PARALLEL MEAN CURVATURE VECTOR SUBMANIFOLDS IN THE HYPERBOLIC SPACE

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DECLARATION

I hereby declare that the thesis entitled 'Parallel Mean Curvature Vector Submanifolds in the Hyperbolic Space' is original and has not been submitted for a degree, or other qualifications in this University or any other institutes. It does not contain any material, partly or wholly, published or written previous by others, except those references quoted in the text.

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Abstract of thesis entitled 'Parallel Mean Curvature Vector Submanifolds in the Hyperbolic Space'

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This thesis concerns with two applications of the Omori-Yau maximum principle for complete non-compact submanifolds whose Ricci curvature are bound from below. The first of these is a pinching theorem for complete parallel mean curvature submanifolds in the standard hyperbolic space while the second one is an extrinsic diameter theorem for bounded mean curvature submanifolds in the standard hyperbolic space.

To obtain the pinching theorem for complete parallel mean curvature submanifolds in the standard hyperbolic space, we generalize the results due to Q.M. Cheng to certain class of submanifolds immersed isometrically in the standard hyperbolic space. In order to do this, we studied carefully the proof of Simons' inequality in the work of Chern, do Carmo and Kobayashi to obtain the generalized Simons' inequality mentioned in Santos' paper. By using this inequality together with the maximum principle of Yau-Omori, we obtained the pinching theorem for parallel mean curvature vector submanifolds in the standard hyperbolic space which parallels the results of Cheng.

On the other hand, we studied the inequality on the Laplacian of the hyperbolic cosine of the distance function for a submanifold in the standard hyperbolic space. We discover that this inequality, when used together

with the Omori type maximum principle, yields extrinsic diameter estimates for submanifolds in the standard hyperbolic space. As a corollary, one can recover the well-known result that there exists no compact constant mean curvature submanifolds in the standard hyperbolic space if $|H| \leq 1$.

TABLE OF CONTENTS

CHAPTER 1 INTRODUCTION

In this dissertation, we study the effect of the maximum principle of Omori [11] on the behavior on parallel mean curvature vector submanifolds in a space form. Historically, there are two main types of results which motivated the theorems proven in this work, both having proofs relying heavily on a maximum principle originally proved by Omori which asserts the existence of almost maximum points on a Riemannian manifold whose Ricci curvature is bounded from below. The first kind of results concerns the extrinsic diameter of a submanifold in ambient manifold. The prototype of which is the classical result of Xavier (see e.g. [1] for a simple proof using maximum principle of Omori) asserting the following:

Theorem.
Suppose M is a minimal submanifold in the Euclidean space such that the Ricci curvature of M is bounded from below, then M cannot be contained in any ball of Euclidean space with finite radius.

The second type of the results based on Omori's maximum principle is a series of pinching theorems for submanifolds originating in the historic works of Simons, Chern et. al. in the sixties. A typical result in this direction is the pinching theorem for parallel mean curvature vector submanifolds in a sphere, provided the length of the second fundamental form satisfies some pointwise bound. Originally these results were proved for compact minimal submanifolds in the sphere, and it was in the late eighties when Cheng managed to generalize them to complete minimal submanifolds in a sphere.

In another direction, Santos [8] considered parallel mean curvature vector submanifolds in the sphere and obtained similar results. Santos [8] proved theorems like the following

Theorem.

Let M^n be a compact submanifold of the unit sphere $S^{n+p}(1)$ with parallel mean curvature vector. Assume that the length of the traceless second fundamental form (to be defined in Chapter 3) is less than or equal to some universal constant depending only on the dimension, the codimension and $|H|$, then either M^n is totally umbilic or M^n is a torus, a minimal Clifford hypersphere or a Vernosese surface.

Analyzing the proofs of the above-mentioned theorems, we see that they have two main ingredients: (i) a differential inequality and (ii) an Omori type maximum principle.

For the first type of theorems, the differential inequality is an inequality on the Laplacian of the extrinsic length function, whereas for the second type of the theorems, the differential inequality is an inequality on the Laplacian of the second fundamental form or some generalization of the second fundamental form (such as the traceless second fundamental form).

Based on this analysis, we are motivated to study related inequalities on the Laplacian of the distance function or the second fundamental form for submanifolds in various ambient spaces. Going through the literature, we found two interesting inequalities, one on the Laplacian of the sinh of the distance function for a submanifold in the hyperbolic space, as proved by

Choe and Gulliver [5]. We discover that this inequality, when used together with the Omori type maximum principle, yields extrinsic diameter estimates for submanifolds in the hyperbolic space. As a corollary, it gives also a simple result that there exists no non-compact constant mean curvature 1 hypersurfaces in the standard hyperbolic space with sectional curvature -1.

In the other direction, we study the Laplacian of the traceless second fundamental form and a generalization of the results of Q.M. Cheng [3] on the pinching of minimal submanifolds in a sphere to the pinching of parallel mean curvature vector submanifolds in the standard hyperbolic space. This kind of inequalities is well-known and is called Simons' inequality. In our work, we study carefully the derivation of Simons' inequality in the work of Chern, do Carmo and Kobayashi [4] to obtain the generalized Simons' inequality mentioned in Santos' paper. Using this inequality, we obtain the pinching theorems for parallel mean curvature vector submanifolds in the standard hyperbolic space which parallels the results of Cheng [3].

The plan of this thesis is as follows. In Chapter 2, preliminary results concerning manifolds and differential forms are discussed. In Chapter 3, we prove step by step the Simons' inequality of Santos needed in the sequel. Moreover, based on P.F. Leung's [7] results, we will give an estimate on the Ricci curvature of a submanifold of the Riemannian manifold. Chapter 4 contains the main results of this thesis, in which we state and prove (i) the extrinsic diameter theorem for bounded mean curvature vector submanifolds in the standard hyperbolic space and (ii) the pinching theorem for parallel mean curvature vector submanifolds in the standard hyperbolic space. In the last chapter, some concluding remarks are given.

CHAPTER 2 STRUCTURE EQUATIONS OF RIEMANNIAN MANIFOLD

In this section, we collect some important definitions and well-known formulas in differential geometry which we will use later. For more details, please see ([2]).

2.1 RIEMANNIAN MANIFOLD

Definition 2.1.1

A topological space M is called a Hausdorff space if given any two distinct points of M, there exist neighborhoods of these points such that they do not intersect.

Definition 2.1.2.

Suppose M is a Hausdorff space. If for any $x \in M$, there exists a neighborhood U of x such that U is homeomorphic to an open set in R^n, then M is called an n-dimensional manifold.

Definition 2.1.3.

An n-dimensional differentiable manifold is a set M together with a family of homeomorphisms $f_\alpha : U_\alpha \subset R^n \to M$ of open sets U_α in R^n into M such that:

1) $\cup_\alpha f_\alpha(U_\alpha) = M$

2) For each pair α, β with $f_\alpha(U_\alpha) \cap f_\beta(U_\beta) = W \neq \phi$, the maps $f_\beta^{-1} \circ f_\alpha$ and $f_\alpha^{-1} \circ f_\beta$ are differentiable.

3)The family $\{(U_\alpha, f_\alpha)\}$ is maximal.

Definition 2.1.4.

A Riemannian manifold is a differentiable manifold M together with a choice, for each point $p \in M$, of a positive definite inner product $\langle , \rangle_p$ in T_pM which varies differentiably with p in the following sense: If X and Y are differentiable vector fields in M, the function $p \mapsto \langle X, Y \rangle_p$ is differentiable in M. The inner product $\langle , \rangle$ is usually called a Riemannian metric on M.

2.2 STRUCTURE EQUATIONS

Next, we restrict our attention to the simple case of R^n. For this simple manifold, we introduce the concept of moving frames and hence define the structure equations. Later on, we will generalize results obtained for R^n to general differentiable manifolds. Our formulations are based on the monographs of do Carmo [2] and Spivak [9].

Let $U \subset R^n$ be an open set and let $e_1, e_2, ..., e_n$ be n differentiable vector fields such that for each $p \in U$, $\langle e_i, e_j \rangle_p = \delta_{ij}$, where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$. Such a set of vector fields is called an orthonormal moving frame. From now on, we will omit the adjective 'orthonormal'.

Given the moving frame $\{e_i\}$, $i = 1, ..., n$, we can define differential 1-forms ω^i by the conditions $\omega^i(e_j) = \delta_{ij}$, $j = 1, ..., n$; in other words, at each p, the basis $\left\{(\omega^i)_p\right\}$ is the dual basis of $\left\{(e_i)_p\right\}$. The set of forms $\{\omega^i\}$ is called the coframe associated to $\{e_i\}$.

Each vector field e_i is a differentiable map $e_i : U \subset R^n \rightarrow R^n$. The differential at $p \in U$, $(de_i)_p : R^n \rightarrow R^n$, is a linear map. Thus, for each p and each $v \in R^n$ we can write

$$(de_i)_p(v) = \sum_j \left(\omega_i^j\right)_p(v)\, e_j$$

It is easily checked that the expression $\left(\omega_i^j\right)_p(v)$ defined above depends linearly on v. Thus $\left(\omega_i^j\right)_p$ is a linear form in R^n and, since e_i is a differentiable vector field, ω_i^j is a differential 1-form. Keeping this in mind, we can write the above as

$$de_i = \sum_j \omega_i^j e_j$$

The n^2 forms ω_i^j so defined are called the connection forms of R^n in the moving frame $\{e_i\}$. Not all the forms ω_i^j are independent. If we differentiate $\langle e_i, e_j\rangle = \delta_{ij}$, we obtain

$$0 = \langle de_i, e_j\rangle + \langle e_i, de_j\rangle = \omega_i^j + \omega_j^i,$$

that is, the connection forms $\omega_i^j = -\omega_j^i$ are antisymmetric in the indices i, j.

Lemma 2.2.1.

Let $\{e_i\}$ be a moving frame in an open set $U \subset R^n$. Let $\{\omega_i\}$ be the coframe associated to $\{e_i\}$ and ω_i^j the connection forms of U in the frame $\{e_i\}$. Then

$$d\omega^i = -\sum_k \omega_k^i \wedge \omega^k, \qquad d\omega_i^j = \sum_k \omega_k^i \wedge \omega_j^k, \qquad i, j, k = 1, ..., n. \qquad (2.2.2)$$

Proof:

If we denote by $x : U \to R^n$ the inclusion map, to say that the forms ω^i are dual to the frame $\{e_i\}$ is equivalent to saying that

$$dx = \sum_i \omega^i e_i, \qquad de_i = \sum_j \omega_i^j e_j, \qquad i = 1, ..., n.$$

For instance, the first structure equation can be obtained as follows:

$$\begin{aligned}
0 &= d\,(dx) \\
&= \sum_i d\omega^i e_i - \sum_i \omega^i \wedge de_i \\
&= \sum_j d\omega^j e_j - \left(\sum_i \omega^i \wedge \sum_j \omega_i^j e_j \right) \\
&= \sum_j d\omega^j e_j - \left(\sum_i \left(-\sum_j \omega_i^j e_j \wedge \omega^i \right) \right) \\
&= \sum_j \left(d\omega_j + \left(\sum_i \omega_j^i \wedge \omega^i \right) \right) e_j,
\end{aligned}$$

hence

$$d\omega^j = -\sum_i \omega_i^j \wedge \omega^i.$$

Similarly, we also have

$$\begin{aligned}
0 &= d\,(de_i) \\
&= \sum_j d\omega_i^j e_j - \sum_k \omega_i^k \wedge de_k \\
&= \sum_j d\omega_i^j e_j - \sum_k \omega_i^k \wedge \sum_j \omega_k^j e_j \\
&= \sum_j d\omega_i^j e_j - \sum_j \sum_k \omega_i^k \wedge \omega_k^j e_j \\
&= \sum_j \left(d\omega_i^j - \sum_k \omega_k^i \wedge \omega_j^k \right) e_j
\end{aligned}$$

from which we immediately deduce the second structural equations.

Lemma 2.2.3.

Let V^n be a vector space of dimension n, and let $\omega^1, ..., \omega^r : V^n \to R$ such that $\sum\limits_{i=1}^{r} \omega^i \wedge \theta_i = 0$. Then

$$\theta_i = \sum_{j=1}^{r} a^i_j \omega^j \qquad \text{with } a^j_i = a^i_j.$$

Proof:

We complete the forms ω^i into a basis $\omega^1, ..., \omega^r, \omega^{r+1}, ..., \omega^n$ of V^* and we write

$$\theta_i = -\sum_{j=1}^{r} a^i_j \omega^j - \sum_{l=r+1}^{n} b^i_l \omega^l, \qquad i = 1, ..., r$$

By using the hypothesis, we obtain

$$\begin{aligned} 0 &= \sum_{j=1}^{r} \omega^i \wedge \theta_i \\ &= -\sum_{i,j=1}^{r} a^i_j \omega^i \wedge \omega^j - \sum_{i,l=1} b^i_l \omega^i \wedge \omega^l \\ &= -\sum_{i<j} \left(a^j_i - a^i_j\right) \omega^i \wedge \omega^j - \sum_{i<l} b^i_l \omega^i \wedge \omega^l \end{aligned}$$

since $\omega^k \wedge \omega^s$, $k < s$, $k, s = 1, ..., n$ are linearly independent, we conclude that $b^l_i = 0$ and $a^j_i = a^i_j$.

Lemma 2.2.4.

Let $U \subset R^n$ and let $\omega^1, ..., \omega^n$ be linearly independent differential 1-forms in U. Assume that there exists a set of differential 1-forms $\{\omega^j_i\}$, $i, j = 1, ..., n$

that satisfy the conditions:

$$\omega_i^j = -\omega_j^i, \qquad d\omega^j = -\sum \omega_k^j \wedge \omega^k.$$

Then such a set is unique.

