)b + c}, \quad i = 1, \cdots, N,$$

and the profits of $\mathcal{A}_i$ under $(\bar{x}_1, \cdots, \bar{x}_N)$ is

$$\mathcal{J}_i(\bar{x}_i; \bar{x}_{-i}) = \frac{a^2 b}{[(N+1)b + c]^2}, \quad i = 1, \cdots, N.$$

Next, some applications of Cournot game or Cournot competition are introduced.

- Cournot model is applied in the international trade problem. In [57], the industrial policies and optimal trade were obtained for a home market that was supplied by a foreign firm and a domestic firm. The consumer's surplus from the consumption was

  $$S(P_d; P_f) = \max_{x_d, x_f} M(x_d; x_f) - P_d x_d - P_f x_f,$$

  where x_d (x_f) was the quantity of the good produced by the home (foreign) firm. M was measured in terms of the numeraire. $P_d(x_d; x_f)$ and $P_f(x_d; x_f)$ were the corresponding inverse demand function of the home firm and the foreign firm, respectively, and

  $$P_d(x_d; x_f) = \frac{\partial M(x_d; x_f)}{\partial x_d}, \quad P_f(x_d; x_f) = \frac{\partial M(x_d; x_f)}{\partial x_f}.$$

  The optimal policy under Cournot competition consisted of a domestic production tax and a tariff. It denoted t as a specific tariff on imports and s as a unit subsidy for the home firm's output. The national welfare in the domestic market was equal to the sum of the consumer's surplus, the home firm's profits, and the tariff revenues, less the production subsidies,

  $$W = S(P_d; P_f) + (P_d - c_d + s)x_d + t x_f - s x_d,$$

where c_d was the home firm's marginal costs. If the goods were homogeneous, then x_d and x_f are determined by an NE of the following conjectural variations model:

$$P_d(x_d; x_f) + x_d(P_d + P_f\gamma) - c_d + s \leq 0,$$

$$P_f(x_d; x_f) + x_f(P_f + P_d\Gamma) - c_f - t \leq 0,$$

where c_f was the foreign firm's marginal costs. $\gamma = \frac{dx_f}{dx_d}$ ($\Gamma = \frac{dx_d}{dx_f}$) was the domestic (foreign) firms' conjectural variations. Under Cournot model, $\gamma = \Gamma = 0$. The total differential of the national welfare was:

$$dW = (P_d - c_d)dx_d + tdx_f - x_f d(P_f - t).$$

When the inverse demand function was linear as

$$P_d = a_d - b_d x_d - k_d x_f, \quad P_f = a_f - k_d x_d - b_f x_f,$$

where all the parameters were positive and $b_d b_f - k_d^2 > 0$, and both the domestic goods and imports were desirable, then the optimal tariff and subsidy were

$$\bar{t} = \frac{1}{3b_d b_f}[b_d b_f(a_f - c_f) - k_d b_f(a_d - c_d)],$$

$$\bar{s} = \frac{1}{3b_d b_f}[(3b_d b_f - 2k_d(a_d - c_d) - b_d k_d(a_f - c_f)],$$

- In [83], the contracting and information sharing under Cournot competition was studied. It contained two competing supply chains, each consisting of one manufacturer and one retailer. The two supply chains were identical except they had different investment costs. First, the manufacturers considered to share information. Secondly, for the given the information structure, the manufacturers made contracts to their retailers and the retailers competed under Cournot model. The linear inverse demand function of retailers was

$$P = a - x_1 - x_2,$$

where P was the market clearing price and $x_1, x_2 \in \mathbb{R}$ were the selling quantities. A was a random variable given by

$$a = \begin{cases} a_h & \text{with probablity } \alpha, \\ a_l & \text{with probablity } 1-\alpha, \end{cases}$$

where a_h and a_l were the high and low demand states, $a_h > a_l > 0$. It denoted the demand state as $d = h$ or l for all the manufacturers and retailers. The manufacturers offered contracts (x_{di}, p_{di}), $i = 1, 2$, to retailers, where x_{di} was the order quantity and p_{di} was the corresponding payment. The profit function of retailer i was

$$(a_d - x_{di} - x_{dj} - w_{di})x_{di},$$

where w_{di} was the whole price of retailer i, $i = 1, 2$, and the profit of manufacturer i was

$$w_{di}\bar{x}_{di}.$$

(1) When the information sharing in both supply chains. The best response function of retailer i and manufacturer i were

$$\bar{x}_{di}(x_{dj}; w_{di}) = \frac{a_d - x_{di} - w_{di}}{2}, \quad \bar{w}_{di}(x_{dj}) = \frac{a_d - x_{dj}}{2}$$

It denoted $\bar{x}_{di}^{e1}$ and $\bar{w}_{di}^{e1}$ be the equilibrium selling quantity and wholesale price, respectively, then

$$\bar{x}_{d1}^{e1} = \bar{x}_{d2}^{e1} = \frac{a_d}{5}, \quad \bar{w}_{d1}^{e1} = \bar{w}_{d2}^{e1} = \frac{2a_d}{5}.$$

(2) When the information sharing in one supply chains, suppose manufacturer 1 did not know the demand state d and the price was changed as w_1. In anticipation of x_{d2}, his expected profit became

$$w_1[\alpha\bar{x}_{h1}(x_{h2}; w_1) + (1-\alpha)\bar{x}_{l1}(x_{l2}; w_1)],$$

and the best response function of retailers and manufacturers were

$$\bar{x}_{d1}(x_{d2}; w_1) = \frac{a_d - x_{d2} - w_1}{2}, \quad \bar{x}_{d2}(x_{d1}; w_{d2}) = \frac{a_d - x_{d1} - w_{d2}}{2},$$

$$\bar{w}_1(x_{h2}; x_{l2}) = \frac{\alpha(a_h - x_{h2}) + (1-\alpha)(a_l - x_{l2})}{2}, \quad \bar{w}_{d2}(x_{d1}) = \frac{a_d - x_{d1}}{2}.$$

It denoted $\bar{x}_{di}^{e2}$ and $\bar{w}_{di}^{e2}$ be the equilibrium selling quantity and wholesale price, respectively, then

$$\begin{cases} \bar{x}_{h1}^{e2} = \frac{a_d}{5} + \frac{8}{35}(1-\alpha)(a_h - a_l), \quad \bar{x}_{h2}^{e2} = \frac{a_d}{5} - \frac{2}{35}(1-\alpha)(a_h - a_l), \\ \bar{x}_{l1}^{e2} = \frac{a_d}{5} - \frac{8}{35}\alpha(a_h - a_l), \quad \bar{x}_{l2}^{e2} = \frac{a_d}{5} + \frac{2}{35}\alpha(a_h - a_l), \\ \bar{w}_{h2}^{e2} = = \frac{2a_d}{5} - \frac{4}{35}(1-\alpha)(a_h - a_l), \quad \bar{w}_{l2}^{e2} = \frac{2a_d}{5} + \frac{4}{35}\alpha(a_h - a_l), \\ \bar{w}_1^{e2} = \frac{2\alpha a_h + 2(1-\alpha)a_l}{5}. \end{cases}$$

(3) When the system had no information sharing, both manufacturers did not know the demand state and the wholesale price did not related to state d. In anticipation of x_{dj}, manufacturer i's expected profit became

$$w_i[\alpha\bar{x}_{hi}(x_{hj}; w_i) + (1-\alpha)\bar{x}_{li}(x_{lj}; w_i)],$$

and the best response function of the retailers and manufacturers were

$$\bar{x}_{di}(x_{dj}; w_i) = \frac{a_d - x_{dj} - w_i}{2}, \quad \bar{w}_i(x_{hj}; x_{lj}) = \frac{\alpha(a_h - x_{hj}) + (1-\alpha)(a_l - x_{lj})}{2}.$$

It denoted $\bar{x}_{di}^{e3}$ and $\bar{w}_{di}^{e3}$ be the equilibrium selling quantity and wholesale price, respectively, then

$$\begin{cases} \bar{x}_{h1}^{e3} = \bar{x}_{h2}^{e3} = \frac{a_d}{5} + \frac{2}{15}(1-\alpha)(a_h - a_l), \quad \bar{x}_{l1}^{e3} = \bar{x}_{l2}^{e3} = \frac{a_d}{5} - \frac{2}{15}\alpha(a_h - a_l), \\ \bar{w}_1^{e3} = \bar{w}_2^{e3} = \frac{2\alpha a_h + 2(1-\alpha)a_l}{5}. \end{cases}$$

The expect profits of the retailers and manufacturers were $\mathcal{J}_{ri}^{k}$ and $\mathcal{J}_{mi}^{k}$, where $i = 1, 2$, $k = e1, e2, e3$. Consequently, it obtained that

$$\mathcal{J}_{mi}^{e1} \geq \mathcal{J}_{m2}^{e2} \geq \mathcal{J}_{mi}^{e3} \geq \mathcal{J}_{m1}^{e2}, \quad \mathcal{J}_{r1}^{e2} \geq \mathcal{J}_{ri}^{e3} \geq \mathcal{J}_{ri}^{e1} \geq \mathcal{J}_{r2}^{e2}.$$

More details about Cournot model combining with Stackelberg game will be introduced in the following sections.

6.1.2 General Stackelberg game

Given NE, we can introduce the LF game in the general multiple-agent context.

Definition 6.3 (LF)**.** *A general* ML/MF *game, is an octuple* $G_{LF} := (\mathcal{A}^L, \mathcal{A}^F; \mathbf{J}^L, \mathbf{J}^F; \Gamma^L, \Gamma^F; \Theta^L, \Theta^F)$*, where* $\mathcal{A}^L := \{\mathcal{A}_i^L\}_{i\in\mathcal{I}^L}$ *is the set of all leaders, where* $\mathcal{A}_i^L$ *denotes the* i^{th} *leader.* $\mathcal{I}^L = \{1, \cdots, N\}$ *is the leader index set.* $\mathcal{A}^F := \{\mathcal{A}_i^F\}_{i\in\mathcal{I}^F}$ *is the set of all followers.* $\mathcal{I}^F = \{1, \cdots, M\}$ *is the follower index set,* $\mathcal{A}_j^F$ *denotes the* j^{th} *follower.*

$\mathbf{J}^L = (\mathcal{J}_1^L, \cdots, \mathcal{J}_N^L)$ *denotes the cost functional of* $\mathcal{A}^L$ *with* $\mathcal{J}_i^L = \mathcal{J}_i^L(x_i; x_{-i}; \mathbf{y}; \theta_i^L)$: $\Gamma_i^L \times \prod_{k\neq i} \Gamma_k^L \times \Gamma^F \times \Theta_i^L \subseteq \mathbb{R}^{n_i} \times \prod_{k\neq i, k\in\mathcal{I}^L} \mathbb{R}^{n_k} \times \mathbb{R}^{\sum_{j=1}^M m_j} \times \mathbb{R}^{l_i} \to \mathbb{R}$ *the individual functional of agent* $\mathcal{A}_i^L$*.* x_i *is the individual strategy taken by agent* $\mathcal{A}_i^L$ *while* $x_{-i} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N)$ *is the decision strategies except that of* $\mathcal{A}_i^L$*.* y_j *is the individual strategy taken by agent* $\mathcal{A}_j^F$*, while* $\mathbf{y} = (y_1, \cdots, y_j, \cdots, y_M)$ *is the strategy profile of all followers.* θ_i^L *is the individual parameter of* $\mathcal{A}_i^L$*.* $\Gamma_i^L \subseteq \mathbb{R}^{n_i}$ *is the individual admissibility of* $\mathcal{A}_i^L$*.* $\Gamma_j^F \subseteq \mathbb{R}^{m_j}$ *is the individual admissibility of* $\mathcal{A}_j^F$*, while* $\Gamma^F = \prod_{j=1}^M \Gamma_j^F$ *the admissible set of all follower strategies.* $\mathcal{P}^L = (\theta_1^L, \cdots \theta_i^L \cdots, \theta_N^L) \in \Theta^L := \prod_{i=1}^N \Theta_i^L \subseteq \prod_{i=1}^N \mathbb{R}^{l_i}$ *denotes the parameter sets amongst all agents while* $\Theta_i^L \subseteq \mathbb{R}^{l_i}$ *the individual parameter support for* $\mathcal{A}_i^L$*.*

$\mathbf{J}^F = (\mathcal{J}_1^F, \cdots, \mathcal{J}_M^F)$ *denotes the cost functional of* $\mathcal{A}^F$ *with* $\mathcal{J}_j^F = \mathcal{J}_j^F(y_j; y_{-j}; \mathbf{x}; \theta_j^F)$: $\Gamma_j^F \times \prod_{k\neq j} \Gamma_k^F \times \Gamma^L \times \Theta_j^F \subseteq \mathbb{R}^{m_j} \times \prod_{k\neq j, k\in\mathcal{I}^F} \mathbb{R}^{m_k} \times \mathbb{R}^{\sum_{i=1}^N n_i} \times \mathbb{R}^{p_i} \to \mathbb{R}$ *the individual*

functional of agent $\mathcal{A}_j^F$. $y_{-j} = (y_1, \cdots, y_{j-1}, y_{j+1}, \cdots, y_M)$ *is the decision strategy except that of* $\mathcal{A}_j^F$. $\mathbf{x} = (x_1, \cdots, x_i, \cdots, x_N)$ *is the strategy profile of all leaders.* θ_j^F *is the individual parameter of* $\mathcal{A}_j^F$. $\Gamma^L = \prod_{i=1}^N \Gamma_i^L$ *is the admissible set of all leader strategies.* $\mathcal{P}^F = (\theta_1^F, \cdots \theta_j^F \cdots, \theta_M^F) \in \Theta^F := \prod_{j=1}^M \Theta_j^F \subseteq \prod_{j=1}^M \mathbb{R}^{p_j}$ *denotes the parameter sets amongst all agents while* $\Theta_j^F \subseteq \mathbb{R}^{p_j}$ *the individual parameter support for* $\mathcal{A}_j^F$.

Thus, in what follows, we may use G_{LF} to represent an ML/MF game. The solvability notion of the LF game is given in the following.

Definition 6.4 (Stackelberg-Nash equilibrium)**.** *A Stackelberg-Nash equilibrium for* G_{LF}, *is an* $(N+M)$*-tuple* $(\bar{x}_1, \cdots, \bar{x}_N; \bar{y}_1(\cdot), \cdots, \bar{y}_M(\cdot))$, *where the best response functional* $\bar{\mathbf{y}}(\cdot) := (\bar{y}_1(\cdot), \cdots, \bar{y}_M(\cdot)) : \Gamma^L \to \Gamma^F$ *satisfies:*

$$\mathcal{J}_j^F(\bar{y}_j(\mathbf{x}); \bar{y}_{-j}(\mathbf{x}); \mathbf{x}; \theta_j^F) \leq \mathcal{J}_j^F(y_j(\mathbf{x}); \bar{y}_{-j}(\mathbf{x}); \mathbf{x}; \theta_j^F), \quad \forall y_j \in \Gamma_j^F, \quad \forall j \in \mathcal{I}^F, \tag{6.1}$$

for any given $\mathbf{x} \in \Gamma^L$, *and* $\bar{\mathbf{x}} := (\bar{x}_1, \cdots, \bar{x}_N) \in \Gamma^L$ *satisfies*

$$\mathcal{J}_i^L(\bar{x}_i; \bar{x}_{-i}; \bar{\mathbf{y}}(\bar{x}_i, \bar{x}_{-i}); \theta_i^L) \leq \mathcal{J}_i^L(x_i; \bar{x}_{-i}; , \bar{\mathbf{y}}(x_i, \bar{x}_{-i}); \theta_i^L), \quad \forall x_i \in \Gamma_i^L, \quad \forall i \in \mathcal{I}^L. \tag{6.2}$$

Remark 6.2. *Noting that* $\bar{\mathbf{y}}(\mathbf{x})$ *is actually an* NE *of the follower subgame* $G_F := (\mathcal{A}^F, \mathbf{J}^F, \Gamma^F, \Theta^F)$, *where* $\mathbf{J}^F$ *is parameterized by the pre-announced leaders' strategy* $\mathbf{x} \in \Gamma^L$. *Thus, the mapping* $\bar{\mathbf{y}}(\cdot) : \Gamma^L \to \Gamma^F$ *is called the* NE *best response of followers. The Stackelberg-Nash equilibrium for* G_{LF} *can thus be written as* $(\bar{x}_1, \cdots, \bar{x}_N; \bar{y}_1(\bar{\mathbf{x}}), \cdots, \bar{y}_M(\bar{\mathbf{x}}))$ *to emphasize its dependence on* $(\bar{y}_1(\bar{\mathbf{x}}), \cdots, \bar{y}_M(\bar{\mathbf{x}}))$.

As we mentioned before, H. Von Stackelberg first put forward Stackelberg game in his book [173], which was based on the market structure theory. In fact, he mentioned three cases in his book. The first case was that each of the two duopolies (firms) was striving for dominating the market and took the position of independence.

Then, both of them achieved the "independent supply" and obtained the greatest profit. This situation is referred as the "Bowley duopoly". The "Bowley duopoly" possibly becomes market dominance for one of the two duopolies in the end, which is impossible that two firms will win forever. The second case was that each of the two duopolies wanted to be the "follower" since that was better for themselves. This situation is referred as the "Cournot duopoly". The difference between the two cases is that each firms in the "Bowley duopoly" is oriented around the change possibilities of the rival supply and does not note the actual rival supply, whereas each firms in the "Cournot duopoly" is oriented around the actual rival supply and ignores its change possibilities. The third case was that one firms strived towards a position of independence when the other firm favours a position of dependence. In this case, an equilibrium occurs as everyone orientates his action to what gives him the greatest profit and no one wants to change the actual price structure. In [173], it was also referred as "asymmetric duopoly".

For example, we continuously use the notations in Cournot game. However, unlike Cournot game, it should first consider the follower's profit (we suppose it is firm 2 here). To calculate the Stackelberg equilibrium, it should first take the derivative of $\mathcal{J}_2$, and set this to zero for maximization (the firms want to maximize their profits, thus the inequality in Definition 6.4 should has an opposite direction here):

$$\frac{\partial \mathcal{J}_2}{\partial x_2} = \frac{\partial P(x_1 + x_2)}{\partial x_2} x_2 + P(x_1 + x_2) - \frac{\partial C_2(x_2)}{\partial x_2} = 0.$$

The best responses of the firm $\mathcal{A}_2$ is the values of $\bar{x}_2(x_1)$ that satisfy above equation and it depends on x_1. Next, it should consider the leader's profit. Putting $\bar{x}_2(x_1)$ into $\mathcal{J}_1$, taking the derivative of $\mathcal{J}_1$, and set this to zero for maximization:

$$\frac{\partial \mathcal{J}_1}{\partial x_1} = \frac{\partial P(x_1 + \bar{x}_2)}{\partial \bar{x}_2} \cdot \frac{\partial \bar{x}_2(x_1)}{\partial x_1} x_1 + \frac{\partial P(x_1 + \bar{x}_2)}{\partial x_1} x_1 + P(x_1 + \bar{x}_2) - \frac{\partial C_1(x_1)}{\partial x_1} = 0.$$

The best responses of the firm $\mathcal{A}_1$ is the values of $\bar{x}_1$ that satisfy above equation and the Stackelberg equilibrium is the pair $(\bar{x}_1, \bar{x}_2)$. More specifically, if the price function is linear $P(x_1 + x_2) = a - b(x_1 + x_2)$ and the cost is quadratic $C_i(x_i) = cx_i^2$, where a, b, $c > 0$. Then, for given x_1,

$$\frac{\partial \mathcal{J}_2}{\partial x_2} = \frac{\partial P(x_1 + x_2)}{\partial x_2} x_2 + P(x_1 + x_2) - \frac{\partial C_2(x_2)}{\partial x_2} = 0$$

$$\Longrightarrow -bx_2 + a - b(x_1 + x_2) - cx_2 = 0.$$

The best responses of $\mathcal{A}_2$ is

$$\bar{x}_2 = \frac{a - bx_1}{2b + c},$$

Putting $\bar{x}_2(x_1)$ into $\mathcal{J}_1$,

$$\frac{\partial \mathcal{J}_1}{\partial x_1} = \frac{\partial P(x_1 + \bar{x}_2)}{\partial \bar{x}_2} \cdot \frac{\partial \bar{x}_2(x_1)}{\partial x_1} x_1 + \frac{\partial P(x_1 + \bar{x}_2)}{\partial x_1} x_1 + P(x_1 + \bar{x}_2) - \frac{\partial C_1(x_1)}{\partial x_1} = 0$$

$$\Longrightarrow a - 2bx_1 - \frac{ab}{2b + c} + \frac{2b^2}{2b + c} x_1 - 2cx_1 = 0.$$

The best responses of $\mathcal{A}_1$ and $\mathcal{A}_2$ are

$$\bar{x}_1 = \frac{a(b + c)}{2(b^2 + 3bc + c^2)},$$

and the best responses of $\mathcal{A}_2$ is

$$\bar{x}_2 = \frac{a - b\bar{x}_1}{2b + c} = \frac{a(b^2 + 5bc + 2c^2)}{2(2b + c)(b^2 + 3bc + c^2)},$$

Thus, the Stackelberg equilibrium is the pair

$$\left(\frac{a(b^2 + 5bc + 2c^2)}{2(2b + c)(b^2 + 3bc + c^2)}, \frac{a(b + c)}{2(b^2 + 3bc + c^2)} \right).$$

Next, a more complicated example of Stackelberg game involving one leader and two followers will be given. Similarly, we assume that the market price is determined

by

$$P(x_1 + x_2 + x_3) = a - b(x_1 + x_2 + x_3),$$

where $P(x_1 + x_2 + x_3)$ is the price of all goods produced in a year, $a, b > 0$ are constants, $x_i \in \mathbb{R}$ is the value of quantity or supply of the firm $\mathcal{A}_i$, $i = 1, 2, 3$. $\mathcal{A}_1$ is assigned to the leader and $\mathcal{A}_2$, $\mathcal{A}_3$ are assigned to the followers. Note that all these parameters are one-dimensional.

