OPTIMAL ERROR BOUNDS FOR CONIC LINEAR FEASIBILITY PROBLEMS AND THEIR APPLICATIONS

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Certificate of Originality

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____________________________(Signature)

Ying LIN (Name of student)

Dedicate to my parents and wife.

Abstract

Error bounds are a requisite for trusting or distrusting solutions in an informed way. Until recently, provable error bounds in the absence of constraint qualifications were unattainable for many classes of cones that do not admit projections with known succinct expressions. In this thesis, we apply a recently developed framework based on facial reduction algorithms and one-step facial residual functions to build up error bounds for two closed convex cones: the generalized power cones and the log-determinant cones.

The generalized power cones admit direct modelling of certain problems and have found applications in geometric programs, generalized location problems, and portfolio optimization, etc. We propose a complete error bound analysis for the conic linear feasibility problems with the generalized power cones without requiring any constraint qualifications. All the error bounds are shown to be tight in the sense of that framework. Besides their utility for understanding solution reliability, the error bounds we discover have additional applications to the algebraic structure of the underlying cone. We then completely determine the automorphism group of the generalized power cones, which was unknown before our work. Based on the automorphism group, we also discuss some other theoretical questions related to homogeneity and perfectness, identifying a set of generalized power cones that are self-dual, irreducible, nonhomogeneous, and perfect.

The log-determinant cone is the closure of the hypograph of the perspective function of the log-determinant function, which has both theoretical and practical importance. Specifically, a problem with a log-determinant term in its objective can be recast as a problem over the log-determinant cone, indicating the significance of the log-determinant cone. As a high-dimensional generalization of the exponential cone, whose error bounds were well studied, the derivation of the error bounds for the log-determinant cone is however not straightforward because of the higher dimension and the more involved facial structure. We establish tight error bounds for the log-determinant cone problem without requiring any constraint qualifications.

Publications Arising from the Thesis

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Contents

List of Notations

$\mathcal{E}$	a finite dimensional Euclidean space
$\mathcal{L}$	a subspace in $\mathcal{E}$
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^n$ ($\mathbb{R}^n_+$)	the set of n-dimensional real vectors (with nonnegative entries)
$\mathbb{R}^{m\times n}$	the set of $m \times n$ real matrices
$\langle \cdot, \cdot \rangle$	the inner product of $\mathcal{E}$
$\|\boldsymbol{x}\|$	the Euclidean norm of a vector $\boldsymbol{x}$
$P_{\mathcal{C}}(\boldsymbol{x})$	the projection of $\boldsymbol{x}$ onto a nonempty closed convex set $\mathcal{C} \subseteq \mathcal{E}$
$\mathrm{dist}(\boldsymbol{x}, \mathcal{C})$	the distance from $\boldsymbol{x}$ to $\mathcal{C}$
$\mathcal{B}(\boldsymbol{x}; \eta)$	the ball centered at $\boldsymbol{x}$ with radius η
$\mathcal{B}(\eta)$	the ball centered at $\mathbf{0}$ with radius η
$\mathrm{Diag}(\boldsymbol{x})$	a diagonal matrix with diagonal vector being $\boldsymbol{x}$
$\mathcal{C}^\perp$	the orthogonal complement of $\mathcal{C}$
$\mathcal{S}^d$	the set of symmetric $d \times d$ matrices
$\mathcal{S}^d_+$, $\mathcal{S}^d_{++}$	the sets of the symmetric $d \times d$ positive semidefinite matrices and positive definite matrices
I_n	the $n \times n$ identity matrix

$X \succeq 0$ ($X \succ 0$)	X is positive semidefinite (positive definite)
$\lambda_i(X)$	the i-th smallest eigenvalue of a matrix X
$\lambda_{\max}(X), \lambda_{\min}(X)$	the maximum and minimum eigenvalues of X
$\mathsf{r}\,(X)$	the rank of X
$\mathrm{tr}(X), \det(X)$	the trace and determinant of X
$\|X\|_F$	the Frobenius norm of X
$\mathcal{K}, \mathcal{K}^*$	a closed convex cone and its dual cone
$\partial\mathcal{K}, \mathrm{ri}\,\mathcal{K}, \mathrm{span}\,\mathcal{K}, \dim\mathcal{K}$	the boundary, relative interior, linear span, dimension of a closed convex cone $\mathcal{K}$
$\mathcal{F} \trianglelefteq \mathcal{K} (\mathcal{F} \mathrel{\ntrianglelefteq} \mathcal{K})$	$\mathcal{F}$ is a (proper) face of $\mathcal{K}$
$\psi_{\mathcal{K},\boldsymbol{n}}$	a one-step facial residual function for $\mathcal{K}$ and $\boldsymbol{n}$
$d_{\mathrm{PPS}}(\mathcal{K}, \boldsymbol{\mathcal{L}} + \boldsymbol{a})$	the distance to the PPS condition of a feasible (Feas)
$\mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$	the generalized power cone where $m \geq 1$, $n \geq 2$ and $\boldsymbol{\alpha} = (\alpha_i)_{i=1}^n \in (0,1)^n$ with $\sum_{i=1}^n \alpha_i = 1$
$\mathrm{Aut}\,(\mathcal{K})$	the automorphism group of a cone $\mathcal{K}$
$\mathrm{Lie\,Aut}\,(\mathcal{K})$	the Lie algebra of the automorphism group of a cone $\mathcal{K}$
$\mathcal{F}_{\boldsymbol{z}}, \mathcal{F}_{\mathrm{r}}$	the two types of faces of the generalized power cone
$\mathcal{K}_{\mathrm{logdet}}$	the log-determinant cone
$\mathcal{F}_{\mathrm{r}}, \mathcal{F}_{\mathrm{d}}, \mathcal{F}_{\#}, \mathcal{F}_{\infty}, \mathcal{F}_{\mathrm{ne}}^{\#}$	the five types of faces of the log-determinant cone

Chapter 1

Introduction

The *convex conic linear feasibility problem* has attracted a lot of attention due to its power in modeling convex problems. Specifically, a convex conic linear feasibility problem admits the following form:

$$\text{Find} \quad \boldsymbol{x} \in (\mathcal{L} + \boldsymbol{a}) \cap \mathcal{K}, \tag{Feas}$$

where $\mathcal{K}$ is a closed convex cone contained in a finite dimensional Euclidean space $\mathcal{E}$, $\mathcal{L} \subseteq \mathcal{E}$ is a subspace and $\boldsymbol{a} \in \mathcal{E}$ is given. Various aspects of (Feas) such as numerical algorithms and applications have been studied in the literature; see e.g., [6, 25]. In this thesis, we focus on the theoretical aspects, particularly *error bounds* for (Feas). To be more precise, assuming the feasibility of (Feas), we want to establish inequalities that give upper bounds on the distance from an arbitrary point to $(\mathcal{L} + \boldsymbol{a}) \cap \mathcal{K}$ based on the individual distances from the point to $\mathcal{L} + \boldsymbol{a}$ and $\mathcal{K}$. As a fundamental topic in optimization [27, 35, 45, 54, 74], error bounds possess a wide range of applications, especially in algorithm design and convergence analysis. A notable application of error bounds for (Feas) is in the design of termination criteria for the celebrated *interior-point method* (IPM), one of the most powerful algorithms that are used in commercial and open source solvers for solving (convex) nonlinear optimization problems; see, for example, [31, 11, 61, 15, 14] and solvers like MOSEK, Alfonso, DDS and Hypatia [12, 30, 55, 47]. The IPM obtains a *Karush-Kuhn-Tucker*

(KKT) point of the optimization problem by iteratively solving a series of *perturbed KKT systems* that approximate the true KKT system. As the KKT system can be formulated as a convex conic linear feasibility problem in the form of (Feas), error bounds for (Feas) provide an upper bound on the distance from a candidate solution to the set of KKT points. This bound is typically of the same order of magnitude as the actual distance, making it an effective tool for designing a robust termination criterion that balances computational efficiency with solution accuracy.

Deducing the error bounds for (Feas) is generally difficult because it involves the projection onto the cone, which can be complicated without an analytic form. When the cone $\mathcal{K}$ is polyhedral, i.e., it can be written as the intersection of finitely many half spaces, the classical Hoffman's error bound [27] provides a comprehensive depiction of a nearly best error bound. In contrast, if the cone is not polyhedral, a similar error bound also holds when some good behaviors (*constraint qualifications*) of the feasible region are assumed. For instance, one of the most commonly seen constraint qualifications is the *Slater's condition*, which is satisfied if the affine space intersects the *relative interior* of the cone [7]. Nonetheless, the Slater's condition is generally not satisfied, and verifying its satisfaction, even when met, presents considerable challenges.

The attempts at the error bounds without constraint qualifications can be traced back to Sturm's pioneering work on the error bounds for positive semidefinite systems [62]. This work is also the first work connecting the error bounds with the *facial reduction algorithms* [9, 56, 70]. In the same year, Luo and Sturm relevantly established a complete picture of error bounds for the cone that is a Cartesian product of *second-order cones* and the positive semidefinite cones [44], generalizing the previous work. Inspired by Sturm's work, Lourenço presented an approach for deducing error bounds for the so-called *amenable cones* without constraint qualifications based on the facial reduction algorithms and the new notion of *facial residual functions* [40].

Lourenço's approach is applied to a more general class of cones, the symmetric cones in the context of Jordan algebras [18, 19] since the symmetric cones are shown to be amenable. This result covers Luo and Sturm's results because both the second-order cone and positive semidefinite cone are symmetric cones.

However, there is no unified framework for conic feasibility problems with general cones. One concrete counterexample such that all previous methods fail to work is the *exponential cone*. While having a simple form, the projection onto the exponential cone requires solving a transcendental equation, which only has numerical solutions. This makes the traditional projection-based method cannot apply. Lourenço's framework also fails to apply to the exponential cone since amenable cones have been proven to be *nice* [42] and so *facially exposed*, while the exponential cone is not. These failures of previous methods motivate Lindstrom, Lourenço and Pong to develop a new framework based on Lourenço's previous work [36]. The new framework is based on the facial reduction algorithms and the *one-step facial residual functions ($\mathbb{1}$-FRFs)* [36, Definition 3.4], and theoretically works for any closed convex cones without any constraint qualifications and avoids the computation of the projection onto the cone.

In this thesis, we apply the recently developed framework to establish the error bounds for (Feas) with two closed convex cones: the generalized power cone and the log-determinant cone. Utilizing the error bounds, we also exploit some interesting applications in algebraic structure.

1.1 Generalized Power Cones

The generalized power cone is defined as

$$\mathcal{P}^{\boldsymbol{\alpha}}_{m,n} = \left\{ \boldsymbol{x} = (\overline{\boldsymbol{x}}, \widetilde{\boldsymbol{x}}) \in \mathbb{R}^{m+n} \;\middle|\; \|\overline{\boldsymbol{x}}\| \leq \prod_{i=1}^{n} \widetilde{x}_i^{\alpha_i},\ \overline{\boldsymbol{x}} \in \mathbb{R}^m,\ \widetilde{\boldsymbol{x}} \in \mathbb{R}^n_+ \right\},$$

where $m \geq 1$, $n \geq 2$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in (0,1)^n$ with $\sum_{i=1}^n \alpha_i = 1$, and $\|\overline{\boldsymbol{x}}\|$ denotes the Euclidean norm of $\overline{\boldsymbol{x}}$. In the specific case when $m \geq 1$, $n = 2$, and $\boldsymbol{\alpha} = (1/2, 1/2)$, $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ is isomorphic to a second-order cone, whose worst-case error bound is known to be Hölderian with exponent 1/2, thanks to the work of Luo and Sturm [44]. The remaining cases, while not as well-known as the second-order cone case, admit more direct modeling of certain problems and have found applications in geometric programs, generalized location problems, and portfolio optimization [11, 47]. More broadly, the inclusion of the power cone[1] (and the exponential cone) makes all the convex instances from the MINLPLib2 benchmark library conic representable [43, 46]. This broad utility has motivated the development of self-concordant barriers [11, 68, 59], and the ongoing development of specialized interior point methods [50, 61]. Optimization with the generalized power cones is implemented in commercial and open source solvers like MOSEK, Alfonso, DDS and Hypatia [12, 30, 55, 47].

We propose in this thesis a complete error bound analysis for the generalized power cone problem (Feas). The generalized power cone cases pose two significant obstructions to error bound analysis that are not present in the second-order cone case. Firstly, known forms for projections onto generalized power cones do not admit simple representations [26]; secondly, their facial structure is more complicated. The first obstruction we obviate via the framework of one-step facial residual functions ($\mathbb{1}$-FRFs), which was established in [36, 37]. The second challenge, facial complexity, we tackle directly. In particular, we build $\mathbb{1}$-FRFs for all faces of $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$. All these $\mathbb{1}$-FRFs are tight in the natural sense of [37]. Consequently, all of the obtained error bounds are tight in this sense.

While error bounds are typically used in convergence analysis and to evaluate the quality of approximate solutions, our approach via $\mathbb{1}$-FRFs admits a surprising additional application to the algebraic structure of the underlying cone. In order to

[1] This refers to $\mathcal{P}^{\boldsymbol{\alpha}}_{1,2}$.

explain our next results, we recall a few concepts. The *automorphism group* of a cone $\mathcal{K}$ is the set of the bijective linear operators $\boldsymbol{A}$ satisfying $\boldsymbol{A}\mathcal{K} = \mathcal{K}$. A cone is said to be *homogeneous* if its automorphism group acts transitively on its relative interior. We say that a cone is *irreducible* if it is not the direct sum of two nontrivial cones whose spans only intersect at the origin.

Because automorphisms of cones must preserve optimal FRFs (up to positively rescaled shifts), we can use our results to establish the automorphism group for $\mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$ and compute its dimension.

This is useful because the automorphism group of a closed convex cone $\mathcal{K}$ has important implications for complementarity problems over $\mathcal{K}$; see [22]. In particular, denoting the dual cone of $\mathcal{K}$ by $\mathcal{K}^*$, a complementarity condition of the form "$\boldsymbol{x} \in \mathcal{K}, \boldsymbol{y} \in \mathcal{K}^*, \langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$" can be split into a square system of equations if and only if the dimension of the automorphism group of $\mathcal{K}$ is at least $\dim \mathcal{K}$, see [53, Theorem 1]. In this case, $\mathcal{K}$ is said to be a *perfect cone*.

Many of the concrete examples of irreducible perfect cones in the literature correspond to homogeneous cones. In this paper we will show that the generalized power cone is irreducible, perfect (when $m \geq 3$) and, except when it reduces to the second order-cone case, always non-homogeneous. This gives an interesting example of an irreducible cone with good complementarity properties that is not a homogeneous cone.

1.2 Log-determinant Cones

The *log-determinant cone* is defined as

$$\mathcal{K}_{\mathrm{logdet}} \coloneqq \left\{(x, y, Z) \in \mathbb{R} \times \mathbb{R}_{++} \times \mathcal{S}^d_{++} : x \leq y \log\det(Z/y)\right\} \cup (\mathbb{R}_- \times \{0\} \times \mathcal{S}^d_+),$$

where $d \geq 1$, $\mathbb{R}_{++}$ is the positive orthant, $\mathcal{S}^d_+$ (resp., $\mathcal{S}^d_{++}$) is the set of $d \times d$ positive semidefinite (resp., positive definite) matrices. We note that the log-determinant cone is the closure of the hypograph of the perspective function of the log-determinant function.

The log-determinant function has both theoretical and practical importance. It is a *self-concordant barrier function* for $\mathcal{S}^d_+$, and hence it is useful for defining the *logarithmically homogeneous self-concordant barrier functions (LHSCBs)* for various matrix cones. LHSCBs are crucial for complexity analysis of the celebrated *primal-dual interior point methods* for solving conic feasibility problems; see, e.g., [51, 12]. In practice, the log-determinant function appears frequently in countless real-world applications, especially in the area of machine learning, to name but a few, the sparse inverse covariance estimation [21], the fused multiple graphical Lasso problem [71, 73], Gaussian process [57, 60], sparse covariance selection [17, 16], finding minimum-volume ellipsoids [1, 65, 69], the determinantal point process [33], kernel learning [5], D-optimal design [2, 10] and so on.

An elementary observation is that

$$t \leq \log\det(Z), Z \in \mathcal{S}^d_{++} \Longleftrightarrow (t, 1, Z) \in \mathcal{K}_{\text{logdet}},$$

in this way, a problem that has a log-determinant term in its objective can be recast as a problem over the log-determinant cone $\mathcal{K}_{\text{logdet}}$. In view of the importance and prevalence of the log-determinant function, the cone $\mathcal{K}_{\text{logdet}}$ can also be used to handle numerous applications.

That said, if one wishes to use conic linear optimization to solve problems involving log-determinants, it is not strictly necessary to use $\mathcal{K}_{\text{logdet}}$. Indeed, it is possible, for example, to consider a reformulation using positive semidefinite cones and exponential cones, e.g., [47, Section 6.2.3].

A natural question then is whether it is more advantageous to use a reformulation or handle $\mathcal{K}_{\text{logdet}}$ directly. Indeed, Hypatia implements the log-determinant cone as a predefined *exotic cone* [12] and their numerical experiments show that the direct use of the log-determinant cone gives numerical advantages compared to the use of reformulations, see [14] and [13, Sections 8.4.1, 8.4.2]. One reason that other formulations may be less efficient is that they increase the dimension of the problem. Another drawback is that they do not capture the geometry of the hypograph of the log determinant function as tightly.

Motivated by these results, we present a study of the facial structure of $\mathcal{K}_{\text{logdet}}$ and the error bounds for (Feas) with $\mathcal{K} = \mathcal{K}_{\text{logdet}}$.

Specifically, we deduce tight error bounds for (Feas) with $\mathcal{K} = \mathcal{K}_{\text{logdet}}$ by deploying the framework in [36, 37]. Although the log-determinant cone is a high-dimensional generalization of the exponential cone, whose error bounds were studied in depth in [36], the derivation of the error bounds for the log-determinant cone is not straightforward. Indeed, the exponential cone is three dimensional and so its facial structure can be visualized explicitly. In contrast, with a higher dimension, the log-determinant cone has a more involved facial structure.

1.3 Contributions

The contributions of this thesis can be summarized as follows:

1. We completely determine the tightest possible error bounds for the generalized power cone.

2. Using our error bounds, we completely determine the automorphism group of $\mathcal{P}^{\alpha}_{m,n}$ and discuss some theoretical questions related to homogeneity and perfectness (in the sense of [23, 22]).

3. We establish the tight error bounds for (Feas) with $\mathcal{K} = \mathcal{K}_{\text{logdet}}$.

Although we do not discuss the details, we mention in passing that determining the error bound associated to conic linear systems makes it possible to compute the KL-exponent of certain functions, as done, for example, in [37, Section 5.1] using results from [72]. See more on the connection between error bounds, KL exponents and convergence rates in [8].

1.4 Organization

This thesis is organized as follows.

- In Chapter 1, we briefly introduce the convex conic linear feasibility problem and the corresponding error bounds. We review the development of the methods to deduce error bounds, especially those do not require constraint qualifications. The two closed convex cones considered in this thesis are also discussed.
- In Chapter 2, we recall notation and preliminaries.
- In Chapters 3 and 4, we establish the error bounds for (Feas) with the generalized power cone and the log-determinant cone, respectively.
- In Chapter 5, we summarize this thesis and discuss the possible future research directions.

Chapter 2

Notation and Preliminaries

We will use plain letters to represent real scalars, bold lowercase letters to denote vectors, bold uppercase letters to stand for matrices,[2] and curly capital letters for (sub)spaces and sets. Let $\mathcal{E}$ be a finite dimensional Euclidean space, $\mathbb{R}_+$ and $\mathbb{R}_-$ be the set of nonnegative and nonpositive real numbers, respectively. The inner product of $\mathcal{E}$ is denoted by $\langle \cdot, \cdot \rangle$ and the induced norm by $\| \cdot \|$. With that, for $\boldsymbol{x} \in \mathcal{E}$ and a closed convex set $\mathcal{C} \subseteq \mathcal{E}$, we denote the projection of $\boldsymbol{x}$ onto $\mathcal{C}$ by $P_{\mathcal{C}}(\boldsymbol{x})$ so that $P_{\mathcal{C}}(\boldsymbol{x}) = \arg\min_{\boldsymbol{y} \in \mathcal{C}} \|\boldsymbol{x} - \boldsymbol{y}\|$ and the distance between $\boldsymbol{x}$ and $\mathcal{C}$ by $\mathrm{dist}(\boldsymbol{x}, \mathcal{C}) = \inf_{\boldsymbol{y} \in \mathcal{C}} \|\boldsymbol{x} - \boldsymbol{y}\| = \|\boldsymbol{x} - P_{\mathcal{C}}(\boldsymbol{x})\|$. For any $\boldsymbol{x} \in \mathcal{E}$ and $\eta \geq 0$, we denote the ball centered at $\boldsymbol{x}$ with radius η by $\mathcal{B}(\boldsymbol{x}; \eta) := \{\boldsymbol{y} \in \mathcal{E} \mid \|\boldsymbol{y} - \boldsymbol{x}\| \leq \eta\}$; we write $\mathcal{B}(\eta)$ for the ball centered at $\mathbf{0}$ with radius η for simplicity. A diagonal matrix with diagonal vector being $\boldsymbol{x}$ is denoted by $\mathrm{Diag}(\boldsymbol{x})$. Meanwhile, we use $\mathcal{C}^{\perp}$ to denote the orthogonal complement of $\mathcal{C}$.

2.1 Matrices

We use $\mathbb{R}^{m \times n}$ to denote the set of all real $m \times n$ matrices and $\mathcal{S}^d$ to denote the set of symmetric $d \times d$ matrices. The $n \times n$ identity matrix will be denoted by I_n. Let $\mathcal{S}^d_+$ and $\mathcal{S}^d_{++}$ be the set of symmetric $d \times d$ *positive semidefinite matrices* and $d \times d$

[2] With an abuse of notation, we use $\mathbf{0}$ to denote a zero vector / matrix, whose dimension should be clear from the context.

positive definite matrices respectively. The interior of $\mathcal{S}^d_+$ is $\mathcal{S}^d_{++}$. We write $X \succ 0$ (resp., $X \succeq 0$) if $X \in \mathcal{S}^d_{++}$ (resp., $X \in \mathcal{S}^d_+$). For any $X \in \mathcal{S}^d$, we let $\lambda_i(X) \in \mathbb{R}$ denote the i-th *eigenvalue* of X such that $\lambda_d(X) \geq \lambda_{d-1}(X) \geq \cdots \geq \lambda_1(X)$. We will use $\lambda_{\max}(X)$ and $\lambda_{\min}(X)$ to denote the maximum and minimum eigenvalues of X, respectively. The *rank* of X is defined by the number of non-zero eigenvalues, denoted by $\mathsf{r}(X)$. The *trace* (resp., *determinant*) of X is defined by $\operatorname{tr}(X) := \sum_{i=1}^d \lambda_i(X)$ (resp., $\det(X) := \prod_{i=1}^d \lambda_i(X)$). With these, we recall that the Frobenius inner product on $\mathcal{S}^d$ is given by $\langle X, Y\rangle := \operatorname{tr}(XY)$ for any $X, Y \in \mathcal{S}^d$, and the Frobenius norm is $\|X\|_F := \sqrt{\operatorname{tr}(X^2)}$. For $X \in \mathbb{R}^{n\times n}$, we denote the nuclear norm and spectral norm of X by $\|X\|_* := \sum_{i=1}^n |\lambda_i(X)|$ and $\|X\|_2 := \max_i |\lambda_i|$, respectively. For any $X \in \mathcal{S}^d_+$ (resp., $X \in \mathcal{S}^d_{++}$), we have $\lambda_i(X) \geq 0$ (resp., $\lambda_i(X) > 0$). We hence also have for any $X, Y \in \mathcal{S}^d_+$ that

$$\operatorname{tr}(XY) \geq \lambda_{\min}(Y)\operatorname{tr}(X) \geq 0 \text{ and moreover, } \operatorname{tr}(XY) = 0 \iff XY = 0. \tag{2.1}$$

For a given non-zero positive semidefinite matrix, the next result connects its determinant with its trace and rank.

Lemma 2.1. *Let $Z \in \mathcal{S}^d_+ \setminus \{\mathbf{0}\}$. Then for any $\eta > 0$, there exists $C > 0$ so that*

$$(\det(R))^{\frac{1}{d}} \leq C[\operatorname{tr}(RZ)]^{\frac{\mathsf{r}(Z)}{d}} \qquad \forall R \in \mathcal{B}(\eta) \cap \mathcal{S}^d_+. \tag{2.2}$$

Proof. Let $Z = Q\Sigma Q^\top$ be an eigendecomposition of Z, where Q is orthogonal and Σ is diagonal, and let r be the rank of Z. Then $\mathsf{r} \geq 1$ since $Z \neq 0$. Without loss of generality, we may suppose that the first r diagonal entries of Σ, denoted as $\sigma_1, \sigma_2, \ldots, \sigma_{\mathsf{r}}$, are nonzero and are arranged in descending order. Then σ_{r} is the smallest positive eigenvalue of Z and we have for any $R \in \mathcal{B}(\eta) \cap \mathcal{S}^d_+$ that

$$\begin{aligned} \operatorname{tr}(RZ) &= \operatorname{tr}(RQ\Sigma Q^\top) = \operatorname{tr}(Q^\top RQ\Sigma) = \operatorname{tr}([Q^\top RQ]_{\mathsf{r}}[\Sigma]_{\mathsf{r}}) \\ &\overset{\text{(a)}}{\geq} \sigma_{\mathsf{r}} \operatorname{tr}([Q^\top RQ]_{\mathsf{r}}) \overset{\text{(b)}}{\geq} \sigma_{\mathsf{r}} \sum_{i=1}^{\mathsf{r}} \lambda_i(Q^\top RQ), \end{aligned} \tag{2.3}$$

where $[A]_{\mathsf{r}}$ is the submatrix of A formed by A_{ij} for $1 \leq i, j \leq \mathsf{r}$, (a) holds since $[Q^\top RQ]_{\mathsf{r}} \succeq 0$ (thanks to $R \succeq 0$), (b) is true because of the interlacing theorem (see [28, Theorem 4.3.8]).

Next, note that we have for any $R \in \mathcal{B}(\eta) \cap \mathcal{S}^d_+$ that

$$\begin{aligned}\det(R) = \det(Q^\top RQ) &= \prod_{i=1}^{d} \lambda_i(Q^\top RQ) \overset{\text{(a)}}{\leq} \eta^{d-\mathsf{r}} \prod_{i=1}^{\mathsf{r}} \lambda_i(Q^\top RQ) \\ &\overset{\text{(b)}}{\leq} \eta^{d-\mathsf{r}} \left(\frac{1}{\mathsf{r}} \sum_{i=1}^{\mathsf{r}} \lambda_i(Q^\top RQ) \right)^{\mathsf{r}},\end{aligned} \tag{2.4}$$

where (a) holds because

(i) $\forall\, i = 1, 2, \ldots, d$, $\lambda_i(Q^\top RQ) = \lambda_i(R)$ since Q is orthogonal.

(ii) $R \in \mathcal{B}(\eta) \cap \mathcal{S}^d_+ \implies \|R\|_F = \sqrt{\operatorname{tr}(R^2)} \leq \eta \implies \forall\, i = 1, 2, \ldots, d,\ \lambda_i(R) \leq \eta.$

and (b) comes from the AM-GM inequality. Combining (2.4) with (2.3) gives

$$(\det(R))^{\frac{1}{d}} \leq \eta^{1-\frac{\mathsf{r}}{d}} \cdot \left(\frac{1}{\mathsf{r}} \sum_{i=1}^{\mathsf{r}} \lambda_i(Q^\top RQ) \right)^{\frac{\mathsf{r}}{d}} \leq \eta^{1-\frac{\mathsf{r}}{d}} \cdot \left(\frac{1}{\mathsf{r}\sigma_{\mathsf{r}}} \operatorname{tr}(RZ) \right)^{\frac{\mathsf{r}}{d}}$$

whenever $R \in \mathcal{B}(\eta) \cap \mathcal{S}^d_+$. Hence, we see that (2.2) holds with $C = \eta^{1-\frac{\mathsf{r}}{d}} (\mathsf{r}\sigma_{\mathsf{r}})^{-\frac{\mathsf{r}}{d}}$. □

2.2 Error bounds for conic feasibility problems

We first recall the definition of error bounds.

Definition 2.2 (Error bounds [38, 54]). *Suppose* (Feas) *is feasible. We say that* (Feas) *satisfies an error bound with a residual function* $r : \mathbb{R}_+ \to \mathbb{R}_+$ *if for every bounded set* $B \subseteq \mathcal{E}$*, there exists a constant* $c_B > 0$ *such that*

$$\operatorname{dist}(\boldsymbol{x}, \mathcal{K} \cap (\mathcal{L} + \boldsymbol{a})) \leq c_B r \left(\max \{\operatorname{dist}(\boldsymbol{x}, \mathcal{K}), \operatorname{dist}(\boldsymbol{x}, \mathcal{L} + \boldsymbol{a})\} \right) \qquad \forall \boldsymbol{x} \in B.$$

We remark that typically it is required that r satisfy $r(0) = 0$, be nondecreasing and be right-continuous at 0. Under these conditions, the error bound in Definition 2.2 can be understood in the context of *consistent error bound functions*; see [38, Definition 3.1]. Specifically, for $B_b = \mathcal{B}(b)$, if Definition 2.2 holds, then c_{B_b} can be taken to be a nondecreasing function of b (since considering a larger constant still preserves the error bound inequality). In this way, the function $\Phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ given by $\Phi(a, b) := c_{B_b} r(a)$ satisfies [38, Definition 3.1], provided that r has the aforementioned properties.

With different residual functions, we will have different error bounds, among which the Lipschitzian and Hölderian error bounds are most widely studied in the literature. Particularly, we say that (Feas) satisfies a *uniform Hölderian error bound* with exponent $\gamma \in (0, 1]$ if Definition 2.2 holds with $r = (\cdot)^\gamma$ for every bounded set B. That is, for every bounded set $B \subseteq \mathcal{E}$, there exists a constant $\kappa_B > 0$ such that

$$\operatorname{dist}(\boldsymbol{x}, \mathcal{K} \cap (\mathcal{L} + \boldsymbol{a})) \leq \kappa_B \max\left\{\operatorname{dist}(\boldsymbol{x}, \mathcal{K}), \operatorname{dist}(\boldsymbol{x}, \mathcal{L} + \boldsymbol{a})\right\}^\gamma,$$

for all $\boldsymbol{x} \in B$. If $\gamma = 1$, then the error bound is said to be *Lipschitzian*. Hölderian error bounds are a particular case of a consistent error bound, see [38, Theorem 3.5].

Let $\mathcal{K}$ be a closed convex cone contained in $\mathcal{E}$ and $\mathcal{K}^*$ be its dual cone. We will denote the boundary, relative interior, linear span, and dimension of $\mathcal{K}$ by $\partial\mathcal{K}, \operatorname{ri}\mathcal{K}, \operatorname{span}\mathcal{K}$ and $\dim\mathcal{K}$, respectively. If $\mathcal{K} \cap -\mathcal{K} = \{\mathbf{0}\}$, then $\mathcal{K}$ is said to be *pointed*. If $\mathcal{F} \subseteq \mathcal{K}$ is a face of $\mathcal{K}$, i.e., for any $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{K}$ such that $\boldsymbol{x} + \boldsymbol{y} \in \mathcal{F}$, we have $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{F}$, then we write $\mathcal{F} \trianglelefteq \mathcal{K}$.[3] If further $\mathcal{F} = \mathcal{K} \cap \{\boldsymbol{n}\}^\perp$ for some $\boldsymbol{n} \in \mathcal{K}^*$, we say that $\mathcal{F}$ is an *exposed face* of $\mathcal{K}$. A face $\mathcal{F}$ is said to be *proper* if $\mathcal{F} \neq \mathcal{K}$, and we denote it by $\mathcal{F} \subsetneq \mathcal{K}$. If $\mathcal{F}$ is proper and $\mathcal{F} \neq \mathcal{K} \cap -\mathcal{K}$, then $\mathcal{F}$ is said to be a *nontrivial* face of $\mathcal{K}$.

The *facial reduction* algorithm [9, 56, 70] and the *FRA-poly* algorithm [41] play

[3] By convention, we only consider nonempty faces.

important roles in making full use of the facial structure of a cone; see also [36, Section 3]. More precisely, assuming (Feas) is feasible, the facial reduction algorithm aims at finding the minimal face that contains the feasible region and satisfies some *constraint qualification*. One of the most commonly used constraint qualification is the so-called *partial-polyhedral Slater (PPS) condition* [40, Definition 3]. For (Feas), if $\mathcal{K}$ and $\mathcal{L}+\boldsymbol{a}$ satisfy the PPS condition, then a Lipschitzian error bound holds for $\mathcal{K}$ and $\mathcal{L}+\boldsymbol{a}$; see [7, Corollary 3] and the discussion preceding [36, Proposition 2.3]. Thanks to this property, we can apply the facial reduction algorithm to deduce the error bounds based on the *one-step facial residual function* [36, Definition 3.4] without requiring any constraint qualifications, as in the framework developed recently in [40]; see also [36, 37]. This framework is highly inspired by the fundamental work of Sturm on error bound for LMIs, see [62]. For the convenience of the reader, we recall the definition of the one-step facial residual function as follows.

Definition 2.3 (One-step facial residual function ($\mathbb{1}$-FRF))**.** *Let $\mathcal{K}$ be a closed convex cone and $\boldsymbol{n} \in \mathcal{K}^*$. Suppose that $\psi_{\mathcal{K},\boldsymbol{n}} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following properties:*

(i) $\psi_{\mathcal{K},\boldsymbol{n}}$ is nonnegative, nondecreasing in each argument and it holds that $\psi_{\mathcal{K},\boldsymbol{n}}(0,t) = 0$ for every $t \in \mathbb{R}_+$.

(ii) The following implication holds for any $\boldsymbol{x} \in \operatorname{span}\mathcal{K}$ and $\epsilon \geq 0$:

$$\operatorname{dist}(\boldsymbol{x},\mathcal{K}) \leq \epsilon, \langle \boldsymbol{x},\boldsymbol{n}\rangle \leq \epsilon \implies \operatorname{dist}(\boldsymbol{x},\mathcal{K}\cap\{\boldsymbol{n}\}^{\perp}) \leq \psi_{\mathcal{K},\boldsymbol{n}}(\epsilon,\|\boldsymbol{x}\|).$$

Then $\psi_{\mathcal{K},\boldsymbol{n}}$ is said to be a one-step facial residual function (FRF) for $\mathcal{K}$ and $\boldsymbol{n}$.

The one-step facial residual function is used in each step of the facial reduction algorithm to connect a face and its subface until a face $\mathcal{F}$ is found such that $\mathcal{F}$ and $\mathcal{L}+\boldsymbol{a}$ satisfy the PPS condition. Then the error bound for $\mathcal{K}$ and $\mathcal{L}+\boldsymbol{a}$ can be obtained as a special composition of those one-step facial residual functions. Due to the importance of the PPS condition in this framework, we shall define the *distance to the PPS condition* of a **feasible** (Feas), denoted by $d_{\rm PPS}(\mathcal{K}, \mathcal{L}+\boldsymbol{a})$, as the length *minus one* of the *shortest* chain of faces (among those chains constructed as in [40, Proposition 5]) such that the PPS condition holds for the final face in the chain and $\mathcal{L}+\boldsymbol{a}$.

Next we present a lemma and a proposition that will help simplify our subsequent analysis.

Lemma 2.4 (Formula of $\|\boldsymbol{w}-\boldsymbol{u}\|$)**.** *Let $\mathcal{K}$ be a closed convex cone and $\boldsymbol{n} \in \partial\mathcal{K}^* \setminus \{\boldsymbol{0}\}$ be such that $\mathcal{F} := \{\boldsymbol{n}\}^\perp \cap \mathcal{K}$ is a nontrivial exposed face of $\mathcal{K}$. Let $\eta > 0$ and let $\boldsymbol{v} \in \partial\mathcal{K} \cap \mathcal{B}(\eta) \setminus \mathcal{F}, \boldsymbol{w} = P_{\{\boldsymbol{n}\}^\perp}(\boldsymbol{v}), \boldsymbol{u} = P_{\mathcal{F}}(\boldsymbol{w})$ and $\boldsymbol{w} \neq \boldsymbol{u}$. Then, we have*

$$\|\boldsymbol{w}-\boldsymbol{u}\|^2 = \|\boldsymbol{v}-\boldsymbol{u}\|^2 - \|\boldsymbol{w}-\boldsymbol{v}\|^2, \tag{2.5}$$

and,

$$\|\boldsymbol{w}-\boldsymbol{u}\| \leq \|\boldsymbol{v}-\boldsymbol{u}\| = \operatorname{dist}(\boldsymbol{v}, \mathcal{F}). \tag{2.6}$$

Proof. Since $\boldsymbol{w} = P_{\{\boldsymbol{n}\}^\perp}(\boldsymbol{v})$, we have

$$\boldsymbol{w} = \boldsymbol{v} - \frac{\langle \boldsymbol{n}, \boldsymbol{v}\rangle}{\|\boldsymbol{n}\|^2}\boldsymbol{n} \quad \text{and} \quad \|\boldsymbol{w}-\boldsymbol{v}\| = \frac{|\langle \boldsymbol{n}, \boldsymbol{v}\rangle|}{\|\boldsymbol{n}\|}.$$

Moreover, we can notice that $\boldsymbol{w} \perp \boldsymbol{n}$ and $\boldsymbol{u} \perp \boldsymbol{n}$.

Now, for any $\widetilde{\boldsymbol{u}} \in \{\boldsymbol{n}\}^\perp$, we have

$$\begin{aligned}
\|\boldsymbol{w}-\widetilde{\boldsymbol{u}}\|^2 &= \left\|\boldsymbol{v} - \frac{\langle \boldsymbol{n}, \boldsymbol{v}\rangle}{\|\boldsymbol{n}\|^2}\boldsymbol{n} - \widetilde{\boldsymbol{u}}\right\|^2 = \left\|\boldsymbol{v} - \frac{\langle \boldsymbol{n}, \boldsymbol{v}\rangle}{\|\boldsymbol{n}\|^2}\boldsymbol{n}\right\|^2 - 2\left\langle \boldsymbol{v} - \frac{\langle \boldsymbol{n}, \boldsymbol{v}\rangle}{\|\boldsymbol{n}\|^2}\boldsymbol{n}, \widetilde{\boldsymbol{u}}\right\rangle + \|\widetilde{\boldsymbol{u}}\|^2 \\
&\overset{\text{(a)}}{=} \|\boldsymbol{v}\|^2 - 2\frac{\langle \boldsymbol{n}, \boldsymbol{v}\rangle}{\|\boldsymbol{n}\|^2}\langle \boldsymbol{n}, \boldsymbol{v}\rangle + \frac{\langle \boldsymbol{n}, \boldsymbol{v}\rangle^2}{\|\boldsymbol{n}\|^2} - 2\langle \boldsymbol{v}, \widetilde{\boldsymbol{u}}\rangle + \|\widetilde{\boldsymbol{u}}\|^2 = \|\boldsymbol{v}\|^2 - \frac{\langle \boldsymbol{n}, \boldsymbol{v}\rangle^2}{\|\boldsymbol{n}\|^2} - 2\langle \boldsymbol{v}, \widetilde{\boldsymbol{u}}\rangle + \|\widetilde{\boldsymbol{u}}\|^2 \\
&= \|\boldsymbol{v}-\widetilde{\boldsymbol{u}}\|^2 - \|\boldsymbol{w}-\boldsymbol{v}\|^2,
\end{aligned}$$

where (a) comes from the fact that $\widetilde{\boldsymbol{u}} \perp \boldsymbol{n}$. This proves (2.5) upon letting $\widetilde{\boldsymbol{u}} = \boldsymbol{u}$.

The above display implies that for any $\widetilde{\boldsymbol{u}} \in \{\boldsymbol{n}\}^\perp$, $\boldsymbol{v}, \boldsymbol{w}, \widetilde{\boldsymbol{u}}$ are three vertices of a right-angled triangle with $\angle \boldsymbol{w}$ being the right angle. This fact also leads us to the observation that $\boldsymbol{u} = P_{\mathcal{F}}(\boldsymbol{v})$. Indeed, suppose not, then there exists $\widehat{\boldsymbol{u}} = P_{\mathcal{F}}(\boldsymbol{v})$ with $\widehat{\boldsymbol{u}} \neq \boldsymbol{u}$ such that $\|\widehat{\boldsymbol{u}} - \boldsymbol{v}\| < \|\boldsymbol{u} - \boldsymbol{v}\|$. Then $\widehat{\boldsymbol{u}} \in \{\boldsymbol{n}\}^\perp$ and hence $\boldsymbol{v}, \boldsymbol{w}, \widehat{\boldsymbol{u}}$ form a new right-angled triangle. Thus,

$$\|\boldsymbol{w} - \widehat{\boldsymbol{u}}\|^2 = \|\boldsymbol{v} - \widehat{\boldsymbol{u}}\|^2 - \|\boldsymbol{w} - \boldsymbol{v}\|^2 < \|\boldsymbol{u} - \boldsymbol{v}\|^2 - \|\boldsymbol{w} - \boldsymbol{v}\|^2 = \|\boldsymbol{w} - \boldsymbol{u}\|^2.$$

Since $\widehat{\boldsymbol{u}} \in \mathcal{F}$, the above display contradicts the fact that $\boldsymbol{u} = P_{\mathcal{F}}(\boldsymbol{w})$. Therefore, $\boldsymbol{u} = P_{\mathcal{F}}(\boldsymbol{v})$ and $\|\boldsymbol{u} - \boldsymbol{v}\| = \operatorname{dist}(\boldsymbol{v}, \mathcal{F})$. □

The next proposition states an error bound result related to the positive semidefinite cone. We present a proof based on the results in [40], although it can also be obtained from Sturm's error bound in [62].

Proposition 2.5 (Error bound for positive semidefinite cones)**.** *Let $Z \in \mathcal{S}^d_+ \setminus \{\mathbf{0}\}$ and $\eta > 0$, then there exists $C_P > 0$ such that*

$$\operatorname{dist}(Y, \mathcal{S}^d_+ \cap \{Z\}^\perp) \leq C_P \operatorname{tr}(YZ)^\alpha \qquad \forall Y \in \mathcal{S}^d_+ \cap \mathcal{B}(\eta), \tag{2.7}$$

where

$$\alpha := \begin{cases} \frac{1}{2} & \text{if } \mathsf{r}(Z) < d, \\ 1 & \text{otherwise.} \end{cases} \tag{2.8}$$

Proof. By [40, Proposition 27, Theorem 37], there exists $C_0 > 0$ such that

$$\operatorname{dist}(Y, \mathcal{S}^d_+ \cap \{Z\}^\perp) \leq C_0 \max\{\operatorname{dist}(Y, \mathcal{S}^d_+), \operatorname{dist}(Y, \{Z\}^\perp)\}^\alpha \quad \text{whenever } Y \in \mathcal{B}(\eta),$$

where α is defined as in (2.8).

