EXISTENCE AND STABILITY OF TRAVELING WAVES TO CONSERVATION LAWS ARISING FROM CHEMOTAXIS

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CERTIFICATE OF ORIGINALITY

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______________________________(Signed)

______________Yan Ran____________(Name of student)

Abstract

This thesis is concerned with the system of conservation laws transformed via a Cole-Hopf transformation from the Keller-Segel model to repulsive chemotaxis which has various applications, such as biological systems that build transport networks (reinforced random walks [27,28]).

Traveling wave solutions to conservation laws arising from chemotaxis with positive chemotactic coefficient (denoted by $\chi > 0$) have been widely studied, while the analysis of traveling waves to conservation laws arising from repulsive chemotaxis (denoted by $\chi < 0$) still remains open. The purpose of this thesis will be to develop some existence and stability theories for such a system of conservation laws to repulsive chemotaxis. We first show the exisence of travaling waves by using the phase plane analysis, where there are three heteroclinic connections in the system. Numerical simulations will be presented to demonstrate the process of wave propagation. We then proceed to prove the stability of traveling waves by employing the method of weighted energy estimates. We get estimates on the types of integrals which are usually interpreted in terms of energies related to the physical problems behind partial differential equations.

This thesis develops the first theoretical results on traveling waves to conservation laws arising from repulsive chemotaxis and more open problems relate to the system will be proposed for future studies.

Key Words: Keller-Segel Model; Conservation Law; Chemotaxis; Traveling Waves; Energy Estimates.

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Chapter 1

Introduction

1.1 Background

The appearance of traveling waves is common in biological phenomena. Traveling wave is a kind of wave, which travels with constant propagation speed and unchanged shape. There are some practical examples demonstrating such waves, including the calcium waves propagating on the surface of the egg of the fish *Medaka* during fertilization [1], the transmission of an advantageous gene in a population [2], the branching pattern formation of colonies of bacteria, *Paenibacillus dendritiformis* [3, 4] and so on. Mathematicians have explored traveling waves for years and have tried to give mathematical modeling and analysis to such an event because the investigation of traveling waves plays an important role in understanding the mechanisms behind various patterns in biology science. Mathematically, if a solution to a partial differential equation is in the form

$$u(x,t) = U(x-ct) = u(z),\ z = x-ct,$$

with $U \in C^\infty(\mathbb{R})$ satisfying conditions

$$U(\pm\infty) = u_\pm,\ U'(\pm\infty) = 0,$$

then $u(x,t)$ is a traveling wave solution that moves at constant speed c in the positive x-direction. z is the wave variable and $u_\pm$ are constants which represent the right and left end states, describing the asymptotic states of U as $z \to \pm\infty$.

In 1937, the Fisher equation, the most fundamental reaction-diffusion model which admits traveling wave solutions, was developed to describe the transmission of advantageous genes in a population. It shows that the combination of reaction and diffusion increases the efficiency of information transferal through traveling waves of concentration movements comparing with the diffusion mechanism alone. The Fisher equation has a wide range of applications in biological science, from bacteria growth to animal dispersal.

However, there are many wave propagating phenomena which cannot be described by the Fisher equation. Instead they can be interpreted by chemotaxis. It describes the movements of bacteria, cells and organisms in response to chemical substances present in the environment. Chemotaxis has many applications in biological and biomedical science, including blood vessel formation, bacterial infection, immune responses, development of cancer and wound healing [5]. Scientists have found traveling wave patterns driven by chemotaxis, for example, bands of motile *Escherichia coli(E.coli)* were observed when the bacteria were placed in one end of a capillary tube containing oxygen and an energy source [6]. Another famous example is the aggregation of *Dictyostelium discoideum* as traveling waves[7]. In 1970s, Evelyn Fox Keller and Lee A. Segel [8, 9] proposed a mathematical model to describe the aggregation of cellular slime molds like *Dictyostelium discoideum*. The Keller-Segel model has become one of the most well-known models in mathematical biology. We study its general form in one dimensional space with logarithmic law,

$$\begin{cases} u_t = [du_x - \chi u(\log\ h)_x]_x, \\ h_t = \epsilon h_{xx} - uf(h), \end{cases} \tag{1.1}$$

with $(x,t) \in \mathrm{R} \times [0,\infty)$. Here $u(x,t)$ and $h(x,t)$ denote the cell density and the chemical concentration, respectively. $d > 0$ and $\epsilon \geq 0$ are the cell and chemical diffusion coefficients. χ is called the chemotactic coefficient measuring the strength of the chemical signals. If $\chi > 0$ (*resp.* $\chi < 0$), the chemical is a chemoattractant (*resp.* chemorepellent). $\log h$ is the chemosensitivity function, the signal detection mechanism. Note that the chemosensitivity functions in other forms, such as kh linear law and $\frac{kh^m}{1+h^m}$ receptor law with $k > 0$ and $m \in \mathbb{N}$ have been widely studied, see [5, 10, 11, 12]. Function $f(h)$ represents the consumption rate of the chemical per cell in the forms shown below

$$f(h) = h^m = \begin{cases} \text{constant rate}, & m = 0, \\ \text{sublinear rate}, & 0 < m < 1, \\ \text{linear rate}, & m = 1, \\ \text{superlinear rate}, & m > 1. \end{cases} \tag{1.2}$$

When $0 \leq m < 1$, Keller and Segel interpret the traveling bands of bacterial chemotaxis experimentally observed in [13] with $\epsilon = 0$ and more works on the existence of traveling bands were done for $\epsilon \geq 0$ [14, 15, 16, 17, 18]. When $m > 1$, H. Schwetlick has proved the non-existence of traveling wave solutions for the model (1.1) in [18]. When $m = 1$, the model (1.1) with nonzero chemical diffusion was used by Rosen to describe the chemotactic movement of motile aerobic bacterial toward oxygen, and later was used to describe the initiation of angiogenesis [19, 20, 21].

The stability of traveling waves remains as a difficult problem, although the existence results has been widely established. When $0 \leq m < 1$, there is no stability result on traveling waves. When the chemical diffusion is nonzero, the linear instability of traveling wave solutions to chemotaxis model was shown by Nagai and Ikeda in [16]. When $m = 1$, the stability of traveling wave solutions for the system with small chemical diffusion or zero chemical diffusion were established for $\chi > 0$ in [23, 24, 22].

The success of these results heavily rely on the following transformation. As can be seen that the singularity lying in the logarithmic law at $h = 0$ has brought difficulties in mathematical analysis, to resolve such a problem, a Cole-Hopf type transformation is applied to model (1.1),

$$v = -\frac{\partial}{\partial x}(\log\ h) = -\frac{h_x}{h}.$$

Model (1.1) with $f(h) = h$ is transformed into the following conservation laws without singularity

$$\begin{cases} u_t - \chi(uv)_x = du_{xx}, \\ v_t + (\epsilon v^2 - u)_x = \epsilon v_{xx}. \end{cases}$$

As we have discussed from the previous context, fruitful results on the traveling waves of the above system were derived. See [25].

1.2 Organization of the Thesis

In this thesis, I will focus on the conservation law system arising from chemotaxis instead of the original model (1.1). The organization of the thesis is as follows.

In the first part of introduction chapter, the concept of traveling waves and chemotaxis has been introduced. Some previous research works relate to traveling waves have been reviewed. In the remaining introduction chapter, I will state the transformed system that we study.

In the second chapter, we introduce methods which will be needed when we analyse the existence and stability of the system and main results of the our study will be illustrated.

In the third chapter, we show the details of the phase plane analysis to prove the existence results in different cases while the non-existence results will also be presented. In addition, we show the numerical simulations of traveling waves for three heteroclinic connection cases by using Matlab Program.

In the fourth chapter, we show the details of weighted energy method to prove the stability results.

In the final chapter, a brief summary of the whole thesis will be given and some research problems that we may explore will be presented.

1.3 Our Considered Problem and Applications

In this thesis, we consider the case of (1.1) when $f(h) = \alpha h$ where $\alpha < 0$ which means that chemical substance is consumed only when cells (bacteria) leave the chemical. We employ the following Cole-Hopf type transformation

$$v = \frac{1}{\alpha}\frac{\partial}{\partial x}(\log\ h) = \frac{1}{\alpha}\frac{h_x}{h}. \tag{1.3}$$

Our considered system is

$$\begin{cases} u_t + (uv)_x = du_{xx}, & x \in \mathbb{R},\ t > 0, \\ v_t + (\sigma v^2 + u)_x = \epsilon v_{xx}, & x \in \mathbb{R},\ t > 0, \end{cases} \tag{1.4}$$

with the initial data

$$(u, v)(x, 0) = (u_0, v_0)(x) \to (u_\pm, v_\pm) \text{ as } x \to \pm\infty. \tag{1.5}$$

The system (1.4) we are concerned with has various applications.

Application 1: The Attraction-repulsion chemotaxis model

Luca et al. [26] proposed the attraction-repulsion chemotaxis model to study the aggregation of *Microglia* observed in Alzhemer's disease.

$$\begin{cases} u_t = d\Delta u - \nabla \cdot (\chi u \nabla s) + \nabla \cdot (\xi u \nabla w), \\ s_t = \epsilon \Delta s + \alpha u - \beta s, \\ w_t = \epsilon \Delta w + \gamma u - \delta w. \end{cases} \tag{1.6}$$

The aggregation of *Microglia* denoted by u due to the interaction with chemoattractant(β-amyloid) denoted by s and chemorepellent (tumor necrosis factor TNF-α) denoted by w. $\chi, \xi, \beta, \delta \geq 0, \alpha, \gamma > 0$ are parameters. When $\beta = \delta = 0$, setting

$$\theta = \chi\alpha - \xi\gamma,$$

where θ is an index measuring the competition between attraction and repulsion. Let

$$h = \chi\nabla s - \xi\nabla w.$$

We have

$$\begin{cases} u_t = d\Delta u - \theta\nabla \cdot (u\nabla h), \\ h_t = \epsilon\Delta h + u. \end{cases}$$

Letting $v = -\nabla h$, we derive

$$\begin{cases} u_t - \theta\nabla \cdot (uv) = d\Delta u, \\ v_t + \nabla u = \epsilon\Delta v. \end{cases}$$

When $\theta = -1$, the above equations are transformed into the system (1.4) with $\sigma = 0$.

Application 2: The Repulsive chemotaxis model with logarithmic sensitivity

Biological systems that build transport networks can be described in terms of reinforced random walks. The repulsive chemotaxis model with logarithmic sensitivity was derived in [27, 28] which is modeled by reinforced random walkers represented by u such as *myxobacteria* and chemical signal denoted by w released from cells. The system reads as follows

$$\begin{cases} u_t = d\Delta u - \nabla \cdot (\chi u\nabla \ln w), \\ w_t = \epsilon\Delta w + uw - \mu w, \end{cases} \tag{1.7}$$

where $\chi < 0$. The Cole-Hopf type transformation $v = -\nabla \ln w = -\frac{\nabla w}{w}$ yields that

$$v_t = \left(-\frac{w_x}{w}\right)_t = \left(\frac{w_t}{w}\right)_x$$

$$
\begin{aligned}
&= -\epsilon \left(\frac{w_{xx}}{w}\right)_x - u_x \\
&= -\epsilon \left[\left(\frac{w_x}{w}\right)_x + \left(\frac{w_x}{w}\right)^2\right]_x - u_x \\
&= -\epsilon \left[(-v)_x + (-v)^2\right]_x - u_x \\
&= \epsilon v_{xx} - \epsilon (v^2)_x - u_x.
\end{aligned}
$$

The system (1.4) can be derived from (1.7) by setting $\chi = -1$ and $\sigma = \epsilon$.

Appliaction 3: The Boussinesq-Burgers system

Our model (1.4) has applications in fluid mechanics. For example, when $\sigma = \frac{1}{2}$ and $w = v$, (1.4) becomes the following Boussinesq-Burgers system which is used to describe the propagation of shallow water waves [29].

$$
\begin{cases}
u_t + (uv)_x = du_{xx}, \\
w_t + \left(u + \frac{w^2}{2}\right)_x = \epsilon w_{xx} + \delta w_{xxt}
\end{cases} \tag{1.8}
$$

In this system, $w = 1 + \rho$ where ρ is the height of the fluid and u represents velocity of the free surface of the fluid above the bottom.

Since my analysis is mainly focus on the applications in biology science, we require $u \geq 0$.

Chapter 2

Methods and Main Results

2.1 Methods

There are various methods to study the existence of traveling waves. For example, topological methods, in particular, the Leray-Schauder method and methods of bifurcation theory. To prove the existence of traveling wave solutions of (1.4), one of the most conventional methods, phase plane analysis (shooting method), is employed. To apply this method, we need to transform the partial differential equations to a system of ordinary differential equations (ODEs). Then we shall reduce the system into a system of first order ODEs. For the resulting system, we can linearize the system by calculating the Jacobian matrix and analyze the properties of equilibrium points. There are four types of critical points, including unstable and stable focus, unstable and stable nodes, centers and saddles. Then a heteroclinic or homoclinic orbit, a path in phase space which joins two different equilibrium points or joins a saddle equilibrium point to itself (unstable manifold and stable manifold), gives existence of a traveling wave solution.

In the phase plane analysis, Poincaré–Bendixson theorem [30] is an important tool applied when proving the existence of traveling waves. The generalized Poincaré–Bendixson theorem is

Proposition 2.1 (Poincaré–Bendixson Theorem)**.** *Let M be an open subset of* R^2

and $f \in C^1(M, \mathrm{R}^2)$. *Fix* $x \in M, \sigma \in \{\pm\}$, *and suppose omega-limit set* $\omega_\sigma(x) \neq \varnothing$ *is compact, connected, and contains only finitely many fixed points. Then one of the following cases holds:*

(*i*) $\omega_\sigma(x)$ *is a fixed orbit;*

(*ii*) $\omega_\sigma(x)$ *is a regular periodic orbit;*

(*iii*) $\omega_\sigma(x)$ *consists of (finitely many) fixed points* $\{x_j\}$ *and non-closed orbits* $\gamma(y)$ *such that* $\omega_\pm(y) \in \{x_j\}$.

To investigate the stability of traveling waves, we are looking for a small perturbation with zero integral from traveling wave solution converges to this traveling wave solution, translated properly by a shift x_0. Mathematically, it means

$$\sup_{x\in\mathbb{R}} |(u,v)(x,t) - (U,V)(x + x_0 - ct)| \to 0, \quad \text{as} \quad t \to +\infty.$$

The weighted energy method is applied for the investigation. Energy method is a very useful technique in PDE analysis. For this method, we get estimates on the types of integrals like $\int |u|^2$ and $\int |\nabla u|^2$ which are usually interpreted in terms of energies related to the physical problems behind PDEs. Therefore, we call methods that involve these kinds of estimates energy methods. Historically, this may come from Hilbert's solution to the Dirichlet problem, which consisted in minimizing $\int |\nabla u|^2$. The method of energy estimates for the nonlinear stability of viscous shock profiles of conservation laws was first introduced by Matsumura and Nishihara in [31] and by Goodman in [32]. In our study, we need to choose a weight function to overcome the singularity occurs during the calculations. As a result, the weighted energy method is applied to study the stability of traveling wave solutions of (1.4).

2.2 Main Results

The first result of our thesis is on the existence of traveling wave solutions of (1.4).

Theorem 2.1. *When $\sigma \geq 1$, there is no traveling wave solution to the system (1.4). When $\sigma \in (0,1)$, we have the following existence results:*

a) Let $(u_+, v_+) = (0,0)$ and $(u_-, v_-) = (c^2(1-\sigma), c)$. Then the system (1.4) has a unique (up to a translation) monotone traveling wave $(U,V)(x-ct)$ satisfying $U_z < 0$ and $V_z < 0$.

b) Let $(u_+, v_+) = (c^2(1-\sigma), c)$ and $(u_-, v_-) = (0, \frac{c}{\sigma})$. Then the system (1.4) has a unique (up to a translation) monotone traveling wave $(U,V)(x-ct)$ satisfying $U_z > 0$ and $V_z < 0$.

c) Let $(u_+, v_+) = (0,0)$ and $(u_-, v_-) = (0, \frac{c}{\sigma})$. Then the system (1.4) has a unique (up to a translation) traveling wave $(U,V)(x-ct)$ satisfying $V_z < 0$.

The main result on the asymptotic stability is as follows

Theorem 2.2. *Let $(U,V)(z)$ be the traveling wave solutions to model (1.4) obtained in Theorem 2.1. If $\epsilon > 0$ is small and $u_+ \geq 0$, then there exists a constant x_0 such that such that the initial perturbation from the spatially shifted traveling waves with shift x_0 is of zero integral, namely $(\phi_0, \psi_0)(\pm\infty) = 0$, where*

$$(\phi_0, \psi_0)(x) = \int_{-\infty}^{x} (u_0(y) - U(y + x_0), v_0(y) - V(y + x_0)\, dy.$$

Then if there exists a constant $\delta_0 > 0$ such that $\|u_0 - U\|_{1,w} + \|v_0 - V\|_{1,w} + \|(\phi_0, \psi_0)\| \leq \delta_0$, then the system (1.4) has a unique solution $(u,v)(x,t)$ satisfying

$$(u - U, v - V) \in C([0,\infty), H_w^1) \cap L^2([0,\infty), H_w^2)$$

and the asymptotic stability follows:

$$\sup_{x\in\mathbb{R}} |(u,v)(x,t) - (U,V)(x + x_0 - ct)| \to 0, \text{ as } t \to +\infty.$$

Chapter 3

Proof of Existence

In this section, the non-existence and existence of traveling wave solutions of system (1.4) with $\sigma > 0$ will be shown by using the phase plane analysis.