Proof: Suppose the existence of another set $\overline{\omega}_i^j$ with $\overline{\omega}_i^j = -\overline{\omega}_j^i$,

$$d\omega^j = -\sum \overline{\omega}_k^j \wedge \omega^k.$$

Then

$$\sum \left(\overline{\omega}_j^k - \omega_j^k\right) \wedge \omega^k = 0,$$

and by Cartan's lemma,

$$\overline{\omega}_k^j - \omega_k^j = \sum_i B_{ki}^j \omega^i, B_{ki}^j = B_{ik}^j,$$

N·tice that

$$\overline{\omega}_k^j - \omega_k^j = \sum_i B_{ki}^j \omega^i = -\left(\overline{\omega}_j^k - \omega_j^k\right) = \sum_i B_{ji}^k \omega^i,$$

since the ω^i are linearly independent, $B_{ki}^j = -B_{ji}^k$. By using the fact that $B_{ki}^j = B_{ik}^j$ and $B_{ki}^j = -B_{ji}^k$, we obtain

$$B_{ji}^k = -B_{ki}^j = -B_{ik}^j = B_{jk}^i = B_{kj}^i = -B_{ij}^k = -B_{ji}^k = 0,$$

that is, $\overline{\omega}_k^j = \omega_k^j$.

Now we will define the structure equations of a Riemannian manifold M.

Definition 2.2.5.

Let ∇_{e_i} be the covariant differentiation with respect to e_i and R be its curvature tensor, then we have

$$\nabla_{e_i} e_j = \sum_{k=1}^{n} \Gamma_{ij}^{k} e_k \tag{2.2.6}$$

$$R\left(e_i, e_j\right) e_k = \sum_{l=1}^{n} R_{kij}^{l} e_l, \tag{2.2.7}$$

where Γ_{ij}^{k} is the Christoffel symbol.

Theorem 2.2.8. (Spivak, [9])

Let $e_1, ..., e_n$ be an orthonormal moving frame on a Riemannian manifold M, and let $\omega^i, \omega_i^j, \Omega_j^i$ be the dual forms, connection forms and curvature forms for this moving frame. Then we have the structure equations of M:

$$d\omega^i = -\sum_{k} \omega_k^i \wedge \omega^k \tag{2.2.9}$$

$$d\omega_i^j = \sum_{k} \omega_k^i \wedge \omega_j^k - \Omega_j^i, \tag{2.2.10}$$

where

$$\omega_i^j = \sum_{k} \Gamma_{ki}^{j} \omega^k, \qquad \Omega_j^i = \sum_{k<l} R_{jkl}^{i} \omega^k \wedge \omega^l. \tag{2.2.11}$$

Proof:

By lemma 2.2.4, we can prove the first structure equation by defining $\omega_i^j = \sum \Gamma_{kj}^{i} \omega^k$.

$$0 = \nabla_{e_k} \left\langle e_i, e_j \right\rangle = \left\langle \nabla_{e_k} e_i, e_j \right\rangle + \left\langle e_i, \nabla_{e_k} e_j \right\rangle = \Gamma_{ki}^{j} + \Gamma_{kj}^{i}$$

this immediately implies that $\omega_i^j = -\omega_j^i$,

$$\begin{aligned}
-\sum_l \omega_l^i \wedge \omega^l (e_j, e_k) &= -\left(\sum_l \omega_l^i (e_j) \omega^l (e_k) - \omega_l^i (e_k) \omega^l (e_j) \right) \\
&= \sum_l \omega_l^i (e_k) \delta_{lj} - \omega_l^j (e_j) \delta_{lk} = \omega_j^i (e_k) - \omega_k^i (e_j) \\
&= \Gamma_{kj}^i - \Gamma_{jk}^i \\
&= 0 - 0 - \omega^i \left(\nabla_{e_j} e_k - \nabla_{e_k} e_j \right) \\
&= \nabla_{e_j} \omega^i (e_k) - \nabla_{e_k} \omega^i (e_j) - \omega^i ([e_j, e_k]) \\
&= d\omega^i (e_j, e_k)
\end{aligned}$$

By lemma 2.2.4, the set $\{\omega_i^j\}$ is unique.

For the second structure equation we expand

$$\begin{aligned}
\sum_{i=1}^n R_{jkl}^i e_i &= R(e_k, e_l) e_j \\
&= \nabla_{e_k} \nabla_{e_l} e_j - \nabla_{e_l} \nabla_{e_k} e_j - \nabla_{[e_k, e_l]} e_j \\
&= \nabla_{e_k} \sum_\mu \Gamma_{lj}^\mu e_\mu - \nabla_{e_l} \sum_\mu \Gamma_{kj}^\mu e_\mu - \left(\nabla_{\nabla_{e_k} e_l} e_j - \nabla_{\nabla_{e_l} e_k} e_j \right) \\
&= \sum_\mu \Gamma_{lj}^\mu \sum_i \Gamma_{k\mu}^i e_i + \sum_\mu \left(\nabla_{e_k} \Gamma_{lj}^\mu \right) e_\mu - \sum_\mu \Gamma_{kj}^\mu \sum_i \Gamma_{l\mu}^i e_i - \sum_\mu \left(\nabla_{e_l} \Gamma_{kj}^\mu \right) e_\mu \\
&\quad - \left(\sum_\mu \Gamma_{kl}^\mu \nabla_{e_\mu} e_j - \sum_\mu \Gamma_{lk}^\mu \nabla_{e_\mu} e_j \right) \\
&= \sum_{i,\mu} \Gamma_{lj}^\mu \Gamma_{k\mu}^i e_i - \sum_{i,\mu} \Gamma_{kj}^\mu \Gamma_{l\mu}^i e_i + \sum_i \left(\nabla_{e_k} \Gamma_{lj}^i \right) e_i - \sum_i \left(\nabla_{e_l} \Gamma_{kj}^i \right) e_i \\
&\quad - \left(\sum_\mu \Gamma_{kl}^\mu \sum_i \Gamma_{\mu j}^i e_i - \sum_\mu \Gamma_{lk}^\mu \sum_i \Gamma_{\mu j}^i e_i \right),
\end{aligned}$$

therefore,

$$R_{jkl}^i = \sum_\mu \left(\Gamma_{k\mu}^i \Gamma_{lj}^\mu - \Gamma_{l\mu}^i \Gamma_{kj}^\mu \right) + \left(\nabla_{e_k} \Gamma_{lj}^i \right) - \left(\nabla_{e_l} \Gamma_{kj}^i \right) - \sum_\mu \left(\Gamma_{kl}^\mu \Gamma_{\mu j}^i - \Gamma_{lk}^\mu \Gamma_{\mu j}^i \right).$$

Comparing with

$$\begin{aligned}
& \left[d\omega_i^j - \sum_\mu \omega_\mu^i \wedge \omega_j^\mu\right](e_k, e_l) \\
= & -\sum_\mu \Gamma_{\mu j}^i d\omega^\mu (e_k, e_l) - \sum_\mu \omega^\mu d\left(\Gamma_{\mu j}^i\right)(e_k, e_l) - \sum_\mu \left[\omega_\mu^i (e_k)\, \omega_j^\mu (e_l) - \omega_\mu^i (e_l)\, \omega_j^\mu (e_k)\right] \\
= & -\sum_\mu \Gamma_{\mu j}^i \left(\Gamma_{lk}^\mu - \Gamma_{kl}^\mu\right) - \left(\nabla_{e_k} \Gamma_{lj}^i\right) + \left(\nabla_{e_l} \Gamma_{kj}^i\right) - \sum_\mu \left[\Gamma_{k\mu}^i \Gamma_{lj}^\mu - \Gamma_{l\mu}^i \Gamma_{kj}^\mu\right] \\
= & -R_{jkl}^i,
\end{aligned}$$

since

$$\Omega_j^i (e_k, e_l) = \sum_{\mu<\phi} R_{j\mu\phi}^i \omega^\mu \wedge \omega^\phi (e_k, e_l) = R_{jkl}^i,$$

we have

$$d\omega_i^j - \sum_\mu \omega_\mu^i \wedge \omega_j^\mu = -\Omega_j^i$$

$$\Rightarrow d\omega_i^j = \sum_\mu \omega_\mu^i \wedge \omega_j^\mu - \Omega_j^i.$$

CHAPTER 3 SIMONS' INEQUALITY

In the first part of this chapter, we shall compute the Laplacian of the second fundamental form of a parallel mean curvature vector submanifold in a symmetric space of constant sectional curvature c. In the last part of this chapter, we will give the definition of Ricci curvature and then give an estimate for the Ricci curvature of an n-dimensional submanifold M immersed in N^{n+p} which was proved in [7].

3.1 THE LAPLACIAN OF THE SECOND FUNDAMENTAL FORM

Let M be an n-dimensional manifold immersed in an $(n+p)$-dimensional Riemannian manifold N. We choose a local field of orthonormal frames $e_1, ..., e_{n+p}$ in N such that, restricted to M, the vectors $e_1, ..., e_n$ are tangent to M and the remaining vectors $e_{n+1}, ..., e_{n+p}$ are normal to M. We shall agree that repeated indices are summed over the respective ranges. With respect to the frame field of N chosen above, let $\omega^1, ..., \omega^{n+p}$ be the field of dual frames. We shall make use of the following convention on the ranges of indices (for more details, please see [4]):

$$1 \le A, B, C, ... \le n+p; \qquad 1 \le i, j, k, ... \le n;$$

$$n+1 \le \alpha, \beta, \gamma, ... \le n+p,$$

Let K and R be the curvature tensors of N and M respectively, then the structure equation of N are given by

$$d\omega^A = -\sum \omega^A_B \wedge \omega^B, \qquad \omega^A_B + \omega^B_A = 0, \tag{3.1.1}$$

$$d\omega^A_B = -\sum \omega^A_C \wedge \omega^C_B + \Psi^A_B, \qquad \Psi^A_B = \frac{1}{2}\sum K^A_{BCD}\omega^C \wedge \omega^D, \tag{3.1.2}$$

$$K^A_{BCD} + K^A_{BDC} = 0. \tag{3.1.3}$$

Restricting these forms to M, then

$$\omega^\alpha = 0.$$

Therefore $0 = d\omega^\alpha = -\sum \omega^\alpha_i \wedge \omega^i$, by Cartan's lemma we may write

$$\omega^\alpha_i = \sum h^\alpha_{ij}\omega^j, \qquad h^\alpha_{ij} = h^\alpha_{ji}. \tag{3.1.4}$$

where $\{h^\alpha_{ij}\}$ is called the second fundamental form of the submanifold M.
From the above formulas, we obtain

$$d\omega^i = -\sum \omega^i_j \wedge \omega^j, \qquad \omega^i_j + \omega^j_i = 0, \tag{3.1.5}$$

$$d\omega^i_j = -\sum \omega^i_k \wedge \omega^k_j + \Omega^i_j, \qquad \Omega^i_j = \frac{1}{2}\sum R^i_{jkl}\omega^k \wedge \omega^l, \tag{3.1.6}$$

$$d\omega^\alpha_\beta = -\sum \omega^\alpha_\gamma \wedge \omega^\gamma_\beta + \Omega^\alpha_\beta, \qquad \Omega^\alpha_\beta = \frac{1}{2}\sum \hat{R}^\alpha_{\beta kl}\omega^k \wedge \omega^l, \tag{3.1.7}$$

where R is the curvature tensor of M and $\hat{R}$ the curvature tensor of the normal bundle to M.
By (3.1.2) we have

$$\frac{1}{2}\sum K^A_{BCD}\omega^C \wedge \omega^D = d\omega^A_B + \sum \omega^A_k \wedge \omega^k_B + \sum \omega^A_\alpha \wedge \omega^\alpha_B \tag{3.1.8}$$

Restricting the indices range to $A = i$ and $B = j$, we get

$$\begin{aligned}\frac{1}{2}\sum K^i_{jCD}\omega^C \wedge \omega^D &= d\omega^i_j + \sum \omega^i_k \wedge \omega^k_j + \sum \omega^i_\alpha \wedge \omega^\alpha_j \\ &= \frac{1}{2}\sum R^i_{jrs}\omega^r \wedge \omega^s - \sum_\alpha h^\alpha_{ir}h^\alpha_{js}\omega^r \wedge \omega^s\end{aligned}$$

Then we evaluate on e_k and e_l to obtain

$$R^i_{jkl} = K^i_{jkl} + \sum_\alpha h^\alpha_{ir} h^\alpha_{js} \left(\delta^r_k \delta^s_l - \delta^r_l \delta^s_k\right)$$

$$\Rightarrow R^i_{jkl} = K^i_{jkl} + \sum_\alpha \left(h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk}\right)$$

Similarly, we restrict the indices range of (3.1.8) to $A = \alpha$ and $B = \beta$, and obtain

$$\begin{aligned}
\frac{1}{2} \sum K^\alpha_{\beta CD} \omega^C \wedge \omega^D &= d\omega^\alpha_\beta + \sum \omega^\alpha_i \wedge \omega^i_\beta + \sum \omega^\alpha_\gamma \wedge \omega^\gamma_\beta \\
&= \frac{1}{2} \sum \hat{R}^\alpha_{\beta rs} \omega^r \wedge \omega^s + \sum_i h^\alpha_{ir} h^\beta_{is} \omega^r \wedge \omega^s
\end{aligned}$$

$$\Rightarrow \hat{R}^\alpha_{\beta kl} = K^\alpha_{\beta kl} + \sum_i h^\alpha_{ir} h^\beta_{is} \left(\delta^r_k \delta^s_l - \delta^r_l \delta^s_k\right)$$

$$\Rightarrow \hat{R}^\alpha_{\beta kl} = K^\alpha_{\beta kl} + \sum_i \left(h^\alpha_{ik} h^\beta_{il} - h^\alpha_{il} h^\beta_{ik}\right) \tag{3.1.10}$$

Next we define the covariant derivative of h^α_{ij} by

$$\nabla h^\alpha_{ij} = dh^\alpha_{ij} - h^\alpha_{ik} \omega^k_j - h^\alpha_{kj} \omega^k_i + h^\beta_{ij} \omega^\alpha_\beta. \tag{3.1.11}$$

Since ∇h^α_{ij} is a 1-form, we set

$$\nabla h^\alpha_{ij} = \sum h^\alpha_{ijk} \omega^k \tag{3.1.12}$$

Multiplying by ω^j and using the wedge product $\wedge$, equation (3.1.11) becomes

$$\nabla h^\alpha_{ij} \wedge \omega^j = dh^\alpha_{ij} \wedge \omega^j - h^\alpha_{ik} \omega^k_j \wedge \omega^j - h^\alpha_{kj} \omega^k_i \wedge \omega^j + h^\beta_{ij} \omega^\alpha_\beta \wedge \omega^j. \tag{3.1.13}$$