The selling quantities or supplies from them are assumed to be directly proportional to the Cobb-Douglas production function (see [7]), which only differ a multiple of unit price. Therefore, their selling quantities or supplies can be given as follows.

$$x_i = \alpha L_i^{\beta} C_i^{\gamma}, \quad i = 1, 2, 3,$$

where α represents the total factor productivity that is assumed to be $\alpha = 1$ for simplicity, L and C represent the labor available and the capital investment, β and γ represent the output elasticities of labor and capital respectively, which are constants depending on the technology levels.

The cost functions of leader $\mathcal{A}_1$ and followers $\mathcal{A}_2$ and $\mathcal{A}_3$, which are related to the labor available and capital invested, are shown as

$$c_i(C_i) = c_i^1 L_i + c_i^2 K_i, \quad i = 1, 2, 3,$$

with constants c_i^1, c_i^2, and the profit functions of the three firms $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ are given as

$$\begin{aligned}\mathcal{J}_i =& P(x_1 + x_2 + x_3)x_i - c_i(K_i) \\ =& L_i^{\beta} C_i^{\gamma}[a - b(L_1^{\beta} C_1^{\gamma} + L_2^{\beta} C_2^{\gamma} + L_3^{\beta} C_3^{\gamma})] - (c_i^1 L_i + c_i^2 C_i), \quad i = 1, 2, 3.\end{aligned}$$

The firms $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ seek their own optimal investment for capital in this Stackelberg game. Therefore, for simplicity, the labor available L_i, $i = 1, 2, 3$, are assumed to be fixed. Similarly, we can also keep the capital investment fixed and solve the optimal input for labor instead.

The leader first announces a strategy. Then, for given C_1, we take the partial derivative of $\mathcal{J}_i$ to C_i, $i = 2$ or 3, and set this to zero,

$$\frac{\partial \mathcal{J}_i}{\partial C_i} = L_i^{\beta}\{\gamma C_i^{\gamma-1}[a - b(L_1^{\beta}C_1^{\gamma} + L_2^{\beta}C_2^{\gamma} + L_3^{\beta}C_3^{\gamma})] + C_i^{\gamma}(-bL_i^{\beta}\gamma C_i^{\gamma-1})\} - c_i^2 = 0.$$

For simplicity, we suppose that $\gamma = 0.5$, then the optimal supplies of the followers (which can be considered as a Cournot equilibrium) are

$$\begin{cases} \bar{C}_2 = \left(\dfrac{L_2^{\beta}(a - bL_1^{\beta}C_1^{\frac{1}{2}})(bL_3^{2\beta} + 2c_3^2)}{4(bL_2^{2\beta} + c_2^2)^2 - b^2L_2^{2\beta}L_3^{2\beta}}\right)^2, \\ \bar{C}_3 = \left(\dfrac{L_3^{\beta}(a - bL_1^{\beta}C_1^{\frac{1}{2}})(bL_2^{2\beta} + 2c_2^2)}{4(bL_3^{2\beta} + c_3^2)^2 - b^2L_3^{2\beta}L_2^{2\beta}}\right)^2. \end{cases}$$

Next, putting $\bar{C}_2$ and $\bar{C}_3$ into the leader's profit function, taking the derivative of $\mathcal{J}_1$, and setting the partial derivative below to zero,

$$\begin{aligned} \frac{\partial \mathcal{J}_1}{\partial C_1} = &- bL_1^{2\beta} - c_1^2 + \frac{1}{2}(aL_1^{\beta} - bL_1^{\beta}L_2^{\beta}C_2^{\frac{1}{2}} - bL_1^{\beta}L_3^{\beta}C_3^{\frac{1}{2}})C_1^{-\frac{1}{2}} \\ &- bL_1^{\beta}[L_2^{\beta}(\frac{\partial C_2^{\frac{1}{2}}}{\partial C_1}) + L_3^{\beta}(\frac{\partial C_3^{\frac{1}{2}}}{\partial C_1})]C_1^{\frac{1}{2}} = 0, \end{aligned}$$

with

$$\begin{cases} \dfrac{\partial C_2^{\frac{1}{2}}}{\partial C_1} = -\dfrac{bL_1^{\beta}L_2^{\beta}}{2}\dfrac{(bL_3^{2\beta} + 2c_3^2)C_1^{-\frac{1}{2}}}{4(bL_2^{2\beta} + c_2^2)^2 - b^2L_2^{2\beta}L_3^{2\beta}}, \\ \dfrac{\partial C_3^{\frac{1}{2}}}{\partial C_1} = -\dfrac{bL_1^{\beta}L_3^{\beta}}{2}\dfrac{(bL_2^{2\beta} + 2c_2^2)C_1^{-\frac{1}{2}}}{4(bL_3^{2\beta} + c_3^2)^2 - b^2L_3^{2\beta}L_2^{2\beta}}. \end{cases}$$

By some elementary calculation, it follows that the Stackelberg equilibrium of the

capital investments are $(\bar{C}_1, \bar{C}_2, \bar{C}_3)$, where

$$\begin{cases} \bar{C}_1 = \left(\dfrac{aL_1^{\beta}\Lambda_2}{\Lambda_1\Lambda_2 + 2bL_1^{\beta}(a - 2bL_1^{\beta})(bL_2^{2\beta}L_3^{2\beta} + c_3^2L_2^{2\beta} + c_2^2L_3^{2\beta})} \right)^2, \\ \bar{C}_2 = a^2L_2^{2\beta}(bL_3^{2\beta} + 2c_3^2)^2 \left(\dfrac{\Lambda_1\Lambda_2 + 2bL_1^{\beta}(a - 2bL_1^{\beta})(bL_2^{2\beta}L_3^{2\beta} + c_3^2L_2^{2\beta} + c_2^2L_3^{2\beta})}{\Lambda_2(\Lambda_1\Lambda_2 + 2bL_1^{\beta}(a - 2bL_1^{\beta})(bL_2^{2\beta}L_3^{2\beta} + c_3^2L_2^{2\beta} + c_2^2L_3^{2\beta}))} \right)^2, \\ \bar{C}_3 = a^2L_3^{2\beta}(bL_2^{2\beta} + 2c_2^2)^2 \left(\dfrac{\Lambda_1\Lambda_3 + 2bL_1^{\beta}(a - 2bL_1^{\beta})(bL_3^{2\beta}L_2^{2\beta} + c_2^2L_3^{2\beta} + c_3^2L_2^{2\beta})}{\Lambda_3(\Lambda_1\Lambda_3 + 2bL_1^{\beta}(a - 2bL_1^{\beta})(bL_3^{2\beta}L_2^{2\beta} + c_2^2L_3^{2\beta} + c_3^2L_2^{2\beta}))} \right)^2, \end{cases}$$

with

$$\begin{cases} \Lambda_1 = 2bL_1^{2\beta} + 2c_1^2, \quad \Lambda_2 = (2bL_2^{2\beta} + 2c_2^2)^2 - b^2L_2^{2\beta}L_3^{2\beta}, \\ \Lambda_3 = (2bL_3^{2\beta} + 2c_3^2)^2 - b^2L_3^{2\beta}L_2^{2\beta}. \end{cases}$$

More details about the Stackelberg game will be introduced in the following sections.

6.1.3 Symmetric game

Next, we present a class of multiple-agent games with a special structure.

Definition 6.5 (Symmetric game)**.** *A multiple-agent game G is called symmetric if for any permutation $\gamma : \mathcal{I} \mapsto \mathcal{I}$, it holds that*

$$\mathcal{J}_i(x_1, \cdots, x_N) = \mathcal{J}_{\gamma^{-1}(i)}(x_{\gamma(1)}, \cdots, x_{\gamma(N)}), \tag{6.3}$$

with $N = |\mathcal{I}|$.

The symmetric games arise naturally from the models of automated-agent interactions where the agents possess identical circumstances, capabilities and perspectives. [36] provided a class of symmetric games and investigated the stability of their NE. [31] analyzed the symmetric equilibrium in repeated games where each stage game is symmetric. [56] showed that a 2-strategy symmetric game must have a pure-strategy NE.

For the asymmetric system with heterogeneous objective functions and parameters (like equation (2.5) and equation (2.6) in [92, Page 899]), when the population gets large, the computational complexity increases rapidly (see [92, Page 912], the number of terms in equation (5.5) has $O(N \times M)$ order). However, for the symmetric games, it can be investigated by specific methods according to their particular structures and the symmetry supports a more compact representation which also brings a good property for computation (see [56]). Thus, it is interesting to introduce some symmetric structure in an ML/MF game context where all leaders/followers may share some identical decision characters. To this end, we may introduce the following definition.

Definition 6.6 (Symmetric LF game)**.** *An* ML/MF *game* G_{LF} *is called symmetric if* (GF1) *and* (GL1) *hold*

(GF1) *For any given pre-committed leaders' strategies* $\mathbf{x}$*, it holds that*

$$\mathcal{J}_j^F(y_j; y_{-j}; \mathbf{x}; \theta_j^F) = \mathcal{J}^F_{(\gamma^F)^{-1}(j)}(y_{\gamma^F(j)}; y_{\gamma^F(-j)}; \mathbf{x}; \theta^F_{(\gamma^F)^{-1}(j)}),$$

for any permutation $\gamma^F : \mathcal{I}^F \to \mathcal{I}^F$*, where* $y_{\gamma^F(-i)} := (y_{\gamma^F(1)}, \cdots, y_{\gamma^F(i-1)}, y_{\gamma^F(i+1)}, \cdots, y_{\gamma^F(N)})$.

(GL1) *For given followers'* NE *best response* $\bar{\mathbf{y}}(\cdot)$ *(if exists), and any admissible leader strategy profile* $(x_1, \cdots, x_N) \in \Gamma^L$*, it holds that*

$$\mathcal{J}_i^L(x_i; x_{-i}; \bar{\mathbf{y}}(x_i, x_{-i}); \theta_i^L) = \mathcal{J}^L_{(\gamma^L)^{-1}(i)}(x_{\gamma^L(i)}; x_{\gamma^L(-i)}; \bar{\mathbf{y}}(x_{\gamma^L(i)} x_{\gamma^L(-i)}); \theta^L_{(\gamma^L)^{-1}(i)}),$$

for any permutation $\gamma^L : \mathcal{I}^L \to \mathcal{I}^L$*, where* $x_{\gamma^L(-i)} := (x_{\gamma^L(1)}, \cdots, x_{\gamma^L(i-1)}, x_{\gamma^L(i+1)}, \cdots, x_{\gamma^L(N)})$.

Based on the aforementioned general games with NE and Stackelberg-Nash equilibrium notions, we are now ready to address the sequential decision structure of the LF game.

6.1.4 Sequential optimization

By Definition 6.4, the scheme for searching the Stackelberg-Nash equilibrium of G_{LF} problem can be decomposed into the following two optimization sub-problems in an "sequential" manner.

(LF1): The Followers' subgame

The crux in this subgame is that the leaders move first by announcing their strategies and anticipate the possible responses from all followers towards such announcements. For any pre-committed leaders' strategy profile $\mathbf{x} \in \Gamma^L$, all the followers $\{\mathcal{A}_j^F\}_{j\in\mathcal{I}^F}$ will face an NE problem, as specified as follows. Actually, the follower $\mathcal{A}_j^F$ will aim to solve

$$\textbf{(LF1)} \begin{cases} \text{minimize} & \mathcal{J}_j^F(y_j; y_{-j}; \mathbf{x}; \theta_j^F), \\ \text{subject to} & y_j \in \Gamma_j^F, \end{cases} \tag{6.4}$$

where θ_j^F is an individual parameter of agent $\mathcal{A}_j^F$, and y_{-j} the strategy profile of all other followers. By Remark 6.2, an M-tuple of followers' strategy profile $(\bar{y}_1(\cdot), \cdots, \bar{y}_M(\cdot))$ is the NE response of followers, if $(\bar{y}_1(\mathbf{x}), \cdots, \bar{y}_M(\mathbf{x}))$ is an NE of **(LF1)** (i.e., satisfies condition (6.1) in Definition 6.4) for $\forall \mathbf{x} \in \Gamma^L$. Noting herein $\mathbf{x}$, θ_j^F are exogenous parameters while y_{-j} is NE decision profile for other peers, and y_j is the principal decision variable for $\mathcal{A}_j^F$.

(LF2): The Leaders' subgame

After solving **(LF1)**, one (not necessary to be unique) NE response $\bar{\mathbf{y}}(\cdot)$, from the standpoint of followers, can be obtained for each pre-announced and parameterized $\mathbf{x} \in \Gamma^L$. Then all leaders $\{\mathcal{A}_i^L\}_{i\in\mathcal{I}^L}$ will face an NE problem. We specify the ith

leader's subgame as follows

$$\textbf{(LF2)} \begin{cases} \text{minimize} & \mathcal{J}_i^L(x_i; x_{-i}; \bar{\mathbf{y}}(x_i, x_{-i}); \theta_i^L), \\ \text{subject to} & x_i \in \Gamma_i^L, \end{cases} \tag{6.5}$$

where θ_i^L is an individual parameter of agent $\mathcal{A}_i^L$. Then the NE response $\bar{\mathbf{x}}$ of **(LF2)** is the Stackelberg-Nash equilibrium strategy profile of leaders (i.e., satisfies condition (6.2) in Definition 6.4). Moreover, the Stackelberg-Nash equilibrium strategy profile of followers can be further determined by $(\bar{y}_1(\bar{\mathbf{x}}), \cdots, \bar{y}_M(\bar{\mathbf{x}}))$. Thus, by Remark 6.2, the Stackelberg-Nash equilibrium for G_{LF} can be denoted by $(\bar{x}_1, \cdots, \bar{x}_N; \bar{y}_1(\bar{\mathbf{x}}), \cdots, \bar{y}_M(\bar{\mathbf{x}}))$ instead of $(\bar{x}_1, \cdots, \bar{x}_N; \bar{y}_1(\cdot), \cdots, \bar{y}_M(\cdot))$.

6.1.5 Information structure

We have specified the scheme of a general LF game, and it is observed that the LF game consists of a lower level subgame **(LF1)** for the followers and an upper level subgame **(LF2)** for the leaders. Moreover, **(LF1)**, **(LF2)** together formalize an iterative decision pattern. Within it, the involved multiple leaders/followers are posed in distinctive hierarchies along with complex interactions in lower and upper levels or between. For further analysis, it becomes necessary to identify the related information structure. In what follows, we also compare the information structures affixed to LF- and Nash-game.

We now specify the information structures between NE of Definition 6.2 and Stackelberg equilibrium of Definition 6.4. We firstly specify the information structure of NE with Definition 6.2.

$$\begin{cases} \mathcal{F}_i = \{\theta_i, \mathcal{J}_i(\cdot), \Gamma_i\}, & \text{the individual information of agent } \mathcal{A}_i, \\ \mathcal{F}_0^G = \{\mathcal{P}, \mathbf{J}(\cdot), \Gamma\}, & \text{the complete information of all agents } \mathcal{A}. \end{cases} \tag{6.6}$$

By Definition 6.2, it follows that each $\mathcal{A}_i$ should access the complete information $\mathcal{F}_0^G$

in order to obtain an *exact* NE in game G. Next, we specify the information structures in the LF game, from the leaders' and the followers' standpoints respectively.

$$\begin{cases} \mathcal{F}_j^F = \left\{\mathbf{x}, \theta_j^F, \mathcal{J}_j^F(\cdot), \Gamma_j^F\right\}, & \text{the individual information of follower } \mathcal{A}_j^F, \\ \mathcal{F}^F = \left\{\mathbf{x}, \mathcal{P}^F, \mathbf{J}^F(\cdot), \Gamma^F\right\}, & \text{the information of all followers } \mathcal{A}^F, \\ \mathcal{F}_i^L = \left\{\theta_i^L, \mathcal{J}_i^L(\cdot), \Gamma_i^L\right\}, & \text{the individual information of leader } \mathcal{A}_i^L, \\ \mathcal{F}_0^{LF} = \left\{\mathcal{P}^F, \mathbf{J}^F(\cdot), \mathcal{P}^L, \mathbf{J}^L(\cdot), \Gamma^F, \Gamma^L\right\}, & \text{the information of all agents.} \end{cases} \tag{6.7}$$

From the standpoint of a generic follower, he will compete with other follower peers to achieve an NE given the pre-committed leader strategy profile $\mathbf{x}$. Thus, in the followers' subgame **(LF1)**, to search the exact equilibrium strategies, each follower $\mathcal{A}_j^F$, $j \in \mathcal{I}^{\mathcal{F}}$ should access the information $\mathcal{F}^F$. This is because for searching equilibrium in **(LF1)**, it is necessary to access the complete information of all other follower peers (especially, their parameters $\mathcal{P}^F = (\theta_1^F, \cdots, \theta_M^F)$). Otherwise, it becomes intractable for the given follower to write down the equilibrium condition and compute the related NE equilibrium strategy. We remark that all followers are competitive in **(LF1)** thus there has no stimulus for them to set some information sharing channel. Also, the followers do not need to access the leaders information i.e., $\mathcal{J}^L(\cdot)$, Γ^L and $\mathcal{P}^L$. In fact, the pre-committed strategy $\mathbf{x}$ is already a sufficient statistic for all necessary information needed on the followers' side. Essentially, this is due to the iterative decision structure between the leaders and the followers.

The situation becomes rather different from the leaders side. Before announcing the pre-committed strategy, each leader $\mathcal{A}_i^L$ should firstly anticipate the possible followers' best responses by solving the followers' subgame **(LF1)**. Hence, information $\mathcal{F}^F$ becomes a must for each $\mathcal{A}_i^L$ to configure such anticipation. Otherwise, it is impossible for the leaders to quantify the best response. In particular, the parameters among all followers should be accessible for each leader when computing the best response functional. Moreover, each leader $\mathcal{A}_i^L$ competes with the other leaders to

achieve an NE. Thus, $\mathcal{A}_i^L$ needs to access the complete information $\mathcal{F}_0^{LF}$ of the whole game. Again, all leaders are also non-cooperative thus there has no motivation for them to share the information via some communication channels.