We define the traveling wave ansatz

$$(u, v)(x, t) = (U, V)(z), z = x - ct, \tag{3.1}$$

where c denotes the wave speed and z is the traveling wave variable. Substituting (3.1) into (1.4) to get the traveling wave equations with boundary conditions

$$\begin{cases} -cU_z + (UV)_z = dU_{zz}, \\ -cV_z + (\sigma V^2 + U)_z = \epsilon V_{zz}, \\ U(\pm\infty) = u_\pm \geq 0, V(\pm\infty) = v_\pm, \\ U_z(\pm\infty) = V_z(\pm\infty) = 0, \end{cases} \tag{3.2}$$

Integrating (3.2) once to reduce the second order ordinary differential equations into first order ODEs,

$$\begin{cases} dU_z = -cU + UV + \rho_1, \\ \epsilon V_z = -cV + (\sigma V^2 + U) + \rho_2, \end{cases} \tag{3.3}$$

where ρ_1 and ρ_2 are constants satisfying

$$\begin{cases} \rho_1 = cu_- - u_- v_- = cu_+ - u_+ v_+, \\ \rho_2 = cv_- - \sigma v_-^2 - u_- = cv_+ - \sigma v_+^2 - u_+, \end{cases} \tag{3.4}$$

In this paper, we assume that ρ_1 and ρ_2 equal to zero. Then from (3.4), we get

$$\begin{cases} u_-(c-v_-) = u_+(c-v_+) = 0, \\ v_-(c-\sigma v_-) - u_- = v_+(c-\sigma v_+) - u_+ = 0, \end{cases} \tag{3.5}$$

The system (3.3) becomes

$$\begin{cases} dU_z = -cU + UV, \\ \epsilon V_z = -cV + (\sigma V^2 + U). \end{cases} \tag{3.6}$$

(3.6) has three equilibrium points $(0,0)$, $(0, \frac{c}{\sigma})$ and $(c^2(1-\sigma), c)$. The Jacobian matrix of the linearized system at (u_c, v_c) is

$$\mathbf{J}_{(u_c,v_c)} = \begin{bmatrix} \frac{-c+v_c}{d} & \frac{u_c}{d} \\ \frac{1}{\epsilon} & \frac{-c+2\sigma v_c}{\epsilon} \end{bmatrix}$$

with eigenvalues λ satisfying

$$\lambda^2 + \left(\frac{c-v_c}{d} + \frac{c-2\sigma v_c}{\epsilon}\right)\lambda + \frac{1}{d\epsilon}\left((v_c - c)(2\sigma v_c - c) - u_c\right) = 0. \tag{3.7}$$

The discriminant of the quadratic equation (3.7) is non-negative in the region $\mathcal{X} = \{(u,v) |\ u \geq 0\}$. Hence all roots of (3.7) are real. Now we can investigate the properties of equilibrium points and apply phase plane analysis to verify the existence results in following three different cases where $\sigma = 1$, $\sigma > 1$ and $0 < \sigma < 1$.

3.1 Case of $\sigma = 1$

When $\sigma = 1$, there are only two equilibria $(0,0)$ and $(0,c)$. For $(0,0)$, we have

$$\mathbf{J}_{(0,0)} = \begin{bmatrix} -\frac{c}{d} & 0 \\ \frac{1}{\epsilon} & -\frac{c}{\epsilon} \end{bmatrix}$$

with $\lambda_1 = -\frac{c}{d}$ and $\lambda_2 = -\frac{c}{\epsilon}$.

Both of the eigenvalues are negative, hence $(0,0)$ is a stable node.

For $(0,c)$, we have

$$\mathbf{J}_{(0,c)} = \begin{bmatrix} 0 & 0 \\ \frac{1}{\epsilon} & \frac{c}{\epsilon} \end{bmatrix}$$

with $\lambda_1 = 0$ and $\lambda_2 = \frac{c}{\epsilon}$.

Thus $(0,c)$ is unstable. Next we prove that there is no heteroclinic orbit connecting $(0,0)$ and $(0,c)$. The corresponding eigenvector of λ_2 is

$$\mathbf{v}_2 = \begin{bmatrix} 0 \\ \alpha \end{bmatrix}$$

where $\alpha \in \mathbb{R}$. It indicates that the eigenvector lies along the V-axis, hence a non-trivial trajectory cannot be formed. We have shown that there is no heteroclinic connection between $(0,0)$ and $(0,c)$ when $\sigma = 1$, therefore, traveling wave solutions for the system (1.4) do not exist when $\sigma = 1$.

3.2 Case of $\sigma > 1$

When $\sigma > 1$, we only consider equilibria $(0,0)$ and $(0, \frac{c}{\sigma})$ since the U-coordinate is smaller than zero at $(c^2(1-\sigma), c)$, this point is not relevant in the context. For $(0,0)$, we have

$$\mathbf{J}_{(0,0)} = \begin{bmatrix} -\frac{c}{d} & 0 \\ \frac{1}{\epsilon} & -\frac{c}{\epsilon} \end{bmatrix}$$

with $\lambda_1 = -\frac{c}{d}$ and $\lambda_2 = -\frac{c}{\epsilon}$.

Both of the eigenvalues are negative, hence $(0,0)$ is a stable node.

For $(0, \frac{c}{\sigma})$, we have

$$\mathbf{J}_{(0,\frac{c}{\sigma})} = \begin{bmatrix} \frac{c(1-\sigma)}{d\sigma} & 0 \\ \frac{1}{\epsilon} & \frac{c}{\epsilon} \end{bmatrix}$$

with $\lambda_1 = \frac{c(1-\sigma)}{d\sigma}$ and $\lambda_2 = \frac{c}{\epsilon}$.

Since $\sigma > 1$, we have $\lambda_1 < 0$ and $\lambda_2 > 0$. Thus $(0, \frac{c}{\sigma})$ is a saddle point. Next we shall prove that there is no heteroclinic orbit connecting $(0,0)$ and $(0, \frac{c}{\sigma})$. The eigenvector of the positive eigenvalue is

$$\mathbf{v}_2 = \begin{bmatrix} 0 \\ \alpha \end{bmatrix}$$

where $\alpha \in \mathbb{R}$. It indicates that the eigenvector lies along the V-axis, hence similar with the case of $\sigma = 1$, traveling wave solutions for the system (1.4) do not exist when $\sigma > 1$. We have shown that there is no traveling wave solution to (1.4) when$\sigma \geq 1$ as stated in Theorem 2.1.

3.3 Case of $0 < \sigma < 1$

There are three possible heteroclinic connections between $(c^2(1-\sigma), c)$ and $(0,0)$,$(c^2(1-\sigma), c)$ and $(0, \frac{c}{\sigma})$ and $(0, \frac{c}{\sigma})$ and $(0,0)$. Setting $c = 1$ and $\sigma = \frac{1}{2}$, we use program Matlab to generate the phase portrait of system (3.6) in $V-U$ plane.(See Fig.1) Then we shall prove the existence of the connections below.

3.3.1 Heteroclinic Connection between $(c^2(1-\sigma), c)$ & $(0,0)$

For $(0,0)$, we have

$$\mathbf{J}_{(0,0)} = \begin{bmatrix} -\frac{c}{d} & 0 \\ \frac{1}{\epsilon} & -\frac{c}{\epsilon} \end{bmatrix}$$

with $\lambda_1 = -\frac{c}{d}$ and $\lambda_2 = -\frac{c}{\epsilon}$.

Since both of the eigenvalues are negative, $(0,0)$ is a stable node.

For $(c^2(1-\sigma), c)$, we have

$$\mathbf{J}_{(c^2(1-\sigma),c)} = \begin{bmatrix} 0 & \frac{c^2(1-\sigma)}{d} \\ \frac{1}{\epsilon} & \frac{-c+2\sigma c}{\epsilon} \end{bmatrix}$$

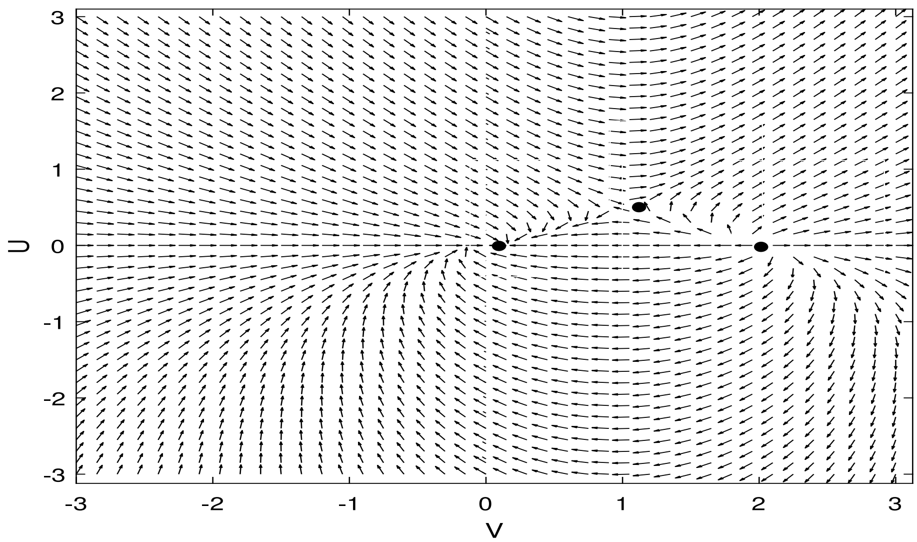

Figure 3.1: Phase portrait of the system (3.6) in $V-U$ plane, setting $c=1$ and $\sigma=0.5$, three critical points are represented by three black points on the figure.

with $\lambda_1 = \frac{c(2\sigma-1)}{2\epsilon} + \frac{\sqrt{\left[\frac{c(1-2\sigma)}{\epsilon}\right]^2 + \frac{4c^2(1-\sigma)}{d\epsilon}}}{2} > 0$ and $\lambda_2 = \frac{c(2\sigma-1)}{2\epsilon} - \frac{\sqrt{\left[\frac{c(1-2\sigma)}{\epsilon}\right]^2 + \frac{4c^2(1-\sigma)}{d\epsilon}}}{2} < 0$ since $\sigma < 1$.

Thus, it is a saddle point.

Next we shall check that there is a heteroclinic orbit connecting $(c^2(1-\sigma), c)$ and $(0,0)$. We have nullclines of the system (3.6) which are given by

$$\begin{cases} U(V-c) = 0 \\ U = V(c-\sigma V) \end{cases} \tag{3.8}$$

The first equation of (3.8) gives two straight lines: $U=0$ and $V=c$, and the second equation gives a parabola. To this end, we can prove that the region bounded by the lines and the parabola is an invariant region of system (3.6). The region is defined by $\mathcal{G} = \{(U,V) |\ 0 \le U \le c^2(1-\sigma), 0 \le V \le c\}$ (see Fig 3.2) and bounded by

$$\Gamma_1 = \{(U,V)| U = V(c-\sigma V), 0 < V < c\}$$

$$\Gamma_2 = \{(U,V)| V = c, 0 < U < c^2(1-\sigma)\}$$

$$\Gamma_3 = \{(U,V)| U = 0, 0 < V < c\}$$

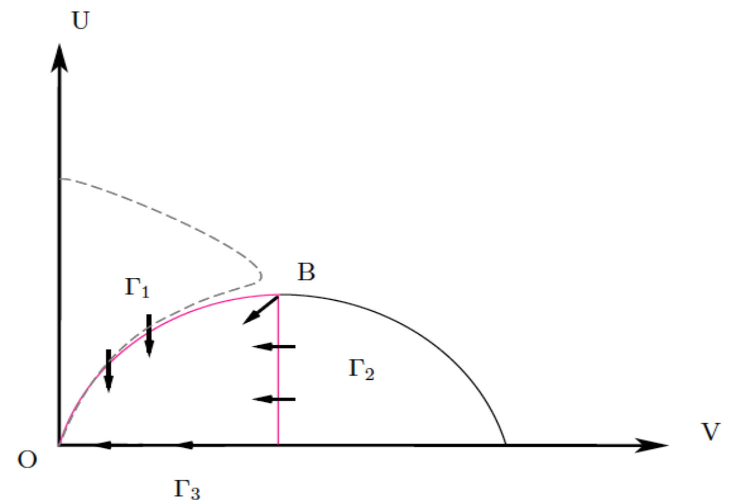

Figure 3.2: A numerical plot of the phase plane of system (3.6), where the deeppink straight line represents $V = c$, $O = (0, 0)$ and $B = (c, c^2(1 - \sigma))$ in $V - U$ plane.

Along Γ_1, $U_z = U(V - c) = V(c - \sigma V)(V - c) < 0$ and $V_z = 0$. Therefore, the direction field along Γ_1 points downward vertically. Along Γ_2, $U_z = 0$ and $V_z = -c^2(1 - \sigma) + U < 0$, thus the direction field along Γ_2 points to the left horizontally. Along Γ_3, $U_z = 0$ and $V_z = V(-c + \sigma V) < 0$. Therefore, the direction field along Γ_3 points to the left horizontally. We conclude that $\mathcal{G}$ is an invariant region from the above analysis. Then we shall prove that the unstable manifold of (3.6) emanating from $(c^2(1 - \sigma), c)$ is trapped inside the invariant region. From direct calculation, we derive that the tangent direction of Γ_1 at $(c^2(1 - \sigma), c)$ is

$$\left.\frac{dU}{dV}\right|_{(c^2(1-\sigma),c)}^{\Gamma_1} = c(1 - 2\sigma)$$

The tangent direction of Γ_2 at $(c^2(1 - \sigma), c)$ is

$$\left.\frac{dU}{dV}\right|_{(c^2(1-\sigma),c)}^{\Gamma_2} = \infty$$

Now we compute the direction of the unstable manifold of (3.6) at $(c^2(1 - \sigma), c)$ and compare with the directions of the boundaries. We consider the associated eigenvector of λ_1,

$$\mathbf{v}_1 = \begin{bmatrix} \frac{c(2\sigma-1)-\epsilon\lambda_1}{\epsilon} \\ -\frac{1}{\epsilon} \end{bmatrix}$$

Tangent to the eigenvector, the direction of the unstable manifold at $(c^2(1-\sigma), c)$ is given by

$$\left.\frac{dU}{dV}\right|_{(c^2(1-\sigma),c)} = c(1-2\sigma) + \epsilon\lambda_1$$

Since ϵ and λ_1 are positive, we have

$$\left.\frac{dU}{dV}\right|_{(c^2(1-\sigma),c)}^{\Gamma_1} < \left.\frac{dU}{dV}\right|_{(c^2(1-\sigma),c)} < \left.\frac{dU}{dV}\right|_{(c^2(1-\sigma),c)}^{\Gamma_2}$$

Therefore we deduce that the direction of the unstable manifold of (3.6) at $(c^2(1-\sigma), c)$ is between the tangent lines of Γ_1 and Γ_2 at $(c^2(1-\sigma), c)$ which points inside the invariant region. By the Poincaré-Bendixson theorem, the unstable manifold has to reach the stable equilibrium $(0,0)$. A solution for the system (3.6) is generated by this trajectory connecting $(c^2(1-\sigma), c)$ and $(0,0)$ with $U_z < 0$ and $V_z < 0$ when $\sigma < 1$. We have proved Theorem 2.1(a).

we shall simulate the traveling wave solutions of system (1.4) numerically by using the "pdepe" Matlab. We set $c = 1$, $\sigma = \frac{1}{2}$, $d = 4$and $\epsilon = 1$ to generate the wave propagation profiles in finite spatial domain, $I = (0, 600)$.

For $(c^2(1-\sigma), c)$ and $(0,0)$, we have

$$\begin{cases} u_- = \frac{1}{2}, \\ u_+ = 0, \end{cases} \text{and} \begin{cases} v_- = 1, \\ v_+ = 0, \end{cases} \tag{3.9}$$

The initial condition is set to be

$$(u_0, v_0) = \left(\frac{1}{2+e^{2(x-100)}}, \frac{1}{1+e^{2(x-100)}}\right).$$

Figure 3.3 and figure 3.4 show the propagation of u and v of the system (3.6). Figure 3.5 and 3.6 show the wave profiles in 3D planes.

