Global dynamics of some predator-prey systems with preytaxis

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CERTIFICATE OF ORIGINALITY

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______________________________(Signed)

LUO Yong (Name of student)

Abstract

Organisms cannot live without food resource as their energy supply, in all probability. The different strategies that they use to forage or to increase their survival rates may result in diverse interactions between or among organisms, amongst which predation as one of fundamental relations exists broadly in nature. This thesis is associated with exploring dynamics of classical solution to two classes of predator-prey models with spatial diffusion and preytaxis effect: direct preytaxis and indirect preytaxis. The preytaxis here refers to that predators have an apparent tendency to move towards the region of higher density of prey. The main difference of being direct or indirect case lies in that predators search for prey directly, or perceive mainly the signals released by prey through which predators may likely find the prey eventually.

In more detail, our results include three parts as below: Firstly, for the direct preytaxis model with no diffusion of prey (i.e., a parabolic-ODE system), we study local-in-time existence and uniqueness of its classical solution by using Banach's fixed-point theory in a suitable Sobolev space as the spatial domain $\Omega \subset \mathbb{R}^n (n \geq 1)$. Also, we derive its global existence by obtaining uniform-in-time boundedness of its solution in norm $L_\infty(\Omega)$, when spatial dimension $n = 2$.

On the other hand, inspired by vanishing viscosity method we explore convergence relationship between the strong solution of a related fully parabolic PDE system and the aforementioned parabolic-ODE system in $\Omega \subset \mathbb{R}^2$, when the diffusion coefficient ε (> 0) of prey density tends to zero. Here the main tools used include analytic semi-group techniques, Aubin-Lions compactness lemma, trace interpolation inequalities, L_p theory and Schauder's estimate of linear parabolic equations, etc.

Finally, for the indirect preytaxis model with density-dependent preytaxis we investigate global-in-time existence, uniqueness and uniform-in-time boundedness of its classical solution in $\Omega \subset \mathbb{R}^n (n \geq 1)$, by a combination of Amann's theory for quasilinear parabolic systems, analytical semigroup techniques and Moser's iteration. In addition, via Lyapunov's function techniques and limit property of dynamical systems we acquire that the classical solution may converge in norm $L_\infty(\Omega)$, as time $t \to +\infty$, to its prey-only state and coexistence state under suitable conditions. The numerical simulations we perform indicate that some density-dependent preytaxis and predators' diffusion may either flatten the spatial one-dimensional patterns which exist in non-density-dependent case, or break the spatial two-dimensional distribution similarity which occurs in non-density-dependent case between predators and chemoattractants (released by prey).

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Contents

List of Notations

Basic Notations

Here we invoke some notations introduced in [1, Appendix A].

Ω (*resp.* $\overline{\Omega}$)	A bounded open (*resp.* closed) domain in n-dimensional Euclidian space $\mathbb{R}^n$ for integer $0 < n \in \mathbb{N}$.
Q_τ (*resp.* $\overline{Q}_\tau$)	A bounded open domain $\Omega \times (0, \tau)$ (*resp.* a bounded closed domain $\overline{\Omega} \times [0, \tau]$) for $0 < \tau < +\infty$. In particular, we let $Q := \Omega \times (0, +\infty)$.
$\vec{\nu}$	The outer normal unit vector to the boundary $\partial\Omega$ of Ω.
∇	The usual gradient operator, i.e., $\nabla := (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n})$ as $x = (x_1, x_2, \cdots, x_n) \in \Omega \subset \mathbb{R}^n$.
$\nabla\cdot$	The usual divergence operator, i.e., $\nabla\cdot := \sum_{i=1}^n \frac{\partial}{\partial x_i}$, as $x = (x_1, x_2, \cdots, x_n) \in \Omega \subset \mathbb{R}^n$.
Δ	The usual Laplace operator, i.e., $\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, as $x = (x_1, x_2, \cdots, x_n) \in \Omega \subset \mathbb{R}^n$.
$D^k f$	The k-times derivative of function $f(x)$ for $k \in \mathbb{N}$. In more detail, we let $D_x^\alpha f = \frac{\partial^{\lvert\alpha\rvert} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$ and $\lvert\alpha\rvert = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ for a multiindex $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ with $\alpha_i \in \mathbb{N}$. Then $D^k f(x) = \{D^\alpha f(x) : \lvert\alpha\rvert = k\}$, and $\lvert D^k f(x)\rvert := \big(\sum_{\lvert\alpha\rvert=k} \lvert D_x^\alpha f\rvert^2\big)^{1/2}$. In particular, $D^2 f$ deonotes the Hessian matrix of f, and we use $Df = \nabla f$ without any confusion.
$\mathbf{a}^\tau$	The transpose of a vector $\mathbf{a}$.

Function Spaces

Below we list several function spaces, with their definitions given in [1, Appendix A], [2, sec.2] or [3, chap.I].

$L_p(\Omega)$ and $L_p(Q_\tau)$	The usual Lebesgue spaces for $1 \le p \le +\infty$ are defined on Ω and Q_τ, respectively.

$W_p^m(\Omega)$	A standard Sobolev space defined on Ω for $1 \le p \le +\infty$ and $m \in \mathbb{N}$, contains the functions whose k-times weak derivatives belong to $L_p(\Omega)$ for all $0 \le k \le m, k \in \mathbb{N}$.
$C^{m+\alpha}(\bar{\Omega})$	A standard m-times Hölder space is defined on $\bar{\Omega}$ for $x \in \bar{\Omega}$, $\alpha \in (0,1)$ and $m \in \mathbb{N}$, and composed of the functions whose k-times classical derivatives are Hölder continuous with index α, for all $0 \le k \le m, k \in \mathbb{N}$.
$C^{m+1-}(X)$	A standard m-times Lipschitz space consists of the functions which have k-times ($0 \le k \le m$, $k, m \in \mathbb{N}$) derivatives being Lipschitz continuous in a metric space X.
$W_p^{2m,m}(Q_\tau)$	A standard t-anisotropic Sobolev space defined on Q_τ for $(x,t) \in Q_\tau = \Omega \times (0,\tau)$, $1 \le p < +\infty$ and $m \in \mathbb{N}$, denotes a set of functions $v(x,t)$ in $L_p(Q_\tau)$ such that the weak derivatives $D_x^r D_t^l v \in L_p(Q_\tau)$ for $r + 2l \le 2m$, l, $r \in \mathbb{N}$.
$C^{2m+\alpha,m+\frac{\alpha}{2}}(\bar{Q}_\tau)$	An usual t-anisotropic Hölder space defined on $\bar{Q}_\tau$ for $(x,t) \in Q_\tau = \Omega \times (0,\tau)$, $\alpha \in (0,1)$ and $m \in \mathbb{N}$, includes the functions $v(x,t)$ whose classical derivatives $D_x^r D_t^l v$ (for $r + 2l \le 2m$, l, $r \in \mathbb{N}$) are Hölder continuous in spatial variable x for index α and in time variable t for index $\frac{\alpha}{2}$, respectively.

Chapter 1

Introduction

Qualitative and quantitative studies and predictions on evolutionary state of species are undoubtedly important to safeguard the balance and biodiversity of ecosystems. To capture this state, it is significant to describe the interactions between and among organisms which may affect the species' survival and reproduction positively, neutrally, or negatively. As classified by ecologists, there are five major types of the interactions inlcuding predation, competition, mutualism, commensalism and amensalism.

As one of the principal themes in ecology, predator-prey relationship exists extensively [4], ranging from macroorganism like lions and gazelles, lynx and snowshoe hare, birds and insects, etc., to microorganism like bacterial predator-prey coevolution [5, 6]. Pioneering works that model the dynamical evolution of predator-prey relation have been made by A. J. Lotka [7], V. Volterra (cf. [8]), A. N. Kolmogorov [9, Chp-II] and [10], G. F. Gauze [11, 12], M. C. Rosenzweig and R. H. MacArthur [13], et al, where they proposed or improved the classical predator-prey models by giving a system of *Ordinary Differential Equations* (ODEs). These models are established usually under three basic theoretical assumptions (cf. [14, Chp.1.1]): (a) abundance: a large number of individuals; (b) uniformity: individuals of the same population are identical in all dynamical aspects; (c) ergodicity: the movement of individuals as a whole population can be treated as a ergodic system. In particular, the (c) implies that each individual "perceives" the same ambient environment due to fast movement and independence from each other, and thus probabilities of the collision or interaction for two individuals are proportional to the production of their densities, i.e., follow 'mass action' type rules. In addition, there is no description on heterogeneous spatial movement of predators and prey in their models. However, spatiotemporal heterogeneity or aggregation is one of essential features of biodiversity of ecosystems (cf. [15, 16]), and to reveal the mechanism behind entails the consideration of their spatial movement. In this way, when considering the spatial diffusion of predators and prey, P. Kareiva and G. Odell [17] introduced a system of parabolic *Partial Differential Equations* (PDEs). A. Stevens and H. G. Othmer [18] came up with a coupled form of PDE-ODEs in which the spatial diffusion of prey is ignorable.

More generally, when spatial movement of organisms is involved, different types of species may display distinguishing biological features of movement in response to various living environment. In field observations there exist preytaxis for insects [17], chemotaxis for monad [19, 20, 21, 22], nutrient taxis for bacteria [23], hypotaxis for cell migration [24], phototaxis or phototropism for plant organs [25], etc. We remark that the term "*A-taxis*" above emphasizes that the movement tendency of a kind of objects (e.g., organisms or some chemicals) is influenced remarkably by another type of objects, condition or substance closely related to "A". For instance, the preytaxis means the predators' movement is highly affected by the prey, chemotaxis means the organisms' movement is largely determined by the chemicals released by the organisms themselves, hypotaxis implies cell migration is directed by its peripheral adhesions, etc.

In this thesis, we shall restrict our attention to some predator-prey systems with spatial diffusion and preytaxis effect. Note that the term *preytaxis* in the literature may refer to two sides: the *attraction* or *repulsion*, of predators along prey density gradients. We tend to the former, that is, we adopt throughout this thesis that predators are inclined to move towards the region of higher density of prey. Moreover, we need to distinguish that: the usual diffusion effect occurring in a region mainly emphasizes the same species from its higher density to its lower density, but the movement tendency between two different species, i.e. predators and prey, happens due to preytaxis effect.

1.1 A Fundamental Equation

To well understand the preytaxis models introduced later, we first turn to a *continuity equation* (cf. [26, Chap.1.1]). In general, for some physical quantity $\mathscr{Q}$ diffusible or conductible in a media $\Omega \subset \mathbb{R}^n (n \geq 1)$, we denote its density by $q(x,t)$ for spatial variable $x \in \Omega$ and time $t \in \mathbb{R}$. Postulate that there are only two ways to change the amount of $\mathscr{Q}$ in a region:

(1) the amount of $\mathscr{Q}$ in a region raises if additional $\mathscr{Q}$ flows inwards through the surface of the region, and drops when it flows outwards;

(2) the amount of $\mathscr{Q}$ in a region increases if new $\mathscr{Q}$ is generated inside the region, and decreases as the $\mathscr{Q}$ is destroyed inside the region.

Then for any subdomain $U \subset \Omega$ with boundary surface ∂U, one may derive

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_U q(x,t)\,\mathrm{d}x = -\int_{\partial U} \mathbf{J} \cdot \vec{\nu}\ \mathrm{d}S + \int_U \varrho(x,t)\,\mathrm{d}x. \tag{1.1}$$

Here $\mathrm{d}S$ is the unit measure of ∂U, $\mathbf{J}$ is the flux density of the quantity $\mathscr{Q}$ which measures the amount of substance that flows through a unit area during a unit time, $\vec{\nu}$ is the outer normal vector to ∂U, and ϱ is the generation of quantity $\mathscr{Q}$ per unit volume in U per unit time (i.e., generation rate). The negative sign in (1.1) shows that the density flows into U through the boundary ∂U.

Together with divergence theorem implying $\int_{\partial U} \mathbf{J} \cdot \vec{\nu}\ \mathrm{d}S = \int_U \nabla \cdot \mathbf{J}\,\mathrm{d}x$, then by the arbitrary $U \subset \Omega$ and by assuming that $q, \varrho, \nabla \cdot \mathbf{J}$ and $\frac{\partial q}{\partial t}$ are continuous in their variables, the integral equation (1.1) changes into the following *continuity equation*

$$\frac{\partial q}{\partial t} = -\nabla \cdot \mathbf{J} + \varrho, \qquad x \in \Omega \subset \mathbb{R}^n\,(n \geq 1), \quad t \in \mathbb{R}, \tag{1.2}$$

where $\nabla\cdot$ is usual divergence operator. This equation as $\varrho = 0$ may be the simplest model capable of generating aggregation phenomenon.

For the generation ϱ in (1.2), one may note that $\varrho > 0$ suggests the sustained creation of $\mathscr{Q}$ and thus it is called source term; $\varrho < 0$ implies the persistent vanishment of $\mathscr{Q}$, thus sink term; $\varrho \equiv 0$ means that the quantity $\mathscr{Q}$ cannot be created or destroyed, hence in this case the equation (1.2) exactly expresses conservation law. For the flux density $\mathbf{J}$ in (1.2), one may invoke either Fick's law in chemical reaction process, Fourier's law in heat conduction, Darcy's law in porous-medium, or Ohm's law in the field of electrical networks, where the flux can be expressed by

$$\mathbf{J} = -D\nabla q$$

for some constant $D > 0$. The zero-flux boundary condition on (1.2) refers usually to

$$\mathbf{J} \cdot \vec{\nu}\big|_{\partial\Omega} = 0 \tag{1.3}$$

which means the change of quantity $\mathscr{Q}$ described by (1.2) in Ω is isolated from its ambient environment (i.e., $\mathbb{R}^n \backslash \Omega$). Finally, it is easy to see that if (1.2) coupled with (1.3) obeys the conservation law, i.e., $\varrho = 0$, then

$$\int_\Omega q(x,t)\,\mathrm{d}x = \int_\Omega q(x,t_0)\,\mathrm{d}x$$

which is obtained by integrating (1.2) with respect to $x \in \Omega$ and setting initial data $q(x, t_0)$. This means the physical quantity $\mathscr{Q}$ of density q will not change over time, due to no inflow and outflow (zero-flux boundary condition) as well as no creation and destruction ($\varrho = 0$).

Below we shall introduce general preytaxis models and the one we considered respectively.

1.2 General Models with Direct Preytaxis

Suppose that in a region the movement of predators and prey under consideration can be viewed as a sort of diffusion. Thus with the above observation on (1.2) and (1.3) at hand, finding the flux density $\mathbf{J}$ and generation rate ϱ is an essential step to derive the desired diffusion equations. To be specific, the classical predator-prey model with preytaxis effect on population level can be traced back to P. Kareiva and G. Odell [17], which generically takes the form

$$\begin{cases} u_t = \nabla \cdot (d(w)\nabla u - u\chi(u,w)\nabla w) + P(u,w), \\ w_t = \varepsilon \Delta w + G(u,w). \end{cases} \tag{1.4}$$

Here $u = u(x,t)$ and $w = w(x,t)$ represent population density of predators and prey at position $x \in \Omega \subset \mathbb{R}^n$ $(n \geq 1)$ and time $t \in \mathbb{R}_+$, respectively, $d(w)$ in diffusion term $\nabla \cdot (d(w)\nabla u)$ depicts the predators' diffusive motility, $\chi(u,w)$ in the preytaxis term $-\nabla(u\chi(u,w)\nabla w)$ measures sensitivity of the preytaxis per unit strength of the gradient ∇w. The negative sign in the preytaxis term means that the direction of predator's movement driven by preytaxis is opposite to its spatial random diffusion, that is, the movement of predators dominated by preytaxis may helpfully form the aggregation of predators, in contrast with its spatial diffusion. The $\varepsilon > 0$ accounts for diffusion rate of the prey species. Interspecific and intraspecific interactions of the predators and prey, for instance, their death, birth, emigration, immigration, etc., may be characterized by $P(u,w)$ and $G(u,w)$, respectively,

$$P(u,w) = \gamma u F(u,w) - h(u), \qquad G(u,w) = wf(w) - g(u,w). \tag{1.5}$$

More precisely, $\gamma u F(u,w)$ (*resp.* $wf(w)$) may characterize birth or arrival (immigration) of the predators (*resp.* of the prey), $h(u)$ (*resp.* $g(u,w)$) refers to death or departure (emigration) of the predators (*resp.* of the prey). Thus one may take

$g(u,w) = uF(u,w)$ if the death or departures of prey is predominantly caused by predation, and we shall adopt this statement in what follows.

The flux density of u and w read $-(d(w)\nabla u - u\chi(u,w)\nabla w)$ and $-\varepsilon\nabla w$, severally. The predator u and prey w governed by (1.4) inhabit in the domain Ω. By an assumption that they cannot come across the boundary of Ω, then it is reasonable to require zero-flux boundary condition in (1.4), that is,

$$(d(w)\nabla u - u\chi(u,w)\nabla w)\cdot\vec{\nu}\big|_{\partial\Omega} = 0, \qquad \varepsilon\nabla w\cdot\vec{\nu}\big|_{\partial\Omega} = 0,$$

in terms of (1.1)–(1.3), or more stronger one, i.e. zero-Neumann boundary

$$\nabla u\cdot\vec{\nu}\big|_{\partial\Omega} = 0, \qquad\qquad \nabla w\cdot\vec{\nu}\big|_{\partial\Omega} = 0.$$

Some popular assumptions and expressions on $d(w)$, $\chi(u,w)$, $f(w)$, $h(u)$ and $F(u,w)$ can be summarized as follow. One may suppose that

$$d'(w) < 0 \quad \text{and} \quad \chi(u,w) = \chi(w) = -d'(w),$$

so $\nabla\cdot\big(d(w)\nabla u - \chi(w)u\nabla w\big) = \Delta(d(w)u)$ and then $d'(w) < 0$ may indicate that predators will slacken their diffusion when perceiving prey signals. This is called "density-suppressed" effect and more detailed discussions can be found in [27, 28, 29, 30, 31] and the references therein. The per capita growth rate of prey population in absence of predators is denoted by $f(w)$ which satisfies

$$f(0) > 0 \quad \text{and} \quad f'(w) < 0,$$

and thus allows logistic growth (growth with a threshold as a result of finite food resources), that is,

$$wf(w) = rw\Big(1 - \frac{w}{K_0}\Big) \quad r,\ K_0 > 0, \tag{1.6}$$

where K_0 represents the carrying capacity (threshold) of the environment and r is speed of growth rate. To describe Allee's effect which states the positive density dependence, or the positive correlation between population density and individual fitness, there is

$$wf(w) = rw\Big(1 - \frac{w}{K_0}\Big)\Big(\frac{w}{a} - 1\Big),\ 0 < a < K_0 \quad \text{(Bistable or Allee effect).} \tag{1.7}$$

As summarized in [32], the death rate of predators may be linear or quadratic expression as

$$h(u) = \theta u, \qquad\qquad h(u) = \theta u + l u^2, \qquad \theta,\ l \geq 0.$$

There are numerous types of functional response function $F(u, w)$ which represent the conversion from intake of prey to new predators, such as Beddington-DeAngelis type (cf. [33, 34, 35])

$$F(u, w) = \frac{b_1 w}{1 + b_2 w + b_3 u}, \qquad b_1,\ b_2,\ b_3 > 0,$$

ratio dependent form (cf. [36])

$$F(u, w) = \frac{w}{u + w},$$

and prey dependent $F(u, w) = F(w)$. In the last case $F(w)$ is often assumed to fulfill

$$F(0) = 0 \quad \text{and} \quad F'(w) > 0,$$

and thus incorporates:

$$\begin{aligned} F(w) &= w \quad \text{(Holling type I or Lotka-Volterra type)},\\ F(w) &= \frac{w}{c + w}, \quad c > 0 \quad \text{(Holling type II)},\\ F(w) &= \frac{w^k}{c^k + w^k}, \quad c > 0, \quad k > 1 \quad \text{(Holling type III)},\\ F(w) &= c(1 - e^{-kw}), \quad c > 0, \quad k > 1 \quad \text{(Ivlev type)}. \end{aligned} \tag{1.8}$$

Note that (1.4) is similar to the systems that may describe chemotaxis [19, 37], nutrient taxis [23], the random walking problem with persistence of direction and external bias for particles [38], etc. In the setting of preytaxis, system (1.4) has been investigated substantially based on diverse assumptions on $d(w), \chi(u,w), P(u,w)$ and $G(u,w)$ in applications. We shall review some typical, not exhausted, results according to hypotheses on $\chi(u,w)$. Firstly, if the preytaxis sensitivity $\chi(u,w) = \chi$ is a constant sufficient small, Wu et al. [32] proved that the unique classical solution exists and is bounded globally in time in $\Omega \subset \mathbb{R}^n (n \geq 1)$ for a large class of F, h and f. Without this smallness on χ, Jin and Wang [39] derived the global boundedness and stability of classical solution in $\Omega \subset \mathbb{R}^2$ regarding Rosenzweig–MacArthur (F of Holling II and f of logistic type) growth terms. Li [40] showed that a unique globally-bounded classical solution for F of Lotka-Volterra type and f of logistic form in $\Omega \subset \mathbb{R}^n (n = 2, 3)$. Cai et al. [41, 42] established the global-in-time existence and boundedness of classical solutions in $\Omega \subset \mathbb{R}^n (n \geq 1)$ and studied its stationary problem as $n = 1$, for ratio-dependent F and logistic f. Secondly, one may suppose $\chi(u,w) = \chi(u)$ with a truncation imposed in response to biological threshold behaviors, for example, there exists a maximal density of the predators due to volume-filling effect or prevention of overcrowding [43]. Under this assumption, Ainseba et al. [44] showed the existence and uniqueness of weak solution; Tao [45] derived the existence of global-in-time classical solutions in $\Omega \subset \mathbb{R}^n (1 \leq n \leq 3)$; He and Zheng [46] further obtained the global-in-time boundedness of the classical solutions; The existence of non-constant steady states was studied in [47, 48] via bifurcation theory and index degree theory. Thirdly, the truncation on $\chi(u,w)$ is not required as $\chi(u,w) = \chi(w)$, provided that the L_∞ boundedness is essentially determined by the w-equation itself. For instance, the growth rate of w, i.e., $f(w)$, is logistic type which may imply that $\|w(\cdot,t)\|_{L_\infty(\Omega)}$ is bounded uniformly in t. With this observation and for $0 \leq \chi(w) \in C^2([0, +\infty))$, Jin and Wang [49] showed the global existence of bounded classical solution in $\Omega \subset \mathbb{R}^2$.

On the other hand, if diffusion strength of prey w is so weak that the diffusion effect can be negligible, that is, one may formally suppose $\varepsilon = 0$, then (1.4) reduces to the following parabolic-ODE system

$$\begin{cases} u_t = \nabla \cdot (d(w)\nabla u - u\chi(u,w)\nabla w) + P(u,w), \\ w_t = G(u,w), \end{cases} \tag{1.9}$$

coupled usually with no-flux boundary condition only on u

$$\big(d(w)\nabla u - u\chi(u,w)\nabla w\big) \cdot \vec{\nu}\big|_{\partial\Omega} = 0.$$

Such a kind of model has been proposed by Stevens and Othmer [18] to account for biological systems where a control species diffuses in response to a non-diffusible signal that may modify the local environment for succeeding passages. For example, myxobacteria travels typically in swarms via gliding and gathers by intercellular molecular signals of negligible diffusion.

The existent results on (1.9) are not as many as that on (1.4), partly because the theories of fully parabolic models may be no longer suitable even for its local-in-time wellposedness, let alone the global wellposedness or uniform-in-time boundedness. Here we review several the most related results. When $d(w) = d$ is a positive constant, in the context of chemotaxis, Friedman and Tello [50] has studied the classical solution and its stability of (1.9) in $\Omega \subset \mathbb{R}^n (n \geq 1)$, when $P = 0$, $\chi(u,w) = \chi(w) \in C^1(\mathbb{R})$ and $G(u,w) = \varphi(u,w)\phi(u,w)$ satisfies $\phi'_u > 0$, $\chi u\phi'_u + \phi'_w < 0$ and $\chi, \varphi > 0$. If $\chi(w)$ and $\varphi(u,w)$ are two positive constants, Negreanu and Tello [51] considered the stationary states and bifurcations under zero-Neumann boundary condition when $P, \phi \in C^2(\mathbb{R}^2)$, $\chi u\phi'_u + \phi'_w > 0$ and $P'_u\phi'_w - P'_w\phi'_u > 0$. They proved global-in-time existence and uniqueness of the classical solution when $\chi u\phi'_u + \phi'_w = 0$ and $P = 0$. Suppose that $P = 0$, $\chi(u,w)\nabla w = \chi\nabla\Gamma(w) > 0$, $\Gamma \in C^2(\mathbb{R})$, and $G \in C^2(\mathbb{R}^2)$. Then Chen et al. [52] derived that the local existence of unique solution (weak solution) in $L_p(0,T;W_p^1(\Omega))$ with $p > \max\{2,n\}$ for both (1.4) and (1.9) in $\Omega \subset \mathbb{R}^n (n \geq 1)$. They showed that the weak solution of (1.4) can converge to that of

(1.9) in a sense when T is small. The local-in-time existence, stability, and blowup results for models similar to (1.9) can be seen [53, 54, 55] and the references therein. In a setting of hypotaxis, Walker [24] considered a system including (1.9) as a part and studied the existence of unique global classical solution.

1.3 Models of Direct Preytaxis under Consideration

As reviewed above, there are lots of hypotheses on $d(w), \chi(u,w)$, $F(u,w)$, $h(u)$ and $f(w)$ in (1.4) and (1.5). For clarity, we shall in both Chapter 2 and Chapter 3 suppose for (1.4) and (1.5) that $g(u,w) = uF(u,w)$,

$$\chi(u,w) = \chi, \quad d(w) = d, \quad F(u,w) = w, \quad h(u) = u(k + lu), \quad f(w) = r\Big(1 - \frac{w}{K_0}\Big)$$

with $\chi, d, k, r, \gamma, K_0 > 0$ and $l \geq 0$. By introducing two nondimensional variables

$$\tilde{t} = k\,t, \qquad \tilde{x} = \sqrt{\frac{k}{d}}\, x,$$

letting

$$\tilde{\varepsilon} = \frac{\varepsilon}{d}, \quad \tilde{\chi} = \frac{\chi K_0}{d}, \quad \tilde{r} = \frac{r}{k}, \quad \tilde{\gamma} = \frac{\gamma K_0}{k}, \quad \tilde{u}(\tilde{x},\tilde{t}) = \frac{u(x,t)}{k}, \quad \tilde{w}(\tilde{x},\tilde{t}) = \frac{w(x,t)}{K_0},$$

and removing the "$\sim$" in the resulting system for brevity of notations, one may see that (1.4) under zero-flux boundary condition changes into

$$\begin{cases} u_t = \Delta u - \nabla\cdot(\chi u \nabla w) + \gamma u w - u(1 + lu), & \text{in } Q, \\ w_t = \varepsilon \Delta w - uw + rw(1-w), & \text{in } Q, \\ (\nabla u - \chi u \nabla w)\cdot \vec{\nu}\big|_{\partial\Omega} = 0, \quad \varepsilon \nabla w \cdot \vec{\nu}\big|_{\partial\Omega} = 0, & t > 0, \\ u(x,0) = u_0(x), \quad w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.10}$$

where $Q := \Omega \times (0, +\infty)$, the open bounded domain $\Omega \subset \mathbb{R}^n (n \geq 1)$ with boundary $\partial\Omega$, $\varepsilon, \chi, \gamma, r > 0$ and $l \geq 0$. Corresponding to (1.9) we therefore have the following parabolic-ODE system:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi u \nabla w) + \gamma u w - u(1 + lu), & \text{in } Q, \\ w_t = -uw + rw(1 - w), & \text{in } Q, \\ (\nabla u - \chi u \nabla w) \cdot \vec{\nu}\big|_{\partial\Omega} = 0, & t > 0, \\ u(x, 0) = u_0(x), \quad w(x, 0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.11}$$

where prey w has no diffusion, like a kind of plant or signaling molecules of insignificant diffusion.

The classical solution of (1.10), as above-mentioned, has been established by Li [40] under zero-Neumann boundary for $n = 2, 3$. To the best of our knowledge, it remains unknown whether the classical solution of (1.11) exists locally or globally in time. If it does, may the classical or strong solution of (1.10) strictly converge to that of (1.11) in a sense, as $\varepsilon \to 0$?

We shall answer these two questions as $\Omega \subset \mathbb{R}^2$ in Chapter 2 and Chapter 3, respectively. Throughout Chapter 2 and Chapter 3 our basic assumptions are:

$$\begin{cases} u_0(x),\ w_0(x) \in C^{2+\beta}(\overline{\Omega}) \quad \text{for some } \beta \in (0, 1),\ \partial\Omega \in C^\infty, \\ u_0(x) \geq (\not\equiv) 0, \quad w_0(x) > 0 \quad \text{for all} \quad x \in \Omega, \\ (\nabla u_0 - \chi u_0 \nabla w_0) \cdot \vec{\nu}\big|_{\partial\Omega} = 0, \\ \big(\text{and } \nabla w_0 \cdot \vec{\nu}\big|_{\partial\Omega} = 0, \quad \varepsilon \in (0, 1) \quad \text{in (1.10)}\big). \end{cases} \tag{1.12}$$

We note that for (1.10) the ε in $\varepsilon \nabla w \cdot \vec{\nu}\big|_{\partial\Omega} = 0$ may be removed as $\varepsilon > 0$ by the linearity of trace operator, but is essentially needed when we take $\varepsilon \to 0$ in the boundary condition (cf. Chapter 3)

To make the boundary conditions standard, we shall introduce a reversible continuous transformation

$$a(x, t) := u(x, t) e^{-\chi w(x,t)}, \qquad \text{i.e.,} \quad u = a e^{\chi w}.$$

Substituting it into (1.10) gives rise to

$$\begin{aligned} a_t =& -\chi a w_t + e^{-\chi w}\nabla\cdot(e^{\chi w}\nabla a) + \gamma a w - a(1+lae^{\chi w}) \\ =& -\chi a w_t + \chi\nabla w\cdot\nabla a + \Delta a + \gamma a w - a(1+lae^{\chi w}), \end{aligned} \tag{1.13}$$

and

$$w_t = \varepsilon\Delta w - awe^{\chi w} + rw(1-w), \tag{1.14}$$

where a and w satisfy zero-Neumann boundary condition, i. e., for $t > 0$,

$$\nabla w\cdot\vec{\nu}\big|_{\partial\Omega} = 0, \quad \nabla a\cdot\vec{\nu}\big|_{\partial\Omega} = e^{-\chi w}(\nabla u - \chi u\nabla w)\cdot\vec{\nu}\big|_{\partial\Omega} = 0, \tag{1.15}$$

in the light of $(\nabla u - \chi u\nabla w)\cdot\vec{\nu}\big|_{\partial\Omega} = 0$, $e^{-\chi w} > 0$, and the definition of trace.

Similarly, we shall state the general systems of indirect preytaxis and the one we consider, severally.

1.4 Generic Models of Indirect Preytaxis

Different from the aforementioned direct search for prey, some predators might start with perceiving chemical signals released by prey, for instance smell of blood or pheromone (trace pheromone, aggregation pheromone, etc.), and then hunt for the prey by tracking such signals, the process of which is called an indirect preytaxis in this case. Similar to a role of direct preytaxis in promoting the heterogeneity of ecosystems, strong indirect preytaxis may also cause spatial heterogeneity (cf. [56]) without considering predator's reproduction, mortality, and random diffusion of the prey. Later Tyutyunov et al. [16] proposed another more general model which reads

$$\begin{cases} u_t = \nabla\cdot\big(d(v)\nabla u - \chi(v)u\nabla v\big) + \gamma uF(w) - \theta u, \\ v_t = d_v\Delta v + \beta w - \sigma v, \\ w_t = d_w\Delta w + wf(w) - uF(w), \end{cases} \tag{1.16}$$

where $u = u(t,x)$ and $w = w(t,x)$ represent population density of predators and prey at position $x \in \Omega \subset \mathbb{R}^n (n \geq 1)$ and time $t \in (0, +\infty)$ severally; $v = v(t,x)$ is concentration of chemicals released by prey which are secreted at a constant rate $\beta > 0$, decay in a fixed rate $\sigma > 0$, and diffuse with a constant diffusivity $d_v > 0$. The $(-d(v)\nabla u + u\chi(v)\nabla v)$ is called the predators' flux density, $d(v)$ is the predators' random-motility function, and $u\chi(v)\nabla v$ means that predators move towards the increasing gradient of the chemical density at an average speed of $\chi(v)\nabla v$ with $\chi(v)$ measuring indirect preytaxis sensitivity per unit strength of the gradient ∇v. In this way the advection term $-\nabla \cdot (u\chi(v)\nabla v)$ is viewed as indirect preytaxis effect of predators.

System (1.16) is usually coupled with zero-Neumann boundary condition

$$\nabla u \cdot \vec{\nu}\big|_{\partial\Omega} = 0, \quad \nabla v \cdot \vec{\nu}\big|_{\partial\Omega} = 0, \quad \nabla w \cdot \vec{\nu}\big|_{\partial\Omega} = 0,$$

which in this case is equivalent to zero-flux boundary condition

$$\big(d(v)\nabla u - \chi(v)u\nabla v\big) \cdot \vec{\nu}\big|_{\partial\Omega} = 0, \quad d_v \nabla v \cdot \vec{\nu}\big|_{\partial\Omega} = 0, \quad d_w \nabla w \cdot \vec{\nu}\big|_{\partial\Omega} = 0,$$

in the sense of trace, since as supposed above $d(v), d_v, d_w$ do not change their signs.

System (1.16) may cover some reaction-diffusion systems used to describe the dynamics amongst the bacterial cell density, concentration of acyl-homoserine lactone, and nutrient density (cf. [27]). In addition, if $\chi(v) = 0$ and d_v and d_w are density-dependent as well, then (1.16) can be used to describe the interactions among uninfected cells, free viruses produced by infected cells, and infected cells (cf. [57]). In this thesis we will understand it in the view of indirect predator-prey relationship. Firstly, when $\chi(v)$ and $d(v)$ are supposed to be constants and $\Omega \subset \mathbb{R}^1$, Tyutyunov et al. [16] studied pattern formation condition on stationary states of (1.16) with zero-Neumann boundary condition. Their numerical analysis illustrated that non-trivial homogeneous stationary state of the model becomes unstable with respect to small perturbation caused by increasing preytaxis strength; Zuo and Song [58] obtained some interesting dynamical behaviors including stability and double-Hopf bifurcation results; Secondly, if $\chi(v)$ and $d(v)$ are constants and $\Omega \subset \mathbb{R}^n (n \geq 1)$, Yoon and Ahn [59] derived the unique global-in-time classical solution to the system (1.16) with functional response functions involving Beddington-DeAngelis type, and showed asymptotic stability of both prey-only and coexistence steady states. They found that preytaxis is an essential factor in generating patterns. Thirdly, when $d(v)$ is a positive constant but $\chi(v)$ is density dependent, Wang and Wang [60] investigated global existence and boundedness of the unique classical solution as well as the asymptotic stabilities of nonnegative and spatial homogeneous equilibria as $\Omega \subset \mathbb{R}^n (n \geq 1)$.

1.5 Models of Indirect Preytaxis under Consideration

In view of the above review a question arises: what will happen when $\chi(v)$ and $d(v)$ are both density-dependent? Relevant results remain unknown before we solve this problem, to the best of our knowledge. This inspires us to study the global-in-time existence, uniqueness and large time behavior of the unique classical solution to

$$\begin{cases} u_t = \nabla \cdot \big(d(v) \nabla u - u\chi(v) \nabla v \big) + \gamma u F(w) - \theta u - \ell u^2, & t > 0,\ x \in \Omega; \\ v_t = d_v \Delta v + \beta w - \sigma v, & t > 0,\ x \in \Omega; \\ w_t = d_w \Delta w + w f(w) - u F(w), & t > 0,\ x \in \Omega; \\ \nabla u \cdot \vec{\mathbf{n}} = 0,\ \nabla v \cdot \vec{\mathbf{n}} = 0,\ \nabla w \cdot \vec{\mathbf{n}} = 0, & t > 0,\ x \in \partial\Omega; \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad w(0,x) = w_0(x), & x \in \Omega, \end{cases} \tag{1.17}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $\vec{\mathbf{n}}$ is the unit outer normal vector towards $\partial\Omega$, $\ell \geq 0$, and d_w, γ, $\theta > 0$.

1.6 Outline of the Thesis

The organization of this thesis is below: We shall in Chapter 2 study the global-in-time existence and uniqueness of classical solution to the model (1.11) (cf. Theorem 2.1). Considering that (1.11) is a parabolic-ODE system, we start with establishing the local-in-time existence and uniqueness of strong solution and then improve the regularity in subsection 2.2.1 in order to make it a local classical solution. Based on some *a prior* estimates in subsection 2.2.2, we shall in subsection 2.2.3 obtain the global-in-time existence, uniqueness, and uniform-in-time boundedness, when $\Omega \subset \mathbb{R}^2$.

We consider in Chapter 3 the limit of the strong solution of (1.10) when the diffusion coefficient ε of prey tends to zero (cf. Theorem 3.1). More precisely, in section 3.2 we first prepare some estimates of the classical solution to (1.10) and then the verify that this classical solution is strong solution by giving corresponding $W_p^{2,1}(Q_T)$ estimates. The main difficulties in this part lie in deriving the upper boundedness of component u in norm $L_\infty(\Omega)$ such that this estimate remains bounded as $\varepsilon \to 0$. With these preparations at hand, we shall in section 3.3 prove that the strong solutions of (1.10) may converge as $\varepsilon \to 0$ to the strong solution of (1.11), by using Aubin-Lions compactness lemma and trace interpolation inequalities in subsection 3.3.1. Then in subsection 3.3.2 we intend to prove that this strong solution of (1.11) fulfills the classical regularity and uniqueness, thus being the unique classical solution of (1.11).

The two chapters above-mentioned pertain to direct preytaxis models (1.10) and (1.11). For the indirect preytaxis model (1.17), we shall in Chapter 4 explore global-in-time existence and uniqueness, by obtaining the uniform-in-time boundedness of the solution in section 4.2. Upon finding suitable Lyapunov's functions in section 4.3, we are to investigate the global asymptotic stability for the prey-only state and coexistence state, by using limit properties of dynamical systems. In addition, we will in section 4.4 derive their linear instability criteria and present some patterns of spatial one-dimensional case in subsection 4.4.2 and two-dimensional case in subsection 4.4.3.

We clarify that the results of Chapter 4 have been published as our paper in [61].

Chapter 2

Global Well-Posedness on Parabolic-ODE System with Direct Preytaxis

2.1 Models and Main Results

As stated in subsection 1.3, we shall in this chapter consider the global-in-time existence and uniqueness of classical solution to the parabolic-ODE system (1.11), that is,

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi u \nabla w) + \gamma u w - u(1 + lu), & \text{in } Q, \\ w_t = -uw + rw(1 - w), & \text{in } Q, \\ (\nabla u - \chi u \nabla w) \cdot \vec{\nu}\big|_{\partial\Omega} = 0, & t > 0, \\ u(x, 0) = u_0(x), \quad w(x, 0) = w_0(x), & x \in \Omega, \end{cases} \tag{2.1}$$

under the condition (1.12), that is,

$$\begin{cases} u_0(x),\ w_0(x) \in C^{2+\beta}(\overline{\Omega}) \quad \text{for some } \beta \in (0,1), \quad \partial\Omega \in C^{2+\beta}, \\ u_0(x) \geq (\not\equiv) 0, \quad w_0(x) > 0 \quad \text{for all} \quad x \in \Omega, \\ (\nabla u_0 - \chi u_0 \nabla w_0) \cdot \vec{\nu}\big|_{\partial\Omega} = 0, \end{cases} \tag{2.2}$$

where $Q = \Omega \times (0, +\infty)$ and $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded open domain. Note that here we need only $\partial\Omega \in C^{2+\beta}$ instead of $\partial\Omega \in C^\infty$ in (1.12).

For the above problem our main result reads as below:

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded open domain and (2.2) hold. Then for any given $T > 0$ there exists a local-in-time positive and unique classical solution (u, w) of (2.1) fulfilling*

$$(u, w)(x, t) \in \Big(C\big(\overline{\Omega} \times [0, T_0)\big) \cap C^{2,1}\big(\overline{\Omega} \times [0, T_0)\big) \Big)^2$$

for some $0 < T_0 < \min\{1, T\}$ which depends on the upper bound of $\|(u_0, w_0)\|_{C^1(\overline{\Omega})}$.

In particular when $n = 2$, the local classical solution is global in time (i.e.,$T_0 = T$), and satisfies

$$\|w(\cdot, t)\|_{L_\infty(\Omega)} + \|u(\cdot, t)\|_{L_\infty(\Omega)} \leq C, \qquad t \in (0, T),$$

where the constant C is independent of T, and $K := \max\{1, \|w_0\|_{L_\infty(\Omega)}\}$.

2.2 Proof of Theorem 2.1

Motivated by [2], we are in this section to prove the Theorem 2.1, that is, when $l > \chi K$ or $\chi > 0$ is small enough, the unique classical solution of (2.1) exists globally in time.

Let $\bar{Q}_T := \overline{\Omega} \times [0, T]$ for any fixed $0 < T < +\infty$. Upon the transformation implemented in (1.13)–(1.15), system (2.1) becomes

$$\begin{cases} a_t = -\chi a w_t + \chi \nabla w \cdot \nabla a + \Delta a + \gamma a w - a(1 + l a e^{\chi w}), & \text{in } Q_T, \\ w_t = -a w e^{\chi w} + r w(1 - w), & \text{in } Q_T, \\ \nabla a \cdot \vec{\nu}\big|_{\partial\Omega} = 0, & t \in (0, T), \\ a(x, 0) = a_0(x) \geq (\not\equiv) 0, \quad w(x, 0) = w_0(x) > 0, & x \in \Omega. \end{cases} \tag{2.3}$$

Note that by the transformation the existence and uniqueness of the classical solution of (2.3) implies that of (2.1). So we next shall focus only on (2.3).

2.2.1 Local Existence and Uniqueness of the Classical Solution to (2.3)

We start with consideration of its strong solution in the following lemma:

Lemma 2.1 (Local existence and uniqueness)**.** *Assume* (2.2) *and* $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$. *Then system* (2.3) *possesses a strong solution, i.e.,*

$$(a, w) \in W_p^{2,1}(Q_{T_0}) \times C^{1,1}(\bar{Q}_{T_0}),$$

for $n + 2 < p < +\infty$, *provided that* $0 < T_0 < 1$ *is sufficient small and depends only on*

$$R \geq 2 + 2\|a_0\|_{C^1(\overline{\Omega})} + 2\|w_0\|_{C^1(\overline{\Omega})}.$$

Moreover,

$$a > 0, \quad 0 < w \leq \max\left\{1, \|w_0\|_{L_\infty(\Omega)}\right\} \qquad \text{in } Q_{T_0}.$$

Proof. Below we shall use Banach's fixed-point theorem to show the local existence of strong solution to (2.3). Taking $0 < T < 1$, we introduce a Banach space X with respect to function (a, w) which is endowed with norm

$$\|(a, w)\|_X := \|a\|_{C^{1,0}(\bar{Q}_T)} + \|w\|_{C^{1,0}(\bar{Q}_T)}$$

and introduce a closed subspace of X by

$$X_R := \Big\{(a, w) \in X : \ a(x, 0) = a_0(x), \ w(x, 0) = w_0(x),$$

$$\nabla a \cdot \vec{\nu}\big|_{\partial\Omega} = 0, \ \nabla a_0 \cdot \vec{\nu}\big|_{\partial\Omega} = 0, \ \|(a, w)\|_X \leq R\Big\}.$$

For any given $(a, w) \in X_R$, to system (2.3) we shall derive a corresponding function pair

$$(\bar{a}, \bar{w}) := \mathscr{F}(a, w),$$

from

$$\begin{cases} \bar{w}_t = \bar{w}\big\{-ae^{\chi w} + r(1 - w)\big\}, & \text{in } Q_T \\ \bar{w}(x, 0) = w_0(x), & x \in \Omega \end{cases} \tag{2.4}$$

and

$$
\begin{cases}
\bar{a}_t - \Delta\bar{a} - \chi\nabla\bar{w}\cdot\nabla\bar{a} + \bar{a}\big\{1-(\chi w - l)ae^{\chi w}\big\} \\
\qquad\qquad\qquad\qquad = \big\{\gamma - r\chi(1-w)\big\}aw, & \text{in } Q_T \\
\nabla\bar{a}\cdot\vec{\nu}\big|_{\partial\Omega} = 0, & t\in(0,T) \\
\bar{a}(x,0) = a_0(x), & x\in\Omega
\end{cases}
\tag{2.5}
$$

by showing that $\mathscr{F}$ is a contraction mapping from X_R into itself.

Indeed, it is easy to see that (2.4) is a linear ODE of $\bar{w}$ and thus its unique solution reads

$$
\bar{w}(x,t) = w_0(x)e^{\int_0^t h_1(x,s)\mathrm{d}s} > 0, \quad \text{with} \quad h_1 := -ae^{\chi w} + r(1-w) \in C^{1,0}(\bar{Q}_T),
$$

by $(a,w)\in C^{1,0}(\bar{Q}_T)$. Taking spatial gradient from both sides may yield

$$
\nabla\bar{w} = \nabla w_0 e^{\int_0^t h_1(x,s)\mathrm{d}s} + w_0(x)e^{\int_0^t h_1(x,s)\mathrm{d}s}\int_0^t \nabla_x h_1(x,s)\mathrm{d}s.
$$

Thereby one may derive from the expression of $\bar{w}$ that

$$
\|\bar{w}\|_{C(\bar{Q}_T)} \le \|w_0\|_{C(\overline{\Omega})}\cdot e^{T\|h_1\|_{C^{1,0}(\bar{Q}_T)}},
$$

obtain from (2.4) that

$$
\|\bar{w}_t\|_{C(\bar{Q}_T)} \le \|\bar{w}\|_{C(\bar{Q}_T)}\cdot\|h_1\|_{C(\bar{Q}_T)},
$$

and infer from the expression of $\nabla\bar{w}$ that

$$
\|\nabla\bar{w}\|_{C(\bar{Q}_T)} \le \|\nabla w_0\|_{C(\overline{\Omega})}\cdot e^{T\|h_1\|_{C^{1,0}(\bar{Q}_T)}} + \|w_0\|_{C(\overline{\Omega})}\cdot e^{T\|h_1\|_{C^{1,0}(\bar{Q}_T)}}\cdot T\|h\|_{C^{1,0}(\bar{Q}_T)}.
$$

Thus for sufficient small $0<T<1$, we may conlude that

$$
\begin{aligned}
\|\bar{w}\|_{C^{1,0}(\bar{Q}_T)} =& \|\bar{w}\|_{C(\bar{Q}_T)} + \|\nabla\bar{w}\|_{C(\bar{Q}_T)} \\
\le& e^{T\|h_1\|_{C^{1,0}(\bar{Q}_T)}}\big\{\|\nabla w_0\|_{C(\overline{\Omega})} + \|w_0\|_{C(\overline{\Omega})}(1+T\|h\|_{C^{1,0}(\bar{Q}_T)})\big\} \\
\le& 2\|\nabla w_0\|_{C(\overline{\Omega})} + 2\|w_0\|_{C(\overline{\Omega})} \\
\le& R
\end{aligned}
\tag{2.6}
$$

in view of $(a,w)\in X_R$.