Subsequently we take exterior differentiation of (3.1.4) to obtain

$$d\omega^\alpha_i = d\left(h^\alpha_{ij} \omega^j\right) = dh^\alpha_{ij} \wedge \omega^j - h^\alpha_{ik} \omega^k_j \wedge \omega^j$$

$$\Rightarrow dh^\alpha_{ij} \wedge \omega^j = d\omega^\alpha_i + h^\alpha_{ik}\omega^k_j \wedge \omega^j. \tag{3.1.14}$$

Substitute back to (3.1.13), we have

$$\sum h^\alpha_{ijk}\omega^k \wedge \omega^j = d\omega^\alpha_i + \sum \omega^\alpha_k \wedge \omega^k_i + \sum \omega^\alpha_\beta \wedge \omega^\beta_i$$

$$\Rightarrow \sum h^\alpha_{ijk}\omega^k \wedge \omega^j = \frac{1}{2}\sum K^\alpha_{ijk}\omega^j \wedge \omega^k$$

$$\Rightarrow \sum \left(h^\alpha_{ijk} + \frac{1}{2}K^\alpha_{ijk}\right)\omega^k \wedge \omega^j = 0.$$

Hence

$$h^\alpha_{ijk} - h^\alpha_{ikj} = -K^\alpha_{ijk} = K^\alpha_{ikj}. \tag{3.1.15}$$

Next we compute the covariant derivative of h^α_{ijk} and try to show a relation concerning h^α_{ijkl} and h^α_{ijlk}. Let us first define the term h^α_{ijkl} by considering the covariant derivative of h^α_{ijk}. Following [3], we define the covariant derivative of h^α_{ljk} by:

$$\nabla h^\alpha_{ijk} = dh^\alpha_{ijk} - h^\alpha_{ljk}\omega^l_i - h^\alpha_{ilk}\omega^l_j - h^\alpha_{ijl}\omega^l_k + h^\beta_{ijk}\omega^\alpha_\beta. \tag{3.1.16}$$

Since the right-hand-side of equation (3.1.16) is a 1-form,we let

$$\nabla h^\alpha_{ijk} = \sum_l h^\alpha_{ijkl}\omega^l$$

$$\Rightarrow \sum_l h^\alpha_{ijkl}\omega^l = dh^\alpha_{ijk} - h^\alpha_{ljk}\omega^l_i - h^\alpha_{ilk}\omega^l_j - h^\alpha_{ijl}\omega^l_k + h^\beta_{ijk}\omega^\alpha_\beta. \tag{3.1.17}$$

Multiplying by ω^k and using the wedge product $\wedge$, we have

$$\sum_{l,k} h^\alpha_{ijkl}\omega^l \wedge \omega^k = dh^\alpha_{ijk} \wedge \omega^k - h^\alpha_{ljk}\omega^l_i \wedge \omega^k - h^\alpha_{ilk}\omega^l_j \wedge \omega^k - h^\alpha_{ijl}\omega^l_k \wedge \omega^k + h^\beta_{ijk}\omega^\alpha_\beta \wedge \omega^k. \tag{3.1.18}$$

Next, we take exterior differentiation d of (3.1.11),

$$d\left(\nabla h_{ij}^{\alpha}\right)=d\left(dh_{ij}^{\alpha}\right)-d\left(h_{lj}^{\alpha}\omega_i^l\right)-d\left(h_{il}^{\alpha}\omega_j^l\right)+d\left(h_{ij}^{\beta}\omega_{\beta}^{\alpha}\right)$$

since $d\left(d\omega\right)=0$ for any form ω and $\nabla h_{ij}^{\alpha}=h_{ijk}^{\alpha}\omega^k$, we have

$$d\left(h_{ijk}^{\alpha}\omega^k\right)=-d\left(h_{lj}^{\alpha}\omega_i^l\right)-d\left(h_{li}^{\alpha}\omega_j^l\right)+d\left(h_{ij}^{\beta}\omega_{\beta}^{\alpha}\right)$$

$$\Rightarrow d\left(h_{ijk}^{\alpha}\right)\wedge\omega^k+h_{ijk}^{\alpha}d\left(\omega^k\right)=-d\left(h_{lj}^{\alpha}\omega_i^l\right)-d\left(h_{li}^{\alpha}\omega_j^l\right)+d\left(h_{ij}^{\beta}\omega_{\beta}^{\alpha}\right)$$

$$\Rightarrow d\left(h_{ijk}^{\alpha}\right)\wedge\omega^k=h_{ijl}^{\alpha}\omega_k^l\wedge\omega^k-d\left(h_{lj}^{\alpha}\omega_i^l\right)-d\left(h_{li}^{\alpha}\omega_j^l\right)+d\left(h_{ij}^{\beta}\omega_{\beta}^{\alpha}\right). \qquad (3.1.19)$$

Substituting (3.1.19) into (3.1.18), we obtain

$$\begin{aligned}
&\sum_{l,k}h_{ijkl}^{\alpha}\omega^l\wedge\omega^k\\
=\;& h_{ijl}^{\alpha}\omega_k^l\wedge\omega^k-d\left(h_{lj}^{\alpha}\omega_i^l\right)-d\left(h_{il}^{\alpha}\omega_j^l\right)+d\left(h_{ij}^{\beta}\omega_{\beta}^{\alpha}\right)\\
&-h_{ljk}^{\alpha}\omega_i^l\wedge\omega^k-h_{ilk}^{\alpha}\omega_j^l\wedge\omega^k-h_{ijl}^{\alpha}\omega_k^l\wedge\omega^k+h_{ijk}^{\beta}\omega_{\beta}^{\alpha}\wedge\omega^k\\
=\;& -d\left(h_{lj}^{\alpha}\omega_i^l\right)-d\left(h_{il}^{\alpha}\omega_j^l\right)+d\left(h_{ij}^{\beta}\omega_{\beta}^{\alpha}\right)-h_{ljk}^{\alpha}\omega_i^l\wedge\omega^k\\
&-h_{ilk}^{\alpha}\omega_j^l\wedge\omega^k+h_{ijk}^{\beta}\omega_{\beta}^{\alpha}\wedge\omega^k\\
=\;& -d\left(h_{lj}^{\alpha}\right)\wedge\omega_i^l-h_{lj}^{\alpha}d\left(\omega_i^l\right)-h_{ljk}^{\alpha}\omega_i^l\wedge\omega^k\\
&-d\left(h_{li}^{\alpha}\right)\wedge\omega_j^l-h_{li}^{\alpha}d\left(\omega_j^l\right)-h_{ilk}^{\alpha}\omega_j^l\wedge\omega^k\\
&+d\left(h_{ij}^{\beta}\right)\wedge\omega_{\beta}^{\alpha}+h_{ij}^{\beta}d\left(\omega_{\beta}^{\alpha}\right)+h_{ijk}^{\beta}\omega_{\beta}^{\alpha}\wedge\omega^k.
\end{aligned}$$

By using the fact that $d\omega_j^i=-\sum\omega_l^i\wedge\omega_j^l+\Omega_j^i$, we have

$$\begin{aligned}
&\sum_{l,k}h_{ijkl}^{\alpha}\omega^l\wedge\omega^k\\
=\;& \left(-d\left(h_{lj}^{\alpha}\right)+h_{\gamma j}^{\alpha}\omega_l^{\gamma}+h_{ljk}^{\alpha}\omega^k\right)\wedge\omega_i^l-h_{lj}^{\alpha}\Omega_i^l\\
&+\left(-d\left(h_{li}^{\alpha}\right)+h_{\gamma i}^{\alpha}\omega_l^{\gamma}+h_{ilk}^{\alpha}\omega^k\right)\wedge\omega_j^l-h_{li}^{\alpha}\Omega_j^l\\
&+\left(d\left(h_{ij}^{\beta}\right)+h_{ij}^{\gamma}\omega_{\gamma}^{\beta}-h_{ijk}^{\beta}\omega^k\right)\wedge\omega_{\beta}^{\alpha}+h_{ij}^{\beta}\Omega_{\beta}^{\alpha}
\end{aligned}$$

$$
\begin{aligned}
= \;& \left(h^{\alpha}_{mj}\omega^{m}_{l} - h^{\alpha}_{sj}\omega^{s}_{l} - h^{\alpha}_{sl}\omega^{s}_{j} + h^{\beta}_{lj}\omega^{\alpha}_{\beta}\right) \wedge \omega^{l}_{i} - h^{\alpha}_{lj}\Omega^{l}_{i} \\
& + \left(h^{\alpha}_{mi}\omega^{m}_{l} - h^{\alpha}_{sl}\omega^{s}_{i} - h^{\alpha}_{si}\omega^{s}_{l} + h^{\beta}_{il}\omega^{\alpha}_{\beta}\right) \wedge \omega^{l}_{j} - h^{\alpha}_{li}\Omega^{l}_{j} \\
& + \left(-h^{\gamma}_{ij}\omega^{\beta}_{\gamma} + h^{\beta}_{sj}\omega^{s}_{i} + h^{\beta}_{si}\omega^{s}_{j} + h^{\gamma}_{ij}\omega^{\beta}_{\gamma}\right) \wedge \omega^{\alpha}_{\beta} + h^{\beta}_{ij}\Omega^{\alpha}_{\beta} \qquad \text{(by (3.1.11) \& (3.1.12))}
\end{aligned}
$$

$$
\begin{aligned}
= \;& \left(-h^{\alpha}_{sl}\omega^{s}_{j} + h^{\beta}_{lj}\omega^{\alpha}_{\beta}\right) \wedge \omega^{l}_{i} - h^{\alpha}_{lj}\Omega^{l}_{i} \\
& + \left(-h^{\alpha}_{sl}\omega^{s}_{i} + h^{\beta}_{il}\omega^{\alpha}_{\beta}\right) \wedge \omega^{l}_{j} - h^{\alpha}_{li}\Omega^{l}_{j} \\
& + \left(h^{\beta}_{sj}\omega^{s}_{i} + h^{\beta}_{si}\omega^{s}_{j}\right) \wedge \omega^{\alpha}_{\beta} + h^{\beta}_{ij}\Omega^{\alpha}_{\beta},
\end{aligned}
$$

By suitably rearranging the indices, we get

$$
\begin{aligned}
& \left(-h^{\alpha}_{sl}\omega^{s}_{j} + h^{\beta}_{lj}\omega^{\alpha}_{\beta}\right) \wedge \omega^{l}_{i} + \left(-h^{\alpha}_{sl}\omega^{s}_{i} + h^{\beta}_{il}\omega^{\alpha}_{\beta}\right) \wedge \omega^{l}_{j} + \left(h^{\beta}_{sj}\omega^{s}_{i} + h^{\beta}_{si}\omega^{s}_{j}\right) \wedge \omega^{\alpha}_{\beta} \\
= \;& \left(h^{\alpha}_{sl} - h^{\alpha}_{sl}\right)\omega^{l}_{i} \wedge \omega^{s}_{j} + \left(h^{\beta}_{lj} - h^{\beta}_{lj}\right)\omega^{\alpha}_{\beta} \wedge \omega^{l}_{j} + \left(h^{\beta}_{il} - h^{\beta}_{il}\right)\omega^{\alpha}_{\beta} \wedge \omega^{l}_{j} = 0.
\end{aligned}
$$

Consequently, we obtain

$$
\sum_{l,k} h^{\alpha}_{ijkl}\omega^{l} \wedge \omega^{k} = -h^{\alpha}_{lj}\Omega^{l}_{i} - h^{\alpha}_{li}\Omega^{l}_{j} + h^{\beta}_{ij}\Omega^{\alpha}_{\beta}. \tag{3.1.20}
$$

Since $\Omega^{i}_{j} = \frac{1}{2}\sum R^{i}_{jkl}\omega^{k} \wedge \omega^{l}$, it follows that

$$
\sum_{k,l} h^{\alpha}_{ijkl}\omega^{l} \wedge \omega^{k} = -\frac{1}{2}\sum h^{\alpha}_{mj}R^{m}_{ikl}\omega^{k} \wedge \omega^{l} - \frac{1}{2}\sum h^{\alpha}_{mi}R^{m}_{jkl}\omega^{k} \wedge \omega^{l} + \frac{1}{2}\sum h^{\beta}_{ij}\widehat{R}^{\alpha}_{\beta kl}\omega^{k} \wedge \omega^{l}
$$

$$
\Rightarrow \left(\sum_{l,k} h^{\alpha}_{ijkl} - \frac{1}{2}\sum h^{\alpha}_{mj}R^{m}_{ikl} - \frac{1}{2}\sum h^{\alpha}_{mi}R^{m}_{jkl} + \frac{1}{2}\sum h^{\beta}_{ij}\widehat{R}^{\alpha}_{\beta kl}\right)\omega^{k} \wedge \omega^{l} = 0
$$

$$
\Rightarrow \sum_{l,k} h^{\alpha}_{ijkl} - \frac{1}{2}\sum h^{\alpha}_{mj}R^{m}_{ikl} - \frac{1}{2}\sum h^{\alpha}_{mi}R^{m}_{jkl} + \frac{1}{2}\sum h^{\beta}_{ij}\widehat{R}^{\alpha}_{\beta kl} = 0.
$$

Therefore,

$$\sum_{l,k} h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum h^\alpha_{mj} R^m_{ikl} + \sum h^\alpha_{mi} R^m_{jkl} - \sum h^\beta_{ij} \widehat{R}^\alpha_{\beta kl}. \tag{3.1.21}$$

Next we want to evaluate the Laplacian Δh^α_{ij} and hence find $\Delta\left(h^\alpha_{ij}\right)^2$.

