The Hong Kong Polytechnic University

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Pricing American Options without Expiry Date

by

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CERTIFICATION OF ORIGINALITY

I hereby declare that this thesis is my own work and that, to the best of my knowledge and belief, it reproduces no material previously published or written, nor material that has been accepted for the award of any other degree or diploma, except where due acknowledgement has been made in the text.

__________________________ (Signed)

______YU Kwok Wai________(Name of student)

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Abstract

The history of options trading started prior to 1973. Many different types of options are regularly traded throughout the world. Options on stocks have been traded in Hong Kong since September 1995. Because of the early exercise opportunity, American-type options are more flexible and popular than European-type options. Although many researchers have contributed to deriving pricing formulas for European options, however there are no closed-form formulas for the prices of American options in most cases. The main difficulty is that it is a free boundary value problem.

To price an American option, it is important to determine the optimal exercise boundary (and the optimal stopping time). For a perpetual American option, the optimal exercise boundary turns out to be constant through time. The word "perpetual" means that the option has no expiry date.

This thesis discusses the martingale approach to pricing perpetual American-type options. A main tool in our approach is the principle of smooth pasting. For simplicity, options in one-stock case are considered first. These options include the perpetual American put option, call option and the perpetual maximum option on one stock. Then we extend our analysis to two-stock case. The perpetual maximum option on two stocks, the perpetual uncapped Margrabe option, the perpetual capped Margrabe options and the perpetual dynamic fund protection are discussed.

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Chapter 1

Introduction

The Chicago Board of Trade was founded in 1848; some futures-type contracts were first introduced within a few years. After that, futures, forward contracts, swaps, options and other derivatives were introduced and traded increasingly in the twentieth century. In financial markets, different types of options are currently actively traded. The Chicago Board of Option Exchange (CBOE) now trades options on over 1,500 stocks and many different stock indices. Options have proved to be very popular contracts.

On 8 March 1995, the Hong Kong Legislative Council passed a motion urging the Government to introduce as expeditiously as possible a mandatory, privately managed retirement protection system with provision for the preservation and portability of benefits. The Government subsequently established the Mandatory Provident Fund (MPF) Authority. MPF was implemented on 1 December 2000. As at the end of March 2004, 97.1 per cent of all employers, 96.0 per cent of relevant employees and 80.3 per cent of all self-employed persons have enrolled in an MPF scheme. Many of the MPF products sold by life insurance companies and banks contain guarantees or options. For example, the return of many MPF products is linked to an equity index, such as the Hang Seng Index; a minimum guarantee (an option) may be provided in order to protect the policyholders from the downside risk of the market. Such guarantees and options should be priced, reserved and hedged using modern option pricing theory.

There is a long history of the theory of option pricing. It began in 1900 when the French mathematician Louis Bachelier deduced an option pricing formula by assuming that stock prices follow a Brownian motion. After that, numerous researchers have contributed in this field. The major breakthrough is the Black-Scholes theory of option pricing (1973).

As demonstrated in Theorem 8.12 of Merton (1990), the value of a European put option is completely determined once the value of the European call option is known if the stock pays no dividends. However, the valuation of European options is not valid for the American put options because of the positive probability of exercising before the expiry date. Myneni (1992) summarized the essential results on the pricing of American options. To price an American option, it is important to determine the optimal exercise boundary (and the optimal stopping time). However, there is no closed-form solution for the price of an American put option since the optimal exercise boundary has no closed-form formula. Since there is no closed-form formula for the problem, Lindberg, Marcusson and Nordman (2002) approximated the optimal exercise boundary with a polynomial. In fact, the asymptotic behavior of the optimal exercise boundary is known as the time to expiry goes to infinity. It is possible to obtain a closed-form formula for the price of a perpetual American option, i.e., the American option without an expiry date. Many researchers analyze and provide new insights in the pricing of perpetual American options. Usually, the formulas for pricing perpetual American options are derived by solving differential equations. Here, the martingale approach is used to avoid differential equations.

This thesis is organized as follows. After giving a brief introduction, Chapter 2 presents some classical assumptions, shows some basic principals of option pricing and discusses the martingale approach. In the first part of this thesis, one-stock cases are considered. Under the geometric Brownian motion assumptions, Chapter 3 discusses the pricing of perpetual American put and call options and the perpetual maximum options on one stock by considering the asymptotic behavior of the optimal exercise boundary. By considering the logarithm of the stock price as a shifted compound Poisson process, Chapter 4 derives pricing formulas for the perpetual American put options with upward jumps and downward jumps.

In the second part of this thesis, option pricing in two-stock cases are discussed. As an extension to Section 3.3, Chapter 5 studies the perpetual maximum options on two stocks. One of the first papers analyzing options on two or more stocks is Margrabe (1978). He extended the Black-Scholes theory of option pricing and derived a closed-form formula for pricing a European option on two stocks driven by geometric Brownian motion. More precisely, he studied an option to exchange one stock for another at the end of some specified period. Such kind of options is called a Margrabe option. Here, Chapter 6 derives explicit formulas for the prices of standard perpetual Margrabe options and perpetual Margrabe options with proportional cap. Chapter 7 discusses the pricing of dynamic fund protection without expiry date. Finally, Chapter 8 draws a conclusion for this thesis.

PART 1 ONE-STOCK CASE

Chapter 2

Fundamentals of Option Pricing

The long history for the theory of option pricing began in 1900. Numerous researchers have contributed in pricing different types of options. Some assumptions are essential to bring into the existence of many option pricing formulas. The risk-neutral probability measure plays an important role in option pricing. This chapter discusses some classical assumptions and the risk-neutral probability measure on option pricing. Further, we introduce some equivalent martingales.

2.1 Some Classical Assumptions

It is assumed that the market is complete and frictionless. In a frictionless market, there are no taxes, no transaction cost and no restriction on borrowing or short sales. Trading is continuous. One of the classical assumptions is that the logarithm of the stock price is assumed to be a Brownian motion.

Let S(t) be the price of a stock at time t and define X(t) by

$$S(t) = S(0)\, e^{X(t)}, \quad t \geq 0. \tag{2.1}$$

We assume that the process $\{X(t),\ t \geq 0\}$ is a Brownian motion (or Wiener process) with instantaneous variance σ^2 and drift parameter μ. The Brownian motion assumption will be applied in the following chapters except Chapter 4 in which $\{X(t),\ t \geq 0\}$ is assumed to be a jump process.

Let r be the risk-free force of interest, ζ be the dividend yield rate of the stock. It is assumed that r and ζ are positive constants, and dividends of amount $\zeta S(t)dt$ are paid between time t and time t + dt.

2.2 Risk-neutral Probability Measure

In mathematical finance, a ***risk-neutral probability measure*** is a probability measure in which today's fair (i.e. no arbitrage) price of a security is equal to the discounted expected value of the future payoffs of the security. The assumption of the completeness of market ensures the existence of a unique risk-neutral probability measure.

As demonstrated in Panjer, et al. (1998), the absence of arbitrage is equivalent to the existence of a risk-neutral probability measure in a discrete model. This is the ***Fundamental Theorem of Asset Pricing.*** A risk-neutral probability measure is also called an equivalent martingale measure. Delbaen and Schachermayer (2004) illustrated the arbitrage opportunity and martingale (equivalent martingale measure) by a simple but apparent "toy example".

2.3 Martingale Approach

In the risk-neutral world, the prices of American options can be calculated as the maximum of the discounted expected values of their corresponding payoffs over all stopping times. Thus, in order to derive pricing formulas, we want to simplify some expectations. Normally, the expectations are calculated by integration or summation. Here, by the martingale approach, we can ignore a sequential complicated calculation. In the following, we introduce a martingale.

Under a risk-neutral measure, the stochastic process $\{e^{-rt}e^{\zeta t}S(t); t \geq 0\}$ is a martingale. The martingale condition is

$$E^*\left[e^{-rt}e^{\zeta t}S(t)\right] = e^{-r(0)+\zeta(0)}S(0) \quad (2.2)$$

or

$$E^*\left[e^{-rt+\zeta t+X(t)}\right] = e^0.$$

Thus, we have

$$(-r+\zeta+(1)\mu^* + \frac{1}{2}(1^2)\sigma^2)t = 0, \quad (2.3)$$

i.e.

$$\mu^* = r - \zeta - \frac{\sigma^2}{2}. \quad (2.4)$$

Here, the asterisk signifies that the expectation is taken with respect to the risk-neutral probability measure. $\{X(t)\}$ is a Wiener process with drift parameter μ^* which is given by (2.4). The diffusion parameter of $\{X(t)\}$ remains σ under the risk-neutral measure.

Chapter 3

Pricing Perpetual American Options under Geometric Brownian Motion

A call (or put) option is a contract that gives its holder the right but not the obligation to buy (or sell) an asset for a certain price (the strike price) within a specific period of time (the expiry date). European options may be exercised only at the expiry date. American options are contracts that can be exercised early, prior to the expiry date. A perpetual American option is a contract without an expiry date. It can be exercised at any time.

Note that an American option always must be worth at least its payoff since it can be exercised at any time prior to the expiry date. An American option is more interesting and complex to evaluate than a European option. In contrast to pricing European options, pricing American options is a challenging problem because of the early exercise opportunity. A main difficulty in pricing American options is to determine the optimal exercise boundary. This problem could be formulated into a free boundary value problem which was observed by McKean (1965). The free boundary problem is to solve a partial differential equation with its Dirichlet conditions and a Neumann condition for the determination of the unknown exercise boundary. See also Allegretto, Barone-Adesi and Elliott (1994). Up to now, closed-form formula for the optimal exercise boundary of the American option has not yet been determined. However, for some perpetual American options, the optimal exercise boundary turns out to be constant through time. This will be illustrated in Section 3.1.

In this chapter, we assume that the logarithm of stock price is a Brownian motion (or Wiener Process). This is a special case of the model in which the logreturn of a stock is driven by a fractional Brownian motion as in Elliott and Chan (2004). After discussing the optimal exercise boundary which is crucial for pricing American options, the pricing formulas for perpetual American put options, call options and perpetual maximum options on one stock are derived in detail in the coming sections. At the end of this chapter, some numerical examples are provided.

3.1 The Optimal Exercise Strategy

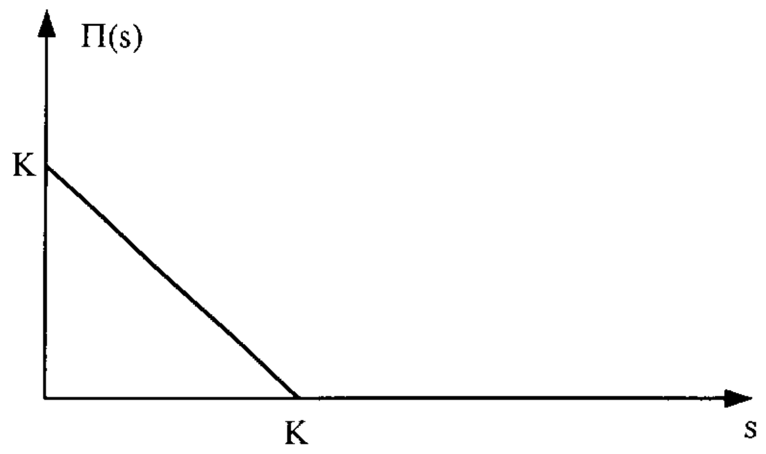
For an American option, the ***optimal exercise strategy*** is the ***stopping time*** for which the maximum value of the expected discounted payoff is attained. (The expectation is taken with respect to the risk-neutral measure.) For some perpetual options, the optimization problem can be simplified to the problem of determining the optimal values of one or two parameters. A detailed derivation of these one or two endpoints of the optimal non-exercise interval is provided.

Now let us look at a put option. If a put option with exercise price K is exercised at time t, the payoff is

$$\Pi(S(t)) = \max(K - S(t), 0),$$

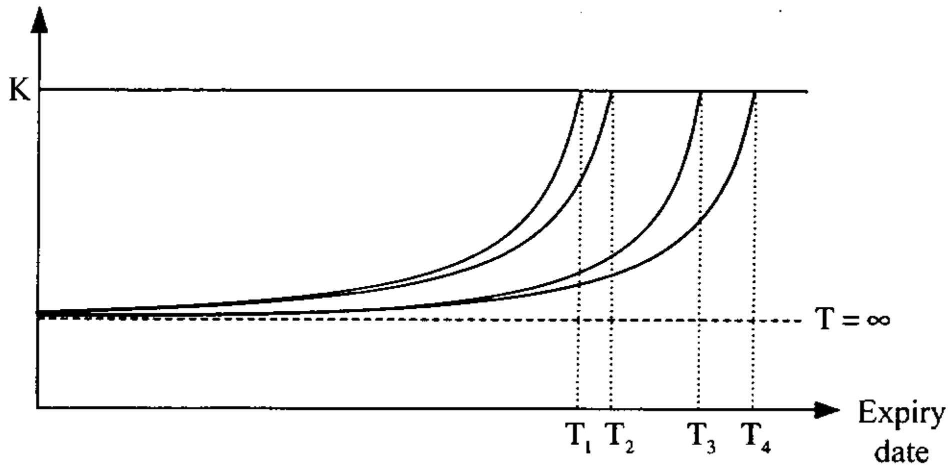
where S(t) is the price of a stock at time t. See Figure 3.1. Since an American option can be exercised at any time prior to the expiry date, choosing the optimal time to exercise is a crucial problem.

Figure 3.1 The Payoff Function of a Put Option



Before discussing on how to choose a stopping time, let us consider the ***optimal exercise boundary*** first. The optimal exercise boundary has to be found for the price of an American put option to be obtained. The optimal exercise boundary separates the region where one should continue to hold the option and the region where one should exercise it. Here, we consider the optimal exercise boundary as a function of the expiry date. As shown in Figure 3.2, there are four optimal exercise boundaries for an American put option corresponding to four finite expiry dates. For a specific expiry date, the corresponding optimal exercise boundary implies that one should continue to hold the option if the stock price is above the boundary curve and one should exercise the option when the stock price falls on or below the optimal exercise boundary. Now, let us consider the asymptotic behavior of the optimal exercise boundary. It is observed that as we extend the expiry date, the curve of the optimal exercise boundary becomes flatter and flatter. This can be explained by the independent increments property of the Brownian motion. By applying the property, as we adjust the expiry date, the optimal exercise boundary can be obtained by shifting along time for different expiry dates. Following the trend, we can at last observe a level boundary as the expiry date tends to infinity. Thus, a level boundary is the optimal exercise boundary for a perpetual American put option. To know more about the analysis of the optimal exercise boundary of an American put option, see , for example, Basso, Nardon and Pianca (2002), Kuske and Keller (1998) and Lindberg, Marcusson and Nordman (2002).

Figure 3.2 Optimal Exercise Boundaries of an American Put Option



3.2 Perpetual American Put Options

Let us illustrate the pricing of a perpetual American put option. For an American put option with exercise price K, its payoff is

$$\Pi(S(t)) = (K - S(t))_+, \tag{3.1}$$

where $m_+ = \text{Max}(m, 0)$. If the owner of the option exercises it at a time t, then he will get $(K - S(t))_+$.

As discussed in the previous section, the optimal exercise boundary of a perpetual American put option is constant through time. Consider the exercise strategy that is to exercise the option as soon as the stock price falls to the level L for the first time. For $0 < L < K$ and $L < S(0)$, define the stopping time T_L as

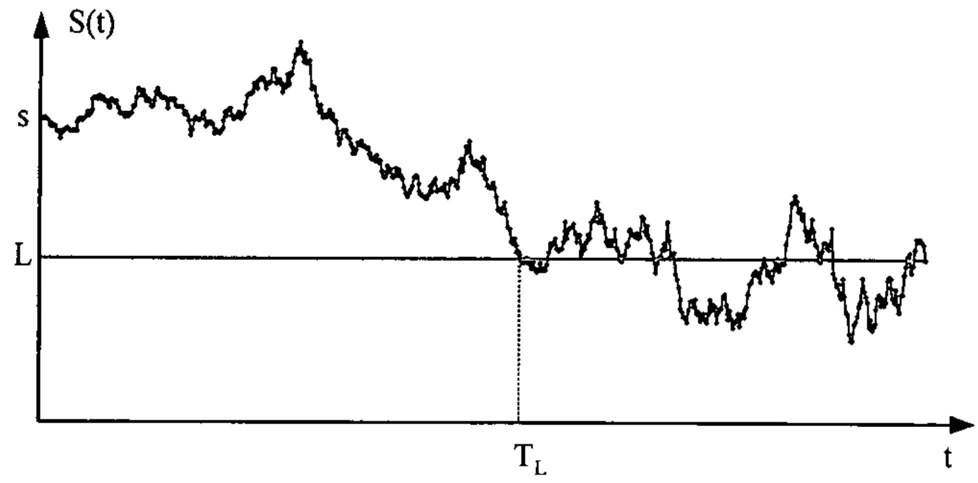
$$T_L = \min\{t \mid S(t) = L\}. \tag{3.2}$$

See Figure 3.3. The value of this exercise strategy T_L is

$$P(s;L) = E^*\left[e^{-rT_L}\Pi(S(T_L)) \,\middle|\, S(0) = s\right], \tag{3.3}$$

where r is the risk-free force of interest.

Figure 3.3 The Stopping Time T_L



Since

$$L = S(T_L) = S(0)e^{X(T_L)} = se^{X(T_L)} \tag{3.4}$$

and

$$\Pi(S(T_L)) = (K - S(T_L))_+ = (K - L)_+ = K - L,$$

formula (3.3) can be simplified to

$$P(s; L) = E^*\left[e^{-rT_L}(K-L)\,\middle|\, S(0)=s\right]$$

$$= (K-L)E^*\left[e^{-rT_L}\,\middle|\, S(0)=s\right]. \tag{3.5}$$

Thus, the problem is to find the value of $E^*\left[e^{-rT_L}\,\middle|\, S(0)=s\right]$.

Let us consider the stochastic process $\left\{e^{-rt+\theta X(t)}\right\}_{t\geq 0}$. This process is a martingale with respect to the risk-neutral measure if

$$E^*\left[e^{-rt+\theta X(t)}\right] = e^0 \tag{3.6}$$

or

$$-rt + \theta\mu^* t + \frac{1}{2}\theta^2\sigma^2 t = 0, \tag{3.7}$$

i.e.,

$$\frac{1}{2}\sigma^2\theta^2 + \mu^*\theta - r = 0, \tag{3.8}$$

where μ^* is given by (2.4). Let θ_1 and θ_2 be the two roots of the quadratic equation (3.8). Since

$$\theta_1\theta_2 = \frac{-r}{\sigma^2/2} = \frac{-2r}{\sigma^2} < 0,$$

one root is negative and the other is positive. Assume that $\theta_1 < 0$ and $\theta_2 > 0$. For the negative root θ_1, the stochastic process $\left\{e^{-rt+\theta_1 X(t)}\right\}_{0\leq t\leq T_L}$ is a martingale bounded between 0 and $\left(\frac{L}{s}\right)^{\theta_1}$. By the optional sampling theorem, we have

$$1 = E^*\left[e^{-rT_L+\theta_1 X(T_L)}\right] = E^*\left[e^{-rT_L}\left(e^{X(T_L)}\right)^{\theta_1}\right] = E^*\left[e^{-rT_L}\left(\frac{L}{s}\right)^{\theta_1}\right] \tag{3.9}$$

or

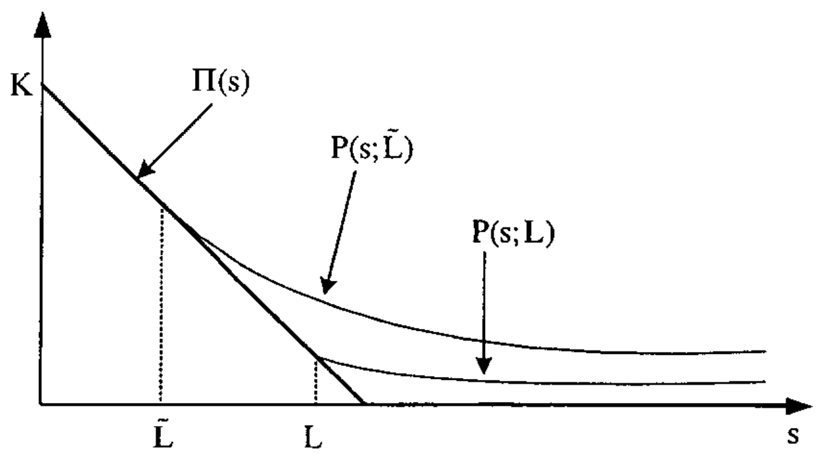
$$E^*\left[e^{-rT_L}\right] = \left(\frac{L}{s}\right)^{-\theta_1}. \tag{3.10}$$

Thus, for $s > L$ and $K > L$, it follows from (3.5) and (3.10) that the value of the strategy T_L is

$$P(s;L) = (K-L)\left(\frac{L}{s}\right)^{-\theta_1}, \tag{3.11}$$

which is shown in Figure 3.4.

Figure 3.4 The Value of the Strategy T_L and the Price of the Perpetual American Put Option



Now, for the optimal exercise strategy, we seek the value L that maximizes $P(s;L)$. This value is denoted by $\tilde{L}$. The optimal value $\tilde{L}$ can be obtained by differentiating P(s; L) with respect to L and setting the derivative equal to zero, i.e.,

$$\left.\frac{\partial P(s;L)}{\partial L}\right|_{L=\tilde{L}} = 0 \tag{3.12}$$

or

$$-\theta_1 K\left(\frac{1}{\tilde{L}}\right)\left(\frac{\tilde{L}}{s}\right)^{-\theta_1} - (-\theta_1+1)\left(\frac{\tilde{L}}{s}\right)^{-\theta_1} = 0. \tag{3.13}$$

Solving equation (3.13) yields the optimal exercise boundary

$$\tilde{L} = \frac{-\theta_1}{1-\theta_1}K, \tag{3.14}$$

which is the same as (3.9) in Gerber and Shiu (1994) if their θ_0 is our θ_1.

