T-PERFECT GRAPHS AND SELF-COMPLEMENTARY GRAPHS

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T-perfect Graphs and Self-complementary Graphs

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CERTIFICATE OF ORIGINALITY

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Abstract

The maximum-weight independent set problem is a fundamental NP-hard problem. To gain a deeper understanding of its complexity, identifying graph classes where the problem can be solved in polynomial time has become a popular research area. Perfect graphs have emerged as one such class, characterized by their independent set polytope being fully described by trivial and clique inequalities. Inspired by the polyhedral characterization of perfect graphs, Chvátal introduced t-perfect graphs, where the independent set polytope is fully described by trivial, edge, and odd-cycle inequalities. This pivotal characteristic enables the development of polynomial-time algorithms to solve the maximum-weight independent set problem specifically for t-perfect graphs. Given that t-perfect graphs are defined from a polyhedral perspective, a profound understanding of their structure is essential.

While a full structural characterization of the class of t-perfect graphs is still at large, substantial advancements have been made for claw-free graphs [Bruhn and Stein, Math. Program. 2012] and P_5-free graphs [Bruhn and Fuchs, SIAM J. Discrete Math. 2017]. We take one more step to characterize t-perfect graphs that are fork-free, and show that they are strongly t-perfect and three-colorable. We also present polynomial-time algorithms for recognizing and coloring these graphs.

Unlike perfect graphs, t-perfect graphs are not closed under substitution or complementation. A full characterization of t-perfection with respect to substitution has been obtained by Benchetrit in his Ph.D. thesis. We attempt to understand t-

perfection with respect to complementation. In particular, we show that there are only five pairs of graphs such that both the graphs and their complements are minimally t-imperfect. We also identify all t-perfect graphs that are self-complementary.

We conduct a more in-depth study of self-complementary graphs. We study split graphs and pseudo-split graphs whose complements are isomorphic to themselves. These special subclasses of self-complementary graphs are actually the core of self-complementary graphs. Indeed, all realizations of forcibly self-complementary degree sequences are pseudo-split graphs. We also give formulas to calculate the number of self-complementary (pseudo-)split graphs of a given order, and show that Trotignon's conjecture holds for all self-complementary split graphs.

Publications Arising from the Thesis

[1] Yixin Cao, Haowei Chen, and Shenghua Wang. Self-complementary (pseudo-)split graphs. Manuscript submitted to *Journal of Graph Theory (JGT).*

[2] Yixin Cao, Haowei Chen, and Shenghua Wang. Self-complementary (pseudo-)split graphs. In *the 16th Latin American Theoretical Informatics Symposium (LATIN)*, 2024.

[3] Yixin Cao and Shenghua Wang. On fork-free t-perfect graphs. Manuscript submitted to *Journal of Graph Theory (JGT).*

[4] Yixin Cao and Shenghua Wang. Complementation in t-perfect graphs. Manuscript submitted to *Discrete Applied Mathematics (DAM).*

[5] Yixin Cao and Shenghua Wang. Complementation in t-perfect graphs. In *the 47th International Workshop on Graph-Theoretic Concepts in Computer Science (WG)*, pages 106–117, 2021.

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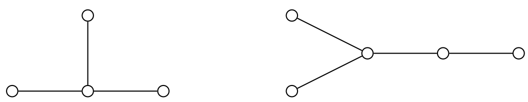
Chapter 1

Introduction

1.1 Background and motivation

Graphs provide a modeling approach for addressing a wide range of real-world problems. Within a graph, an *independent set* (also called *stable set*) is a set of vertices that are pairwise nonadjacent. Surprisingly, the solutions of many real-world problems can be expressed as independent sets in graphs. In practice, the vertices of graphs are often assigned weights to represent their significance, and the objective is to identify an independent set that maximizes the total weight. This fundamental problem, known as the *maximum-weight independent set problem*, holds great importance in graph theory and combinatorial optimization. It is one of the NP-hard problems [71], making it unlikely to find an optimal solution for all instances in an efficient manner. Consequently, it becomes a popular research area to find graph classes where the maximum-weight independent set problem can be solved in polynomial time.

Edmonds' breakthrough paper [45] showed the polynomial-time solvability of the maximum-weight matching problem, which directly implies the tractability of the maximum-weight independent set problem on line graphs. Line graphs have the property of being closed under taking induced subgraphs, classifying the class of line graphs as a hereditary graph class. Hereditary graph classes can be characterized by a set of minimal forbidden induced subgraphs. Beineke [10] provided a complete list of minimal forbidden induced subgraphs for line graphs[1]. It is worth considering whether forbidding certain substructures in graphs could potentially contribute to the design of efficient algorithms for solving the maximum-weight independent set problem. Alekseev [3] observed that when only a finite number of graphs are forbidden, the maximum-weight independent set problem remains NP-hard unless, for at least one graph in the forbidden list, every connected component is a tree with at most three leaves. This motivates people to study the problem on H-free graphs where H is a forest whose every component has at most three leaves.



(a) (b)

Figure 1.1: (a) The claw graph and (b) the fork graph.

With the observation made by Alekseev, it becomes evident that the polynomial-time solvability of the maximum-weight independent set problem on line graphs relies solely on forbidding the $K_{1,3}$ graph, commonly known as the claw graph (see Figure 1.1 (a)). Independently, Minty and Sbihi [90, 112] gave polynomial-time algorithms for solving the maximum-weight independent set problem on graphs that are free of claws. By introducing a subdivision on one of the edges of the claw graph, we obtain the fork graph showed in Figure 1.1 (b). Subsequently, Lozin and Milanič [82] developed a polynomial-time algorithm specifically tailored for solving the maximum-weight independent set problem on graphs that are fork-free. It is worth noting that the class of fork-free graphs is a superclass of the class of claw-free graphs. The class of P_4-free graphs exhibits a simple structure [40], enabling the development of polynomial-time algorithms for solving the maximum-weight independent set problem. By using the concept of potential maximal cliques [13], Lokshtanov et al. [77] introduced a

[1]Refer to here for all minimal forbidden induced subgraphs for line graphs.

polynomial-time algorithm for solving the maximum-weight independent set problem in graphs that are free of P_5's, which has subsequently been extended to graphs that are free of P_6's [64]. For more related results in this line of research, please refer to [2, 4, 8, 15, 16, 26, 52, 53, 57, 67, 80, 81, 83, 84, 92–96, 99, 103].

Another line of research aimed at finding graph classes where the maximum-weight independent set problem can be solved in polynomial time focuses on polyhedral perspectives. To find a maximum-weight independent set in an arbitrary graph G, we can formulate this problem as an integer linear programming problem:

$$\begin{aligned} \max \quad & w^T x \\ \text{subject to} \quad & x_u + x_v \leq 1 \quad \text{for every edge } uv \text{ in } E(G) \\ & x_v \in \{0, 1\} \quad \text{for every vertex } v \text{ in } V(G), \end{aligned} \tag{1.1}$$

where $w : V(G) \to \mathbb{R}_{\geq 0}$ is a weighting of the vertices in the given graph G. The feasible solutions of this integer linear programming problem correspond to the incidence vectors of independent sets of G. The inequalities

$$x_u + x_v \leq 1 \quad \text{for every edge } uv \text{ in } E(G) \qquad \text{(edge inequalities)}$$

are called edge inequalities, since every independent set of G contains at most one end of an edge.

The convex hull of all incidence vectors of independent sets of G forms a bounded polyhedron known as the *independent set polytope* of G, denoted as $P_I(G)$. With this independent set polytope, the problem (1.1) can be equivalently expressed as a linear programming problem:

$$\max\{w^T x \mid x \in P_I(G)\}. \tag{1.2}$$

Solving this linear programming problem is equivalent to finding the maximum-weight independent set in G. However, there are numerous graphs for which the number of inequalities required to describe $P_I(G)$ is exponentially large, making the task of describing the inequalities challenging.

Fortunately, Grötschel et al. [61] demonstrated that despite the exponential number of necessary inequalities, it is still possible to solve (1.2) efficiently as long as the separation problem, which involves determining whether a given vector belongs to a polyhedron and, if not, finding an inequality that is valid for the polyhedron but violated by the vector, can be efficiently performed. Building upon this, multiple linear realizations of the independent set polytope are defined, ensuring that the separation problem for the descriptions of these linear realizations can be effectively solved. For each of these realizations, a graph class can be defined where the independent set polytope is equivalent to the realization. As a result, the maximum-weight independent set problem can be effectively solved within these graph classes.

A natural linear relaxation arises by relaxing the integrality inequalities of (1.1), replacing $x_v \in \{0, 1\}$ with the inequalities

$$0 \leq x_v \leq 1 \quad \text{for every vertex } v \text{ in } V(G). \tag{trivial inequalities}$$

These inequalities are called trivial inequalities. The linear realization, known as the edge polytope and denoted as $P_{\mathrm{E}}(G)$, is described by trivial and edge inequalities. This straightforward realization has a polynomial number of inequalities. The separation problem for these inequalities can be efficiently solved, allowing for effective optimization. Grötschel et al. [63] showed that $P_{\mathrm{E}}(G) = P_I(G)$ if and only if G is a bipartite graph. Consequently, the maximum-weight independent set problem can be efficiently solved in bipartite graphs. Furthermore, a graph is bipartite if and only if it contains no odd cycles. This structural characterization provides a useful insight, facilitating the design of efficient algorithms to determine whether a given graph is bipartite.

Perfect graphs

The graph K_3 serves as the smallest example of a non-bipartite graph. Since the vector $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is in $P_E(K_3)$ but not in $P_I(K_3)$, the polytope $P_I(K_3)$ is not equivalent to the polytope $P_E(K_3)$. It is noteworthy that K_3 is a clique and a clique can intersect at most one vertex with an independent set. Based on this, the clique inequalities are introduced:

$$\sum_{v \in K} x_v \leq 1 \quad \text{for every clique } K \text{ in } G. \tag{clique inequalities}$$

Clearly, each edge in G can be considered as a clique of size two. This indicates that the edge inequalities are encompassed within the clique inequalities. As a result, we can expand the edge polytope by adding additional inequalities for cliques of size three or more. The resulting polytope is referred to as the clique polytope, denoted as $P_{\mathrm{K}}(G)$. For which graph G the clique polytope $P_{\mathrm{K}}(G)$ is equivalent to the independent set polytope $P_I(G)$? This question is answered by Chvátal [29] and Padberg [101] independently. They showed that $P_{\mathrm{K}}(G) = P_I(G)$ if and only if G is a perfect graph.

The concept of perfect graphs was initially proposed by Berge, taking a distinct perspective. It is widely known that determining the chromatic number $\chi(G)$ of a graph G is a challenging task, and obtaining a good lower bound is also difficult. One straightforward lower bound is the clique number $\omega(G)$ because, to color the vertices in the largest clique of G, $\omega(G)$ colors are required. Mycielski [97] demonstrated a method to construct graphs with clique number two and arbitrarily large chromatic number. Consequently, the gap between $\omega(G)$ and $\chi(G)$ can be arbitrarily large. It is natural to investigate which graphs satisfy the equality between these two parameters. There are interesting graphs, such as bipartite graphs. It can also be shown that the equality holds for the complements of bipartite graphs. However, not every graph that satisfies $\chi(G) = \omega(G)$ is noteworthy. By taking the union of two graphs, one being a complete graph with n vertices and the other an arbitrary graph with fewer than

n vertices, we can effortlessly construct numerous graphs that satisfy the equality. Nevertheless, these constructed graphs are generally considered uninteresting. This is because the chromatic number and clique number depend solely on the component corresponding to the large clique, while the other component has no influence on these two parameters. One would like to seek a class of graphs that satisfy $\chi(G) = \omega(G)$, including the aforementioned interesting graphs but excluding the uninteresting ones. Berge proposed a nice class of graphs by making the property hereditary and this class of graphs are just the perfect graphs. In Berge's definition, a graph G is perfect if $\chi(H) = \omega(H)$ for every induced subgraph H of G.

It is worth noting that a graph can contain exponentially many cliques, which implies that the number of inequalities in the description of $P_K(G)$ can also be exponentially large. For example, consider a complete graph G with n vertices. Since every nonempty subset of $V(G)$ is a clique, there are $2^n - 1$ cliques in G. Although this example yields exponentially many inequalities, it is important to observe that every clique inequality induced by a clique that is a proper subset of $V(G)$ is dominated by the inequality $\sum_{v \in V(G)} x_v \leq 1$. Thus, we can hope that we do not need an inequality for each individual clique in G. Padberg [100] demonstrated that it suffices to describe $P_K(G)$ using the clique inequalities corresponding to the maximal cliques in G. This naturally raises the question of how many maximal cliques can exist in a graph and whether the number of maximal cliques is polynomially bounded from above. Moon and Moser [91] showed that every graph has at most $3^{n/3}$ maximal cliques, and they provided examples where this bound is achieved. One of the effective algorithms for listing all maximal cliques was introduced by Bron and Kerbosch [17].

In general graphs, the separation problem over $P_K(G)$ is known to be $\mathcal{NP}$-hard [98, Section 1.6.3]. However, there exists a larger class of inequalities called orthogonality inequalities, which includes clique inequalities and can be separated in polynomial time [62, 79]. The set of vectors satisfying both the orthogonality inequalities and the trivial inequalities is referred to as the theta body, denoted as $\mathrm{TH}(G)$. Grötschel et al. [62] demonstrated that

$$P_I(G) \subseteq \mathrm{TH}(G) \subseteq P_K(G).$$

Notably, in the case of perfect graphs, this theta body becomes a polytope that is equivalent to the independent set polytope [62]. This significant result implies that the maximum-weight independent set problem can be solved in polynomial time for perfect graphs. Additionally, Grötschel et al. [61] developed an efficient algorithm for extracting a maximum-weight independent set from a given graph.

Given a graph, the objective now becomes to determine whether it is perfect. To achieve this, it is desirable to establish a structural characterization similar to that of bipartite graphs. Berge made several observations in this regard. Firstly, he noted that any induced odd cycle with length greater than four is not perfect. This is supported by the fact that such cycles have clique number of two and chromatic number of three. Furthermore, Berge observed that the complement of such a cycle is also not perfect. Consequently, any graph containing an induced odd cycle of length at least five or the complement of such a cycle is not perfect. Despite Berge's extensive search for additional examples of imperfect graphs, he was unable to find any, leading to the formulation of the following conjecture.

Strong perfect graph conjecture. *A graph G is perfect if neither G nor its complement contains an induced odd cycle of length greater than four.*

This conjecture introduced by Berge came to be known as the strong perfect graph conjecture. Realizing the potential difficulty of resolving this conjecture, Berge also formulated a weaker conjecture with the aim of providing a more accessible objective to pursue.

Weak perfect graph conjecture. *If a graph is perfect then so is its complement.*

The weak perfect graph conjecture was resolved by Lovász [78]. However, the strong perfect graph conjecture remained open for over four decades until it was finally resolved by Chudnovsky, Robertson, Seymour, and Thomas in a groundbreaking 150-page paper [27]. Both conjectures now become to theorems. Chudnovsky et al. [24] demonstrated that perfect graphs can be effectively recognized. For additional efficient algorithms that recognize perfect graphs, please refer to [23, 28].

Computing the parameters $\alpha(G)$ (independence number), $\chi(G)$, $\omega(G)$, and $\overline{\chi}(G)$ (clique cover number) for a general graph G is known to be a challenging task. However, if G is a perfect graph, there exist efficient algorithms for computing the independence number $\alpha(G)$. Since $\alpha(G)$ can be computed efficiently and perfect graphs are closed under taking complementation, it follows that the other three parameters can also be computed efficiently for perfect graphs. Consequently, perfect graphs hold a very important place in graph theory and combinatorial optimization.

T-perfect graphs

The polyhedral characterization of perfect graphs has inspired interest in studying various variations. Instead of considering the smallest non-bipartite graph K_3 as a complete graph, it can alternatively be seen as an odd cycle. This observation leads to the introduction of odd-cycle inequalities:

$$\sum_{v \in C} x_v \leq \frac{|V(C)|-1}{2} \quad \text{for every odd-cycle } C \text{ in } G. \qquad \text{(odd-cycle inequalities)}$$

These inequalities stem from the fact that every odd cycle C intersects at most $(|V(C)|-1)/2$ vertices with an independent set. Motivated by the polyhedral characterization of perfect graphs, Chvátal in the same paper [29] proposed a realization of the independent set polytope, denoted as $P_{\mathrm{OC}}(G)$ and called odd cycle polytope, which is described by trivial, edge, and odd-cycle inequalities. It is evident that

$$P_I(G) \subseteq P_{\mathrm{OC}}(G) \subseteq P_E(G).$$

Chvátal became intrigued by the question of which graphs satisfy $P_{\mathrm{OC}}(G) = P_I(G)$. The class of graphs that satisfy this equality later became known as *t-perfect*[2] graphs. Grötschel et al. [62] showed that the separation problem in odd-cycle inequalities can be reduced to finding a shortest path in a specific auxiliary graph. The fact that the shortest path problem can be efficiently solved [43, 115] implies that a maximum-weight independent set in a t-perfect graph can be effectively found. This serves as the core motivation for studying t-perfect graphs. Furthermore, the study of extended formulations for the odd cycle inequalities of the stable set polytope [41, 122] also showed the polynomial-time solvability of the maximum-weight independent set problem on t-perfect graphs. Moreover, Eisenbrand et al. [47] presented a combinatorial polynomial-time algorithm for determining the independence number of a t-perfect graph. Given that t-perfect graphs are defined from a polyhedral perspective, a natural question arises: how can we recognize them? To effectively recognize t-perfect graphs, a profound understanding of their structure is essential.

Like perfect graphs, the class of t-perfect graphs is closed under vertex deletions. In addition, Gerards and Shepherd [56] demonstrated that t-perfection is also preserved under *t-contractions*, where a vertex with the neighborhood forming an independent set is contracted along with all incident edges. A graph obtained through a sequence of vertex deletions and t-contractions is called a t-minor. It is straightforward to see that t-perfection is maintained under taking t-minors. A graph G is considered *minimally t-imperfect* if it is t-imperfect but every t-minor distinct from G is t-perfect. Having a complete list of minimally t-imperfect graphs would allow for the characterization of t-perfection based on these graphs. However, even a conjecture on minimally t-imperfect graphs has yet to be established. To date, known

[2]The t here represents the French word trou which means hole.

minimally t-imperfect graphs include odd wheels [115], even Möbius ladders [116], $(3,3)$-partitionable graphs [21,31], and the complements of certain cycle powers, such as $\overline{C_7}$, $\overline{C_{13}^3}$, $\overline{C_{13}^4}$, and $\overline{C_{19}^7}$ [18,21]. Figures 1.2 and 1.3 illustrate these graphs.

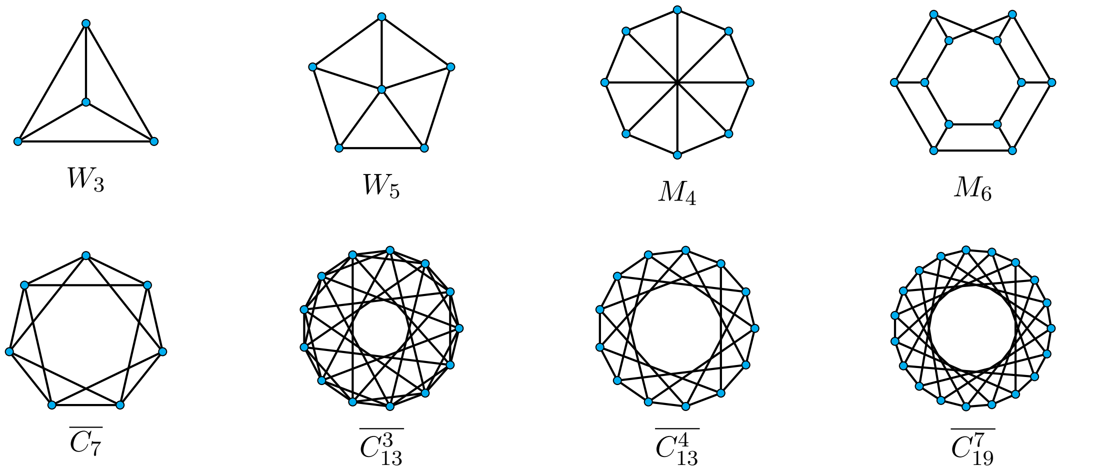


Figure 1.2: The first row shows odd wheels (W_{2k+1}) and even Möbius ladders (M_{2k}) and the second row shows complements of cycle powers.

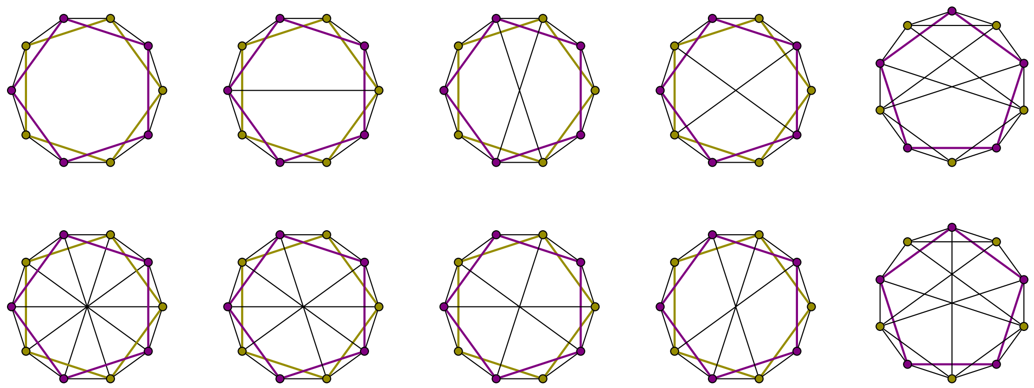


Figure 1.3: The $(3,3)$-partitionable graphs. The first graph is C_{10}^2.

Studying t-perfection is generally a challenging task. To obtain meaningful and achievable results, it is often necessary to investigate t-perfection within specific restricted graph classes. The concept of t-perfect graphs was initially introduced for studying the maximum-weight independent set problem. Motivated by the study of this problem in hereditary graph classes, we focus on exploring t-perfection in H-free graphs, where H is a tree with at most three leaves. Another motivation for studying t-perfection in H-free graphs is that this class of graphs is closed under taking t-minors. Consequently, we can characterize H-free t-perfect graphs using minimally t-imperfect graphs.

If the order of H is at most four, then H is either a path graph or the claw graph. Additionally, if H is a path graph, then H-free graphs are known to be perfect by the strong perfect graph theorem. Moreover, K_4 is the only minimally t-imperfect graph for H-free graphs, as will be explained in the next chapter. For claw-free graphs, Bruhn and Stein provided a complete list of minimally t-imperfect graphs in their work [21]. For graphs H of order five, there are only two graphs under our consideration that are P_5 and fork. While the P_5 is a natural generalization of the P_4, the fork graph is a generalization of both the claw and the P_4. Specifically, a fork can be obtained by attaching a private neighbor to a degree-one vertex of a claw or a degree-two vertex of a P_4. Bruhn and Fuchs showed a complete list of minimally t-imperfect graphs for t-perfect graphs that are free of P_5's [18]. In this thesis, we present all minimally t-imperfect graphs for t-perfect graphs that are fork-free.

A graph G is *strongly t-perfect* if the linear description of $P_{\mathrm{OC}}(G)$ is totally dual integral. It follows from the observation of Edmonds and Giles [46] on totally dual integrality that every strongly t-perfect graph is t-perfect. The other direction remains an open problem. In particular, we do not know whether all P_5-free t-perfect graphs are strongly t-perfect, though it is true for all claw-free t-perfect graphs [20]. In this thesis, we prove that fork-free t-perfect graphs are strongly t-perfect and obtain the following result.

Theorem 1.1. *Let G be a fork-free graph. The following statements are equivalent:*

i) G is t-perfect.

ii) G is strongly t-perfect.

iii) G does not contain the C_7^2, the C_{10}^2, or any odd wheel as a t-minor.

Our structural study toward Theorem 1.1 enables us to develop polynomial-time algorithms for recognizing t-perfect graphs that are fork-free.

Theorem 1.2. *Given a fork-free graph, we can decide in polynomial time whether it is t-perfect.*

It is conjectured that every t-perfect graph is four-colorable [69, 115]. We show that three colors already suffice for a fork-free t-perfect graph.

Theorem 1.3. *Let G be a fork-free graph. If G is t-perfect, then the chromatic number of G is at most three, and an optimal coloring can be found in polynomial time.*

As shown by the weak perfect graph theorem, perfect graphs are closed under complementation. The key step of proving the weak perfect graph theorem is the Replication Lemma: The class of perfect graphs is closed under (clique) substitution. Since K_4 can be obtained by substituting a vertex of a triangle by a K_2 or obtained by taking the complement of $\overline{K_4}$, t-perfection is closed under neither substitution nor complementation. This observation may partially explain the difficulty in characterizing t-perfect graphs. Benchetrit [11] has fully characterized t-perfection with respect to substitution. Our focus is to investigate t-perfection in complementation. Specifically, we want to know whether there exist minimally t-imperfect graphs whose complements are also minimally t-imperfect. Upon careful examination of Figure 1.2, we notice that with few exceptions, (W_3, W_5, W_7, $\overline{C_7}$), the complements of all the others contain a K_4 and therefore not minimally t-imperfect graphs. On the other hand, it is quite obvious that the graphs in the second row of Figure 1.3 are precisely the complements of those in the first. Our result is that the ten $(3, 3)$-partitionable graphs are the all minimally t-imperfect graphs whose complements are also minimally t-imperfect.

Theorem 1.4. *Let G be a minimally t-imperfect graph. The complement of G is minimally t-imperfect if and only if G is a $(3,3)$-partitionable graph.*

A graph is called *self-complementary* if it is isomorphic to its complement. The existence of self-complementary graphs was independently solved by Sachs [111] and Ringel [108]. Their work showed that a self-complementary graph exists with n vertices if and only if $n = 4k$ or $n = 4k + 1$, where k is a positive integer. When considering graphs with a single vertex, the graph is trivially self-complementary. Among graphs with four vertices, there exists only one self-complementary graph that is P_4. As for graphs with five vertices, there are only two self-complementary graphs: C_5 and the bull graph (Figure 1.4). Self-complementary graphs with order at most thirteen were catalogued in [6, 48, 76, 89, 120, 121].

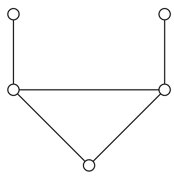


Figure 1.4: The bull graph.

We characterize all self-complementary graphs that are t-perfect: there are 20 of them. In particular, if a self-complement graph is t-perfect but not perfect, then it contains a C_5, and is C_5 itself, or one of graphs in Figure 1.5. All the other self-complementary t-perfect graphs are perfect. Let us remark parenthetically that there is an infinite number of self-complementary graphs that are perfect, e.g., obtained by the 4-path addition [72].

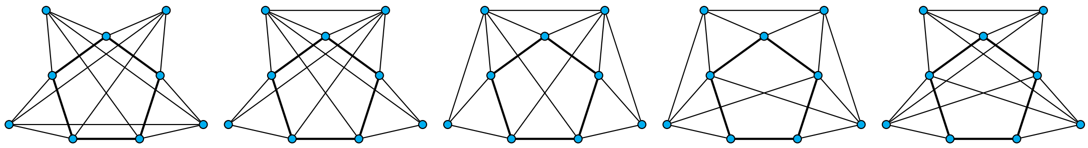


Figure 1.5: Self-complementary graphs that are t-perfect but not perfect ($n > 5$).

Theorem 1.5. *Let G be a self-complementary graph. Then G is t-perfect if and only if G is a K_1, a P_4, a C_5, a bull, or one of the graphs in Figures 1.5 and 1.6.*

Since K_4 is not t-perfect, it suffices to focus on K_4-free graphs. By the Ramsey theorem, a graph on 18 or more vertices contains a K_4 or its complement. Thus, it suffices to consider graphs of no more than 17 vertices[3]. Although this fact narrows the search space down greatly, simple enumeration is not really practical.

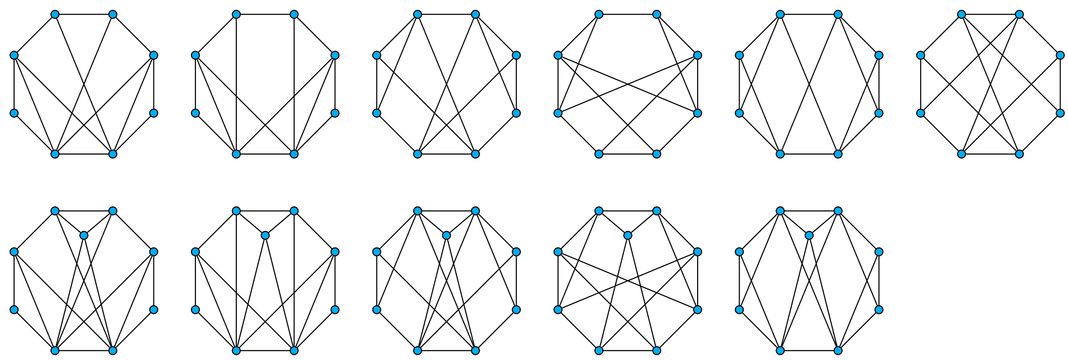


Figure 1.6: Self-complementary graphs that are both perfect and t-perfect ($n > 5$).

From the definition of t-perfect graphs one can easily see that every K_4-free perfect graph is t-perfect. On the other hand, a K_4-free self-complementary perfect graph contains at most nine vertices: it is $\overline{K_4}$-free and 3-colorable. One can thus easily get those self-complementary graphs that are both perfect and t-perfect, the three small ones, (K_1, P_4, and the bull graph), and the eleven graphs in Figure 1.6.

Self-complementary graphs

Self-complementary graphs hold a significant role in graph theory. The relationship between self-complementary graphs and Ramsey numbers is particularly noteworthy. If a self-complementary graph of order n does not contain a clique of size k,

[3]However, there are already 12005168 non-isomorphic graphs of ten vertices (`http://oeis.org/A000088`).

it implies that the Ramsey number $R(k,k)$ is strictly greater than n. This relationship allows for the establishment of bounds on Ramsey numbers through the study of self-complementary graphs [1, 22, 30, 34, 65, 88, 109]. In addition, self-complementary graphs are crucial in the study of the strong perfect graph conjecture. Corneil's work [39] has demonstrated that self-complementary graphs serve as a key point in determining whether the conjecture holds true or not. Furthermore, the study of self-complementary graphs has also shed light on the isomorphism problem. Colbourn and Colbourn [37] showed that the isomorphism problem for (regular) self-complementary graphs is polynomially equivalent to the general isomorphism problem. In fact, simply recognizing whether a graph is self-complementary is also polynomially equivalent to the graph isomorphism problem. In addition to their theoretical significance, self-complementary graphs possess strong structural properties.

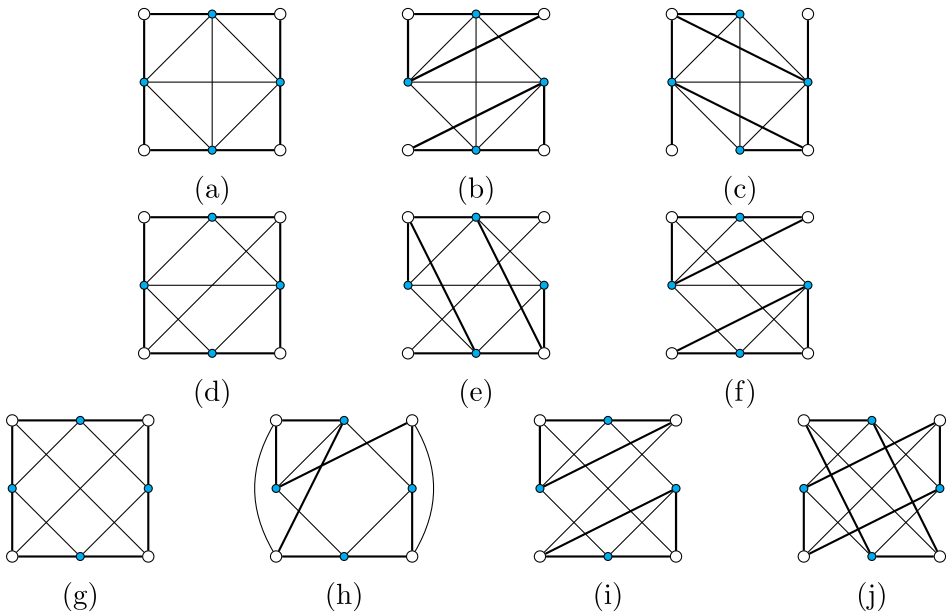
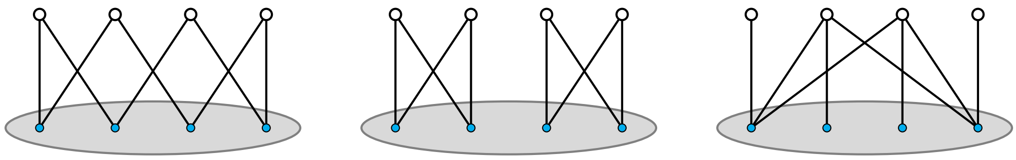


Figure 1.7: All self-complementary graphs on eight vertices.

