A POLYHEDRAL STUDY ON UNIT COMMITMENT WITH A SINGLE TYPE OF BINARY VARIABLES

BIN TIAN

MPhil

The Hong Kong Polytechnic University

2023

The Hong Kong Polytechnic University

Department of Logistics and Maritime Studies

A Polyhedral Study on Unit Commitment with A Single Type of Binary Variables

Bin Tian

A thesis submitted in partial fulfilment of the requirements for the degree of Master of Philosophy

May 2023

CERTIFICATE OF ORIGINALITY

I hereby declare that this thesis is my own work and that, to the best of my knowledge and belief, it reproduces no material previously published or written, nor material that has been accepted for the award of any other degree or diploma, except where due acknowledgement has been made in the text.

__________________________ (Signed)

Bin Tian (Name of student)

Abstract

Efficient power production scheduling is a crucial concern for power system operators aiming to minimize the operational costs. Previous studies have primarily focused on Mixed-integer Linear Programming (MILP) formulations that utilize two or three types of binary variables for Unit Commitment (UC) problems. The investigation of strong formulations with a single type of binary variables has been limited, as it is believed to be challenging to derive strong valid inequalities for them (James Ostrowski, Anjos, and Vannelli 2012) and the improvement of compactness is often accompanied by a compromise in tightness (Ben Knueven, Jim Ostrowski, and J. Wang 2018). To address these difficulties, we consider a compact formulation for the UC problem using a single type of binary variables, which reduces the size of the search tree for the branch-and-cut algorithm. To enhance the tightness of this compact formulation, two-period UC polytope and multi-period strong valid inequalities involving single and two continuous variables are developed. Conditions under which these strong valid inequalities serve as facet-defining inequalities for the multi-period UC polytope are provided. As the number of these inequalities could be large, polynomial separation algorithms are proposed to find most violated inequalities. The efficacy of

the derived strong valid inequalities in tightening the compact formulation is demonstrated through computational experiments on network-constrained UC problems. The result indicates that our strong valid inequalities can speed up the solution process of the compact formulation significantly. Particularly, our strong valid inequalities can also be used to tighten two/three-binary UC formulations.

Acknowledgements

I would like to express my sincere gratitude to my thesis advisor Dr. Kai Pan, whose guidance, support, and encouragement throughout my research journey made this thesis possible. His expertise in the field and his unwavering commitment to my success are invaluable and greatly appreciated.

I would also like to thank Prof. Chung-Lun Li for his insightful comments, helpful suggestions, and rigorous academic standards, which challenged me to refine my ideas and produce high-quality work.

Finally, I would like to acknowledge my friends and family who helped and encouraged me to finish this research.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 The UC Problem

With the increased prevalence of extreme weather events such as drought, wildfires, and flooding around the world in recent years, e.g., the past July ranks as the third hottest July on record in the United States, power consumption in many areas has hit an all-time high last summer. Additionally, continued retirements of coal-fired generating plants, relatively high coal prices, and lower-than-average coal stocks at power plants have limited coal consumption (US EIA 2022). Consequently, the efficient production and distribution of electricity have been identified as the biggest concern for power producers around the globe. The Unit Commitment (UC) problem, which involves determining the scheduling of power generators, has been a challenging optimization problem in the power industry for many years. Due to its computational complexity and practical needs, e.g., the UC problem is repeatedly solved multiple times per day (Xavier, Qiu, and Ahmed 2021), it has received a great deal of attention over the past decades. It concerns scheduling a group of generators subject to their physical and system constraints over a finite time horizon to satisfy the power demand with possibly minimal operational costs. The physical constraints indicate the technical properties of generators, and the most common items are ramp-up/-down constraints, generation lower/upper bound constraints, and minimum-up/-down constraints. It may vary depending on the type of generation unit, such as hydro, thermal, and wind unit, etc. (Ackooij et al. 2018). System constraints typically include the load requirement, the spinning reserve constraint, and the transmission flow limit. All units are coupled by the system constraints to ensure the reliability of the whole system in case of certain contingencies, like outage of machines and overload of transmission lines. The operational costs comprise various factors, including the generation (fuel) cost, start-up, and shut-down costs of generators, with the fuel cost being a significant contributor (Padhy 2004), which is generally assumed to be an increasing convex function of the generation amount (Takriti and Birge 2000). The start-up and shut-down costs are incurred every time when the status of a generator changes, and the latter may be omitted sometimes (Subir Sen and Kothari 1998).

Efficiently solving the UC problem has a great impact on both society and individual consumers. Even small improvements in the quality of solutions for the UC problem can affect the price of electricity over large regions and lead to millions of dollars of savings per day (Damcı-Kurt et al. 2016). For example, a 1% savings in energy markets could result in a 10 million dollar savings annually (Ben Knueven, Jim Ostrowski, and J. Wang 2018).

However, the UC problem still can not be deemed to be well-solved. In general, it is a large-scale, non-convex, Mixed Integer Programming (MIP) problem, which is proven to be NP-hard and difficult to solve when the size of the problem grows large (Bernard Knueven, James Ostrowski, and Watson 2020b; Tejada-Arango et al. 2020; Zheng, J. Wang, and Liu 2015). In a day-ahead deregulated electricity market, the Independent System Operator (ISO) is expected to determine the generation schedule for a power system with hundreds of thermal units and thousands of transmission lines over 24–48 operating hours within an "unreasonable" short time. Specifically, it is desirable for an ISO to obtain a UC solution within 10–15 minutes in practice considering the time spent on verifying the input data, performing postsolve feasibility checks, and pricing run, etc. (Bernard Knueven, James Ostrowski, and Watson 2020b). Therefore, numerous approaches have been devised from both formulation and algorithm perspectives to efficiently solve UC problems over the past decades.

1.2 Solution Approaches to UC

Because of the complexity of UC problems and the practical need for efficiently solving them, various solution methods are developed over the past two decades. For example, these heuristic methods, like priority list, meta-heuristic methods like genetic algorithms, and simulating annealing. Exact methods, such as dynamic programming and the Lagrangian relaxation method are applied to UC problems subsequently. With the advancement of mixed-integer linear programming (MILP) solvers, MILP-based approaches are widely proposed to solve UC problems in an effective way.

Priority list is one of the earliest and simplest methods that was adopted to solve a UC problem (Subir Sen and Kothari 1998). It lists all units by their operational costs and then economically dispatches the system load to units by the pre-determined order. Thus, it can obtain a feasible solution in a very short time period. However, due to its dispatching policy, it tends to result in suboptimal solutions. Heuristic algorithms, such as genetic algorithms and simulating annealing, were frequently used as solution techniques for UC problems in the early times. The heuristic solution technique can easily model both time-dependent and system constraints (Kazarlis, Bakirtzis, and Petridis 1996), and it can also be converted to work on parallel computers. Like the Priority List method, it can lead to suboptimal solutions and the optimality can not be guaranteed (Cohen and Yoshimura 1983). In particular, the execution time leaps dramatically with the number of units to be committed (Subir Sen and Kothari 1998).

Dynamic Programming (DP) based approaches are also widely applied to get UC solutions (Snyder, Powell, and Rayburn 1987; Ouyang and S. M. Shahidehpour 1991; C. Wang and S. M. Shahidehpour 1993). The UC problem is decomposed by time and the state for each time period is represented by the combinations of units. The size of the problem, e.g., the number of states, increases dramatically as the number of units and operation time periods grows, which is known as the ”curse of dimensionality” (Baldick 1995). To address this issue, the DP methods are commonly integrated with the Priority List method or heuristic methods to reduce the dimension and search space. Nevertheless, this could lead to suboptimal solutions (Subir Sen and

Kothari 1998). Compared to the initial studies, DP is recently exploited to solve the subproblems obtained by decomposing the original UC problem or single-unit UC problems. Frangioni and Gentile 2006b proposed a dynamic programming algorithm for solving a single unit UC problem by considering the ramp-up/-down constraints and convex cost functions. They showed that this problem with a quadratic cost function can be solved in $O(n^3)$ time where n is the operating period of the unit. Other studies can be referred to S. J. Wang et al. 1995; Frangioni and Gentile 2006b; Guan, Pan, and Zhou 2018, and Qu et al. 2022, etc.

Considering the large size and nonconvex properties of a general UC problem, Lagrangian Relaxation (LR) based approaches are proposed as an effective way to deal with it. To reduce the scale and complexity of the original problem, the system constraints, such as load requirement, are relaxed through Lagrange multipliers to the objective function. The resulting problem is then decomposed into subproblems either by units or time periods, and these subproblems are solved iteratively by obtaining Lagrange multipliers from the dual problem and then updating them in the primal problem until the primal-dual gap is hard to shrink (Subir Sen and Kothari 1998). Some complicated system constraints that are non-separable are either left in the subproblems or unmodeled (Bendotti, Fouilhoux, and Rottner 2019). It is widely used to solve the large-scale UC problem with complicated constraints within a relatively shorter time. Muckstadt and Koenig 1977 presented a MIP model for the UC problem. The load and spinning reserve constraints were incorporated into the objective function by Lagrange multipliers, and it is then decomposed by units. The obtained subproblem for each unit was solved by the branch-and-bound algorithm and the Lagrange multipliers were updated by the subgradient method in each iteration. Baldick 1995 formulated a generalized MIP model for the UC problem by integrating the power flow, reserve, and fuel constraints. It was then solved based on the LR approach, where the algorithm iterates from dual feasible solutions toward primal feasible solutions. Takriti and Birge 2000 presented a technique to refine the solution for the LR method based on the observation that the resulting schedules for each unit are hard to change after a few iterations. Lu and M. Shahidehpour 2005 considered a UC problem with flexible generation units, which was decomposed into a master problem and a subproblem with coupling constraints by benders decomposition. The master problem was further decomposed into a set of dynamic programming subproblems for each unit based on the LR method. The dynamic programming method was used to solve the subproblem for each unit.

However, due to the nonconvexity of the UC problem, heuristic procedures are needed to obtain feasible solutions, which may be suboptimal (Carrión and Arroyo 2006). It also suffers from slow and unsteady convergence, which could be explained by its nonconvexity, and the primal feasibility is hard to guarantee (Kazarlis, Bakirtzis, and Petridis 1996; Ma and S. M. Shahidehpour 1999; Guan, Pan, and Zhou 2018). To overcome these problems, quadratic terms were added to the objective function to penalize the violation of load constraints and improve its convexity (Subir Sen and Kothari 1998), which refers to the Augmented Lagrangian Relaxation (ALR) method. However, the objective function is no longer separable for each unit and it can lead to local optimal because of the nonconvexity (Ackooij et al.

2018). Therefore, many approaches were utilized to reduce the complexity of it.

Batut and Renaud 1992 developed an ALR-based approach for the UC problem by considering the transmission constraints. The decision variables were duplicated first and the problem was then decomposed into two subproblems, which contain the physical and system constraints, respectively. The Auxiliary Problem Principle was used to linearize the non-separable terms in the objective function. S. J. Wang et al. 1995 discussed a short-term UC problem with transmission and environmental constraints. The nonseparable terms in the objective function were linearized around the solution that was obtained. Quadratic terms were then added to the objective function to improve the steadiness of convergence. The resulting problem was finally decomposed by units and solved by dynamic programming. Ma and S. M. Shahidehpour 1999 solved a MIP model for a UC problem with transmission and voltage constraints by Benders decomposition. It was decomposed into a master problem and two subproblems with transmission and voltage constraints, respectively, which were solved iteratively. The quadratic penalty term associated with power demand is linearized around the previous solution for each iteration to improve its decomposition ability, and the quadratic terms of decision variables were imposed to limit the solution deviation. Takriti and Birge 2000 proposed an Integer Programming model to refine the solution obtained by the LR method.

The extensive advancement of MILP solvers has led to the widespread application of MILP-based approaches in formulating and solving UC problems. Unlike other methods, MILP-based approaches can guarantee convergence to the optimal solution while providing a flexible and accurate modeling framework. Moreover, the optimality gap is easy to obtain. The nonlinear constraints and generation cost functions in UC models can be approximated by linear ones. ISOs are therefore increasingly adopting MILP-based approaches over LR-based approaches to solve large-scale UC problems (K. W. Hedman, O'Neill, and Oren 2009; Wu 2011; James Ostrowski, Anjos, and Vannelli 2012; C. Li, Zhang, and K. Hedman 2021).

Two primary factors are generally considered in evaluations of MILP formulations for UC problems: compactness and tightness. Compactness refers to the size of the problem, which can be quantified by the number of constraints and decision variables. Tightness refers to the proximity of the linear programming (LP) relaxation of the problem to the convex hull of its feasible region. For UC problems, compactness can be achieved by reducing the number of binary variables in an MILP formulation, as fewer binary variables will lead to a reduction in the number of nodes of the search tree for the branch-and-cut method. In terms of compactness, UC formulations can be categorized into three categories, depending on the number of types of binary variables used in the formulation. A single-binary formulation uses a single set of binary variables to denote the on/off status of all units. A two-binary formulation uses an additional set of binary variables to represent the start-up decisions. A three-binary formulation goes one step further by using another set of binary variables for the shut-down decisions. Tightness can be achieved by deriving strong valid inequalities for the MILP formulation to tighten its LP relaxation. A tight formulation can reduce the gap between the objective values of the MILP problem and its LP relaxation by reducing the search space. Most strong valid inequalities are obtained by studying the physical constraints of a single generator (see, e.g., Lee, Leung, and Margot 2004, Rajan and Takriti 2005, Morales-España, Latorre, and Ramos 2013, Damcı-Kurt et al. 2016, Pan and Guan 2016, and Bendotti, Fouilhoux, and Rottner 2018).

Three-binary formulations for UC problems are the most widely studied. Garver 1962 is the first to propose an MILP formulation for a UC problem. In this three-binary formulation, the generation cost function is assumed to be linear with respect to the generation amount. Arroyo and Conejo 2000 introduce a three-binary formulation for a self-scheduling UC problem. They approximate the exponential start-up cost function and the nonconvex generation cost function using stairwise and piecewise functions, respectively. Chang, Aganagic, et al. 2001 present a three-binary formulation for a short-term hydro scheduling UC problem. Chang, Tsai, et al. 2004 put forward a new three-binary formulation for UC problems with thermal generators. They approximate the cubic generation cost function using a piecewise linear one with three breakpoints. T. Li and M. Shahidehpour 2005 compare the LR-based approach with the MILP-based approach in solving a price-based UC problem with various types of generators based on the formulation of Chang, Tsai, et al. 2004. Their numerical results indicate that the MILP-based approach exhibits superior performance on small-scale problems compared with the LR-based approach, and the MILP formulation must be tightened to improve its performance on large-scale problems.

James Ostrowski, Anjos, and Vannelli 2012 consider the formulation of Arroyo and Conejo 2000 and replace their minimum-up/-down constraints with those of Rajan and Takriti 2005 because the latter can reduce the computational time significantly. By studying the physical constraints of a single generator, they derive a class of strong valid inequalities to tighten the MILP formulation. Morales-España, Latorre, and Ramos 2013 propose an alternative three-binary formulation based on the formulation of James Ostrowski, Anjos, and Vannelli 2012. The generation cost function is represented as a linear function with respect to the generation amount. They introduce generation limit constraints to substitute for those in James Ostrowski, Anjos, and Vannelli 2012. They show that the resulting formulation is more compact and tighter than that of James Ostrowski, Anjos, and Vannelli 2012. Morales-España, Gentile, and Ramos 2015 establish a three-binary formulation based on that of Morales-España, Latorre, and Ramos 2013 by considering different start-up/shut-down trajectories, which are ignored in conventional research.

Damcı-Kurt et al. 2016 conduct a polyhedral study of the physical constraints based on the work of James Ostrowski, Anjos, and Vannelli 2012. They derive a convex hull for the two-period case and strong valid inequalities for the multi-period case to tighten the MILP formulation. Because the number of these strong valid inequalities can be exponential, polynomial separation algorithms are provided to apply them in the solution process. Computational results demonstrate that this formulation outperforms the strong formulation of James Ostrowski, Anjos, and Vannelli 2012. Atakan, Lulli, and Suvrajeet Sen 2018 develop a state-transition formulation for UC problems based on the formulations of James Ostrowski, Anjos, and Vannelli 2012 and Morales-España, Latorre, and Ramos 2013. Transmission constraints are not considered in their formulation. Their test results demonstrate that the proposed formulation has a shorter computational time for long-horizon problems than the formulations of James Ostrowski, Anjos, and Vannelli 2012 and Morales-España, Latorre, and Ramos 2013.

Two-binary formulations have also received considerable attention. Rajan and Takriti 2005 study the minimum-up/-down polytope, with only minimum-up/-down constraints, of a single generator using two types of binary variables. They provide a complete description of the convex hull of the polytope. Pan, Guan, et al. 2016 derive several families of strong valid inequalities for UC problems with gas turbine generators. Their two-binary formulation is based on that of Rajan and Takriti 2005. Their strong valid inequalities are facet-defining for the polytope of physical constraints under specific conditions. Pan and Guan 2016 conduct a polyhedral study of physical constraints based on the two-binary formulation of Pan, Guan, et al. 2016. They derive the complete convex hull descriptions for the two- and three-period polytopes under different parameter settings. They also develop strong valid inequalities for the multi-period case and provide polynomial-time separation algorithms for exponentially large valid inequality families.

Bendotti, Fouilhoux, and Rottner 2018 analyze the minimum-up/-down polytope of multiple generators based on the two-binary formulation of Pan and Guan 2016; their generation cost function is linear in the generation amount. They obtain up-set and interval up-set valid inequalities to accelerate the branch-and-cut algorithm. However, given a fractional solution, the problems of separating these two types of inequalities are NP-complete and NP-hard, respectively. Pan, Zhao, et al. 2022 perform a polyhedral study of a single generator by incorporating fuel constraints. They prove that the self-scheduling UC problem with a fuel constraint is NP-hard, and they derive strong valid inequalities to improve the computational performance.

Studies on single-binary formulations are limited. Lee, Leung, and Margot 2004 investigate the minimum-up/-down polytope using a single type of binary variables. They give a complete description of the convex hull of the polytope, obtain valid inequalities, and design an efficient separation procedure for using these valid inequalities. Carrión and Arroyo 2006 propose a single-binary MILP formulation for UC problems. They approximate the generation cost function and the exponential start-up cost function using linear functions as in Arroyo and Conejo 2000. They also establish new minimum-up/-down constraints. They then compare the proposed formulation with the three-binary formulation of Arroyo and Conejo 2000, as well as with its variant in which one type of binary variables in Arroyo and Conejo 2000 is relaxed. The single-binary formulation outperforms the other two formulations significantly, although it is computationally less effective than the three-binary formulation of James Ostrowski, Anjos, and Vannelli 2012.

Frangioni and Gentile 2006a derive perspective cuts for the mixed-integer quadratic programming problem with semi-continuous variables. They test the effectiveness of these cuts by solving a single-binary UC formulation with a quadratic generation cost function. These cuts can substantially improve the performance of the branch-and-cut method. However, their formulation does not consider the ramp-up/-down, spinning reserve, and transmission constraints. Frangioni, Gentile, and Lacalandra 2009 apply the perspective cuts of Frangioni and Gentile 2006a to provide a new piecewise linear approximation of the generation cost function for a short-term UC problem with hydro and thermal generators.

1.3 Motivation

Tighter formulations provide better LP relaxations but usually require more variables or constraints and thus are not compact, whereas compact formulations are generally obtained at the cost of weakening tightness. Therefore, in practice, tightness and compactness must be balanced (Bernard Knueven, James Ostrowski, and Watson 2020b). In most cases, tight formulations are preferred over compact ones because of their shorter solution times, despite the increased complexity resulting from additional binary variables (K. W. Hedman, O'Neill, and Oren 2009; James Ostrowski, Anjos, and Vannelli 2012; Bendotti, Fouilhoux, and Rottner 2018). Moreover, a lack of binary variables for start-up/shut-down decisions makes it difficult to generate strong valid inequalities (James Ostrowski, Anjos, and Vannelli 2012). Thus, few studies examine compact formulations with a single type of binary variables and derive strong valid inequalities to tighten the compact formulations. To bridge this gap, this paper studies a single-binary formulation for a UC problem and derives strong valid inequalities to speed up the solution process. The main contributions of this study are summarized as follows:

- Through an investigation of the physical constraints, we provide a complete description of the convex hull of the two-period UC polytope of the compact formulation.

- We develop strong valid inequality families for the multi-period UC polytope, and we derive the conditions under which the strong valid inequalities are facet-defining. We also develop efficient separation algorithms for determining the most violated inequality in each valid inequality family.

- We demonstrate the effectiveness of our strong valid inequalities in tightening our compact formulation through computational experiments. The results indicate that our strong valid inequalities are effective in solving UC problems and can also be applied to UC formulations that contain more than one type of binary variables.

The structure of the remaining parts is as follows. Section 2 presents our proposed compact formulation for the UC problem and introduces the UC polytope. In Section 3, we give the complete description of the convex hull for the two-period UC polytope and discuss its importance in solving Problem 2.1. Next, Section 4 shows strong valid inequalities with single and two continuous variables, as well as polynomial separation algorithms for applying them to tighten the multi-period UC polytope. In Section 5.1, numerical experiments are carried out to validate our proposed compact formulation and the effectiveness of strong valid inequalities in speeding up the branch-and-cut algorithm. Finally, our discussion is concluded in Section 6. All proofs are provided in the Online Appendix.

Chapter 2

MILP Model for Unit Commitment

In this section, we first present an MILP model for the UC problem, followed by the presentation of a UC polytope for this model. In the UC problem being studied, an ISO plans the generation schedule of a set of generators $\mathcal{G}$ for a number of time periods at minimal operating costs while satisfying physical and system constraints. The system includes a set of buses $\mathcal{B}$ and a set of transmission lines $\mathcal{E}$ that link the buses, allowing surplus power to be distributed. Each bus is equipped with multiple generator units and is responsible for the load requirement of a geographical region. Surplus power at one bus can be transferred to neighboring buses through transmission lines to satisfy the load requirements of other regions. The power flow on each transmission line should not exceed the line's capacity. To ensure the reliability of the power supply, some generation capacity should be reserved for outages. All generators should operate without violating their physical configurations. Every time when a generator starts up or shuts down, a fixed cost is incurred. For each time period, an operational cost is incurred depending on the the generation amount and the online/offline status of the generators.

We let $\mathbb{R}^n$ denote the n dimensional real vector space, $\mathbb{R}^n_+$ denote the n dimensional nonnegative real vector space, and $\mathbb{B}^n$ denote for the n dimensional binary vector space. Given any nonnegative integers a and b, we let $[a,b]_{\mathbb{Z}}$ denote the set of all integers between a and b; that is, $[a,b]_{\mathbb{Z}} = \{a, a+1, \ldots, b\}$ if $a \leq b$, and $[a,b]_{\mathbb{Z}} = \emptyset$ if $a > b$.

Let T be the number of time periods in the operation horizon. For each generator $g \in \mathcal{G}$, let $L^g > 0$ and $\ell^g > 0$ be the minimum-up and minimum-down time requirement, respectively. That is, once the generator starts up, it must stay online for at least L^g time periods, and once it shuts down, it must stay offline for at least ℓ^g time periods. For each $g \in \mathcal{G}$, let $\overline{C}^g$ and $\underline{C}^g$ be the generation upper and lower bounds, where $\overline{C}^g > \underline{C}^g > 0$. For each $g \in \mathcal{G}$, let $V^g > 0$ be the maximum change of the generation amount between two consecutive online time periods, and $\overline{V}^g > 0$ be the start-up/shut-down ramp rate limit. Thus, when a generator g is online, its generation amount should be within the range $[\underline{C}^g, \overline{C}^g]$. When the generator starts up, its generation amount in the start-up period should be within the range $[\underline{C}^g, \overline{V}^g]$. When the generator shuts down, its generation amount in the previous time period should also be within the range $[\underline{C}^g, \overline{V}^g]$. We assume that $\overline{V}^g + V^g \leq \overline{C}^g$ for all $g \in \mathcal{G}$. This condition guarantees that a generator can ramp up at its full rate V^g for at least one period after it starts up. We also assume that $\underline{C}^g < \overline{V}^g < \underline{C}^g + V^g$ for all $g \in \mathcal{G}$, which holds in most industrial settings. For each bus $b \in \mathcal{B}$, let $\mathcal{G}_b$ be the set of generators at bus b (note: $\bigcup_{b \in \mathcal{B}} \mathcal{G}_b = \mathcal{G}$ and $\mathcal{G}_b \cap \mathcal{G}_{b'} = \emptyset$ for all $b, b' \in \mathcal{B}$ such that $b \neq b'$). Other parameters of our model are defined as follows:

- $f^g(\cdot)$: Generation cost function for generator g (for each $g \in \mathcal{G}$, $f^g(\cdot)$ is a non-decreasing convex piecewise linear function with a fixed number of linear segments).
- c^g: Fixed cost incurred if generator g is online ($c^g \geq 0$ for all $g \in \mathcal{G}$).
- ϕ^g: Fixed start-up cost of generator g ($\phi^g \geq 0$ for all $g \in \mathcal{G}$).
- ψ^g: Fixed shut-down cost of generator g ($\psi^g \geq 0$ for all $g \in \mathcal{G}$).
- d_t^b: The load (demand) at bus b in time period t ($d_t^b \geq 0$ for all $t \in [1, T]_{\mathbb{Z}}$ and $b \in \mathcal{B}$).
- C_e: Capacity limit of transmission line e ($C_e \geq 0$ for all $e \in \mathcal{E}$).
- K_e^b: Line flow distribution factor for the flow on transmission line e contributed by the net injection at bus b ($K_e^b \geq 0$ for all $e \in \mathcal{E}$ and $b \in \mathcal{B}$).
- r_t: System reserve factor of time period ($r_t \geq 0$ for all $t \in [1, T]_{\mathbb{Z}}$).

Here, the non-decreasing convex piecewise linear generation cost function $f^g(x)$ is used to approximate the convex quadratic cost function $a^g(x)^2 + b^g x$; see Carrión and Arroyo 2006 and Pan, Zhao, et al. 2022 for similar approximations. Our model has the following decision variables:

- x_t^g: The generation amount of generator $g \in \mathcal{G}$ in period $t \in [1, T]_{\mathbb{Z}}$.
- y_t^g: The online/offline status of generator $g \in \mathcal{G}$ in period $t \in [1, T]_{\mathbb{Z}}$, where $y_t^g = 1$ if g is online in period t, and $y_t^g = 0$ otherwise.

- u_t^g: The start-up cost of generator $g \in \mathcal{G}$ in period $t \in [1,T]_{\mathbb{Z}}$.
- v_t^g: The shut-down cost of generator $g \in \mathcal{G}$ in period $t \in [1,T]_{\mathbb{Z}}$.

Variables x_t^g, u_t^g, and b_t^g are continuous, while variable y_t^g is binary. We assume that the values of $y^g_{-\max\{L^g,\ell^g\}+1}, y^g_{-\max\{L^g,\ell^g\}+2}, \ldots, y^g_{-1}, y^g_0$ and x_0^g (for all $g \in \mathcal{G}$) are given as initial conditions. The UC problem is formulated as follows:

$$
\begin{aligned}
\text{Problem (1):} \quad \min \quad & \textstyle\sum_{g\in\mathcal{G}}\sum_{t=1}^{T}(u_t^g + v_t^g + c^g y_t^g + f^g(x_t^g)) && \text{(2.1a)}\\
\text{s.t.} \quad & -y_{t-1}^g + y_t^g - y_k^g \le 0,\ \forall t\in[-L^g+2,T]_{\mathbb{Z}}, \\
& \qquad \forall k\in[t,\min\{T,t+L^g-1\}]_{\mathbb{Z}},\ \forall g\in\mathcal{G} && \text{(2.1b)}\\
& y_{t-1}^g - y_t^g + y_k^g \le 1,\ \forall t\in[-\ell^g+2,T]_{\mathbb{Z}}, \\
& \qquad \forall k\in[t,\min\{T,t+\ell^g-1\}]_{\mathbb{Z}},\ \forall g\in\mathcal{G} && \text{(2.1c)}\\
& -x_t^g + \underline{C}^g y_t^g \le 0,\ \forall t\in[1,T]_{\mathbb{Z}},\ \forall g\in\mathcal{G} && \text{(2.1d)}\\
& x_t^g - \overline{C}^g y_t^g \le 0,\ \forall t\in[1,T]_{\mathbb{Z}},\ \forall g\in\mathcal{G} && \text{(2.1e)}\\
& x_t^g - x_{t-1}^g \le V^g y_{t-1}^g + \overline{V}^g(1-y_{t-1}^g),\ \forall t\in[1,T]_{\mathbb{Z}}, \forall g\in\mathcal{G} && \text{(2.1f)}\\
& x_{t-1}^g - x_t^g \le V^g y_t^g + \overline{V}^g(1-y_t^g),\ \forall t\in[1,T]_{\mathbb{Z}}, \forall g\in\mathcal{G} && \text{(2.1g)}\\
& u_t^g \ge \phi^g(y_t^g - y_{t-1}^g),\ \forall t\in[1,T]_{\mathbb{Z}}, \forall g\in\mathcal{G} && \text{(2.1h)}\\
& v_t^g \ge \psi^g(y_{t-1}^g - y_t^g),\ \forall t\in[1,T]_{\mathbb{Z}}, \forall g\in\mathcal{G} && \text{(2.1i)}\\
& \textstyle\sum_{g\in\mathcal{G}} x_t^g = \sum_{b\in\mathcal{B}} d_t^b,\ \forall t\in[1,T]_{\mathbb{Z}} && \text{(2.1j)}\\
& \textstyle\sum_{g\in\mathcal{G}} \overline{C}^g y_t^g \ge (1+r_t)\sum_{b\in\mathcal{B}} d_t^b,\ \forall t\in[1,T]_{\mathbb{Z}} && \text{(2.1k)}\\
& -C_e \le \textstyle\sum_{b\in\mathcal{B}} K_e^b\big(\sum_{g\in\mathcal{G}_b} x_t^g - d_t^b\big) \le C_e,\ \forall t\in[1,T]_{\mathbb{Z}}, \forall e\in\mathcal{E} && \text{(2.1l)}\\
& y_t^g\in\{0,1\}, x_t^g\ge 0, u_t^g\ge 0, v_t^g\ge 0,\ \forall t\in[1,T]_{\mathbb{Z}}, \forall g\in\mathcal{G} && \text{(2.1m)}
\end{aligned}
$$

Objective function (2.1a) minimizes the total cost, which includes the start-up costs, shut-down costs, and fixed and variable generation costs. Constraint (2.1b) states the minimum-up requirement for generator g. It requires generator g to stay online in periods $[t, \min\{T, t+L^g-1\}]_{\mathbb{Z}}$ if it starts up in period t. Constraint (2.1c) states the minimum-down requirement for generator g. It requires generator g to stay offline in periods $[t, \min\{T, t+\ell^g-1\}]_{\mathbb{Z}}$ if it shuts down in period t. Constraints (2.1d) and (2.1e) ensure that the generation amount of generator g in period t is 0 if the generator is offline, and is within the range $[\underline{C}^g, \overline{C}^g]$ if the generator is online. Constraints (2.1f) and (2.1g) guarantee that generator g ramps up/down within its limit V^g between two consecutive online time periods. They also guarantee that generator g ramps up by no more than $\overline{V}^g$ units when it starts up, and ramps down by no more than $\overline{V}^g$ units when it shuts down. Constraint (2.1h) and objective function (2.1a), together with the nonnegativity constraint of u_t^g, imply that the start-up cost for generator g in period t is ϕ^g if the generator starts up in period t, and is 0 otherwise. Constraint (2.1i) and objective function (2.1a), together with the nonnegativity constraint of v_t^g, imply that the shut-down cost for generator g in period t is ψ^g if the generator shuts down in period t, and is 0 otherwise. Constraint (2.1j) is the load balance constraint in period t, which requires the total generation amount to satisfy the total demand in the time period. Constraint (2.1k) is the system reserve requirement, which requires the total generation capacity of all online generators to exceed the load requirement by a system reserve factor in order to deal with demand variations. Constraint (2.1l) states the transmission flow limit. In the distribution process, a bus b contributes a factor K_e^b of its net injection $\sum_{g \in \mathcal{G}_b} x_t^g - d_t^b$ to each transmission line e, and the transmission flow limit requires that the absolute value of the total net injection contributed by all buses to each transmission line to stay below its capacity limit in order to prevent it from being overloaded; see Ma and S. M. Shahidehpour 1999, M. Shahidehpour, Yamin, and Z. Li 2002, and Xavier, Qiu, and Ahmed 2021 for similar settings. Constraint (2.1m) states the nonnegativity and binary requirements of the decision variables. Note that the objective function of Problem (2.1) is piecewise linear. Following the existing literature (see, e.g., Arroyo and Conejo 2000), Problem (2.1) can be converted into an MILP.

In Problem (2.1), constraints (2.1b)–(2.1g) specify the physical properties of the generators. Constraints (2.1h) and (2.1i) determine the start-up and shut-down costs. Once the y_t^g values are determined for all $g \in \mathcal{G}$ and $t \in [1, T]_{\mathbb{Z}}$, the u_t^g and v_t^g values can be easily obtained by these constraints. Constraints (2.1j)–(2.1l) are the coupling constraints, or system constraints, which link all generators. Due to the scale and complexity of the UC problem, one way of reducing the solving time is to decompose the problem into smaller subproblems with one subproblem corresponding to each generator (see Bernard Knueven, James Ostrowski, and Watson 2020a). For example, in the Lagrangian Relaxation method, the coupling constraints can be integrated into the objective function through Lagrangian multipliers, and the resulting problem is decomposed into subproblems that contain only the physical constraints (Baldick 1995; Takriti and Birge 2000). Thus, most improvements in UC models are resulted from studying the properties of an individual generator's feasible region (Ben Knueven, Jim Ostrowski, and J. Wang 2018). Moreover, strong valid inequalities for the physical constraints are valid for Problem (2.1) and can be used for tightening its linear relaxation. A tighter linear relaxation can often improve the computational efficiency by reducing the amount of enumeration required to find and prove an optimal solution (Bernard Knueven, James Ostrowski, and Watson 2020b). Hence, in the mathematical analysis presented in Sections 3 and 4, we focus on deriving strong valid inequalities for the physical constraints for the generators in Problem (2.1). Because all the generators have the same set of physical constraints, it suffices to concentrate on the physical constraints of a single generator, and the results obtained can be applied to all other generators. Therefore, in the following analysis, the superscript g in the parameters and decision variables is dropped.

Denote $\mathbf{x} = (x_1, \ldots, x_T)$ and $\mathbf{y} = (y_1, \ldots, y_T)$. Thus, the vector $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^T \times \mathbb{B}^T$ contains the generation amount and on/off status of the generator in the T time periods. The set of $(\mathbf{x}, \mathbf{y})$ values that satisfy the physical constraints of Problem (2.1) is given as

$$
\begin{aligned}
\mathcal{P} = \big\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^T \times \mathbb{B}^T :\\
& -y_{t-1} + y_t - y_k \leq 0, \ \forall t \in [2, T]_{\mathbb{Z}}, \\
& \qquad \forall k \in [t, \min\{T, t + L - 1\}]_{\mathbb{Z}}, && (2.2a) \\
& y_{t-1} - y_t + y_k \leq 1, \ \forall t \in [2, T]_{\mathbb{Z}}, \\
& \qquad \forall k \in [t, \min\{T, t + \ell - 1\}]_{\mathbb{Z}}, && (2.2b) \\
& -x_t + \underline{C} y_t \leq 0, \ \forall t \in [1, T]_{\mathbb{Z}}, && (2.2c) \\
& x_t - \overline{C} y_t \leq 0, \ \forall t \in [1, T]_{\mathbb{Z}}, && (2.2d) \\
& x_t - x_{t-1} \leq V y_{t-1} + \overline{V}(1 - y_{t-1}), \ \forall t \in [2, T]_{\mathbb{Z}}, && (2.2e) \\
& x_{t-1} - x_t \leq V y_t + \overline{V}(1 - y_t), \ \forall t \in [2, T]_{\mathbb{Z}} \big\}. && (2.2f)
\end{aligned}
$$

Here, the assumptions $\overline{C} > \underline{C} > 0$, $V > 0$, $\overline{V} + V \leq \overline{C}$, and $\underline{C} < \overline{V} <$

$\underline{C} + V$ remain valid. Note that inequalities (2.2a)–(2.2f) in $\mathcal{P}$ are the same as inequalities (2.1b)–(2.1g) in Problem (2.1) for a specific generator g, except that t is restricted to the range $[2, T]_{\mathbb{Z}}$ in (2.2a), (2.2b), (2.2e), and (2.2f) (i.e., constraints dependent on the initial conditions are not included in $\mathcal{P}$).

Let $\text{conv}(\mathcal{P})$ denote the convex hull of $\mathcal{P}$, and we refer to $\text{conv}(\mathcal{P})$ as the UC polytope. Obviously, a valid inequality for $\mathcal{P}$ is also valid for Problem (2.1) for any generator g. Hence, the strong valid inequalities developed for $\text{conv}(\mathcal{P})$ can be used for tightening the formulation of Problem (2.1). The following two lemmas provide some important properties of $\mathcal{P}$.

Lemma 1. Consider any point $(\mathbf{x}, \mathbf{y}) \in \mathcal{P}$ and $t \in [2, T]_{\mathbb{Z}}$. (i) If $y_t = 0$, then $y_{t-j} - y_{t-j-1} \leq 0$ for all $j \in [0, \min\{t-2, L-1\}]_{\mathbb{Z}}$. (ii) If $y_t = 1$, then there exists at most one $j \in [0, \min\{t-2, L\}]_{\mathbb{Z}}$ such that $y_{t-j} - y_{t-j-1} = 1$.

Lemma 2. Consider any point $(\mathbf{x}, \mathbf{y}) \in \mathcal{P}$ and $t \in [1, T-1]_{\mathbb{Z}}$. (i) If $y_t = 0$, then $y_{t+j} - y_{t+j+1} \leq 0$ for all $j \in [0, \min\{T-t-1, L-1\}]_{\mathbb{Z}}$. (ii) If $y_t = 1$, then there exists at most one $j \in [0, \min\{T-t-1, L\}]_{\mathbb{Z}}$ such that $y_{t+j} - y_{t+j+1} = 1$.

Chapter 3

The Two-period Convex Hull

In this section, we investigate the properties of set $\mathcal{P}$ when there are only two periods. The strong valid inequalities resulting from our investigation not only can be used for tightening the UC polytope $\text{conv}(\mathcal{P})$ but also can help derive other forms of strong valid inequalities for $\text{conv}(\mathcal{P})$.

Consider any two consecutive periods $t-1$ and t, where $t \in [2, T]_{\mathbb{Z}}$. Denote

$$\mathcal{P}_2 = \big\{(x_{t-1}, x_t, y_{t-1}, y_t) \in \mathbb{R}_+^2 \times \mathbb{B}^2 :$$

$$-x_i + \underline{C} y_i \leq 0, \ \forall i \in \{t-1, t\}, \tag{3.1a}$$

$$x_i - \overline{C} y_i \leq 0, \ \forall i \in \{t-1, t\}, \tag{3.1b}$$

$$x_t - x_{t-1} \leq V y_{t-1} + \overline{V}(1 - y_{t-1}), \tag{3.1c}$$

$$x_{t-1} - x_t \leq V y_t + \overline{V}(1 - y_t)\big\}. \tag{3.1d}$$

Note that when $t = 2$, the set $\mathcal{P}_2$ is the same as the set $\mathcal{P}$ with $T = 2$. Let $\text{conv}(\mathcal{P}_2)$ denote the convex hull of $\mathcal{P}_2$. The following theorem provides a complete description of $\text{conv}(\mathcal{P}_2)$.

Theorem 1. Denote

$$
\begin{aligned}
\mathcal{Q}_2 = \big\{ (x_{t-1}, x_t, y_{t-1}, y_t) \in \mathbb{R}^4 : \\
& y_i \leq 1, \ \forall i \in \{t-1, t\}, && (3.2a) \\
& \underline{C} y_i \leq x_i \leq \overline{C} y_i, \ \forall i \in \{t-1, t\}, && (3.2b) \\
& x_{t-1} \leq \overline{V} y_{t-1} + (\overline{C} - \overline{V}) y_t, && (3.2c) \\
& x_t \leq (\overline{C} - \overline{V}) y_{t-1} + \overline{V} y_t, && (3.2d) \\
& x_t - x_{t-1} \leq (\underline{C} + V) y_t - \underline{C} y_{t-1}, && (3.2e) \\
& x_t - x_{t-1} \leq \overline{V} y_t - (\overline{V} - V) y_{t-1}, && (3.2f) \\
& x_{t-1} - x_t \leq (\underline{C} + V) y_{t-1} - \underline{C} y_t, && (3.2g) \\
& x_{t-1} - x_t \leq \overline{V} y_{t-1} - (\overline{V} - V) y_t \big\}. && (3.2h)
\end{aligned}
$$

Then, $\mathcal{Q}_2 = \text{conv}(\mathcal{P}_2)$.

Theorem 1 implies that any inequality in (3.2a)–(3.2h) is valid for $\text{conv}(\mathcal{P}_2)$. Note that Theorem 1 holds for any $t \in [2, T]_{\mathbb{Z}}$. Thus, for any $t \in [2, T]_{\mathbb{Z}}$, any inequality in (3.2a)–(3.2h) is valid for $\text{conv}(\mathcal{P})$. Note also that inequalities (3.2c)–(3.2h) do not exist in the description of $\mathcal{P}$. Hence, they can be added to the constraint set of $\mathcal{P}$ to tighten the linear relaxation of $\mathcal{P}$. In particular, because $\overline{V} y_t - (\overline{V} - V) y_{t-1} \leq V y_{t-1} + \overline{V}(1 - y_{t-1})$ for any $y_t \leq 1$, the right hand side of (3.2f) is no greater than the right hand side of (2.2e), and thus inequality (3.2f) dominates inequality (2.2e) and can effectively tighten the linear relaxation of $\mathcal{P}$. Similarly, inequality (3.2h) dominates inequality (2.2f) and can effectively tighten the linear relaxation of $\mathcal{P}$. Therefore, the following inequality families can be used as valid inequality for $\mathbf{conv}(\mathcal{P})$:

$$x_t \leq \overline{V} y_t + (\overline{C} - \overline{V}) y_{t+1}, \ \forall t \in [1, T-1]_{\mathbb{Z}}; \tag{3.3}$$

$$x_t \leq (\overline{C} - \overline{V}) y_{t-1} + \overline{V} y_t, \ \forall t \in [2, T]_{\mathbb{Z}}; \tag{3.4}$$

$$x_t - x_{t-1} \leq (\underline{C} + V) y_t - \underline{C} y_{t-1}, \ \forall t \in [2, T]_{\mathbb{Z}}; \tag{3.5}$$

$$x_t - x_{t-1} \leq \overline{V} y_t - (\overline{V} - V) y_{t-1}, \ \forall t \in [2, T]_{\mathbb{Z}}; \tag{3.6}$$

$$x_t - x_{t+1} \leq (\underline{C} + V) y_t - \underline{C} y_{t+1}, \ \forall t \in [1, T-1]_{\mathbb{Z}}; \tag{3.7}$$

$$x_t - x_{t+1} \leq \overline{V} y_t - (\overline{V} - V) y_{t+1}, \ \forall t \in [1, T-1]_{\mathbb{Z}}. \tag{3.8}$$

These valid inequalities provide upper bounds on the generation amount x_t for each time period t, upper bounds on $x_t - x_{t-1}$ for each pair of consecutive time periods t and $t-1$, and upper bounds on $x_t - x_{t+1}$ for each pair of consecutive time periods t and $t+1$.

Inequalities (3.3)–(3.8) also enable us to develop other strong valid inequalities for $\mathbf{conv}(\mathcal{P})$. We demonstrate this by presenting a strong valid inequality derived from (3.5). Consider any point $(\mathbf{x}, \mathbf{y}) \in \mathcal{P}$. For any $k \in [1, T-1]_{\mathbb{Z}}$ and any $t \in [k+1, T]_{\mathbb{Z}}$, because inequality (3.5) is valid for $\mathbf{conv}(\mathcal{P})$, we have

$$\begin{aligned} \sum_{\tau=t-k+1}^{t} (x_\tau - x_{\tau-1}) &\leq \sum_{\tau=t-k+1}^{t} \left[(\underline{C} + V) y_\tau - \underline{C} y_{\tau-1} \right] \\ &= V \sum_{\tau=t-k+1}^{t} y_\tau + \underline{C} \sum_{\tau=t-k+1}^{t} (y_\tau - y_{\tau-1}), \end{aligned}$$

which implies that

$$x_t - x_{t-k} \leq V \sum_{\tau=t-k+1}^{t} y_\tau + \underline{C} y_t - \underline{C} y_{t-k}. \tag{3.9}$$

If $y_t = 1$, then $\sum_{\tau=t-k+1}^{t} y_\tau \leq k y_t$, and by (3.9), $x_t - x_{t-k} \leq (\underline{C} + kV) y_t - \underline{C} y_{t-k}$. If $y_t = 0$, then by (2.2c) and (2.2d), $-x_{t-k} \leq -\underline{C} y_{t-k}$ and $x_t = 0$, which also imply that $x_t - x_{t-k} \leq (\underline{C} + kV) y_t - \underline{C} y_{t-k}$. Thus, in both cases,

$$x_t - x_{t-k} \leq (\underline{C} + kV) y_t - \underline{C} y_{t-k}. \tag{3.10}$$

Hence, (3.10) is a valid inequality for $\mathrm{conv}(\mathcal{P})$ for any $k \in [1, T-1]_{\mathbb{Z}}$ and $t \in [k+1, T]_{\mathbb{Z}}$. It is worth noting that inequality (4.7) presented in Proposition 9 in Section 4.2 is reduced to inequality (3.10) when $m = 0$ and $\mathcal{S} = \emptyset$. Therefore, inequality (3.10) is a special case of the facet-defining valid inequality (4.7).

Chapter 4

Multi-period Strong Valid Inequalities

In this section, we present a collection of strong valid inequalities that can effectively enhance the tightness of Problem (2.1). We provide the validity proofs for these inequalities, and we identify the conditions under which these inequalities are facet-defining for $\text{conv}(\mathcal{P})$. For each family of valid inequalities, we also show that for any given point $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_{+}^{2T}$, an efficient separation algorithm exists for determining the most violated inequality.

4.1 Valid Inequalities with a Single Continuous Variable

In this subsection, we present strong valid inequalities that provide upper bounds on the generation amount x_t for each time period t. Families of such inequalities appear in pairs. The first family consists of inequalities where the upper bound on x_t depends mainly on the values of $y_{t-s} - y_{t-s-1}$ for some $s \geq 0$, while the second family consists of inequalities where the upper bound on x_t depends mainly on the values of $y_{t+s} - y_{t+s+1}$ for some $s \geq 0$.

The following proposition presents a pair of such inequality families.

Proposition 1. Consider any $\mathcal{S} \subseteq [0, \min\{L-1, T-2, \lfloor(\overline{C}-\overline{V})/V\rfloor\}]_{\mathbb{Z}}$. For any $t \in [1,T]_{\mathbb{Z}}$ such that $t \geq s+2$ for all $s \in \mathcal{S}$, the inequality

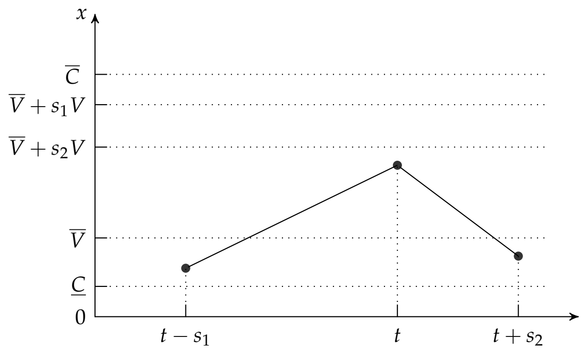
$$x_t \leq \overline{C}y_t - \sum_{s\in\mathcal{S}}(\overline{C}-\overline{V}-sV)(y_{t-s}-y_{t-s-1}) \tag{4.1}$$

is valid and facet-defining for $\text{conv}(\mathcal{P})$. For any $t \in [1,T]_{\mathbb{Z}}$ such that $t \leq T-s-1$ for all $s \in \mathcal{S}$, the inequality

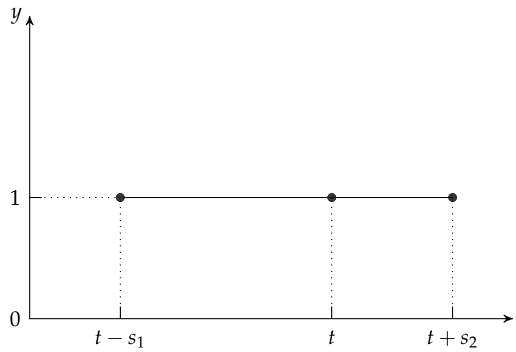
$$x_t \leq \overline{C}y_t - \sum_{s\in\mathcal{S}}(\overline{C}-\overline{V}-sV)(y_{t+s}-y_{t+s+1}) \tag{4.2}$$

is valid and facet-defining for $\text{conv}(\mathcal{P})$.

In Proposition 1, inequalities (4.1) and (4.2) provide upper bounds on the generation amount x_t. These upper bounds can be explained as follows. Let $s_{\max}$ denote the largest element of $\mathcal{S}$. The condition "$\mathcal{S} \subseteq [0, \min\{L-1, T-2, \lfloor(\overline{C}-\overline{V})/V\rfloor\}]_{\mathbb{Z}}$" implies that $s_{\max} \leq L-1$, which in turn implies that there is at most one startup and at most one shutdown during the time interval $[t-s_{\max}, t]$, and that there is at most one shutdown and at most one startup during the time interval $[t+1, t+s_{\max}+1]$. Consider the situation as shown in Figure 4.1, in which a generator starts up in period $t-s_1$, stays online until period $t+s_2$, and shuts down in period $t+s_2+1$, where $s_1, s_2 \in [0, s_{\max}]_{\mathbb{Z}}$, $t-s_1 \geq 2$, and $t+s_2+1 \leq T$. Then, $y_{t-s_1-1} = 0$, $y_{t-s_1} = y_{t-s_1+1} = \cdots = y_{t+s_2} = 1$, and $y_{t+s_2+1} = 0$. If $s_1 \in \mathcal{S}$ and none of the time periods in $\{t-s \geq 2 : s \in \mathcal{S}\}$ is a shut-down period, then the right hand side of inequality (4.1) becomes $\overline{C} - (\overline{C}-\overline{V}-s_1V)$. This upper bound limits the value of x_t to be no more than $\overline{V} + s_1 V$ (see Figure 4.1(a)). Similarly, if $s_2 \in \mathcal{S}$ and none of the periods in $\{t + s + 1 \leq T : s \in \mathcal{S}\}$ is a start-up period, then the right hand side of inequality (4.2) becomes $\overline{C} - (\overline{C} - \overline{V} - s_2 V)$. This upper bound limits the value of x_t to be no more than $\overline{V} + s_2 V$ (see Figure 4.1(a)).



(a) x values



(b) y values

Figure 4.1: The Ramp-up/-down Process of a Generator

In Proposition 1, the set $\mathcal{S}$ only contains elements that are less than L. The following proposition states that under certain conditions, inequalities (4.1) and (4.2) remain valid and facet-defining when $\mathcal{S}$ contains some elements that are greater than or equal to L.

Proposition 2. Consider any integers α, β, and $s_{\max}$ such that (a) $L \leq s_{\max} \leq$

$\min\{T-2, \lfloor(\overline{C}-\overline{V})/V\rfloor\}$, (b) $0 \leq \alpha < \beta \leq s_{\max}$, and (c) $\beta = \alpha+1$ or $s_{\max} \leq L+\alpha$. Let $\mathcal{S} = [0,\alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. For any $t \in [s_{\max}+2, T]_{\mathbb{Z}}$, inequality (4.1) is valid and facet-defining for $\text{conv}(\mathcal{P})$. For any $t \in [1, T - s_{\max} - 1]_{\mathbb{Z}}$, inequality (4.2) is valid and facet-defining for $\text{conv}(\mathcal{P})$.

Example 1. Let $T = 16$, $\overline{C} = 80$, $\underline{C} = 8$, $L = \ell = 5$, $\overline{V} = 15$, and $V = 10$. Then, $\lfloor(\overline{C}-\overline{V})/V\rfloor = 6$. By Proposition 1, we obtain the following pair of valid inequalities if we set $\mathcal{S} = \{0,2,4\}$ and $t = 8$:

$$\begin{cases} x_8 \leq 25y_3 - 25y_4 + 45y_5 - 45y_6 + 65y_7 + 15y_8; \\ x_8 \leq 15y_8 + 65y_9 - 45y_{10} + 45y_{11} - 25y_{12} + 25y_{13}. \end{cases}$$

By Proposition 2, we obtain the following pair of valid inequalities if we set $\mathcal{S} = \{0,1,2,5,6\}$ (i.e., $\alpha = 2$, $\beta = 5$, and $s_{\max} = 6$) and $t = 8$:

$$\begin{cases} x_8 \leq 5y_1 + 10y_2 - 15y_3 + 45y_5 + 10y_6 + 10y_7 + 15y_8; \\ x_8 \leq 15y_8 + 10y_9 + 10y_{10} + 45y_{11} - 15y_{13} + 10y_{14} + 5y_{15}. \end{cases}$$

The next proposition extends Proposition 1 and presents another two families of strong valid inequalities.

Proposition 3. Consider any set $\mathcal{S} \subseteq [0, \min\{L-1, T-3, \lfloor(\overline{C}-\overline{V})/V\rfloor\}]_{\mathbb{Z}}$ and any real number η such that $0 \leq \eta \leq \min\{L-1, (\overline{C}-\overline{V})/V\}$. For any $t \in [1, T-1]_{\mathbb{Z}}$ such that $t \geq s+2$ for all $s \in \mathcal{S}$, the inequality

$$x_t \leq (\overline{C} - \eta V)y_t + \eta V y_{t+1} - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \tag{4.3}$$

is valid for $\text{conv}(\mathcal{P})$. For any $t \in [2, T]_{\mathbb{Z}}$ such that $t \leq T - s - 1$ for all

$s \in \mathcal{S}$, the inequality

$$x_t \leq (\overline{C} - \eta V)y_t + \eta V y_{t-1} - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \tag{4.4}$$

is valid for $\text{conv}(\mathcal{P})$. Furthermore, inequalities (4.3) and (4.4) are facet-defining for $\text{conv}(\mathcal{P})$ when $\eta \in \{0, (\overline{C} - \overline{V})/V\}$ or $\eta = L - 1 \in \mathcal{S}$.

When $t \neq T$, inequality (4.3) is a generalization of inequality (4.1). Specifically, the right hand side of (4.3) differs from the right hand side of (4.1) by $\eta V y_{t+1} - \eta V y_t$, and this difference is zero if $\eta = 0$. Similarly, when $t \neq 1$, inequality (4.4) is a generalization of inequality (4.2), and the right hand side of (4.4) differs from the right hand side of (4.2) by $\eta V y_{t-1} - \eta V y_t$. In Proposition 3, the set $\mathcal{S}$ only contains elements that are less than L. The following proposition, which extends Proposition 2, states that under certain conditions, inequalities (4.3) and (4.4) remain valid and facet-defining when $\mathcal{S}$ contains some elements that are greater than or equal to L.

Proposition 4. Consider any real number η such that $0 \leq \eta \leq \min\{L - 1, (\overline{C} - \overline{V})/V\}$ and any integers α, β, and $s_{\max}$ such that (a) $L \leq s_{\max} \leq \min\{T - 3, \lfloor (\overline{C} - \overline{V})/V \rfloor\}$, (b) $0 \leq \alpha < \beta \leq s_{\max}$, and (c) $\beta = \alpha + 1$ or $s_{\max} \leq L + \alpha$. Let $\mathcal{S} = [0, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. For any $t \in [s_{\max} + 2, T - 1]_{\mathbb{Z}}$, inequality (4.3) is valid for $\text{conv}(\mathcal{P})$. For any $t \in [2, T - s_{\max} - 1]_{\mathbb{Z}}$, inequality (4.4) is valid for $\text{conv}(\mathcal{P})$. Furthermore, (4.3) and (4.4) are facet-defining for $\text{conv}(\mathcal{P})$ when $\eta \in \{0, (\overline{C} - \overline{V})/V\}$ or $\eta = L - 1 \in \mathcal{S}$.

Example 2 (continuation of example 1). In Example 1, if we set $\eta = 2.5$, $\mathcal{S} = \{0, 2, 4\}$, and $t = 8$, then by Proposition 3, we obtain the following pair of valid inequalities:

$$\begin{cases} x_8 \le 25y_3 - 25y_4 + 45y_5 - 45y_6 + 65y_7 - 10y_8 + 25y_9; \\ x_8 \le 25y_7 - 10y_8 + 65y_9 - 45y_{10} + 45y_{11} - 25y_{12} + 25y_{13}. \end{cases}$$

Note that the right hand sides of the first and second inequalities differ from those in the first pair of inequalities in Example 1 by $\eta V y_{t+1} - \eta V y_t$ (i.e., $25y_9 - 25y_8$) and $\eta V y_{t-1} - \eta V y_t$ (i.e., $25y_7 - 25y_8$), respectively. If we set $\eta = 2.5$, $\mathcal{S} = \{0, 1, 2, 5, 6\}$ (i.e., $\alpha = 2$, $\beta = 5$, and $s_{\max} = 6$), and $t = 8$, then by Proposition 4, we obtain the following pair of valid inequalities:

$$\begin{cases} x_8 \le 5y_1 + 10y_2 - 15y_3 + 45y_5 + 10y_6 + 10y_7 - 10y_8 + 25y_9; \\ x_8 \le 25y_7 - 10y_8 + 10y_9 + 10y_{10} + 45y_{11} - 15y_{13} + 10y_{14} + 5y_{15}. \end{cases}$$

Similarly, the right hand sides of the first and second inequalities differ from those in the second pair of inequalities in Example 1 by $\eta V y_{t+1} - \eta V y_t$ (i.e., $25y_9 - 25y_8$) and $\eta V y_{t-1} - \eta V y_t$ (i.e., $25y_7 - 25y_8$), respectively.

The next proposition also extends Proposition 1 and presents another two families of strong valid inequalities.

Proposition 5. Consider any $\mathcal{S} \subseteq [1, \min\{L, T-2, \lfloor (\overline{C} - \overline{V})/V \rfloor\}]_{\mathbb{Z}}$ and any real number η such that $0 \le \eta \le \min\{L, (\overline{C} - \overline{V})/V\}$. For any $t \in [2, T]_{\mathbb{Z}}$ such that $t \ge s + 2$ for all $s \in \mathcal{S}$, the inequality

$$x_t \le (\overline{V} + \eta V) y_t + (\overline{C} - \overline{V} - \eta V) y_{t-1} - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \tag{4.5}$$

is valid for $\mathrm{conv}(\mathcal{P})$. For any $t \in [1, T-1]_{\mathbb{Z}}$ such that $t \le T - s - 1$ for all

$s \in \mathcal{S}$, the inequality

$$x_t \leq (\overline{V} + \eta V)y_t + (\overline{C} - \overline{V} - \eta V)y_{t+1} - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \tag{4.6}$$

is valid for $\text{conv}(\mathcal{P})$. Furthermore, inequalities (4.5) and (4.6) are facet-defining for $\text{conv}(\mathcal{P})$ when $\eta \in \{0, (\overline{C} - \overline{V})/V\}$ or $\eta = L \in \mathcal{S}$.

In Proposition 5, the set $\mathcal{S}$ only contains elements that are less than or equal to L. The following proposition states that under certain conditions, inequalities (4.5) and (4.6) remain valid and facet-defining when $\mathcal{S}$ contains some elements that are greater than L.

Proposition 6. Consider any integers, α, β, and $s_{\max}$ such that (a) $L + 1 \leq s_{\max} \leq \min\{T - 2, \lfloor(\overline{C} - \overline{V})/V\rfloor\}$, (b) $1 \leq \alpha < \beta \leq s_{\max}$, and (c) $\beta = \alpha + 1$ or $s_{\max} \leq L + \alpha$. Let $\mathcal{S} = [1, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. For any $t \in [s_{\max} + 2, T]_{\mathbb{Z}}$, inequality (4.5) is valid for $\text{conv}(\mathcal{P})$. For any $t \in [1, T - s_{\max} - 1]_{\mathbb{Z}}$ inequality (4.6) is valid for $\text{conv}(\mathcal{P})$. Furthermore, (4.5) and (4.6) are facet-defining for $\text{conv}(\mathcal{P})$ when $\eta \in \{0, (\overline{C} - \overline{V})/V\}$ or $\eta = L \in \mathcal{S}$.

Example 3 (continuation of example 1). In Example 1, if we set $\eta = 2.5$, $\mathcal{S} = \{1, 3, 5\}$, and $t = 8$, then by Proposition 5, we obtain the following pair of valid inequalities:

$$\begin{cases} x_8 \leq 15y_2 - 15y_3 + 35y_4 - 35y_5 + 55y_6 - 15y_7 + 40y_8; \\ x_8 \leq 40y_8 - 15y_9 + 55y_{10} - 35y_{11} + 35y_{12} - 15y_{13} + 15y_{14}. \end{cases}$$

If we set $\eta = 2.5$, $\mathcal{S} = \{1, 2, 5, 6\}$ (i.e., $\alpha = 2$, $\beta = 5$, and $s_{\max} = 6$), and

$t = 8$, then by Proposition 6, we obtain the following pair of valid inequalities:

$$\begin{cases} x_8 \leq 5y_1 + 10y_2 - 15y_3 + 45y_5 + 10y_6 - 15y_7 + 40y_8; \\ x_8 \leq 40y_8 - 15y_9 + 10y_{10} + 45y_{11} - 15y_{13} + 10y_{14} + 5y_{15}. \end{cases}$$

Propositions 1–6 have presented different families of valid inequalities. For each family of valid inequalities and any given point $(\mathbf{x}, \mathbf{y})$ with non-binary y values, it is important to have an efficient separation algorithm that can identify the most violated inequality in the family, if such violated inequality exists.

Proposition 7. Let $\hat{\eta}$ and $a_1, \ldots, a_6$ be any real numbers such that $\hat{\eta} \geq 0$. Let $\check{s}$ and $\hat{s}$ be any integers such that $0 \leq \check{s} \leq \hat{s} \leq \min\{T-2, \lfloor(\overline{C} - \overline{V})/V\rfloor\}$. Let $\check{t} = 1$ if $a_1 = a_2 = 0$, and let $\check{t} = 2$ otherwise. Let $\hat{t} = T$ if $a_5 = a_6 = 0$, and let $\hat{t} = T - 1$ otherwise. (i) Consider the following family of inequalities:

$$\begin{aligned} x_t \leq (a_1 + a_2\eta)y_{t-1} + (a_3 + a_4\eta)y_t + (a_5 + a_6\eta)y_{t+1} \\ - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}), \end{aligned}$$

where $\eta \in [0, \hat{\eta}]$, $\mathcal{S} \subseteq [\check{s}, \hat{s}]_{\mathbb{Z}}$, $t \in [\check{t}, \hat{t}]_{\mathbb{Z}}$, and $t \geq s + 2$ for all $s \in \mathcal{S}$. For any given point $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{2T}$, the set $\mathcal{S}$, the real number η, and the integer t corresponding to the most violated inequality can be determined in $O(T)$

time. (ii) Consider the following family of inequalities:

$$x_t \leq (a_1 + a_2\eta)y_{t-1} + (a_3 + a_4\eta)y_t + (a_5 + a_6\eta)y_{t+1} - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}),$$

where $\eta \in [0, \hat{\eta}]$, $\mathcal{S} \subseteq [\check{s}, \hat{s}]_{\mathbb{Z}}$, $t \in [\check{t}, \hat{t}]_{\mathbb{Z}}$, and $t \leq T - s - 1$ for all $s \in \mathcal{S}$. For any given point $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{2T}$, the set $\mathcal{S}$, the real number η, and the integer t corresponding to the most violated inequality can be determined in $O(T)$ time.

In Propositions 1, 3, and 5, the number of combinations of $\mathcal{S}$ and t is exponential in T. Furthermore, in Propositions 3 and 5, η is a real value. However, Proposition 7 implies that given any point $(\mathbf{x}, \mathbf{y})$ with non-binary y values, the most violated inequality in each of the inequality families stated in Propositions 1, 3, and 5 can be determined in linear time. For example, Proposition 7 can be applied to inequality family (4.3) of Proposition 3 by setting $a_1 = a_2 = a_5 = 0$, $a_3 = \overline{C}$, $a_4 = -V$, $a_6 = V$, $\hat{\eta} = \min\{L-1, (\overline{C} - \overline{V})/V\}$, $\check{s} = 0$, and $\hat{s} = \min\{L-1, T-3, \lfloor(\overline{C} - \overline{V})/V\rfloor\}$.

Proposition 8. Let $\hat{\eta}$ and $a_1, \ldots, a_6$ be any real numbers such that $\hat{\eta} \geq 0$. Let $\check{s}$, $\check{s}_{\max}$, and $\hat{s}_{\max}$ be any integers such that $0 \leq \check{s} \leq \check{s}_{\max} \leq \hat{s}_{\max} \leq \min\{T-2, \lfloor(\overline{C} - \overline{V})/V\rfloor\}$. Let $\check{t} = 1$ if $a_1 = a_2 = 0$, and let $\check{t} = 2$ otherwise. Let $\hat{t} = T$ if $a_5 = a_6 = 0$, and let $\hat{t} = T - 1$ otherwise. (i) Consider the following family of inequalities:

$$x_t \leq (a_1 + a_2\eta)y_{t-1} + (a_3 + a_4\eta)y_t + (a_5 + a_6\eta)y_{t+1} \\ - \sum_{s\in\mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}),$$

where $\eta \in [0, \hat{\eta}]$, $\mathcal{S} = [\check{s}, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$, $t \in [s_{\max} + 2, \hat{t}]_{\mathbb{Z}}$, and α, β, and $s_{\max}$ are integers such that (a) $\check{s}_{\max} \leq s_{\max} \leq \hat{s}_{\max}$, (b) $\check{s} \leq \alpha < \beta \leq s_{\max}$, and (c) $\beta = \alpha + 1$ or $s_{\max} \leq L + \alpha$. For any given point $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{2T}$, the integers α, β, $s_{\max}$, t and the real number η corresponding to the most violated inequality can be determined in $O(T^3)$ time. (ii) Consider the following family of inequalities:

$$x_t \leq (a_1 + a_2\eta)y_{t-1} + (a_3 + a_4\eta)y_t + (a_5 + a_6\eta)y_{t+1} \\ - \sum_{s\in\mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}),$$

where $\eta \in [0, \hat{\eta}]$, $\mathcal{S} = [\check{s}, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$, $t \in [\check{t}, T - s_{\max} - 1]_{\mathbb{Z}}$, and α, β, and $s_{\max}$ are integers such that (a) $\check{s}_{\max} \leq s_{\max} \leq \hat{s}_{\max}$, (b) $\check{s} \leq \alpha < \beta \leq s_{\max}$, and (c) $\beta = \alpha + 1$ or $s_{\max} \leq L + \alpha$. For any given point $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{2T}$, the integers α, β, $s_{\max}$, t and the real number η corresponding to the most violated inequality can be determined in $O(T^3)$ time.

In Propositions 2, 4, and 6, the number of combinations of α, β, $s_{\max}$, and t is $O(T^4)$. Furthermore, in Propositions 4 and 6, η is a real value. However, Proposition 8 implies that given any point $(\mathbf{x}, \mathbf{y})$ with non-binary y values, the most violated inequality in each of the inequality families stated in Propositions 2, 4, and 6 can be determined in $O(T^3)$ time. For example, Proposition 8 can be applied to inequality family (4.3) of Proposition 4 by setting $a_1 = a_2 = a_5 = 0$, $a_3 = \overline{C}$, $a_4 = -V$, $a_6 = V$, $\hat{\eta} = \min\{L-1, (\overline{C} - \overline{V})/V\}$, $\check{s} = 0$, $\check{s}_{\max} = L$, and $\hat{s}_{\max} = \min\{T-3, \lfloor(\overline{C} - \overline{V})/V\rfloor\}$.

4.2 Valid Inequalities with Two Continuous Variables

In this subsection, we present strong valid inequalities that provide upper bounds on $x_t - x_{t-k}$ (respectively $x_t - x_{t+k}$) for each pair of time periods t and $t-k$ (respectively t and $t+k$). The following proposition presents a pair of such inequality families.

Proposition 9. Consider any $k \in [1, T-1]_{\mathbb{Z}}$ such that $\overline{C} - \underline{C} - kV > 0$, any $m \in [0, k-1]_{\mathbb{Z}}$, and any $\mathcal{S} \subseteq [0, \min\{k-1, L-m-1\}]_{\mathbb{Z}}$. For any $t \in [k+1, T-m]_{\mathbb{Z}}$, the inequality

$$\begin{aligned} x_t - x_{t-k} \leq (\underline{C} + (k-m)V)y_t + V\sum_{i=1}^{m} y_{t+i} - \underline{C}y_{t-k} \\ - \sum_{s\in\mathcal{S}} (\underline{C} + (k-s)V - \overline{V})(y_{t-s} - y_{t-s-1}) \end{aligned} \tag{4.7}$$

is valid for $\text{conv}(\mathcal{P})$. For any $t \in [m+1, T-k]_{\mathbb{Z}}$, the inequality

$$\begin{aligned} x_t - x_{t+k} \leq (\underline{C} + (k-m)V)y_t + V\sum_{i=1}^{m} y_{t-i} - \underline{C}y_{t+k} \\ - \sum_{s\in\mathcal{S}} (\underline{C} + (k-s)V - \overline{V})(y_{t+s} - y_{t+s+1}) \end{aligned} \tag{4.8}$$

is valid for $\text{conv}(\mathcal{P})$. Furthermore, (4.7) and (4.8) are facet-defining for $\text{conv}(\mathcal{P})$ when $m = 0$ and $s \geq \min\{k-1, 1\}$ for all $s \in \mathcal{S}$.

In Proposition 9, the number of combinations of $\mathcal{S}$, t, k, and m is exponential in T. Thus, the sizes of the inequality families (4.7) and (4.8) are exponential in T. However, the next proposition states that given any point $(\mathbf{x}, \mathbf{y})$ with non-binary y values, the most violated inequalities (4.7) and (4.8) can be determined in polynomial time.

Proposition 10. For any given point $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{2T}$, the most violated inequalities (4.7) and (4.8) can be determined in $O(T^3)$ time, if such violated inequalities exist.

Proposition 11. Consider any $k \in [1, T-1]_{\mathbb{Z}}$ such that $\overline{C} - \underline{C} - kV > 0$, any $m \in [0, k-1]_{\mathbb{Z}}$, and any $\mathcal{S} \subseteq [0, \min\{k-1, L-m-2\}]_{\mathbb{Z}}$. For any $t \in [k+1, T-m-1]_{\mathbb{Z}}$, the inequality

$$\begin{aligned} x_t - x_{t-k} \leq (\underline{C} + (k-m)V - \overline{V})y_{t+m+1} + V\sum_{i=1}^{m} y_{t+i} + \overline{V}y_t - \underline{C}y_{t-k} \\ - \sum_{s \in \mathcal{S}} (\underline{C} + (k-s)V - \overline{V})(y_{t-s} - y_{t-s-1}) \end{aligned} \tag{4.9}$$

is valid and facet-defining for $\mathrm{conv}(\mathcal{P})$. For any $t \in [m+2, T-k]_{\mathbb{Z}}$, the inequality

$$\begin{aligned} x_t - x_{t+k} \leq (\underline{C} + (k-m)V - \overline{V})y_{t-m-1} + V\sum_{i=1}^{m} y_{t-i} + \overline{V}y_t - \underline{C}y_{t+k} \\ - \sum_{s \in \mathcal{S}} (\underline{C} + (k-s)V - \overline{V})(y_{t+s} - y_{t+s+1}) \end{aligned} \tag{4.10}$$

is valid and facet-defining for $\mathrm{conv}(\mathcal{P})$.