For $s > \tilde{L}$, it follows from (3.11) and (3.14) that

$$P(s;\tilde{L}) = (K-\tilde{L})\left(\frac{\tilde{L}}{s}\right)^{-\theta_1}$$

$$= (K - \frac{-\theta_1}{1-\theta_1}K)\left(\frac{1}{s}\frac{-\theta_1}{1-\theta_1}K\right)^{-\theta_1}$$

$$= \frac{K}{1-\theta_1}\left(\frac{-K\theta_1}{s(1-\theta_1)}\right)^{-\theta_1}. \tag{3.15}$$

See Figure 3.4. Note that for $L < K$, as L tends to $\tilde{L}$, the value of P(s; L) increases, i.e., it is approaching the maximum value $P(s;\tilde{L})$. Thus, the price of the perpetual American put option is

$$\begin{cases} K-s & \text{if } s \le \tilde{L}, \\ \dfrac{K}{1-\theta_1}\left(\dfrac{-K\theta_1}{s(1-\theta_1)}\right)^{-\theta_1} & \text{if } s > \tilde{L}. \end{cases} \tag{3.16}$$

Remark: it follows from (3.14) that $\tilde{L} < K$. Now

$$\left.\frac{\partial \Pi(s)}{\partial s}\right|_{s=\tilde{L}} = \left.\frac{\partial (K-s)}{\partial s}\right|_{s=\tilde{L}} = -1 \tag{3.17}$$

and

$$\left.\frac{\partial P(s;\tilde{L})}{\partial s}\right|_{s=\tilde{L}} = \theta_1 (K-\tilde{L}) \frac{1}{s}\left(\frac{\tilde{L}}{s}\right)^{-\theta_1}\Bigg|_{s=\tilde{L}}$$

$$= \frac{\theta_1 K}{1-\theta_1}\frac{1-\theta_1}{-\theta_1 K} = -1. \tag{3.18}$$

Thus, it follows that

$$\left.\frac{\partial P(s;\tilde{L})}{\partial s}\right|_{s=\tilde{L}} = \left.\frac{\partial \Pi(s)}{\partial s}\right|_{s=\tilde{L}}. \tag{3.19}$$

This is known as the ***smooth pasting condition*** or the ***high contact condition***. To know more about the smooth pasting condition, see Dixit (1993). It can be shown that condition (3.19) is equivalent to the first-order condition (3.12).

Since θ_1 can be expressed in terms of the constant dividend yield rate ζ by (2.4) and (3.8), the values of $\tilde{L}$ and $P(s;\tilde{L})$ are affected by ζ. Let us consider a special case that the stock pays no dividends, i.e. $\zeta = 0$. It follows from (2.4) and (3.8) that $\mu^* = r - \frac{\sigma^2}{2}$ and $\theta_1 = -\frac{2r}{\sigma^2}$. Thus, for $s > \tilde{L}$, by (3.14) and (3.15), we obtain

$$\tilde{L} = \frac{\frac{2r}{\sigma^2}}{1+\frac{2r}{\sigma^2}}K = \frac{r}{\frac{\sigma^2}{2}+r}K = \frac{K}{1+\frac{\sigma^2}{2r}} \tag{3.20}$$

which is the same as (37) in Elliott and Chan (2004) with $H = \frac{1}{2}$ and X = K, and

$$P(s;\tilde{L}) = \left(\frac{\sigma^2 K}{\sigma^2 + 2r}\right)\left(\frac{2rK}{s(\sigma^2 + 2r)}\right)^{2r/\sigma^2} \tag{3.21}$$

which is the same as (39) in Elliott and Chan (2004) if K is replaced by X.

3.3 Perpetual American Call Options

After deriving a closed-form formula for the price of the perpetual American put option, the perpetual American call option is discussed in this section. If the underlying stock pays no dividends, it can be proved that it is never optimal to exercise an American call option early and there is no need to determine the optimal exercise boundary since the price of the American call option is equal to the price of its European counterpart. However, this is not the case for an American call option on a dividend-paying stock. In theory, the American call option is liable to be exercised early immediately before any ex-dividend date if the stock pays discrete dividends. Pricing such options is more complicated and requires determining the optimal exercise boundary. Assume that the dividend yield rate is constant. A pricing formula for the perpetual American call option is derived in this section.

Now, let us consider an American call option whose payoff function is

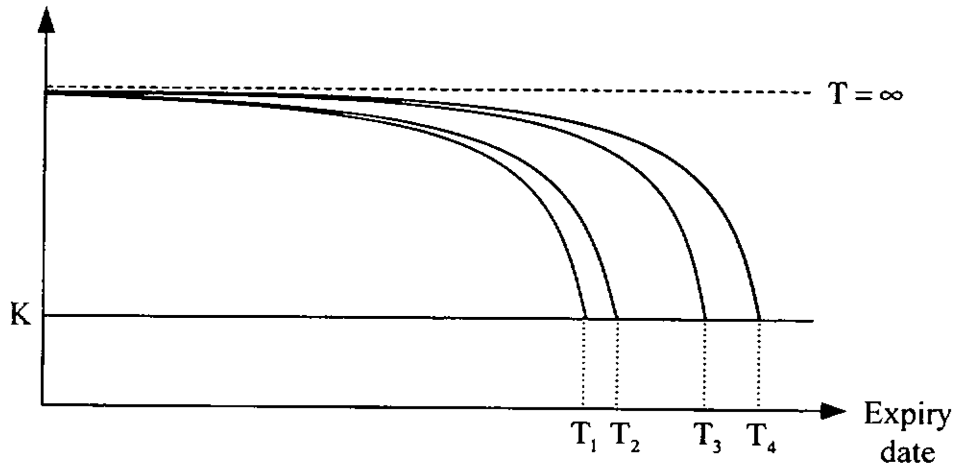
$$\Pi(S(t)) = (S(t) - K)_+, \tag{3.22}$$

where S(t) is the stock price at time t and K is the exercise price. To prevent immediate exercise, we assume that K > S(0). The price of the option at time t is bounded below by its payoff $((S(t) - K)_+)$ and above by the underlying stock price

S(t) at time t. Similar to American put options, a crucial step for pricing American call options is to determine the optimal exercise boundary.

Here, a figure similar to Figure 3.2 can be obtained to illustrate the optimal exercise boundary of an American call option. Figure 3.5 shows the trend of the optimal exercise boundary if we extend the expiry date of an American call option. We observe a level optimal exercise boundary for a perpetual American call option. In contrast to the level optimal exercise boundary for a perpetual American put option, this optimal exercise boundary is an upper bound for those of the American call option with expiry date. See Figure 3.5.

Figure 3.5 Optimal Exercise Boundaries of an American Call Option



For a perpetual American call option, it is sufficient to consider the exercise strategy that is to exercise the option as soon as the stock price rises to a level U for the first time. For U > S(0) and U > K, define a stopping time of the form

$$T_U = \min\{t \mid S(t) = U\}. \tag{3.23}$$

See Figure 3.6. The value of this exercise strategy T_U is

$$C(s;U) = E^*\left[e^{-rT_U}\Pi(S(T_U)) \,\middle|\, S(0) = s\right]. \tag{3.24}$$

Since

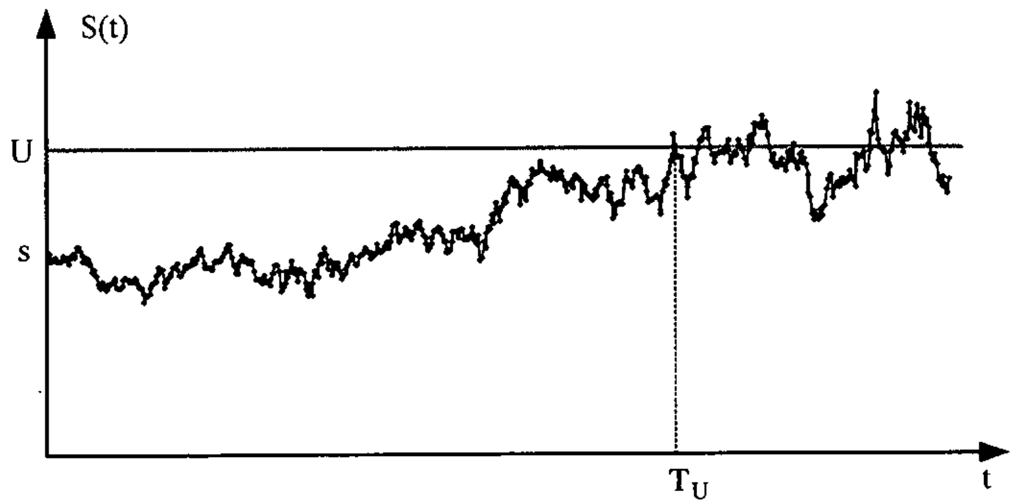
$$\Pi(S(T_U)) = \left(S(T_U) - K\right)_+ = U - K,$$

formula (3.24) can be simplified to

$$C(s;U) = (U - K)E^*\left[e^{-rT_U} \,\middle|\, S(0) = s\right]. \tag{3.25}$$

Now, the problem is simplified to determine the expectation $E^*\left[e^{-rT_U} \,\middle|\, S(0) = s\right]$.

Figure 3.6 The Stopping Time T_U



Let us consider a martingale in the form of $\left\{e^{-rt+\theta X(t)}\right\}_{t\geq 0}$. The martingale condition is given by (3.8) which is a quadratic equation in θ. It was shown in the previous section that there is a negative root θ_1 and a positive root θ_2 for this quadratic equation. The stochastic process $\left\{e^{-rt+\theta_2 X(t)}\right\}_{0\le t\le T_U}$ is a bounded martingale for the positive root θ_2. By the optional sampling theorem, we obtain

$$E^*\left[e^{-rT_U} \middle| S(0)=s\right]=\left(\frac{s}{U}\right)^{\theta_2}. \tag{3.26}$$

Thus, for U > s and U > K, it follows from (3.25) and (3.26) that the value of the exercise strategy T_U is

$$C(s;U)=(U-K)\left(\frac{s}{U}\right)^{\theta_2}. \tag{3.27}$$

Formula (3.27) can be considered as a function of U. Now, let us seek the optimal value of U, denoted by $\tilde{U}$, that maximizes C(s; U). The optimal value can be determined by the first-order condition

$$\left.\frac{\partial C(s;U)}{\partial U}\right|_{U=\tilde{U}}=0, \tag{3.28}$$

or equivalently, the smooth pasting condition

$$\left.\frac{\partial C(s;\tilde{U})}{\partial s}\right|_{s=\tilde{U}}=\left.\frac{\partial \Pi(s)}{\partial s}\right|_{s=\tilde{U}}. \tag{3.29}$$

It follows that the optimal value of U is

$$\tilde{U}=\frac{\theta_2}{\theta_2-1}K, \tag{3.30}$$

which is the same as (3.16) in Gerber and Shiu (1994) if their θ_1 is our θ_2. Thus, for $s<\tilde{U}$, by (3.27) and (3.30),

$$C(s;\tilde{U})=(\tilde{U}-K)\left(\frac{s}{\tilde{U}}\right)^{\theta_2}$$

$$= \frac{K}{\theta_2 - 1}\left(\frac{s(\theta_2 - 1)}{\theta_2 K}\right)^{\theta_2}, \tag{3.31}$$

and the price of the perpetual American call option is

$$\begin{cases} \dfrac{K}{\theta_2 - 1}\left(\dfrac{s(\theta_2 - 1)}{\theta_2 K}\right)^{\theta_2} & \text{if } s < \tilde{U}, \\ s - K & \text{if } s \geq \tilde{U}, \end{cases} \tag{3.32}$$

where K is the exercise price and s = S(0) and θ_2 is the positive root of (3.8).

3.4 Perpetual Maximum Option on One Stock

In this section, we show how to derive pricing formula for a perpetual maximum option on one stock. A maximum option is an option whose payoff is the maximum of two or more stocks or assets, e.g.

$$\Pi(z_1, z_2, z_3, z_4) = \max(z_1, z_2, z_3, z_4), \quad z_1, z_2, z_3, z_4 \geq 0.$$

We also call it an alternative option or greater-of option. Maximum options can be found in firms choosing among mutually exclusive investment alternative, or in employment switching decisions by agents. It also has an application in pension design and valuation (see Sherris (1993)). In some employees' retirement systems, the maximum option is one of the retirement options for the members. Under this option, the retiree receives the largest monthly benefit possible.

Let us consider a perpetual American option with payoff function

$$\Pi(S(t)) = \max(K, S(t)), \tag{3.33}$$

where S(t) is the stock price at time t and K > 0 is the guaranteed price or the floor price. This payoff function can be regarded as a one stock case of the maximum option which is the maximum of a stock and a positive constant K.

Consider an option-exercise strategy of the form:

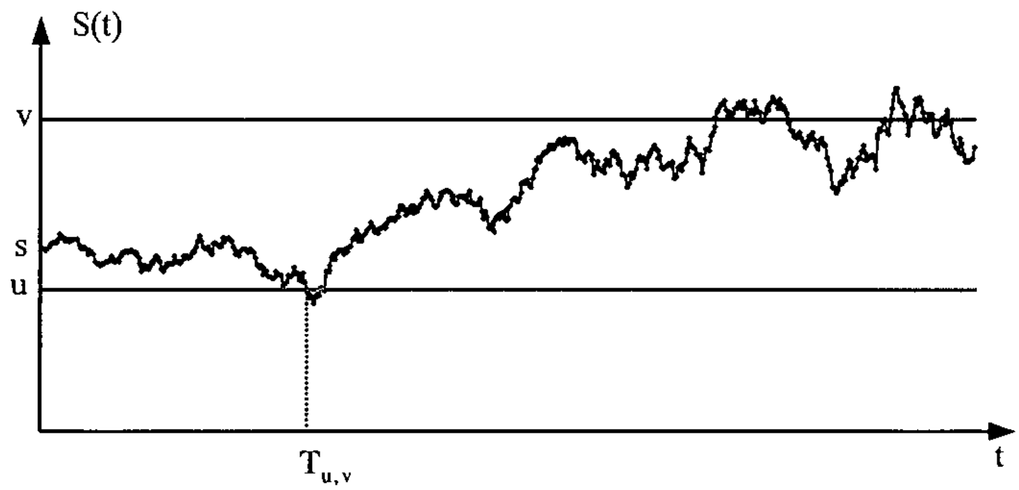
$$T_{u,v} = \min\{t \mid S(t) = u \text{ or } S(t) = v\}, \tag{3.34}$$

with $0 < u \le s = S(0) \le v$. The strategy $T_{u,v}$ is to exercise the option as soon as the stock price rises to a level v or falls to a level u for the first time; the value of this strategy is

$$V(s;u,v) = E^*\left[e^{-rT_{u,v}}\Pi\left(S\left(T_{u,v}\right)\right)\middle|S(0)=s\right], \qquad 0 < u \le s \le v, \tag{3.35}$$

where r is the risk-free force of interest. See Figure 3.7.

Figure 3.7 The Stopping Time $T_{u,v}$



Following Section 10.10 in Panjer, et al. (1998), we express formula (3.35) as

$$V(s;\, u,\, v) \;=\; \Pi(u)\, A(s;\, u,\, v) \;+\; \Pi(v)\, B(s;\, u,\, v), \qquad 0 < u \le s \le v, \qquad (3.36)$$

where

$$A(s; u, v) = E^*\left[e^{-rT_{u,v}} I\left(S\left(T_{u,v}\right) = u\right) \,\Big|\, S(0) = s\right], \qquad (3.37)$$

and

$$B(s; u, v) = E^*\left[e^{-rT_{u,v}} I\left(S\left(T_{u,v}\right) = v\right) \,\Big|\, S(0) = s\right]. \qquad (3.38)$$

Again, we consider a martingale $\left\{e^{-rt+\theta X(t)}\right\}_{t\ge 0}$. For $\theta = \theta_1$ and $\theta = \theta_2$, the stochastic process $\{e^{-rt+\theta X(t)}\}_{0\le t\le T_{u,v}}$ is a bounded martingale. Here, θ_1 and θ_2 are negative and positive roots of the quadratic equation (3.8) with σ being the diffusion coefficient of the Brownian motion $\{\ln S(t)\}$ and ζ the constant dividend-yield rate. By the optional sampling theorem, we have

$$1 = E^*\left[e^{-rT_{u,v} + \theta X(T_{u,v})}\right]. \qquad (3.39)$$

Since

$$1 = I\left(S(T_{u,v}) = u\right) + I\left(S(T_{u,v}) = v\right),$$

(3.39) can be rewritten as

$$1 = E^*\left[I(S(T_{u,v}) = u)\, e^{-rT_{u,v}} \left(\frac{u}{s}\right)^{\theta}\right] + E^*\left[I(S(T_{u,v}) = v)\, e^{-rT_{u,v}} \left(\frac{v}{s}\right)^{\theta}\right]. \qquad (3.40)$$

It follows from (3.37), (3.38) and (3.40) that

$$A(s; u, v)\left(\frac{u}{s}\right)^{\theta} + B(s; u, v)\left(\frac{v}{s}\right)^{\theta} = 1. \qquad (3.41)$$

That is,

$$A(s;u,v)\left(\frac{u}{s}\right)^{\theta_1}+B(s;u,v)\left(\frac{v}{s}\right)^{\theta_1}=1 \qquad (3.42)$$

and

$$A(s;u,v)\left(\frac{u}{s}\right)^{\theta_2}+B(s;u,v)\left(\frac{v}{s}\right)^{\theta_2}=1. \qquad (3.43)$$

Solving from (3.42) and (3.43), we obtain

$$A(s;u,v)=\frac{v^{\theta_2}s^{\theta_1}-v^{\theta_1}s^{\theta_2}}{v^{\theta_2}u^{\theta_1}-v^{\theta_1}u^{\theta_2}} \qquad (3.44)$$

and

$$B(s;u,v)=\frac{s^{\theta_2}u^{\theta_1}-s^{\theta_1}u^{\theta_2}}{v^{\theta_2}u^{\theta_1}-v^{\theta_1}u^{\theta_2}}. \qquad (3.45)$$

Now, we can substitute expressions (3.44) and (3.45) in the right-hand side of (3.36) to get

$$V(s;u,v)=\Pi(u)\frac{v^{\theta_2}s^{\theta_1}-v^{\theta_1}s^{\theta_2}}{v^{\theta_2}u^{\theta_1}-v^{\theta_1}u^{\theta_2}}+\Pi(v)\frac{s^{\theta_2}u^{\theta_1}-s^{\theta_1}u^{\theta_2}}{v^{\theta_2}u^{\theta_1}-v^{\theta_1}u^{\theta_2}}, \qquad (3.46)$$

which is the value of the Strategy $T_{u,v}$ as shown in Figure 3.8.

The next problem is to find $\tilde{u}$ and $\tilde{v}$, the values of u and v that maximize $V(s;u,v)$. Then $V(s;\tilde{u},\tilde{v})$, for $\tilde{u}\le s=S(0)\le\tilde{v}$, is the price of the perpetual American option. The optimal values $\tilde{u}$ and $\tilde{v}$ can be obtained by the first-order condition:

$$V_u(s;\tilde{u},\tilde{v}) = 0, \qquad (3.47)$$

$$V_v(s;\tilde{u},\tilde{v}) = 0; \qquad (3.48)$$

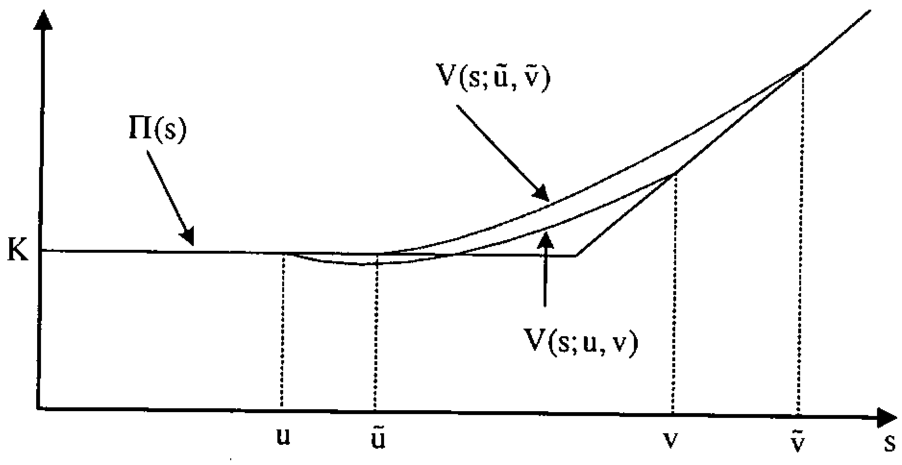
or, by the ***high contact*** or ***smooth pasting*** condition:

$$V_s(\tilde{u};\tilde{u},\tilde{v}) = \Pi'(\tilde{u}), \qquad (3.49)$$

$$V_s(\tilde{v}; \tilde{u}, \tilde{v}) = \Pi'(\tilde{v}). \tag{3.50}$$

We now show that the first-order condition is equivalent to the high contact condition.

