# DATA-DRIVEN ROBUST NETWORK REVENUE MANAGEMENT

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# Data-Driven Robust Network Revenue Management

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CERTIFICATE OF ORIGINALITY

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# Dedication

This thesis is dedicated to my parents.

# Abstract

In this research, we focus on data-driven distributionally robust controls for the quantity-based network revenue management (NRM) problem in which the decision maker accepts or rejects each arriving customer request irrevocably with the goal of maximizing the total expected revenue over a finite selling horizon given limited resources. Instead of the deterministic linear programming (DLP) formulation widely adopted in literature, we approximate the value function of dynamic programming (DP) for NRM problem as probabilistic nonlinear programming (PNLP) in order to capture the randomness in demand. We further take the uncertainty in distribution estimation resulting from either the limited information or the changing environment into account by incorporating the distribution ambiguity into the PNLP formulation. We therefore solve a distributionally robust optimization (DRO) problem to determine an optimal partitioned allocation of capacity to each product against a worst-case distribution in the ambiguity set. We assume that the decision maker does not know the distribution of demand but has access to historical data, which is assumed to be independent and identically distributed (i.i.d.). In this setting, we define our data-driven ambiguity set as a confidence region of a goodness-of-fit (GoF) hypothesis test and then formulate a tractable robust static model.

Furthermore, we extend our robust static NRM model to a multi-stage version. More specifically, we formulate the multi-stage robust NRM model as a robust DP and solve this robust DP using approximate dynamic programming (ADP) approach. The resulting robust ADP model generates robust dynamic bid prices from a conic optimization to help us construct capacity allocation policies. We also provide a constraint generation procedure for solving this robust ADP. To improve the efficiency of problem-solving, we further derive an equivalent reformulation for the robust ADP model, which is computationally tractable and of practical interpretation. By solving this reformulation that approximates the evolution of the selling system under demand uncertainty, we can construct a robust dynamic booking limit policy. We also verify the performance of both our robust static and dynamic policy via numerical experiments.

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# Chapter 1

# Introduction

The area of revenue management (RM) arose in the 1980s from the airline industry. The airlines tactically allocate limited capacity to various classes of customers in order to maximize revenues. Since then, it has been widely studied and combined with other tactics such as dynamic pricing. Also, its spirit has been successfully applied to many industries including railways, hotels and rental cars. The book (K. T. Talluri, Van Ryzin, and Van Ryzin 2004) provides a thorough overview and numerous instances of this area. In cases where there are several different products and each product may consume multiple types of resources (e.g., raw materials, travel legs or hotel booking horizons), the problem described above is often referred to as network revenue management (NRM) (Williamson 1992; Gallego and Van Ryzin 1997).

In the NRM problem, a set of resources with limited capacities are waiting to be allocated to various customers who arrive sequentially over a finite time horizon. Customers are divided into different classes according to their usage of resources and the prices (which are assumed to remain unchanged) that they pay for the corresponding products. Once a customer arrives, the decision maker has to immediately make a decision, either accept or reject the customer. No waiting, backlog or later adjustment is allowed. If the customer is accepted and the remaining capacity is enough to serve, she consumes certain units of each resource requested and pays the corresponding price associated with her class. Otherwise, no resources are consumed while no revenue is generated. The resources are commonly assumed perishable in NRM problem so that unused resources at the end of selling horizon have no salvage value. The decision maker devotes himself to exploring an effective admission control policy to maximize the cumulative expected revenue under the capacity constraints.

Note that the formulation we discussed above is more specifically referred to as the "quantity-based" NRM problem. Another widely studied formulation is known as "price-based" NRM problem where the decision maker tactically adjusts product prices instead of controlling product allocation. (Maglaras and Meissner 2006) showed that these two formulations are actually equivalent to each other in some special cases, such as the single resource multi-product model. In this research, we work on the quantity-based NRM formulation, which is also known as network capacity control, and we will drop the "quantity-based" through the remaining part of this thesis for simplicity of notations.

The NRM problem can be formulated as a dynamic programming (DP). For any network of realistic size, however, a crucial challenge is that computing the value function exactly is impractical because the state space grows exponentially with the size of resources. Consequently, one often approximate the value function. Most approximation methods proposed in the literature are based on one of two basic strategies. One is to decompose the network problem into a collection of single-resource problems. The other strategy is to simplify the network model by posing the problem as a static mathematical programming problem. Notice that the first strategy cannot be appropriately applied to some problems because separating the problems by resources can result in losing important network information. For example, the airline seat inventory control problem we stated above cannot be decomposed for each individual flight leg since some itineraries use multiple resources simultaneously. For the second strategy, three types of approximations are popularly used, which are the deterministic linear program (DLP), probabilistic nonlinear program (PNLP) and randomized linear program (RLP) method. The solutions of these (N)LP-based approximations are usually leveraged to construct heuristic admission control policies and then guide online decision making. Due to its simplicity and computational efficiency, the DLP formulation is fairly popular in literature and practice. An optimal solution to the DLP has been leveraged to design booking limit controls, nested controls (K. Talluri and Van Ryzin 2004), and probabilistic allocation policies (Reiman and Q. Wang 2008; Jasin and Kumar 2012; Bumpensanti and H. Wang 2020). Also, the optimal dual solution of the DLP can serve as bid prices for bid-price control policies (K. Talluri and Van Ryzin 1998). The DLP formulation is obtained by replacing all random demands with their expectations. This exactly makes the DLP simple and efficient while also leads to an upper bound on the optimal revenue since the capacity constraints here are only satisfied in expectation. Instead of ignoring all uncertainty in the forecasts, PNLP captures the randomness in demand by incorporating the expected sales of product $j$, $\mathbb{E}[\min\{D_j, y_j\}]$, into the objective function under this partitioned allocation. This actually leads to a lower bound on the optimal revenue since partitioned allocations are certainly one type of the feasible policies for the network problem. Another approach that can incorporate stochastic information into the DLP is RLP method, where the expectation of demand in the demand constraint of DLP is replaced by the random demand vector itself. The main idea of this approach is to estimate the gradient of approximate value function via demand simulation.

In addition, to include updated information, re-solving heuristics are widely used. This category of techniques re-optimizes the approximate (N)LP formulation over time while substituting the initial capacity with the real remaining capacity at each re-solving epoch. One might expect that re-solving the DLP would lead to higher revenue as it incorporates the most recently updated information acquired through time. However, some works point out that re-solving the DLP may worsen the revenue result (Cooper 2002; L. Chen and Homem-de-Mello 2010; Jasin and Kumar 2013). In general, the performance of re-solving depends heavily on the network type, the demand estimation, and the frequency of re-optimization.

On the other hand, accurate forecasting is key to the success of almost all RM problems. In real-world situations, complete information about the distribution of customer demand is unknown. In this case, most NRM models assume that demand can be characterized by either a stochastic process depicting the customer arrival pattern or a probability distribution describing the aggregate number of demand. However, it is quite often that the actual distribution is not consistent with our assumption when the information about demand is limited. This can result in significant forecasting error and then ineffective RM. One approach taken in the literature to correct such model uncertainty is so-called distributionally robust optimization (DRO). In the framework of DRO, one assumes that the unknown distribution of the random variable belongs to an ambiguity set $\mathcal{A}$ of possible probability distributions and solves the minimax or maximin problem of computing the decision variable, which is optimal against a worst-case distribution in this set. Such a DRO approach is motivated by the reality that perfect knowledge of the exact distribution of a given random process is rarely available (Scarf, Arrow, and Karlin 1957; Žáčková 1966; Dupačová 1987; Prekopa 1995), and we refer the reader to (Rahimian and Mehrotra 2019) for a recent survey on such approaches. Depending on what type of distribution information is available, different forms of ambiguity sets can be constructed. Ambiguity sets having been studied include moment-based, probabilistic metric-based, and goodness-of-fit test-based ambiguity sets, which are not strictly exclusive to other types. More detailed reviews and connections with this research will be provided in Chapter 2.

Combining these, in this study, we investigate the distributionally robust controls for the NRM problem with historical data. Instead of the DLP formulation widely used in literature, we approximate the value function of DP for NRM problem as PNLP in order to capture the randomness in demand. We further take the uncertainty in distribution estimation resulting from either the limited information or the changing environment into account by incorporating the distribution ambiguity into the PNLP formulation. We therefore solve a DRO problem to determine an optimal partitioned allocation of capacity to each product against a worst-case distribution in the ambiguity set. In this study, we assume that the decision maker does not know the distribution of demand but has access to historical data, which is assumed to be independent and identically distributed (i.i.d.). In this setting, we define our data-driven ambiguity set as confidence region of goodness-of-fit (GoF) hypothesis tests and then formulate a tractable robust static model. Furthermore, we extend this robust static NRM model to a multi-stage version. More specifically, we formulate the multi-stage model as a robust DP and solve this robust DP using approximate dynamic programming (ADP) approach. This robust ADP model generates robust dynamic bid prices from a conic optimization to help us construct capacity allocation policies. We also provide a constraint generation procedure for solving this robust ADP. To improve the efficiency of problem-solving, we further derive an equivalent reformulation for the robust ADP model, which is computationally tractable and of practical interpretation. By solving this reformulation that approximates the evolution of the selling system under demand uncertainty, we can construct a robust dynamic booking limit policy. We finally verify the performance of both our robust static and dynamic policy via numerical experiments. To the best of our knowledge, this is the first endeavor to address the demand distribution ambiguity by incorporating distributionally robust approach in the context of quantity-based NRM and further to study the multi-stage robust NRM model.

# Chapter 2

# Literature Review

Our study is mainly related to two streams of research in the literature: NRM and DRO. In what follows, we review each stream separately and then mention the most related works to our research.

## 2.1 Network Revenue Management

In Chapter 1, we actually have reviewed certain amount of works for NRM. Hence, we just make some complements in this section. We study in this research the NRM problem, which can be solved using DP. Because of the curse of dimensionality, however, computing the exact DP solution is often intractable. Therefore, NRM problem is often solved heuristically, either by approximating the value function in the DP (Bertsimas and Popescu 2003; Adelman 2007) or by narrowing down the set of feasible policies. Popular approximate methods have been reviewed in Chapter 1 and here, we briefly review commonly used control policies: partitioned booking limit and bid price control. Partitioned booking limit sets a predetermined quota for requests that can be accepted for each product and accepts customers in a first-come-first-serve (FCFS) fashion (Williamson 1992). The partitioned booking limit is given by DLP solution if the random demands are replaced by their expectations or by PNLP solution if we conserve the randomness in demand. Bid price control leverages the optimal dual variable of the approximate mathematical program to make allocation decisions. Bid prices are defined as the Lagrangian multipliers associated with the capacity constraints of the approximate mathematical program and value the opportunity cost of a unit of capacity. A request for one product is accepted if and only if the collected revenue exceeds the estimated cost of consuming all the requested resources. Although bid price control is in general not optimal (K. Talluri and Van Ryzin 1998), it is widely used in practice.

To take system dynamics into account when studying NRM problem, one widely adopted dynamic framework is approximate dynamic programming (ADP) (Adelman 2007; D. Zhang and Adelman 2009; Topaloglu 2009; Tong and Topaloglu 2014; Kunnumkal and K. Talluri 2016). More specifically, this approach first reformulates the DP of NRM problem as a linear program (LP), where the number of variables and constraints grow exponentially with the number of resources and products. Then the main idea of this approach is approximating the value function as a weighted sum of a collection of basis functions, by which the LP is with far fewer variables. Especially, (Adelman 2007) approximates the value function as an affine function of the state. Furthermore, the coefficients of the affine function are interpreted by Adelman as the marginal resource value in each time period and used to construct bid price control policy where the bid prices are dynamically changing over time. This work laid a foundation for many research focus on methods for computing dynamic bid prices under different problem settings(Tong and Topaloglu 2014; Vossen and D. Zhang 2015; R. Zhang, Samiedaluie, and D. Zhang 2022).

One increasingly popular stream of operations management (OM) and revenue management (RM) literature studies online learning, where the underlying demand function or other item we concerned is unknown and needs to be learned on the fly from realized data. To the best of our knowledge, after the work of (Besbes and Zeevi 2009), many papers consider demand learning settings using the price-based finite-inventory revenue management model from (Gallego and Van Ryzin 1994) as the ground truth model (Besbes and Zeevi 2012; Z. Wang, Deng, and Ye 2014; Boer and Zwart 2015; Cheung, Simchi-Levi, and H. Wang 2017; Ferreira, Simchi-Levi, and H. Wang 2018; Yiwei Chen and Shi 2023). In contrast to the situation of works for price-based RM, to the best of our knowledge, almost no work has been done for quantity-based RM with online learning approach.

## 2.2 Distributionally Robust Optimization

There is a vast body of literature on (distributionally) robust optimization approaches to operations management (OM) and RM including (network) capacity control (Birbil et al. 2009; Perakis and Roels 2010), inventory management (Klabjan, Simchi-Levi, and Song 2013; Das, Dhara, and Natarajan 2021), pricing (Lim and Shanthikumar 2007), and hospital admission (Meng et al. 2015). We refer the reader to an excellent survey by (Lu and Shen 2021) for more information and an overview of robust OM and RM. However, RO is thought to be too conservative in many cases because its optimality is usually attained at the extreme values of the random variable that seldom occur. To circumvent this issue, the DRO approach has received significant attention in recent years. In this framework, the objective is to optimize the worst-case expected cost over a distribution from the ambiguity set, which contains possible distributions of the random variable.

The ambiguity set construction relies on the information about distribution we have. When certain distributional information such as mean and variance is known, the DRO approach can be applied (Perakis and Roels 2008; Bertsimas, Doan, et al. 2010; Delage and Ye 2010; Zymler, Kuhn, and Rustem 2013). This type of ambiguity set is generally referred to moment-based ambiguity set.

An alternating way of constructing ambiguity set is commonly called probabilistic metric-based or statistical distance-based ambiguity set, which generally requires historical realizations. In this setting, the actual distribution is believed to be close to a known nominal or most likely distribution (e.g., empirical distribution fitted from historical data), and the ambiguity set can be constructed to contain all distributions that are close to this distribution with respect to the prescribed probabilistic metric, such as $\phi$-divergence, Kullback-Leibler divergence and Wasserstein distance (Ben-Tal et al. 2013; Z. Wang, Glynn, and Ye 2016; Jiang and Guan 2016; Mohajerin Esfahani and Kuhn 2018). Most papers on data-driven DRO (DD-DRO) also belong to this stream. In addition, in the DD-DRO or DD-RO field, some literature define ambiguity sets as confidence regions of goodness-of-fit (GoF) hypothesis tests. For example, (Klabjan, Simchi-Levi, and Song 2013) studies a stochastic lot-sizing problem under discrete distributional uncertainty described by Pearson's $\chi^2$ GoF test and develops a dynamic programming approach to this particular problem. They also establish conditions for asymptotic convergence for this problem. Also, GoF test can be used to construct uncertainty set for data-driven robust optimization (Bertsimas, Gupta, and Kallus 2018a; Ye Chen et al. 2022). To our knowledge, constructing ambiguity set using statistical hypothesis tests attracts relatively less attention than do the other two approaches to obtain ambiguity set stated above, yet proved to be with nice properties (Bertsimas, Gupta, and Kallus 2018b). More specifically, linking sample average approximation (SAA), DRO, and hypothesis testing of goodness-of-fit, (Bertsimas, Gupta, and Kallus 2018b) systematically studies the theory and properties of DD-DRO approach with confidence region-based ambiguity set, to which they provide a term Robust SAA. They show that Robust SAA not only inherits SAA's favorable asymptotic convergence and tractability, but also enjoys a strong finite sample performance guarantee for a wide class of optimization problems, and demonstrate that solutions of Robust SAA are stable, even for small to moderate $N$. We consider the DD-DRO procedure in our work to be Robust SAA because of its good performance in tractability, stability, asymptotic and finite-sample guarantees. Furthermore, since our approach is based on the univariate test, in terms of methodology, there is no significant difference between the hypothesis-based and statistical distance-based approaches. Actually, in many cases, the statistical distance can be easily translated into a hypothesis GoF test, e.g., GoF test based on Wasserstein distance, especially for univariate random variable. As a result, we can easily incorporate many statistical distance-based ambiguity sets in our framework.

We end the literature review by mentioning some most related papers on our research. (Birbil et al. 2009) studies the robust single-leg revenue management problem. They take into account the estimation error of the demand distributions and arrival probabilities in static and dynamic setting respectively. They propose efficient algorithms to compute robust booking limits under ellipsoidal uncertainty. We note that they also use the PNLP formulation to approximate the original single-leg problem whereas our PNLP is a more complex network version. (Perakis and Roels 2010) develops robust formulations for the NRM problem under polyhedral uncertainty to maximize the worst-case revenue and to minimize the worst-case regret. They also develop an efficient heuristic to compute minimax regret booking limits in a general network. Unlike their assumption that the aggregate demand belongs to a polyhedral uncertainty set, we assume the true distribution of aggregate demand to be in an ambiguity set constructed by given historical data to circumvent the conservative issue. Last but not least, to the best of our knowledge, our data-driven multi-stage robust NRM problem is the first study that lies at the intersection of DRO and ADP with applications in RM field.