In Proposition 11, the number of combinations of $\mathcal{S}$, t, k, and m is exponential in T. Thus, the sizes of the inequality families (4.9) and (4.10) are exponential in T. However, the next proposition states that given any point $(\mathbf{x}, \mathbf{y})$ with non-binary y values, the most violated inequalities (4.9) and (4.10) can be determined in polynomial time.

Proposition 12. For any given point $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{2T}$, the most violated inequalities (4.9) and (4.10) can be determined in $O(T^3)$ time, if such violated inequalities exist.

Chapter 5

Numerical Experiments

5.1 Computational Experiments

We conduct a computational study to evaluate the effectiveness of our strong valid inequalities in tightening the proposed compact MILP formulation for the UC problem. In Section 5.1.1, we describe the problem instances that we use in this computational study. In Section 5.1.2, we present the computational results.

All of the computational experiments are performed on a computer node with Intel(R) Xeon(R) CPU E5-2699 v3 at 2.30GHz and 16 cores. The addressable memory is 32GB. IBM ILOG CPLEX 22.1 is used as the MILP solver to run all of the experiments. The MILP solver is called through its Python application programming interface under the default settings. Note that the performance of a mathematical programming formulation is affected by the inherent random component of the heuristic process used in solvers (Tejada-Arango et al. 2020). Thus, to accurately evaluate the effectiveness of our strong valid inequalities, "traditional branch-and-cut" is chosen to be the search strategy.

5.1.1 Test Instances

We conduct three computational experiments. These experiments are based on a network-constrained UC problem. Recall that in Sections 3 and 4, the superscript g was omitted when we focused on deriving strong valid inequalities for the polytope $\text{conv}(\mathcal{P})$ that consists of a single generator. In the test instances of these three experiments, we reinstate the superscript g in the strong valid inequalities that are used to tighten the UC formulations. Thus, when a strong valid inequality is added to a UC formulation in these three experiments, it will be added to all of the generators at the same time. In all three experiments, the non-decreasing convex piecewise cost function is obtained by approximating the given quadratic cost function $a^g x^2 + b^g x$. We apply the method developed by Frangioni, Gentile, and Lacalandra 2009 to perform this piecewise linear approximation, using nine line segments with the x-coordinates of the breakpoints spread evenly between the lower bound $\underline{C}$ and the upper bound $\overline{C}$.

In the first experiment, we use the data obtained from James Ostrowski, Anjos, and Vannelli 2012 and Pan and Guan 2016. Because of the absence of transmission flow data in this data set, the transmission constraint (2.1l) is not considered in this experiment. The removal of the transmission constraint does not have a major impact on our computational study because we focus primarily on evaluating the effectiveness of the strong valid inequalities in tightening the compact formulation.

Table 5.1: Generator Data (James Ostrowski, Anjos, and Vannelli 2012; Pan and Guan 2016)

Generator Type	$\underline{C}^g$ (MW)	$\overline{C}^g$ (MW)	L^g (h)	ℓ^g (h)	V^g (MW/h)	$\overline{V}^g$ (MW/h)	ϕ^g (\$/h)	ψ^g (\$/h)	a^g (\$/MW^{2}h)	b^g (\$/MWh)	c^g (\$/h)
1	150	455	8	8	91	180	2000	2000	0.00048	16.19	1000
2	150	455	8	8	91	180	2000	2000	0.00031	17.26	970
3	20	130	5	5	26	35	500	500	0.00200	16.6	700
4	20	130	5	5	26	35	500	500	0.00211	16.5	680
5	25	162	6	6	32.4	40	700	700	0.00398	19.7	450
6	20	80	3	3	16	28	150	150	0.00712	22.26	370
7	25	85	3	3	17	33	200	200	0.00079	27.74	480
8	10	55	1	1	11	15	60	60	0.00413	25.92	660

The system contains eight types of generators. Table 5.1 contains the data of these eight generator types. The generation cost function for generator g is $a^g x^2 + b^g x$, where the values of a^g and b^g are provided in the 10th and 11th columns, respectively, of the table. The data set comprises 20 test instances, as shown in Table 5.2. For each instance, the operation horizon is set equal to 24 hours, i.e., $T = 24$, and the system reserve factor is set equal to 3% for all periods, i.e., $r_t = 0.03$ for all $t \in [1, T]_{\mathbb{Z}}$. The system load $\sum_{b \in \mathcal{B}} d_t^b$ in each period t is shown in Table 5.3, and it is expressed as a percentage of the total generation capacity $\sum_{g \in \mathcal{G}} \overline{C}^g$.

In this experiment, we compare the following two formulations:

F1:	minimize	objective function (2.1a)
	subject to	constraints (2.1b)–(2.1k), (2.1m).

Table 5.2: Problem Instances (James Ostrowski, Anjos, and Vannelli 2012; Pan and Guan 2016)

Instance	Number of generators								Total no. of units
	Type 1	Type 2	Type 3	Type 4	Type 5	Type 6	Type 7	Type 8	
1	12	11	0	0	1	4	0	0	28
2	13	15	2	0	4	0	0	1	35
3	15	13	2	6	3	1	1	3	44
4	15	11	0	1	4	5	6	3	45
5	15	13	3	7	5	3	2	1	49
6	10	10	2	5	7	5	6	5	50
7	17	16	1	3	1	7	2	4	51
8	17	10	6	5	2	1	3	7	51
9	12	17	4	7	5	2	0	5	52
10	13	12	5	7	2	5	4	6	54
11	46	45	8	0	5	0	12	16	132
12	40	54	14	8	3	15	9	13	156
13	50	41	19	11	4	4	12	15	156
14	51	58	17	19	16	1	2	1	165
15	43	46	17	15	13	15	6	12	167
16	50	59	8	15	1	18	4	17	172
17	53	50	17	15	16	5	14	12	182
18	45	57	19	7	19	19	5	11	182
19	58	50	15	7	16	18	7	12	183
20	55	48	18	5	18	17	15	11	187

F1-X: minimize objective function (2.1a)
subject to constraints (2.1b)–(2.1k), (2.1m);
constraints (3.3)–(3.8);
user cuts (4.1)–(4.10).

Formulation F1 is the original formulation of Problem 1 with the transmission constraint (2.1l) removed. In F1-X, the strong valid inequality families (3.3)–

Table 5.3: System Load—Percentage of Total Generation Capacity (James Ostrowski, Anjos, and Vannelli 2012; Pan and Guan 2016)

Period	1	2	3	4	5	6	7	8	9	10	11	12
System Load	71%	65%	62%	60%	58%	58%	60%	64%	73%	80%	82%	83%
Period	13	14	15	16	17	18	19	20	21	22	23	24
System Load	82%	80%	79%	79%	83%	91%	90%	88%	85%	84%	79%	74%

(3.8) obtained from the two-period UC polytope are added to the formulation as constraints, and the multi-period strong valid inequality families (4.1)–(4.6) derived in Section 4 are added to the user cut pool of the CPLEX optimizer and are applied at any stage of the optimization. Note that each of the inequality families (4.1)–(4.10) contains a large number of inequalities. Thus, for each of these inequality families, only some of the inequalities are added to F1-X as user cuts. Specifically, for each of the inequality families (4.1)–(4.10), $\mathcal{S}$ is restricted to the empty set and the set that contains all of the elements in its range, and the other parameters such as t, k, and m are allowed to take any values in their respective ranges such that the inequality obtained is facet-defining for $\text{conv}(\mathcal{P})$. For example, for inequality family (4.7), we consider each $k \in [1, T-1]_{\mathbb{Z}}$, $\mathcal{S} = \{\emptyset, [0, \min\{k-1, L-1\}]_{\mathbb{Z}}\}$, $m = 0$, and $t \in [k+1, T]_{\mathbb{Z}}$ such that $s \geq \min\{k-1, 1\}$ for all $s \in \mathcal{S}$.

In the second experiment, we use the same data as in the first experiment. We compare the effectiveness of our strong valid inequalities with that of the valid inequalities in Pan and Guan 2016 in tightening Pan and Guan's two-binary UC formulation. To do so, we solve the following three formulations of the network-constrained UC problem:

F2:	minimize	objective function (38a) in Pan and Guan 2016
	subject to	constraints (38b)–(38i), (38k) in Pan and Guan 2016.

F2-X:	minimize	objective function (38a) in Pan and Guan 2016
	subject to	constraints (38b)–(38i), (38k) in Pan and Guan 2016;
		constraints (3.3)–(3.8) in this paper;
		user cuts (4.1)–(4.10) in this paper.

F2-Y:	minimize	objective function (38a) in Pan and Guan 2016
	subject to	constraints (38b)–(38i), (38k) in Pan and Guan 2016;
		constraints (2d)–(2g) in Pan and Guan 2016;
		user cuts (4)–(7), (10)–(13), (24d), (24h)–(24i), (24o)–(24r), (28)–(36) in Pan and Guan 2016.

Formulation F2 is the two-binary UC formulation in Pan and Guan 2016, except that the transmission constraint (38j) has been excluded. In formulation F2-X, the valid inequalities (3.3)–(3.8) are added as constraints, and the valid inequalities (4.1)–(4.10) are added as user cuts in the same way as in the first experiment. In formulation F2-Y, the strong valid inequalities in Pan and Guan 2016 are used the same way as in Pan and Guan's computational study to tighten formulation F2. Specifically, valid inequalities in the two-period convex hull, (2d)–(2g), are added as constraints, and other valid inequalities are added as user cuts. For inequality families that have an exponential size, the $\mathcal{S}$ set is restricted to the empty set and the set that contains all of the elements in its range. The other parameters, such as t, m, and n, are allowed to take any values in their respective ranges such that the inequality obtained is facet-defining for $\text{conv}(\mathcal{P})$.

The third experiment examines a network-constrained UC problem based on the modified IEEE 118-bus system. The system comprises 54 thermal generators, 118 buses, and 186 transmission lines. System data such as $\overline{C}^g$, $\underline{C}^g$, L^g, ℓ^g, a^g, b^g, c^g, etc., as well as the load amount of each load bus, are obtained from `http://motor.ece.iit.edu/data/SCUC_118/`. Each instance has a 24-hour operation horizon, i.e., $T = 24$. The system reserve factor of each time period is set equal to 3%, as in the first experiment. The maximum hourly load of the system is randomly generated from a uniform distribution on $[0.5 \sum_{g \in \mathcal{G}} \overline{C}^g, \sum_{g \in \mathcal{G}} \overline{C}^g]$. The maximum hourly load of each load bus is then obtained by allocating the maximum hourly load of the system to each load bus in proportion to their load amounts. For each load bus, the loads in different time periods are then obtained by following a daily load profile such that the maximum load of the day is equal to the maximum hourly load. This daily load profile is obtained from `https://www.pjm.com/markets-and-operations/data-dictionary`, which was generated based on the average values of the actual hourly electricity demand over 30 days in the western market. Twenty instances with randomly generated loads are created using this method. Each instance is solved using the following formulations:

F3:	minimize	objective function (2.1a)
	subject to	constraints (2.1b)–(2.1m).

F3-X:	minimize	objective function (2.1a)
	subject to	constraints (2.1b)–(2.1m);
		constraints (3.3)–(3.8);
		user cuts (4.1)–(4.10).

Formulations F3 and F3-X resemble formulations F1 and F1-X, respectively, in the first experiment, with the transmission constraint (2.1l) reinstated. In F3-X, the strong valid inequalities (3.3)–(3.8) and (4.1)–(4.10) are added as constraints and user cuts, respectively, in the same way as in the first experiment.

5.1.2 Computational Results

In this subsection, we report the computational results of the three experiments. In these experiments, each test instance is executed once using each of the formulations in the experiment, and the time limit for each execution is set to one hour. Tables 5.4–5.6 summarize the computational results. In these tables, the "IGap" columns report the root node integrality gaps of the different formulations, where IGap is given as $|Z^* - Z_{\mathrm{LP}}|/Z^* \times 100\%$, where Z^* is the best objective function value obtained by solving the formulations in the experiment and Z_{LP} is the optimal objective function value of the LP relaxation of the formulation concerned. This integrality gap measures the tightness of the formulation. To evaluate the effectiveness of the strong valid inequalities in tightening the formulation, we report the percentage reduction in integrality gap in the "Pct. reduction" columns, where

$$\text{Pct. reduction} = \frac{\mathrm{IGap}_{\text{no valid ineq}} - \mathrm{IGap}_{\text{with valid ineq}}}{\mathrm{IGap}_{\text{no valid ineq}}} \times 100\%,$$

$\mathrm{IGap}_{\text{no valid ineq}}$ is the integrality gap of the formulation with no valid inequality added, and $\mathrm{IGap}_{\text{with valid ineq}}$ is the integrality gap of the current formulation. The "CPU time [TGap]" columns report the computational time (in seconds) required to solve the instance to optimality (with a default optimality gap of 0.01%). Instances that could not be solved to optimality within one hour are marked with "**," and the terminating gaps of those instances are reported (enclosed in square brackets). The "# nodes" columns report the number of branch-and-cut nodes explored. The "# user cuts" columns report the number of user cuts added to each formulation.

Table 5.4: Computational Results of the First Experiment

Instance	IGap		Pct. reduction	CPU time [TGap]		# nodes		# user cuts
	F1	F1-X	F1-X	F1	F1-X	F1	F1-X	F1-X
1	0.45%	0.20%	55.7%	666.8	38.6	289339	11563	211
2	0.39%	0.14%	63.8%	** [0.06%]	301.8	1009424	137851	367
3	0.41%	0.08%	80.4%	** [0.03%]	231.2	596629	53267	554
4	0.36%	0.06%	82.3%	** [0.01%]	1335.1	1584979	494770	299
5	0.51%	0.05%	90.0%	** [0.06%]	703.7	481348	149614	525
6	0.61%	0.04%	93.3%	** [0.07%]	190.5	604379	94790	520
7	0.34%	0.07%	78.7%	** [0.05%]	3194.2	463539	903454	445
8	0.55%	0.06%	89.4%	** [0.09%]	662.4	504831	171291	518
9	0.51%	0.06%	89.1%	** [0.09%]	1630.6	456636	425148	556
10	0.64%	0.04%	93.1%	** [0.09%]	2119.2	410349	433936	697
11	0.32%	0.06%	82.6%	** [0.12%]	** [0.04%]	274195	255185	1006
12	0.32%	0.02%	94.0%	** [0.09%]	** [0.02%]	212690	279909	1923
13	0.40%	0.02%	95.2%	** [0.10%]	927.5	172698	59414	1457
14	0.38%	0.03%	90.9%	** [0.12%]	** [0.01%]	88150	123673	2899
15	0.50%	0.02%	96.4%	** [0.17%]	** [0.01%]	114417	150664	1566
16	0.29%	0.02%	94.1%	** [0.11%]	** [0.01%]	240391	234442	2214
17	0.45%	0.02%	96.1%	** [0.13%]	660.0	177862	15758	1208
18	0.42%	0.02%	95.9%	** [0.13%]	2880.9	207181	130559	1533
19	0.37%	0.02%	94.1%	** [0.09%]	1958.5	208972	54786	2312
20	0.43%	0.02%	96.1%	** [0.12%]	920.3	257848	19072	1198

Table 5.5: Computational Results of the Second Experiment

Instance	IGap			Pct. reduction		CPU time [TGap]			# nodes			# user cuts	
	F2	F2-X	F2-Y	F2-X	F2-Y	F2	F2-X	F2-Y	F2	F2-X	F2-Y	F2-X	F2-Y
1	0.45%	0.20%	0.20%	55.7%	55.7%	710.8	26.0	33.8	365652	12926	25641	168	98
2	0.39%	0.14%	0.14%	64.0%	63.9%	** [0.07%]	819.0	316.2	900272	577704	321040	294	126
3	0.38%	0.07%	0.08%	81.2%	80.3%	** [0.02%]	134.2	104.5	666918	57024	44507	494	252
4	0.35%	0.06%	0.05%	82.2%	84.3%	** [0.01%]	222.7	1675.7	760760	260255	1732340	203	130
5	0.47%	0.05%	0.05%	89.3%	90.3%	** [0.05%]	127.1	458.9	516797	65305	309318	448	243
6	0.60%	0.04%	0.05%	93.2%	92.0%	1908.6	219.9	147.1	375616	268065	213464	515	202
7	0.34%	0.07%	0.08%	79.4%	77.5%	** [0.04%]	** [0.01%]	417.0	618447	3089892	357940	291	132
8	0.52%	0.06%	0.06%	88.8%	88.1%	** [0.05%]	419.0	238.9	526007	240221	147181	407	208
9	0.48%	0.06%	0.05%	88.3%	89.0%	** [0.05%]	930.8	1034.6	434524	440662	495795	535	261
10	0.63%	0.04%	0.05%	93.0%	92.4%	** [0.08%]	1252.6	** [0.02%]	310975	455369	1538372	449	233
11	0.32%	0.05%	0.06%	84.0%	82.5%	** [0.12%]	** [0.03%]	** [0.03%]	180807	256996	570824	806	353
12	0.31%	0.02%	0.02%	93.7%	92.1%	** [0.08%]	** [0.01%]	1739.3	108631	285196	276194	1484	815
13	0.36%	0.02%	0.02%	94.7%	94.5%	** [0.09%]	733.4	887.2	84351	44990	92311	1420	810
14	0.34%	0.02%	0.02%	94.6%	94.1%	** [0.11%]	** [0.01%]	** [0.01%]	62170	179443	412241	2285	1570
15	0.47%	0.02%	0.02%	96.1%	96.4%	** [0.13%]	539.7	180.6	50839	15365	6643	1410	919
16	0.29%	0.02%	0.02%	94.5%	92.0%	** [0.10%]	2790.1	** [0.01%]	142997	191482	249999	1690	762
17	0.41%	0.02%	0.02%	95.6%	96.2%	** [0.10%]	444.3	180.4	70436	14938	7444	1294	757
18	0.40%	0.02%	0.01%	95.7%	96.7%	** [0.10%]	1686.3	239.2	41514	148584	24456	1515	1067
19	0.34%	0.02%	0.02%	93.2%	94.4%	** [0.09%]	1685.8	242.5	93893	136577	19084	1703	1050
20	0.41%	0.02%	0.01%	95.7%	96.3%	** [0.12%]	293.2	245.7	62970	19072	10870	1299	970

Table 5.6: Computational Results of the Third Experiment

Instance	IGap		Pct. reduction	CPU time [TGap]		# nodes		# user cuts
	F3	F3-X	F3-X	F3	F3-X	F3	F3-X	F3-X
1	0.85%	0.36%	57.4%	** [0.15%]	571.7	109080	14069	316
2	1.03%	0.39%	62.2%	** [0.06%]	2324.7	97424	84034	474
3	0.76%	0.39%	49.0%	** [0.02%]	2251.6	158482	31631	445
4	0.93%	0.52%	44.1%	** [0.11%]	2803.9	131942	43108	544
5	0.86%	0.33%	61.9%	** [0.01%]	288.5	175810	7220	394
6	0.88%	0.36%	59.7%	** [0.17%]	1653.5	116598	24813	441
7	1.04%	0.35%	66.3%	** [0.27%]	1800.0	122958	43624	545
8	0.92%	0.42%	54.8%	** [0.11%]	1799.3	180499	22310	591
9	0.91%	0.38%	58.8%	** [0.13%]	851.4	250053	19961	464
10	0.80%	0.38%	52.2%	2938.4	850.7	181549	18889	449
11	0.90%	0.40%	55.9%	** [0.08%]	2425.3	196002	59158	463
12	1.17%	0.47%	60.0%	** [0.15%]	** [0.05%]	287079	44675	619
13	0.87%	0.37%	57.5%	** [0.14%]	3351.3	164549	82738	563
14	0.91%	0.43%	52.2%	** [0.17%]	1969.8	235344	37421	486
15	1.02%	0.44%	57.4%	** [0.30%]	** [0.03%]	266093	111169	696
16	0.89%	0.40%	55.6%	** [0.10%]	1666.2	112045	42146	438
17	0.84%	0.40%	53.0%	** [0.13%]	461.9	139242	13578	448
18	0.85%	0.44%	48.4%	** [0.14%]	2433.3	139375	61670	531
19	0.95%	0.39%	59.0%	** [0.15%]	** [0.02%]	169572	133348	742
20	0.89%	0.40%	55.6%	** [0.21%]	3298.6	158536	50471	615

The computational results of the first experiment are presented in Table 5.4. The integrality gaps generated by formulation F1-X are considerably smaller than those generated by formulation F1, particularly for large instances (i.e., instances 11–20, where the total number of generators exceeds 100). This suggests that formulation F1-X is tighter than formulation F1. Using formulation F1, CPLEX is able to solve only one of the 20 test instances to optimality within one hour. In contrast, using formulation F1-X, CPLEX is able to solve 15 instances to optimality within the same time limit. For instances that cannot be solved to optimality using formulation F1-X, the terminating gaps are all within 0.05% and are much smaller than those using formulation F1. Formulation F1-X tends to explore fewer nodes than formulation F1, and the number of user cuts added by F1-X in the solution process is small compared with the total number of constraints in formulations F1 and 1bin-X. These results demonstrate that our proposed strong valid inequalities can significantly tighten the single-binary formulation of the network-constrained UC problem and thus speed up the solution process.

Table 5.5 presents the computational results of the second experiment, in which three two-binary formulations, namely F2, F2-X, and F2-Y, are used to solve the network-constrained UC problem. Using formulation F2, CPLEX solves only two of the 20 instances within the one-hour time limit. Using formulation F2-X, which includes our proposed valid inequalities, CPLEX is able to solve 16 instances to optimality. Using formulation F2-Y, which includes the valid inequalities developed by Pan and Guan 2016, CPLEX is also able to solve 16 instances to optimality. The integrality gaps and CPU time of formulations F2-X and F2-Y are significantly smaller than those of formulation F2, whereas the integrality gaps and CPU time of formulation F2-X are comparable to those of formulation F2-Y. This demonstrates that strong valid inequalities developed for a single-binary formulation can be used for a two-binary formulation and can achieve comparable effectiveness. Comparing the results presented in Table 5.5 with the results presented in Table 5.4, we observe that formulation F2-X has similar performance as formulation F1-X. This shows that a compact single-binary formulation has similar performance as a two-binary formulation when strong valid inequalities are added to these formulations.

Table 5.6 presents the computational results of the third experiment, in which formulations F3 and F3-X are used to solve the network-constrained UC problem based on the modified IEEE 118-bus system. The integrality gaps generated by formulation F3-X are 44.1% to 66.3% smaller than those generated by formulation F3. Using formulation F3, CPLEX is able to solve only one of the 20 instances to optimality within one hour. In contrast, using formulation F3-X, CPLEX is able to solve 17 instances to optimality within the same time limit. Formulation F3-X explores fewer nodes than formulation F3, and the number of user cuts added by F3-X in the solution process is quite small. These results demonstrate the effectiveness of the strong formulation F3-X.

Chapter 6

Conclusions

This paper considers a compact UC formulation with a single type of binary variables. By analyzing the physical constraints of a single generator, we obtain the convex hull description of the two-period UC polytope, which can be used to tighten the original MILP formulation and derive other strong valid inequalities. For the multi-period UC polytope, we derive strong valid inequalities with one and two continuous variables. Conditions under which these valid inequalities are facet-defining for the multi-period UC polytope are provided. Because the number of inequalities in each valid inequality family is very large, efficient separation algorithms are provided to identify the most violated inequalities. The effectiveness of the proposed strong valid inequalities is demonstrated in solving network-constrained UC problems. Computational results show that our valid inequalities can speed up the solution process significantly. Moreover, these strong valid inequalities exhibit effectiveness comparable to two-binary valid inequalities and thus can be used to tighten two/three-binary formulations.

Various intriguing research directions can be pursued following this line of work. First, it would be interesting to investigate the complete convex hull descriptions of the multi-period UC polytopes, such as the three-period polytope, and derive strong valid inequalities with more than two continuous variables to further tighten Problem 1. In addition, the discussion of the UC polytope can be extended to different parameter settings, such as the case where $\overline{V} \geq \underline{C} + V$ or the case where $\overline{V} + V > \overline{C}$. Second, it would be appealing to incorporate different start-up/shut-down trajectories of generators into the physical constraints to accurately represent the operation of units and to conduct a polyhedral study on the obtained UC polytope to derive strong valid inequalities. Third, considering the demand and electricity price fluctuations that often occur in practice when dealing with UC problems, it would be interesting to formulate the corresponding stochastic UC problems to better reflect real-world scenarios. Fourth, given that different types of electrical generators (e.g., pumped storage hydro units) may have different physical constraints in addition to those considered in this paper, it would be interesting to derive strong valid inequalities for the UC problems with these specific generators. We leave these issues for future research.

Chapter 7

References

Ackooij, Wim van et al. (2018). "Large-scale unit commitment under uncertainty: an updated literature survey". In: Annals of Operations Research 271.1, pp. 11–85.

Arroyo, José M and Antonio J Conejo (2000). "Optimal response of a thermal unit to an electricity spot market". In: IEEE Transactions on Power Systems 15.3, pp. 1098–1104.

Atakan, Semih, Guglielmo Lulli, and Suvrajeet Sen (2018). "A state transition MIP formulation for the unit commitment problem". In: IEEE Transactions on Power Systems 33.1, pp. 736–748.

Baldick, Ross (1995). "The generalized unit commitment problem". In: IEEE Transactions on Power Systems 10.1, pp. 465–475.

Batut, J and A Renaud (1992). "Daily generation scheduling optimization with transmission constraints: A new class of algorithms". In: IEEE Transactions on Power Systems 7.3, pp. 982–989.

Bendotti, Pascale, Pierre Fouilhoux, and Cécile Rottner (2018). "The min-up/min-down unit commitment polytope". In: Journal of Combinatorial Optimization 36.3, pp. 1024–1058.

— (2019). "On the complexity of the unit commitment problem". In: Annals of Operations Research 274.1–2, pp. 119–130.

Carrión, Miguel and José M Arroyo (2006). "A computationally efficient mixed-integer linear formulation for the thermal unit commitment problem". In: IEEE Transactions on Power Systems 21.3, pp. 1371–1378.

Chang, Gary W, Mohamed Aganagic, et al. (2001). "Experiences with mixed integer linear programming based approaches on short-term hydro scheduling". In: IEEE Transactions on Power Systems 16.4, pp. 743–749.

Chang, Gary W, Y D Tsai, et al. (2004). "A practical mixed integer linear programming based approach for unit commitment". In: Proceedings of the IEEE Power Engineering Society General Meeting, 2004. IEEE, pp. 221–225.

Cohen, Arthur I and Miki Yoshimura (1983). "A branch-and-bound algorithm for unit commitment". In: IEEE Transactions on Power Apparatus and Systems PAS-102.2, pp. 444–451.

Damcı-Kurt, Pelin et al. (2016). "A polyhedral study of production ramping". In: Mathematical Programming 158.1–2, pp. 175–205.

Frangioni, Antonio and Claudio Gentile (2006a). "Perspective cuts for a class of convex 0–1 mixed integer programs". In: Mathematical Programming 106.2, pp. 225–236.

— (2006b). "Solving nonlinear single-unit commitment problems with ramping constraints". In: Operations Research 54.4, pp. 767–775.

Frangioni, Antonio, Claudio Gentile, and Fabrizio Lacalandra (2009). "Tighter approximated MILP formulations for unit commitment problems". In: IEEE Transactions on Power Systems 24.1, pp. 105–113.

Garver, Len L (1962). "Power generation scheduling by integer programming—Development of theory". In: Transactions of the American Institute of Electrical Engineers. Part III: Power Apparatus and Systems 81.3, pp. 730–734.

Guan, Yongpei, Kai Pan, and Kezhuo Zhou (2018). "Polynomial time algorithms and extended formulations for unit commitment problems". In: IISE transactions 50.8, pp. 735–751.

Hedman, Kory W, Richard P O'Neill, and Shmuel S Oren (2009). "Analyzing valid inequalities of the generation unit commitment problem". In: 2009 IEEE/PES Power Systems Conference and Exposition. IEEE, pp. 1–6.

Kazarlis, Spyros A, A G Bakirtzis, and Vassilios Petridis (1996). "A genetic algorithm solution to the unit commitment problem". In: IEEE Transactions on Power Systems 11.1, pp. 83–92.

Knueven, Ben, Jim Ostrowski, and Jianhui Wang (2018). "The ramping polytope and cut generation for the unit commitment problem". In: INFORMS Journal on Computing 30.4, pp. 739–749.

Knueven, Bernard, James Ostrowski, and Jean-Paul Watson (2020a). "A novel matching formulation for startup costs in unit commitment". In: Mathematical Programming Computation 12.2, pp. 225–248.

— (2020b). "On mixed-integer programming formulations for the unit commitment problem". In: INFORMS Journal on Computing 32.4, pp. 857–876.

Lee, Jon, Janny Leung, and François Margot (2004). "Min-up/min-down polytopes". In: Discrete Optimization 1.1, pp. 77–85.

Li, Chao, Muhong Zhang, and Kory Hedman (2021). "Extreme ray feasibility cuts for unit commitment with uncertainty". In: INFORMS Journal on Computing 33.3, pp. 1037–1055.

Li, Tao and Mohammad Shahidehpour (2005). "Price-based unit commitment: A case of Lagrangian relaxation versus mixed integer programming". In: IEEE Transactions on Power Systems 20.4, pp. 2015–2025.

Lu, Bo and Mohammad Shahidehpour (2005). "Unit commitment with flexible generating units". In: IEEE Transactions on Power Systems 20.2, pp. 1022–1034.

Ma, Haiti and S M Shahidehpour (1999). "Unit commitment with transmission security and voltage constraints". In: IEEE Transactions on Power Systems 14.2, pp. 757–764.

Morales-España, Germán, Claudio Gentile, and Andres Ramos (2015). "Tight MIP formulations of the power-based unit commitment problem". In: OR Spectrum 37.4, pp. 929–950.

Morales-España, Germán, Jesus M Latorre, and Andres Ramos (2013). "Tight and compact MILP formulation for the thermal unit commitment problem". In: IEEE Transactions on Power Systems 28.4, pp. 4897–4908.

Muckstadt, John A and Sherri A Koenig (1977). "An application of Lagrangian relaxation to scheduling in power-generation systems". In: Operations Research 25.3, pp. 387–403.

Ostrowski, James, Miguel F Anjos, and Anthony Vannelli (2012). “Tight mixed integer linear programming formulations for the unit commitment problem”. In: IEEE Transactions on Power Systems 27.1, pp. 39–46.

Ouyang, Z and S M Shahidehpour (1991). “An intelligent dynamic programming for unit commitment application”. In: IEEE Transactions on Power Systems 6.3, pp. 1203–1209.

Padhy, Narayana Prasad (2004). “Unit commitment—A bibliographical survey”. In: IEEE Transactions on Power Systems 19.2, pp. 1196–1205.

Pan, Kai and Yongpei Guan (2016). “A polyhedral study of the integrated minimum-up/-down time and ramping polytope”. In: arXiv preprint arXiv:1604.02184.

Pan, Kai, Yongpei Guan, et al. (2016). “Strengthened MILP formulation for certain gas turbine unit commitment problems”. In: IEEE Transactions on Power Systems 31.2, pp. 1440–1448.

Pan, Kai, Ming Zhao, et al. (2022). “A Polyhedral Study on Fuel-Constrained Unit Commitment”. In: INFORMS Journal on Computing 34.6, pp. 3309–3324.

Qu, Ming et al. (2022). “Convex Hull Model for a Single-Unit Commitment Problem with Pumped Hydro Storage Unit”. In: IEEE Transactions on Power Systems forthcoming.

Rajan, Deepak and Samer Takriti (2005). Minimum up/down polytopes of the unit commitment problem with start-up costs. Technical Report, IBM, Yorktown Heights, NY.

Sen, Subir and D P Kothari (1998). “Optimal thermal generating unit commitment: a review”. In: International Journal of Electrical Power & Energy Systems 20.7, pp. 443–451.

Shahidehpour, Mohammad, Hatim Yamin, and Zuyi Li (2002). Market Operations in Electric Power Systems: Forecasting, Scheduling, and Risk Management. John Wiley & Sons, New York.

Snyder, Walter L, H David Powell, and John C Rayburn (1987). “Dynamic programming approach to unit commitment”. In: IEEE Transactions on Power Systems 2.2, pp. 339–348.

Takriti, Samer and John R Birge (2000). “Using integer programming to refine Lagrangian-based unit commitment solutions”. In: IEEE Transactions on Power Systems 15.1, pp. 151–156.

Tejada-Arango, Diego A et al. (2020). “Which unit-commitment formulation is best? A comparison framework”. In: IEEE Transactions on Power Systems 35.4, pp. 2926–2936.

US EIA (2022). Daily U.S. electricity generation from natural gas hit a record in mid-July. Accessed April 23, 2023, `https://www.eia.gov/todayinenergy/detail.php?id=53559`.

Wang, Chunyan and S M Shahidehpour (1993). “Effects of ramp-rate limits on unit commitment and economic dispatch”. In: IEEE Transactions on Power Systems 8.3, pp. 1341–1350.

Wang, S J et al. (1995). “Short-term generation scheduling with transmission and environmental constraints using an augmented Lagrangian relaxation”. In: IEEE Transactions on Power Systems 10.3, pp. 1294–1301.

Wu, Lei (2011). “A tighter piecewise linear approximation of quadratic cost curves for unit commitment problems”. In: IEEE Transactions on Power Systems 26.4, pp. 2581–2583.

Xavier, Álinson S, Feng Qiu, and Shabbir Ahmed (2021). "Learning to solve large-scale security-constrained unit commitment problems". In: INFORMS Journal on Computing 33.2, pp. 739–756.

Zheng, Qipeng P, Jianhui Wang, and Andrew L Liu (2015). "Stochastic optimization for unit commitment—A review". In: IEEE Transactions on Power Systems 30.4, pp. 1913–1924.

Chapter 8

Appendices

8.1 Proof of Lemma 1

(i) Consider any $(\mathbf{x}, \mathbf{y}) \in \mathcal{P}$ and any $t \in [2, T]_{\mathbb{Z}}$ such that $y_t = 0$. Suppose, to the contrary, that there exists $j \in [0, \min\{t-2, L-1\}]_{\mathbb{Z}}$ such that $y_{t-j} - y_{t-j-1} = 1$. Then, $t - j \in [2, T]_{\mathbb{Z}}$. Thus, by (2.2a), $y_k = 1$ for all $k \in [t-j, \min\{T, t-j+L-1\}]_{\mathbb{Z}}$. It is easy to check that $t \in [t-j, \min\{T, t-j+L-1\}]_{\mathbb{Z}}$. Hence, $y_t = 1$, which is a contradiction. Therefore, $y_{t-j} - y_{t-j-1} \leq 0$ for all $j \in [0, \min\{t-2, L-1\}]_{\mathbb{Z}}$.

(ii) Consider any $(\mathbf{x}, \mathbf{y}) \in \mathcal{P}$ and any $t \in [2, T]_{\mathbb{Z}}$ such that $y_t = 1$. Suppose, to the contrary, that there exist $j_1, j_2 \in [0, \{t-2, L\}]_{\mathbb{Z}}$ such that $j_1 < j_2$ and $y_{t-j_1} - y_{t-j_1-1} = y_{t-j_2} - y_{t-j_2-1} = 1$. Because $t - j_2 \in [2, T]_{\mathbb{Z}}$, and $y_{t-j_2} - y_{t-j_2-1} = 1$, by (2.2a), $y_k = 1$ for all $k \in [t-j_2, \min\{T, t-j_2+L-1\}]_{\mathbb{Z}}$. Note that $t - j_1 - 1 \geq t - j_2$, $t - j_1 - 1 \leq T$, and $t - j_1 - 1 \leq t - 1 \leq t - j_2 + L - 1$. Thus, $t - j_1 - 1 \in [t-j_2, \min\{T, t-j_2+L-1\}]_{\mathbb{Z}}$. Hence, $y_{t-j_1-1} = 1$, which contradicts that $y_{t-j_1} - y_{t-j_1-1} = 1$. Therefore,

there exists at most one $j \in [0, \min\{t-2, L\}]_{\mathbb{Z}}$ such that $y_{t-j} - y_{t-j-1} = 1$.
□

8.2 Proof of Lemma 2

(i) Consider any $(\mathbf{x}, \mathbf{y}) \in \mathcal{P}$ and any $t \in [1, T-1]_{\mathbb{Z}}$ such that $y_t = 0$. Suppose, to the contrary, that there exists $j \in [0, \min\{T-t-1, L-1\}]_{\mathbb{Z}}$ such that $y_{t+j} - y_{t+j+1} = 1$. Then, $y_{t+j} = 1$ and $y_{t+j+1} = 0$. Because $y_t = 0$ and $y_{t+j} = 1$, there exists $p \in [1, j]_{\mathbb{Z}}$ such that $y_{t+p-1} = 0$ and $y_{t+p} = 1$. Because $t+p \in [2, T]_{\mathbb{Z}}$, $y_{t+p-1} = 0$, and $y_{t+p} = 1$, by (2.2a), $y_k = 1$ for all $k \in [t+p, \min\{T, t+p+L-1\}]_{\mathbb{Z}}$. Note that $j \leq L-1$ and $1 \leq p$, which implies that $t+j+1 \leq t+p+L-1$. Note also that $t+j+1 \geq t+p$ and $t+j+1 \leq T$. Thus, $t+j+1 \in [t+p, \min\{T, t+p+L-1\}]_{\mathbb{Z}}$. Hence, $y_{t+j+1} = 1$, which is a contradiction.

(ii) Consider any $(\mathbf{x}, \mathbf{y}) \in \mathcal{P}$ and any $t \in [1, T-1]_{\mathbb{Z}}$ such that $y_t = 1$. Suppose, to the contrary, that there exist $j_1, j_2 \in [0, \min\{T-t-1, L\}]_{\mathbb{Z}}$ such that $j_1 < j_2$ and $y_{t+j_1} - y_{t+j_1+1} = y_{t+j_2} - y_{t+j_2+1} = 1$. Then, $y_{t+j_1} = 1$, $y_{t+j_1+1} = 0$, $y_{t+j_2} = 1$, and $y_{t+j_2+1} = 0$. Because $y_{t+j_1+1} = 0$ and $y_{t+j_2} = 1$, there exists $p \in [j_1+2, j_2]_{\mathbb{Z}}$ such that $y_{t+p-1} = 0$ and $y_{t+p} = 1$. Because $t+p \in [2, T]_{\mathbb{Z}}$, $y_{t+p-1} = 0$, and $y_{t+p} = 1$, by (2.2a), $y_k = 1$ for all $k \in [t+p, \min\{T, t+p+L-1\}]_{\mathbb{Z}}$. Note that $j_2 \leq L$ and $p \geq 2$, which implies that $t+j_2+1 \leq t+p+L-1$. Note also that $t+j_2+1 \geq t+p$ and $t+j_2+1 \leq T$. Thus, $t+j_2+1 \in [t+p, \min\{T, t+p+L-1\}]_{\mathbb{Z}}$. Hence, $y_{t+j_2+1} = 1$, which is a contradiction. Therefore, there exists at most one $j \in [0, \min\{T-t-1, L\}]_{\mathbb{Z}}$ such that $y_{t+j} - y_{t+j+1} = 1$. □

8.3 Proof of Theorem 1

We divide the proof into two parts. For the sake of simplicity, we let $t = 2$ in $\mathcal{P}_2$ and $\mathcal{Q}_2$.

Part I: $\text{conv}(\mathcal{P}_2) \subseteq \mathcal{Q}_2$.

We prove that the linear inequalities (3.2a)–(3.2h) are valid for $\text{conv}(\mathcal{P}_2)$. To do so, it suffices to show that (3.2a)–(3.2h) are valid for $\mathcal{P}_2$. Clearly, inequalities (3.2a) and (3.2b) hold for any element of $\mathcal{P}_2$. In the following, we show that (3.2c)–(3.2h) hold for any element of $\mathcal{P}_2$.

For inequality (3.2c), consider any element (x_1, x_2, y_1, y_2) of $\mathcal{P}_2$. We consider three different cases. Case 1: $y_1 = 0$. In this case, by (3.1b), $x_1 = 0$. Thus, (x_1, x_2, y_1, y_2) satisfies inequality (3.2c). Case 2: $y_1 = 1$ and $y_2 = 0$. In this case, by (3.1b), $x_2 = 0$. Then, by (3.1d), $x_1 \leq \overline{V}$. Thus, (x_1, x_2, y_1, y_2) satisfies inequality (3.2c). Case 3: $y_1 = 1$ and $y_2 = 1$. In this case, by (3.1b), $x_1 \leq \overline{C}$. Thus, (x_1, x_2, y_1, y_2) satisfies inequality (3.2c).

For inequality (3.2d), consider any element (x_1, x_2, y_1, y_2) of $\mathcal{P}_2$. We consider three different cases. Case 1: $y_2 = 0$. In this case, by (3.1b), $x_2 = 0$. Thus, (x_1, x_2, y_1, y_2) satisfies inequality (3.2d). Case 2: $y_2 = 1$ and $y_1 = 0$. In this case, by (3.1b), $x_1 = 0$. Then, by (3.1c), $x_2 \leq \overline{V}$. Thus, (x_1, x_2, y_1, y_2) satisfies inequality (3.2d). Case 3: $y_2 = 1$ and $y_1 = 1$. In this case, by (3.1b), $x_2 \leq \overline{C}$. Thus, (x_1, x_2, y_1, y_2) satisfies inequality (3.2d).

For inequality (3.2e), consider any element (x_1, x_2, y_1, y_2) of $\mathcal{P}_2$. We consider four different cases. Case 1: $y_1 = y_2 = 0$. In this case, by (3.1b), $x_1 = x_2 = 0$. Thus, (x_1, x_2, y_1, y_2) satisfies inequality (3.2e). Case 2: $y_1 = y_2 = 1$. In this case, by (3.1c), $x_2 - x_1 \leq V$. Thus, (x_1, x_2, y_1, y_2) satisfies inequality (3.2e). Case 3: $y_1 = 1$ and $y_2 = 0$. In this case, by (3.1b), $x_2 = 0$. By (3.1a), $\underline{C} \leq x_1$. Thus, (x_1, x_2, y_1, y_2) satisfies inequality (3.2e). Case 4: $y_1 = 0$ and $y_2 = 1$. In this case, by (3.1b), $x_1 = 0$. Then, by (3.1c), $x_2 \leq \overline{V} < \underline{C} + V$. Thus, (x_1, x_2, y_1, y_2) satisfies inequality (3.2e).

For inequality (3.2f), consider any element (x_1, x_2, y_1, y_2) of $\mathcal{P}_2$. We consider four different cases. Case 1: $y_1 = y_2 = 0$. In this case, by (3.1b), $x_1 = x_2 = 0$. Thus, (x_1, x_2, y_1, y_2) satisfies inequality (3.2f). Case 2: $y_1 = y_2 = 1$. In this case, by (3.1c), $x_2 - x_1 \leq V$. Thus, (x_1, x_2, y_1, y_2) satisfies inequality (3.2f). Case 3: $y_1 = 1$ and $y_2 = 0$. In this case, by (3.1b), $x_2 = 0$. By (3.1a), $\underline{C} \leq x_1$, which implies that $-x_1 < V - \overline{V}$ (because $\overline{V} < \underline{C} + V$). Thus, (x_1, x_2, y_1, y_2) satisfies inequality (3.2f). Case 4: $y_1 = 0$ and $y_2 = 1$. In this case, by (3.1b), $x_1 = 0$. Then, by (3.1c), $x_2 \leq \overline{V}$. Thus, (x_1, x_2, y_1, y_2) satisfies inequality (3.2f).

For inequality (3.2g), consider any element (x_1, x_2, y_1, y_2) of $\mathcal{P}_2$. We consider four different cases. Case 1: $y_1 = y_2 = 0$. In this case, by (3.1b), $x_1 = x_2 = 0$. Thus, (x_1, x_2, y_1, y_2) satisfies inequality (3.2g). Case 2: $y_1 = y_2 = 1$. In this case, by (3.1d), $x_1 - x_2 \leq V$. Thus, (x_1, x_2, y_1, y_2) satisfies inequality (3.2g). Case 3: $y_1 = 1$ and $y_2 = 0$. In this case, by (3.1b), $x_2 = 0$. Then, by (3.1d), $x_1 \leq \overline{V} < \underline{C} + V$. Thus, (x_1, x_2, y_1, y_2) satisfies inequality (3.2g). Case 4: $y_1 = 0$ and $y_2 = 1$. In this case, by (3.1b), $x_1 = 0$. By (3.1a), $\underline{C} \leq x_2$. Thus, (x_1, x_2, y_1, y_2) satisfies inequality (3.2g).

For inequality (3.2h), consider any element (x_1, x_2, y_1, y_2) of $\mathcal{P}_2$. We consider four different cases. Case 1: $y_1 = y_2 = 0$. In this case, by (3.1b), $x_1 = x_2 = 0$. Thus, (x_1, x_2, y_1, y_2) satisfies inequality (3.2h). Case 2: $y_1 = y_2 = 1$. In this case, by (3.1d), $x_1 - x_2 \leq V$. Thus, (x_1, x_2, y_1, y_2) satisfies inequality (3.2h). Case 3: $y_1 = 1$ and $y_2 = 0$. In this case, by (3.1b), $x_2 = 0$. Then, by (3.1d), $x_1 \le \overline{V}$. Thus, (x_1, x_2, y_1, y_2) satisfies inequality (3.2h). Case 4: $y_1 = 0$ and $y_2 = 1$. In this case, by (3.1b), $x_1 = 0$. By (3.1a), $\underline{C} \le x_2$, which implies that $-x_2 < V - \overline{V}$ (because $\overline{V} < \underline{C} + V$). Thus, (x_1, x_2, y_1, y_2) satisfies inequality (3.2h).

Part II: $\mathcal{Q}_2 \subseteq \text{conv}(\mathcal{P}_2)$.

Consider any given $(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2) \in \mathcal{Q}_2$. We have

$$\bar{y}_1 \le 1; \tag{8.1}$$

$$\bar{y}_2 \le 1; \tag{8.2}$$

$$\underline{C}\bar{y}_1 \le \bar{x}_1 \le \overline{C}\bar{y}_1; \tag{8.3}$$

$$\underline{C}\bar{y}_2 \le \bar{x}_2 \le \overline{C}\bar{y}_2; \tag{8.4}$$

$$\bar{x}_1 \le \overline{V}\bar{y}_1 + (\overline{C} - \overline{V})\bar{y}_2; \tag{8.5}$$

$$\bar{x}_2 \le (\overline{C} - \overline{V})\bar{y}_1 + \overline{V}\bar{y}_2; \tag{8.6}$$

$$\bar{x}_2 - \bar{x}_1 \le (\underline{C} + V)\bar{y}_2 - \underline{C}\bar{y}_1; \tag{8.7}$$

$$\bar{x}_2 - \bar{x}_1 \le \overline{V}\bar{y}_2 - (\overline{V} - V)\bar{y}_1; \tag{8.8}$$

$$\bar{x}_1 - \bar{x}_2 \le (\underline{C} + V)\bar{y}_1 - \underline{C}\bar{y}_2; \tag{8.9}$$

$$\bar{x}_1 - \bar{x}_2 \le \overline{V}\bar{y}_1 - (\overline{V} - V)\bar{y}_2. \tag{8.10}$$

We prove that $(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2)$ can be expressed as a convex combination of some elements of $\mathcal{P}_2$. Specifically, we prove that there exist real numbers

$\rho_1, \rho_2, \rho_3, \rho_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$ such that

$$(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2) = \lambda_1(\rho_1, \rho_2, 1, 1) + \lambda_2(\rho_3, 0, 1, 0) + \lambda_3(0, \rho_4, 0, 1) + \lambda_4(0, 0, 0, 0), \tag{8.11}$$

$(\rho_1, \rho_2, 1, 1), (\rho_3, 0, 1, 0), (0, \rho_4, 0, 1), (0, 0, 0, 0) \in \mathcal{P}_2$, and $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$. We set

$$\begin{aligned}
\lambda_1 &= \min\{\bar{y}_1, \bar{y}_2\};\\
\lambda_2 &= \bar{y}_1 - \lambda_1;\\
\lambda_3 &= \bar{y}_2 - \lambda_1;\\
\lambda_4 &= 1 - \bar{y}_1 - \bar{y}_2 + \lambda_1.
\end{aligned}$$

Clearly, $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$. By (8.1)–(8.4), we have $0 \leq \bar{y}_1 \leq 1$ and $0 \leq \bar{y}_2 \leq 1$. It is easy to verify that $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$. We consider five different cases.

Case 1: $\bar{y}_1 = 0$. In this case, $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = \bar{y}_2$, and $\lambda_4 = 1 - \bar{y}_2$. We set $\rho_1 = \rho_2 = \rho_3 = \underline{C}$ and

$$\rho_4 = \begin{cases} \bar{x}_2/\bar{y}_2, & \text{if } \bar{y}_2 > 0;\\ \underline{C}, & \text{if } \bar{y}_2 = 0.\end{cases}$$

By (8.3), $\bar{x}_1 = 0$. It is easy to verify that equation (8.11) holds and that $(\rho_1, \rho_2, 1, 1), (\rho_3, 0, 1, 0), (0, 0, 0, 0) \in \mathcal{P}_2$. Therefore, it suffices to show that $(0, \rho_4, 0, 1) \in \mathcal{P}_2$. Clearly, $(0, \rho_4, 0, 1)$ satisfies (3.1d). By (8.4), $\bar{x}_2 \geq \underline{C}\bar{y}_2$, which implies that $\rho_4 \geq \underline{C}$. Thus, $(0, \rho_4, 0, 1)$ satisfies (3.1a). By (8.4),

$\bar{x}_2 \leq \overline{C}\bar{y}_2$, which implies that $\rho_4 \leq \overline{C}$. Thus, $(0, \rho_4, 0, 1)$ satisfies (3.1b). By (8.8), $\bar{x}_2 \leq \overline{V}\bar{y}_2$, which implies that $\rho_4 \leq \overline{V}$. Thus, $(0, \rho_4, 0, 1)$ satisfies (3.1c). Therefore, $(0, \rho_4, 0, 1) \in \mathcal{P}_2$.

Case 2: $\bar{y}_2 = 0$. In this case, $\lambda_1 = 0$, $\lambda_2 = \bar{y}_1$, $\lambda_3 = 0$, and $\lambda_4 = 1 - \bar{y}_1$. We set $\rho_1 = \rho_2 = \rho_4 = \underline{C}$ and

$$\rho_3 = \begin{cases} \bar{x}_1/\bar{y}_1, & \text{if } \bar{y}_1 > 0; \\ \underline{C}, & \text{if } \bar{y}_1 = 0. \end{cases}$$

By (8.4), $\bar{x}_2 = 0$. It is easy to verify that equation (8.11) holds and that $(\rho_1, \rho_2, 1, 1), (0, \rho_4, 0, 1), (0, 0, 0, 0) \in \mathcal{P}_2$. Therefore, it suffices to show that $(\rho_3, 0, 1, 0) \in \mathcal{P}_2$. Clearly, $(\rho_3, 0, 1, 0)$ satisfies (3.1c). By (8.3), $\bar{x}_1 \geq \underline{C}\bar{y}_1$, which implies that $\rho_3 \geq \underline{C}$. Thus, $(\rho_3, 0, 1, 0)$ satisfies (3.1a). By (8.3), $\bar{x}_1 \leq \overline{C}\bar{y}_1$, which implies that $\rho_3 \leq \overline{C}$. Thus, $(\rho_3, 0, 1, 0)$ satisfies (3.1b). By (8.10), $\bar{x}_1 \leq \overline{V}\bar{y}_1$, which implies that $\rho_3 \leq \overline{V}$. Thus, $(\rho_3, 0, 1, 0)$ satisfies (3.1d). Therefore, $(\rho_3, 0, 1, 0) \in \mathcal{P}_2$.

Case 3: $0 < \bar{y}_1 < \bar{y}_2$. In this case, $\lambda_1 = \bar{y}_1$, $\lambda_2 = 0$, $\lambda_3 = \bar{y}_2 - \bar{y}_1$, and $\lambda_4 = 1 - \bar{y}_2$. We set

$$\begin{aligned}
\rho_1 &= \frac{\bar{x}_1}{\bar{y}_1}; \\
\rho_2 &= \frac{1}{\bar{y}_1}[\bar{x}_2 - (\bar{y}_2 - \bar{y}_1)\rho_4]; \\
\rho_3 &= \underline{C}; \\
\rho_4 &= \min\left\{\frac{\bar{x}_2 - \underline{C}\bar{y}_1}{\bar{y}_2 - \bar{y}_1}, \frac{(\bar{x}_2 - \bar{x}_1) + V\bar{y}_1}{\bar{y}_2 - \bar{y}_1}, \overline{V}\right\}.
\end{aligned}$$

By (8.3), $\rho_1 \geq \underline{C}$. Note that $\rho_4 \leq \frac{\bar{x}_2 - \underline{C}\bar{y}_1}{\bar{y}_2 - \bar{y}_1}$, which implies that $\rho_2 \geq \frac{1}{\bar{y}_1}[\bar{x}_2 -$

$(\bar{y}_2 - \bar{y}_1)\frac{\bar{x}_2 - \underline{C}\bar{y}_1}{\bar{y}_2 - \bar{y}_1}] = \underline{C}$. By (8.4), $\bar{x}_2 \geq \underline{C}\bar{y}_2$, which implies that $\frac{\bar{x}_2 - \underline{C}\bar{y}_1}{\bar{y}_2 - \bar{y}_1} \geq \underline{C}$. By (8.9), $\frac{(\bar{x}_2 - \bar{x}_1) + V\bar{y}_1}{\bar{y}_2 - \bar{y}_1} \geq \underline{C}$. Thus, $\min\{\frac{\bar{x}_2 - \underline{C}\bar{y}_1}{\bar{y}_2 - \bar{y}_1}, \frac{(\bar{x}_2 - \bar{x}_1) + V\bar{y}_1}{\bar{y}_2 - \bar{y}_1}, \overline{V}\} \geq \underline{C}$; that is, $\rho_4 \geq \underline{C}$. Hence, $\rho_1, \rho_2, \rho_3, \rho_4 \geq \underline{C}$. It is easy to verify that equation (8.11) holds and that $(\rho_3, 0, 1, 0), (0, 0, 0, 0) \in \mathcal{P}_2$. Therefore, it suffices to show that $(\rho_1, \rho_2, 1, 1), (0, \rho_4, 0, 1) \in \mathcal{P}_2$.

To show that $(\rho_1, \rho_2, 1, 1) \in \mathcal{P}_2$, we first note that $(\rho_1, \rho_2, 1, 1)$ satisfies (3.1a) (because $\rho_1, \rho_2 \geq \underline{C}$). By (8.3), $\rho_1 \leq \overline{C}$. Note that

$$\begin{aligned}
\rho_2 &= \frac{1}{\bar{y}_1}\left[\bar{x}_2 - (\bar{y}_2 - \bar{y}_1)\min\left\{\frac{\bar{x}_2 - \underline{C}\bar{y}_1}{\bar{y}_2 - \bar{y}_1}, \frac{(\bar{x}_2 - \bar{x}_1) + V\bar{y}_1}{\bar{y}_2 - \bar{y}_1}, \overline{V}\right\}\right] \\
&= \frac{1}{\bar{y}_1}\left[\bar{x}_2 - \min\left\{\bar{x}_2 - \underline{C}\bar{y}_1, (\bar{x}_2 - \bar{x}_1) + V\bar{y}_1, (\bar{y}_2 - \bar{y}_1)\overline{V}\right\}\right] \\
&= \frac{1}{\bar{y}_1}\max\left\{\underline{C}\bar{y}_1, \bar{x}_1 - V\bar{y}_1, \bar{x}_2 - (\bar{y}_2 - \bar{y}_1)\overline{V}\right\} \\
&\leq \frac{1}{\bar{y}_1}\max\left\{\overline{C}\bar{y}_1, \bar{x}_1, \bar{x}_2 - (\bar{y}_2 - \bar{y}_1)\overline{V}\right\} \\
&= \max\left\{\overline{C}, \frac{\bar{x}_1}{\bar{y}_1}, \frac{\bar{x}_2 - (\bar{y}_2 - \bar{y}_1)\overline{V}}{\bar{y}_1}\right\} \\
&\leq \overline{C},
\end{aligned}$$

where the last inequality follows from (8.3) and (8.6). Hence, $(\rho_1, \rho_2, 1, 1)$ satisfies (3.1b). Note that

$$\begin{aligned}
\rho_2 - \rho_1 &= \frac{1}{\bar{y}_1}\left[(\bar{x}_2 - \bar{x}_1) - (\bar{y}_2 - \bar{y}_1)\min\left\{\frac{\bar{x}_2 - \underline{C}\bar{y}_1}{\bar{y}_2 - \bar{y}_1}, \frac{(\bar{x}_2 - \bar{x}_1) + V\bar{y}_1}{\bar{y}_2 - \bar{y}_1}, \overline{V}\right\}\right] \\
&= \frac{1}{\bar{y}_1}\left[(\bar{x}_2 - \bar{x}_1) - \min\left\{\bar{x}_2 - \underline{C}\bar{y}_1, (\bar{x}_2 - \bar{x}_1) + V\bar{y}_1, (\bar{y}_2 - \bar{y}_1)\overline{V}\right\}\right] \\
&= \frac{1}{\bar{y}_1}\max\left\{\underline{C}\bar{y}_1 - \bar{x}_1, -V\bar{y}_1, (\bar{x}_2 - \bar{x}_1) - (\bar{y}_2 - \bar{y}_1)\overline{V}\right\} \\
&= \max\left\{\underline{C} - \frac{\bar{x}_1}{\bar{y}_1}, -V, \frac{(\bar{x}_2 - \bar{x}_1) - (\bar{y}_2 - \bar{y}_1)\overline{V}}{\bar{y}_1}\right\}.
\end{aligned}$$

By (8.3), $\underline{C} - \frac{\bar{x}_1}{\bar{y}_1} \le 0$. By (8.8), $\frac{(\bar{x}_2-\bar{x}_1)-(\bar{y}_2-\bar{y}_1)\overline{V}}{\bar{y}_1} \le V$. Thus, $\rho_2 - \rho_1 \le V$. Hence, $(\rho_1, \rho_2, 1, 1)$ satisfies (3.1c). Because

$$\begin{aligned}\rho_1 - \rho_2 &= \frac{1}{\bar{y}_1}\big[-(\bar{x}_2 - \bar{x}_1) + (\bar{y}_2 - \bar{y}_1)\rho_4\big] \\ &\le \frac{1}{\bar{y}_1}\left[-(\bar{x}_2 - \bar{x}_1) + (\bar{y}_2 - \bar{y}_1)\cdot \frac{(\bar{x}_2 - \bar{x}_1) + V\bar{y}_1}{\bar{y}_2 - \bar{y}_1}\right] = V,\end{aligned}$$

$(\rho_1, \rho_2, 1, 1)$ satisfies (3.1d). Therefore, $(\rho_1, \rho_2, 1, 1) \in \mathcal{P}_2$.

To show that $(0, \rho_4, 0, 1) \in \mathcal{P}_2$, we note that $(0, \rho_4, 0, 1)$ satisfies (3.1a) and (3.1d) (because $\rho_4 \ge \underline{C}$). Because $\rho_4 \le \overline{V} \le \overline{C}$, $(0, \rho_4, 0, 1)$ satisfies (3.1b) and (3.1c). Therefore, $(0, \rho_4, 0, 1) \in \mathcal{P}_2$.

Case 4: $0 < \bar{y}_2 < \bar{y}_1$. In this case, $\lambda_1 = \bar{y}_2$, $\lambda_2 = \bar{y}_1 - \bar{y}_2$, $\lambda_3 = 0$, and $\lambda_4 = 1 - \bar{y}_1$. We set

$$\begin{aligned}\rho_1 &= \frac{1}{\bar{y}_2}[\bar{x}_1 - (\bar{y}_1 - \bar{y}_2)\rho_3]; \\ \rho_2 &= \frac{\bar{x}_2}{\bar{y}_2}; \\ \rho_3 &= \min\left\{\frac{\bar{x}_1 - \underline{C}\bar{y}_2}{\bar{y}_1 - \bar{y}_2}, \frac{(\bar{x}_1 - \bar{x}_2) + V\bar{y}_2}{\bar{y}_1 - \bar{y}_2}, \overline{V}\right\}; \\ \rho_4 &= \underline{C}.\end{aligned}$$

Note that $\rho_3 \le \frac{\bar{x}_1 - \underline{C}\bar{y}_2}{\bar{y}_1 - \bar{y}_2}$, which implies that $\rho_1 \ge \frac{1}{\bar{y}_2}[\bar{x}_1 - (\bar{y}_1 - \bar{y}_2)\frac{\bar{x}_1 - \underline{C}\bar{y}_2}{\bar{y}_1 - \bar{y}_2}] = \underline{C}$. By (8.4), $\rho_2 \ge \underline{C}$. By (8.3), $\bar{x}_1 \ge \underline{C}\bar{y}_1$, which implies that $\frac{\bar{x}_1 - \underline{C}\bar{y}_2}{\bar{y}_1 - \bar{y}_2} \ge \underline{C}$. By (8.7), $\frac{(\bar{x}_1 - \bar{x}_2) + V\bar{y}_2}{\bar{y}_1 - \bar{y}_2} \ge \underline{C}$. Thus, $\min\{\frac{\bar{x}_1 - \underline{C}\bar{y}_2}{\bar{y}_1 - \bar{y}_2}, \frac{(\bar{x}_1 - \bar{x}_2) + V\bar{y}_2}{\bar{y}_1 - \bar{y}_2}, \overline{V}\} \ge \underline{C}$; that is, $\rho_3 \ge \underline{C}$. Hence, $\rho_1, \rho_2, \rho_3, \rho_4 \ge \underline{C}$. It is easy to verify that equation (8.11) holds and that $(0, \rho_4, 0, 1), (0, 0, 0, 0) \in \mathcal{P}_2$. Therefore, it suffices to show that $(\rho_1, \rho_2, 1, 1), (\rho_3, 0, 1, 0) \in \mathcal{P}_2$.

To show that $(\rho_1, \rho_2, 1, 1) \in \mathcal{P}_2$, we first note that $(\rho_1, \rho_2, 1, 1)$ satisfies (3.1a) (because $\rho_1, \rho_2 \geq \underline{C}$). Note that

$$\begin{aligned}
\rho_1 &= \frac{1}{\bar{y}_2}\left[\bar{x}_1 - (\bar{y}_1 - \bar{y}_2)\min\left\{\frac{\bar{x}_1 - \underline{C}\bar{y}_2}{\bar{y}_1 - \bar{y}_2}, \frac{(\bar{x}_1 - \bar{x}_2) + V\bar{y}_2}{\bar{y}_1 - \bar{y}_2}, \overline{V}\right\}\right] \\
&= \frac{1}{\bar{y}_2}\left[\bar{x}_1 - \min\left\{\bar{x}_1 - \underline{C}\bar{y}_2, (\bar{x}_1 - \bar{x}_2) + V\bar{y}_2, (\bar{y}_1 - \bar{y}_2)\overline{V}\right\}\right] \\
&= \frac{1}{\bar{y}_2}\max\left\{\underline{C}\bar{y}_2, \bar{x}_2 - V\bar{y}_2, \bar{x}_1 - (\bar{y}_1 - \bar{y}_2)\overline{V}\right\} \\
&\leq \frac{1}{\bar{y}_2}\max\left\{\overline{C}\bar{y}_2, \bar{x}_2, \bar{x}_1 - (\bar{y}_1 - \bar{y}_2)\overline{V}\right\} \\
&= \max\left\{\overline{C}, \frac{\bar{x}_2}{\bar{y}_2}, \frac{\bar{x}_1 - (\bar{y}_1 - \bar{y}_2)\overline{V}}{\bar{y}_2}\right\} \\
&\leq \overline{C},
\end{aligned}$$

where the last inequality follows from (8.4) and (8.5). By (8.4), $\rho_2 \leq \overline{C}$. Hence, $(\rho_1, \rho_2, 1, 1)$ satisfies (3.1b). Because

$$\rho_2 - \rho_1 = \frac{1}{\bar{y}_2}\left[-(\bar{x}_1 - \bar{x}_2) + (\bar{y}_1 - \bar{y}_2)\rho_3\right] \leq \frac{1}{\bar{y}_2}\left[-(\bar{x}_1 - \bar{x}_2) + (\bar{y}_1 - \bar{y}_2)\cdot\frac{(\bar{x}_1 - \bar{x}_2) + V\bar{y}_2}{\bar{y}_1 - \bar{y}_2}\right] = V,$$

$(\rho_1, \rho_2, 1, 1)$ satisfies (3.1c). Note that

$$\begin{aligned}
\rho_1 - \rho_2 &= \frac{1}{\bar{y}_2}\left[(\bar{x}_1 - \bar{x}_2) - (\bar{y}_1 - \bar{y}_2)\min\left\{\frac{\bar{x}_1 - \underline{C}\bar{y}_2}{\bar{y}_1 - \bar{y}_2}, \frac{(\bar{x}_1 - \bar{x}_2) + V\bar{y}_2}{\bar{y}_1 - \bar{y}_2}, \overline{V}\right\}\right] \\
&= \frac{1}{\bar{y}_2}\left[(\bar{x}_1 - \bar{x}_2) - \min\left\{\bar{x}_1 - \underline{C}\bar{y}_2, (\bar{x}_1 - \bar{x}_2) + V\bar{y}_2, (\bar{y}_1 - \bar{y}_2)\overline{V}\right\}\right] \\
&= \frac{1}{\bar{y}_2}\max\left\{\underline{C}\bar{y}_2 - \bar{x}_2, -V\bar{y}_2, (\bar{x}_1 - \bar{x}_2) - (\bar{y}_1 - \bar{y}_2)\overline{V}\right\} \\
&= \max\left\{\underline{C} - \frac{\bar{x}_2}{\bar{y}_2}, -V, \frac{(\bar{x}_1 - \bar{x}_2) - (\bar{y}_1 - \bar{y}_2)\overline{V}}{\bar{y}_2}\right\}.
\end{aligned}$$

By (8.4), $\underline{C} - \frac{\bar{x}_2}{\bar{y}_2} \leq 0$. By (8.10), $\frac{(\bar{x}_1 - \bar{x}_2) - (\bar{y}_1 - \bar{y}_2)\overline{V}}{\bar{y}_2} \leq V$. Thus, $\rho_1 - \rho_2 \leq V$. Hence, $(\rho_1, \rho_2, 1, 1)$ satisfies (3.1d). Therefore, $(\rho_1, \rho_2, 1, 1) \in \mathcal{P}_2$.

To show that $(\rho_3, 0, 1, 0) \in \mathcal{P}_2$, we note that $(\rho_3, 0, 1, 0)$ satisfies (3.1a) and (3.1c) (because $\rho_3 \geq \underline{C}$). Because $\rho_3 \leq \overline{V} \leq \overline{C}$, $(\rho_3, 0, 1, 0)$ satisfies (3.1b) and (3.1d). Therefore, $(\rho_3, 0, 1, 0) \in \mathcal{P}_2$.

Case 5: $0 < \bar{y}_1 = \bar{y}_2$. In this case, $\lambda_1 = \bar{y}_1 = \bar{y}_2$, $\lambda_2 = 0$, $\lambda_3 = 0$, and $\lambda_4 = 1 - \bar{y}_1 = 1 - \bar{y}_2$. We set

$$\rho_1 = \frac{\bar{x}_1}{\bar{y}_1};\ \rho_2 = \frac{\bar{x}_2}{\bar{y}_2};\ \rho_3 = \underline{C};\ \rho_4 = \underline{C}.$$

Clearly, $\rho_1, \rho_2, \rho_3, \rho_4 \geq 0$. It is easy to verify that equation (8.11) holds and that $(\rho_3, 0, 1, 0), (0, \rho_4, 0, 1), (0, 0, 0, 0) \in \mathcal{P}_2$. Therefore, it suffices to show that $(\rho_1, \rho_2, 1, 1) \in \mathcal{P}_2$. By (8.3) and (8.4), $\rho_1, \rho_2 \geq \underline{C}$ and $\rho_1, \rho_2 \leq \overline{C}$. Thus, $(\rho_1, \rho_2, 1, 1)$ satisfies (3.1a) and (3.1b). By (8.7), $\bar{x}_2 - \bar{x}_1 \leq V\bar{y}_1$, which implies that $\rho_2 - \rho_1 \leq V$. Thus, $(\rho_1, \rho_2, 1, 1)$ satisfies (3.1c). By (8.9), $\bar{x}_1 - \bar{x}_2 \leq V\bar{y}_1$, which implies that $\rho_1 - \rho_2 \leq V$. Thus, $(\rho_1, \rho_2, 1, 1)$ satisfies (3.1d). Therefore, $(\rho_1, \rho_2, 1, 1) \in \mathcal{P}_2$.

Combining Cases 1–5, we conclude that there exist $\rho_1, \rho_2, \rho_3, \rho_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$ that satisfy equation (8.11). Hence, $\mathcal{Q}_2 \subseteq \text{conv}(\mathcal{P}_2)$. □

8.4 Proof of Proposition 1

For notational convenience, we define $s_{\max} = \max\{s : s \in \mathcal{S}\}$ if $\mathcal{S} \neq \emptyset$, and $s_{\max} = -1$ if $\mathcal{S} = \emptyset$. To prove that linear inequalities (4.1) and (4.2) are valid for $\text{conv}(\mathcal{P})$, it suffices to show that they are valid for $\mathcal{P}$. Consider any element $(\mathbf{x}, \mathbf{y})$ of $\mathcal{P}$. We show that $(\mathbf{x}, \mathbf{y})$ satisfies (4.1) and (4.2).

We first show that $(\mathbf{x}, \mathbf{y})$ satisfies (4.1). Consider any $t \in [s_{\max} + 2, T]_{\mathbb{Z}}$ (i.e., $t \in [1, T]_{\mathbb{Z}}$ such that $t \geq s + 2$ for all $s \in \mathcal{S}$). We divide the analysis into three cases.

Case 1: $y_t = 0$. By Lemma 1(i), $y_{t-j} - y_{t-j-1} \leq 0$ for all $j \in [0, \min\{t - 2, L - 1\}]_{\mathbb{Z}}$. Because $\mathcal{S} \subseteq [0, L - 1]_{\mathbb{Z}}$ and $s_{\max} \leq t - 2$, we have $\mathcal{S} \subseteq [0, \min\{t - 2, L - 1\}]_{\mathbb{Z}}$. Thus, $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, \lfloor(\overline{C} - \overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Thus, the right hand side of inequality (4.1) is nonnegative. Because $y_t = 0$, by (2.2d), $x_t = 0$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.1).

Case 2: $y_t = 1$ and $y_{t-s'} - y_{t-s'-1} = 1$ for some $s' \in \mathcal{S}$. By Lemma 1(ii), there exists at most one $j \in [0, \min\{t - 2, L\}]_{\mathbb{Z}}$ such that $y_{t-j} - y_{t-j-1} = 1$. Because $\mathcal{S} \subseteq [0, L]_{\mathbb{Z}}$ and $s_{\max} \leq t - 2$, we have $\mathcal{S} \subseteq [0, \min\{t - 2, L\}]_{\mathbb{Z}}$. This implies that $y_{t-s} - y_{t-s-1} \leq 0$, for all for all $s \in \mathcal{S} \setminus \{s'\}$. For any $s \in \mathcal{S}$, because $s \leq \lfloor(\overline{C} - \overline{V})/V\rfloor$, we have $\overline{C} - \overline{V} - sV \geq 0$. Thus, for any $s \in \mathcal{S} \setminus \{s'\}$, $(\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1})$ is either zero or negative. Hence, $\sum_{s \in \mathcal{S}}(\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \leq \overline{C} - \overline{V} - s'V$. Thus, the right hand side of inequality (4.1) is at least $s'V + \overline{V}$. By (2.2e), $\sum_{\tau=t-s'}^{t}(x_\tau - x_{\tau-1}) \leq \sum_{\tau=t-s'}^{t} Vy_{\tau-1} + \sum_{\tau=t-s_j}^{t} \overline{V}(1 - y_{\tau-1})$, which implies that $x_t - x_{t-s'-1} \leq s'V + \overline{V}$. Because $y_{t-s'} - y_{t-s'-1} = 1$, we have $y_{t-s'-1} = 0$. By (2.2d),

$x_{t-s'-1} = 0$. Hence, $x_t \leq s'V + \overline{V}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.1).