Figure 3.8 The Value of the Strategy $T_{u,v}$ and the Price of the Perpetual Maximum Option on One Stock



By differentiating (3.36) with respective to u and setting s = u, we have

$$V_u(u; u, v) = \Pi'(u) + \Pi(u)\, A_u(u; u, v) + \Pi(v)\, B_u(u; u, v). \tag{3.51}$$

On the other hand, differentiate (3.36) with respective to s and set s = u to obtain

$$V_s(u; u, v) = \Pi(u)\, A_s(u; u, v) + \Pi(v)\, B_s(u; u, v). \tag{3.52}$$

Let us introduce a new parameter x, where $u < x < s < v$. We observe that $A(s; u, v)$ and $B(s; u, v)$ can be factorized as

$$A(s; u, v) = A(s; x, v)\, A(x; u, v). \tag{3.53}$$

and

$$B(s; u, v) = A(s; x, v)\, B(x; u, v) + B(s; x, v). \tag{3.54}$$

Equations (3.53) and (3.54) can be interpreted as follows. For a stock price starting from s, it may reach the level u for the first time in only one way, i.e., it must reach the level x first and then reach the level u. On the other hand, there are two possibilities for a stock price starting from s to reach the level v for the first time. One possibility is that the stock price reaches the level x first and then reaches the level v. Another possibility is that the stock price reaches the level v directly for the first time. We can also check (3.53) and (3.54) by expressions (3.44) and (3.45).

By differentiating (3.53) and (3.54) with respect to x and setting x = s = u, we have

$$A_u(u; u, v) + A_s(u; u, v) = 0, \tag{3.55}$$

and

$$B_u(u; u, v) + B_s(u; u, v) = 0. \tag{3.56}$$

A new identity can be obtained by combining the identities (3.51) and (3.52). Substituting (3.55) and (3.56) into the new identity and simplifying yields

$$V_u(u; u, v) + V_s(u; u, v) = \Pi'(u). \tag{3.57}$$

Likewise, we can obtain

$$V_v(v; u, v) + V_s(v; u, v) = \Pi'(v). \tag{3.58}$$

This shows that the first-order condition (3.47) and (3.48) is equivalent to the high contact condition (3.49) and (3.50).

Now, let us solve for $\tilde{u}$ and $\tilde{v}$. With $\tilde{u} < K < \tilde{v}$, it follows from (3.33), (3.46), (3.49) and (3.50) that

$$\frac{\tilde{u}}{\tilde{v}} = \left(\frac{-\theta_1(\theta_2 - 1)}{\theta_2(1-\theta_1)} \right)^{1/(\theta_2-\theta_1)}, \tag{3.59}$$

which is denoted as $\breve{\varphi}$ in Gerber and Shiu (2003). Further, we can determine $\tilde{u}$ and $\tilde{v}$ in terms of θ_1 and θ_2. We obtain

$$\frac{\tilde{v}}{K}=\left(\frac{-\theta_1}{1-\theta_1}\right)^{-\theta_1/(\theta_2-\theta_1)}\left(\frac{\theta_2}{\theta_2-1}\right)^{\theta_2/(\theta_2-\theta_1)}, \tag{3.60}$$

which is denoted as $\breve{c}$ in Gerber and Shiu (2003), and

$$\frac{\tilde{u}}{K}=\frac{\tilde{v}}{K}\frac{\tilde{u}}{\tilde{v}}=\left(\frac{-\theta_1}{1-\theta_1}\right)^{(1-\theta_1)/(\theta_2-\theta_1)}\left(\frac{\theta_2}{\theta_2-1}\right)^{(\theta_2-1)/(\theta_2-\theta_1)}, \tag{3.61}$$

which is denoted as $\breve{b}$ in Gerber and Shiu (2003). Thus, with $\tilde{u} < K < \tilde{v}$, for the perpetual American option whose payoff is given by (3.33), its price is

$$\begin{cases} K & \text{if } s \le \tilde{u} \\ \Pi(\tilde{u})A(s;\tilde{u},\tilde{v})+\Pi(\tilde{v})B(s;\tilde{u},\tilde{v}) & \text{if } 0<\tilde{u}<s<\tilde{v}, \\ s & \text{if } s \ge \tilde{v} \end{cases} \tag{3.62}$$

or

$$\begin{cases} K & \text{if } s \le \tilde{u} \\ K\dfrac{\theta_2(s/\tilde{u})^{\theta_1}-\theta_1(s/\tilde{u})^{\theta_2}}{\theta_2-\theta_1} & \text{if } 0<\tilde{u}<s<\tilde{v}, \\ s & \text{if } s \ge \tilde{v} \end{cases} \tag{3.63}$$

which is shown in Figure 3.8. For detail derivation of $\tilde{u}$ and $\tilde{v}$, refer to Yu (2003b).

3.5 Numerical Examples

As shown in the previous sections, by assuming the logarithm of the stock price as a geometric Brownian motion, we have derived explicit formulas for pricing perpetual American put options, call options and perpetual maximum options on one stock. This section presents some numerical examples for such options according to the pricing formulas (3.16), (3.32) and (3.63) respectively.

Under the assumption of geometric Brownian motion, only one parameter, i.e. the variance σ^2, has to be estimated. Thus the Brownian motion is simpler than other processes with several parameters. This is the major reason for its popularity. Here, we consider a certain stock with S(0) =100, σ = 0.1, r = 0.1 and ζ = 0.02. The strike prices K ranges between 80 and 120.

Let us consider the perpetual American put option first. It follows from (3.8) that the negative root is $\theta_1 = -16.232$. By applying formulas (3.14) and (3.16), the three columns on the left side of Table 3.1 shows the values of optimal exercise boundary and the prices of the perpetual American put option for various strike prices K. It is obvious that both the value of the optimal exercise boundary and the option price increase as the strike price increases. It is also interesting to explore the option prices corresponding to different values of σ. Now, suppose that the strike price is fixed, e.g. K = 80, and σ varies from 0.1 to 0.3. The values of S(0), r and ζ remain 100, 0.1 and 0.02 respectively. The variations of θ_1, $\tilde{L}$ and the option prices for different σ are shown in the three columns on the right side of Table 3.1. It is observed that the option price increases as σ increases.

Similarly, we can obtain Table 3.2 for the perpetual American call option. From the table, it is easy to observe variations of the value of the optimal exercise boundary and the price of the perpetual American put option corresponding to different strike prices or σ. By comparing Table 3.1 with Table 3.2, we perceive that both the prices of the perpetual American put options and call options are directly proportional to the value of σ. However, the price of perpetual American call options is inversely proportional to the strike price which is opposite for perpetual American put options.

For the perpetual maximum options on one stock, Table 3.3 is constructed to illustrate how the strike price or σ affects the price of the perpetual maximum option on one stock. We assume that S(0) =100, $\sigma = 0.1$, r = 0.1, $\zeta = 0.02$ and the strike price ranges between 80 and 120 for the left four columns in Table 3.3. On the other hand, it is assumed that the strike price is fixed to be 80, σ varies from 0.1 to 0.3, S(0) = 100, r = 0.1 and $\zeta = 0.02$ for the remaining columns in Table 3.3.

Table 3.1 The Perpetual American Put Option

Strike Price K	Optimal Exercise Boundary $\tilde{L}$	Option Price		σ	Negative Root θ_1	Optimal Exercise Boundary $\tilde{L}$	Option Price
80	75.36	0.05		0.100	-16.23	75.36	0.05
85	80.07	0.13		0.125	-10.46	73.02	0.26
90	84.78	0.36		0.150	-7.32	70.39	0.73
95	89.49	0.91		0.175	-5.43	67.55	1.48
100	94.20	2.20		0.200	-4.19	64.59	2.47
105	98.91	5.10		0.225	-3.34	61.58	3.64
110	103.62	10.00		0.250	-2.73	58.56	4.97
115	108.33	15.00		0.275	-2.28	55.59	6.41
120	113.04	20.00		0.300	-1.93	52.69	7.93

Table 3.2 The Perpetual American Call Option

Strike Price K	Optimal Exercise Boundary $\tilde{U}$	Option Price
80	424.64	58.02
85	451.18	57.21
90	477.72	56.45
95	504.26	55.75
100	530.80	55.09
105	557.34	54.47
110	583.88	53.88
115	610.42	53.33
120	636.96	52.81

σ	Positive Root θ_2	Optimal Exercise Boundary $\tilde{U}$	Option Price
0.100	1.23	424.64	58.02
0.125	1.22	438.23	58.77
0.150	1.21	454.61	59.63
0.175	1.20	473.70	60.59
0.200	1.19	495.41	61.61
0.225	1.18	519.67	62.69
0.250	1.17	546.44	63.79
0.275	1.16	575.66	64.91
0.300	1.15	607.31	66.04

Table 3.3 The Perpetual Maximum Option on One Stock

guarantee price K	optimal exercise boundary		option price
	$\tilde{u}$	$\tilde{v}$	
80	78.56	93.97	100.00
85	83.47	99.84	100.00
90	88.38	105.72	98.26
95	93.29	111.59	98.36
100	98.20	117.46	100.30
105	103.11	123.34	105.00
110	108.02	129.21	110.00
115	112.93	135.08	115.00
120	117.84	140.96	120.00

σ	optimal exercise boundary		option price
	$\tilde{u}$	$\tilde{v}$	
0.100	78.56	93.97	100.00
0.125	76.96	97.39	100.00
0.150	74.87	99.94	100.00
0.175	72.37	101.54	99.14
0.200	69.57	102.24	99.88
0.225	66.57	102.17	100.97
0.250	63.47	101.49	102.30
0.275	60.34	100.35	103.81
0.300	57.25	98.89	100.00

Chapter 4

Pricing Perpetual American Options for Jump Processes

Chapter 3 has considered the classical model where the logarithm of the stock price is a Brownian motion. Another model will be introduced here. This chapter considers two models in which the logarithm of the stock price is a shifted compound Poisson process. A great advantage of the Poisson process is its simple sample paths with which it is easy to track the process. Actually, a limiting case of the Poisson process is a Brownian motion. This is shown in Section 4.4.

For the Poisson process, in an infinitesimal time interval, only two possibilities can happen: there is a jump or no jump. For a jump, it could be upward or downward. In the first model, all jumps of the stock price are upwards; and all jumps are downwards in the second model. A model where the stock price has jumps was suggested and discussed by Merton (1975). Gerber and Shiu (1998) have derived some pricing formulas for some perpetual options for jump processes. Recently, Kou (2002) proposed a jump-diffusion model for option pricing, in which the logarithm of the asset price is assumed to follow a Brownian motion plus a compound Poisson process with jump sizes double exponentially distributed, to incorporate the "volatility simile" and to strike a balance between reality and tractability.

This chapter is arranged as follows. Section 4.1 briefly presents the assumptions and the problem to be solved. Sections 4.2 and 4.3 discuss the pricing of perpetual American put options with upward jumps and downward jumps respectively. Finally, Section 4.4 illustrates the limiting cases of those two models discussed in Sections 4.2 and 4.3.

4.1 Assumptions and the Problem

Let S(t) be the price of a stock at time t. We assume that the market is risk-neutral, the stock does not pay any dividends and the risk-free force of interest is a positive constant r. Thus, we assume that the stochastic process $\{e^{-rt}S(t); t \geq 0\}$ is a martingale.

Now, let us define X(t) by

$$S(t) = S(0)\, e^{X(t)}, \quad t \geq 0.$$

We suppose that $\{X(t)\}$ is a process with stationary and independent increments. In the previous chapter, it is assumed that the process $\{X(t),\ t \geq 0\}$ is a Wiener process with instantaneous variance σ^2 and drift parameter $\mu = r - \frac{\sigma^2}{2}$. In this chapter, the assumption is that

$$X(t) = X(0) - ct + Y(t), \tag{4.1}$$

in which the jumps are upward, and

$$X(t) = X(0) + ct - Y(t), \tag{4.2}$$

in which the jumps are downward. Here, c is a positive constant, and $\{Y(t)\}$ is a compound Poisson process with parameter $\lambda > 0$ (expected number of jumps per unit time) and jump amount distribution P(z), $z \geq 0$, where $p(z) = \frac{d}{dz}P(z)$ is the probability density function of jump amounts.

The compound Poisson process assumption means that

$$Y(t) = \sum_{j=1}^{N(t)} Z_j = Z_1 + Z_2 + Z_3 + \cdots + Z_{N(t)},$$

where {N(t)} is a Poisson process with parameter λ, $\{Z_1, Z_2, Z_3, \dots\}$ are independent and identically distributed random variables. {Z} and N(t) are also assumed to be independent of each other.

Our goal is to find the price of a perpetual American put option. Consider an American put option with exercise price K, K < S(0) = s. If the option is exercised at time t, its payoff is given by (3.1). As demonstrated in Section 3.2, an exercise strategy of a perpetual American put option is to exercise the option as soon as the stock price falls to a level L for the first time, where L is a positive constant exercise boundary. For $0 < L < K$ and $L < S(0)$, let us consider the stopping time T_L given by (3.2). The problem is to find the value of the exercise strategy T_L,

$$V(s;L) = E\left[e^{-rT_L}\, \Pi(S(T_L)) \,\middle|\, S(0) = s\right], \qquad s \geq L, \tag{4.3}$$

and to determine the optimal exercise boundary $\tilde{L}$ which maximizes V(s; L). Thus, the price of the option is

$$\begin{cases} V(s;\tilde{L}) & \text{if } s \geq \tilde{L} \\ \Pi(s) & \text{if } s < \tilde{L} \end{cases}.$$

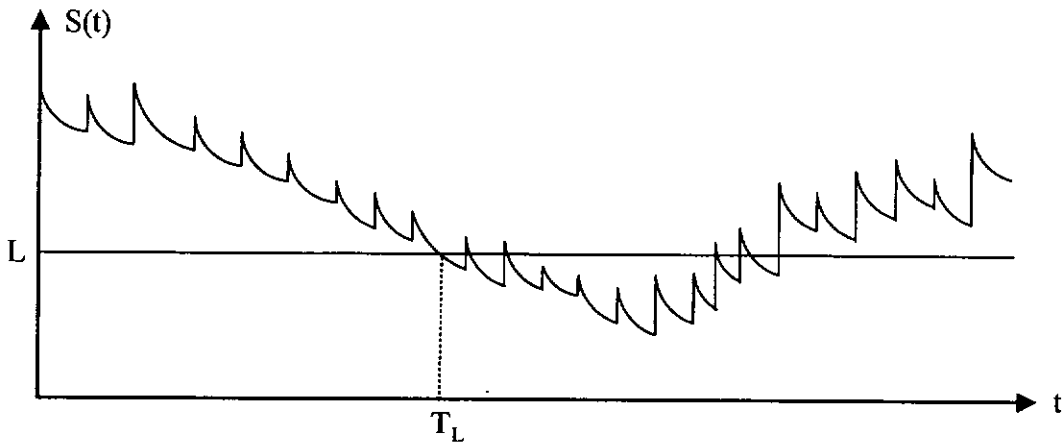
4.2 Perpetual American Put Options with Upward Jumps

For a model with upward jumps defined by (4.1), it is obvious that the sample paths of the stock process $\{S(t)\}$ are free of downward jumps; so we have $S(T_L) = L$. See Figure 4.1 for the stopping time for a model with upward jumps. Hence, formula (4.3) can be simplified to

$$V(s;L) = E\left[e^{-rT_L} \,\middle|\, S(0) = s\right]\Pi(L), \quad s \geq L. \tag{4.4}$$

Now, it remains to find the value of the expectation $E\left[e^{-rT_L} \,\middle|\, S(0) = s\right]$.

Figure 4.1 The Stopping Time T_L for a Model with Upward Jumps



Let us consider the stochastic process $\left\{e^{-rt} S(t)^{\xi}\right\}_{t\geq 0}$. This process is a martingale if

$$E\left[e^{-rt} S(t)^{\xi}\right] = s^{\xi} e^{\xi X(0)}. \tag{4.5}$$

Since

$$e^{-rt} S(t)^{\xi} = e^{-rt} s^{\xi} e^{\xi X(t)} = e^{-rt} s^{\xi} e^{\xi X(0)} e^{\xi Y(t) - \xi ct},$$

the martingale condition (4.5) can be simplified to

$$e^{-rt-c\xi t}E\left[e^{\xi Y(t)}\right]=1. \tag{4.6}$$

Since

$$E\left[e^{\xi N(t)}\right]=\sum_{n=0}^{\infty}e^{\xi n}(\lambda t)^n e^{-\lambda t}/n!=e^{\lambda t e^{\xi}}e^{-\lambda t}=e^{\lambda t\left(e^{\xi}-1\right)},$$

and $\{Z\}$ and $N(t)$ are independent of each other, by the definition of $Y(t)$, we have

$$E\left[e^{\xi Y(t)}\right]=E\left[E\left(\exp\left(\xi\sum_{j=1}^{N(t)}Z_j\right)\middle|N(t)\right)\right]$$

$$=E\left[E\left[e^{\xi Z}\right]^{N(t)}\right]=e^{\lambda t\left[E\left[e^{\xi Z}\right]-1\right]}, \tag{4.7}$$

where

$$E\left[e^{\xi Z}\right]=\int_0^{\infty}e^{\xi z}p(z)\,dz.$$

It follows from (4.6) and (4.7) that

$$e^{-rt-c\xi t+\lambda t\left[E\left[e^{\xi Z}\right]-1\right]}=1,$$

or

$$\lambda\left[E\left[e^{\xi Z}\right]-1\right]-r-c\xi=0, \tag{4.8}$$

which is the same as (5) in Gerber and Shiu (1998) if $z = x$.

Let

$$f(\xi)=\lambda\left[E\left[e^{\xi Z}\right]-1\right]-r-c\xi. \tag{4.9}$$

Differentiate (4.9) with respect to ξ twice to obtain

$$f''(\xi)=\lambda E\left[Z^2 e^{\xi Z}\right],$$

which is positive. Thus, we can conclude that (4.9) is a convex function. There are at most two real solutions for equation (4.8). Since we have assumed that the stochastic process $\{e^{-rt}S(t); t \geq 0\}$ is a martingale, one solution of (4.8) is $\xi_1 = 1$. Because $f(0) = -r$ and $f(\xi) \to \infty$ as $\xi \to -\infty$, the second solution is negative, $\xi_2 = -R < 0$. For the negative root $-R$, the stochastic process $\{e^{-rt}S(t)^{-R}\}_{0 \leq t \leq T_L}$ is a bounded martingale. By the optional sampling theorem, we have

$$s^{-R} = E\left[e^{-rT_L} S(T_L)^{-R} \middle| S(0) = s\right] = E\left[e^{-rT_L} \middle| S(0) = s\right] L^{-R},$$

or

$$E\left[e^{-rT_L} \middle| S(0) = s\right] = \left(\frac{L}{s}\right)^R. \tag{4.10}$$

Thus, it follows from (4.4) and (4.10) that the value of the strategy T_L is

$$V(s;L) = \left(\frac{L}{s}\right)^R \Pi(L), \quad s \geq L, \tag{4.11}$$

which has the same form as (3.11).

Now, for the optimal exercise strategy, we seek $\tilde{L}$, the optimal value of L that maximizes V(s; L). It can be obtained by the first-order condition

$$0 = \left.\frac{\partial V(s;L)}{\partial L}\right|_{L=\tilde{L}} = \frac{R}{\tilde{L}}\left(\frac{\tilde{L}}{s}\right)^R \Pi(\tilde{L}) + \left(\frac{\tilde{L}}{s}\right)^R \Pi'(\tilde{L}),$$

or

$$0 = \frac{R}{\tilde{L}}\Pi(\tilde{L}) + \Pi'(\tilde{L}). \tag{4.12}$$

Note: from (4.11), we have

$$\left.\frac{\partial V(s;L)}{\partial s}\right|_{s=\tilde{L}} = \frac{-R}{\tilde{L}}\left(\frac{\tilde{L}}{s}\right)^{R+1}\Pi(\tilde{L})\Bigg|_{s=\tilde{L}} = \frac{-R}{\tilde{L}}\Pi(\tilde{L}). \tag{4.13}$$

We can see from (4.12) and (4.13) that

$$\left.\frac{\partial V(s;L)}{\partial s}\right|_{s=\tilde{L}} = \Pi'(\tilde{L}). \tag{4.14}$$

This is known as the ***smooth junction condition***. The functions $\Pi(s)$, $s \le \tilde{L}$, and $V(s;\tilde{L})$, $s \ge \tilde{L}$, have a smooth junction at the point $s = \tilde{L}$.

Since L < K, $\Pi(\tilde{L}) = \max(K - \tilde{L}, 0) = K - \tilde{L}$ and $\Pi'(\tilde{L}) = -1$, it follows from (4.12) that

$$\tilde{L} = K\frac{R}{1+R}. \tag{4.15}$$

Thus, the price of a perpetual American put option is

$$\begin{cases} \left(\frac{\tilde{L}}{s}\right)^{R}(K-\tilde{L}) & \text{if } s \ge \tilde{L} \\ K-s & \text{if } s < \tilde{L} \end{cases},$$

or

$$\begin{cases} \left(\frac{RK}{s(1+R)}\right)^{R}\left(\frac{K}{1+R}\right) & \text{if } s \ge \tilde{L} \\ K-s & \text{if } s < \tilde{L} \end{cases}. \tag{4.16}$$

Remark: To determine the negative root –R of (4.8), we need to know the probability density p(z). We suppose that $p(z) = \beta e^{-\beta z}$, z > 0. In this case, we must assumed that $\beta > 1$, otherwise E[S(t)] would be infinite. For this probability density, we have, for $\xi < \beta$,

$$E\left[e^{\xi Z}\right] = \int_0^\infty e^{\xi z} p(z)\,dz = \int_0^\infty e^{\xi z} \beta e^{-\beta z}\,dz = \frac{\beta}{\beta-\xi}. \tag{4.17}$$

It follows from (4.8) and (4.17) that

$$\lambda\left[\frac{\beta}{\beta-\xi}-1\right]-r-c\xi=0,$$

or

$$c\xi^2+(\lambda+r-\beta c)\xi-\beta r=0. \tag{4.18}$$

Since $\xi_1 = 1$ and $\xi_2 = -R$ are roots of (4.18) and $\xi_1\xi_2 = \frac{-\beta r}{c}$, we have

$$R=\frac{\beta r}{c}. \tag{4.19}$$

Thus, we can substitute (4.19) in the right-hand side of (4.15) to obtain

$$\tilde{L}=\frac{r}{r+c/\beta}K. \tag{4.20}$$

4.3 Perpetual American Put Options with Downward Jumps

Now, let us consider the model in which the assumption is

$$X(t) = X(0) + ct - Y(t).$$

For the stochastic process $\left\{e^{-rt}S(t)^{\xi}\right\}_{t\geq 0}$, the martingale condition is

$$E\left[e^{-rt}S(t)^{\xi}\right]=s^{\xi}e^{\xi X(0)},$$

or

$$e^{-rt+c\xi t}E\left[e^{-\xi Y(t)}\right]=1. \tag{4.21}$$

It follows from (4.7) that

$$E\left[e^{-\xi Y(t)}\right] = e^{\lambda t\left[E\left[e^{-\xi Z}\right]-1\right]}. \tag{4.22}$$

Thus, the martingale condition (4.21) can be simplified to

$$e^{-rt+c\xi t+\lambda t\left(E\left[e^{-\xi Z}\right]-1\right)} = 1,$$

or

$$\lambda\left(E\left[e^{-\xi Z}\right]-1\right) - r + c\xi = 0, \tag{4.23}$$

which is the same as (10) in Gerber and Shiu (1998) if z = x.