A graph is a *split graph* if its vertex set can be partitioned into a clique and an independent set. The class of self-complementary graphs and the class of split graphs are connected by the following observation. Consider a self-complementary graph G of order $4k$, where L (resp., H) represents the set of $2k$ vertices with smaller (resp., higher) degrees. Note that $d(x) \leq 2k-1 < 2k \leq d(y)$ for every pair of vertices $x \in L$ and $y \in H$. Xu and Wong [121] observed that the subgraphs of G induced by L and H are complementary to each other. More importantly, the bipartite graph spanned by the edges between L and H is closed under *bipartite complementation* (reverse edges in between but keep both L and H independent). See the thick edges in Figure 1.7. When studying the connection between L and H, it is more convenient to add all the missing edges among H and remove all the edges among L, thereby turning G into a self-complementary split graph. In this sense, every self-complementary graph of order $4k$ can be constructed from a self-complementary split graph of the same order and a graph of order $2k$. For a self-complementary graph of an odd order, the self-complementary split graph is replaced by a self-complementary pseudo-split graph. A pseudo-split graph is either a split graph or a split graph plus a five-cycle such that every vertex on the cycle is adjacent to every vertex in the clique of the split graph and is nonadjacent to any vertex in the independent set of the split graph.



(a) (b) (c)

Figure 1.8: Self-complementary split graphs with eight vertices. Vertices in I are represented by empty nodes on the top, while vertices in K are represented by filled nodes on the bottom. For clarity, edges among vertices in K are omitted. Their degree sequences are (a) $(5^4, 2^4)$, (b) $(5^4, 2^4)$, and (c) $(6^2, 4^2, 3^2, 1^2)$.

The decomposition theorem of Xu and Wong [121] was for the construction of self-complementary graphs, of which another ingredient is their degree sequences (the non-increasing sequence of its vertex degrees). Clapham and Kleitman [33, 36] present a necessary condition for a degree sequence to be that of a self-complementary graph. However, a realization of such a degree sequence may or may not be self-complementary. A natural question is on degree sequences of which all realizations are necessarily self-complementary, called *forcibly self-complementary*. All the degree sequences for self-complementary graphs up to order five, (0^1), $(2^2, 1^2)$, (2^5), and $(3^2, 2^1, 1^2)$, are forcibly self-complementary. Of the four degree sequences for the self-complementary graphs of order eight, only $(5^4, 2^4)$ and $(6^2, 4^2, 3^2, 1^2)$ are focibly self-complementary; see Fig. 1.8. All the realizations of these forcibly self-complementary degree sequences turn out to be pseudo-split graphs. As we will see, this is not incidental.

We take p graphs S_1, S_2, ..., S_p, each being either a four-path or one of the first two graphs in Fig. 1.8. Note that the each of them admits a unique decomposition into a clique K_i and an independent set I_i. For any pair of i, j with $1 \leq i < j \leq p$, we add all possible edges between K_i and $K_j \cup I_j$. It is easy to verify that the resulting graph is self-complementary, and can be partitioned into clique $\bigcup_{i=1}^{p} K_i$ and independent set $\bigcup_{i=1}^{p} I_i$. We use an *elementary self-complementary pseudo-split graph* to such a graph, or one obtained from it by adding a single vertex or a five-cycle and make them complete to $\bigcup_{i=1}^{p} K_i$. For example, we end with the graph in Fig. 1.8(c) with $p = 2$ and both S_1 and S_2 being four-paths. It is a routine exercise to verify that the degree sequence of an elementary self-complementary pseudo-split graph is forcibly self-complementary. We show that the other direction holds as well, thereby fully characterizing forcibly self-complementary degree sequences.

Theorem 1.6. *A degree sequence is forcibly self-complementary if and only if every realization of it is an elementary self-complementary pseudo-split graph.*

Our result also bridges a longstanding gap in the literature on self-complementary graphs. Rao [105] has proposed another characterization for forcibly self-complementary degree sequences (we leave the statement, which is too technical, to Section 5.2). As far as we can check, he never published a proof of his characterization. It follows immediately from Theorem 1.6.

All self-complementary graphs up to order five are pseudo-split graphs, while only three out of the ten self-complementary graphs of order eight are. By examining the list of small self-complementary graphs, Ali [5] counted self-complementary split graphs up to 17 vertices. Whether a graph is a split graph can be determined solely by its degree sequence. However, this approach needs the list of all self-complementary graphs, and hence cannot be generalized to large graphs. Answering a question of Harary [66], Read [106] presented a formula for the number of self-complementary graphs with a specific number of vertices. Clapham [35] simplified Read's formula by studying the isomorphisms between a self-complementary graph and its complement. We take an approach similar to Clapham's for self-complementary split graphs with an even order, which leads to a formula for the number of such graphs. For other self-complementary pseudo-split graphs, we establish a one-to-one correspondence between self-complementary split graphs on $4k$ vertices and those on $4k+1$ vertices, and a one-to-one correspondence between self-complementary pseudo-split graphs of order $4k+1$ that are not split graphs and self-complementary split graphs on $4k-4$ vertices.

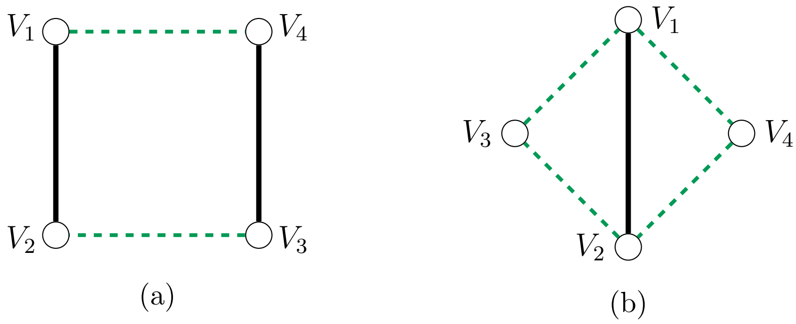
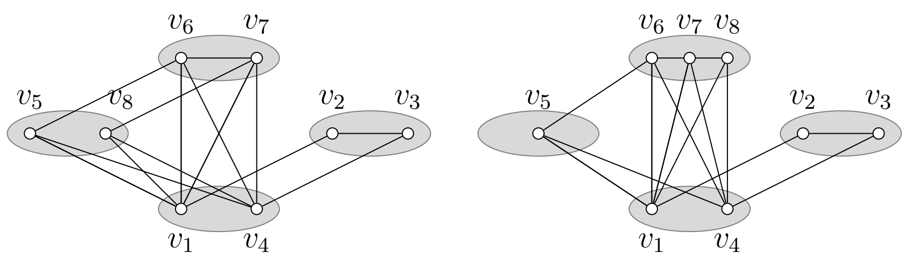


Figure 1.9: The (a) rectangle and (b) diamond partitions. Each node represents one part of the partition. A solid line indicates that all the edges between the two parts are present, a missing line indicates that there are no edges between the two parts, while a dashed line imposes no restrictions on the two parts.

We also study the conjecture of Trotignon [118], which asserts that if a self-complementary graph G does not contain a five-cycle, then its vertex set can be partitioned into four nonempty sets with the adjacency patterns of a rectangle or a diamond, as described in Figure 1.9. He managed to prove certain special graphs satisfy this conjecture. We prove Trotignon's conjecture on self-complementary split graphs, with a stronger statement. We say that a partition of $V(G)$ is *self-complementary* if it forms the same partition in the complement of G, illustrated in Figure 1.10.



(a) (b)

Figure 1.10: Two diamond partitions, of which only the first is self-complementary.

There is another natural motivation of studying self-complementary split graphs. Sridharan and Balaji [117] tried to understand self-complementary graphs that are chordal. They are precisely split graphs [49]. The class of split graphs is *closed under complementation*.[4] We may study self-complementary graphs in other graph classes. Again, for this purpose, it suffices to focus on those closed under complementation. In the simplest case, we can define such a class by forbidding a graph F as well as its complement. It is not interesting when F consists two or three vertices, or is the four-path. When F is the four-cycle, we end with the class of pseudo-split graphs, which is the simplest in this sense.

1.2 Outline and main contributions

This thesis is structured into six chapters, comprising the introductory chapter (Chapter 1), the preliminary chapter (Chapter 2), three main chapters (Chapters 3, 4, and 5), and a concluding chapter (Chapter 6).

- **Chapter 2** In this chapter, we begin by introducing the basics and notations

[4]Some authors call such graph classes "self-complementary," e.g., the influential "Information System on Graph Classes and their Inclusions" (https://www.graphclasses.org).

necessary for our study. We then delve into polyhedral theory, linear programming, and focus on the independent set polytope. Additionally, we present further results in t-perfection that are of great significance for the subsequent chapters. These results will serve as essential tools and insights in our exploration of t-perfection.

- **Chapter 3** In this chapter, our focus lies on the study of t-perfection in fork-free graphs. We prove Theorems 1.1, 1.2, and 1.3 in this chapter. We provide a complete list of minimal forbidden t-minors for fork-free t-perfect graphs. Additionally, we show that every fork-free t-perfect graph is, in fact, strongly t-perfect. We also present polynomial-time algorithms for recognizing and coloring these graphs.

- **Chapter 4** In this chapter, our focus is on the study of complementation in t-perfect graphs. We are particularly interested in graphs G for which both G and its complement are t-perfect or minimally t-imperfect. This motivation leads us to introduce the concept of *core graphs*. A graph G is a core graph if neither G nor its complement contains a t-imperfect graph as a proper t-minor. In Section 4.1, we delve into the investigation of core graphs, exploring their structural properties. Specifically, we show that an imperfect core graph consists of at most ten vertices. Furthermore, we delve into the study of t-perfect core graphs in Section 4.2. By proving Theorem 1.5, we are able to identify all self-complementary t-perfect graphs. Moreover, we shift our focus to study minimally t-imperfect core graphs in Section 4.3. Through the proof of Theorem 1.4, we conclude that they can only be $(3,3)$-partitionable graphs.

- **Chapter 5** In this chapter, we study split graphs and pseudo-split graphs whose complements are isomorphic to themselves. In Section 5.1, we begin by introducing more about antimorphisms. Then we show a connection between self-complementary split graphs and self-complementary pseudo-split graphs.

This connection allows us to narrow our focus to split graphs. Furthermore, we establish a one-to-one correspondence between self-complementary split graphs on $4k$ vertices and those on $4k + 1$ vertices. We also study partitions in self-complementary graphs in this section. Additionally, we give a characterization for forcibly self-complementary degree sequences in Section 5.2. Finally, we tackle the enumeration problem of self-complementary split graphs in Section 5.3.

- **Chapter 6** We conclude this thesis by presenting an overview of open questions and conjectures that have captured our interest in the study of t-perfect graphs and self-complementary graphs. We analyze and discuss these unresolved problems, exploring their significance and potential implications. By presenting these open questions and conjectures, we aim to stimulate further study and foster a deeper understanding of t-perfect graphs and self-complementary graphs.

Chapter 2

Preliminaries

This chapter lays the foundation for understanding the rest of the thesis by introducing essential concepts.

2.1 Basics and notations

In graph theory, a *graph* is a mathematical structure that consists of a set of vertices and a set of edges that connect these vertices. Each edge in a graph represents a relationship or connection between two vertices. All the graphs discussed in this thesis are *finite*; that is, they have a finite number of vertices and edges. Additionally, we only consider *simple graphs*, which have at most one edge connecting any two distinct vertices and no edge that connects a vertex to itself. Furthermore, we focus our attention solely on *undirected graphs*, meaning that the edges do not have any direction associated with them. Conventionally, the vertex set and edge set of a graph G are denoted by, respectively, $V(G)$ and $E(G)$. Graphs are named so because they can be visually represented. Each vertex is depicted as a point, and each edge is represented by a line connecting the points corresponding to its ends. Figure 2.1 illustrates a graph G with $V(G) = \{a, b, c, d\}$ and $E(G) = \{ab, ac, ad, bc, bd, cd\}$.

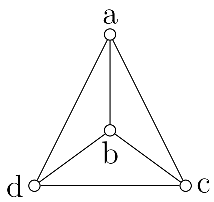


Figure 2.1: A diagram of K_4.

Let G be a graph. The *order* of G refers to the number of vertices in its vertex set, while the *size* of G corresponds to the number of edges in its edge set. Two vertices u and v are *adjacent* in G if there exists an edge connecting them. Conversely, if there is an edge uv in G, we refer to u and v as the *end vertices* or *ends* of uv, and we say that uv is *incident* with both u and v. Since we focus on undirected graphs, the order of vertices in an edge does not matter, so uv is equivalent to vu. The *complement* $\overline{G}$ of G has the same vertex set as G, and two distinct vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in G. The graph G is considered *isomorphic* to another graph H, denoted as $G \cong H$, if there exists a bijection $\phi : V(G) \to V(H)$ such that two vertices u and v are adjacent in G if and only if $\phi(u)$ and $\phi(v)$ are adjacent in H. We say that two sets of vertices are *complete* or *nonadjacent* if there are all possible edges or no edges between them respectively.

Adjacent vertices are commonly referred to as *neighbors* of each other. In the graph G, the set of neighbors of a specific vertex u is known as the *neighborhood* of u, denoted as $N_G(u)$. Additionally, the *closed neighborhood* of u in G is represented as $N_G[u]$, which is defined as the union of $N_G(u)$ and the vertex u itself. The *degree* of vertex u in G, denoted as $d_G(u)$, corresponds to the cardinality of $N_G(u)$. A vertex is considered *isolated* in G if it has no neighbor in G. If every vertex in a graph has the same degree, say k, then we call the graph *k-regular*. In the notations defined in this paragraph, if the graph G is clear from the context, we can remove the subscript G.

If a graph H can be obtained from G by deleting some vertices, we say that G *contains* H, or that H is an *induced subgraph* of G. On the other hand, if H cannot be obtained from G by deleting vertices, we say that G is *H-free*. For a subset $U \subseteq V(G)$, let $G[U]$ denote the subgraph of G induced by U, whose vertex set is U and whose edge set comprises all the edges whose both ends are in U, and let $G - U = G[V(G) \setminus U]$, which is simplified as $G - u$ if U comprises a single vertex u. When a graph H can be obtained from G by deleting some vertices and edges, we say that H is a *subgraph* of G. It is noteworthy that while an induced subgraph of G is a subgraph of G, the reverse is not necessarily true.

A *path* in G is a sequence of distinct vertices $v_1, v_2, \ldots, v_\ell$ of G, where $\ell \geq 1$, such that for every $i = 1, \ldots, \ell - 1$, there is an edge between vertices v_i and v_{i+1} in G. The vertices v_1 and v_ℓ are called the *ends* of the path, while the remaining vertices $v_2, v_3, \ldots, v_{\ell-1}$ are referred to as the *inner vertices*. We say that there exists a path between two vertices u and v if there is a path with ends u and v. A graph is considered *connected* if there is a path between any two vertices in the graph. A *connected component* of a graph G is a subgraph of G that is both connected and inclusion-wise maximal, meaning that it cannot be further enlarged while preserving the property of connectivity. A connected component must have at least one vertex. It is noteworthy that every vertex in G belongs to exactly one connected component. This implies that the vertex set of G can be partitioned into disjoint subsets, each representing a connected component. Furthermore, it is important to emphasize that a graph is not connected if and only if it consists of more than one connected component.

A subset X of the vertex set $V(G)$ is called a *vertex-cut* of graph G if the number of connected components in $G - X$ is greater than the number of connected components in G. Moreover, if the cardinality of X is k, then X is called a *k-vertex-cut*. A graph is called *k-connected* for any positive integer k if it contains more than k vertices and has no k-vertex-cut.

If all vertices in G are pairwise adjacent, then G is referred to as a *complete graph*. Alternatively, if the vertices of G can be arranged in a linear sequence such that two vertices are adjacent if and only if they are consecutive in the sequence, then G is known as a *path graph*. Similarly, if G contains at least three vertices and the vertices of G can be arranged in a cyclic sequence such that two vertices are adjacent if and only if they are consecutive in the sequence, then G is classified as a *cycle graph*. Up to isomorphism, there is a unique complete graph, a unique path graph, and a unique cycle graph on a given number of vertices.

For $\ell \geq 3$, we denote the complete graph, path graph, and cycle graph on ℓ vertices as K_ℓ, P_ℓ, and C_ℓ, respectively. It is worth noting that K_3 is equivalent to C_3 and is commonly referred to as a *triangle graph*. A *hole* is defined as a C_ℓ with $\ell \geq 4$. On the other hand, a *wheel* W_ℓ is obtained by introducing a new vertex to the C_ℓ and connecting it to all the existing vertices of C_ℓ. In the context of cycles, holes, and wheels, an ℓ-cycle, ℓ-hole, or ℓ-wheel is considered *odd* if ℓ is an odd number.

If the vertex set of G can be partitioned into two subsets X and Y, such that every edge of G has one end in X and the other end in Y, then G is called a *bipartite graph*. We can represent G with its bipartition as $G[X, Y]$. In a bipartite graph, there are no edges connecting vertices within the same subset. If every vertex in X is adjacent to every vertex in Y, then G is referred to as a *complete bipartite graph*. For any two positive integers m and n, there exists a unique complete bipartite graph, denoted as $K_{m,n}$, with parts of sizes m and n, respectively (up to isomorphism). Notably, graphs of the form $K_{1,n}$ are called *stars*, and the vertex in the singleton part of $K_{1,n}$ is referred to as the star's *center*. Furthermore, if there exists a vertex in G whose removal leaves a bipartite graph, then G is categorized as an *almost bipartite graph*. On the other hand, if the removal of the closed neighborhood of any vertex in G results in a bipartite graph, then G is known as a *near-bipartite graph*. It is evident that a bipartite graph cannot contain an odd cycle. In fact, the converse is also true [42].

The *line graph* H of G is a graph whose vertex set corresponds to the edge set of G, where two vertices in H are adjacent if their corresponding edges in G share a common vertex. A graph H is considered a line graph if there exists a graph such that H is the line graph of that graph.

A graph is a *tree* if it is connected and does not contain any cycle. For integers $i, j, k \geq 1$, we denote by $S_{i,j,k}$ the tree with exactly three leaves, each at distance i, j, and k from the unique vertex of degree three. The claw graph is isomorphic to $S_{1,1,1}$, the fork graph is isomorphic to $S_{1,1,2}$, and the path graph P_4 is isomorphic to $S_{0,1,2}$.

A subset X of the vertex set $V(G)$ is called a *clique* if the induced subgraph $G[X]$ is a complete graph, meaning that all vertices in X are pairwise adjacent in G. On the other hand, X is called an *independent set* if the complement of the induced subgraph $G[X]$ is a complete graph, meaning that no two vertices in X are adjacent in G. In other words, a clique is a set of vertices in G such that every pair of vertices in the set is adjacent, while an independent set is a set of vertices such that no two vertices in the set are adjacent in G.

A clique (resp., independent set) X of G is said to be *maximal* if $X \cup \{v\}$ is not a clique (resp., independent set) of G for every $v \in V(G) \setminus X$. A *maximum clique* (resp., *maximum independent set*) of G is a clique (resp., independent set) that has the maximum number of vertices compared to all other cliques (resp., independent set) in G. The number of vertices in a maximum clique is called the clique number of G and is denoted as $\omega(G)$, while the number of vertices in a maximum independent set is called the independence number of G and is denoted as $\alpha(G)$. A clique or independent set X of G is said to be *maximum-weight* under the weight function $w : V(G) \to \mathbb{R}_{\geq 0}$ if the sum of weights of vertices in X, denoted by $\sum_{x \in X} w(x)$, is maximized among all cliques or independent sets, respectively, in G with respect to the weight function.

For a positive integer k, a graph G is said to be *k-colorable* if we can partition the vertex set $V(G)$ into k independent sets. In other words, we can assign one of k different colors to each vertex in such a way that no two adjacent vertices have the same color. The smallest value of k for which G is k-colorable is known as the *chromatic number* of G, denoted by $\chi(G)$. On the other hand, the *clique cover number* of G, denoted as $\overline{\chi}(G)$, is the smallest number of cliques needed to cover the vertex set $V(G)$.

A *matching* of G is a set of edges without common vertices. There is a one-to-one correspondence between matchings in G and independent sets in its line graph. Given a matching in G, the corresponding independent set in its line graph consists of the vertices representing the matched edges in G. Conversely, given an independent set in the line graph of G, the corresponding matching in G consists of the edges represented by the vertices in the independent set.

2.2 Polyhedra and linear inequalities

Let $x_0, x_1, \ldots, x_\ell$ be vectors in $\mathbb{R}^n$. If there exist scalars $\lambda_1, \ldots, \lambda_\ell$ such that $x_0 = \sum_{j=1}^{\ell} \lambda_j x_j$, then x_0 is considered a *linear combination* of the other vectors. Furthermore, if $\lambda_1, \ldots, \lambda_\ell$ satisfy the condition $\sum_{j=1}^{\ell} \lambda_j = 1$, then x_0 is an *affine combination* of the other vectors. Moreover, if $\lambda_1, \ldots, \lambda_\ell$ are all nonnegative, then x_0 is a *convex combination* of the other vectors. A set $S \subseteq \mathbb{R}^n$ is said to be linear, affine, or convex if S contains all the linear, affine, or convex combinations of its elements, respectively.

Vectors $x_0, x_1, \ldots, x_\ell$ are *affinely independent* if and only if no vector in $x_0, x_1, \ldots,$ x_ℓ can be written as an affine combination of the other vectors. For a set $S \subseteq \mathbb{R}^n$, the *dimension* of S, denoted as $\dim(S)$, is defined as one less than the maximum number of affinely independent vectors in S. If the dimension of S is n, then S is called *full-dimensional.* This implies that S has $n+1$ affinely independent vectors. The *convex hull* of S, denoted by $\text{conv}(S)$, is the set of all vectors that are convex combinations of elements in S.

Let P be a subset of $\mathbb{R}^n$. If there exists a real matrix A and a real vector b such that

$$P = \{x \in \mathbb{R}^n : Ax \leq b\},$$

then P is a *polyhedron* in $\mathbb{R}^n$. If A and b can be chosen such that all their entries are rational, then P is a *rational polyhedron*. A bounded polyhedron is called a *polytope*. It is noteworthy that the entire space $\mathbb{R}^n$ itself is a polyhedron, as $0^T x \leq 0$ for all $x \in \mathbb{R}^n$. In addition, if $\alpha \in \mathbb{R}^n \setminus \{0\}$ and $\beta \in \mathbb{R}$, the polyhedron $\{x \in \mathbb{R}^n : \alpha^T x \leq \beta\}$ is called a *halfspace*, while the polyhedron $\{x \in \mathbb{R}^n : \alpha^T x = \beta\}$[1] is referred to as a *hyperplane*. From a geometrical point of view, a polyhedron can be understood as the intersection of a finite number of halfspaces.

Let $P \subseteq \mathbb{R}^n$ be a polyhedron. An inequality $\alpha^T x \leq \beta$ is called *valid* for P if it is satisfied by every point in P. A *face* F of P is a subset of P that can be expressed as the intersection of P with a hyperplane defined by a valid inequality. In other words, F is a set of the form

$$F = P \cap \{x \in \mathbb{R}^n : \alpha^T x = \beta\}$$

where $\alpha^T x \leq \beta$ is a valid inequality of P. The face F is also a polyhedron and said to be defined by the valid inequality $\alpha^T x \leq \beta$. A face of P is considered *proper* if it is nonempty and not equivalent to P. The *facets* of P are the inclusion-wise maximal proper faces, and the valid inequality that defines each facet is called a *facet-defining inequality* for P. The *vertices*, also known as *extreme points*, of a polyhedron P are points within P that cannot be expressed as convex combinations of two or more other points in P. The dimension of a facet of P is $\dim(P) - 1$, while the dimension of a vertex is 0.

Inequalities in a linear system $Ax \leq b$ that define the polyhedron P can be categorized as either *redundant* or *irredundant*. A redundant inequality does not

[1]Note that the equality constraint $\alpha^T x = \beta$ can be substituted by two inequality constraints $\alpha^T x \leq \beta$ and $-\alpha^T x \leq -\beta$.

affect the solution set when removed, while an irredundant inequality is necessary to define the polyhedron. Starting from the original system, one can iteratively eliminate redundant inequalities until no further redundancies exist. This process results in a reduced system, known as a *minimal representation*, which precisely describes the polyhedron P without any redundancies. The minimal representation is a concise and efficient description of P in terms of inequalities. It is worth noting that every polyhedron has a unique minimal representation up to multiplying the inequalities by a positive scalar.

Theorem 2.1 ([38]). *For a full-dimensional polyhedron P with a minimal representation $Ax \leq b$, the inequality system $Ax \leq b$ is uniquely defined up to multiplying the inequalities by a positive scalar.*

2.3 Linear programming

Linear programming is the problem of maximizing a linear objective function over a polyhedron. Given a matrix $A \in \mathbb{R}^{m \times n}$ and vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, a linear programming problem can be formulated as:

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0. \end{aligned} \tag{LP}$$

We use P to represent the polyhedron $\{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$, which is also referred to as the *feasible region* of (LP). If P is empty, it indicates that the linear programming problem is *infeasible*. On the other hand, if P is nonempty, every point within the feasible region represents a *feasible solution* to the problem. An *optimal solution*, denoted as x^*, is a feasible solution that satisfies $c^T x^* \geq c^T x$ for all x in P. In other words, it maximizes the objective function over the feasible region. The set of all optimal solutions defines an optimal solution face of P, representing the boundary of the highest values attainable for the objective function within the feasible region. The value of $c^T x^*$ is called *optimal value* of the problem.

The dual problem of (LP), is defined as follows:

$$\begin{aligned} \text{minimize} \quad & y^T b \\ \text{subject to} \quad & y^T A \geq c \\ & y \geq 0. \end{aligned} \tag{DP}$$

The *dual problem* (DP) aims to minimize the value $y^T b$ subject to the constraints $y^T A \geq c$ and $y \geq 0$, where y represents the vector of dual variables. The original problem (LP) from which the dual problem (DP) is derived is commonly known as the *primal problem*. In the realm of linear programming duality, a well-known theorem establishes a strong connection between the primal problem and its dual. For more comprehensive information on this topic, please refer to [38, 113].

Theorem 2.2 ([113, Duality theorem of linear programming])**.** *Given a matrix $A \in \mathbb{R}^{m \times n}$ and vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. Let $P = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$ and $D = \{y \in \mathbb{R}^m : y^T A \geq c, y \geq 0\}$. Then*

$$\max\{c^T x : Ax \leq b, x \geq 0\} = \min\{y^T b : y^T A \geq c, y \geq 0\},$$

if both P and D are nonempty.

The connection between the primal and dual problems in linear programming was initially conjectured by John von Neumann after George Dantzig introduced the linear programming problem. However, it was not until the publication by Gale, Kuhn, and Tucker in [51] that the first rigorous proof of this theorem was provided. Their work established the duality theory in linear programming and provided a solid mathematical foundation for understanding the relationship between the primal and dual problems. This development played a crucial role in advancing the theory and applications of linear programming.

The simplex method, developed by George Dantzig in the 1940s, is a widely used and well-established algorithm for solving linear programming problems. The simplex method begins at one of the extreme points (vertices) of the feasible region. It then iteratively improves the objective function value by moving from one vertex to another along the edges (face of dimension one) of the feasible region. In many practical scenarios, the simplex method exhibits good performance and effectively solves linear programming problems. However, it is important to note that the number of vertices within the feasible region of a given linear programming problem can become quite large. In the worst-case scenario, the running time of the simplex method may experience exponential growth due to the need to explore an extensive number of vertices [60, 74]. To address these limitations, the ellipsoid method was introduced as the first polynomial-time algorithm for solving linear programming problems [73]. The ellipsoid method offers theoretical significance and guarantees polynomial-time complexity. However, its practical implementation has certain drawbacks, leading to the development of alternative methods. One such alternative is the interior point method, pioneered by Narendra Karmarkar in 1984 [70]. Interior point methods have gained popularity for their ability to provide better performance in solving linear programming problems. They use a different approach, focusing on exploring the interior of the feasible region rather than moving along its edges. Interior point methods often exhibit improved efficiency and convergence properties, making them a preferred choice in practical applications. More details of these methods can be found in [63, 119].

In this thesis, we focus on linear programming problems that possess feasible regions, denoted as P, with the properties: (i) P is nonempty; (ii) P is a rational polytope; (iii) P has full-dimensional. Under these conditions, an optimal solution lies at one of the extreme points of the feasible region P.

Consider a linear programming problem represented by (LP). If we impose the condition that x must be integral (every coordinate of x is an integer), the problem transforms into an integer linear programming problem given by:

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0 \\ & x \text{ is integral.} \end{aligned} \tag{ILP}$$

The introduction of the integer constraint transforms the feasible region into a discrete set of points. Specifically, if x is constrained in $\{0, 1\}^n$, the problem is known as a binary integer linear programming problem. It is worth emphasizing that linear programming problems can be efficiently solved using polynomial-time algorithms. However, when integer constraints are introduced, the computational complexity of the problem escalates considerably. In fact, solving integer linear programming problems is classified as $\mathcal{NP}$-hard, denoting the inherent difficulty in finding optimal solutions. This indicates that, in general, there is no known polynomial-time algorithm capable of solving all instances of integer linear programming problems.

Consider the set S which represents the solutions of the integer linear programming problem defined in equation (ILP). The convex hull of S forms a polytope in which every extreme point has integral coordinates. Now, if a matrix A' and a vector b' can be obtained such that the convex hull of S can be expressed as:

$$\text{conv}(S) = \{x : A'x \leq b'\},$$

then the integer linear programming problem can be transformed into an equivalent linear programming problem.

A linear system $Ax \leq b$, where A and b are rational, is called *totally dual integral* (TDI) if for every integral vector c, the optimal value of the dual program is attained by an integeral vector y^* whenever the optimum exists and is finite. Edmonds and Giles [46] showed that if a polyhedron P is the feasible region of a TDI system $Ax \leq b$, where b is a integral vector, then every extreme point of P is integeral.

Theorem 2.3. *If $Ax \leq b$ is TDI and b is integral, then $Ax \leq b$ determines an integral polyhedron.*

2.4 The independent set polytope

For every independent set S of G, we can define an incidence vector χ^S with dimension $|V(G)|$. Each component χ_v^S of the vector is defined as:

$$\chi_v^S = \begin{cases} 1, & \text{if } v \in S, \\ 0, & \text{otherwise.} \end{cases}$$

The incidence vector χ^S provides a binary representation of the independent set S, where each component χ_v^S indicates whether the corresponding vertex v is present ($\chi_v^S = 1$) or absent ($\chi_v^S = 0$) in the independent set S. If we consider the convex hull of the incidence vectors of all the independent sets of G, we obtain a bounded polyhedron. This polyhedron is commonly referred to as an independent set polytope [113]. We denote the independent set polytope of G as $P_I(G)$. Figure 2.2 illustrates the graph P_3 and its independent set polytope.

From Figure 2.2, we can observe that the independent set polytope of P_3 is the intersection of five half-spaces, each of which can be represented as a linear inequality. Consequently, for a graph G, if we can find the defining linear system of $P_I(G)$, we can solve the maximum-weight independent set problem on G by solving the following

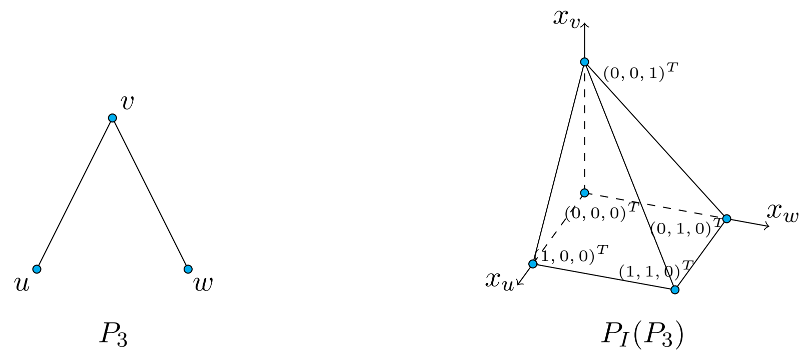


Figure 2.2: The graph P_3 and its independent set polytope $P_I(P_3)$.

linear programming problem:

$$\alpha_w(G) = \max\{w^T x : x \in P_I(G)\}.$$

The optimal value $\alpha_w(G)$ is referred to as the weighted independence number of G.

Consider the incidence vectors of independent sets in G with cardinality at most one. There are precisely $|V(G)| + 1$ such vectors, and they are affinely independent. It implies that the independent set polytope associated with G is full-dimensional. Furthermore, according to Theorem 2.1, the independent set polytope of G has a unique system of linear inequalities (up to multiplying the inequalities by a positive scalar) that describes its facets. For general graphs, a complete description of the facets of $P_I(G)$ is hard to obtain.

2.5 T-perfection

We continue to use the notations introduced in Chapter 1. The graph K_4 is the smallest graph (in terms of the number of vertices and edges) that is not t-perfect, hence not strongly t-perfect. It can be easily verified that the vector $x = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^T$ belongs to $P_{OC}(K_4)$. However, since $1^T x = \frac{4}{3} > 1$, the vector x does not belong to $P_I(K_4)$. A *subdivision* of a graph is obtained by subdividing edges of the graph into paths of length at least one. Figure 2.3 shows a subdivision of K_4. The class of graphs without any subdivision of K_4 as a subgraph is known as series-parallel graphs [44]. Chvátal [29] conjectured that $P_I(G) = P_{OC}(G)$ if G is a series-parallel graph, and this conjecture was later proved by Mahjoub [86]. Thus, series-parallel graphs are t-perfect. Boulala and Uhry [14] showed that series-parallel graphs are also strongly t-perfect. Gerards and Schrijver [55] extended t-perfection to graphs that do not contain an *odd-*K_4 subdivision as a subgraph. An odd-K_4 subdivision is obtained by turning each triangle of K_4 into an odd cycle. Gerards [54] further demonstrated that graphs that do not contain an odd-K_4 subdivision are strongly t-perfect, implying t-perfection and strong t-perfection for almost bipartite graphs. Barahona and Mahjoub [9] studied the independent set polytope of subdivisions of K_4 and provided a characterization for subdivisions of K_4 that are not t-perfect. A subdivision of K_4 is called *bad* if it is not t-perfect. Gerards and Shepherd [56] characterized that every subgraph of a graph G is t-perfect if and only if G contains no *bad-*K_4 subdivision as a subgraph. Schrijver extended this characterization to strongly t-perfect graphs in [114]. It is worth noting that there exist odd-K_4 subdivisions that are not bad. An example is shown in Figure 2.3.

