SOLVING THE CAPACITATED MULTI-TRIP VEHICLE ROUTING PROBLEM WITH TIME WINDOWS

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Solving the Capacitated Multi-Trip Vehicle Routing Problem with Time Windows

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CERTIFICATE OF ORIGINALITY

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Abstract

The capacitated multi-trip vehicle routing problem with time windows is a very challenging well-known vehicle routing problem, where each vehicle is allowed to perform more than one trip to serve customers. This multi-trip characteristic is required in wide practical applications, ensuring better utilization of the vehicles. Such a characteristic, however, makes the problem much more difficult to solve, as in the existing literature, only 14 of the 27 benchmark instances with 100 customers have been solved to optimality.

In this thesis, we develop a novel three-phase exact method to tackle this challenging problem. In the first and the second phases, we utilize both a route-based and a trip-based integer programming models together, solving their linear programming relaxations sequentially through some column-and-cut generation procedures. In the third phase, we then close the integrality gap by solving a trip-based model through a dynamic time discretization technique. Results from extensive computational experiments over benchmark instances demonstrate the effectiveness and efficiency of our newly proposed exact method. For the first time in the literature, all the 27 benchmark instances are solved to optimality in much less average running time than the best-known exact method in the existing literature. Our three-phase exact method is also flexible and can be adapted to solve several other variants of the problem to optimality efficiently.

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Chapter 1

Introduction

Due to its computational challenges and wide applications, the vehicle routing problem (VRP) is one of the most studied problems in operations research since the pioneer study of Dantzig and Ramser [13]. It aims to design trips for available vehicles such that each customer is visited exactly once, vehicle capacity constraints are respected and the total travel cost is minimized. However, since customers may not always be available to be visited all the time, it is often constrained that each customer should be visited within a specified time interval, which is called a time window. Under such a time window constraint, the problem becomes the vehicle routing problem with time windows (VRPTW). Both the VRP and VRPTW are strongly NP-hard since they are generalizations of the traveling salesman problem (TSP), which is well-known to be strongly NP-hard. For comprehensive surveys about the VRP, we refer the reader to Toth and Vigo [43], Laporte [30] and Toth and Vigo [44].

In the settings of both the VRP and VRPTW, each vehicle can perform at most one trip during a planning time horizon (e.g., one day). However, in some practical applications, such as those in last-mile delivery, trips are usually short in terms of travel time and distance. To make better utilization of available vehicles, each vehicle is often allowed to perform more than one trip during the planning time horizon. Under such a setting with multiple trips allowed, the problem becomes the capacitated multi-trip vehicle routing problem with time windows (CMTVRPTW), which is very challenging to solve and attracts a growing number of studies in recent years.

The CMTVRPTW can be formally defined as follows. Consider a graph $G = (\mathbb{V}_0, \mathbb{A})$ where $\mathbb{V}_0 = \mathbb{V} \cup \{0\}$ with $\mathbb{V} = \{1, 2, \ldots, N\}$ denoting a set of customer nodes and with node 0 denoting the depot node, and where $\mathbb{A}$ is a set of arcs. There are K homogeneous vehicles available to deliver goods to serve the customers, with each vehicle having a capacity denoted by Q, so that the total quantity of goods loaded on each vehicle cannot exceed Q. The demand quantity of goods to be delivered to customer $i \in \mathbb{V}$ is denoted by q_i. The planning time horizon is $[a_0, b_0]$. The service for customer $i \in \mathbb{V}$ can only start within a time window, denoted by $[a_i, b_i]$, which implies that customer i cannot be visited after time b_i, and a vehicle should wait until time a_i to start the service if it arrives at customer i before a_i. Let st_i denote the service time at node $i \in \mathbb{V}$. For the sake of presentation, we also denote $st_0 = 0$. Let c_{ij} and $t^{'}_{ij}$ denote the travel cost and travel time associated with arc $(i, j) \in \mathbb{A}$, respectively. Without loss of generality, we can assume that the travel times associated with the arcs satisfy the triangle inequality, i.e., $t^{'}_{ij} + t^{'}_{jk} \geq t^{'}_{ik}$ for each (i, j), (j, k), and (i, k) in $\mathbb{A}$, because otherwise we can define the travel time of each arc (i, j) as the minimum time required to travel between nodes i and j, which satisfies the triangle inequality. As multiple trips are allowed, the number of times that each vehicle can depart from the depot is not limited. Accordingly, the CMTVRPTW aims to design trips for all the vehicles to visit each customer exactly once with the travel cost minimized. This problem is still strongly NP-hard since it generalizes the strongly NP-hard problem VRPTW.

The multi-trip characteristic is required in wide practical applications, such as the last-mile delivery [35], disaster logistics [38], urban waste collection [42], perishable goods delivery [41], biomedical sample transportation [1], and so on. However, such a characteristic makes the problem much more difficult to solve. Specifically, for the CMTVRPTW, only 14 of the 27 benchmark instances with 100 customers, which are adapted from Solomon's instances, are solved to optimality in the existing literature.

As is pointed out in Laporte [30], exact methods and models in the existing literature for the VRP and its variants include branch-and-bound, dynamic programming, vehicle flow formulations, commodity flow formulations, and set partitioning formulations. Among them, methods based on set partitioning formulations are state-of-the-art, including those for the VRP [37], VRPTW [36], and CMTVRPTW [46]. Such exact methods generally consist of two phases, where the linear programming (LP) relaxation of the problem's integer programming (IP) model is solved by column generation or column-and-cut generation in the first phase, and the integrality gap between the LP relaxation and the IP model is closed in the second phase. Furthermore, the solution strategies of the second phase adopted in the existing literature can be mainly classified into two categories. One is to develop efficient branch-and-bound algorithms where the LP relaxation of the problem is solved by column generation at each node of the search tree explored, such as the branch-and-price and the branch-and-price-and-cut algorithms. The other is to enumerate all the possible columns that may appear in an optimal solution and directly apply an IP solver to solve the IP model based on these columns. Both the two categories of solution strategies require tight LP relaxations of the problem.

As the VRP and most of its variants are minimization problems, the tightness of the lower bound provided by the LP relaxation is of great importance for the efficiency of the solution methods that are based on set partitioning formulations. This is because a tighter lower bound leads to fewer search tree nodes that need to be explored in branch-and-price and branch-and-price-and-cut, as well as to fewer columns that need to be generated in the column enumeration procedure. To tighten the LP relaxation of the CMTVRPTW, one can reformulate the problem based on different representations of columns, such as those using columns to represent routes (where a route is a sequence of different nodes), to represent trips (where a trip is a pair of a route and its associated departure time at the depot), and to represent journeys (where a journey is a sequence of routes or trips). However, different reformulations known in the literature have their advantages and disadvantages, in aspects of tightness of the lower bound and required time of computation. Specifically, for models based on routes, based on trips, and based on journeys, the lower bounds obtained from their LP relaxations become tighter, but the time required to compute the lower bounds becomes longer. Therefore, it is challenging to compute a tighter lower bound in a shorter running time. In this thesis, we tackle this challenge from our observation that once the LP relaxation of the route-based model is solved, the LP relaxation of the trip-based model becomes much easier to solve. Based on this observation, we can develop a novel approach to compute a tighter lower bound for the CMTVRPTW efficiently.

Another direction to strengthen the lower bound is to identify valid inequalities of the IP model. Such valid inequalities are satisfied by feasible integer solutions to the IP model but may be violated by feasible fractional solutions to the LP relaxation. Since the CMTVRPTW is a generalization of the VRP, some valid inequalities used in solving the VRP are also valid for the CMTVRPTW. However, not all of them have been applied in the studies of the CMTVRPTW. Moreover, some valid inequalities used in the study of the CMTVRPTW can be further strengthened, to obtain a better lower bound from the LP relaxation. Since the numbers of some valid inequalities grow exponentially with the problem size, it is impractical to enumerate and incorporate all of them into the IP model. Due to this, some cut generation procedures can be developed, so that only a small number of violated inequalities (or cuts) need to be generated.

In addition to the tightness of lower bounds, the procedure for closing the integrality gap is also critical to the computational efficiency of solution methods. Branch-and-price and branch-and-price-and-cut algorithms are often time-consuming, because the pricing problem, which is NP-hard, should be solved at search tree nodes where we need to compute their associated lower bounds, and the number of search tree nodes is usually considerable for large-sized instances. Alternatively, we can utilize the dual solution of the LP relaxation to identify columns that may appear in optimal solutions for the problem. If the LP relaxation is tight, the number of columns identified becomes limited, so that an IP model based on these columns can be solved to optimality by an IP solver directly, closing the integrality gap. This technique is advantageous when the number of identified columns is moderate. Although such a technique has been applied in several recent studies on the CMTVRPTW (see, e.g., Paradiso et al. [35] and Yang [46]), it is still of great interest to further enhance the efficiency.

Moreover, for methods relying on trip-based IP models, their efficiency is significantly affected by how the planning time horizon is discretized into a set of time points. In the VRPTW, since each vehicle can perform at most one trip, we only need to require that the number of trips cannot be greater than K, and thus, one does not need to discretize the planning time horizon. However, in the CMTVRPTW, the number of trips during a planning time horizon is not limited for each vehicle, but no more than K trips can be served by the vehicles at every time point. Since the planning time horizon is an interval that contains countless time points, the complete trip-based IP model has an infinite number of variables and constraints. Therefore, for tractability, one needs to discretize the planning time horizon. Specifically, one can derive a trip-based IP model based on a restricted number of discretized time points as a relaxation of the original problem, so that its optimal solution provides a lower bound on the original problem. This optimal solution is optimal to the original problem only if it is a feasible solution to the original problem. If it is not a feasible solution to the original problem, one needs to identify more time points to strengthen the relaxation. Following this approach, some existing studies, such as Hernandez et al. [23], proposed to discretize the planning time horizon in an equidistant and half-reduced manner, where the set of discrete time points is initialized by selecting equidistant time points within the planning time horizon, and then, the distance between two consecutive time points is reduced by half whenever further time discretization is needed. However, such a time discretization approach often results in a considerable number of time points incorporated into the IP model, so the resulting model is still very difficult to solve.

In this thesis, we propose a novel three-phase exact method for the CMTVRPTW, where LP relaxations of the route-based and trip-based models are solved in the first two phases through some column-and-cut generation procedures, and we close the integrality gap in the third phase by solving the trip-based model with a dynamic time discretization technique. For the first time in the literature, all the 27 benchmark instances with 100 customers are solved to optimality in much less average running time than the best-known method in the existing literature. Other major contributions of this thesis are summarized as follows.

- We propose a more efficient approach to close the integrality gap by solving the trip-based model instead of the route-based model. Because a separation constraint of the route-based model usually involves fewer routes than the trip-based model, so for problem instances with loose time windows (which can be much more difficult than those with tight time windows), maybe much more separation constraints need to be identified to separate infeasible solutions under the route-based model than the trip-based model. This may also explain why the procedure for closing the integrality gap is a critical bottleneck in the up-to-date exact method developed by Yang [46]. We also adopt a dynamic time discretization technique in our solution method, which is more efficient than that in an equidistant and half-reduced style adopted by Hernandez et al. [23].

- For the first time in the literature, our solution method for the CMTVRPTW utilizes the advantages of both the route-based and trip-based models. By solving the LP relaxation of the route-based model, only a set of candidate routes that may appear in optimal solutions are generated, so only trips with their node sequences included in the route set need to be considered in the trip-based model. As a result, the trip-based model and its LP relaxation become much easier to solve, which is critical in tightening the lower bound effectively and closing the integrality gap efficiently.

- We propose a new class of valid constraints, named RWT constraints, which generalize the RSF constraints introduced by Paradiso et al. [35] and strengthen the LP relaxation of the route-based model.

- We introduce a new technique for identifying departure times which cannot lead to any optimal solution, which can effectively reduce the numbers of routes and trips that need to be considered.

- We newly apply some valid inequalities used in the literature on the VRP but never in the existing study on the CMTVRPTW, such as SR5-2, SR5-3, EL, and SRC constraints (see, e.g., [36, 4]), so that the lower bound obtained from the LP relaxation can be further strengthened.

The remainder of this thesis is organized as follows. We review the relevant literature in Chapter 2, formulate the problem by the route-based and trip-based models in Chapter 3, and illustrate our three-phase exact method in Chapter 4. Computational results are presented in Chapter 5, followed by a conclusion in Chapter 6.

Chapter 2

Literature Review

The CMTVRPTW is a generalization of the VRP and VRPTW. In this chapter, we first review several representative works in the literature on the VRP and VRPTW, and then review the literature on the CMTVRPTW. Since our study is to develop exact methods for the CMTVRPTW, only works on exact methods are reviewed here. For the review of heuristic methods of the VRP and its variants, we refer the reader to two recent surveys conducted by Dixit et al. [17] and Elshaer and Awad [19].

2.1 Exact Methods for the VRP

Many exact methods developed for the VRP are branch-and-bound algorithms. It explicitly explores the solution space through a search tree, and prunes branches of the search tree by using lower bounds computed at search tree nodes. Therefore, the efficiency of such branch-and-bound algorithms relies heavily on the tightness of the lower bounds. Christofides et al. [12] developed a branch-and-bound algorithm for the VRP, where lower bounds are derived from k-degree center trees and q-routes. Their computational experiments showed that the lower bounds derived from q-routes are tighter than those derived from k-degree center trees, and instances of the VRP with up to 25 customers can be solved to optimality.

The branch-and-cut algorithms extend the branch-and-bound algorithms by adding valid inequalities to strengthen the lower bounds computed at search tree nodes. Based on an arc-flow model, Lysgaard et al. [33] developed a branch-and-cut algorithm for the VRP, where a variety of valid inequalities were imposed, such as capacity inequalities, framed capacity inequalities, strengthened comb inequalities, multistar inequalities, partial multistar inequalities, and hypotour inequalities. Their methods can solve several large-sized instances of the VRP with up to 134 customers, including three new instances whose optimal solutions were not known before.

The branch-and-price-and-cut algorithms extend the branch-and-cut algorithms in the sense that, unlike the branch-and-cut algorithms, the branch-and-price-and-cut algorithms need to solve both the pricing problems (for adding new variables) and the separation problems (for adding new constraints) at each node of the search tree. Thus, the branch-and-price-and-cut algorithms are suitable for models that consist of both a large number of variables and a large number of constraints. Based on a formulation that combines the arc-flow model and the set-partitioning model, Fukasawa et al. [20] developed a branch-and-price-and-cut algorithm for the VRP which can solve all the benchmark instances in the literature to optimality, with up to 134 customers, where the pricing problem is to find the q-route without 2-cycles that minimizes the reduced cost, and the separation problem is to find valid inequalities that are identified by the CVRPSEP package from [32].

The branch-and-price-and-cut algorithm is based on a set-partitioning model of the VRP, and the associated restricted master problem is usually highly degenerate and has multiple optimal basic solutions. As a result, even after an optimal solution to the restricted master problem is found by a simplex-based method, it may still need to perform a large number of additional iterations in the column generation procedure to prove the optimality. To avoid such a tailing-off effect, Baldacci et al. [4] proposed an additive bounding method that combines three procedures to produce a near-optimal dual solution to the restricted master problem. The first two procedures solve the Lagrangean dual problem of the restricted master problem without clique inequalities where one is based on q-routes and the other is based on elementary routes. The third procedure solves the restricted master problem by a column-and-cut generation procedure with a dual stabilization technique applied. It was shown that only routes with reduced costs no greater than the gap between the lower bound and the upper bound need to be enumerated and kept in the set-partitioning model so that it can be solved by an IP solver directly to produce an optimal solution for the VRP. Computational results showed that their method can produce better lower bounds and is more efficient than that proposed by Fukasawa et al. [20].

The best exact methods developed for the VRP at present are based on the column-and-cut generation procedure. However, introducing those additional cuts in a column-and-cut generation procedure often makes the pricing problem more difficult to solve. Pecin et al. [37] devised a technique to keep the structure of the pricing problem less affected when subset-row cuts are introduced. Moreover, by incorporating some effective optimization techniques, such as dual stabilization, route enumeration, variable fixing, and strong branching, Pecin et al. [37] was able to solve all the benchmark instances with up to 199 customers to optimality.

2.2 Exact Methods for the VRPTW

Similar to the VRP, the state-of-the-art methods for the VRPTW are also based on a set-partitioning model. Desrochers et al. [16] is a pioneer work in this direction, which proposed the first branch-and-price algorithm for the VRPTW and formulated the pricing problem as an elementary shortest path problem with resource constraints (ESPPRC). Since such a pricing problem is NP-hard [18], Desrochers et al. [16] employed a q-route relaxation and developed a dynamic programming algorithm to compute the q-route of the minimum reduced cost with no 2-cycles. The branch-and-price algorithm proposed by Desrochers et al. [16] was able to solve seven benchmark instances of [40] containing 100 customers to optimality.

To improve the efficiency of the branch-and-price algorithm, the LP relaxations of the set-partitioning models of the problem are often strengthened by valid inequalities. Not only pricing problems should be solved to identify variables with negative reduced cost, but also separation problems should be solved to identify violated valid inequalities, which results in a branch-and-price-and-cut algorithm. For the VRPTW, Jepsen et al. [28] introduced the subset-row inequalities and developed a branch-and-price-and-cut algorithm which solved seven new 100-customer instances of [40] to optimality. Based on the k-path inequalities proposed by Kohl et al. [29], Desaulniers et al. [14] introduced the generalized k-path inequalities, and their algorithm solved five new 100-customer instances of [40]. Other valid inequalities, including the rounded capacity constraints [5] and the elementary inequalities [36] have also been introduced in the literature to strengthen the LP relaxation of the set-partitioning model.

Although the LP relaxation of the set-partitioning model can be strengthened by using elementary routes to define columns, the resulting pricing problem ESPPRC is very hard to solve for large-sized instances. However, one can reduce the computational time for solving the pricing problem by relaxing the requirement for routes to be elementary, which is at the cost of weakening the resulting lower bound. Desrochers et al. [16] adopted a q-route relaxation enhanced by a 2-cycle elimination, which was later extended to a k-cycle elimination by Irnich and Villeneuve [26]. Baldacci et al. [5] introduced the ng-route relaxation and was able to solve four new 100-customer instances of [40]. The ng-route relaxation has turned out to be very suitable for the VRPTW, and thus, it has been widely used in recent studies.

The variable fixing technique fixes the non-negative integer variables whose values cannot be positive in any optimal solution to zero. For example, if the reduced cost of a column is greater than the integrality gap between the lower bound and a valid upper bound, then the column cannot appear in any optimal solution [8], and thus, its associated binary variable in the set partitioning model can be fixed to zero. By applying this variable fixing technique, Baldacci et al. [4] proposed to first enumerate all columns (which represent routes) with reduced costs not greater than the integrality gap, then to use these columns to form a set partitioning model, and to solve the resulting model directly by an IP solver. This approach is efficient to obtain the optimal integer solution when the number of the enumerated columns is moderate. The variable fixing technique can not only be applied in reducing the number of columns considered in the set-partitioning model, but also can eliminate arcs of the underlying graph [27] and thus reduce the size of the search tree.

Incorporating some of the above-mentioned techniques and together with three other techniques, including the bidirectional labeling, the decremental state-space relaxation, and the completion bound, Pecin et al. [36] were able to solve all the 56 instances of Solomon [40] that have 100 customers, and 51 of the 60 instances of Gehring and Homberger [21] that have 200 customers, to optimality.

2.3 Exact Methods for the CMTVRPTW and Its Variants

In the literature, different studies sometimes adopt different terms for the same meaning or the same term for different meanings. To avoid ambiguity, in this thesis we define a route as a sequence of nodes, a trip as a time-route pair, and a journey as a sequence of routes or a sequence of trips. Since this thesis focuses on the development of exact methods for the CMTVRPTW, existing works on heuristic methods are not reviewed here. Since time window constraints play a critical role in the CMTVRPTW, which complicates the problem and its solution methods, we only review the studies on the CMTVRPTW and its variants that have time window constraints, and the features of these problems are summarized in Table 2.1. For more comprehensive surveys about multi-trip vehicle routing problems, we refer the reader to Cattaruzza et al. [9] and Cattaruzza et al. [10].

Azi et al. [2] studied a variant of the CMTVRPTW where a single vehicle is allowed to perform multiple trips to visit customers within their time windows. In their settings, each customer can be visited at most once, the loading time at the depot is proportional to the total service time spent during the visit of the customers, and a duration limit is imposed on each route. The objective has two folds, as it aims to maximize the total number of customers visited, under which to minimize the total travel cost. To solve this problem, they developed a two-phase method. They enumerated all non-dominated routes in the first phase and used these routes to construct an instance of the ESPPRC, in which each node of the underlying graph represented a non-dominated route. They then solved this instance of the ESPPRC in the second phase to obtain an optimal journey of the problem. Computational experiments performed on instances derived from the 100-customer instances of Solomon [40] showed that the performance of this solution method was sensitive to the duration limit.

Azi et al. [3] studied a more general problem than that of Azi et al. [2], where multiple vehicles are allowed to perform multiple trips to visit customers within their time windows. The problem also generalizes the CMTVRPTW by taking into account multiple objectives, loading times at the depot, and route duration limits. To solve this problem, Azi et al. [3] also developed a two-phase method. They enumerated all the non-dominated routes in the first phase and used these routes to construct a set-partitioning model for the problem. They then solved the resulting model in the second phase by a branch-and-price algorithm where columns of the restricted master problem represented journeys and nodes on the underlying graph represented non-dominated routes. The computational results indicated that most instances with 25 customers were solved to optimality. This problem was later studied by Macedo et al. [34], and all the non-dominated routes were also enumerated first. By defining nodes as time points and arcs as routes, they proposed a pseudo-polynomial network flow model. Although the size of the network flow model increases with the number of time points, the experimental results on the same set of instances showed that the method of Macedo et al. [34] outperformed that of Azi et al. [3].

By requiring each customer to be visited exactly once and redefining the objective as minimizing the total travel cost, Hernandez et al. [23] considered the CMTVRPTW with limited trip duration (CMTVRPTW-LD). A two-phase method was developed, where in the first phase, non-dominated routes were enumerated to form a set-partitioning model, and then a trip-based branch-and-price algorithm was carried out to solve the model in the second phase. The pricing problem of the branch-and-price algorithm was solved by an inspection method, the running time of which is pseudo-polynomial time and depends on the granularity of the discretized time points. Computational results on instances with 25 and 40 customers showed that their method is on average more efficient than those of Azi et al. [3] and Macedo et al. [34]. Without the duration limit imposed on routes, Hernandez et al. [25] studied the CMTVRPTW with loading times (CMTVRPTW-LT), and they developed two set-covering formulations based on journeys and trips, respectively. Computational results on instances with 25 customers indicated that solutions obtained from the trip-based model were of better quality than those from the journey-based model.

Şahin and Yaman [39] studied another variant of the CMTVRPTW where the multi-depot and heterogeneous fleet characteristics are considered. In their settings, different types of vehicles are associated with different loading capacities, fixed costs, and travel time matrices. There are more than one depot, and vehicle trips can start and end at different depots, but for each vehicle type, the number of vehicles at a depot at the beginning of the planning time horizon should be equal to that at the end of the planning time horizon. To minimize the sum of the fixed cost and travel cost of all vehicles, they adopted a branch-and-price algorithm based on a set-partitioning model, where columns are represented with journeys. An efficient heuristic method was developed to find columns with negative reduced costs, and the exact labeling algorithm is evoked only when the heuristic method fails. Computational results show that their method can solve some instances with up to 40 customers, three depots, and two types of vehicles.

Recently, Paradiso et al. [35] developed a novel route-based model for the CMTVRPTW, where they used a team orienteering problem with time windows (TOPTW) to formulate the separation problem for constraints that ensure no more than K vehicles assigned to the selected routes. To solve the problem, they first applied column generation to compute the lower bound provided by the LP relaxation without considering the constraints on the number of vehicles assigned. They then enumerated all the routes with reduced costs no greater than the integrality gap between the lower bound and an upper bound. They further identified the relaxed structure feasibility (RSF) constraints and SR3 valid inequalities that excluded fractional solutions of the LP relaxation to tighten the lower bound. With these, they reduced the integrality gap, and accordingly, were able to reduce the number of routes to be considered. They then applied a branch-and-cut algorithm to close the integrality gap. In addition to being able to solve almost all benchmark instances with 40 customers and some with 50 customers within a time limit of 3 hours, their solution method can also be adapted to tackle other variants of the CMTVRPTW, including the CMTVRPTW-LD and CMTVRPTW-LT.

Yang [46] still utilized the route-based model to further enhance the solution method proposed by Paradiso et al. [35]. They first applied a heuristic method to compute an upper bound of the problem and applied column-and-cut generation to compute the lower bound provided by the LP relaxation, with a number of RSF and SR3 valid inequalities taken into account. They then enumerated all the routes with reduced costs no greater than the integrality gap. Next, they applied the cutting-plane procedure to solve the strengthened LP relaxation of the set-partitioning model defined on the enumerated routes and added all other violated RSF and SR3 valid inequalities. Since a route can have different reduced costs under different dual solutions, the number of enumerated routes can be different under different dual solutions. Therefore, among all the optimal dual solutions for

Table 2.1: Representative exact methods for the CMTVRPTW and its variants

Reference	Problem abbreviation	The # of vehicles	The # of visit	Duration limit	Loading time	Min cost
Azi et al. [2]	-	$=1$	≤ 1	✓	✓	×
Azi et al. [3]	-	≥ 1	≤ 1	✓	✓	×
Macedo et al. [34]	-	≥ 1	≤ 1	✓	✓	×
Hernandez et al. [23]	CMTVRPTW-LD	≥ 1	$=1$	✓	✓	✓
Hernandez et al. [24]	CMTVRPTW-LT	≥ 1	$=1$	×	✓	✓
Paradiso et al. [35]	CMTVRPTW [1]	≥ 1	$=1$	×	×	✓
Yang [46]	CMTVRPTW [1]	≥ 1	$=1$	×	×	✓

the LP relaxation, they found the one that maximizes the sum of the reduced cost of all the enumerated routes, by solving an LP problem. This reduced the number of enumerated routes. Moreover, by aggregating some routes with certain conditions satisfied, they further reduced the number of columns in the set-partitioning model of the problem. The integrality gap was then closed by a branch-and-cut algorithm. Computational results showed that the improved solution method of Yang [46] was able to solve almost all benchmark instances with 80 customers and some with 100 customers within a time limit of 3 hours to optimality, outperforming the method of Paradiso et al. [35].

From the literature, we can see that the solution methods proposed by Azi et al. [2], Azi et al. [3], Macedo et al. [34] and Hernandez et al. [23] involve enumerating routes without applying the variable fixing technique and utilizing dual information of the LP relaxation, which are doable in the presence of rigid duration limit constraints and difficult to be applied to the CMTVRPTW and other variants without such constraints. Hernandez et al. [25] proposed a trip-based model to solve the CMTVRPTW-LT, but the pricing problem is very difficult to solve and the branch-and-price algorithm that they used to close the integrality gap is also very time-consuming. Relying on a journey-based model, Şahin and Yaman [39]

[1]Their method can also solve CMTVRPTW-LT, CMTVRPTW-LD, CMTVRPTW-R, DRP.

designed a branch-and-price algorithm for a variant of the CMTVRPTW where multi-depot and heterogeneous fleet characteristics are considered, but the pricing problem is very difficult to solve so that only some small-sized instances can be solved. Although the solution method based on route-based models, which was first proposed by Paradiso et al. [35] and later improved by Yang [46], is superior to other exact methods known in the literature, the branch-and-cut algorithm that they used to close the integrality gap is a critical bottleneck. Among the 27 benchmark instances with 100 customers, only 14 instances can be solved to optimality by their solution methods. In this thesis, our new exact method can solve all these benchmark instances to optimality.

Chapter 3

Formulations

In Section 3.1 and Section 3.2 of this chapter, we apply some valid inequalities to strengthen the LP relaxations of the route-based IP model and trip-based IP model of the CMTVRPTW, which were first proposed by Paradiso et al. [35] and Hernandez et al. [23], respectively. These valid inequalities include some newly developed ones, which are named RWT constraints and strengthen the RSF constraints proposed by Paradiso et al. [35]. We then compare the tightness of the LP relaxations of these models in Section 3.3.

3.1 Route-Based IP Model and Its Valid Inequalities

We define a route $r = (0, i_1, i_2, \ldots, i_{n_r}, 0)$ as a sequence of nodes, so that a vehicle that performs the route r departs from the depot 0, visits customer nodes $i_1, i_2, \ldots, i_{n_r}$ sequentially, and then returns back to the depot 0. We use $\mathbb{V}(r) = \{i_1, i_2, \ldots, i_{n_r}\}$ to denote the set of customers visited on route r. Since unnecessary waiting time at customers cannot lead to better solutions, without loss of generality, we can assume that every vehicle serves and then leaves each customer visited as early as possible. In other words, for a vehicle arriving at customer i at time t, it starts serving the customer at time a_i if $t < a_i$, at time t if $a_i \leq t \leq b_i$, and it fails to serve the customer if $t > b_i$. If the vehicle serves the customer successfully, it is assumed to leave for the next node immediately once the service is completed, i.e., at time $(a_i + st_i)$ if $t < a_i$, and at time $(t + st_i)$ if $a_i \leq t \leq b_i$. Accordingly, when a vehicle performs a route, given a departure time from the depot, one can follow the argument above to either determine its arrival time for every customer on this route, or to determine that the vehicle cannot visit all of these customers within their time windows or return back to the depot no later than time b_0. Based on this observation, we define that a route $r = (0, i_1, i_2, \ldots, i_{n_r}, 0)$ is feasible if and only if (i) all the customers $i_1, i_2, \ldots, i_{n_r}$ of route r are different, (ii) the total demand quantity $\sum_{i \in \mathbb{V}(r)} q_i$ of the customers of route r does not exceed the vehicle capacity Q, and (iii) a vehicle that departs from the depot at the earliest time a_0 can visit all the customers $i_1, i_2, \ldots, i_{n_r}$ along route r within their time windows and return back to the depot no later than time b_0. The associated total cost of route r is given by $c_r = c_{0i_1} + \sum_{w=1}^{n_r-1} c_{i_w i_{w+1}} + c_{i_{n_r}0}$.

As shown by Azi et al. [2], for any feasible route r, there exist a time interval $[e_r, l_r]$ and a time duration d_r, such that a vehicle can visit all the customers along route r within their time windows and return back to the depot no later than time b_0, if and only if it departs from the depot at time $t \in [a_0, l_r]$, and returns to the depot at time $t + d_r$ if $t \in [e_r, l_r]$ or at time $e_r + d_r$ if $t \in [a_0, e_r)$. For route r, we thus refer to $[a_0, l_r]$ as the interval of its feasible departure times from the depot, l_r as its latest departure time from the depot, and refer to $e_r + d_r$ and $l_r + d_r$ as its earliest and latest return time to the depot, respectively. It can also be seen that the duration for a vehicle to perform route r is always greater than or equal

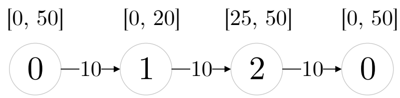


Figure 3.1: A route $r = (0, 1, 2, 0)$ where travel times are on arrows, time windows are above circles.

to d_r. Thus, we refer to d_r as the minimum duration associated with route r.

Figure 3.1 presents an example to illustrate the three parameters, e_r, l_r and d_r, defined above for a route $r = (0, 1, 2, 0)$, where service times are $st_0 = st_1 = st_2 = 0$. Consider a vehicle performing this route and departing from the depot at time t. The time window constraints must be violated if $t > 10$, because the vehicle cannot arrive at customer 1 no later than $b_1 = 20$. If $t \in [5, 10]$, it can be seen that the vehicle must start serving customer 1 at time $(t + 10)$, start serving customer 2 at time $(t + 20)$, and return to the depot at time $(t + 30)$. If $t \in [0, 5)$, it can also be seen that the vehicle must start serving customer 1 at time $(t + 10)$, start serving customer 2 at time 25, and return to the depot at time 35. Therefore, by definition, we have that $e_r = 5$, $l_r = 10$ and $d_r = 30$ for this route r.

