Modeling and Analysis of Some Biological Models with Cross-diffusion

Jiawei Chu

PhD

The Hong Kong Polytechnic University

2025

The Hong Kong Polytechnic University

Department of Applied Mathematics

Modeling and analysis of some biological models with cross-diffusion

Jiawei Chu

A thesis submitted in partial fulfilment of the requirements

for the degree of Doctor of Philosophy

May 2025

CERTIFICATE OF ORIGINALITY

I hereby declare that this thesis is my own work and that, to the best of my knowledge and belief, it reproduces no material previously published or written, nor material that has been accepted for the award of any other degree or diploma, except where due acknowledgement has been made in the text.

______________________________(Signed)

CHU Jiawei (Name of student)

Abstract

Cross-diffusion, a process in which the density gradient of one species induces an advective flux of another species, has been widely applied to model the movement of one species toward or away from the area with higher density of another species (i.e., taxis movement). In this thesis, we first study an indirect prey-taxis model [129] with anti-predation mechanism. Then, we explore another type of anti-predation mechanism: alarm-taxis [46] which described by a three-species Lotka-Volterra food chain model. Next, we apply the cross-diffusion strategy to an SIS epidemic model and numerically explored the effects of cross-diffusion. Finally, we investigate a toxicant-taxis model and theoretically prove the effects of cross-diffusion.

Fundamentally, we establish the global boundedness of classical solutions by using energy estimates. The other main results are as follows:

1. For the indirect prey-taxis model with anti-predation, we prove the global stability of constant steady states by constructing energy functionals. Moreover, when the prey adopts the anti-predation strategy, we establish the existence of non-constant positive steady-state solutions by applying Leray-Schauder degree theory and prove that no Hopf bifurcation occurs. These pattern formation results are different from both indirect prey-taxis (which exhibits Hopf bifurcation) and the case without cross-diffusion (where no patterns emerge).

2. For the three-species Lotka-Volterra food chain model with intraguild predation and taxis mechanisms, we build the global stability of constant steady states by using energy functionals. Moreover, we numerically demonstrate that the combination of taxis mechanisms and intraguild predation can induce rich pattern formations. Notably, our simulations show that prey-taxis may have a destabilizing effect in food chain model with intraguild predation, which contrasts with the well-known stabilizing effect observed in two-species predator-prey systems or the food chain model with alarm-taxis but without intraguild predation.

3. For the SIS model with a cross-diffusion dispersal strategy for the infected individuals, which describes the public health intervention measures, we define the basic reproduc-

tion number R_0. Then we employ a change of variable and apply the index theory along with the principal eigenvalue theory to establish the threshold dynamics in terms of R_0. Moreover, we explore the global stability of constant steady states. Finally, we numerically demonstrate that the cross-diffusion strategy can reduce R_0 and help eradicate the diseases even if the habitat is high-risk in contrast to the situation without cross-diffusion.

4. For the toxicant-taxis model in a time-periodic environment, we prove the existence of positive periodic solutions and the uniform persistence by applying the uniform persistence theory and Principal Floquet bundle theory. Moreover, we establish the global stability of non-constant periodic solutions through energy methods. By studying the effects of the strong toxicant-taxis on the corresponding periodic principal eigenvalue, we theoretically prove that the strong toxicant-taxis (i.e., cross-diffusion) helps aquatic species to survive.

Moreover, we develop new ideas to overcome the difficulties caused by the failure of the comparison principle in cross-diffusion models. For example, the proof ideas developed in Chapter 5 can be applied to prove the existence of time-periodic/non-constant steady-state solutions, and uniform persistence for general cross-diffusion models.

Publications Arising from the Thesis

1. **Jiawei Chu** and Shanbing Li. Global dynamics of an indirect prey-taxis system with an anti-predation mechanism. J. Differential Equations, 385:424–462, 2024.

(The contents of this paper are presented in Chapter 2.)

2. **Jiawei Chu** and Hai-Yang Jin. Global dynamics of a three-species Lotka-Volterra food chain model with intraguild predation and taxis mechanisms. J. Nonlinear Sci., 35(3):Paper No. 56, 47 pages, 2025.

(The contents of this paper are presented in Chapter 3.)

3. **Jiawei Chu** and Zhi-An Wang. Global dynamics of an SIS epidemic model with cross-diffusion: applications to quarantine measures. Nonlinearity, 38(5):Paper No. 055010, 27 pages, 2025.

(The contents of this paper are presented in Chapter 4.)

4. **Jiawei Chu**, King-Yeung Lam and Zhi-An Wang. A diffusive population-toxicant model in a time-periodic environment with negative toxicant-taxis, in preparation.

(The contents of this paper are presented in Chapter 5.)

Acknowledgements

During my three-year Ph.D. journey at The Hong Kong Polytechnic University, I have experienced significant personal and academic growth. This progress is firstly due to the invaluable support and guidance of my Ph.D. supervisor, Professor Zhi-An Wang. His pure passion for academia and rigorous, meticulous attitude to research as well as active and open-minded thinking have profoundly influenced me in every discussion. He has also provided much encouragement, guidance, and assistance throughout various stages of my Ph.D. journey. Whether it was during my application for the Ph.D. program, giving presentations at seminars/conferences, participating in exchange programs, or working on the final thesis topic, his support has been unwavering. I admire and appreciate him deeply.

Additionally, my master's advisor, Professor Haiyang Jin, has been crucial in my research journey. It was his enthusiasm and rigor that guided me into the world of research; his strict requirements helped me develop good basic research habits. His ongoing encouragement and assistance keep energizing me. I have always held him in high regard.

I would also like to express my sincere thanks to other teachers, Professor Shanbing Li and Professor De Tang for their insightful suggestions. Furthermore, I am grateful to Professor King-Yeung Lam for his support and assistance during my six months at The Ohio State University, both in life and research. His approach to problem-solving and high standards in research will always serve as a model for me.

I also deeply thank the postgraduates for their interesting discussions and companionship, and research students for their assistance and camaraderie. Moreover, I am grateful to The Hong Kong Polytechnic University for their financial support throughout my Ph.D program, and the staff in the Department of Applied Mathematics for their kind help.

Last but not least, my thankfulness turns to my family for their love, support, and understanding. I am endlessly blessed to have such an incredible family-they are my safe harbor and the resting place of my soul, no matter where life takes me.

Research is a marathon. I am not be the fastest runner, but I will happily keep moving.

Contents

List of Notations

$\mathbb{R}$ (resp. $\mathbb{R}^+$)	The set of real numbers (resp. non-negative real numbers).
$\mathbb{N}$	The set of non-negative integers.
$\mathbb{R}^n$	The n-dimensional real space with integer $n \geq 1$, $\mathbb{R}^1 =: \mathbb{R}$.
Ω (resp. $\overline{\Omega}$)	A bounded open (resp. closed) domain in $\mathbb{R}^n$ with integer $n \geq 1$. Particularly, denote the boundary of Ω by $\partial\Omega$, $\overline{\Omega} = \Omega \cup \partial\Omega$.
x	The point $x = (x_1, x_2, \cdots, x_n) \in \Omega \subset \mathbb{R}^n$.
Q_T (resp. $\overline{Q}_T$)	A bounded open domain $\Omega \times (0, T)$ (resp. closed domain $\overline{\Omega} \times [0, T]$) with $0 < T < \infty$.
ν	The outward normal unit vector of $\partial\Omega$.
$\partial_\nu f$	The outward normal derivative of u.
∇	The gradient operator: $\nabla := \left(\frac{\partial}{\partial_{x_1}}, \frac{\partial}{\partial_{x_2}}, \cdots, \frac{\partial}{\partial_{x_n}}\right)$ with $x = (x_1, x_2, \cdots, x_n) \in \Omega \subset \mathbb{R}^n$.
$\nabla\cdot$	The divergence operator: $\nabla \cdot := \sum_{i=1}^n \frac{\partial}{\partial x_i^2}$.
Δ	The Laplace operator: $\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$.
$D^k f$	The k-times derivative of function $f(x)$ for integer $k \geq 1$. Specifically, denote $D^\alpha f := \frac{\partial^{\lvert\alpha\rvert} f(x)}{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}}$ where the multiindex $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $\lvert\alpha\rvert = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ with integer $\alpha_i \geq 0$ for integer $i \geq 1$. Then $D^k f := D^k f(x) = \{D^\alpha f : \ \lvert\alpha\rvert = k\}$ and $\lvert D^k f\rvert := (\sum_{\lvert\alpha\rvert=k} \lvert D^\alpha f\rvert^2)^{\frac{1}{2}}$. Particularly, $D^2 f$ represents the Hessian matrix of f.
$\mathbf{v}^{\mathcal{T}}$	The transpose of a vector $\mathbf{v}$.

Chapter 1

Introduction

The reaction-diffusion model is a class of partial differential equations that describe how population densities/concentration in space change over time. It can explain three major spatial phenomena of interest in ecology: (a) the formation of spatial patterns; (b) the impact of spatially environmental characteristics (e.g., size, shape, and heterogeneity) or other factors on species persistence and community structure; (c) waves of invasion by exotic species [17]. These phenomena are respectively addressed in four classical pioneering works on diffusion theory: [133], [72, 116] and [38]. Cross-diffusion, a process in which the density gradient of one species induces an advective flux of another species, has been widely applied in various reaction-diffusion systems to model the movement of one species towards/moves away the area with higher density of another species. Examples include, but not limited to, prey-taxis [68], chemotaxis [69] and toxicant-taxis [32]. Such systems are generally referred to as reaction-cross-diffusion systems or simply cross-diffusion systems without confusion.

1.1 General Cross-diffusion Models

A generic cross-diffusion model can be represented as

$$\begin{cases} (u_i)_t = \nabla \cdot \big(\sum\limits_{j=1}^{m} A_{ij}(u_1, \cdots, u_m) \nabla u_j \big) + f_i(u_1, \cdots, u_m), & x \in \Omega, t > 0, \\ \partial_\nu u_i = 0, & x \in \partial\Omega, t > 0, \end{cases} \tag{1.1}$$

where $i = 1, \cdots, m < \infty$, $u_i := u_i(x, t)$ denote the population densities/ concentration of interacting species at position x and time t, and $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain (habitat) with smooth boundary $\partial\Omega$; $\partial_\nu := \frac{\partial}{\partial \nu}$ with ν denoting the outward normal unit vector of boundary $\partial\Omega$ and the homogeneous Neumann boundary conditions are prescribed to comply with the closed environment where the species cannot cross the habitat boundary $\partial\Omega$. Functions $f_i(u_1, \cdots, u_m)$ represent the intra- or inter-specific population interactions (like cooperation, predator-prey, competition, etc). $A_{ii} := A_{ii}(u_1, \cdots, u_m) > 0$ are the coefficients of diffusion and $A_{ij} := A_{ij}(u_1, \cdots, u_m) \in \mathbb{R}$ $(i \neq j)$ account for the coefficients of cross-diffusion but $\sum_{i,j=1,i\neq j}^{m} A_{ij}^2 \neq 0$. Different forms of A_{ij} represent different diffusion strategies. Note that the system (1.1) encompasses many well-known models, such as chemotaxis [69], prey-taxis [68], bacterial pattern formation [89] for $m = 2$, as well as indirect prey-taxis [129, 135], alarm-taxis [46] for $m = 3$. In particularly, when $A_{i_0 i_0} = d(u_{j_0}) > 0$, $A_{i_0 j_0} = u_{i_0} d'(u_{j_0}) \geq 0$ and $A_{ij} = 0$ $(i \neq j)$ for all $i \neq i_0$ or $j \neq j_0$, the diffusion terms of the population u_{i_0} can be rewritten as $\Delta[d(u_{j_0})u_{i_0}]$, which means the diffusion of the population u_{i_0} depends on the another population density u_{j_0}. This type of diffusion is referred to density/signal-dependent diffusion, and has been received enormous attention (e.g., cf. [39, 40, 63, 128, 138]).

For clarity, our thesis only focuses on two types of cross-diffusion: the density/signal-dependent type cross-diffusion, and the type with constant A_{ii} (such as classical chemotaxis [69] and prey-taxis [68]). By introducing such cross-diffusion strategies of limiting population movement, this thesis incorporates four vital effects: anti-predation tactics of prey, burglar alarm responses, quarantine for infected individuals and toxicant avoidance of species into four reaction-diffusion models: indirect predator-prey model, Lotka-Volterra food chain model, SIS epidemic model, and population-toxicant model, respectively. And our thesis focuses on

- Investigating the effects of the cross-diffusion strategy on pattern formation or species persistence;
- Developing some new ideas/methods to overcome challenges arising from the inapplicability of the comparison principle, an essential tool in reaction-diffusion models without cross-diffusion.

In the following sections, we shall introduce these four mathematical models, research problems and research highlights based on our published journal papers [22, 24, 25].

1.2 An Indirect Predator-prey Model with Cross-diffusion

In ecological systems, some foraging predators may locate the prey by following the substances emitted from prey species, such as pheromones (kairomones) (cf. [154]), chemical alarm cues (cf. [37]), sexual signals (cf. [159]). This type of foraging behavior, called indirect prey-taxis, was first modeled in [129]. Conversely, prey species may exhibit anti-predation mechanisms by releasing the toxic or foul smelling stimulus to drive away their predators [26, 91]. Based on works [26, 91, 129], we focus on the following indirect prey-taxis system with an anti-predation mechanism:

$$\begin{cases} N_t = d_N \Delta N + (\lambda - N)N - NP, & x \in \Omega, t > 0, \\ P_t = \Delta[d(S)P] + (\mu - P)P + \gamma NP, & x \in \Omega, t > 0, \\ S_t = d_S \Delta S + \tau N - \eta S, & x \in \Omega, t > 0, \\ \partial_\nu N = \partial_\nu P = \partial_\nu S = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $N := N(x,t)$ and $P := P(x,t)$ denote the prey and predator densities at position x and time t, respectively. $S := S(x,t)$ is the density of signal released by prey species N with constant production rate $\tau > 0$ and natural decay with rate $\eta > 0$. $\lambda - N$ and $\mu - P$ represent the per-capita growth rate of prey and predators, respectively, where the constant $\lambda > 0$ is the so-called carry capacity and the constant $\mu \neq 0$. The predator is said to be specialist if $\mu < 0$ and generalist if $\mu > 0$. The signal is assumed to undergo random diffusion with a constant rate $d_S > 0$, and $d_N > 0$ is a constant denoting random diffusion of the prey. The predator adopts a signal/density-dependent type cross-diffusion with a positive rate function $d(S)$. Specifically, the term $\Delta[d(S)P]$ represents that the predator's motility is less active when encountering the attractive signals released by the prey if $d'(S) < 0$ and demonstrates the indirect prey-taxis mechanism [129, 135]. When $d'(S) > 0$, it means that the predator will increase its motility if they come into the toxic or foul smelling stimulus released by the prey and demonstrates the anti-predation mechanism of prey [26, 91].

For the predator-prey system, it has been studied for a long time including global dynamics, traveling waves, pattern formation and so on. The research [135] on (1.2) shows that the density-dependent type indirect prey-taxis (i.e., $d'(S) < 0$) can induce the spatio-temporal periodic patterns (Andronov-Hopf bifurcation), this contrasts sharply with direct prey-taxis [65] in which no pattern formation happens. Hence, a critical question arises:

- When the prey act anti-predation behavior (i.e., $d'(S) > 0$), how does the pattern formation differ from the case of the indirect prey-taxis mechanism (i.e., $d'(S) < 0$)?

To explore this question, we shall in Chapter 2 study the global existence of classical solutions, global stability of constant steady states, and bifurcation as well as existence of non-constant positive steady-state solutions. Our results will demonstrate that the anti-predation mechanism induces non-constant steady state patterns without triggering

Hopf bifurcation. This behavior differs from the density-dependent type indirect prey-taxis [135] which exhibits Hopf bifurcation, as well as both direct prey-taxis [65] and the cross-diffusion-free system (1.2), neither of which demonstrates pattern formation.

1.3 A Lotka-Volterra Food Chain Model with Cross-diffusion

Beyond simple predator-prey dynamics (e.g., lion-gazelle system), natural ecosystems exhibit intricate trophic webs. For instance, marine food webs span multiple levels: from plankton to fish, sharks, whales, and ultimately humans, with numerous intermediate species occupying distinct trophic positions. Here, we consider the foundational three-trophic-level food chain model with intraguild predation and taxis mechanisms:

$$\begin{cases} u_t = d_1\Delta u + u(1-u) - b_1 uv - \gamma_1 uw, & x \in \Omega, t > 0, \\ v_t = d_2\Delta v - \xi\nabla\cdot(v\nabla u) + uv - b_2 vw - \theta_1 v, & x \in \Omega, t > 0, \\ w_t = \Delta w - \chi\nabla\cdot[w\nabla\phi(u,v)] + vw + \gamma_2 uw - \theta_2 w, & x \in \Omega, t > 0, \\ \partial_\nu u = \partial_\nu v = \partial_\nu w = 0, & x \in \partial\Omega, t > 0, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, and $u := u(x,t)$, $v := v(x,t)$, $w := w(x,t)$ denote the densities of the prey species, primary and top predators, respectively. The parameters $d_i > 0$ are diffusion coefficients, the cross-diffusion term $-\xi\nabla\cdot(v\nabla u)$ describes the directional movement of primary predators toward their prey density gradient (called prey-taxis mechanism [68]). Similarly, the cross-diffusion term $-\chi\nabla\cdot[w\nabla\phi(u,v)]$ describes the top predators move toward to high gradient of the signal produced as a result of the interaction between the prey and primary predator. The constants $\theta_1 > 0$ and $\theta_2 > 0$ represent the mortality rates of the primary and top predators, respectively. The parameters $b_i > 0$ and $\gamma_i \geq 0$ $(i = 1, 2)$ describe the interaction of interspecies.

For the system (1.3), [66] studied the global dynamics of system (1.3) in a two dimensional bounded domain under the assumptions $\gamma_1 = \gamma_2 = 0$ and $\phi(u,v) = v$, and proved that no pattern formation occurs. When $\gamma_1, \gamma_2 > 0$, the study [46] incorporated the intraspecific competitions for v and w along with $\phi(u,v) = uv$, and studied the global boundedness for $\gamma_1, \gamma_2 \geq 0$, the global stability as well as pattern formation for $\gamma_1 = \gamma_2 = 0$ in one dimensional space. Hence, we ae inspired to investigate

- Whether pattern formation occurs for (1.3) with $\gamma_1, \gamma_2 > 0$ and $\phi(u,v) = v$;
- Whether pattern formation occurs for other forms of $\phi(u,v)$ (instead of $\phi(u,v) = v$) when $\gamma_1 = \gamma_2 = 0$ and no intraspecific competition exists for v and w.

In Chapter 3, we shall provide positive answers to these questions. Our results will demonstrate that prey-taxis can destabilize a positive equilibrium in a three-species Lotka-Volterra model with intraguild predation, which contrasts with the well-known stabilizing effect observed in simpler two-species predator-prey systems or three-species Lotka-Volterra model without intraguild predation.

1.4 An SIS Epidemic Model with Cross-diffusion

To incorporate the effects of human behaviors and public health quarantine measures on the mobility of individuals during the outbreak of disease such as COVID-19 [60, 74, 131]), we introduce the cross-diffusion strategy for the infected individuals into an SIS model:

$$\begin{cases} S_t = d_S \Delta S + \Lambda(x) - \theta S - \alpha(x)\frac{SI}{S+I} + \beta(x) I, & x \in \Omega, t > 0, \\ I_t = d_I \Delta[\gamma(S) I] + \alpha(x)\frac{SI}{S+I} - [\beta(x) + \eta(x)] I, & x \in \Omega, t > 0, \\ \partial_\nu S = \partial_\nu I = 0, & x \in \partial\Omega, t > 0, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary; $S := S(x,t)$ and $I := I(x,t)$ denote the population density of the susceptible and infected individuals at position $x \in \Omega \subset \mathbb{R}^n$ and time $t > 0$, respectively. The susceptible individuals are assumed to move randomly with rate d_S while infected individuals adopt a density-dependent type cross-diffusion with a positive rate function $\gamma(S) \in C^3([0,\infty))$ satisfying

$$\gamma'(S) > 0 \text{ for all } S \in [0,\infty). \tag{1.5}$$

Note that $\Delta[\gamma(S)I] = \nabla \cdot (\gamma(S)\nabla I) + \nabla \cdot (I\gamma'(S)\nabla S)$. The cross-diffusion along with the condition (1.5) indicates that the infected individuals will move away from the area with a higher density of susceptible individuals (like quarantine measure) while dispersing themselves at a rate increasing with respect to the density of susceptible individuals (crowd avoidance). The model (1.4) has included demography changes (recruitment and death of population), where the recruitment of the susceptible population is represented by $\Lambda(x) - \theta S$ with $\Lambda(x)$ being a non-negative Hölder continuous function on $\overline{\Omega}$ and $\theta \geq 0$ being a constant; $\alpha(x)$, $\beta(x)$ and $\eta(x)$ are non-negative Hölder continuous functions on $\overline{\Omega}$ accounting for the disease transmission rate, recovery rate, and death rate of the infected individuals, respectively.

We shall study the SIS epidemic model (1.4) in Chapter 4, and aim to

- explore how the cross-diffusion diffusion strategy can play positive roles in controlling the spread of disease.

Since this cross-diffusion describes the outcome of quarantine measures to the population mobility during the outbreak of infectious disease, our results will elucidate whether the quarantine measures help to control the disease spread from a theoretical perspective. As we know, this is the first work on the SIS epidemic model (1.4) with cross-diffusion (i.e., $\gamma(S)$ is non-constant) and there are no results available for such kind of models.

1.5 A Population-toxicant Model with Cross-diffusion

In aquatic ecosystems, species may detect and avoid toxicant [9, 132]), and the input of toxicant may exhibit temporal periodicity driven by seasonal factors [14]. Therefore, we are inspired to incorporate the negative toxicant-taxis (cf. [32]), and spatially inhomogeneous and time-periodic toxicant input into a population-toxicant system, which reads as

$$\begin{cases} u_t = d_1\Delta u + \chi\nabla\cdot(u\nabla w) + u(r-u-mw), & x\in\Omega,\ t>0,\\ w_t = d_2\Delta w + h(x,t) - \alpha w - \beta uw, & x\in\Omega,\ t>0,\\ \partial_\nu u = \partial_\nu w = 0, & x\in\partial\Omega,\ t>0, \end{cases} \tag{1.6}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain with a smooth boundary. $u(x,t)$ and $v(x,t)$ represent the species and toxicant densities at position x and time t, respectively. The species u diffuses through random diffusion $d_1\Delta u$ with the diffusion rate $d_1>0$, and cross-diffusion $+\chi\nabla\cdot(u\nabla w)$ with the sensitivity parameter $\chi>0$. The term $+\chi\nabla\cdot(u\nabla w)$ describes that species move from areas with high toxicant concentrations to regions with lower toxicant concentration (i.e., negative toxicant-taxis). The toxicant w diffuses randomly with rate $d_2>0$. The positive constants $r,\ m,\ \alpha$ and β represent, respectively, the species' intrinsic growth rate, the toxicant-induced death rate of species, the toxicant's loss rate due to environmental detoxication or microbial degradation, and the toxicant uptake rate by species. The function $h(x,t)$ represents the (spatio-)temporally inhomogeneous input of toxicant into the habit Ω.

For (1.6) with the time-periodic toxicant input $h(x,t)$, the work [86] established the global stability of periodic solutions and explored the asymptotic profiles of positive periodic solutions when diffusion rates are small or large in the absence of toxicant-taxis (i.e., $\chi=0$). Their results indicate that the toxicant input affect the species persistence and extinction.

In Chapter 5, we shall study (1.6) with $\chi>0$ and a more general toxicant input function $h(x,t)$, and investigate

- whether the cross-diffusion diffusion strategy (i.e., toxicant-taxis) enhance aquatic population persistence in heterogeneous polluted environments.

In fact, (1.6) with the cross-diffusion term (i.e., $\chi > 0$) is a non-monotone dynamical system, thus the asymptotic theory of monotone systems (c.f. [157, Chapter 3]) and the comparison principle become inapplicable. As a result, no established methods in the literature can be employed, making the analysis of global dynamics for (1.6) with $\chi > 0$ significantly more challenging. Our proof in Chapter 5 develops some new ideas/outlines to overcome these difficulties, which can be applied to prove the existence of time-periodic or non-constant steady-state solutions, and uniform persistence for general cross-diffusion models.

1.6 Organization of the Thesis

The organization of this thesis is below:

Chapter 2 will explore an indirect prey-taxis system with an anti-predation mechanism (1.2). Section 2.2 will establish the global in-time existence and uniqueness of classical solutions, while Section 2.3 will examine the global stability of constant steady states. In Section 2.4, we shall demonstrate that the anti-predation mechanism can generate steady-state bifurcation but cannot induce Hopf bifurcation. Finally, Section 2.5 will prove the global existence of non-constant positive steady-state solutions.

Chapter 3 will investigate a three-species Lotka-Volterra food chain model with intraguild predation and taxis mechanisms (prey-taxis and alarm-taxis) (1.3) in an open bounded interval. Section 3.2 will prove the existence of global classical solutions with uniform-in-time bounds. Section 3.3 will explore the global stability of constant steady states. In Section 3.4, we shall conduct linear stability and instability analyses to study possible pattern formation. Finally, Section 3.5 will numerically verify theoretical analysis in Section 3.4 and explore the effects of taxis mechanisms.

Chapters 2-3 focus on the effects of cross-diffusion on pattern formation. In the following two chapters, we shift our focus to the effects of cross-diffusion on species persistence.

Chapter 4 will study an SIS model with a cross-diffusion dispersal strategy for infected individuals (1.4). Sections 4.2 and 4.4 will establish the existence of global classical solutions and global stability, respectively. Section 4.3 will give a variational expression of the basic reproduction number R_0, and explore its properties as well as the threshold dynamics in terms of R_0. Section 4.5 will use numerical simulations to illustrate the applications of our analytical results and speculate on unproven results.

Chapter 5 will consider a population-toxicant model in a time-periodic environment with toxicant-taxis (1.6). In Section 5.2, we shall establish the global existence of classical solutions and the *ultimately uniform boundedness*. Section 5.3 will prove the existence of positive periodic solution and uniform persistence, and theoretically show that a large coefficient of cross-diffusion can enlarge the interval of uniform persistence. Section 5.4 will establish the global stability of the semi-trivial periodic solution, as well as the uniqueness and global stability of the positive periodic solutions. Chapter 6 will summarize our results in Chapters 2-5 and list some open questions.

For clarity, we shall abbreviate $\int_\Omega f dx$, $\|f\|_{L^p(\Omega)}$ and $\int_0^T \int_\Omega f dx dt$ as $\int_\Omega f$, $\|f\|_{L^p}$ and $\int_{Q_T} f$, respectively. Additionally, we clarify that the results of Chapters 2-4 have been published as our papers [22, 24, 25].

Chapter 2

Global Dynamics of an Indirect Prey-taxis System with an Anti-predation Mechanism

2.1 Introduction and Main Results

We clarify that the context presented in this chapter has been published in our journal paper [24].

2.1.1 Introduction

Prey-taxis is the direct (attractive or repulsive) movement of predators along prey density gradients. It was first proposed by Kareiva and Odell [68] to explore the consequence of the predator-prey interaction between the ladybug beetle *Coccinella septempunctata* (predator) and the golden aphid *Uroleucon nigrotuberculatum* (prey). The model proposed in [68] in its generalized form can be formulated as

$$\begin{cases} N_t = d_N \Delta N + Nf(N) - PF(N), & x \in \Omega, t > 0, \\ P_t = \nabla \cdot (d(N)\nabla P) - \nabla \cdot (P\xi(N)\nabla N) + \gamma PF(N) + Pg(P), & x \in \Omega, t > 0, \\ \partial_\nu N = \partial_\nu P = 0, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

where $N = N(x,t)$ and $P = P(x,t)$ denote the prey and predator densities at position x and time t, respectively, and $d_N > 0$ is a constant denoting the prey diffusivity. The term $\nabla \cdot (d(N)\nabla P)$ describes the predator's diffusion with a prey-dependent diffusion coefficient $d(N) > 0$, $d'(N) < 0$, and $-\nabla \cdot (P\xi(N)\nabla N)$ accounts for the prey-taxis with a prey-dependent prey-tactic coefficient $\xi(N) \geq 0$. The function $F(N)$ is the so-called functional response function while $f(N)$ and $g(P)$ represent the per-capita growth rate of prey and predators, respectively. The commonly used forms for f and g are $f(N) = \lambda - N$ where $\lambda > 0$ is the so-called carry capacity and $g(P) = \mu - P$ with $\mu \neq 0$ where the predator is said to be specialist if $\mu < 0$ and generalist if $\mu > 0$. The system (2.1) has been extensively studied in recent years, we refer the readers to [16, 49, 64, 65, 79, 80, 122, 152] and references therein for more related works.

Different from the process of direct prey-taxis described by (2.1), some foraging predators may locate the prey by following the density gradient of substances, such as pheromones (kairomones) (cf. [154]), chemical alarm cues (cf. [37]), sexual signals (cf. [159]), smells and so on, which are emitted from prey species. For instance, Parasitoids exploit both plant volatiles and herbivore pheromones to locate their insect prey [35], the wolf spider *Pardosa milvina* moves along stimulus released by crickets [52]. This type of foraging behavior is called indirect prey-taxis, which was first modeled by Tello and Wrzosek [129]. Its general form reads as

$$\begin{cases} N_t = d_N \Delta N + N f(N) - P F(N), & x \in \Omega, t > 0, \\ P_t = \nabla \cdot (d(S) \nabla P) - \nabla \cdot (P \xi(S) \nabla S) + \gamma P F(N) + P g(P), & x \in \Omega, t > 0, \\ S_t = d_S \Delta S + \tau N - \eta S, & x \in \Omega, t > 0, \\ \partial_\nu N = \partial_\nu P = \partial_\nu S = 0, & x \in \partial\Omega. \end{cases} \tag{2.2}$$

All notations and parameters have the same interpretation as in Section 1.2.

Compared to the direct prey-taxis model (2.1), the indirect prey-taxis model (2.2) adds one equation for the released signal S and the movement of predators consists of two parts: diffusion part $\nabla \cdot (d(S)\nabla P)$ and prey-taxis part $-\nabla \cdot (P\xi(S)\nabla S)$ directed by the signal, where both diffusion and prey-taxis coefficients depend on the signal density. The term $\nabla \cdot (d(S)\nabla P)$ means the diffusion of predators with positive random diffusion coefficient $d(S)$. $-\nabla \cdot (P\xi(S)\nabla S)$ is referred to the indirect prey-taxis describes the biased diffusion of predators towards the regions of higher density of stimulus rather than prey with coefficient $\xi(S)$ if $\xi(S) > 0$ [129, 135], and engraves the predators retreat from the area of higher density of stimulus with coefficient $\xi(S)$ if $\xi(S) < 0$, which occurs in the occasion where prey act anti-predation behavior by using stimulus [26, 91].

Some results on (2.2) with $\xi(S) > 0$ (i.e., indirect prey-taxis system) have been developed. We refer to [2, 129, 134, 160] for the case with constant $d(S)$ and $\xi(S)$, and [139] for the case in which $d(S)$ is constant but $\xi(S)$ is non-constant. However, when both $d(S)$ and $\xi(S)$ are non-constant, there are only two recent works: [96] for general $d(S)$ and $\xi(S)$, and [135] for special case where $\xi(S) = -d'(S)$. In such a special case $\xi(S) = -d'(S)$, the diffusion terms of the second equation in (2.2) can be rewritten as $\Delta[d(S)P]$, which denotes that the diffusion of predator P is dependent on the density of signal S and is said to signal/density-dependent diffusion if $d'(S) \neq 0$. Specifically, the term $\Delta[d(S)P]$ represents that the predator's motility is less active when encountering the attractive signals released by the prey and is analogous to "density-suppressed motility" (cf. [39]) if $d'(S) < 0$. When $d'(S) > 0$, it means that the predator will increase its motility if they come into the toxic or foul smelling stimulus released by the prey and demonstrates the anti-predation mechanism of prey. The results in [135] demonstrate that the density-dependent type indirect prey-taxis can induce the time-periodic patterns (Andronov-Hopf bifurcation) even when the predator P adopt the Holling type I functional response (i.e., $F(N) = N$), which contrasts sharply with direct prey-taxis [65]. Hence, it is natural to ask:

- When the prey act anti-predation behavior (i.e., $d'(S) > 0$), how does the long-time behavior/the population distribution differ from the case of the indirect prey-taxis mechanism (i.e., $d'(S) < 0$)?

To explore this question, we focus on the following indirect prey-taxis system with an anti-predation mechanism

$$\begin{cases} N_t = d_N \Delta N + \lambda N - N^2 - NP, & x \in \Omega, t > 0, \\ P_t = \Delta[d(S)P] + \mu P - P^2 + \gamma NP, & x \in \Omega, t > 0, \\ S_t = d_S \Delta S + \tau N - \eta S, & x \in \Omega, t > 0, \\ \partial_\nu N = \partial_\nu P = \partial_\nu S = 0, & x \in \partial\Omega, \\ (N, P, S)(x, 0) = (N_0, P_0, S_0)(x), & x \in \Omega, \end{cases} \tag{2.3}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with smooth boundary and $d'(S) \neq 0$. The parameter λ, γ, τ and η are positive constants and $\mu \in \mathbb{R}$ is more general than the one in [96]. And the density-dependent function $d(S)$ accounts for the dispersal coefficient of the predator and fulfills the assumptions as below

(H_0) $d(S) \in C^3([0, \infty))$ and $d(S) > 0$ on $[0, \infty)$.

Our main goals include the following:

(A.1) Establish the global well-posedness of solutions (global existence and stability) to (2.3) under suitable conditions;

(A.2) Investigate the existence of spatially inhomogeneous patterns bifurcating from constant steady state when the predator employs an anti-predation strategy.

2.1.2 Main Results

The first main theorem on the global boundedness of solutions of (2.3) is given below.

Theorem 2.1 (Global boundedness). *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be a bounded domain with a smooth boundary. Assume that $(N_0, P_0, S_0) \in [W^{1,\infty}(\Omega)]^3$ satisfies $N_0(x), P_0(x), S_0(x) \geq 0$ $(\not\equiv 0)$ in $\bar{\Omega}$ and (H_0) holds. Then the system (2.3) admits a unique classical solution $(N, P, S) \in [C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty))]^3$ with $N, P, S > 0$ in $\Omega \times (0,\infty)$. Moreover, the solution (N, P, S) is uniform-in-time bounded in the following sense*

$$\|N(\cdot,t)\|_{W^{1,\infty}} + \|P(\cdot,t)\|_{L^\infty} + \|S(\cdot,t)\|_{W^{1,\infty}} \leq M \text{ for all } t > 0, \tag{2.4}$$

where the constant $M > 0$ is independent of t. Furthermore, one has

$$\|N(\cdot,t)\|_{L^\infty} \leq M_0 := \max\{\lambda, \|N_0\|_{L^\infty}\}. \tag{2.5}$$

Remark 2.1. *In fact, we may find a constant $K > 0$ defined in (2.25) such that*

$$\|S(\cdot,t)\|_{L^\infty} \leq K, \tag{2.6}$$

where K is particularly independent of t, μ and γ.

Next, we aim to study the global stability of solutions to (2.3). For convenience, we define the regions $\mathcal{R}_i$ $(i = 1,2,3,4)$ (see in Figure 2.1) as below

$$\mathcal{R}_1 := \{(\mu,\lambda) : \mu \leq -\lambda\gamma\};\ \mathcal{R}_2 := \{(\mu,\lambda) : -\lambda\gamma < \mu \leq 0\};$$
$$\mathcal{R}_3 := \{(\mu,\lambda) : 0 < \mu < \lambda\};\ \mathcal{R}_4 := \{(\mu,\lambda) : \mu \geq \lambda\}.$$

The constant steady state (N_c, P_c, S_c) of (2.3) satisfies

$$N_c(\lambda - N_c - P_c) = 0, \quad P_c(\mu - P_c + \gamma N_c) = 0, \quad \tau N_c - \eta S_c = 0.$$

One can easily solve the above equations to obtain

$$(N_c, P_c, S_c) = \begin{cases} (0,0,0) \text{ or } (\lambda, 0, \frac{\tau\lambda}{\eta}), & \text{if } (\mu,\lambda) \in \mathcal{R}_1, \\ (0,0,0) \text{ or } (\lambda, 0, \frac{\tau\lambda}{\eta}) \text{ or } (N^*, P^*, S^*), & \text{if } (\mu,\lambda) \in \mathcal{R}_2, \\ (0,0,0) \text{ or } (0,\mu,0) \text{ or } (\lambda, 0, \frac{\tau\lambda}{\eta}) \text{ or } (N^*, P^*, S^*), & \text{if } (\mu,\lambda) \in \mathcal{R}_3, \\ (0,0,0) \text{ or } (0,\mu,0) \text{ or } (\lambda, 0, \frac{\tau\lambda}{\eta}), & \text{if } (\mu,\lambda) \in \mathcal{R}_4, \end{cases}$$

where

$$(N^*, P^*, S^*) = \left(\frac{\lambda - \mu}{\gamma + 1}, \frac{\lambda\gamma + \mu}{\gamma + 1}, \frac{\tau(\lambda - \mu)}{\eta(\gamma + 1)}\right). \tag{2.7}$$

Then, we have the following results on the global stability of solutions to (2.3).

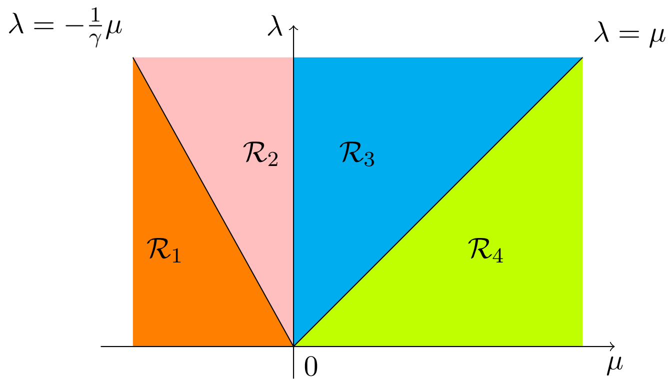


Figure 2.1: The plot of regions $\mathcal{R}_i$ $(i = 1, 2, 3, 4)$.

Theorem 2.2 (Global stability)**.** *Let the conditions in Theorem 2.1 hold and (N, P, S) be the solution of (2.3) obtained in Theorem 2.1. Then we can find constans $C_i > 0$, $\kappa_i > 0$ $(i = 1, 2, 3, 4, 5)$ independent of t and some constant $t_0 > 0$ such that the solution of (2.3) has the following convergence properties.*

(1) If $(\mu, \lambda) \in \mathcal{R}_1$, then $\left(\lambda, 0, \frac{\tau\lambda}{\eta}\right)$ is globally asymptotically stable such that for all $t > t_0$

$$\|N - \lambda\|_{L^\infty} + \|P\|_{L^\infty} + \|S - \frac{\tau\lambda}{\eta}\|_{L^\infty} \leq \begin{cases} C_1 e^{-\kappa_1 t}, & \text{if } \mu < -\lambda\gamma, \\ C_2(1+t)^{-\kappa_2}, & \text{if } \mu = -\lambda\gamma. \end{cases} \tag{2.8}$$

(2) If $(\mu, \lambda) \in \mathcal{R}_2 \cup \mathcal{R}_3$ and

$$\frac{\gamma(\gamma+1)}{\lambda\gamma + \mu} > \frac{\tau^2}{4\eta d_S} \max_{0 \leq S \leq \|S\|_{L^\infty}} \frac{|d'(S)|^2}{d(S)}, \tag{2.9}$$

then (N^*, P^*, S^*) is globally asymptotically stable such that

$$\|N - N^*\|_{L^\infty} + \|P - P^*\|_{L^\infty} + \|S - S^*\|_{L^\infty} \leq C_3 e^{-\kappa_3 t} \text{ for all } t > t_0. \tag{2.10}$$

(3) If $(\mu, \lambda) \in \mathcal{R}_4$ and

$$\gamma > \frac{\tau^2\lambda}{2\eta d_S} \max_{0 \leq S \leq \|S\|_{L^\infty}} \frac{|d'(S)|^2}{d(S)}, \tag{2.11}$$

then $(0, \mu, 0)$ is globally asymptotically stable such that for all $t > t_0$

$$\|N\|_{L^\infty} + \|P - \mu\|_{L^\infty} + \|S\|_{L^\infty} \leq \begin{cases} C_4 e^{-\kappa_4 t}, & \text{if } \mu > \lambda, \\ C_5(1+t)^{-\kappa_5}, & \text{if } \mu = \lambda. \end{cases} \tag{2.12}$$

It is easy to verify that $(0,0,0)$, $\left(\lambda, 0, \frac{\tau\lambda}{\eta}\right)$ and $(0,\mu,0)$ each have the same parameter regions of linear stability in both the system (2.3) and the corresponding the ordinary differential equations (ODE) system (c.f. [24]). This implies no spatial patterns bifurcate from $(0,0,0)$, $\left(\lambda, 0, \frac{\tau\lambda}{\eta}\right)$ and $(0,\mu,0)$ for any $d(S)$ satisfying (H_0). Moreover, for the same reasons, there are no spatial patterns bifurcating from (N^*,P^*,S^*) when the predator takes random dispersal (i.e., $d'(S)=0$). For the case of $d'(S)<0$, which describes the indirect prey-taxis, [135] demonstrated that the density-dependent type indirect prey-taxis in (2.3) can induce the time-periodic patterns, which contrasts sharply with direct prey-taxis [65].

Therefore, a relevant question arises: can spatial patterns bifurcate from (N^*,P^*,S^*) when the predator employs density-dependent dispersal with an anti-predation strategy (i.e., $d'(S)>0$). To give a satisfactory answer to this question for general $d(S)$ is quite hard due to excessive technical computations and abstraction of $d(S)$. Below we shall focus on a specific simple case $d(S)=d_P+\beta S$ to discuss possible bifurcations near (N^*,P^*,S^*) and prove the existence of non-constant steady states. However, the analysis directly extends to other forms of $d(S)$ and results can be obtained similarly.

Before stating our results, we introduce some notations. Let $0=\sigma_1<\sigma_2<\sigma_3<\cdots$ be the eigenvalues of the operator $-\Delta$ on Ω with the homogeneous Neumann boundary condition. Denote

$$\begin{aligned}\beta_m^T := &\frac{1}{|H(\sigma_m)|}[\sigma_m^2 d_N d_S d_P + \sigma_m\left(\eta d_N d_P + d_N d_S P^* + d_P d_S N^*\right)\\ &+ (d_N P^*\eta + d_P N^*\eta + d_S N^* P^* + \gamma N^* P^* d_S) + (1+\gamma)N^*P^*\eta/\sigma_m],\ m\geq 2,\end{aligned} \tag{2.13}$$

where

$$H(z) := z^2 d_N d_S S^* + z(\eta d_N + d_S N^*)S^* + \frac{\tau N^*(\lambda(1-\gamma)-2\mu)}{\gamma+1} \text{ with } z\geq 0, \tag{2.14}$$

and

$$\overline{\sigma} = \frac{-(\eta d_N + d_S N^*)S^* + \sqrt{(\eta d_N + d_S N^*)^2(S^*)^2 + 4 d_N d_S S^* \tau N^* \frac{[2\mu-\lambda(1-\gamma)]}{\gamma+1}}}{2 d_N d_S S^*}, \tag{2.15}$$

which is the positive root of $H(z)$. Then we have the following bifurcation results.

Theorem 2.3 (Bifurcation)**.** *Let σ_m $(m\in\mathbb{N}^+)$ be the eigenvalues of the operator $-\Delta$ with the homogeneous Neumann boundary condition and $\gamma,\tau,\eta,d_N,d_P,d_S$ be fixed parameters. Assume that β_m^T, $H(z)$ and $\overline{\sigma}$ are defined in* (2.13)*,* (2.14) *and* (2.15)*, respectively, and $(\mu,\lambda)\in\mathcal{R}_2\cup\mathcal{R}_3$. Suppose $d(S)=d_p+\beta S$. Then the following conclusions hold.*

(1) System (2.3) *has no Hopf bifurcation arising from* (N^*, P^*, S^*) *for any* $\beta > 0$*;*

(2) System (2.3) *undergoes a steady state bifurcation near* (N^*, P^*, S^*) *at* $\beta = \beta_j^T$ *if the following conditions are satisfied*

(i) $\frac{\lambda(1-\gamma)}{2} < \mu < \lambda$*;*

(ii) there exists an integer $j \geq 2$ *such that* $\sigma_j < \overline{\sigma}$*.*

From Theorem 2.3, we see that the anti-predation mechanism only yields steady state bifurcation arising from (N^*, P^*, S^*), which is different from the indirect prey-taxis. Since the aforementioned steady state bifurcation is just local, we shall show the global existence of non-constant steady state solutions (i.e., stationary patterns) of (2.3) with $d(S) = d_P + \beta S$ by applying Leray-Schauder degree theory.

Therefore, we consider the following stationary problem:

$$\begin{cases} d_N \Delta N + \lambda N - N^2 - NP = 0, & x \in \Omega, \\ \Delta[(d_P + \beta S)P] + \mu P - P^2 + \gamma NP = 0, & x \in \Omega, \\ d_S \Delta S - \eta S + \tau N = 0, & x \in \Omega, \\ \partial_\nu N = \partial_\nu P = \partial_\nu S = 0, & x \in \partial\Omega. \end{cases} \tag{2.16}$$

Theorem 2.4 (Stationary patterns)**.** *Let* σ_m $(m \in \mathbb{N}^+)$ *be the eigenvalues of the operator* $-\Delta$ *with the homogeneous Neumann boundary condition and* $\gamma, \tau, \eta, d_N, d_P, d_S$ *be fixed parameters. Then there is a positive constant* β^* *such that (2.16) has at least one non-constant positive solution if* $\beta \geq \beta^*$ *and the following conditions are satisfied:*

(i) $\frac{\lambda(1-\gamma)}{2} < \mu < \lambda$*;*

(ii) there exist an integer $j \geq 2$ *such that* $\overline{\sigma} \in (\sigma_j, \sigma_{j+1})$*;*

(iii) the sum $\sum_{m=2}^{j} \dim E(\sigma_m)$ *is odd,*

where $\overline{\sigma}$ *is defined in* (2.15) *and* $E(\sigma_m)$ *is the eigenspace corresponding to* σ_m *in* $H^1(\Omega)$*.*

Remark 2.2. *When* $\beta = 0$ *and hence* $d(S) = d_P$ *is constant, we know from Theorem 2.2 that the constant positive solution* (N^*, P^*, S^*) *is globally asymptotically stable for any* $(\mu, \lambda) \in \mathcal{R}_2 \cup \mathcal{R}_3$*, and hence (2.16) has no non-constant positive solution. However, Theorem 2.4 shows that, under suitable additional assumptions, (2.16) has at least one non-constant positive solution for large* $\beta > 0$*. This implies that the anti-predation* β *helps create spatial heterogeneity.*

By Theorem 2.4, we obtain the result on the system (2.16) for the case of one-dimensional space $\Omega = (0, l)$ with the constant $l > 0$.

Proposition 2.1. *Let σ_m ($m \in \mathbb{N}^+$) be the eigenvalues of the operator $-\Delta$ with the homogeneous Neumann boundary condition and $\gamma, \tau, \eta, d_N, d_P, d_S$ be fixed parameters. Then there is a positive constant β^* such that (2.16) has at least one non-constant positive solution if $\beta \geq \beta^*$ and the following conditions are satisfied:*

(i) $\frac{\lambda(1-\gamma)}{2} < \mu < \lambda$;

(ii) there exists an integer $j \geq 2$ such that $\frac{(j-1)^2\pi^2}{l^2} < \overline{\sigma} < \frac{j^2\pi^2}{l^2}$;

(iii) the sum $\sum_{m=2}^{j} \dim E(\sigma_m) = j - 1$ *is odd,*

where $\overline{\sigma}$ is defined in (2.15)*, $E(\sigma_m)$ is the eigenspace corresponding to σ_m in $H^1(\Omega)$.*

2.2 Global Boundedness: Proof of Theorem 2.1

In this section, we shall prove Theorem 2.1 by semigroup estimates and Moser iteration. c_i and M_i ($i = 1, 2, 3, \cdots$) are used to denote the generic positive constants which may vary in the context. First, the local existence of classical solutions to the system (2.3) can be proved by Amann's theorems [7, 8] (see details in [96]).

2.2.1 Local Existence and Preliminaries

Lemma 2.1 (Local existence)**.** *Let the conditions in Theorem 2.1 hold. Then there exists a $T_{\max} \in (0, \infty]$ such that the system* (2.3) *admits a unique classical solution* $(N, P, S) \in [C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max}))]^3$ *with $N, P, S > 0$ in $\Omega \times (0, T_{\max})$. Furthermore, if $T_{\max} < \infty$, then*

$$\lim_{t \nearrow T_{\max}} (\|N(\cdot, t)\|_{L^\infty} + \|P(\cdot, t)\|_{L^\infty} + \|S(\cdot, t)\|_{W^{1,\infty}}) = \infty.$$

Lemma 2.2. *Let (N, P, S) be the solution of* (2.3) *obtained in Lemma 2.1. Then one has*

$$0 < N(\cdot, t) \leq M_0 := \max\{\lambda, \|N_0\|_{L^\infty}\} \text{ for all } x \in \Omega, t \in (0, T_{\max}). \tag{2.17}$$

Proof. Using the same arguments as the proof of [64, Lemma 2.2], we get (2.17). □

Lemma 2.3. *Let (N, P, S) be the solution of (2.3) obtained in Lemma 2.1. Then one has*

$$\int_\Omega P(\cdot, t) \leq M_1 \text{ for all } t \in (0, T_{\max}), \tag{2.18}$$

and

$$\int_\Omega S^n(\cdot, t) \leq M_2 \left((\tau/\eta)^n + 1\right) \text{ for all } t \in (0, T_{\max}), \tag{2.19}$$

where

$$M_1 := \frac{(|\mu| + \gamma M_0 + 1)^2 |\Omega|}{2} + \int_\Omega P_0, \; M_2 := (n-1)^{n-1} \left(\frac{2M_0}{n}\right)^n |\Omega| + \|S_0\|_{L^\infty}^n |\Omega|.$$

Proof. Integrating the second equation of (2.3) over Ω, using (2.17) and Young's inequality, one derives

$$\begin{aligned}
\frac{d}{dt}\int_\Omega P + \int_\Omega P^2 + \int_\Omega P &= \mu \int_\Omega P + \gamma \int_\Omega NP + \int_\Omega P \\
&\leq (|\mu| + \gamma M_0 + 1) \int_\Omega P \\
&\leq \frac{1}{2}\int_\Omega P^2 + \frac{(|\mu| + \gamma M_0 + 1)^2 |\Omega|}{2}.
\end{aligned}$$

Then it follows that

$$\frac{d}{dt}\int_\Omega P + \int_\Omega P \leq \frac{(|\mu| + \gamma M_0 + 1)^2 |\Omega|}{2},$$

which, along with Grönwall's inequality, gives (2.18) directly.

Next, we show (2.19). Multiplying the third equation of (2.3) by S^{n-1} $(n \geq 1)$, and using (2.17) and Young's inequality again, we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega S^n + n(n-1)d_S \int_\Omega S^{n-2}|\nabla S|^2 + \eta n \int_\Omega S^n &\leq \tau M_0 n \int_\Omega S^{n-1} \\
&\leq \frac{\eta n}{2}\int_\Omega S^n + \hat{M},
\end{aligned} \tag{2.20}$$

where $\hat{M} := (2(n-1)/\eta n)^{n-1} (\tau M_0)^n |\Omega|$. Then (2.20) leads to

$$\frac{d}{dt}\int_\Omega S^n + \frac{\eta n}{2}\int_\Omega S^n \leq (2(n-1)/\eta n)^{n-1} (\tau M_0)^n |\Omega|,$$

which alongside Grönwall's inequality implies

$$\begin{aligned}
\int_\Omega S^n &\leq (n-1)^{n-1} (2\tau M_0/\eta n)^n |\Omega| + \int_\Omega S_0^n \\
&\leq M_2 \left((\tau/\eta)^n + 1\right),
\end{aligned}$$

and hence (2.19) holds. The proof of Lemma 2.3 is completed. □

2.2.2 Boundedness of Solutions

Lemma 2.4. *Let* (N, P, S) *be the solution of* (2.3) *obtained in Lemma 2.1. Then, there admits a constant* $M_3 > 0$ *such that*

$$\|S(\cdot,t)\|_{W^{1,\infty}} \leq M_3 \text{ for all } t \in (0, T_{\max}). \tag{2.21}$$

Proof. We rewrite the third equation of (2.3) as

$$S_t = d_S \Delta S - d_S S + \tau N + (d_S - \eta) S. \tag{2.22}$$

Denote the Neumann heat semigroup in Ω by $(e^{\Delta t})_{t>0}$. Then using Duhamel's principle to (2.22), one has

$$\begin{aligned} S(\cdot,t) &= e^{td_S(\Delta-1)} S_0 + \int_0^t e^{(t-s)d_S(\Delta-1)} [\tau N + (d_S - \eta) S](\cdot,s) ds \\ &\leq e^{td_S(\Delta-1)} S_0 + \int_0^t e^{(t-s)d_S(\Delta-1)} (\tau N + d_S S)(\cdot,s) ds. \end{aligned} \tag{2.23}$$

By the well-known semigroup estimate [147, Lemma 1.3], we can find a constant $\sigma_1 > 0$ depending only on Ω such that

$$\begin{aligned} \|S(\cdot,t)\|_{L^\infty} &\leq \|e^{td_S(\Delta-1)} S_0\|_{L^\infty} + \int_0^t \|e^{(t-s)d_S(\Delta-1)} (\tau N + d_S S)(\cdot,s)\|_{L^\infty} ds \\ &\leq \sigma_1 \|S_0\|_{L^\infty} + \sigma_1 \int_0^t \left(1 + (t-s)^{-\frac{1}{2}}\right) e^{-d_S(t-s)} \|\tau N + d_S S\|_{L^n} ds \\ &\leq \sigma_1 \|S_0\|_{L^\infty} + \frac{\sigma_1}{d_S} \left(\tau M_0 |\Omega|^{\frac{1}{n}} + d_S M_2^{\frac{1}{n}} \left((\tau/\eta)^n + 1\right)^{\frac{1}{n}}\right) \cdot \left(1 + d_S^{\frac{1}{2}} \Gamma(1/2)\right), \end{aligned} \tag{2.24}$$

where we have used (2.17), (2.19) and $\Gamma(\cdot)$ denotes the Gamma function. Therefore, (2.6) follows by letting

$$Q := K_0 \big[1 + \big(\tau/d_S + ((\tau/\eta)^n + 1)^{\frac{1}{n}}\big) \cdot \big(1 + d_S^{\frac{1}{2}}\big)\big], \tag{2.25}$$

with $K_0 := \sigma_1 \|S_0\|_{L^\infty} + \sigma_1 \big(M_0 |\Omega|^{\frac{1}{n}} + M_2^{\frac{1}{n}}\big)(1 + \Gamma(1/2))$ independent of t, μ, γ.

Using the semigroup estimate again, we can find a constant $\sigma_2 > 0$ depending only on

Ω such that

$$
\begin{aligned}
\|\nabla S(\cdot,t)\|_{L^\infty} &\le \|\nabla e^{td_S(\Delta-1)}S_0\|_{L^\infty} + \int_0^t \|\nabla e^{(t-s)d_S(\Delta-1)}[\tau N + (d_S-\eta)S](\cdot,s)\|_{L^\infty} ds \\
&\le c_1 + \sigma_2(\tau M_0 + d_S\Gamma_1 + \eta\Gamma_1)\int_0^t \left(1+(t-s)^{-\frac{1}{2}}\right) e^{-(\lambda_1+1)d_S(t-s)} ds \\
&\le c_1 + \frac{\sigma_2(\tau M_0 + d_S\Gamma_1 + \eta\Gamma_1)}{(\lambda_1+1)d_S}\left(1 + (\lambda_1+1)^{\frac{1}{2}} d_S^{\frac{1}{2}} \Gamma(1/2)\right)
\end{aligned}
$$

for all $t \in (0, T_{\max})$, which together with (2.24) gives (2.21) directly. □

Lemma 2.5. *Let (N,P,S) be the solution of* (2.3) *obtained in Lemma 2.1. Then, there admits a constant $M_4 > 0$ such that*

$$
\|P(\cdot,t)\|_{L^\infty} \le M_4 \text{ for all } t \in (0, T_{\max}). \tag{2.26}
$$

Proof. Multiplying the second equation of (2.3) by $P^{k-1} (k \ge 2)$, and integrating the result over Ω, we have

$$
\begin{aligned}
&\frac{1}{k}\frac{d}{dt}\int_\Omega P^k + (k-1)\int_\Omega P^{k-2} d(S)|\nabla P|^2 + \int_\Omega P^{k+1} \\
&\quad = -(k-1)\int_\Omega P^{k-1} d'(S)\nabla P \cdot \nabla S + \mu \int_\Omega P^k + \gamma \int_\Omega N P^k.
\end{aligned} \tag{2.27}
$$

From (H_0) and $0 < S(\cdot,t) \le Q$, there admit constants $\delta_i > 0$ $(i = 1,2,3,4)$ such that

$$
\delta_1 \le d(S) \le \delta_2, \tag{2.28}
$$

and

$$
\delta_3 \le |d'(S)| \le \delta_4. \tag{2.29}
$$

Then applying (2.28), (2.29), (2.5), (2.21) and Young's inequality, we derive from (2.27) that

$$
\begin{aligned}
&\frac{1}{k}\frac{d}{dt}\int_\Omega P^k + \delta_1(k-1)\int_\Omega P^{k-2}|\nabla P|^2 + \int_\Omega P^{k+1} + (k-1)\int_\Omega P^k \\
&\le (k-1)\delta_4 \int_\Omega P^{k-1}|\nabla P||\nabla S| + \mu\int_\Omega P^k + \gamma M_0 \int_\Omega P^k + (k-1)\int_\Omega P^k \\
&\le \frac{\delta_1(k-1)}{2}\int_\Omega P^{k-2}|\nabla P|^2 + \frac{(k-1)\delta_4^2}{2\delta_1}\int_\Omega P^k|\nabla S|^2 + (k-1)(|\mu| + \gamma M_0 + 1)\int_\Omega P^k \\
&\le \frac{\delta_1(k-1)}{2}\int_\Omega P^{k-2}|\nabla P|^2 + (k-1)\left(\frac{\delta_4^2 M_3^2}{2\delta_1} + |\mu| + \gamma M_0 + 1\right)\int_\Omega P^k,
\end{aligned}
$$

which yields

$$\frac{d}{dt}\int_\Omega P^k + k(k-1)\int_\Omega P^k + \frac{\delta_1 k(k-1)}{2}\int_\Omega P^{k-2}|\nabla P|^2 \le c_1 k(k-1)\int_\Omega P^k, \tag{2.30}$$

where $c_1 = \frac{\delta_4^2 M_3^2}{2\delta_1} + |\mu| + \gamma M_0 + 1 > 0$ is independent of t and k.

