ON OPTIMAL CONSUMPTION, INVESTMENT, AND LIFE INSURANCE UNDER CONSUMPTION PATH-DEPENDENT PREFERENCES

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On optimal consumption, investment, and life insurance under consumption path-dependent preferences

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CERTIFICATE OF ORIGINALITY

I hereby declare that this thesis is my own work and that, to the best of my knowledge and belief, it reproduces no material previously published or written, nor material that has been accepted for the award of any other degree or diploma, except where due acknowledgement has been made in the text.

______________________________(Signature)

Qinyi Zhang (Name of student)

Dedicate to my wife, parents, and grandmother.

Abstract

The continuous time optimal consumption and investment problem with path-dependent reference has been extensively investigated by incorporating various model generalizations in the past half-century. On the other hand, optimal life insurance under utility maximization has become a mainstream research topic among academics and practitioners. Different problem formulations and characterizations of consumption behavior pose exciting challenges and opportunities for stochastic control and analysis, coupled with new modelling and computing implementation. The thesis consists of three parts solving different important stochastic control problems, to interpret and guide the consumption, investment, and life insurance premium in the market, either theoretically or computationally.

Part I focuses on an optimal consumption problem for a loss-averse agent with reference to the past consumption maximum. To account for loss aversion on relative consumption, an S-shaped utility is adopted that measures the difference between the nonnegative consumption rate and a fraction of the historical spending peak. We consider the concave envelope of the realization utility with respect to consumption, allowing us to focus on an auxiliary HJB equation on the strength of the concavification principle and dynamic programming arguments. By applying the dual transform and smooth-fit conditions, the auxiliary HJB variational inequality is solved in closed-form piecewisely, and some thresholds of the wealth variable are obtained. The optimal consumption and investment control of the original problem can be derived analytically in piecewise feedback form. Rigorous verification proofs on optimality and concavification principle are provided. Some numerical sensitivity analyses and financial implications are also presented.

Part II focuses on a life-cycle optimal portfolio-consumption problem when the consumption performance is measured by a shortfall aversion preference under an additional drawdown constraint on the consumption rate. Meanwhile, the agent also dynamically chooses her life insurance premium to maximize the expected bequest at death time. By using dynamic programming arguments and the dual transform, we solve the HJB variational inequality explicitly in a piecewise form across different regions and derive some thresholds of the wealth variable for the piecewise optimal feedback controls. Taking advantage of our analytical results, we are able to numerically illustrate some quantitative impacts on optimal consumption and life insurance by model parameters and discuss their financial implications.

Part III focuses on an optimal consumption and life insurance problem under habit formation preference when the return and volatility of the stock price dynamics are unknown. An offline reinforcement learning algorithm is proposed based on a policy improvement result and the evaluation of the policy by minimizing the martingale loss. We illustrate by some simulated examples that the algorithm provides satisfactory performance after combing it with the estimation of volatility. In real data analysis, it is also shown that the proposed algorithm outperforms the conventional least square estimation method on the unknown return and volatility.

Key words: Life insurance, loss aversion, optimal consumption, shortfall aversion, path-dependent consumption, piecewise feedback control, reinforcement learning

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Life is made up of many nodes. Each node ends the previous stage and starts the next journey. Although I usually do not feel it deliberately, when I stand on the node and look back on previous study years, I am very grateful for the help from all sides.

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$\mathbb{R}$	Real number field;
$\mathbb{R}_+$	The subset of $\mathbb{R}$ consisting of nonnegative elements;
$\mathcal{B}(\Omega)$	The Borel set for Ω;
$(\Omega, \mathcal{F}, \mathbb{P})$	The probability space;
$W_\cdot$	One-dimensional Brownian motion;
$\mathbb{F}$	The filtration generated by $W_\cdot$;
$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$	The filtered probability space;
$W_\cdot^{(\zeta)}$	A one-dimensional Brownian motion with drift ζ;
$\left(W_\cdot^{(\zeta)}\right)^*$	The running maximum of $W_t^{(\zeta)}$;
$f(x) = O(g(x))$	There exists a constant C such that $\lvert f(x)\rvert \leqslant C\lvert g(x)\rvert$ for all $x \in \mathbb{R}_+$.

Chapter 1

Introduction

1.1 Introduction

Optimal consumption-investment via utility maximization has been one of the fundamental research topics in mathematical finance. In the seminal works of Merton (1969, 1975), the feedback optimal investment and consumption strategy was first derived by resorting to dynamic programming arguments and the solution of the associated HJB equation. Since then, abundant influential results and methodologies have been rapidly developed to accommodate more general financial market models, trading constraints and other factors in decision making. For instance, Merton's problem has been extended with income (Zeldes (1989), Wang (2009)), assets following diffusion process (Cox and Huang (1989)), stochastic differential utility (Schroder and Skiadas (1999)), transaction costs (Liu (2004)), and drawdown constraints on wealth (Elie and Touzi (2008)).

Some empirical studies have argued that the observed consumption is usually extremely smooth (Campbell and Deaton (1989)), which cannot be reconciled by the optimal solution of some time-separable utility maximization problems. To partially explain the smooth consumption behavior, it has been suggested in the literature to take into account the past consumption decision in the measurement of the utility function. By considering the relative consumption with respect to a reference that depends on past consumption, striking changes in consumption can essentially be ruled out from the optimal solution. Time nonseparable preferences have gained popularity in modelling consumption performance thanks to their capability to explain the observed consumption smoothness and equity premium puzzle. In the literature, there are two major types of time nonseparable preferences involving the information of the past consumption path.

The first type is the so-called habit formation preference, in which utility is generated by the difference between the consumption rate and the weighted integral of past consumption control. Habit formation preference has been widely studied for both discrete-time problems (Abel (1990)) and continuous-time problems (Constantinides (1990), Detemple and Zapatero (1992)). Along this direction, some recent developments can be found in Englezos and Karatzas (2009), Yang and Yu (2022) and references therein. One notable advantage of the habit formation preference is its linear dependence on consumption, which enables one to consider the difference between the consumption rate and habit formation as an auxiliary control in a fictitious market model so that the path-dependence can be hidden. This insightful transform, first observed in Schroder and Skiadas (2002), significantly reduces the complexity of the problem. The martingale and duality approach can be applied by considering the adjusted martingale measure density process essentially based on Fubini's theorem; see Detemple and Karatzas (2003) and Yu (2015, 2017). See also Angoshtari et al. (2022), where habit formation is formulated as a control constraint.

The second type of preference chooses the past consumption maximum as the reference level. Indeed, a large expenditure might signal the turning point of one's standard of living and is usually a decision after careful thought and consideration. Such historical high spending moments are consequent on adequate wealth accumulation and often give rise to some long-term subsequent consumption decisions such as maintenance, repairs and upgrade. To take into account the impacts of the past consumption maximum, some previous studies focus on the Merton optimal consumption problem incorporated with the ratcheting constraint (Dybvig (1995)) and drawdown constraint (Arun (2012), Angoshtari et al. (2019)). Meanwhile, it is also of great importance to understand the consumption behavior when the past spending maximum appears inside the utility. Recently, Deng et al. (2022) adopted the formulation from the habit formation preference where the utility is defined on the difference between the consumption rate and a proportion of the historical consumption maximum. One key feature in Deng et al. (2022) is their allowance of the agent to strategically consume below the reference level due to the use of exponential utility. We also note some fruitful studies on the impact of the past consumption maximum when a ratcheting or a drawdown control constraint was considered under the standard time separable utility on consumption; for example, see Jeon and Park (2021), Jeon and Oh (2022). Although the running maximum term complicates the objective functional, the optimal consumption problems can be approached successfully under the umbrella of dynamic programming. Nevertheless, from the behavioral finance perspective, one shortcoming in these works is their incapability to distinguish an agent's different feelings on the same-sized overperformance and falling behind by consumption. In other words, the psychological loss aversion on relative consumption cannot be reflected in these problems.

Prospect theory utility (Tversky and Kahneman (1992), Kahneman and Tversky (2013)) has been actively applied in behavioral finance dominantly on terminal wealth optimization, see among Berkelaar et al. (2004), Jin and Yu Zhou (2008), He and Zhou (2011, 2014), He and Yang (2019), He and Strub (2022) and references therein. In contrast with neoclassical expected utility theory with classical smooth utility function, prospect theory suggests that the attitudes of utility gains and losses defined are different. Only a handful of papers can be found to encode that the agent may hurt more when consumption falls below a reference, especially when the reference level is endogenously generated by past decisions. Recently, Curatola (2015, 2017) studied a utility maximization problem on consumption for a loss averse agent under an S-shaped utility when the reference is chosen as a specific integral of the past consumption process. Later, van Bilsen et al. (2020) considered a similar problem under a two-part utility when the reference process is defined as the conventional consumption habit formation process. By imposing some artificial lower bounds on consumption control, the martingale and duality approach together with the concavification principle can be employed. Inspired by prospect theory, Guasoni et al. (2020) posited a shortfall aversion preference that reflects the higher utility loss of spending cuts from a reference than the utility gain from similar spending increases. Liang et al. (2022) generalized the preference in Deng et al. (2022) such that the risk aversion differs when the consumption falls below the reference process and an additional drawdown constraint was enforced. However, to the best of our knowledge, none of the literature utilizes past spending maximum to portray reference levels of loss aversion.

Since the seminal work of Yaari (1965), utility-based life-cycle models have become attractive among academics in quantifying the impact of bequest motives, risk aversion, and social security on the decision to purchase life insurance. Richard (1975) proposed the optimal dynamic life insurance problem by combining the portfolio and consumption control under a given distribution of a bounded death time. Pliska and Ye (2007) further studied a similar optimal life insurance and consumption problem for an income earner when the lifetime random variable is unbounded. Later, Ye (2007) extended the model in Pliska and Ye (2007) by considering the dynamic portfolio in a risky asset. Huang and Milevsky (2008) solved a portfolio choice problem that includes mortality-contingent claims and labor income under general hyperbolic absolute risk aversion (HARA) utilities. Duarte et al. (2011) extended Ye (2007) to allow for multiple risky assets. Ekeland et al. (2012) focused on the portfolio management problem for an investor with finite time horizon who is allowed to consume and take out life insurance. Recently, Wei et al. (2020) solved the problem when a couple aims to optimize their consumption, portfolio and life-insurance premium strategies by maximizing the family objective until retirement. Some studies on optimal life insurance in the context of consumption habit-formation can also be found, for example, in Ben-Arab et al. (1996) and Boyle et al. (2022). Only sporadic optimal consumption behavior and life insurance has been studied, and none of the literature considers optimal life insurance with consumption behavior characterized by past spending maximum.

Traditional model-based studies in mathematical finance provide comprehensive theoretical results to illustrate the agent's economic behavior, however, the model-based assumption may not be satisfied most of the time. In practice, before making decisions based on beliefs about future market performance characterized by a parametric stochastic differential equation, one needs to determine the view by observation and experience (Markowitz (1952)). Some machine learning methods have been studied and applied to estimate the market model, for instance, the maximum-likelihood method. However, although the maximum-likelihood method provides almost surely estimation for diffusion coefficients on a time interval (Doob (1953), Genon-Catalot and Jacod (1994)), it is generally too ambitious to expect an effective estimator for translation parameters. Therefore, some more direct methods are expected to overcome the potential error generated by traditional estimation for the market. Reinforcement learning (RL) is currently one of the most active research direction in machine learning. In the RL algorithm, the agent does not pre-specify a known model but, instead, gradually learns the best (or near-best) strategies through trial and error on the basis of existing data (offline) or future data (online). The discrete-time RL technique has been studied and explored to solve large-scale problems during these decades. Recently, RL in continuous-time has become attractive to researchers. Wang et al. (2020) proposed a continuous-time framework of reinforcement learning. The continuous-time framework was extended for the mean-variance portfolio selection problem (Wang and Zhou (2020)) and for corresponding RL techniques: policy evaluation (Jia and Zhou (2022a)), policy gradient (Jia and Zhou (2022b)), and q-learning algorithm (Jia and Zhou (2022c)). Some other scope of RL in continuous-time linear quadratic problems can also be found, for instance, see Li et al. (2022). However, to the best of our knowledge, none of the literatures has focused on the RL technique to obtain optimal consumption and life insurance.

The thesis composes three parts to solve several important optimal consumption investment problems with life insurance under different consumption path preferences. In particular, part I and part II focus on consumption behavior inspired by prospect theory, and use similar techniques to address challenges posed by past spending maximum and nonlinear differential equations. The third part focuses on the problem of data-driven consumer behavior with reference to habit formation by proposing a reinforcement learning algorithm.

Part I focuses on a loss-averse consumer, who invests in a complete market. Our primary scientific interest is to examine the optimal consumption behavior and investment with two-part utility in prospect theory. We solve the optimal consumption behavior and investment choice semi-analytically with the reference level generated by the past consumption peak. In the literature, there are rich works studying optimal consumption and investment for a loss-averse individual, in which the reference level is given by the exponentially weighted past consumption. However, no related works have considered the reference level given by the past spending maximum, which is inspired by the peak-end rule. Therefore a direct solution to optimal consumption behavior and investment choice is important for individuals who feel great pain when remembering happy times in misery.

Part II focuses on a shortfall-averse consumer who invests in the market and purchases life insurance simultaneously. A drawdown constraint on consumption is considered to highlight both the individual's psychologistic and substantial needs. Our contribution lies in solving the life-cycle optimal portfolio-consumption problem explicitly when the preference is measured by a shortfall aversion preference under an additional drawdown constraint on the consumption rate, and the expected bequest at the death time. There are many existing studies on dynamic consumption, investment, and life insurance control but path-dependent consumption behavior is sporadically discussed. Therefore, an explicit result for optimal consumption behavior, investment choice and life insurance purchase is needed for related self-financing individuals.

Part III is motivated by the reality that one cannot have all the knowledge of the market. We focus on a consumer investing in the market and purchasing life insurance under conventional habit formation preference. Her consumption exceeds the reference level treated as standard of living, which is characterized by a backward-looking updating rule used in the habit formation literature since Constantinides (1990). Without enough knowledge of the market, the scientific problem is to compute the optimal consumption, investment and life insurance purchase based on previous market data. Our contribution lies in constructing a reinforcement learning algorithm to solve the control problem without knowledge of the risky asset in the market. Practical strategies and detailed computation procedures to obtain the associated optimal consumption behavior, investment and life insurance purchase seem necessary for the individual.

1.2 Organization of the thesis

The remainder of the thesis is organized as follows.

Chapter 2 summarizes the notations used and important lemmas cited and market models used in the thesis.

Chapter 3 investigates the optimal consumption behavior of a loss-averse agent who feels differently when the consumption is overperforming and falling below a proportion to the past spending maximum. We obtain the optimal portfolio-consumption semi-analytically by dynamic programming arguments and prove the verification theorem.

Chapter 4 works with the shortfall-aversion preference incorporated with the bequest provided by life insurance. We derive the optimal consumption, investment and life insurance purchase explicitly by dynamic programming arguments and prove the verification theorem.

Chapter 5 develops a reinforcement learning algorithm for optimal consumption behavior, investment and life insurance under habit formation by proposing a reinforcement learning algorithm. A martingale loss is minimized to evaluate the policy, and the policy is improved by combining a policy improvement theorem and existing robust statistical estimation. We establish the convergence of the policy improvement algorithm.

Chapter 6 concludes and discusses potential future works.

Appendix A shows all the proofs.

Chapters 3, 4, and 5 are based on manuscripts Li et al. (2021), Li et al. (2023b), and Li et al. (2023a), respectively. Chapter 3 is under review, Chapter 4 is published, and Chapter 5 is prepared to be submitted.

Chapter 2

Preliminary

In this chapter, we first give a brief introduction to the fundamental mathematical concepts, list some necessary results that are used throughout the thesis, and introduce the market model through the thesis.

In the thesis, we use the following notation:

2.1 Probability

We first recall the definition of σ-field and probability space.

Definition 1. *Let a set Ω be nonempty, and let $\mathcal{F} \subset 2^{\Omega}$ (2^{Ω} is the set of all subsets in Ω), called a class, be nonempty. We call $\mathcal{F}$ a σ-field if*

$$\begin{cases} \Omega \in \mathbf{F}; \\ A, B \in \mathcal{F} \Rightarrow B \backslash A \in \mathcal{F}; \\ A_i \in \mathcal{F}, i = 1, 2, \cdots \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}. \end{cases}$$

Definition 2. *Let Ω be a nonempty set and $\mathcal{F}$ a σ-field on Ω. Then $(\Omega, \mathcal{F})$ is called a measurable space. A point $\omega \in \Omega$ is called a sample. A map $\mathbb{P} : \mathcal{F} \to [0, 1]$ is called* *a probability measure on* $(\Omega, \mathcal{F})$ *if*

$$\begin{cases} \mathbb{P}(\varnothing) = 0, \ \mathbb{P}(\Omega) = 1; \\ A_i \in \mathcal{F}, A_i \bigcap A_j = \varnothing, i, j = 1, 2, \cdots, i \neq j, \Rightarrow \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i). \end{cases}$$

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ *is called a probability space. Any* $\mathbb{A} \in \mathcal{F}$ *is called an event, and* $\mathbb{P}(A)$ *represents the probability of event* A. *A set/event* $A \in \mathcal{F}$ *is called a* $\mathbb{P}$*-null set/event if* $\mathbb{P}(A) = 0$. *A probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is said to be complete if for any* $\mathbb{P}$*-null set* $A \in \mathcal{F}$, *one has* $B \in \mathcal{F}$ *whenever* $B \subset A$.

Definition 3. *Let* $(\Omega, \mathcal{F})$ *and* $(\Omega', \mathcal{F}')$ *be two measurable spaces and* $X : \Omega \to \Omega'$ *be a map. The map* X *is said to be* $\mathcal{F}/\mathcal{F}'$*-measurable or simply measurable if* $f^{-1}(\mathcal{F}') = \mathcal{F}$. *We then call* X *an* $\mathcal{F}/\mathcal{F}'$*-random variable, or simply a random variable if there would be no confusion.*

Definition 4. *Next, let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space,* $(\Omega', \mathcal{F}')$ *a measurable space, and* $X : \Omega \to \Omega'$ *a random variable. Then* X *induces a probability measure* $\mathbb{P}_X$ *on* $(\Omega', \mathcal{F}')$ *as follows:*

$$\mathbb{P}_X(A') := \mathbb{P} \circ X^{-1}(A') = \mathbb{P}(X \in A'), \ \forall A' \in \mathcal{F}'.$$

We call $\mathbb{P}_X$ *the distribution of the random variable* X. *Let us define Borel set* $\mathcal{B}(\Omega)$ *as the smallest* σ*-field containing all open sets of* Ω, *then if* $(\Omega', \mathcal{F}') = (\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m))$, $\mathbb{P}_X$ *can be uniquely determined by the following function:*

$$F(x) = F(x_1, \cdots, x_m) := \mathbb{P}(\omega \in \Omega : X_i(\omega) \leqslant x_i, 1 \leqslant i \leqslant m).$$

We call $F(x)$ *the distribution function of* X. *Suppose the following integral exists:*

$$\int_{\Omega} X(\omega) d\mathbb{P}(\omega) = \int_{\mathbb{R}^m} x dF(x) := \mathbb{E}[X],$$

then we say that X *has the mean* $\mathbb{E}[X]$.

2.2 Stochastic Processes

Definition 5. *Let $\mathcal{I}$ be a nonempty index set and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. A family $\{X_t, t \in \mathcal{I}\}$ of random variables from $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathbb{R}^m$ is called a stochastic process. For any $\omega \in \Omega$, the map $t \mapsto X(t, \omega)$ is called a sample path.*

In what follows, we let $\mathcal{I} = [0, \infty)$. We shall interchangeably use $\{X_t, t \in \mathcal{I}\}, X_\cdot, X_t$, or even X to denote a stochastic process.

Next, for a given measurable space $(\Omega, \mathcal{F})$, we introduce a monotone family of sub-σ-fields $\mathcal{F}_t \subset \mathcal{F}$, $t \in [0, \infty)$. Here, by monotonicity we mean

$$\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2},\ 0 \leqslant t_1 \leqslant t_2.$$

Such a family is called a filtration. Set $\mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$ for any $t \in [0, \infty)$, and $\mathcal{F}_{t-} := \bigcup_{s<t} \mathcal{F}_s$ for any $t \in [0, \infty)$. If $\mathcal{F}_{t+} = \mathcal{F}_t$ (resp. $\mathcal{F}_{t-} = \mathcal{F}_t$), we say that $\{\mathcal{F}_t\}_{t \geqslant 0}$ is right (resp. left) continuous. Denoted by $\mathbb{F} = \{\mathcal{F}_t\}_{t \geqslant 0}$, we call $(\Omega, \mathbb{F}, \mathbb{F})$ a filtered measurable space and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a filtered probability space.

Definition 6. *We say that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfies the usual condition if $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets in $\mathcal{F}$, and $\mathbb{F}$ is right continuous.*

Definition 7. *Let $(\Omega, \mathcal{F}, \mathbb{F})$ be a filtered measurable space and X_t a process taking values in a metric space (U, d).*

- *The process X_t is said to be measurable if the map $(t, \omega) \mapsto X_t(\omega)$ is $(\mathcal{B}[0, \infty) \times \mathcal{F})/\mathcal{B}(U)$-measurable.*

- *The process X_t is said to be $\mathbb{F}$-adapted if for all $t \in [0, \infty)$, the map $\omega \mapsto X_t(\omega)$ is $\mathcal{F}_t/\mathcal{B}(U)$-measurable.*

- *The process X_t is $\mathbb{F}$-progressively measurable if for all $t \in [0, \infty)$, the map $(s, \omega) \mapsto X_s(\omega)$ is $\mathcal{B}[0, \infty) \times \mathcal{F}_t/\mathcal{B}(U)$-measurable.*

Let us finally recall the extremely important stochastic process, called Brownian motion.

Definition 8. *Let* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *be a filtered probability space. An* $\mathbb{F}$*-adapted* $\mathbb{R}^m$*-valued process* $X.$ *is called an* m*-dimensional* $\mathbb{F}$*-Brownian motion over* $[0,\infty)$ *if for all* $0 \leqslant s < t$*,* $X_t - X_s$ *is independent of* $\mathcal{F}_s$ *and is normally distributed with mean 0 and covariance* $(t-s)I$*.*

Throughout the thesis, we only consider one-dimensional Brownian motion, that is, $m = 1$. The underlying uncertainty is generated by a fixed filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and $\mathbb{F} = (\mathcal{F})_{t\geqslant 0}$ satisfies the usual conditions. W is an $\mathbb{F}$-adapted Brownian motion. We finally recall some propositions to one-dimensional Brownian motion, which are used later in the thesis.

Lemma 2.1 (Lemma A.5 of Guasoni et al. (2020))**.** *Let* $(W_t)_{t\geqslant 0}$ *be a standard Brownian motion under the probability measure* $\mathbb{P}$*, and denote by* $W_t^* = \sup\limits_{0\leqslant s\leqslant t} W_s$ *its running maximum. Then, for any constants* a, b, k *with* $2a+b \neq 0$*,* $k \geqslant 0$*,*

$$\mathbb{E}\Big[e^{aW_T + bW_T^*}\mathbf{1}_{\{W_T^* > k\}}\Big] = \frac{2(a+b)}{2a+b}\exp\left\{\frac{(a+b)^2}{2}T\right\}\Phi\bigg((a+b)\sqrt{T} - \frac{k}{\sqrt{T}}\bigg) + \frac{2a}{2a+b}\exp\left\{(2a+b)k + \frac{a^2}{2}T\right\}\Phi\bigg(-a\sqrt{T} - \frac{k}{\sqrt{T}}\bigg),$$

and hence

$$\lim_{T\to\infty}\frac{1}{T}\log\mathbb{E}\big[e^{aW_T + bW_T^*}\mathbf{1}_{\{W_T^* > k\}}\big] = \begin{cases} \frac{(a+b)^2}{2}, & \text{if } a+b > 0,\ 2a+b > 0, \\ \frac{a^2}{2}, & \text{if } a < 0,\ 2a+b < 0, \\ 0, & \text{if} a+b \leqslant 0, a \geqslant 0, \end{cases}$$

where $\Phi(\cdot)$ *is the standard normal distribution function.*

Corollary 2.1 (Corollary A.7 of Guasoni et al. (2020))**.** *Let* $W_t^{(\zeta)} = W_t + \zeta t$*, where* W *is a standard Brownian motion under probability measure* $\mathbb{P}$*, and* $(W_t^{(\zeta)})^*$ *is the* *running maximum of* $W_t^{(\zeta)}$. *Then for any constants* a, b, k *with* $2a + b + 2\kappa \neq 0$, $k \geqslant 0$, *the following expectation under* $\mathbb{P}$ *is:*

$$
\begin{aligned}
&\mathbb{E}\left[e^{aB_T^{(\zeta)} + b(B_T^{(\zeta)})^*} \mathbf{1}_{\left\{(W_T^{(\zeta)})^* > k\right\}} \right] \\
=& \frac{2(a+b+\zeta)}{2a+b+2\zeta} \exp\left\{ \frac{(a+b)(a+b+2\zeta)}{2} T \right\} \Phi\left((a+b+\zeta)\sqrt{T} - \frac{k}{\sqrt{T}} \right) \\
&+ \frac{2(a+\zeta)}{2a+b+2\zeta} \exp\left\{ (2a+b+2\zeta)k + \frac{a(a+2\zeta)}{2} T \right\} \Phi\left(-(a+\zeta)\sqrt{T} - \frac{k}{\sqrt{T}} \right).
\end{aligned}
$$

Corollary 2.2 (Equation (9.1) of Rogers and Williams (2000)). *Let* $W_t^{(\zeta)} := W_t + \zeta t$ *be a Brownian motion with drift, and the first hitting time* $H_b := \inf_{t>0}\{W_t^{(\zeta)} = b\}$ *for* $b > 0$. *Then for any* $\nu > 0$, $\beta := \sqrt{\zeta^2 + 2\nu} - \zeta$, *it follows that*

$$
\mathbb{E}\left[e^{-\nu H_b} \right] = \exp\{-b\beta\}.
$$

2.3 Past Consumption Path and Market Models

2.3.1 Past Spending Maximum and Habit Formation

Given a consumption path $(c_t)_{t \geqslant 0}$.

The past spending maximum $H_t := \max\{h, \sup_{s \leqslant t} c_s\}$ denotes the past spending maximum, and $H_0 = h \geqslant 0$ is the initial reference level.

The habit formation Z_t is defined as

$$
Z_t = e^{-\delta t}\left(z + \int_0^t \eta e^{\delta s} c_s ds \right), \quad t \geqslant 0, \tag{2.3.1}
$$

which is equivalently governed by the differential form:

$$
\begin{cases} dZ_t = (\eta c_t - \delta Z_t)dt, & t \geqslant 0, \\ Z_0 = z. \end{cases} \tag{2.3.2}
$$

where z is the initial habit formation, η and δ are the discount factors.

2.3.2 Market Models

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space and $\mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}$ satisfies the usual conditions. The financial market model consists of one riskless asset and one risky asset. The riskless asset price follows $dB_t = rB_t dt$, where $r > 0$ is the interest rate. The risky asset price is governed by the following stochastic differential equation (SDE)

$$dS_t = S_t \mu dt + S_t \sigma dW_t, \quad t \geqslant 0,$$

where W is an $\mathbb{F}$-adapted Brownian motion, $\mu \in \mathbb{R}_+$ and $\sigma > 0$ stand for the drift and volatility. It is assumed that $\mu > r$ and the sharp ratio is denoted by $\kappa := \frac{\mu - r}{\sigma} > 0$.

Let $(\pi_t)_{t \geqslant 0}$ be the amount of wealth that the agent allocates in the risky asset, and let $(c_t)_{t \geqslant 0}$ represent the consumption rate. Similar to Lee (2021), we assume that the life insurance contracts cover mortality risk and are actuarially fair. It is assumed that the individual's death time τ has an exponential distribution with the parameter $\lambda > 0$. Denote by p_t and L_t the instantaneous life insurance premium rate paid by the individual and insurance benefit paid by the insurer, respectively. We have that $p_t = \lambda L_t$, and the bequest b_t received by the individual's heir is given by $b_t = X_t + L_t = X_t + \frac{p_t}{\lambda}$. Thus, the wealth process satisfies

$$\begin{aligned} dX_t &= (rX_t + \pi_t(\mu - r) - c_t - p_t)\, dt + \pi_t \sigma dW_t \\ &= ((r + \lambda)X_t + \pi_t(\mu - r) - c_t - \lambda b_t)\, dt + \pi_t \sigma dW_t, \quad t \geqslant 0, \end{aligned} \tag{2.3.3}$$

with the initial wealth $X_0 = x$. A control variable p_t is transformed to the bequest b_t, which is assumed to be $\mathbb{F}$-adapted.

The control triple (c, π, b) is said to be *admissible* if c is $\mathbb{F}$-predictable and non-negative, π is $\mathbb{F}$-progressively measurable, and (c, π, b) satisfies the integrability condition $\int_0^\infty (c_t + \pi_t^2 + b_t) dt < \infty$ a.s. and the no bankruptcy condition $X_t \geqslant 0$ a.s. for $t \geqslant 0$.

Remark. *The optimal premium p_t is not required to be positive. The wage earner is allowed to purchase a special term pension annuity, and she can receive the premium p_t from the insurance company at time t. However, the wage earner should pay p_t to the company if she dies at time t. This situation is related to the reverse mortgage. Interested readers may refer to Pirvu and Zhang (2012) for more discussion.*

In Chapter 3, we consider the optimal consumption and investment without life insurance, that is, $\lambda = 0$. Then the self-financing wealth process $(X_t)_{t \geqslant 0}$ satisfies

$$dX_t = (rX_t + \pi_t(\mu - r) - c_t)\, dt + \pi_t \sigma dW_t, \quad t \geqslant 0,$$

with the initial wealth $X_0 = x \geqslant 0$. The control pair (c, π) is then said to be *admissible* if c is $\mathbb{F}$-predictable and non-negative, π is $\mathbb{F}$-progressively measurable, and (c, π) satisfies the integrability condition $\int_0^\infty (c_t + \pi_t^2) dt < \infty$ a.s. and the no bankruptcy condition $X_t \geqslant 0$ a.s. for $t \geqslant 0$. The admissible set is denoted by $\mathcal{A}(x, h)$

In Chapter 4, we adopt the consumption drawdown constraint $c_t \geqslant \nu H_t$ where $\nu \in (0, 1)$, and H_t stands for the past spending maximum defined in Section 2.3.1. Similar to the proof of Corollary 1 of Arun (2012), to ensure that the consumption drawdown constraint $c_t \geqslant \nu H_t$ is sustainable for all $t \geqslant 0$, the necessary condition is $X_t \geqslant \frac{\nu H_t}{r+\lambda}$ a.s. for all $t \geqslant 0$. Therefore, from this point onwards, we shall only consider the feasible domain $(x, h) \in [0, +\infty) \times [0, +\infty)$ such that $x \geqslant \frac{\nu h}{r+\lambda}$ for the admissible set $\mathcal{A}(x, h)$.

In Chapter 5, the risky asset price is governed by the SDE

$$dS_t = S_t \big\{ (\mu + r) dt + \sigma dW_t \big\}, \quad t \geqslant 0, \tag{2.3.4}$$

where $\mu > 0$ and $\sigma > 0$ stand for the unknown excess rate of returns and volatility.

As a result, the self-financing wealth process satisfies

$$\begin{aligned} dX_t &= (rX_t + \mu\pi_t - c_t - p_t)\,dt + \pi_t\sigma dW_t \\ &= ((r+\lambda)X_t + \mu\pi_t - c_t - \lambda b_t)\,dt + \pi_t\sigma dW_t, \quad t \geqslant 0, \end{aligned} \tag{2.3.5}$$

with the initial wealth $X_0 = x \geqslant 0$. In addition, we adopt a habit formation preference in Chapter 5, therefore, the admissible set is denoted by $\mathcal{A}(x, z)$.

Chapter 3

Optimal consumption with loss aversion and reference to past spending maximum

This chapter studies an optimal consumption problem for a loss-averse agent with reference to the past consumption maximum. To account for loss aversion on relative consumption, an S-shaped utility is adopted that measures the difference between the nonnegative consumption rate and a fraction of the historical spending peak. We consider the concave envelope of the utility with respect to consumption, allowing us to focus on an auxiliary Hamilton-Jacobi-Bellman (HJB) equation on the strength of the concavification principle and dynamic programming arguments. By applying the dual transform and smooth-fit conditions, the auxiliary HJB equation is solved in closed-form piecewisely and some thresholds of the wealth variable are obtained. The optimal consumption and investment control can be derived in piecewise feedback form. Rigorous verification proofs on optimality and concavification principle are provided. Some numerical sensitivity analyses and financial implications are also presented.

3.1 Model Setup and Problem Formulation

3.1.1 Preference

It is assumed in the present paper that the agent is loss averse on relative consumption in the sense that the agent feels more pain when the consumption is falling below the reference than the same-sized gain. The reference level is chosen as a fraction of the consumption running maximum process λH_t, where $\lambda \in (0,1)$ depicts the degree towards the reference. The utility maximization problem is defined by

$$u(x,h) = \sup_{(\pi,c)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c_t - \lambda H_t)dt\right], \tag{3.1.1}$$

where $U(x)$ is described by the conventional two-part power utility (see Kahneman and Tversky (2013)) that

$$U(x) := \begin{cases} \dfrac{x^{\beta_1}}{\beta_1}, & \text{if } x \geqslant 0, \\ -k\dfrac{(-x)^{\beta_2}}{\beta_2}, & \text{if } x < 0. \end{cases}$$

Here, $k > 0$ stands for the loss aversion degree, and it is assumed in the present paper that $0 < \beta_1, \beta_2 < 1$, which represent the risk aversion parameters over the gain domain $x \geqslant 0$ and the loss domain $x < 0$, respectively. The utility is an S-shaped function on $\mathbb{R}$. The parameter $\lambda \in (0,1)$ reflects the degree of adherence towards the reference level H, which now affects the expected utility directly. $\rho > 0$ is the subjective discount rate to guarantee the convergence of the value function.

Two main challenges in solving (3.1.1) are the path-dependence of $(H_t)_{t\geqslant 0}$ on the control $(c_t)_{t\geqslant 0}$ and the nonconcavity of the S-shaped utility $U(\cdot)$. As a remedy, we propose to consider the concave envelope of the realization utility on consumption by first assuming the validity of the concavification principle (see, for example, Reichlin (2013) and Dong and Zheng (2020)). Later, we plan to characterize the optimal control under the concave envelope function and then verify that the optimal control also attains the value function in the original problem, i.e., the concavification principle indeed holds. Specifically, for each fixed h, let us consider $\tilde{U}(c,h)$ as the concave envelope of $U(c-\lambda h)$ with respect to the variable $c \in [0,h]$ on a constrained domain. That is, for each fixed $h \geqslant 0$, let $\tilde{U}(\cdot,h)$ be the smallest concave function on $[0,h]$ such that $\tilde{U}(c,h) \geqslant U(c,h)$ holds for all $c \in [0,h]$.

3.1.2 Concave envelope of the realization utility

To emphasize the concave envelope only with respect to $c \in [0,h]$ while keeping the variable h fixed, let us consider an equivalent bivariate function

$$U^*(c,h) := U(c-\lambda h),$$

on the domain $\{(c,h) \in \mathbb{R}^2 : c \in [0,h]\}$. Define $U_1^*(c,h) := \frac{1}{\beta_1}(c-\lambda h)^{\beta_1}$ and $U_2^*(c,h) := -\frac{k}{\beta_2}(\lambda h - c)^{\beta_2}$ and denote $U_1^{*\prime}(c,h) := \frac{\partial U_1^*}{\partial c}(c,h)$ and $U_2^{*\prime}(c,h) := \frac{\partial U_2^*}{\partial c}(c,h)$. Note that $U_1^{*\prime}(c,h) \to +\infty$ as $c \to (\lambda h)_+$. As $U_2^{*\prime}(c,h) \to +\infty$ when $c \to (\lambda h)_-$, we have two different subcases:

Subcase (i): If $U_1^*(h,h)+U_2^*(0,h)-hU_1^{*\prime}(h,h) > 0$, there exists a unique solution $z(h) \in (\lambda h, h)$ to the equation

$$U_1^*(z(h),h) + U_2^*(0,h) - z(h)U_1^{*\prime}(z(h),h) = 0. \tag{3.1.2}$$

That is, $z(h)$ is the tangent point of the straight line at $(0, -U_2^*(0,h))$ to the curve $U_1^*(c,h)$ for $c \geqslant \lambda h$. Note that $z(h)$ does not admit an explicit expression in this subcase.

Subcase (ii): If $U_1^*(h,h) + U_2^*(0,h) - hU_1^{*\prime}(h,h) \leqslant 0$, we simply let $z(h) = h$. The concave envelope of $U^*(c,h)$ on $[0,h]$ corresponds to the straight line through two points $(0, U_2^*(0,h))$ and $(h, U_1^*(h,h))$.

Remark. *The condition of **Subcase (ii)** is fulfilled if and only if h and model parameters satisfy one of the following three conditions that*

$$\begin{array}{ll} (S1) & \beta_2 > \beta_1 \geqslant 1-\lambda, \text{ and } h \leqslant \left(\frac{\beta_2(1-\lambda)^{\beta_1-1}(\beta_1+\lambda-1)}{\beta_1 k\lambda^{\beta_2}}\right)^{\frac{1}{\beta_2-\beta_1}}, \\ (S2) & \beta_1 \geqslant 1-\lambda, \ \beta_1 > \beta_2, \text{ and } h \geqslant \left(\frac{\beta_2(1-\lambda)^{\beta_1-1}(\beta_1+\lambda-1)}{\beta_1 k\lambda^{\beta_2}}\right)^{\frac{1}{\beta_2-\beta_1}}, \\ (S3) & \beta_2 = \beta_1 \geqslant 1-\lambda, \ 1 \leqslant \frac{(1-\lambda)^{\beta_1-1}(\beta_1+\lambda-1)}{k\lambda^{\beta_2}}, \ \textit{and } h \geqslant 0. \end{array}$$

Similar to Dong and Zheng (2020), we can define the concave envelope of $U^*(c,h)$ for $c \in [0,h]$ by

$$\tilde{U}(c,h) = \begin{cases} U_2^*(0,h) + \frac{U_1^*(z(h),h)-U_2^*(0,h)}{z(h)}c, & \text{if } 0 \leqslant c < z(h), \\ U_1^*(c,h), & \text{if } z(h) \leqslant c \leqslant h. \end{cases} \tag{3.1.3}$$

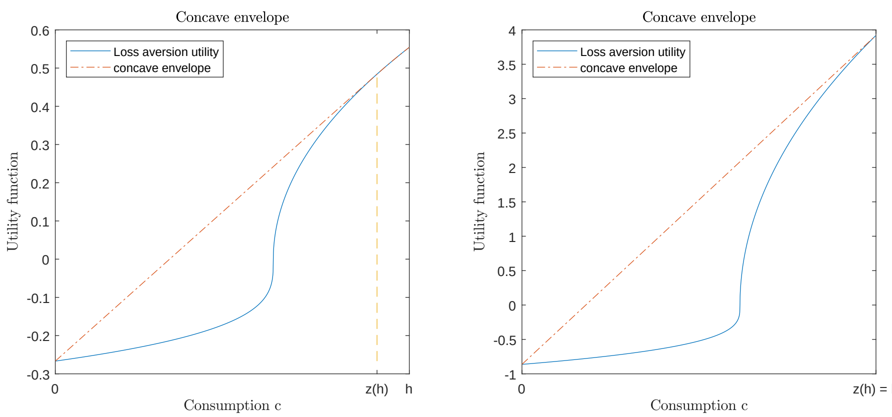

Figure 3.1: Concave envelopes when $0 < \beta_2 < 1$: (left panel) subcase (i) when $z(h) \neq h$; (right panel) subcase (ii) when $z(h) = h$.

Figure 1 illustrates two subcases of the concave envelope of the S-shaped utility $U(c,h)$. We stress that the function $\tilde{U}(c,h)$ is implicit in h, as $z(h)$ is an implicit function in general. To simplify the future presentation, let us also define

$$w(h) := z(h) - \lambda h. \tag{3.1.4}$$

Hence, if $z(h) = h$, then $w(h) = (1-\lambda)h$, i.e., $z(h) = \lambda h + w(h)$.

3.1.3 Equivalent problem

We now consider the auxiliary stochastic control problem

$$\tilde{u}(x,h) = \sup_{(\pi,c)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\tilde{U}(c_t, H_t)dt\right]. \tag{3.1.5}$$

The equivalence between problems (3.1.1) and (3.1.5) is given in the next proposition. Its proof is deferred to Section A.1.4 after we first establish the verification proof on optimality.

Proposition 3.1 (Concavification Principle)**.** *Two problems* (3.1.1) *and* (3.1.5) *admits the same optimal control* (π_t^*, c_t^*) *so that two value functions coincide, i.e.,* $u(x,h) = \tilde{u}(x,h)$ *for any* $(x,h) \in \mathbb{R}_+ \times \mathbb{R}_+$.

Proof. The proof is given in Appendix A.1.4. □

For problem (3.1.5), we can derive the auxiliary HJB equation that

$$\begin{cases} \sup\limits_{c\in[0,h],\pi\in\mathbb{R}} \left[-\rho\tilde{u} + \tilde{u}_x(rx + \pi(\mu - r) - c) + \frac{1}{2}\sigma^2\pi^2\tilde{u}_{xx} + \tilde{U}(c,h)\right] & = & 0, \\ \tilde{u}_h(x,h) & \leqslant & 0, \end{cases} \tag{3.1.6}$$

for $x \geqslant 0$ and $h \geqslant 0$. The free boundary condition $\tilde{u}_h(x,h) = 0$ shall be specified later. Our goal is to find the optimal feedback control $c^*(x,h)$ and $\pi^*(x,h)$. If $\tilde{u}(x,\cdot)$ is C^2 in x, the first order condition gives the optimal portfolio in feedback form by $\pi^*(x,h) = -\frac{\mu-r}{\sigma^2}\frac{\tilde{u}_x}{\tilde{u}_{xx}}$. This implies that HJB equation (3.1.6) can be simplified to

$$\sup_{c\in[0,h]}\left[\tilde{U}(c,h) - c\tilde{u}_x\right] - \rho\tilde{u} + rx\tilde{u}_x - \frac{\kappa^2}{2}\frac{\tilde{u}_x^2}{\tilde{u}_{xx}} = 0, \text{ and } \tilde{u}_h \leqslant 0, \ \forall x \geqslant 0, h \geqslant 0. \tag{3.1.7}$$

3.2 Derivation of the Solution

For ease of presentation and technical convenience, we only consider the case where $\rho = r > 0$ in the present paper. Computations in general cases that $\rho \neq r$ and

$r = 0$ can be conducted similarly. However, some additional sufficient assumptions on model parameters are needed to facilitate the proofs of the verification theorem. Given the implicit concave envelope in (3.1.3), we can still solve the HJB equation in the analytical form. In particular, we plan to characterize some thresholds (depending on h) for wealth level x such that the auxiliary value function, the optimal portfolio and consumption can be expressed analytically in each region.

Let us first introduce the boundary curves $y_1(h) \geqslant y_2(h) > y_3(h)$ by

$$\begin{aligned} y_1(h) &:= \frac{k(\lambda h)^{\beta_2}}{\beta_2 z(h)} + \frac{w(h)^{\beta_1}}{\beta_1 z(h)}, \\ y_2(h) &:= \min\left(y_1(h), \big((1-\lambda)h\big)^{\beta_1 - 1}\right), \\ y_3(h) &:= (1-\lambda)^{\beta_1} h^{\beta_1 - 1}, \end{aligned} \tag{3.2.8}$$

where $z(h)$ and $w(h)$ are defined in Section 3.1.2. Here, $y_1(h)$ and $y_2(h)$ are derivatives of the concave envelope $\tilde{U}(c,h)$ at $c = 0$ and $c = h$ respectively, which are used to simplify the expression of $\sup_{c \in [0,h]}[\tilde{U}(c,h) - c\tilde{u}_x]$ when the maximum occurs at $c = 0$ and $c = h$. We also use $y_3(h)$ to describe the free boundary curve $\tilde{u}_h = 0$. Note that if $w(h) \neq (1-\lambda)h$, we have $y_1(h) > y_2(h) = ((1-\lambda)h)^{\beta_1 - 1} > (1-\lambda)^{\beta_1} h^{\beta_1 - 1} = y_3(h)$ as $0 < \lambda < 1$ by (3.1.2); on the other hand, if $w(h) = (1-\lambda)h$, we have $z(h) = h$, yielding that $y_1(h) = y_2(h) = \frac{k(\lambda h)^{\beta_2}}{\beta_2 z(h)} + \frac{w(h)^{\beta_1}}{\beta_1 z(h)} > \frac{w(h)^{\beta_1}}{\beta_1 z(h)} = \frac{1}{\beta_1}(1-\lambda)^{\beta_1} h^{\beta_1 - 1} > (1-\lambda)^{\beta_1} h^{\beta_1 - 1} = y_3(h)$ as $0 < \beta_1 < 1$.