# Chapter 3

# Robust Static Model

In this chapter, we study the static model, where the random variable is the aggregate demand during the whole selling horizon. In addition, the control policy is determined at the beginning of selling and remains unchanged during the whole time horizon.

Let the number of time periods be $T$. For a positive integer $T$, let $[T]$ denote the set $\{1, ..., T\}$. Note that $T$ is assumed to be finite to ensure our finite time horizon setting. There are $n$ classes of products indexed by $j \in [n]$ and $m$ classes of resources indexed by $l \in [m]$, where resource $l$ has initial capacity $C_l \in \mathbb{R}_+$. We denote the initial capacity vector as $\boldsymbol{C} = [C_1, ..., C_m]^{\mathrm{T}}$. $\boldsymbol{r} = [r_1, ..., r_n]^{\mathrm{T}}$ is the revenue vector, where $r_j \in \mathbb{R}_+$ is the revenue produced by selling a product $j$ for all $j$. If a customer requesting product $j$ is accepted, $A_{l,j} \in \mathbb{R}_+$ units of resource $l$ are used to satisfy this demand. We represent the information of resources required to sell a product $j$ as the vector $\boldsymbol{A}_j = [A_{1,j}, ..., A_{m,j}]^{\mathrm{T}}$, and further define the bill-of-materials (BOM) matrix as $\boldsymbol{A} = [\boldsymbol{A}_1; ...; \boldsymbol{A}_n]$, $\boldsymbol{A} \in \mathbb{R}_+^{m \times n}$. Let the aggregate demand to come during selling horizon for each product $j \in [n]$ be denoted by

$D_j$ (demand over the periods $1, \cdots, T$) with mean $\mu_j$. Let $\boldsymbol{D} = [D_1, \cdots, D_n]^{\mathrm{T}}$ and $\boldsymbol{\mu} = \mathbb{E}[\boldsymbol{D}]$ denote the vector of demands and mean demands, respectively. Given two real vectors $a, b \in \mathbb{R}^n$, let $a \wedge b := [\min\{a_1, b_1\}, \cdots, \min\{a_n, b_n\}]^{\mathrm{T}}$, $a \vee b := [\max\{a_1, b_1\}, \cdots, \max\{a_n, b_n\}]^{\mathrm{T}}$, and $a^+ := a \vee 0$.

## 3.1 Approximations and Control Policies

The NRM problem can be formulated as a dynamic program (DP). Because of the curse of dimensionality, however, computing the exact DP solution is often intractable. Therefore, NRM problem is often solved approximately, either by approximating the value function in the DP or by narrowing down the set of feasible policies. We review some previous work on approximations and control policies in network revenue management applications that are the most pertinent to our own work.

### 3.1.1 Deterministic Linear Program (DLP)

Treating demand as if it were deterministic and equal to its mean $\boldsymbol{\mu}$, the DLP method uses the approximation

$$V^{DLP}(\boldsymbol{C}) = \max_{\boldsymbol{y}} \quad \sum_{j=1}^{n} r_j y_j \tag{3.1a}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} A_{l,j} y_j \leq C_l, \qquad \forall l \in [m], \tag{3.1b}$$

$$0 \leq y_j \leq \mu_j, \qquad \forall j \in [n]. \tag{3.1c}$$

The decision variables $\boldsymbol{y} = (y_1, \cdots, y_n)$ represent a static partitioned allocation of capacity for each of the $n$ products. Capacity constraint (3.1b) specifies that the expected consumption of all $m$ resources cannot exceed their initial capacities, and demand constraint (3.1c) specifies that the number of accepted requests from product $j$ cannot exceed the expected number of total demand, $\mu_j$. Using Jensen's inequality, one can show that the optimal objective value of DLP approximation is an upper bound on the optimal value function (K. T. Talluri, Van Ryzin, and Van Ryzin 2004). Intuitively, DLP is a relaxation of the original problem, because it only requires the capacity constraints to be satisfied in expectation.

### 3.1.2 Probabilistic Nonlinear Program (PNLP)

The main drawback of DLP approximation is that it treats demand as if it equals its mean and does not account for demand uncertainty in the forecasts. Instead of ignoring all other distributional information, the PNLP method uses the approximation

$$\begin{aligned} V^{PNLP}(\boldsymbol{C}) = \max_{\boldsymbol{y}} \quad & \sum_{j=1}^{n} r_j \mathbb{E}\left[\min\left\{D_j, y_j\right\}\right] && (3.2) \\ \text{s.t.} \quad & \sum_{j=1}^{n} A_{l,j} y_j \leq C_l, && \forall l \in [m], \\ & y_j \geq 0, && \forall j \in [n]. \end{aligned}$$

The term $\mathbb{E}\left[\min\left\{D_j, y_j\right\}\right]$ in the objective function, which captures the randomness in demand, is the expected sales of product $j$ under this static partitioned allocation. Notice that solving (3.2) needs more information than solving (3.1). In (3.1), only the mean demands are used as inputs, while in (3.2) the distributions of demands are needed to formulate the objective function. Assuming we have access to historical data samples of size $N$, we can apply a widely adopted and easy-to-implement stochastic optimization method, Sample Average Approximation (SAA), to the PNLP model.

$$\begin{aligned} V_{SAA}^{PNLP}(\boldsymbol{C}) = \max_{\boldsymbol{x},\boldsymbol{y}} \quad & \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{n} r_j x_{j,i} \\ \text{s.t.} \quad & \boldsymbol{y} \in Y, \\ & x_{j,i} \le D_j^i, \;\; x_{j,i} \le y_j, \qquad \forall i \in [N], j \in [n], \end{aligned} \tag{3.3}$$

where $D_j^i$ denotes the $i^{th}$ sample of demand for product $j$ and $Y$ is the feasible region of problem (3.2), i.e.,

$$Y := \left\{ \boldsymbol{y} \in \mathbb{R}_+^n : \sum_{j=1}^{n} A_{l,j} y_j \le C_l, \forall l \in [m] \right\}.$$

We use $Y$ to represent the feasible region of problem (3.2) in the remaining part of the thesis.

### 3.1.3 Control Policies

Suppose that we have obtained an optimal solution $\boldsymbol{y}^*$ for either the DLP or PNLP model. With this solution, we can implement the booking limit (BL) control policy, which is a commonly used admission control policy. Under this policy, the decision maker sets a predetermined quota for each customer class $j$ and accepts requests up to those limits. We assume that each order is for one unit from one customer class, which is without loss of generality since we can treat larger orders as multiple individual orders. Let $\boldsymbol{y}^*$ be the fixed quota vector for the BL policy. The admission policy for determining whether to serve the ${k_j}^{\text{th}}$ order from customer class $j$ is given by:

$$\pi_j^{BL}(k_j) := \mathbb{I}(k_j \leq y_j^*), \qquad \forall j \in [n], \tag{3.4}$$

where $\mathbb{I}(\cdot)$ is the indicator function.

Another widely adopted admission control policy is bid price control (BP), which leverages bid prices to represent the marginal value or opportunity cost of a unit of capacity. Prior to delving into this policy, we introduce the following dual problems of DLP and the SAA version of PNLP, respectively:

$$\begin{aligned}
\min_{\boldsymbol{P},\boldsymbol{W}} \quad & \sum_{l=1}^{m} P_l C_l + \sum_{j=1}^{n} W_j \mu_j && \text{(3.5a)}\\
\text{s.t.} \quad & \sum_{l=1}^{m} A_{l,j} P_l + W_j \geq r_j, \qquad \forall j \in [n], && \text{(3.5b)}\\
& \boldsymbol{P}, \boldsymbol{W} \geq \boldsymbol{0}, && \text{(3.5c)}
\end{aligned}$$

and

$$\begin{aligned}
\min_{\boldsymbol{P},\boldsymbol{W}} \quad & \sum_{l=1}^{m} P_l C_l + \sum_{j=1}^{n} \sum_{i=1}^{N} W_{j,i} D_j^i && \text{(3.6a)}\\
\text{s.t.} \quad & \sum_{l=1}^{m} A_{l,j} P_l + \sum_{i=1}^{N} W_{j,i} \geq r_j, \qquad \forall j \in [n], && \text{(3.6b)}\\
& 0 \leq W_{j,i} \leq \frac{r_j}{N}, \qquad \forall i \in [N], j \in [n], && \text{(3.6c)}\\
& \boldsymbol{P} \geq \boldsymbol{0}. && \text{(3.6d)}
\end{aligned}$$

In the dual formulation of DLP or PNLP, $P_l$ are dual prices on capacity constraints, i.e., $\sum_{j=1}^{n} A_{l,j} y_j \leq C_l, \forall l \in [m]$, and $W_j$ or $W_{j,i}$ are dual prices on demand constraints, i.e., $y_j \leq \mu_j, \forall j \in [n]$ or $x_{j,i} \leq D_j^i, \forall i \in [N], j \in [n]$. Typically, the bid prices are computed as optimal dual prices on capacity constraints, which we denote $\boldsymbol{P}^*$. Based on the optimal dual prices $P_l^*$ for each resource $l$, the decision maker can set the BP policy as

$$\pi_j^{BP}(k_j) := \mathbb{I}\left(r_j \geq \sum_{l=1}^{m} A_{l,j} P_l^*, \boldsymbol{C}_{k_j} \geq \boldsymbol{A}_j\right), \qquad \forall j \in [n], \tag{3.7}$$

where $\boldsymbol{C}_{k_j}$ represents the remaining capacity when the $k_j{}^{\text{th}}$ customer requesting product $j$ arrives. Under this control policy, once a customer for product $j$ arrives, the decision maker accepts the request only if the price the customer pays, $r_j$, is higher than the aggregated bid price, $\sum_{l=1}^{m} A_{l,j} P_l^*$, and the remaining capacity is enough to serve this request.

## 3.2 Distributionally Robust Approach

In practice, the business environment is complex and real demand can be affected by many factors that may change over time. Hence, although we can collect many historical data samples, if nonstationarity is taken into account, there may be only a few samples effective for the coming selling horizon. Under the setting that the only information about demand we have is the limited historical data, to estimate the probability distributions of the total demand for each product while taking into account the inaccurate estimate, we use the framework of distributionally robust optimization (DRO). More specifically, a data-driven (DD) distributionally robust extension of the PNLP model, where instead of assuming knowing the true distribution or simply adopting the empirical distribution, we assume the true distribution lies in a certain ambiguity set $\mathcal{F}$ constructed from the historical data. More specifically, the ambiguity set $\mathcal{F}$ is determined based on a goodness-of-fit (GoF) hypothesis test with respect to the given realized data. Consider the objective to find a static partitioned allocation decision that maximizes the worst-case expected revenue over the ambiguity set $\mathcal{F}$. We obtain the following DRO model:

$$V^{DRO}(\boldsymbol{C}) = \max_{\boldsymbol{y}\in Y} \inf_{F_0\in\mathcal{F}} \sum_{j=1}^{n} r_j \mathbb{E}\left[\min\{D_j, y_j\}\right]. \tag{3.8}$$

Similar to the approaches presented in Section 3.1, the corresponding robust admission control policies rely on the optimal solution to our robust optimization model (3.8).

### 3.2.1 Ambiguity Set

Let the support of $\boldsymbol{D}$ be $\boldsymbol{\Xi} \subseteq \mathbb{R}^n$, which is assumed to be compact. Let $\mathcal{P}(\boldsymbol{\Xi})$ be the set of all probability distributions over $\boldsymbol{\Xi}$. For any probability distribution $F_0 \in \mathcal{P}(\boldsymbol{\Xi})$, we denote $F_0(E)$ as the probability of the event $\boldsymbol{D} \in E$. For the case $n = 1$, let $F_0(a) = F((-\infty, a])$. When $n > 1$, let $F_{0,j}$ be the univariate marginal distribution of the $j^{\text{th}}$ component, i.e., the marginal distribution of demand on product $j$, $F_{0,j}(E) = F_0(\{\boldsymbol{D} : D_j \in E\})$.

The tractability of the robust formulation is essential for success in practical applications, and it depends significantly on the choice of the ambiguity set $\mathcal{F}$. In the data-driven setting, as shown in (Bertsimas, Gupta, and Kallus 2018b), if the ambiguity set is conic representable, then the DD-DRO problem can be formulated as a conic optimization problem, which can be practically solved using commercial solvers such as Gurobi. Therefore, in this research, we utilize the canonical conic representable ambiguity set determined by multiple GoF tests concerning the given observations.

Assuming we have access to independent and identically distributed (i.i.d.) samples $\boldsymbol{D}^1, \cdots, \boldsymbol{D}^N$ and a hypothetical distribution $F_0$ that is chosen a priori and not based on the given demand data samples, we utilize a GoF test that examines the null hypothesis

$$H_0\text{: The samples } \boldsymbol{D}^1, \cdots, \boldsymbol{D}^N \text{ were drawn from } F_0\,,$$

and the alternative hypothesis

$$H_1\text{: The samples } \boldsymbol{D}^1, \cdots, \boldsymbol{D}^N \text{ were not drawn from } F_0.$$

We choose a significance level $\alpha$ for the GoF test, which means the probability of incorrectly rejecting $H_0$ is at most $\alpha$. For our GoF test, we specify a statistic

$$S_N = S_N(F_0, \boldsymbol{D}^1, \cdots, \boldsymbol{D}^N)$$

that depends on the given samples $\boldsymbol{D}^1, \cdots, \boldsymbol{D}^N$ and chosen hypothetical distribution $F_0$. We also specify a threshold $Q_{S_N}(\alpha)$ that does not depend on either the samples or the hypothetical distribution . The GoF test rejects $H_0$, i.e., rejects $F_0$ if $S_N > Q_{S_N}(\alpha)$. To determine $Q_{S_N}(\alpha)$, we can either refer to the tables or compute by simulation.

The set of all distributions $F_0$ that pass a GoF test is called the confidence region of the test and is denoted by

$$\mathcal{F}^{\alpha}_{S_N}(\boldsymbol{D}^1, \cdots, \boldsymbol{D}^N) = \left\{F_0 \in \mathcal{P}(\boldsymbol{\Xi}) : S_N(F_0, \boldsymbol{D}^1, \cdots, \boldsymbol{D}^N) \leq Q_{S_N}(\alpha)\right\}. \tag{3.9}$$

It is used as the data-driven ambiguity set in our DRO model. Note that if the data $\boldsymbol{D}^1, \cdots, \boldsymbol{D}^N$ is indeed drawn from $F_0$, the probability of incorrectly rejecting $F_0$ is at most $\alpha$. Hence, this ambiguity set contains the true, unknown distribution $F$ with probability at least $1 - \alpha$ with respect to the distribution of the data drawn from $F$.

To treat the random demand vector in our model, we adopt the approach based on marginal test in (Bertsimas, Gupta, and Kallus 2018b), which enjoys finite-sample and asymptotic properties under mild assumptions, since our cost function is separable over the components of random demand vector $\boldsymbol{D}$. Assume that each component of the random vector, i.e., the demand of each product, is a general univariate continuous random variable. For univariate distributions, the commonly used GoF tests are, among others, the Kolmogorov-Smirnov(KS) test, the Kuiper test, the Cramér-von Mises(CvM) test, the Watson test, and the Anderson-Darling(AD) test. We let $S_N$ be a statistic of a GoF test for univariate distribution and $Q_{S_N}(\alpha)$ the corresponding threshold. Recall that we can either refer to the tables or compute by simulation to determine $Q_{S_N}(\alpha)$.

Let the significance levels for each product be $\alpha_1, \cdots, \alpha_n > 0$ such that $\alpha = \alpha_1 + \cdots + \alpha_n < 1$. Under the approach based on marginal test, we test the hypothesis for joint distribution, i.e., $F = F_0$, by testing the marginal hypotheses $F_j = F_{0,j}$ for each product $j \in [n]$. For each marginal hypothesis, we apply a certain GoF test with the test statistic $S_N$ at significance level $\alpha_j$ to the given samples $D_j^1, \cdots, D_j^N$ (the historical demand for product $j$). The hypothetical distribution $F_0$ will be rejected if any of the marginal hypotheses fail. The corresponding confidence region is

$$\mathcal{F}^{\alpha}_{\text{marginals}} = \left\{F_0 \in \mathcal{P}(\boldsymbol{\Xi}) : F_{0,j} \in \mathcal{F}^{\alpha_j}_j(D^1_j, \cdots, D^N_j),\ \forall j \in [n]\right\}, \tag{3.10}$$

where $\mathcal{F}^{\alpha_j}_j$ is the confidence region of the test applied on the product $j$.