Then using the Moser iteration process (cf. [123] or [3]) and (2.18), from (2.30) one obtains (2.26) readily. Hence, we complete the proof of Lemma 2.5. □

Proof of Theorem 2.1. The combination of (2.17), (2.21) and (2.26) gives a constant $c_1 > 0$ such that

$$\|N(\cdot,t)\|_{L^\infty} + \|S(\cdot,t)\|_{W^{1,\infty}} + \|P(\cdot,t)\|_{L^\infty} \le c_1. \tag{2.31}$$

Noting (2.31), using Duhamel's principle to the first equation of (2.3) and proceeding with the similar way as the proof in Lemma 2.4 alongside the semigroup estimate [147], one has $\|\nabla N(\cdot,t)\|_{L^\infty} \le c_2$, which together with (2.31) and Lemma 2.1 yields Theorem 2.1. □

2.3 Global Stability: Proof of Theorem 2.2

In this section, we shall show (N, P, S) obtained in Theorem 2.1 will converge to constant steady states and give the convergence rate. We start by presenting a result that will be utilized in the subsequent analysis.

Lemma 2.6. *(Bărălat's Lemma [11]) If $g : [1,\infty) \to \mathbb{R}$ is a uniformly continuous function such that $\lim_{t\to\infty}\int_1^t g(s)ds$ exists, then $\lim_{t\to\infty} g(t) = 0$.*

Next, we improve the regularity of the solution (N, P, S).

Lemma 2.7. *Let (N, P, S) be the solution of (2.3) obtained in Theorem 2.1. Then there exist $\theta \in (0,1)$ and $M_5 > 0$ such that*

$$\|(N,P,S)(\cdot,t)\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar{\Omega}\times[1,\infty))} \le M_5, \ \textit{for all} t \ge 1. \tag{2.32}$$

Proof. Based on (2.4), (2.32) is a consequence of the Hölder estimates for quasilinear parabolic equations (cf. [111, Theorem 1.3 and Remark 1.4]) and the standard parabolic Schauder theory [76]. The proof details can follow the similar procedures as the proof in [145, Lemma 3.4]. □

Proof of Theorem 2.2(1)**.** We consider the following energy functional

$$\mathcal{L}_1(t) := \mathcal{L}_1(N,P,S) := \gamma \int_\Omega \left(N - \lambda - \lambda \ln \frac{N}{\lambda} \right) + \int_\Omega P + \frac{\gamma\eta}{2\tau^2} \int_\Omega \left(S - \frac{\tau\lambda}{\eta} \right)^2. \tag{2.33}$$

Define $f(y) = y - y_* \ln y$. Then $f'(y_*) = 0$ and the Taylor's expansion entails

$$y - y_* - y_* \ln \frac{y}{y_*} = f(y) - f(y_*) = \frac{f''(z)}{2}(y-y_*)^2 = \frac{y_*}{2z^2}(y-y_*)^2 \geq 0 \tag{2.34}$$

for all $y, y_* > 0$ and z is between y and y_*. Then, we take $y = N$, $y_* = \lambda$ in (2.34) to obtain

$$N - \lambda - \lambda \ln \frac{N}{\lambda} = \frac{\lambda}{2z_1^2}(N-\lambda)^2 \geq 0, \tag{2.35}$$

where z_1 is between N and λ. Hence, (2.35) together with (2.33) indicates $\mathcal{L}_1(t) \geq 0$ and "=" holds iff $(N,P,S) = (\lambda, 0, \frac{\tau\lambda}{\eta})$.

Simple calculations along with the fact $(\mu,\lambda) \in \mathcal{R}_1$ (i.e., $\mu + \gamma\lambda \leq 0$) give that

$$\begin{aligned} \frac{d}{dt}\mathcal{L}_1(t) \leq & -\gamma d_N \int_\Omega \frac{|\nabla N|^2}{N^2} - \frac{\gamma\eta d_S}{\tau^2} \int_\Omega |\nabla S|^2 + (\mu + \gamma\lambda) \int_\Omega P \\ & - \frac{\gamma}{2} \int_\Omega (N-\lambda)^2 - \frac{\gamma\eta^2}{2\tau^2} \int_\Omega \big(S - \frac{\tau\lambda}{\eta}\big)^2 - \int_\Omega P^2 \leq -c_1 \mathcal{F}_1(t), \end{aligned} \tag{2.36}$$

where $\mathcal{F}_1(t) := \int_\Omega (N-\lambda)^2 + \int_\Omega \big(S - \frac{\tau\lambda}{\eta}\big)^2 + \int_\Omega P^2$ and $c_1 := \min\big\{\frac{\gamma}{2}, \frac{\gamma\eta^2}{2\tau^2}, 1\big\}$.

Since $\mathcal{L}_1(t) \geq 0$, (2.36) implies that $\int_1^\infty \mathcal{F}_1(t)dt \leq \frac{1}{c_1}\mathcal{L}_1(1) < \infty$. And we deduce from (2.32) that $\mathcal{F}_1(t)$ is uniformly continuous in $[1,\infty)$. Then Lemma 2.6 yields

$$\lim_{t\to\infty} \big(\|N-\lambda\|_{L^2} + \|P\|_{L^1} + \big\|S - \frac{\tau\lambda}{\eta}\big\|_{L^2} \big) = 0.$$

Therefore, following the similar procedures as the proof of [64, Lemma 4.2], we can find positive constants c_i $(i = 1,2,3)$ and t_1 such that

$$\begin{cases} \|N-\lambda\|_{L^2} + \|P\|_{L^1} + \|S - \frac{\tau\lambda}{\eta}\|_{L^2} \leq c_2 e^{-c_1 t}, & \text{if } \mu < -\gamma\lambda, \\ \|N-\lambda\|_{L^2} + \|P\|_{L^1} + \|S - \frac{\tau\lambda}{\eta}\|_{L^2} \leq c_3 (1+t)^{-1}, & \text{if } \mu = -\gamma\lambda, \end{cases}$$

for all $t \geq t_1$. By (2.32), Theorem 2.1 and Gagliardo-Nirenberg inequality, we get (2.8) directly. □

Proof of Theorem 2.2(2)**.** We define the following energy functional

$$\mathcal{L}_2(t) := \mathcal{L}_2(N,P,S) := \gamma \mathcal{F}_N(t) + \mathcal{F}_P(t) + \frac{\gamma\eta}{2\tau^2} \int_\Omega (S - S^*)^2,$$

where $\mathcal{F}_y(t) = \int_\Omega \left(y - y^* - y^* \ln \frac{y}{y^*}\right),\ y = N,\ P$. Proceeding the same procedures as (2.35), we obtain $\mathcal{L}_2(t) \geq 0$. Moreover, $\mathcal{L}_2(t) = 0$ iff $(N, P, S) = (N^*, P^*, S^*)$.

On the other hand, after some calculations, one has

$$\frac{d}{dt}\mathcal{L}_2(t) \leq J_1 - \frac{\gamma}{2}\int_\Omega (N - N^*)^2 - \int_\Omega (P - P^*)^2 - \frac{\gamma\eta^2}{2\tau^2}\int_\Omega (S - S^*)^2, \tag{2.37}$$

where $J_1 := -\int_\Omega Y_1^{\mathcal{T}} A_1 Y_1$ with

$$Y_1 = \begin{pmatrix} \frac{\nabla N}{N} \\ \frac{\nabla P}{P} \\ \frac{\nabla S}{S} \end{pmatrix}, A_1 = \begin{pmatrix} \gamma N^* d_N & 0 & 0 \\ 0 & P^* d(S) & \frac{P^* d'(S) S}{2} \\ 0 & \frac{P^* d'(S) S}{2} & \frac{\gamma\eta d_S S^2}{\tau^2} \end{pmatrix}.$$

Noting $P^* := \frac{\lambda\gamma+\mu}{\gamma+1}$ in (2.7) and calculating directly, we can verify that the matrix A_1 is positive definite iff

$$\frac{\gamma(\gamma+1)}{\lambda\gamma+\mu} > \frac{\tau^2}{4\eta d_S}\frac{|d'(S)|^2}{d(S)}. \tag{2.38}$$

Moreover, (2.38) is ensured by the condition (2.9). Hence, there is a constant $c_1 > 0$ such that

$$J_1 = -\int_\Omega Y_1^{\mathcal{T}} A_1 Y_1 \leq -c_1 \int_\Omega \left(\frac{|\nabla N|^2}{N^2} + \frac{|\nabla P|^2}{P^2} + \frac{|\nabla S|^2}{S^2}\right).$$

Therefore, (2.37) can be updated as

$$\frac{d}{dt}\mathcal{L}_2(t) \leq -\frac{\gamma}{2}\int_\Omega (N - N^*)^2 - \int_\Omega (P - P^*)^2 - \frac{\gamma\eta^2}{2\tau^2}\int_\Omega (S - S^*)^2.$$

Then, following the same way as the proof of Theorem 2.2 (1), one can show that

$$\lim_{t\to\infty} (\|N - N^*\|_{L^2} + \|P - P^*\|_{L^2} + \|S - S^*\|_{L^2}) = 0.$$

We proceed same way as the proof in [64, Lemma 4.2] again to get

$$\|N - N^*\|_{L^2} + \|P - P^*\|_{L^2} + \|S - S^*\|_{L^2} \leq c_3 e^{-c_2 t} \text{ for all } t \geq t_2$$

with some positive constants $c_2,\ c_3$ and t_2. Applying (2.32), Theorem 2.1 and Gagliardo-Nirenberg inequality again, we get (2.10) readily. □

Proof of Theorem 2.2(3)**.** Define the following energy functional

$$\begin{aligned} \mathcal{L}_3(t) &:= \mathcal{L}_3(N, P, S) \\ &:= \frac{\tau^2(\lambda+\mu)}{\eta}\int_\Omega N + \frac{2\tau^2\lambda}{\gamma\eta}\int_\Omega \left(P - \mu - \mu\ln\frac{P}{\mu}\right) + \frac{\mu}{2}\int_\Omega S^2 + \frac{\tau^2}{2\eta}\int_\Omega N^2. \end{aligned}$$

Similar to the proof of (2.35), one obtains that $\mathcal{L}_3(t) \geq 0$ and $\mathcal{L}_3(t) = 0$ iff $(N, P, S) = (0, \mu, 0)$.

Differentating $\mathcal{L}_3(t)$, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{L}_3(t) = J_2 &\underbrace{-\frac{\tau^2\mu}{\eta}\int_\Omega N^2 - \frac{2\tau^2\lambda}{\gamma\eta}\int_\Omega (P-\mu)^2 - \mu\eta\int_\Omega S^2 + \mu\tau\int_\Omega SN}_{J_3} \\ &- \frac{\tau^2\lambda(\mu-\lambda)}{\eta}\int_\Omega N - \frac{\tau^2(\mu-\lambda)}{\eta}\int_\Omega NP - \frac{\tau^2}{\eta}\int_\Omega N^3 - \frac{\tau^2}{\eta}\int_\Omega N^2 P,\end{aligned} \tag{2.39}$$

where $J_2 := -\int_\Omega Y_2^{\mathcal{T}} A_2 Y_2$ with

$$Y_2 = \begin{pmatrix} \frac{\nabla N}{N} \\ \frac{\nabla P}{P} \\ \frac{\nabla S}{S} \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} \frac{\tau^2 d_N N^2}{\eta} & 0 & 0 \\ 0 & \frac{2\tau^2\lambda\mu d(S)}{\gamma\eta} & \frac{\tau^2\lambda\mu d'(S)S}{\gamma\eta} \\ 0 & \frac{\tau^2\lambda\mu d'(S)S}{\gamma\eta} & \mu d_S S^2 \end{pmatrix}.$$

Then A_2 is positive definite iff $\gamma > \frac{\tau^2\lambda|d'(S)|^2}{2\eta d_S d(S)}$, which is ensured by noting (2.11). Hence, there is a constant $c_1 > 0$ such that

$$J_2 = -\int_\Omega Y_2^{\mathcal{T}} A_2 Y_2 \leq -c_1\int_\Omega \left(\frac{|\nabla N|^2}{N^2} + \frac{|\nabla P|^2}{P^2} + \frac{|\nabla S|^2}{S^2}\right) \leq 0. \tag{2.40}$$

As for J_3, using Young's inequality, one has

$$J_3 \leq -\frac{\tau^2\mu}{2\eta}\int_\Omega N^2 - \frac{2\tau^2\lambda}{\gamma\eta}\int_\Omega (P-\mu)^2 - \frac{\mu\eta}{2}\int_\Omega S^2,$$

which together with (2.40), (2.39) and $\lambda \leq \mu$ yeilds

$$\frac{d}{dt}\mathcal{L}_3(t) \leq -\frac{\tau^2\mu}{2\eta}\int_\Omega N^2 - \frac{2\tau^2\lambda}{\gamma\eta}\int_\Omega (P-\mu)^2 - \frac{\mu\eta}{2}\int_\Omega S^2. \tag{2.41}$$

Then, we follow the same way as the proof of Theorem 2.2 (1) to show that

$$\lim_{t\to\infty}(\|N\|_{L^2} + \|P-\mu\|_{L^2} + \|S\|_{L^2}) = 0.$$

Moreover, similar to the proof in [64, Lemma 4.2], we can find some positive constants c_i $(i = 2, 3, 4)$ and t_3 such that for all $t \geq t_3$

$$\begin{cases} \|N\|_{L^2} + \|P-\mu\|_{L^2} + \|S\|_{L^2} \leq c_3 e^{-c_2 t}, & \text{if } \mu > \lambda, \\ \|N\|_{L^2} + \|P-\mu\|_{L^2} + \|S\|_{L^2} \leq c_4(1+t)^{-1}, & \text{if } \mu = \lambda, \end{cases}$$

which alongside (2.32), Theorem 2.1 and Gagliardo-Nirenberg inequality implies (2.12) readily. □

2.4 Bifurcation Analysis: Proof of Theorem 2.3

In this section, we shall analyze the stability of the positive constant steady state (N^*, P^*, S^*) and discuss Hopf/steady state bifurcation arising from (N^*, P^*, S^*) for the predator-prey system with anti-predation $d(S) = d_P + \beta S$. We assume $(\mu, \lambda) \in \mathcal{R}_2 \cup \mathcal{R}_3$ throughout this section.

Before proceeding, we introduce some important notations used in the sequel. Let $0 = \sigma_1 < \sigma_2 < \sigma_3 < \cdots$ be the eigenvalues of the operator $-\Delta$ on Ω with the homogeneous Neumann boundary condition. We denote by $E(\sigma_m)$ the eigenspace corresponding to σ_m in $H^1(\Omega)$. Let $\mathbf{X} = [H^1(\Omega)]^3$ and $\{\theta_{mj} : j = 1, 2, \cdots, \dim E(\sigma_m)\}$ be an orthonormal basis of $E(\sigma_m)$. Then

$$\mathbf{X} = \bigoplus_{m=1}^{\infty} \mathbf{X}_m \quad \text{and} \quad \mathbf{X}_m = \bigoplus_{j=1}^{\dim E(\sigma_m)} \mathbf{X}_{mj}, \tag{2.42}$$

where $\mathbf{X}_{mj} = \{\mathbf{c}\theta_{mj},\ \mathbf{c} \in \mathbb{R}^3\}$. Denote

$$\Phi(\mathbf{u}) = \begin{pmatrix} d_N N \\ d(S)P \\ d_S S \end{pmatrix} \quad \text{and} \quad \Psi(\mathbf{u}) = \begin{pmatrix} \Psi_1(\mathbf{u}) \\ \Psi_2(\mathbf{u}) \\ \Psi_3(\mathbf{u}) \end{pmatrix} = \begin{pmatrix} \lambda N - N^2 - NP \\ \mu P - P^2 + \gamma NP \\ -\eta S + \tau N \end{pmatrix},$$

where $\mathbf{u} = (N, P, S)^{\mathcal{T}}$. Then (2.3) can be rewritten as

$$\frac{\partial \mathbf{u}}{\partial t} = \Delta \Phi(\mathbf{u}) + \Psi(\mathbf{u}).$$

The linearized system of (2.3) at the positive constant steady state $\mathbf{u}^* = (N^*, P^*, S^*)^{\mathcal{T}}$ is:

$$\frac{\partial \mathbf{U}}{\partial t} = \mathcal{L}\mathbf{U} \quad \text{with} \quad \mathcal{L} = \Phi_{\mathbf{u}}(\mathbf{u}^*)\Delta + \Psi_{\mathbf{u}}(\mathbf{u}^*),$$

where $\mathbf{U} := \mathbf{u} - \mathbf{u}^*$ and

$$\Phi_{\mathbf{u}}(\mathbf{u}^*) = \begin{pmatrix} d_N & 0 & 0 \\ 0 & d(S^*) & d'(S^*)P^* \\ 0 & 0 & d_S \end{pmatrix} \quad \text{and} \quad \Psi_{\mathbf{u}}(\mathbf{u}^*) = \begin{pmatrix} -N^* & -N^* & 0 \\ \gamma P^* & -P^* & 0 \\ \tau & 0 & -\eta \end{pmatrix}.$$

By a simple calculation, the characteristic polynomial of the matrix $-\sigma_m \Phi_{\mathbf{u}}(\mathbf{u}^*) + \Psi_{\mathbf{u}}(\mathbf{u}^*)$ is given by

$$\alpha^3 + B_1(\beta, \sigma_m)\alpha^2 + B_2(\beta, \sigma_m)\alpha + B_3(\beta, \sigma_m) = 0, \tag{2.43}$$

where

$$\begin{aligned}
B_1(\beta,\sigma_m) =&\sigma_m\left(d_S+d(S^*)+d_N\right)+\eta+P^*+N^*>0,\\
B_2(\beta,\sigma_m) =&\sigma_m^2\left(d_Sd(S^*)+d_Nd_S+d_Nd(S^*)\right)\\
&+\sigma_m\left(\eta(d(S^*)+d_N)+P^*(d_S+d_N)+N^*(d_S+d(S^*))\right)\\
&+(P^*+N^*)\eta+(1+\gamma)N^*P^*>0,\\
B_3(\beta,\sigma_m) =&\sigma_m^3d_Nd_Sd_P+\sigma_m^2\left(\eta d_Nd_P+d_Nd_SP^*+d_Pd_SN^*\right)+\beta\sigma_mH(\sigma_m)\\
&+\sigma_m\left(d_NP^*\eta+d_PN^*\eta+d_SN^*P^*+\gamma N^*P^*d_S\right)+(1+\gamma)N^*P^*\eta
\end{aligned} \tag{2.44}$$

with

$$H(z):=z^2d_Nd_SS^*+z(\eta d_N+d_SN^*)S^*+\frac{\tau N^*(\lambda(1-\gamma)-2\mu)}{\gamma+1}.$$

A direct calculation yields

$$\begin{aligned}
T(\beta,\sigma_m) :=& B_1(\beta,\sigma_m)B_2(\beta,\sigma_m)-B_3(\beta,\sigma_m)\\
&=b_1\sigma_m^3+b_2\sigma_m^2+b_3\sigma_m+\tau N^*\beta P^*\sigma_m+b_4,
\end{aligned} \tag{2.45}$$

where

$$\begin{aligned}
b_1 =&d_S^2(d(S^*)+d_N)+d^2(S^*)(d_N+d_S)+d_N^2(d_S+d(S^*))+2d_Sd_Nd(S^*)>0,\\
b_2 =&\eta(d(S^*)+d_N)(2d_S+d(S^*)+d_N)+P^*(d_S+d_N)(d_S+2d(S^*)+d_N)\\
&+N^*(d_S+d(S^*))(d_S+d(S^*)+2d_N)>0,\\
b_3 =&d_S(P^*+N^*)\eta+d(S^*)(P^*\eta+N^*P^*+\gamma N^*P^*)+d_N(N^*\eta+N^*P^*+\gamma N^*P^*)\\
&+(\eta+P^*+N^*)(\eta(d(S^*)+d_N)+P^*(d_S+d_N)+N^*(d_S+d(S^*)))>0,\\
b_4 =&\eta^2(P^*+N^*)+\eta(P^*+N^*)^2+(1+\gamma)(P^*+N^*)N^*P^*>0,
\end{aligned} \tag{2.46}$$

and hence $T(\beta,\sigma_m):=B_1(\beta,\sigma_m)B_2(\beta,\sigma_m)-B_3(\beta,\sigma_m)>0$ for each $m\geq 1$. Then we have the following stability result.

Lemma 2.8. *The positive constant steady state* $\mathbf{u}^*$ *of* (2.3) *is linearly stable provided one of the following conditions holds:*

(a) $2\mu\leq\lambda(1-\gamma)$*;*

(b) $2\mu>\lambda(1-\gamma)$ *and* $\sigma_m\geq\bar{\sigma}>0$ *for each* $m\geq 2$*,*

where $\overline{\sigma}$ *is defined in* (2.15)*.*

Proof. From the above analysis, one has $B_j(\beta,\sigma_m) > 0$ $(j = 1, 2)$ and $T(\beta,\sigma_m) > 0$ for each $m \geq 1$. To show that $\mathbf{u}^*$ is linearly stable, it suffices to prove $B_3(\beta,\sigma_m) > 0$ for each $m \geq 1$ based on the well-known Routh-Hurwitz criterion (see Appendix B.1 in [100]).

If $2\mu \leq \lambda(1-\gamma)$, one has $B_3(\beta,\sigma_m) > 0$ for each $m \geq 1$. For the case of $2\mu > \lambda(1-\gamma)$, we obtain $B_3(\beta,\sigma_m) > 0$ directly when $m = 1$. Since $\sigma_m \geq \bar{\sigma}$ for each $m \geq 2$, one obtains that $H(\sigma_m) \geq 0$ for each $m \geq 2$ and hence $B_3(\beta,\sigma_m) > 0$ for each $m \geq 1$. The proof of Lemma 2.8 is finished. □

We are left to discuss the linear stability/instability of $\mathbf{u}^*$ for the parameters satisfying the following assumptions:

(A_1) $2\mu > \lambda(1-\gamma)$ and there exist some $m \geq 2$ such that $\sigma_m < \bar{\sigma}$.

Lemma 2.9. *Let the assumption* (A_1) *hold. Then we have the following statements:*

(1) $\mathbf{u}^*$ *is linearly stable with respect to (2.3) if* $0 < \beta < \min\limits_{m_0}\{\beta^T_{m_0}\}$*;*

(2) $\mathbf{u}^*$ *is linearly unstable with respect to (2.3) if* $\beta > \min\limits_{m_0}\{\beta^T_{m_0}\}$*,*

where β^T_m *and* $\bar{\sigma}$ *are defined in* (2.13) *and* (2.15)*, respectively, and* $m_0 \geq 2$ *satisfying* $\sigma_{m_0} < \bar{\sigma}$*.*

Proof. We first show that $\min\limits_{m_0}\{\beta^T_{m_0}\}$ ($m_0 \geq 2$ satisfying $\sigma_{m_0} < \bar{\sigma}$) exists. For fixed parameters $\gamma, \tau, \eta, d_N, d_P, d_S, \mu$ and λ, (2.15) shows that $\bar{\sigma} > 0$ is also fixed. Since the sequence $\{\sigma_m\}_{m=1}^{\infty}$ is increasing respect to m and satisfies $\sigma_1 = 0$ and $\sigma_m \to \infty$ as $m \to \infty$, then there exists an integer m^* such that $\sigma_{m^*} < \bar{\sigma} < \sigma_{m^*+1}$. Hence, for all $2 \leq m_0 \leq m^* < \infty$, one has $\sigma_{m_0} < \bar{\sigma}$ and such m_0 is finite, which implies that $\min\limits_{m_0}\{\beta^T_{m_0}\}$ exists.

Next, we discuss the stability/ instability of $\mathbf{u}^*$. It follows from (2.44), (2.45) and (2.46) that $B_1 > 0$, $B_2 > 0$ and $B_1B_2 - B_3 > 0$ for any $\beta > 0$. On the other hand, one can check that $B_3(\beta,\sigma_m) > 0$ for any $\beta > 0$ when $m = 1$. When $m \geq 2$ satisfying $\sigma_m \geq \bar{\sigma}$, one has $H(\sigma_m) \geq 0$ and hence $B_3(\beta,\sigma_m) > 0$ for any $\beta > 0$. If $m \geq 2$ satisfying $\sigma_m < \bar{\sigma}$, then $H(\sigma_m) < 0$. It follows from (2.44) that $B_3(\beta,\sigma_m) > 0$ for $0 < \beta < \beta^T_m$ with $m \geq 2$ satisfying $\sigma_m < \bar{\sigma}$. Consequently, with the above three cases, the Routh-Hurwitz criterion implies that $\mathbf{u}^*$ is linearly stable if $0 < \beta < \beta^T_m$ for each $m \geq 2$ satisfying $\sigma_m < \bar{\sigma}$, which together with the existence of $\min\limits_{m_0}\{\beta^T_{m_0}\}$ gives Lemma 2.9 (1).

When $\beta > \beta^T_{m_0}$ for some $m_0 \geq 2$ satisfying $\sigma_{m_0} < \bar{\sigma}$, we get $B_3(\beta, \sigma_{m_0}) < 0$ readily. Then the Routh-Hurwitz criterion indicates that $\mathbf{u}^*$ is linearly unstable. With the existence of $\min_{m_0}\{\beta^T_{m_0}\}$, we obtain Lemma 2.9 (2) directly. The proof of Lemma 2.9 is finished. □

Proof of Theorem 2.3. Now, we discuss the bifurcations from $\mathbf{u}^* = (N^*, P^*, S^*)^{\mathcal{T}}$. Recall that $\mathcal{L}$ has a pair of purely imaginary eigenvalues if and only if $-\sigma_m \Phi_{\mathbf{u}}(\mathbf{u}^*) + \Psi_{\mathbf{u}}(\mathbf{u}^*)$ for some $m \geq 1$ does so. Assume that $-\sigma_m \Phi_{\mathbf{u}}(\mathbf{u}^*) + \Psi_{\mathbf{u}}(\mathbf{u}^*)$ for some $m \geq 1$ has eigenvalues $i\nu$, $-i\nu$ and δ, where $\nu, \delta \in \mathbb{R}$ and $\nu \neq 0$. It follows from the Routh-Hurwitz criterion that

$$B_1(\beta, \sigma_m) = -\delta, \quad B_2(\beta, \sigma_m) = \nu^2, \quad B_3(\beta, \sigma_m) = -\nu^2 \delta. \tag{2.47}$$

Then (2.47) yields

$$T(\beta, \sigma_m) = B_1(\beta, \sigma_m) B_2(\beta, \sigma_m) - B_3(\beta, \sigma_m) = 0.$$

This shows that if $\mathcal{L}$ has a pair of purely imaginary eigenvalues, then $T(\beta, \sigma_m) = 0$. That is, a necessary condition for the Hopf bifurcation to occur is $T(\beta, \sigma_m) = 0$ for some $m \geq 1$. Hence, from (2.45) and (2.46), we know that (2.3) has no Hopf bifurcation arising from $\mathbf{u}^*$ for all $(\mu, \lambda) \in \mathcal{R}_2 \cup \mathcal{R}_3$ and $\beta > 0$. Hence, we complete the proof of Theorem 2.3 (1).

We next consider the possibility of steady state bifurcation arising from $\mathbf{u}^*$. First, we determine the potential steady state bifurcation points. Assume that 0 is an eigenvalue of $-\sigma_m \Phi_{\mathbf{u}}(\mathbf{u}^*) + \Psi_{\mathbf{u}}(\mathbf{u}^*)$ for some $m \geq 1$. It follows from the Routh-Hurwitz criterion that $B_3(\beta, \sigma_m) = 0$, which means that a necessary condition for steady state bifurcation is $B_3(\beta, \sigma_m) = 0$ for some $m \geq 1$. Consequently, noting Lemma 2.8 and Lemma 2.9, if $\frac{\lambda(1-\gamma)}{2} < \mu < \lambda$, then the potential steady state bifurcation points are $\beta = \beta^T_{m_0}$ for some $m_0 \geq 2$ satisfying $\sigma_{m_0} < \bar{\sigma}$.

Second, we verify that the transversality condition holds for the steady state bifurcation. Differentiating the characteristic equation (2.43) with respect to β, we obtain

$$\left(\frac{d\alpha}{d\beta}\right)\Big|_{\beta=\beta^T_{m_0}} = -\frac{\sigma_{m_0} H(\sigma_{m_0})}{B_2(\beta^T_{m_0}, \sigma_m)} > 0,$$

where $H(\sigma_{m_0})$ is defined in (2.14). Thus, the proof of Theorem 2.3 (2) is finished. □

2.5 Stationary Patterns: Proof of Theorem 2.4

In this section, motivated by the ideas in [142, Chapter 6], we shall establish the existence of positive solutions of (2.16) by the Leray-Schauder degree theory. To this end, we first give a priori positive upper and lower bounds for the positive solutions.

2.5.1 Priori Estimates of Positive Solutions

In this subsection, we shall fix the parameters $\lambda, \mu, \gamma, \eta, \tau$ and estimate the upper and lower positive bounds of positive solutions of (2.16) concerning the diffusion coefficients d_N, d_P, d_S and cross-diffusion coefficient β. We first give a priori positive upper bound for the positive solutions of (2.16).

Lemma 2.10. *Let $\varepsilon > 0$ be any fixed constant. Then any positive solution (N, P, S) of (2.16) with $d_N, d_P, d_S \geq \varepsilon$ and $0 \leq \beta \leq 1/\varepsilon$ satisfies*

$$\max_{\overline{\Omega}} N \leq \lambda, \quad \max_{\overline{\Omega}} P \leq (\varepsilon + \tau\lambda/\varepsilon\eta)(\mu + \gamma\lambda)/\varepsilon, \quad \max_{\overline{\Omega}} S \leq \tau\lambda/\eta. \tag{2.48}$$

Furthermore, there exists a positive constant $C = C(\lambda, \mu, \gamma, \eta, \tau, \varepsilon, |\Omega|)$ such that any positive solution (N, P, S) of (2.16) with $d_N, d_P, d_S \geq \varepsilon$ and $0 \leq \beta \leq 1/\varepsilon$ satisfies

$$||(N, P, S)||_{C^{2+\kappa}(\overline{\Omega})} \leq C. \tag{2.49}$$

Proof. Let $x_1 \in \overline{\Omega}$ be a point such that $N(x_1) = \max_{\overline{\Omega}} N(x)$. Applying the maximum principle [94, Lemma 2.1] to the equation of N, it is clear that $N(x_1) \leq \lambda$. Thus, $\max_{\overline{\Omega}} N \leq \lambda$. Let $x_2 \in \overline{\Omega}$ be a point such that $S(x_2) = \max_{\overline{\Omega}} S(x)$. Applying the maximum principle [94, Lemma 2.1] to the equation of S, we have $S(x_2) \leq \tau\lambda/\eta$. Thus, $\max_{\overline{\Omega}} S \leq \tau\lambda/\eta$. Let $\Phi = (d_P + \beta S)P$ and $x_3 \in \overline{\Omega}$ be a point such that $\Phi(x_3) = \max_{\overline{\Omega}} \Phi(x)$. Applying the maximum principle [94, Lemma 2.1] to the equation of P, we have $P(x_3) \leq \mu + \gamma N(x_3) \leq \mu + \gamma\lambda$. Thus,

$$d_P \max_{\overline{\Omega}} P \leq \max_{\overline{\Omega}} \Phi = \Phi(x_3) = (d_P + \beta S(x_3))P((x_3)) \leq (d_P + \beta\tau\lambda/\eta)(\mu + \gamma\lambda).$$

Hence, $\max_{\overline{\Omega}} P \leq (\varepsilon + \tau\lambda/\varepsilon\eta)(\mu + \gamma\lambda)/\varepsilon$. This gives the estimate (2.48).

We now prove the estimate (2.49). Given (2.48), we apply the standard regularity for elliptic equations (see, e.g., [41]) to derive that N, S and $\Phi = (d_P + \beta S)P$ belong to $C^{1+\kappa}(\overline{\Omega})$. Moreover, the $C^{1+\kappa}(\overline{\Omega})$ norms of them depend only on the parameter ε and the parameters $\lambda, \mu, \gamma, \eta, \tau$. Thus, $P \in C^{1+\kappa}(\overline{\Omega})$ and the $C^{1+\kappa}(\overline{\Omega})$ norm of P depends only on the parameter ε and the parameters $\lambda, \mu, \gamma, \eta, \tau$. We again apply the standard regularity for elliptic equations to derive the estimate (2.49). □

We next give a positive lower bound for the positive solutions of (2.16) with respect to the diffusion coefficients d_N, d_P, d_S and cross-diffusion coefficient β. For this, we first prove the following lemma.

Lemma 2.11. *Let $d_{N,m}, d_{P,m}, d_{S,m}, \beta_m \in (0,\infty)$ and (N_m, P_m, S_m) be the corresponding positive solution of (2.16) with $(d_N, d_P, d_S, \beta) = (d_{N,m}, d_{P,m}, d_{S,m}, \beta_m)$. Suppose that $(d_{N,m}, d_{P,m}, d_{S,m}, \beta_m) \rightarrow (d_{N,\infty}, d_{P,\infty}, d_{S,\infty}, \beta_\infty)$ and $(N_m, P_m, S_m) \to (N_\infty, P_\infty, S_\infty)$ uniformly on $\overline{\Omega}$, where $N_\infty, P_\infty, S_\infty$ are constants. Then $(N_\infty, P_\infty, S_\infty)$ satisfies*

$$\begin{cases} \lambda - N_\infty - P_\infty = 0, \\ \mu - P_\infty + \gamma N_\infty = 0, \\ -\eta S_\infty + \tau N_\infty = 0. \end{cases}$$

In particular, if $\lambda > 0, \lambda > \mu$ and $\mu + \gamma\lambda > 0$, then $(N_\infty, P_\infty, S_\infty) = (N^*, P^*, S^*)$, which is the unique positive constant solution of (2.16).

Proof. From the first equation of (2.16), it follows that $\int_\Omega (\lambda - N_m - P_m) N_m = 0$ for all $m \geq 1$. Assume that $\lambda - N_\infty - P_\infty > 0$. Then it is clear that $\lambda - N_m - P_m > 0$ for large m. Thus, $\int_\Omega (\lambda - N_m - P_m) N_m > 0$ for large m due to N_m is positive. This is a contradiction. Similarly, if $\lambda - N_\infty - P_\infty < 0$, we can get a contradiction as above. Therefore, $\lambda - N_\infty - P_\infty = 0$. The same argument shows that $\mu - P_\infty + \gamma N_\infty = 0$. It follows from the first equation of (2.16) that

$$0 = \int_\Omega (-\eta S_m + \tau N_m) dx \to \int_\Omega (-\eta S_\infty + \tau N_\infty) = (-\eta S_\infty + \tau N_\infty)|\Omega|$$

as $m \to \infty$. Thus, $-\eta S_\infty + \tau N_\infty = 0$. This completes the proof for the first part.

Suppose that $N_\infty = 0$. Then we use the proven result to obtain $P_\infty = \lambda$ and $\lambda = \mu$, which is a contradiction to $\lambda > \mu$. Suppose that $P_\infty = 0$. Then $N_\infty = \lambda$ and $\mu + \gamma\lambda = 0$, which is a contradiction to $\mu + \gamma\lambda > 0$. This implies that $N_\infty > 0$ and $P_\infty > 0$, and thus $S_\infty = \frac{\tau}{\eta} N_\infty > 0$. Hence $(N_\infty, P_\infty, S_\infty) = (N^*, P^*, S^*)$. This completes the proof for the second part. □

Lemma 2.12. *Let $\varepsilon > 0$ be any fixed constant. Assume that $\lambda > 0, \lambda > \mu$ and $\mu + \gamma\lambda > 0$. Then there exists a positive constant $C = C(\lambda, \mu, \gamma, \eta, \tau, \varepsilon, |\Omega|)$ such that any positive solution (N, P, S) of (2.16) with $d_N, d_P, d_S \geq \varepsilon$ and $0 \leq \beta \leq 1/\varepsilon$ satisfies*

$$\min_{\overline{\Omega}} N, \quad \min_{\overline{\Omega}} P, \quad \min_{\overline{\Omega}} S \geq C^{-1}(\varepsilon).$$

Proof. Suppose that the conclusion does not hold. Then we may assume there exists a sequence $\{(d_{N,m}, d_{P,m}, d_{S,m}, \beta_m)\}_{m=1}^\infty$, satisfying $d_{N,m}, d_{P,m}, d_{S,m} \geq \varepsilon$ and $\beta_m \leq 1/\varepsilon$, such that the corresponding positive solutions (N_m, P_m, S_m) of (2.16) with $(d_N, d_P, d_S, \beta) = (d_{N,m}, d_{P,m}, d_{S,m}, \beta_m)$ satisfy

$$\min_{\overline{\Omega}} N_m \to 0 \quad \text{or} \quad \min_{\overline{\Omega}} P_m \to 0 \quad \text{or} \quad \min_{\overline{\Omega}} S_m \to 0$$

as $m \to \infty$. Since $d_{N,m}, d_{P,m}, d_{S,m} \geq \varepsilon$, and $0 \leq \beta_m \leq 1/\varepsilon$, subject to a subsequence, we may assume $d_{N,m} \to d_{N,\infty} \in [\varepsilon, \infty]$, $d_{P,m} \to d_{P,\infty} \in [\varepsilon, \infty]$, $d_{S,m} \to d_{S,\infty} \in [\varepsilon, \infty]$ and $\beta_m \to \beta_\infty \in [0, 1/\varepsilon]$. Moreover, it follows from (2.49) that

$$(N_m, P_m, S_m) \to (N_\infty, P_\infty, S_\infty) \quad \text{in} \quad C^{2+\kappa}(\overline{\Omega}) \times C^{2+\kappa}(\overline{\Omega}) \times C^{2+\kappa}(\overline{\Omega})$$

for some nonnegative functions $N_\infty, P_\infty, S_\infty$. It is not hard to see that $(N_\infty, P_\infty, S_\infty)$ also satisfies the estimate (2.49), and

$$\min_{\overline{\Omega}} N_\infty = 0 \quad \text{or} \quad \min_{\overline{\Omega}} P_\infty = 0 \quad \text{or} \quad \min_{\overline{\Omega}} S_\infty = 0.$$

Furthermore, if $d_{N,\infty}, d_{S,\infty}, d_{P,\infty} < \infty$, then $(N_\infty, P_\infty, S_\infty)$ satisfies (2.16). If $d_{N,\infty} = \infty$, then it follows from the estimate (2.48) that N_∞ satisfies $-\Delta N_\infty = 0$ in Ω and $\partial_\nu N_\infty = 0$ on $\partial\Omega$. This means that N_∞ is constant. Likewise, the analogous conclusions hold for $d_{P,\infty} = \infty$ and $d_{S,\infty} = \infty$.

The constants C_i to be used below will depend on the parameters $(\lambda, \mu, \gamma, \eta, \tau, \varepsilon, |\Omega|)$. As they are fixed, this dependence will not be stated explicitly. Due to (2.48), we find that

$$\left\| \frac{\lambda - N_m - P_m}{d_{N,m}} \right\|_{L^\infty} \leq \frac{\lambda + \lambda + (\varepsilon + \tau\lambda/\varepsilon\eta)(\mu + \gamma\lambda)/\varepsilon}{\varepsilon}$$

for all $d_{N,m}, d_{P,m}, d_{S,m} \geq \varepsilon$. We apply Harnack inequality [88, Lemma 4.3] to the equation of N_m to obtain

$$\max_{\overline{\Omega}} N_m \leq C_1 \min_{\overline{\Omega}} N_m. \tag{2.50}$$

Let $\Phi_m = (d_{P,m} + \beta_m S_m) P_m$. Then

$$-\Delta \Phi_m = \frac{\mu - P_m + \gamma N_m}{d_{P,m} + \beta_m S_m} \Phi_m, \quad x \in \Omega, \qquad \partial_\nu \Phi_m = 0, \quad x \in \partial\Omega. \tag{2.51}$$

Since

$$\left\| \frac{\mu - P_m + \gamma N_m}{d_{P,m} + \beta_m S_m} \right\|_{L^\infty} \leq \frac{\mu + (\varepsilon + \tau\lambda/\varepsilon\eta)(\mu + \gamma\lambda)/\varepsilon + \gamma\lambda}{\varepsilon}$$

for all $d_{N,m}, d_{P,m}, d_{S,m} \geq \varepsilon$, we apply Harnack inequality [88, Lemma 4.3] to the equation of Φ_m to obtain

$$\max_{\overline{\Omega}} \Phi_m \leq C_2 \min_{\overline{\Omega}} \Phi_m.$$

Hence,

$$\frac{\max_{\overline{\Omega}} P_m}{\min_{\overline{\Omega}} P_m} \le \frac{\max_{\overline{\Omega}} \Phi_m}{\min_{\overline{\Omega}} \Phi_m} \frac{d_{P,m} + \tau\lambda/\varepsilon\eta}{d_{P,m}} \le C_2 \frac{\varepsilon + \tau\lambda/\varepsilon\eta}{\varepsilon} := C_3,$$

and thus

$$\max_{\overline{\Omega}} P_m \le C_3 \min_{\overline{\Omega}} P_m. \tag{2.52}$$

Let $x_{m,1} \in \overline{\Omega}$ be a point such that $S_m(x_{m,1}) = \max_{\overline{\Omega}} S_m(x)$. Applying the maximum principle [94, Lemma 2.1] to the equation of S_m, we have $S_m(x_{m,1}) \le (\tau/\eta) N_m(x_{m,1})$. Thus,

$$\max_{\overline{\Omega}} S_m \le (\tau/\eta) \max_{\overline{\Omega}} N_m. \tag{2.53}$$

Similarly, we let $x_{m,2} \in \overline{\Omega}$ be a point such that $S_m(x_{m,2}) = \min_{\overline{\Omega}} S_m(x)$. Applying the maximum principle [94, Lemma 2.1] to the equation of S_m, we have $S_m(x_{m,2}) \ge (\tau/\eta) N_m(x_{m,2})$. Thus,

$$\min_{\overline{\Omega}} S_m \ge (\tau/\eta) \min_{\overline{\Omega}} N_m. \tag{2.54}$$

We next complete the proof by considering several different cases.

Case 1: $d_{N,\infty}, d_{S,\infty}, d_{P,\infty} < \infty$. Assume that $\min_{\overline{\Omega}} N_m \to 0$ as $m \to \infty$. Then it follows from (2.50) that $\max_{\overline{\Omega}} N_m \to 0$ as $m \to \infty$, and so $N_\infty = 0$ in $\overline{\Omega}$. Moreover, it follows from (2.53) that $\max_{\overline{\Omega}} S_m \to 0$ as $m \to \infty$, and so $S_\infty = 0$ in $\overline{\Omega}$. Since there is a positive constant C_4 independent of m such that

$$||\Phi_m||_{L^\infty} \le C_4 \quad \text{and} \quad \left|\left|\frac{\mu - P_m + \gamma N_m}{d_{P,m} + \beta_m S_m}\right|\right|_{L^\infty} \le C_4$$

for all $m \ge 1$. Thus, by the standard regularity for elliptic equations (see, e.g., [41]), we may assume that $\Phi_m \to \Phi_\infty$ uniformly in $C^1(\overline{\Omega})$, by passing to a subsequence if necessary. Note that $\max_{\overline{\Omega}} S_m \to 0$ as $m \to \infty$ and $\beta_m \le 1/\varepsilon$. Then

$$P_m = \frac{\Phi_m}{d_{P,m} + \beta_m S_m} \to \frac{\Phi_\infty}{d_{P,\infty}} := P_\infty \quad \text{uniformly in} \quad \overline{\Omega}.$$

Hence, we derive from (2.51) that Φ_∞ satisfies

$$-\Delta\Phi_\infty = \left(\mu - \frac{\Phi_\infty}{d_{P,\infty}}\right)\frac{\Phi_\infty}{d_{P,\infty}}, \quad x \in \Omega, \qquad \partial_\nu \Phi_\infty = 0, \quad x \in \partial\Omega.$$

This implies that either $\Phi_\infty = \mu d_{P,\infty}$ or $\Phi_\infty = 0$, and so $P_\infty = \mu$ or $P_\infty = 0$. Thus, $(N_\infty, P_\infty, S_\infty) = (0, \mu, 0)$ or $(0, 0, 0)$. This is a contradiction to Lemma 2.11. Thus,

$\min_{\overline{\Omega}} N_\infty > 0$. Assume that $\min_{\overline{\Omega}} S_m \to 0$ as $m \to \infty$. Then it follows from (2.54) that $\min_{\overline{\Omega}} N_m \to 0$ as $m \to \infty$, and so (2.50) gives $N_\infty = 0$ in $\overline{\Omega}$. This is a contradiction. Hence, $\min_{\overline{\Omega}} S_\infty > 0$. Assume that $\min_{\overline{\Omega}} P_m \to 0$ as $m \to \infty$. Then it follows from (2.52) that $\max_{\overline{\Omega}} P_m \to 0$ as $m \to \infty$, and so $P_\infty = 0$ in $\overline{\Omega}$. Given the equation of N_m, we apply the standard regularity for elliptic equations (see, e.g., [41]) to get $N_m \to N_\infty$ uniformly in $C^1(\overline{\Omega})$, by passing to a subsequence if necessary. Here $N_\infty \geq 0$ satisfies

$$-d_{N,\infty}\Delta N_\infty = (\lambda - N_\infty)N_\infty, \quad x \in \Omega, \qquad \partial_\nu N_\infty = 0, \quad x \in \partial\Omega.$$

Since we have proved that $\min_{\overline{\Omega}} N_\infty > 0$, it is clear that $N_\infty = \lambda$ in $\overline{\Omega}$. Similarly, we derive from the equation of S_i that $S_\infty = \tau\lambda/\eta$ in $\overline{\Omega}$. Thus, $(N_\infty, P_\infty, S_\infty) = (\lambda, 0, \tau\lambda/\eta)$. This is a contradiction to Lemma 2.11, and we complete the proof of this case.

Case 2: $d_{N,\infty} = \infty$ or $d_{P,\infty} = \infty$ or $d_{S,\infty} = \infty$. If $d_{N,\infty} = \infty$, then N_∞ is a nonnegative constant. Assume that $\min_{\overline{\Omega}} N_m \to 0$ as $m \to \infty$. Then it follows from (2.50) that $\max_{\overline{\Omega}} N_m \to 0$ as $m \to \infty$, and so $N_\infty = 0$ in $\overline{\Omega}$. Moreover, it follows from (2.53) that $\max_{\overline{\Omega}} S_m \to 0$ as $m \to \infty$, and so $S_\infty = 0$ in $\overline{\Omega}$. If $d_{P,\infty} < \infty$, by the same argument as case 1, $(N_\infty, P_\infty, S_\infty) = (0, \mu, 0)$ or $(0, 0, 0)$, which contradicts to Lemma 2.11. If $d_{P,\infty} = \infty$, then P_∞ is a nonnegative constant. Thus, $N_\infty, P_\infty, S_\infty$ are constants and $N_\infty = S_\infty = 0$ in $\overline{\Omega}$. This is a contradiction to Lemma 2.11. Thus, $\min_{\overline{\Omega}} N_\infty > 0$, and so N_∞ is a positive constant. Assume that $\min_{\overline{\Omega}} S_m \to 0$ as $m \to \infty$. Then it follows from (2.54) that $\min_{\overline{\Omega}} N_m \to 0$ as $m \to \infty$, and so (2.50) gives $N_\infty = 0$ in $\overline{\Omega}$. This is a contradiction. Hence, $\min_{\overline{\Omega}} S_\infty > 0$, and so S_∞ is a positive constant. Assume that $\min_{\overline{\Omega}} P_m \to 0$ as $m \to \infty$. Then it follows from (2.52) that $\max_{\overline{\Omega}} P_m \to 0$ as $m \to \infty$, and so $P_\infty = 0$ in $\overline{\Omega}$. Thus, $N_\infty, P_\infty, S_\infty$ are constants and $P_\infty = 0$ in $\overline{\Omega}$. This is a contradiction to Lemma 2.11. Consequently, we completed the proof for $d_{N,\infty} = \infty$.

Similarly, we can derive contradictions for the cases $d_{P,\infty} = \infty$ and $d_{S,\infty} = \infty$. □

2.5.2 Proof of Theorem 2.4

With the priori bounds for the positive solutions in hand, we shall apply the Leray-Schauder degree theory to establish the existence of positive solutions of (2.16).

We use the same notations as before. System (2.16) can be written as

$$\begin{cases} -\Delta\Phi(\mathbf{u}) = \Psi(\mathbf{u}), & x \in \Omega, \\ \partial_\nu \mathbf{u} = 0, & x \in \partial\Omega. \end{cases} \tag{2.55}$$

In what follows, we study the linearization of (2.55) at positive constant steady state $\mathbf{u}^* = (N^*, P^*, S^*)$, and then calculate the fixed point index of $\mathbf{u}^*$.

Since the determinant of $\Phi_{\mathbf{u}}(\mathbf{u})$ is positive for all nonnegative $\mathbf{u}$, a simple calculation shows that $\Phi_{\mathbf{u}}^{-1}(\mathbf{u})$ exists and $\det \Phi_{\mathbf{u}}^{-1}(\mathbf{u})$ is positive. Hence, $\mathbf{u}$ is a positive solution to (2.16) if and only if

$$\mathbf{F}(\mathbf{u}) \triangleq \mathbf{u} - (\mathbf{I} - \Delta)^{-1} \left\{ \Phi_{\mathbf{u}}^{-1}(\mathbf{u}) \left[\Psi(\mathbf{u}) + \nabla \mathbf{u} \Phi_{\mathbf{uu}}(\mathbf{u}) \nabla \mathbf{u} \right] + \mathbf{u} \right\} = 0 \quad \text{in} \quad \mathbf{Y}^{+}.$$

Here $(\mathbf{I} - \Delta)^{-1}$ is the inverse of $\mathbf{I} - \Delta$ under homogeneous Neumann boundary conditions and $\mathbf{Y}^{+} = \{\mathbf{u} \in \mathbf{Y} : N, P, S > 0 \text{ on } \overline{\Omega}\}$, where $\mathbf{Y} = [C^1(\overline{\Omega})]^3$. Since $\mathbf{F}(\cdot)$ is a compact perturbation of the identity operator, the Leray-Schauder degree $\deg(\mathbf{F}(\cdot), 0, B(C))$ is well defined if $\mathbf{F}(\mathbf{u}) \neq 0$ on $\partial B(C)$, where $B(C) = \{\mathbf{u} \in \mathbf{Y} : C^{-1} < N, P, S < C \text{ on } \overline{\Omega}\}$ for $C > 0$. By a straightforward calculation, the linearization of $\mathbf{F}(\mathbf{u})$ at $\mathbf{u}^*$ is given by

$$D_{\mathbf{u}}\mathbf{F}(\mathbf{u}^*) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1} \left\{ \Phi_{\mathbf{u}}^{-1}(\mathbf{u}^*) \Psi_{\mathbf{u}}(\mathbf{u}^*) + \mathbf{I} \right\}.$$

According to the Leray-Schauder index formula [102, Theorem 2.8.1], it is well known that if $D_{\mathbf{u}}\mathbf{F}(\mathbf{u}^*)$ is invertible, then the fixed point index of $\mathbf{F}$ at $\mathbf{u}^*$ is well defined and

$$\operatorname{index}(\mathbf{F}(\cdot), \mathbf{u}^*) = (-1)^{\varsigma},$$

where ς is the number of negative eigenvalues (counting the algebraic multiplicity) of the linearized operator $D_{\mathbf{u}}\mathbf{F}(\mathbf{u}^*)$.

Significantly, the eigenvalues of the linearized operator $D_{\mathbf{u}}\mathbf{F}(\mathbf{u}^*)$ and their algebraic multiplicities are the same regardless of whether it is considered an operator in $\mathbf{X}$ or $\mathbf{Y}$. Hence, it is convenient to use the decomposition (2.42) in our discussion of the eigenvalues of the linearized operator $D_{\mathbf{u}}\mathbf{F}(\mathbf{u}^*)$. By a simple calculation, one sees that $\mathbf{X}_{mj}$ is invariant under $D_{\mathbf{u}}\mathbf{F}(\mathbf{u}^*)$ for each integer $m \geq 1$ and each integer $1 \leq i \leq \dim E(\sigma_m)$. Moreover, α is an eigenvalue of $D_{\mathbf{u}}\mathbf{F}(\mathbf{u}^*)$ if and only if it is an eigenvalue of the matrix

$$\mathbf{K}_m := \mathbf{I} - \frac{1}{1+\sigma_m} \left[\Phi_{\mathbf{u}}^{-1}(\mathbf{u}^*) \Psi_{\mathbf{u}}(\mathbf{u}^*) + \mathbf{I} \right] = \frac{1}{1+\sigma_m} \left[\sigma_m \mathbf{I} - \Phi_{\mathbf{u}}^{-1}(\mathbf{u}^*) \Psi_{\mathbf{u}}(\mathbf{u}^*) \right]$$

for some $m \geq 1$. Consequently, $D_{\mathbf{u}}\mathbf{F}(\mathbf{u}^*)$ is invertible if and only if the matrix $\mathbf{K}_m$ is nonsingular for all $m \geq 1$.

Assume that α is an eigenvalue of $D_{\mathbf{u}}\mathbf{F}(\mathbf{u}^*)$. We next calculate its algebraic multiplicities, which we denote by $\chi(\alpha)$. By definition, it is well known that the algebraic multiplicity of the eigenvalue α is the dimension of the generalized null space E^{α}, where

$$E^{\alpha} := \bigcup_{i=1}^{\infty} \operatorname{Ker} \left[\alpha \mathbf{I} - D_{\mathbf{u}}\mathbf{F}(\mathbf{u}^*) \right]^{i}.$$

Every $\Theta \in \mathbf{X}$ can be uniquely expressed in the form

$$\Theta = \sum_{m=1}^{\infty} \sum_{j=1}^{\dim E(\sigma_m)} C_{mj}\theta_{mj},$$

where $C_{mj} \in \mathbb{R}^3$. Since $\mathbf{X}_{mj}$ is invariant under $D_{\mathbf{u}}\mathbf{F}(\mathbf{u}^*)$, it is also invariant under $[\alpha\mathbf{I} - D_{\mathbf{u}}\mathbf{F}(\mathbf{u}^*)]^i$ for each $i \geq 1$. Consequently, for any fixed $i \geq 1$

$$\Theta \in \operatorname{Ker}[\alpha\mathbf{I} - D_{\mathbf{u}}\mathbf{F}(\mathbf{u}^*)]^i \Longleftrightarrow [\alpha\mathbf{I} - D_{\mathbf{u}}\mathbf{F}(\mathbf{u}^*)]^i C_{mj}\theta_{mj} = 0$$

for all $m \geq 1$ and $1 \leq j \leq \dim E(\sigma_m)$. By a direct calculation, we find that

$$[\alpha\mathbf{I} - D_{\mathbf{u}}\mathbf{F}(\mathbf{u}^*)]^i C_{mj}\theta_{mj} = 0 \Longleftrightarrow [\alpha\mathbf{I} - \mathbf{K}_m]^i C_{mj} = 0.$$

It follows that

$$\dim E^{\alpha} = \sum_{m=1}^{\infty} \left[\dim E(\sigma_m) \times \dim\left(\bigcup_{i=1}^{\infty} \operatorname{Ker}[\alpha\mathbf{I} - \mathbf{K}_m]^i\right)\right].$$

Here $\dim\left(\bigcup_{i=1}^{\infty} \operatorname{Ker}[\alpha\mathbf{I} - \mathbf{K}_m]^i\right)$ is just the algebraic multiplicity of α as an eigenvalue of the matrix $\mathbf{K}_m$. A simple calculation gives that

$$\det\{\mathbf{K}_m\} = \frac{1}{(1+\sigma_m)^3} \det\left\{\sigma_m\mathbf{I} - \Phi_{\mathbf{u}}^{-1}(\mathbf{u}^*)\Psi_{\mathbf{u}}(\mathbf{u}^*)\right\}. \tag{2.56}$$

Moreover, when $\det\{\mathbf{K}_m\} \neq 0$ (i.e., $\det\left\{\sigma_m\mathbf{I} - \Phi_{\mathbf{u}}^{-1}(\mathbf{u}^*)\Psi_{\mathbf{u}}(\mathbf{u}^*)\right\} \neq 0$), the number of negative eigenvalues (counting algebraic multiplicity) of the matrix $\mathbf{K}_m$ is odd if and only if $\det\{\mathbf{K}_m\} < 0$. Consequently,

$$\varsigma = \sum_{\alpha<0} \chi(\alpha) = \sum_{\alpha<0} \dim(E^{\alpha}) = \sum_{m\geq 1, \det\{\mathbf{K}_m\}<0} \dim E(\sigma_m) \pmod 2.$$

In summary, we have the following lemma.