Case 3: $y_t = 1$ and $y_{t-s} - y_{t-s-1} \neq 1$ for all $s \in \mathcal{S}$. In this case, $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S}$. For any $s \in \mathcal{S}$, because $s \leq \lfloor (\overline{C} - \overline{V})/V \rfloor$, we have $\overline{C} - \overline{V} - sV \geq 0$. Thus, the right hand side of inequality (4.1) is at least $\overline{C}$. By (2.2d), $x_t \leq \overline{C}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.1).

Next, we show that $(\mathbf{x}, \mathbf{y})$ satisfies (4.2). Consider any $t \in [1, T - s_{\max} - 1]_{\mathbb{Z}}$ (i.e., $t \in [1, T]_{\mathbb{Z}}$ such that $t \leq T - s - 1$ for all $s \in \mathcal{S}$). We divide the analysis into three cases.

Case 1: $y_t = 0$. In this case, by Lemma 2(i), $y_{t+j} - y_{t+j+1} \leq 0$ for all $j \in [0, \min\{T - t - 1, L - 1\}]_{\mathbb{Z}}$. Because $\mathcal{S} \subseteq [0, L - 1]_{\mathbb{Z}}$ and $s_{\max} \leq T - t - 1$, we have $\mathcal{S} \subseteq [0, \min\{T - t - 1, L - 1\}]_{\mathbb{Z}}$. Thus, $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, \lfloor (\overline{C} - \overline{V})/V \rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Thus, the right hand side of inequality (4.2) is nonnegative. Because $y_t = 0$, by (2.2d), $x_t = 0$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.2).

Case 2: $y_t = 1$ and $y_{t+s'} - y_{t+s'+1} = 1$ for some $s' \in \mathcal{S}$. By Lemma 2(ii), there exists at most one $j \in [0, \min\{T - t - 1, L\}]_{\mathbb{Z}}$ such that $y_{t+j} - y_{t+j+1} = 1$. Because $\mathcal{S} \subseteq [0, L]_{\mathbb{Z}}$ and $s_{\max} \leq T - t - 1$, we have $\mathcal{S} \in [0, \min\{T - t - 1, L\}]_{\mathbb{Z}}$. This implies that $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in [0, s_{\max}]_{\mathbb{Z}} \setminus \{s'\}$. For any $s \in \mathcal{S}$, because $s \leq \lfloor (\overline{C} - \overline{V})/V \rfloor$, we have $\overline{C} - \overline{V} - sV \geq 0$. Thus, for any $s \in \mathcal{S} \setminus \{s'\}$, $(\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1})$ is either zero or negative. Hence, $\sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \leq \overline{C} - \overline{V} - s'V$. Thus, the right hand side of (4.2) is at least $s'V + \overline{V}$. Because $y_{t+s'} - y_{t+s'+1} = 1$, we have $y_{t+s'} = 1$ and $y_{t+s'+1} = 0$. Because $y_t = 1$, $y_{t+s'} = 1$ and $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in [0, s' - 1]_{\mathbb{Z}}$, we have $y_{t+1} = y_{t+2} = \cdots = y_{t+s'} = 1$. By (2.2f),

$\sum_{\tau=t+1}^{t+s'+1}(x_{\tau-1} - x_\tau) \leq \sum_{\tau=t+1}^{t+s'+1} V y_\tau + \sum_{\tau=t+1}^{t+s'+1} \overline{V}(1 - y_\tau)$, which implies that $x_t - x_{t+s'+1} \leq s'V + \overline{V}$. Because $y_{t+s'+1} = 0$, by (2.2d), $x_{t+s'+1} = 0$. Hence, $x_t \leq s'V + \overline{V}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.2).

Case 3: $y_t = 1$ and $y_{t+s} - y_{t+s+1} \neq 1$ for all $s \in \mathcal{S}$. In this case, $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in \mathcal{S}$. For any $s \in \mathcal{S}$, because $s \leq \lfloor (\overline{C} - \overline{V})/V \rfloor$, we have $\overline{C} - \overline{V} - sV \geq 0$. Thus, the right hand side of inequality (4.2) is at least $\overline{C}$. By (2.2d), $x_t \leq \overline{C}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.2).

To prove that inequalities (4.1) and (4.2) are facet-defining for $\text{conv}(\mathcal{P})$, it suffices to show that for each of these two inequalities, there exist $2T$ affinely independent points in $\text{conv}(\mathcal{P})$ that satisfy the inequality at equality. Let $\epsilon = \overline{V} - \underline{C} > 0$.

We first show that inequality (4.1) is facet-defining for $\text{conv}(\mathcal{P})$ by creating $2T$ affinely independent points in $\text{conv}(\mathcal{P})$ that satisfy (4.1) at equality. Because $\mathbf{0} \in \text{conv}(\mathcal{P})$ and $\mathbf{0}$ satisfies (4.1) at equality, it suffices to create the remaining $2T - 1$ nonzero linearly independent points. We denote these $2T - 1$ points as $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ for $r \in [1, T]_{\mathbb{Z}} \setminus \{t\}$, and $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ for $r \in [1, T]_{\mathbb{Z}}$, and we denote the qth component of $\bar{\mathbf{x}}^r$, $\bar{\mathbf{y}}^r$, $\hat{\mathbf{x}}^r$, and $\hat{\mathbf{y}}^r$ as $\bar{x}_q^r$, $\bar{y}_q^r$, $\bar{y}_q^r$, $\hat{x}_q^r$, and $\hat{y}_q^r$, respectively. We divide these $2T - 1$ points into the following five groups:

(A1) For each $r \in [1, T]_{\mathbb{Z}} \setminus \{t\}$, we create a point $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ as follows:

$$\bar{x}_q^r = \begin{cases} \overline{C}, & \text{for } q \in [1, T]_{\mathbb{Z}} \setminus \{r\}; \\ \overline{C} - \epsilon, & \text{for } q = r; \end{cases}$$

and $\bar{y}_q^r = 1$ for all $q \in [1, T]_{\mathbb{Z}}$. It is easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (2.2a)–(2.2f). Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. It is also easy to verify that

$(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (4.1) at equality.

(A2) For each $r \in [1, t - s_{\max} - 2]_{\mathbb{Z}}$, we create a point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as follows:

$$\hat{x}_q^r = \begin{cases} \underline{C}, & \text{for } q = [1, r]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [r+1, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^r = \begin{cases} 1, & \text{for } q \in [1, r]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [r+1, T]_{\mathbb{Z}}. \end{cases}$$

It is easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (2.2a)–(2.2f). Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \operatorname{conv}(\mathcal{P})$. Note that $\hat{x}_t^r = \hat{y}_t^r = 0$. Note also that $t - s - 1 \geq r + 1$ for all $s \in \mathcal{S}$. Thus, $\hat{y}_{t-s}^r - \hat{y}_{t-s-1}^r = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.1) at equality.

(A3) For each $r \in [t - s_{\max} - 1, t - 1]_{\mathbb{Z}}$, we create a point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as follows: If $t - r - 1 \notin \mathcal{S}$, then

$$\hat{x}_q^r = \begin{cases} \underline{C}, & \text{for } q \in [1, r]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [r+1, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^r = \begin{cases} 1, & \text{for } q \in [1, r]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [r+1, T]_{\mathbb{Z}}. \end{cases}$$

If $t-r-1 \in \mathcal{S}$, then

$$\hat{x}_q^r = \begin{cases} 0, & \text{for } q \in [1,r]_{\mathbb{Z}}; \\ \overline{V} + (q-r-1)V, & \text{for } q \in [r+1,t]_{\mathbb{Z}}; \\ \overline{V} + (t-r-1)V, & \text{for } q \in [t+1,T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^r = \begin{cases} 0, & \text{for } q \in [1,r]_{\mathbb{Z}}; \\ 1, & \text{for } q \in [r+1,T]_{\mathbb{Z}}. \end{cases}$$

We first consider the case where $t-r-1 \notin \mathcal{S}$. In this case, it is easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (2.2a)–(2.2f) and is therefore in $\text{conv}(\mathcal{P})$. Note that in this case $\hat{x}_t^r = \hat{y}_t^r = 0$, and $t-s-1 \neq r$ for all $s \in \mathcal{S}$, which implies that $\hat{y}_{t-s}^r - \hat{y}_{t-s-1}^r = 0$ for all $s \in \mathcal{S}$. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.1) at equality. Next, we consider the case where $t-r-1 \in \mathcal{S}$. In this case, it is easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (2.2a) and (2.2b). For each $q \in [1,r]_{\mathbb{Z}}$, we have $\hat{x}_q^r = \hat{y}_q^r = 0$. For each $q \in [r+1,T]_{\mathbb{Z}}$, because $t-r-1 \in \mathcal{S}$, we have $t-r-1 \leq \lfloor (\overline{C} - \overline{V})/V \rfloor$, which implies that $\overline{V} + (t-r-1)V \leq \overline{C}$, which in turn implies that $\underline{C} \leq \hat{x}_q^r \leq \overline{C}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (2.2c) and (2.2d). Note that $\hat{x}_q^r - \hat{x}_{q-1}^r = 0$ when $q \in [2,r]_{\mathbb{Z}}$, $\hat{x}_q^r - \hat{x}_{q-1}^r = \overline{V}$ when $q = r+1$, and $0 \leq \hat{x}_q^r - \hat{x}_{q-1}^r \leq V$ when $q \in [r+2,T]_{\mathbb{Z}}$. Thus, $-V\hat{y}_q^r - \overline{V}(1-\hat{y}_q^r) \leq \hat{x}_q^r - \hat{x}_{q-1}^r \leq V\hat{y}_{q-1}^r + \overline{V}(1-\hat{y}_{q-1}^r)$ for all $q \in [2,T]_{\mathbb{Z}}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (2.2e) and (2.2f). Therefore, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. Note that in this case $\hat{x}_t^r = \overline{V} + (t-r-1)V$, $\hat{y}_t^r = 1$, $\hat{y}_{t-s}^r - \hat{y}_{t-s-1}^r = 1$ when $s = t-r-1$, and $\hat{y}_{t-s} - \hat{y}_{t-s-1}^r = 0$

when $s \neq t-r-1$. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.1) at equality.

(A4) We create a point $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ by setting $\hat{x}^t_q = \overline{C}$ and $\hat{y}^t_q = 1$ for $q \in [1, T]_{\mathbb{Z}}$. It is easy to verify that $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (2.2a)–(2.2f). Thus, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t) \in \text{conv}(\mathcal{P})$. It is also easy to verify that $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (4.1) at equality.

(A5) For each $r \in [t+1, T]_{\mathbb{Z}}$, we create a point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as follows:

$$\hat{x}^r_q = \begin{cases} 0, & \text{for } q \in [1, r-1]_{\mathbb{Z}}; \\ \underline{C}, & \text{for } q \in [r, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}^r_q = \begin{cases} 0, & \text{for } q \in [1, r-1]_{\mathbb{Z}}; \\ 1, & \text{for } q \in [r, T]_{\mathbb{Z}}. \end{cases}$$

It is easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (2.2a)–(2.2f). Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. It is also easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (4.1) at equality.

Table A.1 shows a matrix with $2T-1$ rows, where each row represents a point created by this process. This matrix can be transformed into the matrix in Table A.2 via the following Gaussian elimination process:

Table A.1: A matrix with the rows representing $2T-1$ points in $\mathrm{conv}(\mathcal{P})$ that satisfy inequality (4.1) at equality.

Group	Point	Index r	**x**											**y**											
			1	2	$\cdots$	$t-s_{\max}-2$	$t-s_{\max}-1$	$\cdots$	$t-1$	t	$t+1$	$\cdots$	T	1	2	$\cdots$	$t-s_{\max}-2$	$t-s_{\max}-1$	$\cdots$	$t-1$	t	$t+1$	$\cdots$	T
		1	$\overline{C}-\epsilon$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	$\cdots$	1	1	1	$\cdots$	1
		2	$\overline{C}$	$\overline{C}-\epsilon$	$\cdots$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	$\cdots$	1	1	1	$\cdots$	1
		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-s_{\max}-2$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}-\epsilon$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	$\cdots$	1	1	1	$\cdots$	1
(A1)	$(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$	$t-s_{\max}-1$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}-\epsilon$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	$\cdots$	1	1	1	$\cdots$	1
		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-1$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}-\epsilon$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	$\cdots$	1	1	1	$\cdots$	1
		$t+1$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}-\epsilon$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	$\cdots$	1	1	1	$\cdots$	1
		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		T	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}-\epsilon$	1	1	$\cdots$	1	1	$\cdots$	1	1	1	$\cdots$	1
		1	$\underline{C}$	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	1	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
(A2)		2	$\underline{C}$	$\underline{C}$	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	1	1	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-s_{\max}-2$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}$	0	$\cdots$	0	0	0	$\cdots$	0	1	1	$\cdots$	1	0	$\cdots$	0	0	0	$\cdots$	0
		$t-s_{\max}-1$																						
(A3)	$(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$	$\vdots$	(See Note A.1-1)											(See Note A.1-1)										
		$t-1$																						
(A4)		t	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	$\cdots$	1	1	1	$\cdots$	1
		$t+1$	0	0	$\cdots$	0	0	$\cdots$	0	0	$\underline{C}$	$\cdots$	$\underline{C}$	0	0	$\cdots$	0	0	$\cdots$	0	0	1	$\cdots$	1
(A5)		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
		T	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	$\underline{C}$	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	1

Note A.1-1: For $r \in [t-s_{\max}-1, t-1]_{\mathbb{Z}}$, the $\mathbf{x}$ and $\mathbf{y}$ vectors in group (A3) are given as follows: $\hat{\mathbf{x}}^r = (\underbrace{\underline{C},\ldots,\underline{C}}_{r\text{ terms}}, \underbrace{0,\ldots,0}_{T-r\text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{1,\ldots,1}_{r\text{ terms}}, \underbrace{0,\ldots,0}_{T-r\text{ terms}})$ if $t-r-1 \notin \mathcal{S}$;
$\hat{\mathbf{x}}^r = (\underbrace{0,\ldots,0}_{r\text{ terms}}, \underbrace{\overline{V}, \overline{V}+V, \overline{V}+2V, \ldots, \overline{V}+(t-r-1)V}_{t-r\text{ terms}}, \underbrace{\overline{V}+(t-r-1)V, \ldots, \overline{V}+(t-r-1)V}_{T-t\text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{0,\ldots,0}_{r\text{ terms}}, \underbrace{1,\ldots,1}_{T-r\text{ terms}})$ if $t-r-1 \in \mathcal{S}$.

Table A.2: Lower triangular matrix obtained from Table A.1 via Gaussian elimination.

Group	Point	Index r	$\mathbf{x}$											$\mathbf{y}$										
			1	2	$\cdots$	$t-s_{\max}-2$	$t-s_{\max}-1$	$\cdots$	$t-1$	t	$t+1$	$\cdots$	T	1	2	$\cdots$	$t-s_{\max}-2$	$t-s_{\max}-1$	$\cdots$	$t-1$	t	$t+1$	$\cdots$	T
(B1)	$(\underline{\check{\mathbf{x}}}^r,\underline{\check{\mathbf{y}}}^r)$	1	$-\epsilon$	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		2	0	$-\epsilon$	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-s_{\max}-2$	0	0	$\cdots$	$-\epsilon$	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$t-s_{\max}-1$	0	0	$\cdots$	0	$-\epsilon$	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-1$	0	0	$\cdots$	0	0	$\cdots$	$-\epsilon$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$t+1$	0	0	$\cdots$	0	0	$\cdots$	0	0	$-\epsilon$	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		T	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	$-\epsilon$	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
(B2)	$(\underline{\hat{\mathbf{x}}}^r,\underline{\hat{\mathbf{y}}}^r)$	1	(Omitted)											1	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		2												1	1	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$												$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-s_{\max}-2$												1	1	$\cdots$	1	0	$\cdots$	0	0	0	$\cdots$	0
(B3)		$t-s_{\max}-1$	(Omitted)											(See Note A.2-1)										
		$\vdots$																						
		$t-1$																						
(B4)		t	(Omitted)											1	1	$\cdots$	1	1	$\cdots$	1	1	0	$\cdots$	0
(B5)		$t+1$	(Omitted)											0	0	$\cdots$	0	0	$\cdots$	0	0	1	$\cdots$	0
		$\vdots$												$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
		T												0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	1

Note A.2-1: For $r \in [t-s_{\max}, t-1]_{\mathbb{Z}}$, the $\mathbf{y}$ vector in group (B3) is given as follows: $\underline{\hat{\mathbf{y}}}^r = (\underbrace{1,\ldots,1}_{r\text{ terms}},\underbrace{0,\ldots,0}_{T-r\text{ terms}})$ if $t-r-1 \notin \mathcal{S}$; $\underline{\hat{\mathbf{y}}}^r = (\underbrace{-1,\ldots,-1}_{r\text{ terms}},\underbrace{0,\ldots,0}_{T-r\text{ terms}})$ if $t-r-1 \in \mathcal{S}$.

(i) For each $r \in [1,T]_{\mathbb{Z}} \setminus \{t\}$, the point with index r in group (B1), denoted $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r) = (\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) - (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$. Here, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ is the point with index r in group (A1), and $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ is the point in group (A4).

(ii) For each $r \in [1, t - s_{\max} - 2]_{\mathbb{Z}}$, the point with index r in group (B2), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (A2).

(iii) For each $r \in [t - s_{\max} - 1, t - 1]_{\mathbb{Z}}$, the point with index r in group (B3), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $t - r - 1 \notin \mathcal{S}$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) - (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ if $t - r - 1 \in \mathcal{S}$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (A3), and $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ is the point in group (A4).

(iv) The point in group (B4), denoted $(\underline{\hat{\mathbf{x}}}^t, \underline{\hat{\mathbf{y}}}^t)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^t, \underline{\hat{\mathbf{y}}}^t) = (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t) - (\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$. Here, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ is the point in group (A4), and $(\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$ is the point with index $t + 1$ in group (A5).

(v) For each $r \in [t + 1, T]_{\mathbb{Z}}$, the point with index r in group (B5), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ if $r \neq T$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $r = T$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ and $(\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ are the points with indices r and $r + 1$, respectively, in group (A5).

The matrix shown in Table A.2 is lower triangular; that is, the position of the last nonzero component of a row of the matrix is greater than the position of the last nonzero component of the previous row. This implies that the $2T-1$ points in groups (A1)–(A5) are linearly independent. Therefore, inequality (4.1) is facet-defining for $\text{conv}(\mathcal{P})$.

Next, we show that inequality (4.2) is facet-defining for $\text{conv}(\mathcal{P})$ by creating $2T$ affinely independent points in $\text{conv}(\mathcal{P})$ that satisfy (4.2) at equality. Because $\mathbf{0} \in \text{conv}(\mathcal{P})$ and $\mathbf{0}$ satisfies (4.2) at equality, it suffices to create the remaining $2T-1$ nonzero linearly independent points. We denote these $2T-1$ points as $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ for $r \in [1,T]_{\mathbb{Z}} \setminus \{t\}$, and $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ for $r \in [1,T]_{\mathbb{Z}}$, and we denote the qth component of $\bar{\mathbf{x}}^r$, $\bar{\mathbf{y}}^r$, $\hat{\mathbf{x}}^r$, and $\hat{\mathbf{y}}^r$ as $\bar{x}_q^r$, $\bar{y}_q^r$, $\hat{x}_q^r$, and $\hat{y}_q^r$, respectively. We divide these $2T-1$ points into the following five groups:

(C1) For each $r \in [1,T]_{\mathbb{Z}} \setminus \{t\}$, we create the same point $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ as that in group (A1) for inequality (4.1). Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. It is easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (4.2) at equality.

(C2) For each $r \in [1,t-1]_{\mathbb{Z}}$, we create a point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as follows:

$$\hat{x}_q^r = \begin{cases} \underline{C}, & \text{for } q \in [1,r]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [r+1,T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^r = \begin{cases} 1, & \text{for } q \in [1,r]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [r+1,T]_{\mathbb{Z}}. \end{cases}$$

It is easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (2.2a)–(2.2f). Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. Note that $\hat{x}_t^r = \hat{y}_t^r = 0$. Note also that $t+s \geq r+1$ for all $s \in \mathcal{S}$. Thus, $\hat{y}_{t+s}^r - \hat{y}_{t+s+1}^r = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.2) at equality.

(C3) For each $r \in [t, t+s_{\max}]_{\mathbb{Z}}$, we create a point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as follows: If $r - t \notin \mathcal{S}$, then

$$\hat{x}_q^r = \begin{cases} 0, & \text{for } q \in [1, r]_{\mathbb{Z}}; \\ \underline{C}, & \text{for } q \in [r+1, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^r = \begin{cases} 0, & \text{for } q \in [1, r]_{\mathbb{Z}}; \\ 1, & \text{for } q \in [r+1, T]_{\mathbb{Z}}. \end{cases}$$

If $r - t \in \mathcal{S}$, then

$$\hat{x}_q^r = \begin{cases} \overline{V} + (r-t)V, & \text{for } q \in [1, t-1]_{\mathbb{Z}}; \\ \overline{V} + (r-q)V, & \text{for } q \in [t, r]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [r+1, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^r = \begin{cases} 1, & \text{for } q \in [1, r]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [r+1, T]_{\mathbb{Z}}. \end{cases}$$

We first consider the case where $r - t \notin \mathcal{S}$. In this case, it is easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (2.2a)–(2.2f) and is therefore in $\text{conv}(\mathcal{P})$. Note that in this case, $\hat{x}_t^r = \hat{y}_t^r = 0$, and $t + s \neq r$ for all $s \in \mathcal{S}$, which implies that $\hat{y}_{t+s}^r - \hat{y}_{t+s+1}^r = 0$ for all $s \in \mathcal{S}$. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.2) at equality. Next, we consider the case where $r - t \in \mathcal{S}$. In this case, it is easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (2.2a) and (2.2b). For each $q \in [r+1, T]_{\mathbb{Z}}$, we have $\hat{x}_q^r = \hat{y}_q^r = 0$. For each $q \in [1, r]_{\mathbb{Z}}$, because $r - t \in \mathcal{S}$, we have $r - t \leq \lfloor (\overline{C} - \overline{V})/V \rfloor$, which implies that $\overline{V} + (r-t)V \leq \overline{C}$, which in turn implies that $\underline{C} \leq \hat{x}_q^r \leq \overline{C}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (2.2c) and (2.2d). Note that $\hat{x}_{q-1}^r - \hat{x}_q^r = 0$ when $q \in [r+2, T]_{\mathbb{Z}}$, $\hat{x}_{q-1}^r - \hat{x}_q^r = \overline{V}$ when $q = r+1$, and $0 \leq \hat{x}_{q-1}^r - \hat{x}_q^r \leq V$ when $q \in [1, r]_{\mathbb{Z}}$. Thus, $-V\hat{y}_{q-1}^r - \overline{V}(1 - \hat{y}_{q-1}^r) \leq \hat{x}_{q-1}^r - \hat{x}_q^r \leq V\hat{y}_q^r + \overline{V}(1 - \hat{y}_q^r)$ for all $q \in [2, T]_{\mathbb{Z}}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (2.2e)–(2.2f). Therefore, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. Note that in this case $\hat{x}_t^r = \overline{V} + (r-t)V$, $\hat{y}_t^r = 1$, $\hat{y}_{t+s}^r - \hat{y}_{t+s+1} = 1$ when $s = r-t$, and $\hat{y}_{t+s}^r - \hat{y}_{t+s+1}^r = 0$ when $s \neq r-t$. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.2) at equality.

(C4) We create a point $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ by setting $\hat{x}_q^{t+s_{\max}+1} = \overline{C}$ and $\hat{y}_q^{t+s_{\max}+1} = 1$ for $q \in [1, T]_{\mathbb{Z}}$. It is easy to verify that $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ satisfies (2.2a)–(2.2f). Thus, $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1}) \in \text{conv}(\mathcal{P})$. It is also easy to verify that $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ satisfies inequality (4.2) at equality.

(C5) For each $r \in [t + s_{\max} + 2, T]_{\mathbb{Z}}$, we create a point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as follows:

$$\hat{x}_q^r = \begin{cases} 0, & \text{for } q \in [1, r-1]_{\mathbb{Z}}; \\ \underline{C}, & \text{for } q \in [r, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^r = \begin{cases} 0, & \text{for } q \in [1, r-1]_{\mathbb{Z}}; \\ 1, & \text{for } q \in [r, T]_{\mathbb{Z}}. \end{cases}$$

It is easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (2.2a)–(2.2f). Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. It is also easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.2) at equality.

Table A.3 shows a matrix with $2T - 1$ rows, where each row represents a point created by this process. This matrix can be transformed into the matrix in Table A.4 via the following Gaussian elimination process:

Table A.3: A Matrix with rows representing $2T-1$ points in $\text{conv}(\mathcal{P})$ that satisfy inequality (4.2) at equality.

Group	Point	Index r	**x**												**y**											
			1	2	⋯	$t-1$	t	$t+1$	⋯	$t+s_{\max}+1$	$t+s_{\max}+2$	⋯	T	1	2	⋯	$t-1$	t	$t+1$	⋯	$t+s_{\max}+1$	$t+s_{\max}+2$	⋯	T
(C1)	$(\bar{\mathbf{x}}^r,\bar{\mathbf{y}}^r)$	1	$\overline{C}-\epsilon$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	1	1	⋯	1	1	1	⋯	1	1	⋯	1
		2	$\overline{C}$	$\overline{C}-\epsilon$	⋯	$\overline{C}$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	1	1	⋯	1	1	1	⋯	1	1	⋯	1
		⋮	⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮		⋮
		$t-1$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}-\epsilon$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	1	1	⋯	1	1	1	⋯	1	1	⋯	1
		$t+1$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}$	$\overline{C}-\epsilon$	⋯	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	1	1	⋯	1	1	1	⋯	1	1	⋯	1
		⋮	⋮	⋮		⋮	⋮	⋮	⋱	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮		⋮
		$t+s_{\max}+1$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}-\epsilon$	$\overline{C}$	⋯	$\overline{C}$	1	1	⋯	1	1	1	⋯	1	1	⋯	1
		$t+s_{\max}+2$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}-\epsilon$	⋯	$\overline{C}$	1	1	⋯	1	1	1	⋯	1	1	⋯	1
		⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮		⋮
		T	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}-\epsilon$	1	1	⋯	1	1	1	⋯	1	1	⋯	1
(C2)	$(\hat{\mathbf{x}}^r,\hat{\mathbf{y}}^r)$	1	$\underline{C}$	0	⋯	0	0	0	⋯	0	0	⋯	0	1	0	⋯	0	0	0	⋯	0	0	⋯	0
		2	$\underline{C}$	$\underline{C}$	⋯	0	0	0	⋯	0	0	⋯	0	1	1	⋯	0	0	0	⋯	0	0	⋯	0
		⋮	⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮		⋮	⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮		⋮
		$t-1$	$\underline{C}$	$\underline{C}$	⋯	$\underline{C}$	0	0	⋯	0	0	⋯	0	1	1	⋯	1	0	0	⋯	0	0	⋯	0
(C3)		t ⋮ $t+s_{\max}$	(See Note A.3-1)											(See Note A.3-1)											
(C4)		$t+s_{\max}+1$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	1	1	⋯	1	1	1	⋯	1	1	⋯	1
(C5)		$t+s_{\max}+2$	0	0	⋯	0	0	0	⋯	0	$\underline{C}$	⋯	$\underline{C}$	0	0	⋯	0	0	0	⋯	0	1	⋯	1
		⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋱	⋮
		T	0	0	⋯	0	0	0	⋯	0	0	⋯	$\underline{C}$	0	0	⋯	0	0	0	⋯	0	0	⋯	1

Note A.3-1: For $r \in [t, t+s_{\max}]_{\mathbb{Z}}$, the **x** and **y** vectors in group (C3) as given as follows: $\hat{\mathbf{x}}^r = (\underbrace{0,\ldots,0}_{r\text{ terms}},\underbrace{\underline{C},\ldots,\underline{C}}_{T-r\text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{0,\ldots,0}_{r\text{ terms}},\underbrace{1,\ldots,1}_{T-r\text{ terms}})$ if $r-t \notin \mathcal{S}$;

$\hat{\mathbf{x}}^r = (\underbrace{\overline{V}+(r-t)V,\ldots,\overline{V}+(r-t)V}_{t-1\text{ terms}},\underbrace{\overline{V}+(r-t)V,\overline{V}+(r-t-1)V,\ldots,\overline{V}}_{r-t+1\text{ terms}},\underbrace{0,\ldots,0}_{T-r\text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{1,\ldots,1}_{r\text{ terms}},\underbrace{0,\ldots,0}_{T-r\text{ terms}})$ if $r-t \in \mathcal{S}$.

Table A.4: Lower triangular matrix obtained from Table A.3 via Gaussian elimination.

Group	Point	Index r	$\mathbf{x}$											$\mathbf{y}$										
			1	2	$\cdots$	$t-1$	t	$t+1$	$\cdots$	$t+s_{\max}+1$	$t+s_{\max}+2$	$\cdots$	T	1	2	$\cdots$	$t-1$	t	$t+1$	$\cdots$	$t+s_{\max}+1$	$t+s_{\max}+2$	$\cdots$	T
(D1)	$(\underline{\tilde{\mathbf{x}}}^r, \underline{\tilde{\mathbf{y}}}^r)$	1	$-\epsilon$	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
		2	0	$-\epsilon$	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
		$t-1$	0	0	$\cdots$	$-\epsilon$	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
		$t+1$	0	0	$\cdots$	0	0	$-\epsilon$	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
		$t+s_{\max}+1$	0	0	$\cdots$	0	0	0	$\cdots$	$-\epsilon$	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
		$t+s_{\max}+2$	0	0	$\cdots$	0	0	0	$\cdots$	0	$-\epsilon$	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
		T	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	$-\epsilon$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
(D2)	$(\hat{\underline{\mathbf{x}}}^r, \hat{\underline{\mathbf{y}}}^r)$	1	(Omitted)											1	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
		2												1	1	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
		$\vdots$												$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
		$t-1$												1	1	$\cdots$	1	0	0	$\cdots$	0	0	$\cdots$	0
(D3)		t	(Omitted)											(See Note A.4-1)										
		$\vdots$																						
		$t+s_{\max}$																						
(D4)		$t+s_{\max}+1$	(Omitted)											1	1	$\cdots$	1	1	1	$\cdots$	1	0	$\cdots$	0
(D5)		$t+s_{\max}+2$	(Omitted)											0	0	$\cdots$	0	0	0	$\cdots$	0	1	$\cdots$	0
		$\vdots$												$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\ddots$	$\vdots$
		T												0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	1

Note A.4-1: For $r \in [t, t+s_{\max}]_{\mathbb{Z}}$, the $\mathbf{y}$ vector in group (D3) is given as follows: $\hat{\underline{\mathbf{y}}}^r = (\underbrace{-1,\ldots,-1}_{r\text{ terms}}, \underbrace{0,\ldots,0}_{T-r\text{ terms}})$ if $r-t \notin \mathcal{S}$; $\hat{\underline{\mathbf{y}}}^r = (\underbrace{1,\ldots,1}_{r\text{ terms}}, \underbrace{0,\ldots,0}_{T-r\text{ terms}})$ if $r-t \in \mathcal{S}$.

(i) For each $r \in [1, T]_{\mathbb{Z}} \setminus \{t\}$, the point with index r in group (D1), denoted $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r) = (\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{y}^{t+s_{\max}+1})$. Here, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ is the point with index r in group (C1), and $(\hat{x}^{t+s_{\max}+1}, \hat{y}^{t+s_{\max}+1})$ is the point in group (C4).

(ii) For each $r \in [1, t-1]_{\mathbb{Z}}$, the point with index r in group (D2), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (C2).

(iii) For each $r \in [t, t+s_{\max}]_{\mathbb{Z}}$, the point with index r in group (D3), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $r - t \in \mathcal{S}$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ if $r - t \notin \mathcal{S}$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (C3), and $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ is the point in group (C4).

(iv) The point in group (D4), denoted $(\underline{\hat{\mathbf{x}}}^{t+s_{\max}+1}, \underline{\hat{\mathbf{y}}}^{t+s_{\max}+1})$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^{t+s_{\max}+1}, \underline{\hat{\mathbf{y}}}^{t+s_{\max}+1}) = (\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1}) - (\hat{\mathbf{x}}^{t+s_{\max}+2}, \hat{\mathbf{y}}^{t+s_{\max}+2})$. Here, $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ is the point in group (C4), and $(\hat{\mathbf{x}}^{t+s_{\max}+2}, \hat{\mathbf{y}}^{t+s_{\max}+2})$ is the point with index $t + s_{\max} + 2$ in group (C5).

(v) For each $r \in [t + s_{\max} + 2, T]_{\mathbb{Z}}$, the point with index r in group (D5), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ if $r \neq T$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $r = T$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ and $(\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ are the points with indices r and $r + 1$, respectively, in group (C5).

The matrix shown in Table A.4 is lower triangular; that is, the position of the last nonzero component of a row of the matrix is greater than the position of the last nonzero component of the previous row. This implies that the $2T-1$ points in groups (C1)–(C5) are linearly independent. Therefore, inequality (4.2) is facet-defining for $\text{conv}(\mathcal{P})$. $\square$

8.5 Proof of Proposition 2

To prove that linear inequalities (4.1) and (4.2) are valid for $\mathbf{conv}(\mathcal{P})$ when $\mathcal{S} = [0, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$, it suffices to show that (4.1) and (4.2) are valid for $\mathcal{P}$ when $\mathcal{S} = [0, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. Consider any element $(\mathbf{x}, \mathbf{y})$ of $\mathcal{P}$. We show that $(\mathbf{x}, \mathbf{y})$ satisfies (4.1) and (4.2) when $\mathcal{S} = [0, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$.

We first show that $(\mathbf{x}, \mathbf{y})$ satisfies (4.1) when $\mathcal{S} = [0, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. Consider any $t \in [s_{\max} + 2, T]_{\mathbb{Z}}$. We divide the analysis into four cases.

Case 1: $y_t = 0$ and $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, \lfloor (\overline{C} - \overline{V})/V \rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Thus, in this case, the right hand side of inequality (4.1) is nonnnegative. Because $y_t = 0$, by (2.2d), $x_t = 0$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.1).

Case 2: $y_t = 0$ and $y_{t-s} - y_{t-s-1} > 0$ for some $s \in \mathcal{S}$. Let $\tilde{\mathcal{S}} = \{\sigma \in \mathcal{S} : y_{t-\sigma} - y_{t-\sigma-1} > 0\}$ and $v = |\tilde{\mathcal{S}}|$. Then, $v \geq 1$. Denote $\tilde{\mathcal{S}} = \{\sigma_1, \sigma_2, \ldots, \sigma_v\}$, where $\sigma_1 < \sigma_2 < \cdots < \sigma_v$. Note that $y_{t-\sigma_j-1} = 0$ and $y_{t-\sigma_j} = 1$ for $j = 1, \ldots, v$. Denote $\sigma_0 = -1$. Then for each $j = 1, \ldots, v$, there exists $\sigma'_j \in [\sigma_{j-1} + 1, \sigma_j - 1]_{\mathbb{Z}}$ such that $y_{t-\sigma'_j-1} = 1$ and $y_{t-\sigma'_j} = 0$. Thus,

$$0 \leq \sigma'_1 < \sigma_1 < \sigma'_2 < \sigma_2 \cdots < \sigma'_v < \sigma_v \leq s_{\max}.$$

Because $y_{t-\sigma_v} - y_{t-\sigma_v-1} = 1$ and $t - \sigma_v \in [2, T]_{\mathbb{Z}}$, by (2.2a), $y_k = 1$ for all $k \in [t - \sigma_v, \min\{T, t - \sigma_v + L - 1\}]_{\mathbb{Z}}$, which implies that $t - \sigma'_j \geq t - \sigma_v + L$ for $j = 1, \ldots, v$. Hence, for $j = 1, \ldots, v$, we have $\sigma'_j \leq \sigma_v - L$, which implies that

$$\sigma'_j \leq s_{\max} - L. \tag{8.12}$$

If $\beta = \alpha + 1$, then $\mathcal{S} = [0, s_{\max}]_{\mathbb{Z}}$, which implies that $\sigma'_j \in \mathcal{S}$ for $j = 1, \ldots, v$. If $\beta \neq \alpha + 1$, then condition (c) of Proposition 2 implies that $s_{\max} \leq L + \alpha$, which, by (8.12), implies that $\sigma'_j \leq \alpha$ for $j = 1, \ldots, v$. Thus, in both cases, $\sigma'_j \in \mathcal{S}$ for $j = 1, \ldots, v$. Because $y_t = 0$, by (2.2d), $x_t = 0$. Hence, the left hand side of inequality (4.1) is 0. Because $\mathcal{S} \subseteq [0, \lfloor (\overline{C} - \overline{V})/V \rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Note that $\{\sigma'_1, \ldots, \sigma'_v\} \subseteq \mathcal{S} \setminus \tilde{\mathcal{S}}$ and $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S} \setminus \tilde{\mathcal{S}}$. Thus, $\sum_{s \in \mathcal{S} \setminus \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \leq \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma'_j V)(y_{t-\sigma'_j} - y_{t-\sigma'_j-1})$. Hence, the right hand side of inequality (4.1) is

$$
\begin{aligned}
&\overline{C} y_t - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \\
&= -\sum_{s \in \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) - \sum_{s \in \mathcal{S} \setminus \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \\
&\geq -\sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma_j V)(y_{t-\sigma_j} - y_{t-\sigma_j-1}) - \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma'_j V)(y_{t-\sigma'_j} - y_{t-\sigma'_j-1}) \\
&= -\sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma_j V) + \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma'_j V) \\
&= \sum_{j=1}^{v} (\sigma_j - \sigma'_j) V \\
&> 0.
\end{aligned}
$$

Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.1).

Case 3: $y_t = 1$ and $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, \lfloor (\overline{C} - \overline{V})/V \rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Thus, in this case, the right hand side of inequality (4.1) is at least $\overline{C}$. By (2.2d), $x_t \leq \overline{C}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.1).

Case 4: $y_t = 1$ and $y_{t-s} - y_{t-s-1} > 0$ for some $s \in \mathcal{S}$. Let $\tilde{\mathcal{S}} = \{\sigma \in \mathcal{S} : y_{t-\sigma} - y_{t-\sigma-1} > 0\}$ and $v = |\tilde{\mathcal{S}}|$. Then, $v \geq 1$. Denote $\tilde{\mathcal{S}} = \{\sigma_1, \sigma_2, \ldots, \sigma_v\}$, where $\sigma_1 < \sigma_2 < \cdots < \sigma_v$. Note that $y_{t-\sigma_j-1} = 0$ and $y_{t-\sigma_j} = 1$ for $j = 1, \ldots, v$. Then, for each $j = 2, \ldots, v$, there exists $\sigma'_j \in [\sigma_{j-1}+1, \sigma_j - 1]_{\mathbb{Z}}$ such that $y_{t-\sigma'_j-1} = 1$ and $y_{t-\sigma'_j} = 0$. Thus,

$$0 \leq \sigma_1 < \sigma'_2 < \sigma_2 < \cdots < \sigma'_v < \sigma_v \leq s_{\max}.$$

In addition, $y_k = 1$ for all $k \in [t - \sigma_1, t]_{\mathbb{Z}}$. Because $y_{t-\sigma_v} - y_{t-\sigma_v-1} = 1$ and $t - \sigma_v \in [2, T]_{\mathbb{Z}}$, by (2.2a), $y_k = 1$ for all $k \in [t - \sigma_v, \min\{T, t - \sigma_v + L - 1\}]_{\mathbb{Z}}$, which implies that $t - \sigma'_j \geq t - \sigma_v + L$ for $j = 2, \ldots, v$. Hence, for $j = 2, \ldots, v$, we have $\sigma'_j \leq \sigma_v - L$, which implies that

$$\sigma'_j \leq s_{\max} - L. \tag{8.13}$$

If $\beta = \alpha + 1$, then $\mathcal{S} = [0, s_{\max}]_{\mathbb{Z}}$, which implies that $\sigma'_j \in \mathcal{S}$ for $j = 2, \ldots, v$. If $\beta \neq \alpha + 1$, then condition (c) of Proposition 2 implies that $s_{\max} \leq L + \alpha$, which, by (8.13) implies that $\sigma'_j \leq \alpha$ for all $j = 2, \ldots, v$. Thus, in both cases, $\sigma'_j \in \mathcal{S}$ for $j = 2, \ldots, v$. By (2.2e),

$$\sum_{\tau=t-\sigma_1}^{t} (x_\tau - x_{\tau-1}) \leq \sum_{\tau=t-\sigma_1}^{t} V y_{\tau-1} + \sum_{\tau=t-\sigma_1}^{t} \overline{V}(1 - y_{\tau-1}),$$

which implies that

$$x_t - x_{t-\sigma_1-1} \leq \sum_{\tau=t-\sigma_1}^{t} V y_{\tau-1} + \sum_{\tau=t-\sigma_1}^{t} \overline{V}(1 - y_{\tau-1}) = \sigma_1 V + \overline{V}.$$

Because $y_{t-\sigma_1-1} = 0$, by (2.2d), $x_{t-\sigma_1-1} = 0$. Hence, $x_t \leq \sigma_1 V + \overline{V}$; that is, the left hand side of inequality (4.1) is at most $\sigma_1 V + \overline{V}$. Because $\mathcal{S} \subseteq [0, \lfloor(\overline{C} - \overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Note that $\{\sigma'_2, \ldots, \sigma'_v\} \subseteq \mathcal{S} \setminus \tilde{\mathcal{S}}$ and $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S} \setminus \tilde{\mathcal{S}}$. Thus, $\sum_{s\in\mathcal{S}\setminus\tilde{\mathcal{S}}}(\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \leq \sum_{j=2}^{v}(\overline{C} - \overline{V} - \sigma'_j V)(y_{t-\sigma'_j} - y_{t-\sigma'_j-1})$. Hence, the right hand side of inequality (4.1) is

$$
\begin{aligned}
&\overline{C}y_t - \sum_{s\in\mathcal{S}}(\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \\
&= \overline{C} - \sum_{s\in\tilde{\mathcal{S}}}(\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) - \sum_{s\in\mathcal{S}\setminus\tilde{\mathcal{S}}}(\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \\
&\geq \overline{C} - \sum_{j=1}^{v}(\overline{C} - \overline{V} - \sigma_j V)(y_{t-\sigma_j} - y_{t-\sigma_j-1}) - \sum_{j=2}^{v}(\overline{C} - \overline{V} - \sigma'_j V)(y_{t-\sigma'_j} - y_{t-\sigma'_j-1}) \\
&= \overline{C} - \sum_{j=1}^{v}(\overline{C} - \overline{V} - \sigma_j V) + \sum_{j=2}^{v}(\overline{C} - \overline{V} - \sigma'_j V) \\
&= \sigma_1 V + \overline{V} + \sum_{j=2}^{v}(\sigma_j - \sigma'_j)V \\
&\geq \sigma_1 V + \overline{V}.
\end{aligned}
$$

Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.1).

Next, we show that $(\mathbf{x}, \mathbf{y})$ satisfies (4.2) when $\mathcal{S} = [0, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. Consider any $t \in [1, T - s_{\max} - 1]_{\mathbb{Z}}$. We divide the analysis into four cases.

Case 1: $y_t = 0$ and $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, \lfloor(\overline{C} - \overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Thus, in this case, the right hand side of inequality (4.2) is nonnegative. Because $y_t = 0$, by (2.2d), $x_t = 0$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.2).

Case 2: $y_t = 0$ and $y_{t+s} - y_{t+s+1} > 0$ for some $s \in \mathcal{S}$. Let $\tilde{\mathcal{S}} = \{\sigma \in \mathcal{S} :$

$y_{t+\sigma} - y_{t+\sigma+1} > 0\}$ and $v = |\tilde{\mathcal{S}}|$. Then, $v \geq 1$. Denote $\tilde{\mathcal{S}} = \{\sigma_1, \sigma_2, \ldots, \sigma_v\}$, where $\sigma_1 < \sigma_2 < \cdots < \sigma_v$. Note that $y_{t+\sigma_j} = 1$ and $y_{t+\sigma_j+1} = 0$ for $j = 1, \ldots, v$. Denote $\sigma_0 = -1$. Then, for each $j = 1, \ldots, v$, there exists $\sigma'_j \in [\sigma_{j-1} + 1, \sigma_j - 1]_{\mathbb{Z}}$ such that $y_{t+\sigma'_j} = 0$ and $y_{t+\sigma'_j+1} = 1$. Thus,

$$0 \leq \sigma'_1 < \sigma_1 < \sigma'_2 < \sigma_2 < \cdots < \sigma'_v < \sigma_v \leq s_{\max}.$$

Because $y_{t+\sigma_v+1} = 0$ and $t + \sigma_v + 1 \in [2, T]_{\mathbb{Z}}$, by Lemma 1(i), $y_{t+\sigma_v+1-j} - y_{t+\sigma_v-j} \leq 0$ for all $j \in [0, \min\{t + \sigma_v - 1, L - 1\}]_{\mathbb{Z}}$, which implies that $t + \sigma'_j + 1 \leq t + \sigma_v - L + 1$ for $j = 1, \ldots, v$. Hence, for $j = 1, \ldots, v$, we have $\sigma'_j \leq \sigma_v - L$, which implies that

$$\sigma'_j \leq s_{\max} - L. \tag{8.14}$$

If $\beta = \alpha + 1$, then $\mathcal{S} = [0, s_{\max}]_{\mathbb{Z}}$, which implies that $\sigma'_j \in \mathcal{S}$ for $j = 1, \ldots, v$. If $\beta \neq \alpha + 1$, then condition (c) in Proposition 2 implies that $s_{\max} \leq L + \alpha$, which, by (8.14), implies that $\sigma'_j \leq \alpha$ for $j = 1, \ldots, v$. Thus, in both cases, $\sigma'_j \in \mathcal{S}$ for $j = 1, \ldots, v$. Because $y_t = 0$, by (2.2d), $x_t = 0$. Hence, the left hand side of inequality (4.2) is 0. Because $\mathcal{S} \subseteq [0, \lfloor (\overline{C} - \overline{V})/V \rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Note that $\{\sigma'_1, \ldots, \sigma'_v\} \subseteq \mathcal{S} \setminus \tilde{\mathcal{S}}$ and $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in \mathcal{S} \setminus \tilde{\mathcal{S}}$. Thus, $\sum_{s \in \mathcal{S} \setminus \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \leq \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma'_j V)(y_{t+\sigma'_j} - y_{t+\sigma'_j+1})$. Hence, the right hand side of inequality (4.2) is

$$
\begin{aligned}
&\overline{C}y_t - \sum_{s\in\mathcal{S}}(\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \\
&= -\sum_{s\in\tilde{\mathcal{S}}}(\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) - \sum_{s\in\mathcal{S}\setminus\tilde{\mathcal{S}}}(\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \\
&\geq -\sum_{j=1}^{v}(\overline{C} - \overline{V} - \sigma_j V)(y_{t+\sigma_j} - y_{t+\sigma_j+1}) - \sum_{j=1}^{v}(\overline{C} - \overline{V} - \sigma_j' V)(y_{t+\sigma_j'} - y_{t+\sigma_j'+1}) \\
&= -\sum_{j=1}^{v}(\overline{C} - \overline{V} - \sigma_j V) + \sum_{j=1}^{v}(\overline{C} - \overline{V} - \sigma_j' V) \\
&= \sum_{j=1}^{v}(\sigma_j - \sigma_j')V \\
&> 0.
\end{aligned}
$$

Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.2).

Case 3: $y_t = 1$ and $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, \lfloor(\overline{C} - \overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Thus, in this case, the right hand side of inequality (4.2) is at least $\overline{C}$. By (2.2d), $x_t \leq \overline{C}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.2).

Case 4: $y_t = 1$ and $y_{t+s} - y_{t+s+1} > 1$ for some $s \in \mathcal{S}$. Let $\tilde{\mathcal{S}} = \{\sigma \in \mathcal{S} : y_{t+\sigma} - y_{t+\sigma+1} > 0\}$ and $v = |\tilde{\mathcal{S}}|$. Then $v \geq 1$. Denote $\tilde{\mathcal{S}} = \{\sigma_1, \sigma_2, \ldots, \sigma_v\}$, where $\sigma_1 < \sigma_2 < \cdots < \sigma_v$. Note that $y_{t+\sigma_j} = 1$ and $y_{t+\sigma_j+1} = 0$ for $j = 1, \ldots, v$. Then, for each $j = 2, \ldots, v$, there exists $\sigma_j' \in [\sigma_{j-1} + 1, \sigma_j - 1]_{\mathbb{Z}}$ such that $y_{t+\sigma_j'} = 0$ and $y_{t+\sigma_j'+1} = 1$. Thus,

$$0 \leq \sigma_1 < \sigma_2' < \sigma_2 < \cdots < \sigma_v' < \sigma_v \leq s_{\max}.$$

In addition, $y_k = 1$ for all $k \in [t, t+\sigma_1]_{\mathbb{Z}}$. Because $y_{t+\sigma_v+1} = 0$, by Lemma 1(i), $y_{t+\sigma_v+1-j} - y_{t+\sigma_v-j} \le 0$ for all $j \in [0, \min\{t+\sigma_v-1, L-1\}]_{\mathbb{Z}}$, which implies that $t+\sigma'_j+1 \le t+\sigma_v-L+1$ for $j=2,\ldots,v$. Hence, for $j = 2,\ldots,v$, we have $\sigma'_j \le \sigma_v - L$, which implies that

$$\sigma'_j \le s_{\max} - L. \tag{8.15}$$

If $\beta = \alpha+1$, then $\mathcal{S} = [0, s_{\max}]_{\mathbb{Z}}$, which implies that $\sigma'_j \in \mathcal{S}$ for all $j = 2,\ldots,v$. If $\beta \ne \alpha+1$, then condition (c) of Proposition 2 implies that $s_{\max} \le L+\alpha$, which, by (8.15), implies that $\sigma'_j \le \alpha$ for $j=2,\ldots,v$. Thus, in both cases, $\sigma'_j \in \mathcal{S}$ for $j=2,\ldots,v$. By (2.2f),

$$\sum_{\tau=t+1}^{t+\sigma_1+1}(x_{\tau-1}-x_\tau) \le \sum_{\tau=t+1}^{t+\sigma_1+1} V y_\tau + \sum_{\tau=t+1}^{t+\sigma_1+1}\overline{V}(1-y_\tau),$$

which implies that

$$x_t - x_{t+\sigma_1+1} \le \sum_{\tau=t+1}^{t+\sigma_1+1} V y_\tau + \sum_{\tau=t+1}^{t+\sigma_1+1}\overline{V}(1-y_\tau) = \sigma_1 V + \overline{V}.$$

Because $y_{t+\sigma_1+1} = 0$, by (2.2d), $x_{t+\sigma_1+1} = 0$. Hence, $x_t \le \sigma_1 V + \overline{V}$; that is, the left hand side of inequality (4.2) is at most $\sigma_1 V + \overline{V}$. Because $\mathcal{S} \subseteq [0, \lfloor(\overline{C}-\overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C}-\overline{V}-sV \ge 0$ for all $s \in \mathcal{S}$. Note that $\{\sigma'_2,\ldots,\sigma'_v\} \subseteq \mathcal{S}\setminus\tilde{\mathcal{S}}$ and $y_{t+s}-y_{t+s+1} \le 0$ for all $s \in \mathcal{S}\setminus\tilde{\mathcal{S}}$. Thus, $\sum_{s\in\mathcal{S}\setminus\tilde{\mathcal{S}}}(\overline{C}-\overline{V}-sV)(y_{t+s}-y_{t+s+1}) \le \sum_{j=2}^{v}(\overline{C}-\overline{V}-\sigma'_j V)(y_{t+\sigma'_j} -$

$y_{t+\sigma'_j+1})$. Hence, the right hand side of inequality (4.2) is

$$\begin{aligned}
&\overline{C} - \sum_{s\in\mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \\
&= \overline{C} - \sum_{s\in\tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) - \sum_{s\in\mathcal{S}\setminus\tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \\
&\geq \overline{C} - \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma_j V)(y_{t+\sigma_j} - y_{t+\sigma_j+1}) - \sum_{j=2}^{v} (\overline{C} - \overline{V} - \sigma'_j V)(y_{t+\sigma'_j} - y_{t+\sigma'_j+1}) \\
&= \overline{C} - \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma_j V) + \sum_{j=2}^{v} (\overline{C} - \overline{V} - \sigma'_j V) \\
&= \sigma_1 V + \overline{V} + \sum_{j=2}^{v} (\sigma_j - \sigma'_j) V \\
&\geq \sigma_1 V + \overline{V}.
\end{aligned}$$

Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.2).

It is easy to verify that the proof of facet-defining of inequalities (4.1) and (4.2) in the proof of Proposition 1 remains valid when $\mathcal{S} = [0, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. Therefore, inequalities (4.1) and (4.2) are facet-defining under the conditions stated in Proposition 2. $\square$

8.6 Proof of Proposition 3

For notational convenience, we define $s_{\max} = \max\{s : s \in \mathcal{S}\}$ if $\mathcal{S} \neq \emptyset$, and $s_{\max} = -1$ if $\mathcal{S} = \emptyset$. To prove that linear inequalities (4.3) and (4.4) are valid for $\text{conv}(\mathcal{P})$, it suffices to show that they are valid for $\mathcal{P}$. Consider any element $(\mathbf{x}, \mathbf{y})$ of $\mathcal{P}$. We show that $(\mathbf{x}, \mathbf{y})$ satisfies (4.3) and (4.4).

We first show that $(\mathbf{x}, \mathbf{y})$ satisfies (4.3). Consider any $t \in [s_{\max} + 2, T - 1]_{\mathbb{Z}}$ (i.e., $t \in [1, T-1]_{\mathbb{Z}}$ such that $t \geq s + 2$ for all $s \in \mathcal{S}$). We divide the analysis into three cases.

Case 1: $y_t = 0$. By Lemma 1(i), $y_{t-j} - y_{t-j-1} \leq 0$ for all $j \in [0, \min\{t - 2, L-1\}]_{\mathbb{Z}}$. Because $\mathcal{S} \subseteq [0, L-1]_{\mathbb{Z}}$ and $s_{\max} \leq t - 2$, we have $\mathcal{S} \subseteq [0, \min\{t-2, L-1\}]_{\mathbb{Z}}$. Thus, $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, \lfloor (\overline{C} - \overline{V})/V \rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Thus, the right hand side of (4.3) is at least $\eta V y_{t+1} \geq 0$. Because $y_t = 0$, by (2.2d), $x_t = 0$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.3).

Case 2: $y_t = 1$ and $y_{t-s'} - y_{t-s'-1} = 1$ for some $s' \in \mathcal{S}$. In this case, $y_{t-s'} = 1$ and $y_{t-s'-1} = 0$. Because $s' \leq s_{\max} \leq t - 2$, we have $t - s' \in [2, T]_{\mathbb{Z}}$. By (2.2a), $y_k = 1$ for all $k \in [t - s', \min\{T, t - s' + L - 1\}]_{\mathbb{Z}}$. Because $\mathcal{S} \subseteq [0, L-1]_{\mathbb{Z}}$, we have $s' \leq L - 1$, or equivalently $t - s' + L - 1 \geq t$, and thus $y_{t-s} = 1$ for all $s \in [0, s']_{\mathbb{Z}}$, which implies that $y_{t-s} - y_{t-s-1} = 0$ for all $s \in [0, s'-1]_{\mathbb{Z}}$. Because $s' \leq t-2$, either $s' = t-2$ or $s' \leq t - 3$. If $s' = t - 2$, then it does not exist any $s \in \mathcal{S}$ such that $s > s'$. If $s' \leq t - 3$, then $t - s' - 1 \in [2, T]_{\mathbb{Z}}$, and by Lemma 1(i), $y_{t-s'-j-1} - y_{t-s'-j-2} \leq 0$ for all $j \in [0, \min\{t - s' - 3, L-1\}]_{\mathbb{Z}}$, which implies that $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in [s'+1, \min\{t-2, L+s'\}]_{\mathbb{Z}}$, which in turn implies that $y_{t-s} - y_{t-s-1} \leq 0$

for all $s \in \mathcal{S}$ such that $s > s'$. Hence, $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S} \setminus \{s'\}$. Because $\mathcal{S} \subseteq [0, \lfloor(\overline{C} - \overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Thus,

$$-\sum_{s\in\mathcal{S}}(\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \geq -(\overline{C} - \overline{V} - s'V). \tag{8.16}$$

Because $y_t = 1$, by (8.16), the right hand side of (4.3) is at least $s'V + \overline{V}$ when $y_{t+1} = 1$ and is at least $\overline{V} + (s' - \eta)V$ when $y_{t+1} = 0$. By (2.2d), $x_{t-s'-1} = 0$. By (2.2e), $\sum_{\tau=t-s'}^{t}(x_\tau - x_{\tau-1}) \leq \sum_{\tau=t-s'}^{t} Vy_{\tau-1} + \sum_{\tau=t-s'}^{t} \overline{V}(1 - y_{\tau-1})$, which implies that $x_t \leq s'V + \overline{V}$. If $y_{t+1} = 0$, then by (2.2d) and (2.2f), $x_{t+1} = 0$ and $x_t - x_{t+1} \leq Vy_{t+1} + \overline{V}(1 - y_{t+1})$, which imply that $x_t \leq \overline{V}$. In addition, if $y_{t+1} = 0$, then because $y_k = 1$ for all $k \in [t - s', \min\{T, t - s' + L - 1\}]_{\mathbb{Z}}$, we have $t + 1 \geq t - s' + L$, which implies that $s' \geq L - 1 \geq \eta$. Thus, if $y_{t+1} = 0$, then $x_t \leq \overline{V} + (s' - \eta)V$. Hence, x_t is at most $s'V + \overline{V}$, and it is at most $\overline{V} + (s' - \eta)V$ when $y_{t+1} = 0$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.3).

Case 3: $y_t = 1$ and $y_{t-s} - y_{t-s-1} \neq 1$ for all $s \in \mathcal{S}$. In this case, $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, \lfloor(\overline{C} - \overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Thus, $\sum_{s\in\mathcal{S}}(\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \leq 0$. Because $y_t = 1$, the right hand side of (4.3) is at least $\overline{C}$ when $y_{t+1} = 1$ and is at least $\overline{C} - \eta V \geq \overline{V}$ when $y_{t+1} = 0$ (as $\eta \leq (\overline{C} - \overline{V})/V$). By (2.2d), $x_t \leq \overline{C}$. If $y_{t+1} = 0$, then by (2.2d) and (2.2f), $x_{t+1} = 0$ and $x_t - x_{t+1} \leq Vy_{t+1} + \overline{V}(1 - y_{t+1})$, which imply that $x_t \leq \overline{V}$. Hence, x_t is at most $\overline{C}$, and it is at most $\overline{V}$ when $y_{t+1} = 0$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.3).

Next, we show that $(\mathbf{x}, \mathbf{y})$ satisfies (4.4). Consider any $t \in [2, T - s_{\max} - 1]_{\mathbb{Z}}$ (i.e., $t \in [2, T]_{\mathbb{Z}}$ such that $t \leq T - s - 1$ for all $s \in \mathcal{S}$). We divide the analysis into three cases.

Case 1: $y_t = 0$. In this case, by (2.2d), $x_t = 0$. Thus, the left hand side of (4.4) and the first term on the right hand side of (4.4) are 0. Because $y_t = 0$, by Lemma 2(i), $y_{t+j} - y_{t+j+1} \leq 0$ for all $j \in [0, \min\{T - t - 1, L - 1\}]_{\mathbb{Z}}$. Because $s_{\max} \leq T - t - 1$ and $\mathcal{S} \subseteq [0, L-1]_{\mathbb{Z}}$, we have $\mathcal{S} \subseteq [0, \min\{T - t - 1, L - 1\}]_{\mathbb{Z}}$. Thus, $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, \lfloor (\overline{C} - \overline{V})/V \rfloor]_{\mathbb{Z}}$, the coefficient "$\overline{C} - \overline{V} - sV$" on the right hand side of (4.4) is nonnegative. Thus, the right hand side of (4.4) is at least $\eta V y_{t-1} \geq 0$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.4).

Case 2: $y_t = 1$ and $y_{t+s'} - y_{t+s'+1} = 1$ for some $s' \in \mathcal{S}$. In this case, $y_{t+s'} = 1$ and $y_{t+s'+1} = 0$. Because $s_{\max} \leq T - t - 1$, we have $s' \leq T - t - 1$. If $s' = T - t - 1$, then it does not exist any $s \in \mathcal{S}$ such that $s > s'$. If $s' \leq T - t - 2$, then $t + s' + 1 \in [1, T-1]_{\mathbb{Z}}$, and by Lemma 2(i), $y_{t+s'+j+1} - y_{t+s'+j+2} \leq 0$ for all $j \in [0, \min\{T - t - s' - 2, L - 1\}]_{\mathbb{Z}}$, which implies that $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in [s' + 1, \min\{T - t - 1, L + s'\}]_{\mathbb{Z}}$. Thus, $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in \mathcal{S}$ such that $s > s'$. Because $y_{t+s'+1} = 0$ and $t + s' + 1 \in [2, T]_{\mathbb{Z}}$, by Lemma 1(i), $y_{t+s'-j+1} - y_{t+s'-j} \leq 0$ for all $j \in [0, \min\{t + s' - 1, L - 1\}]_{\mathbb{Z}}$. Hence, $y_\tau \leq y_{\tau-1}$ for all $\tau \in [\max\{2, t + s' - L + 2\}, t + s' + 1]_{\mathbb{Z}}$. Because $y_{t+s'} = 1$, this implies that $y_\tau = 1$ for all $\tau \in [t+1, t+s']_{\mathbb{Z}}$. Thus, $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in [0, s'-1]_{\mathbb{Z}}$, which implies that $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in \mathcal{S}$ such that $s < s'$. Hence, $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in \mathcal{S} \setminus \{s'\}$. Because $\mathcal{S} \subseteq [0, \lfloor (\overline{C} - \overline{V})/V \rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Thus,

$$- \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \geq -(\overline{C} - \overline{V} - s'V). \tag{8.17}$$

Because $y_t = 1$, by (8.17), the right hand side of (4.4) is at least $s'V + \overline{V}$ when $y_{t-1} = 1$ and is at least $\overline{V} + (s' - \eta)V$ when $y_{t-1} = 0$. By (2.2d), $x_{t+s'+1} = 0$. By (2.2f), $\sum_{\tau=t+1}^{t+s'+1}(x_{\tau-1} - x_\tau) \leq \sum_{\tau=t+1}^{t+s'+1} Vy_\tau + \sum_{\tau=t+1}^{t+s'+1} \overline{V}(1 - y_\tau)$, which implies that $x_t \leq s'V + \overline{V}$. If $y_{t-1} = 0$, then by (2.2d) and (2.2e), $x_{t-1} = 0$ and $x_t - x_{t-1} \leq Vy_{t-1} + \overline{V}(1 - y_{t-1})$, which imply that $x_t \leq \overline{V}$. In addition, if $y_{t-1} = 0$, then $y_t > y_{t-1}$, and because $y_\tau \leq y_{\tau-1}$ for all $\tau \in [\max\{2, t + s' - L + 2\}, t + s' + 1]_{\mathbb{Z}}$, we have $t \leq t + s' - L + 1$, which implies that $s' \geq L - 1 \geq \eta$. Thus, if $y_{t-1} = 0$, then $x_t \leq \overline{V} + (s' - \eta)V$. Hence, x_t is at most $s'V + \overline{V}$, and it is at most $\overline{V} + (s' - \eta)V$ when $y_{t-1} = 0$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.4).

Case 3: $y_t = 1$ and $y_{t+s} - y_{t+s+1} \neq 1$ for all $s \in \mathcal{S}$. In this case, $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, \lfloor(\overline{C} - \overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Thus, $\sum_{s \in \mathcal{S}}(\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \leq 0$. Because $y_t = 1$, the right hand side of (4.4) is at least $\overline{C}$ when $y_{t-1} = 1$ and is at least $\overline{C} - \eta V \geq \overline{V}$ when $y_{t-1} = 0$ (as $\eta \leq (\overline{C} - \overline{V})/V$). By (2.2d), $x_t \leq \overline{C}$. If $y_{t-1} = 0$, then by (2.2d) and (2.2e), $x_{t-1} = 0$ and $x_t - x_{t-1} \leq Vy_{t-1} + \overline{V}(1 - y_{t-1})$, which imply that $x_t \leq \overline{V}$. Hence, x_t is at most $\overline{C}$, and it is at most $\overline{V}$ when $y_{t-1} = 0$. Therefore, $(\mathbf{x}, \mathbf{y})$ satisfies (4.4).

To prove that inequalities (4.3) and (4.4) are facet-defining for $\text{conv}(\mathcal{P})$ when $\eta \in \{0, (\overline{C} - \overline{V})/V\}$ or $\eta = L - 1 \in \mathcal{S}$, it suffices to show that for each of these two inequalities, there exist $2T$ affinely independent points in

$\text{conv}(\mathcal{P})$ that satisfy the inequality at equality when $\eta \in \{0, (\overline{C} - \overline{V})/V\}$ or $\eta = L - 1 \in \mathcal{S}$. When $\eta = 0$, inequalities (4.3) and (4.4) become inequalities (4.1) and (4.2), respectively, and by Proposition 1, they are facet-defining for $\text{conv}(\mathcal{P})$. Hence, in the following, we only consider the case where $\eta = (\overline{C} - \overline{V})/V$ or $\eta = L - 1 \in \mathcal{S}$. Let $\epsilon = \overline{V} - \underline{C} > 0$.

We first show that inequality (4.3) is facet-defining for $\text{conv}(\mathcal{P})$ by creating $2T$ affinely independent points in $\text{conv}(\mathcal{P})$ that satisfy (4.3) at equality when $\eta = (\overline{C} - \overline{V})/V$ or $\eta = L - 1 \in \mathcal{S}$. Because $\mathbf{0} \in \text{conv}(\mathcal{P})$ and $\mathbf{0}$ satisfies (4.3) at equality, it suffices to create the remaining $2T - 1$ nonzero linearly independent points. We denote these $2T - 1$ points as $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ for $r \in [1, T]_{\mathbb{Z}} \setminus \{t\}$ and $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ for $r \in [1, T]_{\mathbb{Z}}$, and denote the qth component of $\bar{\mathbf{x}}^r$, $\bar{\mathbf{y}}^r$, $\hat{\mathbf{x}}^r$, and $\hat{\mathbf{y}}^r$ as $\bar{x}^r_q$, $\bar{y}^r_q$, $\hat{x}^r_q$, and $\hat{y}^r_q$, respectively. We divide these $2T - 1$ points into the following six groups:

(A1) For each $r \in [1, T]_{\mathbb{Z}} \setminus \{t\}$, we create the same point $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ as in group (A1) in the proof of Proposition 1. Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. It is easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (4.3) at equality.

(A2) For each $r \in [1, t - s_{\max} - 2]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (A2) in the proof or Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. It is easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.3) at equality.

(A3) For each $r \in [t - s_{\max} - 1, t - 1]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (A3) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. Consider the case where $t - r - 1 \notin \mathcal{S}$. In this case, $\hat{x}^r_t = \hat{y}^r_t = \hat{y}^r_{t+1} = 0$. In addition, $t - s - 1 \neq r$ for all $s \in \mathcal{S}$, which implies that $\hat{y}^r_{t-s} - \hat{y}^r_{t-s-1} = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.3) at equality. Next, consider the case where $t-r-1 \in \mathcal{S}$. In this case, $\hat{x}_t^r = \overline{V} + (t-r-1)V$ and $\hat{y}_t^r = \hat{y}_{t+1}^r = 1$. In addition, $\hat{y}_{t-s}^r - \hat{y}_{t-s-1}^r = 1$ when $s = t-r-1$, and $\hat{y}_{t-s}^r - \hat{y}_{t-s-1}^r = 0$ when $s \neq t-r-1$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.3) at equality.

(A4) We create a point $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ as follows: If $\eta = (\overline{C} - \overline{V})/V$, then

$$\hat{x}_q^t = \begin{cases} \overline{V}, & \text{for } q \in [1,t]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [t+1,T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^t = \begin{cases} 1, & \text{for } q \in [1,t]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [t+1,T]_{\mathbb{Z}}. \end{cases}$$

If $\eta = L-1 \in \mathcal{S}$, then

$$\hat{x}_q^t = \begin{cases} \overline{V}, & \text{for } q \in [t-L+1,t]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [1,t-L]_{\mathbb{Z}} \cup [t+1,T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^t = \begin{cases} 1, & q \in [t-L+1,t]_{\mathbb{Z}}; \\ 0, & q \in [1,t-L]_{\mathbb{Z}} \cup [t+1,T]_{\mathbb{Z}}. \end{cases}$$

We first consider the case where $\eta = (\overline{C} - \overline{V})/V$. It is easy to verify that $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (2.2a)–(2.2f). Thus, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t) \in \text{conv}(\mathcal{P})$. In this case, $\hat{x}_t^t = \overline{V}$, $\hat{y}_t^t = 1$, and $\hat{y}_{t+1}^t = 0$, and $\hat{y}_{t-s}^t - \hat{y}_{t-s-1}^t = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (4.3) at equality. Next, we consider the case where $\eta = L-1 \in \mathcal{S}$. In this case, for any $q \in [2,T]_{\mathbb{Z}}$, $\hat{y}_q^t - \hat{y}_{q-1}^t \leq 0$ if $q \neq t-L+1$, while $\hat{y}_q^t - \hat{y}_{q-1}^t = 1$ and $\hat{y}_k^t = 1$

for all $k \in [q, \min\{T, q+L-1\}]_{\mathbb{Z}}$ if $q = t-L+1$. Thus, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (2.2a). For any $q \in [2,T]_{\mathbb{Z}}$, $\hat{y}^t_{q-1} - \hat{y}^t_q \leq 0$ if $q \neq t+1$, while $\hat{y}^t_{q-1} - \hat{y}^t_q = 1$ and $\hat{y}^t_k = 0$ for all $k \in [q, \min\{T, q+\ell-1\}]_{\mathbb{Z}}$ if $q = t+1$. Thus, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (2.2b). It is easy to verify that $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (2.2c)–(2.2f). Thus, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t) \in \mathrm{conv}(\mathcal{P})$. Note that $\hat{x}^t_t = \overline{V}$, $\hat{y}^t_t = 1$, $\hat{y}^t_{t+1} = 0$, $\hat{y}^t_{t-s} - \hat{y}^t_{t-s-1} = 0$ for all $s \in \mathcal{S} \setminus \{L-1\}$, $\hat{y}^t_{t-L+1} - \hat{y}^t_{t-L} = 1$, and $(\overline{C} - \eta V) - (\overline{C} - \overline{V} - (L-1)V) = \overline{V}$. Thus, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (4.3) at equality.

(A5) We create a point $(\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$ by setting $\hat{x}^{t+1}_q = \overline{C}$ and $\hat{y}^{t+1}_q = 1$ for $q \in [1,T]_{\mathbb{Z}}$. It is easy to verify that $(\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$ satisfies (2.2a)–(2.2f). Thus, $(\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1}) \in \mathrm{conv}(\mathcal{P})$. It is also easy to verify that $(\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$ satisfies (4.3) at equality.

(A6) For each $r \in [t+2, T]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (A5) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \mathrm{conv}(\mathcal{P})$. It is easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.3) at equality.

Table A.5 shows a matrix with $2T-1$ rows, where each row represents a point created by this process. This matrix can be transformed into the matrix in Table A.6 via the following Gaussian elimination process:

(i) For each $r \in [1,T]_{\mathbb{Z}} \setminus \{t\}$, the point with index r in group (B1), denoted $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r) = (\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$. Here, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ is the point with index r in group (A1), and $(\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$ is the point in group (A5).

(ii) For each $r \in [1, t-s_{\max}-2]_{\mathbb{Z}}$, the point with index r in group (B2), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (A2).

(iii) For each $r \in [t - s_{\max} - 1, t - 1]_{\mathbb{Z}}$, the point with index r in group (B3), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $t - r - 1 \notin \mathcal{S}$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$ if $t - r - 1 \in \mathcal{S}$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (A3), and $(\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$ is the point in group (A5).