Similar to (4.8), there are at most two real roots for equation (4.23). Since $\left\{e^{-rt}S(t)^{\xi}\right\}_{t\geq 0}$ is a martingale, one root is $\xi_1 = 1$. And the other root, if it exists, is negative, $\xi_2 = -R$. For the negative root, the stochastic process $\left\{e^{-rt}S(t)^{-R}\right\}_{0\leq t<T_L}$ is a martingale bounded between 0 and L^{-R}. Applying the optional sampling theorem, we obtain

$$s^{-R} = E\left[e^{-rT_L}S(T_L)^{-R}\,\middle|\,S(0)=s\right]. \tag{4.24}$$

As $S(T_L) < L$ and the distribution of $S(T_L)$ is not known in general, the problem here is more complicated than that in the previous section. To get around the difficulty, we assume that the jump amounts have an exponential distribution, i.e., $p(z) = \beta e^{-\beta z}$ for $z > 0$. This distribution has the no-memory property. If the value of T_L and the stock price immediately before the jump occurring at time T_L, the property asserts that the conditional distribution of $\ln L - \ln S(T_L)$ has the same exponential distribution. It follows from (4.24) that

$$s^{-R} = E\left[e^{-rT_L} e^{-R\left[\left(\ln S(T_L) - \ln L\right) + \ln L\right]} \;\middle|\; S(0) = s\right]$$

$$= E\left[e^{-rT_L} e^{-R[-Z]} \;\middle|\; S(0) = s\right] L^{-R}$$

$$= E\left[e^{-rT_L} \;\middle|\; S(0) = s\right] E\left[e^{-R[-Z]}\right] L^{-R}. \tag{4.25}$$

Since the second expectation on the right-hand side of (4.25) is

$$E\left[e^{-R[-Z]}\right] = \int_0^\infty e^{Rz} \beta e^{-\beta z}\, dz = \frac{\beta}{\beta - R},$$

we can rearrange and simplify (4.25) to

$$E\left[e^{-rT_L} \;\middle|\; S(0) = s\right] = \left(\frac{L}{s}\right)^R \frac{\beta - R}{\beta}. \tag{4.26}$$

Thus, it follows from (4.3) and (4.26) that the value of the exercise strategy T_L is

$$V(s;L) = \left(\frac{L}{s}\right)^R \frac{\beta - R}{\beta} E\left[\Pi\left(Le^{-z}\right)\right], \quad s \ge L,$$

or

$$V(s;L) = \left(\frac{L}{s}\right)^R (\beta - R) \int_0^\infty \Pi\left(Le^{-z}\right) e^{-\beta z}\, dz, \quad s \ge L. \tag{4.27}$$

Now, the problem is to find the optimal value $\tilde{L}$ that maximizes V(s;L). The first-order condition is

$$0 = \frac{\partial}{\partial L} V(s;L) \bigg|_{L=\tilde{L}}$$

$$= \left(\frac{\tilde{L}}{s}\right)^R (\beta - R) \left[\frac{R}{\tilde{L}} \int_0^\infty \Pi\left(\tilde{L}e^{-z}\right) e^{-\beta z}\, dz + \int_0^\infty \Pi'\left(\tilde{L}e^{-z}\right) e^{-z} e^{-\beta z}\, dz\right]$$

by differentiating (4.27) with respect to L. The above is equivalent to

$$\frac{R}{\tilde{L}}\int_0^{\infty}\Pi\left(\tilde{L}e^{-z}\right)e^{-\beta z}\,dz = -\int_0^{\infty}\Pi'\left(\tilde{L}e^{-z}\right)e^{-z}e^{-\beta z}\,dz.$$

$$= -\frac{1}{\tilde{L}}\left[\Pi(\tilde{L}) - \beta\int_0^{\infty}\Pi\left(\tilde{L}e^{-z}\right)e^{-\beta z}\,dz\right] \qquad (4.28)$$

by integration by parts. Thus

$$\Pi(\tilde{L}) = (\beta - R)\int_0^{\infty}\Pi\left(\tilde{L}e^{-z}\right)e^{-\beta z}\,dz. \qquad (4.29)$$

It follows from (4.27) and (4.29) that

$$V(\tilde{L};\tilde{L}) = \Pi\left(\tilde{L}\right). \qquad (4.30)$$

This is known as the ***continuous junction condition***. It can be shown that conditions (4.28) and (4.30) are equivalent. Equation (4.30) implies that the functions $\Pi(s)$, $s \le \tilde{L}$, and $V(s;\tilde{L})$, $s \ge \tilde{L}$, match at the point $s = \tilde{L}$.

Since $L < K$ and $z > 0$, we have $\Pi(\tilde{L}) = \max(K - \tilde{L}, 0) = K - \tilde{L}$ and $\Pi(\tilde{L}e^{-z}) = K - \tilde{L}e^{-z}$. It follows from (4.29) that

$$(\beta - R)\int_0^{\infty}\left(K - \tilde{L}e^{-z}\right)e^{-\beta z}\,dz = K - \tilde{L}. \qquad (4.31)$$

The integral on the left-hand side of (4.31) can be simplified to

$$\int_0^{\infty}\left(K - \tilde{L}e^{-z}\right)e^{-\beta z}\,dz = \frac{K}{\beta} - \frac{\tilde{L}}{1+\beta}.$$

Thus, we can simplify and rearrange (4.31) to obtain

$$\tilde{L} = K\frac{R(1+\beta)}{\beta(1+R)}. \qquad (4.32)$$

Comparing (4.32) with (4.15), we can see that the expression of $\tilde{L}$ for a model with downward jumps is quite different from that for a model with upward jumps.

Now, let us determine the negative root $-R$ of equation (4.23). By the assumption of $p(z) = \beta e^{-\beta z}$ for $z > 0$, equation (4.23) can be simplified to

$$c\xi^2 - (\lambda + r - \beta c)\xi - \beta r = 0. \tag{4.33}$$

The product of the roots of the quadratic equation above is $-\beta r / c$. As $\xi_1 = 1$ and $\xi_2 = -R$ are two roots of equation (4.33), it follows that

$$R = \frac{\beta r}{c}. \tag{4.34}$$

By substituting (4.34) in (4.32), we obtain

$$\tilde{L} = K \frac{r}{r + \dfrac{c - r}{\beta + 1}}. \tag{4.35}$$

4.4 Limiting Cases

Based on the formulas obtained in Sections 4.2 and 4.3, we can obtain limiting cases for models with an exponential jump amount distribution. It is shown in this section that a Brownian motion can be obtained as a limit.

Let us consider the model as discussed in Section 4.2. Since $\xi_1 = 1$ is one root of the martingale condition (4.18), it follows that

$$c + (\lambda + r - \beta c) - \beta r = 0. \tag{4.36}$$

Here, we have three parameters c, λ and β. Our goal is to vary one parameter β, while the other two parameters are expressed by β, to obtain the limiting case.

As $\{Y(t)\}$ is a process with stationary and independent increments, we can define

$$\mathrm{Var}\left(Y(t)\right)=\sigma^2 t, \tag{4.37}$$

where σ^2 is the variance per unit time. On the other hand, it follows from (4.7) and (4.17) that the moment generating function of Y(t) is

$$E\left[e^{\xi Y(t)}\right]=\exp\left[\lambda t\left(\frac{\beta}{\beta-\xi}-1\right)\right], \tag{4.38}$$

from which we obtain

$$\mathrm{Var}\left(Y(t)\right)=\frac{2\lambda}{\beta^2}t. \tag{4.39}$$

Thus, we have

$$\frac{2\lambda}{\beta^2}\equiv\sigma^2. \tag{4.40}$$

Now, by (4.36) and (4.40), c and λ can be expressed by β as

$$\lambda=\frac{\beta^2\sigma^2}{2}$$

and

$$c=\frac{\beta^2\sigma^2}{2(\beta-1)}-r,$$

which leads to

$$\frac{c}{\beta}=\frac{\beta\sigma^2}{2(\beta-1)}-\frac{r}{\beta}. \tag{4.41}$$

In the limiting case as $\beta\to\infty$, it is observed from (4.41) that $\frac{c}{\beta}\to\frac{\sigma^2}{2}$. Thus, by (4.19) and (4.20), in the limit $\beta\to\infty$, we have $R=\frac{2r}{\sigma^2}$ and

$$\tilde{L} = \frac{r}{r+\sigma^2/2}K, \tag{4.42}$$

which is the same as the optimal exercise boundary for a perpetual American put option under geometric Brownian motion given by (3.20). Actually, as we increase the value of β, the value of λ is increasing. That is the frequency of the jumps of the Poisson process is increasing by which the Brownian motion will be obtained in the limit.

Similarly, we can show that the limiting case of the model in Section 4.3 is a Brownian motion in an appropriate manner. For the process $\{Y(t)\}$, we also have (4.40) here. On the other hand, for $\xi_1 = 1$, the martingale condition (4.33) becomes

$$c-(\lambda+r-\beta c)-\beta r = 0,$$

or

$$c-r = \frac{\lambda}{1+\beta}. \tag{4.43}$$

Now, by (4.40) and (4.42), we obtain

$$\frac{c-r}{1+\beta} = \frac{\sigma^2}{2}\left(\frac{\beta}{1+\beta}\right)^2, \tag{4.44}$$

by which in the limit $\beta \to \infty$, (4.35) becomes (4.42). Formula 4.42 can also be found in Merton (1973), Gerber and Landry (1998) and in some textbooks, for example Lamberton and Lapeyre (1996).

Now, let us illustrate by a numerical example. Suppose the initial stock price s = 100, r = 0.01, $\sigma = 0.1$ and $\zeta = 0$. Denote the models in Section 4.2 and Section 4.3 by Model I and Model II respectively. Table 4.1 shows $V(s;\tilde{L})$ in Model I and Model II for different values of β and K, and compares with $P(s;\tilde{L})$ under the geometric Brownian motion. From Table 4.1, we observe that the values of a perpetual put option in Model I are quite different from those in Model II when β is small. As β becomes larger and larger, the difference becomes smaller. At last when $\beta \to \infty$, the values of a perpetual put option in Model I and Model II are equal to those under a geometric Brownian motion.

Table 4.1 Prices of Some Perpetual Put Options

beta \ K	90		100		110	
	Model I	Model II	Model I	Model II	Model I	Model II
2	10.80	11.33	14.81	14.29	19.72	17.62
3	8.91	12.00	12.75	15.47	17.63	19.47
4	8.91	12.10	12.75	15.82	17.63	20.14
5	9.10	12.06	12.96	15.90	17.84	20.41
10	9.80	11.66	13.73	15.67	18.62	20.47
20	10.27	11.29	14.24	15.33	19.13	20.21
100	10.69	10.91	14.70	14.93	19.60	19.84
1000	10.79	10.81	14.80	14.83	19.71	19.73
10000	10.80	10.80	14.81	14.82	19.72	19.72
∞	10.800	10.800	14.815	14.815	19.719	19.719
Brownian motion	10.800		14.815		19.719	

PART 2 TWO-STOCK CASE

Chapter 5

Pricing Perpetual Options on Two Stocks

Starting from this chapter, we consider the pricing of perpetual options on two stocks in the rest of this thesis. For a perpetual option, we assume that its payoff function, given by $\Pi(z_1, z_2)$, is homogeneous of degree one. Thus, we have

$$\Pi(z_1, z_2) = z_2 \Pi\left(\frac{z_1}{z_2}, 1\right) \qquad \text{for } z_1, z_2 > 0. \tag{5.1}$$

Here are some examples. For a Maximum option, its payoff is

$$\Pi(z_1, z_2) = \max(z_1, z_2),$$

the pricing of which will be illustrated in Section 5.3;

$$\Pi(z_1, z_2) = (z_1 - z_2)_+,$$

the payoff function for a Margrabe option which is discussed in Section 6.1;

$$\Pi(z_1, z_2) = |z_1 - z_2|,$$

which is the payoff function for a symmetric Margrabe option; and

$$\Pi(z_1, z_2) = \min\left[(z_1 - z_2)_+, kz_2\right],$$

the payoff function for a Margrabe option with proportional cap which is analyzed in Section 6.2.

First of all, Section 5.1 discusses some classical assumptions for option pricing in two-stock case. Under the assumptions, Section 5.2 discusses a general idea of the optimal exercise strategy for perpetual options with homogeneous payoff functions, which provides a broad base for the pricing of perpetual options on two stocks in the following chapters. As an example, Section 5.3 derives a pricing formula for the perpetual maximum option on two stocks. Finally, Section 5.4 illustrates how dividend paying on stocks affects the price of the perpetual maximum option on two stocks.

5.1 Some Classical Assumptions

First of all, let us consider the classical assumption in which the stock price process is assumed to be a geometric Brownian motion. Let $S_i(t)$ be the price of stock i at time t, i = 1, 2. For i = 1, 2, define $X_i(t)$ by

$$S_i(t) = S_i(0)\, e^{X_i(t)}, \quad t \geq 0.$$

We assume that the process $\{X_1(t), X_2(t), t \geq 0\}$ is a bivariate Brownian motion with instantaneous variance σ_1^2, σ_2^2, drift parameter μ_1, μ_2, and correlation coefficient ρ.

Let r be the risk-free force of interest, and ζ_i be the constant dividend yield rate of stock i for i = 1, 2. It is assumed that r, ζ_1 and ζ_2 are positive constants. Dividends of amount $\zeta_i S_i(t)dt$ are paid for stock i between time t and time t + dt. We assume that the stochastic process $\left\{e^{-rt}e^{\zeta_i t}S_i(t); t \geq 0\right\}$ is a martingale. The martingale condition is

$$E^*\left[e^{-rt}e^{\zeta_i t}S_i(t)\right] = e^{-r(0)+\zeta_i(0)}S_i(0) \tag{5.2}$$

or

$$E^*\left[e^{-rt+\zeta_i t+X_i(t)}\right]=e^0.$$

Thus, we have

$$\left[-r+\zeta_i+(1)\mu_i^*+\frac{1}{2}(1^2)\sigma_i^2\right]t=0, \tag{5.3}$$

i.e.,

$$\mu_i^*=r-\zeta_i-\frac{\sigma_i^2}{2}, \tag{5.4}$$

for i = 1, 2. Here, the asterisk signifies that the expectation is taken with respect to the risk-neutral probability measure. Under the risk-neutral measure, $\{X_i(t)\}$ is a Wiener process with drift parameter μ_i^* which is given by (5.4). The diffusion parameter of $\{X_i(t)\}$ remains σ_i under the risk-neutral measure.

5.2 The Optimal Exercise Strategy

Under the non-arbitrage assumption, the price of a perpetual option is the supremum over all stopping time T of

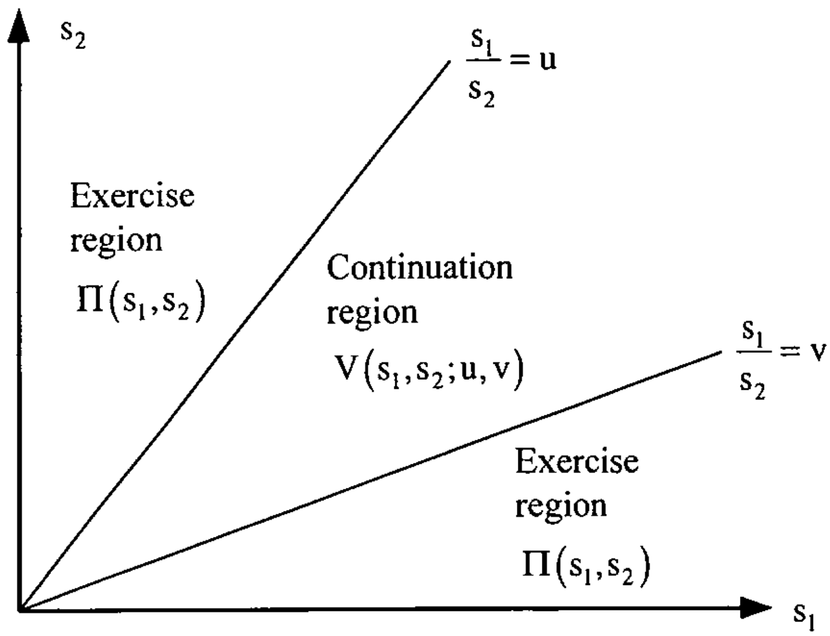
$$E\left[e^{-rT}\Pi\left(S_1(T),S_2(T)\right)\right],$$

which is the expected discounted value of the payoff, i.e. $\Pi\left(S_1(t),S_2(t)\right)$, for stopping time T. Let us denote the price as $V(s_1, s_2)$, where $s_1 = S_1(0) > 0$ and $s_2 = S_2(0) > 0$. As pointed out in Gerber and Shiu (1996), the optimal continuation region (non-exercise region) can be considered as a union of infinite sectors in the first quadrant of the form

$$\{s_1 > 0, s_2 > 0 \mid \tilde{u} < s_1/s_2 < \tilde{v}\}.$$

Here, we consider a single sector which is illustrated in Figure 5.1.

Figure 5.1 Continuation Region and Exercise Region for a Perpetual Option on Two Stocks



Let us consider an exercise strategy (a stopping time) of the form

$$T_{u,v} = \min\left\{ t \,\middle|\, \frac{S_1(t)}{S_2(t)} = u \;\text{ or }\; \frac{S_1(t)}{S_2(t)} = v \right\} \tag{5.5}$$

for $0 < u < s_1/s_2 < v$. The value of this exercise strategy is

$$V(s_1, s_2; u, v) = E^*\left[e^{-rT_{u,v}} \Pi\left(S_1(T_{u,v}), S_2(T_{u,v})\right)\right], \quad u < s_1/s_2 < v. \tag{5.6}$$

It follows from (5.1) and (5.6) that

$$V(s_1,s_2;u,v)=E^*\left[e^{-rT_{u,v}}S_2(T_{u,v})\Pi\left(\frac{S_1(T_{u,v})}{S_2(T_{u,v})},1\right)\right], \tag{5.7}$$

which is a homogeneous function of degree one with respect to the variables s_1 and s_2. Thus, by defining

$$\pi(z)=\Pi(z,1),$$

we can express formula (5.6) as

$$V(s_1,s_2;u,v)=\pi(u)A(s_1,s_2;u,v)+\pi(v)B(s_1,s_2;u,v) \tag{5.8}$$

where

$$A(s_1,s_2;u,v)=E^*\left[e^{-rT_{u,v}}S_2(T_{u,v})\,I\left(\frac{S_1(T_{u,v})}{S_2(T_{u,v})}=u\right)\right] \tag{5.9}$$

and

$$B(s_1,s_2;u,v)=E^*\left[e^{-rT_{u,v}}S_2(T_{u,v})\,I\left(\frac{S_1(T_{u,v})}{S_2(T_{u,v})}=v\right)\right]. \tag{5.10}$$

Now, our goal is to determine the expectations $A(s_1,s_2;u,v)$ and $B(s_1,s_2;u,v)$ in (5.8). These two expectations can be evaluated by considering two appropriate martingales.