When we set our ambiguity set $\mathcal{F}$ to $\mathcal{F}^{\alpha}_{\text{marginals}}$, i.e., $\mathcal{F} = \mathcal{F}^{\alpha}_{\text{marginals}}$, (3.8) can be written as

$$\begin{aligned} V^{DRO}(\boldsymbol{C}) &= \max_{\boldsymbol{y}\in Y} \inf_{F_0 \in \mathcal{F}} \sum_{j=1}^{n} r_j \mathbb{E}\left[\min\{D_j, y_j\}\right] \\ &= \max_{\boldsymbol{y}\in Y} \sum_{j=1}^{n} \inf_{F_{0,j}\in\mathcal{F}^{\alpha_j}_j} \mathbb{E}_{F_{0,j}}\left[r_j \min\{D_j, y_j\}\right] \end{aligned} \tag{3.11}$$

Hence we convert the inner problem of (3.8),i.e., the worst-case expected revenue over GoF test-based ambiguity set, to the summation of $n$ minimization problem. For any $j \in [n]$, the $j^{\text{th}}$ minimization problem is the worst-case expected revenue from selling product $j$ over the ambiguity set $\mathcal{F}^{\alpha_j}_j$.

### 3.2.2 Tractable Reformulation

For any $j \in [n]$, with $D_j$ univariate, we use $D^{(i)}_j$ to denote the observation of demand on product $j$ that is the $i^{th}$ smallest so that $D^{(1)}_j \leq \cdots \leq D^{(N)}_j$. In other words, $D^{(i)}_j$ for all $i \in [N]$ can be obtained by sorting $D^1_j, \cdots, D^N_j$ in ascending order. In addition, for any product $j$, let the lower and upper bounds of random variable $D_j$ be $D^{(0)}_j$ and $D^{(N+1)}_j$, respectively, which are determined before solving our optimization problem and therefore serve as inputs of our model.

We define $\zeta_{j,i}$ as $F_{0,j}(D^{(i)}_j)$, and $\zeta_j$ as the vector $[\zeta_{j,1}, \cdots, \zeta_{j,N}]^{\mathrm{T}}$. Given the observations $D_j^{(1)}, \cdots, D_j^{(N)}$, the test statistic $S_{N_j}$ for an aribitrarily univariate GoF test chosen for $D_j$ can be expressed as a function of $\zeta_j$, and we denote it as $S_{N_j}(\zeta_j)$. The following lemma shows that the ambiguity set of each minimization problem based on any univariate GoF test can be represented as a canonical cone.

**Lemma 1** ((Bertsimas, Gupta, and Kallus 2018b), Theorem 10)**.** *Given $j \in [n]$, for any univariate GoF test with test statistic $S_{N_j}(\zeta_j)$, the condition $S_{N_j}(\zeta_j) \leq Q_{S_{N_j}}(\alpha_j)$ is equivalent to*

$$B_{S_{N_j}} \zeta_j - b_{S_{N_j}, \alpha_j} \in K_{S_{N_j}},$$

*where the convex cone $K_{S_{N_j}}$, the matrix $B_{S_{N_j}}$, and the vector $b_{S_{N_j}, \alpha_j}$ are defined in Theorem 10 of (Bertsimas, Gupta, and Kallus 2018b).*

By Lemma 1 we can reformulate the ambiguity set with respect to demand on product $j$ into a canonical cone. In particular, for any $j \in [n]$, the $j^{\text{th}}$ minimization problem in (3.11), i.e., $\inf_{F_{0,j} \in \mathcal{F}_j^{\alpha_j}} \mathbb{E}_{F_{0,j}} [r_j \min \{D_j, y_j\}]$, is equivalent to

$$\begin{aligned} \min_{\zeta_j} \quad & \mathbb{E}_{\zeta_j} [r_j \min \{D_j, y_j\}] && (3.12a) \\ \text{s.t.} \quad & B_{S_{N_j}} \zeta - b_{S_{N_j}, \alpha_j} \in K_{S_{N_j}}, && (3.12b) \\ & \zeta_{j,i} - \zeta_{j,i-1} \geq 0, \qquad \forall i \in [N+1], && (3.12c) \end{aligned}$$

where we have $\zeta_{j,0} = 0$ and $\zeta_{j,N+1} = 1$ according to the definition of $\zeta_{j,i}$. Model (3.12) is not yet directly solvable because of the nonlinear term $\min \{D_j, y_j\}$. Following the Theorem 11 of (Bertsimas, Gupta, and Kallus 2018b), we further reformulate problem (3.12) into a solvable conic optimization problem in the following theorem.

**Theorem 1.** *The optimization problem (3.12) is equivalent to the following optimization problem:*

$$\max_{\lambda_j, c_j} \quad b_{S_{N_j},\alpha_j}^{\mathrm{T}} \lambda_j + c_{j,N+1} \tag{3.13a}$$

$$\text{s.t.} \quad \lambda_j \in K_{S_{N_j}}^*, \; c_j \in \mathbb{R}^{N+1}, \tag{3.13b}$$

$$\left( B_{S_{N_j}}^{\mathrm{T}} \lambda_j \right)_i = c_{j,i} - c_{j,i+1}, \qquad \forall i \in [N], \tag{3.13c}$$

$$c_{j,i} \le r_j D_j^{(i-1)}, c_{j,i} \le r_j y_j, \qquad \forall i \in [N+1], \tag{3.13d}$$

*where $K_{S_{N_j}}^*$ denotes the dual cone of $K_{S_{N_j}}$.*

*Proof.* Please see Appendix A. □

This theorem offers a tractable reformulation, which yields an explicit conic maximization formulation for each minimization component of the DRO model (3.11). As a result, we can express model (3.11) as a single-level conic optimization problem by inserting all the reformulations of minimization components and changing the order of maximization and summation. The final reformulation of (3.11), which is problem (3.8) with the ambiguity set defined by the confidence region of GoF tests, is presented below.

$$V^{DRO}(\boldsymbol{C})$$

$$= \max_{\boldsymbol{y},\boldsymbol{\lambda},\boldsymbol{c}} \quad \sum_{j=1}^{n} \left( b_{S_{N_j},\alpha_j}^{\mathrm{T}} \lambda_j + c_{j,N+1} \right) \tag{3.14a}$$

$$\text{s.t.} \quad \boldsymbol{y} \in Y,\ \lambda_j \in K_{S_{N_j}}^{*},\ c_j \in \mathbb{R}^{N+1}, \qquad \forall j \in [n], \tag{3.14b}$$

$$\left( B_{S_{N_j}}^{\mathrm{T}} \lambda_j \right)_i = c_{j,i} - c_{j,i+1}, \qquad \forall i \in [N], j \in [n], \tag{3.14c}$$

$$c_{j,i} \le r_j D_j^{(i-1)}, c_{j,i} \le r_j y_j, \qquad \forall i \in [N+1], j \in [n]. \tag{3.14d}$$

Our robust PNLP (3.11) is now solvable by commercial solvers like Gurobi using the conic optimization formulation in (3.14). In addition, the theoretical tractability and statistical properties including asymptotic convergence and finite-sample performance are guaranteed by applying Theorem 13, Theorem 5, Proposition 5 and Proposition 1 of (Bertsimas, Gupta, and Kallus 2018b) to our DD-DRO problem.

Let $\left\{\boldsymbol{y}^{DRO*}, \boldsymbol{\lambda}^{DRO*}, \boldsymbol{c}^{DRO*}\right\}$ be an optimal solution of the robust PNLP reformulation (3.14). Note that $\boldsymbol{y}^{DRO*}$ can be interpreted as an optimal partitioned allocation of capacity to each product against a worst-case distribution in the ambiguity set. Similar to the approaches of DLP and PNLP, we can construct a BL control policy by using $\boldsymbol{y}^{DRO*}$ as the fixed partitioned allocation quota vector. The allocation $\boldsymbol{y}^{DRO*}$ is referred to as the robust optimal PNLP booking limits.

In the meantime, we can consider the following dual problem of the robust

PNLP reformulation (3.14):

$$\min_{\boldsymbol{P},\boldsymbol{W},\boldsymbol{u}} \quad \sum_{l=1}^{m} P_l C_l + \sum_{j=1}^{n} \sum_{i=1}^{N+1} W_{j,i} r_j D_j^{(i-1)} \tag{3.15a}$$

$$\text{s.t.} \quad \sum_{l=1}^{m} A_{l,j} P_l + \sum_{i=1}^{N+1} W_{j,i} r_j \geq r_j, \qquad \forall j \in [n], \tag{3.15b}$$

$$B_{S_{N_j}} \zeta_j - b_{S_{N_j},\alpha_j} \in K_{S_{N_j}}, \qquad \forall j \in [n], \tag{3.15c}$$

$$0 \leq W_{j,i} \leq \zeta_{j,i} - \zeta_{j,i-1}, \qquad \forall i \in [N+1], j \in [n], \tag{3.15d}$$

$$\boldsymbol{P} \geq \boldsymbol{0}, \tag{3.15e}$$

where $\zeta_{j,0} = 0$ and $\zeta_{j,N+1} = 1, \forall j \in [n]$. A robust BP admission policy can be constructed by substituting $P_l^*$ in (3.7) with $P_l^{DRO*}$, which represents the optimal $P_l$ obtained from solving the optimization problem (3.15). We refer to $P_l^{DRO*}$ as the robust optimal PNLP bid prices. These prices capture the robust marginal resource values and are used to evaluate the opportunity cost of a unit of resource $l$ inventory requested when considering inaccurate estimates of demand distributions.

# Chapter 4

# Robust Dynamic Model

The robust models in Chapter 3 are all static models that ignore time dynamics. In those cases, the robust PNLP booking limits and bide prices are determined by a one-time, static rule and therefore do not change as a function of time. In practical applications, static approximation model is usually re-solved frequently to create a dynamic decision rule that changes through time as the system evolves. In this chapter, we aim at exploring a tractable model from which we can derive a dynamically changing robust decision rule as the remaining capacity changes over time.

## 4.1 Robust Dynamic BP Control

To address the curse of dimensionality associated with the dynamic programming (DP) formulation of the NRM problem, one commonly used approach is to use an affine approximation to the value function to obtain a dynamic admission policy. Therefore, in this section, we utilize an affine linear approximation to find a tractable reformulation for our multi-period robust NRM problem. Let $\boldsymbol{D}_t = [D_{t,1}, \cdots, D_{t,n}]^{\mathrm{T}}$ and $\boldsymbol{y}_t = [y_{t,1}, \cdots, y_{t,n}]^{\mathrm{T}}$ be the demand vector and partitioned booking limit vector during time period $t$, respectively. Note that $D_{t,j}, y_{t,j} \in \mathbb{R}_+$ for all $j \in [n], t \in [T]$. We assume that the demand during each time period is i.i.d. We denote the remaining capacity at the beginning of time period $t$ as $\boldsymbol{C}_t$. Let $\mathcal{F}_t$ denote the ambiguity set of $\boldsymbol{D}_t$ and $F_t$ be any distribution in $\mathcal{F}_t$. The optimal worst-case expected revenue from time period $t$ to $T$ given remaining capacity $\boldsymbol{C}_t$ at the beginning of time $t$ is

$$V_t(\boldsymbol{C}_t) = \max_{\pi_t \in \Pi_t} \inf_{F_{[t]} \in \mathcal{F}_{[t]}} \mathbb{E}_{F_{[t]}} \left[ \sum_{\tau=t}^{T} \boldsymbol{r}^{\mathrm{T}} (\boldsymbol{D}_\tau \wedge \boldsymbol{y}_\tau) + V_{T+1}(\boldsymbol{C}_{T+1}) \right], \tag{4.1}$$

where $\boldsymbol{C}_{\tau+1} = \boldsymbol{C}_\tau - \boldsymbol{A}(\boldsymbol{D}_\tau \wedge \boldsymbol{y}_\tau), \tau = t, \cdots, T$ and we assume that $V_{T+1}(\cdot) = 0$. Furthermore, $\pi_t := \{\omega_t, \cdots, \omega_T\}$ is defined as the policy we adopt at the period $t$ and $\omega_\tau, \forall \tau \in \{t, \cdots, T\}$ is the policy for each time periods starting from $t$ respectively; $\Pi_t$ is the set of all admissible policies at period $t$; $\mathcal{F}_{[t]} := \mathcal{F}_t \times \cdots \times \mathcal{F}_T$ is the ambiguity set from period $t$ to $T$ and $F_{[T]}$ is any distribution in the ambiguity set $\mathcal{F}_{[t]}$. As shown in Theorem 2.1 and Theorem 2.2 of (Iyengar 2005), the multi-stage distributionally robust NRM model can be expressed using the following robust Bellman equation:

$$V_t(\boldsymbol{C}_t) = \max_{\boldsymbol{y}_t} \inf_{F_t \in \mathcal{F}_t} \mathbb{E}_{F_t} \left[ \boldsymbol{r}^{\mathrm{T}} (\boldsymbol{D}_t \wedge \boldsymbol{y}_t) + V_{t+1}(\boldsymbol{C}_t - \boldsymbol{A}(\boldsymbol{D}_t \wedge \boldsymbol{y}_t)) \right], \quad \forall t \in [T], \boldsymbol{C}_t \in \mathcal{C}_t, \tag{4.2}$$

where

$$\mathcal{C}_t = \begin{cases} \{\boldsymbol{C}_1\}, & \text{if} \quad t = 1, \\ \{\boldsymbol{C}_t \in \mathbb{R}^m : \boldsymbol{0} \le \boldsymbol{C}_t \le \boldsymbol{C}_1\}, & \text{if} \quad t \ge 2. \end{cases}$$

The boundary condition is $V_{T+1}(\boldsymbol{C}) = 0$ for all $\boldsymbol{C} \in \mathcal{C}_{T+1}$. This formulation can be further equivalently written as follows:

$$\min_{\{V_t(\cdot)\}_{\forall t}} \quad V_1(\boldsymbol{C}_1) \tag{4.3a}$$

$$\text{s.t.} \quad V_t(\boldsymbol{C}_t) \ge \inf_{F_t \in \mathcal{F}_t} \mathbb{E}_{F_t}\left[\boldsymbol{r}^{\mathrm{T}}(\boldsymbol{D}_t \wedge \boldsymbol{y}_t) + V_{t+1}(\boldsymbol{C}_t - \boldsymbol{A}(\boldsymbol{D}_t \wedge \boldsymbol{y}_t))\right], \quad \forall t \in [T], \boldsymbol{C}_t \in \mathcal{C}_t, \boldsymbol{y}_t \in \mathcal{Y}(\boldsymbol{C}_t), \tag{4.3b}$$

where

$$\mathcal{Y}(\boldsymbol{C}_t) = \{\boldsymbol{y}_t \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{y}_t \le \boldsymbol{C}_t, \boldsymbol{y}_t \ge \boldsymbol{0}\}, \qquad \forall t \in [T], \boldsymbol{C}_t \in \mathcal{C}_t.$$

The following proposition shows that any feasible solution to (4.3) provides an upper bound on the optimal worst-case expected revenue starting from any period, which is obtained by solving (4.2).