(iv) The point in group (B4), denoted $(\underline{\hat{\mathbf{x}}}^t, \underline{\hat{\mathbf{y}}}^t)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^t, \underline{\hat{\mathbf{y}}}^t) = (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$. Here, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ is the point in group (A4).

(v) The point in group (B5), denoted $(\underline{\hat{\mathbf{x}}}^{t+1}, \underline{\hat{\mathbf{y}}}^{t+1})$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^{t+1}, \underline{\hat{\mathbf{y}}}^{t+1}) = (\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1}) - (\hat{\mathbf{x}}^{t+2}, \hat{\mathbf{y}}^{t+2})$. Here, $(\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$ is the point in group (A5), and $(\hat{\mathbf{x}}^{t+2}, \hat{\mathbf{y}}^{t+2})$ is the point with index $t+2$ in group (A6).

(vi) For each $r \in [t + 2, T]_{\mathbb{Z}}$, the point with index r in group (B6), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ if $r \neq T$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $r = T$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ and $(\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ are the points with indices r and $r+1$, respectively, in group (A6).

Table A.5: A matrix with the rows representing $2T-1$ points in $\mathrm{conv}(\mathcal{P})$ that satisfy inequality (4.3) at equality.

Group	Point	Index r	**x**																**y**															
			1	2	$\cdots$	$t-s_{\max}-2$	$t-s_{\max}-1$	$t-s_{\max}$	$\cdots$	$t-1$	t	$t+1$	$t+2$	$\cdots$	T	1	2	$\cdots$	$t-s_{\max}-2$	$t-s_{\max}-1$	$t-s_{\max}$	$\cdots$	$t-1$	t	$t+1$	$t+2$	$\cdots$	T
(A1)	$(\bar{\mathbf{x}}^r,\bar{\mathbf{y}}^r)$	1	$\overline{C}-\epsilon$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	$\cdots$	1	1	1	1	$\cdots$	1
		2	$\overline{C}$	$\overline{C}-\epsilon$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	$\cdots$	1	1	1	1	$\cdots$	1
		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-s_{\max}-2$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}-\epsilon$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	$\cdots$	1	1	1	1	$\cdots$	1
		$t-s_{\max}-1$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}-\epsilon$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	$\cdots$	1	1	1	1	$\cdots$	1
		$t-s_{\max}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}-\epsilon$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	$\cdots$	1	1	1	1	$\cdots$	1
		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-1$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}-\epsilon$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	$\cdots$	1	1	1	1	$\cdots$	1
		$t+1$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}-\epsilon$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	$\cdots$	1	1	1	1	$\cdots$	1
		$t+2$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}-\epsilon$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	$\cdots$	1	1	1	1	$\cdots$	1
		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
		T	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}-\epsilon$	1	1	$\cdots$	1	1	1	$\cdots$	1	1	1	1	$\cdots$	1
(A2)	$(\hat{\mathbf{x}}^r,\hat{\mathbf{y}}^r)$	1	$\underline{C}$	0	$\cdots$	0	0	0	$\cdots$	0	0	0	0	$\cdots$	0	1	0	$\cdots$	0	0	0	$\cdots$	0	0	0	0	$\cdots$	0
		2	$\underline{C}$	$\underline{C}$	$\cdots$	0	0	0	$\cdots$	0	0	0	0	$\cdots$	0	1	1	$\cdots$	0	0	0	$\cdots$	0	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-s_{\max}-2$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}$	0	0	$\cdots$	0	0	0	0	$\cdots$	0	1	1	$\cdots$	1	0	0	$\cdots$	0	0	0	0	$\cdots$	0
(A3)		$t-s_{\max}-1$, $\vdots$, $t-1$	(See Note A.5-1)													(See Note A.5-1)												
(A4)		t	(See Note A.5-2)													(See Note A.5-2)												
(A5)		$t+1$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	$\cdots$	1	1	1	1	$\cdots$	1
(A6)		$t+2$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\underline{C}$	$\cdots$	$\underline{C}$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	1	$\cdots$	1
		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
		T	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	0	$\cdots$	$\underline{C}$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	0	$\cdots$	1

Note A.5-1: For $r \in [t-s_{\max}-1, t-1]_{\mathbb{Z}}$, the **x** and **y** vectors in group (A3) are given as follows: $\hat{\mathbf{x}}^r = (\underbrace{\underline{C},\ldots,\underline{C}}_{r \text{ terms}}, \underbrace{0,\ldots,0}_{T-r \text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{1,\ldots,1}_{r \text{ terms}}, \underbrace{0,\ldots,0}_{T-r \text{ terms}})$ if $t-r-1 \notin \mathcal{S}$;

$\hat{\mathbf{x}}^r = (\underbrace{0,\ldots,0}_{r \text{ terms}}, \underbrace{\overline{V}, \overline{V}+V, \overline{V}+2V, \ldots, \overline{V}+(t-r-1)V}_{t-r \text{ terms}}, \underbrace{\overline{V}+(t-r-1)V, \ldots, \overline{V}+(t-r-1)V}_{T-t \text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{0,\ldots,0}_{r \text{ terms}}, \underbrace{1,\ldots,1}_{T-r \text{ terms}})$ if $t-r-1 \in \mathcal{S}$.

Note A.5-2: The **x** and **y** vectors in group (A4) are given as follows: $\hat{\mathbf{x}}^t = (\underbrace{\overline{V},\ldots,\overline{V}}_{t \text{ terms}}, \underbrace{0,\ldots,0}_{T-t \text{ terms}})$ and $\hat{\mathbf{y}}^t = (\underbrace{1,\ldots,1}_{t \text{ terms}}, \underbrace{0,\ldots,0}_{T-t \text{ terms}})$ if $\eta = (\overline{C}-\overline{V})/V$;

$\hat{\mathbf{x}}^t = (\underbrace{0,\ldots,0}_{t-L \text{ terms}}, \underbrace{\overline{V},\ldots,\overline{V}}_{L \text{ terms}}, \underbrace{0,\ldots,0}_{T-t \text{ terms}})$ and $\hat{\mathbf{y}}^t = (\underbrace{0,\ldots,0}_{t-L \text{ terms}}, \underbrace{1,\ldots,1}_{L \text{ terms}}, \underbrace{0,\ldots,0}_{T-t \text{ terms}})$ if $\eta = L-1 \in \mathcal{S}$.

Table A.6: Lower triangular matrix obtained from Table A.5 via Gaussian elimination.

Group	Point	Index r	$\mathbf{x}$														$\mathbf{y}$													
			1	2	⋯	$t-s_{\max}-2$	$t-s_{\max}-1$	$t-s_{\max}$	⋯	$t-1$	t	$t+1$	$t+2$	⋯	T	1	2	⋯	$t-s_{\max}-2$	$t-s_{\max}-1$	$t-s_{\max}$	⋯	$t-1$	t	$t+1$	$t+2$	⋯	T
		1	$-\epsilon$	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0
		2	0	$-\epsilon$	⋯	0	0	0	⋯	0	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0
		⋮	⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮	⋮		⋮
		$t-s_{\max}-2$	0	0	⋯	$-\epsilon$	0	0	⋯	0	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0
		$t-s_{\max}-1$	0	0	⋯	0	$-\epsilon$	0	⋯	0	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0
(B1)	$(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$	$t-s_{\max}$	0	0	⋯	0	0	$-\epsilon$	⋯	0	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0
		⋮	⋮	⋮		⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮	⋮		⋮
		$t-1$	0	0	⋯	0	0	0	⋯	$-\epsilon$	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0
		$t+1$	0	0	⋯	0	0	0	⋯	0	0	$-\epsilon$	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0
		$t+2$	0	0	⋯	0	0	0	⋯	0	0	0	$-\epsilon$	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0
		⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮	⋮		⋮
		T	0	0	⋯	0	0	0	⋯	0	0	0	0	⋯	$-\epsilon$	0	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0
		1														1	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0
(B2)		2														1	1	⋯	0	0	0	⋯	0	0	0	0	⋯	0
		⋮	(Omitted)													⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮	⋮	⋮		⋮
		$t-s_{\max}-2$														1	1	⋯	1	0	0	⋯	0	0	0	0	⋯	0
		$t-s_{\max}-1$																										
(B3)	$(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$	⋮	(Omitted)													(See Note A.6-1)												
		$t-1$																										
(B4)		t	(Omitted)													(See Note A.6-2)												
(B5)		$t+1$	(Omitted)													1	1	⋯	1	1	1	⋯	1	1	1	0	⋯	0
		$t+2$														0	0	⋯	0	0	0	⋯	0	0	0	1	⋯	0
(B6)		⋮	(Omitted)													⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮	⋮	⋱	⋮
		T														0	0	⋯	0	0	0	⋯	0	0	0	0	⋯	1

Note A.6-1: For $r \in [t-s_{\max}-1, t-1]_{\mathbb{Z}}$, the $\mathbf{y}$ vector in group (B3) is given as follows: $\underline{\hat{\mathbf{y}}}^r = (\underbrace{1,\ldots,1}_{r \text{ terms}}, \underbrace{0,\ldots,0}_{T-r \text{ terms}})$ if $t-r-1 \notin \mathcal{S}$; $\underline{\hat{\mathbf{y}}}^r = (\underbrace{-1,\ldots,-1}_{r \text{ terms}}, \underbrace{0,\ldots,0}_{T-r \text{ terms}})$ if $t-r-1 \in \mathcal{S}$.

Note A.6-2: The $\mathbf{y}$ vector in group (B4) is given as follows: $\underline{\hat{\mathbf{y}}}^t = (\underbrace{1,\ldots,1}_{t \text{ terms}}, \underbrace{0,\ldots,0}_{T-t \text{ terms}})$ if $\eta = (\overline{C}-\overline{V})/V$; $\underline{\hat{\mathbf{y}}}^t = (\underbrace{0,\ldots,0}_{t-L \text{ terms}}, \underbrace{1,\ldots,1}_{L \text{ terms}}, \underbrace{0,\ldots,0}_{T-t \text{ terms}})$ if $\eta = L-1 \in \mathcal{S}$.

The matrix shown in Table A.6 is lower triangular; that is, the position of the last nonzero component of a row of the matrix is greater than the position of the last nonzero component of the previous row. This implies that the $2T-1$ points in groups (A1)–(A6) are linearly independent. Therefore, inequality (4.3) is facet-defining for $\text{conv}(\mathcal{P})$.

Next, we show that inequality (4.4) is facet-defining for $\text{conv}(\mathcal{P})$ by creating $2T$ affinely independent points in $\text{conv}(\mathcal{P})$ that satisfy (4.4) at equality when $\eta \in \{0, (\overline{C}-\overline{V})/V\}$ or $\eta = L-1 \in \mathcal{S}$. Because $\mathbf{0} \in \text{conv}(\mathcal{P})$ and $\mathbf{0}$ satisfies (4.4) at equality, it suffices to create the remaining $2T-1$ nonzero linearly independent points. We denote these $2T-1$ points as $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ for $r \in [1,T]_{\mathbb{Z}} \setminus \{t\}$, and $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ for $r \in [1,T]_{\mathbb{Z}}$, and denote the qth component of $\bar{\mathbf{x}}^r$, $\bar{\mathbf{y}}^r$, $\hat{\mathbf{x}}^r$, and $\hat{\mathbf{y}}^r$ as $\bar{x}_q^r$, $\bar{y}_q^r$, $\hat{x}_q^r$, and $\hat{y}_q^r$, respectively. We divided these $2T-1$ points into the following six groups:

(C1) For each $r \in [1,T]_{\mathbb{Z}} \setminus \{t\}$, we create the same point $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ as in group (A1) in the proof of Proposition 1. Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. It is easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (4.4) at equality.

(C2) For each $r \in [1,t-2]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (C2) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. It is easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.4) at equality.

(C3) We create a point $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ as follows: If $\eta = (\overline{C}-\overline{V})/V$, then

$$\hat{x}_q^{t-1} = \begin{cases} 0, & \text{for } q \in [1,t-1]_{\mathbb{Z}}; \\ \overline{V}, & \text{for } q \in [t,T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^{t-1} = \begin{cases} 0, & \text{for } q \in [1, t-1]_{\mathbb{Z}}; \\ 1, & \text{for } q \in [t, T]_{\mathbb{Z}}. \end{cases}$$

If $\eta = L - 1 \in \mathcal{S}$, then

$$\hat{x}_q^{t-1} = \begin{cases} \overline{V}, & \text{for } q \in [t, t+L-1]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [1, t-1]_{\mathbb{Z}} \cup [t+L, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^{t-1} = \begin{cases} 1, & \text{for } q \in [t, t+L-1]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [1, t-1]_{\mathbb{Z}} \cup [t+L, T]_{\mathbb{Z}}. \end{cases}$$

We first consider the case $\eta = (\overline{C} - \overline{V})/V$. It is easy to verify that $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ satisfies (2.2a)–(2.2f). Thus, $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1}) \in \text{conv}(\mathcal{P})$. In this case, $\hat{x}_t^{t-1} = \overline{V}$, $\hat{y}_t^{t-1} = 1$, $\hat{y}_{t-1}^{t-1} = 0$, $\hat{y}_{t+s}^{t-1} - \hat{y}_{t+s+1}^{t-1} = 0$ for all $s \in \mathcal{S}$, and $\overline{C} - \eta V = \overline{V}$. Thus, $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ satisfies (4.4) at equality. Next, we consider the case where $\eta = L - 1 \in \mathcal{S}$. In this case, for any $q \in [2, T]_{\mathbb{Z}}$, $\hat{y}_q^{t-1} - \hat{y}_{q-1}^{t-1} \leq 0$ if $q \neq t$, while $\hat{y}_q^{t-1} - \hat{y}_{q-1}^{t-1} = 1$ and $\hat{y}_k^{t-1} = 1$ for all $k \in [q, \min\{T, q+L-1\}]_{\mathbb{Z}}$ if $q = t$. Thus, $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ satisfies (2.2a). For any $q \in [2, T]_{\mathbb{Z}}$, $\hat{y}_{q-1}^{t-1} - \hat{y}_q^{t-1} \leq 0$ if $q \neq t + L$, while $\hat{y}_{q-1}^{t-1} - \hat{y}_q^{t-1} = 1$ and $\hat{y}_q^{t-1} = 0$ for all $k \in [q, \min\{T, q+\ell-1\}]_{\mathbb{Z}}$ if $q = t + L$. Thus, $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ satisfies (2.2b). It is easy to verify that $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ satisfies (2.2c)–(2.2f). Thus, $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1}) \in \text{conv}(\mathcal{P})$. Note that $\hat{x}_t^{t-1} = \overline{V}$, $\hat{y}_t^{t-1} = 1$, $\hat{y}_{t-1}^{t-1} = 0$, $\hat{y}_{t+s}^{t-1} - \hat{y}_{t+s+1}^{t-1} = 0$ for all $s \in \mathcal{S} \setminus \{L-1\}$, $\hat{y}_{t+L-1}^{t-1} - \hat{y}_{t+L}^{t-1} = 1$, and $(\overline{C} - \eta V) - (\overline{C} - \overline{V} - (L-1)V) = \overline{V}$. Thus, $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ satisfies (4.4) at equality.

(C4) For each $r \in [t, t+s_{\max}]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (C3) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. Consider the case where $r - t \notin \mathcal{S}$. In this case, $\hat{x}^r_t = \hat{y}^r_t = \hat{y}^r_{t-1} = 0$. In addition, $t + s \neq r$ for all $s \in \mathcal{S}$, which implies that $\hat{y}^r_{t+s} - \hat{y}^r_{t+s+1} = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.4) at equality. Next, consider the case where $r - t \in \mathcal{S}$. In this case, $\hat{x}^r_t = \overline{V} + (r-t)V$ and $\hat{y}^r_{t-1} = \hat{y}^r_t = 1$. In addition, $\hat{y}^r_{t+s} - \hat{y}^r_{t+s+1} = 1$ where $s = r - t$, and $\hat{y}^r_{t+s} - \hat{y}^r_{t+s+1} = 0$ where $s \neq r - t$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.4) at equality.

(C5) We create the same point $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ as in group (C4) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1}) \in \text{conv}(\mathcal{P})$. It is easy to verify that $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ satisfies (4.4) at equality.

(C6) For each $r \in [t + s_{\max} + 2, T]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (C5) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. It is easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.4) at equality.

Table A.7 shows a matrix with $2T - 1$ rows, where each row represents a point created by this process. This matrix can be transformed into the matrix in Table A.8 via the following Gaussian elimination process:

(i) For each $r \in [1, T]_{\mathbb{Z}} \setminus \{t\}$, the point with index r in group (D1), denoted $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}) = (\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$. Here, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ is the point with index r in group (C1), and $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ is the point in group (C5).

(ii) For each $r \in [1, t-2]_{\mathbb{Z}}$, the point with index r in group (D2), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (C2).

(iii) The point in group (D3), denoted $(\underline{\hat{\mathbf{x}}}^{t-1}, \underline{\hat{\mathbf{y}}}^{t-1})$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^{t-1}, \underline{\hat{\mathbf{y}}}^{t-1}) = (\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1}) - (\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ if $\eta = (\overline{C} - \overline{V})/V$, and setting $(\underline{\hat{\mathbf{x}}}^{t-1}, \underline{\hat{\mathbf{y}}}^{t-1}) = (\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1}) - (\hat{\mathbf{x}}^{t+L-1}, \hat{\mathbf{y}}^{t+L-1})$ if $\eta = L - 1 \in \mathcal{S}$. Here, $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ is the point in group (C3), $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ is the point in group (C5), and $(\hat{\mathbf{x}}^{t+L-1}, \hat{\mathbf{y}}^{t+L-1})$ is the point with index $t + L - 1$ in group (C4).

(iv) For each $r \in [t, t + s_{\max}]_{\mathbb{Z}}$, the point with index r in group (D4), denoted $(\underline{\hat{\mathbf{x}}}^{r}, \underline{\hat{\mathbf{y}}}^{r})$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^{r}, \underline{\hat{\mathbf{y}}}^{r}) = (\hat{\mathbf{x}}^{r}, \hat{\mathbf{y}}^{r}) - (\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ if $r - t \notin \mathcal{S}$, and setting $(\underline{\hat{\mathbf{x}}}^{r}, \underline{\hat{\mathbf{y}}}^{r}) = (\hat{\mathbf{x}}^{r}, \hat{\mathbf{y}}^{r})$ if $r - t \in \mathcal{S}$. Here, $(\hat{\mathbf{x}}^{r}, \hat{\mathbf{y}}^{r})$ is the point with index r in group (C4), and $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ is the point in group (C5).

(v) The point in group (D5), denoted $(\underline{\hat{\mathbf{x}}}^{t+s_{\max}+1}, \underline{\hat{\mathbf{y}}}^{t+s_{\max}+1})$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^{t+s_{\max}+1}, \underline{\hat{\mathbf{y}}}^{t+s_{\max}+1}) = (\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1}) - (\hat{\mathbf{x}}^{t+s_{\max}+2}, \hat{\mathbf{y}}^{t+s_{\max}+2})$. Here, $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ is the point in group (C5), and $(\hat{\mathbf{x}}^{t+s_{\max}+2}, \hat{\mathbf{y}}^{t+s_{\max}+2})$ is the point with index $t + s_{\max} + 2$ in group (C6).

Table A.7: A matrix with the rows representing $2T-1$ points in $\text{conv}(\mathcal{P})$ that satisfy inequality (4.4) at equality.

Group	Point	Index r	$\mathbf{x}$														$\mathbf{y}$													
			1	2	$\cdots$	$t-2$	$t-1$	t	$t+1$	$\cdots$	$t+s_{\max}$	$t+s_{\max}+1$	$t+s_{\max}+2$	$\cdots$	T	1	2	$\cdots$	$t-2$	$t-1$	t	$t+1$	$\cdots$	$t+s_{\max}$	$t+s_{\max}+1$	$t+s_{\max}+2$	$\cdots$	T
(C1)	$(\bar{\mathbf{x}}^r,\bar{\mathbf{y}}^r)$	1	$\overline{C}-\epsilon$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	1	$\cdots$	1	1	1	$\cdots$	1
		2	$\overline{C}$	$\overline{C}-\epsilon$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	1	$\cdots$	1	1	1	$\cdots$	1
		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-2$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}-\epsilon$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	1	$\cdots$	1	1	1	$\cdots$	1
		$t-1$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}-\epsilon$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	1	$\cdots$	1	1	1	$\cdots$	1
		$t+1$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}-\epsilon$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	1	$\cdots$	1	1	1	$\cdots$	1
		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t+s_{\max}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}-\epsilon$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	1	$\cdots$	1	1	1	$\cdots$	1
		$t+s_{\max}+1$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}-\epsilon$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	1	$\cdots$	1	1	1	$\cdots$	1
		$t+s_{\max}+2$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}-\epsilon$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	1	$\cdots$	1	1	1	$\cdots$	1
		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		T	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}-\epsilon$	1	1	$\cdots$	1	1	1	1	$\cdots$	1	1	1	$\cdots$	1
(C2)	$(\hat{\mathbf{x}}^r,\hat{\mathbf{y}}^r)$	1	$\underline{C}$	0	$\cdots$	0	0	0	0	$\cdots$	0	0	0	$\cdots$	0	1	0	$\cdots$	0	0	0	0	$\cdots$	0	0	0	$\cdots$	0
		2	$\underline{C}$	$\underline{C}$	$\cdots$	0	0	0	0	$\cdots$	0	0	0	$\cdots$	0	1	1	$\cdots$	0	0	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-2$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	1	1	$\cdots$	1	0	0	0	$\cdots$	0	0	0	$\cdots$	0
(C3)		$t-1$	(See Note A.7-1)													(See Note A.7-1)													
(C4)		t	(See Note A.7-2)													(See Note A.7-2)													
		$\vdots$																											
		$t+s_{\max}$																											
(C5)		$t+s_{\max}+1$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	1	$\cdots$	1	1	1	$\cdots$	1
(C6)		$t+s_{\max}+2$	0	0	$\cdots$	0	0	0	0	$\cdots$	0	0	$\underline{C}$	$\cdots$	$\underline{C}$	0	0	$\cdots$	0	0	0	0	$\cdots$	0	0	1	$\cdots$	1
		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
		T	0	0	$\cdots$	0	0	0	0	$\cdots$	0	0	0	$\cdots$	$\underline{C}$	0	0	$\cdots$	0	0	0	0	$\cdots$	0	0	0	$\cdots$	1

Note A.7-1: The $\mathbf{x}$ and $\mathbf{y}$ vectors in group (C3) are given as follows: $\hat{\mathbf{x}}^{t-1} = (\underbrace{0,\ldots,0}_{t-1 \text{ terms}}, \underbrace{\overline{V},\ldots,\overline{V}}_{T-t+1 \text{ terms}})$ and $\hat{\mathbf{y}}^{t-1} = (\underbrace{0,\ldots,0}_{t-1 \text{ terms}}, \underbrace{1,\ldots,1}_{T-t+1 \text{ terms}})$ if $\eta = (\overline{C}-\overline{V})/V$;

$\hat{\mathbf{x}}^{t-1} = (\underbrace{0,\ldots,0}_{t-1 \text{ terms}}, \underbrace{\overline{V},\ldots,\overline{V}}_{L \text{ terms}}, \underbrace{0,\ldots,0}_{T-t-L+1 \text{ terms}})$ and $\hat{\mathbf{y}}^{t-1} = (\underbrace{0,\ldots,0}_{t-1 \text{ terms}}, \underbrace{1,\ldots,1}_{L \text{ terms}}, \underbrace{0,\ldots,0}_{T-t-L+1 \text{ terms}})$ if $\eta = L-1 \in \mathcal{S}$.

Note A.7-2: For $r \in [t, t+s_{\max}]_{\mathbb{Z}}$, the $\mathbf{x}$ and $\mathbf{y}$ vectors in group (C4) are given as follows: $\hat{\mathbf{x}}^r = (\underbrace{0,\ldots,0}_{r \text{ terms}}, \underbrace{\underline{C},\ldots,\underline{C}}_{T-r \text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{0,\ldots,0}_{r \text{ terms}}, \underbrace{1,\ldots,1}_{T-r \text{ terms}})$ if $r-t \notin \mathcal{S}$;

$\hat{\mathbf{x}}^r = (\underbrace{\overline{V}+(r-t)V,\ldots,\overline{V}+(r-t)V}_{t-1 \text{ terms}}, \underbrace{\overline{V}+(r-t)V, \overline{V}+(r-t-1)V, \overline{V}+(r-t-2)V,\ldots,\overline{V}}_{r-t+1 \text{ terms}}, \underbrace{0,\ldots,0}_{T-r \text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{1,\ldots,1}_{r \text{ terms}}, \underbrace{0,\ldots,0}_{T-r \text{ terms}})$ if $r-t \in \mathcal{S}$.

Table A.8: Lower triangular matrix obtained from Table A.7 via Gaussian elimination.

Group	Point	Index r	$\mathbf{x}$													$\mathbf{y}$												
			1	2	⋯	$t-2$	$t-1$	t	$t+1$	⋯	$t+s_{\max}$	$t+s_{\max}+1$	$t+s_{\max}+2$	⋯	T	1	2	⋯	$t-2$	$t-1$	t	$t+1$	⋯	$t+s_{\max}$	$t+s_{\max}+1$	$t+s_{\max}+2$	⋯	T
(D1)		1	$-\epsilon$	0	⋯	0	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0	0	0	⋯	0
		2	0	$-\epsilon$	⋯	0	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0	0	0	⋯	0
		⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮	⋮		⋮	⋮	⋮		⋮
		$t-2$	0	0	⋯	$-\epsilon$	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0	0	0	⋯	0
	$(\underline{\bar{\mathbf{x}}}^r,\underline{\bar{\mathbf{y}}}^r)$	$t-1$	0	0	⋯	0	$-\epsilon$	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0	0	0	⋯	0
		$t+1$	0	0	⋯	0	0	0	$-\epsilon$	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0	0	0	⋯	0
		⋮	⋮	⋮		⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮	⋮		⋮	⋮	⋮		⋮
		$t+s_{\max}$	0	0	⋯	0	0	0	0	⋯	$-\epsilon$	0	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0	0	0	⋯	0
		$t+s_{\max}+1$	0	0	⋯	0	0	0	0	⋯	0	$-\epsilon$	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0	0	0	⋯	0
		$t+s_{\max}+2$	0	0	⋯	0	0	0	0	⋯	0	0	$-\epsilon$	⋯	0	0	0	⋯	0	0	0	0	⋯	0	0	0	⋯	0
		⋮	⋮	⋮		⋮	⋮	⋮	⋮		⋮	⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮	⋮	⋮		⋮	⋮	⋮		⋮
		T	0	0	⋯	0	0	0	0	⋯	0	0	0	⋯	$-\epsilon$	0	0	⋯	0	0	0	0	⋯	0	0	0	⋯	0
(D2)		1	(Omitted)													1	0	⋯	0	0	0	0	⋯	0	0	0	⋯	0
		2														1	1	⋯	0	0	0	0	⋯	0	0	0	⋯	0
		⋮														⋮	⋮	⋱	⋮	⋮	⋮	⋮		⋮	⋮	⋮		⋮
		$t-2$														1	1	⋯	1	0	0	0	⋯	0	0	0	⋯	0
(D3)	$(\underline{\hat{\mathbf{x}}}^r,\underline{\hat{\mathbf{y}}}^r)$	$t-1$	(Omitted)													-1	-1	⋯	-1	-1	0	0	⋯	0	0	0	⋯	0
(D4)		t	(Omitted)													(See Note A.8-1)												
		⋮																										
		$t+s_{\max}$																										
(D5)		$t+s_{\max}+1$	(Omitted)													1	1	⋯	1	1	1	1	⋯	1	1	0	⋯	0
(D6)		$t+s_{\max}+2$	(Omitted)													0	0	⋯	0	0	0	0	⋯	0	0	1	⋯	0
		⋮														⋮	⋮		⋮	⋮	⋮	⋮		⋮	⋮	⋮	⋱	⋮
		T														0	0	⋯	0	0	0	0	⋯	0	0	0	⋯	1

Note A.8-1: For $r \in [t, t+s_{\max}]_{\mathbb{Z}}$, the $\mathbf{y}$ vector in group (D4) is given as follows: $\underline{\hat{\mathbf{y}}}^r = (\underbrace{-1,\ldots,-1}_{r \text{ terms}}, \underbrace{0,\ldots,0}_{T-r \text{ terms}})$ if $r-t \notin \mathcal{S}$; $\underline{\hat{\mathbf{y}}}^r = (\underbrace{1,\ldots,1}_{r \text{ terms}}, \underbrace{0,\ldots,0}_{T-r \text{ terms}})$ if $r-t \in \mathcal{S}$.

(vi) For each $r \in [t + s_{\max} + 2, T]_{\mathbb{Z}}$, the point with index r in group (D6), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ if $r \neq T$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $r = T$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ and $(\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ are the points with indices r and $r + 1$, respectively, in group (C6).

The matrix shown in Table A.8 is lower triangular; that is, the position of the last nonzero component of a row of the matrix is greater than the position of the last nonzero component of the previous row. This implies that these $2T - 1$ points in groups (C1)–(C6) are linearly independent. Therefore, inequality (4.4) is facet-defining for $\text{conv}(\mathcal{P})$. □

8.7 Proof of Proposition 4

To prove that linear inequalities (4.3) and (4.4) are valid for $\mathbf{conv}(\mathcal{P})$ when $\mathcal{S} = [0, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$, it suffices to show that (4.3) and (4.4) are valid for $\mathcal{P}$ when $\mathcal{S} = [0, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. Consider any element $(\mathbf{x}, \mathbf{y})$ of $\mathcal{P}$. We show that $(\mathbf{x}, \mathbf{y})$ satisfies (4.3) and (4.4) when $\mathcal{S} = [0, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$.

We first show that $(\mathbf{x}, \mathbf{y})$ satisfies (4.3) when $\mathcal{S} = [0, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. Consider any $t \in [s_{\max} + 2, T - 1]_{\mathbb{Z}}$. We divide the analysis into four cases.

Case 1: $y_t = 0$ and $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, \lfloor (\overline{C} - \overline{V})/V \rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Thus, in this case, the right hand side of inequality (4.3) is nonnegative. Because $y_t = 0$, by (2.2d), $x_t = 0$. Therefore, $(\mathbf{x}, \mathbf{y})$ satisfies (4.3).

Case 2: $y_t = 0$ and $y_{t-s} - y_{t-s-1} > 0$ for some $s \in \mathcal{S}$. Let $\tilde{\mathcal{S}} = \{\sigma \in \mathcal{S} : y_{t-\sigma} - y_{t-\sigma-1} > 0\}$ and $v = |\tilde{\mathcal{S}}|$. Then, $v \geq 1$. Denote $\tilde{\mathcal{S}} = \{\sigma_1, \sigma_2, \ldots, \sigma_v\}$, where $\sigma_1 < \sigma_2 < \cdots < \sigma_v$. Note that $y_{t-\sigma_j-1} = 0$ and $y_{t-\sigma_j} = 1$ for $j = 1, \ldots, v$. Denote $\sigma_0 = -1$. Then, for each $j = 1, \ldots, v$, there exists $\sigma'_j \in [\sigma_{j-1} + 1, \sigma_j - 1]_{\mathbb{Z}}$ such that $y_{t-\sigma'_j-1} = 1$ and $y_{t-\sigma'_j} = 0$. Thus,

$$0 \leq \sigma'_1 < \sigma_1 < \sigma'_2 < \sigma_2 < \cdots < \sigma'_v < \sigma_v \leq s_{\max}.$$

Because $y_{t-\sigma_v} - y_{t-\sigma_v-1} = 1$ and $t - \sigma_v \in [2, T]_{\mathbb{Z}}$, by (2.2a), $y_k = 1$ for all $k \in [t - \sigma_v, \min\{T, t - \sigma_v + L - 1\}]_{\mathbb{Z}}$, which implies that $t - \sigma'_j \geq t - \sigma_v + L$ for $j = 1, \ldots, v$. Hence, for $j = 1, \ldots, v$, we have $\sigma'_j \leq \sigma_v - L$, which implies that

$$\sigma'_j \leq s_{\max} - L. \tag{8.18}$$

If $\beta = \alpha + 1$, then $\mathcal{S} = [0, s_{\max}]_{\mathbb{Z}}$, which implies that $\sigma'_j \in \mathcal{S}$ for $j = 1, \ldots, v$. If $\beta \neq \alpha + 1$, then condition (c) of Proposition 4 implies that $s_{\max} \leq L + \alpha$, which, by (8.18), implies that $\sigma'_j \leq \alpha$ for $j = 1, \ldots, v$. Thus, in both cases, $\sigma'_j \in \mathcal{S}$ for $j = 1, \ldots, v$. Because $y_t = 0$, by (2.2d), $x_t = 0$. Hence, the left hand side of inequality (4.3) is 0. Because $\mathcal{S} \subseteq [0, \lfloor (\overline{C} - \overline{V})/V \rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Note that $\{\sigma'_1, \ldots, \sigma'_v\} \subseteq \mathcal{S} \setminus \tilde{\mathcal{S}}$ and $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S} \setminus \tilde{\mathcal{S}}$. Thus, $\sum_{s \in \mathcal{S} \setminus \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \leq \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma'_j V)(y_{t-\sigma'_j} - y_{t-\sigma'_j-1})$. Hence, the right hand side of inequality (4.3) is

$$
\begin{aligned}
&(\overline{C} - \eta V) y_t + \eta V y_{t+1} - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \\
&= \eta V y_{t+1} - \sum_{s \in \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \\
&\quad - \sum_{s \in \mathcal{S} \setminus \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \\
&\geq \eta V y_{t+1} - \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma_j V)(y_{t-\sigma_j} - y_{t-\sigma_j-1}) \\
&\quad - \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma'_j V)(y_{t-\sigma'_j} - y_{t-\sigma'_j-1}) \\
&= \eta V y_{t+1} - \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma_j V) + \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma'_j V) \\
&= \eta V y_{t+1} + \sum_{j=1}^{v} (\sigma_j - \sigma'_j) V \\
&> 0.
\end{aligned}
$$

Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.3).

Case 3: $y_t = 1$ and $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq$

$[0, \lfloor(\overline{C}-\overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C}-\overline{V}-sV \geq 0$ for all $s \in \mathcal{S}$. Thus, $\sum_{s\in\mathcal{S}}(\overline{C}-\overline{V}-sV)(y_{t-s}-y_{t-s-1}) \leq 0$. Because $y_t = 1$, the right hand side of (4.3) is at least $\overline{C}$ when $y_{t+1} = 1$ and is at least $\overline{C}-\eta V \geq \overline{V}$ when $y_{t+1} = 0$ (as $\eta \leq (\overline{C}-\overline{V})/V$). By (2.2d), $x_t \leq \overline{C}$. If $y_{t+1} = 0$, then by (2.2d) and (2.2f), $x_{t+1} = 0$ and $x_t - x_{t+1} \leq Vy_{t+1} + \overline{V}(1-y_{t+1})$, which imply that $x_t \leq \overline{V}$. Hence, x_t is at most $\overline{C}$, and it is at most $\overline{V}$ when $y_{t+1} = 0$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.3).

Case 4: $y_t = 1$ and $y_{t-s} - y_{t-s-1} > 0$ for some $s \in \mathcal{S}$. If $y_{t+1} = 1$, then inequality (4.3) becomes inequality (4.1), and by Proposition 1, $(\mathbf{x}, \mathbf{y})$ satisfies the inequality. In the following, we consider the case where $y_{t+1} = 0$. Let $\tilde{\mathcal{S}} = \{\sigma \in \mathcal{S} : y_{t-\sigma} - y_{t-\sigma-1} > 0\}$ and $v = |\tilde{\mathcal{S}}|$. Then, $v \geq 1$. Denote $\tilde{\mathcal{S}} = \{\sigma_1, \sigma_2, \ldots, \sigma_v\}$, where $\sigma_1 < \sigma_2 < \cdots < \sigma_v$. Note that $y_{t-\sigma_j-1} = 0$ and $y_{t-\sigma_j} = 1$ for $j = 1, \ldots, v$. Then, for each $j = 2, \ldots, v$, there exists $\sigma'_j \in [\sigma_{j-1}+1, \sigma_j - 1]_{\mathbb{Z}}$ such that $y_{t-\sigma'_j-1} = 1$ and $y_{t-\sigma'_j} = 0$. Thus,

$$0 \leq \sigma_1 < \sigma'_2 < \sigma_2 < \cdots < \sigma'_v < \sigma_v \leq s_{\max}.$$

In addition, $y_k = 1$ for all $k \in [t-\sigma_1, t]_{\mathbb{Z}}$. Because $y_{t-\sigma_v} - y_{t-\sigma_v-1} = 1$ and $t - \sigma_v \in [2, T]_{\mathbb{Z}}$, by (2.2a), $y_k = 1$ for all $k \in [t-\sigma_v, \min\{T, t-\sigma_v+L-1\}]_{\mathbb{Z}}$, which implies that $t - \sigma'_j \geq t - \sigma_v + L$ for $j = 2, \ldots, v$. Hence, for $j = 2, \ldots, v$, we have $\sigma'_j \leq \sigma_v - L$, which implies that

$$\sigma'_j \leq s_{\max} - L. \tag{8.19}$$

If $\beta = \alpha + 1$, then $\mathcal{S} = [0, s_{\max}]_{\mathbb{Z}}$, which implies that $\sigma'_j \in \mathcal{S}$ for $j = 2, \ldots, v$.

If $\beta \neq \alpha + 1$, then condition (c) of Proposition 4 implies that $s_{\max} \leq L + \alpha$, which, by (8.19), implies that $\sigma'_j \leq \alpha$ for $j = 2, \ldots, v$. Thus, in both cases, $\sigma'_j \in \mathcal{S}$ for $j = 2, \ldots, v$. Because $y_{t+1} = 0$, by (2.2d) and (2.2f), $x_{t+1} = 0$ and $x_t - x_{t+1} \leq V y_{t+1} + \overline{V}(1 - y_{t+1})$, which imply that $x_t \leq \overline{V}$; that is, the left hand side of inequality (4.3) is at most $\overline{V}$. Because $y_{t-\sigma_1} - y_{t-\sigma_1-1} = 1$ and $t - \sigma_1 \in [2, T]_{\mathbb{Z}}$, by (2.2a), $y_k = 1$ for all $k \in [t - \sigma_1, \min\{T, t - \sigma_1 + L - 1\}]_{\mathbb{Z}}$. Because $y_{t+1} = 0$, this implies that $t + 1 \geq t - \sigma_1 + L$, or equivalently, $L - 1 \leq \sigma_1$. Because $\eta \leq L - 1$, we have

$$\eta \leq \sigma_1. \tag{8.20}$$

Because $\mathcal{S} \subseteq [0, \lfloor (\overline{C} - \overline{V})/V \rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Note that $\{\sigma'_2, \ldots, \sigma'_v\} \subseteq \mathcal{S} \setminus \tilde{\mathcal{S}}$ and $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S} \setminus \tilde{\mathcal{S}}$. Thus, $\sum_{s \in \mathcal{S} \setminus \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \leq \sum_{j=2}^{v} (\overline{C} - \overline{V} - \sigma'_j V)(y_{t-\sigma'_j} -$

$y_{t-\sigma'_j-1})$. Hence, the right hand side of inequality (4.3) is

$$
\begin{aligned}
&(\overline{C} - \eta V)y_t + \eta V y_{t+1} - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \\
&= (\overline{C} - \eta V) - \sum_{s \in \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \\
&\quad - \sum_{s \in \mathcal{S} \setminus \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \\
&\geq (\overline{C} - \eta V) - \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma_j V)(y_{t-\sigma_j} - y_{t-\sigma_j-1}) \\
&\quad - \sum_{j=2}^{v} (\overline{C} - \overline{V} - \sigma'_j V)(y_{t-\sigma'_j} - y_{t-\sigma'_j-1}) \\
&= (\overline{C} - \eta V) - \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma_j V) + \sum_{j=2}^{v} (\overline{C} - \overline{V} - \sigma'_j V) \\
&= \overline{V} + (\sigma_1 - \eta)V + \sum_{j=2}^{v} (\sigma_j - \sigma'_j)V \\
&\geq \overline{V} + (\sigma_1 - \eta)V \\
&\geq \overline{V},
\end{aligned}
$$

where the last inequality follows from (8.20). Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.3).

Next, we show that $(\mathbf{x}, \mathbf{y})$ satisfies (4.4) when $\mathcal{S} = [0, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. Consider any $t \in [2, T - s_{\max} - 1]_{\mathbb{Z}}$. We divide the analysis into four cases.

Case 1: $y_t = 0$ and $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, \lfloor (\overline{C} - \overline{V})/V \rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Thus, in this case, the right hand side of inequality (4.4) is nonnegative. Because $y_t = 0$, by (2.2d), $x_t = 0$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.4).

Case 2: $y_t = 0$ and $y_{t+s} - y_{t+s+1} > 0$ for some $s \in \mathcal{S}$. Let $\tilde{\mathcal{S}} = \{\sigma \in \mathcal{S} :$

$y_{t+\sigma} - y_{t+\sigma+1} > 0\}$ and $v = |\tilde{\mathcal{S}}|$. Then, $v \geq 1$. Denote $\tilde{\mathcal{S}} = \{\sigma_1, \sigma_2, \ldots, \sigma_v\}$, where $\sigma_1 < \sigma_2 < \cdots < \sigma_v$. Note that $y_{t+\sigma_j} = 1$ and $y_{t+\sigma_j+1} = 0$ for $j = 1, 2, \ldots, v$. Denote $\sigma_0 = -1$. Then, for each $j = 1, \ldots, v$, there exists $\sigma'_j \in [\sigma_{j-1} + 1, \sigma_j - 1]_{\mathbb{Z}}$ such that $y_{t+\sigma'_j} = 0$ and $y_{t+\sigma'_j+1} = 1$. Thus,

$$0 \leq \sigma'_1 < \sigma_1 < \sigma'_2 < \sigma_2 < \cdots < \sigma'_v < \sigma_v \leq s_{\max}.$$

Because $y_{t+\sigma_v+1} = 0$ and $t + \sigma_v + 1 \in [2, T]_{\mathbb{Z}}$, by Lemma 1(i), $y_{t+\sigma_v+1-j} - y_{t+\sigma_v-j} \neq 1$ for all $j \in [0, \min\{t + \sigma_v - 1, L - 1\}]_{\mathbb{Z}}$, which implies that $t + \sigma'_j + 1 \leq t + \sigma_v - L + 1$ for $j = 1, \ldots, v$. Hence, for $j = 1, \ldots, v$, we have $\sigma'_j \leq \sigma_v - L$, which implies that

$$\sigma'_j \leq s_{\max} - L. \tag{8.21}$$

If $\beta = \alpha + 1$, then $\mathcal{S} = [0, s_{\max}]_{\mathbb{Z}}$, which implies that $\sigma'_j \in \mathcal{S}$ for $j = 1, \ldots, v$. If $\beta \neq \alpha + 1$, then condition (c) of Proposition 4 implies that $s_{\max} \leq L + \alpha$, which, by (8.21), implies that $\sigma'_j \leq \alpha$ for $j = 1, \ldots, v$. Thus, in both cases, $\sigma'_j \in \mathcal{S}$ for $j = 1, \ldots, v$. Because $y_t = 0$, by (2.2d), $x_t = 0$. Hence, the left hand side of inequality (4.4) is 0. Because $\mathcal{S} \subseteq [0, \lfloor (\overline{C} - \overline{V})/V \rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Note that $\{\sigma'_1, \ldots, \sigma'_v\} \subseteq \mathcal{S} \setminus \tilde{\mathcal{S}}$ and $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in \mathcal{S} \setminus \tilde{\mathcal{S}}$. Thus, $\sum_{s \in \mathcal{S} \setminus \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \leq \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma'_j V)(y_{t+\sigma'_j} - y_{t+\sigma'_j+1})$. Hence, the right hand side of inequality (4.4) is

$$
\begin{aligned}
&(\overline{C} - \eta V)y_t + \eta V y_{t-1} - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \\
&= \eta V y_{t-1} - \sum_{s \in \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \\
&\quad - \sum_{s \in \mathcal{S} \setminus \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \\
&\geq \eta V y_{t-1} - \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma_j V)(y_{t+\sigma_j} - y_{t+\sigma_j+1}) \\
&\quad - \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma'_j V)(y_{t+\sigma'_j} - y_{t+\sigma'_j+1}) \\
&= \eta V y_{t-1} - \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma_j V) + \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma'_j V) \\
&= \eta V y_{t-1} + \sum_{j=1}^{v} (\sigma_j - \sigma'_j) V \\
&> 0.
\end{aligned}
$$

Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.4).

Case 3: $y_t = 1$ and $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, \lfloor (\overline{C} - \overline{V})/V \rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Thus, $\sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \leq 0$. Because $y_t = 1$, the right hand side of (4.4) is at least $\overline{C}$ when $y_{t-1} = 1$ and is at least $\overline{C} - \eta V \geq \overline{V}$ when $y_{t-1} = 0$ (as $\eta \leq (\overline{C} - \overline{V})/V$). By (2.2d), $x_t \leq \overline{C}$. If $y_{t-1} = 0$, then by (2.2d) and (2.2e), $x_{t-1} = 0$ and $x_t - x_{t-1} \leq V y_{t-1} + \overline{V}(1 - y_{t-1})$, which imply that $x_t \leq \overline{V}$. Hence, x_t is at most $\overline{C}$, and it is at most $\overline{V}$ when $y_{t-1} = 0$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.4).

Case 4: $y_t = 1$ and $y_{t+s} - y_{t+s+1} > 0$ for some $s \in \mathcal{S}$. If $y_{t-1} = 1$, then inequality (4.4) becomes inequality (4.2), and by Proposition 1, $(\mathbf{x}, \mathbf{y})$ satisfies the inequality. In the following, we consider the case where $y_{t-1} = 0$. Let $\tilde{\mathcal{S}} = \{\sigma \in \mathcal{S} : y_{t+\sigma} - y_{t+\sigma+1} > 0\}$ and $v = |\tilde{\mathcal{S}}|$. Then, $v \geq 1$. Denote $\tilde{\mathcal{S}} = \{\sigma_1, \sigma_2, \ldots, \sigma_v\}$, where $\sigma_1 < \sigma_2 < \cdots < \sigma_v$. Note that $y_{t+\sigma_j} = 1$ and $y_{t+\sigma_j+1} = 0$ for $j = 1, \ldots, v$. Then, for each $j = 2, \ldots, v$, there exists $\sigma'_j \in [\sigma_{j-1} + 1, \sigma_j - 1]_{\mathbb{Z}}$ such that $y_{t+\sigma'_j} = 0$ and $y_{t+\sigma'_j+1} = 1$. Thus,

$$0 \leq \sigma_1 < \sigma'_2 < \sigma_2 < \cdots < \sigma'_v < \sigma_v \leq s_{\max}.$$

In addition, $y_k = 1$ for all $k \in [t, t + \sigma_1]_{\mathbb{Z}}$. Because $y_{t+\sigma_v+1} = 0$ and $t + \sigma_v + 1 \in [2, T]_{\mathbb{Z}}$, by Lemma 1(i), $y_{t+\sigma_v+1-j} - y_{t+\sigma_v-j} \leq 0$ for all $j \in [0, \min\{t + \sigma_v - 1, L - 1\}]_{\mathbb{Z}}$, which implies that $t + \sigma'_j + 1 \leq t + \sigma_v - L + 1$ for $j = 2, \ldots, v$. Hence, for $j = 2, \ldots, v$, we have $\sigma'_j \leq \sigma_v - L$, which implies that

$$\sigma'_j \leq s_{\max} - L. \tag{8.22}$$

If $\beta = \alpha + 1$, then $\mathcal{S} = [0, s_{\max}]_{\mathbb{Z}}$, which implies that $\sigma'_j \in \mathcal{S}$ for all $j = 2, \ldots, v$. If $\beta \neq \alpha + 1$, then condition (c) of Proposition 4 implies that $s_{\max} \leq L + \alpha$, which, by (8.22), implies that $\sigma'_j \leq \alpha$ for $j = 2, \ldots, v$. Thus, in both cases, $\sigma'_j \in \mathcal{S}$ for $j = 2, \ldots, v$. Because $y_{t-1} = 0$, by (2.2d) and (2.2e), $x_{t-1} = 0$ and $x_t - x_{t-1} \leq V y_{t-1} + \overline{V}(1 - y_{t-1})$, which imply that $x_t \leq \overline{V}$; that is, the left hand side of inequality (4.4) is at most $\overline{V}$. Because $y_t - y_{t-1} = 1$, by (2.2a), $y_k = 1$ for all $k \in [t, \min\{T, t + L - 1\}]_{\mathbb{Z}}$. Because $y_{t+\sigma_1+1} = 0$, this implies that $t + \sigma_1 + 1 \geq t + L$, or equivalently, $\sigma_1 \geq L - 1$. Because $\eta \leq L - 1$, we have

$$\sigma_1 \geq \eta. \tag{8.23}$$

Because $\mathcal{S} \subseteq [0, \lfloor(\overline{C} - \overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Note that $\{\sigma'_2, \ldots, \sigma'_v\} \subseteq \mathcal{S} \setminus \tilde{\mathcal{S}}$ and $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in \mathcal{S} \setminus \tilde{\mathcal{S}}$. Thus, $\sum_{s \in \mathcal{S} \setminus \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \leq \sum_{j=2}^{v} (\overline{C} - \overline{V} - \sigma'_j V)(y_{t+\sigma'_j} - y_{t+\sigma'_j+1})$. Hence, the right hand side of inequality (4.4) is

$$
\begin{aligned}
&(\overline{C} - \eta V) y_t + \eta V y_{t-1} - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \\
&= (\overline{C} - \eta V) - \sum_{s \in \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \\
&\quad - \sum_{s \in \mathcal{S} \setminus \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \\
&\geq (\overline{C} - \eta V) - \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma_j V)(y_{t+\sigma_j} - y_{t+\sigma_j+1}) \\
&\quad - \sum_{j=2}^{v} (\overline{C} - \overline{V} - \sigma'_j V)(y_{t+\sigma'_j} - y_{t+\sigma'_j+1}) \\
&= (\overline{C} - \eta V) - \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma_j V) + \sum_{j=2}^{v} (\overline{C} - \overline{V} - \sigma'_j V) \\
&= \overline{V} + (\sigma_1 - \eta) V + \sum_{j=2}^{v} (\sigma_j - \sigma'_j) V \\
&\geq \overline{V} + (\sigma_1 - \eta) V \\
&\geq \overline{V},
\end{aligned}
$$

where the last inequality follows from (8.23). Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.4).

It is easy to verify that the proof of facet-defining of inequalities (4.3) and (4.4) in the proof of Proposition 3 remains valid when $\mathcal{S} = [0, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. Therefore, inequalities (4.3) and (4.4) are facet-defining for $\text{conv}(\mathcal{P})$ under the conditions stated in Proposition 4. $\square$

8.8 Proof of Proposition 5

For notational convenience, we define $s_{\max} = \max\{s : s \in \mathcal{S}\}$ if $\mathcal{S} \neq \emptyset$, and $s_{\max} = 0$ if $\mathcal{S} = \emptyset$. To prove that linear inequalities (4.5) and (4.6) are valid for $\text{conv}(\mathcal{P})$, it suffices to show that they are valid for $\mathcal{P}$. Consider any element $(\mathbf{x}, \mathbf{y})$ of $\mathcal{P}$. We show that $(\mathbf{x}, \mathbf{y})$ satisfies (4.5) and (4.6).

We first show that $(\mathbf{x}, \mathbf{y})$ satisfies (4.5). Consider any $t \in [s_{\max} + 2, T]_{\mathbb{Z}}$ (i.e., $t \in [2, T]_{\mathbb{Z}}$ such that $t \geq s + 2$ for all $s \in \mathcal{S}$). We divide the analysis into three cases.

Case 1: $y_t = 0$. In this case, by (2.2d), $x_t = 0$. Thus, the left hand side of (4.5) and the first term on the right hand side of (4.5) are 0. Because $y_t = 0$, by Lemma 1(i), $y_{t-j} - y_{t-j-1} \leq 0$ for all $j \in [0, \min\{t-2, L-1\}]_{\mathbb{Z}}$. Because $s_{\max} \leq t-2$, we have $\mathcal{S} \subseteq [0, \min\{t-2, L\}]_{\mathbb{Z}}$. Thus, $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S} \setminus \{L\}$. Because $\eta \leq (\overline{C} - \overline{V})/V$, the coefficient "$\overline{C} - \overline{V} - \eta V$" on the right hand side of (4.5) is nonnegative. Because $\mathcal{S} \subseteq [1, \lfloor (\overline{C} - \overline{V})/V \rfloor]_{\mathbb{Z}}$, for any $s \in \mathcal{S}$, the coefficient "$\overline{C} - \overline{V} - sV$" on the right hand side of (4.5) is also nonnegative. Hence, if $s_{\max} \leq L-1$ or $y_{t-L} - y_{t-L-1} \leq 0$, then the right hand side of (4.5) is nonnegative. Now, consider the situation where $s_{\max} = L$ and $y_{t-L} - y_{t-L-1} > 0$. Then, $y_{t-L} = 1$ and $y_{t-L-1} = 0$. By (2.2a), $y_{t-1} = 1$. Thus, the right hand side of (4.5) is at least $(\overline{C} - \overline{V} - \eta V)y_{t-1} - (\overline{C} - \overline{V} - LV)(y_{t-L} - y_{t-L-1}) = (\overline{C} - \overline{V} - \eta V) - (\overline{C} - \overline{V} - LV) = (L - \eta)V \geq 0$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.5).

Case 2: $y_t = 1$ and $y_{t-s'} - y_{t-s'-1} = 1$ for some $s' \in \mathcal{S}$. In this case, $y_{t-s'} = 1$ and $y_{t-s'-1} = 0$. Because $s_{\max} \leq t-2$, we have $s' \leq t-2$. If $s' = t-2$, then it does not exist any $s \in \mathcal{S}$ such that $s > s'$. If $s' \leq t-3$, then $t-s'-1 \in [2,T]_{\mathbb{Z}}$, and by Lemma 1(i), $y_{t-s'-j-1} - y_{t-s'-j-2} \leq 0$ for all $j \in [0, \min\{t-s'-3, L-1\}]_{\mathbb{Z}}$, which implies that $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in [s'+1, \min\{t-2, L+s'\}]_{\mathbb{Z}}$. Thus, $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S}$ such that $s > s'$. Because $y_{t-s'} - y_{t-s'-1} = 1$ and $t-s' \in [2,T]_{\mathbb{Z}}$, by (2.2a), $y_k = 1$ for all $k \in [t-s', \min\{T, t-s'+L-1\}]_{\mathbb{Z}}$. This implies that $y_{t-s} - y_{t-s-1} = 0$ for all $s \in [1, s'-1]_{\mathbb{Z}}$ (as $s' \leq L$). This in turn implies that $y_{t-s} - y_{t-s-1} = 0$ for all $s \in \mathcal{S}$ such that $s < s'$. Hence, $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S} \setminus \{s'\}$. Because $\mathcal{S} \subseteq [1, \lfloor (\overline{C} - \overline{V})/V \rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Thus,

$$-\sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \geq -(\overline{C} - \overline{V} - s'V). \tag{8.24}$$

Note that $t-1 \in [t-s', \min\{T, t-s'+L-1\}]_{\mathbb{Z}}$. Hence, $y_{t-1} = 1$. Because $y_t = 1$ and $y_{t-1} = 1$, by (8.24), the right hand side of inequality (4.5) is at least $s'V + \overline{V}$. By (2.2e), $\sum_{\tau=t-s'}^{t} (x_\tau - x_{\tau-1}) \leq \sum_{\tau=t-s'}^{t} V y_{\tau-1} + \sum_{\tau=t-s'}^{t} \overline{V}(1 - y_{\tau-1})$, which implies that $x_t - x_{t-s'-1} \leq s'V + \overline{V}$. Because $y_{t-s'-1} = 0$, we have $x_{t-s'-1} = 0$. Thus, $x_t \leq s'V + \overline{V}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.5).

Case 3: $y_t = 1$ and $y_{t-s} - y_{t-s-1} \neq 1$ for all $s \in \mathcal{S}$. In this case, $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [1, \lfloor (\overline{C} - \overline{V})/V \rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Thus, $\sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \leq 0$. The right hand side of (4.5) is at least $\overline{V} + \eta V$ when $y_{t-1} = 0$, and is at least $\overline{C}$ when $y_{t-1} = 1$. If $y_{t-1} = 0$, then by (2.2d) and (2.2e), $x_{t-1} = 0$ and $x_t - x_{t-1} \leq \overline{V}$, which imply that $x_t \leq \overline{V}$, and hence, x_t is less than or equal to the right hand side of (4.5). If $y_{t-1} = 1$, then by (2.2d), $x_t \leq \overline{C}$, and hence, x_t is less than or equal to the right hand side of (4.5). Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.5).

Next, we show that $(\mathbf{x}, \mathbf{y})$ satisfies (4.6). Consider any $t \in [1, T - s_{\max} - 1]_{\mathbb{Z}}$ (i.e., $t \in [1, T-1]_{\mathbb{Z}}$ such that $t \leq T - s - 1$ for all $s \in \mathcal{S}$). We divide the analysis into three cases.

Case 1: $y_t = 0$. In this case, by (2.2d), $x_t = 0$. Thus, the left hand side of (4.6) and the first term on the right hand side of (4.6) are 0. Because $y_t = 0$, by Lemma 2(i), $y_{t+j} - y_{t+j+1} \leq 0$ for all $j \in [0, \min\{T - t - 1, L - 1\}]_{\mathbb{Z}}$. Because $s_{\max} \leq T - t - 1$, we have $\mathcal{S} \subseteq [0, \min\{T - t - 1, L\}]_{\mathbb{Z}}$. Thus, $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in \mathcal{S} \setminus \{L\}$. Because $\eta \leq (\overline{C} - \overline{V})/V$, the coefficient "$\overline{C} - \overline{V} - \eta V$" on the right hand side of (4.6) is nonnegative. Because $\mathcal{S} \subseteq [1, \lfloor(\overline{C} - \overline{V})/V\rfloor]_{\mathbb{Z}}$, for any $s \in \mathcal{S}$, the coefficient "$\overline{C} - \overline{V} - sV$" on the right hand side of (4.6) is also nonnegative. Hence, if $s_{\max} \leq L - 1$ or $y_{t+L} - y_{t+L+1} \leq 0$, then the right hand side of (4.6) is nonnegative. Now, consider the situation where $s_{\max} = L$ and $y_{t+L} - y_{t+L+1} > 0$. Then, $t + L + 1 \in [2, T]_{\mathbb{Z}}$, $y_{t+L} = 1$, and $y_{t+L+1} = 0$. By Lemma 1(i), $y_{t+L-j+1} - y_{t+L-j} \leq 0$ for all $j \in [0, L-1]_{\mathbb{Z}}$; that is, $y_{t+1} = y_{t+2} = \cdots = y_{t+L} = 1$. Thus, the right hand side of (4.6) is at least $(\overline{C} - \overline{V} - \eta V)y_{t+1} - (\overline{C} - \overline{V} - LV)(y_{t+L} - y_{t+L+1}) = (\overline{C} - \overline{V} - \eta V) - (\overline{C} - \overline{V} - LV) = (L - \eta)V \geq 0$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.6).

Case 2: $y_t = 1$ and $y_{t+s'} - y_{t+s'+1} = 1$ for some $s' \in \mathcal{S}$. In this case, $y_{t+s'} = 1$ and $y_{t+s'+1} = 0$. Because $s_{\max} \leq T - t - 1$, we have $s' \leq T - t - 1$. If $s' = T - t - 1$, then it does not exist any $s \in \mathcal{S}$ such that $s > s'$. If $s' \leq T - t - 2$, then $t + s' + 1 \in [1, T-1]_{\mathbb{Z}}$, and by Lemma 2(i), $y_{t+s'+j+1} - y_{t+s'+j+2} \leq 0$ for all $j \in [0, \min\{T - t - s' - 2, L - 1\}]_{\mathbb{Z}}$, which implies that $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in [s'+1, \min\{T-t-1, L+s'\}]_{\mathbb{Z}}$. Thus, $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in \mathcal{S}$ such that $s > s'$. Because $y_{t+s'+1} = 0$ and $t + s' + 1 \in [2, T]_{\mathbb{Z}}$, by Lemma 1(i), $y_{t+s'-j+1} - y_{t+s'-j} \leq 0$ for all $j \in [0, \min\{t+s'-1, L-1\}]_{\mathbb{Z}}$. Hence, $y_\tau \leq y_{\tau-1}$ for all $\tau \in [\max\{2, t+s'-L+2\}, t+s'+1]_{\mathbb{Z}}$. Because $y_{t+s'} = 1$ and $s' \leq L$, this implies that $y_\tau = 1$ for all $\tau \in [t+1, t+s']_{\mathbb{Z}}$. Thus, $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in [1, s'-1]_{\mathbb{Z}}$. This implies that $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in \mathcal{S}$ such that $s < s'$. Hence, $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in \mathcal{S} \setminus \{s'\}$. Because $\mathcal{S} \subseteq [1, \lfloor(\overline{C} - \overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Thus,

$$-\sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \geq -(\overline{C} - \overline{V} - s'V). \tag{8.25}$$

Because $y_t = y_{t+1} = 1$, by (8.25), the right hand side of (4.6) is at least $s'V + \overline{V}$. By (2.2f), $\sum_{\tau=t+1}^{t+s'+1} (x_{\tau-1} - x_\tau) \leq \sum_{\tau=t+1}^{t+s'+1} Vy_\tau + \sum_{\tau=t+1}^{t+s'+1} \overline{V}(1 - y_\tau)$, which implies that $x_t - x_{t+s'+1} \leq s'V + \overline{V}$. Because $y_{t+s'+1} = 0$, by (2.2d), $x_{t+s'+1} = 0$. Hence, $x_t \leq s'V + \overline{V}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.6).

Case 3: $y_t = 1$ and $y_{t+s} - y_{t+s+1} \neq 1$ for all $s \in \mathcal{S}$. In this case, $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [1, \lfloor(\overline{C} - \overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Thus, $\sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \leq 0$ for all $s \in \mathcal{S}$. Because $y_t = 1$, the right hand side of (4.6) is at least $\overline{V} + \eta V$ when $y_{t+1} = 0$, and is at least $\overline{C}$ when $y_{t+1} = 1$. If $y_{t+1} = 0$, then by (2.2d) and (2.2f), $x_{t+1} = 0$ and $x_t - x_{t+1} \leq \overline{V}$, which implies that $x_t \leq \overline{V}$, and hence, x_t is less than or equal to the right hand side of (4.6). If $y_{t+1} = 1$, then by (2.2d), $x_t \leq \overline{C}$, and hence, x_t is less than or equal to the right hand side of (4.6). Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.6).

To prove that inequalities (4.5) and (4.6) are facet-defining for $\text{conv}(\mathcal{P})$ when $\eta \in \{0, (\overline{C} - \overline{V})/V\}$ or $\eta = L \in \mathcal{S}$, it suffices to show that for each of these two inequalities, there exist $2T$ affinely independent points in $\text{conv}(\mathcal{P})$ that satisfy the inequality at equality when $\eta \in \{0, (\overline{C} - \overline{V})/V\}$ or $\eta = L \in \mathcal{S}$. Let $\epsilon = \overline{V} - \underline{C} > 0$.

We first show that inequality (4.5) is facet-defining for $\text{conv}(\mathcal{P})$ by creating $2T$ affinely independent points in $\text{conv}(\mathcal{P})$ that satisfy (4.5) at equality when $\eta \in \{0, (\overline{C} - \overline{V})/V\}$ or $\eta = L \in \mathcal{S}$. Because $\mathbf{0} \in \text{conv}(\mathcal{P})$ and $\mathbf{0}$ satisfies (4.5) at equality, it suffices to create the remaining $2T - 1$ nonzero linearly independent points. We denote these $2T - 1$ points as $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ for $r \in [1, T]_{\mathbb{Z}} \setminus \{t\}$ and $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ for $r \in [1, T]_{\mathbb{Z}}$, and denote the qth component of $\bar{\mathbf{x}}^r$, $\bar{\mathbf{y}}^r$, $\hat{\mathbf{x}}^r$, and $\hat{\mathbf{y}}^r$ as $\bar{x}_q^r$, $\bar{y}_q^r$, $\hat{x}_q^r$, and $\hat{y}_q^r$, respectively. We divide these $2T - 1$ points into the following six groups:

(A1) For each $r \in [1, T]_{\mathbb{Z}} \setminus \{t\}$, we create the same point $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ as in group (A1) in the proof of Proposition 1. Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. It is easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (4.5) at equality.

(A2) For each $r \in [1, t - s_{\max} - 2]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (A2) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. It is easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.5) at equality.

(A3) For each $r \in [t - s_{\max} - 1, t - 2]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (A3) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. Consider the case where $t - r - 1 \notin \mathcal{S}$. In this case, $\hat{x}_t^r = \hat{y}_t^r = \hat{y}_{t-1}^r = 0$. In addition, $t - s - 1 \neq r$ for all $s \in \mathcal{S}$, which implies that $\hat{y}^r_{t-s} - \hat{y}^r_{t-s-1} = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.5) at equality. Next, consider the case where $t - r - 1 \in \mathcal{S}$. In this case, $\hat{x}^r_t = \overline{V} + (t - r - 1)V$ and $\hat{y}^r_t = \hat{y}^r_{t-1} = 1$. In addition, $\hat{y}^r_{t-s} - \hat{y}^r_{t-s-1} = 1$ when $s = t - r - 1$, and $\hat{y}^r_{t-s} - \hat{y}^r_{t-s-1} = 0$ when $s \neq t - r - 1$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.5) at equality.

(A4) We create a point $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ as follows: If $\eta = 0$, then

$$\bar{x}^{t-1}_q = \begin{cases} 0, & \text{for } q \in [0, t-1]_{\mathbb{Z}}; \\ \overline{V}, & \text{for } q \in [t, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\bar{y}^{t-1}_q = \begin{cases} 0, & \text{for } q \in [0, t-1]_{\mathbb{Z}}; \\ 1, & \text{for } q \in [t, T]_{\mathbb{Z}}. \end{cases}$$

If $\eta = (\overline{C} - \overline{V})/V$, then

$$\bar{x}^{t-1}_q = \begin{cases} \underline{C}, & \text{for } q \in [1, t-1]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [t, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\bar{y}^{t-1}_q = \begin{cases} 1, & \text{for } q \in [1, t-1]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [t, T]_{\mathbb{Z}}. \end{cases}$$

If $\eta = L \in \mathcal{S}$, then

$$\bar{x}^{t-1}_q = \begin{cases} 0, & \text{for } q \in [1, t-L-1]_{\mathbb{Z}} \cup [t, T]_{\mathbb{Z}}; \\ \overline{V}, & \text{for } q \in [t-L, t-1]_{\mathbb{Z}}; \end{cases}$$

and

$$\bar{y}_q^{t-1} = \begin{cases} 0, & \text{for } q \in [1, t-L-1]_{\mathbb{Z}} \cup [t, T]_{\mathbb{Z}}; \\ 1, & \text{for } q \in [t-L, t-1]_{\mathbb{Z}}. \end{cases}$$

We first consider the case where $\eta = 0$. It is easy to verify that $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ satisfies (2.2a)–(2.2f). Thus, $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1}) \in \text{conv}(\mathcal{P})$. In this case, $\hat{x}_t^{t-1} = \overline{V}$, $\hat{y}_t^{t-1} = 1$, $\hat{y}_{t-1}^{t-1} = 0$, and $\hat{y}_{t-s}^{t-1} - \hat{y}_{t-s-1}^{t-1} = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ satisfies (4.5) at equality. Next, we consider the case where $\eta = (\overline{C} - \overline{V})/V$. It is easy to verify that $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ satisfies (2.2a)–(2.2f). Thus, $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1}) \in \text{conv}(\mathcal{P})$. In this case, $\hat{x}_t^{t-1} = \hat{y}_t^{t-1} = 0$, $\overline{C} - \overline{V} - \eta V = 0$, and $\hat{y}_{t-s}^{t-1} = \hat{y}_{t-s-1}^{t-1} = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ satisfies (4.5) at equality. Next, we consider the case where $\eta = L \in \mathcal{S}$. In this case, for any $q \in [2, T]_{\mathbb{Z}}$, $\hat{y}_q^{t-1} - \hat{y}_{q-1}^{t-1} \leq 0$ if $q \neq t - L$, while $\hat{y}_q^{t-1} - \hat{y}_{q-1}^{t-1} = 1$ and $\hat{y}_k^{t-1} = 1$ for all $k \in [q, \min\{T, q + L - 1\}]_{\mathbb{Z}}$ if $q = t - L$. Thus, $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ satisfies (2.2a). For any $q \in [2, T]_{\mathbb{Z}}$, $\hat{y}_{q-1}^{t-1} - \hat{y}_q^{t-1} \leq 0$ if $q \neq t$, while $\hat{y}_{q-1}^{t-1} - \hat{y}_q^{t-1} = 1$ and $\hat{y}_q^{t-1} = 0$ for all $k \in [q, \min\{T, q + \ell - 1\}]_{\mathbb{Z}}$ if $q = t$. Thus, $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ satisfies (2.2b). It is easy to verify that $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ satisfies (2.2c)–(2.2f). Hence, $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1}) \in \text{conv}(\mathcal{P})$. Note that $\hat{x}_t^{t-1} = \hat{y}_t^{t-1} = 0$, $\hat{y}_{t-1}^{t-1} = 1$, $\hat{y}_{t-s}^{t-1} - \hat{y}_{t-s-1}^{t-1} = 0$ for all $s \in \mathcal{S} \setminus \{L\}$, $\hat{y}_{t-L}^{t-1} - \hat{y}_{t-L-1}^{t-1} = 1$, and $\overline{C} - \overline{V} - \eta V = \overline{C} - \overline{V} - LV$. Thus, $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ satisfies (4.5) at equality.

(A5) We create the same point $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ as in group (A4) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t) \in \text{conv}(\mathcal{P})$. It is easy to verify that $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (4.5) at equality.

(A6) For each $r \in [t+1, T]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (A5) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. It is easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.5) at equality.

Table A.9 shows a matrix with $2T-1$ rows, where each row represents a point created by this process. This matrix can be transformed into the matrix in Table A.10 via the following Gaussian elimination process:

(i) For each $r \in [1, T]_{\mathbb{Z}} \setminus \{t\}$, the point with index r in group (B1), denoted $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r) = (\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) - (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$. Here, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ is the point with index r in group (A1), and $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ is the point in group (A5).

(ii) For each $r \in [1, t-s_{\max}-2]_{\mathbb{Z}}$, the point with index r in group (B2), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (A2).

Table A.9: A matrix with the rows representing $2T-1$ points in $\text{conv}(\mathcal{P})$ that satisfy inequality (4.5) at equality.