In conclusion, to achieve the exact Stackelberg-Nash equilibrium $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$, each follower $\mathcal{A}_j^F$ needs to access $\mathcal{F}^F$ while each leader $\mathcal{A}_i^L$ needs to access $\mathcal{F}_0^{LF}$. Hence, there exhibits some asymmetric information between the leaders and the followers when searching the Stackelberg equilibrium. Especially, the leaders should get to know all the objective functions $\mathcal{J}_j^F(y_j; y_{-j}; \mathbf{x}; \theta_j^F)$ and $\mathcal{J}_i^L(x_i; x_{-i}; \mathbf{y}; \theta_i^L)$, $i \in \mathcal{I}^L$, $j \in \mathcal{I}^F$ when calibrating the NE response of followers, whereas it is not necessary for the followers to get to know $\mathcal{J}_i^L(x_i; x_{-i}; \mathbf{y}; \theta_i^L)$, $i \in \mathcal{I}^L$, when specifying his own strategy. Instead, each follower only needs to know the leader strategy profile $\mathbf{x}$, which suffices to determine its strategy in the Stackelberg equilibrium. In other words, the pre-committed $\mathbf{x}$ should be a sufficient statistic summarizing all information needed from the leaders.

6.2 The Weakly-coupled LF Game with Model Uncertainty

In principle, applying standard optimization results, we can still derive the (exact) equilibrium for **(LF1)**, **(LF2)** respectively. Then, the Stackelberg-Nash equilibrium $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ can thus be designed. However, in presence of a large-population system where $N \gg 2$, $M \gg 2$, the aforementioned standard approach fails to work. Moreover, there arise two significant characteristics when considering LF in a large-population setting.

(1) (**Large scale and curse of dimensionality**). The first character is related to the modeling dimension and computational burden. Unlike the classical LF games (e.g., [4, 190]), in lieu of some *low-dimensional* equilibrium system, we have

to solve a system of equilibrium conditions build on a complex decision mechanism in the current LF setting. Roughly speaking, the number of equilibrium conditions in LF is in the same order to $|\mathcal{I}^L| + |\mathcal{I}^F|$, the sum of all agents' cardinalities. If $|\mathcal{I}^L| + |\mathcal{I}^F|$ is large enough, the high dimensionality of our LF system brings us considerable flexibility to model various applications. However, it also yields a rather heavy computational burden. The so-called the "curse of dimensionality" is used to describe the rapid growth in the difficulty of a problem as the number of the dimension increases (see [117, 150]). For example, a system with 1000 decision markers with binary states needs order $2^{1000} \approx 10^{301}$ computational resources, which is very difficult to tackle. Therefore, it is desirable and necessary to find out an alternative approach to circumvent such computational hurdles. Our main concern here is to treat such large-scale systems by employing a more effective strategy profile with an acceptable computation load. This is valuable for both theoretical analysis and real applications.

(2) (**Non-cooperation, information segmentation, and model uncertainty**) The second character connects to the modeling accuracy. Actually, it is well documented that any decision problems should be subject to possible model uncertainty. In particular, the parameter uncertainty in static optimization. Note that all agents in G_{LF} are non-cooperative thus there is no natural channel for the leaders or the followers to communicate or share their individual information (e.g., individual parameters θ_i^L, θ_j^F and objective functions $\mathcal{J}_i^L(\cdot)$, $\mathcal{J}_j^F(\cdot)$). Thus, there arise some *information segmentation* amongst all involved agents to access the complete information for all population. By segmentation, we mean each agent should know his own individual parameter but has no intention to share with other agents. As a consequence, the complete information set $\mathcal{F}_0^{LF}$ is partitioned into multiple segmented subsets for each agent involved. For a generic agent, his own parameter is totally observable for himself however becomes some hidden variable for other agents.

Subsequently, when we assume such hidden variables follow some statistically independent distributions, then our solution concept should be symmetric or exchangeable. Keep this in mind, in what follows, we might assume $\mathcal{J}_1^L(\cdot) = \cdots = \mathcal{J}_N^L(\cdot)$, $\mathcal{J}_1^F(\cdot) = \cdots = \mathcal{J}_M^F(\cdot)$, which can be denoted by $\mathcal{J}^L$ and $\mathcal{J}^F$ respectively, and $\Gamma_1^L = \cdots = \Gamma_N^L$, $\Gamma_1^F = \cdots = \Gamma_M^F$, which can be denoted by Γ^L and Γ^F respectively. Meanwhile, from now on, we let $n = n_1 = \cdots = n_N$, $m = m_1 = \cdots = m_M$ and $\Gamma_i^L \subseteq \mathbb{R}^n$, $\Gamma_j^F \subseteq \mathbb{R}^m$, for any $i \in \mathcal{I}^L$, $j \in \mathcal{I}^F$. We are now ready to formally introduce the **w-LF**.

Definition 6.7 (w-LF). *A* **w-LF** *is a octuple* $G_{w-LF} :=(\mathcal{A}^L,\ \mathcal{A}^F;\ \mathbf{J}^L,\ \mathbf{J}^F;\ \Gamma^L,\ \Gamma^F;\ \Theta^L,\ \Theta^F)$*, where* $\mathcal{A}^L := \{\mathcal{A}_i^L\}_{i\in\mathcal{I}^L}$ *is the set of all leaders.* $\mathcal{A}_i^L$ *denotes the* i^{th} *leader.* $\mathcal{I}^L = \{1, \cdots, N\}$ *is the leaders' indices set.* $\mathcal{A}^F := \{\mathcal{A}_i^F\}_{i\in\mathcal{I}^F}$ *is the set of all followers.* $\mathcal{I}^F = \{1, \cdots, M\}$ *is the followers' indices set,* $\mathcal{A}_j^F$ *denotes the* j^{th} *follower.*

$\mathbf{J}^L = (J_1^L, \cdots, J_N^L)$ *denotes the cost functional profile of* $\mathcal{A}^L$ *with* $\mathcal{J}_i^L = \mathcal{J}^L(x_i;\ x^{(N)};\ y^{(M)};\ \Pi_i^L) : \Gamma^L \times \Gamma^L \times \Gamma^F \times \Theta^L \subseteq \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}$ *be the individual cost functional of leader* $\mathcal{A}_i^L$*.* x_i *is the individual strategy taken by* $\mathcal{A}_i^L$*, while* $x^{(N)} := \frac{\sum_{i=1}^N x_i}{N}$ *is the strategy average of leaders.* y_j *is the individual strategy taken by follower* $\mathcal{A}_j^F$*, while* $y^{(M)} := \frac{\sum_{j=1}^M y_j}{M}$ *is the strategy average of followers. Random variable* Π_i^L *is the individual parameter of* $\mathcal{A}_i^L$*.* $\Gamma^L \subseteq \mathbb{R}^n$ *is the admissibility of the leaders.* $\Gamma^F \subseteq \mathbb{R}^m$ *is the admissibility of the followers. The parameter support for the leaders is* Θ^L*.*

$\mathbf{J}^F = (J_1^F, \cdots, J_M^F)$ *denotes the cost functional profile of* $\mathcal{A}^F$ *with* $\mathcal{J}_j^F = \mathcal{J}^F(y_j;\ y^{(M)};\ x^{(N)};\ \Pi_j^F) : \Gamma^F \times \Gamma^F \times \Gamma^L \times \Theta^F \subseteq \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ *the individual functional of* $\mathcal{A}_j^F$*. Random variable* Π_j^F *is the individual parameter of* $\mathcal{A}_j^F$*. The parameter support for the followers is* Θ^F*.*

Two salient features in Definition 6.7: one is the introduction of the strategy average terms (also called weak-coupling terms) $x^{(N)}$, $y^{(N)}$; another one is the for-

mulation of random variables Π_i^L, Π_j^F that connects to some subjective probability from the viewpoint of generic agent (leader/follower). We present more justifications to these characters in the following subsections.

6.2.1 The motivations for weak-coupling

Compared with the general LF problem, a crucial distinction in the **w-LF** is that the objective functionals (e.g., profits, utility functions, etc) contain the strategy average terms $x^{(N)}$, $y^{(N)}$. When analyzing the overall equilibrium, the change of an individual strategy of a given agent can be negligible when $|\mathcal{I}^L| + |\mathcal{I}^F|$ is large sufficiently, nevertheless, the change of the strategy average terms across all agents cannot be ignored.

We present some motivations for weak-coupling terms such as $x^{(N)}$, $y^{(N)}$. Virtually, such weak-coupling structures arise naturally from a variety of practical problems such as economics, biology, or management science. In [121], a repeated Cournot oligopoly was considered as the number of firms N in the market grows without boundary, that is $N \to \infty$. Here we still use $\{\mathcal{A}_i\}_{i=1}^N$ to denote the N firms for notation consistency. Firms $\{\mathcal{A}_i\}_{i=1}^N$ chose output levels simultaneously and the price was determined so that all output could be sold. More concretely, x_i was the output of firm $\mathcal{A}_i$, $\mathbf{x} = (x_1, \cdots, x_N)$ was the N-tuple of firms outputs, and

$$x^{(N)} = \frac{\sum_{i=1}^N x_i}{N},$$

was the output average for all firms.

$$P = f(\sum_{i=1}^N x_i/N^s)$$

was the inverse demand function, where the power index $s \geq 0$ allowed demand to increase with the number of firms and all $\{\mathcal{A}_i\}_{i=1}^N$ were weakly-coupled via output

average $x^{(N)}$ when $s = 1$. The demand increased more (less) quickly than the number of firms when $s > 1$ ($s < 1$). Then, the profit of $\mathcal{A}_i$ was

$$\mathcal{J}_i(x_i; x_{-i}) = x_i f(\sum_{i=1}^{N} x_i/N^s) - C(x_i),$$

where $C(x_i)$ was the cost function of $\mathcal{A}_i$. [121] gave out that when $s = 1$, that was $P = f(x^N)$, for given nontrival sequence of trigger strategy equilibrium exists if and only if there was a sequence of collusive output vectors $(\bar{x}_1, \cdots, \bar{x}_N)$ such that

1. $$\frac{\alpha}{1-\alpha}(\mathcal{J}_i^*(x_i; x_{-i}) - \mathcal{J}_i^c(x_i; x_{-i})) \geq \mathcal{J}_i^d(x_i; x_{-i}) - \mathcal{J}_i^*(x_i; x_{-i}) \quad \text{for all } i,$$

   with

   $$\mathcal{J}_i^*(x_i; x_{-i}) = \mathcal{J}_i(\bar{x}_i; \bar{x}_{-i})$$

   represented the collusive profit,

   $$\mathcal{J}_i^c(x_i; x_{-i}) = \mathcal{J}_i(x_i^c; x_{-i}^c)$$

   represented the Cournot profit and

   $$\mathcal{J}_i^d(x_i; x_{-i}) = \max_{x_i} \mathcal{J}_i(x_i; \bar{x}_{-i})$$

   represented the optimal deviation profit. x_i^c, $i = 1, \cdots, N$ were the Cournot output and x_i^*, $i = 1, \cdots, N$ were the trigger strategy equilibrium, which satisfied

   $$\mathcal{J}_i(\bar{x}_i; \bar{x}_{-i}) > \mathcal{J}_i(x_i^c; x_{-i}^c).$$

2. $$f(\sum_{i=1}^{N} \bar{x}_i/N) - f(\sum_{i=1}^{N} x_i^c/N) > K$$

   for all N and some $K > 0$.

This becomes a special case of our **w-LF** that only have the leaders or the followers.

The weak-coupling structure has also been well studied in various other literature, as sketched below.

- In [171], an infinitely repeated Cournot market involving N homogeneous firms $\{\mathcal{A}_i\}_{i=1}^N$ was investigated. It was assumed that there has no demand uncertainty. Thus, the market price P conveyed information about the production decision of each firm by the outputs $\{x_i\}_{i=1}^N$. To represent the relation between the industry output average

$$x^{(N)} = \frac{\sum_{i=1}^N x_i}{N},$$

and the market price, the author used an inverse demand function f satisfying $P = f(x^{(N)})$. Then, the profit of $\mathcal{A}_i$ was

$$\mathcal{J}_i(x_i; x_{-i}) = x_i f(x^{(N)}) - C(x_i),$$

where $C(x_i)$ was the cost function of $\mathcal{A}_i$, and all $\{\mathcal{A}_i\}_{i=1}^N$ were weakly-coupled via output average $x^{(N)}$.

- In [106], an international trade between the firms in home country (which was set as A) and foreign country (which was set as B) was studied. The number of firms in A country was N and the number of firms in B country was M. $\sum_{i=1}^N x_i$ was denoted as the total output in A country, while $\sum_{j=1}^M y_j$ was denoted as the total output in B country. The price in home country and for foreign country were related to their corresponding total output in their own country and defined as

$$P_A = P_A(\sum_{i=1}^N x_i), \quad P_B = P_B(\sum_{j=1}^M y_j),$$

and the profit of firm $\mathcal{A}_i$ or $\mathcal{B}_j$, $i = 1, \cdots, N$, $j = 1, \cdots, M$ was written as

$$\mathcal{J}_i(x_i, \sum_{i=1}^{N} x_i) = x_i P_A - C(x_i), \quad \mathcal{J}_j(y_j, \sum_{j=1}^{M} y_j) = y_j P_B - C(y_j),$$

where C was the firm's cost function. Since opening countries to trade increased the number of firms as they increased their demand, it was more convenient and reasonable to express price with the average output $x^{(N)} = \frac{1}{N}\sum_{i=1}^{N} x_i$ and $y^{(M)} = \frac{1}{M}\sum_{j=1}^{M} y_j$ when N, M became large enough. Then, under this case, the price were rewritten as

$$P_A = P_A(x^{(N)}), \quad P_B = P_B(y^{(M)}),$$

and the profit of firm $\mathcal{A}_i$ or $\mathcal{B}_j$, $i = 1, \cdots, N$, $j = 1, \cdots, M$ were rewritten as

$$\mathcal{J}_i(x_i, x^{(N)}) = x_i P_A - C(x_i), \quad \mathcal{J}_j(y_j, y^{(M)}) = y_j P_B - C(y_j).$$

Comparing to the discussion in [121], the international trade between the firms in home country and foreign country in [106] had a similar result, that was

$$\frac{\alpha_N}{1-\alpha_N} = \frac{\mathcal{J}_i^d(x_i; x^{(N)}) - \mathcal{J}_i^*(x_i; x^{(N)})}{\mathcal{J}_i^*(x_i; x^{(N)}) - \mathcal{J}_i^c(x_i; x^{(N)})}, \quad \text{for all } i,$$

$$\frac{\alpha_M}{1-\alpha_M} = \frac{\mathcal{J}_j^d(y_j; y^{(M)}) - \mathcal{J}_j^*(y_j; y^{(M)})}{\mathcal{J}_j^*(y_j; y^{(M)}) - \mathcal{J}_j^c(y_j; y^{(M)})}, \quad \text{for all } j,$$

with $\mathcal{J}^*$ represented the collusive profit, $\mathcal{J}^c$ represented the Cournot profit and $\mathcal{J}^d$ represented the optimal deviation profit. α_N and α_M were the minimum discount factor.

- In [146], a free-rider problem in lobbying game among N firms $\{\mathcal{A}_i\}_{i=1}^{N}$ was studied. It assumed that the good was traded internationally with a world price of 1 and $f(\sum_{i=1}^{N} x_i)$ was the function of the height of the tariff, where

x_i represented the lobbying contributions of $\mathcal{A}_i$ in the industry. Then the domestic price was given as $P = 1 + f(\sum_{i=1}^{N} x_i)$. The total industry capital was normalized to 1 making each firm of size $\frac{1}{N}$ and the profit of $\mathcal{A}_i$ was

$$p_i = \frac{P}{N} = \frac{1 + f(\sum_{i=1}^{N} x_i)}{N}.$$

Thus, the profit of each firm was weakly-coupled with the others via the lobbying contributions.

- In [49], a competition between two groups $\mathcal{G}_1$ and $\mathcal{G}_2$ in rent-seeking was analyzed. It supposed that $\mathcal{G}_1$ had N players denoted by $\{\mathcal{A}_i^1\}_{i=1}^{N}$ and $\mathcal{G}_2$ had M players denoted by $\{\mathcal{A}_j^2\}_{j=1}^{M}$. x_i and y_j represented the efforts of each player in $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively.

$$c_1(x^{(N)}, y^{(M)}) = \frac{Nx^{(N)}}{Nx^{(N)} + My^{(M)}}, \quad c_2(x^{(N)}, y^{(M)}) = \frac{My^{(M)}}{Nx^{(N)} + My^{(M)}}$$

were the contest success functions of $\mathcal{G}_1$ and $\mathcal{G}_2$, where

$$x^{(N)} = \frac{\sum_{i=1}^{N} x_i}{N}, \quad y^{(M)} = \frac{\sum_{j=1}^{M} y_j}{M}$$

were denoted as the effort average of group $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. The payoff functions of $\mathcal{A}_i^1$ and $\mathcal{A}_j^2$ were given by

$$p_i^1 = \frac{c_1(x^{(N)}, y^{(M)})Z}{N} - x_i, \quad p_j^2 = \frac{c_2(x^{(N)}, y^{(M)})Z}{M} - y_j,$$

where Z was the divisible rent generated by a policy implemented by the government. Again, all members' payoffs were weakly-coupled with the others via effort average.

By aforementioned examples, we can conclude that weakly-coupled structure is indeed widely applied in various scenarios including the repeated Cournot market,

international trade with tariff, rent-seeking activities, etc. Subsequently, introduction of weakly-coupled structure enables us to admit more modeling power for real application.

6.2.2 The motivations for parameter uncertainty

This subsection aims to justify another feature of **w-LF**: parameter uncertainty. Note that through the analysis of information structure in Section 6.1.5, each leader should have complete information $\mathcal{F}_0^{LF}$ about the whole game and each follower has complete information $\mathcal{F}^F$ of all the other followers. This is actually a relatively strong information structure. However, in reality, there is no such strong information structure. Thus, we usually consider another kind of game model in which some uncertain data are involved. Such uncertain information structure in the game model has also attracted great academic attention. There are two routes to tackle the uncertainty: the Bayesian method and the robust method.

For the Bayesian method, [85, 86] studied a game with an incomplete information structure where the agents cannot observe the exact value of some parameters of the game and gave out the assumptions of uncertain parameters on probability distributions called the Bayesian hypothesis. Based on such assumptions, the game can be reformulated as a game with complete information. The reformulated game is called the Bayesian equivalent of the original game.

On the other hand, the distribution-free models based on the worst-case scenario have received attention in recent years [2, 87, 141]. Each agent makes a decision according to the concept of robust optimization (see [20, 21]). Basically, in robust optimization, uncertain data are assumed to belong to some set called an uncertainty set, and then a solution is sought by taking into account the worst case in terms of the objective function value and/or the constraint violation. In an ML/MF game with parameter uncertainy, if each leader and follower has chosen a strategy *pessimistically*

and no agent can obtain more benefits by changing his/her own current strategy unilaterally (i.e., the other agents hold their current strategies), then the tuple of the current strategies of all agents is defined as a robust Stackelberg–Nash equilibrium and the problem of finding such a equilibrium is called a robust ML/MF problem.

A relevant robust equilibrium problem was studied in [92] where an ML/SF problem with model uncertainty was considered with the help of robust analysis. For any strategies $\mathbf{x}$ given by the leaders, the follower chose its strategy by solving the worst-case problem: $\min_y \mathcal{J}^F(x, y), y \in \Gamma^F$. Moreover, each leader $\mathcal{A}_i^L$, $i = 1, \cdots, N$, tried to solve the uncertain optimization problem: $\min_{x_i} \mathcal{J}^L(x_i, x_{-i}, y, \theta_i^L)$ using its own decision variable $x_i \in \Gamma^L$. By defining the worst cost function $\tilde{\mathcal{J}}^L(x_i, x_{-i}) = \sup_{\theta_i \in \theta} \mathcal{J}^L(x_i, x_{-i}, y, \theta_i^L)$, the author reformulated the robust ML/SF problem into a standard NE problem: $\min_{x_i} \tilde{\mathcal{J}}^L(x_i, x_{-i}), x_i \in \Gamma^L$ with complete information.

Based on the above discussions, we conclude that it is also meaningful and promising to incorporate model uncertainty into our LF analysis. In what follows, the information structure of G_{LF} will include some model uncertainty for each leader/follower agent in the context of a large-population system. For sake of simplicity, we focus only on the uncertainty of individual parameter θ_i^L, θ_j^F. The uncertainty arising from objective function $\mathcal{J}_i^L(\cdot)$, $\mathcal{J}_j^F(\cdot)$ can be studied similarly.