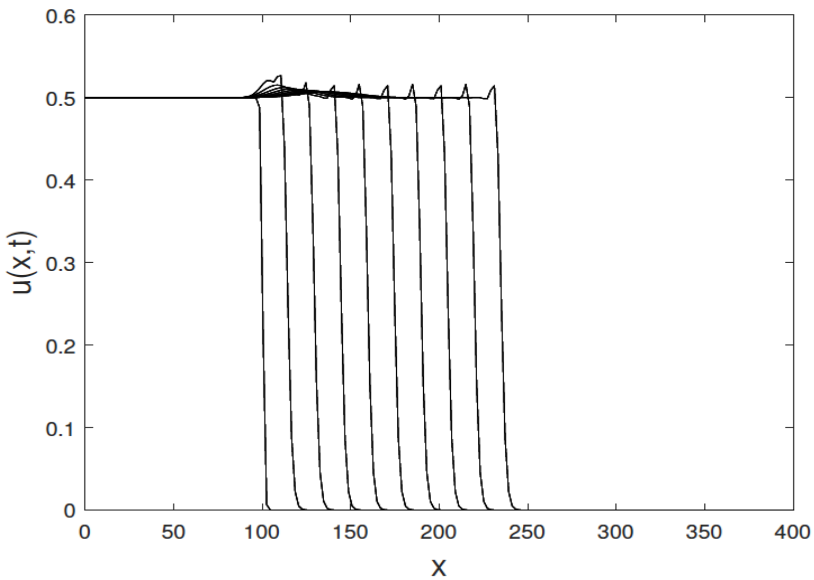

Figure 3.3: Evolutionary wave profile of u

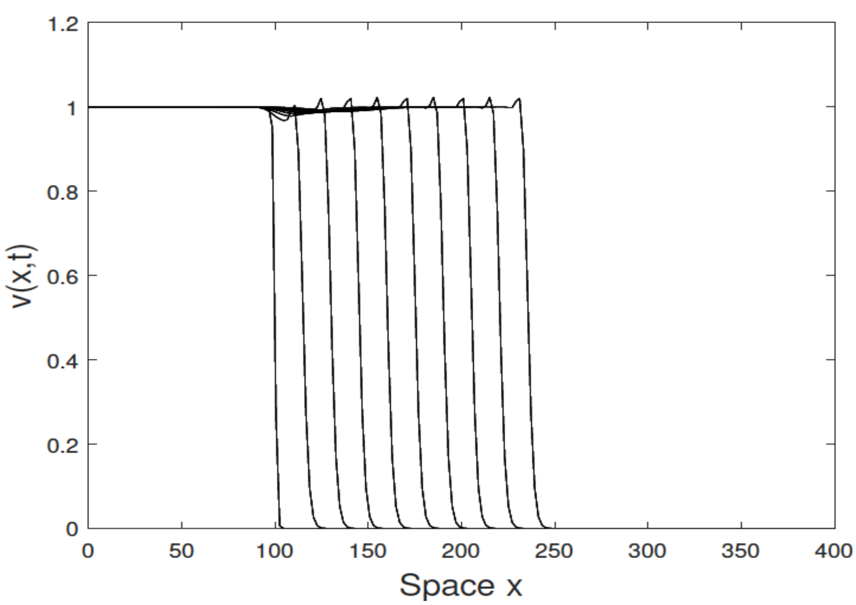

Figure 3.4: Evolutionary wave profile of v

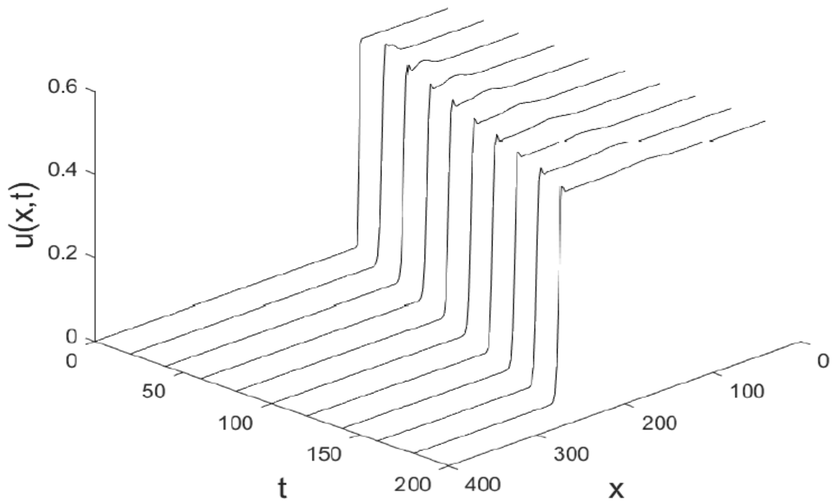

Figure 3.5: Evolutionary wave profile of u in 3D

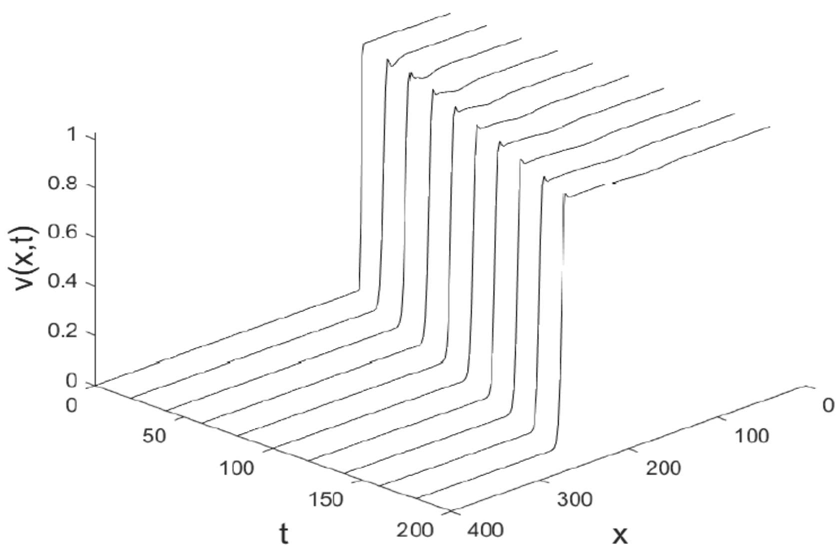

Figure 3.6: Evolutionary wave profile of v in 3D

3.3.2 Heteroclinic Connection between $(0, \frac{c}{\sigma})$ & $(c^2(1-\sigma), c)$

For $(0, \frac{c}{\sigma})$, we have

$$\mathbf{J}_{(0,\frac{c}{\sigma})} = \begin{bmatrix} \frac{c(1-\sigma)}{d\sigma} & 0 \\ \frac{1}{\epsilon} & \frac{c}{\epsilon} \end{bmatrix}$$

with $\lambda_1 = \frac{c(1-\sigma)}{d\sigma}$ and $\lambda_2 = \frac{c}{\epsilon}$.

Since both of the eigenvalues are positive, $(0, \frac{c}{\sigma})$ is an unstable node. For $(c^2(1-\sigma), c)$, we have shown that it is a saddle point.

Next we shall check that there is a heteroclinic orbit connecting $(0, \frac{c}{\sigma})$ and$(c^2(1-\sigma), c)$. Since we can not construct an invariant region for the heteroclinic orbit connecting $(0, \frac{c}{\sigma})$ and $(c^2(1-\sigma), c)$, we reverse the direction of the phase plane. By letting $\bar{z} = -z = -x + ct$, the direction of the orbits in the phase plane will be reversed as shown in the Matlab figure below.

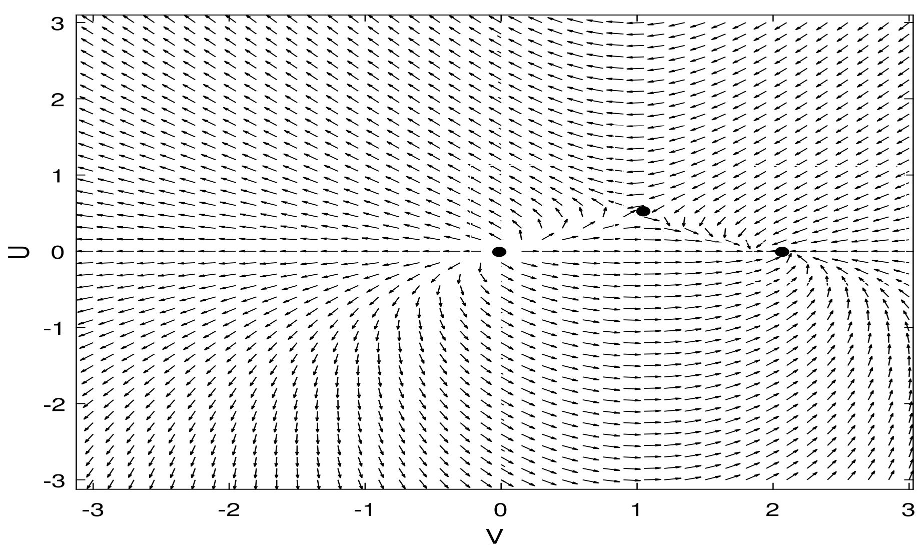

Figure 3.7: Phase portrait of the system (3.6) in $V - U$ plane, setting $c = 1$ and $\sigma = 0.5$ where $\bar{z} = -z = -x + ct$.

Therefore, let $(u,v)(x,t) = (U,V)(\bar{z})$ and substitute into (1.3), then (1.3) becomes

$$\begin{cases} cU_{\bar{z}} - (UV)_{\bar{z}} = dU_{\bar{z}\bar{z}} \\ cV_{\bar{z}} - (\sigma V^2 + U)_{\bar{z}} = \epsilon V_{\bar{z}\bar{z}} \end{cases} \tag{3.10}$$

Integrating (3.9) once gives

$$\begin{cases} cU - (UV) = dU_{\bar{z}} \\ cV - (\sigma V^2 + U) = \epsilon V_{\bar{z}} \end{cases} \tag{3.11}$$

The equilibrium points of the transformed system are the same as the original one, which are $(0, \frac{c}{\sigma})$,$(c^2(1-\sigma), c)$ and $(0,0)$. Since we are looking for the existence of heteroclinic connection between $(0, \frac{c}{\sigma})$ and $(c^2(1-\sigma), c)$, we only need to explore properties of these two points. To begin with, we obtain the Jacobian matrix of (3.10),

$$\mathbf{J}_{(u_c,v_c)} = \begin{bmatrix} \frac{c-v_c}{d} & \frac{-u_c}{d} \\ \frac{-1}{\epsilon} & \frac{c-2\sigma v_c}{\epsilon} \end{bmatrix}$$

At $(c^2(1-\sigma), c)$, we have

$$\mathbf{J}_{(c^2(1-\sigma),c)} = \begin{bmatrix} 0 & \frac{c^2(\sigma-1)}{d} \\ \frac{-1}{\epsilon} & \frac{c(1-2\sigma)}{\epsilon} \end{bmatrix}$$

where the eigenvalues are

$$\lambda_{1,2} = \frac{\frac{c(1-2\sigma)}{\epsilon} \pm \sqrt{\frac{c^2(1-2\sigma)^2}{\epsilon^2} + \frac{c^2(1-\sigma)}{d\epsilon}}}{2},$$

We have $\lambda_1 > 0$ and $\lambda_2 < 0$, so $(c^2(1-\sigma), c)$ is a saddle point.

At $(0, \frac{c}{\sigma})$, we have

$$\mathbf{J}_{(0,\frac{c}{\sigma})} = \begin{bmatrix} \frac{c(\sigma-1)}{d\sigma} & 0 \\ \frac{-1}{\epsilon} & \frac{-c}{\epsilon} \end{bmatrix}$$

with $\lambda_1 = \frac{c(\sigma-1)}{d\sigma}$ and $\lambda_2 = \frac{-c}{\epsilon}$.

Both of the eigenvalues are negative, so $(0, \frac{c}{\sigma})$ is a stable node.

Next we can prove that there is a heteroclinic orbit connecting these two manifolds. We construct a triangle region enclosed by the following lines and show that this region of system (3.10) is invariant. The region is bounded by three lines (see Fig 3.8).

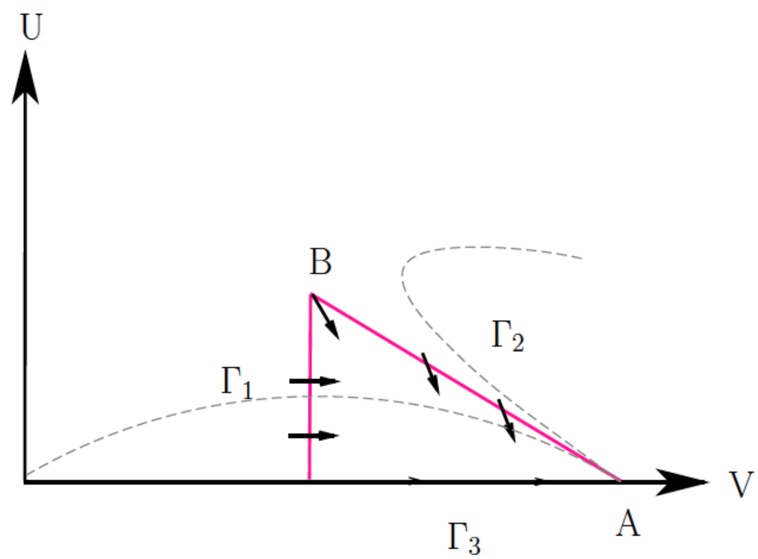

Figure 3.8: phase portrait for $\sigma = 1$ in $V - U$ plane where two deeppink lines represent $V = c$ and $U = c(c - \sigma V)$ respectively, $A = (\frac{c}{\sigma}, 0)$ and $B = (c, c^2(1-\sigma))$.

$$\Gamma_1 = \{(U, V)|0 < U < c^2(1-\sigma), V = c\},$$

$$\Gamma_2 = \{(U, V)|U = c(c - \sigma V), 0 < U < c^2(1-\sigma), c < V < \frac{c}{\sigma}\},$$

$$\Gamma_3 = \{(U, V)|U = 0, c < V < \frac{c}{\sigma}\},$$

Along Γ_1, $U_{\bar{z}} = \frac{U(c-V)}{d} = 0$ since $V = c$ and $V_{\bar{z}} = c^2(1-\sigma) + U > 0$. Therefore, the direction field of (3.10) along Γ_1 points to the right horizontally. For Γ_2, we have $\Gamma_2 = U + c\sigma V - c^2$. If there exists vector fields leave the triangle region by passing through Γ_2, then there exists a point at which $\frac{d\Gamma_2}{d\bar{z}} > 0$. However, by (3.10) and the equation of Γ_2, we derive that

$$\begin{aligned}\frac{d\Gamma_2}{d\bar{z}} &= U_{\bar{z}} + c\sigma V_{\bar{z}} \\ &= \frac{cU - UV}{d} + c\sigma \frac{cV - \sigma V^2 - U}{\epsilon} \\ &= \frac{(-c\sigma V + c^2)(c - V)}{d} + c\sigma \frac{cV - \sigma V^2 + c\sigma V - c^2}{\epsilon}\end{aligned}$$

Since we are interested in the sign of $\frac{d\Gamma_2}{d\bar{z}}$ and the denominators of the above equations are positive, we can neglect the denominators and expand the equation. Then we have

$$F(V) = c\sigma(1-\sigma)V^2 + \left(c^2\sigma(\sigma-1) + c^2(\sigma-1)\right)V + c^3(1-\sigma)$$

We have $\frac{d^2F(V)}{d\bar{z}^2} > 0$, so $F(V)$ is convex. Since $c < V < \frac{c}{\sigma}$, we can show that both $F(c)$ and $F(\frac{c}{\sigma})$ are non-negative to conclude that $F(V) < 0$. Indeed, we have

$$F(c) = c^3\sigma(1-\sigma) + c^3\sigma(\sigma-1) + c^3(\sigma-1) + c^3(1-\sigma) = 0$$

and

$$F(\frac{c}{\sigma}) = \frac{c^3(1-\sigma)}{\sigma} + c^3(\sigma-1) + \frac{c^3(\sigma-1)}{\sigma} + c^3(1-\sigma) = 0$$

Therefore, $F(V) < 0$ and $\frac{d\Gamma_2}{d\bar{z}} < 0$. There has no trajectory leaving the region through Γ_2. Along Γ_3, $U_{\bar{z}} = 0$ and $V_{\bar{z}} = cV - \sigma V^2 > 0$. Therefore, the direction field of (3.10) along Γ_3 points to the right horizontally. Therefore the triangle region is an invariant region. Next we shall prove that the unstable manifold of (3.10) emanating from $(c^2(1-\sigma), c)$ is trapped inside the region. We derive that the tangent direction of Γ_1 is

$$\left.\frac{dU}{dV}\right|_{(c^2(1-\sigma),c)}^{\Gamma_1} = \infty$$

The tangent direction of Γ_2 at $(c^2(1-\sigma), c)$ is

$$\left.\frac{dU}{dV}\right|_{(c^2(1-\sigma),c)}^{\Gamma_2} = -c\sigma$$

Now we can compute the direction of the unstable manifold at $(c^2(1-\sigma), c)$ and compare with the directions calculated above. We consider the associated eigenvector of $\lambda_1 = \frac{\frac{c(1-2\sigma)}{\epsilon} + \sqrt{\frac{c^2(1-2\sigma)^2}{\epsilon^2} + \frac{c^2(1-\sigma)}{d\epsilon}}}{2}$.

$$\mathbf{v}_1 = \begin{bmatrix} \frac{c(1-2\sigma)}{\epsilon} - \lambda_1 \\ \frac{1}{\epsilon} \end{bmatrix}$$

Tangent to the eigenvector, the direction of the unstable manifold at $(c^2(1-\sigma), c)$ is given by

$$\left.\frac{dU}{dV}\right|_{(c^2(1-\sigma),c)} = c(1-2\sigma) - \epsilon\lambda_1$$

When ϵ is small enough, we have

$$c(1-2\sigma) - \epsilon\lambda_1 + c\sigma = c(1-\sigma) > 0$$

Therefore, we get

$$\left.\frac{dU}{dV}\right|_{(c^2(1-\sigma),c)}^{\Gamma_2} < \left.\frac{dU}{dV}\right|_{(c^2(1-\sigma),c)} < \left.\frac{dU}{dV}\right|_{(c^2(1-\sigma),c)}^{\Gamma_1}$$

Hence we conclude that the unstable manifold at $(c^2(1-\sigma), c)$ lies between the tangent lines of Γ_2 and Γ_3. Since the manifold is trapped inside the invariant region, this unstable manifold has to go to the stable equilibrium by the Poincaré-Bendixson theorem.