Firstly, $K^\alpha_{ijk;l}$ is defined by

$$K^\alpha_{ijk;l} = K^\alpha_{ijkl} - \sum K^\alpha_{\beta jk} h^\beta_{il} - \sum K^\alpha_{i\beta k} h^\beta_{jl} - \sum K^\alpha_{ij\beta} h^\beta_{kl} + \sum K^m_{ijk} h^\alpha_{ml}. \tag{3.1.22}$$

Throughout this section, we shall assume that N is locally symmetric, i.e. $K^\alpha_{ijk;l} = 0$. Therefore equation (3.1.22) becomes

$$K^\alpha_{ijkl} = \sum K^\alpha_{\beta jk} h^\beta_{il} + \sum K^\alpha_{i\beta k} h^\beta_{jl} + \sum K^\alpha_{ij\beta} h^\beta_{kl} - \sum K^m_{ijk} h^\alpha_{ml}. \tag{3.1.23}$$

The Laplacian $\triangle h^\alpha_{ij}$ is defined by

$$\triangle h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}. \tag{3.1.24}$$

By (3.1.15), we obtain

$$\triangle h^\alpha_{ij} = \sum_k h^\alpha_{ikjk} - \sum_k K^\alpha_{ijkk} = \sum_k h^\alpha_{kijk} - \sum_k K^\alpha_{ijkk}. \tag{3.1.25}$$

By (3.1.21), we have

$$h^\alpha_{kijk} = h^\alpha_{kikj} + \sum h^\alpha_{km} R^m_{ijk} + \sum h^\alpha_{mi} R^m_{kjk} - \sum h^\beta_{ki} \widehat{R}^\alpha_{\beta jk}. \tag{3.1.26}$$

Using the fact that $h^\alpha_{kikj} = h^\alpha_{kkij} - K^\alpha_{kikj}$ and then substituting the right-hand-side of (3.26) into h^α_{kijk} of (3.1.25). Then

$$\triangle h^\alpha_{ij} = \sum_k \left(h^\alpha_{kkij} - K^\alpha_{kikj} - K^\alpha_{ijkk}\right) + \sum_{k,m} \left(h^\alpha_{km} R^m_{ijk} + h^\alpha_{mi} R^m_{kjk} - h^\beta_{ki} \hat{R}^\alpha_{\beta jk}\right). \tag{3.1.27}$$

From (3.1.9), (3.1.10), (3.1.23) and (3.1.27) we obtain

$$\begin{aligned}
\triangle h_{ij}^{\alpha} &= \sum_{k} h_{kkij}^{\alpha} + \sum_{\beta,k}\left(-K_{ij\beta}^{\alpha}h_{kk}^{\beta} + 2K_{\beta ki}^{\alpha}h_{jk}^{\beta} - K_{k\beta k}^{\alpha}h_{ij}^{\beta} + 2K_{\beta kj}^{\alpha}h_{ki}^{\beta}\right)\\
&+ \sum_{\beta,m,k}\left(h_{mi}^{\alpha}h_{mj}^{\beta}h_{kk}^{\beta} + 2h_{km}^{\alpha}h_{ki}^{\beta}h_{mj}^{\beta} - h_{km}^{\alpha}h_{km}^{\beta}h_{ij}^{\beta} - h_{mi}^{\alpha}h_{mk}^{\beta}h_{kj}^{\beta} - h_{mj}^{\alpha}h_{ki}^{\beta}h_{mk}^{\beta}\right)\\
&+ \sum_{m,k}\left(K_{kik}^{m}h_{mj}^{\alpha} + K_{kjk}^{m}h_{mi}^{\alpha} + 2K_{ijk}^{m}h_{mk}^{\alpha}\right). \qquad (3.1.28)
\end{aligned}$$

By multiplying the term h_{ij}^{α} and rearranging the indices, we get

$$\begin{aligned}
\sum h_{ij}^{\alpha} \triangle h_{ij}^{\alpha} &= \sum_{\alpha,\beta,i,j,k}\left(-K_{ij\beta}^{\alpha}h_{kk}^{\beta}h_{ij}^{\alpha} + 2K_{\beta ki}^{\alpha}h_{jk}^{\beta}h_{ij}^{\alpha} - K_{k\beta k}^{\alpha}h_{ij}^{\beta}h_{ij}^{\alpha}\right) \qquad (3.1.29)\\
&+ \sum_{\alpha,m,i,j,k}\left(2K_{kik}^{m}h_{mj}^{\alpha}h_{ij}^{\alpha} + 2K_{ijk}^{m}h_{mk}^{\alpha}h_{ij}^{\alpha}\right)\\
&- \sum_{\alpha,\beta,i,j,k,l}\left(\left(h_{ik}^{\alpha}h_{jk}^{\beta} - h_{jk}^{\alpha}h_{ik}^{\beta}\right)\left(h_{il}^{\alpha}h_{jl}^{\beta} - h_{jl}^{\alpha}h_{il}^{\beta}\right) + h_{ij}^{\alpha}h_{kl}^{\alpha}h_{ij}^{\beta}h_{kl}^{\beta}\right)\\
&+ \sum_{\alpha,\beta,m,i,j,k} h_{mi}^{\alpha}h_{ij}^{\alpha}h_{mj}^{\beta}h_{kk}^{\beta}.
\end{aligned}$$

Since we assumed that the sectional curvature of N is a constant c, we have the following theorem ([6]).

Theorem 3.1.30

Let N be a space of constant sectional curvature c and R_{BCD}^{A} be the curvature tensor of N, then

$$R_{BCD}^{A} = c\left(\delta_{C}^{A}\delta_{D}^{B} - \delta_{D}^{A}\delta_{C}^{B}\right). \qquad (3.1.31)$$

Proof:

Firstly, we exhibit some formulae for the curvature tensor K (see [10]), (for convenience we adopt the following simplification for the basis vectors - we denote a vector $\frac{\partial}{\partial x_A}$ by A, i.e. the letter A can be interpreted either as an index or as basis vector, depending on the context)

$$R_{BCD}^{A} = -R_{ACD}^{B} \qquad (3.1.32)$$

$$R^A_{BCD} = R^C_{DAB}. \tag{3.1.33}$$

Let $K(A,B) = \langle R(A,B)B,A\rangle$ and $K(A\wedge B)$ be the sectional curvature of the subspace generated by A and B. Following [6] , we have

$$\begin{aligned} R^A_{BCD} &= \langle R(C,D)B,A\rangle \\ &= K(C+A,D+B) - K(C+A,D) \\ &\quad -K(C+A,B) - K(C,D+B) \\ &\quad -K(A,D+B) + K(C,B) + K(A,D) \\ &\quad -K(D+A,C+B) + K(D+A,C) \\ &\quad +K(D+A,B) + K(D,C+B) + K(A,C+B) \\ &\quad -K(D,B) - K(A,C), \end{aligned} \tag{3.1.34}$$

where $K(A,B) = K(A\wedge B)\left(\langle A,A\rangle\langle B,B\rangle - \langle A,B\rangle^2\right)$.

We will prove this lemma by showing that equation (3.1.31) is true for all A, B, C and D.

Case1. $(A = B = C = D)$. Here there is only one type, i.e. R^A_{AAA}.

By equation (3.1.32), we have

$$R^A_{AAA} = -R^A_{AAA} \Rightarrow R^A_{AAA} = 0.$$

Case2. $(A = B = C \neq D)$. Here we have 4 types, i.e. R^A_{AAB}, R^A_{ABA}, R^A_{BAA} and R^B_{AAA}.

By (3.1.33), we have

$$R^A_{AAB} = 0$$

and by (3.1.33),

$$R^A_{BAA} = R^A_{AAB} = 0.$$

Hence

$$R^B_{AAA} = R^A_{ABA} = -R^A_{AAB} = 0.$$

Case3. $(A = B \neq C = D)$. Here we have 3 types, i.e. R^A_{ABB}, R^A_{BAB} and R^A_{BBA}. By (3.1.32)

$$R^A_{ABB} = -R^A_{ABB} \Rightarrow R^A_{ABB} = 0$$

and by definition we have

$$R^A_{BAB} = c.$$

Moreover

$$R^A_{BBA} = -R^A_{BAB} = -c.$$

Case4. $A = B \neq C \neq D$. Here we have 6 types, i.e. R^A_{ABC}, R^B_{ACA}, R^B_{CAA}, R^A_{BAC}, R^A_{BCA} and R^B_{AAC}.
By (3.1.32) and (3.1.33),

$$R^A_{ABC} = R^B_{CAA} = 0$$

and by (3.1.34),

$$\begin{aligned} R^A_{BAC} &= c((4)(2) - 4 - 4 - 2 - 2 + 1 + 1 - (4-1) + (2-1) + 2 + 2 + (2-1) - 1) \\ &= 0. \end{aligned}$$

Hence

$$R^B_{ACA} = -R^B_{AAC} = -R^A_{BCA} = R^A_{BAC} = 0.$$

Case5. All 4 indices are different, i.e. R^A_{BCD}.

By (3.1.34)

$$
\begin{aligned}
R^{A}_{BCD} &= c(4-2-2-2-2+1+1-4+2+2+2+2-1-1) \\
&= 0.
\end{aligned}
$$

It is easy to show that equation (3.1.31) is true for each case by evaluating the right hand side, then the result follows.

With the above theorem, (3.1.29) reduces to

$$
\begin{aligned}
\sum h^{\alpha}_{ij} \bigtriangleup h^{\alpha}_{ij} &= \sum c\left(h^{\alpha}_{kk}h^{\alpha}_{ii} - \delta_{kk}\left(h^{\alpha}_{ij}\right)^2\right) \\
&\quad +2c\left(\delta_{kk}\left(h^{\alpha}_{ij}\right)^2 - \left(h^{\alpha}_{ij}\right)^2 + \left(h^{\alpha}_{jk}\right)^2 - h^{\alpha}_{kk}h^{\alpha}_{jj}\right) - \sum h^{\alpha}_{mi}h^{\alpha}_{ij}h^{\beta}_{mj}h^{\beta}_{kk} \\
&\quad -\sum\left(h^{\alpha}_{ik}h^{\beta}_{jk} - h^{\alpha}_{jk}h^{\beta}_{ik}\right)\left(h^{\alpha}_{il}h^{\beta}_{jl} - h^{\alpha}_{jl}h^{\beta}_{il}\right) + h^{\alpha}_{ij}h^{\alpha}_{kl}h^{\beta}_{ij}h^{\beta}_{kl} \\
&= \sum cn\left(h^{\alpha}_{ij}\right)^2 - \sum ch^{\alpha}_{kk}h^{\alpha}_{ii} + \sum h^{\alpha}_{mi}h^{\alpha}_{ij}h^{\beta}_{mj}h^{\beta}_{kk} \\
&\quad -\sum\left(\left(h^{\alpha}_{ik}h^{\beta}_{jk} - h^{\alpha}_{jk}h^{\beta}_{ik}\right)\left(h^{\alpha}_{il}h^{\beta}_{jl} - h^{\alpha}_{jl}h^{\beta}_{il}\right) + h^{\alpha}_{ij}h^{\alpha}_{kl}h^{\beta}_{ij}h^{\beta}_{kl}\right) \\
&= cn\sum_{\alpha} trA^2_{\alpha} - c\sum_{\alpha}\left(trA_{\alpha}\right)^2 + \sum_{\alpha,\beta}\left(trA_{\alpha}\right)\left(trA_{\alpha}A^2_{\beta}\right) \\
&\quad +\sum_{\alpha,\beta}\left(tr\left[A_{\alpha},A_{\beta}\right]^2\right) - \sum_{\alpha,\beta}\left(trA_{\alpha}A_{\beta}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \sum h^{\alpha}_{ij} \bigtriangleup h^{\alpha}_{ij} &= cn\sum_{\alpha} trA^2_{\alpha} - c\sum_{\alpha}\left(trA_{\alpha}\right)^2 + \sum_{\alpha,\beta}\left(trA_{\alpha}\right)\left(trA_{\alpha}A^2_{\beta}\right) \\
&\quad +\sum_{\alpha,\beta}\left(tr\left[A_{\alpha},A_{\beta}\right]^2\right) - \sum_{\alpha,\beta}\left(trA_{\alpha}A_{\beta}\right)^2. \qquad (3.1.35)
\end{aligned}
$$

It is easy to show that

$$
\bigtriangleup\left(h^{\alpha}_{ij}\right)^2 = 2\left(\sum h^{\alpha}_{ij} \bigtriangleup h^{\alpha}_{ij} + \sum\left(\nabla h^{\alpha}_{ij}\right)^2\right)
$$

$$
\Rightarrow \frac{1}{2}\bigtriangleup|A|^2 = \sum h^{\alpha}_{ij} \bigtriangleup h^{\alpha}_{ij} + \sum\left|\nabla A_{\alpha}\right|^2.
$$

Hence,

$$\frac{1}{2}\triangle|A|^2 = cn\sum_{\alpha} trA_\alpha^2 - c\sum_{\alpha}(trA_\alpha)^2 + \sum_{\alpha,\beta} tr\left[A_\alpha, A_\beta\right]^2 \tag{3.1.36}$$
$$+\sum_{\alpha,\beta}(trA_\alpha)\left(trA_\alpha A_\beta^2\right) - \sum_{\alpha,\beta}(trA_\alpha A_\beta)^2 + \sum|\nabla A_\alpha|^2.$$

3.2 THE TRACELESS SECOND FUNDAMENTAL FORM

Following [8], we will define the linear maps ϕ_α and hence obtain the Simons' inequality for submanifolds with parallel mean curvature vector.
For each α, $n+1 \le \alpha \le n+p$, define linear maps $\phi_\alpha : T_xM \to T_xM$ by

$$\langle \phi_\alpha X, Y\rangle = \langle X, Y\rangle \langle h, e_\alpha\rangle - \langle A_\alpha X, Y\rangle,$$

and a bilinear map $\phi : T_xM \times T_xM \to T_xM^\perp$ by

$$\phi(X,Y) = \sum_{\alpha=n+1}^{n+p} \langle \phi_\alpha X, Y\rangle e_\alpha.$$

It is easy to check that each map ϕ_α is traceless and that

$$|\phi|^2 = \sum_{\alpha=n+1}^{n+p} tr\phi_\alpha^2 = |A|^2 - nH^2. \tag{3.2.1}$$

We will call ϕ the traceless second fundamental form and $|\phi|$ its length.

Next we shall compute the Laplacian of $|\phi|^2$. For that purpose, we choose a local field of orthonormal frame $\{e_1, \ldots.., e_{n+p}\}$ in a such way that $e_{n+1} = \frac{h}{H}$. With this choice,

$$\left\{ \begin{array}{c} \phi_{n+1} = HI - A_{n+1}, \\ \phi_\alpha = -A_\alpha, \quad n+2 \le \alpha \le n+p. \end{array} \right. \tag{3.2.2}$$

Lemma 3.2.3.