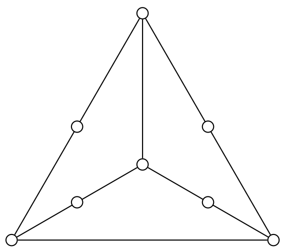


Figure 2.3: An odd-K_4 subdivision that is t-perfect.

A connection between perfect graphs and t-perfect graphs

The classes of t-perfect graphs and perfect graphs are incomparable: C_5 is t-perfect but not perfect, whereas K_4 is perfect but not t-perfect. Despite this, there is an insightful observation that establishes a connection between these two classes of graphs.

Let G be a graph that does not contain K_4. In such a graph, every clique has a size of at most three, implying that any clique constraint in the description of $P_K(G)$ is one of the three kinds of constraints in the description of $P_{OC}(G)$. Additionally, the odd-cycle constraints in description of $P_{OC}(G)$ can be limited to induced odd cycles, as the constraints on non-induced ones are redundant. If G is perfet, then it does not contain odd holes according to the strong perfect graph theorem. Consequently, the odd-cycle constraints in the description of $P_{OC}(G)$ reduce to triangle constraints. Thus, we can deduce that:

$$P_I(G) = P_K(G) = P_{OC}(G).$$

If the linear system that defines $P_K(G)$ is totally dual integral, then G is perfect [29, 46]. Lovász [78] showed that the converse is also true. Therefore, the linear system that defines $P_K(G)$ is totally dual integral if and only if G is perfect; see also [115] for more details. Based on this, we can derive the following result.

Proposition 2.4 (Folklore)**.** *Every K_4-free perfect graph is strongly t-perfect.*

Strong t-perfection

For each weight function $w : V(G) \to \mathbb{Z}_{\geq 0}$, we can make a linear programming problem out of $P_{OC}(G)$ by adding an objective function

$$\max \quad w^T x.$$

The dual of this linear programming problem is a covering problem. A *w-cover* is a family of vertices, edges, and odd cycles in G such that every vertex v in $V(G)$ lies in at least $w(v)$ elements, with repetition allowed. The *cost* of a w-cover is the sum of the costs of its elements, where the cost of a vertex or an edge is one, and the cost of an odd cycle C is $(|V(C)| - 1)/2$. For a vertex set S, we use $w(S)$ to denote $\sum_{v \in S} w(v)$. The following is a consequence of linear programming duality.

Proposition 2.5 ([115])**.** *A graph G is* strongly *t-perfect if and only if there exists a w-cover of cost $\alpha_w(G)$ for every weight function $w : V(G) \to \mathbb{Z}_{\geq 0}$.*

The following observation, implicit from Bruhn and Stein [20], is very helpful in checking the condition of Proposition 2.5. We provide a proof for the sake of completeness. Note that a vertex set K intersects every maximum-weight independent set of G if and only if $\alpha_w(G - K) < \alpha_w(G)$.

Proposition 2.6 ([20])**.** *Let G be a graph and $w : V(G) \to \mathbb{Z}_{\geq 0}$ a weight function. There exists a w-cover of G with cost $\alpha_w(G)$ if*

- *there exists a clique K of at most three vertices such that $\alpha_w(G - K) < \alpha_w(G)$; and*
- *for any weight function $w' : V(G) \to \mathbb{Z}_{\geq 0}$ such that $w'(V(G)) < w(V(G))$, there exists a w'-cover of cost $\alpha_{w'}(G)$.*

Proof. We may assume without loss of generality that K is inclusion-wise minimal satisfying $\alpha_w(G - K) < \alpha_w(G)$. As a result, $w(v) > 0$ for each $v \in K$: a vertex of zero weight has no impact on $\alpha_w(G)$. We can define another weight function $w' : V(G) \to \mathbb{Z}_{\geq 0}$ by setting

$$w'(v) = \begin{cases} w(v) - 1 & v \in K, \\ w(v) & \text{otherwise.} \end{cases}$$

Since $w'(V(G)) < w(V(G))$, there exists a w'-cover $\mathcal{K}$ of cost $\alpha_{w'}(G)$ by assumption. Since $|K| \leq 3$, the set $\mathcal{K} \cup \{K\}$ is a w-cover of G and its cost is $\alpha_{w'}(G) + 1 = \alpha_w(G)$. $\square$

Operations

Let G be a t-perfect graph. For any vertex $v \in V(G)$, the independent set polytope $P_I(G-v)$ is the projection of the intersection of $P_I(G)$ and the hyperplane $x_v = 0$ on $\mathbb{R}^{V(G-v)}$. This implies that t-perfection is preserved under vertex deletions. Gerards and Shepherd [56] showed that t-perfection is also preserved under the following operation:

. *choose a vertex whose neighborhood is an independent set, and contract all edges incident with the vertex.*

This operation is called t-contraction. To illustrate this operation, let's consider the graph shown in Figure 2.4 (a). Note that the neighborhood of u is an independent set. If we perform a t-contraction at vertex u, the resulting graph, shown in Figure 2.4 (b), is isomorphic to K_4.

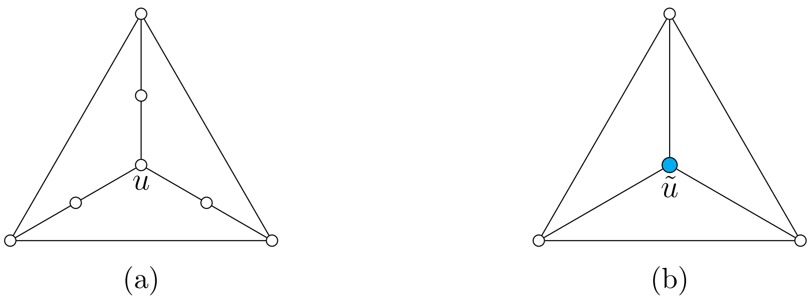


Figure 2.4: A t-contraction at vertex u.

T-contraction preserves t-perfection but not the other way around. After doing t-contraction at a vertex in a t-imperfect graph, the resulting graph can be t-perfect; see Figure 2.5.

Recall that A graph H is called a t-minor of a graph G if H can be obtained from G by a series of vertex deletions and t-contractions. Furthermore, if H is different from G, then H is a *proper* t-minor of G. It is straightforward to check that t-perfection is preserved under taking t-minors. Bruhn and Stein [20] demonstrated

Figure 2.5: (a) A t-imperfect graph and (b) a t-perfect t-minor (by doing t-contraction at the degree-4 vertex).

that this property holds for strong t-perfection as well. Therefore, every t-minor of a strongly t-perfect graph is also strongly t-perfect.

Consider a graph G. Subdividing any of its edges twice produces a new graph G' where both of the newly added vertices are t-contractable. If G is not t-perfect, then G' is t-imperfect either; otherwise, by doing t-contraction on any one of the newly added vertex in G', we obtain a t-minor of G' that is t-perfect and isomorphic to G, a contradiction. Without using the result that t-perfection is closed under t-contractions, we give a simple proof for this.

Proposition 2.7. *If a graph G is not t-perfect, then the graph G' obtained by subdividing an edge of G twice is not t-perfect.*

Proof. Let uv be the edge of G that is subdivided twice. Then the edge uv of G becomes to a path ua_1a_2v in G'. Since G is t-imperfect, $P_{OC}(G)$ has a fractional vertex, say x, that is not in the independent set polytope of G. Let $w : V(G) \to \mathbb{Z}_{\geq 0}$ be a weight function such that $w^T x$ is optimal when we do linear programming over $P_{OC}(G)$. Therefore,

$$\alpha_w(G) < w^T x.$$

Let $M = \max\{w(u), w(v)\}$. We define a weight function $w' : V(G') \to \mathbb{Z}_{\geq 0}$ as

$$w'(p) = \begin{cases} M & \text{if } p \in \{a_1, a_2\}, \\ w(p) & \text{otherwise.} \end{cases}$$

It can be seen that

$$\alpha_{w'}(G') = \alpha_w(G) + M.$$

Let x' be a $|V(G')|$ dimensional vector such that

$$x'_p = \begin{cases} 1 - x_u & \text{if } p = a_1, \\ x_u & \text{if } p = a_2, \\ x_p & \text{otherwise.} \end{cases}$$

It can be checked that $x' \in P_{OC}(G')$ (i.e., x' satisfies all the constraints of $P_{OC}(G')$). Since

$$\begin{aligned} w'^T x' &= w^T x + w(a_1)x_{a_1} + w(a_2)x_{a_2} \\ &= w^T x + M(1 - x_u + x_u) \\ &> \alpha_w(G) + M \\ &= \alpha_{w'}(G'), \end{aligned}$$

the vector x' is not in the independent set polytope of G'. Thus, G' is not t-perfect. $\square$

If G is t-perfect, then G' can be t-imperfect; see Figure 2.6.

By *duplicating* a vertex v of G we introduce copies of v and make them adjacent to every neighbor of v in G. Note that the copies and v itself form an independent set. Benchetrit [11] proved that the class of t-perfect graphs is closed under vertex duplication. Furthermore, he demonstrated that this property holds true even when

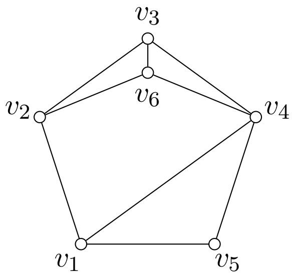


Figure 2.6: The graph is t-perfect (Proposition 2.4). But if we subdivide v_1v_4 twice by introduce two vertices v_7 and v_8, the resulting graph G' is t-imperfect since K_4 is a t-minor of G' obtained by doing t-contraction at v_1 in $G' - \{v_7, v_8\}$.

considering strongly t-perfect graphs.

Lemma 2.8 ([11])**.** *The graph obtained by duplicating any vertex of a (strongly) t-perfect graph is (strongly) t-perfect.*

Chapter 3

T-perfection in Fork-free Graphs

In this chapter, our focus lies on the study of t-perfection in fork-free graphs. We prove Theorems 1.1, 1.2, and 1.3 in this chapter. We provide a complete list of minimal forbidden t-minors for fork-free t-perfect graphs. Additionally, we establish that every fork-free t-perfect graph is, in fact, strongly t-perfect. We also present polynomial-time algorithms for recognizing and coloring these graphs.

3.1 Observations

The main task of this chapter is to prove Theorem 1.1. We recall the theorem here.

Theorem 1.1. *Let G be a fork-free graph. The following statements are equivalent:*

i) G is t-perfect.

ii) G is strongly t-perfect.

iii) G does not contain the C_7^2, the C_{10}^2, or any odd wheel as a t-minor.

Since strong t-perfection implies t-perfection and all the graphs C_7^2, C_{10}^2, and odd wheels are known to be t-imperfect [21, 115], we focus on showing that (iii) implies (ii). To demonstrate the strong t-perfection, we consider the dual of Chvátal's system, which is a covering problem. In particular, it looks for a cover of the vertex set by vertices, edges, and odd cycles. The graph is strongly t-perfect if, for any weight function, there is a cover whose cost equals the maximum weight of independent sets [115]. Let G be a fork-free graph that does not contain a C_7^2, a C_{10}^2, or any odd wheel as a t-minor. We may assume without loss of generality that G is connected. The graph must be strongly t-perfect if it is also claw-free [21], or if it is perfect (note that W_3 is K_4) [29, 78]. Moreover, if G is not perfect, then the nonexistence of K_4's and C_7^2's forces it to contain an odd hole. We may hence assume that G contains a claw and an odd hole. We show that every odd hole H must be a C_5, and every other vertex has either exactly two consecutive neighbors or exactly three nonconsecutive neighbors on H. Based on the adjacency to vertices on H, we can partition $V(G) \setminus V(H)$ into a few sets. A careful inspection of the edges among them shows that there always exists a cover. Therefore, the graph is strongly t-perfect. Our structural study toward Theorem 1.6 enables us to develop polynomial-time algorithms for recognizing and coloring fork-free t-perfect graphs.

3.2 Fork-free imperfect graphs containing a claw

This section is devoted to a structural study of such connected fork-free graphs that (1) do not contain a C_7^2, a C_{10}^2, or any odd wheel as a t-minor, and (2) contain an odd hole and a claw. We use weaker conditions, e.g., dropping the requirement of containing a claw, when a statement may be of independent interest. The first observation is about the neighborhood of an outside vertex on an odd hole.

Proposition 3.1. *Let G be a graph containing an odd hole H and u a vertex in $V(G) \setminus V(H)$.*

i) If u has exactly one neighbor on H, then G contains a fork.

ii) If u has exactly two neighbors on H and they are not consecutive on H, then G contains a fork.

iii) If u has exactly three neighbors on H and they are consecutive on H, then K_4 is a t-minor of G.

iv) If u has exactly four neighbors on H and they form one or two paths on H, then K_4 is a t-minor of G.

Proof. For assertions (i) and (ii), we number the vertices on H as $v_1, v_2, \ldots$. Suppose without loss of generality that $uv_3 \in E(G)$. Then u is adjacent to neither v_2 nor v_4. There is no other neighbor of u on H in (i). In (ii), u cannot be adjacent to both v_1 and v_5; we may assume that $uv_1 \notin E(G)$. Then $\{v_3, v_4, u, v_2, v_1\}$ forms a fork.[1]

For assertions (iii) and (iv), we focus on the subgraph G' induced by $V(H) \cup \{u\}$; see Figure 2.4. Note that any vertex in $V(H) \setminus N(u)$ has only two neighbors in G', and they are not adjacent. We do induction on the length of H. In the base case, $|H| = 5$. (Note that in this case, if u has four neighbors on H, then they must be consecutive.) A t-contraction on a vertex in $V(H) \setminus N(u)$ leads to a K_4. We now consider that $|H| > 5$. We apply a t-contraction on a vertex v in $V(H) \setminus N(u)$, which shortens H into a shorter odd hole, denoted by H'. The length of H' is two shorter than H. If the neighbors of u on H are consecutive, then the two neighbors of v cannot be both adjacent to u (note that $|H| \geq 7$). Thus, u has the same number of neighbors on H' as on H, and they remain consecutive. In the rest, u must have four neighbors on H, and they form two paths. If the two neighbors of v are both adjacent to u, then u has three consecutive neighbors on H'. Otherwise, u still has exactly four neighbors on H' and they form one or two paths on H'. By induction, K_4 is a t-minor of G', hence of G. □

[1]When we list the vertices of a (potential) fork, we always put the degree-three vertex first, followed by its three neighbors, the last of which has degree two.

The following statement further extends Proposition 3.1(ii). The two ends of any edge of an odd hole can only have one private neighbor, which is not adjacent to any other vertex on the hole.

Proposition 3.2. *Let G be a $\{K_4, \text{fork}\}$-free graph containing an odd hole H. For any two vertices on H, at most one of their common neighbors is adjacent to only two vertices on H.*

Proof. Suppose for contradiction that there are two distinct vertices x and y such that they have the same pair of neighbors on H. By Proposition 3.1(ii), the neighbors of x on H have to be consecutive. We number the vertices on H as $v_1, v_2, \ldots$ such that x is adjacent to v_1 and v_2. Since G is K_4-free, $xy \notin E(G)$. Then $\{v_2, x, y, v_3, v_4\}$ forms a fork, a contradiction. □

The existence of claws has another implication on the neighbors of other vertices on a hole H: there must be a vertex adjacent to three or more vertices on H.

Proposition 3.3. *Let G be a connected fork-free graph containing an odd hole H. The graph G is claw-free if*

i) G contains neither K_4 nor W_5; and

ii) every vertex $v \in V(G) \setminus V(H)$ has either zero or two neighbors on H.

Proof. Suppose that G satisfies both conditions (i) and (ii). We number the vertices on H as $v_1, v_2, \ldots$. Since G is a fork-free graph, if a vertex $v \in V(G) \setminus V(H)$ has two neighbors on H, then they are consecutive by Proposition 3.1(ii). Thus, for each i, a neighbor of v_i not on H is adjacent to either v_{i-1} or v_{i+1}. By Proposition 3.2, the degree of v_i is at most four, and it cannot be the center of a claw. Suppose for contradiction that G contains a claw. We take a claw T of G whose center has the shortest distance to H, denoted as d, among all claws of G. Note that $d \geq 1$. Let the vertex set of T be $\{c, x, y, z\}$, where c is the center of T.

Case 1, $d = 1$. The vertex c is adjacent to H, and by assumption, it has exactly two neighbors on H. We first note that we can choose T to intersect H. Suppose that none of x, y, and z is on H, and let v be a neighbor of c on H. Since the degree of v is at most four, v has at most one neighbor, say z, in $\{x, y, z\}$. Then $\{c, x, y, v\}$ is another claw. In the rest, without loss of generality, let the two neighbors of c on H be v_1 and v_2, where $z = v_2$. Since $\{c, x, y, v_2, v_3\}$ cannot induce a fork, at least one of x and y is adjacent to v_3. Since neither x nor y is adjacent to v_2, they cannot be both adjacent to v_3. We may assume that y is adjacent to v_3, hence to v_4 as well but no other vertex on H. Since $\{c, x, v_2, y, v_4\}$ does not induce a fork, x has to be adjacent to v_4 as well, and its other neighbor on H is v_5. But then $\{c, y, z, x, v_5\}$ induces a fork, a contradiction.

Case 2, $d = 2$. We may assume that there is a common neighbor p of c and v_1, and the other neighbor of p on H is v_2. (Note that $cv_1 \notin E(G)$.)

- Subcase 2.1, p has two or more neighbors in $\{x, y, z\}$, say x and y. By the selection of T, there cannot be any claw in G that has p as the center. Thus, x is adjacent to either v_1 or v_2, and so is y. Either cxv_1v_2y or cxv_2v_1y is a five-hole, and p is adjacent to all vertices on it. Therefore, G contains a W_5, a contradiction.

- Subcase 2.2, p has at most one neighbor in $\{x, y, z\}$. (We are in this sub-case when p is one of $\{x, y, z\}$.) Assume without loss of generality that p is adjacent to neither x nor y. Since $\{c, x, y, p, v_1\}$ does not induce a fork, v_1 is adjacent to at least one of x and y. On the other hand, v_2 cannot be adjacent to both x and y. We may assume that $yv_2 \in E(G)$; note that the other neighbor of y on H has to be v_3. Since $\{c, p, x, y, v_3\}$ does not induce a fork, $xv_3 \in E(G)$; note that the other neighbor of x on H has to be v_4. But then $\{c, p, y, x, v_4\}$ induces a fork, a contradiction.

Case 3, $d \geq 3$. Let $cu_1u_2 \cdots$ be a shortest path from c to H. Note that no vertex in $\{x, y, z\}$ is adjacent to u_i with $i > 2$.

- Subcase 3.1, u_1 has at most one neighbor in $\{x, y, z\}$. (We are in this subcase when u_1 is one of $\{x, y, z\}$.) Assume without loss of generality that u_1 is adjacent to neither x nor y. Since $\{c, x, y, u_1, u_2\}$ does not induce a fork, u_2 is adjacent to at least one of x and y, say y. But then $\{u_2, u_1, u_3, y\}$ induces a claw, and its center u_2 has a shorter distance to H than c, a contradiction to the selection of T.

- Subcase 3.2, u_1 has two or more neighbors in $\{x, y, z\}$, say x and y. Note that $u_1 z \notin E(G)$; otherwise $\{u_1, x, y, z\}$ induces a claw, which contradicts the selection of T. If u_2 is adjacent to only x in $\{x, y, z\}$, then $\{c, y, z, x, u_2\}$ induces a fork. If u_2 is adjacent to two vertices in $\{x, y, z\}$, then these two vertices, together with u_2 and u_3, form a claw that is closer to H than T. If u_2 is adjacent to neither x nor y, then $\{u_1, u_2, x, y\}$ induces a claw that is closer to H than T.

Therefore, there cannot be a claw in G. □

The somewhat conflicting requirements in Propositions 3.1 and 3.3 exclude odd holes longer than five, and force every five-hole to be dominating (i.e., every vertex has neighbors on this hole).

Proposition 3.4. *Let G be a connected fork-free graph containing an odd hole H. If G contains a claw and does not contain any odd wheel as a t-minor, then $|H| = 5$, and every vertex in G is adjacent to H.*

Proof. We number the vertices of H as $v_1, \ldots, v_\ell$, where $\ell = 2k+1$. Since G contains a claw, by Proposition 3.1 and Proposition 3.3, we can find a vertex u that has three or more neighbors on H.

We first show $|H| = 5$ by contradiction. On the other hand, u has a non-neighbor on H because G is free of odd wheels. We may assume without loss of generality that u is adjacent to v_4 but not v_5. We argue that $uv_3 \in E(G)$ by contradiction. If $uv_3 \notin E(G)$, then $uv_2 \in E(G)$ because $\{v_4, v_5, u, v_3, v_2\}$ does not induce a fork. By symmetry, $uv_6 \in E(G)$. But then dependent on the adjacency between u and v_1, either $\{v_4, v_3, v_5, u, v_1\}$ or $\{u, v_4, v_6, v_2, v_1\}$ induces a fork. Thus, $uv_3 \in E(G)$.

If u has precisely three neighbors and the only other neighbor of u on H is v_2, then G contains K_4 as a t-minor by Proposition 3.1(iii). Therefore, there is a neighbor v_i of u with $i \notin \{2, 3, 4\}$. We traverse H from v_4, v_5 till we meet the next neighbor of u; let it be v_i. Since $\ell \geq 7$ and $i \neq 2$, one of v_3 and v_4 is nonadjacent to all the vertices in $\{v_{i-1}, v_i, v_{i+1}\}$. Since neither $\{v_i, v_{i-1}, v_{i+1}, u, v_4\}$ nor $\{v_i, v_{i-1}, v_{i+1}, u, v_3\}$ induces a fork, $uv_{i+1} \in E(G)$. If u has precisely four neighbors on H, namely, v_3, v_4, v_i, and v_{i+1}, then G contains K_4 as a t-minor by Proposition 3.1(iv). Therefore, u has at least five neighbors on H. As a result, $6 \leq i \leq \ell$. If $i > 6$, then $\{u, v_4, v_j, v_i, v_{i-1}\}$, where v_j is another neighbor of u on H, induces a fork (note that v_j cannot be adjacent to v_4, v_{i-1}, or v_i). In the rest, $i = 6$. If v_5 is the only non-neighbor of u on H, then G contains an odd wheel as a t-minor. Hence, u has at least one neighbor and one non-neighbor in $\{v_8, v_9, \ldots, v_\ell, v_1, v_2\}$. We can find a j such that u is adjacent to precisely one of $\{v_j, v_{j+1}\}$. But then $\{u, v_4, v_6, v_j, v_{j+1}\}$ induces a fork (note that there cannot be any edge between v_4, v_6 and v_j, v_{j+1}). Therefore, the length of H has to be five.

By Proposition 3.1(iii, iv), the vertex u has exactly three nonconsecutive neighbors on H. Assume without loss of generality that they are v_1, v_3, and v_4. Let $X = V(H) \cup \{u\}$. Suppose that there exists a vertex x that has no neighbor on H. We can find a shortest path $x_1 x_2 \ldots x_p$ between $x_p = x$ and H; hence, x_1 is the only common vertex of this path and H. Note that $p \geq 3$, and the vertex x_3 has no neighbor on H. If $x_3 u \in E(G)$, then $\{v_1, v_2, v_5, u, x_3\}$ forms a fork. Therefore, $N(x_3) \cap X = \emptyset$. Since x_2 is adjacent to both x_3 and H, it is adjacent to all the vertices on H according to Lozin and Milanič [82, Lemma 1]. But then we have an odd wheel, which is impossible. □

Now consider a connected fork-free graph G that contains an odd hole and a claw, and does not contain any odd wheel as a t-minor. It contains a five-hole H by Proposition 3.4. Let us number the vertices on H as $v_1, \ldots, v_5$. For $i = 1, \ldots, 5$, let U_i be the set of common neighbors of v_{i+2} and v_{i+3}. We show that the five sets $U_1, U_2, \ldots, U_5$, together with $V(H)$, partition $V(G)$.

Proposition 3.5. *Let G be a connected fork-free graph containing a five-hole H. If G contains a claw and does not contain any odd wheel as a t-minor, then $\{V(H), U_1, U_2, U_3, U_4, U_5\}$ is a partition of $V(G)$.*

Proof. Let x be an arbitrary vertex in $V(G) \setminus V(H)$. By Proposition 3.4, the vertex x has a neighbor on H. Since G is fork-free and does not contain any odd wheel as a t-minor, x has either exactly two consecutive neighbors on H, or exactly three non-consecutive neighbors on H (Proposition 3.1). Thus, there is a unique $i \in \{1, \ldots, 5\}$ such that $x \in U_i$. On the other hand, no vertex on H is in U_i for all i. □

Since G is K_4-free, for all $i = 1, 2, \ldots, 5$, the set U_i is an independent set. An independent set is *maximal* if it is not a subset of any other independent set. We show that the set $\{v_{i-1}, v_{i+1}\} \cup U_i$ is a maximal independent set of G for every $i = 1, \ldots, 5$.

Proposition 3.6. *Let G be a K_4-free graph containing a five-hole H. If any vertex in $V(G) \setminus V(H)$ has either exactly two consecutive neighbors on H, or exactly three nonconsecutive neighbors on H, then the set $\{v_{i-1}, v_{i+1}\} \cup U_i$ is a maximal independent set of G for every $i = 1, \ldots, 5$. Moreover, they are all the maximal independent sets of G that contain two vertices from H.*

Proof. Let S be a maximal independent set of G. Since H is a C_5, it follows $|S \cap V(H)| \leq 2$. If S contains precisely two vertices from H, they have to be v_{i-1} and

v_{i+1} for some i. By the definitions of U_i, we have $S \setminus V(H) \subseteq U_i$. Thus, $S \subseteq \{v_{i-1}, v_{i+1}\} \cup U_i$. Since there is no edge among vertices in U_i, it must hold by equality by the maximality of S. $\square$

If a vertex in U_i has another neighbor on H, then it has to be v_i by Propositions 3.1(iii). We can thus partition U_i into $U_i^+ = U_i \cap N(v_i)$ and $U_i^- = U_i \setminus N(v_i)$. For any vertex x in U_i^+, the set $\{v_i, v_{i-1}, v_{i+1}, x\}$ induces a claw. According to Proposition 3.2, $|U_i^-| \leq 1$ for $i = 1, \ldots, 5$. By Proposition 3.3, $\bigcup_{i=1}^5 U_i^+$ is not empty. We summarize the adjacency relations among the parts in the following proposition when U_i^+ is not empty.

Proposition 3.7. *Let G be a $\{K_4, W_5, \text{fork}\}$-free graph containing a five-hole H. If any vertex in $V(G) \setminus V(H)$ has either exactly two consecutive neighbors on H, or exactly three nonconsecutive neighbors on H, and U_i^+ is nonempty for some $i = 1, \ldots, 5$, then*

i) U_i is complete to $U_{i-2} \cup U_{i+2}$;

ii) U_i is complete to $U_{i-1}^- \cup U_{i+1}^-$;

iii) U_{i+1}^- is complete to U_{i+2}^-;

iv) at least one of U_{i+2} and U_{i-2} is empty; and

v) a vertex in U_i^+ has at most one non-neighbor in U_{i-1}^+ and at most one non-neighbor in U_{i+1}^+.

Proof. We show the statements for $i = 3$; they hold for other indices by symmetry.

(i) Let x be an arbitrary vertex in U_3 and y an arbitrary vertex in U_5. Suppose first that $x \in U_3^+$. By definition, x is adjacent to both v_1 and v_3 but not v_4, and y is adjacent to v_3 but neither v_1 nor v_4. They have to be adjacent as otherwise $\{v_3, y, v_4, x, v_1\}$ forms a fork. Thus, U_3^+ is complete to U_5, and a similar argument implies that U_3 is complete to U_5^+. The only remaining case is when $x \in U_3^-$ and $y \in U_5^-$ (we have nothing to show if one or both of them are empty). We take an arbitrary vertex $x' \in U_3^+$, which is nonempty by assumption. Note that $xx' \notin E(G)$, and we have seen above that $x'y \in E(G)$. By definition, both x and x' are adjacent to v_5 and neither is adjacent to v_4. Thus, $xy \in E(G)$ as otherwise $\{v_5, v_4, x, x', y\}$ forms a fork. A symmetric argument applies to U_3 and U_1.

(ii) Let x be an arbitrary vertex in U_3. The statement holds vacuously for U_4^- when it is empty. Assume that $U_4^- \neq \emptyset$ and y be the vertex in U_4^-. By definition, y is adjacent to v_2 but none of v_3, v_4, and v_5. If x is in U_3^+, then $xy \in E(G)$ as otherwise $\{v_3, v_4, x, v_2, y\}$ forms a fork. In the remaining case, $x \in U_3^-$. We take an arbitrary vertex $x' \in U_3^+$, which is nonempty by assumption. By the argument above, $x'y \in E(G)$. The vertices $xy \in E(G)$ as otherwise $\{v_5, v_4, x, x', y\}$ forms a fork. A symmetric argument implies that U_3 is complete to U_2^-.

(iii) This assertion holds vacuously when one or both of U_4^- and U_5^- are empty. Hence we may assume otherwise. For $j = 4, 5$, let u_j^- be the vertex in U_j^-. We take an arbitrary vertex $x \in U_3^+$, which is nonempty by assumption. By definition, the vertex x is adjacent to v_5 but not v_4, the vertex u_4^- is adjacent to neither v_5 nor v_4, and the vertex u_5^- is adjacent to neither v_4 nor v_5. By assertions (i, ii), x is adjacent to both u_4^- and u_5^-. Therefore, u_4^- must be adjacent to u_5^- as otherwise $\{x, u_4^-, u_4^-, v_5, v_4\}$ forms a fork.

(iv) Suppose for contradiction that neither U_1 nor U_5 is empty. We pick three arbitrary vertices u_1, u_3^+, and u_5 from U_1, U_3^+, and U_5, respectively. By assertion (i)), u_3^+ is adjacent to both u_1 and u_5. If $u_1u_5 \in E(G)$, then $\{u_3^+, u_5, u_1, v_3\}$ is a clique, a contradiction to that G is K_4-free. In the rest, $u_1u_5 \notin E(G)$. The vertex v_1 must be adjacent to u_1 as otherwise $\{u_3^+, u_5, v_1, u_1, v_4\}$ forms a fork. By symmetry, $v_5u_5 \in E(G)$. But then u_3^+ has five neighbors on the hole $u_5v_5v_2u_1v_3$, contradicting that G is W_5-free.

(v) Let x be an arbitrary vertex in U_3^+. If there are two distinct vertices y and y' in $U_4^+ \setminus N(x)$, then $\{v_2, y, y', v_3, x\}$ forms a fork. A symmetric argument applies to U_2^+. □

3.3 Strong t-perfection

Propositions 3.1–3.5 can be summarized as follows. If a connected fork-free graph G contains a claw and an odd hole and does not contain a C_7^2, a C_{10}^2, or any odd wheel as a t-minor, then every odd hole H in G has length five, and satisfies the following property.

($\star$) A vertex in $V(G) \setminus V(H)$ has either exactly two consecutive neighbors on H, or exactly three nonconsecutive neighbors on H.

Interestingly, the other direction also holds true. The main work of this section is to establish the following lemma.

Lemma 3.8. *Let G be a connected fork-free graph that contains a claw and an odd hole. The following statements are equivalent:*

i) G does not contain a C_7^2, a C_{10}^2, or any odd wheel as a t-minor.

ii) G is $\{K_4, W_5, C_7^2, C_{10}^2\}$-free, and every odd hole in G has length five and satisfies ($\star$).

iii) G is strongly t-perfect.

Before presenting the proof of Lemma 3.8, we use it to prove Theorem 1.1.

Proof of Theorem 1.1. Since strong t-perfection implies t-perfection and C_7^2, C_{10}^2, and all odd wheels are t-imperfect [21,115], it suffices to show that if a fork-free graph does not contain a C_7^2, a C_{10}^2, or any odd wheel as a t-minor, then it is strongly t-perfect. Suppose that G is such a graph. We show that every component of G is strongly t-perfect, and hence G is strongly t-perfect. Let G' be an arbitrary component of G. Note that G' is fork-free and does not contain a C_7^2, a C_{10}^2, or any odd wheel as a t-minor. If G' is claw-free, then it is strongly t-perfect according to Bruhn and Stein [20, Theorem 2] and [21, Theorem 3]. Note that the complement of C_7 is C_7^2, and the complement of an odd hole longer than seven contains a K_4. If G' does not contain an odd hole, then G' is perfect, and hence strongly t-perfect by Proposition 2.4. Now that G' contains a claw and an odd hole, it is strongly t-perfect by Lemma 3.8. $\square$

The rest of the section is devoted to proving Lemma 3.8.

Throughout the rest of this section, G is a $\{$fork, $K_4, W_5, C_7^2, C_{10}^2\}$-free graph that contains a claw and an odd hole, and every odd hole has length five and satisfies $(\star)$. We fix a five-hole H, and partition the vertices $V(G) \setminus V(H)$ into $U_1, \ldots, U_5$. For $i = 1, \ldots, 5$, the set U_i is further partitioned into U_i^+ and U_i^-. Recall that $|U_i^-| \leq 1$ by Proposition 3.2. By Proposition 3.7, the main uncertain adjacencies are between U_i^+ and U_{i+1}^+. Thus, the graph has a very simple structure if only one of U_i^+'s or two nonconsecutive of them are nonempty. Indeed, it can be obtained from one of the small graphs (of order at most ten) in Figure 3.1 by vertex duplications.