Since we have proved that there exists traveling wave solution in the reversed system, there also exists traveling wave solution in the system (3.6). The difference is that the direction of the orbits are reversed. Hence, a solution for the system (3.6) is generated by this trajectory connecting $(c^2(1-\sigma), c)$ and $(0, \frac{c}{\sigma})$ when $\sigma < 1$. We have proved Theorem 2.1(b).

Similar with the simulation of traveling waves for $(c^2(1-\sigma), c)$ and $(0,0)$. For $(c^2(1-\sigma), c)$ and $(0, \frac{c}{\sigma})$, we have

$$\begin{cases} u_- = 0, \\ u_+ = \frac{1}{2}, \end{cases} \text{and} \begin{cases} v_- = 2, \\ v_+ = 1, \end{cases} \tag{3.12}$$

The initial condition is set to be

$$(u_0, v_0) = \left(\frac{1}{2 + e^{2(x-100)}}, \frac{1}{1 + e^{2(x-100)}} \right).$$

Figure 3.9 and figure 3.10 show the propagation of u and v of the system (3.6). Figure 3.11 and 3.12 show the wave profiles in 3D planes.

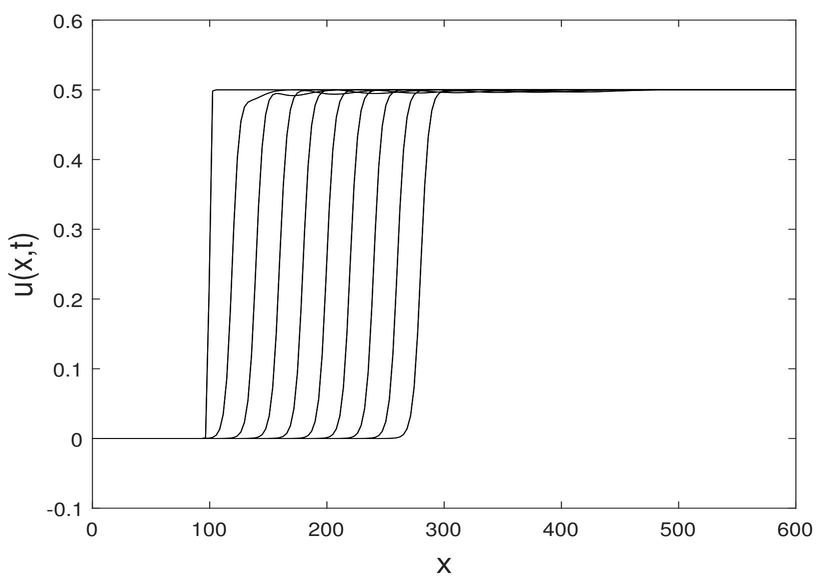

Figure 3.9: Evolutionary wave profile of u

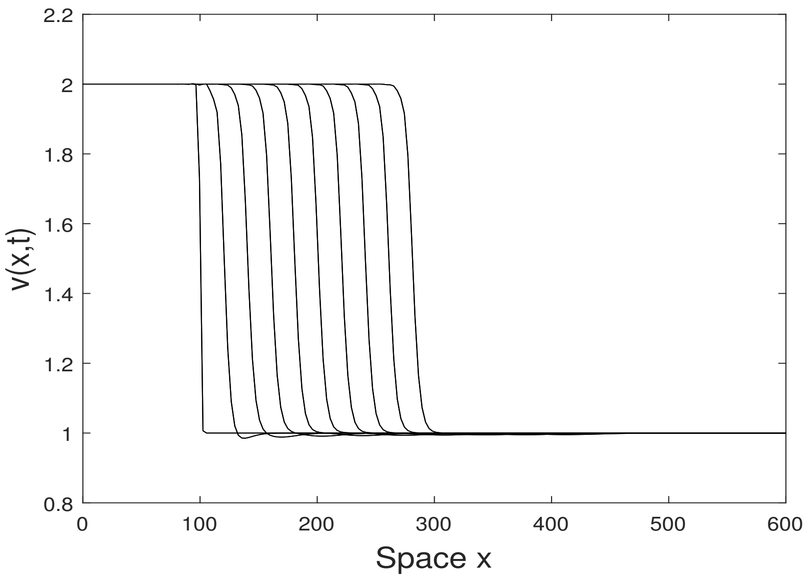

Figure 3.10: Evolutionary wave profile of v

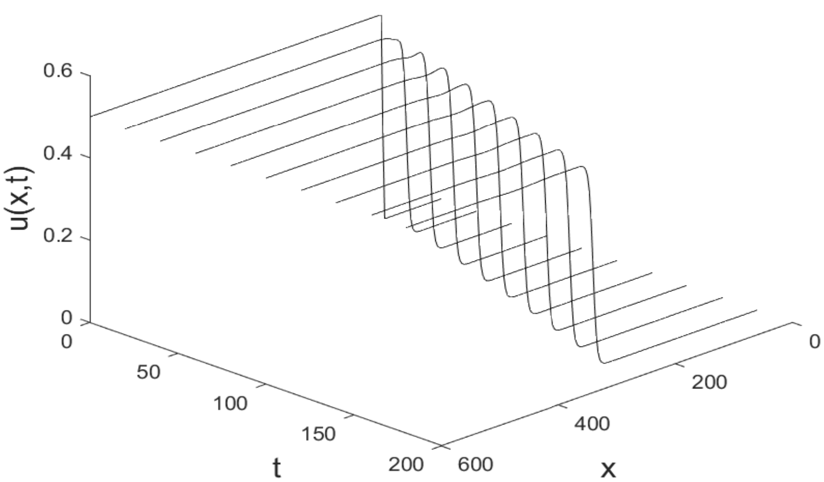

Figure 3.11: Evolutionary wave profile of u in 3D

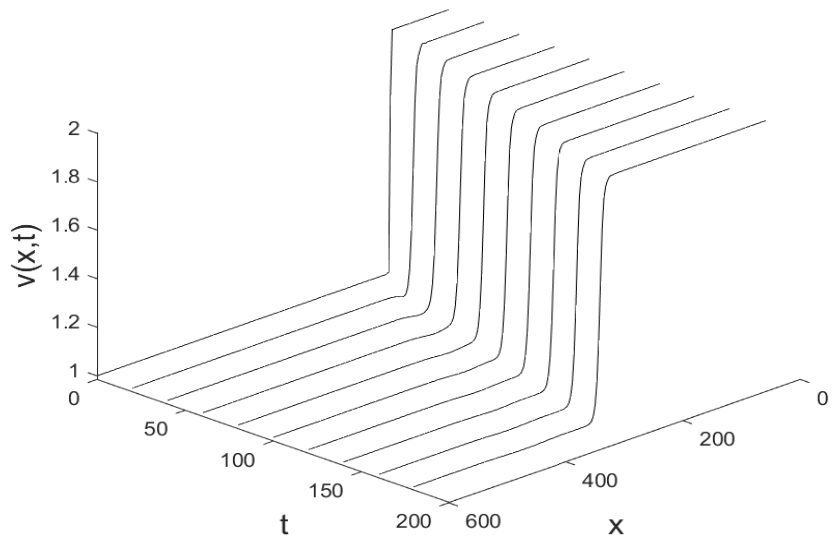

Figure 3.12: Evolutionary wave profile of v in 3D

3.3.3 Heteroclinic Connection between $(0, \frac{c}{\sigma})$ & $(0, 0)$

For $(0, \frac{c}{\sigma})$, we have shown that it is an unstable node. For $(0,0)$, we have shown that it is a stable node.

Next we shall check that there is a heteroclinic orbit connecting $(0, \frac{c}{\sigma})$ and$(0,0)$ by constructing a triangle region OBA where $O = (0,0), A = (0, \frac{c}{\sigma})$ and $B = (m, c)$ where $0 < m \leq c^2(1-\sigma)$. We can prove this region enclosed by the following three lines is an invariant region of system (3.6). (See Fig 3.13)

$$\Gamma_1 = \left\{(U,V)|U = 0, 0 < V < \frac{c}{\sigma}\right\}$$

$$\Gamma_2 = \left\{(U,V)|U = \frac{\sigma m V}{c(\sigma-1)} + \frac{m}{1-\sigma}, 0 < U < c^2(1-\sigma), c < V < \frac{c}{\sigma}\right\}$$

$$\Gamma_3 = \left\{(U,V)|U = \frac{m}{c}V, 0 < U < c^2(1-\sigma), 0 < V < c\right\}$$

Along Γ_1, $U_z = 0$ and $V_z = V(\sigma V - c) < 0$. Therefore, the direction field of (3.6) along Γ_1 points to the left horizontally. For Γ_2, we have $\Gamma_2 = c(1-\sigma)U + m\sigma V - cm$.

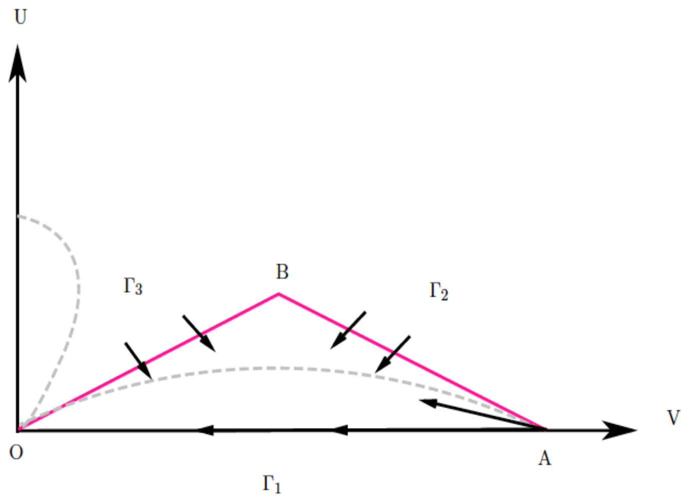

Figure 3.13: phase portrait for $\sigma = 1$ in $V - U$ plane where two deeppink lines represent $U = \frac{m\sigma V}{c(\sigma-1)} + \frac{m}{1-\sigma}$ and $U = \frac{mV}{c}$ respectively, $A = (\frac{c}{\sigma}, 0)$ and $B = (c, m)$.

If there exists vector fields leave the triangle region by passing through Γ_2, then there exists a point at which $\frac{d\Gamma_2}{dz} > 0$. However, by (3.6) and the equation of Γ_2, we derive that

$$\begin{aligned}\frac{d\Gamma_2}{dz} &= c(1-\sigma)U_z + m\sigma V_z \\ &= \frac{c(1-\sigma)(-cU+UV)}{d} + \frac{m\sigma(-cV+\sigma V^2+U)}{\epsilon}\end{aligned}$$

Since we are interested in the positivity of $\frac{d\Gamma_2}{dz}$ and the denominators of the above equations are positive, we can neglect the denominators. Let

$$F(V) = c(1-\sigma)(-cU+UV) + m\sigma(-cV+\sigma V^2+U)$$

Using $U = \frac{\sigma m V}{c(\sigma-1)} + \frac{m}{1-\sigma}$ to expand the equation. Let $m = \frac{c^2(1-\sigma)}{2}$, then we have

$$\frac{dF(V)}{dz} = -c^2(\sigma-1)^2\sigma V^2 - \frac{c^3\sigma^2(1-\sigma)V}{2} - \frac{c^4(1-\sigma)^2}{2} < 0$$

since $0 < \sigma < 1$ and $c > 0$. We get $\frac{d\Gamma_2}{dz} < 0$, therefore, no trajectories can leave the region through Γ_2 with $m = \frac{c^2(1-\sigma)}{2}$.

For Γ_3, we have $\Gamma_3 = mV - cU$. Assume there exists vector fields leaving the triangle region through Γ_3, then there exists a point where $\frac{d\Gamma_3}{dz} < 0$. For $m = \frac{c^2(1-\sigma)}{2}$, we have

$$\begin{aligned}\frac{d\Gamma_3}{dz} &= -cU_z + mV_z \\ &= \frac{-c(U(V-c))}{d} + \frac{m(-cV + \sigma V^2 + U)}{\epsilon}\end{aligned}$$

Similarly, we are only interested in the sign of the above equation, substitute $U = \frac{m}{c}V$ into the equation and simplify it, we have

$$\frac{dF(V)}{dz} = \frac{c^2(1-\sigma)V}{2}(c - V) > 0$$

since $0 < \sigma < 1$ and $0 < V < c$ in this case. Thus, $\frac{d\Gamma_3}{dz} > 0$ which contradicts our assumption, we conclude that no trajectories leaves the region through Γ_3. OBA is an invariant region. Next we shall prove that the unstable manifold of (3.6) emanating from $(0, \frac{c}{\sigma})$ is trapped inside the region. We derive that the tangent direction of Γ_1 is

$$\left.\frac{dU}{dV}\right|_{(0,\frac{c}{\sigma})}^{\Gamma_1} = -\infty$$

The tangent direction of Γ_2 at $(0, \frac{c}{\sigma})$ is

$$\left.\frac{dU}{dV}\right|_{(0,\frac{c}{\sigma})}^{\Gamma_2} = -\frac{c\sigma}{2}$$

Now we can compute the direction of the unstable manifold at $(0, \frac{c}{\sigma})$ and compare with the directions calculated above. We consider the associated eigenvector of $\lambda_1 = \frac{c(1-\sigma)}{d\sigma}$.

$$\mathbf{v}_1 = \begin{bmatrix} \frac{c}{\epsilon} - \lambda_1 \\ \frac{-1}{\epsilon} \end{bmatrix}$$

Tangent to the eigenvector, the direction of the unstable manifold at $(0, \frac{c}{\sigma})$ is given by

$$\left.\frac{dU}{dV}\right|_{(0,\frac{c}{\sigma})} = -c + \epsilon\lambda_1$$

When ϵ is small enough, we have

$$\frac{-c\sigma}{2} + c = \frac{c(2-\sigma)}{2} > 0$$

Therefore, we get

$$\left.\frac{dU}{dV}\right|_{(0,\frac{c}{\sigma})}^{\Gamma_1} < \left.\frac{dU}{dV}\right|_{(0,\frac{c}{\sigma})} < \left.\frac{dU}{dV}\right|_{(0,\frac{c}{\sigma})}^{\Gamma_2}$$

Hence we conclude that the unstable manifold of (3.6) at $(0, \frac{c}{\sigma})$ lies between the tangent lines of Γ_1 and Γ_2. Since the manifold is trapped inside the invariant region, this unstable manifold has to go to the stable equilibrium $(0, 0)$ by the Poincaré-Bendixson theorem. A solution for the system (3.6) is generated by this trajectory connecting $(0, \frac{c}{\sigma})$ and $(0, 0)$ when $\sigma < 1$. We have proved the Theorem 2.1(c).
For the case of $(0, \frac{c}{\sigma})$ and $(0, 0)$, we have

$$\begin{cases} u_- = 0, \\ u_+ = 0, \end{cases} \text{and} \begin{cases} v_- = 2, \\ v_+ = 0, \end{cases} \tag{3.13}$$

The initial condition is set to be

$$(u_0, v_0) = \left(\frac{e^{x-100}}{2 + e^{2(x-100)}}, \frac{2}{1 + e^{2(x-100)}}\right).$$

Figure 3.14 and figure 3.15 show the propagation of u and v of the system (3.6). Figure 3.16 and 3.17 show the wave profiles in 3D planes.