If further $Y \in \mathcal{S}^d_+$, then $\operatorname{dist}(Y, \mathcal{S}^d_+) = 0$; moreover, $\operatorname{dist}(Y, \{Z\}^\perp) = \frac{|\operatorname{tr}(YZ)|}{\|Z\|_F} = \frac{\operatorname{tr}(YZ)}{\|Z\|_F}$. Therefore, letting $C_P := C_0/\|Z\|_F^\alpha$, we can obtain (2.7). □

We end this section with the following lemma, which is useful in the analysis of one-dimensional faces. It will be used repeatedly in our subsequent discussions.

Lemma 2.6 ([37, Lemma 2.5])**.** *Let* $\mathcal{K}$ *be a pointed closed convex cone and let* $\boldsymbol{z} \in \partial\mathcal{K}^* \setminus \{\boldsymbol{0}\}$ *be such that* $\mathcal{F} := \{\boldsymbol{z}\}^\perp \cap \mathcal{K}$ *is a one-dimensional proper face of* $\mathcal{K}$*. Let* $\boldsymbol{f} \in \mathcal{K} \setminus \{\boldsymbol{0}\}$ *be such that* $\mathcal{F} = \{t\boldsymbol{f} \,|\, t \geq 0\}$*. Let* $\eta > 0$ *and* $\boldsymbol{v} \in \partial\mathcal{K} \cap \mathcal{B}(\eta) \setminus \mathcal{F}$*,* $\boldsymbol{w} = P_{\{\boldsymbol{z}\}^\perp}(\boldsymbol{v})$ *and* $\boldsymbol{u} = P_{\mathcal{F}}(\boldsymbol{w})$ *with* $\boldsymbol{u} \neq \boldsymbol{w}$*. Then it holds that* $\langle \boldsymbol{f}, \boldsymbol{z} \rangle = 0$ *and we have*

$$\|\boldsymbol{v} - \boldsymbol{w}\| = \frac{|\,\langle \boldsymbol{z}, \boldsymbol{v} \rangle\,|}{\|\boldsymbol{z}\|}, \qquad \|\boldsymbol{u} - \boldsymbol{w}\| = \begin{cases} \left\| \boldsymbol{v} - \frac{\langle \boldsymbol{z}, \boldsymbol{v} \rangle}{\|\boldsymbol{z}\|^2}\boldsymbol{z} - \frac{\langle \boldsymbol{f}, \boldsymbol{v} \rangle}{\|\boldsymbol{f}\|^2}\boldsymbol{f} \right\| & \text{if } \langle \boldsymbol{f}, \boldsymbol{v} \rangle \geq 0, \\ \left\| \boldsymbol{v} - \frac{\langle \boldsymbol{z}, \boldsymbol{v} \rangle}{\|\boldsymbol{z}\|^2}\boldsymbol{z} \right\| & \text{otherwise .} \end{cases}$$

Moreover, when $\langle \boldsymbol{f}, \boldsymbol{v} \rangle \geq 0$ *(or, equivalently,* $\langle \boldsymbol{f}, \boldsymbol{w} \rangle \geq 0$*), we have* $\boldsymbol{u} = P_{\operatorname{span}\mathcal{F}}(\boldsymbol{w})$*. On the other hand, if* $\langle \boldsymbol{f}, \boldsymbol{v} \rangle < 0$*, we have* $\boldsymbol{u} = \boldsymbol{0}$*.*

Chapter 3

Generalized Power Cones

We consider the generalized power cone and its dual. Let $m \geq 1$, $n \geq 2$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in (0,1)^n$ with $\sum_{i=1}^n \alpha_i = 1$, the generalized power cone $\mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$ and its dual $(\mathcal{P}_{m,n}^{\boldsymbol{\alpha}})^*$ are given respectively by

$$
\begin{aligned}
\mathcal{P}_{m,n}^{\boldsymbol{\alpha}} &= \left\{ \boldsymbol{x} = (\overline{\boldsymbol{x}}, \widetilde{\boldsymbol{x}}) \in \mathbb{R}^{m+n} \;\middle|\; \|\overline{\boldsymbol{x}}\| \leq \prod_{i=1}^n \widetilde{x}_i^{\alpha_i},\ \overline{\boldsymbol{x}} \in \mathbb{R}^m,\ \widetilde{\boldsymbol{x}} \in \mathbb{R}^n_+ \right\}, \\
(\mathcal{P}_{m,n}^{\boldsymbol{\alpha}})^* &= \left\{ \boldsymbol{z} = (\overline{\boldsymbol{z}}, \widetilde{\boldsymbol{z}}) \in \mathbb{R}^{m+n} \;\middle|\; \|\overline{\boldsymbol{z}}\| \leq \prod_{i=1}^n \left(\frac{\widetilde{z}_i}{\alpha_i}\right)^{\alpha_i},\ \overline{\boldsymbol{z}} \in \mathbb{R}^m,\ \widetilde{\boldsymbol{z}} \in \mathbb{R}^n_+ \right\}.
\end{aligned}
\tag{3.1}
$$

Here, given a vector $\boldsymbol{x} \in \mathbb{R}^{m+n}$, we let $\overline{\boldsymbol{x}} \in \mathbb{R}^m$ be the vector corresponding to its first m entries and $\widetilde{\boldsymbol{x}} \in \mathbb{R}^n$ be the vector corresponding to its last n entries.

In this section, we will prove the main result of this chapter: a complete analysis of the error bounds of $\mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$. This will require an analysis of the facial structure of $\mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$ which we will do shortly after the following lemmas.

Lemma 3.1. *Let $n \geq 2$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in (0,1)^n$ with $\sum_{i=1}^n \alpha_i = 1$. Let $\boldsymbol{\zeta} \in \operatorname{int} \mathbb{R}^n_-$ satisfy $\prod_{i=1}^n (-\zeta_i/\alpha_i)^{\alpha_i} = 1$. Define $\widetilde{\boldsymbol{\zeta}} := -\boldsymbol{\alpha} \circ \boldsymbol{\zeta}^{-1}$, where $\circ$ is the Hadamard product and the inverse is taken componentwise. Then there exist $C > 0$ and $\epsilon > 0$ so that*

$$
-1 - \langle \boldsymbol{\zeta}, \boldsymbol{\omega} \rangle \geq C \|\boldsymbol{\omega} - \widetilde{\boldsymbol{\zeta}}\|^2 \text{ whenever } \boldsymbol{\omega} \in \operatorname{int} \mathbb{R}^n_+,\ \|\boldsymbol{\omega} - \widetilde{\boldsymbol{\zeta}}\| \leq \epsilon \text{ and } \prod_{i=1}^n \omega_i^{\alpha_i} = 1. \tag{3.2}
$$

Moreover, for any $\boldsymbol{\omega} \in \operatorname{int} \mathbb{R}^n_+$ *satisfying* $\prod_{i=1}^n \omega_i^{\alpha_i} = 1$*, it holds that* $\langle \boldsymbol{\zeta}, \boldsymbol{\omega} \rangle \leq -1$*; furthermore, we have* $\langle \boldsymbol{\zeta}, \boldsymbol{\omega} \rangle = -1$ *if and only if* $\boldsymbol{\omega} = \widetilde{\boldsymbol{\zeta}}$.

Proof. For each i, we see from the Taylor series of $\ln(\cdot)$ at $\widetilde{\zeta}_i > 0$ that

$$\ln(\omega_i) = \ln(\widetilde{\zeta}_i) + \widetilde{\zeta}_i^{-1}(\omega_i - \widetilde{\zeta}_i) - \widetilde{\zeta}_i^{-2}(\omega_i - \widetilde{\zeta}_i)^2 + O(|\omega_i - \widetilde{\zeta}_i|^3) \text{ as } \omega_i \to \widetilde{\zeta}_i,\ \omega_i > 0.$$

Thus, there exist $\epsilon_i > 0$ and $c_i > 0$ so that

$$\ln(\widetilde{\zeta}_i) \geq \ln(\omega_i) - \widetilde{\zeta}_i^{-1}(\omega_i - \widetilde{\zeta}_i) + c_i(\omega_i - \widetilde{\zeta}_i)^2 \text{ whenever } |\omega_i - \widetilde{\zeta}_i| \leq \epsilon_i \text{ and } \omega_i > 0.$$

Let $\epsilon := \min_{1 \leq i \leq n} \epsilon_i > 0$. Multiplying both sides of the above inequality by α_i and summing the resulting inequalities from $i = 1$ to n, we see that whenever $\boldsymbol{\omega} \in \operatorname{int} \mathbb{R}^n_+$ satisfies $\|\boldsymbol{\omega} - \widetilde{\boldsymbol{\zeta}}\| \leq \epsilon$ and $\prod_{i=1}^n \omega_i^{\alpha_i} = 1$, we have

$$\begin{aligned}
0 &\overset{\text{(a)}}{=} \sum_{i=1}^n \alpha_i \ln(\widetilde{\zeta}_i) \geq \sum_{i=1}^n \alpha_i \ln(\omega_i) - \sum_{i=1}^n \alpha_i \widetilde{\zeta}_i^{-1}(\omega_i - \widetilde{\zeta}_i) + \sum_{i=1}^n \alpha_i c_i (\omega_i - \widetilde{\zeta}_i)^2 \\
&\overset{\text{(b)}}{=} -\sum_{i=1}^n \alpha_i \widetilde{\zeta}_i^{-1}(\omega_i - \widetilde{\zeta}_i) + \sum_{i=1}^n \alpha_i c_i (\omega_i - \widetilde{\zeta}_i)^2 \overset{\text{(c)}}{=} -\sum_{i=1}^n \alpha_i \widetilde{\zeta}_i^{-1}\omega_i + 1 + \sum_{i=1}^n \alpha_i c_i (\omega_i - \widetilde{\zeta}_i)^2 \\
&= \sum_{i=1}^n \zeta_i \omega_i + 1 + \sum_{i=1}^n \alpha_i c_i (\omega_i - \widetilde{\zeta}_i)^2,
\end{aligned}$$

where (a) and (b) hold because $\prod_{i=1}^n \widetilde{\zeta}_i^{\alpha_i} = \prod_{i=1}^n \omega_i^{\alpha_i} = 1$, (c) uses the fact that $\sum_{i=1}^n \alpha_i = 1$, and the last equality follows from the definition of $\widetilde{\boldsymbol{\zeta}}$. Rearranging the above inequality, we conclude that (3.2) holds with $C = \min_{1 \leq i \leq n} \alpha_i c_i > 0$.

Next, let $\boldsymbol{\omega} \in \operatorname{int} \mathbb{R}^n_+$ satisfy $\prod_{i=1}^n \omega_i^{\alpha_i} = 1$. Then $(-1, \boldsymbol{\omega}) \in \mathcal{P}^{\boldsymbol{\alpha}}_{1,n}$. Recall from the assumption that $(1, -\boldsymbol{\zeta}) \in (\mathcal{P}^{\boldsymbol{\alpha}}_{1,n})^*$. From these we deduce $\langle \boldsymbol{\zeta}, \boldsymbol{\omega} \rangle \leq -1$. If $\langle \boldsymbol{\zeta}, \boldsymbol{\omega} \rangle = -1$, then

$$\sum_{i=1}^n \alpha_i \left(\frac{-\zeta_i}{\alpha_i}\right) \omega_i = \sum_{i=1}^n (-\zeta_i)\omega_i = 1 = \prod_{i=1}^n \omega_i^{\alpha_i} = \prod_{i=1}^n \left(\frac{-\zeta_i}{\alpha_i}\right)^{\alpha_i} \prod_{i=1}^n \omega_i^{\alpha_i}.$$

Taking ln on both sides of the above equality, we see that

$$\ln\left[\sum_{i=1}^{n} \alpha_i \left(\frac{-\zeta_i}{\alpha_i}\right) \omega_i\right] = \sum_{i=1}^{n} \alpha_i \ln\left[\left(\frac{-\zeta_i}{\alpha_i}\right) \omega_i\right].$$

Since ln is strictly concave and $\alpha_i \in (0,1)$ for all i, we conclude that there exists $c > 0$ so that $\omega_i \cdot (-\zeta_i/\alpha_i) = c$ for all i. This, together with the facts that $\prod_{i=1}^n \omega_i^{\alpha_i} = \prod_{i=1}^n (-\zeta_i/\alpha_i)^{\alpha_i} = 1$ and $\sum_{i=1}^n \alpha_i = 1$, gives $c = 1$. It thus follows that $\boldsymbol{\omega} = \widetilde{\boldsymbol{\zeta}}$. Conversely, it is routine to check that if $\boldsymbol{\omega} = \widetilde{\boldsymbol{\zeta}}$, then $\prod_{i=1}^n \omega_i^{\alpha_i} = 1$ and $\langle \boldsymbol{\zeta}, \boldsymbol{\omega} \rangle = -1$. □

The next lemma is obtained by applying [37, Lemma 4.1] with $p = q = 2$.

Lemma 3.2. *Let $\boldsymbol{\zeta} \in \mathbb{R}^n$ $(n \geq 1)$ satisfy $\|\boldsymbol{\zeta}\| = 1$. Define $\overline{\boldsymbol{\zeta}} := -\boldsymbol{\zeta}$. Then there exist $C > 0$ and $\epsilon > 0$ so that*

$$1 + \langle \boldsymbol{\zeta}, \boldsymbol{w} \rangle \geq C \sum_{i \in I} |w_i - \overline{\zeta}_i|^2 + \frac{1}{2} \sum_{i \notin I} |w_i|^2 \quad \textit{whenever } \|\boldsymbol{w} - \overline{\boldsymbol{\zeta}}\| \leq \epsilon \quad \textit{and} \quad \|\boldsymbol{w}\| = 1, \tag{3.3}$$

where $I = \{i \,|\, \overline{\zeta}_i \neq 0\}$. Furthermore, for any $\boldsymbol{w}$ satisfying $\|\boldsymbol{w}\| \leq 1$, it holds that $\langle \boldsymbol{\zeta}, \boldsymbol{w} \rangle \geq -1$, with the equality holding if and only if $\boldsymbol{w} = \overline{\boldsymbol{\zeta}}$.

3.1 The facial structure of $\mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$

In this section, we discuss the faces of $\mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$. We first characterize the proper nontrivial exposed faces of $\mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$ in the following proposition.

Proposition 3.3 (Proper nontrivial exposed faces of $\mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$)**.** *Let $\boldsymbol{z} = (\overline{\boldsymbol{z}}, \widetilde{\boldsymbol{z}}) \in \partial(\mathcal{P}_{m,n}^{\boldsymbol{\alpha}})^* \backslash \{\boldsymbol{0}\}$.*

(i) If $\overline{\boldsymbol{z}} \neq \boldsymbol{0}$, then $\boldsymbol{z}$ exposes the following one-dimensional face:

$$\mathcal{F}_{\rm r} := \{\boldsymbol{z}\}^\perp \cap \mathcal{P}_{m,n}^{\boldsymbol{\alpha}} = \{t\boldsymbol{f} \in \mathbb{R}^{m+n} \,|\, t \geq 0\} \textit{ with } \boldsymbol{f} = (-\overline{\boldsymbol{z}}/\|\overline{\boldsymbol{z}}\|^2, \boldsymbol{\alpha} \circ \widetilde{\boldsymbol{z}}^{-1}), \tag{3.4}$$

where the inverse is taken componentwise.

(ii) If $\overline{\boldsymbol{z}} = \mathbf{0}$, then $\boldsymbol{z}$ exposes the following face of dimension $n - |\mathcal{I}|$:

$$\mathcal{F}_{\boldsymbol{z}} := \{\boldsymbol{z}\}^{\perp} \cap \mathcal{P}_{m,n}^{\boldsymbol{\alpha}} = \{\boldsymbol{x} = (\overline{\boldsymbol{x}}, \widetilde{\boldsymbol{x}}) \in \mathbb{R}_{+}^{m+n} \,|\, \overline{\boldsymbol{x}} = \mathbf{0},\, \widetilde{x}_i = 0 \ \forall i \in \mathcal{I}\}, \tag{3.5}$$

where $\mathcal{I} := \{i \,|\, \widetilde{z}_i > 0\} \neq \emptyset$ and $|\mathcal{I}|$ denotes the cardinality of $\mathcal{I}$.

Proof. (i): Notice that $\boldsymbol{x} = (\overline{\boldsymbol{x}}, \widetilde{\boldsymbol{x}}) \in \{\boldsymbol{z}\}^{\perp} \cap \mathcal{P}_{m,n}^{\boldsymbol{\alpha}} \backslash \{\mathbf{0}\}$ if and only if $\boldsymbol{x} \in \partial \mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$, $\boldsymbol{x} \neq \mathbf{0}$ and

$$\langle \overline{\boldsymbol{z}}, \overline{\boldsymbol{x}} \rangle + \langle \widetilde{\boldsymbol{z}}, \widetilde{\boldsymbol{x}} \rangle = 0. \tag{3.6}$$

The above relation yields

$$\sum_{i=1}^{n} \widetilde{z}_i \widetilde{x}_i = -\langle \overline{\boldsymbol{z}}, \overline{\boldsymbol{x}} \rangle \leq \|\overline{\boldsymbol{z}}\| \|\overline{\boldsymbol{x}}\| \leq \prod_{i=1}^{n} \left(\frac{\widetilde{x}_i \widetilde{z}_i}{\alpha_i} \right)^{\alpha_i}, \tag{3.7}$$

where the last inequality follows from the definition of $\mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$ in (3.1).

Note that $\widetilde{x}_i$ cannot be **all** zero, for otherwise $\overline{\boldsymbol{x}}$ will also be zero since $\boldsymbol{x} \in \partial \mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$, which contradicts $\boldsymbol{x} \neq \mathbf{0}$. In addition, we must have $\widetilde{z}_i > 0$ for all i because $\overline{\boldsymbol{z}} \neq \mathbf{0}$ and $\boldsymbol{z} \in \partial (\mathcal{P}_{m,n}^{\boldsymbol{\alpha}})^* \backslash \{\mathbf{0}\}$. Using these observations, we have $\sum_{i=1}^{n} \widetilde{z}_i \widetilde{x}_i > 0$. Combining this with (3.7), we deduce that $\widetilde{x}_i \widetilde{z}_i > 0$ for all i. Now we can take ln on both sides of (3.7) to obtain

$$\ln \left[\sum_{i=1}^{n} \alpha_i \left(\frac{\widetilde{x}_i \widetilde{z}_i}{\alpha_i} \right) \right] \leq \alpha_1 \ln \left(\frac{\widetilde{x}_1 \widetilde{z}_1}{\alpha_1} \right) + \cdots + \alpha_n \ln \left(\frac{\widetilde{x}_n \widetilde{z}_n}{\alpha_n} \right). \tag{3.8}$$

Using this together with the fact that $\ln(\cdot)$ is strictly concave, we deduce that (3.8) holds as an equality. Hence, there exists a constant $c > 0$ so that

$$\widetilde{x}_i = c \alpha_i \widetilde{z}_i^{-1} \ \ \forall i = 1, 2, \ldots, n. \tag{3.9}$$

Plugging (3.9) into (3.6), we obtain

$$\langle \overline{\boldsymbol{z}}, \overline{\boldsymbol{x}} \rangle = -\langle \widetilde{\boldsymbol{z}}, \widetilde{\boldsymbol{x}} \rangle = -c \sum_{i=1}^{n} \alpha_i = -c. \tag{3.10}$$

Moreover, using (3.9) and the last relation in (3.7), we see that

$$\|\overline{\boldsymbol{z}}\|\|\overline{\boldsymbol{x}}\| \leq \prod_{i=1}^{n} \left(\frac{\widetilde{x}_i \widetilde{z}_i}{\alpha_i}\right)^{\alpha_i} = c.$$

The two displayed lines above show that $\|\overline{\boldsymbol{z}}\|\|\overline{\boldsymbol{x}}\| = -\langle \overline{\boldsymbol{z}}, \overline{\boldsymbol{x}} \rangle$, which together with $\overline{\boldsymbol{z}} \neq \boldsymbol{0}$ implies that there exists $\kappa > 0$ so that

$$\overline{\boldsymbol{x}} = -\kappa \overline{\boldsymbol{z}}. \tag{3.11}$$

Plugging (3.11) into (3.10), we obtain that $\kappa = c/\|\overline{\boldsymbol{z}}\|^2$. Using this together with (3.9) and (3.11), we can now conclude that

$$\mathcal{F}_{\mathrm{r}} := \{\boldsymbol{z}\}^{\perp} \cap \mathcal{P}_{m,n}^{\boldsymbol{\alpha}} = \{t\boldsymbol{f} \in \mathrm{I\!R}^{m+n} \,|\, t \geq 0\} \text{ with } \boldsymbol{f} = (-\overline{\boldsymbol{z}}/\|\overline{\boldsymbol{z}}\|^2, \boldsymbol{\alpha} \circ \widetilde{\boldsymbol{z}}^{-1}),$$

where the inverse is taken componentwise.

(ii): In this case, $\overline{\boldsymbol{z}} = \boldsymbol{0}$. Then $\mathcal{I} := \{i \,|\, \widetilde{z}_i > 0\}$ is nonempty because $\boldsymbol{z} \neq \boldsymbol{0}$. Hence, $\boldsymbol{x} = (\overline{\boldsymbol{x}}, \widetilde{\boldsymbol{x}}) \in \{\boldsymbol{z}\}^{\perp} \cap \mathcal{P}_{m,n}^{\boldsymbol{\alpha}} \backslash \{\boldsymbol{0}\}$ if and only if $\boldsymbol{x} \in \partial \mathcal{P}_{m,n}^{\boldsymbol{\alpha}} \backslash \{\boldsymbol{0}\}$ and satisfies

$$\sum_{i \in \mathcal{I}} \widetilde{z}_i \widetilde{x}_i = 0.$$

This means that $\widetilde{x}_i = 0$ whenever $i \in \mathcal{I}$ and hence $\overline{\boldsymbol{x}} = \boldsymbol{0}$. Thus,

$$\mathcal{F}_{\boldsymbol{z}} := \{\boldsymbol{z}\}^{\perp} \cap \mathcal{P}_{m,n}^{\boldsymbol{\alpha}} = \{\boldsymbol{x} = (\overline{\boldsymbol{x}}, \widetilde{\boldsymbol{x}}) \in \mathrm{I\!R}_{+}^{m+n} \,|\, \overline{\boldsymbol{x}} = \boldsymbol{0},\, \widetilde{x}_i = 0 \ \forall i \in \mathcal{I}\}.$$

□

Having characterized the proper exposed faces of $\mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$, we will show that $\mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$ is *projectionally exposed* [9, 63], which means that for every face $\mathcal{F}$ of $\mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$ there is a linear operator $\boldsymbol{P}$ satisfying $\boldsymbol{P}(\mathcal{P}_{m,n}^{\boldsymbol{\alpha}}) = \mathcal{F}$ and $\boldsymbol{P}^2 = \boldsymbol{P}$. In particular, $\boldsymbol{P}$, which depends on $\mathcal{F}$, is a projection that is not necessarily orthogonal. Projectionally exposed cones are both facially exposed [63, Corollary 4.4] and amenable [40, Proposition 9], see also [42].

Proposition 3.4 (Generalized power cones are projectionally exposed)**.** $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ *is projectionally exposed, in particular, all its faces are exposed.*

Proof. Sung and Tam proved in [63, Corollary 4.5] that a sufficient condition for a cone to be projectionally exposed is that all its exposed faces are projectionally exposed. With this in mind, let $\mathcal{F}$ be an exposed face of $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$. If $\mathcal{F} = \{\mathbf{0}\}$ or $\mathcal{F} = \mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$, then the zero map and the identity map are, respectively, projections mapping $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ to $\mathcal{F}$. Otherwise, $\mathcal{F}$ is a nonzero proper face of $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ and is of the form $\{\boldsymbol{z}\}^\perp \cap \mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$, for some $\boldsymbol{z} = (\overline{\boldsymbol{z}}, \widetilde{\boldsymbol{z}}) \in \partial(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})^*\backslash\{\mathbf{0}\}$. By the analysis in cases (i), (ii), we only need to consider two cases.

First, suppose that $\mathcal{F}$ is a one-dimensional face as in (3.4) and let $\boldsymbol{u} \in (\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})^*$ be such that $\langle \boldsymbol{f}, \boldsymbol{u}\rangle = 1$. At least one such $\boldsymbol{u}$ exists, since otherwise we would have $\boldsymbol{f} \in ((\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})^*)^\perp = \{\mathbf{0}\}$. Then, $\boldsymbol{P} = \boldsymbol{f}\boldsymbol{u}^\top$ satisfies $\boldsymbol{P}^2 = \boldsymbol{P}$ and $\boldsymbol{P}(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}) = \mathcal{F}$ as required.

Next, suppose that $\mathcal{F}$ is as in (3.5). Then, we let $\boldsymbol{P}$ be the linear map that maps $(\overline{\boldsymbol{x}}, \widetilde{\boldsymbol{x}})$ to $(\mathbf{0}, \widetilde{\boldsymbol{y}})$ where $\widetilde{y}_i = 0$ if $i \in \mathcal{I}$ and $\widetilde{y}_i = \widetilde{x}_i$ if $i \notin \mathcal{I}$. With that, $\boldsymbol{P}$ is a projection mapping $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ to $\mathcal{F}$. □

3.2 Deducing error bounds and one-step facial residual functions for $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$

We start with the faces $\mathcal{F}_{\rm r}$ that correspond to a $\boldsymbol{z} \in \partial(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})^*\backslash\{\mathbf{0}\}$ with $\overline{\boldsymbol{z}} \neq \mathbf{0}$. We have the following result.

Theorem 3.5. *Let* $\boldsymbol{z} = (\overline{\boldsymbol{z}}, \widetilde{\boldsymbol{z}}) \in \partial(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})^*\backslash\{\mathbf{0}\}$ *with* $\overline{\boldsymbol{z}} \neq \mathbf{0}$ *and let* $\mathcal{F}_{\rm r} := \{\boldsymbol{z}\}^\perp \cap \mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$. *Let* $\eta > 0$ *and define*

$$\gamma_{\boldsymbol{z},\eta} := \inf_{\boldsymbol{v}} \left\{ \frac{\|\boldsymbol{v}-\boldsymbol{w}\|^{\frac{1}{2}}}{\|\boldsymbol{u}-\boldsymbol{w}\|} \;\middle|\; \begin{array}{c} \boldsymbol{v} \in \partial\mathcal{P}^{\boldsymbol{\alpha}}_{m,n} \cap \mathcal{B}(\eta)\backslash\mathcal{F}_{\rm r},\ \boldsymbol{w} = P_{\{\boldsymbol{z}\}^\perp}(\boldsymbol{v}), \\ \boldsymbol{u} = P_{\mathcal{F}_{\rm r}}(\boldsymbol{w}),\ \boldsymbol{u} \neq \boldsymbol{w} \end{array} \right\}. \tag{3.12}$$

Then it holds that $\gamma_{\boldsymbol{z},\eta} \in (0,\infty]$ and that

$$\operatorname{dist}(\boldsymbol{q},\mathcal{F}_{\rm r}) \le \max\{2\sqrt{\eta}, 2\gamma_{\boldsymbol{z},\eta}^{-1}\} \cdot \operatorname{dist}(\boldsymbol{q},\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})^{\frac{1}{2}} \quad \text{whenever } \boldsymbol{q} \in \{\boldsymbol{z}\}^\perp \cap \mathcal{B}(\eta).$$

Proof. Suppose for a contradiction that $\gamma_{\boldsymbol{z},\eta} = 0$. Then, in view of [36, Lemma 3.12], there exist $\widehat{\boldsymbol{v}} \in \mathcal{F}_{\rm r}$ and a sequence $\{\boldsymbol{v}^k\} \subset \partial\mathcal{P}^{\boldsymbol{\alpha}}_{m,n} \cap \mathcal{B}(\eta)\backslash\mathcal{F}_{\rm r}$ such that

$$\lim_{k\to\infty} \boldsymbol{v}^k = \lim_{k\to\infty} \boldsymbol{w}^k = \widehat{\boldsymbol{v}} \text{ and } \lim_{k\to\infty} \frac{\|\boldsymbol{w}^k - \boldsymbol{v}^k\|^{\frac{1}{2}}}{\|\boldsymbol{w}^k - \boldsymbol{u}^k\|} = 0, \tag{3.13}$$

where $\boldsymbol{w}^k = P_{\{\boldsymbol{z}\}^\perp}(\boldsymbol{v}^k)$, $\boldsymbol{u}^k = P_{\mathcal{F}_{\rm r}}(\boldsymbol{w}^k)$ and $\boldsymbol{u}^k \neq \boldsymbol{w}^k$.

Define, for notational simplicity, $z_0 := \|\overline{\boldsymbol{z}}\|$ and $v_0^k := \|\overline{\boldsymbol{v}}^k\|$. Then, since $\{\boldsymbol{v}^k\} \subset \partial\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ and $\boldsymbol{z} \in \partial(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})^*$ with $\overline{\boldsymbol{z}} \neq \boldsymbol{0}$, we have

$$z_0 = \|\overline{\boldsymbol{z}}\| = \prod_{i=1}^n \left(\frac{\widetilde{z}_i}{\alpha_i}\right)^{\alpha_i} > 0 \text{ and } v_0^k = \|\overline{\boldsymbol{v}}^k\| = \prod_{i=1}^n (\widetilde{v}_i^k)^{\alpha_i} \ \forall k. \tag{3.14}$$

If it holds that $v_0^k = 0$ infinitely often, by passing to a further subsequence, we may assume that $v_0^k = 0$ for all k. Then we have in view of Lemma 2.6 that

$$\|\boldsymbol{v}^k - \boldsymbol{w}^k\| = \frac{1}{\|\boldsymbol{z}\|}|\langle\widetilde{\boldsymbol{z}}, \widetilde{\boldsymbol{v}}^k\rangle| \overset{\rm (a)}{=} \frac{1}{\|\boldsymbol{z}\|}\sum_{i=1}^n \widetilde{z}_i\widetilde{v}_i^k \ge \frac{\min_i \widetilde{z}_i}{\|\boldsymbol{z}\|}\|\widetilde{\boldsymbol{v}}^k\|_1 \ge \frac{\min_i \widetilde{z}_i}{\|\boldsymbol{z}\|}\|\widetilde{\boldsymbol{v}}^k\| \overset{\rm (b)}{=} \frac{\min_i \widetilde{z}_i}{\|\boldsymbol{z}\|}\|\boldsymbol{v}^k\|,$$

where (a) holds because $\widetilde{v}_i^k \ge 0$ and $\widetilde{z}_i > 0$ for all i (see (3.14)), and (b) holds since $\|\overline{\boldsymbol{v}}^k\| = 0$. Since $\|\boldsymbol{w}^k - \boldsymbol{u}^k\| = \operatorname{dist}(\boldsymbol{w}^k, \mathcal{F}_{\rm r}) \le \|\boldsymbol{w}^k\| \le \|\boldsymbol{v}^k\|$ as a consequence of the properties of projections, we conclude from this and the above display that $\|\boldsymbol{v}^k - \boldsymbol{w}^k\| \ge \frac{\min_i \widetilde{z}_i}{\|\boldsymbol{z}\|}\|\boldsymbol{w}^k - \boldsymbol{u}^k\|$, contradicting (3.13).

Thus, by considering a further subsequence if necessary, from now on, we assume

$$v_0^k = \|\overline{\boldsymbol{v}}^k\| = \prod_{i=1}^n (\widetilde{v}_i^k)^{\alpha_i} > 0 \ \forall k. \tag{3.15}$$

Using Lemma 2.6, we see that

$$
\begin{aligned}
\|\boldsymbol{v}^k - \boldsymbol{w}^k\| &= \frac{1}{\|\boldsymbol{z}\|}|\langle \boldsymbol{z}, \boldsymbol{v}^k\rangle| = \frac{1}{\|\boldsymbol{z}\|}\left|\sum_{i=1}^m \overline{z}_i \overline{v}_i^k + \sum_{i=1}^n \widetilde{z}_i \widetilde{v}_i^k\right| \\
&= \frac{1}{\|\boldsymbol{z}\|}\left|z_0 v_0^k + \sum_{i=1}^m \overline{z}_i \overline{v}_i^k - \sum_{i=1}^n (-\widetilde{z}_i)\widetilde{v}_i^k - z_0 v_0^k\right| \\
&= \frac{z_0 v_0^k}{\|\boldsymbol{z}\|}\left|1 + \langle z_0^{-1}\overline{\boldsymbol{z}}, (v_0^k)^{-1}\overline{\boldsymbol{v}}^k\rangle - \langle z_0^{-1}(-\widetilde{\boldsymbol{z}}), (v_0^k)^{-1}\widetilde{\boldsymbol{v}}^k\rangle - 1\right| \\
&= \frac{z_0}{\|\boldsymbol{z}\|}\left(1 + \langle z_0^{-1}\overline{\boldsymbol{z}}, (v_0^k)^{-1}\overline{\boldsymbol{v}}^k\rangle - \langle z_0^{-1}(-\widetilde{\boldsymbol{z}}), (v_0^k)^{-1}\widetilde{\boldsymbol{v}}^k\rangle - 1\right) v_0^k,
\end{aligned} \tag{3.16}
$$

where the last equality holds as $\|z_0^{-1}\overline{\boldsymbol{z}}\| = 1$, $\|(v_0^k)^{-1}\overline{\boldsymbol{v}}^k\| = 1$ and $\langle z_0^{-1}\widetilde{\boldsymbol{z}}, (v_0^k)^{-1}\widetilde{\boldsymbol{v}}^k\rangle \geq 1$, thanks to (3.14), (3.15) and Lemma 3.1 applied with $\boldsymbol{\zeta} = -z_0^{-1}\widetilde{\boldsymbol{z}}$.

Let $\boldsymbol{f}$ be defined as in (3.4). We consider two cases:

(I) $\langle \boldsymbol{f}, \boldsymbol{v}^k\rangle \geq 0$ for all sufficiently large k.

(II) $\langle \boldsymbol{f}, \boldsymbol{v}^k\rangle < 0$ infinitely often.

(I): By passing to a further subsequence, we may assume that $\langle \boldsymbol{f}, \boldsymbol{v}^k\rangle \geq 0$ for all k. In this case, if we define

$$
\boldsymbol{Q} = \boldsymbol{I}_{m+n} - \frac{\boldsymbol{z}\boldsymbol{z}^\top}{\|\boldsymbol{z}\|^2} - \frac{\boldsymbol{f}\boldsymbol{f}^\top}{\|\boldsymbol{f}\|^2},
$$

where $\boldsymbol{f}$ is as in (3.4), then we see from Lemma 2.6 and (3.14) that

$$
\begin{aligned}
\|\boldsymbol{u}^k - \boldsymbol{w}^k\| &= \|\boldsymbol{Q}\boldsymbol{v}^k\| = v_0^k \left\|\boldsymbol{Q}\begin{bmatrix}(v_0^k)^{-1}\overline{\boldsymbol{v}}^k \\ (v_0^k)^{-1}\widetilde{\boldsymbol{v}}^k\end{bmatrix}\right\| \\
&\overset{\text{(a)}}{=} v_0^k \left\|\boldsymbol{Q}\begin{bmatrix}(v_0^k)^{-1}\overline{\boldsymbol{v}}^k \\ (v_0^k)^{-1}\widetilde{\boldsymbol{v}}^k\end{bmatrix} - \boldsymbol{Q}\underbrace{\begin{bmatrix}-z_0^{-1}\overline{\boldsymbol{z}} \\ \boldsymbol{\alpha}\circ(z_0\widetilde{\boldsymbol{z}}^{-1})\end{bmatrix}}_{z_0\boldsymbol{f}}\right\| \\
&\leq v_0^k\left[\|(v_0^k)^{-1}\overline{\boldsymbol{v}}^k + z_0^{-1}\overline{\boldsymbol{z}}\| + \|(v_0^k)^{-1}\widetilde{\boldsymbol{v}}^k - \boldsymbol{\alpha}\circ(z_0\widetilde{\boldsymbol{z}}^{-1})\|\right],
\end{aligned} \tag{3.17}
$$

where (a) holds because $\boldsymbol{Q}\boldsymbol{f} = \boldsymbol{0}$ (an identity which is clear from the definitions).

Next, in view of (3.14), we can apply Lemma 3.2 to obtain $C_1 > 0$ and $\epsilon_1 > 0$ so that (3.3) holds with $\overline{\boldsymbol{\zeta}} = -z_0^{-1}\overline{\boldsymbol{z}}$, i.e.,

$$1 + \langle z_0^{-1}\overline{\boldsymbol{z}}, \boldsymbol{\omega}\rangle \geq C_1\|\boldsymbol{\omega} + z_0^{-1}\overline{\boldsymbol{z}}\|^2$$

whenever $\|\boldsymbol{\omega} + z_0^{-1}\overline{\boldsymbol{z}}\| \leq \epsilon_1$ and $\|\boldsymbol{\omega}\| = 1$. On the other hand, in view of the positivity of $1 + \langle z_0^{-1}\overline{\boldsymbol{z}}, \boldsymbol{\omega}\rangle$ when $\|\boldsymbol{\omega}\| = 1$ and $\boldsymbol{\omega} \neq -z_0^{-1}\overline{\boldsymbol{z}}$ (see Lemma 3.2), we know that

$$C_2 := \inf_{\|\boldsymbol{\omega}\|=1}\{1 + \langle z_0^{-1}\overline{\boldsymbol{z}}, \boldsymbol{\omega}\rangle \mid \|\boldsymbol{\omega} + z_0^{-1}\overline{\boldsymbol{z}}\| \geq \epsilon_1\} > 0.$$

This together with the fact $\|z_0^{-1}\overline{\boldsymbol{z}}\| = 1$ (see (3.14)) implies that

$$1 + \langle z_0^{-1}\overline{\boldsymbol{z}}, \boldsymbol{\omega}\rangle \geq C_2 \geq 0.25C_2\|\boldsymbol{\omega} + z_0^{-1}\overline{\boldsymbol{z}}\|^2,$$

whenever $\|\boldsymbol{\omega} + z_0^{-1}\overline{\boldsymbol{z}}\| \geq \epsilon_1$ and $\|\boldsymbol{\omega}\| = 1$. We thus have (with $C_3 := \min\{C_1, C_2/4\}$)

$$1 + \langle z_0^{-1}\overline{\boldsymbol{z}}, \boldsymbol{\omega}\rangle \geq C_3\|\boldsymbol{\omega} + z_0^{-1}\overline{\boldsymbol{z}}\|^2 \quad \text{whenever} \quad \|\boldsymbol{\omega}\| = 1. \tag{3.18}$$

In addition, noting (3.14) again, we can apply Lemma 3.1 with $\boldsymbol{\zeta} = -z_0^{-1}\widetilde{\boldsymbol{z}} \in \operatorname{int}\mathbb{R}^n_-$ to obtain $C_4 > 0$ and $\epsilon > 0$ so that (3.2) holds with $\widetilde{\boldsymbol{\zeta}} = \boldsymbol{\alpha} \circ (z_0\widetilde{\boldsymbol{z}}^{-1})$, i.e.,

$$-1 + \langle z_0^{-1}\widetilde{\boldsymbol{z}}, \boldsymbol{\omega}\rangle \geq C_4\|\boldsymbol{\omega} - \boldsymbol{\alpha} \circ (z_0\widetilde{\boldsymbol{z}}^{-1})\|^2 \tag{3.19}$$

whenever $\|\boldsymbol{\omega} - \boldsymbol{\alpha} \circ (z_0\widetilde{\boldsymbol{z}}^{-1})\| \leq \epsilon$, $\boldsymbol{\omega} \in \operatorname{int}\mathbb{R}^n_+$ and $\prod_{i=1}^n \omega_i^{\alpha_i} = 1$.

Furthermore, consider $h : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ defined by

$$h(\boldsymbol{\omega}) = \begin{cases} \dfrac{\langle z_0^{-1}\widetilde{\boldsymbol{z}}, \boldsymbol{\omega}\rangle - 1}{\|\boldsymbol{\omega} - \boldsymbol{\alpha} \circ (z_0\widetilde{\boldsymbol{z}}^{-1})\|} & \text{if } \|\boldsymbol{\omega} - \boldsymbol{\alpha} \circ (z_0\widetilde{\boldsymbol{z}}^{-1})\| \geq \epsilon,\ \boldsymbol{\omega} \in \Upsilon, \\ \infty & \text{otherwise,} \end{cases} \tag{3.20}$$

where $\Upsilon = \{\boldsymbol{\omega} \in \mathbb{R}^n_+ \mid \prod_{i=1}^n \omega_i^{\alpha_i} = 1\}$. Then we have

$$\begin{aligned} \liminf_{\|\boldsymbol{\omega}\|\to\infty} h(\boldsymbol{\omega}) &= \liminf_{\|\boldsymbol{\omega}\|\to\infty, \boldsymbol{\omega}\in\Upsilon} \frac{\langle z_0^{-1}\widetilde{\boldsymbol{z}}, \boldsymbol{\omega}\rangle - 1}{\|\boldsymbol{\omega} - \boldsymbol{\alpha} \circ (z_0\widetilde{\boldsymbol{z}}^{-1})\|} \\ &\geq \liminf_{\|\boldsymbol{\omega}\|\to\infty, \boldsymbol{\omega}\in\mathbb{R}^n_+} \frac{\langle z_0^{-1}\widetilde{\boldsymbol{z}}, \boldsymbol{\omega}\rangle - 1}{\|\boldsymbol{\omega} - \boldsymbol{\alpha} \circ (z_0\widetilde{\boldsymbol{z}}^{-1})\|} \overset{\text{(a)}}{\geq} \inf_{\|\boldsymbol{\lambda}\|=1, \boldsymbol{\lambda}\in\mathbb{R}^n_+} \langle z_0^{-1}\widetilde{\boldsymbol{z}}, \boldsymbol{\lambda}\rangle \overset{\text{(b)}}{\geq} \min_{1\leq i\leq n} z_0^{-1}\widetilde{z}_i > 0. \end{aligned}$$

Here (a) may be verified by multiplying both numerator and denominator of the left side by $1/\|\boldsymbol{\omega}\|$; (b) holds because $z_0^{-1}\widetilde{z}_i > 0$ for all i (see (3.14)). Since h in (3.20) is also lower semicontinuous on any compact set and **is always positive**,[4] it must then hold that $C_5 := \inf h > 0$. In particular, this means that

$$-1 + \langle z_0^{-1}\widetilde{\boldsymbol{z}}, \boldsymbol{\omega}\rangle \geq C_5\|\boldsymbol{\omega} - \boldsymbol{\alpha} \circ (z_0\widetilde{\boldsymbol{z}}^{-1})\| \tag{3.21}$$

whenever $\|\boldsymbol{\omega} - \boldsymbol{\alpha} \circ (z_0\widetilde{\boldsymbol{z}}^{-1})\| \geq \epsilon$, $\boldsymbol{\omega} \in \operatorname{int} \mathbb{R}^n_+$ and $\prod_{i=1}^n \omega_i^{\alpha_i} = 1$.