Consider the stochastic process $\left\{e^{-rt}S_2(t)\left(\frac{S_1(t)}{S_2(t)}\right)^{\theta}\right\}_{t\geq 0}$. Under the risk-neutral measure, this process is a martingale if

$$e^{-rt}E^*\left[e^{(1-\theta)X_2(t)+\theta X_1(t)}\right]=1$$

or

$$-r+\theta\mu_1^*+(1-\theta)\mu_2^*+\frac{1}{2}\theta^2\sigma_1^2+\frac{1}{2}(1-\theta)^2\sigma_2^2+\theta(1-\theta)\rho\sigma_1\sigma_2=0, \tag{5.11}$$

where μ_1^* and μ_2^* are given by (5.4). It can be shown that (5.11) is a quadratic equation of θ and the left-hand side of (5.11) is a convex function of θ. We observe that this convex function is less than zero for $\theta=0$ or $\theta=1$. Thus, the quadratic equation has one root $\theta_1<0$ and another root $\theta_2>1$. For these two roots $\theta=\theta_1$ and $\theta=\theta_2$, the stochastic process

$$\left\{e^{-rt}S_2(t)\left(\frac{S_1(t)}{S_2(t)}\right)^{\theta}\right\}_{0\le t\le T_{u,v}}, \qquad \text{with } u < s_1/s_2 < v,$$

is a bounded martingale. By the optional sampling theorem, we obtain

$$1=E^*\left[e^{-rT_{u,v}+X_2(T_{u,v})+\theta\left(X_1(T_{u,v})-X_2(T_{u,v})\right)}\right]$$

$$=E^*\left[\left(I\left(\frac{S_1(T_{u,v})}{S_2(T_{u,v})}=u\right)+I\left(\frac{S_1(T_{u,v})}{S_2(T_{u,v})}=v\right)\right)e^{-rT_{u,v}+X_2(T_{u,v})+\theta\left(X_1(T_{u,v})-X_2(T_{u,v})\right)}\right]$$

$$=E^*\left[I\left(\frac{S_1(T_{u,v})}{S_2(T_{u,v})}=u\right)e^{-rT_{u,v}+X_2(T_{u,v})}\left(\frac{us_2}{s_1}\right)^{\theta}\right]+E^*\left[I\left(\frac{S_1(T_{u,v})}{S_2(T_{u,v})}=v\right)e^{-rT_{u,v}+X_2(T_{u,v})}\left(\frac{vs_2}{s_1}\right)^{\theta}\right] \tag{5.12}$$

It follows from (5.9), (5.10) and (5.12) that

$$\left(\frac{us_2}{s_1}\right)^{\theta}\frac{A(s_1,s_2;u,v)}{s_2}+\left(\frac{vs_2}{s_1}\right)^{\theta}\frac{B(s_1,s_2;u,v)}{s_2}=1. \tag{5.13}$$

Since there are two roots $\theta=\theta_1$ and $\theta=\theta_2$, we obtain two formulas

$$\left(\frac{us_2}{s_1}\right)^{\theta_1}\frac{A(s_1,s_2;u,v)}{s_2}+\left(\frac{vs_2}{s_1}\right)^{\theta_1}\frac{B(s_1,s_2;u,v)}{s_2}=1 \tag{5.14}$$

and

$$\left(\frac{us_2}{s_1}\right)^{\theta_2}\frac{A(s_1,s_2;u,v)}{s_2}+\left(\frac{vs_2}{s_1}\right)^{\theta_2}\frac{B(s_1,s_2;u,v)}{s_2}=1. \tag{5.15}$$

Thus, $A(s_1,s_2;u,v)$ and $B(s_1,s_2;u,v)$ can be solved from (5.14) and (5.15), and be expressed as

$$A(s_1,s_2;u,v)=\frac{s_2v^{\theta_2}\left(\frac{s_1}{s_2}\right)^{\theta_1}-s_2v^{\theta_1}\left(\frac{s_1}{s_2}\right)^{\theta_2}}{u^{\theta_1}v^{\theta_2}-u^{\theta_2}v^{\theta_1}} \tag{5.16}$$

and

$$B(s_1,s_2;u,v)=\frac{s_2u^{\theta_1}\left(\frac{s_1}{s_2}\right)^{\theta_2}-s_2u^{\theta_2}\left(\frac{s_1}{s_2}\right)^{\theta_1}}{u^{\theta_1}v^{\theta_2}-u^{\theta_2}v^{\theta_1}}, \tag{5.17}$$

respectively. The solution above can be written as a matrix equation

$$\begin{pmatrix}A(s_1,s_2;u,v)\\B(s_1,s_2;u,v)\end{pmatrix}=\begin{pmatrix}u^{\theta_1} & v^{\theta_1}\\u^{\theta_2} & v^{\theta_2}\end{pmatrix}^{-1}\begin{pmatrix}s_1\left(\frac{s_1}{s_2}\right)^{\theta_1}\\s_1\left(\frac{s_1}{s_2}\right)^{\theta_2}\end{pmatrix}.$$

For $0 < u < 1 < v$, substitute expressions (5.16) and (5.17) in the right-hand side of (5.8) to obtain

$$V(s_1,s_2;u,v)=\pi(u)\frac{s_2v^{\theta_2}\left(\frac{s_1}{s_2}\right)^{\theta_1}-s_2v^{\theta_1}\left(\frac{s_1}{s_2}\right)^{\theta_2}}{u^{\theta_1}v^{\theta_2}-u^{\theta_2}v^{\theta_1}}+\pi(v)\frac{s_2u^{\theta_1}\left(\frac{s_1}{s_2}\right)^{\theta_2}-s_2u^{\theta_2}\left(\frac{s_1}{s_2}\right)^{\theta_1}}{u^{\theta_1}v^{\theta_2}-u^{\theta_2}v^{\theta_1}} \tag{5.18}$$

or

$$V(s_1,s_2;u,v)=\begin{pmatrix}\pi(u) & \pi(v)\end{pmatrix}\begin{pmatrix}u^{\theta_1} & v^{\theta_1}\\ u^{\theta_2} & v^{\theta_2}\end{pmatrix}^{-1}\begin{pmatrix}s_2\left(\dfrac{s_1}{s_2}\right)^{\theta_1}\\ s_2\left(\dfrac{s_1}{s_2}\right)^{\theta_2}\end{pmatrix}, \tag{5.19}$$

which is a formula for the value of the exercise strategy.

Now, our problem is simplified to finding the maximum value of $V(s_1,s_2;u,v)$ in (5.18), i.e. the price of the perpetual option on two stocks. We are to find $\tilde{u}$ and $\tilde{v}$, the optimal values of u and v respectively, that maximize $V(s_1,s_2;u,v)$. One method is by the first-order condition

$$\left.\frac{\partial V(s_1,s_2;u,v)}{\partial u}\right|_{u=\tilde{u},\,v=\tilde{v}}=0, \tag{5.20}$$

$$\left.\frac{\partial V(s_1,s_2;u,v)}{\partial v}\right|_{u=\tilde{u},\,v=\tilde{v}}=0. \tag{5.21}$$

The values of $\tilde{u}$ and $\tilde{v}$ can be determined by solving a system of equations (5.20) and (5.21). However, it is observed from (5.18) that the function $V(s_1,s_2;u,v)$ is complicated and it is difficult to find its partial derivatives with respect to u and v. Let us consider another method by using the ***high contact*** or ***smooth pasting conditions***:

$$V_{s_1}(\tilde{u}s_2,s_2;\tilde{u},\tilde{v})=\Pi_{s_1}(\tilde{u}s_2,s_2), \tag{5.22}$$

$$V_{s_2}(\tilde{u}s_2,s_2;\tilde{u},\tilde{v})=\Pi_{s_2}(\tilde{u}s_2,s_2), \tag{5.23}$$

$$V_{s_1}(\tilde{v}s_2,s_2;\tilde{u},\tilde{v})=\Pi_{s_1}(\tilde{v}s_2,s_2) \tag{5.24}$$

and

$$V_{s_2}(\tilde{v}s_2,s_2;\tilde{u},\tilde{v})=\Pi_{s_2}(\tilde{v}s_2,s_2). \tag{5.25}$$

The subscripts of s_1 or s_2 signify partial differentiations with respect to s_1 or s_2. Since there are two variables to be determined, only two equations are required here. Thus, we can choose any two equations from four smooth pasting conditions above.

5.3 Pricing Perpetual Maximum Options on Two Stocks

In Section 3.4, we have derived a formula for the price of the perpetual maximum option on one stock. Now, let us consider a more general case, the perpetual maximum option on two stocks. Its payoff function is

$$\Pi(z_1, z_2) = \max(z_1, z_2), \quad z_1, z_2 \geq 0.$$

Such a payoff function $\Pi(z_1, z_2)$ is homogeneous of degree one. Thus, we have

$$\Pi(z_1, z_2) = z_2 \max\left\{\frac{z_1}{z_2}, 1\right\}. \tag{5.26}$$

Our goal is to derive an explicit formula for the price of the perpetual maximum option on two stocks, $S_1(t)$ and $S_2(t)$.

Following from Section 5.2, the value of the exercise strategy of a perpetual maximum option can be expressed as (5.18) or (5.19). Here, to obtain the price of the perpetual maximum option, conditions (5.22) and (5.25) will be used since the right-hand sides of (5.22) and (5.25) are equal to zero in the case of the maximum option. Apply (5.19) to conditions (5.22) and (5.25) and combine these two conditions to get

$$\begin{pmatrix}\pi(\tilde{u}) & \pi(\tilde{v})\end{pmatrix}\begin{pmatrix}\tilde{u}^{\theta_1} & \tilde{v}^{\theta_1} \\ \tilde{u}^{\theta_2} & \tilde{v}^{\theta_2}\end{pmatrix}^{-1}\begin{pmatrix}\theta_1\tilde{u}^{\theta_1-1} & (1-\theta_1)\tilde{v}^{\theta_1} \\ \theta_2\tilde{u}^{\theta_2-1} & (1-\theta_2)\tilde{v}^{\theta_2}\end{pmatrix} = \begin{pmatrix}0 & 0\end{pmatrix}. \tag{5.27}$$

Since $\tilde{u}$ and $\tilde{v}$ are nonzero and the matrix $\begin{pmatrix} \tilde{u}^{\theta_1} & \tilde{v}^{\theta_1} \\ \tilde{u}^{\theta_2} & \tilde{v}^{\theta_2} \end{pmatrix}^{-1}$ is non-singular, the

determinant of the matrix $\begin{pmatrix} \theta_1 \tilde{u}^{\theta_1 - 1} & (1-\theta_1)\tilde{v}^{\theta_1} \\ \theta_2 \tilde{u}^{\theta_2 - 1} & (1-\theta_2)\tilde{v}^{\theta_2} \end{pmatrix}$ must be zero, i.e.

$$\theta_1(1-\theta_2)\tilde{u}^{\theta_1-1}\tilde{v}^{\theta_2} - \theta_2(1-\theta_1)\tilde{u}^{\theta_2-1}\tilde{v}^{\theta_1} = 0.$$

Rearrange the equation above to obtain a relationship between $\tilde{u}$ and $\tilde{v}$,

$$\frac{\tilde{u}}{\tilde{v}} = \left(\frac{\theta_1(1-\theta_2)}{\theta_2(1-\theta_1)} \right)^{1/(\theta_2-\theta_1)}. \tag{5.28}$$

Since $\pi(\tilde{u}) = 1$ and $\pi(\tilde{v}) = \tilde{v}$, for condition (5.25), we have

$$\begin{pmatrix} 1 & \tilde{v} \end{pmatrix} \begin{pmatrix} \tilde{u}^{\theta_1} & \tilde{v}^{\theta_1} \\ \tilde{u}^{\theta_2} & \tilde{v}^{\theta_2} \end{pmatrix}^{-1} \begin{pmatrix} (1-\theta_1)\tilde{v}^{\theta_1} \\ (1-\theta_2)\tilde{v}^{\theta_2} \end{pmatrix} = 0,$$

which can be rewritten as

$$\begin{pmatrix} 1 & \tilde{v} \end{pmatrix} \begin{pmatrix} (\tilde{u}/\tilde{v})^{\theta_1} & 1 \\ (\tilde{u}/\tilde{v})^{\theta_2} & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1-\theta_1 \\ 1-\theta_2 \end{pmatrix} = 0. \tag{5.29}$$

By substituting (5.28) into (5.29) and simplifying, we can obtain an expression for $\tilde{v}$, i.e.

$$\tilde{v} = \left(\frac{-\theta_1}{1-\theta_1} \right)^{-\theta_1/(\theta_2-\theta_1)} \left(\frac{\theta_2}{\theta_2-1} \right)^{\theta_2/(\theta_2-\theta_1)}. \tag{5.30}$$

Hence, the expression for $\tilde{u}$ can be obtained by replacing $\tilde{v}$ in (5.28) with the right-hand side of (5.30) and simplifying,

$$\tilde{u} = \left(\frac{-\theta_1}{1-\theta_1} \right)^{(1-\theta_1)/(\theta_2-\theta_1)} \left(\frac{\theta_2}{\theta_2-1} \right)^{(\theta_2-1)/(\theta_2-\theta_1)}. \tag{5.31}$$

For $0 < \tilde{u} < 1 < \tilde{v}$, the price of the perpetual maximum options on two stocks is

$$\begin{cases} s_2 & \text{if } s_1/s_2 \le \tilde{u} \\ V(s_1,s_2;\tilde{u},\tilde{v}) & \text{if } \tilde{u} < s_1/s_2 < \tilde{v}, \\ s_1 & \text{if } s_1/s_2 \ge \tilde{v} \end{cases} \tag{5.32}$$

where $\tilde{u}$ and $\tilde{v}$ are given by (5.30) and (5.31) respectively.

If $\tilde{u} < s_1/s_2 < \tilde{v}$, by (5.18), the price can be rewritten as

$$V(s_1,s_2;\tilde{u},\tilde{v}) = \frac{s_2\left(\frac{s_1}{\tilde{v}s_2}\right)^{\theta_1} - s_2\left(\frac{s_1}{\tilde{v}s_2}\right)^{\theta_2}}{\left(\frac{\tilde{u}}{\tilde{v}}\right)^{\theta_1} - \left(\frac{\tilde{u}}{\tilde{v}}\right)^{\theta_2}} + \tilde{v}\frac{s_2\left(\frac{\tilde{u}}{\tilde{v}}\right)^{\theta_1}\left(\frac{s_1}{\tilde{v}s_2}\right)^{\theta_2} - s_2\left(\frac{\tilde{u}}{\tilde{v}}\right)^{\theta_2}\left(\frac{s_1}{\tilde{v}s_2}\right)^{\theta_1}}{\left(\frac{\tilde{u}}{\tilde{v}}\right)^{\theta_1} - \left(\frac{\tilde{u}}{\tilde{v}}\right)^{\theta_2}},$$

which can be simplified to

$$V(s_1,s_2;\tilde{u},\tilde{v}) = \frac{s_2\left[(\theta_2-1)\left(\frac{s_1}{\tilde{v}s_2}\right)^{\theta_1} + (1-\theta_1)\left(\frac{s_1}{\tilde{v}s_2}\right)^{\theta_2}\right]}{(\theta_2-1)(1-\theta_1)\left[\left(\frac{\tilde{u}}{\tilde{v}}\right)^{\theta_1} - \left(\frac{\tilde{u}}{\tilde{v}}\right)^{\theta_2}\right]} \tag{5.33}$$

by (5.28) and (5.30). By substituting with the right-hand side of (5.28) and rearranging, the denominator of (5.33) can be simplified to

$$(\theta_2-1)(1-\theta_1)\left[\left(\frac{\tilde{u}}{\tilde{v}}\right)^{\theta_1} - \left(\frac{\tilde{u}}{\tilde{v}}\right)^{\theta_2}\right] = (\theta_2-\theta_1)\left(\frac{-\theta_1}{1-\theta_1}\right)^{\theta_1/(\theta_2-\theta_1)}\left(\frac{\theta_2}{\theta_2-1}\right)^{-\theta_2/(\theta_2-\theta_1)}$$

or

$$(\theta_2-1)(1-\theta_1)\left[\left(\frac{\tilde{u}}{\tilde{v}}\right)^{\theta_1} - \left(\frac{\tilde{u}}{\tilde{v}}\right)^{\theta_2}\right] = \frac{\theta_2-\theta_1}{\tilde{v}}. \tag{5.34}$$

by (5.30). Thus, it follows from (5.33) and (5.34) that

$$V(s_1,s_2;\tilde{u},\tilde{v})=\frac{s_2\tilde{v}}{\theta_2-\theta_1}\left[(\theta_2-1)\left(\frac{s_1}{\tilde{v}s_2}\right)^{\theta_1}+(1-\theta_1)\left(\frac{s_1}{\tilde{v}s_2}\right)^{\theta_2}\right]. \quad (5.35)$$

As shown in Gerber and Shiu (1996), four equivalent formulas for the price of the perpetual maximum option have been derived. Formula (5.35) is the same as formula (8.15) in Gerber and Shiu (1996) if $\tilde{v}$, θ_1 and θ_2 are replaced by $\tilde{b}$, θ_0 and θ_1 respectively. By (5.30) and (5.31), formula (5.35) can be rearranged and simplified to

$$V(s_1,s_2;\tilde{u},\tilde{v})=\frac{s_2}{\theta_2-\theta_1}\left[\theta_2\left(\frac{s_1}{\tilde{u}s_2}\right)^{\theta_1}-\theta_1\left(\frac{s_1}{\tilde{u}s_2}\right)^{\theta_2}\right], \quad (5.36)$$

which is the simplest among four equivalent formulas shown in Section 8 of Gerber and Shiu (1996).

5.4 Some Special Cases of Dividend Paying

With the definition of μ_1^* and μ_2^*, quadratic equation (5.11) can be rewritten as

$$a\theta^2+b\theta+c=0,$$

two roots of (5.11), $\theta_1<0$ and $\theta_2>1$, can be expressed as

$$\theta_1=\frac{-b-\sqrt{b^2-4ac}}{2a} \quad (5.37)$$

and

$$\theta_2=\frac{-b+\sqrt{b^2-4ac}}{2a}, \quad (5.38)$$

where

$$a = \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2 - \rho\sigma_1\sigma_2 = \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2\right) = \frac{1}{2}\mathrm{Var}\left[X_1(1) - X_2(1)\right],$$

$$b = \mu_1^* - \mu_2^* - \sigma_2^2 + \rho\sigma_1\sigma_2 = \zeta_2 - \zeta_1 - a$$

and

$$c = -r + \mu_2^* + \frac{1}{2}\sigma_2^2 = -\zeta_2.$$

It is assumed that $\zeta_1 > 0$ and $\zeta_2 > 0$ in the classical assumption. Now, let us see what happen if this assumption is relaxed. Some extreme cases are considered here.

<u>Case I</u>: Consider $\zeta_1 > 0$ and $\zeta_2 = 0$. Here, stock 2 pays no dividends. We observe that $b = -\zeta_1 - a$ and $c = 0$. It follows from (5.37) and (5.38) that

$$\theta_1 = 0 \tag{5.39}$$

and

$$\theta_2 = \frac{-b}{a} = 1 + \frac{\zeta_1}{a}. \tag{5.40}$$

For $\theta_1 = 0$, the stochastic process $\left\{e^{-rt} S_2(t)\left(\frac{S_1(t)}{S_2(t)}\right)^{\theta}\right\}_{t \geq 0}$ readily becomes $\left\{e^{-rt} S_2(t)\right\}_{t \geq 0}$ which is a martingale as $\zeta_2 = 0$. It follows from (5.30), (5.31), (5.39) and (5.40) that

$$\tilde{u} = 0 \tag{5.41}$$

and

$$\tilde{v} \xrightarrow[\text{as } \zeta_2 \to 0]{} \frac{\theta_2}{\theta_2 - 1} = 1 + \frac{a}{\zeta_1}, \tag{5.42}$$

which means that as ζ_2 tends to 0, $\tilde{v}$ tends to $1+\frac{a}{\zeta_1}$. By the definition of the stopping time given by (5.6), for the boundary $\tilde{u}=0$, the perpetual maximum option will never exercise since the price ratio of stock 1 to stock 2 will never fall to the level 0. Now, by (5.35), (5.39), (5.40) and (5.42), it is observed that

$$V(s_1, s_2; \tilde{u}, \tilde{v}) = \frac{s_2 \tilde{v}}{\theta_2}\left[(\theta_2 - 1) + \left(\frac{s_1}{\tilde{v} s_2}\right)^{\theta_2}\right]$$

$$\xrightarrow[\text{as } \zeta_2 \to 0]{} \frac{s_2}{\theta_2 - 1}\left[(\theta_2 - 1) + \left(\frac{(\theta_2 - 1) s_1}{\theta_2 s_2}\right)^{1+\zeta_1/a}\right]$$

$$\xrightarrow[\text{as } \zeta_2 \to 0]{} s_2 + \frac{a s_1}{\zeta_1 + a}\left(\frac{\zeta_1 s_1}{(\zeta_1 + a) s_2}\right)^{\zeta_1/a} \tag{5.43}$$

<u>Case II</u>: Consider $\zeta_1 = 0$ and $\zeta_2 > 0$, i.e. stock 1 pays no dividends. Now, we get $b = \zeta_2 - a$, $\theta_1 = -\frac{\zeta_2}{a}$ and $\theta_2 = 1$. By (5.31), we have

$$\tilde{u} \xrightarrow[\text{as } \zeta_1 \to 0]{} \frac{-\theta_1}{1 - \theta_1} = \frac{\zeta_2}{\zeta_2 + a}.$$

Hence, by (5.36), we obtain

$$V(s_1, s_2; \tilde{u}, \tilde{v}) = \frac{s_2}{1 - \theta_1}\left[\left(\frac{s_1}{\tilde{u} s_2}\right)^{\theta_1} - \theta_1\left(\frac{s_1}{\tilde{u} s_2}\right)\right]$$

$$\xrightarrow[\text{as } \zeta_1 \to 0]{} s_1 + \frac{a s_2}{\zeta_2 + a}\left(\frac{(\zeta_2 + a) s_1}{\zeta_2 s_2}\right)^{-\zeta_2/a} . \qquad (5.44)$$

It is observed that as ζ_1 tends to 0, $\tilde{v}$ tends to ∞; which implies that the perpetual maximum option will never exercise.