Following Paradiso et al. [35] we can obtain a route-based IP model for the CMTVRPTW. For this, we use $\mathbb{R}$ to denote the set of all feasible routes, c_r to denote the cost of each route $r \in \mathbb{R}$, and $\alpha_{ir} = \mathbb{I}_{i \in \mathbb{V}(r)}$ to denote a binary parameter indicating whether or not customer i is visited by route r for $i \in \mathbb{V}$. For a subset $\bar{\mathbb{R}}$ of $\mathbb{R}$, let $\zeta(K, \bar{\mathbb{R}})$ denote the maximum number of routes in $\bar{\mathbb{R}}$ that can be performed by the K vehicles. Define x_r as a binary variable indicating whether or not route r is selected. The route-based IP model for the CMTVRPTW (referred to as model RP in short) can thus be described as follows:

$$\text{(RP)} \quad z_{\text{RP}} = \min \sum_{r \in \mathbb{R}} c_r x_r, \tag{3.1}$$

$$\text{s.t.} \quad \sum_{r\in\mathbb{R}} \alpha_{ir} x_r = 1, \qquad \forall\, i \in \mathbb{V}, \tag{3.2}$$

$$\sum_{r\in\bar{\mathbb{R}}} x_r \leq \zeta(K, \bar{\mathbb{R}}), \qquad \forall\, \bar{\mathbb{R}} \subseteq \mathbb{R}, \tag{3.3}$$

$$x_r \in \{0, 1\}, \qquad \forall\, r \in \mathbb{R}. \tag{3.4}$$

In model RP, the objective (3.1) is to minimize the total travel cost. Constraints (3.2) ensure that each customer is visited exactly once. Constraints (3.3) state that the number of selected routes in $\bar{\mathbb{R}}$ is less than or equal to the maximum number of routes in $\bar{\mathbb{R}}$ that can be performed by the K vehicles. The domain of variables are specified by constraints (3.4). We use LRP to denote the LP relaxation of model RP. Moreover, define $\bar{\mathbb{R}}_{\mathbf{x}} = \{r \in \mathbb{R} : x_r = 1\}$ for each $\mathbf{x}$ satisfying constraints (3.2) and (3.4). It can be seen that $\sum_{r\in\bar{\mathbb{R}}_{\mathbf{x}}} x_r = |\bar{\mathbb{R}}_{\mathbf{x}}|$, $\mathbf{x}$ is feasible to model RP if $|\bar{\mathbb{R}}_{\mathbf{x}}| = \zeta(K, \bar{\mathbb{R}}_{\mathbf{x}})$, and it is infeasible if $|\bar{\mathbb{R}}_{\mathbf{x}}| > \zeta(K, \bar{\mathbb{R}}_{\mathbf{x}})$.

In the literature, Paradiso et al. [35] and Yang [46] applied the RSF and SR3 constraints to strengthen the LP relaxation LRP of model RP. One can further enhance LRP by imposing some other valid inequalities. As illustrated in the remainder of this section, we derive a new class of valid inequalities, named RWT constraints, which strengthen the RSF constraints. We then incorporate these new valid inequalities to strengthen the LRP, together with several other valid inequalities that have not been applied in the studies of the CMTVRPTW, such as the SR5-2, SR5-3, EL, and SRC constraints.

3.1.1 Relaxed Structure Feasibility (RSF) Constraints

The RSF constraints, which were proposed by Paradiso et al. [35], form a relaxation of the constraints that prohibit more than K routes being performed at the same time. Let β_{tr} be a binary parameter to indicate whether route r is always performed at time t for all its feasible departure times from the depot in interval $[a_0, l_r]$. For route $r \in \mathbb{R}$, since l_r is the latest departure time from the depot and $e_r + d_r$ is the earliest return time to the depot, we obtain that $\beta_{tr} = 1$ if and only if $l_r \leq t < e_r + d_r$. Accordingly, the RSF constraints can be represented as follows.

$$\text{(RSF)} \qquad \sum_{r \in \mathbb{R}} \beta_{tr} x_r \leq K, \qquad \forall\ t \in [a_0, b_0]. \tag{3.5}$$

Since $[a_0, b_0]$ is a continuous-time interval, there are an infinite number of RSF constraints in (3.5). Incorporating all of them in model RP makes the model not tractable. To address this issue, we replace $[a_0, b_0]$ in constraints (3.5) with a discrete set $\mathbb{T}$, and strengthen model RP by incorporating only those RSF constraints in $\mathbb{T}$. In a cutting-plane procedure, one can initialize $\mathbb{T}$ with an empty set, and iteratively solves model LRP, identify the violated RSF constraints not in $\mathbb{T}$, and add them into $\mathbb{T}$.

Moreover, given a solution $\mathbf{x}$ of LRP, let $\mathbb{R}_{\mathbf{x}} = \{r \in \mathbb{R} : x_r > 0\}$ denote the set of routes with positive associated variable values. We have $\sum_{r \in \mathbb{R}} \beta_{tr} x_r = \sum_{r \in \mathbb{R}_{\mathbf{x}}} \beta_{tr} x_r$. From the definition of parameter β_{tr} we know that the value of $\sum_{r \in \mathbb{R}_{\mathbf{x}}} \beta_{tr} x_r$ can increase only at $t \in \{l_r : r \in \mathbb{R}_{\mathbf{x}}\}$ and can decrease only at $t \in \{e_r + d_r : r \in \mathbb{R}_{\mathbf{x}}\}$. Thus, $\sum_{r \in \mathbb{R}_{\mathbf{x}}} \beta_{tr} x_r$ is a step function of t. Therefore, to identify RSF constraints that are violated by $\mathbf{x}$, we only need to check inequalities in (3.5) with $t \in \{l_r : r \in \mathbb{R}_{\mathbf{x}}\}$.

3.1.2 Relaxed Working Time (RWT) Constraints

We now derive a new class of valid inequalities, named RWT constraints, for the CMTVRPTW, which strengthen the RSF constraints proposed by Paradiso et al. [35]. Consider any two time points t_1 and t_2 with $t_1 < t_2$, and any route $r \in \mathbb{R}$. Assume that $\sup(\emptyset) = \inf(\emptyset) = 0$. We define a function $F_{t_1t_2r}(\tau)$ as follows:

$$F_{t_1t_2r}(\tau) = \sup([\tau, \tau + d_r] \cap [t_1, t_2]) - \inf([\tau, \tau + d_r] \cap [t_1, t_2]). \tag{3.6}$$

When $\tau \in [e_r, l_r]$, the working time interval for a vehicle to performe route r is $[\tau, \tau + d_r]$ (i.e., such vehicle departs from the depot at time τ and returns to the depot at time $\tau + d_r$), then $F_{t_1t_2r}(\tau)$ is the length of the intersection of $[t_1, t_2]$ and working time interval $[\tau, \tau + d_r]$. When a vehicle performing route r departs from the depot at time $\tau \in [a_0, e_r)$, the working time interval is $[\tau, e_r + d_r]$, then the length of the intersection of $[t_1, t_2]$ and working time interval $[\tau, e_r + d_r]$ is greater than $F_{t_1t_2r}(e_r)$. Therefore, the $\bar{\beta}_{t_1t_2r}$ defined below is a lower bound on the length of the intersection of $[t_1, t_2]$ and the working time interval for a vehicle to perform route r.

$$\bar{\beta}_{t_1t_2r} = \min_{\tau \in [e_r, l_r]} F_{t_1t_2r}(\tau). \tag{3.7}$$

To illustrate the function $F_{t_1t_2r}(\tau)$, consider a route r where, $e_r = 2$, $l_r = 4$ and $d_r = 2$ and assume that $t_1 = 3$, $t_2 = 6$. Form (3.6) we have that $F_{t_1t_2r}(\tau) = \tau - 1$ for $\tau \in [2, 3]$ and $F_{t_1t_2r}(\tau) = 2$ for $\tau \in (3, 4]$. Thus, by (3.7), we obtain that $\bar{\beta}_{t_1t_2r} = \min_{\tau \in [2,4]} F_{t_1t_2r}(\tau) = F_{t_1t_2r}(2) = 1$. This indicates that within the time interval $[3, 6]$, a vehicle performing route r is in working state at least for a time duration of one unit.

Moreover, as illustrated in Figure 3.2, if $t_2 - t_1 \geq d_r$, we can see that $F_{t_1t_2r}(\tau)$

is a piecewise linear function and can be represented as follows:

$$F_{t_1t_2r}(\tau) = \begin{cases} 0, & \text{if } \tau \leq t_1 - d_r, \\ \tau + d_r - t_1, & \text{if } t_1 - d_r < \tau \leq t_1, \\ d_r, & \text{if } t_1 < \tau \leq t_2 - d_r, \\ t_2 - \tau, & \text{if } t_2 - d_r < \tau \leq t_2, \\ 0, & \text{if } \tau > t_2, \end{cases} \tag{3.8}$$

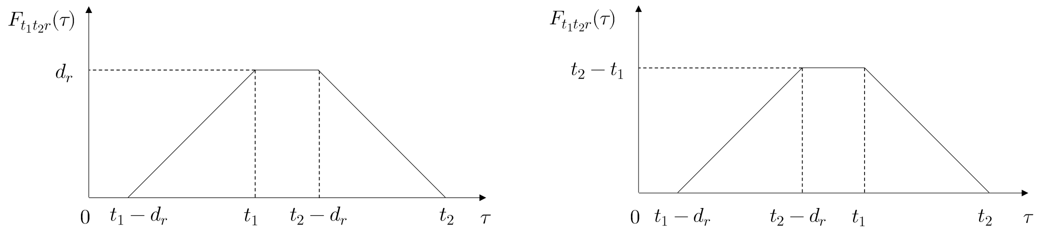
and otherwise, $t_2 - t_1 < d_r$, we can see that $F_{t_1t_2r}(\tau)$ is also a piecewise linear function and can be represented as follows:

$$F_{t_1t_2r}(\tau) = \begin{cases} 0, & \text{if } \tau \leq t_1 - d_r, \\ \tau + d_r - t_1, & \text{if } t_1 - d_r < \tau \leq t_2 - d_r, \\ t_2 - t_1, & \text{if } t_2 - d_r < \tau \leq t_1, \\ t_2 - \tau, & \text{if } t_1 < \tau \leq t_2, \\ 0, & \text{if } \tau > t_2. \end{cases} \tag{3.9}$$

It can thus be verified that $F_{t_1t_2r}(\tau)$ is a quasiconcave function. This, together with (3.7), implies that

$$\bar{\beta}_{t_1t_2r} = \min_{\tau \in [e_r, l_r]} F_{t_1t_2r}(\tau) = \min\{F_{t_1t_2r}(e_r), F_{t_1t_2r}(l_r)\}. \tag{3.10}$$

Moreover, we have $F_{t_1t_2r}(e_r) = \min\{d_r, t_2 - t_1, (e_r + d_r - t_1)^+\}$ and $F_{t_1t_2r}(l_r) = \min\{d_r, t_2 - t_1, (t_2 - l_r)^+\}$. Furthermore, the value of $\bar{\beta}_{t_1t_2r}$ can be computed by equation (3.11) below, which can be derived based on the definitions of t_1, t_2, e_r,



(a) the case for $t_2 - t_1 \geq d_r$ (b) the case for $t_2 - t_1 < d_r$

Figure 3.2: Illustration of function $F_{t_1t_2r}(\tau)$.

l_r, $e_r + d_r$ and $l_r + d_r$ and their relationships:

$$\bar{\beta}_{t_1t_2r} = \begin{cases} t_2 - l_r, & \text{if } t_1 \leq l_r \leq t_2 \leq e_r + d_r, \\ \min\{e_r + d_r - t_1, t_2 - l_r, d_r\}, & \text{if } t_1 \leq l_r \leq e_r + d_r \leq t_2, \\ t_2 - t_1, & \text{if } l_r \leq t_1 \leq t_2 \leq e_r + d_r, \\ e_r + d_r - t_1, & \text{if } l_r \leq t_1 \leq e_r + d_r \leq t_2, \\ (t_2 - l_r)^+, & \text{if } t_1 \leq e_r \leq t_2 \leq l_r + d_r, l_r > e_r + d_r, \\ d_r, & \text{if } t_1 \leq e_r \leq l_r + d_r \leq t_2, l_r > e_r + d_r, \\ \min\left\{(e_r + d_r - t_1)^+, (t_2 - l_r)^+\right\}, & \text{if } e_r \leq t_1 \leq t_2 \leq l_r + d_r, l_r > e_r + d_r, \\ (e_r + d_r - t_1)^+, & \text{if } e_r \leq t_1 \leq l_r + d_r \leq t_2, l_r > e_r + d_r, \\ 0, & \text{otherwise.} \end{cases} \tag{3.11}$$

Consider any feasible solution, for which we know that $\sum_{r \in \mathbb{R}} \bar{\beta}_{t_1t_2r} x_r$ is a lower bound on the total length of durations performed by selected routes in time interval $[t_1, t_2]$. Since there are K vehicles available, this lower bound cannot exceed $K(t_2 - t_1)$. We thus obtain the following valid inequalities to model RP, which is referred to as the RWT constraints:

$$\text{(RWT)} \qquad \sum_{r\in\mathbb{R}} \bar{\beta}_{t_1t_2r}x_r \leq K(t_2 - t_1), \qquad \forall\, t_1, t_2 \in [a_0, b_0], t_1 < t_2. \tag{3.12}$$

Like the RSF constraints, the RWT constraints newly introduced above are also a relaxation of the constraints that prohibit more than K routes from being performed at the same time.

We now show as follows that the RWT constraints strengthen the RSF constraints. For any route r and time point t, by definitions of β and $\bar{\beta}$, if $\beta_{tr} = 1$, then there exists a positive number $\epsilon_{tr} > 0$ such that a vehicle performing route r is in working state during $[t, t + \epsilon_{tr})$ for all its feasible departure times from the depot, which implies that $\bar{\beta}_{t,t+\epsilon_{tr},r} = \epsilon_{tr}$. If $\beta_{tr} = 0$, then there exists a positive number $\epsilon_{tr} > 0$ such that a vehicle performing route r is not in working state during $[t, t + \epsilon_{tr})$ for some of its feasible departure times from the depot, which implies that $\bar{\beta}_{t,t+\epsilon_{tr},r} = 0$. From this we obtain that $\bar{\beta}_{t,t+\epsilon,r} = \epsilon\beta_{tr}$ for all $\epsilon \in (0, \epsilon_{tr}]$. Hence, for any given t and for $\epsilon = \min\{\epsilon_{tr} : r \in \mathbb{R}\} > 0$, we have that

$$\sum_{r\in\mathbb{R}} \beta_{tr}x_r > K \Rightarrow \sum_{r\in\mathbb{R}} \epsilon\beta_{tr}x_r > K\epsilon \Rightarrow \sum_{r\in\mathbb{R}} \bar{\beta}_{t,t+\epsilon,r}x_r > K(t + \epsilon - t). \tag{3.13}$$

This implies that if an RSF constraint is violated, there is an RWT constraint that is also violated. Therefore, the RSF constraints form a relaxation of the RWT constraints.

Moreover, there exist situations where all RSF constraints are satisfied but some RWT constraints are violated. To see this, consider the example shown in Figure 3.3 with three routes, where $e_1 = 0$, $l_1 = 1$, $e_2 = 1$, $l_2 = 2$, $e_3 = 2$, $l_3 = 3$, $d_1 = d_2 = d_3 = 2$, $K = 1$, $x_1 = x_2 = x_3 = 1$. On the one hand, since $\sum_{r=1}^{3} \beta_{tr}x_r = 1$ for $t \in [1, 4)$ and $\sum_{r=1}^{3} \beta_{tr}x_r = 0$ for every other value of t, all

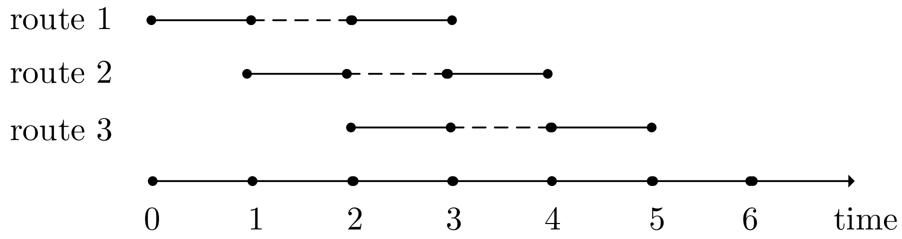


Figure 3.3: An infeasible solution that satisfies all the RSF constraints but violates a RWT constraint, where time intervals $[l_r, e_r + d_r]$ for routes $r = 1, 2, 3$ are shown in dashed lines.

RSF constraints are satisfied. On the other hand, for $t_1 = 0$ and $t_2 = 4$, since $\sum_{r=1}^{3} \bar{\beta}_{0,4,r} x_r = 2 + 2 + 1 = 5 > 1 \times (4 - 0)$, the corresponding RWT constraint is violated.

Therefore, RWT constraints strengthen the RSF constraints, so that they can be applied to improve the lower bound further. Given a solution $\mathbf{x}$ of LRP, we define $\mathbb{T}_{\mathbf{x}}^{\text{RTW}} = \{e_r, l_r, e_r + d_r, l_r + d_r : r \in \mathbb{R}_{\mathbf{x}}\}$. To shorten the running time spent in solving the separation problem for the RWT constraints, one may only check violations of the RWT constraints for some selected pairs of t_1 and t_2 in $\{(t_1, t_2) : t_1, t_2 \in \mathbb{T}_{\mathbf{x}}^{\text{RTW}}, t_1 < t_2\}$.

3.1.3 Subset-Row (SR3, SR5-2, SR5-3) Constraints

The subset-row constraints, introduced by Jepsen et al. [28], are widely applied to the VRP and some of its variants. However, its application to the CMTVRPTW is limited in the existing studies [35, 46]. Following Jepsen et al. [28], the subset-row constraints for the CMTVRPTW can be represented as follows (where $\mathbb{Z}_{++}$ is the set of positive integers):

$$\sum_{r \in \mathbb{R}} \left\lfloor \sum_{i \in \bar{\mathbb{V}}} \alpha_{ir}/k \right\rfloor x_r \leq \left\lfloor |\bar{\mathbb{V}}|/k \right\rfloor, \qquad \forall\, \bar{\mathbb{V}} \subseteq \mathbb{V}, k \in \mathbb{Z}_{++}. \tag{3.14}$$

To derive such constraints, we note that $k \sum_{r \in \mathbb{R}} \left\lfloor \sum_{i \in \bar{\mathbb{V}}} \alpha_{ir}/k \right\rfloor x_r \leq \sum_{r \in \mathbb{R}} \sum_{i \in \bar{\mathbb{V}}} \alpha_{ir} x_r = \sum_{i \in \bar{\mathbb{V}}} \sum_{r \in \mathbb{R}} \alpha_{ir} x_r = |\bar{\mathbb{V}}|$. Thus $\sum_{r \in \mathbb{R}} \left\lfloor \sum_{i \in \bar{\mathbb{V}}} \alpha_{ir}/k \right\rfloor x_r \leq |\bar{\mathbb{V}}|/k$. Due to constraints (3.4), $\sum_{r \in \mathbb{R}} \left\lfloor \sum_{i \in \bar{\mathbb{V}}} \alpha_{ir}/k \right\rfloor x_r$ is an integer and we have $\sum_{r \in \mathbb{R}} \left\lfloor \sum_{i \in \bar{\mathbb{V}}} \alpha_{ir}/k \right\rfloor x_r \leq \left\lfloor |\bar{\mathbb{V}}|/k \right\rfloor$.

Given any customer set $\bar{\mathbb{V}} \subseteq \mathbb{V}$, and any positive integer k, let us define an integer parameter $\eta_{r\bar{\mathbb{V}}k} = \left\lfloor \sum_{i \in \bar{\mathbb{V}}} \alpha_{ir}/k \right\rfloor$. Accordingly, the SR3, SR5-2 and SR5-3 constraints, which are special cases of the subset-row constraints, are shown as follows:

$$\text{(SR3)} \qquad \sum_{r \in \mathbb{R}} \eta_{r\bar{\mathbb{V}}2} x_r \leq 1, \qquad \forall\, \bar{\mathbb{V}} \subseteq \mathbb{V}, |\bar{\mathbb{V}}| = 3, \tag{3.15}$$

$$\text{(SR5-2)} \qquad \sum_{r \in \mathbb{R}} \eta_{r\bar{\mathbb{V}}2} x_r \leq 2, \qquad \forall\, \bar{\mathbb{V}} \subseteq \mathbb{V}, |\bar{\mathbb{V}}| = 5, \tag{3.16}$$

$$\text{(SR5-3)} \qquad \sum_{r \in \mathbb{R}} \eta_{r\bar{\mathbb{V}}3} x_r \leq 1, \qquad \forall\, \bar{\mathbb{V}} \subseteq \mathbb{V}, |\bar{\mathbb{V}}| = 5. \tag{3.17}$$

It can be seen that the number of SR3 constraints is proportional to $|\mathbb{V}|^3$, and their separation problems can be solved efficiently by a direct enumeration, as shown in [35, 46] for the CMTVRPTW. The SR5-2 and SR5-3 constraints have not been applied to the CMTVRPTW in the literature, and the numbers of them are proportional to $|\mathbb{V}|^5$. As a result, separating them by direct enumeration is very time-consuming. For other VRP problems, such as the VRPTW, only some local search heuristics are developed to solve the separation problems for the SR5-2 and SR5-3 constraints [36].

In our study, we apply the SR5-2 and SR5-3 constraints to the CMTVRPTW for the first time in the literature. To solve their separation problems, we first select only some subsets of five customers to verify. If all such SR5-2 or SR5-3 constraints are satisfied, then we propose to solve the IP model SR5-2 below for the SR5-2 constraints and the IP model SR5-3 below for the SR5-3 constraints, respectively, where $\mathbf{x}$ is a given feasible solution to LRP, $\mathbb{R}_{\mathbf{x}} = \{r \in \mathbb{R} : x_r > 0\}$ is a set of routes r with $x_r > 0$ in the given feasible solution to LRP, and $\mathbb{Z}_+$ is the set of all non-negative integers.

$$
\begin{aligned}
\text{(SR5-2)} \qquad & \max z_{\text{SR5-2}}(\mathbf{u}, \mathbf{v}) = \sum_{r \in \mathbb{R}_{\mathbf{x}}} x_r u_r - 2, && && (3.18)\\
& \text{s.t.} \qquad u_r - \frac{1}{2} \sum_{i \in \mathbb{V}} \alpha_{ir} v_i \leq 0, && \forall\, r \in \mathbb{R}_{\mathbf{x}}, && (3.19)\\
& \sum_{i \in \mathbb{V}} v_i = 5, && && (3.20)\\
& u_r \in \mathbb{Z}_+, && \forall\, r \in \mathbb{R}_{\mathbf{x}}, && (3.21)\\
& v_i \in \{0, 1\}, && \forall\, i \in \mathbb{V}. && (3.22)
\end{aligned}
$$

Here, for each route $r \in \mathbb{R}_{\mathbf{x}}$, constraints (3.19) and (3.21) mean that u_r is an integer variable equal to the value of $\eta_{r\bar{\mathbb{V}}2}$. For each $i \in \mathbb{V}$, constraint (3.22) means that v_i is a binary variable indicating whether customer i is in $\bar{\mathbb{V}}$. Constraint (3.20) requires that $|\bar{\mathbb{V}}| = 5$ hold true. The objective function (3.18) is to maximize the violation of a SR5-2 constraint. Specifically, if there exists a feasible solution $(\mathbf{u}, \mathbf{v})$ that satisfies $z_{\text{SR5-2}}(\mathbf{u}, \mathbf{v}) > 0$, then the SR5-2 constraint defined on the set $\bar{\mathbb{V}} = \{i \in \mathbb{V} : v_i = 1\}$ is violated.

$$
\begin{aligned}
\text{(SR5-3)} \qquad & \max z_{\text{SR5-3}}(\mathbf{u}, \mathbf{v}) = \sum_{r \in \mathbb{R}_{\mathbf{x}}} x_r u_r - 1, && && (3.23)\\
& \text{s.t.} \qquad u_r - \frac{1}{3} \sum_{i \in \mathbb{V}} \alpha_{ir} v_i \leq 0, && \forall\, r \in \mathbb{R}_{\mathbf{x}}, && (3.24)
\end{aligned}
$$

$$\sum_{i \in \mathbb{V}} v_i = 5, \tag{3.25}$$

$$u_r \in \mathbb{Z}_+, \qquad \forall\, r \in \mathbb{R}_{\mathbf{x}}, \tag{3.26}$$

$$v_i \in \{0, 1\}, \qquad \forall\, i \in \mathbb{V}. \tag{3.27}$$

Here, for each route $r \in \mathbb{R}_{\mathbf{x}}$, constraints (3.24) and (3.26) mean that u_r is an integer variable equal to the value of $\eta_{r\bar{\mathbb{V}}3}$. For each $i \in \mathbb{V}$, constraint (3.27) means that v_i is a binary variable indicating whether customer i is in $\bar{\mathbb{V}}$. Constraint (3.25) requires that $|\bar{\mathbb{V}}| = 5$ hold true. The objective function (3.23) is to maximize the violation of a SR5-3 constraint. Specifically, if there exists a feasible solution $(\mathbf{u}, \mathbf{v})$ that satisfies $z_{\text{SR5-3}}(\mathbf{u}, \mathbf{v}) > 0$, then the SR5-3 constraint defined on the set $\bar{\mathbb{V}} = \{i \in \mathbb{V} : v_i = 1\}$ is violated. We can set a time limit (e.g., 0.5 second in this study) when applying an optimization solver to solve models SR5-2 and SR5-3.

3.1.4 Elementary (EL) Constraints

The EL constraints, introduced by Pecin et al. [36], are also known to be effective in solving the VRP and some of its variants. However, they have not been applied to the CMTVRPTW in the existing studies. For any route $r \in \mathbb{R}$, any customer $i \in \mathbb{V}$, and any nonempty customer set $\bar{\mathbb{V}} \subseteq \mathbb{V} \setminus \{i\}$, let us define a binary parameter $\lambda_{ir\bar{\mathbb{V}}}$ as follows:

$$\lambda_{ir\bar{\mathbb{V}}} = \left\lfloor \frac{|\bar{\mathbb{V}}| - 1}{|\bar{\mathbb{V}}|} \alpha_{ir} + \sum_{j \in \bar{\mathbb{V}}} \frac{1}{|\bar{\mathbb{V}}|} \alpha_{jr} \right\rfloor. \tag{3.28}$$

By following the literature [36], the EL constraints for the CMTVRPTW can be represented as follows:

$$\sum_{r\in\mathbb{R}} \lambda_{ir\bar{\mathbb{V}}} x_r \leq 1, \qquad \forall\, i \in \mathbb{V}, \bar{\mathbb{V}} \subseteq \mathbb{V} \setminus \{i\}, \bar{\mathbb{V}} \neq \emptyset. \tag{3.29}$$

For any customer i and given nonempty customer subset $\bar{\mathbb{V}} \subseteq \mathbb{V} \setminus \{i\}$, the EL constraint in (3.29) states that the number of routes r that visit all the customers in $\bar{\mathbb{V}}$ or visit customer i together with at least one customer in $\bar{\mathbb{V}}$ cannot be greater than one. Given a fractional solution $\mathbf{x}$ of the LP relaxation LRP of model RP, to solve the separation problems of the EL constraints for the CMTVRPTW efficiently, we only consider the EL constraints for $\bar{\mathbb{V}} \in \{\mathbb{V}(r) : r \in \mathbb{R}_{\mathbf{x}}\}$ in our study.

3.1.5 Strengthened Rounded Capacity (SRC) Constraints

The SRC constraints, which were introduced by Baldacci et al. [4] for the VRP, have not been applied to the CMTVRPTW in the existing studies. For any route $r \in \mathbb{R}$ and any customer set $\bar{\mathbb{V}} \subseteq \mathbb{V}$, define a binary parameter $\mu_{r\bar{\mathbb{V}}} = \mathbb{I}_{\mathbb{V}(r)\cap\bar{\mathbb{V}}\neq\emptyset}$ to indicate whether or not route r visits at least one customer in set $\bar{\mathbb{V}}$. Following Baldacci et al. [4], the SRC constraints for the CMTVRPTW can be represented as follows:

$$\sum_{r\in\mathbb{R}} \mu_{r\bar{\mathbb{V}}} x_r \geq \left\lceil \sum_{i\in\bar{\mathbb{V}}} q_i/Q \right\rceil, \qquad \forall\, \bar{\mathbb{V}} \subseteq \mathbb{V}. \tag{3.30}$$

It states that the number of routes that serve customers of set $\bar{\mathbb{V}}$ cannot be fewer than $\lceil \sum_{i\in\bar{\mathbb{V}}} q_i/Q \rceil$, which is due to the capacity limit of each vehicle.

In the literature, a software package, named CVRPSEP and developed by Lysgaard [32] is widely used to solve the separation problem of the SRC constraints for the VRP and its variations by heuristics. In our study, we solve the separation problem of the SRC constraints for the CMTVRPTW to optimality, so that more violated SRC constraints can be obtained. To achieve this, we propose to solve the IP model SRC below, where $\mathbf{x}$ is a given feasible solution to LRP and $\mathbb{R}_{\mathbf{x}} = \{r \in \mathbb{R} : x_r > 0\}$ is a set of routes r with $x_r > 0$ in the given feasible solution to LRP.

$$\text{(SRC)} \qquad \min z_{\text{SRC}}(\mathbf{u}, \mathbf{v}, w) = \sum_{r \in \mathbb{R}_{\mathbf{x}}} x_r u_r - w, \tag{3.31}$$

$$\text{s.t.} \qquad w - \sum_{i \in \mathbb{V}} q_i v_i / Q \geq 0, \tag{3.32}$$

$$w - \sum_{i \in \mathbb{V}} q_i v_i / Q \leq 1 - \epsilon, \tag{3.33}$$

$$u_r - \alpha_{ir} v_i \geq 0, \qquad \forall\, i \in \mathbb{V}, r \in \mathbb{R}_{\mathbf{x}}, \tag{3.34}$$

$$u_r \in \{0, 1\}, \qquad \forall\, r \in \mathbb{R}_{\mathbf{x}}, \tag{3.35}$$

$$v_i \in \{0, 1\}, \qquad \forall\, i \in \mathbb{V}, \tag{3.36}$$

$$w \in \mathbb{Z}_+. \tag{3.37}$$

Here, for each $i \in \mathbb{V}$, v_i is a binary variable indicating whether customer i is in set $\bar{\mathbb{V}}$. For each route $r \in \mathbb{R}_{\mathbf{x}}$, constraints (3.34) and (3.35) mean that u_r is a binary variable indicating whether $\mu_{r\bar{\mathbb{V}}} = 1$. Constraints (3.32), (3.33) and (3.37) mean that w is an integer variable denoting the value of $\lceil \sum_{i \in \bar{\mathbb{V}}} q_i / Q \rceil$. The parameter ϵ is a very small positive number, which equals 10^{-6} in our study, so as to ensure $w = \lceil \sum_{i \in \bar{\mathbb{V}}} q_i / Q \rceil$. The objective function (3.31) is the violation of a SRC constraint. Specifically, if model SRC above has a feasible solution $(\mathbf{u}, \mathbf{v}, w)$ that satisfies $z_{\text{SRC}}(\mathbf{u}, \mathbf{v}, w) < 0$, then the SRC constraint for $\bar{\mathbb{V}} = \{i \in \mathbb{V} : v_i = 1\}$ must be violated. We can also set a time limit (e.g., 0.5 second in our study) when applying an IP solver to solve model SRC.