**Proposition 1.** *Suppose that $V_t(\cdot)$ solves the robust Bellman equation (4.2) and $\hat{V}_t(\cdot)$ is a feasible solution to (4.3). Then*

$$\hat{V}_t(\boldsymbol{C}_t) \ge V_t(\boldsymbol{C}_t), \qquad \forall t \in [T], \boldsymbol{C}_t \in \mathcal{C}_t.$$

*Proof.* We prove this by induction. For all $\boldsymbol{C}_T \in \mathcal{C}_T, \boldsymbol{y}_T \in \mathcal{Y}(\boldsymbol{C}_T)$, constraint

(4.3b) implies

$$\hat{V}_T(\boldsymbol{C}_T) \geq \inf_{F_T \in \mathcal{F}_T} \mathbb{E}_{F_T}\left[\boldsymbol{r}^{\mathrm{T}}(\boldsymbol{D}_T \wedge \boldsymbol{y}_T) + V_{T+1}(\boldsymbol{C}_T - \boldsymbol{A}(\boldsymbol{D}_T \wedge \boldsymbol{y}_T))\right].$$

As $V_{T+1}(\boldsymbol{C}) = 0$ for all $\boldsymbol{C} \in \mathcal{C}_{T+1}$, we obtain

$$\hat{V}_T(\boldsymbol{C}_T) \geq \max_{\boldsymbol{y}_T \in \mathcal{Y}(\boldsymbol{C}_T)} \inf_{F_T \in \mathcal{F}_T} \mathbb{E}_{F_T}\left[\boldsymbol{r}^{\mathrm{T}}(\boldsymbol{D}_T \wedge \boldsymbol{y}_T)\right] = V_T(\boldsymbol{C}_T),$$

where the equality follows from the robust Bellman equation (4.2) for period $t = T$. Now suppose the result is true for period $t+1$. Then, for all $\boldsymbol{C}_t \in \mathcal{C}_t, \boldsymbol{y}_t \in \mathcal{Y}(\boldsymbol{C}_t)$, constraint (4.3b) yields

$$\begin{aligned}\hat{V}_t(\boldsymbol{C}_t) &\geq \inf_{F_t \in \mathcal{F}_t} \mathbb{E}_{F_t}\left[\boldsymbol{r}^{\mathrm{T}}(\boldsymbol{D}_t \wedge \boldsymbol{y}_t) + \hat{V}_{t+1}(\boldsymbol{C}_t - \boldsymbol{A}(\boldsymbol{D}_t \wedge \boldsymbol{y}_t))\right] \\ &\geq \inf_{F_t \in \mathcal{F}_t} \mathbb{E}_{F_t}\left[\boldsymbol{r}^{\mathrm{T}}(\boldsymbol{D}_t \wedge \boldsymbol{y}_t) + V_{t+1}(\boldsymbol{C}_t - \boldsymbol{A}(\boldsymbol{D}_t \wedge \boldsymbol{y}_t))\right].\end{aligned}$$

As a result,

$$\hat{V}_t(\boldsymbol{C}_t) \geq \max_{\boldsymbol{y}_t \in \mathcal{Y}(\boldsymbol{C}_t)} \inf_{F_t \in \mathcal{F}_t} \mathbb{E}_{F_t}\left[\boldsymbol{r}^{\mathrm{T}}(\boldsymbol{D}_t \wedge \boldsymbol{y}_t) + V_{t+1}(\boldsymbol{C}_t - \boldsymbol{A}(\boldsymbol{D}_t \wedge \boldsymbol{y}_t))\right] = V_t(\boldsymbol{C}_t),$$

where the equality is exactly the robust Bellman equation for time period $t$. □

In general, formulation (4.3) is not computationally tractable when the state and action spaces are continuous, as it involves an infinite number of decision variables and constraints. However, formulation (4.3) provides a starting point for exploring approximations to value function. In order to find a tractable reformulation of problem (4.3), we approximate the value function $V_t(\boldsymbol{C}_t)$ with a linear function of the form

$$V_t(\boldsymbol{C}_t) \approx \theta_t + \sum_{l=1}^{m} P_{t,l} C_{t,l} \qquad \forall t \in [T], \boldsymbol{C}_t \in \mathcal{C}_t, \tag{4.4}$$

where $\{P_{t,l} : t \in [T], l \in [m]\}$ estimates the marginal value of a unit of each resource $l$ in period $t$, and $\{\theta_t : t \in [T]\}$ is a constant offset. Since $V_{T+1}(\boldsymbol{C}) = 0$ for all $\boldsymbol{C} \in \mathcal{C}_{T+1}$, we have $\theta_{T+1} = 0, P_{T+1,l} = 0$ for all $l \in [m]$. The affine linear approximation approach is aimed at computing dynamic bid prices and is actually commonly used in revenue management to address the issue of computational tractability.

Plugging the approximation (4.4) into the reformulation (4.3) of the robust Bellman equation (4.2), we have

$$\min_{\{\theta_t, \boldsymbol{P}_t\}_{\forall t}} \quad \theta_1 + \sum_{l=1}^{m} P_{1,l} C_{1,l} \tag{4.5a}$$

$$\begin{aligned} \text{s.t.} \quad & \theta_t - \theta_{t+1} + \sum_{l=1}^{m} (P_{t,l} - P_{t+1,l}) C_{t,l} \geq \\ & \inf_{F_t \in \mathcal{F}_t} \mathbb{E}_{F_t} \left[ \sum_{j=1}^{n} \left( r_j - \sum_{l=1}^{m} A_{l,j} P_{t+1,l} \right) (D_{t,j} \wedge y_{t,j}) \right], \\ & \forall t \in [T], \boldsymbol{C}_t \in \mathcal{C}_t, \boldsymbol{y}_t \in \mathcal{Y}(\boldsymbol{C}_t). \end{aligned} \tag{4.5b}$$

By substituting the approximation (4.4) into problem (4.3), we restrict (4.3) into an optimization problem over only the parameters $\{P_{t,l} : t \in [T], l \in [m]\}$ and $\{\theta_t : t \in [T]\}$. Let $\left\{P^*_{t,l} : t \in [T], l \in [m]\right\}$ and $\{\theta^*_t : t \in [T]\}$ be the optimal solution of problem (4.5). Proposition 1 immediately yields that $\theta_1 + \sum_{l=1}^{m} P^*_{1,l} C_{1,l}$ is an upper bound on the optimal worst-case expected revenue. Furthermore, we can use the optimal robust solution to construct an approximate dynamic admission policy. Specifically, we use $\left\{P^*_{t,l} : t \in [T], l \in [m]\right\}$ as the optimal robust dynamic bid prices. The admission control policy for deciding whether to accept the ${k_j}^{\text{th}}$ unit of demand for product $j$ in period $t$ is given by:

$$\pi^{DROBP}_{t,j}(k_j) := \mathbb{I}\left(r_j \geq \sum_{l=1}^{m} A_{l,j} P^*_{t+1,l},\ \boldsymbol{C}_{k_j} \geq \boldsymbol{A}_j\right), \quad \forall t \in [T], j \in [n], \tag{4.6}$$

where $\boldsymbol{C}_{k_j}$ is the vector of available capacities when the $k_j^{\text{th}}$ order of product $j$ arrives. This robust policy obtained from the optimal solution of problem (4.5) compares the revenue from selling product $j$ with the total value, or opportunity cost, of the resources consumed by this product. We open product $j$ for sale at time period $t$ if the corresponding revenue justifies the value of the required resources and the remaining capacities are sufficient.

For the rest of this section, we focus on efficient solution techniques for the reformulation (4.5). The following result develops an equivalent formulation of (4.5).

**Proposition 2.** *Problem (4.5) is equivalent to the following problem:*

$$\min_{\{\theta_t, \boldsymbol{P}_t\}_{\forall t}} \quad \theta_1 + \sum_{l=1}^{m} P_{1,l} C_{1,l} \tag{4.7a}$$

$$\begin{aligned} s.t. \quad & \theta_t - \theta_{t+1} + \sum_{l=1}^{m} (P_{t,l} - P_{t+1,l}) C_{t,l} \geq \\ & \inf_{F_t \in \mathcal{F}_t} \left\{ \sum_{j=1}^{n} \mathbb{E}_{F_{t,j}} \left[ \left( r_j - \sum_{l=1}^{m} A_{l,j} P_{t+1,l} \right)^+ (D_{t,j} \wedge y_{t,j}) \right] \right\}, \\ & \forall t \in [T], \boldsymbol{C}_t \in \mathcal{C}_t, \boldsymbol{y}_t \in \mathcal{Y}(\boldsymbol{C}_t). \end{aligned} \tag{4.7b}$$

*Proof.* Note that

$$\begin{aligned}
&\mathbb{E}_{F_t}\left[\sum_{j=1}^{n}\left(r_j-\sum_{l=1}^{m}A_{l,j}P_{t+1,l}\right)(D_{t,j}\wedge y_{t,j})\right]\\
&=\sum_{j=1}^{n}\mathbb{E}_{F_{t,j}}\left[\left(r_j-\sum_{l=1}^{m}A_{l,j}P_{t+1,l}\right)(D_{t,j}\wedge y_{t,j})\right],
\end{aligned}$$

by which we can rewrite formulation (4.5) as

$$\min_{\{\theta_t,\boldsymbol{P}_t\}_{\forall t}} \quad \theta_1+\sum_{l=1}^{m}P_{1,l}C_{1,l} \tag{4.8a}$$

$$\begin{aligned}
\text{s.t.} \quad & \theta_t-\theta_{t+1}+\sum_{l=1}^{m}(P_{t,l}-P_{t+1,l})C_{t,l}\geq\\
& \inf_{F_t\in\mathcal{F}_t}\left\{\sum_{j=1}^{n}\mathbb{E}_{F_{t,j}}\left[\left(r_j-\sum_{l=1}^{m}A_{l,j}P_{t+1,l}\right)(D_{t,j}\wedge y_{t,j})\right]\right\},\\
& \forall t\in[T],\boldsymbol{C}_t\in\mathcal{C}_t,\boldsymbol{y}_t\in\mathcal{Y}(\boldsymbol{C}_t).
\end{aligned} \tag{4.8b}$$

Next, we show the equivalence between (4.7) and (4.8) through the following two parts:

First, consider an arbitrarily feasible solution $\{\theta_t, P_{t,l} : t\in[T], l\in[m]\}$ to (4.7). For any $t\in[T]$, $\boldsymbol{C}_t\in\mathcal{C}_t$, $\boldsymbol{y}_t\in\mathcal{Y}(\boldsymbol{C}_t)$, the feasibility to (4.7) yields

$$\begin{aligned}
&\theta_t-\theta_{t+1}+\sum_{l=1}^{m}(P_{t,l}-P_{t+1,l})C_{t,l}\\
&\geq\inf_{F_t\in\mathcal{F}_t}\left\{\sum_{j=1}^{n}\mathbb{E}_{F_{t,j}}\left[\left(r_j-\sum_{l=1}^{m}A_{l,j}P_{t+1,l}\right)^{+}(D_{t,j}\wedge y_{t,j})\right]\right\}\\
&\geq\inf_{F_t\in\mathcal{F}_t}\left\{\sum_{j=1}^{n}\mathbb{E}_{F_{t,j}}\left[\left(r_j-\sum_{l=1}^{m}A_{l,j}P_{t+1,l}\right)(D_{t,j}\wedge y_{t,j})\right]\right\},
\end{aligned}$$

where the second inequality follows from $D_{t,j} \geq 0$ and $\boldsymbol{y}_t \geq \boldsymbol{0}$ for all $\boldsymbol{y}_t \in \mathcal{Y}(\boldsymbol{C}_t)$. It is straightforward that $\{\theta_t, P_{t,l} : t \in [T], l \in [m]\}$ is also feasible to (4.8).

Next, let $\{\theta_t, P_{t,l} : t \in [T], l \in [m]\}$ be an arbitrary feasible solution to (4.8). Consider any $t \in [T]$, $\boldsymbol{C}_t \in \mathcal{C}_t$, $\boldsymbol{y}_t \in \mathcal{Y}(\boldsymbol{C}_t)$. Note that

$$\begin{aligned}
&\inf_{F_t \in \mathcal{F}_t} \left\{ \sum_{j=1}^{n} \mathbb{E}_{F_{t,j}} \left[ \left( r_j - \sum_{l=1}^{m} A_{l,j} P_{t+1,l} \right)^+ (D_{t,j} \wedge y_{t,j}) \right] \right\} \\
= &\inf_{F_t \in \mathcal{F}_t} \left\{ \sum_{j=1}^{n} \mathbb{E}_{F_{t,j}} \left[ \left( r_j - \sum_{l=1}^{m} A_{l,j} P_{t+1,l} \right) (D_{t,j} \wedge y'_{t,j}) \right] \right\}, \qquad (4.9)
\end{aligned}$$

where

$$y'_{t,j} = \begin{cases} y_{t,j}, & \text{if } r_j - \sum_{l=1}^{m} A_{l,j} P_{t+1,l} \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

for any $j \in [n]$. Since $\boldsymbol{y}'_t = \left[ y'_{t,1}, \cdots, y'_{t,n} \right]^{\mathrm{T}}$ is in $\mathcal{Y}(\boldsymbol{C}_t)$, we obtain from the feasibility to (4.8) that

$$\begin{aligned}
&\theta_t - \theta_{t+1} + \sum_{l=1}^{m} (P_{t,l} - P_{t+1,l}) C_{t,l} \\
\geq &\inf_{F_t \in \mathcal{F}_t} \left\{ \sum_{j=1}^{n} \mathbb{E}_{F_{t,j}} \left[ \left( r_j - \sum_{l=1}^{m} A_{l,j} P_{t+1,l} \right) (D_{t,j} \wedge y'_{t,j}) \right] \right\} \\
= &\inf_{F_t \in \mathcal{F}_t} \left\{ \sum_{j=1}^{n} \mathbb{E}_{F_{t,j}} \left[ \left( r_j - \sum_{l=1}^{m} A_{l,j} P_{t+1,l} \right)^+ (D_{t,j} \wedge y_{t,j}) \right] \right\},
\end{aligned}$$

where the equality follows from (4.9). As the above inequality holds for all $t \in [T]$, $\boldsymbol{C}_t \in \mathcal{C}_t$, $\boldsymbol{y}_t \in \mathcal{Y}(\boldsymbol{C}_t)$, $\{\theta_t, P_{t,l} : t \in [T], l \in [m]\}$ is also feasible to (4.7). This complete the proof. □

For each time period $t$, we can construct the ambiguity set $\mathcal{F}_t$ using the same method based on marginal test as that used to construct $\mathcal{F}$ in the robust static model. Then we have

$$\inf_{F_t \in \mathcal{F}_t} \left\{ \sum_{j=1}^{n} \mathbb{E}_{F_{t,j}} \left[ \left( r_j - \sum_{l=1}^{m} A_{l,j} P_{t+1,l} \right)^+ (D_{t,j} \wedge y_{t,j}) \right] \right\}$$
$$= \sum_{j=1}^{n} \left( r_j - \sum_{l=1}^{m} A_{l,j} P_{t+1,l} \right)^+ \inf_{F_{t,j} \in \mathcal{F}_{t,j}} \mathbb{E}_{F_{t,j}} (D_{t,j} \wedge y_{t,j}),$$

and hence problem (4.7) can be further reformulated as

$$\min_{\{\theta_t, \boldsymbol{P}_t\}_{\forall t}} \quad \theta_1 + \sum_{l=1}^{m} P_{1,l} C_{1,l} \tag{4.10a}$$

$$\text{s.t.} \quad \theta_t - \theta_{t+1} + \sum_{l=1}^{m} (P_{t,l} - P_{t+1,l}) C_{t,l} \geq \sum_{j=1}^{n} \left( r_j - \sum_{l=1}^{m} A_{l,j} P_{t+1,l} \right)^+ \inf_{F_{t,j} \in \mathcal{F}_{t,j}} \mathbb{E}_{F_{t,j}} (D_{t,j} \wedge y_{t,j}),$$
$$\forall t \in [T], \boldsymbol{C}_t \in \mathcal{C}_t, \boldsymbol{y}_t \in \mathcal{Y}(\boldsymbol{C}_t), \tag{4.10b}$$

where the term $\inf_{F_{t,j} \in \mathcal{F}_{t,j}} \mathbb{E}_{F_{t,j}} (D_{t,j} \wedge y_{t,j})$ represents the expected consumption of product $j$ during time period $t$, based on the worst-case distribution of demand for product $j$ in its ambiguity set $\mathcal{F}_{t,j}$, when the pre-allocation quota $y_{t,j}$ is given. Moreover, if the decision maker uses policy (4.6) to determine whether to sell a unit of product $j$ at time period $t$, the term $(r_j - \sum_{l=1}^{m} A_{l,j} P_{t+1,l})^+$ is exactly the net difference between the revenue from selling one unit of product $j$ at time period $t$ and the opportunity cost of the total capacities required to sell it. Here, the opportunity cost of each unit of capacity $l$ is estimated using the marginal value approximation of this resource at time period $t+1$, i.e., $P_{t+1,l}$. We refer to

$(r_j - \sum_{l=1}^{m} A_{l,j} P_{t+1,l})^+$ as the approximate net revenue from one unit of product $j$ at time period $t$. Consequently, the right-hand side of constraint (4.10b) can be interpreted as the worst-case expected approximate total net revenue from all products sold during time period $t$.

For the left-hand side of constraint (4.10b), we have

$$\begin{aligned}
&\theta_t - \theta_{t+1} + \sum_{l=1}^{m} (P_{t,l} - P_{t+1,l}) C_{t,l} \\
&= \theta_t + \sum_{l=1}^{m} P_{t,l} C_{t,l} - \left( \theta_{t+1} + \sum_{l=1}^{m} P_{t+1,l} C_{t,l} \right) \\
&= \theta_t + \sum_{l=1}^{m} P_{t,l} C_{t,l} - \left( \theta_{t+1} + \sum_{l=1}^{m} P_{t+1,l} \left( C_{t+1,l} - C_{t+1,l} + C_{t,l} \right) \right) \\
&= \theta_t + \sum_{l=1}^{m} P_{t,l} C_{t,l} - \left( \theta_{t+1} + \sum_{l=1}^{m} P_{t+1,l} C_{t+1,l} \right) - \sum_{l=1}^{m} P_{t+1,l} \left( C_{t,l} - C_{t+1,l} \right) \\
&\approx V_t(\boldsymbol{C}_t) - V_{t+1}(\boldsymbol{C}_{t+1}) - \sum_{l=1}^{m} P_{t+1,l} \left( C_{t,l} - C_{t+1,l} \right),
\end{aligned}$$

which is the net revenue during time period $t$ under the affine approximation to the robust Bellman equation. Hence, constraint (4.10b) can be interpreted as follows: for each time period $t \in [T]$, starting from any remaining capacities, we consider the approximation of the optimal robust total net revenue generated in this period. This value must be greater than or equal to the worst-case expected approximate total net revenue in period $t$ under any selected pre-allocation quotas, which is obtained by summing the multiplications of the worst-case expected product sales and the net revenue estimation for each product in the corresponding period. Model (4.10) then seeks to find a feasible $\{\theta_t, P_{t,l} : t \in [T], l \in [m]\}$ that achieves the minimal upper bound on the worst-case expected approximate total net revenue for all possible pre-allocation quotas. This allows us to determine an optimal approximate value for the robust revenue over the entire selling horizon.