Lemma 3.9. *If for any $i = 1, \ldots, 5$, one of U_i^+ and U_{i+1}^+ is empty, then G is strongly t-perfect.*

Proof. We may assume without loss of generality that U_2^+ is nonempty, while all of U_1^+, U_3^+, and U_5^+ are empty. Every vertex in U_2^+ is adjacent to v_2, v_4, and v_5 but not v_1 or v_3 by definition.

Suppose first that U_4^+ is also nonempty. Then both U_1 and U_5 are empty by Proposition 3.7 iv). Thus,

$$V(G) \setminus (V(H) \cup U_2^+ \cup U_4^+) = U_2^- \cup U_3^- \cup U_4^-.$$

By Proposition 3.7(i, ii), all the edges between U_2^+ and $U_3^- \cup U_4$ are present. Thus, all vertices in U_2^+ have the same neighborhood in $V(G) \setminus U_2^+$. A symmetric argument applies to U_4^+. Let G_1 be a graph of the pattern in Figure 3.1(a), where for $i = 2, 3, 4$, the optional vertices u_i^- exists if and only if $U_i^- \neq \emptyset$. It is easy to verify that G_1 is strongly t-perfect if it satisfies the condition of Lemma 3.8(ii). We duplicate u_2^+ of G_1 with $|U_2^+|$ vertices, and then duplicate u_4^+ in the resulted graph with $|U_4^+|$ vertices. The final result is G. Therefore, G is strongly t-perfect by Lemma 2.8.

In the rest, U_4^+ is empty. We may assume without loss of generality that $U_5 = \emptyset$. Then

$$V(G) \setminus (V(H) \cup U_2^+) = U_1^- \cup U_2^- \cup U_3^- \cup U_4^-.$$

Every vertex in U_2^+ is adjacent to $U_1^- \cup U_3^- \cup U_4^-$ by Proposition 3.7(i, ii), and nonadjacent to U_2^- by definition. Thus, all vertices in U_2^+ have the same neighborhood in $V(G) \setminus U_2^+$. Let G_2 be a graph of the pattern in Figure 3.1(b), where for $i = 1, 2, 3, 4$, the optional vertices u_i^- exists if and only if $U_i^- \neq \emptyset$. It is easy to verify that G_2 is strongly t-perfect if it satisfies the condition of Lemma 3.8(ii). We duplicate u_2^+ of G_2 with $|U_2^+|$ vertices. The result is G. Therefore, G is strongly t-perfect by Lemma 2.8. $\square$

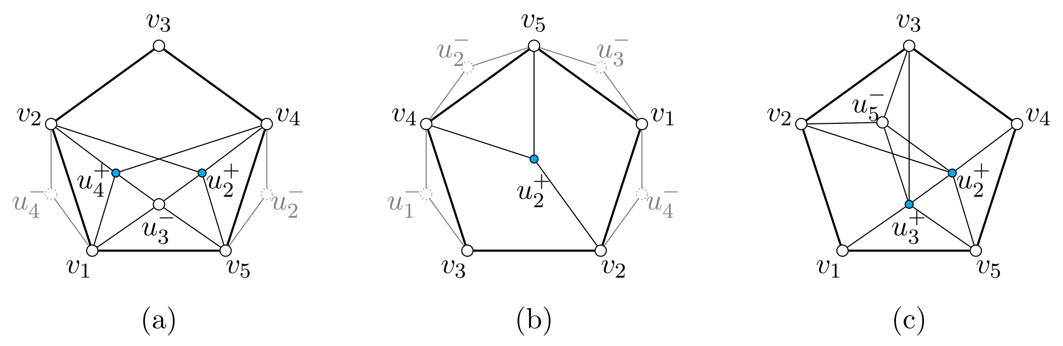


Figure 3.1: Three configurations for Lemma 3.9 and 3.10. The dotted vertices are optional, and their edges, except for H, are not drawn.

Henceforth, we may assume without loss of generality that

$$U_2^+ \neq \emptyset \text{ and } U_3^+ \neq \emptyset.$$

By Proposition 3.7 iv) with $i = 2$, at least one of U_4 and U_5 is empty. For the same reason, at least one of U_1 and U_5 is empty. We note that if neither $U_1 \cup U_5$ nor $U_4 \cup U_5$ is empty, then U_2^+ is complete to U_3^+, and the situation is similar to Lemma 3.9.

Lemma 3.10. *If neither $U_1 \cup U_5$ nor $U_4 \cup U_5$ is empty, then G is strongly t-perfect.*

Proof. We first argue that $U_5 \neq \emptyset$. Suppose otherwise, then neither U_1 nor U_4 is empty. Since both U_3 and U_4 are nonempty, U_1^+ is empty by Proposition 3.7 iv) with $i = 1$. By symmetric, U_4^+ is empty. Therefore $U_1 = U_1^-$ and $U_4 = U_4^-$. Let u_1^- and u_4^- be the only vertex in U_1^- and U_4^-, respectively. By Proposition 3.7 ii) with $i = 3$, the vertex u_3^+ is adjacent to u_4^-. By Proposition 3.7 i) with $i = 3$, the vertex u_3^+ is adjacent to u_1^-. Since $\{u_3^+, u_1^-, v_5, u_4^-, v_2\}$ cannot form a fork, $u_4^- u_1^- \in E(G)$. But then $u_1^- u_4^- v_1 v_5 v_4$ is a five-hole in G and u_3^+ has three consecutive neighbors u_4^-, v_1, and v_5 on it, contradicting $(\star)$.

Since neither U_2 nor U_3 is empty, U_5^+ is empty by Proposition 3.7(iv)) with $i = 5$. Thus, U_5^- is nonempty; let u_5^- be its only vertex. Applying Proposition 3.7(i)) twice, with $i = 2, 3$, respectively, we can conclude that u_5^- is adjacent to all the vertices in $U_2 \cup U_3$. We then argue that U_2^+ is complete to U_3^+. We take an arbitrary vertex u_2^+ from U_2^+ and an arbitrary vertex u_3^+ from U_3^+. If $u_2^+ u_3^+ \notin E(G)$, then $u_2^+ v_2 v_3 u_3^+ v_5$ is a hole in G on which u_5^- has four neighbors. Thus, U_2^+ is complete to U_3^+. We next argue that U_2^- is empty. Suppose otherwise and let u_2^- be the only vertex in U_2^-. Note that $u_5^- u_2^- \in E(G)$. Therefore, $u_5^- u_2^- v_5 v_1 v_2$ is a hole in G. But then, u_3^+ has three consecutive neighbors u_2^-, v_5, and v_1 on the hole, contradicting $(\star)$. Thus, $U_2^- = \emptyset$. A symmetric argument implies U_3^- is empty as well. Since both U_2^+ and U_5^- are nonempty, U_4 is empty by Proposition 3.7(iv)) with $i = 2$. A symmetric argument implies U_1 is empty as well. Therefore, $V(G) \setminus V(H) = U_2^+ \cup U_3^+ \cup U_5^-$, and the three parts U_2^+, U_3^+, and U_5^- are pairwise complete with each other. Let G_1 be the graph in Figure 3.1(c). It is easy to verify that G_1 is strongly t-perfect. We duplicate u_2^+ of G_1 with $|U_2^+|$ vertices, and then duplicate u_3^+ in the resulted graph with $|U_3^+|$ vertices. The final result is G. Therefore, G is strongly t-perfect by Lemma 2.8. □

In the rest, at least one of $U_1 \cup U_5$ and $U_4 \cup U_5$ is empty. We may assume that $U_1 \cup U_5 = \emptyset$; otherwise, we can renumber the vertices on H. We have seen all the maximal independent sets that contains two vertices from H in Proposition 3.6. The following lists other maximal independent sets under this condition.

Proposition 3.11. *If $U_1 \cup U_5$ is empty, then a maximal independent set S of G that contains at most one vertex from H is either*

i) $U_j^- \cup \{v_j\}$ for some $j = 2, 3, 4$; or

ii) a pair of nonadjacent vertices $x \in U_3^+$ and $y \in U_2^+ \cup U_4^+$.

Proof. (i) Suppose first that there is one vertex in $S \cap V(H)$. We first excludes v_1 and v_5. Suppose that $v_1 \in S$. By definition, $U_3 \cup U_4$ is disjoint from S. Thus, $S \subseteq U_2 \cup \{v_1\}$ and cannot be maximal. Likewise, $v_5 \in S$ implies $S \subseteq U_4 \cup \{v_5\}$.

- Case 1, $v_2 \in S$. Then $S \setminus \{v_2\} \subseteq U_2^- \cup U_3$. Since $U_3 \cup \{v_2, v_4\}$ is an independent set, U_2^- cannot be empty, and its only vertex must be in S. It remains to argue that the vertex in U_2^- is adjacent to all the vertices in U_3. We call Proposition 3.7 ii) with $i = 3$.

- Case 2, $v_3 \in S$. Then $S \setminus \{v_3\} \subseteq U_2 \cup U_3^- \cup U_4$. Since U_2 is complete to U_4 by Proposition 3.7 i) with $i = 2$, one of $S \cap U_2$ and $S \cap U_4$ is empty. Since $U_2 \cup \{v_1, v_3\}$ and $U_4 \cup \{v_3, v_5\}$ are independent sets, by the maximality of S, there is a vertex in $U_3^- \cap S$. By Proposition 3.7 ii) with $i = 2$, the vertex in U_3^- is adjacent to all the vertices in U_2. Moreover, the vertex in U_3^- is adjacent to all the vertices in U_4 by Proposition 3.7 iii) with $i = 2$ when $U_4^+ = \emptyset$, or by Proposition 3.7 ii) with $i = 4$ otherwise.

- Case 3, $v_4 \in S$. Then $S \setminus \{v_4\} \subseteq U_4^- \cup U_3$, and the argument is similar to that of case 1.

(ii) Now suppose that S is disjoint from $V(H)$. By assumption, $V(G) \setminus V(H) = U_2 \cup U_3 \cup U_4$. We first argue that

$$S \subseteq U_2^+ \cup U_3^+ \cup U_4^+.$$

For $j = 2, 3, 4$, let x_j be the vertex in U_j^- if $U_j \neq U_j^+$. Applying Proposition 3.7 i)–iii) with $i = 2$, we can conclude that x_2, x_3, and x_4 are pairwise adjacent, when they exist. Therefore, at most one of them is in S. On the other hand, if $x_j \in S$ for $j = 2, 3, 4$, then $S \subseteq U_j$ by Proposition 3.7 i) and ii). Since this contradicts the maximality of S, we must have $S \subseteq U_2^+ \cup U_3^+ \cup U_4^+$. Since U_2^+ is complete to U_4^+ by Proposition 3.7 i) with $i = 2$, the set S is a subset of either $U_2^+ \cup U_3^+$ or $U_3^+ \cup U_4^+$. By Proposition 3.7 v) (with $i = 3$), each vertex in U_3^+ has at most one non-neighbor in U_2^+ and at most one non-neighbor in U_4^+. For the same reason, each vertex in U_2^+ or U_4^+ has at most one non-neighbor in U_3^+. Thus, S is a pair of nonadjacent vertices $x \in U_3^+$ and $y \in U_2^+ \cup U_4^+$. □

The final step of the proof relies on the duality of linear programming; see Proposition 2.5.

Lemma 3.12. *If $U_1 \cup U_5$ is empty, then G is strongly t-perfect.*

Proof. Suppose for contradiction that G is not strongly t-perfect, and assume without loss of generality that G is a counterexample of the minimum number of vertices. Our first claim is that every proper induced subgraph G' of G is strongly t-perfect. If G' does not contain an odd hole, then it is strongly t-perfect (Proposition 2.4). If G' is claw-free, then G is strongly t-perfect [20, 21]. Thus, G' is strongly t-perfect either because it satisfies one of Lemmas 3.9 and 3.10, or by the selection of G.

By Proposition 2.5, there exists a weight function $w : V(G) \to \mathbb{Z}_{\geq 0}$ such that G does not have a w-cover of cost $\alpha_w(G)$. We may take w to be such a function that minimizes $w(V(G))$. The second claim is that the weight is positive. Suppose that $w(v) = 0$ for some vertex $v \in V(G)$. Since every induced subgraph of G is strongly t-perfect, there exists a w-cover $\mathcal{K}$ of $G - v$ with cost $\alpha_w(G - v)$. Since $w(v) = 0$, the cover $\mathcal{K}$ is also a w-cover of G, while $\alpha_w(G - v) = \alpha_w(G)$. But then $\mathcal{K}$ is a w-cover of G with cost $\alpha_w(G)$, a contradiction. As a consequence of the second claim, every maximum-weight independent set is maximal. Recall that all maximal independent sets are listed in Propositions 3.6 and 3.11.

For $j = 2, 3, 4$, let

$$S_j^- = \{v_{j-1}, v_{j+1}\} \cup U_j^-$$

and denote by u_j^- the only vertex contained in U_j^- when it is not empty. For $j = 2, 3$, let u_j^+ be a vertex of the maximum weight from U_j^+, and

$$S_j^+ = \{v_{j-1}, v_{j+1}, u_j^+\}.$$

We define a set $S_4^+ = \{v_3, v_5, u_4^+\}$ when $u_4^+ \neq \emptyset$, with u_4^+ being a vertex of the maximum weight from U_4^+. According to Proposition 3.6, all the nine sets S_j^-, S_j^+, and U_j are independent sets.

From Proposition 2.6 and the selection of the weight function w it can be inferred that $\alpha_w(G - K) = \alpha_w(G)$ for any clique K of at most three vertices. In other words, there exists a maximum-weight independent set S of G disjoint from K. We try to locate a clique of two or three vertices that intersects all maximum-weight independent sets of the graph, thereby producing a contradiction to Proposition 2.6. In the following we consider potential maximum-weight independent sets. By excluding an independent set we mean that we have evidence that it does not have the maximum weight.

Note that U_4 is not empty; otherwise, every odd cycle of G visits v_5, and G is strongly t-perfect according to Gerards [54]. We take an arbitrary vertex u_4 from U_4. Note that $u_4u_2^+ \in E(G)$ by Proposition 3.7 i) with $i = 2$. Let K denote the clique $\{v_2, u_2^+, u_4\}$, and let S be a maximum-weight independent set of G disjoint from K. Note that S has to be $\{v_1, v_4\}$, $\{v_3\} \cup U_3^-$, $\{v_4\} \cup U_4^-$, or one that is disjoint from $V(H)$, i.e., specified in Proposition 3.11(ii).

- Case 1, $S = \{v_1, v_4\}$. (Note that $U_5 = \emptyset$.) Since $\{v_2, v_4, u_3^+\}$ and $\{v_1, v_3, u_2^+\}$ are both independent sets,

$$\begin{aligned}
&w(u_2^+) + w(u_3^+)\\
<&w(v_2) + w(v_4) + w(u_3^+) + w(v_1) + w(v_3) + w(u_2^+) - w(v_4) - w(v_1)\\
=&w(\{v_2, v_4, u_3^+\}) + w(\{v_1, v_3, u_2^+\}) - w(S)\\
\le&\alpha_w(G) + \alpha_w(G) - \alpha_w(G)\\
=&\alpha_w(G).
\end{aligned}$$

  By the selection of u_2^+ and u_3^+, a pair of vertices $x \in U_2^+$ and $y \in U_3^+$ cannot have weight $\alpha_w(G)$. In other words, if a maximum-weight independent set is disjoint from $V(H)$, then it comprises a vertex in U_3^+ and a vertex in U_4^+. On the other hand, from

$$w(v_2) + w(v_3) + w(U_2^- \cup U_3^-) = w(S_2^-) + w(S_3^-) - w(S) \le \alpha_w(G)$$

  we can exclude $\{v_2\} \cup U_2^-$ and $\{v_3\} \cup U_3^-$. Thus, if a maximum-weight independent set contains one vertex from H, then it has to be $\{v_4\} \cup U_4^-$.

– Case 1.1, $\{v_2, v_5\}$ is also a maximum-weight independent set. (Note that $U_1 = \emptyset$.) If $U_4^+ \neq \emptyset$, we use

$$w(u_3^+) + w(u_4^+) < w(S_3^+) + w(S_4^+) - w(\{v_2, v_5\}) \leq \alpha_w(G)$$

to exclude all maximal independent sets disjoint from H. If $U_4^- \neq \emptyset$, we use

$$w(v_4) + w(u_4^-) < w(S_3^-) + w(S_4^-) - w(\{v_2, v_5\}) < \alpha_w(G)$$

to exclude $\{v_4, u_4^-\}$. Since any maximum-weight independent set has to contain two vertices from H, they all intersect the clique $\{v_1, v_5, u_3^+\}$.

– Case 1.2, there exists a maximum-weight independent set $S' = \{x_3, x_4\}$ with $x_3 \in U_3^+$ and $x_4 \in U_4^+$. Note that both $U_3 \cup \{v_2\}$ and $U_4 \cup \{v_3\}$ are not maximal. Therefore, we can use $w(U_3 \cup \{v_2\}) + w(U_4 \cup \{v_3\}) - w(S') < \alpha_w(G)$ to exclude all other pairs $\{x_3', x_4'\}$ with $x_3' \in U_3^+$ and $x_4' \in U_4^+$ (except for S' itself). If U_4^- is empty, then $\{v_1, v_2, x_4\}$ intersects all the possible maximum-weight independent sets. Now that U_4^- is nonempty, we use $w(U_4) + w(S_3^+) - w(S') < \alpha_w(G)$ to exclude $\{v_4, u_4^-\}$. Furthermore, we use $w(S_3^+) + w(S_4^+) - w(S') < \alpha_w(G)$ to exclude $\{v_2, v_5\}$. The clique $\{v_1, v_5, u_3^+\}$ intersects all the remaining maximal independent sets, S, S', $\{v_2, v_4\} \cup U_3$, $\{v_3, v_5\} \cup U_4$, and $\{v_1, v_3\} \cup U_2$.

– Otherwise (neither of cases 1.1 and 1.2 is true), the clique $\{v_3, v_4\}$ intersects all the possible maximum-weight independent sets.

- Case 2, $S = \{v_3, u_3^-\}$. We use

$$w(u_2^+) + w(u_3^+) < w(U_3) + w(S_2^+) - w(S) < \alpha_w(G)$$

to exclude all pairs $\{x_2, x_3\}$ with $x_2 \in U_2^+$ and $x_3 \in U_3^+$. If U_4^+ is nonempty, we use

$$w(u_3^+) + w(u_4^+) < w(U_3) + w(S_4^+) - w(S) < \alpha_w(G)$$

to exclude all pairs $\{x_3, x_4\}$ with $x_3 \in U_3^+$ and $x_4 \in U_4^+$. From $w(S_3^-) + w(S_4^-) - w(S) < \alpha_w(G)$ we can exclude $\{v_2, v_5\}$ and $\{v_4, u_4^-\}$ (when $U_4^- \neq \emptyset$). If $U_2^- \neq \emptyset$, we use $w(S_2^-) + w(S_3^-) - w(S) < \alpha_w(G)$ to exclude $\{v_2, u_2^-\}$. We are left with S, $\{v_2, v_4\} \cup U_3$, $\{v_3, v_5\} \cup U_4$, and $\{v_3, v_1\} \cup U_2$. All of them intersect the clique $\{v_3, v_4\}$.

- Case 3, $S = \{v_4, u_4^-\}$. We can use

  $$w(v_2) + w(v_5) < w(S_3^-) + w(S_4^-) - w(S) \leq \alpha_w(G)$$

  to exclude $\{v_2, v_5\}$. If $U_4^+ \neq \emptyset$, we use

  $$w(u_3^+) + w(u_4^+) < w(U_4) + w(S_3^+) - w(S) < \alpha_w(G)$$

  to exclude all pairs $\{x_3, x_4\}$ with $x_3 \in U_3^+$ and $x_4 \in U_4^+$.

  - Case 3.1, there does not exist a maximum-weight independent set $\{x_2, x_3\}$ with $x_2 \in U_2^+$ and $x_3 \in U_3^+$. If U_2^- is empty, the clique $\{v_3, v_4\}$ intersects all maximum weight independent sets. Now that $U_2^- \neq \emptyset$, we note that $\{u_2^-, u_3^+, u_4^-\}$ intersects all maximum-weight independent sets. To see that it is clique, note that $u_2^- u_4^- \in E(G)$ by Proposition 3.7 i) with $i = 2$, and u_3^+ is adjacent to both u_2^- and u_4^- by Proposition 3.7 ii) with $i = 3$,
  - Case 3.2, there exists a maximum-weight independent set $S' = \{x_2, x_3\}$ with $x_2 \in U_2^+$ and $x_3 \in U_3^+$. We can use $w(U_2 \cup \{v_1\}) + w(U_3 \cup \{v_2\}) - w(S') < \alpha_w(G)$ to exclude all other pairs $\{x'_2, x'_3\}$ with $x'_2 \in U_2^+$ and $x'_3 \in U_3^+$ (except for S' itself). If U_2^- is not empty, then we can use $w(U_2) + w(S_3^+) - w(S') < \alpha_w(G)$ to exclude $\{u_2^-, v_2\}$. Thus, a maximum-weight independent set of G has to be S, S', $\{v_2, v_4\} \cup U_3$, $\{v_3, v_5\} \cup U_4$,

or $\{v_3, v_1\} \cup U_2$. The clique $\{v_4, v_5, x_2\}$ intersects all maximum-weight independent sets.

- Case 4, $S = \{x_2, x_3\}$, where $x_2 \in U_2^+$ and $x_3 \in U_3^+$. Note that $x_2 \neq u_2^+$ because S is disjoint from K. We can use $w(U_2 \cup \{v_1\}) + w(U_3 \cup \{v_2\}) - w(S) < \alpha_w(G)$ to exclude all other pairs $\{x_2', x_3'\}$ with $x_2' \in U_2^+$ and $x_3' \in U_3^+$ (except for S itself). If U_2^- is nonempty, then we can use $w(U_2) + w(S_3^+) - w(S) < \alpha_w(G)$ to exclude $\{u_2^-, v_2\}$. If no maximum-weight independent set intersects U_4^+, then the clique $\{x_2, v_4, v_5\}$ intersects all maximum weight independent sets.

  Suppose that $S' = \{x_3', x_4\}$ is a maximum-weight independent set with $x_3' \in x_3$ and $x_4 \in U_4^+$. We can further use $w(U_3 \cup \{v_4\}) + w(U_4 \cup \{v_3\}) - w(S') < \alpha_w(G)$ to exclude all other pairs $\{x_3'', x_4'\}$ with $x_3'' \in U_3^+$ and $x_4' \in U_4^+$ (except for S' itself). We use $w(S_3^+) + w(S_4^+) - w(S') \leq \alpha_w(G)$ to exclude $\{v_2, v_5\}$. Thus, a maximum-weight independent set of G has to be S, S', $\{v_2, v_4\} \cup U_3$, $\{v_3, v_5\} \cup U_4$, or $\{v_3, v_1\} \cup U_2$. The set $\{v_4, x_2, x_4\}$ intersects all maximum-weight independent sets. Note that $x_2x_4 \in E(G)$ by Proposition 3.7 i) with $i = 2$.

- Case 5, $S = \{x_3, x_4\}$, where $x_3 \in U_3^+$ and $x_4 \in U_4^+$. It is similar to Case 4.

This concludes the proof. □

We now prove Lemma 3.8.

Proof of Lemma 3.8. Since C_7^2, C_{10}^2, and odd wheels are all t-imperfect, (iii) implies (i). By Propositions 3.1–3.5, (i) implies (ii). By Lemmas 3.9, 3.10, and 3.12, (ii) implies (iii). □

3.4 Recognition and coloring

We now describe an algorithm to decide whether a fork-free graph is (strongly) t-perfect. We may assume without loss of generality that the input graph is connected; otherwise, we work on its components one by one and return "yes" if and only if all components return "yes". The algorithm is based on Lemma 3.8. The only condition of Lemma 3.8(ii) that cannot be easily checked in polynomial time is that every odd hole has length five. The following proposition bounds the length of the longest odd holes.

Proposition 3.13. *Let G be a $\{K_4, \text{fork}\}$-free graph containing a five-hole H. If H satisfies ($\star$), then G cannot contain an odd hole with length longer than* 19.

Proof. Let H' be a longest odd hole in G. Suppose for contradiction $|H'| \geq 21$. At least $|H'| - 4$ vertices of H' are in $V(G) \setminus V(H)$. Assume without loss of generality that $|U_i \cap V(H')|$ is maximized with $i = 1$. Then $|U_1 \cap V(H')| \geq \lceil \frac{|H'|-4}{5} \rceil \geq 4$. Since U_1 is an independent set, $|U_1 \cap V(H')| \leq \frac{|H'|-1}{2}$. There exists a vertex x in $V(H') \setminus (V(H) \cup U_1)$. By Propositions 3.7 i), ii), and v) with $i = 1$, the vertex x has at most one non-neighbor in $U_1 \cap V(H')$. But then x has at least three neighbors in $|H'|$, contradicting that H' is a hole. $\square$

We are now ready to present the recognition algorithm and prove Theorem 1.2.

Proof of Theorem 1.2. The input is a fork-free graph G. We start by checking whether it contains a K_4, W_5, C_7^2, or C_{10}^2. Since K_4, W_5, C_7^2, and C_{10}^2 are not t-perfect, we return "no" if any of them is found. If G does not contain a claw, then we call Bruhn–Schaudt algorithm [19] to decide whether G is t-perfect. We then call the algorithm of Chudnovsky et al. [28] to test whether G contains an odd hole. Since G does not contain a K_4, it cannot contain the complement of any odd hole longer than seven. It does not contain a C_7^2, which is the complement of C_7. Therefore, if G does not contain any odd hole, then G is perfect, and t-perfect (Proposition 2.4), and we return "yes." In the rest, G contains a claw and an odd hole, and we check the conditions of Lemma 3.8(ii). We enumerate five-holes, and for each of them, test whether it satisfies ($\star$). If any one does not, then return "no." Finally, we check whether G contains an odd hole of length between 7 and 19. If any is found, then we can return "no." If none is found, every odd hole has length five by Proposition 3.13. Thus, we can return "yes." All the induced subgraphs we need to check have a constant number of vertices, and both algorithms we call take polynomial time [19, 28]. Thus, the whole algorithm runs in polynomial time. $\square$

We finally consider coloring of fork-free t-perfect graphs.

Lemma 3.14. *Let G be a fork-free graph. If G does not contain the C_7^2, the C_{10}^2, or any odd wheel as a t-minor, then the chromatic number of G is at most three, and an optimal coloring of G can be found in polynomial time.*

Proof. We may assume without loss of generality that G is connected; otherwise, we work on its components one by one. Since G does not contain a K_4, it cannot contain the complement of any odd hole longer than seven. It does not contain a C_7^2, which is the complement of C_7. Therefore, if G does not contain any odd hole, then G is perfect, and we can use the algorithm of Chudnovsky et al. [25] to find an optimal coloring. The chromatic number of G is at most three because it is equal to the order of the maximum cliques [27], which is at most three because G is K_4-free. Otherwise, G contains an odd hole, and thus its chromatic number is at least three. Thus, it suffices to find a three coloring, i.e., a partition of $V(G)$ into three (not necessarily maximal) independent sets. If G is claw-free, then we can use the algorithm of Bruhn and Stein [21] to find an optimal coloring. In the rest, G contains a claw. The algorithm now finds a five-hole H, and partition the vertex set $V(G) \setminus V(H)$ according to their adjacencies with H. We may number the vertices on H such that U_1^+ is nonempty and U_4 is empty. This is possible because of Proposition 3.7 iv) with

$i = 1$.

If U_3 is empty, then we partition $V(G)$ into three independent sets $U_5 \cup \{v_4\}$, $U_1 \cup \{v_2, v_5\}$, and $U_2 \cup \{v_1, v_3\}$. If U_5 is empty, then we partition $V(G)$ into three independent sets that are $U_1 \cup \{v_5\}$, $U_2 \cup \{v_1, v_3\}$, and $U_3 \cup \{v_2, v_4\}$. In the rest, neither U_3 nor U_5 is empty. If $U_5^+ \neq \emptyset$, then U_2 is empty because of Proposition 3.7 iv) with $i = 5$. We can partition $V(G)$ into three independent sets $U_1 \cup \{v_2, v_5\}$, $U_3 \cup \{v_3\}$, and $U_5 \cup \{v_1, v_4\}$. The remaining case is when $U_5 = U_5^-$, and we show that this cannot happen. Since neither U_1 nor U_5 is empty, U_3^+ is empty because of Proposition 3.7 iv) with $i = 3$. For $j = 3, 5$, let u_j^- be the only vertex in U_j^-. Let u_1^+ be an arbitrary vertex in U_1^+; it is adjacent to both u_3^-, by Proposition 3.7 i), and u_5^-, by Proposition 3.7 ii), both with $i = 1$. If $u_3^- u_5^- \notin E(G)$, then $\{u_1^+, u_3^-, v_4, u_5^-, v_2\}$ forms a fork; otherwise, u_1^+ has three consecutive neighbors on the hole $u_3^- v_5 v_4 v_3 u_5^-$, contradicting Propositions 3.1(iii). The algorithm is thus complete.

All the induced subgraphs we need to check have a constant number of vertices. Both algorithms we call take polynomial time [21, 25]. The rest is clearly doable in polynomial time. Thus, the whole algorithm runs in polynomial time. $\square$

Theorem 1.3 directly follows from Lemma 3.14 and Theorem 1.1.

Chapter 4

Complementation in T-perfect Graphs

In this chapter, our focus is on the study of complementation in t-perfect graphs. We are particularly interested in graphs G for which both G and its complement are t-perfect or minimally t-imperfect. This motivation leads us to introduce the concept of *core graphs*. In Section 4.1, we delve into the investigation of core graphs, exploring their structural properties. Specifically, we establish that an imperfect core graph consists of at most ten vertices. Furthermore, we delve into the study of t-perfect core graphs in Section 4.2. By proving Theorem 1.5, we are able to identify all self-complementary t-perfect graphs. Moreover, we shift our focus to study minimally t-imperfect core graphs in Section 4.3. Through the proof of Theorem 1.4, we conclude that they can only be $(3, 3)$-partitionable graphs.

4.1 Core graphs

Definition 4.1 (core graphs)**.** *A graph G is a core graph if neither G nor its complement contains a t-imperfect graph as a proper t-minor.*

By definition, any t-minor of a core graph is also a core graph. Moreover, if G is a core graph, then G is either t-perfect or minimally t-imperfect, and so is $\overline{G}$; it is possible that G is t-perfect while $\overline{G}$ is minimally t-imperfect, e.g., C_7 and $\overline{C_7}$. However, there are t-perfect graphs that are not core graphs, e.g., C_9 and $\overline{K_5}$.

Proposition 4.2. *A core graph cannot contain a K_4 or its complement as a proper induced subgraph.*

By Proposition 2.4, any $\{K_4, \overline{K_4}\}$-free perfect graph is a core graph. Therefore, we focus on core graphs that are not perfect. Such a graph cannot contain an odd hole longer than seven or its complement as a proper induced subgraph.

Proposition 4.3. *Let G be a core graph different from C_7 and $\overline{C_7}$. Every odd hole in G is a C_5. Moreover, if G is t-imperfect, then G contains a C_5.*

Proof. For the first assertion, note that $\overline{C_7}$ is t-imperfect, so the only core graph contains C_7 is C_7 itself; and for $k \geq 4$, the hole C_{2k+1} contains a $\overline{K_4}$. For the second assertion, note that if G does not contain a C_5, then G is perfect, hence t-perfect by Propositions 2.4 and 4.2. □

As we will see, five-holes are pivotal in core graphs. First, every C_5 in a core graph different from $\overline{W_5}$ is dominating: every other vertex is adjacent to at least two vertices on it.

Lemma 4.4. *Let G be a core graph different from W_5 and its complement. If G contains a five-hole C, then for every $u \in V(G) \setminus C$, either*

i) u has exactly two neighbors on C, and they are consecutive on C; or

ii) u has exactly three neighbors on C, and they are not consecutive on C.

Proof. We consider the subgraph G' of G induced by u and the five vertices on C. If u is adjacent to all vertices on C, then G' is a W_5. Since W_5 is t-imperfect, $G = G'$, a contradiction. If u is adjacent to four vertices or three consecutive vertices on C, then K_4 is a proper t-minor of G', with t-contraction at a non-neighbor of u on C. Noting that the complement of C is a C_5, we end with the same contradictions on $\overline{G}$ if u has zero or one neighbor on C, or its two neighbors on C are not consecutive. □

The next proposition further stipulates the relationship between a five-hole and other vertices in a core graph.

Proposition 4.5. *In a core graph, every pair of consecutive vertices on a five-hole has at most one common neighbor.*

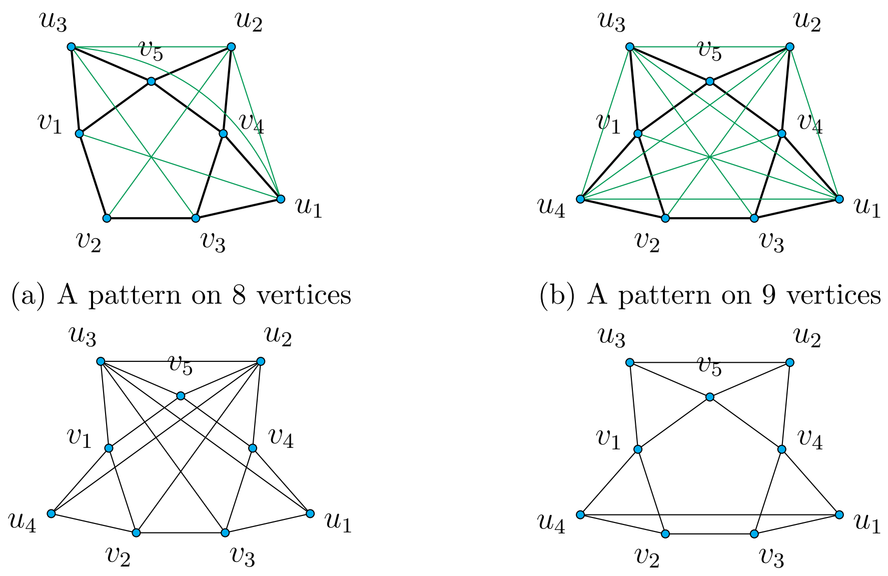
Proof. Let G be a core graph, and let $v_1v_2v_3v_4v_5$ be a five-hole in G. Suppose for contradiction that there are two vertices $x, y \in N(v_2) \cap N(v_3)$. By Lemma 4.4, neither x nor y is adjacent to v_1 or v_4. But then dependent on whether they are adjacent, x and y either form a K_4 with $\{v_2, v_3\}$, or a $\overline{K_4}$ with $\{v_1, v_4\}$, both contradicting Proposition 4.2. The same argument applies to other edges on the 5-cycle. □

As a consequence of Proposition 4.2 and the Ramsey theorem, a core graph has at most 17 vertices. Propositions 4.4 and 4.5 together imply a tighter upper bound on those that are not perfect.