Similar to Deng et al. (2022), we can heuristically decompose the domain into several regions based on the first order condition of c and express the HJB equation (3.1.7) piecewisely. However, the concave envelope of the S-shaped utility complicates the computations here, in which the previous $y_i(h)$ in (3.2.8), $i = 1, 2, 3$, serve as the boundaries of these regions. We can then separate the following regions:

Region I: on the set $\mathcal{R}_1 = \{(x,h) \in \mathbb{R}_+^2 : \tilde{u}_x(x,h) > y_1(h)\}$, $\tilde{U}(c,h) - c\tilde{u}_x$ is decreasing in c, implying that $c^* = 0$ and the HJB equation (3.1.7) becomes

$$-\frac{k}{\beta_2}(\lambda h)^{\beta_2} - r\tilde{u} + rx\tilde{u}_x - \frac{\kappa^2 \tilde{u}_x^2}{2\tilde{u}_{xx}} = 0, \text{ and } \tilde{u}_h \leqslant 0. \tag{3.2.9}$$

Region II: on the set $\mathcal{R}_2 = \{(x,h) \in \mathbb{R}_+^2 : y_2(h) \leqslant \tilde{u}_x(x,h) \leqslant y_1(h)\}$, $\tilde{U}(c,h) - c\tilde{u}_x$ is increasing on $[0, z(h)]$ and concave on $[z(h), h]$, implying that $c^* = \lambda h + \tilde{u}_x^{\frac{1}{\beta_1 - 1}} \geqslant z(h)$ and the HJB equation (3.1.7) becomes

$$\frac{1-\beta_1}{\beta_1}\tilde{u}_x^{\frac{\beta_1}{\beta_1-1}} - \lambda h\tilde{u}_x - r\tilde{u} + rx\tilde{u}_x - \frac{\kappa^2 \tilde{u}_x^2}{2\tilde{u}_{xx}} = 0, \text{ and } \tilde{u}_h \leqslant 0. \tag{3.2.10}$$

Region III: on the set $\mathcal{R}_3 = \{(x,h) \in \mathbb{R}_+ \times \mathbb{R}_+ : \tilde{u}_x(x,h) < y_2(h)\}$, $\tilde{U}(c) - c\tilde{u}_x$ is increasing in c on $[0, h]$, implying that $c^* = h$. To distinguish whether the optimal consumption c_t^* updates the past maximum process H_t^* in this region, one can heuristically substitute $h = c$ in (3.1.7) and apply the first order condition to $\tilde{U}(c,c) - c\tilde{u}_x$ with respect to c and derive the auxiliary singular control $\hat{c}(x) := \tilde{u}_x^{\frac{1}{\beta_1-1}}(1-\lambda)^{-\frac{\beta_1}{\beta_1-1}}$. We then need to split *Region III* further into three subsets:

Region III-(i): on the set $\mathcal{D}_1 = \{(x,h) \in \mathbb{R}_+ \times \mathbb{R}_+ : y_3(h) < \tilde{u}_x < y_2(h)\}$, it is easy to see a contradiction that $\hat{c}(x) < h$, and therefore the optimal consumption c_t^* does not equal $\hat{c}$, and we should follow the previous feedback form $c^*(x,h) = h$, in which h is a previously attained maximum level. The HJB equation is written by

$$\frac{1}{\beta_1}((1-\lambda)h)^{\beta_1} - h\tilde{u}_x - r\tilde{u} + rx\tilde{u}_x - \frac{\kappa^2 \tilde{u}_x^2}{2\tilde{u}_{xx}} = 0, \text{ and } \tilde{u}_h \leqslant 0. \tag{3.2.11}$$

Region III-(ii): on the set $D_2 := \{(x,h) \in \mathbb{R}_+ \times \mathbb{R}_+ : \tilde{u}_x(x,h) = y_3(h)\}$, we obtain $\hat{c}(x) = h$ and the feedback optimal consumption is $c^*(x,h) = \tilde{u}_x^{\frac{1}{\beta_1-1}}(1-\lambda)^{-\frac{\beta_1}{\beta_1-1}} = h$. This corresponds to the singular control c_t^* that creates a new peak for the whole path that $H_t^* > H_s^*$ for any $s < t$. We then impose the free boundary condition $\tilde{u}_h(x,h) = 0$ in this region, and the HJB equation follows the same PDE in (3.2.11).

Region III-(iii): on the set $\mathcal{D}_3 := \{(x,h) \in \mathbb{R}_+ \times \mathbb{R}_+ : \tilde{u}_x(x,h) < y_3(h)\}$, we obtain $\hat{c}(x) > h$. The optimal consumption is again a singular control $c^*(x) = \tilde{u}_x^{\frac{1}{\beta_1-1}}(1-\lambda)^{-\frac{\beta_1}{\beta_1-1}} > h$, pulls the associated H^*_{t-} upward to the new value $\tilde{u}_x(X^*_t, H^*_t)^{\frac{1}{\beta_1-1}}(1-\lambda)^{-\frac{\beta_1}{\beta_1-1}}$, in which $\tilde{u}(x,h)$ is the solution of the HJB equation on set $\mathcal{D}_2$. This suggests that for any given initial value (x,h) in set $\mathcal{D}_3$, the feedback control $c^*(x,h)$ pushes (x,h) to point $(x,\hat{h})$ on the boundary set $\mathcal{D}_2$.

In summary, it is sufficient to consider the effective domain defined by

$$\begin{aligned}\mathcal{C} &:= \{(x,h) \in \mathbb{R}_+ \times \mathbb{R}_+ : \tilde{u}_x(x,h) \geqslant y_3(h)\} \\ &= \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{D}_1 \cup \mathcal{D}_2 \subset \mathbb{R}^2_+,\end{aligned} \tag{3.2.12}$$

and $(x,h) \in \mathcal{D}_3$ can only occur at the initial time $t=0$.

Therefore, the HJB equation (3.1.7) can be written as

$$\begin{aligned}-r\tilde{u} + rx\tilde{u}_x - \frac{\kappa^2\tilde{u}_x^2}{2\tilde{u}_{xx}} = -V(u_x,h), \text{ and } \tilde{u}_h \leqslant 0, \\ \tilde{u}_h = 0, \text{ if } \tilde{u}_x = y_3(h),\end{aligned} \tag{3.2.13}$$

where

$$V(q,h) := \begin{cases} -\frac{k}{\beta_2}(\lambda h)^{\beta_2}, & \text{if } q > y_1(h), \\ -\frac{\beta_1-1}{\beta_1} q^{\frac{\beta_1}{\beta_1-1}} - \lambda h q, & \text{if } y_2(h) \leqslant q \leqslant y_1(h), \\ \frac{1}{\beta_1}((1-\lambda)h)^{\beta_1} - hq, & \text{if } y_3(h) \leqslant q < y_2(h). \end{cases}$$

To solve the above equation, some boundary conditions are also needed. First, to guarantee the global regularity of the solution, we need to impose smooth-fit conditions along two free boundaries that $\tilde{u}_x(x,h) = y_1(h)$ and $\tilde{u}_x(x,h) = y_2(h)$. Next, if we start with 0 initial wealth, to avoid bankruptcy, the optimal investment and the consumption rate should be 0 at all times. Therefore, we have that

$$\lim_{x\to 0} \frac{\tilde{u}_x(x,h)}{\tilde{u}_{xx}(x,h)} = 0 \;\text{ and }\; \lim_{x\to 0} \tilde{u}(x,h) = \int_0^{+\infty} -\frac{k}{\beta_2}(\lambda h)^{\beta_2} e^{-rt} dt = -\frac{k}{r\beta_2}(\lambda h)^{\beta_2}. \tag{3.2.14}$$

On the other hand, when the initial wealth tends to infinity, one can consume as much as possible, leading to an infinitely large consumption rate. In addition, a small variation in initial wealth only leads to a negligible change in the value function. It follows that

$$\lim_{x\to+\infty} \tilde{u}(x,h) = +\infty \quad \text{and} \quad \lim_{x\to+\infty} \tilde{u}_x(x,h) = 0. \tag{3.2.15}$$

We also note that, as the initial value x is large enough, we have $(x,h) \in \mathcal{D}_2$ and thus $c^*(x) = \tilde{u}_x(x,h)^{\frac{1}{\beta_1-1}}(1-\lambda)^{-\frac{\beta_1}{\beta_1-1}}$. Intuitively, our problem is similar to the Merton problem (Merton (1969)) along the free boundary $\mathcal{D}_2$, in which the optimal consumption is asymptotically proportional to the wealth. Therefore, we expect that

$$\lim_{\substack{x\to+\infty \\ (x,h)\in\mathcal{D}_2}} \frac{\tilde{u}_x(x,h)^{\frac{1}{\beta_1-1}}}{x} = c_\infty, \tag{3.2.16}$$

for some constant $c_\infty > 0$. This condition is verified later in Corollary 3.2.

To tackle the nonlinear HJB equation (3.2.13), we employ the dual transform only with respect to the variable x and treat the variable h as a parameter; see similar dual transform arguments in Deng et al. (2022) and Bo et al. (2021). That is, we consider $v(y,h) := \sup_{x\geqslant 0}\{\tilde{u}(x,h) - xy\}$, $y \geqslant y_3(h)$. For a given $(x,h) \in \mathcal{C}$, let us define the variable $y = \tilde{u}_x(x,h)$, and it holds that $\tilde{u}(x,h) = v(y,h) + xy$. We can further deduce that $x = -v_y(y,h)$, $\tilde{u}(x,h) = v(y,h) - yv_y(y,h)$, and $\tilde{u}_{xx}(x,h) = -\frac{1}{v_{yy}(y,h)}$. The nonlinear ODE (3.2.13) can be linearized to

$$\frac{\kappa^2}{2}y^2 v_{yy} - rv = -V(y,h), \tag{3.2.17}$$

and the free boundary condition is transformed to $y = y_3(h)$. As h can be regarded as a parameter, we can study the above equation as an ODE problem of the variable y. Based on the dual transform, the boundary conditions (3.2.15) can be written as

$$\lim_{y\to 0} v_y(y,h) = -\infty, \quad \text{and} \quad \lim_{y\to 0}(v(y,h) - yv_y(y,h)) = +\infty. \tag{3.2.18}$$

The boundary condition (3.2.16) becomes

$$\lim_{y\to 0} \frac{y^{\frac{1}{\beta_1 - 1}}}{v_y(y,h)} = -c_\infty, \tag{3.2.19}$$

along the boundary curve $y_3(h) = (1-\lambda)^{\beta_1} h^{\beta_1 - 1}$. The boundary condition (3.2.14) is equivalent to

$$y v_{yy}(y,h) \to 0 \ \text{ and } \ v(y,h) - y v_y(y,h) \to -\frac{k}{r\beta_2}(\lambda h)^{\beta_2} \ \text{ as } \ v_y(y,h) \to 0. \tag{3.2.20}$$

The dual transform holds that $v_y(y,h) = -x$, and one can derive that $\tilde{u}_h(x,h) = v_h(y,h) + (v_y(y,h) + x)\frac{dy(h)}{dh} = v_h(y,h)$. The free boundary condition (3.2.13) is translated to

$$v_h(y,h) = 0 \ \text{ for } \ y = y_3(h). \tag{3.2.21}$$

Although the dual ODE problem looks similar to the one in Deng et al. (2022), we emphasize that the boundary curves $y_1(h)$ and $y_2(h)$ are implicit functions of h that contains the implicit function $z(h)$. As a result, it becomes more complicated to apply smooth-fit conditions to derive the solution analytically and to prove the verification theorem. It is inevitable that all coefficient functions (in terms of h) in the solution involve $z(h)$. In particular, the following assumption on model parameters is needed, which is used to show that the obtained solution $v(y,h)$ is convex in y and in the verification proof of the optimal control.

Assumption (A1) $\beta_j < -\frac{r_2}{r_1}$, $j = 1,2$, where $r_1 > 1$ and $r_2 < 0$ are two roots to the equation $\eta^2 - \eta - \frac{2r}{\kappa^2} = 0$.

Note that $\beta_j < -\frac{r_2}{r_1}$ implies that $\gamma_j = \frac{\beta_j}{\beta_j - 1} > r_2$, $r_1\beta_j + r_2 = (\gamma_j - r_2)(\beta_j - 1) < 0$, for $j = 1,2$.

Proposition 3.2. *Let **Assumption (A1)** hold. Under boundary conditions* (3.2.18), (3.2.19), (3.2.20)*, the free boundary condition* (3.2.21)*, and the smooth-fit conditions with respect to y along $y = y_1(h)$ and $y = y_2(h)$, ODE* (3.2.17) *in $\{y \in \mathbb{R} : y \geqslant y_3(h)\}$ admits the unique solution that*

$$v(y,h) = \begin{cases} C_2(h)y^{r_2} - \frac{k}{r\beta_2}(\lambda h)^{\beta_2}, & \text{if } y > y_1(h), \\ C_3(h)y^{r_1} + C_4(h)y^{r_2} + \frac{2}{\kappa^2\gamma_1(\gamma_1-r_1)(\gamma_1-r_2)}y^{\gamma_1} - \frac{\lambda h}{r}y, & \text{if } y_2(h) \leqslant y \leqslant y_1(h), \\ C_5(h)y^{r_1} + C_6(h)y^{r_2} + \frac{1}{r\beta_1}((1-\lambda)h)^{\beta_1} - \frac{h}{r}y, & \text{if } y_3(h) \leqslant y < y_2(h), \end{cases} \tag{3.2.22}$$

where $\gamma_1 = \frac{\beta_1}{\beta_1 - 1} < 0$, $w(h)$ is defined in (3.1.4)*, $r_1 > 1$ and $r_2 < 0$ are given in **Assumption (A1)**, $y_1(h), y_2(h)$ and $y_3(h)$ are given in* (3.2.8)*, and functions $C_i(h)$, $i = 2, \dots, 6$, are defined by*

$$C_2(h) := C_4(h) + \frac{y_1(h)^{-r_2}}{r(r_1-r_2)}\left(\frac{kr_1}{\beta_2}(\lambda h)^{\beta_2} + \frac{r_1 r_2}{\gamma_1(\gamma_1-r_2)}y_1(h)^{\gamma_1} + \lambda h r_2 y_1(h)\right),$$

$$C_3(h) := \frac{y_1(h)^{-r_1}}{r(r_1-r_2)}\left(\frac{kr_2}{\beta_2}(\lambda h)^{\beta_2} + \frac{r_1 r_2}{\gamma_1(\gamma_1-r_1)}y_1(h)^{\gamma_1} + \lambda h r_1 y_1(h)\right),$$

$$C_4(h) := C_6(h) + \frac{y_2(h)^{-r_2}}{r(r_1-r_2)}\left(\frac{r_1}{\beta_1}((1-\lambda)h)^{\beta_1} - \frac{r_1 r_2}{\gamma_1(\gamma_1-r_2)}y_2(h)^{\gamma_1} + (1-\lambda)h r_2 y_2(h)\right),$$

$$C_5(h) := C_3(h) + \frac{y_2(h)^{-r_1}}{r(r_1-r_2)}\left(\frac{r_2}{\beta_1}((1-\lambda)h)^{\beta_1} - \frac{r_1 r_2}{\gamma_1(\gamma_1-r_1)}y_2(h)^{\gamma_1} + (1-\lambda)h r_1 y_2(h)\right),$$

$$C_6(h) := \int_h^{+\infty} (1-\lambda)^{(r_1-r_2)\beta_1} C_5'(s) s^{(r_1-r_2)(\beta_1-1)} ds. \tag{3.2.23}$$

Proof. The proof is given in Appendix A.1.1. □

Remark. *Note that all $C_i(h)$, $i = 2, \dots, 6$, are implicit functions of h. In particular, $C_2(h)$, $C_4(h)$ and $C_6(h)$ are written in the integral form. $C_3(h)$ and $C_5(h)$ are written in terms of implicit functions $y_1(h)$ and $y_2(h)$. Some technical efforts are needed to handle these semi-analytical functions in later verification proof.*

Theorem 3.1 (Verification Theorem)**.** *Let $(x,h)\in\mathcal{C}$, $h\in\mathbb{R}$ and $0<\lambda<1$, where $x\geqslant 0$ stands for the initial wealth, $h\geqslant 0$ is the initial reference level, and $\mathcal{C}$ is the effective domain (3.2.12). Let **Assumption (A1)** hold. For $(y,h)\in\{(y,h)\in\mathbb{R}_+^2: y\geqslant y_3(h)\}$, let us define the feedback functions that*

$$c^{\dagger}(y,h)=\begin{cases}0, & \text{if } y>y_1(h),\\ \lambda h+y^{\frac{1}{\beta_1-1}}, & \text{if } y_2(h)\leqslant y\leqslant y_1(h),\\ h, & \text{if } y_3(h)<y<y_2(h),\\ y^{\frac{1}{\beta_1-1}}(1-\lambda)^{-\frac{\beta_1}{\beta_1-1}}, & \text{if } y=y_3(h),\end{cases} \tag{3.2.24}$$

and

$$\pi^{\dagger}(y,h)=\frac{\mu-r}{\sigma^2}yv_{yy}(y,h)$$

$$=\frac{\mu-r}{\sigma^2}\begin{cases}\frac{2r}{\kappa^2}C_2(h)y^{r_2-1}, & \text{if } y>y_1(h),\\ \frac{2r}{\kappa^2}C_3(h)y^{r_1-1}+\frac{2r}{\kappa^2}C_4(h)y^{r_2-1}+\frac{2(\gamma_1-1)}{\kappa^2(\gamma_1-r_1)(\gamma_1-r_2)}y^{\gamma_1-1}, & \text{if } y_2(h)\leqslant y\leqslant y_1(h),\\ \frac{2r}{\kappa^2}C_5(h)y^{r_1-1}+\frac{2r}{\kappa^2}C_6(h)y^{r_2-1}, & \text{if } y_3(h)\leqslant y<y_2(h).\end{cases} \tag{3.2.25}$$

We consider the process $Y_t:=y^*e^{rt}M_t$, where $M_t:=e^{-(r+\frac{\kappa^2}{2})t-\kappa W_t}$ is the discounted rate state price density process, and $y^*=y^*(x,h)$ is the unique solution to the budget constraint $\mathbb{E}[\int_0^\infty c^{\dagger}(Y_t(y),H_t^{\dagger}(y))\ M_t dt]=x$ with

$$H_t^{\dagger}(y):=h\vee\sup_{s\leqslant t}c^{\dagger}(Y_s(y),H_s^{\dagger}(y))=h\vee\left((1-\lambda)^{-\frac{\beta_1}{\beta_1-1}}(\inf_{s\leqslant t}Y_s(y))^{\frac{1}{\beta_1-1}}\right).$$

The value function $\tilde{u}(x,h)$ can be attained by employing the optimal consumption and portfolio strategies in the feedback form that $c_t^*=c^{\dagger}(Y_t^*,H_t^*)$ and $\pi_t^*=\pi^{\dagger}(Y_t^*,H_t^*)$ for all $t\geqslant 0$, where $Y_t^*:=Y_t(y^*)$ and $H_t^*=H_t^{\dagger}(y^*)$.

Proof. The proof is given in Appendix A.1.2. □

Remark. *Note that the optimal consumption c_t^* has a jump when $Y_t^*=y_1(H_t^*)$ and $c_t^*>\lambda H_t^*$ whenever $c_t^*>0$. Meanwhile, we note that the running maximum process*

H_t^* still has continuous paths for $t > 0$. Indeed, from the feedback form, c_{t-}^* jumps only when $c_{t-}^* < H_t^*$ and we also have that $c_t^* \leqslant H_t^*$ after the jump, i.e., the jump never increases H_t^*. Therefore, both X_t^* and H_t^* still have continuous paths.

By the dual representation, we have that $x = g(\cdot, h) := -v_y(\cdot, h)$. Define $f(\cdot, h)$ as the inverse of $g(\cdot, h)$, then $\tilde{u}(x, h) = v \circ (f(x, h), h) + xf(x, h)$. Note that the function f should have a piecewise form across different regions. The invertibility of the map $x \mapsto g(x, h)$ is guaranteed by the next lemma.

Lemma 3.1. *Let **Assumption (A1)** hold. The function $v(y, h)$ is convex in all regions so that the inverse Legendre transform $\tilde{u}(x, h) = \inf_{y \geqslant y_3(h)}[v(y, h) + xy]$ is well defined. Moreover, it implies that the feedback optimal portfolio $\pi^*(y, h) > 0$.*

Proof. The proof is given in Appendix A.1.5. □

Thanks to Lemma 3.1, we can apply the inverse Legendre transform to the solution $v(y, h)$ in (3.2.22). Similar to Section 3.1 in Deng et al. (2022), we can derive the following three boundary curves $x_{\text{zero}}(h)$, $x_{\text{aggr}}(h)$, and $x_{\text{lavs}}(h)$:

$$
\begin{aligned}
&x_{\text{zero}}(h) := -y_1(h)^{r_2-1}C_2(h)r_2,\\
&x_{\text{aggr}}(h) := -C_3(h)r_1y_2(h)^{r_1-1} - C_4(h)r_2y_2(h)^{r_2-1} - \frac{2}{\kappa^2(\gamma_1 - r_1)(\gamma_1 - r_2)}y_2(h)^{\gamma_1-1} + \frac{\lambda h}{r},\\
&x_{\text{lavs}}(h) := -C_5(h)r_1y_3(h)^{r_1-1} - C_6(h)r_2y_3(h)^{r_2-1} + \frac{h}{r},
\end{aligned}
\tag{3.2.26}
$$

and it holds that the feedback function of the optimal consumption satisfies: (i) $c^*(x, h) = 0$ when $0 < x \leqslant x_{\text{zero}}(h)$; (ii) $0 < c^*(x, h) \leqslant h$ when $x_{\text{zero}}(h) \leqslant x \leqslant x_{\text{aggr}}(h)$; (iii) $c^*(x, h) = h$ when $x_{\text{aggr}}(h) < x \leqslant x_{\text{lavs}}(h)$. In particular, the condition $u_x(x, h) \geqslant y_3(h)$ in the effective domain can be explicitly expressed as $x \leqslant x_{\text{lavs}}(h)$. Moreover, the following inverse function is well defined:

$$
\tilde{h}(x) := (x_{\text{lavs}})^{-1}(x), x \geqslant 0. \tag{3.2.27}
$$

Along the boundary $x = x_{\text{lavs}}(h)$, the feedback form of the optimal consumption in (3.2.30) is presented by $c^*(x,h) = (1-\lambda)^{-\frac{\beta_1}{\beta_1-1}} f(x, \tilde{h}(x))^{-\frac{1}{\beta_1-1}}$. Using the dual relationship and Proposition 3.2, the function f can be implicitly determined as follows:

(i) If $x < x_{\text{zero}}(h)$, Lemma 3.1 implies that $v_y(y,h)$ is strictly increasing in y and $f(x,h) = f_1(x,h)$ can be uniquely determined by

$$x = -C_2(h) r_2 (f_1(x,h))^{r_2-1}.$$

(ii) If $x_{\text{zero}}(h) \leqslant x \leqslant x_{\text{aggr}}(h)$, Lemma 3.1 implies that $v_y(y,h)$ is strictly increasing in y and $f(x,h) = f_2(x,h)$ can be uniquely determined by

$$\begin{aligned} x = &- C_3(h) r_1 (f_2(x,h))^{r_1-1} - C_4(h) r_2 (f_2(x,h))^{r_2-1} \\ &- \frac{2}{\kappa^2(\gamma_1 - r_1)(\gamma_1 - r_2)} (f_2(x,h))^{\gamma_1 - 1} + \frac{\lambda h}{r}. \end{aligned} \tag{3.2.28}$$

(iii) If $x_{\text{aggr}}(h) < x \leqslant x_{\text{lavs}}(h)$, Lemma 3.1 implies that $v_y(y,h)$ is strictly increasing in y and $f(x,h) = f_3(x,h)$ can be uniquely determined by

$$x = -C_5(h) r_1 (f_3(x,h))^{r_1-1} - C_6(h) r_2 (f_3(x,h))^{r_2-1} + \frac{h}{r}. \tag{3.2.29}$$

Corollary 3.1. *For $(x,h) \in \mathcal{C}$, $0 < \lambda < 1$, $\beta_1 < 1$ and $\beta_2 < 1$, under* ***Assumption (A1)****, let us define the piecewise function*

$$f(x,h) = \begin{cases} \left(\frac{-x}{C_2(h) r_2}\right)^{\frac{1}{r_2-1}}, & \text{if } x < x_{\text{zero}}(h), \\ f_2(x,h), & \text{if } x_{\text{zero}}(h) \leqslant x \leqslant x_{\text{aggr}}(h), \\ f_3(x,h), & \text{if } x_{\text{aggr}}(h) < x \leqslant x_{\text{lavs}}(h), \end{cases}$$

where $f_2(x,h)$ and $f_3(x,h)$ are defined in (3.2.28) and (3.2.29), respectively.

The value function $\tilde{u}(x,h)$ in (3.1.5) can be written by

$$\tilde{u}(x,h)=\begin{cases} C_2(h)(f(x,h))^{r_2}-\frac{k}{r\beta_2}(\lambda h)^{\beta_2}+xf(x,h), & \text{if } x<x_{\text{zero}}(h),\\ C_3(h)(f(x,h))^{r_1}+C_4(h)(f(x,h))^{r_2} & \\ +\dfrac{2}{\kappa^2\gamma_1(\gamma_1-r_1)(\gamma_1-r_2)}(f(x,h))^{\gamma_1} & \text{if } x_{\text{zero}}(h)\leqslant x\leqslant x_{\text{aggr}}(h),\\ -\dfrac{\lambda h}{r}f(x,h)+xf(x,h), & \\ C_5(h)(f(x,h))^{r_1}+C_6(h)(f(x,h))^{r_2} & \\ +\dfrac{1}{r\beta_1}((1-\lambda)h)^{\beta_1}-\dfrac{h}{r}f(x,h)+xf(x,h), & \text{if } x_{\text{aggr}}(h)<x\leqslant x_{\text{lavs}}(h), \end{cases}$$

where the free boundaries $x_{\text{zero}}(h)$, $x_{\text{aggr}}(h)$, and $x_{\text{lavs}}(h)$ are given explicitly in (3.2.26). The feedback optimal consumption and portfolio can be expressed in terms of primal variables (x,h) that

$$c^*(x,h)=\begin{cases} 0, & \text{if } x<x_{\text{zero}}(h),\\ \lambda h+(f(x,h))^{\frac{1}{\beta_1-1}}, & \text{if } x_{\text{zero}}(h)\leqslant x\leqslant x_{\text{aggr}}(h),\\ h, & \text{if } x_{\text{aggr}}(h)<x<x_{\text{lavs}}(h),\\ (1-\lambda)^{-\frac{\beta_1}{\beta_1-1}}f(x,\tilde{h}(x))^{-\frac{1}{\beta_1-1}}, & \text{if } x=x_{\text{lavs}}(h), \end{cases} \tag{3.2.30}$$

where $\tilde{h}(x)$ is given in (3.2.27), and

$$\begin{aligned}&\pi^*(x,h)\\ =&\frac{\mu-r}{\sigma^2}\begin{cases} (1-r_2)x, & \text{if } x<x_{\text{zero}}(h),\\ \Big(\dfrac{2r}{\kappa^2}C_3(h)f^{r_1-1}(x,h)+\dfrac{2r}{\kappa^2}C_4(h)f^{r_2-1}(x,h) & \\ +\dfrac{2(\gamma_1-1)}{\kappa^2(\gamma_1-r_1)(\gamma_1-r_2)}f^{\gamma_1-1}(x,h)\Big), & \text{if } x_{\text{zero}}(h)\leqslant x\leqslant x_{\text{aggr}}(h),\\ \frac{2r}{\kappa^2}C_5(h)f^{r_1-1}(x,h)+\frac{2r}{\kappa^2}C_6(h)f^{r_2-1}(x,h), & \text{if } x_{\text{aggr}}(h)<x\leqslant x_{\text{lavs}}(h). \end{cases}\end{aligned} \tag{3.2.31}$$

Moreover, for any initial value $X_0^*, H_0^* = (x,h)\in\mathcal{C}$, the stochastic differential equation

$$dX_t^*=rX_t^*dt+\pi^*(\mu-r)dt+\pi^*\sigma dW_t-c^*dt, \tag{3.2.32}$$

has a unique strong solution given the optimal feedback control (c^*, π^*) *as above.*

Proof. The proof is given in Appendix A.1.3. □

3.3 Properties of Optimal Controls

First, compared with the main results in the standard Merton's problem with power utility (see Merton (1969)), our optimal feedback controls $\pi^*(x,h)$ and $c^*(x,h)$ are fundamentally different, which are expressed as the piecewise implicit nonlinear functions of both variables x and h. In particular, our optimal consumption process exhibits jumps when the wealth level crosses the threshold $x_{\text{zero}}(h)$. The more complicated solution structure is rooted in the path-dependent reference process H_t inside the utility and the S-shaped utility accounting for loss aversion.

Moreover, based on Corollary (3.1), we can show some asymptotic results on the optimal consumption-wealth ratio c_t^*/X_t^* and the optimal portfolio-wealth ratio π_t^*/X_t^*, whose proof is given in Appendix A.1.6.

Corollary 3.2. *As* $x \leqslant x_{lavs}(h)$*, the asymptotic behavior of large wealth* $x \to +\infty$ *is equivalent to* $\lim_{h\to+\infty} x_{lavs}(h) = +\infty$*. We then have that*

$$\lim_{h\to+\infty} \frac{c^*(x_{lavs}(h), h)}{x_{lavs}(h)} = L_1, \qquad \lim_{h\to+\infty} \frac{\pi^*(x_{lavs}(h), h)}{x_{lavs}(h)} = L_2,$$

for some constants L_1 *and* L_2*. In addition, as* $\lambda \to 0$*, two limits* L_1 *and* L_2 *coincide with the asymptotic results in the infinite-horizon Merton's problem (Merton (1969)) with power utility* $U^*(x) = \frac{1}{\beta_1} x^{\beta_1}$*. As a result, the boundary conditions* (3.2.16) *and* (3.2.19) *hold valid in our problem.*

Proof. The proof is given in Appendix A.1.6. □

Remark. *As the wealth level becomes sufficiently large, both the optimal consumption and the optimal portfolio amount are asymptotically proportional to the wealth level* *that $c_t^* \approx L_1 X_t^*$ and $\pi_t^* \approx L_2 X_t^*$, in a similar fashion to the asymptotic results in the standard Merton's problem with the power utility. However, it is important to note that our asymptotic limits differ significantly from those in Merton's problem, which now sensitively depends on the reference degree parameter λ and risk aversion parameters β_1, β_2, k from the S-shaped utility. One can see that, even when the agent's wealth level is very high, the impacts from the reference level λH_t and the loss aversion preference do fade out in our model because the large consumption rate c_t^* also lifts up the reference λH_t^* to a new high level. Only in the extreme case when the reference degree $\lambda \to 0$, i.e., there is no reference process, our asymptotic results coincide with those in the standard Merton's problem under power utility.*

Next, we can characterize the average fraction of time that the agent expects to stay in each region.

Corollary 3.3. *The following properties hold:*

1. *The long-run fraction of time that the agent stays in the region $\{x_{\text{aggr}}(H_t^*) \leqslant X_t^* \leqslant x_{\text{lavs}}(H_t^*)\}$ equals the value of $\lim_{h \to +\infty} \frac{y_2(h)}{y_1(h)}$.*

2. *The long-run fraction of time that the agent stays in the region $\{0 \leqslant X_t^* \leqslant x_{\text{zero}}(H_t^*)\}$ equals the value of $1 - \lim_{h \to +\infty} \frac{y_3(h)}{y_1(h)}$.*

3. *Starting from $(x, h) \in \{(x, h) : x \in (x_{\text{zero}}(h), x_{\text{lavs}}(h)]\}$, let us consider the first hitting time of zero consumption that $\tau_{\text{zero}} := \inf\{t \geqslant 0, X_t = x_{\text{zero}}(H_t)\}$. We have that*

$$\mathbb{E}[\tau_{\text{zero}}] = \overline{C}_1(h) f(x, h)^2 + \overline{C}_2(h) + \frac{\log f(x, h)}{\kappa^2},$$

where $\overline{C}_1(h)$ *and* $\overline{C}_2(h)$ *satisfy:*

$$\overline{C}_1(h)y_1(h)^2 + \overline{C}_2(h) + \frac{\log y_1(h)}{\kappa^2} = 0,$$

$$\overline{C}_1'(h)y_3(h)^2 + \overline{C}_2'(h) = 0.$$

4. *Starting from* $(x, h) \in \{(x, h) : x \in [x_{\text{zero}}(h), x_{\text{lavs}}(h))\}$*, let us define the first hitting time to update the historical consumption maximum* $\tau_{\text{lavs}} := \inf\{t \geqslant 0 : X_t = x_{\text{lavs}}(H_t)\}$*. We have that*

$$\mathbb{E}[\tau_{\text{lavs}}] = \frac{2}{\kappa^2} \log\left(\frac{f(x,h)h^{1-\beta_1}}{(1-\lambda)^{\beta_1}}\right).$$

Proof. The proof is given in Appendix A.1.7. □

3.3.1 Boundary Curves

We next present some numerical examples of the thresholds and the optimal feedback functions and discuss some financial implications.

We first plot in Figure 3.2 the boundary curves $x_{\text{zero}}(h)$, $x_{\text{aggr}}(h)$ and $x_{\text{lavs}}(h)$ as functions of h, separating the regions for different feedback forms of the optimal consumption. First, compared with Figure 1 in Deng et al. (2022), it is interesting to note that we need to take into account four different cases in total depending on whether two boundary curves $x_{\text{zero}}(h)$ and $x_{\text{aggr}}(h)$ coincide or not. To be more precise, we know by definition that $x_{\text{zero}}(h) = x_{\text{aggr}}(h)$ if and only if $y_1(h) = y_2(h)$, where $y_1(h)$ and $y_2(h)$ are given in (3.2.8). In view of Remark 3.1.2, $y_1(h) = y_2(h)$ in three different scenarios. The upper left panel in Figure 3.2 corresponds to the case that two boundaries $x_{\text{zero}}(h)$ and $x_{\text{aggr}}(h)$ are completely separated for all $h > 0$, i.e., $y_1(h) > y_2(h)$ for $h > 0$ (with parameters $r = \rho = 0.05$, $\mu = 0.1$, $\sigma = 0.25$, $\beta_1 = 0.2$, $\beta_2 = 0.3$, $k = 1.5$, $\lambda = 0.5$); the upper right panel in Figure 3.2 corresponds to the case that two boundaries $x_{\text{zero}}(h) = x_{\text{aggr}}(h)$ when the reference level is low

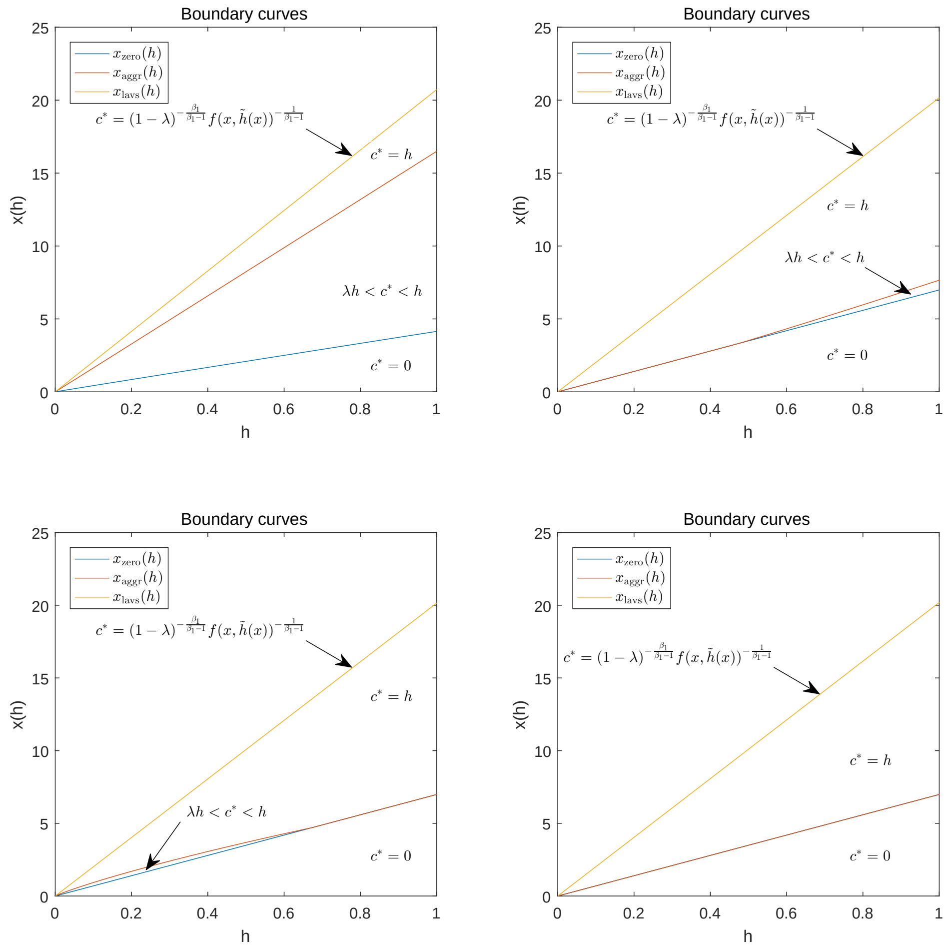

Figure 3.2: Four cases of boundary curves caused by different parameters

that $h \leqslant h^*$ for some critical point $h^* > 0$ (with parameters $r = \rho = 0.05$, $\mu = 0.1$, $\sigma = 0.25$, $\beta_1 = 0.2$, $\beta_2 = 0.3$, $k = 1.5$, $\lambda = 0.92$); the lower left panel in Figure 3.2 corresponds to the case that two boundaries $x_{\text{zero}}(h) = x_{\text{aggr}}(h)$ when the reference level is high that $h \geqslant h^*$ for some $h^* > 0$ (with parameters $r = \rho = 0.05$, $\mu = 0.1$, $\sigma = 0.25$, $\beta_1 = 0.2$, $\beta_2 = 0.1$, $k = 1.5$, $\lambda = 0.973$); and the lower right panel in Figure 3.2 corresponds to the case that $x_{\text{zero}}(h) = x_{\text{aggr}}(h)$ for all $h \geqslant 0$ (with parameters $r = \rho = 0.05$, $\mu = 0.1$, $\sigma = 0.25$, $\beta_1 = 0.2$, $\beta_2 = 0.2$, $k = 1.5$, $\lambda = 0.95$).

Second, Figure 3.2 illustrates again that the positive optimal consumption can never fall below the reference level, i.e., we must have $c^*(x,h) > \lambda h$ if $c^*(x,h) > 0$ so that there exists a jump when the wealth process X_t^* crosses the boundary curve $x_{\text{zero}}(H_t^*)$. In particular, for some value of h such that $x_{\text{zero}}(h) = x_{\text{aggr}}(h)$ hold, the optimal consumption may jump from 0 to the current maximum level $H_t^* = h$ immediately, indicating that the agent consumes at the historical maximum level h if the agent starts to consume. This differs substantially from the continuous optimal consumption process derived in Deng et al. (2022). The jump of consumption is caused by the risk-loving attitude over the loss domain in the S-shaped utility, which corresponds to the linear piece of the concave envelope. In this wealth region, the agent prefers to stop the current consumption if it cannot surpass the reference level. Therefore, our result under the S-shaped utility can depict the extreme behavior of some agents who cannot endure any positive consumption plan below the current reference. We emphasize that, all the boundary curves in Figure 2 are generally nonlinear functions of h, featuring the necessity of two dimensional state processes of X_t and H_t in our control problem. Only in the extreme case when $\beta_1 = \beta_2$, the boundary curves can be expressed in a linear manner, and the dimension reduction can be conducted.

Remark. *When risk aversion parameters satisfy $\beta_1 = \beta_2$, our problem has homogeneous property that $\tilde{u}(x,h) = h^{\beta_1}\tilde{u}(x/h,1)$, and it is sufficient to consider the function $\hat{u}(\omega) := \tilde{u}(\omega,1)$ to reduce the dimension. In this case, the boundary curves degenerate to boundary points for the new state variable $\omega = x/h$.*

3.3.2 Sensitivity Analysis

We now fix the model parameters to $r = 0.05$, $\rho = 0.05$, $\mu = 0.1$, $\sigma = 0.25$, $\beta_1 = 0.2$, $\beta_2 = 0.3$, $k = 1.5$, and reference level $h = 1$. We numerically illustrate the sensitivity with respect to the reference degree $\lambda \in (0,1)$. Let us choose $\lambda \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$. The value function, the optimal feedback consumption, and the optimal feedback portfolio together with marked boundaries x_{zero}, x_{aggr} and x_{lavs} are plotted in Figure 3.3. From all panels, one can observe that the boundary curve $x_{\text{zero}}(1;\lambda)$ is increasing in λ, while boundary curves $x_{\text{aggr}}(1;\lambda)$ and $x_{\text{lavs}}(1;\lambda)$ are both decreasing in λ. On the one hand, one can explain that when the agent has a higher reference with a larger λ and the current wealth is low, it is more likely that the optimal consumption falls below the reference, leading to zero consumption. Therefore, the threshold for positive consumption becomes larger for a larger λ. On the other hand, when the wealth is sufficiently large, larger reference degree λ results in more aggressive consumption (see the middle panel), and it is more likely that the agent lowers the threshold to consume at the global maximum level even by reducing the portfolio amount. Moreover, when λ increases, we can also observe that λH_t^* actually increases faster than the consumption c_t^* during the life cycle, which leads to a drop of $c_t^* - \lambda H_t^*$ and a decline in the value function from the left panel. From the right panel, when wealth decreases to the region $x < x_{\text{zero}}(1;\lambda)$, the optimal consumption stays at 0 due to the linear piece of the concave envelope, but the optimal portfolio is increasing in x with a large slope. This can be interpreted by the fact that the agent needs to invest very aggressively to pull the wealth level back to the threshold $x_{\text{zero}}(1;\lambda)$ driven by the strong desire for positive consumption under the loss aversion preference. When wealth starts to surpass the threshold $x_{\text{zero}}(1;\lambda)$, the agent chooses the positive consumption above the reference level λh, and we can see from the right panel that the agent strategically withdraws some wealth from the risky asset account to support the high consumption plan. In addition, the higher reference degree λ is, the more drastic the decrease in portfolio with respect to x. When wealth tends to be further larger, both the optimal consumption and optimal portfolio become increasing in x. By comparison from the right panel, when wealth is very large, the optimal portfolio is decreasing in the reference degree λ, which is consistent with the fact that the agent needs more cash to support the more aggressive consumption as λ increases.

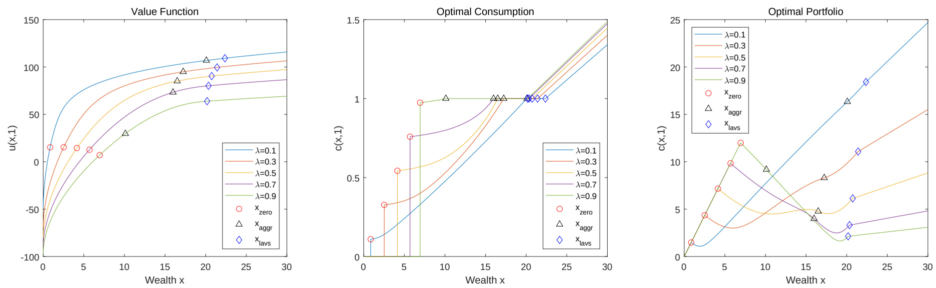

Figure 3.3: Sensitivity analysis on the reference degree λ.

Finally, we numerically illustrate the sensitivity with respect to the expected return μ of the risky asset. Let us fix the model parameters such that $r = 0.05$, $\rho = 0.05$, $\lambda = 0.5$, $\sigma = 0.25$, $\beta_1 = 0.2$, $\beta_2 = 0.3$, $k = 1.5$, $h = 1$ and consider $\mu \in \{0.06, 0.08, 0.1, 0.12, 0.14\}$. The value function and the optimal feedback controls together with marked boundaries are plotted in Figure 3.4. First, from the left panel, one can observe that the value function increases in μ, which matches with the intuition that better market performance guarantees higher wealth and a larger consumption plan. From all panels, it is also interesting to observe that both boundaries $x_{\text{zero}}(1;\mu)$ and $x_{\text{aggr}}(1;\mu)$ are decreasing in μ, while the boundary $x_{\text{lavs}}(1;\mu)$ is increasing in μ. On the one hand, the higher return from the financial market secures better wealth growth, leading to lower thresholds for the agent to start positive consumption and consumption at the historical peak level. On the other hand, a higher return rate μ also motivates the agent to invest more in the risky asset, as one can see that the optimal portfolio is increasing in μ from the right panel. As a result, the agent does not blindly lower the threshold $x_{\text{lavs}}(1;\mu)$ to create the new global consumption peak as it becomes more beneficial in the long run to invest more cash into the risky asset when x is sufficiently large. Therefore, the threshold $x_{\text{lavs}}(1;\mu)$ is actually increased with an increased parameter μ. One can also observe that for $x_{\text{zero}}(1;\mu) \leqslant x \leqslant x_{\text{lavs}}(1;\mu)$, as the expected return μ increases, the agent gradually shifts from the willingness of high consumption plan by sacrificing the portfolio to the more aggressive investment behavior to accumulate the larger wealth. Combining Figure 3.3 and Figure 3.4, we also note that the optimal portfolio $\pi^*(x, 1; \lambda, \mu)$ is decreasing in λ but increasing in μ when x is large, suggesting that some agents with a large reference degree λ are only motivated to invest more wealth in the financial market when the expected return is excessively high. This is consistent with and may partially help to explain the *equity premium puzzle* (see Mehra and Prescott (1985)) that the market premium needs to be very high to attract some agents (possibly those agents with the large reference degree if they adopt our proposed preference on consumption performance) to actively invest in the risky asset.