Formulation (4.10) cannot be directly solved yet. One cause is the piecewise linear functions $(r_j - \sum_{l=1}^{m} A_{l,j} P_{t+1,l})^+$ in constraint (4.10b). Introducing the parameters $\{z_{t,j} : t \in [T], j \in [n]\}$, we have the following equivalent reformulation of (4.10):

$$\min_{\{\theta_t, \boldsymbol{P}_t, \boldsymbol{z}_t\}_{\forall t}} \quad \theta_1 + \sum_{l=1}^{m} P_{1,l} C_{1,l} \tag{4.11a}$$

$$\text{s.t.} \quad \theta_t - \theta_{t+1} + \sum_{l=1}^{m} (P_{t,l} - P_{t+1,l}) C_{t,l} \geq \sum_{j=1}^{n} z_{t,j} \inf_{F_{t,j} \in \mathcal{F}_{t,j}} \mathbb{E}_{F_{t,j}} (D_{t,j} \wedge y_{t,j}), \quad \forall t \in [T], \boldsymbol{C}_t \in \mathcal{C}_t, \boldsymbol{y}_t \in \mathcal{Y}(\boldsymbol{C}_t), \tag{4.11b}$$

$$z_{t,j} \geq r_j - \sum_{l=1}^{m} A_{l,j} P_{t+1,l}, \quad z_{t,j} \geq 0, \qquad \forall t \in [T], j \in [n]. \tag{4.11c}$$

For any given $y_{t,j}$, $\inf_{F_{t,j} \in \mathcal{F}_{t,j}} \mathbb{E}_{F_{t,j}} (D_{t,j} \wedge y_{t,j})$ can be determined using the distributionally robust solution approach discussed in Chapter 3. Therefore, formulation (4.11) is actually a linear programming model with the decision variables $\{\theta_t, P_{t,l}, z_{t,j} : t \in [T], l \in [m], j \in [n]\}$ and an infinite number of constraints. More specifically, while the number of constraints in (4.11c) is $2Tn$, there are infinitely many constraints in (4.11b) due to the continuous state space $\mathcal{C}_t$ and action space $\mathcal{Y}(\boldsymbol{C}_t)$.

To summarize, linear optimization problem (4.11) has manageable variables but an infinite number of constraints. Therefore, we use a constraint generation algorithm to solve this problem. The approach involves iteratively solving a master problem that shares the same decision variables as (4.11) but contains only a subset of the constraints. Once the master problem is solved, we check whether the solution violates any of the constraints in the original problem (4.11). If there are no violations, the solution to the current master problem is optimal for the original problem and we can stop. However, if there is a violation, we add the violated constraint to the current master problem and then start another iteration of the algorithm using the updated master problem.

Given an optimal solution $\left\{\tilde{\theta}_t, \tilde{P}_{t,l}, \tilde{z}_{t,j} : t \in [T], l \in [m], j \in [n]\right\}$ to the current master problem, we check whether the current solution is feasible to the original problem by solving

$$\min_{\boldsymbol{C}_t \in \mathcal{C}_t, \boldsymbol{y}_t \in \mathcal{Y}(\boldsymbol{C}_t)} \left\{ \tilde{\theta}_t - \tilde{\theta}_{t+1} + \sum_{l=1}^{m} (\tilde{P}_{t,l} - \tilde{P}_{t+1,l}) C_{t,l} - \sum_{j=1}^{n} \tilde{z}_{t,j} \inf_{F_{t,j} \in \mathcal{F}_{t,j}} \mathbb{E}_{F_{t,j}} (D_{t,j} \wedge y_{t,j}) \right\} \tag{4.12}$$

for all $t$. According to the distributionally robust solution procedure discussed in Chapter 3, given any $t \in [T]$ and $j \in [n]$, $\inf_{F_{t,j} \in \mathcal{F}_{t,j}} \mathbb{E}_{F_{t,j}} (D_{t,j} \wedge y_{t,j})$ can be equivalently converted to

$$\max_{\lambda_{t,j}, c_{t,j}} \quad b_{S_{N_j}, \alpha_{t,j}}^{\mathrm{T}} \lambda_{t,j} + c_{t,j,N+1} \tag{4.13a}$$

$$\text{s.t.} \quad \lambda_{t,j} \in K_{S_{N_j}}^*, c_{t,j} \in \mathbb{R}^{N+1}, \tag{4.13b}$$

$$\left(B_{S_{N_j}}^{\mathrm{T}} \lambda_{t,j}\right)_i = c_{t,j,i} - c_{t,j,i+1}, \qquad \forall i \in [N], \tag{4.13c}$$

$$c_{t,j,i} \leq D_j^{(i-1)}, c_{t,j,i} \leq y_{t,j}, \qquad \forall i \in [N+1]. \tag{4.13d}$$

Thus, $\sum_{j=1}^{n} \tilde{z}_{t,j} \inf_{F_{t,j} \in \mathcal{F}_{t,j}} \mathbb{E}_{F_{t,j}}(D_{t,j} \wedge y_{t,j})$ is equivalent to

$$\max_{\lambda_{t,j}, c_{t,j}} \quad \sum_{j=1}^{n} \tilde{z}_{t,j} \left( b_{S_{N_j}, \alpha_{t,j}}^{\mathrm{T}} \lambda_{t,j} + c_{t,j,N+1} \right) \tag{4.14a}$$

$$\text{s.t.} \quad \lambda_{t,j} \in K_{S_{N_j}}^{*}, c_{t,j} \in \mathbb{R}^{N+1}, \qquad \forall j \in [n], \tag{4.14b}$$

$$\left( B_{S_{N_j}}^{\mathrm{T}} \lambda_{t,j} \right)_i = c_{t,j,i} - c_{t,j,i+1}, \qquad \forall i \in [N], j \in [n], \tag{4.14c}$$

$$c_{t,j,i} \leq D_j^{(i-1)}, \; c_{t,j,i} \leq y_{t,j}, \qquad \forall i \in [N+1], j \in [n], \tag{4.14d}$$

which is an explicit conic optimization problem. With this tractable reformulation in (4.14), the separate problem (4.12) for any $t$ can be rewritten as

$$\min_{\substack{\boldsymbol{C}_t \in \mathcal{C}_t, \boldsymbol{y}_t \in \mathcal{Y}(\boldsymbol{C}_t), \\ \{\lambda_t, c_t\} \in \Omega(\boldsymbol{y}_t)}} \left\{ \tilde{\theta}_t - \tilde{\theta}_{t+1} + \sum_{l=1}^{m} (\tilde{P}_{t,l} - \tilde{P}_{t+1,l}) C_{t,l} - \sum_{j=1}^{n} \tilde{z}_{t,j} \left( b_{S_{N_j}, \alpha_{t,j}}^{\mathrm{T}} \lambda_{t,j} + c_{t,j,N+1} \right) \right\}, \tag{4.15}$$

where

$$\Omega(\boldsymbol{y}_t) = \left\{ \lambda_t, c_t : (4.14b), (4.14c), (4.14d) \right\}, \qquad \forall t \in [T], \boldsymbol{C}_t \in \mathcal{C}_t, \boldsymbol{y}_t \in \mathcal{Y}(\boldsymbol{C}_t).$$

Let $\left\{ \tilde{\boldsymbol{C}}_t, \tilde{\boldsymbol{y}}_t, \tilde{\lambda}_t, \tilde{c}_t \right\}$ be the optimal solution to (4.15). If the optimal objective value of the single-level conic optimization problem (4.15) is less than 0 for a particular $t$, the constraint corresponding to $\tilde{\boldsymbol{C}}_t \in \mathcal{C}_t$, $\tilde{\boldsymbol{y}}_t \in \mathcal{Y}(\boldsymbol{C}_t)$, and $t \in [T]$ in the original problem (4.11) is violated by the solution to the current master problem. We then update the master problem by adding this constraint to the current master problem and solve the updated master problem. The detailed procedure to implement the constraint generation algorithm is provided in Appendix B.

The efficiency of the constraint generation algorithm is significantly dependent on how quickly we can solve the separate problem. As the sample size $N$ grows and the size of the separate program increases, the proposed constraint generation algorithm may not be sufficiently efficient. Therefore, in the next section, we derive a more efficient and easy-to-implement multi-period robust approximate NRM formulation based on model (4.11).

## 4.2 Robust Dynamic BL Control

In this section, we construct a compact equivalent reformulation of (4.2) that yields a robust dynamic BL policy for the multi-period robust NRM problem.

Consider any period $t \in [T]$. Let $\boldsymbol{P}_t$ and $\boldsymbol{z}_t$ denote the vectors of $P_{t,l}$ for all $l \in [m]$ and $z_{t,j}$ for all $j \in [n]$, respectively. Define

$$
\begin{aligned}
&\phi_t(\boldsymbol{P}_t, \boldsymbol{P}_{t+1}, \boldsymbol{z}_t) \\
= &\sum_{l=1}^{m}(P_{t+1,l} - P_{t,l})C_{t,l} + \sum_{j=1}^{n} z_{t,j} \inf_{F_{t,j} \in \mathcal{F}_{t,j}} \mathbb{E}_{F_{t,j}}(D_{t,j} \wedge y_{t,j}) \\
= &\max_{\substack{\boldsymbol{C}_t \in \mathcal{C}_t, \boldsymbol{y}_t \in \mathcal{Y}(\boldsymbol{C}_t), \\ \{\lambda_t, c_t\} \in \Omega(\boldsymbol{y}_t)}} \left\{ \sum_{l=1}^{m}(P_{t+1,l} - P_{t,l})C_{t,l} + \sum_{j=1}^{n} z_{t,j} \left( b^{\mathrm{T}}_{S_{N_j}, \alpha_{t,j}} \lambda_{t,j} + c_{t,j,N+1} \right) \right\},
\end{aligned}
\tag{4.16}
$$

where the second equality is obtained becuase $\sum_{j=1}^{n} \tilde{z}_{t,j} \inf_{F_{t,j} \in \mathcal{F}_{t,j}} \mathbb{E}_{F_{t,j}}(D_{t,j} \wedge y_{t,j})$ is equivalent to the formulation in (4.14). Using the function $\phi_t(\boldsymbol{P}_t, \boldsymbol{P}_{t+1}, \boldsymbol{z}_t)$, we can equivalently rewrite the robust ADP formulation (4.11) as

$$\min_{\{\theta_t,\boldsymbol{P}_t,\boldsymbol{z}_t\}_{\forall t}} \quad \theta_1 + \sum_{l=1}^{m} P_{1,l} C_{1,l} \tag{4.17a}$$

$$\text{s.t.} \quad \theta_t - \theta_{t+1} \geq \phi_t(\boldsymbol{P}_t, \boldsymbol{P}_{t+1}, \boldsymbol{z}_t) \qquad \forall t \in [T], \tag{4.17b}$$

$$z_{t,j} \geq r_j - \sum_{l=1}^{m} A_{l,j} P_{t+1,l},\ z_{t,j} \geq 0, \quad \forall j \in [n], t \in [T], \tag{4.17c}$$

where the decision variables are still $\{\theta_t, P_{t,l}, z_{t,j} : t \in [T], l \in [m], j \in [n]\}$. Applying the above formulation, we can obtain an equivalent compact formulation of model (4.11).

**Proposition 3.** *Model (4.11) is equivalent to the following conic programming problem:*

$$\min_{\substack{\boldsymbol{P},\boldsymbol{z},\\ \boldsymbol{V},\boldsymbol{W},\boldsymbol{v},\boldsymbol{\zeta}}} \sum_{t=1}^{T}\sum_{l=1}^{m} V_{t,l} C_{1,l} + \sum_{t=1}^{T}\sum_{j=1}^{n}\sum_{i=1}^{N+1} W_{t,j,i} D_j^{(i-1)} + \sum_{l=1}^{m} P_{1,l} C_{1,l} \tag{4.18a}$$

$$\textit{s.t.} \quad V_{t,l} - v_{t,l} \geq P_{t+1,l} - P_{t,l}, \qquad \forall l \in [m], t \in [T], \tag{4.18b}$$

$$\sum_{l=1}^{m} A_{l,j} v_{t,l} + \sum_{i=1}^{N+1} W_{t,j,i} \geq z_{t,j}, \qquad \forall j \in [n], t \in [T], \tag{4.18c}$$

$$B_{S_{N_j}} \zeta_{t,j} - z_{t,j} b_{S_{N_j},\alpha_{t,j}} \in K_{S_{N_j}}, \qquad \forall j \in [n], t \in [T], \tag{4.18d}$$

$$0 \leq W_{t,j,i} \leq \zeta_{t,j,i} - \zeta_{t,j,i-1}, \qquad \forall i \in [N], j \in [n], t \in [T], \tag{4.18e}$$

$$0 \leq W_{t,j,N+1} \leq z_{t,j} - \zeta_{t,j,N}, \qquad \forall j \in [n], t \in [T], \tag{4.18f}$$

$$z_{t,j} \geq r_j - \sum_{l=1}^{m} A_{l,j} P_{t+1,l}, \qquad \forall j \in [n], t \in [T], \tag{4.18g}$$

$$\boldsymbol{z}, \boldsymbol{V}, \boldsymbol{v} \geq \boldsymbol{0}, \tag{4.18h}$$

*where* $\zeta_{t,j,0} = 0$ *for all* $j \in [n], t \in [T]$.

*Proof.* It is straightforward to prove that (4.17) has an optimal solution that satisfies the condition $\theta_t = \theta_{t+1} + \phi_t(\boldsymbol{P}_t, \boldsymbol{P}_{t+1}, \boldsymbol{z}_t)$ for all $t \in [T]$. Also note that $\theta_{T+1} = 0$. Therefore, there exists an optimal solution to (4.17) with $\theta_1 = \sum_{t=1}^{T} \phi_t(\boldsymbol{P}_t, \boldsymbol{P}_{t+1}, \boldsymbol{z}_t)$. This result implies the equivalence between (4.17) and the following optimization problem:

$$\min_{\{\boldsymbol{P}_t, \boldsymbol{z}_t\}_{\forall t}} \quad \sum_{t=1}^{T} \phi_t(\boldsymbol{P}_t, \boldsymbol{P}_{t+1}, \boldsymbol{z}_t) + \sum_{l=1}^{m} P_{1,l} C_{1,l} \tag{4.19a}$$

$$\text{s.t.} \quad z_{t,j} \geq r_j - \sum_{l=1}^{m} A_{l,j} P_{t+1,l}, \;\; z_{t,j} \geq 0, \qquad \forall j \in [n], t \in [T]. \tag{4.19b}$$

Recall that (4.11) is equivalent to (4.17). We obtain the equivalence between (4.11) and (4.19).