The (2.6), combined with known functions

$$h_2:=1-(\chi w-l)ae^{\chi w}\in C^{1,0}(\bar{Q}_T)\quad\text{and}\quad h_3:=\big\{\gamma-r\chi(1-w)\big\}aw\in C^{1,0}(\bar{Q}_T)$$

due to $(a,w)\in X_R$, indicates that (2.5) is a linear parabolic equation of $\bar{a}$, i. e.,

$$\begin{cases}\bar{a}_t-\Delta\bar{a}-\chi\nabla\bar{w}\cdot\nabla\bar{a}+h_2\bar{a}=h_3, & \text{in } Q_T,\\ \nabla\bar{a}\cdot\vec{\nu}\big|_{\partial\Omega}=0, & t\in(0,T),\\ \bar{a}(x,0)=a_0(x), & x\in\Omega.\end{cases}$$

Then L_p-theory of linear parabolic equations immediately implies that (2.5) possesses a unique strong solution $\bar{a}$ which satisfies

$$\|\bar{a}\|_{W_p^{2,1}(Q_T)}\le C(T)\big\{\|h_3\|_{L_p(Q_T)}+\|a_0\|_{W_p^2(\Omega)}\big\}$$

for any $1<p<+\infty$, where $C(T)$ remains bounded for any finite $T>0$. Now applying a Sobolev's embedding

$$W_p^{2,1}(Q_T)\hookrightarrow C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T) \tag{2.7}$$

with $0<\alpha\le 2-\frac{n+2}{p}<+\infty$ (i. e. $\frac{n+2}{2}<p<+\infty$) may show

$$\|\bar{a}\|_{C^{1+\lambda,\frac{1+\lambda}{2}}(\bar{Q}_T)}\le C(n,p,Q_T)\,\|\bar{a}\|_{W_p^{2,1}(Q_T)}, \tag{2.8}$$

with $\lambda\in(0,1)$ and $n+2<p<+\infty$. These estimates combined with

$$\begin{aligned}\|\bar{a}\|_{C^{1,0}(\bar{Q_T})}=&\|\bar{a}\|_{C(\bar{Q}_T)}+\|\nabla\bar{a}\|_{C(\bar{Q}_T)}\\ \le&\|\bar{a}-\bar{a}_0\|_{C(\bar{Q}_T)}+\|\bar{a}_0\|_{C(\bar{Q}_T)}+\|\nabla\bar{a}-\nabla\bar{a}_0\|_{C(\bar{Q}_T)}+\|\nabla\bar{a}_0\|_{C(\bar{Q}_T)}\\ \le&T^{\frac{1+\lambda}{2}}\|\bar{a}\|_{C^{1,\frac{1+\lambda}{2}}(\bar{Q}_T)}+\|a_0\|_{C^1(\bar{\Omega})}\\ \le&T^{\frac{1+\lambda}{2}}\|\bar{a}\|_{C^{1+\lambda,\frac{1+\lambda}{2}}(\bar{Q}_T)}+\|a_0\|_{C^1(\bar{\Omega})}\end{aligned}$$

will indicate

$$\begin{aligned}\|\bar{a}\|_{C^{1,0}(\bar{Q_T})} \le & T^{\frac{1+\lambda}{2}} C(n,p,Q_T)C(T)\big\{\|h_2\|_{L_p(Q_T)} + \|a_0\|_{W_p^2(\Omega)}\big\} + \|a_0\|_{C^1(\overline{\Omega})} \\ \le & 2 + 2\|a_0\|_{C^1(\overline{\Omega})} \\ \le & R\end{aligned} \tag{2.9}$$

for $T > 0$ sufficiently small. So we have proved the mapping

$$\mathscr{F} : X_R \longrightarrow X_R. \tag{2.10}$$

Below we shall show that such a $\mathscr{F}$ is contractive on X_R. Suppose $(\bar{a}_1, \bar{w}_1) = \mathscr{F}(a_1, w_1)$ and $(\bar{a}_2, \bar{w}_2) = \mathscr{F}(a_2, w_2)$ with $(a_1, w_1), (a_2, w_2) \in X_R$. Then we may compute that

$$(\bar{w}_1 - \bar{w}_2)_t = h_4(\bar{w}_1 - \bar{w}_2) + h_5, \quad \text{with} \quad (\bar{w}_1 - \bar{w}_2)(x, 0) = 0$$

where

$$h_4 := -a_1 e^{\chi w_1} + r - r w_1 \quad \text{and} \quad h_5 := \bar{w}_2\big\{a_2 e^{\chi w_2} - a_1 e^{\chi w_1} + r(w_2 - w_1)\big\}.$$

It follows that

$$(\bar{w}_1 - \bar{w}_2)(x, t) = \int_0^t h_5(x, s) e^{\int_s^t h_4(x,\tau)\mathrm{d}\tau} \mathrm{d}s$$

and

$$\begin{aligned}\nabla(\bar{w}_1 - \bar{w}_2)(x, t) = & \int_0^t \nabla_x h_5(x, s) e^{\int_s^t h_4(x,\tau)\mathrm{d}\tau} \mathrm{d}s \\ & + \int_0^t h_5(x, s) e^{\int_s^t h_4(x,\tau)\mathrm{d}\tau} \int_s^t \nabla_x h_4(x, \tau)\mathrm{d}\tau\, \mathrm{d}s.\end{aligned}$$

Note that

$$\begin{aligned}& \|h_5\|_{C(\bar{Q}_T)} \\ = & \|\bar{w}_2\big\{(a_2 - a_1)e^{\chi w_2} + a_1(e^{\chi w_2} - e^{\chi w_1}) + r(w_2 - w_1)\big\}\|_{C(\bar{Q}_T)}\end{aligned}$$

$$\leq \|\bar{w}_2\|_{C(\bar{Q}_T)}\big\{e^{\chi\|w_2\|_{C(\bar{Q}_T)}}\cdot\|a_1-a_2\|_{C(\bar{Q}_T)}+\|(|a_1|+r)e^{\chi(w_1+w_2)}\|_{C(\bar{Q}_T)}\cdot\|w_1-w_2\|_{C(\bar{Q}_T)}\big\}$$

$$\leq \|\bar{w}_2\|_{C(\bar{Q}_T)}\|(|a_1|+r+1)\|_{C(\bar{Q}_T)}e^{\|\chi(w_1+w_2)\|_{C(\bar{Q}_T)}}\big\{\|a_1-a_2\|_{C(\bar{Q}_T)}+\|w_1-w_2\|_{C(\bar{Q}_T)}\big\},$$

$$\begin{aligned}
&\nabla h_5\\
=&\nabla\bar{w}_2\big\{a_2e^{\chi w_2}-a_1e^{\chi w_1}+r(w_2-w_1)\big\}+\bar{w}_2\big\{\nabla a_2e^{\chi w_2}-\nabla a_1e^{\chi w_1}\\
&+\chi a_2e^{\chi w_2}\nabla w_2-\chi a_1e^{\chi w_1}\nabla w_1+r\nabla(w_2-w_1)\big\}\\
=&\nabla\bar{w}_2\big\{a_2e^{\chi w_2}-a_1e^{\chi w_1}+r(w_2-w_1)\big\}+\bar{w}_2\big\{e^{\chi w_2}\nabla(a_2-a_1)+(e^{\chi w_2}-e^{\chi w_1})\nabla a_1\\
&+\chi\nabla w_2e^{\chi w_2}(a_2-a_1)+\chi\nabla w_2a_1(e^{\chi w_2}-e^{\chi w_1})+\chi a_1e^{\chi w_1}\nabla(w_2-w_1)+r\nabla(w_2-w_1)\big\},
\end{aligned}$$

$$\begin{aligned}
&\|\nabla h_5\|_{C(\bar{Q}_T)}\\
\leq&\|\nabla\bar{w}_2\|_{C(\bar{Q}_T)}\|(|a_1|+r+1)\|_{C(\bar{Q}_T)}e^{\|\chi(w_1+w_2)\|_{C(\bar{Q}_T)}}\big\{\|a_1-a_2\|_{C(\bar{Q}_T)}+\|w_1-w_2\|_{C(\bar{Q}_T)}\big\}\\
&+\|\bar{w}_2\|_{C(\bar{Q}_T)}\Big\{e^{\chi\|w_2\|_{C(\bar{Q}_T)}}\|\nabla(a_2-a_1)\|_{C(\bar{Q}_T)}+\chi\|\nabla w_2e^{\chi w_2}\|_{C(\bar{Q}_T)}\|a_1-a_2\|_{C(\bar{Q}_T)}\\
&+\|\nabla a_1+\chi\nabla w_2a_1\|_{C(\bar{Q}_T)}e^{\chi\|w_1+w_2\|_{C(\bar{Q}_T)}}\|w_1-w_2\|_{C(\bar{Q}_T)}\\
&+\|r+\chi a_1e^{\chi w_2}\|_{C(\bar{Q}_T)}\|\nabla(w_1-w_2)\|_{C(\bar{Q}_T)}\Big\}
\end{aligned}$$

and $\|\nabla h_4\|_{C(\bar{Q}_T)}=\|-\nabla a_1e^{\chi w_1}-\chi a_1e^{\chi w_1}\nabla w_1-r\nabla w_1\|_{C(\bar{Q}_T)}$

$$\leq e^{\chi\|w_1\|_{C(\bar{Q}_T)}}\cdot\|\nabla a_1\|_{C(\bar{Q}_T)}+\|r+\chi a_1e^{\chi w_1}\|_{C(\bar{Q}_T)}\cdot\|\nabla w_1\|_{C(\bar{Q}_T)}.$$

These immediately shows that

$$\begin{aligned}
&\|\bar{w}_1-\bar{w}_2\|_{C^{1,0}(\bar{Q}_T)}\\
=&\|\bar{w}_1-\bar{w}_2\|_{C(\bar{Q}_T)}+\|\nabla(\bar{w}_1-\bar{w}_2)\|_{C(\bar{Q}_T)}\\
\leq&Te^{T\|h_4\|_{C(\bar{Q}_T)}}\cdot\big\{\|h_5\|_{C(\bar{Q}_T)}+\|\nabla h_5\|_{C(\bar{Q}_T)}+\|h_5\|_{C(\bar{Q}_T)}\cdot\|h_4\|_{C(\bar{Q}_T)}\big\}\\
\leq&Tc_1(R)\big\{\|a_1-a_2\|_{C^{1,0}(\bar{Q}_T)}+\|w_1-w_2\|_{C^{1,0}(\bar{Q}_T)}\big\}
\end{aligned}\tag{2.11}$$

when $T>0$ is sufficient small.

On the other hand, we see that

$$\begin{cases} (\bar{a}_1-\bar{a}_2)_t-\Delta(\bar{a}_1-\bar{a}_2)-\chi\nabla\bar{w}_1\cdot\nabla(\bar{a}_1-\bar{a}_2) & \\ \qquad\qquad\qquad\qquad +h_2(\bar{a}_1-\bar{a}_2)=h_6, & \text{in } Q_T, \\ \nabla(\bar{a}_1-\bar{a}_2)\cdot\vec{\nu}\big|_{\partial\Omega}=0, & t\in(0,T), \\ (\bar{a}_1-\bar{a}_2)(x,0)=0, & x\in\Omega, \end{cases} \tag{2.12}$$

where

$$h_6:=-\chi\nabla(\bar{w}_1-\bar{w}_2)\cdot\nabla\bar{a}_2-\bar{a}_2\big\{(\chi w_1-l)a_1e^{\chi w_1}-(\chi w_2-l)a_2e^{\chi w_2}\big\}$$
$$+\big\{\gamma-r\chi(1-w_1)\big\}a_1w_1-\big\{\gamma-r\chi(1-w_2)\big\}a_2w_2.$$

Thanks to (2.6) and (2.9), we may consider (2.12) as a linear parabolic equation of $(\bar{a}_1-\bar{a}_2)$. Note that

$$\begin{aligned}&(\chi w_1-l)a_1e^{\chi w_1}-(\chi w_2-l)a_2e^{\chi w_2}\\ =&(\chi w_1e^{\chi w_1}-le^{\chi w_2})(a_1-a_2)+\chi a_2e^{\chi w_1}(w_1-w_2)+(\chi a_2w_2-la_1)(e^{\chi w_1}-e^{\chi w_2})\end{aligned}$$

and

$$\begin{aligned}&\big\{\gamma-r\chi(1-w_1)\big\}a_1w_1-\big\{\gamma-r\chi(1-w_2)\big\}a_2w_2\\ =&\big\{(\gamma-r\chi)w_1+r\chi w_2^2\big\}(a_1-a_2)+\big\{(\gamma-r\chi)a_2+r\chi a_1(w_1+w_2)\big\}(w_1-w_2).\end{aligned}$$

Then for sufficient small $T>0$, we have

$$\begin{aligned}\|h_6\|_{L_p(Q_T)}\le&\chi\|\nabla\bar{a}_2\|_{C(\bar{Q}_T)}\cdot\|\nabla(\bar{w}_1-\bar{w}_2)\|_{L_p(Q_T)}\\ &+\|\bar{a}_2(\chi w_1e^{\chi w_1}-le^{\chi w_2})\|_{C(\bar{Q}_T)}\cdot\|a_1-a_2\|_{L_p(Q_T)}\\ &+\chi\|\bar{a}_2a_2e^{\chi w_1}+|\chi a_2w_2-la_1|e^{\chi(w_1+w_2)}\|_{C(\bar{Q}_T)}\cdot\|w_1-w_2\|_{L_p(Q_T)}\\ &+\|(\gamma-r\chi)w_1+r\chi w_2^2\|_{C(\bar{Q}_T)}\cdot\|a_1-a_2\|_{L_p(Q_T)}\\ &+\|(\gamma-r\chi)a_2+r\chi a_1(w_1+w_2)\|_{C(\bar{Q}_T)}\cdot\|w_1-w_2\|_{L_p(Q_T)}\\ \le&c_2(R)\big\{\|a_1-a_2\|_{C^{1,0}(\bar{Q}_T)}+\|w_1-w_2\|_{C^{1,0}(\bar{Q}_T)}\big\}.\end{aligned}$$

Applying L_p-theory of linear parabolic equations to (2.12) may yield that

$$\begin{aligned}\|\bar{a}_1-\bar{a}_2\|_{W_p^{2,1}(Q_T)} \leq & C(T)\big\{\|h_6\|_{L_p(Q_T)}\big\} \\ \leq & C(T)c_2(R)\big\{\|a_1-a_2\|_{C^{1,0}(\bar{Q}_T)}+\|w_1-w_2\|_{C^{1,0}(\bar{Q}_T)}\big\}.\end{aligned}$$

Similar to the derivation of (2.8), again using the Sobolev embedding leads us to

$$\begin{aligned}\|\bar{a}_1-\bar{a}_2\|_{C^{1+\lambda,\frac{1+\lambda}{2}}(\bar{Q}_T)} \leq & C(n,p,Q_T)\,\|\bar{a}_1-\bar{a}_2\|_{W_p^{2,1}(Q_T)} \\ \leq & C(n,p,Q_T)C(T)c_2(R)\big\{\|a_1-a_2\|_{C^{1,0}(\bar{Q}_T)}+\|w_1-w_2\|_{C^{1,0}(\bar{Q}_T)}\big\}.\end{aligned}$$

So we may derive from $(\bar{a}_1-\bar{a}_2)(x,0)=0$ for $x\in\Omega$ that

$$\begin{aligned}\|\bar{a}_1-\bar{a}_2\|_{C^{1,0}(\bar{Q}_T)} = & \|(\bar{a}_1-\bar{a}_2)(x,t)-(\bar{a}_1-\bar{a}_2)(x,0)\|_{C^{1,0}(\bar{Q}_T)} \\ \leq & T^{\frac{1+\lambda}{2}}\|(\bar{a}_1-\bar{a}_2)(x,t)-(\bar{a}_1-\bar{a}_2)(x,0)\|_{C^{1,\frac{1+\lambda}{2}}(\bar{Q}_T)} \qquad (2.13) \\ \leq & T^{\frac{1+\lambda}{2}}c_3(R)\big\{\|a_1-a_2\|_{C^{1,0}(\bar{Q}_T)}+\|w_1-w_2\|_{C^{1,0}(\bar{Q}_T)}\big\}.\end{aligned}$$

Consequently, by taking $T>0$ to be small enough such that

$$Tc_1(R)+T^{\frac{1+\lambda}{2}}c_3(R)<1,$$

then (2.11) and (2.13) collectively imply that $\mathscr{F}$ in (2.10) is a contraction mapping, which means that (2.3) has a unique strong solution in X_R.

The positivity of a can be obtained by applying comparison principle of linear parabolic to a-equation in (2.3) after treating $\nabla w, w$ as known functions and using the regularity of a. Finally, from (2.4) we infer that

$$w_t \leq rw(1-w) \quad \Rightarrow \quad w(x,t)\leq \frac{1}{(w_0(x)^{-1}-1)e^{-rt}+1} \leq \max\big\{1,\|w_0\|_{L_\infty(\Omega)}\big\}=K$$

upon a comparison principle of ODE. This completes the proof. □

We may improve the regularity of such a strong solution to (2.3) as below.

Lemma 2.2 (Regularity)**.** *The solution derived in Lemma 2.1 possesses the following* *regularity:*

$$(a,w) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T) \times C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T), \qquad \alpha \in (0,1),$$

for any $0 < T \le T_0$*, where the domain* $\Omega \subset \mathbb{R}^n (n \ge 1)$ *is bounded and fulfills that any* $x, y \in \Omega$ *can be connected by finitely many line segments, e.g.,* Ω *to be convex or* $\partial\Omega$ *is sufficient regular, like* $\partial\Omega \in C^{2+\alpha}$.

Proof. We may reformulate the a-equation in (2.3) as

$$\begin{cases} a_t - \Delta a - \chi \nabla w \cdot \nabla a + a = h_7, & \text{in } Q_T, \\ \nabla a \cdot \vec{\nu}\big|_{\partial\Omega} = 0, & t \in (0,T), \\ a(x,0) = a_0(x) > 0, & x \in \Omega, \end{cases} \tag{2.14}$$

with $h_7 := \big\{\gamma - r\chi(1-w)\big\}aw + (\chi w - l)a^2 e^{\chi w}$. For any $0 < T \le T_0$, Lemma 2.1 and Sobolev embedding (2.7) may lead us to that

$$\|w\|_{C^{1,1}(\bar{Q}_T)} \le c_4(R),$$

and for $\lambda = 1 - \frac{n+2}{p} \in (0,1)$ (i. e. $n+2 < p < +\infty$),

$$\|a\|_{C^{1+\lambda,\frac{1+\lambda}{2}}(\bar{Q}_T)} \le C(n,p,Q_T)\,\|a\|_{W_p^{2,1}(Q_T)}.$$

Therefore we have for each fixed $t \in [0,T]$ that the embedding holds:

$$w \in C^1(\bar{\Omega}) \hookrightarrow C^{\alpha}(\bar{\Omega}), \quad \alpha \in (0,1),$$

when Ω satisfies that any $x, y \in \Omega$ can be connected by finitely many line segments. We also have for any fixed $x \in \Omega$ that

$$w \in C^1([0,T]) \hookrightarrow C^{\frac{\alpha}{2}}([0,T]),$$

as $[0,T]$ is a convex domain. These combined with $w \in C^{1,1}(\bar{Q}_T)$ imply $w \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)$, by its definition. Then $h_7 \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)$ since the multiplication of any two elements in a Hölder space remains in this Hölder space and $e^z \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)$ if $z \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)$. Then to (2.14) which may be treated as a linear parabolic equation of a, an application of Schauder estimate gives that

$$\|a\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)} \le C\big\{\|h_7\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)} + \|a_0\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)}\big\}.$$

On the other hand, the w-equation in (2.3) is

$$\begin{cases} w_t = -awe^{\chi w} + rw(1-w), & \text{in } Q_T, \\ w(x,0) = w_0(x) > 0, & x \in \Omega. \end{cases} \tag{2.15}$$

Then we arrive at $w_t \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)$ and

$$(\partial_{x_i} w)_t = h_8\, \partial_{x_i} w + h_9, \quad i = 1, 2, \cdots, n,$$

which means

$$\partial_{x_i} w(x,t) = \partial_{x_i} w(x,0)\, e^{\int_0^t h_8(x,\tau)\,\mathrm{d}\tau} + \int_0^t h_9(x,s) e^{\int_s^t h_8(x,\tau)\,\mathrm{d}\tau}\,\mathrm{d}s \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)$$

where

$$h_8 := -\chi awe^{\chi w} - ae^{\chi w} + r - 2rw \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T), \qquad h_9 := -we^{\chi w}\partial_{x_i} a \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T).$$

So $w \in C^{1+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)$ and $h_8, h_9 \in C^{1+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)$, due to $a \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)$. Moreover, we have

$$\begin{aligned} \partial^2_{x_i x_j} w(x,t) =& \partial^2_{x_i x_j} w(x,0)\, e^{\int_0^t h_8(x,\tau)\,\mathrm{d}\tau} + e^{\int_0^t h_8(x,\tau)\,\mathrm{d}\tau} \partial_{x_i} w(x,0) \int_0^t \partial_{x_j} h_8(x,\tau)\,\mathrm{d}\tau \\ &+ \int_0^t \partial_{x_j} h_9(x,s) e^{\int_s^t h_8(x,\tau)\,\mathrm{d}\tau} + \int_0^t h_9(x,s) e^{\int_s^t h_8(x,\tau)\,\mathrm{d}\tau} \int_s^t \partial_{x_j} h_8(x,\tau)\,\mathrm{d}\tau\,\mathrm{d}s \end{aligned}$$

for $i, j = 1, 2, \cdots, n$, which means

$$w \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T),$$

in virtue of h_8, $\partial_{x_j} h_8$, h_9, $\partial_{x_j} h_9 \in C^{\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)$. This completes our proof. □

The following is the extendability condition on the classical solution of (2.3).

Lemma 2.3 (Extension condition)**.** *The local classical solution obtained in Lemma 2.1 and Lemma 2.2 exists in* $t \in (0,T)$ *for any given* $T > 0$*, provided that*

$$\|a\|_{L_\infty(Q_T)} + \|w\|_{L_\infty(Q_T)} \leq C(T) < +\infty. \tag{2.16}$$

To prove this criteria, we need some *a prior* estimates to be given in the following subsection.

2.2.2 Some *A Prior* Estimates

Throughout this subsection, we always assume that (2.3) has a classical solution

$$(a, w) \in C^{2,1}(\bar{Q}_T) \times C^{2,1}(\bar{Q}_T) \qquad \text{for} \quad T > T_0. \tag{2.17}$$

Moreover, considering the similar structure between the u-equation in (2.1) (i.e. (3.3)) and (3.2), the detailed derivations of the estimates needed in this subsection will be omitted sometimes if they are similar to the corresponding ones in Section 3.2.

Lemma 2.4. *Under the assumption* (2.17)*, we then derive*

$$\|a(\cdot,t)\|_{L_1(\Omega)} + \|w(\cdot,t)\|_{L_1(\Omega)} \leq C_0 \qquad \textit{for} \quad t \in (0,T),$$

where $C_0 := 2\gamma(r+1)|\Omega|K/\min\{1,\gamma\}$ *for* $\gamma > 0$*.*

Proof. We refer readers to the proof of Lemma 3.2, by $\|a(\cdot,t)\|_{L_1(\Omega)} \leq \|u(\cdot,t)\|_{L_1(\Omega)}$ due to $a = ue^{-\chi w}$ and $u, w \geq 0$. □

Lemma 2.5. *Under the assumption* (2.17)*, we may obtain that*

$$\|a(\cdot,t)\|_{L_\infty(\Omega)} \leq \tilde{C}, \qquad t \in (0,T)$$

where the constant $\tilde{C}$ *is independent of* T*.*

Proof. We start with the calculation of $\|a(\cdot,t)\|_{L_p(\Omega)}$ for $2 < p < +\infty$. Indeed, one may compute for $p \geq 2$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega e^{\chi w}a^p + r(p-1)\chi\int_\Omega e^{\chi w}a^p w + p(p-1)\int_\Omega e^{\chi w}a^{p-2}|\nabla a|^2$$

$$+ p\int_\Omega e^{\chi w}a^p + pl\int_\Omega e^{2\chi w}a^{p+1}$$

$$=(p-1)\chi\int_\Omega e^{2\chi w}a^{p+1}w + r(p-1)\chi\int_\Omega e^{\chi w}a^p w^2 + p\gamma\int_\Omega a^p e^{\chi w}w.$$

When $l \geq \chi K$, we see

$$pl\int_\Omega e^{2\chi w}a^{p+1} \geq (p-1)\chi\int_\Omega e^{2\chi w}a^{p+1}w,$$

which means

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega e^{\chi w}a^p + \int_\Omega e^{\chi w}a^p + \frac{p(p-1)}{2}\int_\Omega e^{\chi w}a^{p-2}|\nabla a|^2 \leq p\bar{c}_0\int_\Omega e^{\chi w}a^p \tag{2.18}$$

by $0 < w \leq K$ with $\bar{c}_0 := r\chi K^2 + \gamma K$.

When $l < \chi K$, we note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega e^{\chi w}a^p + p(p-1)\int_\Omega e^{\chi w}a^{p-2}|\nabla a|^2 \leq p\bar{c}_0\int_\Omega e^{\chi w}a^p + (p-1)\bar{c}_1\int_\Omega a^{p+1} \tag{2.19}$$

with $\bar{c}_1 := \chi K e^{2\chi K}$. Using generalized Gagaliardo-Nirenberg interpolation inequality ($n = 2$) [62, Lemma A.5] may lead us to

$$\int_\Omega a^{p+1} =\|a^{\frac{p}{2}}\|_{L_{\frac{2(p+1)}{p}}}^{\frac{2(p+1)}{p}} \leq \eta\|\nabla a^{\frac{p}{2}}\|_{L_2(\Omega)}^{\frac{2(p+1)}{p}-\frac{2}{p}}\|a^{\frac{p}{2}}\big(\ln a^{\frac{p}{2}}\big)^{\frac{p}{2}}\|_{L_{\frac{2}{p}}(\Omega)}^{\frac{2}{p}} + C\|a^{\frac{p}{2}}\|_{L_{\frac{2}{p}}(\Omega)}^{\frac{2(p+1)}{p}} + C_\eta$$

$$=\frac{\eta p\|a\ln a\|_{L_1(\Omega)}}{2}\|\nabla a^{\frac{p}{2}}\|_{L_2(\Omega)}^2 + C\|a\|_{L_1(\Omega)}^{p+1} + C_\eta$$

and thus

$$(p-1)\bar{c}_1\int_\Omega a^{p+1} \leq \frac{p(p-1)}{2}\int_\Omega a^{p-2}|\nabla a|^2 + \bar{c}_2 \leq \frac{p(p-1)}{2}\int_\Omega e^{\chi w}a^{p-2}|\nabla a|^2 + \bar{c}_2$$

where we take $\eta = \frac{4(p-1)\bar{c}_1}{p^2\|a\ln a\|_{L_1(\Omega)}}$ and set $\bar{c}_2 = (p-1)\bar{c}_1\big\{C\|a\|_{L_1(\Omega)}^{p+1} + C_\eta\big\}$.

Therefore, either in this case or in (2.18) we always have for $p \geq 2$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega e^{\chi w}a^p + \int_\Omega e^{\chi w}a^p + \frac{p(p-1)}{2}\int_\Omega e^{\chi w}a^{p-2}|\nabla a|^2 \leq p(\bar{c}_0+1)\int_\Omega e^{\chi w}a^p + \bar{c}_2.$$

Meanwhile, an application of Gagliardo-Nirenberg interpolation inequality ($n = 2$) and Young's inequality with a parameter $\eta > 0$ may yield that

$$\begin{aligned}\|a\|_{L_p(\Omega)}^p =& \|a^{\frac{p}{2}}\|_{L_2(\Omega)}^2 \leq C^2\big\{\|\nabla a^{\frac{p}{2}}\|_{L_2(\Omega)}^{2\theta}\|a^{\frac{p}{2}}\|_{L_{\frac{2}{p}}(\Omega)}^{2(1-\theta)} + \|a^{\frac{p}{2}}\|_{L_{\frac{2}{p}}(\Omega)}^2\big\}\\ \leq& C^2\Big\{\frac{p^2\eta}{4}\int_\Omega a^{p-2}|\nabla a|^2 + (1+\eta^{-\frac{\theta}{1-\theta}})\|a\|_{L_1(\Omega)}^p\Big\}\end{aligned}$$

with $\theta = \frac{p-1}{p}$, which means

$$\begin{aligned}p(\bar{c}_0+1)\int_\Omega e^{\chi w}a^p \leq& p(\bar{c}_0+1)e^{\chi K}C^2\Big\{\frac{p^2\eta}{4}\int_\Omega a^{p-2}|\nabla a|^2 + (1+\eta^{1-p})\|a\|_{L_1(\Omega)}^p\Big\}\\ \leq& \frac{p(p-1)}{4}\int_\Omega e^{\chi w}a^{p-2}|\nabla a|^2 + \bar{c}_3\end{aligned}$$

where we take $\eta = \frac{p-1}{p^2C^2e^{\chi K}(1+\bar{c}_0)}$ and $\bar{c}_3 = p(\bar{c}_0+1)e^{\chi K}C^2\big(1+\eta^{1-p}\big)\|a\|_{L_1(\Omega)}^p$. So we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega e^{\chi w}a^p + \int_\Omega e^{\chi w}a^p \leq \bar{c}_2 + \bar{c}_3$$

which implies

$$\int_\Omega a^p \leq \int_\Omega e^{\chi w}a^p \leq \bar{c}_2 + \bar{c}_3 + \int_\Omega e^{\chi w_0}{a_0}^p, \qquad 2 \leq p < +\infty. \tag{2.20}$$

In view of (2.19) and (2.18), one may obtain an inequality similar to (3.27) by using Gagliardo-Nirenberg interpolation inequality (cf. the part between (3.24) and (3.27)). Then the rest is to conduct Moser's iteration which proceeds as the proof of Lemma 3.8 (i.e., similar to the part after (3.27)). Together with (2.20), one may finally obtain that $\|a(\cdot,t)\|_{L_\infty(\Omega)}$ is upper bounded by some constant $\bar{C}$ which is independent of t and T. Thus we complete this proof. □

Lemma 2.6. *Under assumption* (2.17) *and* $\Omega \subset \mathrm{R}^2$*, we may derive that*

$$\int_\Omega (|\nabla a(\cdot,t)|^2 + |\Delta w(\cdot,t)|^2) \le C(T), \quad \textit{and} \quad \int_0^T \int_\Omega |\Delta a(\cdot,t)|^2 \le C(T)$$

for any $t \in (0,T)$.

Proof. We refer readers to a similar discussion made in the proof of Lemma 3.10, and note that we here have no boundary condition $\nabla w \cdot \vec{\nu}\big|_{\partial\Omega} = 0$, but an inequality similar to (3.34) still holds. □

Lemma 2.6 enables us to derive $\|\nabla w\|_{L_p(\Omega)}$ for $p > 2$ by using Gagliardo-Nirenberg interpolation inequality as

$$\|\nabla w\|_{L_p(\Omega)} \le C\big\{\|\Delta w\|_{L_2(\Omega)}^{\frac{p-2}{p}} \|w\|_{L_\infty(\Omega)}^{\frac{2}{p}} + \|w\|_{L_\infty(\Omega)}\big\}$$

Then, the same as proving Lemma 2.2, one may obtain the following estimate.

Lemma 2.7. *Under assumption* (2.17)*, the solution satisfies*

$$\|(a,w)\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)} \le C(T), \qquad \alpha \in (0,1),$$

for any finite $T > T_0$*, where the domain* $\Omega \subset \mathbb{R}^2$ *is bounded and fulfills that any* $x, y \in \Omega$ *can be connected by finitely many line segments, e.g.,* Ω *to be convex or* $\partial\Omega \in C^2$.

We are now in a position to prove Theorem 2.1 by verifying the extendability condition given in Lemma 2.3 which illustrates that the global-in-time existence of the unique classical solution to (2.3) and thus (2.1).

2.2.3 Proof of Theorem 2.1

By adopting the idea in [2, Theorem 5.3], we may prove the extendability given in Lemma 2.3 as below.

Proof. One may argue the Lemma 2.3 by a contradiction. Suppose that the unique local classical solution (a, w) of (2.3) exists only in $(0, \bar{T}]$ for some $0 < \bar{T} < +\infty$.

Then we may consider $(a, w)(x, \bar{T} - \tau)$ as a new initial value of (2.3) for any $0 < \tau < \bar{T}$. It follows from Lemma 2.1 and Lemma 2.2 that there exists a $0 < \bar{T}_0 < 1$ which depends only on the upper bound of $\|(a, w)(\cdot, \bar{T} - \tau)\|_{C^1(\bar{\Omega})}$, instead of τ, such that the system (2.3) has a unique local classical solution which exists in $(\bar{T} - \tau, \bar{T} - \tau + \bar{T}_0)$. Note that $(a, w)(x, \bar{T} - \tau)$ indeed can be treated as a " point " of $(a, w)(x, t)$ and thus Lemma 2.7 implies that the upper bound of $\|(a, w)(\cdot, \bar{T} - \tau)\|_{C^1(\bar{\Omega})}$ depends only upon $\bar{T}$. So $\bar{T}_0$ is dependent only on $\bar{T}$ (i.e., $\bar{T}_0 = \bar{T}_0(\bar{T})$) between τ and $\bar{T}$.

The above procedure remains true for any $0 < \tau < \bar{T}$. Thereby, one may infer that

$$\bar{T} - \tau + \bar{T}_0 > \bar{T}$$

as long as $\tau < \bar{T}_0$ which can be achieved since $\bar{T}_0$ does not depend on τ. The uniqueness of solution (a, w) to (2.3) means that the solution starting from $(a, w)(x, \bar{T} - \tau)$ and ending up with $(a, w)(x, \bar{T} - \tau + \bar{T}_0)$, is only a section of the solution (a, w) of (2.3), so

$$\bar{T} - \tau + \bar{T}_0 \leq \bar{T}.$$

This is a contradiction. □

Proof of Theorem 2.1: Lemma 2.1 and Lemma 2.2 show that the local-in-time unique classical solution of (2.3), thus of (2.1), exists in $\Omega \subset \mathbb{R}^n (n \geq 1)$. Lemma 2.3 shows that such a local solution exists globally in time for $n = 2$, provided that $l \geq \chi K$ or $\chi > 0$ is small as required in Lemma 2.5. Finally the $L_\infty(\Omega)$ estimate is a consequence of Lemma 2.1 and Lemma 2.5. □

Chapter 3

Vanishing Viscosity Limit on a Fully Parabolic System with Direct Preytaxis

The term *vanishing viscosity limit* orginates from *vanishing viscosity method.* This method may be traced back to M. G. Crandall and P.-L. Lions [63] in 1983 to obtain the *viscosity solutions* (Lipschitz continuous solutions, i.e., W^1_∞) of Dirichlet problem for Hamilton–Jacobi equation $\mathcal{F}(x, u, \nabla u) = 0$ during dealing with the uniqueness of its solution. For clarity in our case, here we review and slightly extend the basic idea of *vanishing viscosity method* as follow: There is a nonlinear parabolic PDE

$$\frac{\partial u}{\partial t} = \mathcal{F}(x, t, u, \nabla u), \qquad x \in \Omega \subset \mathbb{R}^n (n \geq 1), \qquad t > 0, \tag{3.1}$$

subjecting to some suitable initial and boundary conditions, where real-valued function $\mathcal{F}$ is continuous but may not be linear in all its arguments, and ∇u represents the gradient of unknown function $u = u(x, t)$. To qualitatively find the solution of (3.1) in a suitable Sobolev space, one may approximate by solutions $\{u_\varepsilon\}_{\varepsilon \in \mathbb{R}}$ of the following PDE

$$\frac{\partial u}{\partial t} = \varepsilon \Delta u + \mathcal{F}(x, t, u, Du), \qquad x \in \Omega \subset \mathbb{R}^n (n \geq 1), \quad t > 0, \quad \varepsilon \in \mathbb{R}$$

and take the limit of $\{u_\varepsilon\}_{\varepsilon\in\mathbb{R}}$ in a sense, as $\varepsilon \to 0$. The essential part lies in that some estimates for $\{u_\varepsilon\}_{\varepsilon\in\mathbb{R}}$ which do not collapse and blow up as $\varepsilon \to 0$, will allow to pass the limit

$$u_\varepsilon \to u, \qquad \text{in some sense.}$$

Then one may solve the original problem (3.1) if the limit u itself satisfies the equation, required regularity, and the boundary condition (if appropriate).

3.1 Models and Main Results

As stated in subsection 1.3, we shall in this chapter to consider that the strong solution of

$$\begin{cases} u_t = \Delta u - \nabla\cdot(\chi u\nabla w) + \gamma uw - u(1+lu), & \text{in } Q,\\ w_t = \varepsilon\Delta w - uw + rw(1-w), & \text{in } Q,\\ (\nabla u - \chi u\nabla w)\cdot\vec{\nu}\big|_{\partial\Omega} = 0, \quad \varepsilon\nabla w\cdot\vec{\nu}\big|_{\partial\Omega} = 0, & t>0,\\ u(x,0) = u_0(x), \quad w(x,0) = w_0(x), & x\in\Omega, \end{cases} \tag{3.2}$$

will converge, as $\varepsilon \to 0$, to the strong solution of following parabolic-ODE system

$$\begin{cases} u_t = \Delta u - \nabla\cdot(\chi u\nabla w) + \gamma uw - u(1+lu), & \text{in } Q,\\ w_t = -uw + rw(1-w), & \text{in } Q,\\ (\nabla u - \chi u\nabla w)\cdot\vec{\nu}\big|_{\partial\Omega} = 0, & t>0,\\ u(x,0) = u_0(x), \quad w(x,0) = w_0(x), & x\in\Omega, \end{cases} \tag{3.3}$$

where $Q := \Omega\times(0,+\infty)$, the open bounded domain $\Omega\subset\mathbb{R}^2$ with boundary $\partial\Omega$, $\varepsilon,\chi,\gamma,r>0$ and $l\geq 0$. Throughout this chapter, our hypotheses are

$$\begin{cases} u_0(x),\ w_0(x)\in C^{2+\beta}(\overline{\Omega}) \quad \text{for some } \beta\in(0,1), \quad \partial\Omega\in C^\infty,\\ u_0(x)\geq(\not\equiv)0, \quad w_0(x)>0 \quad \text{for all} \quad x\in\Omega,\\ (\nabla u_0 - \chi u_0\nabla w_0)\cdot\vec{\nu}\big|_{\partial\Omega} = 0 \ \ (\text{and } \nabla w_0\cdot\vec{\nu}\big|_{\partial\Omega} = 0,\ \varepsilon\in(0,1) \text{ in (3.2)}). \end{cases} \tag{3.4}$$

For this problem our main results read as below:

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open domain and* (3.4) *hold. For an arbitrarily given $T \in (0, +\infty)$, one may derive:*

(a) For any $\varepsilon > 0$ and $p \in (2, +\infty)$, both the system (3.2) *and* (3.3) *have a unique strong solution denoted by $(u_\varepsilon, w_\varepsilon)$ and (u, w), respectively, which satisfy*

$$(u_\varepsilon, w_\varepsilon)(x,t) \quad \textit{and} \quad (u,w)(x,t) \in \Big(W_p^{2,1}(\Omega \times (0,T))\Big)^2.$$

(b) For $2 < p, q < \infty$, the strong solutions have following convergence relation:

$$u_\varepsilon \to u \quad \textit{in} \quad L_p(0,T; W_q^1(\Omega)) \quad \textit{and} \quad w_\varepsilon \to w \quad \textit{in} \quad L_\infty(0,T; W_q^1(\Omega)).$$

(c) System (3.3) *has a unique classical solution (u, w) fulfilling*

$$(u,w)(x,t) \in \Big(C\big(\overline{\Omega} \times [0,T]\big) \cap C^{2,1}\big(\overline{\Omega} \times [0,T]\big)\Big)^2.$$

We remark that compared with the local weak solution derived in Theorem 1.1, 1.3, and 1.4 of [52], roughly speaking, our Theorem 2.1 and 3.1 can be viewed partially as counterparts in the framework of classical solutions for $\Omega \subset \mathbb{R}^2$. In contrast with [64], we remove the initial boundary condition $\nabla w_0 \cdot \vec{\nu}\big|_{\partial\Omega} = 0$ assumed in the system (3.3), which seems more appropriate for the w-equation as an ODE.

3.2 Global-in-Time Existence of the Classical and Strong Solution to (3.2)

For $\Omega \subset \mathbb{R}^2$ the global-in-time existence of the unique classical solution to system (3.2) given in [40, Theorem 1.1], is obtained actually under zero-Neumann boundary condition, $\Omega \subset \mathbb{R}^2$ being convex, $\varepsilon = 1$, and $l = 0$. This is not sufficient to clarify the dependence on ε in the course of its proof. In addition, the ideas that appeared in [40, 32, 37, 23] in obtaining $\|u\|_{L_p(\Omega)} (2 \le p < +\infty)$, may not be directly applicable in our case, since the relevant estimates may either tend to infinity or give rise to $\chi \to 0$, while $\varepsilon \to 0$. More importantly, these connected estimates in particular when $\varepsilon \to 0$, will play a pivotal role in the proof of Theorem 3.1 in section 3.3. Thus we shall in this section derive the necessary estimates by proving again the following proposition. Based on this, we shall explore that the u-component in (3.2) satisfies $\|u(\cdot,t)\|_{L_\infty(\Omega)} \leq C(T) < +\infty$ as $\varepsilon \to 0$, that is, Lemma 3.9 in section 3.2.3.

Proposition 3.1. *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded open domain and* (3.4) *holds. Then for any given $T, \varepsilon > 0$, system* (3.2) *possesses a global-in-time unique solution*

$$(u,w)(x,t) \in \Big(C\big(\overline{\Omega} \times [0,T]\big) \cap C^{2,1}\big(\overline{\Omega} \times (0,T)\big)\Big)^2.$$

3.2.1 Local Existence

In regard to zero-flux boundary condition, the Amann [65, 66] may yield the following local existence and uniqueness. Alternatively, this can be proved by jointly using Banach's fixed-point theory, semigroup techniques and L_p theory and Schauder's theory of linear parabolic equations. We refer readers to an analogous proof given in [67, Theorem 3.1] and omit its details here for brevity.

Lemma 3.1 (Local existence and uniqueness)**.** *Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain with smooth boundary $\partial\Omega$. If $u_0, w_0 \in C^2(\overline{\Omega})$, then there exists $T_{\max} \in (0,+\infty]$ depending on u_0 and w_0 such that the problem* (3.2) *for each $\varepsilon > 0$ has a unique classical solution on $[0, T_{\max})$ which fulfills*

$$(u,w)(x,t) \in C\big(\overline{\Omega} \times [0,T_{\max})\big) \cap C^{2,1}\big(\overline{\Omega} \times (0,T_{\max})\big). \tag{3.5}$$

If (u_0, w_0) satisfies 0-order compatibility, i.e., $(\nabla u_0 - \chi u_0 \nabla w_0)\cdot \vec{\nu}\big|_{\partial\Omega} = 0$ and $\nabla w_0 \cdot \vec{\nu}\big|_{\partial\Omega} = 0$, then the local solution (u,w) is global-in-time, provided that

$$\lim_{t \nearrow T_{\max}} \big\{ \|u(\cdot,t)\|_{L_\infty(\Omega)} + \|w(\cdot,t)\|_{L_\infty(\Omega)} \big\} < +\infty.$$

Remark 3.1. *For (u, w) obtained in Lemma 3.1, if $u_0, w_0 \geq 0 (\not\equiv 0)$ additionally, then $u(x,t), w(x,t) \geq 0$ for $(x,t) \in \Omega \times [0, T_{\max})$. Furthermore,*

$$u(x,t) > 0, \quad 0 < w(x,t) < \max\{1, \|w_0\|_{L_\infty(\Omega)}\} =: K, \qquad (x,t) \in \Omega \times (0, T_{\max})$$

for each $\varepsilon > 0$.

Indeed, one may infer the positivity of u by applying on any $[0,T] \subset [0, T_{\max})$ the comparison principle of linear parabolic equations to

$$\begin{cases} a_t = \Delta a + \chi \nabla w \cdot \nabla a + a\{\gamma w - \chi w_t - (1 + lae^{\chi w})\}, & \text{in } Q_T, \\ \nabla a \cdot \vec{\nu}\big|_{\partial\Omega} = 0, & \text{in } (0, T], \\ a(x,0) = u_0(x) e^{-\chi w_0(x)} \geq 0 (\not\equiv 0) & \text{in } \Omega, \end{cases}$$

since $w, \nabla w, w_t$, and a here can be treated as known functions with the regularity (3.5). A similar discussion will lead to the positivity of w.

Moreover, one may derive the upper bound of w by applying comparison principle of a single linear parabolic equation to

$$\begin{cases} w_t - \varepsilon \Delta w = -uw + rw(1-w) \leq rw(1-w), & \text{in } \Omega \times (0, T_{\max}) \\ \nabla w \cdot \vec{\nu}\big|_{\partial\Omega} = 0, & \text{in } (0, T_{\max}), \\ w(x,0) = w_0(x), & \text{in } \Omega, \end{cases}$$

since the system composed by the rightmost growth term and coupled with the same initial boundary-value condition, has a upper solution solving the following ODE of Bernoulli type

$$\begin{cases} \frac{\mathrm{d}\bar{w}}{\mathrm{d}t} = r\bar{w}(1-\bar{w}), & t > 0; \\ \bar{w}(0) = \|w_0\|_{L_\infty(\Omega)} > 0, \end{cases}$$

where

$$\bar{w} := \bar{w}(t) = \frac{1}{(\bar{w}_0^{-1}(x) - 1)e^{-rt} + 1} \leq \max\left\{1, \|w_0\|_{L_\infty(\Omega)}\right\}.$$

3.2.2 L_∞ Estimate of Solution to (3.2)

Due to Remark 3.1 and the extendability condition given in Lemma 3.1, it suffices to obtain the L_∞ estimate of u. We start with the following $L_1(\Omega)$ estimate.

Lemma 3.2. *Assume that* (u, w) *is the classical solution to* (3.2)*. Then one may acquire that*

$$0 < \|u(\cdot,t)\|_{L_1(\Omega)} + \|w(\cdot,t)\|_{L_1(\Omega)} \le C_0, \qquad t \in (0, T_{\max})$$

with $C_0 := 2\big\{1 + \gamma(r+1)|\Omega|K + \|u_0\|_{L_1(\Omega)} + \gamma\|w_0\|_{L_1(\Omega)}\big\}/\min\{1,\gamma\}$ *for* $\gamma > 0$.

Proof. For any $0 < T < T_{\max}$ we may compute that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+\gamma w) + \int_\Omega (u+\gamma w) =& \int_\Omega \big\{\gamma r w(1-w) - u(1+lu)\big\} + \int_\Omega (u+\gamma w)\\ \le& \gamma(r+1)\int_\Omega w\end{aligned}$$

according to (3.2) and $u, w \ge 0$. Then from $0 < w < K$ in Remark 3.1 it follows that

$$y'(t) + y(t) \le \gamma(r+1)|\Omega|K \quad \text{with} \quad y(t) := \int_\Omega \{u(x,t) + \gamma w(x,t)\}\,\mathrm{d}x$$

which concludes this proof by solving the above differential inequality in $t \in (0, T)$. □

The next is to find some information on u, like $\int_\Omega u^\mu$ ($\mu \ge 2$) for our purpose later. Instead of directly calculating $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2$ that will yield a right-hand integral $\int_\Omega u\nabla u\cdot\nabla w$ to control with difficulty at present, we thereby shall compute $\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\ln u$ where $\int_\Omega \nabla u\cdot\nabla w$ appears but may be cancelled during computing $\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{|\nabla w|^2}{w}$, which is the purpose of the following two lemmas.

Lemma 3.3. *Assume that* (u, w) *is the classical solution of* (3.2)*. Then we have for*

$l > 0$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\ln u + \int_\Omega u\ln u + \int_\Omega \frac{|\nabla u|^2}{u} + \frac{l}{2}\int_\Omega u^2\ln(u+1) \le \chi\int_\Omega \nabla u\cdot\nabla w + C_1,$$

and for $l = 0$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\ln u + \int_\Omega u\ln u + \int_\Omega \frac{|\nabla u|^2}{u} \le \chi\int_\Omega \nabla u\cdot\nabla w + \gamma K\int_\Omega u\ln u + C_1,$$

where the constant C_1 depends only upon $\gamma, l, K,$ and C_0 from Lemma 3.2.