The Followers' subgame with model uncertainty

Recall our **(LF1)** and **(LF2)** in last section, we can formulate our **w-LF** problem as follows:

$$\textbf{(w-LF1)} \begin{cases} \text{minimize} \quad \mathcal{J}_j^F(y_j; y^{(M)}; x^{(N)}; \theta_j^F), \\ \text{subject to} \quad y_j \in \Gamma^F, \end{cases} \tag{6.8}$$

and

$$\textbf{(w-LF2)} \begin{cases} \text{minimize} & \mathcal{J}_i^L(x_i; x^{(N)}; \bar{y}^{(M)}(x^{(N)}); \theta_i^L), \\ \text{subject to} & x_i \in \Gamma^L, \end{cases} \tag{6.9}$$

where $\bar{y}^{(M)}(x^{(N)})$ is the average of best responses from all followers that depends on $x^{(N)}$. Noting all parameters $\{\theta_i^L\}_{i\in\mathcal{I}^L}$ and $\{\theta_j^F\}_{j\in\mathcal{I}^F}$ here are assumed to be deterministic only. At the moment, we do not introduce any randomness in our models yet hence **(w-LF1)**, **(w-LF2)** essentially are deterministic instead of stochastic optimizations. Nevertheless, as mentioned before, all agents in G_{LF} are competitive, non-cooperative, and have no strong motivation for information sharing. Therefore, a generic leader $\mathcal{A}_i^L$, from his informational point, can only observe his individual parameter θ_i^L. By contrast, all other $N + M - 1$ parameters from other agents become hidden variables and cannot be observed or accessed. A direct consequence is the leaders can no longer anticipate the NE response of followers in an exact sense. Likewise, a generic follower $\mathcal{A}_j^F$ only knows his own parameter θ_j^F and the other $M - 1$ parameters are unavailable to him. This is because each follower only needs to access the information $\mathcal{F}^F$ instead of $\mathcal{F}_0^{LF}$ that we had mentioned in last section. This leads to a model uncertainty among agents in a large-population system along with two hierarchies.

However, we still can study the asymptotic behavior as population M tends to infinity which is essential to consider a family of games with an increasing number of followers. Our analysis below will be based upon the observation that the large population limit may be employed to determine the effect of the mass of the population on any given individual. Specifically, our interest is the case when θ_j^F, $j = 1, \cdots, M$, is adequately randomized in the sense that the population exhibits certain statistical properties.

In this context, the association of the value, the specific index j plays no essential

role. What matters is the frequency of θ_j^F occurred in different segments in the measurable parameter set Θ^F. Along this way, for any given population M, we define the empirical distribution on Θ^F:

$$\pi_M^F(\theta) = \frac{1}{M}\sum_{j=1}^{M} \mathbb{1}_{\Theta^F}(\theta_j^F). \tag{6.10}$$

Within this setup, we make the following assumption:

(**A6.1**) There exists a limiting probability distribution $\pi^F(\theta)$ on Θ^F such that

$$\lim_{M\to+\infty} \pi_M^F(\theta) = \pi^F(\theta), \text{ and } \int_{\Theta^F} d\pi^F(\theta) = 1.$$

By (**A6.1**), we introduce a sequence of independent and identically distributed (i.i.d.) random variables Π_j^F generated by the limiting probability distribution π^F.

With model uncertainty, the follower parameters $\mathcal{P}^F$ are not available for each leader $\mathcal{A}_i^L$. However, through analyzing the whole system, $\mathcal{A}_i^L$ can obtain the empirical distribution π^F of the uncertain parameters as $M \to \infty$. Thus, we consider the case $M \to \infty$ and replace θ_j^F with Π_j^F that is some random variable. For any given pre-committed leader strategy profile $\mathbf{x}$, the followers' subgame becomes

$$\textbf{(w-LF1)}' \begin{cases} \text{minimize} & \mathbb{E}\mathcal{J}_j^F(y_j; y^{(M)}; x^{(N)}; \Pi_j^F), \\ \text{subject to} & y_j \in \Gamma^F. \end{cases} \tag{6.11}$$

Consequently, the leaders can estimate the followers' NE response $\bar{\mathbf{y}}(\cdot)$ by solving **(w-LF1)**$'$.

The Leaders' subgame with model uncertainty

After obtaining the estimation of the NE responses of followers $\bar{\mathbf{y}}(\cdot)$, each leader competes with the other leaders to achieve an NE. Thus, the individual parameter

θ_i^L of $\mathcal{A}_i^L$ is unavailable to the other leaders $\{\mathcal{A}_{i'}^L\}_{i'\neq i}$, and the exact NE is inaccessible either.

However, we still can study the asymptotic behavior as population N tends to infinity similarly. For any given population N, we define the empirical distribution on Θ^L:

$$\pi_N^L(\theta) = \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\Theta^L}(\theta_i^L). \tag{6.12}$$

We also make the following assumption:

(**A6.2**) There exists a limiting probability distribution $\pi^L(\theta)$ on Θ^L such that

$$\lim_{N\to+\infty} \pi_N^L(\theta) = \pi^L(\theta), \text{ and } \int_{\Theta^L} d\pi^L(\theta) = 1.$$

Through analyzing the whole system, the leaders can obtain the limiting empirical distribution π^L as $N \to \infty$. We introduce a sequence of i.i.d. random variables Π_i^L derived by the limiting probability distribution π^L. Thus, when $N \to \infty$, θ_i^L can be replaced by Π_i^L and the leaders' subgame becomes

$$\textbf{(w-LF2)}' \begin{cases} \text{minimize} & \mathbb{E}\mathcal{J}_i^L(x_i; x^{(N)}; \bar{y}^{(M)}(x^{(N)}); \Pi_i^L), \\ \text{subject to} & x_i \in \Gamma^L. \end{cases} \tag{6.13}$$

Remark 6.3. *Unlike the classical* NE *problem,* **(w-LF2)**$'$ *here is idiosyncratic from the perspective of each leader* $\mathcal{A}_i^L$*. More specifically, for the leader* $\mathcal{A}_i^L$*,* Π_i^L *follows Dirac distribution* $\delta_{\theta_i^L}$ *and* $\Pi_1^L, \cdots, \Pi_{i-1}^L, \Pi_{i+1}^L, \cdots \Pi_N^L$ *all follow the limiting distribution* π^L*.* $\mathcal{A}_i^L$ *knows its own individual parameter* $\Pi_i^L = \theta_i^L$ *exactly and* $\theta_1^L, \cdots, \theta_{i-1}^L, \theta_{i+1}^L, \cdots, \theta_N^L$ *cannot be observed by* $\mathcal{A}_i^L$ *thus become hidden (random) variables. Instead,* $\mathcal{A}_i^L$ *can only calibrate limiting distribution* π^L *across all such random variables from a macro-scale, although it is impossible to calibrate their exact values in a micro-scale. Equivalently, this distribution can be transformed into the real distribution of* $\theta_1^L, \cdots, \theta_{i-1}^L, \theta_{i+1}^L, \cdots, \theta_N^L$ *under subjective probabilistic reasoning.*

The next section will focus on applying the MFG approach to study the **w-LF**. In fact, the mean field analysis is based on some exchangeability characteristic among all the agents in a large-population system. By de Finetti theorem, any exchangeable sequence should be conditional independent w.r.t some tail σ-algebra. Hence, by conditional law of large number, the weakly-coupled term, when the population size is sufficiently large, can be approximated by some limiting process driven by such tail algebra. In particular, this may be approximated by the conditional expectation of a generic agent on such tail σ-algebra.

6.3 The Weakly-coupled LF Problem with the MFG Analysis: the General Case

As discussed before, it becomes quite intractable to compute the exact equilibrium of **w-LF** because of the "curse of dimensionality." As an alternative resolution, MFG theory provides one effective methodology to derive an equilibrium with reduced computation complexity but in an approximate or asymptotic sense.

We recall that in a large-population system, each agent interacts with others via the strategy average $x^{(N)}$ across the whole population. To effectively handle the associated weak-coupling, MFG will first construct an auxiliary optimization problem for a generic agent using a mean field heuristic (i.e., fixing the strategy average $x^{(N)}$ by its asymptotic limit x^0). In this way, each agent needs only consider a low-dimensional off-line optimization parameterized by pre-fixed term x^0. Unlike exact equilibrium calling for global/complete information, only local information is required to quantify such approximated equilibrium. Secondly, MFG continues to employ some fixed point argument to determine the frozen x^0. This is also called the CC system since the realized optimal state/decision should replicate the state-average limit pre-fixed. Again, only local information (e.g., $\mathcal{F}_i^L$ and $\mathcal{F}_j^F$) is needed

to achieve such asymptotic/approximated equilibrium (see [104, 138]) in a large-population system. Often, the asymptotic equilibrium is also called "ε-equilibrium" with more details. Besides, we assume that all leaders and followers are homogeneous and the agents in the same hierarchy play a Nash-Cournot game. Confining to the **w-LF** here, we may introduce the notion of ε-Stackelberg-Nash-Cournot (ε-SNC) equilibrium as below.

Definition 6.8 (ε-SNC equilibrium)**.** *An* ε-SNC *equilibrium of* **w-LF** *is a* $(N+M)$*-tuple* $(\bar{x}_1, \cdots, \bar{x}_N; \bar{y}_1(\cdot), \cdots, \bar{y}_M(\cdot))$*, where* $\bar{\mathbf{y}}(\cdot) : \Gamma^L \to \Gamma^F$ *satisfies that, for any given* $\mathbf{x} \in \Gamma^L$*,*

$$\begin{aligned} &\mathbb{E}\mathcal{J}_j^F(\bar{y}_j(\mathbf{x}); \bar{y}_{-j}(\mathbf{x}); \mathbf{x}; \Pi_j^F) \\ &\qquad \leq \min_{y_j \in \Gamma_j^F} \mathbb{E}\mathcal{J}_j^F(y_j(\mathbf{x}); \bar{y}_{-j}(\mathbf{x}); \mathbf{x}; \Pi_j^F) + \varepsilon(M), \quad \forall j \in \mathcal{I}^F. \end{aligned} \tag{6.14}$$

Meanwhile, find a $\bar{\mathbf{x}} \in \Gamma^L$ *satisfies*

$$\begin{aligned} &\mathbb{E}\mathcal{J}_i^L(\bar{x}_i; \bar{x}_{-i}; \bar{\mathbf{y}}(\bar{x}_i, \bar{x}_{-i}); \Pi_i^L) \\ &\qquad \leq \min_{x_i \in \Gamma_i^L} \mathbb{E}\mathcal{J}_i^L(x_i; \bar{x}_{-i}; \bar{\mathbf{y}}(x_i, \bar{x}_{-i}); \Pi_i^L) + \varepsilon(M, N), \quad \forall i \in \mathcal{I}^L, \end{aligned} \tag{6.15}$$

where $\varepsilon(M), \varepsilon(M, N) \to 0$ *a.s. as* $M, N \to \infty$*.*

To proceed, let us recall some standard notations. We denote vector as $v = (v_1, \cdots, v_n)$, $w = (w_1, \cdots, w_m)$ and sometimes abuse the formations of column- or row-vector. For a differentiable function $f(v) : \mathbb{R}^n \to \mathbb{R}$, $\frac{\partial f}{\partial v_i} : \mathbb{R}^n \to \mathbb{R}$ denotes the partial derivatives of f with respect to (w.r.t.) argument v_i, $i = 1, \cdots, n$. The gradient of f is denoted by $\nabla f = (\frac{\partial f}{\partial v_1}, \cdots, \frac{\partial f}{\partial v_n}) : \mathbb{R}^n \to \mathbb{R}^n$. For a continuously differentiable function $f(v, w) = (f_1(v, w), \cdots, f_m(v, w)) : \mathbb{R}^{n+m} \to \mathbb{R}^m$, where $f_i(v, w) : \mathbb{R}^{n+m} \to \mathbb{R}$, $i = 1, \cdots, m$ are the components of f, the Jacobian matrix of

f w.r.t. w is denoted by

$$\frac{\partial f}{\partial w} = \begin{pmatrix} \frac{\partial f_1}{\partial w_1} & \cdots & \frac{\partial f_1}{\partial w_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial w_1} & \cdots & \frac{\partial f_m}{\partial w_m} \end{pmatrix}. \tag{6.16}$$

6.3.1 The MFG scheme

In what follows, we apply the MFG approach to search for the approximated equilibrium of the **w-LF**. Recall the index sets $\mathcal{I}^F = \{1, 2, \cdots, M\}$ and $\mathcal{I}^L = \{1, 2, \cdots, N\}$, then the **w-LF** can be formulated as **(w-LF1)** and **(w-LF2)**:

$$\textbf{(w-LF1)} \begin{cases} \text{minimize} \quad \mathcal{J}_j^F(y_j; y^{(M)}; x^{(N)}; \theta_j^F), \\ \text{subject to} \quad y_j \in \Gamma^F, \quad j \in \mathcal{I}^F, \end{cases} \tag{6.17}$$

for the followers' problem and

$$\textbf{(w-LF2)} \begin{cases} \text{minimize} \quad \mathcal{J}_i^L(x_i; x^{(N)}; \bar{y}^{(M)}(x^{(N)}); \theta_i^L), \\ \text{subject to} \quad x_i \in \Gamma^L, \quad i \in \mathcal{I}^L, \end{cases} \tag{6.18}$$

for the leaders' problem. Recall that $\bar{y}^{(M)}(x^{(N)})$ is the average of the best responses of all followers that depends on $x^{(N)}$. The following assumptions are technical.

(**A6.3**)(Integrability) From the aspect of the ith leader and the jth follower, the sets of admissible strategies of the other leaders $\{\mathcal{A}_{i'}^L\}_{i' \neq i, i' \in \mathcal{I}^L}$ and followers $\{\mathcal{A}_{j'}^F\}_{j' \neq j, j' \in \mathcal{I}^F}$ satisfy:

$$x_{i'} \in L^1(\Omega; \mathbb{R}^n), \quad i' \neq i, \ i' \in \mathcal{I}^L, \qquad y_{j'} \in L^1(\Omega; \mathbb{R}^m), \quad j' \neq j, \ j' \in \mathcal{I}^F.$$

(**A6.4**)(Boundness, closeness and convexity) The sets $\Gamma^L \subseteq \mathbb{R}^n$ and $\Gamma^F \subseteq \mathbb{R}^m$ are non-empty, bounded, closed and convex.

We now proceed to analyze our **w-LF**. We first treat the sub-problem **(w-LF1)** (6.17) from the follower's perspective. To this end, we construct the following aux-

iliary problem for generic follower $\mathcal{A}_j^F$, $j \in \mathcal{I}^F$, denoted as **(w-LFA1)**:

$$\textbf{(w-LFA1)} \begin{cases} \text{minimize} & J_j^F(y_j; y^{(0,x^{(N)})}; x^{(N)}; \theta_j^F), \\ \text{subject to} & y_j \in \Gamma^F \subseteq \mathbb{R}^m, \quad j \in \mathcal{I}^F, \end{cases} \tag{6.19}$$

by letting $M \to \infty$ in (6.17) and freezing $y^{(M)}$ using its limit $y^{(0,x^{(N)})}$ (it only takes limits in M and still depends on $x^{(N)}$). Later, we will determine $y^{(0,x^{(N)})}$ using mean field reasoning. For further analysis, we give the following assumption:

(**A6.5**) For any given $x^{(N)} \in \mathbb{R}^n$, $y^{(0,x^{(N)})} \in \mathbb{R}^m$, J_j^F is strictly convex w.r.t y_j.

Since $(x_1, x_2, \cdots, x_N) \in \mathbb{R}^{nN}$ is pre-fixed, $x^{(N)}$ becomes some endogenous parameter. Then, by (**A6.4**), (**A6.5**), there exists a minimizer of (6.19) and the minimizer is unique since J_j^F is strictly convex. Therefore, the optimal strategy of (6.19) is:

$$\bar{y}_j(y^{(0,x^{(N)})}, x^{(N)}, \theta_j^F) = \underset{y_j \in \Gamma^F}{\arg\min}\, J_j^F(y_j; y^{(0,x^{(N)})}; x^{(N)}; \theta_j^F), \quad j \in \mathcal{I}^F. \tag{6.20}$$

To determine the NE, it is necessary for j^{th} follower to quantify optimal strategies of other agents. By symmetry, other agents should take parallel strategies alike $\{\bar{y}_{j'}(y^{(0,x^{(N)})}, x^{(N)}, \theta_{j'}^F)\}_{j' \neq j}$. However, noting all agents are non-cooperative, no one would like to share their own information (e.g., parameter) with others. In particular, the jth follower cannot observe the parameters $\{\theta_{j'}^F\}_{j' \neq j, j' \in \mathcal{I}^F}$ of other agents. Thus, as mentioned in last section, he cannot quantify the optimal strategies of others hence fails to compute the relevant exact equilibrium. However, the MFG approach provides some alternative resolution: instead of searching or estimating exact parameter for a given specific agent, the jth follower may estimate the empirical distribution across all agents in a micro-scale by π^F under (**A6.1**). Therefore, from $\mathcal{A}_j^F$'s aspect, the parameters $\{\theta_{j'}^F\}_{j' \neq j, j' \in \mathcal{I}^F}$ of other followers are unknown and can be treated as the random variables $\{\Pi_{j'}^F\}_{j' \neq j, j' \in \mathcal{I}^F}$. For this reason, the optimizer

$\bar{y}_{j'}$ can be reformulated as

$$\bar{y}_{j'}(y^{(0,x^{(N)})}, x^{(N)}, \Pi^F_{j'}) = \arg\min_{y_{j'}\in\Gamma^F} J^F_{j'}(y_{j'}; y^{(0,x^{(N)})}; x^{(N)}; \Pi^F_{j'}), \quad j' \neq j,\ j' \in \mathcal{I}^F. \tag{6.21}$$

If we further assume that $J^F_{j'}(\cdot, y^{(0,x^{(N)})}, x^{(N)}, \Pi^F_{j'})$ is continuous and $J^F_{j'}(y_j, y^{(0,x^{(N)})}, x^{(N)}, \cdot)$ is measurable, then by Kuratowski and Ryll-Nardzewski measurable selection theorem (see [3, 114, 161]) and noting $\Pi^F_{j'}$ is a random variable, then $\bar{y}_{j'}(y^{(0,x^{(N)})}, x^{(N)}, \Pi^F_{j'})$ is a random variable here. $\bar{y}_{j'}$ depends on $y^{(0,x^{(N)})}$ that is to be determined by a CC system. Also, $\bar{y}_j$, $\bar{y}_{j'}$ are un-determined since they depend on $x^{(N)}$ and $\{\Pi^F_{j'}\}_{j'\neq j, j'\in\mathcal{I}^F}$ are i.i.d. under $x^{(N)}$.

Applying the mean-field reasoning, and taking the conditional expectation on $\bar{y}_{j'}$ under $x^{(N)}$ in (6.21), we can obtain the CC system of our followers' problem

$$y^{(0,x^{(N)})} = \mathbb{E}(\bar{y}_{j'}(y^{(0,x^{(N)})}, x^{(N)}, \Pi^F_{j'})|x^{(N)}). \tag{6.22}$$

Here, the conditional expectation operator $\mathbb{E}(\cdot|x^{(N)})$ is due to the de Finetti theorem by noting all $\{\bar{y}_{j'}(y^{(0,x^{(N)})}, x^{(N)}, \Pi^F_{j'})\}_{j'\in\mathcal{I}_F}$ have the tail-sigma algebra generated by common term $x^{(N)}$ that might be treated as random variables. Also, recall $y^{(0,x^{(N)})}$ depends on pre-fixed profile $x^{(N)}$ that is to be determined. In this sense, (6.22) does not completely characterize $y^{(0,x^{(N)})}$ yet. By (**A6.3**), $\bar{y}_{j'} \in L^1(\Omega; \mathbb{R}^m)$ and the expectation of $\bar{y}_{j'}$ is well-defined.