For any $t \in [T]$, observed that $\phi_t(\boldsymbol{P}_t, \boldsymbol{P}_{t+1}, \boldsymbol{z}_t)$ defined in (4.16) is the optimal value of a conic programming problem. Applying strong duality, we obtain

$$\phi_t(\boldsymbol{P}_t, \boldsymbol{P}_{t+1}, \boldsymbol{z}_t)$$

$$= \min_{\boldsymbol{V}_t, \boldsymbol{W}_t, \boldsymbol{v}_t, \boldsymbol{\zeta}_t} \quad \sum_{l=1}^{m} V_{t,l} C_{1,l} + \sum_{j=1}^{n} \sum_{i=1}^{N+1} W_{t,j,i} D_j^{(i-1)} \tag{4.20a}$$

$$\text{s.t.} \quad V_{t,l} - v_{t,l} \geq P_{t+1,l} - P_{t,l}, \qquad \forall l \in [m], \tag{4.20b}$$

$$\sum_{l=1}^{m} A_{l,j} v_{t,l} + \sum_{i=1}^{N+1} W_{t,j,i} \geq z_{t,j}, \qquad \forall j \in [n], \tag{4.20c}$$

$$B_{S_{N_j}} \zeta_{t,j} - z_{t,j} b_{S_{N_j}, \alpha_{t,j}} \in K_{S_{N_j}}, \qquad \forall j \in [n], \tag{4.20d}$$

$$0 \leq W_{t,j,i} \leq \zeta_{t,j,i} - \zeta_{t,j,i-1}, \qquad \forall i \in [N], j \in [n], \tag{4.20e}$$

$$0 \leq W_{t,j,N+1} \leq z_{t,j} - \zeta_{t,j,N}, \qquad \forall j \in [n], \tag{4.20f}$$

$$\boldsymbol{V}_t, \boldsymbol{v}_t \geq \boldsymbol{0}, \tag{4.20g}$$

where $\zeta_{t,j,0} = 0$ for all $j \in [n]$. We can obtain the desired result by substituting the above formulation of $\phi_t(\boldsymbol{P}_t, \boldsymbol{P}_{t+1}, \boldsymbol{z}_t)$ into (4.19). $\square$

Then according to strong duality, problem (4.18) is equivalent to

$$\max_{\substack{\boldsymbol{y},\boldsymbol{\sigma},\\ \boldsymbol{a},\boldsymbol{\lambda},\boldsymbol{h}}} \quad \sum_{t=1}^{T}\sum_{j=1}^{n} r_j \sigma_{t,j} \tag{4.21a}$$

$$\text{s.t.} \quad a_{1,l} = C_{1,l}, \qquad \forall l \in [m], \tag{4.21b}$$

$$a_{t,l} = a_{t-1,l} - \sum_{j=1}^{n} A_{l,j}\sigma_{t-1,j}, \qquad \forall l \in [m], t \in \{2, \cdots, T\}, \tag{4.21c}$$

$$\sum_{j=1}^{n} A_{l,j} y_{t,j} \le a_{t,l}, \qquad \forall l \in [m], t \in [T], \tag{4.21d}$$

$$y_{t,j} \le D_j^{(i-1)} + h_{t,j,i}, \qquad \forall i \in [N+1], j \in [n], t \in [T], \tag{4.21e}$$

$$(B_{S_{N_j}}^{\mathrm{T}} \lambda_{t,j})_i = h_{t,j,i+1} - h_{t,j,i}, \qquad \forall i \in [N], j \in [n], t \in [T], \tag{4.21f}$$

$$\sigma_{t,j} \le y_{t,j} + b_{S_{N_j},\alpha_{t,j}}^{\mathrm{T}} \lambda_{t,j} - h_{t,j,N+1}, \qquad \forall j \in [n], t \in [T]. \tag{4.21g}$$

$$\lambda_{t,j} \in K_{S_{N_j}}^{*}, \qquad \forall j \in [n], t \in [T], \tag{4.21h}$$

$$\boldsymbol{y}, \boldsymbol{\sigma}, \boldsymbol{a}, \boldsymbol{h} \ge \boldsymbol{0}. \tag{4.21i}$$

Model (4.21) approximates the robust Bellman equation (4.2) based on the robust ADP (4.11) derived in Section 4.1. When $T = 1$, it becomes the same as the robust static model (3.14) discussed in Chapter 3. In the multi-period setting, it approximates how the system evolves against a worst-case demand distribution from the ambiguity set.

The decision variables $\{a_{t,l} : l \in [m], t \in [T]\}$ in model (4.21) correspond to the dual variables for constraint (4.18b). They can be seen as the approximate worst-case expected remaining capacities at the beginning of each time period $t$.

The constraints (4.21b) are associated with the decision variable $\{P_{1,l} : l \in [m]\}$ in (4.18), and specify the initial available capacity for each resource. Similarly, the constraints (4.21c) correspond to the decision variables $\{P_{t,l} : l \in [m], t \in 2, ..., T\}$ in (4.18). The term $\sum_{j=1}^{n} A_{l,j}\sigma_{t-1,j}$ in constraints (4.21c) represents the approximate worst-case expected capacity consumption for resource $l$ during period $t-1$, where $\sigma_{t,j}$, as aforementioned, is the expected demand of product $j$ fulfilled in period $t$. Therefore, constraints (4.21b) and (4.21c) ensure the flow balance of available capacities over time under the worst-case expectation. In addition, for any $l \in [m]$ and $t \in [T]$, the constraint associated with the decision variable $V_{t,l}$ in model (4.18) is $a_{t,l} \le C_{1,l}$. However, we can already deduce from constraints (4.21b) and (4.21c) that $C_{1,l} = a_{1,l} \ge a_{2,l} \ge \cdots \ge a_{T,l}$. Therefore, we can drop the redundant constraints $a_{t,l} \le C_{1,l}$ associated with the decision variables $V_{t,l}$.

The decision variables $\{y_{t,j} : j \in [n], t \in [T]\}$ in problem (4.21) correspond to the dual variables of constraint (4.18c). They represent the approximate worst-case expected pre-allocation quota for product $j$ at time period $t$. According to constraint (4.21d), which is associated with the decision variables $\{v_{t,l} : l \in [m], t \in [T]\}$ in (4.18), the amount of resources consumed by the pre-allocation decision $y_{t,j}$ at time period $t$ does not exceed the corresponding expected remaining capacity $a_{t,l}$. Constraint (4.21e) is associated with the decision variables $\{W_{t,j,i} : j \in [n], i \in [N+1], t \in [T]\}$, and imposes an upper bound $D_j^{(i-1)} + h_{t,j,i}$ on $y_{t,j}$. Here, $D_j^{(i-1)}$ is the $(i-1)^{\text{th}}$ smallest demand data sample for product $j$, and $h_{t,j,i}$ can be viewed as a constant offset.

The decision variables $\left\{\lambda_{t,j} \in K^*_{S_{N_j}} : j \in [n], t \in [T]\right\}$ are the dual prices of constraint (4.18d) in (4.18), which characterizes the ambiguity set for the unknown demand distribution. Constraint (4.21f) shows how the constant offsets $h_{t,j,i}$ are determined using $\left\{\lambda_{t,j} \in K^*_{S_{N_j}} : j \in [n], t \in [T]\right\}$. Notice that constraint (4.21f) is associated with the decision variables $\{\zeta_{t,j,i} : j \in [n], i \in [N], t \in [T]\}$, which represent the worst-case distribution function of the demand of product $j$ in period $t$.

The decision variables $\{\sigma_{t,j} : j \in [n], t \in [T]\}$ in model (4.21) correspond to the dual prices of the constraints (4.18g). They represent the expected amount of demand for product $j$ satisfied during time period $t$. Constraint (4.21g), associated with decision variables $\{z_{t,j} : j \in [n], t \in [T]\}$ in (4.18), relates the expectation of the fulfilled demand $\sigma_{t,j}$ to the pre-allocation quota $y_{t,j}$, with the demand uncertainty captured by $b^{\mathrm{T}}_{S_{N_j},\alpha_{t,j}} \lambda_{t,j} - h_{t,j,N+1}$. Based on this interpretation of $\sigma_{t,j}$, the objective function of model (4.21) can be viewed as the approximate worst-case total expected revenue from all products throughout the selling horizon.

The formulation (4.21) is computationally tractable, with problem size increasing linearly in the number of resources $m$, the number of products $n$, the sample size $N$, and the length of the selling horizon $T$. Furthermore, an optimal solution to problem (4.21) can be used to construct our robust dynamic BL policy. More specifically, our booking limits can be set to $\boldsymbol{y}^*$, which is an optimal $\boldsymbol{y}$ to (4.21). Under this policy, every request is accepted as long as the booking limit for that period is not reached, and there is sufficient capacity to serve the demand. For example, for the $k_{t,j}^{\text{th}}$ demand of product $j$ in period $t$, the decision to serve the request is based on the policy:

$$\pi_{t,j}^{DROBL}(k_{t,j}) := \mathbb{I}\left(k_{t,j} \le y^*_{t,j}, \boldsymbol{C}_{k_{t,j}} \ge \boldsymbol{A}_j\right), \qquad \forall t \in [T], j \in [n], \quad (4.22)$$

where $\boldsymbol{C}_{k_{t,j}}$ is the remaining capacity when the $k_{t,j}^{\text{th}}$ demand of product $j$ occurs.

We want to emphasize that when implementing the robust dynamic BL policy, it is necessary to check if there is enough remaining capacity to fulfill a unit of demand for a particular product, even if the book limiting for that product has not been reached. This is because constraint (4.21d) only ensures that the resources consumed by the booking limits $\{y_{t,j}^* : j \in [n]\}$ do not exceed the expected available capacities $\{a_{t,l} : l \in [m]\}$ in period $t$, but the actual available capacity may deviate from $\{a_{t,l} : l \in [m]\}$ due to demand uncertainty. Therefore, it may not be possible to fulfill all demands within the booking limits. To address this issue, we make the following adjustments to the robust booking limits $\boldsymbol{y}_t^*$ in our numerical study. For any period $t$, we can observe the actual available capacity of resource $l$ at the beginning of the period, denoted by $C_{t,l}$, before any demands occur. Given the robust booking limits $\{y_{t,j}^* : j \in [n]\}$ and the actual capacities $\{C_{t,l} : l \in [m]\}$, the following optimization problem can be solved at the beginning of period $t$:

$$\max_{\boldsymbol{y}^C} \quad \sum_{j=1}^{n} r_j y_{t,j}^C \tag{4.23a}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} A_{l,j} y_{t,j}^C \le C_{t,l}, \qquad \forall l \in [m], \tag{4.23b}$$

$$y_{t,j}^C \le y_{t,j}^*, \qquad \forall j \in [n]. \tag{4.23c}$$

Let $\{y_{t,j}^C : j \in [n]\}$ denote an optimal solution to this problem. Instead of using $\{y_{t,j}^* : j \in [n]\}$ as the booking limits for period $t$ in our implementation of the robust dynamic BL policy, we use $\{y_{t,j}^C : j \in [n]\}$. By doing so, there will be enough resources to fulfill all demands within the corresponding booking limits $y_{t,j}^C$.

# Chapter 5

# Numerical Experiments

In this chapter, we report the numerical experiments conducted to verify the contribution that our distributionally robust approach can make to quantity-based NRM problems.

We consider a line structure that is common in NRM problem and consider the case that is, $m = 5$ resource classes with $n = 15$ product classes (line structure, $\sum_{l=1}^{5} l = 15$). Figure 5.1 shows the line structure. The parameter generating

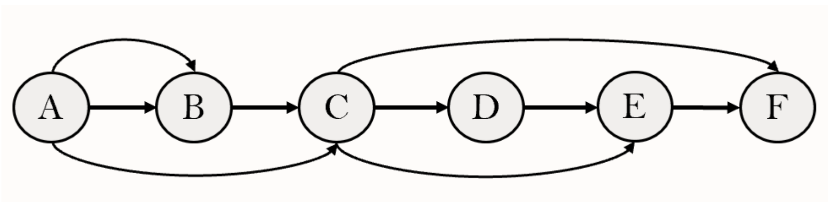


Figure 5.1: Line structure with $5$ classes of resources

methods for the relevant parameters in our numerical studies are detailed as follows:

$\boldsymbol{r}$: the revenue vector from selling one product. We first generate the revenue from selling one product that uses only one class of resource from a discrete uniform distribution on the interval $[10, 60]$. Then we generate the revenues for those products that use more than one class of resources from a discrete uniform distribution on the interval $[0.9R, R]$, where $R$ is the summation of the revenue from corresponding single-resource products. In practice, it is common that the price of a bundle product is less than the sum of the prices of each component.

$\boldsymbol{A}$: the $m \times n$ BOM matrix contains elements only $1$ and $0$, which represents whether a resource class is a component of a product or not.

$\boldsymbol{C}$: the initial capacity for each resources are all $1000$.

$T$: the number of time periods (opportunities of changing control policies). We consider the total selling horizon is $30$ units. For the static models including DLP, SAA and Robust Static Model, we determine a policy at the beginning of selling and never change during these $30$ units selling time. On the other hand, for the Robust Dynamic Model, we consider $T = \{2, 5, 10, 30\}$. For example, if $T$=2, we have two opportunities of changing our admission policy, one is at the beginning of time period $1$ and the other is at the beginning of time period $16$, and each policy will be adopted and unchanged during each selling horizon with $15$ units selling time.

$\boldsymbol{D}$: the demand for each product. In this research, we assume we could collect the data which represents the demand during one selling unit and the number of samples we can obtain is limited, $10$ or $20$ in this numerical study. Note that for the robust dynamic model when $T = 30$, we can simply use the unit demand data for each time period because in this case, the length of each time period equals one unit selling time. However, for other cases in our research, we can no longer conduct like that. Therefore, we propose a bootstrapping method to generate some relevant data to approximate multi-unit demand distribution. For example, suppose we have $N = 10$ unit demand samples and the number of time periods is $T = 2$, which implies we need to estimate the total demand during $30/T = 15$ unit selling time using these $10$ data samples representing demand during one unit selling time. According to the discrete uniform distribution on the interval $[1, N]$, we randomly sample $15$ data from these $10$ data samples with replacement and then the sum of these $15$ samples will be viewed as a sample drawn from the $15 - unit$ demand distribution. Repeating this procedure $10$ times we can obtain $10$ samples to approximate the distribution of demand during $15$ selling units. In addition, for DLP, we simply scale up the mean of one unit demand to estimate the mean of demand during $15$ selling units.

On the other hand, in our research, the historical data and test data are all generated from normal distribution and details about the data generation will be discussed later in the numerical performance part.

$\alpha$: the significance level for our selected GoF test. In this research, we simply use the same $\alpha$ for any $j \in [n], t \in [T]$, i.e., $\alpha_{t,j} = \alpha,\ \forall j \in [n], t \in [T]$. Also, it is possible to set different $\alpha$ across different products and time periods. Note that the choice of $\alpha$ is relevant to the number of product classes. More specifically, consider the case in our numerical study, we have $15$ classes of products, suppose we set our $\alpha = 0.01$ for each class of product, then the total significance level will be $0.01 \times 15 = 0.15$, which satisfies the definition of significance level, i.e., $0 \leq \alpha \leq 1$. However, consider $\alpha = 0.1$, the total significance level exceeds $1$ and therefore is not valid. According to the statistical test table, $\alpha = 0.001, 0.01, 0.02, 0.05$ are valid for our case and we choose $\alpha = 0.01, 0.02$ in this numerical study. In addition, without referring tables, we can choose other value for $\alpha$ and conduct simulation to compute corresponding statistics threshold

$Q_{S_N}(\alpha)$, where $S_N$ is the statistics of the selected GoF test. In this research, we use the widely adopted KS test in statistical hypothesis testing field as our GoF test.

We computed our instances using Gurobi $10.0.1$ and Table 5.1 shows the optimal objective value and CPU time for each model when the number of data samples is $10$. The models shown in the first column of Table 5.1 is:

- DLP: deterministic linear program (3.1);
- SAA: (3.3), sample average approximation applied to PNLP;
- RS: robust static model (3.14);
- RD: robust dynamic model (4.21).

Recall that each sample is a vector with dimension $n = 15$. Given any $j \in [n]$, we generate $N$ samples from a truncated normal distribution (which ensures our generated data are all greater than or equal to zero) whose original mean (we denote $\mu_j^G$) is randomly generated from a discrete uniform distribution on the interval $[10, 25]$ and whose original standard deviation (we denote $std_j^G$) is $\mu_j^G/\rho_j$, where $\rho_j$ is randomly generated from a discrete uniform distribution on the interval $[2, 6]$. Our historical and testing data are all generated by this method. Note that the historical observations applied to all the models in Table 5.1 are the same, i.e., for each $j$, one particular sample path from the truncated normal distribution $\tilde{n}\ (\hat{\mu}_j^G, \hat{std}_j^G)$, where $\hat{\mu}_j^G, \hat{std}_j^G$ are one pair of generated particular mean and standard deviation for original normal distribution. Based on these generated $10$ one unit demand samples, note that again we use a bootstrapping method to generate data for multi-unit demand samples for model SAA, RS and RD $(T = 2, 5, 10)$

while we use these $10$ one unit demand observations directly for RD $(T = 30)$, and simply scale up the mean of these $10$ one unit demand samples for model DLP.