Proof. Considering the positivity stated in Remark 3.1, one may multiply the u-equation of (3.2) by $(1+\ln u)$ and integrate the resulting equation with respect to $x\in\Omega$, to obtain that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\ln u + \int_\Omega u\ln u + \int_\Omega \frac{|\nabla u|^2}{u}\\ =&\chi\int_\Omega \nabla u\cdot\nabla w + \gamma\int_\Omega uw(1+\ln u) - \int_\Omega (u + lu^2 + lu^2\ln u),\end{aligned}$$

which completes the proof of the case $l = 0$ by $0 < w < K$. When $l > 0$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\ln u + \int_\Omega u\ln u + \int_\Omega \frac{|\nabla u|^2}{u}\\ \le&\chi\int_\Omega \nabla u\cdot\nabla w + \gamma K\int_\Omega (u + u\ln u) - \int_\Omega (u + lu^2 + lu^2\ln u)\\ \le&\chi\int_\Omega \nabla u\cdot\nabla w + \int_\Omega \{\gamma K u\ln u - lu^2 - lu^2\ln u\} + \gamma K\int_\Omega u\\ \le&\chi\int_\Omega \nabla u\cdot\nabla w - \frac{l}{2}\int_\Omega u^2\ln(u+1) + \hat{C} + \gamma K\|u\|_{L_1(\Omega)}.\end{aligned}$$

Note that the last inequality above is obtained by [62, Lemma 3.1] when $l > 0$. More precisely, a continuous function $\varphi : [0,+\infty) \to \mathbb{R}$, is defined by

$$\varphi(z) := \begin{cases} az^2 + bz\ln z - cz^2\ln z + \lambda z^2\ln(z+1), & z > 0;\\ 0, & z = 0,\end{cases} \tag{3.6}$$

with $c>0, \lambda<c$ and $a,b\in\mathbb{R}$, and then it satisfies

$$\frac{\varphi(z)}{z^2\ln z}\longrightarrow\ \lambda-c<0,\quad \text{as}\quad z\to+\infty.$$

So there exists a $z_0\in(0,+\infty)$ making $\varphi(z)\le 0$ on $(z_0,+\infty)$. Then it follows that as $z\ge 0$,

$$az^2+bz\ln z-cz^2\ln z\le -\lambda^2z^2\ln(z+1)+\hat{C},\quad \text{with}\quad \varphi(z)\le\max_{[0,z_0]}|\varphi(z)|=:\hat{C}.$$

This completes the proof. □

To counteract the $\chi\int_\Omega\nabla u\cdot\nabla w$, we give the following integral inequality.

Lemma 3.4. *Assume (u,w) is the classical solution to (3.2) in Q_T. One may acquire that*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{|\nabla w|^2}{w}+3r\int_\Omega|\nabla w|^2+\frac{\varepsilon}{2}\int_\Omega w|D^2\ln w|^2\\ \le&\frac{\|w(\cdot,t)\|_{L_1(\Omega)}}{2\delta_0}+\int_\Omega\frac{|\nabla w|^2}{w}(r-u)-2\int_\Omega\nabla u\cdot\nabla w\end{aligned}$$

for any constant $\delta_0>0$ to be small enough and independent of ε.

Proof. Observe that

$$\frac{\partial}{\partial t}\left(\frac{|\nabla w|^2}{w}\right)=-\frac{|\nabla w|^2}{w^2}w_t+\frac{2\nabla w\cdot\nabla w_t}{w},$$

and $\int_\Omega\frac{2\nabla w\cdot\nabla w_t}{w}=2\int_\Omega\nabla\ln w\cdot\nabla w_t=-2\int_\Omega w_t\Delta\ln w=-2\int_\Omega\left(\frac{\Delta w}{w}-\frac{|\nabla w|^2}{w^2}\right)w_t$. Together with the w-equation of (3.2) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{|\nabla w|^2}{w}=-\int_\Omega\frac{|\nabla w|^2}{w^2}w_t-2\int_\Omega\left(\frac{\Delta w}{w}-\frac{|\nabla w|^2}{w^2}\right)w_t$$

$$=\int_\Omega\frac{|\nabla w|^2}{w^2}w_t-2\int_\Omega\frac{\Delta w}{w}w_t$$

$$
\begin{aligned}
&= \int_\Omega \frac{|\nabla w|^2}{w^2}\{\varepsilon\Delta w - uw + rw(1-w)\} - 2\int_\Omega \frac{\Delta w}{w}\{\varepsilon\Delta w - uw + rw(1-w)\} \\
&= \varepsilon\int_\Omega \frac{|\nabla w|^2}{w^2}\Delta w + \int_\Omega \frac{|\nabla w|^2}{w}(r-u) - r\int_\Omega |\nabla w|^2 - 2\varepsilon\int_\Omega \frac{|\Delta w|^2}{w} \\
&\quad - 2\int_\Omega (r-u)\Delta w + 2r\int_\Omega w\Delta w \\
&= \varepsilon\int_\Omega \frac{|\nabla w|^2}{w^2}\Delta w - 2\varepsilon\int_\Omega \frac{|\Delta w|^2}{w} + \int_\Omega \frac{|\nabla w|^2}{w}(r-u) - 3r\int_\Omega |\nabla w|^2 - 2\int_\Omega \nabla u\cdot\nabla w.
\end{aligned}
$$

Concerning the first two terms in the last equality, one may acquire in light of Lemma A.3 that

$$
-\int_\Omega \frac{|\nabla w|^2}{w^2}\Delta w = -\frac{2}{3}\int_\Omega \frac{|\Delta w|^2}{w} + \frac{2}{3}\int_\Omega \frac{|D^2 w|^2}{w} - \frac{2}{3}\int_\Omega \frac{|\nabla w|^4}{w^3} - \frac{1}{3}\int_{\partial\Omega} \frac{1}{w}\frac{\partial|\nabla w|^2}{\partial\vec{\nu}}
$$

which gives

$$
\begin{aligned}
&\varepsilon\int_\Omega \frac{|\nabla w|^2}{w^2}\Delta w - 2\varepsilon\int_\Omega \frac{|\Delta w|^2}{w} \\
=& -\frac{4\varepsilon}{3}\int_\Omega \frac{|\Delta w|^2}{w} - \frac{2\varepsilon}{3}\int_\Omega \frac{|D^2 w|^2}{w} + \frac{2\varepsilon}{3}\int_\Omega \frac{|\nabla w|^4}{w^3} + \frac{\varepsilon}{3}\int_{\partial\Omega} \frac{1}{w}\frac{\partial|\nabla w|^2}{\partial\vec{\nu}}.
\end{aligned}
$$

Here $\int_\Omega \frac{|\nabla w|^4}{w^3}$ can be bounded by $\int_\Omega w|D^2\ln w|^2$ due to Lemma A.3, but a further control of $\int_\Omega w|D^2\ln w|^2$ needs to relate $\int_\Omega w|D^2\ln w|^2$ to the rest terms in the above equality. That is,

$$
\begin{aligned}
|D^2\ln w|^2 =& \frac{1}{w^2}\sum_{i,j=1}^n (w_{x_ix_j})^2 + \frac{1}{w^4}\sum_{i=1}^n (w_{x_i})^2\sum_{j=1}^n (w_{x_j})^2 - \frac{2}{w^3}\sum_{i,j=1}^n w_{x_ix_j}w_{x_i}w_{x_j} \\
=& \frac{|D^2 w|^2}{w^2} + \frac{|\nabla w|^4}{w^4} - \frac{2\nabla w\cdot(D^2 w\cdot\nabla w)}{w^3} \\
=& \frac{|D^2 w|^2}{w^2} + \frac{|\nabla w|^4}{w^4} - \frac{\nabla w\cdot\nabla(|\nabla w|^2)}{w^3},
\end{aligned}
$$

and then we may show through integration by parts that

$$\int_\Omega w|D^2 \ln w|^2 = \int_\Omega \frac{|D^2 w|^2}{w} + \int_\Omega \frac{|\nabla w|^4}{w^3} + \int_\Omega \nabla\Big(\frac{1}{w}\Big) \cdot \nabla(|\nabla w|^2)$$

$$= \int_\Omega \frac{|D^2 w|^2}{w} + \int_\Omega \frac{|\nabla w|^4}{w^3} + \int_\Omega \frac{|\nabla w|^2}{w^2}\Delta w - 2\int_\Omega \frac{|\nabla w|^4}{w^3}$$

$$= \int_\Omega \frac{|D^2 w|^2}{w} + \int_\Omega \frac{|\nabla w|^2}{w^2}\Delta w - \int_\Omega \frac{|\nabla w|^4}{w^3}.$$

This gives

$$\varepsilon \int_\Omega \frac{|\nabla w|^2}{w^2}\Delta w - 2\varepsilon \int_\Omega \frac{|\Delta w|^2}{w}$$

$$= -\frac{4\varepsilon}{3}\int_\Omega \frac{|\Delta w|^2}{w} - \frac{2\varepsilon}{3}\int_\Omega w|D^2 \ln w|^2 + \frac{2\varepsilon}{3}\int_\Omega \frac{|\nabla w|^2}{w^2}\Delta w + \frac{\varepsilon}{3}\int_{\partial\Omega} \frac{1}{w}\frac{\partial |\nabla w|^2}{\partial \vec{\nu}}$$

$$= \frac{2}{3}\Big(\varepsilon \int_\Omega \frac{|\nabla w|^2}{w^2}\Delta w - 2\varepsilon \int_\Omega \frac{|\Delta w|^2}{w}\Big) - \frac{2\varepsilon}{3}\int_\Omega w|D^2 \ln w|^2 + \frac{\varepsilon}{3}\int_{\partial\Omega} \frac{1}{w}\frac{\partial |\nabla w|^2}{\partial \vec{\nu}}$$

which indicates

$$\varepsilon \int_\Omega \frac{|\nabla w|^2}{w^2}\Delta w - 2\varepsilon \int_\Omega \frac{|\Delta w|^2}{w} = -2\varepsilon \int_\Omega w|D^2 \ln w|^2 + \varepsilon \int_{\partial\Omega} \frac{1}{w}\frac{\partial |\nabla w|^2}{\partial \vec{\nu}}.$$

Substituting this equality into $\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{|\nabla w|^2}{w}$ yields immediately that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{|\nabla w|^2}{w}$$

$$= \varepsilon \int_\Omega \frac{|\nabla w|^2}{w^2}\Delta w - 2\varepsilon \int_\Omega \frac{|\Delta w|^2}{w} + \int_\Omega \frac{|\nabla w|^2}{w}(r-u) - 3r\int_\Omega |\nabla w|^2 - 2\int_\Omega \nabla u \cdot \nabla w$$

$$= -2\varepsilon \int_\Omega w|D^2 \ln w|^2 + \varepsilon \int_{\partial\Omega} \frac{1}{w}\frac{\partial |\nabla w|^2}{\partial \vec{\nu}} + \int_\Omega \frac{|\nabla w|^2}{w}(r-u)$$

$$- 3r\int_\Omega |\nabla w|^2 - 2\int_\Omega \nabla u \cdot \nabla w.$$

We may finally combine $\int_{\partial\Omega} \frac{1}{w}\frac{\partial |\nabla w|^2}{\partial \vec{\nu}}$ with $\int_\Omega w|D^2 \ln w|^2$. Applying Lemma A.4, Lemma 2.2 of [68] (or cf. (3.13)) and Young's inequality with small constant $\delta > 0$, one has

$$\varepsilon \int_{\partial\Omega} \frac{1}{w} \frac{\partial |\nabla w|^2}{\partial \vec{\nu}} \leq 2\kappa\varepsilon \int_{\partial\Omega} \frac{1}{w} |\nabla w|^2 = 2\kappa\varepsilon \int_{\partial\Omega} |w^{\frac{1}{2}} \nabla \ln w|^2$$

$$\leq 2\kappa\varepsilon\bar{C}\Big(\|D(w^{\frac{1}{2}} \nabla \ln w)\|_{L_2(\Omega)} + \|w^{\frac{1}{2}} \nabla \ln w\|_{L_2(\Omega)} \Big) \cdot \|w^{\frac{1}{2}} \nabla \ln w\|_{L_2(\Omega)}$$

$$\leq 2\kappa\varepsilon\bar{C}\left[\frac{\delta}{2}\Big(\|D(w^{\frac{1}{2}} \nabla \ln w)\|_{L_2(\Omega)} + \|w^{\frac{1}{2}} \nabla \ln w\|_{L_2(\Omega)} \Big)^2 + \frac{1}{2\delta} \|w^{\frac{1}{2}} \nabla \ln w\|^2_{L_2(\Omega)} \right]$$

$$\leq 2\kappa\varepsilon\bar{C}\Big(\delta \int_\Omega |D(w^{\frac{1}{2}} \nabla \ln w)|^2 + \delta \int_\Omega |w^{\frac{1}{2}} \nabla \ln w|^2 + \frac{1}{2\delta} \int_\Omega |w^{\frac{1}{2}} \nabla \ln w|^2 \Big)$$

$$= 2\kappa\varepsilon\bar{C}\delta \int_\Omega |D(w^{\frac{1}{2}} \nabla \ln w)|^2 + 2\kappa\varepsilon\bar{C}(\delta + \frac{1}{2\delta}) \int_\Omega |w^{\frac{1}{2}} \nabla \ln w|^2$$

$$= 2\kappa\varepsilon\bar{C}\delta \int_\Omega |\frac{1}{2} w^{-\frac{1}{2}} \nabla w \cdot \nabla \ln w + w^{\frac{1}{2}} D(\nabla \ln w)|^2 + 2\kappa\varepsilon\bar{C}(\delta + \frac{1}{2\delta}) \int_\Omega |w^{\frac{1}{2}} \nabla \ln w|^2$$

$$\leq 4\kappa\varepsilon\bar{C}\delta \int_\Omega \Big(\frac{1}{4} \frac{|\nabla w|^4}{w^3} + |w^{\frac{1}{2}} D(\nabla \ln w)|^2 \Big) + 2\kappa\varepsilon\bar{C}(\delta + \frac{1}{2\delta}) \int_\Omega |w^{\frac{1}{2}} \nabla \ln w|^2$$

$$\leq \kappa\varepsilon\bar{C}\delta \int_\Omega \frac{|\nabla w|^4}{w^3} + 4\kappa\varepsilon\bar{C}\delta \int_\Omega w|D^2 \ln w|^2 + 2\kappa\varepsilon\bar{C}(\delta + \frac{1}{2\delta}) \int_\Omega \frac{|\nabla w|^2}{w}$$

$$\leq \kappa\varepsilon\bar{C}\delta[(2+\sqrt{n})^2 + 4] \int_\Omega w|D^2 \ln w|^2 + 2\kappa\varepsilon\bar{C}(\delta + \frac{1}{2\delta}) \int_\Omega \frac{|\nabla w|^2}{w}$$

where $\bar{C}$ is a constant independent of ε and the last inequality is obtained by Lemma A.3. Moreover, again using Young's inequality and Lemma A.3, there exists a $\delta_0 > 0$ such that

$$\int_\Omega \frac{|\nabla w|^2}{w} = \int_\Omega \frac{|\nabla w|^2}{w^{\frac{3}{2}}} w^{\frac{1}{2}} \leq \frac{\delta_0}{2} \int_\Omega \frac{|\nabla w|^4}{w^3} + \frac{1}{2\delta_0} \int_\Omega w$$

$$\leq \frac{\delta_0}{2}(2+\sqrt{n})^2 \int_\Omega w|D^2 \ln w|^2 + \frac{\|w\|_{L_1(\Omega)}}{2\delta_0}.$$

Then one may complete this proof by taking the $\delta > 0$ to be small enough such that

$$\kappa\bar{C}\delta[(2+\sqrt{n})^2 + 4] \leq 1,$$

fixing such a δ, and then by taking above $\delta_0 > 0$ to be small which fulfills

$$\frac{\delta_0}{2}(2+\sqrt{n})^2 \cdot 2\kappa \bar{C}(\delta + \frac{1}{2\delta}) \leq \frac{1}{2}.$$

□

A combination of Lemma 3.3 and Lemma 3.4 yields the following results.

Corollary 3.1. *If the estimates in Lemma 3.3 and Lemma 3.4 hold, then we may acquire that*

$$\int_\Omega u(\cdot,t)\ln u(\cdot,t), \quad \frac{\chi}{2}\int_\Omega \frac{|\nabla w(\cdot,t)|^2}{w(\cdot,t)} \leq c_1 e^{rt}, \qquad t \in (0,T), \tag{3.7}$$

$$\int_0^T \int_\Omega \Big(\frac{|\nabla u|^2}{u} + \frac{\chi\varepsilon}{4} w|D^2 \ln w|^2\Big) \leq (c_2 T + c_1)e^{rT}, \tag{3.8}$$

for any $n \geq 1$, *where we set* $c_0 := \frac{\chi C_0}{4\delta_0} + C_1 + r|\Omega|$, $c_1 := 1 + \frac{c_0}{r} + \int_\Omega \big(u_0 \ln u_0 + \frac{\chi}{2}\frac{|\nabla w_0|^2}{w_0}\big)$, *and* $c_2 := c_0 + rc_1$ *(when* $l = 0$ *we replace all* r *by* $r + \gamma K$*). Moreover, we have that*

$$\int_0^T \int_\Omega u^2 \leq \big(c_3 T + (2C)^4 C_0^2 (c_1 + 2)\big)e^{rT}, \tag{3.9}$$

where we set $c_3 := c_2 (2C)^4 C_0^2$ *when* $n = 1, 2$, $l \geq 0$ *or* $c_3 := \gamma K C_0 / l$ *when* $n \geq 1$, $l > 0$*. Here the constant* C *is from Gagliardo-Nirenberg interpolation inequality.*

Proof. When $l > 0$, Lemma 3.3 and Lemma 3.4 jointly indicate that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\ln u + \int_\Omega u \ln u + \int_\Omega \frac{|\nabla u|^2}{u} + \frac{l}{2}\int_\Omega u^2 \ln(u+1) \\
&+ \frac{\chi}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{|\nabla w|^2}{w} + \frac{3r\chi}{2}\int_\Omega |\nabla w|^2 + \frac{\chi\varepsilon}{4}\int_\Omega w|D^2 \ln w|^2 \\
\leq& \frac{\chi}{2}\int_\Omega \frac{|\nabla w|^2}{w}(r-u) + C_2,
\end{aligned} \tag{3.10}$$

where $C_2 := \frac{\chi \|w(\cdot,t)\|_{L_1(\Omega)}}{4\delta_0} + C_1 \leq \frac{\chi C_0}{4\delta_0} + C_1$ by Lemma 3.2. Then it follows from

$z\ln z > -\frac{1}{e}$ for $z \in \mathbb{R}_+$ and positivity of (u,w) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\Big(u\ln u + \frac{\chi}{2}\frac{|\nabla w|^2}{w}\Big) \leq & r\int_\Omega\Big(u\ln u + \frac{\chi}{2}\frac{|\nabla w|^2}{w}\Big) - r\int_\Omega u\ln u + C_2 \\ \leq & r\int_\Omega\Big(u\ln u + \frac{\chi}{2}\frac{|\nabla w|^2}{w}\Big) + c_0,\end{aligned}$$

which implies (3.7). In conjunction with (3.7), the inequality (3.10) also indicates by $z\ln z > -e^{-1}(z \in \mathbb{R}_+)$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\Big(u\ln u + \frac{\chi}{2}\frac{|\nabla w|^2}{w}\Big) + \int_\Omega\frac{|\nabla u|^2}{u} + \frac{\chi\varepsilon}{4}\int_\Omega w|D^2\ln w|^2 \leq rc_1e^{rT} + c_0.$$

Integrating it from both sides with respect to t in $(0,T)$ may conclude (3.8). When $l=0$, in a similar way it suffices to replace r in (3.7) and (3.8) by $r+\gamma K$.

Finally, an application of Ggaliardo-Nirenberg interpolation gives that

$$\begin{aligned}\|u\|^2_{L_2(\Omega)} = \|\sqrt{u}\|^4_{L_4(\Omega)} \leq & (2C)^4\big(\|\nabla\sqrt{u}\|^{4\theta}_{L_2(\Omega)}\|\sqrt{u}\|^{4(1-\theta)}_{L_2(\Omega)} + \|\sqrt{u}\|^4_{L_2(\Omega)}\big) \\ \leq & (2C)^4C_0^4\big\{\|\nabla\sqrt{u}\|^n_{L_2(\Omega)} + 1\big\}\end{aligned}$$

with $\theta = \frac{n}{4} \in (0,1)$ as $n=1,2$. Then using (3.8) as well as Lemma 3.2 may yield (3.9) when $n=1,2$ and $l=0$. This is also true if $l>0$. Alternatively, if $l>0$ one may see for any $n \geq 1$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u + \int_\Omega u(1+lu) = \int_\Omega \gamma uw \leq \gamma KC_0.$$

Integrating this inequality with respect to $t \in (0,T)$ may give (3.9) as well. $\square$

Note that below we shall no longer distinguish between the constant C of Gagliardo-Nirenberg interpolation inequality and its certain powers or multiples, for simplicity. We are in a position to estimate $\int_\Omega|\nabla w|^p (p \geq 2)$ as usual, and will display the dependence on ε during the derivations. Directly one may estimate $\int_\Omega|\nabla w|^2$ in order to obtain $\int_0^T\int_\Omega|\Delta w|^2$.

Lemma 3.5. *Suppose $\Omega \subset \mathbb{R}^n (n \geq 1)$. Let (u, w) be the local classical solution of (3.2) in $(0, T) \subset (0, T_{\max})$. Then one may have*

$$\int_\Omega |\nabla w(\cdot, t)|^2 + \varepsilon \int_0^T \int_\Omega |\Delta w(\cdot, t)|^2 \leq \|\nabla w_0\|^2_{L_2(\Omega)} + \Big(\frac{4c_1}{\chi} + K^2(c_2 T + c_1)\Big) e^{rT}.$$

Proof. By $\frac{\mathrm{d}}{\mathrm{d}t}|\nabla w| = \frac{\nabla w \cdot \nabla w_t}{|\nabla w|}$ and zero-flux boundary condition on w we compute that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla w|^2 + 2r\int_\Omega |\nabla w|^2 + 2\varepsilon \int_\Omega |\Delta w|^2 + 2\int_\Omega u|\nabla w|^2 + 4r\int_\Omega w|\nabla w|^2 \\
=&4r\int_\Omega |\nabla w|^2 - 2\int_\Omega w\nabla u \cdot \nabla w \\
\leq&2r\int_\Omega w|\nabla w|^2 + 2r\int_\Omega \frac{|\nabla w|^2}{w} + K^2\int_\Omega \frac{|\nabla u|^2}{u} + \int_\Omega u|\nabla w|^2,
\end{aligned}$$

where the last inequality is obtained by applying Young's inequality to the two right-hand terms of the equality to obtain that

$$4r\int_\Omega |\nabla w|^2 = 4r\int_\Omega \frac{|\nabla w|}{\sqrt{w}} \cdot \sqrt{w}|\nabla w| \leq 2r\int_\Omega \frac{|\nabla w|^2}{w} + 2r\int_\Omega w|\nabla w|^2$$

and that by $0 < w < K$

$$2\int_\Omega w\nabla u \cdot \nabla w \leq 2K\frac{|\nabla u|}{\sqrt{u}}\sqrt{u}|\nabla w| \leq \frac{4K^2}{2\eta}\frac{|\nabla u|^2}{u} + \frac{\eta}{2}u|\nabla w|^2.$$

By setting $\eta = 2$, we may derive that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla w|^2 + 2r\int_\Omega |\nabla w|^2 + 2\varepsilon\int_\Omega |\Delta w|^2 + \int_\Omega u|\nabla w|^2 + 2r\int_\Omega w|\nabla w|^2 \\
\leq&2r\int_\Omega \frac{|\nabla w|^2}{w} + K^2\int_\Omega \frac{|\nabla u|^2}{u}
\end{aligned}$$

which means

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla w|^2 + 2r\int_\Omega |\nabla w|^2 + 2\varepsilon\int_\Omega |\Delta w|^2 \leq 2r\int_\Omega \frac{|\nabla w|^2}{w} + K^2\int_\Omega \frac{|\nabla u|^2}{u} \tag{3.11}$$

and thus

$$\begin{aligned}\int_\Omega |\nabla w|^2 \le & e^{-2rt}\int_\Omega |\nabla w_0|^2 + 2r\int_0^t\int_\Omega \frac{|\nabla w|^2}{w} + K^2\int_0^t\int_\Omega \frac{|\nabla u|^2}{u}\\ \le & e^{-2rt}\int_\Omega |\nabla w_0|^2 + \frac{4c_1}{\chi}e^{rt} + K^2(c_2 t + c_1)e^{rt}\end{aligned}$$

by (3.7) and (3.8). More generally, integrating (3.11) in $t \in (0, T)$ and using (3.7) and (3.8) again may conclude that

$$\begin{aligned}&\int_\Omega |\nabla w|^2 + 2r\int_0^T\int_\Omega |\nabla w|^2 + 2\varepsilon\int_0^T\int_\Omega |\Delta w|^2\\ \le & \int_\Omega |\nabla w_0|^2 + 2r\int_0^T\int_\Omega \frac{|\nabla w|^2}{w} + K^2\int_0^T\int_\Omega \frac{|\nabla u|^2}{u}\\ \le & \int_\Omega |\nabla w_0|^2 + \left(\frac{4c_1}{\chi} + K^2(c_2 t + c_1)\right)e^{rt}.\end{aligned}$$

This completes the proof. □

For any $q > 2$, we have the following estimate.

Lemma 3.6. *Suppose that (u, w) is the classical solution of (3.2) in Q_T for any $0 < T < T_{\max}$. Then one may have that*

$$\begin{aligned}&\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla w|^q + \frac{\varepsilon(q-2)}{8}\int_\Omega |\nabla w|^{q-2}(\nabla|\nabla w|)^2 + \frac{\varepsilon}{2}\int_\Omega |\nabla w|^{q-2}|D^2 w|^2\\ \le & \frac{nq}{\varepsilon}\int_\Omega |\nabla w|^{(q-2)\lambda} + \frac{C(T,q)}{\varepsilon}\left(1 + \int_\Omega u^{\frac{2\lambda}{\lambda-1}}\right)\end{aligned}$$

where $q \in (2, +\infty), \lambda \in (1, +\infty)$ and $C(T, q)$ is independent of ε provided that ε is bounded from above.

Proof. By $\frac{\mathrm{d}}{\mathrm{d}t}|\nabla w| = \frac{\nabla w\cdot\nabla w_t}{|\nabla w|}$ and zero-flux boundary condition we compute

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla w|^q = \int_\Omega |\nabla w|^{q-2}\nabla w\cdot\nabla w_t = -\int_\Omega \nabla\cdot(|\nabla w|^{q-2}\nabla w)w_t$$

$$=-\int_\Omega \nabla\cdot(|\nabla w|^{q-2}\nabla w)\big\{\varepsilon\Delta w-uw+rw(1-w)\big\}$$

$$=-\,\varepsilon\int_\Omega \nabla\cdot(|\nabla w|^{q-2}\nabla w)\Delta w-\int_\Omega \nabla\cdot(|\nabla w|^{q-2}\nabla w)\big\{-uw+rw(1-w)\big\}$$

$$=:I+II.$$

For I, we observe that

$$I=-\,\varepsilon\int_\Omega \nabla\cdot(|\nabla w|^{q-2}\nabla w)\Delta w$$

$$=-\,\varepsilon(q-2)\int_\Omega(|\nabla w|^{q-3}\Delta w\nabla w)\cdot\nabla|\nabla w|-\varepsilon\int_\Omega|\nabla w|^{q-2}|\Delta w|^2$$

$$=:I_A+I_B.$$

We keep I_B unchanged. Herein using integration by parts and $\nabla w\cdot\vec{\nu}\big|_{\partial\Omega}=0$, one may obtain

$$I_A=-\,\varepsilon(q-2)\int_\Omega(|\nabla w|^{q-3}\Delta w\nabla w)\cdot\nabla|\nabla w|$$

$$=\varepsilon(q-2)\int_\Omega\nabla\cdot(|\nabla w|^{q-3}\Delta w\nabla w)|\nabla w|-\varepsilon(q-2)\int_{\partial\Omega}(|\nabla w|^{q-3}\Delta w|\nabla w|)\nabla w\cdot\vec{\nu}$$

$$=\varepsilon(q-2)(q-3)\int_\Omega|\nabla w|^{q-3}(\Delta w\nabla w)\cdot\nabla|\nabla w|+\varepsilon(q-2)\int_\Omega|\nabla w|^{q-2}(\nabla\Delta w\cdot\nabla w)$$

$$+\,\varepsilon(q-2)\int_\Omega|\nabla w|^{q-2}|\Delta w|^2$$

that is, for $q>2$

$$-\varepsilon(q-2)\int_\Omega(|\nabla w|^{q-3}\Delta w\nabla w)\cdot\nabla|\nabla w|=\varepsilon\int_\Omega|\nabla w|^{q-2}(\nabla\Delta w\cdot\nabla w)+\varepsilon\int_\Omega|\nabla w|^{q-2}|\Delta w|^2.$$

Then applying $\Delta|\nabla w|^2=2\nabla\Delta w\cdot\nabla w+2|D^2w|^2$ and again integration by parts to the first right-hand term above may yield that

$$\varepsilon\int_\Omega|\nabla w|^{q-2}(\nabla\Delta w\cdot\nabla w)$$

$$=\frac{\varepsilon}{2}\int_\Omega |\nabla w|^{q-2}\Delta|\nabla w|^2 - \varepsilon\int_\Omega |\nabla w|^{q-2}|D^2 w|^2$$

$$=-\frac{\varepsilon}{2}\int_\Omega \nabla|\nabla w|^{q-2}\cdot\nabla|\nabla w|^2 + \frac{\varepsilon}{2}\int_{\partial\Omega}|\nabla w|^{q-2}\nabla|\nabla w|^2\cdot\vec{\nu} - \varepsilon\int_\Omega |\nabla w|^{q-2}|D^2 w|^2$$

$$=-\varepsilon(q-2)\int_\Omega |\nabla w|^{q-2}(\nabla|\nabla w|)^2 + \frac{\varepsilon}{2}\int_{\partial\Omega}|\nabla w|^{q-2}\nabla|\nabla w|^2\cdot\vec{\nu} - \varepsilon\int_\Omega |\nabla w|^{q-2}|D^2 w|^2.$$

It follows that for $q > 2$,

$$I_A = -\varepsilon(q-2)\int_\Omega |\nabla w|^{q-2}(\nabla|\nabla w|)^2 + \frac{\varepsilon}{2}\int_{\partial\Omega}|\nabla w|^{q-2}\nabla|\nabla w|^2\cdot\vec{\nu} - \varepsilon\int_\Omega |\nabla w|^{q-2}|D^2 w|^2$$
$$+\varepsilon\int_\Omega |\nabla w|^{q-2}|\Delta w|^2,$$

which implies for $q > 2$ that

$$I = -\varepsilon(q-2)\int_\Omega |\nabla w|^{q-2}(\nabla|\nabla w|)^2 + \frac{\varepsilon}{2}\int_{\partial\Omega}|\nabla w|^{q-2}\nabla|\nabla w|^2\cdot\vec{\nu} - \varepsilon\int_\Omega |\nabla w|^{q-2}|D^2 w|^2.$$

In regard to II, letting $h(u,w) := -uw + rw(1-w)$ and using Young's inequality with parameter $\frac{2nq}{\delta}$ ($\delta > 0$), one may calculate that

$$II = -\int_\Omega \nabla\cdot(|\nabla w|^{q-2}\nabla w)h \le \int_\Omega |\nabla w|^{q-2}\big((q-2)|\nabla|\nabla w|| + |\Delta w|\big)|h|$$
$$\le\frac{nq}{\delta}\int_\Omega |\nabla w|^{q-2}h^2 + \frac{\delta}{4nq}\int_\Omega |\nabla w|^{q-2}\big((q-2)|\nabla|\nabla w|| + |\Delta w|\big)^2$$
$$\le\frac{nq}{\delta}\int_\Omega |\nabla w|^{q-2}h^2 + \frac{\delta(q-2)}{2}\int_\Omega |\nabla w|^{q-2}(\nabla|\nabla w|)^2 + \frac{\delta}{2}\int_\Omega |\nabla w|^{q-2}|D^2 w|^2$$

where we have used $|\Delta u|^2 \le n|D^2 u|^2$. Hence combined with I and II, letting $\delta = \varepsilon$ will yield

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla w|^q + \frac{\varepsilon(q-2)}{2}\int_\Omega |\nabla w|^{q-2}(\nabla|\nabla w|)^2 + \frac{\varepsilon}{2}\int_\Omega |\nabla w|^{q-2}|D^2 w|^2$$
$$\le\frac{nq}{\varepsilon}\int_\Omega |\nabla w|^{q-2}h^2 + \frac{\varepsilon}{2}\int_{\partial\Omega}|\nabla w|^{q-2}\nabla|\nabla w|^2\cdot\vec{\nu}$$

$$\leq \frac{nq}{\varepsilon}\int_\Omega |\nabla w|^{(q-2)\lambda} + \frac{nq}{\varepsilon}\int_\Omega h^{\frac{2\lambda}{\lambda-1}} + \kappa\varepsilon\int_{\partial\Omega}|\nabla w|^q \tag{3.12}$$

by Lemma A.4 and via Young's inequality with index λ and $\frac{\lambda}{\lambda-1}$ for any $1 < \lambda < \infty$. Below we shall deal with the last two terms in (3.12).

For the $\kappa\varepsilon\int_{\partial\Omega}|\nabla w|^q$ in (3.12), using interpolation-trace inequality in [68, Lemma 2.2], one may find that

$$\|\nabla w\|_{L_q(\partial\Omega)} \leq \bar{C}(\|\nabla|\nabla w|\|_{L_q(\Omega)} + \|\nabla w\|_{L_q(\Omega)})^{\frac{1}{q}}\|\nabla w\|_{L_q(\Omega)}^{1-\frac{1}{q}}, \quad 1 < q < \infty, \tag{3.13}$$

where $\bar{C}$ depends on Ω and q. Then using Young's inequality with parameter $\eta > 0$ and with index q and $\frac{q}{q-1}$ may yield that

$$\begin{aligned}
&\kappa\varepsilon\int_{\partial\Omega}|\nabla w|^q = \kappa\varepsilon\||\nabla w|^{\frac{q}{2}}\|^2_{L_2(\partial\Omega)}\\
\leq&\kappa\varepsilon\bar{C}^2\big(\|\nabla|\nabla w|^{\frac{q}{2}}\|_{L_2(\Omega)} + \||\nabla w|^{\frac{q}{2}}\|_{L_2(\Omega)}\big)^{2(1-\frac{1}{q})}\||\nabla w|^{\frac{q}{2}}\|^{\frac{2}{q}}_{L_2(\Omega)}\\
\leq&\kappa\varepsilon\bar{C}^2\eta\big(\|\nabla|\nabla w|^{\frac{q}{2}}\|_{L_2(\Omega)} + \||\nabla w|^{\frac{q}{2}}\|_{L_2(\Omega)}\big)^2 + \kappa\varepsilon\bar{C}^2\eta^{-\frac{1}{q-1}}\||\nabla w|^{\frac{q}{2}}\|^2_{L_2(\Omega)}\\
\leq&2\kappa\varepsilon\bar{C}^2\eta\|\nabla|\nabla w|^{\frac{q}{2}}\|^2_{L_2(\Omega)} + \kappa\varepsilon\bar{C}^2\big(2\eta + \eta^{-\frac{1}{q-1}}\big)\||\nabla w|^{\frac{q}{2}}\|^2_{L_2(\Omega)}\\
\leq&\frac{\varepsilon(q-2)}{4}\int_\Omega|\nabla w|^{q-2}(\nabla|\nabla w|)^2 + \varepsilon c_4\int_\Omega|\nabla w|^q,
\end{aligned}$$

where we take $\eta = \frac{q-2}{2\kappa\bar{C}^2q^2} < \frac{1}{2\kappa\bar{C}^2}$ and $c_4 := 1+\kappa\bar{C}^2\Big(\frac{2\kappa\bar{C}^2q^2}{q-2}\Big)^{\frac{1}{q-1}} = 1+\kappa\bar{C}^2e^{\frac{1}{q-1}\ln\frac{2\kappa\bar{C}^2q^2}{q-2}}$ will be bounded uniformly as $q \in (2+\iota_0, +\infty)$ for any fixed $\iota_0 > 0$.

For $\int_\Omega h^{\frac{2\lambda}{\lambda-1}}$ in (3.12), one may generally compute that for any $1 < \sigma < \infty$,

$$\begin{aligned}
\frac{nq}{\varepsilon}\int_\Omega h^\sigma =&\frac{nq}{\varepsilon}\int_\Omega\big(uw + rw(1+w)\big)^\sigma\\
\leq&\frac{c_5}{\varepsilon}\int_\Omega u^\sigma + \frac{c_5}{\varepsilon}
\end{aligned}$$

with $c_5 := \frac{nq}{2}\cdot\max\big\{(2K)^\sigma,\ (2rK(1+K))^\sigma|\Omega|\big\}$ as a result of $0 < w < K$.

Then we may update (3.12) by taking $\sigma = \frac{2\lambda}{\lambda-1}$ and $1 < \lambda < +\infty$ that

$$\begin{aligned}&\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla w|^q + \frac{\varepsilon(q-2)}{4}\int_\Omega |\nabla w|^{q-2}(\nabla|\nabla w|)^2 + \frac{\varepsilon}{2}\int_\Omega |\nabla w|^{q-2}|D^2 w|^2 \\ \le & \frac{nq}{\varepsilon}\int_\Omega |\nabla w|^{(q-2)\lambda} + \varepsilon c_4 \int_\Omega |\nabla w|^q + \frac{c_5}{\varepsilon}\big(1+\int_\Omega u^{\frac{2\lambda}{\lambda-1}}\big).\end{aligned} \tag{3.14}$$

We proceed to decompose $\varepsilon c_4 \int_\Omega |\nabla w|^q$ in (3.14). Indeed, when $4 < q < +\infty$ one may use generalized Gagliardo-Nirenberg interpolation inequality in Lemma A.2 to show that

$$\begin{aligned}\int_\Omega |\nabla w|^q =& \||\nabla w|^{\frac{q}{2}}\|^2_{L_2(\Omega)} \\ \le & C\|\nabla|\nabla w|^{\frac{q}{2}}\|^{2\theta}_{L_2(\Omega)} \ \ \||\nabla w|^{\frac{q}{2}}\|^{2(1-\theta)}_{L_{\frac{4}{q}}(\Omega)} + C\||\nabla w|^{\frac{q}{2}}\|^2_{L_{\frac{4}{q}}(\Omega)} \\ \le & C\eta\|\nabla|\nabla w|^{\frac{q}{2}}\|^2_{L_2(\Omega)} + C(\eta^{-\frac{\theta}{1-\theta}}+1)\||\nabla w|^{\frac{q}{2}}\|^2_{L_{\frac{4}{q}}(\Omega)}\end{aligned}$$

with $\theta = \frac{\frac{q}{4}-\frac{1}{2}}{\frac{q}{4}-\frac{1}{2}+\frac{1}{n}} \in (0,1)$ as long as $2 < q < +\infty$, where we have used Young's inequality with parameter $\eta > 0$ and with index $\frac{1}{\theta}$ and $\frac{1}{1-\theta}$. It follows from $\eta = \frac{q-2}{2q^2 c_4 C}$ that

$$\varepsilon c_4 \int_\Omega |\nabla w|^q \le \frac{\varepsilon(q-2)}{8}\int_\Omega |\nabla w|^{q-2}(\nabla|\nabla w|)^2 + \varepsilon c_6 \|\nabla w\|^q_{L_2(\Omega)} \tag{3.15}$$

with $c_6 := c_4 C(\eta^{-\frac{\theta}{1-\theta}}+1)$ and $\frac{\theta}{1-\theta} = \frac{n(q-2)}{4}$.

On the other hand, when $1 < q \le 4$, using the Gagliardo-Nirenberg interpolation inequality in Lemma A.1 and Young's inequality with parameter $\delta > 0$ and with index $\frac{1}{\beta}$ and $\frac{1}{1-\beta}$ to show that

$$\begin{aligned}\int_\Omega |\nabla w|^q =& \||\nabla w|^{\frac{q}{2}}\|^2_{L_2(\Omega)} \\ \le & C\|\nabla|\nabla w|^{\frac{q}{2}}\|^{2\beta}_{L_2(\Omega)} \cdot \||\nabla w|^{\frac{q}{2}}\|^{2(1-\beta)}_{L_1(\Omega)} + C\||\nabla w|^{\frac{q}{2}}\|^2_{L_1(\Omega)}\end{aligned}$$

$$\leq C\delta \|\nabla|\nabla w|^{\frac{q}{2}}\|_{L_2(\Omega)}^2 + C(\delta^{-\frac{\beta}{1-\beta}}+1)\||\nabla w|^{\frac{q}{2}}\|_{L_1(\Omega)}^2,$$

with $\beta = \frac{n}{n+2}$. Thus taking $\delta = \frac{q-2}{2q^2 c_4 C}$ will yield that

$$\varepsilon c_4 \int_\Omega |\nabla w|^q \leq \frac{\varepsilon(q-2)}{8}\int_\Omega |\nabla w|^{q-2}(\nabla|\nabla w|)^2 + \varepsilon c_7 \||\nabla w|^{\frac{q}{2}}\|_{L_1(\Omega)}^2, \tag{3.16}$$

with $c_7 := c_4 C(\delta^{-\frac{\beta}{1-\beta}}+1)$ and $\frac{\beta}{1-\beta} = \frac{n}{2}$.

For any $0 < T < T_{\max}$ one may further control the $\|\nabla w\|_{L_2(\Omega)}^q$ in (3.15) and the $\||\nabla w|^{\frac{q}{2}}\|_{L_1(\Omega)}^2$ in (3.16). Indeed, it follows from (3.7) that

$$\|\nabla w\|_{L_2(\Omega)}^q \leq c_8^{\frac{q}{2}} \quad \text{as} \quad 4 < q < +\infty, \quad \text{and} \quad \||\nabla w|^{\frac{4}{2}}\|_{L_1(\Omega)}^2 \leq c_8^2 \quad \text{as} \quad q = 4,$$

where $c_8 := 2Kc_1 e^{rT}/\chi$. Using Hölder's inequality and again (3.7) may yield that

$$\||\nabla w|^{\frac{q}{2}}\|_{L_1(\Omega)}^2 = \Big(\int_\Omega |\nabla w|^{\frac{q}{2}}\Big)^2 \leq |\Omega|^{\frac{4-q}{2}}\Big(\int_\Omega |\nabla w|^2\Big)^{\frac{q}{2}} \leq (1+c_8)|\Omega|^{\frac{4-q}{2}}, \quad 1 < q < 4.$$

Thereupon, substituting (3.15) and (3.16) into (3.14) may lead us to

$$\begin{aligned}&\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla w|^q + \frac{\varepsilon(q-2)}{8}\int_\Omega |\nabla w|^{q-2}(\nabla|\nabla w|)^2 + \frac{\varepsilon}{2}\int_\Omega |\nabla w|^{q-2}|D^2 w|^2 \\ \leq & \frac{nq}{\varepsilon}\int_\Omega |\nabla w|^{(q-2)\lambda} + \frac{c_9}{\varepsilon}\big(1 + \int_\Omega u^{\frac{2\lambda}{\lambda-1}}\big)\end{aligned} \tag{3.17}$$

where either $c_9 := \varepsilon^2 c_6 c_8^{\frac{q}{2}} + c_5$ as $q > 4$ or $c_9 := \varepsilon^2 c_7 \cdot \max\{c_8^2,\, (1+c_8)|\Omega|^{\frac{4-q}{2}}\} + c_5$ as $q \in (1,4]$. □

Remark 3.2. *In addition, setting $\lambda(q-2) \leq 2$ in (3.17) and combining $q > 2$ therein will yield $2 < q \leq 2 + \frac{2}{\lambda}$ for $1 < \lambda < \infty$. Then a joint use of Hölder's inequality and (3.7) gives that*

$$\int_\Omega |\nabla w|^{\lambda(q-2)} \leq |\Omega|^{1-\frac{\lambda(q-2)}{2}}\Big(\int_\Omega |\nabla w|^2\Big)^{\frac{\lambda(q-2)}{2}} \leq |\Omega|^{1-\frac{\lambda(q-2)}{2}} c_8^{\frac{\lambda(q-2)}{2}}$$

when $\lambda(q-2) < 2$, *and the upper bound is still valid as* $\lambda(q-2) = 2$. *We thus have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla w|^q \le \frac{c_{10}}{\varepsilon}\Big(1+\int_\Omega u^{\frac{2\lambda}{\lambda-1}}\Big),$$

with $c_{10} := nq|\Omega|^{1-\frac{\lambda(q-2)}{2}} c_8^{\frac{\lambda(q-2)}{2}} + c_9$. *This inequality immediately shows*

$$\int_\Omega |\nabla w|^q \le \frac{c_{11}}{\varepsilon}, \tag{3.18}$$

where $2 < q < 4 - \frac{4}{\sigma}$ *for* $\sigma > 2$ *and* $c_{11} := c_{10}\int_0^T \big(1+\int_\Omega u^\sigma\big)$.

After these preparations, we next may derive $\|u\|_{L_\infty(\Omega)}$ through $\|a(\cdot,t)\|_{L_\infty(\Omega)}$ in the following two lemmas.

Lemma 3.7. *Assume that* (a, w) *is a local classical solution of* (1.13)– (1.15) *in* $t \in (0, T_{\max})$. *Then one may have for* $p \ge 2$ *that*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega e^{\chi w}a^p + r(p-1)\chi\int_\Omega e^{\chi w}a^p w + \frac{p(p-1)}{2}\int_\Omega e^{\chi w}a^{p-2}|\nabla a|^2\\ &+ p\int_\Omega e^{\chi w}a^p + pl\int_\Omega e^{2\chi w}a^{p+1} \\ \le&\varepsilon p^2\bar{c}_1\int_\Omega a^p|\nabla w|^2 + p\bar{c}_2\int_\Omega a^{p+1} + p\bar{c}_3\int_\Omega a^p\end{aligned} \tag{3.19}$$

where $\bar{c}_1 := e^{\chi K}\chi^2\big(1+\frac{\varepsilon}{2}\big)$, $\bar{c}_2 := \chi K e^{2\chi K}$, *and* $\bar{c}_3 := r\chi K^2 e^{\chi K} + \gamma e^{\chi K}K$. *Furthermore, assume that for some* $\sigma \in (2, +\infty)$

$$\int_0^T\int_\Omega u^\sigma \le c \tag{3.20}$$

where c *may depend on* $T \in (0, T_{\max})$ *and other parameters of the system* (3.2) *but remains bounded as* $\varepsilon \to 0$. *Then we have for* $\Omega \subset \mathbb{R}^2$ *that*

$$\int_\Omega a^p(x,t) \le \bar{c}(p)e^{\bar{c}(p)\cdot t} \tag{3.21}$$

where the constant $\bar{c}(p)$ is independent of $\varepsilon > 0$.