Lemma 2.13. *Assume that* $\det\left\{\sigma_m\mathbf{I} - \Phi_{\mathbf{u}}^{-1}(\mathbf{u}^*)\Psi_{\mathbf{u}}(\mathbf{u}^*)\right\} \neq 0$ *for all* $m \geq 1$. *Then*

$$\operatorname{index}(\mathbf{F}(\cdot), \mathbf{u}^*) = (-1)^{\varsigma}, \quad \textit{where} \quad \varsigma = \sum_{m\geq 1, \det\{\mathbf{K}_m\}<0} \dim E(\sigma_m).$$

Given Lemma 2.13, to facilitate our computation of $\operatorname{index}(\mathbf{F}(\cdot), \mathbf{u}^*)$, we next determine the sign of $\det\{\mathbf{K}_m\}$. By virtue of (2.56), we see that

$$\det\{\mathbf{K}_m\} = \det\{\sigma_m\Phi_{\mathbf{u}}(\mathbf{u}^*) - \Psi_{\mathbf{u}}(\mathbf{u}^*)\} \cdot \det\{\Phi_{\mathbf{u}}^{-1}(\mathbf{u}^*)\}/(1+\sigma_m)^3. \tag{2.57}$$

As we have known for all $\sigma \geq 0$, $\det\{\Phi_{\mathbf{u}}^{-1}(\mathbf{u}^*)\}/(1+\sigma)^3$ is positive, we shall consider $\det\{\sigma\Phi_{\mathbf{u}}(\mathbf{u}^*) - \Psi_{\mathbf{u}}(\mathbf{u}^*)\}$. A direct calculation yields

$$\Phi_{\mathbf{u}}(\mathbf{u}^*) = \begin{pmatrix} d_N & 0 & 0 \\ 0 & d_P + \beta S^* & \beta P^* \\ 0 & 0 & d_S \end{pmatrix} \quad \text{and} \quad \Psi_{\mathbf{u}}(\mathbf{u}^*) = \begin{pmatrix} -N^* & -N^* & 0 \\ \gamma P^* & -P^* & 0 \\ \tau & 0 & -\eta \end{pmatrix}.$$

Thus,

$$\sigma\Phi_{\mathbf{u}}(\mathbf{u}^*) - \Psi_{\mathbf{u}}(\mathbf{u}^*) = \begin{pmatrix} \sigma d_N + N^* & N^* & 0 \\ -\gamma P^* & \sigma(d_P + \beta S^*) + P^* & \sigma\beta P^* \\ -\tau & 0 & \sigma d_S + \eta \end{pmatrix}.$$

Furthermore, we have

$$\begin{aligned} \det\{\sigma\Phi_{\mathbf{u}}(\mathbf{u}^*) - \Psi_{\mathbf{u}}(\mathbf{u}^*)\} &= \mathfrak{C}_3(\beta)\sigma^3 + \mathfrak{C}_2(\beta)\sigma^2 + \mathfrak{C}_1(\beta)\sigma + \mathfrak{C}_0(\beta) \\ &\triangleq \mathfrak{C}(\beta, \sigma), \end{aligned} \tag{2.58}$$

where

$$\begin{aligned} &\mathfrak{C}_3(\beta) = d_N d_S (d_P + \beta S^*) > 0, \\ &\mathfrak{C}_2(\beta) = \eta d_N (d_P + \beta S^*) + d_N d_S P^* + N^* d_S (d_P + \beta S^*) > 0, \\ &\mathfrak{C}_1(\beta) = \eta d_N P^* + \eta N^* (d_P + \beta S^*) + d_S N^* P^* - \tau\beta N^* P^* + \gamma N^* P^* d_S, \\ &\mathfrak{C}_0(\beta) = N^* P^* \eta + \gamma N^* P^* \eta > 0. \end{aligned} \tag{2.59}$$

Next, we discuss the dependence of $\mathfrak{C}(\beta, \sigma)$ on the cross-diffusion coefficient β. Suppose that $\widetilde{\sigma}_1(\beta)$, $\widetilde{\sigma}_2(\beta)$, $\widetilde{\sigma}_3(\beta)$ are the three roots of $\mathfrak{C}(\beta, \sigma) = 0$ and satisfy $\mathrm{Re}\{\widetilde{\sigma}_1(\beta)\} \leq \mathrm{Re}\{\widetilde{\sigma}_2(\beta)\} \leq \mathrm{Re}\{\widetilde{\sigma}_3(\beta)\}$. Notice that $\mathfrak{C}_3(\beta) > 0$ and $\mathfrak{C}_0(\beta) > 0$. It follows from the Routh-Hurwitz criterion [100, Appendix B.1] that $\widetilde{\sigma}_1(\beta)\widetilde{\sigma}_2(\beta)\widetilde{\sigma}_3(\beta) < 0$. Thus, at least one of $\widetilde{\sigma}_1(\beta)$, $\widetilde{\sigma}_2(\beta)$, $\widetilde{\sigma}_3(\beta)$ is real and negative, and the product of the other two are positive. Moreover, we have the following limits:

$$\begin{aligned} &\lim_{\beta\to\infty} \mathfrak{C}_3(\beta)/\beta = d_N d_S S^* \triangleq \mathbb{C}_3 > 0, \\ &\lim_{\beta\to\infty} \mathfrak{C}_2(\beta)/\beta = \eta d_N S^* + d_S N^* S^* \triangleq \mathbb{C}_2 > 0, \\ &\lim_{\beta\to\infty} \mathfrak{C}_1(\beta)/\beta = \tau N^*[\lambda(1-\gamma) - 2\mu]/(\gamma+1) \triangleq \mathbb{C}_1, \\ &\lim_{\beta\to\infty} \mathfrak{C}_0(\beta)/\beta = 0. \end{aligned}$$

When $2\mu > \lambda(1-\gamma)$, it is clear that $\mathbb{C}_1 < 0$, and thus $\mathfrak{C}_1(\beta) < 0$ for large β. By virtue of (2.58) and (2.59), we obtain

$$\lim_{\beta\to\infty} \frac{\mathfrak{C}(\beta,\sigma)}{\beta} = \sigma[\mathbb{C}_3\sigma^2 + \mathbb{C}_2\sigma + \mathbb{C}_1] \triangleq \mathbb{C}(\sigma). \tag{2.60}$$

Obviously, $\mathbb{C}(\sigma)$ has three real roots

$$0, \quad \frac{-\mathbb{C}_2 + \sqrt{\mathbb{C}_2^2 - 4\mathbb{C}_3\mathbb{C}_1}}{2\mathbb{C}_3} > 0, \quad \frac{-\mathbb{C}_2 - \sqrt{\mathbb{C}_2^2 - 4\mathbb{C}_3\mathbb{C}_1}}{2\mathbb{C}_3} < 0. \tag{2.61}$$

Hence, when $2\mu > \lambda(1-\gamma)$ and β is sufficiently large, a continuity argument shows that $\widetilde{\sigma}_1(\beta)$ is real and negative. Moreover, $\widetilde{\sigma}_2(\beta)$ and $\widetilde{\sigma}_3(\beta)$ are real and positive since $\widetilde{\sigma}_2(\beta)\widetilde{\sigma}_3(\beta) > 0$. In particular, $\widetilde{\sigma}_1(\beta)$, $\widetilde{\sigma}_2(\beta)$ and $\widetilde{\sigma}_3(\beta)$ satisfy

$$\begin{cases} \lim\limits_{\beta\to\infty} \widetilde{\sigma}_1(\beta) = \frac{-\mathbb{C}_2-\sqrt{\mathbb{C}_2^2-4\mathbb{C}_3\mathbb{C}_1}}{2\mathbb{C}_3} < 0, \\ \lim\limits_{\beta\to\infty} \widetilde{\sigma}_2(\beta) = 0, \\ \lim\limits_{\beta\to\infty} \widetilde{\sigma}_3(\beta) = \frac{-\mathbb{C}_2+\sqrt{\mathbb{C}_2^2-4\mathbb{C}_3\mathbb{C}_1}}{2\mathbb{C}_3} = \overline{\sigma} > 0. \end{cases} \tag{2.62}$$

Summarizing, we have the following lemma.

Lemma 2.14. *Let $\widetilde{\sigma}_1(\beta)$, $\widetilde{\sigma}_2(\beta)$ and $\widetilde{\sigma}_3(\beta)$ be the three roots of $\mathfrak{C}(\beta,\sigma) = 0$ (see* (2.58)*). Assume $2\mu > \lambda(1-\gamma)$. Then there is a positive number β^* such that, for all $\beta \geq \beta^*$, $\widetilde{\sigma}_1(\beta)$, $\widetilde{\sigma}_2(\beta)$ and $\widetilde{\sigma}_3(\beta)$ are all real and satisfy (2.62). Moreover, if $\beta \geq \beta^*$, then*

$$\begin{cases} -\infty < \widetilde{\sigma}_1(\beta) < 0 < \widetilde{\sigma}_2(\beta) < \widetilde{\sigma}_3(\beta), \\ \mathfrak{C}(\beta,\sigma) < 0 \quad \textit{when} \quad \sigma \in (-\infty, \widetilde{\sigma}_1(\beta)) \cup (\widetilde{\sigma}_2(\beta), \widetilde{\sigma}_3(\beta)), \\ \mathfrak{C}(\beta,\sigma) > 0 \quad \textit{when} \quad \sigma \in (\widetilde{\sigma}_1(\beta), \widetilde{\sigma}_2(\beta)) \cup (\widetilde{\sigma}_3(\beta), \infty). \end{cases} \tag{2.63}$$

Proof of Theorem 2.4. In view of the assumption on $\overline{\sigma}$, Lemma 2.14 shows that there is a positive constant β^* such that, for all $\beta \geq \beta^*$, (2.63) holds and

$$\widetilde{\sigma}_1(\beta) < 0 = \sigma_1 < \widetilde{\sigma}_2(\beta) < \sigma_2, \quad \widetilde{\sigma}_3(\beta) \in (\sigma_j, \sigma_{j+1}). \tag{2.64}$$

It follows from (2.57) and (2.58) that

$$\det\{\mathbf{K}_m\} = \frac{\det\{\Phi_{\mathbf{u}}^{-1}(\mathbf{u}^*)\}}{(1+\sigma_m)^3} \cdot \det\{\sigma_m\Phi_{\mathbf{u}}(\mathbf{u}^*) - \Psi_{\mathbf{u}}(\mathbf{u}^*)\} = \frac{\det\{\Phi_{\mathbf{u}}^{-1}(\mathbf{u}^*)\} \cdot \mathfrak{C}(\beta,\sigma_m)}{(1+\sigma_m)^3},$$

which, along with (2.63) and (2.64), gives

$$\begin{cases} \det\{\mathbf{K}_1\} = \det\{\Phi_{\mathbf{u}}^{-1}(\mathbf{u}^*)\}(1+\gamma)N^*P^*\eta > 0, & \\ \det\{\mathbf{K}_m\} < 0, & 2 \le m \le j, \\ \det\{\mathbf{K}_m\} > 0, & m \ge j+1. \end{cases}$$

Hence, 0 is not an eigenvalue of the matrix $\sigma_m \mathbf{I} - \Phi_{\mathbf{u}}^{-1}(\mathbf{u}^*)\Psi_{\mathbf{u}}(\mathbf{u}^*)$ for any $m \ge 1$, and thus, Lemma 2.13 shows that

$$\varsigma = \sum_{m\ge 1, \det\{\mathbf{K}_m\}<0} \dim E(\sigma_m) = \sum_{m=2}^{j} \dim E(\sigma_m), \quad \text{which is odd,}$$

and

$$\text{index}\,(\mathbf{F}(\cdot), \mathbf{u}^*) = (-1)^{\varsigma} = -1. \tag{2.65}$$

Based on the homotopy invariance of the topological degree, we shall complete the proof by contradiction. Assume that the conclusion is false for some $\beta = \overline{\beta} \ge \beta^*$. In the following, we will fix $\beta = \overline{\beta}$. For $t \in [0,1]$, we define

$$\Phi(t, \mathbf{u}) = (d_N N, d_P P + t\beta SP, d_S S)^{\mathcal{T}}$$

and consider

$$-\Delta\Phi(t, \mathbf{u}) = \Psi(\mathbf{u}), \quad x \in \Omega, \qquad \partial_\nu \mathbf{u} = 0, \quad x \in \partial\Omega. \tag{2.66}$$

Clearly, $\mathbf{u}$ is a positive non-constant solution of (2.16) if and only if it is such a solution of (2.66) for $t = 1$. For any $0 \le t \le 1$, $\mathbf{u}^*$ is the unique constant positive solution of (2.66). As we observed above, $\mathbf{u}$ is a positive solution to (2.66) if and only if

$$\mathbf{F}(t; \mathbf{u}) \triangleq \mathbf{u} - (\mathbf{I} - \Delta)^{-1} \left\{ \Phi_{\mathbf{u}}^{-1}(t; \mathbf{u}) \left[\Psi(\mathbf{u}) + \nabla\mathbf{u}\Phi_{\mathbf{uu}}(t; \mathbf{u})\nabla\mathbf{u} \right] + \mathbf{u} \right\} = 0 \ \text{ in } \ \mathbf{Y}^+.$$

Obviously, $\mathbf{F}(1; \mathbf{u}) = \mathbf{F}(\mathbf{u})$. It follows from Theorem 2.2 that $\mathbf{u}^*$ is the only solution of $\mathbf{F}(0; \mathbf{u}) = 0$ in $\mathbf{Y}^+$. By a straightforward calculation, the linearization of $\mathbf{F}(t; \mathbf{u})$ at $\mathbf{u}^*$ is given by

$$D_{\mathbf{u}}\mathbf{F}(t; \mathbf{u}^*) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1} \left\{ \Phi_{\mathbf{u}}^{-1}(t; \mathbf{u}^*)\Psi_{\mathbf{u}}(\mathbf{u}^*) + \mathbf{I} \right\}.$$

In particular,

$$D_{\mathbf{u}}\mathbf{F}(0; \mathbf{u}^*) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1} \left\{ \mathcal{D}^{-1}\Psi_{\mathbf{u}}(\mathbf{u}^*) + \mathbf{I} \right\}$$

and

$$D_{\mathbf{u}}\mathbf{F}(1; \mathbf{u}^*) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1} \left\{ \Phi_{\mathbf{u}}^{-1}(\mathbf{u}^*)\Psi_{\mathbf{u}}(\mathbf{u}^*) + \mathbf{I} \right\} = D_{\mathbf{u}}\mathbf{F}(\mathbf{u}^*),$$

where $\mathcal{D} = \text{diag}(d_N, d_P, d_S)$. Hence, noting the facts $\mathbf{F}(1;\mathbf{u}) = \mathbf{F}(\mathbf{u})$ and $D_{\mathbf{u}}\mathbf{F}(1;\mathbf{u}^*) = D_{\mathbf{u}}\mathbf{F}(\mathbf{u}^*)$, from (2.65), one has

$$\text{index}\,(\mathbf{F}(1;\cdot), \mathbf{u}^*) = \text{index}\,(\mathbf{F}(\cdot), \mathbf{u}^*) = -1.$$

And by virtue of the stability of $\mathbf{u}^*$, we can easily show that

$$\text{index}\,(\mathbf{F}(0;\cdot), \mathbf{u}^*) = (-1)^0 = 1.$$

In addition, according to Lemmas 2.10 and 2.12, for all $0 \le t \le 1$, there exists a positive constant C such that any positive solution of (2.16) satisfies $1/C < N, P, S < C$. Consequently, for all $0 \le t \le 1$, $\mathbf{F}(t;\mathbf{u}) \neq 0$ on $\partial B(C)$. By the homotopy invariance of the topological degree [5, Theorem 11.1], we see that

$$\deg\,(\mathbf{F}(1;\cdot), 0, B(C)) = \deg\,(\mathbf{F}(0;\cdot), 0, B(C))\,. \tag{2.67}$$

Since Theorem 2.2 shows that $\mathbf{u}^*$ is the only solution of $\mathbf{F}(0;\mathbf{u}) = 0$ in $B(C)$, the excision property [5, Corollary 11.2] implies that

$$\deg\,(\mathbf{F}(0;\cdot), 0, B(C)) = \text{index}\,(\mathbf{F}(0;\cdot), \mathbf{u}^*) = 1.$$

On the other hand, our supposition implies that the equation $\mathbf{F}(1;\mathbf{u}) = 0$ has only the positive solution $\mathbf{u}^*$ in $B(C)$. Thus, the excision property yields

$$\deg\,(\mathbf{F}(1;\cdot), 0, B(C)) = \text{index}\,(\mathbf{F}(1;\cdot), \mathbf{u}^*) = -1.$$

This contradicts (2.67), and the proof is complete. □

Chapter 3

Global Dynamics of a Three-Species Lotka-Volterra Food Chain Model with Intraguild Predation and Taxis Mechanisms

3.1 Introduction and Main results

Before presenting our context, we clarify that the results stated in this chapter have been published in our journal paper [22].

3.1.1 Introduction

To understand the complex ecological interactions, various ordinary differential equation (ODE) type food chain models have been proposed, and some interesting and impressive results have been established on the dynamics of three species food chain model (e.g., [47, 53, 75, 120, 137]). In particular, the chaos phenomenon can be found for the three species food chain models with nonlinear functional responses [47, 73] or for the simple Lotka-Volterra type functional responses with intraguild predation (i.e., a simple kind of omnivory in which a predator and a prey share a common resource) [120]. As we know, the spatial movement plays an indispensable role for the population species to survive and thrive. However, compared with the well-known results on the temporal three species predator-prey systems (e.g., [47, 53, 75, 120, 137]), few results are available for the food chain model with spatial movement. Here, we shall consider the three-species

Lotka-Volterra food chain model with spatial movement:

$$\begin{cases} u_t = d_1\Delta u + u(1-u) - b_1uv - \gamma_1uw, & x\in\Omega, t>0,\\ v_t = d_2\Delta v - \xi\nabla\cdot(v\nabla u) + uv - b_2vw - \theta_1 v, & x\in\Omega, t>0,\\ w_t = \Delta w - \chi\nabla\cdot[w\nabla\phi(u,v)] + vw + \gamma_2uw - \theta_2 w, & x\in\Omega, t>0,\\ \partial_\nu u = \partial_\nu v = \partial_\nu w = 0, & x\in\partial\Omega, t>0. \end{cases} \tag{3.1}$$

All notations and parameters have the same interpretation as in Section 1.3.

Related works on the system (3.1)**.** To put our research into perspective, we first recall some related results for (3.1). If $w\equiv 0$, (3.1) becomes the two species predator-prey system with prey-taxis (called the prey-taxis system), which was first proposed by Kareiva and Odell to interpret the heterogeneous aggregative patterns due to the area-restricted search strategy [68] and has been extensively studied (cf. [16, 64, 65, 68, 148, 151] and references therein).

Different from the substantial results on the two-species predator-prey systems with various taxis mechanisms, limited attention has been paid to the three-species spatial food chain model (3.1) (i.e., $w\neq 0$). Recently, the authors in [66] studied the global dynamics of system (3.1) in a two dimensional bounded domain under the following assumptions:

$$\gamma_1 = \gamma_2 = 0 \quad\text{and}\quad \phi(u,v) = v. \tag{3.2}$$

The ideas/methods used in [66] depend on that the system (3.1) with (3.2) has a nice entropy estimate, which was first developed in [124] for the classical chemotaxis system with consumption of chemoattractant and later was used to study the prey-taxis system [64].

If $\gamma_1,\ \gamma_2>0$, the ODE counterpart of (3.1), termed the intraguild predation (IGP) model, exhibits complex dynamics and was extensively studied (see [53, 98, 110, 120] and references therein). For the spatial model (3.1) with intraguild predation (i.e., $\gamma_1,\gamma_2>0$), the study [46] incorporated the intraspecific competitions for v and w along with the signal intensity function $\phi(u,v)=uv$, termed the alarm-taxis, which was proposed to test the "burglar alarm" hypothesis (cf. [15]): a prey species renders itself dangerous to a primary predator by generating an alarm call to attract a second predator at higher trophic levels in the food chain that preys on the primary predator. In [46], the authors established the global boundedness for $\gamma_1,\gamma_2\geq 0$ and pattern formations for $\gamma_1=\gamma_2=0$ in one dimensional space. Motivated by the work [46], the authors in [67] considered the ratio-dependent functional response (i.e., replacing $\gamma_i uw$ by $\gamma_i\frac{uw}{u+w}$ for $i=1,\ 2$) and established the global boundedness and stability for $\gamma_1,\gamma_2\geq 0$ in two dimensions. No results exist for the spatial Lotka-Volterra food chain model (3.1) with intraguild predation (i.e., $\gamma_1, \gamma_2 > 0$) and more general signal functional $\phi(u, v)$.

Consequently, our goal is to study (3.1) with $\gamma_1, \gamma_2 > 0$ and more general signal functional $\phi(u, v)$. To explore the combined effects of the intraguild predation and taxis mechanisms more clearly, we focus on studying the global dynamics of the system (3.1) in an open interval $\Omega \subset \mathbb{R}$:

$$\begin{cases} u_t = d_1 u_{xx} + u(1-u) - b_1 uv - \gamma_1 uw, & x \in \Omega, t > 0, \\ v_t = d_2 v_{xx} - \xi(v u_x)_x + uv - b_2 vw - \theta_1 v, & x \in \Omega, t > 0, \\ w_t = w_{xx} - \chi(w \phi(u,v)_x)_x + vw + \gamma_2 uw - \theta_2 w, & x \in \Omega, t > 0, \\ u_x = v_x = w_x = 0, & x \in \partial\Omega, t > 0, \\ (u, v, w)(x, 0) = (u_0, v_0, w_0)(x), & x \in \Omega. \end{cases} \tag{3.3}$$

For more generally, we assume that the signal intensity function $\phi(u, v)$ satisfies the following conditions:

(H0) $\phi(y, z) : (0, \infty) \times (0, \infty) \to \mathbb{R}$ is positive and it belongs to $C^2([0, \infty) \times [0, \infty))$.

Specifically, our objectives include the following:

(B.1) Establish the global well-posedness of solutions (global existence and stability) to (3.1) under suitable conditions;

(B.2) Explore the effects of the intraguild predation and/or taxis mechanisms (prey-taxis and alarm-taxis) on pattern formations.

The main challenge in the analyses is that, if $\gamma_1, \gamma_2 > 0$ or $\phi(u, v) \neq v$, the ideas used in [66] are not available anymore. Moreover, due to the lack of quadratic decay terms (i.e., intraspecific competitions) for v and w, the methods developed in [67] are also inapplicable, which motivates us to develop new ideas to study this model.

3.1.2 Main Results

We first show the global existence of classical solution as follows.

Theorem 3.1 (Global boundedness)**.** *Let $\Omega \subset \mathbb{R}$ be a bounded open interval. Suppose that the initial data $0 \lneqq (u_0, v_0, w_0) \in [W^{1,\infty}(\Omega)]^3$ and (H0) holds. Then (3.3) admits a unique global classical solution (u, v, w) fulfilling $u, v, w > 0$. Moreover, there exists a constant $M > 0$ independent of t such that*

$$\|u(\cdot, t)\|_{W^{1,2}} + \|v(\cdot, t)\|_{W^{1,2}} + \|w(\cdot, t)\|_{L^\infty} \leq M.$$

Remark 3.1. *The upper bounds of* $\|u(\cdot,t)\|_{L^\infty}$ *and* $\|v(\cdot,t)\|_{L^\infty}$ *play an important role in studying the large time behavior of solutions. In fact, we can show that*

$$\|u(\cdot,t)\|_{L^\infty} \le M_0 := \max\{1, \|u_0\|_{L^\infty}\}, \tag{3.4}$$

and

$$\|v(\cdot,t)\|_{L^\infty} \le K_0 := C[1+\xi(\xi^6+1)^{\frac{1}{2}}], \tag{3.5}$$

where the constant $C>0$ *depends on the parameters* $u_0, v_0, \gamma_i, \theta_i, b_i, d_i$ $(i=1,2)$ *and* $|\Omega|$ *but it is independent of* ξ *and* χ.

A central question in population dynamics is whether the interacting species population will arrive at the coexistence, exclusion or extinction eventually. When $\gamma_1=\gamma_2=0$ and $\phi(u,v)=v$, it has been proved in [66] that the globally bounded solution will converge to the constant steady state as $t\to\infty$ and no pattern formation occurs. Hence, there exist some interesting questions:

(i) How about the global dynamics of solution for the system (3.3) with $\gamma_1,\gamma_2>0$? Whether or not pattern formation occurs?

(ii) If $\gamma_1=\gamma_2=0$, whether or not pattern formation occurs for other forms of $\phi(u,v)$ instead of $\phi(u,v)=v$?

To answer the above questions, we first classify the constant steady state (u_c,v_c,w_c) of the system (3.3) with $\gamma_1,\gamma_2>0$, which satisfies

$$u_c(1-u_c-b_1v_c-\gamma_1w_c)=0, \quad v_c(u_c-b_2w_c-\theta_1)=0, \quad w_c(v_c+\gamma_2u_c-\theta_2)=0. \tag{3.6}$$

A direct calculation implies that the constant steady state (u_c,v_c,w_c) takes the following five cases:

- Trivial steady states: $E_0:=(0,0,0)$ and $E_1:=(1,0,0)$;
- Semi-trivial steady states: $E_{12}:=\left(\theta_1, \frac{1-\theta_1}{b_1}, 0\right)$ and $E_{13}:=\left(\frac{\theta_2}{\gamma_2}, 0, \frac{\gamma_2-\theta_2}{\gamma_1\gamma_2}\right)$;
- Coexistence steady state: $E_*:=(u_*,v_*,w_*)$, where

$$\begin{cases} u_*=\frac{b_2(1-b_1\theta_2)+\gamma_1\theta_1}{b_2+\gamma_1-b_1b_2\gamma_2}>0,\\ v_*=\frac{\gamma_1(\theta_2-\gamma_2\theta_1)+b_2(\theta_2-\gamma_2)}{b_2+\gamma_1-b_1b_2\gamma_2}>0,\\ w_*=\frac{b_1(\gamma_2\theta_1-\theta_2)+(1-\theta_1)}{b_2+\gamma_1-b_1b_2\gamma_2}>0. \end{cases} \tag{3.7}$$

One can check that the coexistence steady state $E_* := (u_*, v_*, w_*)$ is linearly unstable if $b_2 + \gamma_1 - b_1 b_2 \gamma_2 < 0$. Therefore, for the case of coexistence steady state (u_*, v_*, w_*), we only focus on studying the dynamics in the following range of parameters

$$\begin{cases} b_2 + \gamma_1 - b_1 b_2 \gamma_2 > 0, \\ \gamma_1(\theta_2 - \gamma_2\theta_1) + b_2(\theta_2 - \gamma_2) > 0, \\ b_1(\gamma_2\theta_1 - \theta_2) + (1 - \theta_1) > 0, \end{cases} \iff \begin{cases} b_2 + \gamma_1 - b_1 b_2 \gamma_2 > 0, \\ \theta_2 > \frac{\gamma_1\gamma_2}{b_2+\gamma_1}\theta_1 + \frac{b_2\gamma_2}{b_2+\gamma_1}, \\ \theta_2 < \frac{b_1\gamma_2-1}{b_1}\theta_1 + \frac{1}{b_1}. \end{cases} \tag{3.8}$$

Then by constructing some appropriate energy functionals, we can derive the global stability of the constant steady states as follows.

Theorem 3.2 (Global stability)**.** *Assume M_0 and K_0 are defined in* (3.4) *and* (3.5)*, respectively. Then the solution (u, v, w) of* (3.3) *obtained in Theorem 3.1 has the following convergence properties:*

(1) If $\theta_1 > 1$ and $\theta_2 > \gamma_2$, then it holds that

$$\lim_{t\to\infty} (\|u - 1\|_{L^\infty} + \|v\|_{L^\infty} + \|w\|_{L^\infty}) = 0.$$

(2) If $0 < \theta_1 < 1$ and $\theta_2 > \ell_1$ with

$$\ell_1 := \frac{\gamma_1}{b_1 b_2}\theta_1 - \frac{\theta_1}{b_1} + \frac{1}{b_1} + \frac{\max\{b_1 b_2\gamma_2 - \gamma_1, 0\}}{b_1 b_2}, \tag{3.9}$$

then there exists $\xi_0 > 0$ such that whenever $\xi \in (0, \xi_0)$, it holds that

$$\lim_{t\to\infty} \left(\|u - \theta_1\|_{L^\infty} + \|v - \frac{1-\theta_1}{b_1}\|_{L^\infty} + \|w\|_{L^\infty} \right) = 0.$$

(3) If $\theta_1 > 1$, $\theta_2 < \min\{\gamma_2, \ell_2\}$ with

$$\ell_2 := \frac{\gamma_1\gamma_2}{b_1 b_2 \gamma_2 + b_2}\theta_1 + \frac{b_2\gamma_2}{b_1 b_2\gamma_2 + b_2} + \frac{\gamma_2 \min\{b_1 b_2 \gamma_2 - \gamma_1, 0\}}{b_1 b_2 \gamma_2 + b_2}, \tag{3.10}$$

then there exist $\xi_1 > 0$ and $\chi_1 > 0$ such that whenever $\xi \in (0, \xi_1)$ and $\chi \in (0, \chi_1)$, it holds that

$$\lim_{t\to\infty} \left(\|u - \frac{\theta_2}{\gamma_2}\|_{L^\infty} + \|v\|_{L^\infty} + \|w - \frac{\gamma_2 - \theta_2}{\gamma_1\gamma_2}\|_{L^\infty} \right) = 0.$$

(4) If (3.8) *and $\gamma_1 = b_1 b_2 \gamma_2$ hold, then there exist $\xi_2 > 0$ and $\chi_2 > 0$ such that whenever $\xi \in (0, \xi_2)$ and $\chi \in (0, \chi_2)$, it holds that*

$$\lim_{t\to\infty} (\|u - u_*\|_{L^\infty} + \|v - v_*\|_{L^\infty} + \|w - w_*\|_{L^\infty}) = 0,$$

where the coexistence steady state (u_*, v_*, w_*) is defined in (3.7)*.*

In view of the results obtained in Theorem 3.2, it is natural to ask whether or not pattern formations (non-constant steady states) are possible when parameters outside the stability regimes found in Theorem 3.2. To answer this question, we first do some linearly stable analysis (see Proposition 3.1), which together with the global stability results for the corresponding space-absent ODE system obtained in [55], implies that the pattern (if any) can only arise from the homogeneous coexistence steady state (u_*, v_*, w_*). In Section 3.4, we shall use linear stability analysis to find the conditions on parameters for the instability of coexistence steady state. Then we perform numerical simulations to illustrate that spatially inhomogeneous patterns indeed can be found under certain conditions in Section 3.5, and give positive answers to aforementioned questions (i) and (ii).

3.2 Global Boundedness: Proof of Theorem 3.1

This section will prove the boundedness of the global classical solution to (3.3) as stated in Theorem 3.1. In the following context, the constants k_i and M_i $(i = 1, 2, 3\cdots)$ represent generic positive constants independent of t and will vary line-by-line.

3.2.1 Local Existence and Preliminaries

Firstly, the local existence of solutions can be proved by using the Amann's theorem [7, Theorem 7.3], we omit the proof details for brevity.

Lemma 3.1 (Local existence)**.** *Let the conditions in Theorem 3.1 hold. Then there admits* $T_{\max} \in (0, \infty]$ *such that the system* (3.3) *has a unique classical solution*

$$(u, v, w) \in \left[C^0([0, T_{\max}); W^{1,2}(\Omega)] \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max}))\right]^3$$

satisfying $u, v, w > 0$ *for all* $t > 0$*. Moreover, it holds that if* $T_{\max} < \infty$, *then*

$$\limsup_{t \nearrow T_{\max}} (\|u(\cdot, t)\|_{W^{1,p}} + \|v(\cdot, t)\|_{W^{1,p}} + \|w(\cdot, t)\|_{L^\infty}) = \infty, \forall p > 1.$$

Using similar arguments as in [64, Lemma 2.2], we obtain the boundedness of u immediately as follows.

Lemma 3.2. *Suppose the assumptions in Lemma 3.1 hold. Then it holds that*

$$0 < u(x, t) \leq M_0 := \max\{1, \|u_0\|_{L^\infty}\} \quad \text{for all } (x, t) \in \Omega \times (0, T_{\max}); \tag{3.11}$$

Moreover, one has

$$\limsup_{t \to \infty} u(x, t) \leq 1 \quad \text{for all } \; x \in \bar{\Omega}. \tag{3.12}$$

Lemma 3.3. *Let (u,v,w) be a solution to the system (3.3) obtained in Lemma 3.1. Then there exist two constants $M_1>0$ and $M_2>0$ independent of ξ and χ such that for all $t\in(0,T_{\max})$*

$$\|v(\cdot,t)\|_{L^1}\le M_1:=\frac{\theta_1\|u_0\|_{L^1}+\theta_1 b_1\|v_0\|_{L^1}+(1+\theta_1)M_0|\Omega|}{\theta_1 b_1}, \tag{3.13}$$

and

$$\|w(\cdot,t)\|_{L^1}\le M_2:=\begin{cases}\frac{\gamma_0(\|u_0\|_{L^1}+b_1\|v_0\|_{L^1}+b_1b_2\|w_0\|_{L^1})+2M_0|\Omega|}{b_1b_2\gamma_0}, & \text{if } \gamma_i=0,\\ \frac{\gamma_0(b_2\gamma_2\|u_0\|_{L^1}+b_2\gamma_1\|w_0\|_{L^1}+\gamma_1\|v_0\|_{L^1})+2b_2\gamma_2M_0|\Omega|+M_0M_1\gamma_1}{\gamma_0b_2\gamma_1}, & \text{if } \gamma_i>0,\end{cases} \tag{3.14}$$

where $i=1,2$.

Proof. Using the first and second equations of (3.3) and applying the homogeneous Neumann boundary conditions, we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega(u+b_1v)+\int_\Omega u^2&=\int_\Omega u-b_1\theta_1\int_\Omega v-\gamma_1\int_\Omega uw-b_1b_2\int_\Omega vw,\\ &\le\int_\Omega u-b_1\theta_1\int_\Omega v,\end{aligned}$$

which, along with $\theta_1>0$ and (3.11), can be updated as

$$\frac{d}{dt}\int_\Omega(u+b_1v)+\theta_1\int_\Omega(u+b_1v)+\int_\Omega u^2\le(1+\theta_1)\int_\Omega u\le(1+\theta_1)M_0|\Omega|,$$

and hence applying Grönwall's inequality, one has

$$\|v(\cdot,t)\|_{L^1}\le\frac{\|u_0\|_{L^1}}{b_1}+\|v_0\|_{L^1}+\frac{(1+\theta_1)M_0|\Omega|}{\theta_1b_1}=:M_1. \tag{3.15}$$

Next, we shall show the boundedness of $\|w(\cdot,t)\|_{L^1}$. To this end, we divide our proof into two cases: $\gamma_1=\gamma_2=0$ and $\gamma_1,\gamma_2>0$.

Case 1: $\gamma_1=\gamma_2=0$. In this case, we deduce from the equations of (3.3) that

$$\frac{d}{dt}\int_\Omega(u+b_1v+b_1b_2w)+\int_\Omega u^2+b_1\theta_1\int_\Omega v+b_1b_2\theta_2\int_\Omega w=\int_\Omega u. \tag{3.16}$$

Denoting $\gamma_0:=\min\{1,\theta_1,\theta_2\}$ and using (3.11), it follows from (3.16) that

$$\frac{d}{dt}\int_\Omega(u+b_1v+b_1b_2w)+\gamma_0\int_\Omega(u+b_1v+b_1b_2w)\le 2M_0|\Omega|,$$

which, together with Grönwall's inequality, gives

$$\|w(\cdot,t)\|_{L^1} \le \frac{\gamma_0(\|u_0\|_{L^1} + b_1\|v_0\|_{L^1} + b_1 b_2\|w_0\|_{L^1}) + 2M_0|\Omega|}{b_1 b_2 \gamma_0}. \tag{3.17}$$

Case 2: $\gamma_1, \gamma_2 > 0$**.** Using the equations of (3.3), one has

$$\frac{d}{dt}\int_\Omega \left(\gamma_2 u + \gamma_1 w + \frac{\gamma_1}{b_2} v\right) + \gamma_2 \int_\Omega u^2 + \theta_2\gamma_1 \int_\Omega w + \frac{\theta_1\gamma_1}{b_2}\int_\Omega v \le \gamma_2 \int_\Omega u + \frac{\gamma_1}{b_2}\int_\Omega uv,$$

which together with (3.11) and (3.15) derives

$$\frac{d}{dt}\int_\Omega \left(\gamma_2 u + \gamma_1 w + \frac{\gamma_1}{b_2} v\right) + \gamma_0 \int_\Omega \left(\gamma_2 u + \gamma_1 w + \frac{\gamma_1}{b_2} v\right) \le 2\gamma_2 M_0|\Omega| + \frac{\gamma_1 M_0 M_1}{b_2},$$

and hence using Grönwall's inequality, we have

$$\|w(\cdot,t)\|_{L^1} \le \frac{\gamma_0(b_2\gamma_2\|u_0\|_{L^1} + b_2\gamma_1\|w_0\|_{L^1} + \gamma_1\|v_0\|_{L^1}) + 2b_2\gamma_2 M_0|\Omega| + M_0 M_1\gamma_1}{\gamma_0 b_2 \gamma_1},$$

which combined with (3.17) gives (3.14). Then, the proof of Lemma 3.3 is completed. □

Next, we can use the semigroup estimates to obtain the boundedness of $\|u_x(\cdot,t)\|_{L^q}$ for any $q > 1$ in one dimensional space.

Lemma 3.4. *Let* (u, v, w) *be the solution to the system* (3.3) *obtained in Lemma 3.1. Then for any* $q > 1$*, it holds that*

$$\|u_x(\cdot,t)\|_{L^q} \le M_3 := M_3(q), \text{ for all } \; t \in (0, T_{\max}), \tag{3.18}$$

where the constant $M_3(q) > 0$ *is defined in* (3.22)*, and is independent of* ξ *and* χ*.*

Proof. The first equation of (3.3) can be rewritten as

$$u_t - d_1(u_{xx} - u) = f(x,t), \tag{3.19}$$

where $f(x,t) = (d_1 + 1 - u - b_1 v - \gamma_1 w)u$. Using Hölder inequality, the facts $0 < u \le M_0$ in (3.11), $\|v(\cdot,t)\|_{L^1} \le M_1$ in (3.13) and $\|w(\cdot,t)\|_{L^1} \le M_2$ in (3.14), one has

$$\begin{aligned} \|f(\cdot,t)\|_{L^1} &= \|(d_1 + 1 - u - b_1 v - \gamma_1 w)u\|_{L^1} \\ &\le M_0\left(|\Omega|(d_1 + 1 + M_0) + M_1 b_1 + M_2\gamma_1\right) =: \ell_3. \end{aligned} \tag{3.20}$$

We denote the Neumann heat semigroup in Ω by $(e^{\Delta t})_{t>0}$. Applying Duhamel's principle to (3.19) and using the semigroup estimates (e.g., see [147, Lemma 1.3]) and (3.20) guarantee that there exist two constants $\sigma_1 > 0$ and $\sigma_2 > 0$ depending only on Ω such that

$$\begin{aligned}
\|u_x(\cdot,t)\|_{L^q} &\le \|\partial_x e^{td_1(\Delta-1)}u_0\|_{L^q} + \int_0^t \|\partial_x e^{(t-s)d_1(\Delta-1)} f(\cdot,s)\|_{L^q} ds \\
&\le \sigma_1\|\partial_x u_0\|_{L^q} + \sigma_2 \int_0^t e^{-(\lambda_1+1)d_1(t-s)} \left(1+(t-s)^{-1+\frac{1}{2q}}\right) \|f(\cdot,s)\|_{L^1} ds \\
&\le \sigma_1\|\partial_x u_0\|_{L^q} + \sigma_2 \ell_3 \int_0^\infty e^{-(\lambda_1+1)d_1 z} \left(1+z^{-1+\frac{1}{2q}}\right) ds \\
&\le \sigma_1\|\partial_x u_0\|_{L^q} + \frac{\sigma_2\ell_3}{(\lambda_1+1)d_1}\left(1+\Gamma(1/2q)\left((\lambda_1+1)d_1\right)^{1-\frac{1}{2q}}\right),
\end{aligned} \tag{3.21}$$

where $\Gamma(\cdot)$ represents the Gamma function defined by $\Gamma(y) := \int_0^\infty t^{-1+y}e^{-t}dt$, and $\lambda_1 > 0$ denotes the first nonzero eigenvalue of $-\Delta$ under Neumann boundary conditions. Then (3.18) follows directly from (3.21) by choosing

$$\begin{aligned}
M_3(q) :=& \frac{\sigma_2 M_0\left(|\Omega|(d_1+1+M_0)+M_1 b_1 + M_2\gamma_1\right)}{(\lambda_1+1)d_1}\left(1+\Gamma(1/2q)\left((\lambda_1+1)d_1\right)^{1-\frac{1}{2q}}\right) \\
&+ \sigma_1\|\partial_x u_0\|_{L^q},
\end{aligned} \tag{3.22}$$

which is independent of t, ξ and χ. Then the proof of Lemma 3.4 is completed. □

The following is an auxiliary result that will be used later.

Lemma 3.5. *[119, Lemma 3.4] Let $T > 0$ and $T_0 \in (0,T)$ and suppose $f(t) : [0,T) \to [0,\infty)$ is an absolutely continuous function and satisfies*

$$f'(t) + \alpha f(t) \le h(t) \text{ for all } t \in (0,T),$$

where constant $\alpha > 0$ and the nonnegative function $h \in L^1_{loc}([0,T))$ fulfilling

$$\int_t^{t+T_0} h(s)ds \le \beta \text{ for all } t \in [0, T-T_0).$$

Then

$$f(t) \le \max\left\{f(0)+\beta, \frac{\beta}{\alpha T_0} + 2\beta\right\} \text{ for all } t \in (0,T).$$

3.2.2 Boundedness of $\|v(\cdot,t)\|_{L^\infty}$

Since the upper bound of $\|v(\cdot,t)\|_{L^\infty}$ plays a vital role in studying the global stability of coexistence steady state, in the following, we shall give the explicit relation between the upper bound of $\|v(\cdot,t)\|_{L^\infty}$ and ξ.

Lemma 3.6. *Let (u, v, w) be the solution of the system* (3.3) *obtained in Lemma 3.1. Then*

$$\int_\Omega v^2(\cdot, t) \le M_4(\xi^6 + \xi^2 + 1), \qquad \text{for all } t \in (0, T_{\max}), \tag{3.23}$$

and

$$\int_t^{t+\tau} \int_\Omega v_x^2(\cdot, s) dx ds \le \frac{2M_4}{d_2}(\xi^6 + \xi^2 + 1), \quad \text{for all } t \in (0, T_{\max} - \tau), \tag{3.24}$$

where $\tau = \min\{1, \frac{T_{\max}}{2}\}$ and $M_4 > 0$ defined in (3.30)*, is independent of χ, ξ and t.*

Proof. Multiplying v-equation in (3.3) by v, and using Young's inequality and $0 < u(\cdot, t) \le M_0$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega v^2 + d_2 \int_\Omega v_x^2 + b_2 \int_\Omega w v^2 + \theta_1 \int_\Omega v^2 &= \xi \int_\Omega v v_x \cdot u_x + \int_\Omega u v^2 \\ &\le \xi \|v\|_{L^\infty} \|v_x\|_{L^2} \|u_x\|_{L^2} + M_0 \|v\|_{L^2}^2.\end{aligned} \tag{3.25}$$

Taking $q = 2$ in (3.18), it follows that

$$\begin{aligned}\|u_x(\cdot t)\|_{L^2} \le & \frac{\sigma_2 M_0 \left(|\Omega|(d_1 + 1 + M_0) + M_1 b_1 + M_2 \gamma_1\right)}{(\lambda_1 + 1) d_1} \left(1 + \Gamma\left(1/4\right)\left((\lambda_1 + 1) d_1\right)^{\frac{3}{4}}\right) \\ & + \sigma_1 \|\partial_x u_0\|_{L^2} \\ =: & \Gamma_1,\end{aligned} \tag{3.26}$$

and then applying Gagliardo-Nirenberg inequality, Young's inequality as well as $\|v(\cdot, t)\|_{L^1} \le M_1$ in (3.13), one derives

$$\begin{aligned}\xi \|v\|_{L^\infty} \|v_x\|_{L^2} \|u_x\|_{L^2} &\le k_1 \xi (\|v_x\|_{L^2}^{\frac{2}{3}} \|v\|_{L^1}^{\frac{1}{3}} + \|v\|_{L^1}) \|v_x\|_{L^2} \|u_x\|_{L^2} \\ &\le k_1 \xi M_1^{\frac{1}{3}} \Gamma_1 \|v_x\|_{L^2}^{\frac{5}{3}} + k_1 \xi M_1 \Gamma_1 \|v_x\|_{L^2} \\ &\le \frac{d_2}{4} \|v_x\|_{L^2}^2 + k_2(\xi^6 + \xi^2),\end{aligned} \tag{3.27}$$

where $k_2 := \left\{\left(\frac{20}{3d_2}\right)^5 \frac{k_1^6}{6} + \frac{2k_1^2}{d_2}\right\} M_1^2 \Gamma_1^2 (1 + \Gamma_1^4)$. Similarly, using Gagliardo-Nirenberg inequality and the fact $\|v(\cdot, t)\|_{L^1} \le M_1$ again, we have

$$\begin{aligned}\left(1/2 + M_0\right) \|v\|_{L^2}^2 &\le k_3 \left(1/2 + M_0\right) \left(\|v_x\|_{L^2}^{\frac{2}{3}} \|v\|_{L^1}^{\frac{4}{3}} + \|v\|_{L^1}^2\right) \\ &\le k_3 \left(1/2 + M_0\right) M_1^{\frac{4}{3}} \|v_x\|_{L^2}^{\frac{2}{3}} + k_3 \left(1/2 + M_0\right) M_1^2\end{aligned} \tag{3.28}$$

$$\leq \frac{d_2}{4}\|v_x\|_{L^2}^2 + k_4,$$

where $k_4 := k_3\big(\frac{1}{2} + M_0\big)M_1^2\big\{1 + \big(\frac{k_3}{3d_2}\big)^{\frac{1}{2}}\frac{4}{3}\left(\frac{1}{2} + M_0\right)^{\frac{1}{2}}\big\}$ is independent of ξ and χ. Substituting (3.27), (3.28) into (3.25) ensures a constant $k_5 := 2(k_2 + k_4)$ such that

$$\frac{d}{dt}\int_\Omega v^2 + \int_\Omega v^2 + d_2\int_\Omega v_x^2 \leq 2k_2(\xi^6 + \xi^2) + 2k_4 \leq k_5(\xi^6 + \xi^2 + 1), \tag{3.29}$$

which along with Grönwall's inequality gives

$$\|v(\cdot,t)\|_{L^2}^2 \leq k_5(\xi^6 + \xi^2 + 1) + \|v_0\|_{L^2}^2,$$

and hence (3.23) follows by taking

$$M_4 := k_5 + \|v_0\|_{L^2} = 2(k_2 + k_4) + \|v_0\|_{L^2}. \tag{3.30}$$

Finally, we integrate (3.29) with respect to t to obtain that for all $t \in (0, T_{\max} - \tau)$,

$$\begin{aligned} d_2\int_t^{t+\tau}\int_\Omega v_x^2(\cdot,s)dxds &\leq k_5(\xi^6 + \xi^2 + 1) + \int_\Omega v^2(\cdot,t) \\ &\leq 2k_5(\xi^6 + \xi^2 + 1) + \|v_0\|_{L^2} \\ &\leq 2M_4(\xi^6 + \xi^2 + 1), \end{aligned}$$

and hence (3.24) follows directly. Then the proof of Lemma 3.6 is completed. □

Lemma 3.7. *Let (u, v, w) be the solution of the system (3.3) obtained in Lemma 3.1. Then there exists a positive constant M_5 defined in (3.36), which is independent of ξ, χ, such that*

$$\|v(\cdot,t)\|_{L^\infty} \leq M_5[1 + \xi(\xi^6 + \xi^2 + 1)^{\frac{1}{2}}], \text{ for all } t \in (0, T_{\max}). \tag{3.31}$$

Proof. We rewrite the second equation of (3.3) as

$$v_t = d_2v_{xx} - d_2v - (\xi vu_x)_x + (d_2 + u)v - (b_2w + \theta_1)v. \tag{3.32}$$

Applying Duhamel's principle to (3.32), one has

$$\begin{aligned} v(\cdot,t) =& e^{td_2(\Delta-1)}v_0 - \xi\int_0^t e^{(t-s)d_2(\Delta-1)}(vu_x)_xds + \int_0^t e^{(t-s)d_2(\Delta-1)}(d_2 + u)vds \\ &- \int_0^t e^{(t-s)d_2(\Delta-1)}(b_2w + \theta_1)vds, \end{aligned}$$

which, combined with the facts $b_2, w, v > 0$ and the semigroup estimates [147, Lemma 1.3], entails us to find two constants $\sigma_3 > 0$ and $\sigma_4 > 0$ depending only on Ω such that

$$\begin{aligned}
\|v(\cdot,t)\|_{L^\infty} \le & \|e^{td_2(\Delta-1)}v_0\|_{L^\infty} + \xi\int_0^t \|e^{(t-s)d_2(\Delta-1)}(vu_x)_x\|_{L^\infty}ds \\
& + \int_0^t \|e^{(t-s)d_2(\Delta-1)}(u+d_1-\theta_1)v\|_{L^\infty}ds \\
\le & \sigma_3\|v_0\|_{L^\infty} + \xi\sigma_4\int_0^t e^{-(\lambda_1+1)d_2(t-s)}(1+(t-s)^{-\frac{5}{6}})\|vu_x\|_{L^{\frac{3}{2}}}ds \\
& + \sigma_3\int_0^t e^{-(\lambda_1+1)d_2(t-s)}(1+(t-s)^{-\frac{1}{2}})\|(u+d_2)v\|_{L^1}ds \\
=: & \sigma_3\|v_0\|_{L^\infty} + J_1 + J_2.
\end{aligned} \tag{3.33}$$

Choosing $q = 6$ in (3.22), we can find a constant $\Gamma_2 > 0$ independent of χ and ξ such that

$$\begin{aligned}
\|u_x(\cdot . t)\|_{L^6} \le & \frac{\sigma_2 M_0\left(|\Omega|(d_1+1+M_0)+M_1b_1+M_2\gamma_1\right)}{(\lambda_1+1)d_1}\left(1+\Gamma\left(1/12\right)\left((\lambda_1+1)d_1\right)^{\frac{11}{12}}\right) \\
& + \sigma_1\|\partial_x u_0\|_{L^6} \\
=: & \Gamma_2,
\end{aligned}$$

which, along with Hölder inequality, and (3.23), indicates

$$\|vu_x\|_{L^{\frac{3}{2}}} \le \|v\|_{L^2}\|u_x\|_{L^6} \le M_4^{\frac{1}{2}}(\xi^6+\xi^2+1)^{\frac{1}{2}}\Gamma_2,$$

and hence

$$\begin{aligned}
J_1 :&= \sigma_4\xi\int_0^t e^{-d_2(\lambda_1+1)(t-s)}(1+(t-s)^{-\frac{5}{6}})\|vu_x\|_{L^{\frac{3}{2}}}ds \\
&\le \sigma_4 M_4^{\frac{1}{2}}\xi(\xi^6+\xi^2+1)^{\frac{1}{2}}\Gamma_2\int_0^t e^{-d_2(\lambda_1+1)(t-s)}(1+(t-s)^{-\frac{5}{6}})ds \\
&\le \sigma_4 M_4^{\frac{1}{2}}\xi(\xi^6+\xi^2+1)^{\frac{1}{2}}\Gamma_2\int_0^\infty e^{-d_2(\lambda_1+1)z}\left(1+z^{-1+\frac{1}{6}}\right)ds \\
&\le k_1\xi(\xi^6+\xi^2+1)^{\frac{1}{2}},
\end{aligned} \tag{3.34}$$

where

$$k_1 := \frac{\sigma_4 M_4^{\frac{1}{2}}\Gamma_2}{d_2(\lambda_1+1)}\left(1+\Gamma\left(1/6\right)d_2^{\frac{5}{6}}(\lambda_1+1)^{\frac{5}{6}}\right)$$

is independent of χ and ξ. Noting the facts $0 < u \le M_0$ and $\|v(\cdot,t)\|_{L^1} \le M_1$, one derives

$$\begin{aligned} J_2 :&= \sigma_3 \int_0^t e^{-d_2(\lambda_1+1)(t-s)}(1+(t-s)^{-\frac{1}{2}})\|(u+d_2)v\|_{L^1} ds \\ &\le \sigma_3 \int_0^t e^{-d_2(\lambda_1+1)(t-s)}(1+(t-s)^{-\frac{1}{2}})\|u+d_2\|_{L^\infty}\|v\|_{L^1} ds \\ &\le \sigma_3(M_0+d_2)M_1 \int_0^t e^{-d_2(\lambda_1+1)(t-s)}(1+(t-s)^{-\frac{1}{2}}) ds \\ &\le k_2, \end{aligned} \tag{3.35}$$

where

$$k_2 := \frac{\sigma_3(M_0+d_2)M_1}{\lambda_1 d_2 + d_2}\left(1+\Gamma(1/2)(\lambda_1 d_2 + d_2)^{\frac{1}{2}}\right).$$

Then substituting (3.34) and (3.35) into (3.33), we have

$$\|v(\cdot,t)\|_{L^\infty} \le \sigma_3\|v_0\|_{L^\infty} + k_1\xi(\xi^6+\xi^2+1)^{\frac{1}{2}} + k_2,$$

which gives (3.31) by choosing

$$\begin{aligned} M_5 := &\frac{\sigma_4 M_4^{\frac{1}{2}}\Gamma_2}{d_2(\lambda_1+1)}\left(1+\Gamma(1/6)\, d_2^{\frac{5}{6}}(\lambda_1+1)^{\frac{5}{6}}\right) \\ &+ \frac{\sigma_3(M_0+d_2)M_1}{\lambda_1 d_2 + d_2}\left(1+\Gamma(1/2)(\lambda_1 d_2+d_2)^{\frac{1}{2}}\right) + \sigma_3\|v_0\|_{L^\infty}. \end{aligned} \tag{3.36}$$

Hence the proof of Lemma 3.7 is finished. □

3.2.3 Boundedness of $\|w(\cdot,t)\|_{L^\infty}$

To establish the boundedness of $\|w(\cdot,t)\|_{L^\infty}$, we first prove the space-time bound for w based on some ideas in [125].

Lemma 3.8. *Let (u,v,w) be the solution of the system (3.3) obtained in Lemma 3.1. Then there exists a constant $M_6 > 0$ such that*

$$\int_t^{t+\tau}\int_\Omega w^2(\cdot,s)dxds \le M_6, \ \textit{for all} \ \ t \in (0, T_{\max}-\tau), \tag{3.37}$$

where $\tau = \min\{1, \frac{T_{\max}}{2}\}$.