By passing to suitable subsequences, we will end up with one of the following two cases:

Case 1: $\|(v_0^k)^{-1}\widetilde{\boldsymbol{v}}^k - \boldsymbol{\alpha} \circ (z_0\widetilde{\boldsymbol{z}}^{-1})\| \leq \epsilon$ for all k. Then we have from (3.18) (with $\boldsymbol{\omega} = (v_0^k)^{-1}\overline{\boldsymbol{v}}^k$) and (3.19) (with $\boldsymbol{\omega} = (v_0^k)^{-1}\widetilde{\boldsymbol{v}}^k$) that for these k

$$\begin{aligned}
&1 + \langle z_0^{-1}\overline{\boldsymbol{z}}, (v_0^k)^{-1}\overline{\boldsymbol{v}}^k\rangle - \langle z_0^{-1}(-\widetilde{\boldsymbol{z}}), (v_0^k)^{-1}\widetilde{\boldsymbol{v}}^k\rangle - 1\\
&\geq \min\{C_3, C_4\}(\|(v_0^k)^{-1}\overline{\boldsymbol{v}}^k + z_0^{-1}\overline{\boldsymbol{z}}\|^2 + \|(v_0^k)^{-1}\widetilde{\boldsymbol{v}}^k - \boldsymbol{\alpha} \circ (z_0\widetilde{\boldsymbol{z}}^{-1})\|^2).
\end{aligned}$$

Combining this with (3.16) and (3.17), we see further that

$$\begin{aligned}
&\|\boldsymbol{v}^k - \boldsymbol{w}^k\|\\
&\geq \frac{z_0}{\|\boldsymbol{z}\|}\min\{C_3, C_4\}(\|(v_0^k)^{-1}\overline{\boldsymbol{v}}^k + z_0^{-1}\overline{\boldsymbol{z}}\|^2 + \|(v_0^k)^{-1}\widetilde{\boldsymbol{v}}^k - \boldsymbol{\alpha} \circ (z_0\widetilde{\boldsymbol{z}}^{-1})\|^2)v_0^k\\
&\geq \frac{z_0}{2\|\boldsymbol{z}\|}\min\{C_3, C_4\}(\|(v_0^k)^{-1}\overline{\boldsymbol{v}}^k + z_0^{-1}\overline{\boldsymbol{z}}\| + \|(v_0^k)^{-1}\widetilde{\boldsymbol{v}}^k - \boldsymbol{\alpha} \circ (z_0\widetilde{\boldsymbol{z}}^{-1})\|)^2 v_0^k\\
&\geq \frac{z_0\min\{C_3, C_4\}}{2\|\boldsymbol{z}\|v_0^k}\|\boldsymbol{u}^k - \boldsymbol{w}^k\|^2 \geq \frac{z_0\min\{C_3, C_4\}}{2\|\boldsymbol{z}\|\eta}\|\boldsymbol{u}^k - \boldsymbol{w}^k\|^2,
\end{aligned}$$

where the last inequality holds because $\boldsymbol{v}^k \in \mathcal{B}(\eta)$. The above display contradicts (3.13) and hence Case 1 cannot happen.

[4] The positivity can be seen by applying Lemma 3.1 with $\boldsymbol{\zeta} = -z_0^{-1}\widetilde{\boldsymbol{z}}$.

Case 2: $\|(v_0^k)^{-1}\widetilde{\boldsymbol{v}}^k - \boldsymbol{\alpha} \circ (z_0 \widetilde{\boldsymbol{z}}^{-1})\| \geq \epsilon$ for all k. Then we have from (3.18) and (3.21) that for these k

$$\begin{aligned}
&1 + \langle z_0^{-1}\overline{\boldsymbol{z}}, (v_0^k)^{-1}\overline{\boldsymbol{v}}^k\rangle - \langle z_0^{-1}(-\widetilde{\boldsymbol{z}}), (v_0^k)^{-1}\widetilde{\boldsymbol{v}}^k\rangle - 1 \\
&\geq \min\{C_3, C_5\}(\|(v_0^k)^{-1}\overline{\boldsymbol{v}}^k + z_0^{-1}\overline{\boldsymbol{z}}\|^2 + \|(v_0^k)^{-1}\widetilde{\boldsymbol{v}}^k - \boldsymbol{\alpha} \circ (z_0 \widetilde{\boldsymbol{z}}^{-1})\|).
\end{aligned}$$

Using this together with (3.16), we deduce that for all large k,

$$\|\boldsymbol{v}^k - \boldsymbol{w}^k\| \geq \frac{z_0 \min\{C_3, C_5\}}{\|\boldsymbol{z}\|}(\|(v_0^k)^{-1}\overline{\boldsymbol{v}}^k + z_0^{-1}\overline{\boldsymbol{z}}\|^2 + \|(v_0^k)^{-1}\widetilde{\boldsymbol{v}}^k - \boldsymbol{\alpha} \circ (z_0 \widetilde{\boldsymbol{z}}^{-1})\|)v_0^k.$$

This implies that

$$\begin{aligned}
\|(v_0^k)^{-1}\overline{\boldsymbol{v}}^k + z_0^{-1}\overline{\boldsymbol{z}}\| &\leq M_1\sqrt{(v_0^k)^{-1}\|\boldsymbol{v}^k - \boldsymbol{w}^k\|}, \\
\|(v_0^k)^{-1}\widetilde{\boldsymbol{v}}^k - \boldsymbol{\alpha} \circ (z_0 \widetilde{\boldsymbol{z}}^{-1})\| &\leq M_1 (v_0^k)^{-1}\|\boldsymbol{v}^k - \boldsymbol{w}^k\|,
\end{aligned} \tag{3.22}$$

where $M_1 := \max\left\{\left(\frac{z_0}{\|\boldsymbol{z}\|}\min\{C_3, C_5\}\right)^{-1}, \left(\frac{z_0}{\|\boldsymbol{z}\|}\min\{C_3, C_5\}\right)^{-1/2}\right\}$. Using (3.22) together with (3.17), we obtain that

$$\begin{aligned}
\|\boldsymbol{u}^k - \boldsymbol{w}^k\| &\leq M_1 v_0^k\left[\sqrt{(v_0^k)^{-1}\|\boldsymbol{v}^k - \boldsymbol{w}^k\|} + (v_0^k)^{-1}\|\boldsymbol{v}^k - \boldsymbol{w}^k\|\right] \\
&\overset{\text{(a)}}{\leq} M_1\sqrt{\eta}\sqrt{\|\boldsymbol{v}^k - \boldsymbol{w}^k\|} + M_1\|\boldsymbol{v}^k - \boldsymbol{w}^k\| \\
&\overset{\text{(b)}}{\leq} 3M_1\sqrt{\eta}\sqrt{\|\boldsymbol{v}^k - \boldsymbol{w}^k\|},
\end{aligned} \tag{3.23}$$

where (a) holds since $\boldsymbol{v}^k \in \mathcal{B}(\eta)$ wherefore $v_0^k \leq \eta$, and (b) holds because $\|\boldsymbol{w}^k\| \leq \|\boldsymbol{v}^k\| \leq \eta$ (because the projection onto $\mathcal{K}$ is nonexpansive and $\mathbf{0} \in \mathcal{K}$), wherefore

$$\|\boldsymbol{w}^k - \boldsymbol{v}^k\| = \sqrt{\|\boldsymbol{w}^k - \boldsymbol{v}^k\|}\sqrt{\|\boldsymbol{w}^k - \boldsymbol{v}^k\|} \leq 2\sqrt{\eta}\sqrt{\|\boldsymbol{w}^k - \boldsymbol{v}^k\|}.$$

Altogether, (3.23) contradicts (3.13) and hence Case 2 cannot happen.

Summarizing the above discussions, we see that Case (I) cannot happen.

(II): By passing to a further subsequence, we may assume that $\langle \boldsymbol{f}, \boldsymbol{v}^k\rangle < 0$ for all k. This together with the definition of $\boldsymbol{f}$ gives

$$\frac{v_0^k}{z_0}[\langle -z_0^{-1}\overline{\boldsymbol{z}}, (v_0^k)^{-1}\overline{\boldsymbol{v}}^k\rangle + \langle \boldsymbol{\alpha}\circ(z_0\widetilde{\boldsymbol{z}}^{-1}), (v_0^k)^{-1}\widetilde{\boldsymbol{v}}^k\rangle] < 0.$$

Since $\frac{v_0^k}{z_0} > 0$, we deduce that $\langle -z_0^{-1}\overline{\boldsymbol{z}}, (v_0^k)^{-1}\overline{\boldsymbol{v}}^k\rangle + \langle \boldsymbol{\alpha}\circ(z_0\widetilde{\boldsymbol{z}}^{-1}), (v_0^k)^{-1}\widetilde{\boldsymbol{v}}^k\rangle < 0$ for all k. Then it must hold that

$$\lim_{k\to\infty} \|(v_0^k)^{-1}\overline{\boldsymbol{v}}^k + z_0^{-1}\overline{\boldsymbol{z}}\| + \|(v_0^k)^{-1}\widetilde{\boldsymbol{v}}^k - \boldsymbol{\alpha}\circ(z_0\widetilde{\boldsymbol{z}}^{-1})\| \neq 0;$$

otherwise, we have $(v_0^k)^{-1}\overline{\boldsymbol{v}}^k \to -z_0^{-1}\overline{\boldsymbol{z}}$ and $(v_0^k)^{-1}\widetilde{\boldsymbol{v}}^k \to \boldsymbol{\alpha}\circ(z_0\widetilde{\boldsymbol{z}}^{-1})$, which further gives $\langle -z_0^{-1}\overline{\boldsymbol{z}}, (v_0^k)^{-1}\overline{\boldsymbol{v}}^k\rangle + \langle \boldsymbol{\alpha}\circ(z_0\widetilde{\boldsymbol{z}}^{-1}), (v_0^k)^{-1}\widetilde{\boldsymbol{v}}^k\rangle \to \|z_0^{-1}\overline{\boldsymbol{z}}\|^2 + \|\boldsymbol{\alpha}\circ(z_0\widetilde{\boldsymbol{z}}^{-1})\|^2 = \|z_0\boldsymbol{f}\|^2 > 0$, a contradiction.

Consequently, there exists $\epsilon > 0$ such that for all sufficiently large k,

$$\|(v_0^k)^{-1}\overline{\boldsymbol{v}}^k + z_0^{-1}\overline{\boldsymbol{z}}\| + \|(v_0^k)^{-1}\widetilde{\boldsymbol{v}}^k - \boldsymbol{\alpha}\circ(z_0\widetilde{\boldsymbol{z}}^{-1})\| \geq \epsilon. \tag{3.24}$$

Consider the function $G : \mathbb{R}^{m+n} \to \mathbb{R}\cup\{\infty\}$ defined by

$$G(\boldsymbol{\xi},\boldsymbol{\omega}) := \begin{cases} \dfrac{|\langle z_0^{-1}\overline{\boldsymbol{z}}, \boldsymbol{\xi}\rangle + \langle z_0^{-1}\widetilde{\boldsymbol{z}}, \boldsymbol{\omega}\rangle|}{\sqrt{1+\|\boldsymbol{\omega}\|^2}} & \text{if } (\boldsymbol{\xi},\boldsymbol{\omega})\in\Xi,\ \|\boldsymbol{\xi}\| = 1, \text{ and } \boldsymbol{\omega}\in\Upsilon, \\ \infty & \text{otherwise,} \end{cases}$$

where $\Upsilon = \{\boldsymbol{\omega}\in\mathbb{R}^n_+ \mid \prod_{i=1}^n \omega_i^{\alpha_i} = 1\}$ and $\Xi = \{(\boldsymbol{\xi},\boldsymbol{\omega}) \mid \|\boldsymbol{\xi} + z_0^{-1}\overline{\boldsymbol{z}}\| + \|\boldsymbol{\omega} - \boldsymbol{\alpha}\circ(z_0\widetilde{\boldsymbol{z}}^{-1})\| \geq \epsilon\}$. Since $\langle z_0^{-1}\overline{\boldsymbol{z}}, \boldsymbol{\xi}\rangle + \langle z_0^{-1}\widetilde{\boldsymbol{z}}, \boldsymbol{\omega}\rangle = 1 + \langle z_0^{-1}\overline{\boldsymbol{z}}, \boldsymbol{\xi}\rangle - \langle z_0^{-1}(-\widetilde{\boldsymbol{z}}), \boldsymbol{\omega}\rangle - 1$, we see from (3.14), (3.24), Lemma 3.2 and Lemma 3.1 that G is never zero. Moreover, it is clearly lower semicontinuous on any compact set, and

$$\liminf_{\|(\boldsymbol{\xi},\boldsymbol{\omega})\|\to\infty} G(\boldsymbol{\xi},\boldsymbol{\omega}) = \liminf_{\|\boldsymbol{\omega}\|\to\infty,\boldsymbol{\omega}\in\Upsilon} \frac{|\langle z_0^{-1}\widetilde{\boldsymbol{z}}, \boldsymbol{\omega}\rangle|}{\sqrt{1+\|\boldsymbol{\omega}\|^2}} \overset{\text{(a)}}{\geq} \inf_{\|\boldsymbol{\lambda}\|=1,\boldsymbol{\lambda}\in\mathbb{R}^n_+} |\langle z_0^{-1}\widetilde{\boldsymbol{z}}, \boldsymbol{\lambda}\rangle| \overset{\text{(b)}}{\geq} \min_i |z_0^{-1}\widetilde{z}_i| > 0,$$

where (a) may be verified by multiplying numerator and denominator by $1/\|\boldsymbol{\omega}\|$ and (b) holds since $z_0^{-1}\widetilde{z}_i > 0$ for all i. Thus, $C_6 := \inf G > 0$ and we have for all large k,

$$\begin{aligned}\frac{\|\boldsymbol{v}^k - \boldsymbol{w}^k\|}{\|\boldsymbol{u}^k - \boldsymbol{w}^k\|} &\overset{\text{(a)}}{=} \frac{\|\boldsymbol{v}^k - \boldsymbol{w}^k\|}{\|\boldsymbol{w}^k\|} \overset{\text{(b)}}{\geq} \frac{\|\boldsymbol{v}^k - \boldsymbol{w}^k\|}{\|\boldsymbol{v}^k\|} \\ &\overset{\text{(c)}}{=} \frac{z_0}{\|\boldsymbol{z}\|} \frac{|\langle z_0^{-1}\overline{\boldsymbol{z}}, (v_0^k)^{-1}\overline{\boldsymbol{v}}^k\rangle + \langle z_0^{-1}\widetilde{\boldsymbol{z}}, (v_0^k)^{-1}\widetilde{\boldsymbol{v}}^k\rangle| v_0^k}{\sqrt{(v_0^k)^2 + \|\widetilde{\boldsymbol{v}}^k\|^2}} \\ &= \frac{z_0}{\|\boldsymbol{z}\|} \frac{|\langle z_0^{-1}\overline{\boldsymbol{z}}, (v_0^k)^{-1}\overline{\boldsymbol{v}}^k\rangle + \langle z_0^{-1}\widetilde{\boldsymbol{z}}, (v_0^k)^{-1}\widetilde{\boldsymbol{v}}^k\rangle|}{\sqrt{1 + \|(v_0^k)^{-1}\widetilde{\boldsymbol{v}}^k\|^2}} \overset{\text{(d)}}{\geq} \frac{C_6 z_0}{\|\boldsymbol{z}\|},\end{aligned}$$

where (a) follows from Lemma 2.6, which states that $\boldsymbol{u}^k = \boldsymbol{0}$ in Case (II), (b) holds because the projection onto the cone is nonexpansive and $\boldsymbol{0}$ is in the cone, (c) follows from (3.16) and (d) follows from (3.24), (3.15) and the definitions of G and C_6. The above display contradicts (3.13). Thus, Case (II) also cannot happen.

Summarizing the above, we conclude that (3.13) cannot happen. Thus, in view of [36, Lemma 3.12], we must indeed have $\gamma_{\boldsymbol{z},\eta} \in (0, \infty]$ and that the desired error bound follows from [36, Theorem 3.10]. □

Remark 3.6 (Optimality of the error bound in Theorem 3.5)**.** *Let $\boldsymbol{z} \in \partial(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})^* \backslash \{\boldsymbol{0}\}$ with $\overline{\boldsymbol{z}} \neq \boldsymbol{0}$ and let $\mathcal{F}_{\mathrm{r}} := \{\boldsymbol{z}\}^\perp \cap \mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$. Then necessarily $\widetilde{z}_i > 0$ for all i. Moreover, we also know from the definition that $\alpha_i > 0$ for all i. Now, consider the continuous function $\boldsymbol{q} : (0, \alpha_1) \to \{\boldsymbol{z}\}^\perp$ defined by $\epsilon \mapsto \boldsymbol{q}_\epsilon := (\bar{\boldsymbol{q}}_\epsilon, \widetilde{\boldsymbol{q}}_\epsilon)$ where*

$$\bar{\boldsymbol{q}}_\epsilon = -\overline{\boldsymbol{z}}/\|\overline{\boldsymbol{z}}\|^2,\ (\widetilde{\boldsymbol{q}}_\epsilon)_1 = (\alpha_1 - \epsilon)\widetilde{z}_1^{-1},\ (\widetilde{\boldsymbol{q}}_\epsilon)_2 = (\alpha_2 + \epsilon)\widetilde{z}_2^{-1},\ \textit{and}\ (\widetilde{\boldsymbol{q}}_\epsilon)_i = \alpha_i \widetilde{z}_i^{-1},\ \forall i \geq 3.$$

Notice that $\boldsymbol{q}_\epsilon$ only differs from the $\boldsymbol{f}$ in (3.4) in two entries. One can check that

$\langle \boldsymbol{z}, \boldsymbol{q}_\epsilon \rangle = 0$ *and* $\boldsymbol{q}_\epsilon \to \boldsymbol{f} \in \mathcal{F}_{\rm r}\backslash\{\boldsymbol{0}\}$ *as* $\epsilon \downarrow 0$. *Moreover, we have*

$$\begin{aligned}
&\prod_{i=1}^n (\widetilde{\boldsymbol{q}}_\epsilon)_i^{\alpha_i} = (\alpha_1 - \epsilon)^{\alpha_1}(\alpha_2+\epsilon)^{\alpha_2}\widetilde{z}_1^{-\alpha_1}\widetilde{z}_2^{-\alpha_2}\prod_{i=3}^n\left(\frac{\alpha_i}{\widetilde{z}_i}\right)^{\alpha_i} \\
&= \left(1-\frac{\epsilon}{\alpha_1}\right)^{\alpha_1}\left(1+\frac{\epsilon}{\alpha_2}\right)^{\alpha_2}\prod_{i=1}^n\left(\frac{\alpha_i}{\widetilde{z}_i}\right)^{\alpha_i} \overset{\rm (a)}{=} \left(1-\frac{\epsilon}{\alpha_1}\right)^{\alpha_1}\left(1+\frac{\epsilon}{\alpha_2}\right)^{\alpha_2}\|\overline{\boldsymbol{z}}\|^{-1} \\
&= \left(1-\frac{\epsilon}{\alpha_1}\right)^{\alpha_1}\left(1+\frac{\epsilon}{\alpha_2}\right)^{\alpha_2}\|\bar{\boldsymbol{q}}_\epsilon\|,
\end{aligned}$$

where (a) holds because $\boldsymbol{z} \in \partial(\mathcal{P}_{m,n}^{\boldsymbol{\alpha}})^*\backslash\{\boldsymbol{0}\}$ *with* $\overline{\boldsymbol{z}} \neq \boldsymbol{0}$. *In view of this, if we define a continuous function* $\boldsymbol{p} : (0,\alpha_1) \to \mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$ *by* $\epsilon \mapsto \boldsymbol{p}_\epsilon := (\bar{\boldsymbol{p}}_\epsilon, \widetilde{\boldsymbol{p}}_\epsilon)$ *where*

$$\bar{\boldsymbol{p}}_\epsilon := -\left(1-\frac{\epsilon}{\alpha_1}\right)^{\alpha_1}\left(1+\frac{\epsilon}{\alpha_2}\right)^{\alpha_2}\frac{\overline{\boldsymbol{z}}}{\|\overline{\boldsymbol{z}}\|^2} \quad \text{and} \quad \widetilde{\boldsymbol{p}}_\epsilon := \widetilde{\boldsymbol{q}}_\epsilon;$$

then it is clear that $\boldsymbol{p}_\epsilon \in \mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$, *and we can compute that*

$$\begin{aligned}
\operatorname{dist}(\boldsymbol{q}_\epsilon, \mathcal{P}_{m,n}^{\boldsymbol{\alpha}}) \le \|\boldsymbol{q}_\epsilon - \boldsymbol{p}_\epsilon\| &= \frac{1}{\|\overline{\boldsymbol{z}}\|}\left|\left(1-\frac{\epsilon}{\alpha_1}\right)^{\alpha_1}\left(1+\frac{\epsilon}{\alpha_2}\right)^{\alpha_2} - 1\right| \\
&= \frac{1}{\|\overline{\boldsymbol{z}}\|}\left|(1-\epsilon+O(\epsilon^2))(1+\epsilon+O(\epsilon^2)) - 1\right| = O(\epsilon^2).
\end{aligned} \tag{3.25}$$

Next, we estimate $\operatorname{dist}(\boldsymbol{q}_\epsilon, \mathcal{F}_{\rm r})$. *Notice that* $\langle \boldsymbol{q}_\epsilon, \boldsymbol{f} \rangle > 0$ *for all sufficiently small* ϵ *because* $\boldsymbol{q}_\epsilon \to \boldsymbol{f}$. *Hence, using the definition of* $\mathcal{F}_{\rm r}$ *and Lemma 2.6, we see that*

$$\operatorname{dist}(\boldsymbol{q}_\epsilon, \mathcal{F}_{\rm r})^2 = \left\|\boldsymbol{q}_\epsilon - \frac{\langle \boldsymbol{q}_\epsilon, \boldsymbol{f}\rangle}{\|\boldsymbol{f}\|^2}\boldsymbol{f}\right\|^2 = \|\boldsymbol{q}_\epsilon\|^2 - \frac{(\langle \boldsymbol{q}_\epsilon, \boldsymbol{f}\rangle)^2}{\|\boldsymbol{f}\|^2}.$$

A direct computation then shows that

$$\begin{aligned}
\|\boldsymbol{q}_\epsilon\|^2 &= \frac{1}{\|\overline{\boldsymbol{z}}\|^2} + (\alpha_1-\epsilon)^2\widetilde{z}_1^{-2} + (\alpha_2+\epsilon)^2\widetilde{z}_2^{-2} + \sum_{i=3}^n \alpha_i^2\widetilde{z}_i^{-2} \\
&= \frac{1}{\|\overline{\boldsymbol{z}}\|^2} + \sum_{i=1}^n \alpha_i^2\widetilde{z}_i^{-2} + 2\epsilon(\alpha_2\widetilde{z}_2^{-2} - \alpha_1\widetilde{z}_1^{-2}) + \epsilon^2(\widetilde{z}_1^{-2} + \widetilde{z}_2^{-2})
\end{aligned}$$

$$= \|\boldsymbol{f}\|^2 + 2\epsilon(\alpha_2 \widetilde{z}_2^{-2} - \alpha_1 \widetilde{z}_1^{-2}) + \epsilon^2(\widetilde{z}_1^{-2} + \widetilde{z}_2^{-2}),$$

where the last equality follows from the definition of $\boldsymbol{f}$ *in* (3.4)*. Furthermore,*

$$\begin{aligned}
(\langle \boldsymbol{q}_\epsilon, \boldsymbol{f} \rangle)^2 &= \left(\tfrac{1}{\|\overline{\boldsymbol{z}}\|^2} + \alpha_1(\alpha_1 - \epsilon)\widetilde{z}_1^{-2} + \alpha_2(\alpha_2 + \epsilon)\widetilde{z}_2^{-2} + \textstyle\sum_{i=3}^n \alpha_i^2 \widetilde{z}_i^{-2} \right)^2 \\
&= \left[\|\boldsymbol{f}\|^2 + \epsilon(\alpha_2 \widetilde{z}_2^{-2} - \alpha_1 \widetilde{z}_1^{-2}) \right]^2 \\
&= \|\boldsymbol{f}\|^4 + 2\epsilon\|\boldsymbol{f}\|^2(\alpha_2 \widetilde{z}_2^{-2} - \alpha_1 \widetilde{z}_1^{-2}) + \epsilon^2(\alpha_2 \widetilde{z}_2^{-2} - \alpha_1 \widetilde{z}_1^{-2})^2.
\end{aligned}$$

Combining the above three identities, we deduce further that

$$\begin{aligned}
\operatorname{dist}(\boldsymbol{q}_\epsilon, \mathcal{F}_{\mathrm{r}})^2 &= \epsilon^2 \left(\widetilde{z}_1^{-2} + \widetilde{z}_2^{-2} - \frac{(\alpha_2 \widetilde{z}_2^{-2} - \alpha_1 \widetilde{z}_1^{-2})^2}{\|\boldsymbol{f}\|^2} \right) \\
&\geq \epsilon^2 \left(\widetilde{z}_1^{-2} + \widetilde{z}_2^{-2} - \frac{(\alpha_2 \widetilde{z}_2^{-2} - \alpha_1 \widetilde{z}_1^{-2})^2}{\alpha_2^2 \widetilde{z}_2^{-2} + \alpha_1^2 \widetilde{z}_1^{-2}} \right),
\end{aligned} \tag{3.26}$$

where the inequality follows from the definition of $\boldsymbol{f}$*. Now, notice that in* (3.26)*, the scalar term is strictly greater than zero, because*

$$(\alpha_2 \widetilde{z}_2^{-2} - \alpha_1 \widetilde{z}_1^{-2})^2 < (\alpha_2 \widetilde{z}_2^{-2} + \alpha_1 \widetilde{z}_1^{-2})^2 \leq (\widetilde{z}_1^{-2} + \widetilde{z}_2^{-2})(\alpha_2^2 \widetilde{z}_2^{-2} + \alpha_1^2 \widetilde{z}_1^{-2}),$$

where the strict inequality holds because $\alpha_i \widetilde{z}_i^{-2} > 0$ *for* $i = 1, 2$*, and the last inequality follows from the Cauchy-Schwarz inequality. This together with* (3.26) *shows that* $\operatorname{dist}(\boldsymbol{q}_\epsilon, \mathcal{F}_{\mathrm{r}}) = \Omega(\epsilon)$*. Combining this with* (3.25)*, we obtain* $\limsup_{\epsilon \downarrow 0} \frac{\operatorname{dist}(\boldsymbol{q}_\epsilon, \mathcal{P}_{m,n}^{\boldsymbol{\alpha}})^{\frac{1}{2}}}{\operatorname{dist}(\boldsymbol{q}_\epsilon, \mathcal{F}_{\mathrm{r}})} < \infty$*. Thus* $|\cdot|^{\frac{1}{2}}$ *satisfies the asymptotic optimality criterion (cf. [37, Definition 3.1]) for* $\mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$ *and* $\boldsymbol{z}$*, which implies that the error bound is optimal in the sense of [37, Theorem 3.2(b)].*

We now look at the faces that are exposed by $\boldsymbol{z} \in \partial(\mathcal{P}_{m,n}^{\boldsymbol{\alpha}})^* \backslash \{\boldsymbol{0}\}$ with $\overline{\boldsymbol{z}} = \boldsymbol{0}$.

Theorem 3.7. *Let* $\boldsymbol{z} \in \partial(\mathcal{P}_{m,n}^{\boldsymbol{\alpha}})^* \backslash \{\boldsymbol{0}\}$ *with* $\overline{\boldsymbol{z}} = \boldsymbol{0}$ *and let* $\mathcal{F}_{\boldsymbol{z}} := \{\boldsymbol{z}\}^\perp \cap \mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$*. Let* $\mathcal{I} := \{i \,|\, \widetilde{z}_i > 0\}$*,*[5] $\eta > 0$ *and define* $\beta := \sum_{i \in \mathcal{I}} \alpha_i$ *and*

$$\gamma_{\boldsymbol{z},\eta} := \inf_{\boldsymbol{v}} \left\{ \frac{\|\boldsymbol{v} - \boldsymbol{w}\|^\beta}{\|\boldsymbol{u} - \boldsymbol{w}\|} \;\middle|\; \begin{array}{c} \boldsymbol{v} \in \partial\mathcal{P}_{m,n}^{\boldsymbol{\alpha}} \cap \mathcal{B}(\eta) \backslash \mathcal{F}_{\boldsymbol{z}},\ \boldsymbol{w} = P_{\{\boldsymbol{z}\}^\perp}(\boldsymbol{v}), \\ \boldsymbol{u} = P_{\mathcal{F}_{\boldsymbol{z}}}(\boldsymbol{w}),\ \boldsymbol{u} \neq \boldsymbol{w} \end{array} \right\}. \tag{3.27}$$

[5] Since $\overline{\boldsymbol{z}} = \boldsymbol{0}$ and $\boldsymbol{z} \in \partial(\mathcal{P}_{m,n}^{\boldsymbol{\alpha}})^* \backslash \{\boldsymbol{0}\}$, we must have $\emptyset \neq \mathcal{I} \subsetneq \{1, 2, \ldots, n\}$.

Then it holds that $\gamma_{\boldsymbol{z},\eta} \in (0, \infty]$ *and that*

$$\operatorname{dist}(\boldsymbol{q}, \mathcal{F}_{\boldsymbol{z}}) \le \max\{2\eta^{1-\beta}, 2\gamma_{\boldsymbol{z},\eta}^{-1}\} \cdot \operatorname{dist}(\boldsymbol{q}, \mathcal{P}_{m,n}^{\boldsymbol{\alpha}})^{\beta} \quad \textit{whenever } \boldsymbol{q} \in \{\boldsymbol{z}\}^{\perp} \cap \mathcal{B}(\eta).$$

Proof. In view of [36, Theorem 3.10], we need only show that $\gamma_{\boldsymbol{z},\eta} > 0$. To that end, let $\boldsymbol{v} \in \partial\mathcal{P}_{m,n}^{\boldsymbol{\alpha}} \cap \mathcal{B}(\eta) \backslash \mathcal{F}_{\boldsymbol{z}}$, $\boldsymbol{w} = P_{\{\boldsymbol{z}\}^{\perp}}(\boldsymbol{v})$, $\boldsymbol{u} = P_{\mathcal{F}_{\boldsymbol{z}}}(\boldsymbol{w})$, and $\boldsymbol{u} \neq \boldsymbol{w}$. Then a direct computation shows that

$$\|\boldsymbol{w} - \boldsymbol{v}\| = \frac{1}{\|\boldsymbol{z}\|}|\langle \boldsymbol{z}, \boldsymbol{v}\rangle| \overset{\text{(a)}}{=} \frac{1}{\|\boldsymbol{z}\|}\sum_{i\in\mathcal{I}} \widetilde{z}_i\widetilde{v}_i \overset{\text{(b)}}{\ge} \frac{\min_{i\in\mathcal{I}}\widetilde{z}_i}{\|\boldsymbol{z}\|}\sum_{i\in\mathcal{I}}\widetilde{v}_i \overset{\text{(c)}}{\ge} \frac{\min_{i\in\mathcal{I}}\widetilde{z}_i}{\|\boldsymbol{z}\|}\|\widetilde{\boldsymbol{v}}_{\mathcal{I}}\|, \tag{3.28}$$

where (a), (b) and (c) hold because $\widetilde{v}_i \ge 0$ and $\widetilde{z}_i > 0$ for all $i \in \mathcal{I}$, with $\|\widetilde{\boldsymbol{v}}_{\mathcal{I}}\| := \sqrt{\sum_{i\in\mathcal{I}}\widetilde{v}_i^2}$ (note that $\mathcal{I} \neq \emptyset$, thanks to $\overline{\boldsymbol{z}} = 0$ and $\boldsymbol{z} \neq \boldsymbol{0}$). Next, notice that $\boldsymbol{w} = \boldsymbol{v} - \frac{\langle \boldsymbol{z},\boldsymbol{v}\rangle}{\|\boldsymbol{z}\|^2}\boldsymbol{z}$. Using this and the definitions of $\boldsymbol{z}$ and $\mathcal{I}$, we deduce that

$$\overline{\boldsymbol{w}} = \overline{\boldsymbol{v}}, \;\; \widetilde{w}_i = \widetilde{v}_i - \frac{\widetilde{z}_i}{\|\boldsymbol{z}\|^2}\left(\sum_{j\in\mathcal{I}}\widetilde{z}_j\widetilde{v}_j\right) \;\; \forall i \in \mathcal{I} \quad \text{and} \quad \widetilde{w}_i = \widetilde{v}_i \ge 0 \;\; \forall i \notin \mathcal{I}. \tag{3.29}$$

In view of this and the definition of $\mathcal{F}_{\boldsymbol{z}}$ in (3.5), we see that $\widetilde{u}_i = \widetilde{w}_i$ whenever $i \notin \mathcal{I}$, and hence

$$\|\boldsymbol{w} - \boldsymbol{u}\| = \sqrt{\|\overline{\boldsymbol{w}}\|^2 + \sum_{i\in\mathcal{I}}\widetilde{w}_i^2} \le \sqrt{\|\overline{\boldsymbol{v}}\|^2 + n(1+\sqrt{n})^2\|\widetilde{\boldsymbol{v}}_{\mathcal{I}}\|^2}, \tag{3.30}$$

where the inequality follows from (3.29) and the fact that for each $i \in \mathcal{I}$,

$$\begin{aligned} |\widetilde{w}_i| &= \left|\widetilde{v}_i - \frac{\widetilde{z}_i}{\|\boldsymbol{z}\|^2}\left(\sum_{j\in\mathcal{I}}\widetilde{z}_j\widetilde{v}_j\right)\right| \le \left(1 + \frac{|\widetilde{z}_i|}{\|\boldsymbol{z}\|^2}\sum_{j\in\mathcal{I}}|\widetilde{z}_j|\right)\|\widetilde{\boldsymbol{v}}_{\mathcal{I}}\| \\ &\le \left(1 + \frac{\sqrt{n}|\widetilde{z}_i|}{\|\boldsymbol{z}\|}\right)\|\widetilde{\boldsymbol{v}}_{\mathcal{I}}\| \le (1+\sqrt{n})\|\widetilde{\boldsymbol{v}}_{\mathcal{I}}\|. \end{aligned}$$

Next, note that we have

$$\|\overline{\boldsymbol{v}}\| = \prod_{i=1}^{n}(\widetilde{v}_i)^{\alpha_i} = \prod_{i\notin\mathcal{I}}\widetilde{v}_i^{\alpha_i} \cdot \prod_{i\in\mathcal{I}}\widetilde{v}_i^{\alpha_i} \le \prod_{i\notin\mathcal{I}}\eta^{\alpha_i} \cdot \prod_{i\in\mathcal{I}}\|\widetilde{\boldsymbol{v}}_{\mathcal{I}}\|^{\alpha_i} = \eta^{1-\beta}\|\widetilde{\boldsymbol{v}}_{\mathcal{I}}\|^{\beta}, \tag{3.31}$$

where the inequality holds because $\boldsymbol{v} \in \mathcal{B}(\eta)$. Combining (3.28), (3.30) and (3.31), we deduce

$$
\begin{aligned}
\|\boldsymbol{w} - \boldsymbol{u}\| &\le \sqrt{\|\overline{\boldsymbol{v}}\|^2 + n(1+\sqrt{n})^2\|\widetilde{\boldsymbol{v}}_{\mathcal{I}}\|^2} \le \|\overline{\boldsymbol{v}}\| + (n+\sqrt{n})\|\widetilde{\boldsymbol{v}}_{\mathcal{I}}\| \\
&\le \eta^{1-\beta}\|\widetilde{\boldsymbol{v}}_{\mathcal{I}}\|^{\beta} + (n+\sqrt{n})\|\widetilde{\boldsymbol{v}}_{\mathcal{I}}\| = (\eta^{1-\beta} + (n+\sqrt{n})\|\widetilde{\boldsymbol{v}}_{\mathcal{I}}\|^{1-\beta})\|\widetilde{\boldsymbol{v}}_{\mathcal{I}}\|^{\beta} \\
&\overset{\text{(a)}}{\le} \eta^{1-\beta}(n+1+\sqrt{n})\|\widetilde{\boldsymbol{v}}_{\mathcal{I}}\|^{\beta} \overset{\text{(b)}}{\le} \frac{\eta^{1-\beta}(n+1+\sqrt{n})\|\boldsymbol{z}\|^{\beta}}{(\min_{i\in\mathcal{I}} \widetilde{z}_i)^{\beta}}\|\boldsymbol{w} - \boldsymbol{v}\|^{\beta}.
\end{aligned}
$$

Here (a) holds since $\boldsymbol{v} \in \mathcal{B}(\eta)$ and $\beta \in (0,1)$; (b) is true because of (3.28). Thus, $\gamma_{\boldsymbol{z},\eta} \ge \frac{(\min_{i\in\mathcal{I}} \widetilde{z}_i)^{\beta}}{\eta^{1-\beta}(n+1+\sqrt{n})\|\boldsymbol{z}\|^{\beta}} > 0$, and the desired error bound follows from [36, Theorem 3.10]. □

Remark 3.8 (Optimality of the error bound in Theorem 3.7)**.** *Let* $\boldsymbol{z} \in \partial(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})^* \backslash \{\boldsymbol{0}\}$ *with* $\overline{\boldsymbol{z}} = \boldsymbol{0}$ *and let* $\mathcal{F}_{\boldsymbol{z}} := \{\boldsymbol{z}\}^{\perp} \cap \mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$*. Let* $\mathcal{I} := \{i \,|\, \widetilde{z}_i > 0\} \ne \emptyset$ *and define*

$$\beta := \sum_{i\in\mathcal{I}} \alpha_i \in (0,1).$$

Fix any $\boldsymbol{u} \in \mathbb{R}^m$ *with* $\|\boldsymbol{u}\| = 1$ *and define the continuous function* $\boldsymbol{q} : (0,1) \to \{\boldsymbol{z}\}^{\perp}$ *by* $\epsilon \mapsto \boldsymbol{q}_{\epsilon} := (\bar{\boldsymbol{q}}_{\epsilon}, \widetilde{\boldsymbol{q}}_{\epsilon})$ *where*

$$\bar{\boldsymbol{q}}_{\epsilon} = \epsilon^{\beta}\boldsymbol{u},\ (\widetilde{\boldsymbol{q}}_{\epsilon})_i = 0\ \forall i \in \mathcal{I},\ \textit{and}\,(\widetilde{\boldsymbol{q}}_{\epsilon})_i = 1,\ \forall i \notin \mathcal{I}.$$

It is clear that for all ϵ, $\langle \boldsymbol{z}, \boldsymbol{q}_{\epsilon}\rangle = 0$ *and* $\operatorname{dist}(\boldsymbol{q}_{\epsilon}, \mathcal{F}_{\boldsymbol{z}}) \to 0$ *as* $\epsilon \downarrow 0$ *. Now, define the function* $\boldsymbol{p} : (0,1) \to \mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ *by* $\epsilon \mapsto \boldsymbol{p}_{\epsilon} := (\bar{\boldsymbol{p}}_{\epsilon}, \widetilde{\boldsymbol{p}}_{\epsilon})$ *where*

$$\bar{\boldsymbol{p}}_{\epsilon} = \epsilon^{\beta}\boldsymbol{u},\ (\widetilde{\boldsymbol{p}}_{\epsilon})_i = \epsilon\ \forall i \in \mathcal{I},\ \textit{and}\,(\widetilde{\boldsymbol{p}}_{\epsilon})_i = 1,\ \forall i \notin \mathcal{I}.$$

Clearly $\boldsymbol{p}_{\epsilon}$ *lies in* $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$*, and we have that* $\operatorname{dist}(\boldsymbol{q}_{\epsilon}, \mathcal{P}^{\boldsymbol{\alpha}}_{m,n}) \le \|\boldsymbol{q}_{\epsilon} - \boldsymbol{p}_{\epsilon}\| \le |\mathcal{I}| \cdot \epsilon$*. On the other hand, we have in view of* (3.5) *that* $\operatorname{dist}(\boldsymbol{q}_{\epsilon}, \mathcal{F}_{\boldsymbol{z}}) = \epsilon^{\beta} > 0$*. Hence,* $\limsup_{\epsilon\downarrow 0} \frac{\operatorname{dist}(\boldsymbol{q}_{\epsilon}, \mathcal{P}^{\boldsymbol{\alpha}}_{m,n})^{\beta}}{\operatorname{dist}(\boldsymbol{q}_{\epsilon}, \mathcal{F}_{\boldsymbol{z}})} \le |\mathcal{I}|^{\beta} < \infty$*. Thus* $|\cdot|^{\beta}$ *satisfies the asymptotic optimality criterion (cf. [37, Definition 3.1]) for* $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ *and* $\boldsymbol{z}$*, which implies that the error bound is optimal in the sense of [37, Theorem 3.2(b)].*

Using Theorems 3.5 and 3.7 together with [36, Lemma 3.9], we have the following result concerning one-step facial residual functions.

Corollary 3.9. *Consider $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ and its dual cone $(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})^*$.*

(i) Let $\boldsymbol{z} \in \partial(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})^* \backslash \{\boldsymbol{0}\}$ with $\overline{\boldsymbol{z}} \neq \boldsymbol{0}$ and let $\mathcal{F}_{\rm r} := \{\boldsymbol{z}\}^\perp \cap \mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$. Let $\gamma_{\boldsymbol{z},t}$ be defined as in (3.12). Then the function $\psi_{\mathcal{P}^{\boldsymbol{\alpha}}_{m,n},\boldsymbol{z}} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\psi_{\mathcal{P}^{\boldsymbol{\alpha}}_{m,n},\boldsymbol{z}}(\epsilon, t) := \max\{\epsilon, \epsilon/\|\boldsymbol{z}\|\} + \max\{2\sqrt{t}, 2\gamma_{\boldsymbol{z},t}^{-1}\}(\epsilon + \max\{\epsilon, \epsilon/\|\boldsymbol{z}\|\})^{\frac{1}{2}} \quad (3.32)$$

is a one-step facial residual function for $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ and $\boldsymbol{z}$.

(ii) Let $\boldsymbol{z} \in \partial(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})^* \backslash \{\boldsymbol{0}\}$ with $\overline{\boldsymbol{z}} = \boldsymbol{0}$ and let $\mathcal{F}_{\boldsymbol{z}} := \{\boldsymbol{z}\}^\perp \cap \mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$. Let $\gamma_{\boldsymbol{z},t}$ be defined as in (3.27), where $\beta := \sum_{i:\widetilde{z}_i > 0} \alpha_i$. Then the function $\psi_{\mathcal{P}^{\boldsymbol{\alpha}}_{m,n},\boldsymbol{z}} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\psi_{\mathcal{P}^{\boldsymbol{\alpha}}_{m,n},\boldsymbol{z}}(\epsilon, t) := \max\{\epsilon, \epsilon/\|\boldsymbol{z}\|\} + \max\{2t^{1-\beta}, 2\gamma_{\boldsymbol{z},t}^{-1}\}(\epsilon + \max\{\epsilon, \epsilon/\|\boldsymbol{z}\|\})^{\beta} \quad (3.33)$$

is a one-step facial residual function for $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ and $\boldsymbol{z}$.

We now collect these results to show the tight error bounds for $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$.

Theorem 3.10 (Error bounds for the generalized power cone and their optimality)**.** *Consider $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ and its dual cone $(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})^*$. Let $\mathcal{L} \subseteq \mathbb{R}^{m+n}$ be a subspace and $\boldsymbol{a} \in \mathbb{R}^{m+n}$ be given. Suppose that $(\mathcal{L} + \boldsymbol{a}) \cap \mathcal{P}^{\boldsymbol{\alpha}}_{m,n} \neq \emptyset$. Then the following items hold.*

(i) $d_{\rm PPS}(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}, \mathcal{L} + \boldsymbol{a}) \leq 1$.

(ii) If $d_{\rm PPS}(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}, \mathcal{L} + \boldsymbol{a}) = 0$, then a Lipschitzian error bound holds.