Proof. For any $p \geq 2$ and (1.13)– (1.15), we may compute that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega e^{\chi w}a^p \\
=&\chi\int_\Omega e^{\chi w}a^p w_t + p\int_\Omega e^{\chi w}a^{p-1}\big\{-\chi a w_t + e^{-\chi w}\nabla\cdot(e^{\chi w}\nabla a) + \gamma a w - a(1+lae^{\chi w})\big\} \\
=&-(p-1)\chi\int_\Omega e^{\chi w}a^p w_t + p\int_\Omega a^{p-1}\nabla\cdot(e^{\chi w}\nabla a) + p\gamma\int_\Omega a^p e^{\chi w}w - p\int_\Omega a^p e^{\chi w}(1+lae^{\chi w}) \\
=&-(p-1)\chi\int_\Omega e^{\chi w}a^p\big\{\varepsilon\Delta w - ae^{\chi w}w + rw(1-w)\big\} + p\int_\Omega a^{p-1}\nabla\cdot(e^{\chi w}\nabla a) \\
&+ p\gamma\int_\Omega a^p e^{\chi w}w - p\int_\Omega a^p e^{\chi w}(1+lae^{\chi w}) \\
=&\varepsilon\chi^2(p-1)\int_\Omega e^{\chi w}a^p|\nabla w|^2 + \varepsilon\chi p(p-1)\int_\Omega e^{\chi w}a^{p-1}\nabla a\cdot\nabla w + (p-1)\chi\int_\Omega e^{2\chi w}a^{p+1}w \\
&- r(p-1)\chi\int_\Omega e^{\chi w}a^p w(1-w) - p(p-1)\int_\Omega e^{\chi w}a^{p-2}|\nabla a|^2 \\
&+ p\gamma\int_\Omega a^p e^{\chi w}w - p\int_\Omega a^p e^{\chi w} - pl\int_\Omega e^{2\chi w}a^{p+1}
\end{aligned}
$$

that is,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega e^{\chi w}a^p + r(p-1)\chi\int_\Omega e^{\chi w}a^p w + p(p-1)\int_\Omega e^{\chi w}a^{p-2}|\nabla a|^2 + p\int_\Omega e^{\chi w}a^p + pl\int_\Omega e^{2\chi w}a^{p+1} \\
=&\varepsilon\chi^2(p-1)\int_\Omega e^{\chi w}a^p|\nabla w|^2 + \varepsilon\chi p(p-1)\int_\Omega e^{\chi w}a^{p-1}\nabla a\cdot\nabla w + (p-1)\chi\int_\Omega e^{2\chi w}a^{p+1}w \\
&+ r(p-1)\chi\int_\Omega e^{\chi w}a^p w^2 + p\gamma\int_\Omega a^p e^{\chi w}w.
\end{aligned}
$$

An application of Young's inequality leads us to

$$
\varepsilon\chi p(p-1)\int_\Omega e^{\chi w}a^{p-1}\nabla a\cdot\nabla w \leq \frac{p(p-1)}{2}\int_\Omega e^{\chi w}a^{p-2}|\nabla a|^2 + \frac{p(p-1)(\varepsilon\chi)^2}{2}\int_\Omega e^{\chi w}a^p|\nabla w|^2.
$$

Combined with $0 < w < K$, therefore, the integral inequality (3.19) is obtained.

Furthermore, since assumption (3.20) indicates the estimate (3.18), then by (3.19) a joint application of Hölder's inequality and Gagliardo-Nirenberg inequality means that

$$\begin{aligned}\varepsilon p^2\bar{c}_1\int_\Omega a^p|\nabla w|^2 \leq&\varepsilon p^2\bar{c}_1\|\nabla w\|^2_{L_{2\tau'}(\Omega)}\|a^{\frac{p}{2}}\|^2_{L_{2\tau}(\Omega)}\\ \leq&\varepsilon p^2\bar{c}_1(2C)^2\|\nabla w\|^2_{L_{2\tau'}(\Omega)}\big(\|\nabla a^{\frac{p}{2}}\|^{2\theta}_{L_2(\Omega)}\|a^{\frac{p}{2}}\|^{2(1-\theta)}_{L_{\frac{2}{p}}(\Omega)}+\|a^{\frac{p}{2}}\|^2_{L_{\frac{2}{p}}(\Omega)}\big)\end{aligned}$$

with $\theta=\frac{p-\frac{1}{\tau}}{p-1+\frac{2}{n}}\in(0,1)$ as $1\leq n<\frac{2\tau}{\tau-1}$ and $\tau'=\frac{\tau}{\tau-1}$ for $\tau\in(1,+\infty)$. Then taking $2\tau'=q$ in (3.18) which means $\tau>2+\frac{2}{\sigma-2}$, and using Young's inequality with one parameter $\eta>0$ will result in

$$\begin{aligned}\varepsilon p^2\bar{c}_1\int_\Omega a^p|\nabla w|^2 \leq&p^2\bar{c}_4\big(\eta\|\nabla a^{\frac{p}{2}}\|^2_{L_2(\Omega)}+\eta^{-\frac{\theta}{1-\theta}}\|a\|^p_{L_1(\Omega)}+\|a\|^p_{L_1(\Omega)}\big)\\ \leq&\frac{p(p-1)}{8}\int_\Omega a^{p-2}|\nabla a|^2+p^2\bar{c}_4(\eta^{-\frac{\theta}{1-\theta}}\|a\|^p_{L_1(\Omega)}+\|a\|^p_{L_1(\Omega)}\big)\end{aligned}\tag{3.22}$$

with $\bar{c}_4:=\varepsilon^{\frac{1}{\tau}}\bar{c}_1(2C)^2c_9^{\frac{1}{\tau'}}$ and $\eta=\frac{p-1}{2p^3\bar{c}_4}$.

Similarly, applying Gagliardo-Nirenberg interpolation inequality ($n=2$) given in [62, Lemma A.5] may yield that

$$\begin{aligned}p\bar{c}_2\int_\Omega a^{p+1} =&p\bar{c}_2\|a^{\frac{p}{2}}\|^{\frac{2(p+1)}{p}}_{L_{\frac{2(p+1)}{p}}(\Omega)}\\ \leq&p\bar{c}_5\big(\delta\|\nabla a^{\frac{p}{2}}\|^{\frac{2(p+1)}{p}-\frac{2}{p}}_{L_2(\Omega)}\ \|a^{\frac{p}{2}}\ln^{\frac{p}{2}}\big(a^{\frac{p}{2}}\big)\|^{\frac{2}{p}}_{L_{\frac{2}{p}}(\Omega)}+C\|a^{\frac{p}{2}}\|^{\frac{2(p+1)}{p}}_{L_{\frac{2}{p}}(\Omega)}+C_\delta\big)\\ =&p\bar{c}_5\big(\delta\|\nabla a^{\frac{p}{2}}\|^2_{L_2(\Omega)}\ \|a\ln a\|_{L_1(\Omega)}+C\|a\|^{p+1}_{L_1(\Omega)}+C_\delta\big)\\ \leq&\frac{p(p-1)}{8}\int_\Omega a^{p-2}|\nabla a|^2+p\bar{c}_5\big(C\|a\|^{p+1}_{L_1(\Omega)}+C_\delta\big)\end{aligned}$$

where we take $\delta=\frac{p-1}{2p^2\bar{c}_5(1+\|a\ln a\|_{L_1(\Omega)})}$ which makes sense by $\|a\ln a\|_{L_1(\Omega)}\leq\|u\ln u\|_{L_1(\Omega)}$.

Consequently, we may derive from (3.19) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega e^{\chi w}a^p + p\int_\Omega e^{\chi w}a^p \le p^2\bar{c}_6 + p\bar{c}_3\int_\Omega a^p \le p^2\bar{c}_6 + p\bar{c}_3\int_\Omega e^{\chi w}a^p$$

where $\bar{c}_6 := \bar{c}_4(\eta^{-\frac{\theta}{1-\theta}}\|a\|^p_{L_1(\Omega)} + \|a\|^p_{L_1(\Omega)}) + \bar{c}_5\big(C\|a\|^{p+1}_{L_1(\Omega)} + C_\delta\big)$. Solving this inequality may conclude that

$$\int_\Omega a^p \le \int_\Omega e^{\chi w}a^p \le e^{p\bar{c}_3\cdot t}\Big(\frac{p\bar{c}_6}{\bar{c}_3} + \|e^{\chi w_0}a_0^p\|_{L_1(\Omega)}\Big).$$

Thus we complete the proof. □

Remark 3.3. *We need some illustrations on the sufficient condition* (3.20) *as follow:*

(i) Without condition (3.20)*, it is not difficult to derive*

$$\|u(\cdot,t)\|_{L_2(\Omega)} \le \frac{\bar{M}(T)}{\varepsilon}\cdot e^{\frac{M(T)}{\varepsilon}}, \qquad t\in(0,T)$$

for some constants $M(T)$ and $\bar{M}(T)$ independent of $\varepsilon > 0$, which is enough to obtain the $\|u(\cdot,t)\|_{L_\infty(\Omega)}$ for any fixed $\varepsilon > 0$. Based on this, we shall in Lemma 3.9 prove that $\|u(\cdot,t)\|_{L_\infty(\Omega)}$ remains bounded uniformly in $\varepsilon > 0$.

(ii) Condition (3.20) *can be relaxed to allow*

$$\int_0^T\int_\Omega u^\sigma \le \frac{c}{\varepsilon^\rho}$$

for some $\rho < 1 - \frac{2}{\sigma}$ and $\sigma > 2$, where constant c remains bounded as $\varepsilon \to 0$.

(iii) Also, condition (3.20) *can be replaced by*

$$\|u\|_{L_2(\Omega)} \le c$$

where constant c remains bounded as $\varepsilon \to 0$.

Proof. For (i), one may compute that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2 =& \int_\Omega u\big\{\Delta u - \chi\nabla\cdot(u\nabla w) + \gamma uw - u(1+lu)\big\}\\ &= -\int_\Omega |\nabla u|^2 + \chi\int_\Omega u\nabla u\cdot\nabla w + \gamma\int_\Omega u^2 w - \int_\Omega u^2(1+lu).\\ &\leq -\frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\chi^2}{2}\int_\Omega u^2|\nabla w|^2 + \gamma K\int_\Omega u^2 - \int_\Omega u^2 - l\int_\Omega u^3\end{aligned}$$

Applying Hölder's inequality and Gagliardo-Nirenberg interpolation inequality ($n = 2$) may yield that

$$\begin{aligned}\frac{\chi^2}{2}\int_\Omega u^2|\nabla w|^2 \leq& \frac{\chi^2}{2}\|u\|^2_{L_4(\Omega)}\|\nabla w\|^2_{L_4(\Omega)}\\ \leq& 2\chi^2 C^2\big(\|\nabla u\|_{L_2(\Omega)}\|u\|_{L_2(\Omega)} + \|u\|^2_{L_1(\Omega)}\big)\|\nabla w\|^2_{L_4(\Omega)}.\end{aligned}$$

Applying Young's inequality gives that

$$2\chi^2 C^2\|\nabla u\|_{L_2(\Omega)}\|u\|_{L_2(\Omega)}\|\nabla w\|^2_{L_4(\Omega)} \leq \frac{1}{4}\|\nabla u\|^2_{L_2(\Omega)} + 4\chi^4 C^4\|u\|^2_{L_2(\Omega)}\|\nabla w\|^4_{L_4(\Omega)}.$$

This means

$$\frac{\chi^2}{2}\int_\Omega u^2|\nabla w|^2 \leq \frac{1}{4}\|\nabla u\|^2_{L_2(\Omega)} + 4\chi^4 C^4\|\nabla w\|^4_{L_4(\Omega)}\|u\|^2_{L_2(\Omega)} + 2\chi^2 C^2\|u\|^2_{L_1(\Omega)}\|\nabla w\|^2_{L_4(\Omega)},$$

thus

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L_2(\Omega)} + \|u\|^2_{L_2(\Omega)} + \frac{1}{4}\|\nabla u\|^2_{L_2(\Omega)} + l\|u\|^3_{L_3(\Omega)}\\ \leq& \big\{\gamma K + 4\chi^4 C^4\|\nabla w\|^4_{L_4(\Omega)}\big\}\|u\|^2_{L_2(\Omega)} + 2\chi^2 C^2\|u\|^2_{L_1(\Omega)}\|\nabla w\|^2_{L_4(\Omega)}.\end{aligned}$$

Thereupon, we may find that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L_2(\Omega)} \leq& 2\big\{\gamma K + 4\chi^4 C^4\|\nabla w\|^4_{L_4(\Omega)}\big\}\|u\|^2_{L_2(\Omega)}\\ &+ 4\chi^2 C^2\|u\|^2_{L_1(\Omega)}\|\nabla w\|^2_{L_4(\Omega)}.\end{aligned}$$

Solving this integral inequality shows that

$$
\begin{aligned}
&\|u(\cdot,t)\|^2_{L_2(\Omega)} \\
\le& e^{2\int_0^T\{\gamma K+4\chi^4C^4\|\nabla w\|^4_{L_4(\Omega)}\}}\Big\{\|u(\cdot,0)\|^2_{L_2(\Omega)}+2\chi^2C^2\|u\|^2_{L_1(\Omega)}\int_0^T\big(1+\|\nabla w\|^4_{L_4(\Omega)}\big)\Big\} \\
\le& \frac{\bar M(T)}{\varepsilon}\cdot e^{\frac{M(T)}{\varepsilon}}
\end{aligned}
$$

where $M(T) := 2(\gamma KT + (4C)^4(\chi K)^3(c_2T+c_1)e^{rT})$, $\bar M(T) := \|u(\cdot,0)\|^2_{L_2(\Omega)} + 2\chi C^2\|u\|^2_{L_1(\Omega)}\big(\chi T + 4^3(c_2T+c_1)K^3e^{rT}\big)$, and we have used

$$
\int_0^T\int_\Omega|\nabla w|^4 \le K^3\int_0^T\int_\Omega\frac{|\nabla w|^4}{w^3}\le K^3(2+\sqrt{2})^2\int_0^T\int_\Omega w|D^2\ln w|^2\le\frac{4^3K^3(c_2T+c_1)e^{rT}}{\chi\varepsilon}
$$

by Lemma A.3 and estimate (3.8), or by Lemma 3.5 and Gagliardo-Nirenberg interpolation inequality.

For (ii), when it is used in (3.22) and later in (3.23), one may require the power of ε to be nonnegative in order to take $\varepsilon\to 0$, that is, by the expression of c_{12} in (3.23) with the c_{11} defined in (3.18), one have

$$
1-\frac{1+\rho}{\tau'}\ge 0,\quad\text{with}\quad \tau'=\frac{\tau}{\tau-1}\quad\text{and}\quad 2+\frac{2}{\sigma-2}<\tau<+\infty,
$$

so $\rho < 1-\frac{2}{\sigma}$.

For (iii), one may use Gagliardo-Nirenberg interpolation inequality ($n=2$) to show that

$$
\|u\|^3_{L_3(\Omega)}=\|\sqrt{u}\|^6_{L_6(\Omega)}\le C^6\big\{\|\nabla\sqrt{u}\|^2_{L_2(\Omega)}\|\sqrt{u}\|^4_{L_4(\Omega)}+\|\sqrt{u}\|^6_{L_4(\Omega)}\big\}
$$

and thus by the estimate (3.8) one may see that

$$
\int_0^T\int_\Omega u^3\le C^6\|u\|^2_{L_2(\Omega)}\int_0^T\int_\Omega\frac{|\nabla u|^2}{u}+C^6\|u\|^3_{L_2(\Omega)}\le C^6\|u\|^2_{L_2(\Omega)}\big\{(c_2T+c_1)e^{rT}+\|u\|_{L_2(\Omega)}\big\}
$$

which satisfies the condition (3.20).

□

Lemma 3.8. *For $\Omega \subset \mathbb{R}^2$ and $\varepsilon \in (0,1)$, under assumption (3.20) one may derive*

$$\|a(\cdot,t)\|_{L_\infty(\Omega)} \le \max\big\{1,\ \ C_0(T)\big\}, \qquad t \in (0,T) \subset (0,T_{\max}).$$

Note that $C_0(T)$ is additionally increasing in $1/\varepsilon$ if Remark 3.3 (i) is satisfied, and depend on $\max\{1,\varepsilon\}$ if Remark 3.3 (ii) or (iii) holds.

Proof. We proceed to compute the right-hand terms in (3.19). Firstly, we note that the only difference amongst using assumption (3.20), Remark 3.3 (i), (ii), and (iii) lies in the obtained coefficients in dealing with $\varepsilon p^2 \bar{c}_1 \int_\Omega a^p |\nabla w|^2$.

Indeed, if assumption (3.20) holds, then a joint application of Hölder's inequality and Gagliardo-Nrienberg interpolation inequality yield that

$$\begin{aligned}\varepsilon p^2 \bar{c}_1 \int_\Omega a^p |\nabla w|^2 \le& \varepsilon p^2 \bar{c}_1 \|\nabla w\|^2_{L_{2\tau'}(\Omega)} \|a^{\frac{p}{2}}\|^2_{L_{2\tau}(\Omega)} \\ \le& \varepsilon p^2 \bar{c}_1 C^2 \|\nabla w\|^2_{L_{2\tau'}(\Omega)} \big(\|\nabla a^{\frac{p}{2}}\|^{2\theta}_{L_2(\Omega)} \|a^{\frac{p}{2}}\|^{2(1-\theta)}_{L_1(\Omega)} + \|a^{\frac{p}{2}}\|^2_{L_1(\Omega)}\big)\end{aligned}$$

where $\theta = 1 - \frac{1}{2\tau} \in (0,1)$ and $\tau' = \frac{\tau}{\tau-1}$ for $\tau \in (1,+\infty)$. Taking $2\tau' = q$, by (3.18) and Young's inequality with parameter $\eta > 0$, we know that

$$\begin{aligned}\varepsilon p^2 \bar{c}_1 \int_\Omega a^p |\nabla w|^2 \le& p^2 c_{12} \big(\|\nabla a^{\frac{p}{2}}\|^{2\theta}_{L_2(\Omega)} \|a^{\frac{p}{2}}\|^{2(1-\theta)}_{L_1(\Omega)} + \|a^{\frac{p}{2}}\|^2_{L_1(\Omega)}\big) \\ \le& p^2 c_{12} \big(\eta \|\nabla a^{\frac{p}{2}}\|^2_{L_2(\Omega)} + (\eta^{-\frac{\theta}{1-\theta}} + 1) \|a^{\frac{p}{2}}\|^2_{L_1(\Omega)}\big)\end{aligned} \tag{3.23}$$

where $c_{12} := 1 + C^2 \varepsilon^{\frac{1}{\tau}} \bar{c}_1 c_{11}^{\frac{1}{\tau'}}$. Setting $\eta = \frac{p-1}{2p^3 c_{12}}$ leads us to that

$$\varepsilon p^2 \bar{c}_1 \int_\Omega a^p |\nabla w|^2 \le \frac{p(p-1)}{8} \int_\Omega a^{p-2} |\nabla a|^2 + p^{4\tau} c_{13} \|a^{\frac{p}{2}}\|^2_{L_1(\Omega)} \tag{3.24}$$

with $c_{13} := ((4c_{12})^{2\tau-1} + 1) c_{12}$ and $\tau > 2 + \frac{2}{\sigma - 2}$.

The above computations remain almost unchanged if Remark 3.3 (ii) or (iii) holds alternatively, that is, c_{12} depends on $\max\{1,\varepsilon\}$ instead of $1/\varepsilon$. On the other hand, when Remark 3.3 (i) holds, the coefficient c_{12} and thus c_{13} may be increasing in $1/\varepsilon$. This difference does not occur in the rest right-hand terms of (3.19). Consequently, we below shall no longer distinguish the assumption (3.20), Remark 3.3 (i), (ii), and (iii).

Secondly, by applying Gagliardo-Nirenberg interpolation inequality and Young's inequality with parameter $\delta > 0$ and with index $\frac{2p}{p+2}$ and $\frac{2p}{p-2}$, we have

$$\begin{aligned}
\int_\Omega a^{p+1} =& \|a^{\frac{p}{2}}\|^{\frac{2(p+1)}{p}}_{L_{\frac{2(p+1)}{p}}(\Omega)} \\
\leq & c_{14}\big(\|\nabla a^{\frac{p}{2}}\|^{\frac{2(p+1)\theta}{p}}_{L_2(\Omega)} \|a^{\frac{p}{2}}\|^{\frac{2(p+1)(1-\theta)}{p}}_{L_1(\Omega)} + \|a^{\frac{p}{2}}\|^{\frac{2(p+1)}{p}}_{L_1(\Omega)}\big) \\
= & c_{14}\big(\|\nabla a^{\frac{p}{2}}\|^{\frac{p+2}{p}}_{L_2(\Omega)} \|a^{\frac{p}{2}}\|_{L_1(\Omega)} + \|a^{\frac{p}{2}}\|^{\frac{2(p+1)}{p}}_{L_1(\Omega)}\big) \\
\leq & c_{14}\big(\delta\|\nabla a^{\frac{p}{2}}\|^2_{L_2(\Omega)} + \delta^{-\frac{p+2}{p-2}}\|a^{\frac{p}{2}}\|^{\frac{2p}{p-2}}_{L_1(\Omega)} + \|a^{\frac{p}{2}}\|^{\frac{2(p+1)}{p}}_{L_1(\Omega)}\big)
\end{aligned}$$

where $\theta = \frac{p+2}{2(p+1)} \in (0,1)$ and $c_{14} = (1+C)^4 \geq C^{\frac{2(p+1)}{p}}$ with $p > 2$. Then taking $\delta = \frac{p-1}{2p^2\bar{c}_2 c_{14}}$ will yield that

$$\begin{aligned}
p\bar{c}_2\int_\Omega a^{p+1} \leq & \frac{p(p-1)}{8}\int_\Omega a^{p-2}|\nabla a|^2 + p\bar{c}_2 c_{14}\big(\delta^{-\frac{p+2}{p-2}} + 1\big)\big(\|a^{\frac{p}{2}}\|_{L_1(\Omega)} + 1\big)^{\frac{2p}{p-2}} \\
\leq & \frac{p(p-1)}{8}\int_\Omega a^{p-2}|\nabla a|^2 + c_{15}^{\frac{2p}{p-2}} \cdot p^{1+\frac{2p}{p-2}}\big(\|a^{\frac{p}{2}}\|_{L_1(\Omega)} + 1\big)^{\frac{2p}{p-2}}
\end{aligned} \tag{3.25}$$

with $c_{15} := 4\bar{c}_2 c_{14} + 1$, in light of $\frac{2(p+1)}{p} = \frac{2p}{p-2}\cdot\frac{p^2-p-2}{p^2} \leq \frac{2p}{p-2}$ and $\frac{p+2}{p-2} = \frac{2p}{p-2}\cdot\frac{p+2}{2p} \leq \frac{2p}{p-2}$.

Thirdly, similar to the derivation of (3.24), we know that

$$\begin{aligned}
p\bar{c}_3\int_\Omega a^p \leq & p\bar{c}_3 C^2\big(\|\nabla a^{\frac{p}{2}}\|_{L_2(\Omega)}\|a^{\frac{p}{2}}\|_{L_1(\Omega)} + \|a^{\frac{p}{2}}\|^2_{L_1(\Omega)}\big) \\
\leq & p\bar{c}_3 C^2\big(\rho\|\nabla a^{\frac{p}{2}}\|^2_{L_2(\Omega)} + (1+\rho^{-1})(1 + \|a^{\frac{p}{2}}\|_{L_1(\Omega)})^2\big) \\
\leq & \frac{p(p-1)}{8}\int_\Omega a^{p-2}|\nabla a|^2 + c_{16}\cdot p^2(1 + \|a^{\frac{p}{2}}\|_{L_1(\Omega)})^2
\end{aligned} \tag{3.26}$$

with $\rho = \frac{p-1}{2p^2\bar{c}_3C^2}$ and $c_{16} := 2c_5C^2(1+4\bar{c}_3C^2)$.

Then in conjunction with (3.24)–(3.26), we may derive from (3.19) and $e^{\chi w} \geq 1$ that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega e^{\chi w}a^p + \int_\Omega e^{\chi w}a^p \\
\leq & \Big(p^{4\tau}c_{13} + c_{15}^{\frac{2p}{p-2}} \cdot p^{1+\frac{2p}{p-2}} + c_{16}\cdot p^2\Big)\big(\|a^{\frac{p}{2}}\|_{L_1(\Omega)} + 1\big)^{\frac{2p}{p-2}} \qquad (3.27)\\
\leq & c_{17}\cdot p^\mu \cdot \big(\|a^{\frac{p}{2}}\|_{L_1(\Omega)}^{\frac{2p}{p-2}} + 1\big)
\end{aligned}$$

where $c_{17} := 2^6 \cdot 3 \cdot \max\{c_{13}, c_{15}^6, c_{16}\}$ and $\mu := \max\{4\tau, 7\}$ with $\tau > 2 + \frac{2}{\sigma-2}$, in view of $\frac{p}{p-2} \leq 3$ for $p \geq 3$. Immediately it follows that

$$\int_\Omega a^p \leq \int_\Omega e^{\chi w}a^p \leq c_{18}\cdot \|a_0\|_{L_\infty(\Omega)}^p + p^\mu \cdot c_{17} \cdot \left(1 + \sup_{t\in(0,T)}\Big(\int_\Omega a^{\frac{p}{2}}\Big)^{\frac{2p}{p-2}}\right)$$

with $a_0 = u_0e^{-\chi w_0}$ and $c_{18} := |\Omega| \cdot e^{\chi\|w_0\|_{L_\infty(\Omega)}}$. Now we let $p_k = 3\cdot 2^k$, $\sigma_k = \frac{2p_k}{p_k-2}$, and

$$M_k := \max\Big\{1, \sup_{t\in(0,T)}\int_\Omega a^{p_k}\Big\}, \qquad k = 0, 1, 2, \cdots.$$

Therefore, we have for $k \geq 1$ that

$$M_k \leq c_{18}\|a_0\|_{L_\infty(\Omega)}^{p_k} + c_{19}\, p_k^\mu\, M_{k-1}^{\sigma_k} \qquad (3.28)$$

with $c_{19} := 2c_{17}$.

Now if there exist infinitely many $k \geq 1$ such that $p_k^\mu c_{19} M_{k-1}^{\sigma_k} < c_{18}\|a_0\|_{L_\infty(\Omega)}^{p_k}$, then

$$\Big(\int_\Omega a^{p_k}\Big)^{\frac{1}{p_k}} \leq M_k^{\frac{1}{p_k}} \leq (2c_{18})^{\frac{1}{p_k}}\|a_0\|_{L_\infty(\Omega)}, \qquad k \geq 1,$$

which means $\|a(\cdot,t)\|_{L_\infty(\Omega)} \leq \|a_0\|_{L_\infty(\Omega)}$ for any $t \in (0,T)$, by letting $k \to +\infty$.

On the contrary, there is a set I of finite many integers such that

$$p_k^\mu c_{19} M_{k-1}^{\sigma_k} < c_{18}\|a_0\|_{L_\infty(\Omega)}^{p_k} \quad \text{for } k \in I,$$

thus

$$M_k \le 2c_{18}\|a_0\|^{p_k}_{L_\infty(\Omega)} \le 2c_{18}\max_{k\in I}\left\{\|a_0\|^{p_k}_{L_\infty(\Omega)}\right\} =: N, \qquad k \in I,$$

Meanwhile, we have $p_k^\mu c_{19} M_{k-1}^{\sigma_k} \ge c_{18}\|a_0\|^{p_k}_{L_\infty(\Omega)}$ as long as $k \notin I$ and $k \ge 1$, then

$$M_k \le p_k^\mu \cdot (2c_{19}) \cdot M_{k-1}^{\sigma_k} \le \iota^k M_{k-1}^{\sigma_k}, \qquad \text{for all } k \ge 1,$$

with some $\iota > 1$ fulfilling $\iota^k > \max\{N^k, 3^\mu \cdot (2c_{19}) \cdot (2^\mu)^k\} > 1$ for all $k \ge 0$. A simple induction shows that

$$\begin{aligned} M_k \le & \iota^k (M_{k-1})^{\sigma_k} \le \iota^{k+(k-1)\sigma_k}(M_{k-2})^{\sigma_k\sigma_{k-1}} \le \cdots \\ \le & \iota^{k+\sum_{j=k_0+1}^{k}\left((j-1)\prod_{i=j}^{k}\sigma_i\right)} \cdot M_{k_0-1}^{\prod_{j=k_0}^{k}\sigma_j}, \qquad k_0 \ge 1. \end{aligned}$$

Noting that for all $k \ge 0$, $\sigma_k \ge \sigma_{k+1} > 1$ and $\sigma_k = \frac{2p_k}{p_k-2} = 2(1+\frac{2}{p_k-2}) \le 2 + \frac{4}{2^k}$, thus one may obtain that

$$\prod_{i=j}^{k}\sigma_i \le \prod_{i=j}^{k} 2\big(1+\frac{2}{2^i}\big) = 2^{k-j+1}e^{\sum_{i=j}^{k}\ln\left(1+\frac{2}{2^i}\right)} \le 2^{k-j+1}e^{\sum_{i=j}^{k}\frac{2}{2^i}} \le 2^{k-j+1}e^4$$

and

$$M_k^{\frac{1}{p_k}} \le \iota^{\frac{k}{p_k}+\frac{2^{k+1}e^4}{p_k}\sum_{j=2}^{k}\frac{j-1}{2^j}} \cdot M_{k_0-1}^{\frac{2^{k+1}e^4}{p_k 2^{k_0}}}$$

that is, by letting $k \to +\infty$,

$$\|a(\cdot,t)\|_{L_\infty(\Omega)} \le \iota^{\frac{e^4}{3}} \cdot M_{k_0-1}^{\frac{2e^4}{3\cdot 2^{k_0}}}, \qquad t \in (0,T).$$

Then together with (3.21) and the definition of M_k we may complete this proof. □

Proof of Proposition 3.1: Since Remark 3.3 (i) holds for any finite $\varepsilon > 0$, then Lemma 3.8 implies

$$\|a(\cdot,t)\|_{L_\infty(\Omega)} \le C(1/\varepsilon, T), \quad t \in (0,T) \subset (0,T_{\max}) \tag{3.29}$$

where the constant $C(1/\varepsilon,T)$ is increasing in $1/\varepsilon$ and T. Alternatively, Remark 3.3

(i) and Remark A.1 jointly show that

$$\|\nabla w(\cdot,t)\|_{L_p(\Omega)} \le \frac{C(T)}{\varepsilon} \cdot e^{\frac{C(T)}{\varepsilon}} \qquad \text{for} \quad p \in [1,+\infty), \quad t \in (0,T) \subset (0,T_{\max})$$

and for some constant $C(T) > 0$ which increases in T and depends on $\max\{1,\varepsilon\}$. This certainly supports the derivation of (3.24) in the proof of Lemma 3.8 with corresponding c_{12} and c_{13} increasing in $1/\varepsilon$ in such a case. Hence we have (3.29) as well.

Thus the local unique classical solution of (3.2) claimed in Lemma 3.1 exists globally in time, for any given $\varepsilon > 0$. This completes the proof. □

3.2.3 The Strong Solution of (3.2)

We first prove a critical lemma which enables us to remove the dependence on $1/\varepsilon$ in the upper bound of $\|a(\cdot,t)\|_{L_\infty(\Omega)}$, based on Remark 3.3 (i).

Lemma 3.9. *For system* (3.2) *and any given* $0 < T < +\infty$*, there exists a finite* $M = M(T) > 0$ *such that*

$$\|u(\cdot,t)\|_{L_\infty(Q_T)} \le M(T), \qquad t \in (0,T)$$

where $M(T)$ *remains bounded as* $\varepsilon \to 0$.

Proof. By Remark 3.3 (i), Remark A.1, Lemma 3.7 and Lemma 3.8, for $0 < w < K$ and the given T, $\varepsilon > 0$, one may derive from the derivation of (3.29) that

$$\|u(\cdot,t)\|_{L_\infty(Q_T)} \le C(1/\varepsilon, T) < +\infty,$$

where the constant $C(1/\varepsilon,T) \to +\infty$ as $\varepsilon \to 0$. Obviously, this upper bound is rather rough. So we shall below prove that there may be no $1/\varepsilon$ included in this upper bound, as $\varepsilon \to 0$.

Let $b \in (0,1/2)$. A nonempty measurable set

$$\mathcal{Q}_b := \Big\{(x,t) \in \Omega \times (0,T) : u(x,t) > b \cdot \|u(\cdot,t)\|_{L_\infty(Q_T)}\Big\} \subset Q_T = \Omega \times (0,T)$$

satisfies $Q_{b_1} \subsetneq Q_{b_2}$ as $0 < b_2 < b_1 < 1/2$, and its measure is defined by $|\mathcal{Q}_b| \le |Q_T| = T|\Omega|$. Then by (3.9) we know that

$$\begin{aligned}\big(c_3 T + (2C)^4 C_0^2 (c_1 + 2)\big) e^{rT} \ge \int_0^T \int_\Omega u^2 \ge \int_{\mathcal{Q}_b} u^2(x,t) \\ \ge |\mathcal{Q}_b| b^2 \|u(\cdot,t)\|^2_{L_\infty(Q_T)}.\end{aligned} \tag{3.30}$$

In addition, for each $\varepsilon > 0$ one may have

$$\bigcup_{b \in (0,1/2)} \mathcal{Q}_b = Q_T,$$

where $\mathcal{Q}_b \neq Q_T$ for all $b \in (0,1/2)$ due to $u > 0$ in Q_T and $u \ge 0$ in $\Omega \times [0,T)$ (cf. Remark 3.1). In general, if there is a $b_0 \in (0,1)$ such that $\mathcal{Q}_b = Q_T$ for $b \in (0, b_0]$ (i.e. u has a positive lower bound), one may replace all the intervals $(0,1/2)$ by (b_0, κ) with $0 < b_0 < \kappa < 1$.

Now for any fixed T, we have $|Q_T|$ fixed. If $\|u(\cdot,t)\|_{L_\infty(Q_T)} \to +\infty$ as $\varepsilon \to 0$, then letting $\varepsilon \to 0$ in (3.30) gives that

$$|\mathcal{Q}_b| \to 0, \qquad \text{for each } b \in (0,1/2),$$

and thus

$$|Q_T| = \Big| \bigcup_{b \in (0,1/2)} \mathcal{Q}_b \Big| \to 0, \qquad \text{as } \varepsilon \to 0,$$

which is a contradiction. Then there exists some constant $M(T) > 0$ bounded as $\varepsilon \to 0$, such that

$$\|u\|_{L_\infty(Q_T)} \le M(T), \qquad \text{for any } \varepsilon > 0.$$

This completes the proof. □

Lemma 3.9 means that for any given $0 < T < +\infty$, one may find

$$\|u(\cdot,t)\|_{L_\infty(\Omega)} \le \sup_{t \in (0,T)} \|u(\cdot,t)\|_{L_\infty(\Omega)} \le C(T) \quad \text{and} \quad \|a(\cdot,t)\|_{L_\infty(\Omega)} \le C(T), \tag{3.31}$$

where $C(T)$ is increasing in T and can be independent of $\varepsilon \in (0,1)$.

With this result at hand, we may estimate uniformly in ε the second order derivatives of a and w as below.

Lemma 3.10. *Let $\Omega \subset \mathbb{R}^2$. We may derive that*

$$\int_\Omega (|\nabla a(\cdot,t)|^2 + |\Delta w(\cdot,t)|^2) \le C(T), \quad \text{and} \quad \int_0^T \int_\Omega |\Delta a(\cdot,t)|^2 \le C(T)$$

for any $t \in (0,T) \subset (0,T_{\max})$, where $C(T)$ is increasing in T and independent of ε.

Proof. Invoking $a_t = -\chi a\{\varepsilon\Delta w - ae^{\chi w}w + rw(1-w)\} + \chi\nabla w\cdot\nabla a + \Delta a + \gamma aw - a(1+lae^{\chi w})$ and using zero-flux boundary condition on a, we may calculate that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla a|^2 = \int_\Omega \nabla a\cdot\nabla a_t = -\int_\Omega \Delta a\ a_t\\
=&-\int_\Omega \Delta a\big\{-\chi a\{\varepsilon\Delta w - ae^{\chi w}w + rw(1-w)\} + \chi\nabla w\cdot\nabla a + \Delta a + \gamma aw - a(1+lae^{\chi w})\big\}\\
=&-\int_\Omega \Delta a\big\{\chi a^2 e^{\chi w}w - r\chi aw(1-w) + \gamma aw - a(1+lae^{\chi w})\big\}\\
&-\int_\Omega |\Delta a|^2 + \chi\varepsilon\int_\Omega a\Delta a\Delta w - \chi\int_\Omega \Delta a\nabla w\cdot\nabla a\\
\le& C(T)\int_\Omega |\Delta a| - \int_\Omega |\Delta a|^2 + \chi\varepsilon\int_\Omega a\Delta a\Delta w - \chi\int_\Omega \Delta a\nabla w\cdot\nabla a
\end{aligned}$$

with $\| -\chi a^2 e^{\chi w}w + r\chi aw(1-w) + \gamma aw - a(1+lae^{\chi w})\|_{L_\infty(\Omega)} \le C(T)$.

Furthermore, by Young's inequality with parameter $\eta > 0$ one may derive that

$$C(T)\int_\Omega |\Delta a| \le \frac{C^2(T)|\Omega|\eta}{2} + \frac{1}{2\eta}\int_\Omega |\Delta a|^2 \le 2C^2(T)|\Omega| + \frac{1}{8}\int_\Omega |\Delta a|^2, \quad \eta = 4,$$

$$\chi(\varepsilon+1)\int_\Omega a\Delta a\Delta w \le 2\big(\chi(1+\varepsilon)\|a\|_{L_\infty(\Omega)}\big)^2\int_\Omega |\Delta w|^2 + \frac{1}{8}\int_\Omega |\Delta a|^2, \quad \eta = 4\chi(1+\varepsilon)\|a\|_{L_\infty(\Omega)},$$

$$-\chi\int_\Omega \Delta a\nabla w\cdot\nabla a \le 2\chi^2\int_\Omega |\nabla w\cdot\nabla a|^2 + \frac{1}{8}\int_\Omega |\Delta a|^2, \quad \eta = 4\chi,$$

$$2\chi^2\int_\Omega|\nabla w\cdot\nabla a|^2\le\chi^2\eta\int_\Omega|\nabla w|^4+\frac{\chi^2}{\eta}\int_\Omega|\nabla a|^4\le\hat{c}_0\int_\Omega|\Delta w|^2+\frac{1}{8}\int_\Omega|\Delta a|^2+\hat{c}_1$$

with $\eta=8\chi^2(2C)^4\|a\|^2_{L_\infty(\Omega)}$, $\hat{c}_0=\eta\chi^2(2C)^4K^2$, $\hat{c}_1=\frac{1}{8}\|a\|^2_{L_\infty(\Omega)}+\eta\chi^2(2C)^4K^4$, where we have applied Gagliardo-Nirenberg interpolation inequality ($n=2$) to $\int_\Omega|\nabla w|^4$ and $\int_\Omega|\nabla a|^4$, that is,

$$\int_\Omega|\nabla A|^4=\|\nabla A\|^4_{L_4(\Omega)}\le(2C)^4\big\{\|\Delta A\|^2_{L_2(\Omega)}\|A\|^2_{L_\infty(\Omega)}+\|A\|^4_{L_\infty(\Omega)}\big\}. \tag{3.32}$$

Combining these estimates may yield that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla a|^2+\frac{1}{2}\int_\Omega|\Delta a|^2\le\hat{c}_2\int_\Omega|\Delta w|^2+\hat{c}_3 \tag{3.33}$$

with $\hat{c}_2:=\hat{c}_0+2\big(\chi(1+\varepsilon)\|a\|_{L_\infty(\Omega)}\big)^2$ and $\hat{c}_3:=\hat{c}_1+2C^2(T)|\Omega|$.

Below we proceed to estimate $\int_\Omega|\Delta w|^2$. Indeed, by $\nabla w\cdot\vec{\nu}\big|_{\partial\Omega}=0$, we have

$$\int_{\partial\Omega}\Delta w\nabla w_t\cdot\vec{\nu}=\int_{\partial\Omega}\Delta w\frac{\mathrm{d}}{\mathrm{d}t}(\nabla w\cdot\vec{\nu})=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial\Omega}\Delta w(\nabla w\cdot\vec{\nu})-\int_{\partial\Omega}(\nabla w\cdot\vec{\nu})\frac{\mathrm{d}}{\mathrm{d}t}\Delta w=0,$$

and thus by $\frac{\mathrm{d}}{\mathrm{d}t}|\Delta w|=\frac{\Delta w}{|\Delta w|}\Delta w_t$ and $w_t=\varepsilon\Delta w-ae^{\chi w}w+rw(1-w)$, one may compute that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\Delta w|^2=\int_\Omega\Delta w\Delta w_t=-\int_\Omega\nabla\Delta w\cdot\nabla w_t+\int_{\partial\Omega}\Delta w\nabla w_t\cdot\vec{\nu}\\
=&-\int_\Omega\nabla\Delta w\cdot\big\{\varepsilon\nabla\Delta w-e^{\chi w}w\nabla a-\chi ae^{\chi w}w\nabla w-ae^{\chi w}\nabla w+r\nabla w-2rw\nabla w)\big\}\\
=&-\varepsilon\int_\Omega|\nabla\Delta w|^2+\int_\Omega\Delta w\nabla\cdot\big\{-e^{\chi w}w\nabla a-\chi ae^{\chi w}w\nabla w-ae^{\chi w}\nabla w+r\nabla w-2rw\nabla w)\big\}\\
=&-\varepsilon\int_\Omega|\nabla\Delta w|^2+\int_\Omega\Delta w\big\{-2(\chi w+1)e^{\chi w}\nabla w\cdot\nabla a-(\chi w+2)\chi ae^{\chi w}|\nabla w|^2-2r|\nabla w|^2\\
&-(\chi ae^{\chi w}w+ae^{\chi w}+2rw)\Delta w+r\Delta w-e^{\chi w}w\Delta a\big\}.
\end{aligned}$$

A joint use of $0<w<K$ and Young's inequality with parameter $\eta>0$ will produce that

$$-2\int_\Omega \Delta w(\chi w+1)e^{\chi w}\nabla w\cdot\nabla a \le \hat c_4\int_\Omega|\Delta w|^2+\frac{\hat c_4}{2\eta}\int_\Omega|\nabla w|^4+\frac{\eta\hat c_4}{2}\int_\Omega|\nabla a|^4$$

with $\hat c_4 := (\chi K+1)e^{\chi K}$. Then using (3.32) may yield that

$$\frac{\eta\hat c_4}{2}\int_\Omega|\nabla a|^4\le\frac18\int_\Omega|\Delta a|^2+\hat c_5,\quad\text{and}\quad\frac{\hat c_4}{2\eta}\int_\Omega|\nabla w|^4\le\hat c_6\int_\Omega|\Delta w|^2+\hat c_7$$

where we take $\eta=\frac{1}{4\hat c_4(2C)^4\|a\|^2_{L_\infty(\Omega)}}$, $\hat c_5:=\frac18\|a\|^2_{L_\infty(\Omega)}$, $\hat c_6:=\frac{\hat c_4}{2\eta}(2C)^4K^2$, and $\hat c_7:=\frac{\hat c_4}{2\eta}(2C)^4K^4$. Another application of (3.32) yields that

$$-\int_\Omega\big((\chi w+2)\chi ae^{\chi w}+2r\big)|\nabla w|^2\Delta w\le\hat c_8\int_\Omega|\Delta w|^2+\hat c_9$$

with $\hat c_8:=\frac{\iota}{2}\big(1+(2C)^4K^2\big)$ and $\hat c_9:=\frac{\iota}{2}(2C)^4K^4$, and $\iota:=2r+\chi(\chi K+2)e^{\chi K}\|a\|_{L_\infty(\Omega)}$. Similarly,

$$-\int_\Omega e^{\chi w}w\Delta w\Delta a\le\frac18\int_\Omega|\Delta a|^2+\hat c_{10}\int_\Omega|\Delta w|^2$$

with $\hat c_{10}:=2(e^{\chi K}K)^2$.

Thereupon, substituting these estimates into $\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\Delta w|^2$ will lead us to

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\Delta w|^2+\varepsilon\int_\Omega|\nabla\Delta w|^2+\int_\Omega(\chi ae^{\chi w}w+ae^{\chi w}+2rw)|\Delta w|^2\\ \le&\hat c_{11}\int_\Omega|\Delta w|^2+\frac14\int_\Omega|\Delta a|^2+\hat c_{12}\end{aligned}\tag{3.34}$$

with $\hat c_{11}:=\hat c_4+\hat c_6+\hat c_8+\hat c_{10}+r$ and $\hat c_{12}:=\hat c_5+\hat c_7+\hat c_9$. In conjunction with (3.33), we thus may infer that

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|\nabla a|^2+|\Delta w|^2)+\frac14\int_\Omega|\Delta a|^2\le(\hat c_2+\hat c_{11})\int_\Omega(|\nabla a|^2+|\Delta w|^2)+(\hat c_3+\hat c_{12}).$$

This implies for any $t\in(0,T)$ that

$$\int_\Omega (|\nabla a(\cdot,t)|^2 + |\Delta w(\cdot,t)|^2) \le \hat{c}_{13}, \quad \text{and thus} \quad \int_0^T \int_\Omega |\Delta a(\cdot,t)|^2 \le \hat{c}_{14}$$

with $\hat{c}_{13} := e^{2(\hat{c}_2+\hat{c}_{11})T}\big\{\frac{\hat{c}_3+\hat{c}_{12}}{\hat{c}_2+\hat{c}_{11}} + \int_\Omega(|\nabla a_0|^2+|\Delta w_0|^2)\big\}$ and $\hat{c}_{14} := 2\int_\Omega(|\nabla a_0|^2+|\Delta w_0|^2)+$ $2T\big(\hat{c}_{13}(\hat{c}_2+\hat{c}_{11})+\hat{c}_3+\hat{c}_{12}\big)$. This completes the proof. □

Corollary 3.2. *Let $\Omega \subset \mathbb{R}^2$. Then we have*

$$\int_\Omega |w_t|^2 + \int_0^T \int_\Omega (|u_t|^2 + |\Delta u|^2) \le C(T), \quad \textit{for any } t \in (0,T),$$

where $C(T)$ is independent of ε.

Proof. By $u = ae^{\chi w}$, we know that $\nabla u = e^{\chi w}\nabla a + \chi e^{\chi w} a \nabla w$ and

$$\Delta u = e^{\chi w}\Delta a + 2\chi e^{\chi w}\nabla a \cdot \nabla w + \chi^2 e^{\chi w} a |\nabla w|^2 + \chi e^{\chi w} a \Delta w.$$

In light of Lemma 3.1, Lemma 3.8, and Lemma 3.9, one may deduce from system (3.2) that

$$\int_\Omega |u_t|^2 \le 3\int_\Omega \Big(|\Delta u|^2 + 2\chi^2(|\nabla u\cdot\nabla w|^2 + u^2|\Delta w|^2) + (\gamma u w + u + l u^2)^2\Big),$$

where

$$3\int_\Omega |\Delta u|^2 \le 12e^{2\chi K}\int_\Omega \Big(|\Delta a|^2 + 2\chi^2|\nabla w|^4 + 2\chi^2|\nabla a|^4 + \chi^4\|a\|^2_{L_\infty(\Omega)}|\nabla w|^4 + \chi^2\|a\|^2_{L_\infty(\Omega)}|\Delta w|^2\Big)$$

and

$$6\chi^2\int_\Omega |\nabla u\cdot\nabla w|^2 \le 6\chi^2\Big\{2e^{2\chi K}\int_\Omega|\nabla a|^4 + 2e^{2\chi K}(1+\chi^2\|a\|^2_{L_\infty(\Omega)})\int_\Omega|\nabla w|^4\Big\}.$$

Thereupon, we derive that

$$\int_\Omega |u_t|^2 \le \hat{c}_{15}\int_\Omega|\Delta a|^2 + \hat{c}_{16}\int_\Omega|\Delta w|^2 + \hat{c}_{17}\int_\Omega|\nabla a|^4 + \hat{c}_{18}\int_\Omega|\nabla w|^4 + \hat{c}_{19}$$

where $\hat{c}_{15} := 12e^{2\chi K}$, $\hat{c}_{16} := 18\chi^2 e^{2\chi K}\|a\|^2_{L_\infty(\Omega)}$, $\hat{c}_{17} := 36\chi^2 e^{2\chi K}$, $\hat{c}_{18} := 12\chi^2 e^{2\chi K}(3+$

$2\chi^2\|a\|^2_{L_\infty(\Omega)})$, and $\hat{c}_{19} := 3|\Omega|e^{2\chi K}\big(\gamma K + 1 + le^{\chi K}\|a\|_{L_\infty(\Omega)}\big)^2\|a\|^2_{L_\infty(\Omega)}$. In addition, applying (3.32) yields

$$\hat{c}_{17}\int_\Omega |\nabla a|^4 \le \hat{c}_{17}(2C)^4\|a\|^2_{L_\infty(\Omega)}\int_\Omega |\Delta a|^2 + \hat{c}_{17}(2C)^4\|a\|^4_{L_\infty(\Omega)}$$

and

$$\hat{c}_{18}\int_\Omega |\nabla w|^4 \le \hat{c}_{19}(2C)^4K^2\int_\Omega |\Delta w|^2 + \hat{c}_{18}(2C)^4K^4.$$

It follows that

$$\int_\Omega |u_t|^2 \le \hat{c}_{20}\int_\Omega |\Delta a|^2 + \hat{c}_{21}\int_\Omega |\Delta w|^2 + \hat{c}_{22} \tag{3.35}$$

with $\hat{c}_{20} := \hat{c}_{15} + \hat{c}_{17}(2C)^4\|a\|^2_{L_\infty(\Omega)}$, $\hat{c}_{21} := \hat{c}_{16} + \hat{c}_{18}(2C)^4K^2$, and $\hat{c}_{22} := \hat{c}_{19} + \hat{c}_{17}(2C)^4\|a\|^4_{L_\infty(\Omega)} + \hat{c}_{18}(2C)^4K^4$. Likewise, one may derive that

$$\int_\Omega |\Delta u|^2 \le \hat{c}_{23}\int_\Omega |\Delta a|^2 + \hat{c}_{24}\int_\Omega |\Delta w|^2 + \hat{c}_{25} \tag{3.36}$$

with $\hat{c}_{23} := 4e^{2\chi K}\big(1 + 2\chi^2(2C)^4\|a\|^2_{L_\infty(\Omega)}\big)$, $\hat{c}_{24} := 4e^{2\chi K}\big(\|a\|^2_{L_\infty(\Omega)} + (2+\chi^2\|a\|^2_{L_\infty(\Omega)})$ $\cdot\,(2C)^4K^2\big)$, and $\hat{c}_{25} := 8\chi^2e^{2\chi K}(2C\|a\|_{L_\infty(\Omega)})^4 + 4\chi^2e^{2\chi K}(2+\chi^2\|a\|^2_{L_\infty(\Omega)})(2CK)^4$.