For ease of comparison, in this numerical study, we view DLP as the benchmark for any other models mentioned in Table 5.1 and show their ratio to DLP. From Table 5.1, the optimal objective value of SAA is the closest to that of DLP while the optimal objective of RS is the furthest from that of DLP. Also, the optimal objective value of robust model is increasing with the number of time periods, which actually does not hold in policy performance.

| Model | Optimal Objective | | Ratio to DLP (%) | | Time Used ($\times 10^{-2} s$) | |
|---|---|---|---|---|---|---|
| DLP | 177673 | | 100 | | 2.337 | |
| SAA | 177535 | | 99.92 | | 3.372 | |
| | $\alpha = 0.01$ | $\alpha = 0.02$ | $\alpha = 0.01$ | $\alpha = 0.02$ | $\alpha = 0.01$ | $\alpha = 0.02$ |
| RS | 90729 | 96469 | 51.07 | 54.30 | 2.032 | 2.271 |
| RD (T=2) | 133303 | 138698 | 75.03 | 78.06 | 2.862 | 2.791 |
| RD (T=5) | 165649 | 167510 | 93.23 | 94.28 | 6.130 | 5.488 |
| RD (T=10) | 171264 | 172149 | 96.39 | 96.89 | 11.818 | 12.508 |
| RD (T=30) | 173106 | 173598 | 97.43 | 97.71 | 39.482 | 43.361 |

Table 5.1: Optimal objective value and CPU time of each model when $N$=10

In Table 5.2, we provide the numerical results for policy performance with testing data from the same distribution as historical realizations when the number of realizations is $10$. More specifically, we generate 10 more different sample paths (each path is of size $30$, which is the total units of selling time) from the truncated normal distribution $\tilde{n}\ (\hat{\mu}_j^G, \hat{std}_j^G)$, which is actually the distribution that we generate our historical data from. Therefore, Table 5.2 shows the numerical testing results under the assumption that the historical data and the testing data are exactly drawn from the same distribution, which in practice implies that the distribution remains unchanged from the $N$ days before our selling horizon to the end of our selling horizon (assuming one unit selling time is one day and we use the most recent daily data in our numerical experiments). In addition, we use the solution of model to construct admission control policy and therefore we name the policies the names of the corresponding models for clarity in Table 5.2.

Although separating the whole selling horizon into more time periods generates higher revenue from the perspective of optimal objective value for our robust approaches, we can not obtain such a result from the perspective of policy performance. The revenue from testing experiments is actually not strictly increasing with the number of time periods. We observe that among all the policies in Table 5.2, the average revenue generated by RS is the highest, even beyond that of DLP and SAA. Also, policy RS performs better than DLP and SAA in minimal revenue and equally well in maximal revenue as SAA. However, other robust policies all perform not as well as DLP and SAA in this same distribution case, from perspectives of average, minimal and maximal revenue. Furthermore, the average standard deviations of the robust policies are all less than those of DLP and SAA.

Unlike we did in Table 5.2, we generate testing data from different distribution from $\tilde{n}$ $(\hat{\mu}_j^G, \hat{std}_j^G)$, which we used to generate historical observations and testing data for Table 5.2, to conduct numerical experiments in Table 5.3. To be more detailed, we generate $10$ more different $(\mu_j^G, std_j^G)$ pairs using the same generating method as $(\hat{\mu}_j^G, \hat{std}_j^G)$ to obtain $10$ more different distributions and from each distribution we generate randomly one sample path with size $30$. Hence, we use these $10$ sets of testing data to evaluate the performance of our policies when the demand distribution during the selling horizon is different from the distribution of the historical data.

Table 5.3 shows the average revenue on the $10$ testing instances and mini-

| $\alpha$ | Policy | Revenue (Mean(Std),[Min,Max]) | | Ratio to DLP (%) | |
|---|---|---|---|---|---|
| | DLP | 176573 (825) | [175013, 177484] | 100 (100) | [100, 100] |
| | SAA | 177061 (598) | [175912, 177535] | 100.28 (72.48) | [100.51, 100.03] |
| | RS | 177248 (390) | [176296, 177535] | 100.38 (47.27) | [100.73, 100.03] |
| | RD (T=2) | 174045 (45) | [174030, 174179] | 98.57 (5.45) | [99.44, 98.14] |
| 0.01 | RD (T=5) | 171234 (85) | [171090, 171411] | 96.98 (10.30) | [97.76, 96.58] |
| | RD (T=10) | 173444 (71) | [173319, 173610] | 98.23 (8.61) | [99.03, 97.82] |
| | RD (T=30) | 171826 (65) | [171674, 171915] | 97.31 (7.88) | [98.09, 96.86] |
| | RS | 177248 (390) | [176296, 177535] | 100.38 (47.27) | [100.73, 100.03] |
| | RD (T=2) | 174113 (182) | [173952, 174518] | 98.61 (22.06) | [99.39, 98.33] |
| 0.02 | RD (T=5) | 171633 (75) | [171513, 171793] | 97.20 (9.09) | [98.00, 96.79] |
| | RD (T=10) | 173482 (78) | [173362, 173614] | 98.25 (9.45) | [99.06, 97.82] |
| | RD (T=30) | 171906 (70) | [171804, 172039] | 97.36 (8.48) | [98.17, 96.93] |

Table 5.2: Policy performance with testing data from the same distribution as historical data when $N$=10

| $\alpha$ | Policy | Revenue (Mean(Std),[Min,Max]) | | Ratio to DLP (%) | |
|---|---|---|---|---|---|
| | DLP | 166042 (9099) | [149147, 177673] | 100 (100) | [100, 100] |
| | SAA | 165323 (7802) | [150430, 177163] | 99.57 (85.75) | [100.86, 99.71] |
| | RS | 167016 (8517) | [150430, 177535] | 100.59 (93.60) | [100.86, 99.92] |
| | RD (T=2) | 173497 (936) | [170798, 174035] | 104.49 (10.29) | [114.52, 97.95] |
| 0.01 | RD (T=5) | 171548 (649) | [170130, 172378] | 103.32 (7.13) | [114.07, 97.02] |
| | RD (T=10) | 169708 (4036) | [163176, 173608] | 102.21 (44.36) | [109.41, 97.71] |
| | RD (T=30) | 170450 (2495) | [164859, 172262] | 102.65 (27.42) | [110.53, 96.95] |
| | RS | 167016 (8517) | [150430, 177535] | 100.59 (93.60) | [100.86, 99.92] |
| | RD (T=2) | 172625 (2023) | [167161, 174050] | 103.96 (22.23) | [112.08, 97.96] |
| 0.02 | RD (T=5) | 171625 (681) | [169882, 172374] | 103.36 (7.48) | [113.90, 97.02] |
| | RD (T=10) | 169677 (4039) | [163292, 173609] | 102.19 (44.39) | [109.48, 97.71] |
| | RD (T=30) | 169623 (3521) | [162236, 172553] | 102.16 (38.70) | [108.78, 97.12] |

Table 5.3: Policy performance with testing data from different distribution from historical data when $N$=10

mal and maximal revenue among the 10 instances. Similar to the performance in "same distribution" case, in "different distribution" case, the RS policy performs slightly better than DLP and SAA in average and minimal revenue. Remarkably, in contrast to performing not well in "same distribution" case, RD policies generate higher average revenue than DLP and SAA. More specifically, for $T = 2, 5, 10, 30$, the average revenue generated by RD policy is $2.21 \sim 4.49$ % ($\alpha = 0.01$) or $2.16 \sim 3.96$ % ($\alpha = 0.02$) higher than that of DLP while SAA generates less average revenue than that of DLP. It is also noteworthy that although RD policies generate less revenue than that of DLP from perspective of the maximal revenue, the minimal revenue generated by RD policies is significantly higher than that of DLP. Specifically, $9.41 \sim 14.52$ % when $\alpha = 0.01$ and $8.78 \sim 13.90$ % when $\alpha = 0.02$.

| Model | Optimal Objective | | Ratio to DLP (%) | | Time Used ($\times 10^{-2}s$) | |
|---|---|---|---|---|---|---|
| DLP | 177635 | | 100 | | 1.787 | |
| SAA | 177416 | | 99.88 | | 2.152 | |
| | $\alpha = 0.01$ | $\alpha = 0.02$ | $\alpha = 0.01$ | $\alpha = 0.02$ | $\alpha = 0.01$ | $\alpha = 0.02$ |
| RS | 114890 | 119102 | 64.68 | 67.05 | 2.519 | 2.467 |
| RD (T=2) | 153525 | 156394 | 86.43 | 88.04 | 4.326 | 3.477 |
| RD (T=5) | 171676 | 172321 | 96.65 | 97.01 | 10.174 | 11.093 |
| RD (T=10) | 173830 | 174088 | 97.86 | 98.00 | 45.545 | 33.530 |
| RD (T=30) | 174469 | 174687 | 98.22 | 98.34 | 112.366 | 76.903 |

Table 5.4: Optimal objective value and CPU time of each model when $N$=20

Then supposing we could collect $N = 20$ data samples, we conduct the same experiments again except the number of historical data and the results are shown in Table 5.4, 5.5, 5.6. From Table 5.4, with the number of available data increasing, the optimal objective value of both DLP and SAA decrease while that of robust policies increases. Also, the optimal objective value obtained with more historical data shares the same monotonicity property in the number of time periods. Similar to the case with $10$ data samples, we conduct numerical experiments in "same distribution" and "different distribution" settings and the testing data are the same as those we used in experiments for $N = 10$. From Table 5.5 and 5.6, the average revenue of RS is still always higher than that of DLP while sometimes slightly less than that of SAA. Almost consistent with the results in the $N = 10$ setting, although RD policies generate less average revenue than that of DLP and SAA in "same distribution" setting, they generate higher average revenue in "different distribution" setting except in the case $T = 10, \alpha = 0.02$. Furthermore, in "different distribution" setting, the performance on the minimal revenue of RD policies is still significantly higher than that shown by DLP. More specifically, $6.50 \sim 13.90$ % when $\alpha = 0.01$ and $3.29 \sim 12.80$ % when $\alpha = 0.02$.

We also consider the case where the actual demand distribution even falls into a different class of distribution from that of the distribution of historical data. Specifically, we conduct more numerical experiments where the testing data for each product is drawn from the uniform distribution on the interval $[5, 30]$ rather than the normal distribution we used in the experiments above and the results are shown in Table 5.7 for $N = 10$ and Table 5.8 for $N = 20$ respectively. Note that we generate 10 different sample paths as the testing data. When $N = 10$, the performance of RS policy is better than that of both DLP and SAA policy in terms of the average, minimal and maximal revenue. In addition, the performance of SAA policy is even worse than that of DLP policy. However, when the available historical data increases to $20$, the advantage of RS policy over SAA policy becomes no longer clear and the performance of SAA policy surpasses that of DLP policy. Unlike the RS policy, RD policies perform quite well in both the

| $\alpha$ | Policy | Revenue (Mean(Std),[Min,Max]) | | Ratio to DLP (%) | |
|---|---|---|---|---|---|
| | DLP | 176904 (580) | [175425, 177477] | 100 (100) | [100, 100] |
| | SAA | 177442 (0) | [177442, 177442] | 100.30 (0) | [101.15, 99.98] |
| | | | | | |
| | RS | 177420 (0) | [177420, 177420] | 100.29 (0) | [101.14, 99.97] |
| | RD (T=2) | 173927 (77) | [173748, 173983] | 98.32 (13.28) | [99.04, 98.03] |
| 0.01 | RD (T=5) | 173635 (74) | [173540, 173770] | 98.15 (12.76) | [98.93, 97.91] |
| | RD (T=10) | 173726 (397) | [173119, 174343] | 98.20 (68.45) | [98.69, 98.23] |
| | RD (T=30) | 172399 (89) | [172193, 172480] | 97.45 (15.34) | [98.16, 97.18] |
| | | | | | |
| | RS | 177420 (0) | [177420, 177420] | 100.29 (0) | [101.14, 99.97] |
| | RD (T=2) | 173703 (117) | [173467, 173886] | 98.19 (20.17) | [98.88, 97.98] |
| 0.02 | RD (T=5) | 173960 (63) | [173866, 174054] | 98.34 (10.86) | [99.11, 98.07] |
| | RD (T=10) | 164615 (1388) | [162811, 167134] | 93.05 (239.31) | [92.81, 94.17] |
| | RD (T=30) | 171655 (50) | [171560, 171723] | 97.03 (8.62) | [97.80, 96.76] |

Table 5.5: Policy performance with testing data from the same distribution as historical data when $N$=20

| $\alpha$ | Policy | Revenue (Mean(Std),[Min,Max]) | | Ratio to DLP (%) | |
|---|---|---|---|---|---|
| | DLP | 166603 (8743) | [150012, 177635] | 100 (100) | [100, 100] |
| | SAA | 167810 (7943) | [152002, 177442] | 100.72 (90.85) | [101.33, 99.89] |
| | RS | 167574 (7611) | [152339, 177420] | 100.58 (87.05) | [101.55, 99.88] |
| | RD (T=2) | 173015 (1150) | [170860, 174215] | 103.85 (13.15) | [113.90, 98.07] |
| 0.01 | RD (T=5) | 171775 (2316) | [166739, 173592] | 103.10 (26.49) | [111.15, 97.72] |
| | RD (T=10) | 168155 (6010) | [159770, 174366] | 100.93 (68.74) | [106.50, 98.16] |
| | RD (T=30) | 169593 (3951) | [162924, 172911] | 101.79 (45.19) | [108.61, 97.34] |
| | RS | 168040 (7814) | [152339, 177420] | 100.86 (89.37) | [101.55, 99.88] |
| | RD (T=2) | 172527 (1597) | [169210, 174037] | 103.56 (18.27) | [112.80, 97.97] |
| 0.02 | RD (T=5) | 171413 (3208) | [164052, 173878] | 102.89 (36.69) | [109.36, 97.88] |
| | RD (T=10) | 164953 (6201) | [154946, 172772] | 99.01 (70.93) | [103.29, 97.26] |
| | RD (T=30) | 170538 (2460) | [165170, 172304] | 102.36 (28.14) | [110.10, 97.00] |

Table 5.6: Policy performance with testing data from different distribution from historical data when $N$=20

$N = 10$ and $20$ cases. Both the average and minimal revenue produced by the RD policies are significantly higher than those generated by DLP, SAA and RS policy. More specifically, with respect to the average revenue, $5.13 \sim 6.79$ % ($N = 10, \alpha = 0.01$), $4.64 \sim 6.75$ % ($N = 10, \alpha = 0.02$),$5.37 \sim 5.90$ % ($N = 20, \alpha = 0.01$) and $4.71 \sim 5.90$ % ($N = 20, \alpha = 0.02$); with respect to the minimal revenue, $6.59 \sim 8.35$ % ($N = 10, \alpha = 0.01$), $4.03 \sim 8.26$ % ($N = 10, \alpha = 0.02$),$5.68 \sim 7.33$ % ($N = 20, \alpha = 0.01$) and $4.66 \sim 6.80$ % ($N = 20, \alpha = 0.02$). More notably, in contrast to the results of the numerical experiments using truncated normal testing data, the RD policies take an advantage over DLP, SAA and RS policy not only in the average and minimal revenue but also in the maximal revenue. Specifically, $3.55 \sim 5.15$ % ($N = 10, \alpha = 0.01$), $3.81 \sim 5.15$ % ($N = 10, \alpha = 0.02$),$3.81 \sim 4.69$ % ($N = 20, \alpha = 0.01$) and $3.50 \sim 4.50$ % ($N = 20, \alpha = 0.02$).

Comparing all the tables, we summarize the observations mentioned above and provide some other findings:

(1) With the number of historical data increasing, the optimal object value of model RS and RD become closer to that of DLP, which is the upper bound. Despite the fact that the optimal objective value of our robust models is less than that of DLP and SAA, the policy performance does not follow this. Since when we solve our robust model, we are optimizing the pre-allocation quota decisions supposing we are facing the worst-case distribution from the ambiguity set. This particular distribution occurs with a very small probability in our testing experiments and also the practical applications. Therefore, the revenue generated from our robust policy is believed to be higher than the corresponding optimal objective value of the optimization problem from which we determine the pre-allocation quota

| $\alpha$ | Policy | Revenue (Mean(Std),[Min,Max]) | | Ratio to DLP (%) | |
|---|---|---|---|---|---|
| | DLP | 163196 (1803) | [160621, 165958] | 100 (100) | [100, 100] |
| | SAA | 160802 (2227) | [157594, 165078] | 98.53 (123.52) | [98.12, 99.47] |
| | RS | 164480 (1803) | [161904, 167241] | 100.79 (100) | [100.80, 100.77] |
| | RD (T=2) | 174269 (164) | [174030, 174497] | 106.79 (9.10) | [108.35, 105.15] |
| 0.01 | RD (T=5) | 171574 (192) | [171202, 171856] | 105.13 (10.65) | [106.59, 103.55] |
| | RD (T=10) | 173677 (148) | [173372, 173901] | 106.42 (8.21) | [107.94, 104.79] |
| | RD (T=30) | 172055 (131) | [171880, 172261] | 105.43 (7.27) | [107.01, 103.80] |
| | RS | 164480 (1803) | [161904, 167241] | 100.79 (100) | [100.80, 100.77] |
| | RD (T=2) | 174211 (216) | [173889, 174509] | 106.75 (11.98) | [108.26, 105.15] |
| 0.02 | RD (T=5) | 171851 (239) | [171477, 172284] | 105.30 (13.26) | [106.76, 103.81] |
| | RD (T=10) | 173682 (154) | [173372, 173927] | 106.43 (8.54) | [107.94, 104.80] |
| | RD (T=30) | 170774 (1747) | [167099, 172704] | 104.64 (96.89) | [104.03, 104.06] |

Table 5.7: Policy performance with testing data from an uniform distribution when $N$=10

| $\alpha$ | Policy | Revenue (Mean(Std),[Min,Max]) | | Ratio to DLP (%) | |
|---|---|---|---|---|---|
| | DLP | 164061 (1803) | [161486, 166823] | 100 (100) | [100, 100] |
| | SAA | 166051 (1803) | [163476, 168813] | 101.21 (100) | [101.23, 101.19] |
| | | | | | |
| | RS | 165661 (2090) | [162656, 169150] | 100.98 (115.92) | [100.72, 101.39] |
| | RD (T=2) | 173025 (616) | [172125, 174056] | 105.46 (34.17) | [106.59, 104.34] |
| 0.01 | RD (T=5) | 173739 (164) | [173326, 173937] | 105.90 (9.10) | [107.33, 104.26] |
| | RD (T=10) | 173680 (1390) | [170666, 174652] | 105.86 (77.09) | [105.68, 104.69] |
| | RD (T=30) | 172865 (188) | [172523, 173171] | 105.37 (10.43) | [106.83, 103.81] |
| | | | | | |
| | RS | 166389 (1803) | [163814, 169150] | 101.42 (100) | [101.44, 101.39] |
| | RD (T=2) | 173034 (948) | [171195, 174335] | 105.47 (52.58) | [106.01, 104.50] |
| 0.02 | RD (T=5) | 173739 (511) | [172462, 174194] | 105.90 (28.34) | [106.80, 104.42] |
| | RD (T=10) | 171795 (1214) | [169009, 173270] | 104.71 (67.33) | [104.66, 103.86] |
| | RD (T=30) | 172271 (223) | [171895, 172658] | 105 (12.37) | [106.45, 103.50] |

Table 5.8: Policy performance with testing data from an uniform distribution when $N$=20

decisions if the ambiguity set is appropriately constructed. On the other hand, the DLP or SAA policy does not have this guarantee and the actual revenue produced by the DLP or SAA model in testing experiments or practice may be less than the corresponding optimal objective value.