Corollary 4.6. *If a core graph contains a C_5, then it has at most ten vertices.*

Let G be a core graph that contains a five-hole, and we use the following notations for its vertices and edges, where the indices are always understood as modulo 5. We fix a five-hole C and number its vertices as $v_1, \ldots, v_5$ in order, and let $U = V(G) \setminus C$. According to Lemma 4.4, each vertex in U is adjacent to two consecutive vertices on C. If a vertex in U is adjacent to v_i and v_{i+1}, $i = 1, \ldots, 5$, then we denote it as u_{i+3}; by Lemma 4.4, this is well defined. The five edges on C are all the edges among $v_1, \ldots, v_5$. For each u_i, the two edges u_iv_{i+2} and u_iv_{i+3} must exist in G. Apart from these $2|U| + 5$ edges, by Lemma 4.4, the other possible edges are among U or

u_iv_i, $i = 1, \ldots, 5$; they are called *potential edges*. Shown in Figures 4.1(a, b) are two *pattern graphs*, from which we can obtain different particular graphs, with different materializations of potential edges. We use (1324) to denote the graph of pattern Figure 4.1(b) in which U induces a path, with edges u_1u_3, u_2u_3, and u_2u_4. In case that $G[U]$ is not connected, we use $\|$ to separate its components, e.g., $(14\|23)$ in Figure 4.1(d). Moreover, we cap an index i with $\circ$ to denote the present of the edge u_iv_i, e.g., $(1\mathring{3}\mathring{2}4)$ in Figure 4.1(c).



(a) A pattern on 8 vertices

(b) A pattern on 9 vertices

(c) $(1\mathring{3}\mathring{2}4)$

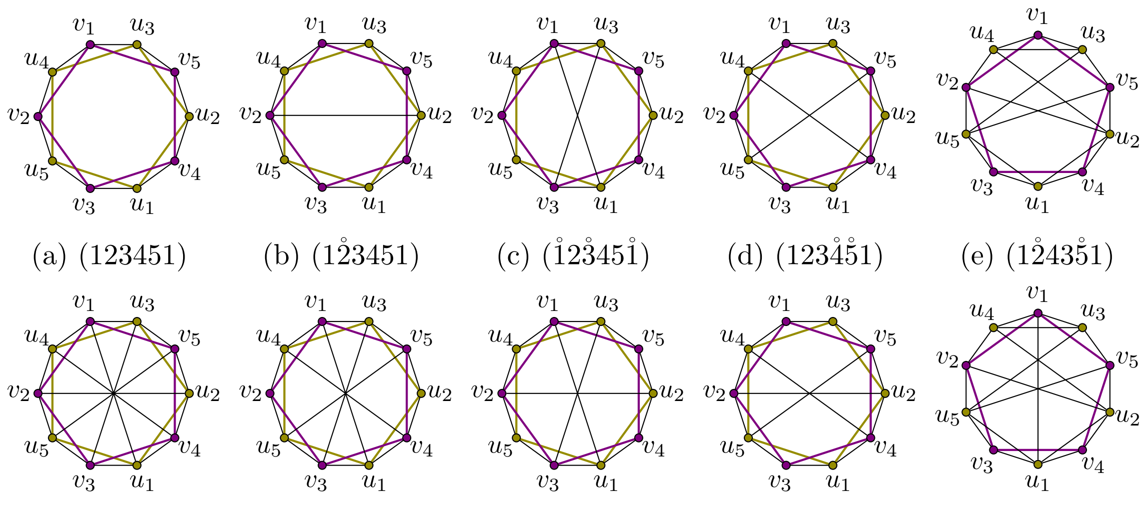
(d) $(14\|23)$

Figure 4.1: Two patterns (a, b) and two particular graphs (c, d) of the second pattern. In the patterns, potential edges are depicted as thin green lines, while normal ones as thick black lines; no other edges can exist.

The $(3, 3)$-partitionable graphs, as illustrated in Figure 1.3, are graphs of the pattern on ten vertices. Similarly, the graphs illustrated in Figure 1.5 are graphs of the pattern on nine vertices. To refer to these graphs, we assign labels to their vertices and use our notation; see Figures 4.2 and 4.3.

Benchetrit proposed the following sufficient condition for t-perfection.

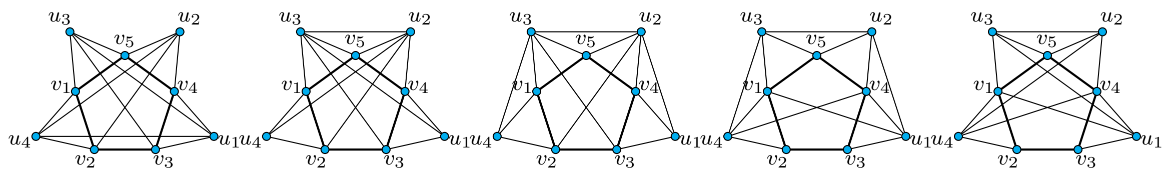
Proposition 4.7 ([11])**.** *Let K be a clique that intersects every inclusion-wise maximal independent set of a graph G. If $G - v$ is t-perfect for every $v \in K$, then G is*



(a) (123451) (b) $(1\mathring{2}3451)$ (c) $(\mathring{1}2\mathring{3}45\mathring{1})$ (d) $(123\mathring{4}\mathring{5}1)$ (e) $(1\mathring{2}43\mathring{5}1)$

(f) $(\mathring{1}\mathring{2}\mathring{3}\mathring{4}\mathring{5}\mathring{1})$ (g) $(\mathring{1}2\mathring{3}\mathring{4}\mathring{5}\mathring{1})$ (h) $(\mathring{1}\mathring{2}\mathring{3}45\mathring{1})$ (i) $(1\mathring{2}3\mathring{4}\mathring{5}1)$ (j) $(\mathring{1}\mathring{2}43\mathring{5}\mathring{1})$

Figure 4.2: The $(3,3)$-partitionable graphs.



(a) $(\mathring{2}41\mathring{3})$ (b) $(1\mathring{3}\mathring{2}4)$ (c) $(1\mathring{2}\mathring{3}4)$ (d) $(\mathring{1}23\mathring{4})$ (e) $(\mathring{1}32\mathring{4})$

Figure 4.3: Self-complementary graphs that are t-perfect but not perfect ($n > 5$).

also t-perfect.

We present a collection of t-perfect graphs that will be used in the subsequent sections. These graphs are illustrated in Figure 4.4.

Proposition 4.8. *The following graphs are t-perfect:* (12), $(1\|\mathring{2})$, $(1\mathring{2})$, $(\mathring{1}\|\mathring{2})$, $(\mathring{1}\mathring{2})$, $(1\|23)$, $(\mathring{3}1\mathring{2})$, $(\mathring{1}\mathring{3}\|\mathring{2})$, $(1\|\mathring{2}4)$, $(14\mathring{2})$, $(1\|\mathring{2}\mathring{4})$, $(1\mathring{4}\mathring{2})$, $(\mathring{1}\|\mathring{2}\|\mathring{4})$, $(\mathring{1}\mathring{2}4)$, $(\mathring{1}\mathring{2}\mathring{4})$, $(1\mathring{3}\mathring{4}\mathring{2})$, $(1\mathring{3}\mathring{4}2)$, $(\mathring{1}\mathring{3}\mathring{4}2)$, $(\mathring{1}\mathring{2}\mathring{4}3)$, $(2\mathring{3}14)$, $(2\mathring{3}\mathring{1}4)$, $(23\mathring{1}4)$, $(1\mathring{4}32)$, $(\mathring{1}\mathring{2}\mathring{4}\mathring{3}\mathring{1})$, $(1\mathring{2}\mathring{4}3)$, $(1\mathring{3}\mathring{2}41)$, $(1\mathring{3}\mathring{2}4)$, $(14\|23)$, $(1\mathring{4}\mathring{3}2)$, $(1\mathring{3}\|\mathring{2}4)$, *and* $(\mathring{2}41\mathring{3})$.

Proof. Graphs (12), $(1\|\mathring{2})$, $(1\mathring{2})$, $(\mathring{1}\|\mathring{2})$, and $(\mathring{1}\mathring{2})$ are almost bipartite graph, hence t-perfect. For each of the other graphs, we find a 3-clique K and then use Proposition 4.7. To show $G - v$ is t-perfect for every $v \in K$, we either directly show that it is isomorphic to a t-perfect graph, or show that it is a K_4-free perfect graph (Proposition 2.4). The details are listed in Table 4.1, where $\star$ means that the graph is a K_4-free perfect graph. $\square$

As easy consequences of Lemma 4.4, we have the following observations on core graphs that contain a C_5. Here $i = 1, \ldots, 5$.

Ob.1) If both u_iv_i and $u_{i+1}u_{i+2}$ are in $E(G)$, then at least one of u_iu_{i+1} and u_iu_{i+2} is in $E(G)$; otherwise, v_{i+4} has four neighbors on the 5-cycle $u_iv_iu_{i+2}u_{i+1}v_{i+3}$. By symmetry, if both u_iv_i and $u_{i-1}u_{i-2}$ are in $E(G)$, then at least one of u_iu_{i-2} and u_iu_{i-1} is in $E(G)$.

Ob.2) If both u_iu_{i+1} and u_iu_{i+3} are in $E(G)$, then at least one of u_iv_i and $u_{i+1}u_{i+3}$ is in $E(G)$; otherwise, v_{i+3} has three consecutive neighbors on the 5-cycle $u_iu_{i+1}v_{i+4}v_iu_{i+3}$. By symmetry, if both u_iu_{i-1} and u_iu_{i-3} are in $E(G)$, then at least one of u_iv_i and $u_{i-1}u_{i-3}$ is in $E(G)$.

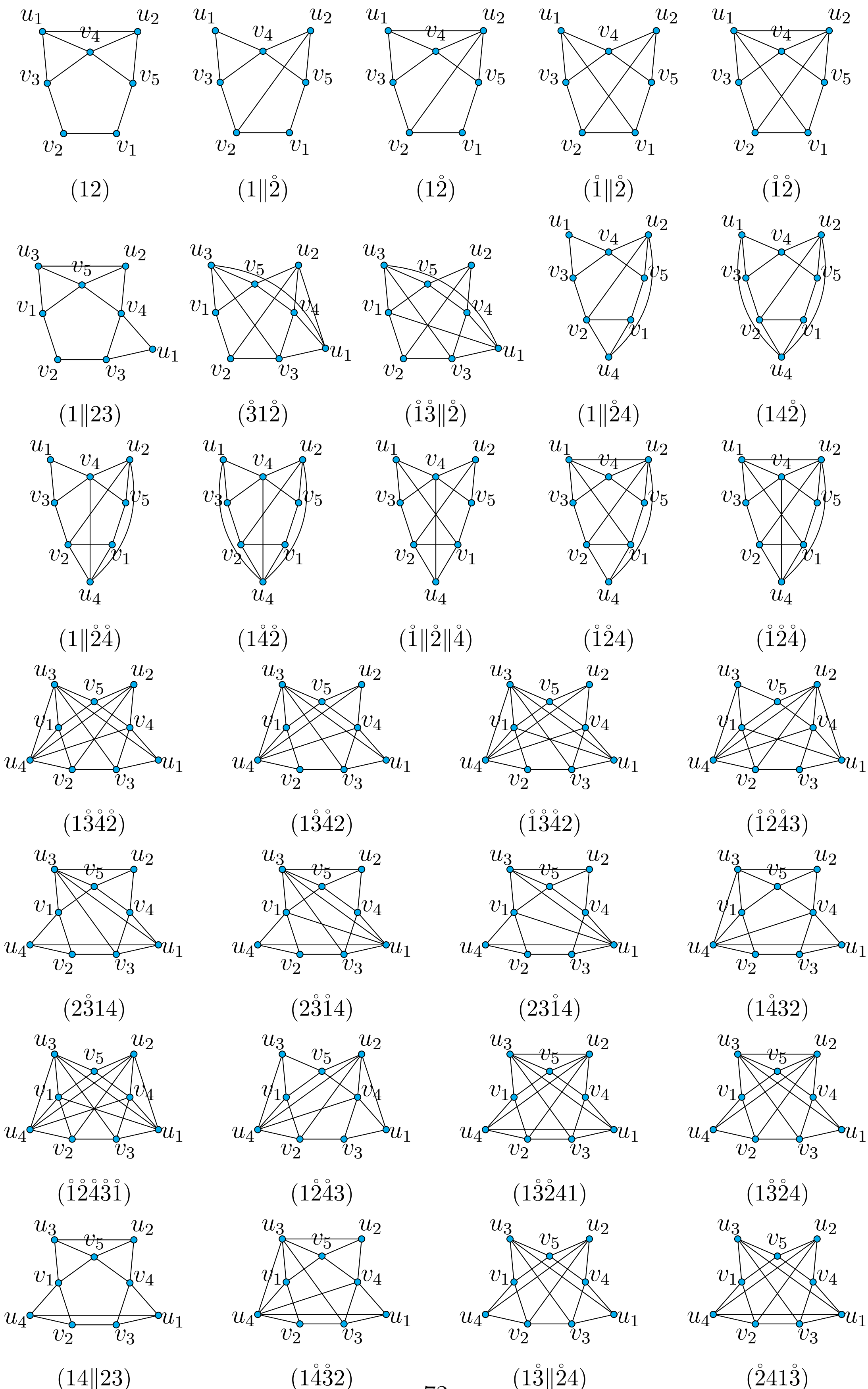


Figure 4.4: Some t-perfect graphs.

Table 4.1: For the proof of Proposition 4.8

	$K = \{a, b, c\}$	$G - a$	$G - b$	$G - c$
$(1\Vert 23)$	$\{v_3, v_4, u_1\}$	$\star$	$\star$	(12)
$(1\Vert\mathring{2}4)$	$\{v_1, v_2, u_4\}$	$\star$	$\star$	$(1\Vert\mathring{2})$
$(14\mathring{2})$	$\{v_1, v_2, u_4\}$	$\star$	$(1\Vert\mathring{2})$	$(1\Vert\mathring{2})$
$(1\Vert\mathring{2}\mathring{4})$	$\{v_1, v_2, u_4\}$	$\star$	$\star$	$(1\Vert\mathring{2})$
$(1\mathring{4}\mathring{2})$	$\{v_1, v_2, u_4\}$	$\star$	$\star$	$(1\Vert\mathring{2})$
$(\mathring{1}\Vert\mathring{2}\Vert\mathring{4})$	$\{v_1, v_2, u_4\}$	$\star$	$\star$	$(\mathring{1}\Vert\mathring{2})$
$(\mathring{1}\mathring{2}4)$	$\{v_1, v_2, u_4\}$	$\star$	$\star$	$(\mathring{1}\mathring{2})$
$(\mathring{1}\mathring{2}\mathring{4})$	$\{v_4, v_5, u_2\}$	$\star$	$\star$	$(\mathring{1}\Vert\mathring{2}\Vert\mathring{4} - u_2)$
$(\mathring{3}1\mathring{2})$	$\{v_1, v_5, u_3\}$	$(\mathring{1}\mathring{2}\mathring{4} - u_1)$	$(1\mathring{2})$	$(1\mathring{2})$
$(\mathring{1}\mathring{3}\Vert\mathring{2})$	$\{v_4, v_5, u_2\}$	$(1\mathring{2})$	$(1\mathring{2})$	$(\mathring{1}\mathring{2}\mathring{4} - u_1)$
$(1\mathring{3}\mathring{4}\mathring{2})$	$\{v_4, v_5, u_2\}$	$(1\Vert\mathring{2}\mathring{4})$	$\star$	$(\mathring{1}\mathring{2}4)$
$(1\mathring{3}\mathring{4}2)$	$\{v_4, v_5, u_2\}$	$\star$	$\star$	$(\mathring{1}\mathring{2}4)$
$(\mathring{1}\mathring{3}\mathring{4}2)$	$\{v_4, v_5, u_2\}$	$\star$	$\star$	$(\mathring{1}\mathring{2}\mathring{4})$
$(\mathring{1}\mathring{2}\mathring{4}3)$	$\{v_1, v_5, u_3\}$	$\star$	$\star$	$(\mathring{1}\mathring{2}\mathring{4})$
$(2\mathring{3}14)$	$\{v_4, v_5, u_2\}$	$\star$	$\star$	$(14\mathring{2})$
$(2\mathring{3}\mathring{1}4)$	$\{v_4, v_5, u_2\}$	$\star$	$\star$	$(1\mathring{4}\mathring{2})$
$(23\mathring{1}4)$	$\{v_3, v_4, u_1\}$	$\star$	$\star$	$(1\Vert 23)$
$(1\mathring{4}32)$	$\{v_3, v_4, u_1\}$	$\star$	$\star$	$(1\mathring{2}3451) - \{u_3, u_4\}$
$(\mathring{1}\mathring{2}\mathring{4}\mathring{3}\mathring{1})$	$\{v_4, v_5, u_2\}$	$(\mathring{1}\mathring{2}\mathring{4})$	$\star$	$(\mathring{1}\mathring{2}\mathring{4})$
$(1\mathring{2}\mathring{4}3)$	$\{v_3, v_4, u_1\}$	$\star$	$\star$	$(\mathring{1}\mathring{2}\mathring{4}3) - u_1$
$(1\mathring{3}\mathring{2}41)$	$\{v_4, v_5, u_2\}$	$\star$	$\star$	$(14\mathring{2})$
$(1\mathring{3}\mathring{2}4)$	$\{v_4, v_5, u_2\}$	$\star$	$\star$	$(1\Vert\mathring{2}4)$
$(14\Vert 23)$	$\{v_3, v_4, u_1\}$	$(1\Vert 23)$	$\star$	$(1\Vert 23)$
$(1\mathring{4}\mathring{3}2)$	$\{v_3, v_4, u_1\}$	$\star$	$\star$	$(123\mathring{4}\mathring{5}1) - \{u_1, u_2\}$
$(1\mathring{3}\Vert\mathring{2}4)$	$\{v_4, v_5, u_2\}$	$(1\Vert\mathring{2}4)$	$\star$	$(1\Vert\mathring{2}4)$
$(\mathring{2}41\mathring{3})$	$\{v_4, v_5, u_2\}$	$(14\mathring{2})$	$\star$	$(14\mathring{2})$

Ob.3) Suppose, all of $u_{i-2}u_{i-1}$, $u_{i-1}u_{i+1}$, and $u_{i+1}u_{i+2}$ are in $E(G)$. If $u_{i-1}v_{i-1}$ or $u_{i+1}v_{i+1}$ is in $E(G)$, then at least one of $u_{i-1}u_{i+2}$, $u_{i-2}u_{i+1}$, and $u_{i-2}u_{i+2}$ is in $E(G)$; otherwise, v_{i-1} or v_{i+1} has four neighbors on the 5-cycle $u_{i-2}u_{i-1}u_{i+1}u_{i+2}v_i$.

Ob.4) If $u_{i-1}u_{i+1} \in E(G)$ and $u_{i-1}v_{i-1}, u_{i+1}v_{i+1} \notin E(G)$, then $u_{i+1}u_{i+2}, u_{i-1}u_{i-2} \notin E(G)$, and $u_iu_{i-1}, u_iu_{i+1} \in E(G)$; otherwise, the neighborhood of u_{i-2}, u_{i+2}, or, respectively, u_i on the 5-cycle $u_{i-1}u_{i+1}v_{i-1}v_iv_{i+1}$ does not satisfy Lemma 4.4.

Ob.5) If $u_iu_{i+1} \notin E(G)$ and at least one of u_i and u_{i+1} is adjacent to u_{i+3}, then at most one of u_iv_i and $u_{i+1}v_{i+1}$ can be in $E(G)$; otherwise, u_{i+3} has three consecutive neighbors on the 5-cycle $u_iv_iv_{i+1}u_{i+1}v_{i+3}$.

Ob.6) If $u_{i+1}v_{i+1}$ is in $E(G)$ and none of $u_{i+1}u_{i+2}$, $u_{i+2}u_{i-2}$, and $u_{i-1}u_{i-2}$ is in $E(G)$, then $u_{i+1}u_{i-2}$, $u_{i+2}u_{i-1}$, and $u_{i+1}u_{i-1}$ cannot be all present in G; otherwise, v_{i+1} has four neighbors on the 5-cycle $u_{i-1}u_{i+2}v_iu_{i-2}u_{i+1}$. By symmetry, if $u_{i-1}v_{i-1}$ is in $E(G)$ and none of $u_{i+1}u_{i+2}$, $u_{i+2}u_{i-2}$, and $u_{i-1}u_{i-2}$ is in $E(G)$, then $u_{i+1}u_{i-2}$, $u_{i+2}u_{i-1}$, and $u_{i+1}u_{i-1}$ cannot be all present in G.

All graphs of pattern Figure 4.1(a) are summarized in Table 4.2 and characterized in Lemma 4.9.

Lemma 4.9. *Let G be an imperfect core graph of order eight. At least one of G and $\overline{G}$*

i) is t-perfect; or

ii) has a degree-2 vertex in U.

Proof. Since G is imperfect, it contains a C_5 by Proposition 4.3. In particular, G or $\overline{G}$ is of the pattern in Figure 4.1(a). Note that if the degree of a vertex is five in G, then its degree in $\overline{G}$ is two. According to Table 4.2, it suffices to show that graphs $(\mathring{1}\mathring{3}\|\mathring{2})$, $(\mathring{1}3\|\mathring{2})$, $(\mathring{1}2\mathring{3})$, $(\mathring{3}1\mathring{2})$, $(\mathring{1}23)$, $(\mathring{3}12)$, $(\mathring{1}\|\mathring{2}\|\mathring{4})$, and (123) are t-perfect. We have seen

Table 4.2: Graphs of pattern Figure 4.1(a). The columns are for combinations of edges among U; the cases with only u_2u_3 and only $\{u_1u_3, u_2u_3\}$ are omitted because they are symmetric to respectively, u_1u_2 and $\{u_1u_2, u_1u_3\}$. The rows are possible combinations of edges between U and C. The invocation of an observation means that this configuration violates this observation.

	all	$\{u_1u_2, u_1u_3\}$	$\{u_1u_2, u_2u_3\}$	$\{u_1u_2\}$	$\{u_1u_3\}$
all	$d(u_1) = 5$	$d(u_1) = 5$	$d(u_2) = 5$	Ob.1) $(i = 3)$	$(\mathring{1}\mathring{3}\|\mathring{2})$
$\{u_1v_1, u_2v_2\}$	$d(u_1) = 5$	$d(u_1) = 5$	$d(u_2) = 5$	$d(u_3) = 2$	$(\mathring{1}3\|\mathring{2})$
$\{u_1v_1, u_3v_3\}$	$d(u_1) = 5$	$d(u_1) = 5$	$(\mathring{1}2\mathring{3})$	Ob.1) $(i = 3)$	$d(u_2) = 2$
$\{u_2v_2, u_3v_3\}$	$d(u_2) = 5$	$(\mathring{3}1\mathring{2})$	$d(u_2) = 5$	Ob.1) $(i = 3)$	$\cong (\mathring{1}3\|\mathring{2})$
$\{u_1v_1\}$	$d(u_1) = 5$	$d(u_1) = 5$	$(\mathring{1}23)$	$d(u_3) = 2$	$d(u_2) = 2$
$\{u_2v_2\}$	$d(u_2) = 5$	Ob.4) $(i = 2)$	$d(u_2) = 5$	$d(u_3) = 2$	Ob.4) $(i = 2)$
$\{u_3v_3\}$	$d(u_3) = 5$	$(\mathring{3}12)$	$\cong (\mathring{1}23)$	Ob.1) $(i = 3)$	$d(u_2) = 2$
none	$\overline{G} \cong (\mathring{1}\|\mathring{2}\|\mathring{4})$	Ob.4) $(i = 2)$	(123)	$d(u_3) = 2$	$d(u_2) = 2$

in Proposition 4.8 that $(\mathring{1}\|\mathring{2}\|\mathring{4})$, $(\mathring{3}1\mathring{2})$, and $(\mathring{1}\mathring{3}\|\mathring{2})$ are t-perfect. The graph $(\mathring{1}3\|\mathring{2})$ is t-perfect because $(\mathring{1}3\|\mathring{2})$ is isomorphic to $(\mathring{2}41\mathring{3}) - u_1$, and $(\mathring{2}41\mathring{3})$ is t-perfect. On the other hand, $(\mathring{3}12)$, $(\mathring{1}2\mathring{3})$, $(\mathring{1}23)$, and (123) are isomorphic to, respectively, $(1\mathring{2}43\mathring{5}1) - \{u_1, u_2\}$, $(1\mathring{2}43\mathring{5}1) - \{u_3, u_4\}$, $(1\mathring{2}3451) - \{u_1, u_5\}$, and $(123451) - \{u_1, u_5\}$, all t-perfect. $\square$

4.2 T-perfect core graphs

4.2.1 Degree-bounded graphs

According to Propositions 4.4 and 4.5, every core graph of order nine is of the pattern in Figure 4.1(b). Throughout this section, let G denote a core graph of order nine where the degree of every vertex is between three and five. (The reason of imposing degree constraints will become clear shortly.) We consider whether edges u_iu_{i+1}, $i = 1, 2, 3$ are present in G.

Proposition 4.10. *Let G be a degree-bounded core graph on nine vertices. If for all $i = 1, 2, 3$, the edge $u_i u_{i+1}$ is in $E(G)$, then G is an induced subgraph of a $(3, 3)$-partitionable graph.*

Proof. We argue first that none of u_1u_4, u_1u_3, and u_2u_4 can be present in G; i.e., $u_1u_2u_3u_4$ is an induced path in G. Suppose that $u_1u_4 \in E(G)$, then by Ob.4) (with $i = 5$), at least one of u_4v_4 and u_1v_1 is in $E(G)$. We may assume that $u_4v_4 \in E(G)$, and the other case is symmetric. Since $\{u_1, u_4, u_2, v_4\}$ is not a clique, $u_2u_4 \notin E(G)$. By Ob.2) (with $i = 1$), $u_1v_1 \in E(G)$, and then since $\{u_1, u_3, u_4, v_1\}$ is not a clique, u_1u_3 cannot be present. But then $G - \{v_2, v_3\}$ is isomorphic to $\overline{C_7}$, a contradiction. Thus, $u_1u_4 \notin E(G)$. By Ob.2) (with $i = 2$), (noting $u_1u_2 \in E(G)$,) the presence of u_2u_4 would imply the presence of u_2v_2, but then $d(u_2) = 6$. Thus, $u_2u_4 \notin E(G)$, and by a symmetric argument, $u_1u_3 \notin E(G)$.

Now that none of u_1u_4, u_1u_3, and u_2u_4 is present, we consider all possible combinations of edges $\{u_iv_i \mid i = 1, \ldots, 4\} \cap E(G)$. If none of them is in $E(G)$, then G is isomorphic to $(123451) - u_1$. If all of them are in $E(G)$, then G is isomorphic to $(\mathring{1}\mathring{2}\mathring{3}\mathring{4}\mathring{5}\mathring{1}) - u_1$. If only one u_iv_i is in $E(G)$, then G is isomorphic to $(\mathring{1}2\mathring{3}45\mathring{1}) - u_3$ or $(123\mathring{4}\mathring{5}1) - u_4$. If only one u_iv_i is absent, then G is isomorphic to $(\mathring{1}\mathring{2}\mathring{3}45\mathring{1}) - u_4$ or $(1\mathring{2}3\mathring{4}\mathring{5}1) - u_1$. Otherwise, exact two of edges u_iv_i are in $E(G)$, then G is isomorphic to one of $(\mathring{1}2\mathring{3}45\mathring{1}) - u_4$, $(\mathring{1}\mathring{2}\mathring{3}45\mathring{1}) - u_1$, $(\mathring{1}2\mathring{3}45\mathring{1}) - u_2$, and $(123\mathring{4}\mathring{5}1) - u_2$. □

In the rest, for at least one of $i = 1, 2, 3$, the edge u_iu_{i+1} is absent from G. In the second case, we assume that both u_1u_2 and u_2u_3 are absent from G; see Figure 4.5(b).

Proposition 4.11. *Let G be a degree-bounded core graph on nine vertices. If both u_1u_2 and u_2u_3 are absent from G, then G is isomorphic to one of* $(1\mathring{3}\mathring{4}\mathring{2})$, $(1\mathring{3}\mathring{4}2)$, $(\mathring{1}\mathring{3}\mathring{4}2)$, $(1\mathring{3}\|\mathring{2}4)$, *and* $(\mathring{2}41\mathring{3})$.

Proof. We first argue that the edge u_1u_3 must be present. Suppose for contradiction that u_1u_3 is absent. Note that $u_2v_2 \in E(G)$, as otherwise $\{u_1, u_2, u_3, v_2\}$ forms an

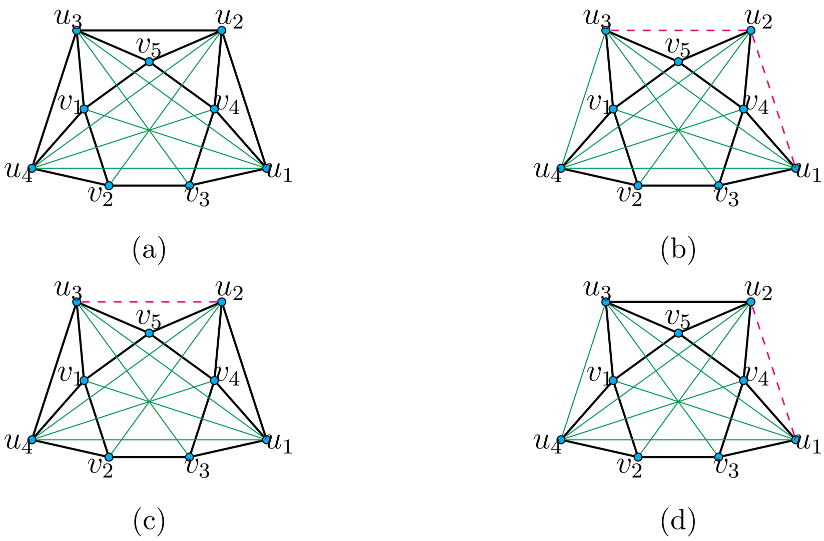


Figure 4.5: Refined patterns on nine vertices. Potential edges in Figure 4.1(b) but absent here are emphasized by red dashed lines. (a) all the three edges u_1u_2, u_2u_3, and u_3u_4 are present; (b) both u_1u_2 and u_2u_3 are absent; (c) u_2u_3 is absent but both u_1u_2 and u_3u_4 are present; (d) u_1u_2 is absent but u_2u_3 is present.

independent set. The edge u_3v_3 cannot be in $E(G)$, as otherwise u_1 has only one neighbor on the 5-cycle $u_3v_3v_2u_2v_5$, contradicting Lemma 4.4. Then $d(u_3) > 2$ forces $u_3u_4 \in E(G)$. By Ob.1) (with $i = 2$), $u_2u_4 \in E(G)$, and by Ob.5) (with $i = 1$), $u_1v_1 \notin E(G)$. Since $d(u_1) > 2$, the edge u_1u_4 must be present. Now that $u_3u_4 \in E(G)$ and $u_1v_1 \notin E(G)$, Ob.4) (with $i = 5$) implies $u_4v_4 \in E(G)$. But then $d(u_4) = 6$, contradicting that G is degree-bounded.

Now that $u_1u_3 \in E(G)$, by Ob.4) (with $i = 2$), at least one of u_1v_1 and u_3v_3 is present. Assume first that $u_1v_1 \in E(G)$. By Proposition 4.2, at least one of u_3u_4, u_2u_4, and u_3v_3 is in $E(G)$, as otherwise $\{u_2, u_3, u_4, v_3\}$ forms an independent set. We argue that $u_3u_4 \in E(G)$. Suppose for contradiction that $u_3u_4 \notin E(G)$. If $u_2u_4 \in E(G)$, then by Ob.5) (with $i = 1$), $u_2v_2 \notin E(G)$; and by Ob.6) (with $i = 5$), Ob.4) (with $i = 3$), and Ob.5) (with $i = 3$), $u_1u_4 \notin E(G)$, $u_4v_4 \in E(G)$, and $u_3v_3 \notin E(G)$. Then $u_1v_3v_2u_4u_2v_5u_3$ is a 7-cycle in G, contradicting Proposition 4.3. Thus, $u_2u_4 \notin E(G)$, and $u_3v_3 \in E(G)$. By Ob.5) (with $i = 3$), $u_4v_4 \notin E(G)$. Since $d(u_4) > 2$, $u_1u_4 \in E(G)$ and by Ob.5) (with $i = 1$), $u_2v_2 \notin E(G)$. But then $d(u_2) = 2$, a contradiction. Now that $u_3u_4 \in E(G)$, the edge u_1u_4 cannot exist, as otherwise $\{u_1, u_3, u_4, v_1\}$ forms a clique. By Ob.2) ($i = 3$), $u_3v_3 \in E(G)$. If $u_2v_2 \in E(G)$, then by Ob.1) (with $i = 2$), $u_2u_4 \in E(G)$, which violates Ob.5) (with $i = 1$). Since $d(u_2) > 2$, the edge u_2u_4 must be present. Ob.4) (with $i = 3$), together with the fact that $u_2v_2 \notin E(G)$, implies $u_4v_4 \in E(G)$. Thus, G is $(\mathring{1}\mathring{3}\mathring{4}2)$.