Moreover,

$$\int_\Omega |w_t|^2 \le \int_\Omega |\varepsilon\Delta w - awe^{\chi w} + rw(1-w)|^2 \le (1+2\varepsilon^2)\int_\Omega |\Delta w|^2 + \hat{c}_{26} \tag{3.37}$$

with $\hat{c}_{26} := 2K^2\big(\|a\|_{L_\infty(\Omega)}e^{\chi K} + r + rK\big)^2$.

Consequently, we may complete this proof by integrating (3.35), (3.36), and (3.37) with respect to $t \in (0,T)$, then by making use of Lemma 3.10. □

Remark 3.4. *Some illustrations are needed here:*

(i) Estimate (3.31)*, Lemma 3.10, and Corollary 3.2 are used to discuss, as* $\varepsilon \to 0$*, the convergence of the strong solution from* (3.2) *to* (3.3) *in section 3.3.*

(ii) As a simple consequence, the following estimate (3.38) *implies that the classical*

solution of (3.2)*, for each fixed* $\varepsilon > 0$*, is a strong solution. However, it is hard to ensure that the* $c(T)$ *in* (3.38) *remains bounded as* $\varepsilon \to 0$.

Lemma 3.11. *Assume that* (a, w) *is the global-in-time classical solution of* (1.13)—(1.15)*. Then under assumption* (3.4) *and for each fixed* $\varepsilon > 0$*, one may have*

$$\|w\|_{W_p^{2,1}(Q_T)} \le c(T), \qquad 2 < p < +\infty, \tag{3.38}$$

and for any $\varepsilon \in (0, 1)$*,*

$$\|\nabla w(\cdot, t)\|_{L_\infty(\Omega)} \le C(T),\ t \in (0, T), \quad \|a\|_{W_p^{2,1}(Q_T)} \le C(T),\ 2 < p < +\infty, \tag{3.39}$$

where two $C(T)$ *in* (3.39) *remain bounded for finite values of* T *and are independent of* ε.

Proof. Note that for any $\varepsilon > 0$ fixed and any given $0 < T < +\infty$, the w fulfills

$$\begin{cases} w_t - \varepsilon \Delta w = -a w e^{\chi w} + r w(1 - w), & \text{in } Q_T, \\ \nabla w \cdot \vec{\nu}\big|_{\partial\Omega} = 0, \nabla w_0 \cdot \vec{\nu}\big|_{\partial\Omega} = 0, & t \in (0, T), \\ w(x, 0) = w_0(x) > 0, & x \in \Omega, \end{cases}$$

which, combined with L_p-theory of linear parabolic equations, shows that

$$\|w\|_{W_p^{2,1}(Q_T)} \le c(T)\big\{\|w_0\|_{W_p^2(\Omega)} + \| - a w e^{\chi w} + r w(1 - w)\|_{L_p(Q_T)}\big\}$$

where $c(T)$ is bounded for any finite values of T but we may not ensure the boundedness as $\varepsilon \to 0$.

On the other hand, by semigroup theory (cf. the proof of Lemma A.5) the solution w fulfills

$$w(x, t) = e^{-t(1 - \varepsilon\Delta)} w_0(x) + \int_0^t e^{-(t-s)(1-\varepsilon\Delta)} g(w, s) \mathrm{d}s,$$

where $g(w, t)(x) = -a(x, t) w e^{\chi w} + (r + 1) w - r w^2$ is locally Lipschitz continuous in w and locally Hölder continuous in t, i.e., $\|g(w, t) - g(v, s)\|_{W_p^2(\Omega)} \le C(g)\{|t - s|^\beta + \|w - v\|_{W_p^2(\Omega)}\}$ for some $\beta \in (0, 1)$ and $w, v \in D(1 - \varepsilon\Delta)$, and the constant $C(g)$

remains bounded as $\varepsilon \to 0$. This is not difficult to obtain if we can show the local Hölder's continuity in t (note $0 < w < K$ in this case). Indeed, for any $t > 0$, one may compute the Hölder constant for a as

$$< a >_t = \sup_{x,t \in Q_T} \frac{|a(x,t) - a(x,0)|}{|t-0|^\beta} \leq \frac{1}{t_0^\beta} \sup_{t \in (0,T)} \Big\{ \|a(\cdot,t)\|_{L_\infty(\Omega)} + \|a_0\|_{L_\infty(\Omega)} \Big\}$$

for some $t_0 \in (0, T]$ by regularity (3.5), and $< a >_t$ remains bounded as $\varepsilon \to 0$ by (3.31).

Thus by Remark A.1, one may obtain

$$\|\nabla w(\cdot,t)\|_{L_\infty(\Omega)} \leq C(T)$$

due to the boundedness of $\|w_0\|_{C^2(\overline{\Omega})}$ supposed in (3.4) and of $\| - awe^{\chi w} + rw(1 - w)\|_{L_\infty(\Omega)}$, where $C(T)$ is dependent of ε as $\varepsilon \in (0, 1)$.

Taking derivative from both sides with respect to t shows that

$$\begin{aligned} w_t =& - (1-\varepsilon\Delta)e^{-t(1-\varepsilon\Delta)} w_0(x) - \int_0^t (1-\varepsilon\Delta)e^{-(t-s)(1-\varepsilon\Delta)} g(w,s)\mathrm{d}s + g(w,t) \\ =& - (1-\varepsilon\Delta)e^{-t(1-\varepsilon\Delta)} w_0(x) - \int_0^t (1-\varepsilon\Delta)e^{-(t-s)(1-\varepsilon\Delta)} g(w,s)\mathrm{d}s \\ &+ \int_0^t (1-\varepsilon\Delta)e^{-(t-s)(1-\varepsilon\Delta)} g(w,t)\mathrm{d}s + e^{-t(1-\varepsilon\Delta)} g(w,t) \\ =& - (1-\varepsilon\Delta)e^{-t(1-\varepsilon\Delta)} w_0(x) + e^{-t(1-\varepsilon\Delta)} g(w,t) \\ &+ \int_0^t (1-\varepsilon\Delta)e^{-(t-s)(1-\varepsilon\Delta)} \big\{ g(w,t) - g(w,s) \big\} \mathrm{d}s. \end{aligned}$$

and thus for $p \in (1, +\infty)$ there exists some $0 < \delta < \mathrm{Re}(\sigma(1 - \varepsilon\Delta))$ such that

$$\begin{aligned} \|w_t\|_{L_p(\Omega)} \leq& \|(1-\varepsilon\Delta)e^{-t(1-\varepsilon\Delta)} w_0\|_{L_p(\Omega)} + \|e^{-t(1-\varepsilon\Delta)} g(w,t)\|_{L_p(\Omega)} \\ &+ \int_0^t \|(1-\varepsilon\Delta)e^{-(t-s)(1-\varepsilon\Delta)}\| \cdot \|g(w,t) - g(w,s)\|_{L_p(\Omega)} \mathrm{d}s \end{aligned}$$

$$\leq C\Big\{t^{-1}e^{-\delta t}\|w_0\|_{L_p(\Omega)} + e^{-\delta t}\|g(w,t)\|_{L_p(\Omega)} + C(g)\int_0^t (t-s)^{\beta-1}e^{-\delta(t-s)}\mathrm{d}s\Big\}$$

$$< +\infty,$$

for any $\varepsilon \in (0,1)$.

This combined with Lemma 3.10, means that $w_t, \nabla w \in L_p(Q_T)$ for any finite $p \geq 2$. Together with $a, w \in L_\infty(Q_T)$ and thus $g \in L_p(Q_T)$, we may apply L_p-theory of linear parabolic equations to

$$\begin{cases} a_t - \Delta a - \chi \nabla w \cdot \nabla a = -\chi a w_t + \gamma a w - a(1 + lae^{\chi w}), & \text{in } Q_T, \\ \nabla a \cdot \vec{\nu}\big|_{\partial\Omega} = 0, & t \in (0,T), \\ a(x,0) = a_0(x) > 0, & x \in \Omega, \end{cases}$$

to acquire

$$\|a\|_{W_p^{2,1}(Q_T)} \leq c(T)\big\{\|a_0\|_{W_p^2(\Omega)} + \| -\chi a w_t + \gamma a w - a(1 + lae^{\chi w})\|_{L_p(Q_T)}\big\},$$

where $c(T)$ is bounded for any finite values of T.

Together with the $L_\infty(\Omega)$ boundedness of a given in (3.31) and $0 < w < K$, one may complete this proof.

□

3.3 Convergence of (3.2) to the System (3.3)

In section 3.2 we have proved, for any fixed $\varepsilon > 0$, that the unique classical solution to (3.2) exits globally in time. Denote by $(u_\varepsilon, w_\varepsilon)$ the classical solution of (3.2), thus a strong solution of (3.2) due to Lemma 3.11. In this section we shall check the convergence of $(u_\varepsilon, w_\varepsilon)$, as $\varepsilon \to 0$.

3.3.1 Passing to the Limit as $\varepsilon \to 0$

For any given $0 < T < +\infty$, any $(x,t) \in Q_T = \Omega \times (0,T)$ and any $\varepsilon > 0$, one may derive from Lemma 3.9 that:

- by Lemma 3.7, and Lemma 3.8, we know $u_\varepsilon, w_\varepsilon \in L_\infty(Q_T)$ and

$$\sup_{t\in(0,T]} \|(u_\varepsilon, w_\varepsilon)(\cdot,t)\|_{L_\infty(\Omega)} \le C(T);$$

- by Lemma 3.10, Corollary 3.2, and Lemma 3.11, we see u_ε, $w_\varepsilon \in W_2^{2,1}(Q_T)$ and

$$\|(u_\varepsilon, w_\varepsilon)\|_{W_2^{2,1}(Q_T)} \le C(T), \qquad \|\nabla w_\varepsilon\|_{L_\infty(Q_T)} \le C(T),$$

where $C(T)$ is independent of ε.

Below we still denoted by $\{u_\varepsilon\}_{\varepsilon>0}$ and $\{w_\varepsilon\}_{\varepsilon>0}$ their corresponding subsequences as we desire, for brevity of notations. Then it is readily to see that there exists only one pair of $(u, w) \in W_2^{2,1}(Q_T) \times W_2^{2,1}(Q_T)$ (here (u, w) may not be the classical solution of (3.3)) such that

$$u_\varepsilon \rightharpoonup u \ \text{ in } \ W_2^{2,1}(Q_T), \quad w_\varepsilon \rightharpoonup w \ \text{ in } \ W_2^{2,1}(Q_T), \quad \text{as } \varepsilon \to 0, \tag{3.40}$$

by the weak compactness of a reflexive Banach space, and

$$\|(u, w)\|_{W_2^{2,1}(Q_T)} \le \liminf_{\varepsilon\to 0} \|(u_\varepsilon, w_\varepsilon)\|_{W_2^{2,1}(Q_T)} \le C(T) \tag{3.41}$$

by boundedness of a weakly converging sequence. Furthermore, there exist

$$u_\varepsilon \overset{*}{\rightharpoonup} u \quad \text{in } L_\infty(Q_T) \qquad \text{and} \qquad w_\varepsilon \overset{*}{\rightharpoonup} w \quad \text{in } L_\infty(Q_T), \tag{3.42}$$

by $L_\infty(Q_T) = \big(L_1(Q_T)\big)^*$ in the sense of isometric isomorphism and $*$-weak compactness of normed linear space $L_\infty(Q_T)$. One may infer

$$\|(u, w)\|_{L_\infty(Q_T)} \le \liminf_{\varepsilon\to 0} \|(u_\varepsilon, w_\varepsilon)\|_{L_\infty(Q_T)} \le C(T) \tag{3.43}$$

from the boundedness of a $*$-weak convergent sequence in $L_\infty(Q_T)$.

Lemma 3.12. *One may derive that the above (u, w) satisfies*

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla w) + \gamma uw - u(1+lu), & a.e.\ (x,t) \in Q_T, \\ w_t = -uw + rw(1-w), & a.e.\ (x,t) \in Q_T, \\ (\nabla u - \chi u\nabla w)\cdot\vec{\nu} = 0, & a.e.,\ (x,t) \in \partial\Omega\times(0,T). \end{cases}$$

Proof. We shall first prove that for any $\phi \in L_2(Q_T)$ that

$$\int_{Q_T} \big(u_t - \Delta u + \chi \nabla \cdot (u \nabla w) - \gamma u w + u(1+lu)\big)\phi = 0$$
$$\int_{Q_T} \big(w_t + uw - rw(1-w)\big)\phi = 0, \tag{3.44}$$

by taking $\varepsilon \to 0$ in

$$\int_{Q_T} \big(u_{\varepsilon t} - \Delta u_\varepsilon + \chi \nabla \cdot (u_\varepsilon \nabla w_\varepsilon) - \gamma u_\varepsilon w_\varepsilon + u_\varepsilon(1+lu_\varepsilon)\big)\phi = 0,$$
$$\int_{Q_T} \big(w_{\varepsilon t} - \varepsilon \Delta w_\varepsilon + u_\varepsilon w_\varepsilon - r w_\varepsilon(1-w_\varepsilon)\big)\phi = 0. \tag{3.45}$$

This can be divided into the following four parts:

(i) Indeed, (3.40) allows us to collect that

$$\int_{Q_T} u_{\varepsilon t}\phi \to \int_{Q_T} u_t \phi, \quad \int_{Q_T} \Delta u_\varepsilon \phi \to \int_{Q_T} \Delta u \phi,$$

and

$$\int_{Q_T} w_{\varepsilon t}\phi \to \int_{Q_T} w_t \phi, \quad \int_{Q_T} \varepsilon \Delta w_\varepsilon \phi \to 0.$$

(ii) Thanks to (3.42), one may readily get that

$$\int_{Q_T} u_\varepsilon \phi \to \int_{Q_T} u\phi \quad \text{and} \quad \int_{Q_T} w_\varepsilon \phi \to \int_{Q_T} w\phi,$$

for any $\phi \in L_2(Q_T) \subset L_1(Q_T)$ for $0 < T < \infty$.

(iii) By Lemma 3.10, Corollary 3.2, and Lemma 3.11, we see that for $p \in [2, +\infty)$, $u_\varepsilon \in L_p(0,T; W_2^2(\Omega))$, $w_\varepsilon \in L_\infty(0,T; W_2^2(\Omega))$ and $u_{\varepsilon t}, w_{\varepsilon t} \in L_2(0,T; L_2(\Omega))$. In addition, we know the compact Sobolev embeddings (cf. [69, Thm. 6.3]) that for $n = 2$

$$W_2^2(\Omega) \hookrightarrow\hookrightarrow W_q^1(\Omega), \quad 1 \le q < \infty \tag{3.46}$$

and that for $q > n \in \mathbb{N}_+$,

$$W_q^1(\Omega) \hookrightarrow\hookrightarrow L_p(\Omega), \quad 1 \le p < +\infty. \tag{3.47}$$

Then we invoke Aubin-Lions lemma [70, 71, 72] (e.g. [70, Thm. 5 and Corol.4, in Sect.8]) to infer that for $2 \le p < +\infty$ and $2 = n < q < \infty$,

$$u_\varepsilon \to u \quad \text{in} \quad L_p(0,T; W_q^1(\Omega)) \quad \text{and} \quad w_\varepsilon \to w \quad \text{in} \quad L_\infty(0,T; W_q^1(\Omega)). \tag{3.48}$$

Hence by (3.43) and Hölder's inequality, we have for $\varepsilon \to 0$ that

$$\begin{aligned}
&\int_{Q_T} |(u_\varepsilon w_\varepsilon - uw)\phi| \\
\le& \|w_\varepsilon\|_{L_\infty(Q_T)} \int_{Q_T} |u_\varepsilon - u||\phi| + \|u\|_{L_\infty(Q_T)} \int_{Q_T} |w_\varepsilon - w||\phi| \\
\le& \|w_\varepsilon\|_{L_\infty(Q_T)} \cdot \|u_\varepsilon - u\|_{L_2(Q_T)} \cdot \|\phi\|_{L_2(Q_T)} \\
&\qquad + \|u\|_{L_\infty(Q_T)} \cdot \|w_\varepsilon - w\|_{L_2(Q_T)} \cdot \|\phi\|_{L_2(Q_T)} \\
\le& \big\{ \|w_\varepsilon\|_{L_\infty(Q_T)} \cdot \|u_\varepsilon - u\|_{L_2(0,T;W_4^1(\Omega))} \\
&\qquad + T^{1/2} \|u\|_{L_\infty(Q_T)} \cdot \|w_\varepsilon - w\|_{L_\infty(0,T;W_4^1(\Omega))} \big\} |\Omega|^{1/4} \|\phi\|_{L_2(Q_T)} \\
\to& 0,
\end{aligned}$$

and similarly

$$\int_{Q_T} |(u_\varepsilon^2 - u^2)\phi| \le \|u_\varepsilon - u\|_{L_2(Q_T)} \cdot \|u_\varepsilon + u\|_{L_\infty(Q_T)} \cdot \|\phi\|_{L_2(Q_T)} \to 0,$$

as well as

$$\int_{Q_T} |(w_\varepsilon^2 - w^2)\phi| \le \|w_\varepsilon - w\|_{L_2(Q_T)} \cdot \|w_\varepsilon + w\|_{L_\infty(Q_T)} \cdot \|\phi\|_{L_2(Q_T)} \to 0.$$

In addition, we may deduce that

$$\int_{Q_T} |(\nabla u_\varepsilon \nabla w_\varepsilon - \nabla u \cdot \nabla w)\phi| \leq \int_{Q_T} |\nabla u_\varepsilon - \nabla u||\nabla w_\varepsilon|\,|\phi| + \int_{Q_T} |\nabla w_\varepsilon - \nabla w||\nabla u|\,|\phi|$$

where by Hölder's inequality one has

$$\begin{aligned}\int_{Q_T} |\nabla u_\varepsilon - \nabla u||\nabla w_\varepsilon|\,|\phi| \leq& \|\nabla u_\varepsilon - \nabla u\|_{L_2(Q_T)} \cdot \|\nabla w_\varepsilon\|_{L_\infty(Q_T)} \cdot \|\phi\|_{L_2(Q_T)} \\ \leq& \|u_\varepsilon - u\|_{L_2(0,T;W_4^1(\Omega))} \cdot \|\nabla w_\varepsilon\|_{L_\infty(Q_T)} \cdot \|\phi\|_{L_2(Q_T)} |\Omega|^{1/4} \\ \to& 0\end{aligned}$$

and similarly

$$\begin{aligned}&\int_{Q_T} |\nabla w_\varepsilon - \nabla w||\nabla u|\,|\phi| \leq \|(\nabla w_\varepsilon - \nabla w)\nabla u\|_{L_2(Q_T)} \|\phi\|_{L_2(Q_T)} \\ \leq& \sup_{t \in (0,T)} \|w_\varepsilon - w\|_{W_4^1(\Omega)} \cdot \|u\|_{L_2(0,T;W_4^1(\Omega))} \cdot \|\phi\|_{L_2(Q_T)} \\ \to& 0,\end{aligned}$$

by the uniform (in $\varepsilon > 0$) upper bound of $\|\nabla w_\varepsilon\|_{L_\infty(Q_T)}$ and $\|u\|_{L_2(0,T;W_4^1(\Omega))}$.

(iv) The last one is to prove that

$$\int_{Q_T} u_\varepsilon \Delta w_\varepsilon \phi \to \int_{Q_T} u \Delta w \phi, \qquad \text{as } \varepsilon \to 0. \tag{3.49}$$

In fact, by $u_\varepsilon \Delta w_\varepsilon - u\Delta w = u_\varepsilon \Delta w_\varepsilon - u\Delta w_\varepsilon + u\Delta w_\varepsilon - u\Delta w$ then we have

$$\int_{Q_T} (u\Delta w_\varepsilon - u\Delta w)\phi \to 0$$

in view of $u\phi \in L_2(Q_T)$ by $u \in L_\infty(Q_T)$, and of weak compactness of a sequence in $W_2^{2,1}(Q_T)$. In addition, we have

$$\int_{Q_T} |(u_\varepsilon - u)\Delta w_\varepsilon \phi| \leq \|\phi\|_{L_2(Q_T)} \|(u_\varepsilon - u)\Delta w_\varepsilon\|_{L_2(Q_T)}$$

$$\leq \|\phi\|_{L_2(Q_T)} \sup_{t\in(0,T)} \|u_\varepsilon - u\|_{L_\infty(\Omega)} \cdot \|\Delta w_\varepsilon\|_{L_2(Q_T)}$$

$$\leq \|\phi\|_{L_2(Q_T)} \|u_\varepsilon(\cdot,\tilde{t}) - u(x,\tilde{t})\|_{W_q^1(\Omega)} \cdot \|w_\varepsilon\|_{W_2^{2,1}(Q_T)} \to 0.$$

We remark that $\tilde{t} \in [0,T]$ exists since the uniform (in $\varepsilon > 0$) boundedness of $\|u\|_{L_\infty(Q_T)}$ and $\|u_\varepsilon\|_{L_\infty(Q_T)}$ given in (3.42) will make sense $\|u_\varepsilon - u\|_{L_\infty(Q_T)}$ and (3.39) will indicate the continuity in t. Then the Sobolev embedding (cf. [69, II of Thm. 6.3] makes sense the last inequality above. Finally a joint use of (3.40) (where Q_T can be replaced by $[0,T]\times\Omega$) and compact embedding (3.46) will show that this estimate tends to zero. Therefore, we may arrive at (3.49), i. e.,

$$\int_{Q_T} (u_\varepsilon \Delta w_\varepsilon - u\Delta w_\varepsilon)\phi \to 0.$$

Together (i)-(iv), we thus can take $\varepsilon \to 0$ in (3.45) to obtain (3.44). Due to the arbitrary $\phi \in L_2(Q_T)$, one may readily conclude that (u,w) in (3.40) satisfies

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla w) + \gamma uw - u(1+lu), & \text{a.e. } (x,t)\in Q_T \\ w_t = -uw + rw(1-w), & \text{a.e. } (x,t)\in Q_T. \end{cases} \tag{3.50}$$

We are next to verify that this (u,w) fulfills the zero-flux boundary, i.e.,

$$(\nabla u - \chi u\nabla w)\cdot\vec{\nu} = 0, \quad \text{a.e., } (x,t)\in\partial\Omega\times(0,T) \tag{3.51}$$

by showing

$$\int_0^T \int_{\partial\Omega} |(\nabla u - \chi u\nabla w)\cdot\vec{\nu}| = 0.$$

As a matter of fact, an application of $(\nabla u_\varepsilon - \chi u_\varepsilon\nabla w_\varepsilon)\cdot\vec{\nu}\big|_{\partial\Omega} = 0$ may yield that

$$\begin{aligned} &\|(\nabla u - \chi u\nabla w)\cdot\vec{\nu}\|_{L_1(\partial\Omega)} \\ =&\|(\nabla u - \chi u\nabla w)\cdot\vec{\nu} - (\nabla u_\varepsilon - \chi u_\varepsilon\nabla w_\varepsilon)\cdot\vec{\nu}\|_{L_1(\partial\Omega)} \\ \leq&|\partial\Omega|^{\frac{1}{2}}\|\nabla u - \nabla u_\varepsilon\|_{L_2(\partial\Omega)} + \chi\|u_\varepsilon\nabla w_\varepsilon - u\nabla w\|_{L_1(\partial\Omega)}. \end{aligned}$$

Here we invoke a trace-interpolation inequality (cf. [64, Lemma 2.5] for $n = 2$), that is,

$$\|\nabla A\|_{L_q(\partial\Omega)} \leq C\|A\|^{\theta}_{W^2_q(\Omega)} \cdot \|A\|^{1-\theta}_{L_p(\Omega)} \quad \text{with} \quad \theta = \frac{\frac{1}{2}(1-\frac{1}{q})+\frac{1}{p}}{1-\frac{1}{q}+\frac{1}{p}} \in (0,1) \tag{3.52}$$

for $q > 1$ and $p > 0$. Therefore, by Hölder's inequality with index $\frac{3}{2}$ and 3 one may calculate that

$$\begin{aligned}\int_0^T \|\nabla u - \nabla u_\varepsilon\|^2_{L_2(\partial\Omega)} \leq & C^2 \int_0^T \|u-u_\varepsilon\|^{\frac{4}{3}}_{W^2_2(\Omega)} \cdot \|u-u_\varepsilon\|^{\frac{2}{3}}_{L_4(\Omega)} \\ \leq & C^2 \Big(\int_0^T \|u-u_\varepsilon\|^2_{W^2_2(\Omega)}\Big)^{\frac{2}{3}} \cdot \Big(\int_0^T \|u-u_\varepsilon\|^2_{L_4(\Omega)}\Big)^{\frac{1}{3}} \\ \leq & T^{\frac{1}{3}} C^2 \|u-u_\varepsilon\|^{\frac{4}{3}}_{W^{2,1}_2(Q_T)} \cdot \|u-u_\varepsilon\|^{\frac{2}{3}}_{L_2(0,T;W^1_4(\Omega))} \to 0\end{aligned} \tag{3.53}$$

as $\varepsilon \to 0$, in light of (3.48) and $u_\varepsilon, u \in W^{2,1}_2(Q_T)$.

In addition, it is easy to see that

$$\begin{aligned}&\|u_\varepsilon \nabla w_\varepsilon - u\nabla w\|_{L_1(\partial\Omega)} \\ \leq & \|(u_\varepsilon - u)\nabla w_\varepsilon\|_{L_1(\partial\Omega)} + \|u(\nabla w_\varepsilon - \nabla w)\|_{L_1(\partial\Omega)} \\ \leq & \|u_\varepsilon - u\|_{L_2(\partial\Omega)} \cdot \|\nabla w_\varepsilon\|_{L_2(\partial\Omega)} + \|u\|_{L_2(\partial\Omega)} \cdot \|\nabla w_\varepsilon - \nabla w\|_{L_2(\partial\Omega)}\end{aligned}$$

Then an application of [68, Lemma 2.2] will yield that

$$\|u\|_{L_2(\partial\Omega)} \leq C\|u\|^{\frac{1}{2}}_{W^1_2(\Omega)} \cdot \|u\|^{\frac{1}{2}}_{L_2(\Omega)}$$

and

$$\|u_\varepsilon - u\|_{L_2(\partial\Omega)} \leq C\|u_\varepsilon - u\|^{\frac{1}{2}}_{W^1_2(\Omega)} \cdot \|u_\varepsilon - u\|^{\frac{1}{2}}_{L_2(\Omega)}.$$

Again making use of (3.52) shows that $\|\nabla w_\varepsilon\|_{L_2(\partial\Omega)} \leq C\|w_\varepsilon\|^{\frac{2}{3}}_{W^2_2(\Omega)} \cdot \|w_\varepsilon\|^{\frac{1}{3}}_{L_4(\Omega)}$ and

$$\|\nabla w_\varepsilon - \nabla w\|_{L_2(\partial\Omega)} \leq C\|w_\varepsilon - w\|^{\frac{2}{3}}_{W^2_2(\Omega)} \cdot \|w_\varepsilon - w\|^{\frac{1}{3}}_{L_4(\Omega)}.$$

Thereupon, by Hölder's inequality and (3.48) we immediately arrive at

$$\int_0^T \|u_\varepsilon - u\|_{L_2(\partial\Omega)} \cdot \|\nabla w_\varepsilon\|_{L_2(\partial\Omega)} \leq \Big(\int_0^T \|\nabla w_\varepsilon\|^2_{L_2(\partial\Omega)}\Big)^{\frac{1}{2}} \Big(\int_0^T \|u_\varepsilon - u\|^2_{L_2(\partial\Omega)}\Big)^{\frac{1}{2}}$$

$$\leq C\Big(\int_0^T \|w_\varepsilon\|^{\frac{4}{3}}_{W_2^2(\Omega)} \cdot \|w_\varepsilon\|^{\frac{2}{3}}_{L_2(\Omega)}\Big)^{\frac{1}{2}} \Big(\int_0^T \|u_\varepsilon - u\|_{W_2^1(\Omega)} \cdot \|u_\varepsilon - u\|_{L_2(\Omega)}\Big)^{\frac{1}{2}}$$

$$\leq C\Big(\int_0^T \|w_\varepsilon\|^2_{W_2^2(\Omega)}\Big)^{\frac{1}{3}} \Big(\int_0^T \|w_\varepsilon\|^2_{L_2(\Omega)}\Big)^{\frac{1}{6}} \Big(\int_0^T \|u_\varepsilon - u\|^2_{W_2^1(\Omega)}\Big)^{\frac{1}{2}}$$

$$\leq C\|w_\varepsilon\|^{\frac{2}{3}}_{W_2^{2,1}(Q_T)} \big(\|w_\varepsilon\|^2_{L_\infty(\Omega)}|\Omega|T\big)^{\frac{1}{6}} \Big(\int_0^T \|u_\varepsilon - u\|^2_{W_4^1(\Omega)}|\Omega|^{\frac{1}{2}}\Big)^{\frac{1}{2}} \to 0,$$

and similarly

$$\int_0^T \|u\|_{L_2(\partial\Omega)} \cdot \|\nabla w_\varepsilon - \nabla w\|_{L_2(\partial\Omega)} \leq \Big(\int_0^T \|u\|^2_{L_2(\partial\Omega)}\Big)^{\frac{1}{2}} \Big(\int_0^T \|\nabla w_\varepsilon - \nabla w\|^2_{L_2(\partial\Omega)}\Big)^{\frac{1}{2}}$$

$$\leq C\Big(\int_0^T \|u\|_{W_2^1(\Omega)} \cdot \|u\|_{L_2(\Omega)}\Big)^{\frac{1}{2}} \Big(\int_0^T \|w_\varepsilon - w\|^{\frac{4}{3}}_{W_2^2(\Omega)} \cdot \|w_\varepsilon - w\|^{\frac{2}{3}}_{L_4(\Omega)}\Big)^{\frac{1}{2}}$$

$$\leq C\Big(\int_0^T \|u\|^2_{W_2^1(\Omega)}\Big)^{\frac{1}{4}} \Big(\int_0^T \|u\|^2_{L_2(\Omega)}\Big)^{\frac{1}{4}} \Big(\int_0^T \|w_\varepsilon - w\|^2_{W_2^2(\Omega)}\Big)^{\frac{1}{3}} \Big(\int_0^T \|w_\varepsilon - w\|^2_{L_4(\Omega)}\Big)^{\frac{1}{6}}$$

$$\leq 2C\|u\|^{\frac{1}{2}}_{W_2^{2,1}(Q_T)} \big(\|u\|_{L_\infty(Q_T)}|\Omega|T\big)^{\frac{1}{4}} \Big(\int_0^T \|w_\varepsilon\|^2_{W_2^2(\Omega)} + \|w\|^2_{W_2^2(\Omega)}\Big)^{\frac{1}{3}} \big(T\|w_\varepsilon - w\|^2_{L_\infty(0,T;W_4^1(\Omega))}\big)^{\frac{1}{6}}$$

$$\leq 4C\|u\|^{\frac{1}{2}}_{W_2^{2,1}(Q_T)} \big(\|u\|_{L_\infty(Q_T)}|\Omega|T\big)^{\frac{1}{4}} \|w_\varepsilon\|^{\frac{2}{3}}_{W_2^2(Q_T)} \big(T\|w_\varepsilon - w\|^2_{L_\infty(0,T;W_4^1(\Omega))}\big)^{\frac{1}{6}} \to 0$$

as $\varepsilon \to 0$, by means of (3.48).

We thus have

$$\int_0^T \|(\nabla u - \chi u \nabla w) \cdot \vec{\nu}\|_{L_1(\partial\Omega)} \leq \int_0^T \|\nabla u - \chi u \nabla w\|_{L_1(\partial\Omega)} = 0,$$

which means $(\nabla u - \chi u \nabla w) \cdot \vec{\nu}\big|_{\partial\Omega} = 0$ for almost everywhere $(x,t) \in \partial\Omega \times (0,T)$. This proves (3.51).

On the other hand, replacing $u_\varepsilon - u$ in (3.53) by $w_\varepsilon - w$ will lead us to

$$\int_0^T \|\nabla w_\varepsilon - \nabla w\|^2_{L_2(\partial\Omega)} \to 0, \quad \text{as} \quad \varepsilon \to 0$$

which combined with $\nabla w_\varepsilon \cdot \vec{\nu}\big|_{\partial\Omega} = 0$, instantly indicates $\nabla w \cdot \vec{\nu} = 0$ for almost everywhere $(x,t) \in \partial\Omega \times (0,T)$. Thus we have

$$\int_0^T \|\varepsilon\nabla w_\varepsilon\|^2_{L_2(\partial\Omega)} \le \int_0^T \|\varepsilon\nabla w_\varepsilon - \varepsilon\nabla w\|^2_{L_2(\partial\Omega)} \to 0, \quad \text{as} \quad \varepsilon \to 0$$

which means

$$\varepsilon\nabla w_\varepsilon \cdot \vec{\nu} \to 0, \quad \text{a.e. } (x,t) \in \partial\Omega \times (0,T).$$

A combination of (3.50) and (3.51) may complete this proof. □

3.3.2 Proof of Theorem 3.1

By L_p-theory and Schauder theory of linear parabolic equations, one may derive the following regularity result.

Lemma 3.13. *For any given $T > 0$, the (u,w) which is defined by (3.40) and satisfies the system in Lemma 3.12, is a strong solution to (3.3), that is,*

$$\|(u,w)\|_{W_p^{2,1}(Q_T)} \le C(T), \qquad \textit{for} \quad p \in (2,+\infty).$$

Furthermore, the strong solution (u,w) is a classical solution of (3.3) fulfilling

$$u,\, w \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T), \qquad \textit{for some} \quad \alpha \in (0,1),$$

where $\bar{Q}_T = \overline{\Omega} \times [0,T]$.

Proof. We first prove that (u,w) is a strong solution of (3.3). Indeed, in light of the transformation given in (1.13)–(1.15), we can convert the u-component in (3.3) into the following a-component

$$\begin{cases} a_t - \Delta a + a = g(x,t), & (x,t) \in Q_T, \\ \nabla a \cdot \vec{\nu}\big|_{\partial\Omega} = 0, & t \in (0,T), \\ a(x,0) = a_0(x), & x \in \Omega. \end{cases} \tag{3.54}$$

with $a_0(x) := e^{-\chi w_0(x)} u_0(x)$ and

$$g(x,t) = \chi \nabla w \cdot \nabla a + \gamma a w + \chi a^2 e^{\chi w} w - r\chi a w(1-w) - l a^2 e^{\chi w}.$$

We first have $\nabla a_0 \cdot \vec{\nu}\big|_{\partial\Omega} = 0$ owning to $(\nabla u_0 - \chi \nabla w_0) \cdot \vec{\nu}\big|_{\partial\Omega} = 0$ assumed in (3.4). Furthermore, in order to make use of L_p theory of linear parabolic equations which concludes that

$$\|a\|_{W_p^{2,1}(Q_T)} \le c(T)\big\{\|g\|_{L_p(Q_T)} + \|a_0\|_{W_p^2(\Omega)}\big\}, \quad 2 < p + \infty,$$

we need to estimate

$$\|g\|^p_{L_p(Q_T)} \le c_1(T) + \chi \int_0^T \|\nabla w \cdot \nabla a\|^p_{L_p(\Omega)}$$

with $c_1(T) = |Q_T| \|\chi a^2 e^{\chi w} w - r\chi a w(1-w) - l a^2 e^{\chi w} + \gamma a w\|^p_{L_\infty(Q_T)} < +\infty$ by (3.43). Indeed, we use Young's inequality with index q and $\frac{q}{q-1}$ for $1 < q < +\infty$ and with a parameter $\eta > 0$, to calculate that

$$\|\nabla w \cdot \nabla a\|^p_{L_p(\Omega)} = \int_\Omega |\nabla w \cdot \nabla a|^p \le \eta \int_\Omega |\nabla a|^{qp} + \eta^{-\frac{1}{q-1}} \int_\Omega |\nabla w|^{\frac{qp}{q-1}}$$

where an application of Gagliardo-Nirenberg interpolation inequality produces

$$\|\nabla a\|^{pq}_{L_{pq}(\Omega)} \le (2C)^{pq} \|\Delta a\|^{pq\theta}_{L_p(\Omega)} \cdot \|a\|^{pq(1-\theta)}_{L_\infty(\Omega)} + (2C\|a\|_{L_\infty(\Omega)})^{pq},$$

with $\theta = \frac{\frac{1}{2} - \frac{1}{pq}}{1 - \frac{1}{p}} \in (0,1)$ as long as $p > 2$. Letting $pq\theta = p$ will yield $q = 2$ and thus $\theta = \frac{1}{2}$. Then we see that

$$\eta \int_\Omega |\nabla a|^{2p} \le \eta (4C^2 \|a\|_{L_\infty(\Omega)})^p \|\Delta a\|^p_{L_p(\Omega)} + \eta (2C\|a\|_{L_\infty(\Omega)})^{2p}$$

and

$$\chi\int_0^T \|\nabla w\cdot\nabla a\|^p_{L_p(\Omega)} \leq \eta\chi T(2C\|a\|_{L_\infty(\Omega)})^{2p} + \eta\chi(4C^2\|a\|_{L_\infty(\Omega)})^p\int_0^T \|\Delta a\|^p_{L_p(\Omega)}$$
$$+\frac{\chi}{\eta}\int_0^T \|\nabla w\|^{2p}_{L_{2p}(\Omega)}.$$

In conjunction with these inequalities, we therefore have

$$\begin{aligned}\|a\|^p_{W^{2,1}_p(Q_T)} \leq & 2^p c(T)(\|g\|^p_{L_p(Q_T)} + \|a_0\|^p_{L_p(\Omega)})\\ \leq & 2^p c(T)\Big(c_1(T) + \eta\chi T(2C\|a\|_{L_\infty(\Omega)})^{2p} + \|a_0\|^p_{L_p(\Omega)}\\ & + \eta\chi(4C^2\|a\|_{L_\infty(\Omega)})^p\|a\|^p_{W^{2,1}_p(Q_T)} + \frac{\chi}{\eta}\|\nabla w\|^{2p}_{L_{2p}(Q_T)}\Big)\end{aligned}$$

and then taking $\eta = \frac{1}{\chi 2^{p+1}c(T)(4C_1^2\|a\|_{L_\infty(\Omega)})^p}$ may produce that

$$\begin{aligned}\|a\|^p_{W^{2,1}_p(Q_T)} \leq & 2^{p+1}c(T)\Big(c_1(T) + \eta\chi(2C\|a\|_{L_\infty(\Omega)})^{2p} + \|a_0\|^p_{L_p(\Omega)} + \frac{\chi}{\eta}\|\nabla w\|^{2p}_{L_{2p}(Q_T)}\Big)\\ =: & c_2(T) + c_3(T)\|\nabla w\|^{2p}_{L_{2p}(Q_T)}\end{aligned}$$

with $c_2(T) := 2^{p+1}c(T)\big(c_1(T)+\|a_0\|^p_{L_p(\Omega)}\big)+T\big(\|a\|^{1/2}_{L_\infty(\Omega)}\big)^{2p}$ and $c_3(T) := (\chi 2^{p+1}c(T))^2\cdot$ $(4C_1^2\|a\|_{L_\infty(\Omega)})^p$. Then one may have $\|\nabla w\|_{L_{2p}(Q_T)} \leq T\cdot\sup_{t\in(0,T)}\|w\|_{W^1_{2p}(\Omega)} < +\infty$ by (3.48), and thus

$$\|a\|_{W^{2,1}_p(Q_T)} \leq C(T) < +\infty, \qquad 2 < p < +\infty. \tag{3.55}$$

On the other hand, w-equation in (3.3) satisfies

$$\begin{cases} w_t = -ae^{\chi w}w - rw(1-w), & \text{in } Q_T,\\ w(x,0) = w_0(x), & \text{in } \Omega,\end{cases}$$

which has a unique solution and thus

$$\|w_t\|_{L_\infty(Q_T)} \leq T\big(\|a\|_{L_\infty(\Omega)}e^{\chi\|w\|_{L_\infty(\Omega)}}\|w\|_{L_\infty(\Omega)} + r\|w\|_{L_\infty(\Omega)}(1+\|w\|_{L_\infty(\Omega)})\big).$$

Taking derivatives from both sides may yield that

$$\nabla w_t = h\nabla w + \hat{g} \text{ or } \nabla w(x,t) = \nabla w_0(x) e^{\int_0^t h(x,s)\mathrm{d}s} + \int_0^t \hat{g}(x,\tau) e^{\int_\tau^t h(x,s)\mathrm{d}s} \mathrm{d}\tau \quad (3.56)$$

with $h := -\chi a e^{\chi w} w - a e^{\chi w} + r - 2rw \in L_\infty(Q_T)$ by (3.43), and $\hat{g} = -w e^{\chi w} \nabla a \in L_p(Q_T)$ by (3.55), for $p > 2$, which gives

$$\|\nabla w\|_{L_p(Q_T)} \leq C(T), \quad \text{for } 2 < p < +\infty.$$

Furthermore, taking divergence from both sides of (3.56) will give us that

$$\begin{aligned} &\Delta w(x,t) = \nabla \cdot (\nabla w(x,t)) \\ =&\Delta w_0(x) e^{\int_0^t h(x,s)\mathrm{d}s} + \nabla w_0(x) e^{\int_0^t h(x,s)\mathrm{d}s} \int_0^t \nabla h(x,s)\mathrm{d}s \\ &+ \int_0^t \nabla \hat{g}(x,\tau) e^{\int_\tau^t h(x,s)\mathrm{d}s} \mathrm{d}\tau + \int_0^t \hat{g}(x,\tau) e^{\int_\tau^t h(x,s)\mathrm{d}s} \int_\tau^t \nabla h(x,s)\mathrm{d}s \, \mathrm{d}\tau \end{aligned} \quad (3.57)$$

where $\nabla h \in L_p(Q_T)$ and thus $\nabla \hat{g} \in L_p(Q_T)$ for $p > 2$ by (3.55), which concludes

$$\|\Delta w\|_{L_p(Q_T)} \leq C(T), \quad \text{for } 2 < p < +\infty.$$

Therefore, we have

$$\|w\|_{W_p^{2,1}(Q_T)} \leq C(T), \quad \text{for } 2 < p < +\infty.$$

This combined with (3.55) illustrates that (a, w), thus (u, w) obtained in Lemma 3.12, is a strong solution.

Next we show that the strong solution (a, w) is classical solution by applying Schauder theory to this a-component and by directly estimating w. Note that α as follow may be different from line to line but we shall not change this notation for brevity, based on a fact that two Hölder spaces differing only in Hölder indices $\alpha \in (0,1)$ are equivalent to each other.

In fact, using the Sobolev's embedding $W_p^{2,1}(Q_T) \hookrightarrow C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)$ for $0 < \alpha \leq 2 - \frac{4}{p}$

and $p > 2$, may lead us to $a, w \in C^{1+\alpha,\frac{1+\alpha}{2}}(\bar{Q}_T)$ for some $\alpha \in (0,1)$, when the above $p > 4$.

This immediately implies $\chi\nabla w \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)$ and $g - \chi\nabla w \cdot \nabla a = \gamma aw + \chi a^2 e^{\chi w} w - r\chi aw(1-w) - la^2 e^{\chi w} \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)$. Consequently, together with $\partial\Omega \in C^{2+\alpha}$ and $a_0 \in C^{2+\alpha}(\overline{\Omega})$ assumed in (3.4), an application of Schauder theory to (3.54) enables us to infer that $a \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)$ for some $\alpha \in (0,1)$.

Since $a, w \in C^{1+\alpha,\frac{1+\alpha}{2}}(\bar{Q}_T)$, we may derive $w_t \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)$ due to $w_t = -ae^{\chi w} w + rw(1-w)$. Furthermore, (3.57) may indicates that $\Delta w \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)$ by $a \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)$ and $w, \nabla w \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)$. Thus we have $w \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)$. □

Remark 3.5. *From the* (3.56)*, one may derive that*

$$\nabla w \cdot \vec{\nu} = e^{\int_0^t h(x,s)\mathrm{d}s}\, \nabla w_0(x) \cdot \vec{\nu} + \int_0^t e^{\int_\tau^t h(x,s)\mathrm{d}s}\, \hat{g}(x,\tau) \cdot \vec{\nu}\, \mathrm{d}\tau$$

with $\hat{g} = -we^{\chi w}\nabla a$, which combined with $\nabla a \cdot \vec{\nu}\big|_{\partial\Omega} = 0$ and $w_0, a \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)$, shows that

$$\nabla w \cdot \vec{\nu}\big|_{\partial\Omega} = 0$$

if $\nabla w_0 \cdot \vec{\nu}\big|_{\partial\Omega} = 0$ is additionally assumed for (3.3)*.*

Lemma 3.14 (Uniqueness)**.** *The classical solution derived in Lemma 3.13 is unique.*

Proof. We first note that the Remark 3.1 holds for the classical solution (u, w) to (3.3) after a similar discussion, where w-equation is an ODE in this case.

Assume that there exist two classical solutions of (3.3) in $Q_T = \Omega \times (0, T)$ (for any $0 < T < \infty$), denoted by (u_1, w_1) and (u_2, w_2), respectively. Letting $u := u_1 - u_2$

and $w := w_1 - w_2$, we observe that (u, w) should fulfill that

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u_1 \nabla w_1 - u_2 \nabla w_2) \\ \qquad\qquad + \gamma(u_1 w_1 - u_2 w_2) - u - lu(u_1 + u_2), & \text{in } Q_T \\ w_t = -(u_1 w_1 - u_2 w_2) + rw - rw(w_1 + w_2), & \text{in } Q_T \\ (\nabla u_i - \chi u_i \nabla w_i) \cdot \vec{\nu}\big|_{\partial\Omega} = 0, \quad i = 1, 2, & \text{in } (0, T) \\ u(x, 0) = 0, \quad w(x, 0) = 0, & \text{in } \Omega. \end{cases} \tag{3.58}$$

Then the uniqueness holds by showing $u \equiv 0$ and $w \equiv 0$ through an inequality

$$f'(t) \leq cf(t) \qquad \text{with} \quad c > 0 \quad \text{and} \quad f(t) \geq 0, \quad \text{for} \quad t \in [0, T).$$

Since upon a comparison of ODE it gives $0 \leq f(t) \leq f(0)e^{ct}$ which means $f(t) \equiv 0$ for $t \in (0, T)$ as $f(0) = 0$.