Case III: Let us consider the case that both stock 1 and stock 2 pay no dividends, i.e. $\zeta_1 = 0$ and $\zeta_2 = 0$. Following from (5.37) and (5.38), it is obvious that $\theta_1 = 0$ and $\theta_2 = 1$. By (5.28), we obtain $\frac{\tilde{u}}{\tilde{v}} = 0$. And it follows from (5.35), (5.39), (5.40) and (5.42) that

$$V(s_1, s_2; \tilde{u}, \tilde{v}) \xrightarrow[\text{as } \zeta_2 \to 0]{} \frac{s_2 \tilde{v}}{\theta_2}\left[(\theta_2 - 1) + \left(\frac{s_1}{\tilde{v} s_2}\right)^{\theta_2}\right]$$

$$\xrightarrow[\text{as } \zeta_2 \to 0]{} s_2 + \frac{s_2}{\theta_2 - 1}\left(\frac{(\theta_2 - 1) s_1}{\theta_2 s_2}\right)^{\theta_2}$$

$$\xrightarrow[\text{as } \zeta_2 \to 0,\, \zeta_1 \to 0]{} s_2 + s_1 . \qquad (5.45)$$

In this case, the perpetual maximum option will never be exercised.

Case IV: Consider the case that $\zeta_1 = \zeta_2$. Since b = – a, it follows from (5.37) and (5.38) that $\theta_1 + \theta_2 = 1$. By (5.30) and (5.31), $\tilde{u}$ and $\tilde{v}$ can be simplified to

$$\tilde{v} = \left(\frac{-\theta_1}{1 - \theta_1}\right)^{-1/(1 - 2\theta_1)} \qquad (5.46)$$

and

$$\tilde{u} = \left(\frac{-\theta_1}{1-\theta_1}\right)^{1/(1-2\theta_1)} = \frac{1}{\tilde{v}}. \tag{5.47}$$

Thus, it follows from (5.35) and (5.46) that

$$V(s_1, s_2; \tilde{u}, \tilde{v}) = \frac{s_2}{1-2\theta_1}\left[(-\theta_1)^{-\theta_1}(1-\theta_1)^{1-\theta_1}\right]^{1/(1-2\theta_1)}\left[\left(\frac{s_1}{s_2}\right)^{\theta_1} + \left(\frac{s_1}{s_2}\right)^{1-\theta_1}\right]. \tag{5.48}$$

Now, let us illustrate by a numerical example for Case I and Case II. Suppose that the initial prices of stock 1 and stock 2 are 100 and 95 respectively; $r = 0.1$, $\sigma_1 = 0.2$, $\sigma_2 = 0.1$ and $\rho = 0.5$. We want to check the limits (5.42) and (5.43) in Case I and (5.44) in Case II. Tables 5.1 and 5.2 show $V(s_1, s_2; \tilde{u}, \tilde{v})$ for different values of ζ_2 and ζ_1 respectively, and compare with the limits in Case I and Case II.

We observe from Table 5.1 and Table 5.2 that $V(s_1, s_2; \tilde{u}, \tilde{v})$ increases when either ζ_2 or ζ_1 decreases. In Table 5.1, as ζ_2 is very small and approaching 0, the values of $\tilde{v}$ and $V(s_1, s_2; \tilde{u}, \tilde{v})$ are approaching those calculated by (5.42) and (5.43) in the last row. Similarly, when ζ_1 is very small and approaching 0, the value of $V(s_1, s_2; \tilde{u}, \tilde{v})$ is approaching that calculated by (5.44) in the last row of Table 5.2.

Table 5.1 Price of the Perpetual Maximum Option for Different Values of ζ_2

ζ_1	ζ_2	θ_1	θ_2	$\tilde{u}$	$\tilde{v}$	$V(s_1,s_2;\tilde{u},\tilde{v})$
0.03	0.02	-0.591	2.257	0.745	1.295	104.420
0.03	0.015	-0.414	2.414	0.707	1.319	105.122
0.03	0.01	-0.257	2.591	0.652	1.350	106.097
0.03	0.005	-0.120	2.786	0.555	1.397	107.623
0.03	0.001	-0.023	2.956	0.354	1.464	110.009
0.03	0.0005	-0.011	2.978	0.286	1.478	110.558
0.03	0.00001	0.000	3.000	0.079	1.499	111.380
0.03	0.0000001	0.000	3.000	0.017	1.500	111.415
In the limiting case as ζ_2 tends to 0:						
0.03	0	0.000	3.000	0.000	1.500	111.415

Table 5.2 Price of the Perpetual Maximum Option for Different Values of ζ_1

ζ_1	ζ_2	θ_1	θ_2	$\tilde{u}$	$\tilde{v}$	$V(s_1,s_2;\tilde{u},\tilde{v})$
0.03	0.02	-0.591	2.257	0.745	1.295	104.420
0.025	0.02	-0.667	2.000	0.731	1.337	105.085
0.02	0.02	-0.758	1.758	0.716	1.397	105.929
0.01	0.02	-1.000	1.333	0.673	1.641	108.632
0.005	0.02	-1.155	1.155	0.639	2.000	111.189
0.0005	0.02	-1.314	1.014	0.585	4.636	116.406
0.000001	0.02	-1.333	1.000	0.571	64.364	118.021
0.00000001	0.02	-1.333	1.000	0.571	463.151	118.030
In the limiting case as ζ_1 tends to 0:						
0	0.02	-1.333	1.000	0.571	∞	118.030

Chapter 6

Pricing Perpetual Margrabe Options

In 1978, Margrabe (1978) extended the Black-Scholes theory of option pricing and derived a closed-form formula for the price of a European option to exchange one asset for another driven by geometric Brownian motion. Such option is called Margrabe option or exchange option. The Margrabe option was originated from currency option which gives the holder the right to exchange one currency for another at a prearranged exchange rate on a specific date. Magrabe options have many applications in many fields, e.g. banks, fund companies and insurance companies. Consider a life insurance company holding a fixed-rate asset has agreed to pay a floating-rate interest to annuity policyholders. The company may use a Margrabe option to convert its floating-rate liability to fixed-rate liability, which can protect it from an increase in the floating rate without changing the cost for its asset. The main advantage of these options is that they allow investors more efficient transfer of risk and greater flexibility in managing portfolio.

The payoff function of a Margrabe option is

$$\Pi(z_1, z_2) = (z_1 - z_2)_+ = \max(z_1 - z_2, 0), \quad z_1, z_2 \geq 0, \tag{6.1}$$

which can be rewritten as

$$\Pi(z_1, z_2) = \max(z_1, z_2) - z_2, \tag{6.2}$$

where $\max(z_1, z_2)$ is the payoff function of a maximum option. Thus, under the same assumptions and with the same maturity date, the price of a European Margrabe option is equal to the price of a European maximum option minus the expected discounted value of stock 2. However, this relationship is not true for their American counterparts, except for the case that stock 2 pays no dividends. It is discussed in details later in this chapter.

6.1 Standard Perpetual Margrabe Options

In this section, our goal is to derive a closed-form formula for the price of a perpetual Margrabe option, i.e. American Margrabe option without expiry date. A key for the problem is to determine the stopping time which is equivalent to determine the optimal exercise boundary. For two stocks $S_1(t)$ and $S_2(t)$, by assuming the homogeneity of the payoff function given by (6.1), we have

$$\Pi\left(S_1(t),S_2(t)\right)=S_2(t)\max\left(\frac{S_1(t)}{S_2(t)}-1,0\right). \tag{6.3}$$

Now, it is sufficient to consider the stopping time of the form

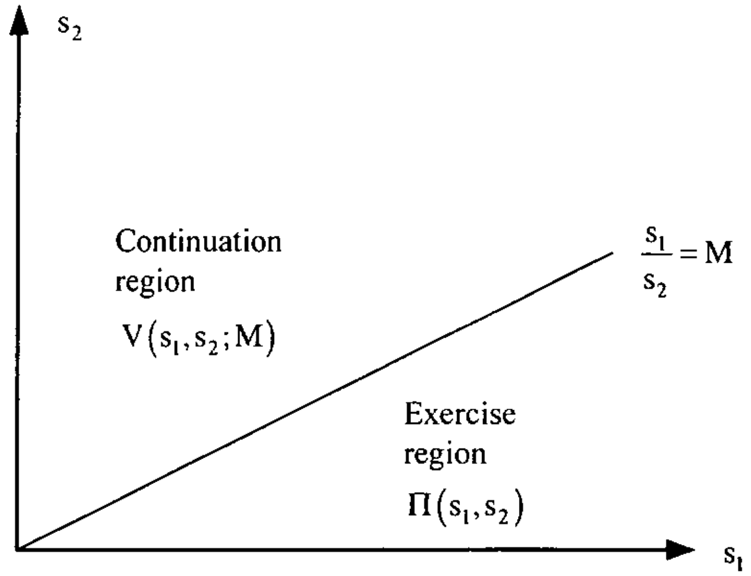
$$T_M=\min\left\{t\ \middle|\ \frac{S_1(t)}{S_2(t)}=M\right\}, \tag{6.4}$$

where $M>1$ and $\frac{S_1(0)}{S_2(0)}=\frac{s_1}{s_2}<M$. The exercise strategy is to exercise the option as soon as the ratio of the prices of stock 1 to stock 2 rises to a level M for the first time. Thus, the value of this exercise strategy T_M is

$$V(s_1,s_2;M)=E^*\left[e^{-rT_M}\,\Pi(S_1(T_M),S_2(T_M))\right], \tag{6.5}$$

where r is the constant risk-free force of interest. See Figure 6.1.

Figure 6.1 Continuation Region and Exercise Region for a Perpetual Margrabe Option



Since $\frac{S_1(T_M)}{S_2(T_M)} = M > 1$, it follows from (6.3) and (6.5) that

$$\Pi\left(S_1(T_M), S_2(T_M)\right) = S_2(T_M)\max\left(\frac{S_1(T_M)}{S_2(T_M)} - 1, 0\right) = (M-1)S_2(T_M) \qquad (6.6)$$

and

$$V(s_1, s_2; M) = s_2(M-1)E^*\left[e^{-rT_M + X_2(T_M)}\right]. \qquad (6.7)$$

Thus, the problem is simplified to finding the value of the expectation $E^*\left[e^{-rT_M + X_2(T_M)}\right]$.

Let us consider a stochastic process $\left\{e^{-rt}S_2(t)\left(\frac{S_1(t)}{S_2(t)}\right)^{\theta}\right\}_{t\geq 0}$. If this process is a martingale, the martingale condition is given by (5.11). There are two real roots $\theta_1 < 0$ and $\theta_2 > 1$ for the quadratic equation (5.11). For the positive root θ_2, the stochastic process $\left\{e^{-rt}S_2(t)\left(\frac{S_1(t)}{S_2(t)}\right)^{\theta_2}\right\}_{0\leq t\leq T_M}$ is a martingale. Note that $\lim_{t\to\infty} e^{-rt}S_2(t) = 0$ almost surely, and $\left(\frac{S_1(t)}{S_2(t)}\right)^{\theta_2}$ is bounded above by M^{θ_2} for $t < T_M$. (See page 308 of Gerber and Shiu (1996).) Thus, we can apply the optional sampling theorem to obtain

$$1 = E^*\left[e^{-rT_M + X_2(T_M) + \theta_2(X_1(T_M) - X_2(T_M))}\right] = E^*\left[e^{-rT_M + X_2(T_M)}\left(\frac{Ms_2}{s_1}\right)^{\theta_2}\right]$$

or

$$E^*\left[e^{-rT_M + X_2(T_M)}\right] = \left(\frac{s_1}{Ms_2}\right)^{\theta_2}. \tag{6.8}$$

It follows from (6.7) and (6.8) that the value of the exercise strategy T_M is

$$V(s_1, s_2; M) = s_2(M-1)\left(\frac{s_1}{Ms_2}\right)^{\theta_2}. \tag{6.9}$$

For the optimal exercise strategy, the price of the perpetual Margrabe option is the maximum value of (6.9). Let us seek $\tilde{M}$ which is the optimal value of M that maximizes $V(s_1, s_2; M)$. By the first-order condition, we have

$$\left.\frac{\partial V(s_1,s_2;M)}{\partial M}\right|_{M=\tilde{M}}=0$$

or

$$\tilde{M}=\frac{\theta_2}{\theta_2-1}. \tag{6.10}$$

Thus, for $\frac{s_1}{s_2}<\tilde{M}$, it follows from (6.9) and (6.10) that the price of the perpetual Margrabe option is

$$V(s_1,s_2;\tilde{M})=\left(\frac{s_1}{\theta_2}\right)^{\theta_2}\left(\frac{\theta_2-1}{s_2}\right)^{\theta_2-1}. \tag{6.11}$$

Now, let us compare the price of a perpetual Margrabe option with that of a perpetual maximum option under the same assumptions, i.e., compare formula (6.11) with (5.35). The difference between the prices of these two perpetual options can be obtained by subtraction, i.e.,

$$V(s_1,s_2;\tilde{u},\tilde{v})-V(s_1,s_2;\tilde{M})$$

$$=\frac{s_2\tilde{v}}{\theta_2-\theta_1}\left[(\theta_2-1)\left(\frac{s_1}{\tilde{v}s_2}\right)^{\theta_1}+(1-\theta_1)\left(\frac{s_1}{\tilde{v}s_2}\right)^{\theta_2}\right]-\left(\frac{s_1}{\theta_2}\right)^{\theta_2}\left(\frac{\theta_2-1}{s_2}\right)^{\theta_2-1}. \tag{6.12}$$

The above formula is a bit complicated. A simpler form can be obtained in the special case that stock 2 pays no dividends, i.e. $\zeta_2=0$. In this case, we have

$$\theta_2=1+\frac{\zeta_1}{a}$$

where

$$a=\frac{1}{2}\left(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2\right).$$

By (6.11), we have

$$V(s_1,s_2;\tilde{M})=\left(\frac{as_1}{a+\zeta_1}\right)^{1+\zeta_1/a}\left(\frac{\zeta_1}{as_2}\right)^{\zeta_1/a}$$

or

$$V(s_1,s_2;\tilde{M})=\frac{as_1}{\zeta_1+a}\left(\frac{\zeta_1 s_1}{(\zeta_1+a)s_2}\right)^{\zeta_1/a}. \tag{6.13}$$

Thus, it follows from (5.43) and (6.13) that the difference between the price of a perpetual Margrabe option and that of a perpetual maximum option is

$$V(s_1,s_2;\tilde{u},\tilde{v})-V(s_1,s_2;\tilde{M})=s_2, \tag{6.14}$$

which is the expected discounted value of stock 2, i.e. the initial price of stock 2.

Note that the call and put options on one stock are special cases of the Margrabe option. Considering the payoff function of a Margrabe option, if $S_2(t)$ is replaced by a constant K in the payoff function, we obtain

$$\Pi(S_1(t))=(S_1(t)-K)_+,$$

which is the payoff function of a call option on one stock. One the other hand, if we replace $S_1(t)$ by a constant K in the payoff function, we have

$$\Pi(S_2(t))=(K-S_2(t))_+,$$

which is the payoff function of a put option on one stock.

Now, let us illustrate by a numerical example. Suppose that the initial prices of stock 1 and stock 2 are 100 and 95 respectively; r = 0.1, $\sigma_1=0.2$, $\sigma_2=0.1$ and $\rho=0.5$. Table 6.1 shows the value of $V(s_1,s_2;\tilde{M})$ for different values of ζ_2. By comparing the last row of Table 5.1 with that of Table 6.1, we observe that as ζ_2

tends to 0, the difference between the prices of the perpetual maximum option and the perpetual Margrabe option is 95 which is exactly the initial price of stock 2.

Table 6.1 Price of the Perpetual Margrabe Option for Different Values of ζ_2

ζ_1	ζ_2	θ_2	$\tilde{M}$	$V(s_1,s_2;\tilde{M})$
0.03	0.02	2.257	1.795	22.640
0.03	0.015	2.414	1.707	20.906
0.03	0.01	2.591	1.629	19.278
0.03	0.005	2.786	1.560	17.778
0.03	0.001	2.956	1.511	16.677
0.03	0.0005	2.978	1.506	16.545
0.03	0.00001	3.000	1.500	16.418
0.03	0.0000001	3.000	1.500	16.415
In the limiting case as ζ_2 tends to 0:				
0.03	0	3.000	1.500	16.415

6.1.1 Alternative Derivations for the Pricing Formula

Alternatively, the optimal exercise boundary and the pricing formula for the perpetual Margrabe option can be obtained by two other methods. One alternative method is following the optimal exercise strategy in Section 5.2. The value of the exercise strategy to exercise the perpetual Margrabe option in two boundary levels can also be expressed as (5.18) or (5.19). Since we have $\pi(u)=0$ and $\pi(v)=v-1$, it follows from (5.18) that

$$V(s_1,s_2;u,v)=(v-1)\frac{s_2u^{\theta_1}\left(\frac{s_1}{s_2}\right)^{\theta_2}-s_2u^{\theta_2}\left(\frac{s_1}{s_2}\right)^{\theta_1}}{u^{\theta_1}v^{\theta_2}-u^{\theta_2}v^{\theta_1}}$$

$$=(v-1)\frac{s_2\left(\frac{s_1}{s_2}\right)^{\theta_2}-s_2u^{\theta_2-\theta_1}\left(\frac{s_1}{s_2}\right)^{\theta_1}}{v^{\theta_2}-u^{\theta_2-\theta_1}v^{\theta_1}}. \qquad (6.15)$$

Note that the lower optimal exercise boundary is $\tilde{u}=0$. Since $\theta_1<0$ and $\theta_2>1$ are two roots of (5.11), it follows from (6.15) that

$$V(s_1,s_2;0,v)=(v-1)\frac{s_2\left(\frac{s_1}{s_2}\right)^{\theta_2}}{v^{\theta_2}}, \qquad (6.16)$$

which is a function of v. By the first-order condition $\left.\frac{\partial V(s_1,s_2;0,v)}{\partial v}\right|_{v=\tilde{v}}=0$, we can obtain an expression for the optimal value of v that maximizes $V(s_1,s_2;0,v)$ as

$$\tilde{v}=\frac{\theta_2}{\theta_2-1}, \qquad (6.17)$$

which is the same as $\tilde{M}$ in (6.10). Hence, by (6.16) and (6.17), we can obtain an expression for the price of the perpetual Margrabe option $V(s_1,s_2;0,\tilde{v})$ which is the same as $V(s_1,s_2;\tilde{M})$ in (6.11).

Another alternative method is discussed in the following. Let us rewrite the payoff function of the standard Margrabe option as

$$\Pi(S_1(t),S_2(t))=S_1(t)\max\left(1-\frac{S_2(t)}{S_1(t)},0\right). \qquad (6.18)$$

We consider the stopping time of the form

$$T_N = \min\left\{ t \;\middle|\; \frac{S_2(t)}{S_1(t)} = N \right\}, \tag{6.19}$$

where $0 < N < 1$. Now, the exercise strategy is to exercise the option as soon as the price ratio of stock 2 to stock 1 falls to a level N for the first time. The value of this exercise strategy can be expressed as (6.5) where M is replaced by N.

Since $\frac{S_2(T_N)}{S_1(T_N)} = N < 1$, we have

$$\Pi\big(S_1(T_N), S_2(T_N)\big) = (1-N)S_1(T_N), \tag{6.20}$$

and the value of the exercise strategy T_N is

$$V(s_1, s_2; N) = s_1(1-N)E^*\left[e^{-rT_N + X_1(T_N)}\right]. \tag{6.21}$$

The expectation on the right-hand side of (6.21) is to be determined.

Consider a martingale of the form $\left\{ e^{-rt} S_1(t)\left(\frac{S_2(t)}{S_1(t)}\right)^{\xi} \right\}_{t\geq 0}$. The martingale condition is

$$-r + \xi\mu_2^* + (1-\xi)\mu_1^* + \frac{1}{2}\xi^2\sigma_2^2 + \frac{1}{2}(1-\xi)^2\sigma_1^2 + \xi(1-\xi)\rho\sigma_1\sigma_2 = 0, \tag{6.22}$$

which is the same as (5.11) if $\xi = 1-\theta$. Since there are two roots for (5.11), i.e., $\theta_1 < 0$ and $\theta_2 > 1$; there are also two roots for (6.21), $\xi_1 = 1-\theta_1 > 1$ and $\xi_2 = 1-\theta_2 < 0$. For the negative root ξ_2, the stochastic process $\left\{ e^{-rt} S_1(t)\left(\frac{S_2(t)}{S_1(t)}\right)^{\xi_2} \right\}_{0\leq t\leq T_N}$ is a martingale. Since $\lim_{t\to\infty} e^{-rt} S_1(t) = 0$ almost surely, and $\left(\frac{S_2(t)}{S_1(t)}\right)^{\xi_2}$ is bounded above by N^{ξ_2} for $t < T_N$, the optional sampling theorem can be applied to obtain

$$1 = E^*\left[e^{-rT_N + X_1(T_N) + \xi_2\left(X_2(T_N) - X_1(T_N)\right)}\right] = E^*\left[e^{-rT_N + X_1(T_N)}\left(\frac{N s_1}{s_2}\right)^{\xi_2}\right]$$

or

$$E^*\left[e^{-rT_N + X_1(T_N)}\right] = \left(\frac{s_2}{N s_1}\right)^{\xi_2}. \tag{6.23}$$

Substituting (6.23) in the right-hand side of (6.21) yields

$$V(s_1, s_2; N) = s_1(1-N)\left(\frac{s_2}{N s_1}\right)^{\xi_2}. \tag{6.24}$$

Now, it is ready to find the price of the perpetual Margrabe option which is the maximum value of (6.24). $V(s_1, s_2; N)$ can be viewed as a function of N. Hence, the price can be obtained by determining the optimal value of N, denoted by $\tilde{N}$, which maximizes $V(s_1, s_2; N)$. By the first-order condition, we have

$$\tilde{N} = \frac{-\xi_2}{1-\xi_2}. \tag{6.25}$$

Thus, for $\frac{s_1}{s_2} > \tilde{N}$, by (6.24) and (6.25), the price of the perpetual Margrabe option is

$$V(s_1, s_2; \tilde{N}) = \left(\frac{s_1}{1-\xi_2}\right)^{1-\xi_2}\left(\frac{-\xi_2}{s_2}\right)^{-\xi_2} \tag{6.26}$$

which is the same as (6.11) since $\xi_2 = 1 - \theta_2$.