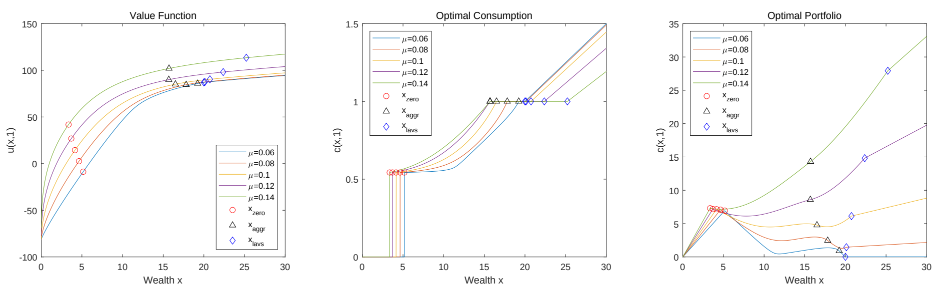

Figure 3.4: Sensitivity analysis on the expected return μ.

Chapter 4

Optimal consumption and life insurance under shortfall aversion and a drawdown constraint

In this chapter, we adopt the shortfall aversion preference proposed in Guasoni et al. (2020) together with dynamic life insurance control, and enforce an additional drawdown constraint on the consumption rate as a subsistence consumption requirement. The objective function of the control problem also involves the expected bequest from life insurance, which renders the dimension reduction in Guasoni et al. (2020) not applicable in our problem. Instead, we encounter a two-dimensional HJB equation. Similar to Deng et al. (2022), taking the wealth level and reference level as two state variables, we can derive the value function and optimal strategies in analytical form by solving the associated HJB equation with some boundary conditions. The HJB equation can be expressed in a piecewise form based on the decomposition of the state domain such that the feedback optimal consumption: (1) equals the drawdown constraint rate, (2) lies between the drawdown constraint and the past spending maximum, and (3) attains the past consumption peak. By using the dual transform and some smooth-fit conditions, the HJB equation is linearized to a parameterized ODE, which can be solved in closed-form. The desired feedback form of optimal consumption, investment and insurance strategies can be obtained by the inverse transform. Contrary to Guasoni et al. (2020), our boundary curves for the wealth variable to distinguish different optimal feedback controls are all nonlinear functions due to the additional life insurance control. Our analytical results allow us to numerically illustrate how the model parameters affect the optimal decision on consumption and life insurance. By comparing with some existing results without shortfall aversion, we can also illustrate how the reference of past spending maximum motivates the insurance purchase. Some interesting financial implications induced by the shortfall aversion preference and the drawdown constraint are discussed therein.

The remainder of this chapter is organized as follows. Section 4.1 introduces the market model with mortality risk and the stochastic control problem under the shortfall aversion preference. Section 4.2 gives some heuristic arguments to solve the HJB equation and presents theS main results on the optimal feedback consumption, portfolio and life insurance controls. Section 4.3 presents several numerical examples to illustrate some sensitivity analysis results and their financial implications.

4.1 Model Setup and Problem Formulation

4.1.1 Shortfall Aversion Preference and Control Problem

It is assumed in the present paper that the agent is shortfall averse on relative consumption in the sense that utility losses of spending cuts from a reference. The reference process is chosen as the running maximum consumption process $H_t := \max\{h,\ \sup_{s\leqslant t} c_s\}$, and $H_0 = h \geqslant 0$ is the initial reference level. We adopt the shortfall aversion preference proposed in Guasoni et al. (2020) on consumption and consider the expected utility on bequest at the time of death. The objective function of the control problem is defined by

$$\begin{aligned}&\mathbb{E}\left[\int_0^\tau e^{-\rho t}U(c_t, H_t)dt + e^{-\rho\tau}V(b_\tau)\right]\\ =&\mathbb{E}\left[\int_0^\infty e^{-(\rho+\lambda)t}U(c_t, H_t)dt + \lambda\int_0^\infty e^{-(\rho+\lambda)t}V(b_t)dt\right],\end{aligned} \tag{4.1.1}$$

where $U(c, h)$ is the so-called shortfall aversion preference that satisfies

$$U(c,h) = \begin{cases} \frac{1}{\gamma_1}\left(\frac{c}{h^\alpha}\right)^{\gamma_1}, & \text{if } \nu h \leqslant c < h, \\ \frac{1}{\gamma_1}\left(c^{1-\alpha}\right)^{\gamma_1}, & \text{if } c \geqslant h, \end{cases}$$

with $0 < \gamma_1 < 1$, and $V(b)$ is a standard constant relative risk aversion (CRRA) utility that

$$V(b) = K\frac{b^{\gamma_2}}{\gamma_2},\ 0 < \gamma_2 < 1,\ K > 0,$$

and K stands for the bequest motive level. According to Figure 4.1, the utility function $U(c, h)$ has a kink at $c = h$.

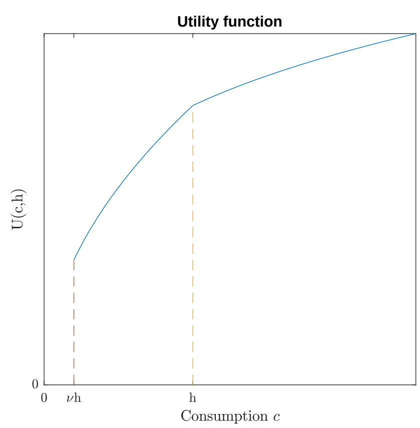

Figure 4.1: Utility $U(c, h)$ for a consumption rate c, with reference point h

The agent aims to maximize the expected utility under a shortfall aversion preference subject to a drawdown constraint on consumption control that

$$\max_{(c,\pi,b)\in\mathcal{A}(x,h)} \mathbb{E}\left[\int_0^\infty e^{-(\rho+\lambda)t}U(c_t, H_t)dt + \lambda\int_0^\infty e^{-(\rho+\lambda)t}V(b_t)dt\right]. \tag{4.1.2}$$

For ease of presentation, it is assumed that the discount factor ρ equals the risk-free rate r.

4.2 Main Results

4.2.1 The HJB Equation

For problem (4.1.2), we can derive the auxiliary HJB equation on the feasible domain $\{(x,h) \in [0,+\infty) \times [0,\infty) : x \geqslant \frac{\nu h}{r+\lambda}\}$ using some heuristic arguments that

$$\begin{aligned}\sup_{c\in[\nu h,h],\pi\in\mathbb{R},b\geqslant 0}\Big[&-(r+\lambda)u + u_x\left((r+\lambda)x + \pi(\mu-r) - c - \lambda b\right) \\ &+\frac{1}{2}\sigma^2\pi^2 u_{xx} + U(c,h) + \lambda V(b)\Big] = 0, \\ &u_h(x,h) \leqslant 0,\end{aligned} \tag{4.2.3}$$

for $x \geqslant \frac{\nu h}{r+\lambda}$ and $h \geqslant 0$. The free boundary condition $u_h(x,h) = 0$ shall be specified later. Our goal is to find the optimal feedback control $c^*(x,h)$, $\pi^*(x,h)$, and $b^*(x,h)$. If $u(x,\cdot)$ is C^2 in x, the first order condition gives the optimal portfolio and optimal bequest in feedback form that $\pi^*(x,h) = -\frac{\mu-r}{\sigma^2}\frac{u_x}{u_{xx}}$ and $b^*(x,h) = \left(\frac{u_x}{K}\right)^{\frac{1}{\gamma_2-1}}$, respectively. The HJB equation (4.2.3) can be simplified to

$$\begin{aligned}\sup_{c\in[\nu h,h]}[U(c,h) - cu_x] - (r+\lambda)u + (r+\lambda)xu_x - \lambda K^{-\frac{1}{\gamma_2-1}}\frac{1-\gamma_2}{\gamma_2}u_x^{\frac{\gamma_2}{\gamma_2-1}} - \frac{\kappa^2}{2}\frac{u_x^2}{u_{xx}} = 0, \\ u_h \leqslant 0, \quad \forall x \geqslant \frac{\nu h}{r+\lambda}.\end{aligned} \tag{4.2.4}$$

4.2.2 Some Heuristic Results

We aim to solve the HJB equation in analytical form. In particular, we plan to characterize some thresholds (depending on h) for wealth level x such that the auxiliary value function, the optimal portfolio and consumption can be expressed analytically in each region.

Similar to Deng et al. (2022) and Li et al. (2021), we can heuristically decompose the domain based on the first order condition with respect to c and express the HJB equation (4.2.4) piecewisely. In particular, we have the following disjoint regions:

Region I: on the set $\mathcal{R}_1 = \{(x,h) \in \mathbb{R}_+^2 : x \geqslant \frac{\nu h}{r+\lambda}, u_x > \nu^{\gamma_1 - 1} h^{(1-\alpha)\gamma_1 - 1}\}$, $U(c,h) - cu_x$ is decreasing in c on $[\nu h, h]$, implying that $c^* = \nu h$.

Region II: on the set $\mathcal{R}_2 = \{(x,h) \in \mathbb{R}_+^2 : x \geqslant \frac{\nu h}{r+\lambda}, h^{(1-\alpha)\gamma_1 - 1} \leqslant u_x \leqslant \nu^{\gamma_1 - 1} h^{(1-\alpha)\gamma_1 - 1}\}$, $U(c,h) - cu_x$ attains its maximum in $[\nu h, h]$, implying that $c^* = h^{\frac{\alpha\gamma_1}{\gamma_1 - 1}} u_x^{\frac{1}{\gamma_1 - 1}}$.

Region III: on the set $\mathcal{R}_3 = \{(x,h) \in \mathbb{R}_+^2 : x \geqslant \frac{\nu h}{r+\lambda}, u_x < h^{(1-\alpha)\gamma_1 - 1}\}$, $U(c,h) - cu_x$ is increasing in c on $[\nu h, h]$, implying that $c^* = h$. To distinguish whether the optimal consumption c_t^* updates the past maximum process H_t^* in this region, we need to split *Region III* into three subregions:

Region III-(i): on the set $\mathcal{D}_1 = \{(x,h) \in \mathbb{R}_+ \times \mathbb{R}_+ : x \geqslant \frac{\nu h}{r+\lambda}, (1-\alpha) h^{(1-\alpha)\gamma_1 - 1} < u_x < h^{(1-\alpha)\gamma_1 - 1}\}$, we have a contradiction that $\hat{c}(x) = \left(\frac{u_x(x,h)}{1-\alpha}\right)^{\frac{1}{(1-\alpha)\gamma_1 - 1}} < h$, and therefore c_t^* is not a singular control. We still need to follow the previous feedback form $c^*(x,h) = h$, in which h is a previously attained maximum level. The corresponding running maximum process remains flat at the instant time. In this region, we only know that $u_h(x,h) \leqslant 0$ as we have $dH_t = 0$.

Region III-(ii): on the set $D_2 := \{(x,h) \in \mathbb{R}_+ \times \mathbb{R}_+ : x \geqslant \frac{\nu h}{r+\lambda}, u_x = (1-\alpha) h^{(1-\alpha)\gamma_1 - 1}\}$, we obtain $\hat{c}(x) = \left(\frac{u_x(x,h)}{1-\alpha}\right)^{\frac{1}{(1-\alpha)\gamma_1 - 1}} = h$ and the feedback optimal consumption $c^*(x,h) = \left(\frac{u_x(x,h)}{1-\alpha}\right)^{\frac{1}{(1-\alpha)\gamma_1 - 1}}$. This corresponds to the singular control c_t^* that creates a new peak for the whole path and $H_t^* = c_t^* = \left(\frac{u_x(X_t^*, H_t^*)}{1-\alpha}\right)^{\frac{1}{(1-\alpha)\gamma_1 - 1}}$ is strictly increasing at the instant time so that $H_t^* > H_s^*$ for any $s < t$ and we must require the following free boundary condition that $u_h(x,h) = 0$. In this region, it is noted that $c^*(x,h) = h$, therefore, the HJB equation follows the same PDE with in Region I but together with the new free boundary condition.

Region III-(iii): on the set $\mathcal{D}_3 := \{(x,h) \in \mathbb{R}_+ \times \mathbb{R}_+ : u_x(x,h) < (1-\alpha)h^{(1-\alpha)\gamma_1 - 1}\}$, we obtain $\hat{c}(x) = \left(\frac{u_x(x,h)}{1-\alpha}\right)^{\frac{1}{(1-\alpha)\gamma_1 - 1}} > h$. This indicates that the initial reference level h is below the feedback control $\hat{c}(x)$, and the optimal consumption is again a singular control $c^*(x) > h$, which creates a new consumption peak. As the running maximum process H_t^* is updated immediately by c_t^*, the feedback optimal consumption pulls the associated H_{t-}^* upward from its original value to the new value in the direction of h and X_t^* remains the same, in which $u(x,h)$ is the solution of the HJB equation on set $\mathcal{D}_2$. This suggests that for any given initial value (x,h) in set $\mathcal{D}_3$, the feedback control $c^*(x,h)$ pushes (x,h) to jump immediately to the point $(x,\hat{h})$ on the boundary set $\mathcal{D}_2$ for the given level of x, where $\hat{h} = \left(\frac{u_x(x,\hat{h})}{1-\alpha}\right)^{\frac{1}{(1-\alpha)\gamma_1 - 1}}$.

Therefore, it is sufficient to consider the effective domain defined by

$$\begin{aligned} \mathcal{C} &:= \left\{(x,h) \in \mathbb{R}_+ \times \mathbb{R}_+ : x \geqslant \frac{\nu h}{r+\lambda}, u_x(x,h) \geqslant (1-\alpha)h^{(1-\alpha)\gamma_1 - 1}\right\} \\ &= \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{D}_1 \cup \mathcal{D}_2 \subset \mathbb{R}_+^2. \end{aligned} \tag{4.2.5}$$

The only possibility for $(x,h) \in \mathcal{D}_3$ occurs at the initial time $t = 0$. If (X_0^*, H_0^*) starts from $\mathcal{C}$, then the controlled process (X_t^*, H_t^*) always stays inside region $\mathcal{C}$ and either reflects at the boundary or moves along boundary $\mathcal{D}_2$ after visiting boundary $\mathcal{D}_2$. On the other hand, if the process (X_0^*, H_0^*) starts from the value (x,h) inside region $\mathcal{D}_3$, the optimal control enforces an instant jump (and the only jump) of the process H from $H_{0-} = h$ to $H_0 = \hat{h}$ on boundary $\mathcal{D}_2$ and both processes X_t and H_t become continuous processes diffusing inside the effective domain $\mathcal{C}$ afterwards for $t < 0$.

Therefore, the HJB equation (4.2.4) can be written as

$$
\begin{aligned}
-(r+\lambda)u + (r+\lambda)xu_x - \frac{\kappa^2}{2}\frac{u_x^2}{u_{xx}} &= -\tilde{V}(u_x, h), \text{ and } u_h \leqslant 0,\\
u_h &= 0, \text{ if } u_x = (1-\alpha)h^{(1-\alpha)\gamma_1 - 1},
\end{aligned} \tag{4.2.6}
$$

where we define

$$
\tilde{V}(q,h) := \begin{cases} \lambda K^{-\frac{1}{\gamma_2-1}}\frac{1-\gamma_2}{\gamma_2}q^{\frac{\gamma_2}{\gamma_2-1}} + \frac{\nu^{\gamma_1}}{\gamma}h^{(1-\alpha)\gamma_1} - \nu hq, & \text{if } q > \nu^{\gamma_1-1}h^{(1-\alpha)\gamma_1-1},\\ \lambda K^{-\frac{1}{\gamma_2-1}}\frac{1-\gamma_2}{\gamma_2}q^{\frac{\gamma_2}{\gamma_2-1}} + \frac{1-\gamma_1}{\gamma_1}h^{\frac{\alpha\gamma_1}{\gamma_1-1}}q^{\frac{\gamma_1}{\gamma_1-1}}, & \text{if } h^{(1-\alpha)\gamma_1-1} \leqslant q \leqslant \nu^{\gamma_1-1}h^{(1-\alpha)\gamma_1-1},\\ \lambda K^{-\frac{1}{\gamma_2-1}}\frac{1-\gamma_2}{\gamma_2}q^{\frac{\gamma_2}{\gamma_2-1}} + \frac{1}{\gamma_1}h^{(1-\alpha)\gamma_1} - hq, & \text{if } (1-\alpha)h^{(1-\alpha)\gamma_1-1} \leqslant q < h^{(1-\alpha)\gamma_1-1}. \end{cases} \tag{4.2.7}
$$

To solve the equation, some boundary conditions are needed. First, to guarantee the desired global regularity of the solution, we need to impose the smooth-fit condition along two free boundaries such that $u_x(x,h) = \nu^{\gamma_1-1}h^{(1-\alpha)\gamma_1-1}$ and $u_x(x,h) = h^{(1-\alpha)\gamma_1-1}$. Next, note that if we start with initial wealth $x = \frac{\nu h}{r+\lambda}$, to confront the risk of bankruptcy, the optimal investment $\pi^*(x) = -\frac{\mu-r}{\sigma^2}\frac{u_x}{u_{xx}}$ should always be 0. The wealth level never changes as there is no trading strategy, the consumption rate should also be $c_t = \nu h$, and the optimal bequest should also be 0 all the time. Therefore, we can conclude that

$$
\lim_{x\to\frac{\nu h}{r+\lambda}}\frac{u_x(x,h)}{u_{xx}(x,h)} = 0 \text{ and } \lim_{x\to\frac{\nu h}{r+\lambda}} u(x,h) = \int_0^{+\infty} e^{-(r+\lambda)t}\frac{1}{\gamma_1}\left(\frac{\nu h}{h^\alpha}\right)^{\gamma_1} dt = \frac{\nu^{\gamma_1}}{(r+\lambda)\gamma_1}h^{(1-\alpha)\gamma_1}. \tag{4.2.8}
$$

On the other hand, when the initial wealth tends to infinity, one can consume as much as possible, which leads to an infinitely large consumption rate and bequest. A small variation in initial wealth only leads to a negligible change in the value function. In addition, the optimal consumption rate should be proportional to the wealth level in region $\mathcal{D}_2$. It follows that

$$
\lim_{x\to+\infty} u_x(x,h) = 0, \quad \text{and} \quad \lim_{x\to+\infty,\ (x,h)\in\mathcal{D}_2}\frac{h}{x} = C_\infty, \tag{4.2.9}
$$

where $C_\infty > 0$ is a constant. See Corollary 4.1 for the verification of the last boundary condition .

To tackle the nonlinear HJB equation (4.2.6), we employ the dual transform only with respect to the variable x and treat the variable h as a parameter; see similar dual transform arguments in Bo et al. (2021), Deng et al. (2022) and Li et al. (2021). That is, we consider $v(y,h) := \sup_{x \geqslant \frac{\nu h}{r+\lambda}} \{u(x,h) - xy\}$, $y \geqslant (1-\alpha)h^{(1-\alpha)\gamma_1 - 1}$. For a given $(x,h) \in \mathcal{C}$, let us define the variable $y = u_x(x,h)$ and it holds that $u(x,h) = v(y,h) + xy$. We can further deduce that

$$x = -v_y(y,h),\ u(x,h) = v(y,h) - yv_y(y,h) \text{ and } u_{xx}(x,h) = -\frac{1}{v_{yy}(y,h)}.$$

The nonlinear equation (4.2.6) can be reduced to

$$\frac{\kappa^2}{2} y^2 v_{yy} - (r+\lambda)v = -\tilde{V}(y,h), \tag{4.2.10}$$

where $\tilde{V}(\cdot,\cdot)$ is defined in (4.2.7), and the free boundary condition is transformed to the point $y = (1-\alpha)h^{(1-\alpha)\gamma_1 - 1}$. As h can be regarded as a parameter, we can study the above equation as the ODE problem of the variable y. Based on the dual transform, the boundary conditions (4.2.9) can be written as

$$\lim_{y \to 0} v_y(y,h) = -\infty, \quad \text{and} \quad \lim_{h \to \infty} \frac{h}{v_y(y,h)} = -C_\infty, \tag{4.2.11}$$

on free boundary $y = (1-\alpha)h^{(1-\alpha)\gamma_1 - 1}$. The boundary condition (4.2.8) is equivalent to

$$yv_{yy}(y,h) \to 0 \text{ and } v(y,h) - yv_y(y,h) \to \frac{\nu^{\gamma_1}}{(r+\lambda)\gamma_1} h^{(1-\alpha)\gamma_1} \text{ as } v_y(y,h) \to -\frac{\nu h}{r+\lambda}. \tag{4.2.12}$$

The dual transform holds that $v_y(y,h) = -x$, and one can derive that $u_h(x,h) = v_h(y,h) + (v_y(y,h) + x)\frac{dy(h)}{dh} = v_h(y,h)$. The free boundary condition (4.2.6) is written by

$$v_h(y,h) = 0 \ \text{ as } \ y = (1-\alpha)h^{(1-\alpha)\gamma_1 - 1}. \tag{4.2.13}$$

In particular, to facilitate some mathematical arguments, we need to impose the following technical assumption on model parameters. This assumption is needed in deriving the explicit form of coefficient functions $C_i(h)$, $i = 1, ..., 6$, in Proposition 4.1 below. It is also needed in the proof of Lemma 4.1 when we verify that the obtained solution $v(y,h)$ is convex in the variable y and in the proof of the verification theorem on optimality.

Assumption (A1) $\gamma_2 \leqslant (1-\alpha)\gamma_1 < -\frac{r_2}{r_1} \neq \gamma_1$, where $r_1 > 1$ and $r_2 < 0$ are two solutions to the equation $\eta^2 - \eta - \frac{2(r+\lambda)}{\kappa^2} = 0$.

Proposition 4.1. *Under* ***Assumption (A1)****, boundary conditions* (4.2.11)*,* (4.2.12)*, the free boundary condition* (4.2.13)*, and the smooth-fit conditions with respect to y at free boundary points $y = \nu^{\gamma_1 - 1}h^{(1-\alpha)\gamma_1 - 1}$ and $y = h^{(1-\alpha)\gamma_1 - 1}$, the ODE* (4.2.10) *in the domain $\{y \in \mathbb{R} : y \geqslant (1-\alpha)h^{(1-\alpha)\gamma_1 - 1}\}$ admits the unique solution given explicitly by*

$$v(y,h) = \begin{cases} C_2(h)y^{r_2} + \dfrac{2\lambda K^{1-\beta_2}}{\kappa^2\beta_2(\beta_2 - r_1)(\beta_2 - r_2)}y^{\beta_2} & \\ \quad + \dfrac{\nu^{\gamma_1}}{(r+\lambda)\gamma_1}h^{(1-\alpha)\gamma_1} - \dfrac{\nu h}{r+\lambda}y, & \text{if } y > \nu^{\gamma_1 - 1}h^{(1-\alpha)\gamma_1 - 1}, \\ C_3(h)y^{r_1} + C_4(h)y^{r_2} + \dfrac{2\lambda K^{1-\beta_2}}{\kappa^2\beta_2(\beta_2 - r_1)(\beta_2 - r_2)}y^{\beta_2} & \\ \quad + \dfrac{2h^{\alpha\beta_1}}{\kappa^2\beta_1(\beta_1 - r_1)(\beta_1 - r_2)}y^{\beta_1}, & \text{if } h^{(1-\alpha)\gamma_1 - 1} \leqslant y \leqslant \nu^{\gamma_1 - 1}h^{(1-\alpha)\gamma_1 - 1}, \\ C_5(h)y^{r_1} + C_6(h)y^{r_2} + \dfrac{2\lambda K^{1-\beta_2}}{\kappa^2\beta_2(\beta_2 - r_1)(\beta_2 - r_2)}y^{\beta_2} & \\ \quad + \dfrac{1}{(r+\lambda)\gamma_1}h^{(1-\alpha)\gamma_1} - \dfrac{h}{r+\lambda}y, & \text{if } (1-\alpha)h^{(1-\alpha)\gamma_1 - 1} \leqslant y < h^{(1-\alpha)\gamma_1 - 1}, \end{cases} \tag{4.2.14}$$

where $\beta_1 = \frac{\gamma_1}{\gamma_1 - 1}$, $\beta_2 = \frac{\gamma_2}{\gamma_2 - 1}$, *and functions* $C_2(h), C_3(h), \cdots, C_6(h)$ *are given by*

$$\begin{aligned}
C_2(h) &= C_4(h) + \frac{1-\beta_1}{(r+\lambda)(r_1-r_2)(\beta_1-r_2)}\nu^{r_1\gamma_1+r_2}h^{r_1(1-\alpha)\gamma_1+r_2},\\
C_3(h) &= \frac{1-\beta_1}{(r+\lambda)(r_1-r_2)(\beta_1-r_1)}\nu^{r_2\gamma_1+r_1}h^{r_2(1-\alpha)\gamma_1+r_1},\\
C_4(h) &= C_6(h) + \frac{\beta_1-1}{(r+\lambda)(r_1-r_2)(\beta_1-r_2)}h^{r_1(1-\alpha)\gamma_1+r_2},\\
C_5(h) &= C_3(h) - \frac{1-\beta_1}{(r+\lambda)(r_1-r_2)(\beta_1-r_1)}h^{r_2(1-\alpha)\gamma_1+r_1},\\
C_6(h) &= \frac{(1-\alpha)^{r_1-r_2}(1-\beta_1)(r_2(1-\alpha)\gamma_1+r_1)}{(r+\lambda)(r_1-r_2)(\beta_1-r_1)(r_1(1-\alpha)\gamma_1+r_2)}(1-\nu^{r_2\gamma_1+r_1})h^{r_1(1-\alpha)\gamma_1+r_2}
\end{aligned} \tag{4.2.15}$$

where $r_1 > 1$ *and* $r_2 < 0$ *are two roots to the quadratic equation* $\eta^2 - \eta - \frac{2(r+\lambda)}{\kappa^2} = 0$.

Proof. The proof is given in Appendix A.2.1. □

Theorem 4.1 (Verification Theorem)**.** *Let* $(x,h) \in \mathcal{C}$, $h \in \mathbb{R}$ *and* $0 < \lambda < 1$, *where* $x \geqslant 0$ *stands for the initial wealth,* $h \geqslant 0$ *is the initial reference level, and* $\mathcal{C}$ *stands for the effective domain* (4.2.5)*. For* $(y,h) \in \{(y,h) \in \mathbb{R}_+^2 : y \geqslant (1-\alpha)h^{(1-\alpha)\gamma_1-1}\}$, *let us define the feedback functions that*

$$c^\dagger(y,h) = \begin{cases} \nu h, & \text{if } y > \nu^{\gamma_1-1}h^{(1-\alpha)\gamma_1-1},\\ h^{\frac{\alpha\gamma_1}{\gamma_1-1}}u_x^{\frac{1}{\gamma_1-1}}, & \text{if } h^{(1-\alpha)\gamma_1-1} \leqslant y \leqslant \nu^{\gamma_1-1}h^{(1-\alpha)\gamma_1-1},\\ h, & \text{if } (1-\alpha)h^{(1-\alpha)\gamma_1-1} < y < h^{(1-\alpha)\gamma_1-1},\\ \left(\frac{y}{1-\alpha}\right)^{\frac{1}{(1-\alpha)\gamma_1-1}}, & \text{if } y = (1-\alpha)h^{(1-\alpha)\gamma_1-1}, \end{cases} \tag{4.2.16}$$

$$\pi^\dagger(y,h) = \frac{\mu - r}{\sigma^2} y v_{yy}(y,h)$$

$$= \frac{\mu - r}{\sigma^2} \begin{cases} \frac{2(r+\lambda)}{\kappa^2} C_2(h) y^{r_2-1} + \frac{2\lambda K^{1-\beta_2}(\beta_2-1)}{\kappa^2(\beta_2-r_1)(\beta_2-r_2)} y^{\beta_2-1}, & \text{if } y > \nu^{\gamma_1-1} h^{(1-\alpha)\gamma_1-1}, \\ \dfrac{2(r+\lambda)}{\kappa^2} C_3(h) y^{r_1-1} + \dfrac{2(r+\lambda)}{\kappa^2} C_4(h) y^{r_2-1} \\ + \dfrac{2\lambda K^{1-\beta_2}(\beta_2-1)}{\kappa^2(\beta_2-r_1)(\beta_2-r_2)} y^{\beta_2-1} & \text{if } h^{(1-\alpha)\gamma_1-1} \leqslant y \leqslant \nu^{\gamma_1-1} h^{(1-\alpha)\gamma_1-1}, \\ + \dfrac{2(\beta_1-1) h^{\alpha\gamma_1}}{\kappa^2(\beta_1-r_1)(\beta_1-r_2)} y^{\beta_1-1}, \\ \dfrac{2(r+\lambda)}{\kappa^2} C_5(h) y^{r_1-1} + \dfrac{2(r+\lambda)}{\kappa^2} C_6(h) y^{r_2-1} & \text{if } (1-\alpha) h^{(1-\alpha)\gamma_1-1} \leqslant y < h^{(1-\alpha)\gamma_1-1}, \\ + \dfrac{2\lambda K^{1-\beta_2}(\beta_2-1)}{\kappa^2(\beta_2-r_1)(\beta_2-r_2)} y^{\beta_2-1}, \end{cases} \tag{4.2.17}$$

and

$$b^\dagger(y,h) = \left(\frac{y}{K}\right)^{\frac{1}{\gamma_2-1}}. \tag{4.2.18}$$

We consider the process $Y_t(y) := y e^{(r+\lambda)t} M_t$*, where* $M_t := e^{-(r+\lambda+\frac{\kappa^2}{2})t - \kappa W_t}$ *is the discounted rate state price density process, and* $y^* = y^*(x,h)$ *is the unique solution to the budget constraint* $\mathbb{E}[\int_0^\infty (c^\dagger(Y_t(y), H_t^\dagger(y)) + \lambda b^\dagger(Y_t(y), H_t^\dagger(y))) M_t dt] = x$*, where*

$$H_t^\dagger(y) := h \vee \sup_{s\leqslant t} c^\dagger(Y_s(y), H_s^\dagger(y)) = h \vee \left(\inf_{s\leqslant t} Y_s(y)/(1-\alpha)\right)^{\frac{1}{(1-\alpha)\gamma_1-1}},$$

is the optimal reference process corresponding to any fixed $y > 0$*. The value function* $u(x,h)$ *can be attained by employing the optimal consumption and portfolio strategies in the feedback form that* $c_t^* = c^\dagger(Y_t^*, H_t^*)$ *and* $\pi_t^* = \pi^\dagger(Y_t^*, H_t^*)$ *for all* $t \geqslant 0$*, where* $Y_t^* := Y_t(y^*)$ *and* $H_t^* = H_t^\dagger(y^*)$*.*

The process H_t^* *is strictly increasing if and only if* $Y_t^* = (1-\alpha) H_t^{*(1-\alpha)\gamma_1-1}$*. If we have* $y^*(x,h) < (1-\alpha) h^{(1-\alpha)\gamma_1-1}$ *at the initial time, the optimal consumption creates a new peak and brings* $H_{0-}^* = h$ *jumping immediately to a higher level* $H_0^* = \left(\frac{y^*(x,h)}{1-\alpha}\right)^{\frac{1}{(1-\alpha)\gamma_1-1}}$ *such that* $t = 0$ *becomes the only jump time of* H_t^**.*

Proof. The proof is given in Appendix A.2.2. □

Using the dual relationship between u and v, we have the optimal $x = g(\cdot, h) := -v_y(\cdot, h)$. Define $f(\cdot, h)$ as the inverse of $g(\cdot, h)$, then $u(x, h) = v(f(x, h), h) + xf(x, h)$. Note that v has different expressions in the regions $c = 0$, $0 < c < h$ and $c = h$, and the function f should also have piecewise across these regions. By the definition of g, the invertibility of the map $x \mapsto g(x, h)$ is guaranteed by the following lemma.

Lemma 4.1. *Under* ***Assumption (A1)****, the value function* $v(y, h)$ *in* (4.2.14) *is convex in all regions so that the inverse Legendre transform* $u(x, h) = \inf_{y \geqslant (1-\alpha)h^{(1-\alpha)\gamma-1}} [v(y, h) + xy]$ *is well defined. Moreover, it implies that the feedback optimal portfolio* $\pi^*(y, h) > 0$ *all the time.*

Proof. See Appendix A.2.3. □

4.2.3 Optimal Feedback Controls

The main result in this subsection is based on **Assumption (A1)**. Thanks to Lemma 4.1, we can apply the inverse Legendre transform to the solution $v(y, h)$ in (4.2.14). Similar to Section 3.1 in Deng et al. (2022), we can characterize the following four boundary curves $x_{\text{bound}}(h)$, $x_{\text{low}}(h)$, $x_{\text{aggr}}(h)$, and $x_{\text{lavs}}(h)$:

$$
\begin{aligned}
x_{\text{bound}}(h) &:= \frac{\nu h}{r+\lambda},\\
x_{\text{low}}(h) &:= -C_2(h) r_2 \nu^{-r_1(\gamma_1-1)} h^{-r_1((1-\alpha)\gamma_1-1)} - \frac{2\lambda K^{1-\beta_2}\nu^{(\beta_2-1)(\gamma_1-1)}}{\kappa^2(\beta_2-r_1)(\beta_2-r_2)} h^{(\beta_2-1)((1-\alpha)\gamma_1-1)} + \frac{\nu h}{r+\lambda},\\
x_{\text{aggr}}(h) &:= -C_3(h) r_1 h^{-r_2((1-\alpha)\gamma_1-1)} - C_4(h) r_2 h^{-r_1((1-\alpha)\gamma_1-1)}\\
&\quad - \frac{2\lambda K^{1-\beta_2}}{\kappa^2(\beta_2-r_1)(\beta_2-r_2)} h^{(\beta_2-1)((1-\alpha)\gamma_1-1)} - \frac{2}{\kappa^2(\beta_1-r_1)(\beta_1-r_2)} h,\\
x_{\text{lavs}}(h) &:= -C_5(h) r_1 (1-\alpha)^{r_1-1} h^{-r_2((1-\alpha)\gamma_1-1)} - C_6(h) r_2 (1-\alpha)^{r_2-1} h^{-r_1((1-\alpha)\gamma_1-1)}\\
&\quad - \frac{2\lambda(1-\alpha)^{\beta_2-1} K^{1-\beta_2}}{\kappa^2(\beta_2-r_1)(\beta_2-r_2)} h^{(\beta_2-1)((1-\alpha)\gamma_1-1)} + \frac{h}{r+\lambda},
\end{aligned}
\tag{4.2.19}
$$

and it holds that the feedback function of the optimal consumption satisfies: (i) $c^*(x,h) = \nu h$ when $x_{\text{bound}}(h) \leqslant x < x_{\text{low}}(h)$; (ii) $\nu h < c^*(x,h) < h$ when $x_{\text{low}}(h) \leqslant x \leqslant x_{\text{aggr}}(h)$; (iii) $c^*(x,h) = h$ when $x_{\text{aggr}}(h) < x \leqslant x_{\text{lavs}}(h)$. In particular, the condition $u_x(x,h) \geqslant (1-\alpha)h^{(1-\alpha)\gamma_1-1}$ in the effective domain $\mathcal{C}$ in (4.2.5) now can be explicitly expressed as $x \leqslant x_{\text{lavs}}(h)$.

We also define functions $f_1(x,h)$, $f_2(x,h)$ and $f_3(x,h)$ to be the respective solutions to three equations that

$$
\begin{aligned}
x &= -C_2(h) r_2 (f_1(x,h))^{r_2-1} - \frac{2\lambda K^{1-\beta_2} f_1(x,h)^{\beta_2-1}}{\kappa^2(\beta_2-r_1)(\beta_2-r_2)} + \frac{\nu h}{r+\lambda}, \quad \text{if } x_{\text{bound}}(h) \leqslant x < x_{\text{low}}(h),\\
x &= -C_3(h) r_1 (f_2(x,h))^{r_1-1} - C_4(h) r_2 (f_2(x,h))^{r_2-1}\\
&\quad - \frac{2\lambda K^{1-\beta_2} f_2(x,h)^{\beta_2-1}}{\kappa^2(\beta_2-r_1)(\beta_2-r_2)} - \frac{2h^{\alpha\beta_1} f_2(x,h)^{\beta_1-1}}{\kappa^2(\beta_1-r_1)(\beta_1-r_2)}, \qquad \text{if } x_{\text{low}}(h) \leqslant x \leqslant x_{\text{aggr}}(h),\\
x &= -C_5(h) r_1 (f_3(x,h))^{r_1-1} - C_6(h) r_2 (f_3(x,h))^{r_2-1}\\
&\quad - \frac{2\lambda K^{1-\beta_2} f_3(x,h)^{\beta_2-1}}{\kappa^2(\beta_2-r_1)(\beta_2-r_2)} + \frac{h}{r+\lambda}, \qquad \text{if } x_{\text{aggr}}(h) < x \leqslant x_{\text{lavs}}(h).
\end{aligned}
\tag{4.2.20}
$$

The following proposition shows the semi-analytical form for the value function, optimal consumption, and optimal portfolio.

Theorem 4.2. *For $(x,h) \in \{(x,h) \in \mathbb{R}_+^2 : x \geqslant x_{\text{bound}}(h)\}$, $0 < \nu < 1$, $\gamma_1, \gamma_2 > 0$, the value function $u(x,h)$ in (4.1.1) can be expressed in a piecewise form that*

$$u(x,h) = \begin{cases} C_2(h) f_1(x,h)^{r_2} + \dfrac{2\lambda K^{1-\beta_2}}{\kappa^2 \beta_2(\beta_2 - r_1)(\beta_2 - r_2)} f_1(x,h)^{\beta_2} \\ + \dfrac{\nu^\gamma}{(r+\lambda)\gamma_1} h^{(1-\alpha)\gamma} - \dfrac{\nu h}{r+\lambda} f_1(x,h), & \text{if } x_{\text{bound}}(h) \leqslant x < x_{\text{low}}(h), \\ C_3(h) f_2(x,h)^{r_1} + C_4(h) f_2(x,h)^{r_2} + \dfrac{2\lambda K^{1-\beta_2}}{\kappa^2 \beta_2(\beta_2 - r_1)(\beta_2 - r_2)} f_2(x,h)^{\beta_2} \\ + \dfrac{2h^{\alpha\gamma_1}}{\kappa^2 \beta_1(\beta_1 - r_1)(\beta_1 - r_2)} f_2(x,h)^{\beta_1}, & \text{if } x_{\text{low}}(h) \leqslant x \leqslant x_{\text{aggr}}(h), \\ C_5(h) f_3(x,h)^{r_1} + C_6(h) f_3(x,h)^{r_2} + \dfrac{2\lambda K^{1-\beta_2}}{\kappa^2 \beta_2(\beta_2 - r_1)(\beta_2 - r_2)} f_3(x,h)^{\beta_2} \\ + \dfrac{1}{(r+\lambda)\gamma_1} h^{(1-\alpha)\gamma_1} - \dfrac{h}{r+\lambda} f_3(x,h), & \text{if } x_{\text{aggr}}(h) < x \leqslant x_{\text{lavs}}(h), \end{cases} \tag{4.2.21}$$

where the boundaries $x_{\text{bound}}(h)$, $x_{\text{low}}(h)$, $x_{\text{aggr}}(h)$, and $x_{\text{lavs}}(h)$ are given in (4.2.19). Moreover, the feedback optimal consumption and portfolio can also be given in terms of primal variables (x,h) accordingly:

$$c^*(x,h) = \begin{cases} \nu h, & \text{if } x_{\text{bound}}(h) \leqslant x < x_{\text{low}}(h), \\ h^{\frac{\alpha\gamma_1}{\gamma_1 - 1}} f_2(x,h)^{\frac{1}{\gamma_1 - 1}}, & \text{if } x_{\text{low}}(h) \leqslant x \leqslant x_{\text{aggr}}(h), \\ h, & \text{if } x_{\text{aggr}}(h) < x < x_{\text{lavs}}(h), \\ \left(\frac{f_3(x,\tilde{h}(x))}{1-\alpha}\right)^{\frac{1}{(1-\alpha)\gamma_1 - 1}}, & \text{if } x = x_{\text{lavs}}(h), \end{cases} \tag{4.2.22}$$

where $\tilde{h}(x) := x_{\text{lavs}}^{-1}(x)$, *the optimal portfolio*

$$
\begin{aligned}
&\pi^*(x,h)\\
&=\frac{\mu-r}{\sigma^2}\begin{cases}
\frac{2(r+\lambda)}{\kappa^2}C_2(h)f_1(x,h)^{r_2-1}+\frac{2\lambda K^{1-\beta_2}(\beta_2-1)}{\kappa^2(\beta_2-r_1)(\beta_2-r_2)}f_1(x,h)^{\beta_2-1}, & \textit{if } x_{\text{bound}}(h)\leqslant x<x_{\text{low}}(h),\\
\dfrac{2(r+\lambda)}{\kappa^2}C_3(h)f_2(x,h)^{r_1-1}+\dfrac{2(r+\lambda)}{\kappa^2}C_4(h)f_2(x,h)^{r_2-1} & \\
\quad+\dfrac{2\lambda K^{1-\beta_2}(\beta_2-1)}{\kappa^2(\beta_2-r_1)(\beta_2-r_2)}f_2(x,h)^{\beta_2-1} & \textit{if } x_{\text{low}}(h)\leqslant x\leqslant x_{\text{aggr}}(h),\\
\quad+\dfrac{2(\beta_1-1)h^{\alpha\gamma_1}}{\kappa^2(\beta_1-r_1)(\beta_1-r_2)}f_2(x,h)^{\beta_1-1}, & \\
\dfrac{2(r+\lambda)}{\kappa^2}C_5(h)f_3(x,h)^{r_1-1}+\dfrac{2(r+\lambda)}{\kappa^2}C_6(h)f_3(x,h)^{r_2-1} & \\
\quad+\dfrac{2\lambda K^{1-\beta_2}(\beta_2-1)}{\kappa^2(\beta_2-r_1)(\beta_2-r_2)}f_3(x,h)^{\beta_2-1}, & \textit{if } x_{\text{aggr}}(h)<x\leqslant x_{\text{lavs}}(h),
\end{cases}
\end{aligned}
\tag{4.2.23}
$$

and the optimal bequest

$$
b^*(x,h)=\begin{cases}
\left(\frac{f_1(x,h)}{K}\right)^{\frac{1}{\gamma_2-1}}, & \textit{if } x_{\text{bound}}\leqslant x<x_{\text{low}}(h),\\
\left(\frac{f_2(x,h)}{K}\right)^{\frac{1}{\gamma_2-1}}, & \textit{if } x_{\text{low}}\leqslant x<x_{\text{aggr}}(h),\\
\left(\frac{f_3(x,h)}{K}\right)^{\frac{1}{\gamma_2-1}}, & \textit{if } x_{\text{aggr}}<x\leqslant x_{\text{lavs}}(h).
\end{cases}
\tag{4.2.24}
$$

Moreover, for any initial value $(X_0^*, H_0^*) = (x,h) \in \mathcal{C}$, *the stochastic differential equation*

$$
dX_t^* = (r+\lambda)X_t^*dt + \pi^*(\mu-r)dt - c^*dt - \lambda b_t^*dt + \pi^*\sigma dW_t, \tag{4.2.25}
$$

has a unique strong solution under the optimal feedback control (c^*, π^*).

Proof. The proof of Theorem 4.2 is trivial under the results of Theorem 4.1 and the inverse Legendre transform. Moreover, the existence and uniqueness of the strong solution to SDE (4.2.25) follows the same argument in the proof of Proposition 5.1 of Deng et al. (2022). □

Based on Theorem 4.2, we can derive some asymptotic results of the optimal consumption-wealth ratio c_t^*/X_t^* and the invest fraction π_t^*/X_t^* when the wealth is sufficiently large. As wealth $x \to +\infty$, the running maximum h updates to $h = x_{\text{lavs}}^{-1}(x)$ and tends to infinity. Therefore, from the constraint that $x \leqslant x_{\text{lavs}}(h)$, the asymptotic properties of optimal controls as $x \to +\infty$ should be restrained along the boundary curve $x = x_{\text{lavs}}(h)$ as $h \to +\infty$.