Indeed, multiplying the u-equation here by u and using integration by parts one may infer that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2 + \int_\Omega u^2 + l\int_\Omega (u_1 + u_2)u^2 + \int_\Omega |\nabla u|^2 \\
=&\chi \int_\Omega \nabla u \cdot (u_1 \nabla w_1 - u_2 \nabla w_2) + \int_{\partial\Omega} (u_1 - u_2)\Big(\nabla(u_1 - u_2) - \chi(u_1 \nabla w_1 - u_2 \nabla w_2)\Big) \cdot \vec{\nu} \\
&+ \gamma \int_\Omega (u_1 - u_2)(u_1 w_1 - u_2 w_2) \\
=&\chi \int_\Omega \nabla u \cdot (u_1 \nabla w + u \nabla w_2) + \gamma \int_\Omega u(u_1 w + u w_2) \\
\leq&\chi \|u_1\|_{L_\infty(\Omega)} \int_\Omega \big(\frac{|\nabla u|^2}{\beta} + \beta |\nabla w|^2\big) + \chi \int_\Omega |u \nabla u \cdot \nabla w_2| \\
&+ \gamma \|u_1\|_{L_\infty(\Omega)} \int_\Omega uw + \gamma \|w_2\|_{L_\infty(\Omega)} \int_\Omega u^2
\end{aligned}$$

where we have used $\|u\|_{L_\infty(\Omega)} \leq \|u_i\|_{L_\infty(\Omega)}$ by the nonnegativity of $u_i, i = 1, 2$, and Young's inequality with parameter $\beta > 0$. Making use of Hölder's inequality, Gagliardo-Nirenberg interpolation inequality and of Young's inequality with parameter $\eta, \delta > 0$, one may proceed to calculate that

$$\begin{aligned}
&\chi \int_\Omega |u\nabla u \cdot \nabla w_2| \leq \chi \|\nabla u\|_{L_2(\Omega)} \|\nabla w_2\|_{L_4(\Omega)} \|u\|_{L_4(\Omega)} \\
\leq &\chi C \|\nabla w_2\|_{L_4(\Omega)} \|\nabla u\|_{L_2(\Omega)} (\|\nabla u\|^{\frac{1}{2}}_{L_2(\Omega)} \|u\|^{\frac{1}{2}}_{L_2(\Omega)} + \|u\|_{L_2(\Omega)}) \\
\leq &\chi C \|\nabla w_2\|_{L_4(\Omega)} (\|\nabla u\|^{\frac{3}{2}}_{L_2(\Omega)} \|u\|^{\frac{1}{2}}_{L_2(\Omega)} + \|\nabla u\|_{L_2(\Omega)} \|u\|_{L_2(\Omega)}) \\
\leq &\chi C \|\nabla w_2\|_{L_4(\Omega)} \Big((\eta + \delta) \|\nabla u\|^2_{L_2(\Omega)} + (\frac{1}{\eta^3} + \frac{1}{\delta}) \|u\|^2_{L_2(\Omega)} \Big).
\end{aligned} \tag{3.59}$$

Now taking $\beta = 8\chi \|u_1\|_{L_\infty(\Omega)}$ and $\eta = \delta = \frac{1}{8\chi C \sup_{t\in[0,T]} \|\nabla w_2\|_{L_4(\Omega)}}$ which makes sense due to (3.56) or $w_i \in C^{2,1}(\bar{Q}_T)$, one may show that

$$\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^2 + \int_\Omega u^2 + l \int_\Omega (u_1 + u_2) u^2 + \frac{5}{8} \int_\Omega |\nabla u|^2 \\
= &8\chi^2 \|u_1\|^2_{L_\infty(\Omega)} \int_\Omega |\nabla w|^2 + \tilde{c}_0 \int_\Omega u^2 + \gamma \|u_1\|_{L_\infty(\Omega)} \int_\Omega w^2,
\end{aligned} \tag{3.60}$$

with $\tilde{c}_0 := \gamma(\|u_1\|_{L_\infty(\Omega)} + \|w_2\|_{L_\infty(\Omega)}) + \chi C \sup_{[0,T]} \|\nabla w_2\|_{L_4(\Omega)} \big(\frac{1}{\eta^3} + \frac{1}{\delta}\big)$.

In addition, multiplying the w-equation in (3.58) by w may lead us to

$$\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega w^2 + \int_\Omega u_1 w^2 + r \int_\Omega (w_1 + w_2) w^2 = -\int_\Omega w_2 u w + r \int_\Omega w^2 \\
\leq &\|w_2\|_{L_\infty(\Omega)} \int_\Omega u^2 + (r + \|w_2\|_{L_\infty(\Omega)}) \int_\Omega w^2.
\end{aligned} \tag{3.61}$$

Taking gradient ∇ from both sides of the w-equation in (3.58) and multiplying the resulting equation by ∇w illustrate that

$$\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla w|^2 + \int_\Omega u_1 |\nabla w|^2 + r \int_\Omega (w_1 + w_2) |\nabla w|^2 \\
= &-\int_\Omega w \nabla u_1 \cdot \nabla w - \int_\Omega w_2 \nabla u \cdot \nabla w - \int_\Omega u \nabla w_2 \cdot \nabla w + r \int_\Omega |\nabla w|^2
\end{aligned}$$

$$
\begin{aligned}
&- r\int_\Omega w\nabla w\cdot(\nabla w_1+\nabla w_2)\\
\le &\int_\Omega |w\nabla u_1\cdot\nabla w| + \|w_2\|_{L_\infty(\Omega)}\int_\Omega |\nabla u\cdot\nabla w| + \int_\Omega |u\nabla w_2\cdot\nabla w| + r\int_\Omega |\nabla w|^2\\
&+ r\int_\Omega |w\nabla w\cdot(\nabla w_1+\nabla w_2)|.
\end{aligned}
$$

Similar to (3.59), we may obtain that

$$
\int_\Omega |w\nabla u_1\cdot\nabla w| \le 2C\sup_{(0,T)}\|\nabla u_1\|_{L_4(\Omega)}\cdot\int_\Omega \big(|\nabla w|^2+w^2\big),
$$

$$
r\int_\Omega |w\nabla w\cdot(\nabla w_1+\nabla w_2)| \le 2rC\sup_{(0,T)}\|\nabla w_1+\nabla w_2\|_{L_4(\Omega)}\cdot\int_\Omega \big(|\nabla w|^2+w^2\big),
$$

$$
\|w_2\|_{L_\infty(\Omega)}\int_\Omega |\nabla u\cdot\nabla w| \le \frac{1}{8}\int_\Omega |\nabla u|^2 + 2\|w_2\|^2_{L_\infty(\Omega)}\int_\Omega |\nabla w|^2,
$$

and

$$
\begin{aligned}
&\int_\Omega |u\nabla w_2\cdot\nabla w| \le C\|\nabla w_2\|_{L_4(\Omega)}\|\nabla w\|_{L_2(\Omega)}\big(\|\nabla w\|^{\frac{1}{2}}_{L_2(\Omega)}\|w\|^{\frac{1}{2}}_{L_2(\Omega)} + \|w\|_{L_2(\Omega)}\big)\\
\le &C^2\|\nabla w_2\|^2_{L_4(\Omega)}\|\nabla w\|^2_{L_2(\Omega)} + 2\|\nabla w\|_{L_2(\Omega)}\|w\|_{L_2(\Omega)} + 2\|w\|^2_{L_2(\Omega)}\\
\le &\big(C^2\sup_{(0,T)}\|\nabla w_2\|^2_{L_4(\Omega)} + 2\big)\int_\Omega |\nabla w|^2 + 4\int_\Omega w^2.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla w|^2 + \int_\Omega u_1|\nabla w|^2 + r\int_\Omega (w_1+w_2)|\nabla w|^2\\
\le &\tilde{c}_1\int_\Omega |\nabla w|^2 + \tilde{c}_2\int_\Omega w^2 + \frac{1}{8}\int_\Omega |\nabla u|^2,
\end{aligned}
$$

with $\tilde{c}_1 := M + 2\|w_2\|^2_{L_\infty(\Omega)} + r + 2 + C^2\sup_{(0,T)}\|\nabla w_2\|^2_{L_4(\Omega)}$, $\tilde{c}_2 := M+4$, and $M := 2C\sup_{(0,T)}\|\nabla u_1\|_{L_4(\Omega)} + 2rC\sup_{(0,T)}\|\nabla w_1+\nabla w_2\|_{L_4(\Omega)}$ which is finite by the regularity $a, w \in C^{2,1}(\bar{Q}_T)$. This inequality combined with (3.60) and (3.61), may indicate that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u^2 + w^2 + |\nabla w|^2) \le \tilde{c}_3 \int_\Omega (u^2 + w^2 + |\nabla w|^2)$$

where $\tilde{c}_3 := \max\left\{\tilde{c}_0 + \|w_2\|_{L_\infty(\Omega)},\ \tilde{c}_1 + 8\chi^2\|u_1\|^2_{L_\infty(\Omega)},\ \tilde{c}_2 + \gamma\|u_1\|_{L_\infty(\Omega)} + r + \|w_2\|_{L_\infty(\Omega)}\right\}$.

Then we may complete this proof by letting

$$f(t) := \int_\Omega (u^2 + w^2 + |\nabla w|^2)(x,t)\,\mathrm{d}x$$

which is a continuous function in $t \in [0, T)$ owning to the solution (u, w) being continuous to its initial value, and thus $f(0) = \int_\Omega (u(x,0)^2 + w(x,0)^2 + |\nabla w(x,0)|^2)\,\mathrm{d}x = 0$ as a result of (3.58). □

Proof of Theorem 3.1: Lemma 3.13 implies that the solution (u, w) obtained in Lemma 3.12 is a strong solution of (3.3), which combined with Lemma 3.11 proves Theorem 3.1 (a). Consequently, the convergence given in (3.48) occurs between the strong solution of (3.2) and that of (3.3), which proves Theorem 3.1 (b). In addition, Lemma 3.13 shows that the strong solution of (3.3) is the classical solution of (3.3). This alone with the uniqueness derived in Lemma 3.14 proves the Theorem 3.1 (c). So we complete the proof. □

Chapter 4

Global Dynamics on Fully Parabolic System with Density-Dependent Indirect Preytaxis

4.1 Models and Main results

In Chapter 2 and Chapter 3 we have studied the global-in-time existence and uniqueness of classical solution to direct preytaxis models (1.10) and (1.11). We shall in this chapter consider the global-in-time existence and large time behaviors of the unique classical solution to indirect preytaxis model (1.17), that is,

$$\begin{cases} u_t = \nabla \cdot \big(d(v)\nabla u - u\chi(v)\nabla v\big) + \gamma u F(w) - \theta u - \ell u^2, & t>0,\ x\in\Omega; \\ v_t = d_v \Delta v + \beta w - \sigma v, & t>0,\ x\in\Omega; \\ w_t = d_w \Delta w + w f(w) - uF(w), & t>0,\ x\in\Omega; \\ \nabla u\cdot \vec{\mathbf{n}} = 0,\ \nabla v\cdot \vec{\mathbf{n}} = 0,\ \nabla w\cdot \vec{\mathbf{n}} = 0, & t>0,\ x\in\partial\Omega; \\ u(0,x)=u_0(x),\quad v(0,x)=v_0(x),\quad w(0,x)=w_0(x), & x\in\Omega, \end{cases} \tag{4.1}$$

where $\Omega\subset\mathbb{R}^n (n\geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $\vec{\mathbf{n}}$ is the unit outer normal vector towards $\partial\Omega$, $\ell\geq 0$, and d_w, γ, $\theta>0$.

Before specifying our main results, several notations need to be explained. Let X

be a metric space. We denote by $C^{m+1-}(X)$ the set of functions with their k-times ($0 \le k \le m$, $k, m \in \mathbb{N}$) derivatives being Lipschitz continuous in X. Note that the k-times derivatives are Lipschitz continuous if $(k+1)$-times derivatives are bounded in X and the boundary of X is regular enough, e.g., Hölder space $C^{2+\alpha}, \alpha \in (0,1)$.

To ensure the existence of solutions to (1.16) and (4.1), the real-valued functions $d(v), \chi(v), f(w)$, and $F(w)$ should satisfy that

(H1) $d(v), \chi(v) \in C^{1+1-}([0,+\infty))$ and for $v \in [0,+\infty)$, $\chi(v) \ge 0$, $d(v) > 0$ and $d'(v) \le 0$;

(H2) $f \in C^{1+1-}\big([0,+\infty)\big)$ and there exists a constant $K_0 > 0$ such that $f(K_0) = 0$ and $f(w) < 0$ for all $w > K_0$ and $f(w) > 0$ for $w \in (0, K_0)$;

(H3) $F(w) \in C^{1+1-}([0,+\infty))$ and there is a constant $C_F > 0$ such that $0 \le F(w) \le C_F|w|$. Moreover, $F'(w) > 0$ for all $w \in [0,+\infty)$.

Thus (H2) allows logistic $f(w)$ and all $F(w)$ in (1.8) support (H3). The (H1) is more general than that of [73][1].

Note that our results are applicable to (1.16) since system (4.1) can reduce to (1.16) when $\ell = 0$. We have published these results in our paper [61]. We first derive the existence of global-in-time classical solution to (4.1) as below:

Theorem 4.1. *Let* $\Omega \subset \mathbb{R}^n$ $(n \ge 1)$ *be a bounded domain with smooth boundary* $\partial\Omega$. *Under the hypotheses* $(H1)$–$(H3)$, *if* $(u_0, v_0, w_0) \in C^2(\overline{\Omega}, \mathbb{R}^3)$ *with* $u_0, v_0, w_0 \ge 0\,(\not\equiv 0)$ *and fulfills* 0*-order compatibility condition* (*i.e.* $\nabla u_0\big|_{\partial\Omega} = \nabla v_0\big|_{\partial\Omega} = \nabla w_0\big|_{\partial\Omega} = 0$), *then the system* (4.1) *has a unique nonnegative* (*resp. positive*) *classical solution on* $[0,\infty)$ (*resp. on* $(0,\infty)$) *satisfying*

$$(u, v, w)(t, x) \in C\Big([0,+\infty) \times \overline{\Omega},\, \mathbb{R}^3\Big) \cap C^{1,2}\Big((0,+\infty) \times \overline{\Omega},\, \mathbb{R}^3\Big). \tag{4.2}$$

[1] We remark that [73] is published independently at almost the same time as our paper [61] but there is no any related information received by us until the publishing of [61].

Furthermore, there is a constant $\mathcal{C} > 0$ *independent of* t *such that*

$$\|u(t,\cdot)\|_{L_\infty(\Omega)} + \|v(t,\cdot)\|_{W^1_\infty(\Omega)} + \|w(t,\cdot)\|_{W^1_\infty(\Omega)} \leq \mathcal{C} \quad \textit{for all} \quad t > 0, \tag{4.3}$$

where $0 < w(t,x) \leq \max\left\{K_0, \|w_0\|_{L_\infty(\Omega)}\right\}$ *for all* $(t,x) \in (0,+\infty) \times \Omega$.

We next investigate the asymptotic behaviors of such a classical solution. Suppose that (4.1) has a constant steady state denoted by (u_c, v_c, w_c), then

$$\begin{cases} \gamma u_c F(w_c) = u_c(\theta + \ell u_c), \\ \beta w_c = \sigma v_c, \\ w_c f(w_c) = u_c F(w_c). \end{cases} \tag{4.4}$$

If in addition each component of (u_c, v_c, w_c) is nonnegative, three possible constant steady states may be formulated as follow:

- extinction state: if $u_c = 0$ and $w_c = 0$ then $(u_c, v_c, w_c) = (0,0,0)$;
- exclusion (prey-only) state: if $u_c = 0$ but $w_c > 0$ then $w_c = K_0$ and $(u_c, v_c, w_c) = (0, \frac{\beta K_0}{\sigma}, K_0)$;
- coexistence state: $u_c, w_c > 0$ thus $v_c = \frac{\beta w_c}{\sigma} > 0$, $u_c = \frac{w_c f(w_c)}{F(w_c)}$ and $\gamma F(w_c) = \theta + \ell u_c$. Denote by (u_*, v_*, w_*) this positive constant solution.

To construct appropriate Lyapunov functions we desire, we have to impose that

(H4) for any $w \in [0, +\infty)$, $\varphi(w) := \frac{w f(w)}{F(w)}$ is continuously differentiable, $\varphi'(w) < 0$ and $0 < \varphi(0) = \lim\limits_{w \to 0^+} \varphi(w)$ exists.

This is not very stringent and can be achieved if $f(w) = r(1 - \frac{w}{K_0})$ and $F(w)$ is Holling type I or II with $0 < K_0 \leq c$ given in (1.8).

After these preparations, we can formulate our second result as below.

Theorem 4.2. *Suppose that* (u,v,w) *is a global classical solution to the system* (4.1) *fulfilling* $(H1)$*–*$(H4)$*. Let* K_0 *be defined in (H2).*

1) *If* $\gamma F(K_0) \le \theta$*, then the prey-only state* $(0, \frac{\beta K_0}{\sigma}, K_0)$ *exists and is globally asymptotic stable. Furthermore, if* $\gamma F(K_0) < \theta$*, there are constants* $\hat{c}_1, \hat{c}_2, T_0 > 0$ *such that*

$$\|u(t,\cdot)\|_{L_\infty(\Omega)} + \left\|v(t,\cdot) - \tfrac{\beta K_0}{\sigma}\right\|_{L_\infty(\Omega)} + \|w(t,\cdot) - K_0\|_{L_\infty(\Omega)} \le \hat{c}_2 e^{-\hat{c}_1 t}, \quad t > T_0.$$

2) *If the coexistence steady state* (u_*, v_*, w_*) *exists and*

$$\max_{0\le v\le K_2} \frac{\chi(v)^2}{d(v)} \le \frac{16 d_v \gamma\sigma}{\beta^2 u_*} \min_{w_1\in[0,C_1]}\{-\varphi'(w_1)\} \min_{w_2\in[0,C_1]}\{F'(w_2)\}, \tag{4.5}$$

with K_2 *from Remark 4.2 and* $C_1 := \max\left\{K_0, \|w_0\|_{L_\infty(\Omega)}\right\}$*, then* (u_*, v_*, w_*) *is globally asymptotic stable. Moreover, there are constants* $\bar{c}_1, \bar{c}_2, T_1 > 0$ *such that*

$$\|u(t,\cdot)-u_*\|_{L_\infty(\Omega)}+\|v(t,\cdot)-v_*\|_{L_\infty(\Omega)}+\|w(t,\cdot)-w_*\|_{L_\infty(\Omega)} \le \bar{c}_1 e^{-\bar{c}_2 t}, \qquad t > T_1.$$

Note that there is no $\gamma F(K_0) > \theta$ (biologically interpreted as "strong predation") assumed in 2) of Theorem 4.2 since it has been ensured by the existence of the coexistence steady state along with (H2) and (H3). In fact, (H2) and (H3) imply $0 < w_* < K_0$ and then $\gamma F(K_0) > \gamma F(w_*) = \theta + \ell u_* \ge \theta$ by $F'(w) > 0$ in (H3). Also, (4.5) might be simplified by specific f and F, for example:

Corollary 4.1. *If* $f(w) = r(1 - \frac{w}{K_0})$ *and* $F(w) = \frac{w}{c+w}$ *with* $0 < K_0 < c$ *then the coexistence steady state exits and* (4.5) *becomes*

$$\max_{0<v\le K_2} \frac{\chi(v)^2}{d(v)} \le \frac{16 d_v \gamma\sigma (c - K_0)}{c K_0 \beta^2 u_*},$$

with K_2 *from Remark 4.2. Then the asymptotic stability above-mentioned remains unchanged.*

4.2 Global Existence of the Classical Solution

We shall apply the celebrated results developed by H. Amann [65, 66] to derive local and global existence of classical solution to (4.1). The conclusions and proofs can be applied to (1.16) after slight modifications.

4.2.1 Local Existence

Lemma 4.1 (Local existence and uniqueness). *Let $\Omega \subset \mathbb{R}^n\,(n \geq 1)$ be a bounded open domain. If $(H1)$–$(H3)$ hold, $(u_0, v_0, w_0) \in C^2(\Omega, \mathbb{R}^3)$ with $u_0, v_0, w_0 \geq 0\,(\not\equiv 0)$, then there exists a $T_{\max} \in (0, +\infty]$ depending on (u_0, v_0, w_0) such that the system (4.1) has a unique nonnegative (resp. positive) classical solution on $[0, T_{\max})$ (resp. $(0, T_{\max})$) satisfying*

$$(u, v, w)(t, x) \in C\big([0, T_{\max}) \times \overline{\Omega},\, \mathbb{R}^3\big) \cap C^{1,2}\big((0, T_{\max}) \times \overline{\Omega},\, \mathbb{R}^3\big). \tag{4.6}$$

Proof. Note that we first strengthen the conditions in $(H1) - (H3)$ by replacing the interval $[0, +\infty)$ with $\mathbb{R}$. Finally we will see the obtained results still make sense without this enhancement. For clarity, we reformulate system (4.1) as

$$\begin{cases} \mathbf{w}_t = \nabla \cdot (\mathbf{A}(\mathbf{w})\nabla \mathbf{w}) + \Psi(\mathbf{w}), & x \in \Omega,\ t > 0, \\ \nabla u \cdot \vec{\mathbf{n}} = 0,\ \nabla v \cdot \vec{\mathbf{n}} = 0,\ \nabla w \cdot \vec{\mathbf{n}} = 0, & x \in \partial\Omega,\ t > 0, \\ \mathbf{w}(x, 0) = \mathbf{w}_0(x), & x \in \Omega, \end{cases} \tag{4.7}$$

where for $x \in \Omega$ and $t \geq 0$, $\mathbf{w} = (u, v, w)^\tau$ and $\mathbf{w_0} = (u_0, v_0, w_0)^\tau \in \mathbb{R}^3$ (τ denoting transposition) are two vector-valued functions, $\nabla \mathbf{w} = (\nabla u, \nabla v, \nabla w)^\tau$,

$$\mathbf{A}(\mathbf{w}) = \begin{pmatrix} d(v) & -u\chi(v) & \\ & d_v & \\ & & d_w \end{pmatrix}_{3\times 3} \text{ and } \Psi(\mathbf{w}) = \begin{pmatrix} \gamma u F(w) - \theta u - \ell u^2 \\ \beta w - \sigma v \\ w f(w) - u F(w) \end{pmatrix}.$$

It is easy to see that $d(v) > 0$ for $v \in \mathbb{R}$ by (H1). Then along with $d_v, d_w > 0$, all ordering principal minor determinants of $\mathbf{A}(\mathbf{x})$ are positive, which implies that

$\mathbf{A}(\mathbf{x})$ is positively definite for all $\mathbf{x} \in \mathbb{R}^3$. Thus we know for all $t > 0, x \in \Omega$, $\mathbf{w}_t - \nabla \cdot (\mathbf{A}(\mathbf{w})\nabla \mathbf{w})$ is Petrowskii parabolic (cf. Eq (50) in [74]) and $\nabla \cdot (\mathbf{A}(\mathbf{w})\nabla \mathbf{w})$ is normally elliptic (cf. p.16 or Theorem 4.4 in [65]) with separated divergence form. Moreover, $\nabla \cdot (\mathbf{A}(\mathbf{w})\nabla \mathbf{w})$ coupled with the boundary condition in (4.7) is normally elliptic as well.

By (H1) all elements of $\mathbf{A}(\mathbf{x})$ are in $C^{1+1-}(\mathbb{R})$ (functions and their first-order derivatives being Lipschitz continuous on $\mathbb{R}$). Similarly the regularity conditions in (H2) and (H3) show every component of $\Psi(\mathbf{w})$ is $C^{1+1-}(\mathbb{R}^3)$. In terms of Theorem 7.3-(ii), Theorem 9.2, and Corollary 9.3 of H. Amann [65], we know that given $\mathbf{w}_0 \in W_p^2(\Omega, \mathbb{R}^3)$ with $p > n$ and $p \geq 2$, there exist a $T_{\max} \in (0, +\infty]$ relating to $\mathbf{w}_0$ and $0 < 2\epsilon < \min\{2 - n/p, 1\}$ such that (4.1) has a unique (cf. the Corollary 9.3) maximal classical solution on $[0, T_{\max}) \times \Omega$ satisfying

$$(u, v, w) \in B\big(J', C^{2+2\epsilon}(\overline{\Omega}, \mathbb{R}^3)\big) \cap C^{0+\epsilon}\big((0, T_{\max}), C^2(\overline{\Omega}, \mathbb{R}^3)\big) \cap C^{1+\epsilon}\big((0, T_{\max}), C(\overline{\Omega}, \mathbb{R}^3)\big)$$

for every compact subinterval J' of $(0, T_{\max})$, where $B(X, Y)$ (*resp.* $C^m(X, Y)$) denotes the set of all bounded mappings (*resp.* all *m-th* continuously differentiable functions) from X to Y, and $C^{m+\iota}(X, Y)$ is the set of all mappings from X to Y which up to their *m-th* derivatives are ι- Hölder continuous on X with $\iota \in (0, 1)$ and $m \in \mathbb{N}$. Moreover, if $\mathbf{w}_0 \in C^2(\Omega, \mathbb{R}^3)$, then by Theorem 1 of [66] we know that the system (4.7) has a unique maximal classical solution

$$(u, v, w) \in C\big([0, T_{\max}), C(\overline{\Omega}, \mathbb{R}^3)\big) \cap C\big((0, T_{\max}), C^2(\overline{\Omega}, \mathbb{R}^3)\big) \cap C^1\big((0, T_{\max}), C(\overline{\Omega}, \mathbb{R}^3)\big) \tag{4.8}$$

As a result, (4.8) implies (4.6).

Then we may find this unique local classical solution is nonnegative on $[0, T_{\max})$. Indeed, we may first rewrite the u-equation in system (4.1) as

$$u_t = d(v)\Delta u + [d'(v)\nabla v - \chi(v)\nabla v] \cdot \nabla u - [\chi'(v)(\nabla v \cdot \nabla v) + \chi(v)\Delta v]u + \gamma u F(w) - \theta u - \ell u^2.$$

By the regularity (4.8), v, w in u-equation can be treated as known functions at present. Then within any $[0, T] \subset [0, T_{\max})$ one can apply comparison principle of linear parabolic equations to such a equation coupled with $\nabla u \cdot \vec{\mathbf{n}} = 0$ and $u_0(x) \geq 0(\not\equiv 0)$. Thus we derive $u \geq 0$ in $[0, T_{\max}) \times \Omega$ and $u > 0$ in $(0, T_{\max}) \times \Omega$. Similarly, one may acquire that $v, w > 0$ in $(0, T_{\max}) \times \Omega$, and $v, w \geq 0$ in $[0, T_{\max}) \times \Omega$. Therefore, $\mathbb{R}$ in $(H1) - (H3)$ as supposed at the very beginning of this proof can be replaced by $[0, +\infty)$. This completes the proof. □

By Theorem 1 of [66], it suffices to verify that $\|(u, v, w)(t, \cdot)\|_{H_p^s(\Omega)} \leq C(T) < +\infty$ for any $t \in (0, T) \subset (0, T_{\max})$, $p > n$ and $p \geq 2$ as well as some s satisfying $1 < s < \min\left\{1 + \frac{1}{p}, 2 - \frac{n}{p}\right\}$, in order to extend such a local unique classical solution to a global one. To make this extendability criteria easier to verify (i.e., to weaken this H_p^s-topology, the Bessel potential space), we resort to Theorem 5.2 of [66] at the cost of imposing an extra condition on the initial data. This can be formulated in the following lemma.

Lemma 4.2. *Suppose that $(u_0, v_0, w_0) \in C^2(\overline{\Omega}, \mathbb{R}^3)$ additionally fulfills 0-order compatibility condition (i.e., $\nabla u_0\big|_{\partial\Omega} = \nabla v_0\big|_{\partial\Omega} = \nabla w_0\big|_{\partial\Omega} = 0$). Then the above local classical solution is global if*

$$\limsup_{t \nearrow T_{\max}} \left\{\|u(t, \cdot)\|_{L_\infty(\Omega)} + \|v(t, \cdot)\|_{L_\infty(\Omega)} + \|w(t, \cdot)\|_{L_\infty(\Omega)}\right\} < +\infty.$$

4.2.2 L_∞ Estimate on $w(t, x), v(t, x)$ and $u(t, x)$

Lemma 4.3. *Under the conditions in Lemma 4.1, it holds that*

$$0 < w(t, x) \leq \max\left\{K_0, \|w_0\|_{L_\infty(\Omega)}\right\}, \qquad \text{for any} \quad (t, x) \in (0, T_{\max}) \times \Omega,$$

where K_0 is from $(H2)$ and is independent of $T_{\max}$.

Proof. One may use comparison principle to prove this result and more details can be seen in Lemma 2.2 of [39]. □

Remark 4.1. *Under the conditions in Lemma 4.1, if* (u, v, w) *is a nonnegative classical solution to system* (4.1) *on* $(0, T_{\max}) \times \Omega$*, then*

$$\|u(t,\cdot)\|_{L_1(\Omega)} + \|v(t,\cdot)\|_{L_1(\Omega)} + \|w(t,\cdot)\|_{L_1(\Omega)} \leq C \tag{4.9}$$

where C *is a positive constant and independent of* $T_{\max}$.

It is easy to see that the solution to v-equation of system (4.1) can be formally expressed via heat semigroup theory with zero-Neumann boundary condition. Precisely, the estimation on $v(t,x)$ follows from Lemma 1 of Kowalczyk and Szymańska [75] or Lemma A.5 as below.

Lemma 4.4. *Assume that* $\Omega \subset \mathbb{R}^n (n \geq 1)$, $v_0(x) \in W^1_\infty(\Omega)$ *and*

$$\|w(t,\cdot)\|_{L_p(\Omega)} < C \quad \text{for all} \quad t \in (0, T_{\max}).$$

Then for every $t \in (0, T_{\max})$*, the classical solution* $v(t,x)$ *of the* v*-equation in system* (4.1) *satisfies*

$$\|v(t,\cdot)\|_{W^1_q(\Omega)} \leq \mathcal{C} \qquad \text{when} \quad \begin{cases} q < \frac{np}{n-p}, & p < n; \\ q < +\infty, & p = n; \\ q = +\infty, & p > n. \end{cases}$$

Here C *and* $\mathcal{C}$ *are positive constants and independent of* $T_{\max}$.

In conjunction with Lemma 4.3 we thus have the following W^1_∞ estimate on $v(t,x)$.

Remark 4.2. *There exists a constant* $K_2 > 0$ *independent of* $T_{\max}$ *such that if* $v_0 \in W^1_\infty(\Omega)$*, then* $\|v(t,\cdot)\|_{W^1_\infty(\Omega)} \leq K_2$ *for all* $t \in (0, T_{\max})$.

The next lemma is to show L_∞ estimate on $u(t,x)$.

Lemma 4.5. *Let* $(H1)$*–*$(H3)$ *hold. Suppose that* (u, v, w) *is the solution of system* (4.1) *obtained in Lemma 4.1. Then there exists a positive constant* $\tilde{C}$ *independent of* $T_{\max}$ *such that*

$$\|u(t,\cdot)\|_{L_\infty(\Omega)} \leq \tilde{C} \quad \text{for all } t \in (0, T_{\max}).$$

Proof. Here we adopt Moser's iteration method. Indeed, we assume $t \in (0,T) \subset (0,T_{\max})$ with $0 < T < T_{\max}$. Multiplying the first equation in system (4.1) by $u^{p-1}(p \geq 1)$ and integrating the result with respect to x in Ω may yield

$$\begin{aligned}&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + (p-1)\int_\Omega d(v)u^{p-2}|\nabla u|^2 + \theta\int_\Omega u^p + \ell\int_\Omega u^{p+1}\\ =&(p-1)\int_\Omega u^{p-1}\chi(v)\nabla u\cdot\nabla v + \gamma\int_\Omega u^p F(w).\end{aligned}$$

Lemma 4.1 shows $u(t,x), v(t,x), w(t,x) > 0$ for all $(t,x) \in (0,T_{\max}) \times \Omega$. In addition, Remark 4.2 concludes that $\|\nabla v\|_{L_\infty(\Omega)} \leq \|v(t,\cdot)\|_{W^1_\infty(\Omega)} \leq K_2$ (independent of $T_{\max}$). Thus (H1) implies $d(v) \geq d(K_2) =: c_0$ and $|\chi(v)| \leq \max_{0<v\leq K_2}\chi(v) =: c_1$. By $0 \leq F(w) \leq C_F w$ in (H3) we then may obtain

$$\begin{aligned}&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + (p-1)c_0\int_\Omega u^{p-2}|\nabla u|^2 + \theta\int_\Omega u^p + \ell\int_\Omega u^{p+1}\\ \leq&(p-1)c_1\int_\Omega u^{p-1}|\nabla u||\nabla v| + C_F\gamma\int_\Omega u^p w.\end{aligned}$$

Applying Cauchy's inequality to the first right-hand term may lead us to

$$(p-1)c_1\int_\Omega u^{p-1}|\nabla u||\nabla v| \leq \frac{(p-1)c_0}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{(p-1)c_1^2}{2c_0}\int_\Omega u^p|\nabla v|^2.$$

Hence

$$\begin{aligned}&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + \frac{(p-1)c_0}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \theta\int_\Omega u^p + \ell\int_\Omega u^{p+1}\\ \leq&\frac{(p-1)c_1^2K_2^2}{2c_0}\int_\Omega u^p + C_F\gamma\int_\Omega u^p w.\end{aligned}$$

Below by setting $p \geq 2$ and due to $u^{p-2}|\nabla u|^2 = |u^{\frac{p}{2}-1}\nabla u|^2 = |\frac{2}{p}\nabla u^{\frac{p}{2}}|^2 = \frac{4}{p^2}|\nabla u^{\frac{p}{2}}|^2$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + \frac{2(p-1)c_0}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + p\theta\int_\Omega u^p + p\ell\int_\Omega u^{p+1}$$

$$\leq \frac{p(p-1)c_1^2K_2^2}{2c_0}\int_\Omega u^p + pC_FC_1\gamma\int_\Omega u^p, \qquad \ell \geq 0,$$

with $C_1 = \max\left\{K_0, \|w_0\|_{L_\infty(\Omega)}\right\}$. So it remains to consider: (I) $\theta - C_FC_1\gamma > 0$ and (II) $\theta - C_FC_1\gamma \leq 0$. For the case (I), one may have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + \frac{2(p-1)c_0}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + p(\theta - C_FC_1\gamma)\int_\Omega u^p \\
\leq &\frac{p(p-1)c_1^2K_2^2}{2c_0}\int_\Omega u^p;
\end{aligned} \tag{4.10}$$

and for the case (II),

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + \frac{2(p-1)c_0}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + p\theta\int_\Omega u^p \tag{4.11}$$

$$\leq \frac{p(p-1)c_1^2K_2^2}{2c_0}\int_\Omega u^p + p(C_FC_1\gamma + \theta)\int_\Omega u^p. \tag{4.12}$$

To conduct Moser's iteration, we use Gagliardo-Nirenberg interpolation to decompose the right-hand $\int_\Omega u^p$ into $\int_\Omega |u^{\frac{p}{2}}|$ and $\int_\Omega |\nabla u^{\frac{p}{2}}|^2$ so that the latter one can be cancelled if its coefficient is set appropriately.

Indeed, by Gagliardo-Nirenberg interpolation inequality and Young's inequality with parameter $\eta > 0$ and with index $\frac{1}{\alpha}$ and $\frac{1}{1-\alpha}$ one may have

$$\begin{aligned}
\int_\Omega |u|^p = &\|u^{\frac{p}{2}}\|^2_{L_2(\Omega)} \leq c_2\|\nabla u^{\frac{p}{2}}\|^{2\alpha}_{L_2(\Omega)}\|u^{\frac{p}{2}}\|^{2(1-\alpha)}_{L_q(\Omega)} + c_3\|u^{\frac{p}{2}}\|^2_{L_s(\Omega)} \\
&\leq c_2\eta\|\nabla u^{\frac{p}{2}}\|^2_{L_2(\Omega)} + c_2\Big(\frac{1}{\eta}\Big)^{\frac{\alpha}{1-\alpha}}\|u^{\frac{p}{2}}\|^2_{L_q(\Omega)} + c_3\|u^{\frac{p}{2}}\|^2_{L_s(\Omega)}
\end{aligned} \tag{4.13}$$

with $\alpha = \frac{\frac{1}{q}-\frac{1}{2}}{\frac{1}{n}+\frac{1}{q}-\frac{1}{2}} \in (0,1)$ as $1 \leq q < 2$. Then associated with (4.10), by taking $q = s = 1$ in (4.13) we may infer that $\alpha = \frac{1}{\frac{2}{n}+1}$, $\frac{\alpha}{1-\alpha} = \frac{n}{2}$, and

$$\frac{p(p-1)c_1^2K_2^2}{2c_0}\int_\Omega |u|^p \leq \frac{(p-1)c_0}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + p^{n+2}\, c_4\Big(\int_\Omega |u^{\frac{p}{2}}|\Big)^2 \tag{4.14}$$

where we have taken $\eta = \frac{2c_0^2}{p^2 c_2 (c_1 K_2)^2}$ and $c_4 = \big(c_3 + \frac{c_2 (c_1 K_2 \sqrt{c_2})^n}{(\sqrt{2} c_0)^n}\big) \cdot \frac{(c_1 K_2)^2}{2c_0}$. Therefore, we derive

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^p + p(\theta - C_F C_1 \gamma) \int_\Omega u^p \le p^{n+2}\, c_4 \Big(\int_\Omega |u^{\frac{p}{2}}| \Big)^2.$$

In regard to (4.11), taking $q = s = 1$ in (4.13) again will produce that

$$p(C_F C_1 \gamma + \theta) \int_\Omega u^p \le \frac{(p-1)c_0}{p} \int_\Omega |\nabla u^{\frac{p}{2}}|^2 + p^{n+2}\, c_5 \Big(\int_\Omega |u^{\frac{p}{2}}| \Big)^2 \tag{4.15}$$

where we have set $\eta = \frac{(p-1)c_0}{p^2 c_2 (C_F C_1 \gamma + \theta)}$ and $c_5 = \big(c_3 + \frac{(c_2 \sqrt{C_F C_1 \gamma + \theta})^n}{(\sqrt{c_0})^n}\big) \cdot (C_F C_1 \gamma + \theta)$. Hence (4.14) and (4.15) jointly show that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^p + p\theta \int_\Omega u^p \le p^{n+2}\, c_6 \Big(\int_\Omega |u^{\frac{p}{2}}| \Big)^2 \tag{4.16}$$

with $c_6 = c_4 + c_5$.

To sum up, by letting $\kappa := \theta - C_F C_1 \gamma > 0$ in case (I) or $\kappa := \theta > 0$ in case (II), there is a constant $c_7 := \max\{c_4, c_6\}$ which is independent of p, such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^p + p\kappa \int_\Omega u^p \le c_7 p^{n+2} \Big(\int_\Omega |u^{\frac{p}{2}}| \Big)^2 \le c_7 p^{n+2} \sup_{t \in [0,T)} \Big(\int_\Omega |u^{\frac{p}{2}}| \Big)^2, \qquad p \ge 2,$$

on $(0, T) \subset (0, T_{\max})$. Notice that the rightmost term above is unrelated to time variable t. Then solving this inequality with respect to t on $(0,T) \subset (0, T_{\max})$ gives

$$\begin{aligned} \int_\Omega u^p(t,x)\,\mathrm{d}x \le& \int_\Omega u_0^p(x)\,\mathrm{d}x + \frac{c_7}{\kappa} p^{n+1} \sup_{t\in[0,T)} \Big(\int_\Omega |u^{\frac{p}{2}}(t,x)|\,\mathrm{d}x \Big)^2 \\ \le& \big(|\Omega| + \frac{c_7}{\kappa} + 1\big) p^{n+1} \max\Big\{ \|u_0\|_{L_\infty(\Omega)},\ \sup_{t\in[0,T)} \Big(\int_\Omega |u^{\frac{p}{2}}(t,x)|\,\mathrm{d}x \Big)^{\frac{2}{p}} \Big\}^p. \end{aligned}$$

This indicates

$$\tilde{F}(p) \le \big(|\Omega| + \frac{c_7}{\kappa} + 1\big)^{\frac{1}{p}} p^{\frac{n+1}{p}}\, \tilde{F}(p/2)$$

with $\tilde{F}(p) := \max\Big\{ \|u_0\|_{L_\infty(\Omega)},\ \sup_{t\in[0,T)} \Big(\int_\Omega u^p(t,x)\,\mathrm{d}x \Big)^{\frac{1}{p}} \Big\}$. Denoting $c_8 = |\Omega| +$

$\frac{c_7}{\kappa} + 1$ and setting $p = 2^i, i = 1, 2, 3, \cdots$, then we have

$$\tilde{F}(2^i) \le c_8^{\sum_{k=1}^{i} 2^{-k}}\ 2^{\sum_{k=1}^{i} \frac{k}{2^{(n+1)k}}}\ \tilde{F}(1) \le c_8\ 2^{\frac{2^{n+1}}{(2^{n+1}-1)^2}}\ \tilde{F}(1)$$

and $\tilde{F}(1) = \{\|u_0\|_{L_\infty(\Omega)},\ \sup_{t\in[0,T)} \int_\Omega u(t,x)\,\mathrm{d}x\} \le \{\|u_0\|_{L_\infty(\Omega)},\ C\}$ where C is from (4.9) and thus is independent of $i, T_{\max}$ and T. Finally, letting $i \to +\infty$ concludes that for all $t \in (0,T) \subset (0,T_{\max})$,

$$\|u(t,\cdot)\|_{L_\infty(\Omega)} \le c_8\ 2^{\frac{2^{n+1}}{(2^{n+1}-1)^2}}\ \{\|u_0\|_{L_\infty(\Omega)},\ C\}.$$

Hence such an estimate holds for all $t \in (0,T_{\max})$ due to T arbitrarily in $(0,T_{\max})$. This completes the proof. □

Remark 4.3. *By rewriting the third component in system* (4.1) *as*

$$w_t = d_w \Delta w - w + R$$

with $R = w + wf(w) - uF(w)$, *then one may apply Lemma 4.4 to this equation after a rescaling. Since in view of Lemma 4.3 and Lemma 4.5, one may infer that*

$$\|R(t,\cdot)\|_{L_\infty(\Omega)} \le C(\|w(t,\cdot)\|_{L_\infty(\Omega)} + \|u(t,\cdot)\|_{L_\infty(\Omega)}) \le \mathcal{C} \quad \text{for all } t \in (0,T_{\max}),$$

with constants $C, \mathcal{C}$ *independent of* $T_{\max}$. *It follows that*

$$\|w(t,\cdot)\|_{W^1_\infty(\Omega)} \le \mathfrak{C} \quad \text{for all } t \in (0,T_{\max}),$$

if $w_0(x) \in W^1_\infty(\Omega)$ *where constant* $\mathfrak{C}$ *is independent of* $T_{\max}$.

This remark is useful to prove asymptotic stability in the next section.

4.2.3 Proof of Theorem 4.1

Proof. Lemma 4.1 has shown the existence of local unique classical solution to system (4.1). The extendability standard of such a classical solution in Lemma 4.2 can be satisfied by Lemma 4.3, Lemma 4.5, and Remark 4.2. So one can obtain the global existence of unique classical solution to system (4.1), and the regularity (4.2). Finally the estimate (4.3) holds for all $t > 0$ by Remark 4.2, Lemma 4.5 and Remark 4.3. $\square$

4.3 Global Asymptotic Stability

In the last section we have proved that system (4.1) possesses a unique global-in-time classical solution under (H1)–(H3). In this section we concentrate on its longtime behaviors if (H4) holds in addition. To this end, we introduce the following two basic lemmas.

Lemma 4.6. *If F fulfills condition $(H3)$, then a function*

$$\zeta(z) := \int_{\kappa}^{z} \frac{F(\eta) - F(\kappa)}{F(\eta)} \, \mathrm{d}\eta \tag{4.17}$$

is nonnegative and convex. Furthermore, if $z \to \kappa$ then

$$\frac{F'(\kappa)}{4F(\kappa)}(z-\kappa)^2 \le \zeta(z) \le \frac{F'(\kappa)}{F(\kappa)}(z-\kappa)^2.$$

This lemma can be proved by doing Talyor's expansion of $\zeta(z)$ with respect to z up to its second order derivative at $z = \kappa$ ($\zeta(\kappa) = \zeta'(\kappa) = 0$). One may refer to Lemma 4.1 in [39] for more details.

We below summarize limit property of a dynamic system (cf. Chap.4 in [76]) that we will use later. Given a dynamic system (nonlinear semigroup) $\{S(t) : t \ge 0\}$ on a complete metric space $(\mathcal{M}, \|\cdot\|)$. Then for a real-valued continuous function $L(\mathbf{x})$, $\mathbf{x} \in \mathcal{M}$, we say $L(\mathbf{x})$ is a Lyapunov function of this dynamic system if for all $t \ge 0, \mathbf{x} \in \mathcal{M}$ and $\delta \in \mathbb{R}$,

$$\frac{\mathrm{d}L(S(t)\mathbf{x})}{\mathrm{d}t} := \lim_{\delta \to 0^+} \sup \frac{L(S(t+\delta)\mathbf{x}) - L(S(t)\mathbf{x})}{\delta} \le 0.$$

For any $\mathbf{x} \in \mathcal{M}$, $\Gamma(\mathbf{x}) := \{S(t)\mathbf{x} : t \ge 0\}$ denotes the trajectory through $\mathbf{x}$. In particular, we call $\mathbf{x}$ is an equilibrium point of the dynamic system if $\Gamma(\mathbf{x}) = \{\mathbf{x}\}$.

Lemma 4.7. *Let* $\mathcal{E} := \{\mathbf{x} \in \mathcal{M} : \frac{\mathrm{d}L(S(t)\mathbf{x})}{\mathrm{d}t} = 0\}$. *Denote by* $\mathcal{Z} := \{\mathbf{x} \in \mathcal{E} : S(t)\mathbf{x} \in \mathcal{E}$ *for all* $t \geq 0\}$ *the largest invariant subset of* $\mathcal{E}$. *For some* $\mathbf{x}_0 \in \mathcal{M}$, *if the trajectory* $\Gamma(\mathbf{x}_0) = \{S(t)\mathbf{x}_0 : t \geq 0\}$ *is contained in a compact set of* $\mathcal{M}$, *then there are two properties for the* ω*-limit set* $\mathcal{V}_\omega(\mathbf{x}_0)$ *of* $\Gamma(\mathbf{x}_0)$ *(or* $\mathbf{x}_0$*) as:*

(i) $\mathcal{V}_\omega(\mathbf{x}_0) \subset \mathcal{Z}$*;*

(ii) $S(t)\mathbf{x}_0 \to \mathcal{Z}$ *as* $t \to \infty$,

where $\mathcal{V}_\omega(\mathbf{x}_0) := \{\lim_{k\to+\infty} S(t_k)\mathbf{x}_0 \in \mathcal{M} :\ \exists\, t_k > 0, \lim_{k\to+\infty} t_k = +\infty\} = \bigcap_{\tau\geq 0} \overline{\{S(t)\mathbf{x}_0 :\ t \geq \tau\}}$.