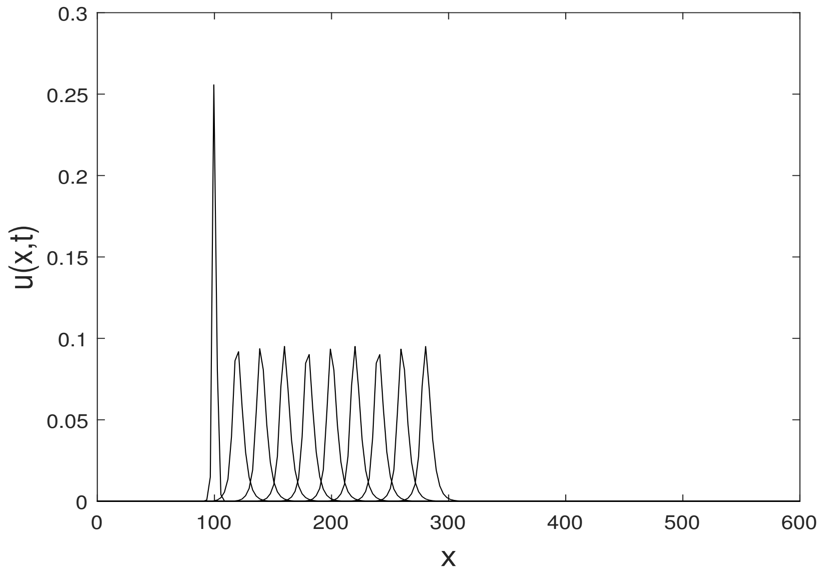

Figure 3.14: Evolutionary wave profile of u

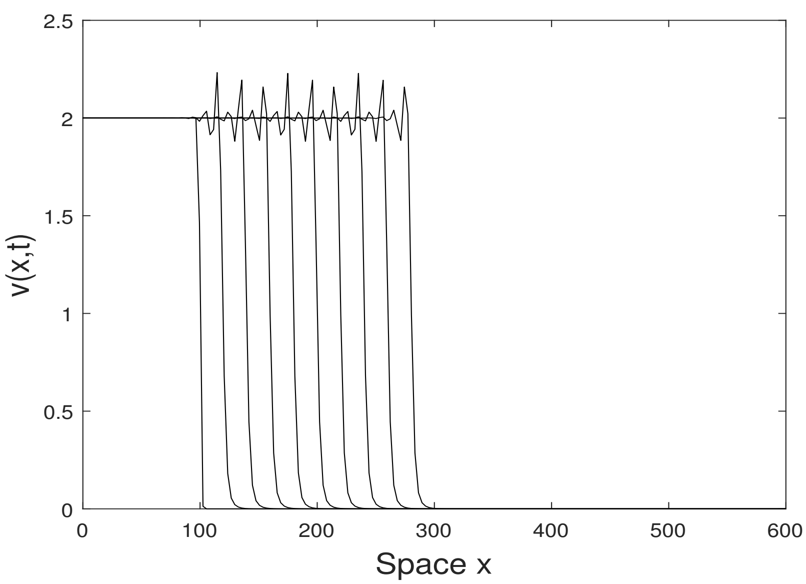

Figure 3.15: Evolutionary wave profile of v

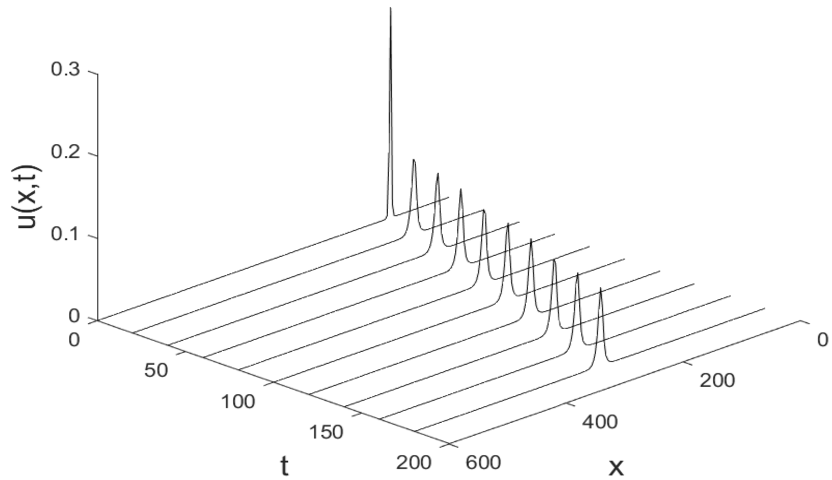

Figure 3.16: Evolutionary wave profile of u in 3D

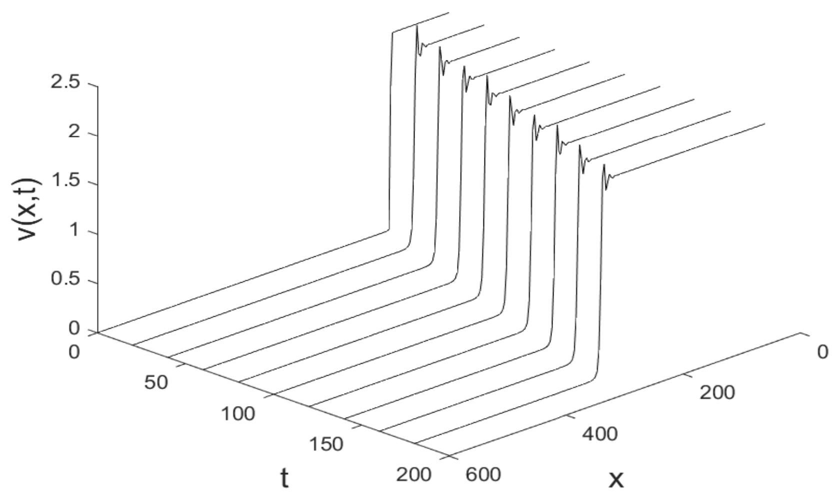

Figure 3.17: Evolutionary wave profile of v in 3D

Chapter 4

Proof of Stability

4.1 Energy Estimates

Next we investigate the stability of traveling wave solutions to the system (1.4). We will choose suitable weight functions according to different cases. Before stating our expected stability result, we introduce the following notations.

Notations: Let Ω be a domain, $L^2(\Omega)$ denotes the space of square integrable functions defined in Ω, $H^k(\Omega)$ the Sobolev space of the L^2 functions $f(x)$ defined in Ω whose derivatives $\frac{\partial^n}{\partial x^n} f(n = 1, 2, ..., k)$ belong to $L^2(\Omega)$. Let $H^k_w(\Omega)$ be the weighted Sobolev space with the norm given by

$$\|f\|_{H^k_w(\Omega)} = \left(\sum_{i=0}^{k} \int_{\Omega} w(x) \left| \frac{\partial^i f(x)}{\partial x^i} \right|^2 dx \right)^{\frac{1}{2}}.$$

$\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_k := \|\cdot\|_{H^k(\Omega)}$ and $\|\cdot\|_{k,w} := \|\cdot\|_{H^k_w(\Omega)}$ will be used for simplicity.

Apply the anti-derivative technique to decompose the solution since (1.4) is a system of conservation laws.

$$(u, v)(x, t) = (U, V)(x + x_0 - ct) + (\phi_z, \psi_z)(z, t), \tag{4.1}$$

Then

$$(\phi(z,t),\psi(z,t)) = \int_{-\infty}^{z} (u(y,t) - U(y+x_0-ct), v(y,t) - V(y+x_0-ct)\, dy, \quad (4.2)$$

where the initial data is

$$(\phi_0,\psi_0)(z) = \int_{-\infty}^{x} (u_0(y) - U(y+x_0), v_0(y) - V(y+x_0)\, dy,$$

We have

$$(\phi_0,\psi_0)(\pm\infty) = 0, \quad (4.3)$$

by the following assumption

$$\int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x) - U(x+x_0) \\ v_0(x) - V(x+x_0) \end{pmatrix} dx = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

since we assume that the initial perturbation is a spatially shifted traveling wave with an amount x_0 with zero integral.

Moreover, we have

$$\phi(\pm\infty,t) = \psi(\pm\infty,t) = 0, \quad (4.4)$$

for all $z \in \mathbb{R}$ and $t > 0$. since

$$\begin{pmatrix} \phi(\pm\infty,t) \\ \psi(\pm\infty,t) \end{pmatrix} = \int_{-\infty}^{+\infty} \begin{pmatrix} u(x,t) - U(x+x_0-ct) \\ u(x,t) - U(x+x_0-ct) \end{pmatrix} dx = \int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x) - U(x+x_0) \\ v_0(x) - V(x+x_0) \end{pmatrix} dx$$

Substituting (4.1) into (1.4), integrating the resulting equations with respect to z and using the wave equations in (U,V), we obtain the equations for the perturbation

$$\begin{cases} \phi_t = d\phi_{zz} + (c-V)\phi_z - U\psi_z - \phi_z\psi_z, \\ \psi_t = \epsilon\psi_{zz} + (c-2\sigma V)\psi_z - \phi_z - \sigma\psi_z^2, \end{cases} \quad (4.5)$$

We look for solutions to the reformulated system in the following solution space

$$X(0,T) := \{(\phi(z,t),\psi(z,t) : \phi \in C([0,T]; H^2_w, \phi_z \in L^2((0,T); H^2_w),$$
$$\psi \in C([0,T]; H^2_w, \psi_z \in C([0,T]; H^1_w \cap L^2((0,T); H^2_w)\}$$

Define

$$N(t) := \sup_{\tau\in[0,t]} (\|\phi(\cdot,\tau)\|_{2,w} + \|\psi(\cdot,\tau)\|_2 + \|\psi_z(\cdot,\tau)\|_{1,w}). \tag{4.6}$$

By the Sobolev embedding theorem, it holds that

$$\sup_{\tau\in[0,t]} \{\|\phi(\cdot,\tau)\|_{L^\infty}, \|\phi_z(\cdot,\tau)\|_{L^\infty}, \|\psi(\cdot,\tau)\|_{L^\infty}, \|\psi_z(\cdot,\tau)\|_{L^\infty}\} \le N(t). \tag{4.7}$$

For (4.5), we shall prove the following statements.

Proposition 4.1. *If $\epsilon > 0$, there exists a constant $\delta > 0$ such that if $N(0) \le \delta$, then the Cauchy problem (4.5) has a unique global solution $(\phi,\psi) \in \mathrm{X}(0,\infty)$ satisfying*

$$\begin{aligned} &\|\phi\|^2_{2,w} + \|\psi\|^2_2 + \|\psi_z\|^2_{1,w} + \int_0^t (\|\phi_z(\cdot,\tau)\|^2_{2,w} + \|\psi_z(\cdot,\tau)\|^2_{2,w} d\tau \\ &\le C(\|\phi_0\|^2_{2,w} + \|\psi_0\|^2_2 + \|\psi_{0z}\|^2_{1,w}) \le CN^2(0), \end{aligned} \tag{4.8}$$

for all $t \in [0,\infty)$. Moreover, it holds that

$$\sup_{z\in\mathbb{R}} |(\phi_z,\psi_z)(z,t)| \to 0, \ \text{as } t \to +\infty, \tag{4.9}$$

Theorem 2.2 is a direct consequence of the above Proposition. To prove Proposition 4.1, we first need to prove the local existence of a unique solution to system (4.5) and then prove the global existence of (ϕ,ψ) by establishing some *a priori* estimates.

Proposition 4.2. *(Local existence) For any $\delta_1 > 0$, there exists a constant $T_0 > 0$ depending on δ_1 such that if $(\phi_0,\psi_0) \in \mathrm{H}^2_w$ with $N(0) \le \delta_1$, then (4.5) has a unique solution $(\phi,\psi) \in \mathrm{X}(0,T_0)$ satisfying $N(t) \le 2N(0)$ for any $t \in [0,T_0]$.*

Proposition 4.2 can be proved by the well-known fixed point theorem. [See [33] for details].

Proposition 4.3. *(A priori estimates) Assume that* $(\phi, \psi) \in \mathrm{X}(0, T)$ *is a solution to (4.5) obtained from Proposition 4.2 for some positive* T*.Then there exists a constant* $\delta_2 > 0$ *independent of* T*, such that if* $N(t) \leq \delta_2$ *for all* $t \in [0, T]$*, then the solution* (ϕ, ψ) *satisifies (4.8) for any* $t \in [0, T]$*.*

The proof of Proposition 4.3 is based on the following lemmas. We assume that $N(t) < 1$ in the context. $\int_{-\infty}^{\infty} f(x,t)dx$ and $\int_0^t \int_{-\infty}^{\infty} f(x,\tau)dxd\tau$ will be abbereviated as $\int f$ and $\int_0^t \int f$ for simplification. We will prove it in the following three different cases.

4.2 Case of $(u_-, v_-) = (c^2(1-\sigma), c)$ and $(u_+, v_+) = (0, 0)$

When $(u_-, v_-) = (c^2(1-\sigma), c)$ and $(u_+, v_+) = (0, 0)$, the traveling wave solution component U has the following asymptotic behavior

$$\begin{aligned} U(z) - u_- &\sim Ce^{\lambda_1 z}, \text{ as } z \to -\infty, \\ U(z) &\sim Ce^{-\frac{c}{d}z}, \text{ as } z \to \infty, \end{aligned} \tag{4.10}$$

where C is a positive constant. Since singularity is arised from $u_+ = 0$, we choose the following weight function

$$w(z) = e^{\frac{c}{d}z} + 1, \ z \in \mathbb{R}, \tag{4.11}$$

Lemma 4.1. *Let the assumptions in Proposition 4.3 hold, there exists a positive* *constant C such that*

$$\begin{aligned}
&\|\phi\|_w^2 + \|\psi\|^2 + \frac{d}{4}\int_0^t \|\phi_z(\cdot,\tau)\|_w^2 d\tau + 2\epsilon\int_0^t \|\psi_z(\cdot,\tau)\|^2 d\tau \\
&+ \frac{d}{3}\int_0^t \|U_z(\cdot)\phi(\cdot,\tau)\|^2 d\tau \qquad (4.12)\\
\leq& C\Big(\|\phi_0\|_w^2 + \|\psi_0\|^2 + N(t)\int_0^t\int w\psi_z^2\Big).
\end{aligned}$$

Proof. Multiplying the first equation of (4.5) by ϕ/U and the second one by ψ, then add together obtaining the following

$$\begin{aligned}
&\frac{1}{2}\Big(\frac{\phi^2}{U} + \psi^2\Big)_t - \Big[\frac{d\phi\phi_z}{U} + \frac{(c-V)\phi^2}{2U} + \epsilon\phi\phi_z + (\frac{c}{2} - \sigma V)\psi^2 - \phi\psi\Big]_z \\
&+ \frac{d\phi_z^2}{U} + \frac{\phi\phi_z\psi_z}{U} + \epsilon\psi_z^2 = \frac{dU_z\phi\phi_z}{U^2} + \frac{U_z(c-V)\phi^2}{2U^2} + \frac{V_z\phi^2}{2U} + \sigma V_z\psi^2 - \sigma\psi_z^2\psi,
\end{aligned}$$

Since $V_z < 0$ from the existence result, we have

$$\begin{aligned}
&\frac{1}{2}\Big(\frac{\phi^2}{U} + \psi^2\Big)_t - \Big[\frac{d\phi\phi_z}{U} + \frac{(c-V)\phi^2}{2U} + \epsilon\phi\phi_z + (\frac{c}{2} - \sigma V)\psi^2 - \phi\psi\Big]_z \\
&+ \frac{d\phi_z^2}{U} + \epsilon\psi_z^2 \leq \frac{dU_z\phi\phi_z}{U^2} + \frac{U_z(c-V)\phi^2}{2U^2} - \sigma\psi_z^2\psi - \frac{\phi\phi_z\psi_z}{U},
\end{aligned} \qquad (4.13)$$

From (3.6), $dU_z = U(V-c)$, we have

$$\frac{U_z(c-V)\phi^2}{2U^2} = \frac{-dU_z^2\phi^2}{2U^3}, \qquad (4.14)$$

By Young's inequality, it holds that

$$\frac{dU_z\phi\phi_z}{U^2} \le \frac{3d\phi_z^2}{4U} + \frac{dU_z^2\phi^2}{3U^3},$$

$$\begin{aligned} \frac{\phi\phi_z\psi_z}{U} &\le \frac{d}{8U}\|\phi(\cdot,t)\|_{L^\infty}\phi_z^2 + \frac{2}{dU}\|\psi(\cdot,t)\|_{L^\infty}\psi_z^2 \\ &\le \frac{dN(t)}{8U}\phi_z^2 + \frac{2N(t)}{dU}\psi_z^2, \end{aligned} \tag{4.15}$$

where we have assumed that $\|\phi(\cdot,t)\|_{L^\infty} \le N(t)$ and $\|\psi(\cdot,t)\|_{L^\infty} \le N(t)$. Substitute the above inequalities into (4.13) and integrate the resultant inequality with respect to z and t.

Note that

$$\int \left[\frac{d\phi\phi_z}{U} + \frac{(c-V)\phi^2}{2U} + \epsilon\phi\phi_z + (\frac{c}{2} - \sigma V)\psi^2 - \phi\psi\right]_z = 0,$$

We get

$$\begin{aligned} &\frac{1}{2}\int\Big(\frac{\phi^2}{U} + \psi^2\Big) + \frac{d}{4}\int_0^t\int\frac{\phi_z^2}{U} + \epsilon\int_0^t\int\psi_z^2 + \frac{d}{6}\int_0^t\int\frac{U_z^2\phi^2}{U^3} \\ &\le \frac{dN(t)}{8}\int_0^t\int\frac{\phi_z^2}{U} + \frac{2N(t)}{d}\int_0^t\int\frac{\psi_z^2}{U} + \sigma N(t)\int_0^t\int\psi_z^2 + \frac{1}{2}\int\Big(\frac{\phi_0^2}{U} + \psi_0^2\Big), \end{aligned}$$

Since U is monotone decreasing in $(-\infty,\infty)$, we have $u_- \ge U$. Then $\psi_z^2 \le \frac{u_-\psi_z^2}{U}$ and using $\sigma < 1$, assume $N(t)$ is small enough, we derive that

$$\begin{aligned} &\int\Big(\frac{\phi^2}{U} + \psi^2\Big) + \frac{d}{4}\int_0^t\int\frac{\phi_z^2}{U} + 2\epsilon\int_0^t\int\psi_z^2 + \frac{d}{3}\int_0^t\int\frac{U_z^2\phi^2}{U^3} \\ &\le CN(t)\int_0^t\int\frac{\psi_z^2}{U} + \int\Big(\frac{\phi_0^2}{U} + \psi_0^2\Big), \end{aligned} \tag{4.16}$$

Now we need to bound $1/U$ in terms of our weight function (4.11). Since $U(z)$ has the asymptotic behavior $U(z) \sim Ce^{-\frac{c}{d}z}$ as $z \to \infty$, there is a constant $M > 0$ such that $\frac{1}{U} \sim Ce^{\frac{c}{d}z}$ for any $z \geq M$. We have chosen $w(z) = e^{\frac{c}{d}z} + 1$ as our weight function, so we can find two constants $\alpha > \beta > 0$ such that

$$\beta w \leq \frac{1}{U} \leq \alpha w,$$

for any $z \geq M$.