Let ϕ be the bilinear map defined above, then we have

$$\frac{1}{2}\triangle|\phi|^2 = \sum_{\alpha=n+1}^{n+p}|\nabla\phi_\alpha|^2+n\left(c+H^2\right)|\phi|^2+\sum_{\alpha,\beta>n+1}^{n+p} tr\left[\phi_\alpha,\phi_\beta\right]^2-\sum_{\alpha,\beta=n+1}^{n+p}\left(tr\phi_\alpha\phi_\beta\right)^2$$
$$-nH\sum_{\alpha,\beta=n+1}^{n+p}\left(tr\phi_{n+1}\phi_\alpha^2\right). \qquad (3.2.4)$$

Proof:

Firstly, we simplify the terms of (3.1.36) in terms of ϕ and ϕ_α.

$$\frac{1}{2}\triangle|A|^2=\frac{1}{2}\triangle\left(|\phi|^2+nH^2\right)=\frac{1}{2}\triangle|\phi|^2$$

$$\sum_{\alpha=n+1}^{n+p}|\nabla A_\alpha|^2=\sum_{\alpha=n+2}^{n+p}|\nabla(-\phi_\alpha)|^2+\left|\nabla\left(HI-\phi_{n+1}\right)\right|^2=\sum_{\alpha=n+1}^{n+p}|\nabla\phi_\alpha|^2$$

$$cn\sum_{\alpha=n+1}^{n+p}trA_\alpha^2=cn|A|^2=cn|\phi|^2+cn^2H^2$$

$$-c\sum_{\alpha=n+1}^{n+p}(trA_\alpha)^2=-cn^2H^2 \qquad \left(\text{since } h=\frac{1}{n}\sum_{\alpha=n+1}^{n+p}(trA_\alpha)e_\alpha \text{ and } |h|=H\right)$$

$$\sum_{\alpha,\beta=n+1}^{n+p}tr[A_\alpha,A_\beta]^2=\sum_{\alpha,\beta>n+1}^{n+p}tr[A_\alpha,A_\beta]^2+\sum_{\alpha>n+1}^{n+p}tr[A_\alpha,A_{n+1}]^2+\sum_{\beta>n+1}^{n+p}tr[A_{n+1},A_\beta]^2.$$

But $tr[A_\alpha,A_{n+1}]=0$ for all α, therefore

$$\sum_{\alpha,\beta=n+1}^{n+p}tr[A_\alpha,A_\beta]^2=\sum_{\alpha,\beta>n+1}^{n+p}tr[A_\alpha,A_\beta]^2$$

and

$$\sum_{\alpha,\beta>n+1}^{n+p}tr[A_\alpha,A_\beta]^2=\sum_{\alpha,\beta>n+1}^{n+p}tr\left[-\phi_\alpha,-\phi_\beta\right]^2=\sum_{\alpha,\beta>n+1}^{n+p}tr\left[\phi_\alpha,\phi_\beta\right]^2.$$

$$
\begin{aligned}
&\sum_{\alpha,\beta=n+1}^{n+p} (trA_\alpha)\left(trA_\alpha A_\beta^2\right) \\
=\ & \sum_{\substack{\alpha=n+2\\ \beta=n+1}}^{n+p} tr\left(-\phi_\alpha\right) tr\left(-\phi_\alpha A_\beta^2\right) + \sum_{\alpha=n+1}^{n+p} trA_{n+1} tr\left(A_{n+1}A_\alpha^2\right) \\
=\ & \left(trA_{n+1}\right) \sum_{\alpha=n+1}^{n+p} tr\left(\left(HI-\phi_{n+1}\right)A_\alpha^2\right) \qquad (\text{since } tr\left(\phi_\alpha\right)=0) \\
=\ & nH^2\left|A\right|^2 - nH \sum_{\alpha=n+1}^{n+p} tr\left(\phi_{n+1}A_\alpha^2\right) \\
=\ & nH^2\left|A\right|^2 - nH\left(\sum_{\alpha=n+2}^{n+p} tr\left(\phi_{n+1}\phi_\alpha^2\right) + tr\left(\phi_{n+1}\left(HI-\phi_{n+1}\right)^2\right)\right) \\
=\ & nH^2\left(\left|\phi\right|^2 + nH^2\right) - nH\left(\sum_{\alpha=n+1}^{n+p} tr\left(\phi_{n+1}\phi_\alpha^2\right) - 2Htr\left(\phi_{n+1}^2\right)\right) \\
=\ & nH^2\left|\phi\right|^2 + n^2H^4 - nH\sum_{\alpha=n+1}^{n+p} tr\left(\phi_{n+1}\phi_\alpha^2\right) + 2nH^2 tr\left(\phi_{n+1}^2\right),
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{\alpha,\beta=n+1}^{n+p} \left(trA_\alpha A_\beta\right)^2 \\
=\ & \sum_{\alpha,\beta>n+1}^{n+p} \left(tr\phi_\alpha\phi_\beta\right)^2 + \sum_{\alpha=n+2}^{n+p} \left(trA_\alpha A_{n+1}\right)^2 + \sum_{\beta=n+2}^{n+p} \left(trA_{n+1}A_\beta\right)^2 + \left(trA_{n+1}^2\right)^2 \\
=\ & \sum_{\alpha,\beta>n+1}^{n+p} \left(tr\phi_\alpha\phi_\beta\right)^2 + \sum_{\alpha=n+2}^{n+p} \left(tr\phi_\alpha\left(HI-\phi_{n+1}\right)\right)^2 + \sum_{\beta=n+2}^{n+p} \left(tr\left(HI-\phi_{n+1}\right)\phi_\beta\right)^2 \\
& + \left(tr\left(H^2I - 2H\phi_{n+1} + \phi_{n+1}^2\right)\right)^2 \\
=\ & \sum_{\alpha,\beta>n+1}^{n+p} \left(tr\phi_\alpha\phi_\beta\right)^2 + \sum_{\alpha=n+2}^{n+p} \left(tr\phi_\alpha\phi_{n+1}\right)^2 + \sum_{\beta=n+2}^{n+p} \left(tr\phi_{n+1}\phi_\beta\right)^2 + \left(nH^2 + tr\phi_{n+1}^2\right)^2 \\
=\ & \sum_{\alpha,\beta=n+1}^{n+p} \left(tr\phi_\alpha\phi_\beta\right)^2 + n^2H^4 + 2nH^2 tr\left(\phi_{n+1}^2\right).
\end{aligned}
$$

$$\therefore\ \frac{1}{2}\Delta|\phi|^2=\sum_{\alpha=n+1}^{n+p}|\nabla\phi_\alpha|^2+n\left(c+H^2\right)|\phi|^2+\sum_{\alpha,\beta>n+1}^{n+p}tr\left[\phi_\alpha,\phi_\beta\right]^2$$
$$-\sum_{\alpha,\beta=n+1}^{n+p}\left(tr\phi_\alpha\phi_\beta\right)^2-nH\sum_{\alpha=n+1}^{n+p}tr\left(\phi_{n+1}\phi_\alpha^2\right).$$

Lemma 3.2.5.

Let x_i, $i=1,...,n$, be real numbers such that $\sum_i x_i=0$ and $\sum_i x_i^2=1$. Then

$$\sum_i^n x_i^4\le\frac{(n-2)^2}{n(n-1)}+\frac{1}{n}. \tag{3.2.6}$$

Proof:

It is easy to see that (3.2.6) is true for $n=2,3$. If $n\ge 4$, consider an n-tuple $(x_1,...x_n)$ such that $\sum_i x_i=0$ and $\sum_i x_i^2=1$. We define a polynomial $P(x)$,

$$\begin{aligned}P(x)&=\prod_{i=1}^n(x-x_i)=x^n-\frac{1}{2}x^{n-2}+\left(\sum_i x_i^3\right)x^{n-3}\\&+\frac{1}{8}\left(1-2\sum_i x_i^4\right)x^{n-4}+...\end{aligned}$$

such that $P(x)$ has n real roots. Hence the $(n-4)$-th derivative $P^{(n-4)}(x)$ of $P(x)$ has four real roots. Since we have

$$\begin{aligned}P^{(n-4)}(x)&=\frac{n!}{4!}x^4-\frac{(n-1)!}{2(2!)}x^2+\frac{(n-2)!}{3}\left(\sum_i x_i^3\right)x\\&+\frac{(n-3)!}{8}\left(1-2\sum_i x_i^4\right),\end{aligned}$$

this implies that the equation

$$x^4-\frac{6}{n(n-1)}x^2+\frac{8\left(\sum_i x_i^3\right)}{n(n-1)(n-2)}x+\frac{3\left(1-2\sum_i x_i^4\right)}{n(n-1)(n-2)(n-3)}=0$$

has only real roots. Since an equation of the form $x^4+6Ax^2+4Bx+C=0$ has only real roots then $C+3A^2\ge 0$, we have

$$\frac{3\left(1-2\sum_i x_i^4\right)}{n(n-1)(n-2)(n-3)}+3\left(\frac{1}{n(n-1)}\right)^2\ge 0$$

$$\Rightarrow \sum_i x_i^4 \leq \frac{n(n-1)(n-2)(n-3)}{2}\left(\left(\frac{1}{n(n-1)}\right)^2+\frac{1}{n(n-1)(n-2)(n-3)}\right)$$

$$\Rightarrow \sum_i x_i^4 \leq \left(\frac{(n-2)^2}{n(n-1)}-\frac{(n-2)^2}{2n(n-1)}+\frac{(n^2-2n+2)}{2n(n-1)}\right)$$

$$\Rightarrow \sum_i x_i^4 \leq \left(\frac{(n-2)^2}{n(n-1)}+\frac{1}{n}\right).$$

Lemma 3.2.7.

Let A and B be symmetric matrices such that $[A,B]=0$ and $trA=trB=0$. Then

$$trA^2B \leq \left|\frac{n-2}{\sqrt{n(n-1)}}\left(trA^2\right)\left(trB^2\right)^{\frac{1}{2}}\right|. \tag{3.2.8}$$

Proof:

Since $[A,B]=0$, we can choose an orthonormal basis of R^n which simultaneously diagonalize A and B, i.e. $P^TAP=D_1$ and $P^TBP=D_2$ where D_1 and D_2 are diagonal matrices. So we must show that the eigenvalues $\{x_1,...x_n\}$ of A and $\{y_1,...y_n\}$ of B satisfy

$$\sum_i x_i^2 y_i \leq \left|\frac{n-2}{\sqrt{n(n-1)}}\left(\sum_i x_i^2\right)\left(\sum_i y_i^2\right)^{\frac{1}{2}}\right|. \tag{3.2.9}$$

Without loss of generality we may assume

$$\sum_i x_i^2 = \sum_i y_i^2 = 1.$$

Since P is orthonormal, we have

$$trA = \sum_{ij} P_{ji}x_jP_{ij} = \sum_j\left(\sum_i P_{ij}^2\right)x_j = \sum_j x_j = 0.$$

Similarly,

$$\sum_j y_j = 0.$$

So we are looking for the extreme values of the function $f(x_i, y_i) = \sum_i x_i^2 y_i$, with the constraints $\sum_i x_i = \sum_i y_i = 0$ and $\sum_i x_i^2 = \sum_i y_i^2 = 1$. By the method of Langrange's multipliers, we let

$$g = \sum_i x_i^2 y_i + A \sum_i x_i + B \sum_i y_i + C \left(\sum_i x_i^2 - 1 \right) + D \left(\sum_i y_i^2 - 1 \right). \quad (3.2.10)$$

Setting the partial derivatives of g with respect to the variables x_i and y_i to zero we obtain the following system of equations:

$$2x_i y_i + A + 2Cx_i = 0 \qquad \text{for} \qquad i = 1, ..., n, \quad (3.2.11)$$

$$x_i^2 + B + 2Dy_i = 0 \qquad \text{for} \qquad i = 1, ..., n. \quad (3.2.12)$$

Summing up (3.2.12) for all i we have $B = -\frac{1}{n}$. Moreover, it is easy to see that $C = 2D$, therefore we have

$$x_i y_i = \lambda x_i + \mu \quad (3.2.13)$$

$$x_i^2 = \lambda y_i + \frac{1}{n}. \quad (3.2.14)$$

Multiplying (3.2.13) by x_i and summing up for all i, we have

$$\lambda = \sum_i x_i^2 y_i. \quad (3.2.15)$$

Therefore λ is the extreme value we are looking for. Squaring both sides of (3.2.14) and summing up over i, we obtain

$$\sum_i x_i^4 = \lambda^2 \sum_i x_i^2 + \frac{2\lambda}{n} \sum_i x_i + \frac{1}{n}$$

$$\Rightarrow \lambda^2 = \sum_i x_i^4 - \frac{1}{n}.$$

Lemma 3.2.5 implies that

$$\lambda^2 \leq \frac{(n-2)^2}{n(n-1)}. \tag{3.2.16}$$

Hence,

$$-\frac{n-2}{\sqrt{n(n-1)}} \leq \sum_i x_i^2 y_i \leq \frac{n-2}{\sqrt{n(n-1)}} \tag{3.2.17}$$

implying

$$\sum_i x_i^2 y_i \leq \left| \frac{n-2}{\sqrt{n(n-1)}} \left(\sum_i x_i^2\right) \left(\sum_i y_i^2\right)^{\frac{1}{2}} \right|. \tag{3.2.18}$$

Recall that $trA^2 = tr\left(P^T D_1^2 P\right) = \sum_j \left(\sum_i P_{ij}^2\right) x_j^2 = \sum_j x_j^2$. Similarly, $trB^2 = \sum_j y_j^2$ and $trA^2B = \sum_i x_i^2 y_i$. Substitute these to (3.2.18) we obtain (3.2.8).