Additionally if all $\mathbf{y} \in \mathcal{E}$ *are equilibria and all elements of* $\mathcal{E}$ *are isolated from each other, then* $\mathcal{V}_\omega(\mathbf{x}_0)$ *consists of equilibria and contains only one element.*

Lemma 4.6 may help us to construct Lyapunov functions we need. Lemma 4.7 indicates that one may apply Lemma 4.7 to corresponding Lyapunov functions, in order to investigate the global asymptotic stability of the prey–only state $(0, \frac{\beta K_0}{\sigma}, K_0)$ and the coexistence state (u_*, v_*, w_*). Indeed, Theorem 4.1 means that system (4.1) has the unique global-in-time classical solution $(u, v, w) \in C^2(\overline{\Omega}, \mathbb{R}^3)$ which is continuous to its initial value $(u_0, v_0, w_0) =: \mathbf{u}_0(x) \in C^2(\overline{\Omega}, \mathbb{R}^3)$. This indicates that system (4.1) can generate a dynamic system on $C^2(\overline{\Omega}, \mathbb{R}^3)$, still denoted by $\{S(t) : t \geq 0\}$, such that $S(t)\mathbf{u}_0 := \mathbf{u}(t, x; \mathbf{u}_0(x)) := (u(t, x; u_0(x)), v(t, x; v_0(x)), w(t, x; w_0(x))) \in C^2(\overline{\Omega}, \mathbb{R}^3)$, and $S(0)$ is an identity, i.e., $S(0)\mathbf{u}_0(x) = \mathbf{u}_0(x)$ for any $\mathbf{u}_0(x) \in C^2(\overline{\Omega}, \mathbb{R}^3)$. Then $\{S(t)\mathbf{u}_0 :\ t \geq 0\}$ is a trajectory through $\mathbf{u}_0(x)$ which can be contained in a compact subset of $C^2(\overline{\Omega}, \mathbb{R}^3)$ by (4.3) and one estimation similar to the (46) and (48) in Theorem 4.1 of [67]. The $(0, \frac{\beta K_0}{\sigma}, K_0)$ and (u_*, v_*, w_*) can be viewed as two equilibria of this dynamic system.

In addition, (H2) and (H4) indicate that $F'(w) > 0$ and $-\varphi'(w) > 0$ are continuous on $[0, C]$ for any $C > 0$. Thus

$$\min_{w\in[0,C]} F'(w) \min_{w\in[0,C]} (-\varphi'(w)) \tag{4.18}$$

exists and is strictly positive.

With these preparations at hand, we below prove 1) of the Theorem 4.2.

4.3.1 Asymptotical Stability of Prey-only Steady State

Lemma 4.8. *Let* $(H1)-(H4)$ *hold and* (u, v, w) *be a global-in-time classical solution of system* (4.1) *obtained in Theorem 4.1. Then the prey–only state* $(0, \frac{\beta K_0}{\sigma}, K_0)$ *is globally asymptotic stable provided that* $F(K_0) \le \frac{\theta}{\gamma}$*. Furthermore, if* $F(K_0) < \frac{\theta}{\gamma}$*, there are constants* $\bar{c}_1, \bar{c}_2, T_0 > 0$ *such that for* $t > T_0 > 0$

$$\|u(t,\cdot)\|_{L_\infty(\Omega)} + \left\|v(t,\cdot) - \tfrac{\beta K_0}{\sigma}\right\|_{L_\infty(\Omega)} + \|w(t,\cdot) - K_0\|_{L_\infty(\Omega)} \le \bar{c}_2 e^{-\frac{\bar{c}_1 t}{2(n+1)}}.$$

Proof. We may construct a function for $t > 0$ that

$$\begin{aligned}\mathcal{L}_1(t) :=& \mathcal{L}_1(u(t), v(t), w(t)) \\ :=& \frac{1}{\gamma}\int_\Omega u + \frac{M}{2}\int_\Omega \Big(v - \frac{\beta K_0}{\sigma}\Big)^2 + \int_\Omega \int_{K_0}^{w} \frac{F(\eta) - F(K_0)}{F(\eta)}\,\mathrm{d}\eta\end{aligned}$$

where (u, v, w) is the classical solution to system (4.1) and the constant $M > 0$ is to be determined after (4.21).

Next we show that $\mathcal{L}_1$ is a Lyapunov function, i.e., $\frac{\mathrm{d}\mathcal{L}_1}{\mathrm{d}t} \le 0$ for all (u, v, w) solving system (4.1). Indeed, under the zero-Neumann boundary condition in system (4.1), one has

$$\begin{aligned}\frac{\mathrm{d}\mathcal{L}_1}{\mathrm{d}t} =& \frac{1}{\gamma}\int_\Omega u_t + M\int_\Omega \Big(v - \frac{\beta K_0}{\sigma}\Big)v_t + \int_\Omega \frac{F(w) - F(K_0)}{F(w)} w_t \\ =& \frac{1}{\gamma}\int_\Omega (\gamma u F(w) - \theta u - \ell u^2) + \int_\Omega \frac{F(w) - F(K_0)}{F(w)} w_t \\ &+ M\int_\Omega \Big(v - \frac{\beta K_0}{\sigma}\Big)v_t.\end{aligned} \tag{4.19}$$

Moreover, for the second right-hand term one may further infer that

$$\int_\Omega \frac{F(w) - F(K_0)}{F(w)} w_t$$

$$=\int_\Omega \frac{F(w)-F(K_0)}{F(w)}\big(d_w\Delta w + wf(w) - uF(w)\big)$$

$$=-\int_\Omega d_w F(K_0)F'(w)\frac{|\nabla w|^2}{F^2(w)} + \int_\Omega \frac{F(w)-F(K_0)}{F(w)}(wf(w)-uF(w))$$

and from $f(K_0)=0$ in (H2) we may derive that

$$\int_\Omega \frac{F(w)-F(K_0)}{F(w)}wf(w) = \int_\Omega \left(\frac{wf(w)}{F(w)} - \frac{K_0 f(K_0)}{F(K_0)}\right)(F(w)-F(K_0))$$

$$= \int_\Omega \varphi'(w_1)F'(w_2)(w-K_0)^2$$

where $\varphi(w)=\frac{wf(w)}{F(w)}$, w_i is between w and K_0, $i=1,2$, in addition to

$$-\int_\Omega \frac{F(w)-F(K_0)}{F(w)}uF(w) = \int_\Omega F(K_0)u - \int_\Omega F(w)u.$$

On the other hand, by $\beta w_c=\sigma v_c$ and $w_c=K_0$, one may infer that

$$M\int_\Omega \Big(v-\frac{\beta K_0}{\sigma}\Big)v_t$$

$$=M\int_\Omega \Big(v-\frac{\beta K_0}{\sigma}\Big)(d_v\Delta v + \beta w - \sigma v)$$

$$=-Md_v\int_\Omega \nabla\Big(v-\frac{\beta K_0}{\sigma}\Big)\nabla v + M\int_\Omega \Big(v-\frac{\beta K_0}{\sigma}\Big)(\beta w-\sigma v)$$

$$=-Md_v\int_\Omega \Big|\nabla\Big(v-\frac{\beta K_0}{\sigma}\Big)\Big|^2 + M\beta\int_\Omega \Big(v-\frac{\beta K_0}{\sigma}\Big)(w-K_0) - M\sigma\int_\Omega \Big(v-\frac{\beta K_0}{\sigma}\Big)^2$$

and using the Young's inequality with ε will yield

$$M\beta\int_\Omega \Big(v-\frac{\beta K_0}{\sigma}\Big)(w-K_0) \le M\beta\int_\Omega \left[\frac{\varepsilon}{2}\Big(v-\frac{\beta K_0}{\sigma}\Big)^2 + \frac{(w-K_0)^2}{2\varepsilon}\right] \tag{4.20}$$

for any $\varepsilon>0,\ M\beta>0$.

Then by using the assumption $F(K_0)\le\frac{\theta}{\gamma}$, setting $0<\varepsilon\le\frac{2\sigma}{\beta}$, and by invoking the estimates from (4.19) and (4.20) one may update (4.19) that

$$
\begin{aligned}
\frac{\mathrm{d}\mathcal{L}_1}{\mathrm{d}t} =& \int_\Omega \Big(F(K_0) - \frac{\theta}{\gamma}\Big)u - \int_\Omega \frac{\ell u^2}{\gamma} - d_w F(K_0)\int_\Omega F'(w)\frac{|\nabla w|^2}{F^2(w)} \\
&+ \int_\Omega \varphi'(w_1)F'(w_2)(w-K_0)^2 - Md_v\int_\Omega \Big|\nabla\Big(v - \frac{\beta K_0}{\sigma}\Big)\Big|^2 \\
&- M\sigma\int_\Omega\Big(v-\frac{\beta K_0}{\sigma}\Big)^2 + M\beta\int_\Omega\Big(v - \frac{\beta K_0}{\sigma}\Big)(w-K_0) \\
\le& \int_\Omega \Big(F(K_0) - \frac{\theta}{\gamma}\Big)u + \int_\Omega\Big(\varphi'(w_1)F'(w_2) + \frac{M\beta}{2\varepsilon}\Big)(w-K_0)^2 \\
&- M(\sigma - \frac{\varepsilon\beta}{2})\int_\Omega\Big(v - \frac{\beta K_0}{\sigma}\Big)^2 \\
\le& \int_\Omega\Big(\varphi'(w_1)F'(w_2) + \frac{M\beta}{2\varepsilon}\Big)(w-K_0)^2.
\end{aligned}
\tag{4.21}
$$

In light of Lemma 4.3 we know $0 < w_1, w_2 \le C_1$ with $C_1 = \max\big\{K_0, \|w_0\|_{L_\infty(\Omega)}\big\}$. Hence making use of (4.18) and taking

$$
0 < M \le \frac{4\sigma}{\beta^2}\min_{w\in[0,C_1]} F'(w)\min_{w\in[0,C_1]}(-\varphi'(w))
$$

will conclude that $\frac{\mathrm{d}\mathcal{L}_1}{\mathrm{d}t} \le 0$.

For each $t > 0$, we let

$$
\mathcal{L}_1(t) = \int_\Omega \frac{u}{\gamma} + \int_\Omega \frac{M}{2}\Big(v - \frac{\beta K_0}{\sigma}\Big)^2 + \int_\Omega\int_{K_0}^{w}\frac{F(\eta)-F(K_0)}{F(\eta)}\,\mathrm{d}\eta =: \int_\Omega \mathcal{H}_1(u,v,w).
$$

Here $\mathcal{H}_1(u,v,w) := \frac{u}{\gamma} + \frac{M}{2}\big(v - \frac{\beta K_0}{\sigma}\big)^2 + \zeta(w)$ is a convex function of (u,v,w) in view of Lemma 4.6 with $\kappa = K_0$. $\mathcal{H}_1(u,v,w)$ has no more than one minimum point, so does $\mathcal{L}_1(t)$ in the sense of (u,v,w). The equation $\frac{\mathrm{d}\mathcal{L}_1(t)}{\mathrm{d}t} = 0$ thus has at most one solution in the sense of (u,v,w), which implies that $\frac{\mathrm{d}\mathcal{L}_1(t)}{\mathrm{d}t} = 0$ if and only if $(u,v,w) = (0, \frac{\beta K_0}{\sigma}, K_0)$. Then Lemma 4.7 concludes that the solution of (4.1) which is bounded will approach $(0, \frac{\beta K_0}{\sigma}, K_0)$ as $t \to \infty$. In other words, $(0, \frac{\beta K_0}{\sigma}, K_0)$ is globally asymptotic stable.

We can further ascertain the corresponding convergent rate. Due to $F(K_0) < \frac{\theta}{\gamma}$, the first inequality in (4.21), and Lemma 4.6, there exists a constant $\hat{c}_1 > 0$ such that

$$\frac{\mathrm{d}\mathcal{L}_1(t)}{\mathrm{d}t} \leq -\hat{c}_1 \mathcal{L}_1(t), \qquad \text{for} \quad t > 0.$$

Solving this inequality shows

$$\mathcal{L}_1(t) \leq \hat{c}_2 e^{-\hat{c}_1 t}, \qquad \text{for} \quad t > 0$$

where the constant $\hat{c}_2 > 0$ depends only on $\mathcal{L}_1(0)$. Lemma 4.6 also signifies that there is a $T_1 > 0$ such that

$$\frac{1}{\gamma}\int_\Omega u + \frac{M}{2}\int_\Omega \Big(v - \frac{\beta K_0}{\sigma}\Big)^2 + \int_\Omega \frac{F'(K_0)}{4F(K_0)}(w - K_0)^2 \leq \hat{c}_2 e^{-\hat{c}_1 t}, \qquad \text{for} \quad t \geq T_1$$

which means

$$\|u(t,\cdot)\|_{L_1(\Omega)} + \big\|v(t,\cdot) - \tfrac{\beta K_0}{\sigma}\big\|_{L_2(\Omega)} + \|w(t,\cdot) - K_0\|_{L_2(\Omega)} \leq \hat{c}_3 e^{-\frac{\hat{c}_1}{2}t}, \qquad \text{for} \quad t \geq T_1$$

with $\hat{c}_3 = 3\max\big\{\hat{c}_2\gamma,\ (\frac{2\hat{c}_2}{M})^{1/2},\ \big(\frac{4F(K_0)\hat{c}_2}{F'(K_0)}\big)^{1/2}\big\}$.

We next may strengthen this convergence rate. Since (u, v, w) is a classical solution to (4.1), then by (4.3) there exists a constant $\hat{c}_4 > 0$ such that $\|u\|_{L_\infty(\Omega)}$, $\|\nabla v\|_{L_\infty(\Omega)}$, $\|\nabla w\|_{L_\infty(\Omega)} \leq \hat{c}_4$ when $t > T_1 > 0$. Similar to the estimation of (46) and (48) in Theorem 4.1 of [67] and by semigroup theory and L^p-L^q estimate, there exist $\hat{c}_4' > 0$ and $\epsilon \in (0, 1)$ such that $\|w(t,\cdot)\|_{C^{2+\epsilon}(\bar{\Omega})}, \|v(t,\cdot)\|_{C^{2+\epsilon}(\bar{\Omega})} \leq \hat{c}_4'$ for all $t > T_1' > 0$. Denote $T_0 = \max\{T_1, T_1'\}$. One can apply the Theorem 7.2 or 7.4 in Chap.V of [3] to the first equation of (4.1) which can be rewritten as $u_t - d(v)\Delta u + b(t, x, u, \nabla u) = 0$ with

$$b(t, x, u, \nabla u) = -[d'(v)\nabla v - \chi(v)\nabla v]\cdot\nabla u + [\chi'(v)(\nabla v\cdot\nabla v) + \chi(v)\Delta v]u - \gamma u F(w) + \theta u + \ell u^2.$$

Then there exits another constant, still denoted by $\hat{c}_4$, such that $\|\nabla u\|_{L_\infty(\Omega)} \le \hat{c}_4$ for all $t > T_0$.

An application of Gagliardo–Nirenberg interpolation inequality may yield that for all $t > T_0$,

$$\|u\|_{L_\infty(\Omega)} \le \hat{c}_5(\|\nabla u\|^{\frac{n}{n+1}}_{L_\infty(\Omega)}\|u\|^{\frac{1}{n+1}}_{L_1(\Omega)} + \|u\|_{L_1(\Omega)}) \le \hat{c}_6 e^{-\frac{\hat{c}_1 t}{2(n+1)}},$$

$$\big\|v - \tfrac{\beta K_0}{\sigma}\big\|_{L_\infty(\Omega)} \le \hat{c}_7\big(\big\|\nabla(v - \tfrac{\beta K_0}{\sigma})\big\|^{\frac{n}{n+2}}_{L_\infty(\Omega)}\big\|v - \tfrac{\beta K_0}{\sigma}\big\|^{\frac{2}{n+2}}_{L_2(\Omega)} + \big\|v - \tfrac{\beta K_0}{\sigma}\big\|_{L_2(\Omega)}\big) \le \hat{c}_8 e^{-\frac{\hat{c}_1 t}{n+2}},$$

$$\|w - K_0\|_{L_\infty(\Omega)} \le \hat{c}_9\big(\|\nabla(w - K_0)\|^{\frac{n}{n+2}}_{L_\infty(\Omega)}\|w - K_0\|^{\frac{2}{n+2}}_{L_2(\Omega)} + \|w - K_0\|_{L_2(\Omega)}\big) \le \hat{c}_{10} e^{-\frac{\hat{c}_1 t}{n+2}},$$

where $\hat{c}_6 := \hat{c}_5(\hat{c}_4^{\frac{n}{n+1}}\hat{c}_3^{\frac{1}{n+1}} + \hat{c}_3)$, $\hat{c}_8 := \hat{c}_7(\hat{c}_4^{\frac{n}{n+2}}\hat{c}_3^{\frac{2}{n+2}} + \hat{c}_3)$, and $\hat{c}_{10} := \hat{c}_9(\hat{c}_4^{\frac{n}{n+2}}\hat{c}_3^{\frac{2}{n+2}} + \hat{c}_3)$. Therefore,

$$\|u\|_{L_\infty(\Omega)} + \big\|v - \tfrac{\beta K_0}{\sigma}\big\|_{L_\infty(\Omega)} + \|w - K_0\|_{L_\infty(\Omega)} \le \hat{c}_{11} e^{-\frac{\hat{c}_1 t}{2(n+1)}}, \qquad t > T_0$$

with $\hat{c}_{11} := \hat{c}_6 + \hat{c}_8 + \hat{c}_{10}$. This completes the proof. □

4.3.2 Asymptotical Stability of Coexistence Steady State

As is shown in (4.4), the positive coexistence state (u_*, v_*, w_*) should satisfy

$$u_* F(w_*) = w_* f(w_*) = \frac{u_*(\theta + \ell u_*)}{\gamma} > 0\ , \qquad v_* = \frac{\beta w_*}{\sigma} > 0, \qquad w_* > 0.$$

We are now in a position to prove part 2) of the Theorem 4.2.

Lemma 4.9. *Let $(H1) - (H4)$ hold and (u, v, w) be the global classical solution of system (4.1) obtained in Theorem 4.1. If the coexistence steady state (u_*, v_*, w_*) exists and*

$$\max_{0<v\le K_2} \frac{\chi(v)^2}{d(v)} \le \frac{16 d_v \gamma\sigma}{\beta^2 u_*} \min_{\tilde{w}_1\in[0,C_1]}\{-\varphi'(\tilde{w}_1)\} \min_{\tilde{w}_2\in[0,C_1]}\{F'(\tilde{w}_2)\}, \tag{4.22}$$

with K_2 from Remark 4.2 and $C_1 = \max\left\{K_0, \|w_0\|_{L_\infty(\Omega)}\right\}$ and $\varphi(w) = \frac{wf(w)}{F(w)}$, then *the* (u_*, v_*, w_*) *is globally asymptotic stable. Moreover, there are three constants* $\tilde{c}_1, \tilde{c}_2, T_1 > 0$ *such that*

$$\|u(t,\cdot)-u_*\|_{L_\infty(\Omega)} + \left\|v(t,\cdot)-v_*\right\|_{L_\infty(\Omega)} + \|w(t,\cdot)-w_*\|_{L_\infty(\Omega)} \le \tilde{c}_1 e^{-\frac{\tilde{c}_2 t}{n+2}}, \qquad t > T_1.$$

Proof. We may construct the following function for $t > 0$ that

$$\begin{aligned}\mathcal{L}_2(t) :=& \mathcal{L}_2(u(t), v(t), w(t))\\ :=& \frac{1}{\gamma}\int_\Omega \big(u - u_* - u_* \ln \frac{u}{u_*}\big) + \frac{M}{2}\int_\Omega (v-v_*)^2 + \int_\Omega \int_{w_*}^{w} \frac{F(\eta)-F(w_*)}{F(\eta)}\,\mathrm{d}\eta\end{aligned}$$

where (u, v, w) is the global classical solution of system (4.1) and $M > 0$ is a constant to be determined in system (4.25). Similar to Lemma 4.8, we first verify $\frac{\mathrm{d}\mathcal{L}_2(t)}{\mathrm{d}t} \le 0$. Replacing u_t, v_t, w_t in the following equality may yield

$$\begin{aligned}&\frac{\mathrm{d}\mathcal{L}_2(t)}{\mathrm{d}t}\\ =&\frac{1}{\gamma}\int_\Omega \big(u_t - \frac{u_*}{u}u_t\big) + M\int_\Omega (v-v_*)v_t + \int_\Omega \frac{F(w)-F(w_*)}{F(w)}w_t\\ =&\frac{1}{\gamma}\int_\Omega (\gamma u F(w) - \theta u - \ell u^2)\\ &-\frac{u_*}{\gamma}\int_\Omega \Big(d(v)\frac{|\nabla u|^2}{u^2} - \chi(v)\frac{\nabla u\cdot\nabla v}{u} + (\gamma F(w) - \theta - \ell u)\Big)\\ &+M\int_\Omega \Big(-d_v|\nabla v|^2 + (v-v_*)(\beta w - \sigma v)\Big) - \int_\Omega d_w F(w_*)\frac{F'(w)}{F^2(w)}|\nabla w|^2\\ &+\int_\Omega \Big(\frac{wf(w)}{F(w)} - u\Big)(F(w)-F(w_*)).\end{aligned} \tag{4.23}$$

For the terms above involving ∇u and ∇v and for $u \neq 0$, one may notice that

$$\begin{aligned}&-\frac{u_*}{\gamma}\int_\Omega \big(d(v)\frac{|\nabla u|^2}{u^2} - \chi(v)\frac{\nabla u\cdot\nabla v}{u}\big) - M\int_\Omega d_v|\nabla v|^2\\ =&-\frac{u_*}{\gamma}\int_\Omega \Big(\frac{\nabla u}{u}, \nabla v\Big)\mathbf{H}\Big(\frac{\nabla u}{u}, \nabla v\Big)^\tau \le 0\end{aligned}$$

where τ refers to transpose and

$$\mathbf{H} := \begin{pmatrix} d(v) & -\frac{\chi(v)}{2} \\ -\frac{\chi(v)}{2} & \frac{\gamma M d_v}{u_*} \end{pmatrix}$$

is positive semi-definite when

$$M \geq \max_{0<v\leq \mathcal{C}_1} \frac{u_* \chi(v)^2}{4 d_v \gamma d(v)}. \tag{4.24}$$

Here $0 < v(t,x) < \mathcal{C}_1$ for all $t > 0$ and all $x \in \Omega$, owning to Remark 4.2 and the regularity (4.2) in Theorem 4.1. In terms of $u_* = \frac{w_* f(w_*)}{F(w_*)}$ one may obtain that

$$\int_\Omega \Big(\frac{wf(w)}{F(w)} - u\Big)(F(w) - F(w_*)) = \int_\Omega \Big(\frac{wf(w)}{F(w)} - \frac{w_* f(w_*)}{F(w_*)} + u_* - u\Big)(F(w) - F(w_*))$$

$$= \int_\Omega \varphi'(\tilde{w}_1) F'(\tilde{w}_2)(w - w_*)^2 - \int_\Omega (F(w) - F(w_*))(u - u_*)$$

where $\varphi(w) = \frac{wf(w)}{F(w)}$, $\tilde{w}_i$ lies between w and w_*, $i = 1, 2$. In light of $\ell u_* + \theta = \gamma F(w_*)$, we have

$$\frac{1}{\gamma}\int_\Omega (\gamma u F(w) - \theta u - \ell u^2) - \frac{u_*}{\gamma}\int_\Omega (\gamma F(w) - \theta - \ell u) = \frac{1}{\gamma}\int_\Omega (u - u_*)(\gamma F(w) - \theta - \ell u)$$

$$= \frac{1}{\gamma}\int_\Omega (u - u_*)\big[\gamma F(w) - \theta - \ell u - (\gamma F(w_*) - \theta - \ell u_*)\big]$$

$$= \int_\Omega (u - u_*)(F(w) - F(w_*)) - \frac{\ell}{\gamma}\int_\Omega (u - u_*)^2.$$

Note that $(v - v_*)(\beta w - \sigma v) = \beta(v - v_*)(w - w_*) - \sigma(v - v_*)^2$ by $v_* = \frac{\beta w_*}{\sigma}$. One can derive from Young's inequality that

$$\begin{aligned} & M\int_\Omega (v - v_*)(\beta w - \sigma v) \\ = & -M\sigma \int_\Omega (v - v_*)^2 + M\beta \int_\Omega (v - v_*)(w - w_*) \end{aligned}$$

$$\leq M\Big(\frac{\beta\varepsilon}{2}-\sigma\Big)\int_\Omega (v-v_*)^2+\frac{M\beta}{2\varepsilon}\int_\Omega (w-w_*)^2$$
$$\leq \frac{M\beta}{2\varepsilon}\int_\Omega (w-w_*)^2,$$

for $0\leq \frac{\beta\varepsilon}{2}\leq \sigma$ or $0<\varepsilon\leq \frac{2\sigma}{\beta}$. Consequently, we know

$$\frac{\mathrm{d}\mathcal{L}_2(t)}{\mathrm{d}t}\leq -d_wF(w_*)\int_\Omega \frac{F'(w)}{F^2(w)}|\nabla w|^2+\int_\Omega \Big(\varphi'(\tilde{w}_1)F'(\tilde{w}_2)+\frac{M\beta}{2\varepsilon}\Big)(w-w_*)^2.$$

Lemma 4.3 shows $0<\tilde{w}_1,\tilde{w}_2\leq C_1$ with $C_1=\max\big\{K_0,\|w_0\|_{L_\infty(\Omega)}\big\}$. Thus by (4.18) we can set

$$0<M\leq \frac{4\sigma}{\beta^2}\min_{[0,C_1]}\big\{-\varphi'(\tilde{w}_1)\big\}\min_{[0,C_1]}\big\{F'(\tilde{w}_2)\big\}. \tag{4.25}$$

Then (4.22) implies there exists a $M>0$ such that both (4.24) and (4.25) hold, which means $\frac{\mathrm{d}\mathcal{L}_2(t)}{\mathrm{d}t}\leq 0$.

Next we claim that $\frac{\mathrm{d}\mathcal{L}_2(t)}{\mathrm{d}t}=0$ will lead to $(u,v,w)=(u_*,v_*,w_*)$. In fact,

$$\mathcal{L}_2(t)=\frac{1}{\gamma}\int_\Omega \big(u-u_*-u_*\ln\frac{u}{u_*}\big)+\frac{M}{2}\int_\Omega (v-v_*)^2+\int_\Omega\int_{w_*}^{w}\frac{F(\eta)-g(w_*)}{F(\eta)}\,\mathrm{d}\eta=:\int_\Omega \mathcal{H}_2(u,v,w)$$

and $\mathcal{H}_2(u,v,w)=\frac{1}{\gamma}\big(u-u_*-u_*\ln\frac{u}{u_*}\big)+\frac{M}{2}(v-v_*)^2+\int_{w_*}^{w}\frac{F(\eta)-F(w_*)}{F(\eta)}\,\mathrm{d}\eta$ is a nonnegative convex function of (u,v,w) based on Lemma 4.6, the expansion (4.26), and on (4.27) as below. So the equation $\frac{\mathrm{d}\mathcal{L}_2(t)}{\mathrm{d}t}=0$ has at most one minimum point in the sense of (u,v,w). Together with $(u,v,w)=(u_*,v_*,w_*)$ leading to $\frac{\mathrm{d}\mathcal{L}_2(t)}{\mathrm{d}t}=0$, thus we may infer that the equation $\frac{\mathrm{d}\mathcal{L}_2(t)}{\mathrm{d}t}=0$ indicates $(u,v,w)=(u_*,v_*,w_*)$, which concludes that $\frac{\mathrm{d}\mathcal{L}_2(t)}{\mathrm{d}t}=0$ if and only if $(u,v,w)=(u_*,v_*,w_*)$. Then by Lemma 4.7 the solution (u,v,w) of system (4.1) which is bounded will converges to (u_*,v_*,w_*) as $t\to\infty$. In other words, (u_*,v_*,w_*) is globally asymptotic stable.

We can further acquire its the convergent rate. Indeed, letting $\kappa=w_*$ in

Lemma 4.6 means

$$
\begin{aligned}
\zeta(w) =&\zeta(w_*) + \zeta'(w_*)(w-w_*) + \frac{1}{2}\zeta''(\tilde{w})(w-w_*)^2 \\
=&\frac{F(w_*)F'(\tilde{w})}{2F^2(\tilde{w})}(w-w_*)^2 \geq 0,
\end{aligned}
\tag{4.26}
$$

with $\tilde{w}$ lying between w and w_*. Furthermore, denoting $\rho(u) = u - u_* - u_* \ln \frac{u}{u_*}$ and doing its Taylor expansion at $u = u_*$ may show

$$
\rho(u) = \rho(u_*) + \rho'(u_*)(u-u_*) + \frac{1}{2}\rho''(\tilde{u})(u-u_*)^2 = \frac{u_*}{2\tilde{u}^2}(u-u_*)^2 \geq 0, \tag{4.27}
$$

where $\tilde{u}$ lies between u and u_*. Lemma 4.5 and the regularity (4.2) jointly show that there exists a $\tilde{T}_1 > 0$ such that $0 < \delta \leq u \leq \mathcal{C}_2 < \infty$ as $t > \tilde{T}_1$, which means $\frac{u_*}{2\mathcal{C}_2^2} \leq \frac{u_*}{2\tilde{u}^2} \leq \frac{u_*}{2\delta^2}$. Again observing the derivations from (4.23) to (4.25), there is a constant $\tilde{c}_0 > 0$ such that

$$
\frac{\mathrm{d}\mathcal{L}_2(t)}{\mathrm{d}t} \leq -\tilde{c}_0\mathcal{L}_2(t), \qquad \text{for all } t > \tilde{T}_1.
$$

Analogous to the corresponding parts in proving Lemma 4.8, there exist two constants $\tilde{c}_1, \tilde{c}_2 > 0$ and $T_1 \geq \tilde{T}_1 > 0$ such that

$$
\|u(t,\cdot) - u_*\|_{L_\infty(\Omega)} + \left\|v(t,\cdot) - v_*\right\|_{L_\infty(\Omega)} + \|w(t,\cdot) - w_*\|_{L_\infty(\Omega)} \leq \tilde{c}_1 e^{-\frac{\tilde{c}_2 t}{n+2}}, \qquad t > T_1.
$$

□

4.4 Linear Instability and Presentation of Patterns

The previous sections involve that there exists a unique global classical solution to system (4.1) and it may approach its steady states exponentially under suitable conditions. However, there is no discussion of instability on its steady states. To figure this out, we below shall analyse linear instability of these constant steady states and then numerically explore the impact of density-dependent $d(v)$ and $\chi(v)$ on the patterns.

4.4.1 Linear Instability

Proposition 4.1. *Assume that* (u_c, v_c, w_c) *is the constant steady state of the system* (4.1). *Then the* (u_c, v_c, w_c) *is linearly instable if there exists at least one* λ_j *in* (4.31) *having strictly positive real part (viz. one of* (4.32)–(4.34) *holds); It is linearly stable if all the real parts of* λ_j *are strictly negative.*

Proof. We first linearize the system (4.1) at (u_c, v_c, w_c) as

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} d(v_c)\Delta + B_1 & -u_c\chi(v_c)\Delta & B_2 \\ 0 & d_v\Delta - \sigma & \beta \\ B_3 & 0 & d_w\Delta + B_4 \end{pmatrix}\begin{pmatrix} u - u_c \\ v - v_c \\ w - w_c \end{pmatrix} \tag{4.28}$$

$$=:\mathbf{B}\tilde{\mathbf{w}}$$

where $\tilde{\mathbf{w}} := (u - u_c, v - v_c, w - w_c)^\tau$,

$$B_1 := \gamma F(w_c) - \theta - 2\ell u_c, \qquad B_2 := \gamma u_c F'_w(w_c),$$
$$B_3 := -F(w_c), \qquad B_4 := f(w_c) + w_c f'(w_c) - u_c F'_w(w_c).$$

In order to obtain the eigenvalues (denoted by $\{\lambda_j\}_{j=0}^{\infty}$) of the linear operator $\mathbf{B}$, we invoke the following eigenvalue problem:

$$\begin{cases} -\Delta\Phi(x) = \mu\Phi(x), & x \in \Omega, \\ \nabla\Phi(x)\cdot\vec{\mathbf{n}} = 0, & x \in \partial\Omega, \end{cases}$$

the eigenvalues $\{\mu_j\}_{j=0}^{\infty}$ of which, without counting the finite multiplicities, can be formulated as

$$0 = \mu_0 < \mu_1 < \mu_2 < \cdots < \mu_m < \cdots.$$

Then to $\{\mu_j\}_{j=0}^{\infty}$ the corresponding eigenfunctions, denoted by $\{\phi_j(x)\}_{j=0}^{\infty}$ in $L^2(\Omega)$, constitute an orthonormal basis of $L^2(\Omega)$. Plus $\frac{\partial\tilde{\mathbf{w}}}{\partial t} = \frac{\partial\mathbf{w}}{\partial t}$, we thus can formulate a general solution $\tilde{\mathbf{w}}$ to (4.28) (note $\frac{\partial\tilde{\mathbf{w}}}{\partial t} = \mathbf{B}\tilde{\mathbf{w}} = \lambda\tilde{\mathbf{w}}$) in the form of components (in particular spatial parts in $L^2(\Omega)$) as

$$u - u_c = \sum_{j=0}^{\infty} u_j \phi_j(x) e^{\lambda_j t},\ v - v_c = \sum_{j=0}^{\infty} v_j \phi_j(x) e^{\lambda_j t},\ w - w_c = \sum_{j=0}^{\infty} w_j \phi_j(x) e^{\lambda_j t}, \quad (4.29)$$

where u_j, v_j, w_j are constants for all j. Note that if there is a j such that $u_j = v_j = w_j = 0$, one may automatically remove the corresponding terms in (4.29). In this fashion we have

$$P_j \tilde{\mathbf{w}} := \begin{pmatrix} -d(v_c)\mu_j + B_1 & -u_c \chi(v_c)\mu_j & B_2 \\ 0 & -d_v \mu_j - \sigma & \beta \\ B_3 & 0 & -d_w \mu_j + B_4 \end{pmatrix} \tilde{\mathbf{w}} = \lambda_j \tilde{\mathbf{w}}, \qquad (4.30)$$

which is equivalent to

$$\det(\lambda_j I - P_j) = 0, \qquad j = 0, 1, 2, \dots$$

or the eigenpolynomial

$$\lambda_j^3 + a_1 \lambda_j^2 + a_2 \lambda_j + a_3 = 0, \qquad j = 0, 1, 2, \dots \qquad (4.31)$$

where I is a 3×3 unit matrix and other real-valued coefficients are:

$$a_1 = -\operatorname{Trace}(P_j) = \big(d(v_c) + d_v + d_w\big)\mu_j + \sigma - B_1 - B_4,$$

$$a_2 = \det \begin{pmatrix} -d(v_c)\mu_j + B_1 & -u_c \chi(v_c)\mu_j \\ 0 & -d_v \mu_j - \sigma \end{pmatrix} + \det \begin{pmatrix} -d_v \mu_j - \sigma & \beta \\ 0 & -d_w \mu_j + B_4 \end{pmatrix},$$

$$a_3 = -\det(P_j) = (B_1 - d(v_c)\mu_j)(\sigma + d_v \mu_j)(B_4 - d_w \mu_j) \\ - B_3 B_2 (\sigma + d_v \mu_j) + B_3 \beta u_c \chi(v_c)\mu_j.$$

Denote $\mathbf{p} = a_2 - \frac{{a_1}^2}{3}$, $\mathbf{q} = \frac{2{a_1}^3}{27} - \frac{a_1 a_2}{3} + a_3$, $\vartheta = e^{\mathbf{i}\frac{2\pi}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}$ with $\mathbf{i} = \sqrt{-1}$, and $\Xi = \frac{\mathbf{q}^2}{4} + \frac{\mathbf{p}^3}{27}$. Then by Cardano's formula for every j one can specify three roots of (4.31) as:

$$\lambda_j^{(1)} = -\frac{a_1}{3} + \sqrt[3]{\frac{-\mathbf{q}}{2} + \sqrt{\Xi}} + \sqrt[3]{\frac{-\mathbf{q}}{2} - \sqrt{\Xi}},$$

$$\lambda_j^{(2)} = -\frac{a_1}{3} + \vartheta \sqrt[3]{\frac{-\mathbf{q}}{2} + \sqrt{\Xi}} + \vartheta^2 \sqrt[3]{\frac{-\mathbf{q}}{2} - \sqrt{\Xi}},$$

$$\lambda_j^{(3)} = -\frac{a_1}{3} + \vartheta^2 \sqrt[3]{\frac{-\mathbf{q}}{2} + \sqrt{\Xi}} + \vartheta \sqrt[3]{\frac{-\mathbf{q}}{2} - \sqrt{\Xi}}.$$

Consequently, we may identify the linear instability by requiring one of the real parts of these roots to be strictly positive in the following cases:

- When $\Xi > 0$, one may readily see that $\frac{-\mathbf{q}}{2} \pm \sqrt{\Xi} \in \mathbb{R}$ and thus $\lambda_j^{(1)}$ is real and $\lambda_j^{(2)}$, $\lambda_j^{(3)}$ are complex numbers. So we require

$$\max\left\{\mathrm{Re}(\lambda_j^{(1)}),\, \mathrm{Re}(\lambda_j^{(2)}),\, \mathrm{Re}(\lambda_j^{(3)})\right\} = -\frac{a_1}{3} + \max\left\{\Lambda,\, \frac{-\Lambda}{2}\right\} > 0 \tag{4.32}$$

  with $\Lambda := \sqrt[3]{\frac{-\mathbf{q}}{2} + \sqrt{\Xi}} + \sqrt[3]{\frac{-\mathbf{q}}{2} - \sqrt{\Xi}}$;

- When $\Xi = 0$ then $\lambda_j^{(1)}$, $\lambda_j^{(2)}$ and $\lambda_j^{(3)}$ are real (by $\vartheta + \vartheta^2 = -1$) and $\lambda_j^{(2)} = \lambda_j^{(3)}$. Then we demand

$$\max\left\{\mathrm{Re}(\lambda_j^{(1)}),\, \mathrm{Re}(\lambda_j^{(2)}),\, \mathrm{Re}(\lambda_j^{(3)})\right\} = -\frac{a_1}{3} + \max\left\{2\Lambda_0,\, -\Lambda_0\right\} > 0 \tag{4.33}$$

  with $\Lambda_0 := \sqrt[3]{\frac{-\mathbf{q}}{2}}$;

- When $\Xi < 0$, $\lambda_j^{(1)}$, $\lambda_j^{(2)}$, and $\lambda_j^{(3)}$ are real but different from each other. So we need

$$\max\left\{\lambda_j^{(1)}, \lambda_j^{(2)}, \lambda_j^{(3)}\right\} > 0. \tag{4.34}$$

This completes the proof. □

Note that Proposition 4.1 does not concisely show how the density-dependent $d(v)$ and $\chi(v)$ directly affect the patterns. So we next resort to numerical simulations with parameters taken hypothetically. The units of these parameters can be inferred from pp.252–262 of [17].

4.4.2 Numerical Simulation in One-Dimensional Case

When motility function $d(v)$ and prey-taxis sensitivity function $\chi(v)$ are constants, one may find (cf. [16]) that the coexistence state of spatial one-dimensional model (1.16) (i.e., system (4.1) with $\ell = 0$) becomes unstable regarding small perturbation (by increasing prey-taxis coefficient). In this subsection, we shall show that some density-dependent $d(v)$ and $\chi(v)$ can stabilize such a stationary state but this stabilization effect can be weakened by enhancement of conversion rate.

To show this difference, we remain unchanged some parameters and functions taken in [16], except for $d(v)$, $\chi(v)$ and conversion rate γ. Specifically, the growth rate function of prey is Θ-logistic type

$$f(w) = r\Big(1 - \Big(\frac{w}{K}\Big)^{\Theta}\Big), \qquad r,\ K > 0,\ \Theta \geq 1,$$

and the functional response function is Ivlev type

$$F(w) = c(1 - e^{-\varsigma w}), \quad \varsigma > 1, \quad c > 0.$$

Let $\Omega = (0, L)$ and take other parameters in Table 4.1. Thus we derive from (4.4) (with $\ell = 0$) that $(u_*, v_*, w_*) \approx (1.2599, 1.3787, 0.6267)$. In addition, we set initial data as $u_0(x) = u_* + 0.01 \cdot \cos(\pi x), v_0(x) = v_* + 0.01 \cdot \sin(\pi x), w_0(x) = w_* + 0.01 \cdot \cos(\pi x)$.

Table 4.1: Parameters selection–I.

γ	θ	ℓ	d_v	d_w	σ	β	K	r	c	Θ	ς	L
1.2	0.45	0	0.0001	0.09	0.2	0.44	1	1	1	3	0.75	1

When $d(v) = 0.002533$ and $\chi(v) = 1$, one can still derive the patterns (cf. (a) in Figure 4.1) that are analogous to the first row of Figure 7 in [16]. However, if we replace them by density-dependent forms such as $d(v) = \frac{1}{1+e^{8v-1}}$ or $\frac{1}{1+8v}$, things will become different.

Precisely, it is not difficult to see from (a) to (d) in Figure 4.1 that some density-dependent $d(v)$ and $\chi(v)$ of exponential or algebraic form may flatten, or we say stabilize, the pattern bifurcating from the coexistence steady state (u_*, v_*, w_*) under small perturbations. However, this effect might be suppressed by increasing conversion rate. For example, after resetting conversion rate γ, approximate time-periodic patterns can appear, like the change from (d) to (e) in Figure 4.1. In addition, by enhancing γ in Figure 1(b), (c) (for instance, by letting $\gamma = 26$), the system may produce patterns like Figure 1(d) as well.

4.4.3 Numerical Simulation in Two-Dimensional Case

An individual-based modelling method to simulate one population whose individuals undergo density-dependent movement in 2-dimensional spatial domain can be see in [77]. For two populations spatially in a 2-dimensional disc, i.e., one predator and one prey considered in the system (4.1) with $\ell > 0$, some density-dependent $d(v)$ and $\chi(v)$ may help to change the spatial distribution similarity which exists in non-density-dependent case between predators and signals of prey.

We herein set the growth rate function of prey as

$$f(w) = r\Big(1 - \frac{w}{K_0}\Big), \qquad r,\ K_0 > 0,$$

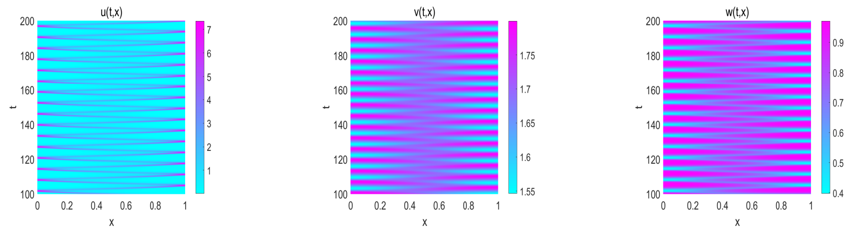
and take the functional response function to be the Holling type II

$$F(w) = \frac{w}{c + w}, \quad c > 0,$$

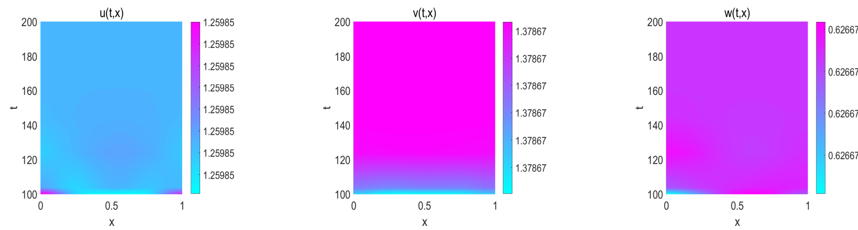
together with different values of r and different forms of $d(v)$ and $\chi(v)$ specified below Figures 4.2 and 4.3. Without loss of generality, we may adopt initial values as

$$u_0(x,y) = u_c + Q(x,y), \qquad v_0(x,y) = v_c + Q(x,y), \qquad w_0(x,y) = w_c + Q(x,y),$$

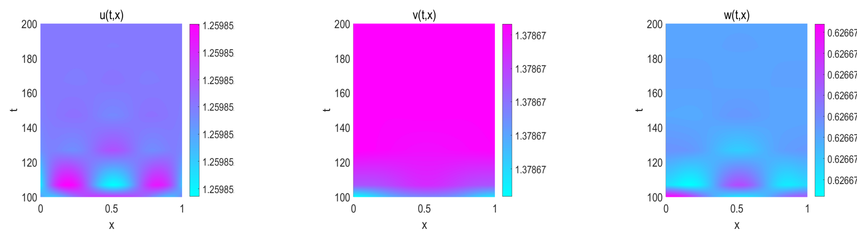
where $Q(x,y) = \cos \pi x + \cos \pi y$, $(x,y) \in \mathcal{B}_3(\mathbf{0})$–a circle with radius 3 and centre



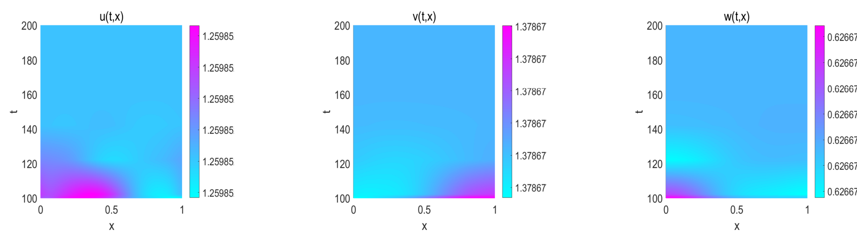
(a) $\chi(v) = 1, \quad d(v) = 0.002533$



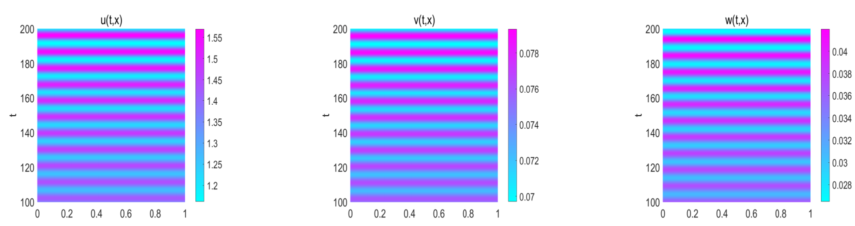
(b) $\chi(v) = 8/(1+8v)^2, \quad d(v) = 1/(1+8v)$



(c) $\chi(v) = 8e^{8(v-1)}/(1+e^{8(v-1)})^2, \quad d(v) = 1/(1+e^{8(v-1)})$



(d) $\chi(v) = 8/(1+8v)^2, \quad d(v) = 1/(1+e^{8(v-1)})$



(e) $\chi(v) = 8/(1+8v)^2, \quad d(v) = 1/(1+e^{8(v-1)}), \quad \gamma = 18$

Figure 4.1: Here $(u_*, v_*, w_*) \approx (1.2599, 1.3787, 0.6267)$ from (a) to (d) and $(u_*, v_*, w_*) \approx (1.3502, 0.0743, 0.0338)$ in (e).

at the origin, (u_c, v_c, w_c) may equal to $(0,0,0), (0, \frac{\beta K_0}{\sigma}, K_0)$ or (u_*, v_*, w_*), the last of which exists as $\gamma r > \theta$, $u_* = w_* = \frac{K_0(\gamma r - \theta)}{K_0 \ell + \gamma r}$ and $v_* = \frac{K_0 \beta(\gamma r - \theta)}{\sigma(K_0 \ell + \gamma r)}$. Other specific parameters are given in Table 4.2.

Table 4.2: Parameters selection–II.