Proof. Applying Gagliardo-Nirenberg inequality, Cauchy-Schwarz inequality, and the fact $\|\sqrt{w+1}\|_{L^2}^2 = \int_\Omega (w+1) \leq M_2 + |\Omega|$, we obtain

$$\begin{aligned}
\int_\Omega w^2 \leq &\int_\Omega (w+1)^2 \\
=&\|\sqrt{w+1}\|_{L^4}^4 \\
\leq& k_1 \|\partial_x \sqrt{w+1}\|_{L^1}^2 \|\sqrt{w+1}\|_{L^2}^2 + k_1 \|\sqrt{w+1}\|_{L^2}^4 \\
\leq& \frac{k_1(M_2+|\Omega|)}{4} \left(\int_\Omega \frac{|w_x|}{\sqrt{w+1}} \right)^2 + k_1(M_2+|\Omega|)^2 \\
\leq& \frac{k_1(M_2+|\Omega|)^2}{4} \int_\Omega \frac{w_x^2}{(w+1)^2} + k_1(M_2+|\Omega|)^2.
\end{aligned} \tag{3.38}$$

On the other hand, we use the third equation of (3.3), (3.13) and Young's inequality to derive that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \ln(w+1) =& \int_\Omega \frac{w_t}{w+1} \\
=& \int_\Omega \frac{w_x^2}{(w+1)^2} - \chi \int_\Omega \frac{w\phi(u,v)_x \cdot w_x}{(w+1)^2} + \int_\Omega \frac{(v+\gamma_2 u)w}{w+1} - \theta_2 \int_\Omega \frac{w}{w+1} \\
\geq& \int_\Omega \frac{w_x^2}{(w+1)^2} - \chi \int_\Omega \frac{w\phi(u,v)_x \cdot w_x}{(w+1)^2} - \theta_2|\Omega| \\
\geq& \frac{1}{2}\int_\Omega \frac{w_x^2}{(w+1)^2} - \frac{\chi^2}{2}\int_\Omega \frac{w^2|\phi(u,v)_x|^2}{(w+1)^2} - \theta_2|\Omega|.
\end{aligned}$$

Noting the facts $0 \leq \ln(w+1) \leq w$ and $\frac{w^2}{(w+1)^2} \leq 1$ for all $w \geq 0$ and integrating (3.39) from t to $(t+\tau)$, one has

$$\begin{aligned}
\int_t^{t+\tau}\int_\Omega \frac{w_x^2}{(w+1)^2} \leq& 2\theta_2|\Omega| + \chi^2 \int_t^{t+\tau}\int_\Omega \frac{w^2|\phi(u,v)_x|^2}{(w+1)^2} + 2\int_\Omega \ln(w+1)(\cdot, t+\tau) \\
\leq& 2\theta_2|\Omega| + 2M_2 + \chi^2 \int_t^{t+\tau}\int_\Omega |\phi_u u_x + \phi_v v_x|^2.
\end{aligned} \tag{3.39}$$

Furthermore, by (H0) and the L^∞-boundedness of u, v (see (3.11) and (3.31)), there exists a constant $\gamma > 0$ independent of t such that

$$|\phi_u| + |\phi_v| \leq \gamma \quad \text{for all } t \in (0, T_{\max}), \tag{3.40}$$

and then using (3.24) and (3.26), one derives

$$\begin{aligned}
\chi^2 \int_t^{t+\tau}\int_\Omega |\phi_u u_x + \phi_v v_x|^2 \leq& 2\chi^2\gamma^2 \int_t^{t+\tau}\int_\Omega (u_x^2 + v_x^2) \\
\leq& 2\chi^2\gamma^2 \left(\Gamma_1^2 + \frac{2M_4}{d_2}(\xi^6 + \xi^2 + 1) \right).
\end{aligned} \tag{3.41}$$

We substitute (3.41) into (3.39) to obtain that for all $t \in (0, T_{\max} - \tau)$

$$\int_t^{t+\tau} \int_\Omega \frac{w_x^2}{(w+1)^2} \leq 2\theta_2|\Omega| + 2M_2 + 2\chi^2\gamma^2 \left(\Gamma_1^2 + \frac{2M_4}{d_2}(\xi^6 + \xi^2 + 1)\right). \tag{3.42}$$

Hence, integrating (3.38) from t to $(t+\tau)$ and applying (3.42), we get (3.37) directly. Then the proof of Lemma 3.8 is finished. □

Lemma 3.9. *Let (u, v, w) be the solution to the system (3.3) obtained in Lemma 3.1. Then there exists a positive constant M_7 such that*

$$\int_t^{t+\tau} \int_\Omega u_{xx}^2(\cdot, s) dx ds \leq M_7, \text{ for all } t \in (0, T_{\max} - \tau), \tag{3.43}$$

where $\tau := \min\{1, \frac{1}{2}T_{\max}\}$.

Proof. We multiply the first equation by $-u_{xx}$, and use Young's inequality and (3.26) to derive

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega u_x^2 + d_1\int_\Omega u_{xx}^2 + 2\int_\Omega uu_x^2 &= \int_\Omega u_x^2 + b_1\int_\Omega uvu_{xx} + \gamma_1\int_\Omega uwu_{xx} \\
&\leq \int_\Omega u_x^2 + \frac{d_1}{2}\int_\Omega u_{xx}^2 + \frac{b_1^2}{d_1}\int_\Omega u^2v^2 + \frac{\gamma_1^2}{d_1}\int_\Omega u^2w^2 \\
&\leq \frac{d_1}{2}\int_\Omega u_{xx}^2 + \frac{\gamma_1^2M_0^2}{d_1}\int_\Omega w^2 + \frac{b_1^2M_0^2M_4(\xi^6+\xi^2+1)}{d_1} + \Gamma_1^2,
\end{aligned}$$

which gives

$$\frac{d}{dt}\int_\Omega u_x^2 + d_1\int_\Omega u_{xx}^2 \leq \frac{2\gamma_1^2M_0^2}{d_1}\int_\Omega w^2 + \frac{2b_1^2M_0^2M_4(\xi^6+\xi^2+1)}{d_1} + 2\Gamma_1^2. \tag{3.44}$$

Then integrating (3.44) with respect to t, and using (3.37) and (3.26) imply that for all $t \in (0, T_{\max} - \tau)$,

$$\begin{aligned}
\int_t^{t+\tau}\int_\Omega u_{xx}^2(\cdot, s)dxds &\leq \frac{2\gamma_1^2M_0^2}{d_1^2}\int_t^{t+\tau}\int_\Omega w^2 + \frac{1}{d_1}\int_\Omega u_x^2(\cdot, t) + \frac{2b_1^2M_0^2M_4(\xi^6+1)}{d_1^2} + \frac{2\Gamma_1^2}{d_1} \\
&\leq \frac{2\gamma_1^2M_0^2M_6}{d_1^2} + \frac{3\Gamma_1^2}{d_1} + \frac{2b_1^2M_0^2M_4(\xi^6+\xi^2+1)}{d_1^2} \\
&=: M_7,
\end{aligned}$$

which entails (3.43) immediately. Then the proof of Lemma 3.9 is completed. □

Lemma 3.10. *Let (u, v, w) be the solution of the system (3.3) obtained in Lemma 3.1. Then there exists a positive constant M_8 such that for all $t \in (0, T_{\max})$,*

$$\int_\Omega v_x^2(\cdot, t) \le M_8, \ \textit{for all} \ \ t \in (0, T_{\max} - \tau). \tag{3.45}$$

Proof. Multiplying the second equation of (3.3) by $-v_{xx}$, integrating the result over Ω, and using Hölder inequality and $\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{L^2} + \|v(\cdot,t)\|_{L^\infty} \le k_1$, one obtains

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega v_x^2 + 2d_2 \int_\Omega v_{xx}^2 \\
&= 2\xi \int_\Omega v u_{xx} v_{xx} + 2\xi \int_\Omega v_x u_x v_{xx} + 2\theta_1 \int_\Omega v v_{xx} + 2b_2 \int_\Omega v w v_{xx} - 2\int_\Omega u v v_{xx} \\
&\le 2\xi k_1 \|u_{xx}\|_{L^2} \|v_{xx}\|_{L^2} + 2\xi \|v_x u_x\|_{L^2} \|v_{xx}\|_{L^2} \\
&\quad + 2k_1(\theta_1 + k_1)\|v_{xx}\|_{L^2} |\Omega|^{\frac{1}{2}} + 2b_2 k_1 \|w\|_{L^2} \|v_{xx}\|_{L^2} \\
&\le d_2 \|v_{xx}\|_{L^2}^2 + \frac{4\xi^2 k_1^2}{d_2}\|u_{xx}\|_{L^2}^2 + \frac{4\xi^2}{d_2}\|v_x u_x\|_{L^2}^2 + \frac{4b_2^2 k_1^2}{d_2}\|w\|_{L^2}^2 \\
&\quad + \frac{4k_1^2(\theta_1 + k_1)^2 |\Omega|}{d_2},
\end{aligned}$$

which yields

$$\frac{d}{dt}\int_\Omega v_x^2 + d_2 \int_\Omega v_{xx}^2 \le \frac{4\xi^2 k_1^2}{d_2}\|u_{xx}\|_{L^2}^2 + \frac{4\xi^2}{d_2}\|v_x u_x\|_{L^2}^2 + \frac{4b_2^2 k_1^2}{d_2}\|w\|_{L^2}^2 + \frac{4k_1^2(\theta_1 + k_1)^2 |\Omega|}{d_2}. \tag{3.46}$$

Furthermore, choosing $q = 4$ in Lemma 3.4, and using Hölder inequality and Gagliardo-Nirenberg inequality, we derive

$$\frac{4\xi^2}{d_2}\|v_x u_x\|_{L^2}^2 \le \frac{4\xi^2}{d_2}\|u_x\|_{L^4}^2 \|v_x\|_{L^4}^2 \le k_2 \|v_{xx}\|_{L^2}\|v\|_{L^\infty} + k_2 \|v\|_{L^\infty}^2 \le \frac{d_2}{2}\|v_{xx}\|_{L^2}^2 + k_3, \tag{3.47}$$

and

$$\int_\Omega v_x^2 = \|v_x\|_{L^2}^2 \le k_4 \left(\|v_{xx}\|_{L^2} \|v\|_{L^2} + \|v\|_{L^2}^2 \right) \le \frac{d_2}{2}\|v_{xx}\|_{L^2}^2 + k_5. \tag{3.48}$$

Substituting (3.47) and (3.48) into (3.46), one has

$$\frac{d}{dt}\int_\Omega v_x^2 + \int_\Omega v_x^2 \le \frac{4\xi^2 k_1^2}{d_2}\|u_{xx}\|_{L^2}^2 + \frac{4b_2^2 k_1^2}{d_2}\|w\|_{L^2}^2 + k_6, \tag{3.49}$$

with $k_6 = k_3 + k_5 + \frac{4k_1^2(\theta_1 + k_1)^2|\Omega|}{d_2}$. Letting

$$h(t) := \frac{4\xi^2 k_1^2}{d_2}\|u_{xx}\|_{L^2}^2 + \frac{4b_2^2 k_1^2}{d_2}\|w\|_{L^2}^2 + k_6,$$

then using Lemma 3.9 and Lemma 3.8, we have

$$\int_t^{t+\tau} h(s)ds = \frac{4\xi^2 k_1^2}{d_2}\int_t^{t+\tau}\int_\Omega u_{xx}^2(\cdot,s)dxds + \frac{4b_2^2k_1^2}{d_2}\int_t^{t+\tau}\int_\Omega w^2(\cdot,s)dxds + k_6\tau \le k_7. \tag{3.50}$$

Applying Lemma 3.5 to (3.49) and using (3.50), one gets (3.45). Then, we complete the proof of Lemma 3.10. □

Lemma 3.11. *Let (u, v, w) be the solution of the system (3.3) obtained in Lemma 3.1. Then it holds that*

$$\|w(\cdot,t)\|_{L^4} \le M_9, \qquad \textit{for all } t \in (0, T_{\max}), \tag{3.51}$$

where $M_9 > 0$ is a constant independent of t.

Proof. We multiply the third equation of (3.3) by w^3, integrate the results over Ω and use Young's inequality with the boundedness of $\|u(\cdot,t)\|_{L^\infty}$ and $\|v(\cdot,t)\|_{L^\infty}$ to derive

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}\int_\Omega w^4 + 3\int_\Omega w^2w_x^2 &= 3\chi\int_\Omega w^3(\phi_u u_x\cdot w_x + \phi_v v_x \cdot w_x) + \int_\Omega w^4(v+\gamma_2 u - \theta_2)\\ &\le 3\chi\int_\Omega w^3(|\phi_u||u_x| + |\phi_v||v_x|)|w_x| + k_1\int_\Omega w^4\\ &\le \frac{3}{2}\int_\Omega w^2w_x^2 + \frac{3\chi^2}{2}\int_\Omega w^4(|\phi_u||u_x|+|\phi_v||v_x|)^2 + k_1\int_\Omega w^4,\end{aligned}$$

which, together with the basic inequality $(y+z)^2 \le 2(y^2+z^2)$ and the fact $\frac{1}{4}\int_\Omega |(w^2)_x|^2 = \int_\Omega w^2w_x^2$, gives

$$\begin{aligned}&\frac{d}{dt}\int_\Omega w^4 + \int_\Omega w^4 + \frac{3}{2}\int_\Omega |(w^2)_x|^2\\ &\le 12\chi^2\int_\Omega w^4(\phi_u^2u_x^2 + \phi_v^2v_x^2) + (4k_1+1)\int_\Omega w^4\\ &\le \|w\|_{L^\infty}^4\left(12\chi^2\|\phi_u\|_{L^\infty}^2\|u_x\|_{L^2}^2 + \|\phi_v\|_{L^\infty}^2\|v_x\|_{L^2}^2 + (4k_1+1)|\Omega|\right)\\ &\le k_2\|w\|_{L^\infty}^4,\end{aligned} \tag{3.52}$$

where we have used Hölder inequality, (3.40) and (3.45) as well as (3.26). By Gagliardo-Nirenberg inequality, Young's inequality and (3.13), one has

$$\begin{aligned}k_2\|w\|_{L^\infty}^4 = k_2\|w^2\|_{L^\infty}^2 &\le k_3\|(w^2)_x\|_{L^2}^{\frac{8}{5}}\|w^2\|_{L^{\frac{1}{2}}}^{\frac{2}{5}} + k_3\|w^2\|_{L^{\frac{1}{2}}}^2\\ &= k_3\|(w^2)_x\|_{L^2}^{\frac{8}{5}}\|w\|_{L^1}^{\frac{4}{5}} + k_3\|w\|_{L^1}^4\\ &\le \frac{3}{2}\|(w^2)_x\|_{L^2}^2 + k_4,\end{aligned}$$

which, substituted into (3.52), gives

$$\frac{d}{dt}\int_\Omega w^4 + \int_\Omega w^4 \le k_4,$$

and then (3.51) follows by Grönwall's inequality. Hence, the proof of Lemma 3.11 is completed. □

Lemma 3.12. *Let (u, v, w) be the solution of the system (3.3) obtained in Lemma 3.1. Then there exists a constant $M_{10} > 0$ independent of t such that*

$$\|w(\cdot,t)\|_{L^\infty} \le M_{10}, \quad \text{for all } t \in (0, T_{\max}). \tag{3.53}$$

Proof. Applying Duhamel's principle to the third equation of (3.3), and using the well-known semigroup estimates, we have

$$\begin{aligned}\|w(\cdot,t)\|_{L^\infty} \le& k_1\|w_0\|_{L^\infty} + k_2\int_0^t e^{-(\lambda_1+1)(t-s)}(1+(t-s)^{-\frac{7}{8}})\|\phi(u,v)_x w\|_{L^{\frac{4}{3}}}ds \\ &+ k_3\int_0^t e^{-(\lambda_1+1)(t-s)}(1+(t-s)^{-\frac{1}{4}})\|(v+\gamma_2 u + 1 - \theta_2)w\|_{L^2}ds \\ \le& k_1\|w_0\|_{L^\infty} + I_1 + I_2.\end{aligned} \tag{3.54}$$

Noting the facts $\|w(\cdot,t)\|_{L^4} \le M_9$, (3.40), (3.45) and (3.26), and using Hölder inequality, one has

$$\begin{aligned}\|\phi(u,v)_x w\|_{L^{\frac{4}{3}}} =& \|(\phi_u u_x + \phi_v v_x)w\|_{L^{\frac{4}{3}}} \\ \le& \|\phi_u u_x + \phi_v v_x\|_{L^2}\|w\|_{L^4} \\ \le& \frac{M_9^2}{2} + \|\phi_u\|_{L^\infty}^2\|u_x\|_{L^2}^2 + \|\phi_v\|_{L^\infty}^2\|v_x\|_{L^2}^2 \\ \le& \frac{M_9^2}{2} + \gamma^2(M_8 + \Gamma_1^2) =: k_4,\end{aligned}$$

and hence

$$I_1 \le k_2 k_4\int_0^t e^{-(\lambda_1+1)(t-s)}(1+(t-s)^{-\frac{7}{8}})ds \le k_5. \tag{3.55}$$

On the other hand, using Hölder inequality and the boundedness of u, v and $\|w\|_{L^4}$, we can find a constant $k_6 > 0$ such that

$$\|(v+\gamma_2 u - \theta_2 + 1)w\|_{L^2} \le \|v + \gamma_2 u - \theta_2 + 1\|_{L^4}\|w\|_{L^4} \le k_6,$$

and hence

$$I_2 \leq k_3 k_6 \int_0^t e^{-(\lambda_1+1)(t-s)}(1+(t-s)^{-\frac{1}{4}})ds \leq k_7. \tag{3.56}$$

Substituting (3.55) and (3.56) into (3.54) gives (3.53), and hence the proof of Lemma 3.12 is completed. □

Proof of Theorem 3.1. Noting (3.11) and (3.26), we derive

$$\|u(\cdot,t)\|_{W^{1,2}} \leq k_1, \quad \text{for all } t \in (0, T_{\max}). \tag{3.57}$$

And the combination of (3.31) and (3.45) gives

$$\|v(\cdot,t)\|_{W^{1,2}} \leq k_2, \quad \text{for all } t \in (0, T_{\max}). \tag{3.58}$$

Then combining (3.57), (3.58) and (3.53), and using Lemma 3.1, we directly prove Theorem 3.1. □

3.3 Global Stability: Proof of Theorem 3.2

In this section, we use Lyapunov functionals and LaSalle's invariant principle to establish global stability of constant steady states for the system (3.3).

3.3.1 Case of Prey-only

In this subsection, we shall study the global stability of $(1,0,0)$ (i.e., prey-only steady state) provided $\theta_1 > 1$ and $\theta_2 > \gamma_2$. To this end, we introduce the energy functional as below:

$$\mathcal{F}_1(t) := \mathcal{F}_1(u,v,w) = \alpha_1 \int_\Omega (u - 1 - \ln u) + b_1 \int_\Omega v + b_1 b_2 \int_\Omega w,$$

where

$$\alpha_1 := \begin{cases} 1, & \text{if } \gamma_1 = \gamma_2 = 0, \\ \min\left\{\frac{\theta_1 - 1}{4}, \frac{b_1 b_2(\theta_2-\gamma_2)}{4\gamma_1}\right\}, & \text{if } \gamma_1, \gamma_2 > 0. \end{cases}$$

Proof Theorem 3.2(1). By same way as proof in Theorem 2.2 (1), we derive that $\mathcal{F}_1(t) \geq 0$ and $\mathcal{F}_1(t) = 0$ if and only if $(u,v,w) = (1,0,0)$. Moreover, some calculations give

$$\begin{aligned} \frac{d}{dt}\mathcal{F}_1(t) = & -\alpha_1 d_1 \int_\Omega \frac{u_x^2}{u^2} - \alpha_1 \int_\Omega (u-1)^2 - \alpha_1 b_1 \int_\Omega uv - \alpha_1 \gamma_1 \int_\Omega uw \\ & + b_1 \int_\Omega (u - \theta_1 + \alpha_1)v + \int_\Omega (b_1 b_2 \gamma_2 u - b_1 b_2 \theta_2 + \alpha_1 \gamma_1) w. \end{aligned} \tag{3.59}$$

Case 1: $\gamma_1 = \gamma_2 = 0$. In this case, substituting $\alpha_1 = 1$ and $\gamma_1 = \gamma_2 = 0$ into (3.59), and using $\theta_1 > 1$, one has

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_1(t) &= -d_1\int_\Omega \frac{u_x^2}{u^2} - \int_\Omega (u-1)^2 - b_1(\theta_1 - 1)\int_\Omega v - b_1 b_2\theta_2 \int_\Omega w \\ &\leq -\int_\Omega (u-1)^2 - b_1(\theta_1-1)\int_\Omega v - b_1b_2\theta_2\int_\Omega w \leq 0.\end{aligned} \tag{3.60}$$

Case 2: $\gamma_1, \gamma_2 > 0$. Noting the facts $\limsup_{t\to\infty} u(x,t) \leq 1$ in (3.12) and $\theta_1 > 1$ as well as $\theta_2 > \gamma_2$, for $\varepsilon_1 := \min\left\{\frac{\theta_1 - 1}{2}, \frac{\theta_2-\gamma_2}{2\gamma_2}\right\}$, we can find a $t_1 > 0$ such that

$$u(x,t) \leq 1 + \varepsilon_1 \text{ for any } x \in \bar{\Omega} \text{ and } t > t_1,$$

which, together with $\alpha_1 := \min\left\{\frac{\theta_1-1}{4}, \frac{b_1b_2(\theta_2-\gamma_2)}{4\gamma_1}\right\}$, entails

$$\begin{aligned} u - \theta_1 + \alpha_1 &\leq 1 + \varepsilon_1 + \alpha_1 - \theta_1 \\ &\leq 1 + \frac{\theta_1-1}{2} - \theta_1 + \frac{\theta_1 - 1}{4} \\ &= -\frac{\theta_1 - 1}{4} < 0 \text{ for all } t > t_1, \end{aligned} \tag{3.61}$$

and

$$\begin{aligned} b_1b_2\gamma_2 u - b_1b_2\theta_2 + \alpha_1\gamma_1 &\leq b_1b_2\gamma_2(1+\varepsilon_1) - b_1b_2\theta_2 + \alpha_1\gamma_1 \\ &\leq b_1b_2(\gamma_2 - \theta_2) + b_1b_2\gamma_2\frac{\theta_2-\gamma_2}{2\gamma_2} + \frac{b_1b_2(\theta_2-\gamma_2)}{4\gamma_1}\gamma_1 \\ &= -\frac{b_1b_2(\theta_2-\gamma_2)}{4} < 0 \text{ for all } t > t_1. \end{aligned} \tag{3.62}$$

The combination of (3.59), (3.61) and (3.62) gives that for all $t > t_1$

$$\frac{d}{dt}\mathcal{F}_1(t) \leq -\alpha_1\int_\Omega (u-1)^2 - \frac{b_1(\theta_1-1)}{4}\int_\Omega v - \frac{b_1b_2(\theta_2-\gamma_2)}{4}\int_\Omega w \leq 0. \tag{3.63}$$

Furthermore, all the above cases indicates that $\frac{d}{dt}\mathcal{F}_1(t) = 0$ iff $(u,v,w) = (1,0,0)$. Hence, by LaSalle's invariance principle (e.g. see [115, pp.198-199, Theorem 5.24]), we know that (u,v,w) converges to $(1,0,0)$ in L^∞ as $t \to \infty$. □

3.3.2 Case of Semi-coexistence

In this subsection, we first study the global stability of semi-coexistence $E_{12} := \left(\theta_1, \frac{1-\theta_1}{b_1}, 0\right)$. Denote $V := \frac{1-\theta_1}{b_1}$, we introduce the following energy functional:

$$\mathcal{F}_2(t) := \mathcal{F}_2(u,v,w) = \int_\Omega \left(u - \theta_1 - \theta_1 \ln\frac{u}{\theta_1}\right) + b_1\int_\Omega\left(v - V - V\ln\frac{v}{V}\right) + b_1b_2\int_\Omega w.$$

Proof of Theorem 3.2(2). Following same way as the proof in Theorem 2.2 (1), we can check that $\mathcal{F}_2(t) \geq 0$ and $\mathcal{F}_2(t) = 0$ iff $(u, v, w) = \big(\theta_1, \frac{1-\theta_1}{b_1}, 0\big)$. Applying the equations of (3.3) and using the fact $1 = \theta_1 + b_1 V$, one has

$$\frac{d}{dt}\mathcal{F}_2(t) = -\int_\Omega Y_1^T B_1 Y_1 + \int_\Omega h_1(x,t)w - \int_\Omega (u-\theta_1)^2, \tag{3.64}$$

where

$$Y_1 = \begin{pmatrix} \frac{u_x}{v} \\ \frac{v_x}{v} \end{pmatrix}, \quad B_1 := \begin{pmatrix} \theta_1 d_1 & -\frac{b_1 V \xi u}{2} \\ -\frac{b_1 V \xi u}{2} & b_1 d_2 V \end{pmatrix}$$

and

$$h_1(x,t) := (b_1 b_2 \gamma_2 - \gamma_1)u + b_1 b_2 V + \gamma_1 \theta_1 - b_1 b_2 \theta_2.$$

After some calculations, one can check that B_1 is a positive definite matrix provided that

$$\xi^2 (1-\theta_1)\|u\|^2_{L^\infty} < 4\theta_1 d_1 d_2. \tag{3.65}$$

Since $0 < \theta_1 < 1$ and $\|u\|_{L^\infty}$ is independent of ξ, we can find an appropriate constant $\xi_0 > 0$ such that if $0 < \xi < \xi_0$, then (3.65) holds, which entails us to find a constant $k_1 > 0$ such that

$$-\int_\Omega Y_1^T B_1 Y_1 \leq -k_1 \int_\Omega \left(\frac{u_x^2}{u^2} + \frac{v_x^2}{v^2}\right). \tag{3.66}$$

Next, we shall show that under condition $\theta_2 > \ell_1$ with ℓ_1 defined in (3.9), there exists a constant $k_2 > 0$ such that

$$\int_\Omega h_1(x,t)w \leq -\frac{b_1 b_2 k_2}{2}\int_\Omega w. \tag{3.67}$$

We divide our proof into two cases: $b_1 b_2 \gamma_2 \leq \gamma_1$ and $b_1 b_2 \gamma_2 > \gamma_1$.

Case 1: $b_1 b_2 \gamma_2 \leq \gamma_1$**.** In this case, from $\theta_2 > \ell_1$, one has $\theta_2 > \frac{\gamma_1}{b_1 b_2}\theta_1 - \frac{\theta_1}{b_1} + \frac{1}{b_1}$, which indicates

$$h_1(x,t) \leq b_1 b_2 \frac{1-\theta_1}{b_1} + \gamma_1\theta_1 - b_1 b_2 \theta_2 = -b_1 b_2\left(\theta_2 - \frac{\gamma_1}{b_1 b_2}\theta_1 + \frac{\theta_1}{b_1} - \frac{1}{b_1}\right) < 0. \tag{3.68}$$

Case 2: $b_1 b_2 \gamma_2 > \gamma_1$**.** For this case, $\theta_2 > \ell_1$ and the fact $\limsup_{t\to\infty} u(x,t) \leq 1$ in (3.12) can guarantee that for the positive constant $\varepsilon_2 := \frac{b_1 b_2}{2(b_1 b_2 \gamma_2 - \gamma_1)}(\theta_2 - \ell_1)$, there exists a constant $t_2 > 0$ such that $u(x,t) \leq 1 + \varepsilon_2$ for any $x \in \bar{\Omega}$ and $t > t_2$, and hence

$$\begin{aligned} h_1(x,t) &\leq (b_1 b_2 \gamma_2 - \gamma_1) + \frac{b_1 b_2}{2}(\theta_2 - \ell_1) + b_1 b_2 \frac{1-\theta_1}{b_1} + \gamma_1\theta_1 - b_1 b_2 \theta_2 \\ &= -\frac{b_1 b_2}{2}(\theta_2 - \ell_1) < 0. \end{aligned} \tag{3.69}$$

Combining (3.68) with (3.69) and letting

$$k_2 = \theta_2 - \frac{\gamma_1}{b_1 b_2}\theta_1 + \frac{\theta_1}{b_1} - \frac{1}{b_1} - \frac{\max\{b_1 b_2 \gamma_2 - \gamma_1, 0\}}{b_1 b_2},$$

we directly obtain (3.67). Then substituting (3.66) and (3.67) into (3.64), one has

$$\frac{d}{dt}\mathcal{F}_2(t) \le -k_1 \int_\Omega \left(\frac{u_x^2}{u^2} + \frac{v_x^2}{v^2}\right) - \int_\Omega (u-\theta_1)^2 - \frac{b_1 b_2 k_2}{2}\int_\Omega w \le 0,$$

and "=" holds iff $(u, v_x, w) = (\theta_1, 0, 0)$. Furthermore, the fact $v_x = 0$ entails $v = \tilde{v}$, where $\tilde{v}$ is a positive constant. Hence, $(u, v, w) = (\theta_1, \tilde{v}, 0)$ satisfies $0 = \theta_1(1 - \theta_1 - b_1\tilde{v})$, which yields $\tilde{v} = \frac{1-\theta_1}{b_1} = V$. Then $\frac{d}{dt}\mathcal{F}_2(t) = 0$ implies $(u, v, w) = \left(\theta_1, \frac{1-\theta_1}{b_1}, 0\right)$.

Applying LaSalle's invariance principle, one obtains that the semi-coexistence $\left(\theta_1, \frac{1-\theta_1}{b_1}, 0\right)$ is globally asymptotically stable, which proves Theorem 3.2 (2). □

Next, we shall study the global stability of $\left(\frac{\theta_2}{\gamma_2}, 0, \frac{\gamma_2-\theta_2}{\gamma_1\gamma_2}\right)$. Denote $W := \frac{\gamma_2-\theta_2}{\gamma_1\gamma_2}$, then we introduce the following energy functional:

$$\begin{aligned}\mathcal{F}_3(t) := \mathcal{F}_3(u, v, w) =& \frac{b_1 b_2 \gamma_2}{\gamma_1}\int_\Omega \left(u - \frac{\theta_2}{\gamma_2} - \frac{\theta_2}{\gamma_2}\ln\frac{u\gamma_2}{\theta_2}\right) + b_1 b_2 \int_\Omega \left(w - W - W\ln\frac{w}{W}\right)\\ &+ b_1\int_\Omega v + \int_\Omega v^2.\end{aligned} \tag{3.70}$$

Proof of Theorem 3.2(3). We follow the same way as the proof in Theorem 2.2 (1) to get that $\mathcal{F}_3(t) \ge 0$ and "=" holds iff $(u, v, w) = \left(\frac{\theta_2}{\gamma_2}, 0, \frac{\gamma_2-\theta_2}{\gamma_1\gamma_2}\right)$. Moreover, by the definition of $\mathcal{F}_3(t)$ in (3.70), we utilize the equations of (3.3) and the fact $1 = \frac{\theta_2}{\gamma_2} + \gamma_1 W$ to derive

$$\frac{d}{dt}\mathcal{F}_3(t) = -\int_\Omega Y_2^T B_2 Y_2 - \frac{b_1 b_2 \gamma_2}{\gamma_1}\int_\Omega \left(u - \frac{\theta_2}{\gamma_2}\right)^2 + b_1 \int_\Omega v h_2(x,t) + 2\int_\Omega v^2 h_3(x,t), \tag{3.71}$$

where

$$Y_2 = \begin{pmatrix} \frac{u_x}{u} \\ \frac{v_x}{v} \\ \frac{w_x}{w} \end{pmatrix}, \quad B_2 := \begin{pmatrix} \frac{b_1 b_2 d_1 \theta_2}{\gamma_1} & -\xi u v^2 & -\frac{b_1 b_2 \chi W u \phi_u}{2} \\ -\xi u v^2 & 2 d_2 v^2 & -\frac{b_1 b_2 \chi W v \phi_v}{2} \\ -\frac{b_1 b_2 \chi W u \phi_u}{2} & -\frac{b_1 b_2 \chi W v \phi_v}{2} & b_1 b_2 W, \end{pmatrix}$$

and

$$h_2(x,t) := \left(1 - \frac{b_1 b_2 \gamma_2}{\gamma_1}\right)u - \theta_1 - b_2 W + \frac{b_1 b_2 \theta_2}{\gamma_1}, \qquad h_3(x,t) := u - b_2 w - \theta_1. \tag{3.72}$$

After some calculations, we can check that B_2 is positive definite if

$$\begin{vmatrix} \frac{b_1 b_2 d_1 \theta_2}{\gamma_1} & -\xi u v^2 \\ -\xi u v^2 & 2d_2 v^2 \end{vmatrix} = \left(\frac{2b_1 b_2 d_1 d_2 \theta_2}{\gamma_1} - \xi^2 u^2 v^2 \right) v^2 > 0, \tag{3.73}$$

and

$$\begin{aligned} |B_2| =& b_1 b_2 W v^2 \left(\frac{2b_1 b_2 d_1 d_2 \theta_2}{\gamma_1} - \xi^2 u^2 v^2 \right) \\ & - \frac{b_1^2 b_2^2 \chi^2 W^2 v^2}{4} \left(2uv\phi_u \phi_v \xi u + 2u^2 \phi_u^2 d_2 + \phi_v^2 \frac{d_1 \theta_2 b_1 b_2}{\gamma_1} \right) > 0. \end{aligned} \tag{3.74}$$

Indeed, it can be verified that (3.73) and (3.74) hold if

$$2b_1 b_2 d_1 d_2 \theta_2 > \xi^2 \gamma_1 M_0^2 K_0^2 + \chi^2 M_*^c, \tag{3.75}$$

where M_0 and K_0 are defined in (3.4) and (3.5), respectively, and

$$M_*^c := \frac{b_1 b_2 (\gamma_2 - \theta_2)}{4\gamma_2} \left(2\xi M_0^2 K_0 \|\phi_v\|_{L^\infty} \|\phi_u\|_{L^\infty} + 2d_2 M_0^2 \|\phi_u\|_{L^\infty}^2 + \frac{d_1 \theta_2 b_1 b_2}{\gamma_1} \|\phi_v\|_{L^\infty}^2 \right).$$

Since $M_0 \geq \|u\|_{L^\infty}$ is independent of ξ, χ and $K_0 \geq \|v\|_{L^\infty}$ is independent of χ, moreover, for any given $\phi(u, v) \in C^2([0, \infty))$, we can obtain the upper bounds of $\|\phi_u\|_{L^\infty}$ and $\|\phi_v\|_{L^\infty}$ are independent of χ. Then there exist $\xi_1 > 0$ and $\chi_1 > 0$ such that (3.75) holds if $\xi \in (0, \xi_1)$ and $\chi \in (0, \chi_1)$. Hence, we can find a constant $k_1 > 0$ such that

$$-\int_\Omega Y_2^T B_2 Y_2 \leq -k_1 \int_\Omega \left(\frac{u_x^2}{u^2} + \frac{v_x^2}{v^2} + \frac{w_x^2}{w^2} \right). \tag{3.76}$$

Next, we shall show $h_3(x, t) < 0$ and $h_2(x, t) < 0$, respectively. Noting $\theta_1 > 1$ and $\theta_2 < \ell_2$ with ℓ_2 defined in (3.10), we can take

$$\varepsilon_3 := \begin{cases} \frac{\theta_1 - 1}{2}, & \text{if } \gamma_1 \leq b_1 b_2 \gamma_2, \\ \min\left\{ \frac{\theta_1 - 1}{2}, \frac{(\ell_2 - \theta_2)(b_1 b_2 \gamma_2 + b_2)}{2(\gamma_1 - b_1 b_2 \gamma_2)\gamma_2} \right\}, & \text{if } \gamma_1 > b_1 b_2 \gamma_2. \end{cases}$$

From (3.12), we can find a constant $t_3 > 0$ such that

$$u(x, t) \leq 1 + \varepsilon_3 \text{ for all } x \in \bar{\Omega} \text{ and } t > t_3, \tag{3.77}$$

and hence

$$h_3(x, t) := u - b_2 w - \theta_1 \leq 1 + \varepsilon_3 - \theta_1 \leq \frac{\theta_1 - 1}{2} + 1 - \theta_1 = -\frac{\theta_1 - 1}{2} < 0. \tag{3.78}$$

As for h_2, we need to distinguish in two cases:

Case 1: $\gamma_1 \le b_1 b_2 \gamma_2$. This case means $1 - \frac{b_1 b_2 \gamma_2}{\gamma_1} \le 0$, thus it follows from $\theta_2 < \ell_2$ and (3.72) that

$$h_2(x,t) \le -\theta_1 - \frac{b_2\gamma_2 - b_2\theta_2}{\gamma_1\gamma_2} + \frac{b_1 b_2 \theta_2}{\gamma_1} = -\frac{b_1 b_2 \gamma_2 + b_2}{\gamma_1\gamma_2}(\ell_2 - \theta_2) < 0. \tag{3.79}$$

Case 2: $\gamma_1 > b_1 b_2 \gamma_2$. In this case, we have $1 - \frac{b_1 b_2 \gamma_2}{\gamma_1} > 0$, which along with (3.72), (3.77) and $\theta_2 < \ell_2$ gives

$$\begin{aligned} h_2(x,t) &\le (1 - \frac{b_1 b_2 \gamma_2}{\gamma_1}) + (1 - \frac{b_1 b_2 \gamma_2}{\gamma_1})\frac{(\ell_2 - \theta_2)(b_1 b_2 \gamma_2 + b_2)}{2(\gamma_1 - b_1 b_2 \gamma_2)\gamma_2} - \theta_1 - b_2 W + \frac{b_1 b_2 \theta_2}{\gamma_1} \\ &= \frac{(b_1 b_2 \gamma_2 + b_2)(\ell_2 - \theta_2)}{2\gamma_1\gamma_2} + \frac{\gamma_1 - b_1 b_2 \gamma_2}{\gamma_1} - \theta_1 - \frac{b_2\gamma_2 - b_2\theta_2}{\gamma_1\gamma_2} + \frac{b_1 b_2 \theta_2}{\gamma_1} \\ &= -\frac{b_1 b_2 \gamma_2 + b_2}{2\gamma_1\gamma_2}(\ell_2 - \theta_2) < 0. \end{aligned} \tag{3.80}$$

Then combining (3.78), (3.79) and (3.80), we derive that

$$b_1 \int_\Omega v h_2(x,t) + 2\int_\Omega v^2 h_3(x,t) \le -\frac{b_1 b_2 (b_1\gamma_2 + 1)(\ell_2 - \theta_2)}{2\gamma_1\gamma_2} \int_\Omega v,$$

which, along with (3.76) and (3.71), gives

$$\begin{aligned} &\frac{d}{dt}\mathcal{F}_3(t) \\ &\le -k_1 \int_\Omega \left(\frac{u_x^2}{u^2} + \frac{v_x^2}{v^2} + \frac{w_x^2}{w^2}\right) - \frac{b_1 b_2 \gamma_2}{\gamma_1}\int_\Omega \left(u - \frac{\theta_2}{\gamma_2}\right)^2 - \frac{b_1(b_1 b_2 \gamma_2 + b_2)(\ell_2 - \theta_2)}{2\gamma_1\gamma_2}\int_\Omega v \\ &\le 0. \end{aligned}$$

Thus, $\frac{d}{dt}\mathcal{F}_3(t) = 0$ iff $(u, v, w_x) = \left(\frac{\theta_2}{\gamma_2}, 0, 0\right)$. This indicates $w = \tilde{w}$, where $\tilde{w} > 0$ is a constant. Since $\left(\frac{\theta_2}{\gamma_2}, 0, \tilde{w}\right)$ is a solution of (3.6), then one has $\frac{\theta_2}{\gamma_2}\left(1 - \frac{\theta_2}{\gamma_2} - \gamma_1\tilde{w}\right) = 0$, which implies $\tilde{w} = \frac{\gamma_2 - \theta_2}{\gamma_1\gamma_2}$. Hence, $\frac{d}{dt}\mathcal{F}_3(t) = 0$ iff $(u, v, w) = \left(\frac{\theta_2}{\gamma_2}, 0, \frac{\gamma_2 - \theta_2}{\gamma_1\gamma_2}\right)$. Then, one obtains that $\left(\frac{\theta_2}{\gamma_2}, 0, \frac{\gamma_2 - \theta_2}{\gamma_1\gamma_2}\right)$ is globally asymptotically stable by applying LaSalle's invariance principle. This proves Theorem 3.2 (3). □

3.3.3 Case of Coexistence

In this subsection, we shall study the global stability of coexistence steady state (u_*, v_*, w_*) defined in (3.7) under the condition (3.8). We first introduce the following energy function

$$\mathcal{F}_4(t) := \mathcal{F}_4(u, v, w) = \mathcal{F}_u(t) + b_1\mathcal{F}_v(t) + b_1 b_2 \mathcal{F}_w(t),$$

where $\mathcal{F}_y(t) = \int_\Omega \left(y - y_* - y_* \ln \frac{y}{y_*}\right), \ y = u, v, w.$

Proof of Theorem 3.2(4). Using the same way as the proof in Theorem 2.2 (1), we can check that $\mathcal{F}(t) \geq 0$ and $\mathcal{F}(t) = 0$ iff $(u, v, w) = (u_*, v_*, w_*)$.

Next, we shall show $\frac{d}{dt}\mathcal{F}_4(t) \leq 0$ under certain conditions for the parameters. In fact, using the first equation of (3.3) and $u_* + b_1 v_* + \gamma_1 w_* = 1$, we derive

$$\begin{aligned} \frac{d}{dt}\mathcal{F}_u(t) = &-u_* d_1 \int_\Omega \frac{u_x^2}{u^2} - \int_\Omega (u - u_*)^2 - b_1 \int_\Omega (u - u_*)(v - v_*) \\ &- \gamma_1 \int_\Omega (u - u_*)(w - w_*). \end{aligned} \tag{3.81}$$

Applying $u_* - b_2 w_* = \theta_1$ and the second equation of (3.3), one has

$$\begin{aligned} b_1 \frac{d}{dt}\mathcal{F}_v(t) = &-b_1 v_* d_2 \int_\Omega \frac{v_x^2}{v^2} + b_1 \xi v_* \int_\Omega \frac{u_x \cdot v_x}{v} + b_1 \int_\Omega (v - v_*)(u - u_*) \\ &- b_1 b_2 \int_\Omega (v - v_*)(w - w_*). \end{aligned} \tag{3.82}$$

Similarly, noting $v_* + \gamma_2 u_* = \theta_2$ and applying the third equation of (3.3), we derive that

$$\begin{aligned} b_1 b_2 \frac{d}{dt}\mathcal{F}_w(t) = &-b_1 b_2 w_* \int_\Omega \frac{w_x^2}{w^2} + b_1 b_2 w_* \chi \int_\Omega \frac{\phi_u u_x \cdot w_x + \phi_v v_x \cdot w_x}{w} \\ &+ b_1 b_2 \int_\Omega (w - w_*)(v - v_*) + b_1 b_2 \gamma_2 \int_\Omega (w - w_*)(u - u_*). \end{aligned} \tag{3.83}$$

We combine (3.81), (3.82) and (3.83) and use $b_1 b_2 \gamma_2 - \gamma_1 = 0$ to obtain

$$\frac{d}{dt}\mathcal{F}_4(t) = -\int_\Omega Y_3^T B_3 Y_3 - \int_\Omega (u - u_*)^2, \tag{3.84}$$

where

$$Y_3 = \begin{pmatrix} \frac{u_x}{u} \\ \frac{v_x}{v} \\ \frac{w_x}{w} \end{pmatrix} \text{ and } B_3 = \begin{pmatrix} u_* d_1 & -\frac{b_1 \xi v_* u}{2} & -\frac{\chi b_1 b_2 w_* \phi_u u}{2} \\ -\frac{b_1 \xi v_* u}{2} & b_1 v_* d_2 & -\frac{\chi b_1 b_2 w_* \phi_v v}{2} \\ -\frac{\chi b_1 b_2 w_* \phi_u u}{2} & -\frac{\chi b_1 b_2 w_* \phi_v v}{2} & b_1 b_2 w_* \end{pmatrix}.$$

After some calculations, one can verify that the matrix B_3 is positive definite if and only if

$$\begin{vmatrix} u_* d_1 & -\frac{b_1 \xi v_* u}{2} \\ -\frac{b_1 \xi v_* u}{2} & b_1 v_* d_2 \end{vmatrix} = \frac{v_* b_1 (4 u_* d_1 d_2 - b_1 v_* \xi^2 u^2)}{4} > 0, \tag{3.85}$$

and

$$
\begin{aligned}
|B_3| = & \frac{b_1^2 b_2 w_*}{4}(4d_1 d_2 u_* v_* - b_1 \xi^2 v_*^2 u^2) \\
& - \frac{b_1^2 b_2 w_* \chi^2}{4}(u_* d_1 b_2 w_* \phi_v^2 v^2 + \xi v_* u w_* \phi_v v \cdot b_1 b_2 \phi_u u + b_1 b_2 v_* d_2 w_* \phi_u^2 u^2) \\
& > 0.
\end{aligned} \tag{3.86}
$$

Since $M_0 \geq \|u\|_{L^\infty}$ is independent of ξ, χ and $K_0 \geq \|v\|_{L^\infty}$ is independent of χ (see Remark 3.1), we can find appropriate numbers $\xi_2 > 0$ and $\chi_2 > 0$ such that if $\xi \in (0, \xi_2)$ and $\chi \in (0, \chi_2)$, then

$$
4d_1 d_2 u_* v_* > b_1 v_*^2 M_0^2 \xi^2 + \chi^2 M_*(\xi, u, v),
$$

where

$$
\begin{aligned}
M_*(\xi, u, v) := & \; u_* w_* b_2 d_1 \|\phi_v\|_{L^\infty}^2 K_0^2 + \xi v_* w_* b_1 b_2 \|\phi_v\|_{L^\infty} \|\phi_u\|_{L^\infty} M_0^2 K_0 \\
& + b_1 b_2 v_* w_* d_2 \|\phi_u\|_{L^\infty}^2 K_0^2,
\end{aligned}
$$

which gives (3.85) and (3.86). Hence, there exists a constant $k_1 > 0$ such that (3.84) can be updated as

$$
\frac{d}{dt}\mathcal{F}_4(t) \leq -k_1 \int_\Omega \left(\frac{u_x^2}{u^2} + \frac{v_x^2}{v^2} + \frac{w_x^2}{w^2} \right) - \int_\Omega (u - u_*)^2 \leq 0. \tag{3.87}
$$

Then (3.87) implies $\frac{d}{dt}\mathcal{F}_4(t) \leq 0$ and "=" holds iff $(u, v_x, w_x) = (u_*, 0, 0)$, this indicates $v = \tilde{v}_*$ and $w = \tilde{w}_*$, where $\tilde{v}_*$ and $\tilde{w}_*$ are positive constants satisfying

$$
u_*(1 - u_* - b_1 \tilde{v}_* - \gamma_1 \tilde{w}_*) = 0, \quad \tilde{v}_*(u_* - b_2 \tilde{w}_* - \theta_1) = 0, \quad \tilde{w}_*(\tilde{v}_* + \gamma_2 u_* - \theta_2) = 0.
$$

This together with the definition of u_* in (3.7) gives

$$
\tilde{v}_* = \frac{\gamma_1(\theta_2 - \gamma_2 \theta_1) + b_2(\theta_2 - \gamma_2)}{b_2} = v_*, \quad \tilde{w}_* = \frac{b_1(\gamma_2 \theta_1 + \theta_2) + (1 - \theta_1)}{b_2} = w_*.
$$

Therefore, we conclude that $\frac{d}{dt}\mathcal{F}_4(t) \leq 0$ and $\frac{d}{dt}\mathcal{F}_4(t) = 0$ iff $(u, v, w) = (u_*, v_*, w_*)$. Then, LaSalle's invariance principle yields that (u_*, v_*, w_*) is globally asymptotically stable. This proves Theorem 3.2 (4). □

3.4 Linear Stability/Instability Analysis

In this section, we shall study the possible pattern formation for the system (3.3). In fact, for the space-absent ODE system of (3.3)

$$\begin{cases} u_t = u(1-u) - b_1 uv - \gamma_1 uw, \\ v_t = uv - b_2 vw - \theta_1 v, \\ w_t = vw + \gamma_2 uw - \theta_2 w, \end{cases}$$

it has been proved in [55] that:

(1) The trivial steady state $E_0 := (0,0,0)$ is always linearly unstable.

(2) The prey-only steady state $E_1 := (1,0,0)$ is linearly stable if $\theta_1 > 1$ and $\theta_2 > \gamma_2$.

(3) The semi-coexistence steady state $E_{12} := \left(\theta_1, \frac{1-\theta_1}{b_1}, 0\right)$ exists if $\theta_1 < 1$ and it is linearly stable provided

$$\theta_2 > (b_1\gamma_2 - 1)\theta_1/b_1 + 1/b_1. \tag{3.88}$$

(4) The semi-coexistence steady state $E_{13} := \left(\frac{\theta_2}{\gamma_2}, 0, \frac{\gamma_2-\theta_2}{\gamma_1\gamma_2}\right)$ exists if $\theta_2 < \gamma_2$ and it is linearly stable provided

$$\theta_2 < \gamma_1\gamma_2\theta_1/(b_2+\gamma_1) + b_2\gamma_2/(b_2+\gamma_1). \tag{3.89}$$

For the system (3.3) with spatial movement, by the linear analysis, we can show that the steady states E_1, E_{12} and E_{13} are still linearly stable and hence no pattern formation occurs. More precisely, we have the following results:

Proposition 3.1. *Assume (H0) and $\phi_u \geq 0, \phi_v \geq 0$ hold. Then for the system* (3.3)*, E_1 is linearly stable if $\theta_1 > 1$ and $\theta_2 > \gamma_2$; E_{12} is linearly stable if $\theta_1 < 1$ and* (3.88) *hold; E_{13} is linearly stable if $\theta_2 < \gamma_2$ and* (3.89) *hold.*

Proof. The proof can be found in the Appendix, see Section 3.6. □

And it has been shown in [55] that if (u_*, v_*, w_*) exists for the corresponding ODE system of (3.3), then it is linearly stable if and only if

$$\begin{cases} b_2 + \gamma_1 - b_1 b_2 \gamma_2 > 0, \\ \gamma_1\gamma_2 u_* w_* + b_1 u_* v_* > (\gamma_1 - \gamma_2 b_1 b_2) v_* w_*. \end{cases} \tag{3.90}$$

Hence, in the following, we focus only on whether pattern formation emerges from the coexistence steady state (u_*, v_*, w_*) under the conditions (3.90) and (3.8).

As discussed in Section 3.6, the linear stability/instability of (u_*, v_*, w_*) are determined by the eigenvalue of the following characteristic equation

$$\mu^3 + P_1(\chi, \lambda_k)\mu^2 + P_2(\chi, \lambda_k)\mu + P_3(\chi, \lambda_k) = 0,$$

where $\{\lambda_k\}_{k=0}^{\infty} : 0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \lambda_3 \dots$ denote the sequence of eigenvalues of $-\Delta$ under Neumann boundary conditions and $P_i(\chi, \lambda_k)$ $(i = 1, 2, 3)$ are given as follows

$$\begin{aligned}
P_1(\chi, \lambda_k) :=& \lambda_k(d_1 + d_2 + 1) + u_* > 0,\\
P_2(\chi, \lambda_k) :=& \lambda_k^2(d_1 d_2 + d_1 + d_2) + \lambda_k[(d_2 + 1)u_* + \chi\phi_u^*\gamma_1 u_* w_* + \chi\phi_v^* b_2 v_* w_* + \xi b_1 u_* v_*]\\
&+ \gamma_1\gamma_2 u_* w_* + b_2 v_* w_* + b_1 u_* v_*,\\
P_3(\chi, \lambda_k) :=& \lambda_k^3 d_1 d_2\\
&+ \lambda_k^2\left(d_2 u_* + \chi\phi_u^* d_2\gamma_1 u_* w_* + \chi\phi_v^* d_1 b_2 v_* w_* + \xi b_1 u_* v_* + \chi\phi_v^*\xi\gamma_1 u_* v_* w_*\right)\\
&+ \lambda_k\left[\gamma_1\xi u_* v_* w_* + \chi(b_2\phi_v^* + \gamma_1\phi_v^* - \phi_u^* b_1 b_2)u_* v_* w_*\right]\\
&+ \lambda_k(\gamma_1\gamma_2 d_2 u_* w_* + b_2 d_1 v_* w_* + b_1 u_* v_*)\\
&+ (b_2 + \gamma_1 - \gamma_2 b_1 b_2)u_* v_* w_*,
\end{aligned} \tag{3.91}$$

with $\phi_u^* = \phi_u(u_*, v_*)$ and $\phi_v^* = \phi_v(u_*, v_*)$. From Routh-Hurwitz criterion (e.g., Appendix B.1 in [100]), (u_*, v_*, w_*) is linearly stable iff for each $k \in \mathbb{N}$, it holds that

$$P_1(\chi, \lambda_k) > 0, \quad P_3(\chi, \lambda_k) > 0, \quad P_1(\chi, \lambda_k)P_2(\chi, \lambda_k) - P_3(\chi, \lambda_k) > 0.$$

A direct calculation gives

$$H(\chi, \lambda_k) := P_1(\chi, \lambda_k)P_2(\chi, \lambda_k) - P_3(\chi, \lambda_k) = \lambda_k^3 K_1 + \lambda_k^2 K_2 + \lambda_k K_3 + K_4, \tag{3.92}$$

where

$$\begin{aligned}
K_1 :=& (d_1 d_2 + d_1 + d_2 + 1)(d_1 + d_2) > 0,\\
K_2 :=& (d_1 d_2 + d_1)u_* + \xi(d_1 + d_2)b_1 u_* v_* + (d_1 + d_2 + 1)(d_2 + 1)u_*\\
&+ (d_1 + 1)\chi\phi_u^*\gamma_1 u_* w_* + (d_2 + 1)\chi\phi_v^* b_2 v_* w_* - \chi\phi_v^*\xi\gamma_1 u_* v_* w_*,\\
K_3 :=& (d_2 + 1)u_*^2 + (d_1 + 1)\gamma_1\gamma_2 u_* w_* + (d_2 + 1)b_2 v_* w_* + (d_1 + d_2)b_1 u_* v_* + b_1\xi u_*^2 v_*\\
&+ \chi\phi_u^*\gamma_1 u_*^2 w_* + \chi\phi_u^* b_1 b_2 u_* v_* w_* - (\chi\phi_v^* + \xi)\gamma_1 u_* v_* w_*,\\
K_4 :=& u_*[\gamma_1\gamma_2 u_* w_* + b_1 u_* v_* - (\gamma_1 - \gamma_2 b_1 b_2)v_* w_*].
\end{aligned} \tag{3.93}$$

When $\chi = \xi = 0$, one can easily check that $P_3(\chi, \lambda_k) > 0$ and $H(\chi, \lambda_k) > 0$ for all $k \in \mathbb{N}$, which indicates that (u_*, v_*, w_*) is linearly stable. Hence, in the following, we will study whether or not the taxis mechanisms can induce the pattern formations. Since $H(\chi, \lambda_k)$ depends on the values of $\phi_u^* = \phi_u(u_*, v_*)$, $\phi_v^* = \phi_v(u_*, v_*)$, γ_1 and γ_2. For a better understanding of the difference between the effect of prey-taxis and alarm-taxis in the food chain model with/without intraguild predation, we shall focus on the linear stability/instability of coexistence steady state for two types of $\phi(u, v)$: $\phi(u, v) = v$ and $\phi(u, v) = uv$, both under the conditions $\gamma_1, \gamma_2 \geq 0$.

3.4.1 Linear Stability/Instability Analysis: $\gamma_1 = \gamma_2 = 0$

In this subsection, we shall study the linear stability/instability of (u_*, v_*, w_*) to (3.3) with $\phi(u, v) = v$ or $\phi(u, v) = uv$ in the case of $\gamma_1 = \gamma_2 = 0$. In this case, (3.3) can be simplified as

$$\begin{cases} u_t = d_1 u_{xx} + u(1-u) - b_1 uv, \\ v_t = d_2 v_{xx} - \xi(v u_x)_x + uv - b_2 vw - \theta_1 v, \\ w_t = w_{xx} - \chi(w\phi(u,v)_x)_x + vw - \theta_2 w, \end{cases} \tag{3.94}$$

which is the classical Lotka-Volterra food chain model with taxis mechanisms (i.e., $\xi, \chi > 0$). And $(u_*, v_*, w_*) = (1 - b_1\theta_2, \theta_2, \frac{1-\theta_1-b_1\theta_2}{b_2})$ exists provided

$$\theta_1 + b_1\theta_2 < 1. \tag{3.95}$$

It has been proved in [66] that if $\phi(v) = v$, the coexistence steady state of the system (3.94) is globally stable if $\xi > 0$ and $\chi > 0$ are both small. Thus, it is natural to ask whether or not (u_*, v_*, w_*) is linearly unstable and pattern formation occurs for large ξ and χ. In fact, we have the following results.

Lemma 3.13 (Linear stability: $\phi(u, v) = v$). *Let $\phi(u, v) = v$ and assume* (3.95) *holds, then (u_*, v_*, w_*) of* (3.94) *is linearly stable for all $\chi, \xi \geq 0$.*

Proof. Since $\phi(u, v) = v$, we have $\phi_u^* = 0$ and $\phi_v^* = 1$. Then noting $\gamma_1 = \gamma_2 = 0$, it follows from (3.91) that for each $k \in \mathbb{N}$

$$\begin{aligned} P_3(\chi, \lambda_k) =& \lambda_k^3 d_1 d_2 + \lambda_k^2 \left(d_2 u_* + \chi d_1 b_2 v_* w_* + \xi b_1 u_* v_*\right) \\ &+ \lambda_k \left(b_2 d_1 v_* w_* + b_1 u_* v_* + \chi b_2 u_* v_* w_*\right) + b_2 u_* v_* w_* > 0. \end{aligned}$$

On the other hand, by $K_i (i = 1, 2, 3, 4)$ in (3.93), one can check that $K_i > 0$ for $i = 1, 2, 3, 4$, which implies for each $k \in \mathbb{N}$

$$H(\chi, \lambda_k) = P_1(\chi, \lambda_k) P_2(\chi, \lambda_k) - P_3(\chi, \lambda_k) = \lambda_k^3 K_1 + \lambda_k^2 K_2 + \lambda_k K_3 + K_4 > 0.$$

Then Routh-Hurwitz criterion implies that (u_*, v_*, w_*) is linearly stable. □

Remark 3.2. *Lemma 3.13 implies that no pattern formation occurs for the classical Lotka-Volterra food chain model with prey-taxis mechanisms for any* $\xi, \chi \geq 0$.

Next, we shall study the possibility of pattern formation for the Lotka-Volterra food chain model incorporating the alarm-taxis. The main results are as follows.

Lemma 3.14 (Linear stability/instability: $\phi(u, v) = uv$)**.** *Let* $\phi(u, v) = uv$ *and assume* (3.95) *holds. It holds that*

(1) If $2b_1\theta_2 \leq 1$*, then* (u_*, v_*, w_*) *is linearly stable for all* $\chi > 0$.