When $z < M$, $\frac{1}{U}$ is monotone increasing in $(-\infty, \infty)$ and let $1 < w \leq 2e^{\frac{c}{d}M}$, we have

$$\frac{w}{2u_- e^{\frac{c}{d}M}} \leq \frac{1}{u_-} \leq \frac{1}{U} \leq \frac{1}{U(M)} \leq \frac{w}{U(M)},$$

for any $z < M$.

Therefore, for any $z \in \mathbb{R}$, we can find two constants $C_2 > C_1 > 0$ such that

$$C_1 w \leq \frac{1}{U} \leq C_2 w. \tag{4.17}$$

Lemma 4.1 is proved by (4.16) and (4.17). $\square$

We next estimate the first-order derivatives of(ϕ, ψ)

Lemma 4.2. *Let the assumptions in Proposition 4.3 hold, then there exists a positive constant C such that*

$$\begin{aligned} &\|\phi_z\|_w^2 + \|\psi_z\|^2 + \|\psi_z\|_w^2 + d\int_0^t \|\phi_{zz}(\cdot,\tau)\|_w^2 d\tau + \int_0^t \|\psi_z(\cdot,\tau)\|^2 d\tau \\ &+ \int_0^t \|\psi_{zz}(\cdot)\phi(\cdot,\tau)\|_w^2 d\tau \leq C\Big(\|\phi_0\|_{1,w}^2 + \|\psi_0\|_1^2 + \|\psi_{0z}\|_w^2\Big). \end{aligned} \tag{4.18}$$

Proof. Differentiate (4.5) with respect to z, we have

$$\begin{cases} \phi_{tz} = d\phi_{zzz} + (c-V)\phi_{zz} - V_z\phi_z - U\psi_{zz} - U_z\psi_z - (\phi_z\psi_z)_z, \\ \psi_{tz} = \epsilon\psi_{zzz} + c\psi_{zz} - 2\sigma(V\psi_z)_z - \phi_{zz} - \sigma(\psi_z^2)_z, \end{cases} \tag{4.19}$$

Multiplying the first equation of (4.19) by ϕ_z/U and the second equation by ψ_z, we have

$$\begin{cases} \frac{\phi_{tz}\phi_z}{U} = \frac{d\phi_{zzz}\phi_z}{U} + \frac{(c-V)\phi_{zz}\phi_z}{U} - \frac{V_z\phi_z^2}{U} - \psi_{zz}\phi_z - \frac{U_z\psi_z\phi_z}{U} - \frac{(\phi_z\psi_z)_z\phi_z}{U}, \\ \psi_{tz}\psi_z = \epsilon\psi_{zzz}\psi_z + (c-2\sigma V)\psi_{zz}\psi_z - 2\sigma V_z\psi_z^2 - \phi_{zz}\psi_z - \sigma(\psi_z^2)_z\psi_z, \end{cases}$$

Since we have

$$\frac{d\phi_{zzz}\phi_z}{U} = d\Big(\frac{\phi_{zz}\phi_z}{U}\Big)_z - \frac{d\phi_{zz}^2}{U} - \Big[\frac{d\phi_z^2}{2}\Big(\frac{1}{U}\Big)_z\Big]_z + \frac{\phi_z^2}{2}\Big(\frac{d}{U}\Big)_{zz},$$

$$\frac{((c-V)\phi_{zz}\phi_z}{U} = \Big(\frac{(c-V)\phi_z^2}{2U}\Big)_z - \frac{\phi_z^2}{2}\Big(\frac{c-V}{U}\Big)_z,$$

$$\frac{(\phi_z\psi_z)_z\phi_z}{U} = \Big(\frac{\phi_z^2\psi_z}{U}\Big)_z - \frac{\phi_z\psi_z\phi_{zz}}{U} + \frac{U_z\phi_z^2\psi_z}{U^2},$$

$$\psi_{zz}\phi_z + \phi_{zz}\psi_z = (\phi_z\psi_z)_z,$$

$$\psi_{zzz}\psi_z = (\psi_{zz}\psi_z)_z - \psi_{zz}^2,$$

$$[C\psi_{zz} - 2\sigma(V\psi_z)_z]\psi_z = \Big[\Big(\frac{c}{2} - \sigma V\Big)\psi_z^2\Big]_z - \sigma V_z\psi_z^2,$$

$$-\sigma(\psi_z^2)_z\psi_z = \frac{-2\sigma}{3}(\psi_z^3)_z,$$

integrating the equations with respect to z and adding them together, we get the following equation

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int\Big(\frac{\phi_z^2}{U} + \psi_z^2\Big) + d\int\frac{\phi_{zz}^2}{U} + \epsilon\int\psi_{zz}^2 \\ &= -\sigma\int V_z\psi_z^2 + \frac{1}{2}\int\phi_z^2\Big[\Big(\frac{d}{U}\Big)_{zz} - \Big(\frac{c-V}{U}\Big)_z\Big] - \int\frac{V_z\phi_z^2}{U}, \\ &-\int\frac{U_z\psi_z\phi_z}{U} + \int\frac{\phi_z\psi_z\phi_{zz}}{U} - \int\frac{U_z\phi_z^2\psi_z}{U^2}, \end{aligned} \tag{4.20}$$

Using the first equation of (3.6), we have

$$\begin{aligned} &\Big(\frac{d}{U}\Big)_{zz} - \Big(\frac{c-V}{U}\Big)_z \\ &= \frac{2dU_z^2}{U^3} - \frac{dU_{zz}}{U^2} + \frac{dU_{zz}}{U^2} - \frac{2dU_z^2}{U^3} \\ &= 0, \end{aligned} \tag{4.21}$$

By Cauchy-Schwarz inequality and $\|\psi_z(\cdot,t)\|_{L^\infty} \le N(t)$, we have

$$\int \left|\frac{\phi_z\psi_z\phi_{zz}}{U}\right| \le \frac{d}{4}\int \frac{\phi_{zz}^2}{U} + \frac{N(t)}{d}\int \frac{\phi_z^2}{U},$$

$$\int \left|\frac{U_z\phi_z\psi_z}{U}\right| = \left|\frac{U_z}{U}\right| \int |\psi_z\phi_z| \le C\int U\psi_z^2 + C\int \frac{\phi_z^2}{U}$$

$$\int \left|\frac{U_z\phi_z^2\psi_z}{U^2}\right| \le CN(t)\int \frac{\phi_z^2}{U},$$

Since $V_z = \frac{-cV+\sigma V+U}{\epsilon}$ where U and V are bounded, we have $|V_z|$ is bounded by a constant $C > 0$. Integrating (4.20) with respect to t, we have

$$\begin{aligned}
&\frac{1}{2}\int \left(\frac{\phi_z^2}{U} + \psi_z^2\right) + d\int_0^t \int \frac{\phi_{zz}^2}{U} + \epsilon\int_0^t \int \psi_{zz}^2 \\
&\le \frac{1}{2}\int \left(\frac{\phi_{0z}^2}{U} + \psi_{0z}^2\right) + C\int_0^t \int \psi_z^2 + C\int_0^t \int \frac{\phi_z^2}{U} + C\int_0^t \int U\psi_z^2 \\
&+ C\int_0^t \int \frac{\phi_z^2}{U} + CN(t)\int_0^t \int \frac{\phi_z^2}{U} + \frac{d}{4}\int_0^t \int \frac{\phi_{zz}^2}{U} + \frac{N(t)}{d}\int_0^t \int \frac{\phi_z^2}{U},
\end{aligned}$$

The above inequality combines with lemma 4.1 gives

$$\begin{aligned}
&\int \left(\frac{\phi_z^2}{U} + \psi_z^2\right) + d\int_0^t \int \frac{\phi_{zz}^2}{U} + \epsilon\int_0^t \int \psi_{zz}^2 \\
&\le C\left(\|\phi_0\|_{1,w}^2 + \|\psi_0\|_1^2 + \int_0^t \int U\psi_z^2 + N(t)\int_0^t \int \frac{\psi_z^2}{U}\right)
\end{aligned} \tag{4.22}$$

Now we need to estimate $\int_0^t \int U\psi_z^2$. Multiplying the first equation of (4.5) by ψ_z leads to

$$U\psi_z^2 = d\phi_{zz}\psi_z + c\phi_z\psi_z - V\phi_z\psi_z - \phi_z\psi_z^2 - \phi_t\psi_z, \tag{4.23}$$

The second equation of (4.19) gives,

$$\begin{aligned}\phi_t\psi_z &= (\phi\psi_z)_t - \phi\psi_{zt} \\ &= (\phi\psi_z)_t - \phi[\epsilon\psi_{zzz} + c\psi_{zz} - 2\sigma(V\psi_z)_z - \phi_{zz} - \sigma(\psi_z^2)_z] \\ &= (\phi\psi_z)_t - \epsilon(\phi\psi_{zz})_z + \epsilon\phi_z\psi_{zz} - c(\phi\psi_z)_z + c\phi_z\psi_z + 2\sigma(\phi V\psi_z)_z \\ &- 2\sigma\phi_z V\psi_z + (\phi\phi_z)_z - \phi_z^2 + \sigma(\phi\psi_z^2)_z - \sigma\phi_z\psi_z^2,\end{aligned} \tag{4.24}$$

and

$$\begin{aligned}d\phi_{zz}\psi_z &= d\psi_z[\epsilon\psi_{zzz} + c\psi_{zz} - 2\sigma(V\psi_z)_z - \sigma(\psi_z^2)_z - \psi_{tz}] \\ &= d\Big[-\frac{1}{2}(\psi_z^2)_t + \epsilon(\psi_z\psi_{zz})_z - \epsilon\psi_{zz}^2 + \frac{c}{2}(\psi_z^2)_z - \sigma(V\psi_z^2)_z \\ &- \sigma V_z\psi_z^2 - \frac{2\sigma}{3}(\psi_z^3)_z\Big],\end{aligned} \tag{4.25}$$

We substitute the above equations into (4.23) and then integrate the resultant equation with respect to z, we have

$$\begin{aligned}&\int U\psi_z^2 + \frac{d}{2}\int(\psi_z^2)_t + d\epsilon\int\psi_{zz}^2 + d\sigma\int V_z\psi_z^2 \\ &= -\int V\psi_z\phi_z - (1-\sigma)\int\phi_z\psi_z^2 - \int(\phi\psi_z)_t \\ &- \epsilon\int\phi_z\psi_{zz} + 2\sigma\int\phi_z V\psi_z + \int\phi_z^2\end{aligned}$$

Integrating with respect to t, noting that

$$\int_0^t\int\phi_z\psi_z^2 \leq N(t)\int_0^t\int\psi_z^2,$$

by $\|\phi_z(\cdot,t)\|_{L^\infty} \leq N(t)$ and using Cauchy-Schwarz inequality, V and $|V_z|$ are bounded, we derive

$$
\begin{aligned}
&\frac{d}{2}\int \psi_z^2 + \epsilon d\int_0^t\int \psi_{zz}^2 + \sigma d\int_0^t\int V_z\psi_z^2 + \int_0^t\int U\psi_z^2 \\
&= \frac{d}{2}\int \psi_{0z}^2 - \int_0^t\int V\phi_z\psi_z - \int_0^t\int \phi_z\psi_z^2 - \int \phi\psi_z + \int \phi_0\psi_{z0} - \epsilon\int_0^t\int \phi_z\psi_{zz} \\
&+ 2\sigma\int_0^t\int V\phi_z\psi_z + \int_0^t\int \phi_z^2 + \sigma\int_0^t\int \phi_z\psi_z^2 \\
&\le \frac{d}{2}\int \psi_{0z}^2 + \int \phi_0\psi_{0z} - \int \phi\psi_z + (2\sigma-1)C\int_0^t\int U\psi_z^2 + (2\sigma-1)C\int_0^t\int \frac{\phi_z^2}{U} \\
&+ (\sigma-1)N(t)\int_0^t\int \psi_z^2 + (\frac{\epsilon}{2d}+1)\int_0^t\int \phi_z^2 + \frac{\epsilon d}{2}\int_0^t\int \psi_{zz}^2,
\end{aligned}
\tag{4.26}
$$

Since $\phi_z^2 \le \frac{u_-}{U}\phi_z^2$, combining with lemma 4.1 yields

$$
\begin{aligned}
&\int \psi_z^2 + \epsilon\int_0^t\int \psi_{zz}^2 + \int_0^t\int U\psi_z^2 \\
&\le \int \psi_{0z}^2 + \frac{2}{d}\int \phi_0\psi_{0z} + C\int_0^t\int \frac{\phi_z^2}{U} + \frac{2(\sigma-1)N(t)}{d}\int_0^t\int \psi_z^2 \\
&\le C\left(\int \phi_0^2 + \int \psi_{z0}^2 + \int_0^t\int \frac{\phi_z^2}{U} + N(t)\int_0^t\int \psi_z^2\right) \\
&\le C\left(\|\phi_0\|_w^2 + \|\psi_0\|_1^2 + N(t)\int_0^t\int \frac{\psi_z^2}{U}\right),
\end{aligned}
\tag{4.27}
$$

Together with (4.22), we have

$$
\begin{aligned}
&\int\left(\frac{\phi_z^2}{U} + \psi_z^2\right) + d\int_0^t\int \frac{\phi_{zz}^2}{U} + \epsilon\int \psi_{zz}^2 \\
&\le C\left(\|\phi_0\|_{1,w}^2 + \|\psi_0\|_1^2 + N(t)\int_0^t\int \frac{\psi_z^2}{U}\right),
\end{aligned}
\tag{4.28}
$$

Next we shall estimate $\int_0^t\int \frac{\psi_z^2}{U}$. Multiplying the second equation of (4.19) by $e^{\frac{c}{d}z}\psi_z$, we derive

$$\psi_{tz}e^{\frac{c}{d}z}\psi_z = \epsilon\psi_{zzz}e^{\frac{c}{d}z}\psi_z + c\psi_{zz}e^{\frac{c}{d}z}\psi_z - 2\sigma(V\psi_z)_z e^{\frac{c}{d}z}\psi_z$$
$$- \phi_{zz}e^{\frac{c}{d}z}\psi_z - \sigma(\psi_z^2)_z e^{\frac{c}{d}z}\psi_z,$$

Note that

$$\epsilon\psi_{zzz}e^{\frac{c}{d}z}\psi_z = (\epsilon\psi_{zz}e^{\frac{c}{d}z}\psi_z)_z - \epsilon\psi_{zz}\frac{c}{d}e^{\frac{c}{d}z}\psi_z - \epsilon e^{\frac{c}{d}z}\psi_{zz}^2,$$

$$[c\psi_{zz} - 2\sigma(V\psi_z)_z]e^{\frac{c}{d}z}\psi_z = \left[\left(\frac{c}{2} - \sigma V\right)e^{\frac{c}{d}z}\psi_z^2\right]_z$$
$$- \sigma V_z e^{\frac{c}{d}z}\psi_z^2 - \frac{c}{2}\frac{c}{d}e^{\frac{c}{d}z}\psi_z^2 + \sigma V\frac{c}{d}e^{\frac{c}{d}z}\psi_z^2,$$

$$-\sigma(\psi_z^2)_z e^{\frac{c}{d}z}\psi_z = \left[-\frac{2\sigma}{3}e^{\frac{c}{d}z}\psi_z^3\right]_z + \frac{2\sigma}{3}\psi_z^3\frac{c}{d}e^{\frac{c}{d}z},$$

We have

$$\left(\frac{e^{\frac{c}{d}z}}{2}\psi_z^2\right)_t + \epsilon\psi_{zz}^2 e^{\frac{c}{d}z} + e^{\frac{c}{d}z}\psi_z^2\left[\sigma V_z + \frac{c^2}{2d} - \frac{\sigma V c}{d}\right]$$
$$= (\epsilon\psi_{zz}e^{\frac{c}{d}z}\psi_z)_z - \epsilon\psi_{zz}\frac{c}{d}e^{\frac{c}{d}z}\psi_z + \left[\left(\frac{c}{2} - \sigma V\right)e^{\frac{c}{d}z}\psi_z^2\right]_z - \phi_{zz}e^{\frac{c}{d}z}\psi_z \tag{4.29}$$
$$+ \left[-\frac{2\sigma}{3}e^{\frac{c}{d}z}\psi_z^3\right]_z + \frac{2\sigma}{3}\psi_z^3\frac{c}{d}e^{\frac{c}{d}z},$$

By Young's inequality, we get

$$\left|\epsilon\psi_{zz}\frac{c}{d}e^{\frac{c}{d}z}\psi_z\right| \le \frac{\epsilon}{2}e^{\frac{c}{d}z}\psi_{zz}^2 + \frac{\epsilon c^2}{2d^2}e^{\frac{c}{d}z}\psi_z^2,$$

and

$$\left|\phi_{zz}e^{\frac{c}{d}z}\psi_z\right| \le \frac{c^2}{4d}e^{\frac{c}{d}z}\psi_z^2 + \frac{d}{c^2}e^{\frac{c}{d}z}\phi_{zz}^2,$$