γ	θ	ℓ	β	σ	K_0	d_v	d_w	c
10	1	1	10	12	10	0.1	0.1	1

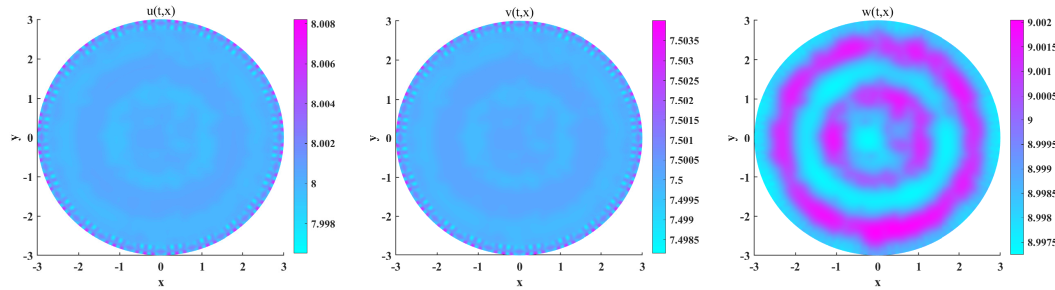
Figures 4.2 and 4.3 present the spatial distribution of predator, chemicals released by prey and of prey, in a circular domain at time $t = 50$ and $t = 500$. We may observe that:

(i) non-constant steady states exist since the corresponding patterns have few changes from time $t = 50$ to $t = 500$. Parameter r seems important in producing more abundant patterns after other parameters are fixed, for example (a) and (b) in Figure 4.2 and that in Figure 4.3, or (c) and (d) in Figure 4.2 and that in Figure 4.3;

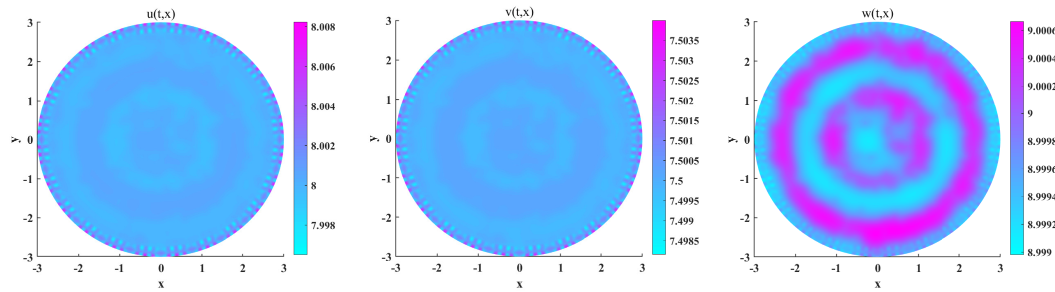
(ii) if $d(v)$ and $\chi(v)$ are constants, spatial distribution of predators and chemoattractant are very similar; The density-dependent decays of $d(v)$ and $\chi(v)$ may lower the similarities, but the extent may be effected by other parameters, like r in f.

4.4.4 Biological Explanation of the Simulations

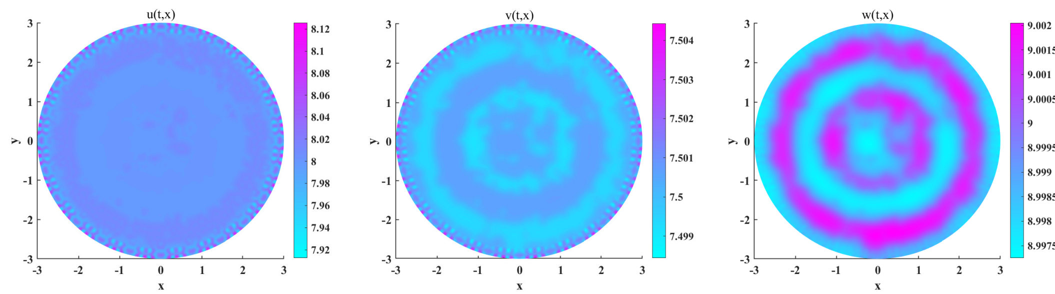
System (4.1) describes a spatiotemporal evolution process of an isolated ecosystem within a domain Ω, which includes two populations i.e., one predator and one prey. The most arresting feature in system (4.1) is that the predators may search for the prey as their food, mainly through chemoattractants released by the prey, since some factors including natural camouflage, the environment of the prey, range of visibility of the predators, etc., result in many difficulties for the predators in finding



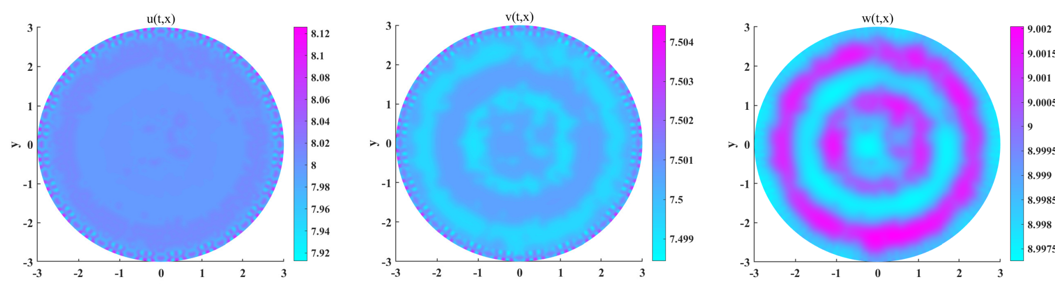
(a) Time t=50, $\chi(v) = 1$, $d(v) = 4$, $r = 8$



(b) Time t=500, $\chi(v) = 1$, $d(v) = 4$, $r = 8$

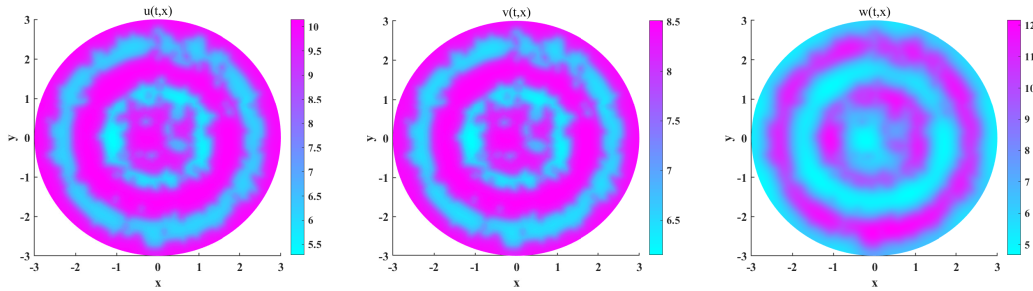


(c) Time t=50, $\chi(v) = \frac{10}{(1+10v)^2}$, $d(v) = \frac{1}{1+e^{10v-1}}$, $r = 8$

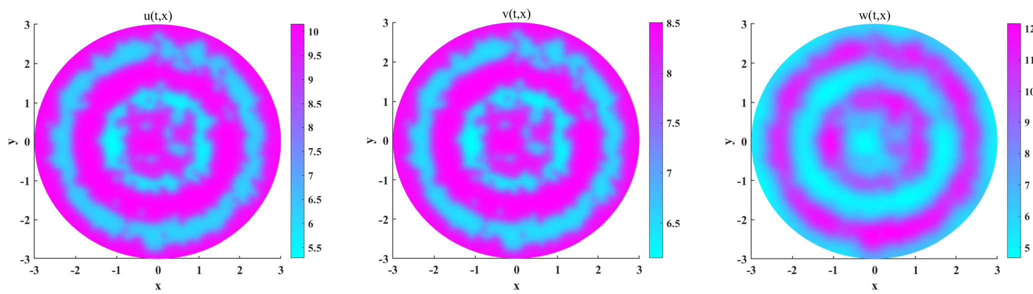


(d) Time t=500, $\chi(v) = \frac{10}{(1+10v)^2}$, $d(v) = \frac{1}{1+e^{10v-1}}$, $r = 8$

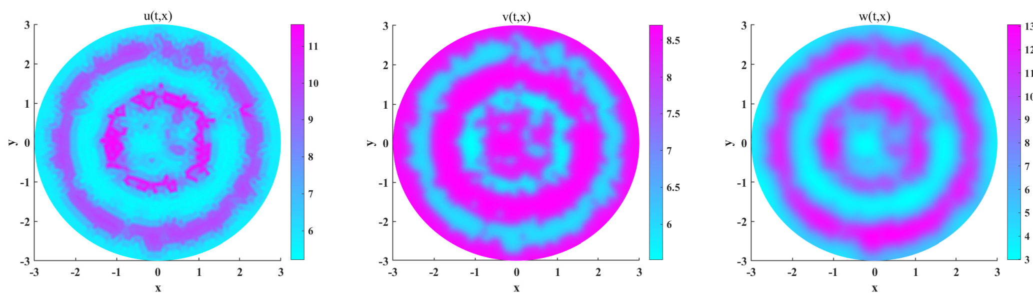
Figure 4.2: By numerical simulation, different values of constant steady state (u_c, v_c, w_c) give the analogous resulting graphics. Here we take $(u_*, v_*, w_*) = (8.7778, 7.3148, 8.7778)$ for example. Density dependent $d(v)$ and $\chi(v)$ may change the patterns of the predator density u and the prey signal density v but have little effect on that of prey density w.



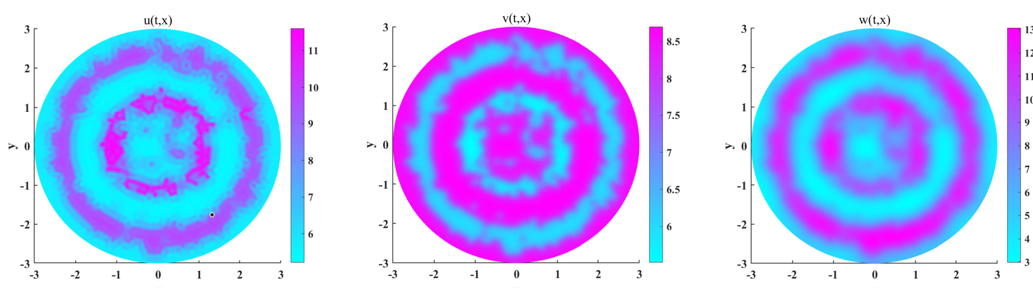
(a) Time t=50, $\chi(v) = 1$, $d(v) = 4$, $r = 10$



(b) Time t=500, $\chi(v) = 1$, $d(v) = 4$, $r = 10$



(c) Time t=50, $\chi(v) = \frac{10}{(1+10v)^2}$, $d(v) = \frac{1}{1+e^{10v-1}}$, $r = 10$



(d) Time t=500, $\chi(v) = \frac{10}{(1+10v)^2}$, $d(v) = \frac{1}{1+e^{10v-1}}$, $r = 10$

Figure 4.3: Here $(u_*, v_*, w_*) = (9, 7.5, 9)$. Compared with Figure 4.2, we only change the value of r and readily see that the impact of the density-dependent $d(v)$ and $\chi(v)$ on patterns of u and v, in particular for v, may be subjected to the value of r. Still the $d(v)$ and $\chi(v)$ cannot distinctly affect that of prey density w.

the prey directly. So the chemoattractants usually have diffused relatively far from the prey before they are perceived by the predators. Here $u(x,t)$, $v(x,t)$, and $w(x,t)$ refer to population density of the predators, concentration of the chemical signals, and population density of the prey, respectively. The system being isolated means that there might be negligible quantities of the predators, the prey, and the prey signals crossing the boundary of Ω, compared with overwhelming majorities of them (the predators, the prey, and the prey signals) within Ω. Other organisms living in Ω are not taken into consideration in the system (4.1).

Theorem 4.1 states that the system (4.1) has a global-in-time classical solution which is continuous to its initial value, when (H1)–(H3) are satisfied. As a result, for given initial densities $u_0(x), v_0(x)$ and $w_0(x)$, one can predict by the unique classical solution of system (4.1) the density of the predators, the prey signals and the prey, at any time $0 < t < \infty$ and any spatial position $x \in \Omega$. The obtained L_∞ bound in Theorem 4.1 signifies that there is a maximal density for all three of them.

Theorem 4.2 illustrates that in some cases (if (H4) holds), the spatial distributions of the predators, the prey signals, and the prey in Ω may be approximately homogeneous as the time goes by. This is, as it should be, a much ideal case, but at least the large-time behavior of such a solution indicates a trend through which one can foresee whether this ecosystem can evolve into exclusion state (prey being extinct in Ω) or coexistence state over time. So this tendency which can be viewed as an early warning, makes significantly biological sense to protect the biodiversity and ecological balance in the domain Ω.

For simplicity, in regard to numerical simulations we only list the patterns which bifurcate from coexistence steady state in Subsections 4.4.2 and 4.4.3 (the case of exclusion state is similar). In Subsection 4.4.2, (a) of Figure 4.1 recovers the pattern corresponding to the point A in Figure 7 obtained in [16] with $d(v)$ and $\chi(v)$ being constants, which is the starting point of our simulations. Then in (b) and (c) of Figure 4.1, we set $\chi(v) = -d'(v)$ with $d(v)$ satisfying algebraic decay and exponential decay respectively. Finally in Figure 4.1 (d) and (e) we remove the relation $\chi(v) = -d'(v)$ and take $\chi(v)$ and $d(v)$ to be algebraic and exponential decay severally. From this process we see that random motility function $d(v)$ and indirect prey-taxis sensitivity $\chi(v)$, being density-dependent form, may help the spatial distribution (of the predators, the prey signals, and the prey) to be approximately homogeneous. Because one may observe that the spatial distributions of Figure 4.1 (b)–(e) become more even than that of Figuere 4.1 (a), although the approximate time-periodic pattern may appear when the conversation rate γ is increased.

All simulations in Subsection 4.4.2 are spatial 1-dimensional case, which matter in theory. What will happen in spatial 2-dimensional case makes more realistic sense, which is the aim of Subsection 4.4.3. Firstly, we see the spatial distribution of high density for both the prey signals and the prey, either in Figures 4.2 or 4.3, stagger a little bit each other (instead of being overlap) in position (this point can also be seen in Figure 4.1 but it is not so distinct). This is consistent with the feature of indirect prey-taxis that signals of the prey have diffused a distance far from the prey before they are captured by the predators. Secondly, when $\chi(v)$ and $d(v)$ are constants (cf. (a), (b) in Figures 4.2 and 4.3), we find that the spatial distribution of the predators and of the prey signals are highly similar, since the predators conduct the signals-based (indirect prey-taxis) foraging strategy to search for the prey. However, the $\chi(v)$ and $d(v)$ in density-dependent form (cf. (c), (d) in Figures 4.2 and 4.3) may lower similarity of spatial distribution between the predators and the prey chemicals. Finally, increasing the value of r (from $f(w)$) in Figure 4.2 may yield Figure 4.3 from which one may infer that some parameters in system (4.1), like r, are important to produce abundant patterns.

Chapter 5

Conclusions and Future Works

5.1 Conclusions in Biological Sense

In this thesis, we have studied the existence, uniqueness, and uniform boundedness of the positive classical solution to the direct preytaxis model (1.11) in Chapter 2, which indicates the unique evolutionary state on predators and prey governed by (1.11), and their densities are upper bounded uniformly in time. In particular, the uniform-in-time boundedness of densities of predators and prey suggests prevention of overcrowding, and thus the theoretical controllability of predators and prey (as pests) [78]. Moreover, we have established in two-dimensional case in Chapter 3 the convergence relation between the strong solution of (1.10) and of (1.11) as the diffusion coefficient of prey tends to zero. The convergence relation strictly proves that there exists some spatiotemporal similarity of spatial distributions between the densities described by (1.10) and by (1.11), when the diffusion rate of prey becomes much weak.

On the other hand, we have in Chapter 4 explored the existence, uniqueness, uniform boundedness, and large-time behaviors of the classical solution to the indirect preytaxis system (1.17). These results show that for density-dependent preytaxis sensitivity, the evolutionary state of predators and prey is unique and their densities are the bounded uniformly in time, when the predators conduct signal-based foraging strategy. The global asymptotic behaviors show that if $\gamma F(K_0) \leq \theta$, then the predators will be extinct over time. If (4.4) has positive constant solution for suitable growth rate function f, functional response function F and death rate h, then the predators and prey may coexist over time. This asymptotical tendency in theory can be used, as a biological warning or theoretical explorations on properties of the predator-prey system, to predict the development of biodiversity of ecosystems, when the predator-prey relation in the field experiment can be depicted by (1.17).

5.2 Future Works

Except for the problems addressed in this thesis, there are some other relevant questions of interest left open. We below display some of them to pursue in the future:

(1) For direct preytaxis model (2.1), the local-in-time existence of its classical solution holds in $\Omega \subset \mathbb{R}^n (n \geq 1)$ by Theorem 2.1. Also, Theorem 2.1 or Theorem 3.1-(c) shows that the global-in-time existence holds for $n = 2$ and for any constant preytaxis sensitivity $\chi > 0$. Then we may ask:

   a) Does the global existence and convergence relation similar to Theorem 3.1-(b) remain true for any finite $n > 2$ and for any $\chi > 0$?

   b) May a) be true when the $d = d(w)$ and $\chi = \chi(u, w)$ are density-dependent (non-constant)?

(2) For indirect preytaxis system (4.1), the assumption (H2) supports logistic growth (1.6) but excludes Allee's effect (1.7). So what will happen if f is replaced by (1.7)?

On the other hand, the spatiotemporal aggregation or heterogeneity of species corresponds in a sense to the stable nonconstant steady states of related evolutionary systems. So one may wonder:

(3) Does the nonconstant steady state of (4.1) exist? If it exists, can the classical solution of (4.1) converge to such a nonconstant steady state in a sense?

(4) Similarly, what about the large time behaviors of the classical solution of (2.1)?

Appendix A

Auxiliary Inequalities

For clarity and completeness, we shall in this section list out some inequalities that are frequently invoked in the main content. Firstly, the convexity of the function: $s \in \mathbb{R} \longmapsto s^2$ indicates that

$$|\Delta u|^2 = \Big(\sum_{i=1}^{n} u_{x_i x_i}\Big)^2 \leq n \sum_{i=1}^{n} u_{x_i x_i}^2 \leq n|D^2 u|^2, \quad \text{i.e.,} \quad |\Delta u| \leq \sqrt{n}|D^2 u|.$$

Moreover, it follows from [79, Prop.3, pp.58-59] or [80, pp.230-235] that there exists a constant $C(n,p)$ such that

$$\|D^2 u\|_{L_p(\Omega)} \leq C(n,p)\|\Delta u\|_{L_p(\Omega)}, \qquad \text{for} \quad 1 < p < +\infty, \tag{A.1}$$

where $u \in W_p^2(\Omega)$ [1]and Ω is bounded. In particular, when $u\big|_{\partial\Omega} = 0$ or $\nabla u \cdot \vec{\nu}\big|_{\partial\Omega} = 0$, one may find $\|\Delta u\|_{L_2(\Omega)}^2 = \|D^2 u\|_{L_2(\Omega)}^2$ through integration by parts (cf. [1, p.326]). More generalizations can be seen in [81].

Below is the well-known Gagliardo-Nirenberg interpolation inequality in a bounded domain (cf. [82] and [83]).

Lemma A.1. *Assume that a m-times differentiable function $u : \Omega \to \mathbb{R}$ is defined on a bounded domain $\Omega \subset \mathbb{R}^n (n \geq 1)$ with Lipschitz continuous boundary $\partial\Omega$. Then*

[1] Note that in the origin inequality $u \in C_0^2(\Omega)$, here $u \in W_p^2(\Omega)$ since $C_0^\infty(\Omega)$ is dense in $W_p^m(\Omega)$ when Ω is bounded and sufficient regular for $1 \leq p < \infty, m \in \mathbb{N}$.

it holds that

$$\|D^j u\|_{L_p(\Omega)} \le C\big(\|D^m u\|^{\alpha}_{L_r(\Omega)} \|u\|^{1-\alpha}_{L_q(\Omega)} + \|u\|_{L_s(\Omega)}\big),$$

where $1 \le q, r \le \infty$, $\frac{j}{m} \le \alpha \le 1$,

$$\frac{1}{p} = \frac{j}{n} + \left(\frac{1}{r} - \frac{m}{n}\right)\alpha + \frac{1-\alpha}{q},$$

$s > 0$ *is arbitrary, and constants* C *depend upon the domain* Ω *as well as* m, n, *etc. Moreover, one may replace* $\|D^2 u\|_{L_r(\Omega)}$ *by* $\|\Delta u\|_{L_r(\Omega)}$ *in the above inequality due to* (A.1).

Note that by Hölder's inequality, if $1 \le p < q < \infty$ and $|\Omega| := \int_\Omega \mathrm{d}x < +\infty$, then

$$\Big(\int_\Omega u^p\Big)^{\frac{1}{p}} \le \Big(\int_\Omega u^q\Big)^{\frac{1}{q}} |\Omega|^{1-\frac{p}{q}}.$$

Take $0 < s < \min\{r, q\}$, one may rewrite the right hand of Gagliardo-Nirenberg interpolation inequality as

$$\begin{aligned}\|D^j u\|_{L_p(\Omega)} \le & C\|D^m u\|^{\alpha}_{L_r(\Omega)} \|u\|^{1-\alpha}_{L_q(\Omega)} + C\|u\|^{\alpha}_{L_s(\Omega)} \|u\|^{1-\alpha}_{L_s(\Omega)} \\ \le & C\|D^m u\|^{\alpha}_{L_r(\Omega)} \|u\|^{1-\alpha}_{L_q(\Omega)} + C\|u\|^{\alpha}_{W^m_s(\Omega)} \|u\|^{1-\alpha}_{L_s(\Omega)} \\ \le & C_1 \|u\|^{\alpha}_{W^m_r(\Omega)} \|u\|^{1-\alpha}_{L_q(\Omega)}\end{aligned}$$

with $C_1 = C(1 + |\Omega|^{(1-\frac{s}{r})\alpha + (1-\frac{s}{q})(1-\alpha)})$.

Next, we use this lemma to see that q in Lemma A.1 may take values less than 1. When Ω is replaced by $\mathbb{R}^n (n \ge 1)$, one may refer to [84, Lemma 3.2]. When $\Omega \subset \mathbb{R}^n$ is an open bounded domain, there is no complete proof in the literature as far as we know, and thus we supplement it as follow for the sake of completeness.

Lemma A.2. *Suppose that* $0 < \mu \le r \le \sigma < +\infty$ *and* $0 < \mu < 1 < \sigma < +\infty$. *Let* $\varphi \in L_s(\Omega) \cap L_\mu(\Omega)$ *and* $D\varphi \in L_\sigma(\Omega)$ *for any* $s > 0$, *and* $\Omega \subset \mathbb{R}^n$ *is an open and* *bounded domain with a Lipschitz continuous boundary. Then*

$$\|\varphi\|_{L_r(\Omega)} \le c_0 \|D\varphi\|^{\rho}_{L_\sigma(\Omega)} \|\varphi\|^{1-\rho}_{L_\mu(\Omega)} + c_1 \|\varphi\|^{\theta}_{L_\mu(\Omega)} \|\varphi\|^{1-\theta}_{L_s(\Omega)}$$

where constants c_0 *and* c_1 *may depend on* σ, μ, r, n *and* Ω*, and* $\rho, \theta \in (0,1)$ *satisfy*

$$-\frac{n}{r} = \rho\big(1 - \frac{n}{\sigma}\big) - (1-\rho)\frac{n}{\mu}, \qquad \frac{r\theta}{\mu} + \frac{r(1-\theta)}{\sigma} = 1.$$

Proof. Suppose $\theta \in (0,1)$. By Hölder's inequality with index relation $\frac{r\theta}{\mu} + \frac{r(1-\theta)}{\sigma} = 1$, we have

$$\begin{aligned}\int_\Omega |\varphi|^r = \int_\Omega |\varphi|^{r\theta + (1-\theta)r} &\le \Big(\int_\Omega |\varphi|^{r\theta \cdot \frac{\mu}{r\theta}}\Big)^{\frac{r\theta}{\mu}} \cdot \Big(\int_\Omega |\varphi|^{(1-\theta)r \cdot \frac{\sigma}{(1-\theta)r}}\Big)^{\frac{(1-\theta)r}{\sigma}} \\ &= \Big(\int_\Omega |\varphi|^{\mu}\Big)^{\frac{r\theta}{\mu}} \cdot \Big(\int_\Omega |\varphi|^{\sigma}\Big)^{\frac{(1-\theta)r}{\sigma}}\end{aligned}$$

that is,

$$\|\varphi\|_{L_r(\Omega)} \le \|\varphi\|^{\theta}_{L_\mu(\Omega)} \ \|\varphi\|^{1-\theta}_{L_\sigma(\Omega)}, \qquad \theta = \frac{\mu(\sigma - r)}{r(\sigma - \mu)},$$

with $\mu > r\theta$ and $\sigma > r(1-\theta)$. Suppose $\frac{r}{\mu} \ge 1$ and thus $\frac{r}{\sigma} \le 1$, which means $0 < \mu \le r \le \sigma < \infty$.

Below we restrict $0 < \mu < 1$. If taking $r = 1$ one may get that

$$\|\varphi\|_{L_1(\Omega)} \le \|\varphi\|^{\beta}_{L_\mu(\Omega)} \ \|\varphi\|^{1-\beta}_{L_\sigma(\Omega)}, \qquad \beta = \frac{\mu(\sigma - 1)}{(\sigma - \mu)} \in (0,1).$$

By Gagliardo-Nirenberg interpolation inequality with $1 < \sigma < \infty$,

$$\|\varphi\|_{L_\sigma(\Omega)} \le C_1 \|D\varphi\|^{\alpha}_{L_\sigma(\Omega)} \|\varphi\|^{1-\alpha}_{L_1(\Omega)} + C_2 \|\varphi\|_{L_s(\Omega)}, \qquad \frac{1}{\sigma} = \Big(\frac{1}{\sigma} - \frac{1}{n}\Big)\alpha + 1 - \alpha,$$

and together with the last inequality,

$$\begin{aligned}\|\varphi\|_{L_\sigma(\Omega)} \le& C_1 \|D\varphi\|^{\alpha}_{L_\sigma(\Omega)} \big(\|\varphi\|^{\beta}_{L_\mu(\Omega)} \ \|\varphi\|^{1-\beta}_{L_\sigma(\Omega)}\big)^{1-\alpha} + C_2 \|\varphi\|_{L_s(\Omega)} \\ \le& C_1 \|D\varphi\|^{\alpha}_{L_\sigma(\Omega)} \ \|\varphi\|^{\beta(1-\alpha)}_{L_\mu(\Omega)} \ \|\varphi\|^{(1-\beta)(1-\alpha)}_{L_\sigma(\Omega)} + C_2 \|\varphi\|_{L_s(\Omega)}.\end{aligned}$$

Now we may invoke an elementary function to establish the upper bound of $\|\varphi\|_{L_\sigma(\Omega)}$ from this inequality. Indeed, for any $\tau \in (0,1)$, and a, b being two positive constants, a function $h(x) := ax^\tau - x + b : x \in \mathbb{R}_+ \longmapsto \mathbb{R}$ has properties that $h(x)' = a\tau x^{\tau-1} - 1 > 0$ if $0 < x < (a\tau)^{\frac{1}{1-\tau}}$; $h'(x) < 0$ if $(a\tau)^{\frac{1}{1-\tau}} < x < +\infty$; and $h(x) = ax(\frac{1}{x^{1-\tau}} - 1) + b \to -\infty$ if $x \to +\infty$. So by $\lim_{x\to 0_+} h(x) = b > 0$ there exists only one point $x_0 \in \left((a\tau)^{\frac{1}{1-\tau}}, +\infty\right)$ such that $h(x_0) = 0$ and $h(x) \geq 0$ if $x \in (0, x_0)$.

Furthermore, we may give a rough upper bound for this x_0. To this end, by noting that $h'(x) = \frac{a}{x^{1-\tau}} \cdot \tau - 1$ is decreasing in $x > (a\tau)^{\frac{1}{1-\tau}}$, one may observe that there exists a $x_1 > 0$ such that $h'(x_1) = \tau - 1 < 0$, which indicates $x_1 = a^{\frac{1}{1-\tau}} \geq (a\tau)^{\frac{1}{1-\tau}}$ and $h(x_1) = b = \lim_{x\to 0_+} h(x)$. This observation implies that there is a line: $f(x) = b + (\tau - 1)(x - x_1)$ with slope $\tau - 1$ which intersects $h(x)$ at $(x_1, h(x_1))$ and fulfills $h(x) \leq f(x)$, as $x > x_1$. Then for some $x_2 > (a\tau)^{\frac{1}{1-\tau}}$ fulfilling $f(x_2) = 0$ if and only if $x_2 = x_1 + \frac{b}{1-\tau}$. As a result, if $h(x_0) = 0$ for the $x_0 > (a\tau)^{\frac{1}{1-\tau}}$ shall imply $x_0 \leq x_2 = a^{\frac{1}{1-\tau}} + \frac{b}{1-\tau}$.

Then by letting $\tau = (1-\beta)(1-\alpha)$, $a = C_1\|D\varphi\|^\alpha_{L_\sigma(\Omega)} \, \|\varphi\|^{\beta(1-\alpha)}_{L_\mu(\Omega)}$, and $b = C_2\|\varphi\|_{L_s(\Omega)}$, we know that

$$\|\varphi\|_{L_\sigma(\Omega)} \leq \left(C_1\|D\varphi\|^\alpha_{L_\sigma(\Omega)} \, \|\varphi\|^{\beta(1-\alpha)}_{L_\mu(\Omega)}\right)^{\frac{1}{1-(1-\beta)(1-\alpha)}} + \frac{C_2}{1-(1-\beta)(1-\alpha)}\|\varphi\|_{L_s(\Omega)}.$$

Therefore, combining all these inequalities may conclude that

$$\begin{aligned}
&\|\varphi\|_{L_r(\Omega)} \leq \|\varphi\|^\theta_{L_\mu(\Omega)} \, \|\varphi\|^{1-\theta}_{L_\sigma(\Omega)} \\
\leq &\|\varphi\|^\theta_{L_\mu(\Omega)} \, \left(\left(C_1\|D\varphi\|^\alpha_{L_\sigma(\Omega)} \, \|\varphi\|^{\beta(1-\alpha)}_{L_\mu(\Omega)}\right)^{\frac{1}{1-(1-\beta)(1-\alpha)}} + \frac{C_2}{1-(1-\beta)(1-\alpha)}\|\varphi\|_{L_s(\Omega)}\right)^{1-\theta} \\
\leq &C_1^{\frac{1-\theta}{1-(1-\beta)(1-\alpha)}}\|D\varphi\|^{\frac{\alpha(1-\theta)}{1-(1-\beta)(1-\alpha)}}_{L_\sigma(\Omega)}\|\varphi\|^{\frac{\beta(1-\alpha)(1-\theta)}{1-(1-\beta)(1-\alpha)}+\theta}_{L_\mu(\Omega)} + \left(\frac{C_2}{1-(1-\beta)(1-\alpha)}\right)^{1-\theta}\|\varphi\|^{1-\theta}_{L_s(\Omega)} \, \|\varphi\|^\theta_{L_\mu(\Omega)}
\end{aligned}$$

and

$$\frac{\alpha(1-\theta)}{1-(1-\beta)(1-\alpha)}+\frac{\beta(1-\alpha)(1-\theta)}{1-(1-\beta)(1-\alpha)}+\theta=1.$$

Letting $\rho := \frac{\alpha(1-\theta)}{1-(1-\beta)(1-\alpha)} \in (0,1)$ and substituting the expressions of θ, α, β into ρ may show that

$$-\frac{n}{r}=\rho\big(1-\frac{n}{\sigma}\big)-(1-\rho)\frac{n}{\mu}, \qquad \text{for } \mu<1<\sigma \text{ and } \mu<r<\sigma.$$

Finally, by Young's inequality ($a^\theta b^{1-\theta} \leq \theta a+(1-\theta)b$ with $a, b \geq 0$ and any $\theta \in (0,1)$) one may find that

$$\Big(\frac{C_2}{1-(1-\beta)(1-\alpha)}\Big)^{1-\theta}=\Big(\frac{C_2}{1-(1-\beta)(1-\alpha)}\Big)^{1-\theta}\cdot 1^\theta \leq \frac{C_2}{1-(1-\beta)(1-\alpha)}+1=: c_0$$

and similarly

$$C_1^{\frac{1}{\alpha}\cdot\frac{\alpha(1-\theta)}{1-(1-\beta)(1-\alpha)}} \leq C_1^{\frac{1}{\alpha}}+1=: c_1.$$

This completes the proof. □

Now we introduce another powerful lemma, the proof details of which can be found in [85, Lemma 3.1 and 3.3].

Lemma A.3. *Suppose that $h \in C^2(\mathbb{R})$. Then for all $\phi \in C^2(\overline{\Omega})$ satisfying $\frac{\partial\phi}{\partial\vec{\nu}}=0$ on $\partial\Omega \subset \mathbb{R}^{n-1}(n \geq 2)$, we have*

$$\int_\Omega h'(\phi)|\nabla\phi|^2\Delta\phi=-\frac{2}{3}\int_\Omega h(\phi)|\Delta\phi|^2+\frac{2}{3}\int_\Omega h(\phi)|D^2\phi|^2-\frac{1}{3}\int_\Omega h''(\phi)|\nabla\phi|^4 -\frac{1}{3}\int_{\partial\Omega} h(\phi)\frac{\partial}{\partial\vec{\nu}}|\nabla\phi|^2,$$

and

$$\int_\Omega \frac{|\nabla\phi|^4}{\phi^3} \leq (2+\sqrt{n})^2\int_\Omega \phi|D^2\ln\phi|^2,$$

where $D^2\phi$ denotes the Hessian of ϕ.

Note that when $n = 1$, the first equality actually is $\int_\Omega h'(\phi)\phi_x^2\phi_{xx} = -\frac{1}{3}\int_\Omega h''(\phi)\phi_x^4$, and when $n \geq 2$, one should observe $\nabla|\nabla z|^2 = 2D^2 z \cdot \nabla z$ for any $z \in C^2(\overline{\Omega})$. The following lemma is often used to control outer normal derivatives (cf. [86, Lemma 4.2]) on the boundary.

Lemma A.4. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain and $w \in C^2(\overline{\Omega})$ fulfill $\frac{\partial w}{\partial \vec{\nu}}\big|_{\partial\Omega} = 0$. Then one may have*

$$\frac{\partial |\nabla w|^2}{\partial \vec{\nu}} \leq 2\kappa |\nabla w|^2, \qquad \text{on } \partial\Omega,$$

where $\kappa = \kappa(\Omega) > 0$ is an upper bound for the curvatures of $\partial\Omega$.

The following estimate can be achieved by using the Lemma 1 in [75]. Here we give its generalization and display the proof details in order to show the dependence on the diffusion coefficient ε in the upper bound of $\|v\|_{W_q^1(\Omega)}$, for our purpose.

Lemma A.5. *For any $\varepsilon > 0$, a bounded open domain $\Omega \subset \mathbb{R}^n (n \geq 1)$, suppose $v(x,t)$ is the classical solution of*

$$\begin{cases} v_t - \varepsilon\Delta v = g(v,t), & \Omega \times (t_0, T), \\ \nabla v \cdot \vec{\nu}\big|_{\partial\Omega} = 0, & t \in (t_0, T), \end{cases}$$

where g is locally Lipschitz continuous function in v and locally Hölder continuous in t. Then for all $t \in (t_0, T)$ and any $\varepsilon > 0$,

$$\|v(\cdot,t)\|_{W_q^1(\Omega)} \leq \hat{c}\Big\{\|v(\cdot,t_0)\|_{W_q^1(\Omega)} + \sup_{t\in(t_0,T)} \|v(\cdot,t) + g(v,t)\|_{L_p(\Omega)}\Big\} \tag{A.2}$$

where the constant $\hat{c}$ depends on $\max\{1,\varepsilon\}$, provided that

$$\begin{cases} q < \frac{np}{n-p}, & \text{as } p < n, \\ q < +\infty, & \text{as } p = n, \\ q = +\infty, & \text{as } p > n. \end{cases}$$

Moreover, if $v(x,t_0) \in W_q^2(\Omega)$ *satisfies* $\nabla v(x,t_0)\cdot \vec{\nu}\big|_{\partial\Omega} = 0$*, then* (A.2) *holds for* $t \to t_0$ *provided that we replace* $\|v(\cdot,t_0)\|_{W_q^1(\Omega)}$ *in* (A.2)*, by* $\|v(\cdot,t_0)\|_{W_q^2(\Omega)}$ *when* $q \in [1,+\infty)$*, and by* $v(\cdot,t_0) \in W_{\bar{q}}^2(\Omega)$ *for some* $\bar{q} > n$ *when* $q = +\infty$*, respectively.*

Proof. We follow the ideas of [75, Lemma 1] and [67, Lemma 4.1] that let $A := A_p$ be the sectorial operator in $L_p(\Omega)\,(1 \le p < +\infty)$ which is given by

$$Av := -\varepsilon\Delta v \qquad \text{for} \quad v \in D(A) := \left\{v \in W_p^2(\Omega);\ \nabla v \cdot \vec{\nu}\big|_{\partial\Omega} = 0\right\}. \tag{A.3}$$

Then the p-independent spectra of A (denoted by $\sigma(A)$), without counting the multiplications, are

$$0 = \Lambda_0 < \Lambda_1 < \Lambda_2 < \cdots < \Lambda_m < \cdots$$

where $\Lambda_j = \varepsilon\lambda_j$ and λ_j is the *j-th* eigenvalue of the problem

$$\begin{cases} -\Delta v = \lambda v, & x \in \Omega, \\ \nabla v \cdot \vec{\nu} = 0, & x \in \partial\Omega. \end{cases}$$

Hence as obtained in [87, Lemma 1, p.15], $-A$ is the infinitesimal generator of the analytical semigroup $(e^{-tA})_{t\ge 0}$. Moreover, $A+1$ is also a sectorial in $L_p(\Omega)$ with its domain given by $D(A)$ (cf. [88, p.418] or [76, Theorem 1.3.2, p.19]). Thus $-(A+1)$ is the infinitesimal generator of the analytical semigroup of $(e^{-t(A+1)})_{t\ge 0}$ and in this case $(A+1)^\beta$ exists for $\beta \in \mathbb{R}$ since $\sigma(A+1) \ge 1 > 0$. Now letting $h := v + g(v,t)$ one may have the representation of v as

$$v(x,t) = e^{-(t-t_0)(A+1)}v(x,t_0) + \int_{t_0}^{t} e^{-(t-\tau)(A+1)}h(x,\tau)\mathrm{d}\tau =: E_0(t) + E_1(t)$$

for $t_0 < t \le T$, thanks to $g(v,t)$ being locally Lipschitz continuous function in v and locally Hölder continuous in t. We shall below deal with $E_0(t)$ and $E_1(t)$, respectively.

(**1**) Estimate of $E_0(t)$ for any $t > t_0$. Indeed, when $q \in [1,+\infty)$ and $\beta > 1/2$, we have

$$\|E_0(t)\|_{W_q^1(\Omega)} \le c\|(A+1)^\beta e^{-(t-t_0)(A+1)}v(\cdot,t_0)\|_{L_q(\Omega)}$$

$$\leq c\|(A+1)^{\beta}e^{-(t-t_0)(A+1)}\| \cdot \|v(\cdot,t_0)\|_{L_q(\Omega)}$$
$$\leq c_\beta (t-t_0)^{-\beta} e^{-\delta(t-t_0)} \|v(\cdot,t_0)\|_{L_q(\Omega)}$$

for some $0 < \delta < \mathrm{Re}(\sigma(A+1))$, where we use the boundedness of linear operator $(A+1)^{\beta}e^{-t(A+1)}$ and a fact (cf. [76, Theorem 1.6.1]) that for $\beta \in (0,1)$

$$D((A_q+1)^{\beta}) \hookrightarrow \begin{cases} W_q^1(\Omega), & \text{if } \beta > \frac{1}{2}; \\ C^{\delta}(\overline{\Omega}), & \text{if } 2\beta - \frac{n}{q} > \delta \geq 0. \end{cases} \tag{A.4}$$

Moreover, when $q = +\infty$, by the embedding (A.4) we know for $2\beta - \frac{n}{p} > 1$ that

$$\begin{aligned} \|E_0(t)\|_{W^1_\infty(\overline{\Omega})} \leq c\|E_0(t)\|_{C^1(\overline{\Omega})} \leq & c\|(A_p+1)^{\beta}e^{-(t-t_0)(A_p+1)}v(\cdot,t_0)\|_{L_p(\Omega)} \\ \leq & c_\beta (t-t_0)^{-\beta} e^{-\delta(t-t_0)} \|v(\cdot,t_0)\|_{L_p(\Omega)}. \end{aligned} \tag{A.5}$$

(**2**) Estimate of $E_0(t)$ when $t \to t_0$. For any $\beta \in (1/2,1)$ we may have

$$\begin{aligned} \lim_{t\to t_0} \|E_0(t)\|_{W^1_q(\Omega)} \leq & c \lim_{t\to t_0} \|(A+1)^{\beta}e^{-(t-t_0)(A+1)}v(\cdot,t_0)\|_{L_q(\Omega)} = c\|(A+1)^{\beta}v(\cdot,t_0)\|_{L_q(\Omega)} \\ \leq & 2c\bar{C}_0 \|v(\cdot,t_0)\|^{\beta}_{W^2_q(\Omega)} \|(A+1)v(\cdot,t_0)\|^{1-\beta}_{L_q(\Omega)} \leq 2\bar{c}\bar{C}_0\|v(\cdot,t_0)\|_{W^2_q(\Omega)} \end{aligned}$$

where $\bar{C}_0$ is from the interpolation inequality of sectorial operator (cf. [76, Theorem 1.4.4]) and $\bar{c}$ depends on $\max\{1,\varepsilon\}$.

Taking $p > n$ and $\beta \in (1/2,1)$ such that $2\beta - \frac{n}{p} > 1$, then by the first row of (A.5), we have

$$\begin{aligned} \lim_{t\to t_0} \|E_0(t)\|_{W^1_\infty(\Omega)} \leq & \lim_{t\to t_0} c\|(A_p+1)^{\beta}e^{-(t-t_0)(A_p+1)}v(\cdot,t_0)\|_{L_p(\Omega)} = c\|(A_p+1)^{\beta}v(\cdot,t_0)\|_{L_p(\Omega)} \\ \leq & 2c\bar{C}\|v(\cdot,t_0)\|^{\beta}_{W^2_p(\Omega)} \|(A+1)v(\cdot,t_0)\|^{1-\beta}_{L_p(\Omega)} \leq 2\bar{c}\bar{C}\|v(\cdot,t_0)\|_{W^2_p(\Omega)}, \end{aligned}$$

where $\bar{c}$ may depend on $\max\{1,\varepsilon\}$.

(**3**) The estimate of $E_1(t)$ for $t > t_0$. For $1 < p < q < +\infty$ and any $z \in L_p(\Omega)$ we invoke a fact (cf. [89, Lemma 4.1] or [90, Proposition 3.1]) that

$$\|e^{-tA}z\|_{L_q(\Omega)} \le C(p,q)t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|z\|_{L_p(\Omega)}$$

for $t \in (0,T]$, where $C(p,q)$ is from Gagliardo-Nirenberg interpolation inequality. Moreover, by the fact (cf. [76, Theorem 1.4.3]) that

$$\|(A+1)^{\beta}e^{-t(A+1)}\| \le C_{\beta}t^{-\beta}e^{-\delta_0 t}, \qquad t>0,$$

for any $\beta \ge 0$ and some $0<\delta_0<\mathrm{Re}\,(\sigma(A+1))$, we may compute that

$$\begin{aligned}
&\|(A+1)^{\beta}E_1(t)\|_{L_q(\Omega)} \\
\le&\int_{t_0}^{t}\|(A+1)^{\beta}e^{-(t-\tau)(A+1)}h(x,\tau)\|_{L_q(\Omega)}\,\mathrm{d}\tau \qquad \text{(by Minkoswki's inequality)} \\
\le&\int_{t_0}^{t}\|(A+1)^{\beta}e^{-\frac{(t-\tau)}{2}(A+1)}e^{-\frac{(t-\tau)}{2}(A+1)}h(x,\tau)\|_{L_q(\Omega)}\,\mathrm{d}\tau \\
\le&2^{\beta}C_{\beta}\int_{t_0}^{t}(t-\tau)^{-\beta}e^{-\frac{\delta}{2}(t-\tau)}\|e^{-\frac{(t-\tau)}{2}(A+1)}h(x,\tau)\|_{L_q(\Omega)}\,\mathrm{d}\tau \\
\le&2^{\beta+\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}C(p,q)C_{\beta}\int_{t_0}^{t}(t-\tau)^{-\beta-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}e^{-\frac{\delta}{2}(t-\tau)}\|h(x,\tau)\|_{L_p(\Omega)}\,\mathrm{d}\tau \\
\le&2^{\beta+\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}C(p,q)C_{\beta}\sup_{t\in(0,T)}\|h(x,t)\|_{L_p(\Omega)}\int_{0}^{t-t_0}\varsigma^{-\beta-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}e^{-\mu\varsigma}\,\mathrm{d}\varsigma
\end{aligned} \tag{A.6}$$

for $p \in (1,q)$ and some $0<\delta<\mathrm{Re}\,(\sigma(A+1))$, thus $-\beta-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})<0$ for any $n \in \mathbb{N}_+$. Note that the last integral is essentially a Γ function which converges absolutely when

$$1-\beta-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})>0 \quad \text{and} \quad \mu>0.$$

Therefore, by (A.4) if $\beta \in (\frac{1}{2},1)$ then (A.6) means

$$\begin{aligned}
&\|E_1(t)\|_{W_q^1(\Omega)} \\
\le&C\|(A_q+1)^{\beta}E_1(t)\|_{L_q(\Omega)}
\end{aligned}$$

$$\leq C(p,q,\beta,n)C_\beta \sup_{t\in(0,T)} \|h(x,t)\|_{L_p(\Omega)} \int_0^\infty \varsigma^{-\beta-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} e^{-\mu\varsigma}\,\mathrm{d}\varsigma$$

$$< +\infty$$

which holds for

$$q < \frac{1}{\frac{2(\beta-1)}{n} + \frac{1}{p}} = \frac{1}{\frac{1}{p} - \frac{1}{n} + \frac{2}{n}(\beta - \frac{1}{2})}, \quad \text{as} \quad p \leq n.$$

On the other hand, when q is large enough, by (A.4) with $2\beta - \frac{n}{q} > 1$ one may have by (A.6) that

$$\|E_1(t)\|_{W^1_\infty(\Omega)} \leq \|E_1(t)\|_{C^1(\overline{\Omega})} \leq C\|(A_q+1)^\beta E_1(t)\|_{L_q(\Omega)}$$

$$\leq C(p,q,\beta,n)C_\beta \sup_{t\in(0,T)} \|h(x,t)\|_{L_p(\Omega)} \int_0^\infty \varsigma^{-\beta-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} e^{-\mu\varsigma}\,\mathrm{d}\varsigma < +\infty$$

as long as

$$\frac{n}{2q} + \frac{1}{2} < \beta < 1 - \frac{n}{2}(\frac{1}{p} - \frac{1}{q})$$

which is satisfied as $p > n$.

(**4**) Estimate of $E_1(t)$ when $t \to t_0$. Here either $p \leq n$ or $p > n$, the (A.6) implies that

$$\lim_{t\to t_0} \left\{\|E_1(t)\|_{W^1_q(\Omega)} + \|E_1(t)\|_{W^1_\infty(\Omega)}\right\} \leq 2C(p,q,\beta,n)C_\beta \sup_{t\in(0,T)} \|h(x,t)\|_{L_p(\Omega)}.$$

This completes the proof. □

Remark A.1. *This result shows that when $p = n = 2$ and (u,w) is the local classical solution of* (1.10)*, if*

$$\|u(\cdot,t)\|_{L_p(\Omega)} \leq c,$$

then $\|h\|_{L_p(\Omega)} = \|w - uw + rw(1-w)\|_{L_p(\Omega)} \leq c\|u\|_{L_p(\Omega)}$ by $0 < w < K$ given in Remark 3.1. Thus one may derive that for any $t \in (0,T) \subset (0,T_{\max})$,

$$\|w(\cdot,t)\|_{W^1_q(\Omega)} \leq \hat{c}, \qquad q \in [1,+\infty).$$

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