Corollary 4.1. *Two limits* $\lim_{h\to+\infty} \frac{c^*(x_{\text{lavs}}(h),h)}{x_{\text{lavs}}(h)}$ *and* $\lim_{h\to+\infty} \frac{\pi^*(x_{\text{lavs}}(h),h)}{x_{\text{lavs}}(h)}$ *exist and are both positive. Meanwhile, the asymptotic behavior of the optimal bequest* $\lim_{h\to+\infty} \frac{b^*(x_{\text{lavs}}(h),h)}{x_{\text{lavs}}(h)}$ *also exists, and is positive if and only if* $\gamma_2 = (1-\alpha)\gamma_1$.

Proof. See Appendix A.2.4. □

Remark. *Contrary to Guasoni et al. (2020), all boundary curves* $x_{\text{low}}(h), x_{\text{aggr}}(h)$ *and* $x_{\text{lavs}}(h)$ *in the present paper are all nonlinear functions of* h*, because the expected bequest and the optimal life insurance control are considered in our problem. If* $\lambda = 0$ *such that there is no life insurance control, the boundary curves become linear functions of the reference variable* h*, and the results are similar to those in Guasoni et al. (2020).*

Remark. *Under optimal control* (c^*, π^*, b^*)*, the wealth process* X_t^* *satisfies the constraint that* $X_t^* \geqslant \frac{\nu H_t^*}{r+\lambda}$ *if the initial condition* $X_0^* = x \geqslant \frac{\nu h}{r+\lambda}$ *is satisfied. Indeed, let* $Z_t^* := X_t^* - \frac{\nu H_t^*}{r+\lambda}$*. If* $Z_t^* = 0$ *at some* $t \geqslant 0$*, the optimal feedback controls satisfy that* $c_t^* = \nu H_t^*$*,* $\pi_t^* = 0$*, and* $b_t^* = 0$*, indicating that* $Z_s^* = 0$ *and* $c_s^* = \nu H_s^*$ *for all* $s \geqslant t$*. That is, the optimal wealth* X_t^* *stays at the level* $\frac{\nu H_t^*}{r+\lambda}$ *once this level is hit.*

Remark. *As wealth* x *tends to the lower bound* $\frac{\nu h}{r+\lambda}$*, the optimal bequest* $b^* \to 0$*, and thus the optimal premium* $p^* = \lambda(b^* - x) < 0$*. If the parameters satisfy* $\kappa^2(\beta_2^2 - 1) \geqslant 2r$*, the optimal premium shall always be negative. If the parameters satisfy*

$\kappa^2(\beta_2^2 - 1) < 2r$, the optimal premium would be positive if $x > x^*$, where x^* satisfies $f(x^*, h) < h^{(1-\alpha)\gamma_1 - 1}$ and

$$\frac{\kappa^2(\beta_2^2 - 1) - 2r}{\kappa^2(\beta_2 - r_1)(\beta_2 - r_2)K^{\beta_2 - 1}} f(x^*, h)^{\beta_2 - 1} = -r_1 C_5(h) f(x^*, h)^{r_1 - 1} - r_2 C_6(h) f(x^*, h)^{r_2 - 1} + \frac{h}{r + \lambda}.$$

4.3 Numerical Illustrations

In this section, we numerically illustrate some quantitative properties of the feedback functions of optimal consumption, investment, and life insurance premium policy established in Theorem 4.2. Let us choose the following values of the model parameters: $r = 0.05$, $\mu = 0.1$, $\sigma = 0.25$, $\rho = 0.05$, $\lambda = 0.03$, $\nu = 0.2$, $\gamma_1 = 0.5$, $\gamma_2 = 0.1$, $\alpha = 0.7$, $K = 5$, and reference level $h = 1$. In the following figures, we only change the value of one parameter (while keeping other parameters fixed) to show some sensitivity results with respect to that parameter.

4.3.1 Boundary Curves

The left panel of Figure 4.2 shows that three boundary curves $x_{\text{low}}(h)$, $x_{\text{aggr}}(h)$, and $x_{\text{lavs}}(h)$ are increasing nonlinear functions of h. The graphs are consistent with the intuition that if the past reference level is higher, the investor would expect larger wealth thresholds to trigger the change of consumption from the low constraint $c = \nu h$ to $c > \nu h$, and from $c < h$ to the historical maximum $c = h$, respectively. From the middle panel, the higher mortality probability motivates the agent to reduce all thresholds and consume more aggressively before the death occurs. It can be seen from the right panel of Figure 4.2 that x_{low}, x_{aggr} and x_{lavs} are all decreasing in the shortfall aversion parameter α, indicating that the more shortfall averse the agent is, the more eager the agent is to consume at the historical maximum level by lowering the corresponding thresholds.

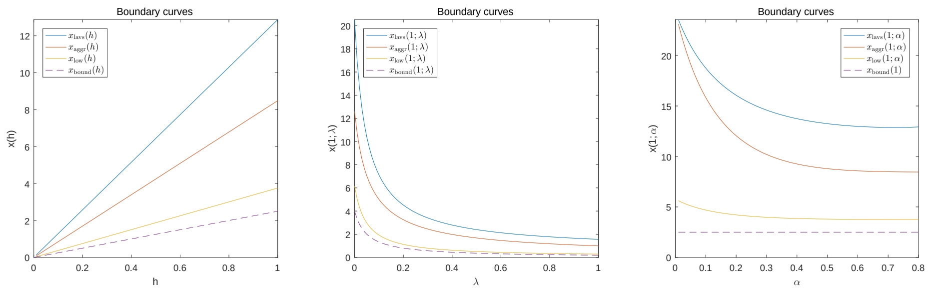

Figure 4.2: Boundary curves x_{bound}, x_{low}, x_{aggr} and x_{lavs} with respect to the reference variable h (left), the force of mortality λ (middle), and the shortfall aversion parameter α (right), respectively.

4.3.2 Sensitivity Analysis

Figures 4.3 to 4.5 show the sensitivity results of optimal controls on the force of mortality λ, the shortfall aversion α and the bequest motive K, respectively. From Figure 4.3, when the wealth level x is sufficiently large, the higher force of mortality motivates the larger optimal consumption and higher optimal insurance premium but results in the lower portfolio allocation in the risky asset. These observations can be explained by the real life situation that the agent spends more cash from the financial market to consume more and purchase more life insurance in view of the higher probability of death. It is interesting to see from Figure 4.4 that a larger shortfall aversion parameter α (i.e., the stronger desire to consume at the historical peak level), leads to a larger optimal insurance premium, which is similar to the observation made in Ben-Arab et al. (1996) that higher consumption habits would increase the demand for life insurance. It is also consistent with two real life observations: (i) the agent who develops a higher standard of living due to a larger α would purchase more life insurance, possibly to ensure that the left family members can afford the high living standard after the death of the agent; (ii) when the agent has sufficient wealth, purchasing more life insurance can also be an effective instrument to reduce some spared cash and smooth out the consumption path so that the reference level does not increase significantly. From Figure 4.5, it is natural to see that the higher bequest motive K yields higher demand for life insurance and lower portfolio allocation. We stress that a higher bequest goal also lowers all consumption thresholds and increases the consumption level. This can be explained by the real life observation that the agent who cares more about life insurance protection is more likely to develop a higher standard of living and consume more aggressively due to a higher reference level.

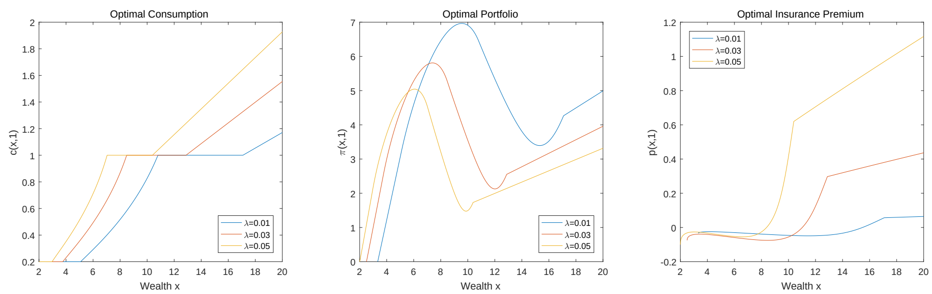

Figure 4.3: Optimal consumption, portfolio and insurance premium for various forces of mortality.

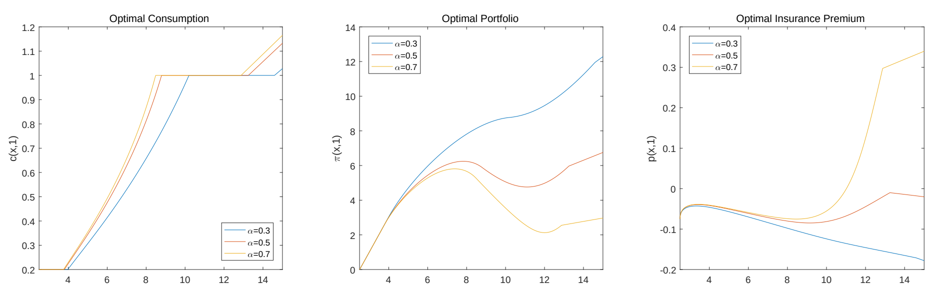

Figure 4.4: Optimal consumption, portfolio and insurance premium for various shortfall aversion.

Figure 4.6 shows the sensitivity results of optimal controls on the drawdown constraint parameter ν. When the wealth level is sufficient such that the drawdown constraint on the consumption rate can be supported, the larger parameter ν increases all thresholds for the consumption plan and leads to a higher past spending maximum when the wealth level is large. Due to the higher minimum consumption

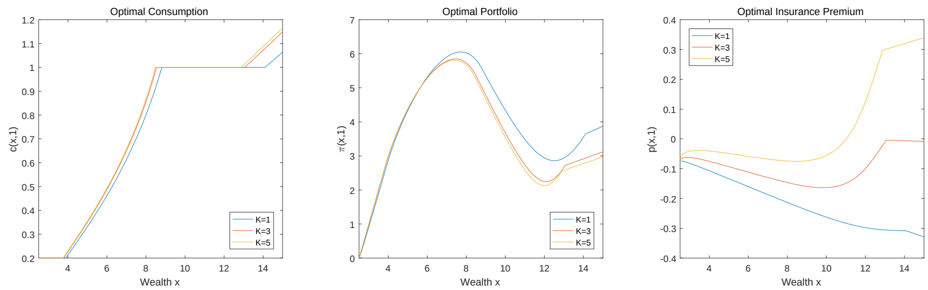

Figure 4.5: Optimal consumption, portfolio and insurance premium for various bequest motives.

rate at the drawdown constraint level and higher consumption when the wealth level is large, it is reasonable to observe that the larger parameter ν reduces the incentives of portfolio allocation and life insurance when wealth is sufficient.

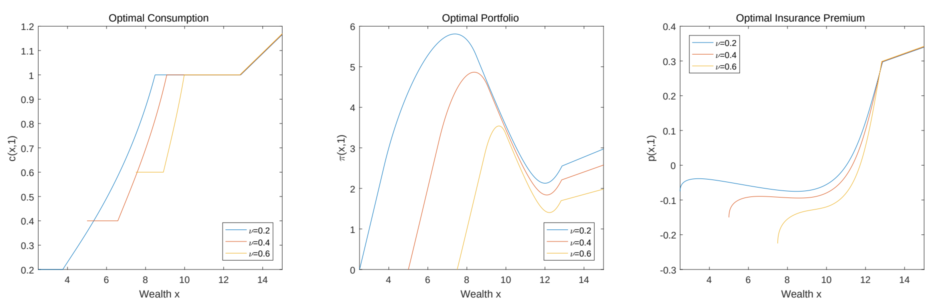

Figure 4.6: Optimal consumption, portfolio and insurance premium for various drawdown constraint parameters.

Figures 4.7 and 4.8 present the simulated paths of the optimal wealth, the optimal consumption, the optimal portfolio, and the optimal life insurance premium in ten years in three different models: 1) our proposed model with life insurance, reference to past spending maximum and drawdown constraint (our model), 2) the shortfall aversion model in Guasoni et al. (2020), and 3) the standard optimal consumption and life insurance model. If we do not consider life insurance and drawdown constraints, that is, $\lambda = 0$ and $\nu = 0$, our model is equivalent to Guasoni's model (Guasoni). Moreover, a nonhabit individual would not be affected by the consumption path in her model and can be characterized by our model if shortfall aversion

$\alpha = 0$ (non-habit). We set the initial wealth to be $X_0 = 3.5$. One can observe that the optimal wealth sample path in the nonhabit model dominates the other two counterparts, and the optimal wealth path in our model grows slowest due to the life insurance purchase and the consumption reference. For the same reasoning, the portfolio allocation in our model is also the least. Regarding the optimal consumption paths, the simulated path in our model is smoother than the other two paths, and the overall consumption level is also highest due to the drawdown constraint. Finally, comparing the demand for life insurance between our model and the nonhabit model, our life insurance premium path becomes much smoother, indicating that the reference to past consumption not only leads to stable consumption behavior, but also helps to smooth out the optimal premium plan.

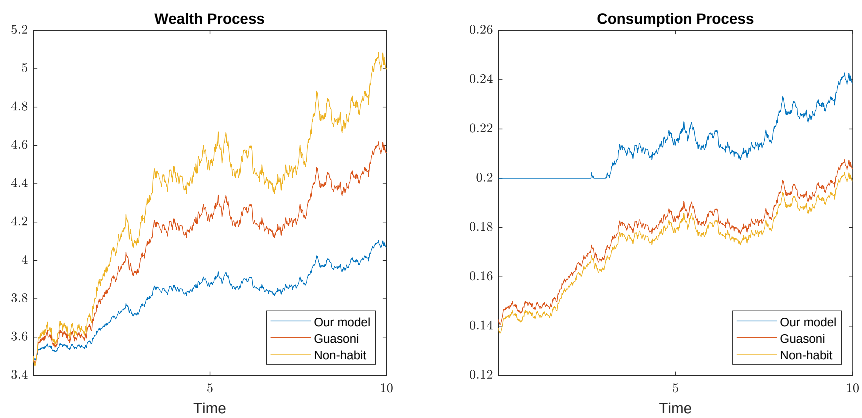

Figure 4.7: Wealth and consumption processes ($X_0 = 3.5$).

4.4 Conclusion

In this chapter, we study the optimal life insurance problem together with dynamic portfolio and consumption plans in a new framework under the shortfall aversion preference and a drawdown constraint on consumption. For the infinite horizon stochastic control problem, we can find the classical solution to the associated HJB equation in piecewise form and characterize the optimal feedback controls explic-

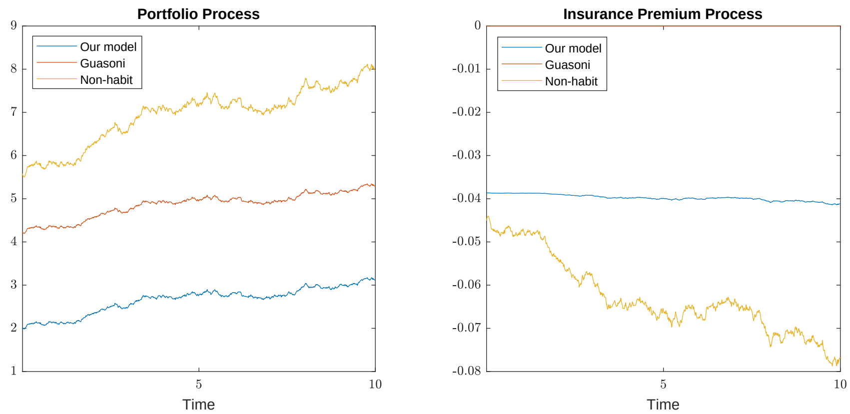

Figure 4.8: Portfolio and life insurance premium processes ($X_0 = 3.5$).

itly across different regions. Thanks to our analytical results, we can numerically illustrate the sensitivity results of the optimal strategies on model parameters and conclude with some interesting financial implications.

Several directions of future research can be considered. For example, one may consider the problem in the market model with regime switching, and discuss some quantitative changes in optimal strategies in bull and bear regime states. It is also appealing to study the more challenging problem over a finite horizon, in which the analytical characterization of the value function is not promising and all boundary curves to distinguish different optimal feedback controls are time-dependent. Some new techniques are needed to tackle the parabolic PDE problem.

Chapter 5

On the Policy Improvement Algorithm for Optimal Consumption and Life Insurance with Habit Formation

In this chapter, we propose a reinforcement learning method to choose the optimal relative consumption, portfolio and life insurance purchasing under habit formation preference. In particular, the risky asset in the market is unknown. The preference measures both the difference between consumption and living standard and expected bequest at the time of death. Any given policy is evaluated by minimizing the martingale loss rather than the classical temporal difference (TD) error. With some necessary estimation for some market coefficient, the policy is updated by our proposed policy improvement theorem (PIT). Our proposed algorithm and results are applied to obtain the optimal portfolio, consumption, and life insurance purchase in the real market.

The remainder of this chapter is organized as follows. Section 5.1 introduces the market model, habit formation, mortality risk, and the stochastic control problem. Section 5.2 proposes the reinforcement learning algorithm based on the solution to the classical stochastic control problem. Simulation studies are conducted to assess the performance of our proposed algorithm in Section 5.3. In section 5.4, we further apply our algorithm to consume, invest and purchase life insurance from July 2021 to June 2022 based on the market in the previous 12 months.

5.1 Problem Formulation

The value function is defined by

$$\begin{aligned} J(x,z;c,\pi,b) &= \mathbb{E}^{\mathcal{F}_0}\left[\int_0^\tau e^{-\rho s}U(c_s - Z_s)ds + Ke^{-\rho\tau}U(b_\tau)ds\right] \\ &= \mathbb{E}^{\mathcal{F}_0}\left[\int_0^\infty e^{-(\rho+\lambda)s}\big(U(c_s - Z_s) + \lambda K U(b_s)\big)ds\right], \end{aligned} \tag{5.1.1}$$

where $\rho > 0$ is the discount factor, $U(x) := \frac{1}{\gamma}x^\gamma$ is the constant relative risk aversion (CRRA) utility function where parameter $\gamma < 1$, and $K > 0$ stands for the bequest motive level. Our problem is to find a control triple $(c^*, \pi^*, b^*) \in \mathcal{A}(x,z)$ such that

$$J(x,z;c^*,\pi^*,b^*) = \sup_{(c,\pi,b)\in\mathcal{A}(x,z)} J(x,z;c,\pi,b) := u(x,z),$$

where $u(x,z)$ is called the value function of our problem.

To ensure that under the optimal policy, the expected utility of consumption flow grows at a rate that is lower than the time preference, then the problem is well-posed, we need the following conditions:

$$0 < \eta < r+\lambda+\delta, \ \ \rho+\lambda-\gamma(r+\lambda)-\frac{\gamma\mu^2}{2(1-\gamma)\sigma^2} > 0, \ \ \text{and} \ \ \lambda K^{\frac{1}{1-\gamma}}\left(\frac{r+\lambda+\delta-\eta}{r+\lambda+\delta}\right)^{\frac{\gamma}{\gamma-1}} < 1. \tag{5.1.2}$$

5.2 Algorithm Design

In this section, we propose an algorithm to obtain the optimal value function, the optimal feedback consumption and investment strategies with unknown μ and σ in the stock price dynamics (2.3.4). To be more specific, we first refer to the classical optimal consumption-investment problem with life insurance if we have knowledge of all the parameters in the model, then propose a policy improvement theorem (PIT) inspired by the solution to the classical problem, and evaluate the policy by minimizing the martingale loss. In practice, the policy is improved by the combination of PIT and the estimation of some model parameters.

5.2.1 Optimality of the classical problem

In this section, we refer to the solution to the classical optimal consumption and life insurance problem with habit formation.

Theorem 5.1. *The optimal value function of problem (5.1.1) is given by*

$$u(x,z) = \frac{dh^{\gamma-1}}{(d+\eta)\gamma}\left(x - \frac{z}{d}\right)^{\gamma}, \tag{5.2.3}$$

where $d := r + \lambda + \delta - \eta$ and h is defined as

$$h := \left(1 - \lambda K^{\frac{1}{1-\gamma}}\left(\frac{d}{d+\eta}\right)^{\frac{\gamma}{\gamma-1}}\right)^{-1}\left(\frac{d}{(d+\eta)(1-\gamma)}\right)\left(\rho + \lambda - \gamma(r+\lambda) - \frac{\gamma\mu^2}{2(1-\gamma)\sigma^2}\right). \tag{5.2.4}$$

Moreover, the optimal feedback control triple is

$$\begin{aligned} c^*(x,z) &= z + h\left(x - \frac{z}{d}\right), \\ \pi^*(x,z) &= \frac{\mu}{(1-\gamma)\sigma^2}\left(x - \frac{z}{d}\right), \\ b^*(x,z) &= \left(\frac{d}{(d+\eta)K}\right)^{\frac{1}{\gamma-1}} h\left(x - \frac{z}{d}\right). \end{aligned} \tag{5.2.5}$$

The associated optimal wealth and habit formation processes under (c^*, π^*) are the *unique solution of the SDE*

$$\begin{aligned} dX_t^* &= (rX_t^* + \pi_t^*\mu - c_t^* - \lambda b_t^*)dt + \pi_t^*\sigma dW_t, \\ dZ_t^* &= (\eta c_t^* - \delta Z_t^*)dt, \quad t \geqslant 0, \end{aligned} \tag{5.2.6}$$

with initial pair $(X_0^*, Z_0^*) = (x, z)$ *satisfying* $x \geqslant \frac{z}{d}$.

Proof. The proof is similar to the proof of Theorem 1 in Constantinides (1990), so we omit it here. □

5.2.2 A policy improvement theorem

In this section, we provide a policy improvement theorem that guarantees that the iterated value functions are nondecreasing, and ultimately converge to the optimal value function. Jacka and Mijatovi (2017) proved the policy improvement theorem for continuous-time stochastic control problems. However, for the completeness of this paper, we still show the proof in this section.

Theorem 5.2 (Policy Improvement Theorem). *Let* $c = c(\cdot,\cdot)$, $\pi = \pi(\cdot,\cdot)$ *and* $b = b(\cdot,\cdot)$ *be a triple of arbitrarily given admissible feedback control policies. Suppose that the corresponding value function* $u^{c,\pi,b}(\cdot,\cdot) \in C^{2,1}(\{(x,z) : x \geqslant \frac{z}{d}\})$ *satisfies* $u_{xx}^{c,\pi,b}(x,z) < 0$, *for any* $x \geqslant \frac{z}{d}$. *Suppose further that the feedback policy triplet* $(\tilde{c}, \tilde{\pi}, \tilde{b})$ *defined by*

$$\tilde{c}(x,z) = z + \left(u_x^{c,\pi,b} - \eta u_z^{c,\pi,b}\right)^{\frac{1}{\gamma-1}}, \quad \tilde{\pi}(x,z) = -\frac{\mu}{\sigma^2}\frac{u_x^{c,\pi,b}}{u_{xx}^{c,\pi,b}}, \quad \tilde{b}(x,z) = \left(\frac{u_x^{c,\pi,b}}{K}\right)^{\frac{1}{\gamma-1}}, \tag{5.2.7}$$

is admissible. Then

$$u^{\tilde{c},\tilde{\pi},\tilde{b}}(x,z) \geqslant u^{c,\pi,b}(x,z), \quad x \geqslant \frac{z}{d} > 0. \tag{5.2.8}$$

Proof. See Appendix A.3.1. □

The optimal feedback control (5.2.5) in Theorem 5.1 suggests that a candidate initial feedback policy triple may take the form $c_0(x,z) = z + a_1\big(x - \frac{z}{d}\big)$, $\pi_0(x,z) = a_2\big(x - \frac{z}{d}\big)$, and $b_0 = a_3\big(x - \frac{z}{d}\big)$. Therefore, such a choice leads to the convergence of both the value functions and the policies in a finite number of iterations theoretically if r, λ, δ, η are known and thus d is known.

Theorem 5.3. *Consider the initial control triple* $c_0(x,z) = z + a_1\big(x - \frac{z}{d}\big)$, $\pi_0(x,z) = a_2\big(x - \frac{z}{d}\big)$, *with* $a_1, a_2 > 0$, *and* $b_0 = a_3\big(x - \frac{z}{d}\big)$. *Denote by* $\{(c_n(x,z), \pi_n(x,z), b_n(x,z)),\ x \geqslant \frac{z}{d} > 0, n \geqslant 1\}$ *the sequence of feedback triples by the policy improvement scheme* (5.2.7), *and* $\{u^{c_n,\pi_n,b_n}, x \geqslant \frac{z}{d} > 0, n \geqslant 1\}$ *is the sequence of the corresponding value functions. Then*

$$\lim_{n\to\infty} c_n(x,z) = c^*(x,z), \quad \lim_{n\to\infty} \pi_n(x,z) = \pi^*(x,z), \quad \lim_{n\to\infty} b_n(x,z) = b^*(x,z), \tag{5.2.9}$$

and

$$\lim_{n\to\infty} u^{c_n,\pi_n,b_n}(x,z) = u(x,z), \tag{5.2.10}$$

for any $x \geqslant \frac{z}{d} > 0$, *where* (c^*, π^*, b^*) *is the optimal policy triplet in* (5.2.5) *and* u *is the value function* (5.2.3).

Proof. See Appendix A.3.2. □

5.2.3 TD error and martingale loss for policy evaluation

For policy evaluation to learn the value function $u^{c,\pi,b}$ under any arbitrarily given admissible feedback policy pair (c, π), we first review the temporal difference (TD) methods, and show that mean square TD error (MSTDE) is not applicable for our problem. We then extend the martingale loss proposed by Jia and Zhou (2022a) in finite horizon for our infinite horizon problem.

By Bellman's consistency, we have

$$e^{-\rho t}u^{c,\pi,b}(x,z) = \mathbb{E}^{\mathcal{F}_t}\left[e^{-\rho s}u^{c,\pi,b}(X_s,Z_s) + \int_t^s e^{-\rho v}\frac{1}{\gamma}\big((c_v - Z_v)^\gamma dv + \lambda K b_v^\gamma\big)dv\right], \quad s \geqslant t, \tag{5.2.11}$$

for $x \geqslant \frac{z}{d} > 0$. Rearranging this equation and dividing both sides by $s-t$, we obtain

$$\mathbb{E}^{\mathcal{F}_t}\left[\frac{e^{-\rho(s-t)}u^{c,\pi,b}(X_s,Z_s) - u^{c,\pi,b}(X_t,Z_t)}{s-t} + \frac{1}{s-t}\int_t^s e^{-\rho(v-t)}\frac{1}{\gamma}\big((c_v - Z_v)^\gamma dv + \lambda K b_v^\gamma\big)dv\right] = 0.$$

Let $s \to t$ on the left-hand side, then we obtain the TD error

$$\epsilon_t := -\rho u^{c,\pi,b}(X_t,Z_t) + \dot{u}^{c,\pi,b}(X_t,Z_t) + \frac{1}{\gamma}(c_t - Z_t)^\gamma + \frac{\lambda K}{\gamma}b_t^\gamma, \tag{5.2.12}$$

where $\dot{u}^{c,\pi,b}(X_t,Z_t) = \frac{u^{c,\pi,b}(X_{t+\triangle t},Z_{t+\triangle t}) - u^{c,\pi,b}(X_t,Z_t)}{\triangle t}$ is the total derivative and $\triangle t$ is the discretization step for the learning algorithm. The objective of the policy evaluation is to minimize the sum square of the TD error ϵ_t. Denote by J^θ and $(c^\varphi, \pi^\varphi, b^\varphi)$ the value function and policy, respectively with θ and φ being the vector of weights to be learned. The objective function is defined as

$$\begin{aligned}\mathcal{R}(\theta,\varphi) &= \frac{1}{2}\mathbb{E}\left[\int_0^\infty e^{-\rho t}|\epsilon_t|^2 dt\right] \\ &= \frac{1}{2}\mathbb{E}\left[\int_0^\infty e^{-\rho t}\left| -\rho J^\theta(X_t,Z_t) + \dot{J}^\theta(X_t,Z_t) + \frac{1}{\gamma}(c_t^\varphi - Z_t)^\gamma + \frac{\lambda K}{\gamma}(b_t^\varphi)^\gamma\right|^2 dt\right],\end{aligned}$$

where $(c^\varphi, \pi^\varphi) = \{c_t^\varphi, \pi_t^\varphi, b_t^\varphi, t \geqslant 0\}$ is generated from $(c^\varphi, \pi^\varphi, b^\varphi)$ in an implementable algorithm.

To make the minimization process applicable, one needs to discretize an infinite time horizon into small equal-length intervals $[t_i, t_{i+1}]$ for $i = 0, 1, \cdots$, with $t_0 = 0$. Then a set of samples $D = \{(x_i, z_i) : i = 0, 1, \cdots\}$ are collected in the following way. The initial sample is (x_0, z_0) for $i = 0$. Then at each time t_i, $i = 0, \cdots, l$, one applies $(c_{t_i}^\varphi, \pi_{t_i}^\varphi)$ to be the consumption and investment in the risky asset, and then observe the wealth x_{i+1} at the next time instant t_{i+1}. Therefore, the discretization version of the infinite horizon mean square TD error is defined as

$$MSTDE_{\triangle t}(\theta,\varphi) := \frac{1}{2}\mathbb{E}\Big[\sum_{i=0}^{\infty} e^{-\rho t_i}\Big(-\rho J^{\theta}(X_{t_i},Z_{t_i}) + \frac{J^{\theta}(X_{t_{i+1}},Z_{t_{i+1}}) - J^{\theta}(X_{t_i},Z_{t_i})}{t_{i+1}-t_i} + \frac{1}{\gamma}(c^{\varphi}_{t_i} - Z_{t_i})^{\gamma} + \frac{\lambda K}{\gamma}(b^{\varphi}_{t_i})^{\gamma}\Big)^2 \triangle t\Big] \tag{5.2.13}$$

Denote by $M^{\varphi}_t = e^{-\rho t}u^{c^{\varphi},\pi^{\varphi}}(X_t,Z_t) + \int_0^t e^{-\rho s}\frac{1}{\gamma}\big((c^{\varphi}_s - Z_s)^{\gamma} + \lambda K(b^{\varphi}_s)^{\gamma}\big)ds$, then

$$\begin{aligned}
&\sum_{i=0}^{\infty} e^{-\rho t_i}\Big(-\rho u^{c^{\varphi},\pi^{\varphi},b^{\varphi}}(X_{t_i},Z_{t_i}) + \frac{u^{c^{\varphi},\pi^{\varphi}}(X_{t_{i+1}},Z_{t_{i+1}}) - u^{c^{\varphi},\pi^{\varphi}}(X_{t_i},Z_{t_i})}{t_{i+1}-t_i} \\
&+ \frac{1}{\gamma}\big((c^{\varphi}_{t_i} - Z_{t_i})^{\gamma} + \lambda K b^{\gamma}_{t_i}\big)\Big)^2 \triangle t \\
=&\frac{1}{\triangle t}\sum_{i=0}^{\infty} e^{-\rho t_i}\Big(-\rho u^{c^{\varphi},\pi^{\varphi}}(X_{t_i},Z_{t_i})\triangle t + u^{c^{\varphi},\pi^{\varphi}}(X_{t_{i+1}},Z_{t_{i+1}}) - u^{c^{\varphi},\pi^{\varphi}}(X_{t_i},Z_{t_i}) \\
&+ \int_{t_i}^{t_{i+1}} \frac{1}{\gamma}\big((c^{\varphi}_s - Z_s)^{\gamma} + \lambda K b^{\varphi}_s\big)ds + O(\triangle t)^2\Big)^2 \\
\approx&\frac{1}{\triangle t}\langle M\rangle_{\infty} = \frac{\sigma^2}{\triangle t}\langle u_x^{c^{\varphi},\pi^{\varphi},b^{\varphi}}(X_t,Z_t)\rangle,
\end{aligned}$$

which is the quadratic variation and thus nonzero. Thus, the minimizer of $MSTDE_{\triangle t}(\theta,\varphi)$ cannot provide a good approximation of the value function.

Because the minimizer of MSTDE minimizes the quadratic variation of the martingale M^{φ}_t, we aim to apply the martingale loss proposed in Jia and Zhou (2022a) to approximate the value function. According to the martingale condition that

$M_t = \mathbb{E}[M_s|\mathcal{F}_t]$ for all $s \geqslant t$, we aim to minimize the following martingale loss

$$
\begin{aligned}
ML(\theta) :=& \frac{1}{2}\|M_\cdot^{\varphi}(\theta) - M_s^{\varphi}(\theta)\|_2^2 \\
=& \frac{1}{2}\mathbb{E}\int_0^{\infty} |M_t^{\varphi}(\theta) - M_s^{\varphi}(\theta)|^2 dt \\
\approx& \frac{1}{2}\mathbb{E}\Bigg[\sum_{i=0}^{\infty}\Bigg(e^{-\rho t_i} J^{\theta}(X_{t_i}, Z_{t_i}) - \sum_{j=i+1}^{\infty} e^{-\rho t_j}\frac{1}{\gamma}\big((c_{t_j}^{\varphi} - Z_{t_j})^{\gamma} + \lambda K b_{t_j}^{\gamma}\big)\Bigg)^2 \triangle t\Bigg] \\
:=& ML_{\triangle t}(\theta),
\end{aligned}
\tag{5.2.14}
$$

as $s \to \infty$, where t_i is a mesh grid in time. Loss function $ML_{\triangle}(\theta)$ does not rely on the knowledge of μ and σ in the system, it is implementable in the algorithm. Moreover, the time horizon we observe cannot be infinite, but we can select a truncated time T that is sufficiently large.

5.2.4 The habit formation algorithms

In this section, we present an algorithm to solve (5.1.1). The algorithm consists of two procedures: policy evaluation and policy improvement. Policy is evaluated by minimizing the martingale loss, and is improved by PIT. By virtue of Theorem 5.1 and Theorem 5.3, we focus on the optimal policy triplet taking the form $c(x, z) = z + \varphi_1\big(x - \frac{z}{d}\big)$ and $\pi(x, z) = \varphi_2\big(x - \frac{z}{d}\big)$, $b(x, z) = \varphi_3(x - \frac{z}{d})$, and denoted by $\varphi = (\varphi_1, \varphi_2, \varphi_3)^{\top}$ the parameters to be learned.

For policy evaluation, as suggested by the theoretical optimal value function (5.2.3) in Theorem 5.2.3, we consider the parameterized J^{θ}, where $\theta > 0$, by

$$
J^{\theta}(x, z) = \theta\left(x - \frac{z}{d}\right)^{\gamma}. \tag{5.2.15}
$$

We can minimize $ML_{\triangle t}(\theta)$ using the gradient descent algorithms to devise the updating rules for θ. Precisely ,we compute

$$\frac{\partial ML_{\triangle t}(\theta)}{\partial\theta}=\sum_{i=0}^{\infty}\Big(e^{-\rho t_i}J^{\theta}(X_{t_i},Z_{t_i})-\sum_{j=i}^{\infty}e^{-\rho t_j}\frac{1}{\gamma}\big((c^{\varphi}_{t_j}-Z_{t_j})^{\gamma}+\lambda K(b^{\varphi}_{t_j})^{\gamma}\big)\Big)\triangle t$$
$$\cdot\, e^{-\rho t_i}\frac{\partial J^{\theta}(X_{t_i},Z_{t_i})}{\partial\theta}, \tag{5.2.16}$$

where $\frac{\partial J^{\theta}(x,z)}{\partial\theta}=\frac{J^{\theta}(x,z)}{\theta}$ due to the form $J^{\theta}(x,z)=\theta\big(x-\frac{z}{d}\big)^{\gamma}$.

From the policy improvement updating scheme (5.2.7), it follows that the optimal consumption $c(x,z)=z+\big(\frac{\theta\gamma(d+\eta)}{d}\big)^{\frac{1}{\gamma-1}}\big(x-\frac{z}{d}\big)$ and the optimal bequest $b(x,z)=\big(\frac{\theta\gamma}{K}\big)^{\frac{1}{\gamma-1}}\big(x-\frac{z}{d}\big)$, indicating that $\varphi_1=\big(\frac{\theta\gamma(d+\eta)}{d}\big)^{\frac{1}{\gamma-1}}$ and $\varphi_3=\big(\frac{\theta\gamma}{K}\big)^{\frac{1}{\gamma-1}}$. Parameter φ_2 may not change by the policy improvement theorem 5.2, however, by the expression $\varphi_2^*=\frac{\mu}{(1-\gamma)\sigma^2}$, a basic idea is to make some estimation for μ or σ and then update φ_2 while updating φ_1 and θ. Statistical estimation is the first choice to estimate μ and σ by the trajectories D. However, although the classical estimator σ is accurate, the estimator for μ may have large variance and is thus not robust, therefore, we aim to update φ_2 under φ_1,θ, and estimation for σ. To be more specific, for wealth, consumption and investment trajectories $\{X_{t_i}\}_{i=0}^{l}$, $\{c^{\varphi}_{t_i}\}_{i=0}^{l}$, $\{\pi^{\varphi}_{t_i}\}_{i=0}^{l}$ and $\{b^{\varphi}_{t_i}\}_{i=0}^{l}$ with $t_l=T>0$, we can first estimate σ^2 by

$$\hat{\sigma}^2=\sum_{i=0}^{l-1}\Big(\frac{X_{t_{i+1}}-(1+r\triangle t)X_{t_i}-c^{\varphi}_{t_i}\triangle t-\lambda b^{\varphi}_{t_i}\triangle t}{\pi^{\varphi}_{t_i}}-\frac{1}{l}\sum_{j=0}^{l-1}\frac{X_{t_{j+1}}-(1+r\triangle t)X_{t_j}-c^{\varphi}_{t_j}\triangle t-\lambda b^{\varphi}_{t_j}\triangle t}{\pi^{\varphi}_{t_j}}\Big)^2, \tag{5.2.17}$$

and define

$$\hat{\mu}^2=\frac{2(1-\gamma)\hat{\sigma}^2}{\gamma d}\cdot\bigg(\rho+\lambda-\gamma(r+\lambda)-\bigg(1-\lambda K^{\frac{1}{1-\gamma}}\Big(\frac{d}{d+\eta}\Big)^{\frac{\gamma}{\gamma-1}}\bigg)(d+\eta)(1-\gamma)\varphi_1\bigg), \tag{5.2.18}$$

by virtue of (5.2.4). Then φ_2 can be updated by $\varphi_2=\frac{\hat{\mu}}{(1-\gamma)\hat{\sigma}^2}$. Based on the discussion above, we propose algorithm 1 to obtain the optimal θ, φ_1, φ_2, and φ_3, and thus the corresponding value function and optimal policy triplet.

Algorithm 1 Algorithm based on PIT

Inputs: Initial wealth and habit formation (x_0, z_0), horizon T, discretization $\triangle t$, learning rate η_θ, and number of episodes n.
Required program: an wealth and reference simulator $(x', z') = Environment_{\triangle t}(t; x, z; c, \pi, b)$ that takes current time-wealth and action (c, π, b) as inputs and generates wealth and reference (x', z') at time $t + \triangle t$.
Learning procedure:

Initialize θ, φ
for $i \in \{1, \cdots, n\}$ **do**
Obtain two local state trajectories $\{(X_{t_i}, Z_{t_i}), i = 0, 1, \cdots, l\}$ by running system (2.3.5) with parameter triplet $\varphi = (\varphi_1, \varphi_2, \varphi_3)$
Update θ (policy evaluation) using (5.2.16) by

$$\theta \leftarrow \theta - \eta_\theta \frac{\partial ML_{\triangle t}(\theta)}{\partial \theta}$$

Update φ_1 and φ_3 (policy improvement) by

$$\varphi_1 \leftarrow \left(\frac{\theta\gamma(d+\eta)}{d}\right)^{\frac{1}{\gamma-1}}, \text{ and } \varphi_3 \leftarrow \left(\frac{\theta\gamma}{K}\right)^{\frac{1}{\gamma-1}}$$

Estimate $\hat{\sigma}$ and determine $\hat{\mu}^2$ using (5.2.17) and (5.2.18) respectively
Update φ_2 (policy improvement) by

$$\varphi_2 \leftarrow \frac{\hat{\mu}}{(1-\gamma)\hat{\sigma}^2}$$

end for

5.3 Simulation Studies

In this section, we use a simulated example to illustrate the feasibility and advantages of our proposed learning algorithm. We compare the performance of our habit formation algorithm (HF) with the conventional least square method (LS). Recall that in the classical LS method, the estimators for μ and σ can be plugged into (5.2.5) and then the corresponding policy is obtained.

We generate 100 datasets from model (2.3.4). This is repeated by the following combinations of μ and σ from the sets $\mu \in \{0.05, 0.1, 0.15\}$ and $\sigma \in \{0.2, 0.25, 0.3\}$.

	μ	σ	θ	φ_1	φ_2	φ_3
HF algorithm			0.3868(0.0007)	4.0087(0.1342)	3.3997(2.8084)	26.7565(0.8957)
LS method	0.05	0.2	0.4166(0.0056)	3.5593(0.5317)	5.6471(6.5414)	23.7573(3.5500)
Theoretical			0.3841	4.0617	2.5	27.1100
HF algorithm			0.3865(0.0007)	4.0166(0.1420)	2.6397(2.2169)	26.8095(0.9475)
LS method	0.05	0.25	0.4144(0.0051)	3.5818(0.5074)	4.2800(5.0979)	23.9071(3.3865)
Theoretical			0.3837	4.0708	1.6	27.1713
HF algorithm			0.3865(0.0007)	4.0146(0.1319)	2.1897(1.8494)	26.7959(0.8805)
LS method	0.05	0.3	0.4131(0.0048)	3.5961(0.4915)	3.4376(4.1725)	24.0024(3.2804)
Theoretical			0.3834	4.0758	1.1111	27.2046
HF algorithm			0.3876(0.0009)	3.9950(0.1697)	3.8156(2.9471)	26.6651(1.1328)
LS method	0.1	0.2	0.4233(0.0053)	3.4544(0.5898)	7.0726(7.0602)	23.0571(3.9367)
Theoretical			0.3878	3.9851	5	26.5992
HF algorithm			0.3877(0.0009)	3.9928(0.1619)	3.0072(2.4254)	26.6505(1.0807)
LS method	0.1	0.25	0.4250(0.0083)	3.4845(0.6088)	5.2825(5.6165)	23.2575(4.0629)
Theoretical			0.3860	4.0219	3.2	26.8444
HF algorithm			0.3867(0.0009)	4.0141(0.1851)	2.3446(1.9179)	26.7929(1.2352)
LS method	0.1	0.3	0.4208(0.0069)	3.5192(0.5738)	4.1081(4.5445)	23.4893(3.8300)
Theoretical			0.3851	4.0418	2.2222	26.9776
HF algorithm			0.3882(0.0009)	3.9825(0.1815)	4.2310(2.9906)	26.5819(1.2111)
LS method	0.15	0.2	0.4405(0.0084)	3.2912(0.7314)	8.9684(7.6968)	21.9676(4.8821)
Theoretical			0.3941	3.8576	7.5	25.7479
HF algorithm			0.3880(0.0008)	3.9864(0.1512)	3.1194(2.3816)	26.6075(1.0090)
LS method	0.15	0.25	0.4292(0.0063)	3.3938(0.6449)	6.4216(5.8667)	22.6521(4.3046)
Theoretical			0.3900	3.9402	4.8	26.2996
HF algorithm			0.3876(0.0009)	3.9950(0.1697)	2.5437(1.9647)	26.6651(1.1328)
LS method	0.15	0.3	0.4233(0.0053)	3.4544(0.5898)	4.7151(4.7068)	23.0571(3.9367)
Theoretical			0.3878	3.9851	3.3333	26.5992

Table 5.1: Learned/Estimated value function parameters and policies (standard errors in brackets) in 100 simulations by habit-formation algorithm and LS method with initial wealth and living standard pair $(x_0, z_0) = (1, 0.02)$.

We choose

$$\lambda = 0.02,\ \rho = 5,\ r = 0.0063,\ \eta = 0.2,\ \delta = 0.3,\ \gamma = 0.5.$$

We generate prices of the risky asset by taking $T = 1$ and $\triangle t = \frac{1}{252}$ with daily rebalancing in horizon $[0, T]$ for the training set and testing set, respectively. We consider the habit formation problem with initial wealth $x_0 = 1$ and living standard $z_0 = 0.02$, and thus $\frac{z_0}{r+\lambda+\delta-\eta} \approx 0.188 < x_0$. Across all the simulations in this section, the learning rate is fixed as $\alpha = 0.02$. Moreover, in some iterated steps using PIT, $\hat{\mu}^2$ may be computed as negative, and we simply let $\hat{\mu} = 0$ in this step.

Table 5.3 summarizes the simulation results for the habit-formation method and LS method, under market scenarios different combinations of μ's and σ's. In each scenario, the value function parameter θ learned by our proposed algorithm is nearer to the true value than that given by the classical LS method and is more robust with lower standard errors. The HF algorithm also proposes optimal consumption and life insurance premium nearer to the true value and lower standard deviation than the LS method. In most scenarios, the HF algorithm also provides a better investment policy with a lower standard deviation. As the expected return μ increases or volatility σ decreases, the theoretical value function becomes slightly larger.

5.4 Real Data Analysis

In this section, we apply the habit-formation algorithm to a real dataset. We also compare the HF algorithm with the policy based on the LS method.