3.2 Trip-Based IP Model and Its Valid Inequalities

We define a trip $s = (\tau, r)$ as a time-route pair where $r = (0, i_1, i_2, \ldots, i_{n_r}, 0) \in \mathbb{R}$ is a feasible route and $\tau \in [a_0, l_r]$ is a feasible departure time from the depot of route r. Such a representation of a trip indicates that a vehicle departs from the depot at time τ, serves customers $i_1, i_2, \ldots, i_{n_r}$ along the route r, and then returns to the depot. As mentioned in Section 3.1, since unnecessary waiting time for customers cannot lead to better solutions, when visiting a customer, each vehicle can be assumed to start its service and then leave for the next node as early as possible. Therefore, the arrival time for each customer along the route can be determined by the departure time τ from the depot. Moreover, it can be seen that the total cost of the trip $s = (\tau, r)$, denoted by c_s, is the same as the total cost c_r of the associated route r.

Consider any solution to CMTVRPTW. If it contains a trip (τ, r) with $\tau \in [a_0, e_r)$, then the trip (τ, r), which departs from the depot earlier than e_r, can be replaced by another trip (e_r, r), which departs from the depot at time e_r, without impairing the feasibility and total cost of the solution. This is followed by our argument in Section 3.1, which implies that vehicles perform such two trips during $[\tau, e_r + d_r)$ and $[e_r, e_r + d_r)$ respectively, so that the latter trip dominates the former. Therefore, without loss of generality, we require every trip (τ, r) satisfying $\tau \in [e_r, l_r]$ and refer to $[e_r, l_r]$ as the interval of its non-dominated departure times from the depot.

Hernandez et al. [23] proposed a trip-based IP model for the CMTVRPTW-LD, which can also be applied to formulate the CMTVRPTW as follows. Let $\mathbb{S}$ denote the set of all trips. For each trip $s \in \mathbb{S}$, let r_s denote its associated route,

$\tau_s \in [e_{r_s}, l_{r_s}]$ denote its associated departure time from the depot, and c_s denote its associated cost. For each customer $i \in \mathbb{V}$ and trip $s \in \mathbb{S}$, let $\alpha_{is} = \mathbb{I}_{i \in \mathbb{V}(r_s)}$ denote a binary parameter indicating whether customer i is visited by trip s. Let γ_{ts} denote a binary parameter indicating whether trip s is performed at time t. This implies that $\gamma_{ts} = 1$ if and only if $\tau_s \leq t < \tau_s + d_{r_s}$. Define y_s as a binary variable indicating whether trip s appears in the solution. We can formulate the CMTVRPTW as the following trip-based IP model.

$$\text{(SP)} \qquad z_{\text{SP}} = \min \sum_{s \in \mathbb{S}} c_s y_s, \tag{3.38}$$

$$\text{s.t.} \qquad \sum_{s \in \mathbb{S}} \alpha_{is} y_s = 1, \qquad \forall\, i \in \mathbb{V}, \tag{3.39}$$

$$\sum_{s \in \mathbb{S}} \gamma_{ts} y_s \leq K, \qquad \forall\, t \in [a_0, b_0], \tag{3.40}$$

$$y_s \in \{0, 1\}, \qquad \forall\, s \in \mathbb{S}. \tag{3.41}$$

In model SP, the objective (3.38) is to minimize the total travel cost. Constraints (3.39) ensure that each customer must be visited exactly once. Constraints (3.40), which are referred to as working time (WT) constraints, state that at most K trips can be performed at any time. Constraints (3.41) specify the domain of variables. We denote the LP relaxation of model SP as LSP.

Since $[a_0, b_0]$ is a continuous time interval, there are an infinite number of WT constraints in (3.40). Incorporating all of them in model SP makes the model not tractable. To address this issue, we replace $[a_0, b_0]$ in (3.40) with a discrete set $\mathbb{T}$, as shown below.

$$\sum_{s \in \mathbb{S}} \gamma_{ts} y_s \leq K, \qquad \forall\, t \in \mathbb{T}. \tag{3.42}$$

From this, we obtain a relaxation of model SP, denoted by SP($\mathbb{T}$). We can then strengthen such a relaxation of model SP through a cutting-plane procedure, which initializes $\mathbb{T}$ with an empty set, iteratively solves and strengthens the relaxation SP($\mathbb{T}$) of model SP, and identifies violated WT constraints not in $\mathbb{T}$, and add them into $\mathbb{T}$. This process ends when the optimal solution of the relaxation SP($\mathbb{T}$) is feasible to model SP, i.e., the WT constraints are satisfied for all $t \in [a_0, b_0]$.

Given a solution $\mathbf{y}$ of relaxation LSP or relaxation SP($\mathbb{T}$), let $\mathbb{S}_{\mathbf{y}} = \{s \in \mathbb{S} : y_s > 0\}$ denote the set of trips with positive variable values. We have that $\sum_{s\in\mathbb{S}} \gamma_{ts} y_s = \sum_{s\in\mathbb{S}_{\mathbf{y}}} \gamma_{ts} y_s$. From the definition of parameter γ_{ts} we know that the value of $\sum_{s\in\mathbb{S}_{\mathbf{y}}} \gamma_{ts} y_s$ can increase only at $t \in \{\tau_s : s \in \mathbb{S}_{\mathbf{y}}\}$ and can decrease only at $t \in \{\tau_s + d_{r_s} : s \in \mathbb{S}_{\mathbf{y}}\}$. Thus, $\sum_{s\in\mathbb{S}_{\mathbf{y}}} \gamma_{ts} y_s$ is a step function of t. Therefore, to identify WT constraints that are violated by $\mathbf{y}$, we only need to check inequalities in (3.40) with $t \in \{\tau_s : s \in \mathbb{S}_{\mathbf{y}}\}$.

Moreover, by definitions of the SR3, SR5-2, SR5-3, EL and SRC constraints, it can be seen that these constraints can also be applied to strengthen the LP relaxation of the trip-based model. Let us define parameters $\eta_{s\bar{\mathbb{V}}k} = \eta_{r_s\bar{\mathbb{V}}k}$, $\lambda_{is\bar{\mathbb{V}}} = \lambda_{ir_s\bar{\mathbb{V}}}$ and $\mu_{s\bar{\mathbb{V}}} = \mu_{r_s\bar{\mathbb{V}}}$, then following the inequalities introduced in Section 3.42 for the route-based IP model, we can obtain the following valid inequalities for the trip-based IP model SP.

$$\text{(SR3)} \qquad \sum_{s\in\mathbb{S}} \eta_{s\bar{\mathbb{V}}2} y_s \leq 1, \qquad \forall\, \bar{\mathbb{V}} \subseteq \mathbb{V}, |\bar{\mathbb{V}}| = 3, \tag{3.43}$$

$$\text{(SR5-2)} \qquad \sum_{s\in\mathbb{S}} \eta_{s\bar{\mathbb{V}}2} y_s \leq 2, \qquad \forall\, \bar{\mathbb{V}} \subseteq \mathbb{V}, |\bar{\mathbb{V}}| = 5, \tag{3.44}$$

$$\text{(SR5-3)} \qquad \sum_{s\in\mathbb{S}} \eta_{s\bar{\mathbb{V}}3} y_s \leq 1, \qquad \forall\, \bar{\mathbb{V}} \subseteq \mathbb{V}, |\bar{\mathbb{V}}| = 5, \tag{3.45}$$

$$\text{(EL)} \qquad \sum_{s\in\mathbb{S}} \lambda_{is\bar{\mathbb{V}}} y_s \leq 1, \qquad \forall\, i \in \mathbb{V}, \bar{\mathbb{V}} \subseteq \mathbb{V} \setminus \{i\}, \bar{\mathbb{V}} \neq \emptyset, \tag{3.46}$$

$$\text{(SRC)} \qquad \sum_{s\in\mathbb{S}} \mu_{s\bar{\mathbb{V}}} y_s \geq \left\lceil \sum_{i\in\bar{\mathbb{V}}} q_i/Q \right\rceil, \qquad \forall\, \bar{\mathbb{V}} \subseteq \mathbb{V}. \tag{3.47}$$

To separate these valid inequalities of model SP, we can apply the methods similar to those developed for separating valid inequalities of model RP in Section 3.1 with notations $\mathbb{R}_{\mathbf{x}}$, x_r, α_{ir}, and u_r replaced by $\mathbb{S}_{\mathbf{y}}$, y_s, α_{is}, and u_s.

3.3 LP Relaxation Models

We can now strengthen the LP relaxations, LPR and LSP, of model RP and model SP, respectively, by incorporating the valid inequalities derived in Section 3.1 and Section 3.2. As a result, we obtain the following two strengthened LP relaxations of model RP and model SP:

$$
\begin{aligned}
\text{(SLRP)} \qquad & z_{\text{SLRP}} = \min \sum_{r \in \mathbb{R}} c_r x_r, \\
\text{s.t.} \qquad & (3.2), (3.3), (3.5), (3.12), (3.15), (3.16), (3.17), (3.29), (3.30), \\
& x_r \geq 0, \qquad \forall\ r \in \mathbb{R}, \\
\text{(SLSP)} \qquad & z_{\text{SLSP}} = \min \sum_{s \in \mathbb{S}} c_s y_s, \\
\text{s.t.} \qquad & (3.39), (3.40), (3.43), (3.44), (3.45), (3.46), (3.47), \\
& y_s \geq 0, \qquad \forall\ s \in \mathbb{S}.
\end{aligned}
$$

As shown in Paradiso et al. [35], to solve a separation problem of constraint (3.3), one needs to solve a TOPTW on the route set $\mathbb{R}_{\mathbf{x}}$. It is known that the TOPTW is NP-hard and $|\mathbb{R}_{\mathbf{x}}|$ is often very large for a fractional solution $\mathbf{x}$. Moreover, with SRC constraints (3.30) being further relaxed, the pricing problems of the column generation approach in Chapter 4 become much easier to solve, since a stronger dominance rule can be developed.

Due to the observations above, in our solution method to be presented in Chapter 4, we will utilize three LP relaxations, including SLSP and two relaxations of SLRP. One relaxation of SLRP, referred to as SLRP1, relaxes constraints (3.3). The other relaxation of SLRP, referred to as SLRP2, relaxes both constraints (3.3) and constraints (3.30). Let z_{SLSP}, z_{SLRP1}, and z_{SLRP2} denote the objective values of these three LP relaxations. We can now establish Proposition 1 below to compare the tightness of the three LP relaxations to be utilized in our solution method.

Proposition 1. $z_{\text{SLSP}} \geq z_{\text{SLRP1}} \geq z_{\text{SLRP2}}$.

Proof. By definition, SLRP2 is a relaxation of SLRP1, implying that $z_{\text{SLRP1}} \geq z_{\text{SLRP2}}$. Thus, to prove Proposition 1 we only need to show that $z_{\text{SLSP}} \geq z_{\text{SLRP1}}$.

To show $z_{\text{SLSP}} \geq z_{\text{SLRP1}}$, consider any optimal solution $\mathbf{y}$ of SLSP. Consider the trip subset $\mathbb{S}_{\mathbf{y}} = \{s \in \mathbb{S} : y_s > 0\}$ and the route subset $\mathbb{R}_{\mathbf{y}} = \{r_s : s \in \mathbb{S}_{\mathbf{y}}\}$. We can now construct $\mathbf{x}$ for SLRP1 by setting $x_r = \sum_{s \in \mathbb{S}_{\mathbf{y}}: r_s = r} y_s$ for $r \in \mathbb{R}_{\mathbf{y}}$ and setting $x_r = 0$ for $r \in \mathbb{R} \setminus \mathbb{R}_{\mathbf{y}}$. Since $\gamma_{ts} \geq \beta_{tr_s}$, $\alpha_{is} = \alpha_{ir_s}$, $\eta_{s\bar{\mathbb{V}}k} = \eta_{r_s\bar{\mathbb{V}}k}$, $\lambda_{is\bar{\mathbb{V}}} = \lambda_{ir_s\bar{\mathbb{V}}}$ and $\mu_{s\bar{\mathbb{V}}} = \mu_{r_s\bar{\mathbb{V}}}$, it can be seen that $\mathbf{x}$ satisfies constraints (3.2), (3.5), (3.15), (3.16), (3.17), (3.29) and (3.30).

Moreover, for any $t_1, t_2 \in [a_0, b_0]$, $t_1 < t_2$, according to definitions of $\mathbb{R}_{\mathbf{y}}$ and $\mathbf{x}$, we have

$$\sum_{r \in \mathbb{R}} \bar{\beta}_{t_1 t_2 r} x_r = \sum_{r \in \mathbb{R}_{\mathbf{y}}} \bar{\beta}_{t_1 t_2 r} x_r = \sum_{r \in \mathbb{R}_{\mathbf{y}}} \bar{\beta}_{t_1 t_2 r} \sum_{s \in \mathbb{S}_{\mathbf{y}}: r_s = r} y_s = \sum_{s \in \mathbb{S}_{\mathbf{y}}} \bar{\beta}_{t_1 t_2 r_s} y_s. \tag{3.48}$$

Using the definition of $\bar{\beta}_{t_1 t_2 r}$, we have

$$\sum_{s \in \mathbb{S}_{\mathbf{y}}} \bar{\beta}_{t_1 t_2 r_s} y_s = \sum_{s \in \mathbb{S}_{\mathbf{y}}} \min_{\tau \in [e_{r_s}, l_{r_s}]} \{\sup([\tau, \tau + d_{r_s}] \cap [t_1, t_2]) - \inf([\tau, \tau + d_{r_s}] \cap [t_1, t_2])\} y_s$$
$$\leq \sum_{s \in \mathbb{S}_{\mathbf{y}}} \{\sup([\tau_s, \tau_s + d_{r_s}] \cap [t_1, t_2]) - \inf([\tau_s, \tau_s + d_{r_s}] \cap [t_1, t_2])\} y_s. \tag{3.49}$$

According to the definition of γ_{ts}, we have

$$\sup([\tau_s, \tau_s + d_{r_s}] \cap [t_1, t_2]) - \inf([\tau_s, \tau_s + d_{r_s}] \cap [t_1, t_2]) = \int_{t=t_1}^{t_2} \gamma_{ts} dt, \tag{3.50}$$

and

$$\begin{aligned} &\sum_{s \in \mathbb{S}_{\mathbf{y}}} \{\sup([\tau_s, \tau_s + d_{r_s}] \cap [t_1, t_2]) - \inf([\tau_s, \tau_s + d_{r_s}] \cap [t_1, t_2])\} \, y_s \\ =& \sum_{s \in \mathbb{S}_{\mathbf{y}}} \left(\int_{t=t_1}^{t_2} \gamma_{ts} dt \right) y_s = \sum_{s \in \mathbb{S}_{\mathbf{y}}} \int_{t=t_1}^{t_2} (\gamma_{ts} y_s) \, dt = \int_{t=t_1}^{t_2} \sum_{s \in \mathbb{S}_{\mathbf{y}}} (\gamma_{ts} y_s) \, dt. \end{aligned} \tag{3.51}$$

Since $\mathbf{y}$ is feasible for SLSP, $\sum_{s \in \mathbb{S}_{\mathbf{y}}} (\gamma_{ts} y_s) \leq K$, then we have

$$\sum_{r \in \mathbb{R}} \bar{\beta}_{t_1 t_2 r} x_r \leq \int_{t=t_1}^{t_2} \sum_{s \in \mathbb{S}_{\mathbf{y}}} (\gamma_{ts} y_s) \, dt \leq \int_{t=t_1}^{t_2} K dt = K(t_2 - t_1). \tag{3.52}$$

Thus, we obtain that $\mathbf{x}$ also satisfies the constraints (3.12), which implies that $\mathbf{x}$ is a feasible solution to SLRP1. Since $c_s = c_{r_s}$ for each $s \in \mathbb{S}$, it is easy to show that $\mathbf{x}$ and $\mathbf{y}$ have the same objective value. Because the objective value of $\mathbf{y}$ equals z_{SLSP}, and the objective value of $\mathbf{x}$ cannot be less than z_{SLRP1}, we obtain that $z_{\text{SLSP}} \geq z_{\text{SLRP1}}$, which, together with $z_{\text{SLRP1}} \geq z_{\text{SLRP2}}$, completes the proof of Proposition 1. □

Chapter 4

The Three-Phase Exact Method

In this chapter, we illustrate the three-phase solution method for the CMTVRPTW in detail, where LP relaxations of the route-based and trip-based models are solved in the first two phases respectively, and the integrality gap is closed in the last phase to obtain an optimal integer solution. The outline of the solution method is given in Section 4.1, followed by the illustration of each phase in other sections. Specifically, Step 1 of Phase 1 for solving SLRP2 is interpreted in Section 4.2, Step 2 of Phase 1 for route enumeration is described in Section 4.3 and Step 3 of Phase 1 for solving SLRP1 is presented in Section 4.4. Phase 2 for solving the strengthened LP relaxation of the trip-based model SLSP is illustrated in Section 4.5, and Phase 3 for closing the integrality gap is presented in Section 4.6. We then explain how to apply the variable fixing technique for shrinking the departure time interval in Section 4.7.

To simplify the representation of labeling algorithms in solving SLRP2 and in route enumeration, we define $t_{ij} = t'_{ij} + st_i$ for each arc $(i, j) \in \mathbb{A}$. For each (i, j), (j, k), and (i, k) in $\mathbb{A}$, it can be seen that the triangle inequality $t_{ij} + t_{jk} \geq t_{ik}$ also holds. This is because $t'_{ij} + t'_{jk} \geq t'_{ik}$ holds for each (i, j), (j, k), and (i, k) in

$\mathbb{A}$ under our assumption, and because the service time st_i is nonnegative for each $i \in \mathbb{V}_0$.

4.1 Outline of Solution Method

The variable fixing (also called variable elimination) technique aims to remove non-negative integer variables whose values cannot be positive in any optimal solution to an IP model. It has been widely applied to algorithms for various combinatorial optimization problems, such as the 0-1 knapsack problem [45], the VRP [22, 4, 27, 15], and so on. In our solution method, the variable fixing technique is applied in route enumeration, route elimination, trip elimination, and Proposition 8. We formally state this technique in the following lemma.

Lemma 1. *Consider any feasible IP model (named model P) with non-negative integer variables and bounded optimal objective value, and the dual of its LP relaxation (named model D), which are formulated as follows, where* $\mathbb{Z}_+^n$ *is the set of non-negative integer vectors in dimension* n*.*

$$\text{(P)} \qquad z_{\text{P}} = \min_{\mathbf{x} \in \mathbb{Z}_+^n} \left\{ \mathbf{c}^T \mathbf{x} : \mathbf{A}_1 \mathbf{x} = \mathbf{w}_1, \mathbf{A}_2 \mathbf{x} \leq \mathbf{w}_2, \mathbf{A}_3 \mathbf{x} \geq \mathbf{w}_3 \right\}.$$

$$\text{(D)} \qquad z_{\text{D}} = \max_{\boldsymbol{\pi}_1, \boldsymbol{\pi}_2 \leq \mathbf{0}, \boldsymbol{\pi}_3 \geq \mathbf{0}} \left\{ \boldsymbol{\pi}_1^T \mathbf{w}_1 + \boldsymbol{\pi}_2^T \mathbf{w}_2 + \boldsymbol{\pi}_3^T \mathbf{w}_3 : \boldsymbol{\pi}_1^T \mathbf{A}_1 + \boldsymbol{\pi}_2^T \mathbf{A}_2 + \boldsymbol{\pi}_3^T \mathbf{A}_3 \leq \mathbf{c}^T \right\}.$$

For any upper bound UB *on* z_{P} *and any feasible solution* $(\bar{\boldsymbol{\pi}}_1, \bar{\boldsymbol{\pi}}_2, \bar{\boldsymbol{\pi}}_3)$ *to model D, if the reduced cost* $c_i^{'}$ *of* x_i *w.r.t.* $(\bar{\boldsymbol{\pi}}_1, \bar{\boldsymbol{\pi}}_2, \bar{\boldsymbol{\pi}}_3)$ *satisfies* $c_i^{'} > UB - \bar{\boldsymbol{\pi}}_1^T \mathbf{w}_1 - \bar{\boldsymbol{\pi}}_2^T \mathbf{w}_2 - \bar{\boldsymbol{\pi}}_3^T \mathbf{w}_3$*, i.e.,* $(\mathbf{c}^T - \bar{\boldsymbol{\pi}}_1^T \mathbf{A}_1 - \bar{\boldsymbol{\pi}}_2^T \mathbf{A}_2 - \bar{\boldsymbol{\pi}}_3^T \mathbf{A}_3)_i > UB - \bar{\boldsymbol{\pi}}_1^T \mathbf{w}_1 - \bar{\boldsymbol{\pi}}_2^T \mathbf{w}_2 - \bar{\boldsymbol{\pi}}_3^T \mathbf{w}_3$*, then there exists no optimal solution* $\bar{\mathbf{x}}$ *to model P with* $\bar{x}_i > 0$ *(or* $\bar{x}_i \geq 1$*).*

We omit the proof of Lemma 1 since it can be directly derived from the proof in Hadjar et al. [22], where only constraints with equal signs and with less than or equal signs are considered. In Lemma 1, we explicitly consider three sets of constraints in model P (including constraints with equal signs, with less than or equal signs, and with greater than or equal signs). This is because the route-based and trip-based models studied in this work for the CMTVRPTW contain these three sets of constraints.

With the variable fixing technique, only routes with reduced cost no greater than the integrality gap between a valid upper bound and the lower bound need to be considered. This can reduce the number of variables significantly. The resulting IP model can be solved much more efficiently when the integrality gap is small (see, e.g., its applications in Baldacci et al. [4, 5], Paradiso et al. [35] and Yang [46]). In contrast, without applying the variable fixing technique, exact algorithms, such as the branch-and-price and branch-and-price-and-cut algorithms, can be very time-consuming, since the number of routes that need to be considered grows exponentially on the problem size, and the pricing problem to be solved at each search tree node is often strongly NP-hard and difficult to solve.

To apply the variable fixing technique, a valid upper bound UB is needed. Baldacci et al. [4] set UB as the best upper bound known, while Pecin et al. [36] and Yang [46] computed a valid upper bound through heuristic methods. In this study, we follow an approach applied in Baldacci et al. [6] and Paradiso et al. [35] to use a guessed upper bound UB_g, which is initially set to be slightly greater than the lower bound $LB1$ provided by SLRP2, and then increased whenever UB_g is found to be an invalid upper bound. We follow such a way due to its efficiency and also because the lower bound $LB1$ provided by SLRP2 is usually very tight.

To present our three-phase solution method, we introduce some additional notations as follows. We use gap_{ini} to denote an initial guessed gap between the lower bound $LB1$ and the optimal objective value, use Δ_{gap} to denote the increment of the guessed gap at each iteration, and use gap_{max} to denote the maximum value of the guessed gap. These are hyperparameters of our solution method. Moreover, in addition to SP($\mathbb{T}$) defined in Section 3.2, we use SP(UB_g), SP($\mathbb{T}, UB_g$), SLRP1(UB_g), and SLSP(UB_g) to denote restricted models of SP, SP($\mathbb{T}$), SLRP1, and SLSP, where the guessed upper bound UB_g is applied in variable fixing.

The outline of our solution method can be described in the following three phases, which are also illustrated in Figure 4.1.

- Phase 1 (Solve the LP relaxation of the route-based model).
  - Step 1 (Solve SLRP2). Initialize the guessed gap as $gap \leftarrow gap_{ini}$. Solve SLRP2 through a column-and-cut generation procedure. Denote its objective value as $LB1$ and the associated dual solution as $\boldsymbol{\pi}^1$.
  - Step 2 (Route enumeration). If $gap > gap_{max}$, terminate the algorithm with no optimal solutions found but with the lower bound $LB1$ returned. Otherwise, set the guessed upper bound as $UB_g \leftarrow LB1 * (1 + gap)$. Enumerate the set $\mathbb{R}_1$ of all routes with reduced costs no greater than $UB_g - LB1$ w.r.t. $\boldsymbol{\pi}^1$.
  - Step 3 (Solve SLRP1(UB_g)). Solve SLRP1(UB_g) based on $\mathbb{R}_1$ through a column-and-cut generation procedure. Denote its objective value as $LB2$ and the associated dual solution as $\boldsymbol{\pi}^2$. Remove from $\mathbb{R}_1$ all routes with reduced costs greater than $UB_g - LB2$ w.r.t. $\boldsymbol{\pi}^2$, and denote the set of remained routes as $\mathbb{R}_2$.
- Phase 2 (Solve the LP relaxation of the trip-based model). Solve SLSP(UB_g) based on $\mathbb{R}_2$ through a column-and-cut generation procedure. Denote its objective value as $LB3$ and the associated dual solution as $\boldsymbol{\pi}^3$. Remove some routes from $\mathbb{R}_2$ by applying the variable fixing technique, and denote

the set of remained routes as $\mathbb{R}_3$.

- Phase 3 (Close the integrality gap). Solve $\text{SP}(\mathbb{T}, UB_g)$ through an IP solver directly. (i) If $\text{SP}(\mathbb{T}, UB_g)$ is infeasible or $z_{\text{SP}(\mathbb{T},UB_g)} > UB_g$, then we let $gap \leftarrow gap + \Delta_{gap}$ and turn to Step 2 of Phase 1. (ii) If the optimal solution of $\text{SP}(\mathbb{T}, UB_g)$ is infeasible for SP, then we separate and add violated WT constraints to enlarge $\mathbb{T}$, and then repeat Phase 3. (iii) Otherwise, the optimal solution of $\text{SP}(\mathbb{T}, UB_g)$ is also optimal for SP, and thus, we terminate the algorithm.

We define the optimal objective value of an infeasible minimization problem to be $+\infty$. Note that $\text{SP}(UB_g)$ can be derived from SP by replacing $\mathbb{R}$ with $\mathbb{R}_3$, and that $\text{SP}(\mathbb{T}, UB_g)$ is a relaxation of $\text{SP}(UB_g)$ by considering WT constraints only on a discrete set of time points $\mathbb{T}$. It can thus be seen that $z_{\text{SP}(UB_g)} \geq z_{\text{SP}}$ (since $\mathbb{R}_3 \subseteq \mathbb{R}$), and that $z_{\text{SP}(UB_g)} \geq z_{\text{SP}(\mathbb{T},UB_g)}$ (since $\mathbb{T} \subset [a_0, b_0]$).

Moreover, when UB_g is a valid upper bound (i.e., $z_{\text{SP}} \leq UB_g$), $\mathbb{R}_3$ must contain all routes that appear in all optimal solutions to the CMTVRPTW, which implies that $z_{\text{SP}(UB_g)} = z_{\text{SP}} \leq UB_g$. When UB_g is not a valid upper bound (i.e., $z_{\text{SP}} > UB_g$), some routes that appear in optimal solutions to the CMTVRPTW may not be included in $\mathbb{R}_3$, and thus, the optimal solution of $\text{SP}(UB_g)$ may not be optimal for SP, implying that $z_{\text{SP}(UB_g)} \geq z_{\text{SP}} > UB_g$. For this reason, if condition (i) of Phase 3 is satisfied, i.e., $z_{\text{SP}(\mathbb{T},UB_g)} > UB_g$, then we konw that $z_{\text{SP}(UB_g)} \geq z_{\text{SP}(\mathbb{T},UB_g)} > UB_g$, implying that UB_g is not a valid upper bound, and thus, we need to increase the guessed value of UB_g, accordingly.

The efficiency of the newly proposed solution method above relies on the tightness of lower bounds and the efficiency of closing the integrality gap. To ensure the tightness of the lower bounds, in addition to introducing and applying those valid inequalities, we also solve the LP relaxation of a trip-based model, which can

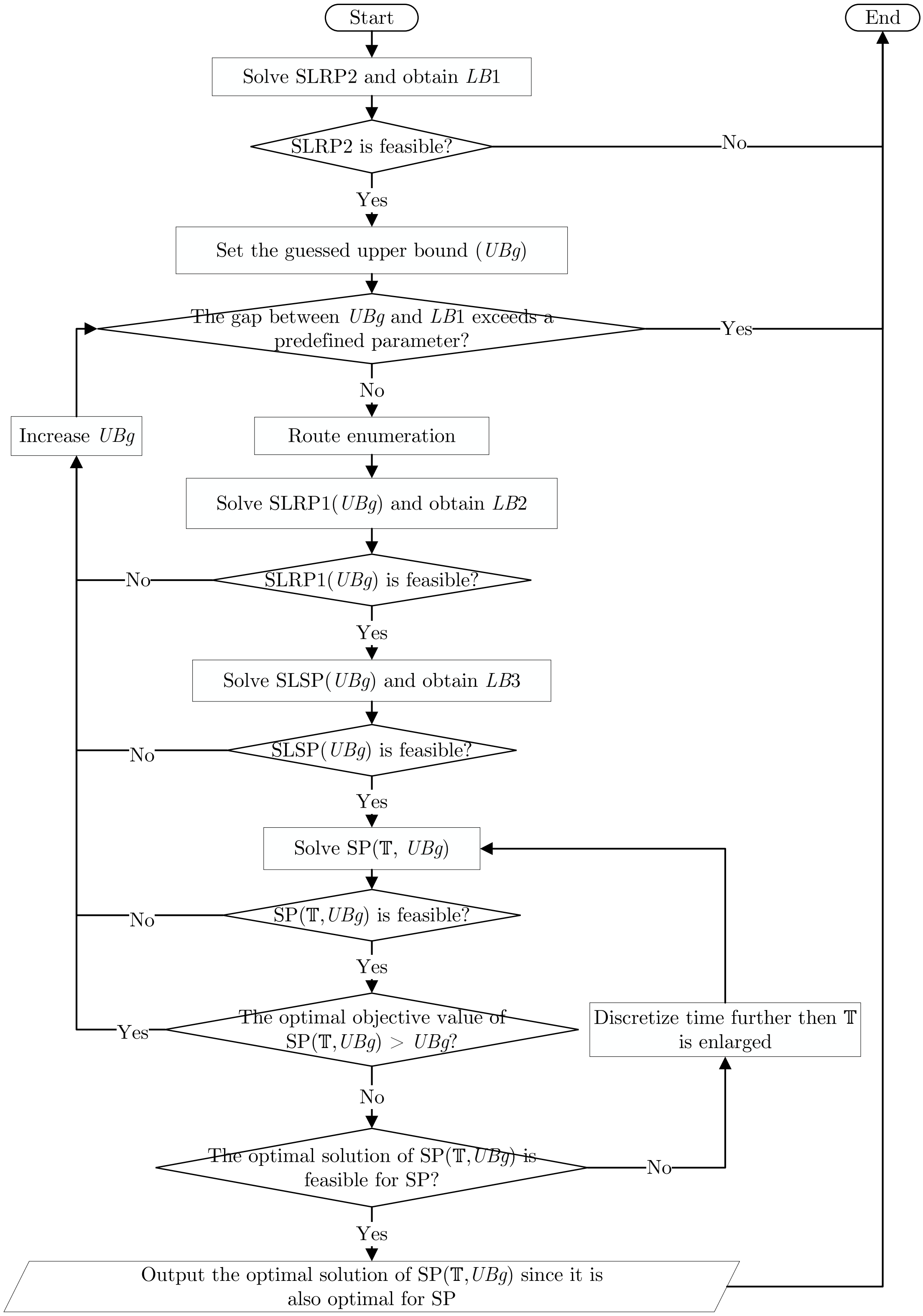


Figure 4.1: Outline of the solution method.

be tighter than that of the route-based model in Paradiso et al. [35] and Yang [46]. To ensure the efficiency of closing the integrality gap, we utilize the enumerated routes and close the integrality gap by solving trip-based models through an IP solver directly and by applying a dynamic time discretization technique.

Our approach to closing the integrality gap is different from that of Paradiso et al. [35] and Yang [46], which is based on the branch-and-cut algorithms and relies on the route-based model. Constraints (3.3) of the route-based model are weak, and the number of these constraints that need to be identified can be very large for some instances. Although Hernandez et al. [23] and Hernandez et al. [25] adopted a trip-based model for two variants of the CMTVRPTW, they closed the integrality gap by applying branch-and-price algorithms, which can be very time-consuming, since the NP-hard pricing problems need to be solved at a lot of search tree nodes.