6.2 Perpetual Margrabe Options with Proportional Cap

As discussed at the beginning of this chapter, Margrabe options have many uses. They are common in financial markets. Based on the standard Margrabe option, we introduce an option with a payoff function of the form

$$\Pi(z_1, z_2) = \min\left[(z_1 - z_2)_+, kz_2\right], \quad z_1, z_2 > 0, \tag{6.27}$$

where $k > 0$ is a constant and $(z_1 - z_2)_+$ is the payoff function of a standard Margrabe option. This option is called a Margrabe option with proportional cap kz_2. The option with cap is designed to limit the payment amount. Obviously, the payoff of this option is always less or equal to the payoff of a standard Margrabe option.

For a capped Margrabe option on two stocks $S_1(t)$ and $S_2(t)$, its price is the supremum of

$$E\left[e^{-rT}\Pi(S_1(T), S_2(T))\right]$$

over all stopping time T. It is crucial to determine the stopping time T which is equivalent to determine the exercise boundary of the option. Once we have determined the optimal exercise boundary, it is straightforward to obtain the price of the option. Our goal is to derive an explicit formula for the price of a perpetual Margrabe option on two stocks $S_1(t)$ and $S_2(t)$ with proportional cap $kS_2(t)$.

According to Broadie and Detemple (1997), the immediate exercise boundary of an American Margrabe option with proportional cap $kS_2(t)$ is the minimum of the exercise boundary of an uncapped Margrabe option and $1 + k$. This is shown later. Broadie and Detemple (1997) also represented the value of a capped American Margrabe option in terms of the value of its uncapped counterpart and the payoff at the cap. However, they had not derived a closed-form solution for the optimal exercise boundary of a capped American Margrabe option since the optimal exercise boundary of an uncapped American Margrabe option has not been determined. Although the exercise boundary of an American Margrabe option is difficult to determine, however the exercise boundary of a perpetual Margrabe option is comparably easy to observe. The optimal exercise boundary of a perpetual Margrabe option is constant through time. Section 6.1 has derived a closed-form formula for the optimal exercise boundary of a perpetual Margrabe option. Thus, the optimal exercise boundary of a perpetual Margrabe option with proportional cap can also be obtained.

Since the homogeneity of the payoff function $\Pi(z_1,z_2)$ given by (6.27), it can be rewritten as

$$\Pi(z_1,z_2)=\min\left[z_2\max\left(\frac{z_1}{z_2}-1,0\right),\ kz_2\right]$$

$$=z_2\left\{\min\left[\max\left(\frac{z_1}{z_2},1\right),1+k\right]-1\right\}. \tag{6.28}$$

Let us consider the payoff function $\Pi(S_1(t),S_2(t))$ of a Margrabe option with proportional cap $kS_2(t)$. It is sufficient to consider the stopping time of the form

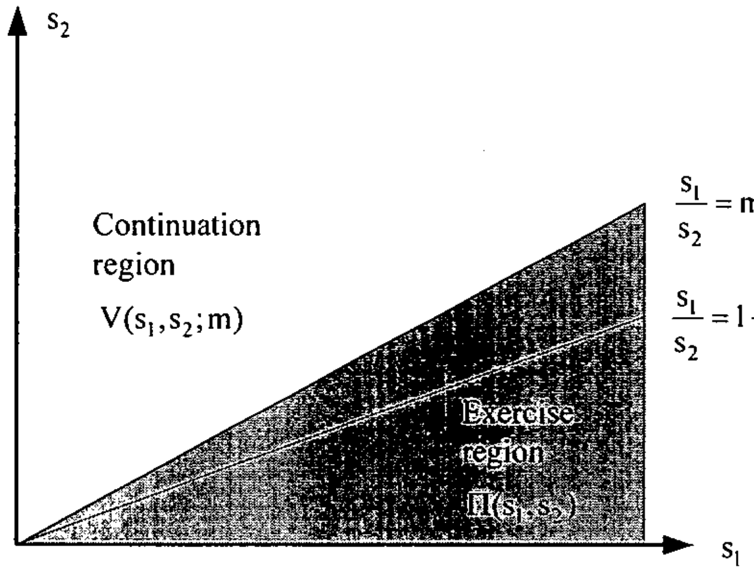
$$T_m=\min\left\{t\ \middle|\ \frac{S_1(t)}{S_2(t)}=m\right\} \tag{6.29}$$

for $\frac{S_1(0)}{S_2(0)}=\frac{s_1}{s_2}<m$. The exercise strategy is to exercise the option as soon as the stock price ratio rises to a level m for the first time. As shown in Figure 6.2, for

$m < 1+k$, the shaded region is the exercise region. The value of exercise strategy T_m is

$$V(s_1,s_2;m)=E^*\left[e^{-rT_m}\Pi\left(S_1\left(T_m\right),S_2\left(T_m\right)\right)\right]. \tag{6.30}$$

Figure 6.2 Continuation Region and Exercise Region for a Perpetual Margrabe Option with Proportional Cap



It is observed from (6.28) that m should be larger than one; otherwise the payoff of the option will be zero. Since $\frac{S_1(T_m)}{S_2(T_m)}=m>1$, it follows from (6.28) and (6.30) that

$$\Pi(S_1(T_m),S_2(T_m))=\{\min[m,1+k]-1\}S_2(T_m) \tag{6.31}$$

and

$$V(s_1,s_2;m)=s_2\{\min[m,1+k]-1\}E^*\left[e^{-rT_m+X_2(T_m)}\right]. \tag{6.32}$$

Now, the problem is simplified to finding the value of the expectation $E^*\left[e^{-rT_m+X_2(T_m)}\right]$. Similar to the derivation of $E^*\left[e^{-rT_M+X_2(T_M)}\right]$ as shown in Section 6.1, we can obtain an expression for $E^*\left[e^{-rT_m+X_2(T_m)}\right]$ as

$$E^*\left[e^{-rT_m+X_2(T_m)}\right]=\left(\frac{s_1}{ms_2}\right)^{\theta_2}, \tag{6.33}$$

where θ_2 is the positive root of the quadratic equation (5.11). Thus, the value of the exercise strategy T_m is

$$V(s_1,s_2;m)=s_2\{\min[m,1+k]-1\}\left(\frac{s_1}{ms_2}\right)^{\theta_2}, \tag{6.34}$$

which is illustrated by a numerical example later in this section.

Since k is constant, we seek the optimal value of m, denoted by $\tilde{m}$, that maximizes $V(s_1,s_2;m)$ which can be considered as a function of m. Hence, we can obtain $V(s_1,s_2;\tilde{m})$ which is the price of a perpetual Margrabe option with proportional cap. Normally, the optimal value of m can be obtained by the first-order condition. However, the function $V(s_1,s_2;m)$ given by (6.34) is a bit complicated. Let us discuss the function $V(s_1,s_2;m)$ first.

It is obvious that there are only two possible values for $\min[m,1+k]$, i.e., 1+k and m. Considering $\min[m,1+k]=1+k$, by (6.34), we have

$$V(s_1, s_2; m) = ks_2 \left(\frac{s_1}{ms_2} \right)^{\theta_2} \tag{6.35}$$

which is a decreasing function of m. Since $\min[m, 1+k] = 1+k$ implies that $m \geq 1+k$, it follows that the optimal exercise boundary is $\tilde{m} = 1+k$. Thus, for $\frac{s_1}{s_2} < \tilde{m}$, the price of a perpetual Margrabe option with proportional cap is

$$V(s_1, s_2; \tilde{m}) = k s_2 \left(\frac{s_1}{(1+k)s_2} \right)^{\theta_2}, \tag{6.36}$$

in which the option is exercised at the cap.

On the other hand, for $\min[m, 1+k] = m$, it follows from (6.26) that

$$V(s_1, s_2; m) = s_2 (m-1) \left(\frac{s_1}{ms_2} \right)^{\theta_2} \tag{6.37}$$

in which by applying the first-order condition, we obtain the optimal exercise boundary of a perpetual Margrabe option with proportional cap as

$$\tilde{m} = \frac{\theta_2}{\theta_2 - 1} = \tilde{M} \tag{6.38}$$

where $\tilde{M}$ is the optimal exercise boundary of a perpetual uncapped Margrabe option. Thus, for $\frac{s_1}{s_2} < \tilde{M}$, the price of a perpetual Margrabe option with proportional cap is

$$V(s_1, s_2; \tilde{m}) = s_2 (\tilde{M} - 1) \left(\frac{s_1}{\tilde{M}s_2} \right)^{\theta_2} \tag{6.39}$$

which is the same as a perpetual uncapped Margrabe option under the same assumptions. Note that to obtain (6.39), we should make sure that $\tilde{m} = \tilde{M} < 1+k$; otherwise, for $\tilde{M} \geq 1+k$, we will obtain (6.36).

We conclude from above that the perpetual Margrabe option with proportional cap is exercised either at the cap or on the optimal exercise boundary of a perpetual uncapped Margrabe option. Since we have either $\tilde{m} = \tilde{M}$ if $\tilde{M} < 1 + k$, or $\tilde{m} = 1 + k$ if $1 + k \leq \tilde{M}$, the optimal exercise boundary of a perpetual Margrabe option with proportional cap can be expressed as

$$\tilde{m} = \min\left\{\tilde{M}, 1 + k\right\}, \tag{6.40}$$

which has been proved by Broadie and Detemple (1997).

Now, we consider another capped Margrabe option with payoff function

$$\Pi(z_1, z_2) = \min\left[(z_1 - z_2)_+, kz_1\right], \quad z_1, z_2 > 0, \tag{6.41}$$

which is obtained by replacing the cap kz_2 in (6.27) by kz_1. By the assumption of homogeneity of degree one for the payoff function, (6.41) can be rewritten as

$$\Pi(z_1, z_2) = z_1 \min\left[\left(1 - \frac{z_2}{z_1}\right)_+, 1 - (1 - k)\right]. \tag{6.42}$$

For an option with this payoff, it is sufficient to consider an exercise strategy (stopping time) that exercises the option as soon as the price ratio of stock 2 to stock 1 falls to a level n for the first time. The exercise strategy can be denoted as T_n and in the form of (6.19) if N is replaced by n, for $0 < n < 1$. Thus, the value of exercise strategy T_n is $V(s_1, s_2; n)$ and can be expressed as (6.5) where M is replaced by n.

In order to derive an explicit expression for $V(s_1, s_2; n)$, we need to simplify an expectation. Since $\frac{S_2(T_n)}{S_1(T_n)} = n < 1$, we have

$$\Pi\left(S_1\left(T_n\right),S_2\left(T_n\right)\right)=S_1\left(T_n\right)\min\left[1-n\ ,\ k\right]. \tag{6.43}$$

Hence, $V\left(s_1,s_2;n\right)$ can be simplified to

$$V\left(s_1,s_2;n\right)=s_1\min\left[1-n\ ,\ k\right]E^*\left[e^{-rT_n+X_1\left(T_n\right)}\right]. \tag{6.44}$$

The expectation on the right-hand side of (6.44) can be derived in the same way as for $E^*\left[e^{-rT_N+X_1\left(T_N\right)}\right]$ in Section 6.1.1. By replacing N by n in (6.23), we obtain a simple expression for $E^*\left[e^{-rT_n+X_1\left(T_n\right)}\right]$. Substitute the expression in (6.44) to get

$$V\left(s_1,s_2;n\right)=s_1\min\left[1-n\ ,\ k\right]\left(\frac{s_2}{n\,s_1}\right)^{\xi_2}. \tag{6.45}$$

It remains to seek the maximum value of $V\left(s_1,s_2;n\right)$.

Let us consider $V\left(s_1,s_2;n\right)$ as a function of n and determine the optimal value of n that maximizes the function. Since there are two values for $\min\left[1-n\ ,\ k\right]$ in (6.45), function $V\left(s_1,s_2;n\right)$ can be distinguished into two situations. For $\min\left[1-n\ ,\ k\right]=k$, it follows from (6.45) that

$$V\left(s_1,s_2;n\right)=k\,s_1\left(\frac{s_2}{n\,s_1}\right)^{\xi_2}, \tag{6.46}$$

which is an increasing function of n. It implies that $n\le 1-k$ in this situation. Thus the optimal value of n is $\tilde{n}=1-k$ and the price of the perpetual capped Margrabe option with payoff (6.41) is

$$V\left(s_1,s_2;\tilde{n}\right)=k\,s_1\left(\frac{s_2}{(1-k)s_1}\right)^{\xi_2}, \qquad \text{for } \frac{s_1}{s_2}>\tilde{n}. \tag{6.47}$$

In this situation, the option is exercised at the cap.

For another value $\min[1-n , k] = 1-n$, we have (6.24) if n is instead of N. Finally, we obtain $\tilde{n} = \tilde{N}$ where $\tilde{N}$ is given by (6.25). And the price of a perpetual Margrabe option with proportion cap $kS_1(t)$ is the same as the price of a perpetual uncapped Margrabe option given by (6.26). It turns out that the option is exercised at the optimal exercise boundary of a perpetual uncapped Margrabe option. To make this result feasible, we must check for $\tilde{N} > 1-k$. If $\tilde{N} \le 1-k$, it goes to the result in the previous situation.

From the above, we conclude that if $\tilde{N} > 1-k$, we have $\tilde{n} = \tilde{N}$; else, we have $\tilde{n} = 1-k$. Thus, a perpetual Margrabe option with proportional cap $kS_1(t)$ is exercised either at the cap or on the exercise boundary of its uncapped counterpart. Its optimal exercise boundary can be expressed as

$$\tilde{n} = \max\left\{\tilde{N}, 1-k\right\}, \tag{6.48}$$

which is illustrated in Section 6.2.1.

6.2.1 Numerical Examples

Now, we illustrate by some numerical examples. Suppose that the initial prices of stock 1 and stock 2 are 100 and 95 respectively; $r = 0.1$, $\sigma_1 = 0.2$, $\sigma_2 = 0.1$, $\rho = 0.5$, $\zeta_1 = 0.03$ and $\zeta_2 = 0.02$. Under these assumptions, we obtain $\tilde{M} = 1.795$. The values of the exercise strategy T_m, given by (6.34), for different values of m and k are displayed in Table 6.2 and Figure 6.3. As shown in Table 6.2, the maximum value in each column is shaded. Actually, these shaded values are the prices of the perpetual Margrabe option with proportional cap $kS_2(t)$ for different values of k. We can also observe from Table 6.2 and Figure 6.3 that for k = 0.4 and k = 0.6, i.e. $1+k \le \tilde{M}$, the maximum value is obtained when m reaches 1+k; for k = 0.8, 1.0 and 1.2, i.e. $1+k > \tilde{M}$, the maximum value is obtained when m reaches $\tilde{M}$.

Similarly, for a perpetual Margrabe option with proportional cap $kS_1(t)$, we can also calculate the values of the exercise strategy T_n for different values of n and k, which are displayed in Table 6.3 and Figure 6.4. The maximum value in each column is also shaded. Here, $\tilde{N} = 0.557$. We observe from Table 6.3 that whenever $1-k < \tilde{N}$, the option is exercised at the boundary $\tilde{N}$.

Table 6.2 Values of the Exercise Strategy T_m in (6.34) for Different Values of m and k

m \ k	0.2	0.4	0.6	0.8	1.0	1.2
1.1	8.6015	8.6015	8.6015	8.6015	8.6015	8.6015
1.2	14.1351	14.1351	14.1351	14.1351	14.1351	14.1351
1.3	11.7986	17.6979	17.6979	17.6979	17.6979	17.6979
1.4	9.9811	19.9622	19.9622	19.9622	19.9622	19.9622
1.5	8.5417	17.0833	21.3541	21.3541	21.3541	21.3541
1.6	7.3837	14.7673	22.1510	22.1510	22.1510	22.1510
1.7	6.4393	12.8786	19.3179	22.5376	22.5376	22.5376
1.8	5.6598	11.3197	16.9795	22.6393	22.6393	22.6393
1.9	5.0096	10.0191	15.0287	20.0382	22.5430	22.5430
2.0	4.4618	8.9237	13.3855	17.8473	22.3092	22.3092
2.2	3.5981	7.1963	10.7944	14.3925	17.9906	21.5888
2.4	2.9565	5.9130	8.8694	11.8259	14.7824	17.7389
2.6	2.4678	4.9355	7.4033	9.8711	12.3389	14.8066
2.8	2.0876	4.1753	6.2629	8.3505	10.4382	12.5258
$\tilde{M}$	5.6931	5.6931	17.0793	22.6395	22.6395	22.6395

Figure 6.3 Value of V(s₁, s₂; m) as a Function of m

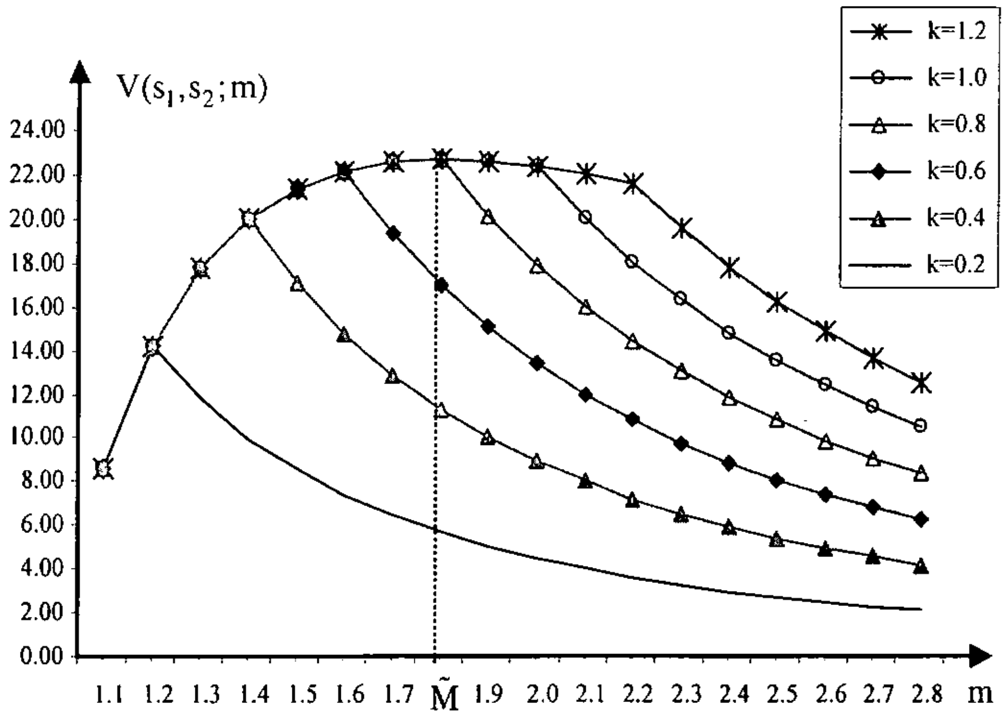


Figure 6.4 Value of V(s₁, s₂; n) as a Function of n

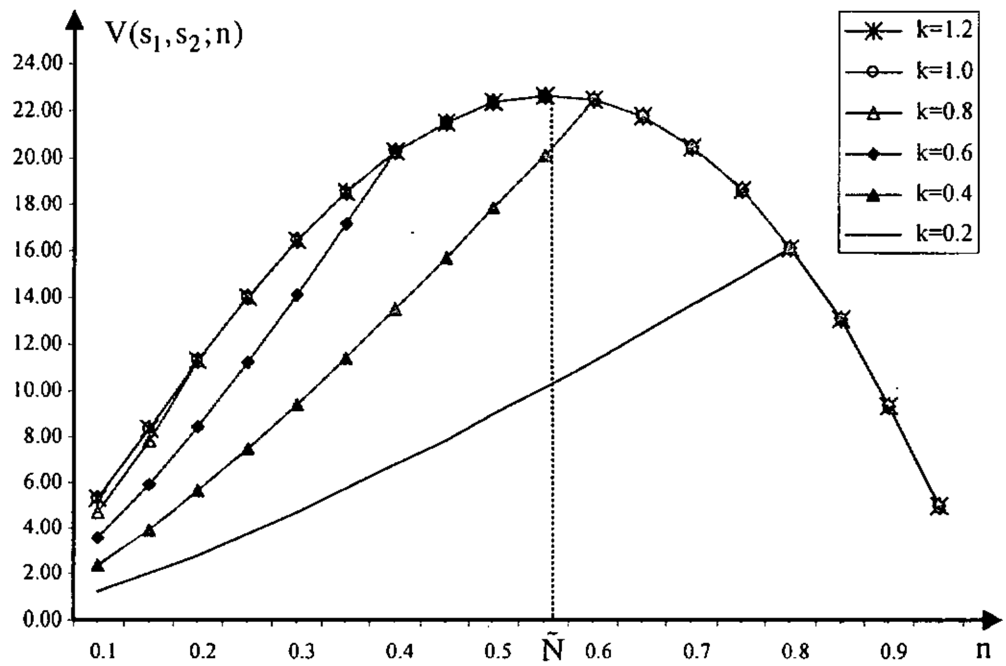


Table 6.3 Values of the Exercise Strategy T_n in (6.45) for Different Values of n and k

n \ k	0.2	0.4	0.6	0.8	1.0	1.2
0.10	1.1795	2.3590	3.5386	4.7181	5.3078	5.3078
0.15	1.9639	3.9277	5.8916	7.8554	8.3464	8.3464
0.20	2.8197	5.6394	8.4590	11.2787	11.2787	11.2787
0.25	3.7329	7.4658	11.1988	13.9984	13.9984	13.9984
0.30	4.6947	9.3894	14.0840	16.4314	16.4314	16.4314
0.35	5.6988	11.3975	17.0963	18.5210	18.5210	18.5210
0.40	6.7406	13.4811	20.2217	20.2217	20.2217	20.2217
0.45	7.8165	15.6330	21.4953	21.4953	21.4953	21.4953
0.50	8.9237	17.8473	22.3092	22.3092	22.3092	22.3092
0.60	11.2228	22.4456	22.4456	22.4456	22.4456	22.4456
0.70	13.6231	20.4346	20.4346	20.4346	20.4346	20.4346
0.80	16.1135	16.1135	16.1135	16.1135	16.1135	16.1135
0.90	9.3428	9.3428	9.3428	9.3428	9.3428	9.3428
0.95	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000
$\tilde{N}$	10.2210	20.4420	22.6395	22.6395	22.6395	22.6395

Chapter 7

Pricing Perpetual Dynamic Fund Protection

Up to now, we have derived explicit formulas for several perpetual options on one stock or on two stocks. In practice, options can be acted as protection tools. For example, for a investor who holds a certain amount of a stock, he may buy a European put option to protect himself from suffering a great loss due to an unexpected drop of the stock price. However, a drawback for a put option is that its holder has little hope of having more than K at the expiry date. In this situation, dynamic protections can provide a better protection to the investor.