All data used can be freely downloaded from Kenneth French's website `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`. The riskless asset is set to be the treasury bill, and the risky asset is set to be the value-weighted CRSP firms incorporated in the US and listed on the NYSE, AMEX, or NASDAQ that have a CRSP share code of 10 or 11 at the beginning of day t, good shares and price data at the beginning of t, and good return data for t. The train dataset consists of the daily risk-free and excess return rates from July 1, 2020 to June 30, 2021, and the test dataset consists of the return rates from July 1, 2021 to June 30, 2022. To have a better idea about what the data are like, we plot the observations from July 2020 to June 2022, of the value-weighted stock index and treasury bill in Figure 5.1.

We compare the HF algorithm and LS method to consume and invest from July 2021 to June 2022. The risk-free rate is estimated as $r = 0.0063$ using both the

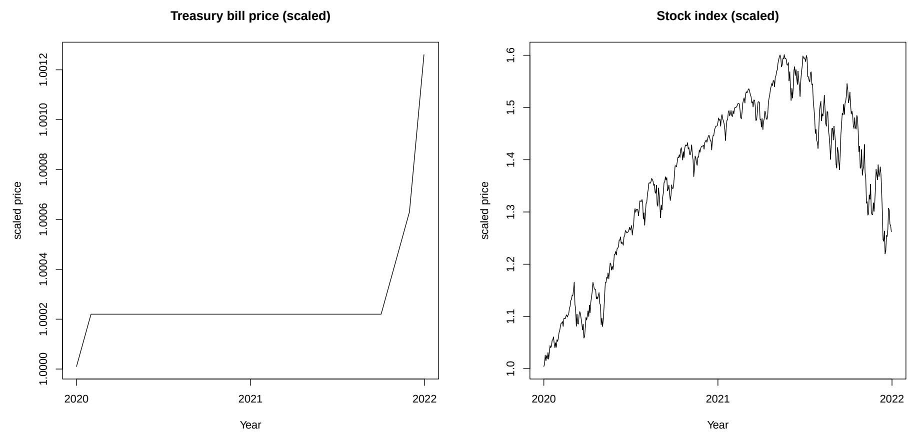

Figure 5.1: Scaled prices of the treasury bill and stock index from July 2020 to June 2022

train and test data, and we simply treat it as a constant in our numerical study. The mortality risk is selected as $\lambda = 0.02$, and the model parameters are selected as $\rho = 5,\ \eta = 0.2,\ \delta = 0.3$, and $\gamma = 0.5$. The learned and estimated optimal policy is trained from July 2020 to June 2021. For each year, we trade on each trading day, which is approximately 252 trading days per year. At the beginning of the year, we assume we have an initial balance of \$100 and a living standard of \$2. Although this initial choice is arbitrary due to the homogeneous property of our problem, it is a useful way of comparing the performance during the course of a year. Similar to section 5.3, time discretization is also selected to be $\triangle t = \frac{1}{252}$, and the time horizon in our study is $T = 1$.

The wealth and living standard trajectories under optimal policies obtained by the HF algorithm and the LS method are plotted in Figure 5.2. The wealth process obtained by the HF algorithm fluctuates less than that obtained by the LS method and has a higher ending than that obtained by the LS method. Moreover, the living standard obtained by the HF algorithm dominates that obtained using the LS method. The consumption investment and premium processes are plotted in Figure 5.3. The optimal consumption, investment, and premium processes all decreased near

0 after the end of 2021, which may due to the fact that the stock index increases in the training year, and decreases in the testing year. Moreover, the consumption and premium processes obtained by the HF algorithm almost dominate those generated by the LS method respectively. The utility obtained by the HF algorithm is 3.8166, which is larger than that obtained by the LS method (3.0377). Although both approaches are too optimistic about the bull market from July 2020 to June 2021 to resist the impact of the bear market from July 2021 to June 2022 on wealth, the HF algorithm still provides higher utility on consumption and bequest.

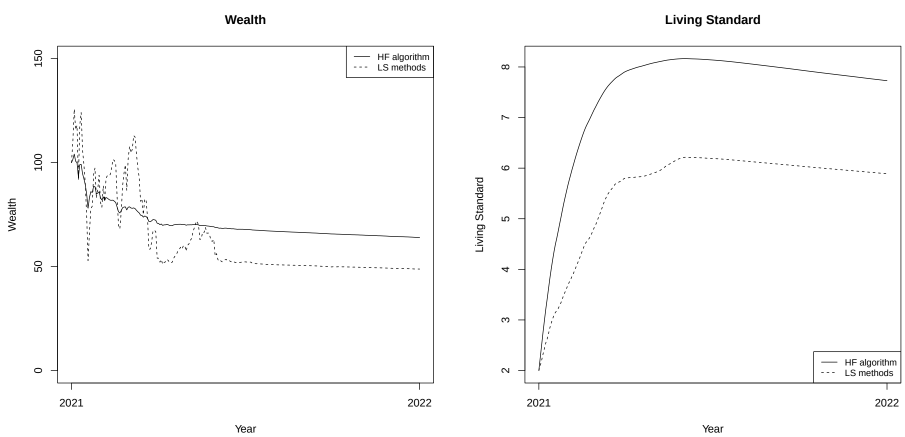

Figure 5.2: Wealth and living standard trajectories from July 2021 to June 2022

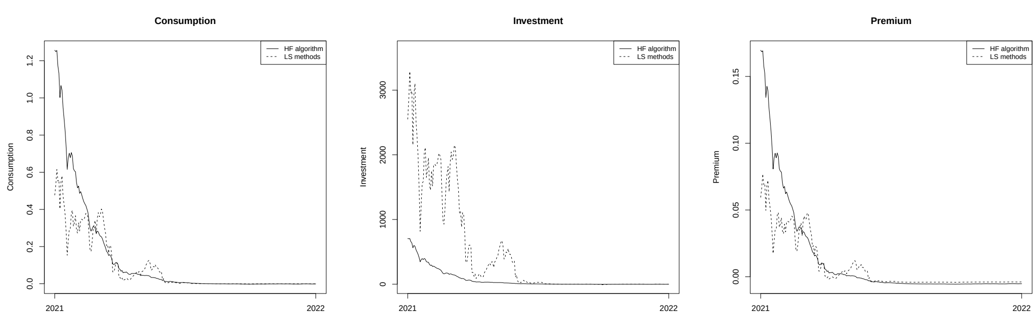

Figure 5.3: Consumption, investment and premium processes from July 2021 to June 2022

Chapter 6

Concluding Remarks

The thesis focuses on three problems related to optimal consumption, investment, and life insurance purchase decisions from the perspective of behavioral finance. The three problems are formulated via utility maximization, in which utility depends on the agents' consumption path. Inspired by prospect theory, the past spending maximum was first considered to be the reference level by the loss aversion agent in Chapter 3. Then, the past spending maximum was applied by a shortfall aversion agent, who also consumes life insurance in her life cycle in Chapter 4. Dynamic programming and the dual transform technique play important roles in addressing both problems. Finally, the classical exponentially weighted average consumption path was considered to be the living standard, and the agent also purchases life insurance during her life without knowing model-based knowledge to the market in Chapter 5. We proposed an actor-critic algorithm under the framework of reinforcement learning, where the value function is learned by minimizing the martingale loss, and a specific policy improvement result.

In Chapter 3, the consumption is limited to be nonnegative, which cannot guarantee the living standard to survive. Therefore, we may further add some constraints on consumption to ensure that the agent can enjoy her living standard. In Chapter 4, the agent is allowed to purchase life insurance from the market, and the drawdown constraint is considered to guarantee the agent's living standard. Both chapters derive the optimal consumption and portfolio for a self-financed agent in the complete market, whose assumptions may not be satisfied in the real world. Therefore, some more realistic extensions can be considered in the future, for instance, similar problems with income, assets following the diffusion process, transaction costs, and borrowing constraints. Moreover, data-driven problems with past spending maximum are also attractive. All these open problems deserve more future efforts separately. In Chapter 5, we tried to obtain the optimal consumption, portfolio, and life insurance purchase without any model-based knowledge of the market. We need to point out that our algorithm is not direct, because our algorithm has to estimate the market parameters in each iteration and then update the parameters in the value function and policy. Some direct reinforcement learning algorithms can be developed and designed in this framework without any estimation for the model parameters, for example, the continuous time reinforcement learning algorithms in Wang and Zhou (2020), Jia and Zhou (2022a), Jia and Zhou (2022b), and Jia and Zhou (2022c) may also be applied to formulate and solve problems in more comprehensive market settings under consumption habit formation preferences.

Appendix A

Proofs

We then show all the proofs of the thesis.

A.1 Proofs for Chapter 3

A.1.1 Proof of Proposition 3.2

It is straightforward to see that the linear ODE (3.2.17) admits the general solution

$$v(y,h) = \begin{cases} C_1(h)y^{r_1} + C_2(h)y^{r_2} - \frac{k}{r\beta_2}(\lambda h)^{\beta_2}, & \text{if } y > y_1(h), \\ C_3(h)y^{r_1} + C_4(h)y^{r_2} + \frac{2}{\kappa^2\gamma_1(\gamma_1-r_1)(\gamma_1-r_2)}y^{\gamma_1} - \frac{\lambda h}{r}y, & \text{if } y_2(h) \leqslant y \leqslant y_1(h), \\ C_5(h)y^{r_1} + C_6(h)y^{r_2} + \frac{1}{r\beta_1}((1-\lambda)h)^{\beta_1} - \frac{h}{r}y, & \text{if } y_3(h) \leqslant y < y_2(h), \end{cases}$$

where $C_1(\cdot), \cdots, C_6(\cdot)$ are functions of h to be determined.

The free boundary condition $v_y(y,h) \to 0$ in (3.2.20) implies that $y \to +\infty$. Together with the free boundary conditions in (3.2.20) and the formula of $v(y,h)$ in the region $y \geqslant y_1(h)$, we deduce that $C_1(h) \equiv 0$. To determine the remaining parameters, we consider the smooth-fit conditions with respect to the variable y

along $y = y_1(h)$ and $y = y_2(h)$ that

$$\begin{aligned}
& - C_3(h)y_1(h)^{r_1} + (C_2(h) - C_4(h))y_1(h)^{r_2} \\
=& \frac{k}{r\beta_2}(\lambda h)^{\beta_2} + \frac{2}{\kappa^2\gamma_1(\gamma_1 - r_1)(\gamma_1 - r_2)} y_1(h)^{\gamma_1} - \frac{\lambda h}{r} y_1(h), \\
& - r_1 C_3(h)y_1(h)^{r_1 - 1} + r_2(C_2(h) - C_4(h))y_1(h)^{r_2 - 1} \\
=& \frac{2}{\kappa^2(\gamma_1 - r_1)(\gamma_1 - r_2)} y_1(h)^{\gamma_1 - 1} - \frac{\lambda h}{r}, \\
& (C_3(h) - C_5(h))y_2(h)^{r_1} + (C_4(h) - C_6(h))y_2(h)^{r_2} \\
=& \frac{1}{r\beta_1}((1-\lambda)h)^{\beta_1} - \frac{2}{\kappa^2\gamma_1(\gamma_1 - r_1)(\gamma_1 - r_2)} y_2(h)^{\gamma_1} - \frac{(1-\lambda)h}{r} y_2(h), \\
& r_1(C_3(h) - C_5(h))y_2(h)^{r_1 - 1} + r_2(C_4(h) - C_6(h))y_2(h)^{r_2 - 1} \\
=& - \frac{2}{\kappa^2(\gamma_1 - r_1)(\gamma_1 - r_2)} y_2(h)^{\gamma_1 - 1} - \frac{(1-\lambda)h}{r}.
\end{aligned} \tag{A.1.1}$$

The equations in (A.1.1) can be treated as linear equations for $C_3(h)$, $C_2(h) - C_4(h)$, and $C_3(h) - C_5(h)$ and $C_4(h) - C_6(h)$. By solving the system of equations, we can obtain

$$\begin{aligned}
C_3(h) &= \frac{y_1(h)^{-r_1}}{r(r_1 - r_2)}\bigg(\frac{kr_2}{\beta_2}(\lambda h)^{\beta_2} + \frac{r_1 r_2}{\gamma_1(\gamma_1 - r_1)} y_1(h)^{\gamma_1} + \lambda h r_1 y_1(h)\bigg), \\
C_2(h) - C_4(h) &= \frac{y_1(h)^{-r_2}}{r(r_1 - r_2)}\bigg(\frac{kr_1}{\beta_2}(\lambda h)^{\beta_2} + \frac{r_1 r_2}{\gamma_1(\gamma_1 - r_2)} y_1(h)^{\gamma_1} + \lambda h r_2 y_1(h)\bigg), \\
C_3(h) - C_5(h) &= \frac{y_2(h)^{-r_1}}{r(r_1 - r_2)}\bigg(- \frac{r_2}{\beta_1}((1-\lambda)h)^{\beta_1} + \frac{r_1 r_2}{\gamma_1(\gamma_1 - r_1)} y_2(h)^{\gamma_1} - (1-\lambda)h r_1 y_2(h)\bigg), \\
C_4(h) - C_6(h) &= \frac{y_2(h)^{-r_2}}{r(r_1 - r_2)}\bigg(\frac{r_1}{\beta_1}((1-\lambda)h)^{\beta_1} - \frac{r_1 r_2}{\gamma_1(\gamma_1 - r_2)} y_2(h)^{\gamma_1} + (1-\lambda)h r_2 y_2(h)\bigg).
\end{aligned}$$

Therefore, $C_2(h)$ to $C_5(h)$ can be expressed by (3.2.23). To solve $C_2(h), C_4(h)$ and $C_6(h)$, we shall find $C_6(h)$ first, and $C_4(h)$ and $C_2(h)$ can then be determined.

Indeed, as $h \to +\infty$, we obtain $y \to 0$ in the region $y_3(h) \leqslant y < y_2(h)$, and the boundary condition (3.2.19) leads to $\lim_{h\to+\infty} \frac{h}{v_y(y_3(h),h)} = C$, where C is a positive constant. Along the free boundary, we have $v_y(y_3(h),h) = r_1 C_5(h) y_3(h)^{r_1-1} + r_2 C_6(h) y_3(h)^{r_2-1} + \frac{h}{r}$. It follows from $\lim_{h\to+\infty} \frac{h}{v_y(y_3(h),h)} > 0$ that $v_y(y_3(h),h) = O(h)$ as $h \to +\infty$. Therefore, we can deduce that $C_6(h) = O(C_5(h) h^{(r_1-r_2)(\beta_1-1)}) + O(h^{r_1(\beta_1-1)+1})$. By Lemma A.1 and the definition $y_3(h) = (1-\lambda)^{\beta_1} h^{\beta_1-1}$, it follows that

$$\begin{aligned} C_6(h) &= O(C_5(h) h^{(r_1-r_2)(\beta_1-1)}) + O(h^{r_1(\beta_1-1)+1}) \\ &= O(h^{(r_1-r_2)(\beta_1-1)+r_2\beta_1+r_1+(\beta_2-\beta_1)}) + O(h^{(r_1-r_2)(\beta_1-1)+r_2\beta_1+r_1}) + O(h^{r_1\beta_1+r_2}) \\ &= O(h^{r_1\beta_2+r_2}) + O(h^{r_1\beta_1+r_2}), \end{aligned}$$

as $h \to +\infty$, where the last equation holds because

$$\min(r_1\beta_1 + r_2, r_1\beta_2 + r_2) \leqslant r_1\beta_1 + r_2 + (\beta_2 - \beta_1) \leqslant \max(r_1\beta_1 + r_2, r_1\beta_2 + r_2).$$

From **Assumption (A1)**, it follows that $\lim_{h\to+\infty} C_6(h) = 0$. Therefore, we can write $C_6(h) = -\int_h^\infty C_6'(s)ds$. We then apply the free boundary condition (3.2.21) at $y_3(h) = (1-\lambda)^{\beta_1} h^{\beta_1-1}$ that

$$C_5'(h) y_3(h)^{r_1} + C_6'(h) y_3(h)^{r_2} + \frac{1}{r}(1-\lambda)^{\beta_1} h^{\beta_1-1} - \frac{1}{r} y_3(h) = 0,$$

which implies the desired result of $C_6(h)$ in (3.2.23).

A.1.2 Proof of Theorem 3.1 (Verification Theorem)

The proof of the verification theorem boils down to show that the solution of the auxiliary HJB equation (3.1.6) coincides with the value function, i.e. there exists $(\pi^*, c^*) \in \mathcal{A}(x)$ such that $\tilde{u}(x,h) = \mathbb{E}[\int_0^\infty e^{-rt} \tilde{U}(c_t^*, H_t^*)dt]$. For any admissible strategy $(\pi, c) \in \mathcal{A}(x)$, we have $\mathbb{E}[\int_0^\infty c_t M_t dt] \leqslant x$ by the supermartingale property and the standard budget constraint argument, see Karatzas et al. (1991).

Let (λ, h) be regarded as parameters, the dual transform of U with respect to c in the constrained domain that $V(q,h) := \sup_{c\geqslant 0}[\tilde{U}(c,h) - cq]$ defined in (3.2.17). Moreover, V can be attained by the construction of the feedback optimal control $c^{\dagger}(y,h)$ in (3.2.24).

In what follows, we distinguish the two reference processes, namely $H_t := h \vee \sup_{s\leqslant t} c_s$ and $H_t^{\dagger}(y) := h \vee \sup_{s\leqslant t} c^{\dagger}(Y_s(y), H_s^{\dagger}(y))$ that correspond to the reference process under an arbitrary consumption process c_t and under the optimal consumption process $c^{\dagger}$ with an arbitrary $y > 0$. Note that the global optimal reference process shall be defined later by $H_t^* := H_t^{\dagger}(y^*)$ with $y^* > 0$ to be determined. Let us now further introduce

$$\hat{H}_t(y) := h \vee \left((1-\lambda)^{-\frac{\beta_1}{\beta_1 - 1}} (\inf_{s\leqslant t} Y_s(y))^{\frac{1}{\beta_1 - 1}} \right), \tag{A.1.2}$$

where $Y_t(y) = ye^{rt}M_t$ is the discounted martingale measure density process.

For any admissible controls $(\pi, c) \in \mathcal{A}(x)$, recall the reference process $H_t = h \vee \sup_{s\leqslant t} c_s$, and for all $y > 0$, we see that

$$\begin{aligned} \mathbb{E}\left[\int_0^{\infty} e^{-rt}\tilde{U}(c_t, H_t)dt\right] &= \mathbb{E}\left[\int_0^{\infty} e^{-rt}(\tilde{U}(c_t, H_t) - Y_t(y)c_t)dt\right] + y\mathbb{E}\left[\int_0^{\infty} c_t M_t dt\right] \\ &\leqslant \mathbb{E}\left[\int_0^{\infty} e^{-rt} V(Y_t(y), H_t^{\dagger}(y))dt\right] + yx \\ &= \mathbb{E}\left[\int_0^{\infty} e^{-rt} V(Y_t(y), \hat{H}_t(y))dt\right] + yx \\ &= v(y,h) + yx, \end{aligned} \tag{A.1.3}$$

the third equation holds because of Lemma A.3, and the last equation is verified by Lemma A.2. In addition, Lemma A.4 guarantees equality with the choice of $c_t^* = c^{\dagger}(Y_t(y^*), H_t^{\dagger}(y^*))$, in which y^* satisfies that $\mathbb{E}\big[\int_0^{\infty} c^{\dagger}(Y_t(y^*), H_t^{\dagger}(y^*))M_t dt\big] = x$

for a given $x \in \mathbb{R}_+$ and $h \geqslant 0$. In conclusion, we have

$$\sup_{(\pi,c)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^\infty e^{-rt}\tilde{U}(c_t, H_t)dt\right] = \inf_{y>0}(v(y,h) + yx) = \tilde{u}(x,h).$$

Then we prove some auxiliary results that have been used above. We first need some asymptotic results on the coefficients in Proposition 3.2.

Lemma A.1. *Based on the semi-analytical forms in Proposition* (3.2)*, we have that*

$$C_2(h) = O\big(h^{\beta_2}w(h)^{-r_2(\beta_1-1)}\big) + O\big(w(h)^{r_1\beta_1+r_2}\big) + O\big(w(h)^{(\gamma_2-r_2)(\beta_1-1)}\big),$$

$$C_3(h) = O\big(w(h)^{r_2\beta_1+r_1+(\beta_2-\beta_1)}\big) + O\big(w(h)^{r_2\beta_1+r_1}\big),$$

$$C_4(h) = O\big(h^{r_1\beta_1+r_2+(\beta_2-\beta_1)}\big) + O\big(h^{r_1\beta_1+r_2}\big),$$

$$C_5(h) = O\big(h^{r_2\beta_1+r_1+(\beta_2-\beta_1)}\big) + O\big(h^{r_2\beta_1+r_1}\big),$$

$$C_6(h) = O\big(h^{r_1\beta_1+r_2+(\beta_2-\beta_1)}\big) + O\big(h^{r_1\beta_1+r_2}\big),$$

where $\gamma_2 = \frac{\beta_2}{\beta_2-1}$*, and* $w(h)$ *is defined in* (3.1.4)*. Moreover, the function* $w(h)$ *satisfies*

$$w(h) = O(h),\ \ w(h)^{-1} = O(h^{-1}) + O(h^{-\frac{\beta_2-1}{\beta_1-1}}),\ \ h = O(w(h)) + O(w(h)^{\frac{\beta_2-1}{\beta_1-1}}),\ \ h^{-1} = O(w(h)^{-1}).$$

Proof. In what follows, C denotes a positive constant, whose value may change from line to line. We first discuss the asymptotic results of $y_1(h)$ and $y_2(h)$. It is easy to see that $(1-\lambda)^{\beta_1}h^{\beta_1-1} = y_3(h) < y_2(h) \leqslant ((1-\lambda)h)^{\beta_1-1}$ and thus $y_2(h) = O(h^{\beta_1-1})$, $y_2(h)^{-1} = O(h^{1-\beta_1})$. Moreover, if $y_1(h) > y_2(h)$, we have $y_1(h) = w(h)^{\beta_1-1}$; if $y_1(h) = y_2(h)$, indicating that $w(h) = (1-\lambda)h$, thus we have $y_1(h) \leqslant ((1-\lambda)h)^{\beta_1-1} = w(h)^{\beta_1-1}$ and $y_1(h) > y_3(h) = (1-\lambda)^{\beta_1}h^{\beta_1-1} = (1-\lambda)w(h)^{\beta_1-1}$. Therefore, we have $y_1(h) = O(w(h)^{\beta_1-1})$ and $y_1(h)^{-1} = O(w(h)^{1-\beta_1})$.

To obtain the asymptotic properties of $C_2(h)$ to $C_6(h)$, we need to derive the asymptotic property of $w(h)$. If $y_1(h) > y_2(h)$, equation (3.1.2) indicates that

$$\frac{1-\beta_1}{\beta_1}\left(\frac{w(h)}{h}\right)^{\beta_1} + \frac{k}{\beta_2}\lambda^{\beta_2}h^{\beta_2-\beta_1} - \lambda\left(\frac{w(h)}{h}\right)^{\beta_1-1} = 0, \qquad \text{(A.1.4)}$$

where $0 < \beta_1 < 1$, $0 < \beta_2 < 1$, $0 < \lambda < 1$ and $h > 0$. We shall obtain the asymptotic property of w in two cases: $\beta_1 < \beta_2$ and $\beta_1 > \beta_2$. In the sequel of the proof below, let $C > 0$ be a generic positive constant independent of (x, h), which may be different from line to line. If $\beta_1 < \beta_2$, as $h \to +\infty$, the second term of equation (A.1.4) goes to infinity, yielding $(\frac{w(h)}{h})^{\beta_1-1} - Ch^{\beta_2-\beta_1} \to 0$ and thus $w \geqslant Ch^{\frac{\beta_2-1}{\beta_1-1}}$; as $h \to 0$, the second term goes to 0, yielding $(\frac{w(h)}{h})^{\beta_1} - C(\frac{w(h)}{h})^{\beta_1-1} \to 0$ and thus $w(h) \geqslant Ch$. If $\beta_1 > \beta_2$, we can similarly obtain that $w(h) \geqslant Ch$ and $w(h) \geqslant Ch^{\frac{\beta_2-1}{\beta_1-1}}$ as h goes to infinity and 0 respectively. Together with the fact that $w(h) \leqslant (1-\lambda)h$, we deduce that

$$h = O(w(h)) + O(w(h)^{\frac{\beta_1-1}{\beta_2-1}}), \text{ and } h^{-1} = O(w(h)^{-1}).$$

If $y_1(h) = w(h)^{\beta_1-1} > y_2(h) = ((1-\lambda)h)^{\beta_1-1}$, then $y_1'(h) = (\beta_1-1)w(h)^{\beta_1-2}w'(h) = O\big(w(h)^{\beta_1-2}w'(h)\big)$, and $y_2'(h) = O\big(h^{\beta_1-2}\big)$. If $y_1(h) = y_2(h) = \frac{k}{\beta_2}\lambda^{\beta_2}h^{\beta_2-1} + \frac{1}{\beta_1}(1-\lambda)^{\beta_1}h^{\beta_1-1}$, then $w(h) = (1-\lambda)h$, $w'(h) = 1-\lambda$, and thus $y_1'(h) = \frac{k}{\gamma_2}\lambda^{\beta_2}h^{\beta_2-2} + \frac{1}{\gamma_1}(1-\lambda)^{\beta_1}h^{\beta_1-1} = O\big(h^{-1}y_1(h)\big) = O(h^{-1}w(h)^{\beta_1-1}) = O(w(h)^{\beta_1-2}w'(h))$, and $y_2'(h) = O\big(\frac{y_2(h)}{h}\big) = O\big(h^{\beta_1-2}\big)$. In summary, we have $y_1'(h) = O(w(h)^{\beta_1-2}w'(h))$ and $y_2'(h) = O\big(h^{\beta_1-2}\big)$.

We further discuss the asymptotic property of $w'(h)$. If $w(h) = (1-\lambda)h$, it is obvious that $w'(h) = 1-\lambda = O(1)$. Otherwise, we have

$$w'(h) = \frac{\lambda}{1-\beta_1} \cdot \frac{w(h)^{\beta_1-1} - k(\lambda h)^{\beta_2-1}}{w(h)^{\beta_1-1} + \lambda h w(h)^{\beta_1-2}}.$$

Since $\lambda h w(h)^{\beta_1-1} > \frac{k}{\beta_2}(\lambda h)^{\beta_2} > k(\lambda h)^{\beta_2}$, we can derive that $w'(h) > 0$, $w'(h) < C$, $w'(h) = O(1)$, and $hw'(h) = O(w(h))$.

Based on the asymptotic property of $y_1(h)$ and $y_2(h)$, we shall find the asymptotic results of $C_2(h)$ to $C_6(h)$. Let us begin with $C_3(h)$ and $C_5(h)$. It is easy to see that

$$C_3(h) = O(w(h)^{r_2\beta_1+r_1+(\beta_2-\beta_1)}) + O(w(h)^{r_2\beta_1+r_1}).$$

Note that

$$C_3(h) = \frac{1}{r(r_1-r_2)}\left\{\frac{kr_2}{\beta_2}(\lambda h)^{\beta_2}y_1(h)^{-r_1} + \frac{r_1r_2}{(\gamma_1-r_1)\gamma_1}y_1(h)^{\gamma_1-r_1} + \lambda hr_1y_1(h)^{r_2}\right\}$$
$$= C^1h^{\beta_2}y_1(h)^{-r_1} + C^2y_1(h)^{\gamma_1-r_1} + C^3hy_1(h)^{r_2},$$

and

$$C_3(h) - C_5(h) = \frac{y_2(h)^{-r_1}}{r(r_1-r_2)}\left(-\frac{r_2}{\beta_1}((1-\lambda)h)^{\beta_1} + \frac{r_1r_2}{\gamma_1(\gamma_1-r_1)}y_2(h)^{\gamma_1} - (1-\lambda)hr_1y_2(h)\right)$$
$$= C^1h^{\beta_1}y_2(h)^{-r_1} + C^2y_2(h)^{\gamma_1-r_1} + C^3hy_2(h)^{r_2},$$

where C^1 to C^3 are discriminant constants in each equation. Then by $y_1(h) = O\big(w(h)^{\beta_1-1}\big)$, $y_1(h)^{-1} = O\big(w(h)^{1-\beta_1}\big) = O\big(h^{1-\beta_1}\big)$, $y_1'(h) = O\big(w(h)^{\beta_1-2}w'(h)\big)$, $y_2(h) = O\big(h^{\beta_1-1}\big)$, $y_2(h)^{-1} = O\big(h^{1-\beta_1}\big)$, $y_2'(h) = O\big(h^{\beta_1-2}\big)$, $w(h) = O(h)$, $w'(h) = O(1)$ and $hw'(h) = O(w(h))$, we have

$$C_3'(h) = C^1h^{\beta_2-1}y_1(h)^{-r_1} + C^2h^{\beta_2}y_1(h)^{-r_1-1}y_1'(h) + C^3y_1(h)^{\gamma_1-r_1-1}y_1'(h)$$
$$+ C^4y_1(h)^{r_2} + hy_1(h)^{r_2-1}y_1'(h)$$
$$= O\big(h^{r_2(\beta_1-1)+(\beta_2-\beta_1)}\big) + O\big(h^{r_2(\beta_1-1)}\big),$$

and

$$C_3'(h) - C_5'(h) = C^1h^{\beta_1-1}y_2(h)^{-r_1} + C^2h^{\beta_1}y_2(h^{-r_1-1}y_2'(h)) + C^3y_2(h)^{\gamma_1-r_1-1}y_2'(h)$$
$$+ C^4y_2(h)^{r_2} + C^5hy_2(h)^{-r_2-1}y_2'(h)$$
$$= O(h^{r_2(\beta_1-1)+(\beta_2-\beta_1)}) + O(h^{r_2(\beta_1-1)}),$$

where C^1 to C^5 are discriminant constants, and thus

$$C_5'(h) = O(h^{r_2(\beta_1-1)+(\beta_2-\beta_1)}) + O(h^{r_2(\beta_1-1)}).$$

Recall that

$$C_6'(h) = -(1-\lambda)^{(r_1-r_2)\beta_1}C_5'(h)h^{(r_1-r_2)(\beta_1-1)} = O(h^{r_1(\beta_1-1)+(\beta_2-\beta_1)}) + O(h^{r_1(\beta_1-1)}).$$

We can obtain the asymptotic property of $C_6(h)$ that

$$C_6(h) = -\int_h^\infty C_6'(h)dh = O(h^{r_1\beta_1+r_2+(\beta_2-\beta_1)}) + O(h^{r_1\beta_1+r_2}).$$

Finally, it follows that

$$C_4(h) = O(h^{r_1\beta_1+r_2+(\beta_2-\beta_1)}) + O(h^{r_1\beta_1+r_2}).$$

and

$$C_2(h) = O(h^{\beta_2}w(h)^{-r_2(\beta_1-1)}) + O(w(h)^{r_1\beta_1+r_2}) + O(w(h)^{(\gamma_2-r_2)(\beta_1-1)}),$$

in view that $h = O(w(h)) + O\big(w(h)^{\frac{\beta_1-1}{\beta_2-1}}\big)$. □

Following similar proofs of Lemma 5.1 and Lemma 5.2 in Deng et al. (2022) and using asymptotic results in Lemma A.1, we can readily obtain the next two lemmas.

Lemma A.2. *For any $y > 0$ and $h \geqslant 0$, the dual transform $v(y,h)$ of the value function $\tilde{u}(x,h)$ satisfies*

$$v(y,h) = \mathbb{E}\bigg[\int_0^\infty e^{-rt}V(Y_t(y), \hat{H}_t(y))dt\bigg],$$

where $V(\cdot,\cdot)$ is defined in (3.2.17), and $Y_t(\cdot)$ and $\hat{H}_t(\cdot)$ are defined in (A.1.2).

Lemma A.3. *Let $V(\cdot,\cdot)$, Y_t, H_t^* and $\hat{H}_t$ be the same as in Lemma A.2, then for all $y > 0$, we have $H_t^\dagger = \hat{H}_t(y)$, $t \geqslant 0$, and hence*

$$\mathbb{E}\bigg[\int_0^\infty e^{-rt}V(Y_t(y), H_t^\dagger(y))dt\bigg] = \mathbb{E}\bigg[\int_0^\infty e^{-rt}V(Y_t(y), \hat{H}_t(y))dt\bigg].$$

Let us then continue to prove some other auxiliary results.

Lemma A.4. *The inequality in (A.1.3) becomes equality with $c_t^* = c^\dagger(Y_t(y^*), \hat{H}_t(y^*))$, $t \geqslant 0$, with $y^* = y^*(x,h)$ as the unique solution to*

$$\mathbb{E}\bigg[\int_0^\infty c^\dagger(Y_t(y^*), \hat{H}_t(y^*))M_t dt\bigg] = x. \tag{A.1.5}$$

Proof. By the definition of V, it is obvious that for all $(\pi, c) \in \mathcal{A}(x)$, $\tilde{U}(c_t, H_t) - Y_t(y)c_t \leqslant V(Y_t(y), H_t)$. Moreover, the inequality becomes an equality with the optimal feedback $c^\dagger(Y_t(y), H_t^\dagger(y))$. Thus, it follows that

$$\int_0^\infty e^{-rt}(\tilde{U}(c_t, H_t) - Y_t(y)c_t)dt \leqslant \int_0^\infty e^{-rt}V(Y_t(y), H_t^\dagger(y))dt.$$

To turn (A.1.3) into an equality, the equality of (A.1.5) needs to hold with some $y^* > 0$ to be determined later, and

$$\tilde{U}(c_t, H_t) - Y_t(y)c_t = V(Y_t(y), H_t) \tag{A.1.6}$$

also needs to hold. Hence, we choose to employ $c^\dagger(Y_t(y), \hat{H}_t(y)) := \hat{H}_t(y)F_t(y, Y_t(y))$, where

$$F_t(y, z) := \mathbb{I}_{\{y_3(\hat{H}_t(y)) \leqslant z < y_2(\hat{H}_t(y))\}} + \left(\lambda + \frac{z^{\frac{1}{\beta_1 - 1}}}{\hat{H}_t(y)}\right)\mathbb{I}_{\{y_2(\hat{H}_t(y)) \leqslant z < y_1(\hat{H}_t(y))\}}.$$

It follows from definition that: (i) If $y \to 0_+$, then $\hat{H}_t(y) \to +\infty$ and $F_t(y, Y_t(y)) > 0$, it indicates that $\mathbb{E}[\int_0^\infty M_t c^\dagger(Y_t(y), \hat{H}_t(y))dt] \to +\infty$; (ii) If $y \to +\infty$, then $\hat{H}_t(y) \to h_+$ and $F_t(y, Y_t(y)) \to 0_+$, it indicates that $\mathbb{E}[\int_0^\infty M_t c^\dagger(Y_t(y), \hat{H}_t(y))dt] \to 0_+$. The existence of y^* can thus be verified if $\mathbb{E}[\int_0^\infty M_t c^\dagger(Y_t(y), \hat{H}_t(y))dt]$ is continuous in y.

Indeed, let $c^\ddagger(Y_t(y), \hat{H}_t(y)) = \max(c^\dagger(Y_t(y), \lambda\hat{H}_t(y)))$, then $\mathbb{E}[\int_0^\infty M_t c^\ddagger(Y_t(y), \hat{H}_t(y))dt]$ exists and is continuous in y, and

$$\mathbb{E}[\int_0^\infty M_t c^\dagger(Y_t(y), \hat{H}_t(y))dt] = \mathbb{E}[\int_0^\infty M_t c^\ddagger(Y_t(y), \hat{H}_t(y))\mathbf{1}\{Y_t(y) \leqslant y_1(\hat{H}_t(y))\}dt].$$

Therefore, $\mathbb{E}[\int_0^\infty M_t c^\dagger(Y_t(y), \hat{H}_t(y))dt]$ is also continuous in y. □

Lemma A.5. *The following transversality condition holds that for all $y > 0$,*

$$\lim_{T \to +\infty} \mathbb{E}\left[e^{-rT}v(Y_T(y), \hat{H}_T(y))\right] = 0.$$

Proof. Recall that $\hat{H}_t(y) := h \vee \left((1-\lambda)^{-\frac{\beta_1}{\beta_1-1}}(\inf_{s\leqslant t} Y_s(y))^{\frac{1}{\beta_1-1}}\right)$. In this proof, the results in Lemma A.7 and Lemma A.8 are applied repeatedly, therefore, we omit the illustrations if there is no ambiguity. In more detail, we use Lemma A.7 with $\beta = \beta_2 \geqslant \min(\beta_1, \beta_2)$, and use Lemma A.8 with $\gamma = \gamma_1, \gamma_2$, and $\frac{\beta_2}{\beta_1-1}$ since $r_1 > 0 > \frac{\beta_2}{\beta_1-1} \geqslant \min(\gamma_1, \gamma_2) > r_2$, which can be obtained by some simple computations.

Let us first consider the case $c_T = 0$. We first write that

$$e^{-rT}\mathbb{E}[v(Y_T(y), \hat{H}_T(y))] = e^{-rT}\mathbb{E}\left[C_2(\hat{H}_T(y))Y_T(y)^{r_2} - \frac{k}{r\beta_2}(\lambda \hat{H}_T(y))^{\beta_2}\right], \quad \text{(A.1.7)}$$

in which the second term converges to 0 as $T \to +\infty$ due to Lemma A.7. For the first term in (A.1.7), since $Y_T(y) > y_1(\hat{H}_T(y)) \geqslant w_T(y)^{\beta_1-1}$, we have

$$\begin{aligned} e^{-rT}\mathbb{E}\left[C_2(\hat{H}_T(y))(Y_T(y))^{r_2}\right] &= e^{-rT}O(\mathbb{E}[\hat{H}_T^{\beta_2}(y) w_T^{-r_2(\beta_1-1)} Y_T(y)^{r_2}]) \\ &\quad + e^{-rT}O(\mathbb{E}[w_T(y)^{r_1\beta_1+r_2} Y_T(y)^{r_2}]) \\ &\quad + e^{-rT}O(\mathbb{E}[w_T^{(\gamma_2-r_2)(\beta_1-1)} Y_T(y)^{r_2}]) \\ &= e^{-rT}O(\mathbb{E}[\hat{H}_T^{\beta_2}(y)]) + e^{-rT}O(\mathbb{E}[Y_T(y)^{\gamma_1}]) \\ &\quad + e^{-rT}O(\mathbb{E}[Y_T(y)^{\gamma_2}]), \end{aligned}$$

which vanishes as $T \to +\infty$ due to Lemma A.7 and Lemma A.8.

We then consider the case $0 < c_T < \hat{H}_T(y)$. In this case, $y_2(\hat{H}_T(y)) \leqslant Y_T(y) \leqslant y_1(\hat{H}_T(y))$, and thus

$$\begin{aligned} e^{-rT}v(Y_T(y), \hat{H}_T(y)) = e^{-rT}\Bigg[&C_3(\hat{H}_T(y))Y_T(y)^{r_1} + C_4(\hat{H}_T(y))Y_T(y)^{r_2} \\ &+ \frac{2}{\kappa^2\gamma_1(\gamma_1-r_1)(\gamma_1-r_2)}Y_T(y)^{\gamma_1} - \frac{\lambda \hat{H}_T(y)}{r}Y_T(y)\Bigg]. \end{aligned} \quad \text{(A.1.8)}$$

We consider asymptotic behavior of the above equation term by term as $T \to +\infty$.

The third term in (A.1.8) clearly converges to 0 by Lemma A.7. For the fourth term in (A.1.8), since $Y_T(y) \leqslant y_1(\hat{H}_T(y)) = O(\hat{H}_T(y)^{\beta_2-1}) + O(\hat{H}_T(y)^{\beta_1-1})$, we have

$$\mathbb{E}[e^{-rT}Y_T(y)\hat{H}_T(y)] = e^{-rT}O(\mathbb{E}[\hat{H}_T(y)^{\beta_2}]) + e^{-rT}O(\mathbb{E}[\hat{H}_T(y)^{\beta_1}]),$$

which also vanishes as $T \to +\infty$ by Lemma A.7.

Let us continue to consider the terms containing $C_3(\hat{H}_T(y))$ and $C_4(\hat{H}_T(y))$ in equation (A.1.8). Because of the constraint $w_T(y) = O(Y_t(y)^{\frac{1}{\beta_1-1}})$ due to $Y_t(y) \leqslant y_1(\hat{H}_T(y)) \leqslant w_T(y)^{\beta_1-1}$ which is discussed in the proof of Remark (A.1), we can deduce that

$$\begin{aligned}
&e^{-rT}\mathbb{E}\bigg[C_3(\hat{H}_T(y))(Y_T(y))^{r_1}\bigg] \\
=&e^{-rT}O(\mathbb{E}[w_T(y)^{r_1+r_2\beta_1+(\beta_2-\beta_1)}Y_T(y)^{r_1}]) + e^{-rT}O(\mathbb{E}[w_T(y)^{r_1+r_2\beta_1}(Y_T(y))^{r_1}]) \\
=&e^{-rT}O(\mathbb{E}[Y_T(y)^{\frac{\beta_2}{\beta_1-1}}]) + e^{-rT}O(\mathbb{E}[Y_T(y)^{\gamma_1}]),
\end{aligned}$$

which converges to 0 by Lemma A.8.

In addition, from $Y_T(y) \geqslant (1-\lambda)^{\beta_1}\hat{H}_T(y)^{\beta_1-1}$, it follows that $\hat{H}_T(y)^{-1} = O(Y_T(y)^{\frac{1}{1-\beta_1}})$, and thus

$$\begin{aligned}
&e^{-rT}\mathbb{E}\bigg[C_4(\hat{H}_T(y))(Y_T(y))^{r_2}\bigg] \\
=&e^{-rT}O(\mathbb{E}[\hat{H}_T(y)^{r_1\beta_1+r_2+(\beta_2-\beta_1)}Y_T(y)^{r_2}]) + e^{-rT}O(\mathbb{E}[\hat{H}_T(y)^{r_1\beta_1+r_2}(Y_T(y))^{r_2}]) \\
=&e^{-rT}O(\mathbb{E}[Y_T(y)^{\frac{\beta_2}{\beta_1-1}}]) + e^{-rT}O(\mathbb{E}[Y_T(y)^{\gamma_1}]),
\end{aligned}$$

which vanishes as $T \to +\infty$ by Lemma A.8.

Finally, we consider the case $c_T = \hat{H}_T(y)$ and write that

$$\begin{aligned}
e^{-rT}v(Y_T(y), \hat{H}_T(y)) = e^{-rT}\bigg(&C_5(\hat{H}_T(y))Y_T(y)^{r_1} + C_6(\hat{H}_T(y))Y_T(y)^{r_2} \\
&+ \frac{1}{r\beta_1}((1-\lambda)\hat{H}_T(y))^{\beta_1} - \frac{1}{r}\hat{H}_T(y)Y_T(y)\bigg).
\end{aligned} \tag{A.1.9}$$

In this case, similar to the discussion for (A.1.8), we have $w_T(y) = O(Y_T(y)^{\frac{1}{\beta_1 - 1}})$.

The last two terms of the right-hand side in equation (A.1.9), similar to the last two terms of the right-hand side in equation (A.1.8), also converge to 0 as $T \to +\infty$.

For the first term in (A.1.9), by Remark A.1, we have

$$\begin{aligned} e^{-rT} C_5(\hat{H}_T(y)) Y_T(y)^{r_1} &= e^{-rT} \Big(O(w_T(y)^{r_2\beta_1 + r_1 + (\beta_2 - \beta_1)}) + O(w_T(y)^{r_2\beta_1 + r_1}) \Big) Y_T(y)^{r_1} \\ &= e^{-rT} O\Big(Y_T(y)^{\frac{\beta_2}{\beta_1 - 1}} \Big) + e^{-rT} O\Big(Y_T(y)^{\gamma_1} \Big), \end{aligned}$$

which converges to 0 as $T \to +\infty$ by Lemma A.8.

For the second term in (A.1.9), by Remark A.1, we have

$$\begin{aligned} e^{-rT} C_6(\hat{H}_T(y)) Y_T(y)^{r_2} &= e^{-rT} \Big(O(\hat{H}_T(y)^{r_1\beta_1 + r_2 + (\beta_2 - \beta_1)}) + O(\hat{H}_T(y)^{r_1\beta_1 + r_2})) \Big) Y_T(y)^{r_1} \\ &= e^{-rT} O\Big(Y_T(y)^{\frac{\beta_2}{\beta_1 - 1}} \Big) + e^{-rT} O\Big(Y_T(y)^{\gamma_1} \Big), \end{aligned}$$

which also vanishes as $T \to +\infty$ by Lemma A.8. Therefore, we obtain the desired result. □

Lemma A.6. *For any* $T > 0$, *we have*

$$\lim_{n \to +\infty} \mathbb{E}\big[e^{-r\tau_n} v(Y_{\tau_n}(y), \hat{H}_{\tau_n}(y)) \mathbf{1}_{\{T > \tau_n\}} \big] = 0. \tag{A.1.10}$$

Proof. By the definition of τ_n, for all $t \leqslant \tau_n$, we have $Y_t(y) \in \big[\frac{1}{n}, n\big]$, and thus

$$\hat{H}_t(y) \leqslant \max(h, (1-\lambda)^{-\frac{\beta_1}{\beta_1 - 1}} n^{1-\beta_1}) = O(1) + O(n^{1-\beta_1}).$$

Therefore, we have that $Y_t(y)^{r_1} \leqslant n^{r_1}$, $Y_t(y)^{r_2} \leqslant \big(\frac{1}{n}\big)^{r_2} = n^{-r_2}$. Together with the fact that $r_1 > 0 > \max\{\gamma_1, \gamma_2\} \geqslant \min\{\gamma_1, \gamma_2\} > r_2$ by Assumption (A1), we shall show the order of $v(Y_{\tau_n}(y), \hat{H}_{\tau_n}(y))$ in cases when $c^*_{\tau_n} = 0$, $0 < c^*_{\tau_n} < \hat{H}_{\tau_n}(y)$, and $c^*_{\tau_n} = \hat{H}_{\tau_n}(y)$.