Lemma 3.2.19. ([4], Lemma 1, p.65)
Let A and B be symmetric $(n \times n)$-matrices. Then

$$N(AB - BA) \leq 2N(A)N(B). \tag{3.2.20}$$

Proof:
Since B is symmetric we can find an orthonormal matrix P such that $P^T BP = D$. By using the fact that $N(A) = N\left(P^T AP\right)$ for any orthogonal matrix P, we have

$$N(AB - BA) = N\left(P^T (AB - BA) P\right) = N\left(P^T APD - DP^T AP\right).$$

Let $b_1, \ldots, b_n$ be the diagonal entries in D and $P^T AP = (a_{ij})$. It is easy to show that

$$N(AB - BA) = N\left(P^T APD - DP^T AP\right) = \sum_{i \neq j} a_{ij}^2 (b_i - b_j)^2.$$

Since $(b_i - b_j)^2 \leq 2\left(b_i^2 + b_j^2\right)$, we obtain

$$\begin{aligned} & N(AB - BA) \\ \leq\ & 2\sum_{i \neq j} a_{ij}^2 \left(b_i^2 + b_j^2\right) \\ \leq\ & 2\left(\sum_{i,j} a_{ij}^2\right)\left(\sum_i b_i^2\right) = 2N\left(P^T AP\right) N(D) = 2N(A)\, N(B). \end{aligned}$$

With the above lemmas, we can prove the following assertions.

Assertion 1: ([8], p.8)

$$\sum_{\alpha,\beta>n+1}^{n+p} tr\left[\phi_\alpha, \phi_\beta\right]^2 - \sum_{\alpha,\beta>n+1}^{n+p} \left(tr\phi_\alpha\phi_\beta\right)^2 \geq -\left(2 - \frac{1}{p-1}\right)\left(|\phi|^2 - \left|\phi_{n+1}\right|^2\right)^2. \tag{3.2.21}$$

Proof:

For a matrix $A = (a_{ij})$ we denote by $N(A)$ the square of the norm of A, i.e.

$$N(A) = tr\left(A^T A\right) = \sum_{ij} (a_{ij})^2.$$

By direct calculation we have

$$tr\left[\phi_\alpha, \phi_\beta\right]^2 = -N\left(\phi_\alpha\phi_\beta - \phi_\beta\phi_\alpha\right). \tag{3.2.22}$$

By lemma 3.2.19, we have

$$tr\left[\phi_\alpha, \phi_\beta\right]^2 = -N\left(\phi_\alpha\phi_\beta - \phi_\beta\phi_\alpha\right) \geq -2N\left(\phi_\alpha\right) N\left(\phi_\beta\right) = -2tr\left(\phi_\alpha^2\right) tr\left(\phi_\beta^2\right)$$

$$\Rightarrow \sum_{\alpha,\beta>n+1}^{n+p} tr\left[\phi_\alpha, \phi_\beta\right]^2 \geq \sum_{\alpha\neq\beta>n+1}^{n+p} -2tr\left(\phi_\alpha^2\right) tr\left(\phi_\beta^2\right). \tag{3.2.23}$$

Let S_α and $S_{\alpha\beta}$ be defined by

$$S_\alpha = \sum_{i,j} \left(h_{ij}^\alpha\right)^2 \qquad and \qquad S_{\alpha\beta} = \sum_{i,j} h_{ij}^\alpha h_{ij}^\beta,$$

respectively. Then we have

$$\sum_{\alpha,\beta>n+1}^{n+p} \left(tr\phi_\alpha\phi_\beta\right)^2 = \sum_{\alpha,\beta>n+1} (S_{\alpha\beta})^2 \tag{3.2.24}$$
$$\geq \sum_{\alpha>n+1} S_\alpha^2.$$

By (3.2.23) and (3.2.24), we have

$$\sum_{\alpha,\beta>n+1}^{n+p} tr\left[\phi_\alpha,\phi_\beta\right]^2 - \sum_{\alpha,\beta>n+1}^{n+p} \left(tr\phi_\alpha\phi_\beta\right)^2 \tag{3.2.25}$$
$$\begin{aligned}
&\leq -\left(\sum_{\alpha\neq\beta>n+1}^{n+p} 2S_\alpha S_\beta + \sum_{\alpha>n+1} S_\alpha^2\right) \\
&= -\left(2\sum_{\beta>\alpha>n+1}^{n+p} S_\alpha S_\beta + 2\sum_{\beta>\alpha>n+1}^{n+p} S_\alpha S_\beta + \sum_{\alpha>n+1} S_\alpha^2\right) \\
&= -\left(2\sum_{\beta>\alpha>n+1}^{n+p} S_\alpha S_\beta + \left(\sum_{\alpha>n+1} S_\alpha\right)^2\right).
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{\beta>\alpha>n+1} (S_\alpha - S_\beta)^2 &= -2\sum_{\beta>\alpha>n+1} S_\alpha S_\beta + \sum_{\beta>\alpha>n+1} (S_\alpha^2 + S_\beta^2) \\
&= -2\sum_{\beta>\alpha>n+1} S_\alpha S_\beta + (p-2)\sum_{\alpha>n+1} S_\alpha^2 \\
&= -2\sum_{\beta>\alpha>n+1} S_\alpha S_\beta + (p-2)\left(\left(\sum_{\alpha>n+1} S_\alpha\right)^2 - 2\sum_{\beta>\alpha>n+1} S_\alpha S_\beta\right) \\
&= -2(p-1)\sum_{\beta>\alpha>n+1} S_\alpha S_\beta + (p-2)\left(\sum_{\alpha>n+1} S_\alpha\right)^2,
\end{aligned}$$

this implies

$$-2\sum_{\beta>\alpha>n+1} S_\alpha S_\beta = \frac{1}{p-1}\sum_{\beta>\alpha>n+1} (S_\alpha - S_\beta)^2 - \frac{p-2}{p-1}\left(\sum_{\alpha>n+1} S_\alpha\right)^2. \tag{3.2.26}$$

Substitute back into (3.2.25), we have

$$\begin{aligned}
&\sum_{\alpha,\beta>n+1}^{n+p} tr\left[\phi_\alpha,\phi_\beta\right]^2 - \sum_{\alpha,\beta=n+1}^{n+p}\left(tr\phi_\alpha\phi_\beta\right)^2 \\
\geq\ & \frac{1}{p-1}\sum_{\beta>\alpha>n+1}\left(S_\alpha - S_\beta\right)^2 - \frac{p-2}{p-1}\left(\sum_{\alpha>n+1} S_\alpha\right)^2 - \left(\sum_{\alpha>n+1} S_\alpha\right)^2 \\
\geq\ & \left(-\frac{p-2}{p-1}-1\right)\left(\sum_{\alpha>n+1} S_\alpha\right)^2 \\
\geq\ & -\left(2-\frac{1}{p-1}\right)\left(|\phi|^2 - |\phi_{n+1}|^2\right)^2.
\end{aligned}$$

Assertion 2: ([?], p.8)

$$\sum_{\alpha=n+1}^{n+p} tr\left(\phi_{n+1}\phi_\alpha^2\right) \leq \frac{n-2}{\sqrt{n(n+1)}}\left|\phi_{n+1}\right||\phi|^2. \tag{3.2.27}$$

Proof:

Recall that $\left[\phi_\alpha,\phi_{n+1}\right]=0$ and $tr\phi_\alpha = 0$ for all α. Hence for each α, $n+1\leq\alpha\leq n+p$, we can apply lemma 3.2.7 to ϕ_α and $\phi_{n+1}$to obtain

$$tr\phi_{n+1}\phi_\alpha^2 \leq \frac{n-2}{\sqrt{n(n-1)}}\left|\phi_\alpha\right|^2\left|\phi_{n+1}\right|.$$

summing up for all α, we obtain (3.2.27).

Assertion 3: ([8], p.9)

$$\sum_{\alpha=n+2}^{n+p}\left(tr\phi_{n+1}\phi_\alpha\right)^2 \leq \left|\phi_{n+1}\right|^2\left(|\phi|^2 - \left|\phi_{n+1}\right|^2\right). \tag{3.2.28}$$

Proof:

By Cauchy-Schwarz's inequality,

$$\left(tr\phi_{n+1}\phi_\alpha\right)^2 = \left(\sum_{i,j}\phi_{ij}^{n+1}\phi_{ji}^\alpha\right)^2 \leq \sum_{i,j}\left(\phi_{ij}^{n+1}\right)^2\sum_{i,j}\left(\phi_{ij}^\alpha\right)^2 = \left|\phi_{n+1}\right|^2\left|\phi_\alpha\right|^2$$

$$\Rightarrow \sum_{\alpha=n+2}^{n+p}\left(tr\phi_{n+1}\phi_\alpha\right)^2 \leq \left|\phi_{n+1}\right|^2\left(|\phi|^2 - \left|\phi_{n+1}\right|^2\right).$$

By putting (3.2.21), (3.2.27) and (3.2.28) into (3.2.4), we obtain

$$\begin{aligned}\frac{1}{2}\triangle|\phi|^2 &\geq n\left(c+H^2\right)|\phi|^2 - nH\left(\frac{n-2}{\sqrt{n(n+1)}}\left|\phi_{n+1}\right||\phi|^2\right)\\ &-\left(2-\frac{1}{p-1}\right)\left(|\phi|^2-\left|\phi_{n+1}\right|^2\right)^2 - 2\sum_{\alpha=n+2}^{n+p}\left(tr\phi_{n+1}\phi_\alpha\right)^2 - \left(tr\phi_{n+1}^2\right)^2\end{aligned}$$

$$\begin{aligned}\Rightarrow\ &\frac{1}{2}\triangle|\phi|^2 \geq n\left(c+H^2\right)|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n+1)}}H\left|\phi_{n+1}\right||\phi|^2 - \left(2-\frac{1}{p-1}\right)|\phi|^4\\ &+\left(2-\frac{1}{p-1}\right)\left|\phi_{n+1}\right|^2\left(2|\phi|^2-\left|\phi_{n+1}\right|^2\right) - 2\left|\phi_{n+1}\right|^2\left(|\phi|^2-\left|\phi_{n+1}\right|^2\right) - \left|\phi_{n+1}\right|^4\end{aligned}$$

$$\begin{aligned}\Rightarrow\ &\frac{1}{2}\triangle|\phi|^2 \geq n\left(c+H^2\right)|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n+1)}}H|\phi|^3 - \left(2-\frac{1}{p-1}\right)|\phi|^4\\ &+\left(1-\frac{1}{p-1}\right)\left|\phi_{n+1}\right|^2\left(2|\phi|^2-\left|\phi_{n+1}\right|^2\right).\end{aligned}$$

Since

$$\left(1-\frac{1}{p-1}\right)\left|\phi_{n+1}\right|^2\left(2|\phi|^2-\left|\phi_{n+1}\right|^2\right)\geq 0.$$

We obtain

$$\frac{1}{2}\triangle|\phi|^2 \geq |\phi|^2\left(n\left(c+H^2\right) - \frac{n(n-2)}{\sqrt{n(n+1)}}H|\phi| - \left(2-\frac{1}{p-1}\right)|\phi|^2\right). \quad (3.2.29)$$

3.3 THE RICCI CURVATURE OF A SUBMANIFOLD

Definition 3.3.1. (for more details, please see [7])

Given a point p and a unit tangent vector X at that point, we can define the Ricci curvature in this direction by averaging all the sectional curvatures of the two dimensional tangent planes which contain this tangent vector. This is given by

$$Ric(X) = \sum_{i=1}^{n} R\left(e_i, X, e_i, X\right). \quad (3.3.2)$$

From (3.1.9), we have

$$\begin{aligned} Ric\,(e_k) &= \sum K^i_{kik} + \sum \left(h^\alpha_{ii} h^\alpha_{kk} - (h^\alpha_{ik})^2\right) && (3.3.3) \\ &= \sum K^i_{kik} + \sum_\alpha nHh^\alpha_{kk} - \sum_{\alpha,i} (h^\alpha_{ik})^2 \\ &= \sum K^i_{kik} + nHh^{n+1}_{kk} - \sum_{\alpha,i} (h^\alpha_{ik})^2 . \end{aligned}$$

Lemma 3.3.4. ([7])

Let (h_{ij}); $i,j = 1,...,n$ be a symmetric $n \times n$ matrix such that $n \geq 2$ and let $\sum h_{ii} = nH$ and $\sum h^2_{ij} = S$. Then

$$\begin{aligned} nh_{nn}H - \sum_i (h_{in})^2 &\geq \frac{1}{n^2}\{2\,(n-1)\,n^2H^2 - n\,(n-2)\,H\sqrt{n\,(n-1)\,(S-nH^2)} \\ &\quad -n\,(n-1)\,S\}, && (3.3.5) \end{aligned}$$

where H is the norm of the mean curvature vector h, i.e. $H = |h| \geq 0$.

Proof:

We shall look for the minimum value of $nh_{nn}H - \sum_i (h_{in})^2$ among all $n \times n$ symmetric matrices which have the same trace and the same square norm. Thus we prove the lemma by solving the following problem:

Minimize

$$f = nh_{nn}H - \sum_i (h_{in})^2 \qquad (3.3.6)$$

subject to the constraints

$$\sum h_{ii} = nH \qquad (3.3.7)$$

and

$$\sum h^2_{ii} + 2\sum_{i<j} h^2_{ij} = S, \qquad (3.3.8)$$

where (h_{ij}) is a symmetric $n \times n$ matrix.

Let

$$g = nh_{nn}H - \sum_i (h_{in})^2 + \lambda \left(\sum h_{ii} - nH \right) + \mu \left(\sum h_{ii}^2 + 2 \sum_{i<j} h_{ij}^2 - S \right) \quad (3.3.9)$$

where λ and μ are the Lagrange multipliers. Setting the partial derivatives of g with respect to the variables h_{ij} to zero we obtain the following system of equations:

$$\lambda + 2\mu h_{ii} = 0 \qquad \text{for} \qquad i = 1, ..., n-1. \quad (3.3.10)$$

$$nH - 2h_{nn} + \lambda + 2\mu h_{nn} = 0. \quad (3.3.11)$$

$$4\mu h_{ij} = 0 \qquad \text{for} \qquad i < j < n. \quad (3.3.12)$$

$$-2h_{in} + 4\mu h_{in} = 0 \qquad \text{for} \qquad i = 1, ..., n-1. \quad (3.3.13)$$

Now we shall consider three cases.

Case 1: $\mu = 0$.

By (3.3.10) we have $\lambda = 0$.Therefore (3.3.11) and (3.3.12) imply that

$$h_{in} = 0 \qquad \text{for} \qquad i = 1, ..., n-1$$

and

$$h_{nn} = \frac{1}{2} nH.$$

Hence

$$f = \frac{1}{2} n^2 H^2 - \frac{1}{4} n^2 H^2 = \frac{1}{4} n^2 H^2. \quad (3.3.14)$$

Case 2: $\mu = \frac{1}{2}$.