(iii) If $d_{\rm PPS}(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}, \mathcal{L} + \boldsymbol{a}) = 1$, consider the chain of faces $\mathcal{F} \subsetneq \mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ with length being 2.

(a) If $\mathcal{F} = \mathcal{F}_{\rm r}$, then a Hölderian error bound with exponent $1/2$ holds.

(b) *If* $\mathcal{F} = \mathcal{F}_{\boldsymbol{z}}$ *with* $\boldsymbol{z} \in (\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})^* \cap \mathcal{L}^\perp \cap \{\boldsymbol{a}\}^\perp$*, then a Hölderian error bound with exponent* $\beta := \sum_{i:\widetilde{z}_i>0} \alpha_i$ *holds.*

(c) *If* $\mathcal{F} = \{\mathbf{0}\}$*, then a Lipschitzian error bound holds.*

(iv) *All these error bounds are the best in the sense stated in [37, Theorem 3.2(b)].*

Proof. As is shown in Section 4.1, all the proper exposed faces of the generalized power cone are polyhedral. Then the process of facial reduction needs at most one step to reach the PPS condition. Hence, $d_{\mathrm{PPS}}(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}, \mathcal{L}+\boldsymbol{a}) \leq 1$. This shows item (i).

If $d_{\mathrm{PPS}}(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}, \mathcal{L}+\boldsymbol{a}) = 0$, i.e., (Feas) satisfies the PPS condition, then by [7, Corollary 3], a Lipschitzian error bound holds. This shows item (ii).

Next, let $d_{\mathrm{PPS}}(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}, \mathcal{L}+\boldsymbol{a}) = 1$; i.e., we need one step to reach the PPS condition. In this case, the error bound depends on the exposed face $\mathcal{F}$ that contains the feasible region. If $\mathcal{F} = \mathcal{F}_{\mathrm{r}}$, then by Corollary 3.9i, we conclude that a Hölderian error bound with exponent 1/2 holds. Remark 3.6 implies that $\mathfrak{g} = |\cdot|^{\frac{1}{2}}$ satisfies the asymptotic optimality criterion for $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ and $\boldsymbol{z}$ with $\overline{\boldsymbol{z}} \neq \mathbf{0}$. Hence, by [37, Theorem 3.2], the obtained Hölderian error bound with exponent 1/2 is the best error bound.

If $\mathcal{F} = \mathcal{F}_{\boldsymbol{z}}$ with $\boldsymbol{z} \in (\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})^* \cap \mathcal{L}^\perp \cap \{\boldsymbol{a}\}^\perp$, then using Corollary 3.9(ii), we conclude that a Hölderian error bound with exponent $\beta := \sum_{i\in\mathcal{I}} \alpha_i$ holds, where $\mathcal{I} = \{i \,|\, \widetilde{z}_i > 0\}$. The optimality of this error bound comes from Remark 3.8 and [37, Theorem 3.2]. If $\mathcal{F} = \{\mathbf{0}\}$, which means the feasible region is $\{\mathbf{0}\}$, then a Lipschitzian error bound holds automatically and it is naturally tight, see [40, Proposition 27]. □

3.3 Application: Self-duality, homogeneity, irreducibility and perfectness of $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$

In this section, we consider the *self-duality*, *homogeneity*, *irreducibility* and *perfectness* of $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$. We first briefly explain the importance of those questions.

In what follows, we need the following concepts. We will denote by $\mathrm{Aut}(\mathcal{K})$ the group of automorphisms of $\mathcal{K}$ which are the linear bijections $\boldsymbol{M} : \mathcal{E} \to \mathcal{E}$ such that $\boldsymbol{M}\mathcal{K} = \mathcal{K}$. Then, the Lie algebra of $\mathrm{Aut}(\mathcal{K})$ denoted by $\mathrm{Lie}\,\mathrm{Aut}(\mathcal{K})$ corresponds to the linear maps $\boldsymbol{L}$ for which $e^{t\boldsymbol{L}} \in \mathrm{Aut}(\mathcal{K})$ for all $t \in \mathbb{R}$ or, equivalently, is the tangent space at the identity element when $\mathrm{Aut}(\mathcal{K})$ is seen as a Lie group.

Recall that a cone $\mathcal{K}$ is called *self-dual* if there exists a positive definite matrix $\boldsymbol{Q}$ such that $\boldsymbol{Q}\mathcal{K} = \mathcal{K}^*$. This is equivalent to the existence of *some* inner product under which $\mathcal{K}$ becomes self-dual, e.g., [29, Proposition 1]. A cone is *homogeneous* if for every $\boldsymbol{x}, \boldsymbol{y} \in \mathrm{ri}\,\mathcal{K}$, there is a matrix $\boldsymbol{A} \in \mathrm{Aut}(\mathcal{K})$ such that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$. A homogeneous and self-dual cone is called *symmetric* [18].

If a closed convex cone $\mathcal{K}$ can be expressed as a direct sum of two nonempty and nontrivial sets $\mathcal{K}_1, \mathcal{K}_2 \subset \mathcal{K}$, i.e., $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$ with $\mathcal{K}_1 \neq \{\mathbf{0}\}, \mathcal{K}_2 \neq \{\mathbf{0}\}$ and $\mathrm{span}(\mathcal{K}_1) \cap \mathrm{span}(\mathcal{K}_2) = \{\mathbf{0}\}$, then $\mathcal{K}$ is said to be *reducible*; it might not be immediately obvious, but this forces $\mathcal{K}_1$ and $\mathcal{K}_2$ to be convex cones, e.g., [39, Lemma 3.2]. Otherwise, $\mathcal{K}$ is said to be *irreducible* or *indecomposable*, e.g., [39, 3, 23].

3.3.1 Some theoretical context

It is relatively recent that the power cone has been a subject of research in optimization. However, the power cone was first considered in the 50's by Max Koecher in the context of the so-called *domains of positivity*, see [32]. More precisely, Koecher proposed a family of 3D cones in [32, Section 11,d)] which corresponds to $\mathcal{P}^{\boldsymbol{\alpha}}_{1,2}$, with $\alpha \in (0, 1)$. After that, the power cone languished in relative obscurity inside the optimization community, although it was discussed briefly in [67] and in [66] under the name of *Koecher cone*. As indicated in the introduction, several works helped to revitalize the interest in power cones by showcasing modelling applications, algorithms and software [11, 47, 61, 30, 55, 12].

When the power cone is bundled together in the class of "non-symmetric cones", it might be interesting to take a step back and understand two points: (a) how exactly the power cone fails to be symmetric and (b) why one should care about this.

Starting from the latter, what is special about symmetric cones is that they are supported by a powerful theory of *Jordan algebras* [18]. Being a symmetric cone is a *very* favourable property which was heavily exploited to develop efficient primal-dual interior point algorithms, e.g., [19]. However, being a symmetric cone is also restrictive for it is known that, up to linear isomorphism, each symmetric cone is a direct product of only five types of cones. The most remarkable examples of symmetric cones are the $\mathbb{R}^n_+$, the real symmetric positive semidefinite matrices $\mathcal{S}^n_+$, the second-order cone and the direct products of those three.

As for item (a), examining (3.1), we immediately see that the dual of $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ under the Euclidean inner product is just $\boldsymbol{D}\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$, where $\boldsymbol{D}$ is a diagonal matrix with positive entries, so $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ is indeed self-dual in the sense above. Thus the only gap between $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ and the class of symmetric cones is the homogeneity.

Given that being symmetric is very advantageous, one may reasonably wonder if the family of cones $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ parametrized by $\boldsymbol{\alpha}$ and m and n are indeed non-homogeneous in general. To the best of our knowledge, although it is well-known (e.g., see comments in [67, Section 4]) that $\mathcal{P}^{\boldsymbol{\alpha}}_{1,2}$ is non-homogeneous except when $\boldsymbol{\alpha} = (1/2, 1/2)$, there is no result on the generalized power cone regarding which combination of the parameters m, n and $\boldsymbol{\alpha}$ leads to homogeneity or not. We fill this gap with Theorem 3.12 and Corollary 3.14, which tells us precisely which of the generalized power cones are homogeneous or not.

We also completely determine the automorphism group of $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$. While this may seem an esoteric question, the automorphism group of a cone $\mathcal{K}$ is intimately connected to complementarity questions over $\mathcal{K}$. For example, it is known that $\boldsymbol{L}$ belongs to the Lie algebra of $\operatorname{Aut}(\mathcal{K})$ if and only if the following implication holds

$$\boldsymbol{x} \in \mathcal{K}, \boldsymbol{y} \in \mathcal{K}^*, \langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0 \Rightarrow \langle \boldsymbol{L}\boldsymbol{x}, \boldsymbol{y} \rangle = 0,$$

see [22]. If a cone has "enough" automorphisms then a complementarity problem can be rewritten as a square system using the matrices from the Lie algebra of $\operatorname{Aut}(\mathcal{K})$. In particular, when the dimension of $\operatorname{Aut}(\mathcal{K})$ is at least $\dim \mathcal{K}$, then the cone is said to be *perfect*, see [22, Page 5] and [53, Theorem 1]. An example of this phenomenon is how the conditions $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n_+, \langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$ imply n equations $x_i y_i = 0$ which is useful in several contexts.

The quantity $\dim \operatorname{Aut} \mathcal{K}$ is called the *Lyapunov rank of* $\mathcal{K}$ [22, 23] and is additive with respect to direct sums [22, Proposition 1]. Since any cone can be written as a direct sum of irreducible cones, it becomes important to identify *which* irreducible cones are perfect.

It is interesting to note that many of the examples of irreducible perfect cones in the literature (e.g., [22, 23, 53]) seem to be homogeneous. In addition, every homogeneous cone is perfect, which follows by known results about Lie groups, e.g., see [34, Theorem 21.20] or Section 2 in [52] which summarizes useful results. The final observation we will make in this chapter is that, surprisingly, for some choices of parameters, $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ is perfect but non-homogeneous, see Corollary 3.14. We note that in [64], Sznajder showed that there are choices of parameters for which the so-called *extended second order cone* is irreducible and perfect. This corresponds to a family of cones proposed by Németh and Zhang that contains the second order cones [49]. However, as far as we know, the homogeneity of those cones (or the lack thereof) was not discussed in general.

3.3.2 Automorphisms of the generalized power cone

In this section, we will prove our main results regarding $\mathrm{Aut}(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})$. The basic strategy is simple: if $\boldsymbol{A} \in \mathrm{Aut}(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})$, then $\boldsymbol{A}$ must map a face $\mathcal{F}_1$ of $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ to another face $\mathcal{F}_2$ of $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ with the same properties such as the dimension. More than that, the optimal exponents associated to FRFs of $\mathcal{F}_1$ and $\mathcal{F}_2$ must be the same. These conditions impose enough restrictions on $\boldsymbol{A}$ that we are able to completely determine its shape. Note that when $n = 2$ and $\boldsymbol{\alpha} = (1/2, 1/2)$, $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ is isomorphic to the second-order cone, whose automorphism group is well-known. Below, we focus on the complementary cases.

Theorem 3.11 (Automorphisms of $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$)**.** *For $m \geq 1, n > 2$ and any $\boldsymbol{\alpha} \in (0,1)^n$ such that $\sum_{i=1}^n \alpha_i = 1$, or for $m \geq 1, n = 2$ and any $\boldsymbol{\alpha} \in (0,1)^2$ such that $\alpha_1 \neq \alpha_2$ and $\alpha_1 + \alpha_2 = 1$, it holds that $\boldsymbol{A} \in \mathrm{Aut}(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})$ if and only if*

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{E} \end{bmatrix} \tag{3.34}$$

for some (invertible) generalized permutation matrix[6] *$\boldsymbol{E} \in \mathbb{R}^{n\times n}$ with positive nonzero entries and invertible matrix $\boldsymbol{B} \in \mathbb{R}^{m\times m}$ satisfying $\|\boldsymbol{B}\boldsymbol{x}\| = \prod_{k=1}^n (E_{k,l_k})^{\alpha_{l_k}} \|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \mathbb{R}^m$, where E_{k,l_k} is the nonzero element in the k-th row of $\boldsymbol{E}$ and $\alpha_{l_k} = \alpha_k$.*

Proof. Suppose that there exists a matrix

$$\boldsymbol{A} := \begin{bmatrix} \boldsymbol{B} & \boldsymbol{C} \\ \boldsymbol{D} & \boldsymbol{E} \end{bmatrix} \text{ with } \boldsymbol{B} \in \mathbb{R}^{m\times m}, \boldsymbol{C} \in \mathbb{R}^{m\times n}, \boldsymbol{D} \in \mathbb{R}^{n\times m}, \boldsymbol{E} \in \mathbb{R}^{n\times n}$$

such that $\boldsymbol{A}\mathcal{P}^{\boldsymbol{\alpha}}_{m,n} = \mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$.

First note that the entries of $\boldsymbol{E}$ must all be nonnegative, for if the (i,j)-th entry was negative, then we could pick a vector $\boldsymbol{q} := (\boldsymbol{0}, \boldsymbol{c}) \in \mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ with $c_j = 1$ and $c_k = 0$ for $k \neq j$, wherefore $\boldsymbol{A}\boldsymbol{q} \notin \mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$, which is a contradiction.

[6] A generalized permutation matrix is a matrix where in each column and each row there is exactly one nonzero entry.

Additionally, such a matrix $\boldsymbol{A}$ must be invertible and if ψ is an FRF for a face $\mathcal{F}_1 \trianglelefteq \mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$, then $\boldsymbol{A}$ must map $\mathcal{F}_1$ onto a face $\mathcal{F}_2 \trianglelefteq \mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ which has the same dimension and admits an FRF that is a positively rescaled shift of ψ; see [40, Proposition 17].

Observe from Section 4.1 that the generalized power cone has two types of faces defined in (3.4) and (3.5) (denoted by $\mathcal{F}_{\rm r}$ and $\mathcal{F}_{\boldsymbol{z}}$ respectively with an abuse of notation) with the corresponding (optimal) one-step facial residual functions in (3.32) and (3.33), respectively. We also notice that the dimension of the faces of the first type is 1, while the dimension of a face of the second type is $n - |\mathcal{I}|$. These lead to the following observations:

(I) Given an $\mathcal{I}$ with $\beta_{\mathcal{I}} := \sum_{i\in\mathcal{I}} \alpha_i$, if $|\mathcal{I}| < n-1$, i.e., the dimension of the corresponding face is larger than 1, then $\boldsymbol{A}$ must map the face associated with $\mathcal{I}$ to a face associated with an $\bar{\mathcal{I}}$ where $|\mathcal{I}| = |\bar{\mathcal{I}}|$ and $\beta_{\bar{\mathcal{I}}} = \beta_{\mathcal{I}}$.

(II) In the case when $n = 2$, since we assumed $\alpha_1 \neq \alpha_2$ and thus $\alpha_1 \neq 1/2$, $\boldsymbol{A}$ cannot map a one-dimensional face of type $\mathcal{F}_{\boldsymbol{z}}$ (whose FRF admits an optimal exponent of α_1 or α_2) to one of type $\mathcal{F}_{\rm r}$ (whose FRF admits an optimal exponent of 1/2).

Thus, a face of type $\mathcal{F}_{\boldsymbol{z}}$ with $|\mathcal{I}| = 1$ must be mapped to a face of the same type. From now on, for each $k \in \{1, 2, \ldots, n\}$, we let i_k and l_k be such that $\boldsymbol{A}\mathcal{F}_{\{k\}} = \mathcal{F}_{\{i_k\}}$ and $\boldsymbol{A}\mathcal{F}_{\{l_k\}} = \mathcal{F}_{\{k\}}$, where $\mathcal{F}_{\{k\}}$ denotes the face of type $\mathcal{F}_{\boldsymbol{z}}$ associated with $\mathcal{I} = \{k\}$. We deduce immediately from the above discussions that $\{1, 2, \ldots, n\} = \{i_1, i_2, \ldots, i_n\} = \{l_1, l_2, \ldots, l_n\}$ and $\alpha_k = \alpha_{i_k} = \alpha_{l_k}$.

Now, fix any $k \in \{1, 2, \ldots, n\}$. Then for any $\widetilde{\boldsymbol{x}}_{\mathcal{I}} := (c_1, \ldots, c_{k-1}, 0, c_{k+1}, \ldots, c_n)$ with $c_i > 0$ for all $i \neq k$, it must hold that $\boldsymbol{A}$ maps $\boldsymbol{x}_{\mathcal{I}} := (\boldsymbol{0}, \widetilde{\boldsymbol{x}}_{\mathcal{I}})$ to some $\boldsymbol{x}_{\hat{\mathcal{I}}} := (\boldsymbol{0}, \widetilde{\boldsymbol{x}}_{\hat{\mathcal{I}}})$ with $\hat{\mathcal{I}} = \{i_k\}$, $\alpha_k = \alpha_{i_k}$ and $(\widetilde{\boldsymbol{x}}_{\hat{\mathcal{I}}})_{i_k} = 0$. Thus,

$$\begin{bmatrix} \boldsymbol{B} & \boldsymbol{C} \\ \boldsymbol{D} & \boldsymbol{E} \end{bmatrix} \begin{bmatrix} \boldsymbol{0} \\ \widetilde{\boldsymbol{x}}_{\mathcal{I}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \widetilde{\boldsymbol{x}}_{\hat{\mathcal{I}}} \end{bmatrix}.$$

Therefore, we have $\boldsymbol{C}\widetilde{\boldsymbol{x}}_{\mathcal{I}} = \mathbf{0}$. This together with the arbitrariness of $c_i > 0$ shows that all except possibly the k-th column of $\boldsymbol{C}$ are $\mathbf{0}$. Since k is arbitrary, then we conclude that $\boldsymbol{C} = \mathbf{0}$.

Next, notice that we also have $\boldsymbol{E}\widetilde{\boldsymbol{x}}_{\mathcal{I}} = \widetilde{\boldsymbol{x}}_{\hat{\mathcal{I}}}$. Since $(\widetilde{\boldsymbol{x}}_{\hat{\mathcal{I}}})_{i_k} = 0$, we see that $E_{i_k}\widetilde{\boldsymbol{x}}_{\mathcal{I}} = 0$, where E_{i_k} is the i_k-th row of $\boldsymbol{E}$. Using again the arbitrariness of $c_i > 0$ in the definition of $\widetilde{\boldsymbol{x}}_{\mathcal{I}}$, we conclude that all entries of E_{i_k} are 0 except possibly for the k-th entry, i.e., E_{i_k} has only one possibly nonzero entry and that entry is nonnegative. From the arbitrariness of k and the fact that $\{i_1, i_2, \ldots, i_n\} = \{1, 2, \ldots, n\}$, we immediately obtain that every entry of the i_k-th row $\boldsymbol{E}$ has all of its entries equal to zero except possibly for the k-th, which is nonnegative.

Taking into account of the fact that $\boldsymbol{A}$ is invertible and $\boldsymbol{C} = \mathbf{0}$, we know that none of the columns of $\boldsymbol{E}$ can be identically zero, and so we altogether have that each of the rows and columns of $\boldsymbol{E}$ consists of one strictly positive entry, with all other entries identically zero. Then, we have shown that

$$E_{s,r} \neq 0 \quad \Leftrightarrow \quad (s, r) = (i_k, k) \text{ for some } k \in \{1, 2, \ldots, n\}, \tag{3.35}$$

where the latter condition is also equivalent to $(s, r) = (k, l_k)$ for some $k \in \{1, 2, \ldots, n\}$.

We next claim that $\boldsymbol{A}$ must map faces of type $\mathcal{F}_{\mathrm{r}}$ to a face of type $\mathcal{F}_{\mathrm{r}}$. Since $\boldsymbol{A}$ must permute faces whose FRFs admit the same optimal exponent, we only need to consider the extreme case that there exists a face of type $\mathcal{F}_{\boldsymbol{z}}$ corresponding to an $\mathcal{I} := \{1, 2, \ldots, i-1, i+1, \ldots, n\}$ for some i (i.e., the dimension of the corresponding face is 1) with $\beta_{\mathcal{I}} = 1/2$, and argue that $\boldsymbol{A}$ cannot map $\mathcal{F}_{\mathrm{r}}$ onto such $\mathcal{F}_{\boldsymbol{z}}$. Suppose for contradiction that this happens; then there must exist $\boldsymbol{x} = (\overline{\boldsymbol{x}}, \widetilde{\boldsymbol{x}})$ in some face of type $\mathcal{F}_{\mathrm{r}}$ with $\overline{\boldsymbol{x}} \neq \mathbf{0}$ and $\widetilde{x}_i > 0$ for all i such that

$$\begin{bmatrix} \boldsymbol{B} & \mathbf{0} \\ \boldsymbol{D} & \boldsymbol{E} \end{bmatrix} \begin{bmatrix} \overline{\boldsymbol{x}} \\ \widetilde{\boldsymbol{x}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{e}_i \end{bmatrix},$$

where $\boldsymbol{e}_i \in \mathbb{R}^n$ is the vector whose elements are all zero except for the i-th element being 1. However, this cannot happen because $\boldsymbol{B}\overline{\boldsymbol{x}} = \boldsymbol{0}$ and the invertibility of $\boldsymbol{B}$ (a consequence of invertibility of $\boldsymbol{A}$) implies $\overline{\boldsymbol{x}} = \boldsymbol{0}$, leading to a contradiction. Hence, $\boldsymbol{A}$ must map faces of type $\mathcal{F}_{\mathrm{r}}$ onto a face of type $\mathcal{F}_{\mathrm{r}}$.

Thus, for any $\boldsymbol{x} = (\overline{\boldsymbol{x}}, \widetilde{\boldsymbol{x}})$ in one of the type $\mathcal{F}_{\mathrm{r}}$ faces with $\overline{\boldsymbol{x}} \neq \boldsymbol{0}$, $\min_i\{\widetilde{x}_i\} > 0$ and $\|\overline{\boldsymbol{x}}\| = \prod_{i=1}^n \widetilde{x}_i^{\alpha_i}$, there must be $\boldsymbol{y} = (\overline{\boldsymbol{y}}, \widetilde{\boldsymbol{y}})$ in one of the type $\mathcal{F}_{\mathrm{r}}$ faces with $\overline{\boldsymbol{y}} \neq \boldsymbol{0}$, $\min_i\{\widetilde{y}_i\} > 0$ and $\|\overline{\boldsymbol{y}}\| = \prod_{i=1}^n \widetilde{y}_i^{\alpha_i}$ such that

$$\begin{bmatrix} \boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{D} & \boldsymbol{E} \end{bmatrix} \begin{bmatrix} \overline{\boldsymbol{x}} \\ \widetilde{\boldsymbol{x}} \end{bmatrix} = \begin{bmatrix} \overline{\boldsymbol{y}} \\ \widetilde{\boldsymbol{y}} \end{bmatrix}.$$

Recall that there is exactly one nonzero element in each row of $\boldsymbol{E}$, and this element is positive. From the definition of l_k, this nonzero element is E_{k,l_k}; see (3.35).

Fix any j and $k \in \{1, \ldots, n\}$. Pick any $(\overline{\boldsymbol{x}}, \widetilde{\boldsymbol{x}}) \in \mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ such that $\overline{\boldsymbol{x}} = \boldsymbol{e}_j$ and $\prod_{i=1}^n \widetilde{x}_i^{\alpha_i} = 1$. For any $t > 0$, one can check that $(t^{\alpha_{l_k}}\overline{\boldsymbol{x}}, \widetilde{x}_1, \cdots, t\widetilde{x}_{l_k}, \cdots, \widetilde{x}_n) \in \mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ belongs to a face of type $\mathcal{F}_{\mathrm{r}}$. Thus, there exists $(\overline{\boldsymbol{y}}, \widetilde{\boldsymbol{y}})$ such that

$$t^{\alpha_{l_k}}\boldsymbol{B}\boldsymbol{e}_j = \overline{\boldsymbol{y}} \quad \text{and} \quad t^{\alpha_{l_k}} D_{k,j} + t E_{k,l_k}\widetilde{x}_{l_k} = \widetilde{y}_k > 0.$$

The second relation implies that $D_{k,j} + t^{1-\alpha_{l_k}} E_{k,l_k}\widetilde{x}_{l_k} > 0$. Letting $t \downarrow 0$, we conclude that $D_{k,j} \geq 0$. As the choices of j and k were arbitrary, we see that all entries of $\boldsymbol{D}$ are nonnegative. Considering $\overline{\boldsymbol{x}} = -\boldsymbol{e}_j$, a similar argument shows that all entries of $\boldsymbol{D}$ are nonpositive. Hence, $\boldsymbol{D} = \boldsymbol{0}$.

Now, for any $\overline{\boldsymbol{x}} \in \mathbb{R}^m$, pick any $(\overline{\boldsymbol{x}}, \widetilde{\boldsymbol{x}}) \in \partial\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$. Because $\boldsymbol{A}$ is invertible and $\boldsymbol{A}\mathcal{P}^{\boldsymbol{\alpha}}_{m,n} = \mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$, which implies $\boldsymbol{A}\,\mathrm{ri}\,\mathcal{P}^{\boldsymbol{\alpha}}_{m,n} = \mathrm{ri}\,\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ and $\boldsymbol{A}\partial\mathcal{P}^{\boldsymbol{\alpha}}_{m,n} = \partial\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$, then there exists $(\overline{\boldsymbol{y}}, \widetilde{\boldsymbol{y}}) \in \partial\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ so that

$$\boldsymbol{B}\overline{\boldsymbol{x}} = \overline{\boldsymbol{y}} \quad \text{and} \quad E_{k,l_k}\widetilde{x}_{l_k} = \widetilde{y}_k \ \text{ for } k = 1, 2, \ldots, n.$$

Thus,

$$\|\boldsymbol{B}\overline{\boldsymbol{x}}\| = \|\overline{\boldsymbol{y}}\| = \prod_{k=1}^{n} \widetilde{y}_k^{\alpha_k} = \prod_{k=1}^{n} (E_{k,l_k}\widetilde{x}_{l_k})^{\alpha_k} \overset{\text{(a)}}{=} \prod_{k=1}^{n} (E_{k,l_k}\widetilde{x}_{l_k})^{\alpha_{l_k}}$$

$$= \prod_{k=1}^{n} E_{k,l_k}^{\alpha_{l_k}} \prod_{i=1}^{n} \widetilde{x}_i^{\alpha_i} = \prod_{k=1}^{n} E_{k,l_k}^{\alpha_{l_k}} \|\overline{\boldsymbol{x}}\|.$$

where (a) holds as $\alpha_k = \alpha_{l_k}$ for all k. The above shows the necessity of the form in (3.34).

Conversely, if $\boldsymbol{A}$ is a matrix of the form (3.34), then $\boldsymbol{A}$ must be invertible since $\boldsymbol{B}$ and $\boldsymbol{E}$ are invertible. For any $\boldsymbol{x} = (\overline{\boldsymbol{x}}, \widetilde{\boldsymbol{x}}) \in \mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$, we have $\boldsymbol{A}\boldsymbol{x} = (\boldsymbol{B}\overline{\boldsymbol{x}}, \boldsymbol{E}\widetilde{\boldsymbol{x}})$. Hence,

$$\|\boldsymbol{B}\overline{\boldsymbol{x}}\| = \prod_{k=1}^{n} E_{k,l_k}^{\alpha_{l_k}} \|\overline{\boldsymbol{x}}\| \le \prod_{k=1}^{n} E_{k,l_k}^{\alpha_{l_k}} \prod_{i=1}^{n} \widetilde{x}_i^{\alpha_i} = \prod_{k=1}^{n} \left(E_{k,l_k}\widetilde{x}_{l_k}\right)^{\alpha_{l_k}},$$

where the last equality holds as $\{1,\ldots,n\} = \{l_1,\ldots,l_n\}$. This implies $\boldsymbol{A}\mathcal{P}_{m,n}^{\boldsymbol{\alpha}} \subseteq \mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$.

We claim

$$\text{(i)}\ \left(\boldsymbol{E}^{-1}\right)_{i,j} = \begin{cases} 0, & E_{j,i} = 0, \\ \frac{1}{E_{j,i}}, & E_{j,i} \neq 0. \end{cases} \quad \text{(ii)}\ \|\boldsymbol{B}^{-1}\boldsymbol{x}\| = \prod_{k=1}^{n} E_{k,l_k}^{-\alpha_{l_k}} \|\boldsymbol{x}\| \quad \forall \boldsymbol{x} \in \mathbb{R}^m. \quad (3.36)$$

Granting these, we have that for any $\boldsymbol{x} = (\overline{\boldsymbol{x}}, \widetilde{\boldsymbol{x}}) \in \mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$, $\boldsymbol{A}^{-1}\boldsymbol{x} = (\boldsymbol{B}^{-1}\overline{\boldsymbol{x}}, E^{-1}\widetilde{\boldsymbol{x}})$ satisfies

$$\prod_{i=1}^{n} (\boldsymbol{E}^{-1}\widetilde{\boldsymbol{x}})_i^{\alpha_i} = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} (\boldsymbol{E}^{-1})_{i,j}\widetilde{x}_j\right)^{\alpha_i} \overset{\text{(a)}}{=} \prod_{k=1}^{n} \left((\boldsymbol{E}^{-1})_{l_k,k}\widetilde{x}_k\right)^{\alpha_{l_k}} = \prod_{k=1}^{n} (E_{k,l_k}^{-1}\widetilde{x}_k)^{\alpha_{l_k}}$$

$$= \prod_{k=1}^{n} E_{k,l_k}^{-\alpha_{l_k}} \prod_{i=1}^{n} \widetilde{x}_i^{\alpha_{l_i}} \overset{\text{(b)}}{=} \prod_{k=1}^{n} E_{k,l_k}^{-\alpha_{l_k}} \prod_{i=1}^{n} \widetilde{x}_i^{\alpha_i} \ge \prod_{k=1}^{n} E_{k,l_k}^{-\alpha_{l_k}} \|\overline{\boldsymbol{x}}\| \overset{\text{(c)}}{=} \|\boldsymbol{B}^{-1}\overline{\boldsymbol{x}}\|,$$

where (a) is true thanks to the fact that in the sum there is only one nonzero term, which comes from identity (i) and (3.35); (b) holds because $\alpha_k = \alpha_{l_k}$ for all k; (c) comes from identity (ii). Hence, $\boldsymbol{A}^{-1}\boldsymbol{x} \in \mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$. This implies $\boldsymbol{A}\mathcal{P}^{\boldsymbol{\alpha}}_{m,n} \supseteq \mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ and consequently $\boldsymbol{A}\mathcal{P}^{\boldsymbol{\alpha}}_{m,n} = \mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$.

Now, it remains to show (3.36). Since $\boldsymbol{E}$ is a generalized permutation matrix with all nonzero elements being positive, then we immediately have (i) from $\boldsymbol{E}\boldsymbol{E}^{-1} = \boldsymbol{I}_n$. Recall that, by assumption, $\|\boldsymbol{B}\boldsymbol{x}\| = \prod_{k=1}^{n} E_{k,l_k}^{\alpha_{l_k}} \|\boldsymbol{x}\|$ for any $\boldsymbol{x} \in \mathbb{R}^m$ and $\boldsymbol{B}$ is invertible. Using these, we can deduce (ii) in (3.36) as follows: for any $\boldsymbol{x} \in \mathbb{R}^m$,

$$\|\boldsymbol{x}\| = \|\boldsymbol{B}\boldsymbol{B}^{-1}\boldsymbol{x}\| = \prod_{k=1}^{n} E_{k,l_k}^{\alpha_{l_k}} \|\boldsymbol{B}^{-1}\boldsymbol{x}\|.$$

□

The next theorem is about the dimension of $\operatorname{Aut}(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})$.

Theorem 3.12. *Let $m \geq 1$, $n \geq 2$ and $\boldsymbol{\alpha} \in (0,1)^n$ such that $\sum_{i=1}^{n} \alpha_i = 1$, then we have the following statements about* $\dim \operatorname{Aut}(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})$.

(i) If $m \geq 1$, $n = 2$ and $\boldsymbol{\alpha} := (1/2, 1/2)$, then $\dim \operatorname{Aut}(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}) = (m^2 + 3m + 4)/2$.

(ii) If $m \geq 1$, $n > 2$ and $\sum_{i=1}^{n} \alpha_i = 1$ or $m \geq 1$, $n = 2$, $\alpha_1 \neq \alpha_2$ and $\alpha_1 + \alpha_2 = 1$, then:

$$\operatorname{Lie} \operatorname{Aut}(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}) = \left\{ \begin{bmatrix} \boldsymbol{G} & \boldsymbol{0} \\ \boldsymbol{0} & \operatorname{Diag}(\boldsymbol{h}) \end{bmatrix} \;\middle|\; \begin{array}{l} \boldsymbol{G} + \boldsymbol{G}^\top = 2\boldsymbol{\alpha}^\top \boldsymbol{h} \boldsymbol{I}_m, \\ \boldsymbol{G} \in \mathbb{R}^{m \times m},\ \boldsymbol{h} \in \mathbb{R}^n \end{array} \right\}. \tag{3.37}$$

Hence, $\dim \operatorname{Aut}(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}) = \dim \operatorname{Lie} \operatorname{Aut}(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}) = n + m(m-1)/2$.

Proof. (i) If $m \geq 1$, $n = 2$ and $\boldsymbol{\alpha} := (1/2, 1/2)$, then $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ is isomorphic to a second-order cone; see, [47, Section 3.1.2]. Hence, we know from [22, Page 12 (v)] that

$$\dim \operatorname{Aut}(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}) = \frac{(m+2)^2 - m}{2} = \frac{m^2 + 3m + 4}{2}.$$

(ii) By [24, Corollary 3.45], $\dim \operatorname{Aut}(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}) = \dim \operatorname{Lie} \operatorname{Aut}(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})$. This in addition to [24, Corollary 3.46] show that it suffices to calculate the dimension of the tangent space at the identity of $\operatorname{Aut}(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})$ to obtain $\dim \operatorname{Aut}(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})$.

First, we compute $\mathrm{Lie\,Aut}\,(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})$ and for that we consider an arbitrary continuously differentiable curve $\boldsymbol{F} : (-1,1) \to \mathrm{Aut}\,(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})$ with $\boldsymbol{F}(0) = \boldsymbol{I}_{m+n}$ and $\boldsymbol{F}(t) \in \mathrm{Aut}\,(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})$ for any $t \in (-1,1)$. We further denote

$$\boldsymbol{F}(t) = \begin{bmatrix} \boldsymbol{G}_t & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{H}_t \end{bmatrix} \quad \text{and} \quad \dot{\boldsymbol{F}}(t) = \begin{bmatrix} \dot{\boldsymbol{G}}_t & \boldsymbol{0} \\ \boldsymbol{0} & \dot{\boldsymbol{H}}_t \end{bmatrix},$$

where $\boldsymbol{G}_t \in \mathbb{R}^{m\times m}$ and $\boldsymbol{H}_t \in \mathbb{R}^{n\times n}$ are both invertible; $\boldsymbol{G}_0 = \boldsymbol{I}_m$, $\boldsymbol{H}_0 = \boldsymbol{I}_n$; $\boldsymbol{H}_t$ is a generalized permutation matrix with all nonzero elements being strictly positive (which we assume, by suitably shrinking the neighborhood of definition of $\boldsymbol{F}$ and reparameterizing, to be only nonzero along the diagonal); $\dot{\boldsymbol{F}}(0)$ lies in the tangent space of $\mathrm{Aut}\,(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})$ at $\boldsymbol{I}$, that is,

$$\dot{\boldsymbol{F}}(0) = \begin{bmatrix} \dot{\boldsymbol{G}}_0 & \boldsymbol{0} \\ \boldsymbol{0} & \dot{\boldsymbol{H}}_0 \end{bmatrix} \in \mathrm{Lie\,Aut}\,(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}); \tag{3.38}$$

$\dot{\boldsymbol{G}}_t$ and $\dot{\boldsymbol{H}}_t$ refer to the componentwise derivative of $\boldsymbol{G}$ and $\boldsymbol{H}$ with respect to t, respectively.

Since $\boldsymbol{H}_t$ and $\dot{\boldsymbol{H}}_t$ are diagonal, we let $\boldsymbol{h}_t$ and $\dot{\boldsymbol{h}}_t$ be the diagonal vectors of $\boldsymbol{H}_t$ and $\dot{\boldsymbol{H}}_t$, respectively, i.e., $\boldsymbol{H}_t = \mathrm{Diag}(\boldsymbol{h}_t)$ and $\dot{\boldsymbol{H}}_t = \mathrm{Diag}(\dot{\boldsymbol{h}}_t)$. We also let h_t^k and $\dot{h}_t^k$ denote the k-th element of the vectors $\boldsymbol{h}_t$ and $\dot{\boldsymbol{h}}_t$ respectively. Then, from Theorem 3.11,

$$\|\boldsymbol{G}_t\boldsymbol{x}\|^2 = \prod_{k=1}^{n} (h_t^k)^{2\alpha_k} \|\boldsymbol{x}\|^2, \quad \forall\, \boldsymbol{x} \in \mathbb{R}^m, \forall\, t \in (-1,1). \tag{3.39}$$

Differentiating[7] both sides of (3.39) with respect to t, we can obtain

$$2\boldsymbol{x}^\top \boldsymbol{G}_t^\top \dot{\boldsymbol{G}}_t \boldsymbol{x} = \boldsymbol{x}^\top \boldsymbol{x} \sum_{k=1}^{n} 2\alpha_k \left(h_t^k\right)^{2\alpha_k - 1} \dot{h}_t^k \prod_{j\neq k} \left(h_t^j\right)^{2\alpha_j} = \boldsymbol{x}^\top \boldsymbol{x} \sum_{k=1}^{n} 2 \frac{\alpha_k}{h_t^k} \dot{h}_t^k \prod_{j=1}^{n} \left(h_t^j\right)^{2\alpha_j}$$

$$= 2\left(\boldsymbol{x}^\top \boldsymbol{x} \prod_{j=1}^{n} (h_t^j)^{2\alpha_j}\right) \sum_{k=1}^{n} \frac{\alpha_k}{h_t^k} \dot{h}_t^k \overset{\text{(a)}}{=} 2\boldsymbol{x}^\top \boldsymbol{G}_t^\top \boldsymbol{G}_t \boldsymbol{x} \left(\boldsymbol{\alpha} \circ (\boldsymbol{h}_t)^{-1}\right)^\top \dot{\boldsymbol{h}}_t,$$

[7] This calculation simply uses the chain rule to differentiate $(h_t^k)^{2\alpha_k}$ for a given k, and then applies the product rule for the product over all k.

where the inverse is taken componentwise, and the rest of (a) comes from (3.39). Notice that $\left(\boldsymbol{\alpha} \circ (\boldsymbol{h}_t)^{-1}\right)^\top \dot{\boldsymbol{h}}_t$ is a scalar, by rearranging terms, one has

$$\boldsymbol{x}^\top \left[\boldsymbol{G}_t^\top \dot{\boldsymbol{G}}_t - \left(\boldsymbol{\alpha} \circ (\boldsymbol{h}_t)^{-1}\right)^\top \dot{\boldsymbol{h}}_t \boldsymbol{G}_t^\top \boldsymbol{G}_t\right] \boldsymbol{x} = 0, \quad \forall\, \boldsymbol{x} \in \mathbb{R}^m,\ \forall\, t \in (-1, 1).$$

Letting $t = 0$ and recalling $\boldsymbol{G}_0 = \boldsymbol{I}_m, \boldsymbol{H}_0 = \boldsymbol{I}_n$, we have

$$\boldsymbol{x}^\top \left(\dot{\boldsymbol{G}}_0 - \boldsymbol{\alpha}^\top \dot{\boldsymbol{h}}_0 I_m\right) \boldsymbol{x} = 0, \quad \forall\, \boldsymbol{x} \in \mathbb{R}^m. \tag{3.40}$$

Recall that $2\boldsymbol{x}^\top \dot{\boldsymbol{G}}_0 \boldsymbol{x} = \boldsymbol{x}^\top (\dot{\boldsymbol{G}}_0 + \dot{\boldsymbol{G}}_0^\top) \boldsymbol{x}$. We can thus rewrite (3.40) as

$$\boldsymbol{x}^\top \left(\dot{\boldsymbol{G}}_0 + \dot{\boldsymbol{G}}_0^\top - 2\boldsymbol{\alpha}^\top \dot{\boldsymbol{h}}_0 I_m\right) \boldsymbol{x} = 0, \quad \forall\, \boldsymbol{x} \in \mathbb{R}^m.$$

Since the matrix in the parentheses is zero, the above display implies that

$$\dot{\boldsymbol{G}}_0 + \dot{\boldsymbol{G}}_0^\top = 2\boldsymbol{\alpha}^\top \dot{\boldsymbol{h}}_0 I_m.$$

The above derivation and (3.38) show that any matrix in $\operatorname{Lie}\operatorname{Aut}(\mathcal{P}_{m,n}^{\boldsymbol{\alpha}})$ satisfies the above display.

Conversely, suppose that $\boldsymbol{G}$ and $\operatorname{Diag}(\boldsymbol{h})$ are such that $\boldsymbol{G} + \boldsymbol{G}^\top = 2\boldsymbol{\alpha}^\top \boldsymbol{h} I_m$ and $\boldsymbol{U} := \begin{bmatrix} \boldsymbol{G} & \boldsymbol{0} \\ \boldsymbol{0} & \operatorname{Diag}(\boldsymbol{h}) \end{bmatrix}$. We need to show that the matrix exponential $e^{t\boldsymbol{U}}$ belongs to $\operatorname{Aut}(\mathcal{P}_{m,n}^{\boldsymbol{\alpha}})$ for every $t \in \mathbb{R}$. To this end, recall that $e^{\boldsymbol{X}+\boldsymbol{Y}} = e^{\boldsymbol{X}} e^{\boldsymbol{Y}}$ if $\boldsymbol{X}\boldsymbol{Y} = \boldsymbol{Y}\boldsymbol{X}$, we have

$$e^{t\boldsymbol{G}} = e^{2t\boldsymbol{\alpha}^\top \boldsymbol{h} I_m - t\boldsymbol{G}^\top} = e^{2t\boldsymbol{\alpha}^\top \boldsymbol{h} \boldsymbol{I}_m} e^{-t\boldsymbol{G}^\top} = e^{2t\boldsymbol{\alpha}^\top \boldsymbol{h}} e^{-t\boldsymbol{G}^\top},$$

since $2t\boldsymbol{\alpha}^\top \boldsymbol{h} \boldsymbol{I}_m$ and $-t\boldsymbol{G}^\top$ commute. This shows that $(e^{t\boldsymbol{G}})^\top e^{t\boldsymbol{G}} = e^{t\boldsymbol{G}^\top} e^{t\boldsymbol{G}} = e^{2t\boldsymbol{\alpha}^\top \boldsymbol{h}} \boldsymbol{I}_m$, i.e., $e^{t\boldsymbol{G}}$ is an orthogonal matrix multiplied by the scalar $e^{t\boldsymbol{\alpha}^\top \boldsymbol{h}}$. Then

$$\|e^{t\boldsymbol{G}} \boldsymbol{x}\| = e^{t\boldsymbol{\alpha}^\top \boldsymbol{h}} \|\boldsymbol{x}\| = e^{\sum_{i=1}^n t h_i \alpha_i} \|\boldsymbol{x}\| = \prod_{i=1}^n \left(e^{t h_i}\right)^{\alpha_i} \|\boldsymbol{x}\| \quad \forall \boldsymbol{x} \in \mathbb{R}^m. \tag{3.41}$$

Since

$$e^{t\boldsymbol{U}} = \begin{bmatrix} e^{t\boldsymbol{G}} & \boldsymbol{0} \\ \boldsymbol{0} & e^{\mathrm{Diag}(t\boldsymbol{h})} \end{bmatrix} = \begin{bmatrix} e^{t\boldsymbol{G}} & \boldsymbol{0} \\ \boldsymbol{0} & \mathrm{Diag}(e^{t\boldsymbol{h}}) \end{bmatrix},$$

where $e^{t\boldsymbol{h}}$ corresponds to the vector such that its i-th component is e^{th_i} and h_i is the i-th component of $\boldsymbol{h}$, we conclude from (3.41) and Theorem 3.11 that $e^{t\boldsymbol{U}} \in \mathrm{Aut}\,(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})$.