Next, we analyze the leader's optimization problem **(w-LF2)** (6.18). With the followers' best response (6.20), the leaders continue to design their optimal strategies in an iterative manner. Because we had frozen $y^{(M)}$ by $y^{(0,x^{(N)})}$ in **(w-LFA1)** (6.19), (6.18) can be rewritten as

$$\begin{cases} \text{minimize} \quad J^L_i(x_i; x^{(N)}; y^{(0,x^{(N)})}; \theta^L_i), \\ \text{subject to} \quad x_i \in \Gamma^L, \quad i \in \mathcal{I}^L, \end{cases} \tag{6.23}$$

where $y^{(0,x^{(N)})}$ is given by (6.22). Similarly, we construct the following auxiliary problem for a generic leader $\mathcal{A}_i^L$, $i \in \mathcal{I}^L$, denoted as **(w-LFA2)** :

$$\textbf{(w-LFA2)} \begin{cases} \text{minimize} \quad \bar{J}_i^L(x_i; x^0; y^0; \Pi_i^L), \\ \text{subject to} \quad x_i \in \Gamma^L, \quad i \in \mathcal{I}^L, \end{cases} \tag{6.24}$$

by letting $N \to \infty$ in equation (6.23) and freezing $x^{(N)}$ by some (deterministic) quantity x^0 using the mean field reasoning. Note that $x^{(N)}$ has been frozen by x^0 in (6.24), thus $y^{(0,x^{(N)})}$ is replaced by y^0 by sending $N \to \infty$. For further analysis, we give the following assumption:

(**A6.6**) For any given $x^0 \in \Gamma^L$, $\bar{J}_i^L$ is strictly convex w.r.t x_i.

Under (**A6.4**), (**A6.6**), by similar argument as (6.20), there exists a minimizer of (6.24) that is also unique as $\bar{J}_i^L$ is strictly convex. Thus, the optimal strategy $\bar{x}_i$ satisfying:

$$\bar{x}_i(x^0, y^0, \theta_i^L) = \arg\min_{x_i \in \Gamma^L} \bar{J}_i^L(x_i; x^0; y^0; \theta_i^L), \quad i \in \mathcal{I}^L. \tag{6.25}$$

After obtaining his optimal strategy, the ith leader begins to estimate his peers' optimal strategy. Noting all leaders are non-cooperative and would no like to share their own information (parameters), so a generic agent $\mathcal{A}_i^L$ cannot completely figure out the others' strategies because $\{\theta_{i'}^L\}_{i' \neq i, i' \in \mathcal{I}^L}$ are hidden variables for him. Instead, by the discussion in last section, he would treat $\{\theta_{i'}^L\}_{i' \neq i, i' \in \mathcal{I}^L}$ using the limiting distribution π^L under (**A6.2**). Therefore, from $\mathcal{A}_i^L$'s aspect, the other leaders' individual parameters are unknown and can be treated as the random variables $\{\Pi_{i'}^L\}_{i' \neq i, i' \in \mathcal{I}^L}$. For this reason, the optimizer $\bar{x}_{i'}$ can be recast as

$$\bar{x}_{i'}(x^0, y^0, \Pi_{i'}^L) = \arg\min_{x_{i'} \in \Gamma^L} \bar{J}_{i'}^L(x_{i'}; x^0; y^0; \Pi_{i'}^L), \quad i' \neq i, i' \in \mathcal{I}^L. \tag{6.26}$$

Applying the mean field reasoning, and taking the expectation on $\bar{x}_i$ in (6.26), we

can obtain the CC condition from the leaders' perspective:

$$x^0 = \mathbb{E}\bar{x}_{i'}(x^0, y^0, \Pi^L_{i'}), \tag{6.27}$$

where y^0 inside depends on x^0. By (**A6.3**), $x_{i'} \in L^1(\Omega; \mathbb{R}^n)$ thus the expectation operator on $\bar{x}_i$ is meaningful. Combining with (6.22), we can obtain the CC system of **w-LF**:

$$\text{(CC)} \begin{cases} x^0 = \mathbb{E}\bar{x}_{i'}(x^0, y^0, \Pi^L_{i'}), \\ y^0 = \mathbb{E}(\bar{y}_{j'}(y^0, x^0, \Pi^F_{j'})|x^0). \end{cases} \tag{6.28}$$

Once the CC system (6.28) is solved, we might design the distributed strategies $\bar{y}_{j'} = \bar{y}_{j'}(y^0, x^0, \Pi^F_{j'})$ and $\bar{x}_{i'} = \bar{x}_{i'}(x^0, y^0, \Pi^L_{i'})$ for the leaders $\{\mathcal{A}^L_i\}_{i\in\mathcal{I}^L}$ and followers $\{\mathcal{A}^F_j\}_{j\in\mathcal{I}^F}$, respectively in the **w-LF**. Moreover, we can continue to verify that they form an ε-SNC equilibrium. Note that through the whole MFG procedure mentioned above, each agent can obtain its own MFG strategy without knowing the exact information of the others'.

6.3.2 Fixed point analysis

The CC system (6.28) plays an essential role to the design decentralized strategy of **w-LF**. To this end, we discuss the uniqueness and existence of our CC system (6.28). To begin with, we introduce two preliminary results and one assumption.

Proposition 6.1 (Brouwer fixed-point theorem)**.** *If Γ is a non-empty, compact, convex subset of $\mathbb{R}^n$ and $f : \Gamma \to \Gamma$ is a continuous function, then f has a fixed point $v_0 \in \Gamma$ such that $f(v_0) = v_0$.*

Proposition 6.2 (Implicit function theorem)**.** *Let $v = (v_1, \cdots, v_n)$, $w = (w_1, \cdots, w_m)$ and $f(v, w)$=$(f_1(v, w), \cdots, f_m(v, w)) : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a continuously differentiable function. Let $(v^0, w^0) = (v^0_1, \cdots, v^0_n, w^0_1, \cdots, w^0_m)$ with $f(v^0, w^0) = 0$. If the Jacobian matrix: $\frac{\partial f}{\partial w}(v^0, w^0)$ is invertible, then there exists an open set $U \subseteq \mathbb{R}^n$ containing*

"v^0" such that there exists a unique continuously differentiable function g such that $g(v^0) = w^0$, and $f(v, g(v)) = 0$ for $\forall v \in U$.

(**A6.7**)

(i) $\bar{J}^L_{i'}(x_{i'}; x^0; y^0; \Pi^L_{i'})$ is twice continuously differentiable w.r.t. $x_{i'}$ and x^0.

(ii) For any given $x^0 \in \Gamma^L$, let $\bar{J}^L_{i'}$ is strictly convex, and there is a $\bar{x}_{i'} \in \Gamma^L_{i'}$ s.t. $\nabla_{x_{i'}} \bar{J}^L_{i'}(x_{i'}; x^0; y^0; \Pi^L_{i'})|_{x_{i'}=\bar{x}_{i'}} = 0$.

(iii) For any given $x^0 \in \Gamma^L$ and the corresponding $\bar{x}_{i'} \in \Gamma^L$, the matrix $\frac{\partial \nabla_{x_{i'}} \bar{J}^L_{i'}}{\partial x_{i'}}(\bar{x}_{i'}, x^0) \neq 0$.

Since y^0 depends on the x^0 and π^F, we denote y^0 as $y^0(x^0, \pi^F)$ and $\bar{x}_{i'}(x^0, y^0(x^0, \pi^F), \Pi^L_{i'})$ as $\bar{x}_{i'}(x^0, \Pi^L_{i'})$. We first study the uniqueness and existence of x^0. To further analysis, by [96, Section 2], we introduce the following assumptions.

(**A6.8**) For each $x^0 \in \mathbb{R}^n$, $\bar{x}_{i'}(x^0, \cdot) \in \Gamma^L$, there exists a constant $L_1 \geq 0$ which is independent of $\Pi^L_{i'}$ such that $\langle \bar{x}_{i'}(x^0_1, \Pi^L_{i'}) - \bar{x}_{i'}(x^0_2, \Pi^L_{i'}), x^0_1 - x^0_2 \rangle \leq -L_1 |x^0_1 - x^0_2|^2$, for any $x^0_1, x^0_2 \in \mathbb{R}^n$.

Based on the assumptions above we have the following result of x^0.

Theorem 6.1. *Under* (**A6.1**), (**A6.2**), (**A6.7**) *and* (**A6.8**), *the* CC *system* (6.28) *admits at most one solution for x^0.*

Proof According to (**A6.7**), there exists a unqiue $\bar{x}_{i'}(x^0, y^0(x^0, \pi^F), \Pi^L_{i'})$ such that $\nabla_{x_{i'}} \bar{J}^L_{i'}(x_{i'}; x^0; y^0(x^0, \pi^F); \Pi^L_{i'})|_{x_{i'}=\bar{x}_{i'}} = 0$. By equation (6.27), it follows that

$$\langle \mathbb{E}\bar{x}_{i'}(x^0_1, \Pi^L_{i'}) - \mathbb{E}\bar{x}_{i'}(x^0_2, \Pi^L_{i'}), x^0_1 - x^0_2 \rangle = \langle x^0_1 - x^0_2, x^0_1 - x^0_2 \rangle = |x^0_1 - x^0_2|^2 \geq 0.$$

By (**A6.8**), for any $x^0_1, x^0_2 \in \Gamma^L \subseteq \mathbb{R}^n$ (note that x^0_1 and x^0_2 are deterministic here),

$$\begin{aligned} |x^0_1 - x^0_2|^2 =& \langle \mathbb{E}\bar{x}_{i'}(x^0_1, \Pi^L_{i'}) - \mathbb{E}\bar{x}_{i'}(x^0_2, \Pi^L_{i'}), x^0_1 - x^0_2 \rangle \\ =& \mathbb{E}\langle \bar{x}_{i'}(x^0_1, \Pi^L_{i'}) - \bar{x}_{i'}(x^0_2, \Pi^L_{i'}), x^0_1 - x^0_2 \rangle \leq -L_1 |x^0_1 - x^0_2|^2 \leq 0. \end{aligned}$$

Then, we have $|x_1^0 - x_2^0|^2 = 0$. Hence, the uniqueness follows. $\square$

Theorem 6.2. *Under* (**A6.1**)-(**A6.8**)*, there exists a unique solution satisfying the* CC *system* (6.28)*.*

Proof For any $x \in \mathbb{R}^n$, we denote $T = T(x) := \mathbb{E}\bar{x}_{i'}(x, \Pi_{i'}^L)$. First, under (**A6.4**), $\Gamma^L \in \mathbb{R}^n$ is non-empty, convex, and compact, then for each $x^0 \in \Gamma^L$, $T(x^0) \in \Gamma^L$. Thus T is a stable mapping on Γ^L. Second, under (**A6.7**) and Proposition 6.2, there exists a unique continuously differentiable function $\bar{x}_{i'}(x^0, \Pi_{i'}^L) \in \Gamma^L$ w.r.t. x^0 such that $\nabla_{x_{i'}} \bar{J}_{i'}^L(x_{i'}; x^0; y^0; \Pi_{i'}^L)|_{x_{i'}=\bar{x}_{i'}} = 0$. Thus, T is continuous on Γ^L. Thus, by Proposition 6.1 and Theorem 6.1, there exists a unique solution for x^0 such that $x^0 = \mathbb{E}\bar{x}_{i'}(x^0, \Pi_{i'}^L)$. Since $y^0 = \mathbb{E}(x^0, \Pi_{j'}^F)|x^0)$ depends on x^0, therefore the existence and uniqueness of y^0 follows. $\square$

In what follows, we present a special case to illustrate how the MFG scheme works.

6.4 The Weakly-coupled LF Problem: the Quadratic Functional Case

6.4.1 The procedure of quadratic weakly-coupled LF problem

This section considers an important and special case of **(LF1)** and **(LF2)** in which the functional takes quadratic-form, more specifically,

$$\begin{aligned} \mathcal{J}_j^F(y_j; y^{(M)}; x^{(N)}; \theta_j^F) =& |\Lambda_F^1 y_j - \Lambda_F^2 y^{(M)} - \Lambda_F^3 x^{(N)} - \theta_j^F|_{Q_F}^2, \quad j \in \mathcal{I}^F, \\ \mathcal{J}_i^L(x_i; x^{(N)}; y^{(M)}; \theta_i^L) =& |\Lambda_L^1 x_i - \Lambda_L^2 x^{(N)} - \Lambda_L^3 y^{(M)} - \theta_i^L|_{Q_L}^2, \quad i \in \mathcal{I}^L, \end{aligned} \tag{6.29}$$

with $\Lambda_F^1 \in \mathbb{R}^{m_1 \times m}$, $\Lambda_F^2 \in \mathbb{R}^{m_1 \times m}$, $\Lambda_F^3 \in \mathbb{R}^{m_1 \times n}$, $\Lambda_L^1 \in \mathbb{R}^{m_1 \times n}$, $\Lambda_L^2 \in \mathbb{R}^{m_1 \times n}$, $\Lambda_L^3 \in \mathbb{R}^{m_1 \times m}$, $Q_F \in \mathbb{S}^{m_1}$, $Q_L \in \mathbb{S}^{m_1}$. Moreover, $\mathcal{I}^F = \{1, 2, \cdots, M\}$ and $\mathcal{I}^L = \{1, 2, \cdots, N\}$

are the agent index sets for the followers and the leaders, respectively. For dimensional consistency, we set $n = m = m_1 = l = p$. We emphasize that quadratic optimization has always been a prototype in optimization theory (e.g., see [33, 60, 92, 148] for more recent studies) because of its structural tractability and amenable approximation to general nonlinear functional. To be consistent, we denote the related problem as **q-LF** which can be further formulated sequentially as **(q-LF1)**, **(q-LF2)**:

$$\textbf{(q-LF1)} \begin{cases} \text{minimize} & \mathcal{J}_j^F(y_j; y^{(M)}; x^{(N)}; \theta_j^F), \\ \text{subject to} & y_j \in \mathbb{R}^m, \quad j \in \mathcal{I}^F, \end{cases} \tag{6.30}$$

for the followers with decision variables $\{y_j\}_{j\in\mathcal{I}^F}$ and

$$\textbf{(q-LF2)} \begin{cases} \text{minimize} & \mathcal{J}_i^L(x_i; x^{(N)}; y^{(M)}; \theta_i^L), \\ \text{subject to} & x_i \in \bar{\Gamma} \subseteq \mathbb{R}^n, \quad i \in \mathcal{I}^L, \end{cases} \tag{6.31}$$

for the leaders with decision variables $\{x_i\}_{i\in\mathcal{I}^L}$ and $\bar{\Gamma} \subseteq \mathbb{R}^n$ is a closed convex set. Moreover, we set the following assumptions on model. For further analysis, we introduce

(**A6.9**)(Definiteness) $Q_F, Q_L > 0$.

(**A6.10**)(Full rank) Λ_F^1, Λ_L^1 are of full-rank.

(**A6.11**)(Invertibility) $(\Lambda_F^1)^T Q_F(\Lambda_F^1 - \Lambda_F^2)$ is invertible.

We point out (**A6.9**) connects to the (strict) convexity, (**A6.10**), (**A6.11**) are related to invertibility. Note that (**A6.11**) is not redundant because it cannot be implied from (**A6.9**), (**A6.10**). Here is one counter-example:

$$\Lambda_F^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad Q_F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Lambda_F^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix},$$

following (**A6.9**), (**A6.10**), however

$$(\Lambda_F^1)^\top Q_F(\Lambda_F^1 - \Lambda_F^2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix},$$

which does not have full rank. Thus (**A6.11**) is not true. Moreover, (**A6.11**) implies that $\Lambda_F^1 \neq \Lambda_F^2$, otherwise $(\Lambda_F^1)^\top Q_F(\Lambda_F^1 - \Lambda_F^2) = 0$. We now proceed to analyze **q-LF**.

As discussed in the general case, we first deal with the optimization problem **(q-LF1)** (6.30) from the standpoint of followers. To this end, we should introduce the following auxiliary problem for a generic follower $\mathcal{A}_j^F$, $j \in \mathcal{I}^F$, denoted as **(q-LFA1)**:

$$\textbf{(q-LFA1)} \begin{cases} \text{minimize} \quad J_j^F(y_j; y^{(0,x^{(N)})}; x^{(N)}; \theta_j^F) \\ \qquad\qquad := |\Lambda_F^1 y_j - \Lambda_F^2 y^{(0,x^{(N)})} - \Lambda_F^3 x^{(N)} - \theta_j^F|_{Q_F}^2, \\ \text{subject to} \quad y_j \in \mathbb{R}^m, \quad j \in \mathcal{I}^F, \end{cases} \tag{6.32}$$

by letting $M \to \infty$ in (6.30) and freezing $y^{(M)}$ using the term $y^{(0,x^{(N)})}$. Note that $y^{(0,x^{(N)})}$ will be affected by $x^{(N)}$ which is given by the leaders. Moreover, the influence of an individual follower on $y^{(0,x^{(N)})}$ may be negligible.

We examine the convexity of functional (6.32) in (**q-LFA1**). To this end, we compute the first-order (gradient vector) and second-order partial derivative (Hessian matrix) w.r.t. decision variables:

$$\begin{cases} \nabla_{y_j} J_j^F(y_j; y^{(0,x^{(N)})}; x^{(N)}; \theta_j^F) \\ \qquad = 2\Big[(\Lambda_F^1)^\top Q_F\Big(\Lambda_F^1 \bar{y}_j - \Lambda_F^2 y^{(0,x^{(N)})} - \Lambda_F^3 x^{(N)} - \theta_j^F\Big)\Big], \\ \nabla_{y_j}^2 J_j^F(y_j; y^{(0,x^{(N)})}; x^{(N)}; \theta_j^F) = 2(\Lambda_F^1)^\top Q_F \Lambda_F^1. \end{cases} \tag{6.33}$$

By (**A6.9**) and (**A6.10**), $(\Lambda_F^1)^\top Q_F \Lambda_F^1 > 0$, thus the functional J_j^F in (6.32) is strictly convex w.r.t. y_j. In fact, by (**A6.9**), all spectrum values of Q_F are positive. Then,

combining (**A6.10**) implies that the positivity of all spectrum of $(\Lambda_F^1)^\top Q_F \Lambda_F^1$. This also implies the invertibility of $(\Lambda_F^1)^\top Q_F \Lambda_F^1$.

Recall that $(x_1, x_2, \cdots, x_N) \in \mathbb{R}^{nN}$ is pre-fixed so $x^{(N)}$ becomes some endogenous parameter. Thus, **(q-LFA1)** is a strictly (uniformly) convex optimization to decision y_j, thus the optimal decision exists and should be unique, denoted by $\bar{y}_j$. In fact, the uniform convexity implies the coercivity and it is obvious any quadratic functional including (6.32) is always continuous, thus (6.32) always admits a minimizer in the full space $\mathbb{R}^m$. Moreover, the uniform convexity implies the strict convexity thus such minimizer, if exists, should be unique.

Also, note that the decision variable y_j is in full space, therefore **(q-LFA1)** is an optimization problem without constraint. Hence, the optimizer $\bar{y}_j$ can be characterized sufficiently by *zero-gradient condition*:

$$\begin{aligned} &\nabla_{y_j} J_j^F(y_j; y^{(0,x^{(N)})}; x^{(N)}; \theta_j^F) = 0 \\ \Longrightarrow &\bar{y}_j = \Theta_1(\Lambda_F^2 y^{(0,x^{(N)})} + \Lambda_F^3 x^{(N)} + \theta_j^F), \quad j \in \mathcal{I}^F, \end{aligned} \tag{6.34}$$

where $\Theta_1 = \Theta_1(\Lambda_F^1, Q_F) := [(\Lambda_F^1)^\top Q_F \Lambda_F^1]^{-1} (\Lambda_F^1)^\top Q_F$. After obtaining his optimal strategy, the jth follower begins to estimate his peers' optimal strategy. Since the agents are non-cooperative, no one will share their own information with the others and the jth follower will not know his peer's personal parameter $\{\theta_{j'}^F\}_{j' \neq j, j' \in \mathcal{I}^F}$ exactly. As we mentioned before, he can only estimate $\{\theta_{j'}^F\}_{j' \neq j, j' \in \mathcal{I}^F}$ by the limiting distribution π^F under (**A6.1**). Therefore, from $\mathcal{A}_j^F$'s aspect, the parameters of other followers are unknown and can be treated as the random variables $\{\Pi_{j'}^F\}_{j' \neq j, j' \in \mathcal{I}^F}$. For this reason, the optimizer $\bar{y}_{j'}$ can be presented as

$$\bar{y}_{j'} = \Theta_1(\Lambda_F^2 y^{(0,x^{(N)})} + \Lambda_F^3 x^{(N)} + \Pi_{j'}^F), \quad j' \neq j, \ j' \in \mathcal{I}^F. \tag{6.35}$$

We remark that $\bar{y}_j$ and $\bar{y}_{j'}$ depend on $y^{(0,x^{(N)})}$ that is to be determined by the CC system. Also, $\bar{y}_j$ and $\bar{y}_{j'}$ are un-determined since they depend on $x^{(N)}$ and $\{\Pi_{j'}^F\}_{j' \neq j, j' \in \mathcal{I}^F}$

are i.i.d. under $x^{(N)}$.