(2) If $2b_1\theta_2 > 1$*, then* (u_*, v_*, w_*) *is linearly unstable provided* $\chi > 0$ *is large enough and there exists some* $k \in \mathbb{N}^+$ *such that*

$$0 < \lambda_k < (2b_1\theta_2 - 1)/d_1. \tag{3.96}$$

Proof. For $\phi(u, v) = uv$, one has $\phi_u^* = \phi_u(u_*, v_*) = v_*$ and $\phi_v^* = \phi_v(u_*, v_*) = u_*$. Noting $\gamma_1 = \gamma_2 = 0$ and the definitions of $K_i (i = 1, 2, 3, 4)$ in (3.93), we have $K_i > 0$ for all $i = 1, 2, 3, 4$, which implies that for each $k \in \mathbb{N}$

$$H(\chi, \lambda_k) = P_1(\chi, \lambda_k)P_2(\chi, \lambda_k) - P_3(\chi, \lambda_k) > 0.$$

Moreover, using $u_* - b_1v_* = 1 - 2b_1\theta_2$ and the facts $\gamma_1 = \gamma_2 = 0$, $\phi_u^* = v_*$, $\phi_v^* = u_*$ again, we deduce from (3.91) that

$$\begin{aligned} P_3(\chi, \lambda_k) =& \lambda_k^3 d_1 d_2 + \lambda_k^2 (d_2 + \xi b_1 v_*) u_* + \lambda_k (d_1 b_2 w_* + b_1 u_*) v_* + b_2 u_* v_* w_* \\ &+ \lambda_k \chi b_2 u_* v_* w_* (\lambda_k d_1 + 1 - 2b_1\theta_2). \end{aligned} \tag{3.97}$$

Then if $2b_1\theta_2 \leq 1$, one has $P_3(\chi, \lambda_k) > 0$ for any $k \in \mathbb{N}$, and hence (u_*, v_*, w_*) is linearly stable by Routh-Hurwitz criterion.

On the other hand, if $2b_1\theta_2 > 1$ and (3.96) holds, we get that $\lambda_k d_1 + 1 - 2b_1\theta_2 < 0$ for some $k \in \mathbb{N}^+$. Since λ_k, u_*, v_*, w_* are independent of χ, it follows that $P_3(\chi, \lambda_k) \leq 0$ for sufficiently large $\chi > 0$. Therefore, according to Routh-Hurwitz criterion, (u_*, v_*, w_*) is linearly unstable. □

Remark 3.3. *For the Lotka-Volterra food chain model* (3.94)*, our results imply that* $\phi(u, v)$ *plays an important role on the pattern formation. If* $\phi(u, v) = v$ *(i.e., prey-taxis mechanism), no pattern formation occurs. If* $\phi(u, v) = uv$ *(i.e., alarm-taxis mechanism), the potential steady state bifurcations generating from* (u_*, v_*, w_*) *may happen. Compared with the results in [46], our results confirm that the alarm-taxis can trigger the pattern formation by itself even without logistic growth source.*

3.4.2 Linear Stability/Instability Analysis: $\gamma_1, \gamma_2 > 0$

In this subsection, we shall study the possibility of pattern formation for the system (3.3) with intraguild predation (i.e., $\gamma_1, \gamma_2 > 0$). To this end, we analyze the linear stability/instability of (u_*, v_*, w_*) defined in (3.7). When $\gamma_1, \gamma_2 > 0$, we rewrite $P_3(\chi, \lambda_k)$ in (3.91) as follows:

$$\begin{aligned} P_3(\chi, \lambda_k) =& \lambda_k^3 d_1 d_2 + \lambda_k^2 (d_2 u_* + \xi b_1 u_* v_*) \\ & + \lambda_k (\gamma_1 u_* w_* \gamma_2 d_2 + b_2 v_* w_* d_1 + b_1 u_* v_* + \gamma_1 u_* v_* w_* \xi) \\ & + \lambda_k^2 \chi (\phi_u^* d_2 \gamma_1 u_* w_* + \phi_v^* d_1 b_2 v_* w_* + \phi_v^* \xi \gamma_1 u_* v_* w_*) \\ & + \lambda_k \chi u_* v_* w_* (b_2 \phi_v^* + \gamma_1 \phi_v^* - \phi_u^* b_1 b_2) + (b_2 + \gamma_1 - \gamma_2 b_1 b_2) u_* v_* w_*. \end{aligned} \tag{3.98}$$

Lemma 3.15 (Linear stability/instability: $\phi(u,v) = v$)**.** *Let $\phi(u,v) = v$ and assume* (3.8) *and* (3.90) *hold. Then we have the following results:*

(1) (u_*, v_*, w_*) *is linearly stable provided*

$$\chi + \xi \le \widetilde{K_3}/\gamma_1 u_* v_* w_* \quad \text{and} \quad d_2 + 1 \ge \xi \gamma_1 u_* / b_2, \tag{3.99}$$

with $\widetilde{K_3} > 0$ defined in (3.103)*.*

(2) (u_*, v_*, w_*) *is linearly unstable provided $\chi > 0$ large enough and one of the following conditions holds:*

$$\begin{cases} d_2 + 1 > \frac{\xi \gamma_1 u_*}{b_2}, \\ 0 < \lambda_k < \frac{\gamma_1 u_*}{(d_2+1) b_2 - \xi \gamma_1 u_*} \quad \textit{for some } k \in \mathbb{N}^+, \end{cases} \tag{3.100}$$

or

$$d_2 + 1 \le \frac{\xi \gamma_1 u_*}{b_2} \quad \textit{for all } k \in \mathbb{N}^+. \tag{3.101}$$

Proof. Since $\phi(u,v) = v$, one has $\phi_v^* = 1$ and $\phi_u^* = 0$. Noting $b_2 + \gamma_1 - \gamma_2 b_1 b_2 > 0$, it follows from (3.98) that $P_3(\chi, \lambda_k) > 0$ for all $k \in \mathbb{N}$.

Since (3.8) and (3.90) hold, we derive from (3.93) that $K_1 > 0$ and $K_4 > 0$. Hence, to determine the sign of $H(\chi, \lambda_k)$, we only need to consider the values of K_2 and K_3. Using the facts $\phi_v^* = 1$ and $\phi_u^* = 0$, we rewrite K_2 and K_3 defined in (3.93) as follows:

$$K_2 = \widetilde{K_2} + \chi v_* w_* [(d_2+1) b_2 - \xi \gamma_1 u_*] \quad \text{and} \quad K_3 = \widetilde{K_3} - (\chi + \xi) \gamma_1 u_* v_* w_*,$$

where $\widetilde{K_2} > 0$ and $\widetilde{K_3} > 0$ are defined by

$$\widetilde{K_2} := (d_1 d_2 + d_1) u_* + \xi (d_1 + d_2) b_1 u_* v_* + (d_1 + d_2 + 1)(d_2 + 1) u_*, \tag{3.102}$$

$$\widetilde{K_3} := (d_2+1)u_*^2 + (d_1+1)\gamma_1\gamma_2 u_* w_* + (d_2+1)b_2 v_* w_* + (d_1+d_2)b_1 u_* v_* + b_1\xi u_*^2 v_*. \quad (3.103)$$

Then we can derive from (3.99) that K_2 and K_3 are positive and hence $H(\chi, \lambda_k) > 0$ for all $k \in \mathbb{N}$, which implies that (u_*, v_*, w_*) is linearly stable by using Routh-Hurwitz criterion.

Next, we shall show that (u_*, v_*, w_*) is linearly unstable for large χ under conditions (3.100) or (3.101). To this end, we rewrite $H(\chi, \lambda_k)$ (see in (3.92)) as follows:

$$\begin{aligned} H(\chi, \lambda_k) = &\lambda_k^3 K_1 + \lambda_k^2 \widetilde{K_2} + \lambda_k \widetilde{K_3} + K_4 \\ &+ \lambda_k \chi v_* w_* \left(\lambda_k[(d_2+1)b_2 - \xi\gamma_1 u_*] - \gamma_1 u_*\right) - \lambda_k \xi \gamma_1 u_* v_* w_*, \end{aligned} \quad (3.104)$$

where $\widetilde{K_2}$ and $\widetilde{K_3}$ are defined by (3.102) and (3.103), respectively.

Since λ_k and the value of (u_*, v_*, w_*) are independent of χ, then if (3.100) or (3.101) holds, we can find $\chi > 0$ large enough such that $H(\chi, \lambda_k) \leq 0$, and hence (u_*, v_*, w_*) is linearly unstable by applying Routh-Hurwitz criterion again. □

Remark 3.4. *Compared with Lemma 3.13 and Lemma 3.15, we found that the intraguild predation (i.e., $\gamma_1, \gamma_2 > 0$) plays an important role for the pattern formation.*

Next, we shall study the possible pattern formation in the system (3.3) with alarm-taxis in the sense of $\phi(u, v) = uv$.

Lemma 3.16 (Linear stability/instability: $\phi(u, v) = uv$)**.** *Let $\phi(u, v) = uv$, $\chi > 0$ and $\xi \geq 0$. Assume* (3.8) *and* (3.90) *hold. Then it holds that:*

(1) (u_*, v_*, w_*) is linearly stable provided

$$b_2 u_* + \gamma_1 u_* - v_* b_1 b_2 \geq 0 \quad (3.105)$$

and

$$0 < \xi \leq \min\left\{\widetilde{K_3}/\gamma_1 u_* v_* w_*, (d_1+1)/u_* + (d_2+1)b_2/u_*\gamma_1\right\}, \quad (3.106)$$

where $\widetilde{K_3} > 0$ defined in (3.103)*.*

(2) (u_*, v_*, w_*) is linearly unstable provided $\chi > 0$ large enough and one of the following conditions holds:

$$b_2 u_* + \gamma_1 u_* - v_* b_1 b_2 < 0 \quad \text{and} \quad 0 < \lambda_{k_0} < \frac{|b_2 u_* + \gamma_1 u_* - v_* b_1 b_2|}{d_2\gamma_1 + d_1 b_2 + u_*\xi\gamma_1} \text{ for some } k_0 \in \mathbb{N}^+, \quad (3.107)$$

or for some $k_0 \in \mathbb{N}^+$

$$\xi > \frac{d_1+1}{u_*} + \frac{(d_2+1)b_2}{u_*\gamma_1} \text{ and } \lambda_{k_0} > \frac{v_* b_1 b_2}{|(d_1+1)\gamma_1 + (d_2+1)b_2 - u_*\xi\gamma_1|}. \quad (3.108)$$

Proof. From $\phi(u,v) = uv$, one has $\phi_v^* = u_*$ and $\phi_u^* = v_*$. Hence we can derive that

$$\begin{aligned}
&P_3(\chi,\lambda_k)\\
&= \lambda_k^3 d_1 d_2 + \lambda_k^2(d_2 u_* + \xi b_1 u_* v_*) + \lambda_k(\gamma_1 u_* w_* \gamma_2 d_2 + b_2 v_* w_* d_1 + b_1 u_* v_* + \gamma_1 u_* v_* w_* \xi)\\
&+ \lambda_k \chi u_* v_* w_* [\lambda_k(d_2\gamma_1 + d_1 b_2 + u_*\xi\gamma_1) + (b_2 u_* + \gamma_1 u_* - v_* b_1 b_2)]\\
&+ (b_2 + \gamma_1 - \gamma_2 b_1 b_2) u_* v_* w_*,
\end{aligned} \tag{3.109}$$

and

$$\begin{aligned}
H(\chi,\lambda_k) = {}& \lambda_k^3 K_1 + \lambda_k^2 \widetilde{K_2} + \lambda_k(\widetilde{K_3} - \xi\gamma_1 u_* v_* w_*) + K_4\\
&+ \lambda_k \chi u_* v_* w_* \left(\lambda_k[(d_1+1)\gamma_1 + (d_2+1)b_2 - u_*\xi\gamma_1] + v_* b_1 b_2\right).
\end{aligned} \tag{3.110}$$

Then if (3.105) and (3.106) hold, one can verify that $P_3(\chi,\lambda_k) > 0$ and $H(\chi,\lambda_k) > 0$ for each $k \in \mathbb{N}$, and hence by applying Routh-Hurwitz criterion, we obtain that (u_*, v_*, w_*) is linearly stable .

On the contrary, if (3.107) holds, we can choose χ large enough such that $P_3(\chi,\lambda_k) < 0$. Thus, we derive from Routh-Hurwitz criterion that (u_*, v_*, w_*) is linearly unstable. Similarly, if (3.108) holds, we have $H(\chi,\lambda_k) < 0$ for large χ, and hence (u_*, v_*, w_*) is linearly unstable. □

Remark 3.5. *Compared with the the Lotka-Volterra food chain model* (3.94) *with* $\phi(u,v) = uv$*, the intraguild predation model (i.e.,* $\gamma_1, \gamma_2 > 0$*) has richer dynamics. Specifically, the intraguild predation model has not only the potential of steady state bifurcations but also that of Hopf bifurcations.*

Remark 3.6. *The instability results of the intraguild predation model with* $\phi(u,v) = uv$ *indicate that the alarm taxis mechanism can promote potential steady state bifurcations, which can not be induced by the intraguild predation model with* $\phi(u,v) = v$.

3.5 Pattern Formations: Numerical Simulations

In this section, we shall give some numerical simulations to verify our theoretical analysis in Section 3.4.

3.5.1 Food Chain Model with Alarm-taxis: $\gamma_1 = \gamma_2 = 0$ and $\phi(u,v) = uv$

In this case, we fix the value of the parameters in all simulations as follows:

$$d_1 = 0.1,\ d_2 = b_1 = b_2 = 1, \theta_1 = 0.1,\ \theta_2 = 0.7,\ \gamma_1 = \gamma_2 = 0,$$

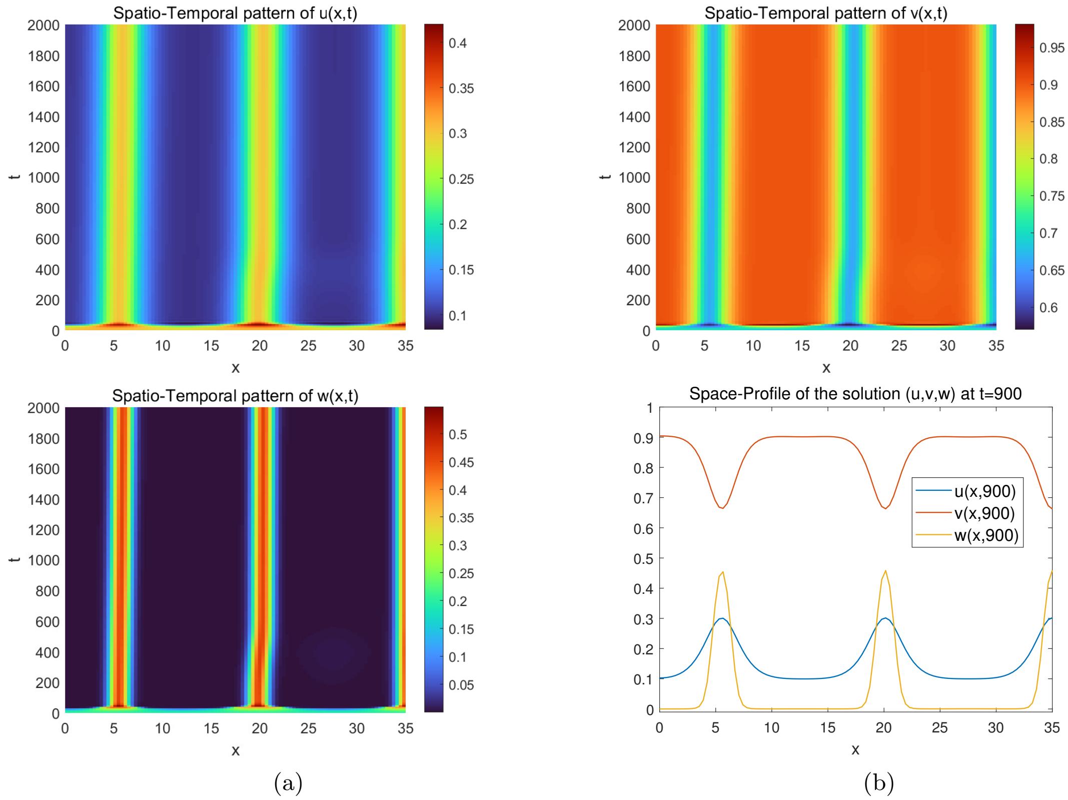


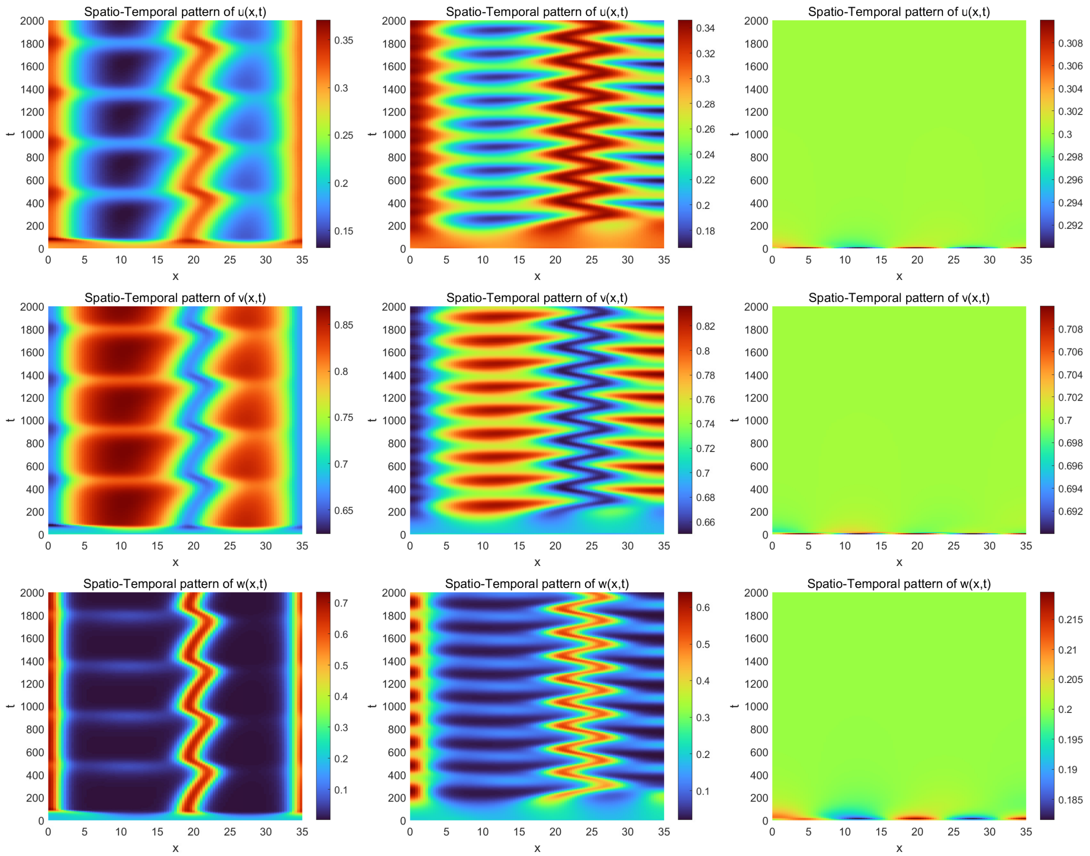
Figure 3.1: Numerical simulation of spatio-temporal patterns generated by (3.3) with $\phi(u,v) = uv$ and $\gamma_1 = \gamma_2 = 0$. The parameter values are: $\chi = 80, \xi = 0,\ d_1 = 0.1,\ d_2 = b_1 = b_2 = 1, \theta_1 = 0.1,\ \theta_2 = 0.7$. The initial datum (u_0, v_0, w_0) is set as a small random perturbation of the homogeneous coexistence steady state $(0.3, 0.7, 0.2)$.

which gives $(u_*, v_*, w_*) = (0.3, 0.7, 0.2)$ and $\theta_1 + b_1\theta_2 < 1$ as well as $2b_1\theta_2 > 1$. Hence, by Lemma 3.14 and the fact $H(\chi, \lambda_k) > 0$, we expect only the spatio-temporal steady state (aggregation) pattern occurs when

$$\chi \geq \chi_k^{\mathcal{S}_1}(\xi) := \frac{5}{21(4-\lambda_k)}\left(100\lambda_k^2 + 30(10+7\xi)\lambda_k + 224 + \frac{42}{\lambda_k}\right), \tag{3.111}$$

for some $k \in \mathbb{N}^+$ such that $0 < \lambda_k < 4$ and here $\chi_k^{\mathcal{S}_1}(\xi)$ is the root of $P_3(\chi, \lambda_k) = 0$ in (3.97). Taking $\Omega = (0, 10\pi)$, with allowable wavenumber satisfying $0 < \lambda_k = (k/10)^2 < 4$, we get the allowable unstable modes for $k = 1, 2, 3 \cdots, 18, 19$. We choose $\lambda_k = (5/10)^2$, then $\chi_k^{\mathcal{S}_1}(\xi)$ in (3.111) can be updated as $\chi_5^{\mathcal{S}_1}(\xi) = \frac{631+70\xi}{21}$.

We first pick $\xi = 0$ to find a value $\chi_5^{\mathcal{S}_1}(0) \approx 30.0476$ for the possibility of pattern formations. As shown in Figure 3.1, by letting $\chi = 80 > 30.0476$ and we can find the steady state patterns (see Figure 3.1): the time evolutionary profiles of solutions are horizontal



(a) $\chi = 80$, ξ=10 (b) $\chi = 80$, ξ=20 (c) $\chi = 80$, ξ=40

Figure 3.2: Numerical simulation of spatio-temporal patterns for (3.3) with $\phi(u, v) = uv$. The fixed parameter values are: $d_1 = 0.1$, $d_2 = b_1 = b_2 = 1, \theta_1 = 0.1$, $\theta_2 = 0.7$ and $\gamma_1 = \gamma_2 = 0$. The initial datum (u_0, v_0, w_0) is set as a small random perturbation of the homogeneous coexistence steady state $(0.3, 0.7, 0.2)$.

lines, and the space-profiles show that all species reach an inhomogeneous coexistence state in space.

The expression in (3.111) implies that the critical value $\chi_k^{S_1}(\xi) > 0$ is increasing in terms of $\xi \geq 0$, the spatio-temporal patterns generated due to any fixed large χ and fixed mode k will disappear as the value of $\xi \geq 0$ increases, which implies the prey-taxis has a stabilization effect on the homogeneous steady state. To verify this fact, we use numerical simulations to find that the spatio-temporal patterns gradually evolve into the spatially homogeneous patterns as ξ increases from 0 to 10, then to 20, and finally disappear at $\xi = 40$, see more details in Figure 3.2.

3.5.2 Food Chain Model with Intraguild Predation and Prey-taxis: $\gamma_1 > 0$, $\gamma_2 > 0$ and $\phi(u, v) = v$

In this case, we fix the value of the parameters as follows:

$$d_1 = 0.1,\ d_2 = b_1 = b_2 = \gamma_2 = 1,\ \gamma_1 = 2,\ \theta_1 = 0.1,\ \theta_2 = 0.9. \tag{3.112}$$

Then $(u_*, v_*, w_*) = (0.15, 0.75, 0.05)$. As discussed in Lemma 3.15, only Hopf bifurcations can occur by noting the fact $P_3(\chi, \lambda_k) > 0$.

We derive from (3.112) that $H(\chi, \lambda_k) = 0$ in (3.104) is equivalent to

$$\chi = \chi_k^{\mathcal{H}_1}(\xi) := \frac{9680\lambda_k^2 + (2640 + 495\xi)\lambda_k + \frac{54}{\lambda_k} + 1041 + 90\xi}{15(3 + 3\lambda_k\xi - 20\lambda_k)}, \tag{3.113}$$

which is positive provided $\lambda_k(20-3\xi) < 3$. Taking $\Omega = (0, 10\pi)$, the allowable wavenumber $\lambda_k = (k/10)^2$ satisfying $\lambda_k(20 - 3\xi) < 3$, then $k = 1, 2, 3$ are allowable unstable modes for any $\xi \geq 0$. Fixing $k = 2$ and (3.113) can be simplified as

$$\chi_2^{\mathcal{H}_1}(\xi) = 61 + \frac{62386}{75(55 + 3\xi)}. \tag{3.114}$$

We first choose $\xi = 0$ to obtain a value $\chi_2^{\mathcal{H}_1}(0) \approx 76.124$ for possible pattern formations. As shown in Figure 3.3(a), with $\chi = 100 > 76.124$ in hand, we can find the spatio-temporal patterns. In particular, the time evolutionary profiles of solutions are periodically oscillatory, which indicates the bifurcation might be of Hopf bifurcation type (see the last picture in Figure 3.3(a)). Moreover, (3.114) indicates that for fixed unstable mode $k = 2$, the critical value $\chi_2^{\mathcal{H}_1}(\xi) > 0$ is decreasing about $\xi \geq 0$, which implies the prey-taxis might have a destabilization effect on patterns. This is an interesting phenomenon, which is different from the food chain model without intraguild predation.

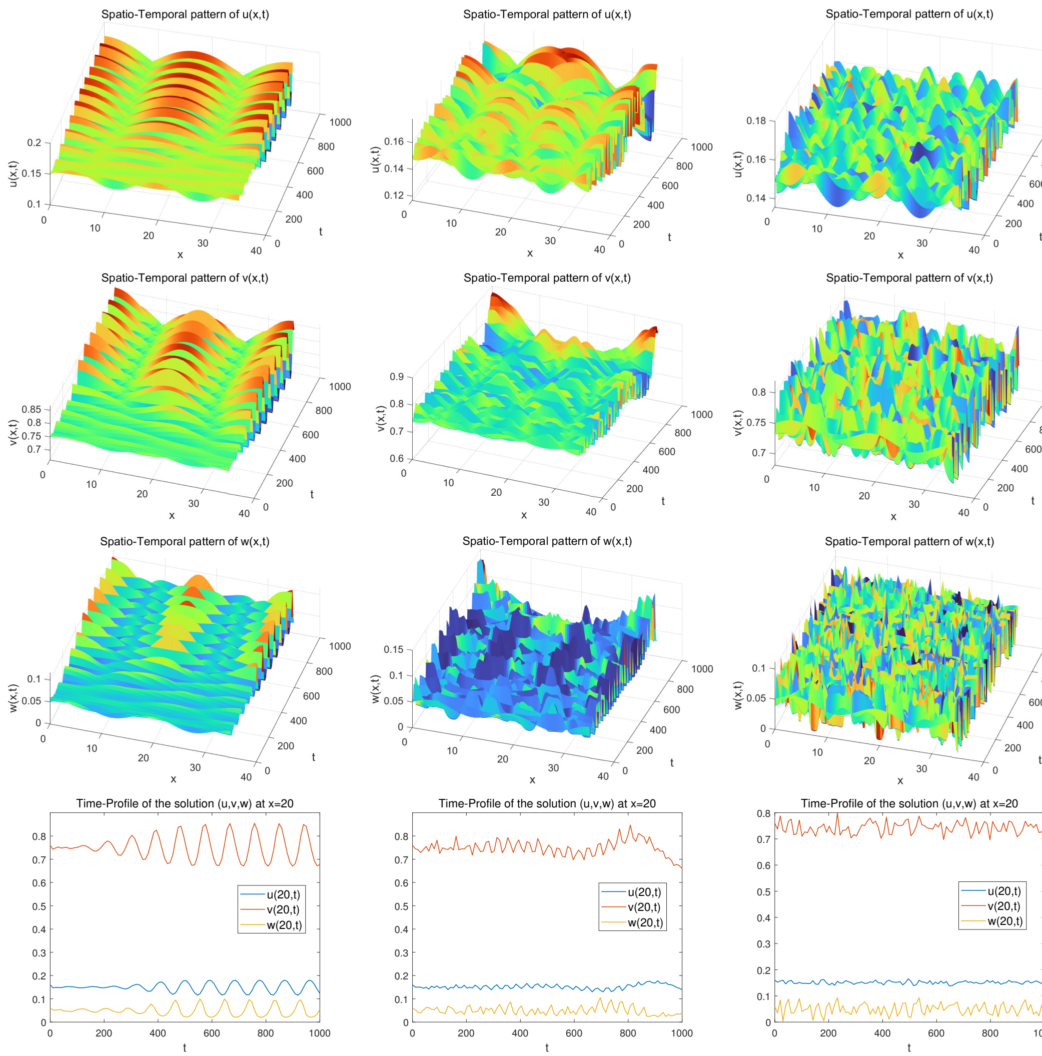
To verify this fact, we take $\xi = 10$ and $\xi = 20$ and find that the patterns become unstable as ξ increasing from 0 to 10 and then to 20, and the chaotic spatio-temporal patterns may happen, see Figure 3.3(c).

3.5.3 Food Chain Model with Intraguild Predation and Alarm-taxis: $\gamma_1, \gamma_2 > 0$ and $\phi(u, v) = uv$

In this case, we fix the parameters as follows for simulations:

$$d_1 = 0.1,\ d_2 = b_1 = b_2 = \gamma_2 = 1,\ \gamma_1 = 2, \theta_1 = 0.1,\ \theta_2 = 0.9, \tag{3.115}$$

this implies $(u_*, v_*, w_*) = (0.15, 0.75, 0.05)$. From Lemma 3.16, we know that the steady state and Hopf bifurcations are both possible.



(a) $\chi = 100,\ \xi$=0 (b) $\chi = 100,\ \xi$=10 (c) $\chi = 100,\ \xi$=20

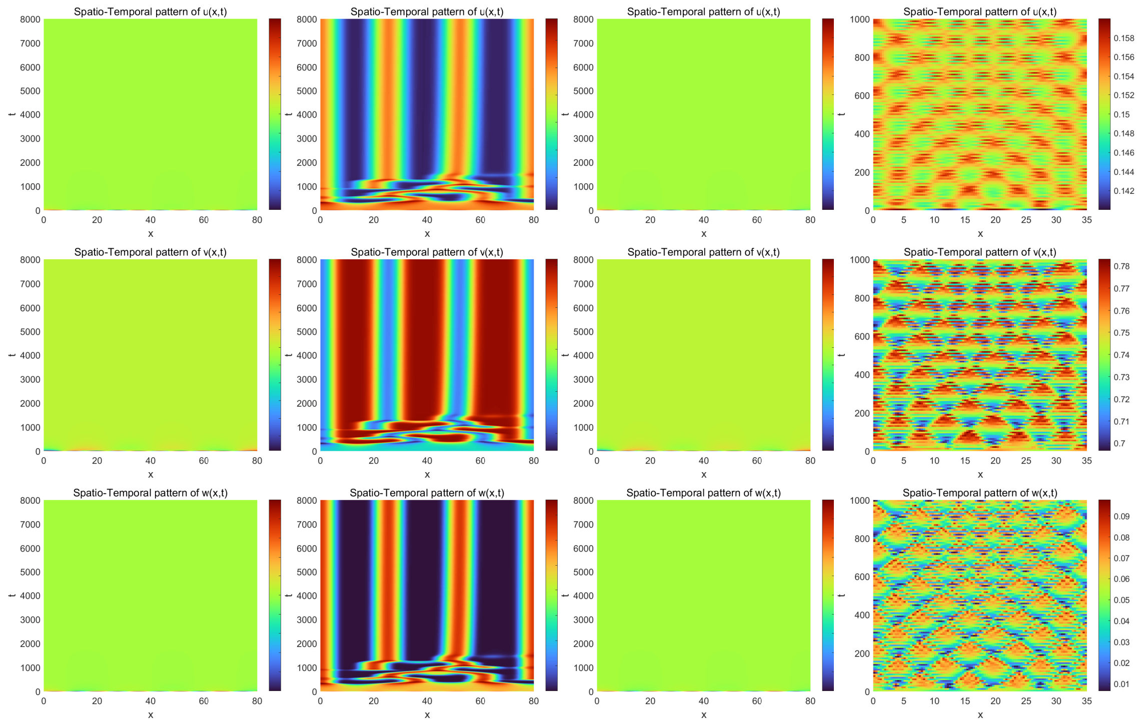
Figure 3.3: Numerical simulation of spatio-temporal patterns generated by (3.3) with $\phi(u, v) = v$ and $\gamma_1, \gamma_2 > 0$. The parameter values are: $d_1 = 0.1,\ d_2 = b_1 = b_2 = \gamma_2 = 1,\ \gamma_1 = 2,\ \theta_1 = 0.1,\ \theta_2 = 0.9$. The initial datum (u_0, v_0, w_0) is set as a small random perturbation of the homogeneous coexistence steady state $(0.15, 0.75, 0.05)$.

By (3.115), we first derive from (3.109) and (3.110) in Lemma 3.16 to obtain that $P_3(\chi, \lambda_k) = 0$ and $H(\chi, \lambda_k) = 0$ are, respectively, equivalent to

$$\chi = \chi_k^{\mathcal{S}_2}(\xi) := \frac{1600\lambda_k^2 + 600\lambda_k(4 + 3\xi) + \frac{180}{\lambda_k} + 2100 + 180\xi}{27 - 27\lambda_k(7 + \xi)}, \tag{3.116}$$

and

$$\chi = \chi_k^{\mathcal{H}_2}(\xi) := \frac{77440\lambda_k^2 + 120\lambda_k(176 + 33\xi) + \frac{432}{\lambda_k} + 8328 + 180\xi}{54\lambda_k(\xi - 14) - 135}. \tag{3.117}$$



(a) $\chi = 0, \xi$=0 (b) $\chi = 450, \xi$=0 (c) $\chi = 450, \xi$=4 (d) $\chi = 450, \xi$=45

Figure 3.4: Numerical simulation of spatio-temporal patterns generated by (3.3) with $\phi(u,v) = uv$ and $\gamma_1, \gamma_2 > 0$. The parameter values are: $d_1 = 0.1$, $d_2 = b_1 = b_2 = \gamma_2 = 1$, $\gamma_1 = 2, \theta_1 = 0.1$, $\theta_2 = 0.9$. The initial datum (u_0, v_0, w_0) is set as a small random perturbation of the homogeneous coexistence steady state $(0.15, 0.75, 0.05)$.

We deduce from (3.110) in Lemma 3.16 that if

$$0 \leq \xi \leq \min\left\{\frac{\widetilde{K_3}}{\gamma_1 u_* v_* w_*}, \frac{d_1+1}{u_*} + \frac{(d_2+1)b_2}{u_* \gamma_1}\right\} = \min\left\{\frac{347}{15} + \frac{3\xi}{2}, 14\right\} = 14,$$

then $H(\chi, \lambda_k) > 0$ for any $k \in \mathbb{N}$ and hence no Hopf bifurcation occurs, which motivates us to study the possibility of steady state pattern formation. To illustrate this case, we take $\Omega = (0, 10\pi)$, then from (3.107), the allowable unstable modes $k \in \mathbb{N}^+$ must satisfy $0 < \lambda_k = (k/10)^2 < \frac{1}{7+\xi}$.

We take $k = 3$ and $\xi = 0$, then (3.116) implies that $\chi_3^{\mathcal{S}_2}(0) \approx 433.329$, which is a value for possible pattern formations. As shown in Figure 3.4(b), choosing $\chi = 450 > 433.329$, we can find the steady state patterns. Furthermore, for the fixed unstable mode $k = 3$, the pattern formations will disappear as ξ increasing from 0 to 4, see Figure 3.4(c).

For relatively large $\xi > 14$, from Lemma 3.16 and the definition of $\chi_k^{\mathcal{H}_2}$ in (3.117), the Hopf bifurcations possibly occur as long as the allowable unstable modes $k \in \mathbb{N}^+$ satisfying

$\lambda = (k/10)^2 > \frac{5}{2(\xi-14)}$. With $\chi = 450$ in hand, for the same unstable mode $k = 3$, we pick $\xi = 45$ to find the spatio-temporal patterns, see Figure 3.4(d).

Our results demonstrate that for the fixed large $\chi = 450$, as the parameter ξ increases, the steady state patterns (see Figure 3.4(b)) evolve first into the constant state (see Figure 3.4(c)) and then further develop into the Hopf bifurcation patterns (see Figure 3.4(d)). Moreover, from Figure 3.4(a), we observe that no pattern formation occurs when $\chi = \xi = 0$ and $\gamma_1, \gamma_2 > 0$. This, together with Figure 3.1, Figure 3.3(a), Figure 3.4(b) and Lemma 3.13, indicates that the signal taxis mechanism plays an essential role in promoting pattern formation.

3.6 Appendix: Proof of Proposition 3.1

In this section, we are devoted to giving some basic linear analysis on the linear stability/instability of constant steady state for the system (3.3). To this end, we first linearize the system (3.3) at constant steady state (u_c, v_c, w_c) to obtain

$$\begin{cases} \Psi_t = \mathcal{A}\Delta\Psi + \mathcal{B}\Psi, & x \in \Omega, \quad t > 0, \\ \nabla\Psi \cdot \nu = 0, & x \in \partial\Omega, \quad t > 0, \\ \Psi(x, 0) = (u_0 - u_c, v_0 - v_c, w_0 - w_c)^{\mathcal{T}}, & x \in \Omega, \end{cases}$$

where

$$\Psi := \begin{pmatrix} u - u_c \\ v - v_c \\ w - w_c \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} d_1 & 0 & 0 \\ -\xi v_c & d_2 & 0 \\ -\chi w_c \phi_u^c & -\chi w_c \phi_v^c & 1 \end{pmatrix} \text{ and } \mathcal{B} = \begin{pmatrix} -u_c & -b_1 u_c & -\gamma_1 u_c \\ v_c & B_{22} & -b_2 v_c \\ \gamma_2 w_c & w_c & B_{33} \end{pmatrix},$$

with $\phi_u^c := \phi_u(u_c, v_c)$, $\phi_v^c := \phi_v(u_c, v_c)$ and

$$B_{22} := u_c - b_2 w_c - \theta_1 \quad \text{and} \quad B_{33} := v_c + \gamma_2 u_c - \theta_2. \tag{3.118}$$

Then, the linear stability of (u_c, v_c, w_c) is determined by the eigenvalues of the matrix $(-\lambda_k \mathcal{A} + \mathcal{B})$, which satisfies the characteristic equation $\mu^3 + P_1\mu^2 + P_2\mu + P_3 = 0$, where $P_i := P_i(\lambda_k)$ $(i = 1, 2, 3)$ are defined as below

$$\begin{aligned} P_1(\lambda_k) :=& \lambda_k(d_1 + d_2 + 1) + u_c - B_{22} - B_{33}, \\ P_2(\lambda_k) :=& \lambda_k^2(d_1 d_2 + d_1 + d_2) + \lambda_k \{(d_2 + 1)u_c - (d_1 + 1)B_{22} - (d_1 + d_2)B_{33}\} \\ &+ \lambda_k(\chi\phi_u^c\gamma_1 u_c w_c + \chi\phi_v^c b_2 v_c w_c + \xi b_1 u_c v_c) + \gamma_1\gamma_2 u_c w_c + b_2 v_c w_c + b_1 u_c v_c \\ &- u_c B_{22} - u_c B_{33} + B_{22}B_{33}, \end{aligned}$$

$$
\begin{aligned}
P_3(\lambda_k) :=& \lambda_k^3 d_1 d_2 + \lambda_k^2(-d_1 d_2 B_{33} + d_2 u_c - d_1 B_{22}) \\
&+ \lambda_k^2(\chi\phi_u^c d_2\gamma_1 u_c w_c + \chi\phi_v^c d_1 b_2 v_c w_c + \xi b_1 u_c v_c + \chi\phi_v^c \xi\gamma_1 u_c v_c w_c) \\
&+ \lambda_k(-u_c B_{22} - d_2 u_c B_{33} + d_1 B_{22} B_{33}) \\
&+ \lambda_k \{\gamma_1 u_c w_c(\gamma_2 d_2 - \chi\phi_u^c B_{22}) + b_2 v_c w_c(d_1 + \chi\phi_v^c u_c) + b_1 u_c v_c(1 - \xi B_{33}) \\
&\quad + \gamma_1 u_c v_c w_c(\chi\phi_v^c + \xi) - \chi\phi_u^c b_1 b_2 u_c v_c w_c\} \\
&+ u_c B_{22} B_{33} - \gamma_1\gamma_2 u_c w_c B_{22} - b_1 u_c v_c B_{33} + (b_2 + \gamma_1 - \gamma_2 b_1 b_2) u_c v_c w_c. \quad (3.119)
\end{aligned}
$$

Based on Routh-Hurwitz criterion (e.g., Appendix B.1 in [100]), the nonnegative constant steady states (u_c, v_c, w_c) are linearly stable if and only if for each $k \in \mathbb{N}$, it holds that

$$
P_1 > 0, \quad P_3 > 0, \quad P_1 P_2 - P_3 > 0.
$$

Calculating directly, one obtains

$$
P_1 P_2 - P_3 =: \lambda_k^3 K_1^c + \lambda_k^2 K_2^c + \lambda_k K_3^c + K_4^c + \chi(\lambda_k^2 K_5^c + \lambda_k K_6^c),
$$

where

$$
\begin{aligned}
K_1^c :=& (d_1 d_2 + d_1 + d_2 + 1)(d_1 + d_2) > 0, \\
K_2^c :=& (d_1 d_2 + d_1) u_c + (d_1 + d_2)(-B_{33}) + (d_1 d_2 + d_2)(-B_{22}) + \xi(d_1 + d_2) b_1 u_c v_c \\
&+ (d_1 + d_2 + 1)\{(d_2 + 1) u_c - (d_1 + 1) B_{22} - (d_1 + d_2) B_{33}\}, \\
K_3^c :=& (u_c - B_{22} - B_{33})\{(d_2 + 1) u_s - (d_1 + d_2) B_{33} - (d_1 + 1) B_{22}\} \\
&+ (d_2 + 1) B_{22} B_{33} - (d_1 + 1) u_c B_{33} - (d_1 + d_2) u_s B_{22} \\
&+ [(d_1 + 1)\gamma_2 - \xi]\gamma_1 u_c w_c + (d_2 + 1) b_2 v_c w_c + (d_1 + d_2) b_1 u_c v_c + (u_c - B_{22}) b_1 \xi u_c v_c, \\
K_4^c :=& -(B_{22} + B_{33})(B_{22} B_{33} + b_2 v_s w_s) - u_c(u_c - B_{22} - B_{33})(B_{22} + B_{33}) \\
&+ (u_c - B_{33})\gamma_1\gamma_2 u_c w_c + (u_c - B_{22}) b_1 u_c v_c - (\gamma_1 - \gamma_2 b_1 b_2) u_c v_c w_c.
\end{aligned}
$$

Also

$$
K_5^c := (d_1 + 1)\phi_u^c \gamma_1 u_c w_c + (d_2 + 1)\phi_v^c b_2 v_c w_c - \phi_v^c \xi\gamma_1 u_c v_c w_c, \quad (3.120)
$$

and

$$
K_6^c := (u_c - B_{33})\phi_u^c \gamma_1 u_c w_c + (-B_{22} - B_{33})\phi_v^c b_2 v_c w_c + \phi_u^c b_1 b_2 u_c v_c w_c - \phi_v^c \gamma_1 u_c v_c w_c. \quad (3.121)
$$

Proof of Proposition 3.1. For the corresponding ODE system of (3.3), it has been proved in [55] that the constant steady state (u_c, v_c, w_c) is linearly stable under the following conditions:

$$(u_c, v_c, w_c) = \begin{cases} (1,0,0), & \text{if } \theta_1 > 1 \text{ and } \theta_2 > \gamma_2, \\ \left(\theta_1, \frac{1-\theta_1}{b_1}, 0\right), & \text{if } \theta_1 < 1 \text{ and } \theta_2 > \frac{b_1\gamma_2 - 1}{b_1}\theta_1 + \frac{1}{b_1}, \\ \left(\frac{\theta_2}{\gamma_2}, 0, \frac{\gamma_2-\theta_2}{\gamma_1\gamma_2}\right), & \text{if } \theta_2 < \gamma_2 \text{ and } \theta_2 < \frac{\gamma_1\gamma_2}{b_2+\gamma_1}\theta_1 + \frac{b_2\gamma_2}{b_2+\gamma_1}. \end{cases} \tag{3.122}$$

Under the conditions (3.122), we can derive from (3.118) that $B_{22} \leq 0$ and $B_{33} \leq 0$, which gives $K_j^c > 0$ $(j = 1, 2, 3, 4)$.

For $(1, 0, 0)$ or $\left(\theta_1, \frac{1-\theta_1}{b_1}, 0\right)$, one obtains $w_c = 0$, which together with the facts $B_{22} \leq 0$ and $B_{33} \leq 0$ substituted into P_3 in (3.119) implies that for any $k \in \mathbb{N}$

$$\begin{aligned} P_3 =& \lambda_k^3 d_1 d_2 + \lambda_k^2(-d_1 d_2 B_{33} + d_2 u_c - d_1 B_{22} + \xi b_1 u_c v_c) + u_c B_{22} B_{33} - b_1 u_c v_c B_{33} \\ &+ \lambda_k[-u_c B_{22} - d_2 u_c B_{33} + d_1 B_{22} B_{33} + b_1 u_c v_c(1 - \xi B_{33})] > 0. \end{aligned}$$

Since $w_c = 0$, by (3.120)-(3.121), one has $K_5^c = K_6^c = 0$, which together with $K_i^c > 0$ $(i = 1, 2, 3, 4)$ implies $P_1 P_2 - P_3 > 0$. Hence, by Routh-Hurwitz criterion, E_1 and E_{12} are linearly stable.

As for $E_{13} := \left(\frac{\theta_2}{\gamma_2}, 0, \frac{\gamma_2-\theta_2}{\gamma_1\gamma_2}\right)$, one has $v_c = 0$ which together with $\phi_u \geq 0$, gives $K_5^c = (d_1 + 1)\gamma_1 \chi \phi_u^c u_c w_c \geq 0$ and $K_6^c = (u_c - B_{33})\chi\phi_u^c \gamma_1 u_c w_c \geq 0$. Using $K_i^c > 0$ $(j = 1, 2, 3, 4)$ again, one obtains $P_1 P_2 - P_3 > 0$ for each $k \in \mathbb{N}$. On the other hand, noting the facts $B_{22} \leq 0, B_{33} \leq 0$, $v_c = 0$ and $\phi_u^c \geq 0$, $\phi_v^c \geq 0$, from (3.119), we get that

$$\begin{aligned} 0 < P_3 :=& \lambda_k^3 d_1 d_2 + \lambda_k^2(-d_1 d_2 B_{33} + d_2 u_c - d_1 B_{22} + \chi\phi_u^c d_2 \gamma_1 u_c w_c) \\ &+ \lambda_k\{-u_c B_{22} - d_2 u_c B_{33} + d_1 B_{22} B_{33} + (\gamma_2 d_2 - \chi\phi_u^c B_{22})\gamma_1 u_c w_c\} \\ &+ u_c B_{22} B_{33} - \gamma_1\gamma_2 u_c w_c B_{22}. \end{aligned}$$

Therefore, $\left(\frac{\theta_2}{\gamma_2}, 0, \frac{\gamma_2-\theta_2}{\gamma_1\gamma_2}\right)$ is linearly stable by applying Routh-Hurwitz criterion. Then we complete the proof of Proposition 3.1. □

Chapter 4

Global Dynamics of an SIS Epidemic Model with Cross-diffusion: Applications to Quarantine Measures

4.1 Introduction and Main Results

Before presenting our context, we clarify that the results presented in this chapter have been published in our journal paper [25].

4.1.1 Introduction

Infectious diseases [10, 13, 60] have brought in tremendous impacts on public health and the global economy such as the unprecedented novel coronavirus disease 2019 (COVID-19). Mathematical modelings and analysis of infectious diseases have had a long history and numerous results are available (cf. [33, 51, 97]). In epidemiology, the basic reproduction number of an infection, denoted by R_0, is the expected number of cases directly generated by one case in a population where all individuals are susceptible to infection. This number is the threshold determining if an emerging infectious disease can spread in a population. Specifically, the infection persists if $R_0 > 1$ while becomes extinct in the long run if $R_0 < 1$. Generally, the larger the value of R_0, the harder it is to control the epidemic. It is therefore of mathematical and biological importance to properly define and give explicit estimates of R_0 (cf. [50, 136]). It is noteworthy that the value of R_0 can vary, even for the same disease strain, depending on external factors such as environmental conditions, public health policy governing the detection and movement pattern of the infected population, and so on. Among a large number of mathematical works based on reaction-diffusion (or with advection) models (cf. [4, 82, 99, 144] and reference therein), most (if not all) mathematical models have assumed that both susceptible and infected individuals employ homogeneous diffusive movements. However, this assumption leaves out the effects of human behaviors and public health quarantine measures on the mobility of individuals during the outbreak of disease such as COVID-19 [60, 74, 131]).

To fill this gap, we shall introduce the cross-diffusion for the infected individuals (i.e., the diffusion of the infected individuals depend on the density of the susceptible population) into the SIS model and explore the effect of the human intervention on the propagation of infectious diseases, particularly on the basic reproduction number R_0. There are many SIS epidemic models, we choose, among others, the SIS model with frequency-dependent transmission mechanism (cf. [30]) and demographic change (i.e., population growth/recruitment and death). That is, denoting the population density of the susceptible and infected individuals at position $x \in \Omega \subset \mathbb{R}^n$ and time $t > 0$ by $S(x,t)$ and $I(x,t)$, respectively, we consider the following SIS model with cross-diffusion on I:

$$\begin{cases} S_t = d_S \Delta S + \Lambda(x) - \theta S - \alpha(x)\frac{SI}{S+I} + \beta(x) I, & x \in \Omega, t > 0, \\ I_t = d_I \Delta[\gamma(S) I] + \alpha(x)\frac{SI}{S+I} - [\beta(x) + \eta(x)] I, & x \in \Omega, t > 0, \\ \partial_\nu S = \partial_\nu I = 0, & x \in \partial\Omega, t > 0. \end{cases} \tag{4.1}$$

All the other notations and parameters have the same interpretation as in Section 1.4.

Relevant results on (4.1) **with** $\gamma(S) = 1$.We recall some related results developed for the SIS model (4.1) with $\gamma(S) = 1$. When the demographic changes are not considered (i.e., $\Lambda(x) = \theta = \eta(x) = 0$), by integrating the sum of the two equations of (4.1), one immediately finds that the total population is conserved, namely

$$\int_\Omega [S(x,t) + I(x,t)] dx = \int_\Omega (S_0 + I_0) dx =: N, \quad \forall\ t > 0,$$

where the constant $N > 0$ denotes the number of total population. For this case, Allen et al. [4] first introduced the basic reproduction number R_0 via a variational formula and established the existence, uniqueness and global stability of the *disease-free equilibrium* (DFE) if $R_0 < 1$. When $R_0 > 1$, they proved the existence and uniqueness of the *endemic equilibrium* (EE), and explored the asymptotic behavior of the unique EE as $d_S \to 0$. Particularly, they conjectured that this unique EE is globally stable, which was later confirmed by [104] for the cases of $d_I = d_S$ or $\alpha(x) = r\beta(x)$ with constant $r > 1$. The results in [104] imply that the disease will persist in the high-risk domain Ω (namely $\int_\Omega \alpha(x)dx > \int_\Omega \beta(x)dx$). When α and β are temporally and spatially inhomogeneous, Peng and Zhao [107] showed that the disease will persist in the high-risk domain Ω, and the joint effect of spatial heterogeneity and temporal periodicity may enhance the persistence of the disease. In addition, [34] explored the existence of traveling wave solutions.

When the demographic changes are included (i.e., $\Lambda(x),\ \theta,\ \eta(x) > 0$), the total population is no longer conserved and the analysis will be more involved. The first result seemed to be obtained by Li et al. in [81] where the global existence and boundedness of classical solutions as well as the threshold dynamics in terms of the basic reproduction number R_0 were studied. By the uniform persistence theory, they showed that the disease will persist uniformly and hence at least one EE exists in the high-risk domain. The asymptotic profiles of EE for large and/or small of d_S or d_I were further obtained in [81]. These findings imply that a varying total population may enhance disease persistence, thereby posing greater challenges to the disease control. In addition, Li et al. [83] introduced an infectious population oriented taxis advection term for S (i.e., the susceptible moves away from the density gradient of the infected individuals) with varying/conserved total population and showed that such a cross-diffusion does not contribute to eradication of the disease. Last but not least, we refer readers to [84, 126, 127] for some results on SIS models with taxis movement in the S-equation, and [19, 31, 82, 105, 146, 153] and the references therein for more results on various SIS epidemic models with random diffusion.

We aim to study the SIS epidemic model (4.1) with cross-diffusion for I and explore how the cross-diffusion diffusion strategy can play positive roles in controlling the spread of disease. Our main goals include the following:

(S.1) Establish the global well-posedness of solutions (global existence and stability) to (4.1) under suitable conditions;

(S.2) Investigate the effects of cross-diffusion on the persistence and extinction of the infectious disease.

The main challenge in the analyses arises from the cross-diffusion structure in the I-equation. The SIS model with taxis-like advection in the S-equation considered in [83] is significantly different from (4.1) with the cross-diffusion in the I-equation. For the model of [83], the L^∞ boundedness of I can be directly obtained from the I-equation based on the boundedness of L^1 by using the result "L^1-boundedness implies L^∞-boundedness" for classical reaction-diffusion equations proved in [3]. But for the cross-diffusion SIS system (4.1), the boundedness of I can not be obtained directly from the I-equation alone. This needs more complicated coupling estimates to establish the global boundedness of solutions under the structural hypothesis (H2), as shown in Section 4.2.

4.1.2 Main Results

Throughout this chapter, we suppose that the initial value (S_0, I_0) satisfies

$$0 \le S_0 \in W^{1,\infty}(\Omega),\ I_0 \in C(\overline{\Omega}) \text{ with } I_0 \ge 0 \text{ and } \int_\Omega I_0(x)dx > 0, \tag{4.2}$$

and the following conditions hold:

(H0) The functions $\Lambda(x),\ \alpha(x),\ \beta(x),\ \eta(x)$ are positive and Hölder continuous on $\overline{\Omega}$, and θ is a positive constant.

Moreover, $\gamma(S)$ is assumed to fulfill the following conditions:

(H1) $\gamma(S) \in C^3([0,\infty))$, $\gamma'(S) > 0$ and $\gamma(0) = 1$;

(H2) There exist some positive constants $\mathcal{K}_0$ and $\mathcal{K}_1$ such that $\gamma(S) \le \mathcal{K}_0$ and $\gamma'(S) \le \mathcal{K}_1$.

Note that in (H1), $\gamma(0)$ can be any positive constant, which however can be absorbed into d_I. Hence, we simply assume $\gamma(0) = 1$ without loss of generality.

Our first result concerning the global boundedness of solutions is given below.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and hypotheses (H0)-(H2) hold. Then (4.1) with (4.2) admits a unique classical solution $(S, I) \in [C(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty))]^2$ satisfying $S, I > 0$ on $\overline{\Omega} \times (0,\infty)$. Moreover, there exists a constant $C > 0$ independent of $\mathcal{K}_1$ such that*

$$\|S\|_{W^{1,\infty}} + \|I\|_{L^\infty} \le C(1 + \mathcal{K}_1^{12})e^{C(1+\mathcal{K}_1^4)} =: M(\mathcal{K}_1) \text{ for all } t > 0. \tag{4.3}$$

Remark 4.1. *When considering the mass action infection mechanism (cf. [70]), namely $\frac{SI}{S+I}$ is replaced by SI in (4.1), Theorem 4.1 can hold without the assumption (H2) since the boundedness of S can follow directly from the comparison principle.*

Next we shall explore how the cross-diffusion affects the basic reproduction number R_0. To this end, we consider the stationary problem

$$\begin{cases} d_S\Delta S + \Lambda(x) - \theta S - \alpha(x)\frac{SI}{S+I} + \beta(x)I = 0, & x \in \Omega, \\ d_I\Delta[\gamma(S)I] + \alpha(x)\frac{SI}{S+I} - [\beta(x) + \eta(x)]I = 0, & x \in \Omega, \\ \partial_\nu S = \partial_\nu I = 0, & x \in \partial\Omega. \end{cases} \tag{4.4}$$

It is easy to check that (4.4) has a unique semi-trivial solution $(\widetilde{S}(x), 0) =: (\widetilde{S}, 0)$ satisfying $0 < \widetilde{S} \le \frac{1}{\theta} \max_{x\in\overline{\Omega}} \Lambda$ and

$$d_S \Delta S + \Lambda(x) - \theta S = 0 \text{ in } \Omega;\ \partial_\nu S = 0 \text{ on } \partial\Omega.$$

$(\widetilde{S}, 0)$ is called the *disease-free equilibrium* (DFE). An *endemic equilibrium* (EE), denoted by $(\hat{S}(x), \hat{I}(x))$, is a solution of (4.4) satisfying $\hat{I}(x) \ge 0$ and $\hat{I}(x) \not\equiv 0$ on Ω. In fact, if EE exists, then the maximum principle and the Hopf boundary lemma for elliptic equations assert that $\hat{S}(x) > 0, \hat{I}(x) > 0$ in $\overline{\Omega}$. By the nomenclature from [4], we define the low-risk site Ω^- and the high-risk site Ω^+ as:

$$\Omega^- = \{x \in \Omega :\ \alpha(x) < \beta(x) + \eta(x)\}, \quad \Omega^+ = \{x \in \Omega :\ \alpha(x) > \beta(x) + \eta(x)\}.$$

The domain Ω is called a low-risk domain if $\int_\Omega \alpha(x)dx < \int_\Omega [\beta(x) + \eta(x)]dx$ and a high-risk domain if $\int_\Omega \alpha(x)dx \ge \int_\Omega [\beta(x) + \eta(x)]dx$.

Now we define the basic reproduction number R_0 of (4.1) by the following variational form (see the motivation detailed in Section 4.3.1):

$$R_0 := R_0(d_I, \gamma(\widetilde{S})) = \sup_{0\ne w\in H^1(\Omega)} \frac{\int_\Omega \alpha(x) w^2 dx}{\int_\Omega \{d_I |\nabla(\gamma^{\frac{1}{2}}(\widetilde{S})w)|^2 + (\beta(x) + \eta(x))w^2\}dx}. \tag{4.5}$$

When the infected individuals take random movement (i.e., $\gamma(S) = 1$), we denote the basic reproduction number by $\hat{R}_0$ given in [81]. Below we present some qualitative properties of R_0 in terms of d_I, which can be readily proved by the proofs of [4, Lemma 2.2] and [92, Lemma 3.1]. We skip the details here for brevity.