Finally, a direct computation shows that the dimension of the right-hand side of (3.37) is $n + m(m-1)/2$, which is just the claimed dimension. □

3.3.3 Homogeneity, irreducibility and perfectness of generalized power cone

In this section, we will use Theorem 3.11 to prove the homogeneity, irreducibility and perfectness of $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$. Before moving on, we recall the following lemma.

Lemma 3.13. *(i) If a closed convex pointed cone $\mathcal{K}$ is reducible, i.e., $\mathcal{K}$ is a direct sum of two nonempty, nontrivial sets $\mathcal{K}_1$ and $\mathcal{K}_2$, then we have $\mathcal{K}_1 \precneqq \mathcal{K}$, $\mathcal{K}_2 \precneqq \mathcal{K}$ and $\dim(\mathcal{K}) = \dim(\mathcal{K}_1) + \dim(\mathcal{K}_2)$.*

(ii) A proper cone $\mathcal{K} \subseteq \mathbb{R}^p$ is perfect if and only if $\dim \mathrm{Lie}\,\mathrm{Aut}\,(\mathcal{K}) \geq p$.

Proof. (i) The fact that $\mathcal{K}_1$ and $\mathcal{K}_2$ are faces is well-known, see [39, Lemma 3.2]. The conclusion on dimensions follows directly from the definition of direct sum.

(ii) This fact comes from [53, Theorem 1] and the first display on [22, Page 4]. □

Using Lemma 3.13, Theorems 3.11 and 3.12, we have the following corollary.

Corollary 3.14. *Let $m \geq 1$, $n \geq 2$ and $\boldsymbol{\alpha} \in (0,1)^n$ such that $\sum_{i=1}^n \alpha_i = 1$, then the following statements hold for the generalized power cone $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$.*

(i) $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ is irreducible.

(ii) If $m \geq 1, n = 2$ and $\boldsymbol{\alpha} := (1/2, 1/2)$, then $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ is homogeneous and perfect.

(iii) If $m \geq 1$, $n > 2$ and $\sum_{i=1}^n \alpha_i = 1$ or $m \geq 1$, $n = 2$, $\alpha_1 \neq \alpha_2$ and $\alpha_1 + \alpha_2 = 1$, then $\mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$ is nonhomogeneous. In addition, if $1 \leq m \leq 2$, then $\mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$ is not perfect; if $m \geq 3$, then $\mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$ is perfect.

Proof. (i) Recall that the two types of faces of $\mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$ are defined as in (3.4) and (3.5), with dimensions being 1 and $n - |\mathcal{I}|$, respectively. Since $\mathcal{I} \neq \emptyset$ and so $|\mathcal{I}| \geq 1$, for any possible pair of nontrivial faces $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$, we have $\dim(\mathcal{F}_1) + \dim(\mathcal{F}_2) < m+n = \dim(\mathcal{P}_{m,n}^{\boldsymbol{\alpha}})$. This together with Lemma 3.13(i) show that $\mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$ is irreducible.

(ii) If $m \geq 1, n = 2$ and $\boldsymbol{\alpha} := (1/2, 1/2)$, $\mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$ is isomorphic to a second-order cone and so is homogeneous; see, for example, [47, Section 3.1.2]. The perfectness holds by Theorem 3.12(i) and Lemma 3.13(ii).

(iii) Take any $m \geq 1, n > 2$ with any $\boldsymbol{\alpha} \in (0,1)^n$ such that $\sum_{i=1}^n \alpha_i = 1$ or $m \geq 1, n = 2$ with any $\boldsymbol{\alpha} \in (0,1)^2$ such that $\alpha_1 \neq \alpha_2$, consider $\boldsymbol{x} = (\boldsymbol{0}, \widetilde{\boldsymbol{x}}) \in \operatorname{ri} \mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$ and $\boldsymbol{y} = (\overline{\boldsymbol{y}}, \widetilde{\boldsymbol{y}}) \in \operatorname{ri} \mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$, where $\min_i\{\widetilde{x}_i\} > 0$, $\min_i\{\widetilde{y}\} > 0$ and $\overline{\boldsymbol{y}} \neq \boldsymbol{0}$, $\|\overline{\boldsymbol{y}}\| < \prod_{i=1}^n \widetilde{y}_i^{\alpha_i}$. Using (3.34), for all $\boldsymbol{A}$ such that $\boldsymbol{A}\mathcal{P}_{m,n}^{\boldsymbol{\alpha}} = \mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$, we have $\boldsymbol{Ax} \neq \boldsymbol{y}$ because $\boldsymbol{B0} = \boldsymbol{0} \neq \overline{\boldsymbol{y}}$ for all possible $\boldsymbol{B}$. Then by definition, $\mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$ with $m \geq 1$, $n = 2$ and $\sum_{i=1}^n \alpha_i = 1$ or $m \geq 1$, $n = 2$, $\alpha_1 \neq \alpha_2$ and $\alpha_1 + \alpha_2 = 1$ is nonhomogeneous. By Theorem 3.12(ii), we have $\dim \operatorname{Lie} \operatorname{Aut}(\mathcal{P}_{m,n}^{\boldsymbol{\alpha}}) = n + \frac{m(m-1)}{2} \geq m + n$ if and only if $m \geq 3$. The conclusion concerning perfectness now follows from this and Lemma 3.13(ii). $\square$

Chapter 4

Log-determinant Cones

In this section, we will compute the one-step facial residual functions for the log-determinant cones, and obtain error bounds. Let d be a positive integer and $\mathrm{sd}(d) := \frac{d(d+1)}{2}$ be the dimension of $\mathcal{S}^d$, we consider the $(\mathrm{sd}(d)+2)$-dimensional space $\mathbb{R}\times\mathbb{R}\times\mathcal{S}^d$. We let $\boldsymbol{x} := (\boldsymbol{x}_x, \boldsymbol{x}_y, \boldsymbol{x}_Z)$ denote an element of $\mathbb{R}\times\mathbb{R}\times\mathcal{S}^d$, where $\boldsymbol{x}_x \in \mathbb{R}, \boldsymbol{x}_y \in \mathbb{R}$ and $\boldsymbol{x}_Z \in \mathcal{S}^d$, and equip $\mathbb{R}\times\mathbb{R}\times\mathcal{S}^d$ with the following inner product:

$$\langle \boldsymbol{x}, \boldsymbol{z}\rangle = \boldsymbol{x}_x\boldsymbol{z}_x + \boldsymbol{x}_y\boldsymbol{z}_y + \mathrm{tr}(\boldsymbol{x}_Z\boldsymbol{z}_Z) \quad \text{for any} \quad \boldsymbol{x}, \boldsymbol{z} \in \mathbb{R}\times\mathbb{R}\times\mathcal{S}^d.$$

Recall that the log-determinant cone is defined as follows.

$$\mathcal{K}_{\mathrm{logdet}} := \left\{(x,y,Z)\in\mathbb{R}\times\mathbb{R}_{++}\times\mathcal{S}^d_{++} : x \le y\log\det(Z/y)\right\} \cup (\mathbb{R}_-\times\{0\}\times\mathcal{S}^d_+) \tag{4.1}$$

$$= \left\{(x,y,Z)\in\mathbb{R}\times\mathbb{R}_{++}\times\mathcal{S}^d_{++} : y^d e^{x/y} \le \det(Z)\right\} \cup (\mathbb{R}_-\times\{0\}\times\mathcal{S}^d_+). \tag{4.2}$$

Its dual cone is given by

$$\mathcal{K}^*_{\mathrm{logdet}} := \left\{(x,y,Z)\in\mathbb{R}_{--}\times\mathbb{R}\times\mathcal{S}^d_{++} : y \ge x(\log\det(-Z/x)+d)\right\}\cup(\{0\}\times\mathbb{R}_+\times\mathcal{S}^d_+) \tag{4.3}$$

$$= \left\{(x,y,Z)\in\mathbb{R}_{--}\times\mathbb{R}\times\mathcal{S}^d_{++} : (-x)^d e^{y/x} \le e^d\det(Z)\right\}\cup(\{0\}\times\mathbb{R}_+\times\mathcal{S}^d_+). \tag{4.4}$$

In terms of the derivation of the dual cone, here is a sketch. Let $f : \mathcal{S}^d_{++} \to \mathbb{R}$ be such that $f(Z) = -d - \log\det(Z)$ and let $\mathcal{K}$ be the closed convex cone generated by the set $C := \{(1, y, Z) \mid f(Z) \le y\}$. We have $\mathcal{K} = \mathrm{cl}\,\{(x, y, Z) \in \mathbb{R}_{++}\times\mathbb{R}\times\mathcal{S}^d_{++} \mid xf(Z/x) \le y\}$. That is, $\mathcal{K}$ is the closure of $\{(x, y, Z) \in \mathbb{R}_{++}\times\mathbb{R}\times\mathcal{S}^d_{++} \mid y \ge$

$x(-\log\det(Z/x)-d)\}$. By [58, Theorem 14.4], the closed convex cone $\bar{\mathcal{K}}$ generated by $\{(1,v,W) \mid v \geq f^*(W)\}$ satisfies $\bar{\mathcal{K}} = \{(u,v,W) \mid (-v,-u,W) \in \mathcal{K}^\circ\}$, where $\mathcal{K}^\circ$ is the polar of $\mathcal{K}$. The conjugate of f is $-\log\det(-W)$ for $W \in -\mathcal{S}^d_{++}$. Overall, we conclude that $(x,y,Z) \in \mathcal{K}^*$ iff $(-x,-y,-Z) \in \mathcal{K}^\circ$ iff $(y,x,-Z)$ is in the closure of $\{(u,v,W) \in \mathbb{R}_{++} \times \mathbb{R} \times -\mathcal{S}^d_{++} \mid v \geq -u\log\det(-W/u)\}$. Finally, this implies that $(x,y,Z) \in \mathcal{K}^*$ if and only if (x,y,Z) is in the closure of $\{(x,y,Z) \in \mathbb{R} \times \mathbb{R}_{++} \times \mathcal{S}^d_{++} \mid -x \leq y\log\det(Z/y)\}$. This means $(x,y,Z) \in \mathcal{K}^*$ iff $(-x,y,Z) \in \mathcal{K}_{\text{logdet}}$. Thus, we conclude that the cones in (4.1) and (4.3) are dual to each other.

One should notice that if $d = 1$, then the log-determinant cone reduces to the exponential cone, whose corresponding error bound results were discussed in [36]. Hence, without loss of generality, **we assume that $d > 1$ in the rest of this paper**. Notice from (4.2) and (4.4) that $\mathcal{K}^*_{\text{logdet}}$ is a scaled and rotated version of $\mathcal{K}_{\text{logdet}}$.

For convenience, we further define

$$\begin{aligned}
\mathcal{K}^1_{\text{logdet}} &:= \left\{(x,y,Z) \in \mathbb{R} \times \mathbb{R}_{++} \times \mathcal{S}^d_{++} : x \leq y\log\det(Z/y)\right\};\\
\mathcal{K}^{1\text{e}}_{\text{logdet}} &:= \left\{(x,y,Z) \in \mathbb{R} \times \mathbb{R}_{++} \times \mathcal{S}^d_{++} : x = y\log\det(Z/y)\right\};\\
\mathcal{K}^2_{\text{logdet}} &:= \mathbb{R}_- \times \{0\} \times \mathcal{S}^d_+;\\
\mathcal{K}^{*1}_{\text{logdet}} &:= \left\{(x,y,Z) \in \mathbb{R}_{--} \times \mathbb{R} \times \mathcal{S}^d_{++} : y \geq x(\log\det(-Z/x)+d)\right\};\\
\mathcal{K}^{*1\text{e}}_{\text{logdet}} &:= \left\{(x,y,Z) \in \mathbb{R}_{--} \times \mathbb{R} \times \mathcal{S}^d_{++} : y = x(\log\det(-Z/x)+d)\right\};\\
\mathcal{K}^{*2}_{\text{logdet}} &:= \{0\} \times \mathbb{R}_+ \times \mathcal{S}^d_+. \qquad (4.5)
\end{aligned}$$

With that, we have

$$\partial\mathcal{K}_{\text{logdet}} = \mathcal{K}^{1\text{e}}_{\text{logdet}} \cup \mathcal{K}^2_{\text{logdet}} \qquad (4.6)$$

and

$$\partial\mathcal{K}^*_{\text{logdet}} = \mathcal{K}^{*1\text{e}}_{\text{logdet}} \cup \mathcal{K}^{*2}_{\text{logdet}}.$$

Before moving on, we present several inequalities, which will be useful for our subsequent analysis.

1. Let $\eta > 0$, and let $\boldsymbol{x} = (\boldsymbol{x}_y \log\det(\boldsymbol{x}_Z/\boldsymbol{x}_y), \boldsymbol{x}_y, \boldsymbol{x}_Z) \in \mathcal{K}^{\rm 1e}_{\rm logdet} \cap \mathcal{B}(\eta)$ with $\boldsymbol{x}_y > 0$ and $\boldsymbol{x}_Z \succ 0$ and satisfy $\boldsymbol{x}_y \log\det(\boldsymbol{x}_Z/\boldsymbol{x}_y) \geq 0$. Then, we have

$$0 \leq \boldsymbol{x}_y \log\det(\boldsymbol{x}_Z/\boldsymbol{x}_y) \leq \boldsymbol{x}_y \log\det(\eta I_d/\boldsymbol{x}_y) \leq d\boldsymbol{x}_y|\log(\eta)| - d\boldsymbol{x}_y \log(\boldsymbol{x}_y). \quad (4.7)$$

2. Let $\alpha > 0$ and $s > 0$. The following inequalities hold for all sufficiently small $t > 0$,

$$t \leq \sqrt{t}, \quad -t^\alpha \log(t) \leq t^{\alpha/2}, \quad t^\alpha \leq -\frac{1}{\log(t)}, \quad -t^\alpha \log(t) \leq -\frac{1}{\log(st)}. \quad (4.8)$$

4.1 Facial structure

In general, we are more interested in nontrivial faces, especially nontrivial exposed faces. Recall that if there exists $\boldsymbol{n} := (\boldsymbol{n}_x, \boldsymbol{n}_y, \boldsymbol{n}_Z) \in \partial\mathcal{K}^*_{\rm logdet} \setminus \{\boldsymbol{0}\}$ such that $\mathcal{F} = \mathcal{K}_{\rm logdet} \cap \{\boldsymbol{n}\}^\perp$, then $\mathcal{F}$ is a nontrivial exposed face of $\mathcal{K}_{\rm logdet}$. Different nonzero $\boldsymbol{n}$'s along $\partial\mathcal{K}^*_{\rm logdet}$ will induce different nontrivial exposed faces.

The next proposition completely characterizes the facial structure of the log-determinant cone.

Proposition 4.1 (Facial structure of $\mathcal{K}_{\rm logdet}$)**.** *All nontrivial faces of the log-determinant cone can be classified into the following types:*

(a) infinitely many 1-dimensional faces exposed by $\boldsymbol{n} = (\boldsymbol{n}_x, \boldsymbol{n}_x(\log\det(-\boldsymbol{n}_Z/\boldsymbol{n}_x) + d), \boldsymbol{n}_Z)$ *with* $\boldsymbol{n}_x < 0, \boldsymbol{n}_Z \succ 0$*,*

$$\mathcal{F}_{\rm r} := \left\{(y\log\det(-\boldsymbol{n}_x\boldsymbol{n}_Z^{-1}), y, -y\boldsymbol{n}_x\boldsymbol{n}_Z^{-1}) : y \in \mathbb{R}_+\right\} = \{y\boldsymbol{f}_{\rm r} : y \in \mathbb{R}_+\}, \quad (4.9)$$

where

$$\boldsymbol{f}_{\rm r} = (\log\det(-\boldsymbol{n}_x\boldsymbol{n}_Z^{-1}), 1, -\boldsymbol{n}_x\boldsymbol{n}_Z^{-1}). \quad (4.10)$$

(b) a single $(\mathrm{sd}(d)+1)$*-dimensional exposed face exposed by* $\boldsymbol{n} = (0, \boldsymbol{n}_y, \boldsymbol{0})$ *with* $\boldsymbol{n}_y > 0$,

$$\mathcal{F}_{\mathrm{d}} := \mathbb{R}_- \times \{0\} \times \mathcal{S}^d_+ = \mathcal{K}^2_{\mathrm{logdet}}. \tag{4.11}$$

(c) infinitely many $(\mathrm{sd}(d - \mathsf{r}(\boldsymbol{n}_Z)) + 1)$*-dimensional exposed faces given by*

$$\mathcal{F}_{\#} := \mathbb{R}_- \times \{0\} \times (\mathcal{S}^d_+ \cap \{\boldsymbol{n}_Z\}^\perp), \tag{4.12}$$

which are exposed by

$$\boldsymbol{n} = (0, \boldsymbol{n}_y, \boldsymbol{n}_Z) \text{ with } \boldsymbol{n}_y \geq 0, \boldsymbol{n}_Z \succeq 0, 0 < \mathsf{r}(\boldsymbol{n}_Z) < d. \tag{4.13}$$

(d) a single 1*-dimensional exposed face exposed by*

$$\boldsymbol{n} = (0, \boldsymbol{n}_y, \boldsymbol{n}_Z) \text{ with } \boldsymbol{n}_y \geq 0, \boldsymbol{n}_Z \succ 0,$$

that is, $\mathsf{r}(\boldsymbol{n}_Z) = d$,

$$\mathcal{F}_{\infty} := \mathbb{R}_- \times \{0\} \times \{\boldsymbol{0}\}. \tag{4.14}$$

(e) infinitely many non-exposed faces defined by

$$\mathcal{F}^{\#}_{\mathrm{ne}} := \{0\} \times \{0\} \times (\mathcal{S}^d_+ \cap \{\boldsymbol{n}_Z\}^\perp), \tag{4.15}$$

which are proper subfaces of exposed faces of the form $\mathcal{F}_{\#}$ *or* $\mathcal{F}_{\mathrm{d}}$ *(see* (4.11) *and* (4.12)*), and* $\boldsymbol{n}_Z$ *comes from the* $\boldsymbol{n}$ *that exposes* $\mathcal{F}_{\#}$ *or* $\mathcal{F}_{\mathrm{d}}$*, i.e.,* $0 \leq \mathsf{r}(\boldsymbol{n}_Z) < d$.

Proof. Let $\boldsymbol{n} := (\boldsymbol{n}_x, \boldsymbol{n}_y, \boldsymbol{n}_Z) \in \mathcal{K}^*_{\mathrm{logdet}}$ be such that $\{\boldsymbol{n}\}^\perp \cap \mathcal{K}_{\mathrm{logdet}}$ is a nontrivial face of $\mathcal{K}_{\mathrm{logdet}}$. Recall that $\mathcal{K}_{\mathrm{logdet}}$ is pointed, so $\boldsymbol{n} \in \partial\mathcal{K}^*_{\mathrm{logdet}} \setminus \{\boldsymbol{0}\}$. By (4.5), $\boldsymbol{n}_x \leq 0$ and we can determine whether $\boldsymbol{n} \in \mathcal{K}^{*1}_{\mathrm{logdet}}$ or $\boldsymbol{n} \in \mathcal{K}^{*2}_{\mathrm{logdet}}$ by checking whether $\boldsymbol{n}_x < 0$ or not. Therefore, we shall consider the following cases.

<u>$\boldsymbol{n}_x < 0$:</u> $\boldsymbol{n}_x < 0$ indicates that $\boldsymbol{n} \in \mathcal{K}^{*1e}_{\mathrm{logdet}}$, then we must have

$$\boldsymbol{n} = (\boldsymbol{n}_x, \boldsymbol{n}_x(\log\det(-\boldsymbol{n}_Z/\boldsymbol{n}_x) + d), \boldsymbol{n}_Z) \text{ with } \boldsymbol{n}_x < 0,\ \boldsymbol{n}_Z \succ 0.$$

For any $\boldsymbol{q} := (\boldsymbol{q}_x, \boldsymbol{q}_y, \boldsymbol{q}_Z) \in \partial\mathcal{K}_{\text{logdet}}$, since $\boldsymbol{n}_x < 0$, we can see that $\boldsymbol{q} \in \{\boldsymbol{n}\}^{\perp}$ if and only if

$$\boldsymbol{q}_x + \boldsymbol{q}_y(\log\det(-\boldsymbol{n}_Z/\boldsymbol{n}_x) + d) + \operatorname{tr}(\boldsymbol{n}_Z\boldsymbol{q}_Z)/\boldsymbol{n}_x = 0. \tag{4.16}$$

If $\boldsymbol{q}_y = 0$, then $\boldsymbol{q} \in \mathcal{K}^2_{\text{logdet}}$, and so $\boldsymbol{q}_x \leq 0, \boldsymbol{q}_Z \succeq 0$. This together with $\boldsymbol{n}_Z \succ 0$ and (2.1) imply that $\operatorname{tr}(\boldsymbol{n}_Z\boldsymbol{q}_Z) \geq 0$. Since $\boldsymbol{n}_x < 0$, we observe that

$$0 \leq -\boldsymbol{q}_x = \operatorname{tr}(\boldsymbol{n}_Z\boldsymbol{q}_Z)/\boldsymbol{n}_x \leq 0.$$

Thus, $\boldsymbol{q}_x = 0$ and $\operatorname{tr}(\boldsymbol{n}_Z\boldsymbol{q}_Z) = 0$. The latter relation leads to $\boldsymbol{q}_Z = \boldsymbol{0}$. Consequently, $\boldsymbol{q} = \boldsymbol{0}$.

If $\boldsymbol{q}_y \neq 0$, then $\boldsymbol{q}_y > 0$ by the definition of the log-determinant cone and hence $\boldsymbol{q} \in \mathcal{K}^{1e}_{\text{logdet}}$. Then, we know that $\boldsymbol{q}_x = \boldsymbol{q}_y \log\det(\boldsymbol{q}_Z/\boldsymbol{q}_y)$, $\boldsymbol{q}_Z \succ 0$ and hence (4.16) becomes

$$\log\det\left(\frac{\boldsymbol{q}_Z}{\boldsymbol{q}_y}\right) + \log\det\left(-\frac{\boldsymbol{n}_Z}{\boldsymbol{n}_x}\right) + d + \operatorname{tr}\left(\frac{\boldsymbol{n}_Z\boldsymbol{q}_Z}{\boldsymbol{n}_x\boldsymbol{q}_y}\right) = 0.$$

After rearranging terms, we have

$$\log\det\left(-\frac{\boldsymbol{n}_Z\boldsymbol{q}_Z}{\boldsymbol{n}_x\boldsymbol{q}_y}\right) + d + \operatorname{tr}\left(\frac{\boldsymbol{n}_Z\boldsymbol{q}_Z}{\boldsymbol{n}_x\boldsymbol{q}_y}\right) = 0. \tag{4.17}$$

Note also that

$$\det\left(-\frac{\boldsymbol{n}_Z\boldsymbol{q}_Z}{\boldsymbol{n}_x\boldsymbol{q}_y}\right) = \det\left(-\frac{\boldsymbol{n}_Z^{\frac{1}{2}}\boldsymbol{q}_Z\boldsymbol{n}_Z^{\frac{1}{2}}}{\boldsymbol{n}_x\boldsymbol{q}_y}\right) \quad \text{and} \quad \operatorname{tr}\left(\frac{\boldsymbol{n}_Z\boldsymbol{q}_Z}{\boldsymbol{n}_x\boldsymbol{q}_y}\right) = \operatorname{tr}\left(\frac{\boldsymbol{n}_Z^{\frac{1}{2}}\boldsymbol{q}_Z\boldsymbol{n}_Z^{\frac{1}{2}}}{\boldsymbol{n}_x\boldsymbol{q}_y}\right),$$

where $\boldsymbol{n}_Z^{\frac{1}{2}}\boldsymbol{q}_Z\boldsymbol{n}_Z^{\frac{1}{2}} \succ 0$ and $-\frac{\boldsymbol{n}_Z^{\frac{1}{2}}\boldsymbol{q}_Z\boldsymbol{n}_Z^{\frac{1}{2}}}{\boldsymbol{n}_x\boldsymbol{q}_y} \succ 0$.

Let $f(x) = \log(x) - x + 1$, we can rewrite (4.17) as follows,

$$\begin{aligned}
&\sum_{i=1}^{d} f\left(\lambda_i\left(-\frac{\boldsymbol{n}_Z^{\frac{1}{2}}\boldsymbol{q}_Z\boldsymbol{n}_Z^{\frac{1}{2}}}{\boldsymbol{n}_x\boldsymbol{q}_y}\right)\right) \\
=&\sum_{i=1}^{d}\left(\log\left(\lambda_i\left(-\frac{\boldsymbol{n}_Z^{\frac{1}{2}}\boldsymbol{q}_Z\boldsymbol{n}_Z^{\frac{1}{2}}}{\boldsymbol{n}_x\boldsymbol{q}_y}\right)\right) + 1 - \lambda_i\left(-\frac{\boldsymbol{n}_Z^{\frac{1}{2}}\boldsymbol{q}_Z\boldsymbol{n}_Z^{\frac{1}{2}}}{\boldsymbol{n}_x\boldsymbol{q}_y}\right)\right) = 0.
\end{aligned} \tag{4.18}$$

Since $f(x) \leq 0$ for all $x > 0$ and $f(x) = 0$ if and only if $x = 1$, (4.18) holds if and only if

$$\lambda_i \left(-\frac{\boldsymbol{n}_Z^{\frac{1}{2}} \boldsymbol{q}_Z \boldsymbol{n}_Z^{\frac{1}{2}}}{\boldsymbol{n}_x \boldsymbol{q}_y} \right) = 1 \quad \forall i \in \{1, 2, \ldots, d\}.$$

This illustrates that all the eigenvalues of $-\frac{\boldsymbol{n}_Z^{\frac{1}{2}} \boldsymbol{q}_Z \boldsymbol{n}_Z^{\frac{1}{2}}}{\boldsymbol{n}_x \boldsymbol{q}_y}$ are 1. Hence, one can immediately see $\boldsymbol{n}_Z^{\frac{1}{2}} \boldsymbol{q}_Z \boldsymbol{n}_Z^{\frac{1}{2}} = -\boldsymbol{n}_x \boldsymbol{q}_y I_d$ and so $\boldsymbol{q}_Z = -\boldsymbol{q}_y \boldsymbol{n}_x \boldsymbol{n}_Z^{-1}$. By substituting this expression of $\boldsymbol{q}_Z$ into $\boldsymbol{q} = (\boldsymbol{q}_y \log\det(\boldsymbol{q}_Z/\boldsymbol{q}_y), \boldsymbol{q}_y, \boldsymbol{q}_Z)$, we obtain (4.9).

$\underline{\boldsymbol{n}_x = 0}$: $\boldsymbol{n}_x = 0$ indicates that $\boldsymbol{n} \in \mathcal{K}^{*2}_{\text{logdet}}$, then $\boldsymbol{n}_y \geq 0$ and $\boldsymbol{n}_Z \succeq 0$. Now, for any $\boldsymbol{q} \in \partial\mathcal{K}_{\text{logdet}}$, we have $\boldsymbol{q} \in \{\boldsymbol{n}\}^\perp$ if and only if

$$\boldsymbol{n}_y \boldsymbol{q}_y + \operatorname{tr}(\boldsymbol{n}_Z \boldsymbol{q}_Z) = 0. \tag{4.19}$$

Since $\boldsymbol{n}_y \geq 0, \boldsymbol{q}_y \geq 0, \boldsymbol{n}_Z \succeq 0$ and $\boldsymbol{q}_Z \succeq 0$, we observe that both summands on the left hand side of (4.19) are nonnegative. Therefore, (4.19) holds if and only if

$$\boldsymbol{n}_y \boldsymbol{q}_y = 0, \quad \operatorname{tr}(\boldsymbol{n}_Z \boldsymbol{q}_Z) = 0. \tag{4.20}$$

These together with (2.1) make it clear the cases we need to consider.

Specifically, if $\boldsymbol{n}_x = 0$, we consider the following four cases.

1. If $\mathsf{r}(\boldsymbol{n}_Z) = 0$ and $\boldsymbol{n}_y = 0$, then $\boldsymbol{n} = \mathbf{0}$, which contradicts our assumption. This case is hence impossible.

2. If $\mathsf{r}(\boldsymbol{n}_Z) = 0$ and $\boldsymbol{n}_y > 0$, then by (4.20), $\boldsymbol{q}_y = 0$. This corresponds to (4.11).

3. If $0 < \mathsf{r}(\boldsymbol{n}_Z) < d$, then $\boldsymbol{q}_Z \succeq 0$ but $\boldsymbol{q}_Z$ is not definite, so $(\boldsymbol{q}_x, \boldsymbol{q}_y, \boldsymbol{q}_Z) \in \mathcal{K}^2_{\text{logdet}}$. Since $\boldsymbol{q}_Z \in \{\boldsymbol{n}_Z\}^\perp$ holds, this corresponds to (4.12).

4. If $\mathsf{r}(\boldsymbol{n}_Z) = d$, i.e., $\boldsymbol{n}_Z \succ 0$, then $\boldsymbol{q}_Z = \mathbf{0}$. This corresponds to (4.14).

Therefore, we obtain the exposed faces defined as in (4.11), (4.12) and (4.14).

We now show that all nontrivial faces of $\mathcal{K}_{\text{logdet}}$ were accounted for (4.9), (4.11), (4.12), (4.14) and (4.15). First of all, by the previous discussion, all nontrivial exposed faces must be among the ones in (4.9), (4.11), (4.12), and (4.14). Suppose $\mathcal{F}$ is a non-exposed face of $\mathcal{K}_{\text{logdet}}$. Then it must be contained in a nontrivial exposed face $\widehat{\mathcal{F}}$ of $\mathcal{K}_{\text{logdet}}$, e.g., [9, Proposition 3.6] or [42, Proposition 2.1]. The faces in (4.9) and (4.14) are one-dimensional, so the only candidates for $\widehat{\mathcal{F}}$ are the faces as in (4.11) and (4.12).

So suppose that $\widehat{\mathcal{F}}$ is as in (4.11) or (4.12). Recalling the list of nontrivial exposed faces described so far, the only nontrival faces of $\widehat{\mathcal{F}}$ that have not appeared yet are the ones of the form $\mathcal{F}_{\text{ne}}^{\#}$ (as in (4.15)) for some $\boldsymbol{n}_Z$ with $0 \leq \mathsf{r}\,(\boldsymbol{n}_Z) < d$. This shows the completeness of the classification. □

It is worth noting that when $d = 1$, the case corresponding to $\mathcal{F}_{\#}$ does not occur. We also have the following relationships between these nontrivial faces. Let $\boldsymbol{n} \neq \boldsymbol{0}$ with $0 \leq \mathsf{r}\,(\boldsymbol{n}_Z) < d$ be given. If $\mathsf{r}\,(\boldsymbol{n}_Z) > 0$, then the corresponding faces $\mathcal{F}_{\#}$ and $\mathcal{F}_{\text{ne}}^{\#}$ satisfy the following inclusion

$$\mathcal{F}_{\text{ne}}^{\#} \subsetneq \mathcal{F}_{\#} \subsetneq \mathcal{F}_{\text{d}} \quad \text{and} \quad \mathcal{F}_{\infty} \subsetneq \mathcal{F}_{\#} \subsetneq \mathcal{F}_{\text{d}}. \tag{4.21}$$

If $\mathsf{r}\,(\boldsymbol{n}_Z) = 0$, then we have

$$\mathcal{F}_{\text{ne}}^{\#} \subsetneq \mathcal{F}_{\text{d}}. \tag{4.22}$$

For distinct $\boldsymbol{n}^1 := (\boldsymbol{n}_x^1, \boldsymbol{n}_y^1, \boldsymbol{n}_Z^1)$ and $\boldsymbol{n}^2 := (\boldsymbol{n}_x^2, \boldsymbol{n}_y^2, \boldsymbol{n}_Z^2)$ with $0 < \mathsf{r}\,(\boldsymbol{n}_Z^1) < d$ and $0 < \mathsf{r}\,(\boldsymbol{n}_Z^2) < d$, suppose $\boldsymbol{n}^1$ and $\boldsymbol{n}^2$ expose $\mathcal{F}_{\#}^1$ and $\mathcal{F}_{\#}^2$, respectively. If $\text{range}(\boldsymbol{n}_Z^1) \supsetneq \text{range}(\boldsymbol{n}_Z^2)$, then $\mathcal{F}_{\#}^1 \subsetneq \mathcal{F}_{\#}^2$ (see, e.g., [4, Section 6]). A similar result also holds for non-exposed faces, that is, denote the non-exposed faces by $\mathcal{F}_{\text{ne}}^{\#1}$ and $\mathcal{F}_{\text{ne}}^{\#2}$, respectively, with respect to $\boldsymbol{n}^1$ and $\boldsymbol{n}^2$, if $\text{range}(\boldsymbol{n}_Z^1) \supsetneq \text{range}(\boldsymbol{n}_Z^2)$, then $\mathcal{F}_{\text{ne}}^{\#1} \subsetneq \mathcal{F}_{\text{ne}}^{\#2}$.

4.2 One-step facial residual functions

In this section, we shall apply the strategy in [36, Section 3.1] to compute the corresponding one-step facial residual functions for nontrivial exposed faces of the log-determinant cone. Put concretely, consider $\mathcal{F} = \mathcal{K}_{\text{logdet}} \cap \{\boldsymbol{n}\}^\perp$ with $\boldsymbol{n} \in \partial\mathcal{K}^*_{\text{logdet}} \setminus \{\boldsymbol{0}\}$. For $\eta > 0$ and some nondecreasing function $\mathfrak{g} : \mathbb{R}_+ \to \mathbb{R}_+$ with $\mathfrak{g}(0) = 0$ and $\mathfrak{g} \geq |\cdot|^\alpha$ for some $\alpha \in (0, 1]$, we define

$$\gamma_{\boldsymbol{n},\eta} := \inf_{\boldsymbol{v}} \left\{ \frac{\mathfrak{g}(\|\boldsymbol{v} - \boldsymbol{w}\|)}{\|\boldsymbol{u} - \boldsymbol{w}\|} \;\middle|\; \begin{array}{c} \boldsymbol{v} \in \partial\mathcal{K}_{\text{logdet}} \cap \mathcal{B}(\eta) \setminus \mathcal{F},\ \boldsymbol{w} = P_{\{\boldsymbol{n}\}^\perp}(\boldsymbol{v}), \\ \boldsymbol{u} = P_{\mathcal{F}}(\boldsymbol{w}),\ \boldsymbol{u} \neq \boldsymbol{w} \end{array} \right\}. \tag{4.23}$$

In view of [36, Theorem 3.10] and [36, Lemma 3.9], if $\gamma_{\boldsymbol{n},\eta} \in (0, \infty]$ then we can use $\gamma_{\boldsymbol{n},\eta}$ and $\mathfrak{g}$ to construct a one-step facial residual function for $\mathcal{K}_{\text{logdet}}$ and $\boldsymbol{n}$. In [36], the positivity of $\gamma_{\boldsymbol{n},\eta}$ (with the exponential cone in place of $\mathcal{K}_{\text{logdet}}$ and some properly selected $\mathfrak{g}$) was shown by contradiction. Here, we will follow a similar strategy and make extensive use of the following fact from [36, Lemma 3.12]: if $\gamma_{\boldsymbol{n},\eta} = 0$, then there exist $\widehat{\boldsymbol{v}} \in \mathcal{F}$ and a sequence $\{\boldsymbol{v}^k\} \subset \partial\mathcal{K}_{\text{logdet}} \cap \mathcal{B}(\eta) \setminus \mathcal{F}$ such that

$$\lim_{k\to\infty} \boldsymbol{v}^k = \lim_{k\to\infty} \boldsymbol{w}^k = \widehat{\boldsymbol{v}} \text{ and } \lim_{k\to\infty} \frac{\mathfrak{g}(\|\boldsymbol{w}^k - \boldsymbol{v}^k\|)}{\|\boldsymbol{u}^k - \boldsymbol{w}^k\|} = 0, \tag{4.24}$$

where $\boldsymbol{w}^k = P_{\{\boldsymbol{n}\}^\perp}(\boldsymbol{v}^k)$, $\boldsymbol{u}^k = P_{\mathcal{F}}(\boldsymbol{w}^k)$ and $\boldsymbol{u}^k \neq \boldsymbol{w}^k$.

4.2.1 $\mathcal{F}_{\text{d}}$: the unique $(\text{sd}(d) + 1)$-dimensional faces

We define the piecewise modified Boltzmann-Shannon entropy $\mathfrak{g}_{\text{d}} : \mathbb{R}_+ \to \mathbb{R}_+$ as follows:

$$\mathfrak{g}_{\text{d}}(t) := \begin{cases} 0 & \text{if } t = 0, \\ -t\log(t) & \text{if } 0 < t \leq \frac{1}{e^2}, \\ t + \frac{1}{e^2} & \text{if } t > \frac{1}{e^2}. \end{cases} \tag{4.25}$$

Note that $\mathfrak{g}_{\text{d}}$ is nondecreasing with $\mathfrak{g}_{\text{d}}(0) = 0$ and $|t| \leq \mathfrak{g}_{\text{d}}(t)$ for any $t \in \mathbb{R}_+$.

The next theorem shows that $\gamma_{\boldsymbol{n},\eta} \in (0, \infty]$ for $\mathcal{F}_{\text{d}}$, which implies that an entropic error bound holds.

Theorem 4.2 (Entropic error bound concerning $\mathcal{F}_{\rm d}$)**.** *Let* $\boldsymbol{n} = (0, \boldsymbol{n}_y, \boldsymbol{0}) \in \partial\mathcal{K}^*_{\rm logdet}$ *with* $\boldsymbol{n}_y > 0$ *such that* $\mathcal{F}_{\rm d} = \mathcal{K}_{\rm logdet} \cap \{\boldsymbol{n}\}^\perp$*. Let* $\eta > 0$ *and let* $\gamma_{\boldsymbol{n},\eta}$ *be defined as in* (4.23) *with* $\mathcal{F} = \mathcal{F}_{\rm d}$ *and* $\mathfrak{g} = \mathfrak{g}_{\rm d}$*. Then* $\gamma_{\boldsymbol{n},\eta} \in (0,\infty]$ *and*

$$\operatorname{dist}(\boldsymbol{q}, \mathcal{F}_{\rm d}) \le \max\{2, 2\gamma_{\boldsymbol{n},\eta}^{-1}\} \cdot \mathfrak{g}_{\rm d}(\operatorname{dist}(\boldsymbol{q}, \mathcal{K}_{\rm logdet})) \qquad \forall \boldsymbol{q} \in \{\boldsymbol{n}\}^\perp \cap \mathcal{B}(\eta). \tag{4.26}$$

Proof. If $\gamma_{\boldsymbol{n},\eta} = 0$, in view of [36, Lemma 3.12], there exist $\widehat{\boldsymbol{v}} \in \mathcal{F}_{\rm d}$ and a sequence $\{\boldsymbol{v}^k\} \subset \partial\mathcal{K}_{\rm logdet} \cap \mathcal{B}(\eta) \setminus \mathcal{F}_{\rm d}$ such that (4.24) holds with $\mathfrak{g} = \mathfrak{g}_{\rm d}$ and $\mathcal{F} = \mathcal{F}_{\rm d}$.

By (4.11), $\widehat{\boldsymbol{v}} = (\widehat{\boldsymbol{v}}_x, 0, \widehat{\boldsymbol{v}}_Z)$ with $\widehat{\boldsymbol{v}}_Z \succeq 0$. Since $\boldsymbol{v}^k \in \partial\mathcal{K}_{\rm logdet} \cap \mathcal{B}(\eta) \setminus \mathcal{F}_{\rm d}$ for all k, we have $\boldsymbol{v}^k_y > 0$ and $\boldsymbol{v}^k \in \mathcal{K}^{\rm 1e}_{\rm logdet}$ for all k. Hence, $\boldsymbol{v}^k = (\boldsymbol{v}^k_y \log\det(\boldsymbol{v}^k_Z/\boldsymbol{v}^k_y), \boldsymbol{v}^k_y, \boldsymbol{v}^k_Z)$ with $\boldsymbol{v}^k_y > 0, \boldsymbol{v}^k_Z \succ 0$ for all k.