Therefore, by applying the mean field reasoning and taking the conditional expectation on both sides of the expression of $\bar{y}_{j'}$ under $x^{(N)}$ in (6.35),

$$\begin{aligned} y^{(0,x^{(N)})} =& \mathbb{E}(\bar{y}_{j'}|x^{(N)}) = \mathbb{E}\Big(\Theta_1(\Lambda_F^2 y^{(0,x^{(N)})} + \Lambda_F^3 x^{(N)} + \Pi_{j'}^F)\Big| x^{(N)}\Big) \\ =& \Theta_1\Lambda_F^2 y^{(0,x^{(N)})} + \Theta_1\Lambda_F^3 x^{(N)} + \Theta_1\alpha, \end{aligned} \tag{6.36}$$

where α is denoted as the conditional expectations of $\{\Pi_{j'}^F\}_{j'\neq j, j'\in\mathcal{I}^F}$ under $x^{(N)}$. To be verified soon, $\bar{y}_j \in L^1(\Omega;\mathbb{R}^m)$, thus the above conditional expectation is meaningful. Note that $x^{(N)}$ is fixed but not asymptotic yet since till now, we only apply the limiting scheme on $M \to \infty$ instead N. Let $\Theta_2 = \Theta_2(\Lambda_F^1, \Lambda_F^2, Q_F) := I - \Theta_1\Lambda_F^2$, then we have

$$\Theta_2 = I - [(\Lambda_F^1)^\top Q_F \Lambda_F^1]^{-1}(\Lambda_F^1)^\top Q_F \Lambda_F^2 = \Theta_1(\Lambda_F^1 - \Lambda_F^2).$$

By (**A6.11**) and the discussion below equation (6.33), $(\Lambda_F^1)^\top Q_F \Lambda_F^1$ and $(\Lambda_F^1)^\top Q_F(\Lambda_F^1 - \Lambda_F^2)$ are invertible, thus Θ_2 is invertible. Then, we can obtain the CC system of problem (**q-LFA1**)

$$y^{(0,x^{(N)})} = \Theta_2^{-1}\Theta_1(\alpha + \Lambda_F^3 x^{(N)}), \tag{6.37}$$

and equation (6.35) can be rewritten as

$$\bar{y}_{j'} = \Theta_1\Big[(\Lambda_F^2\Theta_2^{-1}\Theta_1 + I)\Lambda_F^3 x^{(N)} + \Lambda_F^2\Theta_2^{-1}\Theta_1\alpha + \Pi_{j'}^F\Big], \quad j' \neq j,\ j' \in \mathcal{I}^F. \tag{6.38}$$

Noticing $y^{(0,x^{(N)})}$ depends on the pre-fixed strategy profile $x^{(N)}$, thus it is still undetermined. We continue to analyze the leader's optimization problem **(q-LF2)** (6.31). Based on the followers' best responses (6.34) and (6.38), the leaders begin to make their own optimal decisions. Since we had frozen $y^{(M)}$ by $y^{(0,x^{(N)})}$ in problem

(**q-LFA1**)(6.32), (6.31) can be rewritten as

$$\begin{cases} \text{minimize} \quad J_i^L(x_i; x^{(N)}; y^{(0,x^{(N)})}; \theta_i^L) \\ \qquad\qquad := |\Lambda_L^1 x_i - \Lambda_L^2 x^{(N)} - \Lambda_L^3 y^{(0,x^{(N)})} - \theta_i^L|_{Q_L}^2, \\ \text{subject to} \quad x_i \in \bar{\Gamma} \subseteq \mathbb{R}^n, \quad i \in \mathcal{I}^L, \end{cases} \tag{6.39}$$

where $y^{(0,x^{(N)})}$ is given by (6.37). Similarly, we let $N \to \infty$ in equation (6.39), by the mean field approximation, $x^{(N)}$ can be approximated by some deterministic quantity x^0. Subsequently, we obtain the following auxiliary problem of leaders **(q-LFA2)** :

$$\textbf{(q-LFA2)} \begin{cases} \text{minimize} \quad \bar{J}_i^L(x_i; x^0; y^0; \theta_i^L) := |\Lambda_L^1 x_i - (\Lambda_L^2 + \Lambda_L^3 \Theta_2^{-1} \Theta_1 \Lambda_F^3) x^0 \\ \qquad\qquad - \Lambda_L^3 \Theta_2^{-1} \Theta_1 \alpha - \theta_i^L|_{Q_L}^2, \\ \text{subject to} \quad x_i \in \bar{\Gamma} \subseteq \mathbb{R}^n, \quad i \in \mathcal{I}^L. \end{cases} \tag{6.40}$$

Note that $x^{(N)}$ has been frozen by x^0 in (6.40), thus $y^{(0,x^{(N)})}$ is replaced by y^0. We can compute the sub-gradient and second-order partial derivative of (6.40),

$$\begin{cases} \nabla_{x_i} \bar{J}_i^L(x_i; x^0; y^0; \theta_i^L) = 2\Big[(\Lambda_L^1)^\top Q_L \Big(\Lambda_L^1 x_i - (\Lambda_L^2 \\ \qquad\qquad + \Lambda_L^3 \Theta_2^{-1} \Theta_1 \Lambda_F^3) x^0 - \Lambda_L^3 \Theta_2^{-1} \Theta_1 \alpha - \theta_i^L\Big)\Big], \\ \nabla_{x_i}^2 \bar{J}_i^L(x_i; x^0; y^0; \theta_i^L) = 2(\Lambda_L^1)^\top Q_L \Lambda_L^1. \end{cases} \tag{6.41}$$

Again, by (**A6.9**) and (**A6.10**), $(\Lambda_L^1)^\top Q_L \Lambda_L^1 > 0$ and the functional $\bar{J}_i^L$ in (6.40) is uniformly convex w.r.t. x_i. The uniform convexity implies the coercivity thus it suffices to search the minimizer over a compact level set, even though the convex closed set $\bar{\Gamma}$ is not necessary to be bounded (compact). Moreover, the uniform convexity implies the strict convexity thus such minimizer, if exists, should be unique.

Moreover, for convex optimization, the local minimizer is also a global minimizer. Thus, the unique optimizer $\bar{x}_i$ can be further characterized by the following sub-

gradient inequality:

$$
\begin{aligned}
&2\Big\langle (\Lambda_L^1)^\top Q_L\Big(\Lambda_L^1\bar{x}_i - (\Lambda_L^2 + \Lambda_L^3\Theta_2^{-1}\Theta_1\Lambda_F^3)x^0 - \Lambda_L^3\Theta_2^{-1}\Theta_1\alpha - \theta_i^L\Big), x_i - \bar{x}_i\Big\rangle \\
=&2\Big\langle (\Lambda_L^1)^\top Q_L\Lambda_L^1\Big(\bar{x}_i - [(\Lambda_L^1)^\top Q_L\Lambda_L^1]^{-1}(\Lambda_L^1)^\top Q_L(\Lambda_L^2 + \Lambda_L^3\Theta_2^{-1}\Theta_1\Lambda_F^3)x^0 \\
&\quad - [(\Lambda_L^1)^\top Q_L\Lambda_L^1]^{-1}(\Lambda_L^1)^\top Q_L\Lambda_L^3\Theta_2^{-1}\Theta_1\alpha - [(\Lambda_L^1)^\top Q_L\Lambda_L^1]^{-1}(\Lambda_L^1)^\top Q_L\theta_i^L\Big), \\
&\quad x_i - \bar{x}_i\Big\rangle > 0, \quad \text{for } \forall\ x_i \in \bar{\Gamma},
\end{aligned}
\tag{6.42}
$$

or equivalently (noticing $\Psi := (\Lambda_L^1)^\top Q_L\Lambda_L^1 > 0$, thus $\Psi^{\frac{1}{2}} = \big((\Lambda_L^1)^\top Q_L\Lambda_L^1\big)^{\frac{1}{2}} > 0$ is well-defined (see Theorem 7.2.6 in [90, Chapter 7])),

$$
\begin{aligned}
&2\Big\langle \Psi^{\frac{1}{2}}\Big(\bar{x}_i - [(\Lambda_L^1)^\top Q_L\Lambda_L^1]^{-1}(\Lambda_L^1)^\top Q_L(\Lambda_L^2 + \Lambda_L^3\Theta_2^{-1}\Theta_1\Lambda_F^3)x^0 \\
&\quad - [(\Lambda_L^1)^\top Q_L\Lambda_L^1]^{-1}(\Lambda_L^1)^\top Q_L\Lambda_L^3\Theta_2^{-1}\Theta_1\alpha - [(\Lambda_L^1)^\top Q_L\Lambda_L^1]^{-1}\times \\
&\quad (\Lambda_L^1)^\top Q_L\theta_i^L\Big), \Psi^{\frac{1}{2}}(x_i - \bar{x}_i)\Big\rangle > 0, \quad \text{for } \forall\ x_i \in \bar{\Gamma}.
\end{aligned}
\tag{6.43}
$$

Since $\Psi > 0$, we take the following norm on $\mathbb{R}^n$ (see [93, 94, 183] for more details):

$$
|x|_\Psi^2 = \langle x, x\rangle_\Psi := x^\top\Psi x = \Big\langle \Psi^{\frac{1}{2}}x, \Psi^{\frac{1}{2}}x\Big\rangle.
$$

Moreover, it is easy to verify that $\langle x, y\rangle_\Psi = \Big\langle \Psi^{\frac{1}{2}}x, \Psi^{\frac{1}{2}}y\Big\rangle$ is an inner product when $\Psi > 0$. Note that such positive definite condition cannot be relaxed to $\Psi \geq 0$. This implies that (**A6.9**), (**A6.10**) should be necessary. It follows that (6.43) is equivalent to

$$
\bar{x}_i = \mathbf{P}_{\bar{\Gamma}}\Big[\Theta_3\Big((\Lambda_L^2 + \Lambda_L^3\Theta_2^{-1}\Theta_1\Lambda_F^3)x^0 + \Lambda_L^3\Theta_2^{-1}\Theta_1\alpha + \theta_i^L\Big)\Big], \quad i \in \mathcal{I}^L, \tag{6.44}
$$

with $\Theta_3 = [(\Lambda_L^1)^\top Q_L\Lambda_L^1]^{-1}(\Lambda_L^1)^\top Q_L$ and $\mathbf{P}_{\bar{\Gamma}}(\cdot) : \mathbb{R}^n \to \bar{\Gamma}$ is the projection operator under the norm $|\cdot|_\Psi$. Note that for any convex closed set $\Gamma \subseteq \mathbb{R}^n$, the project

operator $\mathbf{P}_\Gamma(\cdot)$ is always well-defined (see e.g., Section 1.5.2 of Chapter 1 in [74] and [93]) for the inner product $\langle x, y\rangle_S = \left\langle S^{\frac{1}{2}}x, S^{\frac{1}{2}}y\right\rangle$ for $\forall S > 0$. Naturally, this is also the case with $S = \Psi$. After obtaining his optimal strategy, the ith leader begins to estimate his peers' optimal strategy. Since the leaders are non-cooperative and will not announce their own information, $\mathcal{A}_i^L$ cannot obtain the exact value of his peer's personal parameter $\{\theta_{i'}^L\}_{i'\neq i, i'\in\mathcal{I}^L}$. As we mentioned before, he can only estimate $\{\theta_{i'}^L\}_{i'\neq i, i'\in\mathcal{I}^L}$ by the limiting distribution π^L under (**A6.2**). Therefore, from $\mathcal{A}_i^L$'s aspect, the other leaders' personal parameters are unknown and can be treated as the random variables $\{\Pi_{i'}^L\}_{i'\neq i, i'\in\mathcal{I}^L}$. For this reason, the optimizer $\bar{x}_{i'}$ can be presented as

$$\bar{x}_{i'} = \mathbf{P}_{\bar{\Gamma}}\Big[\Theta_3\Big((\Lambda_L^2 + \Lambda_L^3\Theta_2^{-1}\Theta_1\Lambda_F^3)x^0 + \Lambda_L^3\Theta_2^{-1}\Theta_1\alpha + \Pi_{i'}^L\Big)\Big], \quad i' \neq i,\ i' \in \mathcal{I}^L. \tag{6.45}$$

Moreover, by the mean field reasoning, taking the expectation on both sides of (6.45), we have

$$x^0 = \mathbb{E}\bar{x}_{i'} = \mathbb{E}\Big\{\mathbf{P}_{\bar{\Gamma}}\Big[\Theta_3\Big((\Lambda_L^2 + \Lambda_L^3\Theta_2^{-1}\Theta_1\Lambda_F^3)x^0 + \Lambda_L^3\Theta_2^{-1}\Theta_1\alpha + \Pi_{i'}^L\Big)\Big]\Big\}. \tag{6.46}$$

Combining with (6.36), we can obtain the CC system of **q-LF**:

$$(\mathrm{CC})\begin{cases} x^0 = \mathbb{E}\Big\{\mathbf{P}_{\bar{\Gamma}}\Big[\Theta_3\Big((\Lambda_L^2 + \Lambda_L^3\Theta_2^{-1}\Theta_1\Lambda_F^3)x^0 + \Lambda_L^3\Theta_2^{-1}\Theta_1\alpha + \Pi_{i'}^L\Big)\Big]\Big\}, \\ y^0 = \Theta_2^{-1}\Theta_1\alpha + \Theta_2^{-1}\Theta_1\Lambda_F^3 x^0. \end{cases} \tag{6.47}$$

For sake of presentation, we denote $\mathcal{A} := \Theta_3(\Lambda_L^2 + \Lambda_L^3\Theta_2^{-1}\Theta_1\Lambda_F^3)$ and $\mathcal{B} := \Theta_3\Lambda_L^3\Theta_2^{-1}\Theta_1$ hereafter. Noticing the matrix space $\mathbb{R}^{n\times n}$ and vector space $\mathbb{R}^n$ are both of finite dimension. Thus, it makes no difference for which matrix norm $|\cdot|_{n\times n}$ to be adopted in $L^\infty(\Omega;\mathbb{R}^{n\times n})$ because all these norms are equivalent. Likewise, for $S, S' > 0$, the associated norms $|x|_S^2 = x^\top Sx$ and $|x|_{S'}^2 = x^\top S'x$ are equivalent on $\mathbb{R}^n$, therefore $L_S^1(\Omega;\mathbb{R}^n) = L_{S'}^1(\Omega;\mathbb{R}^n)$. Hence, we need only simply write $L^1(\Omega;\mathbb{R}^n) = L_S^1(\Omega;\mathbb{R}^n)$

for all $S > 0$. We remark that in general, $L^\infty(\Omega;\mathbb{R}^{n\times n})$ and $L^1(\Omega;\mathbb{R}^n)$ are not finite dimensional space. Next, we present some preliminary results.

Proposition 6.3. *Let Γ be a non-empty closed convex subset in $\mathbb{R}^n$. Then, for any $A \in L^\infty(\Omega;\mathbb{R}^{n\times n})$, $b \in L^1(\Omega;\mathbb{R}^n)$, $X \in L^1(\Omega;\mathbb{R}^n)$, we have $\mathbf{P}_\Gamma(AX+b) \in L^1(\Omega;\mathbb{R}^n)$, where the projection operator $\mathbf{P}_\Gamma(\cdot) : \mathbb{R}^n \to \Gamma$ is defined under inner product $\langle x, y\rangle_S = \left\langle S^{\frac{1}{2}}x, S^{\frac{1}{2}}y\right\rangle$ for any $S > 0$.*

Proof First, we point out the above result is rather general since A, X, b can all be random matrices and vectors. Second, recall that for any $S > 0$, $\langle x, y\rangle_S = \left\langle S^{\frac{1}{2}}x, S^{\frac{1}{2}}y\right\rangle$ is an inner product and the Euclidean space $\mathbb{R}^n$ is a Hilbert space under the associated norm $|x|_S^2 = \langle x, x\rangle_S$. Hence, the projection operator $\mathbf{P}_\Gamma(A(\omega)X(\omega) + b(\omega)) : \mathbb{R}^n \to \Gamma$ is well-defined under $|x|_S^2$ for each $\omega \in \Omega$. Moreover, $\mathbf{P}_\Gamma(A(\cdot)X(\cdot) + b(\cdot))$ is a random variable because of the continuity property of projection. Third, in case Γ is bounded, $\mathbf{P}_\Gamma(A(\cdot)X(\cdot) + b(\cdot))$ is also bounded because of $\mathbf{P}_\Gamma(A(\omega)X(\omega) + b(\omega)) \in \Gamma$ for $\omega \in \Omega$. Hence, $\mathbf{P}_\Gamma(AX + b) \in L^1(\Omega;\mathbb{R}^n)$. Last, in case Γ is unbounded, noting that it is non-empty, thus there exists at least $z \in \Gamma$ with $\mathbf{P}_\Gamma(z) = z$. By Theorem 1.5.5 of Chapter 1 in [74], the projection operator is Lipschitz continuous with Lipschitz constant 1. Thus, for any $S > 0$,

$$|\mathbf{P}_\Gamma(AX + b)| \leq 2|z| + 2\mathbb{E}|AX| + \mathbb{E}|b| < +\infty. \tag{6.48}$$

Hence the result. Recall the definition of $L^1(\Omega;\mathbb{R}^n)$ does not depend on the choice of specific norm. Thus, the above norms on the right hand side can be understood to the standard norm with $S = I$ (identity matrix). In addition, the above estimate still holds true for any other $S > 0$ such as $S = \Psi$ with modified coefficients. □

Remark 6.4. *By Proposition 6.3, the expectation definition in the* CC *system* (6.47) *is well-posed by setting $A = \mathcal{A}$, $b = \mathcal{B}\alpha + \Theta_3\Pi_{i'}^L$.*

Remark 6.5. *Note that the projection operators under different inner products are different. For example, we let* Γ_1 *is a* r*-dimensional subspace in* $\mathbb{R}^n$ *and* $r \leq n$ *with* $(v_1, \cdots, v_r)$ *as basis.* $\mathbf{P}^1_{\Gamma_1}(\cdot) : \mathbb{R}^n \to \Gamma_1$ *and* $\mathbf{P}^2_{\Gamma_1}(\cdot) : \mathbb{R}^n \to \Gamma_1$ *are two projection operators defined under* $\langle \cdot, \cdot \rangle_{S_1} = \langle S_1^{\frac{1}{2}} \cdot, S_1^{\frac{1}{2}} \cdot \rangle$ *and* $\langle \cdot, \cdot \rangle_{S_2} = \langle S_2^{\frac{1}{2}} \cdot, S_2^{\frac{1}{2}} \cdot \rangle$*, respectively,* S_1*,* $S_2 > 0$*. We denote* $V = (v_1, \cdots, v_r)$*, then the projection operators can be expressed as* $\mathbf{P}^1_{\Gamma_1} = V(V^\top S_1 V)^{-1} V^\top S_1$ *and* $\mathbf{P}^2_{\Gamma_1} = V(V^\top S_2 V)^{-1} V^\top S_2$ *(see [183, Section 4.1] or Chapter 2). Suppose that*

$$V = \begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix} > 0, \quad S_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} > 0,$$

we can obtain that

$$\mathbf{P}^1_{\Gamma_1} = \begin{pmatrix} 0.7826 & 0.2174 & -0.4348 \\ 0.4348 & 0.5652 & 0.8696 \\ -0.1739 & 0.1739 & 0.6522 \end{pmatrix}, \quad \mathbf{P}^2_{\Gamma_1} = \begin{pmatrix} 0.8333 & 0.1667 & -0.3333 \\ 0.1667 & 0.8333 & 0.3333 \\ -0.3333 & 0.3333 & 0.3333 \end{pmatrix}.$$