The variable fixing technique and column generation procedure can be applied to models with a finite number of variables, which is not the case in the trip-based model since the domain of the departure time at the depot for any trip is an interval and the number of variables is infinite in the trip-based model. However, if the travel time $t^{'}_{ij}$ for all $(i, j) \in \mathbb{A}$, the service time st_i for all $i \in \mathbb{V}$ and the endpoints of the time window $[a_i, b_i]$ for all $i \in \mathbb{V}_0$ are rational numbers, then there exists an optimal solution where departure times of selected trips at the depot are all rational numbers. Furthermore, there must exist a positive integer M such that by multiplying M all the departure times become integers. Thus, we can obtain an optimal solution for the trip-based model by only considering departure times at the depot in the form k/M, where k is a non-negative integer and $a_0 \leq k/M \leq b_0$. Because the number of values of k satisfying such conditions is finite, the route set $\mathbb{R}$ has a finite cardinality, and each trip can be determined by a route and the associated departure time at the depot, we can obtain an optimal solution of the trip-based model by only considering a finite number of variables (trips). Therefore, the variable fixing technique and column generation procedure can be applied to the trip-based model by only considering such variables.

4.2 Step 1 of Phase 1: Solve SLRP2

We solve SLRP2 by a column-and-cut generation procedure where column generation and cut generation proceed alternately. To restrict the running time for solving SLRP2, we impose a limit on the maximum number of valid inequalities that can be added to the model, and the cut generation will not be called if the number of added valid inequalities reaches this limit. We terminate the procedure and obtain a lower bound $LB1$, if (i) no new cut is identified in a call of cut generation, or (ii) the elapsed running time is greater than a predefined time limit at the end of a call of column generation, or (iii) the difference between the objective values yield by two successive calls of column generation is less than a predefined threshold.

We define the restricted master problem of SLRP2 by considering only a subset of columns and constraints in SLRP2. Let $\mathbb{T}$, $\mathbb{C}_{rwt}$, $\mathbb{C}_{sr3}$, $\mathbb{C}_{sr5\text{-}2}$, $\mathbb{C}_{sr5\text{-}3}$ and $\mathbb{C}_{el}$ denote sets of indices associated with constraints (3.5), (3.12), (3.15), (3.16), (3.17) and (3.29), respectively. They are initialized to be empty, and then enlarged iteratively in cut generation by separating violated valid inequalities through methods illustrated in Section 3.1.

To start column generation, an initial feasible solution is needed. For this, we define a dummy feasible route r_0 with $c_{r_0} = M$ where M is a large positive constant, $\alpha_{ir_0} = 1$ for all $i \in \mathbb{V}$, $e_{r_0} = l_{r_0} = a_0$, $d_{r_0} = 0$ and $\mu_{r_0\bar{\mathbb{V}}} = \lceil \sum_{i\in\mathbb{V}} q_i/Q \rceil$

for all $\bar{\mathbb{V}} \subseteq \mathbb{V}$. In the restricted master problem, we replace $\mathbb{R}$, which consists of an exponential number of routes, with a route subset $\mathbb{R}'$ which is initialized to be $\{r_0\}$. Let x_0 be the variable associated with route r_0. The solution $\mathbf{x}$ where $x_0 = 1$ and $x_r = 0$ for all $r \in \mathbb{R}' \setminus \{r_0\}$ is always feasible to the restricted master problem. However, when M is sufficiently large and the column generation terminates with $x_0 > 0$ in the optimal solution to the last restricted master problem, it can be guaranteed that the original problem must have no feasible solution [7].

At each iteration of column generation, we denote the dual solution of the current restricted master problem as $\boldsymbol{\pi}^1 = (\mathbf{f}, \mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{m}, \mathbf{o}, \mathbf{u})$, where $\mathbf{f}, \mathbf{g} \leq \mathbf{0}, \mathbf{h} \leq \mathbf{0}, \mathbf{k} \leq \mathbf{0}, \mathbf{m} \leq \mathbf{0}, \mathbf{o} \leq \mathbf{0}, \mathbf{u} \leq \mathbf{0}$ are associated with constraints (3.2), (3.5), (3.12), (3.15), (3.16), (3.17), (3.29) respectively. The reduced cost of a route $r \in \mathbb{R}$ w.r.t. $\boldsymbol{\pi}^1$ can thus be represented by

$$\begin{aligned} c_r^{'}(\boldsymbol{\pi}^1) = c_r - \sum_{i \in \mathbb{V}} f_i \alpha_{ir} - \sum_{t \in \mathbb{T}} g_t \beta_{tr} - \sum_{(t_1,t_2) \in \mathbb{C}_{rwt}} h_{t_1 t_2} \bar{\beta}_{t_1 t_2 r} - \sum_{\bar{\mathbb{V}} \in \mathbb{C}_{sr3}} k_{\bar{\mathbb{V}}} \eta_{r\bar{\mathbb{V}}2} \\ - \sum_{\bar{\mathbb{V}} \in \mathbb{C}_{sr5\text{-}2}} m_{\bar{\mathbb{V}}} \eta_{r\bar{\mathbb{V}}2} - \sum_{\bar{\mathbb{V}} \in \mathbb{C}_{sr5\text{-}3}} o_{\bar{\mathbb{V}}} \eta_{r\bar{\mathbb{V}}3} - \sum_{(i,\bar{\mathbb{V}}) \in \mathbb{C}_{el}} u_{i\bar{\mathbb{V}}} \lambda_{ir\bar{\mathbb{V}}}. \end{aligned} \tag{4.1}$$

Accordingly, we can formulate the pricing problem of SLRP2 as follows:

$$\text{(SubSLRP2)} \qquad z_{\text{SubSLRP2}} = \min \left\{ c_r^{'}(\boldsymbol{\pi}^1) : r \in \mathbb{R} \right\}. \tag{4.2}$$

It can be seen that this pricing problem is an ESPPRC, aiming to find an elementary route $r \in \mathbb{R}$ such that the reduced cost $c_r^{'}(\boldsymbol{\pi}^1)$ is minimized. By solving this pricing problem, if we obtain a route with negative reduced cost, i.e., $z_{\text{SubSLRP2}} < 0$, we add the route to $\mathbb{R}'$, and otherwise, we terminate column generation, as the SLRP2 based on the complete route set $\mathbb{R}$ and cut index sets $\mathbb{T}$, $\mathbb{C}_{rwt}$, $\mathbb{C}_{sr3}$, $\mathbb{C}_{sr5\text{-}2}$, $\mathbb{C}_{sr5\text{-}3}$, $\mathbb{C}_{el}$ has been solved to optimality.

4.2.1 Labeling Algorithm for Pricing Problem SubSLRP2

We solve the pricing problem SubSLRP2 by a labeling algorithm similar to that of Yang [46], since the master problem SLRP2 is an extension of the LP relaxation of their route-based model where additional constraints (3.12), (3.16), (3.17) and (3.29) are taken into account. In our labeling algorithm, we adopt label representation and state transition equations similarly to theirs, and we also show that their dominance rules are still valid for solving SubSLRP2. However, we apply a rollback pruning technique in our labeling algorithm, which has not been applied in Yang [46], as well as utilize a new completion bound, to effectively prune the labels.

Label Representation

Except for a different computation of the total reduced cost for each backward path, our label representation is the same as that of Yang [46] and its detail is described as follows for the completion of our presentation.

We define an elementary backward path $p = (i_0, i_1, \ldots, i_n)$ as a node sequence indicating that a vehicle performing this backward path departs from node i_n, visits $i_{n-1}, \ldots, i_1$ sequentially and arrives at i_0 eventually, where $i_0 = 0$, $\{i_1, \ldots, i_{n-1}\} \subseteq \mathbb{V}$, and $i_1, \ldots, i_{n-1}, i_n$ are different from each other. Such a backward path is associated with a label, which is defined as $L = (p, i, q, e, l, d, \xi, \rho)$, where $i = i_n$, $q = \sum_{w=1}^{n} q_{i_w}$ is the cumulative demand quantity, $[e, l]$ is the non-dominated departure time set and d is the minimum duration. It indicates that for any vehicle that performs the backward path p and departs from node i at time t, the time window constraints are satisfied if and only if $t \in [a_0, l]$, and the vehicle arrives at node i_0 at time $t + d$ if $t \in [e, l]$ and at time $e + d$ if $t \in [a_0, e)$. Denote $\mathbb{V}(p) = \mathbb{V} \cap \{i_0, i_1, \ldots, i_n\}$, $e_p = e$, $l_p = l$, and $d_p = d$. Parameters β_{tp}, $\bar{\beta}_{t_1 t_2 p}$, $\eta_{p\bar{\mathbb{V}}2}$,

$\eta_{p\bar{\mathbb{V}}3}$, $\lambda_{jp\bar{\mathbb{V}}}$ can thus be defined in the same way as parameters β_{tr}, $\bar{\beta}_{t_1t_2r}$, $\eta_{r\bar{\mathbb{V}}2}$, $\eta_{r\bar{\mathbb{V}}3}$, $\lambda_{jr\bar{\mathbb{V}}}$. Accordingly, the arc-based reduced cost ξ and the total reduced cost ρ of backward path p can be represented as follows.

$$\xi = \sum_{w=1}^{n}(c_{i_w i_{w-1}} - f_{i_w}), \tag{4.3}$$

$$\begin{aligned}\rho = \xi - \sum_{t\in\mathbb{T}} g_t\beta_{tp} - \sum_{(t_1,t_2)\in\mathbb{C}_{rwt}} h_{t_1t_2}\bar{\beta}_{t_1t_2p} - \sum_{\bar{\mathbb{V}}\in\mathbb{C}_{sr3}} k_{\bar{\mathbb{V}}}\eta_{p\bar{\mathbb{V}}2} - \sum_{\bar{\mathbb{V}}\in\mathbb{C}_{sr5\text{-}2}} m_{\bar{\mathbb{V}}}\eta_{p\bar{\mathbb{V}}2} \\ - \sum_{\bar{\mathbb{V}}\in\mathbb{C}_{sr5\text{-}3}} o_{\bar{\mathbb{V}}}\eta_{p\bar{\mathbb{V}}3} - \sum_{(j,\bar{\mathbb{V}})\in\mathbb{C}_{el}} u_{j\bar{\mathbb{V}}}\lambda_{jp\bar{\mathbb{V}}}.\end{aligned} \tag{4.4}$$

A label $L = (p, i, q, e, l, d, \xi, \rho)$ is feasible if and only if $q \leq Q$ and $l \geq a_i$. We call L a complete label if $i = 0$ and $n \geq 1$, for which p is a route. Otherwise, L is called an incomplete label. Thus, finding feasible routes with negative reduced cost is equivalent to finding labels that are feasible, complete, and with $\rho < 0$.

State Transition Equations

It is known that a label-setting algorithm for solving ESPPRC is a dynamic programming algorithm, where starting from the initial label $(\{0\}, 0, 0, a_0, b_0, 0, 0, 0)$, unexplored labels are generated and explored by extending previously explored labels. By extending a feasible and incomplete label $L = (p, i, q, e, l, d, \xi, \rho)$ toward a node $j \in \mathbb{V}_0 \setminus \mathbb{V}(p)$, the newly generated label $L' = (p', j, q', e', l', d', \xi', \rho')$ can be determined by the following state transition equations, which are almost the same with that of Yang [46] except that ρ' is computed in a different way.

$$p' = (i_0, i_1, \ldots, i_n = i, i_{n+1} = j), \tag{4.5}$$

$$q' = q + q_j, \tag{4.6}$$

$$e' = \begin{cases} b_j, & \text{if } e > b_j + t_{ji}, \\ e - t_{ji}, & \text{if } a_j + t_{ji} \le e \le b_j + t_{ji}, \\ a_j, & \text{otherwise,} \end{cases} \tag{4.7}$$

$$l' = \min\{l - t_{ji}, b_j\}, \tag{4.8}$$

$$d' = d + \max\{t_{ji}, e - b_j\}, \tag{4.9}$$

$$\xi' = \xi + c_{ji} - f_j, \tag{4.10}$$

$$\rho' \text{ is computed according to (4.4).} \tag{4.11}$$

According to (4.7) and (4.9), we can obtain that

$$e' + d' = \max\{e + d, a_j + t_{ji} + d\}. \tag{4.12}$$

Dominance Rule

In our labeling algorithm, a dominance rule is applied to significantly reduce the number of labels that need to be explored. We define that label L_1 dominates label L_2 if for every feasible route associated with a label that equals or extends L_2, there exists a feasible route associated with a label that equals or extends L_1 but has an equal or smaller reduced cost. This implies that discarding label L_2 but keeping label L_1 will not prevent the labeling algorithm from finding an optimal solution to the pricing problem.

We can establish Lemma 2, which indicates that the dominance rule adopted in the labeling algorithm of Yang [46] for solving their pricing problem is still valid in our labeling algorithm for solving SubSLRP2.

Lemma 2. *Let* $L_1 = (p_1, i_1, q_1, e_1, l_1, d_1, \xi_1, \rho_1)$ *and* $L_2 = (p_2, i_2, q_2, e_2, l_2, d_2, \xi_2, \rho_2)$ *be two feasible labels, then* L_1 *dominates* L_2 *if all the following relations are satis-*

fied: (i) $i_1 = i_2$, (ii) $\mathbb{V}(p_1) \subseteq \mathbb{V}(p_2)$, (iii) $l_1 \geq l_2$, (iv) $e_1 + d_1 \leq e_2 + d_2$, (v) $d_1 \leq d_2$, *and* (vi) $\xi_1 \leq \xi_2$.

Proof. Relations (iii), (iv), (v) imply that $\beta_{tp_1} \leq \beta_{tp_2}$ and $\bar{\beta}_{t_1t_2p_1} \leq \bar{\beta}_{t_1t_2p_2}$. Relation (ii) implies that $\eta_{p_1\bar{\mathbb{V}}2} \leq \eta_{p_2\bar{\mathbb{V}}2}$, $\eta_{p_1\bar{\mathbb{V}}3} \leq \eta_{p_2\bar{\mathbb{V}}3}$ and $\lambda_{kp_1\bar{\mathbb{V}}} \leq \lambda_{kp_2\bar{\mathbb{V}}}$. This, together with relation (vi) and $\mathbf{g} \leq \mathbf{0}, \mathbf{h} \leq \mathbf{0}, \mathbf{k} \leq \mathbf{0}, \mathbf{m} \leq \mathbf{0}, \mathbf{o} \leq \mathbf{0}, \mathbf{u} \leq \mathbf{0}$, implies that $\rho_1 \leq \rho_2$.

Moreover, Yang [46] showed that for any node $j \in \mathbb{V}_0 \setminus \mathbb{V}(p_2)$ such that L_2 can be extended to a feasible label $L_2' = (p_2', i_2', q_2', e_2', l_2', d_2', \xi_2', \rho_2')$ by visiting j, L_1 can be extended to a feasible label $L_1' = (p_1', i_1', q_1', e_1', l_1', d_1', \xi_1', \rho_1')$ by visiting j with relations (i)–(vi) still valid for L_1' and L_2'. Because relations (ii)–(vi) are valid for L_1' and L_2', we obtain that $\rho_1' \leq \rho_2'$ for the same reason to $\rho_1 \leq \rho_2$. It shows that for every feasible route associated with L_2, the route associated with L_1 is also feasible and has an equal or smaller reduced cost. It also shows that for every feasible route associated with label L_2' that extends L_2, the route associated with label L_1' that extends L_1 in the same way as L_2' extends L_2 is also feasible and has an equal or smaller reduced cost. This proves that L_1 dominates L_2. □

Rollback Pruning

One can apply the dominance rule to compare every pair of labels to prune dominated labels as many as possible. However, this is very time-consuming since the total number of label pairs can be considerably large.

To apply the dominance rule more efficiently, we utilize a rollback pruning approach introduced by Lozano et al. [31] for a relatively simple elementary shortest path problem with resource constraints. In this approach, some sufficient conditions need to be derived for a new label L to be dominated by some generated labels that visit fewer nodes, and such sufficient conditions must be easy to be verified, often involving a small number of simple arithmetic operations. As a result, one can first evaluate whether a new label L satisfies these sufficient conditions. If these sufficient conditions are satisfied, rollback pruning is applied to prune label L. Otherwise, one then applies the dominance rule by comparing label L with other generated labels and prune label L if it is dominated.

The problem considered by Lozano et al. [31] is much easier than SubSLRP2 studied here. The rollback pruning has never been applied in the development of exact methods for the CMTVRPTW. We now establish Proposition 2 below to show that the rollback pruning is valid to be applied in our labeling algorithm for solving SubSLRP2.

Proposition 2. *For a feasible label $L_2 = (p_2, i_n, q_2, e_2, l_2, d_2, \xi_2, \rho_2)$ where $p_2 = (i_0, i_1, \ldots, i_{n-1}, i_n)$, if there exists an integer $0 \leq \bar{w} \leq n-2$, such that $c_{i_n i_{\bar{w}}} - f_{i_n} \leq \sum_{w=\bar{w}+1}^{n} (c_{i_w i_{w-1}} - f_{i_w})$, then label L_2 can be pruned since it is dominated by another label $L_1 = (p_1, i_n, q_1, e_1, l_1, d_1, \xi_1, \rho_1)$ where $p_1 = (i_0, i_1, \ldots, i_{\bar{w}-1}, i_{\bar{w}}, i_n)$.*

Proof. To prove this proposition, we show as follows that relations (ii)–(vi) in Lemma 2 are satisfied for labels L_1 and L_2.

Since backward path p_1 can be derived by removing nodes $i_{\bar{w}+1}, i_{\bar{w}+2}, \ldots, i_{n-1}$ from p_2, we have $\mathbb{V}(p_1) \subset \mathbb{V}(p_2)$, implying that (ii) in Lemma 2 is satisfied.

Consider a label $L = (p, i_{\bar{w}}, q, e, l, d, \xi, \rho)$ where $p = (i_0, i_1, \ldots, i_{\bar{w}})$, which can be extended to L_1 and L_2. Due to the triangle inequality, $t_{i_n i_{\bar{w}}} \leq \sum_{w=\bar{w}+1}^{n} t_{i_w i_{w-1}}$.

From equation (4.8) we have that $l_1 = \min\{l - t_{i_n i_{\bar{w}}}, b_{i_n}\}$, $l_2 \leq b_{i_n}$, and $l_2 \leq l - \sum_{w=\bar{w}+1}^{n} t_{i_w i_{w-1}} \leq l - t_{i_n i_{\bar{w}}}$. Thus, it can be seen that $l_1 \geq l_2$, implying that (iii) in Lemma 2 is satisfied.

From (4.9) and (4.12) we know that d and $e + d$ are non-decreasing after labels are extended. It can also be seen that $e_1 + d_1 = \max\{e + d, a_{i_n} + t_{i_n i_{\bar{w}}} + d\}$,

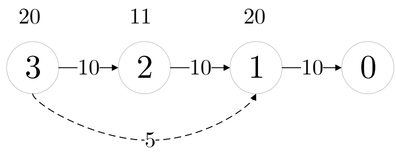


Figure 4.2: A backward path where travel costs are on arrows and dual values f_i for $i = 1, 2, 3$ are above circles.

$e_2 + d_2 \geq e + d$, and $e_2 + d_2 \geq a_{i_n} + \sum_{w=\bar{w}+1}^{n} t_{i_w i_{w-1}} + d \geq a_{i_n} + t_{i_n i_{\bar{w}}} + d$. Thus, $e_1 + d_1 \leq e_2 + d_2$, implying that (iv) in Lemma 2 is satisfied.

From (4.9) we know that $d_1 = \max\{d + t_{i_n i_{\bar{w}}}, d + e - b_{i_n}\}$, $d_2 \geq d + \sum_{w=\bar{w}+1}^{n} t_{i_w i_{w-1}} \geq d + t_{i_n i_{\bar{w}}}$, and $d_2 \geq d + e - b_{i_n}$. Thus, $d_1 \leq d_2$, implying that (v) in Lemma 2 is satisfied.

Moreover, from (4.10), we have that $\xi_1 = \xi + c_{i_n i_{\bar{w}}} - f_{i_n}$ and $\xi_2 = \xi + \sum_{w=\bar{w}+1}^{n} (c_{i_w i_{w-1}} - f_{i_w})$. Since $c_{i_n i_{\bar{w}}} - f_{i_n} \leq \sum_{w=\bar{w}+1}^{n} (c_{i_w i_{w-1}} - f_{i_w})$, we obtain that $\xi_1 \leq \xi_2$. Thus (vi) in Lemma 2 is satisfied.

Therefore, by Lemma 2, we obtain that L_2 is dominated by L_1, and thus, L_2 can be pruned. This completes the proof of Propostion 2. □

To illustrate the application of Proposition 2, let us consider a label L_2 with backward path $p_2 = (0, 1, 2, 3)$ shown in Figure 4.2. Note that $c_{31} - f_3 = -15 < (c_{32} - f_3) + (c_{21} - f_2) = -11$. Thus, due to Proposition 2, label L_2 is dominated by another label L_1 with $p_1 = (0, 1, 3)$, and can be pruned by rollback pruning. Following this way, label L_2 is pruned by a small number of simple arithmetic operations and with no needs to compare L_2 with other labels.

Completion Bound

The completion bound is widely used to prune labels [6, 31, 46], which can reduce the number of explored labels significantly. Let input parameters Δ_t and Δ_q be step sizes associated with time and demand quantity respectively, and denote

$\mathbb{Z}$ as the set of integers, then the state space for computing the completion bound is as follows.

$$\Omega = \Big\{(t,q,i) : i \in \mathbb{V}_0, q \in \{q_i\} \cup \{k\Delta_q : q_i \le k\Delta_q \le Q, k \in \mathbb{Z}\}, \\ t \in \{a_i, b_i\} \cup \{k\Delta_t : a_i \le k\Delta_t \le b_i, k \in \mathbb{Z}\}\Big\}. \quad (4.13)$$

For each $(t,q,i) \in \Omega$, we compute a value $\underline{\xi}(t,q,i)$ as follows, which is the completion bound. Completion bound $\underline{\xi}(t,q,i)$ is initialized with zero if $i = 0$, and initialized with $+\infty$ otherwise. Let $floor(q) = \max\{\underline{q} : \underline{q} \le q, \underline{q} \in \{q_i\} \cup \{k\Delta_q : q_i \le k\Delta_q \le Q, k \in \mathbb{Z}\}\}$, $ceil(t) = \min\{\bar{t} : \bar{t} \ge t, \bar{t} \in \{a_i, b_i\} \cup \{k\Delta_t : a_i \le k\Delta_t \le b_i, k \in \mathbb{Z}\}\}$. Moreover, we define $\mathbb{L}(t,q,i)$ as the set of all labels in the form $(p', i', q', e', l', d', \xi', \rho')$ which can be obtained by extending the label $((i), i, q, a_i, t, 0, 0, 0)$, where either $i' = 0$, or $floor(q') - floor(q) > n_q\Delta_q$ and $floor(q' - q_{i'}) - floor(q) \le n_q\Delta_q$. And n_q is an input non-negative integer parameter which can balance the tightness of the completion bound and the computing time for obtaining such bound. Then the completion bound $\underline{\xi}(t,q,i)$ for all $(t,q,i) \in \Omega$ is determined by

$$\underline{\xi}(t,q,i) = \min_{(p',i',q',e',l',d',\xi',\rho') \in \mathbb{L}(t,q,i)} \left\{\xi' + \underline{\xi}(ceil(l'), floor(q'), i')\right\}. \quad (4.14)$$

To compute the completion bound by utilizing completion bounds computed previously, the computation is performed with three-nested loops, where i iterates through each customer $i \in \mathbb{V}$ in the most inner loop, q goes from large to small in the second loop, and t goes from small to large in the most outer loop. For a state $(t,q,i) \in \Omega$, we compute the completion bound $\underline{\xi}(t,q,i)$ by running the labeling algorithm where the starting label is $((i), i, q, a_i, t, 0, 0, 0)$, and labels out-side $\mathbb{L}(t,q,i)$ will not be explored since completion bounds computed previously can be utilized with equation (4.14).

According to the above description, it is easy to know that the completion bound $\underline{\xi}(t,q,i)$ is a lower bound on the minimum arc-based reduced cost of all labels that can be derived by extending the label $((i),i,q,a_i,t,0,0,0)$ and are in the form of $L' = (p',0,q',e',l',d',\xi',\rho')$ (i.e., the associated backward path p' ends at the depot). For any $\bar{t}$ and $\underline{q}$ satisfying $t \leq \bar{t} \leq b_i$ and $q_i \leq \underline{q} \leq q$, it is easy to know that $\underline{\xi}(\bar{t},\underline{q},i) \leq \underline{\xi}(t,q,i)$, since any label derived by extending $((i),i,q,a_i,t,0,0,0)$ can also be derived by extending $((i),i,\underline{q},a_i,\bar{t},0,0,0)$.

Although the completion bound is applied in Paradiso et al. [35], they have not illustrated it in detail. Yang [46] applied the completion bound in the form of $\underline{\xi}(q,i)$, not depending on time t and might be loose. We apply the completion bound in the form of $\underline{\xi}(t,q,i)$, compute it in a different way, and strike a balance between the tightness of the completion bound and the computational efficiency through some input parameters.

Let $c_{ub}^{'}$ be an upper bound on the optimal objective value of SubSLRP2, which can be initialized as zero and be updated whenever a better solution to SubSLRP2 is found. As indicated in Proposition 3 below, we can apply the completion bound to prune labels.

Proposition 3. *For any feasible incomplete label $L = (p,i,q,e,l,d,\xi,\rho)$, if there exist $\underline{q}$ and $\bar{t}$ such that $q_i \leq \underline{q} \leq q$, $l \leq \bar{t} \leq b_i$, $(\bar{t},\underline{q},i) \in \Omega$ and $\rho + \underline{\xi}(\bar{t},\underline{q},i) > c_{ub}^{'}$, then label L can be pruned, since no complete labels with reduced cost less than or equal to $c_{ub}^{'}$ can be derived by extending L.*

Proof. Consider any complete label $L' = (p',0,q',e',l',d',\xi',\rho')$ which can be derived by extending L. To prove Proposition 3, we only need to show that $\rho' > c_{ub}^{'}$.

First, we are going to prove as follows that $\rho' - \xi' \geq \rho - \xi$. Without loss of generality, we assume that $p = (i_0, i_1, \ldots, i_n = i)$ and $p' = (i_0, i_1, \ldots, i_n = i, j_1, j_2, \ldots, j_{n'-n} = 0)$ where $n < n'$. Since L can be extended to L', we have $\mathbb{V}(p) \subset \mathbb{V}(p')$. From (4.8), (4.9) and (4.12), we can see that $l \geq l'$, $e + d \leq e' + d'$ and $d \leq d'$. Therefore, $\beta_{tp} \leq \beta_{tp'}$, $\bar{\beta}_{t_1t_2p} \leq \bar{\beta}_{t_1t_2p'}$, $\eta_{p\bar{\mathbb{V}}2} \leq \eta_{p'\bar{\mathbb{V}}2}$, $\eta_{p\bar{\mathbb{V}}3} \leq \eta_{p'\bar{\mathbb{V}}3}$, and $\lambda_{jp\bar{\mathbb{V}}} \leq \lambda_{jp'\bar{\mathbb{V}}}$. Since $\mathbf{g} \leq \mathbf{0}, \mathbf{h} \leq \mathbf{0}, \mathbf{k} \leq \mathbf{0}, \mathbf{m} \leq \mathbf{0}, \mathbf{o} \leq \mathbf{0}$ and $\mathbf{u} \leq \mathbf{0}$, from (4.4) we obtain that $\rho' - \xi' \geq \rho - \xi$.

Next, consider a label $L'' = (p'', 0, q'', e'', l'', d'', \xi'', \rho'')$ which is derived by extending label $((i), i, q, a_i, l, 0, 0, 0)$, where $p'' = (i, j_1, j_2, \ldots, j_{n'-n} = 0)$. From equation (4.10), it can be seen that $\xi' = \xi + \xi''$. Moreover, since $\underline{\xi}(l, q, i)$ is a completion bound for labels extended from $((i), i, q, a_i, l, 0, 0, 0)$, we have that $\xi'' \geq \underline{\xi}(l, q, i)$, which, together with $\underline{\xi}(l, q, i) \geq \underline{\xi}(\bar{t}, \underline{q}, i)$ shown earlier, implies that $\xi'' \geq \underline{\xi}(l, q, i) \geq \underline{\xi}(\bar{t}, \underline{q}, i)$. Hence, we obtain that $\rho' = (\rho' - \xi') + \xi' \geq (\rho - \xi) + \xi + \xi'' \geq \rho + \underline{\xi}(l, q, i) \geq \rho + \underline{\xi}(\bar{t}, \underline{q}, i) > c'_{ub}$. This completes the proof of Proposition 3. □

A Heuristic Method for the Pricing Problem

For column-generation, it is not necessary to always find an optimal solution to SubSLRP2 except for the last iteration. Due to this, we execute the exact labeling algorithm only when a heuristic method fails to find routes with negative reduced costs. It is known that the labeling algorithm is a dynamic programming algorithm where each stage is defined by the number of arcs included in the backward path associated with each label. Accordingly, we develop a heuristic method to solve SubSLRP2 by adapting the exact labeling algorithm with only a limited number of (e.g., 2000) labels that are of relatively small reduced costs kept at each stage for further extensions.

4.3 Step 2 of Phase 1: Route Enumeration

If UB_g is a valid upper bound, then according to Lemma 1, only routes with reduced cost no greater than $(UB_g - LB_1)$ w.r.t. $\boldsymbol{\pi}^1$ can appear in optimal solutions. Let $\mathbb{R}_1$ denote the set of all such routes. Accordingly, the set $\mathbb{R}$, which may consist of an exponential number of routes, can be replaced by $\mathbb{R}_1$. For a backward path $p = (i_0, i_1, \ldots, i_n)$, let $c_p = \sum_{w=1}^{n} c_{i_w i_{w-1}}$ denote its associated travel cost. The aforementioned labeling algorithm for solving SubSLRP2 can then be adapted to enumerate all routes of $\mathbb{R}_1$.

Similar to the labeling algorithm for solving SubSLRP2, we can drive a dominance rule to reduce the number of labels explored. However, the definition of the dominance relation is different from that of the labeling algorithm for solving SubSLRP2. Here, label L_1 dominates label L_2 if for any feasible route r_2 associated with a label that equals or extends L_2, there exists a feasible route r_1 associated with a label that equals or extends L_1 but having an equal or smaller travel cost, such that any feasible solution of model RP containing r_2 is still feasible when r_2 is replaced by r_1. The following lemma, established by Yang [46], is valid for the labeling algorithm for route enumeration.

Lemma 3. *Let* $L_1 = (p_1, i_1, q_1, e_1, l_1, d_1, \xi_1, \rho_1)$ *and* $L_2 = (p_2, i_2, q_2, e_2, l_2, d_2, \xi_2, \rho_2)$ *be two feasible labels, then* L_1 *dominates* L_2 *if all the following relations are satisfied:* (i) $i_1 = i_2$, (ii) $\mathbb{V}(p_1) = \mathbb{V}(p_2)$, (iii) $l_1 \geq l_2$, (iv) $e_1 + d_1 \leq e_2 + d_2$, (v) $d_1 \leq d_2$, *and* (vi) $c_{p_1} \leq c_{p_2}$.

Unlike the labeling algorithm for solving SubSLRP2, which aims to find a route with the minimum reduced cost, the route enumeration here aims to find all routes with reduced costs no greater than $(UB_g - LB_1)$ w.r.t. $\boldsymbol{\pi}^1$. Thus, the completion bound can also be applied by setting c'_{ub} in Proposition 3 as $(UB_g - LB_1)$. However, due to relation (ii) in Lemma 3, which requires the dominated label and dominating label visit the same set of customers, the rollback pruning approach introduced earlier for solving SubSLRP2 cannot be adapted for route enumeration here.