Recall that $\boldsymbol{n} = (0, \boldsymbol{n}_y, \boldsymbol{0})$ with $\boldsymbol{n}_y > 0$, then $\|\boldsymbol{n}\| = \boldsymbol{n}_y$ and $\langle \boldsymbol{n}, \boldsymbol{v}^k \rangle = \boldsymbol{n}_y \boldsymbol{v}^k_y > 0$. Since $\boldsymbol{w}^k = P_{\{\boldsymbol{n}\}^\perp}(\boldsymbol{v}^k)$ and $\{\boldsymbol{n}\}^\perp$ is a hyperplane, one can immediately see that for all k,

$$\boldsymbol{w}^k = \boldsymbol{v}^k - \frac{\langle \boldsymbol{n}, \boldsymbol{v}^k \rangle}{\|\boldsymbol{n}\|^2}\boldsymbol{n} = (\boldsymbol{v}^k_y \log\det(\boldsymbol{v}^k_Z/\boldsymbol{v}^k_y), 0, \boldsymbol{v}^k_Z) \quad \text{and} \quad \|\boldsymbol{w}^k - \boldsymbol{v}^k\| = \frac{|\langle \boldsymbol{n}, \boldsymbol{v}^k \rangle|}{\|\boldsymbol{n}\|} = \boldsymbol{v}^k_y.$$

Using (4.11), $\boldsymbol{u}^k = P_{\mathcal{F}_{\rm d}}(\boldsymbol{w}^k)$ and $\boldsymbol{u}^k \neq \boldsymbol{w}^k$, we see that $\boldsymbol{v}^k_y \log\det(\boldsymbol{v}^k_Z/\boldsymbol{v}^k_y) > 0$ and $\boldsymbol{u}^k = (0, 0, \boldsymbol{v}^k_Z)$. We thus obtain that for all k,

$$\|\boldsymbol{w}^k - \boldsymbol{u}^k\| = \boldsymbol{v}^k_y \log\det(\boldsymbol{v}^k_Z/\boldsymbol{v}^k_y).$$

Because $\lim_{k\to\infty} \boldsymbol{v}^k_y = 0$, for sufficiently large k, we have $0 < \boldsymbol{v}^k_y < \frac{1}{e^2}$. Hence,

$$\lim_{k\to\infty} \frac{\mathfrak{g}_{\rm d}(\|\boldsymbol{w}^k - \boldsymbol{v}^k\|)}{\|\boldsymbol{w}^k - \boldsymbol{u}^k\|} \overset{\rm (a)}{\ge} \lim_{k\to\infty} \frac{-\boldsymbol{v}^k_y \log(\boldsymbol{v}^k_y)}{d\boldsymbol{v}^k_y |\log(\eta)| - d\boldsymbol{v}^k_y \log(\boldsymbol{v}^k_y)} = \lim_{k\to\infty} \frac{1}{d - d\frac{|\log(\eta)|}{\log(\boldsymbol{v}^k_y)}} = \frac{1}{d} > 0,$$

where (a) comes from the fact $\boldsymbol{v}^k \in \mathcal{B}(\eta)$ and (4.7). This contradicts (4.24) with $\mathfrak{g}_{\rm d}$ in place of $\mathfrak{g}$ and hence this case cannot happen. Therefore, we conclude that $\gamma_{\boldsymbol{n},\eta} \in (0,\infty]$, with which and [36, Theorem 3.10], (4.26) holds. ☐

Remark 4.3 (Tightness of (4.26)). *We claim that for* $\mathcal{F}_{\rm d}$*, there is a specific choice of sequence* $\{\boldsymbol{w}^k\}$ *in* $\{\boldsymbol{n}\}^\perp$ *with* $\operatorname{dist}(\boldsymbol{w}^k, \mathcal{K}_{\rm logdet}) \to 0$ *along which both sides of* (4.26) *vanish at the same order of magnitude. Recall that we assumed that* $d > 1$*; see the discussions following* (4.4).[8] *Let* $\boldsymbol{n} = (0, \boldsymbol{n}_y, \boldsymbol{0})$ *with* $\boldsymbol{n}_y > 0$ *so that* $\{\boldsymbol{n}\}^\perp \cap \mathcal{K}_{\rm logdet} = \mathcal{F}_{\rm d}$*. Define* $\boldsymbol{w}^k = (d\log(k)/k, 0, I_d)$ *for every* $k \in \mathbb{N}$*. Then* $\{\boldsymbol{w}^k\} \subseteq \{\boldsymbol{n}\}^\perp$*. Since* $\log(k)/k > 0$ *for any* $k \geq 2$ *and* $\log(k)/k \to 0$ *as* $k \to \infty$*, there exists* $\eta > 0$ *such that* $\left\{\boldsymbol{w}^k\right\} \subseteq \mathcal{B}(\eta)$*. Thus, applying* (4.26)*, there exists* $\kappa_B > 0$ *such that*

$$\operatorname{dist}\left(\boldsymbol{w}^k, \mathcal{F}_{\rm d}\right) \leq \kappa_B \mathfrak{g}_{\rm d}\left(\operatorname{dist}\left(\boldsymbol{w}^k, \mathcal{K}_{\rm logdet}\right)\right) \quad \textit{for all sufficiently large } k.$$

Noticing that the projection of $\boldsymbol{w}^k$ *onto* $\mathcal{F}_{\rm d}$ *(see* (4.11)*) is given by* $(0, 0, I_d)$*, we obtain*

$$\frac{d\log(k)}{k} = \operatorname{dist}(\boldsymbol{w}^k, \mathcal{F}_{\rm d}) \leq \kappa_B \mathfrak{g}_{\rm d}\left(\operatorname{dist}(\boldsymbol{w}^k, \mathcal{K}_{\rm logdet})\right).$$

Let $\boldsymbol{v}^k = (d\log(k)/k, 1/k, I_d)$ *for every* k*. Then* $\operatorname{dist}(\boldsymbol{w}^k, \mathcal{K}_{\rm logdet}) \leq 1/k$ *since* $\boldsymbol{v}^k \in \mathcal{K}_{\rm logdet}$*. In view of the definition of* $\mathfrak{g}_{\rm d}$ *(see* (4.25)*) and its monotonicity, we conclude that for large enough* k *we have*

$$\frac{d\log(k)}{k} = \operatorname{dist}(\boldsymbol{w}^k, \mathcal{F}_{\rm d}) \leq \kappa_B \mathfrak{g}_{\rm d}(\operatorname{dist}(\boldsymbol{w}^k, \mathcal{K}_{\rm logdet})) \leq \kappa_B \frac{\log(k)}{k}.$$

That means it holds that for all sufficiently large k*,*

$$d \leq \frac{\operatorname{dist}(\boldsymbol{w}^k, \mathcal{F}_{\rm d})}{\mathfrak{g}_{\rm d}(\operatorname{dist}(\boldsymbol{w}^k, \mathcal{K}_{\rm logdet}))} \leq \kappa_B.$$

Consequently, for any given nonnegative function $\mathfrak{g} : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\lim_{t\downarrow 0} \frac{\mathfrak{g}(t)}{\mathfrak{g}_{\rm d}(t)} = 0$*, we have upon noting* $\operatorname{dist}(\boldsymbol{w}^k, \mathcal{K}_{\rm logdet}) \to 0$ *that*

$$\frac{\operatorname{dist}(\boldsymbol{w}^k, \mathcal{F}_{\rm d})}{\mathfrak{g}(\operatorname{dist}(\boldsymbol{w}^k, \mathcal{K}_{\rm logdet}))} = \frac{\operatorname{dist}(\boldsymbol{w}^k, \mathcal{F}_{\rm d})}{\mathfrak{g}_{\rm d}(\operatorname{dist}(\boldsymbol{w}^k, \mathcal{K}_{\rm logdet}))} \frac{\mathfrak{g}_{\rm d}(\operatorname{dist}(\boldsymbol{w}^k, \mathcal{K}_{\rm logdet}))}{\mathfrak{g}(\operatorname{dist}(\boldsymbol{w}^k, \mathcal{K}_{\rm logdet}))} \to \infty,$$

which shows that the choice of $\mathfrak{g}_{\rm d}$ *in* (4.26) *is tight.*

[8] When $d = 1$, the log-determinant cone reduces to the exponential cone studied in [36], where the tightness of the corresponding error bounds was shown in Remark 4.14 therein.

Upon invoking Theorem 4.2 and [36, Lemma 3.9], we obtain the following one-step facial residual function for $\mathcal{K}_{\rm logdet}$ and $\boldsymbol{n}$.

Corollary 4.4. *Let* $\boldsymbol{n} = (0, \boldsymbol{n}_y, \boldsymbol{0}) \in \partial\mathcal{K}^*_{\rm logdet}$ *with* $\boldsymbol{n}_y > 0$ *such that* $\mathcal{F}_{\rm d} = \mathcal{K}_{\rm logdet} \cap \{\boldsymbol{n}\}^\perp$*. Let* $\gamma_{\boldsymbol{n},t}$ *be defined as in* (4.23) *with* $\mathcal{F} = \mathcal{F}_{\rm d}$ *and* $\mathfrak{g} = \mathfrak{g}_{\rm d}$ *in* (4.25)*. Then the function* $\psi_{\mathcal{K},\boldsymbol{n}} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *defined by*

$$\psi_{\mathcal{K},\boldsymbol{n}}(\epsilon, t) := \max\{\epsilon, \epsilon/\|\boldsymbol{n}\|\} + \max\left\{2, 2\gamma_{\boldsymbol{n},t}^{-1}\right\} \mathfrak{g}_{\rm d}\left(\epsilon + \max\{\epsilon, \epsilon/\|\boldsymbol{n}\|\}\right)$$

is a one-step facial residual function for $\mathcal{K}_{\rm logdet}$ *and* $\boldsymbol{n}$*.*

4.2.2 $\mathcal{F}_\#$: the family of $(\mathrm{sd}(d - \mathsf{r}(\boldsymbol{n}_Z)) + 1)$-dimensional faces

Let $\eta > 0$ and let $\boldsymbol{n} \in \partial\mathcal{K}^*_{\rm logdet}$ be such that $\mathcal{F}_\# = \mathcal{K}_{\rm logdet} \cap \{\boldsymbol{n}\}^\perp$. Let $\gamma_{\boldsymbol{n},\eta}$ be defined as in (4.23) with $\mathcal{F} = \mathcal{F}_\#$ and some nondecreasing function $\mathfrak{g} : \mathbb{R}_+ \to \mathbb{R}_+$ with $\mathfrak{g}(0) = 0$ and $\mathfrak{g} \geq |\cdot|^\alpha$ for some $\alpha \in (0, 1]$. If $\gamma_{\boldsymbol{n},\eta} = 0$, in view of [36, Lemma 3.12], there exists $\widehat{\boldsymbol{v}} \in \mathcal{F}_\#$ and a sequence $\{\boldsymbol{v}^k\} \subset \partial\mathcal{K}_{\rm logdet} \cap \mathcal{B}(\eta) \setminus \mathcal{F}_\#$ such that (4.24) holds. As we will see later in the proofs of Theorem 4.6 and Theorem 4.8 below, we will encounter the following three cases:

(I) $\boldsymbol{n}_y \geq 0$ and $\boldsymbol{v}^k \in \mathcal{F}_{\rm d} \cap \mathcal{B}(\eta) \setminus \mathcal{F}_\#$ for all large k;

(II) $\boldsymbol{n}_y > 0$ and $\boldsymbol{v}^k \in \partial\mathcal{K}_{\rm logdet} \cap \mathcal{B}(\eta) \setminus \mathcal{F}_{\rm d}$ infinitely often;

(III) $\boldsymbol{n}_y = 0$ and $\boldsymbol{v}^k \in \partial\mathcal{K}_{\rm logdet} \cap \mathcal{B}(\eta) \setminus \mathcal{F}_{\rm d}$ infinitely often.

For case (I), we have the following lemma which will aid in our further analysis. One should notice that this lemma holds for both $\mathcal{F}_\#$ and $\mathcal{F}_\infty$.

Lemma 4.5. *Let* $\boldsymbol{n} = (0, \boldsymbol{n}_y, \boldsymbol{n}_Z) \in \partial\mathcal{K}^*_{\rm logdet} \setminus \{\boldsymbol{0}\}$ *with* $\boldsymbol{n}_y \geq 0$ *and* $\boldsymbol{n}_Z \succeq 0$ *such that* $\mathcal{F} = \mathcal{K}_{\rm logdet} \cap \{\boldsymbol{n}\}^\perp$ *with* $\mathcal{F} = \mathcal{F}_\#$ *or* $\mathcal{F}_\infty$*. Let* $\overline{\boldsymbol{v}} \in \mathcal{F}$ *be arbitrary and* $\{\boldsymbol{v}^k\} \subset \mathcal{F}_{\rm d} \cap \mathcal{B}(\eta) \setminus \mathcal{F}$ *be such that*

$$\lim_{k\to\infty} \boldsymbol{v}^k = \lim_{k\to\infty} \boldsymbol{w}^k = \overline{\boldsymbol{v}},$$

where $\boldsymbol{w}^k = P_{\{\boldsymbol{n}\}^\perp}(\boldsymbol{v}^k), \boldsymbol{u}^k = P_{\mathcal{F}}(\boldsymbol{w}^k)$ *and* $\boldsymbol{w}^k \neq \boldsymbol{u}^k$*. Then*

$$\liminf_{k\to\infty} \frac{\|\boldsymbol{w}^k - \boldsymbol{v}^k\|^\alpha}{\|\boldsymbol{w}^k - \boldsymbol{u}^k\|} \in (0, \infty],$$

where α *is defined as in* (2.8) *with* Z *being* $\boldsymbol{n}_Z$*.*

Proof. Note that $\{\boldsymbol{v}^k\} \subset \mathcal{F}_{\rm d} \cap \mathcal{B}(\eta) \setminus \mathcal{F}$ implies $\boldsymbol{v}^k = (\boldsymbol{v}_x^k, 0, \boldsymbol{v}_Z^k)$ with $\boldsymbol{v}_x^k \leq 0$ and $\boldsymbol{v}_Z^k \in \mathcal{S}_+^d$ for all k. Then, $\langle \boldsymbol{n}, \boldsymbol{v}^k\rangle = \operatorname{tr}(\boldsymbol{v}_Z^k \boldsymbol{n}_Z)$, which is nonnegative since both $\boldsymbol{v}_Z^k$ and $\boldsymbol{n}_Z$ are positive semidefinite. Because $\boldsymbol{w}^k = P_{\{\boldsymbol{n}\}^\perp}(\boldsymbol{v}^k)$ and $\{\boldsymbol{n}\}^\perp$ is a hyperplane, one can immediately see that for all k,

$$\|\boldsymbol{w}^k - \boldsymbol{v}^k\| = \frac{|\langle \boldsymbol{n}, \boldsymbol{v}^k\rangle|}{\|\boldsymbol{n}\|} = \frac{\operatorname{tr}(\boldsymbol{v}_Z^k \boldsymbol{n}_Z)}{\|\boldsymbol{n}\|}.$$

On the other hand, by Lemma 2.4 and the formula of $\mathcal{F}$, we obtain that for all k,

$$\|\boldsymbol{w}^k - \boldsymbol{u}^k\| \leq \operatorname{dist}(\boldsymbol{v}^k, \mathcal{F}) = \operatorname{dist}(\boldsymbol{v}_Z^k, \mathcal{S}_+^d \cap \{\boldsymbol{n}_Z\}^\perp) \leq C_P \operatorname{tr}(\boldsymbol{v}_Z^k \boldsymbol{n}_Z)^\alpha,$$

where the final inequality comes from Proposition 2.5 and α is defined as in (2.8) with Z being $\boldsymbol{n}_Z$.

Now, we can conclude that

$$\liminf_{k\to\infty} \frac{\|\boldsymbol{w}^k - \boldsymbol{v}^k\|^\alpha}{\|\boldsymbol{w}^k - \boldsymbol{u}^k\|} \geq \frac{1}{C_P \|\boldsymbol{n}\|^\alpha} > 0.$$

This completes the proof. □

Now, we are ready to show the error bound concerning $\mathcal{F}_\#$. We first show that we have a Hölderian error bound concerning $\mathcal{F}_\#$ when $\boldsymbol{n}_y > 0$.

Theorem 4.6 (Hölderian error bound concerning $\mathcal{F}_\#$ if $\boldsymbol{n}_y > 0$)**.** *Let* $\boldsymbol{n} = (0, \boldsymbol{n}_y, \boldsymbol{n}_Z) \in \partial\mathcal{K}_{\rm logdet}^*$ *with* $\boldsymbol{n}_y > 0$*,* $\boldsymbol{n}_Z \succeq 0$ *and* $0 < \mathsf{r}(\boldsymbol{n}_Z) < d$ *such that* $\mathcal{F}_\# = \mathcal{K}_{\rm logdet} \cap \{\boldsymbol{n}\}^\perp$*. Let* $\eta > 0$ *and let* $\gamma_{\boldsymbol{n},\eta}$ *be defined as in* (4.23) *with* $\mathcal{F} = \mathcal{F}_\#$ *and* $\mathfrak{g} = |\cdot|^{\frac{1}{2}}$*. Then* $\gamma_{\boldsymbol{n},\eta} \in (0, \infty]$ *and*

$$\operatorname{dist}(\boldsymbol{q}, \mathcal{F}_\#) \leq \max\{2\eta^{\frac{1}{2}}, 2\gamma_{\boldsymbol{n},\eta}^{-1}\} \cdot (\operatorname{dist}(\boldsymbol{q}, \mathcal{K}_{\rm logdet}))^{\frac{1}{2}} \qquad \forall \boldsymbol{q} \in \{\boldsymbol{n}\}^\perp \cap \mathcal{B}(\eta). \quad (4.27)$$

Proof. If $\gamma_{\boldsymbol{n},\eta} = 0$, in view of [36, Lemma 3.12], there exist $\widehat{\boldsymbol{v}} \in \mathcal{F}_{\#}$ and a sequence $\{\boldsymbol{v}^k\} \subset \partial\mathcal{K}_{\text{logdet}} \cap \mathcal{B}(\eta) \setminus \mathcal{F}_{\#}$ such that (4.24) holds with $\mathfrak{g} = |\cdot|^{\frac{1}{2}}$ and $\mathcal{F} = \mathcal{F}_{\#}$. Since $\{\boldsymbol{v}^k\} \subset \partial\mathcal{K}_{\text{logdet}} \cap \mathcal{B}(\eta) \setminus \mathcal{F}_{\#}$, the equation for the boundary of $\mathcal{K}_{\text{logdet}}$ (see (4.6) and (4.21)) implies that we have the following two cases:

(i) $\boldsymbol{v}^k \in \partial\mathcal{K}_{\text{logdet}} \cap \mathcal{B}(\eta) \setminus \mathcal{F}_{\text{d}}$ infinitely often;

(ii) $\boldsymbol{v}^k \in \mathcal{F}_{\text{d}} \cap \mathcal{B}(\eta) \setminus \mathcal{F}_{\#}$ for all large k.

(i) Passing to a subsequence if necessary, we can assume that $\boldsymbol{v}^k \in \partial\mathcal{K}_{\text{logdet}} \cap \mathcal{B}(\eta) \setminus \mathcal{F}_{\text{d}}$ for all k, that is,

$$\boldsymbol{v}^k = (\boldsymbol{v}_y^k \log\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k), \boldsymbol{v}_y^k, \boldsymbol{v}_Z^k) \text{ with } \boldsymbol{v}_y^k > 0, \boldsymbol{v}_Z^k \succ 0, \quad \text{for all } k.$$

Then, $\langle \boldsymbol{n}, \boldsymbol{v}^k \rangle = \boldsymbol{n}_y \boldsymbol{v}_y^k + \operatorname{tr}(\boldsymbol{v}_Z^k \boldsymbol{n}_Z)$, which is positive since $\boldsymbol{n}_y > 0, \boldsymbol{v}_y^k > 0$ and both $\boldsymbol{v}_Z^k, \boldsymbol{n}_Z$ are positive semidefinite.

Now, one can check that

$$\|\boldsymbol{w}^k - \boldsymbol{v}^k\| = \frac{\langle \boldsymbol{n}, \boldsymbol{v}^k \rangle}{\|\boldsymbol{n}\|} = \frac{\boldsymbol{n}_y \boldsymbol{v}_y^k + \operatorname{tr}(\boldsymbol{v}_Z^k \boldsymbol{n}_Z)}{\|\boldsymbol{n}\|}. \tag{4.28}$$

On the other hand, by Lemma 2.4, the formula of $\mathcal{F}_{\#}$ and Proposition 2.5, we obtain the following inequality for all k,

$$\|\boldsymbol{w}^k - \boldsymbol{u}^k\| \le \operatorname{dist}(\boldsymbol{v}^k, \mathcal{F}_{\#}) \le (\boldsymbol{v}_y^k \log\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k))_+ + \boldsymbol{v}_y^k + C_P \operatorname{tr}(\boldsymbol{v}_Z^k \boldsymbol{n}_Z)^{\frac{1}{2}}. \tag{4.29}$$

Let $\tau^k := \operatorname{tr}(\boldsymbol{v}_Z^k \boldsymbol{n}_Z)$ and $\mathsf{r} := \mathsf{r}(\boldsymbol{n}_Z)$.

If $\boldsymbol{v}_y^k \log\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k) \ge 0$ infinitely often, then by extracting a subsequence if necessary, we may assume that $\boldsymbol{v}_y^k \log\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k) \ge 0$ for all k. Then we have from (4.29) and (4.7) that for all large k,

$$\begin{aligned}\|\boldsymbol{w}^k - \boldsymbol{u}^k\| &\le d|\log(\eta)|\boldsymbol{v}_y^k - d\boldsymbol{v}_y^k \log(\boldsymbol{v}_y^k) + \boldsymbol{v}_y^k + C_P(\tau^k)^{\frac{1}{2}} \\ &\overset{\text{(a)}}{\le} (d|\log(\eta)| + 1)(\boldsymbol{v}_y^k)^{\frac{1}{2}} + d(\boldsymbol{v}_y^k)^{\frac{1}{2}} + C_P(\tau^k)^{\frac{1}{2}}\end{aligned}$$

$$\overset{\text{(b)}}{\leq} \frac{(d|\log(\eta)|+d+1)\|\boldsymbol{n}\|^{\frac{1}{2}}}{(\boldsymbol{n}_y)^{\frac{1}{2}}}\|\boldsymbol{w}^k-\boldsymbol{v}^k\|^{\frac{1}{2}} + C_P\|\boldsymbol{n}\|^{\frac{1}{2}}\|\boldsymbol{w}^k-\boldsymbol{v}^k\|^{\frac{1}{2}}$$

$$= \left[\frac{(d|\log(\eta)|+d+1)\|\boldsymbol{n}\|^{\frac{1}{2}}}{(\boldsymbol{n}_y)^{\frac{1}{2}}} + C_P\|\boldsymbol{n}\|^{\frac{1}{2}}\right]\|\boldsymbol{w}^k-\boldsymbol{v}^k\|^{\frac{1}{2}},$$

where (a) holds by (4.8) with $\alpha = 1$ and the fact that $\boldsymbol{v}_y^k \to 0$ (since $\boldsymbol{v}^k \to \widehat{\boldsymbol{v}} \in \mathcal{F}_{\#}$), (b) is true since $\|\boldsymbol{w}^k-\boldsymbol{v}^k\|^{\frac{1}{2}} \geq (\boldsymbol{n}_y\boldsymbol{v}_y^k)^{\frac{1}{2}}/(\|\boldsymbol{n}\|)^{\frac{1}{2}}$ and $\|\boldsymbol{w}^k-\boldsymbol{v}^k\|^{\frac{1}{2}} \geq (\tau^k)^{\frac{1}{2}}/(\|\boldsymbol{n}\|)^{\frac{1}{2}}$ for all k thanks to (4.28).

This contradicts (4.24) with $|\cdot|^{\frac{1}{2}}$ in place of $\mathfrak{g}$ and hence this case cannot happen.

If $\boldsymbol{v}_y^k \log\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k) < 0$ infinitely often, then by extracting a subsequence if necessary, we may assume that $\boldsymbol{v}_y^k \log\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k) < 0$ for all k. Similar to the previous analysis, we have from (4.29), (4.8) and (4.28) that for all large k,

$$\|\boldsymbol{w}^k-\boldsymbol{u}^k\| \leq \boldsymbol{v}_y^k + C_P(\tau^k)^{\frac{1}{2}} \leq (\boldsymbol{v}_y^k)^{\frac{1}{2}} + C_P(\tau^k)^{\frac{1}{2}} \leq \left[(\|\boldsymbol{n}\|/\boldsymbol{n}_y)^{\frac{1}{2}} + C_P\|\boldsymbol{n}\|^{\frac{1}{2}}\right]\|\boldsymbol{w}^k-\boldsymbol{v}^k\|^{\frac{1}{2}}.$$

The above display contradicts (4.24) with $|\cdot|^{\frac{1}{2}}$ in place of $\mathfrak{g}$ and hence this case cannot happen.

(ii) By Lemma 4.5, case (ii) also cannot happen.

Hence, we conclude that $\gamma_{\boldsymbol{n},\eta} \in (0,\infty]$. In view of [36, Theorem 3.10], we deduce that (4.27) holds. □

Remark 4.7 (Tightness of (4.27))**.** *Fix any* $0 < \mathsf{r} < d$ *(recall that we assumed* $d \geq 2$*; see the discussions following* (4.4)*). Let* $\boldsymbol{n} = (0, \boldsymbol{n}_y, \boldsymbol{n}_Z)$ *with* $\boldsymbol{n}_y > 0$, $\boldsymbol{n}_Z \succeq 0$ *and* $\mathsf{r}(\boldsymbol{n}_Z) = \mathsf{r}$. *Then, we have* $\mathcal{F}_{\#} = \mathcal{K}_{\rm logdet} \cap \{\boldsymbol{n}\}^\perp$ *from* (4.12)*. Let* $R \in \mathbb{R}^{d\times d}$ *be such that* $\boldsymbol{n}_Z = R\begin{bmatrix}\boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \Sigma_{\mathsf{r}}\end{bmatrix}R^\top$ *where* $\Sigma_{\mathsf{r}} \in \mathcal{S}^{\mathsf{r}}$ *is diagonal,* $\Sigma_{\mathsf{r}} \succ 0$ *and* $RR^\top = I_d$*. Then*

$$\mathcal{F}_{\#} = \mathbb{R}_- \times \{0\} \times (\mathcal{S}_+^d \cap \{\boldsymbol{n}_Z\}^\perp) = \mathbb{R}_- \times \{0\} \times \left\{R\begin{bmatrix}A & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}R^\top \,:\, A \in \mathcal{S}_+^{d-\mathsf{r}}\right\}. \quad (4.30)$$

Fix a $Q \in \mathbb{R}^{\mathsf{r}\times(d-\mathsf{r})}$ *with* $0 < \lambda_{\max}(Q^\top Q) \le 1$. *For every* $k > 0$, *we define*

$$\boldsymbol{w}^k = \left(-1, 0, R\begin{bmatrix} I_{d-\mathsf{r}} & \frac{Q^\top}{k} \\ \frac{Q}{k} & \mathbf{0} \end{bmatrix} R^\top\right) \text{ and } \boldsymbol{v}^k = \left(-1, 0, R\begin{bmatrix} I_{d-\mathsf{r}} & \frac{Q^\top}{k} \\ \frac{Q}{k} & \frac{I_\mathsf{r}}{k^2} \end{bmatrix} R^\top\right).$$

Then there exists $\eta > 0$ *such that* $\{\boldsymbol{w}^k\} \subset \{\boldsymbol{n}\}^\perp \cap \mathcal{B}(\eta)$. *We also observe that* $R\begin{bmatrix} I_{d-\mathsf{r}} & \frac{Q^\top}{k} \\ \frac{Q}{k} & \frac{I_\mathsf{r}}{k^2} \end{bmatrix} R^\top \succeq 0$ *for all* k *based on standard arguments involving the Schur complement. Then* $\{\boldsymbol{v}^k\} \subset \mathcal{F}_\mathrm{d} \subset \mathcal{K}_\mathrm{logdet}$. *With that, we have*

$$\mathrm{dist}(\boldsymbol{w}^k, \mathcal{K}_\mathrm{logdet}) \le \|\boldsymbol{w}^k - \boldsymbol{v}^k\| = \left\|\frac{I_\mathsf{r}}{k^2}\right\|_F = \frac{\sqrt{\mathsf{r}}}{k^2}.$$

Therefore, by applying (4.27) *and using* (4.30), *there exists* $\kappa_B > 0$ *such that*

$$0 < \frac{\sqrt{2}\|Q\|_F}{k} = \mathrm{dist}(\boldsymbol{w}^k, \mathcal{F}_\#) \le \kappa_B \mathrm{dist}(\boldsymbol{w}^k, \mathcal{K}_\mathrm{logdet})^{\frac{1}{2}} \le \frac{\kappa_B \mathsf{r}^{\frac{1}{4}}}{k}.$$

Consequently, for all k, *we have*

$$0 < \frac{\sqrt{2}\|Q\|_F}{\mathsf{r}^{\frac{1}{4}}} \le \frac{\mathrm{dist}(\boldsymbol{w}^k, \mathcal{F}_\#)}{\mathrm{dist}(\boldsymbol{w}^k, \mathcal{K}_\mathrm{logdet})^{\frac{1}{2}}} \le \kappa_B.$$

Similar to the argument in Remark 4.3, we conclude that the choice of $|\cdot|^{\frac{1}{2}}$ *is tight.*

Next, we consider the case where $\boldsymbol{n}_y = 0$. Define $\mathfrak{g}_\mathrm{log}$ as follows

$$\mathfrak{g}_\mathrm{log}(t) := \begin{cases} 0 & \text{if } t = 0, \\ -\frac{1}{\log(t)} & \text{if } 0 < t \le \frac{1}{e^2}, \\ \frac{1}{4} + \frac{1}{4}e^2 t & \text{if } t > \frac{1}{e^2}. \end{cases} \tag{4.31}$$

We note that $\mathfrak{g}_\mathrm{log}$ is increasing with $\mathfrak{g}_\mathrm{log}(0) = 0$ and $|t| \le \mathfrak{g}_\mathrm{log}(t)$ for all $t \in \mathbb{R}_+$. Moreover, $\mathfrak{g}_\mathrm{log}(t) > \mathfrak{g}_\mathrm{d}(t)$ for any $t \in (0, \frac{1}{e^2})$. With $\mathfrak{g}_\mathrm{log}$, the next theorem shows that $\gamma_{\boldsymbol{n},\eta} \in (0, \infty]$ for $\mathcal{F}_\#$, which implies that a log-type error bound holds.

Theorem 4.8 (Log-type error bound concerning $\mathcal{F}_\#$ if $\boldsymbol{n}_y = 0$)**.** *Let* $\boldsymbol{n} = (0, 0, \boldsymbol{n}_Z) \in \partial\mathcal{K}^*_{\text{logdet}}$ *with* $\boldsymbol{n}_Z \succeq 0$ *and* $0 < \mathsf{r}(\boldsymbol{n}_Z) < d$ *such that* $\mathcal{F}_\# = \mathcal{K}_{\text{logdet}} \cap \{\boldsymbol{n}\}^\perp$*. Let* $\eta > 0$ *and let* $\gamma_{\boldsymbol{n},\eta}$ *be defined as in* (4.23) *with* $\mathcal{F} = \mathcal{F}_\#$ *and* $\mathfrak{g} = \mathfrak{g}_{\log}$ *in* (4.31)*. Then* $\gamma_{\boldsymbol{n},\eta} \in (0, \infty]$ *and*

$$\operatorname{dist}(\boldsymbol{q}, \mathcal{F}_\#) \leq \max\{2, 2\gamma_{\boldsymbol{n},\eta}^{-1}\} \cdot \mathfrak{g}_{\log}(\operatorname{dist}(\boldsymbol{q}, \mathcal{K}_{\text{logdet}})) \qquad \forall \boldsymbol{q} \in \{\boldsymbol{n}\}^\perp \cap \mathcal{B}(\eta). \tag{4.32}$$

Proof. If $\gamma_{\boldsymbol{n},\eta} = 0$, in view of [36, Lemma 3.12], there exists $\widehat{\boldsymbol{v}} \in \mathcal{F}_\#$ and sequences $\{\boldsymbol{v}^k\}, \{\boldsymbol{w}^k\}, \{\boldsymbol{u}^k\}$ being defined as those therein, with the cone being $\mathcal{K}_{\text{logdet}}$ and the face being $\mathcal{F}_\#$, such that (4.24) holds with $\mathfrak{g} = \mathfrak{g}_{\log}$ as in (4.31). As in the proof of Theorem 4.6, the condition $\{\boldsymbol{v}^k\} \subset \partial\mathcal{K}_{\text{logdet}} \cap \mathcal{B}(\eta) \setminus \mathcal{F}_\#$ means that we need to consider the following two cases:

(i) $\boldsymbol{v}^k \in \partial\mathcal{K}_{\text{logdet}} \cap \mathcal{B}(\eta) \setminus \mathcal{F}_{\text{d}}$ infinitely often;

(ii) $\boldsymbol{v}^k \in \mathcal{F}_{\text{d}} \cap \mathcal{B}(\eta) \setminus \mathcal{F}_\#$ for all large k.

(i) Passing to a subsequence if necessary, we can assume that $\boldsymbol{v}^k \in \partial\mathcal{K}_{\text{logdet}} \cap \mathcal{B}(\eta) \setminus \mathcal{F}_{\text{d}}$ for all k, that is,

$$\boldsymbol{v}^k = (\boldsymbol{v}^k_y \log\det(\boldsymbol{v}^k_Z/\boldsymbol{v}^k_y), \boldsymbol{v}^k_y, \boldsymbol{v}^k_Z) \text{ with } \boldsymbol{v}^k_y > 0, \boldsymbol{v}^k_Z \succ 0, \quad \text{for all } k.$$

Then $\langle \boldsymbol{n}, \boldsymbol{v}^k \rangle = \operatorname{tr}(\boldsymbol{v}^k_Z \boldsymbol{n}_Z)$, which is nonnegative since $\boldsymbol{n}_Z \succeq 0, \boldsymbol{v}^k_Z \succ 0$.

Now, one can check that for all k,

$$\|\boldsymbol{w}^k - \boldsymbol{v}^k\| = \frac{\langle \boldsymbol{n}, \boldsymbol{v}^k \rangle}{\|\boldsymbol{n}\|} = \frac{\operatorname{tr}(\boldsymbol{v}^k_Z \boldsymbol{n}_Z)}{\|\boldsymbol{n}\|}. \tag{4.33}$$

On the other hand, by Lemma 2.4, the formula of $\mathcal{F}_\#$ and Proposition 2.5, we obtain that for all k,

$$\|\boldsymbol{w}^k - \boldsymbol{u}^k\| \leq \operatorname{dist}(\boldsymbol{v}^k, \mathcal{F}_\#) \leq (\boldsymbol{v}^k_y \log\det(\boldsymbol{v}^k_Z/\boldsymbol{v}^k_y))_+ + \boldsymbol{v}^k_y + C_P \operatorname{tr}(\boldsymbol{v}^k_Z \boldsymbol{n}_Z)^{\frac{1}{2}}. \tag{4.34}$$

Let $\tau^k := \operatorname{tr}(\boldsymbol{v}^k_Z \boldsymbol{n}_Z)$ and $\mathsf{r} := \mathsf{r}(\boldsymbol{n}_Z)$.

If $\boldsymbol{v}_y^k \log\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k) \geq 0$ infinitely often, then, by passing to a subsequence if necessary, we may assume that $\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k) \geq 1$ for all k, and hence $(\boldsymbol{v}_y^k)^d \leq \det(\boldsymbol{v}_Z^k)$ for all k. Thus, upon invoking Lemma 2.1, we obtain that for all k,

$$\boldsymbol{v}_y^k \leq (\det(\boldsymbol{v}_Z^k))^{\frac{1}{d}} \leq C(\tau^k)^{\frac{\mathsf{r}}{d}}. \tag{4.35}$$

Then, for all sufficiently large k,

$$\begin{aligned}
\|\boldsymbol{w}^k - \boldsymbol{u}^k\| &\overset{\text{(a)}}{\leq} d\boldsymbol{v}_y^k |\log(\eta)| - d\boldsymbol{v}_y^k \log(\boldsymbol{v}_y^k) + \boldsymbol{v}_y^k + C_P(\tau^k)^{\frac{1}{2}} \\
&\overset{\text{(b)}}{\leq} (d|\log(\eta)| + 1)C(\tau^k)^{\frac{\mathsf{r}}{d}} - dC(\tau^k)^{\frac{\mathsf{r}}{d}} \log(C(\tau^k)^{\frac{\mathsf{r}}{d}}) + C_P(\tau^k)^{\frac{1}{2}} \\
&= (d|\log(\eta)| + 1)C(\tau^k)^{\frac{\mathsf{r}}{d}} - Cd\log(C)(\tau^k)^{\frac{\mathsf{r}}{d}} - C\mathsf{r}(\tau^k)^{\frac{\mathsf{r}}{d}} \log(\tau^k) + C_P(\tau^k)^{\frac{1}{2}} \\
&\overset{\text{(c)}}{\leq} (Cd|\log(\eta)| + C - Cd\log(C))(\tau^k)^{\frac{\mathsf{r}}{d}} + C\mathsf{r}(\tau^k)^{\frac{r}{2d}} + C_P(\tau^k)^{\frac{1}{2}} \\
&\leq \Big|Cd|\log(\eta)| + C - Cd\log(C)\Big|(\tau^k)^{\rho} + C\mathsf{r}(\tau^k)^{\rho} + C_P(\tau^k)^{\rho} \\
&= C_{\#}(\tau^k)^{\rho},
\end{aligned}$$

where $\rho = \min\{\frac{\mathsf{r}}{2d}, \frac{1}{2}\}$ and $C_{\#} := \Big|Cd|\log(\eta)| + C - Cd\log(C)\Big| + C\mathsf{r} + C_P > 0$, (a) comes from (4.34) and (4.7), (b) holds because of (4.35) and the fact that $x \mapsto -x\log(x)$ is increasing for all sufficiently small positive x, (c) is true by (4.8) (with $\alpha = \mathsf{r}/d > 0$).

Therefore, we conclude that

$$\lim_{k\to\infty} \frac{\mathfrak{g}_{\log}(\|\boldsymbol{w}^k - \boldsymbol{v}^k\|)}{\|\boldsymbol{w}^k - \boldsymbol{u}^k\|} \geq \liminf_{k\to\infty} \frac{\|\boldsymbol{w}^k - \boldsymbol{v}^k\|^{\rho}}{\|\boldsymbol{w}^k - \boldsymbol{u}^k\|} \geq \lim_{k\to\infty} \frac{(\tau^k)^{\rho}}{\|\boldsymbol{n}\|^{\rho} C_{\#}(\tau^k)^{\rho}} = \frac{1}{\|\boldsymbol{n}\|^{\rho} C_{\#}} > 0.$$

This contradicts (4.24) with $\mathfrak{g}_{\log}$ in place of $\mathfrak{g}$ and hence this case cannot happen.

If $\boldsymbol{v}_y^k \log\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k) < 0$ infinitely often, then by passing to a subsequence if necessary, we may assume that $\boldsymbol{v}_y^k \log\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k) < 0$ for all large k. Moreover, recalling the exponential form of $\mathcal{K}_{\text{logdet}}$ in (4.2), we have $(\boldsymbol{v}_y^k)^d e^{\boldsymbol{v}_x^k/\boldsymbol{v}_y^k} = \det(\boldsymbol{v}_Z^k)$ for all k. Invoking Lemma 2.1, we then see that for all k,

$$\boldsymbol{v}_y^k e^{\boldsymbol{v}_x^k/(d\boldsymbol{v}_y^k)} = (\det(\boldsymbol{v}_Z^k))^{\frac{1}{d}} \leq C(\tau^k)^{\frac{\mathsf{r}}{d}}.$$

Thus, by taking logarithm on both sides, the above inequality becomes

$$\log(\boldsymbol{v}_y^k) + \frac{\boldsymbol{v}_x^k}{d\boldsymbol{v}_y^k} \leq \log(C) + \frac{\mathsf{r}}{d}\log(\tau^k).$$

Since $\boldsymbol{v}_y^k \to 0$, $\tau^k \to 0$, and both sequences are positive, we note that $-\boldsymbol{v}_y^k \log(\tau^k) > 0$ for all large k. After multiplying $-\boldsymbol{v}_y^k$ on both sides of the above display and rearranging terms, we see that for all large k,

$$0 < -\boldsymbol{v}_y^k \log(\tau^k) \leq \frac{d\log(C)\boldsymbol{v}_y^k}{\mathsf{r}} - \frac{d\boldsymbol{v}_y^k \log(\boldsymbol{v}_y^k)}{\mathsf{r}} - \frac{\boldsymbol{v}_x^k}{\mathsf{r}}.$$

Then, by passing to the limit on both sides of the above display, we obtain that

$$\begin{aligned} 0 &\leq \limsup_{k\to\infty} -\boldsymbol{v}_y^k \log(\tau^k) \leq \limsup_{k\to\infty} \frac{d\log(C)\boldsymbol{v}_y^k}{\mathsf{r}} - \frac{d\boldsymbol{v}_y^k \log(\boldsymbol{v}_y^k)}{\mathsf{r}} - \frac{\boldsymbol{v}_x^k}{\mathsf{r}} \\ &= -\lim_{k\to\infty} \frac{\boldsymbol{v}_x^k}{\mathsf{r}} = -\frac{\widehat{\boldsymbol{v}}_x}{\mathsf{r}}. \end{aligned} \tag{4.36}$$

Therefore, we conclude that

$$\begin{aligned} \lim_{k\to\infty} \frac{\mathfrak{g}_{\log}(\|\boldsymbol{w}^k - \boldsymbol{v}^k\|)}{\|\boldsymbol{w}^k - \boldsymbol{u}^k\|} &\overset{\text{(a)}}{\geq} \liminf_{k\to\infty} -\frac{1}{\log(\tau^k) - \log(\|\boldsymbol{n}\|)} \frac{1}{\boldsymbol{v}_y^k + C_P(\tau^k)^{\frac{1}{2}}} \\ &= \liminf_{k\to\infty} \frac{1}{\log(\|\boldsymbol{n}\|)(\boldsymbol{v}_y^k + C_P(\tau^k)^{\frac{1}{2}}) - \boldsymbol{v}_y^k \log(\tau^k) - C_P(\tau^k)^{\frac{1}{2}} \log(\tau^k)} \\ &\overset{\text{(b)}}{\geq} \lim_{k\to\infty} \frac{-\mathsf{r}}{\boldsymbol{v}_x^k} \in (0,\infty], \end{aligned}$$

where (a) is true owing to (4.33) and (4.34), (b) comes from (4.36) and the fact $\boldsymbol{v}_y^k \to 0, \tau^k \to 0$, the last inequality holds because $\widehat{\boldsymbol{v}}_x \leq 0$ thanks to $\boldsymbol{v}_y^k \log\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k) < 0$ for all large k. The above display contradicts (4.24) with $\mathfrak{g}_{\log}$ in place of $\mathfrak{g}$ and so this case cannot happen.

(ii) In this case, we have from Lemma 4.5 that $\liminf_{k\to\infty} \frac{\|\boldsymbol{w}^k - \boldsymbol{v}^k\|^{1/2}}{\|\boldsymbol{w}^k - \boldsymbol{u}^k\|} \in (0, \infty]$, which implies that

$$\lim_{k\to\infty} \frac{\mathfrak{g}_{\log}(\|\boldsymbol{w}^k - \boldsymbol{v}^k\|)}{\|\boldsymbol{w}^k - \boldsymbol{u}^k\|} \geq \liminf_{k\to\infty} \frac{\|\boldsymbol{w}^k - \boldsymbol{v}^k\|^{1/2}}{\|\boldsymbol{w}^k - \boldsymbol{u}^k\|} \in (0, \infty],$$

where we recall that $|t|^{1/2} \leq \mathfrak{g}_{\log}(t)$ for t sufficiently small. In view of the definition of $\gamma_{\boldsymbol{n},\eta}$, case (ii) also cannot happen.

Hence, we conclude that $\gamma_{\boldsymbol{n},\eta} \in (0, \infty]$. Using this together with [36, Theorem 3.10], we deduce that (4.32) holds. □

Remark 4.9 (Tightness of (4.32))**.** *Let $\boldsymbol{n} = (0, 0, \boldsymbol{n}_Z)$ with $\boldsymbol{n}_Z \succeq 0$, $0 < \mathsf{r}(\boldsymbol{n}_Z) < d$. Then, we have $\mathcal{F}_\# = \{\boldsymbol{n}\}^\perp \cap \mathcal{K}_{\text{logdet}}$ from* (4.12)*. Consider the sequence $\boldsymbol{w}^k = (-1, 1/k, \boldsymbol{0})$, $\boldsymbol{v}^k = (-1, 1/k, I_d/(ke^{\frac{k}{d}}))$ and $\boldsymbol{u}^k = (-1, 0, \boldsymbol{0})$ for every k, we note that $\boldsymbol{w}^k \in \{\boldsymbol{n}\}^\perp, \boldsymbol{v}^k \in \mathcal{K}_{\text{logdet}}$ and $\boldsymbol{u}^k = P_{\mathcal{F}_\#}(\boldsymbol{w}^k)$ for every k. Moreover, there exists $\eta > 0$ such that $\{\boldsymbol{w}^k\} \subseteq \mathcal{B}(\eta)$. Therefore, applying* (4.32)*, there exists $\kappa_B > 0$ such that*

$$\frac{1}{k} = \operatorname{dist}(\boldsymbol{w}^k, \mathcal{F}_\#) \leq \kappa_B \mathfrak{g}_{\log}(\operatorname{dist}(\boldsymbol{w}^k, \mathcal{K}_{\text{logdet}})) \leq \kappa_B \mathfrak{g}_{\log}\left(\frac{\sqrt{d}}{ke^{\frac{k}{d}}}\right) \quad \forall k \in \mathbb{N}.$$

In view of the definition of $\mathfrak{g}_{\log}$ (see (4.31)*) and its monotonicity, for large enough k we have*

$$\frac{1}{k} = \operatorname{dist}(\boldsymbol{w}^k, \mathcal{F}_\#) \leq \kappa_B \mathfrak{g}_{\log}(\operatorname{dist}(\boldsymbol{w}^k, \mathcal{K}_{\text{logdet}})) \leq \frac{\kappa_B}{\log k + (k/d) - \log\sqrt{d}} \leq \kappa_B \frac{2d}{k}.$$

Consequently, it holds that for all sufficiently large k,

$$\frac{1}{2d} \leq \frac{\operatorname{dist}(\boldsymbol{w}^k, \mathcal{F}_\#)}{\mathfrak{g}_{\log}(\operatorname{dist}(\boldsymbol{w}^k, \mathcal{K}_{\text{logdet}}))} \leq \kappa_B.$$

Similar to the argument in Remark 4.3, we conclude that the choice of $\mathfrak{g}_{\log}$ is tight.