By (3.3.10), (3.3.11) and (3.3.12), we have

$$h_{ii} = -\lambda \qquad \text{for} \qquad 1 = 1, ..., n-1, \tag{3.3.15}$$

$$h_{nn} = nH + \lambda, \tag{3.3.16}$$

$$h_{ij} = 0 \qquad \text{for} \qquad i < j < n. \tag{3.3.17}$$

Substitute (3.3.7) into (3.3.16) and by (3.3.15), we obtain

$$h_{nn} = \left(-\left(n-1\right)\lambda + h_{nn}\right) + \lambda = h_{nn} + \left(n-2\right)\lambda$$

$$\Rightarrow \left(n-2\right)\lambda = 0. \tag{3.3.18}$$

1) If $n = 2$, then

$$\begin{aligned} f &= h_{22}\left(h_{11} + h_{22}\right) - h_{12}^2 - h_{22}^2 \\ &= h_{11}h_{22} - h_{12}^2 \\ &= \frac{1}{2}\left(h_{11}^2 + 2h_{11}h_{22} + h_{22}^2 - h_{11}^2 - h_{22}^2 - h_{12}^2\right). \end{aligned}$$

By using (3.3.7) and (3.3.8), we have

$$f = \frac{1}{2}\left(n^2H^2 - S\right). \tag{3.3.19}$$

2) If $n \geq 3$, then (3.3.18) imply $\lambda = 0$. By (3.3.15) and (3.3.17) we have

$$h_{ij} = 0 \qquad \text{for} \qquad i, j = 1, ..., n-1.$$

Then (3.3.8) becomes

$$h_{nn}^2 + 2\sum_i h_{in}^2 = S.$$

$$\Rightarrow \sum_i h_{in}^2 = \frac{1}{2}\left(S - n^2H^2\right). \tag{3.3.20}$$

Since the left hand side of (3.3.20) is non-negative, we have $S \geq n^2H^2$.Also we obtain

$$\begin{aligned} f &= n^2H^2 - \sum_i (h_{in})^2 \\ &= n^2H^2 - \frac{1}{2}\left(S - n^2H^2\right) \\ &= \frac{1}{2}\left(n^2H^2 - S\right). \end{aligned}$$

Thus in this case, we have

$$\text{If } n = 2, \text{ then } f = \frac{1}{2}\left(n^2H^2 - S\right) \tag{3.3.21}$$

$$\text{and if } n \geq 3 \text{ and } S \geq n^2H^2, \text{ then } f = \frac{1}{2}\left(n^2H^2 - S\right) \leq 0. \tag{3.3.22}$$

Case 3.$\mu \neq 0$ and $\mu \neq \frac{1}{2}$

From (3.3.10), (3.3.11) and (3.3.12), we have

$$h_{11} = \ldots = h_{n-1n-1}, \tag{3.3.23}$$

$$h_{ij} = 0 \qquad \text{for} \qquad i < j < n, \tag{3.3.24}$$

$$h_{in} = 0 \qquad \text{for} \qquad i = 1, \ldots, n-1. \tag{3.3.25}$$

For convenience we put $h_{nn} = x$. Then from (3.3.8), (3.3.23), (3.3.24) and (3.3.25), we have

$$(n-1)\, h_{11}^2 + x^2 = S. \tag{3.3.26}$$

However, by (3.3.7)

$$(n-1)\, h_{11} = nH - x$$

$$\Rightarrow (n-1)\,h_{11}^2 = \frac{1}{n-1}\,(nH-x)^2\,. \tag{3.3.27}$$

Therefore,

$$\frac{1}{n-1}\,(nH-x)^2 + x^2 = S$$

$$\Rightarrow nx^2 - 2nHx + n^2H^2 - (n-1)\,S = 0. \tag{3.3.28}$$

We also have

$$f = nHx - x^2. \tag{3.3.29}$$

By solving (3.3.28) we have

$$x = \frac{nH - \sqrt{n\,(n-1)\,(S-nH^2)}}{n} \qquad \text{or} \qquad x = \frac{nH + \sqrt{n\,(n-1)\,(S-nH^2)}}{n}. \tag{3.3.30}$$

Substitute these values into (3.3.29), we obtain

$$f = \frac{1}{n^2}\left\{n^2\,(n-1)\,H^2 - \left(n\,(n-1)\,\left(S-nH^2\right)\right) - n\,(n-2)\,H\sqrt{n\,(n-1)\,(S-nH^2)}\right\} \tag{3.3.31}$$

or

$$f = \frac{1}{n^2}\left\{n^2\,(n-1)\,H^2 - \left(n\,(n-1)\,\left(S-nH^2\right)\right) + n\,(n-2)\,H\sqrt{n\,(n-1)\,(S-nH^2)}\right\}. \tag{3.3.32}$$

Since we have assumed that $H \geq 0$, we find that (3.3.31) is smaller than (3.3.32). Hence the minimum value of f in this case is

$$f = \frac{1}{n^2}\left\{2n^2\,(n-1)\,H^2 - n\,(n-2)\,H\sqrt{n\,(n-1)\,(S-nH^2)} - n\,(n-1)\,S\right\}. \tag{3.3.33}$$

Finally, we complete the proof by comparing the value of (3.3.14), (3.3.21), (3.3.22) and (3.3.33). If $n = 2$, then (3.3.21) = (3.3.33) < (3.3.14). If $n \geq 3$, and $n^2H^2 > S$, then (3.3.22) does not occur and (3.3.33) < (3.3.14), so (3.3.33) is the minimum. If $n \geq 3$ and $n^2H^2 \leq S$ then

$$\begin{aligned}
& \frac{1}{n^2}\left\{2n^2(n-1)H^2 - n(n-2)H\sqrt{n(n-1)(S-nH^2)} - n(n-1)S\right\} \\
\leq & \frac{1}{n^2}\left\{2n^2(n-1)H^2 - n(n-2)H\sqrt{n(n-1)(n^2H^2-nH^2)} - n(n-1)S\right\} \\
= & \frac{1}{n^2}\left\{2n^2(n-1)H^2 - n^2(n-1)(n-2)H^2 - n(n-1)S\right\} \\
< & \frac{1}{n^2}\left\{2n^2(n-1)H^2 + n^2(n-1)\left(\frac{1}{2}n-2\right)H^2 - n(n-1)S\right\} \\
= & \frac{1}{2}n(n-1)H^2 - \frac{n-1}{n}S \\
< & \frac{1}{2}n^2H^2 - \frac{1}{2}S.
\end{aligned}$$

Therefore (3.3.14) > (3.3.22) > (3.3.33) and hence (3.3.33) is the minimum, thus completing the proof.

It is easy to see that when $H = 0$, (3.3.5) becomes

$$-\sum_i (h_{in})^2 \geq \frac{n-1}{n}|A|^2. \tag{3.3.34}$$

Theorem 3.3.35 ([7])

Let M^n be a submanifold immersed in a Riemannian manifold N^{n+p} of constant sectional curvature c. Let Ric, $|A|^2$ and H be the functions which assign to each point p_0 of M the minimum Ricci curvature, the square length of the second fundamental form and the mean curvature respectively of M at p_0. We have

$$\begin{aligned}
Ric \geq & (n-1)c + \frac{1}{n^2}\{2n^2(n-1)H^2 \\
& -n(n-2)H\sqrt{n(n-1)\left(|A|^2-nH^2\right)} - n(n-1)|A|^2\}.
\end{aligned} \tag{3.3.36}$$

Proof:

The Ricci curvature in the direction e_k at p_0 is

$$Ric\,(e_k, e_k) = \sum_i K^i_{kik} + nHh^{n+1}_{kk} - \sum_{\alpha,i} (h^\alpha_{ik})^2$$

$$\Rightarrow Ric\,(e_k, e_k) = \sum_i K^i_{kik} + nHh^{n+1}_{kk} - \sum_i \left(h^{n+1}_{ik}\right)^2 - \sum_{\alpha\geq n+2,i} (h^\alpha_{ik})^2. \qquad (3.3.37)$$

Let

$$|A_\alpha|^2 = \sum_{i,j} \left(h^\alpha_{ij}\right)^2 \qquad \text{for} \qquad \alpha = n+1, ..., n+p. \qquad (3.3.38)$$

Since $H \geq 0$, we apply lemma 3.3.4 to the matrix $\left(h^\alpha_{ij}\right)$ to obtain

$$\begin{aligned} nh^{n+1}_{kk}H - \sum_i \left(h^{n+1}_{ik}\right)^2 \;&\geq\; \frac{1}{n^2}\{2\,(n-1)\,n^2H^2 \qquad (3.3.39) \\ - \;& n\,(n-2)\,H\sqrt{n\,(n-1)\left(|A_{n+1}|^2 - nH^2\right)} - n\,(n-1)\,|A_{n+1}|^2\}. \end{aligned}$$

Since $tr\left(h^\alpha_{ij}\right) = 0$ for $\alpha = n+2, ..., n+p$, we can apply (3.3.34) to these matrices to obtain

$$-\sum_i (h^\alpha_{ik})^2 \geq -\frac{n-1}{n}\,|A_\alpha|^2 \qquad for \qquad \alpha = n+2, ..., n+p$$

and hence

$$-\sum_{\alpha\geq n+2,i} (h^\alpha_{ik})^2 \geq -\frac{n-1}{n} \sum_{\alpha\geq n+2} |A_\alpha|^2. \qquad (3.3.40)$$

Recall that

$$|A|^2 = \sum_\alpha |A_\alpha|^2. \qquad (3.3.41)$$

Therefore $|A_{n+1}|^2 \leq |A|^2$ and putting this into (3.3.39) we have

$$\begin{aligned} nh^{n+1}_{kk}H - \sum_i \left(h^{n+1}_{ik}\right)^2 \;&\geq\; \frac{1}{n^2}\{2\,(n-1)\,n^2H^2 - n\,(n-2)\,H\sqrt{n\,(n-1)\left(|A|^2 - nH^2\right)} \\ & -n\,(n-1)\,|A_{n+1}|^2\}. \qquad (3.3.42) \end{aligned}$$

Now adding (3.3.40) and (3.3.42) and using (3.3.41) we obtain

$$\begin{aligned}
& nh_{kk}^{n+1}H - \sum_i \left(h_{ik}^{n+1}\right)^2 - \sum_{\alpha\geq n+2,i} \left(h_{ik}^{\alpha}\right)^2 \\
\geq\ & \frac{1}{n^2}\{2\,(n-1)\,n^2H^2 - n\,(n-2)\,H\sqrt{n\,(n-1)\left(|A|^2 - nH^2\right)} \\
& -n\,(n-1)\,|A_{n+1}|^2\} - \frac{n-1}{n}\sum_{\alpha\geq n+2} |A_\alpha|^2 \\
=\ & \frac{1}{n^2}\{2\,(n-1)\,n^2H^2 - n\,(n-2)\,H\sqrt{n\,(n-1)\left(|A|^2 - nH^2\right)} \\
& -n\,(n-1)\,|A|^2\}.
\end{aligned}$$

Since N is a constant sectional curvature space we have

$$\sum_i K_{kik}^i = (n-1)\,c. \tag{3.3.43}$$

Therefore we have

$$Ric \geq (n-1)\,c + \frac{1}{n^2}\{2n^2\,(n-1)\,H^2 - n\,(n-2)\,H\sqrt{n\,(n-1)\left(|A|^2 - nH^2\right)} - n\,(n-1)\,|A|^2$$

CHAPTER 4 MAIN RESULTS

In this chapter, we prove two results mentioned in chapter 1. The first of them is a pinching theorem for parallel mean curvature vector submanifolds in the standard hyperbolic space. The second is an extrinsic diameter theorem for bounded mean curvature vector submanifolds in the hyperbolic space. Both of these theorems make essential use of the maximum principle for certain class of complete Riemannian manifolds.

4.1 THE PINCHING THEOREM FOR PARALLEL MEAN CURVATURE VECTOR SUBMANIFOLDS IN THE STANDARD HYPERBOLIC SPACE

In this section, we shall generalize the results due to Q.M. Cheng [3] to certain class of submanifolds immersed isometrically in the standard hyperbolic space.

First we recall the following maximum principle of Omori type (see [11])

Lemma 4.1.1.

Let M^n be a complete Riemannian manifold with Ricci curvature bounded from below. Let f be a C^2 function bounded from above on M^n, then for all $\varepsilon > 0$ there exists a point x in M^n such that at x (x depends on ε),

$$f > \sup f - \varepsilon, \tag{4.1.2}$$

$$\|\nabla f\| < \varepsilon, \tag{4.1.3}$$

$$\Delta f < \varepsilon. \tag{4.1.4}$$

We will apply the maximum principle of Omori-Yau to the differential inequality on the traceless second fundamental form obtained in the preceeding chapter to get the following

Theorem 4.1.5.

Let M^n be a complete non-compact parallel mean curvature vector submanifold $(p > 1)$ immersed isometrically in the standard hyperbolic space $\mathcal{H}^{n+p}(-1)$ and R be the scalar curvature. Suppose that $|H| > 1$, then either M^n is totally umbilic or

$$\inf R \leq n(n-1)\left(|H|^2 - 1\right) - \tilde{\phi}^2,$$

where

$$\tilde{\phi} = \frac{(p-1)}{2(2p-3)}\left[\sqrt{\frac{n(n-2)^2}{n-1}|H|^2 + 4n\left(-1+|H|^2\right)\left(\frac{2p-3}{p-1}\right)} - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\right] \tag{4.1.6}$$

Proof:

By (3.2.29) we have

$$\frac{1}{2}\Delta|\phi|^2 \geq |\phi|^2\left\{n(c+|H|^2) - \frac{n(n-2)}{\sqrt{n(n-1)}}|\phi||H| - \frac{2p-3}{p-1}|\phi|^2\right\}. \tag{4.1.7}$$

1) If $\inf R \leq n(n-1)\left(|H|^2-1\right) - \tilde{\phi}^2$, then theorem 4.1.5 is true.