Proposition 4.1. *Let $q_1(x) := \alpha(x)\gamma^{-1}(\widetilde{S})$, $q_2(x) := [\beta(x)+\eta(x)]\gamma^{-1}(\widetilde{S})$ and $q(x) := \frac{q_1(x)}{q_2(x)}$ with $\gamma^{-1}(\widetilde{S}) = 1/\gamma(\widetilde{S})$. Under hypotheses (H0)-(H1), the following results hold.*

(i) R_0 is strictly decreasing in d_I provided that Ω^- and Ω^+ are nonempty. Moreover, $R_0 \to \max\{q(x) :\ x \in \overline{\Omega}\}$ as $d_I \to 0$ and $R_0 \to \int_\Omega q_1(x)dx / \int_\Omega q_2(x)dx$ as $d_I \to \infty$;

(ii) If $\int_\Omega q_1(x)dx > \int_\Omega q_2(x)dx$, then $R_0 > 1$ for all $d_I > 0$;

(iii) If $\int_\Omega q_1(x)dx < \int_\Omega q_2(x)dx$, then there admits a unique positive constant d_I^* such that $R_0 > 1$ (resp. $R_0 < 1$) for $d_I < d_I^*$ (reps. $d_I > d_I^*$) when Ω^- and Ω^+ are nonempty.

Remark 4.2. *If Ω^- and Ω^+ are nonempty and $\Lambda(x)$ is a constant, then $\widetilde{S} > 0$ is a constant. This along with the monotonicity of R_0 in Proposition 4.1-(i) yields $R_0 < \hat{R}_0$, and hence* *implies that the cross-diffusion can reduce the value of R_0. In other words, the intervention measures are effective for controlling the spread of diseases (see more discussion in Section 4.5).*

Remark 4.3. *If Ω is a high-risk domain, namely $\int_\Omega \alpha(x)dx \geq \int_\Omega [\beta(x) + \eta(x)]dx$, we can choose a rate function $\gamma(S)$ such that $\int_\Omega \alpha(x)\gamma^{-1}(\widetilde{S})dx < \int_\Omega [\beta(x) + \eta(x)]\gamma^{-1}(\widetilde{S})dx$ (see a specific example in Section 4.5). By Proposition 4.1-(iii), there exists a unique d_I^* such that $R_0 < 1$ whenever $d_I > d_I^*$, which is substantially different from the well-known results with random diffusion (i.e., $\gamma(S) = 1$) for which the basic reproduction number $\hat{R}_0 > 1$ for all $d_I > 0$ (e.g., [81, Proposition 3.2 (c)], [107, Theorem 2.5 (a)]).*

The basic reproduction number R_0 normally can determine threshold dynamics. Specifically, if $R_0 > 1$ (resp. $R_0 < 1$), the disease persists (resp. becomes extinct). The following theorem indicates that R_0 defined in (4.5) can determine the threshold dynamics locally.

Theorem 4.2. *Let hypotheses (H0)-(H2) hold. Then the following statements hold.*

(i) If $R_0 < 1$, then DFE $(\widetilde{S}, 0)$ is linearly stable;

(ii) If $R_0 > 1$, then DFE $(\widetilde{S}, 0)$ is linearly unstable and (4.1) admits at least one EE.

Remark 4.4. *The uniqueness of non-trivial EE in general and the existence of non-trivial EE when $R_0 \leq 1$ remain open.*

Finally we prove the global stability of DFE and EE depending on the sign of $1 - R_0$.

Theorem 4.3. *Let (S, I) be the solution obtained in Theorem 4.1. The following statements hold.*

(i) If $\alpha(x) \leq \beta(x) + \varepsilon\eta(x)$ with fixed constant $0 < \varepsilon < 1$, then $R_0 < 1$ and DFE is globally asymptotically stable with

$$\|S - \widetilde{S}(x)\|_{L^\infty} + \|I\|_{L^\infty} \leq M_1 e^{-\kappa_1 t} \text{ for all } t > 1. \tag{4.6}$$

(ii) Assume that Λ, α, β, η are all positive constants. If $\alpha > \beta + \eta$ (i.e., $R_0 > 1$), then the unique constant EE $(\hat{S}, \hat{I})$ defined in (4.72) is globally asymptotically stable provided that $\theta = \eta$ and

$$2d_S d_I + 4d_S d_I M_2 > d_I^2 \mathcal{K}_0^2 + d_S^2 + d_I \mathcal{K}_1 H(\mathcal{K}_1), \tag{4.7}$$

where $H(\mathcal{K}_1) = M(\mathcal{K}_1)[1+M_2(M(\mathcal{K}_1)+1)^2]\{2d_S+\mathcal{K}_1 M(\mathcal{K}_1)d_I[1+M_2(M(\mathcal{K}_1)+1)^2]\}$ with $M_2 := \frac{4\eta[\Lambda(\alpha-\beta-\eta)+\eta\alpha]}{(\alpha-\beta-\eta)^2(\Lambda+\eta)}$.

Remark 4.5. *If $\gamma(S) = 1$, the global stability of DEF can be proved with the mere condition $R_0 < 1$ (see [81]). Here we give a more sufficient condition. If Λ, α, β, η are positive constant, it follows from Proposition 4.1-(i) that $\mathcal{R}_0 = \frac{\alpha}{\beta+\eta}$. Thus, $\alpha > \beta + \eta$ is equivalent to $\mathcal{R}_0 > 1$. In addition, $\theta = \eta$ is a technical assumption, which is not needed in the case $\gamma(S) = 1$ and $d_I = d_S$ (see [81]).*

Remark 4.6. *Since $M(\mathcal{K}_1) > 0$ is an increasing function of $\mathcal{K}_1$ and M_2 is independent of $\mathcal{K}_1$, the condition (4.7) can be achieved by choosing $\mathcal{K}_1$ small. For example, fixing $\Lambda = \eta = \theta = d_S = 1$, $\alpha = d_I = 2.5$, $\beta = 0.5$ and taking $\gamma(S) = \mathcal{K}_0 - \frac{\mathcal{K}_0 - 1}{S+1}$ with $1 < \mathcal{K}_0 \ll 2$, then $\gamma'(S) = \frac{\mathcal{K}_0-1}{(S+1)^2} \le (\mathcal{K}_0 - 1) =: \mathcal{K}_1$. Let $\mathcal{K}_0$ be close to 1 (i.e., $\mathcal{K}_1$ be close to 0) such that $d_I\mathcal{K}_1 H(\mathcal{K}_1) \ll 1$, then $2d_S d_I + 4d_S d_I M_2 = 75 > d_I^2 2^2 + d_S^2 + 1 = 27 > d_I^2\mathcal{K}_0^2 + d_S^2 + d_I\mathcal{K}_1 H(\mathcal{K}_1)$, and thus (4.7) holds.*

4.2 Global Boundedness and Existence: Proof of Theorem 4.1

In this section, we will study the global existence and boundedness of solutions to (4.1). Throughout this chapter, c_i and C_i $(i = 1, 2, 3, \cdots)$ are used to denote generic positive constants, which may vary in the context and are independent of t and $\mathcal{K}_1$.

4.2.1 Local Existence and Preliminaries

Firstly, the local solvability of (4.1) can be proved by using the Amann's theorem [7, Theorem 7.3], and the positivity of S and I follows from the strong maximum principle, see e.g. [65, Lemma 2.1]. We omit the proof details for brevity..

Lemma 4.1 (Local existence)**.** *Let the conditions in Theorem 4.1 hold. Then there admits a $T_{\max} \in (0, \infty]$ such that (4.1) has a unique classical solution $(S, I) \in [C(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))]^2$ with $S, I > 0$ on $\overline{\Omega} \times (0, T_{\max})$. Moreover,*

$$\text{if } T_{\max} < \infty, \text{ then } \lim_{t \nearrow T_{\max}} (\|S\|_{W^{1,\infty}} + \|I\|_{L^\infty}) = \infty. \tag{4.8}$$

In the sequel, we denote

$$g_* = \min_{x \in \overline{\Omega}} g \text{ and } g^* = \max_{x \in \overline{\Omega}} g \text{ for } g \in \{\Lambda(x), \alpha(x), \beta(x), \eta(x)\}. \tag{4.9}$$

Lemma 4.2. *Let (S, I) be the solution obtained in Lemma 4.1. Then there exists a constant $C_1 > 0$ such that*

$$\|S(\cdot,t)\|_{L^1} + \|I(\cdot,t)\|_{L^1} \leq C_1 \quad \text{for all } t \in (0, T_{\max}).$$

Proof. Adding the first two equations of (4.1) and integrating the result by parts, we get

$$\frac{d}{dt}\int_\Omega (S+I) + \min\{\theta, \eta_*\}\int_\Omega (S+I) \leq \Lambda^*|\Omega|.$$

This along with Grönwall's inequality indicates

$$\int_\Omega (S+I) \leq \frac{\Lambda^*|\Omega|}{\min\{\theta,\eta_*\}} + \int_\Omega (S_0 + I_0) =: C_1,$$

where η_* and Λ^* are defined in (4.9). Hence, the proof of Lemma 4.2 is completed. □

Lemma 4.3. *Let (S, I) be the solution obtained in Lemma 4.1. Then there exists a constant $C_2 > 0$ such that*

$$\int_t^{t+\tau}\int_\Omega I^2 \leq C_2 \quad \text{for all } t \in (0, \widetilde{T}_{\max}), \tag{4.10}$$

where τ is a constant such that

$$0 < \tau < \min\{1, T_{\max}\} \quad \text{and } \widetilde{T}_{\max} := T_{\max} - \tau. \tag{4.11}$$

Proof. We add the first equation of (4.1) with the second one to get

$$(S+I)_t = \Delta\left(d_S S + d_I\gamma(S)I\right) + \Lambda(x) - \theta S - \eta(x)I,$$

which, along with hypothesis (H2), can be rewritten as

$$\begin{aligned}(S+I)_t + \mathcal{A}\left(d_S S + d_I\gamma(S)I\right) &= (\delta d_I\gamma(S) - \eta(x))I + (\delta d_S - \theta)S + \Lambda(x)\\ &\leq (\delta d_I\mathcal{K}_0 - \eta_*)I + (\delta d_S - \theta)S + \Lambda^* \leq \Lambda^*,\end{aligned} \tag{4.12}$$

where $\delta := \min\left\{\frac{\eta_*}{d_I\mathcal{K}_0}, \frac{\theta}{d_S}\right\} > 0$, and $\mathcal{A}$ is the self-adjoint realisation of $-\Delta + \delta$ subject to homogeneous Neumann boundary conditions in $L^2(\Omega)$. Then $\mathcal{A}$ is invertible with bounded inverse by the Fredholm alternative theorem. Hence there is a constant $c_1 > 0$ such that

$$\|\mathcal{A}^{-1}\phi\|_{L^2} \leq c_1\|\phi\|_{L^2} \text{ for all } \phi \in L^2(\Omega), \tag{4.13}$$

and

$$\|\mathcal{A}^{-\frac{1}{2}}\phi\|_{L^2}^2 = \int_\Omega \phi\cdot\mathcal{A}^{-1}\phi dx \leq c_1\|\phi\|_{L^2}^2 \text{ for all } \phi \in L^2(\Omega). \tag{4.14}$$

We multiply (4.12) by $\mathcal{A}^{-1}(S+I) \geq 0$ to get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathcal{A}^{-\frac{1}{2}}(S+I)|^2 + \int_\Omega (d_S S + d_I\gamma(S)I)(S+I) \leq \Lambda^* \int_\Omega \mathcal{A}^{-1}(S+I),$$

which together with hypothesis (H1) gives a constant $c_2 := \min\{d_S, d_I\}$ such that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathcal{A}^{-\frac{1}{2}}(S+I)|^2 + c_2\int_\Omega (S+I)^2 \leq \Lambda^* \int_\Omega \mathcal{A}^{-1}(S+I). \tag{4.15}$$

Using (4.13), (4.14) along with Hölder inequality and Young's inequality yields

$$\Lambda^* \int_\Omega \mathcal{A}^{-1}(S+I) \leq \Lambda^* |\Omega|^{\frac{1}{2}} c_1 \|S+I\|_{L^2} \leq \frac{c_2}{4}\|S+I\|_{L^2}^2 + \frac{(\Lambda^*)^2|\Omega|c_1^2}{c_2}, \tag{4.16}$$

and

$$\frac{c_2}{4c_1}\int_\Omega |\mathcal{A}^{-\frac{1}{2}}(S+I)|^2 \leq \frac{c_2}{4}\|S+I\|_{L^2}^2. \tag{4.17}$$

We substitute (4.16) and (4.17) into (4.15) to obtain

$$\frac{d}{dt}\int_\Omega |\mathcal{A}^{-\frac{1}{2}}(S+I)|^2 + \frac{c_2}{2c_1}\int_\Omega |\mathcal{A}^{-\frac{1}{2}}(S+I)|^2 + c_2\int_\Omega (S+I)^2 \leq \frac{2(\Lambda^*)^2|\Omega|c_1^2}{c_2} =: c_3. \tag{4.18}$$

Applying Grönwall's inequality to (4.18) and using (4.14) again, one has

$$\int_\Omega |\mathcal{A}^{-\frac{1}{2}}(S+I)|^2 \leq \frac{2c_1c_3}{c_2} + c_1(\|S_0\|_{L^2}^2 + \|I_0\|_{L^2}^2) =: c_4. \tag{4.19}$$

We integrate (4.18) over $(t, t+\tau)$ and apply (4.19) to get

$$c_2\int_t^{t+\tau}\int_\Omega I^2 \leq c_2\int_t^{t+\tau}\int_\Omega (S+I)^2 \leq c_3 + c_4,$$

which gives (4.10) by letting $C_2 := \frac{c_3+c_4}{c_2}$. This finishes the proof of Lemma 4.3. □

Lemma 4.4. *Let (S, I) be the solution obtained in Lemma 4.1. Then there exist two positive constants C_3 and C_4 such that*

$$\|\nabla S(\cdot,t)\|_{L^2} \leq C_3 \quad \text{for all } t \in (0, T_{\max}), \tag{4.20}$$

and

$$\int_t^{t+\tau}\int_\Omega |\Delta S|^2 \leq C_4 \quad \text{for all } t \in (0, \widetilde{T}_{\max}), \tag{4.21}$$

where τ and $\widetilde{T}_{\max}$ are defined in (4.11).

Proof. We multiply the first equation of (4.1) by $-\Delta S$ and apply Young's inequality to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla S|^2 + d_S\int_\Omega |\Delta S|^2 \\
&= -\int_\Omega \Lambda(x)\Delta S + \theta\int_\Omega S\Delta S + \int_\Omega \Big(\alpha(x)\frac{SI}{S+I} - \beta(x)I\Big)\Delta S \\
&\le \Lambda^*\int_\Omega |\Delta S| - \theta\int_\Omega |\nabla S|^2 + (\alpha^*+\beta^*)\int_\Omega I|\Delta S| \\
&\le \frac{d_S}{2}\int_\Omega |\Delta S|^2 - \theta\int_\Omega |\nabla S|^2 + \frac{(\alpha^*+\beta^*)^2}{d_S}\int_\Omega I^2 + \frac{(\Lambda^*)^2|\Omega|}{d_S},
\end{aligned}$$

which indicates

$$\frac{d}{dt}\int_\Omega |\nabla S|^2 + 2\theta\int_\Omega |\nabla S|^2 + d_S\int_\Omega |\Delta S|^2 \le c_1\int_\Omega I^2 + c_2 =: h(t), \tag{4.22}$$

where $c_1 = \frac{2(\alpha^*+\beta^*)^2}{d_S}$ and $c_2 = \frac{2(\Lambda^*)^2|\Omega|}{d_S}$. Moreover, it follows from (4.10) that $\int_t^{t+\tau} h(s)ds \le c_1C_2 + c_2 =: c_3$. This along with [61, Lemma 2.4] gives

$$\int_\Omega |\nabla S|^2 \le c_3 + 2(||\nabla S_0||_{L^2}^2 + 3c_3 + 6\theta\tau + c_3/2\theta\tau + 1) =: C_3^2,$$

which implies (4.20) directly. Integrating (4.22) over $(t, t+\tau)$ yields

$$d_S\int_t^{t+\tau}\int_\Omega |\Delta S|^2 \le c_3 + C_3^2.$$

Thus (4.21) holds with $C_4 := (c_3 + C_3^2)/d_S$ and the proof of Lemma 4.4 is finished. □

4.2.2 Boundedness of solutions

We first derive the *a priori* L^2-estimate of I.

Lemma 4.5. *Let (S, I) be the solution obtained in Lemma 4.1. Then there exists a constant $C_5 > 0$ such that*

$$\|I(\cdot,t)\|_{L^2} \le e^{C_5(\mathcal{K}_1^2+1)^2} \quad \textit{for all } t \in (0, T_{\max}). \tag{4.23}$$

Proof. Multiplying I-equation in (4.1) by I and integrating the result by parts, one has

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega I^2 &= -d_I\int_\Omega \gamma(S)|\nabla I|^2 - d_I\int_\Omega I\gamma'(S)\nabla S\cdot\nabla I + \int_\Omega \frac{\alpha(x)SI^2}{S+I} - \int_\Omega [\beta+\eta](x)I^2 \\
&\le -d_I\int_\Omega \gamma(S)|\nabla I|^2 + d_I\int_\Omega \gamma'(S)I|\nabla S||\nabla I| + \alpha^*\int_\Omega I^2 - (\beta_*+\eta_*)\int_\Omega I^2,
\end{aligned}$$

which, along with hypotheses (H1) and (H2), gives

$$\frac{d}{dt}\int_\Omega I^2 + 2d_I\int_\Omega |\nabla I|^2 + 2(\beta_* + \eta_*)\int_\Omega I^2 \le 2d_I\mathcal{K}_1\int_\Omega I|\nabla S||\nabla I| + 2\alpha^*\int_\Omega I^2. \tag{4.24}$$

With Young's inequality and Hölder inequality, we have

$$2d_I\mathcal{K}_1\int_\Omega I|\nabla S||\nabla I| \le d_I\int_\Omega |\nabla I|^2 + d_I\mathcal{K}_1^2\|I\|_{L^4}^2\|\nabla S\|_{L^4}^2,$$

which, substituted into (4.24), gives

$$\frac{d}{dt}\int_\Omega I^2 + d_I\int_\Omega |\nabla I|^2 + 2(\beta_* + \eta_*)\int_\Omega I^2 \le d_I\mathcal{K}_1^2\|I\|_{L^4}^2\|\nabla S\|_{L^4}^2 + 2\alpha^*\|I\|_{L^2}^2. \tag{4.25}$$

On the other hand, we use Gagliardo-Nirenberg inequality in two dimensions to get

$$\|I\|_{L^4}^2 \le c_1\left(\|\nabla I\|_{L^2}\|I\|_{L^2} + \|I\|_{L^2}^2\right), \tag{4.26}$$

and the estimate (cf. [63, Lemma 2.5])

$$\|\nabla S\|_{L^4}^2 \le c_2\left(\|\Delta S\|_{L^2}\|\nabla S\|_{L^2} + \|\nabla S\|_{L^2}^2\right) \le c_2C_3\left(\|\Delta S\|_{L^2} + C_3\right), \tag{4.27}$$

where we have used (4.20). The combination of (4.26) with (4.27) yields

$$\begin{aligned} d_I\mathcal{K}_1^2\|I\|_{L^4}^2\|\nabla S\|_{L^4}^2 \le& d_I\mathcal{K}_1^2c_1c_2C_3\left(\|\nabla I\|_{L^2}\|I\|_{L^2} + \|I\|_{L^2}^2\right)\left(\|\Delta S\|_{L^2} + C_3\right) \\ \le& d_I\mathcal{K}_1^2c_1c_2C_3\|\nabla I\|_{L^2}\|I\|_{L^2}\|\Delta S\|_{L^2} + d_I\mathcal{K}_1^2c_1c_2C_3^2\|\nabla I\|_{L^2}\|I\|_{L^2} \\ &+ d_I\mathcal{K}_1^2c_1c_2C_3\|I\|_{L^2}^2\|\Delta S\|_{L^2} + d_I\mathcal{K}_1^2c_1c_2C_3^2\|I\|_{L^2}^2 \\ \le& d_I\|\nabla I\|_{L^2}^2 + c_3\mathcal{K}_1^4\|I\|_{L^2}^2\|\Delta S\|_{L^2}^2 + c_4(1+\mathcal{K}_1^2)^2\|I\|_{L^2}^2 \end{aligned} \tag{4.28}$$

with $c_3 := d_Ic_1^2c_2^2C_3^2$ and $c_4 := \frac{d_I(1+c_1c_2C_3^2)^2}{2}$. Substituting (4.28) into (4.25) gives a constant $c_5 := c_4 + 2\alpha^*$ such that

$$\frac{d}{dt}\|I\|_{L^2}^2 \le [c_3\mathcal{K}_1^4\|\Delta S\|_{L^2}^2 + c_5(1+\mathcal{K}_1^2)^2]\|I\|_{L^2}^2. \tag{4.29}$$

Furthermore, (4.10) motivates us to find a positive constant $t_1 \in [(t-\tau)_+, t)$ for any $t \in (0, T_{\max})$ such that

$$\|I(\cdot, t_1)\|_{L^2}^2 \le \max\{\|I_0\|_{L^2}^2, C_2/\tau\} =: c_6. \tag{4.30}$$

It then follows from (4.21) that

$$\int_{t_1}^{t_1+\tau}\int_\Omega |\Delta S|^2 \le C_4. \tag{4.31}$$

Noting $t_1 < t \le t_1 + \tau \le t_1 + 1$, (4.30) and (4.31), we integrate (4.29) over (t_1, t) and get

$$\|I(\cdot,t)\|_{L^2}^2 \le \|I(\cdot,t_1)\|_{L^2}^2 \, e^{c_3\mathcal{K}_1^4 \int_{t_1}^t \|\Delta S(\cdot,s)\|_{L^2}^2 ds + c_5(1+\mathcal{K}_1^2)^2}$$
$$\le c_6 e^{c_3\mathcal{K}_1^4 C_4 + c_5(1+\mathcal{K}_1^2)^2} \le e^{(c_6+c_3C_4+c_5)(1+\mathcal{K}_1^2)^2}.$$

Hence (4.23) follows by letting $C_5 := (c_6 + c_3 C_4 + c_5)/2$ and the proof is finished. □

Lemma 4.6. *Let (S, I) be the solution obtained in Lemma 4.1. Then there exists a constant $C_6 > 0$ such that*

$$\|\nabla S(\cdot,t)\|_{L^4} \le e^{C_6(1+\mathcal{K}_1^2)^2} \text{ for all } t \in (0, T_{\max}). \tag{4.32}$$

Proof. We rewrite the first equation of (4.1) as

$$S_t - d_S\Delta S + \theta S = \Lambda(x) - \alpha(x)\frac{SI}{S+I} + \beta(x)I =: H(x,t). \tag{4.33}$$

Applying (4.23) gives

$$\|H(\cdot,t)\|_{L^2} \le \|\Lambda^* + \alpha^* I + \beta^* I\|_{L^2} \le c_1 e^{C_5(1+\mathcal{K}_1^2)^2}, \tag{4.34}$$

where $c_1 = \Lambda^*|\Omega|^{\frac{1}{2}} + (\alpha^* + \beta^*)$. By $(e^{t\Delta})_{t>0}$ we denote the Neumann heat semigroup in Ω. Then applying Duhamel's principle to (4.33) yields that

$$S(\cdot,t) = e^{t(d_S\Delta-\theta)}S_0 + \int_0^t e^{(t-s)(d_S\Delta-\theta)}H(\cdot,s)ds, \tag{4.35}$$

which, along with (4.34) and well-known semigroup estimates (see e.g., [18, Lemma 2.1]), gives

$$\begin{aligned}\|\nabla S(\cdot,t)\|_{L^4} &\le \|\nabla e^{t(d_S\Delta-\theta)}S_0\|_{L^4} + \int_0^t \|\nabla e^{(t-s)(d_S\Delta-\theta)}H(\cdot,s)\|_{L^4}ds\\
&\le \kappa_1 e^{-d_S\lambda_1 t}\|\nabla S_0\|_{L^4} + \kappa_2\int_0^t \left(1+(t-s)^{-\frac{3}{4}}\right)e^{-d_S\lambda_1(t-s)}\|H(\cdot,s)\|_{L^2}ds\\
&\le \kappa_1\|\nabla S_0\|_{L^4} + \kappa_2 c_1 e^{C_5(1+\mathcal{K}_1^2)^2}[1+\Gamma(1/4)(d_S\lambda_1)^{\frac{3}{4}}]\\
&\le c_2 e^{C_5(1+\mathcal{K}_1^2)^2},\end{aligned}$$

where positive constants κ_i $(i = 1, 2)$ and λ_1 are independent of $\mathcal{K}_1$, and $c_2 := \kappa_1\|\nabla S_0\|_{L^4} + \kappa_2 c_1[1 + \Gamma(1/4)(d_S\lambda_1)^{\frac{3}{4}}]$. Here $\Gamma(\cdot)$ denotes the Gamma function defined by $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$. Hence, (4.32) follows by letting $C_6 := c_2 + C_5$ and we complete the proof of Lemma 4.6. □

Lemma 4.7. *Let (S, I) be the solution obtained in Lemma 4.1. Then there exists a constant $C_7 > 0$ such that*

$$\|I(\cdot,t)\|_{L^3} \le (1+\mathcal{K}_1^2)e^{C_7(1+\mathcal{K}_1^2)^2} \text{ for all } t \in (0, T_{\max})\,. \tag{4.36}$$

Proof. We multiply the second equation of (4.1) by I^2 and integrate the result to get

$$\frac{1}{3}\frac{d}{dt}\int_\Omega I^3 = -2d_I \int_\Omega \gamma(S)I|\nabla I|^2 - 2d_I \int_\Omega I^2\gamma_1'(S)\nabla S\cdot\nabla I + \int_\Omega \frac{\alpha(x)SI^3}{S+I} - \int_\Omega [\beta+\eta](x)I^3,$$

which, together with hypotheses (H1) and (H2), gives

$$\frac{d}{dt}\int_\Omega I^3 + 6d_I\int_\Omega I|\nabla I|^2 + 3(\beta_*+\eta_*)\int_\Omega I^3 \le 6d_I\mathcal{K}_1\int_\Omega I^2|\nabla S||\nabla I| + 3\alpha^*\int_\Omega I^3. \tag{4.37}$$

Applying Young's inequality, Hölder inequality and (4.32), one has

$$\begin{aligned} 6d_I\mathcal{K}_1\int_\Omega I^2|\nabla S||\nabla I| + 3\alpha^*\int_\Omega I^3 &\le 3d_I\int_\Omega I|\nabla I|^2 + 3d_I\mathcal{K}_1^2\int_\Omega I^3|\nabla S|^2 + 3\alpha^*\int_\Omega I^3 \\ &\le 3d_I\int_\Omega I|\nabla I|^2 + 3d_I\mathcal{K}_1^2\|I\|_{L^6}^3\|\nabla S\|_{L^4}^2 + 3\alpha^*|\Omega|^{\frac{1}{2}}\|I\|_{L^6}^3 \\ &\le 3d_I\int_\Omega I|\nabla I|^2 + c_1\sigma_1(\mathcal{K}_1)\|I\|_{L^6}^3, \end{aligned}$$

which substituted into (4.37) gives

$$\frac{d}{dt}\int_\Omega I^3 + 3d_I\int_\Omega I|\nabla I|^2 + 3(\beta_*+\eta_*)\int_\Omega I^3 \le c_1\sigma_1(\mathcal{K}_1)\|I\|_{L^6}^3, \tag{4.38}$$

where $c_1 := 3d_I + 3\alpha^*|\Omega|^{\frac{1}{2}}$ and $\sigma_1(\mathcal{K}_1) := 1 + \mathcal{K}_1^2 e^{2C_6(1+\mathcal{K}_1^2)^2} > 1$.

From (4.23), we have $\|I^{\frac{3}{2}}(\cdot,t)\|_{L^{\frac{4}{3}}} = \|I(\cdot,t)\|_{L^2}^{\frac{3}{2}} \le e^{\frac{3}{2}C_5(1+\mathcal{K}_1^2)^2}$. Then using Gagliardo-Nirenberg inequality in two dimensions and Young's inequality, one derives

$$\begin{aligned} c_1\sigma_1(\mathcal{K}_1)\|I\|_{L^6}^3 = c_1\sigma_1(\mathcal{K}_1)\|I^{\frac{3}{2}}\|_{L^4}^2 &\le c_1c_2\sigma_1(\mathcal{K}_1)\Big(\|\nabla I^{\frac{3}{2}}\|_{L^2}^{\frac{4}{3}}\|I^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^{\frac{2}{3}} + \|I^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^2\Big) \\ &\le c_3\sigma_2(\mathcal{K}_1)\Big(\|\nabla I^{\frac{3}{2}}\|_{L^2}^{\frac{4}{3}} + 1\Big) \\ &\le \frac{4d_I}{3}\|\nabla I^{\frac{3}{2}}\|_{L^2}^2 + \frac{c_3^3\sigma_2^3(\mathcal{K}_1)}{12d_I^2} + c_3\sigma_2(\mathcal{K}_1), \end{aligned} \tag{4.39}$$

where $c_3 := c_1c_2$ and $\sigma_2(\mathcal{K}_1) := (1+\mathcal{K}_1^2)e^{(2C_6+3C_5)(1+\mathcal{K}_1^2)^2} > 1$. The combination of (4.39) with (4.38) implies

$$\frac{d}{dt}\|I\|_{L^3}^3 + 3(\beta_*+\eta_*)\|I\|_{L^3}^3 \le c_4\sigma_2^3(\mathcal{K}_1), \tag{4.40}$$

where $c_4 := c_3 + \frac{c_3^3}{12d_I^2}$. Then (4.40) gives

$$\|I\|_{L^3}^3 \le e^{-3(\beta_*+\eta_*)t}\|I_0\|_{L^3}^3 + \frac{c_4\sigma_2^3(\mathcal{K}_1)}{3(\beta_*+\eta_*)}(1-e^{-3(\beta_*+\eta_*)t}) \le c_8\sigma_2^3(\mathcal{K}_1)$$

with $c_8 := c_4/(3\beta_*+3\eta_*)+\|I_0\|_{L^3}^3$. Therefore, (4.36) follows by letting $C_7 := c_8^{\frac{1}{3}}+2C_6+3C_5$ and the proof of Lemma 4.7 is completed. □

Lemma 4.8. *Let (S,I) be the solution obtained in Lemma 4.1. Then there exist two positive constants C_8 and C_9 such that*

$$\|S(\cdot,t)\|_{W^{1,\infty}} \le (1+\mathcal{K}_1^2)e^{C_8(1+\mathcal{K}_1^2)^2} \text{ for all } t\in(0,T_{\max}), \tag{4.41}$$

and

$$\|I(\cdot,t)\|_{L^\infty} \le (1+\mathcal{K}_1^2)^6 e^{C_9(1+\mathcal{K}_1^2)^2} \text{ for all } t\in(0,T_{\max}). \tag{4.42}$$

Proof. By (4.36), we conclude from (4.33) that

$$\|H(\cdot,t)\|_{L^3} \le \|\Lambda^*+\alpha^* I+\beta^* I\|_{L^3} \le c_1(1+\mathcal{K}_1^2)e^{C_7(1+\mathcal{K}_1^2)^2} =: c_1\sigma_3(\mathcal{K}_1), \tag{4.43}$$

where $c_1 := \Lambda^*|\Omega|^{\frac{1}{3}}+\alpha^*+\beta^*$. Applying the semigroup estimates to (4.35) and using (4.43), one has

$$\begin{aligned}\|S(\cdot,t)\|_{L^\infty} &\le \kappa_3 e^{-\theta t}\|S_0\|_{L^\infty} + \kappa_4\int_0^t \left(1+(t-s)^{-\frac{1}{3}}\right)e^{-\theta(t-s)}\|H(\cdot,s)\|_{L^3}ds\\ &\le \kappa_3\|S_0\|_{L^\infty} + \kappa_4 c_1\sigma_3(\mathcal{K}_1)\int_0^t\left(1+(t-s)^{-\frac{1}{3}}\right)e^{-\theta(t-s)}ds\\ &\le c_2\sigma_3(\mathcal{K}_1),\end{aligned} \tag{4.44}$$

where $c_2 := \kappa_3\|S_0\|_{L^\infty}+\kappa_4 c_1[1+\Gamma(2/3)\theta^{\frac{1}{3}}]$ with constants κ_3 and κ_4 independent of $\mathcal{K}_1$. Similarly, (4.43) along with the semigroup estimates yields

$$\begin{aligned}\|\nabla S(\cdot,t)\|_{L^\infty} &\le c_3\|S_0\|_{W^{1,\infty}} + \kappa_2\int_0^t\left(1+(t-s)^{-\frac{5}{6}}\right)e^{-d_S\lambda_1(t-s)}\|H(\cdot,s)\|_{L^3}ds\\ &\le c_4\sigma_3(\mathcal{K}_1).\end{aligned} \tag{4.45}$$

Here $c_4 := c_3\|S_0\|_{W^{1,\infty}}+\kappa_2 c_1[1+\Gamma(1/6)(d_S\lambda)^{\frac{5}{6}}]$. Then, (4.45) alongside (4.44) gives (4.41) by letting $C_8 := c_2+c_4+C_7$.

Multiplying the second equation of (4.1) by $I^{p-1}(p \geq 2)$ and integrating the result, one derives

$$\frac{1}{p}\frac{d}{dt}\int_\Omega I^p = -\, d_I(p-1)\int_\Omega \gamma(S)I^{p-2}|\nabla I|^2 - d_I(p-1)\int_\Omega I^{p-1}\gamma'(S)\nabla S\cdot\nabla I + \int_\Omega \alpha(x)\frac{SI^p}{S+I} - \int_\Omega [\beta(x)+\eta(x)]I^p,$$

which, along with hypotheses (H1) and (H2) and (4.45), gives

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega I^p + d_I(p-1)\int_\Omega I^{p-2}|\nabla I|^2 + (\beta_* + \eta_*)\int_\Omega I^p \\ &\leq d_I(p-1)\sigma_4(\mathcal{K}_1)\int_\Omega I^{p-1}|\nabla I| + \alpha^*\int_\Omega I^p \\ &\leq \frac{d_I(p-1)}{2}\int_\Omega I^{p-2}|\nabla I|^2 + \sigma_5(\mathcal{K}_1)\int_\Omega I^p,\end{aligned}$$

where $\sigma_4(\mathcal{K}_1) := c_4\mathcal{K}_1(1+\mathcal{K}_1^2)e^{C_7(1+\mathcal{K}_1^2)^2}$ and $\sigma_5(\mathcal{K}_1) := \frac{d_I(p-1)\sigma_4^2(\mathcal{K}_1)+2\alpha^*}{2}$. Hence, we obtain

$$\frac{d}{dt}\int_\Omega I^p + \frac{p(p-1)d_I}{2}\int_\Omega I^{p-2}|\nabla I|^2 \leq \sigma_5(\mathcal{K}_1)p\int_\Omega I^p \leq c_5\sigma_6(\mathcal{K}_1)p(p-1)\int_\Omega I^p, \quad (4.46)$$

where $\sigma_6(\mathcal{K}_1) := (1+\mathcal{K}_1^2)^3 e^{2C_7(1+\mathcal{K}_1^2)^2} > 1$ and $c_5 := \frac{d_I c_4^2 + 2\alpha^*}{2}$ are independent of p. We add $p(p-1)\int_\Omega I^p$ to the both sides of (4.46) and denote $c_6 := c_5 + 1$. Then the inequality (4.46) can be rewritten as

$$\frac{d}{dt}\int_\Omega I^p + p(p-1)\int_\Omega I^p \leq -\frac{p(p-1)d_I}{2}\int_\Omega I^{p-2}|\nabla I|^2 + c_6\sigma_6(\mathcal{K}_1)p(p-1)\int_\Omega I^p. \quad (4.47)$$

Based on (4.47), we can proceed with the same procedure as the proof in [23, Lemma 3.6] to find a constant $c_7 > 0$ only depending on Ω such that

$$\|I(\cdot,t)\|_{L^\infty} \leq 2^6 c_8 \max\{C_1, \|I_0\|_{L^\infty}\} \leq c_9(1+\mathcal{K}_1^2)^6 e^{4C_7(1+\mathcal{K}_1^2)^2}$$

with $c_8 := c_6\sigma_6(\mathcal{K}_1)c_7 \max\left\{1, \frac{c_6\sigma_6(\mathcal{K}_1)}{2d_I}\right\} + |\Omega| + 1$ and $c_9 := 2^6(C_1 + \|I_0\|_{L^\infty})(c_6c_7 + \frac{c_6^2c_7}{2d_I} + |\Omega| + 1)$. Hence (4.42) holds with $C_9 := c_9 + 4C_7$, and we finish the proof. □

Proof of Theorem 4.1. The combination of Lemma 4.8 with Lemma 4.1 yields Theorem 4.1. □

4.3 Basic Reproduction Number R_0: Proof of Theorem 4.2

In this section, we study the properties of R_0 and the threshold dynamics of (4.1) in terms of R_0. Below we always suppose that hypotheses (H0)-(H1) hold.

4.3.1 Properties of R_0 and Stability of DFE

Motivated by the ideas in [4], we consider the linearized eigenvalue problem of (4.1) at $(\widetilde{S}, 0)$:

$$\begin{cases} d_S\Delta\phi - \theta\phi + [\beta(x) - \alpha(x)]\psi + \lambda\phi = 0, & x \in \Omega, \\ d_I\Delta[\gamma(\widetilde{S})\psi] + [\alpha(x) - \beta(x) - \eta(x)]\psi + \lambda\psi = 0, & x \in \Omega, \\ \partial_\nu\phi = \partial_\nu\psi = 0, & x \in \partial\Omega. \end{cases} \tag{4.48}$$

Obviously, the differential operator defined in (4.48) is not self-adjoint and hence inconvenient to be studied by the conventional variational approach. To treat (4.48) variationally, we introduce a change of variable $u = \gamma(\widetilde{S})\psi$, which, along with the fact that the mapping $\psi \mapsto \gamma(\widetilde{S})\psi$ is bijective due to $1 \le \gamma(\widetilde{S}) \le \gamma\left(\frac{\Lambda^*}{\theta}\right)$, reformulates (4.48) as

$$d_S\Delta\phi - \theta\phi + [\beta(x) - \alpha(x)]\gamma^{-1}(\widetilde{S})u + \lambda\phi = 0, \quad x \in \Omega, \tag{4.49}$$

$$d_I\Delta u + [\alpha(x) - \beta(x) - \eta(x)]\gamma^{-1}(\widetilde{S})u + \lambda\gamma^{-1}(\widetilde{S})u = 0, \quad x \in \Omega, \tag{4.50}$$

$$\partial_\nu\phi = \partial_\nu u = 0, \quad x \in \partial\Omega, \tag{4.51}$$

where we denote $\gamma^{-1}(\widetilde{S}) = 1/\gamma(\widetilde{S})$ hereafter. The reformulated eigenvalue problem (4.49)-(4.51) is an elliptic system with self-adjoint operators and a weight function $\gamma^{-1}(\widetilde{S})$. For the weighted eigenvalue problem (4.50) with $\partial_\nu u = 0$, it follows from [77, Remark 1.3.8] that there exists a principal eigenvalue $\lambda^* \in \mathbb{R}$, which is simple and corresponds to a unique positive eigenfunction u^* up to a constant multiple. Since the weight function $\gamma^{-1}(\widetilde{S})$ is strictly positive, we may use the variational formula (e.g., [17, pp. 102] and [27]) to characterize λ^* as

$$\lambda^* = \inf_{0 \ne w \in H^1(\Omega)} \frac{\int_\Omega d_I|\nabla w|^2 + [\beta(x) + \eta(x) - \alpha(x)]\gamma^{-1}(\widetilde{S})w^2 dx}{\int_\Omega \gamma^{-1}(\widetilde{S})w^2 dx}.$$

This inspires us to define the basic reproduction number

$$R_0 = \sup_{0 \ne w \in H^1(\Omega)} \frac{\int_\Omega \alpha(x)\gamma^{-1}(\widetilde{S})w^2 dx}{\int_\Omega [d_I|\nabla w|^2 + (\beta(x) + \eta(x))\gamma^{-1}(\widetilde{S})w^2] dx} > 0, \tag{4.52}$$

which is equivalent to (4.5). The above transformation makes the analysis on the properties of R_0 more tractable. To explore the threshold dynamics in terms of R_0, we establish the following property of R_0 in addition to those stated in Proposition 4.1:

$$R_0 > 1 \text{ iff } \lambda^* < 0,\ R_0 = 1 \text{ iff } \lambda^* = 0 \text{ and } R_0 < 1 \text{ iff } \lambda^* > 0, \tag{4.53}$$

which can be proved by the same argument of the proof of [4, Lemma 2.3].

Next, we shall show that the linear stability of DFE $(\widetilde{S}, 0)$ can be classified by the value of R_0.

Lemma 4.9. *The DFE $(\widetilde{S}, 0)$ is linearly stable if $R_0 < 1$, and unstable if $R_0 > 1$.*

Proof. We first show the linear stability of $(\widetilde{S}, 0)$ under the assumption $R_0 < 1$. This amounts to show that if (λ, ϕ, u) is a solution to (4.49)-(4.51) with $\phi \not\equiv 0$ or $u \not\equiv 0$, then $Re(\lambda) > 0$. We have two cases to proceed.

Case 1: $u \equiv 0$ and $\phi \not\equiv 0$. Hence (λ, ϕ) is an eigenpair of the following eigenvalue problem

$$d_S \Delta\phi - \theta\phi + \lambda\phi = 0,\ x \in \Omega;\ \ \partial_\nu \phi = 0,\ x \in \partial\Omega. \tag{4.54}$$

Since the Laplacian operator Δ in (4.54) is self-adjoint, λ is real. Multiplying the first equation of (4.54) by ϕ and integrating the result, we immediately get $\lambda \geq \theta > 0$.

Case 2: $u \not\equiv 0$. In this case, (λ, u) is an eigenpair of the eigenvalue problem (4.50) with $\partial_\nu u = 0$. It follows from (4.53) and $R_0 < 1$ that $\mathrm{Re}(\lambda) \geq \lambda^* > 0$. Therefore, DFE $(\widetilde{S}, 0)$ is stable if $R_0 < 1$.

We now show that $(\widetilde{S}, 0)$ is linearly unstable if $R_0 > 1$. First (4.53) indicates that $\lambda^* < 0$. On the other hand, one can easily check that

$$d_S \Delta\phi - \theta\phi + [\beta(x) - \alpha(x)]\gamma^{-1}(\widetilde{S})u^* + \lambda^*\phi = 0,\ x \in \Omega; \quad \partial_\nu \phi = 0,\ x \in \partial\Omega$$

has a solution ϕ^*. Then (λ^*, ϕ^*, u^*) is a solution to (4.49)-(4.51) with $u^* > 0$ and $\lambda^* < 0$, which shows that $(\widetilde{S}, 0)$ is linearly unstable. □

4.3.2 Existence of EE with $R_0 > 1$

In this subsection, we shall establish the existence of EE for $R_0 > 1$. Usually the existence of EE can be established based on the uniform persistence theory. But this is inapplicable here due to the cross-diffusion structure in the I-equation. Below we shall directly explore the existence of positive solutions to (4.4).

To this end, we introduce a change of variable $Z = \gamma(S)I$, and reformulate (4.4) into the following problem without cross-diffusion

$$\begin{cases} d_S \Delta S + \Lambda(x) - \theta S - \alpha(x)\frac{SZ\gamma^{-1}(S)}{S+Z\gamma^{-1}(S)} + \beta(x)Z\gamma^{-1}(S) = 0, & x \in \Omega, \\ d_I \Delta Z + \alpha(x)\frac{SZ\gamma^{-1}(S)}{S+Z\gamma^{-1}(S)} - [\beta(x) + \eta(x)]Z\gamma^{-1}(S) = 0, & x \in \Omega, \\ \partial_\nu S = \partial_\nu Z = 0, & x \in \partial\Omega. \end{cases} \tag{4.55}$$

Thus, (4.4) admits a positive solution if and only if (4.55) admits a positive solution. In the spatially homogeneous environment, it is easy to verify that (4.4) admits a unique constant EE if $R_0 > 1$. For the spatially inhomogeneous environment, to establish the existence of EE for $R_0 > 1$, we first prove (4.55) admits a positive solution by applying the index theory and principal eigenvalue theory.

We start by giving a result on the eigenvalue problem, which will be used later.

Lemma 4.10 ([29, 85, 121])**.** *Let $\lambda_1(d, r)$ be the principal eigenvalue of*

$$d\Delta u + r(x)u + \lambda u = 0, \quad x \in \Omega; \quad \partial_\nu u = 0, \quad x \in \partial\Omega. \tag{4.56}$$

Consider the weighted eigenvalue problem

$$-d\Delta u + Mu = \mu(M + r)u, \quad x \in \Omega; \quad \partial_\nu u = 0, \quad x \in \partial\Omega, \tag{4.57}$$

where function $r(x) \in C(\Omega)$, $d > 0$, $M > 0$ and $M + r > 0$ on Ω. Then the following statements hold:

(i) If $\lambda_1(d, r) < 0$, (4.57) has an eigenvalue μ smaller than 1;

(ii) If $\lambda_1(d, r) > 0$, (4.57) has no eigenvalue μ smaller than or equal to 1.

Next we derive *a priori* estimates for the positive solutions of (4.55).

Lemma 4.11. *Let (S, Z) be a positive solution of (4.55) and assumptions (H1)-(H2) hold. Then*

$$S \leq \frac{\Lambda^*}{c_0 d_S} =: C_S \text{ and } Z \leq \frac{\Lambda^*}{c_0 d_I} =: C_Z \text{ in } \Omega, \tag{4.58}$$

where the constant $c_0 := \min\left\{\frac{\theta}{d_S}, \frac{\eta_*}{\mathcal{K}_0 d_I}\right\}$.

Proof. Adding the first two equations of (4.55), one gets

$$\Delta(d_S S + d_I Z) + \Lambda(x) - \theta S - \eta(x) Z \gamma^{-1}(S) = 0,$$

which, along with hypotheses (H1)-(H2) and (4.55), gives

$$\begin{cases} \Delta(d_S S + d_I Z) + \Lambda^* - c_0(d_S S + d_I Z) \geq 0, & x \in \Omega, \\ \partial_\nu(d_S S + d_I Z) = 0, & x \in \partial\Omega. \end{cases} \tag{4.59}$$

Denoting $v := d_S S + d_I Z$ and applying the maximum principle [93, Proposition 2.2] to (4.59), we get $\max\limits_{\overline{\Omega}}(d_S S + d_I Z) = \max\limits_{\overline{\Omega}} v \leq \frac{\Lambda^*}{c_0}$. This gives (4.58) and the proof of Lemma 4.11 is finished. $\square$

With Lemma 4.11 in hand, we introduce some notations as in [121]:

$$X = \{\phi \in C^1(\overline{\Omega}) \cap C^2(\Omega) \mid \partial_\nu \phi = 0 \text{ on } \partial\Omega\}, \quad E = C(\overline{\Omega}) \times C(\overline{\Omega}),$$
$$W = C^+(\overline{\Omega}) \times C^+(\overline{\Omega}) \text{ with } C^+(\overline{\Omega}) = \{\phi \in C(\overline{\Omega}) \mid \phi \geq 0\},$$
$$D = \{(S, Z) \in W \mid S < 1 + C_S,\ Z < 1 + C_Z\} \subset W.$$

Then for any constant $\delta \in [0, 1]$, we define a operator $T_\delta :\ D\ \to W$ by

$$T_\delta(S,\ Z) \triangleq \left(\begin{array}{c} \mathcal{T}_1^{-1}\big[\Lambda(x) - \alpha(x)\frac{SZ\gamma^{-1}(S)}{S+Z\gamma^{-1}(S)} + \beta(x)Z\gamma^{-1}(S) + (m-\theta)S\big] \\ \mathcal{T}_2^{-1}\big[mZ + \delta\alpha(x)\frac{SZ\gamma^{-1}(S)}{S+Z\gamma^{-1}(S)} - (\beta(x)+\eta(x))Z\gamma^{-1}(S)\big] \end{array} \right),$$

where $m > 0$ is a large constant such that $m - [\beta(x) + \eta(x)]\gamma^{-1}(S) - \theta > 0$ for all $(S, Z) \in D$, and $\mathcal{T}_i^{-1}$ $(i = 1, 2)$ denote the inverse operators of $\mathcal{T}_i$ under homogeneous Neumann boundary conditions, respectively, with $\mathcal{T}_1(S) := -d_S\Delta S + mS$ for $S \in X$ and $\mathcal{T}_2(Z) := -d_I\Delta Z + mZ$ for $Z \in X$. Lemma 4.11 shows that (4.55) admits a positive solution if and only if T_1 has a positive fixed point on D. Moreover, one can check that the operator T_1 is compact and $T_1(D) \subseteq W$ by applying the elliptic regularity theory and compact embedding theorem, and $(\widetilde{S}, 0)$ is the unique non-positive fixed point of T_1 on D.

Then, we shall show that $\text{index}_W(T_1, (\widetilde{S}, 0))$, as defined in [71, Definition I.2.1], exists and compute it.

Lemma 4.12. *Let the conditions in Lemma 4.11 hold and assume* $\lambda_1(d_I, m_2(x)) \not\equiv 0$. *Then*

$$\text{index}_W(T_1, (\widetilde{S}, 0)) = \begin{cases} 0, & \text{if } \lambda_1(d_I, m_2(x)) < 0, \\ 1, & \text{if } \lambda_1(d_I, m_2(x)) > 0, \end{cases}$$

where $m_2(x) := [\alpha(x) - \beta(x) - \eta(x)]\gamma^{-1}(\widetilde{S})$.

Proof. By a straightforward calculation, the Fréchet derivative $DT_1(\widetilde{S}, 0)$ of T_1 at $(\widetilde{S}, 0)$ is given by

$$DT_1(\widetilde{S}, 0)(\phi, \psi) = \left(\begin{array}{c} \mathcal{T}_1^{-1}\big[(m-\theta)\phi + m_1(x)\psi\big] \\ \mathcal{T}_2^{-1}\big[(m + m_2(x))\psi\big] \end{array} \right),$$

where $m_1(x) := [\beta(x) - \alpha(x)F(\widetilde{S}, 0)]\gamma^{-1}(\widetilde{S})$. We shall prove that $DT_1(\widetilde{S}, 0)$ has no non-zero fixed point in $C(\overline{\Omega}) \times C^+(\overline{\Omega})$. If not, then we obtain

$$\begin{cases} d_S\Delta\phi - \theta\phi + m_1(x)\psi = 0, & x \in \Omega, \\ d_I\Delta\psi + m_2(x)\psi = 0, & x \in \Omega, \\ \partial_\nu\phi = \partial_\nu\psi = 0, & x \in \partial\Omega. \end{cases} \tag{4.60}$$

It follows from the first equation of (4.60) that $\phi = 0$ if $\psi = 0$. Hence, $\psi \in C^+(\overline{\Omega})\backslash\{0\}$, this along with [77, Theorem 1.3.6] gives $\lambda_1(d_I, m_2(x)) \equiv 0$, which contradicts the assumption $\lambda_1(d_I, m_2(x)) \not\equiv 0$. Therefore, $DT_1(\widetilde{S}, 0)$ has no non-zero fixed point in $C(\overline{\Omega}) \times C^+(\overline{\Omega})$, this means that $\text{index}_W(T_1, (\widetilde{S}, 0))$ exists.

To compute $\text{index}_W(T_1, (\widetilde{S}, 0))$, we shall employ principal eigenvalue result given in Lemma 4.10 and the index theory (see [28, 113]), which is presented in [121, Lemma 3.1]. Choose $W_{(\widetilde{S},0)} = C(\overline{\Omega}) \times C^+(\overline{\Omega})$, $H_{(\widetilde{S},0)} = C(\overline{\Omega}) \times \{0\}$, $E_{(\widetilde{S},0)} = \{0\} \times C(\overline{\Omega})$ such that $E = H_y \oplus E_y$ and $W_{(\widetilde{S},0)}$ is a generating cone. Then it follows from [121, Lemma 3.1] that $P \circ DT_1(\widetilde{S}, 0) = \mathcal{T}_2^{-1}[m + m_2(x)]$, where $P: E \to E_y$ is a projection operator. If $\lambda_1(d_I, m_2(x)) < 0$, by Lemma 4.10, we know that $\mathcal{T}_2^{-1}[m + m_2(x)]$ has an eigenvalue bigger than 1. This along with [121, Lemma 3.1] gives $\text{index}_W(T_1, (\widetilde{S}, 0)) = 0$. If $\lambda_1(d_I, m_2(x)) > 0$, Lemma 4.10 shows that all eigenvalues of the operator $\mathcal{T}_2^{-1}[m + m_2(x)]$ are smaller than 1. Thus, [121, Lemma 3.1] yields

$$\text{index}_W(T_1, (\widetilde{S}, 0)) = (-1)^{\ell},$$

where ℓ denotes the sum of algebraic multiplicities of the eigenvalues of $DT_1(\widetilde{S}, 0)$ restricted in $H_{(\widetilde{S},0)}$ which are greater than 1.

We next prove that $DT_1(\widetilde{S}, 0)$ restricted in $H_{(\widetilde{S},0)}$ does not have eigenvalues greater than or equal to 1. Assume that $DT_1(\widetilde{S}, 0)$ has an eigenvalue $\mu_0 \geq 1$ associated with eigenfunction $(\phi, \psi) = (\phi, 0) \in H_{(\widetilde{S},0)}$ fulfilling $\|\phi\|_{L^2} = 1$. Then we have

$$-d_S \Delta\phi + m\phi = \frac{1}{\mu_0}(m - \theta)\phi, \quad x \in \Omega; \quad \partial_\nu \phi = 0, \quad x \in \partial\Omega.$$

Since $\lambda_1(d_S, -\theta) > 0$, Lemma 4.10 gives $\frac{1}{\mu_0} > 1$. This contradicts $\mu_0 \geq 1$. Hence $\text{index}_W(T, (\widetilde{S}, 0)) = (-1)^{\ell} = (-1)^0 = 1$ and the proof of Lemma 4.12 is completed. □

Lemma 4.13. *Let the conditions in Lemma 4.11 hold. Then* (4.55) *admits at least one positive solution when* $\lambda_1(d_I, m_2(x)) < 0$.

Proof. Assume that (4.55) has no positive solution, then $(\widetilde{S}, 0)$ is the unique fixed point of T_1 on D. Lemma 4.11 indicates that T_1 has no fixed point on ∂D (i.e., $(I - T_1)(\partial D) \neq 0$), and thus $\deg_W(I - T_1, D, 0)$ is well-defined (see the definition in [71, Definition II.2.2]). Then the excision property [5, Corollary 11.2] shows that

$$\deg_W(I - T_1, D, 0) = \text{index}_W(T_1, (\widetilde{S}, 0)),$$

which, along with $\lambda_1(d_I, m_2(x)) < 0$ and Lemma 4.12, gives

$$\deg_W(I - T_1, D, 0) = 0. \tag{4.61}$$

On the other hand, for each $\delta \in [0,1]$, T_δ has a fixed point (S,Z) iff (S,Z) is a solution of the following problem

$$\begin{cases} d_S\Delta S + \Lambda(x) - \theta S - \alpha(x)\frac{SZ\gamma^{-1}(S)}{S+Z\gamma^{-1}(S)} + \beta(x)Z\gamma^{-1}(S) = 0, & x \in \Omega, \\ d_I\Delta Z + \delta\alpha(x)\frac{SZ\gamma^{-1}(S)}{S+Z\gamma^{-1}(S)} - [\beta(x)+\eta(x)]Z\gamma^{-1}(S) = 0, & x \in \Omega, \\ \partial_\nu S = \partial_\nu Z = 0, & x \in \partial\Omega. \end{cases} \tag{4.62}$$

Proceeding with the similar procedure as the proof in Lemma 4.11, we get that all fixed points of T_δ satisfy (4.58) for each $\delta \in [0,1]$, which means that $(I - T_\delta)(\partial D) \neq 0$. Hence, the homotopy invariance of the topological degree [5, Theorem 11.1] implies

$$\deg_W(I - T_\delta, D, 0) = \deg_W(I - T_1, D, 0) = \deg_W(I - T_0, D, 0). \tag{4.63}$$

When $\delta = 0$, (4.62) only has a unique solution, which is denoted by $(\widetilde{S}^0, 0)$. Hence, the excision property implies that

$$\deg_W(I - T_0, D, 0) = \mathrm{index}_W(T_0, (\widetilde{S}^0, 0)). \tag{4.64}$$

Following the same proof as in Lemma 4.12, one can check that

$$\mathrm{index}_W(T_0, (\widetilde{S}^0, 0)) = 1,$$

which, together with (4.63) and (4.64), gives $\deg_W(I - T_1, D, 0)) = \deg_W(I - T_0, D, 0)) = 1$. This contradicts (4.61). Hence (4.55) admits at least one positive solution and the proof of Lemma 4.13 is completed. □

Using Lemma 4.13, we further establish the existence of EE when $R_0 > 1$. To achieve this goal, we show that the principal eigenvalues of the weighted and unweighted eigenvalue problems have the same sign.