Then, for any vector $w = (1\ 2\ 3)^\top \in \mathbb{R}^3$*, the corresponding projections are*

$$\mathbf{P}^1_{\Gamma_1} w = \begin{pmatrix} -0.0870 \\ 4.1739 \\ 2.1304 \end{pmatrix}, \quad \mathbf{P}^2_{\Gamma_1} w = \begin{pmatrix} 0.1667 \\ 2.8333 \\ 1.3333 \end{pmatrix},$$

which shows that $\mathbf{P}^1_{\Gamma_1} w \neq \mathbf{P}^2_{\Gamma_1} w$*. This implies that, for a vector, the projection operators under different norm will lead to different projections. Therefore, in what follows, we will focus on the projection operator* $\mathbf{P}_{\bar{\Gamma}}(\cdot) : \mathbb{R}^n \to \bar{\Gamma}$ *under the norm* $|\cdot|_\Psi$*.*

Proposition 6.4. *The projection operator* $\mathbf{P}_{\bar{\Gamma}}(\cdot) : \mathbb{R}^n \to \bar{\Gamma}$ *under the norm* $|\cdot|_\Psi$ *in* (6.44) *is monotonic.*

Proof For any u_1, $u_2 \in \mathbb{R}^n$, we have the two-sided variational inequalities:

$$\langle \mathbf{P}_{\bar{\Gamma}}(u_2) - \mathbf{P}_{\bar{\Gamma}}(u_1), \mathbf{P}_{\bar{\Gamma}}(u_1) - u_1 \rangle_\Psi \geq 0, \ \langle \mathbf{P}_{\bar{\Gamma}}(u_1) - \mathbf{P}_{\bar{\Gamma}}(u_2), \mathbf{P}_{\bar{\Gamma}}(u_2) - u_2 \rangle_\Psi \geq 0, \tag{6.49}$$

where the norm $|\cdot|_\Psi^2 = \langle \cdot, \cdot \rangle_\Psi = \langle \Psi^{\frac{1}{2}} \cdot, \Psi^{\frac{1}{2}} \cdot \rangle$, $\Psi > 0$ (see Appendix in [93]). Adding up these two inequalities, we have

$$\begin{aligned} 0 \leq & \langle \mathbf{P}_{\bar{\Gamma}}(u_2) - \mathbf{P}_{\bar{\Gamma}}(u_1), \mathbf{P}_{\bar{\Gamma}}(u_1) - u_1 + u_2 - \mathbf{P}_{\bar{\Gamma}}(u_2) \rangle_\Psi \\ = & \langle \mathbf{P}_{\bar{\Gamma}}(u_2) - \mathbf{P}_{\bar{\Gamma}}(u_1), u_2 - u_1 \rangle_\Psi - \langle \mathbf{P}_{\bar{\Gamma}}(u_2) - \mathbf{P}_{\bar{\Gamma}}(u_1), \mathbf{P}_{\bar{\Gamma}}(u_2) - \mathbf{P}_{\bar{\Gamma}}(u_1) \rangle_\Psi. \end{aligned} \tag{6.50}$$

Then, it follows that

$$\langle \mathbf{P}_{\bar{\Gamma}}(u_2) - \mathbf{P}_{\bar{\Gamma}}(u_1), u_2 - u_1 \rangle_\Psi \geq |\mathbf{P}_{\bar{\Gamma}}(u_2) - \mathbf{P}_{\bar{\Gamma}}(u_1)|_\Psi^2 \geq 0, \tag{6.51}$$

which means that $\mathbf{P}_{\bar{\Gamma}}$ is monotonic. □

6.4.2 The well-posedness of the CC system

A crux is the well-posedness of the above CC system (6.47) since it plays a central role in designing the decentralized LF strategy for **q-LF**. This may be proceeded using the fixed point analysis based on Banach contraction mapping. With this, the existence and uniqueness of parameter pair (x^0, y^0) can be ensured under some norm estimates. Here, we prefer to address this issue using a different analysis. To this end, we may establish the uniqueness and existence separately. We discuss the uniqueness first by introducing the following assumption:

(**A6.12**) $\Psi\mathcal{A} + \mathcal{A}^\top \Psi < 0$.

Then, we introduce the following theorem:

Theorem 6.3. *Under* (**A6.9**) *to* (**A6.12**), *the* CC *system* (6.47) *admits at most one solution.*

Proof We introduce a mapping: for any $x \in \mathbb{R}^n$, $T = T(x) := \mathbb{E}\ \mathbf{P}_{\bar{\Gamma}}(\mathcal{A}x + b)$ with $b = \mathcal{B}\alpha + \Theta_3 \Pi_{i'}^L$. Note that $b \in L^1(\Omega; \mathbb{R}^n)$ due to Proposition 6.3. Then, the CC system (6.47) can be reformulated as some fixed point relation: $x = T(x)$. Suppose that there are two solutions for it, denoted respectively by $x_1 = T(x_1), x_2 = T(x_2)$.

Then, on the one hand, by Proposition 6.4, we know for any $\omega \in \Omega$,

$$\begin{aligned}0 \leq & \Big\langle \mathbf{P}_{\bar{\Gamma}}(\mathcal{A}x_1 + b(\omega)) - \mathbf{P}_{\bar{\Gamma}}(\mathcal{A}x_2 + b(\omega)), (\mathcal{A}x_1 + b(\omega)) - (\mathcal{A}x_2 + b(\omega)) \Big\rangle_{\Psi} \\ = & \Big\langle \mathbf{P}_{\bar{\Gamma}}(\mathcal{A}x_1 + b(\omega)) - \mathbf{P}_{\bar{\Gamma}}(\mathcal{A}x_2 + b(\omega)), \mathcal{A}(x_1 - x_2) \Big\rangle_{\Psi}.\end{aligned}$$

On the other hand, by the monotonicity of expectation, we have

$$\begin{aligned}0 \leq & \mathbb{E}\Big\langle \mathbf{P}_{\bar{\Gamma}}(\mathcal{A}x_1 + b) - \mathbf{P}_{\bar{\Gamma}}(\mathcal{A}x_2 + b), \mathcal{A}(x_1 - x_2) \Big\rangle_{\Psi} \\ = & \Big\langle \mathbb{E}\ \mathbf{P}_{\bar{\Gamma}}(\mathcal{A}x_1 + b) - \mathbb{E}\ \mathbf{P}_{\bar{\Gamma}}(\mathcal{A}x_2 + b), \mathcal{A}(x_1 - x_2) \Big\rangle_{\Psi} \\ = & \Big\langle T(x_1) - T(x_2), \mathcal{A}(x_1 - x_2) \Big\rangle_{\Psi} = \Big\langle x_1 - x_2, \mathcal{A}(x_1 - x_2) \Big\rangle_{\Psi} \\ = & \Big\langle x_1 - x_2, \Psi\mathcal{A}(x_1 - x_2) \Big\rangle = \Big\langle x_1 - x_2, \frac{\Psi\mathcal{A} + \mathcal{A}^\top\Psi}{2}(x_1 - x_2) \Big\rangle \leq 0,\end{aligned}$$

where the last equality is due to the transpose invariance of quadratic form. Noting that $\widehat{\mathcal{A}} = \frac{\Psi\mathcal{A}+\mathcal{A}^\top\Psi}{2}$ is symmetric, then we have

$$0 = \Big\langle x_1 - x_2, \frac{\Psi\mathcal{A} + \mathcal{A}^\top\Psi}{2}(x_1 - x_2) \Big\rangle = |x_1 - x_2|^2_{\widehat{\mathcal{A}}}. \tag{6.52}$$

Hence the uniqueness of x^0 by noting $\widehat{\mathcal{A}} < 0$ due to (**A6.12**). Since $y^0 = \Theta_2^{-1}\Theta_1\alpha + \Theta_2^{-1}\Theta_1\Lambda_F^3 x^0$, the uniqueness of y^0 follows. □

Recall that

$$\begin{cases}\Theta_3 = [(\Lambda_L^1)^\top Q_L \Lambda_L^1]^{-1}(\Lambda_L^1)^\top Q_L, \quad \Theta_2 = I - \Theta_1\Lambda_F^2, \\ \Theta_1 = [(\Lambda_F^1)^\top Q_F \Lambda_F^1]^{-1}(\Lambda_F^1)^\top Q_F,\end{cases}$$

we have the following more detailed computation for $\Psi\mathcal{A}$ and $\mathcal{A}^T\Psi$:

$$\begin{cases}\Psi\mathcal{A} = (\Lambda_L^1)^\top Q_L \Big[\Lambda_L^2 + \Lambda_L^3 \Big((\Lambda_F^1)^\top Q_F(\Lambda_F^1 - \Lambda_F^2)\Big)^{-1}(\Lambda_F^1)^\top Q_F \Lambda_F^3\Big], \\ \mathcal{A}^\top\Psi = \Big[\Lambda_L^2 + \Lambda_L^3 \Big((\Lambda_F^1)^\top Q_F(\Lambda_F^1 - \Lambda_F^2)\Big)^{-1}(\Lambda_F^1)^\top Q_F \Lambda_F^3\Big]^T Q_L \Lambda_L^1.\end{cases} \tag{6.53}$$

Thus, (**A6.12**) can be rewritten with the original matrices as follows:

(**A6.13**)

$$(\Lambda_L^1)^\top Q_L\Big[\Lambda_L^2+\Lambda_L^3\Big((\Lambda_F^1)^\top Q_F(\Lambda_F^1-\Lambda_F^2)\Big)^{-1}(\Lambda_F^1)^\top Q_F\Lambda_F^3\Big]$$
$$+\Big[\Lambda_L^2+\Lambda_L^3\Big((\Lambda_F^1)^\top Q_F(\Lambda_F^1-\Lambda_F^2)\Big)^{-1}(\Lambda_F^1)^\top Q_F\Lambda_F^3\Big]^T Q_L\Lambda_L^1<0.$$

Obviously, (**A6.13**) is tedious and complicated, thus we introduce the following interesting and simple cases:

Case 1 $\Lambda_L^3=0$, $\Lambda_L^1=I$. In this case, the leaders only care about the peers' strategies and ignore the followers', while the followers consider both hierarchies' strategies. Then, (**A6.13**) could be simplified as

$$\Psi\mathcal{A}+\mathcal{A}^\top\Psi=Q_L\Lambda_L^2+(\Lambda_L^2)^TQ_L<0.$$

Case 2 $\Lambda_F^3=0$, $\Lambda_L^3=0$. In this case, the followers only care about the peer's strategies and ignore the leaders', while the leaders ignore the followers' strategies and only care about their peers'. Then

$$\Psi\mathcal{A}+\mathcal{A}^\top\Psi=(\Lambda_L^1)^\top Q_L\Lambda_L^2+(\Lambda_L^2)^TQ_L\Lambda_L^1<0.$$

Case 3 $\Lambda_F^2=0$, $\Lambda_F^1=I$. In this case, the followers ignore the peer's strategies and only focus on the leaders' strategies. Then, (**A6.13**) will be

$$\Psi\mathcal{A}+\mathcal{A}^\top\Psi=(\Lambda_L^1)^\top Q_L\Big(\Lambda_L^2+\Lambda_L^3\Lambda_F^3\Big)+\Big(\Lambda_L^2+\Lambda_L^3\Lambda_F^3\Big)^TQ_L\Lambda_L^1<0.$$

It seems that no matter the leaders or the followers ignore another hierarchy's strategies, the form of $\Psi\mathcal{A}+\mathcal{A}^\top\Psi$ will simplify to the form in Case 2.

Next, we establish the existence of parameter pair (x^0,y^0) in our CC system (6.47). Before that, we first give an assumption:

(**A6.14**) $\bar{\Gamma}$ is bounded (compact).

We first present the following existence result.

Proposition 6.5. *Suppose that* (**A6.1**)-(**A6.2**), (**A6.9**)-(**A6.14**) *hold. Then the* CC *system* (6.47) *admits at least one solution.*

Proof Recall the mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(x) = \mathbb{E}\ \mathbf{P}_{\bar{\Gamma}}(\mathcal{A}x + b)$ with $b = \mathcal{B}\alpha + \Theta_3 \Pi_{i'}^L$. Then, to verify the existence, it suffices to show the mapping T admits at least one fixed point. We examine the mapping T confined on $\bar{\Gamma}$. First, under (**A6.14**), it is obvious that $\bar{\Gamma}$ is compact and convex, and for $x \in \bar{\Gamma}, T(x) \in \bar{\Gamma}$, thus T is a stable mapping on $\bar{\Gamma}$. Second, T is continuous by noting the Lipschitz continuity of projection operator, expectation operation, and affine transformation. Therefore, by the Brouwer fixed-point theorem (see Proposition 6.1), the CC system (6.47) admits at least one solution x^0. Hence the existence for y^0 follows. □

Combining Proposition 6.5, we have the following result.

Proposition 6.6. *Under* (**A6.1**)-(**A6.2**) *and* (**A6.9**)-(**A6.14**), *the* CC *system* (6.47) *admits a unique solution.*

Proof By Proposition 6.5, there exists at least one solution for the CC system (6.47). Combining this with Theorem 6.3, we know that the solution is unique. The proposition follows. □

An interesting observation is that (**A6.12**) is imposed on the definitiveness of the weight matrix, while (**A6.14**) is on the compactness of domain constraint. These two assumptions are in different directions but jointly ensure the well-posedness (existence, uniqueness) of the CC system (6.47).

Sometimes, (**A6.14**) seems somewhat restrictive, and we have the following unbounded extension.

(**A6.15**) $\mathcal{A}^\top \Psi \mathcal{A} < \Psi$.

We have the following existence result.

Proposition 6.7. *Suppose that* (**A6.1**)-(**A6.2**), (**A6.9**)-(**A6.14**), *and* (**A6.15**) *hold. Then the* CC *system* (6.47) *admits at least one solution.*

Proof Since $\bar{\Gamma}$ is unbounded, thus for any $\delta > 0$, $C^\delta := \bar{\Gamma} \cap B(0,\delta) \neq \emptyset$ where $B(0,\delta)$ denotes the ball of $\mathbb{R}^n$ centered by the origin with radius $\delta > 0$. Moreover, for any $\delta > 0$, C^δ is convex and compact (bounded, closed). We now verify that the mapping T will be stable for some C^δ with sufficiently large $\delta > 0$. Noting that for $\mathcal{A}x + b$ in the definition of T, only b is random depending on $\omega \in \Omega$. Then, for any $x \in C^\delta$, we have $|\mathcal{A}x + b(\omega)|^2_\Psi \leq 2(x^\top \mathcal{A}^\top \Psi \mathcal{A}x) + 2|b(\omega)|^2_\Psi$ for a.s. $\omega \in \Omega$. Since $\bar{\Gamma}$ is non-empty, there exists at least one $z \in \bar{\Gamma}$ such that $z = \mathbf{P}_{\bar{\Gamma}}(z)$. Thus, we have, for a.s. $\omega \in \Omega$:

$$\begin{aligned}
|\mathbf{P}_{\bar{\Gamma}}(\mathcal{A}x + b(\omega))|_\Psi =& |\mathbf{P}_{\bar{\Gamma}}(\mathcal{A}x + b(\omega)) - z + z|_\Psi \\
\leq& |\mathbf{P}_{\bar{\Gamma}}(\mathcal{A}x + b(\omega)) - \mathbf{P}_{\bar{\Gamma}}(z)|_\Psi + |z|_\Psi \\
\leq& |(\mathcal{A}x + b(\omega)) - z|_\Psi + |z|_\Psi \leq |\mathcal{A}x|_\Psi + |b(\omega)|_\Psi + 2|z|_\Psi.
\end{aligned}$$

Then, taking the expectation on both sides:

$$\begin{aligned}
\mathbb{E}|\mathbf{P}_{\bar{\Gamma}}(\mathcal{A}x + b)|_\Psi \leq& |\mathcal{A}x|_\Psi + \mathbb{E}|b|_\Psi + 2|z|_\Psi = (x^\top \mathcal{A}^\top \Psi \mathcal{A}x)^{\frac{1}{2}} + \mathbb{E}|b|_\Psi + 2|z|_\Psi \\
<& |x|_\Psi + \mathbb{E}|b|_\Psi + 2|z|_\Psi \leq |x|_\Psi.
\end{aligned}$$

Hence, $T(x) = \mathbb{E}\ \mathbf{P}_{\bar{\Gamma}}(\mathcal{A}x + b)$ is stable on C^δ for sufficiently large δ. Then, the continuous mapping T on C^δ admits at least one solution x^0 by the Brouwer fixed-point theorem, thus the existence for y^0 follows. Hence the result. □

6.4.3 ε-Stackelberg-Nash-Cournot equilibrium

Next, we give the proof of the ε-SNC equilibrium (see Definition 6.8) for our strategies in (6.38) and (6.45). For sake of notation simplicity, we will use K to denote a generic constant in the following discussion. The value of K may be different at

different places and it only depends on the coefficients. Before that, we first give two preliminary results.

Lemma 6.1. *Suppose that* (**A6.1**)-(**A6.2**), (**A6.9**)-(**A6.14**), *and* (**A6.15**) *hold. Then for given* $x^{(N)} \in \mathbb{R}^n$,

$$|\bar{y}^{(M)} - y^0| \to \ 0 \ a.s., \ as \ \ M \to \infty.$$

Proof For the pre-fixed $x^{(N)} \in \mathbb{R}^n$ and by equation (6.34) and (6.38), $\bar{y}^{(M)} = \Theta_1[(\Lambda_F^2\Theta_2^{-1}\Theta_1 + I)\Lambda_F^3 x^{(N)} + \Lambda_F^2\Theta_2^{-1}\Theta_1\alpha + \frac{\theta_j^F}{M} + \frac{\sum_{j'=1,j'\neq j}^M \Pi_{j'}^F}{M}]$, then we have

$$|\bar{y}^{(M)} - y^0| = \Big|\Theta_1[(\Lambda_F^2\Theta_2^{-1}\Theta_1 + I)\Lambda_F^3 x^{(N)} + \Lambda_F^2\Theta_2^{-1}\Theta_1\alpha + \frac{\theta_j^F}{M} + \frac{\sum_{j'=1,j'\neq j}^M \Pi_{j'}^F}{M}]$$
$$- \Theta_2^{-1}\Theta_1(\alpha + \Lambda_F^3 x^{(N)})\Big| = \Big|\Theta_1(\frac{\theta_j^F}{M} + \frac{\sum_{j'=1,j'\neq j}^M \Pi_{j'}^F}{M} - \alpha)\Big|.$$

By equation (6.37), $x^{(N)}$ is pre-fixed, there exists a constant K such that $|y^0|^2 \leq K$, thus by equation (6.34), $|\bar{y}_j|^2 \leq K$ and $|\bar{y}^{(M)}|^2 \leq K$ (the detailed proof is similar to [101, Lemma 5.4]), for some K is independent of M. Then, we can obtain that, there exists a constant K such that $\mathbb{E}|\Pi_{j'}^F|^2 \leq K$. Since $\{\Pi_j^F\}_{j=1}^M$ are i.i.d. random variables, by the Kolmogorov's strong law of large numbers, $\frac{\sum_{j'=1,j'\neq j}^M \Pi_{j'}^F}{M} \to \alpha \ a.s.$ under $x^{(N)}$ as $M \to \infty$, hence $|\bar{y}^{(M)} - y^0| \to 0 \ a.s.$ as $M \to \infty$. □