Group	Point	Index r	**x**													**y**												
			1	2	⋯	$t-s_{\max}-2$	$t-s_{\max}-1$	$t-s_{\max}$	⋯	$t-2$	$t-1$	t	$t+1$	⋯	T	1	2	⋯	$t-s_{\max}-2$	$t-s_{\max}-1$	$t-s_{\max}$	⋯	$t-2$	$t-1$	t	$t+1$	⋯	T
(A1)	$(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$	1	$\overline{C}-\epsilon$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	1	1	⋯	1	1	1	⋯	1	1	1	1	⋯	1
		2	$\overline{C}$	$\overline{C}-\epsilon$	⋯	$\overline{C}$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	1	1	⋯	1	1	1	⋯	1	1	1	1	⋯	1
		⋮	⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮	⋮		⋮
		$t-s_{\max}-2$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}-\epsilon$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	1	1	⋯	1	1	1	⋯	1	1	1	1	⋯	1
		$t-s_{\max}-1$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}-\epsilon$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	1	1	⋯	1	1	1	⋯	1	1	1	1	⋯	1
		$t-s_{\max}$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}$	$\overline{C}-\epsilon$	⋯	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	1	1	⋯	1	1	1	⋯	1	1	1	1	⋯	1
		⋮	⋮	⋮		⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮	⋮		⋮
		$t-2$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}-\epsilon$	$\overline{C}$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	1	1	⋯	1	1	1	⋯	1	1	1	1	⋯	1
		$t-1$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}-\epsilon$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	1	1	⋯	1	1	1	⋯	1	1	1	1	⋯	1
		$t+1$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}-\epsilon$	⋯	$\overline{C}$	1	1	⋯	1	1	1	⋯	1	1	1	1	⋯	1
		⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮	⋮		⋮
		T	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}-\epsilon$	1	1	⋯	1	1	1	⋯	1	1	1	1	⋯	1
(A2)	$(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$	1	$\underline{C}$	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0	1	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0
		2	$\underline{C}$	$\underline{C}$	⋯	0	0	0	⋯	0	0	0	0	⋯	0	1	1	⋯	0	0	0	⋯	0	0	0	0	⋯	0
		⋮	⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮	⋮	⋮		⋮	⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮	⋮	⋮		⋮
		$t-s_{\max}-2$	$\underline{C}$	$\underline{C}$	⋯	$\underline{C}$	0	0	⋯	0	0	0	0	⋯	0	1	1	⋯	1	0	0	⋯	0	0	0	0	⋯	0
(A3)		$t-s_{\max}-1$, ⋮, $t-2$	(See Note A.9-1)													(See Note A.9-1)												
(A4)		$t-1$	(See Note A.9-2)													(See Note A.9-2)												
(A5)		t	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	⋯	$\overline{C}$	1	1	⋯	1	1	1	⋯	1	1	1	1	⋯	1
(A6)		$t+1$	0	0	⋯	0	0	0	⋯	0	0	0	$\underline{C}$	⋯	$\underline{C}$	0	0	⋯	0	0	0	⋯	0	0	0	1	⋯	1
		⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮	⋮	⋱	⋮
		T	0	0	⋯	0	0	0	⋯	0	0	0	0	⋯	$\underline{C}$	0	0	⋯	0	0	0	⋯	0	0	0	0	⋯	1

Note A.9-1: For $r \in [t-s_{\max}-1, t-2]_{\mathbb{Z}}$, the **x** and **y** vectors in group (A3) are given as follows: $\hat{\mathbf{x}}^r = (\underbrace{\underline{C},\ldots,\underline{C}}_{r \text{ terms}}, \underbrace{0,\ldots,0}_{T-r \text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{1,\ldots,1}_{r \text{ terms}}, \underbrace{0,\ldots,0}_{T-r \text{ terms}})$ if $t-r-1 \notin \mathcal{S}$;

$\hat{\mathbf{x}}^r = (\underbrace{0,\ldots,0}_{r \text{ terms}}, \underbrace{\overline{V}, \overline{V}+V, \overline{V}+2V, \ldots, \overline{V}+(t-r-1)V}_{t-r \text{ terms}}, \underbrace{\overline{V}+(t-r-1)V, \ldots, \overline{V}+(t-r-1)V}_{T-t \text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{0,\ldots,0}_{r \text{ terms}}, \underbrace{1,\ldots,1}_{T-r \text{ terms}})$ if $t-r-1 \in \mathcal{S}$.

Note A.9-2: The **x** and **y** vectors in group (A4) are given as follows: $\hat{\mathbf{x}}^{t-1} = (\underbrace{0,\ldots,0}_{t-1 \text{ terms}}, \underbrace{\overline{V},\ldots,\overline{V}}_{T-t+1 \text{ terms}})$ and $\hat{\mathbf{y}}^{t-1} = (\underbrace{0,\ldots,0}_{t-1 \text{ terms}}, \underbrace{1,\ldots,1}_{T-t+1 \text{ terms}})$ if $\eta = 0$;

$\hat{\mathbf{x}}^{t-1} = (\underbrace{\underline{C},\ldots,\underline{C}}_{t-1 \text{ terms}}, \underbrace{0,\ldots,0}_{T-t+1 \text{ terms}})$ and $\hat{\mathbf{y}}^{t-1} = (\underbrace{1,\ldots,1}_{t-1 \text{ terms}}, \underbrace{0,\ldots,0}_{T-t+1 \text{ terms}})$ if $\eta = (\overline{C}-\overline{V})/V$; $\hat{\mathbf{x}}^{t-1} = (\underbrace{0,\ldots,0}_{t-L-1 \text{ terms}}, \underbrace{\overline{V},\ldots,\overline{V}}_{L \text{ terms}}, \underbrace{0,\ldots,0}_{T-t+1 \text{ terms}})$ and $\hat{\mathbf{y}}^{t-1} = (\underbrace{0,\ldots,0}_{t-L-1 \text{ terms}}, \underbrace{1,\ldots,1}_{L \text{ terms}}, \underbrace{0,\ldots,0}_{T-t+1 \text{ terms}})$ if $\eta = L \in \mathcal{S}$.

Table A.10: Lower triangular matrix obtained from Table A.9 via Gaussian elimination.

Group	Point	Index r	$\mathbf{x}$														$\mathbf{y}$													
			1	2	$\cdots$	$t-s_{\max}-2$	$t-s_{\max}-1$	$t-s_{\max}$	$\cdots$	$t-2$	$t-1$	t	$t+1$	$\cdots$	T	1	2	$\cdots$	$t-s_{\max}-2$	$t-s_{\max}-1$	$t-s_{\max}$	$\cdots$	$t-2$	$t-1$	t	$t+1$	$\cdots$	T
(B1)	$(\underline{\mathbf{x}}^r,\underline{\mathbf{y}}^r)$	1	$-\epsilon$	0	$\cdots$	0	0	0	$\cdots$	0	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	0	$\cdots$	0
		2	0	$-\epsilon$	$\cdots$	0	0	0	$\cdots$	0	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-s_{\max}-2$	0	0	$\cdots$	$-\epsilon$	0	0	$\cdots$	0	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	0	$\cdots$	0
		$t-s_{\max}-1$	0	0	$\cdots$	0	$-\epsilon$	0	$\cdots$	0	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	0	$\cdots$	0
		$t-s_{\max}$	0	0	$\cdots$	0	0	$-\epsilon$	$\cdots$	0	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-2$	0	0	$\cdots$	0	0	0	$\cdots$	$-\epsilon$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	0	$\cdots$	0
		$t-1$	0	0	$\cdots$	0	0	0	$\cdots$	0	$-\epsilon$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	0	$\cdots$	0
		$t+1$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$-\epsilon$	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
		T	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	0	$\cdots$	$-\epsilon$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	0	$\cdots$	0
(B2)	$(\hat{\underline{\mathbf{x}}}^r,\hat{\underline{\mathbf{y}}}^r)$	1	(Omitted)													1	0	$\cdots$	0	0	0	$\cdots$	0	0	0	0	$\cdots$	0
		2														1	1	$\cdots$	0	0	0	$\cdots$	0	0	0	0	$\cdots$	0
		$\vdots$														$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-s_{\max}-2$														1	1	$\cdots$	1	0	0	$\cdots$	0	0	0	0	$\cdots$	0
(B3)		$t-s_{\max}-1$	(Omitted)													(See Note A.10-1)												
		$\vdots$																										
		$t-2$																										
(B4)		$t-1$	(Omitted)													(See Note A.10-2)												
(B5)		t	(Omitted)													1	1	$\cdots$	1	1	1	$\cdots$	1	1	1	0	$\cdots$	0
(B6)		$t+1$	(Omitted)													0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	1	$\cdots$	0
		$\vdots$														$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
		T														0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	0	$\cdots$	1

Note A.10-1: For $r \in [t-s_{\max}-1, t-2]_{\mathbb{Z}}$, the $\mathbf{y}$ vector in group (B3) is given as follows: $\hat{\underline{\mathbf{y}}}^r = (\underbrace{1,\ldots,1}_{r\text{ terms}}, \underbrace{0,\ldots,0}_{T-r\text{ terms}})$ if $t-r-1 \notin \mathcal{S}$; $\hat{\underline{\mathbf{y}}}^r = (\underbrace{-1,\ldots,-1}_{r\text{ terms}}, \underbrace{0,\ldots,0}_{T-r\text{ terms}})$ if $t-r-1 \in \mathcal{S}$.

Note A.10-2: The $\mathbf{y}$ vector in group (B4) is given as follows: $\hat{\underline{\mathbf{y}}}^{t-1} = (\underbrace{-1,\ldots,-1}_{t-1\text{ terms}}, \underbrace{0,\ldots,0}_{T-t+1\text{ terms}})$ if $\eta = 0$; $\hat{\underline{\mathbf{y}}}^{t-1} = (\underbrace{1,\ldots,1}_{t-1\text{ terms}}, \underbrace{0,\ldots,0}_{T-t+1\text{ terms}})$ if $\eta = (\overline{C}-\overline{V})/V$; $\hat{\underline{\mathbf{y}}}^{t-1} = (\underbrace{0,\ldots,0}_{t-L-1\text{ terms}}, \underbrace{1,\ldots,1}_{L\text{ terms}}, \underbrace{0,\ldots,0}_{T-t+1\text{ terms}})$ if $\eta = L \in \mathcal{S}$.

(iii) For each $r \in [t - s_{\max} - 1, t - 2]_{\mathbb{Z}}$, the point with index r in group (B3), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $t - r - 1 \notin \mathcal{S}$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) - (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ if $t - r - 1 \in \mathcal{S}$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (A3), and $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ is the point in group (A5).

(iv) The point in group (B4), denoted $(\underline{\hat{\mathbf{x}}}^{t-1}, \underline{\hat{\mathbf{y}}}^{t-1})$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^{t-1}, \underline{\hat{\mathbf{y}}}^{t-1}) = (\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1}) - (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ if $\eta = 0$, and setting $(\underline{\hat{\mathbf{x}}}^{t-1}, \underline{\hat{\mathbf{y}}}^{t-1}) = (\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ if $\eta = (\overline{C} - \overline{V})/V$ or $\eta = L \in \mathcal{S}$. Here, $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ is the point in group (A4), and $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ is the point in group (A5).

(v) The point in group (B5), denoted $(\underline{\hat{\mathbf{x}}}^t, \underline{\hat{\mathbf{y}}}^t)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^t, \underline{\hat{\mathbf{y}}}^t) = (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t) - (\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$. Here, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ is the point in group (A5), and $(\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$ is the point with index $t + 1$ in group (A6).

(vi) For each $r \in [t + 1, T]_{\mathbb{Z}}$, the point with index r, denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ if $r \neq T$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $r = T$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ and $(\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ are the points with indices r and $r + 1$, respectively, in group (A6).

The matrix shown in Table A.10 is lower triangular; that is, the position of the last nonzero component of a row of the matrix is greater than the position of the last nonzero component of the previous row. This implies that the $2T - 1$ points in groups (A1)–(A6) are linearly independent. Therefore, inequality (4.5) is facet-defining for $\text{conv}(\mathcal{P})$.

Next, we show that inequality (4.6) is facet-defining for $\text{conv}(\mathcal{P})$ by creating $2T$ affinely independent points in $\text{conv}(\mathcal{P})$ that satisfy (4.6) at equality when $\eta \in \{0, (\overline{C} - \overline{V})/V\}$ or $\eta = L \in \mathcal{S}$. Because $\mathbf{0} \in \text{conv}(\mathcal{P})$ and $\mathbf{0}$ satisfies (4.6) at equality, it suffices to create the remaining $2T - 1$ nonzero linearly independent points. We denoted these $2T - 1$ points as $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ for $r \in [1, T]_{\mathbb{Z}} \setminus \{t\}$ and $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ for $r \in [1, T]_{\mathbb{Z}}$, and denote the qth component of $\bar{\mathbf{x}}^r$, $\bar{\mathbf{y}}^r$, $\hat{\mathbf{x}}^r$, and $\hat{\mathbf{y}}^r$ as $\bar{x}_q^r$, $\bar{y}_q^r$, $\hat{x}_q^r$, and $\hat{y}_q^r$, respectively. We divide these $2T - 1$ points into the following six groups:

(C1) For each $r \in [1, T]_{\mathbb{Z}} \setminus \{t\}$, we create the same point $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ as that in group (A1) in the proof of Proposition 1. Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. It is easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (4.6) at equality.

(C2) For each $r \in [1, t-1]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as that in group (C2) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. It is easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.6) at equality.

(C3) We create a point $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ as follows: If $\eta = 0$, then

$$\hat{x}_q^t = \begin{cases} \overline{V}, & \text{for } q \in [1, t]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [t+1, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^t = \begin{cases} 1, & \text{for } q \in [1, t]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [t+1, T]_{\mathbb{Z}}. \end{cases}$$

If $\eta = (\overline{C} - \overline{V})/V$, then

$$\hat{x}_q^t = \begin{cases} 0, & \text{for } q \in [1, t]_{\mathbb{Z}}; \\ \underline{C}, & \text{for } q \in [t+1, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^t = \begin{cases} 0, & \text{for } q \in [1, t]_{\mathbb{Z}}; \\ 1, & \text{for } q \in [t+1, T]_{\mathbb{Z}}. \end{cases}$$

If $\eta = L \in \mathcal{S}$, then

$$\hat{x}_q^t = \begin{cases} 0, & \text{for } q \in [1, t]_{\mathbb{Z}} \cup [t+L+1, T]_{\mathbb{Z}}; \\ \overline{V}, & \text{for } q \in [t+1, t+L]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^t = \begin{cases} 0, & \text{for } q \in [1, t]_{\mathbb{Z}} \cup [t+L+1, T]_{\mathbb{Z}}; \\ 1, & \text{for } q \in [t+1, t+L]_{\mathbb{Z}}; \end{cases}$$

We first consider the case where $\eta = 0$. It is easy to verify that $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (2.2a)–(2.2f). Thus, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t) \in \text{conv}(\mathcal{P})$. In this case, $\hat{x}_t^t = \overline{V}$, $\hat{y}_t^t = 1$, $\hat{y}_{t+1}^t = 0$, and $\hat{y}_{t+s}^t - \hat{y}_{t+s+1}^t = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (4.6) at equality. Next, we consider the case where $\eta = (\overline{C} - \overline{V})/V$. It is easy to verify that $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (2.2a)–(2.2f). Thus, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t) \in \text{conv}(\mathcal{P})$. In this case, $\hat{x}_t^t = \hat{y}_t^t = 0$, $\overline{C} - \overline{V} - \eta V = 0$, and $\hat{y}_{t+s}^t - \hat{y}_{t+s+1}^t = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (4.6) at equality. Next, we consider the case where $\eta = L \in \mathcal{S}$. In this case, for any $q \in [2, T]_{\mathbb{Z}}$, $\hat{y}_q^t - \hat{y}_{q-1}^t \leq 0$ if $q \neq t+1$, while $\hat{y}_q^t - \hat{y}_{q-1}^t = 1$ and $\hat{y}_k^t = 1$ for all $k \in [q, \min\{T, q+L-1\}]_{\mathbb{Z}}$ if $q = t+1$. Thus, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (2.2a). For any $q \in [2, T]_{\mathbb{Z}}$, $\hat{y}_{q-1}^t - \hat{y}_q^t \leq 0$ if $q \neq t+L+1$, while $\hat{y}_{q-1}^t - \hat{y}_q^t = 1$ and $\hat{y}_k^t = 0$ for all $k \in [q, \min\{T, q+\ell-1\}]_{\mathbb{Z}}$ if $q = t+L+1$. Thus, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (2.2b). It is easy to verify that $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (2.2c)–(2.2f). Hence, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t) \in \text{conv}(\mathcal{P})$. Note that $\hat{x}^t_t = \hat{y}^t_t = 0$, $\hat{y}^t_{t+1} = 1$, $\hat{y}^t_{t+s} - \hat{y}^t_{t+s+1} = 0$ for all $s \in \mathcal{S} \setminus \{L\}$, $\hat{y}^t_{t+L} - \hat{y}^t_{t+L+1} = 1$, and $\overline{C} - \overline{V} - \eta V = \overline{C} - \overline{V} - LV$. Thus, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (4.6) at equality.

(C4) For each $r \in [t+1, t+s_{\max}]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (C3) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. Consider the case where $r - t \notin \mathcal{S}$. In this case, $\hat{x}^r_t = \hat{y}^r_t = \hat{x}^r_{t+1} = 0$. In addition, $t + s \neq r$ for all $s \in \mathcal{S}$, which implies that $\hat{y}^r_{t+s} - \hat{y}^r_{t+s+1} = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.6) at equality. Next, consider the case where $r - t \in \mathcal{S}$. In this case, $\hat{x}^r_t = \overline{V} + (r-t)V$ and $\hat{y}^r_t = \hat{y}^r_{t+1} = 1$. In addition, $\hat{y}^r_{t+s} - \hat{y}^r_{t+s+1} = 1$ when $s = r - t$, and $\hat{y}^r_{t+s} - \hat{y}^r_{t+s+1} = 0$ when $s \neq r - t$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.6) at equality.

(C5) We create the same point $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ as in group (C4) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1}) \in \text{conv}(\mathcal{P})$. It is easy to verify that $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ satisfies (4.6) at equality.

(C6) For each $r \in [t+s_{\max}+2, T]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (C5) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. It is easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.6) at equality.

Table A.11: A matrix with the rows representing $2T-1$ points in $\text{conv}(\mathcal{P})$ that satisfy inequality (4.6) at equality.

Group	Point	Index r	x													y												
			1	2	$\cdots$	$t-1$	t	$t+1$	$t+2$	$\cdots$	$t+s_{\max}$	$t+s_{\max}+1$	$t+s_{\max}+2$	$\cdots$	T	1	2	$\cdots$	$t-1$	t	$t+1$	$t+2$	$\cdots$	$t+s_{\max}$	$t+s_{\max}+1$	$t+s_{\max}+2$	$\cdots$	T
(C1)	$(\bar{\mathbf{x}}^r,\bar{\mathbf{y}}^r)$	1	$\overline{C}-\epsilon$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	1	$\cdots$	1	1	1	$\cdots$	1
		2	$\overline{C}$	$\overline{C}-\epsilon$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	1	$\cdots$	1	1	1	$\cdots$	1
		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-1$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}-\epsilon$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	1	$\cdots$	1	1	1	$\cdots$	1
		$t+1$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}-\epsilon$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	1	$\cdots$	1	1	1	$\cdots$	1
		$t+2$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}-\epsilon$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	1	$\cdots$	1	1	1	$\cdots$	1
		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t+s_{\max}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}-\epsilon$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	1	$\cdots$	1	1	1	$\cdots$	1
		$t+s_{\max}+1$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}-\epsilon$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	1	$\cdots$	1	1	1	$\cdots$	1
		$t+s_{\max}+2$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}-\epsilon$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	1	$\cdots$	1	1	1	$\cdots$	1
		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		T	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}-\epsilon$	1	1	$\cdots$	1	1	1	1	$\cdots$	1	1	1	$\cdots$	1
(C2)	$(\hat{\mathbf{x}}^r,\hat{\mathbf{y}}^r)$	1	$\underline{C}$	0	$\cdots$	0	0	0	0	$\cdots$	0	0	0	$\cdots$	0	1	0	$\cdots$	0	0	0	0	$\cdots$	0	0	0	$\cdots$	0
		2	$\underline{C}$	$\underline{C}$	$\cdots$	0	0	0	0	$\cdots$	0	0	0	$\cdots$	0	1	1	$\cdots$	0	0	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-1$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	1	1	$\cdots$	1	0	0	0	$\cdots$	0	0	0	$\cdots$	0
(C3)		t	(See Note A.11-1)													(See Note A.11-1)												
(C4)		$t+1$	(See Note A.11-2)													(See Note A.11-2)												
		$\vdots$																										
		$t+s_{\max}$																										
(C5)		$t+s_{\max}+1$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	$\overline{C}$	$\overline{C}$	$\cdots$	$\overline{C}$	1	1	$\cdots$	1	1	1	1	$\cdots$	1	1	1	$\cdots$	1
(C6)		$t+s_{\max}+2$	0	0	$\cdots$	0	0	0	0	$\cdots$	0	0	$\underline{C}$	$\cdots$	$\underline{C}$	0	0	$\cdots$	0	0	0	0	$\cdots$	0	0	1	$\cdots$	1
		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
		T	0	0	$\cdots$	0	0	0	0	$\cdots$	0	0	0	$\cdots$	$\underline{C}$	0	0	$\cdots$	0	0	0	0	$\cdots$	0	0	0	$\cdots$	1

Note A.11-1: The $\mathbf{x}$ and $\mathbf{y}$ vectors in group (C3) are given as follows: $\hat{\mathbf{x}}^t = (\underbrace{\overline{V},\dots,\overline{V}}_{t\text{ terms}}, \underbrace{0,\dots,0}_{T-t\text{ terms}})$ and $\hat{\mathbf{y}}^t = (\underbrace{1,\dots,1}_{t\text{ terms}}, \underbrace{0,\dots,0}_{T-t\text{ terms}})$ if $\eta = 0$;
$\hat{\mathbf{x}}^t = (\underbrace{0,\dots,0}_{t\text{ terms}}, \underbrace{\underline{C},\dots,\underline{C}}_{T-t\text{ terms}})$ and $\hat{\mathbf{y}}^t = (\underbrace{0,\dots,0}_{t\text{ terms}}, \underbrace{1,\dots,1}_{T-t\text{ terms}})$ if $\eta = (\overline{C}-\overline{V})/V$; $\hat{\mathbf{x}}^t = (\underbrace{0,\dots,0}_{t\text{ terms}}, \underbrace{\overline{V},\dots,\overline{V}}_{L\text{ terms}}, \underbrace{0,\dots,0}_{T-t-L\text{ terms}})$ and $\hat{\mathbf{y}}^t = (\underbrace{0,\dots,0}_{t\text{ terms}}, \underbrace{1,\dots,1}_{L\text{ terms}}, \underbrace{0,\dots,0}_{T-t-L\text{ terms}})$ if $\eta = L \in \mathcal{S}$.

Note A.11-2: For $r \in [t+1, t+s_{\max}]_{\mathbb{Z}}$, the $\mathbf{x}$ and $\mathbf{y}$ vectors in group (C4) are given as follows: $\hat{\mathbf{x}}^r = (\underbrace{0,\dots,0}_{r\text{ terms}}, \underbrace{\underline{C},\dots,\underline{C}}_{T-r\text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{0,\dots,0}_{r\text{ terms}}, \underbrace{1,\dots,1}_{T-r\text{ terms}})$ if $r-t \notin \mathcal{S}$;
$\hat{\mathbf{x}}^r = (\underbrace{\overline{V}+(r-t)V,\dots,\overline{V}+(r-t)V}_{t-1\text{ terms}}, \underbrace{\overline{V}+(r-t)V, \overline{V}+(r-t-1)V, \overline{V}+(r-t-2)V,\dots,\overline{V}}_{r-t+1\text{ terms}}, \underbrace{0,\dots,0}_{T-r\text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{1,\dots,1}_{r\text{ terms}}, \underbrace{0,\dots,0}_{T-r\text{ terms}})$ if $r-t \in \mathcal{S}$.

Table A.12: Lower triangular matrix obtained from Table A.11 via Gaussian elimination.

Group	Point	Index r	**x**													**y**												
			1	2	⋯	$t-1$	t	$t+1$	$t+2$	⋯	$t+s_{\max}$	$t+s_{\max}+1$	$t+s_{\max}+2$	⋯	T	1	2	⋯	$t-1$	t	$t+1$	$t+2$	⋯	$t+s_{\max}$	$t+s_{\max}+1$	$t+s_{\max}+2$	⋯	T
(D1)	$(\underline{\mathbf{x}}^r, \underline{\mathbf{y}}^r)$	1	$-\epsilon$	0	⋯	0	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0	0	0	⋯	0
		2	0	$-\epsilon$	⋯	0	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0	0	0	⋯	0
		⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮	⋮		⋮	⋮	⋮		⋮
		$t-1$	0	0	⋯	$-\epsilon$	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0	0	0	⋯	0
		$t+1$	0	0	⋯	0	0	$-\epsilon$	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0	0	0	⋯	0
		$t+2$	0	0	⋯	0	0	0	$-\epsilon$	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0	0	0	⋯	0
		⋮	⋮	⋮		⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮	⋮		⋮	⋮	⋮		⋮
		$t+s_{\max}$	0	0	⋯	0	0	0	0	⋯	$-\epsilon$	0	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0	0	0	⋯	0
		$t+s_{\max}+1$	0	0	⋯	0	0	0	0	⋯	0	$-\epsilon$	0	⋯	0	0	0	⋯	0	0	0	0	⋯	0	0	0	⋯	0
		$t+s_{\max}+2$	0	0	⋯	0	0	0	0	⋯	0	0	$-\epsilon$	⋯	0	0	0	⋯	0	0	0	0	⋯	0	0	0	⋯	0
		⋮	⋮	⋮		⋮	⋮	⋮	⋮		⋮	⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮	⋮	⋮		⋮	⋮	⋮		⋮
		T	0	0	⋯	0	0	0	0	⋯	0	0	0	⋯	$-\epsilon$	0	0	⋯	0	0	0	0	⋯	0	0	0	⋯	0
(D2)	$(\hat{\underline{\mathbf{x}}}^r, \hat{\underline{\mathbf{y}}}^r)$	1	(Omitted)													1	0	⋯	0	0	0	0	⋯	0	0	0	⋯	0
		2														1	1	⋯	0	0	0	0	⋯	0	0	0	⋯	0
		⋮														⋮	⋮	⋱	⋮	⋮	⋮	⋮		⋮	⋮	⋮		⋮
		$t-1$														1	1	⋯	1	0	0	0	⋯	0	0	0	⋯	0
(D3)		t	(Omitted)													(See Note A.12-1)												
(D4)		$t+1$	(Omitted)													(See Note A.12-2)												
		⋮																										
		$t+s_{\max}$																										
(D5)		$t+s_{\max}+1$	(Omitted)													1	1	⋯	1	1	1	1	⋯	1	1	0	⋯	0
(D6)		$t+s_{\max}+2$	(Omitted)													0	0	⋯	0	0	0	0	⋯	0	0	1	⋯	0
		⋮														⋮	⋮		⋮	⋮	⋮	⋮		⋮	⋮	⋮	⋱	⋮
		T														0	0	⋯	0	0	0	0	⋯	0	0	0	⋯	1

Note A.12-1: The **y** vector in group (D3) is given as follows: $\hat{\underline{\mathbf{y}}}^t = (\underbrace{1,\ldots,1}_{t \text{ terms}}, \underbrace{0,\ldots,0}_{T-t \text{ terms}})$ if $\eta = 0$; $\hat{\underline{\mathbf{y}}}^t = (\underbrace{-1,\ldots,-1}_{t \text{ terms}}, \underbrace{0,\ldots,0}_{T-t \text{ terms}})$ if $\eta = (\overline{C}-\overline{V})/V$ or $\eta = L \in \mathcal{S}$.

Note A.12-2: For $r \in [t+1, t+s_{\max}]_{\mathbb{Z}}$, the **y** vector in group (D4) is given as follows: $\hat{\underline{\mathbf{y}}}^r = (\underbrace{-1,\ldots,-1}_{r \text{ terms}}, \underbrace{0,\ldots,0}_{T-r \text{ terms}})$ if $r-t \notin \mathcal{S}$; $\hat{\underline{\mathbf{y}}}^r = (\underbrace{1,\ldots,1}_{r \text{ terms}}, \underbrace{0,\ldots,0}_{T-r \text{ terms}})$ if $r-t \in \mathcal{S}$.

Table A.11 shows a matrix with $2T-1$ rows, where each row represents a point created by this process. This matrix can be transformed into the matrix in Table A.12 via the following Gaussian elimination process:

(i) For each $r \in [1,T]_{\mathbb{Z}} \setminus \{t\}$, the point with index r in group (D1), denoted $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r) = (\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$. Here, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ is the point with index r in group (C1), and $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ is the point in group (C5).

(ii) For each $r \in [1,t-1]_{\mathbb{Z}}$, the point with index r in group (D2), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (C2).

(iii) The point in group (D3), denoted $(\underline{\hat{\mathbf{x}}}^t, \underline{\hat{\mathbf{y}}}^t)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^t, \underline{\hat{\mathbf{y}}}^t) = (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ if $\eta = 0$, setting $(\underline{\hat{\mathbf{x}}}^t, \underline{\hat{\mathbf{y}}}^t) = (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t) - (\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ if $\eta = (\overline{C} - \overline{V})/V$, and setting $(\underline{\hat{\mathbf{x}}}^t, \underline{\hat{\mathbf{y}}}^t) = (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t) - (\hat{\mathbf{x}}^{t+L}, \hat{\mathbf{y}}^{t+L})$ if $\eta = L \in \mathcal{S}$. Here, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ is the point in group (C3), $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ is the point in group (C5), and $(\hat{\mathbf{x}}^{t+L}, \hat{\mathbf{y}}^{t+L})$ is the point with index $t+L$ in group (C4).

(iv) For each $r \in [t+1, t+s_{\max}]_{\mathbb{Z}}$, the point with index r in group (D4), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $r - t \in \mathcal{S}$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ if $r - t \notin \mathcal{S}$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (C4), and $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ is the point in group (C5).

(v) The point in group (D5), denoted $(\underline{\hat{\mathbf{x}}}^{t+s_{\max}+1}, \underline{\hat{\mathbf{y}}}^{t+s_{\max}+1})$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^{t+s_{\max}+1}, \underline{\hat{\mathbf{y}}}^{t+s_{\max}+1}) = (\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1}) - (\hat{\mathbf{x}}^{t+s_{\max}+2}, \hat{\mathbf{y}}^{t+s_{\max}+2})$.

Here, $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ is the point in group (C5), and $(\hat{\mathbf{x}}^{t+s_{\max}+2}, \hat{\mathbf{y}}^{t+s_{\max}+2})$ is the point with index $t + s_{\max} + 2$ in group (C6).

(vi) For each $r \in [t + s_{\max} + 2, T]_{\mathbb{Z}}$, the point with index r in group (D6), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ if $r \neq T$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $r = T$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ and $(\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ are the points with indices r and $r + 1$, respectively, in group (C6).

The matrix shown in Table A.12 is lower triangular; that is, the position of the last nonzero component of a row of the matrix is greater than the position of the last nonzero component of the previous row. This implies that the $2T - 1$ points in groups (C1)–(C6) are linearly independent. Therefore, inequality (4.6) is facet-defining for $\text{conv}(\mathcal{P})$. $\square$

8.9 Proof of Proposition 6

To prove that linear inequality (4.5) and (4.6) are valid for $\text{conv}(\mathcal{P})$ when $\mathcal{S} = [1, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$, it suffices to show that (4.5) and (4.6) are valid for $\mathcal{P}$ when $\mathcal{S} = [1, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. Consider any element $(\mathbf{x}, \mathbf{y})$ of $\mathcal{P}$. We show that $(\mathbf{x}, \mathbf{y})$ satisfies (4.5) and (4.6) when $\mathcal{S} = [1, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$.

We first show that $(\mathbf{x}, \mathbf{y})$ satisfies (4.5) when $\mathcal{S} = [1, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. Consider any $t \in [s_{\max} + 2, T]_{\mathbb{Z}}$. We divide the analysis into four cases.

Case 1: $y_t = 0$ and $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [1, \lfloor (\overline{C} - \overline{V})/V \rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Because $\eta \leq (\overline{C} - \overline{V})/V$, we have $\overline{C} - \overline{V} - \eta V \geq 0$. Thus, in this case, the right hand side of inequality (4.5) is nonnegative. Because $y_t = 0$, by (2.2d), $x_t = 0$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.5).

Case 2: $y_t = 0$ and $y_{t-s} - y_{t-s-1} > 0$ for some $s \in \mathcal{S}$. Let $\tilde{\mathcal{S}} = \{\sigma \in \mathcal{S} : y_{t-\sigma} - y_{t-\sigma-1} > 0\}$ and $v = |\tilde{\mathcal{S}}|$. Then, $v \geq 1$. Denote $\tilde{\mathcal{S}} = \{\sigma_1, \sigma_2, \ldots, \sigma_v\}$, where $\sigma_1 < \sigma_2 < \cdots < \sigma_v$. Note that $y_{t-\sigma_j-1} = 0$ and $y_{t-\sigma_j} = 1$ for $j = 1, \ldots, v$. Denote $\sigma_0 = -1$. Then, for each $j = 1, \ldots, v$, there exists $\sigma'_j \in [\sigma_{j-1} + 1, \sigma_j - 1]_{\mathbb{Z}}$ such that $y_{t-\sigma'_j-1} = 1$ and $y_{t-\sigma'_j} = 0$. Thus,

$$0 \leq \sigma'_1 < \sigma_1 < \sigma'_2 < \sigma_2 < \cdots < \sigma'_v < \sigma_v \leq s_{\max}.$$

Because $y_{t-\sigma_v} - y_{t-\sigma_v-1} = 1$ and $t - \sigma_v \in [2, T]_{\mathbb{Z}}$, by (2.2a), $y_k = 1$ for all $k \in [t - \sigma_v, \min\{T, t - \sigma_v + L - 1\}]_{\mathbb{Z}}$, which implies that $t - \sigma'_j \geq t - \sigma_v + L$ for $j = 1, \ldots, v$. Hence, for $j = 1, \ldots, v$, we have $\sigma'_j \leq \sigma_v - L$, which implies that

$$\sigma'_j \leq s_{\max} - L. \tag{8.26}$$

If $\beta = \alpha + 1$, then $\mathcal{S} = [1, s_{\max}]_{\mathbb{Z}}$, which implies that $\sigma'_j \in \mathcal{S}$ for $j = 2, \ldots, v$. If $\beta \neq \alpha + 1$, then condition (c) of Proposition 6 implies that $s_{\max} \leq L + \alpha$, which, by (8.26), implies that $\sigma'_j \leq \alpha$ for $j = 1, \ldots, v$. Because $\sigma'_2 > \sigma_1 \geq 1$, we have $\sigma'_j \in \mathcal{S}$ for $j = 2, \ldots, v$. Thus, in both cases, $\sigma'_j \in \mathcal{S}$ for $j = 2, \ldots, v$. Because $y_{t-\sigma_1} - y_{t-\sigma_1-1} = 1$ and $t - \sigma_1 \in [2, T]_{\mathbb{Z}}$, by (2.2a), $y_k = 1$ for all $k \in [t - \sigma_1, \min\{T, t - \sigma_1 + L - 1\}]_{\mathbb{Z}}$. Because $y_t = 0$, , this implies that $t \geq t - \sigma_1 + L$, or equivalent, $\sigma_1 \geq L$. Because $\eta \leq L$, we have

$$\eta \leq \sigma_1. \tag{8.27}$$

Because $y_t = 0$, by (2.2d), $x_t = 0$. Hence, the left hand side of inequality (4.5) is 0. Because $s_{\max} \leq \lfloor (\overline{C} - \overline{V})/V \rfloor$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Note that if $y_{t-1} = 0$, then $\sigma'_1 \geq 1$ and $\sigma'_1 \in \mathcal{S}$; if $y_{t-1} = 1$, then $\sigma'_1 = 0$ and $\sigma'_1 \notin \mathcal{S}$. Note that $\{\sigma'_2, \ldots, \sigma'_v\} \subseteq \mathcal{S} \setminus \tilde{\mathcal{S}}$ and $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S} \setminus \tilde{\mathcal{S}}$. Thus, $\sum_{s \in \mathcal{S} \setminus \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \leq \sum_{j=2}^{v} (\overline{C} - \overline{V} - \sigma'_j V)(y_{t-\sigma'_j} - y_{t-\sigma'_j-1}) + (\overline{C} - \overline{V} - \sigma'_1 V)(y_{t-\sigma'_1} - y_{t-\sigma'_1-1})(1 - y_{t-1})$. Hence, the right hand side of inequality (4.5) is

$$
\begin{aligned}
&(\overline{V}+\eta V)y_t+(\overline{C}-\overline{V}-\eta V)y_{t-1}-\sum_{s\in\mathcal{S}}(\overline{C}-\overline{V}-sV)(y_{t-s}-y_{t-s-1})\\
&=(\overline{C}-\overline{V}-\eta V)y_{t-1}-\sum_{s\in\tilde{\mathcal{S}}}(\overline{C}-\overline{V}-sV)(y_{t-s}-y_{t-s-1})\\
&\quad-\sum_{s\in\mathcal{S}\setminus\tilde{\mathcal{S}}}(\overline{C}-\overline{V}-sV)(y_{t-s}-y_{t-s-1})\\
&\geq(\overline{C}-\overline{V}-\eta V)y_{t-1}-\sum_{j=1}^{v}(\overline{C}-\overline{V}-\sigma_j V)(y_{t-\sigma_j}-y_{t-\sigma_j-1})\\
&\quad-\sum_{j=2}^{v}(\overline{C}-\overline{V}-\sigma_j' V)(y_{t-\sigma_j'}-y_{t-\sigma_j'-1})\\
&\quad-(\overline{C}-\overline{V}-\sigma_1' V)(y_{t-\sigma_1'}-y_{t-\sigma_1'-1})(1-y_{t-1})\\
&=(\overline{C}-\overline{V}-\eta V)y_{t-1}-(\overline{C}-\overline{V}-\sigma_1' V)y_{t-1}-(\overline{C}-\overline{V}-\sigma_1 V)+(\overline{C}-\overline{V}-\sigma_1' V)\\
&\quad-\sum_{j=2}^{v}(\overline{C}-\overline{V}-\sigma_j V)+\sum_{j=2}^{v}(\overline{C}-\overline{V}-\sigma_j' V)\\
&=(\sigma_1'-\eta)Vy_{t-1}+(\sigma_1-\sigma_1')V+\sum_{j=2}^{v}(\sigma_j-\sigma_j')V\\
&\geq(\sigma_1'-\eta)Vy_{t-1}+(\sigma_1-\sigma_1')V\\
&\geq 0.
\end{aligned}
$$

where the last inequality follows from $y_{t-1}\in\{0,1\}$, $\sigma_1>\sigma_1'$, and (8.27). Therefore, in this case, $(\mathbf{x},\mathbf{y})$ satisfies (4.5).

Case 3: $y_t=1$ and $y_{t-s}-y_{t-s-1}\leq 0$ for all $s\in\mathcal{S}$. Because $\mathcal{S}\subseteq[0,\lfloor(\overline{C}-\overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C}-\overline{V}-sV\geq 0$ for all $s\in\mathcal{S}$. Thus, $\sum_{s\in\mathcal{S}}(\overline{C}-\overline{V}-sV)(y_{t-s}-y_{t-s-1})\leq 0$. Because $y_t=1$, the right hand side of inequality (4.5) is at least $\overline{C}$ when $y_{t-1}=1$ and is at least $\overline{V}+\eta V\geq\overline{V}$ when

$y_{t-1} = 0$ (as $\eta \geq 0$). By (2.2d), $x_t \leq \overline{C}$. If $y_{t-1} = 0$, then by (2.2d) and (2.2e), $x_{t-1} = 0$ and $x_t - x_{t-1} \leq Vy_{t-1} + \overline{V}(1 - y_{t-1})$, which imply that $x_t \leq \overline{V}$. Hence, x_t is at most $\overline{C}$, and is at most $\overline{V}$ when $y_{t-1} = 0$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.5).

Case 4: $y_t = 1$ and $y_{t-s} - y_{t-s-1} > 0$ for some $s \in \mathcal{S}$. Let $\tilde{\mathcal{S}} = \{\sigma \in \mathcal{S} : y_{t-\sigma} - y_{t-\sigma-1} > 0\}$ and $v = |\tilde{\mathcal{S}}|$. Then, $v \geq 1$. Denote $\tilde{\mathcal{S}} = \{\sigma_1, \sigma_2, \ldots, \sigma_v\}$, where $\sigma_1 < \sigma_2 < \cdots < \sigma_v$. Note that $y_{t-\sigma_j-1} = 0$ and $y_{t-\sigma_j} = 1$ for $j = 1, \ldots, v$. Then, for each $j = 2, \ldots, v$, there exists $\sigma'_j \in [\sigma_{j-1} + 1, \sigma_j - 1]_{\mathbb{Z}}$ such that $y_{t-\sigma'_j-1} = 1$ and $y_{t-\sigma'_j} = 0$. Thus,

$$1 \leq \sigma_1 < \sigma'_2 < \sigma_2 < \cdots < \sigma'_v < \sigma_v \leq s_{\max}.$$

Because $y_{t-\sigma_v} - y_{t-\sigma_v-1} = 1$ and $t - \sigma_v \in [2, T]_{\mathbb{Z}}$, by (2.2a), $y_k = 1$ for all $k \in [t - \sigma_v, \min\{T, t - \sigma_v + L - 1\}]_{\mathbb{Z}}$, which implies that $t - \sigma'_j \geq t - \sigma_v + L$ for $j = 2, \ldots, v$. Hence, for $j = 2, \ldots, v$, we have $\sigma'_j \leq \sigma_v - L$, which implies that

$$\sigma'_j \leq s_{\max} - L. \tag{8.28}$$

If $\beta = \alpha + 1$, then $\mathcal{S} = [1, s_{\max}]_{\mathbb{Z}}$, which implies that $\sigma'_j \in \mathcal{S}$ for $j = 2, \ldots, v$. If $\beta \neq \alpha + 1$, then condition (c) of Proposition 6 implies that $s_{\max} \leq L + \alpha$, which, by (8.28), implies that $1 < \sigma'_j \leq \alpha$ for $j = 2, \ldots, v$. Thus, in both cases, $\sigma'_j \in \mathcal{S}$ for $j = 2, \ldots, v$. If $y_{t-1} = 0$, by (2.2d) and (2.2e), then $x_{t-1} = 0$ and $x_t - x_{t-1} \leq Vy_{t-1} + \overline{V}(1 - y_{t-1}) = \overline{V}$, which implies that $x_t \leq \overline{V}$. In addition, there exists $\sigma'_1 \in [1, \sigma_1 - 1]_{\mathbb{Z}}$ such that $y_{t-\sigma'_1} = 0$, $y_{t-\sigma'_1-1} = 1$, and $\sigma'_1 \in \mathcal{S}$. If $y_{t-1} = 1$, then we have $y_k = 1$ for all $k \in [t - \sigma_1, t]_Z$. Because

$y_{t-\sigma_1-1} = 0$, by (2.2d) and (2.2e), we have $x_{t-\sigma_1-1} = 0$ and

$$\sum_{\tau=t-\sigma_1}^{t} (x_\tau - x_{\tau-1}) \le \sum_{\tau=t-\sigma_1}^{t} V y_{\tau-1} + \sum_{\tau=t-\sigma_1}^{t} \overline{V}(1 - y_{\tau-1}),$$

which implies that

$$x_t - x_{t-\sigma_1-1} \le \sum_{\tau=t-\sigma_1}^{t} V y_{\tau-1} + \sum_{\tau=t-\sigma_1}^{t} \overline{V}(1 - y_{\tau-1}) = \sigma_1 V + \overline{V}.$$

Thus, the left hand side of inequality (4.5) is at most $\sigma_1 V + \overline{V}$. Because $\mathcal{S} \subseteq [0, \lfloor(\overline{C} - \overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \ge 0$ for all $s \in \mathcal{S}$. Note that $\{\sigma'_2, \ldots, \sigma'_v\} \subseteq \mathcal{S} \setminus \tilde{\mathcal{S}}$. Thus, $\sum_{s\in\mathcal{S}\setminus\tilde{\mathcal{S}}}(\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \le \sum_{j=2}^{v}(\overline{C} - \overline{V} - \sigma'_j V)(y_{t-\sigma'_j} - y_{t-\sigma'_j-1})$. Hence, when $y_{t-1} = 0$, the right hand side of inequality (4.5) is

$$
\begin{aligned}
&(\overline{V} + \eta V)y_t + (\overline{C} - \overline{V} - \eta V)y_{t-1} - \sum_{s\in\mathcal{S}}(\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \\
&= \overline{V} + \eta V - \sum_{s\in\tilde{\mathcal{S}}}(\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) - \sum_{s\in\mathcal{S}\setminus\tilde{\mathcal{S}}}(\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \\
&\ge \overline{V} + \eta V - \sum_{j=1}^{v}(\overline{C} - \overline{V} - \sigma_j V)(y_{t-\sigma_j} - y_{t-\sigma_j-1}) - \sum_{j=1}^{v}(\overline{C} - \overline{V} - \sigma'_j V)(y_{t-\sigma'_j} - y_{t-\sigma'_j-1}) \\
&= \overline{V} + \eta V + \sum_{j=1}^{v}(\sigma_j - \sigma'_j)V \\
&\ge \overline{V} + \eta V \\
&\ge \overline{V}.
\end{aligned}
$$

When $y_t = 1$, the right hand side of inequality (4.5) is

$$
\begin{aligned}
&(\overline{V} + \eta V)y_t + (\overline{C} - \overline{V} - \eta V)y_{t-1} - \sum_{s\in\mathcal{S}}(\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \\
&= \overline{C} - \sum_{s\in\tilde{\mathcal{S}}}(\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) - \sum_{s\in\mathcal{S}\setminus\tilde{\mathcal{S}}}(\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}) \\
&\geq \overline{C} - \sum_{j=1}^{v}(\overline{C} - \overline{V} - \sigma_j V)(y_{t-\sigma_j} - y_{t-\sigma_j-1}) - \sum_{j=2}^{v}(\overline{C} - \overline{V} - \sigma'_j V)(y_{t-\sigma'_j} - y_{t-\sigma'_j-1}) \\
&= \overline{C} - (\overline{C} - \overline{V} - \sigma_1 V) - \sum_{j=2}^{v}(\overline{C} - \overline{V} - \sigma_j V) + \sum_{j=2}^{v}(\overline{C} - \overline{V} - \sigma'_j V) \\
&= \sigma_1 V + \overline{V} + \sum_{j=2}^{v}(\sigma_j - \sigma'_j)V \\
&\geq \sigma_1 V + \overline{V}.
\end{aligned}
$$

Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.5).

Next, we show that $(\mathbf{x}, \mathbf{y})$ satisfies (4.6) when $\mathcal{S} = [1, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. Consider any $t \in [1, T - s_{\max} - 1]_{\mathbb{Z}}$. We divide the analysis into four cases.

Case 1: $y_t = 0$ and $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [1, \lfloor(\overline{C} - \overline{V})/V\rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Because $\eta \leq (\overline{C} - \overline{V})/V$, we have $\overline{C} - \overline{V} - \eta V \geq 0$. Thus, in this case, the right hand side of inequality (4.6) is nonnegative. Because $y_t = 0$, by (2.2d), $x_t = 0$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.6).

Case 2: $y_t = 0$ and $y_{t+s} - y_{t+s+1} > 0$ for some $s \in \mathcal{S}$. Let $\tilde{\mathcal{S}} = \{\sigma \in \mathcal{S} : y_{t+\sigma} - y_{t+\sigma+1} > 0\}$ and $v = |\tilde{\mathcal{S}}|$. Then, $v \geq 1$. Denote $\tilde{\mathcal{S}} = \{\sigma_1, \sigma_2, \ldots, \sigma_v\}$, where $\sigma_1 < \sigma_2 < \cdots < \sigma_v$. Note that $y_{t+\sigma_j} = 1$ and $y_{t+\sigma_j+1} = 0$ for $j = 1, \ldots, v$. Denote $\sigma_0 = -1$. Then, for each $j = 1, \ldots, v$, there exists

$\sigma'_j \in [\sigma_{j-1}+1, \sigma_j - 1]_{\mathbb{Z}}$ such that $y_{t+\sigma'_j} = 0$ and $y_{t+\sigma'_j+1} = 1$. Thus,

$$0 \leq \sigma'_1 < \sigma_1 < \sigma'_2 < \sigma_2 < \cdots < \sigma'_v < \sigma_v \leq s_{\max}.$$

Because $y_{t+\sigma_v+1} = 0$ and $t + \sigma_v + 1 \in [2, T]_{\mathbb{Z}}$, by Lemma 1(i), $y_{t+\sigma_v+1-j} - y_{t+\sigma_v-j} \leq 0$ for all $j \in [0, \min\{t+\sigma_v-1, L-1\}]_{\mathbb{Z}}$, which implies that $t + \sigma'_j + 1 \leq t + \sigma_v - L + 1$ for $j = 1, \ldots, v$. Hence, for $j = 1, \ldots, v$, we have $\sigma'_j \leq \sigma_v - L$, which implies that

$$\sigma'_j \leq s_{\max} - L. \tag{8.29}$$

If $\beta = \alpha + 1$, then $\mathcal{S} = [1, s_{\max}]_{\mathbb{Z}}$, which implies that $\sigma'_j \in \mathcal{S}$ for $j = 2, \ldots, v$. If $\beta \neq \alpha + 1$, then condition (c) of Proposition 6 implies that $s_{\max} \leq L + \alpha$, which, by (8.29), implies that $\sigma'_j \leq \alpha$ for $j = 2, \ldots, v$. Because $\sigma'_2 > \sigma_1 \geq 1$, we have $\sigma'_j \in \mathcal{S}$ for $j = 2, \ldots, v$. Thus, in both cases, $\sigma'_j \in \mathcal{S}$ for $j = 2, \ldots, v$. Because $y_t = 0$ and $t \in [1, T-1]_{\mathbb{Z}}$, by Lemma 2(i), $y_{t+j} - y_{t+j+1} \leq 0$ for all $j \in [0, \min\{T-t-1, L-1\}]_{\mathbb{Z}}$, which implies that $t + \sigma_1 \geq t + L$, or equivalently, $\sigma_1 \geq L$. Because $\eta \leq L$, we have

$$\eta \leq \sigma_1. \tag{8.30}$$

Because $y_t = 0$, by (2.2d), $x_t = 0$. Hence, the left hand side of inequality (4.6) is 0. Because $\mathcal{S} \subseteq [0, \lfloor (\overline{C} - \overline{V})/V \rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Note that if $y_{t+1} = 0$, then $\sigma'_1 \in \mathcal{S}$; if $y_{t+1} = 1$, then $\sigma'_1 = 0$ and $\sigma'_1 \notin \mathcal{S}$. Note that $\{\sigma'_2, \ldots, \sigma'_v\} \subseteq \mathcal{S} \setminus \tilde{\mathcal{S}}$ and $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in \mathcal{S} \setminus \tilde{\mathcal{S}}$. Thus, $\sum_{s \in \mathcal{S} \setminus \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \leq \sum_{j=2}^{v} (\overline{C} - \overline{V} -$

$\sigma_j' V)(y_{t+\sigma_j'} - y_{t+\sigma_j'+1}) + (\overline{C} - \overline{V} - \sigma_1' V)(y_{t+\sigma_1'} - y_{t+\sigma_1'+1})(1 - y_{t+1})$. Hence, the right hand side of inequality (4.6) is

$$
\begin{aligned}
&(\overline{V} + \eta V)y_t + (\overline{C} - \overline{V} - \eta V)y_{t+1} - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \\
&= (\overline{C} - \overline{V} - \eta V)y_{t+1} - \sum_{s \in \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \\
&\quad - \sum_{s \in \mathcal{S} \setminus \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \\
&\geq (\overline{C} - \overline{V} - \eta V)y_{t+1} - \sum_{j=1}^{v} (\overline{C} - \overline{V} - \sigma_j V)(y_{t+\sigma_j} - y_{t+\sigma_j+1}) \\
&\quad - \sum_{j=2}^{v} (\overline{C} - \overline{V} - \sigma_j' V)(y_{t+\sigma_j'} - y_{t+\sigma_j'+1}) \\
&\quad - (\overline{C} - \overline{V} - \sigma_1' V)(y_{t+\sigma_1'} - y_{t+\sigma_1'+1})(1 - y_{t+1}) \\
&= (\overline{C} - \overline{V} - \eta V)y_{t+1} - (\overline{C} - \overline{V} - \sigma_1' V)y_{t+1} - (\overline{C} - \overline{V} - \sigma_1 V) + (\overline{C} - \overline{V} - \sigma_1' V) \\
&\quad - \sum_{j=2}^{v} (\overline{C} - \overline{V} - \sigma_j V) + \sum_{j=2}^{v} (\overline{C} - \overline{V} - \sigma_j' V) \\
&= (\sigma_1' - \eta)V y_{t+1} + (\sigma_1 - \sigma_1')V + \sum_{j=2}^{v} (\sigma_j - \sigma_j')V \\
&\geq (\sigma_1' - \eta)V y_{t+1} + (\sigma_1 - \sigma_1')V \\
&\geq 0.
\end{aligned}
$$

where the last inequality follows from $y_{t+1} \in \{0, 1\}$, $\sigma_1 > \sigma_1'$, and (8.30). Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.6).

Case 3: $y_t = 1$ and $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, \lfloor (\overline{C} - \overline{V})/V \rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Thus, $\sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}) \leq 0$. Because $y_t = 1$, the right hand side of inequality (4.6) is at least $\overline{C}$ when $y_{t+1}=1$ and is at least $\overline{V}+\eta V \geq \overline{V}$ when $y_{t+1}=0$ (as $\eta \geq 0$). By (2.2d), $x_t \leq \overline{C}$. If $y_{t+1}=0$, then by (2.2d) and (2.2f), $x_{t+1}=0$ and $x_t - x_{t+1} \leq V y_{t+1} + \overline{V}(1-y_{t+1})$, which imply that $x_t \leq \overline{V}$. Hence, x_t is at most $\overline{C}$, and is at most $\overline{V}$ when $y_{t+1}=0$. Therefore, in both cases, $(\mathbf{x},\mathbf{y})$ satisfies (4.6).

Case 4: $y_t=1$ and $y_{t+s}-y_{t+s+1}>0$ for some $s \in \mathcal{S}$. Let $\tilde{\mathcal{S}}=\{\sigma \in \mathcal{S} : y_{t+\sigma}-y_{t+\sigma+1}>0\}$ and $v=|\tilde{\mathcal{S}}|$. Then, $v \geq 1$. Denote $\tilde{\mathcal{S}}=\{\sigma_1,\sigma_2,\ldots,\sigma_v\}$, where $\sigma_1<\sigma_2<\cdots<\sigma_v$. Note that $y_{t+\sigma_j}=1$ and $y_{t+\sigma_j+1}=0$ for $j=1,\ldots,v$. Then, for each $j=2,\ldots,v$, there exists $\sigma'_j \in [\sigma_{j-1}+1,\sigma_j-1]_{\mathbb{Z}}$ such that $y_{t+\sigma'_j}=0$ and $y_{t+\sigma'_j+1}=1$. Thus,

$$1 \leq \sigma_1 < \sigma'_2 < \sigma_2 < \cdots < \sigma'_v < \sigma_v \leq s_{\max}.$$

Because $y_{t+\sigma_v+1}=0$ and $t+\sigma_v+1 \in [2,T]_{\mathbb{Z}}$, by Lemma 1(i), $y_{t+\sigma_v+1-j}-y_{t+\sigma_v-j} \leq 0$ for all $j \in [0,\min\{t+\sigma_v-1,L-1\}]_{\mathbb{Z}}$, which implies that $t+\sigma'_j+1 \leq t+\sigma_v-L+1$ for $j=2,\ldots,v$. Hence, for $j=2,\ldots,v$, we have $\sigma'_j \leq \sigma_v - L$, which implies that,

$$\sigma'_j \leq s_{\max}-L. \tag{8.31}$$

If $\beta=\alpha+1$, then $\mathcal{S}=[1,s_{\max}]_{\mathbb{Z}}$, which implies that $\sigma'_j \in \mathcal{S}$ for $j=2,\ldots,v$. If $\beta \neq \alpha+1$, then condition (c) of Proposition 6 implies that $s_{\max} \leq L+\alpha$, which, by (8.31), implies that $\sigma'_j \leq \alpha$ for $j=2,\ldots,v$. Thus, in both cases, $\sigma'_j \in \mathcal{S}$ for $j=2,\ldots,v$. If $y_{t+1}=0$, by (2.2d) and (2.2f), then $x_{t+1}=0$ and $x_t-x_{t+1} \leq V y_{t+1}+\overline{V}(1-y_{t+1})=\overline{V}$, which implies that $x_t \leq \overline{V}$. In addition, there exists $\sigma_1' \in [1, \sigma_1 - 1]_{\mathbb{Z}}$ such that $y_{t+\sigma'} = 0$, $y_{t+\sigma_1'+1} = 1$, and $\sigma_1' \in \mathcal{S}$. If $y_{t+1} = 1$, then we have $y_k = 1$ for all $k \in [t, t+\sigma_1]_{\mathbb{Z}}$. Because $y_{t+\sigma_1+1} = 0$, by (2.2d) and (2.2f), we have $x_{t+\sigma_1+1} = 0$ and

$$\sum_{\tau=t+1}^{t+\sigma_1+1} (x_{\tau-1} - x_\tau) \leq \sum_{\tau=t+1}^{t+\sigma_1+1} V y_\tau + \sum_{\tau=t+1}^{t+\sigma_1+1} \overline{V}(1 - y_\tau),$$

which implies that

$$x_t - x_{t+\sigma_1+1} \leq \sum_{\tau=t+1}^{t+\sigma_1+1} V y_\tau + \sum_{\tau=t+1}^{t+\sigma_1+1} \overline{V}(1 - y_\tau) = \sigma_1 V + \overline{V}.$$

Thus, the left hand side of inequality (4.6) is at most $\sigma_1 V + \overline{V}$. Because $\mathcal{S} \subseteq [0, \lfloor (\overline{C} - \overline{V})/V \rfloor]_{\mathbb{Z}}$, we have $\overline{C} - \overline{V} - sV \geq 0$ for all $s \in \mathcal{S}$. Note that $\{\sigma_2', \ldots, \sigma_v'\} \subseteq \mathcal{S} \setminus \tilde{\mathcal{S}}$. Thus, $\sum_{s \in \mathcal{S} \setminus \tilde{\mathcal{S}}} (\overline{C} - \overline{V} - sV) \leq \sum_{j=2}^{v} (\overline{C} - \overline{V} - \sigma_j' V)(y_{t+\sigma_j'} - y_{t+\sigma_j'+1})$. Hence, when $y_{t+1} = 0$, the right hand side of inequality (4.6) is

$$
\begin{aligned}
&(\overline{V}+\eta V)y_t+(\overline{C}-\overline{V}-\eta V)y_{t+1}-\sum_{s\in\mathcal{S}}(\overline{C}-\overline{V}-sV)(y_{t+s}-y_{t+s+1})\\
&=(\overline{V}+\eta V)-\sum_{s\in\tilde{\mathcal{S}}}(\overline{C}-\overline{V}-sV)(y_{t+s}-y_{t+s+1})-\sum_{s\in\mathcal{S}\setminus\tilde{\mathcal{S}}}(\overline{C}-\overline{V}-sV)(y_{t+s}-y_{t+s+1})\\
&\geq(\overline{V}+\eta V)-\sum_{j=1}^{v}(\overline{C}-\overline{V}-\sigma_jV)(y_{t+\sigma_j}-y_{t+\sigma_j+1})-\sum_{j=1}^{v}(\overline{C}-\overline{V}-\sigma_j'V)(y_{t+\sigma_j'}-y_{t+\sigma_j'+1})\\
&=(\overline{V}+\eta V)-\sum_{j=1}^{v}(\overline{C}-\overline{V}-\sigma_jV)+\sum_{j=1}^{v}(\overline{C}-\overline{V}-\sigma_j'V)\\
&=(\overline{V}+\eta V)+\sum_{j=1}^{v}(\sigma_j-\sigma_j')V\\
&>(\overline{V}+\eta V)\\
&\geq\overline{V}.
\end{aligned}
$$

When $y_t = 1$, the right hand side of inequality (4.6) is

$$
\begin{aligned}
&(\overline{V}+\eta V)y_t+(\overline{C}-\overline{V}-\eta V)y_{t+1}-\sum_{s\in\mathcal{S}}(\overline{C}-\overline{V}-sV)(y_{t+s}-y_{t+s+1})\\
&=\overline{C}-\sum_{s\in\tilde{\mathcal{S}}}(\overline{C}-\overline{V}-sV)(y_{t+s}-y_{t+s+1})-\sum_{s\in\mathcal{S}\setminus\tilde{\mathcal{S}}}(\overline{C}-\overline{V}-sV)(y_{t+s}-y_{t+s+1})\\
&\geq\overline{C}-\sum_{j=1}^{v}(\overline{C}-\overline{V}-\sigma_jV)(y_{t+\sigma_j}-y_{t+\sigma_j+1})-\sum_{j=2}^{v}(\overline{C}-\overline{V}-\sigma_j'V)(y_{t+\sigma_j'}-y_{t+\sigma_j'+1})\\
&=\overline{C}-(\overline{C}-\overline{V}-\sigma_1V)-\sum_{j=2}^{v}(\overline{C}-\overline{V}-\sigma_jV)+\sum_{j=1}^{v}(\overline{C}-\overline{V}-\sigma_j'V)\\
&=\sigma_1V+\overline{V}+\sum_{j=2}^{v}(\sigma_j-\sigma_j')V\\
&\geq\sigma_1V+\overline{V}.
\end{aligned}
$$

Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.6).

It is easy to verify that the proof of facet-defining of inequalities (4.5) and (4.6) in the proof of Proposition 5 remains valid when $\mathcal{S} = [1, \alpha]_{\mathbb{Z}} \cup [\beta, s_{\max}]_{\mathbb{Z}}$. Therefore, inequalities (4.5) and (4.6) are facet-defining under the conditions stated in Proposition 6. $\square$

8.10 Proof of Proposition 7

(i) Consider the inequality

$$x_t \leq (a_1 + a_2\eta)y_{t-1} + (a_3 + a_4\eta)y_t + (a_5 + a_6\eta)y_{t+1} - \sum_{s\in\mathcal{S}}(\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}), \tag{8.32}$$

and consider any given point $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2T}_{+}$. For any integer $t \leq T$, let

$$\theta(t) = \sum_{\tau=2}^{t} \max\{y_\tau - y_{\tau-1}, 0\}.$$

Then, for any $t \in [2, T]_{\mathbb{Z}}$,

$$\sum_{\substack{s\in[\check{s},\hat{s}]_{\mathbb{Z}} \\ t-s\geq 2}} \max\{y_{t-s} - y_{t-s-1}, 0\} = \theta(t - \check{s}) - \theta(t - \hat{s} - 1) \tag{8.33}$$

and

$$\sum_{\substack{s\in[\check{s}+1,\hat{s}+1]_{\mathbb{Z}} \\ t-s\geq 2}} \max\{y_{t-s} - y_{t-s-1}, 0\} = \theta(t - \check{s} - 1) - \theta(t - \hat{s} - 2). \tag{8.34}$$

Furthermore, for any $t \in [2, T]_{\mathbb{Z}}$,

$$\sum_{\substack{s \in [\check{s}, \hat{s}]_{\mathbb{Z}} \\ t-s \geq 2}} s \max\{y_{t-s} - y_{t-s-1}, 0\} - \sum_{\substack{s \in [\check{s}+1, \hat{s}+1]_{\mathbb{Z}} \\ t-s \geq 2}} s \max\{y_{t-s} - y_{t-s-1}, 0\}$$

$$= \begin{cases} \check{s} \max\{y_{t-\check{s}} - y_{t-\check{s}-1}, 0\} - (\hat{s}+1) \max\{y_{t-\hat{s}-1} - y_{t-\hat{s}-2}, 0\}, \\ \qquad\qquad\qquad\qquad\qquad \text{if } 2 \leq t - \hat{s} - 1; \\ \check{s} \max\{y_{t-\check{s}} - y_{t-\check{s}-1}, 0\}, \qquad \text{if } t - \hat{s} - 1 < 2 \leq t - \check{s}; \\ 0, \qquad\qquad\qquad\qquad\qquad \text{if } t - \check{s} < 2. \end{cases}$$

Note that

$$\theta(t - \check{s}) - \theta(t - \check{s} - 1) = \begin{cases} \max\{y_{t-\check{s}} - y_{t-\check{s}-1}, 0\}, & \text{if } t - \check{s} \geq 2; \\ 0, & \text{if } t - \check{s} < 2. \end{cases}$$

and

$$\theta(t - \hat{s} - 1) - \theta(t - \hat{s} - 2) = \begin{cases} \max\{y_{t-\hat{s}-1} - y_{t-\hat{s}-2}, 0\}, & \text{if } t - \hat{s} - 1 \geq 2; \\ 0, & \text{if } t - \hat{s} - 1 < 2. \end{cases}$$

Hence, for any $t \in [2, T]_{\mathbb{Z}}$,

$$\sum_{\substack{s \in [\check{s}, \hat{s}]_{\mathbb{Z}} \\ t-s \geq 2}} s \max\{y_{t-s} - y_{t-s-1}, 0\} - \sum_{\substack{s \in [\check{s}+1, \hat{s}+1]_{\mathbb{Z}} \\ t-s \geq 2}} s \max\{y_{t-s} - y_{t-s-1}, 0\}$$

$$= \check{s}[\theta(t - \check{s}) - \theta(t - \check{s} - 1)] - (\hat{s}+1)[\theta(t - \hat{s} - 1) - \theta(t - \hat{s} - 2)]. \tag{8.35}$$

For any $\eta \in [0, \hat{\eta}]$, $\mathcal{S} \subseteq [\check{s}, \hat{s}]_{\mathbb{Z}}$, and $t \in [\check{t}, \hat{t}]_{\mathbb{Z}}$ such that $t \geq s + 2\ \forall s \in \mathcal{S}$, let

$$\begin{aligned}\tilde{v}(\eta, \mathcal{S}, t) =& x_t - (a_1 + a_2\eta)y_{t-1} - (a_3 + a_4\eta)y_t - (a_5 + a_6\eta)y_{t+1} \\ &+ \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}).\end{aligned}$$

If $\tilde{v}(\eta, \mathcal{S}, t) > 0$, then $\tilde{v}(\eta, \mathcal{S}, t)$ is the amount of violation of inequality (8.32). If $\tilde{v}(\eta, \mathcal{S}, t) \leq 0$, then there is no violation of inequality (8.32). For any $\eta \in [0, \hat{\eta}]$ and $t \in [\check{t}, \hat{t}]_{\mathbb{Z}}$, let

$$v(\eta, t) = \max_{\mathcal{S} \subseteq [\check{s}, \min\{\hat{s}, t-2\}]_{\mathbb{Z}}} \{\tilde{v}(\eta, \mathcal{S}, t)\}.$$

If $v(\eta, t) > 0$, then $v(\eta, t)$ is the largest possible violation of inequality (8.32) for this combination of η and t. If $v(\eta, t) \leq 0$, then the largest possible violation of inequality (8.32) is zero for this combination of η and t.

Note that $\overline{C} - \overline{V} - sV \geq 0$ for any $s \in [\check{s}, \hat{s}]_{\mathbb{Z}}$. Thus, for any given $\eta \in [0, \hat{\eta}]$ and $t \in [\check{t}, \hat{t}]_{\mathbb{Z}}$, $\tilde{v}(\eta, \mathcal{S}, t)$ is maximized when $\mathcal{S}$ contains all $s \in [\check{s}, \min\{\hat{s}, t-2\}]_{\mathbb{Z}}$ such that $y_{t-s} - y_{t-s-1} > 0$ (if any). If it does not exist any $s \in [\check{s}, \min\{\hat{s}, t-2\}]_{\mathbb{Z}}$ such that $y_{t-s} - y_{t-s-1} > 0$, then $\tilde{v}(\eta, \mathcal{S}, t)$ is maximized when $\mathcal{S} = \emptyset$, and $v(\eta, t) = x_t - (a_1 + a_2\eta)y_{t-1} - (a_3 + a_4\eta)y_t -$

$(a_5 + a_6\eta)y_{t+1}$. Hence, for any $\eta \in [0, \hat{\eta}]$ and $t \in [\check{t}, \hat{t}]_{\mathbb{Z}}$,

$$v(\eta, t) =$$
$$x_t - (a_1 + a_2\eta)y_{t-1} - (a_3 + a_4\eta)y_t - (a_5 + a_6\eta)y_{t+1}$$
$$+ (\overline{C} - \overline{V}) \sum_{\substack{s \in [\check{s}, \hat{s}]_{\mathbb{Z}} \\ t-s \geq 2}} \max\{y_{t-s} - y_{t-s-1}, 0\} - V \sum_{\substack{s \in [\check{s}, \hat{s}]_{\mathbb{Z}} \\ t-s \geq 2}} s \max\{y_{t-s} - y_{t-s-1}, 0\}.$$

When $t = \check{t}$, we have

$$v(\eta, \check{t}) = \begin{cases} x_1 - (a_3 + a_4\eta)y_1 - (a_5 + a_6\eta)y_2, & \text{if } \check{t} = 1; \\ x_2 - (a_1 + a_2\eta)y_1 - (a_3 + a_4\eta)y_2 - (a_5 + a_6\eta)y_3 & \\ \qquad + (\overline{C} - \overline{V}) \max\{y_2 - y_1, 0\}, & \text{if } \check{t} = 2 \text{ and } \check{s} = 0; \\ x_2 - (a_1 + a_2\eta)y_1 - (a_3 + a_4\eta)y_2 - (a_5 + a_6\eta)y_3, & \text{otherwise.} \end{cases}$$

For any $\eta \in [0, \hat{\eta}]$ and $t \in [\check{t} + 1, \hat{t}]_{\mathbb{Z}}$,

$$v(\eta, t) - v(\eta, t-1) =$$
$$(x_t - x_{t-1})$$
$$- (a_1 + a_2\eta)(y_{t-1} - y_{t-2}) - (a_3 + a_4\eta)(y_t - y_{t-1}) - (a_5 + a_6\eta)(y_{t+1} - y_t)$$
$$+ (\overline{C} - \overline{V}) \left[\sum_{\substack{s \in [\check{s}, \hat{s}]_{\mathbb{Z}} \\ t-s \geq 2}} \max\{y_{t-s} - y_{t-s-1}, 0\} - \sum_{\substack{s \in [\check{s}+1, \hat{s}+1]_{\mathbb{Z}} \\ t-s \geq 2}} \max\{y_{t-s} - y_{t-s-1}, 0\} \right]$$
$$- V \left[\sum_{\substack{s \in [\check{s}, \hat{s}]_{\mathbb{Z}} \\ t-s \geq 2}} s \max\{y_{t-s} - y_{t-s-1}, 0\} - \sum_{\substack{s \in [\check{s}+1, \hat{s}+1]_{\mathbb{Z}} \\ t-s \geq 2}} (s-1) \max\{y_{t-s} - y_{t-s-1}, 0\} \right],$$

which implies that

$$\begin{aligned}
&v(\eta,t) = \\
&v(\eta,t-1) + (x_t - x_{t-1}) \\
&- (a_1 + a_2\eta)(y_{t-1} - y_{t-2}) - (a_3 + a_4\eta)(y_t - y_{t-1}) - (a_5 + a_6\eta)(y_{t+1} - y_t) \\
&+ (\overline{C} - \overline{V}) \left[\sum_{\substack{s\in[\check{s},\hat{s}]_{\mathbb{Z}} \\ t-s\geq 2}} \max\{y_{t-s} - y_{t-s-1}, 0\} - \sum_{\substack{s\in[\check{s}+1,\hat{s}+1]_{\mathbb{Z}} \\ t-s\geq 2}} \max\{y_{t-s} - y_{t-s-1}, 0\} \right] \\
&- V \sum_{\substack{s\in[\check{s}+1,\hat{s}+1]_{\mathbb{Z}} \\ t-s\geq 2}} \max\{y_{t-s} - y_{t-s-1}, 0\} \\
&- V \left[\sum_{\substack{s\in[\check{s},\hat{s}]_{\mathbb{Z}} \\ t-s\geq 2}} s \max\{y_{t-s} - y_{t-s-1}, 0\} - \sum_{\substack{s\in[\check{s}+1,\hat{s}+1]_{\mathbb{Z}} \\ t-s\geq 2}} s \max\{y_{t-s} - y_{t-s-1}, 0\} \right].
\end{aligned}$$

Thus, by (8.33), (8.34), and (8.35),

$$\begin{aligned}
&v(\eta,t) = \\
&v(\eta,t-1) + (x_t - x_{t-1}) \\
&- (a_1 + a_2\eta)(y_{t-1} - y_{t-2}) - (a_3 + a_4\eta)(y_t - y_{t-1}) - (a_5 + a_6\eta)(y_{t+1} - y_t) \\
&+ (\overline{C} - \overline{V})[\theta(t-\check{s}) - \theta(t-\hat{s}-1) - \theta(t-\check{s}-1) + \theta(t-\hat{s}-2)] \\
&- V[\check{s}\theta(t-\check{s}) - (\check{s}-1)\theta(t-\check{s}-1) - (\hat{s}+1)\theta(t-\hat{s}-1) + \hat{s}\theta(t-\hat{s}-2)]
\end{aligned} \tag{8.36}$$

for any $\eta \in [0, \hat{\eta}]$ and $t \in [\check{t}+1, \hat{t}]_{\mathbb{Z}}$. Note that $\tilde{v}(\eta, \mathcal{S}, t)$ is linear in η. Thus, for any given t, $v(\eta, t)$ is maximized when $\eta = 0$ or $\eta = \hat{\eta}$. That is, the largest possible value of $v(\eta, t)$ is equal to $v(0, t)$ if $a_2 y_{t-1} + a_4 y_t + a_6 y_{t+1} \geq 0$, and

Algorithm 1 Determination of the most violated inequality (8.32) for any given $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_{+}^{2T}$

1: $\theta(t) \leftarrow 0 \;\forall\; t \in [-\hat{s}, 1]_{\mathbb{Z}}$
2: **for** $t = 2, \ldots, \hat{t}$ **do**
3: $\quad \theta(t) \leftarrow \theta(t-1) + \max\{y_t - y_{t-1}, 0\}$
4: **end for**
5: **for** $\eta = 0, \hat{\eta}$ **do**
6: $\quad$ **if** $\check{t} = 1$ **then**
7: $\quad\quad v(\eta, \check{t}) \leftarrow x_1 - (a_3 + a_4\eta)y_1 - (a_5 + a_6\eta)y_2$
8: $\quad$ **else if** $\check{s} = 0$ **then**
9: $\quad\quad v(\eta, \check{t}) \leftarrow x_2 - (a_1 + a_2\eta)y_1 - (a_3 + a_4\eta)y_2 - (a_5 + a_6\eta)y_3 + (\overline{C} - \overline{V}) \max\{y_2 - y_1, 0\}$
10: $\quad$ **else**
11: $\quad\quad v(\eta, \check{t}) \leftarrow x_2 - (a_1 + a_2\eta)y_1 - (a_3 + a_4\eta)y_2 - (a_5 + a_6\eta)y_3$
12: $\quad$ **end if**
13: $\quad$ **for** $t = \check{t} + 1, \ldots, \hat{t}$ **do**
14: $\quad\quad v(\eta, t) \leftarrow v(\eta, t-1) + (x_t - x_{t-1})$ $- (a_1 + a_2\eta)(y_{t-1} - y_{t-2}) - (a_3 + a_4\eta)(y_t - y_{t-1}) - (a_5 + a_6\eta)(y_{t+1} - y_t)$ $+ (\overline{C} - \overline{V})[\theta(t-\check{s}) - \theta(t-\hat{s}-1) - \theta(t-\check{s}-1) + \theta(t-\hat{s}-2)]$ $- \overline{V}[\check{s}\theta(t-\check{s}) - (\check{s}-1)\theta(t-\check{s}-1) - (\hat{s}+1)\theta(t-\hat{s}-1) + \hat{s}\theta(t-\hat{s}-2)]$
15: $\quad$ **end for**
16: **end for**
17: $(\eta^*, t^*) \leftarrow \text{argmax}_{(\eta,t) \in \{0,\hat{\eta}\} \times [\check{t},\hat{t}]_{\mathbb{Z}}} \{v(\eta, t)\}$
18: $\mathcal{S}^* \leftarrow \emptyset$
19: **for** $s = \check{s}, \ldots, \min\{\hat{s}, t^* - 2\}$ **do**
20: $\quad$ **if** $y_{t^*-s} - y_{t^*-s-1} > 0$ **then** $\mathcal{S}^* \leftarrow \mathcal{S}^* \cup \{s\}$
21: **end for**

the largest possible value of $v(\eta, t)$ is equal to $v(\hat{\eta}, t)$ if $a_2 y_{t-1} + a_4 y_t + a_6 y_{t+1} < 0$. Hence, to determine the η and t values corresponding to the largest violation of inequality (8.32), it suffices to determine $v(0, \check{t}), v(0, \check{t} + 1), \ldots, v(0, \hat{t})$ and $v(\hat{\eta}, \check{t}), v(\hat{\eta}, \check{t} + 1), \ldots, v(\hat{\eta}, \hat{t})$. Algorithm 1 performs this computation.