4.4 Step 3 of Phase 1: Solve SLRP1(UB_g)

We solve SLRP1(UB_g) through a column-and-cut generation procedure, where the initial feasible solution is the optimal solution of SLRP2. If the guessed value UB_g is a valid upper bound on the optimal objective value, then routes that are not in $\mathbb{R}_1$ cannot appear in any optimal solution. Thus, in this case, we can solve the pricing problem in column generation directly by computing the reduced cost of each route in $\mathbb{R}_1$ and finding the one with the smallest reduced cost.

Being different from the method for solving SLRP2, here we incorporate SRC constraints (3.30). Let $\mathbb{C}_{src}$ and $\mathbf{v} \geq \mathbf{0}$ denote the set of indices and dual value vector associated with constraints (3.30), respectively. Let $\boldsymbol{\pi}^2 = (\mathbf{f}, \mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{m}, \mathbf{o}, \mathbf{u}, \mathbf{v})$ denote the optimal dual solution of the restricted master problem. For each route $r \in \mathbb{R}_1$, its reduced cost w.r.t. $\boldsymbol{\pi}^2$ can be represented as follows:

$$\begin{aligned} c_r^{'}(\boldsymbol{\pi}^2) = c_r - \sum_{i \in \mathbb{V}} f_i \alpha_{ir} - \sum_{t \in \mathbb{T}} g_t \beta_{tr} - \sum_{(t_1,t_2) \in \mathbb{C}_{rwt}} h_{t_1 t_2} \bar{\beta}_{t_1 t_2 r} - \sum_{\bar{\mathbb{V}} \in \mathbb{C}_{sr3}} k_{\bar{\mathbb{V}}} \eta_{r\bar{\mathbb{V}}2} \\ - \sum_{\bar{\mathbb{V}} \in \mathbb{C}_{sr5\text{-}2}} m_{\bar{\mathbb{V}}} \eta_{r\bar{\mathbb{V}}2} - \sum_{\bar{\mathbb{V}} \in \mathbb{C}_{sr5\text{-}3}} o_{\bar{\mathbb{V}}} \eta_{r\bar{\mathbb{V}}3} - \sum_{(i,\bar{\mathbb{V}}) \in \mathbb{C}_{el}} u_{i\bar{\mathbb{V}}} \lambda_{ir\bar{\mathbb{V}}} - \sum_{\bar{\mathbb{V}} \in \mathbb{C}_{src}} v_{\bar{\mathbb{V}}} \mu_{r\bar{\mathbb{V}}}. \end{aligned} \quad (4.15)$$

Accordingly, the pricing problem of SLRP1(UB_g) can be formulated as

$$(\text{SubSLRP1}(UB_g)) \qquad z_{\text{SubSLRP1}(UB_g)} = \min_{r \in \mathbb{R}_1} \left\{ c_r^{'}(\boldsymbol{\pi}^2) \right\}. \quad (4.16)$$

After solving SLRP1(UB_g), we remove routes with reduced cost greater than ($UB_g - LB_2$) w.r.t. $\boldsymbol{\pi}^2$ from set $\mathbb{R}_1$, and the set of remaining routes is denoted by $\mathbb{R}_2$.

4.5 Phase 2: Solve SLSP(UB_g)

We newly develop a column-and-cut generation procedure to solve SLSP(UB_g) based on the route set $\mathbb{R}_2$ in Phase 2. For the cut generation, we initialize the constraint index sets, $\mathbb{C}_{sr3}$, $\mathbb{C}_{sr5\text{-}2}$, $\mathbb{C}_{sr5\text{-}3}$, $\mathbb{C}_{el}$ and $\mathbb{C}_{src}$, to their values obtained in Phase 1, and initialize the set $\mathbb{T}$ to an empty set. For the column generation, an initial feasible solution is needed. For this, we define a dummy feasible trip $s_0 = (a_0, r_0)$ with $c_{s_0} = M$, where M is a large positive constant, and r_0 is a dummy feasible route as defined in Section 4.2. The restricted master problem of SLSP(UB_g) contains only a subset of columns and constraints in SLSP(UB_g), where the complete trip set $\mathbb{S}$ is replaced by a trip subset $\mathbb{S}'$, which is initialized to $\{s_0\}$. Let y_0 denote a binary decision variable associated with trip s_0. Accordingly, the solution $\mathbf{y}$ with $y_0 = 1$ and $y_s = 0$ for all $s \in \mathbb{S}' \setminus \{s_0\}$ is always feasible to the restricted master problem. However, when the column generation terminates with $y_0 > 0$ in the optimal solution of the last restricted master problem, and when M is sufficiently large, it can be guaranteed that either the original problem is infeasible or UB_g is not a valid upper bound [7].

Consider each iteration of the column generation. With a slight abuse of notations, we use $\boldsymbol{\pi}^3 = (\mathbf{f}, \mathbf{g}, \mathbf{k}, \mathbf{m}, \mathbf{o}, \mathbf{u}, \mathbf{v})$ to denote the dual solution of the current restricted master problem, where $\mathbf{f}, \mathbf{g} \leq \mathbf{0}, \mathbf{k} \leq \mathbf{0}, \mathbf{m} \leq \mathbf{0}, \mathbf{o} \leq \mathbf{0}, \mathbf{u} \leq \mathbf{0}, \mathbf{v} \geq \mathbf{0}$ are associated with constraints (3.39), (3.40), (3.43), (3.44), (3.45), (3.46), (3.47) respectively. The reduced cost of a trip $s \in \mathbb{S}$ w.r.t. $\boldsymbol{\pi}^3$ can thus be represented as follows:

$$\begin{aligned} c_s^{'}(\boldsymbol{\pi}^3) = c_s - \sum_{i\in\mathbb{V}} f_i\alpha_{is} - \sum_{t\in\mathbb{T}} g_t\gamma_{ts} - \sum_{\bar{\mathbb{V}}\in\mathbb{C}_{sr3}} k_{\bar{\mathbb{V}}}\eta_{s\bar{\mathbb{V}}2} - \sum_{\bar{\mathbb{V}}\in\mathbb{C}_{sr5\text{-}2}} m_{\bar{\mathbb{V}}}\eta_{s\bar{\mathbb{V}}2} \\ - \sum_{\bar{\mathbb{V}}\in\mathbb{C}_{sr5\text{-}3}} o_{\bar{\mathbb{V}}}\eta_{s\bar{\mathbb{V}}3} - \sum_{(i,\bar{\mathbb{V}})\in\mathbb{C}_{el}} u_{i\bar{\mathbb{V}}}\lambda_{is\bar{\mathbb{V}}} - \sum_{\bar{\mathbb{V}}\in\mathbb{C}_{src}} v_{\bar{\mathbb{V}}}\mu_{s\bar{\mathbb{V}}}. \end{aligned} \tag{4.17}$$

For each route $r \in \mathbb{R}_2$, let $\mathbb{S}(r)$ denote the set of trips with their routes equal to r, i.e., $\mathbb{S}(r) = \{s \in \mathbb{S} : r_s = r\} = \{(\tau, r) : \tau \in [e_r, l_r]\}$. If UB_g is a valid upper bound, routes that are not in $\mathbb{R}_2$ cannot appear in any optimal solution, and thus, the trip set $\mathbb{S}$ in SLSP(UB_g) can be substituted with $\bigcup_{r\in\mathbb{R}_2}\mathbb{S}(r)$. Accordingly, the pricing problem of SLSP(UB_g) can be formulated as follows:

$$(\text{SubSLSP}(UB_g)) \qquad z_{\text{SubSLSP}(UB_g)} = \min_{r\in\mathbb{R}_2}\min_{s\in\mathbb{S}(r)}\left\{c_s^{'}(\boldsymbol{\pi}^3)\right\}. \tag{4.18}$$

According to equation (4.17) we can reformulate $\min_{s\in\mathbb{S}(r)}\left\{c_s^{'}(\boldsymbol{\pi}^3)\right\}$ as follows

$$\begin{aligned} \min_{s\in\mathbb{S}(r)}\left\{c_s^{'}(\boldsymbol{\pi}^3)\right\} = \min_{s\in\mathbb{S}(r)}\left\{-\sum_{t\in\mathbb{T}} g_t\gamma_{ts}\right\} + c_r - \sum_{i\in\mathbb{V}} f_i\alpha_{ir} - \sum_{\bar{\mathbb{V}}\in\mathbb{C}_{sr3}} k_{\bar{\mathbb{V}}}\eta_{r\bar{\mathbb{V}}2} - \sum_{\bar{\mathbb{V}}\in\mathbb{C}_{sr5\text{-}2}} m_{\bar{\mathbb{V}}}\eta_{r\bar{\mathbb{V}}2} \\ - \sum_{\bar{\mathbb{V}}\in\mathbb{C}_{sr5\text{-}3}} o_{\bar{\mathbb{V}}}\eta_{r\bar{\mathbb{V}}3} - \sum_{(i,\bar{\mathbb{V}})\in\mathbb{C}_{el}} u_{i\bar{\mathbb{V}}}\lambda_{ir\bar{\mathbb{V}}} - \sum_{\bar{\mathbb{V}}\in\mathbb{C}_{src}} v_{\bar{\mathbb{V}}}\mu_{r\bar{\mathbb{V}}}. \end{aligned} \tag{4.19}$$

We next show that the pricing problem SubSLSP(UB_g) can be solved by inspection based on $\mathbb{R}_2$ and some discrete sets of time points. To achieve this, due to (4.17) and (4.18) we only need to show that $\min_{s\in\mathbb{S}(r)}\left\{-\sum_{t\in\mathbb{T}} g_t\gamma_{ts}\right\}$ can be solved by inspection based on a discrete set of time points for each route $r \in \mathbb{R}_2$.

First, for $\mathbb{T}$ and each route $r \in \mathbb{R}_2$, we define a discrete set of time points, $\mathbb{DT}_r = \{l_r\} \cup \{e_r \leq t \leq l_r : t \in \mathbb{T}\} \cup \{e_r \leq t - d_r \leq l_r : t \in \mathbb{T}\}$, and sort elements in $\mathbb{DT}_r$ in a non-decreasing order so that $\mathbb{DT}_r = \{t_1^r, t_2^r, \ldots, t_n^r\}$ with $e_r \leq t_1^r \leq t_2^r \leq \cdots \leq t_n^r = l_r$. Then, we establish Proposition 4 to facilitate the proofs of Proposition 5, Proposition 7 and Proposition 8.

Proposition 4. *Consider any feasible route r, any discrete set of time points $\mathbb{T}$ and the associated $\mathbb{DT}_r = \{t_1^r, t_2^r, \ldots, t_n^r\}$ defined above. For any $1 \leq w \leq n-1$, any $\tau \in (t_w^r, t_{w+1}^r]$ and any $t \in \mathbb{T}$, $\tau \leq t < \tau + d_r$ holds true if and only if $t_{w+1}^r \leq t < t_{w+1}^r + d_r$. For any $\tau \in [e_r, t_1^r]$ and any $t \in \mathbb{T}$, $\tau \leq t < \tau + d_r$ holds true if and only if $t_1^r \leq t < t_1^r + d_r$.*

Proof. For any $1 \leq w \leq n-1$, any $\tau \in (t_w^r, t_{w+1}^r]$ and any $t \in \mathbb{T}$, according to the definition of $\mathbb{DT}_r$, we have $(t_w^r, t_{w+1}^r) \cap \mathbb{T} = \emptyset$ and $(t_w^r, t_{w+1}^r) \cap \{t' - d_r : t' \in \mathbb{T}\} = \emptyset$.

(i) if $\tau \leq t$ then $t_{w+1}^r \leq t$, since $(t_w^r, t_{w+1}^r) \cap \mathbb{T} = \emptyset$;

(ii) if $t < \tau + d_r$ then $t < t_{w+1}^r + d_r$ since $\tau \leq t_{w+1}^r$;

(iii) if $t_{w+1}^r \leq t$ then $\tau \leq t$ since $\tau \leq t_{w+1}^r$;

(iv) if $t < t_{w+1}^r + d_r$ then $t < \tau + d_r$, since $(t_w^r, t_{w+1}^r) \cap \{t' - d_r : t' \in \mathbb{T}\} = \emptyset$.

Thus $\tau \leq t < \tau + d_r$ holds true if and only if $t_{w+1}^r \leq t < t_{w+1}^r + d_r$.

For any $\tau \in [e_r, t_1^r]$ and any $t \in \mathbb{T}$, the statement in this Proposition holds true naturally if $e_r = t_1^r$, and can be verified in the same way with the above part if $e_r < t_1^r$, which completes the proof of Proposition 4. □

Now we establish Proposition 5 to simplify the computation of $\min_{s \in \mathbb{S}(r)} \left\{-\sum_{t \in \mathbb{T}} g_t \gamma_{ts}\right\}$ for each route $r \in \mathbb{R}_2$.

Proposition 5. *Consider any feasible route r, any discrete set of time points $\mathbb{T}$ and the associated $\mathbb{DT}_r = \{t_1^r, t_2^r, \ldots, t_n^r\}$ defined above, we have*

$$\min_{s\in\mathbb{S}(r)}\left\{-\sum_{t\in\mathbb{T}} g_t\gamma_{ts}\right\} = \min_{\tau\in\mathbb{DT}_r}\left\{-\sum_{t\in\mathbb{T}:\tau\le t<\tau+d_r} g_t\right\}. \tag{4.20}$$

Proof. According to Proposition 4, we have

$$-\sum_{t\in\mathbb{T}:\tau\le t<\tau+d_r} g_t = -\sum_{t\in\mathbb{T}:t_1^r\le t<t_1^r+d_r} g_t, \quad \forall\ \tau\in[e_r, t_1^r], \tag{4.21}$$

$$-\sum_{t\in\mathbb{T}:\tau\le t<\tau+d_r} g_t = -\sum_{t\in\mathbb{T}:t_{w+1}^r\le t<t_{w+1}^r+d_r} g_t, \quad \forall\ 1\le w\le n-1, \tau\in(t_w^r, t_{w+1}^r]. \tag{4.22}$$

By the definition of γ_{ts}, we have

$$\begin{aligned}&\min_{s\in\mathbb{S}(r)}\left\{-\sum_{t\in\mathbb{T}} g_t\gamma_{ts}\right\} = \min_{\tau\in[e_r,l_r]}\left\{-\sum_{t\in\mathbb{T}:\tau\le t<\tau+d_r} g_t\right\}\\ &=\min\left\{\min_{\tau\in[e_r,t_1^r]}\left\{-\sum_{t\in\mathbb{T}:\tau\le t<\tau+d_r} g_t\right\}, \min_{1\le w\le n-1,\tau\in(t_w^r,t_{w+1}^r]}\left\{-\sum_{t\in\mathbb{T}:\tau\le t<\tau+d_r} g_t\right\}\right\}.\end{aligned} \tag{4.23}$$

According to equations (4.21) and (4.22), we have

$$\begin{aligned}&\min\left\{\min_{\tau\in[e_r,t_1^r]}\left\{-\sum_{t\in\mathbb{T}:\tau\le t<\tau+d_r} g_t\right\}, \min_{1\le w\le n-1,\tau\in(t_w^r,t_{w+1}^r]}\left\{-\sum_{t\in\mathbb{T}:\tau\le t<\tau+d_r} g_t\right\}\right\}\\ &=\min_{0\le w\le n-1}\left\{-\sum_{t\in\mathbb{T}:t_{w+1}^r\le t<t_{w+1}^r+d_r} g_t\right\}.\end{aligned} \tag{4.24}$$

With the definition of $\mathbb{DT}_r$, we have

$$\min_{0\le w\le n-1}\left\{-\sum_{t\in\mathbb{T}:t^r_{w+1}\le t<t^r_{w+1}+d_r} g_t\right\}=\min_{\tau\in\mathbb{DT}_r}\left\{-\sum_{t\in\mathbb{T}:\tau\le t<\tau+d_r} g_t\right\}, \tag{4.25}$$

which completes the proof of Proposition 5. □

Thus the pricing problem SubSLSP(UB_g) can be reformulated as follows:

$$\begin{aligned} z_{\text{SubSLSP}(UB_g)} = \min_{r\in\mathbb{R}_2}\Bigg\{ & \min_{\tau\in\mathbb{DT}_r}\left\{-\sum_{t\in\mathbb{T}:\tau\le t<\tau+d_r} g_t\right\} + c_r - \sum_{i\in\mathbb{V}} f_i\alpha_{ir} - \sum_{\bar{\mathbb{V}}\in\mathbb{C}_{sr3}} k_{\bar{\mathbb{V}}}\eta_{r\bar{\mathbb{V}}2} \\ & - \sum_{\bar{\mathbb{V}}\in\mathbb{C}_{sr5\text{-}2}} m_{\bar{\mathbb{V}}}\eta_{r\bar{\mathbb{V}}2} - \sum_{\bar{\mathbb{V}}\in\mathbb{C}_{sr5\text{-}3}} o_{\bar{\mathbb{V}}}\eta_{r\bar{\mathbb{V}}3} - \sum_{(i,\bar{\mathbb{V}})\in\mathbb{C}_{el}} u_{i\bar{\mathbb{V}}}\lambda_{ir\bar{\mathbb{V}}} - \sum_{\bar{\mathbb{V}}\in\mathbb{C}_{src}} v_{\bar{\mathbb{V}}}\mu_{r\bar{\mathbb{V}}}\Bigg\}, \end{aligned} \tag{4.26}$$

which can be solved by enumerating all routes $r\in\mathbb{R}_2$ and all departure times $\tau\in\mathbb{DT}_r$, since both $\mathbb{R}_2$ and $\mathbb{DT}_r$ for each $r\in\mathbb{R}_2$ are finite and are often of small or moderate sizes.

Furthermore, the dual solution $\boldsymbol{\pi}^3$ of SLSP(UB_g) can be utilized to eliminate routes that cannot appear in any optimal solution from $\mathbb{R}_2$. To achieve this, we define $c'_r(\boldsymbol{\pi}^3)$ for each route $r\in\mathbb{R}_2$ as follows.

$$\begin{aligned} c'_r(\boldsymbol{\pi}^3) = c_r & - \sum_{i\in\mathbb{V}} f_i\alpha_{ir} - \sum_{t\in\mathbb{T}} g_t\beta_{tr} - \sum_{\bar{\mathbb{V}}\in\mathbb{C}_{sr3}} k_{\bar{\mathbb{V}}}\eta_{r\bar{\mathbb{V}}2} - \sum_{\bar{\mathbb{V}}\in\mathbb{C}_{sr5\text{-}2}} m_{\bar{\mathbb{V}}}\eta_{r\bar{\mathbb{V}}2} \\ & - \sum_{\bar{\mathbb{V}}\in\mathbb{C}_{sr5\text{-}3}} o_{\bar{\mathbb{V}}}\eta_{r\bar{\mathbb{V}}3} - \sum_{(i,\bar{\mathbb{V}})\in\mathbb{C}_{el}} u_{i\bar{\mathbb{V}}}\lambda_{ir\bar{\mathbb{V}}} - \sum_{\bar{\mathbb{V}}\in\mathbb{C}_{src}} v_{\bar{\mathbb{V}}}\mu_{r\bar{\mathbb{V}}}. \end{aligned} \tag{4.27}$$

Proposition 6 below indicates that every route r with $c'_r(\boldsymbol{\pi}^3) > UB_g - LB_3$ can be eliminated from $\mathbb{R}_2$. As a result, the set of remaining routes is denoted by $\mathbb{R}_3$.

Proposition 6. *If UB_g is a valid upper bound, then every route $r\in\mathbb{R}_2$ with $c'_r(\boldsymbol{\pi}^3) > UB_g - LB_3$ cannot appear in any optimal solution.*

Proof. Consider the case where UB_g is a valid upper bound of the problem. For any trip $s = (\tau_s, r_s)$ and any time point $t \in \mathbb{T}$, by definition, we have $\gamma_{ts} \geq \beta_{tr_s}$. This, together with $\mathbf{g} \leq \mathbf{0}$, implies that $-\sum_{t\in\mathbb{T}} g_t\gamma_{ts} \geq -\sum_{t\in\mathbb{T}} g_t\beta_{tr_s}$. Moreover, by definition we know that $c_s = c_{r_s}$, $\alpha_{is} = \alpha_{ir_s}$, $\eta_{s\bar{\mathbb{V}}k} = \eta_{r_s\bar{\mathbb{V}}k}$, $\lambda_{is\bar{\mathbb{V}}} = \lambda_{ir_s\bar{\mathbb{V}}}$ and $\mu_{s\bar{\mathbb{V}}} = \mu_{r_s\bar{\mathbb{V}}}$. Thus, it can be seen from (4.17) and (4.27) that

$$c_s'(\boldsymbol{\pi}^3) \geq c_{r_s}'(\boldsymbol{\pi}^3). \tag{4.28}$$

According to the variable fixing technique indicated in Lemma 1, every trip s with its reduced cost $c_s'(\boldsymbol{\pi}^3)$ greater than $(UB_g - LB_3)$ w.r.t. $\boldsymbol{\pi}^3$ cannot appear in any optimal solution. Thus, if $c_r'(\boldsymbol{\pi}^3) > UB_g - LB_3$, then by (4.28) we have $c_s'(\boldsymbol{\pi}^3) > UB_g - LB_3$ for every trip s with $r_s = r$, and all such trips cannot appear in any optimal solution. Hence, route r cannot appear in any optimal solution, which completes the proof of Proposition 6. □

4.6 Phase 3: Close the Integrality Gap

Consider relaxation SP($\mathbb{T}$) defined in Section 3.2 and relaxations SP(UB_g) and SP($\mathbb{T}, UB_g$) defined in Section 4.1. We are now going to illustrate the details on how to close the integrality gap in Phase 3 of our solution method.

First, we initialize set $\mathbb{T}$ with the time points that are obtained in the cut generation for solving SLSP(UB_g) in Phase 2. As introduced in Section 4.1, we need to solve SP($\mathbb{T}, UB_g$) in Phase 3, which results in the following three cases:

(i) If SP($\mathbb{T}, UB_g$) is infeasible or $z_{\text{SP}(\mathbb{T},UB_g)} > UB_g$, then we know that either the CMTVRPTW instance is infeasible or UB_g is not a valid upper bound. Accordingly, we increase the value of UB_g and turn to Step 2 of Phase 1, so as to repeat the route enumeration if the gap between UB_g and $LB1$ does not exceed gap_{max}.

(ii) If $\text{SP}(\mathbb{T}, UB_g)$ is feasible and $z_{\text{SP}(\mathbb{T},UB_g)} \leq UB_g$, but the optimal solution $\mathbf{y}$ of $\text{SP}(\mathbb{T}, UB_g)$ is infeasible to SP, then there must exist some WT constraints that are violated by $\mathbf{y}$. Accordingly, we separate such constraints, add them to enlarged set $\mathbb{T}$, and repeat Phase 3 to solve $\text{SP}(\mathbb{T}, UB_g)$ again.

(iii) If $\text{SP}(\mathbb{T}, UB_g)$ is feasible, $z_{\text{SP}(\mathbb{T},UB_g)} \leq UB_g$, and the optimal solution $\mathbf{y}$ of $\text{SP}(\mathbb{T}, UB_g)$ is feasible to SP, then we obtain that $z_{\text{SP}} \leq z_{\text{SP}(\mathbb{T},UB_g)} \leq UB_g$ since $\mathbf{y}$ is feasible to SP. Moreover, since $\text{SP}(\mathbb{T})$ is a relaxation of SP, we have that $z_{\text{SP}(\mathbb{T})} \leq z_{\text{SP}} \leq UB_g$. That it, UB_g is also a valid upper bound on $\text{SP}(\mathbb{T})$, so we have $z_{\text{SP}(\mathbb{T},UB_g)} = z_{\text{SP}(\mathbb{T})}$. Therefore, it can be seen that $z_{\text{SP}(\mathbb{T},UB_g)} = z_{\text{SP}(\mathbb{T})} \leq z_{\text{SP}}$. Hence, $z_{\text{SP}(\mathbb{T},UB_g)} = z_{\text{SP}}$ and $\mathbf{y}$ is optimal for SP.

We next illustrate how to solve $\text{SP}(\mathbb{T}, UB_g)$, which contains an infinite number of variables since the domain of the departure time at the depot of each trip is a continuous interval. To tackle this challenge, we establish Proposition 7 below, where the discrete set of time points $\mathbb{DT}_r$ is defined as in Section 4.5. It indicates that only a limited number of trips need to be considered when solving $\text{SP}(\mathbb{T}, UB_g)$.

Proposition 7. *If a trip $s = (\tau, r)$, where $\tau \in [e_r, l_r] \setminus \mathbb{DT}_r$, appears in an optimal solution to $\text{SP}(\mathbb{T}, UB_g)$, then trip s can be substituted with another trip $s' = (\tau', r)$, where $\tau' = \min\{t \in \mathbb{DT}_r : t \geq \tau\}$, without impairing the optimality of the solution.*

Proof. According to Proposition 4, for any $t \in \mathbb{T}$, $\tau \leq t < \tau + d_r$ if and only if $\tau' \leq t < \tau' + d_r$, thus we have $\gamma_{ts} = \gamma_{ts'}$ according to the definition of parameter γ. Moreover, by definition it can be seen that $\alpha_{is} = \alpha_{ir} = \alpha_{is'}$ for $i \in \mathbb{V}$ and $c_s = c_r = c_{s'}$. For any optimal solution to $\text{SP}(\mathbb{T}, UB_g)$ that contains trip s, one can replace s with s' without violating constraints (3.39) and (3.42) or changing the objective value, and thus, resulting in another optimal solution to SP($\mathbb{T}, UB_g$). This proves Proposition 7. $\square$

By Proposition 7, to solve SP($\mathbb{T}, UB_g$), we only need to consider the trips in $\mathbb{S}(\mathbb{T}, \mathbb{R}_3) = \{(\tau, r) : r \in \mathbb{R}_3, \tau \in \mathbb{DT}_r\}$, which is a discrete set with a finite number of elements, so the resulting SP($\mathbb{T}, UB_g$) can be solved directly by an IP solver. Furthermore, the cardinality of $\mathbb{S}(\mathbb{T}, \mathbb{R}_3)$ can further be reduced by removing trips with reduced cost greater than $(UB_g - LB_3)$ w.r.t. $\boldsymbol{\pi}^3$.

For the time discretization in our three-phase solution method, we remove all the elements in $\mathbb{T}$ when Phase 1 is completed, and initialize $\mathbb{T}$ with time points separated in Phase 2 at the beginning of Phase 3. In case (ii) considered after solving SP($\mathbb{T}, UB_g$) in Phase 3, we apply the separation method described in Section 3.2 to separate only a small number of time points (five at most) that correspond to violated WT constraints. Following this way, $|\mathbb{T}|$ can be kept small and $|\mathbb{S}(\mathbb{T}, \mathbb{R}_3)|$ is in a moderate size, so that SP($\mathbb{T}, UB_g$) can be solved more efficiently.

Note that SP($\mathbb{T}, UB_g$) may have multiple optimal solutions, and maybe some of them are infeasible to SP but others are feasible to SP. So in case (ii) considered after solving SP($\mathbb{T}, UB_g$) in Phase 3, i.e., when SP($\mathbb{T}, UB_g$) is feasible, $z_{\text{SP}(\mathbb{T}, UB_g)} \leq UB_g$ and an optimal solution $\mathbf{y}$ of SP($\mathbb{T}, UB_g$) is infeasible to SP, we try to determine whether there exists another optimal solution of SP($\mathbb{T}, UB_g$) feasible to SP as follows. Denote $\mathbb{R}^{\mathbf{y}} = \{r_s : s \in \mathbb{S}, y_s = 1\}$ and $\mathbf{x}^{\mathbf{y}}$ where $x_r^{\mathbf{y}} = 1$ for all $r \in \mathbb{R}^{\mathbf{y}}$ and $x_r^{\mathbf{y}} = 0$ for all $r \in \mathbb{R} \setminus \mathbb{R}^{\mathbf{y}}$. If $|\mathbb{R}^{\mathbf{y}}| = \zeta(K, \mathbb{R}^{\mathbf{y}})$, then $\mathbf{x}^{\mathbf{y}}$ is a feasible solution of RP, and there exists a feasible solution $\hat{\mathbf{y}}$ of SP where $\mathbb{R}^{\hat{\mathbf{y}}} = \mathbb{R}^{\mathbf{y}}$, which can be obtained by setting departure times associated with routes in $\mathbb{R}^{\mathbf{y}}$ appropriately. Note that $\hat{\mathbf{y}}$ is also feasible to SP($\mathbb{T}, UB_g$) since $\mathbb{T} \subset [a_0, b_0]$ and fewer WT constraints are considered in SP($\mathbb{T}, UB_g$) than in SP. Because the objective values of solutions $\mathbf{y}$, $\mathbf{x}^{\mathbf{y}}$ and $\hat{\mathbf{y}}$ are equal, we conclude that $\hat{\mathbf{y}}$ is an optimal solution of

SP($\mathbb{T}, UB_g$) feasible to SP. To determine whether $|\mathbb{R}^{\mathbf{y}}| = \zeta(K, \mathbb{R}^{\mathbf{y}})$ is satisfied, we develop a set-partitioning model of the TOPTW as Paradiso et al. [35], and solve the model with an IP solver where columns are enumerated directly since $|\mathbb{R}^{\mathbf{y}}|$ and the number of columns is often small.

4.7 Variable Fixing for the Departure Time

In this section, we apply the variable fixing technique to shrink the domain of departure times that need to be considered in solving the CMTVRPTW. For any feasible route $r \in \mathbb{R}$, with the definition of $t_0^r = e_r$ and the discrete set of time points $\mathbb{DT}_r = \{t_1^r, t_2^r, \ldots, t_n^r\}$ defined in Section 4.5, we can establish Proposition 8 to identify departure time intervals that need not to be considered.

Proposition 8. *For any feasible route r, any $w \in \{0, 1, \ldots, n\}$, let $c_{rw}^{'}$ denote the reduced cost of trip $s = (t_w^r, r)$ w.r.t. $\boldsymbol{\pi}^3$. If UB_g is a valid upper bound and $LB_3 + c_{rw}^{'} > UB_g$, then trip (e_r, r) cannot appear in any optimal solution if $w = 0$, and trip (τ, r) for any $\tau \in (t_{w-1}^r, t_w^r]$ cannot appear in any optimal solution if $w \in \{1, \ldots, n\}$.*

Proof. If $w = 0$ and $LB_3 + c_{rw}^{'} > UB_g$, according to the variable fixing technique, the trip (t_0^r, r), i.e., (e_r, r), cannot appear in any optimal solution.

For any $w \in \{1, \ldots, n\}$ and $\tau \in (t_{w-1}^r, t_w^r]$, denote trip $s = (\tau, r)$ with reduced cost $c_s^{'}$ w.r.t. $\boldsymbol{\pi}^3$ and trip $s' = (t_w^r, r)$ with reduced cost $c_{s'}^{'} = c_{rw}^{'}$ w.r.t. $\boldsymbol{\pi}^3$. Because $r_s = r = r_{s'}$, we have $c_s = c_{s'}$, $\alpha_{is} = \alpha_{is'}$, $\eta_{s\bar{\mathbb{V}}2} = \eta_{s'\bar{\mathbb{V}}2}$, $\eta_{s\bar{\mathbb{V}}3} = \eta_{s'\bar{\mathbb{V}}3}$, $\lambda_{is\bar{\mathbb{V}}} = \lambda_{is'\bar{\mathbb{V}}}$, $\mu_{s\bar{\mathbb{V}}} = \mu_{s'\bar{\mathbb{V}}}$. Moreover, according to Proposition 4, for any $t \in \mathbb{T}$, $\tau \leq t < \tau + d_r$ if and only if $t_w^r \leq t < t_w^r + d_r$, so $\gamma_{ts} = \gamma_{ts'}$. Therefore, we have $c_s^{'} = c_{s'}^{'} = c_{rw}^{'}$. Thus, if $LB_3 + c_{rw}^{'} > UB_g$, then $LB_3 + c_s^{'} > UB_g$, which, together with Lemma 1, implies that trip $s = (\tau, r)$ cannot appear in any optimal solution to the CMTVRPTW. This completes the proof of Proposition 8. $\square$

The variable fixing technique indicated in Proposition 8 can be applied to remove some trips from $\mathbb{S}(\mathbb{T}, \mathbb{R}_3)$ and some routes from $\mathbb{R}_3$. In particular, if $LB_3 + c'_{rw} > UB_g$, we can remove the trip (t^r_w, r) from $\mathbb{S}(\mathbb{T}, \mathbb{R}_3)$. If $LB_3 + c'_{rw} > UB_g$ for all $t^r_w \in \{e_r\} \cup \mathbb{DT}_r$, we can remove the route r from $\mathbb{R}_3$.