In the rest of the proof, $u_1v_1 \notin E(G)$ and $u_3v_3 \in E(G)$. Since $d(u_2) > 2$, at least one of u_2v_2 and u_2u_4 needs to be present. Note that the presence of u_2v_2 implies the presence of u_2u_4, by Lemma 4.4 applied on vertex u_4 and the 5-cycle $u_3v_3v_2u_2v_5$. If $u_2u_4 \in E(G)$, but u_2v_2 is not, then by Ob.4) (with $i = 3$), $u_4v_4 \in E(G)$. The edge u_3u_4 is in $E(G)$, as otherwise u_1 has three consecutive neighbors on the 5-cycle $u_3v_3v_4u_4v_1$, contradicting Lemma 4.4. Since $d(u_4) < 6$, the edge u_1u_4 cannot be present. Then G is $(1\mathring{3}\mathring{4}2)$. Now that both u_2u_4 and u_2v_2 are present, the only potential edges that have not been excluded are u_4v_4, u_3u_4, and u_1u_4. Note that the presence of u_3u_4 implies the presence of u_4v_4; otherwise, by Ob.4) (with $i = 5$), $u_1u_4 \notin E(G)$, but then v_3 has three consecutive neighbors on the 5-cycle $u_1v_4u_2u_4u_3$, contradicting Lemma 4.4.

- If none of u_4v_4, u_3u_4, and u_1u_4 is present, then G is $(1\mathring{3}\|\mathring{2}4)$.
- If u_1u_4 is in $E(G)$ but u_4v_4 and u_3u_4 are not, then G is $(\mathring{2}41\mathring{3})$.
- Otherwise, we must have $u_4v_4 \in E(G)$. Then u_3u_4 must be present as well, as otherwise $u_3v_3v_4u_4v_1$ is a 5-cycle on which u_1 has three consecutive neighbors, contradicting Lemma 4.4. Since $d(u_4) < 6$, the edge u_1u_4 cannot be present, and G is $(1\mathring{3}\mathring{4}\mathring{2})$. □

Note that it is symmetric to Proposition 4.11 if both u_2u_3 and u_3u_4 are absent. Next we consider the situation that u_2u_3 is absent but both u_1u_2 and u_3u_4 are present; see Figure 4.5(c).

Proposition 4.12. *Let G be a degree-bounded imperfect core graph on nine vertices.*

If both u_1u_2 and u_3u_4 are in $E(G)$ but u_2u_3 is not, then G is isomorphic to one of $(1\mathring{2}\mathring{4}3)$, $(\mathring{1}\mathring{2}\mathring{4}\mathring{3}\mathring{1})$, $(\mathring{1}\mathring{2}\mathring{4}3)$, $(1\mathring{2}43\mathring{5}1) - u_2$, *and* $(\mathring{1}\mathring{2}43\mathring{5}\mathring{1}) - u_5$.

Proof. We start by arguing that u_1u_4 is absent, and at least one of u_2v_2 and u_3v_3 is present. Suppose for contradiction that $u_1u_4 \in E(G)$. By Ob.4) (with $i = 5$), at least one of u_4v_4 and u_1v_1 is in $E(G)$. If u_4v_4 is in $E(G)$ but u_1v_1 is not, then by Ob.2) (with $i = 1$), $u_2u_4 \in E(G)$; then $d(u_4) = 6$, a contradiction. A symmetric argument applies if u_1v_1 is in $E(G)$ but u_4v_4 is not. Now that both u_1v_1 and u_4v_4 are in $E(G)$, neither u_1u_3 nor u_2u_4 can be in $E(G)$, as otherwise $\{u_3, u_1, u_4, v_1\}$ or, respectively, $\{u_1, u_4, u_2, v_4\}$ forms a clique. But then v_4 has four neighbors on the 5-cycle $u_1u_4u_3v_5u_2$, contradicting Lemma 4.4. In the rest, $u_1u_4 \notin E(G)$. For u_2v_2 and u_3v_3, if both of them are absent, then by Ob.2) (with $i = 3$), u_1u_3 has to be absent as well (note that u_3u_4 is in $E(G)$ while u_3v_3 and u_1u_4 are absent). By a symmetric argument, the edge u_2u_4 is also absent. But then $u_1v_3v_2u_4u_3v_5u_2$ is a 7-cycle in G, contradicting Proposition 4.3.

Since u_2v_2 and u_3v_3 are symmetric, it suffices to consider $u_2v_2 \in E(G)$. By Ob.1) (with $i = 2$), $u_2u_4 \in E(G)$. If none of the remaining undecided potential edges, u_1v_1, u_3v_3, u_4v_4, and u_1u_3, is in $E(G)$, then G is isomorphic to $(1\mathring{2}43\mathring{5}1) - u_2$. If $u_1u_3 \in E(G)$, then by Ob.2) (with $i = 3$), $u_3v_3 \in E(G)$. The edge u_1v_1 is in $E(G)$, as otherwise u_4 has four neighbors on the 5-cycle $u_1u_3v_1v_2u_2$, contradicting Lemma 4.4. A symmetric argument enables us conclude that $u_4v_4 \in E(G)$. Then G is $(\mathring{1}\mathring{2}\mathring{4}\mathring{3}\mathring{1})$. Now that $u_1u_3 \notin E(G)$, which implies u_3v_3 is not in $E(G)$ either, as otherwise, u_1u_2 is in $E(G)$ but neither u_2u_3 nor u_1u_3 is, contradicting Ob.1) (with $i = 3$). If u_1v_1 is in $E(G)$ but u_4v_4 is not, then G is isomorphic to $(\mathring{1}\mathring{2}43\mathring{5}\mathring{1}) - u_5$; if u_4v_4 is in $E(G)$ but u_1v_1 is not, then G is $(1\mathring{2}\mathring{4}3)$; otherwise, both u_1v_1 and u_4v_4 are in $E(G)$, and G is $(\mathring{1}\mathring{2}\mathring{4}3)$. □

In the last case, u_2u_3 is in $E(G)$, but at least one of u_1u_2 and u_3u_4 is not. We may assume without loss of generality that u_1u_2 is absent; see Figure 4.5(d).

Proposition 4.13. *Let G be a degree-bounded imperfect core graph on nine vertices. If u_2u_3 is in $E(G)$ but u_1u_2 is not, then G is isomorphic to one of* $(2\mathring{3}14)$, $(2\mathring{3}\mathring{1}4)$, $(23\mathring{1}4)$, $(1\mathring{4}32)$, $(1\mathring{3}\mathring{2}41)$, $(1\mathring{3}\mathring{2}4)$, $(14\|23)$, $(1\mathring{4}\mathring{3}2)$, $(\mathring{1}\mathring{2}43\mathring{5}\mathring{1}) - u_3$, $(1\mathring{2}43\mathring{5}1) - u_3$, *and* $(1\mathring{2}43\mathring{5}1) - u_1$.

Proof. Consider first that u_3u_4 is in $E(G)$. We argue that neither u_1v_1 nor u_1u_3 can be present. If u_1v_1 is in $E(G)$, then by Ob.1) (with $i = 1$), $u_1u_3 \in E(G)$. As a result, $u_1u_4 \notin E(G)$, as otherwise $\{u_1, u_3, u_4, v_1\}$ is a clique. But then $u_3v_3 \in E(G)$ by Ob.2) (with $i = 3$), and $d(u_3) = 6$, a contradiction. Likewise, the existence of u_1u_3 would force $u_3v_3 \in E(G)$ by Ob.4) (with $i = 2$), then $d(u_3) = 6$. Now u_1 is adjacent to neither u_3 nor v_1, the edge u_1u_4 must be present to avoid $d(u_1) > 2$. Moreover, $u_4v_4 \in E(G)$ by Ob.4) (with $i = 5$), and then from $d(u_4) < 6$ it can be inferred $u_2u_4 \notin E(G)$. If neither of the undecided potential edges, u_2v_2 and u_3v_3, is present, then G is $(1\mathring{4}32)$; if only u_2v_2 is present, then G is isomorphic to $(1\mathring{2}43\mathring{5}1) - u_3$; if only u_3v_3 is present, then G is $(1\mathring{4}\mathring{3}2)$; otherwise, both are present, and G is isomorphic to $(\mathring{1}\mathring{2}43\mathring{5}\mathring{1}) - u_3$.

In the rest, u_3u_4 is not in $E(G)$. We consider the potential edges incident to u_1 and u_4; note that their degrees are at least three. By Ob.1) (with $i = 1$), the presence of u_1v_1 implies the existence of u_1u_3; likewise, u_4v_4 implies $u_2u_4 \in E(G)$.

- Case 1, u_1v_1 is in $E(G)$. Note that if u_2u_4 is in $E(G)$, then u_4v_4 must be in $E(G)$ as well; otherwise $u_2v_2 \in E(G)$ by Ob.4) (with $i = 3$), contradicting Ob.5) (with $i = 1$). First, if $u_2v_4 \in E(G)$, then by Ob.5) (with $i = 1$), $u_2v_2 \notin E(G)$. A symmetric argument implies $u_3v_3 \notin E(G)$. Note that $u_1u_4 \notin E(G)$, as otherwise $G - \{v_2, v_3\}$ is isomorphic to $\overline{C_7}$. Then G is isomorphic to $(1\mathring{2}43\mathring{5}1) - u_1$. Second, if u_1u_4 is in $E(G)$ but u_4v_4 and u_2u_4 are not, then by Ob.5) (with $i = 1$), $u_2v_2 \notin E(G)$. Dependent on whether u_3v_3 is present or not, G is either $(2\mathring{3}\mathring{1}4)$ or $(23\mathring{1}4)$.

- Case 2, u_4v_4 is in $E(G)$. It is symmetric to case 1.

- Case 3, u_1u_3 is in $E(G)$ but u_1v_1 and u_4v_4 are not. By Ob.4) (with $i = 2$), $u_3v_3 \in E(G)$. If u_2u_4 is in $E(G)$, then by Ob.4) (with $i = 3$), $u_2v_2 \in E(G)$. Dependent on whether u_1u_4 is in $E(G)$ or not, G is either $(1\mathring{3}\mathring{2}4)$ or $(1\mathring{3}\mathring{2}41)$. If u_1u_4 is in $E(G)$ but u_2u_4 is not, then $u_2v_2 \notin E(G)$, as otherwise $u_2u_3u_1u_4v_2$ is a 5-cycle, on which v_4 has two non-consecutive neighbors, contradicting Lemma 4.4. Then G is $(2\mathring{3}14)$.

- Case 4, u_2u_4 is in $E(G)$ but u_1v_1 and u_4v_4 are not. It is symmetric to case 3.

Now that all of u_1v_1, u_4v_4, u_1u_3, and u_2u_4 are absent, the edge u_1u_4 must be present to ensure $d(u_1) > 2$. Then $u_3v_3 \notin E(G)$, as otherwise u_2 has only one neighbor on the 5-cycle $u_1u_4v_1u_3v_3$, contradicting Lemma 4.4. A symmetric argument implies $u_2v_2 \notin E(G)$. Thus, G is $(14\|23)$. □

By Propositions 4.10–4.13, a degree-bounded imperfect core graph of order nine is one of $(1\mathring{3}\mathring{4}\mathring{2})$, $(1\mathring{3}\mathring{4}2)$, $(\mathring{1}\mathring{3}\mathring{4}2)$, $(\mathring{1}\mathring{2}\mathring{4}3)$, $(\mathring{1}\mathring{2}\mathring{4}\mathring{3}\mathring{1})$, $(1\mathring{2}\mathring{4}3)$, $(1\mathring{3}\mathring{2}4)$, $(1\mathring{3}\mathring{2}41)$, $(2\mathring{3}14)$, $(2\mathring{3}\mathring{1}4)$, $(23\mathring{1}4)$, $(\mathring{2}41\mathring{3})$, $(1\mathring{4}32)$, $(14\|23)$, $(1\mathring{4}\mathring{3}2)$, $(1\mathring{3}\|\mathring{2}4)$, or a proper induced subgraph of a $(3,3)$-partitionable graph.

Lemma 4.14. *All degree-bounded imperfect core graphs of order nine are t-perfect. Only* $(\mathring{1}2\mathring{3}45\mathring{1}) - u_2$*,* $(123\mathring{4}\mathring{5}1) - u_2$*,* $(1\mathring{2}43\mathring{5}1) - u_1$*,* $(1\mathring{3}\mathring{2}4)$*, and* $(\mathring{2}41\mathring{3})$ *of them are self-complementary graphs.*

4.2.2 Self-complementary graphs

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. We first consider the sufficiency. One may easily verify that all the graphs in Figures 4.3 and 1.6 are self-complementary. We have seen that C_5, $(\mathring{2}41\mathring{3})$, and $(1\mathring{3}\mathring{2}4)$ are t-perfect; $(1\mathring{2}\mathring{3}4)$, $(\mathring{1}23\mathring{4})$, and $(\mathring{1}32\mathring{4})$ are isomorphic to $(123\mathring{4}\mathring{5}1) - u_2$, $(\mathring{1}2\mathring{3}45\mathring{1}) - u_2$, and $(1\mathring{2}43\mathring{5}1) - u_1$ respectively, hence t-perfect as well.

On the other hand, every graph in Figure 1.6 is perfect and K_4-free, and hence t-perfect by Proposition 2.4; so are K_1, K_4, and the bull graph.

For the necessity, suppose that a graph G is both self-complementary and t-perfect. We argue that $n \leq 9$ where n is the order of G.

- Case 1, G is perfect. Since G is t-perfect, it is K_4-free. Since G is self-complementary, it is also $\overline{K_4}$-free. In other words, a maximum independent set of G consists of at most three vertices. Since G is perfect, its chromatic number is at most three. Thus, $n \leq 3 \times 3 = 9$.
- Case 2, G is not perfect. Since both G and $\overline{G}$ are t-perfect, G is a core graph. Note that G is not C_7, and thus it contains a C_5 by Proposition 4.3. Thus, $n \leq 10$ by Corollary 4.6.

In particular, $n \in \{1, 4, 5, 8, 9\}$ sicne the order of a self-complementary graph is either $4k$ or $4k + 1$ for some nonnegative integer k. There are 49 self-complementary graphs of order at most nine, and they have been explicitly constructed by Xu and Wong [121]. Of these 49 graphs, 36 are K_4-free, of which 14 are perfect: K_1, P_4, the bull graph, and the eleven graphs in Figure 1.6. In the rest we focus on K_4-free self-complementary graphs G that are not perfect.

Since G is not perfect, it contains an odd hole, and by Proposition 4.3, every odd hole in G is a 5-cycle. If $n = 5$, then G is C_5. If $n = 9$, then G is of the pattern in Fig. 4.1(b). We argue that G is degree bounded. Every vertex in C has degree at least three and at most five. Suppose that one vertex $u \in U$ has degree two, then it is not adjacent to any other vertex in U. But then the degree of u in $\overline{G}$ is six; thus there is a degree-6 vertex, which has to be in U. But then we have a vertex in U that is nonadjacent to others in U, and another vertex in U that is adjacent to all of the others in U, a contradiction. By Lemma 4.14, G is one of the graphs in Figure 4.3; i.e., $(\mathring{1}23\mathring{4})$, $(\mathring{2}41\mathring{3})$, $(\mathring{1}32\mathring{4})$, $(1\mathring{2}\mathring{3}4)$, and $(1\mathring{3}\mathring{2}4)$.

It remains to show that there is no graph of order 8 satisfying the conditions. Let G be an imperfect core graph of order 8. We may assume that the indices for the three vertices in U are not consecutive: If G is of the pattern in Figure 4.1(a), then we can consider its complement. (With different choices of 5-cycles, a core graph may be of more than one patterns.) If there is a vertex x of degree 2, then $x \in U$, and the two neighbors of x are adjacent. Then in $\overline{G}$, every vertex in U has degree at least three, which means x is mapped to a vertex y in C. However, if y has degree two, then its two neighbors are not adjacent in $\overline{G}$, a contradiction. Therefore, the minimum degree is at least three, and since G is self-complementary, the maximum degree is at most four. By Lemma 4.9, G can only be one of $(\mathring{1}\mathring{3}\|\mathring{2})$, $(\mathring{1}3\|\mathring{2})$, $(\mathring{1}2\mathring{3})$, $(\mathring{3}1\mathring{2})$, $(\mathring{1}23)$, $(\mathring{3}12)$, $\overline{(\mathring{1}\|\mathring{2}\|\mathring{4})}$, and (123), but none of them is self-complementary. $\square$

4.3 Minimally t-imperfect core graphs

4.3.1 The proof of Theorem 1.4

Bruhn and Stein [20] showed that the $(3, 3)$-partitionable graphs are minimally t-imperfect. Therefore, we only need to show the sufficiency in Theorem 1.4. We say that a clique K of a connected graph G is a *clique separator* of G if $G - K$ is not connected.

Lemma 4.15 (Chvátal [29], Gerards [56]). *No minimally t-imperfect graph contains a clique separator.*

Throughout this section, we assume that both G and its complement $\overline{G}$ are minimally t-imperfect graphs. By Lemma 4.15, neither G nor $\overline{G}$ can have a clique separator. Thus, for each vertex $u \in U$, we have

$$2 < d(u) < n - 3. \tag{4.1}$$

Note that if $d(u) = n - 3$, then u has two neighbors in $\overline{G}$, which is a clique separator.

Note that G is a core graph. By Proposition 4.3 and Corollary 4.6, the order of G is between five and ten. Fonlupt and Hadjar showed that every almost bipartite graph is t-perfect [50]. The only imperfect core graph of order five is C_5. Both imperfect core graphs of order six, (1) and $(\mathring{1})$, are almost bipartite, e.g., removing v_3. There are 16 core graphs of order seven that are different from C_7 and $\overline{C_7}$, namely, (12), $(\mathring{1}2)$, $(1\mathring{2})$, $(\mathring{1}\mathring{2})$, $(1\|2)$, $(\mathring{1}\|2)$, $(1\|\mathring{2})$, $(\mathring{1}\|\mathring{2})$, and their complements. All the listed eight graphs become bipartite after removing v_4, hence almost bipartite. By Lemma 4.9 and the degree requirements (4.1), G cannot have order eight either. Likewise, by Lemma 4.14, all core graphs of order nine satisfying (4.1) are t-perfect. Therefore, we are only left with $n = 10$.

In the rest of this section, the order of G is ten. Our analysis is based on whether (123451) is a (not necessarily induced) subgraph of G. The arguments here are somewhat similar to that in Section 4.2.1. Let us start with an easy case, where all the five edges u_iu_{i+1} for $i = 1, \ldots, 5$ are in $E(G)$; see Figure 4.6(a). Recall that all the indices are understood as modulo 5.

Proposition 4.16. *If for all $i = 1, \ldots, 5$, the edge u_iu_{i+1} is in $E(G)$, then G is one of the $(3,3)$-partitionable graphs.*

Proof. We first argue that U induces a cycle. Suppose for contradiction that u_1u_3 is present. By Ob.4) (with $i = 2$), at least one of u_1v_1 and u_3v_3 is in $E(G)$. Since they are symmetric, we consider $u_1v_1 \in E(G)$. Since $\{u_1, u_3, u_4, v_1\}$ is not a clique, $u_1u_4 \notin E(G)$. Then by Ob.2) (with $i = 3$), $u_3v_3 \in E(G)$, and since $\{u_1, u_3, u_5, v_3\}$ is not a clique, $u_3u_5 \notin E(G)$. But then $G - \{v_4, v_5, u_2\}$ is isomorphic to $\overline{C_7}$, and G is not minimally t-imperfect.

Now that $G[U]$ is a C_5, dependent on the combination of edges $u_iv_i, i = 1, \ldots, 5$, we are in one of the $(3,3)$-partitionable graphs that contain (123451). □

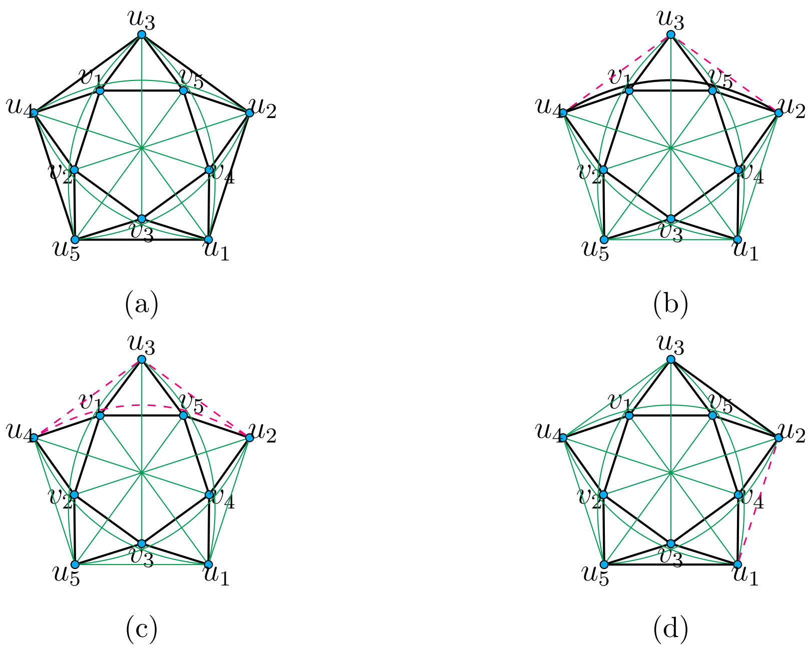


Figure 4.6: (a) All the five edges u_iu_{i+1} for $i = 1, \ldots, 5$ are present; (b) both u_2u_3 and u_3u_4 are absent while u_2u_4 is present; (c) all the edges among u_2, u_3, u_4 are absent; (d) u_1u_2 is absent, while only u_2u_3 and u_1u_5 are present.

The following two propositions deal with the case where for some $i = 1, \ldots, 5$, both edges u_iu_{i-1} and u_iu_{i+1} are absent, Proposition 4.17 for $u_{i-1}u_{i+1}$ being present, and Proposition 4.18 for otherwise; see Figure 4.6(b, c).

Proposition 4.17. *Let $i = 1, \ldots, 5$. If neither u_iu_{i-1} nor u_iu_{i+1} is in $E(G)$, then $u_{i-1}u_{i+1}$ cannot be in $E(G)$ either.*

Proof. Assume without loss of generality $i = 3$; i.e., both u_2u_3 and u_3u_4 are absent, and we show by contradiction that u_2u_4 cannot be in $E(G)$. By Ob.4) (with $i = 3$), at least one of u_2v_2 and u_4v_4 is in $E(G)$. Since they are symmetric, we may consider $u_2v_2 \in E(G)$.

Suppose that $u_4u_5 \in E(G)$. Then $u_2u_5 \notin E(G)$, as otherwise, $\{u_2, u_4, u_5, v_2\}$ is a K_4. By Ob.2) (with $i = 4$), $u_4v_4 \in E(G)$. If $u_1u_2 \in E(G)$, then $u_1u_4 \notin E(G)$ because $\{u_2, u_1, u_4, v_4\}$ cannot be a K_4; by Ob.3) (with $i = 3$), $u_1u_5 \in E(G)$, but then $G - \{u_3, v_1, v_5\}$ is a $\overline{C_7}$, a contradiction to Proposition 4.3. Now that $u_1u_2 \notin E(G)$.

The edge u_1u_5 is not in $E(G)$, as otherwise u_2v_2 is in $E(G)$ and both u_1u_2 and u_2u_5 are not, a contradiction to Ob.1) (with $i = 2$). By Ob.5) (with $i = 1$), u_1v_1 cannot be in $E(G)$ either. The set $\{u_2, u_1, u_5, v_1\}$ is an independent set in G, a contradiction. In the rest, $u_4u_5 \notin E(G)$.

Suppose $u_3u_5 \in E(G)$. By Ob.5) (with $i = 2$), $u_3v_3 \notin E(G)$. Further, By Ob.6), Ob.4), and Ob.5) (with $i = 1$, $i = 4$, and $i = 4$ respectively), $u_2u_5 \notin E(G)$, $u_5v_5 \in E(G)$, and $u_4v_4 \notin E(G)$. Hence, $u_2v_4v_3u_5u_3v_1u_4$ is a 7-cycle in G, contradicting Proposition 4.3. Thus, $u_3u_5 \notin E(G)$.

Suppose $u_4v_4 \in E(G)$. By Ob.5) (with $i = 4$), $u_5v_5 \notin E(G)$. Since $\{u_1, u_5, u_4, v_5\}$ cannot be an independent set, at least one of u_1u_4 and u_1u_5 needs to be present. By Ob.1) (with $i = 4$), if $u_1u_5 \in E(G)$, then $u_1u_4 \in E(G)$ as well. Therefore, we always have $u_1u_4 \in E(G)$. Since $\{u_1, u_2, u_4, v_4\}$ cannot induce a K_4 in G, $u_1u_2 \notin E(G)$. By Ob.5) (with $i = 1$), $u_1v_1 \notin E(G)$. Since $\{u_2, u_1, u_5, v_1\}$ is not an independent set, at least one of u_1u_5 and u_2u_5 is in $E(G)$. If u_1u_5 is in $E(G)$, then by Ob.1) (with $i = 2$), $u_2u_5 \in E(G)$ and $G - \{u_3, v_1, v_5\}$ is isomorphic to $\overline{C_7}$. Otherwise, by Ob.6) (with $i = 3$), u_2u_5 has to be absent as well, and then $\{u_2, u_1, u_5, v_1\}$ is an independent set.

Therefore, none of u_3u_5, u_4u_5, and u_4v_4 can be in $E(G)$, and then $\{u_3, u_4, u_5, v_4\}$ forms an independent set, contradicting Proposition 4.2. $\square$

Proposition 4.18. *For all $i = 1, \ldots, 5$, at least one of u_iu_{i-1} and u_iu_{i+1} is in $E(G)$.*

Proof. Assume without loss of generality, let $i = 3$. Suppose for contradiction that neither u_2u_3 nor u_3u_4 is in $E(G)$. By Proposition 4.17, $u_2u_4 \notin E(G)$. Thus, $u_3v_3 \in E(G)$, as otherwise $\{u_2, u_3, u_4, v_3\}$ forms an independent set. As a result, $u_2v_2 \notin E(G)$, as otherwise u_4 has only one neighbor on the 5-cycle $u_3v_3v_2u_2v_5$. Moreover, u_1u_2 must be in G: Otherwise, by Proposition 4.17, (noting that $u_2u_3 \notin E(G)$,) u_1u_3 cannot be in $E(G)$ either, then $\{u_1, u_2, u_3, v_2\}$ forms an independent set. By Ob.1) (with $i = 3$), $u_1u_3 \in E(G)$, and then by Ob.5) (with $i = 3$), $u_4v_4 \notin E(G)$. Since $\{u_1, u_5, u_4, v_5\}$ does not induce an independent set, at least one of u_1u_5, u_4u_5, u_5v_5, and u_1u_4 is in $E(G)$.

First, suppose that u_1u_5 is in $E(G)$. Then u_3u_5 is not in $E(G)$, as otherwise $\{u_3, u_5, u_1, v_3\}$ induces a K_4. By Ob.2) (with $i = 1$), $u_1v_1 \in E(G)$. The edge $u_4u_5 \notin E(G)$, as otherwise contradicting Ob.1) (with $i = 3$). But then $\{u_3, u_4, u_5, v_4\}$ forms an independent set.

Second, suppose that u_4u_5 is in $E(G)$. By Ob.1) (with $i = 3$), $u_3u_5 \in E(G)$. If u_1v_1 is in $E(G)$, then by Ob.1) (with $i = 1$), $u_1u_4 \in E(G)$, which means that $G - \{u_2, v_4, v_5\}$ is isomorphic to $\overline{C_7}$. Thus, $u_1v_1 \notin E(G)$; a symmetric argument enables us to conclude that $u_5v_5 \notin E(G)$. Since neither u_2u_4 nor u_5v_5 is in $E(G)$, from Ob.2) (with $i = 5$) we can conclude that, $u_2u_5 \notin E(G)$. By a symmetric argument we have u_1u_4 is not in $E(G)$ either. Now that none of u_1v_1, u_5v_5, u_2u_5, and u_1u_4 is in $E(G)$, there is a 7-cycle $u_5u_4v_1v_5u_2u_1v_3$. Therefore, $u_4u_5 \notin E(G)$.

Third, suppose u_5v_5 is in $E(G)$. By Ob.1) (with $i = 5$), $u_2u_5 \in E(G)$. The edge $u_3u_5 \in E(G)$, as otherwise $\{u_3, u_4, u_5, v_4\}$ forms an independent set. But then $G - \{v_1, v_2, u_4\}$ is isomorphic to $\overline{C_7}$. Therefore, $u_5v_5 \notin E(G)$.

Last, suppose u_1u_4 is in $E(G)$. By Ob.6) (with $i = 2$), $u_3u_5 \notin E(G)$. But then $\{u_3, u_4, u_5, v_4\}$ forms an independent set.

In summary, none of u_1u_5, u_4u_5, u_5v_5, and u_1u_4 can be in $E(G)$, and thus $\{u_1, u_5, u_4, v_5\}$ forms an independent set. □

In the remaining case, u_iu_{i+1} for some $i = 1, \ldots, 5$ is absent, but both $u_{i+1}u_{i+2}$ and u_iu_{i-1} are present. Moreover, by Proposition 4.18, at least one of $u_{i+2}u_{i+3}$ and $u_{i-1}u_{i-2}$ is in $E(G)$. See Figure 4.6(d).

Proposition 4.19. *If there is an $i = 1, \ldots, 5$ such that u_iu_{i+1} is not in $E(G)$, then G is one of the $(3,3)$-graphs.*

Proof. Without loss of generality, let $i = 1$. Then $u_1u_2 \notin E(G)$, u_2u_3 and u_1u_5 are in

$E(G)$, and at least one of u_3u_4 and u_4u_5 is in $E(G)$. We show by contradiction that u_3u_4 and u_4u_5 cannot be both in $E(G)$. In particular, we show that none of u_1v_1, u_2v_2, u_1u_3, u_2u_5, and u_3u_5 is in $E(G)$, and then $u_1v_4u_2u_3v_1v_2u_5$ is a 7-cycle.

- If u_3u_5 is in $E(G)$, then by Ob.4) (with $i = 4$), at least one of u_3v_3 and u_5v_5 is in $E(G)$. If u_3v_3 is in $E(G)$ but u_5v_5 is not, then by Ob.2) (with $i = 5$), $u_1u_3 \in E(G)$, which means $d(u_3) = 7$, a contradiction. A symmetric argument applies if u_5v_5 is in $E(G)$ but u_3v_3 is not. Hence, both u_3v_3 and u_5v_5 are in $E(G)$. As a result, neither u_1u_3 nor u_2u_5 can be in $E(G)$, as otherwise $\{u_3, u_1, u_5, v_3\}$ or, respectively, $\{u_5, u_2, u_3, v_5\}$ forms a clique. However, the vertex v_3 has four neighbors on the 5-cycle $u_3u_5u_1v_4u_2$. Therefore, $u_3u_5 \notin E(G)$.

- If u_1v_1 is in $E(G)$, then by Ob.1) (with $i = 1$), $u_1u_3 \in E(G)$. Note that $u_1u_4 \notin E(G)$, as otherwise $\{u_1, u_3, u_4, v_1\}$ forms a cliqued. By Ob.2) (with $i = 3$), $u_3v_3 \in E(G)$. But then $G - \{v_4, v_5, u_2\}$ is isomorphic to $\overline{C_7}$. Therefore, $u_1v_1 \notin E(G)$. By a symmetric argument, $u_2v_2 \notin E(G)$.

- Now that none of u_1v_1, u_2v_2, and u_3u_5 is in $E(G)$, from Ob.2) (with $i = 1$) it can be inferred $u_1u_3 \notin E(G)$, and then by Ob.2) (with $i = 2$), $u_2u_5 \notin E(G)$.

Thus, at most one of u_3u_4 and u_4u_5 is in $E(G)$. We may assume without loss of generality that u_3u_4 is in $E(G)$ and u_4u_5 is not; the other case is symmetric.

We argue that none of u_1u_3, u_3u_5, u_1v_1, and u_5v_5 can be in $E(G)$. Suppose that u_1u_3 is in $E(G)$. By Ob.2) (with $i = 1$), at least one of u_1v_1 and u_3u_5 is in $E(G)$. If $u_3u_5 \in E(G)$, then $u_3v_3 \notin E(G)$, as otherwise $\{u_3, u_1, u_5, v_3\}$ forms a clique. On the other hand, by Ob.4) (with $i = 2$), at least one of u_1v_1 and u_3v_3 is in $E(G)$. Therefore, we always have $u_1v_1 \in E(G)$. Then $u_1u_4 \notin E(G)$, as otherwise $\{u_1, u_3, u_4, v_1\}$ forms a clique. By Ob.2) (with $i = 3$), $u_3v_3 \in E(G)$, which further implies $u_3u_5 \notin E(G)$ because $d(u_3) < 6$. But then all of u_1u_5, u_1u_3, u_3u_4, and u_3v_3 are in $E(G)$ and none of u_1u_4, u_3u_5, and u_4u_5 is in $E(G)$, contradicting Ob.3) (with $i = 2$). Therefore,

$u_1u_3 \notin E(G)$. By a symmetric argument, we can conclude that u_3u_5 cannot be in $E(G)$ either. Now that none of u_1u_3, u_3u_5, u_1u_2, and u_4u_5 is in $E(G)$, together with the fact that both u_2u_3 and u_3u_4 are in $E(G)$, from Ob.1) (with $i = 1$ and $i = 5$), it can be inferred that both u_1v_1 and u_5v_5 cannot be in $E(G)$.