In Algorithm 1, step 1 sets $\theta(t)$ to zero when $t \leq 1$. Steps 2–4 determine the $\theta(t)$ values recursively for $t = 2, 3, \ldots, \hat{t}$. These steps require $O(T)$ time.

Steps 5–16 consider the case $\eta = 0$ and the case $\eta = \hat{\eta}$. For each of these two η values, these steps first determine $v(\eta, \check{t})$, and then determine $v(\eta, \check{t}+1), v(\eta, \check{t}+2), \ldots, v(\eta, \hat{t})$ recursively using equation (8.36). These steps require $O(T)$ time. Steps 17–21 identify the most violated inequality (8.32) by setting the η and t values to $(\eta^*, t^*) = \text{argmax}_{(\eta,t)\in\{0,\hat{\eta}\}\times[\check{t},\hat{t}]_{\mathbb{Z}}}\{v(\eta, t)\}$ and setting $\mathcal{S}$ equal to the set of s values such that $s \in [\check{s}, \min\{\hat{s}, t^* - 2\}]_{\mathbb{Z}}$ and $y_{t^*-s} - y_{t^*-s-1} > 0$. These steps also require $O(T)$ time. Therefore, the total computational time of Algorithm 1 is $O(T)$.

(ii) Consider the inequality

$$\begin{aligned} x_t \leq &(a_1 + a_2\eta)y_{t-1} + (a_3 + a_4\eta)y_t + (a_5 + a_6\eta)y_{t+1} \\ &- \sum_{s\in\mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}), \end{aligned} \tag{8.37}$$

and consider any given point $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{2T}$. For any integer $t \geq 1$, let

$$\theta'(t) = \sum_{\tau=t}^{T-1} \max\{y_\tau - y_{\tau+1}, 0\}.$$

Then, for any $t \in [1, T-1]_{\mathbb{Z}}$,

$$\sum_{\substack{s\in[\check{s},\hat{s}]_{\mathbb{Z}} \\ t+s\leq T-1}} \max\{y_{t+s} - y_{t+s+1}, 0\} = \theta'(t + \check{s}) - \theta'(t + \hat{s} + 1) \tag{8.38}$$

and

$$\sum_{\substack{s\in[\check{s}+1,\hat{s}+1]_{\mathbb{Z}} \\ t+s\leq T-1}} \max\{y_{t+s} - y_{t+s+1}, 0\} = \theta'(t + \check{s} + 1) - \theta'(t + \hat{s} + 2). \tag{8.39}$$

Furthermore, for any $t \in [1, T-1]_{\mathbb{Z}}$,

$$\sum_{\substack{s\in[\check{s},\hat{s}]_{\mathbb{Z}}\\ t+s\leq T-1}} s\max\{y_{t+s}-y_{t+s+1},0\} - \sum_{\substack{s\in[\check{s}+1,\hat{s}+1]_{\mathbb{Z}}\\ t+s\leq T-1}} s\max\{y_{t+s}-y_{t+s+1},0\}$$

$$= \begin{cases} \check{s}\max\{y_{t+\check{s}}-y_{t+\check{s}+1},0\} - (\hat{s}+1)\max\{y_{t+\hat{s}+1}-y_{t+\hat{s}+2},0\}, \\ \qquad\qquad \text{if } t+\hat{s}+1 \leq T-1; \\ \check{s}\max\{y_{t+\check{s}}-y_{t+\check{s}+1},0\}, \qquad \text{if } t+\check{s} \leq T-1 < t+\hat{s}+1; \\ 0, \qquad\qquad \text{if } T-1 < t+\check{s}. \end{cases}$$

Note that

$$\theta'(t+\check{s}) - \theta'(t+\check{s}+1) = \begin{cases} \max\{y_{t+\check{s}}-y_{t+\check{s}+1},0\}, \\ \qquad \text{if } t+\check{s} \leq T-1; \\ 0, \quad \text{if } t+\check{s} > T-1. \end{cases}$$

and

$$\theta'(t+\hat{s}+1) - \theta'(t+\hat{s}+2) = \begin{cases} \max\{y_{t+\hat{s}+1}-y_{t+\hat{s}+2},0\}, \\ \qquad \text{if } t+\hat{s}+1 \leq T-1; \\ 0, \qquad \text{if } t+\hat{s}+1 > T-1. \end{cases}$$

Hence, for any $t \in [1, T-1]_{\mathbb{Z}}$,

$$\sum_{\substack{s\in[\check{s},\hat{s}]_{\mathbb{Z}}\\ t+s\leq T-1}} s\max\{y_{t+s}-y_{t+s+1}0,\} - \sum_{\substack{s\in[\check{s}+1,\hat{s}+1]_{\mathbb{Z}}\\ t+s\leq T-1}} s\max\{y_{t+s}-y_{t+s+1},0\}$$

$$= \check{s}[\theta'(t+\check{s}) - \theta'(t+\check{s}+1)] - (\hat{s}+1)[\theta'(t+\hat{s}+1) - \theta'(t+\hat{s}+2)]. \tag{8.40}$$

For any $\eta \in [0, \hat{\eta}]$, $\mathcal{S} \subseteq [\check{s}, \hat{s}]_{\mathbb{Z}}$, and $t \in [\check{t}, \hat{t}]_{\mathbb{Z}}$ such that $t \leq T - s - 1$ $\forall s \in \mathcal{S}$, let

$$\begin{aligned}\tilde{v}'(\eta, \mathcal{S}, t) = & x_t - (a_1 + a_2\eta)y_{t-1} - (a_3 + a_4\eta)y_t - (a_5 + a_6\eta)y_{t+1} \\ & + \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}).\end{aligned}$$

If $\tilde{v}'(\eta, \mathcal{S}, t) > 0$, then $\tilde{v}'(\eta, \mathcal{S}, t)$ is the amount of violation of inequality (8.37). If $\tilde{v}'(\eta, \mathcal{S}, t) \leq 0$, then there is no violation of inequality (8.37). For any $\eta \in [0, \hat{\eta}]$ and $t \in [\check{t}, \hat{t}]_{\mathbb{Z}}$, let

$$v'(\eta, t) = \max_{\mathcal{S} \subseteq [\check{s}, \min\{\hat{s}, T-t-1\}]_{\mathbb{Z}}} \{\tilde{v}'(\eta, \mathcal{S}, t)\}.$$

If $v'(\eta, t) > 0$, then $v'(\eta, t)$ is the largest possible violation of inequality (8.37) for this combination of η and t. If $v'(\eta, t) \leq 0$, then for any given η and t, the largest possible violation of inequality (8.37) is zero for this combination of η and t.

Note that $\overline{C} - \overline{V} - sV \geq 0$ for any $s \in [\check{s}, \hat{s}]_{\mathbb{Z}}$. Thus, for any given $\eta \in [0, \hat{\eta}]$ and $t \in [\check{t}, \hat{t}]_{\mathbb{Z}}$, $\tilde{v}'(\eta, \mathcal{S}, t)$ is maximized when $\mathcal{S}$ contains all $s \in [\check{s}, \min\{\hat{s}, T - t - 1\}]_{\mathbb{Z}}$ such that $y_{t+s} - y_{t+s+1} > 0$ (if any). If it does not exist any $s \in [\check{s}, \min\{\hat{s}, T - t - 1\}]_{\mathbb{Z}}$ such that $y_{t+s} - y_{t+s+1} > 0$, then $\tilde{v}'(\eta, \mathcal{S}, t)$ is maximized when $\mathcal{S} = \emptyset$, and $v'(\eta, t) = x_t - (a_1 + a_2\eta)y_{t-1} -$

$(a_3 + a_4\eta)y_t - (a_5 + a_6\eta)y_{t+1}$. Hence, for any $\eta \in [0, \hat{\eta}]$ and $t \in [\check{t}, \hat{t}]_{\mathbb{Z}}$,

$$\begin{aligned} v'(\eta, t) = x_t &- (a_1 + a_2\eta)y_{t-1} - (a_3 + a_4\eta)y_t - (a_5 + a_6\eta)y_{t+1} \\ &+ (\overline{C} - \overline{V}) \sum_{\substack{s \in [\check{s}, \hat{s}]_{\mathbb{Z}} \\ t+s \le T-1}} \max\{y_{t+s} - y_{t+s+1}, 0\} \\ &- V \sum_{\substack{s \in [\check{s}, \hat{s}]_{\mathbb{Z}} \\ t+s \le T-1}} s \max\{y_{t+s} - y_{t+s+1}, 0\}. \end{aligned}$$

When $t = \hat{t}$, we have

$$v'(\eta, \hat{t}) = \begin{cases} x_T - (a_1 + a_2\eta)y_{T-1} - (a_3 + a_4\eta)y_T, & \text{if } \hat{t} = T; \\ x_{T-1} - (a_1 + a_2\eta)y_{T-2} - (a_3 + a_4\eta)y_{T-1} - (a_5 + a_6\eta)y_T & \\ \quad +(\overline{C} - \overline{V}) \max\{y_{T-1} - y_T, 0\}, & \text{if } \hat{t} = T-1 \text{ and } \check{s} = 0; \\ x_{T-1} - (a_1 + a_2\eta)y_{T-2} - (a_3 + a_4\eta)y_{T-1} - (a_5 + a_6\eta)y_T, & \text{otherwise.} \end{cases}$$

For any $\eta \in [0, \hat{\eta}]$ and $t \in [\check{t}, \hat{t} - 1]_{\mathbb{Z}}$,

$$\begin{aligned} &v'(\eta, t) - v'(\eta, t+1) = \\ &(x_t - x_{t+1}) \\ &- (a_1 + a_2\eta)(y_{t-1} - y_t) - (a_3 + a_4\eta)(y_t - y_{t+1}) - (a_5 + a_6\eta)(y_{t+1} - y_{t+2}) \\ &+ (\overline{C} - \overline{V}) \left[\sum_{\substack{s \in [\check{s}, \hat{s}]_{\mathbb{Z}} \\ t+s \le T-1}} \max\{y_{t+s} - y_{t+s+1}, 0\} - \sum_{\substack{s \in [\check{s}+1, \hat{s}+1]_{\mathbb{Z}} \\ t+s \le T-1}} \max\{y_{t+s} - y_{t+s+1}, 0\} \right] \\ &- V \left[\sum_{\substack{s \in [\check{s}, \hat{s}]_{\mathbb{Z}} \\ t+s \le T-1}} s \max\{y_{t+s} - y_{t+s+1}, 0\} - \sum_{\substack{s \in [\check{s}+1, \hat{s}+1]_{\mathbb{Z}} \\ t+s \le T-1}} (s-1) \max\{y_{t+s} - y_{t+s+1}, 0\} \right], \end{aligned}$$

which implies that

$$
\begin{aligned}
&v'(\eta,t) = \\
&v'(\eta,t+1) + (x_t - x_{t+1}) \\
&- (a_1 + a_2\eta)(y_{t-1} - y_t) - (a_3 + a_4\eta)(y_t - y_{t+1}) - (a_5 + a_6\eta)(y_{t+1} - y_{t+2}) \\
&+ (\overline{C} - \overline{V}) \left[\sum_{\substack{s\in[\check{s},\hat{s}]_{\mathbb{Z}} \\ t+s\le T-1}} \max\{y_{t+s} - y_{t+s+1}, 0\} - \sum_{\substack{s\in[\check{s}+1,\hat{s}+1]_{\mathbb{Z}} \\ t+s\le T-1}} \max\{y_{t+s} - y_{t+s+1}, 0\} \right] \\
&- V \sum_{\substack{s\in[\check{s}+1,\hat{s}+1]_{\mathbb{Z}} \\ t+s\le T-1}} \max\{y_{t+s} - y_{t+s+1}, 0\} \\
&- V \left[\sum_{\substack{s\in[\check{s},\hat{s}]_{\mathbb{Z}} \\ t+s\le T-1}} s\max\{y_{t+s} - y_{t+s+1}, 0\} - \sum_{\substack{s\in[\check{s}+1,\hat{s}+1]_{\mathbb{Z}} \\ t+s\le T-1}} s\max\{y_{t+s} - y_{t+s+1}, 0\} \right].
\end{aligned}
$$

Thus, by (8.38), (8.39), and (8.40),

$$
\begin{aligned}
&v'(\eta,t) = \\
&v'(\eta,t+1) + (x_t - x_{t+1}) \\
&- (a_1 + a_2\eta)(y_{t-1} - y_t) - (a_3 + a_4\eta)(y_t - y_{t+1}) - (a_5 + a_6\eta)(y_{t+1} - y_{t+2}) \\
&+ (\overline{C} - \overline{V})[\theta'(t+\check{s}) - \theta'(t+\hat{s}+1) - \theta'(t+\check{s}+1) + \theta(t+\hat{s}+2)] \\
&- V[\check{s}\theta'(t+\check{s}) - (\check{s}-1)\theta'(t+\check{s}+1) - (\hat{s}+1)\theta'(t+\hat{s}+1) + \hat{s}\theta'(t+\hat{s}+2)]
\end{aligned}
\tag{8.41}
$$

for any $\eta \in [0,\hat{\eta}]$ and $t \in [\check{t},\hat{t}-1]_{\mathbb{Z}}$. Note that $\tilde{v}'(\eta,\mathcal{S},t)$ is linear in η. Thus, for any given t, $v'(\eta,t)$ is maximized when $\eta = 0$ or $\eta = \hat{\eta}$. That is, the largest possible value of $v'(\eta,t)$ is equal to $v'(0,t)$ if $a_2y_{t-1} + a_4y_t +$

Algorithm 2 Determination of the most violated inequality (8.37) for any given $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{2T}$

1: $\theta'(t) \leftarrow 0 \ \forall\ t \in [T, T + \hat{s} + 1]_{\mathbb{Z}}$
2: **for** $t = T - 1, \ldots, \check{t}$ **do**
3: $\quad \theta'(t) \leftarrow \theta'(t+1) + \max\{y_t - y_{t+1}, 0\}$
4: **end for**
5: **for** $\eta = 0, \hat{\eta}$ **do**
6: $\quad$ **if** $\hat{t} = T$ **then**
7: $\quad\quad v'(\eta, \hat{t}) \leftarrow x_T - (a_1 + a_2\eta)y_{T-1} - (a_3 + a_4\eta)y_T$
8: $\quad$ **else if** $\hat{t} = T - 1$ and $\check{s} = 0$ **then**
9: $\quad\quad v'(\eta, \hat{t}) \leftarrow x_{T-1} - (a_1 + a_2\eta)y_{T-2} - (a_3 + a_4\eta)y_{T-1} - (a_5 + a_6\eta)y_T + (\overline{C} - \overline{V}) \max\{y_{T-1} - y_T, 0\}$
10: $\quad$ **else**
11: $\quad\quad v'(\eta, \hat{t}) \leftarrow x_{T-1} - (a_1 + a_2\eta)y_{T-2} - (a_3 + a_4\eta)y_{T-1} - (a_5 + a_6\eta)y_T$
12: $\quad$ **end if**
13: $\quad$ **for** $t = \hat{t} - 1, \ldots, \check{t}$ **do**
14: $\quad\quad v'(\eta, t) \leftarrow v'(\eta, t+1) + (x_t - x_{t+1})$
$- (a_1 + a_2\eta)(y_{t-1} - y_t) - (a_3 + a_4\eta)(y_t - y_{t+1}) - (a_5 + a_6\eta)(y_{t+1} - y_{t+2})$
$+ (\overline{C} - \overline{V})[\theta'(t + \check{s}) - \theta'(t + \hat{s} + 1) - \theta'(t + \check{s} + 1) + \theta'(t + \hat{s} + 2)]$
$- V[\check{s}\theta'(t + \check{s}) - (\check{s} - 1)\theta'(t + \check{s} + 1) - (\hat{s} + 1)\theta'(t + \hat{s} + 1) + \hat{s}\theta'(t + \hat{s} + 2)]$
15: $\quad$ **end for**
16: **end for**
17: $(\eta^*, t^*) \leftarrow \text{argmax}_{(\eta,t) \in \{0,\hat{\eta}\} \times [\check{t},\hat{t}]_{\mathbb{Z}}} \{v'(\eta, t)\}$
18: $\mathcal{S}^* \leftarrow \emptyset$
19: **for** $s = \check{s}, \ldots, \min\{\hat{s}, T - t^* - 1\}$ **do**
20: $\quad$ **if** $y_{t^*+s} - y_{t^*+s+1} > 0$ **then** $\mathcal{S}^* \leftarrow \mathcal{S}^* \cup \{s\}$
21: **end for**

$a_6 y_{t+1} \geq 0$, and the largest possible value of $v'(\eta, t)$ is equal to $v'(\hat{\eta}, t)$ if $a_2 y_{t-1} + a_4 y_t + a_6 y_{t+1} < 0$. Hence, to determine the η and t values corresponding to the largest violation of inequality (8.37), it suffices to determine $v'(0, \check{t}), v'(0, \check{t} + 1), \ldots, v'(0, \hat{t})$ and $v'(\hat{\eta}, \check{t}), v'(\hat{\eta}, \check{t} + 1), \ldots, v'(\hat{\eta}, \hat{t})$. Algorithm 2 performs this computation.

In Algorithm 2, step 1 sets $\theta'(t)$ to zero when $t \geq T$. Steps 2–4 determine the $\theta'(t)$ values recursively for $t = T - 1, T - 2, \ldots, \check{t}$. These steps require

$O(T)$ time. Steps 5–16 consider the case $\eta = 0$ and the case $\eta = \hat{\eta}$. For each of these two η values, these steps first determine $v'(\eta, \hat{t})$, and then determine $v'(\eta, \hat{t} - 1), v'(\eta, \hat{t} - 2), \ldots, v'(\eta, \check{t})$ recursively using equation (8.41). These steps require $O(T)$ time. Steps 17–21 identify the most violated inequality (8.37) by setting the η and t values to $(\eta^*, t^*) = \text{argmax}_{(\eta,t)\in\{0,\hat{\eta}\}\times[\check{t},\hat{t}]_{\mathbb{Z}}}\{v'(\eta, t)\}$ and setting $\mathcal{S}$ equal to the set of s values such that $s \in [\check{s}, \min\{\hat{s}, T - t^* - 1\}]_{\mathbb{Z}}$ and $y_{t^*+s} - y_{t^*+s+1} > 0$. These steps also require $O(T)$ time. Therefore, the total computational time of Algorithm 2 is $O(T)$. $\square$

8.11 Proof of Proposition 8

(i) Consider the inequality

$$
\begin{aligned}
x_t \leq &(a_1 + a_2\eta)y_{t-1} + (a_3 + a_4\eta)y_t + (a_5 + a_6\eta)y_{t+1} \\
&- \sum_{s\in\mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}),
\end{aligned} \tag{8.42}
$$

and consider any given point $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{2T}$. For any $\eta \in [0, \hat{\eta}]_{\mathbb{Z}}$, $s_{\max} \in [\check{s}_{\max}, \hat{s}_{\max}]_{\mathbb{Z}}$, $\mathcal{S} \subseteq [\check{s}, s_{\max}]_{\mathbb{Z}}$, $t \in [s_{\max} + 2, \hat{t}]_{\mathbb{Z}}$, let

$$
\begin{aligned}
\tilde{v}(\eta, \mathcal{S}, t) = &x_t - (a_1 + a_2\eta)y_{t-1} - (a_3 + a_4\eta)y_t - (a_5 + a_6\eta)y_{t+1} \\
&+ \sum_{s\in\mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}).
\end{aligned}
$$

If $\tilde{v}(\eta, \mathcal{S}, t) > 0$, then $\tilde{v}(\eta, \mathcal{S}, t)$ is the amount of violation of inequality (8.42). If $\tilde{v}(\eta, \mathcal{S}, t) \leq 0$, then there is no violation of inequality (8.42). Note that $\tilde{v}(\eta, \mathcal{S}, t)$ is linear in η. Thus, for any given $\mathcal{S}$ and t, the function $\tilde{v}(\eta, \mathcal{S}, t)$ is maximized at $\eta = 0$ if $a_2y_{t-1} + a_4y_t + a_6y_{t+1} \geq 0$, and is maximized at $\eta = \hat{\eta}$ if $a_2y_{t-1} + a_4y_t + a_6y_{t+1} < 0$. For any $s_{\max} \in [\check{s}_{\max}, \hat{s}_{\max}]_{\mathbb{Z}}$, $t \in [s_{\max} + 2, \hat{t}]_{\mathbb{Z}}$, and $i \in [\check{s}, s_{\max}]_{\mathbb{Z}}$, let

$$
v_1(s_{\max}, t, i) = \begin{cases} \tilde{v}(0, [\check{s}, i]_{\mathbb{Z}}, t), & \text{if } a_2y_{t-1} + a_4y_t + a_6y_{t+1} \geq 0; \\ \tilde{v}(\hat{\eta}, [\check{s}, i]_{\mathbb{Z}}, t), & \text{if } a_2y_{t-1} + a_4y_t + a_6y_{t+1} < 0; \end{cases}
$$

that is,

$$v_1(s_{\max}, t, i) = \begin{cases} x_t - a_1 y_{t-1} - a_3 y_t - a_5 y_{t+1} \\ \qquad + \sum_{s=\check{s}}^{i} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}), \\ \qquad\qquad \text{if } a_2 y_{t-1} + a_4 y_t + a_6 y_{t+1} \geq 0; \\ x_t - (a_1 + a_2 \hat{\eta}) y_{t-1} - (a_3 + a_4 \hat{\eta}) y_t - (a_5 + a_6 \hat{\eta}) y_{t+1} \\ \qquad + \sum_{s=\check{s}}^{i} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}), \\ \qquad\qquad \text{if } a_2 y_{t-1} + a_4 y_t + a_6 y_{t+1} < 0. \end{cases}$$

For any $s_{\max} \in [\check{s}_{\max}, \hat{s}_{\max}]_{\mathbb{Z}}$, $t \in [s_{\max} + 2, \hat{t}]_{\mathbb{Z}}$, and $j \in [\check{s} + 1, s_{\max}]_{\mathbb{Z}}$, let

$$v_2(s_{\max}, t, j) = \sum_{s=j}^{s_{\max}} (\overline{C} - \overline{V} - sV)(y_{t-s} - y_{t-s-1}).$$

Note that

$$v_1(s_{\max}, t, i) = \begin{cases} v_1(s_{\max}, t, i-1) + (\overline{C} - \overline{V} - iV)(y_{t-i} - y_{t-i-1}), \quad \text{if } i \geq \check{s} + 1; \\ x_t - a_1 y_{t-1} - a_3 y_t - a_5 y_{t+1} + (\overline{C} - \overline{V} - \check{s}V)(y_{t-\check{s}} - y_{t-\check{s}-1}), \\ \qquad\qquad \text{if } i = \check{s} \text{ and } a_2 y_{t-1} + a_4 y_t + a_6 y_{t+1} \geq 0; \\ x_t - (a_1 + a_2 \hat{\eta}) y_{t-1} - (a_3 + a_4 \hat{\eta}) y_t - (a_5 + a_6 \hat{\eta}) y_{t+1} \\ \quad + (\overline{C} - \overline{V} - \check{s}V)(y_{t-\check{s}} - y_{t-\check{s}-1}), \\ \qquad\qquad \text{if } i = \check{s} \text{ and } a_2 y_{t-1} + a_4 y_t + a_6 y_{t+1} < 0. \end{cases}$$

Thus, for each $s_{\max}$ and t, the values of $v_1(s_{\max}, t, \check{s}), v_1(s_{\max}, t, \check{s}+1), \ldots, v_1(s_{\max}, t, s_{\max})$ can be determined recursively in $O(T)$ time. This implies that the $v_1(s_{\max}, t, i)$ values (for all $s_{\max}$, t, and i) can be determined in $O(T^3)$ time. Similarly, the $v_2(s_{\max}, t, j)$ values (for all $s_{\max}$, t, and j) can be determined in $O(T^3)$

time. For each $s_{\max}$, t, and j, let

$$\hat{v}_2(s_{\max}, t, j) = \max_{\beta \in [j, s_{\max}]_{\mathbb{Z}}} \{v_2(s_{\max}, t, \beta)\}$$

and

$$\hat{\beta}(s_{\max}, t, j) = \operatorname*{argmax}_{\beta \in [j, s_{\max}]_{\mathbb{Z}}} \{v_2(s_{\max}, t, \beta)\}.$$

Note that for each $s_{\max}$ and t, the values of $\hat{v}_2(s_{\max}, t, \check{s}+1), \hat{v}_2(s_{\max}, t, \check{s}+2), \ldots, \hat{v}_2(s_{\max}, t, s_{\max})$ and $\hat{\beta}_2(s_{\max}, t, \check{s}+1), \hat{\beta}_2(s_{\max}, t, \check{s}+2), \ldots, \hat{\beta}_2(s_{\max}, t, s_{\max})$ can be determined in $O(T)$ time by setting

$$\hat{v}_2(s_{\max}, t, j) = \begin{cases} \max\{\hat{v}_2(s_{\max}, t, j+1), v_2(s_{\max}, t, j)\}, & \text{if } j \leq s_{\max} - 1; \\ v_2(s_{\max}, t, s_{\max}), & \text{if } j = s_{\max}; \end{cases}$$

and

$$\hat{\beta}(s_{\max}, t, j) = \begin{cases} \hat{\beta}(s_{\max}, t, j+1), & \text{if } \hat{v}_2(s_{\max}, t, j+1) \geq v_2(s_{\max}, t, j); \\ j, & \text{if } \hat{v}_2(s_{\max}, t, j+1) < v_2(s_{\max}, t, j). \end{cases}$$

This implies that the $\hat{v}_2(s_{\max}, t, j)$ and $\hat{\beta}(s_{\max}, t, j)$ values (for all $s_{\max}$, t, and j) can be determined in $O(T^3)$ time. Note that the condition "$\beta = \alpha + 1$ or $s_{\max} \leq L + \alpha$" in Proposition 8 implies that "$\mathcal{S} = [\check{s}, s_{\max}]_{\mathbb{Z}}$" or "$s_{\max} \leq L + \alpha$." For any $s_{\max} \in [\check{s}_{\max}, \hat{s}_{\max}]_{\mathbb{Z}}$ and $t \in [s_{\max} + 2, \hat{t}]_{\mathbb{Z}}$, if $\mathcal{S} = [\check{s}, s_{\max}]_{\mathbb{Z}}$, then the largest possible amount of violation of inequality (8.42) is equal to $v_1(s_{\max}, t, s_{\max})$. For any $s_{\max} \in [\check{s}_{\max}, \hat{s}_{\max}]_{\mathbb{Z}}$, $t \in [s_{\max} + 2, \hat{t}]_{\mathbb{Z}}$, and $\alpha \in [\check{s}, s_{\max} - 1]_{\mathbb{Z}}$, if $s_{\max} \leq L + \alpha$, then the largest possible amount of violation of inequality (8.42) is equal to $v_1(s_{\max}, t, \alpha) + v_2(s_{\max}, t, \hat{\beta}(s_{\max}, t, \alpha + 1)) =$

$v_1(s_{\max}, t, \alpha) + \hat{v}_2(s_{\max}, t, \alpha + 1)$.

To determine the most violated inequality (8.42) that satisfies the conditions in Proposition 8, we first determine all $v_1(s_{\max}, t, i)$, $v_2(s_{\max}, t, j)$, $\hat{v}_2(s_{\max}, t, j)$, and $\hat{\beta}(s_{\max}, t, j)$ values, which requires $O(T^3)$ time. Next, we search for the $s_{\max}$ and t values such that $v_1(s_{\max}, t, s_{\max})$ is the largest possible. This requires $O(T^2)$ time. Let $s^*_{\max}$ and t^* be the $s_{\max}$ and t values obtained, and let $\mathcal{S}^* = [\check{s}, s^*_{\max}]_{\mathbb{Z}}$. Let $\eta^* = 0$ if $a_2 y_{t^*-1} + a_4 y_{t^*} + a_6 y_{t^*+1} \geq 0$, and $\eta^* = \hat{\eta}$ otherwise. Next, we search for the $s_{\max}$, t, and α values, where $\alpha \in [s_{\max} - L, s_{\max} - 1]_{\mathbb{Z}}$, such that $v_1(s_{\max}, t, \alpha) + \hat{v}_2(s_{\max}, t, \alpha + 1)$ is the largest possible. This requires $O(T^3)$ time. Let $s^{**}_{\max}$, t^{**}, and α^{**} be the $s_{\max}$, t, and α values obtained, and let $\mathcal{S}^{**} = [\check{s}, \alpha^{**}]_{\mathbb{Z}} \cup [\beta^{**}, s^{**}_{\max}]_{\mathbb{Z}}$, where $\beta^{**} = \hat{\beta}(s^{**}_{\max}, t^{**}, \alpha^{**} + 1)$. Let $\eta^{**} = 0$ if $a_2 y_{t^{**}-1} + a_4 y_{t^{**}} + a_6 y_{t^{**}+1} \geq 0$, and $\eta^{**} = \hat{\eta}$ otherwise. If $v_1(s^*_{\max}, t^*, s^*_{\max}) > v_1(s^{**}_{\max}, t^{**}, \alpha^{**}) + \hat{v}_2(s^{**}_{\max}, t^{**}, \alpha^{**} + 1)$, then the most violated inequality (8.42) is obtained by setting $\mathcal{S} = \mathcal{S}^*$, $\eta = \eta^*$, and $t = t^*$. Otherwise, it is obtained by setting $\mathcal{S} = \mathcal{S}^{**}$, $\eta = \eta^{**}$, and $t = t^{**}$. The overall computational time of this process is $O(T^3)$.

(ii) Consider the inequality

$$
\begin{aligned}
x_t \leq & (a_1 + a_2\eta) y_{t-1} + (a_3 + a_4\eta) y_t + (a_5 + a_6\eta) y_{t+1} \\
& - \sum_{s \in \mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}),
\end{aligned}
\tag{8.43}
$$

and consider any given point $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2T}_+$. For any $\eta \in [0, \hat{\eta}]_{\mathbb{Z}}$, $s_{\max} \in$

$[\check{s}_{\max}, \hat{s}_{\max}]_{\mathbb{Z}}$, $\mathcal{S} \subseteq [\check{s}, s_{\max}]_{\mathbb{Z}}$, and $t \in [\check{t}, T - s_{\max} - 1]_{\mathbb{Z}}$, let

$$\tilde{v}'(\eta, \mathcal{S}, t) = x_t - (a_1 + a_2\eta)y_{t-1} - (a_3 + a_4\eta)y_t - (a_5 + a_6\eta)y_{t+1} + \sum_{s\in\mathcal{S}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}).$$

If $\tilde{v}'(\eta, \mathcal{S}, t) > 0$, then $\tilde{v}(\eta, \mathcal{S}, t)$ is the amount of violation of inequality (8.43). If $\tilde{v}'(\eta, \mathcal{S}, t) \leq 0$, then there is no violation of inequality (8.43). Note that $\tilde{v}'(\eta, \mathcal{S}, t)$ is linear in η. Thus, for any given $\mathcal{S}$ and t, the function $\tilde{v}'(\eta, \mathcal{S}, t)$ is maximized at $\eta = 0$ if $a_2y_{t-1} + a_4y_t + a_6y_{t+1} \geq 0$, and is maximized at $\eta = \hat{\eta}$ if $a_2y_{t-1} + a_4y_t + a_6y_{t+1} < 0$. For any $s_{\max} \in [\check{s}_{\max}, \hat{s}_{\max}]_{\mathbb{Z}}$, $t \in [\check{t}, T - s_{\max} - 1]_{\mathbb{Z}}$, and $i \in [\check{s}, s_{\max}]_{\mathbb{Z}}$, let

$$v_1'(s_{\max}, t, i) = \begin{cases} \tilde{v}'(0, [\check{s}, i]_{\mathbb{Z}}, t), & \text{if } a_2y_{t-1} + a_4y_t + a_6y_{t+1} \geq 0; \\ \tilde{v}'(\hat{\eta}, [\check{s}, i]_{\mathbb{Z}}, t), & \text{if } a_2y_{t-1} + a_4y_t + a_6y_{t+1} < 0; \end{cases}$$

that is,

$$v_1'(s_{\max}, t, i) = \begin{cases} x_t - a_1y_{t-1} - a_3y_t - a_5y_{t+1} \\ \quad + \sum_{s=\check{s}}^{i} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}), \\ \quad\quad \text{if } a_2y_{t-1} + a_4y_t + a_6y_{t+1} \geq 0;, \\ x_t - (a_1 + a_2\hat{\eta})y_{t-1} - (a_3 + a_4\hat{\eta})y_t - (a_5 + a_6\hat{\eta})y_{t+1} \\ \quad + \sum_{s=\check{s}}^{i} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}), \\ \quad\quad \text{if } a_2y_{t-1} + a_4y_t + a_6y_{t+1} < 0. \end{cases}$$

For any $s_{\max} \in [\check{s}_{\max}, \hat{s}_{\max}]_{\mathbb{Z}}$, $t \in [\check{t}, T - s_{\max} - 1]_{\mathbb{Z}}$, and $j \in [\check{s} + 1, s_{\max}]_{\mathbb{Z}}$, let

$$v_2'(s_{\max}, t, j) = \sum_{s=j}^{s_{\max}} (\overline{C} - \overline{V} - sV)(y_{t+s} - y_{t+s+1}).$$

Similar to $v_1(s_{\max}, t, i)$ and $v_2(s_{\max}, t, j)$, the $v_1'(s_{\max}, t, i)$ and $v_2'(s_{\max}, t, j)$ values (for all $s_{\max}$, t, i, and j) can be determined in $O(T^3)$ time. For each $s_{\max}$, t, and j, let

$$\hat{v}_2'(s_{\max}, t, j) = \max_{\beta \in [j, s_{\max}]_{\mathbb{Z}}} \{v_2'(s_{\max}, t, \beta)\}$$

and

$$\hat{\beta}'(s_{\max}, t, j) = \operatorname*{argmax}_{\beta \in [j, s_{\max}]_{\mathbb{Z}}} \{v_2'(s_{\max}, t, \beta)\}.$$

Similar to $\hat{v}_2(s_{\max}, t, j)$ and $\hat{\beta}_2(s_{\max}, t, j)$, the $\hat{v}_2'(s_{\max}, t, j)$ and $\hat{\beta}_2'(s_{\max}, t, j)$ values can be determined recursively in $O(T^3)$ time. For any $s_{\max} \in [\check{s}_{\max}, \hat{s}_{\max}]_{\mathbb{Z}}$ and $t \in [\check{t}, T - s_{\max} - 1]_{\mathbb{Z}}$, if $\mathcal{S} = [\check{s}, s_{\max}]_{\mathbb{Z}}$, then the largest possible amount of violation of inequality (8.43) is equal to $v_1'(s_{\max}, t, s_{\max})$. For any $s_{\max} \in [\check{s}_{\max}, \hat{s}_{\max}]_{\mathbb{Z}}$, $t \in [\check{t}, T - s_{\max} - 1]_{\mathbb{Z}}$, and $\alpha \in [\check{s}, s_{\max} - 1]_{\mathbb{Z}}$, if $s_{\max} \leq L + \alpha$, then the largest possible amount of violation of inequality (8.43) is equal to $v_1'(s_{\max}, t, \alpha) + v_2'(s_{\max}, t, \hat{\beta}'(s_{\max}, t, \alpha + 1)) = v_1'(s_{\max}, t, \alpha) + \hat{v}_2'(s_{\max}, t, \alpha + 1)$.

To determine the most violated inequality (8.43) that satisfies the conditions in Proposition 8, we first determine all $v_1'(s_{\max}, t, i)$, $v_2'(s_{\max}, t, j)$, $\hat{v}_2'(s_{\max}, t, j)$, and $\hat{\beta}'(s_{\max}, t, j)$ values. Next, we search for the $s_{\max}$ and t values such that $v_1'(s_{\max}, t, s_{\max})$ is the largest possible. Let $s^*_{\max}$ and t^* be the $s_{\max}$ and t values obtained, and let $\mathcal{S}^* = [\check{s}, s^*_{\max}]_{\mathbb{Z}}$. Let $\eta^* = 0$ if $a_2 y_{t^*-1} + a_4 y_{t^*} + a_6 y_{t^*+1} \geq 0$, and $\eta^* = \hat{\eta}$ otherwise. Next, we search for the $s_{\max}$,

t, and α values, where $\alpha \in [s_{\max} - L, s_{\max} - 1]_{\mathbb{Z}}$, such that $v_1'(s_{\max}, t, \alpha) + \hat{v}_2'(s_{\max}, t, \alpha + 1)$ is the largest possible. Let $s_{\max}^{**}$, t^{**}, and α^{**} be the $s_{\max}$, t, and α values obtained, and let $\mathcal{S}^{**} = [\check{s}, \alpha^{**}]_{\mathbb{Z}} \cup [\beta^{**}, s_{\max}^{**}]_{\mathbb{Z}}$, where $\beta^{**} = \hat{\beta}'(s_{\max}^{**}, t^{**}, \alpha^{**} + 1)$. Let $\eta^{**} = 0$ if $a_2 y_{t^{**}-1} + a_4 y_{t^{**}} + a_6 y_{t^{**}+1} \geq 0$, and $\eta^{**} = \hat{\eta}$ otherwise. If $v_1'(s_{\max}^{*}, t^{*}, s_{\max}^{*}) > v_1'(s_{\max}^{**}, t^{**}, \alpha^{**}) + \hat{v}_2'(s_{\max}^{**}, t^{**}, \alpha^{**} + 1)$, then the most violated inequality (8.43) is obtained by setting $\mathcal{S} = \mathcal{S}^{*}$, $\eta = \eta^{*}$, and $t = t^{*}$. Otherwise, it is obtained by setting $\mathcal{S} = \mathcal{S}^{**}$, $\eta = \eta^{**}$, and $t = t^{**}$. The overall computational time of this process is $O(T^3)$. □

8.12 Proof of Proposition 9

For notational convenience, we define $s_{\max} = \max\{s : s \in \mathcal{S}\}$ if $\mathcal{S} \neq \emptyset$, and $s_{\max} = -1$ if $\mathcal{S} = \emptyset$. To prove that linear inequalities (4.7) and (4.8) are valid for $\text{conv}(\mathcal{P})$, it suffices to show that they are valid for $\mathcal{P}$. Consider any element $(\mathbf{x}, \mathbf{y})$ of $\mathcal{P}$. We show that $(\mathbf{x}, \mathbf{y})$ satisfies (4.7) and (4.8).

We first show that $(\mathbf{x}, \mathbf{y})$ satisfies (4.7). Consider any $t \in [k+1, T-m]_{\mathbb{Z}}$. We divide the analysis into three cases:

Case 1: $y_t = 0$. In this case, by (2.2c) and (2.2d), $-x_{t-k} \leq -\underline{C}y_{t-k}$ and $x_t = 0$. Thus, The left hand side of (4.7) is at most $-\underline{C}y_{t-k}$ and the first term on the right hand side of (4.7) is 0. Because $y_t = 0$ and $t \in [2, T]_{\mathbb{Z}}$, by Lemma 1(i), we have $y_{t-j} - y_{t-j-1} \leq 0$ for all $j \in [0, \min\{t-2, L-1\}]_{\mathbb{Z}}$. Because $\mathcal{S} \subseteq [0, \min\{k-1, L-m-1\}]_{\mathbb{Z}}$, $m \geq 0$ and, $t \geq k+1$, we have $\mathcal{S} \subseteq [0, \min\{t-2, L-1\}]_{\mathbb{Z}}$. Thus, $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, k-1]_{\mathbb{Z}}$ and $\underline{C} + V > \overline{V}$, for any $s \in \mathcal{S}$, the coefficient "$\underline{C} + (k-s)V - \overline{V}$" on the right hand side of (4.7) is positive. Hence, the right hand side of (4.7) is at least $-\underline{C}y_{t-k}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.7).

Case 2: $y_t = 1$ and $y_{t-s'} - y_{t-s'-1} = 1$ for some $s' \in \mathcal{S}$. In this case, $y_{t-s'} = 1$ and $y_{t-s'-1} = 0$. Because $y_t = 1$ and $t \in [2, T]_{\mathbb{Z}}$, by Lemma 1(ii), there exists at most one $j \in [0, \min\{t-2, L\}]_{\mathbb{Z}}$ such that $y_{t-j} - y_{t-j-1} = 1$. Because $\mathcal{S} \subseteq [0, \min\{k-1, L-m-1\}]_{\mathbb{Z}}$, $m \geq 0$ and, $t \geq k+1$, we have $\mathcal{S} \subseteq [0, \min\{t-2, L\}]_{\mathbb{Z}}$. Thus, we have $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S} \setminus \{s'\}$. Because $y_{t-s'} - y_{t-s'-1} = 1$ and $t - s' \in [2, T]_{\mathbb{Z}}$, by (2.2a), we have $y_k = 1$ for all $k \in [t-s', \min\{T, t-s'+L-1\}]_{\mathbb{Z}}$. Because $\mathcal{S} \subseteq [0, L-m-1]_{\mathbb{Z}}$, we have $t - s' + L - 1 \geq t + m$. Thus, $y_\tau = 1$ for all

$\tau \in [t-s', t+m]_{\mathbb{Z}}$. Because $\mathcal{S} \subseteq [0, k-1]_{\mathbb{Z}}$ and $\underline{C} + V > \overline{V}$, the coefficient "$\underline{C} + (k-s)V - \overline{V}$" on the right hand side of (4.7) is positive for all $s \in \mathcal{S}$. Hence, the right hand side of inequality (4.7) is at least $s'V + \overline{V} - \underline{C}y_{t-k}$. By (2.2e), we have $x_{t-s'-1} = 0$ and $\sum_{\tau=t-s'}^{t}(x_\tau - x_{\tau-1}) \leq \sum_{\tau=t-s'}^{t} Vy_{\tau-1} + \sum_{\tau=t-s'}^{t} \overline{V}(1-y_{\tau-1})$, which implies that $x_t - x_{t-s'-1} \leq s'V + \overline{V}$. Because $y_{t-s'-1} = 0$, by (2.2d), we have $x_{t-s'-1} = 0$. Thus, we have $x_t \leq s'V + \overline{V}$. By (2.2c), we have $-x_{t-k} \geq -\underline{C}y_{t-k}$. Hence, $x_t - x_{t-k} \leq s'V + \overline{V} - \underline{C}y_{t-k}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.7).

Case 3: $y_t = 1$ and $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, k-1]_{\mathbb{Z}}$ and $\underline{C} + V > \overline{V}$, for any $s \in \mathcal{S}$, the coefficient "$\underline{C} + (k-s)V - \overline{V}$" on the right hand side of (4.7) is positive. If there does not exist $i \in [0, m-1]_{\mathbb{Z}}$ such that $y_{t+i} - y_{t+i+1} = 1$, then $y_{t+i} = 1$ for all $i \in [1, m]_{\mathbb{Z}}$. Thus, the right hand side of inequality (4.7) is at least $\underline{C} + kV - \underline{C}y_{t-k}$. Let $t' = \max\{\tau \in [2, t]_{\mathbb{Z}} : y_\tau - y_{\tau-1} = 1\}$. When $t' \leq t-k$ or t' does not exist, we have $y_\tau = 1$ for all $\tau \in [t-k, t]_{\mathbb{Z}}$. By (2.2e), we have $\sum_{\tau=t-k+1}^{t}(x_\tau - x_{\tau-1}) \leq \sum_{\tau=t-k+1}^{t} Vy_{\tau-1} + \sum_{\tau=t-k+1}^{t} \overline{V}(1-y_{\tau-1})$, which implies that $x_t - x_{t-k} \leq kV$. When $t' > t-k$, we have $y_{t'} = 1$ and $y_{t'-1} = 0$. By (2.2e), we have $\sum_{\tau=t'}^{t}(x_\tau - x_{\tau-1}) \leq \sum_{\tau=t'}^{t} Vy_{\tau-1} + \overline{V}(1-y_{\tau-1})$, which implies that $x_t - x_{t'-1} \leq (t-t')V + \overline{V}$. Because $y_{t'-1} = 0$, by (2.2d), we have $x_{t'-1} = 0$. Thus, we have $x_t \leq (t-t')V + \overline{V} < \underline{C} + kV$ as $t' > t-k$ and $\underline{C} + V > \overline{V}$. By (2.2c), we have $-x_{t-k} \leq -\underline{C}y_{t-k}$. Hence, in both cases, the left hand side of inequality (4.7) is no larger than $\underline{C} + kV - \underline{C}y_{t-k}$. Now, consider the case where there exists some $i \in [0, m-1]_{\mathbb{Z}}$ such that $y_{t+i} - y_{t+i+1} = 1$. Let $i^* = \min\{i \in [0, m]_{\mathbb{Z}} : y_{t+i} - y_{t+i+1} = 1\}$. Thus, $y_\tau = 1$ for all $\tau \in [t, t+i^*]_{\mathbb{Z}}$ and $y_{t+i^*+1} = 0$. Hence, the right hand side of inequality (4.7) is at least $\underline{C} + (k-m)V + i^*V - \underline{C}y_{t-k} > \overline{V} + i^*V - \underline{C}y_{t-k}$ as $m \le k-1$ and $\underline{C} + V > \overline{V}$. By (2.2f), we have $\sum_{\tau=t+1}^{t+i^*+1}(x_{\tau-1} - x_\tau) \le \sum_{\tau=t+1}^{t+i^*+1} Vy_\tau + \sum_{\tau=t+1}^{t+i^*+1} \overline{V}(1-y_\tau)$, which implies that $x_t - x_{t+i^*+1} \le \overline{V} + i^*V$. Because $y_{t+i^*+1} = 0$, by (2.2d), we have $x_{t+i^*+1} = 0$. Thus, $x_t \le \overline{V} + i^*V$. By (2.2c), we have $x_{t-k} \le \underline{C}y_{t-k}$. Hence, $x_t - x_{t-k} \le \overline{V} + i^*V - \underline{C}y_{t-k}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.7).

Next, we show that $(\mathbf{x}, \mathbf{y})$ satisfies (4.8). Consider any $t \in [m+1, T-k]_{\mathbb{Z}}$. We divide the analysis into three cases.

Case 1: $y_t = 0$. In this case, by (2.2c) and (2.2d), $-x_{t+k} \le -\underline{C}y_{t+k}$ and $x_t = 0$. Thus, the left hand side of (4.8) is at most $-\underline{C}y_{t+k}$ and the first term on the right hand side of (4.8) is 0. Because $y_t = 0$ and $t \in [1, T-1]_{\mathbb{Z}}$, by Lemma 2(i), $y_{t+j} - y_{t+j+1} \le 0$ for all $j \in [0, \min\{T-t-1, L-1\}]_{\mathbb{Z}}$. Because $m \ge 0$ and $t \le T-k$, we have $\mathcal{S} \subseteq [0, \min\{T-t-1, L-1\}]_{\mathbb{Z}}$. Thus, $y_{t+s} - y_{t+s+1} \le 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \in [0, k-1]_{\mathbb{Z}}$ and $\underline{C} + V > \overline{V}$, for any $s \in \mathcal{S}$, the coefficient "$\underline{C} + (k-s)V - \overline{V}$" on the right hand side of (4.8) is positive. Hence, the right hand side of (4.8) is at least $-\underline{C}y_{t+k}$. Therefore in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.8).

Case 2: $y_t = 1$ and $y_{t+s'} - y_{t+s'+1} = 1$ for some $s' \in \mathcal{S}$. In this case, $y_{t+s'} = 1$ and $y_{t+s'+1} = 0$. Because $y_t = 1$ and $t \in [1, T-1]_{\mathbb{Z}}$, by Lemma 2(ii), there exists at most one $j \in [0, \min\{T-t-1, L\}]_{\mathbb{Z}}$ such that $y_{t+j} - y_{t+j+1} = 1$. Because $\mathcal{S} \subseteq [0, \min\{k-1, L-m-1\}]_{\mathbb{Z}}$, $m \ge 0$, and $t \le T-k$, we have $\mathcal{S} \subseteq [0, \min\{T-t-1, L\}]_{\mathbb{Z}}$. Thus, $y_{t+s} - y_{t+s+1} \le 0$ for all $s \in \mathcal{S} \setminus \{s'\}$. Because $y_{t+s'+1} = 0$ and $t+s'+1 \in [2, T]_{\mathbb{Z}}$, by Lemma 1(i), we have $y_{t+s'+1-j} - y_{t+s'-j} \le 0$ for all $j \in [0, \min\{t+s'-1, L-1\}]_{\mathbb{Z}}$. Because $\mathcal{S} \subseteq [0, \min\{k-1, L-m-1\}]_{\mathbb{Z}}$, we have $t+s-L+2 \le t-m+1$. Thus,

$y_\tau - y_{\tau-1} \le 0$ for all $\tau \in [t-m+1, t+s'+1]_{\mathbb{Z}}$. Because $y_{t+s'} = 1$, we have $y_\tau = 1$ for all $\tau \in [t-m, t+s']_{\mathbb{Z}}$. Because $\mathcal{S} \subseteq [0, k-1]_{\mathbb{Z}}$ and $\underline{C} + V > \overline{V}$, the coefficient "$\underline{C} + (k-s)V - \overline{V}$" on the right hand side of inequality (4.8) is positive. Hence, the right hand side of inequality (4.8) is at least $s'V + \overline{V} - \underline{C}y_{t+k}$. By (2.2f), we have $\sum_{\tau=t+1}^{t+s'+1}(x_{\tau-1} - x_\tau) \le \sum_{\tau=t+1}^{t+s'+1} Vy_\tau + \overline{V}(1-y_\tau)$, which implies that $x_t - x_{t+s'+1} \le s'V + \overline{V}$. Because $y_{t+s'+1} = 0$, by (2.2d), we have $x_{t+s'+1} = 0$. Thus, we have $x_t \le s'V + \overline{V}$. By (2.2c), we have $-x_{t+k} \le -\underline{C}y_{t+k}$. Hence, $x_t - x_{t+k} \le s'V + \overline{V} - \underline{C}y_{t+k}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.8).

Case 3: $y_t = 1$ and $y_{t+s} - y_{t+s+1} \le 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, k-1]_{\mathbb{Z}}$ and $\underline{C} + V > \overline{V}$, for any $s \in \mathcal{S}$, the coefficient "$\underline{C} + (k-s)V - \overline{V}$" on the right hand side of (4.8) is positive. If there does not exist $i \in [0, m-1]_{\mathbb{Z}}$ such that $y_{t-i} - y_{t-i-1} = 1$, then $y_{t-i} = 1$ for all $i \in [1, m]_{\mathbb{Z}}$. Thus, the right hand side of inequality (4.8) is at least $\underline{C} + kV - \underline{C}y_{t+k}$. Let $t' = \min\{\tau \in [t, T-1]_{\mathbb{Z}} : y_\tau - y_{\tau+1} = 1\}$. When $t' \ge t+k$ or t' does not exist, we have $y_\tau = 1$ for all $\tau \in [t, t+k]_{\mathbb{Z}}$. By (2.2f), we have $\sum_{\tau=t+1}^{t+k-1}(x_{\tau-1} - x_\tau) \le \sum_{\tau=t+1}^{t+k-1} Vy_\tau + \sum_{\tau=t+1}^{t+k-1} \overline{V}(1-y_\tau)$, which implies that $x_t - x_{t+k} \le kV$. When $t' < t+k$, we have $y_{t'} = 1$ and $y_{t'+1} = 0$. By (2.2f), we have $\sum_{\tau=t+1}^{t'+1}(x_{\tau-1} - x_\tau) \le \sum_{\tau=t+1}^{t'+1} Vy_\tau + \sum_{\tau=t+1}^{t'+1} \overline{V}(1-y_\tau)$, which implies that $x_t - x_{t'+1} \le (t'-t)V + \overline{V}$. Because $y_{t'+1} = 0$, by (2.2d), we have $x_{t'+1} = 0$. Thus, we have $x_t \le (t'-t)V + \overline{V} < \underline{C} + kV$ as $t' < t+k$ and $\underline{C} + V > \overline{V}$. By (2.2c), we have $-x_{t+k} \le -\underline{C}y_{t+k}$. Hence, in both cases, the left hand side of inequality (4.8) is no larger than $\underline{C} + kV - \underline{C}y_{t+k}$. Now, consider the case where there exists $i \in [0, m-1]_{\mathbb{Z}}$ such that $y_{t-i} - y_{t-i-1} = 1$. Let $i^* = \min\{i \in [0, m-1]_{\mathbb{Z}} : y_{t-i} - y_{t-i-1} = 1\}$. Thus, $y_\tau = 1$ for all

$\tau \in [t-i^*, t]_{\mathbb{Z}}$ and $y_{t-i^*-1} = 0$. Hence, the left hand side of inequality (4.8) is at least $\underline{C} + (k-m)V + i^*V - \underline{C}y_{t+k} > \overline{V} + i^*V - \underline{C}y_{t+k}$ as $m \leq k-1$ and $\underline{C} + V > \overline{V}$. By (2.2e), we have $\sum_{\tau=t-i^*+1}^{t}(x_\tau - x_{\tau-1}) \leq \sum_{\tau=t-i^*+1}^{t} Vy_\tau + \sum_{\tau=t-i^*+1}^{t} \overline{V}(1-y_\tau)$, which implies that $x_t - x_{t-i^*-1} \leq \overline{V} + i^*V$. Because $y_{t-i^*-1} = 0$, by (2.2d), $x_{t-i^*-1} = 0$. Thus, $x_t \leq \overline{V} + i^*V$. By (2.2c), $-x_{t+k} \leq -\underline{C}y_{t+k}$. Hence, $x_t - x_{t+k} \leq \overline{V} + i^*V - \underline{C}y_{t+k}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.8).

To prove that inequalities (4.7) and (4.8) are facet-defining for $\text{conv}(\mathcal{P})$ when $m = 0$ and $s \geq \min\{k-1, 1\}$ for all $s \in \mathcal{S}$, it suffices to show that for each of these two inequalities, there exist $2T$ affinely independent points in $\text{conv}(\mathcal{P})$ that satisfy the inequality at equality when $m = 0$ and $s \geq \min\{k-1, 1\}$ for all $s \in \mathcal{S}$. Let $\epsilon = \min\{\overline{V} - \underline{C}, \overline{C} - \underline{C} - kV\} > 0$.

We first show that inequality (4.7) is facet-defining for $\text{conv}(\mathcal{P})$ by creating $2T$ affinely independent points in $\text{conv}(\mathcal{P})$ that satisfy (4.7) at equality when $m = 0$ and $s \geq \min\{k-1, 1\}$ for all $s \in \mathcal{S}$. Because $\mathbf{0} \in \text{conv}(\mathcal{P})$ and $\mathbf{0}$ satisfies (4.7) at equality, it suffices to create the remaining $2T-1$ linearly independent points. We denote these $2T-1$ points as $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ for $r \in [1, T]_{\mathbb{Z}} \setminus \{t-k\}$, and $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ for $r \in [1, T]_{\mathbb{Z}}$, and denote the qth component of $\bar{\mathbf{x}}^r$, $\bar{\mathbf{y}}^r$, $\hat{\mathbf{x}}^r$, and $\hat{\mathbf{y}}^r$ as $\bar{x}_q^r$, $\bar{y}_q^r$, $\hat{x}_q^r$, and $\hat{y}_q^r$, respectively. We divide these $2T-1$ points into the following eight groups:

(A1) For each $r \in [1, t-k-1]_{\mathbb{Z}}$, we create a point $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ as follows:

$$\bar{x}_q^r = \begin{cases} \underline{C}, & \text{for } q \in [1, r-1]_{\mathbb{Z}}; \\ \underline{C} + \epsilon, & \text{for } q = r; \\ 0, & \text{for } q \in [r+1, T]; \end{cases}$$

and

$$\bar{y}_q^r = \begin{cases} 1, & \text{for } q \in [1, r]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [r+1, T]_{\mathbb{Z}}. \end{cases}$$

It is easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (2.2a)–(2.2f). Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. Note that $\bar{x}_t^r = \bar{x}_{t-k}^r = \bar{y}_t^r = \bar{y}_{t-k}^r = 0$ and $m = 0$. Because $t - s - 1 \neq r$ for all $s \in \mathcal{S}$, we have $\bar{y}_{t-s}^r - \bar{y}_{t-s-1}^r = 0$ for all $s \in \mathcal{S}$. Hence, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (4.8) at equality.

(A2) For each $r \in [t-k+1, t-1]_{\mathbb{Z}}$, we create a point $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ as follows:

$$\bar{x}_q^r = \begin{cases} \underline{C}, & \text{for } q \in [1, t-1]_{\mathbb{Z}} \setminus \{r\}; \\ \underline{C} + \epsilon, & \text{for } q = r; \\ 0, & \text{for } q \in [t, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\bar{y}_q^r = \begin{cases} 1, & \text{for } q \in [1, t-1]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [t, T]_{\mathbb{Z}}. \end{cases}$$

It is easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (2.2a)–(2.2f). Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. Note that $\bar{x}_t^r = \bar{y}_t^r = 0$, $\bar{x}_{t-k}^r = \underline{C}$, $\bar{y}_{t-k}^r = 1$, and $m = 0$. The existence of $r \in [t-k+1, t-1]_{\mathbb{Z}}$ implies that $k \geq 2$, which in turn implies that $s \geq 1$ for all $s \in \mathcal{S}$. Hence, $\bar{y}_{t-s}^r - \bar{y}_{t-s-1}^r = 0$ for all $s \in \mathcal{S}$. Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (4.8) at equality.

(A3) We create a point $(\bar{\mathbf{x}}^t, \bar{\mathbf{y}}^t)$ as follows:

$$\bar{x}^t_q = \begin{cases} \underline{C}, & \text{for } q \in [1, t-k-1]_{\mathbb{Z}}; \\ \underline{C} + (q-t+k)V + \epsilon, & \text{for } q \in [t-k, t]_{\mathbb{Z}}; \\ \underline{C} + kV, & \text{for } q \in [t+1, T]_{\mathbb{Z}}; \end{cases}$$

and $\bar{y}^t_q = 1$ for all $q \in [1, T]_{\mathbb{Z}}$. It is easy to verify that $(\bar{\mathbf{x}}^t, \bar{\mathbf{y}}^t)$ satisfies (2.2a)–(2.2d). Note that $\bar{x}^t_q - \bar{x}^t_{q-1} = 0$ when $q \in [2, t-k-1]_{\mathbb{Z}}$, $0 < \bar{x}^t_q - \bar{x}^t_{q-1} \leq V$ when $q \in [t-k, t]_{\mathbb{Z}}$, and $-\epsilon \leq \bar{x}^t_q - \bar{x}^t_{q-1} \leq 0$ when $q \in [t+1, T]_{\mathbb{Z}}$. Thus, $-V\bar{y}^t_q - \overline{V}(1-\bar{y}^t_q) \leq \bar{x}^t_q - \bar{x}^t_{q-1} \leq V\bar{y}^t_{q-1} + \overline{V}(1-\bar{y}^t_{q-1})$ for all $q \in [2, T]_{\mathbb{Z}}$. Hence, $(\bar{\mathbf{x}}^t, \bar{\mathbf{y}}^t)$ satisfies (2.2e) and (2.2f). Therefore, $(\bar{\mathbf{x}}^t, \bar{\mathbf{y}}^t) \in \text{conv}(\mathcal{P})$. Note that $\bar{x}^t_t = \underline{C} + kV + \epsilon$, $\bar{x}^t_{t-k} = \underline{C} + \epsilon$, $\bar{y}^t_t = \bar{y}^t_{t-k} = 1$, $m = 0$, and $\bar{y}^t_{t-s} - \bar{y}^t_{t-s-1} = 0$ for all $s \in \mathcal{S}$. Thus, $(\bar{\mathbf{x}}^t, \bar{\mathbf{y}}^t)$ satisfies (4.7) at equality.

(A4) For each $r \in [t+1, T]_{\mathbb{Z}}$, we create a point $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ as follows:

$$\bar{x}^r_q = \begin{cases} 0, & \text{for } q \in [1, t]_{\mathbb{Z}}; \\ \underline{C}, & \text{for } q \in [t+1, T]_{\mathbb{Z}} \setminus \{r\}; \\ \underline{C} + \epsilon, & \text{for } q = r; \end{cases}$$

and

$$\bar{y}^r_q = \begin{cases} 0, & \text{for } q \in [1, t]_{\mathbb{Z}}; \\ 1, & \text{for } q \in [t+1, T]_{\mathbb{Z}}. \end{cases}$$

It is easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (2.2a)–(2.2f). Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. Note that $\bar{x}^r_t = \bar{x}^r_{t-k} = \bar{y}^r_t = \bar{y}^t_{t-k} = 0$, $m = 0$, and $\bar{y}^r_{t-s} - \bar{y}^r_{t-s-1} = 0$ for all $s \in \mathcal{S}$. Hence, $(\bar{\mathbf{x}}^t, \bar{\mathbf{y}}^t)$ satisfies (4.7) at equality.

(A5) For each $r \in [1, t - s_{\max} - 2]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (A2) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \mathrm{conv}(\mathcal{P})$. If $r \geq t - k$, then $\hat{x}^r_t = \hat{y}^r_t = 0$, $\hat{x}^r_{t-k} = \underline{C}$, $\hat{y}^r_{t-k} = 1$, $m = 0$, and $\hat{y}^r_{t-s} - \hat{y}^r_{t-s-1} = 0$ for all $s \in \mathcal{S}$. If $r < t - k$, then $\hat{x}^r_t = \hat{x}^r_{t-k} = \hat{y}^r_t = \hat{y}^r_{t-k} = 0$, $m = 0$, and $\hat{y}^r_{t-s} - \hat{y}^r_{t-s-1} = 0$ for all $s \in \mathcal{S}$. Hence, in both cases, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.7) at equality.

(A6) For each $r \in [t - s_{\max} - 1, t - 1]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (A3) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \mathrm{conv}(\mathcal{P})$. To show that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.7) at equality, we first consider the case where $t - r - 1 \notin \mathcal{S}$. In this case, $\hat{x}^r_t = \hat{y}^r_t = 0$ and $m = 0$. Because $t - k \leq t - s_{\max} - 1 \leq r$, we have $\hat{x}^r_{t-k} = \underline{C}$ and $\hat{y}^r_{t-k} = 1$. Because $t - s - 1 \neq r$ for all $s \in \mathcal{S}$, we have $\hat{y}^r_{t-s} - \hat{y}^r_{t-s-1} = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.7) at equality. Next, we consider the case where $t - r - 1 \in \mathcal{S}$. In this case, $\hat{x}^r_t = \overline{V} + (t - r - 1)V$, $\hat{y}^r_t = 1$, $\hat{x}^r_{t-k} = \hat{y}^r_{t-k} = 0$, and $m = 0$. In addition, $\hat{y}^r_{t-s} - \hat{y}^r_{t-s-1} = 1$ when $s = t - r - 1$, and $\hat{y}^r_{t-s} - \hat{y}^r_{t-s-1} = 0$ when $s \neq t - r - 1$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.7) at equality.

(A7) We create a point $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ as follows:

$$\hat{\mathbf{x}}^t_q = \begin{cases} \underline{C}, & \text{for } q \in [1, t - k - 1]_{\mathbb{Z}}; \\ \underline{C} + (q - t + k)V, & \text{for } q \in [t - k, t]_{\mathbb{Z}}; \\ \underline{C} + kV, & \text{for } q \in [t + 1, T]_{\mathbb{Z}}; \end{cases}$$

and $\hat{y}^t_q = 1$ for all $q \in [1, T]_{\mathbb{Z}}$. It is easy to verify that $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (2.2a)–(2.2d). Note that $\hat{x}^t_q - \hat{x}^t_{q-1} = 0$ when $q \in [2, t - k]_{\mathbb{Z}}$, $\hat{x}^t_q -$

$\hat{x}^t_{q-1} = V$ when $q \in [t-k+1, t]_{\mathbb{Z}}$, and $\hat{x}^t_q - \hat{x}^t_{q-1} = 0$ when $q \in [t+1, T]_{\mathbb{Z}}$. Thus, $-V\hat{y}^t_q - \overline{V}(1-\hat{y}^t_q) \leq \hat{x}^t_q - \hat{x}^t_{q-1} \leq V\hat{y}^t_{q-1} + \overline{V}(1-\hat{y}^t_{q-1})$ for all $q \in [2, T]_{\mathbb{Z}}$. Hence, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (2.2e) and (2.2f). Therefore, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t) \in \text{conv}(\mathcal{P})$. Note that $\hat{x}^t_t = \underline{C} + kV$, $\hat{x}^t_{t-k} = \underline{C}$, $\hat{y}^t_t = \hat{y}^t_{t-k} = 1$, $m = 0$, and $\hat{y}^t_{t-s} - \hat{y}^t_{t-s-1} = 0$ for all $s \in \mathcal{S}$, Thus, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ satisfies (4.7) at equality.

(A8) For each $r \in [t+1, T]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (A5) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. Note that $\hat{x}^r_t = \hat{x}^r_{t-k} = \hat{y}^r_t = \hat{y}^r_{t-k} = 0$, $m = 0$, and $\hat{y}^r_{t-s} - \hat{y}^r_{t-s-1} = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.7) at equality.

Table A.13 shows a matrix with $2T-1$ rows, where each row represents a point created by this process. This matrix can be transformed into the matrix in Table A.14 via the following Gaussian elimination process:

(i) For each $r \in [1, t-k-1]_{\mathbb{Z}}$, the point with index r in group (B1), denoted $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r) = (\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) - (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$. Here, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ is the point with index r in group (A1), and $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (A5).

(ii) For each $r \in [t-k+1, t-1]_{\mathbb{Z}}$, the point with index r in group (B2), denoted $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r) = (\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$. Here, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ is the point in group (A2), and $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ is the point with index $t-1$ in group (A6). Note that because $s \geq 1$ for all $s \in \mathcal{S}$, the point with index $t-1$ in group (A6) is given by $\hat{x}^{t-1}_q = \underline{C}$ and $\hat{y}^{t-1}_q = 1$ for $q \in [1, t-1]_{\mathbb{Z}}$, and $\hat{x}^{t-1}_q = \hat{y}^{t-1}_q = 0$ for $q \in [t, T]_{\mathbb{Z}}$.

(iii) The point in group (B3), denoted $(\underline{\bar{\mathbf{x}}}^t, \underline{\bar{\mathbf{y}}}^t)$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^t, \underline{\bar{\mathbf{y}}}^t) = (\bar{\mathbf{x}}^t, \bar{\mathbf{y}}^t) - (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$. Here, $(\bar{\mathbf{x}}^t, \bar{\mathbf{y}}^t)$ is the point in group (A3), and $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ is the point in group (A7).

(iv) For each $r \in [t+1, T]_{\mathbb{Z}}$, the point with index r in group (B4), denoted $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r) = (\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) - (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$. Here, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ is the point with index r in group (A4), and $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point in group (A8).

(v) For each $r \in [1, t - s_{\max} - 2]_{\mathbb{Z}}$, the point with index r in group (B5), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (A5).

(vi) For each $r \in [t - s_{\max} - 1, t-1]_{\mathbb{Z}}$, the point with index r in group (B6), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $t - r - 1 \notin \mathcal{S}$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) - (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ if $t - r - 1 \in \mathcal{S}$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (A6), and $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ is the point in group (A7).