Chapter 5

Computational Results

In this chapter, we report our computational results to demonstrate the effectiveness and efficiency of the newly proposed three-phase exact method for solving the CMTVRPTW and its variants. We implemented the method in C++ and applied the optimization solver of CPLEX 12.10 to solve LP and IP models. The computational experiments were conducted on a desktop PC with Intel(R) Core(TM) i7-10700 CPU @ 2.90GHz (8 cores) and 32GB of RAM.

We compared the performance of our three-phase method (or TPM in short) with that of Yang [46] (or EPCEM in short), as the EPCEM of Yang [46] is the best exact method known in the literature for the CMTVRPTW and its variants. They solve benchmark instances with no less than 70 customers on a workstation running Red Hat Enterprise Linux 8.1 with Intel(R) Core(TM) i9-9900K CPU @ 3.60GHz (8 cores) and 64GB of RAM. That is, they use a device with higher performance than ours. Their program is implemented in C++, and Gurobi 9.1.1 is used as the LP and IP solver.

Each benchmark instance reported in this chapter is the same as that used by

Yang [46], and has 70 or 100 customer nodes. All the 25-customer, 40-customer and 50-customer benchmark instances are not reported in this chapter because they can be solved very efficiently by both the two solution methods. To make the comparisons concise, we either do not show computational results on 80-customer instances because they are similar to that on 70-customer instances.

To find all the benchmark instances reported in this chapter, we refer the reader to the link https://github.com/Yu1423/CMTVRPTW/tree/main/Data. From each instance, we can obtain the following data directly or by computation: the set of customers $\mathbb{V}$, the set of arcs $\mathbb{A}$, the fleet size K, the vehicle capacity Q, the demand quantity q_i, time window $[a_i, b_i]$ and service time st_i for each customer $i \in \mathbb{V}$, the planning time horizon (the time window for the depot) $[a_0, b_0]$, the travel cost c_{ij} and travel time t'_{ij} for each arc $(i, j) \in \mathbb{A}$.

We set some critical parameters as follows. When solving a pricing problem, at most 50 columns with negative reduced cost are added into the restricted master problem at each iteration of column generation. The time for solving a separation problem is limited by 0.5 seconds, and at most 5 valid inequalities of each category are added into the restricted master problem at each iteration of cut generation. To control the time for solving SLRP2, the number of SR3 constraints separated is limited by 1000, and the numbers of SR5-2, SR5-3, and EL valid inequalities separated are limited by 100. However, when solving SLRP1 and SLSP, the maximum number of valid inequalities separated of each category is limited by 2000. At the beginning of column generation, columns (excluding r_0 and s_0) in $\mathbb{R}'$ and $\mathbb{S}'$ with reduced costs greater than the integrality gap are removed according to the variable fixing technique, thus $|\mathbb{R}'|$ and $|\mathbb{S}'|$ are kept small and restricted master problems can be solved efficiently. When solving SLRP2, if the elapsed time is greater than 1800 seconds at the end of column generation, we will terminate the column-and-cut generation procedure and turn to Step 2 of Phase 1. In Phase 3, if the number of candidate trips $|\mathbb{S}(\mathbb{T}, \mathbb{R}_3)|$ is greater than two million, we terminate the algorithm without finding an optimal solution. Moreover, for guessing the upper bound, we set $\Delta_{gap} = 0.3\%$ and $gap_{max} = 12\%$. To strike a balance between the tightness of the valid upper bound and the computing time for obtaining such a valid upper bound, we set $gap_{ini} = 1.2\%$ for the CMTVRPTW and CMTVRPTW-LT instances, $gap_{ini} = 1.0\%$ for the CMTVRPTW-LD, CMTVRPTW-R and DRP instances. In computational results reported in this chapter, we did not apply the variable fixing technique for departure time (Proposition 8) since it leads to longer computing time for some instances.

All parameters about Cplex are set to default, except that the tolerance on the optimality gap is set as 10^{-5}, because its default value (10^{-4}) leads to premature terminations without achieving optimal objective values on a few instances. Yang [46] set such parameter to its default value, which might be the reason for which the objective values of four instances reported by Yang [46] are greater slightly than that reported by us although Yang [46] stated that these instances were solved to optimality (see Appendix A.2).

In the following sections, we report the computational results for the CMTVRPTW and four variants on benchmark instances, where input values of travel cost and time are all truncated to one decimal digit as is done in Paradiso et al. [35] and Yang [46]. To tackle these variants, only the labeling algorithm for solving the pricing problem SubSLRP2 should be adapted to handle side constraints, and other components of the solution method need not be changed. Specifically, we adapt feasibility conditions, state transition equations, and dominance rules of the labeling algorithm the same with Yang [46].

Tables 5.1 to 5.5 compare the performance of EPCEM and TPM on the same set of instances, and the following information is included: the name of the group (Group), the number of customers in each instance (N), the number of instances solved to optimality (Solved), the average computing time for closing the integrality gap (T_{close}), the average computing time for solving an instance (T_{total}), the average gap between the lower bound and the optimal objective value ($LB\%$). The average numbers are computed over instances that are solved to optimality by the two solution methods respectively, and all the numbers for the EPCEM are copied from or computed according to Yang [46]. For our detailed computational results, we refer the reader to Appendix A.2.

There are 216 70-customer instances and 216 100-customer instances reported in this chapter. With a time limit of 3 hours for each instance, Yang [46] can solve 214 70-customer instances and 168 100-customer instances, whereas we can solve all the 216 70-customer instances and 213 100-customer instances. Such advantage is much more obvious in difficult instances such as the CMTVRPTW and CMTVRPTW-LT instances. Moreover, we can solve the 100-customer CMTVRPTW instance R206 and the 100-customer CMTVRPTW-LT instance R206 within 5 and 6 hours respectively, so we are able to solve 215 100-customer instances.

5.1 Comparison on the CMTVRPTW

This test set includes 54 instances derived from type 2 Solomon instances for the VRPTW using a procedure adopted by Yang [46], where the vehicle capacity is changed to $Q = 100$, the numbers of available vehicles are changed to $K = 6$ and $K = 8$ for instances with 70 and 100 customers respectively. The reason for not considering type 1 Solomon instances is that their short planning time horizon and tight time windows prevent the vehicles from performing multiple trips [35].

Table 5.1: Comparison on the CMTVRPTW

Group_N	EPCEM				TPM			
	Solved	T_{close}	T_{total}	LB%	Solved	T_{close}	T_{total}	LB%
C2_70	8/8	3125.3	3198.9	1.1	8/8	1.6	240.2	0.12
R2_70	11/11	452.9	538.7	1.0	11/11	33.1	659.5	0.53
RC2_70	8/8	310.1	420.4	1.2	8/8	4.2	408.3	0.19
All	27/27	1202.4	1291.9	1.1	27/27	15.2	460.8	0.31
C2_100	5/8	6148.3	6693.1	1.0	8/8	8.3	424.5	0.01
R2_100	5/11	3428.4	4025.1	0.9	11/11	2901.1	4400.7	0.49
RC2_100	4/8	1539.9	1982.7	0.8	8/8	118.1	1151.3	0.37
All	14/27	3860.2	4394.4	0.9	27/27	1219.4	2259.8	0.31

The format "Group_N" of the instance name contains two parts, where C2, R2, and RC2 represent that they are derived from groups C, R, and RC of type 2 Solomon instances. In addition to the depot node, we consider the first 70 and all the 100 customer nodes when N is set as 70 and 100 respectively.

Table 5.1 summarizes computational results on the CMTVRPTW. With a time limit of 3 hours, EPCEM can solve only 14 of the 27 100-customer instances whereas our TPM can solve 26 of them, and TPM can solve the remaining one R206 within 5 hours (see Appendix A.2). For the 70-customer and 100-customer instances, the average computing times (T_{total}) of TPM is about 36% and 52% of that for EPCEM. Moreover, our procedure for closing the integrality gap is much more efficient since the fractions $\frac{T_{close}}{T_{total}}$ in TPM are about 3% and 54% in 70-customer and 100-customer instances respectively, whereas the fractions $\frac{T_{close}}{T_{total}}$ in EPCEM are about 93% and 88% in 70-customer and 100-customer instances respectively. In addition, the average value of LB% in TPM is much less than the average value of LB% in EPCEM, so we offer much tighter lower bounds. Based on the above observations, we can conclude that TPM performs better than EPCEM on the CMTVRPTW instances.

5.2 Comparison on the CMTVRPTW-LT

Introduced by Hernandez et al. [25], the CMTVRPTW-LT differs from the CMTVRPTW in the sense that the time for loading goods at the depot is considered. Specifically, let lt_i be the time at the depot for loading goods which will be delivered to customer $i \in \mathbb{V}$, then the vehicle performing route r or trip $s = (\tau_s, r_s)$ where $r_s = r$ will spend a loading time $\sum_{i \in \mathbb{V}(r)} lt_i$ at the depot before departure. Following Hernandez et al. [25], instances for the CMTVRPTW-LT are the same with that for the CMTVRPTW, except that the loading time for each customer $i \in \mathbb{V}$ is set as 20% of the service time st_i, i.e., $lt_i = 0.2st_i$.

As is shown in Table 5.2, with a time limit of 3 hours, EPCEM can solve only 12 of the 27 100-customer instances for the CMTVRPTW-LT whereas our TPM can solve 26 of them, and TPM can solve the remaining one R206 within 6 hours (see Appendix A.2). For the 70-customer and 100-customer instances, the average computing times (T_{total}) of TPM is about 28% and 71% of that for EPCEM, which shows that TPM is more efficient. Also, the average value of LB% in TPM is about one-third of that in EPCEM, so TPM offers much tighter lower bounds than EPCEM. So we conclude that TPM outperforms EPCEM on the CMTVRPTW-LT instances.

5.3 Comparison on the CMTVRPTW-LD

The CMTVRPTW-LD studied by Hernandez et al. [23] differs from the CMTVRPTW-LT in the sense that a limit $\bar{d}$ is imposed on the time that goods can be on board for each trip, where the loading time at the depot, the service time for the last customer on the trip, and the travel time from the last customer to the depot are

Table 5.2: Comparison on the CMTVRPTW-LT

Group_N	EPCEM				TPM			
	Solved	T_{close}	T_{total}	LB%	Solved	T_{close}	T_{total}	LB%
C2_70	8/8	3941.6	4017.7	1.2	8/8	8.4	319.4	0.20
R2_70	11/11	164.8	253.7	1.0	11/11	29.0	517.1	0.48
RC2_70	8/8	693.2	938.4	1.3	8/8	5.9	460.6	0.26
All	27/27	1440.4	1571.8	1.1	27/27	16.0	441.8	0.33
C2_100	3/8	4858.7	5244.1	0.9	8/8	16.7	425.6	0.03
R2_100	5/11	1362.3	2237.1	0.9	11/11	3386.2	4958.8	0.50
RC2_100	4/8	3176.7	3810.4	0.8	8/8	242.2	1205.9	0.40
All	12/27	2841.2	3513.3	0.9	27/27	1456.3	2503.6	0.33

not included. That is, for each trip, the time difference between the departure time at the depot and the arrival time at the last customer cannot be greater than $\bar{d}$. Following Yang [46], the benchmark instances are derived from type 2 Solomon instances, where $lt_i = 0.2st_i$ for all $i \in \mathbb{V}$, $\bar{d} \in \{220, 250\}$ for instances in group C2 and $\bar{d} \in \{75, 100\}$ for instances in groups R2 and RC2. In these instances, the vehicle capacities are the same as that of Solomon instances, i.e., $Q = 700$ for group C2 and $Q = 1000$ for groups R2 and RC2.

Table 5.3 reports computational results on the CMTVRPTW-LD. Due to the rigid duration limit parameters $\bar{d}$, most of these benchmark instances can be solved very efficiently by both the two solution methods. Still, our TPM can solve three more 100-customer instances than EPCEM and yield tighter lower bounds. So TPM is slightly superior to EPCEM on these CMTVRPTW-LD instances.

5.4 Comparison on the CMTVRPTW-R

Introduced by Hernandez et al. [25], neither considering the loading time nor duration limit, the CMTVRPTW with release dates (CMTVRPTW-R) takes the

Table 5.3: Comparison on the CMTVRPTW-LD

Group_N_$\bar{d}$	EPCEM				TPM			
	Solved	T_{close}	T_{total}	LB%	Solved	T_{close}	T_{total}	LB%
C2_70_220	8/8	0.1	17.9	0.5	8/8	1.6	6.0	0.39
C2_70_250	8/8	0.4	12.2	0.3	8/8	3.9	9.3	0.19
R2_70_75	11/11	0.1	13.2	0.4	11/11	3.1	17.2	0.16
R2_70_100	11/11	0.1	37.4	0.3	11/11	0.3	30.1	0.14
RC2_70_75	8/8	0.3	25.6	0.7	8/8	4.6	37.5	0.35
RC2_70_100	8/8	2.4	40.8	0.7	8/8	3.1	85.8	0.47
All	54/54	0.5	24.6	0.5	54/54	2.7	30.2	0.27
C2_100_220	8/8	6.9	360.6	0.6	8/8	6.8	16.5	0.43
C2_100_250	8/8	5.6	61.5	0.8	8/8	32.1	65.0	0.51
R2_100_75	11/11	0.9	106.1	0.4	11/11	52.3	111.3	0.20
R2_100_100	7/11	3.2	639.2	0.3	10/11	23.2	306.5	0.24
RC2_100_75	8/8	68.4	116.7	0.3	8/8	24.5	47.4	0.19
RC2_100_100	8/8	26.9	213.0	0.4	8/8	8.9	129.6	0.22
All	50/54	17.9	233.1	0.5	53/54	26.2	120.0	0.29

release dates into account. Specifically, the release date rd_i is the time at which the goods for customer i become available at the depot, and the vehicle performing route r or trip $s = (\tau_s, r_s)$ where $r_s = r$ can depart from the depot only when the goods for all customers on route r are available, i.e., no earlier than $\max_{i \in \mathbb{V}(r)} rd_i$. Following Yang [46], the instances for the CMTVRPTW-R are the same as that for the CMTVRPTW, except that the former include release dates which are set by the procedure proposed in Hernandez et al. [25] relying on the release date parameter $\kappa \in \{0.25, 0.5, 0.75\}$.

Table 5.4 summarizes computational results on the CMTVRPTW-R, which shows that TPM can solve all the 81 100-customer instances. However, EPCEM can solve only 67 of them and cannot solve one 70-customer instance. Moreover, the average time for closing the integrality gap (T_{close}) in TPM is about 3% and 13% of that in EPCEM on 70-customer and 100-customer instances respectively.

Table 5.4: Comparison on the CMTVRPTW-R

Group_N_κ	EPCEM				TPM			
	Solved	T_{close}	T_{total}	LB%	Solved	T_{close}	T_{total}	LB%
C2_70_0.25	8/8	113.4	166.4	1.0	8/8	1.8	210.8	0.00
C2_70_0.5	8/8	44.4	83.3	0.9	8/8	1.1	51.2	0.00
C2_70_0.75	8/8	4.0	39.5	0.6	8/8	1.3	70.0	0.00
R2_70_0.25	11/11	5.9	248.1	0.5	11/11	3.6	222.5	0.20
R2_70_0.5	11/11	4.3	169.4	0.4	11/11	3.8	246.3	0.13
R2_70_0.75	10/11	26.6	187.1	0.5	11/11	5.0	302.8	0.21
RC2_70_0.25	8/8	300.6	385.1	1.0	8/8	1.2	346.0	0.17
RC2_70_0.5	8/8	271.7	357.9	1.2	8/8	1.9	355.3	0.23
RC2_70_0.75	8/8	14.3	115.2	0.6	8/8	2.5	221.4	0.16
All	80/81	79.6	195.5	0.7	81/81	2.6	228.7	0.13
C2_100_0.25	7/8	3773.9	4181.5	1.4	8/8	2.0	693.2	0.02
C2_100_0.5	8/8	2640.6	2949.0	1.5	8/8	0.8	417.4	0.00
C2_100_0.75	6/8	1568.3	2279.7	1.2	8/8	6.4	346.0	0.00
R2_100_0.25	9/11	450.4	1806.9	0.5	11/11	411.7	1401.0	0.33
R2_100_0.5	8/11	57.5	1799.6	0.5	11/11	267.0	1366.2	0.35
R2_100_0.75	8/11	12.5	1284.7	0.4	11/11	182.3	1116.8	0.20
RC2_100_0.25	7/8	1298.4	1639.7	0.7	8/8	31.4	869.8	0.23
RC2_100_0.5	7/8	705.1	1122.4	0.7	8/8	190.4	1066.2	0.29
RC2_100_0.75	7/8	21.0	637.0	0.5	8/8	59.1	544.8	0.22
All	67/81	1130.4	1959.3	0.8	81/81	145.6	916.3	0.19

The average computing time (T_{total}) in TPM is about the half of that in EPCEM on 100-customer instances, and the average gap between the lower bound and the optimal objective value ($LB\%$) provided by TPM is less than a quarter of that provided by EPCEM. So we conclude that TPM performs better than EPCEM on the CMTVRPTW-R instances.

5.5 Comparison on the DRP

Taking the energy cost and battery capacity into account, the drone-routing problem (DRP) considered by Cheng et al. [11] is a variant of the CMTVRPTW. Specifically, the objective function is the sum of the travel cost and the energy cost which depends on arcs traversed and load carried. Moreover, drones should return to the depot before depleting the battery. Due to the change of objective function, rollback pruning should be adapted according to Proposition 9 in Appendix A.1.

Two sets (A and B) of benchmark instances can be produced by the procedure provided by Cheng et al. [11], where demand quantities of all customers are multiplied by 0.03 for instances in set B. Since instances in set A can be solved extremely quickly by both EPCEM and TPM, we only compare their performance on set B, and provide the performance of TPM on set A in Appendix A.2.

Table 5.5 shows computational results on the DRP instances (set B), where TPM can solve all benchmark instances to optimality whereas EPCEM cannot solve one 70-customer instance and two 100-customer instances. For 100-customer instances, the average time for closing the integrality gap ($\mathrm{T}_{\mathrm{close}}$) and the average time for solving an instance ($\mathrm{T}_{\mathrm{total}}$) in TPM are about 21% and 35% of that in EPCEM. Furthermore, the average gap between the lower bound and the optimal objective value ($LB\%$) provided by TPM is about one-fifth of that provided by EPCEM. So we conclude that TPM outperforms EPCEM on the DRP (set B) instances.

Table 5.5: Comparison on the DRP (set B)

Group_N	EPCEM				TPM			
	Solved	T_{close}	T_{total}	LB%	Solved	T_{close}	T_{total}	LB%
C2_70	8/8	0.3	6.1	0.4	8/8	8.2	20.0	0.00
R2_70	11/11	4.2	15.0	0.6	11/11	8.7	95.8	0.23
RC2_70	7/8	72.8	87.7	0.9	8/8	1.5	48.3	0.06
All	26/27	21.5	31.8	0.6	27/27	6.4	59.3	0.11
C2_100	8/8	328.9	409.8	0.9	8/8	10.9	65.5	0.00
R2_100	10/11	17.8	52.8	0.4	11/11	268.7	454.3	0.21
RC2_100	7/8	2238.1	2314.0	0.5	8/8	150.2	267.5	0.10
All	25/27	739.0	800.2	0.6	27/27	157.2	283.8	0.11

Chapter 6

Conclusions and Future Research

6.1 Conclusions

In this thesis, we propose a three-phase exact method for solving the CMTVRPTW and relevant variants. The first phase aims to solve the LP relaxation of the route-based model and enumerate a set of candidate routes containing those selected in optimal solutions. Based on these enumerated routes, we solve the LP relaxation of the trip-based model which can provide tighter lower bounds in the second phase. In the last phase, the integrality gap is closed by solving the trip-based IP model with dynamic time discretization. In this way, we can utilize both advantages of the route-based and trip-based models. Specifically, the trip-based LP and IP models have advantages in tightening the lower bound and closing the integrality gap respectively, but solving them directly is very difficult. However, the route-based model is relatively easy to solve, and once it is solved we can perform the route-enumeration, then the trip-based LP and IP models become much easier to solve based on these enumerated routes. Thus we solve the route-based LP model, the trip-based LP model, and the trip-based IP model successively.

In addition to adopting valid inequalities used for solving the CMTVRPTW in the literature, we also apply some other valid inequalities used for solving the VRP but not used in studies about the CMTVRPTW, and newly introduce RWT constraints which strengthen RSF constraints. Moreover, we propose IP models for separating some of these valid inequalities exactly when heuristic methods fail to identify violated constraints. Strengthened by these valid inequalities, the newly developed LP relaxations can provide much tighter lower bounds on the optimal objective value of the CMTVRPTW.

Results from extensive computational experiments over benchmark instances demonstrate that our newly proposed exact method is effective and efficient. We solve all the 27 benchmark instances (including 13 open ones) with 100 customers for the CMTVRPTW to optimality with a much shorter average computing time compared with the best-known exact method in the existing literature. The solution method is also adapted to solve four variants of the CMTVRPTW, and it also performs better on the benchmark instances of these variants than the best-known exact method in the existing literature. In addition to the number of solved instances and the relatively short average computing time on each instance, our solution method achieves much tighter lower bounds on the optimal objective values and much higher efficiency for closing the integrality gap.

6.2 Future Research

Potential future research directions are as follows. Firstly, we observe that for some instances, the relative gaps between the lower bounds obtained and the optimal objective values are larger than that for other instances. To reduce the relative gaps for these instances, we need to identify new valid inequalities which can strengthen the lower bounds effectively.

Secondly, from this thesis, we know that different models can provide different lower bounds, and models which can provide tighter lower bounds can be more difficult to solve. So a potential direction for future research is to develop new models for this problem, or design more efficient methods to solve existing models which can provide better lower bounds (e.g., the journey-based model).

Moreover, to further demonstrate the effectiveness and efficiency of our solution method, more extensive computational experiments should be performed. On the one hand, we should try to generate and solve instances with more than 100 customers. On the other hand, for each component of the method, we need to identify the associated effectiveness and computing effort.

To further demonstrate the robustness of our solution method and distinguish bottlenecks of instances, intensive sensitivity experiments should be done. we can compare the performance of the method on instances with changed data, such as modified time windows, fleet size, vehicle capacity, and demand quantities.

Lastly, we should consider more practical considerations in the future, such as very large numbers of customers, consistency requirements between successive planning time horizons, soft time windows, and stochastic factors (e.g., random demands and travel times). For example, consider a delivery system in a district with thousands of or even more customers, it might be impractical to apply our solution method directly to obtain an optimal solution. However, a feasible solution of good quality is often acceptable in practice even if it is not optimal. Accordingly, the very large problem size can be handled by firstly clustering customers according to their addresses and time windows, then our solution method can be applied based on these clusters.

Appendix A

A.1 Rollback Pruning for the DRP

In the labeling algorithm for the DRP, state transition equations and dominance rules introduced by Yang [46] are still valid. However, rollback pruning is not applied by them. Moreover, Proposition 2 is not valid for the DRP because the objective function includes not only travel cost but also energy cost. Instead, we apply rollback pruning with the following proposition for the DRP, where ce_{ijq} is the energy cost for a drone to travel along arc (i, j) with load q.

Proposition 9. *For a feasible label* $L_2 = (p_2, i_n, q_2, e_2, l_2, d_2, \xi_2, \rho_2)$ *where* $p_2 = (i_0, i_1, \ldots, i_n)$*, denote* $q_{\bar{w}} = \sum_{w=1}^{\bar{w}} q_{i_w}$*. If there exists an integer* $0 \leq \bar{w} \leq n - 2$ *, such that* $c_{i_n i_{\bar{w}}} + ce_{i_n i_{\bar{w}} q_{\bar{w}}} - f_{i_n} \leq \sum_{w=\bar{w}+1}^{n} (c_{i_w i_{w-1}} + ce_{i_w i_{w-1} q_{w-1}} - f_{i_w})$*, then label* L_2 *can be pruned since it is dominated by another label* $L_1 = (p_1, i_n, q_1, e_1, l_1, d_1, \xi_1, \rho_1)$ *where* $p_1 = (i_0, i_1, \ldots, i_{\bar{w}}, i_n)$*.*

We omit the proof for Proposition 9 since it can be derived in the same way as Proposition 2.

A.2 Detailed Computational Results

Detailed computational results are shown in Tables A.1 to A.6, where the following information is included: the instance name (Name), the number of customers (N), the duration parameter in the CMTVRPTW-LD ($\bar{d}$), the release date parameter in the CMTVRPTW-R (κ), the relative gap between the valid upper bound and the first lower bound $\frac{UB_g - LB1}{LB1}$ ($UB_g\%$), the optimal objective value (optimal), the relative gap between the first lower bound and the optimal objective value $\frac{\text{optimal}-LB1}{\text{optimal}}$ ($LB1\%$), the cardinality of $\mathbb{R}_1$ ($|\mathbb{R}_1|$), the relative gap between the second lower bound and the optimal objective value $\frac{\text{optimal}-LB2}{\text{optimal}}$ ($LB2\%$), the cardinality of $\mathbb{R}_2$ ($|\mathbb{R}_2|$), the relative gap between the third lower bound and the optimal objective value $\frac{\text{optimal}-LB3}{\text{optimal}}$ ($LB3\%$), the cardinality of $\mathbb{R}_3$ ($|\mathbb{R}_3|$), the computing time of Phase 3 under the valid upper bound for closing the integrality gap in seconds (T_{close}), the sum of the computing time of Step 1 of Phase 1 and the computing time from Step 2 of Phase 1 to Phase 3 under the valid upper bound in seconds (T_{valid}), and the computing time for solving the instance in seconds (T_{total}). Note that $T_{\text{total}} - T_{\text{valid}}$ is the computing time from Step 2 of Phase 1 to Phase 3 under the invalid upper bounds in seconds. For instances which are not solved to optimality, all values are reported as "-".

The objective values of four instances (the 70-customer CMTVRPTW-LD instance C206 with $\bar{d} = 250$, the 100-customer CMTVRPTW-LD instance C202 with $\bar{d} = 220$, the 70-customer DRP instance R203 and the DRP instance Set_A1_Cust_45_5) reported by Yang [46] are greater slightly than that reported by us although Yang [46] stated that these instances were solved to optimality. Particularly, the DRP instance Set_A1_Cust_45_5 was also solved by Paradiso et al. [35], and the objective value reported by them is 18727.9, which is the same as that reported by us.

However, the value reported by Yang [46] is 18728.7. For other instances that Yang [46] stated that they were solved to optimality, the objective values reported by us and Yang [46] are the same. For the remaining instances shown in this appendix, the objective values reported by us are less than or equal to that reported by Yang [46].

Table A.1: Detailed computational results for the CMTVRPTW

Name	N	$UB_g\%$	optimal	$LB1\%$	$\lvert\mathbb{R}_1\rvert$	$LB2\%$	$\lvert\mathbb{R}_2\rvert$	$LB3\%$	$\lvert\mathbb{R}_3\rvert$	T_{close}	T_{valid}	T_{total}
C201	70	1.20	1052.2	0.66	198,834	0.00	9226	0.00	6077	0.7	41.9	41.9
C202	70	1.20	1047.7	1.16	204,256	0.21	3077	0.21	2880	1.5	353.5	353.5
C203	70	1.20	1040.4	0.98	396,955	0.12	8041	0.12	7663	1.9	200.5	200.5
C204	70	1.20	1036.8	1.04	575,382	0.10	6848	0.10	6466	1.1	216.3	216.3
C205	70	1.50	1047.9	1.27	599,850	0.17	15,118	0.15	11,030	3.3	337.3	553.9
C206	70	1.20	1042.0	1.05	295,799	0.08	2258	0.08	1772	0.7	201.8	201.8
C207	70	1.20	1040.3	1.09	296,863	0.18	4773	0.18	3389	2.1	226.7	226.7
C208	70	1.20	1040.3	1.11	271,326	0.12	1772	0.12	1468	1.2	127.2	127.2
R201	70	1.20	1118.4	0.72	17,306	0.38	8714	0.38	7951	4.8	158.2	158.2
R202	70	1.50	1041.1	1.34	83,295	1.00	38,439	1.00	37,452	53.7	385.4	639.0
R203	70	1.20	958.0	0.66	67,558	0.36	27,804	0.31	25,630	28.0	812.5	812.5
R204	70	1.20	921.8	0.81	148,267	0.28	26,397	0.28	25,973	16.7	592.2	592.2
R205	70	1.50	1033.4	1.46	110,719	1.04	37,813	1.03	36,130	66.2	523.9	872.2
R206	70	1.50	985.9	1.31	182,113	0.88	58,031	0.85	51,178	84.7	749.3	1258.1
R207	70	1.20	942.0	0.84	85,283	0.44	29,013	0.44	27,697	28.5	596.7	596.7
R208	70	1.20	917.5	0.90	127,662	0.41	36,953	0.41	28,787	17.4	424.0	424.0
R209	70	1.20	955.3	1.02	43,116	0.56	11,286	0.48	8623	10.3	403.7	403.7
R210	70	1.20	980.4	0.95	71,183	0.57	27,578	0.51	23,100	41.4	834.4	834.4
R211	70	1.20	914.8	0.71	81,675	0.12	15,659	0.12	12,689	12.5	663.1	663.1
RC201	70	1.20	1364.5	1.01	9152	0.25	1544	0.19	1076	0.3	152.2	152.3
RC202	70	1.20	1284.6	0.92	13,472	0.29	2565	0.29	2410	1.1	251.1	251.1
RC203	70	1.20	1230.5	0.53	31,975	0.00	9537	0.00	9217	2.1	85.8	85.8
RC204	70	1.20	1206.6	0.97	65,790	0.01	2522	0.01	2231	0.3	398.3	398.3
RC205	70	1.20	1335.3	0.84	19,276	0.43	7729	0.42	7358	8.1	330.0	330.0
RC206	70	1.50	1285.5	1.27	32,861	0.41	6339	0.38	5181	3.5	261.2	607.5
RC207	70	1.50	1236.5	1.19	68,673	0.05	3265	0.01	2158	0.7	394.1	677.2
RC208	70	1.50	1208.2	1.29	140,284	0.22	7912	0.20	5860	17.7	524.6	764.3
C201	100	1.20	1473.3	0.89	738,835	0.00	15,990	0.00	11,227	2.8	382.1	382.1
C202	100	1.20	1464.1	0.94	921,353	0.03	16,427	0.03	14,368	5.1	211.8	211.8
C203	100	1.20	1456.3	0.96	1,343,280	0.00	16,693	0.00	15,744	3.1	808.6	808.6
C204	100	1.20	1448.7	0.96	2,288,342	0.01	11,772	0.01	10,703	31.6	332.3	332.3
C205	100	1.20	1460.2	0.93	863,489	0.00	13,918	0.00	7184	1.0	269.1	269.1
C206	100	1.20	1455.1	1.02	970,975	0.00	8920	0.00	3306	0.6	716.1	716.1
C207	100	1.20	1454.5	0.73	1,267,310	0.02	93,595	0.01	56,113	20.1	346.8	346.8
C208	100	1.20	1451.9	0.95	989,453	0.00	8396	0.00	7388	1.8	329.4	329.4
R201	100	1.20	1399.6	1.13	210,163	0.90	109,846	0.88	99,742	235.1	679.8	679.8
R202	100	1.20	1304.7	1.13	975,352	0.85	402,949	0.82	348,670	5770.3	7310.9	7310.9
R203	100	1.20	1204.8	0.91	1,550,844	0.55	457,082	0.53	410,661	1954.4	3363.6	3363.6
R204	100	1.20	1162.2	0.68	3,714,822	0.20	532,665	0.19	465,467	402.7	2418.1	2418.1
R205	100	1.20	1267.3	1.12	458,574	0.90	237,194	0.83	160,142	807.3	1490.0	1490.0
R206	100	1.20	1220.9	1.08	1,385,502	0.74	453,397	0.69	341,222	13852.4	14799.9	14799.9
R207	100	1.20	1182.5	0.77	2,802,273	0.35	506,777	0.35	496,572	2116.4	4420.1	4420.1