Similar to the proof of Lemma A.5, if $c^*_{\tau_n} = 0$, we have that

$$\begin{aligned} v(Y_{\tau_n}(y), \hat{H}_{\tau_n}(y)) &= C_2(\hat{H}_{\tau_n}(y))Y_{\tau_n}(y)^{r_2} - \frac{k}{r\beta_2}(\lambda \hat{H}_{\tau_n}(y))^{\beta_2} \\ &= O(1) + O(n^{\beta_2(1-\beta_1)}) + O(n^{-\gamma_1}) + O(n^{-\gamma_2}) \\ &= O(n^{-r_2}). \end{aligned}$$

If $0 < c^*_{\tau_n} < \hat{H}_{\tau_n}(y)$, we have that

$$\begin{aligned} v(Y_{\tau_n}(y), \hat{H}_{\tau_n}(y)) =& C_3(\hat{H}_{\tau_n}(y))Y_{\tau_n}(y)^{r_1} + C_4(\hat{H}_{\tau_n}(y))Y_{\tau_n}(y)^{r_2} \\ &+ \frac{2}{\kappa^2\gamma_1(\gamma_1 - r_1)(\gamma_1 - r_2)}Y_{\tau_n}(y)^{\gamma_1} - \frac{\lambda \hat{H}_{\tau_n}(y)}{r}Y_{\tau_n}(y) \\ &= O(1) + O(n^{\beta_2(1-\beta_1)}) + O(n^{\beta_1(1-\beta_1)}) + O(n^{\frac{\beta_2}{1-\beta_1}}) + O(n^{-\gamma_1}) \\ &= O(n^{-r_2}). \end{aligned}$$

If $c_{\tau_n} = \hat{H}_{\tau_n}(y)$, we have that

$$\begin{aligned} v(Y_{\tau_n}(y), \hat{H}_{\tau_n}(y)) &= C_5(\hat{H}_{\tau_n}(y))Y_{\tau_n}(y)^{r_1} + C_6(\hat{H}_{\tau_n}(y))Y_{\tau_n}(y)^{r_2} \\ &\quad + \frac{1}{r\beta_1}((1-\lambda)\hat{H}_{\tau_n}(y))^{\beta_1} - \frac{1}{r}\hat{H}_{\tau_n}(y)Y_{\tau_n}(y) \\ &= O(n^{-r_2}). \end{aligned}$$

In conclusion, in all cases, $v(Y_{\tau_n}(y), \hat{H}_{\tau_n}(y)) = O(n^{-r_2})$. In addition, similar to the proof of (A.25) in Guasoni et al. (2020), there exists some constant C such that $\mathbb{E}[\mathbf{1}_{\{\tau \leqslant T\}}] \leqslant n^{-2\xi}(1 + y^{2\xi})e^{CT}$, for any $\xi \geqslant 1$. Putting all the pieces together, we obtain the desired claim (A.1.10). □

Lemma A.7. *For $\beta \in \{\beta_1, \beta_2\}$, we have*

$$\lim_{T\to+\infty} \mathbb{E}\left[e^{-rT}\hat{H}_T(y)^{\beta}\right] = 0. \tag{A.1.11}$$

Proof. It is obvious that

$$e^{-rT}\mathbb{E}\Big[\hat{H}_T(y)^\beta\Big] \leqslant e^{-rT}\mathbb{E}\Big[\sup_{s\leqslant T} Y_s(y)^{\frac{\beta}{\beta_1-1}}(1-\lambda)^{-\frac{\beta_1\beta}{\beta_1-1}}\Big] + e^{-rT}\mathbb{E}[h^\beta],$$

and it is clear that $e^{-rT}\mathbb{E}[h^\beta] = O(e^{-rT})$ as $T \to +\infty$.

Let us define $W_t^{(\frac{1}{2}\kappa)} := W_t + \frac{1}{2}\kappa t$ with its running maximum $\Big(W_t^{(\frac{1}{2}\kappa)}\Big)^*$. It follows that

$$\begin{aligned} & e^{-rT}\mathbb{E}\Big[\sup_{s\leqslant T} Y_s(y)^{\frac{\beta}{\beta_1-1}}(1-\lambda)^{-\frac{\beta_1\beta}{\beta_1-1}}\Big] \\ =& e^{-rT} O\bigg(\mathbb{E}\bigg[\exp\bigg\{aW_T^{(\zeta)} + b\Big(W_T^{(\zeta)}\Big)^* \mathbb{I}\bigg\{\Big(W_T^{(\zeta)}\Big)^* \geqslant k\bigg\}\bigg\}\bigg]\bigg), \end{aligned}$$

where $a = 0$, $b = -\frac{\beta}{\beta_1-1}\kappa > 0$, $\zeta = \frac{1}{2}\kappa > 0$, and $k = 0$. Note that $2a + b + 2\zeta > 2a + b + \zeta > 0$, thanks to Corollary A.7 in Guasoni et al. (2020), we have that

$$\begin{aligned} & \mathbb{E}\bigg[\exp\bigg\{aW_T^{(\zeta)} + b\Big(W_T^{(\zeta)}\Big)^*\bigg\}\mathbf{1}_{\{(W_T^{(\zeta)})^* \geqslant k\}}\bigg] \\ =& \frac{2(a+b+\zeta)}{2a+b+2\zeta}\exp\bigg\{\frac{(a+b)(a+b+2\zeta)}{2}T\bigg\}\Phi\bigg((a+b+\zeta)\sqrt{T} - \frac{k}{\sqrt{T}}\bigg) \\ &+ \frac{2(a+\zeta)}{2a+b+2\zeta}\exp\bigg\{(2a+b+2\zeta)k + \frac{a(a+2\zeta)}{2}T\bigg\}\Phi\bigg(-(a+\zeta)\sqrt{T} - \frac{k}{\sqrt{T}}\bigg), \end{aligned}$$

and therefore

$$\begin{aligned} & \lim_{T\to+\infty}\frac{1}{T}\log\mathbb{E}\bigg[\exp\bigg\{aW_T^{(\zeta)} + b\Big(W_T^{(\zeta)}\Big)^*\bigg\}\mathbf{1}_{\{(W_T^{(\zeta)})^* \geqslant k\}}\bigg] - r \\ =& \frac{(a+b)(a+b+2\xi)}{2} - r = \frac{\kappa^2}{2}\gamma_0(\gamma_0 - 1) - r = \frac{\kappa^2}{2}(\gamma_0 - r_1)(\gamma_0 - r_2), \end{aligned}$$

where $\gamma_0 = \frac{\beta}{\beta_1 - 1}$. It thus holds that

$$
\begin{aligned}
& e^{-rT}\mathbb{E}\Bigg[\exp\left\{aW_T^{(\zeta)} + b\Big(W_T^{(\zeta)}\Big)^*\mathbf{1}_{\left\{\left(W_T^{(\zeta)}\right)^* \geqslant k\right\}}\right\}\Bigg] \\
= & \exp\left\{\left(\frac{1}{T}\log\mathbb{E}\Bigg[\exp\left\{aW_T^{(\zeta)} + b\Big(W_T^{(\zeta)}\Big)^*\right\}\mathbf{1}_{\left\{\left(W_T^{(\zeta)}\right)^* \geqslant k\right\}}\Bigg] - r\right)T\right\} \\
= & O\left(\exp\left\{\frac{\kappa^2}{2}(\gamma_0 - r_1)(\gamma_0 - r_2)T\right\}\right),
\end{aligned}
$$

as $T \to +\infty$. Thanks to Assumption (A1), we have $r_1 > 0 > \gamma_0 \geqslant \min(\gamma_1, \gamma_2) > r_2$. It follows that $(\gamma_0 - r_1)(\gamma_0 - r_2) < 0$ and thus

$$
\mathbb{E}\left[e^{-rT}\hat{H}_T(y)^\beta\right] = O\left(\exp\left\{\frac{\kappa^2}{2}(\gamma_0 - r_1)(\gamma_0 - r_2)T\right\}\right) + O(e^{-rT}),
$$

which tends to 0 as $T \to +\infty$. □

Lemma A.8. *For any $r_2 < \gamma < r_1$, we have*

$$
\lim_{T \to +\infty} \mathbb{E}\left[e^{-rT}Y_T(y)^\gamma\right] = 0. \tag{A.1.12}
$$

Proof. In fact, we have that

$$
\begin{aligned}
\mathbb{E}\left[e^{-rT}Y_T(y)^\gamma\right] &= e^{-rT}\mathbb{E}\left[(ye^{rT} \cdot e^{-(r+\frac{\kappa^2}{2})T - \kappa W_T})^\gamma\right] \\
&= y^\gamma e^{-rT}\mathbb{E}\left[e^{\gamma(-\frac{\kappa^2}{2}T - \kappa W_T)}\right] = O\left(e^{(\gamma - r_1)(\gamma - r_2)\frac{\kappa^2}{2}T}\right),
\end{aligned}
$$

which converges to 0 in view that $r_2 < \gamma < r_1$ by Assumption (A1). □

A.1.3 Proof of Corollary 3.1

To conclude the main results in Corollary 3.1, it is sufficient to prove that the SDE (3.2.32) has a unique strong solution (X_t^*, H_t^*) for any initial value $(x, h) \in \mathcal{C}$. To this end, we can essentially follow the arguments in the proof of Proposition 5.9 in Deng et al. (2022). However, due to more complicated expressions of $C_2(h)$-$C_6(h)$ in (3.2.23) and different feedback functions, we need to prove the following auxiliary lemmas to conclude Corollary 3.1.

Lemma A.9. *The function f is C^1 within each of the subsets of $\mathbb{R}_+^2$: $x \leqslant x_1(h)$, $x_1(h) < x < x_2(h)$ and $x_2(h) \leqslant x \leqslant x_3(h)$, and it is continuous at the boundary of $x = x_2(h)$ and $x = x_3(h)$. Moreover, we have that*

$$f_x(x,h) = \frac{1}{g(y,h)}$$
$$= \begin{cases} \Big(- C_2(h) r_2 (r_2 - 1)(f(x,h))^{r_2 - 2}\Big)^{-1}, & \text{if } x \leqslant x_{\text{zero}}(h), \\ \Big(- C_3(h) r_1 (r_1 - 1)(f(x,h))^{r_1 - 2} \\ \quad - C_4(h) r_2 (r_2 - 1)(f(x,h))^{r_2 - 2} & \text{if } x_{\text{zero}}(h) < x < x_{\text{aggr}}(h), \\ \quad - \frac{2(\gamma_1 - 1)}{\kappa^2(\gamma_1 - r_1)(\gamma_1 - r_2)}(f(x,h))^{\gamma_1 - 2}\Big)^{-1}, \\ \Big(- C_5(h) r_1 (r_1 - 1)(f(x,h))^{r_1 - 2} \\ \quad - C_6(h) r_2 (r_2 - 1)(f(x,h))^{r_2 - 2}\Big)^{-1}, & \text{if } x_{\text{aggr}}(h) \leqslant x \leqslant x_{\text{lavs}}(h), \end{cases} \tag{A.1.13}$$

and

$$f_h(x,h) = -g_h(f(x,h),h) \cdot f_x(x,h). \tag{A.1.14}$$

Proof. The proof is the same as Lemma 5.6 in Deng et al. (2022), so we omit it. □

Lemma A.10. *The function π^* is Lipschitz on $\mathcal{C}$.*

Proof. By (3.2.24), (3.2.25) and the inverse transform, we can express c^* and π^* in terms of the primal variables as in (3.2.30) and (3.2.31). Combining the expressions of c^* and π^* with Proposition 3.2 which implies that the coefficients $(C_i)_{2 \leqslant i \leqslant 5}$ are C^1, Lemma A.9 implies that the C^1 regularity of f, together with the continuity of f at the boundary between the three regions, we can draw the conclusion that $(x,h) \to c^*(x,h)$ and $(x,h) \to \pi^*(x,h)$ are locally Lipschitz on $\mathcal{C}$.

(i) Boundedness of $\frac{\partial \pi^*}{\partial x}$.

First using π^* in (3.2.31), we have

$$\frac{\partial \pi^*}{\partial x}(x,h) = \frac{\mu - r}{\sigma^2}$$

$$\times \begin{cases} 1 - r_2, & \text{if } x < x_{\text{zero}}(h), \\ \Big(\frac{2r}{\kappa^2} C_3(h)(r_1 - 1) f_2^{r_1 - 2}(x,h) \frac{\partial f_2}{\partial x} \\ \quad + \frac{2r}{\kappa^2} C_4(h)(r_2 - 1) f_2^{r_2 - 2}(x,h) \frac{\partial f_2}{\partial x} & \text{if } x_{\text{zero}}(h) \leqslant x \leqslant x_{\text{aggr}}(h), \\ \quad + \frac{2(\gamma_1 - 1)^2}{\kappa^2(\gamma_1 - r_1)(\gamma_1 - r_2)} f_2^{\gamma_1 - 2}(x,h) \frac{\partial f_2}{\partial x}\Big), \\ \frac{2r}{\kappa^2} C_5(h)(r_1 - 1) f^{r_1 - 2}(x,h) \frac{\partial f_2}{\partial x} & \text{if } x_{\text{aggr}}(h) < x \leqslant x_{\text{lavs}}(h). \\ \quad + \frac{2r}{\kappa^2} C_6(h)(r_2 - 1) f^{r_2 - 2}(x,h) \frac{\partial f_2}{\partial x}, \end{cases} \tag{A.1.15}$$

Note that the first line is constant and hence bounded. For the second line, by differentiating (3.2.28) and using the fact that $r_1(r_1 - 1) = r_2(r_2 - 1) = \frac{2r}{\kappa^2}$, we have that

$$1 = -\frac{2r}{\kappa^2} C_3(h) f_2(x,h)^{r_1 - 2} \frac{\partial f_2}{\partial x}(x,h) - \frac{2r}{\kappa^2} C_4(h) f_2(x,h)^{r_2 - 2} \frac{\partial f_2}{\partial x}(x,h)$$
$$- \frac{2(\gamma_1 - 1)}{\kappa^2(\gamma_1 - r_1)(\gamma_1 - r_2)} f_2(x,h)^{\gamma_1 - 2} \frac{\partial f_2}{\partial x}(x,h).$$

Plugging this back to $\frac{\partial \pi^*}{\partial x}$, we can obtain

$$\frac{\partial \pi^*}{\partial x}(x,h) = \frac{\mu - r}{\sigma^2} \left\{ \left(\frac{2r}{\kappa^2} C_3(h) f_2^{r_1 - 1}(x,h)(r_1 - r_2) + \frac{2(\gamma_1 - 1)}{\kappa^2(\gamma_1 - r_1)} f_2^{\gamma_1 - 1}(x,h) \right) \frac{1}{f_2} \frac{\partial f_2}{\partial x} + (1 - r_2) \right\}$$

Combined with Lemma A.9, we can obtain $\frac{\partial \pi^*}{\partial x}(x,h) = \frac{\mu - r}{\sigma}(\frac{A_1}{B_1} + (1-r_2))$, where

$$
\begin{aligned}
A_1 &:= \frac{2r}{\kappa^2} C_3(h) f_2^{r_1-1}(x,h)(r_1-r_2) + \frac{2(\gamma_1-1)}{\kappa^2(\gamma_1-r_1)} f_2^{\gamma_1-1}(x,h), \\
B_1 &:= -\frac{2r}{\kappa^2} C_3(h) f_2(x,h)^{r_1-1} - \frac{2r}{\kappa^2} C_4(h) f_2(x,h)^{r_2-1} - \frac{2(\gamma_1-1)}{\kappa^2(\gamma_1-r_1)(\gamma_1-r_2)} f_2(x,h)^{\gamma_1-1}.
\end{aligned}
\tag{A.1.16}
$$

We shall show that $A_1 > 0$ and $B_1 < 0$, and $\frac{A_1}{B_1}$ is bounded. We only need to discuss the case that $y_1(h) > y_2(h) = ((1-\lambda)h)^{\beta_1 - 1}$, because the second region reduces to a point for any fixed h if $y_1(h) = y_2(h)$. Indeed, it is obvious that $A_1 > 0$ since $C_3(h) > 0$ according to the proof of Lemma 3.1 and $\gamma_1 < 0$. Moreover, we have that

$$
\begin{aligned}
A_1 &= \frac{2}{\kappa^2}\left(r(r_1-r_2)C_3(h) + \frac{\gamma_1-1}{\gamma_1-r_1} f_2^{\gamma_1-r_1}(x,h) \right) f_2^{r_1-1}(x,h) \\
&= \frac{2}{\kappa^2}\Bigg(\frac{r_2}{\gamma_1-r_1} y_1(h)^{\gamma_1-r_1} + \frac{\lambda}{(1-\lambda)^{(\gamma_1-1)(\beta_1-1)}} y_2(h)^{\gamma_1-1} y_1(h)^{1-r_1} \\
&\quad + \frac{\gamma_1-1}{\gamma_1-r_1} \cdot f_2^{\gamma_1-r_1}(x,h) \Bigg) f_2^{r_1-1}(x,h) \\
&\leqslant K_1\big(y_1(h)^{1-r_1} y_2(h)^{\gamma_1-1} f_2(x,h)^{r_1-1} + f_2(x,h)^{\gamma_1-1} \big),
\end{aligned}
$$

where K_1 is a positive constant. For B_1, according to the proof of Lemma 3.1, we have that

$$
\begin{aligned}
B_1 &= -\frac{2r}{\kappa^2} C_3(h) f_2(x,h)^{r_1-1} - \frac{2r}{\kappa^2} C_4(h) f_2(x,h)^{r_2-1} - \frac{2(\gamma_1-1)}{\kappa^2(\gamma_1-r_1)(\gamma_1-r_2)} f_2(x,h)^{\gamma_1-1} \\
&\leqslant -\frac{2r}{\kappa^2} C_3(h) f_2(x,h)^{r_1-1} - C f_2(x,h)^{\gamma_1-1} \\
&\leqslant -K_2 (y_1(h)^{1-r_1} y_2(h)^{\gamma_1-1} f_2(x,h)^{r_1-1} + f_2(x,h)^{\gamma_1-1}),
\end{aligned}
$$

where K_2 is some positive constant. Therefore, $0 > \frac{A_1}{B_1} \geqslant -C$ for some positive constant independent of h, and thus $\frac{\partial \pi^*}{\partial x}$ in the second line of (A.1.15) is bounded.

For the third line, by differentiating (3.2.29) and using the fact that $r_1(r_1-1) = r_2(r_2-1) = \frac{2r}{\kappa^2}$, we have that $1 = -\frac{2r}{\kappa^2}C_5(h)f_3(x,h)^{r_1-2}\frac{\partial f_3}{\partial x}(x,h) - \frac{2r}{\kappa^2}C_6(h)f_3(x,h)^{r_2-2}\frac{\partial f_3}{\partial x}(x,h)$. Putting this back to the third line of (A.1.15), we can obtain $\frac{\partial \pi^*}{\partial x}(x,h) = \frac{\mu-r}{\sigma^2}\{\frac{A_2}{B_2} + (1-r_2)\}$, where

$$\begin{aligned} A_2 &:= \frac{2r(r_1-r_2)}{\kappa^2}C_5(h)f_3^{r_1-1}(x,h), \\ B_2 &:= -\frac{2r}{\kappa^2}C_5(h)f_3(x,h)^{r_1-1} - \frac{2r}{\kappa^2}C_6(h)f_3(x,h)^{r_2-1}, \end{aligned} \tag{A.1.17}$$

by combining with the results of Lemma A.9. In fact, by the proof of Lemma 3.1, we have $C_5(h) > 0$ and $C_6(h) > 0$, therefore, $A_2 > 0$ and $B_2 < 0$. Moreover, we have that

$$B_2 = -\frac{2r}{\kappa^2}C_5(h)f_3(x,h)^{r_1-1} - \frac{2r}{\kappa^2}C_6(h)f_3(x,h)^{r_2-1} \leqslant -\frac{2r}{\kappa^2}C_5(h)f_3(x,h)^{r_1-1},$$

and thus $0 > \frac{A_2}{B_2} \geqslant r_2 - r_1$, indicating that $\frac{\partial \pi^*}{\partial x}$ is bounded in the third line.

(ii) Boundedness of $\frac{\partial \pi^*}{\partial h}$.

First, using equations (A.1.13) and (A.1.14) and the definition of $g(\cdot,h) = -v_y(\cdot,h)$, we have

$$\begin{aligned} &f_h(x,h) = -g_h(f,h)\cdot f_x(x,h) \\ &= \begin{cases} C_2'(h)r_2f_1(x,h)^{r_2-1}\cdot\Big(-\frac{2r}{\kappa^2}C_2(h)f_1(x,h)^{r_2-2}\Big)^{-1}, & \text{if } x < x_{\text{zero}}(h), \\ \Big(C_3'(h)r_1f_2(x,h)^{r_1-1} + C_4'(h)r_2f_2(x,h)^{r_2-1} - \dfrac{\lambda}{r}\Big) \\ \times\Big(-\dfrac{2r}{\kappa^2}C_3(h)f_2(x,h)^{r_1-2} - \dfrac{2r}{\kappa^2}C_4(h)f_2(x,h)^{r_2-2} & \text{if } x_{\text{zero}}(h) \leqslant x \leqslant x_{\text{aggv}}(h), \\ \quad - \dfrac{2(\gamma_1-1)}{\kappa^2(\gamma_1-r_1)(\gamma_1-r_2)}f_2(x,h)^{\gamma_1-2}\Big)^{-1}, \\ \Big(C_5'(h)r_1f_3(x,h)^{r_1-1} + C_6'(h)r_2f_3(x,h)^{r_2-1} - \dfrac{1}{r}\Big) \\ \times\Big(-\dfrac{2r}{\kappa^2}C_5(h)f_3(x,h)^{r_1-2} - \dfrac{2r}{\kappa^2}C_6(h)f_3(x,h)^{r_2-2}\Big)^{-1}, & \text{if } x_{\text{aggv}}(h) \leqslant x \leqslant x_{\text{lavs}}(h). \end{cases} \end{aligned}$$

We analyze the derivative $\frac{\partial \pi^*}{\partial h}$ in different regions separately. In the region $x < x_{\text{zero}}(h)$, $\frac{\partial \pi^*}{\partial h} = 0$, hence it is bounded. In the region $x_{\text{zero}}(h) \leqslant x(h) \leqslant x_{\text{aggv}}(h)$, we also only need to discuss the case that $y_1(h) > y_2(h) = ((1-\lambda)h)^{\beta_1 - 1}$, and

$$\begin{aligned}\frac{\partial \pi^*}{\partial h} = \frac{\mu - r}{\sigma^2}\bigg(&\frac{2r}{\kappa^2}C_3'(h)f_2(x,h)^{r_1-1} + \frac{2r}{\kappa^2}C_3(h)(r_1-1)f_2(x,h)^{r_2-2}\frac{\partial f_2}{\partial h}\\ &+\frac{2r}{\kappa^2}C_4'(h)f_2(x,h)^{r_2-1} + \frac{2r}{\kappa^2}C_4(h)(r_2-1)f_2(x,h)^{r_2-2}\frac{\partial f_2}{\partial h}\\ &+\frac{2(\gamma_1-1)^2}{\kappa^2(\gamma_1-r_1)(\gamma_1-r_2)}f_2^{\gamma_1-2}(x,h)\frac{\partial f_2}{\partial h}\bigg).\end{aligned}$$

By differentiating (3.2.28) and using the fact that $r_1(r_1-1) = r_2(r_2-1) = \frac{2r}{\kappa^2}$, we have that

$$\begin{aligned}&C_4'(h)\frac{2r}{\kappa^2}f_2(x,h)^{r_2-1} + C_4(h)\frac{2r}{\kappa^2}(r_2-1)\frac{\partial f_2}{\partial h}f_2(x,h)^{r_2-2}\\ =&-C_3'(h)r_1(r_2-1)f_2(x,h)^{r_1-1} - C_3(h)\frac{2r}{\kappa^2}(r_2-1)\frac{\partial f_2}{\partial h}f_2(x,h)^{r_1-2}\\ &-\frac{2(\gamma_1-1)(r_2-1)}{\kappa^2(\gamma_1-r_1)(\gamma_1-r_2)}\frac{\partial f_2}{\partial h}f_2(x,h)^{\gamma_1-2} + (r_2-1)\frac{\lambda}{r}.\end{aligned}$$

Putting this back to the previous expression of $\frac{\partial \pi^*}{\partial h}$, we can obtain that

$$\begin{aligned}\frac{\partial \pi^*}{\partial h} =& \frac{\mu-r}{\sigma^2}\bigg(\bigg[\frac{2r}{\kappa^2}(r_1-r_2)C_3(h)f_2(x,h)^{r_1-1} - \frac{2(\gamma_1-1)(r_2-1)}{\kappa^2(\gamma_1-r_1)(\gamma_1-r_2)}f_2(x,h)^{\gamma_1-1}\bigg]\frac{1}{f_2}\frac{\partial f_2}{\partial h}\\ &+r_1(r_1-r_2)C_3'(h)f_2(x,h)^{r_1-1} + (r_2-1)\frac{\lambda}{r}\bigg)\\ =&\frac{\mu-r}{\sigma^2}\bigg(A_1\cdot\frac{1}{f_2}\frac{\partial f_2}{\partial h} + r_1(r_1-r_2)C_3'(h)f_2(x,h)^{r_1-1} - \frac{\lambda}{r}(1-r_2)\bigg),\end{aligned} \tag{A.1.18}$$

where A_1 is defined in (A.1.16). In (A.1.18), the third term is a constant. For the second term, by the proof of Lemma 3.1, we first have $C_3'(h) > 0$, and

$$C_3'(h) = \frac{1}{r(r_1 - r_2)h}(kr_2(\lambda h)^{\beta_2} + \lambda h r_1 w(h)^{\beta_1 - 1})w(h)^{-r_1(\beta_1 - 1)}$$
$$= \frac{1}{r(r_1 - r_2)}\left(\lambda(r_1 + r_2\beta_2)y_1(h)^{1-r_1} + \frac{r_2\beta_2}{\gamma_1 h}y_1(h)^{\gamma_1 - r_1}\right).$$

In the sequel of the proof below, let $K_1 > 0$ be a generic positive constant independent of (x, h), which may be different from line to line. Hence, we have that

$$\begin{aligned} r_1(r_1 - r_2)C_3'(h)f_2(x,h)^{r_1-1} &= \left(\lambda(r_1 + r_2\beta_2)y_1(h)^{1-r_1} + \frac{r_2\beta_2}{\gamma_1 h}y_1(h)^{\gamma_1-r_1}\right)f_2(x,h)^{r_1-1} \\ &\leqslant \left(\lambda(r_1 + r_2\beta_2)y_1(h)^{1-r_1} + \frac{r_2\beta_2}{\gamma_1 h}y_1(h)^{\gamma_1-r_1}\right)y_1(x,h)^{r_1-1} \\ &\leqslant K_1\left(1 + \left(\frac{y_2(h)}{y_1(h)}\right)^{1-\gamma_1}\right) \leqslant K_1. \end{aligned}$$

Therefore, the second term is bounded.

For the first term in (A.1.18), by virtue of $A_1 \leqslant C(y_1(h)^{1-r_1}y_2(h)^{\gamma_1-1}f_2(x,h)^{r_1-1} + f_2(x,h)^{\gamma_1-1})$, it is sufficient to show that $\frac{1}{f_2}\frac{\partial f_2}{\partial h} \geqslant C(y_1(h)^{1-r_1}y_2(h)^{\gamma_1-1}f_2(x,h)^{r_1-1} + f_2(x,h)^{\gamma_1-1})$ for some positive constant C. Indeed, we have that

$$\frac{1}{f_2}\frac{\partial f_2}{\partial h} = \left(C_3'(h)r_1 f_2(x,h)^{r_1-1} + C_4'(h)r_2 f_2(x,h)^{r_2-1} - \frac{\lambda}{r}\right) \times \frac{1}{B_1},$$

where B_1 is defined in (A.1.16). As $C_3'(h)r_1f_2(x,h)^{r_1-1}$ and $\frac{\lambda}{r}$ are bounded, it is sufficient to show that $C_4'(h)r_2f_2(x,h)^{r_2-1}$ is bounded. As $C_6'(h) = -C_5'(h)y_3(h)^{r_1-r_2}$, similar to K_1, constant K_2 in the following equation may differ from line to line, and we have that

$$
\begin{aligned}
|C_4'(h)| &= |C_4'(h) - C_6'(h) + C_6'(h)| \\
&= \left| \frac{(\gamma_1 - r_2)(\beta_1 - 1)}{(r_2 - r_1)(1-\beta_1)(\gamma_1 - r_2)} (1-\lambda)^{(\gamma_1 - r_2)(\beta_1 - 1)} h^{r_1(\beta_1 - 1)} - C_5'(h) y_3(h)^{r_1 - r_2} \right| \\
&= \left| K_2 h^{r_1(\beta_1 - 1)} - \big(C_3'(h) - (C_3'(h) - C_5'(h))\big) y_3(h)^{r_1 - r_2} \right| \\
&= |K_2 h^{r_1(\beta_1 - 1)} - C_3'(h) y_3(h)^{r_1 - r_2}| \\
&\leqslant K_2 h^{r_1(\beta_1 - 1)} + C\big(y_1(h)^{1-r_1} + y_2(h)^{1-\gamma_1} y_1(h)^{\gamma_1 - r_1}\big) y_2(h)^{r_1 - r_2} \\
&\leqslant K_2 \big(y_2(h)^{r_1} + y_2(h)^{r_1 - r_2} y_1(h)^{1-r_1} + y_2(h)^{2r_1 - \gamma_1} y_1(h)^{\gamma_1 - r_1}\big),
\end{aligned}
$$

where the first inequality holds because $y_3(h) = (1-\lambda) y_2(h)$. Similar to K_1, constant K_3 in the following equation may differ from line to line, and it follows that

$$
\begin{aligned}
& |C_4'(h) r_2 (f_2(x,h)^{r_2 - 1})| \\
=& K_3 |C_4'(h)| f_2(x,h)^{r_2 - 1} \\
\leqslant& K_3 |C_4'(h)| y_2(h)^{r_2 - 1} \\
\leqslant& K_3 \big(y_2(h)^{r_1} + y_2(h)^{r_1 - r_2} y_1(h)^{1-r_1} + y_2(h)^{2r_1 - \gamma_1} y_1(h)^{\gamma_1 - r_1}\big) y_2(h)^{r_2 - 1} \\
=& K_3 \left(1 + \left(\frac{y_2(h)}{y_1(h)}\right)^{r_1 - 1} + \left(\frac{y_2(h)}{y_1(h)}\right)^{r_1 - \gamma_1}\right),
\end{aligned}
$$

which is bounded as $y_1(h) > y_2(h)$.

In the region $x_{\text{aggr}}(h) \leqslant x(h) \leqslant x_{\text{lavs}}(h)$, similar computations yield that

$$
\frac{\partial \pi^*}{\partial h} = \frac{\mu - r}{\sigma^2} \left(A_2 \cdot \frac{1}{f_3} \frac{\partial f_3}{\partial h} + r_1(r_1 - r_2) C_5'(h) f_3(x,h)^{r_1 - 1} - \frac{1}{r}(1 - r_2) \right),
$$

where A_2 is defined in (A.1.17). For the term $r_1(r_1 - r_2)C_5'(h) f_3(x,h)^{r_1 - 1}$, due to $C_5'(h) < C_3'(h)$, we have that $r(r_1 - r_2)C_5'(h) f_3(x,h)^{r_1-1} \leqslant r(r_1 - r_2) C_3'(h) f_2(\bar{x}, h)^{r_1 - 1}$, which is bounded as $f_3(x,h) < y_2(h) \leqslant f_2(\bar{x}, h)$, where $\bar{x}$ is chosen such that $y_2(h) \leqslant$

$f_2(\bar{x}, h) \leqslant y_1(h)$. Moreover, for the term $A_2 \cdot \frac{1}{f_3} \frac{\partial f_3}{\partial h}$, similar to the proof in the region $y_2(h) \leqslant f_2(x, h) \leqslant y_1(h)$, it is enough to check that $C_6'(h) f_3(x, h)^{r_2 - 1}$ is bounded. Indeed, we can obtain

$$|C_6'(h) f_3(x, h)^{r_2-1}| = |C_5'(h) y_3(h)^{r_1 - r_2} f_3(x, h)^{r_2 - 1}| \leqslant C_5'(h) f_3(x, h)^{r_1 - 1},$$

which is shown to be bounded. Putting all the pieces together completes the proof.

□

A.1.4 Proof of Proposition 3.1 (Concavification Principle)

To prove this proposition, we claim that under the optimal controls c_t^* and π_t^*, it holds that $\tilde{U}(c_t^*, H_t^*) = U(c_t^* - \lambda H_t^*)$ all the time. In fact, for any $(x, h) \in \mathcal{C}$, according to the definition of concave envelop $\tilde{U}(x, h)$ of $U^*(x, h)$ in $x \in [0, h]$ in (3.1.3), we can easily see that $\tilde{U}(x, h) = U^*(x, h)$ if $x \in \mathcal{C}_h := \{0\} \cup [z(h), h]$, where $z(h)$ is defined in Section 3.1.2. We shall interpret the claim in all the regions of wealth X_t^*.

If $X_t^* < x_{\text{zero}}(H_t^*)$, then $c_t^* = 0 \in \mathcal{C}_{H_t^*}$, indicating that $\tilde{U}(c_t^*, H_t^*) = U(c_t^* - \lambda H_t^*)$.

If $x_{\text{zero}}(H_t^*) \leqslant X_t^* \leqslant x_{\text{aggr}}(H_t^*)$, yielding the existence of the solution $z(H_t^*)$ for equation (3.1.2) with $h = H_t^*$. Moreover, the optimal consumption satisfies that $z(H_t^*) \leqslant c_t^* = \lambda H_t^* + (f(X_t^*, H_t^*))^{\frac{1}{\beta_1 - 1}} \leqslant H_t^*$, where $f(x, h)$ is defined in Corollary 3.1. This leads to the fact that $c_t^* \in \mathcal{C}_{H_t^*}$ and thus $\tilde{U}(c_t^*, H_t^*) = U(c_t^* - \lambda H_t^*)$.

If $x_{\text{aggr}}(H_t^*) < X_t^* \leqslant x_{\text{lavs}}(H_t^*)$, then $c_t^* = H_t^* \in \mathcal{C}_{H_t^*}$, indicating that $\tilde{U}(c_t^*, H_t^*) = U(c_t^* - \lambda H_t^*)$.

Therefore, we have verified that the optimal consumption rate c_t^* always leads to $U(c_t^* - \lambda H_t^*) = \tilde{U}(c_t^*, H_t^*)$. Thus, given the optimal portfolio π_t^* and c_t^* for the stochastic control problem (3.1.5), based on the fact that $\tilde{U}(x, h) \geqslant U(x - \lambda h)$ everywhere and corresponding $\tilde{u} \geqslant u$, we have

$$\tilde{u}(x, h) = \mathbb{E}\left[\int_0^\infty e^{-rt} \tilde{U}(c_t^*, H_t^*) dt\right] = \mathbb{E}\left[\int_0^\infty e^{-rt} U(c_t^* - \lambda H_t^*) dt\right] \leqslant u(x, h) \leqslant \tilde{u}(x, h),$$

that is, $\tilde{u} = u$, and the optimal portfolio and consumption for (3.1.1) are the same as (3.1.5).

A.1.5 Proof of Lemma 3.1

We prove $v_y y(y,h) > 0$ in three regions: $y > y_1(h)$, $y_2(h) \leqslant y \leqslant y_1(h)$, and $y_3(h) \leqslant y < y_2(h)$, respectively.

(i) In the region $y_3(h) \leqslant y < y_2(h)$, $v_{yy}(y,h) = r_1(r_1-1)C_5(h)y^{r_1-2} + r_2(r_2-1)C_6(h)y^{r_2-2}$.

As $r_1(r_1-1) = r_2(r_2-1) = \frac{2r}{\kappa^2} > 0$, we only need to prove that $C_5(h) > 0$ and $C_6(h) > 0$. We shall separate the proof into two cases: the case that $y_1(h) > y_2(h)$ and that $y_1(h) = y_2(h)$. If $y_1(h) = w(h)^{\beta_1-1} > y_2(h) = ((1-\lambda)h)^{\beta_1-1}$, we can deduce that

$$\begin{aligned}
C_3(h) &= \frac{y_1(h)^{-r_1}}{r(r_1-r_2)}\left(\frac{kr_2}{\beta_2}(\lambda h)^{\beta_2} + \frac{r_1 r_2}{(\gamma_1-r_1)\gamma_1}y_1(h)^{\gamma_1} + \lambda h r_1 y_1(h)\right),\\
&= \frac{w(h)^{-r_1(\beta_1-1)}}{r(r_1-r_2)}\left(\frac{r_2}{\gamma_1}w(h)^{\beta_1} + \lambda h r_2 w(h)^{\beta_1-1} + \frac{r_1 r_2}{(\gamma_1-r_1)\gamma_1}w(h)^{\beta_1} + \lambda h r_1 w(h)^{\beta_1-1}\right)\\
&= \frac{w(h)^{-r_1(\beta_1-1)}}{r(r_1-r_2)}\left(\frac{r_2}{\gamma_1-r_1}w(h)^{\beta_1} + \lambda h w(h)^{\beta_1-1}\right) > 0,
\end{aligned}$$

and

$$\begin{aligned}
C_3(h) - C_5(h) &= \frac{y_2(h)^{-r_1}}{r(r_1-r_2)}\left(-\frac{r_2}{\beta_1}((1-\lambda)h)^{\beta_1} + \frac{r_1 r_2}{\gamma_1(\gamma_1-r_1)}y_2(h)^{\gamma_1} - (1-\lambda)h r_1 y_2(h)\right)\\
&= \frac{1}{r(r_1-r_2)(1-\beta_1)(\gamma_1-r_1)}((1-\lambda)h)^{r_2\beta_1+r_1} < 0,
\end{aligned}$$

therefore, we have $C_5(h) = C_3(h) - (C_3(h) - C_5(h)) > 0$.

We next prove that $C_6'(h) < 0$, and hence $C_6(h) = -\int_h^\infty C_6'(s)ds > 0$. It is easy to see that $C_3'(h) - C_5'(h) < 0$, and hence $C_5'(h) > C_3'(h) > 0$, where the second inequality follows from

$$C_3'(h) = \frac{1}{r(r_1-r_2)}\bigg(-\frac{k}{\beta_2}(\lambda h)^{\beta_2} + \frac{1}{\gamma_1}w(h)^{\beta_1} + \lambda h w(h)^{\beta_1-1}\bigg) r_1 r_2(\beta_1-1)w(h)^{-r_1\beta_1-r_2}w'(h)$$
$$+ \frac{1}{r(r_1-r_2)h}(kr_2(\lambda h)^{\beta_2} + \lambda h r_1 w(h)^{\beta_1-1})w(h)^{-r_1(\beta_1-1)} > 0,$$

thanks to $\frac{k}{\beta_2}(\lambda h)^{\beta_2} - \frac{1}{\gamma_1}w(h)^{\beta_1} - \lambda h w(h)^{\beta_1-1} \leqslant 0$ and $w'(h) > 0$.

Along the free boundary condition (3.2.21), we have $C_5'(h)y_3(h)^{r_1} + C_6'(h)y_3(h)^{r_2} = 0$, therefore, we can deduce that $C_6'(h) = -C_5'(h)y_3(h)^{r_1-r_2} < 0$.

We then consider the case that $y_1(h) = y_2(h) = \frac{k}{\beta_2}\lambda^{\beta_2}h^{\beta_2-1} + \frac{1}{\beta_1}(1-\lambda)^{\beta_1}h^{\beta_1-1} \leqslant ((1-\lambda)h)^{\beta_1-1}$, in which we have that

$$C_5(h) = \frac{y_1(h)^{r_2-1}}{r(r_1-r_2)}\bigg(\frac{kr_2}{\beta_2}(\lambda h)^{\beta_2} + \frac{r_2}{\beta_1}((1-\lambda)h)^{\beta_1} + hr_1 y_1(h)\bigg) = \frac{h}{r(r_1-r_2)}y_1(h)^{r_2} > 0,$$

and $C_5'(h) = \frac{y_1(h)^{r_2-1}}{r(r_1-r_2)}\bigg(y_1(h) + r_2 h y_1'(h)\bigg) > 0$. Thus, it holds that $C_6'(h) = -C_5'(h)y_3(h)^{r_1-r_2} < 0$, implying that $C_6(h) > 0$ when $y_1(h) = y_2(h)$.

(ii) In the region $y_2(h) \leqslant y \leqslant y_1(h)$, we only need to consider the case that $y_1(h) = w(h)^{\beta_1-1} > y_2(h) = ((1-\lambda)h)^{\beta_1-1}$, otherwise the second-order derivative of $v(y,h)$ in y is trivial because this region reduces to a point.

Because $C_3(h) > 0$, $C_4(h) > C_4(h) - C_6(h)$, $r_1(r_1-1) = r_2(r_2-1) = \frac{2r}{\kappa^2}$, we can deduce that

$$v_{yy}(y,h) = \frac{2r}{\kappa^2}\bigg(C_3(h)y^{r_1-\gamma_1} + C_4(h)y^{r_2-\gamma_1} + \frac{\gamma_1-1}{r(\gamma_1-r_1)(\gamma_1-r_2)}\bigg)y^{\gamma_1-2}$$
$$> \frac{2r}{\kappa^2}\bigg((C_4(h)-C_6(h))y^{r_2-\gamma_1} + \frac{\gamma_1-1}{r(\gamma_1-r_1)(\gamma_1-r_2)}\bigg)y^{\gamma_1-2}$$
$$\geqslant \frac{2r}{\kappa^2}\bigg((C_4(h)-C_6(h))((1-\lambda)h)^{(r_2-\gamma_1)(\beta_1-1)} + \frac{\gamma_1-1}{r(\gamma_1-r_1)(\gamma_1-r_2)}\bigg)y^{\gamma_1-2},$$

where the last inequality holds because $y \geqslant ((1-\lambda)h)^{\beta_1-1}$, $\gamma_1 > r_2$, and $C_4(h) -$

$C_6(h) < 0$. Moreover, we have that

$$(C_4(h) - C_6(h))((1-\lambda)h)^{(r_2-\gamma_1)(\beta_1-1)} + \frac{\gamma_1 - 1}{r(\gamma_1 - r_1)(\gamma_1 - r_2)} = \frac{\gamma_1 - 1}{r(\gamma_1 - r_1)(r_1 - r_2)} > 0.$$

Thus, we can deduce that $v_{yy}(y, h) > 0$.

(iii) In the region $y > y_1(h)$, $v_{yy}(y, h) = r_2(r_2 - 1)C_2(h)y^{r_2-2}$. Since $r_2(r_2 - 1) = \frac{2r}{\kappa^2} > 0$, we only need to prove that $C_2(h) > 0$. We shall also discuss $C_2(h) > 0$ for two cases that $y_1(h) > y_2(h)$ or $y_1(h) = y_2(h)$.

If $y_1(h) > y_2(h)$, indicating that $y_1(h) = w(h)^{\beta_1-1}$, we have $\frac{k}{\beta_2}(\lambda h)^{\beta_2} - \frac{1}{\gamma_1}w(h)^{\beta_1} - \lambda h w(h)^{\beta_1-1} = 0$. Similar to the proof of $C_5(h) > 0$, we have

$$\begin{aligned}
C_2(h) > C_2(h) - C_6(h) &= (C_2(h) - C_4(h)) + (C_4(h) - C_6(h)) \\
&= \frac{w(h)^{-r_2(\beta_1-1)}}{r(r_1 - r_2)}\left(\frac{r_1}{\gamma_1 - r_2}w(h)^{\beta_1} + \lambda h w(h)^{\beta_1-1}\right) \\
&\quad - \frac{1}{r(r_2 - r_1)(1-\beta_1)(\gamma_1 - r_2)}y_2(h)^{\gamma_1 - r_2} \\
&> \frac{r_1}{r(r_1 - r_2)(\gamma_1 - r_2)}y_1(h)^{\gamma_1 - r_2} + \frac{\gamma_1 - 1}{r(r_1 - r_2)(\gamma_1 - r_2)}y_2(h)^{\gamma_1 - r_2} \\
&> \frac{1}{r(r_1 - r_2)}y_2(h)^{\gamma_1 - r_2} > 0.
\end{aligned}$$

If $y_1(h) = y_2(h)$, similar to the proof of $C_5(h) > 0$, we can obtain that $C_2(h) > C_2(h) - C_6(h) = \frac{h}{r(r_1 - r_2)}y_1(h)^{r_1} > 0$.