Using Theorems 4.6 and 4.8 in combination with [36, Lemma 3.9], we obtain the following one-step facial residual functions for $\mathcal{K}_{\rm logdet}$ and $\boldsymbol{n}$.

Corollary 4.10. *Let* $\boldsymbol{n} = (0, \boldsymbol{n}_y, \boldsymbol{n}_Z) \in \partial\mathcal{K}^*_{\rm logdet}$ *with* $\boldsymbol{n}_y \geq 0$*,* $\boldsymbol{n}_Z \succeq 0$ *and* $0 < \mathsf{r}(\boldsymbol{n}_Z) < d$ *such that* $\mathcal{F}_\# = \mathcal{K}_{\rm logdet} \cap \{\boldsymbol{n}\}^\perp$*.*

(i) If $\boldsymbol{n}_y > 0$*, let* $\gamma_{\boldsymbol{n},t}$ *be as in* (4.23) *with* $\mathcal{F} = \mathcal{F}_\#$ *and* $\mathfrak{g} = |\cdot|^{\frac{1}{2}}$*. Then the function* $\psi_{\mathcal{K},\boldsymbol{n}} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *defined by*

$$\psi_{\mathcal{K},\boldsymbol{n}}(\epsilon, t) := \max\{\epsilon, \epsilon/\|\boldsymbol{n}\|\} + \max\left\{2t^{\frac{1}{2}}, 2\gamma_{\boldsymbol{n},t}^{-1}\right\}(\epsilon + \max\{\epsilon, \epsilon/\|\boldsymbol{n}\|\})^{\frac{1}{2}}$$

is a one-step facial residual function for $\mathcal{K}_{\rm logdet}$ *and* $\boldsymbol{n}$*.*

(ii) If $\boldsymbol{n}_y = 0$*, let* $\gamma_{\boldsymbol{n},t}$ *be as in* (4.23) *with* $\mathcal{F} = \mathcal{F}_\#$ *and* $\mathfrak{g} = \mathfrak{g}_{\log}$ *in* (4.31)*. Then the function* $\psi_{\mathcal{K},\boldsymbol{n}} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *defined by*

$$\psi_{\mathcal{K},\boldsymbol{n}}(\epsilon, t) := \max\{\epsilon, \epsilon/\|\boldsymbol{n}\|\} + \max\left\{2, 2\gamma_{\boldsymbol{n},t}^{-1}\right\}\mathfrak{g}_{\log}(\epsilon + \max\{\epsilon, \epsilon/\|\boldsymbol{n}\|\})$$

is a one-step facial residual function for $\mathcal{K}_{\rm logdet}$ *and* $\boldsymbol{n}$*.*

4.2.3 $\mathcal{F}_\infty$: the exceptional 1-dimensional face

We first show a Lipschitz error bound concerning $\mathcal{F}_\infty$ if $\boldsymbol{n}_y > 0$.

Theorem 4.11 (Lipschitz error bound concerning $\mathcal{F}_\infty$ if $\boldsymbol{n}_y > 0$)**.** *Let* $\boldsymbol{n} = (0, \boldsymbol{n}_y, \boldsymbol{n}_Z) \in \partial\mathcal{K}^*_{\rm logdet}$ *with* $\boldsymbol{n}_y > 0$ *and* $\boldsymbol{n}_Z \succ 0$ *such that* $\mathcal{F}_\infty = \mathcal{K}_{\rm logdet} \cap \{\boldsymbol{n}\}^\perp$*. Let* $\eta > 0$ *and let* $\gamma_{\boldsymbol{n},\eta}$ *be defined as in* (4.23) *with* $\mathcal{F} = \mathcal{F}_\infty$ *and* $\mathfrak{g} = |\cdot|$*. Then* $\gamma_{\boldsymbol{n},\eta} \in (0, \infty]$ *and*

$$\operatorname{dist}(\boldsymbol{q}, \mathcal{F}_\infty) \leq \max\{2, 2\gamma_{\boldsymbol{n},\eta}^{-1}\} \cdot \operatorname{dist}(\boldsymbol{q}, \mathcal{K}_{\rm logdet}) \qquad \forall \boldsymbol{q} \in \{\boldsymbol{n}\}^\perp \cap \mathcal{B}(\eta). \tag{4.37}$$

Proof. If $\gamma_{\boldsymbol{n},\eta} = 0$, in view of [36, Lemma 3.12], there exists $\widehat{\boldsymbol{v}} \in \mathcal{F}_\infty$ and sequences $\{\boldsymbol{v}^k\}, \{\boldsymbol{w}^k\}, \{\boldsymbol{u}^k\}$ being defined as those therein, with the cone being $\mathcal{K}_{\rm logdet}$ and the face being $\mathcal{F}_\infty$, such that (4.24) holds with $\mathfrak{g} = |\cdot|$. Note that $\{\boldsymbol{v}^k\} \subset \partial\mathcal{K}_{\rm logdet} \cap \mathcal{B}(\eta) \setminus \mathcal{F}_\infty$ means that we need to consider the following two cases:

(i) $\boldsymbol{v}^k \in \partial\mathcal{K}_{\text{logdet}} \cap \mathcal{B}(\eta) \setminus \mathcal{F}_{\text{d}}$ infinitely often;

(ii) $\boldsymbol{v}^k \in \mathcal{F}_{\text{d}} \cap \mathcal{B}(\eta) \setminus \mathcal{F}_\infty$ for all large k.

(i) Without loss of generality, we assume that $\boldsymbol{v}^k \in \partial\mathcal{K}_{\text{logdet}} \cap \mathcal{B}(\eta) \setminus \mathcal{F}_{\text{d}}$ for all k by passing to a subsequence if necessary, that is,

$$\boldsymbol{v}^k = (\boldsymbol{v}_y^k \log\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k), \boldsymbol{v}_y^k, \boldsymbol{v}_Z^k) \text{ with } \boldsymbol{v}_y^k > 0, \boldsymbol{v}_Z^k \succ 0 \quad \text{for all } k.$$

Then, $\langle \boldsymbol{n}, \boldsymbol{v}^k \rangle = \boldsymbol{n}_y \boldsymbol{v}_y^k + \operatorname{tr}(\boldsymbol{v}_Z^k \boldsymbol{n}_Z) > 0$ and

$$\|\boldsymbol{w}^k - \boldsymbol{v}^k\| = \frac{\boldsymbol{n}_y \boldsymbol{v}_y^k + \operatorname{tr}(\boldsymbol{v}_Z^k \boldsymbol{n}_Z)}{\|\boldsymbol{n}\|}.$$

On the other hand, by Lemma 2.4, we obtain that for all k,

$$\|\boldsymbol{w}^k - \boldsymbol{u}^k\| \le \operatorname{dist}(\boldsymbol{v}^k, \mathcal{F}_\infty) \le (\boldsymbol{v}_y^k \log\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k))_+ + \boldsymbol{v}_y^k + \|\boldsymbol{v}_Z^k\|_F. \tag{4.38}$$

If $\boldsymbol{v}_y^k \log\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k) \ge 0$ infinitely often, by passing to a subsequence if necessary, we may assume that $\boldsymbol{v}_y^k \log\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k) \ge 0$ for all large k and hence, recalling that $\|\boldsymbol{v}_Z^k\|_F \le \operatorname{tr}(\boldsymbol{v}_Z^k)$ (since $\boldsymbol{v}_Z^k \succ 0$), we obtain

$$\begin{aligned}
\|\boldsymbol{w}^k - \boldsymbol{u}^k\| &\le \boldsymbol{v}_y^k \log\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k) + \boldsymbol{v}_y^k + \operatorname{tr}(\boldsymbol{v}_Z^k) = \boldsymbol{v}_y^k + \operatorname{tr}(\boldsymbol{v}_Z^k) + \boldsymbol{v}_y^k \log(\prod_{i=1}^d \lambda_i(\boldsymbol{v}_Z^k)/\boldsymbol{v}_y^k) \\
&= \boldsymbol{v}_y^k + \operatorname{tr}(\boldsymbol{v}_Z^k) + \sum_{i=1}^d \boldsymbol{v}_y^k \log(\lambda_i(\boldsymbol{v}_Z^k)/\boldsymbol{v}_y^k) \\
&\overset{\text{(a)}}{\le} \boldsymbol{v}_y^k + \operatorname{tr}(\boldsymbol{v}_Z^k) + \sum_{i=1}^d \boldsymbol{v}_y^k (\lambda_i(\boldsymbol{v}_Z^k)/\boldsymbol{v}_y^k + 1) \\
&= \boldsymbol{v}_y^k + \operatorname{tr}(\boldsymbol{v}_Z^k) + \operatorname{tr}(\boldsymbol{v}_Z^k) + d\boldsymbol{v}_y^k = (1+d)\boldsymbol{v}_y^k + 2\operatorname{tr}(\boldsymbol{v}_Z^k),
\end{aligned}$$

where (a) holds because $\log(x) \le x + 1$ for all $x > 0$.

Combining these identities and using (2.1) yields:

$$\lim_{k\to\infty}\frac{\|\boldsymbol{w}^k-\boldsymbol{v}^k\|}{\|\boldsymbol{w}^k-\boldsymbol{u}^k\|}\geq\liminf_{k\to\infty}\frac{\frac{\boldsymbol{n}_y}{\|\boldsymbol{n}\|(1+d)}(1+d)\boldsymbol{v}_y^k+\frac{\lambda_{\min}(\boldsymbol{n}_Z)}{2\|\boldsymbol{n}\|}2\operatorname{tr}(\boldsymbol{v}_Z^k)}{(1+d)\boldsymbol{v}_y^k+2\operatorname{tr}(\boldsymbol{v}_Z^k)}$$

$$\geq\min\left\{\frac{\boldsymbol{n}_y}{\|\boldsymbol{n}\|(1+d)},\frac{\lambda_{\min}(\boldsymbol{n}_Z)}{2\|\boldsymbol{n}\|}\right\}>0.$$

This contradicts (4.24) with $|\cdot|$ in place of $\mathfrak{g}$ and hence this case cannot happen.

If $\boldsymbol{v}_y^k\log\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k)<0$ infinitely often, then by extracting a subsequence if necessary, we may assume that $\boldsymbol{v}_y^k\log\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k)<0$ for all large k and hence (4.38) becomes

$$\|\boldsymbol{w}^k-\boldsymbol{u}^k\|\leq\boldsymbol{v}_y^k+\operatorname{tr}(\boldsymbol{v}_Z^k).$$

Therefore,

$$\lim_{k\to\infty}\frac{\|\boldsymbol{w}^k-\boldsymbol{v}^k\|}{\|\boldsymbol{w}^k-\boldsymbol{u}^k\|}\geq\liminf_{k\to\infty}\frac{\frac{\boldsymbol{n}_y}{\|\boldsymbol{n}\|}\boldsymbol{v}_y^k+\frac{\lambda_{\min}(\boldsymbol{n}_Z)}{\|\boldsymbol{n}\|}\operatorname{tr}(\boldsymbol{v}_Z^k)}{\boldsymbol{v}_y^k+\operatorname{tr}(\boldsymbol{v}_Z^k)}\geq\min\left\{\frac{\boldsymbol{n}_y}{\|\boldsymbol{n}\|},\frac{\lambda_{\min}(\boldsymbol{n}_Z)}{\|\boldsymbol{n}\|}\right\}>0.$$

The above inequality contradicts (4.24) with $|\cdot|$ in place of $\mathfrak{g}$ and hence this case cannot happen.

(ii) By Lemma 4.5, case (ii) also cannot happen.

Overall, we conclude that $\gamma_{\boldsymbol{n},\eta}\in(0,\infty]$, and so by [36, Theorem 3.10], (4.37) holds. □

Note that a Lipschitz error bound is always tight up to a constant, so (4.37) is tight.

If $\boldsymbol{n}_y=0$, we have the following Log-type error bound for $\mathcal{K}_{\text{logdet}}$.

Theorem 4.12 (Log-type error bound concerning $\mathcal{F}_\infty$ if $\boldsymbol{n}_y=0$)**.** *Let* $\boldsymbol{n}=(0,0,\boldsymbol{n}_Z)\in\partial\mathcal{K}^*_{\text{logdet}}$ *with* $\boldsymbol{n}_Z\succ 0$ *such that* $\mathcal{F}_\infty=\mathcal{K}_{\text{logdet}}\cap\{\boldsymbol{n}\}^\perp$*. Let* $\eta>0$ *and let* $\gamma_{\boldsymbol{n},\eta}$ *be defined as in* (4.23) *with* $\mathcal{F}=\mathcal{F}_\infty$ *and* $\mathfrak{g}=\mathfrak{g}_{\log}$ *in* (4.31)*. Then* $\gamma_{\boldsymbol{n},\eta}\in(0,\infty]$ *and*

$$\operatorname{dist}(\boldsymbol{q},\mathcal{F}_\infty)\leq\max\{2,2\gamma_{\boldsymbol{n},\eta}^{-1}\}\cdot\mathfrak{g}_{\log}(\operatorname{dist}(\boldsymbol{q},\mathcal{K}_{\text{logdet}}))\qquad\forall\boldsymbol{q}\in\{\boldsymbol{n}\}^\perp\cap\mathcal{B}(\eta).\quad(4.39)$$

Proof. If $\gamma_{\boldsymbol{n},\eta} = 0$, in view of [36, Lemma 3.12], there exists $\widehat{\boldsymbol{v}} \in \mathcal{F}_\infty$ and sequences $\{\boldsymbol{v}^k\}, \{\boldsymbol{w}^k\}, \{\boldsymbol{u}^k\}$ being defined as those therein, with the cone being $\mathcal{K}_{\rm logdet}$ and the face being $\mathcal{F}_\infty$, such that (4.24) holds with $\mathfrak{g} = \mathfrak{g}_{\log}$ as in (4.31). Note that $\{\boldsymbol{v}^k\} \subset \partial\mathcal{K}_{\rm logdet} \cap \mathcal{B}(\eta) \setminus \mathcal{F}_\infty$ means that we need to consider the following two cases:

(i) $\boldsymbol{v}^k \in \partial\mathcal{K}_{\rm logdet} \cap \mathcal{B}(\eta) \setminus \mathcal{F}_{\rm d}$ infinitely often;

(ii) $\boldsymbol{v}^k \in \mathcal{F}_{\rm d} \cap \mathcal{B}(\eta) \setminus \mathcal{F}_\infty$ for all large k.

(i) Without loss of generality, we assume that $\boldsymbol{v}^k \in \partial\mathcal{K}_{\rm logdet} \cap \mathcal{B}(\eta) \setminus \mathcal{F}_{\rm d}$ for all k by passing to a subsequence if necessary, that is,

$$\boldsymbol{v}^k = (\boldsymbol{v}_y^k \log\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k), \boldsymbol{v}_y^k, \boldsymbol{v}_Z^k) \text{ with } \boldsymbol{v}_y^k > 0, \boldsymbol{v}_Z^k \succ 0 \quad \text{for all } k.$$

Then $\langle \boldsymbol{n}, \boldsymbol{v}^k \rangle = \operatorname{tr}(\boldsymbol{v}_Z^k \boldsymbol{n}_Z) \geq 0$ and

$$\|\boldsymbol{w}^k - \boldsymbol{v}^k\| = \frac{\langle \boldsymbol{n}, \boldsymbol{v}^k \rangle}{\|\boldsymbol{n}\|} = \frac{\operatorname{tr}(\boldsymbol{v}_Z^k \boldsymbol{n}_Z)}{\|\boldsymbol{n}\|}. \tag{4.40}$$

In addition, by Lemma 2.4, we obtain that for all k,

$$\|\boldsymbol{w}^k - \boldsymbol{u}^k\| \leq \operatorname{dist}(\boldsymbol{v}^k, \mathcal{F}_\infty) \leq (\boldsymbol{v}_y^k \log\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k))_+ + \boldsymbol{v}_y^k + \|\boldsymbol{v}_Z^k\|_F. \tag{4.41}$$

Let $\tau^k := \operatorname{tr}(\boldsymbol{v}_Z^k \boldsymbol{n}_Z)$.

If $\boldsymbol{v}_y^k \log\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k) \geq 0$ infinitely often, then by passing to a subsequence if necessary, we may assume that $\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k) \geq 1$ for all k. Hence we have $(\boldsymbol{v}_y^k)^d \leq \det(\boldsymbol{v}_Z^k)$. Thus, combining Lemma 2.1 with $\mathsf{r}(\boldsymbol{n}_Z) = d$, we obtain that for all k,

$$\boldsymbol{v}_y^k \leq (\det(\boldsymbol{v}_Z^k))^{\frac{1}{d}} \leq C\tau^k. \tag{4.42}$$

Then, for sufficiently large k,

$$
\begin{aligned}
&\|\boldsymbol{w}^k - \boldsymbol{u}^k\| \overset{\text{(a)}}{\leq} d|\log(\eta)|\boldsymbol{v}_y^k - d\boldsymbol{v}_y^k \log(\boldsymbol{v}_y^k) + \boldsymbol{v}_y^k + \operatorname{tr}(\boldsymbol{v}_Z^k) \\
&= (d|\log(\eta)| + 1)\boldsymbol{v}_y^k - d\boldsymbol{v}_y^k \log(\boldsymbol{v}_y^k) + \frac{1}{\lambda_{\min}(\boldsymbol{n}_Z)}\lambda_{\min}(\boldsymbol{n}_Z)\operatorname{tr}(\boldsymbol{v}_Z^k) \\
&\overset{\text{(b)}}{\leq} (Cd|\log(\eta)| + C)\tau^k - Cd\tau^k \log(C\tau^k) + \frac{1}{\lambda_{\min}(\boldsymbol{n}_Z)}\tau^k \\
&= (Cd|\log(\eta)| + C - Cd\log(C))\tau^k - Cd\tau^k \log(\tau^k) + \frac{1}{\lambda_{\min}(\boldsymbol{n}_Z)}\tau^k \\
&\overset{\text{(c)}}{\leq} \left(\Big|Cd|\log(\eta)| + C - Cd\log(C)\Big| + Cd + \frac{1}{\lambda_{\min}(\boldsymbol{n}_Z)}\right)(-\tau^k \log(\tau^k)) \\
&= C_\infty(-\tau^k \log(\tau^k)),
\end{aligned}
\tag{4.43}
$$

where $C_\infty := \Big|Cd|\log(\eta)| + C - Cd\log(C)\Big| + Cd + \frac{1}{\lambda_{\min}(\boldsymbol{n}_Z)} > 0$, (a) comes from (4.41) and (4.7), (b) holds because of (2.1), (4.42) and the fact that $x \mapsto -x\log(x)$ is increasing for all sufficiently small positive x, (c) is true because $x \leq -x\log(x)$ for sufficiently small x and $\tau^k \to 0$ because $\boldsymbol{v}_Z^k \to 0$.

Hence,

$$
\begin{aligned}
\lim_{k\to\infty} \frac{\mathfrak{g}_{\log}(\|\boldsymbol{w}^k - \boldsymbol{v}^k\|)}{\|\boldsymbol{w}^k - \boldsymbol{u}^k\|} &\geq \liminf_{k\to\infty} -\frac{1}{\log\left(\frac{\tau^k}{\|\boldsymbol{n}\|}\right)} \frac{1}{C_\infty(-\tau^k \log(\tau^k))} \\
&\geq \lim_{k\to\infty} \frac{-\tau^k \log(\tau^k)}{C_\infty(-\tau^k \log(\tau^k))} = \frac{1}{C_\infty} > 0,
\end{aligned}
$$

where the first inequality comes from (4.40) and (4.43), the second inequality comes from (4.8) (with $\alpha = 1$ and $s = \frac{1}{\|\boldsymbol{n}\|}$). This contradicts (4.24) with $\mathfrak{g}_{\log}$ in place of $\mathfrak{g}$ and hence this case cannot happen.

If $\boldsymbol{v}_y^k \log\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k) < 0$ infinitely often, then by passing to a subsequence if necessary, we may assume that that $\boldsymbol{v}_y^k \log\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k) < 0$ for all k. Moreover, recalling the exponential form of $\mathcal{K}_{\text{logdet}}$ in (4.2), we have $(\boldsymbol{v}_y^k)^d e^{\boldsymbol{v}_x^k/\boldsymbol{v}_y^k} = \det(\boldsymbol{v}_Z^k)$ for all k. Upon invoking Lemma 2.1 with $\mathsf{r}\,(\boldsymbol{n}_Z) = d$, we then see that for all k, we have

$$\boldsymbol{v}_y^k e^{\boldsymbol{v}_x^k/(d\boldsymbol{v}_y^k)} = (\det(\boldsymbol{v}_Z^k))^{\frac{1}{d}} \leq C\tau^k.$$

Thus, by taking the logarithm on both sides, the above inequality becomes

$$\log(\boldsymbol{v}_y^k) + \frac{\boldsymbol{v}_x^k}{d\boldsymbol{v}_y^k} \leq \log(C) + \log(\tau^k).$$

Since $\boldsymbol{v}_y^k \to 0$, $\tau^k \to 0$ and $\{\boldsymbol{v}_y^k\}$, $\{\tau^k\}$ are positive sequences, we note that $-\boldsymbol{v}_y^k \log(\tau^k) > 0$ for all large k. After multiplying $-\boldsymbol{v}_y^k$ on both sides of the above display and rearranging terms, we see that for all large k,

$$0 < -\boldsymbol{v}_y^k \log(\tau^k) \leq -\boldsymbol{v}_y^k \log(\boldsymbol{v}_y^k) - \frac{\boldsymbol{v}_x^k}{d} + \log(C)\boldsymbol{v}_y^k.$$

Then, by passing to the limit on both sides of the above display, we obtain that

$$\begin{aligned} 0 &\leq \limsup_{k\to\infty} -\boldsymbol{v}_y^k \log(\tau^k) \leq \limsup_{k\to\infty} -\boldsymbol{v}_y^k \log(\boldsymbol{v}_y^k) - \frac{\boldsymbol{v}_x^k}{d} + \log(C)\boldsymbol{v}_y^k \\ &= -\lim_{k\to\infty} \frac{\boldsymbol{v}_x^k}{d} = -\frac{\widehat{\boldsymbol{v}}_x}{d}. \end{aligned} \tag{4.44}$$

Note also that since $\boldsymbol{n}_Z$ has full rank, we have upon invoking the equivalence in (2.1) that $\{\boldsymbol{n}_Z\}^\perp \cap \mathcal{S}_+^d = \{\boldsymbol{0}\}$. Then Proposition 2.5 guarantees that $C_P\tau^k \geq \|\boldsymbol{v}_Z^k\|_F$.

Therefore, altogether we conclude that

$$\begin{aligned} &\lim_{k\to\infty} \frac{\mathfrak{g}_{\log}(\|\boldsymbol{w}^k - \boldsymbol{v}^k\|)}{\|\boldsymbol{w}^k - \boldsymbol{u}^k\|} \overset{\text{(a)}}{\geq} \liminf_{k\to\infty} -\frac{1}{\log(\tau^k) - \log(\|\boldsymbol{n}\|)} \frac{1}{\boldsymbol{v}_y^k + C_P\tau^k} \\ &\geq \liminf_{k\to\infty} \frac{1}{\log(\|\boldsymbol{n}\|)(\boldsymbol{v}_y^k + C_P\tau^k) - \boldsymbol{v}_y^k \log(\tau^k) - C_P\tau^k \log(\tau^k)} \overset{\text{(b)}}{\geq} \lim_{k\to\infty} \frac{-d}{\boldsymbol{v}_x^k} \in (0,\infty], \end{aligned}$$

where (a) is true owing to (4.40), (4.41), (b) comes from (4.44), $\tau^k \log(\tau^k) \to 0$ and $\boldsymbol{v}_y^k + C_P\tau^k \to 0$, the last inequality holds because $\widehat{\boldsymbol{v}}_x \leq 0$. The above display contradicts (4.24) with $\mathfrak{g}_{\log}$ in place of $\mathfrak{g}$ and hence this case cannot happen.

(ii) Analogously to the proof of Theorem 4.8, by Lemma 4.5, case (ii) cannot happen.

Therefore, we obtain that $\gamma_{\boldsymbol{n},\eta} \in (0, \infty]$. Using this together with [36, Theorem 3.10], we deduce that (4.32) holds. $\square$

Remark 4.13 (Tightness of (4.39))**.** *Let* $\boldsymbol{n} = (0, 0, \boldsymbol{n}_Z)$ *with* $\boldsymbol{n}_Z \succ 0$. *Then,* $\mathcal{F}_\infty = \{\boldsymbol{n}\}^\perp \cap \mathcal{K}_{\mathrm{logdet}}$. *Consider the same sequences* $\{\boldsymbol{v}^k\}, \{\boldsymbol{w}^k\}, \{\boldsymbol{u}^k\}$ *in Remark 4.9, i.e., for every* k,

$$\boldsymbol{v}^k = (-1, 1/k, I_d/(ke^{\frac{k}{d}})), \quad \boldsymbol{w}^k = (-1, 1/k, \boldsymbol{0}), \quad \boldsymbol{u}^k = (-1, 0, \boldsymbol{0}).$$

Note that there exists $\eta > 0$ *such that* $\boldsymbol{w}^k \in \{\boldsymbol{n}\}^\perp \cap \mathcal{B}(\eta)$, $\boldsymbol{v}^k \in \mathcal{K}_{\mathrm{logdet}}$ *and* $\boldsymbol{u}^k = P_{\mathcal{F}_\infty}(\boldsymbol{w}^k)$ *for any* k. *Therefore, applying* (4.39), *there exists* $\kappa_B > 0$ *such that*

$$\frac{1}{k} = \mathrm{dist}(\boldsymbol{w}^k, \mathcal{F}_\infty) \leq \kappa_B \mathfrak{g}_{\log}(\mathrm{dist}(\boldsymbol{w}^k, \mathcal{K}_{\mathrm{logdet}})) \leq \kappa_B \mathfrak{g}_{\log}\left(\frac{\sqrt{d}}{ke^{\frac{k}{d}}}\right) \quad \forall k \in \mathbb{N}.$$

In view of the definition of $\mathfrak{g}_{\log}$ *(see* (4.31)*) and its monotonicity, for large enough* k *we have*

$$\frac{1}{k} = \mathrm{dist}(\boldsymbol{w}^k, \mathcal{F}_\infty) \leq \kappa_B \mathfrak{g}_{\log}(\mathrm{dist}(\boldsymbol{w}^k, \mathcal{K}_{\mathrm{logdet}})) \leq \frac{\kappa_B}{\log k + (k/d) - \log\sqrt{d}} \leq \kappa_B \frac{2d}{k}.$$

Consequently, it holds that for all sufficiently large k,

$$\frac{1}{2d} \leq \frac{\mathrm{dist}(\boldsymbol{w}^k, \mathcal{F}_\infty)}{\mathfrak{g}_{\log}(\mathrm{dist}(\boldsymbol{w}^k, \mathcal{K}_{\mathrm{logdet}}))} \leq \kappa_B.$$

Similar to the argument in Remark 4.3, we conclude that the choice of $\mathfrak{g}_{\log}$ *is tight.*

Using Theorem 4.11 and Theorem 4.12 in combination with [36, Lemma 3.9], we deduce the following one-step facial residual function for $\mathcal{K}_{\mathrm{logdet}}$ and $\boldsymbol{n}$.

Corollary 4.14. *Let* $\boldsymbol{n} = (0, \boldsymbol{n}_y, \boldsymbol{n}_Z) \in \partial\mathcal{K}^*_{\mathrm{logdet}}$ *with* $\boldsymbol{n}_y \geq 0$ *and* $\boldsymbol{n}_Z \succ 0$ *such that* $\mathcal{F}_\infty = \mathcal{K}_{\mathrm{logdet}} \cap \{\boldsymbol{n}\}^\perp$.

(i) If $\boldsymbol{n}_y > 0$*, let* $\gamma_{\boldsymbol{n},t}$ *be as in* (4.23) *with* $\mathcal{F} = \mathcal{F}_\infty$ *and* $\mathfrak{g} = |\cdot|$*. Then the function*

$$\psi_{\mathcal{K},\boldsymbol{n}}(\epsilon, t) := \max\{\epsilon, \epsilon/\|\boldsymbol{n}\|\} + \max\left\{2, 2\gamma_{\boldsymbol{n},t}^{-1}\right\}(\epsilon + \max\{\epsilon, \epsilon/\|\boldsymbol{n}\|\})$$

is a one-step facial residual function for $\mathcal{K}_{\text{logdet}}$ *and* $\boldsymbol{n}$*.*

(ii) If $\boldsymbol{n}_y = 0$*, let* $\gamma_{\boldsymbol{n},t}$ *be as in* (4.23) *with* $\mathcal{F} = \mathcal{F}_\infty$ *and* $\mathfrak{g}_{\log}$ *defined in* (4.31)*. Then the function*

$$\psi_{\mathcal{K},\boldsymbol{n}}(\epsilon, t) := \max\{\epsilon, \epsilon/\|\boldsymbol{n}\|\} + \max\left\{2, 2\gamma_{\boldsymbol{n},t}^{-1}\right\}\mathfrak{g}_{\log}(\epsilon + \max\{\epsilon, \epsilon/\|\boldsymbol{n}\|\})$$

is a one-step facial residual function for $\mathcal{K}_{\text{logdet}}$ *and* $\boldsymbol{n}$*.*

4.2.4 $\mathcal{F}_{\text{r}}$: the family of 1-dimensional faces

Theorem 4.15 (Hölderian error bound concerning $\mathcal{F}_{\text{r}}$)**.** *Let* $\boldsymbol{n} = (\boldsymbol{n}_x, \boldsymbol{n}_x(\log\det(-\boldsymbol{n}_Z/\boldsymbol{n}_x) + d), \boldsymbol{n}_Z) \in \partial\mathcal{K}^*_{\text{logdet}}$ *with* $\boldsymbol{n}_x < 0$ *and* $\boldsymbol{n}_Z \succ 0$ *such that* $\mathcal{F}_{\text{r}} = \mathcal{K}_{\text{logdet}} \cap \{\boldsymbol{n}\}^\perp$*. Let* $\eta > 0$ *and let* $\gamma_{\boldsymbol{n},\eta}$ *be defined as* (4.23) *with* $\mathcal{F} = \mathcal{F}_{\text{r}}$ *and* $\mathfrak{g} = |\cdot|^{\frac{1}{2}}$*. Then* $\gamma_{\boldsymbol{n},\eta} \in (0, \infty]$ *and*

$$\text{dist}(\boldsymbol{q}, \mathcal{F}_{\text{r}}) \le \max\{2\eta^{\frac{1}{2}}, 2\gamma_{\boldsymbol{n},\eta}^{-1}\} \cdot (\text{dist}(\boldsymbol{q}, \mathcal{K}_{\text{logdet}}))^{\frac{1}{2}} \qquad \forall \boldsymbol{q} \in \{\boldsymbol{n}\}^\perp \cap \mathcal{B}(\eta). \quad (4.45)$$

Proof. If $\gamma_{\boldsymbol{n},\eta} = 0$, in view of [36, Lemma 3.12], there exists $\widehat{\boldsymbol{v}} \in \mathcal{F}_{\text{r}}$ and sequences $\{\boldsymbol{v}^k\}, \{\boldsymbol{w}^k\}, \{\boldsymbol{u}^k\}$ being defined as those therein, with the cone being $\mathcal{K}_{\text{logdet}}$ and the face being $\mathcal{F}_{\text{r}}$, such that (4.24) holds with $\mathfrak{g} = |\cdot|^{\frac{1}{2}}$. We consider two different cases.

(i) $\boldsymbol{v}^k \in \mathcal{F}_{\text{d}}$ infinitely often, i.e., $\boldsymbol{v}^k_y = 0$ infinitely often (wherefore $\widehat{\boldsymbol{v}} = \boldsymbol{0}$);

(ii) $\boldsymbol{v}^k \notin \mathcal{F}_{\text{d}}$ for all large k, i.e., $\boldsymbol{v}^k_y > 0$ for all large k.

(i) If $\boldsymbol{v}^k_y = 0$ infinitely often, by extracting a subsequence if necessary, we may assume that

$$\boldsymbol{v}^k = (\boldsymbol{v}^k_x, 0, \boldsymbol{v}^k_Z) \text{ with } \boldsymbol{v}^k_x \le 0, \boldsymbol{v}^k_Z \succeq 0 \quad \text{for all } k.$$

Combining this with the definition of $\boldsymbol{n}$, we have

$$|\langle \boldsymbol{n}, \boldsymbol{v}^k\rangle| = |\boldsymbol{n}_x\boldsymbol{v}^k_x + \text{tr}(\boldsymbol{n}_Z\boldsymbol{v}^k_Z)| = -\boldsymbol{n}_x|\boldsymbol{v}^k_x| + \text{tr}(\boldsymbol{n}_Z\boldsymbol{v}^k_Z) \ge -\boldsymbol{n}_x|\boldsymbol{v}^k_x| + \lambda_{\min}(\boldsymbol{n}_Z)\,\text{tr}(\boldsymbol{v}^k_Z)$$

$$\geq \min\{-\boldsymbol{n}_x, \lambda_{\min}(\boldsymbol{n}_Z)\}(|\boldsymbol{v}_x^k| + \operatorname{tr}(\boldsymbol{v}_Z^k)) \geq \min\{-\boldsymbol{n}_x, \lambda_{\min}(\boldsymbol{n}_Z)\}\|\boldsymbol{v}^k\|.$$

Here, we recall that $\operatorname{tr}(\boldsymbol{n}_Z\boldsymbol{v}_Z^k) \geq 0, \lambda_{\min}(\boldsymbol{n}_Z) > 0, \operatorname{tr}(\boldsymbol{v}_Z^k) \geq 0$.

Since projections are non-expansive, we have $\|\boldsymbol{w}^k\| \leq \|\boldsymbol{v}^k\|$. Moreover, since $\mathbf{0} \in \mathcal{F}_{\mathrm{r}}$, we have $\operatorname{dist}(\cdot, \mathcal{F}_{\mathrm{r}}) \leq \|\cdot\|$. Thus,

$$\begin{aligned}\|\boldsymbol{w}^k - \boldsymbol{u}^k\| =& \operatorname{dist}(\boldsymbol{w}^k, \mathcal{F}_{\mathrm{r}}) \leq \|\boldsymbol{w}^k\| \leq \|\boldsymbol{v}^k\| \\ &\leq \frac{1}{\min\{-\boldsymbol{n}_x, \lambda_{\min}(\boldsymbol{n}_Z)\}}|\langle \boldsymbol{n}, \boldsymbol{v}^k\rangle| = \frac{\|\boldsymbol{n}\|}{\min\{-\boldsymbol{n}_x, \lambda_{\min}(\boldsymbol{n}_Z)\}}\|\boldsymbol{w}^k - \boldsymbol{v}^k\|.\end{aligned}$$

This display shows that (4.24) for $\mathfrak{g} = |\cdot|$ does not hold in this case. Since $|t|^{1/2} \geq |t|$ holds for small $t > 0$, we conclude that (4.24) for $\mathfrak{g} = |\cdot|^{1/2}$ does not hold as well.

(ii) If $\boldsymbol{v}_y^k > 0$ for all large k, by passing to a subsequence if necessary, we can assume that

$$\boldsymbol{v}^k = (\boldsymbol{v}_y^k \log\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k), \boldsymbol{v}_y^k, \boldsymbol{v}_Z^k) \text{ with } \boldsymbol{v}_y^k > 0, \boldsymbol{v}_Z^k \succ 0, \quad \text{for all } k.$$

Thus, we have

$$\|\boldsymbol{w}^k - \boldsymbol{v}^k\| = \frac{|\langle \boldsymbol{n}, \boldsymbol{v}^k\rangle|}{\|\boldsymbol{n}\|}, \tag{4.46}$$

and

$$\begin{aligned}\langle \boldsymbol{n}, \boldsymbol{v}^k\rangle &= \boldsymbol{n}_x\boldsymbol{v}_y^k \log\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k) + \boldsymbol{n}_x\boldsymbol{v}_y^k(\log\det(-\boldsymbol{n}_Z/\boldsymbol{n}_x) + d) + \operatorname{tr}(\boldsymbol{n}_Z\boldsymbol{v}_Z^k) \\
&= \boldsymbol{n}_x\boldsymbol{v}_y^k \left(\log\det\left(-\frac{\boldsymbol{v}_Z^k\boldsymbol{n}_Z}{\boldsymbol{v}_y^k\boldsymbol{n}_x}\right) + d + \operatorname{tr}\left(\frac{\boldsymbol{v}_Z^k\boldsymbol{n}_Z}{\boldsymbol{v}_y^k\boldsymbol{n}_x}\right)\right) \\
&= \boldsymbol{n}_x\boldsymbol{v}_y^k \left(\log\det\left(-\frac{\boldsymbol{n}_Z^{\frac{1}{2}}\boldsymbol{v}_Z^k\boldsymbol{n}_Z^{\frac{1}{2}}}{\boldsymbol{v}_y^k\boldsymbol{n}_x}\right) + d + \operatorname{tr}\left(\frac{\boldsymbol{n}_Z^{\frac{1}{2}}\boldsymbol{v}_Z^k\boldsymbol{n}_Z^{\frac{1}{2}}}{\boldsymbol{v}_y^k\boldsymbol{n}_x}\right)\right) \\
&= \boldsymbol{n}_x\boldsymbol{v}_y^k \sum_{i=1}^d \left(\log\left(\lambda_i\left(-\frac{\boldsymbol{n}_Z^{\frac{1}{2}}\boldsymbol{v}_Z^k\boldsymbol{n}_Z^{\frac{1}{2}}}{\boldsymbol{v}_y^k\boldsymbol{n}_x}\right)\right) + 1 + \lambda_i\left(\frac{\boldsymbol{n}_Z^{\frac{1}{2}}\boldsymbol{v}_Z^k\boldsymbol{n}_Z^{\frac{1}{2}}}{\boldsymbol{v}_y^k\boldsymbol{n}_x}\right)\right) \\
&= \boldsymbol{n}_x\boldsymbol{v}_y^k \sum_{i=1}^d \left(t_i^k + 1 - e^{t_i^k}\right) \geq 0, \qquad (4.47)\end{aligned}$$

where $t_i^k := \log\left(\lambda_i\left(-\frac{\boldsymbol{n}_Z^{\frac{1}{2}}\boldsymbol{v}_Z^k\boldsymbol{n}_Z^{\frac{1}{2}}}{\boldsymbol{v}_y^k\boldsymbol{n}_x}\right)\right)$ for $i = 1, 2, \ldots, d$ and $k \geq 1$, and the nonnegativity comes from the observation that $t + 1 - e^t \leq 0$ for all $t \in \mathbb{R}$ and the facts that $\boldsymbol{n}_x < 0$ and $\boldsymbol{v}_y^k > 0$; recall that here $\boldsymbol{v}_Z^k \succ 0, \boldsymbol{n}_Z \succ 0, \boldsymbol{n}_x < 0$, and $\boldsymbol{v}_y^k > 0$, then $\lambda_i(-\frac{\boldsymbol{n}_Z^{\frac{1}{2}}\boldsymbol{v}_Z^k\boldsymbol{n}_Z^{\frac{1}{2}}}{\boldsymbol{v}_y^k\boldsymbol{n}_x}) > 0$ for all i, and hence t_i^k is well-defined.