Name	N	$UB_g\%$	optimal	$LB1\%$	$\lvert\mathbb{R}_1\rvert$	$LB2\%$	$\lvert\mathbb{R}_2\rvert$	$LB3\%$	$\lvert\mathbb{R}_3\rvert$	T_{close}	T_{valid}	T_{total}
R208	100	1.20	1157.5	0.60	4,074,012	0.17	803,247	0.17	790,201	954.3	3110.6	3110.6
R209	100	1.20	1205.4	0.60	1,029,587	0.34	431,264	0.23	284,053	469.6	1604.1	1604.1
R210	100	1.20	1211.8	0.75	1,461,854	0.44	534,895	0.40	453,080	1216.4	2849.3	2849.3
R211	100	1.20	1160.6	0.85	3,673,428	0.34	408,525	0.29	324,276	4133.6	6361.2	6361.2
RC201	100	1.20	1806.8	0.58	103,287	0.34	43,495	0.34	41,777	35.8	314.1	314.1
RC202	100	1.20	1680.2	1.13	213,240	0.70	52,440	0.68	45,983	64.7	635.7	635.7
RC203	100	1.20	1601.0	0.78	765,378	0.34	133,681	0.26	102,707	79.0	1197.4	1197.4
RC204	100	1.20	1574.6	0.78	1,366,597	0.05	44,000	0.05	42,851	20.1	1052.2	1052.2
RC205	100	1.20	1732.6	1.16	277,102	0.76	70,026	0.76	67,277	464.8	1046.2	1046.2
RC206	100	1.50	1698.1	1.30	588,193	0.64	86,631	0.57	65,438	185.1	842.4	1330.9
RC207	100	1.50	1640.7	1.24	1,362,430	0.40	66,198	0.30	40,634	64.9	1097.2	2077.8
RC208	100	1.20	1570.7	0.78	1,740,493	0.00	86,410	0.00	44,611	30.5	1556.3	1556.3

Table A.2: Detailed computational results for the CMTVRPTW-LT

Name	N	$UB_g\%$	optimal	$LB1\%$	$\lvert\mathbb{R}_1\rvert$	$LB2\%$	$\lvert\mathbb{R}_2\rvert$	$LB3\%$	$\lvert\mathbb{R}_3\rvert$	T_{close}	T_{valid}	T_{total}
C201	70	1.50	1063.2	1.19	332,772	0.35	50,073	0.35	34,837	19.3	309.6	523.0
C202	70	1.20	1053.4	0.92	185,676	0.34	37,512	0.34	34,244	34.2	399.7	399.7
C203	70	1.20	1045.2	1.18	304,808	0.32	8936	0.32	7971	2.2	255.7	255.7
C204	70	1.20	1038.4	1.07	532,293	0.07	3977	0.07	3870	0.8	249.1	249.1
C205	70	1.20	1048.2	1.16	322,982	0.08	450	0.00	175	0.0	80.1	80.2
C206	70	1.50	1044.1	1.23	665,788	0.23	18,493	0.23	16,421	9.6	336.5	655.9
C207	70	1.20	1040.3	1.08	330,538	0.17	3740	0.15	2267	0.9	172.5	172.5
C208	70	1.20	1040.3	1.11	277,232	0.12	1846	0.12	1490	0.5	219.1	219.1
R201	70	1.20	1118.4	0.49	17,977	0.19	8797	0.18	8494	5.5	246.5	246.5
R202	70	1.20	1041.1	1.18	40,766	0.90	16,313	0.90	15,044	13.8	293.2	293.2
R203	70	1.20	959.5	0.76	60,554	0.40	29,440	0.39	27,935	33.2	700.5	700.6
R204	70	1.20	921.8	0.83	143,764	0.28	25,920	0.28	24,459	20.0	620.8	620.8
R205	70	1.20	1033.4	1.17	45,761	0.80	15,941	0.79	14,980	26.8	405.5	405.5
R206	70	1.20	985.9	1.16	67,499	0.72	19,371	0.71	17,131	17.5	492.5	492.5
R207	70	1.20	942.0	0.87	84,615	0.44	27,042	0.44	25,122	28.4	637.1	637.1
R208	70	1.20	917.5	0.93	134,754	0.41	28,052	0.41	26,388	20.5	465.6	465.6
R209	70	1.20	955.9	0.96	43,104	0.55	11,927	0.46	10,305	22.9	438.4	438.4
R210	70	1.20	983.4	1.04	73,423	0.69	25,140	0.65	22,980	120.7	892.2	892.2
R211	70	1.20	914.8	0.59	83,071	0.08	21,019	0.06	18,090	9.3	495.6	495.6
RC201	70	1.20	1367.5	1.00	8675	0.41	2590	0.36	2209	1.1	217.1	217.2
RC202	70	1.20	1284.6	1.04	11,456	0.13	777	0.11	580	0.2	223.8	223.8
RC203	70	1.20	1230.5	0.46	29,449	0.00	13,110	0.00	12,003	2.4	181.7	181.7
RC204	70	1.20	1206.6	1.00	64,849	0.00	1738	0.00	1420	0.2	268.2	268.2
RC205	70	1.20	1340.4	1.05	19,058	0.47	5022	0.45	4417	7.7	450.7	450.7
RC206	70	1.80	1290.2	1.54	52,672	0.63	12,100	0.60	10,196	9.0	258.9	747.9
RC207	70	1.50	1241.1	1.37	73,555	0.34	4611	0.29	3521	4.8	443.4	691.7
RC208	70	1.50	1209.4	1.30	140,840	0.30	10,676	0.26	8329	21.5	608.4	903.7
C201	100	1.50	1480.6	1.28	1,432,654	0.25	84,652	0.22	46,948	88.5	732.6	998.7
C202	100	1.20	1465.5	0.93	943,036	0.01	17,133	0.01	14,864	7.1	542.5	542.5
C203	100	1.20	1459.6	1.12	1,417,634	0.02	2119	0.02	1583	0.6	636.4	636.4
C204	100	1.20	1448.7	0.94	2,285,232	0.00	10,179	0.00	7766	0.8	289.7	289.7
C205	100	1.20	1461.9	0.90	1,045,308	0.06	29,847	0.00	10,619	2.1	289.7	289.7
C206	100	1.20	1456.9	1.11	1,079,269	0.02	639	0.02	419	0.1	248.3	248.3
C207	100	1.20	1454.8	0.64	1,216,803	0.04	221,335	0.01	163,144	31.5	220.5	220.5
C208	100	1.20	1451.9	0.76	1,019,353	0.00	32,071	0.00	20,536	2.8	178.7	178.7
R201	100	1.20	1403.1	0.98	216,125	0.79	116,005	0.79	107,809	188.4	619.6	619.6
R202	100	1.20	1305.8	0.99	897,474	0.71	366,804	0.69	346,213	1056.8	2605.7	2605.7

Name	N	$UB_g\%$	optimal	$LB1\%$	$\lvert\mathbb{R}_1\rvert$	$LB2\%$	$\lvert\mathbb{R}_2\rvert$	$LB3\%$	$\lvert\mathbb{R}_3\rvert$	T_{close}	T_{valid}	T_{total}
R203	100	1.20	1206.4	0.96	1,423,648	0.59	395,109	0.59	372,522	3837.8	5376.4	5376.4
R204	100	1.20	1162.2	0.63	3,406,203	0.16	530,040	0.16	528,161	406.4	2271.8	2271.8
R205	100	1.20	1267.7	0.94	407,646	0.74	219,668	0.66	151,966	313.8	1097.9	1097.9
R206	100	1.20	1222.9	1.14	1,457,583	0.80	447,567	0.73	336,429	19601.6	20799.9	20799.9
R207	100	1.20	1182.5	0.76	2,834,595	0.37	629,831	0.37	576,178	1824.8	3724.7	3724.7
R208	100	1.20	1157.5	0.63	3,740,853	0.15	586,457	0.15	584,941	492.9	2808.9	2808.9
R209	100	1.20	1207.8	0.68	943,117	0.41	434,327	0.29	248,450	534.2	1811.7	1811.7
R210	100	1.20	1215.8	0.90	1,407,792	0.61	545,758	0.59	480,160	3497.6	5291.2	5291.2
R211	100	1.20	1164.0	1.07	3,574,570	0.50	292,393	0.44	198,917	5493.6	8138.4	8138.4
RC201	100	1.20	1809.5	0.57	98,430	0.33	46,103	0.32	42,769	19.1	338.7	338.7
RC202	100	1.50	1689.2	1.46	510,938	0.92	118,315	0.90	95,711	241.7	806.0	1215.7
RC203	100	1.20	1601.0	0.73	634,617	0.24	108,136	0.21	84,464	50.3	978.9	978.9
RC204	100	1.20	1574.6	0.79	1,408,969	0.04	38,021	0.03	33,757	23.2	985.3	985.3
RC205	100	1.20	1737.7	1.13	240,527	0.77	73,469	0.77	70,620	1062.1	1737.6	1737.6
RC206	100	1.50	1702.5	1.36	592,529	0.79	111,039	0.68	73,621	223.2	792.8	1334.8
RC207	100	1.20	1641.7	1.13	517,064	0.32	13,035	0.23	7948	25.8	1019.3	1019.3
RC208	100	1.20	1572.7	0.89	1,796,006	0.12	44,842	0.08	28,929	292.3	2036.7	2036.7

Table A.3: Detailed computational results for the CMTVRPTW-LD

Name	N	$\bar{d}$	$UB_g\%$	optimal	$LB1\%$	$\lvert\mathbb{R}_1\rvert$	$LB2\%$	$\lvert\mathbb{R}_2\rvert$	$LB3\%$	$\lvert\mathbb{R}_3\rvert$	T_{close}	T_{valid}	T_{total}
C201	70	220	1.00	1918.7	0.00	270	0.00	264	0.00	263	0.0	1.3	1.3
C201	70	250	1.00	1587.5	0.00	368	0.00	348	0.00	346	0.0	1.4	1.4
C202	70	220	1.00	1896.6	0.30	785	0.16	655	0.16	651	0.0	2.2	2.2
C202	70	250	1.00	1582.7	0.03	2003	0.00	1871	0.00	1861	0.0	2.9	2.9
C203	70	220	1.00	1835.9	0.27	1708	0.05	1394	0.05	1386	0.1	4.3	4.3
C203	70	250	1.00	1571.6	0.77	5095	0.46	3229	0.46	3080	0.2	15.5	15.5
C204	70	220	1.00	1774.3	0.43	2846	0.39	2755	0.39	2716	12.6	18.1	18.1
C204	70	250	1.00	1557.8	0.96	7588	0.82	6575	0.82	6203	30.9	39.0	39.0
C205	70	220	1.00	1846.3	0.98	618	0.79	534	0.79	529	0.0	1.5	1.5
C205	70	250	1.00	1580.5	0.26	1011	0.11	903	0.11	896	0.0	3.5	3.5
C206	70	220	1.60	1842.3	1.37	1320	1.08	1229	1.08	1229	0.1	4.7	10.2
C206	70	250	1.00	1573.0	0.23	1997	0.05	1752	0.05	1752	0.1	4.5	4.6
C207	70	220	1.00	1798.6	0.41	1457	0.29	1357	0.29	1343	0.1	5.4	5.4
C207	70	250	1.00	1568.6	0.08	3774	0.04	3625	0.04	3618	0.1	4.0	4.0
C208	70	220	1.00	1815.5	0.41	1340	0.35	1271	0.35	1269	0.1	4.5	4.5
C208	70	250	1.00	1568.6	0.08	3087	0.04	2975	0.04	2975	0.1	3.7	3.7
R201	70	75	1.00	1838.1	0.06	807	0.00	756	0.00	709	0.5	3.4	3.4
R201	70	100	1.00	1597.0	0.00	1050	0.00	932	0.00	901	0.1	1.9	1.9
R202	70	75	1.00	1708.5	0.05	1949	0.00	1861	0.00	1822	0.0	3.3	3.3
R202	70	100	1.00	1469.4	0.00	2326	0.00	2187	0.00	2106	0.1	3.3	3.3
R203	70	75	1.00	1559.1	0.16	2716	0.00	1979	0.00	1929	0.0	6.3	6.3
R203	70	100	1.00	1305.6	0.00	5108	0.00	4157	0.00	3939	0.1	6.8	6.8
R204	70	75	1.00	1390.4	0.45	3785	0.29	2983	0.29	2776	7.5	19.7	19.7
R204	70	100	1.00	1110.3	0.03	10,192	0.00	8762	0.00	8330	0.3	25.0	25.0
R205	70	75	1.00	1608.9	0.66	1699	0.00	450	0.00	402	0.1	8.8	8.8
R205	70	100	1.00	1358.8	0.22	3110	0.22	2885	0.22	2803	0.1	5.7	5.7
R206	70	75	1.00	1531.3	0.22	2116	0.01	1449	0.01	1371	0.3	5.1	5.1
R206	70	100	1.00	1278.6	0.49	5389	0.34	3773	0.34	3695	0.5	33.5	33.5
R207	70	75	1.00	1454.8	0.21	3777	0.03	2328	0.03	2258	0.1	21.3	21.3
R207	70	100	1.00	1186.1	0.66	5902	0.32	2932	0.32	2872	0.5	74.0	74.0
R208	70	75	1.00	1376.1	0.40	4548	0.22	3471	0.22	3310	10.0	31.0	31.0
R208	70	100	1.00	1087.6	0.08	10,757	0.00	8919	0.00	7754	0.3	53.0	53.0

Name	N	$\bar{d}$	$UB_g\%$	optimal	$LB1\%$	$\lvert\mathbb{R}_1\rvert$	$LB2\%$	$\lvert\mathbb{R}_2\rvert$	$LB3\%$	$\lvert\mathbb{R}_3\rvert$	T_{close}	T_{valid}	T_{total}
R209	70	75	1.00	1476.4	0.44	2778	0.15	1664	0.15	1599	0.2	19.7	19.7
R209	70	100	1.00	1211.5	0.59	4010	0.52	3513	0.52	3512	0.6	42.2	42.2
R210	70	75	1.00	1543.5	0.78	2390	0.41	1254	0.41	1224	0.4	31.8	31.8
R210	70	100	1.00	1299.0	0.41	6015	0.12	3463	0.11	3187	0.4	51.4	51.4
R211	70	75	1.00	1375.4	0.77	5059	0.71	4655	0.65	4195	15.4	38.6	38.6
R211	70	100	1.00	1082.0	0.10	13,538	0.10	7390	0.00	5982	0.6	34.0	34.0
RC201	70	75	1.00	2392.0	0.09	407	0.00	374	0.00	353	0.2	2.8	2.8
RC201	70	100	1.00	1798.5	0.00	611	0.00	576	0.00	554	0.0	2.5	2.5
RC202	70	75	1.00	2167.3	0.50	691	0.43	645	0.43	619	0.1	3.2	3.2
RC202	70	100	1.00	1664.8	0.58	1291	0.42	1101	0.42	1076	0.2	8.3	8.3
RC203	70	75	1.00	1986.1	0.47	1071	0.21	786	0.21	772	0.3	4.5	4.5
RC203	70	100	1.00	1482.0	0.00	3545	0.00	3109	0.00	3048	0.1	7.3	7.3
RC204	70	75	1.00	1843.6	0.49	2414	0.20	1818	0.20	1614	3.8	29.7	29.7
RC204	70	100	1.00	1290.9	0.79	6460	0.54	4175	0.54	4137	1.5	118.1	118.1
RC205	70	75	1.00	2197.3	0.26	702	0.03	534	0.03	484	0.1	5.4	5.4
RC205	70	100	1.00	1723.5	0.04	1447	0.00	1125	0.00	1095	0.1	3.4	3.4
RC206	70	75	1.00	2095.4	0.79	729	0.35	497	0.35	454	0.6	14.8	14.8
RC206	70	100	1.30	1582.4	1.11	3508	0.86	2729	0.85	2510	1.5	33.5	57.4
RC207	70	75	1.60	1924.7	1.56	2755	0.86	1488	0.86	1449	0.6	38.8	103.0
RC207	70	100	1.60	1367.3	1.45	8060	1.01	3377	1.01	3308	0.1	13.6	29.5
RC208	70	75	1.30	1818.1	1.25	3811	0.88	2730	0.73	2326	31.2	73.0	136.3
RC208	70	100	1.60	1249.3	1.55	29,675	1.08	12,706	0.97	10,329	20.9	194.5	460.1
C201	100	220	1.00	2902.4	0.50	591	0.24	530	0.24	519	0.1	4.6	4.6
C201	100	250	1.00	2335.4	0.32	876	0.13	775	0.13	771	0.1	20.3	20.3
C202	100	220	1.00	2830.4	0.50	2131	0.46	2051	0.46	2051	0.2	7.2	7.2
C202	100	250	1.00	2311.8	0.56	8122	0.40	6386	0.40	6330	0.7	16.8	16.8
C203	100	220	1.00	2763.0	0.52	3912	0.45	3679	0.45	3652	16.4	22.1	22.1
C203	100	250	1.00	2292.2	0.78	16,724	0.56	11,612	0.56	11,311	42.4	67.4	67.4
C204	100	220	1.00	2704.4	0.56	5652	0.27	4560	0.27	4470	36.8	57.3	57.3
C204	100	250	1.00	2283.6	0.74	23,175	0.64	20,499	0.64	19,743	210.2	277.8	277.8
C205	100	220	1.00	2793.2	0.97	1352	0.63	1156	0.63	1153	0.1	10.4	10.4
C205	100	250	1.00	2320.4	0.92	2338	0.49	1612	0.49	1585	0.6	30.9	30.9
C206	100	220	1.30	2770.6	1.04	2296	0.88	2179	0.88	2179	0.2	6.9	11.3
C206	100	250	1.30	2308.8	1.00	5837	0.68	4345	0.68	4288	0.7	26.8	49.7
C207	100	220	1.00	2743.2	0.65	2423	0.26	1814	0.26	1721	0.1	12.5	12.5
C207	100	250	1.00	2305.7	0.78	6434	0.43	4682	0.43	4636	0.7	36.8	36.8
C208	100	220	1.00	2738.9	0.39	2575	0.22	2424	0.22	2424	0.3	6.7	6.7
C208	100	250	1.00	2302.2	0.93	6297	0.75	5535	0.75	5483	1.7	20.2	20.2
R201	100	75	1.00	2273.4	0.28	3157	0.10	2454	0.10	2299	0.1	6.4	6.4
R201	100	100	1.00	1916.9	0.00	6816	0.00	6215	0.00	6067	0.3	14.6	14.6
R202	100	75	1.00	2100.3	0.34	20,461	0.12	12,619	0.12	12,435	0.5	32.8	32.8
R202	100	100	1.00	1756.3	0.08	69,656	0.00	55,747	0.00	55,387	4.5	120.2	120.2
R203	100	75	1.00	1869.9	0.44	44,642	0.27	34,926	0.27	34,824	3.6	100.9	100.9
R203	100	100	1.00	1548.9	0.51	152,855	0.38	108,692	0.38	105,478	28.4	346.9	346.9
R204	100	75	1.00	1712.0	0.00	52,037	0.00	45,757	0.00	44,986	7.0	27.5	27.5
R204	100	100	1.00	1361.0	0.59	220,753	0.44	150,159	0.44	149,758	79.3	622.6	622.6
R205	100	75	1.00	1961.7	0.10	9199	0.03	8234	0.03	8150	1.4	36.2	36.2
R205	100	100	1.00	1604.5	0.39	24,961	0.22	16,466	0.22	16,026	3.4	117.8	117.8
R206	100	75	1.00	1854.1	0.47	27,476	0.25	19,852	0.25	19,766	7.3	65.9	65.9
R206	100	100	1.00	1518.9	0.26	116,993	0.22	104,714	0.22	104,367	16.8	235.3	235.3
R207	100	75	1.00	1771.5	0.57	40,936	0.43	34,368	0.43	34,274	6.2	102.4	102.4
R207	100	100	1.00	1411.7	0.70	125,968	0.45	63,377	0.45	62,130	18.7	407.0	407.0
R208	100	75	1.00	1687.7	0.37	49,796	0.27	39,364	0.27	38,287	231.1	318.7	318.7
R208	100	100	1.00	1322.7	0.47	267,824	0.23	113,929	0.23	113,150	64.6	700.2	700.2
R209	100	75	1.00	1833.2	0.53	21,958	0.41	18,251	0.41	17,978	2.2	83.5	83.5

Name	N	$\bar{d}$	$UB_g\%$	optimal	$LB1\%$	$\lvert\mathbb{R}_1\rvert$	$LB2\%$	$\lvert\mathbb{R}_2\rvert$	$LB3\%$	$\lvert\mathbb{R}_3\rvert$	T_{close}	T_{valid}	T_{total}
R209	100	100	1.00	1462.5	0.33	57,892	0.02	22,032	0.02	21,618	2.7	226.9	226.9
R210	100	75	1.00	1841.6	0.12	26,984	0.07	24,873	0.07	24,587	1.0	58.0	58.0
R210	100	100	1.00	1532.4	0.58	95,885	0.41	60,420	0.41	58,982	13.2	273.3	273.3
R211	100	75	1.00	1678.9	0.37	73,420	0.30	64,983	0.28	61,139	315.1	392.3	392.3
R211	100	100	-	-	-	-	-	-	-	-	-	-	-
RC201	100	75	1.00	3120.3	0.11	1437	0.04	1333	0.04	1236	0.5	7.8	7.8
RC201	100	100	1.00	2370.2	0.02	2929	0.00	2609	0.00	2404	0.3	6.4	6.4
RC202	100	75	1.00	2819.5	0.63	3263	0.56	2892	0.56	2829	0.9	14.3	14.3
RC202	100	100	1.00	2148.6	0.48	7949	0.36	6219	0.36	6160	0.7	44.3	44.3
RC203	100	75	1.00	2550.5	0.21	6137	0.06	4975	0.06	4759	0.1	21.6	21.6
RC203	100	100	1.00	1896.4	0.21	18,733	0.10	14,567	0.10	14,398	0.8	98.1	98.1
RC204	100	75	1.00	2430.3	0.50	11,611	0.31	8478	0.31	8115	0.8	42.1	42.1
RC204	100	100	1.00	1725.8	0.56	35,503	0.36	22,248	0.36	21,931	6.4	199.6	199.6
RC205	100	75	1.00	2874.8	0.13	3338	0.05	3042	0.05	2779	0.4	15.1	15.1
RC205	100	100	1.00	2206.2	0.13	8825	0.08	7582	0.08	7444	0.4	34.0	34.0
RC206	100	75	1.00	2724.7	0.09	3642	0.00	2976	0.00	2724	0.2	7.9	7.9
RC206	100	100	1.00	2064.1	0.35	11,184	0.13	7718	0.13	7549	0.6	68.9	68.9
RC207	100	75	1.00	2612.7	0.69	8071	0.47	5692	0.47	5195	3.2	41.3	41.3
RC207	100	100	1.00	1876.2	0.74	13,649	0.21	3940	0.21	3734	0.8	158.2	158.2
RC208	100	75	1.00	2381.3	0.23	15,025	0.09	11,634	0.04	10,516	189.9	229.0	229.0
RC208	100	100	1.00	1667.7	0.88	46,453	0.65	23,497	0.53	17,948	61.5	427.3	427.3

Table A.4: Detailed computational results for the CMTVRPTW-R

Name	N	κ	$UB_g\%$	optimal	$LB1\%$	$\lvert\mathbb{R}_1\rvert$	$LB2\%$	$\lvert\mathbb{R}_2\rvert$	$LB3\%$	$\lvert\mathbb{R}_3\rvert$	T_{close}	T_{valid}	T_{total}
C201	70	0.25	1.60	1068.7	1.47	59,084	0.00	1056	0.00	415	0.0	185.1	533.6
C201	70	0.5	1.30	1072.0	1.24	21,592	0.00	143	0.00	123	0.0	21.7	30.4
C201	70	0.75	1.00	1080.9	0.33	9393	0.00	2780	0.00	2700	0.1	24.4	24.4
C202	70	0.25	1.00	1121.0	0.38	16,169	0.00	5885	0.00	5386	0.5	40.5	40.5
C202	70	0.5	1.00	1121.0	0.29	9581	0.00	5112	0.00	4756	0.3	14.9	14.9
C202	70	0.75	1.00	1121.0	0.33	8210	0.00	3317	0.00	3112	0.2	20.5	20.5
C203	70	0.25	1.00	1156.3	0.24	52,070	0.00	24,806	0.00	21,918	2.2	102.5	102.5
C203	70	0.5	1.00	1156.3	0.15	55,248	0.00	44,208	0.00	33,954	3.1	69.9	69.9
C203	70	0.75	1.00	1156.3	0.15	90,253	0.00	36,093	0.00	26,426	2.2	40.4	40.4
C204	70	0.25	1.00	1145.6	0.25	198,991	0.00	122,939	0.00	88,349	11.6	288.9	288.9
C204	70	0.5	1.00	1145.6	0.24	210,202	0.00	54,105	0.00	40,899	5.0	81.7	81.7
C204	70	0.75	1.00	1145.6	0.17	227,950	0.00	71,736	0.00	47,602	6.8	269.9	269.9
C205	70	0.25	1.60	1063.2	1.42	61,672	0.00	1451	0.00	1118	0.1	177.7	564.7
C205	70	0.5	1.00	1066.6	0.77	13,235	0.00	525	0.00	462	0.0	31.1	31.1
C205	70	0.75	1.00	1075.9	0.86	12,737	0.00	425	0.00	266	0.0	30.2	30.2
C206	70	0.25	1.60	1053.4	1.39	73,902	0.00	671	0.00	473	0.0	55.5	85.1
C206	70	0.5	1.60	1062.3	1.41	58,649	0.00	607	0.00	571	0.1	47.8	79.4
C206	70	0.75	1.30	1072.5	1.15	33,376	0.00	613	0.00	440	0.0	59.7	76.8
C207	70	0.25	1.00	1047.2	0.88	29,157	0.00	349	0.00	330	0.0	37.2	37.2
C207	70	0.5	1.00	1051.9	0.81	21,738	0.00	493	0.00	396	0.0	41.6	41.6
C207	70	0.75	1.00	1060.6	0.29	23,794	0.00	8521	0.00	8364	0.6	36.7	36.7
C208	70	0.25	1.60	1050.6	1.32	77,926	0.00	1020	0.00	909	0.1	22.6	33.8
C208	70	0.5	1.60	1055.9	1.31	50,908	0.00	1001	0.00	854	0.1	29.8	60.3
C208	70	0.75	1.00	1058.5	0.54	23,454	0.00	3679	0.00	2592	0.2	60.7	60.7
R201	70	0.25	1.00	1159.1	0.67	7323	0.47	3215	0.47	3107	1.0	81.7	81.7
R201	70	0.5	1.00	1173.9	0.62	4968	0.26	2405	0.26	1979	0.4	74.8	74.8
R201	70	0.75	1.00	1214.4	0.58	6143	0.38	4234	0.38	3833	1.6	94.3	94.3
R202	70	0.25	1.00	1115.4	0.00	12,145	0.00	4996	0.00	4787	0.3	48.6	48.6

Name	N	κ	$UB_g\%$	optimal	$LB1\%$	$\lvert\mathbb{R}_1\rvert$	$LB2\%$	$\lvert\mathbb{R}_2\rvert$	$LB3\%$	$\lvert\mathbb{R}_3\rvert$	T_{close}	T_{valid}	T_{total}
R202	70	0.5	1.00	1125.5	0.00	13,441	0.00	6306	0.00	5247	0.3	65.3	65.3
R202	70	0.75	1.00	1125.5	0.00	4432	0.00	2917	0.00	2859	0.1	4.5	4.5
R203	70	0.25	1.00	1113.0	0.13	18,453	0.00	9911	0.00	9841	1.1	99.9	99.9
R203	70	0.5	1.00	1123.8	0.00	26,078	0.00	18,140	0.00	16,783	1.4	89.3	89.3
R203	70	0.75	1.00	1148.1	0.55	24,584	0.36	14,936	0.36	14,541	3.9	239.1	239.1
R204	70	0.25	1.00	1057.7	0.65	41,297	0.03	5259	0.03	5003	1.1	324.7	324.7
R204	70	0.5	1.00	1057.7	0.64	39,937	0.03	4917	0.03	4788	1.2	393.4	393.4
R204	70	0.75	1.00	1079.8	0.74	62,184	0.24	11,127	0.24	10,778	2.3	407.6	407.6
R205	70	0.25	1.00	1073.5	0.60	12,495	0.47	9452	0.47	8944	3.1	103.0	103.0
R205	70	0.5	1.00	1083.0	0.31	12,179	0.13	5697	0.13	5082	0.6	97.9	97.9
R205	70	0.75	1.00	1084.6	0.00	13,484	0.00	7582	0.00	7010	0.5	98.8	98.8
R206	70	0.25	1.00	1039.6	0.00	17,335	0.00	10,996	0.00	9976	0.7	140.4	140.4
R206	70	0.5	1.00	1059.3	0.51	22,212	0.20	8578	0.20	8090	3.4	440.1	440.1
R206	70	0.75	1.00	1070.6	0.45	21,856	0.24	9944	0.24	8998	3.8	336.9	336.9
R207	70	0.25	1.00	1049.3	0.65	24,721	0.17	5027	0.17	4880	0.8	155.1	155.1
R207	70	0.5	1.00	1056.5	0.34	36,566	0.00	15,798	0.00	14,783	2.2	222.0	222.0
R207	70	0.75	1.00	1056.5	0.33	26,619	0.00	10,653	0.00	9198	0.7	121.9	121.9
R208	70	0.25	1.00	997.4	0.00	79,976	0.00	37,224	0.00	32,199	4.9	332.3	332.3
R208	70	0.5	1.00	997.4	0.00	66,625	0.00	32,575	0.00	29,443	5.8	330.5	330.5
R208	70	0.75	1.00	997.4	0.00	76,866	0.00	34,134	0.00	31,093	5.3	367.2	367.3
R209	70	0.25	1.00	995.4	0.87	25,206	0.47	7655	0.47	6361	11.2	241.4	241.4
R209	70	0.5	1.00	997.4	0.73	19,766	0.40	8211	0.39	7183	6.0	232.4	232.4
R209	70	0.75	1.00	1033.8	0.60	13,071	0.42	8164	0.41	7589	6.7	323.4	323.4
R210	70	0.25	1.00	1026.5	0.45	18,768	0.23	8565	0.23	8382	2.5	204.1	204.1
R210	70	0.5	1.00	1032.7	0.10	15,450	0.00	8133	0.00	7557	0.6	138.6	138.6
R210	70	0.75	1.00	1094.5	0.33	16,267	0.02	10,084	0.02	9561	2.4	266.3	266.3
R211	70	0.25	1.00	930.4	0.85	30,249	0.40	6413	0.40	5372	12.4	716.1	716.1
R211	70	0.5	1.00	930.4	0.81	30,881	0.37	7614	0.37	6762	19.6	625.0	625.0
R211	70	0.75	1.30	958.8	1.09	67,007	0.66	22,094	0.66	21,151	27.8	574.6	1071.2
RC201	70	0.25	1.00	1367.5	0.07	3275	0.00	2169	0.00	2064	0.1	32.5	32.5
RC201	70	0.5	1.30	1397.6	1.07	5509	0.40	1700	0.40	1423	0.7	75.2	139.4
RC201	70	0.75	1.00	1434.6	0.82	3287	0.28	1134	0.28	1042	0.2	111.5	111.5
RC202	70	0.25	1.90	1409.8	1.59	18,011	0.77	6201	0.77	5566	2.6	86.0	294.4
RC202	70	0.5	2.20	1413.9	1.99	21,966	0.52	3366	0.52	3129	1.1	106.0	405.8
RC202	70	0.75	1.00	1438.3	0.23	2607	0.11	2028	0.11	1690	0.2	57.5	57.5
RC203	70	0.25	1.00	1397.9	0.21	10,442	0.00	5799	0.00	5127	0.5	156.9	156.9
RC203	70	0.5	1.00	1407.7	0.82	11,402	0.00	613	0.00	146	0.0	116.7	116.7
RC203	70	0.75	1.00	1483.9	0.00	9042	0.00	4979	0.00	4580	0.2	32.0	32.0
RC204	70	0.25	1.00	1354.0	0.86	27,186	0.00	1044	0.00	821	0.1	100.5	100.5
RC204	70	0.5	1.00	1354.0	0.88	27,335	0.00	836	0.00	623	0.1	97.8	97.8
RC204	70	0.75	1.00	1409.5	0.72	23,744	0.00	2088	0.00	1643	0.2	120.1	120.1
RC205	70	0.25	1.00	1361.5	0.51	5603	0.00	1725	0.00	1584	0.1	70.0	70.0
RC205	70	0.5	1.00	1433.0	0.56	6177	0.40	4907	0.40	4483	2.3	142.7	142.7
RC205	70	0.75	1.00	1474.6	0.42	3120	0.22	2415	0.22	2254	0.3	29.2	29.2
RC206	70	0.25	1.00	1309.1	0.51	5838	0.00	1712	0.00	1245	0.1	40.6	40.6
RC206	70	0.5	1.00	1309.9	0.51	5304	0.00	1659	0.00	1017	0.1	56.1	56.1
RC206	70	0.75	1.00	1347.7	0.10	4415	0.00	2901	0.00	2304	0.1	39.2	39.3
RC207	70	0.25	1.60	1281.8	1.49	37,993	0.31	2688	0.31	2561	2.9	302.6	685.1
RC207	70	0.5	1.60	1281.8	1.45	37,945	0.31	3482	0.31	3262	2.7	288.3	643.0
RC207	70	0.75	1.00	1382.5	0.66	10,482	0.34	5004	0.34	4609	5.1	200.5	200.5
RC208	70	0.25	1.60	1216.4	1.57	74,984	0.47	7771	0.30	3045	3.4	853.0	1388.0
RC208	70	0.5	1.60	1216.4	1.34	87,116	0.37	11,640	0.21	6630	7.9	720.5	1241.2
RC208	70	0.75	1.30	1235.3	1.02	36,259	0.37	9838	0.36	9138	13.5	652.5	1181.2
C201	100	0.25	1.90	1500.6	1.74	266,891	0.00	3412	0.00	2404	0.1	484.6	1314.6
C201	100	0.5	1.90	1500.6	1.80	178,432	0.00	727	0.00	248	0.0	106.1	206.8