2) If $\inf R > n(n-1)\left(|H|^2-1\right) - \tilde{\phi}^2$ then we get by using Codazzi equation for submanifolds in a space of constant curvature c (see Willmore [10] p.123, equation (4.13)) that

$$
\begin{aligned}
R &= cn\,(n-1) - |A|^2 + n^2\,|H|^2 \\
&= cn\,(n-1) - |\phi|^2 - n\,|H|^2 + n^2\,|H|^2 \\
&= cn\,(n-1) - |\phi|^2 + n\,(n-1)\,|H|^2 \\
&= -n\,(n-1) - |\phi|^2 + n\,(n-1)\,|H|^2 \quad \text{(since in our case } c = -1\text{)}
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow \inf\left(-n\,(n-1) - |\phi|^2 + n\,(n-1)\,|H|^2\right) > n\,(n-1)\left(|H|^2 - 1\right) - \bar{\phi}^2 \\
&\Rightarrow -n\,(n-1) + n\,(n-1)\,|H|^2 + \inf\left(-\,|\phi|^2\right) > n\,(n-1)\left(|H|^2 - 1\right) - \bar{\phi}^2 \\
&\Rightarrow -\sup|\phi|^2 > -\bar{\phi}^2 \\
&\Rightarrow \bar{\phi}^2 > \sup|\phi|^2
\end{aligned}
$$

therefore $|\phi|$ is bounded.

By (3.3.36) and using the identity (3.2.1), we get

$$
\begin{aligned}
Ric\,(x,x) \;\geq\;& (n-1)\,c \\
&+\frac{1}{n^2}\left\{2n^2\,(n-1)\,|H|^2 - n\,(n-2)\,|H|\,\sqrt{n\,(n-1)\left(|A|^2 - n\,|H|^2\right)} - n\,(n-1\right.
\end{aligned}
$$

$$
\Rightarrow\; Ric(x,x) \geq (n-1)(|H|^2 - 1) - \frac{|\phi|}{n}\left\{(n-2)\,|H|\,\sqrt{n(n-1)} + (n-1)\,|\phi|\right\}. \tag{4.1.8}
$$

Therefore $Ric(x,x)$ of M^n is bounded from below since ϕ is bounded from above.

We define $|\phi| = f, F = (f^2 + a)^{\frac{1}{2}}$(where $a > 0$ is any positive constant). F is bounded because f is bounded.

By simple calculation we obtain

$$\begin{aligned}
\Delta F &= \nabla \cdot \nabla \left((f^2 + a)^{\frac{1}{2}} \right) \\
&= \nabla \cdot \left\{ \left(\frac{1}{2} (f^2 + a)^{\frac{-1}{2}} \right) \nabla f^2 \right\} \\
&= \frac{1}{2}(f^2 + a)^{\frac{-1}{2}} \Delta f^2 - \frac{1}{4}(f^2 + a)^{\frac{-3}{2}} \left| \nabla f^2 \right|^2 \\
&= \frac{1}{2F}\Delta f^2 - \frac{1}{F} \left| \nabla F \right|^2,
\end{aligned}$$

therefore

$$\frac{1}{2}\Delta f^2 = F \Delta F + \left| \nabla F \right|^2. \tag{4.1.9}$$

Applying lemma 4.1.1 to F, we can find for each $\varepsilon > 0$ a point x in M^n such that at x

$$|\nabla F(x)| < \varepsilon, \tag{4.1.10}$$

$$\Delta F(x) < \varepsilon, \tag{4.1.11}$$

$$F(x) > \sup F - \varepsilon. \tag{4.1.12}$$

and by (4.1.9) we have

$$\frac{1}{2}\Delta f^2 < \varepsilon^2 + \varepsilon F = \varepsilon(\varepsilon + F) \qquad \text{(since } F > 0\text{)}. \tag{4.1.13}$$

We take a sequence $\{\varepsilon_m\}$ such that $\varepsilon_m \to 0$(as $m \to \infty$). There exists a point x_m in M^n such that (4.1.10), (4.1.11) and (4.1.12) hold. Hence, $\varepsilon_m(\varepsilon_m + F(x_m)) \to 0$ for $m \to \infty$ since $F(x)$ is bounded.

On the other hand, from (4.1.12)

$$F(x_m) > \sup F - \varepsilon_m.$$

Since F is bounded,$\{F(x_m)\}$ is a bounded sequence and

$$F(x_m) \to F_0$$

where we can choose subsequence if necessary. Hence

$$F_0 \geq \sup F.$$

According to the definition of supremum,we have

$$F_0 = \sup F.$$

From the definition of F, we get

$$f(x_m) \to f_0 = \sup f.$$

From (4.1.7) and (4.1.13), we have

$$f^2 \left\{ n(-1+H^2) - \frac{n\,(n-2)}{\sqrt{n(n-1)}} f\,|H| - \frac{2p-3}{p-1} f^2 \right\} \leq \frac{1}{2}\Delta f^2 < \varepsilon^2 + \varepsilon F$$

$$\begin{aligned}
&\Rightarrow \quad f(x_m)^2 \left\{ n(-1+H^2) - \frac{n\,(n-2)}{\sqrt{n(n-1)}} f(x_m)\,|H| - \frac{2p-3}{p-1} f(x_m)^2 \right\} \\
&< \quad \varepsilon_m^2 + \varepsilon_m F(x_m) \\
&\leq \quad \varepsilon_m^2 + \varepsilon_m F_0.
\end{aligned}$$

Let $m \to \infty$, we have $\varepsilon_m \to 0$ and $f(x_m) \to f_0$. Hence

$$f_0^2 \left\{ n(-1+|H|^2) - \frac{n\,(n-2)}{\sqrt{n(n-1)}} f_0\,|H| - \frac{2p-3}{p-1} f_0^2 \right\} \leq 0$$

$$\Rightarrow f_0 = 0 \text{ or } \left\{ n(-1+|H|^2) - \frac{n\,(n-2)}{\sqrt{n(n-1)}} f_0\,|H| - \frac{2p-3}{p-1} f_0^2 \right\} \leq 0$$

if $f_0 = 0 \Rightarrow M^n$ is totally umbilic.

On the other hand, if $\left\{ n(-1+|H|^2) - \frac{n\,(n-2)}{\sqrt{n(n-1)}} f_0\,|H| - \frac{2p-3}{p-1} f_0^2 \right\} \leq 0$

$$\Rightarrow \sup|\phi| \geq \frac{(p-1)}{2\,(2p-3)} \left\{ \sqrt{\frac{n\,(n-2)^2}{n-1}\,|H|^2 + 4n\left(-1+|H|^2\right)\left(\frac{2p-3}{p-1}\right)} - \frac{n\,(n-2)}{\sqrt{n\,(n-1)}}\,|H| \right.$$
$$= \tilde{\phi}$$

which contradicts our assumption $\sup|\phi| < \tilde{\phi}$. Therefore $\sup|\phi| \geq \tilde{\phi}$ or M^n is totally umbilic.

By the identity $|\phi|^2 = -R + n\,(n-1)\left(|H|^2 - 1\right)$, we have

$$\inf R \leq n(n-1)\left(|H|^2 - 1\right) - \tilde{\phi}^2$$

or M^n is totally umbilic.

Remark

A similar result can be proved for the codimensin 1 case by using the corresponding differential inequality.

4.2 THE EXTRINSIC DIAMETER THEOREM FOR PARALLEL MEAN CURVATURE VECTOR SUBMANIFOLDS IN THE STANDARD HYPERBOLIC SPACE

In this section, we prove an extrinsic diameter theorem for bounded mean curvature vector submanifolds in the standard hyperbolic space. As a consequence of it, we show the non-existence of parallel mean curvature vector surfaces in the standard hyperbolic space if $|H| < 1$.

First we recall an important identity relating the Laplacian on a submanifold to the Laplacian of the ambient manifold due to Choe and Gulliver [5]

Lemma 4.2.1.
Let $f \in C^\infty(N^m)$ where N^m is an m-dimensional Riemannian manifold and suppose Q^n is an n-dimensional submanifold of N^m. Denote by $\overline{\nabla}$, $\overline{\Delta}$ the connection and Laplacian on N^m and Δ the Laplacian on Q^n, then the following formula relating $\overline{\Delta}$ to Δ holds:

$$\Delta\left(f\mid_Q\right) = \left(\overline{\Delta} f\right)\mid_Q + \left\langle nh, \overline{\nabla} f\right\rangle\mid_Q - \sum_{k=n+1}^{m} \overline{\nabla}^2 f\left(e_k, e_k\right)\mid_Q . \tag{4.2.2}$$

Applying this identity to a submanifold (not necessarily with parallel mean curvature vector) of the standard hyperbolic space, one gets

Lemma 4.2.3. ([5])
Let M^n be a submanifold of the hyperbolic space in $\mathcal{H}^m(-1)$, then

$$\Delta r = \left(n - \|\nabla r\|^2\right)\coth r + \left\langle nh, \overline{\nabla} r\right\rangle\mid_M \tag{4.2.4}$$

where r is the distance function on $\mathcal{H}^m(-1)$ measured from a fixed point in $\mathcal{H}^m(-1)\backslash M^n$.

Proof:

Applying Lemma 4.2.1 to the function $\cosh r$, we have

$$\begin{aligned}
\Delta\left(\cosh r\mid_M\right) &= \left(\overline{\Delta}\cosh r\right)\mid_M + \left\langle nh, \overline{\nabla}\cosh r\right\rangle\mid_M - \sum_{k=n+1}^{m}\overline{\nabla}^2\cosh r\,(e_k,e_k)\mid_M \\
&= m\cosh r + \left\langle nh, \overline{\nabla}\cosh r\right\rangle\mid_M - (m-n)\cosh r \\
&= n\cosh r + \sinh r\left\langle nh, \overline{\nabla} r\right\rangle\mid_M . \qquad (4.2.5)
\end{aligned}$$

On the other hand,

$$\Delta\cosh r = div(\nabla\cosh r) = div((\sinh r)\nabla r) = (\cosh r)\,\|\nabla r\|^2 + (\sinh r)\Delta r \qquad (4.2.6)$$

From (4.2.5) and (4.2.6) we obtain

$$(\cosh r)\,\|\nabla r\|^2 + (\sinh r)\Delta r = n\cosh r + \sinh r\left\langle nh, \overline{\nabla} r\right\rangle\mid_M$$

$$\Rightarrow \Delta r = (n - \|\nabla r\|^2)\coth r + \left\langle nh, \overline{\nabla} r\right\rangle\mid_M .$$

Combining the preceeding lemma with Lemma 4.1.1, one obtains

Theorem 4.2.7.

Let M^n be a complete bounded parallel mean curvature vector submanifold in the standard hyperbolic space $\mathcal{H}^m(-1)$, with Ricci curvature bounded from below and suppose that it is contained in a bounded extrinsic ball, then $\sup|H| > 1$.

Proof:

Recall that

$$\Delta r = (n - \|\nabla r\|^2)\coth r + \left\langle nh, \overline{\nabla} r\right\rangle\mid_M .$$

Since

$$\begin{aligned} \left|\langle nh, \overline{\nabla} r\rangle\right| &\leq n \sup |H| \left|\overline{\nabla} r\right| \\ &\leq n \sup |H| (1) \qquad (\because \left|\overline{\nabla} r\right| \leq 1) \\ &\leq n \sup |H| \end{aligned}$$

$$\Rightarrow -n \sup |H| \leq \langle nh, \overline{\nabla} r\rangle \leq n \sup |H|$$

and from (4.2.4) we have

$$\triangle r \geq (n - \|\nabla r\|^2) \coth r - n \sup |H| . \tag{4.2.8}$$

Since M^n is a complete Riemannian manifold with Ricci curvature bounded from below and r is bounded from above on M^n, applying lemma 4.1.1 to r, we have for each $\varepsilon > 0$, there exists a point x in M^n such that at x,

$$|\nabla r(x)| < \varepsilon, \tag{4.2.9}$$

$$\triangle r(x) < \varepsilon, \tag{4.2.10}$$

$$r(x) > (\sup r - \varepsilon). \tag{4.2.11}$$

We take a sequence $\{\varepsilon_m\}$ such that $\varepsilon_m \to 0 (m \to \infty)$ and for all m, there exists a point x_m in M^n such that (4.2.9), (4.2.10) and (4.2.11) hold good.

$$\therefore \triangle r(x_m) \geq (n - \|\nabla r(x_m)\|^2) \coth r(x_m) - n \sup |H|$$

$$\Rightarrow \varepsilon_m > \triangle r(x_m) > (n - (\varepsilon_m)^2) \coth r(x_m) - n \sup |H| .$$

Since $n - (\varepsilon_m)^2 > 0$ for $\varepsilon_m \to 0$ and $\coth r(x_m) > 1$ because $0 < r(x_m) < \infty$.

$$\Rightarrow \varepsilon_m > (n - (\varepsilon_m)^2) - n \sup |H| \qquad for \qquad m \to \infty$$

$$\Rightarrow 0 > (n - 0^2) - n \sup |H| = n - n \sup |H|$$

$$\Rightarrow \sup |H| > 1.$$

Remark

As an interesting corollary, one proves that there exists no compact parallel mean curvature vector $|H| = 1$ submanifold in the standard hyperbolic space.

CHAPTER 5 CONCLUSION

This study attempts to apply the method of Q.M. Cheng for certain submanifolds in the hyperbolic space. In [3] Cheng proved a pinching theorem for minimal submanifolds in a sphere. His method consists of two main ingredients: a differential inequality and the maximum principle of Omori. From this we are motivated to study related inequalities on variants of second fundamental form for other kinds of submanifolds in different ambient spaces. In line with this, we studied the Laplacian of the traceless second fundamental form. In order to do this, we studied carefully the derivation of Simons' inequality in the work of Chern, do Carmo and Kobayashi [4] to obtain the generalized Simons' inequality mentioned in Santos' paper [8]. By using this inequality together with the maximum principle of Omori, we obtained the pinching theorem for parallel mean curvature vector submanifolds in the standard hyperbolic space which parallels the results of Cheng.

On the other hand, we studied the inequality on the Laplacian of the cosh function of the distance function for a submanifold in the hyperbolic space, as proved by Choe and Gulliver [5] and proved a result concerning bounded mean curvature submanifolds in the hyperbolic space. As a corollary of this result, we obtained also a simple result which asserts that there exists no non-compact constant mean curvature 1 hypersurfaces in the standard hyperbolic space with sectional curvature -1.

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