Lemma 4.14. *Assume that $d > 0$, $r(x) \in C(\overline{\Omega})$, and the positive function $a(x) \in C(\overline{\Omega})$. Let $\lambda_1(d,r)$ and ς^* be the principal eigenvalue of (4.56) and*

$$d\Delta u + r(x)u + \varsigma a(x)u = 0, \quad x \in \Omega; \quad \partial_\nu u = 0, \quad x \in \partial\Omega,$$

respectively. Then it follows that

$$\mathrm{sign}(\varsigma^*) = \mathrm{sign}[\lambda_1(d,r)]. \tag{4.65}$$

Proof. Denote the positive eigenfunctions associated with $\lambda_1(d,r)$ and ς^* by u^* and w^*, respectively, satisfying $\|u^*\|_{L^\infty} = \|w^*\|_{L^\infty} = 1$. Then we have

$$\begin{cases} d\Delta u^* + r(x)u^* + \lambda_1(d,r)u^* = 0, & x \in \Omega, \\ d\Delta w^* + r(x)w^* + \varsigma^* a(x)w^* = 0, & x \in \Omega, \\ \partial_\nu u^* = \partial_\nu w^* = 0, & x \in \partial\Omega. \end{cases} \tag{4.66}$$

We multiply the first equation of (4.66) by w^* and the second by u^*, and integrate the results by parts. Then subtract the resulting equation, we get

$$\varsigma^* \int_\Omega a(x)w^*u^* = \lambda_1(d,r)\int_\Omega w^*u^*.$$

This along with the fact that $a(x)$, w^*, u^* are positive gives (4.65) directly and hence completes the proof of Lemma 4.14. □

Lemma 4.15. *Let the conditions in Lemma 4.11 hold. Then* (4.4) *admits at least an EE when* $R_0 > 1$.

Proof. Taking $d = d_I$, $r(x) = m_2(x)$ and $a(x) = \gamma^{-1}(\widetilde{S})$ in Lemma 4.14, then (4.65) along with (4.53) indicates that $\text{sign}(\lambda_1(d_I, m_2(x))) = \text{sign}(\lambda^*) = \text{sign}(1 - R_0) < 0$. Thus, Lemma 4.13 implies Lemma 4.15 directly. □

Proof of Theorem 4.2. Combining Lemma 4.9 with Lemma 4.15, we get Theorem 4.2. □

4.4 Global Stability: Proof of Theorem 4.3

In this section, we shall explore the globally asymptotical stability of non-negative steady states of (4.1). We first improve the regularity of the solution (S, I).

Lemma 4.16. *Let* (S, I) *be the solution obtained in Theorem 4.1. Then there exist constants* $\kappa \in (0,1)$ *and* $C_{10} > 0$ *such that*

$$\|(S,I)(\cdot,t)\|_{C^{2+\kappa,1+\frac{\kappa}{2}}(\overline{\Omega}\times[1,\infty))} \leq C_{10}. \tag{4.67}$$

Proof. The result is obtained by the Hölder estimates for quasilinear parabolic equations (cf. [111, Theorem 1.3 and Remark 1.4]) and the standard parabolic Schauder theory [76]. The proof details can follow the similar procedures as the proof in [145, Lemma 3.4], we omit for brevity. □

Proof of Theorem 4.3. We first prove the results claimed in Theorem 4.3-(i). With the given condition, it is obvious $R_0 < 1$ by the definition (4.5). Integrating the second equation of (4.1) by parts yields

$$\frac{d}{dt}\int_\Omega I + (1-\varepsilon)\int_\Omega \eta(x)I = \int_\Omega \big[\frac{\alpha(x)S}{S+I} - \beta(x) - \varepsilon\eta(x)\big]I \le \int_\Omega \left[\alpha - \beta - \varepsilon\eta\right](x)I,$$

which, along with $\alpha(x) \le \beta(x) + \varepsilon\eta(x)$ and $\varepsilon \in [0,1)$, implies

$$\frac{d}{dt}\int_\Omega I + (1-\varepsilon)\eta_*\int_\Omega I \le 0.$$

This indicates that for all $t > 0$

$$\|I\|_{L^1} \le e^{-(1-\varepsilon)\eta_* t}\|I_0\|_{L^1}. \tag{4.68}$$

We utilize Gagliardo-Nirenberg inequality in two dimensions to find a constant $c_1 > 0$ such that

$$\|I\|_{L^\infty} \le c_1\Big(\|\nabla I\|_{L^\infty}^{\frac{2}{3}}\|I\|_{L^1}^{\frac{1}{3}} + \|I\|_{L^1}\Big) \le c_2 e^{-\frac{(1-\varepsilon)\eta_*}{3}t}, \quad \forall t > 1, \tag{4.69}$$

where we have used (4.67) and (4.68).

It follows from the first equation of (4.1) that

$$(S-\widetilde{S})_t = d_S\Delta(S-\widetilde{S}) - \theta(S-\widetilde{S}) - \alpha(x)\frac{SI}{S+I} + \beta(x)I. \tag{4.70}$$

Applying Duhamel's principle to (4.70), one has

$$S - \widetilde{S} = e^{(t-1)(d_S\Delta-\theta)}(S(\cdot,1)-\widetilde{S}) + \int_1^t e^{(t-z)(d_S\Delta-\theta)}\Big[\beta(x) - \alpha(x)\frac{S}{S+I}\Big]I(\cdot,z)dz.$$

By the standard heat Neumann semigroup estimates (see e.g., [18, Lemma 2.1]), we get from (4.69) that

$$\begin{aligned}
\|S-\widetilde{S}\|_{L^\infty} &\le c_3 e^{-\theta t}\|S(\cdot,1)-\widetilde{S}\|_{L^\infty} \\
&\quad + c_3\int_1^t e^{-\theta(t-z)}\Big(1+(t-z)^{-\frac{1}{2}}\Big)\left\|\left(\beta(x) - \frac{\alpha(x)S}{S+I}\right)I(\cdot,z)\right\|_{L^2} dz \\
&\le c_3 e^{-\theta t}\|S(\cdot,1)-\widetilde{S}\|_{L^\infty} + c_4\int_1^t e^{-\theta(t-z)}\left(1+(t-z)^{-\frac{1}{2}}\right)\|I(\cdot,z)\|_{L^2}dz \\
&\le c_5 e^{-\theta t} + c_4 c_2|\Omega|^{\frac{1}{2}}\int_1^t e^{-\theta(t-z)}\left(1+(t-z)^{-\frac{1}{2}}\right)e^{-\frac{(1-\varepsilon)\eta_*}{3}z}dz
\end{aligned} \tag{4.71}$$

$$\leq c_5 e^{-\theta t} + c_4 c_2 |\Omega|^{\frac{1}{2}} \int_1^t e^{-\theta(t-z)} \left(1 + (t-z)^{-\frac{1}{2}}\right) e^{-c_6 z} dz$$
$$\leq c_7 e^{-c_6 t},$$

where $c_4 := c_3(\beta^* + \alpha^*)$ and $c_6 := \frac{1}{2}\min\{\theta, (1-\varepsilon)\eta_*/3\}$. Therefore, combining (4.69) with (4.71) indicates (4.6) directly. This completes the proof of Theorem 4.3-(i).

Next we proceed to prove Theorem 4.3-(ii). When $\Lambda(x)$, $\alpha(x)$, $\beta(x)$ and $\eta(x)$ are positive constants, it follows from Proposition 4.1-(i) that $R_0 = \frac{\alpha}{\beta+\eta}$. Clearly there exists a unique constant EE $(\hat{S}, \hat{I})$ iff $R_0 > 1$, where

$$\hat{S} = \frac{\Lambda(\beta+\eta)}{\eta(\alpha-\beta-\eta)+\theta(\beta+\eta)} \quad \text{and} \quad \hat{I} = \frac{\Lambda(\alpha-\beta-\eta)}{\eta(\alpha-\beta-\eta)+\theta(\beta+\eta)}. \tag{4.72}$$

We define

$$\mathcal{E}(t) := \int_\Omega \left\{ (S+I+1) - (\hat{S}+\hat{I}+1) - (\hat{S}+\hat{I}+1)\ln\frac{S+I+1}{\hat{S}+\hat{I}+1} \right\}$$
$$+ \frac{4\eta\alpha}{(\alpha-\beta-\eta)^2} \int_\Omega \left[(I+1) - (\hat{I}+1) - (\hat{I}+1)\ln\frac{I+1}{\hat{I}+1} \right].$$

Following the same way as the proof of Theorem 2.2 (1), one can directly check that $\mathcal{E}(t) \geq 0$ where "=" holds iff $(S, I) = (\hat{S}, \hat{I})$. Next, we show that $\frac{d}{dt}\mathcal{E}(t) \leq -c_1\mathcal{F}(t)$ for some $c_1 > 0$ and function $\mathcal{F}(t) \geq 0$. For simplicity, we denote

$$E := E(S,I) := (S+I+1) - (\hat{S}+\hat{I}+1) - (\hat{S}+\hat{I}+1)\ln\frac{S+I+1}{\hat{S}+\hat{I}+1}$$
$$+ \frac{4\eta\alpha}{(\alpha-\beta-\eta)^2} \left[(I+1) - (\hat{I}+1) - (\hat{I}+1)\ln\frac{I+1}{\hat{I}+1} \right],$$

and

$$h_1 := h_1(S,I) = \Lambda - \theta S - \alpha\frac{SI}{S+I} + \beta I, \quad h_2 := h_2(S,I) = \alpha\frac{SI}{S+I} - (\beta+\eta)I.$$

Hence, one gets

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(t) &= \int_\Omega E_S S_t + E_I I_t \\ &= \int_\Omega [E_S h_1 + E_I h_2] + \int_\Omega [d_S E_S \Delta S + d_I E_I \Delta(\gamma(S)I)] =: J_1 + J_2, \end{aligned} \tag{4.73}$$

where $E_S := \frac{\partial E}{\partial S}$ and $E_I := \frac{\partial E}{\partial I}$. Noting $\Lambda = \theta\hat{S} + \eta\hat{I}$, $\beta + \eta = \frac{\alpha\hat{S}}{\hat{S}+\hat{I}}$ and $\theta = \eta$, we have

$$J_1 = \int_\Omega \left(1 - \frac{\hat{I}+\hat{S}+1}{S+I+1}\right) h_1 + \int_\Omega \left[1 - \frac{\hat{I}+\hat{S}+1}{S+I+1} + \frac{4\eta\alpha}{(\alpha-\beta-\eta)^2}\left(1 - \frac{\hat{I}+1}{I+1}\right)\right] h_2$$

$$= \int_\Omega \frac{S-\hat{S}+I-\hat{I}}{S+I+1}(\Lambda-\theta S-\eta I) + \frac{4\eta\alpha}{(\alpha-\beta-\eta)^2}\int_\Omega \frac{I-\hat{I}}{I+1}\left(\frac{\alpha S}{S+I}-\beta-\eta\right)I$$
$$= -\int_\Omega \Phi B_1 \Phi^{\mathcal{T}},$$

where $\Phi = \left(\frac{S-\hat{S}}{\sqrt{S+I+1}}, \frac{I-\hat{I}}{\sqrt{S+I+1}}\right)$ and

$$B_1 = \begin{pmatrix} \eta & \frac{1}{2}\left[2\eta - \frac{4\eta\alpha}{(\alpha-\beta-\eta)}\cdot\frac{I(S+I+1)}{(I+1)(S+I)}\right] \\ \frac{1}{2}\left[2\eta - \frac{4\eta\alpha}{(\alpha-\beta-\eta)}\cdot\frac{I(S+I+1)}{(I+1)(S+I)}\right] & \frac{4\eta\alpha(\beta+\eta)}{(\alpha-\beta-\eta)^2}\cdot\frac{I(S+I+1)}{(I+1)(S+I)}+\eta \end{pmatrix}.$$

A direct calculation gives that $|B_1| = \frac{4\eta^2\alpha^2}{(\alpha-\beta-\eta)^2}\cdot\frac{I(S+I+1)}{(I+1)(S+1)}\left(1-\frac{I(S+I+1)}{(I+1)(S+I)}\right) > 0$, which yields a constant $c_1 > 0$ such that

$$J_1 = -\int_\Omega \Phi B_1 \Phi^{\mathcal{T}} \le -c_1\int_\Omega \left(\frac{(S-\hat{S})^2}{S+I+1}+\frac{(I-\hat{I})^2}{S+I+1}\right) \le 0. \tag{4.74}$$

With simple calculations, we find $E_{SS} = E_{SI} = E_{IS} = \frac{\hat{S}+\hat{I}+1}{(S+I+1)^2}$, and

$$E_{II} = \frac{\hat{S}+\hat{I}+1}{(S+I+1)^2}\left[1+\frac{4\eta\alpha}{(\alpha-\beta-\eta)^2}\frac{\hat{I}+1}{\hat{S}+\hat{I}+1}\left(\frac{S+I+1}{I+1}\right)^2\right]$$
$$=: \frac{(\hat{S}+\hat{I}+1)[1+M_0 f(S,I)]}{(S+I+1)^2},$$

where $M_0 := \frac{4\eta[\Lambda(\alpha-\beta-\eta)+\eta\alpha]}{(\alpha-\beta-\eta)^2(\Lambda+\eta)}$ and $f(S,I) := \left(\frac{S+I+1}{I+1}\right)^2$. Thus, J_2 can be rewritten as

$$J_2 = -d_S\int_\Omega E_{SS}|\nabla S|^2 - d_S\int_\Omega E_{SI}\nabla I\cdot\nabla S - d_I\int_\Omega \gamma(S)E_{IS}\nabla I\cdot\nabla S$$
$$- d_I\int_\Omega \gamma(S)E_{II}|\nabla I|^2 - d_I\int_\Omega I\gamma'(S)E_{IS}|\nabla S|^2 - d_I\int_\Omega I\gamma'(S)E_{II}\nabla I\cdot\nabla S$$
$$= -\int_\Omega \Psi B_2 \Psi^{\mathcal{T}}$$

with $\Psi = (\nabla S, \nabla I)$ and

$$B_2 = \begin{pmatrix} d_S E_{SS} + d_I I\gamma'(S)E_{IS} & \frac{d_S E_{SI}+d_I\gamma(S)E_{IS}+d_I I\gamma'(S)E_{II}}{2} \\ \frac{d_S E_{SI}+d_I\gamma(S)E_{IS}+d_I I\gamma'(S)E_{II}}{2} & d_I\gamma(S)E_{II} \end{pmatrix}.$$

With direct computation, we can show that B_2 is positive definite iff

$$g_1(S,I) > g_2(S,I), \tag{4.75}$$

where

$$g_1(S,I) = 2d_S d_I \gamma(S) + 2d_I^2 I\gamma'(S)\gamma(S)[1+M_0 f(S,I)] + 4d_S d_I \gamma(S) M_0 f(S,I),$$
$$g_2(S,I) = d_I^2[\gamma(S)]^2 + d_S^2 + d_I\gamma'(S)I[1+M_0 f(S,I)]\{2d_S + d_I I\gamma'(S)[1+M_0 f(S,I)]\}.$$

By hypothesis (H2), (4.3) and $1 < f(S,I) < (S+I+1)^2$, (4.7) ensures that (4.75) holds. Therefore, there is a constant $c_2 > 0$ such that

$$J_2 = -\int_\Omega \Psi B_2 \Psi^{\mathcal{T}} \le -c_2 \int_\Omega (|\nabla S|^2 + |\nabla I|^2) \le 0,$$

which along with (4.74) substituted into (4.73) gives

$$\frac{d}{dt}\mathcal{E}(t) \le -c_1 \int_\Omega \left(\frac{(S-\hat{S})^2}{S+I+1} + \frac{(I-\hat{I})^2}{S+I+1} \right) =: -c_1 \mathcal{F}(t).$$

Based on Lemma 4.16, following the same way as the proof of Theorem 2.2 (1), one can get

$$\lim_{t\to\infty} (\|S-\hat{S}\|_{L^2} + \|I-\hat{I}\|_{L^2}) = 0. \tag{4.76}$$

Applying the Gagliardo-Nirenberg inequality in two dimensions for any $u \in W^{1,\infty}$:

$$\|u\|_{L^\infty} \le c_3 \big(\|\nabla u\|_{L^\infty}^{\frac{1}{2}} \|u\|_{L^2}^{\frac{1}{2}} + \|u\|_{L^2} \big),$$

we obtain from (4.67) and (4.76) that $\lim_{t\to\infty} (\|S-\hat{S}\|_{L^\infty} + \|I-\hat{I}\|_{L^\infty}) = 0$. This finishes the proof of Theorem 4.3-(ii). □

4.5 Numerical Simulations and Discussion

This chapter investigates an SIS model with cross-diffusion dispersal strategy for the infected individuals describing the public health intervention measures (like quarantine) during the outbreak of infectious diseases. The considered SIS model adopts the frequency-dependent transmission mechanism and includes demographic changes (i.e., population recruitment and death). Apart from the global boundedness of solutions established in Theorem 4.1, we define the basic reproduction number R_0 by a variational formula and study the threshold dynamics of the model based on R_0 (see Theorem 4.2 and Theorem 4.3). Below we shall use numerical simulations to illustrate the applications of our analytical results and speculate some results not proved in this chapter. We set $\Omega = (0,2)$ in all simulations.

In a special case where the recruitment rate $\Lambda(x)$ of susceptible individuals is constant, we see that cross-diffusion dispersal strategy (see Remark 4.2) reduces the value of R_0, namely the basic reproduction number R_0 for $\gamma'(S) \neq 0$ is less than $\hat{R}_0$, where $\hat{R}_0$ is the basic reproduction number when $\gamma(S) = 1$. We can see a numerical example shown in Figure 4.1-(a). This implies that public health intervention measures limiting the mobility of infected individuals is effective in controlling the spread of infectious diseases. However, if $\Lambda(x)$ is not constant, we are unable to prove $R_0 < \hat{R}_0$ analytically. Below we use an example to illustrate this conclusion numerically for non-constant $\Lambda(x)$. To this end, we take

$$\gamma(S) = e^S \tag{4.77}$$

satisfying hypothesis (H1) and

$$d_S = \theta = 1, \tag{4.78}$$

as well as

$$\Lambda(x) = -\frac{1}{3}x^3 + x^2 + 2x, \quad \alpha(x) = 2x + 1, \quad \beta(x) = x, \quad \eta(x) = 1.8. \tag{4.79}$$

Then $\Lambda(x)$ is positive on Ω and one can check that $\widetilde{S} = -\frac{1}{3}x^3 + x^2 + 2 > 0$. The graphs of functions R_0 and $\hat{R}_0$ are numerically plotted in Figure 4.1-(b), where we observe that $R_0 < \hat{R}_0$. However, whether or not the cross-diffusion dispersal strategy reduces the basic reproduction number so that $R_0 < \hat{R}_0$ for all $\gamma(S)$ satisfying the hypothesis (H1) remains an outstanding theoretical question for future efforts.

When $\gamma(S)$ is constant, namely the infected individuals undergo random dispersal, the classical results showed that the disease would persist in the high-risk domain Ω (cf. [81, Proposition 3.2, Theorem 3.1], [107, Theorem 2.5, Theorem 3.3]), as numerically shown in Figure 4.2-(a) where we assume $\gamma(S) = 1$ and $d_I = 0.2$ while other functions and parameter values are given by (4.78) and (4.79). The results in Proposition 4.1 along with Theorem 4.2 and Theorem 4.3 indicate that the cross-diffusion dispersal strategy will help eradicate the infectious disease even in the high-risk domain. To illustrate this result, we use the functions and parameter values given in (4.77)-(4.79). With them, we can verify that

$$\int_0^2 [\alpha(x) - \beta(x) - \eta(x)]dx = 0.4 > 0,$$

which means that Ω is a high-risk domain. In this case, the asymptotically stable spatial profile of (S, I) is numerically plotted in Figure 4.2-(b) which demonstrates that the disease will be eradicated in the whole domain. By (4.79), we find $\int_0^2 [\alpha(x) - \beta(x) - \eta(x)]\gamma^{-1}(\widetilde{S})dx \approx$

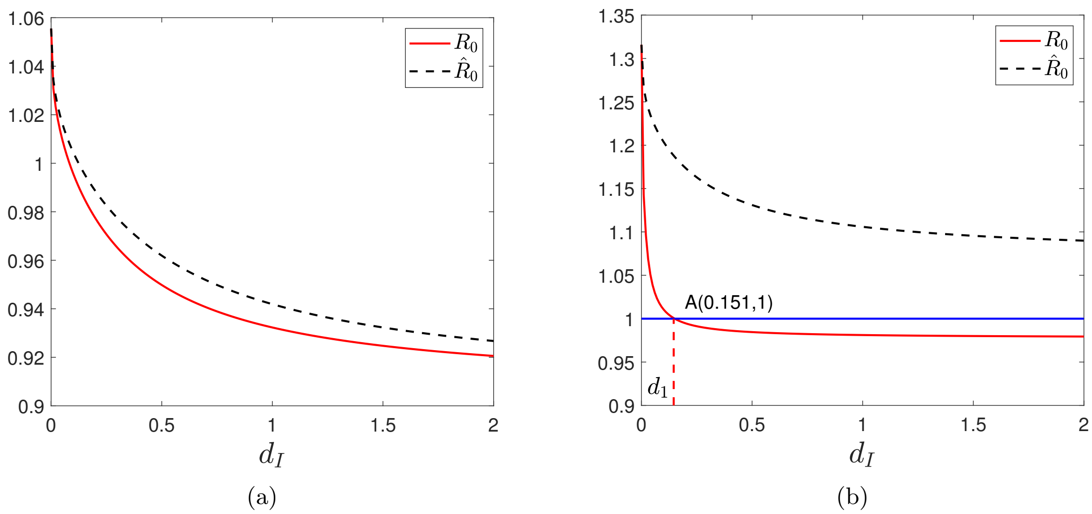


Figure 4.1: Graphs of functions R_0 and $\hat{R}_0$ versus $d_I > 0$, where functions and parameters are taken as follows: (a) $\gamma(S) = 2 - (S+1)^{-1}, d_S = \theta = \Lambda(x) = 1$, and $\alpha(x) = x^2 + 2x + 1.5, \beta(x) = x^2 + 0.5, \eta(x) = x + 2.5$; (b) The functions and parameter values are given in (4.77), (4.78) and (4.79).

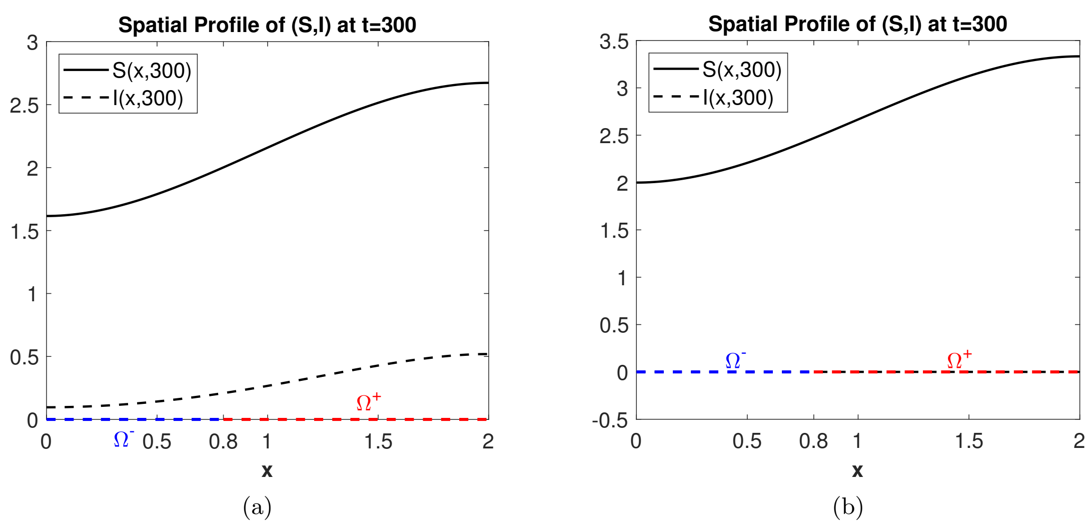


Figure 4.2: The profile of susceptible and infected populations with $d_I = 0.2$. (a): $\gamma(S) = 1$; (b): $\gamma(S) = e^S$. Other functions and parameter values are given in (4.78) and (4.79). The initial value (S_0, I_0) is set as a small random perturbation of (2,1).

$-0.0083 < 0$, $\Omega^- = \{x : \ 0 < x < 0.8\}$ and $\Omega^+ = \{x : \ 0.8 < x < 2\}$ are nonempty. This alongside Proposition 4.1-(iii) and the Figure 4.1-(b) show that $R_0 < 1$ if $d_I = 0.2 > d_1 \approx 0.151$. Therefore, it follows from Theorem 4.2 that DFE is linearly stable, which implies that the disease may be eradicated. This is well supported by numerical results shown in Figure 2-(b). However, we can not conclude the global stability of DFE based on Theorem 4.3-(i) since one can check that the condition $\alpha(x) \leq \beta(x) + \varepsilon\eta(x)$ for all $x \in \Omega$ with some $\varepsilon \in [0, 1)$ is not satisfied by the functions chosen in (4.79). The numerical simulation of the asymptotically stable spatial profile shown in Figure 4.2-(b) indicates that DFE may be globally asymptotically stable even if the condition in Theorem 4.3-(i) is not fulfilled. Therefore how to relax the condition of Theorem 4.3-(i) is another interesting question remaining open in this chapter. The best situation we anticipate is to replace the condition of Theorem 4.3-(i) by $R_0 < 1$, but this can not be proven based on the method in this chapter.

Chapter 5

A Diffusive Population-toxicant Model in a Time-periodic Environment with Negative Toxicant-taxis

5.1 Introduction and Main Results

5.1.1 Introduction

In aquatic ecosystems, toxicant have detrimental effects on biological systems at various levels [103, 108, 114]. Investigating their impact on aquatic population dynamics and identifying the key factors determining species persistence or extinction are vital to protect aquatic species and preserve ecosystem diversity. This topic has been extensively studied in early modeling settings, including matrix population models (e.g., [36, 48, 118]), ordinary differential equation models (e.g., [45, 56, 57]), and reaction-advection-diffusion equation models [143, 155, 158]. However, these settings leave out the fact that aquatic species may detect and avoid toxicant [9, 132]. On the other hand, the input of toxicant into aquatic ecosystems may exhibit temporal periodicity due to seasonal factors such as variation in rainfall, surface water, and temperature [14]. For example, during the wet seasons, increased rainfall may lead to more runoff, carrying nitrogen and phosphorus from human activities, such as agricultural practices or fuel combustion, into water bodies, causing seasonal pollution peaks [1, 112].

Therefore, we are inspired to incorporate the negative toxicant-taxis (cf. [32]), and spatially heterogeneous and time-periodic toxicant input into a population-toxicant system, which reads as

$$
\begin{cases}
u_t = d_1\Delta u + \chi\nabla\cdot(u\nabla w) + u(r - u - mw), & x\in\Omega,\ t>0,\\
w_t = d_2\Delta w + h(x,t) - \alpha w - \beta uw, & x\in\Omega,\ t>0,\\
\partial_\nu u = \partial_\nu w = 0, & x\in\partial\Omega,\ t>0,\\
(u,w)(x,0) = (u_0,w_0)(x), & x\in\Omega,
\end{cases}
\tag{5.1}
$$

All notations and parameters have the same interpretation as in Section 1.5.

Research on the spatiotemporal model (5.1) is still in its formative stage. The first study on (5.1) with $\chi > 0$ was conducted in [32], where the authors established the global existence of classical solutions to (5.1) with $h(x,t) = h(x)$. When $h(x,t) = h_0$ for a constant h_0, they proved the global stability of constant steady states and numerically illustrated the occurrence of spatially heterogeneous coexistence for large χ. The theoretical existence of such spatially heterogeneous coexistence was later rigorously established in the work [21] for $h(x,t) = h_0$ by using Leray-Schauder degree theory. When the toxicant input is time-periodic, the work [86] established the global stability of periodic solutions and explored the asymptotic profiles of positive periodic solutions when diffusion rates are small or large in the absence of toxicant-taxis (i.e., $\chi = 0$). Their results indicate that the toxicant input affect the species persistence or extinction. In fact, (5.1) with $\chi = 0$ is a monotone dynamical system, which allow the asymptotic theory of monotone systems [157, Chapter 3] to be applied in studying the global dynamics, as shown in [86]. In contrast, the system (5.1) with $\chi > 0$ is non-monotone, and the comparison principle becomes inapplicable. As a result, no established methods in the literature can be employed, making the analysis of global dynamics for (5.1) with $\chi > 0$ significantly more challenging.

Therefore, we shall focus on (5.1) with $\chi > 0$ and a more general toxicant input function. To overcome the aforementioned challenge, we effectively employ the principal Floquet bundle theory [77, Chapter 4] and persistence theory [157, Chapter 3] as well as the energy functional method. Our main objectives are as follows:

(T.1) Identify the conditions for the periodic solution exists, its locally/globally stability, and the uniform persistence of species;

(T.2) Explore whether the toxicant-taxis (i.e., avoidance of toxicant) helps aquatic species to survive in a polluted environment.

5.1.2 Main Results

Throughout this chapter, we denote $\Omega\times(0,T) =: Q_T$, and assume that

(H0) nonconstant $h(x,t) \in C^{\kappa_0,\frac{\kappa_0}{2}}\left(\overline{\Omega}\times[0,\infty)\right)$, $\partial_t h \not\equiv 0$ and $h(x,t) = h(x,t+T) \geq (\not\equiv)\ 0$ with constants $\kappa_0 \in (0,1)$ and period $T > 0$.

We begin by stating the global existence and boundedness of solutions to (5.1).

Theorem 5.1 (Global boundedness)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and the hypothesis (H0) hold. Assume that $u_0 \in C(\overline{\Omega}), w_0 \in C^1(\overline{\Omega})$ with $u_0, w_0 \geq 0\ (\not\equiv 0)$.*

(i) Then system (5.1) *has a unique global classical solution*

$$(u,w) \in \left[C\left(\overline{\Omega}\times[0,\infty)\right) \cap C^{2,1}\left(\overline{\Omega}\times(0,\infty)\right)\right]^2$$

satisfying $u, w > 0$ for all $t > 0$ and

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C_0, \quad \forall t > 0. \tag{5.2}$$

where $C_0 := C_0(u_0, v_0) > 0$ is a constant independent of t.

(ii) There exists a constant M_0 independent of (u_0, v_0) and m such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq M_0, \quad \forall t > T_0, \tag{5.3}$$

for some constant $T_0 > 0$ depending on initial data (u_0, w_0).

Remark 5.1. *Theorem 5.1(ii) establishes the ultimately uniform boundedness (see [78, Definition 2.1]) of the solution to* (5.1)*. This is important to study the uniform persistence of* (5.1)*.*

In fact, the system (5.1) may have two types of nonnegative T-periodic solutions: positive T-periodic solution which exists in some circumstance, and the semi-trivial T-periodic solution $(0, \hat{w}^h(x,t))$ which always exists. Here, $\hat{w}^h(x,t) =: \hat{w}(x,t)$ is the unique solution of the following equation

$$\begin{cases} w_t = d_2\Delta w + h(x,t) - \alpha w, & x \in \Omega, t > 0, \\ \partial_\nu w = 0, & x \in \partial\Omega, t > 0, \\ w(x,t) = w(x,t+T), & x \in \Omega, t \geq 0, \end{cases} \tag{5.4}$$

and it is bounded and positive [95, Proposition 4.4.8]. Applying the maximum principle [141, Theorem 7.1] to (5.4) gives

$$\frac{h^*}{\alpha} := \frac{1}{\alpha}\max_{(t,x)\in Q_T} h(x,t) \geq \hat{w}(x,t) \geq \frac{1}{\alpha}\min_{(t,x)\in Q_T} h(x,t) =: \frac{h_*}{\alpha} \geq 0, \ \ \forall (x,t) \in Q_T.$$

Hence,

$$\frac{h^*}{\alpha} \geq \max_{(x,t)\in Q_T} \hat{w}(x,t) =: \hat{w}^* \geq \hat{w}_* := \min_{(x,t)\in Q_T} \hat{w}(x,t) \geq \frac{h_*}{\alpha}, \tag{5.5}$$

where "=" holds iff $h(x,t)$ is constant, and $\hat{w}^*$, $\hat{w}_* > 0$ are independent of β, m.

We establish the following results on the uniform persistence and the existence of positive T-periodic solutions to (5.1).

Theorem 5.2 (Uniform persistence and existence)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and (H0) hold. Then there exists a constant $m^* := m^*(\chi)$ satisfies*

$$\frac{r}{\hat{w}^*} < m^*(\chi) < \frac{r}{\hat{w}_*}, \quad \forall \chi \geq 0,$$

where the positive constants $\hat{w}^*$, $\hat{w}_*$ are defined in (5.5)*, such that the following statements hold:*

(i) If $m > m^*$, then $(0, \hat{w})$ is linearly stable;

(ii) If $m < m^*$, then $(0, \hat{w})$ is linearly unstable; moreover, the system (5.1) *admits at least one positive T-periodic solution, and the species u is uniformly persistent, i.e., there exists $\eta_0 > 0$ independent of initial data (u_0, w_0) such that*

$$\liminf_{t\to\infty} u(x,t) \geq \eta_0 \text{ uniformly for } x \in \overline{\Omega}. \tag{5.6}$$

The following result concerns the effects of toxicant taxis χ on the threshold value m^*.

Theorem 5.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and let $h(x,t) = a(x) + b(x,t) \geq 0$ satisfy* (H0)*. Then for given $a(x) \in C^{\kappa_0}(\overline{\Omega})$ and $\eta > 0$, there exists a small $\sigma(a,\eta) > 0$ such that if $0 < \|b\|_{L^\infty(Q_T)} < \sigma(a,\eta)$, then*

$$\liminf_{\chi\to\infty} m^*(\chi) \geq \frac{r}{\hat{w}_*^h} - \eta.$$

Remark 5.2. *When $\chi = 0$ and $\partial_t h \not\equiv 0$, Lemma 2.1 and Remark 3.1 in [86] demonstrate that the species u is uniformly persistent for $m \in (0, m^*(0))$, and (3.6) in [86] gives $\frac{r}{\hat{w}^*} < m^*(0) < \frac{r}{\hat{w}_*}$. Thus, Theorem 5.3 indicates that a large χ can enlarge the interval of uniform persistence, i.e., $(0, m^*(0)) \subset (0, m^*(\chi))$. This demonstrates that strong toxicant-taxis χ destabilizes the semi-trivial T-periodic solution $(0, \hat{w})$, and helps aquatic species to survive in a polluted environment.*

Next, we employ the energy estimates method to establish the global stability of T-periodic solutions of (5.1).

Theorem 5.4 (Global stability)**.** *Let (u, v) be the solution of (5.1) obtained in Theorem 5.1. The following results hold.*

(i) If $m > \frac{r(\alpha+\beta M_0)}{h_*}$ $(> m^*)$ with M_0 given in (5.3) and $h(x,t) > 0$, then there exist constants $C_1 > 0$ and $\theta_1 > 0$ independent of t such that

$$\|u\|_{L^\infty(\Omega)} + \|w - \hat{w}\|_{L^\infty(\Omega)} \leq C_1 e^{-\theta_1 t}, \quad \forall t > \tilde{t}_1, \tag{5.7}$$

for some constant $\tilde{t}_1 > 0$.

(ii) Assume $0 < m < \frac{\alpha r}{h^*}$ $(\leq m^*)$, $h(x,t) \equiv h(t)$ and

$$0 \leq \chi \leq \sqrt{\frac{4d_1 d_2 m}{\beta \max_{t\in(0,T)}\{u_*(t)\} \max_{t\in(0,T)}\{w_*(t)\}}} =: \chi_*, \tag{5.8}$$

as well as

$$0 < \beta < \beta_0 = \frac{4\alpha^2(\alpha r - mh*)^2}{(2\alpha r - mh^*)^2 h^* m}. \tag{5.9}$$

Then (5.1) admits a unique positive T-periodic solution $(u_*, w_*) \in [C^1([0,T])]^2$ depending on t only, such that

$$\|u - u_*\|_{L^\infty(\Omega)} + \|w - w_*\|_{L^\infty(\Omega)} \leq C_2 e^{-\theta_2 t}, \quad \forall t > \tilde{t}_2, \tag{5.10}$$

for some constants $C_2 > 0$ and $\tilde{t}_2 > 0$. Here, $C_2, \theta_2 > 0$ are independent of t.

Remark 5.3. *Global stability for constant h with $\chi > 0$ and nonconstant h with $\chi \equiv 0$ were established in [32] and [86], respectively. However, the case of nonconstant h with $\chi > 0$ remains open. Theorem 5.4 (in which $m \in (0, \frac{\alpha r}{h^*}) \cup (\frac{r(\alpha+\beta C_0)}{h_*}, \infty)$) provides a preliminary exploration (see a schematic in Figure 5.1), while the global dynamics for spatially nonconstant input rate $h(x,t) \not\equiv h(t)$ and intermediate $m \in (\frac{\alpha r}{h^*}, \frac{r(\alpha+\beta C_0)}{h_*})$ remain unknown.*

Throughout this chapter, c_i, C_i, m_i and $M_i (i = 1, 2, 3, \cdots)$ denote generic positive constants, which may vary in the context and are independent of t. Particularly, c_i, C_i may depend on the initial data (u_0, w_0) but m_i and M_i are independent of (u_0, w_0) and m.

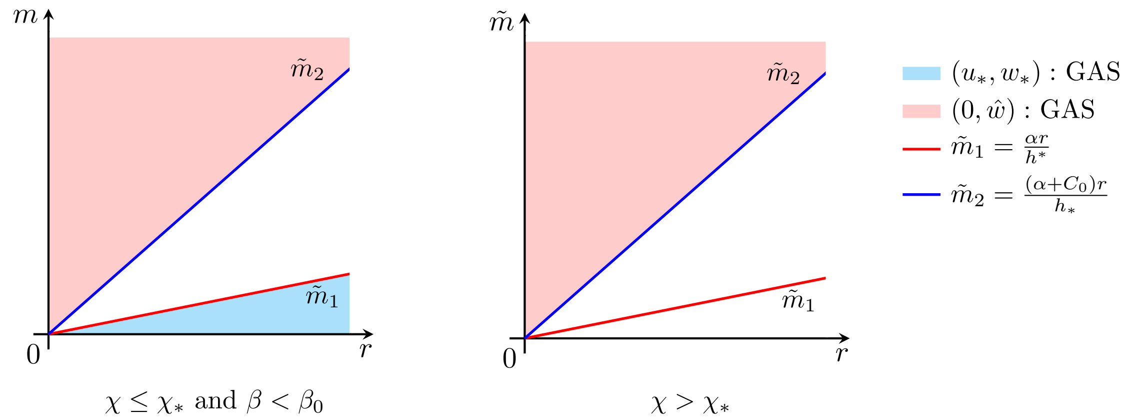


Figure 5.1: A graph of global stability regions for $(0, \hat{w})$ and the positive T-periodic solution (u_*, w_*), where GAS represents globally asymptotical stability.

5.2 Global Boundedness: Proof of Theorem 5.1

5.2.1 Local Existence and Preliminaries

We establish the existence and uniqueness of local classical solutions based on the classical Amann's theorem [6–8]. The positivity of u and w follows from the strong maximum principle.

Lemma 5.1. *Let the conditions in Theorem 5.1 hold. Then there is a* $T_{\max} \in (0, \infty]$ *such that* (5.1) *has a unique classical solution* $(u, w) \in [C(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))]^2$ *with* $u, w > 0$ *in* $\overline{\Omega} \times (0, T_{\max})$. *Moreover,*

$$\textit{if } T_{\max} < \infty, \ \textit{then} \lim_{t \nearrow T_{\max}} (\|w\|_{W^{1,\infty}} + \|u\|_{L^\infty}) = \infty. \tag{5.11}$$

Proof. The proof of Lemma 5.1 follows the same way as proof in [32, Lemma 3.1]. □

Lemma 5.2. *Let* (u, w) *be the solution of* (5.1) *obtained in Lemma 5.1. Then there exists a constant* $C_2 := C_2(u_0, v_0) > 0$ *such that*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \le C_2, \quad \forall t \in (0, T_{max}).$$

Proof. By slightly modifying the proof in [32] (replacing $\overline{h}$ in [32] with h^*), one gets Lemma 5.2. We omit the proof for brevity. □

Proof of Theorem 5.1(i). Theorem 5.1(i) is a consequence of the combination of Lemma 5.2 with Lemma 5.1. □

Next, we shall show that the solutions obtained in Theorem 5.1(i) is *ultimately uniformly bounded* in $C(\overline{\Omega})$. To this end, we start with the following estimate of w.

Lemma 5.3. *There exists a constant $M_1 > 0$ such that, for every classical solution (u, w) of (5.1), there exits a constant $T_1 > 0$ such that*

$$0 < w \leq M_1, \quad \forall t > T_1. \tag{5.12}$$

Proof. By the second equation in (5.1), one has

$$\frac{d}{dt}\big(\sup_{x\in\Omega} w\big) \leq h^* - \alpha \sup_{x\in\Omega} w. \tag{5.13}$$

Solving (5.13) directly yields $\sup_{x\in\Omega} w \leq e^{-\alpha t}\|w_0\|_{L^\infty} + \frac{h^*}{\alpha}$, which immediately implies the statement in Lemma 5.3. □

Lemma 5.4. *There exists a constant $M_2 > 0$ such that, for every classical solution (u, w) of (5.1), there exits a constant $T_2 > 0$ such that*

$$\|u(\cdot, t)\|_{L^1} \leq M_2, \quad \forall t > T_2, \tag{5.14}$$

and

$$\int_t^{t+1}\int_\Omega u^2(x, s)dxds \leq M_2, \quad \forall t > T_2. \tag{5.15}$$

Proof. We integrate u-equation in (5.1) over Ω by parts, and use Young's inequality to get

$$\begin{aligned}\frac{d}{dt}\int_\Omega u + \int_\Omega u + \int_\Omega u^2 + m\int_\Omega uw &= (r+1)\int_\Omega u \\ &\leq \frac{1}{2}\int_\Omega u^2 + \frac{(r+1)^2|\Omega|}{2}.\end{aligned}$$

This gives

$$\frac{d}{dt}\int_\Omega u + \int_\Omega u + \frac{1}{2}\int_\Omega u^2 \leq \frac{(r+1)^2|\Omega|}{2} =: m_1. \tag{5.16}$$

Then we have $\frac{d}{dt}\big(e^t\int_\Omega u\big) \leq m_1 e^t$, which implies

$$\int_\Omega u \leq m_1(1 - e^{-t}) + e^{-t}\int_\Omega u_0 \leq m_1 + e^{-t}\int_\Omega u_0. \tag{5.17}$$

For $m_1 > 0$, there is a constant $t_1 > 0$ such that $e^{-t}\int_\Omega u_0 \leq m_1$ for all $t > t_1$. Hence, (5.17) gives

$$\int_\Omega u \leq 2m_1, \quad \forall t > t_1.$$

Integrating (5.16) over $(t, t+1)$ yields

$$\frac{1}{2}\int_t^{t+1}\int_\Omega u^2(x,s)dxds \le m_1 + \int_\Omega u(\cdot,t) \le 3m_1, \forall t > t_1.$$

Then (5.14) and (5.15) are derived by letting $M_2 = 3(r+1)^2|\Omega|$ and $T_2 = t_1$. □

Lemma 5.5. *There exist constants $M_i > 0$ $(i = 3,4)$ such that, for every classical solution (u, w) of (5.1), there exist constants $T_i > 0$ $(i = 3,4)$ such that*

$$\|\nabla w(\cdot,t)\|_{L^2}^2 \le M_3, \quad \forall t > T_3, \tag{5.18}$$

and

$$\int_t^{t+1}\int_\Omega |\Delta w(x,s)|^2 dxds \le M_4, \quad \forall t > T_4. \tag{5.19}$$

Proof. We multiply the second equation in (5.1) by w and integrate the result equation over Ω by parts to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2 + d_2\int_\Omega |\nabla w|^2 + \alpha\int_\Omega w^2 + \beta\int_\Omega uw^2 = \int_\Omega h(x,t)w \le h^* M_1 \Omega,$$

which gives

$$\int_t^{t+1}\int_\Omega |\nabla w|^2 \le \frac{h^* M_1 \Omega}{d_2} + \frac{1}{2d_2}\int_\Omega w^2(\cdot, t+1) \le m_1, \ \forall t > T_1, \tag{5.20}$$

where $m_1 := \frac{2h^* M_1\Omega + M_1^2|\Omega|}{2d_2}$. Multiplying w-equation in (5.1) by $-\Delta w$, integrating the result by parts and applying Young's inequality, one obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla w|^2 + d_2\int_\Omega |\Delta w|^2 &= \int_\Omega [-h(x,t) + \alpha w + \beta uw]\Delta w \\ &\le \int_\Omega (h^* + \alpha w + \beta uw)|\Delta w| \\ &\le \frac{d_2}{2}\int_\Omega |\Delta w|^2 + m_2 + \frac{3\beta^2 M_1^2}{2d_2}\int_\Omega u^2, \forall t > T_1,\end{aligned} \tag{5.21}$$

where $m_2 := \frac{3(h^*)^2|\Omega| + 3\alpha^2 M_1^2|\Omega|}{2d_2}$. Then (5.21) implies

$$\frac{d}{dt}\int_\Omega |\nabla w|^2 + d_2\int_\Omega |\Delta w|^2 \le \int_\Omega |\nabla w|^2 + 2m_2 + \frac{3\beta^2 M_1^2}{d_2}\int_\Omega u^2, \forall t > T_1. \tag{5.22}$$

Combining (5.20), (5.22) with (5.15), and applying the uniform Gönwall inequality in [130, Lemma 1.1 in Chap.3] yield (5.18) by letting $M_3 := (m_1 + 2m_2 + 3\beta^2 M_1^2 M_2)e$ and $T_3 := \max\{T_1, T_2\}$.

Integrating (5.22) over $(t, t+1)$ along with (5.18) and (5.15), we obtain (5.19) by taking $M_4 := 2m_2/d_2 + 3\beta^2 M_1 M_2/d_2^2 + M_3/d_2$ and $T_4 :== \max\{T_1, T_2\}$. □

Lemma 5.6. *There exists a constant $M_5 > 0$ such that, for every classical solution (u, w) of* (5.1)*, there exists constant $T_5 > 0$ such that*

$$\int_t^{t+1} \int_\Omega |\nabla w(x,s)|^4 dxds \leq M_5, \quad \forall t > T_5. \tag{5.23}$$

Proof. Applying Gagliardo-Nirenberg inequality in two dimensional space, an equality in [12, Lemma 1] and (5.18), one gets

$$\begin{aligned} \|\nabla w\|_{L^4}^4 &\leq m_1 \|\nabla w\|_{L^2}^2 (\|\Delta w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2) \\ &\leq m_1 M_3 \|\Delta w\|_{L^2}^2 + m_1 M_3^2, \quad \forall t > T_3. \end{aligned} \tag{5.24}$$

Integrating (5.24) and applying (5.19), one derives (5.23) by taking $M_5 := m_1 M_3(M_4 + M_3)$ and $T_5 := \max\{T_3, T_4\}$. □

5.2.2 Ultimately Uniform Boundedness

Lemma 5.7. *There exists a constant $M_6 > 0$ such that, for every classical solution (u, w) of* (5.1)*, there exists constant $T_6 > 0$ such that*

$$\|u(\cdot, t)\|_{L^2} \leq M_6, \quad \forall t > T_6. \tag{5.25}$$

Proof. Multiplying u-equation in (5.1) by u and using Young's inequality yield

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u^2 + 2d_1 \int_\Omega |\nabla u|^2 + 2\int_\Omega u^3 + 2m \int_\Omega u^2 w \\ &= -2\chi \int_\Omega u \nabla w \cdot \nabla u + 2r \int_\Omega u^2 \\ &\leq d_1 \int_\Omega |\nabla u|^2 + \frac{\chi^2}{d_1}\int_\Omega u^2 |\nabla w|^2 + 2r \int_\Omega u^2. \end{aligned}$$

This gives

$$\frac{d}{dt}\int_\Omega u^2 + d_1 \int_\Omega |\nabla u|^2 + 2\int_\Omega u^3 + 2m \int_\Omega uw \leq \frac{\chi^2}{d_1}\int_\Omega u^2 |\nabla w|^2 + 2r \int_\Omega u^2. \tag{5.26}$$

To estimate the right terms in (5.26), we shall use the variant of the Poincaré inequality [54] as below:

$$\|v\|_{W^{1,p}} \leq c(\|\nabla v\|_{L^p(\Omega)} + \|v\|_{L^q(\Omega)}), \forall v \in W^{1,q}(\Omega) \tag{5.27}$$

for some constants $c > 0$, $p > 1$ and $q > 0$. And another inequality that Gagliardo-Nirenberg interpolation inequality [54, 101]:

$$\|v\|_{L^p} \leq \hat{c}\|v\|^{\lambda}_{W^{1,q}(\Omega)}\|v\|^{1-\lambda}_{L^{\theta}(\Omega)}, \forall v \in W^{1,q}(\Omega), \tag{5.28}$$

where constants $\hat{c} > 0$, $p, q \geq 1$ satisfying $p(n-q) < nq$, $\theta \in (0,p)$ with

$$\lambda = \frac{\frac{n}{\theta} - \frac{n}{p}}{\frac{n}{\theta} + 1 - \frac{n}{q}} \in (0,1).$$

Then applying (5.28), (5.27), (5.14) and Young's inequality, one has

$$\begin{aligned} 2r\int_{\Omega} u^2 = 2r\|u\|^2_{L^2} &\leq 2rm_1\|u\|_{W^{1,2}}\|u\|_{L^1} \\ &\leq 2rm_2\big(\|\nabla u\|_{L^2} + \|u\|_{L^1}\big)\|u\|_{L^1} \\ &\leq 2rm_2M_2\big(\|\nabla u\|_{L^2} + M_2\big) \\ &\leq \frac{d_1}{4}\|\nabla u\|^2_{L^2} + m_3, \quad \forall t > T_2, \end{aligned} \tag{5.29}$$

where $m_3 := \frac{4r^2m_2^2M_2^2}{d_1} + 2rm_2M_2^2$. On the other hand, using Hölder inequality, (5.28), (5.27) and (5.14), we obtain

$$\begin{aligned} \frac{\chi^2}{d_1}\int_{\Omega} u^2|\nabla w|^2 &\leq \frac{\chi^2}{d_1}\|u\|^2_{L^4}\|\nabla w\|^2_{L^4} \\ &\leq \frac{\chi^2}{d_1}m_4\|u\|_{L^2}\|u\|_{W^{1,2}}\|\nabla w\|^2_{L^4} \\ &\leq \frac{\chi^2}{d_1}m_5\|u\|_{L^2}(\|\nabla u\|_{L^2} + \|u\|_{L^1})\|\nabla w\|^2_{L^4} \\ &\leq \frac{d_1}{4}\|\nabla u\|^2_{L^2} + \frac{d_1}{4}\|u\|^2_{L^1} + \frac{2m_5^2\chi^4}{d_1^3}\|u\|^2_{L^2}\|\nabla w\|^4_{L^4} \\ &\leq \frac{d_1}{4}\|\nabla u\|^2_{L^2} + m_6 + m_6\chi^4\|u\|^2_{L^2}\|\nabla w\|^4_{L^4}, \quad \forall t > T_2, \end{aligned} \tag{5.30}$$

where $m_6 := \frac{d_1M_2^2}{4} + \frac{2m_5^2}{d_1^3}$.

Then the combination of (5.29) with (5.30) updates (5.26) as

$$\frac{d}{dt}\|u\|^2_{L^2} + \frac{d_1}{2}\|\nabla u\|^2_{L^2} \leq m_6\chi^4\|u\|^2_{L^2}\|\nabla w\|^4_{L^4} + m_3 + m_6. \tag{5.31}$$

Since $\int_t^{t+1}||\nabla w(x,s)||^4_{L^4} \leq M_5$ for all $t > T_5$ (see (5.23)) and $\int_t^{t+1}||u(\cdot,t)||^2_{L^2} \leq M_2$ for all $t > T_2$ (see (5.15)), we apply the uniform Grönwall inequality in [130, Lemma 1.1 in

Chap.3] to (5.31) and then obtain

$$\int_\Omega u^2(\cdot,t+1) \le (m_3+m_6+M_2)e^{m_6\chi^4 M_5} =: M_6, \quad \forall t > T_6 = \max\{T_2, T_5\},$$

this implies (5.25) directly. □

Lemma 5.8. *There exists a constant $M_7 > 0$ such that, for every classical solution (u, w) of (5.1), there exists constant $T_7 > 0$ such that*

$$\|\nabla w(\cdot,t)\|_{L^4} \le M_7, \quad \forall t > T_7. \tag{5.32}$$

Proof. We rewrite the second equation of (5.1) as

$$w_t - d_2\Delta w + \alpha w = h - \beta u w =: H(x,t). \tag{5.33}$$

Applying (5.12) and (5.25), one obtains that for all $t > \max\{T_1, T_6\} =: t_1$

$$\|H(\cdot,t)\|_{L^2} = \big\|h - \beta u w\big\|_{L^2} \le h^*|\Omega|^{\frac{1}{2}} + \beta M_1 M_6. \tag{5.34}$$

Denote the Neumann heat semigroup in Ω by $(e^{t\Delta})_{t>0}$. We apply Duhamel's principle to (5.33) and then get

$$w(\cdot,t) = e^{(t-t_1)(d_2\Delta-\alpha)}w(\cdot,t_1) + \int_{t_1}^t e^{(t-s)(d_2\Delta-\alpha)}H(\cdot,s)ds. \tag{5.35}$$

Using (5.34) and well-known semigroup estimate (see e.g., [18, Lemma 2.1]), if follows from (5.35) that

$$\begin{aligned}
\|\nabla w(\cdot,t)\|_{L^4} &\le \big\|\nabla e^{(t-t_1)(d_2\Delta-\alpha)}w(\cdot,t_1)\big\|_{L^4} + \int_{t_1}^t \big\|\nabla e^{(t-s)(d_2\Delta-\alpha)}H(\cdot,s)\big\|_{L^4}ds \\
&\le m_1 e^{-d_2\lambda_1(t-t_1)}\|\nabla w(\cdot,t_1)\|_{L^4} \\
&\quad + m_2\int_{t_1}^t \big(1+(t-s)^{-\frac{3}{4}}\big)e^{-d_2\lambda_1(t-s)}\|H(\cdot,s)\|_{L^2}ds \\
&\le m_1 e^{-d_2\lambda_1(t-t_1)}\|\nabla w(\cdot,t_1)\|_{L^4} \\
&\quad + m_2\big(h^*|\Omega|^{\frac{1}{2}} + \beta M_1 M_6\big)\int_0^\infty \big(1+z^{-\frac{3}{4}}\big)e^{-d_2\lambda_1 z}dz \\
&\le m_1 e^{-d_2\lambda_1(t-t_1)}\|\nabla w(\cdot,t_1)\|_{L^4} + m_3,
\end{aligned}$$

where λ_1 is the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions.