A.1.6 Proof of Corollary 3.2

Proof. We first have that

$$\lim_{h\to+\infty} C_6(h)h^{-r_2-r_1\beta_1} = \frac{1}{r_2+r_1\beta_1}\lim_{h\to+\infty}\frac{C_6'(h)}{h^{r_1(\beta_1-1)}}$$

$$= -\frac{1}{r_2+r_1\beta_1}(1-\lambda)^{\beta_1(r_1-r_2)}\lim_{h\to+\infty}\frac{C_5'(h)}{h^{r_2(\beta_1-1)}}$$

$$= -\frac{\gamma_1-r_1}{\gamma_1-r_2}(1-\lambda)^{\beta_1(r_1-r_2)}\lim_{h\to+\infty} C_5(h)h^{-r_1-r_2\beta_1},$$

by L'Hôpital's rule. To compute $\lim_{h\to+\infty} C_5(h)h^{-r_1-r_2\beta_1}$, we need to consider two cases that $y_1(h) > y_2(h)$ and $y_1(h) = y_2(h)$.

We first consider the case that $y_1(h) = y_2(h)$ as $h \to +\infty$, indicating that $\beta_1 > 1-\lambda$ and $\beta_2 \leqslant \beta_1$ in condition (S2) or (S3), therefore, $C_5(h) = \frac{h}{r(r_1-r_2)}y_1(h)^{r_2}$, and thus

$$\lim_{h\to+\infty} C_5(h)h^{-r_2\beta_1-r_1} = \frac{1}{r(r_1-r_2)}\lim_{h\to+\infty}\left(\frac{k}{\beta_2}\lambda^{\beta_2}h^{\beta_2-\beta_1}+\frac{1}{\beta_1}(1-\lambda)^{\beta_1}\right)^{r_2}$$

$$= \frac{1}{r(r_1-r_2)}\left(\frac{k}{\beta_2}\lambda^{\beta_2}\mathbf{1}_{\{\beta_2=\beta_1\}}+\frac{1}{\beta_1}(1-\lambda)^{\beta_1}\right)^{r_2}.$$

Therefore, we can derive that

$$\lim_{h\to+\infty}\frac{c^*(x_{\text{lavs}}(h),h)}{x_{\text{lavs}}(h)} = \lim_{h\to+\infty}\frac{h}{x_{\text{lavs}}(h)}$$

$$= \lim_{h\to+\infty}\frac{h}{-C_5(h)r_1(1-\lambda)^{\beta_1(r_1-1)}h^{(\beta_1-1)(r_1-1)}-C_6(h)r_2(1-\lambda)^{\beta_1(r_2-1)}h^{(\beta_1-1)(r_2-1)}+\frac{h}{r}}$$

$$=\left(1-\frac{(1-\lambda)^{\beta_1(r_1-1)}\gamma_1}{\gamma_1-r_2}\left(\frac{k}{\beta_2}\lambda^{\beta_2}\mathbf{1}_{\{\beta_2=\beta_1\}}+\frac{1}{\beta_1}(1-\lambda)^{\beta_1}\right)^{r_2}\right)^{-1}r,$$

and

$$\lim_{h\to+\infty}\frac{\pi^*(x_{\text{lavs}}(h),h)}{x_{\text{lavs}}(h)}=\lim_{h\to+\infty}\frac{\pi^*(x_{\text{lavs}}(h),h)}{h}\cdot\frac{h}{x_{\text{lavs}}(h)}$$

$$=\frac{2r}{\mu-r}\lim_{h\to+\infty}\frac{h}{x_{\text{lavs}}(h)}\times\lim_{h\to+\infty}\frac{(1-\lambda)^{\beta_1(r_1-1)}C_5(h)h^{-r_1-r_2\beta_1}+(1-\lambda)^{\beta_1(r_2-1)}C_6(h)h^{-r_2-r_1\beta_1}}{h}$$

$$=\frac{2r}{\mu-r}\times\left(1-\frac{(1-\lambda)^{\beta_1(r_1-1)}\gamma_1}{\gamma_1-r_2}\left(\frac{k}{\beta_2}\lambda^{\beta_2}\mathbf{1}_{\{\beta_2=\beta_1\}}+\frac{1}{\beta_1}(1-\lambda)^{\beta_1}\right)^{r_2}\right)^{-1}$$
$$\times\frac{(1-\lambda)^{\beta_1(r_1-1)}}{\gamma_1-r_2}\left(\frac{k}{\beta_2}\lambda^{\beta_2}\mathbf{1}_{\{\beta_2=\beta_1\}}+\frac{1}{\beta_1}(1-\lambda)^{\beta_1}\right)^{r_2}.$$

Let us then consider the other case when $y_1(h)>y_2(h)$. If $\beta_2<\beta_1$, the second term in (A.1.4) converges to 0, and thus $\frac{w(h)}{h}$ converges to a constant $-\lambda\gamma_1$. If $\beta_2=\beta_1$, the second term in (A.1.4) equals a constant, and $\frac{w(h)}{h}$ becomes a constant $w(1)$ that is the unique solution to $-\frac{1}{\gamma_1}w(1)^{\beta_1}+\frac{k}{\beta_2}\lambda^{\beta_2}-\lambda w(1)^{\beta_1-1}=0$. Otherwise, if $\beta_2>\beta_1$, the second term in (A.1.4) goes to infinity as $h\to+\infty$, indicating that $\frac{w(h)}{h}$ converges to 0.

Thus, we always have that

$$\lim_{h\to+\infty}C_3(h)h^{-r_1-r_2\beta_1}=\lim_{h\to+\infty}\left[\frac{r_2}{r(r_1-r_2)(\gamma_1-r_1)}\left(\frac{w}{h}\right)^{r_1+r_2\beta_1}+\frac{\lambda}{r(r_1-r_2)}\left(\frac{w}{h}\right)^{r_2(\beta_1-1)}\right]$$
$$=\frac{w_0^{r_2(\beta_1-1)}(r_2w_0+\lambda(\gamma_1-r_1))}{r(r_1-r_2)(\gamma_1-r_1)},$$

where $w_0:=\lim_{h\to+\infty}\frac{w(h)}{h}$. It holds that

$$\lim_{h\to+\infty}C_5(h)h^{-r_1-r_2\beta_1}=\frac{w_0^{r_2(\beta_1-1)}(r_2w_0+\lambda(\gamma_1-r_1))+(\gamma_1-1)(1-\lambda)^{r_2\beta_1+r_1}}{r(r_1-r_2)(\gamma_1-r_1)}.$$

Then we can deduce that

$$\lim_{h\to+\infty}\frac{c^*(x_{\text{lavs}}(h),h)}{x_{\text{lavs}}(h)}=\lim_{h\to+\infty}\frac{1}{-(1-\lambda)^{\beta_1(r_1-1)}\frac{\gamma_1(r_1-r_2)}{\gamma_1-r_2}C_5(h)h^{-r_2\beta_1-r_1}+\frac{1}{r}}$$

$$=\left(1-\frac{\gamma_1(1-\lambda)^{-r_2\beta_1}}{(\gamma_1-r_1)(\gamma_1-r_2)}\left(w_0^{r_2(\beta_1-1)}(r_2w_0+\lambda(\gamma_1-r_1))+(\gamma_1-1)(1-\lambda)^{r_2\beta_1+r_1}\right)\right)^{-1}r,$$

and

$$\lim_{h\to+\infty}\frac{\pi^*(x_{\text{lavs}}(h),h)}{x_{\text{lavs}}(h)}$$

$$=\frac{2r}{\mu-r}\lim_{h\to+\infty}\frac{h}{x_{\text{lavs}}(h)}\cdot(1-\lambda)^{\beta_1(r_1-1)}\frac{r_1-r_2}{\gamma_1-r_2}$$

$$\cdot\frac{w_0^{r_2(\beta_1-1)}(r_2w_0+\lambda(\gamma_1-r_1))+(\gamma_1-1)(1-\lambda)^{r_2\beta_1+r_1}}{r(r_1-r_2)(\gamma_1-r_1)}$$

$$=\frac{2r(1-\lambda)^{-r_2\beta_1}}{\mu-r}$$

$$\cdot\left(1-\frac{\gamma_1(1-\lambda)^{-r_2\beta_1}}{(\gamma_1-r_1)(\gamma_1-r_2)}\left(w_0^{r_2(\beta_1-1)}(r_2w_0+\lambda(\gamma_1-r_1))+(\gamma_1-1)(1-\lambda)^{r_2\beta_1+r_1}\right)\right)^{-1}$$

$$\cdot\frac{w_0^{r_2(\beta_1-1)}(r_2w_0+\lambda(\gamma_1-r_1))+(\gamma_1-1)(1-\lambda)^{r_2\beta_1+r_1}}{(\gamma_1-r_1)(\gamma_1-r_2)},$$

where

$$w_0=\begin{cases}-\lambda\gamma_1, & \text{if } \beta_2<\beta_1\leqslant 1-\lambda,\\ w(1), & \text{if } \beta_2=\beta_1,\\ 0, & \text{if } \beta_2>\beta_1.\end{cases}$$

Recall that $\frac{\pi^*(x)}{x}=\frac{\mu-r}{\sigma^2(1-\beta_1)}$ and $\frac{c^*(x)}{x}=\frac{(\gamma_1-r_1)(\gamma_1-r_2)}{r_1r_2}r$ in Merton's problem. In our setting, as $\lambda\to 0$, it is obvious that $\beta_1<1-\lambda$. On the other hand, similar to the discussion of the limit of $\frac{w(h)}{h}$ as $h\to+\infty$, we have that $w(1)\to 0$ as $\lambda\to 0$, and thus $w_0\to 0$ as $\lambda\to 0$ in all three scenarios when $y_1(h)>y_2(h)$ as $h\to+\infty$.

Therefore, we can deduce that

$$\lim_{\lambda\to 0}\lim_{h\to+\infty}\frac{c^*(x_{\text{lavs}}(h),h)}{x_{\text{lavs}}(h)}$$

$$=\lim_{\lambda\to 0}\left(1-\frac{\gamma_1(1-\lambda)^{-r_2\beta_1}}{(\gamma_1-r_1)(\gamma_1-r_2)}\left(w_0^{r_2(\beta_1-1)}(r_2w_0+\lambda(\gamma_1-r_1))+(\gamma_1-1)(1-\lambda)^{r_2\beta_1+r_1}\right)\right)^{-1}r$$

$$=\frac{(\gamma_1-r_1)(\gamma_1-r_2)}{r_1r_2}r,$$

and

$$\lim_{\lambda\to 0}\lim_{h\to+\infty}\frac{\pi^*(x_{\text{lavs}}(h),h)}{x_{\text{lavs}}(h)}$$

$$=\lim_{\lambda\to 0}\frac{2r(1-\lambda)^{-r_2\beta_1}}{\mu-r}\times\left(1-\frac{\gamma_1(1-\lambda)^{-r_2\beta_1}}{(\gamma_1-r_1)(\gamma_1-r_2)}\left(w_0^{r_2(\beta_1-1)}(r_2w_0+\lambda(\gamma_1-r_1))\right.\right.$$

$$\left.\left.+(\gamma_1-1)(1-\lambda)^{r_2\beta_1+r_1}\right)\right)^{-1}\times\frac{w_0^{r_2(\beta_1-1)}(r_2w_0+\lambda(\gamma_1-r_1))+(\gamma_1-1)(1-\lambda)^{r_2\beta_1+r_1}}{(\gamma_1-r_1)(\gamma_1-r_2)}$$

$$=\frac{2r(\gamma_1-1)}{(\mu-r)r_1r_2}=\frac{2r(\gamma_1-1)}{-\frac{2r}{\kappa^2}(\mu-r)}=\frac{\mu-r}{\sigma^2(1-\beta_1)},$$

which completes the proof. □

A.1.7 Proof of Corollary 3.3

Proof. Let us consider the auxiliary process $Y_t^* := Y_t(y^*)$ and H_t^* defined in Theorem 3.1.

(i) The long-run fraction of time that the agent stays in the region $\{x_{\text{aggr}}(H_t^*) \leqslant$

$X_t^* \leqslant x_{\text{lavs}}(H_t^*)\}$ can be computed by

$$
\begin{aligned}
&\lim_{T\to+\infty}\frac{1}{T}\mathbb{E}\bigg[\int_0^T \mathbf{1}_{\{x_{\text{aggr}}(H_t^*)<X_t\leqslant x_{\text{lavs}}(H_t^*)\}}dt\bigg]\\
=&\lim_{T\to+\infty}\frac{1}{T}\mathbb{E}\bigg[\int_0^T \mathbf{1}_{\{y_3(H_t^*)\leqslant Y_t(y^*)<y_2(H_t^*)\}}dt\bigg]\\
=&\lim_{T\to+\infty}\frac{1}{T}\mathbb{E}\bigg[\int_0^T \mathbf{1}_{\{\inf_{s\leqslant t}Y_s(y^*)\leqslant Y_t(y^*)<\lim_{h\to+\infty}\frac{y_2(h)}{y_3(h)}\inf_{s\leqslant t}Y_s(y^*)\}}dt\bigg]\\
=&1-\lim_{h\to+\infty}\frac{y_3(h)}{y_2(h)},
\end{aligned}
$$

where the last equation holds by the same argument to prove Theorem 5.1 in Guasoni et al. (2020).

(ii) The long-run fraction of time that the agent stays in the region $\{0 \leqslant X_t^* \leqslant x_{\text{zero}}(H_t^*)\}$ can be computed by

$$
\begin{aligned}
&\lim_{T\to+\infty}\frac{1}{T}\mathbb{E}\bigg[\int_0^T \mathbf{1}_{\{X_t<x_{\text{zero}}(H_t^*)\}}dt\bigg]\\
=&1-\lim_{T\to+\infty}\frac{1}{T}\mathbb{E}\bigg[\int_0^T \mathbf{1}_{\{x_{\text{zero}}(H_t^*)\leqslant X_t\leqslant x_{\text{lavs}}(H_t^*)\}}dt\bigg]\\
=&1-\lim_{T\to+\infty}\frac{1}{T}\mathbb{E}\bigg[\int_0^T \mathbf{1}_{\{y_3(H_t^*)\leqslant Y_t(y^*)\leqslant y_1(H_t^*)\}}dt\bigg]\\
=&1-\lim_{T\to+\infty}\frac{1}{T}\mathbb{E}\bigg[\int_0^T \mathbf{1}_{\{\inf_{s\leqslant t}Y_s(y^*)\leqslant Y_t(y^*)\leqslant\lim_{h\to+\infty}\frac{y_1(h)}{y_3(h)}\cdot\inf_{s\leqslant t}Y_s(y^*)\}}dt\bigg]\\
=&\lim_{h\to+\infty}\frac{y_3(h)}{y_1(h)}.
\end{aligned}
$$

(iii) Let $\tilde{V}(y,h)$ be the solution to the following PDE:

$$
\begin{aligned}
\frac{\kappa^2}{2}y^2\tilde{V}_{yy}(y,h)-\frac{\kappa^2}{2}y\tilde{V}_y(y,h)&=-1,\quad \text{for}\quad (y,h)\in\Omega,\\
\tilde{V}(y_1(h),h)=0,\quad \tilde{V}_h(y_3(h),h)&=0,
\end{aligned}
$$

where $\Omega = \{(y,h) \in \mathbb{R}_+^2 : y_3(h) \leqslant y \leqslant y_1(h)\}$. It holds that $\tilde{V}(y,h) = \overline{C}_1(h)y^2 + \overline{C}_2(h) + \frac{\log y}{\kappa^2}$, where $\overline{C}_1(h)$ and $\overline{C}_2(h)$ satisfy

$$\overline{C}_1(h)y_1(h)^2 + \overline{C}_2(h) + \frac{\log y_1(h)}{\kappa^2} = 0,$$

$$\overline{C}_1'(h)y_3(h)^2 + \overline{C}_2'(h) = 0.$$

Applying Itô's formula to $\tilde{V}(Y_t(y^*), H_t^*)$, and integrating from 0 to τ_{zero}, we have that

$$\begin{aligned}
&\tilde{V}(Y_{\tau_{\text{zero}}}(y^*), H^*_{\tau_{\text{zero}}}) - \tilde{V}(y^*, H_0^*) \\
=& -\tau_{\text{zero}} - \kappa \int_0^{\tau_{\text{zero}}} Y_s(y^*)\tilde{V}_y(Y_s(y^*), H_s^*)dWs + \int_0^{\tau_{\text{zero}}} \tilde{V}_h(Y_s(y^*), H_s^*)dH_s^*.
\end{aligned}$$

Note that $\tilde{V}(Y_{\tau_{\text{zero}}}(y^*), H^*_{\tau_{\text{zero}}}) = 0$, the stochastic integral is square-integrable and thus a martingale with zero mean, and H_t^* only increases when $\tilde{V}_h(Y_s(y^*), H_s^*) = 0$, implying $\int_0^{\tau_{\text{zero}}} \tilde{V}_h(Y_s(y^*), H_s^*)dH_s^* = 0$. Together with the fact that $y^* = f(x,h)$, we can finally deduce that $\mathbb{E}[\tau_{\text{zero}}] = \tilde{V}(f(x,h),h) = \overline{C}_1(h)f(x,h)^2 + \overline{C}_2(h) + \frac{\log f(x,h)}{\kappa^2}$.

(iv) Before time τ_{lavs}, the historical consumption peak $H_t^* = h$ does not increase, and

$$\{Y_t(y^*) \leqslant y_3(h)\} = \left\{ -\kappa W_t - \frac{\kappa^2}{2}t \leqslant -\log\left(\frac{y^* h^{1-\beta_1}}{(1-\lambda)^{\beta_1}}\right)\right\}.$$

Then, by equation (9.1) in Rogers and Williams (2000), let $b = \frac{1}{\kappa}\log\left(\frac{y^* h^{1-\beta_1}}{(1-\lambda)^{\beta_1}}\right)$, $c = \frac{\kappa}{2}$, $\beta = \sqrt{c^2 + 2\nu} - c$, it follows that for any $\nu > 0$: $\mathbb{E}[e^{-\nu\tau_{\text{lavs}}}] = e^{-b\beta}$. Then, it holds that

$$\mathbb{E}[\tau_{\text{lavs}}] = -\frac{d\mathbb{E}[e^{-\nu\tau_{\text{lavs}}}]}{d\nu}\bigg|_{\nu\downarrow 0} = \frac{b}{c} = \frac{2}{\kappa^2}\log\left(\frac{y^* h^{1-\beta_1}}{(1-\lambda)^{\beta_1}}\right) = \frac{2}{\kappa^2}\log\left(\frac{f(x,h)h^{1-\beta_1}}{(1-\lambda)^{\beta_1}}\right).$$

□

A.2 Proofs for Chaptper 4

A.2.1 Proof of Proposition 4.1

It is easy to show that the general solution of linear ODE (4.2.10) admits the piecewise form in each region that

$$v(y,h)=\begin{cases} C_1(h)y^{r_1}+C_2(h)y^{r_2}+\frac{(\nu h)^{\gamma_1}}{(r+\lambda)\gamma_1 h^{\alpha\gamma_1}}-\frac{\nu h}{r+\lambda}y, & \text{if } >\nu^{\gamma_1-1}h^{(1-\alpha)\gamma_1-1} \\ C_3(h)y^{r_1}+C_4(h)y^{r_2}+\frac{2h^{\alpha\beta_1}}{\kappa^2\beta_1(\beta_1-r_1)(\beta_1-r_2)}y^{\beta_1}, & \text{if } h^{(1-\alpha)\gamma_1-1}\leqslant y\leqslant \nu^{\gamma_1-1}h^{(1-\alpha)\gamma_1-1}, \\ C_5(h)y^{r_1}+C_6(h)y^{r_2}+\frac{1}{(r+\lambda)\gamma_1}h^{(1-\alpha)\gamma}-\frac{h}{r+\lambda}y, & \text{if } (1-\alpha)h^{(1-\alpha)\gamma_1-1}\leqslant y< h^{(1-\alpha)\gamma_1-1}, \end{cases} \tag{A.2.19}$$

where $C_1(\cdot),\cdots,C_6(\cdot)$ are functions of h to be determined.

The free boundary condition $v_y(y,h)\to -\frac{\nu h}{r+\lambda}$ in (4.2.12) implies that $y\to+\infty$. Together with the free boundary conditions in (4.2.12) and the formula of $v(y,h)$ in the region $y>\nu^{\gamma_1-1}h^{(1-\alpha)\gamma_1-1}$, we deduce $C_1(h)\equiv 0$. To determine the left parameters, we consider the smooth-fit conditions with respect to the variable y at two free boundary points $y=y_1(h)=\nu^{\gamma_1-1}h^{(1-\alpha)\gamma_1-1}$ and $y=y_2(h)=h^{(1-\alpha)\gamma-1}$, that is,

$$
\begin{aligned}
&- C_3(h)y_1(h)^{r_1} + (C_2(h) - C_4(h))y_1(h)^{r_2} \\
&= \frac{2h^{\alpha\beta_1}}{\kappa^2\beta_1(\beta_1 - r_1)(\beta_1 - r_2)}y_1(h)^{\beta_1} + \frac{\nu h}{r+\lambda}y_1(h) - \frac{(\nu h)^\gamma}{(r+\lambda)\gamma h^{\alpha\gamma}}, \\
&- r_1 C_3(h)y_1(h)^{r_1-1} + r_2(C_2(h) - C_4(h))y_1(h)^{r_2-1} \\
&= \frac{2h^{\alpha\beta_1}}{\kappa^2(\beta_1 - r_1)(\beta_1 - r_2)}y_1(h)^{\beta_1-1} + \frac{\nu h}{r+\lambda}, \\
&(C_3(h) - C_5(h))y_2(h)^{r_1} + (C_4(h) - C_6(h))y_2(h)^{r_2} \\
&= -\frac{2h^{\alpha\beta_1}}{\kappa^2\beta_1(\beta_1 - r_1)(\beta_1 - r_2)}y_2(h)^{\beta_1} + \frac{1}{(r+\lambda)\gamma}h^{(1-\alpha)\gamma} - \frac{h}{r+\lambda}y_2(h), \\
&r_1(C_3(h) - C_5(h))y_2(h)^{r_1-1} + r_2(C_4(h) - C_6(h))y_2(h)^{r_2-1} \\
&= -\frac{2h^{\alpha\beta_1}}{\kappa^2(\beta_1 - r_1)(\beta_1 - r_2)}y_2(h)^{\beta_1-1} - \frac{h}{r+\lambda}.
\end{aligned}
\tag{A.2.20}
$$

Then the equations (A.2.20) are linear equations for $C_3(h)$, $C_2(h) - C_4(h)$, and $C_3(h) - C_5(h)$ and $C_4(h) - C_6(h)$. By solving the above two systems, we can obtain

$$
\begin{aligned}
C_3(h) &= \frac{1-\beta_1}{(r+\lambda)(r_1 - r_2)(\beta_1 - r_1)}(\nu h)^{r_2\gamma_1 + r_1}h^{-r_2\alpha\gamma_1}, \\
C_2(h) - C_4(h) &= \frac{1-\beta_1}{(r+\lambda)(r_1 - r_2)(\beta_1 - r_2)}(\nu h)^{r_1\gamma_1 + r_2}h^{-r_1\alpha\gamma_1}, \\
C_3(h) - C_5(h) &= \frac{1-\beta_1}{(r+\lambda)(r_1 - r_2)(\beta_1 - r_1)}h^{r_2(1-\alpha)\gamma_1 + r_1}, \\
C_4(h) - C_6(h) &= \frac{\beta_1 - 1}{(r+\lambda)(r_1 - r_2)(\beta_1 - r_2)}h^{r_1(1-\alpha)\gamma_1 + r_2},
\end{aligned}
\tag{A.2.21}
$$

therefore, $C_2(h)$ to $C_5(h)$ can be written by (4.2.14).

To obtain $C_2(h), C_4(h)$ and $C_6(h)$, we aim to find $C_6(h)$ first, and then $C_4(h)$ and $C_2(h)$ can be determined. Indeed, as $h \to +\infty$, we obtain $y \to 0$ in the region

$(1-\alpha)h^{(1-\alpha)\gamma_1-1} \leqslant y < h^{(1-\alpha)\gamma_1-1}$, and the boundary condition (4.2.11) leads to

$$\lim_{h\to+\infty} \frac{h}{v_y((1-\alpha)h^{(1-\alpha)\gamma_1-1}, h)} = C,$$

where C is a negative constant. Along the free boundary, we have

$$v_y((1-\alpha)h^{(1-\alpha)\gamma_1-1}, h) = r_1 C_5(h)\big((1-\alpha)h^{(1-\alpha)\gamma_1-1}\big)^{r_1-1} + r_2 C_6(h)\big((1-\alpha)h^{(1-\alpha)\gamma_1-1}\big)^{r_2-1} + \frac{h}{r+\lambda}.$$

It follows from $\lim_{h\to+\infty} \frac{h}{v_y((1-\alpha)h^{(1-\alpha)\gamma-1}, h)} < 0$ that $v_y((1-\alpha)h^{(1-\alpha)\gamma-1}, h) = O(h)$ as $h \to +\infty$. Therefore, we can deduce that

$$C_6(h) = O(C_5(h)h^{(r_1-r_2)((1-\alpha)\gamma_1-1)}) + O(h^{r_1(1-\alpha)\gamma_1+r_2}).$$

From the asymptotic property of $C_5(h)$ in Lemma A.2.2, it follows that

$$C_6(h) = O(C_5(h)h^{(r_1-r_2)((1-\alpha)\gamma_1-1)}) + O(h^{r_1(1-\alpha)\gamma_1+r_2}) = O(h^{r_1(1-\alpha)\gamma_1+r_2}),$$

as $h \to +\infty$. By **Assumption (A1)**, we have $\lim_{h\to+\infty} C_6(h) = 0$, and thus we have $C_6(h) = -\int_h^\infty C_6'(s)ds$.

In addition, to obtain $C_6'(h)$, we apply the free boundary condition (4.2.13) at point $y = (1-\alpha)h^{(1-\alpha)\gamma_1-1}$ such that

$$C_5'(h)\big((1-\alpha)h^{(1-\alpha)\gamma_1-1}\big)^{r_1} + C_6'(h)\big((1-\alpha)h^{(1-\alpha)\gamma_1-1}\big)^{r_2} + \frac{1-\alpha}{r+\lambda}h^{(1-\alpha)\gamma_1-1} - \frac{1-\alpha}{r+\lambda}h^{(1-\alpha)\gamma_1-1} = 0,$$

which yields

$$\begin{aligned} C_6'(h) &= -(1-\alpha)^{r_1-r_2} C_5'(h) h^{(r_1-r_2)((1-\alpha)\gamma_1-1)} \\ &= \frac{(1-\alpha)^{r_1-r_2}(1-\beta_1)(r_2(1-\alpha)\gamma_1+r_1)}{(r+\lambda)(r_1-r_2)(\beta_1-r_1)}(1-\nu^{r_2\gamma_1+r_1})h^{r_1((1-\alpha)\gamma_1-1)}. \end{aligned}$$

As a result, we conclude that

$$C_6(h) = -\int_h^\infty C_6'(s)ds = \frac{(1-\alpha)^{r_1-r_2}(1-\beta_1)(r_2(1-\alpha)\gamma_1+r_1)}{(r+\lambda)(r_1-r_2)(\beta_1-r_1)(r_1(1-\alpha)\gamma_1+r_2)}(1-\nu^{r_2\gamma_1+r_1})h^{r_1(1-\alpha)\gamma_1+r_2}.$$

A.2.2 Proof of Theorem 4.1

Similar to Deng et al. (2022), we need to show that the solution of the HJB equation (4.2.3) coincides with the value function, i.e. there exists $(\pi^*, c^*, b^*) \in \mathcal{A}(x)$ such that $u(x,h) = \mathbb{E}\Big[\int_0^\infty e^{-rt} u(c_t^*, H_t^*) dt\Big]$. For any admissible strategy $(\pi, c) \in \mathcal{A}(x)$, similar to the proof of Lemma 1 in Arun (2012), we have

$$\mathbb{E}\left[\int_0^\infty (c_t + \lambda b_t) M_t dt\right] \leqslant x. \tag{A.2.22}$$

Let h be the fixed parameter, the dual transform of $U(c,h) + \lambda V(b)$ with respect to c and b in the constrained domain that $\widetilde{V}(q,h) := \sup_{c \in [\nu h, h]} \big[U(c,h) - cq\big] + \lambda \sup_{b \geqslant 0} \big[V(b) - bq\big]$ defined in (4.2.7). Moreover, $\widetilde{V}$ can be attained by the construction of the feedback optimal control $c^\dagger(y,h)$ in (4.2.16).

In what follows, we distinguish the two reference processes, namely $H_t := h \vee \sup_{s \leqslant t} c_s$ and $H_t^\dagger(y) := h \vee \sup_{s \leqslant t} c^\dagger(Y_s(y), H_s^\dagger(y))$ that correspond to the reference process under an arbitrary consumption process c_t and under the optimal consumption process $c^\dagger$ with an arbitrary $y > 0$. Note that the global optimal reference process shall be defined later by $H_t^* := H_t^\dagger(y^*)$ with $y^* > 0$ to be determined. Let us now further introduce

$$\hat{H}_t(y) := h \vee \left((1-\alpha)^{-\frac{1}{(1-\alpha)\gamma_1 - 1}} \Big(\inf_{s \leqslant t} Y_s(y) \Big)^{\frac{1}{(1-\alpha)\gamma_1 - 1}} \right), \tag{A.2.23}$$

where $Y_t(y) = y e^{rt} M_t$ is the discounted martingale measure density process.

For any admissible controls $(\pi, c) \in \mathcal{A}(x)$, recall the reference process $H_t = h \vee$

$\sup_{s\leqslant t} c_s$, and for all $y>0$, we see that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\infty e^{-(r+\lambda)t}U(c_t,H_t)dt+\lambda\int_0^\infty e^{-(r+\lambda)t}V(b_t)dt\right]\\
&=\mathbb{E}\bigg[\int_0^\infty e^{-(r+\lambda)t}(U(c_t,H_t)-Y_t(y)c_t)dt\bigg]+\lambda\mathbb{E}\bigg[\int_0^\infty e^{-(r+\lambda)t}(V(b_t)-Y_t(y)b_t)dt\bigg]\\
&\quad+y\mathbb{E}\bigg[\int_0^\infty(c_t+\lambda b_t)M_tdt\bigg]\\
&\leqslant\mathbb{E}\bigg[\int_0^\infty e^{-(r+\lambda)t}\tilde{V}(Y_t(y),H_t^\dagger(y))dt\bigg]+yx\\
&=\mathbb{E}\bigg[\int_0^\infty e^{-(r+\lambda)t}\tilde{V}(Y_t(y),\hat{H}_t(y))dt\bigg]+yx\\
&=v(y,h)+yx.
\end{aligned}\tag{A.2.24}$$

The third equation holds because of Lemma A.12, and the last equation is verified by Lemma A.11. In addition, Lemma A.13 guarantees the inequality, and shows that it becomes an equality with the choices of $c_t^*=c^\dagger(Y_t(y^*),H_t^\dagger(y^*))$ and $b_t^*=b^\dagger(Y_t(y^*),H_t^\dagger(y^*))$, in which y^* is the solution to the equation $\mathbb{E}\big[\int_0^\infty(c^\dagger(Y_t(y^*),H_t^\dagger(y^*))+\lambda b^\dagger(Y_t(y^*),H_t^\dagger(y^*)))\ M_tdt\big]=x$ for a given $x\geqslant\frac{\nu h}{r+\lambda}$. In conclusion, we have

$$\begin{aligned}
&\sup_{(\pi,c)\in\mathcal{A}(x)}\mathbb{E}\left[\int_0^\infty e^{-(r+\lambda)t}U(c_t,H_t)dt+\lambda\int_0^\infty e^{-(r+\lambda)t}V(b_t)dt\right]\\
&=\inf_{y>0}(v(y,h)+yx)=u(x,h).
\end{aligned}$$

The proof of the theorem is also based on some auxiliary results. We present some asymptotic results on the coefficients in Proposition 4.1, whose proof is straightforward and hence omitted.

Remark. *Based on the semi-analytical forms in Proposition (4.1), we note that*

$$C_2(h)=O(h^{r_1(1-\alpha)\gamma_1+r_2}),\ C_3(h)=O(h^{r_2(1-\alpha)\gamma_1+r_1}),\ C_4(h)=O(h^{r_1(1-\alpha)\gamma_1+r_2}),$$

$$C_5(h)=O(h^{r_2(1-\alpha)\gamma_1+r_1}),\ C_6(h)=O(h^{r_1(1-\alpha)\gamma_1+r_2}),$$

as $h \to +\infty$, which are used in later proofs.

By following similar proofs of Lemma 5.1 to Lemma 5.3 in Deng et al. (2022) and using asymptotic results in Remark A.2.2, we can readily obtain the next three lemmas.

Lemma A.11. *For any $y > 0$ and $h \geqslant 0$, the dual transform $v(y, h)$ of the value function $u(x, h)$ satisfies*

$$v(y,h) = \mathbb{E}\bigg[\int_0^\infty e^{-rt}[\tilde{V}(Y_t(y), \hat{H}_t(y)) + \bar{V}(Y_t(y))]dt\bigg],$$

where $Y_t(\cdot)$ and $\hat{H}_t(\cdot)$ are defined in (A.2.23).

Lemma A.12. *For all $y > 0$, we have $H_t^\dagger = \hat{H}_t(y)$, $t \geqslant 0$, and hence*

$$\mathbb{E}\bigg[\int_0^\infty e^{-rt}\tilde{V}(Y_t(y), H_t^\dagger(y))dt\bigg] = \mathbb{E}\bigg[\int_0^\infty e^{-rt}\tilde{V}(Y_t(y), \hat{H}_t(y))dt\bigg].$$

Lemma A.13. *The inequality in (A.2.24) becomes equality with $c_t^* = c^\dagger(Y_t(y^*), \hat{H}_t(y^*))$ and $b_t^* = b^\dagger(Y_t(y^*), \hat{H}_t(y^*))$, $t \geqslant 0$, with $y^* = y^*(x, h)$ as the unique solution to*

$$\mathbb{E}[\int_0^\infty (c^\dagger(Y_t(y^*), \hat{H}_t(y^*)) + \lambda b^\dagger(Y_t(y^*), \hat{H}_t(y^*)))M_t dt] = x.$$

Let us continue to prove some other auxiliry results.

Lemma A.14. *The following transversality condition holds that for all $y > 0$,*

$$\lim_{T\to+\infty} \mathbb{E}\bigg[e^{-rT}v(Y_T(y), \hat{H}_T(y))\bigg] = 0.$$

Proof. Let us recall that

$$\hat{H}_t(y) := h \vee \bigg((1-\alpha)^{-\frac{1}{(1-\alpha)\gamma_1 - 1}}(\inf_{s\leqslant t} Y_s(y))^{\frac{1}{(1-\alpha)\gamma_1 - 1}}\bigg).$$

Let us first consider the case $c_T = 0$. We first write that

$$e^{-rT}\mathbb{E}[v(Y_T(y), \hat{H}_T(y))] = e^{-rT}\mathbb{E}\Big[C_2(\hat{H}_T(y))Y_T(y)^{r_2} + \frac{2\lambda K^{1-\beta_2}}{\kappa^2\beta_2(\beta_2 - r_1)(\beta_2 - r_2)}Y_T(y)^{\beta_2} + \frac{\nu^\gamma}{(r+\lambda)\gamma_1}\hat{H}_T(y)^{(1-\alpha)\gamma} - \frac{\nu\hat{H}_T(y)}{r+\lambda}Y_T(y)\Big], \tag{A.2.25}$$

where the last two terms can vanish due to Lemma A.16 and Lemma A.18 respectively, and the last third term can also vanish because of Lemma A.17 and the fact $\beta_2 > r_2$ by **Assumption (A1)**. For the first term in (A.2.25), since $Y_T(y) > \hat{H}_T(y)^{(1-\alpha)\gamma_1 - 1}$, we have

$$\begin{aligned} e^{-rT}\mathbb{E}\Big[C_2(\hat{H}_T(y))(Y_T(y))^{r_2}\Big] &= O\big(e^{-rT}C_2(\hat{H}_T(y))\hat{H}_T(y)^{r_2((1-\alpha)\gamma_1-1)}\big) \\ &= O\big(e^{-rT}\hat{H}_T(y)^{r_1(1-\alpha)\gamma_1+r_2}\hat{H}_T(y)^{r_2((1-\alpha)\gamma_1-1)}\big) \\ &= O\big(e^{-rT}\hat{H}_T(y)^{(1-\alpha)\gamma_1}\big), \end{aligned}$$

which vanishes as $T \to +\infty$ due to Lemma A.16.

We then consider the case $0 < c_T < \hat{H}_T(y)$.

$$\begin{aligned} &\mathbb{E}[e^{-rT}v(Y_T(y), \hat{H}_T(y))] \\ =&e^{-rT}\mathbb{E}\Big[C_3(\hat{H}_T(y))Y_T(y)^{r_1} + C_4(\hat{H}_T(y))Y_T(y)^{r_2} \\ &+ \frac{2\lambda K^{1-\beta_2}}{\kappa^2\beta_2(\beta_2 - r_1)(\beta_2 - r_2)}Y_T(y)^{\beta_2} + \frac{2\hat{H}_T(y)^{\alpha\beta_1}}{\kappa^2\beta_1(\beta_1 - r_1)(\beta_1 - r_2)}Y_T(y)^{\beta_1}\Big]. \end{aligned} \tag{A.2.26}$$

We consider asymptotic behavior of the above equation term by term as $T \to +\infty$.

Thanks to **Assumption (A1), $\beta_2 > r_2$,** and the third term can vanish due to Lemma A.17. For the fourth term in (A.2.26), since $Y_T(y) \geqslant \hat{H}_T(y)^{(1-\alpha)\gamma_1 - 1}$ and $\beta_1 = \frac{\gamma_1}{\gamma_1 - 1} < 0$, we have

$$\mathbb{E}[e^{-rT}\hat{H}_T(y)^{\alpha\beta_1}Y_T(y)^{\beta_1}] = O\big(e^{-rT}\mathbb{E}\big[\hat{H}_T(y)^{\alpha\beta_1+\beta_1((1-\alpha)\gamma_1-1)}\big]\big) = O\big(e^{-rT}\mathbb{E}\big[\hat{H}_T(y)^{(1-\alpha)\gamma_1}\big]\big),$$

which also vanishes as $T \to +\infty$ due to Lemma A.16.

Let us consider the terms containing $C_3(\hat{H}_T(y))$ and $C_4(\hat{H}_T(y))$ in equation (A.2.26). Because of the constraint $Y_t(y) = O(\hat{H}_T(y)^{(1-\alpha)\gamma_1 - 1})$, we can deduce that

$$\begin{aligned} e^{-rT}\mathbb{E}\Big[C_3(\hat{H}_T(y))(Y_T(y))^{r_1}\Big] &= O\big(e^{-rT}C_3(\hat{H}_T(y))\hat{H}_T(y)^{r_1((1-\alpha)\gamma_1-1)}\big) \\ &= O\big(e^{-rT}\hat{H}_T(y)^{r_2(1-\alpha)\gamma_1+r_1}\hat{H}_T(y)^{r_1((1-\alpha)\gamma_1-1)}\big) \\ &= O\big(e^{-rT}\hat{H}_T(y)^{(1-\alpha)\gamma_1}\big), \end{aligned}$$

which converges to 0 by Lemma A.16.

In addition, since $Y_T(y) \geqslant \hat{H}_T(y)^{(1-\alpha)\gamma_1-1}$ and $r_2 < 0$, we obtain

$$\begin{aligned} e^{-rT}\mathbb{E}\Big[C_4(\hat{H}_T(y))(Y_T(y))^{r_2}\Big] &= O\big(e^{-rT}C_4(\hat{H}_T(y))\hat{H}_T(y)^{r_2((1-\alpha)\gamma_1-1)}\big) \\ &= O\big(e^{-rT}\hat{H}_T(y)^{r_1(1-\alpha)\gamma_1+r_2}\hat{H}_T(y)^{r_2((1-\alpha)\gamma_1-1)}\big) \\ &= O\big(e^{-rT}\hat{H}_T(y)^{(1-\alpha)\gamma_1}\big), \end{aligned}$$

which vanishes as $T \to +\infty$ by Lemma A.16.

Finally, we consider the case $C_T = \hat{H}_T(y)$ and write that

$$\begin{aligned} &\mathbb{E}[e^{-rT}v(Y_T(y), \hat{H}_T(y))] \\ =&e^{-rT}\mathbb{E}\Big[C_5(\hat{H}_T(y))Y_T(y)^{r_1} + C_6(\hat{H}_T(y))Y_T(y)^{r_2} \\ &+ \frac{2\lambda K^{1-\beta_2}}{\kappa^2\beta_2(\beta_2-r_1)(\beta_2-r_2)}Y_T(y)^{\beta_2} + \frac{1}{(r+\lambda)\gamma_1}\hat{H}_T(y)^{(1-\alpha)\gamma_1} - \frac{\hat{H}_T(y)}{(r+\lambda)}Y_T(y)\Big], \end{aligned} \tag{A.2.27}$$

where the last three terms converge to 0 by Lemma A.17 with **Assumption (A1)**, Lemma A.16, and Lemma A.18, respectively.

For the first term in (A.2.27), since $Y_T(y) \leqslant \hat{H}_T(y)^{(1-\alpha)\gamma_1-1}$, we have

$$e^{-rT}\mathbb{E}\Big[C_5(\hat{H}_T(y))(Y_T(y))^{r_1}\Big] = O\big(e^{-rT}C_5(\hat{H}_T(y))\hat{H}_T(y)^{r_1((1-\alpha)\gamma_1-1)}\big)$$
$$= O\big(e^{-rT}\hat{H}_T(y)^{r_2(1-\alpha)\gamma_1+r_1}\hat{H}_T(y)^{r_1((1-\alpha)\gamma_1-1)}\big)$$
$$= O\big(e^{-rT}\hat{H}_T(y)^{(1-\alpha)\gamma_1}\big),$$

which converges to 0 as $T \to +\infty$ by Lemma A.16.

For the second term in (A.2.27), by $Y_T(y) \geqslant (1-\alpha)\hat{H}_T(y)^{(1-\alpha)\gamma_1-1}$ and $r_2 < 0$, we have

$$e^{-rT}\mathbb{E}\Big[C_6(\hat{H}_T(y))(Y_T(y))^{r_2}\Big] = O\big(e^{-rT}C_6(\hat{H}_T(y))\hat{H}_T(y)^{r_2((1-\alpha)\gamma_1-1)}\big)$$
$$= O\big(e^{-rT}\hat{H}_T(y)^{r_1(1-\alpha)\gamma_1+r_2}\hat{H}_T(y)^{r_2((1-\alpha)\gamma_1-1)}\big)$$
$$= O\big(e^{-rT}\hat{H}_T(y)^{(1-\alpha)\gamma_1}\big),$$

which also vanishes as $T \to +\infty$ by Lemma A.16. Therefore, we obtain the desired result. □

Lemma A.15. *For any $T > 0$, we have*

$$\lim_{n\to+\infty}\mathbb{E}\big[e^{-r\tau_n}v(Y_{\tau_n}(y),\hat{H}_{\tau_n}(y))\mathbf{1}_{\{T>\tau_n\}}\big] = 0,$$

where τ_n is defined by

$$\tau_n = \inf\{t \geqslant 0 | Y_t(y) \geqslant n, \hat{H}_t(y) \geqslant \big((1-\alpha)n\big)^{-\frac{1}{(1-\alpha)\gamma_1-1}}\}.$$

Proof. By the definition of τ_n, for all $t \leqslant \tau_n$, we have $Y_t(y) \in \big[\frac{1}{n}, n\big]$, and thus

$$h \leqslant \hat{H}_t(y) \leqslant \max(h, \big((1-\alpha)n\big)^{-\frac{1}{(1-\alpha)\gamma_1-1}}) = O(1) + O(n^{-\frac{1}{(1-\alpha)\gamma_1-1}}).$$

Therefore, we have $Y_t(y)^{r_1} \leqslant n^{r_1}$, $Y_t(y)^{r_2} \leqslant \big(\frac{1}{n}\big)^{r_2} = n^{-r_2}$. Then we shall obtain the order of $v(Y_{\tau_n}(y), \hat{H}_{\tau_n}(y))$ in three cases, in the sense that $c^*_{\tau_n} = 0$, $0 < c^*_{\tau_n} < \hat{H}_{\tau_n}(y)$, and $c^*_{\tau_n} = \hat{H}_{\tau_n}(y)$.