Next, we turn to compute $\|\boldsymbol{w}^k - \boldsymbol{u}^k\|$. Using Lemma 2.4, (4.9) and (4.10), one can see for all k,

$$\begin{aligned}
&\|\boldsymbol{w}^k - \boldsymbol{u}^k\| \leq \mathrm{dist}(\boldsymbol{v}^k, \mathcal{F}_{\mathrm{r}}) \overset{\text{(a)}}{\leq} \|\boldsymbol{v}^k - \boldsymbol{v}_y^k\boldsymbol{f}_{\mathrm{r}}\| \\
&= \|(\boldsymbol{v}_y^k \log\det(\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k) - \boldsymbol{v}_y^k \log\det(-\boldsymbol{n}_x\boldsymbol{n}_Z^{-1}), 0, \boldsymbol{v}_Z^k + \boldsymbol{v}_y^k\boldsymbol{n}_x\boldsymbol{n}_Z^{-1})\| \\
&\leq \boldsymbol{v}_y^k(|\log\det(-(\boldsymbol{n}_Z^{\frac{1}{2}}\boldsymbol{v}_Z^k\boldsymbol{n}_Z^{\frac{1}{2}})/(\boldsymbol{v}_y^k\boldsymbol{n}_x))| + \|\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k + \boldsymbol{n}_x\boldsymbol{n}_Z^{-1}\|_F) \\
&\leq \boldsymbol{v}_y^k\left(\left(\sum_{i=1}^d |t_i^k|\right) + \|\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k + \boldsymbol{n}_x\boldsymbol{n}_Z^{-1}\|_F\right),
\end{aligned}$$

where (a) holds because $\boldsymbol{v}_y^k\boldsymbol{f}_{\mathrm{r}} \in \mathcal{F}_{\mathrm{r}}$. We remark that $\boldsymbol{v}_Z^k/\boldsymbol{v}_y^k + \boldsymbol{n}_x\boldsymbol{n}_Z^{-1}$ is a symmetric matrix. Let

$$A_k := \frac{\boldsymbol{v}_Z^k}{\boldsymbol{v}_y^k} + \boldsymbol{n}_x\boldsymbol{n}_Z^{-1}, \quad B := -\boldsymbol{n}_x\boldsymbol{n}_Z^{-1}, \quad D_k := \frac{\boldsymbol{v}_Z^k\boldsymbol{n}_Z}{\boldsymbol{v}_y^k\boldsymbol{n}_x}, \quad \widehat{D}_k := \frac{\boldsymbol{n}_Z^{\frac{1}{2}}\boldsymbol{v}_Z^k\boldsymbol{n}_Z^{\frac{1}{2}}}{\boldsymbol{v}_y^k\boldsymbol{n}_x}.$$

We notice that $D_k = \boldsymbol{n}_Z^{-\frac{1}{2}}\widehat{D}_k\boldsymbol{n}_Z^{\frac{1}{2}}$ and $e^{t_i^k} = \lambda_i(-\widehat{D}_k)$ for $i = 1, 2, \ldots, d$ and $k \geq 1$. Then, we have for all k,

$$\begin{aligned}
&\|A_k\|_F \leq \|A_k\|_* = \|A_k(\boldsymbol{n}_Z/\boldsymbol{n}_x)(\boldsymbol{n}_x\boldsymbol{n}_Z^{-1})\|_* = \|(D_k + I)B\|_* \\
&\overset{\text{(a)}}{=} \sup_{\|W\|_2 \leq 1} \mathrm{tr}(W(D_k + I)B) = \sup_{\|W\|_2 \leq 1} \mathrm{tr}\left(BW\left(\boldsymbol{n}_Z^{-\frac{1}{2}}\widehat{D}_k\boldsymbol{n}_Z^{\frac{1}{2}} + I\right)\right) \\
&= \sup_{\|W\|_2 \leq 1} \mathrm{tr}\left(\boldsymbol{n}_Z^{\frac{1}{2}}BW\boldsymbol{n}_Z^{-\frac{1}{2}}(\widehat{D}_k + I)\right) \overset{\text{(b)}}{\leq} \|\widehat{D}_k + I\|_* \sup_{\|W\|_2 \leq 1} \|\boldsymbol{n}_Z^{\frac{1}{2}}BW\boldsymbol{n}_Z^{-\frac{1}{2}}\|_2 \\
&= \beta\sum_{i=1}^d |\lambda_i(\widehat{D}_k + I)| = \beta\sum_{i=1}^d |\lambda_i(\widehat{D}_k) + 1| = \beta\sum_{i=1}^d |\lambda_i(-\widehat{D}_k) - 1| = \beta\sum_{i=1}^d |e^{t_i^k} - 1|,
\end{aligned}$$

where $\beta := \sup_{\|W\|_2 \le 1} \|\boldsymbol{n}_Z^{\frac{1}{2}} B W \boldsymbol{n}_Z^{-\frac{1}{2}}\|_2 \in (0, \infty)$, and (a) and (b) hold since the dual norm of nuclear norm $\|\cdot\|_*$ is the spectral norm $\|\cdot\|_2$. Hence, we obtain that for all k,

$$\|\boldsymbol{w}^k - \boldsymbol{u}^k\| \le \boldsymbol{v}_y^k \left(\sum_{i=1}^{d} |t_i^k| + \beta |e^{t_i^k} - 1| \right). \tag{4.48}$$

Before moving on, we define two auxiliary functions and discuss some useful properties. Define

$$h(t) := t + 1 - e^t \quad \text{ and } \quad g(t) := |t| + \beta |e^t - 1|. \tag{4.49}$$

We observe that

$$\begin{aligned} h(t) = 0 &\Longleftrightarrow t = 0, \\ \lim_{t \to \infty} h(t) = \lim_{t \to -\infty} h(t) &= -\infty, \\ h'(t) = 1 - e^t, &\quad h'(0) = 0, \\ h''(t) = -e^t, &\quad h''(0) = -1. \end{aligned} \tag{4.50}$$

In addition, $g(t) \ge 0$ for all $t \in \mathbb{R}$ and $g(t) = 0$ if and only if $t = 0$.

Now, recall from the setting of $\{\boldsymbol{v}^k\}$ that $\boldsymbol{v}^k \to \widehat{\boldsymbol{v}}$ and $\langle \boldsymbol{n}, \boldsymbol{v}^k \rangle \to 0$. This and the formula of $\langle \boldsymbol{n}, \boldsymbol{v}^k \rangle$ in (4.47) reveal that we need to consider the following two cases:

(I) $\liminf_{k \to \infty} \sum_{i=1}^{d} h(t_i^k) = 0$;

(II) $\liminf_{k \to \infty} \sum_{i=1}^{d} h(t_i^k) \in [-\infty, 0)$.

For notational simplicity, we define $\boldsymbol{t}^k := (t_i^k)_{i=1}^d$ for all k.

(I) Without loss of generality, by passing to a further subsequence, we assume that $\lim_{k \to \infty} \sum_{i=1}^{d} h(t_i^k) = 0$. Combining this assumption and the fact that $h(t) \le 0$ for all $t \in \mathbb{R}$ with (4.50), we know that $\boldsymbol{t}^k \to \boldsymbol{0}$. Now, consider the Taylor expansion of $h(t)$ at $t = 0$, that is,

$$h(t) = -0.5t^2 + O(|t|^3), \quad t \to 0.$$

It follows that there exists $\epsilon > 0$ such that for any t satisfying $|t| < \epsilon$, $h(t) \leq -0.25t^2 \leq 0$. Thus, we have for all large k that,

$$0 \leq \sum_{i=1}^{d} 0.25(t_i^k)^2 \leq \sum_{i=1}^{d} |h(t_i^k)|. \tag{4.51}$$

We can deduce the lower bound of $\|\boldsymbol{w}^k - \boldsymbol{v}^k\|^{\frac{1}{2}}$ for sufficiently large k as follows:

$$\begin{aligned}\|\boldsymbol{w}^k - \boldsymbol{v}^k\|^{\frac{1}{2}} &\overset{\text{(a)}}{=} \frac{|\langle \boldsymbol{n}, \boldsymbol{v}^k\rangle|^{\frac{1}{2}}}{\|\boldsymbol{n}\|^{\frac{1}{2}}} \overset{\text{(b)}}{=} \frac{|\boldsymbol{n}_x|^{\frac{1}{2}}|\boldsymbol{v}_y^k|^{\frac{1}{2}}}{\|\boldsymbol{n}\|^{\frac{1}{2}}} \left(\sum_{i=1}^{d} |h(t_i^k)|\right)^{\frac{1}{2}} \\ &\overset{\text{(c)}}{\geq} \frac{|\boldsymbol{n}_x|^{\frac{1}{2}}|\boldsymbol{v}_y^k|^{\frac{1}{2}}}{2\|\boldsymbol{n}\|^{\frac{1}{2}}} \left(\sum_{i=1}^{d} (t_i^k)^2\right)^{\frac{1}{2}} \overset{\text{(d)}}{\geq} \frac{(|\boldsymbol{n}_x||\boldsymbol{v}_y^k|)^{\frac{1}{2}}}{2(d\|\boldsymbol{n}\|)^{\frac{1}{2}}} \left(\sum_{i=1}^{d} |t_i^k|\right),\end{aligned}$$

where (a) comes from (4.46), (b) comes from (4.47) and (4.49), (c) holds by (4.51), (d) comes from the root-mean inequality.

Next, to derive a bound for $\|\boldsymbol{w}^k - \boldsymbol{u}^k\|$, we shall relate $|e^{t_i^k} - 1|$ to $|t_i^k|$. To this end, notice that $\lim_{t\to 0}(e^t - 1)/t = 1$. Then, there exists $C_1 > 0$ such that for any $i = 1, 2, \ldots, d$,

$$|e^{t_i^k} - 1| \leq C_1|t_i^k| \quad \text{for sufficiently large } k.$$

Therefore, by (4.48), for all sufficiently large k,

$$\|\boldsymbol{w}^k - \boldsymbol{u}^k\| \leq \boldsymbol{v}_y^k(\beta C_1 + 1)\sum_{i=1}^{d} |t_i^k|.$$

We thus conclude that

$$\begin{aligned}\lim_{k\to\infty} \frac{\|\boldsymbol{w}^k - \boldsymbol{v}^k\|^{\frac{1}{2}}}{\|\boldsymbol{w}^k - \boldsymbol{u}^k\|} &\geq \liminf_{k\to\infty} \frac{(|\boldsymbol{n}_x||\boldsymbol{v}_y^k|)^{\frac{1}{2}}}{2(d\|\boldsymbol{n}\|)^{\frac{1}{2}}} \frac{(\sum_{i=1}^{d} |t_i^k|)}{\boldsymbol{v}_y^k(\beta C_1 + 1)(\sum_{i=1}^{d} |t_i^k|)} \\ &\overset{\text{(a)}}{\geq} \frac{|\boldsymbol{n}_x|^{\frac{1}{2}}}{2(d\|\boldsymbol{n}\|\eta)^{\frac{1}{2}}(\beta C_1 + 1)} > 0,\end{aligned}$$

where (a) holds since $0 < \boldsymbol{v}_y^k \leq \|\boldsymbol{v}^k\| \leq \eta$. This contradicts (4.24) with $|\cdot|^{\frac{1}{2}}$ in place of $\mathfrak{g}$ and hence this case cannot happen.

(II) In this case, in view of (4.50), by passing to a further subsequence if necessary, we can assume that there exist $\epsilon > 0$ and i_0 such that $|t_{i_0}^k| \geq \epsilon$ for all large k, that is, $g(t_{i_0}^k) > 0$ for all large k. Then, $\|\boldsymbol{t}^k\|_\infty \geq \epsilon$ for all large k. Now, consider the following function

$$H(\boldsymbol{t}) := \begin{cases} \frac{\sum_{i=1}^d |h(t_i)|}{\sum_{i=1}^d g(t_i)} & \text{if } \|\boldsymbol{t}\|_\infty \geq \epsilon, \\ \infty & \text{otherwise,} \end{cases}$$

where h is defined as in (4.49). Since $\|\boldsymbol{t}\|_\infty \geq \epsilon$ implies $g(t_i) > 0$ for some i, we see that H is well-defined. Moreover, one can check that H is lower semi-continuous and never zero.

We claim that $\inf H > 0$. Granting this, we have

$$\lim_{k\to\infty} \frac{\|\boldsymbol{w}^k - \boldsymbol{v}^k\|^{\frac{1}{2}}}{\|\boldsymbol{w}^k - \boldsymbol{u}^k\|} \geq \liminf_{k\to\infty} \frac{\|\boldsymbol{w}^k - \boldsymbol{v}^k\|}{\|\boldsymbol{w}^k - \boldsymbol{u}^k\|} \overset{\text{(a)}}{\geq} \liminf_{k\to\infty} \frac{|\boldsymbol{n}_x|}{\|\boldsymbol{n}\|} \frac{\sum_{i=1}^d |h(t_i^k)|}{\sum_{i=1}^d g(t_i^k)}$$
$$\overset{\text{(b)}}{=} \liminf_{k\to\infty} \frac{|\boldsymbol{n}_x|}{\|\boldsymbol{n}\|} H(\boldsymbol{t}^k) \geq \frac{|\boldsymbol{n}_x|}{\|\boldsymbol{n}\|} \inf H > 0,$$

where (a) comes from (4.46), (4.47), (4.48) and the definition of h and g in (4.49), (b) holds thanks to the definition of H. The above display contradicts (4.24) with $|\cdot|^{\frac{1}{2}}$ in place of $\mathfrak{g}$ and hence this case cannot happen. Therefore, we obtain that $\gamma_{\boldsymbol{n},\eta} \in (0, \infty]$ with $\mathfrak{g} = |\cdot|^{\frac{1}{2}}$. Together with [36, Theorem 3.10], we deduce that (4.45) holds.

Now, it remains to show that $\inf H > 0$. We claim that it suffices to prove $\liminf_{\|\boldsymbol{t}\|\to\infty} H(\boldsymbol{t}) > 0$ because H is lower semi-continuous and never zero. Suppose, for the sake of contradiction, that $\inf H = 0$. Then, there exists a sequence $\{\boldsymbol{\zeta}^l\}$ such that $H(\boldsymbol{\zeta}^l) \to 0$. If $\{\boldsymbol{\zeta}^l\}$ is unbounded, we can find a subsequence $\{\boldsymbol{\zeta}^{l_k}\}$ such that $\|\boldsymbol{\zeta}^{l_k}\| \to \infty$ and $H(\boldsymbol{\zeta}^{l_k}) \to 0$ holds, which would contradict $\liminf_{\|\boldsymbol{t}\|\to\infty} H(\boldsymbol{t}) > 0$. So $\{\boldsymbol{\zeta}^l\}$ must be bounded and passing to a subsequence we may assume it converges to some $\bar{\boldsymbol{\zeta}}$. By lower semicontinuity, we have $H(\bar{\boldsymbol{\zeta}}) \leq \liminf_{\boldsymbol{t}\to\bar{\boldsymbol{\zeta}}} H(\boldsymbol{t}) \leq \lim_{l\to\infty} H(\boldsymbol{\zeta}^l) = 0$. However, H is always positive, so this cannot happen either. Therefore, $\liminf_{\|\boldsymbol{t}\|\to\infty} H(\boldsymbol{t}) > 0$ implies $\inf H > 0$.

To this end, consider a sequence $\{\boldsymbol{\zeta}^l\}$ such that $\|\boldsymbol{\zeta}^l\| \to \infty$ and

$$\lim_{l\to\infty} H(\boldsymbol{\zeta}^l) = \liminf_{\|\boldsymbol{t}\|\to\infty} H(\boldsymbol{t}),$$

then there exists at least one $i_0 \in \{1, 2, \dots, d\}$ such that $|\zeta_{i_0}^l| \to \infty$. Consequently, $|h(\zeta_{i_0}^l)| \to \infty$ and $g(\zeta_{i_0}^l) \to \infty$, and so both $\sum_{i=1}^d |h(\zeta_i^l)|$ and $\sum_{i=1}^d g(\zeta_i^l)$ tend to ∞. Passing to a subsequence, we can assume that for each i, $\lim_{l\to\infty} \zeta_i^l \in [-\infty, \infty]$ exists and we can split $\boldsymbol{\zeta}^l$ into three parts:

(1) $\zeta_i^l \to \overline{\zeta}_i \in \mathbb{R} \setminus \{0\}$, then $|h(\zeta_i^l)| \to |h(\overline{\zeta}_i)| \neq 0$, $g(\zeta_i^l) \to g(\overline{\zeta}_i) \neq 0$. Denote the set of indices of these components by $\mathcal{I}_{\boldsymbol{\zeta}}^C$ where C refers to "constant".

For any $i \in \mathcal{I}_{\boldsymbol{\zeta}}^C$, we have

$$\lim_{l\to\infty} \frac{|h(\zeta_i^l)|}{g(\zeta_i^l)} = \frac{|h(\overline{\zeta}_i)|}{g(\overline{\zeta}_i)} > 0.$$

Thus, there exists a constant $C_C > 0$ such that for all sufficiently large l and all $i \in \mathcal{I}_{\boldsymbol{\zeta}}^C$,

$$|h(\zeta_i^l)| \geq C_C g(\zeta_i^l).$$

(2) $\zeta_i^l \to 0$, then $|h(\zeta_i^l)| \to 0$, $g(\zeta_i^l) \to 0$. Denote the set of indices of these components by $\mathcal{I}_{\boldsymbol{\zeta}}^0$.

(3) $|\zeta_i^l| \to \infty$, then $|h(\zeta_i^l)| \to \infty$, $g(\zeta_i^l) \to \infty$. Denote the set of these components by $\mathcal{I}_{\boldsymbol{\zeta}}^\infty$. We have $\mathcal{I}_{\boldsymbol{\zeta}}^\infty \neq \emptyset$, since otherwise $\|\boldsymbol{\zeta}^l\| \not\to \infty$.

For any $i \in \mathcal{I}_{\boldsymbol{\zeta}}^\infty$, we notice that

$$\liminf_{l\to\infty} \frac{|h(\zeta_i^l)|}{g(\zeta_i^l)} \geq \min\left\{\liminf_{t\to-\infty} \frac{|h(t)|}{g(t)}, \liminf_{t\to\infty} \frac{|h(t)|}{g(t)}\right\} = \min\left\{1, \frac{1}{\beta}\right\} := \widehat{\beta} > 0.$$

Thus, for all sufficiently large l and all $i \in \mathcal{I}_{\boldsymbol{\zeta}}^C$,

$$|h(\zeta_i^l)| \geq \frac{\widehat{\beta}}{2} g(\zeta_i^l).$$

Combining the above three cases, we obtain

$$\liminf_{\|\boldsymbol{t}\|\to\infty} H(\boldsymbol{t}) = \lim_{l\to\infty} H(\boldsymbol{\zeta}^l)$$

$$\geq \lim_{l\to\infty} \frac{C_C \sum_{i\in\mathcal{I}_\xi^C} g(\zeta_i^l) + \frac{\widehat{\beta}}{2}\sum_{i\in\mathcal{I}_\xi^\infty} g(\zeta_i^l) + \sum_{i\in\mathcal{I}_\xi^0} g(\zeta_i^l)}{\sum_{i=1}^d g(\zeta_i^l)} + \frac{\sum_{i\in\mathcal{I}_\xi^0} |h(\zeta_i^l)| - \sum_{i\in\mathcal{I}_\xi^0} g(\zeta_i^l)}{\sum_{i=1}^d g(\zeta_i^l)}$$

$$\overset{\text{(a)}}{\geq} \min\left\{C_C, \frac{\widehat{\beta}}{2}, 1\right\} > 0, \tag{4.52}$$

where (a) comes from the fact that

$$\lim_{l\to\infty} \frac{\sum_{i\in\mathcal{I}_\xi^0} |h(\zeta_i^l)| - \sum_{i\in\mathcal{I}_\xi^0} g(\zeta_i^l)}{\sum_{i=1}^d g(\zeta_i^l)} = 0,$$

which holds because the numerator tends to 0 while the denominator tends to infinity.

□

Remark 4.16 (Tightness of (4.45))**.** *Let $\boldsymbol{n}$ be defined as in Proposition 4.1.(a)) and*

$$\boldsymbol{v}^k = \left(\log\det(-\boldsymbol{n}_x\boldsymbol{n}_Z^{-1}) + \frac{1}{k}, 1, e^{\frac{1}{dk}}(-\boldsymbol{n}_x\boldsymbol{n}_Z^{-1})\right), \quad \boldsymbol{w}^k = P_{\{\boldsymbol{n}\}^\perp}(\boldsymbol{v}^k), \quad \boldsymbol{u}^k = P_{\mathcal{F}_{\rm r}}(\boldsymbol{w}^k),$$

so that $\mathcal{F}_{\rm r} = \mathcal{K}_{\rm logdet} \cap \{\boldsymbol{n}\}^\perp$, $\{\boldsymbol{v}^k\} \subset \mathcal{K}_{\rm logdet}$ and there exists $\eta > 0$ such that $\{\boldsymbol{w}^k\} \subseteq \mathcal{B}(\eta)$. Then we have

$$\|\boldsymbol{w}^k - \boldsymbol{v}^k\| = \frac{|\langle \boldsymbol{n}, \boldsymbol{v}^k\rangle|}{\|\boldsymbol{n}\|} = \frac{-\boldsymbol{n}_x|\frac{1}{k} + d - de^{\frac{1}{dk}}|}{\|\boldsymbol{n}\|}.$$

For the sake of notational simplicity, we denote $\xi := \frac{1}{k}$, then

$$\|\boldsymbol{w}^k - \boldsymbol{v}^k\| = \frac{-\boldsymbol{n}_x|\xi + d - de^{\frac{\xi}{d}}|}{\|\boldsymbol{n}\|}. \tag{4.53}$$

Consider the Taylor expansion of $\xi + d - de^{\frac{\xi}{d}}$ with respect to ξ at 0, we have

$$\xi + d - de^{\frac{\xi}{d}} = \xi + d - d\left(1 + \frac{\xi}{d} + \frac{\xi^2}{2d^2}\right) + o(\xi^2) = -\frac{\xi^2}{2d} + o(\xi^2), \text{ as } \xi \to 0. \tag{4.54}$$

Next, upon invoking the definitions of $\mathcal{F}_{\rm r}$ and $\boldsymbol{f}_{\rm r}$ (see (4.9) *and* (4.10)*, respectively), we can see that*

$$\|\boldsymbol{v}^k-\boldsymbol{u}^k\|^2=\operatorname{dist}^2(\boldsymbol{v}^k,\mathcal{F}_{\rm r})=\min_{y\geq 0}\|\boldsymbol{v}^k-y\boldsymbol{f}_{\rm r}\|^2$$

$$=\min_{y\geq 0}\left\|\left((1-y)\log\det(-\boldsymbol{n}_x\boldsymbol{n}_Z^{-1})+\xi,1-y,-(e^{\frac{\xi}{d}}-y)\boldsymbol{n}_x\boldsymbol{n}_Z^{-1}\right)\right\|^2$$

$$=\min_{y\geq 0}\left\{\underbrace{\left[(1-y)\log\det(-\boldsymbol{n}_x\boldsymbol{n}_Z^{-1})+\xi\right]^2+(1-y)^2+(e^{\frac{\xi}{d}}-y)^2\boldsymbol{n}_x^2\|\boldsymbol{n}_Z^{-1}\|_F^2}_{F(y)}\right\}$$

For the sake of brevity, we denote $\mu:=\log\det(-\boldsymbol{n}_x\boldsymbol{n}_Z^{-1})$ and $\nu:=\boldsymbol{n}_x^2\|\boldsymbol{n}_Z^{-1}\|_F^2$. Then we have

$$F(y)=(\mu(y-1)-\xi)^2+(y-1)^2+\nu(y-e^{\frac{\xi}{d}})^2.$$

Noting

$$\begin{aligned}F'(y)&=2\mu\left(\mu(y-1)-\xi\right)+2(y-1)+2\nu(y-e^{\frac{\xi}{d}})\\&=(2\mu^2+2+2\nu)y-(2\mu^2+2+2\mu\xi+2\nu e^{\xi/d}),\end{aligned}$$

we know that F attains its minimum at $\bar{y}=\frac{\mu^2+1+\mu\xi+\nu e^{\xi/d}}{\mu^2+\nu+1}$, which is larger than 0 for sufficiently large k (or, equivalently, sufficiently small ξ).

Next we move towards the analysis of $\|\boldsymbol{v}^k-\boldsymbol{u}^k\|^2=F(\bar{y})$. Consider the Taylor expansion of $\bar{y}-1$ and $\bar{y}-e^{\frac{\xi}{d}}$ with respect to ξ at 0, we have that

$$\bar{y}-1=\frac{\mu^2+1+\mu\xi+\nu e^{\xi/d}}{\mu^2+\nu+1}-1=\frac{\mu\xi+\nu(e^{\frac{\xi}{d}}-1)}{\mu^2+\nu+1}=\frac{\mu\xi+\nu\frac{\xi}{d}}{\mu^2+\nu+1}+o(\xi),$$

and

$$\bar{y}-e^{\frac{\xi}{d}}=\bar{y}-1-\frac{\xi}{d}+o(\xi)=\left(\frac{\mu+\frac{\nu}{d}}{\mu^2+\nu+1}-\frac{1}{d}\right)\xi+o(\xi)=-\left(\frac{\frac{\mu^2}{d}-\mu+\frac{1}{d}}{\mu^2+\nu+1}\right)\xi+o(\xi).$$

Then, for all sufficiently large k,

$$\|\boldsymbol{v}^k-\boldsymbol{u}^k\|^2=F(\bar{y})$$

$$= (\mu(\bar{y}-1)-\xi)^2 + (\bar{y}-1)^2 + \nu(\bar{y}-e^{\frac{\xi}{d}})^2$$

$$= \left(\frac{\mu^2+\mu\frac{\nu}{d}}{\mu^2+\nu+1}-1\right)^2\xi^2 + \left(\frac{\mu+\frac{\nu}{d}}{\mu^2+\nu+1}\right)^2\xi^2 + \nu\left(\frac{\frac{\mu^2}{d}-\mu+\frac{1}{d}}{\mu^2+\nu+1}\right)^2\xi^2 + o(\xi^2)$$

$$= \Bigg[\underbrace{\left(\frac{\nu(1-\frac{\mu}{d})+1}{\mu^2+\nu+1}\right)^2 + \left(\frac{\mu+\frac{\nu}{d}}{\mu^2+\nu+1}\right)^2 + \nu\left(\frac{\frac{\mu^2}{d}-\mu+\frac{1}{d}}{\mu^2+\nu+1}\right)^2}_{C_{\rm r}\geq 0}\Bigg]\xi^2 + o(\xi^2). \tag{4.55}$$

Next we show $C_{\rm r} > 0$. Suppose that $\frac{\mu^2}{d} - \mu + \frac{1}{d} = 0$, then[9] $\mu = \frac{d\pm\sqrt{d^2-4}}{2} > 0$ *and* $\mu(1-\frac{\mu}{d}) = \frac{1}{d}$. *This implies that* $1-\frac{\mu}{d} > 0$ *and hence* $\nu(1-\frac{\mu}{d})+1 > 0$ *thanks to* $\nu > 0$. *Therefore, we can see that* $C_{\rm r} > 0$ *because either* $\frac{\mu^2}{d} - \mu + \frac{1}{d} \neq 0$ *or* $\nu(1-\frac{\mu}{d})+1 \neq 0$.

Using Lemma 2.4, (4.53), (4.54) *and* (4.55), *we deduce that*

$$L_{\rm r} := \lim_{k\to\infty} \frac{\|\boldsymbol{w}^k-\boldsymbol{u}^k\|}{\|\boldsymbol{w}^k-\boldsymbol{v}^k\|^{\frac{1}{2}}} = \lim_{k\to\infty}\frac{\sqrt{\|\boldsymbol{v}^k-\boldsymbol{u}^k\|^2-\|\boldsymbol{w}^k-\boldsymbol{v}^k\|^2}}{\|\boldsymbol{w}^k-\boldsymbol{v}^k\|^{\frac{1}{2}}}$$

$$= \lim_{k\to\infty}\frac{\|\boldsymbol{n}\|^{\frac{1}{2}}}{\left(-\boldsymbol{n}_x|\frac{1}{k}+d-de^{\frac{1}{dk}}|\right)^{\frac{1}{2}}}\frac{\sqrt{\|\boldsymbol{n}\|^2\|\boldsymbol{v}^k-\boldsymbol{u}^k\|^2-(\frac{1}{k}+d-de^{\frac{1}{dk}})^2\boldsymbol{n}_x^2}}{\|\boldsymbol{n}\|}$$

$$= \frac{1}{\|\boldsymbol{n}\|^{\frac{1}{2}}}\lim_{k\to\infty}\sqrt{\frac{\|\boldsymbol{n}\|^2}{-\boldsymbol{n}_x}\frac{\|\boldsymbol{v}^k-\boldsymbol{u}^k\|^2}{|\frac{1}{k}+d-de^{\frac{1}{dk}}|}+\left|\frac{1}{k}+d-de^{\frac{1}{dk}}\right|\cdot\boldsymbol{n}_x}$$

$$= \frac{1}{\|\boldsymbol{n}\|^{\frac{1}{2}}}\lim_{\xi\to 0}\sqrt{\frac{\|\boldsymbol{n}\|^2}{-\boldsymbol{n}_x}\frac{C_{\rm r}\xi^2+o(\xi^2)}{\frac{\xi^2}{2d}+o(\xi^2)}+\boldsymbol{n}_x\left(\frac{\xi^2}{2d}+o(\xi^2)\right)}$$

$$= \frac{1}{\|\boldsymbol{n}\|^{\frac{1}{2}}}\sqrt{\frac{2\|\boldsymbol{n}\|^2C_{\rm r}d}{-\boldsymbol{n}_x}} = \sqrt{\frac{2\|\boldsymbol{n}\|C_{\rm r}d}{-\boldsymbol{n}_x}} > 0. \tag{4.56}$$

By contrast, applying (4.45), *there exists* $\kappa_B > 0$ *such that*

$$\|\boldsymbol{w}^k-\boldsymbol{u}^k\| = \mathrm{dist}(\boldsymbol{w}^k,\mathcal{F}_{\rm r}) \leq \kappa_B\mathrm{dist}(\boldsymbol{w}^k,\mathcal{K}_{\rm logdet})^{\frac{1}{2}} \leq \kappa_B\|\boldsymbol{w}^k-\boldsymbol{v}^k\|^{\frac{1}{2}}.$$

This shows that $L_{\rm r} \leq \kappa_B < \infty$. *Moreover, from* (4.56), *for large enough* k, *we have*

[9] Note that this quadratic in μ has real roots because $d \geq 2$; see the discussions following (4.4).

$\|\boldsymbol{w}^k - \boldsymbol{u}^k\| \geq \frac{L_{\mathrm{r}}}{2}\|\boldsymbol{w}^k - \boldsymbol{v}^k\|^{\frac{1}{2}}$. *Therefore, for sufficiently large k, we have*

$$\frac{L_{\mathrm{r}}}{2}\|\boldsymbol{w}^k - \boldsymbol{v}^k\|^{\frac{1}{2}} \leq \mathrm{dist}(\boldsymbol{w}^k, \mathcal{F}_{\mathrm{r}}) \leq \kappa_B \mathrm{dist}(\boldsymbol{w}^k, \mathcal{K}_{\mathrm{logdet}})^{\frac{1}{2}} \leq \kappa_B \|\boldsymbol{w}^k - \boldsymbol{v}^k\|^{\frac{1}{2}}.$$

Consequently, it holds that for all large enough k,

$$\frac{L_{\mathrm{r}}}{2} \leq \frac{\mathrm{dist}(\boldsymbol{w}^k, \mathcal{F}_{\mathrm{r}})}{\mathrm{dist}(\boldsymbol{w}^k, \mathcal{K}_{\mathrm{logdet}})^{\frac{1}{2}}} \leq \kappa_B.$$

Similar to the argument in Remark 4.3, we conclude that the choice of $|\cdot|^{\frac{1}{2}}$ is tight.

By Theorem 4.15, we have the following one-step facial residual function for $\mathcal{K}_{\mathrm{logdet}}$ and $\boldsymbol{n}$.

Corollary 4.17. *Let $\boldsymbol{n} = (\boldsymbol{n}_x, \boldsymbol{n}_x(\log\det(-\boldsymbol{n}_Z/\boldsymbol{n}_x)+d), \boldsymbol{n}_Z) \in \partial\mathcal{K}^*_{\mathrm{logdet}}$ with $\boldsymbol{n}_x < 0$ and $\boldsymbol{n}_Z \succ 0$ such that $\mathcal{F}_{\mathrm{r}} = \mathcal{K}_{\mathrm{logdet}} \cap \{\boldsymbol{n}\}^{\perp}$. Let $\gamma_{\boldsymbol{n},t}$ be as in (4.23) with $\mathcal{F} = \mathcal{F}_{\mathrm{r}}$ and $\mathfrak{g} = |\cdot|^{\frac{1}{2}}$. Then the function $\psi_{\mathcal{K},\boldsymbol{n}} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by*

$$\psi_{\mathcal{K},\boldsymbol{n}}(\epsilon, t) := \max\{\epsilon, \epsilon/\|\boldsymbol{n}\|\} + \max\left\{2t^{\frac{1}{2}}, 2\gamma_{\boldsymbol{n},t}^{-1}\right\}(\epsilon + \max\{\epsilon, \epsilon/\|\boldsymbol{n}\|\})^{\frac{1}{2}}$$

is a one-step facial residual function for $\mathcal{K}_{\mathrm{logdet}}$ and $\boldsymbol{n}$.

4.3 Error bounds

In this section, we combine all the previous analysis to deduce the error bound concerning (Feas) with $\mathcal{K} = \mathcal{K}_{\mathrm{logdet}}$. We proceed as follows.

We consider (Feas) with $\mathcal{K} = \mathcal{K}_{\mathrm{logdet}}$ and we suppose (Feas) is feasible. We also let $\mathfrak{d} := d_{\mathrm{PPS}}(\mathcal{K}_{\mathrm{logdet}}, \mathcal{L} + \boldsymbol{a})$, where we recall that d_{PPS} denotes the distance to the PPS condition, i.e., the minimum number of facial reduction steps necessary to find a face $\mathcal{F}$ such that $\mathcal{F}$ and $\mathcal{L} + \boldsymbol{a}$ satisfy the PPS condition; see [40, Section 2.4.1].

In particular, invoking [40, Proposition 5], there exists a chain of faces

$$\mathcal{F}_{\mathfrak{d}+1} \subsetneq \mathcal{F}_{\mathfrak{d}} \subsetneq \cdots \subsetneq \mathcal{F}_2 \subsetneq \mathcal{F}_1 = \mathcal{K}_{\mathrm{logdet}} \tag{4.57}$$

together with $\boldsymbol{n}^1, \dots, \boldsymbol{n}^{\mathfrak{d}}$ satisfying the following properties:

(a) For all $i \in \{1, \dots, \mathfrak{d}\}$ we have

$$\boldsymbol{n}^i \in \mathcal{F}_i^* \cap \mathcal{L}^\perp \cap \{\boldsymbol{a}\}^\perp \;\text{ and }\; \mathcal{F}_{i+1} = \mathcal{F}_i \cap \{\boldsymbol{n}^i\}^\perp.$$

(b) $\mathcal{F}_{\mathfrak{d}+1} \cap (\mathcal{L} + \boldsymbol{a}) = \mathcal{K}_{\mathrm{logdet}} \cap (\mathcal{L} + \boldsymbol{a})$ and $\mathcal{F}_{\mathfrak{d}+1}$ and $\mathcal{L} + \boldsymbol{a}$ satisfy the PPS condition.

In order to get the final error bound for (Feas) we aggregate the one-step facial residual functions for each of $\mathcal{F}_i$ and $\boldsymbol{n}^i$ using the recipe described in [36, Theorem 3.8].

So far, we only computed facial residual functions for $\mathcal{F}_1 = \mathcal{K}_{\mathrm{logdet}}$ and $\boldsymbol{n}^1 \in \mathcal{K}_{\mathrm{logdet}}^*$, but we need the ones for the other $\mathcal{F}_i$ and $\boldsymbol{n}^i$. Fortunately, thanks to the facial structure of $\mathcal{K}_{\mathrm{logdet}}$, if $\mathfrak{d} \geq 2$, then $\mathcal{F}_2$ must be a face of the form $\mathcal{F}_{\mathrm{d}}$ or $\mathcal{F}_{\#}$ (see (4.11) and (4.12)). This is because all other possibilities correspond to non-exposed faces or faces of dimension 1 (for which the PPS condition is automatically satisfied).

$\mathcal{F}_{\#}$ and $\mathcal{F}_{\mathrm{d}}$ are symmetric cones [18, 19] since they are linearly isomorphic to a direct product of $\mathbb{R}_-$ and a face of a positive semidefinite cone (which are symmetric cones on their own right, e.g., [40, Proposition 31]). The conclusion is that for the faces "down the chain" we can compute the one-step facial residual functions using the general result for symmetric cones given in [40, Theorem 35]. We note this as a lemma.

Lemma 4.18. *Let $\bar{\mathcal{F}}$ be a face of $\mathcal{F}_{\mathrm{d}}$. Let $\boldsymbol{n} \in \bar{\mathcal{F}}^* \cap \mathcal{L}^\perp \cap \{\boldsymbol{a}\}^\perp$. Then, there exists a constant $\kappa > 0$ such that the function*

$$\psi_{\bar{\mathcal{F}},\boldsymbol{n}}(\epsilon, t) \coloneqq \kappa\epsilon + \kappa\sqrt{\epsilon t}$$

is a one-step facial residual function for $\bar{\mathcal{F}}$ and $\boldsymbol{n}$.

Proof. Follows by invoking [40, Theorem 35] with $\mathcal{K} \coloneqq \bar{\mathcal{F}}$, $\mathcal{F} \coloneqq \bar{\mathcal{F}}$ and $z \coloneqq \boldsymbol{n}$. □

We are now positioned to prove our main result in this paper.

Theorem 4.19 (Error bounds for (Feas) with $\mathcal{K} = \mathcal{K}_{\text{logdet}}$)**.** *Consider* (Feas) *with* $\mathcal{K} = \mathcal{K}_{\text{logdet}}$*. Suppose* (Feas) *is feasible and let* $\mathfrak{d} := d_{\text{PPS}}(\mathcal{K}_{\text{logdet}}, \mathcal{L} + \boldsymbol{a})$ *and consider a chain of faces as in* (4.57)*. Then* $\mathfrak{d} \leq \min\{d-1, \dim(\mathcal{L}^\perp \cap \{\boldsymbol{a}\}^\perp)\} + 1$ *and the following items hold:*

(i) If $\mathfrak{d} = 0$*, then* (Feas) *satisfies a Lipschitzian error bound.*

(ii) If $\mathfrak{d} = 1$*, we have* $\mathcal{F}_2 = \{\mathbf{0}\}$ *or* $\mathcal{F}_2 = \mathcal{F}_{\text{d}}$ *or* $\mathcal{F}_2 = \mathcal{F}_{\#}$ *or* $\mathcal{F}_2 = \mathcal{F}_{\text{r}}$ *or* $\mathcal{F}_2 = \mathcal{F}_{\infty}$*.*

(a) If $\mathcal{F}_2 = \{\mathbf{0}\}$*, then* (Feas) *satisfies a Lipschitzian error bound.*

(b) If $\mathcal{F}_2 = \mathcal{F}_{\text{d}}$*, then* (Feas) *satisfies an entropic error bound.*[10]

(c) If $\mathcal{F}_2 = \mathcal{F}_{\#}$ *and* $\boldsymbol{n}_y^1 > 0$*, then* (Feas) *satisfies a Hölderian error bound with exponent* $\frac{1}{2}$*. If* $\mathcal{F}_2 = \mathcal{F}_{\#}$ *and* $\boldsymbol{n}_y^1 = 0$*, then* (Feas) *satisfies a log-type error bound.*[10]

(d) If $\mathcal{F}_2 = \mathcal{F}_{\text{r}}$*, then* (Feas) *satisfies a Hölderian error bound with exponent* $\frac{1}{2}$*.*

(e) If $\mathcal{F}_2 = \mathcal{F}_{\infty}$ *and* $\boldsymbol{n}_y^1 > 0$*, then* (Feas) *satisfies a Lipschitzian error bound. If* $\mathcal{F}_2 = \mathcal{F}_{\infty}$ *and* $\boldsymbol{n}_y^1 = 0$*, then* (Feas) *satisfies a log-type error bound.*[10]

(iii) If $\mathfrak{d} \geq 2$ *we have* $\mathcal{F}_2 = \mathcal{F}_{\text{d}}$ *or* $\mathcal{F}_2$ *is of form* $\mathcal{F}_{\#}$*. Then, an error bound with residual function* $\underbrace{\mathfrak{h} \circ \mathfrak{h} \circ \cdots \circ \mathfrak{h}}_{\mathfrak{d}-1} \circ \bar{\mathfrak{g}}$ *holds, where* $\mathfrak{h} = |\cdot|^{\frac{1}{2}}$ *and*

$$\bar{\mathfrak{g}} = \begin{cases} \mathfrak{g}_{\text{d}} & \text{if } \mathcal{F}_2 = \mathcal{F}_{\text{d}}, \\ \mathfrak{g}_{\log} & \text{if } \mathcal{F}_2 = \mathcal{F}_{\#} \text{ and } \boldsymbol{n}_y^1 = 0, \\ |\cdot|^{\frac{1}{2}} & \text{if } \mathcal{F}_2 = \mathcal{F}_{\#} \text{ and } \boldsymbol{n}_y^1 > 0. \end{cases} \tag{4.58}$$

[10] An entropic error bound is an error bound with the residual function being $\mathfrak{g}_{\text{d}}$, see Definition 2.2. A log-type error bound refers to an error bound with the residual function being $\mathfrak{g}_{\log}$. See (4.25) and (4.31) for the definitions of $\mathfrak{g}_{\text{d}}$ and $\mathfrak{g}_{\log}$, respectively.

Proof. Following the discussion so far, if $\mathfrak{d} \geq 2$, it is because $\mathcal{F}_2 = \mathcal{F}_{\mathrm{d}}$ or $\mathcal{F}_2$ is of the form $\mathcal{F}_{\#}$. Also, as remarked previously, in this case, $\mathcal{F}_2$ is a symmetric cone that is a direct product of a polyhedral cone (of rank at most 1) and a symmetric cone of rank at most d. Considering the conic feasibility problem with $\mathcal{K} = \mathcal{F}_2$, it follows from [40, Proposition 24, Remark 39] that

$$d_{\mathrm{PPS}}(\mathcal{F}_2, \mathcal{L} + \boldsymbol{a}) \leq \min\{d - 1, \dim(\mathcal{L}^\perp \cap \{\boldsymbol{a}\}^\perp)\}.$$

Hence, by adding the first facial reduction step to get $\mathcal{F}_2$, we obtain the bound on $\mathfrak{d}$. Next, we examine the possibilities for $\mathfrak{d}$.

(*i*) If $\mathfrak{d} = 0$, then (Feas) satisfies the PPS condition and so a Lipschitzian error bound holds because of [7, Corollary 6].

(*ii*) If $\mathfrak{d} = 1$, then the possibilities for $\mathcal{F}_2$ are $\{\mathbf{0}\}$, $\mathcal{F}_{\mathrm{d}}$, $\mathcal{F}_{\#}$, $\mathcal{F}_\infty$ or $\mathcal{F}_{\mathrm{r}}$. Then, except for the case $\{\mathbf{0}\}$, the error bound then follows from [36, Theorem 3.8] and the facial residual functions computed in Corollaries 4.4, 4.10, 4.14 and 4.17. The case $\mathcal{F}_2 = \{\mathbf{0}\}$ follows from [40, Proposition 27].

(*iii*) In this case, it must hold that $\mathcal{F}_2 = \mathcal{F}_{\mathrm{d}}$ or $\mathcal{F}_2$ is of form $\mathcal{F}_{\#}$. Both cases, as discussed previously, correspond to symmetric cones. The error bound is obtained by invoking [36, Theorem 3.8] and using the facial residual functions constructed in Corollaries 4.4 and 4.10 and Lemma 4.18.

□

From Theorem 4.19 we see the presence of *non-Hölderian* behaviour in the cases of entropic and logarithmic error bounds. A similar phenomenon was observed in the study of error bounds for the exponential cone, see [36, Section 4.4]. The analysis of convergence rates of algorithms under non-Hölderian error bounds is still a challenge (see [38, Sections 5 and 6]) and $\mathcal{K}_{\mathrm{logdet}}$ is thus another interesting test bed for research ideas on this topic.

Chapter 5

Concluding Remarks

In this thesis, we establish the optimal error bounds for conic linear feasibility problems involving generalized power cones and log-determinant cones. Their applications in algebraic structures are also explored. Specifically, we characterize the automorphism group of the generalized power cone, which was unknown until our work. Utilizing the automorphism group, we investigate other algebraic properties of the generalized power cone, including homogeneity, reducibility, and perfectness.

The characterization of the automorphism group of the generalized power cone is particularly notable as it bridges analysis, geometry, and algebra. Furthermore, the generalized power cone is closely related to *nonnegativity problems* [48], providing possible directions for future research related to the generalized power cone.

Similar exploration in geometry also apply to the log-determinant cone. For instance, the likelihood geometry of determinantal point processes [20] requires extensive calculations and cannot be easily extended to high-dimensional cases. Could our results help progress this problem? Additionally, how can we generalize the error bounds for the log-determinant cone from positive semidefinite cones to *Euclidean Jordan algebras* [19]?

Returning to the framework employed in this thesis, several potential directions emerge. One natural question is:

> *Can we establish some* ***calculus rules*** *for one-step facial residual functions for certain closed convex cones?*

For instance, the generalized power cone can be viewed as a specific composition of two closed convex cones, and the log-determinant cone can be considered a *spectral extension* of the exponential cone. Are there any connections between one-step facial residual functions for these cones?

The generalization of the classical facial reduction algorithms, typically applicable to the intersection of an affine subspace and a closed convex cone, to the intersections of two general closed convex sets is an interesting prospect. Achieving this would simplify the regularization of convex optimization problems, as lifting would no longer be necessary. Additionally, it might enable us to establish an *extended dual*, as discussed in [56].

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