Table A.13: A matrix with the rows representing $2T-1$ linearly independent points in $\mathrm{conv}(\mathcal{P})$ satisfying inequality (4.7) at equality.

x

Group	Point	Index r	1	$\cdots$	$t-k-1$	$t-k$	$t-k+1$	$\cdots$	$t-s_{\max}-2$	$t-s_{\max}-1$	$\cdots$	$t-1$	t	$t+1$	$\cdots$	T
(A1)	$(\bar{\mathbf{x}}^r,\bar{\mathbf{y}}^r)$	1	$\underline{C}+\epsilon$	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-k-1$	$\underline{C}$	$\cdots$	$\underline{C}+\epsilon$	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
(A2)		$t-k+1$	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\underline{C}+\epsilon$	$\cdots$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}$	0	0	$\cdots$	0
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-s_{\max}-2$	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}+\epsilon$	$\underline{C}$	$\cdots$	$\underline{C}$	0	0	$\cdots$	0
		$t-s_{\max}-1$	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}+\epsilon$	$\cdots$	$\underline{C}$	0	0	$\cdots$	0
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-1$	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}+\epsilon$	0	0	$\cdots$	0
(A3)		t	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}+\epsilon$	$\underline{C}+V+\epsilon$	$\cdots$	$\underline{C}+(k-s_{\max}-2)V+\epsilon$	$\underline{C}+(k-s_{\max}-1)V+\epsilon$	$\cdots$	$\underline{C}+(k-1)V+\epsilon$	$\underline{C}+kV+\epsilon$	$\underline{C}+kV$	$\cdots$	$\underline{C}+kV$
(A4)		$t+1$	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	$\underline{C}+\epsilon$	$\cdots$	$\underline{C}$
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
		T	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	$\underline{C}+\epsilon$
(A5)	$(\hat{\mathbf{x}}^r,\hat{\mathbf{y}}^r)$	1	$\underline{C}$	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-k-1$	$\underline{C}$	$\cdots$	$\underline{C}$	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$t-k$	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$t-k+1$	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\underline{C}$	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-s_{\max}-2$	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}$	0	$\cdots$	0	0	0	$\cdots$	0
(A6)		$t-s_{\max}-1$, $\vdots$, $t-1$	(See Note A.13-1)													
(A7)		t	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\underline{C}+V$	$\cdots$	$\underline{C}+(k-s_{\max}-2)V$	$\underline{C}+(k-s_{\max}-1)V$	$\cdots$	$\underline{C}+(k-1)V$	$\underline{C}+kV$	$\underline{C}+kV$	$\cdots$	$\underline{C}+kV$
(A8)		$t+1$	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	$\underline{C}$	$\cdots$	$\underline{C}$
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
		T	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	$\underline{C}$

y

Group	Point	Index r	1	$\cdots$	$t-k-1$	$t-k$	$t-k+1$	$\cdots$	$t-s_{\max}-2$	$t-s_{\max}-1$	$\cdots$	$t-1$	t	$t+1$	$\cdots$	T
(A1)	$(\bar{\mathbf{x}}^r,\bar{\mathbf{y}}^r)$	1	1	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-k-1$	1	$\cdots$	1	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
(A2)		$t-k+1$	1	$\cdots$	1	1	1	$\cdots$	1	1	$\cdots$	1	0	0	$\cdots$	0
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-s_{\max}-2$	1	$\cdots$	1	1	1	$\cdots$	1	1	$\cdots$	1	0	0	$\cdots$	0
		$t-s_{\max}-1$	1	$\cdots$	1	1	1	$\cdots$	1	1	$\cdots$	1	0	0	$\cdots$	0
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-1$	1	$\cdots$	1	1	1	$\cdots$	1	1	$\cdots$	1	0	0	$\cdots$	0
(A3)		t	1	$\cdots$	1	1	1	$\cdots$	1	1	$\cdots$	1	1	1	$\cdots$	1
(A4)		$t+1$	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	1	$\cdots$	1
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		T	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	1
(A5)	$(\hat{\mathbf{x}}^r,\hat{\mathbf{y}}^r)$	1	1	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-k-1$	1	$\cdots$	1	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$t-k$	1	$\cdots$	1	1	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$t-k+1$	1	$\cdots$	1	1	1	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-s_{\max}-2$	1	$\cdots$	1	1	1	$\cdots$	1	0	$\cdots$	0	0	0	$\cdots$	0
(A6)		$t-s_{\max}-1$, $\vdots$, $t-1$	(See Note A.13-1)													
(A7)		t	1	$\cdots$	1	1	1	$\cdots$	1	1	$\cdots$	1	1	1	$\cdots$	1
(A8)		$t+1$	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	1	$\cdots$	1
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
		T	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	1

Note A.13-1: For $r \in [t-s_{\max}-1, t-1]_{\mathbb{Z}}$, the $\mathbf{x}$ and $\mathbf{y}$ vectors in group (A6) are given as follows: $\hat{\mathbf{x}}^r = (\underbrace{\underline{C},\ldots,\underline{C}}_{r \text{ terms}}, \underbrace{0,\ldots,0}_{T-r \text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{1,\ldots,1}_{r \text{ terms}}, \underbrace{0,\ldots,0}_{T-r \text{ terms}})$ if $t-r-1 \notin \mathcal{S}$;
$\hat{\mathbf{x}}^r = (\underbrace{0,\ldots,0}_{r \text{ terms}}, \underbrace{\overline{V}, \overline{V}+V, \overline{V}+2V, \ldots, \overline{V}+(t-r-1)V}_{t-r \text{ terms}}, \underbrace{\overline{V}+(t-r-1)V, \ldots, \overline{V}+(t-r-1)V}_{T-t \text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{0,\ldots,0}_{r \text{ terms}}, \underbrace{1,\ldots,1}_{T-r \text{ terms}})$ if $t-r-1 \in \mathcal{S}$.

Table A.14: Lower triangular matrix obtained from Table A.13 via Gaussian elimination.

x

Group	Point	Index r	1	$\cdots$	$t{-}k{-}1$	$t{-}k$	$t{-}k{+}1$	$\cdots$	$t{-}s_{\max}{-}2$	$t{-}s_{\max}{-}1$	$\cdots$	$t{-}1$	t	$t{+}1$	$\cdots$	T
(B1)	$(\underline{\mathbf{x}}^r, \underline{\mathbf{y}}^r)$	1	ϵ	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t{-}k{-}1$	0	$\cdots$	ϵ	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
(B2)		$t{-}k{+}1$	0	$\cdots$	0	0	ϵ	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t{-}s_{\max}{-}2$	0	$\cdots$	0	0	0	$\cdots$	ϵ	0	$\cdots$	0	0	0	$\cdots$	0
		$t{-}s_{\max}{-}1$	0	$\cdots$	0	0	0	$\cdots$	0	ϵ	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t{-}1$	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	ϵ	0	0	$\cdots$	0
(B3)		t	0	$\cdots$	0	ϵ	ϵ	$\cdots$	ϵ	ϵ	$\cdots$	ϵ	ϵ	0	$\cdots$	0
(B4)		$t{+}1$	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	ϵ	$\cdots$	0
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
		T	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	ϵ
(B5)	$(\hat{\underline{\mathbf{x}}}^r, \hat{\underline{\mathbf{y}}}^r)$	1 … $t{-}s_{\max}{-}2$	(Omitted)													
(B6)		$t{-}s_{\max}{-}1$ … $t-1$	(Omitted)													
(B7)		t	(Omitted)													
(B8)		$t{+}1$ … T	(Omitted)													

y

Group	Point	Index r	1	$\cdots$	$t{-}k{-}1$	$t{-}k$	$t{-}k{+}1$	$\cdots$	$t{-}s_{\max}{-}2$	$t{-}s_{\max}{-}1$	$\cdots$	$t{-}1$	t	$t{+}1$	$\cdots$	T
(B1)	$(\underline{\mathbf{x}}^r, \underline{\mathbf{y}}^r)$	1	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t{-}k{-}1$	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
(B2)		$t{-}k{+}1$	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t{-}s_{\max}{-}2$	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$t{-}s_{\max}{-}1$	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t{-}1$	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
(B3)		t	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
(B4)		$t{+}1$	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		T	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
(B5)	$(\hat{\underline{\mathbf{x}}}^r, \hat{\underline{\mathbf{y}}}^r)$	1	1	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t{-}k{-}1$	1	$\cdots$	1	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$t{-}k$	1	$\cdots$	1	1	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$t{-}k{+}1$	1	$\cdots$	1	1	1	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t{-}s_{\max}{-}2$	1	$\cdots$	1	1	1	$\cdots$	1	0	$\cdots$	0	0	0	$\cdots$	0
(B6)		$t{-}s_{\max}{-}1$ … $t-1$	(See Note A.14-1)													
(B7)		t	1	$\cdots$	1	1	1	$\cdots$	1	1	$\cdots$	1	1	0	$\cdots$	0
(B8)		$t{+}1$	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	1	$\cdots$	0
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
		T	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	1

Note A.14-1: For $r \in [t-s_{\max}-1, t-1]_{\mathbb{Z}}$, the **y** vector in group (B6) is given as follows: $\hat{\underline{\mathbf{y}}}^r = (\underbrace{1,\dots,1}_{r \text{ terms}}, \underbrace{0,\dots,0}_{T-r \text{ terms}})$ if $t-r-1 \notin \mathcal{S}$; $\hat{\underline{\mathbf{y}}}^r = (\underbrace{-1,\dots,-1}_{r \text{ terms}}, \underbrace{0,\dots,0}_{T-r \text{ terms}})$ if $t-r-1 \in \mathcal{S}$.

(vii) The point in group (B7), denoted $(\underline{\hat{\mathbf{x}}}^t, \underline{\hat{\mathbf{y}}}^t)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^t, \underline{\hat{\mathbf{y}}}^t) = (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t) - (\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$. Here, $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ is the point in group (A7), and $(\hat{\mathbf{x}}^{t+1}, \hat{\mathbf{y}}^{t+1})$ is the point with index $t+1$ in group (A8).

(viii) For each $r \in [t+1, T]_{\mathbb{Z}}$, the point with index r in group (B8), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ if $r \neq T$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $r = T$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ and $(\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ are the points with indices r and $r+1$, respectively, in group (A8).

The matrix shown in Table A.14 is lower triangular; that is, the position of the last nonzero component of a row of the matrix is greater than the position of the last nonzero component of the previous row. This implies that these $2T-1$ points in groups (A1)–(A8) are linearly independent. Therefore, inequality (4.7) is facet-defining for $\text{conv}(\mathcal{P})$.

Next, we show that inequality (4.8) is facet-defining for $\text{conv}(\mathcal{P})$ by creating $2T$ affinely independent points in $\text{conv}(\mathcal{P})$ that satisfy (4.8) at equality when $m = 0$ and $s \geq \min\{k-1, 1\}$ for all $s \in \mathcal{S}$. Because $\mathbf{0} \in \text{conv}(\mathcal{P})$ and $\mathbf{0}$ satisfies (4.8) at equality, it suffices to create the remaining $2T-1$ linearly independent points. We denote these $2T-1$ points as $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ for $r \in [1, T]_{\mathbb{Z}} \setminus \{t\}$, and $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ for all $r \in [1, T]_{\mathbb{Z}}$, and denote the qth component of $\bar{\mathbf{x}}^r$, $\bar{\mathbf{y}}^r$, $\hat{\mathbf{x}}^r$, and $\hat{\mathbf{y}}^r$ as $\bar{x}^r_q$, $\bar{y}^r_q$, $\hat{x}^r_q$, and $\hat{y}^r_q$, respectively. We divide these $2T-1$ points into the following eight groups:

(C1) For each $r \in [1, t-1]_{\mathbb{Z}}$, we create a point $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ as follows:

$$\bar{x}_q^r = \begin{cases} \underline{C}, & \text{for } q \in [1, r-1]_{\mathbb{Z}}; \\ \underline{C} + \epsilon, & \text{for } q = r; \\ 0, & \text{for } q \in [r+1, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\bar{y}_q^r = \begin{cases} 1, & \text{for } q \in [1, r]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [r+1, T]_{\mathbb{Z}}. \end{cases}$$

It is easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (2.2a)–(2.2f). Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. Note that $\bar{x}_t^r = \bar{x}_{t+k}^r = \bar{y}_t^r = \bar{y}_{t+k}^r = 0$, $m = 0$, and $\bar{y}_{t+s}^r - \bar{y}_{t+s+1}^r = 0$ for all $s \in \mathcal{S}$. Hence, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (4.8) at equality.

(C2) For each $r \in [t+1, t+k-1]_{\mathbb{Z}}$, we create a point $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ as follows:

$$\bar{x}_q^r = \begin{cases} 0, & \text{for } q \in [1, t]_{\mathbb{Z}}; \\ \underline{C} + \epsilon, & \text{for } q = r; \\ \underline{C}, & \text{for } q \in [t+1, T]_{\mathbb{Z}} \setminus \{r\}; \end{cases}$$

and

$$\bar{y}_q^r = \begin{cases} 0, & \text{for } q \in [1, t]_{\mathbb{Z}}; \\ 1, & \text{for } q \in [t+1, T]_{\mathbb{Z}}. \end{cases}$$

It is easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (2.2a)–(2.2f). Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. Note that $\bar{x}_t^r = \bar{y}_t^r = 0$, $\bar{x}_{t+k}^r = \underline{C}$, $\bar{y}_{t+k}^r = 1$, and $m = 0$. The existence of $r \in [t+1, t+k-1]_{\mathbb{Z}}$ implies that $k \geq 2$, which in turn implies that $s \geq 1$ for all $s \in \mathcal{S}$. Hence, $\bar{y}_{t-s}^r - \bar{y}_{t-s-1}^r = 0$ for all $s \in \mathcal{S}$. Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (4.8) at equality.

(C3) We create a point $(\bar{\mathbf{x}}^{t+k}, \bar{\mathbf{y}}^{t+k})$ as follows:

$$\bar{x}_q^{t+k} = \begin{cases} \underline{C} + kV, & \text{for } q \in [1, t-1]_{\mathbb{Z}}; \\ \underline{C} + (t+k-q)V + \epsilon, & \text{for } q \in [t, t+k]_{\mathbb{Z}}; \\ \underline{C}, & \text{for } q \in [t+k+1, T]_{\mathbb{Z}}; \end{cases}$$

and $\bar{y}_q^{t+k} = 1$ for all $q \in [1, T]_{\mathbb{Z}}$. It is easy to verify that $(\bar{\mathbf{x}}^{t+k}, \bar{\mathbf{y}}^{t+k})$ satisfies (2.2a)–(2.2d). Note that $\bar{x}_q^{t+k} - \bar{x}_{q-1}^{t+k} = 0$ when $q \in [2, t-1]_{\mathbb{Z}}$, $0 < \bar{x}_q^{t+k} - \bar{x}_{q-1}^{t+k} \leq V$ when $q \in [t, t+k]_{\mathbb{Z}}$, and $-\epsilon \leq \bar{x}_q^{t+k} - \bar{x}_{q-1}^{t+k} \leq 0$ when $q \in [t+k+1, T]_{\mathbb{Z}}$. Thus, $-V\bar{y}_q^{t+k} - \overline{V}(1 - \bar{y}_q^{t+k}) \leq \bar{x}_q^{t+k} - \bar{x}_{q-1}^{t+k} \leq V\bar{y}_{q-1}^{t+k} + \overline{V}(1 - \bar{y}_{q-1}^{t+k})$ for all $q \in [2, T]_{\mathbb{Z}}$. Hence, $(\bar{\mathbf{x}}^{t+k}, \bar{\mathbf{y}}^{t+k})$ satisfies (2.2e)–(2.2f). Therefore, $(\bar{\mathbf{x}}^{t+k}, \bar{\mathbf{y}}^{t+k}) \in \text{conv}(\mathcal{P})$. Note that $\bar{x}_t^{t+k} = \underline{C} + kV + \epsilon$, $\bar{x}_{t+k}^{t+k} = \underline{C} + \epsilon$, $\bar{y}_t^{t+k} = \bar{y}_{t+k}^{t+k} = 1$, $m = 0$, and $\bar{y}_{t+s}^{t+k} - \bar{y}_{t+s+1}^{t+k} = 0$ for all $s \in \mathcal{S}$. Thus, $(\bar{\mathbf{x}}^{t+k}, \bar{\mathbf{y}}^{t+k})$ satisfies (4.8) at equality.

(C4) For each $r \in [t+k+1, T]_{\mathbb{Z}}$, we create a point $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ as follows:

$$\bar{x}_q^r = \begin{cases} 0, & \text{for } q \in [1, r-1]_{\mathbb{Z}}; \\ \underline{C} + \epsilon, & \text{for } q = r; \\ \underline{C}, & \text{for } q \in [r+1, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\bar{y}_q^r = \begin{cases} 0, & \text{for } q \in [1, r-1]_{\mathbb{Z}}; \\ 1, & \text{for } q \in [r, T]_{\mathbb{Z}}. \end{cases}$$

It is easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (2.2a)–(2.2f). Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. Note that $\bar{x}_t^r = \bar{x}_{t+k}^r = \bar{y}_t^r = \bar{y}_{t+k}^r = 0$, $m = 0$, and $\bar{y}_{t+s}^r -$

$\bar{y}^r_{t+s+1} = 0$ for all $s \in \mathcal{S}$. Hence, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (4.8) at equality.

(C5) For each $r \in [1, t-1]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (C2) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \mathrm{conv}(\mathcal{P})$. Note that $\hat{x}^r_t = \hat{x}^r_{t+k} = \hat{y}^r_t = \hat{y}^r_{t+k} = 0$, $m = 0$, and $\hat{y}^r_{t+s} - \hat{y}^r_{t+s+1} = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.8) at equality.

(C6) For each $r \in [t, t+s_{\max}]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (C3) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \mathrm{conv}(\mathcal{P})$. To show that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.8) at equality, we first consider the case where $r - t \notin \mathcal{S}$. In this case, $\hat{x}^r_t = \hat{y}^r_t = 0$, $\hat{x}^r_{t+k} = \underline{C}$, $\hat{y}^r_{t+k} = 1$, and $m = 0$. Because $t + s \neq r$ for all $s \in \mathcal{S}$, we have $\hat{y}^r_{t+s} - \hat{y}^r_{t+s+1} = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.8) at equality. Next, we consider the case where $r - t \in \mathcal{S}$. In this case, $\hat{x}^r_t = \overline{V} + (r-t)V$, $\hat{y}^r_t = 1$, $\hat{x}^r_{t+k} = \hat{y}^r_{t+k} = 0$, and $m = 0$. In addition, $\hat{y}^r_{t+s} - \hat{y}^r_{t+s+1} = 1$ when $s = r - t$, and $\hat{y}^r_{t+s} - \hat{y}^r_{t+s+1} = 0$ when $s \neq r - t$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.8) at equality.

(C7) We create a point $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ as follows:

$$\hat{x}^{t+s_{\max}+1}_q = \begin{cases} \underline{C} + kV, & \text{for } q \in [1, t-1]_{\mathbb{Z}}; \\ \underline{C} + (t+k-q)V, & \text{for } q \in [t, t+k]_{\mathbb{Z}}; \\ \underline{C}, & \text{for } q \in [t+k+1, T]_{\mathbb{Z}}; \end{cases}$$

and $\hat{y}^{t+s_{\max}+1}_q = 1$ for all $q \in [1, T]_{\mathbb{Z}}$. It is easy to verify that $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ satisfies (2.2a)–(2.2d). Note that $\hat{x}^{t+s_{\max}+1}_q - \hat{x}^{t+s_{\max}+1}_{q-1} = 0$ when $q \in [2, t]_{\mathbb{Z}}$, $\hat{x}^{t+s_{\max}+1}_q - \hat{x}^{t+s_{\max}+1}_{q-1} = -V$ when

$q \in [t+1, t+k]_{\mathbb{Z}}$, and $\hat{x}_q^{t+s_{\max}+1} - \hat{x}_{q-1}^{t+s_{\max}+1} = 0$ when $q \in [t+k+1, T]_{\mathbb{Z}}$. Thus, $-V\hat{y}_q^{t+s_{\max}+1} - \overline{V}(1-\hat{y}_q^{t+s_{\max}+1}) \leq \hat{x}_q^{t+s_{\max}+1} - \hat{x}_{q-1}^{t+s_{\max}+1} \leq V\hat{y}_{q-1}^{t+s_{\max}+1} + \overline{V}(1-\hat{y}_{q-1}^{t+s_{\max}+1})$ for all $q \in [2, T]_{\mathbb{Z}}$. Hence, $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ satisfies (2.2e)–(2.2f). Therefore, $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1}) \in \operatorname{conv}(\mathcal{P})$. Note that $\hat{x}_t^{t+s_{\max}+1} = \underline{C} + kV$, $\hat{x}_{t+k}^{t+s_{\max}+1} = \underline{C}$, $\hat{y}_t^{t+s_{\max}+1} = \hat{y}_{t+k}^{t+s_{\max}+1} = 1$, $m = 0$, and $\hat{y}_{t+s}^{t+s_{\max}+1} - \hat{y}_{t+s+1}^{t+s_{\max}+1} = 0$ for all $s \in \mathcal{S}$. Thus, $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ satisfies (4.8) at equality.

(C8) For each $r \in [t+s_{\max}+2, T]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (C5) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \operatorname{conv}(\mathcal{P})$. If $r \leq t+k$, then $\hat{x}_t^r = \hat{y}_t^r = 0$, $\hat{x}_{t+k}^r = \underline{C}$, $\hat{y}_{t+k}^r = 1$, $m = 0$, and $\hat{y}_{t+s}^r - \hat{y}_{t+s+1}^r = 0$ for all $s \in \mathcal{S}$. If $r > t+k$, then $\hat{x}_t^r = \hat{x}_{t+k}^r = \hat{y}_t^r = \hat{y}_{t+k}^r = 0$, $m = 0$, and $\hat{y}_{t+s}^r - \hat{y}_{t+s+1}^r = 0$ for all $s \in \mathcal{S}$. Hence, in both cases, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.8) at equality.

Table A.15 shows a matrix with $2T-1$ rows, where each row represents a point created by this process. This matrix can be transformed into the matrix in Table A.16 via the following Gaussian elimination process:

(i) For each $r \in [1, t-1]_{\mathbb{Z}}$, the point with index r in group (D1), denoted $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r) = (\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) - (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$. Here, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ is the point with index r in group (C1), and $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (C5).

(ii) For each $r \in [t+1, t+k-1]_{\mathbb{Z}}$, the point with index r in group (D2), denoted $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r) = (\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) - (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$. Here, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ is the point with index r in group (C2), and $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ is the point with index t in group (C6). Note that because $s \geq 1$ for all $s \in \mathcal{S}$, the point with index t in group (C6) is given by $\hat{x}_q^t = \hat{y}_q^t = 0$ for $q \in [1,t]_{\mathbb{Z}}$, and $\hat{x}_q^t = \underline{C}$ and $\hat{y}_q^t = 1$ for $q \in [t+1,T]_{\mathbb{Z}}$.

(iii) The point in group (D3), denoted $(\underline{\bar{\mathbf{x}}}^{t+k}, \underline{\bar{\mathbf{y}}}^{t+k})$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^{t+k}, \underline{\bar{\mathbf{y}}}^{t+k}) = (\bar{\mathbf{x}}^{t+k}, \bar{\mathbf{y}}^{t+k}) - (\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$. Here, $(\bar{\mathbf{x}}^{t+k}, \bar{\mathbf{y}}^{t+k})$ is the point in group (C3), and $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ is the point in group (C7).

(iv) For each $r \in [t+k+1,T]_{\mathbb{Z}}$, the point with index r in group (D4), denoted $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r) = (\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) - (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$. Here, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ is the point with index r in group (C4), and $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (C8).

(v) For each $r \in [1,t-1]_{\mathbb{Z}}$, the point with index r in group (D5), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (C5).

(vi) For each $r \in [t, t+s_{\max}]_{\mathbb{Z}}$, the point with index r in group (D6), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ if $r - t \notin \mathcal{S}$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $r - t \in \mathcal{S}$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (C6), and $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ is the point in group (C7).

Table A.15: A matrix with the rows representing $2T-1$ linearly independent points in $\text{conv}(\mathcal{P})$ satisfying inequality (4.8) at equality.

Group	Point	Index r	**x**													**y**												
			1	$\cdots$	$t-1$	t	$t+1$	$\cdots$	$t+s_{\max}+2$	$\cdots$	$t+k-1$	$t+k$	$t+k+1$	$\cdots$	T	1	$\cdots$	$t-1$	t	$t+1$	$\cdots$	$t+s_{\max}+2$	$\cdots$	$t+k-1$	$t+k$	$t+k+1$	$\cdots$	T
(C1)	$(\bar{\mathbf{x}}^r,\bar{\mathbf{y}}^r)$	1	$\underline{C}+\epsilon$	$\cdots$	0	0	0	$\cdots$	0	$\cdots$	0	0	0	$\cdots$	0	1	$\cdots$	0	0	0	$\cdots$	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-1$	$\underline{C}$	$\cdots$	$\underline{C}+\epsilon$	0	0	$\cdots$	0	$\cdots$	0	0	0	$\cdots$	0	1	$\cdots$	1	0	0	$\cdots$	0	$\cdots$	0	0	0	$\cdots$	0
(C2)		$t+1$	0	$\cdots$	0	0	$\underline{C}+\epsilon$	$\cdots$	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}$	0	$\cdots$	0	0	1	$\cdots$	1	$\cdots$	1	1	1	$\cdots$	1
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t+s_{\max}+2$	0	$\cdots$	0	0	$\underline{C}$	$\cdots$	$\underline{C}+\epsilon$	$\cdots$	$\underline{C}$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}$	0	$\cdots$	0	0	1	$\cdots$	1	$\cdots$	1	1	1	$\cdots$	1
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t+k-1$	0	$\cdots$	0	0	$\underline{C}$	$\cdots$	$\underline{C}$	$\cdots$	$\underline{C}+\epsilon$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}$	0	$\cdots$	0	0	1	$\cdots$	1	$\cdots$	1	1	1	$\cdots$	1
(C3)		$t+k$	$\underline{C}+kV$	$\cdots$	$\underline{C}+kV$	$\underline{C}+kV+\epsilon$	$\underline{C}+(k-1)V+\epsilon$	$\cdots$	$\underline{C}+(k-s_{\max}-2)V+\epsilon$	$\cdots$	$\underline{C}+\epsilon+V$	$\underline{C}+\epsilon$	$\underline{C}$	$\cdots$	$\underline{C}$	1	$\cdots$	1	1	1	$\cdots$	1	$\cdots$	1	1	1	$\cdots$	1
(C4)		$t+k+1$	0	$\cdots$	0	0	0	$\cdots$	0	$\cdots$	0	0	$\underline{C}+\epsilon$	$\cdots$	$\underline{C}$	0	$\cdots$	0	0	0	$\cdots$	0	$\cdots$	0	0	1	$\cdots$	1
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		T	0	$\cdots$	0	0	0	$\cdots$	0	$\cdots$	0	0	0	$\cdots$	$\underline{C}+\epsilon$	0	$\cdots$	0	0	0	$\cdots$	0	$\cdots$	0	0	0	$\cdots$	1
(C5)	$(\hat{\mathbf{x}}^r,\hat{\mathbf{y}}^r)$	1	$\underline{C}$	$\cdots$	0	0	0	$\cdots$	0	$\cdots$	0	0	0	$\cdots$	0	1	$\cdots$	0	0	0	$\cdots$	0	$\cdots$	0	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t-1$	$\underline{C}$	$\cdots$	$\underline{C}$	0	0	$\cdots$	0	$\cdots$	0	0	0	$\cdots$	0	1	$\cdots$	1	0	0	$\cdots$	0	$\cdots$	0	0	0	$\cdots$	0
(C6)		t, $\vdots$, $t+s_{\max}$	(See Note A.15-1)													(See Note A.15-1)												
(C7)		$t+s_{\max}+1$	$\underline{C}+kV$	$\cdots$	$\underline{C}+kV$	$\underline{C}+kV$	$\underline{C}+(k-1)V$	$\cdots$	$\underline{C}+(k-s_{\max}-2)V$	$\cdots$	$\underline{C}+V$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}$	1	$\cdots$	1	1	1	$\cdots$	1	$\cdots$	1	1	1	$\cdots$	1
(C8)		$t+s_{\max}+2$	0	$\cdots$	0	0	0	$\cdots$	$\underline{C}$	$\cdots$	$\underline{C}$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}$	0	$\cdots$	0	0	0	$\cdots$	1	$\cdots$	1	1	1	$\cdots$	1
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
		$t+k-1$	0	$\cdots$	0	0	0	$\cdots$	0	$\cdots$	$\underline{C}$	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}$	0	$\cdots$	0	0	0	$\cdots$	0	$\cdots$	1	1	1	$\cdots$	1
		$t+k$	0	$\cdots$	0	0	0	$\cdots$	0	$\cdots$	0	$\underline{C}$	$\underline{C}$	$\cdots$	$\underline{C}$	0	$\cdots$	0	0	0	$\cdots$	0	$\cdots$	0	1	1	$\cdots$	1
		$t+k+1$	0	$\cdots$	0	0	0	$\cdots$	0	$\cdots$	0	0	$\underline{C}$	$\cdots$	$\underline{C}$	0	$\cdots$	0	0	0	$\cdots$	0	$\cdots$	0	0	1	$\cdots$	1
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
		T	0	$\cdots$	0	0	0	$\cdots$	0	$\cdots$	0	0	0	$\cdots$	$\underline{C}$	0	$\cdots$	0	0	0	$\cdots$	0	$\cdots$	0	0	0	$\cdots$	1

Note A.15-1: For $r \in [t, t+s_{\max}]_{\mathbb{Z}}$, the **x** and **y** vectors in group (C6) are given as follows: $\hat{\mathbf{x}}^r = (\underbrace{0,\ldots,0}_{r \text{ terms}},\underbrace{\underline{C},\ldots,\underline{C}}_{T-r \text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{0,\ldots,0}_{r \text{ terms}},\underbrace{1,\ldots,1}_{T-r \text{ terms}})$ if $r-t \notin \mathcal{S}$;

$\hat{\mathbf{x}}^r = (\underbrace{\overline{V}+(r-t)V,\ldots,\overline{V}+(r-t)V}_{t-1 \text{ terms}},\underbrace{\overline{V}+(r-t)V,\overline{V}+(r-t-1)V,\overline{V}+(r-t-2)V,\ldots,\overline{V}}_{r-t+1 \text{ terms}},\underbrace{0,\ldots,0}_{T-r \text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{1,\ldots,1}_{r \text{ terms}},\underbrace{0,\ldots,0}_{T-r \text{ terms}})$ if $r-t \in \mathcal{S}$.

Table A.16: Lower triangular matrix obtained from Table A.15 via Gaussian elimination.

Group	Point	Index r	$\mathbf{x}$														$\mathbf{y}$									
			1	$\cdots$	$t-1$	t	$t+1$	$\cdots$	$t+s_{\max}+1$	$t+s_{\max}+2$	$\cdots$	$t+k-1$	$t+k$	$t+k+1$	$\cdots$	T	1	$\cdots$	$t-1$	t	$t+1$	$\cdots$	$t+s_{\max}+1$	$t+s_{\max}+2$	$\cdots$	T
(D1)		1	ϵ	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
		$t-1$	0	$\cdots$	ϵ	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
(D2)		$t+1$	0	$\cdots$	0	0	ϵ	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
	$(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$	$t+s_{\max}+1$	0	$\cdots$	0	0	0	$\cdots$	ϵ	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
		$t+s_{\max}+2$	0	$\cdots$	0	0	0	$\cdots$	0	ϵ	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
		$t+k-1$	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	ϵ	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
(D3)		$t+k$	0	$\cdots$	0	ϵ	ϵ	$\cdots$	ϵ	ϵ	$\cdots$	ϵ	ϵ	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
(D4)		$t+k+1$	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	ϵ	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
		T	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$	ϵ	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
(D5)		1	(Omitted)														1	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0
		$\vdots$															$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
		$t-1$															1	$\cdots$	1	0	0	$\cdots$	0	0	$\cdots$	0
(D6)		t	(Omitted)														(See Note A.16-1)									
	$(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$	$\vdots$																								
		$t+s_{\max}$																								
(D7)		$t+s_{\max}+1$	(Omitted)														1	$\cdots$	1	1	1	$\cdots$	1	0	$\cdots$	0
(D8)		$t+s_{\max}+2$	(Omitted)														0	$\cdots$	0	0	0	$\cdots$	0	1	$\cdots$	0
		$\vdots$															$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\ddots$	$\vdots$
		T															0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	1

Note A.16-1: For $r \in [t, t+s_{\max}]_{\mathbb{Z}}$, the $\mathbf{y}$ vector in group (D6) is given as follows: $\underline{\hat{\mathbf{y}}}^r = (\underbrace{-1,\ldots,-1}_{r\text{ terms}}, \underbrace{0,\ldots,0}_{T-r\text{ terms}})$ if $r-t \notin \mathcal{S}$; $\underline{\hat{\mathbf{y}}}^r = (\underbrace{1,\ldots,1}_{r\text{ terms}}, \underbrace{0,\ldots,0}_{T-r\text{ terms}})$ if $r-t \in \mathcal{S}$.

(vii) The point in group (D7), denoted $(\underline{\hat{\mathbf{x}}}^{t+s_{\max}+1}, \underline{\hat{\mathbf{y}}}^{t+s_{\max}+1})$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^{t+s_{\max}+1}, \underline{\hat{\mathbf{y}}}^{t+s_{\max}+1}) = (\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1}) - (\hat{\mathbf{x}}^{t+s_{\max}+2}, \hat{\mathbf{y}}^{t+s_{\max}+2})$. Here, $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ is the point in group (C7), and $(\hat{\mathbf{x}}^{t+s_{\max}+2}, \hat{\mathbf{y}}^{t+s_{\max}+2})$ is the point with index $t + s_{\max} + 2$ in group (C8).

(viii) For each $r \in [t + s_{\max} + 2, T]_{\mathbb{Z}}$, the point with index r in group (D8), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ if $r \neq T$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $r = T$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ and $(\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ are the points with indices r and $r + 1$, respectively, in group (C8).

The matrix in Table A.16 is lower triangular; that is, the position of the last nonzero component of a row of the matrix is greater than the position of the last nonzero component of the previous row. This implies that these $2T - 1$ points in groups (C1)–(C8) are linearly independent. Therefore, inequality (4.8) is facet-defining for $\text{conv}(\mathcal{P})$. □

8.13 Proof of Proposition 10

For notational convenience, denote $\hat{k} = \max\{k \in [1, T-1]_{\mathbb{Z}} : \overline{C} - \underline{C} - kV > 0\}$, and denote $\hat{s}_{km} = \min\{k-1, L-m-1\}$ for any $k \in [1, \hat{k}]_{\mathbb{Z}}$ and $m \in [0, k-1]_{\mathbb{Z}}$.

We first consider inequality (4.7). Consider any given $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{2T}$. For any $t \in [1, T]_{\mathbb{Z}}$, let

$$\theta(t) = \sum_{\tau=2}^{t} \max\{y_\tau - y_{\tau-1}, 0\}.$$

Then, for any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, and $t \in [k+1, T-m]_{\mathbb{Z}}$,

$$\sum_{s=1}^{\hat{s}_{km}} \max\{y_{t-s} - y_{t-s-1}, 0\} = \sum_{\tau=t-\hat{s}_{km}}^{t-1} \max\{y_\tau - y_{\tau-1}, 0\} = \theta(t-1) - \theta(t - \hat{s}_{km} - 1). \tag{8.44}$$

For any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, $t \in [k+1, T-m]_{\mathbb{Z}}$, and $\mathcal{S} \subseteq [0, \hat{s}_{km}]_{\mathbb{Z}}$, let

$$\begin{aligned} \tilde{v}_{km}(\mathcal{S}, t) = & x_t - x_{t-k} - (\underline{C} + (k-m)V)y_t - V\sum_{i=1}^{m} y_{t+i} + \underline{C}y_{t-k} \\ & + \sum_{s\in\mathcal{S}} (\underline{C} + (k-s)V - \overline{V})(y_{t-s} - y_{t-s-1}). \end{aligned}$$

If $\tilde{v}_{km}(\mathcal{S}, t) > 0$, then $\tilde{v}_{km}(\mathcal{S}, t)$ is the amount of violation of inequality (4.7). If $\tilde{v}_{km}(\mathcal{S}, t) \leq 0$, there is no violation of inequality (4.7). For any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, and $t \in [k+1, T-m]_{\mathbb{Z}}$, let

$$v_{km}(t) = \max_{\mathcal{S} \subseteq [0, \hat{s}_{km}]_{\mathbb{Z}}} \{\tilde{v}_{km}(\mathcal{S}, t)\}.$$

If $v_{km}(t) > 0$, then $v_{km}(t)$ is the largest possible violation of inequality (4.7) for this combination of k, m, and t. If $v_{km}(t) \leq 0$, the largest possible violation of inequality (4.7) is zero for this combination of k, m, and t. Because $\underline{C} + V > \overline{V}$, we have $\underline{C} + (k-s)V - \overline{V} > 0$ for all $k \in [1, \hat{k}]_{\mathbb{Z}}$, $s \in [0, \hat{s}_{km}]_{\mathbb{Z}}$, and $m \in [0, k-1]_{\mathbb{Z}}$. Thus, for any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, and $t \in [k+1, T-m]_{\mathbb{Z}}$, $\tilde{v}_{km}(\mathcal{S}, t)$ is maximized when $\mathcal{S}$ contains all $s \in [0, \hat{s}_{km}]_{\mathbb{Z}}$ such that $y_{t-s} - y_{t-s-1} > 0$ (if any). If it does not exist any $s \in [0, \hat{s}]_{\mathbb{Z}}$ such that $y_{t-s} - y_{t-s-1} > 0$, then $\tilde{v}_{km}(\mathcal{S}, t)$ is maximized when $\mathcal{S} = \emptyset$, and $v_{km}(t) = x_t - x_{t-k} - (\underline{C} + (k-m)V)y_t - V\sum_{i=1}^{m} y_{t+i} + \underline{C}y_{t-k}$. Hence, for any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, and $t \in [k+1, T-m]_{\mathbb{Z}}$,

$$\begin{aligned} v_{km}(t) = & x_t - x_{t-k} - (\underline{C} + (k-m)V)y_t - V\sum_{i=1}^{m} y_{t+i} + \underline{C}y_{t-k} \\ & + \sum_{s=0}^{\hat{s}_{km}} (\underline{C} + (k-s)V - \overline{V}) \max\{y_{t-s} - y_{t-s-1}, 0\}. \end{aligned}$$

Determining $\theta(t)$ for all $t \in [1, T]_{\mathbb{Z}}$ can be done recursively in $O(T)$ time by setting $\theta(1) = 0$ and setting $\theta(t) = \theta(t-1) + \max\{y_t - y_{t-1}, 0\}$ for $t = 2, \ldots, T$. Clearly, for each $k \in [1, \hat{k}]_{\mathbb{Z}}$ and each $m \in [0, k-1]_{\mathbb{Z}}$, the value of $v_{km}(k+1)$ can be determined in $O(T)$ time. For any $k \in [1, \hat{k}]_{\mathbb{Z}}$,

$m \in [0, k-1]_{\mathbb{Z}}$, and $t \in [k+2, T-m]_{\mathbb{Z}}$,

$$\begin{aligned}
&v_{km}(t) - v_{km}(t-1) \\
&= (x_t - x_{t-1}) - (x_{t-k} - x_{t-k-1}) - (\underline{C} + (k-m)V)(y_t - y_{t-1}) \\
&\quad - V\left[\sum_{i=1}^{m} y_{t+i} - \sum_{i=1}^{m} y_{t+i-1}\right] + \underline{C}(y_{t-k} - y_{t-k-1}) \\
&\quad + (\underline{C} + kV - \overline{V})\left[\sum_{s=0}^{\hat{s}_{km}} \max\{y_{t-s} - y_{t-s-1}, 0\} - \sum_{s=0}^{\hat{s}_{km}} \max\{y_{t-s-1} - y_{t-s-2}, 0\}\right] \\
&\quad - V\left[\sum_{s=0}^{\hat{s}_{km}} s \max\{y_{t-s} - y_{t-s-1}, 0\} - \sum_{s=0}^{\hat{s}_{km}} s \max\{y_{t-s-1} - y_{t-s-2}, 0\}\right] \\
&= (x_t - x_{t-1}) - (x_{t-k} - x_{t-k-1}) - (\underline{C} + (k-m)V)(y_t - y_{t-1}) \\
&\quad - V(y_{t+m} - y_t) + \underline{C}(y_{t-k} - y_{t-k-1}) \\
&\quad + (\underline{C} + kV - \overline{V})\left[\max\{y_t - y_{t-1}, 0\} - \max\{y_{t-\hat{s}_{km}-1} - y_{t-\hat{s}_{km}-2}, 0\}\right] \\
&\quad - V\left[\sum_{s=1}^{\hat{s}_{km}} \max\{y_{t-s} - y_{t-s-1}, 0\} - \hat{s}_{km} \max\{y_{t-\hat{s}_{km}-1} - y_{t-\hat{s}_{km}-2}, 0\}\right].
\end{aligned}$$

This, together with (8.44), implies that

$$\begin{aligned}
v_{km}(t) = v_{km}(t-1) &+ (x_t - x_{t-1}) - (x_{t-k} - x_{t-k-1}) - (\underline{C} + (k-m)V)(y_t - y_{t-1}) \\
&- V(y_{t+m} - y_t) + \underline{C}(y_{t-k} - y_{t-k-1}) \\
&+ (\underline{C} + kV - \overline{V})\left[\max\{y_t - y_{t-1}, 0\} - \max\{y_{t-\hat{s}_{km}-1} - y_{t-\hat{s}_{km}-2}, 0\}\right] \\
&- V\left[\theta(t-1) - \theta(t - \hat{s}_{km} - 1) - \hat{s}_{km} \max\{y_{t-\hat{s}_{km}-1} - y_{t-\hat{s}_{km}-2}, 0\}\right].
\end{aligned}$$

Thus, for each $k \in [1, \hat{k}]_{\mathbb{Z}}$ and $m \in [0, k-1]_{\mathbb{Z}}$, the values of $v_{km}(k+1), v_{km}(k+2), \ldots, v_{km}(T-m)$ can be determined recursively in $O(T)$ time. Hence, the values of k, m, t and the set $\mathcal{S}$ corresponding to the largest possible violation of inequality (4.7) can be obtained in $O(T^3)$ time.

Next, we consider inequality (4.8). Consider any given point $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{2T}$. For any $t \in [1, T]_{\mathbb{Z}}$, let

$$\theta'(t) = \sum_{\tau=t}^{T-1} \max\{y_\tau - y_{\tau+1}, 0\}.$$

Then, for any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, and $t \in [m+1, T-k]_{\mathbb{Z}}$,

$$\sum_{s=1}^{\hat{s}_{km}} \max\{y_{t+s} - y_{t+s+1}, 0\} = \sum_{\tau=t+1}^{t+\hat{s}_{km}} \max\{y_\tau - y_{\tau+1}, 0\} = \theta'(t+1) - \theta'(t + \hat{s}_{km} + 1). \tag{8.45}$$

For any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, $t \in [m+1, T-k]_{\mathbb{Z}}$, and $\mathcal{S} \subseteq [0, \hat{s}_{km}]_{\mathbb{Z}}$, let

$$\begin{aligned} \tilde{v}'_{km}(\mathcal{S}, t) =& x_t - x_{t+k} - (\underline{C} + (k-m)V)y_t - V\sum_{i=1}^{m} y_{t-i} + \underline{C}y_{t+k} \\ &+ \sum_{s \in \mathcal{S}} (\underline{C} + (k-s)V - \overline{V})(y_{t+s} - y_{t+s+1}). \end{aligned}$$

If $\tilde{v}'_{km}(\mathcal{S}, t) > 0$, then $\tilde{v}'_{km}(\mathcal{S}, t)$ is the amount of violation of inequality (4.8). If $\tilde{v}'_{km}(\mathcal{S}, t) \leq 0$, there is no violation of inequality (4.8). For any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, and $t \in [m+1, T-k]_{\mathbb{Z}}$, let

$$v'_{km}(t) = \max_{\mathcal{S} \subseteq [0, \hat{s}_{km}]_{\mathbb{Z}}} \{\tilde{v}'_{km}(\mathcal{S}, t)\}.$$

If $v'_{km}(t) > 0$, then $v'_{km}(t)$ is the largest possible violation of inequality (4.8) for this combination of k, m, and t. If $v'_{km}(t) \leq 0$, the largest possible violation of inequality (4.8) is zero for this combination of k, m, and t. Because

$\underline{C} + V > \overline{V}$, we have $\underline{C} + (k-s)V - \overline{V} > 0$ for all $k \in [1, \hat{k}]_{\mathbb{Z}}$, $s \in [0, \hat{s}_{km}]_{\mathbb{Z}}$, and $m \in [0, k-1]_{\mathbb{Z}}$. Thus, for any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, and $t \in [m+1, T-k]_{\mathbb{Z}}$, $\tilde{v}'_{km}(\mathcal{S}, t)$ is maximized when $\mathcal{S}$ contains all $s \in [0, \hat{s}_{km}]_{\mathbb{Z}}$ such that $y_{t+s} - y_{t+s+1} > 0$ (if any). If it does not exist any $s \in [0, \hat{s}]_{\mathbb{Z}}$ such that $y_{t+s} - y_{t+s+1} > 0$, then $\tilde{v}'_{km}(\mathcal{S}, t)$ is maximized when $\mathcal{S} = \emptyset$, and $v'_{km}(t) = x_t - x_{t+k} - (\underline{C} + (k-m)V)y_t - V\sum_{i=1}^{m} y_{t-i} + \underline{C}y_{t+k}$. Hence, for any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, and $t \in [m+1, T-k]_{\mathbb{Z}}$,

$$v'_{km}(t) = x_t - x_{t+k} - (\underline{C} + (k-m)V)y_t - V\sum_{i=1}^{m} y_{t-i} + \underline{C}y_{t+k} + \sum_{s=0}^{\hat{s}_{km}} (\underline{C} + (k-s)V - \overline{V}) \max\{y_{t+s} - y_{t+s+1}, 0\}.$$

Determining $\theta'(t)$ for all $t \in [1, T]_{\mathbb{Z}}$ can be done recursively in $O(T)$ time by setting $\theta'(T) = 0$ and setting $\theta'(t) = \theta'(t+1) + \max\{y_t - y_{t+1}, 0\}$ for $t = T-1, T-2, \ldots, 1$. Clearly, for each $k \in [1, \hat{k}]_{\mathbb{Z}}$ and each $m \in [0, k-1]_{\mathbb{Z}}$, the value of $v'_{km}(T-k)$ can be determined in $O(T)$ time. For any $k \in [1, \hat{k}]_{\mathbb{Z}}$,

$m \in [0, k-1]_{\mathbb{Z}}$, and $t \in [m+1, T-k-1]_{\mathbb{Z}}$,

$$\begin{aligned}
& v'_{km}(t) - v'_{km}(t+1) \\
& = (x_t - x_{t+1}) - (x_{t+k} - x_{t+k+1}) - (\underline{C} + (k-m)V)(y_t - y_{t+1}) \\
& \quad - V\left[\sum_{i=1}^{m} y_{t-i} - \sum_{i=1}^{m} y_{t-i+1}\right] + \underline{C}(y_{t+k} - y_{t+k+1}) \\
& \quad + (\underline{C} + kV - \overline{V})\left[\sum_{s=0}^{\hat{s}_{km}} \max\{y_{t+s} - y_{t+s+1}, 0\} - \sum_{s=0}^{\hat{s}_{km}} \max\{y_{t+s+1} - y_{t+s+2}, 0\}\right] \\
& \quad - V\left[\sum_{s=0}^{\hat{s}_{km}} s \max\{y_{t+s} - y_{t+s+1}, 0\} - \sum_{s=0}^{\hat{s}_{km}} s \max\{y_{t+s+1} - y_{t+s+2}, 0\}\right] \\
& = (x_t - x_{t+1}) - (x_{t+k} - x_{t+k+1}) - (\underline{C} + (k-m)V)(y_t - y_{t+1}) \\
& \quad - V(y_{t-m} - y_t) + \underline{C}(y_{t+k} - y_{t+k+1}) \\
& \quad + (\underline{C} + kV - \overline{V})\left[\max\{y_t - y_{t+1}, 0\} - \max\{y_{t+\hat{s}_{km}+1} - y_{t+\hat{s}_{km}+2}, 0\}\right] \\
& \quad - V\left[\sum_{s=1}^{\hat{s}_{km}} \max\{y_{t+s} - y_{t+s+1}, 0\} - \hat{s}_{km} \max\{y_{t+\hat{s}_{km}+1} - y_{t+\hat{s}_{km}+2}, 0\}\right].
\end{aligned}$$

This, together with (8.45), implies that

$$\begin{aligned}
v'_{km}(t) = {} & v'_{km}(t+1) + (x_t - x_{t+1}) - (x_{t+k} - x_{t+k+1}) - (\underline{C} + (k-m)V)(y_t - y_{t+1}) \\
& - V(y_{t-m} - y_t) + \underline{C}(y_{t+k} - y_{t+k+1}) \\
& + (\underline{C} + kV - \overline{V})\left[\max\{y_t - y_{t+1}, 0\} - \max\{y_{t+\hat{s}_{km}+1} - y_{t+\hat{s}_{km}+2}, 0\}\right] \\
& - V\left[\theta'(t+1) - \theta'(t + \hat{s}_{km} + 1) - \hat{s}_{km} \max\{y_{t+\hat{s}_{km}+1} - y_{t+\hat{s}_{km}+2}, 0\}\right].
\end{aligned}$$

Thus, for each $k \in [1, \hat{k}]_{\mathbb{Z}}$ and $m \in [0, k-1]_{\mathbb{Z}}$, the values of $v'_{km}(m+1), v'_{km}(m+2), \ldots, v'_{km}(T-k)$ can be determined recursively in $O(T)$ time. Hence, the values of k, m, t and the set $\mathcal{S}$ corresponding to the largest possible violation of inequality (4.8) can be obtained in $O(T^3)$ time. □

8.14 Proof of Proposition 11

For notational convenience, we define $s_{\max} = \max\{s : s \in \mathcal{S}\}$ if $\mathcal{S} \neq \emptyset$, and $s_{\max} = -1$ if $\mathcal{S} = \emptyset$. To prove that linear inequalities (4.9) and (4.10) are valid for $\text{conv}(\mathcal{P})$, it suffices to show that they are valid for $\mathcal{P}$. Consider any element $(\mathbf{x}, \mathbf{y})$ of $\mathcal{P}$. We show that $(\mathbf{x}, \mathbf{y})$ satisfies (4.9) and (4.10).

We first show that $(\mathbf{x}, \mathbf{y})$ satisfies (4.9). Consider any $t \in [k+1, T-m-1]_{\mathbb{Z}}$. We divide the analysis into three cases:

Case 1: $y_t = 0$. In this case, by (2.2c) and (2.2d), $-x_{t-k} \leq -\underline{C}y_{t-k}$ and $x_t = 0$. Thus, the left hand side of (4.9) is at most $-\underline{C}y_{t-k}$ and the third term on the right hand side of (4.9) is 0. Because $y_t = 0$ and $t \in [2, T]_{\mathbb{Z}}$, by Lemma 1(i), $y_{t-j} - y_{t-j-1} \leq 0$ for all $j \in [0, \min\{t-2, L-1\}]_{\mathbb{Z}}$. Because $\mathcal{S} \subseteq [0, \min\{k-1, L-m-2\}]_{\mathbb{Z}}$, $m \geq 0$, and $t \geq k+1$, we have $\mathcal{S} \subseteq [0, \min\{t-2, L-1\}]_{\mathbb{Z}}$. Thus, $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S}$. Because $m \leq k-1$, $\mathcal{S} \subseteq [0, k-1]_{\mathbb{Z}}$, and $\underline{C} + V > \overline{V}$, the coefficients "$\underline{C} + (k-m)V - \overline{V}$" and "$\underline{C} + (k-s)V - \overline{V}$" on the right hand side of (4.9) are positive for any $s \in \mathcal{S}$. Thus, the right hand side of (4.9) is at least $-\underline{C}y_{t-k}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.9).

Case 2: $y_t = 1$ and $y_{t-s'} - y_{t-s'-1} = 1$ for some $s' \in \mathcal{S}$. In this case, $y_{t-s'} = 1$ and $y_{t-s'-1} = 0$. Because $y_t = 1$ and $t \in [2, T]_{\mathbb{Z}}$, by Lemma 1(ii), there exists at most one $j \in [0, \min\{t-2, L\}]_{\mathbb{Z}}$ such that $y_{t-j} - y_{t-j-1} = 1$. Because $\mathcal{S} \subseteq [0, \min\{k-1, L-m-2\}]_{\mathbb{Z}}$, $m \geq 0$, and $t \geq k+1$, we have $\mathcal{S} \subseteq [0, \min\{t-2, L\}]_{\mathbb{Z}}$. Thus, $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S} \setminus \{s'\}$. Because $y_{t-s'} - y_{t-s'-1} = 1$ and $t - s' \in [2, T]_{\mathbb{Z}}$, by (2.2a), we have $y_\tau = 1$ for all $\tau \in [t-s', \min\{T, t-s'+L-1\}]_{\mathbb{Z}}$. Because $\mathcal{S} \subseteq [0, L-m-2]_{\mathbb{Z}}$, we have $t - s' + L - 1 \geq t + m + 1$. Thus, $y_\tau = 1$ for all $\tau \in [t - s', t + m + 1]_{\mathbb{Z}}$. Because $\mathcal{S} \subseteq [0, k-1]_{\mathbb{Z}}$ and $\underline{C} + V > \overline{V}$, for any $s \in \mathcal{S}$, the coefficient "$\underline{C} + (k - s)V - \overline{V}$" on the right hand side of inequality (4.9) is positive. Hence, the right hand side of (4.9) is at least $s'V + \overline{V} - \underline{C}y_{t-k}$. By (2.2e), $\sum_{\tau=t-s'}^{t}(x_\tau - x_{\tau-1}) \leq \sum_{\tau=t-s'}^{t} Vy_{\tau-1} + \sum_{\tau=t-s'}^{t} \overline{V}(1 - y_{\tau-1})$, which implies that $x_t - x_{t-s'-1} \leq s'V + \overline{V}$. Because $y_{t-s'-1} = 0$, by (2.2d), $x_{t-s'-1} = 0$. Thus, $x_t \leq s'V + \overline{V}$. By (2.2c), $-x_{t-k} \leq -\underline{C}y_{t-k}$. Thus, $x_t - x_{t-k} \leq s'V + \overline{V} - \underline{C}y_{t-k}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.9).

Case 3: $y_t = 1$ and $y_{t-s} - y_{t-s-1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, k-1]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, and $\underline{C} + V > \overline{V}$, the coefficients "$\underline{C} + (k - m)V - \overline{V}$" and "$\underline{C} + (k - s)V - \overline{V}$", for any $s \in \mathcal{S}$, are positive. If there exists some $i \in [0, m]_{\mathbb{Z}}$ such that $y_{t+i} - y_{t+i+1} = 1$, we have $y_{t+i} = 1$ and $y_{t+i+1} = 0$. Let $i^* = \min\{i \in [0, m]_{\mathbb{Z}} : y_{t+i} - y_{t+i+1} = 1\}$. Thus, $y_\tau = 1$ for all $\tau \in [t, t + i^*]_{\mathbb{Z}}$. Hence, the right hand side of inequality (4.9) is at least $V\sum_{i=1}^{m} y_{t+i} + \overline{V} - \underline{C}y_{t-k}$. By (2.2f), we have $\sum_{\tau=t+1}^{t+i^*+1}(x_{\tau-1} - x_\tau) \leq \sum_{\tau=t+1}^{t+i^*+1} Vy_\tau + \sum_{\tau=t+1}^{t+i^*+1} \overline{V}(1 - y_\tau)$, which implies that $x_t - x_{t+i^*+1} \leq i^*V + \overline{V}$. Because $y_{t+i^*+1} = 0$, by (2.2d), we have $x_{t+i^*+1} = 0$. Thus, $x_t \leq i^*V + \overline{V}$. By (2.2c), we have $-x_{t-k} \leq -\underline{C}y_{t-k}$. Thus, $x_t - x_{t-k} \leq i^*V + \overline{V} - \underline{C}y_{t-k} \leq V\sum_{i=1}^{m} y_{t+i} + \overline{V} - \underline{C}y_{t-k}$ as $i^* \in [0, m]_{\mathbb{Z}}$. Now, we consider the case where there does not exist $i \in [0, m]_{\mathbb{Z}}$ such that $y_{t+i} - y_{t+i+1} = 1$. Thus, $y_\tau = 1$ for all $\tau \in [t, t + m + 1]_{\mathbb{Z}}$. The right hand side of inequality (4.9) is then at least $\underline{C} + kV - \underline{C}y_{t-k}$. Let $t' = \max\{\tau \in [2, t]_{\mathbb{Z}} : y_\tau - y_{\tau-1} = 1\}$. Then, we have $y_{t'-1} = 0$, $y_{t'} = 1$, and $y_\tau = 1$ for all $\tau \in [t', t]_{\mathbb{Z}}$. When $t' \leq t - k$ or t' does not exist, by (2.2e), we have $\sum_{\tau=t-k+1}^{t}(x_\tau - x_{\tau-1}) \leq \sum_{\tau=t-k+1}^{t} y_{\tau-1} + \sum_{\tau=t-k+1}^{t} \overline{V}(1 - y_{\tau-1})$, which implies that $x_t - x_{t-k} \leq kV = \underline{C} + kV - \underline{C}y_{t-k}$. When $t' > t-k$, by (2.2e), we have $\sum_{\tau=t'}^{t}(x_\tau - x_{\tau-1}) \leq \sum_{\tau=t'}^{t} Vy_{\tau-1} + \sum_{\tau=t'}^{t} \overline{V}(1-y_{\tau-1})$, which implies that $x_t - x_{t'} \leq (t-t')V + \overline{V} < \underline{C} + kV$ as $t' > t-k$ and $\underline{C} + V > \overline{V}$. By (2.2c), we have $-x_{t-k} \leq -\underline{C}y_{t-k}$. Thus, $x_t - x_{t-k} \leq (t-t')V + \overline{V} - \underline{C}y_{t-k} < \underline{C} + kV - \underline{C}y_{t-k}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.9).

Next, we show that $(\mathbf{x}, \mathbf{y})$ satisfies (4.10). Consider any $t \in [m+2, T-k]_{\mathbb{Z}}$. We divide the analysis into three cases.

Case 1: $y_t = 0$. In this case, by (2.2c) and (2.2d), $-x_{t+k} \leq -\underline{C}y_{t+k}$ and $x_t = 0$. Thus, the left hand side of (4.10) is at most $-\underline{C}y_{t+k}$ and the third term on the right hand side of (4.10) is 0. Because $y_t = 0$ and $t \in [1, T-1]_{\mathbb{Z}}$, by Lemma 2(i), $y_{t+j} - y_{t+j+1} \leq 0$ for all $j \in [0, \min\{T-t-1, L-1\}]_{\mathbb{Z}}$. Because $\mathcal{S} \subseteq [0, \min\{k-1, L-m-2\}]_{\mathbb{Z}}$, $m \geq 0$, and $t \leq T-k$, we have $\mathcal{S} \subseteq [0, \min\{T-t-1, L-1\}]_{\mathbb{Z}}$. Thus, $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, k-1]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, and $\underline{C} + V > \overline{V}$, the coefficients "$\underline{C} + (k-m)V - \overline{V}$" and "$\underline{C} + (k-s)V - \overline{V}$", for any $s \in \mathcal{S}$, on the right hand side of (4.10) are positive. Thus, the right hand side of (4.10) is at least $-\underline{C}y_{t+k}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.10).

Case 2: $y_t = 1$ and $y_{t+s'} - y_{t+s'+1} = 1$ for some $s' \in \mathcal{S}$. In this case, $y_{t+s'} = 1$ and $y_{t+s'+1} = 0$. Because $y_{t+s'+1} = 0$ and $t+s'+1 \in [2, T]_{\mathbb{Z}}$, by Lemma 1(i), we have $y_{t+s'+1-j} - y_{t+s'-j} \leq 0$ for all $j \in [0, \min\{t+s'-1, L-1\}]_{\mathbb{Z}}$. Because $\mathcal{S} \subseteq [0, L-m-2]_{\mathbb{Z}}$, we have $t+s'-L+2 \leq t-m$. Thus, $y_\tau - y_{\tau-1} \leq 0$ for all $\tau \in [t-m, t+s'+1]_{\mathbb{Z}}$. Because $y_{t+s'} = 1$, we have $y_{\tau=1}$ for all $\tau \in [t-m-1, t+s']_{\mathbb{Z}}$. By (2.2f), we have $\sum_{\tau=t+1}^{t+s'+1}(x_{\tau-1} - x_\tau) \leq \sum_{\tau=t+1}^{t+s'+1} Vy_\tau + \sum_{\tau=t+1}^{t+s'+1} \overline{V}(1-y_\tau)$, which implies that $x_t - x_{t+s'+1} \leq s'V + \overline{V}$.

Because $y_{t+s'+1} = 0$, by (2.2d), we have $x_{t+s'-1} = 0$. Thus, $x_t \leq s'V + \overline{V}$. By (2.2c), $-x_{t-k} \leq -\underline{C}y_{t-k}$. Thus, the left hand side of inequality (4.10) is at most $s'V + \overline{V} - \underline{C}y_{t-k}$. Because $y_t = 1$ and $t \in [1, T-1]_{\mathbb{Z}}$, by Lemma 2(ii), there exists at most one $j \in [0, \min\{T-t-1, L\}]_{\mathbb{Z}}$ such that $y_{t+j} - y_{t+j+1} = 1$. Because $\mathcal{S} \subseteq [0, \min\{k-1, L-m-2\}]$, $m \geq 0$, and $t \leq T-k$, we have $\mathcal{S} \subseteq [0, \min\{T-t-1, L\}]_{\mathbb{Z}}$. Thus, $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in \mathcal{S} \setminus \{s'\}$. Because $\mathcal{S} \subseteq [0, k-1]_{\mathbb{Z}}$ and $\underline{C} + V > \overline{V}$, for any $s \in \mathcal{S}$, the coefficient "$\underline{C} + (k-s)V - \overline{V}$" on the right hand side of inequality (4.10) is positive. Hence, the right hand side of inequality (4.10) is at least $s'V + \overline{V} - \underline{C}y_{t-k}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.10).

Case 3: $y_t = 1$ and $y_{t+s} - y_{t+s+1} \leq 0$ for all $s \in \mathcal{S}$. Because $\mathcal{S} \subseteq [0, k-1]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, and $\underline{C} + V > \overline{V}$, the coefficients "$\underline{C} + (k-m)V - \overline{V}$" and "$\underline{C} + (k-s)V - \overline{V}$", for any $s \in \mathcal{S}$, on the right hand side of (4.10) are positive. If there exists some $i \in [0, m]_{\mathbb{Z}}$ such that $y_{t-i} - y_{t-i-1} = 1$, we have $y_{t-i} = 1$ and $y_{t-i-1} = 0$. Let $i^* = \min\{i \in [0, m]_{\mathbb{Z}} : y_{t-i} - y_{t-i-1} = 1\}$. Thus, $y_\tau = 1$ for all $\tau \in [t - i^*, t]_{\mathbb{Z}}$. Hence, the right hand side of inequality (4.10) is at least $V \sum_{i=1}^{m} y_{t-i} + \overline{V} - \underline{C}y_{t+k}$. By (2.2e), we have $\sum_{\tau=t-i^*-1}^{t}(x_\tau - x_{\tau-1}) \leq \sum_{\tau=t-i^*-1}^{t} Vy_{\tau-1} + \sum_{\tau=t-i^*-1}^{t} \overline{V}(1 - y_{\tau-1})$, which implies that $x_t - x_{t-i^*-1} \leq i^*V + \overline{V}$. Because $y_{t-i^*-1} = 0$, by (2.2d), we have $x_{t-i^*-1} = 0$. Thus, $x_t \leq i^*V + \overline{V}$. By (2.2c), we have $-x_{t+k} \leq -\underline{C}y_{t+k}$. Thus, $x_t - x_{t+k} \leq i^*V + \overline{V} - \underline{C}y_{t+k} \leq V \sum_{i=1}^{m} y_{t-i} + \overline{V} - \underline{C}y_{t+k}$ as $i^* \in [0, m]_{\mathbb{Z}}$. Now, we consider the case where there does not exist $i \in [0, m]_{\mathbb{Z}}$ such that $y_{t-i} - y_{t-i-1} = 1$. Thus, $y_\tau = 1$ for all $\tau \in [t-m-1, t]_{\mathbb{Z}}$. The right hand side of inequality (4.10) is then at least $\underline{C} + kV - \underline{C}y_{t+k}$. Let $t' = \min\{\tau \in [t, T-1]_{\mathbb{Z}} : y_\tau - y_{\tau+1} = 1\}$. Then, we have $y_{t'} = 1$, $y_{t'+1} = 0$, and

$y_\tau = 1$ for all $\tau \in [t, t']_{\mathbb{Z}}$. When $t' \geq t + k$ or t' does not exist, by (2.2f), we have $\sum_{\tau=t+1}^{t+k}(x_{\tau-1} - x_\tau) \leq \sum_{\tau=t+1}^{t+k} V y_\tau + \sum_{\tau=t+1}^{t+k} \overline{V}(1 - y_\tau)$, which implies that $x_t - x_{t+k} \leq kV = \underline{C} + kV - \underline{C}y_{t+k}$. When $t' < t + k$, by (2.2f), we have $\sum_{\tau=t+1}^{t'+1}(x_{\tau-1} - x_\tau) \leq \sum_{\tau=t+1}^{t'+1} V y_\tau + \sum_{\tau=t+1}^{t'+1} \overline{V}(1 - y_\tau)$, which implies that $x_t - x_{t'+1} \leq (t' - t)V + \overline{V}$. Because $y_{t'+1} = 0$, by (2.2d), we have $x_{t'+1} = 0$. Thus, $x_t \leq (t' - t)V + \overline{V}$. By (2.2c), we have $-x_{t+k} \leq -\underline{C}y_{t+k}$. Thus, $x_t - x_{t+k} \leq (t' - t)V + \overline{V} - \underline{C}y_{t+k} < \underline{C} + kV - \underline{C}y_{t+k}$ as $t' < t + k$ and $\underline{C} + V > \overline{V}$. Therefore, in this case, $(\mathbf{x}, \mathbf{y})$ satisfies (4.10).

To prove that inequalities (4.9) and (4.10) are facet-defining for $\text{conv}(\mathcal{P})$, it suffices to show that for each of these two inequalities, there exist $2T$ affinely independent points in $\text{conv}(\mathcal{P})$ that satisfy the inequality at equality. Let $\epsilon = \min\{\overline{V} - \underline{C}, \overline{C} - \underline{C} - kV\} > 0$.

We first show that inequality (4.9) is facet-defining for $\text{conv}(\mathcal{P})$ by creating $2T$ affinely independent points in $\text{conv}(\mathcal{P})$ that satisfy (4.9) at equality. Because $\mathbf{0} \in \text{conv}(\mathcal{P})$ and $\mathbf{0}$ satisfies (4.9) at equality, it suffices to create the remaining $2T - 1$ nonzero linearly independent points. We denote these $2T - 1$ points as $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ for $r \in [1, T]_{\mathbb{Z}} \setminus \{t - k\}$, and $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ for $r \in [1, T]_{\mathbb{Z}}$, and denote the qth component of $\bar{\mathbf{x}}^r$, $\bar{\mathbf{y}}^r$, $\hat{\mathbf{x}}^r$, and $\hat{\mathbf{y}}^r$ as $\bar{x}_q^r$, $\bar{y}_q^r$, $\hat{x}_q^r$, and $\hat{y}_q^r$, respectively. We divide these $2T - 1$ points into the following eight groups.

(A1) For each $r \in [1, t-1]_{\mathbb{Z}} \setminus \{t - k\}$, we create a point $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ as follows:

$$\bar{x}_q^r = \begin{cases} \underline{C}, & \text{for } q \in [1, t-1]_{\mathbb{Z}} \setminus \{r\}; \\ \underline{C} + \epsilon, & \text{for } q = r; \\ \overline{V}, & \text{for } q = t; \\ 0, & \text{for } q \in [t+1, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\bar{y}_q^r = \begin{cases} 1, & \text{for } q \in [1, t]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [t+1, T]_{\mathbb{Z}}. \end{cases}$$

It is easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (2.2a)–(2.2f). Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. Note that $\bar{x}_t^r = \overline{V}$, $\bar{x}_{t-k}^r = \underline{C}$, $\bar{y}_t^r = \bar{y}_{t-k}^r = 1$, $\bar{y}_{t+m+1}^r = 0$, $\sum_{i=1}^m \bar{y}_{t+i}^r = 0$, and $\bar{y}_{t-s}^r - \bar{y}_{t-s-1}^r = 0$ for all $s \in \mathcal{S}$. Hence, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (4.9) at equality.

(A2) We create the same point $(\bar{\mathbf{x}}^t, \bar{\mathbf{y}}^t)$ as in group (A3) in the proof of Proposition 9. Thus, $(\bar{\mathbf{x}}^t, \bar{\mathbf{y}}^t) \in \text{conv}(\mathcal{P})$. Note that $\bar{x}_t^t = \underline{C} + kV + \epsilon$, $\bar{x}_{t-k}^t = \underline{C} + \epsilon$, $\bar{y}_t^t = \bar{y}_{t-k}^t = \bar{y}_{t+m+1}^t = 1$, $\sum_{i=1}^m \bar{y}_{t+i}^t = m$, and $\bar{y}_{t-s}^t - \bar{y}_{t-s-1}^t = 0$ for all $s \in \mathcal{S}$. Hence, $(\bar{\mathbf{x}}^t, \bar{\mathbf{y}}^t)$ satisfies (4.9) at equality.

(A3) For each $r \in [t+1, T]_{\mathbb{Z}}$, we create a point $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ as follows:

$$\bar{x}_q^r = \begin{cases} \underline{C}, & \text{for } q \in [1, t-k-1]_{\mathbb{Z}}; \\ \underline{C} + (q - t + k)V, & \text{for } q \in [t-k, t]_{\mathbb{Z}}; \\ \underline{C} + kV, & \text{for } q \in [t+1, T]_{\mathbb{Z}} \setminus \{r\}; \\ \underline{C} + kV + \epsilon, & \text{for } q = r; \end{cases}$$

and $\bar{y}_q^r = 1$ for all $q \in [1, T]_{\mathbb{Z}}$. It is easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (2.2a)–(2.2d). Note that $\bar{x}_q^r - \bar{x}_{q-1}^r = 0$ when $q \in [2, t-k]_{\mathbb{Z}}$, $\bar{x}_q^r - \bar{x}_{q-1}^r = V$ when $q \in [t-k+1, t]_{\mathbb{Z}}$, and $-\epsilon \leq \bar{x}_q^r - \bar{x}_{q-1}^r \leq \epsilon$ when $q \in [t+1, T]_{\mathbb{Z}}$. Thus, $-V\bar{y}_q^r - \overline{V}(1 - \bar{y}_q^r) \leq \bar{x}_q^r - \bar{x}_{q-1}^r \leq V\bar{y}_{q-1}^r + \overline{V}(1 - \bar{y}_{q-1}^r)$ for all $q \in [2, T]_{\mathbb{Z}}$. Hence, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (2.2e) and (2.2f). Therefore, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. Note that $\bar{x}_t^r = \underline{C} + kV$, $\bar{x}_{t-k}^r = \underline{C}$, $\bar{y}_t^r = \bar{y}_{t-k}^r = \bar{y}_{t+m+1}^r = 1$, $\sum_{i=1}^m \bar{y}_{t+i}^r = m$, and $\bar{y}_{t-s}^r - \bar{y}_{t-s-1}^r = 0$ for all

$s \in \mathcal{S}$. Hence, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (4.9) at equality.

(A4) For each $r \in [1, t - s_{\max} - 2]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (A2) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. If $r \geq t - k$, then $\hat{x}^r_t = \hat{y}^r_t = 0$, $\hat{x}^r_{t-k} = \underline{C}$, $\hat{y}^r_{t-k} = 1$, $\sum_{i=1}^{m} \hat{y}^r_{t+i} = 0$, $\hat{y}^r_{t+m+1} = 0$, and $\hat{y}^t_{t-s} - \hat{y}^t_{t-s-1} = 0$ for all $s \in \mathcal{S}$. If $r < t - k$, then $\hat{x}^r_t = \hat{x}^r_{t-k} = \hat{y}^r_t = \hat{y}^r_{t-k} = 0$, $\sum_{i=1}^{m} \hat{y}^r_{t+i} = 0$, $\hat{y}^r_{t+m+1} = 0$, and $\hat{y}^t_{t-s} - \hat{y}^t_{t-s-1} = 0$ for all $s \in \mathcal{S}$. Hence, in both cases, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.9) at equality.