Similar to the proof of Lemma A.14, if $c^*_{\tau_n} = 0$, we have

$$
\begin{aligned}
v(Y_{\tau_n}(y), \hat{H}_{\tau_n}(y)) &= C_2(\hat{H}_{\tau_n}(y))Y_{\tau_n}(y)^{r_2} + \frac{2\lambda K^{1-\beta_2}}{\kappa^2\beta_2(\beta_2 - r_1)(\beta_2 - r_2)}Y_{\tau_n}(y)^{\beta_2} \\
&\quad + \frac{\nu^\gamma}{(r+\lambda)\gamma_1}\hat{H}_{\tau_n}(y)^{(1-\alpha)\gamma_1} - \frac{\nu \hat{H}_{\tau_n}(y)}{r+\lambda}Y_{\tau_n}(y) \\
&= O(n^{-r_2}) + O(n^{-\beta_2}) + O(n^{-\frac{(1-\alpha)\gamma_1}{(1-\alpha)\gamma_1 - 1}}) + O(n^{\frac{(1-\alpha)\gamma_1 - 2}{(1-\alpha)\gamma_1 - 1}}) \\
&= O(n^{r^*}),
\end{aligned}
$$

where $r^* := \max\big\{ -r_2, -\beta_1, \frac{(1-\alpha)\gamma_1-2}{(1-\alpha)\gamma_1-1}, \frac{(r_1-r_2)(1-\alpha)\gamma_1-2r_1}{(\alpha-1)\gamma_1-1} \big\}$. Here, we have $-\beta_2 < -r_2$ by **Assumption (A1)**. If $0 < c^*_{\tau_n} < \hat{H}_{\tau_n}(y)$, we have

$$
\begin{aligned}
v(Y_{\tau_n}(y), \hat{H}_{\tau_n}(y)) =& C_3(\hat{H}_{\tau_n}(y))Y_{\tau_n}(y)^{r_1} + C_4(\hat{H}_{\tau_n}(y))Y_{\tau_n}(y)^{r_2} \\
& + \frac{2\lambda K^{1-\beta_2}}{\kappa^2\beta_2(\beta_2 - r_1)(\beta_2 - r_2)}Y_{\tau_n}(y)^{\beta_2} + \frac{2\hat{H}_{\tau_n}(y)^{\alpha\beta_1}}{\kappa^2\beta_1(\beta_1 - r_1)(\beta_1 - r_2)}Y_{\tau_n}(y)^{\beta_1} \\
=& O(n^{\frac{(r_1-r_2)(1-\alpha)\gamma_1-2r_1}{(\alpha-1)\gamma_1-1}}) + O(n^{-r_2}) + O(n^{-\beta_2}) + O(n^{-\beta_1}) \\
=& O(n^{r^*}).
\end{aligned}
$$

If $c_{\tau_n} = \hat{H}_{\tau_n}(y)$, we have

$$
\begin{aligned}
v(Y_{\tau_n}(y), \hat{H}_{\tau_n}(y)) &= C_5(\hat{H}_{\tau_n}(y))Y_{\tau_n}(y)^{r_1} + C_6(\hat{H}_{\tau_n}(y))Y_{\tau_n}(y)^{r_2} \\
&\quad + \frac{2\lambda K^{1-\beta_2}}{\kappa^2\beta_2(\beta_2 - r_1)(\beta_2 - r_2)}Y_{\tau_n}(y)^{\beta_2} + \frac{1}{(r+\lambda)\gamma_1}\hat{H}_{\tau_n}(y)^{(1-\alpha)\gamma_1} - \frac{\hat{H}_{\tau_n}(y)}{r+\lambda}Y_T(y) \\
&= O(n^{\frac{(r_1-r_2)(1-\alpha)\gamma_1-2r_1}{(\alpha-1)\gamma_1-1}}) + O(n^{-r_2}) + O(n^{-\beta_2}) + O(n^{\frac{-(1-\alpha)\gamma_1}{(1-\alpha)\gamma_1-1}}) + O(n^{\frac{(1-\alpha)\gamma_1-2}{(1-\alpha)\gamma_1-1}}) \\
&= O(n^{r^*}).
\end{aligned}
$$

Therefore, in all the cases, $v(Y_{\tau_n}(y), \hat{H}_{\tau_n}(y)) = O(n^{r^*})$. In addition, similar to the proof of (A.25) in Guasoni et al. (2020), there exists some constant C such that

$$
\mathbb{E}[\mathbf{1}_{\{\tau_n \leqslant T\}}] \leqslant n^{-2\xi}(1 + y^{2\xi})e^{CT},
$$

for any $\xi \geqslant 1$. Putting all the pieces together, the desired claim holds that

$$\lim_{n\to+\infty} \mathbb{E}\big[e^{-r\tau_n} v(Y_{\tau_n}(y), \hat{H}_{\tau_n}(y))\mathbf{1}_{\{T>\tau_n\}}\big] = 0.$$

□

Lemma A.16. *For γ_1 that satisfies **Assumption (A1)**, we have*

$$\lim_{T\to+\infty} \mathbb{E}\Big[e^{-rT}\hat{H}_T(y)^{\gamma_1^*}\Big] = 0, \tag{A.2.28}$$

where $\gamma_1^* := (1-\alpha)\gamma_1$.

Proof. Let $\beta_1^* := \frac{\gamma_1^*}{\gamma_1^*-1}$. It is obvious that

$$\begin{aligned} e^{-rT}\mathbb{E}\Big[\hat{H}_T(y)^{(1-\alpha)\gamma_1}\Big] &\leqslant e^{-rT}\mathbb{E}\Big[\sup_{s\leqslant T}(1-\alpha)^{-\frac{(1-\alpha)\gamma_1}{(1-\alpha)\gamma_1-1}} Y_s(y)^{\frac{(1-\alpha)\gamma_1}{(1-\alpha)\gamma_1-1}}\Big] + e^{-rT}\mathbb{E}[h^{(1-\alpha)\gamma_1}] \\ &= e^{-rT}\mathbb{E}\Big[\sup_{s\leqslant T}(1-\alpha)^{-\beta_1^*}Y_s(y)^{\beta_1^*}\Big] + e^{-rT}\mathbb{E}[h^{(1-\alpha)\gamma_1}], \end{aligned}$$

in which it is clear that $e^{-rT}\mathbb{E}[h^{(1-\alpha)\gamma_1}] = O(e^{-rT})$ as $T\to+\infty$.

Then we consider the first term $e^{-rT}\mathbb{E}[\sup_{s\leqslant T}(1-\alpha)^{-\beta_1^*}Y_s(y)^{\beta_1^*}]$. Define $W_t^{(\frac{1}{2}\kappa)} = W_t + \frac{1}{2}\kappa t$, which is also a Brownian motion under the equivalent measure $\mathbb{Q}$, with its running maximum $(W_t^{(\frac{1}{2}\kappa)})^*$. It follows that

$$\begin{aligned} e^{-rT}\mathbb{E}\Big[\hat{H}_T(y)^{(1-\alpha)\gamma_1}\Big] &\leqslant e^{-rT}\mathbb{E}\Big[\sup_{s\leqslant T}(1-\alpha)^{-\beta_1^*}Y_s(y)^{\beta_1^*}\Big] \\ =& e^{-rT}O\bigg(\mathbb{E}\bigg[\exp\bigg\{-\beta_1^*\sup_{s\leqslant T}\big(\frac{1}{2}\kappa^2 s + \kappa W_s\big)\bigg\}\bigg]\bigg) \\ =& e^{-rT}O\bigg(\mathbb{E}\bigg[\exp\bigg\{-\kappa\beta_1^*\sup_{s\leqslant T}W_s^{(\frac{1}{2}\kappa)}\bigg\}\bigg]\bigg) \\ :=& e^{-rT}O\bigg(\mathbb{E}\bigg[\exp\bigg\{aW_T^{(\zeta)} + b\Big(W_T^{(\zeta)}\Big)^*\bigg\}\mathbf{1}_{\{(W_T^{(\zeta)})^*\geqslant k\}}\bigg]\bigg), \end{aligned}$$

where $a=0$, $b=-\beta_1^*\kappa>0$, $\zeta=\frac{1}{2}\kappa>0$, and $k=0$. Note that $2a+b+2\zeta>2a+b+\zeta>0$, thanks to Corollary A.7 in Guasoni et al. (2020), we have

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left\{aW_T^{(\zeta)}+b\left(W_T^{(\zeta)}\right)^*\right\}\mathbf{1}_{\left\{\left(W_T^{(\zeta)}\right)^*\geqslant k\right\}}\right]\\
=&\frac{2(a+b+\zeta)}{2a+b+2\zeta}\exp\left\{\frac{(a+b)(a+b+2\zeta)}{2}T\right\}\Phi\left((a+b+\zeta)\sqrt{T}-\frac{k}{\sqrt{T}}\right)\\
+&\frac{2(a+\zeta)}{2a+b+2\zeta}\exp\left\{(2a+b+2\zeta)k+\frac{a(a+2\zeta)}{2}T\right\}\Phi\left(-(a+\zeta)\sqrt{T}-\frac{k}{\sqrt{T}}\right),
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\lim_{T\to+\infty}\frac{1}{T}\log\mathbb{E}\left[\exp\left\{aW_T^{(\zeta)}+b\left(W_T^{(\zeta)}\right)^*\right\}\mathbf{1}_{\left\{\left(W_T^{(\zeta)}\right)^*\geqslant k\right\}}\right]-r\\
=&\frac{(a+b)(a+b+2\xi)}{2}-r=\frac{\kappa^2}{2}\beta_1^*(\beta_1^*-1)-r=\frac{\kappa^2}{2}(\beta_1^*-r_1)(\beta_1^*-r_2).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&e^{-rT}\mathbb{E}\left[\exp\left\{aW_T^{(\zeta)}+b\left(W_T^{(\zeta)}\right)^*\mathbf{1}_{\left\{\left(W_T^{(\zeta)}\right)^*\geqslant k\right\}}\right\}\right.\\
&=\exp\left\{\left(\frac{1}{T}\log\mathbb{E}\left[\exp\left\{aW_T^{(\zeta)}+b\left(W_T^{(\zeta)}\right)^*\right\}\mathbf{1}_{\left\{\left(W_T^{(\zeta)}\right)^*\geqslant k\right\}}\right]-r\right)T\right\}\\
&=O\left(\exp\left\{\frac{\kappa^2}{2}(\beta_1^*-r_1)(\beta_1^*-r_2)T\right\}\right),
\end{aligned}
$$

as $T\to+\infty$. Together with the fact that $r_2<\beta_1^*<r_1$ under **Assumption (A1)**, we have $(\beta_1^*-r_1)(\beta_1^*-r_2)<0$ and thus

$$
\begin{aligned}
\mathbb{E}\left[e^{-rT}\hat{H}_T(y)^{(1-\alpha)\gamma_1}\right]&=O\left(\exp\left\{\frac{\kappa^2}{2}(\beta_1^*-r_1)(\beta_1^*-r_2)T\right\}\right)+O(e^{-rT})\\
&=O\left(\exp\left\{\frac{\kappa^2}{2}(\beta_1^*-r_1)(\beta_1^*-r_2)T\right\}\right),
\end{aligned}
$$

which tends to 0 as $T\to+\infty$. □

Lemma A.17. *For $r_2 < \beta_0 < r_1$, we have*

$$\lim_{T\to+\infty} \mathbb{E}\Big[e^{-rT} Y_T(y)^{\beta_0}\Big] = 0. \tag{A.2.29}$$

Proof. In fact,

$$\begin{aligned}\mathbb{E}\Big[e^{-rT} Y_T(y)^{\beta_0}\Big] &= e^{-rT}\mathbb{E}\Big[(ye^{rT}\cdot e^{-(r+\frac{\kappa^2}{2})T-\kappa W_T})^{\beta_0}\Big]\\ &= y_1^{\gamma} e^{-rT}\mathbb{E}\big[e^{\beta_0(-\frac{\kappa^2}{2}T-\kappa W_T)}\big]\\ &= O\Big(e^{(\beta_0-r_1)(\beta_0-r_2)\frac{\kappa^2}{2}T}\Big),\end{aligned}$$

which converges to 0 in view that $r_2 < \beta_0 < r_1$. □

Lemma A.18. *For $\beta_1^* = \frac{\gamma_1^*}{\gamma_1^*-1} < 0$ with $\gamma_1^* := (1-\alpha)\gamma_1$, we have*

$$\lim_{T\to+\infty} \mathbb{E}\Big[e^{-rT}\hat{H}_T(y) Y_T(y)\Big] = 0. \tag{A.2.30}$$

Proof. In fact,

$$\mathbb{E}\Big[e^{-rT}\hat{H}_T(y)Y_T(y)\Big] \leqslant \mathbb{E}\Big[e^{-rT}hY_T(y)\Big] + \mathbb{E}\Big[e^{-rT}Y_T(y)\sup_{s\leqslant T}(1-\alpha)^{-\frac{1}{(1-\alpha)\gamma_1-1}}Y_s(y)^{\frac{1}{(1-\alpha)\gamma_1-1}}\Big],$$

where the first term converges to 0 by Lemma A.17. For the second term,

$$\begin{aligned}&\mathbb{E}\Big[e^{-rT}Y_T(y)\sup_{s\leqslant T}(1-\alpha)^{-\frac{1}{(1-\alpha)\gamma_1-1}}Y_s(y)^{\frac{1}{(1-\alpha)\gamma_1-1}}\Big]\\ &= O\bigg(\mathbb{E}\Big[e^{-rT}Y_T(y)\sup_{s\leqslant T}Y_s(y)^{\frac{1}{\gamma_1^*-1}}\Big]\bigg)\\ &= e^{-rT}O\bigg(\mathbb{E}\bigg[\exp\bigg\{-\frac{\kappa^2}{2}T-\kappa W_T-\frac{1}{\gamma_1^*-1}\sup_{s\leqslant T}\Big(\frac{\kappa^2}{2}s+\kappa W_s\Big)\bigg\}\bigg]\bigg)\\ &= e^{-rT}O\bigg(\mathbb{E}\bigg[\exp\bigg\{-\kappa W^{(\zeta)}-\frac{\kappa}{\gamma_1^*-1}\Big(W_T^{(\zeta)}\Big)^*\bigg\}\bigg]\bigg)\\ &= e^{-rT}O\bigg(\mathbb{E}\bigg[\exp\bigg\{a_1W^{(\zeta)}+b_1\Big(W_T^{(\zeta)}\Big)^*\bigg\}\mathbf{1}_{\{(W_T^{(\zeta)})^*\geqslant k\}}\bigg]\bigg),\end{aligned}$$

where $a_1 = -\kappa$, $b_1 = -\frac{\kappa}{\gamma_1^*-1} > 0$, $\zeta = \frac{1}{2}\kappa$, and $k = 0$. Note that $2a_1 + b_1 + 2\zeta = \frac{\gamma_1^*}{1-\gamma_1^*} > 0$ and $a_1 + \zeta < 0$, thanks to Corollary A.7 in Guasoni et al. (2020), we can derive

$$e^{-rT}O\Bigg(\mathbb{E}\Bigg[\exp\left\{a_1 W^{(\zeta)} + b_1\left(W_T^{(\zeta)}\right)^*\right\}\mathbf{1}_{\left\{\left(W_T^{(\zeta)}\right)^* \geqslant k\right\}}\Bigg]\Bigg)$$
$$=O\left(\exp\left\{\left(\frac{(a_1+b_1)(a_1+b_1+2\zeta)}{2} - r\right)T\right\}\right) + O\left(\exp\left\{\left(\frac{a_1(a_1+2\zeta)}{2} - r\right)T\right\}\right),$$

where the second term equals $O(\exp\{-rT\})$ as $a_1 + 2\zeta = 0$. For the first term,

$$\frac{(a_1+b_1)(a_1+b_1+2\zeta)}{2} - r = \frac{\kappa^2}{2}\left(\frac{\gamma_1^*}{\gamma_1^*-1}\cdot\frac{1}{\gamma_1^*-1}\right) - r$$
$$=\frac{\kappa^2}{2}\left(\beta_1^*(\beta_1^*-1) - \frac{2(r+\lambda)}{\kappa^2}\right) = \frac{\kappa^2}{2}(\beta_1^*-r_1)(\beta_1^*-r_2).$$

Thanks to **Assumption (A1)**, we have $\beta_1^* > r_2$, and therefore $\frac{\kappa^2}{2}(\beta_1^*-r_1)(\beta_1^*-r_2) < 0$. In summary, we complete the proof. □

A.2.3 Proof of Lemma 4.1

We prove $v_{yy}(y,h) > 0$ the three regions: $y > \nu^{\gamma_1-1}h^{(1-\alpha)\gamma_1-1}$, $h^{(1-\alpha)\gamma_1-1} \leqslant y \leqslant \nu^{\gamma_1-1}h^{(1-\alpha)\gamma_1-1}$, and $(1-\alpha)h^{(1-\alpha)\gamma-1} \leqslant y < h^{(1-\alpha)\gamma-1}$, respectively. To be more specific, we first analyze $v_{yy}(y,h)$ in the region $(1-\alpha)h^{(1-\alpha)\gamma-1} \leqslant y < h^{(1-\alpha)\gamma-1}$, then the region $h^{(1-\alpha)\gamma-1} \leqslant y \leqslant \nu^{\gamma_1-1}h^{(1-\alpha)\gamma_1-1}$, and finally the region $y > \nu^{\gamma_1-1}h^{(1-\alpha)\gamma_1-1}$.

(i) In the region $(1-\alpha)h^{(1-\alpha)\gamma-1} \leqslant y < h^{(1-\alpha)\gamma-1}$, $v_{yy}(y,h) = r_1(r_1-1)C_5(h)y^{r_1-2} + r_2(r_2-1)C_6(h)y^{r_2-2} + \frac{2\lambda(\beta_2-1)K^{1-\beta_2}}{\kappa^2(\beta_2-r_1)(\beta_2-r_2)}y^{\beta_2-2}$. Since $r_1(r_1-1) = r_2(r_2-1) = \frac{2(r+\lambda)}{\kappa^2} > 0$ and $\frac{2\lambda(\beta_2-1)K^{1-\beta_2}}{\kappa^2(\beta_2-r_1)(\beta_2-r_2)} > 0$, we only need to prove $C_5(h) \geqslant 0$ and $C_6(h) > 0$. According to (4.2.22), we can easily deduce that $C_5(h) > 0$ and $C_6(h) > 0$.

(ii) In the region $h^{(1-\alpha)\gamma_1-1} \leqslant y \leqslant \nu^{\gamma_1-1}h^{(1-\alpha)\gamma_1-1}$, because $r_1(r_1-1) = r_2(r_2-1) = \frac{2(r+\lambda)}{\kappa^2}$, we can deduce that

$$\begin{aligned}
v_{yy}(y,h) &= r_1(r_1-1)C_3(h)y^{r_1-2} + r_2(r_2-1)C_4(h)y^{r_2-2} \\
&\quad + \frac{2\lambda(\beta_2-1)K^{1-\beta_2}}{\kappa^2(\beta_2-r_1)(\beta_2-r_2)}y^{\beta_2-2} + \frac{2(\beta_1-1)h^{\alpha\beta_1}}{\kappa^2(\beta_1-r_1)(\beta_1-r_2)}y^{\beta_1-2} \\
&= \frac{2(r+\lambda)}{\kappa^2}\left(C_3(h)y^{r_1-\beta_1} + C_4(h)y^{r_2-\beta_1} + \frac{(\beta_1-1)h^{\alpha\beta_1}}{(r+\lambda)(\beta_1-r_1)(\beta_1-r_2)}\right)y^{\beta_1-2} \\
&\quad + \frac{2\lambda(\beta_2-1)K^{1-\beta_2}}{\kappa^2(\beta_2-r_1)(\beta_2-r_2)}y^{\beta_2-2}.
\end{aligned}$$

Let us define $\varphi(y) := C_3(h)y^{r_1-\beta_1} + C_4(h)y^{r_2-\beta_1} + \frac{(\beta_1-1)h^{\alpha\beta_1}}{(r+\lambda)(\beta_1-r_1)(\beta_1-r_2)}$. Because the last term in the above equation is positive, it is sufficient to verify that $\varphi(y) > 0$. We separate the proof into the following steps: (1) showing $\varphi(y)$ is either monotone or first increasing and then decreasing; (2) show $\varphi(y) > 0$ at two points $y = \nu^{\gamma_1-1}h^{(1-\alpha)\gamma_1-1}$ and $y = h^{(1-\alpha)\gamma_1-1}$.

Indeed, the extreme point $y^\dagger$ of $\varphi(y)$ should satisfy the first order condition $\varphi'(y^\dagger) = 0$, i.e.,

$$C_3(h)(r_1-\beta_1)(y^\dagger)^{r_1-\beta_1-1} + C_4(h)(r_2-\beta_1)(y^\dagger)^{r_2-\beta_1-1} = 0.$$

We remark that $C_3(h) < 0$, $r_1 - \beta_1 > 0$, while $C_4(h)(r_2-\beta_1)$ can be negative or positive. If $C_4(h)(r_2-\beta_1) \leqslant 0$, there is no solution for $y^\dagger$, hence $\varphi(y)$ is monotone. If $C_4(h)(r_2-\beta_1) > 0$, there exists a unique real solution to the above equation

$$y^\dagger = \left(\frac{(\beta_1-r_2)C_4(h)}{(r_1-\beta_1)C_3(h)}\right)^{\frac{1}{r_1-r_2}},$$

which might fall into the interval $\left[h^{(1-\alpha)\gamma-1}, \nu^{\gamma_1-1}h^{(1-\alpha)\gamma_1-1}\right]$. Noticing that $C_3(h) < 0$, $(r_1-\beta_1) > 0$, and

$$\varphi'(y) = C_3(h)(r_1-\beta_1)y^{r_1-\beta_1-1} + C_4(h)(r_2-\beta_1)y^{r_2-\beta_1-1},$$

it follows that when $y \leqslant y^{\dagger}$, $\varphi'(y) \geqslant 0$; when $y > y^{\dagger}$, $\varphi'(y) \leqslant 0$. Hence $\varphi(y)$ increases before reaching $y^{\dagger}$, and then decreases after exceeding $y^{\dagger}$.

Then we aim to prove $\varphi(\nu^{\gamma_1-1}h^{(1-\alpha)\gamma_1-1}) \geqslant 0$ and $\varphi(h^{(1-\alpha)\gamma_1-1}) \geqslant 0$. Indeed, if $y = \nu^{\gamma_1-1}h^{(1-\alpha)\gamma_1-1}$, we obtain

$$\begin{aligned}
&\varphi(\nu^{\gamma_1-1}h^{(1-\alpha)\gamma_1-1})\\
&= C_3(h)y^{r_1-\beta_1} + C_4(h)y^{r_2-\beta_1} + \frac{(\beta_1-1)h^{\alpha\beta_1}}{(r+\lambda)(\beta_1-r_1)(\beta_1-r_2)}\\
&\geqslant C_3(h)y^{r_1-\beta_1} + (C_4(h)-C_6(h))y^{r_2-\beta_1} + \frac{(\beta_1-1)h^{\alpha\beta_1}}{(r+\lambda)(\beta_1-r_1)(\beta_1-r_2)}\\
&= \frac{1-\beta_1}{(r+\lambda)(r_1-r_2)(\beta_1-r_1)}h^{\alpha\beta_1} + \frac{\beta_1-1}{(r+\lambda)(r_1-r_2)(\beta_1-r_2)}\frac{h^{r_1\gamma_1+r_2+\alpha\beta_1}}{(\nu h)^{r_1\gamma_1+r_2}}\\
&\quad + \frac{\beta_1-1}{(r+\lambda)(\beta_1-r_1)(\beta_1-r_2)}h^{\alpha\beta_1}\\
&\geqslant \frac{(\beta_1-1)h^{\alpha\beta_1}}{r}\left(-\frac{1}{(r_1-r_2)(\beta_1-r_1)} + \frac{1}{(r_1-r_2)(\beta_1-r_2)} + \frac{1}{(\beta_1-r_1)(\beta_1-r_2)}\right)\\
&= 0,
\end{aligned}$$

where the last second inequality holds because $(\beta_1 - r_2)(r_1\gamma_1 + r_2) < 0$ and

$0 < \nu < 1$. On the other hand, if $y = h^{(1-\alpha)\gamma-1}$, we can obtain

$$
\begin{aligned}
&\varphi(h^{(1-\alpha)\gamma_1-1}) \\
&\geqslant C_3(h)y^{r_1-\beta_1} + (C_4(h) - C_6(h))y^{r_2-\beta_1} + \frac{(\beta_1-1)h^{\alpha\beta_1}}{(r+\lambda)(\beta_1-r_1)(\beta_1-r_2)} \\
&= \frac{1-\beta_1}{(r+\lambda)(r_1-r_2)(\beta_1-r_1)}\frac{(\nu h)^{r_2\gamma_1+r_1}}{h^{r_2\gamma_1+r_1-\alpha\beta_1}} + \frac{\beta_1-1}{(r+\lambda)(r_1-r_2)(\beta_1-r_2)}h^{\alpha\beta_1} \\
&\quad + \frac{\beta_1-1}{(r+\lambda)(\beta_1-r_1)(\beta_1-r_2)}h^{\alpha\beta_1} \\
&\geqslant \frac{(\beta_1-1)h^{\alpha\beta_1}}{r+\lambda}\left(-\frac{1}{(r_1-r_2)(\beta_1-r_1)} + \frac{1}{(r_1-r_2)(\beta_1-r_2)} + \frac{1}{(\beta_1-r_1)(\beta_1-r_2)}\right) \\
&= 0.
\end{aligned}
$$

(iii) In the region $y > (\nu h)^{\gamma-1}h^{\alpha\gamma}$, similar to the proof of $C_5(h) \geqslant 0$, we can obtain

$$C_2(h) > C_2(h) - C_6(h) \geqslant 0.$$

Therefore, $v_{yy}(y,h) = r_2(r_2-1)C_2(h)y^{r_2-2} + \frac{2\lambda(\beta_2-1)K^{1-\beta_2}}{\kappa^2(\beta_2-r_1)(\beta_2-r_2)}y^{\beta_2-2} > 0$.

A.2.4 Proof of Corollary 4.1

Along the boundary $x_{\text{lavs}}(h)$, we first have $\frac{c^*(x_{\text{lavs}}(h),h)}{x_{\text{lavs}}(h)} = \frac{h}{x_{\text{lavs}}(h)}$, where $x_{\text{lavs}}(h)$ is defined in (4.2.19):

$$
\begin{aligned}
x_{\text{lavs}}(h) :=& -C_5(h)r_1(1-\alpha)^{r_1-1}h^{-r_2((1-\alpha)\gamma_1-1)} - C_6(h)r_2(1-\alpha)^{r_2-1}h^{-r_1((1-\alpha)\gamma_1-1)} \\
&- \frac{2\lambda(1-\alpha)^{\beta_2-1}K^{1-\beta_2}}{\kappa^2(\beta_2-r_1)(\beta_2-r_2)}h^{(\beta_2-1)((1-\alpha)\gamma_1-1)} + \frac{h}{r+\lambda}.
\end{aligned}
$$

In addition, we have

$$
\begin{aligned}
C_5(h)r_1(1-\alpha)^{r_1-1}h^{-r_2((1-\alpha)\gamma_1-1)} &= \frac{r_1(1-\alpha)^{-r_2}(\nu^{r_2\gamma_1+r_1}-1)(1-\beta_1)}{(r+\lambda)(r_1-r_2)(\beta_1-r_1)}h, \\
C_6(h)r_2(1-\alpha)^{r_2-1}h^{-r_1((1-\alpha)\gamma_1-1)} &= \frac{r_2(1-\alpha)^{-r_2}(1-\nu^{r_2\gamma_1+r_1})(1-\beta_1)(r_2(1-\alpha)\gamma_1+r_1)}{(r+\lambda)(r_1-r_2)(\beta_1-r_1)(r_1(1-\alpha)\gamma_1+r_2)}h,
\end{aligned}
$$

and

$$(\beta_2 - 1)((1-\alpha)\gamma_1 - 1) \leqslant 1,$$

thanks to **Assumption (A1)**, and the equality holds if and only if $\gamma_2 = (1-\alpha)\gamma_1$. Therefore, we have

$$\begin{aligned}\lim_{h\to+\infty} \frac{x_{\text{lavs}}(h)}{h} &= -\frac{r_1(1-\alpha)^{-r_2}(\nu^{r_2\gamma_1+r_1}-1)(1-\beta_1)}{(r+\lambda)(r_1-r_2)(\beta_1-r_1)} \\ &\quad - \frac{r_2(1-\alpha)^{-r_2}(1-\nu^{r_2\gamma_1+r_1})(1-\beta_1)(r_2(1-\alpha)\gamma_1+r_1)}{(r+\lambda)(r_1-r_2)(\beta_1-r_1)(r_1(1-\alpha)\gamma_1+r_2)} \\ &\quad - \frac{2\lambda(1-\alpha)^{\beta_2-1}K^{1-\beta_2}}{\kappa^2(\beta_2-r_1)(\beta_2-r_2)}\mathbf{1}_{\{\gamma_2=(1-\alpha)\gamma_1\}} + \frac{1}{r+\lambda}.\end{aligned}$$

The optimal investment on $x_{\text{lavs}}(h)$ is

$$\begin{aligned}\pi^*(x_{\text{lavs}}(h),h) &= \frac{2(r+\lambda)}{\kappa^2}C_5(h)f_3(x_{\text{lavs}}(h),h)^{r_1-1} + \frac{2(r+\lambda)}{\kappa^2}C_6(h)f_3(x_{\text{lavs}}(h),h)^{r_2-1} \\ &\quad + \frac{2\lambda K^{1-\beta_2}(\beta_2-1)}{\kappa^2(\beta_2-r_1)(\beta_2-r_2)}f_3(x_{\text{lavs}}(h),h)^{\beta_2-1} \\ &= \frac{2(r+\lambda)(1-\alpha)^{-r_2}}{\kappa^2}C_5(h)h^{-r_2((1-\alpha)\gamma_1-1)} \\ &\quad + \frac{2(r+\lambda)(1-\alpha)^{-r_1}}{\kappa^2}C_6(h)h^{-r_1((1-\alpha)\gamma_1-1)} \\ &\quad + \frac{2\lambda K^{1-\beta_2}(\beta_2-1)(1-\alpha)^{\beta_2-1}}{\kappa^2(\beta_2-r_1)(\beta_2-r_2)}h^{(\beta_2-1)((1-\alpha)\gamma_1-1)}.\end{aligned}$$

Therefore, we conclude

$$\lim_{h\to+\infty}\frac{\pi^*(x_{\text{lavs}}(h),h)}{x_{\text{lavs}}(h)} = \lim_{h\to+\infty}\frac{\pi^*(x_{\text{lavs}}(h),h)}{h}\cdot\lim_{h\to+\infty}\frac{h}{x_{\text{lavs}}(h)},$$

which also exists.

The optimal bequest on $x_{\text{lavs}}(h)$ is

$$b^*(x_{\text{lavs}}(h),h) = K^{-\frac{1}{\gamma_2-1}}\left((1-\alpha)h^{(1-\alpha)\gamma_1-1}\right)^{\frac{1}{\gamma_2-1}} = \left(\frac{1-\alpha}{K}\right)^{\frac{1}{\gamma_2-1}}h^{\frac{(1-\alpha)\gamma_1-1}{\gamma_2-1}}.$$

Therefore, we conclude

$$\lim_{h\to+\infty}\frac{b^*(x_{\text{lavs}}(h),h)}{x_{\text{lavs}}(h)}=\lim_{h\to+\infty}\left(\frac{1-\alpha}{K}\right)^{\frac{1}{\gamma_2-1}}\frac{h}{x_{\text{lavs}}(h)}\cdot h^{\frac{(1-\alpha)\gamma_1-\gamma_2}{\gamma_2-1}}$$
$$=\mathbf{1}_{\{\gamma_2=(1-\alpha)\gamma_1\}}\left(\frac{1-\alpha}{K}\right)^{\frac{1}{\gamma_2-1}}\lim_{h\to+\infty}\frac{h}{x_{\text{lavs}}(h)}$$

is positive if $\gamma_2=(1-\alpha)\gamma_1$, and equals 0 otherwise.

A.3 Proofs for Chaptper 5

A.3.1 Proof of Theorem 5.2

Fix (x,z) satisfying $x\geqslant \frac{z}{d}>0$. By assumption, the feedback control pair $(\tilde{c},\tilde{\pi}\tilde{b})$ is admissible, the open-loop control strategy, $(\tilde{c}_s,\tilde{\pi}_s,b_s)_{s\geqslant t}$, generated from $(\tilde{c},\tilde{\pi})$ with respect to the initial condition $(X_t^{\tilde{c},\tilde{\pi},\tilde{b}},Z_t^{\tilde{c},\tilde{\pi},\tilde{b}})=(x,z)$ is admissible. Let $\{(X_s^{\tilde{c},\tilde{\pi},\tilde{b}},Z_s^{\tilde{c},\tilde{\pi},\tilde{b}}),s\geqslant t\}$ be the corresponding wealth and habit formation process under $(\tilde{c},\tilde{\pi},\tilde{b})$. Applying Itô's formula, we have

$$\begin{aligned}
&e^{-\rho s}u^{c,\pi,b}(X_s^{\tilde{c},\tilde{\pi},\tilde{b}},Z_s^{\tilde{c},\tilde{\pi},\tilde{b}})\\
=&e^{-\rho t}u^{c,\pi}(x,z)+\int_t^s e^{-\rho v}\bigg(-\rho u^{c,\pi,b}(X_v^{\tilde{c},\tilde{\pi},\tilde{b}},Z_v^{\tilde{c},\tilde{\pi},\tilde{b}})\\
&+u_x^{c,\pi,b}(X_v^{\tilde{c},\tilde{\pi},\tilde{b}},Z_v^{\tilde{c},\tilde{\pi},\tilde{b}})\cdot(rX_v^{\tilde{c},\tilde{\pi},\tilde{b}}+\mu\tilde{\pi}_v-\tilde{c}_v-\lambda\tilde{b}_v)+u_z^{c,\pi,b}(X_v^{\tilde{c},\tilde{\pi},\tilde{b}},Z_v^{\tilde{c},\tilde{\pi},\tilde{b}})\cdot(\eta\tilde{c}_v-\delta Z_v^{\tilde{c},\tilde{\pi},\tilde{b}})\\
&+\frac{1}{2}(\pi_v^{\tilde{c},\tilde{\pi},\tilde{b}})^2\sigma^2u_{xx}^{c,\pi,b}(X_v^{\tilde{c},\tilde{\pi},\tilde{b}},Z_v^{\tilde{c},\tilde{\pi},\tilde{b}})\bigg)dv+\int_t^s e^{-\rho v}\tilde{\pi}_v\sigma u^{c,\pi,b}(X_v^{\tilde{c},\tilde{\pi},\tilde{b}},Z_v^{\tilde{c},\tilde{\pi},\tilde{b}})dW_v,\quad s\geqslant t.
\end{aligned}\tag{A.3.31}$$

Define the stopping times $\tau_n := \inf\{s \geqslant t : \int_t^s e^{-2\rho v}\tilde{\pi}_v^2\sigma^2(u^{c,\pi,b}(X_v^{\tilde{c},\tilde{\pi},\tilde{b}}, Z_v^{\tilde{c},\tilde{\pi},\tilde{b}}))^2 \geqslant n\}$, for $n \geqslant 1$. Then from (A.3.31), we obtain

$$\begin{aligned} e^{-\rho t}u^{c,\pi,b}(x,z) = \mathbb{E}^{\mathcal{F}_t}\bigg[& e^{-\rho(s\wedge\tau_n)}u^{c,\pi,b}(X_{s\wedge\tau_n}^{\tilde{c},\tilde{\pi},\tilde{b}}, Z_{s\wedge\tau_n}^{\tilde{c},\tilde{\pi},\tilde{b}}) - \int_t^{s\wedge\tau_n} e^{-\rho v}\bigg(-\rho u^{c,\pi,b}(X_v^{\tilde{c},\tilde{\pi},\tilde{b}}, Z_v^{\tilde{c},\tilde{\pi},\tilde{b}}) \\ & + u_x^{c,\pi,b}(X_v^{\tilde{c},\tilde{\pi},\tilde{b}}, Z_v^{\tilde{c},\tilde{\pi},\tilde{b}})\cdot(rX_v^{\tilde{c},\tilde{\pi},\tilde{b}} + \mu\tilde{\pi}_v - \tilde{c}_v - \lambda\tilde{b}_v) \\ & + u_z^{c,\pi,b}(X_v^{\tilde{c},\tilde{\pi},\tilde{b}}, Z_v^{\tilde{c},\tilde{\pi},\tilde{b}})\cdot(\eta\tilde{c}_v - \delta Z_v^{\tilde{c},\tilde{\pi},\tilde{b}}) + \frac{1}{2}(\pi_v^{\tilde{c},\tilde{\pi},\tilde{b}})^2\sigma^2 u_{xx}^{c,\pi,b}(X_v^{\tilde{c},\tilde{\pi},\tilde{b}}, Z_v^{\tilde{c},\tilde{\pi},\tilde{b}})\bigg)dv\bigg]. \end{aligned} \tag{A.3.32}$$

On the other hand, by standard arguments and the assumption that $u^{c,\pi,b}$ is smooth, we have

$$-\rho u^{c,\pi,b}(x,z) + u_x^{c,\pi,b}(x,z)\cdot(rx+\mu\pi-c-\lambda b) + u_z^{c,\pi,b}(x,z)\cdot(\eta c-\delta z) + \frac{1}{2}\pi^2\sigma^2 u_{xx}^{c,\pi,b}(x,z) + \frac{1}{\gamma}(c-z)^\gamma + \frac{\lambda K}{\gamma}b^\gamma = 0,$$

for any $x \geqslant \frac{z}{d} > 0$. It follows that

$$\begin{aligned} -\rho u^{c,\pi,b}(x,z) + \sup_{c'\in\mathbb{R},\pi'\in\mathbb{R},b'\in\mathbb{R}_+} \bigg\{ & u_x^{c,\pi,b}(x,z)\cdot(rx+\mu\pi'-c'-b') + u_z^{c,\pi,b}(x,z)\cdot(\eta c'-\delta z) \\ & + \frac{1}{2}(\pi')^2\sigma^2 u_{xx}^{c,\pi,b}(x,z) + \frac{1}{\gamma}(c'-z)^\gamma + \frac{\lambda K}{\gamma}(b')^\gamma\bigg\} \geqslant 0. \end{aligned} \tag{A.3.33}$$

Note that the maximizer of the Hamiltonian in (A.3.33) is given by the feedback policy $(\tilde{c},\tilde{\pi},\tilde{b})$ in (5.2.7). Therefore, equation (A.3.31) implies that

$$e^{-\rho t}u^{c,\pi,b}(x,z) \leqslant \mathbb{E}^{\mathcal{F}_t}\bigg[e^{-\rho(s\wedge\tau_n)}u^{c,\pi,b}(X_{s\wedge\tau_n}^{\tilde{c},\tilde{\pi},\tilde{b}}, X_{s\wedge\tau_n}^{\tilde{c},\tilde{\pi},\tilde{b}}) + \int_t^{s\wedge\tau_n} e^{-\rho v}\frac{1}{\gamma}\{(\tilde{c}_v - Z_v^{\tilde{c},\tilde{\pi},\tilde{b}})^\gamma + \lambda K\tilde{b}_v^\gamma\}dv\bigg],$$

for $x \geqslant \frac{z}{d}$ and $s \geqslant t$. Now sending $n \to \infty$, we obtain

$$e^{-\rho t}u^{c,\pi,b}(x,z) \leqslant \mathbb{E}^{\mathcal{F}_t}\bigg[e^{-\rho s}u^{c,\pi,b}(X_s^{\tilde{c},\tilde{\pi},\tilde{b}}, Z_s^{\tilde{c},\tilde{\pi},\tilde{b}}) + \int_t^s e^{-\rho v}\frac{1}{\gamma}\{(\tilde{c}_v - Z_v^{\tilde{c},\tilde{\pi},\tilde{b}})^\gamma + \lambda K\tilde{b}_v^\gamma\}dv\bigg],$$

taking $s \to +\infty$, we have

$$u^{c,\pi,b}(x,z) \leqslant \mathbb{E}^{\mathcal{F}_t}\bigg[\int_t^\infty e^{-\rho(v-t)}\frac{1}{\gamma}\{(\tilde{c}_v - Z_v^{\tilde{c},\tilde{\pi},\tilde{b}})^\gamma + \lambda K\tilde{b}_v^\gamma\}dv\bigg].$$

A.3.2 Proof of Theorem 5.3

For feedback policy pair $c_0(x,z) = z + a_1\big(x - \frac{z}{d}\big)$, $\pi_0(x,z) = a_2\big(x - \frac{z}{d}\big)$ and $b_0(x,z) = a_3\big(x - \frac{z}{d}\big)$ that is admissible with respect to initial (x,z). It follows from the standard argument that the corresponding value function u^{c_0,π_0} satisfies the PDE

$$
\begin{aligned}
&-\rho u^{c_0,\pi_0}(x,z) + u_x^{c_0,\pi_0}(x,z)(rx + \mu\pi_0(x,z) - c_0(x,z) - \lambda b_0(x,z))\\
&+ u_z^{c_0,\pi_0}(x,z)(\eta c_0(x,z) - \delta z) + \frac{1}{2}\pi_0^2\sigma^2 u_{xx}^{c_0,\pi_0}(x,z) + \frac{1}{\gamma}(c_0 - z)^\gamma + \frac{\lambda K}{\gamma} b_0^\gamma = 0,
\end{aligned}
\tag{A.3.34}
$$

with initial condition $u(x, bx) = 0$ for any $x > 0$. Solving this equation, we obtain

$$
u^{c_0,\pi_0,b_0} = L_0\left(x - \frac{z}{d}\right)^\gamma,
$$

where L_0 is some constant related to a_1, a_2 and a_3. By Theorem 5.2, we can obtain the optimal feedback control triple $(c_1(x,z), \pi_1(x,z), b_1(x,z))$

$$
\begin{aligned}
c_1(x,z) &= z + \left(\frac{L_0\gamma(d+\eta)}{d}\right)^{\frac{1}{\gamma-1}}\left(x - \frac{z}{d}\right),\\
\pi_1(x,z) &= \frac{\mu}{(1-\gamma)\sigma^2}\left(x - \frac{z}{d}\right),\\
b_1(x,z) &= \left(\frac{L_0\gamma}{K}\right)^{\frac{1}{\gamma-1}}\left(x - \frac{z}{d}\right),
\end{aligned}
$$

and we can obtain the associated value function

$$
u^{c_1,\pi_1,b_1} = L_1\left(x - \frac{z}{d}\right)^\gamma,
$$

for some constant $L_1 > 0$. Following this process, we can obtain a sequence $\{L_n, n \geqslant 0\}$ as the coefficient of the value function. By Theorem 5.2, the sequence $\{L_n, n \geqslant 0\}$ is indeed a non-decreasing sequence, and has an upper bound L^* by our assumption that the solution exists. Because the value function has a limit $u^{c_\infty,\pi_\infty,b_\infty}$, the optimal control triplet $(c_n(x,z),\pi_n(x,z),b_n(x,z))$ also has the limit $(c_\infty(x,z),\pi_\infty(x,z),b_\infty(x,z))$. By the standard argument, the value function $u^{c_\infty,\pi_\infty,b_\infty} = L^*\big(x-\frac{z}{d}\big)^\gamma$ satisfies the following PDE

$$
\begin{aligned}
&-\rho u^{c_\infty,\pi_\infty,b_\infty}(x,z) + u_x^{c_\infty,\pi_\infty,b_\infty}(x,z)(rx+\mu\pi_\infty(x,z)-c_\infty(x,z))\\
&+u_z^{c_\infty,\pi_\infty,b_\infty}(x,z)(\eta c_\infty(x,z)-\delta z)+\frac{1}{2}\pi_\infty^2\sigma^2 u_{xx}^{c_\infty,\pi_\infty,b_\infty}(x,z)+\frac{1}{\gamma}(c_\infty-z)^\gamma+\frac{\lambda K}{\gamma}b_\infty^\gamma=0,
\end{aligned}
$$

for any $x \geqslant \frac{z}{d} > 0$ with initial condition $u(x,bx)=0$ for all $x>0$. Moreover, limit of the optimal control triples

$$
\begin{aligned}
c_\infty(x,z) &= z+\left(\frac{L^*\gamma(d+\eta)}{d}\right)^{\frac{1}{\gamma-1}}\left(x-\frac{z}{d}\right),\\
\pi_\infty(x,z) &= \frac{\mu}{(1-\gamma)\sigma^2}\left(x-\frac{z}{d}\right),\\
b_\infty(x,z) &= \left(\frac{L^*\gamma}{K}\right)^{\frac{1}{\gamma-1}}\left(x-\frac{z}{d}\right).
\end{aligned}
$$

Combining these equations, we have $L^* = \frac{d}{\gamma(b+\eta)}h^{\gamma-1}$ and thus $u^{c_\infty,\pi_\infty,b_\infty}(x,z) = u^*(x,z)$, $(c_\infty(x,z),\pi_\infty(x,z),b_\infty(x,z)) = (c^*(x,z),\pi^*(x,z),b^*(x,z))$.

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