(2) No matter whether the demand distribution during the selling horizon is the same as the distribution from which our historical data are drawn, in the setting of limited data, RS policy performs slightly better than DLP and SAA from perspectives of average, minimal and maximal revenue. However, with the increasing amount of data, the advantages of RS over SAA become weaker or even comparable.

(3) Although RD policies generate less average revenue than DLP, SAA and RS in "same distribution" cases and less maximal revenue in both "same distribution" and "different distribution" cases, they generate higher average revenue than DLP, SAA and RS policy and significantly higher minimal revenue in "different distribution" cases. In addition, if the actual distribution falls into a different class of distribution from that of the distribution of historical data, RD policies perform better than DLP, SAA and RS policy in terms of the average, minimal as well as maximal revenue.

(4) The optimal objective value of model RD is increasing with the number of time periods while the average revenue in policy performance testing does not follow this pattern. Especially, considering RD policies for $T = 2, 5, 10$, the average revenue is decreasing with the number of time periods in "different distribution" cases. One potential cause is the effect of the bootstrapping method we used to generate multi-unit demand data based on given one unit demand data.

The numerical results also provide some implications for applying our method in practice. When the demand distribution during the coming selling season is believed to be different from that of the historical data (this kind of case can happen when the business environment changes), RD policy is more recommended because of its good performance under distributional ambiguity. On the other hand, when the environment is stationary and the demand distribution is believed to be the same as that of the historical data, the RS policy is a more suitable method when we have only a limited amount of historical observations since it performs well in such a circumstance. In addition, the SAA policy is also a good choice in stationary environment if enough amount of historical data is available.

# Chapter 6

# Conclusions

In this research, we incorporate demand uncertainty into the canonical quantity-based NRM problem. Assuming we have no information about the demand distribution but limited historical data (i.i.d.), we develop a distributionally robust PNLP model where the ambiguity set is constructed based on a statistical GoF test with the given data. We show that this data-driven robust PNLP can be formulated as a conic optimization problem, and it is tractable and enjoys the finite-sample and asymptotic performance guarantees according to theorems from (Bertsimas, Gupta, and Kallus 2018b). Specially, taking time dynamics into account, we extend our robust static model to dynamic version. More specifically, we first develop an approximate formulation, robust ADP, for DP of NRM problem that uniquely lies at the intersection of DRO and ADP. To facilitate the computational tractability, we further provide an equivalent reformulation of our robust ADP, which approximates the evolution of the selling system under demand uncertainty. Both our robust static and dynamic models are solved to determine an optimal partitioned allocation of capacity to each product against a worst-case distribution in the ambiguity set. Based on an optimal solution of our robust static or dynamic model, we construct a robust static or dynamic booking limit policy to help the firm make capacity allocation decisions.

Numerically, we conduct empirical experiments to validate the performance of our robust approach and the main conclusions are: (1) With the number of historical data increasing, the optimal object value of model RS and RD become closer to that of DLP, which is the upper bound. Despite the fact that the optimal objective value of our robust models is less than that of DLP and SAA, the policy performance does not follow this; (2) No matter whether the demand distribution during the selling horizon is the same as the distribution from which our historical data are drawn, in the setting of limited data, RS policy performs slightly better than DLP and SAA from perspectives of average, minimal and maximal revenue. However, with the increasing amount of data, the advantages of RS over SAA become weaker or even comparable; (3) Although RD policies generate less average revenue than DLP, SAA and RS in the testing instances using data drawn from the same distribution as the historical data and less maximal revenue in both "same distribution" and "different distribution" testing cases, they generate higher average revenue than DLP, SAA and RS policy and significantly higher minimal revenue in "different distribution" cases. In addition, if the actual distribution falls into a different class of distribution from that of the distribution of historical data, RD policies perform better than DLP, SAA and RS policy in terms of the average, minimal as well as maximal revenue; (4) The optimal objective value of model RD is increasing with the number of time periods while the average revenue in policy performance testing does not follow this pattern. Especially, considering RD policies for $T = 2, 5, 10$, the average revenue is decreasing with the number of time periods in “different distribution” cases. One potential cause is the effect of the bootstrapping method we used to generate multi-unit demand data based on given one unit demand data.

We finally propose some potential extensions to this work. First, we could conduct more research on the robust dynamic policies both theoretically and empirically in order to explore some interesting conclusions on the relationships between the number of time periods and the corresponding policy performance. In addition, as mentioned above, we may have to take the effect of the bootstrapping method we used to generate the data we need into consideration. Secondly, it would be interesting to consider more types of GoF test, especially those recast from some commonly used statistical distance and then compare the performance of the control policies obtained from them with those of the control policies proposed in our thesis. Instead of the marginal test approach adopted in our thesis, it would be worthwhile to explore other formulations in the framework of our approach when we consider joint distribution setting, where we could capture more information about the demand on all products. Thirdly, to include updated information, i.e., the selling time periods left and the real remaining capacities we have at the beginning of each time period, one commonly used approach in both theoretical and practical research in RM area is re-solving. Taking system dynamics into consideration, the new solution to the updated model is then used to adjust control policies. Therefore, we could re-solve our robust static and dynamic model over time in experiment implementations to seek for a set of more accurate and effective booking limits. Fourthly, it is possible to extend our approach to online version, where we assume we have no access to the historical data or the historical data is not suitable for us to estimate the demand distribution during the coming selling horizon and we could only collect the realizations coming "on the fly" after the selling horizon begins. Combing with the third potential extension above, we could also simultaneously update the system information and ambiguity set, which is constructed based on all the past realizations we collected till the current decision-making time period, when we re-solve the model.

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# Appendix

## Appendix A: Proof of Theorem 1

*Proof.* To simplify notation, we omit the index $j$ for each product in this proof. The optimization problem (3.12) can be rewritten as follows:

$$
\begin{aligned}
\min_{\zeta} \quad & \mathbb{E}_{\zeta}[r \min\{D, y\}] && && \text{(1a)}\\
\text{s.t.} \quad & B_{S_N}\zeta - b_{S_N,\alpha} \in K_{S_N}, && && \text{(1b)}\\
& \zeta_i - \zeta_{i-1} \geq 0, && \forall i \in [N+1]. && \text{(1c)}
\end{aligned}
$$

Recall that $\zeta_i = F_0(D^{(i)})$. We can further rewrite problem (1) as

$$
\begin{aligned}
\min_{\zeta} \quad & \sum_{i=1}^{N+1} \left( \inf_{D \in (D^{(i-1)}, D^{(i)}]} r \min\{D, y\} \right) (\zeta_i - \zeta_{i-1}) && && \text{(2a)}\\
\text{s.t.} \quad & B_{S_N}\zeta - b_{S_N,\alpha} \in K_{S_N}, && && \text{(2b)}\\
& \zeta_i - \zeta_{i-1} \geq 0, && \forall i \in [N+1]. && \text{(2c)}
\end{aligned}
$$

According to strong duality, (2) is equal to

$$
\begin{aligned}
&\max_{\lambda\in K^*_{S_N},\boldsymbol{\mu}\geq\mathbf{0}} \quad \min_{\zeta} \quad \sum_{i=1}^{N+1}\left(\inf_{D\in(D^{(i-1)},D^{(i)}]} r\min\{D,y\}\right)(\zeta_i-\zeta_{i-1}) \\
&\qquad\qquad + \lambda^{\mathrm{T}}(b_{S_N,\alpha}-B_{S_N}\zeta)+\sum_{i=1}^{N+1}\mu_i(\zeta_{i-1}-\zeta_i) \\
=&\max_{\lambda\in K^*_{S_N},\boldsymbol{\mu}\geq\mathbf{0}} \quad \min_{\zeta} \quad \sum_{i=1}^{N+1}\left\{\left(\inf_{D\in(D^{(i-1)},D^{(i)}]} r\min\{D,y\}\right)-\mu_i\right\}(\zeta_i-\zeta_{i-1}) \\
&\qquad\qquad + \lambda^{\mathrm{T}}(b_{S_N,\alpha}-B_{S_N}\zeta) \\
=&\max_{\lambda\in K^*_{S_N},\boldsymbol{\mu}\geq\mathbf{0}} \quad \min_{\zeta} \quad \sum_{i=1}^{N+1} c_i(\zeta_i-\zeta_{i-1})+\lambda^{\mathrm{T}}(b_{S_N,\alpha}-B_{S_N}\zeta) \\
&\qquad \text{s.t.} \quad c_i=\left(\inf_{D\in(D^{(i-1)},D^{(i)}]} r\min\{D,y\}\right)-\mu_i, \qquad \forall i\in[N+1], \\
=&\max_{\lambda\in K^*_{S_N},c} \quad \min_{\zeta} \quad \sum_{i=1}^{N+1} c_i(\zeta_i-\zeta_{i-1})+\lambda^{\mathrm{T}}(b_{S_N,\alpha}-B_{S_N}\zeta) \\
&\qquad \text{s.t.} \quad c_i\leq \inf_{D\in(D^{(i-1)},D^{(i)}]} r\min\{D,y\}, \qquad \forall i\in[N+1], \\
=&\max_{\lambda\in K^*_{S_N},c} \quad b^{\mathrm{T}}_{S_N,\alpha}\lambda+\min_{\zeta} \quad \sum_{i=1}^{N+1} c_i(\zeta_i-\zeta_{i-1})-\sum_{i=1}^{N}(B^{\mathrm{T}}_{S_N}\lambda)_i\zeta_i \\
&\qquad \text{s.t.} \quad c_i\leq \inf_{D\in(D^{(i-1)},D^{(i)}]} r\min\{D,y\}, \qquad \forall i\in[N+1].
\end{aligned}
$$

Note that $\sum_{i=1}^{N+1} c_i(\zeta_i-\zeta_{i-1}) = c_{N+1}\zeta_{N+1}+\sum_{i=1}^{N}(c_i-c_{i+1})\zeta_i-c_1\zeta_0$ with $\zeta_{N+1}=1$

and $\zeta_0 = 0$. Hence, the dual of (2) can be further reduced to

$$
\begin{aligned}
\max_{\lambda \in K^*_{S_N}, c} \quad & b^{\mathrm{T}}_{S_N,\alpha}\lambda + c_{N+1} + \min_{\zeta} \sum_{i=1}^{N} \left[(c_i - c_{i+1}) - (B^{\mathrm{T}}_{S_N}\lambda)_i\right] \zeta_i \\
\text{s.t.} \quad & c_i \le \inf_{D \in (D^{(i-1)}, D^{(i)}]} r \min\{D, y\}, \qquad \forall i \in [N+1], \\
= \max_{\lambda \in K^*_{S_N}, c} \quad & b^{\mathrm{T}}_{S_N,\alpha}\lambda + c_{N+1} \\
\text{s.t.} \quad & (B^{\mathrm{T}}_{S_N}\lambda)_i = c_i - c_{i+1}, \qquad \forall i \in [N], \\
& c_i \le \inf_{D \in (D^{(i-1)}, D^{(i)}]} r \min\{D, y\}, \qquad \forall i \in [N+1].
\end{aligned}
$$

As $r \geq 0$, the constraint $c_i \leq \inf_{D\in(D^{(i-1)},D^{(i)}]} r \min\{D, y\}$ is equivalent to $c_i \leq rD^{(i-1)}$ and $c_i \leq ry$. This yields the equivalence between (1) and

$$
\begin{aligned}
\max_{\lambda, c} \quad & b^{\mathrm{T}}_{S_N,\alpha}\lambda + c_{N+1} \\
\text{s.t.} \quad & \lambda \in K^*_{S_N}, c \in \mathbb{R}^{N+1}, \\
& (B^{\mathrm{T}}_{S_N}\lambda)_i = c_i - c_{i+1}, \qquad \forall i \in [N], \\
& c_i \le rD^{(i-1)}, \ c_i \le ry, \qquad \forall i \in [N+1].
\end{aligned}
$$

□

# Appendix B: Constraint Generation Algorithm

Assuming that we can add a maximum of $v$ constraints in each iteration of the constraint generation algorithm, the procedure is presented in Algorithm 1.

**Algorithm 1:** Constraint Generation for Model (4.11)

**Input:** $\boldsymbol{r}, \boldsymbol{A}, \boldsymbol{C}_1, \theta_{T+1}, \boldsymbol{P}_{T+1}, (\boldsymbol{D}^{(i)})_{\forall i \in [N+1]}, (K^*_{S_{N_j}}, B_{S_{N_j}}, (b_{S_{N_j}, \alpha_{t,j}})_{\forall t})_{\forall j}, v$;

**Output:** $(\theta^*_t, \boldsymbol{P}^*_t, \boldsymbol{z}^*_t)_{\forall t}$

1 Set $\mathscr{C} \leftarrow \{(\theta_t, \boldsymbol{P}_t, \boldsymbol{z}_t)_{\forall t} : (4.11c)\}$.

2 Solve the master problem $\min_{(\theta_t, \boldsymbol{P}_t, \boldsymbol{z}_t)_{\forall t} \in \mathscr{C}} \{\theta_1 + \sum_{l=1}^m P_{1,l} C_{1,l}\}$. Let $(\tilde{\theta}_t, \tilde{\boldsymbol{P}}_t, \tilde{\boldsymbol{z}}_t)_{\forall t}$ denote the corresponding optimal solution.

3 Set $k \leftarrow 0, s \leftarrow 0$.

4 **for** $t \leftarrow 1$ *to* $T$ **do**

5 Solve the separation problem (4.15). Let $(\tilde{\boldsymbol{C}}_t, \tilde{\boldsymbol{y}}_t, \tilde{\lambda}_t, \tilde{c}_t)$ and $\rho_t$ denote the corresponding optimal solution and optimal value, respectively.

6 **if** $\rho_t \geq 0$ **then**

7 Set $k \leftarrow k + 1$.

8 **else**

9 Update $\mathscr{C} \leftarrow \mathscr{C} \cap$
$$\left\{\theta_t - \theta_{t+1} + \sum_{l=1}^{m} (P_{t,l} - P_{t+1,l}) \tilde{C}_{t,l} \geq \sum_{j=1}^{n} z_{t,j} \left( b^{\mathrm{T}}_{S_{N_j}, \alpha_{t,j}} \tilde{\lambda}_{t,j} + \tilde{c}_{t,j,N+1} \right) \right\}.$$

10 Set $s \leftarrow s + 1$.

11 **if** $s = v$ **then**

12 Go to Step 16.

13 **end**

14 **end**

15 **end**

16 **if** $k = T$ **then**

17 Return $(\theta^*_t, \boldsymbol{P}^*_t, \boldsymbol{z}^*_t)_{\forall t} \leftarrow (\tilde{\theta}_t, \tilde{\boldsymbol{P}}_t, \tilde{\boldsymbol{z}}_t)_{\forall t}$.

18 **else**

19 Go to Step 2.

20 **end**