(A5) For each $r \in [t - s_{\max} - 1, t - 1]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (A3) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. To show that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.9) at equality, we first consider the case where $t - r - 1 \notin \mathcal{S}$. In this case, $\hat{x}^r_q = \hat{y}^r_q = 0$ for all $q \in [t, t + m + 1]_{\mathbb{Z}}$. Because $t - k \leq t - s_{\max} - 1 \leq r$, we have $\hat{x}^r_{t-k} = \underline{C}$, and $\hat{y}^r_{t-k} = 1$. Because $t - s - 1 \neq r$ for all $s \in \mathcal{S}$, we have $\hat{y}^r_{t-s} - \hat{y}^r_{t-s-1} = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.9) at equality. Next, we consider the case where $t - r - 1 \in \mathcal{S}$. In this case, $\hat{x}^r_t = \overline{V} + (t - r - 1)V$ and $\hat{y}^r_q = 1$ for all $q \in [t, t + m + 1]_{\mathbb{Z}}$. Because $t - k \leq t - s_{\max} - 1 \leq r$, we have $\hat{x}^r_{t-k} = \hat{y}^r_{t-k} = 0$. In addition, $\hat{y}^r_{t-s} - \hat{y}^r_{t-s-1} = 1$ when $s = t - r - 1$, and $\hat{y}^r_{t-s} - \hat{y}^r_{t-s-1} = 0$ when $s \neq t - r - 1$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.9) at equality.

(A6) For each $r \in [t, t+m]_{\mathbb{Z}}$, we create a point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as follows:

$$\hat{x}_q^r = \begin{cases} \underline{C}, & \text{for } q \in [1, 2t-r-1]_{\mathbb{Z}}; \\ \overline{V} + (q+r-2t)V, & \text{for } q \in [2t-r, t-1]_{\mathbb{Z}}; \\ \overline{V} + (r-q)V, & \text{for } q \in [t, r]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [r+1, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^r = \begin{cases} 1, & \text{for } q \in [1, r]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [r+1, T]_{\mathbb{Z}}. \end{cases}$$

It is easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (2.2a)–(2.2d). Note that $-V \leq \hat{x}_q^r - \hat{x}_{q-1}^r \leq V$ when $q \in [2, r]_{\mathbb{Z}}$, $\hat{x}_q^r - \hat{x}_{q-1}^r = -\overline{V}$ when $q = r+1$, and $\hat{x}_q^r - \hat{x}_{q-1}^r = 0$ when $q \in [r+2, T]_{\mathbb{Z}}$. Thus, $-V\hat{y}_q^r - \overline{V}(1-\hat{y}_q^r) \leq \hat{x}_q^r - \hat{x}_{q-1}^r \leq V\hat{y}_{q-1}^r + \overline{V}(1-\hat{y}_{q-1}^r)$ for all $q \in [2, T]_{\mathbb{Z}}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (2.2e)–(2.2f). Therefore, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. Note that $\hat{x}_t^r = \overline{V} + (r-t)V$ and $\hat{y}_t^r = 1$. Note also that $\hat{y}_{t+m+1}^r = 0$ and $V\sum_{i=1}^{m} y_{t+i}^r = (r-t)V$. Because $t-k \leq t-m-1 \leq 2t-r-1$, we have $\hat{x}_{t-k}^r = \underline{C}$ and $\hat{y}_{t-k}^r = 1$. For any $s \in \mathcal{S}$, because $t-s \leq t \leq r$, we have $\hat{y}_{t-s}^r - \hat{y}_{t-s-1}^r = 0$. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.9) at equality.

(A7) We create a point $(\hat{\mathbf{x}}^{t+m+1}, \hat{\mathbf{y}}^{t+m+1})$ as follows:

$$\hat{\mathbf{x}}_q^{t+m+1} = \begin{cases} \underline{C}, & \text{for } q \in [1, t-k-1]_{\mathbb{Z}}; \\ \underline{C} + (q-t+k)V, & \text{for } q \in [t-k, t]_{\mathbb{Z}}; \\ \underline{C} + kV, & \text{for } q \in [t+1, T]_{\mathbb{Z}}; \end{cases}$$

and $\hat{y}_q^{t+m+1} = 1$ for all $q \in [1, T]_{\mathbb{Z}}$. It is easy to verify that $(\hat{\mathbf{x}}^{t+m+1}, \hat{\mathbf{y}}^{t+m+1})$

satisfies (2.2a)–(2.2d). Note that $\hat{x}_q^{t+m+1} - \hat{x}_{q-1}^{t+m+1} = 0$ when $q \in [2, t-k]_{\mathbb{Z}}$, $\hat{x}_q^{t+m+1} - \hat{x}_{q-1}^{t+m+1} = V$ when $q \in [t-k+1, t]_{\mathbb{Z}}$, and $\hat{x}_q^{t+m+1} - \hat{x}_{q-1}^{t+m+1} = 0$ when $q \in [t+1, T]_{\mathbb{Z}}$. Thus, $-V\hat{y}_q^{t+m+1} - \overline{V}(1-\hat{y}_q^{t+m+1}) \leq \hat{x}_q^{t+m+1} - \hat{x}_{q-1}^{t+m+1} \leq V\hat{y}_{q-1}^{t+m+1} + \overline{V}(1-\hat{y}_{q-1}^{t+m+1})$ for all $q \in [2, T]_{\mathbb{Z}}$. Hence, $(\hat{\mathbf{x}}^{t+m+1}, \hat{\mathbf{y}}^{t+m+1})$ satisfies (2.2e) and (2.2f). Therefore, $(\hat{\mathbf{x}}^{t+m+1}, \hat{\mathbf{y}}^{t+m+1}) \in \text{conv}(\mathcal{P})$. Note that $\hat{x}_t^{t+m+1} = \underline{C} + kV$, $\hat{x}_{t-k}^{t+m+1} = \underline{C}$, $\hat{y}_{t+m+1}^{t+m+1} = \hat{y}_t^{t+m+1} = \hat{y}_{t-k}^{t+m+1} = 1$, $\sum_{i=1}^{m} y_{t+i}^{t+m+1} = m$, and $\hat{y}_{t-s}^{t+m+1} - \hat{y}_{t-s-1}^{t+m+1} = 0$ for all $s \in \mathcal{S}$, Thus, $(\hat{\mathbf{x}}^{t+m+1}, \hat{\mathbf{y}}^{t+m+1})$ satisfies (4.9) at equality.

(A8) For each $r \in [t+m+2, T]_{\mathbb{Z}}$, we create a point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as follows:

$$\hat{x}_q^r = \begin{cases} 0, & \text{for } q \in [1, r-1]_{\mathbb{Z}}; \\ \underline{C}, & \text{for } q \in [r, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^r = \begin{cases} 0, & \text{for } q \in [1, r-1]_{\mathbb{Z}}; \\ 1, & \text{for } q \in [r, T]_{\mathbb{Z}}. \end{cases}$$

It is easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (2.2a)–(2.2f). Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. Note that $\hat{x}_t^r = \hat{x}_{t-k}^r = 0$, $\hat{y}_t^r = \hat{y}_{t-k}^r = \hat{y}_{t+m+1}^r = 0$, $\sum_{i=1}^{m} y_{t+i}^r = 0$, and $\hat{y}_{t-s}^r - \hat{y}_{t-s-1}^r = 0$ for all $s \in \mathcal{S}$, Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.9) at equality.

Table A.17 shows a matrix with $2T-1$ rows, where each row represents a point created by this process. This matrix can be transformed into the matrix in Table A.18 via the following Gaussian elimination process:

(i) For each $r \in [1, t-1]_{\mathbb{Z}} \setminus \{t-k\}$, the point with index r in group (B1), denoted $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r) = (\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) - (\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$.

Here, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ is the point with index r in group (A1), and $(\hat{\mathbf{x}}^t, \hat{\mathbf{y}}^t)$ is the point with index t in group (A6).

(ii) The point in group (B2), denoted $(\underline{\bar{\mathbf{x}}}^t, \underline{\bar{\mathbf{y}}}^t)$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^t, \underline{\bar{\mathbf{y}}}^t) = (\bar{\mathbf{x}}^t, \bar{\mathbf{y}}^t) - (\hat{\mathbf{x}}^{t+m+1}, \hat{\mathbf{y}}^{t+m+1})$. Here, $(\bar{\mathbf{x}}^t, \bar{\mathbf{y}}^t)$ is the point in group (A2), and $(\hat{\mathbf{x}}^{t+m+1}, \hat{\mathbf{y}}^{t+m+1})$ is the point in group (A7).

Table A.17: A matrix with the rows representing $2T-1$ linearly independent points in $\mathrm{conv}(\mathcal{P})$ satisfying inequality (4.9) at equality.

Group	Point	Index r	**x**: 1	⋯	$t-k-1$	$t-k$	$t-k+1$	⋯	$t-1$	t	$t+1$	⋯	$t+m$	$t+m+1$	$t+m+2$	⋯	T
(A1)		1	$\underline{C}+\epsilon$	⋯	$\underline{C}$	$\underline{C}$	$\underline{C}$	⋯	$\underline{C}$	$\overline{V}$	0	⋯	0	0	0	⋯	0
		⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮
		$t-k-1$	$\underline{C}$	⋯	$\underline{C}+\epsilon$	$\underline{C}$	$\underline{C}$	⋯	$\underline{C}$	$\overline{V}$	0	⋯	0	0	0	⋯	0
		$t-k+1$	$\underline{C}$	⋯	$\underline{C}$	$\underline{C}$	$\underline{C}+\epsilon$	⋯	$\underline{C}$	$\overline{V}$	0	⋯	0	0	0	⋯	0
	$(\bar{\mathbf{x}}^r,\bar{\mathbf{y}}^r)$	⋮	⋮		⋮	⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮	⋮		⋮
		$t-1$	$\underline{C}$	⋯	$\underline{C}$	$\underline{C}$	$\underline{C}$	⋯	$\underline{C}+\epsilon$	$\overline{V}$	0	⋯	0	0	0	⋯	0
(A2)		t	$\underline{C}$	⋯	$\underline{C}$	$\underline{C}+\epsilon$	$\underline{C}+V+\epsilon$	⋯	$\underline{C}+(k-1)V+\epsilon$	$\underline{C}+kV+\epsilon$	$\underline{C}+kV$	⋯	$\underline{C}+kV$	$\underline{C}+kV$	$\underline{C}+kV$	⋯	$\underline{C}+kV$
(A3)		$t+1$	$\underline{C}$	⋯	$\underline{C}$	$\underline{C}$	$\underline{C}+V$	⋯	$\underline{C}+(k-1)V$	$\underline{C}+kV$	$\underline{C}+kV+\epsilon$	⋯	$\underline{C}+kV$	$\underline{C}+kV$	$\underline{C}+kV$	⋯	$\underline{C}+kV$
		⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮	⋱	⋮	⋮	⋮		⋮
		$t+m$	$\underline{C}$	⋯	$\underline{C}$	$\underline{C}$	$\underline{C}+V$	⋯	$\underline{C}+(k-1)V$	$\underline{C}+kV$	$\underline{C}+kV$	⋯	$\underline{C}+kV+\epsilon$	$\underline{C}+kV$	$\underline{C}+kV$	⋯	$\underline{C}+kV$
		$t+m+1$	$\underline{C}$	⋯	$\underline{C}$	$\underline{C}$	$\underline{C}+V$	⋯	$\underline{C}+(k-1)V$	$\underline{C}+kV$	$\underline{C}+kV$	⋯	$\underline{C}+kV$	$\underline{C}+kV+\epsilon$	$\underline{C}+kV$	⋯	$\underline{C}+kV$
		$t+m+2$	$\underline{C}$	⋯	$\underline{C}$	$\underline{C}$	$\underline{C}+V$	⋯	$\underline{C}+(k-1)V$	$\underline{C}+kV$	$\underline{C}+kV$	⋯	$\underline{C}+kV$	$\underline{C}+kV$	$\underline{C}+kV+\epsilon$	⋯	$\underline{C}+kV$
		⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮	⋱	⋮
		T	$\underline{C}$	⋯	$\underline{C}$	$\underline{C}$	$\underline{C}+V$	⋯	$\underline{C}+(k-1)V$	$\underline{C}+kV$	$\underline{C}+kV$	⋯	$\underline{C}+kV$	$\underline{C}+kV$	$\underline{C}+kV$	⋯	$\underline{C}+kV+\epsilon$
(A4)		1	$\underline{C}$	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0
		⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮
		$t-k-1$	$\underline{C}$	⋯	$\underline{C}$	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0
		$t-k$	$\underline{C}$	⋯	$\underline{C}$	$\underline{C}$	0	⋯	0	0	0	⋯	0	0	0	⋯	0
		$t-k+1$	$\underline{C}$	⋯	$\underline{C}$	$\underline{C}$	$\underline{C}$	⋯	0	0	0	⋯	0	0	0	⋯	0
		⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮
	$(\hat{\mathbf{x}}^r,\hat{\mathbf{y}}^r)$	$t-s_{\max}-2$	$\underline{C}$	⋯	$\underline{C}$	$\underline{C}$	$\underline{C}$	⋯	0	0	0	⋯	0	0	0	⋯	0
(A5)		$t-s_{\max}-1$															
		⋮	(See Note A.17-1)														
		$t-1$															
(A6)		t	$\underline{C}$	⋯	$\underline{C}$	$\underline{C}$	$\underline{C}$	⋯	$\underline{C}$	$\overline{V}$	0	⋯	0	0	0	⋯	0
		$t+1$	$\underline{C}$	⋯	$\underline{C}$	$\underline{C}$	$\underline{C}$	⋯	$\overline{V}$	$\overline{V}+V$	$\overline{V}$	⋯	0	0	0	⋯	0
		⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮	⋱	⋮	⋮	⋮		⋮
		$t+m$	$\underline{C}$	⋯	$\underline{C}$	$\underline{C}$	(See Note A.17-2)	⋯	$\overline{V}+(m-1)V$	$\overline{V}+mV$	$\overline{V}+(m-1)V$	⋯	$\overline{V}$	0	0	⋯	0
(A7)		$t+m+1$	$\underline{C}$	⋯	$\underline{C}$	$\underline{C}$	$\underline{C}+V$	⋯	$\underline{C}+(k-1)V$	$\underline{C}+kV$	$\underline{C}+kV$	⋯	$\underline{C}+kV$	$\underline{C}+kV$	$\underline{C}+kV$	⋯	$\underline{C}+kV$
(A8)		$t+m+2$	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	$\underline{C}$	⋯	$\underline{C}$
		⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮	⋱	⋮
		T	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	$\underline{C}$

Group	Point	Index r	**y**: 1	⋯	$t-k-1$	$t-k$	$t-k+1$	⋯	$t-1$	t	$t+1$	⋯	$t+m$	$t+m+1$	$t+m+2$	⋯	T
(A1)		1	1	⋯	1	1	1	⋯	1	1	0	⋯	0	0	0	⋯	0
		⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮
		$t-k-1$	1	⋯	1	1	1	⋯	1	1	0	⋯	0	0	0	⋯	0
		$t-k+1$	1	⋯	1	1	1	⋯	1	1	0	⋯	0	0	0	⋯	0
	$(\bar{\mathbf{x}}^r,\bar{\mathbf{y}}^r)$	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮
		$t-1$	1	⋯	1	1	1	⋯	1	1	0	⋯	0	0	0	⋯	0
(A2)		t	1	⋯	1	1	1	⋯	1	1	1	⋯	1	1	1	⋯	1
(A3)		$t+1$	1	⋯	1	1	1	⋯	1	1	1	⋯	1	1	1	⋯	1
		⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮
		$t+m$	1	⋯	1	1	1	⋯	1	1	1	⋯	1	1	1	⋯	1
		$t+m+1$	1	⋯	1	1	1	⋯	1	1	1	⋯	1	1	1	⋯	1
		$t+m+2$	1	⋯	1	1	1	⋯	1	1	1	⋯	1	1	1	⋯	1
		⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮
		T	1	⋯	1	1	1	⋯	1	1	1	⋯	1	1	1	⋯	1
(A4)		1	1	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0
		⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮
		$t-k-1$	1	⋯	1	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0
		$t-k$	1	⋯	1	1	0	⋯	0	0	0	⋯	0	0	0	⋯	0
		$t-k+1$	1	⋯	1	1	1	⋯	0	0	0	⋯	0	0	0	⋯	0
		⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮
	$(\hat{\mathbf{x}}^r,\hat{\mathbf{y}}^r)$	$t-s_{\max}-2$	1	⋯	1	1	1	⋯	0	0	0	⋯	0	0	0	⋯	0
(A5)		$t-s_{\max}-1$															
		⋮	(See Note A.17-1)														
		$t-1$															
(A6)		t	1	⋯	1	1	1	⋯	1	1	0	⋯	0	0	0	⋯	0
		$t+1$	1	⋯	1	1	1	⋯	1	1	1	⋯	0	0	0	⋯	0
		⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮	⋱	⋮	⋮	⋮		⋮
		$t+m$	1	⋯	1	1	1	⋯	1	1	1	⋯	1	0	0	⋯	0
(A7)		$t+m+1$	1	⋯	1	1	1	⋯	1	1	1	⋯	1	1	1	⋯	1
(A8)		$t+m+2$	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	1	⋯	1
		⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮	⋱	⋮
		T	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	1

Note A.17-1: For $r \in [t-s_{\max}-1, t-1]_{\mathbb{Z}}$, the $\mathbf{x}$ and $\mathbf{y}$ vectors in group (A5) are given as follows: $\hat{\mathbf{x}}^r = (\underbrace{\underline{C},\dots,\underline{C}}_{r \text{ terms}},\underbrace{0,\dots,0}_{T-r \text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{1,\dots,1}_{r \text{ terms}},\underbrace{0,\dots,0}_{T-r \text{ terms}})$ if $t-r-1 \notin \mathcal{S}$;
$\hat{\mathbf{x}}^r = (\underbrace{0,\dots,0}_{r \text{ terms}},\underbrace{\overline{V},\overline{V}+V,\overline{V}+2V,\dots,\overline{V}(r-t-1)V}_{t-r \text{ terms}},\underbrace{\overline{V}+(t-r-1)V,\dots,\overline{V}+(t-r-1)V}_{T-t \text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{0,\dots,0}_{r \text{ terms}},\underbrace{1,\dots,1}_{T-r \text{ terms}})$ if $t-r-1 \in \mathcal{S}$.

Note A.17-2: In group (A6), $\hat{x}^{t+m}_{t-k+1} = \underline{C}$ if $m < k-1$, and $\hat{x}^{t+m}_{t-k+1} = \overline{V}$ if $m = k-1$.

Table A.18: Lower triangular matrix obtained from Table A.17 via Gaussian elimination.

Group	Point	Index r	$\mathbf{x}$															$\mathbf{y}$												
			1	⋯	$t-k-1$	$t-k$	$t-k+1$	⋯	$t-1$	t	$t+1$	⋯	$t+m$	$t+m+1$	$t+m+2$	⋯	T	1	⋯	$t-s_{\max}-2$	$t-s_{\max}-1$	⋯	t	$t+1$	⋯	$t+m$	$t+m+1$	$t+m+2$	⋯	T
(B1)	$(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$	1	ϵ	0	⋯	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	⋯	0	0	⋯	0	0	⋯	0	0	0	⋯	0
		⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮		⋮	⋮		⋮	⋮		⋮	⋮	⋮		⋮
		$t-k-1$	0	⋯	ϵ	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	⋯	0	0	⋯	0	0	⋯	0	0	0	⋯	0
		$t-k+1$	0	⋯	0	0	ϵ	⋯	0	0	0	⋯	0	0	0	⋯	0	0	⋯	0	0	⋯	0	0	⋯	0	0	0	⋯	0
		⋮	⋮	⋮		⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮		⋮	⋮		⋮	⋮		⋮	⋮	⋮		⋮
		$t-1$	0	⋯	0	0	0	⋯	ϵ	0	0	⋯	0	0	0	⋯	0	0	⋯	0	0	⋯	0	0	⋯	0	0	0	⋯	0
(B2)		t	0	0	⋯	ϵ	ϵ	⋯	ϵ	ϵ	0	⋯	0	0	0	⋯	0	0	⋯	0	0	⋯	0	0	⋯	0	0	0	⋯	0
(B3)		$t+1$	0	0	⋯	0	0	⋯	0	0	ϵ	⋯	0	0	0	⋯	0	0	⋯	0	0	⋯	0	0	⋯	0	0	0	⋯	0
		⋮	⋮	⋮		⋮	⋮		⋮	⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮		⋮	⋮		⋮	⋮		⋮	⋮	⋮		⋮
		$t+m$	0	0	⋯	0	0	⋯	0	0	0	⋯	ϵ	0	0	⋯	0	0	⋯	0	0	⋯	0	0	⋯	0	0	0	⋯	0
		$t+m+1$	0	0	⋯	0	0	⋯	0	0	0	⋯	0	ϵ	0	⋯	0	0	⋯	0	0	⋯	0	0	⋯	0	0	0	⋯	0
		$t+m+2$	0	0	⋯	0	0	⋯	0	0	0	⋯	0	0	ϵ	⋯	0	0	⋯	0	0	⋯	0	0	⋯	0	0	0	⋯	0
		⋮	⋮	⋮		⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮	⋱	⋮	⋮		⋮	⋮		⋮	⋮		⋮	⋮	⋮		⋮
		T	0	0	⋯	0	0	⋯	0	0	0	⋯	0	0	0	⋯	ϵ	0	⋯	0	0	⋯	0	0	⋯	0	0	0	⋯	0
(B4)	$(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$	1	(Omitted)															1	⋯	0	0	⋯	0	0	⋯	0	0	0	⋯	0
		⋮																⋮	⋱	⋮	⋮		⋮	⋮		⋮	⋮	⋮		⋮
		$t-s_{\max}-2$																1	⋯	1	0	⋯	0	0	⋯	0	0	0	⋯	0
(B5)		$t-s_{\max}-1$	(Omitted)															(See Note A.18-1)												
		⋮																												
		$t-1$																												
(B6)		t	(Omitted)															1	⋯	1	1	⋯	1	0	⋯	0	0	0	⋯	0
		$t+1$																1	⋯	1	1	⋯	1	1	⋯	0	0	0	⋯	0
		⋮																⋮		⋮	⋮		⋮	⋮	⋱	⋮	⋮	⋮		⋮
		$t+m$																1	⋯	1	1	⋯	1	1	⋯	1	0	0	⋯	0
(B7)		$t+m+1$	(Omitted)															1	⋯	1	1	⋯	1	1	⋯	1	1	0	⋯	0
(B8)		$t+m+2$	(Omitted)															0	⋯	0	0	⋯	0	0	⋯	0	0	1	⋯	0
		⋮																⋮		⋮	⋮		⋮	⋮		⋮	⋮	⋮	⋱	⋮
		T																0	⋯	0	0	⋯	0	0	⋯	0	0	0	⋯	1

Note A.18-1: For $r \in [t-s_{\max}-1, t-1]_{\mathbb{Z}}$, the $\mathbf{y}$ vector in group (B5) is given as follows: $\underline{\hat{\mathbf{y}}}^r = (\underbrace{1,\ldots,1}_{r \text{ terms}},\underbrace{0,\ldots,0}_{T-r \text{ terms}})$ if $t-r-1 \notin \mathcal{S}$; $\underline{\hat{\mathbf{y}}}^r = (\underbrace{-1,\ldots,-1}_{r \text{ terms}},\underbrace{0,\ldots,0}_{T-r \text{ terms}})$ if $t-r-1 \in \mathcal{S}$.

(iii) For each $r \in [t+1, T]_{\mathbb{Z}}$, the point with index r in group (B3), denoted $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r) = (\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{t+m+1}, \hat{\mathbf{y}}^{t+m+1})$. Here, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ is the point with index r in group (A3), and $(\hat{\mathbf{x}}^{t+m+1}, \hat{\mathbf{y}}^{t+m+1})$ is the point in group (A7).

(iv) For each $r \in [1, t-s_{\max}-2]_{\mathbb{Z}}$, the point with index r in group (B4), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (A4).

(v) For each $r \in [t-s_{\max}-1, t-1]_{\mathbb{Z}}$, the point with index r in group (B5), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $t-r-1 \notin \mathcal{S}$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{t+m+1}, \hat{\mathbf{y}}^{t+m+1})$ if $t-r-1 \in \mathcal{S}$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (A5), and $(\hat{\mathbf{x}}^{t+m+1}, \hat{\mathbf{y}}^{t+m+1})$ is the point in group (A7).

(vi) For each $r \in [t, t+m]_{\mathbb{Z}}$, the point with index r in group (B6), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (A6).

(vii) The point in group (B7), denoted $(\underline{\hat{\mathbf{x}}}^{t+m+1}, \underline{\hat{\mathbf{y}}}^{t+m+1})$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^{t+m+1}, \underline{\hat{\mathbf{y}}}^{t+m+1}) = (\hat{\mathbf{x}}^{t+m+1}, \hat{\mathbf{y}}^{t+m+1}) - (\hat{\mathbf{x}}^{t+m+2}, \hat{\mathbf{y}}^{t+m+2})$. Here, $(\hat{\mathbf{x}}^{t+m+1}, \hat{\mathbf{y}}^{t+m+1})$ is the point in group (A7), and $(\hat{\mathbf{x}}^{t+m+2}, \hat{\mathbf{y}}^{t+m+2})$ is the point with index $t+m+2$ in group (A8).

(viii) For each $r \in [t+m+2, T]_{\mathbb{Z}}$, the point with index r in group (B8), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ if $r \neq T$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $r = T$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ and

$(\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ are the points with indices r and $r+1$, respectively, in group (A8).

The matrix shown in Table A.18 is lower triangular; that is, the position of the last nonzero component of a row of the matrix is greater than the position of the last nonzero component of the previous row. This implies that the $2T-1$ points in groups (A1)–(A8) are linearly independent. Therefore, inequality (4.9) is facet-defining for $\text{conv}(\mathcal{P})$.

Next, we show that inequality (4.10) is facet-defining for $\text{conv}(\mathcal{P})$ by creating $2T$ affinely independent points in $\text{conv}(\mathcal{P})$ that satisfy (4.10) at equality. Because $\mathbf{0} \in \text{conv}(\mathcal{P})$ and $\mathbf{0}$ satisfies (4.10) at equality, it suffices to create the remaining $2T-1$ nonzero linearly independent points. We denote these $2T-1$ points as $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ for $r \in [1,T]_{\mathbb{Z}} \setminus \{t\}$, and $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ for all $r \in [1,T]_{\mathbb{Z}}$, and denote the qth component of $\bar{\mathbf{x}}^r$, $\bar{\mathbf{y}}^r$, $\hat{\mathbf{x}}^r$, and $\hat{\mathbf{y}}^r$ as $\bar{x}^r_q$, $\bar{y}^r_q$, $\hat{x}^r_q$, and $\hat{y}^r_q$, respectively. We divide these $2T-1$ points into the following nine groups:

(C1) For each $r \in [1, t-1]_{\mathbb{Z}}$, we create a point $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ as follows:

$$\bar{x}^r_q = \begin{cases} \underline{C} + kV, & \text{for } q \in [1, t-1]_{\mathbb{Z}} \setminus \{r\}; \\ \underline{C} + kV + \epsilon, & \text{for } q = r; \\ \underline{C} + (t+k-q)V, & \text{for } q \in [t, t+k]_{\mathbb{Z}}; \\ \underline{C}, & \text{for } q \in [t+k+1, T]_{\mathbb{Z}}; \end{cases}$$

and $\bar{y}^r_q = 1$ for all $q \in [1,T]_{\mathbb{Z}}$. It is easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (2.2a)–(2.2d). Note that $-\epsilon \leq \bar{x}^r_q - \bar{x}^r_{q-1} \leq \epsilon$ when $q \in [2,t]_{\mathbb{Z}}$, $\bar{x}^r_q - \bar{x}^r_{q-1} = -V$ when $q \in [t+1, t+k]_{\mathbb{Z}}$, and $\bar{x}^r_q - \bar{x}^r_{q-1} = 0$ when $q \in$

$[t+k+1,T]_{\mathbb{Z}}$. Thus, $-V\bar{y}^r_q - \overline{V}(1-\bar{y}^r_q) \leq \bar{x}^r_q - \bar{x}^r_{q-1} \leq V\bar{y}^r_{q-1} + \overline{V}(1-\bar{y}^r_{q-1})$ for all $q \in [2,T]_{\mathbb{Z}}$. Hence, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (2.2e)–(2.2f). Therefore, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. Note that $\bar{x}^r_t = \underline{C} + kV$, $\bar{x}^r_{t+k} = \underline{C}$, $\bar{y}^r_t = \bar{y}^r_{t+k} = \bar{y}^r_{t-m-1} = 1$, $\sum_{i=1}^m \bar{y}^r_{t-i} = m$, and $\bar{y}^r_{t+s} - \bar{y}^r_{t+s+1} = 0$ for all $s \in \mathcal{S}$. Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (4.10) at equality.

(C2) For each $r \in [t+1, t+k-1]_{\mathbb{Z}}$, we create a point $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ as follows:

$$\bar{x}^r_q = \begin{cases} 0, & \text{for } q \in [1, t-1]_{\mathbb{Z}}; \\ \overline{V}, & \text{for } q = t; \\ \underline{C} + \epsilon, & \text{for } q = r; \\ \underline{C}, & \text{for } q \in [t+1, T]_{\mathbb{Z}} \setminus \{r\}; \end{cases}$$

and

$$\bar{y}^r_q = \begin{cases} 0, & \text{for } q \in [q, t-1]_{\mathbb{Z}}; \\ 1, & \text{for } q \in [t, T]_{\mathbb{Z}}. \end{cases}$$

It is easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (2.2a)–(2.2f). Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. Note that $\bar{x}^r_t = \overline{V}$, $\bar{x}^r_{t+k} = \underline{C}$, $\bar{y}^r_t = \bar{y}^r_{t+k} = 1$, $\bar{y}^r_{t-m-1} = 0$, $\sum_{i=1}^m \bar{y}^r_{t-i} = 0$, and $\bar{y}^r_{t+s} - \bar{y}^r_{t+s+1} = 0$ for all $s \in \mathcal{S}$. Hence, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (4.10) at equality.

(C3) We create the same point $(\bar{\mathbf{x}}^{t+k}, \bar{\mathbf{y}}^{t+k})$ as in group (C3) in the proof of Proposition 9. Thus, $(\bar{\mathbf{x}}^{t+k}, \bar{\mathbf{y}}^{t+k}) \in \text{conv}(\mathcal{P})$. Note that $\bar{x}^{t+k}_t = \underline{C} + kV + \epsilon$, $\bar{x}^{t+k}_{t+k} = \underline{C} + \epsilon$, $\bar{y}^{t+k}_t = \bar{y}^{t+k}_{t+k} = \bar{y}^{t+k}_{t-m-1} = 1$, $\sum_{i=1}^m \bar{y}^{t+k}_{t-i} = m$, and $\bar{y}^{t+k}_{t+s} - \bar{y}^{t+k}_{t+s+1} = 0$ for all $s \in \mathcal{S}$. Hence, $(\bar{\mathbf{x}}^{t+k}, \bar{\mathbf{y}}^{t+k})$ satisfies (4.10) at equality.

(C4) For each $r \in [t+k+1, T]_{\mathbb{Z}}$, we create a point $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ as follows:

$$\bar{x}_q^r = \begin{cases} 0, & \text{for } q \in [1, t-1]_{\mathbb{Z}}; \\ \overline{V}, & \text{for } q = t; \\ \underline{C} + \epsilon, & \text{for } q = r; \\ \underline{C}, & \text{for } q \in [t+1, T]_{\mathbb{Z}} \setminus \{r\}; \end{cases}$$

and

$$\bar{y}_q^r = \begin{cases} 0, & \text{for } q \in [q, t-1]_{\mathbb{Z}}; \\ 1, & \text{for } q \in [t, T]_{\mathbb{Z}}. \end{cases}$$

It is easy to verify that $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (2.2a)–(2.2f). Thus, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. Note that $\bar{x}_t^r = \overline{V}$, $\bar{x}_{t+k}^r = \underline{C}$, $\bar{y}_t^r = \bar{y}_{t+k}^r = 1$, $\bar{y}_{t-m-1}^r = 0$, $\sum_{i=1}^m \bar{y}_{t-i}^r = 0$, and $\bar{y}_{t+s}^r - \bar{y}_{t+s+1}^r = 0$ for all $s \in \mathcal{S}$. Hence, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ satisfies (4.10) at equality.

(C5) For each $r \in [1, t-m-2]_{\mathbb{Z}}$, we create a point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as follows:

$$\hat{x}_q^r = \begin{cases} \underline{C}, & \text{for } q \in [1, r]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [r+1, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}_q^r = \begin{cases} 1, & \text{for } q \in [1, r]_{\mathbb{Z}}; \\ 0, & \text{for } q \in [r+1, T]_{\mathbb{Z}}. \end{cases}$$

It is easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (2.2a)–(2.2f). Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. Note that $\hat{x}_t^r = \hat{x}_{t+k}^r = \hat{y}_t^r = \hat{y}_{t+k}^r = \hat{y}_{t-m-1}^r = 0$, $\sum_{i=1}^m \hat{y}_{t-i}^r = 0$, and $\hat{y}_{t+s}^r - \hat{y}_{t+s+1}^r = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.10) at equality.

(C6) For each $r \in [t-m-1, t-1]_{\mathbb{Z}}$, we create a point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as follows:

$$\hat{x}^r_q = \begin{cases} 0, & \text{for } q \in [1, r]_{\mathbb{Z}}; \\ \overline{V} + (q-r-1)V, & \text{for } q \in [r+1, t]_{\mathbb{Z}}; \\ \overline{V} + (2t-r-q-1)V, & \text{for } q \in [t+1, 2t-r-1]_{\mathbb{Z}}; \\ \underline{C}, & \text{for } q \in [2t-r, T]_{\mathbb{Z}}; \end{cases}$$

and

$$\hat{y}^r_q = \begin{cases} 0, & \text{for } q \in [1, r]_{\mathbb{Z}}; \\ 1, & \text{for } q \in [r+1, T]_{\mathbb{Z}}. \end{cases}$$

It is easy to verify that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (2.2a)–(2.2d). Note that $\hat{x}^r_q - \hat{x}^r_{q-1} = 0$ when $q \in [2, r]_{\mathbb{Z}}$, $\hat{x}^r_q - \hat{x}^r_{q-1} = \overline{V}$ when $q = r+1$, and $-V \leq \hat{x}^r_q - \hat{x}^r_{q-1} \leq V$ when $q \in [r+2, T]_{\mathbb{Z}}$. Thus, $-V\hat{y}^r_q - \overline{V}(1-\hat{y}^r_q) \leq \hat{x}^r_q - \hat{x}^r_{q-1} \leq V\hat{y}^r_{q-1} + \overline{V}(1-\hat{y}^r_{q-1})$ for all $q \in [2, T]_{\mathbb{Z}}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (2.2e)–(2.2f). Therefore, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. Note that $\hat{x}^r_t = \overline{V} + (t-r-1)V$, $\hat{y}^r_t = 1$, $\hat{x}^r_{t-m-1} = \hat{y}^r_{t-m-1} = 0$, $\sum_{i=1}^m \hat{y}^r_{t-i} = t-r-1$, and $\hat{y}^r_{t+s} - \hat{y}^r_{t+s+1} = 0$ for all $s \in \mathcal{S}$. Because $t+k > t+m \geq 2t-r-1$, we have $\hat{x}^r_{t+k} = \underline{C}$ and $\hat{y}^r_{t+k} = 1$. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.10) at equality.

(C7) For each $r \in [t, t+s_{\max}]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (C3) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. To show that $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.10) at equality, we first consider the case where $r - t \notin \mathcal{S}$. In this case, $\hat{x}^r_t = \hat{y}^r_t = \hat{y}^r_{t-m-1} = 0$, $\hat{x}^r_{t+k} = \underline{C}$, $\hat{y}^r_{t+k} = 1$, and $\sum_{i=1}^m \hat{y}^r_{t-i} = 0$. Because $t+s \neq r$ for all $s \in \mathcal{S}$, we have $\hat{y}^r_{t+s} - \hat{y}^r_{t+s+1} = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.10) at equality. Next, we consider the case where $r - t \in \mathcal{S}$. In this case, $\hat{x}_t^r = \overline{V} + (r-t)V$, $\hat{y}_t^r = 1$, $\hat{x}_{t+k}^r = \hat{y}_{t+k}^r = 0$, $\hat{y}_{t-m-1}^r = 1$, and $\sum_{i=1}^m \hat{y}_{t-i}^r = m$. In addition, $\hat{y}_{t+s}^r - \hat{y}_{t+s+1}^r = 1$ when $s = r - t$, and $\hat{y}_{t+s}^r - \hat{y}_{t+s+1}^r = 0$ when $s \neq r - t$. Hence, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.10) at equality.

(C8) We create the same point $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ as in group (C7) in the proof of Proposition 9. Thus, $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1}) \in \text{conv}(\mathcal{P})$. Note that $\hat{x}_t^{t+s_{\max}+1} = \underline{C} + kV$, $\hat{x}_{t+k}^{t+s_{\max}+1} = \underline{C}$, $\hat{y}_t^{t+s_{\max}+1} = \hat{y}_{t+k}^{t+s_{\max}+1} = \hat{y}_{t-m-1}^{t+s_{\max}+1} = 1$, $\sum_{i=1}^m \hat{y}_{t-i}^{t+s_{\max}+1} = m$, and $\hat{y}_{t+s}^{t+s_{\max}+1} - \hat{y}_{t+s+1}^{t+s_{\max}+1} = 0$ for all $s \in \mathcal{S}$. Hence, $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ satisfies (4.10) at equality.

(C9) For each $r \in [t + s_{\max} + 2, T]_{\mathbb{Z}}$, we create the same point $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ as in group (C5) in the proof of Proposition 1. Thus, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) \in \text{conv}(\mathcal{P})$. If $r \leq t + k$, then $\hat{x}_t^r = \hat{y}_t^r = 0$, $\hat{x}_{t+k}^r = \underline{C}$, $\hat{y}_{t+k}^r = 1$, $\hat{y}_{t-m-1}^r = 0$, $\sum_{i=1}^m \hat{y}_{t-i}^r = 0$, and $\hat{y}_{t+s}^r - \hat{y}_{t+s+1}^r = 0$ for all $s \in \mathcal{S}$. If $r > t + k$, then $\hat{x}_t^r = \hat{x}_{t+k}^r = \hat{y}_t^r = \hat{y}_{t+k}^r = 0$, $\hat{y}_{t-m-1}^r = 0$, $\sum_{i=1}^m \hat{y}_{t-i}^r = 0$, and $\hat{y}_{t+s}^r - \hat{y}_{t+s+1}^r = 0$ for all $s \in \mathcal{S}$. Hence, in both cases, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ satisfies (4.10) at equality.

Table A.19 shows a matrix with $2T - 1$ rows, where each row represents a point created by this process. This matrix can be transformed into the matrix in Table A.20 via the following Gauissian elimination process:

(i) For each $r \in [1, t-1]_{\mathbb{Z}}$, the point in group (D1), denoted $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r) = (\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$. Here, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ is the point with index r in group (C1), and $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ is the point in group (C8).

Table A.19: A matrix with the rows representing $2T-1$ linearly independent points in $\text{conv}(\mathcal{P})$ satisfying inequality (4.10) at equality.

Group	Point	Index r	$\mathbf{x}$																	$\mathbf{y}$											
			1	⋯	$t-m-2$	$t-m-1$	$t-m$	⋯	$t-1$	t	$t+1$	⋯	$t+s_{\max}+1$	⋯	$t+k$	$t+k+1$	⋯	T	1	⋯	$t-m-2$	$t-m-1$	$t-m$	⋯	$t-1$	t	⋯	$t+s_{\max}+2$	⋯	T
		1	$\underline{C}+kV+\epsilon$	⋯	$\underline{C}+kV$	$\underline{C}+kV$	$\underline{C}+kV$	⋯	$\underline{C}+kV$	$\underline{C}+kV$	$\underline{C}+(k-1)V$	⋯	$\underline{C}+(k-s_{\max}-1)V$	⋯	$\underline{C}$	$\underline{C}$	⋯	$\underline{C}$	1	⋯	1	1	1	⋯	1	1	⋯	1	⋯	1
		⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮	⋮		⋮		⋮	⋮		⋮	⋮		⋮	⋮	⋮		⋮	⋮		⋮		⋮
(C1)		$t-m-2$	$\underline{C}+kV$	⋯	$\underline{C}+kV+\epsilon$	$\underline{C}+kV$	$\underline{C}+kV$	⋯	$\underline{C}+kV$	$\underline{C}+kV$	$\underline{C}+(k-1)V$	⋯	$\underline{C}+(k-s_{\max}-1)V$	⋯	$\underline{C}$	$\underline{C}$	⋯	$\underline{C}$	1	⋯	1	1	1	⋯	1	1	⋯	1	⋯	1
		$t-m-1$	$\underline{C}+kV$	⋯	$\underline{C}+kV$	$\underline{C}+kV+\epsilon$	$\underline{C}+kV$	⋯	$\underline{C}+kV$	$\underline{C}+kV$	$\underline{C}+(k-1)V$	⋯	$\underline{C}+(k-s_{\max}-1)V$	⋯	$\underline{C}$	$\underline{C}$	⋯	$\underline{C}$	1	⋯	1	1	1	⋯	1	1	⋯	1	⋯	1
		$t-m$	$\underline{C}+kV$	⋯	$\underline{C}+kV$	$\underline{C}+kV$	$\underline{C}+kV+\epsilon$	⋯	$\underline{C}+kV$	$\underline{C}+kV$	$\underline{C}+(k-1)V$	⋯	$\underline{C}+(k-s_{\max}-1)V$	⋯	$\underline{C}$	$\underline{C}$	⋯	$\underline{C}$	1	⋯	1	1	1	⋯	1	1	⋯	1	⋯	1
		⋮	⋮		⋮	⋮	⋮	⋱	⋮	⋮	⋮		⋮		⋮	⋮	⋮		⋮	⋮		⋮	⋮	⋮		⋮		⋮		⋮
	$(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$	$t-1$	$\underline{C}+kV$	⋯	$\underline{C}+kV$	$\underline{C}+kV$	$\underline{C}+kV$	⋯	$\underline{C}+kV+\epsilon$	$\underline{C}+kV$	$\underline{C}+(k-1)V$	⋯	$\underline{C}+(k-s_{\max}-1)V$	⋯	$\underline{C}$	$\underline{C}$	⋯	$\underline{C}$	1	⋯	1	1	1	⋯	1	1	⋯	1	⋯	1
		$t+1$	0	⋯	0	0	0	⋯	0	$\overline{V}$	$\underline{C}+\epsilon$	⋯	$\underline{C}$	⋯	$\underline{C}$	$\underline{C}$	⋯	$\underline{C}$	0	⋯	0	0	0	⋯	0	1	⋯	1	⋯	1
(C2)		⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮	⋱	⋮		⋮	⋮		⋮	⋮		⋮	⋮	⋮		⋮	⋮		⋮		⋮
		$t+s_{\max}+1$	0	⋯	0	0	0	⋯	0	$\overline{V}$	$\underline{C}$	⋯	$\underline{C}+\epsilon$	⋯	$\underline{C}$	$\underline{C}$	⋯	$\underline{C}$	0	⋯	0	0	0	⋯	0	1	⋯	1	⋯	1
		⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮	⋮	⋮		⋮	⋮		⋮	⋮		⋮	⋮	⋮		⋮	⋮		⋮		⋮
		$t+k-1$	0	⋯	0	0	0	⋯	0	$\overline{V}$	$\underline{C}$	⋯	$\underline{C}$	⋯	$\underline{C}$	$\underline{C}$	⋯	$\underline{C}$	0	⋯	0	0	0	⋯	0	1	⋯	1	⋯	1
(C3)		$t+k$	$\underline{C}+kV$	⋯	$\underline{C}+kV$	$\underline{C}+kV$	$\underline{C}+kV$	⋯	$\underline{C}+kV$	$\underline{C}+kV+\epsilon$	$\underline{C}+(k-1)V+\epsilon$	⋯	$\underline{C}+(k-s_{\max}-1)V+\epsilon$	⋯	$\underline{C}+\epsilon$	$\underline{C}$	⋯	$\underline{C}$	1	⋯	1	1	1	⋯	1	1	⋯	1	⋯	1
		$t+k+1$	0	⋯	0	0	0	⋯	0	$\overline{V}$	$\underline{C}$	⋯	$\underline{C}$	⋯	$\underline{C}$	$\underline{C}+\epsilon$	⋯	$\underline{C}$	0	⋯	0	0	0	⋯	0	1	⋯	1	⋯	1
(C4)		⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮		⋮	⋮	⋱	⋮	⋮		⋮	⋮	⋮		⋮	⋮		⋮		⋮
		T	0	⋯	0	0	0	⋯	0	$\overline{V}$	$\underline{C}$	⋯	$\underline{C}$	⋯	$\underline{C}$	$\underline{C}$	⋯	$\underline{C}+\epsilon$	0	⋯	0	0	0	⋯	0	1	⋯	1	⋯	1
		1	$\underline{C}$	⋯	0	0	0	⋯	0	0	0	⋯	0	⋯	0	0	⋯	0	1	⋯	0	0	0	⋯	0	0	⋯	0	⋯	0
(C5)		⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮	⋮		⋮		⋮	⋮		⋮	⋮	⋱	⋮	⋮	⋮		⋮	⋮		⋮		⋮
		$t-m-2$	$\underline{C}$	⋯	$\underline{C}$	0	0	⋯	0	0	0	⋯	0	⋯	0	0	⋯	0	1	⋯	1	0	0	⋯	0	0	⋯	0	⋯	0
		$t-m-1$	0	⋯	0	0	$\overline{V}$	⋯	$\overline{V}+(m-1)V$	$\overline{V}+mV$	$\overline{V}+(m-1)V$	⋯	(See Note A.19-1)	⋯	$\underline{C}$	$\underline{C}$	⋯	$\underline{C}$	0	⋯	0	0	1	⋯	1	1	⋯	1	⋯	1
(C6)	$(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$	⋮	⋮		⋮	⋮	⋮	⋱	⋮	⋮	⋮		⋮		⋮	⋮		⋮	⋮		⋮	⋮	⋮	⋱	⋮	⋮		⋮		⋮
		$t-2$	0	⋯	0	0	0	⋯	$\overline{V}$	$\overline{V}+V$	$\overline{V}$	⋯	$\underline{C}$	⋯	$\underline{C}$	$\underline{C}$	⋯	$\underline{C}$	0	⋯	0	0	0	⋯	1	1	⋯	1	⋯	1
		$t-1$	0	⋯	0	0	0	⋯	0	$\overline{V}$	$\underline{C}$	⋯	$\underline{C}$	⋯	$\underline{C}$	$\underline{C}$	⋯	$\underline{C}$	0	⋯	0	0	0	⋯	0	1	⋯	1	⋯	1
		t																												
(C7)		⋮	(See Note A.19-2)																(See Note A.19-2)											
		$t+s_{\max}$																												
(C8)		$t+s_{\max}+1$	$\underline{C}+kV$	⋯	$\underline{C}+kV$	$\underline{C}+kV$	$\underline{C}+kV$	⋯	$\underline{C}+kV$	$\underline{C}+kV$	$\underline{C}+(k-1)V$	⋯	$\underline{C}+(k-s_{\max}-1)V$	⋯	$\underline{C}$	$\underline{C}$	⋯	$\underline{C}$	1	⋯	1	1	1	⋯	1	1	⋯	1	⋯	1
		$t+s_{\max}+2$	0	⋯	0	0	0	⋯	0	0	0	⋯	0	⋯	$\underline{C}$	$\underline{C}$	⋯	$\underline{C}$	0	⋯	0	0	0	⋯	0	0	⋯	1	⋯	1
(C9)		⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮		⋮		⋮	⋮		⋮	⋮		⋮	⋮	⋮		⋮	⋮		⋮	⋱	⋮
		T	0	⋯	0	0	0	⋯	0	0	0	⋯	0	⋯	0	0	⋯	$\underline{C}$	0	⋯	0	0	0	⋯	0	0	⋯	0	⋯	1

Note A.19-1: In group (C6), $\hat{x}^{t-m-1}_{t+s_{\max}+1} = \overline{V} + (m - s_{\max} - 1)V$ if $m > s_{\max}$, and $\hat{x}^{t-m-1}_{t+s_{\max}+1} = \underline{C}$ if $m \leq s_{\max}$.

Note A.19-2: For $r \in [t, t+s_{\max}]_{\mathbb{Z}}$, the $\mathbf{x}$ and $\mathbf{y}$ vectors in group (C7) are given as follows: $\hat{\mathbf{x}}^r = (\underbrace{0,\ldots,0}_{r \text{ terms}}, \underbrace{\underline{C},\ldots,\underline{C}}_{T-r \text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{0,\ldots,0}_{r \text{ terms}}, \underbrace{1,\ldots,1}_{T-r \text{ terms}})$ if $r - t \notin \mathcal{S}$;

$\hat{\mathbf{x}}^r = (\underbrace{\overline{V} + (r-t)V, \ldots, \overline{V} + (r-t)V}_{t-1 \text{ terms}}, \underbrace{\overline{V} + (r-t)V, \overline{V} + (r-t-1)V, \overline{V} + (r-t-2)V, \ldots, \overline{V}}_{r-t+1 \text{ terms}}, \underbrace{0,\ldots,0}_{T-r \text{ terms}})$ and $\hat{\mathbf{y}}^r = (\underbrace{1,\ldots,1}_{r \text{ terms}}, \underbrace{0,\ldots,0}_{T-r \text{ terms}})$ if $r - t \in \mathcal{S}$.

Table A.20: Lower triangular matrix obtained from Table A.19 via Gaussian elimination.

Group	Point	Index r	$\mathbf{x}$											$\mathbf{y}$											
			1	$\cdots$	$t-1$	t	$t+1$	$\cdots$	$t+k-1$	$t+k$	$t+k+1$	$\cdots$	T	1	$\cdots$	$t-m-2$	$t-m-1$	$\cdots$	$t-1$	t	$\cdots$	$t+s_{\max}+1$	$t+s_{\max}+2$	$\cdots$	T
(D1)		1	ϵ	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0
		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
		$t-1$	0	$\cdots$	ϵ	0	0	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0
(D2)		$t+1$	0	$\cdots$	0	0	ϵ	$\cdots$	0	0	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0
	$(\underline{\tilde{\mathbf{x}}}^r, \underline{\tilde{\mathbf{y}}}^r)$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
		$t+k-1$	0	$\cdots$	0	0	0	$\cdots$	ϵ	0	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0
(D3)		$t+k$	0	$\cdots$	0	ϵ	ϵ	$\cdots$	ϵ	ϵ	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0
(D4)		$t+k+1$	0	$\cdots$	0	0	0	$\cdots$	0	0	ϵ	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0
		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
		T	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	ϵ	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0
(D5)		1												1	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0
		$\vdots$	(Omitted)											$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
		$t-m-2$												1	$\cdots$	1	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0
(D6)		$t-m-1$												-1	$\cdots$	-1	-1	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0
		$\vdots$	(Omitted)											$\vdots$		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
	$(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$	$t-1$												-1	$\cdots$	-1	-1	$\cdots$	-1	0	$\cdots$	0	0	$\cdots$	0
(D7)		t																							
		$\vdots$	(Omitted)											(See Note A.20-1)											
		$t+s_{\max}$																							
(D8)		$t+s_{\max}+1$	(Omitted)											1	$\cdots$	1	1	$\cdots$	1	1	$\cdots$	1	0	$\cdots$	0
(D9)		$t+s_{\max}+2$												0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0	1	$\cdots$	0
		$\vdots$	(Omitted)											$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\ddots$	$\vdots$
		T												0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	1

Note A.20-1: For $r \in [t, t+s_{\max}]_{\mathbb{Z}}$, the $\mathbf{y}$ vector in group (D7) is given as follows: $\underline{\hat{\mathbf{y}}}^r = (\underbrace{-1,\ldots,-1}_{r \text{ terms}}, \underbrace{0,\ldots,0}_{T-r \text{ terms}})$ if $r - t \notin \mathcal{S}$; $\underline{\hat{\mathbf{y}}}^r = (\underbrace{1,\ldots,1}_{r \text{ terms}}, \underbrace{0,\ldots,0}_{T-r \text{ terms}})$ if $r - t \in \mathcal{S}$.

(ii) For each $r \in [t+1, t+k-1]_{\mathbb{Z}}$, the point in group (D2), denoted $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r) = (\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$. Here, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ is the point with index r in group (C2), and $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ is the point with index $t-1$ in group (C6).

(iii) The point in group (D3), denoted $(\underline{\bar{\mathbf{x}}}^{t+k}, \underline{\bar{\mathbf{y}}}^{t+k})$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^{t+k}, \underline{\bar{\mathbf{y}}}^{t+k}) = (\bar{\mathbf{x}}^{t+k}, \bar{\mathbf{y}}^{t+k}) - (\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$. Here, $(\bar{\mathbf{x}}^{t+k}, \bar{\mathbf{y}}^{t+k})$ is the point in group (C3), and $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ is the point in group (C8)

(iv) For each $r \in [t+k+1, T]_{\mathbb{Z}}$, the point in group (D4), denoted $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\bar{\mathbf{x}}}^r, \underline{\bar{\mathbf{y}}}^r) = (\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$. Here, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ is the point with index r in group (C4), and $(\hat{\mathbf{x}}^{t-1}, \hat{\mathbf{y}}^{t-1})$ is the point with index $t-1$ in group (C6).

(v) For each $r \in [1, t-m-2]_{\mathbb{Z}}$, the point in group (D5), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (C5).

(vi) For each $r \in [t-m-1, t-1]_{\mathbb{Z}}$, the point in group (D6), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ is the point with index r in group (C6), and $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ is the point in group (C8).

(vii) For each $r \in [t, t+s_{\max}]_{\mathbb{Z}}$, the point with index r in group (D7), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ if $r - t \notin \mathcal{S}$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $r - t \in \mathcal{S}$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$

is the point with index r in group (C7), and $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ is the point in group (C8).

(viii) the point in group (D8), denoted $(\underline{\hat{\mathbf{x}}}^{t+s_{\max}+1}, \underline{\hat{\mathbf{y}}}^{t+s_{\max}+1})$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^{t+s_{\max}+1}, \underline{\hat{\mathbf{y}}}^{t+s_{\max}+1}) = (\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1}) - (\hat{\mathbf{x}}^{t+s_{\max}+2}, \hat{\mathbf{y}}^{t+s_{\max}+2})$. Here, $(\hat{\mathbf{x}}^{t+s_{\max}+1}, \hat{\mathbf{y}}^{t+s_{\max}+1})$ is the point in group (C8), and $(\hat{\mathbf{x}}^{t+s_{\max}+2}, \hat{\mathbf{y}}^{t+s_{\max}+2})$ is the point with index $t + s_{\max} + 2$ in group (C9).

(ix) For each $r \in [t + s_{\max} + 2, T]_{\mathbb{Z}}$, the point in group (D9), denoted $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r)$, is obtained by setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r) - (\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ if $r \neq T$, and setting $(\underline{\hat{\mathbf{x}}}^r, \underline{\hat{\mathbf{y}}}^r) = (\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ if $r = T$. Here, $(\hat{\mathbf{x}}^r, \hat{\mathbf{y}}^r)$ and $(\hat{\mathbf{x}}^{r+1}, \hat{\mathbf{y}}^{r+1})$ are the points with indices r and $r + 1$, respectively, in group (C9).

The matrix in Table A.20 is lower triangular; that is, the position of the last nonzero component of a row of the matrix is greater than the position of the last nonzero component of the previous row. This implies that these $2T - 1$ points in groups (C1)–(C9) are linearly independent. Therefore, inequality (4.10) is facet-defining for $\text{conv}(\mathcal{P})$. $\square$

8.15 Proof of Proposition 12

For notational convenience, denote $\hat{k} = \max\{k \in [1, T-1]_{\mathbb{Z}} : \overline{C} - \underline{C} - kV > 0\}$, and denote $\hat{s}_{km} = \min\{k-1, L-m-2\}$ for any $k \in [1, \hat{k}]_{\mathbb{Z}}$ and $m \in [0, k-1]_{\mathbb{Z}}$. We first consider inequality (4.9). Consider any given $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_{+}^{2T}$. For any $t \in [1, T]_{\mathbb{Z}}$, let

$$\theta(t) = \sum_{\tau=2}^{t} \max\{y_\tau - y_{\tau-1}, 0\}.$$

Then, for any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, and $t \in [k+1, T-m-1]_{\mathbb{Z}}$,

$$\sum_{s=1}^{\hat{s}_{km}} \max\{y_{t-s} - y_{t-s-1}, 0\} = \sum_{\tau=t-\hat{s}_{km}}^{t-1} \max\{y_\tau - y_{\tau-1}, 0\} = \theta(t-1) - \theta(t - \hat{s}_{km} - 1). \tag{8.46}$$

For any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, $t \in [k+1, T-m-1]_{\mathbb{Z}}$, and $\mathcal{S} \subseteq [0, \hat{s}_{km}]_{\mathbb{Z}}$, let

$$\begin{aligned}\tilde{v}_{km}(\mathcal{S}, t) = {} & x_t - x_{t-k} - (\underline{C} + (k-m)V - \overline{V})y_{t+m+1} - V\sum_{i=1}^{m} y_{t+i} - \overline{V}y_t + \underline{C}y_{t-k} \\ & + \sum_{s \in \mathcal{S}} (\underline{C} + (k-s)V - \overline{V})(y_{t-s} - y_{t-s-1}).\end{aligned}$$

If $\tilde{v}_{km}(\mathcal{S}, t) > 0$, then $\tilde{v}_{km}(\mathcal{S}, t)$ is the amount of violation of inequality (4.9). If $\tilde{v}_{km}(\mathcal{S}, t) \leq 0$, there is no violation of inequality (4.9). For any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, and $t \in [k+1, T-m-1]_{\mathbb{Z}}$, let

$$v_{km}(t) = \max_{\mathcal{S} \subseteq [0, \hat{s}_{km}]_{\mathbb{Z}}} \{\tilde{v}_{km}(\mathcal{S}, t)\}.$$

If $v_{km}(t) > 0$, then $v_{km}(t)$ is the largest possible violation of inequality (4.9) for this combination of k, m, and t. If $v_{km}(t) \leq 0$, the largest possible violation of inequality (4.9) is zero for this combination of k, m, and t. Because $\underline{C} + V > \overline{V}$, we have $\underline{C} + (k-s)V - \overline{V} > 0$ for all $k \in [1, \hat{k}]_{\mathbb{Z}}$, $s \in [0, \hat{s}_{km}]_{\mathbb{Z}}$, and $m \in [0, k-1]_{\mathbb{Z}}$. Thus, for any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, and $t \in [k+1, T-m-1]_{\mathbb{Z}}$, $\tilde{v}_{km}(\mathcal{S}, t)$ is maximized when $\mathcal{S}$ contains all $s \in [0, \hat{s}_{km}]_{\mathbb{Z}}$ such that $y_{t-s} - y_{t-s-1} > 0$ (if any). If it does not exist any $s \in [0, \hat{s}]_{\mathbb{Z}}$ such that $y_{t-s} - y_{t-s-1} > 0$, then $\tilde{v}_{km}(\mathcal{S}, t)$ is maximized when $\mathcal{S} = \emptyset$, and $v_{km}(t) = x_t - x_{t-k} - (\underline{C} + (k-m)V - \overline{V})y_{t+m+1} - V\sum_{i=1}^{m} y_{t+i} - \overline{V}y_t + \underline{C}y_{t-k}$. Hence, for any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, and $t \in [k+1, T-m-1]_{\mathbb{Z}}$,

$$
\begin{aligned}
v_{km}(t) = x_t - x_{t-k} - (\underline{C} + (k-m)V - \overline{V})y_{t+m+1} - V\sum_{i=1}^{m} y_{t+i} - \overline{V}y_t + \underline{C}y_{t-k} \\
+ \sum_{s=0}^{\hat{s}_{km}} (\underline{C} + (k-s)V - \overline{V}) \max\{y_{t-s} - y_{t-s-1}, 0\}.
\end{aligned}
$$

Determining $\theta(t)$ for all $t \in [1, T]_{\mathbb{Z}}$ can be done recursively in $O(T)$ time by setting $\theta(1) = 0$ and setting $\theta(t) = \theta(t-1) + \max\{y_t - y_{t-1}, 0\}$ for $t = 2, \ldots, T$. Clearly, for each $k \in [1, \hat{k}]_{\mathbb{Z}}$ and each $m \in [0, k-1]_{\mathbb{Z}}$, the value of $v_{km}(k+1)$ can be determined in $O(T)$ time. For any $k \in [1, \hat{k}]_{\mathbb{Z}}$,

$m \in [0, k-1]_{\mathbb{Z}}$, and $t \in [k+2, T-m-1]_{\mathbb{Z}}$,

$$
\begin{aligned}
&v_{km}(t) - v_{km}(t-1) \\
&= (x_t - x_{t-1}) - (x_{t-k} - x_{t-k-1}) - (\underline{C} + (k-m)V - \overline{V})(y_{t+m+1} - y_{t+m}) \\
&\quad - V\left[\sum_{i=1}^{m} y_{t+i} - \sum_{i=1}^{m} y_{t+i-1}\right] - \overline{V}(y_t - y_{t-1}) + \underline{C}(y_{t-k} - y_{t-k-1}) \\
&\quad + (\underline{C} + kV - \overline{V})\left[\sum_{s=0}^{\hat{s}_{km}} \max\{y_{t-s} - y_{t-s-1}, 0\} - \sum_{s=0}^{\hat{s}_{km}} \max\{y_{t-s-1} - y_{t-s-2}, 0\}\right] \\
&\quad - V\left[\sum_{s=0}^{\hat{s}_{km}} s\max\{y_{t-s} - y_{t-s-1}, 0\} - \sum_{s=0}^{\hat{s}_{km}} s\max\{y_{t-s-1} - y_{t-s-2}, 0\}\right] \\
&= (x_t - x_{t-1}) - (x_{t-k} - x_{t-k-1}) - (\underline{C} + (k-m)V - \overline{V})(y_{t+m+1} - y_{t+m}) \\
&\quad - V(y_{t+m} - y_t) - \overline{V}(y_t - y_{t-1}) + \underline{C}(y_{t-k} - y_{t-k-1}) \\
&\quad + (\underline{C} + kV - \overline{V})\left[\max\{y_t - y_{t-1}, 0\} - \max\{y_{t-\hat{s}_{km}-1} - y_{t-\hat{s}_{km}-2}, 0\}\right] \\
&\quad - V\left[\sum_{s=1}^{\hat{s}_{km}} \max\{y_{t-s} - y_{t-s-1}, 0\} - \hat{s}_{km}\max\{y_{t-\hat{s}_{km}-1} - y_{t-\hat{s}_{km}-2}, 0\}\right].
\end{aligned}
$$

This, together with (8.46), implies that

$$
\begin{aligned}
&v_{km}(t) = \\
&v_{km}(t-1) + (x_t - x_{t-1}) - (x_{t-k} - x_{t-k-1}) - (\underline{C} + (k-m)V - \overline{V})(y_{t+m+1} - y_{t+m}) \\
&- V(y_{t+m} - y_t) - \overline{V}(y_t - y_{t-1}) + \underline{C}(y_{t-k} - y_{t-k-1}) \\
&+ (\underline{C} + kV - \overline{V})\left[\max\{y_t - y_{t-1}, 0\} - \max\{y_{t-\hat{s}_{km}-1} - y_{t-\hat{s}_{km}-2}, 0\}\right] \\
&- V\left[\theta(t-1) - \theta(t-\hat{s}_{km}-1) - \hat{s}_{km}\max\{y_{t-\hat{s}_{km}-1} - y_{t-\hat{s}_{km}-2}, 0\}\right].
\end{aligned}
$$

Thus, for each $k \in [1, \hat{k}]_{\mathbb{Z}}$ and $m \in [0, k-1]_{\mathbb{Z}}$, the values of $v_{km}(k+1), v_{km}(k+2), \ldots, v_{km}(T-m-1)$ can be determined recursively in $O(T)$

time. Hence, the values of k, m, t and the set $\mathcal{S}$ corresponding to the largest possible violation of inequality (4.9) can be obtained in $O(T^3)$ time.

Next, we consider inequality (4.10). Consider any given point $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{2T}$. For any $t \in [1, T]_{\mathbb{Z}}$, let

$$\theta'(t) = \sum_{\tau=t}^{T-1} \max\{y_\tau - y_{\tau+1}, 0\}.$$

Then, for any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, and $t \in [m+2, T-k]_{\mathbb{Z}}$,

$$\sum_{s=1}^{\hat{s}_{km}} \max\{y_{t+s} - y_{t+s+1}, 0\} = \sum_{\tau=t+1}^{t+\hat{s}_{km}} \max\{y_\tau - y_{\tau+1}, 0\} = \theta'(t+1) - \theta'(t+\hat{s}_{km}+1). \tag{8.47}$$

For any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, $t \in [m+2, T-k]_{\mathbb{Z}}$, and $\mathcal{S} \subseteq [0, \hat{s}_{km}]_{\mathbb{Z}}$, let

$$\begin{aligned}\tilde{v}'_{km}(\mathcal{S}, t) = x_t - x_{t+k} - (\underline{C} + (k-m)V - \overline{V})y_{t-m-1} - V\sum_{i=1}^{m} y_{t-i} - \overline{V}y_t + \underline{C}y_{t+k} \\ + \sum_{s\in\mathcal{S}} (\underline{C} + (k-s)V - \overline{V})(y_{t+s} - y_{t+s+1}).\end{aligned}$$

If $\tilde{v}'_{km}(\mathcal{S}, t) > 0$, then $\tilde{v}'_{km}(\mathcal{S}, t)$ is the amount of violation of inequality (4.10). If $\tilde{v}'_{km}(\mathcal{S}, t) \le 0$, there is no violation of inequality (4.10). For any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, and $t \in [m+2, T-k]_{\mathbb{Z}}$, let

$$v'_{km}(t) = \max_{\mathcal{S} \subseteq [0, \hat{s}_{km}]_{\mathbb{Z}}} \{\tilde{v}'_{km}(\mathcal{S}, t)\}.$$

If $v'_{km}(t) > 0$, then $v'_{km}(t)$ is the largest possible violation of inequality (4.10) for this combination of k, m, and t. If $v'_{km}(t) \le 0$, the largest possible violation of inequality (4.10) is zero for this combination of k, m, and t. Because $\underline{C} + V > \overline{V}$, we have $\underline{C} + (k-s)V - \overline{V} > 0$ for all $k \in [1, \hat{k}]_{\mathbb{Z}}$, $s \in [0, \hat{s}_{km}]_{\mathbb{Z}}$, and $m \in [0, k-1]_{\mathbb{Z}}$. Thus, for any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, and $t \in [m+2, T-k]_{\mathbb{Z}}$, $\tilde{v}'_{km}(\mathcal{S}, t)$ is maximized when $\mathcal{S}$ contains all $s \in [0, \hat{s}_{km}]_{\mathbb{Z}}$ such that $y_{t+s} - y_{t+s+1} > 0$ (if any). If it does not exist any $s \in [0, \hat{s}]_{\mathbb{Z}}$ such that $y_{t+s} - y_{t+s+1} > 0$, then $\tilde{v}'_{km}(\mathcal{S}, t)$ is maximized when $\mathcal{S} = \emptyset$, and $v'_{km}(t) = x_t - x_{t+k} - (\underline{C} + (k-m)V - \overline{V})y_{t-m-1} - V\sum_{i=1}^{m} y_{t-i} - \overline{V}y_t + \underline{C}y_{t+k}$. Hence, for any $k \in [1, \hat{k}]_{\mathbb{Z}}$, $m \in [0, k-1]_{\mathbb{Z}}$, and $t \in [m+2, T-k]_{\mathbb{Z}}$,

$$v'_{km}(t) = x_t - x_{t+k} - (\underline{C} + (k-m)V - \overline{V})y_{t-m-1} - V\sum_{i=1}^{m} y_{t-i} - \overline{V}y_t + \underline{C}y_{t+k} + \sum_{s=0}^{\hat{s}_{km}} (\underline{C} + (k-s)V - \overline{V}) \max\{y_{t+s} - y_{t+s+1}, 0\}.$$

Determining $\theta'(t)$ for all $t \in [1, T]_{\mathbb{Z}}$ can be done recursively in $O(T)$ time by setting $\theta'(T) = 0$ and setting $\theta'(t) = \theta'(t+1) + \max\{y_t - y_{t+1}, 0\}$ for $t = T-1, T-2, \ldots, 1$. Clearly, for each $k \in [1, \hat{k}]_{\mathbb{Z}}$ and each $m \in [0, k-1]_{\mathbb{Z}}$, the value of $v'_{km}(T-k)$ can be determined in $O(T)$ time. For any $k \in [1, \hat{k}]_{\mathbb{Z}}$,

$m \in [0, k-1]_{\mathbb{Z}}$, and $t \in [m+2, T-k-1]_{\mathbb{Z}}$,

$$\begin{aligned}
&v'_{km}(t) - v'_{km}(t+1) \\
&= (x_t - x_{t+1}) - (x_{t+k} - x_{t+k+1}) - (\underline{C} + (k-m)V - \overline{V})(y_{t-m-1} - y_{t-m}) \\
&\quad - V\left[\sum_{i=1}^{m} y_{t-i} - \sum_{i=1}^{m} y_{t-i+1}\right] - \overline{V}(y_t - y_{t+1}) + \underline{C}(y_{t+k} - y_{t+k+1}) \\
&\quad + (\underline{C} + kV - \overline{V})\left[\sum_{s=0}^{\hat{s}_{km}} \max\{y_{t+s} - y_{t+s+1}, 0\} - \sum_{s=0}^{\hat{s}_{km}} \max\{y_{t+s+1} - y_{t+s+2}, 0\}\right] \\
&\quad - V\left[\sum_{s=0}^{\hat{s}_{km}} s\max\{y_{t+s} - y_{t+s+1}, 0\} - \sum_{s=0}^{\hat{s}_{km}} s\max\{y_{t+s+1} - y_{t+s+2}, 0\}\right] \\
&= (x_t - x_{t+1}) - (x_{t+k} - x_{t+k+1}) - (\underline{C} + (k-m)V - \overline{V})(y_{t-m-1} - y_{t-m}) \\
&\quad - V(y_{t-m} - y_t) - \overline{V}(y_t - y_{t+1}) + \underline{C}(y_{t+k} - y_{t+k+1}) \\
&\quad + (\underline{C} + kV - \overline{V})\left[\max\{y_t - y_{t+1}, 0\} - \max\{y_{t+\hat{s}_{km}+1} - y_{t+\hat{s}_{km}+2}, 0\}\right] \\
&\quad - V\left[\sum_{s=1}^{\hat{s}_{km}} \max\{y_{t+s} - y_{t+s+1}, 0\} - \hat{s}_{km}\max\{y_{t+\hat{s}_{km}+1} - y_{t+\hat{s}_{km}+2}, 0\}\right].
\end{aligned}$$

This, together with (8.47), implies that

$$\begin{aligned}
&v'_{km}(t) = \\
&v'_{km}(t+1) + (x_t - x_{t+1}) - (x_{t+k} - x_{t+k+1}) - (\underline{C} + (k-m)V - \overline{V})(y_{t-m-1} - y_{t-m}) \\
&- V(y_{t-m} - y_t) - \overline{V}(y_t - y_{t+1}) + \underline{C}(y_{t+k} - y_{t+k+1}) \\
&+ (\underline{C} + kV - \overline{V})\left[\max\{y_t - y_{t+1}, 0\} - \max\{y_{t+\hat{s}_{km}+1} - y_{t+\hat{s}_{km}+2}, 0\}\right] \\
&- V\left[\theta'(t+1) - \theta'(t+\hat{s}_{km}+1) - \hat{s}_{km}\max\{y_{t+\hat{s}_{km}+1} - y_{t+\hat{s}_{km}+2}, 0\}\right].
\end{aligned}$$

Thus, for each $k \in [1, \hat{k}]_{\mathbb{Z}}$ and $m \in [0, k-1]_{\mathbb{Z}}$, the values of $v'_{km}(m+2), v'_{km}(m+3), \ldots, v'_{km}(T-k)$ can be determined recursively in $O(T)$ time.

Hence, the values of k, m, t and the set $\mathcal{S}$ corresponding to the largest possible violation of inequality (4.10) can be obtained in $O(T^3)$ time. □