Integrating (4.29) over $\mathbb{R} \times [0, t]$ and using $\|\psi_z(\cdot, t)\|_{L^\infty} \le N(t)$, we have

$$\int e^{\frac{c}{d}z}\psi_z^2 + 2\epsilon\int_0^t\int e^{\frac{c}{d}z}\psi_{zz}^2 + 2\int_0^t\int\left[\sigma V_z + \frac{c^2}{2d} - \frac{\sigma V c}{d}\right]e^{\frac{c}{d}z}\psi_z^2$$

$$
\begin{aligned}
&\leq \int e^{\frac{c}{d}z}\psi_{0z}^2 + \epsilon \int_0^t \int e^{\frac{c}{d}z}\psi_{zz}^2 + \frac{\epsilon c^2}{d^2}\int_0^t \int e^{\frac{c}{d}z}\psi_z^2 + \frac{c^2}{2d}\int_0^t \int e^{\frac{c}{d}z}\psi_z^2 \\
&+ \frac{2d}{c^2}\int_0^t \int e^{\frac{c}{d}z}\phi_{zz}^2 + \frac{4\sigma c N(t)}{3d}\int_0^t \int \psi_z^2 e^{\frac{c}{d}z},
\end{aligned}
$$

Rearranging the equation and using (4.28), we derive

$$
\begin{aligned}
&\int e^{\frac{c}{d}z}\psi_z^2 + \int_0^t \int \left[\frac{c^2}{2d} + 2\sigma V_z - 2\sigma V\frac{c}{d} - \frac{\epsilon c^2}{d^2}\right] e^{\frac{c}{d}z}\psi_z^2 + \epsilon \int_0^t \int e^{\frac{c}{d}z}\psi_{zz}^2 \\
&\leq \int e^{\frac{c}{d}z}\psi_{0z}^2 + \frac{2d}{c^2}\int_0^t \int e^{\frac{c}{d}z}\phi_{zz}^2 + \frac{4\sigma c N(t)}{3d}\int_0^t \int e^{\frac{c}{d}z}\psi_z^2 \\
&\leq \int w\psi_{0z}^2 + C\int_0^t \int \frac{\phi_{zz}^2}{U} + CN(t)\int_0^t \int w\psi_z^2 \\
&\leq C\left(\|\phi_0\|_{1,w}^2 + \|\psi_0\|_1^2 + \|\psi_{0z}\|_w^2 + N(t)\int_0^t \int w\psi_z^2\right)
\end{aligned}
\tag{4.30}
$$

where we have used $e^{\frac{c}{d}z} \leq w \leq \frac{1}{C_1 U}$ for $z \in \mathbb{R}$. When $\epsilon > 0$ is small enough such that $\frac{c^2}{d}(\frac{1}{2} - \frac{\epsilon}{d}) > 0$, we require $\epsilon \leq \frac{d}{2}$, then it follows from (4.30) that

$$
\begin{aligned}
&\int_0^{+\infty} e^{\frac{c}{d}z}\psi_z^2 + \int_0^t \int_0^{+\infty} e^{\frac{c}{d}z}\psi_z^2 + \epsilon \int_0^t \int_0^{+\infty} e^{\frac{c}{d}z}\psi_{zz}^2 \\
&\leq C\left(\|\phi_0\|_{1,w}^2 + \|\psi_0\|_1^2 + \|\psi_{0z}\|_w^2 + N(t)\int_0^t \int w\psi_z^2\right).
\end{aligned}
\tag{4.31}
$$

Since the weight function is chosen to be $e^{\frac{c}{d}z} + 1$, therefore we have $e^{\frac{c}{d}z} \geq \frac{w}{2}$ in the domain of $[0, +\infty)$. Then it follows that

$$
\begin{aligned}
&\int_0^{+\infty} w\psi_z^2 + \int_0^t \int_0^{+\infty} w\psi_z^2 + \epsilon \int_0^t \int_0^{+\infty} w\psi_{zz}^2 \\
&\leq C\left(\|\phi_0\|_{1,w}^2 + \|\psi_0\|_1^2 + \|\psi_{0z}\|_w^2 + N(t)\int_0^t \int w\psi_z^2\right)
\end{aligned}
$$

For the domain of $z \in (-\infty, 0)$, we have $1 < w < 2$. Recalling that U is monotone decreasing, $U(0) < U(z)$. Therefore $\frac{U(0)w}{2} < U(0) < U(z)$. Thus, from (4.17) and

(4.27), we get

$$\int_{-\infty}^{0} w\psi_z^2 + \int_0^t \int_{-\infty}^{0} w\psi_z^2 + \epsilon \int_0^t \int_{-\infty}^{0} w\psi_{zz}^2$$

$$\leq C \left(\|\phi_0\|_w^2 + \|\psi_0\|_1^2 + N(t) \int_0^t \int w\psi_z^2 \right)$$

For $z \in (-\infty, \infty)$, we have

$$\int w\psi_z^2 + \int_0^t \int w\psi_z^2 + \epsilon \int_0^t \int w\psi_{zz}^2$$

$$\leq C \left(\|\phi_0\|_{1,w}^2 + \|\psi_0\|_1^2 + \|\psi_{0z}\|_w^2 + N(t) \int_0^t \int w\psi_z^2 \right)$$

Thus

$$\int w\psi_z^2 + (1 - CN(t)) \int_0^t \int w\psi_z^2 + \epsilon \int_0^t \int w\psi_{zz}^2$$

$$\leq C \left(\|\phi_0\|_{1,w}^2 + \|\psi_0\|_1^2 + \|\psi_{0z}\|_w^2 \right)$$

When $N(t)$ is small enough, we have

$$\int w\psi_z^2 + \int_0^t \int w\psi_z^2 + \epsilon \int_0^t \int w\psi_{zz}^2 \leq C \left(\|\phi_0\|_{1,w}^2 + \|\psi_0\|_1^2 + \|\psi_{0z}\|_w^2 \right) \tag{4.32}$$

Therefore, by (4.12), (4.28) and (4.32), we have proved lemma 4.2 $\square$

We next estimate the second-order derivatives of (ϕ, ψ).

Lemma 4.3. *Let the assumptions in Proposition 4.3 hold, then there exists a constant $C > 0$ such that*

$$\begin{aligned} &\|\phi_{zz}\|_w^2 + \|\psi_{zz}\|^2 + d\int_0^t \|\phi_{zzz}(\cdot,\tau)\|_w^2 d\tau \\ &+ \int_0^t \|\psi_{zz}(\cdot,\tau)\|^2 d\tau + \epsilon \int_0^t \|\psi_{zzz}(\cdot,\tau)\|^2 d\tau \\ \leq & C\Big(\|\phi_0\|_{2,w}^2 + \|\psi_0\|_2^2 + \|\psi_{0,z}\|_{1,w}^2 \Big). \end{aligned} \tag{4.33}$$

Proof. Differentiating (4.19) with respect to z gives

$$\begin{cases} \phi_{tzz} = d\phi_{zzzz} + (c - V)\phi_{zzz} - V_z\phi_{zz} - (V_z\phi_z)_z - U_z\psi_{zz} - U\psi_{zzz} \\ \qquad\quad -(\phi_z\psi_z)_{zz} - (U_z\psi_z)_z, \\ \psi_{tzz} = \epsilon\psi_{zzzz} + c\psi_{zzz} - 2\sigma(V\psi_z)_{zz} - \phi_{zzz} - \sigma(\psi_z^2)_{zz}, \end{cases} \tag{4.34}$$

Multiplying the first equation of (4.34) by ϕ_{zz}/U and the second by ψ_{zz} and using

$$\begin{aligned} \frac{d\phi_{zzzz}\phi_{zz}}{U} &= d\Big(\frac{\phi_{zzz}\phi_{zz}}{U}\Big)_z - \frac{d\phi_{zzz}^2}{U} - \frac{d}{2}\Big(\phi_{zz}^2\Big(\frac{1}{U}\Big)_z\Big)_z + \frac{d}{2}\phi_{zz}^2\Big(\frac{1}{U}\Big)_{zz}, \\ \frac{(c-V)\phi_{zzz}\phi_{zz}}{U} &= \Big[\frac{(c-V)\phi_{zz}^2}{2U}\Big]_z - \frac{\phi_{zz}^2}{2}\Big(\frac{(c-V)}{U}\Big)_z, \\ -\psi_{zzz}\phi_{zz} - \phi_{zzz}\psi_{zz} &= -(\psi_{zz}\phi_{zz})_z, \\ \psi_{zzzz}\psi_{zz} &= (\psi_{zzz}\psi_{zz})_z - \psi_{zzz}^2, \\ [c\psi_{zzz} - 2\sigma(V\psi_z)_{zz}]\psi_{zz} &= \Big[\frac{c}{2}(\psi_{zz}^2) - 2\sigma(V\psi_z)_z\psi_{zz}\Big]_z + 2\sigma V_z\psi_z\psi_{zzz} \\ &\quad + \sigma(V\psi_{zz}^2)_z - \sigma V_z\psi_{zz}^2, \\ -\sigma(\psi_z^2)_{zz}\psi_{zz} &= -\sigma[(\psi_z^2]_z\psi_{zz}]_z + 2\sigma\psi_z\psi_{zz}\psi_{zzz}, \end{aligned}$$

we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int\Big(\frac{\phi_{zz}^2}{U} + \psi_{zz}^2\Big) + d\int\frac{\phi_{zzz}^2}{U} + \epsilon\int\psi_{zzz}^2 + \sigma\int V_z\psi_{zz}^2 \\ &= -\int\frac{V_z\phi_{zz}^2}{U} - \int\frac{(V_z\phi_z)_z\phi_{zz}}{U} - \int\frac{(U_z\psi_z)_z\phi_{zz}}{U} - \int\frac{U_z\psi_{zz}\phi_{zz}}{U} \\ &\quad - \int\frac{(\phi_z\psi_z)_{zz}\phi_{zz}}{U} + 2\sigma\int V_z\psi_z\psi_{zzz} + 2\sigma\int\psi_z\psi_{zz}\psi_{zzz}, \end{aligned} \tag{4.35}$$

Note that

$$\begin{aligned} \frac{(V_z\phi_z)_z\phi_{zz}}{U} &= \Big(\frac{V_z\phi_z\phi_{zz}}{U}\Big)_z - \frac{V_z\phi_z\phi_{zzz}}{U} + \frac{U_zV_z\phi_z\phi_{zz}}{U^2}, \\ \frac{(U_z\psi_z)_z\phi_{zz}}{U} &= \Big(\frac{U_z\psi_z\phi_{zz}}{U}\Big)_z - \frac{U_z\psi_z\phi_{zzz}}{U} + \frac{U_z^2\psi_z\phi_{zz}}{U^2}, \end{aligned}$$

$$\frac{(\phi_z\psi_z)_{zz}\phi_{zz}}{U} = \left(\frac{(\phi_z\psi_z)_z\phi_{zz}}{U}\right)_z - \frac{(\phi_z\psi_z)_z\phi_{zzz}}{U} + \frac{(\phi_z\psi_z)_z\phi_{zz}U_z}{U^2},$$

$$\psi_z\psi_{zzz} + \psi_{zz}\psi_{zzz} = (\psi_z\psi_{zz})_z - \psi_{zz}^2 + (\frac{1}{2}\psi_{zz}^2)_z,$$

Since $|V_z \leq C|$ and $\left|\frac{U_z}{U} \leq C\right|$, we get by Cauchy-Schwarz inequality

$$\int\left|\frac{(V_z\phi_z)_z\phi_{zz}}{U}\right| = \left|-\int\frac{V_z\phi_z\phi_{zzz}}{U} + \int\frac{V_zU_z\phi_z\phi_{zz}}{U^2}\right|$$
$$\leq \frac{d}{4}\int\frac{\phi_{zzz}^2}{U} + \frac{C}{d}\int\frac{\phi_z^2}{U} + d\int\frac{\phi_{zz}^2}{U} + \frac{C}{d}\int\frac{\phi_z^2}{U},$$

$$\left|\int\frac{U_z\psi_{zz}\phi_{zz}}{U}\right| \leq \frac{d}{4}\int\frac{\phi_{zz}^2}{U} + \frac{C}{d}\int\psi_{zz}^2U$$

$$\int\left|\frac{(U_z\psi_z)_z\phi_{zz}}{U}\right| = \left|\int\frac{U_z\psi_z\phi_{zzz}}{U} + \int\frac{U_z^2\psi_z\phi_{zz}}{U^2}\right|$$
$$\leq \frac{d}{4}\int\frac{\phi_{zzz}^2}{U} + \frac{C}{d}\int U\psi_z^2 + d\int\frac{\phi_{zz}^2}{U} + \frac{C}{d}\int\psi_z^2U,$$

$$\int\left|\frac{(\phi_z\psi_z)_{zz}\phi_{zz}}{U}\right| = -\int\frac{(\phi_z\psi_z)_z\phi_{zzz}}{U} + \int\frac{(\phi_z\psi_z)_z\phi_{zz}U_z}{U^2}$$
$$\leq \frac{dN(t)}{4}\int\frac{\phi_{zzz}^2}{U} + \frac{2CN(t)}{d}\int\frac{\phi_{zz}^2}{U} + \frac{2CN(t)}{d}\int\frac{\psi_{zz}^2}{U},$$

Integrate (4.35) over $(0,t)$, we have

$$\frac{1}{2}\int\Big(\frac{\phi_{zz}^2}{U} + \psi_{zz}^2\Big) + \frac{d}{4}\int_0^t\int\frac{\phi_{zzz}^2}{U} + \epsilon\int\psi_{zzz}^2$$
$$\leq \frac{1}{2}\int\left(\frac{\phi_{0zz}^2}{U} + \psi_{0zz}^2\right) + \sigma C\int_0^t\int\psi_{zz}^2 + C\int_0^t\int\frac{\phi_{zz}^2}{U} + \frac{2C}{d}\int_0^t\int\psi_z^2U \tag{4.36}$$
$$+ \frac{2C}{d}\int_0^t\int\frac{\phi_z^2}{U} + \frac{C}{d}\int_0^t\int\psi_{zz}^2U + \frac{2CN(t)}{d}\int_0^t\int\frac{\psi_{zz}^2}{U},$$

Using lemma 4.2, we derive

$$
\begin{aligned}
&\int \left(\frac{\phi_{zz}^2}{U} + \psi_{zz}^2\right) + d\int_0^t \int \frac{\phi_{zzz}^2}{U} + \epsilon \int \psi_{zzz}^2 \\
&\leq C\left(\|\phi_0\|_{2,w}^2 + \|\psi_0\|_2^2 + \|\psi_{0z}\|_w^2\right) + N(t)\int_0^t \int \frac{\psi_{zz}^2}{U} + \int_0^t \int U\psi_{zz}^2,
\end{aligned}
\tag{4.37}
$$

We next estimate $\int_0^t \int U\psi_{zz}^2$. Multiplying the first equation of (4.19) by ψ_{zz}, we get

$$
U\psi_{zz}^2 = [d\phi_{zzz} + (c-V)\phi_{zz} - V_z\phi_z - U_z\psi_z - (\phi_z\psi_z)_z]\psi_{zz} - \phi_{tz}\psi_{zz}, \tag{4.38}
$$

Noting

$$
\begin{aligned}
d\phi_{zzz}\psi_{zz} &= d[-\psi_{tzz} + \epsilon\psi_{zzzz} + c\psi_{zzz} - 2\sigma(V\psi_z)_{zz} - \sigma(\psi_z^2)_{zz}]\psi_{zz} \\
&= d\Big[-\frac{1}{2}(\psi_{zz}^2)_t + \epsilon(\psi_{zz}\psi_{zzz})_z - \epsilon\psi_{zzz}^2 + \frac{c}{2}(\psi_{zz}^2)_z - 2\sigma[\psi_{zz}(V\psi_z)_z]_z \\
&\quad + 2\sigma\psi_{zzz}(V\psi_z)_z - \sigma[\psi_{zz}(\psi_z^2)_z]_z + \sigma\psi_{zzz}(\psi_z^2)_z\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
\phi_{tz}\psi_{zz} &= (\phi_z\psi_{zz})_t - \phi_z\psi_{zzt} \\
&= (\phi_z\psi_{zz})_t - [\epsilon\psi_{zzzz} + c\psi_{zzz} - 2\sigma(V\psi_z)_{zz} - \sigma(\psi_z^2)_{zz} - \phi_{zzz}]\phi_z \\
&= (\phi_z\psi_{zz})_t - \epsilon(\phi_z\psi_{zzz})_z + \epsilon\phi_{zz}\psi_{zzz} - c(\phi_z\psi_{zz})_z + c\psi_{zz}\phi_{zz} + 2\sigma(\phi_z(V\psi_z)_z)_z \\
&\quad - 2\sigma\phi_{zz}(V\psi_z)_z + \sigma(\phi_z(\psi_z^2)_z)_z - \sigma\phi_{zz}(\psi_z^2)_z + (\phi_z\phi_{zz})_z - \phi_{zz}^2,
\end{aligned}
$$

We substitute the above equations into (4.38) and integrate the resultant equation with respect to z, we have

$$
\begin{aligned}
&\int U\psi_{zz}^2 + \frac{d}{2}\int (\psi_{zz}^2)_t + d\epsilon \int \psi_{zzz}^2 \\
&= 2\sigma d \int \psi_{zzz}(V\psi_z)_z + d\sigma \int \psi_{zzz}(\psi_z^2)_z - \int V\psi_{zz}\phi_{zz} - \int V_z\psi_{zz}\phi_z - \int U_z\psi_z\psi_{zz} \\
&- \int \phi_z\psi_z)_z\psi_{zz} - \int (\phi_z\psi_{zz})_t - \epsilon \int \phi_{zz}\psi_{zzz} \\
&+ 2\sigma \int \phi_{zz}(V\psi_z)_z + \sigma \int \phi_{zz}(\psi_z^2)_z + \int \phi_{zz}^2
\end{aligned}
\tag{4.39}
$$