Name	N	κ	$UB_g\%$	optimal	$LB1\%$	$\lvert\mathbb{R}_1\rvert$	$LB2\%$	$\lvert\mathbb{R}_2\rvert$	$LB3\%$	$\lvert\mathbb{R}_3\rvert$	T_{close}	T_{valid}	T_{total}
C201	100	0.75	1.30	1504	1.19	62,083	0.00	949	0.00	326	0.0	268.0	314.9
C202	100	0.25	1.30	1545.4	1.15	123,174	0.00	1333	0.00	1289	0.2	248.3	492.8
C202	100	0.5	1.00	1547.3	0.86	59,277	0.00	971	0.00	752	0.1	138.7	138.7
C202	100	0.75	1.30	1552.9	1.04	135,850	0.00	3806	0.00	3224	0.5	209.0	337.9
C203	100	0.25	1.30	1577.7	1.02	518,024	0.14	30,419	0.14	25,983	7.6	476.4	765.4
C203	100	0.5	1.30	1578.7	1.08	580,730	0.00	6832	0.00	6454	1.5	528.5	728.7
C203	100	0.75	1.30	1579.6	0.99	530,572	0.00	20,879	0.00	12,335	1.0	362.1	580.6
C204	100	0.25	1.00	1560.5	0.56	362,339	0.00	33,728	0.00	27,143	7.7	359.5	359.5
C204	100	0.5	1.00	1560.9	0.59	419,947	0.00	72,126	0.00	25,859	4.4	564.9	564.9
C204	100	0.75	1.00	1569.1	0.52	774,086	0.00	111,504	0.00	77,579	49.5	724.4	724.4
C205	100	0.25	1.90	1488.2	1.75	311,372	0.00	2212	0.00	1275	0.1	303.1	1067.6
C205	100	0.5	1.90	1490	1.84	230,873	0.00	272	0.00	126	0.0	98.9	162.3
C205	100	0.75	1.60	1491.7	1.56	131,531	0.00	265	0.00	116	0.1	110.4	151.3
C206	100	0.25	1.90	1476	1.59	429,401	0.00	9003	0.00	3940	0.4	239.0	597.0
C206	100	0.5	2.20	1481.7	1.89	425,267	0.00	3207	0.00	2874	0.3	134.5	198.7
C206	100	0.75	1.30	1490.5	1.15	104,412	0.00	667	0.00	592	0.0	114.1	141.5
C207	100	0.25	1.60	1472.8	1.38	250,704	0.00	4536	0.00	2179	0.2	347.2	797.3
C207	100	0.5	1.60	1474.4	1.33	200,045	0.00	3939	0.00	1697	0.2	145.6	259.8
C207	100	0.75	1.60	1480.4	1.36	194,882	0.00	1829	0.00	1493	0.2	150.2	217.4
C208	100	0.25	1.60	1471.2	1.57	319,713	0.00	139	0.00	108	0.1	122.8	151.0
C208	100	0.5	1.90	1477.4	1.76	346,233	0.00	1052	0.00	659	0.1	321.9	1079.6
C208	100	0.75	1.60	1481.2	1.54	180,788	0.00	453	0.00	157	0.1	180.8	300.2
R201	100	0.25	1.00	1435.6	0.65	59,336	0.42	31,118	0.42	28,847	26.5	418.5	418.5
R201	100	0.5	1.00	1442.6	0.63	51,206	0.47	31,105	0.47	27,801	14.0	464.7	464.7
R201	100	0.75	1.00	1483.6	0.64	43,259	0.29	17,008	0.29	16,696	9.8	439.9	439.9
R202	100	0.25	1.00	1401.4	0.53	154,117	0.23	57,261	0.22	55,014	26.8	568.2	568.2
R202	100	0.5	1.00	1413.8	0.68	175,594	0.46	102,061	0.45	93,429	97.5	750.6	750.6
R202	100	0.75	1.00	1429	0.38	217,895	0.26	164,761	0.25	154,330	145.7	760.8	760.8
R203	100	0.25	1.00	1370.9	0.14	312,566	0.00	226,472	0.00	216,761	44.5	461.1	461.1
R203	100	0.5	1.00	1372.8	0.14	331,679	0.00	266,036	0.00	231,170	46.5	562.8	562.8
R203	100	0.75	1.00	1394.7	0.26	344,209	0.07	189,723	0.07	188,698	66.5	654.7	654.7
R204	100	0.25	1.00	1324.6	0.76	793,150	0.42	221,122	0.41	203,329	107.9	1245.1	1245.1
R204	100	0.5	1.00	1324.6	0.75	882,365	0.42	226,484	0.41	200,874	212.4	1650.4	1650.4
R204	100	0.75	1.00	1334.6	0.41	800,625	0.10	241,278	0.10	235,647	192.9	1633.8	1633.8
R205	100	0.25	1.00	1314.4	0.61	120,733	0.45	69,084	0.45	66,784	40.3	651.5	651.5
R205	100	0.5	1.00	1332.3	0.60	105,529	0.41	49,592	0.41	49,401	34.5	576.0	576.0
R205	100	0.75	1.00	1361.8	0.54	116,846	0.37	71,661	0.37	68,376	51.6	572.4	572.4
R206	100	0.25	1.00	1274.8	0.01	297,371	0.00	246,956	0.00	241,390	41.8	640.3	640.3
R206	100	0.5	1.00	1298.1	0.34	395,190	0.21	198,079	0.20	175,261	88.3	1106.7	1106.7
R206	100	0.75	1.00	1323.5	0.51	403,273	0.21	110,954	0.21	103,722	111.0	951.0	951.0
R207	100	0.25	1.00	1286.7	0.68	442,145	0.45	221,834	0.45	218,842	196.6	1149.6	1149.6
R207	100	0.5	1.00	1297.3	0.50	371,829	0.18	154,100	0.18	136,334	82.8	1220.7	1220.7
R207	100	0.75	1.00	1304.7	0.27	622,611	0.04	316,142	0.04	298,663	313.0	1452.7	1452.7
R208	100	0.25	1.00	1253.1	0.44	1,082,095	0.24	422,997	0.24	417,102	341.9	1905.9	1905.9
R208	100	0.5	1.00	1253.1	0.46	1,033,092	0.25	400,897	0.25	392,067	295.1	1766.4	1766.4
R208	100	0.75	1.00	1253.1	0.34	996,291	0.17	507,153	0.17	492,997	711.6	2353.5	2353.5
R209	100	0.25	1.00	1255.8	0.92	280,085	0.67	122,158	0.66	117,610	488.7	1404.7	1404.7
R209	100	0.5	1.30	1258.8	1.04	882,673	0.84	446,688	0.84	439,714	1318.5	2458.1	3651.2
R209	100	0.75	1.00	1288.6	0.11	197,181	0.00	137,257	0.00	127,899	42.7	720.6	720.6
R210	100	0.25	1.30	1277.3	1.01	1,168,370	0.76	584,171	0.76	575,272	2704.8	3577.3	4790.6
R210	100	0.5	1.00	1283.7	0.58	325,645	0.34	175,314	0.34	173,704	180.9	1004.6	1004.6
R210	100	0.75	1.00	1341.5	0.44	282,194	0.26	170,330	0.26	152,454	121.9	972.7	972.7
R211	100	0.25	1.00	1171.4	0.40	545,818	0.10	185,060	0.06	161,465	509.1	2175.3	2175.4
R211	100	0.5	1.00	1175	0.61	591,090	0.26	141,473	0.26	133,176	566.4	2273.9	2273.9
R211	100	0.75	1.00	1199.3	0.90	508,339	0.41	52,886	0.41	50,783	238.9	1773.0	1773.0

Name	N	κ	$UB_g\%$	optimal	$LB1\%$	$\lvert\mathbb{R}_1\rvert$	$LB2\%$	$\lvert\mathbb{R}_2\rvert$	$LB3\%$	$\lvert\mathbb{R}_3\rvert$	T_{close}	T_{valid}	T_{total}
RC201	100	0.25	1.00	1839.1	0.97	29,557	0.51	6035	0.51	5445	3.7	169.8	169.8
RC201	100	0.5	1.00	1849.6	0.45	26,968	0.05	7541	0.05	6997	1.3	210.4	210.4
RC201	100	0.75	1.00	1871.2	0.32	23,307	0.00	9344	0.00	7802	0.8	109.8	109.8
RC202	100	0.25	1.00	1790.8	0.47	47,343	0.34	29,532	0.34	27,778	9.1	207.6	207.6
RC202	100	0.5	1.00	1813.4	0.43	47,706	0.28	29,624	0.28	27,437	8.6	209.1	209.1
RC202	100	0.75	1.00	1841.7	0.88	46,445	0.58	16,947	0.58	15,264	7.7	210.7	210.7
RC203	100	0.25	1.00	1808.2	0.31	355,647	0.12	146,937	0.12	140,917	60.6	1070.2	1070.2
RC203	100	0.5	1.00	1831.1	0.44	294,346	0.29	189,999	0.28	168,469	62.2	715.4	715.4
RC203	100	0.75	1.00	1880.7	0.55	382,517	0.43	210,105	0.43	208,814	183.1	852.4	852.4
RC204	100	0.25	1.00	1749.4	0.41	332,579	0.09	121,265	0.09	100,009	43.9	670.3	670.3
RC204	100	0.5	1.00	1749.4	0.39	325,110	0.09	122,194	0.09	99,858	70.7	818.2	818.2
RC204	100	0.75	1.00	1780.4	0.18	440,550	0.07	297,943	0.07	270,346	212.4	1249.3	1249.3
RC205	100	0.25	1.00	1760.4	0.54	64,928	0.27	26,984	0.26	25,717	25.7	594.8	594.8
RC205	100	0.5	1.00	1819	0.80	60,026	0.51	26,736	0.51	25,383	22.7	371.6	371.6
RC205	100	0.75	1.00	1877.8	0.61	51,447	0.30	19,702	0.30	18,877	21.0	319.8	319.8
RC206	100	0.25	1.00	1734.1	0.45	77,313	0.07	21,869	0.07	19,894	10.0	368.7	368.7
RC206	100	0.5	1.00	1746.9	0.57	59,056	0.27	27,001	0.27	25,132	13.0	309.2	309.2
RC206	100	0.75	1.00	1793.6	0.31	44,928	0.11	25,349	0.11	24,698	3.4	196.8	196.8
RC207	100	0.25	1.30	1694.4	1.07	367,965	0.31	26,329	0.31	22,825	54.5	498.4	871.0
RC207	100	0.5	1.30	1694.4	1.03	348,674	0.29	31,239	0.29	26,903	71.8	518.9	919.0
RC207	100	0.75	1.00	1780.4	0.37	88,065	0.09	36,129	0.08	33,255	18.4	355.4	355.4
RC208	100	0.25	1.30	1595.5	1.07	853,055	0.40	76,026	0.12	17,775	43.3	1947.3	3005.6
RC208	100	0.5	1.60	1602.5	1.49	2,150,557	0.75	258,446	0.56	91,396	1273.3	2834.3	4976.6
RC208	100	0.75	1.00	1620.1	0.95	221,868	0.27	10,571	0.19	5742	26.3	1064.0	1064.0

Table A.5: Detailed computational results for the DRP on Set A

Name	$UB_g\%$	optimal	$LB1\%$	$\lvert\mathbb{R}_1\rvert$	$LB2\%$	$\lvert\mathbb{R}_2\rvert$	$LB3\%$	$\lvert\mathbb{R}_3\rvert$	T_{close}	T_{valid}	T_{total}
DRP_A1_10_1	4.30	3132.0	4.00	30	3.86	25	3.27	22	0.0	0.2	1.6
DRP_A1_10_2	1.00	4738.9	0.00	17	0.00	13	0.00	13	0.0	0.0	0.0
DRP_A1_10_3	3.10	4556.3	2.85	48	1.42	25	0.00	14	0.0	0.2	0.9
DRP_A1_10_4	9.10	4391.5	8.19	89	3.74	60	0.23	20	0.0	0.4	6.1
DRP_A1_10_5	1.00	4524.2	0.19	19	0.12	12	0.00	5	0.0	0.2	0.2
DRP_A1_15_1	1.00	7072.0	0.70	48	0.39	39	0.31	30	0.0	0.6	0.6
DRP_A1_15_2	1.00	4397.8	0.15	49	0.00	38	0.00	24	0.0	0.4	0.4
DRP_A1_15_3	6.40	5968.2	5.81	425	5.56	396	5.53	393	0.1	2.9	47.4
DRP_A1_15_4	1.00	5491.0	0.81	43	0.00	25	0.00	16	0.0	0.2	0.2
DRP_A1_15_5	3.40	7383.4	3.09	145	2.83	120	2.80	118	0.0	1.0	8.7
DRP_A1_20_1	1.00	8284.9	0.00	48	0.00	31	0.00	27	0.0	0.1	0.1
DRP_A1_20_2	1.00	9548.0	0.81	88	0.66	73	0.66	70	0.0	0.8	0.8
DRP_A1_20_3	1.90	8816.1	1.71	174	0.40	57	0.09	33	0.0	0.3	0.7
DRP_A1_20_4	1.30	6693.8	0.00	85	0.00	79	0.00	39	0.0	0.5	0.7
DRP_A1_20_5	3.10	7782.1	2.73	290	1.74	196	1.45	171	0.1	2.2	10.8
DRP_A1_25_1	2.20	10680.0	2.05	494	0.12	51	0.12	32	0.0	0.4	1.2
DRP_A1_25_2	1.30	8636.2	1.10	166	0.76	85	0.43	48	0.0	3.1	4.8
DRP_A1_25_3	2.50	10094.5	2.18	452	1.30	272	0.68	120	0.0	1.5	4.8
DRP_A1_25_4	1.00	10146.6	0.00	130	0.00	123	0.00	103	0.1	0.4	0.4
DRP_A1_25_5	1.60	11166.0	1.40	319	0.61	179	0.61	170	0.0	2.3	5.7
DRP_A1_30_1	1.00	9831.6	0.00	230	0.00	207	0.00	185	0.0	2.2	2.2
DRP_A1_30_2	1.00	12665.0	0.19	284	0.00	209	0.00	128	0.0	1.7	1.7
DRP_A1_30_3	2.20	12359.4	2.08	798	1.25	485	0.95	367	0.2	12.8	51.3
DRP_A1_30_4	1.00	12512.8	0.94	320	0.30	99	0.23	74	0.0	2.5	2.5
DRP_A1_30_5	1.30	12086.1	0.81	501	0.00	122	0.00	91	0.0	1.1	1.9

Name	UB_g%	optimal	$LB1$%	$\lvert\mathbb{R}_1\rvert$	$LB2$%	$\lvert\mathbb{R}_2\rvert$	$LB3$%	$\lvert\mathbb{R}_3\rvert$	T_{close}	T_{valid}	T_{total}
DRP_A1_35_1	1.60	12434.0	1.56	2320	0.91	1102	0.91	1025	0.8	13.0	35.9
DRP_A1_35_2	1.00	13020.8	0.84	572	0.54	313	0.53	304	0.2	9.3	9.3
DRP_A1_35_3	1.00	13230.4	0.00	293	0.00	293	0.00	265	0.0	3.3	3.3
DRP_A1_35_4	1.00	13863.5	0.00	382	0.00	362	0.00	340	0.0	0.4	0.4
DRP_A1_35_5	1.00	13281.6	0.51	339	0.43	299	0.43	288	0.2	12.5	12.5
DRP_A1_40_1	1.00	15540.1	0.49	776	0.10	419	0.10	404	0.2	7.7	7.7
DRP_A1_40_2	1.00	16881.3	0.35	384	0.11	323	0.07	309	0.2	11.0	11.0
DRP_A1_40_3	1.00	14178.4	0.22	692	0.08	585	0.08	540	0.1	15.0	15.0
DRP_A1_40_4	1.00	16286.8	0.21	532	0.02	476	0.00	452	0.5	3.1	3.1
DRP_A1_40_5	1.00	15620.2	0.48	414	0.25	324	0.03	237	0.6	17.9	17.9
DRP_A1_45_1	1.00	14569.0	0.31	2158	0.23	1945	0.21	1833	0.7	52.0	52.0
DRP_A1_45_2	1.00	19727.5	0.88	952	0.55	632	0.45	487	1.9	19.8	19.8
DRP_A1_45_3	1.00	18825.4	0.21	1425	0.07	1005	0.01	896	1.0	21.4	21.4
DRP_A1_45_4	1.00	16298.5	0.60	1262	0.47	1109	0.37	913	0.2	29.9	29.9
DRP_A1_45_5	1.00	18727.9	0.44	774	0.19	550	0.19	524	0.5	13.9	13.9
DRP_A2_10_1	4.30	4999.0	3.98	24	2.11	17	0.00	11	0.0	0.1	0.2
DRP_A2_10_2	1.00	5825.5	0.00	14	0.00	10	0.00	8	0.0	0.1	0.1
DRP_A2_10_3	1.00	5269.9	0.00	9	0.00	9	0.00	6	0.0	0.1	0.1
DRP_A2_10_4	5.20	6157.2	4.85	36	2.97	33	0.43	21	0.0	0.2	1.0
DRP_A2_10_5	1.00	5534.0	0.78	13	0.59	13	0.00	12	0.0	0.3	0.3
DRP_A2_15_1	1.00	6869.6	0.19	22	0.10	20	0.00	17	0.0	0.2	0.2
DRP_A2_15_2	1.60	8535.0	1.43	43	1.43	43	1.42	43	0.1	0.3	0.8
DRP_A2_15_3	1.00	6612.0	0.00	23	0.00	22	0.00	17	0.0	0.1	0.1
DRP_A2_15_4	8.50	8777.9	7.69	78	7.69	78	5.46	51	0.0	0.1	2.6
DRP_A2_15_5	1.00	8672.1	0.00	45	0.00	45	0.00	45	0.0	0.1	0.1
DRP_A2_20_1	3.70	11422.7	3.35	107	1.73	80	0.39	41	0.0	0.7	5.1
DRP_A2_20_2	3.70	9730.0	3.50	154	2.80	124	2.24	78	0.0	0.4	3.7
DRP_A2_20_3	1.00	10093.7	0.00	37	0.00	37	0.00	31	0.0	0.2	0.2
DRP_A2_20_4	1.00	9492.4	0.00	39	0.00	39	0.00	37	0.0	0.1	0.1
DRP_A2_20_5	1.90	8299.5	1.73	126	1.47	101	1.43	99	0.0	0.7	3.1
DRP_A2_25_1	1.00	11436.3	0.01	72	0.01	61	0.00	54	0.0	0.3	0.3
DRP_A2_25_2	1.00	12426.4	0.00	57	0.00	52	0.00	50	0.0	0.2	0.2
DRP_A2_25_3	4.00	10973.4	3.71	210	3.11	170	2.47	109	0.5	2.0	12.5
DRP_A2_25_4	1.00	12275.4	0.69	65	0.01	46	0.01	41	0.1	0.5	0.5
DRP_A2_25_5	1.00	11788	0.15	70	0.13	50	0.00	49	0.0	0.4	0.4
DRP_A2_30_1	1.00	14997.4	0.18	112	0.00	101	0.00	91	0.1	1.4	1.4
DRP_A2_30_2	1.00	12794.3	0.96	140	0.96	110	0.96	101	0.3	1.7	1.7
DRP_A2_30_3	1.00	12234.4	0.79	224	0.10	100	0.08	80	0.0	2.5	2.5
DRP_A2_30_4	1.00	11587.3	0.19	103	0.00	78	0.00	61	0.0	0.5	0.5
DRP_A2_30_5	1.30	13261.5	1.23	224	1.23	206	1.04	168	0.3	2.5	4.5
DRP_A2_35_1	1.00	14282.9	0.42	398	0.00	271	0.00	209	0.0	1.8	1.8
DRP_A2_35_2	1.00	17443.4	0.27	114	0.13	100	0.03	89	0.2	3.2	3.2
DRP_A2_35_3	1.00	14691.3	0.92	211	0.92	193	0.73	140	0.0	2.7	2.7
DRP_A2_35_4	1.30	17689.3	1.20	330	0.94	274	0.72	200	0.1	5.7	10.2
DRP_A2_35_5	1.00	16812.4	0.45	275	0.40	243	0.29	190	0.1	6.5	6.5
DRP_A2_40_1	1.00	17002.9	0.06	328	0.00	293	0.00	269	0.2	4.6	4.6
DRP_A2_40_2	1.30	17949	1.07	423	0.36	213	0.36	213	0.5	6.5	12.9
DRP_A2_40_3	1.00	18078.8	0.16	196	0.00	178	0.00	168	0.2	2.2	2.2
DRP_A2_40_4	1.00	18559.6	0.59	263	0.45	240	0.24	177	0.5	5.2	5.2
DRP_A2_40_5	1.00	13798.5	0.51	250	0.38	207	0.19	127	0.1	7.6	7.6
DRP_A2_45_1	1.00	18654.7	0.01	344	0.01	320	0.00	309	0.4	2.6	2.6
DRP_A2_45_2	1.00	19590.6	0.21	438	0.11	412	0.08	383	1.0	7.0	7.0
DRP_A2_45_3	1.00	20207.2	0.73	490	0.22	218	0.13	174	0.3	4.2	4.2
DRP_A2_45_4	1.00	17306.9	0.52	553	0.09	364	0.08	321	0.5	12.4	12.4
DRP_A2_45_5	1.00	24311.7	0.72	299	0.01	131	0.00	111	0.4	2.8	2.8

Name	$UB_g\%$	optimal	$LB1\%$	$\lvert\mathbb{R}_1\rvert$	$LB2\%$	$\lvert\mathbb{R}_2\rvert$	$LB3\%$	$\lvert\mathbb{R}_3\rvert$	T_{close}	T_{valid}	T_{total}
DRP_A2_50_1	1.30	24698.5	0.76	1901	0.32	1342	0.22	1104	2.5	21.2	37.5
DRP_A2_50_2	1.00	21939.4	0.83	534	0.23	221	0.12	182	0.2	23.7	23.7
DRP_A2_50_3	1.00	19700	0.19	672	0.18	625	0.18	555	0.6	15.8	15.8
DRP_A2_50_4	1.00	19841.8	0.12	416	0.05	327	0.00	262	0.6	11.9	11.9
DRP_A2_50_5	1.00	23721.5	0.58	392	0.16	267	0.10	224	0.8	8.1	8.1

Table A.6: Detailed computational results for the DRP on Set B

Name	N	$UB_g\%$	optimal	$LB1\%$	$\lvert\mathbb{R}_1\rvert$	$LB2\%$	$\lvert\mathbb{R}_2\rvert$	$LB3\%$	$\lvert\mathbb{R}_3\rvert$	T_{close}	T_{valid}	T_{total}
C201	70	1.00	1862.8	0.48	10,372	0.00	3006	0.00	2251	0.4	13.0	13.0
C202	70	1.00	1856.9	0.28	12,198	0.00	6015	0.00	5538	0.5	19.3	19.3
C203	70	1.00	1852.1	0.34	17,104	0.00	7045	0.00	6797	0.5	12.2	12.2
C204	70	1.00	1851.6	0.34	23,939	0.00	8645	0.00	7560	58.6	69.1	69.1
C205	70	1.00	1853.0	0.30	11,965	0.00	5308	0.00	4507	2.1	18.5	18.5
C206	70	1.00	1851.9	0.29	13,931	0.00	6297	0.00	4861	0.9	12.5	12.5
C207	70	1.00	1851.9	0.36	13,929	0.00	5835	0.00	5086	1.1	7.3	7.3
C208	70	1.00	1851.9	0.36	12,619	0.00	5217	0.00	4387	1.5	7.7	7.7
R201	70	1.00	1531.5	0.63	4277	0.42	2830	0.42	2602	2.4	53.1	53.1
R202	70	1.00	1462.2	0.13	8207	0.03	6091	0.03	5877	1.2	122.0	122.0
R203	70	1.00	1412.9	0.39	14,174	0.21	9363	0.17	8111	2.8	87.4	87.4
R204	70	1.00	1393.1	0.51	19,248	0.29	11,293	0.29	11,068	1.7	87.8	87.8
R205	70	1.00	1458.7	0.39	7155	0.39	5694	0.39	5434	1.0	54.9	54.9
R206	70	1.00	1429.2	0.38	13,315	0.26	8554	0.09	5817	3.6	101.4	101.4
R207	70	1.00	1403.0	0.66	17,100	0.15	3838	0.15	3385	7.6	111.8	111.8
R208	70	1.00	1390.4	0.66	21,954	0.23	5886	0.23	5767	62.1	154.4	154.4
R209	70	1.00	1417.8	0.96	10,894	0.40	2359	0.38	1861	2.6	87.0	87.0
R210	70	1.00	1433.9	0.49	13,300	0.23	7704	0.22	6641	3.0	93.9	93.9
R211	70	1.00	1390.2	0.65	18,140	0.21	5564	0.21	4976	8.2	100.4	100.4
RC201	70	1.30	2328.5	1.18	3664	0.67	2079	0.36	1049	0.8	78.9	154.5
RC202	70	1.00	2253.2	0.93	3386	0.26	915	0.08	403	0.2	41.6	41.6
RC203	70	1.00	2227.5	0.68	5727	0.00	1586	0.00	1540	0.1	13.9	13.9
RC204	70	1.00	2225.4	0.74	6092	0.00	1489	0.00	1135	5.6	42.0	42.0
RC205	70	1.00	2270.7	0.77	3610	0.16	1456	0.03	756	0.8	42.4	42.4
RC206	70	1.00	2259.0	0.55	3701	0.12	1700	0.03	1373	0.3	40.9	40.9
RC207	70	1.00	2233.0	0.81	4668	0.01	785	0.00	644	0.9	24.1	24.1
RC208	70	1.00	2225.4	0.74	7596	0.00	1601	0.00	1274	3.0	27.2	27.2
C201	100	1.00	2733.4	0.91	33,519	0.00	551	0.00	545	0.1	41.8	41.8
C202	100	1.00	2729.1	0.84	61,406	0.00	1661	0.00	1599	0.2	37.4	37.4
C203	100	1.00	2725.8	0.84	93,269	0.00	2071	0.00	1963	0.2	39.0	39.0
C204	100	1.00	2720.8	0.83	112,111	0.00	2445	0.00	1872	86.4	206.0	206.0
C205	100	1.00	2726.9	0.91	45,593	0.00	677	0.00	448	0.1	36.8	36.8
C206	100	1.00	2722.3	0.80	51,165	0.00	1929	0.00	1530	0.2	54.1	54.1
C207	100	1.00	2720.9	0.80	50,728	0.00	1980	0.00	1950	0.2	21.8	21.8
C208	100	1.00	2720.7	0.81	53,587	0.00	1571	0.00	1249	0.2	87.4	87.4
R201	100	1.00	1974.3	0.34	39,996	0.22	28,017	0.22	26,987	7.3	143.0	143.0
R202	100	1.00	1919	0.80	91,207	0.60	48,069	0.47	23,962	38.3	205.3	205.3
R203	100	1.00	1845.7	0.51	215,733	0.39	152,251	0.37	141,314	798.4	1003.2	1003.2
R204	100	1.00	1819.2	0.14	360,427	0.03	271,302	0.03	268,692	47.9	288.9	288.9
R205	100	1.00	1884.4	0.42	84,548	0.26	58,097	0.22	48,458	125.4	293.1	293.1
R206	100	1.00	1852.8	0.41	168,921	0.20	92,802	0.19	86,235	188.8	365.5	365.5
R207	100	1.00	1831.5	0.33	265,423	0.16	144,097	0.16	134,790	259.3	489.1	489.1
R208	100	1.00	1815.5	0.13	458,365	0.00	292,576	0.00	287,701	31.2	129.9	129.9
R209	100	1.00	1846.0	0.48	150,953	0.28	73,870	0.27	69,280	57.8	260.6	260.6

Name	N	$UB_g\%$	optimal	$LB1\%$	$\lvert\mathbb{R}_1\rvert$	$LB2\%$	$\lvert\mathbb{R}_2\rvert$	$LB3\%$	$\lvert\mathbb{R}_3\rvert$	T_{close}	T_{valid}	T_{total}
R210	100	1.00	1853.6	0.52	179,653	0.37	111,594	0.35	92,755	517.4	707.3	707.3
R211	100	1.00	1815.5	0.19	487,283	0.00	240,795	0.00	230,958	884.0	1111.0	1111.0
RC201	100	1.00	2960.3	0.62	24,489	0.45	15,954	0.18	6274	8.7	99.0	99.0
RC202	100	1.00	2870.7	0.41	50,079	0.09	29,823	0.08	28,022	8.8	136.3	136.3
RC203	100	1.00	2853.0	0.42	94,980	0.08	47,050	0.08	44,204	46.4	206.4	206.4
RC204	100	1.00	2847.0	0.40	113,726	0.05	45,706	0.05	44,129	592.3	704.1	704.1
RC205	100	1.00	2898.2	0.52	39,082	0.28	25,855	0.22	21,968	16.3	134.0	134.0
RC206	100	1.00	2886.3	0.37	45,816	0.16	32,036	0.08	23,760	7.8	99.1	99.1
RC207	100	1.00	2854.6	0.42	58,647	0.05	29,176	0.04	26,238	174.0	281.3	281.3
RC208	100	1.00	2846.7	0.39	130,401	0.04	61,533	0.04	58,147	347.6	480.0	480.0

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