Lemma 6.2. *Suppose that* (**A6.1**)-(**A6.2**), (**A6.9**)-(**A6.14**), *and* (**A6.15**) *hold. Then*

$$|\bar{x}^{(N)} - x^0| \to \ 0 \ a.s., \ as \ \ N \to \infty.$$

Proof Recall the fixed point in our CC system (6.47) $x^0 = \mathbb{E}\ \mathbf{P}_{\bar{\Gamma}}(\mathcal{A}x^0 + b_{i'})$ with $b_{i'} = \mathcal{B}\alpha + \Theta_3\Pi_{i'}^L$ and $\bar{x}_{i'} = \mathbf{P}_{\bar{\Gamma}}(\mathcal{A}x^0 + b_{i'})$, then we have $\bar{x}^{(N)} = \frac{1}{N}\mathbf{P}_{\bar{\Gamma}}(\mathcal{A}x^0 + b_i) + \frac{1}{N}\sum_{i'=1,i'\neq i}^N \mathbf{P}_{\bar{\Gamma}}(\mathcal{A}x^0 + b_{i'})$ and

$$|\bar{x}^{(N)} - x^0| = \Big|\frac{1}{N}\mathbf{P}_{\bar{\Gamma}}(\mathcal{A}x^0 + b_i) + \frac{1}{N}\sum_{i'=1,i'\neq i}^N \mathbf{P}_{\bar{\Gamma}}(\mathcal{A}x^0 + b_{i'}) - x^0\Big|.$$

Note that $\mathbf{P}_{\bar{\Gamma}}(\mathcal{A}x^0 + b_i)$ is fixed. By the existence and uniqueness of the CC system (6.47), for some constant K, $|x^0|^2 \leq K$. Then, by equation (6.44), $|\bar{x}_i|^2 \leq K$ and $|\bar{x}^{(N)}|^2 \leq K$ for some K is independent of N. Thus, there exists a constant K such that $\mathbb{E}|\mathbf{P}_{\bar{\Gamma}}(\mathcal{A}x^0 + b_{i'})|^2 \leq K$. Since $\{\mathcal{A}x^0 + b_{i'}\}_{j'\neq j, j'=1}^{M}$ are i.i.d. random variables, by the similar argument in [94, Lemma 5.2] and the Kolmogorov's strong law of large numbers, the result is straightforward. □

Theorem 6.4. *Suppose that* (**A6.1**)-(**A6.2**), (**A6.9**)-(**A6.14**), *and* (**A6.15**) *hold. Then the strategies* $(\bar{x}_1, \cdots, \bar{x}_N, \bar{y}_1, \cdots, \bar{y}_M)$ *given in* (6.38) *and* (6.44) *satisfy* ε-SNC *equilibrium.*

Proof By the definition 6.8, we first consider the followers' subgame. For given $(x_1, x_2, \cdots, x_N) \in \mathbb{R}^{nN}$, $x^{(N)}$ is pre-fixed and

$$\begin{aligned}
\inf_{y_j\in\mathbb{R}^m} \mathcal{J}_j^F(y_j; \bar{y}_{-j}; x^{(N)}; \theta_j^F) =& \Big|\Lambda_F^1 y_j - \frac{\Lambda_F^2 y_j}{M} - \frac{\Lambda_F^2}{M}\sum_{j'\neq j}^{M} \bar{y}_{j'} - \Lambda_F^3 x^{(N)} - \theta_j^F\Big|_{Q_F}^2 \\
=& \Big|\Lambda_F^1(\bar{y}_j + \delta y_j) - \Lambda_F^2(\bar{y}^{(M)} + \frac{\delta y_j}{M}) - \Lambda_F^3 x^{(N)} - \theta_j^F\Big|_{Q_F}^2 \\
=& \mathcal{J}_j^F(\bar{y}_j; \bar{y}^{(M)}; x^{(N)}; \theta_j^F) + U_1 + U_2 + U_3,
\end{aligned} \tag{6.54}$$

where δy_j is the variation of $y_j - \bar{y}_j$ and

$$\begin{cases}
U_1 = 2\langle Q_F(\Lambda_F^1 \bar{y}_j - \Lambda_F^2 \bar{y}^{(M)} - \Lambda_F^3 x^{(N)} - \theta_j^F), \Lambda_F^1 \delta y_j\rangle, \\
U_2 = -\dfrac{2}{M}\langle Q_F(\Lambda_F^1 \bar{y}_j - \Lambda_F^2 \bar{y}^{(M)} - \Lambda_F^3 x^{(N)} - \theta_j^F), \Lambda_F^2 \delta y_j\rangle,\ U_3 = |\Lambda_F^1 \delta y_j - \Lambda_F^2 \dfrac{\delta y_j}{M}|_{Q_F}^2.
\end{cases}$$

By equation (6.34), we can obtain that

$$\begin{aligned}
U_1 =& 2\langle (\Lambda_F^1)^\top Q_F \Lambda_F^1(\Theta_1 \Lambda_F^2 y^0 + \Theta_1 \Lambda_F^3 x^{(N)} + \Theta_1 \theta_j^F) - (\Lambda_F^1)^\top Q_F \Lambda_F^2 \bar{y}^{(M)} \\
&- (\Lambda_F^1)^\top Q_F \Lambda_F^3 x^{(N)} - (\Lambda_F^1)^\top Q_F \theta_j^F, \delta y_j\rangle = 2\langle (\Lambda_F^1)^\top Q_F \Lambda_F^2 (y^0 - \bar{y}^{(M)}), \delta y_j\rangle.
\end{aligned} \tag{6.55}$$

By the similar discussion in Lemma 6.1, under the pre-fixed $x^{(N)} \in \mathbb{R}^n$, $|y^0|^2 \leq K$, $|\bar{y}_j|^2 \leq K$, and $|\bar{y}^{(M)}|^2 \leq K$ for some K is independent of M. Thus, there exists a constant K independent of M such that

$$\inf_{y_j \in \mathbb{R}^m} \mathcal{J}_j^F(y_j; \bar{y}_{-j}; x^{(N)}; \theta_j^F) \leq \mathcal{J}_j^F(\bar{y}_j; \bar{y}^{(M)}; x^{(N)}; \theta_j^F) \leq K. \tag{6.56}$$

Since $U_3 \geq 0$, by (6.54) and (6.55), it follows that

$$\mathcal{J}_j^F(\bar{y}_j; \bar{y}^{(M)}; x^{(N)}; \theta_j^F) - \inf_{y_j \in \mathbb{R}^m} \mathcal{J}_j^F(y_j; \bar{y}_{-j}; x^{(N)}; \theta_j^F) \leq -U_1 - U_2.$$

By Lemma 6.1 and (6.56), we have U_1, $U_2 \to 0$ *a.s.* under $x^{(N)}$ when $M \to \infty$. Denote $\varepsilon(M) = -(U_1 + U_2)$, then

$$\mathcal{J}_j^F(\bar{y}_j; \bar{y}_{-j}; x^{(N)}; \theta_j^F) \leq \inf_{y_j \in \mathbb{R}^m} \mathcal{J}_j^F(y_j; \bar{y}_{-j}; x^{(N)}; \theta_j^F) + \varepsilon(M),$$

where $\varepsilon(M) \to 0$ *a.s.* as $M \to \infty$.

After the followers give out their best response, we consider the leaders' subgame. By the similar argument in (6.54),

$$\inf_{x_i \in \Gamma_i^L} \mathcal{J}_i^L(x_i; \bar{x}_{-i}; \bar{y}^{(M)}; \theta_i^L) = \mathcal{J}_i^L(\bar{x}_i; \bar{x}^{(N)}; \bar{y}^{(M)}; \theta_i^L) + \bar{U}_1 + \bar{U}_2 + \bar{U}_3 + \bar{U}_4, \tag{6.57}$$

where δx_i is the variation of $x_i - \bar{x}_i$ and

$$\begin{cases} \bar{U}_1 = 2\langle Q_L(\Lambda_L^1 \bar{x}_i - \Lambda_L^2 \bar{x}^{(N)} - \Lambda_L^3 \bar{y}^{(M)} - \theta_i^L), \Lambda_L^1 \delta x_i \rangle, \\ \bar{U}_2 = -\dfrac{2}{N} \langle Q_L(\Lambda_L^1 \bar{x}_i - \Lambda_L^2 \bar{x}^{(N)} - \Lambda_L^3 \bar{y}^{(M)} - \theta_i^L), \Lambda_L^2 \delta x_i \rangle, \\ \bar{U}_3 = -\dfrac{2}{N} \langle Q_L(\Lambda_L^1 \bar{x}_i - \Lambda_L^2 \bar{x}^{(N)} - \Lambda_L^3 \bar{y}^{(M)} - \theta_i^L), \Lambda_L^3 \Theta_1 (\Lambda_F^2 \Theta_2^{-1} \Theta_1 + I) \Lambda_F^3 \delta x_i \rangle, \\ \bar{U}_4 = \| \Lambda_L^1 \delta x_i - \Lambda_L^2 \dfrac{\delta x_i}{N} - \Lambda_L^3 \Theta_1 (\Lambda_F^2 \Theta_2^{-1} \Theta_1 + I) \Lambda_F^3 \dfrac{\delta x_i}{N} \|_{Q_L}^2. \end{cases}$$

Note that, by the similar argument in (6.55), $\bar{U}_1$ can be written as

$$\begin{aligned} \bar{U}_1 &= 2\langle (\Lambda_L^1)^\top Q_L (\Lambda_L^1 \bar{x}_i - (\Lambda_L^2 + \Lambda_L^3 \Theta_2^{-1} \Theta_1 \Lambda_F^3) \bar{x}^0 - \Lambda_L^3 \Theta_2^{-1} \Theta_1 \alpha - \theta_i^L), \delta x_i \rangle \\ &+ 2\langle (\Lambda_L^1)^\top Q_L \Lambda_L^2 (x^0 - \bar{x}^{(N)}) + (\Lambda_L^1)^\top Q_L \Lambda_L^3 (y^0 - \bar{y}^{(M)}), \delta x_i \rangle := \bar{U}_{11} + \bar{U}_{12}. \end{aligned} \tag{6.58}$$

By equation (6.43), we have

$$\bar{U}_{11} = 2\langle (\Lambda_L^1)^\top Q_L[\Lambda_L^1 \bar{x}_i - (\Lambda_L^2 + \Lambda_L^3 \Theta_2^{-1}\Theta_1 \Lambda_F^3)\bar{x}^0 - \Lambda_L^3 \Theta_2^{-1}\Theta_1 \alpha - \theta_i^L], \delta x_i\rangle > 0.$$

According to the discussion in Lemma 6.2, $|x^0|^2 \leq K$, $|\bar{x}_i|^2 \leq K$, and $|\bar{x}^{(N)}|^2 \leq K$ for some K is independent of N. Thus, there exists a constant K independent of N such that

$$\inf_{x_i \in \Gamma_i^L} \mathcal{J}_i^L(x_i; \bar{x}_{-i}; \bar{y}^{(M)}; \theta_i^L) \leq \mathcal{J}_i^L(\bar{x}_i; \bar{x}^{(N)}; y^{(M)}; \theta_i^L) \leq K. \tag{6.59}$$

Since $\bar{U}_{11}, \bar{U}_4 \geq 0$, by (6.57) and (6.58), it follows that

$$\mathcal{J}_i^L(\bar{x}_i; \bar{x}^{(N)}; y^{(M)}; \theta_i^L) - \inf_{x_i \in \Gamma_i^L} \mathcal{J}_i^L(x_i; \bar{x}_{-i}; \bar{y}^{(M)}; \theta_i^L) \leq -\bar{U}_{12} - \bar{U}_2 - \bar{U}_3.$$

By Lemma 6.1, Lemma 6.2 and (6.59), we have $\bar{U}_{12}$, $\bar{U}_2$, $\bar{U}_3 \to 0$ *a.s.* when $M, N \to \infty$. Denote $\varepsilon(M, N) = -(\bar{U}_{12} + \bar{U}_2 + \bar{U}_3)$, then

$$\mathcal{J}_i^L(\bar{x}_i; \bar{x}^{(N)}; y^{(M)}; \theta_i^L) \leq \inf_{x_i \in \Gamma_i^L} \mathcal{J}_i^L(x_i; \bar{x}_{-i}; \bar{y}^{(M)}; \theta_i^L) + \varepsilon(M, N),$$

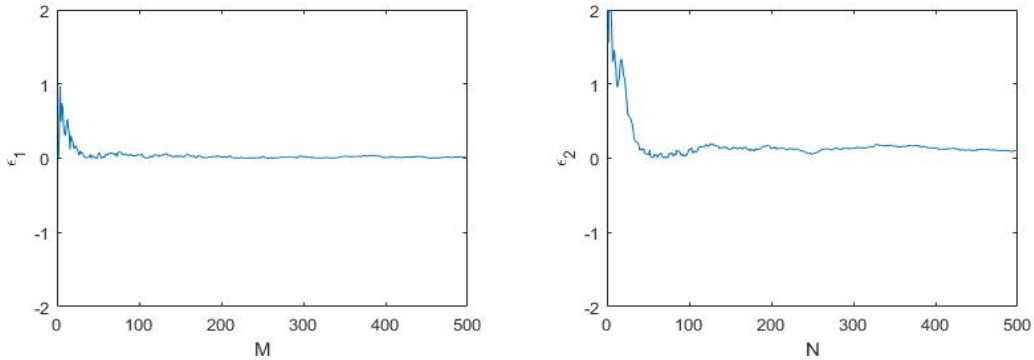
where $\varepsilon(M, N) \to 0$ *a.s.* as $M, N \to \infty$. The theorem follows. □

6.5 Numerical Example

In this section, we give a numerical example. Suppose that there are 500 leaders and 500 followers with $\Lambda_F^1 = 3$, $\Lambda_F^2 = 1$, $\Lambda_F^3 = 2$, $\Lambda_L^1 = 2$, $\Lambda_L^2 = -2$, $\Lambda_L^3 = 1$, $Q_F = 2$, $Q_L = 1$. Moreover, $\mathcal{I}^F = \{1, 2, \cdots, 500\}$, $\mathcal{I}^L = \{1, 2, \cdots, 500\}$, and for dimensional consistency, we set $n = m = m_1 = l = p = 1$. Then the assumptions **??**-**??**, **??** are satisfied and the Eq. (6.30), (6.31) can be rewritten as

$$\begin{cases} \text{minimize} \quad \mathcal{J}_j^F(y_j; y^{(M)}; x^{(N)}; \theta_j^F) = 2|3y_j - y^{(M)} - 2x^{(N)} - \theta_j^F|^2, \\ \text{subject to} \quad y_j \in \mathbb{R}^m, \quad j \in \mathcal{I}^F, \end{cases} \tag{6.60}$$

for the followers with decision variables $\{y_j\}_{j \in \mathcal{I}^F}$ and



(a) (b)

Figure 6.1: (a) is the curve of convergence in Lemma 6.1 and (b) is the curve of convergence in Lemma 6.2

$$\begin{cases} \text{minimize} \quad \mathcal{J}_i^L(x_i; x^{(N)}; y^{(M)}; \theta_i^L) = |2x_i + 2x^{(N)} - y^{(M)} - \theta_i^L|^2, \\ \text{subject to} \quad x_i \in \bar{\Gamma} \subseteq \mathbb{R}^n, \quad i \in \mathcal{I}^L, \end{cases} \tag{6.61}$$

for the leaders with decision variables $\{x_i\}_{i\in\mathcal{I}^L}$. Therefore, by the discussion in Section 5.1 and 5.2, we can obtain that the CC system (6.47) of **q-LF** as follows

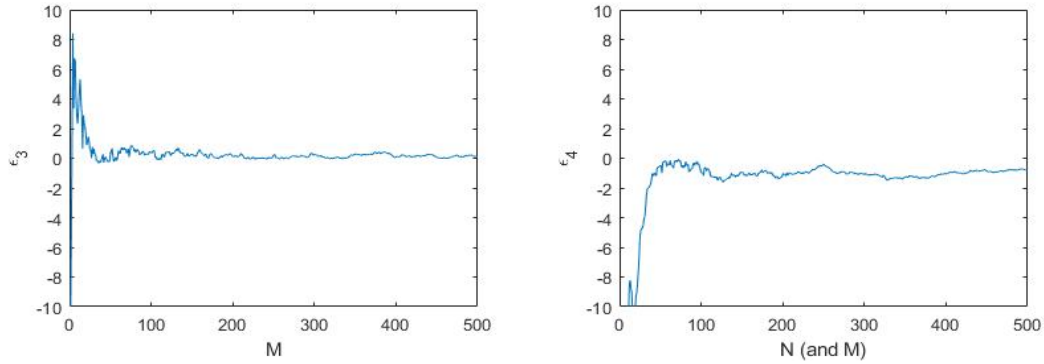
$$\begin{cases} x^0 = \mathbb{E}\Big\{\mathbf{P}_{\bar{\Gamma}}\Big[0.5 \times \Big(-0.5x^0 + 0.5\alpha + \Pi_{i'}^L\Big)\Big]\Big\}, \\ y^0 = 0.5\alpha + x^0. \end{cases} \tag{6.62}$$

Now, we consider the case that the leaders' strategies are unconstrained. Suppose that the conditional expectations of $\{\Pi_j^F\}_{j=1}^M$ under $x^{(N)}$ is $\alpha = 1$ and the expectation of $\{\Pi_i^L\}_{i=1}^N$ is 0. Then the above system can be rewritten as

$$x^0 = 0.2, \quad y^0 = 0.7. \tag{6.63}$$

Next, we simulate the results of our ε-SNC equilibrium in Section 6.4.3. We first defined

$$\begin{cases} \varepsilon_1 = |\bar{y}^{(M)} - y^0|, \quad \varepsilon_2 = |\bar{x}^{(N)} - x^0|, \\ \varepsilon_3 = \mathcal{J}_j^F(\bar{y}_j; \bar{y}^{(M)}; x^{(N)}; \theta_j^F) - \inf\limits_{y_j\in\mathbb{R}^m} \mathcal{J}_j^F(y_j; \bar{y}_{-j}; x^{(N)}; \theta_j^F), \\ \varepsilon_4 = \mathcal{J}_i^L(\bar{x}_i; \bar{x}^{(N)}; y^{(M)}; \theta_i^L) - \inf\limits_{x_i\in\Gamma_i^L} \mathcal{J}_i^L(x_i; \bar{x}_{-i}; \bar{y}^{(M)}; \theta_i^L), \end{cases}$$



(a) (b)

Figure 6.2: (a) is the curve of convergence in the followers' subgame and (b) is the curve of convergence in the leaders' subgame.

where ε_1 represents the process of convergence in Lemma 6.1, while ε_2 represents the process of convergence in Lemma 6.2. ε_3 and ε_4 are corresponding to the processes of convergence of our ε-SNC equilibrium.

The curves of ε_1, ε_2 are shown in Figure 1, and ε_3, ε_4 are shown in Figure 2 as M and N increase from 1 to 500. The X-axis indicates the number of agents M or N, and the Y-axis indicates $\varepsilon_i, i = 1, 2, 3, 4$. It can be seen that they are approaching zero when M or N is growing larger and larger.

6.6 Conclusion

In this chapter, we study a class of **w-LF** games with model uncertainty under a large-population system. The leaders or followers play a Nash game with each other in their hierarchy and play a Stackelberg game between two hierarchies. Applying the mean field approximations, we obtain an asymptotic SNC equilibrium of our problem. Finally, we give out a quadratic case and a numerical simulation.

Chapter 7

Conclusions and Future work

This chapter concludes the thesis and gives out some possible future works that are related to the topics in this thesis.

7.1 Conclusions

In this thesis, the problems related to the mean field game and team with leader-follower interaction are introduced step by step. A mixed leader-follower problem between two players with input constraints is first studied by using the maximum principle and the minimizing sequence method. Then a robust mean field team control problem under a large population system is considered by utilizing the mean field heuristics. These two chapters introduce the background, fundamentals, and some techniques of the Stackelberg game and the mean field game (or team) respectively. After that, the principle of the leader-follower game, the maximum principle, and the mean field approximation methods are applied to investigate a leader-follower mean field team problem with one leader and N followers. A pair of decentralized optimal control laws are obtained and proved to satisfy the asymptotic Stackelberg equilibrium. Meanwhile, a Stackelberg-Nash-Cournot equilibrium between N leaders and M followers under a static optimization context is introduced as a supplement for the leader-follower mean field game topic of the thesis. Moreover, some numerical exam-

ples are provided to simulate the asymptotic result of the mean field approximation at the end of some chapters.

7.2 Future Work

The Related future works are listed below.

1. Some techniques and results of the mean field leader-follower problem we obtained in this thesis can be applied in the financial market, especially combining them with the mean-variance model.

2. Limitations and constraints appear commonly in the real world. Thus, it is more meaningful and practical to study the leader-follower game with not only the input constraints but also some "hard-constraints". Also, the related results may apply to management science or deep learning.

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