Rearranging the equation, we have

$$
\begin{aligned}
&\frac{d}{2}\frac{d}{dt}\int \psi_{zz}^2 + \epsilon d \int \psi_{zzz}^2 + \int U\psi_{zz}^2 + \frac{d}{dt}\int \phi_z\psi_{zz} \\
&= -\epsilon \int \phi_{zz}\psi_{zzz} + \int \phi_{zz}^2 + 2\sigma \int \phi_{zz}(V\psi_z)_z + \sigma \int \phi_{zz}(\psi_z^2)_z + 2\sigma d \int \psi_{zzz}(V\psi_z)_z \\
&+ \sigma d \int \psi_{zzz}(\psi_z^2)_z - \int [V\phi_{zz} + V_z\phi_z + U_z\psi_z + (\phi_z\psi_z)_z]\psi_{zz}
\end{aligned}
\tag{4.40}
$$

By applying Cauchy-Schwarz inequality

$$
\begin{aligned}
\int [V\phi_{zz} + V_z\phi_z + U_z\psi_z]\psi_{zz} &\le \frac{1}{4}\int U\psi_{zz}^2 \\
&+ C\left(\int \frac{V^2\phi_{zz}^2}{U} + \int \frac{V_z^2\phi_z^2}{U} + \int \frac{U_z^2\psi_z^2}{U}\right), \\
\int (\phi_z\psi_z)_z\psi_{zz} &= \int \phi_z\psi_{zz}^2 + \int \psi_z\phi_{zz}\psi_{zz} \\
&\le N(t)\int \psi_{zz}^2 + N(t)\int \frac{\phi_{zz}^2}{U} + \frac{N(t)}{4}\int U\psi_{zz}^2.
\end{aligned}
$$

and integrating (4.40) over $[0, t]$, we have

$$
\begin{aligned}
&\int \psi_{zz}^2 + \epsilon \int_0^t \int \psi_{zzz}^2 + \int_0^t \int U\psi_{zz}^2 \\
&\leq C\left(\|\phi_0\|_{2,w}^2 + \|\psi_0\|_2^2 + \|\psi_{0z}\|_w^2 + N(t) \int_0^t \int \frac{\psi_{zz}^2}{U} \right)
\end{aligned} \tag{4.41}
$$

Substitute (4.41) into (4.37) gives

$$
\begin{aligned}
&\int \left(\frac{\phi_{zz}^2}{U} + \psi_{zz}^2 \right) + d\int_0^t \int \frac{\phi_{zzz}^2}{U} + \epsilon \int \psi_{zzz}^2 \\
&\leq C\left(\|\phi_0\|_{2,w}^2 + \|\psi_0\|_2^2 + \|\psi_{0z}\|_w^2 + N(t) \int_0^t \int \frac{\psi_{zz}^2}{U} \right)
\end{aligned} \tag{4.42}
$$

Next we apply similar procedure to estimate the last term. Multiplying the second equation of (4.19) by $e^{\frac{c}{d}z}\psi_{zz}$, we have

$$
\begin{aligned}
&e^{\frac{c}{d}z}\psi_{zz}\psi_{tzz} = e^{\frac{c}{d}z}\psi_{zz}\epsilon\psi_{zzzz} + e^{\frac{c}{d}z}\psi_{zz}c\psi_{zzz} - e^{\frac{c}{d}z}\psi_{zz}2\sigma(V\psi_z)_{zz} \\
&- e^{\frac{c}{d}z}\psi_{zz}\phi_{zzz} - e^{\frac{c}{d}z}\psi_{zz}\sigma(\psi_z^2)_{zz},
\end{aligned}
$$

We derive

$$
\begin{aligned}
&\left(\frac{e^{\frac{c}{d}z}\psi_{zz}^2}{2} \right)_t + \frac{c^2}{2d} e^{\frac{c}{d}z}\psi_{zz}^2 + \epsilon e^{\frac{c}{d}z}\psi_{zzz}^2 \\
&= [\epsilon\psi_{zzz}e^{\frac{c}{d}z}\psi_{zz} + \frac{c}{2}e^{\frac{c}{d}z}\psi_{zz}^2 - 2\sigma(V\psi_z)_z e^{\frac{c}{d}z}\psi_{zz} - \sigma(\psi_z^2)_z e^{\frac{c}{d}z}\psi_{zz}]_z \\
&- \frac{c\epsilon}{d}e^{\frac{c}{d}z}\psi_{zz}\psi_{zzz} + 2\sigma(V\psi_z)_z e^{\frac{c}{d}z}\psi_{zzz} + 2\sigma(V\psi_z)_z \frac{c}{d}e^{\frac{c}{d}z}\psi_{zz} - e^{\frac{c}{d}z}\psi_{zz}\phi_{zzz} \\
&+ \frac{\sigma c}{d}(\psi_z^2)_z e^{\frac{c}{d}z}\psi_{zz} + \sigma(\psi_z^2)_z e^{\frac{c}{d}z}\psi_{zzz}
\end{aligned} \tag{4.43}
$$

By Young's inequality

$$
\begin{aligned}
&|-\frac{c\epsilon}{d}e^{\frac{c}{d}z}\psi_{zz}\psi_{zzz} + 2\sigma(V\psi_z)_z e^{\frac{c}{d}z}\psi_{zzz}| \leq \frac{\epsilon}{4}e^{\frac{c}{d}z}\psi_{zzz}^2 + C\epsilon e^{\frac{c}{d}z}(\psi_{zz}^2 + \psi_z^2), \\
&|e^{\frac{c}{d}z}\psi_{zz}\phi_{zzz}| \leq e^{\frac{c}{d}z}\phi_{zzz}^2 + Ce^{\frac{c}{d}z}\psi_{zz}^2,
\end{aligned}
$$

and integrating (4.43) over $\mathbb{R} \times [0,t]$,

$$
\begin{aligned}
&\int e^{\frac{c}{d}z}\psi_{zz}^2 + \frac{c^2}{d}\int_0^t\int e^{\frac{c}{d}z}\psi_{zz}^2 + \frac{3\epsilon}{2}\int_0^t\int e^{\frac{c}{d}z}\psi_{zzz}^2 \\
&\leq \int e^{\frac{c}{d}z}\psi_{0zz}^2 + C\int_0^t\int e^{\frac{c}{d}z}\psi_{zz}^2 + C\int_0^t\int e^{\frac{c}{d}z}\psi_z^2 \\
&+ 2\int_0^t\int e^{\frac{c}{d}z}\psi_{zzz}^2 + \frac{\sigma N(t)}{2}\int_0^t\int e^{\frac{c}{d}z}\psi_{zzz}^2 \\
&\leq C\left(\|\phi_0\|_{2,w}^2 + \|\psi_0\|_2^2 + \|\psi_{0z}\|_{1,w}^2 + N(t)\int_0^t\int w\psi_{zz}^2\right)
\end{aligned}
\tag{4.44}
$$

where we have used that V_z and V are bounded and $\|\psi_z(\cdot,t)\|_{L^\infty} \leq N(t)$ to get the first inequality, and used Lemma 2 with the inequality (4.42) to get the second inequality.

It follows that

$$
\begin{aligned}
&\int_0^{+\infty} e^{\frac{c}{d}z}\psi_{zz}^2 + \int_0^t\int_0^{+\infty} e^{\frac{c}{d}z}\psi_{zz}^2 + \epsilon\int_0^t\int_0^{+\infty} e^{\frac{c}{d}z}\psi_{zzz}^2 \\
&\leq C\left(\|\phi_0\|_{2,w}^2 + \|\psi_0\|_2^2 + \|\psi_{0z}\|_{1,w}^2 + N(t)\int_0^t\int w\psi_{zz}^2\right).
\end{aligned}
\tag{4.45}
$$

Since we have $e^{\frac{c}{d}z} \geq \frac{w}{2}$ in the domain of $[0,+\infty)$, then it follows that

$$
\begin{aligned}
&\int_0^{+\infty} w\psi_{zz}^2 + \int_0^t\int_0^{+\infty} w\psi_{zz}^2 + \epsilon\int_0^t\int_0^{+\infty} w\psi_{zzz}^2 \\
&\leq C\left(\|\phi_0\|_{2,w}^2 + \|\psi_0\|_2^2 + \|\psi_{0z}\|_{1,w}^2 + N(t)\int_0^t\int w\psi_{zz}^2\right)
\end{aligned}
$$

For the domain of $z \in (-\infty, 0)$, we apply the same argument, then we get

$$
\begin{aligned}
&\int_{-\infty}^0 w\psi_{zz}^2 + \int_0^t\int_{-\infty}^0 w\psi_{zz}^2 + \epsilon\int_0^t\int_{-\infty}^0 w\psi_{zzz}^2 \\
&\leq C\left(\|\phi_0\|_{2,w}^2 + \|\psi_0\|_2^2 + \|\psi_{0z}\|_w^2 + N(t)\int_0^t\int w\psi_{zz}^2\right)
\end{aligned}
\tag{4.46}
$$

Together we have

$$\begin{aligned}
&\int w\psi_{zz}^2 + \int_0^t \int w\psi_{zz}^2 + \epsilon \int_0^t \int w\psi_{zzz}^2 \\
&\le C\left(\|\phi_0\|_{2,w}^2 + \|\psi_0\|_2^2 + \|\psi_{0z}\|_{1,w}^2 + N(t)\int_0^t \int w\psi_{zz}^2 \right)
\end{aligned} \tag{4.47}$$

When $N(t)$ is small enough, we have

$$\int w\psi_{zz}^2 + \int_0^t \int w\psi_{zz}^2 + \epsilon \int_0^t \int w\psi_{zzz}^2 \le C\left(\|\phi_0\|_{2,w}^2 + \|\psi_0\|_2^2 + \|\psi_{0z}\|_{1,w}^2 \right) \tag{4.48}$$

Therefore, lemma 4.3 is proved by using (4.12),(4.18),(4.37) and (4.48). □

4.3 Case of $(u_-, v_-) = (0, \frac{c}{\sigma})$ and $(u_+, v_+) = (c^2(1-\sigma), c)$

Apply the same technique, we can get (4.16). Note that U is monotone increasing in $(-\infty, \infty)$, we have $u_+ \ge U$, so $\psi_z^2 \le \frac{u_+\psi_z^2}{U}$. Now we bound $1/U$ in terms of weight function. For $(u_-, v_-) = (0, \frac{c}{\sigma})$ and $(u_+, v_+) = (c^2(1-\sigma), c)$, we have the following asymptotic behavior for component U

$$\begin{aligned}
U(z) &\sim Ce^{\frac{c}{d}z}, \text{ as } z \to -\infty, \\
U(z) - u_+ &\sim Ce^{-\lambda_2 z}, \text{ as } z \to \infty,
\end{aligned} \tag{4.49}$$

where C is a positive constant. Since singularity is arised from $u_- = 0$, we shall choose weight function $w(z)$ as

$$w(z) = e^{-\frac{c}{d}z} + 1, \ z \in \mathbb{R}, \tag{4.50}$$

Since $\frac{1}{U} \sim Ce^{-\frac{c}{d}z}$ for $z < M$ where $M > 0$, we can find two constants $\beta > \alpha > 0$ such that

$$\alpha w \le \frac{1}{U} \le \beta w, \text{ for any } z < M,$$

When $z > M$, $\frac{1}{U}$ is monotone decreasing in $(-\infty, \infty)$ and $1 < w(z) \leq 2e^{-\frac{c}{d}M}$, we have

$$\frac{w}{2u_+ e^{-\frac{c}{d}M}} \leq \frac{1}{u_+} \leq \frac{1}{U} \leq \frac{1}{U(M)} \leq \frac{w}{U(M)},$$

for any $z > M$.

Therefore, we get (4.17) for $z \in \mathbb{R}$. The desired lemma 4.1 follows from (4.16) and (4.17). Noting that $\phi_z \leq \frac{u_+}{U}\phi_z^2$. When estimating $\int_0^t \int \frac{\psi_z^2}{U}$, we multiply the second equation of (4.19) by $e^{-\frac{c}{d}z}\psi_z$. We apply the similar procedure to get lemma 4.2. When estimating $\int_0^t \int \frac{\psi_{zz}^2}{U}$, we multiply the second equation of (4.19) by $e^{-\frac{c}{d}z}\psi_{zz}$. Lemma 4.3 is derived by using the similar procedure.

4.4 Case of $(u_-, v_-) = (0, \frac{c}{\sigma})$ and $(u_+, v_+) = (0, 0)$

When $(u_-, v_-) = (0, \frac{c}{\sigma})$ and $(u_+, v_+) = (0, 0)$, the traveling wave solution component U has the following asymptotic behavior

$$\begin{aligned} &U(z) \sim Ce^{\frac{c}{d}z}, \text{ as } z \to -\infty, \\ &U(z) \sim Ce^{-\frac{c}{d}z}, \text{ as } z \to \infty, \end{aligned} \tag{4.51}$$

where C is a positive constant. Since singularities are arised from $u_+ = 0$ and $u_- = 0$, we choose the following weight function

$$w(z) = \begin{cases} e^{-\frac{c}{d}z} + 1, & z \in (-\infty, 0), \\ e^{\frac{c}{d}z} + 1, & z \in [0, +\infty), \end{cases} \tag{4.52}$$

Applying the same technique and using the fact that U is monotone increasing in $(-\infty, 0)$ $(u_+ \geq U)$ and is monotone decreasing in $[0, \infty)$ $(u_- \geq U)$ to get (4.16) where we have used $\psi_z^2 \leq \frac{u_\pm \psi_z^2}{U}$. Next we bound $1/U$ in terms of weight function. Since $\frac{1}{U} \sim Ce^{-\frac{c}{d}z}$ for $z < 0$, we can find two constants $\beta > \alpha > 0$ such that

$$\alpha w \leq \frac{1}{U} \leq \beta w, \text{ for any } z < 0,$$

Similarly, we can find two constants $\mu > \nu > 0$ such that

$$\nu w \leq \frac{1}{U} \leq \mu w, \text{ for any } z \geq 0,$$

where $\frac{1}{U} \sim Ce^{\frac{c}{d}z}$ for $z \geq 0$. Therefore, we get (4.17) for $z \in \mathbb{R}$. The desired lemma 4.1 follows from (4.16) and (4.17).Noting that $\phi_z \leq \frac{u_\pm}{U}\phi_z^2$. When estimating $\int_0^t \int \frac{\psi_z^2}{U}$, we multiply the second equation of (4.19) by $e^{-\frac{c}{d}z}\psi_z$ for the domain of $(-\infty, 0)$ and multiply the second equation of (4.19) by $e^{\frac{c}{d}z}\psi_z$ for the domain of $[0, \infty)$. Applying the similar procedure as we done for the case of $(u_-, v_-) = (c^2(1-\sigma), c)$ and $(u_+, v_+) = (0, 0)$ to get lemma 4.2. When estimating $\int_0^t \int \frac{\psi_{zz}^2}{U}$, we multiply the second equation of (4.19) by $e^{-\frac{c}{d}z}\psi_{zz}$ for the domain of $(-\infty, 0)$ and multiply the second equation of (4.19) by $e^{\frac{c}{d}z}\psi_{zz}$ for the domain of $[0, \infty)$. Lemma 4.3 is derived by using the similar procedure.

4.5 Proof of the Main Result

Proof of Proposition 4.1. Since (4.8) has been implied by proposition 4.3, we only need to prove (4.9). From global estimate (4.8) we have

$$\|\phi_z(\cdot, t), \psi_z(\cdot, t)\|_{1,w} \to 0, as\ t \to +\infty$$

Hence, for all $z \in \mathbb{R}$, it follows that

$$\phi_z^2(z, t) = 2\int_{-\infty}^{z} \phi_z \phi_{zz}(y, t)dy$$

$$\leq 2\left(\int_{-\infty}^{\infty} \phi_z^2 dy\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \phi_{zz}^2 dy\right)^{\frac{1}{2}}$$

$$\leq \|\phi_z(\cdot, t)\|_{1,w} \to 0,\ as\ t \to +\infty$$

Applying the same method to ψ_z leads to

$$\psi_z(z, t) \to 0,\ as\ t \to +\infty,$$

for all $z \in \mathbb{R}$.

Hence, (4.9) is proved. $\square$

Theorem 2.2 is a direct consequence of Proposition 4.1 where (4.8) guarantees that $N(t)$ is small for all $t > 0$ when $N(0)$ is small.

Chapter 5

Summary and Prospects for the Future

We show that traveling wave solutions of the system (3.6) do not exist when $\sigma \geq 1$. There are three heteroclinic orbits connecting the critical points $(U, V) = (0, 0), (c^2(1-\sigma), c)$ and $(0, \frac{c}{\sigma})$ in the (U, V) phase plane of (3.6) deriving from the chemotaxis model when $0 \leq \sigma < 1$ under the assumption of $\rho_1 = \rho_2 = 0$. This is proved by using the phase plane analysis. Furthermore, we have established the stability result of the traveling waves of (3.6) by applying the weighted energy method.

However, there are many limitations in this study. Firstly, we only consider the case when $\rho_1 = \rho_2 = 0$, the existence of traveling waves is worth studying when ρ_1 and ρ_2 not equal to zero after finishing the previous work. In addition, we are now exploring traveling waves to the chemotaxis model in one dimensional space. The study of two-dimensional traveling wave solutions (planar waves) is a challenging topic and has many open problems and is worthwhile to be solved in the future.

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