In this chapter, we discuss the dynamic fund protection which was proposed by Gerber and Pafumi (2000). The concept is that the primary fund is upgraded to a protected fund. It guarantees that the protected fund does not fall below a guaranteed level at all time on or before the expiry date. The dynamic fund protection has also been investigated in a number of research articles. Imai and Boyle (2001) studied the protection in which the primary fund price follows a constant elasticity of variance (CEV) process. Explicit pricing formulas for European protections and perpetual protections have been derived by Gerber and Shiu (2003a, 2003b). As a discussion, Yu (2003a) provided an alternative evaluation of the expectation (89) which is the price of a European protection in Gerber and Shiu (2003a). For numerical analysis, Fung and Li (2003) illustrated the pricing of dynamic fund protections under discrete monitoring. Recently, Chu and Kwok (2004) investigated the price of the dynamic fund protection by considering its reset and withdraw rights.

As studied in Gerber and Pafumi (2000), the guaranteed level for an investment fund was considered to be constant in all times. Gerber and Shiu (2003b) suggested that the guaranteed level is not necessarily constant or exponential, but can be stochastic. They considered the guaranteed level as a stock price or stock index. This chapter considers the guarantee level as a stock price and expresses the dynamic fund protection in terms of two stocks. Our goal is to derive a pricing formula for a perpetual dynamic fund protection by the exercise strategy stated in Section 5.2.

7.1 Derivation of the Pricing Formula

Let $S_1(t)$ and $S_2(t)$ be the prices of stock 1 and stock 2 at time t, $t \geq 0$. We assume that the stock price process is a geometric Brownian motion. Let us define the time-t value of the upgraded fund as follows

$$\tilde{S}_2(t) = S_2(t) \max\left\{1,\ \max_{0 \leq \tau \leq t} \frac{S_1(\tau)}{S_2(\tau)}\right\}, \qquad \text{for } t \geq 0. \tag{7.1}$$

This can be viewed as the payoff function of a dynamic fund protection option. Here, $S_1(t)$ and $S_2(t)$ can be viewed as the guaranteed level at time t and the time-t value of the primary fund respectively. Note that the payoff is path-dependent which is different from those path-independent payoffs discussed in the previous chapters, because the payoff depends on the maximum value of the stock-price ratio during the monitoring period in the option life.

We assume that $S_1(0) < S_2(0)$, i.e. $\frac{S_1(0)}{S_2(0)} < 1$. At time 0, the value of the protected fund is equal to $S_2(0)$. Since

$$\max\left\{1, \max_{0\le\tau\le t}\frac{S_1(\tau)}{S_2(\tau)}\right\} \ge 1, \tag{7.2}$$

We always have $\tilde{S}_2(t) \ge S_2(t)$. With the guarantee, the time-t value of the protected fund is prevented from falling below the guaranteed level $S_1(t)$. Similar to the interpretation of (2.1) in Gerber and Shiu (2003b), (7.1) can be interpreted as follow. Whenever the value of the protected fund threatens to fall below the guaranteed level, just enough funds will be added to prevent this from happening.

The price of a protected fund, i.e. the price of an option with payoff function (7.1), is the supremum of

$$E\left[e^{-rT}\,\tilde{S}_2(T)\right]$$

over all stopping times T, where r is assumed to be the constant risk-free force of interest. Here, the price is denoted by $V(s_1, s_2)$, where $s_1 = S_1(0) > 0$ and $s_2 = S_2(0) > 0$. Note that $V(s_1, s_2)$ is homogenous of degree one.

As proposed in Section 5.2, we obtain a general form of the exercise strategy for a perpetual option on two stocks with a homogenous payoff function. That is to consider the exercise strategy in the form of $T_{u,v}$ given by (5.5) in which to exercise the option as soon as the stock-price ratio falls to a level u or rises to a level v for the first time. The value of this exercise strategy is $V(s_1, s_2; u, v)$ which is given by (5.6). The exercise strategy $T_{u,v}$ can also be applied here. Thus, for the perpetual dynamic fund protection, the value of the exercise strategy is

$$V(s_1,s_2;u,v)=E^*\left[e^{-rT_{u,v}}S_2(T_{u,v})\max\left\{1,\max_{0\le\tau\le T_{u,v}}\frac{S_1(\tau)}{S_2(\tau)}\right\}\right]. \tag{7.3}$$

Following from Section 5.2, by the definition of $A(s_1,s_2;u,v)$ and $B(s_1,s_2;u,v)$ in (5.9) and (5.10), we can express (7.3) as

$$V(s_1,s_2;u,v)=V(u,1;u,v)A(s_1,s_2;u,v)+V(v,1;u,v)B(s_1,s_2;u,v), \tag{7.4}$$

Here, $A(s_1,s_2;u,v)$ and $B(s_1,s_2;u,v)$ can be solved and expressed as (5.16) and (5.17).

Similar to the derivation of $\tilde{u}$ and $\tilde{v}$ in Chapter 5, we can also obtain the relationship between $\tilde{u}$ and $\tilde{v}$. Does this mean that the expressions of the optimal exercise boundaries for a perpetual dynamic fund protection option are the same as those for a perpetual maximum option on two stocks? The answer is No because there are more restrictions on the exercise region for the dynamic fund protection. It will be explained in the following.

If we compare (7.1) with (5.26) which is the payoff function of a maximum option on two stocks, we observe that although they look quite similar; however, we always have

$$\max_{0\le\tau\le t}\frac{S_1(\tau)}{S_2(\tau)}\ge\frac{S_1(t)}{S_2(t)}$$

which reveals that the dynamic fund protection is more expensive than the maximum option.

At time 0, we have $\frac{s_1}{s_2} < 1$ by assumption. The continuation region is

$$\{(s_1,s_2) \mid V(s_1,s_2) > s_2\}. \tag{7.5}$$

Since $V(s_1,s_2)$ is a homogenous function of degree one, it follows from (7.5) that

$$\left\{(s_1,s_2) \;\middle|\; V\left(\frac{s_1}{s_2},1\right) > 1\right\}.$$

Because $V\left(\frac{s_1}{s_2},1\right)$ is a non-decreasing function of $\frac{s_1}{s_2}$, we have

$$\left\{(s_1,s_2) \;\middle|\; \frac{s_1}{s_2} > \omega\right\} \tag{7.6}$$

for some ω where $0 < \omega < 1$. On the other hand, we know that the value of the protected fund is prevented to fall below the guaranteed level. Whenever the price of stock 2 reaches the price of stock 1, the dynamic fund protection option will be exercised. It follows from (7.6) that the continuation region for the perpetual dynamic fund protection is

$$\left\{(s_1,s_2) \;\middle|\; \omega < \frac{s_1}{s_2} \le 1\right\}. \tag{7.7}$$

Thus, for the perpetual dynamic fund protection, the exercise boundaries, u and v, are replaced by 1 and ω respectively. It follows from (7.4) that the value of the exercise strategy is

$$V(s_1,s_2;\omega,1) = V(\omega,1;\omega,1)A(s_1,s_2;\omega,1) + V(1,1;\omega,1)B(s_1,s_2;\omega,1), \tag{7.8}$$

which is similar to expression (6.3) in Gerber and Pafumi (2000).

Note that $V(\omega,1;\omega,1) = 1$ and $V(1,1;\omega,1)$ can be derived and expressed as

$$V(1,1;\omega,1)=\frac{\theta_2-\theta_1}{(\theta_2-1)\omega^{\theta_1}+(1-\theta_1)\omega^{\theta_2}}, \tag{7.9}$$

which is the same as (3.15) in Gerber and Shiu (2003b). Here, $\theta_1<0$ and $\theta_2>1$ are two roots of the quadratic equation (5.11). For detail derivation of (7.9), see Gerber and Shiu (2003b). Thus, it follows from (5.16), (5.17), (7.8) and (7.9) that

$$V(s_1,s_2;\omega,1)=\frac{s_2\left(\frac{s_1}{s_2}\right)^{\theta_1}-s_2\left(\frac{s_1}{s_2}\right)^{\theta_2}}{\omega^{\theta_1}-\omega^{\theta_2}}$$

$$+\frac{\theta_2-\theta_1}{(\theta_2-1)\omega^{\theta_1}+(1-\theta_1)\omega^{\theta_2}}\frac{s_2\omega^{\theta_1}\left(\frac{s_1}{s_2}\right)^{\theta_2}-s_2\omega^{\theta_2}\left(\frac{s_1}{s_2}\right)^{\theta_1}}{\omega^{\theta_1}-\omega^{\theta_2}}, \tag{7.10}$$

which can be simplified to

$$V(s_1,s_2;\omega,1)=\frac{s_2\left[(\theta_2-1)\left(\frac{s_1}{s_2}\right)^{\theta_1}+(1-\theta_1)\left(\frac{s_1}{s_2}\right)^{\theta_2}\right]}{(\theta_2-1)\omega^{\theta_1}+(1-\theta_1)\omega^{\theta_2}}. \tag{7.11}$$

Let us consider (7.11) as a function of ω and determine the optimal value of ω, denoted by $\tilde{\omega}$, that maximizes $V(s_1,s_2;\omega,1)$. It is observed from the right-hand side of (7.11) that only the denominator depends on ω and the maximum value of $V(s_1,s_2;\omega,1)$ can be obtained by minimizes the denominator. By applying the first-order condition to the denominator, we have

$$\theta_1(\theta_2-1)\tilde{\omega}^{\theta_1-1}+\theta_2(1-\theta_1)\tilde{\omega}^{\theta_2-1}=0,$$

or

$$\tilde{\omega}=\left(\frac{\theta_1(1-\theta_2)}{\theta_2(1-\theta_1)}\right)^{1/(\theta_2-\theta_1)}, \tag{7.12}$$

which is the optimal value of ω that minimizes the denominator and maximizes $V(s_1, s_2; \omega, 1)$. Note that the ratio of $\tilde{\omega}$ to 1 is

$$\frac{\tilde{\omega}}{1} = \left(\frac{\theta_1 (1-\theta_2)}{\theta_2 (1-\theta_1)} \right)^{1/(\theta_2-\theta_1)},$$

which is the same as the ratio of $\tilde{u}$ to $\tilde{v}$ given by (5.28).

Since the value of the protected fund is prevented from falling below the guaranteed level, the region $\frac{s_1}{s_2} > 1$ is not applicable here. Thus, for $0 < \tilde{\omega} < 1$, the price of the perpetual dynamic fund protection is

$$\begin{cases} s_2 & \text{if } s_1 / s_2 \le \tilde{\omega} \\ V(s_1, s_2; \tilde{\omega}, 1) & \text{if } \tilde{\omega} < s_1 / s_2 \le 1 \end{cases}, \tag{7.13}$$

or

$$\begin{cases} s_2 & \text{if } s_1 / s_2 \le \tilde{\omega} \\ \dfrac{s_2 \left[(\theta_2 - 1) \left(\frac{s_1}{s_2} \right)^{\theta_1} + (1-\theta_1) \left(\frac{s_1}{s_2} \right)^{\theta_2} \right]}{(\theta_2 - 1) \tilde{\omega}^{\theta_1} + (1-\theta_1) \tilde{\omega}^{\theta_2}} & \text{if } \tilde{\omega} < s_1 / s_2 \le 1 \end{cases}, \tag{7.14}$$

which is the expansion of (2.17) in Gerber and Shiu (2003b) if $\tilde{\omega} = \tilde{\varphi}$. See also Young (2003) for alternative method of deriving the pricing function V in (2.17).

7.2 Comparison with the Price of a Perpetual Maximum Option

This section compares the price of a perpetual dynamic fund protection with that of a perpetual maximum option under the same assumptions. As shown in Section 5.3, some formulas for the price of the perpetual maximum option have been derived. It follows from (5.32) and (5.36) that the price of a perpetual maximum option is

$$\begin{cases} s_2 & \text{if } s_1/s_2 \le \tilde{u} \\ \dfrac{s_2}{\theta_2-\theta_1}\left[\theta_2\left(\dfrac{s_1}{\tilde{u}s_2}\right)^{\theta_1}-\theta_1\left(\dfrac{s_1}{\tilde{u}s_2}\right)^{\theta_2}\right] & \text{if } \tilde{u} < s_1/s_2 < \tilde{v}. \\ s_1 & \text{if } s_1/s_2 \ge \tilde{v} \end{cases} \quad (7.15)$$

Our goal is to find the difference between (7.14) and (7.15).

First of all, let us learn about the relationship between $\tilde{\omega}$, $\tilde{u}$ and $\tilde{v}$. We already have $0<\tilde{\omega}<1$ and $0<\tilde{u}<1<\tilde{v}$. Since $\dfrac{\tilde{\omega}}{1}=\dfrac{\tilde{u}}{\tilde{v}}$, we have $\tilde{u}=\tilde{\omega}\tilde{v}>\tilde{\omega}$. Thus, it follows that

$$0<\tilde{\omega}<\tilde{u}<1<\tilde{v}. \quad (7.16)$$

Now, we can subtract (7.15) from (7.14) to obtain the price difference between the perpetual dynamic fund protection and the perpetual maximum option as

$$\begin{cases} 0 & \text{if } s_1/s_2 \le \tilde{\omega} \\ s_2\dfrac{\left[(\theta_2-1)\left(\dfrac{s_1}{s_2}\right)^{\theta_1}+(1-\theta_1)\left(\dfrac{s_1}{s_2}\right)^{\theta_2}\right]}{(\theta_2-1)\tilde{\omega}^{\theta_1}+(1-\theta_1)\tilde{\omega}^{\theta_2}}-s_2 & \text{if } \tilde{\omega}<s_1/s_2\le\tilde{u} \\ s_2\dfrac{\left[(\theta_2-1)\left(\dfrac{s_1}{s_2}\right)^{\theta_1}+(1-\theta_1)\left(\dfrac{s_1}{s_2}\right)^{\theta_2}\right]}{(\theta_2-1)\tilde{\omega}^{\theta_1}+(1-\theta_1)\tilde{\omega}^{\theta_2}}-\dfrac{s_2}{\theta_2-\theta_1}\left[\theta_2\left(\dfrac{s_1}{\tilde{u}s_2}\right)^{\theta_1}-\theta_1\left(\dfrac{s_1}{\tilde{u}s_2}\right)^{\theta_2}\right] & \text{if } \tilde{u}<s_1/s_2\le 1. \end{cases}$$

(7.17)

Let us use a numerical example for illustration. Assume that r = 0.1, $\sigma_1 = 0.2$, $\sigma_2 = 0.1$, $\zeta_1 = 0.03$, $\zeta_2 = 0.02$ and $\rho = 0.5$. We construct two tables. Table 7.1 displays the price of a perpetual dynamic fund protection given by (7.14) for different values of s_1 and s_2. It also shows that the price of a perpetual dynamic fund protection increases as either of initial stock price increases. The hyphens signify that it is not applicable to calculate the price of the protection in those situations. According to (7.17), Table 7.2 shows the variation of the price difference between the perpetual dynamic fund protection and the perpetual maximum option when we modify the values of s_1 and s_2. We observe that as s_1 increases, the price difference increases; however, as s_2 increases, the price difference decreases.

Table 7.1 Price of the Perpetual Dynamic Fund Protection for Different Values of s_1 and s_2

$s_2 \backslash s_1$	**100**	**105**	**110**	**115**	**120**	**125**	**130**	**135**
100	129.48	—	—	—	—	—	—	—
105	129.79	135.96	—	—	—	—	—	—
110	130.67	136.25	144.18	—	—	—	—	—
115	132.06	137.09	144.59	148.90	—	—	—	—
120	133.90	138.42	145.53	149.17	155.38	—	—	—
125	136.15	140.19	146.94	149.95	155.64	161.85	—	—
130	138.77	142.36	148.78	151.18	156.38	162.10	168.33	—
135	141.72	144.88	151.01	152.82	157.57	162.82	168.57	174.80
140	145.00	147.74	153.59	154.83	159.15	163.96	169.26	175.03
145	148.56	150.90	156.50	157.19	161.09	165.49	170.36	175.70
150	152.38	154.35	159.71	159.86	163.38	167.37	171.84	176.77
155	156.46	158.05	163.21	162.83	165.97	169.58	173.66	178.20
160	160.78	162.01	166.96	166.07	168.84	172.09	175.81	179.97
165	165.31	166.20	170.96	169.56	171.99	174.89	178.25	182.05
170	170.06	170.60	175.19	173.29	175.38	177.94	180.96	184.42
175	175	175.22	179.64	177.25	179.01	181.24	183.93	187.06
180	180	180.02	184.29	181.41	182.86	184.78	187.14	189.95
185	185	185	189.15	185.78	186.92	188.53	190.58	193.08
190	190	190	194.18	190.34	191.18	192.48	194.24	196.43
195	195	195	195	195.08	195.62	196.63	198.10	200.00

Table 7.2 **Price Difference between the Perpetual Dynamic Fund Protection and the Perpetual Maximum Option for Different Values of s_1 and s_2**

$s_2 \setminus s_1$	100	105	110	115	120	125	130	135
100	22.53	—	—	—	—	—	—	—
105	19.87	23.66	—	—	—	—	—	—
110	17.38	20.99	24.79	—	—	—	—	—
115	15.04	18.49	22.12	25.91	—	—	—	—
120	12.82	16.13	19.60	23.24	27.04	—	—	—
125	10.70	13.89	17.23	20.72	24.36	28.17	—	—
130	8.67	11.75	14.97	18.33	21.83	25.49	29.29	—
135	6.72	9.71	12.81	16.05	19.43	22.95	26.61	30.42
140	5.00	7.73	10.74	13.88	17.14	20.53	24.06	27.73
145	3.56	5.90	8.75	11.79	14.95	18.23	21.64	25.18
150	2.38	4.35	6.85	9.77	12.84	16.02	19.32	22.75
155	1.46	3.05	5.18	7.83	10.81	13.89	17.09	20.41
160	0.78	2.01	3.78	6.07	8.84	11.84	14.95	18.17
165	0.31	1.20	2.62	4.56	6.99	9.85	12.88	16.01
170	0.06	0.60	1.69	3.29	5.38	7.94	10.88	13.93
175	0.00	0.22	0.97	2.25	4.01	6.24	8.93	11.91
180	0.00	0.02	0.46	1.41	2.86	4.78	7.14	9.94
185	0.00	0.00	0.14	0.78	1.92	3.53	5.58	8.08
190	0.00	0.00	0.01	0.34	1.18	2.48	4.24	6.43
195	0.00	0.00	0.00	0.08	0.62	1.63	3.10	5.00

Chapter 8

Conclusions

This thesis has derived closed-form formulas for pricing several perpetual options. In the first part of this thesis, we have discussed the pricing of perpetual options on one stock. After given a brief introduction in Chapter 1 and discussed the fundamentals of option pricing in Chapter 2, we illustrated the pricing of perpetual American put options, perpetual American call options and the perpetual maximum option on one stock in Chapter 3. All the options discussed in Chapter 3 are under geometric Brownian motion. For comparison, Chapter 4 discussed the pricing of perpetual American put options for jump processes. It was shown that the limiting case of the Poisson process is a Brownian motion.

In the second part of this thesis, we considered perpetual options on two stocks. Chapter 5 derived explicit pricing formulas for perpetual maximum options on two stocks. Chapter 6 studied the pricing of perpetual uncapped Margrabe options and perpetual Margrabe options with proportional cap. Finally, Chapter 7 discussed the pricing of perpetual dynamic fund protection options in which its payoff is path-dependent. This chapter will conclude the thesis and give some suggestions for further research.

For further research, the optimal exercise strategy discussed in Section 5.2 can be applied to price perpetual options with homogeneous payoffs. Also, this strategy can be extended and applied to price perpetual options on two or more stocks. Moreover, we can use the exercise strategy to price some more complicated perpetual options, such as "path-dependent options" or "exotic options".

As discussed in Chapter 7, we introduced a dynamic fund protection option with a path-dependent payoff. A path-dependent payoff depends on the maximum or minimum of the stock price during the monitoring period in the option life. Since path-dependent options are more flexible than path-independent options, they become more and more popular in financial markets. Other examples of the path-dependent options are lookback options and Asian options. Although pricing formulas for some European path-dependent options have been derived by some researchers, see for example, Goldman et al. (1979) and Lee (2002); however there are no explicit formulas for pricing their American counterparts. As illustrated in Hull (2002), American path-dependent options could be priced by binomial trees method. But this method is time consuming and not efficient. In the risk-neutral world, the martingale approach is more efficient and can be extended to price perpetual American path-dependent options.

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