At least one of u_4v_4 and u_1u_4 is in $E(G)$, as otherwise $u_1v_4v_5u_3u_4v_2u_5$ is a 7-cycle. If u_4v_4 is in $E(G)$, then Ob.1) (with $i = 4$) will force u_1u_4 in $E(G)$ as well. On the other hand, u_1u_4 is in $E(G)$ and Ob.4) (with $i = 5$) will force u_4v_4 in $E(G)$ as well. Therefore, both u_4v_4 and u_1u_4 are in $E(G)$. Moreover, at least one of u_2v_2 and u_2u_5 is in $E(G)$, as otherwise $u_5v_2v_1u_3u_2v_4u_1$ is a 7-cycle. By a symmetric argument, both u_2v_2 and u_2u_5 are in $E(G)$. Note that u_2u_4 cannot be in $E(G)$, as otherwise $G - \{v_1, v_5, u_3\}$ is isomorphic to $\overline{C_7}$. Dependent on whether u_3v_3 is in $E(G)$, the graph is isomorphic to either $(1\mathring{2}43\mathring{5}1)$ or its complement. $\square$

The discussion on the order of G, and Propositions 4.16–4.19 imply Theorem 1.4.

Bruhn and Stein showed that those $(3, 3)$-partitionable graphs containing C_{10}^2 as a subgraph are minimally t-imperfect, while the minimally t-imperfection of $(1\mathring{2}43\mathring{5}1)$ and $(\mathring{1}\mathring{2}43\mathring{5}\mathring{1})$ are referred to an unpublished manuscript of Bruhn. For the sake of completeness, we provide a proof here.

Lemma 4.20. $(1\mathring{2}43\mathring{5}1)$ *and* $(\mathring{1}\mathring{2}43\mathring{5}\mathring{1})$ *are minimally t-imperfect.*

Proof. Let graph G be one of $(1\mathring{2}43\mathring{5}1)$ and $(\mathring{1}\mathring{2}43\mathring{5}\mathring{1})$, and we show that G is t-imperfect at first. For vector $x \in \mathbb{R}^{V(G)}$ with $x_v = \frac{1}{3}$ for all $v \in V(G)$, it is not difficult to check that x is in $P_{OC}(G)$. However, x is not in the independent set polytope of the graph G (note that $\mathbf{1}^T x > \alpha(G)$).

For G to be minimally t-imperfect, we argue that every proper t-minor of G is t-perfect. Note that for every $v \in V(G)$, the neighbors of v could not form an independent set. We only need to check that $G - v$ is t-perfect, for every $v \in V(G)$. Moreover, for every $i = 1, \ldots, 5$, $G - u_i$ is isomorphic to $G - v_i$ and by symmetry,

$G-u_1$ is isomorphic to $G-u_5$ and $G-u_2$ is isomorphic to $G-u_4$. Therefore, it suffices to show that $G-u_1$, $G-u_2$, and $G-u_3$ are t-perfect. Since $G-u_1$ is isomorphic to $(\mathring{1}32\mathring{4})$, $G-u_2$ is isomorphic to $(21\mathring{3}4)$ or $(21\mathring{3}\mathring{4})$, and $G-u_3$ is isomorphic to $(1\mathring{4}3\mathring{2})$ or $(1\mathring{4}\mathring{3}\mathring{2})$, we should show that all of $(\mathring{1}32\mathring{4})$, $(21\mathring{3}4)$, $(21\mathring{3}\mathring{4})$, $(1\mathring{4}3\mathring{2})$, and $(1\mathring{4}\mathring{3}\mathring{2})$ are t-perfect. We use the same argument for proving Proposition 4.8 to show these five graphs are t-perfect. The details are listed in Table 4.3. □

Table 4.3: For the proof of Lemma 4.20

	$K=\{a,b,c\}$	$G-a$	$G-b$	$G-c$
$(\mathring{1}32\mathring{4})$	$\{v_3,v_4,u_1\}$	$\star$	$\star$	$(23\mathring{1}4)-u_4$
$(21\mathring{3}4)$	$\{v_1,v_2,u_4\}$	$\star$	$\star$	$(23\mathring{1}4)-u_4$
$(21\mathring{3}\mathring{4})$	$\{v_1,v_2,u_4\}$	$\star$	$\star$	$(23\mathring{1}4)-u_4$
$(1\mathring{4}3\mathring{2})$	$\{v_3,v_4,u_1\}$	$\star$	$\star$	$(\mathring{1}2\mathring{3}45\mathring{1})-\{u_4,u_5\}$
$(1\mathring{4}\mathring{3}\mathring{2})$	$\{v_3,v_4,u_1\}$	$\star$	$\star$	$(\mathring{1}\mathring{2}\mathring{3}45\mathring{1})-\{u_4,u_5\}$

Chapter 5

Self-complementary (Pseudo-)Split Graphs

In this chapter, we study split graphs and pseudo-split graphs whose complements are isomorphic to themselves. In Section 5.1, we begin by introducing more about antimorphisms. Then we show a connection between self-complementary split graphs and self-complementary pseudo-split graphs. This connection allows us to narrow our focus to split graphs. Furthermore, we establish a one-to-one correspondence between self-complementary split graphs on $4k$ vertices and those on $4k+1$ vertices. We also study partitions in self-complementary graphs in this section. Additionally, we give a characterization for forcibly self-complementary degree sequences in Section 5.2. Finally, we tackle the enumeration problem of self-complementary split graphs in Section 5.3.

5.1 Preliminaries

An *isomorphism* between two graphs G_1 and G_2 is a bijection between their vertex sets, i.e., $\sigma\colon V(G_1) \to V(G_2)$, such that two vertices u and v are adjacent in G_1 if and only if $\sigma(u)$ and $\sigma(v)$ are adjacent in G_2. Two graphs with an isomorphism are *isomorphic*. A graph is *self-complementary* if it is isomorphic to its *complement* $\overline{G}$. An isomorphism between G and $\overline{G}$ is a permutation of $V(G)$, called an *antimorphism*.

We represent an antimorphism as the product of disjoint cycles $\sigma = \sigma_1\sigma_2\cdots\sigma_p$, where $\sigma_i = (v_{i1}v_{i2}\cdots)$ for all i. Sachs and Ringel [108, 111] independently showed that there can be at most one vertex v fixed by an antimorphism σ, i.e., $\sigma(v) = v$. For any other vertex u, the smallest number k satisfying $\sigma^k(u) = u$ has to be a multiplier of four. Gibbs [58] observed that if a vertex v has d neighbors in G, then the degree of $\sigma(v)$ in G is $n - 1 - d$ where n is the order of G. It implies that if v is fixed by σ, then its degree in G is $(n - 1)/2$. The vertices in every cycle of σ with a length of more than one alternate in degrees d and $n - 1 - d$.

Lemma 5.1 ([108, 111])**.** *In an antimorphism of a self-complementary graph, the length of each cycle is either* 1 *or* $4p$ *for some positive integer* p*. Moreover, there is a unique cycle of length one if and only if the order of the graph is odd.*

For any subset of cycles in σ, the vertices within those cycles induce a subgraph that is self-complementary. Indeed, the selected cycles themselves act as an antimorphism for the subgraph.

Proposition 5.2 ([58])**.** *Let* G *be a self-complementary graph and* σ *an antimorphism of* G*. For any subset of cycles in* σ*, the vertices within those cycles induce a self-complementary graph.*

A graph is a *split graph* if its vertex set can be partitioned into a clique and an independent set. We use $K \uplus I$, where K being a clique and I an independent set, to denote a *split partition*. A split graph may have more than one split partition.

Lemma 5.3. *A self-complementary split graph on* $4k$ *vertices has a unique split partition* $\{v \mid d(v) \geq 2k\} \uplus \{v \mid d(v) < 2k\}$*.*

Proof. Let G be a self-complementary split graph with $4k$ vertices, and σ an antimorphism of G. By definition, for any vertex $v \in V(G)$, we have $d(v) + d(\sigma(v)) = 4k - 1$. Thus,

$$\min(d(v), d(\sigma(v))) \leq 2k - 1 < 2k \leq \max(d(v), d(\sigma(v))).$$

As a result, G does not contain any clique or independent set of order $2k+1$. Suppose for contradiction that there exists a split partition $K \uplus I$ of G different from the given. There must be a vertex $x \in I$ with $d(x) \geq 2k$. We must have $d(x) = 2k$ and $N(x) \subseteq K$. But then there are at least $|N[x]| = 2k + 1$ vertices having degree at least $2k$, a contradiction. $\square$

We correlate self-complementary split graphs having even and odd orders.

Proposition 5.4. *Let G be a split graph on $4k+1$ vertices. If G is self-complementary, then G has exactly one vertex v of degree $2k$, and $G - v$ is also self-complementary.*

Proof. Let σ be an antimorphism of G. By Lemma 5.1, there exists a cycle of length one in σ; let it be (v). We can write $\sigma = \sigma_1 \ldots \sigma_p(v)$. By Proposition 5.2, $G - v$ is self-complementary with $\sigma = \sigma_1 \ldots \sigma_p$ as an antimorphism. Since it is an induced subgraph of a split graph, it is a self-complementary split graph, and has a unique split partition $K \uplus I$ by Lemma 5.3. The degree of v is $|K| = 2k$. On the other hand, every vertex in K has at least one neighbor in I: otherwise, we can move it from K to I to get another split partition of $G - v$. Thus, $d(x) > 2k$ for each vertex $x \in K$. In a similar way, we can conclude that $d(x) < 2k$ for each vertex $x \in I$. $\square$

A *pseudo-split graph* is either a split graph, or a graph whose vertex set can be partitioned into a clique K, an independent set I, and a set C that (1) induces a C_5; (2) is complete to K; and (3) is nonadjacent to I. We say that $K \uplus I \uplus C$ is a *pseudo-split partition* of the graph, where C may or may not be empty. If C is empty, then $K \uplus I$ is a split partition of the graph. Otherwise, the graph has a unique pseudo-split partition. Similar to split graphs, the complement of a pseudo-split graph remains a pseudo-split graph.

Proposition 5.5. *Let G be a self-complementary pseudo-split graph with a pseudo-split partition $K \uplus I \uplus C$. If $C \neq \emptyset$, then $G - C$ is a self-complementary split graph with an even order.*

Proof. Let σ be an antimorphism of G. In both G and its complement, the only C_5 is induced by C. Thus, $\sigma(C) = C$. Since C is complete to K and nonadjacent to I, it follows that $\sigma(K) = I$ and $\sigma(I) = K$. Thus, $G - C$ is a self-complementary graph. It is clearly a split graph and has an even order. □

In the rest of this section, we are exclusively concerned with partitions of the vertex set of a graph G into four nonempty subsets. A partition $\mathcal{P} = \{V_1, V_2, V_3, V_4\}$ of $V(G)$ is a *rectangle partition* if V_1 is complete to V_2 and nonadjacent to V_3, while V_4 is complete to V_3 and nonadjacent to V_2, or a *diamond partition* if V_1 is complete to V_2 while V_3 is nonadjacent to V_4. See Fig. 1.9. Trotignon [118] conjectured that every C_5-free self-complementary graph G admits one of the two partitions.

Lemma 5.6. *Every self-compelemtary split graph G admits a diamond partition. If G has an even order, then it admits a diamond partition that is self-complementary.*

Proof. Let $K \uplus I$ be a split partition of G. For any proper and nonempty subset $K' \subseteq K$ and proper and nonempty subset $I' \subseteq I$, the partition

$$K', K \setminus K', I', I \setminus I'$$

is a diamond partition.

Now suppose that the order of G is $4k$. We fix an arbitrary antimorphism $\sigma = \sigma_1 \sigma_2 \cdots \sigma_p$ of G. We may assume without loss of generality that for all $i = 1, \ldots, p$, the first vertex in σ_i is in K. For $j = 1, \ldots, |\sigma_i|$, we assign the jth vertex of σ_i to

$V_{j \pmod 4}$. For $j = 1, \ldots, 4$, we have $\sigma(V_j) = V_{j+1 \pmod 4}$. Moreover, $V_1 \cup V_3 = K$ and $V_2 \cup V_4 = I$. Thus, $\{V_1, V_3, V_2, V_4\}$ is a self-complementary diamond partition of G. □

For a positive integer k, let Z_k denote the graph obtained from a P_4 as follows. We substitute each degree-one vertex with an independent set of k vertices, and each degree-two vertex with a clique of k vertices. For example, P_4 itself is Z_1 and depicted in Figure 1.7(b) is Z_2.

Lemma 5.7. *A self-complementary split graph admits a rectangle partition if and only if it is an Z_k.*

Proof. The sufficiency is trivial, and we consider the necessity. Suppose that G is a self-complementary split graph and it has a rectangle partition $\{V_1, V_2, V_3, V_4\}$. Let $K \uplus I$ be a split partition of G. There are at least one edge and at least one missing edge between any three parts. Thus, vertices in K are assigned to precisely two parts in the partition. By the definition of rectangle partition, K is either $V_2 \cup V_3$ or $V_1 \cup V_4$. Assume without loss of generality that $K = V_2 \cup V_3$. Since V_2 is complete to V_1 and nonadjacent to V_4, any antimorphism of G maps V_2 to either V_1 or V_4. If $|V_2| \neq |V_3|$, then the numbers of edges between K and I in G and $\overline{G}$ are different. This is impossible. It further implies $|V_1| = |V_4|$, and hence G is precisely $Z_{|V_1|}$. □

5.2 Forcibly self-complementary degree sequences

The *degree sequence* of a graph G is the sequence of degrees of all vertices, listed in non-increasing order, and G is a *realization* of this degree sequence. For our purpose, it is more convenient to use a compact form of degree sequences where the same degrees are grouped:

$$(d_i^{n_i})_{i=1}^{\ell} = (d_1^{n_1}, \ldots, d_\ell^{n_\ell}) = \left(\underbrace{d_1, \ldots, d_1}_{n_1}, \underbrace{d_2, \ldots, d_2}_{n_2}, \ldots, \underbrace{d_\ell, \ldots, d_\ell}_{n_\ell} \right).$$

Note that we always have $d_1 > d_2 > \cdots > d_\ell$. For example, the degree sequences of the first two graphs in Fig. 1.8 are both

$$\left(5^4, 2^4\right) = (5, 5, 5, 5, 2, 2, 2, 2).$$

These two graphs are not isomorphic; thus, a degree sequence may have non-isomorphic realizations.

For four vertices v_1, v_2, v_3, and v_4 such that v_1 is adjacent to v_2 not v_3 while v_4 is adjacent to v_3 not v_2, the operation of replacing v_1v_2 and v_3v_4 with v_1v_3 and v_2v_4 is a *2-switch*, denoted as $(v_1v_2, v_3v_4) \to (v_1v_3, v_2v_4)$. See Fig. 5.1. It is easy to check that this operation does not change the degree of any vertex.

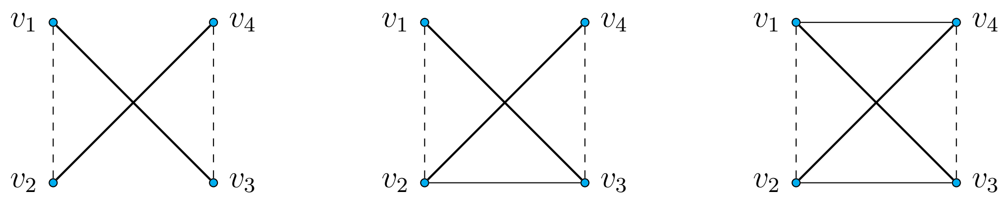


Figure 5.1: Illustrations for 2-switches.

Lemma 5.8 ([110]). *Two graphs have the same degree sequence if and only if they can be transformed to each other by a series of 2-switches.*

The subgraph induced by the four vertices involved in a 2-switch operation must be a $2K_2$, P_4, or C_4. Moreover, after the operation, the four vertices induce an isomorphic subgraph. Since a split graph G cannot contain any $2K_2$ or C_4 [49], a 2-switch must be done on a P_4. In any split partition $K \uplus I$ of G, the two degree-one vertices of P_4 are from I, while the others from K. The graph remains a split graph after this operation. Thus, if a degree sequence has a realization that is a split graph, then all its realizations are split graphs [49]. A similar statement holds for pseudo-split graphs [85].

We do not have a similar claim on degree sequences of self-complementary graphs. Clapham and Kleitman [36] have fully characterized all such degree sequences, called *potentially self-complementary degree sequences*. For simplicity, we only need a simpler statement on even-order graphs.

Theorem 5.9 ([33, 36]). *A degree sequence $(d_i^{n_i})_{i=1}^{\ell}$ of even order n is potentially self-complementary if and only if ℓ is even, and for all $i = 1, \ldots, \ell/2$,*

- $d_i + d_{\ell+1-i} = n - 1$, *and*
- $n_i = n_{\ell+1-i}$ *is even.*

Moreover, for all $p = 1, \ldots, \ell/2$

$$\sum_{i=1}^{p} n_i d_i \leq \left(\sum_{i=1}^{p} n_i\right)\left(n - 1 - \sum_{i=1}^{p} \frac{n_i}{2}\right).$$

A degree sequence is *forcibly self-complementary* if all of its realizations are self-complementary. We refer to the graph in Figure 1.7(a) as a *trampoline* graph.

Proposition 5.10. *The following degree sequences are all forcibly self-complementary: $(2^2, 1^2)$, (2^5), and $(5^4, 2^4)$.*

Proof. Applying a 2-switch operation to a realization of $(2^2, 1^2)$ or (2^5) leads to an isomorphic graph. A 2-switch operation transforms a Z_2 into a trampoline, and vice versa. Thus, the statement follows from Lemma 5.8. $\square$

We take p vertex-disjoint graphs S_1, S_2, …, S_p, each of which is isomorphic to P_4, Z_2, or trampoline. For $i = 1, \ldots, p$, let $H_i \uplus L_i$ denote the unique split partition of S_i (Lemma 5.3). Let C be another set of 0, 1, or 5 vertices. We add all possible edges among $\bigcup_{i=1}^{p} H_i$ to make it a clique, and for each $i = 1, \ldots, p$, add all possible edges between H_i and $\bigcup_{j=i+1}^{p} L_j$.[1] Finally, we add all possible edges between C and $\bigcup_{i=1}^{p} H_i$, and add edges to make C a cycle if $|C| = 5$. Let $\mathcal{E}$ denote the set of graphs that can be constructed above.

Lemma 5.11. *All graphs in $\mathcal{E}$ are self-complementary pseudo-split graphs, and their degree sequences are forcibly self-complementary.*

Proof. Let G be any graph in $\mathcal{E}$. It has a split partition $(\bigcup_{i=1}^{p} H_i \cup C) \uplus \bigcup_{i=1}^{p} L_i$ when $|C| \leq 1$, and a pseudo-split partition $(\bigcup_{i=1}^{p} H_i) \uplus (\bigcup_{i=1}^{p} L_i) \uplus C$ otherwise. To show that it is self-complementary, we construct an antimorphism σ for it. For each $i = 1, \ldots, p$, we take an antimorphism σ_i of S_i, and set $\sigma(x) = \sigma_i(x)$ for all $x \in V(S_i)$. If C consists of a single vertex v, we set $\sigma(v) = v$. If $|C| = 5$, we take an antimorphism σ_{p+1} of C_5 and set $\sigma(x) = \sigma_{p+1}(x)$ for all $x \in C$. It is easy to verify that a pair of vertices u, v are adjacent in G if and only if $\sigma(u)$ and $\sigma(v)$ are adjacent in $\overline{G}$.

For the second assertion, we show that applying a 2-switch to G in $\mathcal{E}$ leads to another graph in $\mathcal{E}$. Since G is a split graph, a 2-switch can only be applied to a P_4. For two vertices $v_1 \in H_i$ and $v_2 \in H_j$ with $i < j$, we have $N[v_2] \subseteq N[v_1]$. Thus, there cannot be any P_4 involving both v_1 and v_2. A similar argument applies to two vertices in L_i and L_j with $i \neq j$. Therefore, a 2-switch can be applied either *inside* C or *inside* S_i for some $i \in \{1, \ldots, p\}$. By Proposition 5.10, the resulting graph is in $\mathcal{E}$, hence self-complementary. Thus, the degree sequence of G is forcibly self-complementary by Lemma 5.8. □

We refer to graphs in $\mathcal{E}$ as *elementary self-complementary pseudo-split graphs.* The rest of this section is devoted to showing that all realizations of forcibly self-

[1] The reader familiar with threshold graphs may note its use here. If we contract H_i and L_i into two vertices, the graph we constructed is a threshold graph. Threshold graphs have a stronger characterization by degree sequences. Since a threshold graph free of $2K_2$, P_4, and C_4, no 2-switch is possible on it. Thus, the degree sequence of a threshold graph has a unique realization.

complementary degree sequences are elementary self-complementary pseudo-split graphs. We start with a simple observation on potentially self-complementary degree sequences with two different degrees. It can be derived from Clapham and Kleitman [36]. We provide a direct and simple proof here.

Proposition 5.12. *There is a self-complementary graph of the degree sequence* $(d^{2k}, (4k-1-d)^{2k})$ *if and only if* $2k \leq d \leq 3k-1$*. Moreover, there exists a self-complementary graph with a one-cycle antimorphism.*

Proof. Necessity. By the definition of degree sequences, $d > 4k-1-d$. Therefore, $d \geq 2k$. Let H be the set of vertices of degree d and L the set of vertices of degree $4k-1-d$. Each vertex in H has at most $|H|-1 = 2k-1$ neighbors in H. Thus, the number of edges between H and L is at least $2k(d-2k+1)$. On the other hand, the number of edges between H and L is at most $2k(4k-1-d)$. Thus, $4k-1-d \geq d-2k+1$, and the claim follows.

Sufficiency. We construct a self-complementary graph that has an antimorphism with exactly one cycle $(v_1 v_2 \cdots, v_{4k})$ by using the method of Gibbs [58]. Note that the adjacencies between the first vertex and the other vertices decide the graph. We set the neighborhood of v_1 to be $\{v_2, v_6, \ldots, v_{4k-2}\} \cup X$, where

$$X = \begin{cases} \{v_3, v_5, \ldots, v_{d-k}\} \cup \{v_{2k+1}\} \cup \{v_{4k-1}, v_{4k-3}, \ldots, v_{5k-d+2}\}, & d \not\equiv k \pmod 2 \\ \{v_3, v_5, \ldots, v_{d-k+1}\} \cup \{v_{4k-1}, v_{4k-3}, \ldots, v_{5k-d+1}\}, & d \equiv k \pmod 2 \end{cases}$$

In the constructed graph, all odd-number vertices have degree d, and the others $4k-d-1$. □

The next proposition considers the parity of the number of vertices with a specific degree. It directly follows from Clapham and Kleitman [36], and Xu and Wong [121, Theorem 4.4].

Proposition 5.13 ([36,121])**.** *Let G be a graph of order $4k$ and v an arbitrary vertex of G. Let H and L be the $2k$ vertices of the largest and smallest degrees, respectively in G. If G is self-complementary, then all the following are even: the number of vertices with degree $d_G(v)$ in G, the number of vertices with degree $d_{G[H]}(v)$ in $G[H]$, and the number of vertices with degree $d_{G[L]}(v)$ in $G[L]$.*

In general, it is quite challenging to verify that a degree sequence is indeed forcibly self-complementary. On the other hand, to show that a degree sequence is not forcibly self-complementary, it suffices to construct a realization that is not self-complementary. We have seen that degree sequences (2^5), $(2^2, 1^2)$, and $(5^4, 2^4)$ are forcibly self-complementary. They are the only ones of these forms.

Proposition 5.14. *The following degree sequences are not forcibly self-complementary.*

i) $((2k)^{4k+1})$, *where* $k \geq 2$.

ii) $(d^{2k}, (n-1-d)^{2k})$, *where* $k \geq 2$ *and* $d \neq 5$.

iii) $(d^{2k_1}, (d-1)^{2k_2}, (n-d)^{2k_2}, (n-1-d)^{2k_1})$, *where* $k_1, k_2 > 0$.

Proof. The statement holds vacuously if the degree sequence is not potentially self-complementary. Henceforth, we assume that they are.

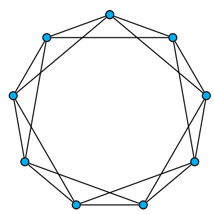


Figure 5.2: The graph C_9^2, with degree sequence (4^9), is not self-complementary.

(i) We start from a cycle graph on $4k+1$ vertices, and add an edge between every pair of vertices with distance at most k on this cycle. The resulting graph is denoted as C_{4k+1}^k. As an example, the graph for $k = 2$ is in Fig. 5.2. To see that the graph

C^k_{4k+1} is not self-complementary, note that for any vertex v, there are $3k(k-1)/2$ edges among $N(v)$ and $k(k-1)/2$ missing edges among $V(G) \setminus N[v]$.

(ii) By Proposition 5.12, we have that $2k \le d \le 3k-1$. The graph in Fig. 5.3 has degree sequence $(4^4, 3^4)$ and is not self-complementary. In the rest, $k \ge 3$.

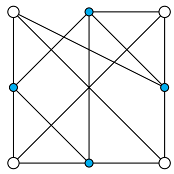


Figure 5.3: A graph, with degree sequences $(4^4, 3^4)$, is not self-complementary.

Case 1: $d = 3k-1$. Starting with a P_4, we substitute each degree-one vertex with an independent set of k vertices, and each degree-two vertex with a clique of k vertices. The degree sequence is $((3k-1)^{2k}, (k)^{2k})$. We label the vertices of degree $3k-1$ as $u_1, \ldots, u_{2k}$ and vertices of degree k as $v_1, \ldots, v_{2k}$. For $i = 1, \ldots, k$, we conduct $(u_k v_i, u_{k+i} v_{k+i}) \to (u_k v_{k+i}, u_{k+i} v_i)$. See Fig. 5.4 for the example of $k = 3$. We show that the resulting graph is not self-complementary. Note that the $k-1$ vertices $u_1, \ldots, u_{k-1}$ are twins (having the same neighborhood). It suffices to argue that there are no twins in $v_1, \ldots, v_{2k}$. Since $N(u_k) = \{v_{k+1}, \ldots, v_{2k}\}$, we separate them into $v_1, \ldots, v_k$ and $v_{k+1}, \ldots, v_{2k}$. For $1 \le i < j \le k$, vertices v_i and v_j are not twins because u_{k+i} is adjacent to v_i but not v_j. For $k+1 \le i < j \le 2k$, vertices v_i and v_j are not twins because u_i is adjacent to v_j but not v_i.

(a) (b)

Figure 5.4: Two graphs with degree sequence $(8^6, 3^6)$, where (a) is self-complementary but (b) not.

Case 2: $d < 3k-1$. Using the method shown in Proposition 5.12, we can construct a realization G of $(d^{2k}, (n-1-d)^{2k})$. Note that G is self-complementary with an antimorphism $\sigma = (v_1 v_2 \cdots, v_{4k})$. Let $H = \{v_1, v_3, v_5, v_7, \ldots, v_{4k-1}\}$. Note that the vertices in H share the same degree d.

If v_1 is adjacent to v_{2k+1}, then it is not adjacent to v_{2k-1}; otherwise, from our construction, v_{2k-1} must be v_{d-k} and it implies that $d = 3k - 1$, a contradiction. The fact that v_1 is adjacent to v_2 implies that v_{2k-1} is adjacent to v_{2k} and v_{2k} is not adjacent to v_{2k+1}. We conduct the 2-switch $(v_1 v_{2k+1}, v_{2k-1} v_{2k}) \to (v_1 v_{2k-1}, v_{2k} v_{2k+1})$, and denote by G' the resulting graph. It can be observed that

$$|N_{G'}(v) \cap H| = \begin{cases} |N_G(v) \cap H| + 1 & \text{if } v = v_{2k-1}, \\ |N_G(v) \cap H| - 1 & \text{if } v = v_{2k+1}, \text{ and} \\ |N_G(v) \cap H| & \text{if } v \in H \setminus \{2k-1, 2k+1\}. \end{cases}$$

The graph G' is not self-complementary by Proposition 5.13.

We now consider the case that v_1 is not adjacent to v_{2k+1}. From our construction, we know that $d - k$ is even and v_1 is adjacent to v_{d-k+1} and not adjacent to v_{d-k+3}. The fact that v_1 is adjacent to v_2 and not adjacent to v_4 implies v_{d-k+3} is adjacent to v_{d-k+4} and v_{d-k+1} is not adjacent to v_{d-k+4}. By conducting the 2-switch $(v_1 v_{d-k+1}, v_{d-k+3} v_{d-k+4}) \to (v_1 v_{d-k+3}, v_{d-k+1} v_{d-k+4})$, the resulting graph G' have the same degree sequence as G. By using arguments similar to the previous paragraph, it can be shown that G' is not self-complementary.

(iii) We use τ to denote the degree sequence $(d^{2k_1}, (d-1)^{2k_2}, (n-d)^{2k_2}, (n-1-d)^{2k_1})$. Since τ is potentially self-complementary, the inequality

$$k_1 d + k_2(d-1) \leq (k_1 + k_2)(n - 1 - (k_1 + k_2))$$

should be satisfied by Theorem 5.9. Therefore,

$$d \leq n - 1 - (k_1 + k_2) + \frac{k_2}{k_1 + k_2} < n - 1 - (k_1 + k_2).$$

By using the same theorem, it can be seen that the integer sequence $(d^{2k_1+2k_2}, (n-1-d)^{2k_1+2k_2})$ is potentially self-complementary.

Let $k = k_1 + k_2$. We can construct a realization G of $(d^{2k}, (n-1-d)^{2k})$ by using the method shown in Proposition 5.12. Note that G is self-complementary with an antimorphism $\sigma = (v_1 v_2 \cdots, v_{4k})$ and all odd-numbered vertices have degree d, and the others have degree $4k-d-1$. The fact that v_1 is adjacent to v_3 implies $\sigma^{4i}(v_1)$ is adjacent to $\sigma^{4i}(v_3)$ for all $i = 1, 2, \ldots, k-1$. Furthermore, since v_1 is adjacent to v_2, the vertex v_3 is adjacent to v_4 and v_4 is not adjacent to v_5. Moreover, we can further deduce that $\{v_5\}$ is complete to $\{v_2, v_6, \ldots, v_{4k-2}\}$ since $\{v_1\}$ is complete to $\{v_2, v_6, \ldots, v_{4k-2}\}$.

We claim that v_1 is adjacent to v_5 in G. Suppose v_1 is not adjacent to v_5. Then v_1 is only adjacent to v_3 and v_{4k-1} in $\{v_3, v_5, v_7, \ldots, v_{4k-1}\}$. Since $d > n-1-d$, we have that n can only be eight and the degree sequence of G is $(4^4, 3^4)$. Note that $d > d-1 > n-2 > n-1-d$. The difference between d and $n-1-d$ is at least three. We encounter a contradiction.

We now remove the edge $\sigma^{4i}(v_1)\sigma^{4i}(v_3)$ and add edge $\sigma^{4i+1}(v_1)\sigma^{4i+1}(v_3)$ for all $i = 0, 1, 2, \ldots, k_2 - 1$. The resulting graph G' is a realization of the degree sequence τ. In G', the vertex v_1 is adjacent to v_5 and not adjacent to v_3. The vertex v_4 is adjacent to v_3 and not adjacent to v_5. By conducting the 2-switch $(v_1v_5, v_3v_4) \to (v_1v_3, v_4v_5)$, the resulting graph G'' have the same degree sequence as G'.

We show that G'' is not self-complementary. Let $H = \{v_1, v_3, v_5, v_7, \ldots, v_{4k-1}\}$ and $L = \{v_2, v_4, v_6, \ldots, v_{4k}\}$. Suppose G'' is a self-complementary graph. Then any antimorphism σ' of G'' maps H to L and vice versa. Since v_5 is adjacent to v_4 and $\{v_5\}$ is complete to $\{v_2, v_6, \ldots, v_{4k-2}\}$, the vertex v_5 has $k+1$ neighbors in L. Therefore, $\sigma'(v_5)$ is in L and it has $k+1$ non-neighbors in H. Every vertex in L has k neighbors in H and $|H| = 2k$. No vertex in L can have $k+1$ non-neighbors in H. We encounter a contradiction. $\square$

We fix a forcibly self-complementary degree sequence $\tau = (d_1^{n_1}, \dots, d_\ell^{n_\ell})$ and a realization G of τ. For each $i = 1, \dots, \ell$, let

$$V_i = \{v \in V(G) \mid d(v) = d_i\}, \quad V_i^+ = V_i \cup V_{\ell+1-i},$$

and we define the *ith slice* of G as the induced subgraph $G[V_i^+]$. Note that $V_i = V_{\ell+1-i}$ and $V_i^+ = V_i$ when ℓ is odd and $i = (\ell+1)/2$.

Each slice must be self-complementary, and more importantly, its degree sequence is forcibly self-complementary.

Lemma 5.15. *For all $i = 1, \dots, \ell$, the degree sequence of the subgraph $G[V_i^+]$ is forcibly self-complementary.*

Proof. Let σ be an antimorphism of G. Since $d_1 > d_2 > \cdots > d_\ell$, we have $\sigma(V_i) = V_{\ell+1-i}$ and $\sigma(V_{\ell+1-i}) = V_i$ (note that V_i and $V_{\ell+1-i}$ are either identical or disjoint). Therefore, $n_i = n_{\ell+1-i}$. By Proposition 5.4b, the cycles of σ consisting of vertices from V_i^+ is an antimorphism of $G[V_i^+]$, and $G[V_i^+]$ is self-complementary. To show that the degree sequence of $G[V_i^+]$ is forcibly self-complementary, let S be any other realization of the same degree sequence. By Lemma 5.8, we can transform $G[V_i^+]$ to S by a sequence of 2-switches applied on vertices in V_i^+. We can apply the same sequence of 2-switches to G, and denote by G' the resulting graph. By Lemma 5.8, the degree sequence of G' is also τ, and S is the ith slice of G'. By the first assertion, S is self-complementary. $\square$

Lemma 5.15 imposes limitations on possible 2-switches applicable to G.

Corollary 5.16. *For all $i = 1, \dots, \ell$, the number of edges in $G[V_i^+]$ or between V_i and $V_{\ell+1-i}$ cannot be changed by any sequence of 2-switches.*

Proof. Let G' be the graph obtained from G by a sequence of 2-swithces. By the definition of 2-swithces, every vertex has the same degree in G and G'. Since G' is a realization of τ, the subgraph $G'[V_i^+]$ is self-complementary. Thus, the number of edges in $G'[V_i^+]$ is the same as in $G[V_i^+]$. Since there are an antimorphism σ of G and an antimorphism σ' of G' such that $\sigma(V_i) = \sigma'(V_i) = V_{\ell+1-i}$, the number of edges between V_i and $V_{\ell+1-i}$ are the same. $\square$

All the vertices in V_i share the same degree in the ith slice. In other words, the ith slice has at most two distinct degrees.

Lemma 5.17. *For each $i \in \{1, \ldots, \ell\}$, the vertices in V_i have the same degree in $G[V_i^+]$.*

Proof. Suppose for contradiction that vertices in V_i have different degrees in $G[V_i^+]$.

Case 1, there are two vertices v_1 and v_2 in V_i such that

$$d = d_{G[V_i^+]}(v_1) > d_{G[V_i^+]}(v_2) + 1.$$

There exists a vertex $x_1 \in V_i^+$ adjacent to v_1 but not to V_2. On the other hand, since $d_G(v_1) = d_G(v_2)$, there must be a vertex

$$x_2 \in N(v_2) \setminus (N(v_1) \cup v_i^+).$$

We apply the 2-switch $(x_1v_1, x_2v_2) \to (x_1v_2, x_2v_1)$ to G and denote by G' the resulting graph. By Lemma 5.15, $G[V_i^+]$ is self-complementary, and hence there are an even number of vertices with degree d in $G[V_i^+]$ by Theorem 5.9. The degree of a vertex x in $G'[V_i^+]$ is

$$\begin{cases} d_{G[V_i^+]}(x) - 1 & x = v_1, \\ d_{G[V_i^+]}(x) + 1 & x = v_2, \\ d_{G[V_i^+]}(x) & \text{otherwise}. \end{cases}$$

Thus, the number of vertices with degree d in $G'[V_i^+]$ is odd. Hence, $G'[V_i^+]$ is not self-complementary by Theorem 5.9. By Lemma 5.8, G' is also a realization of τ, and hence $G'[V_i^+]$ is self-compelemtary by Lemma 5.15. We end with a contradiction.

Case 2, the degree of vertices in V_i is either d or $d-1$ for some d in $G[V_i^+]$. By Lemma 5.15, the degree sequence of $G[V_i^+]$ is forcibly self-complementary. It cannot be of the form $(d^{2k_1}, (d-1)^{2k_2}, (n-d)^{2k_2}, (n-1-d)^{2k_1})$ by Proposition 5.14(iii). Thus, the degree sequence of $G[V_i^+]$ must be $(d^{2k}, (d-1)^{2k})$ for some k. By Proposition 5.14(ii), $k=1$ and $d=2$. Let $v_1v_2v_3v_4$ denote the path induced by V_i^+. By the applicability of the 2-switch $(v_1v_2, v_3v_4) \to (v_1v_3.v_2v_4)$ and Corollary 5.16, we must have $i = \ell+1-i$. Also note that $\ell > 1$ because vertices in V_i have different degrees in $G[V_i]$. Let σ be an antimorphism of G. In every cycle disjoint from V_i, the neighbors of v_1 and v_2 differ by an even number. Thus, $d_G(v_1) \neq d_G(v_2)$, a contradiction. □

We can now settle the interval structure of each slice.

Lemma 5.18. *For all $i = 1, \ldots, \lfloor \ell/2 \rfloor$,*

i) the slice $G[V_i^+]$ is isomorphic to either a P_4, a Z_2, or a trampoline, and

ii) $V_i \uplus V_{\ell+1-i}$ is a split partition of $G[V_i^+]$.

Moreover, if ℓ is odd, the slice $G[V_{(\ell+1)/2}]$ is either a C_5 or consists of a single vertex.

Proof. For all $i = 1, \ldots, \ell$, the induced subgraph $G[V_i^+]$ of G is self-complementary by Lemma 5.15. Furthermore,$G[V_i^+]$ is either a regular graph or has two different degrees (Lemma 5.17). For all $i = 1, \ldots, \lfloor \ell/2 \rfloor$, the sets V_i and $V_{\ell+1-i}$ are disjoint. Hence, $|V_i^+|$ is $4k$ for some positive k, and the degree sequence of $G[V_i^+]$ is of the form $(d^{2k}, (4k-1-d)^{2k})$. By Lemma 5.15 and Proposition 5.14(ii), $k=1$ and $d=5$. Thus, the degree sequence of $G[V_i]$ is either $(2^2, 1^2)$ or $(5^4, 2^4)$, whose realizations are either a a P_4, a Z_2, or a trampoline. Let $H_i \uplus L_i$ be the unique split partition of $G[V_i^+]$. Suppose to the contradiction of (ii) that there is a vertex $v_1 \in V_i \cap L_i$. We can find a vertex $v_2 \in H_i \setminus N(v_1)$ and a vertex $x_2 \in N(v_2) \cap L_i$. Note that x_2 is not adjacent to v_1. Since $d_G(v_1) > d_G(v_2)$ while $d_{G[V_i^+]}(v_1) < d_{G[V_i^+]}(v_2)$, we can find a vertex x_1 in $V(G) \setminus V_i^+$ that is adjacent to v_1 but not to v_2. The applicability of the 2-switch $(x_1v_1, x_2v_2) \to (x_1v_2, x_2v_1)$ violates Corollary 5.16.

If ℓ is odd, then $G[V_{(\ell+1)/2}]$ is a regular graph. Hence, the degree sequence of $G[V_{(\ell+1)/2}]$ is $((2k)^{4k+1})$, where $k = (|V_{(\ell+1)/2}| - 1)/4$. By Lemma 5.15 and Proposition 5.14(i), $k \leq 1$. The statement follows. $\square$

The next is on edges between different slices.

Lemma 5.19. *For every $i \in \{1, 2, \ldots, \lfloor \ell/2 \rfloor\}$, if a vertex in $V(G) \setminus V_i$ has a neighbor in $V_{\ell+1-i}$, then it is adjacent to all the vertices in V_i^+.*

Proof. Let $x_1 \in V(G) \setminus V_i$ be adjacent to $v_1 \in V_{\ell+1-i}$. Since $V_{\ell+1-i}$ is an independent set, it dose not contain x_1. Suppose for contradiction that $V_i^+ \not\subseteq N(x_1)$, and let v_2 be a vertex in $V_i^+ \setminus N(x_1)$. If $v_2 \in V_i$, we can find a vertex $x_3 \in V_i \setminus N(v_1)$ by Lemma 5.18. The applicability of the 2-switch $(x_1v_1, v_2x_3) \to (x_1v_2, v_1x_3)$ violates Corollary 5.16. In the rest, $x_2 \in V_{\ell+1-i}$.

If there exists a vertex $x_2 \in V_i \cap N(v_2) \setminus N(v_1)$, then we can conduct the 2-switch $(x_1v_1, x_2v_2) \to (x_1v_2, x_2v_1)$, but the ith slice of the resulting graph cannot be isomorphic to P_4, Z_2, or trampoline, contradicting Lemma 5.18(i). Therefore, $V_i \cap N(v_2) \subseteq N(v_1)$, and $G[V_i^+]$ must be isomorphic to Z_2. We can find a vertex x_3 in $V_i \setminus N(v_1)$ and a vertex v_3 in $V_{\ell+1-i} \cap N(x_3)$. Note that neither x_2v_3 nor x_3v_1 is an edge. We may either conduct the 2-switch $(x_1v_3, x_2v_2) \to (x_1v_2, x_2v_3)$ or $(x_1v_1, x_3v_3) \to (x_1v_3, x_3v_1)$ to G, depending on whether x_1 is adjacent to v_3. In either case, the ith slice of the resulting graph contradicts Lemma 5.18(i). These contradictions conclude the proof. $\square$

We are now ready to prove the main lemma.

Lemma 5.20. *The graph G is an elementary self-complementary pseudo-split graph.*

Proof. Let σ be an antimorphism of G. For each $i \in \{1, 2, \ldots, \lfloor \ell/2 \rfloor\}$, we denote $H_i = V_i$ and $L_i = V_{\ell+1-i}$. By Lemma 5.18, $H_i \uplus L_i$ is a split partition of $G[V_i^+]$. Let i, j be two distinct indices in $\{1, 2, \ldots, \lfloor \ell/2 \rfloor\}$. We argue that there cannot be any edge between H_i and L_j if $i > j$. Suppose for contradiction that there exists $x \in H_i$ that is adjacent to $y \in L_j$ for some $i > j$. By Lemma 5.19, x is adjacent to all the vertices in $G[V_j^+]$. Consequently, $\sigma(x)$ is in L_i and has no neighbor in $G[V_j^+]$. Let v_1 be a vertex in H_j. Since v_1 is not adjacent to $\sigma(x)$, it has no neighbor in L_i by Lemma 5.19. Note that $G[V_i^+]$ is either a P_4, a Z_2, or a trampoline, and so dose $G[V_j^+]$. If we focus on the graph induced by $V_i^+ \cup V_j^+$, we can observe that

$$d_{G[V_i^+ \cup V_j^+]}(v_1) < d_{G[V_i^+ \cup V_j^+]}(x).$$

Since $d_G(v_1) > d_G(x)$, we can find a vertex x_1 in $V(G) \setminus (V_i^+ \cup V_j^+)$ that is adjacent to v_1 but not x. Let v_2 be a neighbor of x in L_i. Note that v_2 is not adjacent to v_1. We can conduct the 2-switch $(x_1v_1, xv_2) \to (x_1x, v_1v_2)$, violating Corollary 5.16. Therefore, L_i is nonadjacent to $\bigcup_{p=i+1}^{\lfloor \ell/2 \rfloor} H_p$ for all $i = 1, \ldots, \lfloor \ell/2 \rfloor$. Since $\sigma(L_i) = H_i$ and $\sigma(\bigcup_{p=i+1}^{\lfloor \ell/2 \rfloor} H_p) = \bigcup_{p=i+1}^{\lfloor \ell/2 \rfloor} L_p$, we can obtain that K_i is complete to $\bigcup_{p=i+1}^{\lfloor \ell/2 \rfloor} I_p$. Moreover, H_i is complete to $\bigcup_{p=i+1}^{\lfloor \ell/2 \rfloor} H_p$ by Lemma 5.19, and hence L_i is nonadjacent to $\bigcup_{p=i+1}^{\lfloor \ell/2 \rfloor} L_p$.

We are done if ℓ is even. In the rest, we assume that ℓ is odd. By Lemma 5.18, the subgraph induceded by $V_{(\ell+1)/2}$ is either a C_5 or contains exactly one vertex. It suffices to show that $V_{(\ell+1)/2}$ is complete to H_i and nonadjacent to L_i for every $i \in \{1, 2, \ldots, \lfloor \ell/2 \rfloor\}$. Suppose $\sigma(v) = v$. When $V_{(\ell+1)/2} = \{v\}$, the claim follows from Lemma 5.19 and that $\sigma(v) = v$ and $\sigma(V_i) = V_{\ell+1-i}$. Now $|V_{(\ell+1)/2}| = 5$. Suppose for contradiction that there is a pair of adjacent vertices $v_1 \in V_{(\ell+1)/2}$ and $x \in L_i$. Let $v_2 = \sigma(v_1)$. By Lemma 5.19(ii), v_1 is adjacent to all the vertices in $G[V_i^+]$. Accordingly, v_2 has no neighbor in $G[V_i^+]$. Since $G[V_{(\ell+1)/2}]$ is a C_5 , we can find $v_3 \in V_{(\ell+1)/2}$ that is adjacent to v_2 but not v_1. We can conduct the 2-switch $(xv_1, v_2v_3) \to (xv_2, v_1v_3)$ and denote by G' as the resulting graph. It can be seen that $G'[V_{(\ell+1)/2}]$ is not a C_5, contradicting Lemma 5.18. □

Lemmas 5.11 and 5.20 imply Theorem 1.6 and Rao's characterization of forcibly self-complementary degree sequences [105].

Theorem 5.21 ([105]). *A degree sequence $(d_i^{n_i})_{i=1}^{\ell}$ is forcibly self-complementary if and only if for all $i = 1, \ldots, \lfloor \ell/2 \rfloor$,*

$$n_{\ell+1-i} = n_i \qquad \in \{2, 4\}, \tag{5.1}$$

$$d_{\ell+1-i} = n - 1 - d_i = \sum_{j=1}^{i} n_j - \frac{1}{2} n_i, \tag{5.2}$$

and $n_{(\ell+1)/2} \in \{1, 5\}$ and $d_{(\ell+1)/2} = \frac{1}{2}(n-1)$ when ℓ is odd.

Proof. The sufficiency follows from Lemma 5.11: note that an elementary self-complementary pseudo-split graph in which $G[V_i^+]$ has $2n_i$ vertices satisfies the conditions. The necessity follows from Lemma 5.20. □

5.3 Enumeration

In this section, we consider the enumeration of self-complementary pseudo-split graphs and self-complementary split graphs. The following corollary of Propositions 5.4 and 5.5 focuses us on self-complementary split graphs of even orders. Let λ_n and λ'_n denote the number of split graphs and pseudo-split graphs, respectively, of order n that are self-complementary. For convenience, we set $\lambda_0 = 1$.

Corollary 5.22. *For each $k \geq 1$, it holds $\lambda_{4k+1} = \lambda_{4k}$. For each $n > 0$,*

$$\lambda'_n = \begin{cases} \lambda_n & n \equiv 0 \pmod 4, \\ \lambda_{n-1} + \lambda_{n-5} & n \equiv 1 \pmod 4. \end{cases}$$

Let $\sigma = \sigma_1 \dots \sigma_p$ be an antimorphism of a self-complementary graph with $4k$ vertices. We find the number of ways in which edges can be introduced so that the result is a self-complementary split graph with σ as an antimorphism. We need to consider adjacencies among vertices in the same cycle and the adjacencies between vertices from different cycles of σ. For the second part, we further separate into two cases depending on whether the cycles have the same length. We use G to denote a resulting graph and denote by G_i the graph induced by the vertices in the ith cycle, for $i = 1, \dots, p$. By Lemma 5.3, G has a unique split partition and we refer to it as $K \uplus I$.

(i) The subgraph G_i is determined if it has been decided whether v_{i1} is to be adjacent or not adjacent to each of the following $\frac{|\sigma_i|}{2}$ vertices in σ_i. Among those $\frac{|\sigma_i|}{2}$ vertices, half of them are odd-numbered in σ_i. Therefore, v_{i1} is either adjacent to all of them or adjacent to none of them by Lemma 5.3. The number of adjacencies to be decided is $\frac{|\sigma_i|}{4} + 1$.

(ii) The adjacencies between two subgraphs G_i and G_j of the same order are determined if it has been decided whether v_{i1} is to be adjacent or not adjacent to each of the vertices in G_j. By Lemma 5.3, the vertex v_{i1} and half of vertices of G_j are decided in K or in I after (i). The number of adjacencies to be decided is $\frac{|\sigma_j|}{2}$.

(iii) We now consider the adjacencies between two subgraphs G_i and G_j of different orders. We use $\gcd(x, y)$ to denote the greatest common factor of two integers x and y. The adjacencies between G_i and G_j are determined if it has been decided whether v_{i1} is to be adjacent or not adjacent to each of the first $\gcd(|\sigma_i|, |\sigma_j|)$ vertices of G_j. Among those $\gcd(|\sigma_i|, |\sigma_j|)$ vertices of G_j, half of them are decided in the same part of $K \uplus I$ as v_{i1} after (i). The number of adjacencies to be decided is $\frac{1}{2}\gcd(|\sigma_i|, |\sigma_j|)$.

By Lemma 5.1, $|\sigma_i| \equiv 0 \pmod 4$ for every $i = 1, \dots, p$. Let c be the cycle structure of σ. We use c_q to denote the number of cycles in c with length $4q$ for every

$q = 1, 2, \ldots, k$. The total number of adjacencies to be determined is

$$\begin{aligned} P &= \sum_{q=1}^{k}(c_q(q+1) + \frac{1}{2}c_q(c_q-1)\cdot 2q) + \sum_{1\le r<s\le k} c_r c_s \cdot \frac{1}{2}\mathrm{gcd}(4r,4s) \\ &= \sum_{q=1}^{k}(qc_q^2 + c_q) + 2\sum_{1\le r<s\le k} c_r c_s \mathrm{gcd}(r,s)\,. \end{aligned}$$

For each adjacency, there are two choices. Therefore, the number of labeled self-complementary split graphs with this σ as an antimorphism is 2^P.

The number of distinct permutations of the cycle structure c consisting of c_q cycles of length $4q$ for every $q = 1, 2, \ldots, k$ is

$$\frac{(4k)!}{\prod_{q=1}^{k}(4q)^{c_q}\cdot c_q!}\,,$$

and it is the number of possible choices for σ [35]. Let C_{4k} be the set that contains all cycle structures c that satisfy $\sum_{q=1}^{k} c_q \cdot 4q = 4k$. Then the number of antimorphisms with all possible labeled self-complementary split graphs with $4k$ vertices corresponding to each is

$$\sum_{c\in C_{4k}} \frac{(4k!)}{\prod_{q=1}^{k}(4q)^{c_q}\cdot c_q!}2^P\,. \tag{5.3}$$

For a graph G with $4k$ vertices, let A_G be the set of automorphisms of G. Then, the number of different labelings of G is $(4k)!/|A_G|$. If G is self-complementary, then the number of antimorphisms of G is equal to the number of automorphisms of G. Let S be the set of all non-isomorphic self-complementary split graphs with $4k$ vertices and let $\lambda_{4k} = |S|$. The number of labeled self-complementary split graphs with all possible antimorphisms corresponding to each is equal to

$$\sum_{G\in S}|A_G|\frac{(4k)!}{|A_G|} = \lambda_{4k}\,(4k)!. \tag{5.4}$$

Let Equation (5.3) equals to Equation (5.4) and we solve for λ_{4k}:

$$\lambda_{4k} = \sum_{c \in C_{4k}} \frac{2^P}{\prod_{q=1}^{k} (4q)^{c_q} \cdot c_q!} .$$

We list below the numbers of self-complementary (pseudo-)split graphs on up to 20 vertices.

n	4	5	8	9	12	13	16	17	20
split graphs	1	1	3	3	16	16	218	218	9,608
pseudo-split graphs	1	2	3	4	16	19	218	234	9,608
all	1	2	10	36	720	5,600	703,760	11,220,000	9,168,331,776

Chapter 6

Conclusions

We conclude this thesis by presenting an overview of open questions and conjectures that have captured our interest in the study of t-perfect graphs and self-complementary graphs. We discuss these problems, exploring their significance and potential implications. By presenting these open questions and conjectures, we aim to stimulate further study and foster a deeper understanding of t-perfect graphs and self-complementary graphs.

T-perfect graphs

Similar to the structural characterization of perfect graphs (the strong perfect graph theorem), one may want to characterize t-perfect graphs by minimal forbidden t-minors that are graphs minimally t-imperfect. T-perfect graphs are arised from the odd cycle polytope, suggesting that odd cycles may hold key insights for understanding t-perfection. Exploring the properties of odd cycles within minimally t-imperfect graphs may provide valuable clues to unravel the underlying structure of these graphs.

We say an odd cycle in a graph is *dominating* if every vertex in the graph has a neighbor on the cycle. In a graph G where every odd cycle is dominating, removing the closed neighborhood of any vertex will result in a graph without any odd cycles, and hence G is a near-bipartite graph. Upon checking the known minimally t-imperfect graphs, all the graphs illustrated in Figure 1.2 are near-bipartite. Among the $(3,3)$-partitionable graphs shown in Figures 1.2, the graph $\overline{C_{10}^2}$ (the first graph in the second row) is near-bipartite as well. However, upon checking the nine $(3,3)$-partitionable graphs that are not near-bipartite, it is evident that the only odd cycle that is not dominating in those graphs is C_3. Consequently, in every known minimally t-imperfect graph, odd holes are dominating. This observation holds significant importance in proving the main results presented in Chapters 3 and 4. We propose the following conjecture regarding odd cycles in minimally t-imperfect graphs.

Conjecture 6.1. *Every odd hole is dominating in a minimally t-imperfect graph.*

We have seen that the $(3,3)$-partitionable graph $\overline{C_{10}^2}$ is near-bipartite. Actually, the near-bipartite graphs $\overline{C_7}$ and $\overline{C_{13}^3}$ are $(2,3)$-partitionable and $(4,3)$-partitionable graphs, respectively. From this point of view, it may possible to find new minimally t-imperfect graphs in the class of partitionable graphs.

For any (p,q)-partitionable graph, it can be verified that $p \geq 2$ and $q \geq 2$. In the case where a (p,q)-partitionable graph is a minimally t-imperfect graph, we have observed that q must be greater than three. This is due to the fact that every $(p,2)$-partitionable graph is an odd hole, which is known to be t-perfect. Moreover, if q were larger than four, then the (p,q)-partitionable graph would contain a K_4, which would contradict its status as a minimally t-imperfect graph. As a result, p can only be three. We now focus on $(p,3)$-partitionable graphs. The graph $\overline{C_7}$ is the only $(2,3)$-partitionable graph and all $(3,3)$-partitionable graphs are shown in Figure 4.2. Some $(4,3)$-partitionable graphs are found by Chvatal [32]. We have inspected those graphs shown in Figures 3–6 of the paper [32], and confirmed that they are not minimally t-imperfect. The graph $\overline{C_{13}^3}$ is the only known minimally t-imperfect $(4,3)$-partitionable graph. A method to construct all $(p,3)$-partitionable graphs with $p \leq 9$ has been introduced by Boros[12]. We pose an open question regarding $(p, 3)$-partitionable graphs.

Question 6.2. *Is there a $(p, 3)$-partitionable graph with $p \geq 5$ that is minimally t-imperfect?*

The current list of minimally t-imperfect graphs has not been updated for around six years. Every existing minimally t-imperfect graphs can be classified as a near-bipartite graphs or a partitionable graph. We may want to know whether there exists a minimally t-imperfect graph that is not in these two graph classes. According to this, we propose the following conjecture.

Conjecture 6.3. *A minimally t-imperfect graph is a near-bipartite graph or a $(p, 3)$-partitionable graph with some $p \geq 2$.*

It is worth noting that all minimally t-imperfect graphs that are near-bipartite have already been obtained. They are $\overline{C_7}$, $\overline{C_{10}^2}$, $\overline{C_{13}^3}$, $\overline{C_{13}^4}$, $\overline{C_{19}^7}$, odd wheels, and even Möbius ladders. More results on this topic can be found in [18, 68, 116]. If Conjecture 6.3 holds, then for any minimally t-imperfect graph, every vertex must satisfy one of two conditions: either its neighbors form an independent set (i.e., for graphs $\overline{C_{13}^4}$, $\overline{C_{19}^7}$, and even Möbius ladders), or the vertex is contained in a triangle (i.e., for odd wheels and $(p, 3)$-partitionable graphs). In other words, every vertex is either contractable or not contractable. To disprove Conjecture 6.3, it may be sufficient to find a minimally t-imperfect graph that contains both a contractable vertex and a vertex that is not contractable. Therefore, we tend to pose the following question.

Question 6.4. *Can a minimally t-imperfect graph have a contractable vertex and a vertex that is not contractable?*

Chvátal [29] showed that a minimally t-imperfect graph cannot contain a clique separator. Therefore, cut vertices are absent in such graphs, implying that every vertex in a minimally t-imperfect graph has a degree of at least two. Upon further examination of the known minimally t-imperfect graphs, it can be observed that every vertex in these graphs has a degree of at least three. An open question arises as follows.

Question 6.5 ([21]). *Is there a minimally t-imperfect graph that contains a vertex of degree two?*

A *separation* of a graph G is denoted by (G_1, G_2) and is defined as follows: G_1 and G_2 are induced subgraphs of G; $G = G_1 \cup G_2$; and $G - G_1 \neq \emptyset \neq G - G_2$. The order of the separation, denoted by $|V(G_1) \cap V(G_2)|$, is the number of vertices shared by both G_1 and G_2. Bruhn found a structural property for minimally t-imperfect graphs that possess a separation of order two.

Lemma 6.6 ([21]). *Let G be a minimally t-imperfect graph, and (G_1, G_2) a separation of G with order at most two. Then exactly one of G_1 and G_2 is a path between two vertices u and v in $V(G_1) \cap V(G_2)$. Moreover, (G_1, G_2) has order two, and u is not adjacent to v in G.*

Assuming a negative answer to Question 6.5, it follows that no minimally t-imperfect graph can have a separation of order two. Consequently, every minimally t-imperfect graph must be 3-connected. This motivates the following openquestion:

Question 6.7 ([21]). *Are all minimally t-imperfect graphs 3-connected?*

We next consider the number of vertices in minimally t-imperfect graphs. Our observation of known minimally t-imperfect graphs indicates that only odd wheels, even Möbius ladders, and $(3,3)$-partitionable graphs have an even number of vertices. This observation motivates the following question.

Question 6.8. *Let G be a minimally t-imperfect graph. If G is not an odd wheel, an even Möbius ladder, or a $(3,3)$-partitionable graph, is the order of G necessarily odd?*

While discovering all minimally t-imperfect graphs may seem challenging, significant progress has been made by imposing restrictions on the graph structures. For instance, the complete characterization of minimally t-imperfect graphs has been attained for classes such as $S_{1,1,1}$-free graphs [21], $S_{1,1,2}$-free Theorem 1.1, and P_5-free graphs [18]. The class of $S_{1,1,3}$-free graphs is a generalization of both $S_{1,1,2}$-free graphs and P_5-free graphs, as is the class of $S_{1,2,2}$-free graphs. It is worth exploring if the existing techniques can be used to find all minimally t-imperfect graphs that are $S_{1,1,3}$-free or $S_{1,2,2}$-free.

Question 6.9. *Could we characterize t-perfection on $S_{1,1,3}$-free or $S_{1,2,2}$-free graphs?*

Padberg [102] proved that for any minimally imperfect graph G, the clique polytope $P_K(G)$ has exactly one non-integral extreme point, with all coordinates equal to $\frac{1}{\omega(G)}$. Additionally, the independent set polytope of a minimally imperfect graph G can be obtained by adding the *full-rank inequality* $x(V(G)) \leq \alpha(G)$ to the description of $P_K(G)$.

Motivated by perfect graphs, we aim to investigate whether the independent set polytope of a minimally t-imperfect graph G can be obtained by adding the full-rank inequality into the description of the odd cycle polytope $P_{OC}(G)$. Benchetrit [11] studied the independent set polytope of several minimally t-imperfect graphs, including odd wheels, even Möbius ladders, and $(p, 3)$-partitionable graphs with p equals to two and three. His results revealed that the independent set polytope of these graphs can be derived by adding the full-rank inequality to the description of $P_{OC}(G)$, with the exception of odd wheels having more than four vertices. This motivates us to pose the following conjecture.

Conjecture 6.10 ([11])**.** *Let G be a minimally t-imperfect graph such that G is not an odd wheel with more than four vertices. Then, the independent set polytope of G can be described by the description of $P_{OC}(G)$ together with the full-rank inequality.*

We are also interested in the following question:

Question 6.11 ([21]). *Does the odd cycle polytope $P_{OC}(G)$ on a minimally t-imperfect graph G have precisely one extreme point that is not integral?*

It follows from the observation of Edmonds and Giles [46] on total dual integrality that every strongly t-perfect graph is t-perfect. The other direction remains an open problem. In particular, we do not know whether all P_5-free t-perfect graphs are strongly t-perfect, though it is true for all claw-free t-perfect graphs [20] and fork-free t-perfect graphs 1.1.

Conjecture 6.12 ([115]). *Every t-perfect graph is strongly t-perfect.*

To refute a graph G being strongly t-perfect, it suffices to identify a weight function w such that the costs of all w-covers of G are greater than $\alpha_w(G)$. For all the known minimally t-imperfect graphs, the unit-weight function is a certificate. Note that $\alpha_w(G) = \alpha(G)$ for unit weighting w.

Conjecture 6.13. *Let w be the unit weight function to a graph G.*

i) If G is t-perfect, then there is a w-cover of cost $\alpha(G)$.

ii) If G is a minimally t-imperfect graph, then there cannot be a w-cover of cost $\alpha(G)$.

Note that the second statement of Conjecture 6.13 cannot be generalized to t-imperfect graphs that are not minimal; e.g., the graph in Figure 6.1. For the unit-weight function w, we have $\alpha_w(G) = 3$. On the other hand,

$$\{\{v_1v_2v_3v_4v_5\}, \{v_6v_7\}\}$$

is a w-cover of cost 3. It is t-imperfect since K_4 is a t-minor of G (doing t-contraction at v_3). To produce a certificate, we increase $w(v_3)$ to two (the weights of other vertices

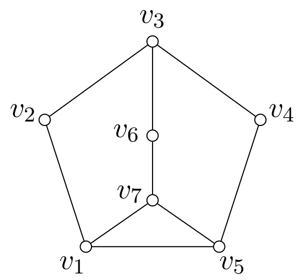


Figure 6.1: A counterexample.

remain one). The value of $\alpha_w(G)$ remains 3, while no w-cover of G has a cost smaller than 4.

An open question independent to Conjecture 6.13 is for which class of t-imperfect graphs a unit-weight function is a negative certificate.

Question 6.14. *Let w be the unit weight function. For what t-imperfect graph G, the cost of a minimum w-cover of G is strictly greater than $\alpha_w(G)$?*

Shepherd [69] conjectured that every t-perfect graph is three-colorable, which was refuted by Laurent and Seymour [115]. The example of Laurent and Seymour [115] needs four colors. It is known that a t-perfect graph can be three-colored if it is claw-free, P_5-free, or fork-free.

Conjecture 6.15 ([69, 115])**.** *Every t-perfect graph is* 4*-colorable.*

If Conjecture 6.15 is refuted, is there a constant bound for the chromatic number?

Question 6.16. *Is there a constant k such that every t-perfect graph has chromatic number at most k?*

Bounding the chromatic number $\chi(G)$ of a graph G in terms of other graph invariants, such as the clique number $\omega(G)$ and the maximum degree $\Delta(G)$, has a long tradition. One well-established proposition in this field is that for any graph G, it holds that $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$. In 1998, Reed [107] put forward a conjecture suggesting that the chromatic number $\chi(G)$ is upper-bounded by the average of the lower bound $\omega(G)$ and the upper bound $\Delta(G)+1$:

Conjecture 6.17 (Reed's Conjecture [107])**.** *Every graph G satisfies $\chi(G) \leq \lceil \frac{\omega(G)+\Delta(G)+1}{2} \rceil$.*

Reed's Conjecture is true in various graph classes, such as perfect graphs, graphs with disconnected complements [104], certain types of triangle-free graphs [75], odd hole-free graphs [7], and specific classes of P_5-free graphs [7]. However, for the family of triangle-free graphs, it is not yet known if the conjecture holds. In cases where $\omega(G) = 2$, the conjecture simplifies to the following:

Conjecture 6.18. *If G is a triangle-free graph, then $\chi(G) \leq \frac{\Delta(G)}{2} + 2$.*

Previous findings have demonstrated that Conjecture 6.18 is true when $\Delta(G)$ meets a particular requirement [75], or when the number of vertices in the graph is at most 24 [59]. However, our concern lies with t-perfect graphs. Thus, it remains to be determined if Reed's conjecture holds true for this particular class of graphs.

Question 6.19. *Does Reed's conjecture hold for (triangle-free) t-perfect graphs?*

Several results on the chromatic number of triangle-free t-perfect graphs are in Marcus' thesis [87] but no constant bound is known.

Question 6.20. *Is there a constant k such that every triangle-free t-perfect graph has chromatic number at most k?*

Self-complementary graphs

The problem of determining whether two given graphs are isomorphic or not is known as the graph isomorphism problem. Colbourn and Colbourn [37] showed that the graph isomorphism problem for (regular) self-complementary graphs is polynomially equivalent to the general graph isomorphism problem, making it *GI-complete.*

Interestingly, even the task of determining whether a graph is self-complementary or not falls under the GI-complete complexity class. Split graphs, on the other hand, possess a remarkable property: they can be recognized solely based on their degree sequences. However, despite this advantage, the isomorphism problem for split graphs remains GI-complete. If we consider the graph isomorphism problem and recognition problem on graphs that are not only self-complementary but also admit a split partition, could the two problem be solved by an efficient algorithm? We are interested in the following to questions.

Question 6.21. *Is the graph isomorphism problem on self-complementary split graphs GI-complete?*

Question 6.22. *Is the recognition problem on self-complementary split graphs GI-complete?*

Finaly, we recall the conjectures about partitions in self-complementary graphs.

Conjecture 6.23 (Trotignon [118])**.** *Let G be a self-complementary graph of even order. If G is C_5-free, then G admits a rectangle or diamond partition.*

We generalize Trotignon's conjecture based on our study of self-complementary partitions.

Conjecture 6.24. *Let G be a self-complementary graph of even order. If G is C_5-free, then G admits a rectangle or diamond partition that is self-complementary.*

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