Since $\|\nabla w\|_{L^\infty} \le C_0$, then for $m_3 > 0$, we can take $T_7 > 0$ sufficiently large so that, for all $t > T_7 > t_1$, one has $\|\nabla w(\cdot,t)\|_{L^4} \le 2m_3$. Then (5.32) follows by letting $M_7 := 2m_3$. □

Lemma 5.9. *There exists a constant $M_8 > 0$ such that, for every classical solution (u, w) of (5.1), there exists constant $T_8 > 0$ such that*

$$\|u(\cdot,t)\|_{L^3} \le M_8, \quad \forall t > T_8. \tag{5.36}$$

Proof. We multiply u-equation in (5.1) by u^2 and integrate the result over Ω by parts to obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u^3 + 6d_1\int_\Omega u|\nabla u|^2 + 3m\int_\Omega wu^3 + 3\int_\Omega u^4 + 3\int_\Omega u^3 \\
&= -6\chi\int_\Omega u^2\nabla w\cdot\nabla u + 3(r+1)\int_\Omega u^3 \\
&\le 6\chi\int_\Omega u^2|\nabla w||\nabla u| + 3(r+1)\int_\Omega u^3.
\end{aligned} \tag{5.37}$$

Applying Young's inequality, Hölder inequality and (5.32), and using the fact $3d_1\int_\Omega u|\nabla u|^2 = \frac{4d_1}{3}\int_\Omega |\nabla u^{\frac{3}{2}}|^2 = \frac{4d_1}{3}\|\nabla u^{\frac{3}{2}}\|_{L^2}^2$, one derives that for all $t > T_7$

$$6\chi\int_\Omega u^2|\nabla w||\nabla u| \le 3d_1\int_\Omega u|\nabla u|^2 + \frac{3\chi^2}{d_1}\|u\|_{L^6}^3\|\nabla w\|_{L^4}^2 \le \frac{4d_1}{3}\|\nabla u^{\frac{3}{2}}\|_{L^2}^2 + \frac{3\chi^2M_7^2}{d_1}\|u\|_{L^6}^3,$$

which, along with Hölder inequality, upates (5.37) as

$$\frac{d}{dt}\|u\|_{L^3}^3 + 3\|u\|_{L^3}^3 + \frac{4d_1}{3}\left\|\nabla u^{\frac{3}{2}}\right\|_{L^2}^2 \le m_1\|u\|_{L^6}^3, \quad \forall t > T_7, \tag{5.38}$$

where $m_1 := \frac{3\chi^2M_7^2}{d_1} + 3(r+1)|\Omega|^{\frac{1}{2}}$. Applying Gagliardo-Nirenberg inequality in two dimensional space and Young's inequality derives

$$\begin{aligned}
m_1\|u\|_{L^6}^3 = m_1\|u^{\frac{3}{2}}\|_{L^4}^2 \le& m_2\big(\|\nabla u^{\frac{3}{2}}\|_{L^2}^{\frac{4}{3}}\|u^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^{\frac{2}{3}} + \|u^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^2\big) \\
=& m_2\big(\|\nabla u^{\frac{3}{2}}\|_{L^2}^{\frac{4}{3}}\|u\|_{L^2} + \|u\|_{L^2}^3\big) \\
\le& m_2M_6\|\nabla u^{\frac{3}{2}}\|_{L^2}^{\frac{4}{3}} + m_2M_6^3 \\
\le& \frac{4d_1}{3}\|\nabla u^{\frac{3}{2}}\|_{L^2}^2 + m_3, \quad \forall t > T_6,
\end{aligned} \tag{5.39}$$

where $m_3 := \big(\frac{m_2^3}{12d_1^2} + m_2\big)M_6^3$. We substitute (5.39) into (5.38) to obtain

$$\frac{d}{dt}\|u\|_{L^3}^3 + 3\|u\|_{L^3}^3 \le m_3, \forall t > \max\{T_6, T_7\} =: t_1,$$

which implies

$$\|u\|_{L^3}^3 \le e^{-3(t-t_1)}\|u(\cdot,t_1)\|_{L^3}^3 + \frac{m_3}{3}.$$

Since $\|u\|_{L^\infty} \le C_0$ (see (5.2)), for $\frac{m_3}{3}$, we can find a $T_8 > 0$ sufficiently large so that, for all $t > T_8 \ge t_1$, it holds that $\|u\|_{L^3} \le \big(2m_3/3\big)^{\frac{1}{3}} =: M_8$. This gives (5.36) directly. □

Lemma 5.10. *There exist constants $M_i > 0$ $(i = 9, 10)$ such that, for every classical solution (u, w) of (5.1), there exist constants $T_i > 0$ $(i = 9, 10)$ such that*

$$\|\nabla w(\cdot, t)\|_{L^\infty} \le M_9, \quad \forall t > T_9. \tag{5.40}$$

and

$$\|u(\cdot, t)\|_{L^\infty} \le M_{10}, \quad \forall t > T_{10}. \tag{5.41}$$

Proof. By (5.12) and (5.36), one has

$$\|H(\cdot, t)\|_{L^3} = \|h - \beta uw\|_{L^3} \quad \le h^*|\Omega|^{\frac{1}{3}} + \beta M_1 M_8 =: m_1, \forall t > \max\{T_1.T_8\} =: t_1. \tag{5.42}$$

Applying the semigroup estimate (see e.g., [18, Lemma 2.1]) to (5.35) and using (5.42) and (5.2), we obtain

$$\begin{aligned}
\|\nabla w(\cdot, t)\|_{L^\infty} &\le \|\nabla e^{(t-t_1)(d_2\Delta - \alpha)} w(\cdot, t_1)\|_{L^\infty} + \int_{t_1}^{t} \big\|\nabla e^{(t-s)(d_1\Delta - \alpha)} H(\cdot, s)\big\|_{L^\infty} ds \\
&\le m_2 e^{-d_2\lambda_1(t-t_1)}\big(1 + (t-t_1)^{-\frac{1}{2}}\big)\big\|\nabla w(\cdot, t_1)\big\|_{L^\infty} \\
&\quad + m_3 \int_{t_1}^{t} \big(1 + (t-s)^{-\frac{5}{6}}\big) e^{-d_2\lambda_1(t-s)} \|H(\cdot, s)\|_{L^3} ds \\
&\le m_2 e^{-d_2\lambda_1(t-t_1)}\big(1 + (t-t_1)^{-\frac{1}{2}}\big)\|\nabla w(\cdot, t_1)\|_{L^\infty} + m_4 \\
&\le 2m_4, \quad \forall t > T_9 > t_1,
\end{aligned} \tag{5.43}$$

for some large constant $T_9 > 0$. Then (5.43) gives (5.40) with $M_9 := 2m_4$.

Next, multiplying u-equation in (5.1) by $u^{p-1} (p \ge 2)$, integrating the result over Ω, and applying (5.40) as well as Young's inequality, we obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + d_1(p-1)\int_\Omega u^{p-2}|\nabla u|^2 + m\int_\Omega wu^p + \int_\Omega u^{p+1} \\
&= -\chi(p-1)\int_\Omega u^{p-1}\nabla u \cdot \nabla w + r\int_\Omega u^p \\
&\le \chi(p-1)M_9 \int_\Omega u^{p-1}|\nabla u| + r\int_\Omega u^p \\
&\le \frac{d_1(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + m_1(p-1)\int_\Omega u^p, \forall t > T_9,
\end{aligned}$$

where $m_1 := \frac{\chi^2 M_9^2}{2d_1} + r$ is independent of p. Hence for any $t > T_9$,

$$\frac{d}{dt}\int_\Omega u^p + p(p-1)\int_\Omega u^p \le -\frac{p(p-1)d_1}{2}\int_\Omega u^{p-2}|\nabla u|^2 + m_2 p(p-1)\int_\Omega u^p, \tag{5.44}$$

where $m_2 = m_1 + 1$ is independent of p. Then, we follow the same way as the proof in [23, Lemma 3.6] to find a constant m_3 such that

$$\begin{aligned} \|u(\cdot,t)\|_{L^\infty} &\le m_3 \max\left\{e^{-(t-T_9)}\|u(\cdot,T_9)\|_{L^\infty}, \|u(\cdot,t)\|_{L^1}\right\} \\ &\le 2m_3M_2, \quad \forall t > T_{10} \ge T_9, \end{aligned} \tag{5.45}$$

for some large constant $T_{10} > 0$. Hence, (5.41) follows by taking $M_{10} := 2m_3M_2$. □

Proof of Theorem 5.1(ii). The combination of Lemma 5.3 and Lemma 5.10 implies the statement in Theorem 5.1(ii). □

5.3 Uniform Persistence and Existence: Proof of Theorems 5.2 and 5.3

The objective of this section is to study the local stability of $(0,\hat{w})$, and to explore the uniform persistence and the existence of positive T-periodic solutions. Finally, we shall explore the effects of cross-diffusion on the critical point $m^*(\chi)$

5.3.1 Local Stability of $(0,\hat{w})$

In this subsection, we study the local stability of $(0,\hat{w})$, where $\hat{w}$ is the unique solution to (5.4) and is independent of m and χ. Then it follows from [62, Lemma 2.2] that

$$\|\hat{w}\|_{C^{2+\kappa_0,1+\frac{\kappa_0}{2}}(\overline{Q}_T)} \le C_3\|h\|_{C^{\kappa_0,\frac{\kappa_0}{2}}(\overline{Q}_T)}. \tag{5.46}$$

Next, we linearize (5.1) at $(0,\hat{w})$ to get

$$\begin{cases} \phi_t = d_1\Delta\phi + \chi\nabla\cdot(\phi\nabla\hat{w}) + (r - m\hat{w})\phi, & x\in\Omega, t>0, \\ \psi_t = d_2\Delta\psi - \alpha\psi - \beta\phi\hat{w}, & x\in\Omega, t>0, \\ \partial_\nu\phi = \partial_\nu\psi = 0, & x\in\partial\Omega, t>0. \end{cases} \tag{5.47}$$

To obtain the linear stability of $(0,\hat{w})$ to (5.1), it suffices to study the linearized eigenvalue problem as follows:

$$\begin{cases} \phi_t - d_1\Delta\phi - \chi\nabla\cdot(\phi\nabla\hat{w}) - (r - m\hat{w})\phi = \lambda\phi, & \text{in } \Omega\times[0,T], \\ \psi_t - d_2\Delta\psi + \alpha\psi + \beta\phi\hat{w} = \lambda\psi, & \text{in } \Omega\times[0,T], \\ \partial_\nu\phi = \partial_\nu\psi = 0, & \text{on } \partial\Omega\times[0,T], \\ \phi(x,0) = \phi(x,T), \psi(x,0) = \psi(x,T), & \text{in } \Omega. \end{cases} \tag{5.48}$$

It follows from [77, Theorem 2.1.1] that the first equation in (5.48) with $\partial_\nu \phi = 0$ and $\phi(x, t+T) = \phi(x,t)$ admits a unique principal eigenvalue $\lambda_* := \lambda_*(\chi, m) \in \mathbb{R}$ with a positive eigenfunction ϕ_*, which is unique up to constant multiple. Moreover, using the similar discussion in [59, Lemma 3.2], we know that the linear stability of $(0, \hat{w})$ is determined by the sign of λ_*, i.e., $(0, \hat{w})$ is linearly stable if $\lambda_* > 0$ and unstable if $\lambda_* < 0$.

Lemma 5.11. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and (H0) holds. There admits a unique constant $m^* := m^*(\chi) > 0$ such that $(0, \hat{w})$ is linearly stable if $m > m^*$, and is linearly unstable if $0 < m < m^*$.*

Proof. Note that $\lambda_* \in \mathbb{R}$ and $\phi_* > 0$ satisfy

$$\begin{cases} (\phi_*)_t - d_1 \Delta \phi_* - \chi \nabla \cdot (\phi_* \nabla \hat{w}) - (r - m\hat{w})\phi_* = \lambda_* \phi_*, & \text{in } \Omega \times [0, T], \\ \partial_\nu \phi_* = 0, & \text{on } \partial\Omega \times [0, T], \\ \phi_*(x, 0) = \phi_*(x, T), & \text{in } \Omega. \end{cases} \tag{5.49}$$

We integrate the first equation in (5.49) over Q_T by parts to get

$$-\int_{Q_T} (r - m\hat{w})\phi_* = \lambda_* \int_{Q_T} \phi_*. \tag{5.50}$$

Since $\phi_* > 0$, by (5.50) and the definitions of $\hat{w}_*,\ \hat{w}^*$ in (5.5), one has

$$\lambda_* > 0, \text{ if } m > \frac{r}{\hat{w}_*}; \quad \lambda_* < 0, \text{ if } 0 < m < \frac{r}{\hat{w}^*}. \tag{5.51}$$

We deduce from [17, Lemma 2.15] that if $m < \hat{m}$, $\lambda_*(\chi, m) < \lambda_*(\chi, \hat{m})$. This combined with (5.51) shows that there is a unique $m^* := m^*(\chi)$ satisfying $\frac{r}{\hat{w}^*} < m^* < \frac{r}{\hat{w}_*}$ such that

$$\lambda_* > 0 \text{ if } m > m^*; \quad \lambda_* < 0 \text{ if } 0 < m < m^*.$$

This proves Lemma 5.11. □

Next, we shall prove uniform persistence and the existence of positive T-periodic solutions. To this end, we first improve the regularity of (u, w), a key result will be used in later.

Lemma 5.12. *Let (u, w) be the solution obtained in Theorem 5.1.Then there exist constants $\gamma \in (0, 1)$ and $M_7 > 0$ independent of u_0, w_0 such that*

$$\|(u, w)\|_{C^{2+\gamma, 1+\frac{\gamma}{2}}(\overline{\Omega} \times [T_0+1, \infty))} \le M_7. \tag{5.52}$$

Proof. Rewriting (1.6) as

$$\begin{cases} u_t = d_1\Delta u + \chi\nabla w\cdot\nabla u + G_1(x,t), & x\in\Omega, t>0,\\ w_t = d_2\Delta w + G_2(x,t), & x\in\Omega, t>0,\\ \partial_\nu u = \partial_\nu w = 0, & x\in\partial\Omega, t>0,\\ (u,w)(x,0) = (u_0,w_0), & x\in\Omega, \end{cases}$$

where $G_1(x,t) := u(r - mw - u - \chi\Delta w)$ and $G_2(x,t) := h - \alpha w - \beta uw$. Then it is easy to check that (5.52) is a consequence of the interior L^p estimate ([87, Theorems 7.30 and 7.35]), interior Schauder estimate for parabolic equation [76] as well as the *eventual uniform boundedness* (see (5.68)). The proof details can follow the similar procedures as the proof in [140, Theorem 2.1], we omit here for brevity. □

5.3.2 Uniform Persistence and Positive T-Periodic Solution

In this subsection, we establish the uniform persistence and the existence of positive T-periodic solutions. The proof mainly based on the results of uniform persistence and coexistence states for general dynamical systems developed in [156, 157] (see also [117]) as well as the Principal Floquet bundle theory [77, Chapter 4]. To proceed, we first recall some basic notations and definitions.

Let X be a complete metric space with a metric d. By [43] and [157, Chapter 3.1], we say that $\{\Psi(t) : X \to X\}_{t\geq 0}$ is a T-periodic (autonomous) semiflow on X if there is a $T > 0$ (for every $T > 0$) such that $(t, z) \mapsto \Psi(t)z : [0,\infty) \times X \to X$ is jointly continuous in (t, z), $\Psi(0)z = z$ for all $z \in X$ and $\Psi(t + T) = \Psi(t)\Psi(T)$ for all $t \geq 0$. Assume that $X_0 \subset X$ and $\partial X_0 \subset X$ are open and closed sets, respectively, satisfying $X_0 \cap \partial X_0 = \emptyset$ and $X = X_0 \cup \partial X_0$. Let $\Psi(t) : X \to X (t \geq 0)$ be a semiflow and $\Psi(t)X_0 \subset X_0, t \geq 0$, $\Psi(t)$ is **uniformly persistent** with respect to $(X_0, \partial X_0)$ if there is $\eta > 0$ such that

$$\liminf_{t\to\infty} d\big(\Psi(t)v, \partial X_0\big) \geq \eta, \quad \forall v \in X_0.$$

Here, we choose

$$X = \{(v,z)(x) | v \in C(\overline{\Omega}),\ z \in C^1(\overline{\Omega}),\ v(x) \geq 0, z(x) \geq 0, \forall x \in \Omega\}$$

with the norm $\|(v,z)\|_X = \|v\|_{C(\overline{\Omega})} + \|z\|_{C^1(\overline{\Omega})}$, and

$$X_0 = \{(v,z)(x) \in X |\ v(x) \not\equiv 0\}, \quad \partial X_0 = \{(v,z)(x) \in X |\ v(x) \equiv 0\},$$

then $X = X_0 \cup \partial X_0$. By Theorem 5.1(i), for every $(u_0, w_0) \in X$, (5.1) admits a unique global classical solution $\Theta(t, (u_0, w_0)) := (u, w)(x, t) \in X$. Hence, when $\partial_t h \not\equiv 0$, we can define semiflow $\Psi(t) : X \to X$ by

$$\Psi(t)(u_0, w_0) = \Theta(t, (u_0, w_0)), \quad (u_0, w_0) \in X, t \geq 0.$$

Clearly, $\Psi(t)X_0 \subset X_0$ and $\Psi(t)\partial X_0 \subset \partial X_0$ for any $t \geq 0$. For a given periodic system, from [157, Theorem 3.1.1], we know that studying the uniform persistence can be reduced to study the uniform persistence of its corresponding Poincaré map.

Since the toxicant input rate $h(x, t)$ in (5.1) is T-periodic, then the associated semiflow $\Psi(t)$ is also T-periodic, and the Poincaré map $S : X \to X$ can be defined by $S(v, z) = \Psi(T)(v, z) = \Theta(T, (v, z))$ for any $(v, z) \in X$. We shall demonstrate that the linear instability of the semi-trivial T-periodic solution $(0, \hat{w})$ implies that it is a uniform weak repeller by using the principal Floquet bundle theory [77, Chapter 4].

Proposition 5.1. *If $\lambda_* < 0$, then $(0, \hat{w})$ is a uniform weak repeller for X_0 in the sense that there exists a positive constant $\delta > 0$ such that*

$$\limsup_{j \to \infty} \|S^j(u_0, w_0) - (0, \hat{w})\|_X \geq \delta, \ \forall (u_0, w_0) \in X_0.$$

Proof. Suppose, by contradiction, that there exists $(\tilde{u}_0, \tilde{w}_0) \in X_0$ such that

$$\limsup_{j \to \infty} \|S^j(\tilde{u}_0, \tilde{w}_0) - (0, \hat{w})\|_X = 0. \tag{5.53}$$

Next, we shall divide our proof as three steps.

Step 1. For any $t \geq 0$, there exists some $j \in N^+$ such that $t = jT + t'$ with $t' \in [0, T)$, we have

$$\begin{aligned}(\tilde{u}, \tilde{w})(x, t) := \Theta(t, (\tilde{u}_0, \tilde{w}_0)) &= \Theta(jT + t', (\tilde{u}_0, \tilde{w}_0)) \\ &= \Theta(t', \Theta(jT, (\tilde{u}_0, \tilde{w}_0))) \\ &= \Theta(t', S^j(\tilde{u}_0, \tilde{w}_0)).\end{aligned} \tag{5.54}$$

The continuous dependence of solution on initial data, together with (5.53) and (5.54), implies

$$\lim_{t \to \infty} (\|\tilde{u}(\cdot, t)\|_{L^\infty} + \|\tilde{w}(\cdot, t) - \hat{w}\|_{L^\infty}) = 0. \tag{5.55}$$

On the other hand, (5.52) and (5.46) implies that for $\gamma_1 := \min\{\gamma, \kappa_0\}$,

$$\|(\tilde{u}, \tilde{w} - \hat{w})\|_{C^{2+\gamma_1, 1+\frac{\gamma_1}{2}}(\bar{\Omega} \times [t, t+1])} \leq c_1, \quad \forall t > T_0 + 1.$$

By [109, Lemma 4], we get that $C^{2+\gamma_1,1+\frac{\gamma_1}{2}}(\overline{\Omega}\times[t,t+1])$ is compactly embedded into $C^{2+\gamma_0,1+\frac{\gamma_0}{2}}(\overline{\Omega}\times[t,t+1])$ for any $\gamma_0 : 0<\gamma_0<\gamma_1$. This together with the (5.55) gives that (taking a subsequence if necessary)

$$\lim_{t\to\infty}\left(\|\tilde{u}\|_{C^{2+\gamma_0,1+\frac{\gamma_0}{2}}(\overline{\Omega}\times[t,t+1])}+\|\tilde{w}(\cdot,t)-\hat{w})\|_{C^{2+\gamma_0,1+\frac{\gamma_0}{2}}(\overline{\Omega}\times[t,t+1])}\right)=0,\ \ \forall t>T_0+1. \quad (5.56)$$

Step 2. Consider the following equation

$$\begin{cases} z_t=d_1\Delta z+\chi\nabla\cdot(z\nabla\hat{w})+z(r-m\hat{w}), & x\in\Omega, t>0,\\ \partial_\nu z=0, & x\in\partial\Omega, t>0,\\ z(x,0)=z_0(x), & x\in\Omega. \end{cases} \quad (5.57)$$

Let (λ_*,ϕ_*) be the eigenpair of (5.49), then it is clear that $\phi_* e^{-\lambda_* t}$ satisfies the first two equations of (5.57).

Now, by (5.46), we get $\hat{w}\in C^{2+\gamma_0,1+\frac{\gamma_0}{2}}(\bar{\Omega}\times[0,T])$. Hence, it is clear that $\hat{w}$ extends smoothly (and periodically) to $C^{2+\gamma_0,1+\frac{\gamma_0}{2}}(\bar{\Omega}\times\mathbb{R})$, this smooth extension of $\hat{w}$ is denoted by $\underline{\hat{w}}$. We then fix a smooth extension $\underline{w}\in C^{2+\gamma_0,1+\frac{\gamma_0}{2}}(\bar{\Omega}\times\mathbb{R})$ of $\tilde{w}(x,t)$ satisfying

$$\underline{w}(x,t)=\tilde{w}(x,t), \quad x\in\Omega, t\in[t_0,\infty). \quad (5.58)$$

Moreover, using (5.56), for arbitrary $\delta_0>0$ (which will be specified later), we can choose $t_0>1$ in (5.58) to ensure that

$$\|\underline{\hat{w}}\|_{C^{2+\gamma_0,1+\frac{\gamma_0}{2}}(\overline{\Omega}\times\mathbb{R})}\le c_1 \quad \text{and} \quad \|\underline{w}-\underline{\hat{w}}\|_{C^{2+\gamma_0,1+\frac{\gamma_0}{2}}(\overline{\Omega}\times\mathbb{R})}\le\delta_0. \quad (5.59)$$

For this given $\underline{w}(x,t)\in C^{2+\gamma_0,1+\frac{\gamma_0}{2}}(\overline{\Omega}\times\mathbb{R})$, we deduce from [77, Theorem 4.2.2] that there exists a unique ordered triple

$$(P(x,t),I(x,t),\Lambda^{\tilde{w}}(t))\in[C^{2+\gamma_0,1+\frac{\gamma_0}{2}}(\overline{\Omega}\times\mathbb{R})]^2\times C^{\frac{\gamma_0}{2}}(\mathbb{R})$$

fulfilling

$$\begin{cases} P_t=d_1\Delta P+\chi\nabla\cdot(P\nabla\underline{w})+(r-m\underline{w})P+\Lambda^{\tilde{w}}(t)P, & x\in\Omega, t\in\mathbb{R},\\ \partial_\nu P=0, & x\in\partial\Omega, t\in\mathbb{R},\\ P>0,\ (x,t)\in\Omega\times\mathbb{R},\ \text{and}\ \int_\Omega P(x,t)dx=1, & t\in\mathbb{R}, \end{cases} \quad (5.60)$$

and

$$\begin{cases} -I_t=d_1\Delta I-\chi\nabla\underline{w}\cdot\nabla I+(r-m\underline{w})I+\Lambda^{\tilde{w}}(t)I, & x\in\Omega, t\in\mathbb{R},\\ \partial_\nu I=0, & x\in\partial\Omega, t\in\mathbb{R},\\ I>0,\ (x,t)\in\Omega\times\mathbb{R},\ \text{and}\ \int_\Omega P(x,t)I(x,t)dx=1, & t\in\mathbb{R}. \end{cases} \quad (5.61)$$

Moreover, there is a constant $c_2 > 0$ such that

$$\frac{1}{c_2} \le P(x,t) \le c_2, \quad \frac{1}{c_2} \le I(x,t) \le c_2, \quad \forall (x,t) \in \Omega \times \mathbb{R}. \tag{5.62}$$

Notice that if we replace $\underline{w}$ by the periodic function $\hat{w}$, then it can be verified that

$$P(x,t) = q(t)\phi_*(x,t), \quad I(x,t) = p(t)\psi_*(x,t), \quad \Lambda^{\hat{w}}(t) = \lambda_* + \frac{q'(t)}{q(t)},$$

where $(\lambda_*, \phi_*(x,t))$ are the eigenpair of (5.49) and $\psi_*(x,t)$ is the eigenfunction of the adjoint T-periodic problem, $q(t) = \frac{1}{\int_\Omega \phi_*(x,t)dx}$ and $p(t) = \frac{\int_\Omega \phi_*(x,t)\,dx}{\int_\Omega \phi_*(x,t)\psi_*(x,t)\,dx}$. It is also obvious that $q(t)$ is T-periodic and so that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{q'(s)}{q(s)}\,ds = 0. \tag{5.63}$$

Recall also the main hypothesis of the theorem that $\lambda_* < 0$.

By the smooth dependence of $\Lambda^{\hat{w}}(t)$ on the coefficients in (5.60) (cf. [77, Theorem 4.34]), from (5.56), (5.58) and the equations (5.49), (5.60), we deduce that if $\delta_0 > 0$ in (5.59) is chosen small enough, then the principal Floquet bundle with $w = \tilde{w}$ and $w = \hat{w}$ is close to each other. In particular,

$$\left\|\Lambda^{\tilde{w}}(t) - \lambda_* - \frac{q'(t)}{q(t)}\right\|_{C^{\frac{\gamma_0}{2}}(\mathbb{R})} \le -\frac{\lambda_*}{2},$$

this and (5.63) implies

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t -\Lambda^{\tilde{w}}(s)\,ds \ge -\frac{\lambda_*}{2} > 0. \tag{5.64}$$

Step 3. Multiplying $\tilde{u}$-equation in (5.1) by I and I-equation in (5.61) by $\tilde{u}$, and then integrating the result over Ω by parts, we have

$$\frac{d}{dt}\int_\Omega \tilde{u}I = \int_\Omega \tilde{u}I_t - d_1\int_\Omega \nabla\tilde{u}\cdot\nabla I - \chi\int_\Omega \tilde{u}\nabla\tilde{w}\cdot\nabla I + \int_\Omega \tilde{u}I(r - m\tilde{w} - \tilde{u}), \quad \forall t > 0, \tag{5.65}$$

and

$$\int_\Omega I_t\tilde{u} - d_1\int_\Omega \nabla\tilde{u}\cdot\nabla I - \chi\int_\Omega \tilde{u}\nabla\tilde{w}\cdot\nabla I + \int_\Omega (r - m\tilde{w})I\tilde{u} = -\Lambda^{\tilde{w}}(t)\int_\Omega I\tilde{u}, \quad \forall t > 0,$$

which combined with (5.65) derives

$$\frac{d}{dt}\int_\Omega \tilde{u}I = -\Lambda^{\tilde{w}}(t)\int_\Omega \tilde{u}I - \int_\Omega \tilde{u}^2 I. \tag{5.66}$$

Using $\tilde{u} \to 0$ uniformly, we may choose $t_1 \in [t_0, \infty)$ such that $\sup_{[t_1,\infty)} \|\tilde{u}\|_\infty < -\frac{\lambda_*}{4}$, which gives

$$\frac{d}{dt}\int_\Omega \tilde{u}I \geq \left(-\Lambda^{\tilde{w}}(t) + \frac{\lambda_*}{4}\right)\int_\Omega \tilde{u}I \quad \text{for } t \in [t_1, \infty). \tag{5.67}$$

Solving (5.67) directly gives

$$\begin{aligned}\int_\Omega \tilde{u}I &\geq \exp\left((t-t_1)\left(\frac{1}{t-t_1}\int_{t_1}^t -\Lambda^{\tilde{w}}(s)\,ds + \frac{\lambda_*}{4}\right)\right)\int_\Omega \tilde{u}(x,t_1)I(x,t_1)\,dx \\ &= \exp((t-t_1)\left(-\frac{\lambda_*}{4} + o(1)\right)\int_\Omega \tilde{u}(x,t_1)I(x,t_1)\,dx.\end{aligned}$$

Since $\lambda_* < 0$, this contradicts that $\int_\Omega \tilde{u}I \to 0$ (due to (5.55) and (5.62)) as $t \to \infty$. Therefore, the proof of Proposition 5.1 is finished. □

Proof of Theorem 5.2. Lemma 5.11 gives a description of the linear stability and instability of $(0, \hat{w})$. In what follows, we focus on showing the uniform persistence and the existence of positive T-periodic solutions.

With (5.3), [78, Theorem 2.2] implies that there exist some constants $\kappa > 1$ and $M_{11} > 0$ independent of initial data (u_0, w_0) such that

$$\|u(\cdot,t)\|_{C^\kappa(\overline{\Omega})} + \|w(\cdot,t)\|_{C^\kappa(\overline{\Omega})} \leq M_{11}, \quad \forall t > T_{11}, \tag{5.68}$$

for some constants $T_{11} > 0$ and $\kappa > 1$. Since $\kappa > 1$, it follows from Arzelà-Ascoli theorem that $C^\kappa(\overline{\Omega})$ is embedded compactly into $C^1(\overline{\Omega})$ and $C(\overline{\Omega})$, this along with (5.68) indicates that

$$\Psi(t) \text{ is point dissipative in } X, \text{ and it is compact in } X \text{ for each } t > T_{11}. \tag{5.69}$$

Thus, [44, Theorem 2.2] (or [42, Theorem 2.4.7]) implies that

$$\Psi(t) \text{ has a global attractor in } X. \tag{5.70}$$

To prove the uniform persistence of the T-periodic semiflow $\Phi(t)$, it suffices to show that the Poincaré map $S : X \to X$ is uniformly persistent (cf. [157, Theorem 3.1.1]). From the above statements, one can see that $S : X \to X$ is point dissipative, continuous, $S(X_0) \subset X_0$ and S^{n_0} is compact for some integer $n_0 \geq 1$. Then we deduce from [157, Theorem 1.1.3] that $S : X \to X$ has a global attractor B.

Denote A_∂ as the maximal compact invariant sets of S in ∂X_0. We claim that A_∂ consists of a single point. Indeed, if $(u_0, w_0) \in A_\partial$, then $u_0 \equiv 0$, this gives the existence of an entire solution w to

$$\begin{cases} w_t = d_2 \Delta w + h(x,t) - \alpha w, & x \in \Omega, t \in \mathbb{R}, \\ \partial_\nu w = 0, & x \in \partial\Omega, t \in \mathbb{R}, \\ w(x,0) = w_0, \quad x \in \Omega \quad \text{and} & \sup_{t\in\mathbb{R}} \|w\|_{L^\infty} \le C. \end{cases} \tag{5.71}$$

On the other hand, $\hat{w}$ satisfies (5.4), which along with (5.71) indicates that $W := w - \hat{w}$ satisfies

$$\begin{cases} W_t = d_2 \Delta W - \alpha W, & x \in \Omega, t \in \mathbb{R}, \\ \partial_\nu W = 0, & x \in \partial\Omega, t \in \mathbb{R}, \\ W(x,0) = w_0(x) - \hat{w}(x,0), & x \in \Omega. \end{cases} \tag{5.72}$$

Multiplying the first equation in (5.72) by W and integrating the result over Ω by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega W^2 + \alpha \int_\Omega W^2 = -d_2 \int_\Omega |\nabla W|^2 \le 0 \quad \text{ for } t \in \mathbb{R}.$$

This motives us to find a constant $\kappa_1 > 0$ such that

$$\|w - \hat{w}\|_{L^2} = \|W\|_{L^2} \le \|w(\cdot, \underline{t}) - \hat{w}(\cdot, \underline{t})\|_{L^2} e^{-\kappa_1 (t - \underline{t})}, \quad \forall t \ge \underline{t}. \tag{5.73}$$

Letting $\underline{t} \to -\infty$, we deduce that $w \equiv \hat{w}$. This shows $A_\partial = \{(0, \hat{w})\}$.

Next, we claim that if $\lambda_* < 0$, then S is uniformly weakly persistent with respect to $(X_0, \partial X_0)$, i.e., there exists $\eta_1 > 0$ such that $\limsup_{j\to\infty} \|S^j(v,z) - \partial X_0\|_X \ge \eta_1$ for all $(v,z) \in X_0$. Indeed, suppose not, then same arguments as (5.54) and (5.55) imply $\lim_{t\to\infty} \|\tilde{u}\|_{L^\infty} = 0$ for some initial data $(\tilde{v}, \tilde{z}) \in X_0$, and hence the omega limit set $\omega(\tilde{v}, \tilde{z})$ of S is a subset of ∂X_0. Then $\omega(\tilde{v}, \tilde{z}) \subset A_\partial$. This implies that $\lim_{j_k\to\infty} \|S^{j_k}(\tilde{v}, \tilde{z}) - (0, \hat{w})\|_X = 0$, which is impossible in view of Proposition 5.1. Since $S : X \to X$ has a global attractor B and S is uniformly weakly persistent, then we can apply [157, Theorem 1.3.3] to conclude that S is uniform persistent with respect to $(X_0, \partial X_0)$ and hence [156, Theorem 2.1] indicates that T-periodic semiflow $\{\Psi(t)\}_{t\ge 0}$ is uniformly persistent in the following sense

$$\liminf_{t\to\infty} \|u(x,t)\|_{C(\Omega)} \ge m_1. \tag{5.74}$$

On the other hand, we derive from (5.1) that

$$\begin{cases} u_t = d_1 \Delta u + \chi \nabla w \cdot \nabla u + u(r - u - mw + \chi \Delta w), & x \in \Omega, \ t > 0, \\ \partial_\nu u = 0, & x \in \partial\Omega, \ t > 0. \end{cases} \tag{5.75}$$

Then (5.52) implies that for all $t \geq T_0 + 1$, one has

$$\|\nabla w\|_{L^\infty} \leq m_2, \quad \text{and } \|r - u - mw + \chi \Delta w\|_{L^\infty} \leq m_3.$$

This combined with Harnack inequality [58, Theorem 2.5] gives that

$$\sup_{x \in \Omega} u(x,t) \leq m_4 \inf_{x \in \Omega} u(x,t), \tag{5.76}$$

for some constant $m_4 > 0$. Taking the inferior limit in time t on both sides of (5.76) and applying (5.74) directly yields $\liminf_{t \to \infty} \inf_{x \in \Omega} u \geq \frac{m_1}{m_4}$. This proves (5.6).

Finally, we prove the existence of positive T-periodic solutions. From Theorems 1.3.7 in [157], we get that $S: X_0 \to X_0$ has a global attractor $A_0 \subset X_0$, and a fixed point ρ_0 of S in A_0 exists (see [157, Theorem 1.3.8]). Therefore, (1.6) has a T-periodic solution $\rho_*(x,t) := \Psi(t)\rho_0$ in X_0. Since $A_0 \subset X_0$ and $S = \Psi(T)$, it holds that $A_0 = S(A_0) = \Psi(T)A_0$, which along with the strong maximum principle yields $A_0 \subset \mathrm{Int}(X_0)$. Then $\rho_*(x,t) = \Psi(t)\rho_0 \in \Psi(t)A_0 \subset \mathrm{Int}(X_0)$ and hence $\rho_*(x,t)$ is a positive T-periodic solution. Furthermore, the standard regularity theory for parabolic equations yields that $\rho_*(x,t) \in [C^{2+\gamma, 1+\frac{\gamma}{2}}(\overline{\Omega} \times (0,T])]^2$ (see e.g., Lemma 5.12). Hence, the proof of Theorem 5.2 is finished. □

Next, we investigate the effects of negative toxicant-taxis χ on the threshold value $m^*(\chi)$, which is characterized by the effects of χ on λ^*. In the sequel, we sometimes denote $\hat{w} =: \hat{w}^h$ and its minimum value $\hat{w}_* =: \hat{w}_*^h$ to emphasize the dependence of the solution $\hat{w}$ of (5.4) on the coefficient $h = h(x,t)$. Particularly, when $h(x,t) = a(x) \geq 0$, then $\hat{w}^a$ is the unique positive solution to

$$d_2 \Delta w + a(x) - \alpha w = 0, \quad x \in \Omega, \quad \partial_\nu w = 0, \quad x \in \partial\Omega. \tag{5.77}$$

Lemma 5.13. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and assume that* (H) *holds.*

(i) If $h(x,t) = a(x) \geq 0$ and $a(x)$ is nonconstant, then

$$\limsup_{\chi \to \infty} \lambda_* \leq \min_{x \in \overline{\Omega}} \{m\hat{w}^a(x) - r\} =: m\hat{w}_*^a - r. \tag{5.78}$$

(ii) Fix $a(x) \in C^{\kappa_0}(\overline{\Omega})$ and let $\hat{w}^a(x)$ be the unqiue positive solution of (5.77) *and $m\hat{w}_*^a - r < 0$. Let $h(x,t) = a(x) + b(x,t)$ with $b(x,t)$ being a T-periodic function, then there*

exists a small $\sigma(a) > 0$ such that if $0 < \|b(x,t)\|_{L^\infty(Q_T)} \le \sigma(a)$, then

$$\limsup_{\chi\to\infty} \lambda_* < 0, \tag{5.79}$$

i.e., we show that $b(x,t)$ is a small perturbation which is uniform for large χ.

Proof. We first use the conditions $h(x,t) = a(x)$, $m\hat{w}_*^a < r$ and Sard's theorem to prove the statement (i). When $h = a(x)$, one can check that the eigenpair (λ_*, ϕ_*) satisfies the elliptic problem

$$\begin{cases} -d_1\Delta\phi - \chi\nabla\cdot(\phi\nabla\hat{w}^a) - (r - m\hat{w}^a)\phi = \lambda\phi, & x \in \Omega, \\ \partial_\nu\phi = 0, & x \in \partial\Omega, \end{cases} \tag{5.80}$$

where $\hat{w}^a := \hat{w}^a(x)$ is the unique solution of (5.77). To prove (5.78), it suffices to prove, for each $\varepsilon > 0$, that

$$\limsup_{\chi\to\infty} \lambda_* < m\hat{w}_*^a - r + \varepsilon. \tag{5.81}$$

For convenience, considering the following adjoint eigenvalue problem of (5.80)

$$\begin{cases} -d_1\Delta\psi + \chi\nabla\hat{w}^a\cdot\nabla\psi - (r - m\hat{w}^a)\psi = \lambda\psi, & x \in \Omega, \\ \partial_\nu\psi = 0, & x \in \partial\Omega. \end{cases} \tag{5.82}$$

We deduce from [17, Corollary 2.13] that λ_* is also the principal eigenvalue for (5.82). Then to prove (5.78), by [77, Lemma 1.3.13], we only need to construct a nonnegative, nontrivial subsolution $\underline{\psi}$ satisfying

$$-d_1\Delta\underline{\psi} + \chi\nabla\hat{w}^a\cdot\nabla\underline{\psi} + mE\underline{\psi} - \varepsilon\underline{\psi} \le 0, \qquad x \in \Omega, \tag{5.83}$$

$$\partial_\nu\underline{\psi} = 0, \qquad x \in \partial\Omega \tag{5.84}$$

in the generalized sense (see [77, Definition 1.1.1]). Here $E := \hat{w}^a(x) - \hat{w}_*^a \ge 0$ in $\overline{\Omega}$.

Let $\varepsilon > 0$ be fixed. Since $\hat{w}^a(x) \in C^2(\overline{\Omega})$, by Sard's theorem, we can fix $s_1, s_2 \in \mathbb{R}^+$ such that

$$\hat{w}_*^a < s_1 < s_2 < \hat{w}_*^a + \frac{\varepsilon}{4m} \quad \text{and} \quad \inf_{U'} |\nabla\hat{w}^a| > 0, \tag{5.85}$$

where we define $U' := U(s_2)\setminus U(s_1)$ and that

$$U(s) := \{x \in \overline{\Omega}: \ \hat{w}^a(x) \le s\}. \tag{5.86}$$

Next, we choose a smooth cut-off function $G(s) = G(s; s_1, s_2) : \mathbb{R} \to \mathbb{R}$ satisfying

$$\begin{cases} G(s) = s_2 - s \quad \text{ for } s \in [(s_1+s_2)/2, \infty), \\ G'(s) < 0 \quad \text{ for } s \in (s_1, (s_1+s_2)/2], \quad G'(s) \equiv 0 \quad \text{ for } s \in (-\infty, s_1], \\ \|\!|G(s)|\!\| := \sup_{s\in\mathbb{R}} \left[|G'(s)| + |G''(s)|\right] < +\infty. \end{cases} \tag{5.87}$$

Define

$$H_1(s) := \sup_{U(s)} \left[-d_1 \frac{G'(\hat{w}^a)}{G(\hat{w}^a)} \Delta \hat{w}^a\right] \quad \text{ and } \quad H_2(s) := \sup_{U(s)} \left[-\frac{d_1 G''(\hat{w}^a)}{G(\hat{w}^a)} |\nabla \hat{w}^a|^2\right].$$

We claim that there is $\hat{s} \in (s_1, s_2]$ such that

$$H_i(\hat{s}) < \frac{\varepsilon}{8} \quad \text{ for } i = 1, 2. \tag{5.88}$$

Indeed, the existence of $\hat{s}$ follows from the fact that (a) $\|\hat{w}^a\|_{C^2(\overline{\Omega})} < +\infty$ is given, (b) $H_i(s)$ are continuous and finite-valued in $[s_1, s_2)$, and (c) $H_i(s) \searrow 0$ as $s \searrow s_1$, for $i = 1, 2$.

Now, define

$$\psi(x) = G(\hat{w}^a(x))), \tag{5.89}$$

then by a direct computation, we get

$$-d_1 \Delta\psi + \chi \nabla \hat{w}^a \cdot \nabla \psi = G\left[-d_1 \frac{G'}{G} \Delta \hat{w}^a - \frac{d_1 G''}{G} |\nabla \hat{w}^a|^2 + \chi \frac{G'}{G} |\nabla \hat{w}^a|^2\right], \tag{5.90}$$

where $G := G(\hat{w}^a(x))$, $G' := G'(\hat{w}^a(x))$ and $G'' := G''(\hat{w}^a(x))$. We shall verify that $\psi(x)$ satisfies (5.83) in classical sense for $x \in U(s_2)$ by proceeding two cases:

$$x \in U(\hat{s}) \quad \text{and} \quad x \in U(s_2) \setminus U(\hat{s}).$$

Case 1: $x \in U(\hat{s})$. Following from (5.88), (5.90), $G(s) > 0$, $G'(s) \le 0$ (see (5.87)), $\chi \ge 0$ and $mE < \frac{\varepsilon}{4}$, one has

$$-d_1 \Delta\psi + \chi \nabla \hat{w}^a \cdot \nabla \psi + (mE - \varepsilon)\psi \le G[H_1(\hat{s}) + H_2(\hat{s}) + (mE - \varepsilon)] \le 0. \tag{5.91}$$

Case 2: $x \in U(s_2) \setminus U(\hat{s}) = \{x : \ \hat{w}^a(x) \in (\hat{s}, s_2]\}$. Then it follows from (5.87) that $G' < 0$ and $G > 0$. Note (5.90), $\inf_{U(s_2)\setminus U(\hat{s})} |\nabla \hat{w}^a| \ge \inf_{U'} |\nabla \hat{w}^a| > 0$ and $mE < \frac{\varepsilon}{4}$, one has

$$\begin{aligned} &- d_1 \Delta\psi + \chi \nabla \hat{w}^a \cdot \nabla \psi + (mE - \varepsilon)\psi \\ &= -G' d_1 \Delta \hat{w}^a - G'' d_1 |\nabla \hat{w}^a|^2 + \chi G' |\nabla \hat{w}^a|^2 + (mE - \varepsilon) G \\ &\le \|\!|G|\!\| \left[d_1 \|\Delta \hat{w}^a\|_{L^\infty} + d_1 \|\nabla \hat{w}^a\|_{L^\infty}^2\right] + \chi G' \inf_{U(s_2)\setminus U(\hat{s})} |\nabla \hat{w}^a|^2 < 0, \end{aligned}$$

provided that $\chi \geq \tilde{\chi}$, for some $\tilde{\chi} \geq 1$. In fact, for each $\chi \geq \tilde{\chi}$, the differential inequality holds in some relatively open set $U \subset \overline{\Omega}$ containing $\overline{U}(s_2)$. Now, note that $\underline{\psi}(x) := \max\{\psi, 0\}$ is continuous, and

$$\underline{\psi}(x) = \psi(x) \quad \text{in } U(s_2), \quad \underline{\psi}(x) = 0 \quad \text{in } \Omega \setminus U(s_2).$$

It follows that $\underline{\psi}(x)$ satisfies the first differential inequality of (5.84) in the generalized sense (see [77, Definition 1.1.1]). Also, note that ψ and 0 both satisfies the Neumann boundary condition classically, so $\underline{\psi}$ also satisfies the Neumann boundary condition in the generalized sense. By the eigenvalue comparison lemma [77, Lemma 1.3.13], this implies that (5.81) holds. Since $\varepsilon > 0$ is arbitrary, it follows that $\limsup_{\chi\to\infty} \lambda_* \leq m\hat{w}_*^a - r$. This proves the statement (i).

For the statement (ii), let $h = a(x) + b(x,t)$ with $b(x,t)$ being time-periodic, then it holds that $\lambda_* \in \mathbb{R}$ and $\phi_* > 0$ satisfy (5.49). Following [17, Lemma 2.15], we get that λ_* is also the principal eigenvalue of the following adjoint problem

$$\begin{cases} -\zeta_t - d_1\Delta\zeta + \chi\nabla\zeta \cdot \nabla\hat{w}^h - (r - m\hat{w}^h)\zeta = \lambda\zeta, & \text{in } \Omega \times [0,T], \\ \partial_\nu \zeta = 0, & \text{on } \partial\Omega \times [0,T], \\ \zeta(x,0) = \zeta(x,T), & \text{in } \Omega. \end{cases} \tag{5.92}$$

To achieve goal (5.79), for $\varepsilon = \frac{|r - m\hat{w}_*^a|}{2} > 0$, we first construct a nontrivial function $\underline{\zeta} \geq 0$ satisfying

$$\begin{cases} -\underline{\zeta}_t - d_1\Delta\underline{\zeta} + \chi\nabla\underline{\zeta} \cdot \nabla\hat{w}^h + [m(\hat{w}^a - \hat{w}_*^a) - \varepsilon]\underline{\zeta} \leq 0, & \text{in } \Omega \times [0,T], \\ \partial_\nu \underline{\zeta} = 0, & \text{on } \partial\Omega \times [0,T], \\ \underline{\zeta}(x,0) = \underline{\zeta}(x,T), & \text{in } \Omega \end{cases} \tag{5.93}$$

in the generalized sense. In fact, for each fixed $a(x)$, one can choose $\sigma(a)$ such that if $0 < \|b(x,t)\|_{L^\infty(Q_T)} \leq \sigma(a)$, then the above argument in statement (i) can be repeated to show that for $\chi \gg 1$, $\underline{\psi}(x) := \max\{\psi, 0\}$ qualifies again as a generalized subsolution of the periodic eigenvalue problem (5.92). To prove this, we shall divide our proof into two steps:

Step 1: We claim that for each fixed $a(x)$, there exists a constant $\sigma(a) > 0$ so that if $0 < \|b(x,t)\|_{L^\infty(Q_T)} \leq \sigma(a)$, then

$$\|\hat{w}^h(x,t) - \hat{w}^a(x)\|_{C^{1+\beta_0, \frac{1+\beta_0}{2}}(\overline{Q}_T)} \leq c_0\sigma(a) \quad \text{and} \quad \inf_{U'} \nabla\hat{w}^h \cdot \nabla\hat{w}^a > 0. \tag{5.94}$$

Indeed, denote $\ell(x,t) := \hat{w}^h(x,t) - \hat{w}^a(x)$, we deduce from $\hat{w}^a$-equation, $\hat{w}^h$-equation, the condition $h(x,t) = a(x) + b(x,t)$ and [87, Theorem 7.35] that for some constant $p \geq 1$

$$\|l(x,t)\|_{W^{2,1}_{2p}(Q_T)} \leq c_1\|b(x,t)\|_{L^{2p}(Q_T)},$$

which along with the Sobolev embedding theorem $W^{2,1}_{2p} \hookrightarrow C^{1+\beta_0,\frac{1+\beta_0}{2}}$ with $0 < \beta_0 < 1 - \frac{n+2}{2p}$, gives

$$\|\ell(x,t)\|_{C^{1+\beta_0,\frac{1+\beta_0}{2}}(\overline{Q}_T)} \leq c_2\|\ell(x,t)\|_{L^{2p}(Q_T)} \leq c_2c_1\|b\|_{L^{2p}(Q_T)} \leq c_2c_1|\Omega|^{\frac{1}{2p}}\|b\|_{L^\infty(Q_T)},$$

this yields

$$\|\ell(x,t)\|_{C^{1+\beta_0,\frac{1+\beta_0}{2}}(\overline{Q}_T)} \leq c_0\|b\|_{L^\infty(Q_T)} \tag{5.95}$$

by letting $c_0 := c_1c_2|\Omega|^{\frac{1}{2p}}$. Taking

$$\sigma(a) := \frac{c_*}{2c_0\|\nabla \hat{w}^a(x)\|_{L^\infty}} \tag{5.96}$$

such that

$$\|b\|_{L^\infty(Q_T)} \leq \sigma(a), \tag{5.97}$$

where the constant $c_* := \inf_{x\in U'} |\nabla \hat{w}^a(x)|^2 > 0$ (see (5.85)). Applying (5.95), (5.97) and (5.96), we know that the first inequality in (5.94) holds, and

$$\begin{aligned}\inf_{U'} \nabla\hat{w}^h\cdot\nabla\hat{w}^a &= \inf_{U'}|\nabla\hat{w}^a|^2 + \inf_{U'}\nabla\ell\cdot\nabla\hat{w}^a\\ &\geq \inf_{U'}|\nabla\hat{w}^a|^2 - \|\ell\|_{L^\infty(Q_T)}\|\nabla\hat{w}^a\|_{L^\infty}\\ &\geq \inf_{U'}|\nabla\hat{w}^a|^2 - c_0\sigma(a,\eta)\|\nabla\hat{w}^a\|_{L^\infty} = \inf_{U'}|\nabla\hat{w}^a|^2/2 > 0.\end{aligned} \tag{5.98}$$

Hence, we finish the proof of (5.94).

Step 2: For $x \in U(\hat{s})$, using (5.98), (5.94), $G'(s) = 0$ for $s \leq s_1$ and the fact $mE < \varepsilon/4$, one has

$$\begin{aligned}&-\underline{\psi}_t - d_1\Delta\underline{\psi} + \chi\nabla\hat{w}^h\cdot\nabla\underline{\psi} + m(\hat{w}^a - \hat{w}^a_*)\underline{\psi} - \varepsilon\underline{\psi}\\ &= -G'd_1\Delta\hat{w}^a - G''d_1|\nabla\hat{w}^a|^2 + G'\chi\nabla\hat{w}^a\cdot\nabla w^h + [m(\hat{w}^a - \hat{w}^a_*) - \varepsilon]G\\ &\leq G[H_1(\hat{s}) + H_2(\hat{s}) + m(\hat{w}^a - \hat{w}^a_*) - \varepsilon] + G'\chi\nabla\hat{w}^a\cdot\nabla w^h\\ &\leq -G\varepsilon/4 + \chi G' \inf_{U(\hat{s})\setminus U(s_1)} \nabla\hat{w}^a\cdot\nabla w^h \leq \chi G'\inf_{U'}\nabla\hat{w}^a\cdot\nabla w^h < 0,\ \forall x \in U(\hat{s}).\end{aligned}$$

For $x \in U(s_2) \setminus U(\hat{s})$, note (5.87), (5.90), $\inf_{U(s_2)\setminus U(\hat{s})}\nabla\hat{w}^h\cdot\nabla\hat{w}^a \geq \inf_{U'}\nabla\hat{w}^h\cdot\nabla\hat{w}^a > 0$ and $mE < \frac{\varepsilon}{4}$, we obtain

$$\begin{aligned}&-\underline{\psi}_t - d_1\Delta\underline{\psi} + \chi\nabla\hat{w}^h\cdot\nabla\underline{\psi} + m(\hat{w}^a - \hat{w}^a_*)\underline{\psi} - \varepsilon\underline{\psi}\\ &= -G'd_1\Delta\hat{w}^a - G''d_1|\nabla\hat{w}^a|^2 + G'\chi\nabla\hat{w}^h\cdot\nabla w^a + [m(\hat{w}^a - \hat{w}^a_*) - \varepsilon]G\end{aligned}$$

$$\leq \|G\| \left[d_1 \|\Delta \hat{w}^a\|_{L^\infty} + d_1 \|\nabla \hat{w}^a\|^2_{L^\infty}\right] + \chi G' \inf_{U(s_2)\setminus U(\hat{s})} \nabla \hat{w}^h \cdot \nabla w^a < 0,$$

provided that $\chi > 0$ is sufficiently large. Therefore, similar to the proof of Lemma 5.13(i), we get that $\underline{\psi} := \max\{0, \psi\}$ is a generalized subsolution of (5.92). Then it follows from [90, Proposition A.1] that

$$\limsup_{\chi\to\infty} \lambda_* \leq m\hat{w}^a_* - r + m\|\ell\|_{L^\infty(Q_T)} + |m\hat{w}^a_* - r|/2. \tag{5.99}$$

Hence, by (5.94) and the condition $m\hat{w}^a_* - r < 0$, we can choose

$$\sigma(a) = \min\left\{\frac{c_*}{2c_0\|\nabla \hat{w}^a(x)\|_{L^\infty}}, \frac{r - m\hat{w}^a_*}{4c_0 m}\right\}$$

so that $\limsup_{\chi\to\infty} \lambda_* < 0$. This proves the statement (ii). □

Remark 5.4. *In Lemma 5.13, $\hat{w}$ appears in both advection and linear terms. The results in Lemma 5.13 include the scenario where the set of local minimum for $\hat{w}$ may contain some flat pieces, a case excluded in [20, 106] for high-dimensional settings. And the proof of Lemma 5.13(ii) removes the nondegeneracy conditions imposed on $\hat{w}$ in [20, 106] in high-dimensional spaces.*

Proof of Theorem 5.3. We apply (5.99) to get that there exists some constant $c_1 > 0$ independent of b, χ such that

$$\liminf_{\chi\to\infty} m^*(\chi) \geq \frac{r - c_1 m\|\ell\|_{L^\infty(Q_T)}}{\hat{w}^a_*} \geq \frac{r - c_1 m\|\ell\|_{L^\infty(Q_T)}}{\hat{w}^h_* + \|\ell\|_{L^\infty(Q_T)}}.$$

Hence, for each $\eta > 0$, one can choose $\sigma(a, \eta)$ sufficiently small to ensure that $\|\ell\|_\infty$ is small enough (thanks to (5.94)), so that

$$\liminf_{\chi\to\infty} m^*(\chi) \geq \frac{r}{\hat{w}^h_*} - \eta.$$

Therefore, the proof of Theorem 5.3 is complete. □

5.4 Global Stability: Proof of Theorem 5.4

In this section, we explore the global dynamics of the system (5.1). We start by proving the global stability of the semi-trivial T-periodic solution $(0, \hat{w}(x,t))$.

Proof of Theorem 5.4(i). We first show the convergence of $\|u\|_{L^\infty}$ as $t \to \infty$. To this end, we consider the following equation

$$\begin{cases} v_t = d_2\Delta v + h - (\alpha + \beta M_0)v, & x \in \Omega,\ t > T_0, \\ \partial_\nu v = 0, & x \in \partial\Omega,\ t > T_0, \\ v(x, T_0) = w(x, T_0), & x \in \Omega, \end{cases} \tag{5.100}$$

where constants $T_0 > 0$ and $M_0 > 0$ are introduced in Theorem 5.1(ii). Then, the comparison principle yields

$$w(x,t) \geq v(x,t), \quad \forall t \geq T_0. \tag{5.101}$$

Proceeding the similar procedures as in proof in (5.73), we obtain

$$\lim_{t\to\infty} \|v(\cdot,t) - \hat{v}\|_{L^\infty} = 0, \tag{5.102}$$

where $0 < \hat{v}(x,t) \in C^{2+\gamma, 1+\frac{\gamma}{2}}(\overline{\Omega} \times [0,T])$ is the unique solution (see, e.g., [95, Proposition 4.4.8] and [62, Lemma 2.2]) of the following equation

$$\begin{cases} \hat{v}_t = d_2\Delta \hat{v} + h - (\alpha + \beta M_0)\hat{v}, & x \in \Omega,\ t > T_0, \\ \partial_\nu \hat{v} = 0, & x \in \partial\Omega,\ t > T_0, \\ \hat{v}(x, T_0) = \hat{v}(x, T_0 + T), & x \in \Omega. \end{cases} \tag{5.103}$$

On the other hand, we apply the maximum principle [141, Theorem 7.1] for periodic parabolic equations to (5.103) to get

$$\hat{v}(x,t) \geq \frac{h_*}{\alpha + \beta M_0} := m_{M_0} > 0. \tag{5.104}$$

With (5.101), (5.102) and (5.104), we can find a constant $t_1 \geq T_0$ such that

$$w(x,t) \geq v(x,t) \geq \hat{v}(x,t) - \frac{mm_{M_0} - r}{2m} \geq m_{M_0} - \frac{mm_{M_0} - r}{2m} = \frac{m_{M_0}}{2} + \frac{r}{2m} > 0, \quad \forall t > t_1. \tag{5.105}$$

Next, we integrate the first equation of (5.1) and use (5.105) to get that for all $t > t_1$,

$$\frac{d}{dt}\int_\Omega u = \int_\Omega u(r - mw - u) \leq \frac{r - mm_{M_0}}{2}\int_\Omega u,$$

this derives

$$\|u(\cdot,t)\|_{L^1} \le e^{-\frac{r-mm_{M_0}}{2}(t-t_1)}\int_\Omega u(\cdot,t_1) \le c_1 e^{-\frac{r-mm_{M_0}}{2}(t-t_1)}, \quad \forall t>t_1. \tag{5.106}$$

Then (5.106) together with Gagliardo-Nirenberg inequality in two dimensional space and (5.52) gives that for all $t>\max\{t_1, T_0+1\} =: t_2$

$$\|u(\cdot,t)\|_{L^\infty} \le c_2\|u(\cdot,t)\|_{W^{1,\infty}}^{\frac{2}{3}}\|u(\cdot,t)\|_{L^1}^{\frac{1}{3}} \le c_3 e^{-\frac{(mm_{M_0}-r)}{6}(t-t_2)}. \tag{5.107}$$

Next, we shall prove the convergence of $\|w(\cdot,t)-\hat{w}(\cdot,t)\|_{L^\infty}$ as $t\to\infty$. Let $W := w-\hat{w}$, then we deduce from (5.1) and (5.4) that

$$\begin{cases} W_t = d_2\Delta W - \alpha W - \beta uW - \beta\hat{w}u, & x\in\Omega,\ t>0,\\ \partial_\nu W = 0, & x\in\partial\Omega,\ t>0,\\ W(x,0) = w_0 - \hat{w}(\cdot,0), & x\in\Omega. \end{cases} \tag{5.108}$$

Multiplying the first equation of (5.108) by W, integrating the result and using (5.5), one obtains

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega W^2 + d_2\int_\Omega|\nabla W|^2 + \alpha\int_\Omega W^2 + \beta\int_\Omega uW^2 &= -\beta\int_\Omega \hat{w}uW \\ &\le \beta c_4\int_\Omega u|W| \\ &\le \frac{\alpha}{2}\int_\Omega W^2 + \frac{\beta^2c_4^2}{2\alpha}\int_\Omega u^2, \end{aligned}$$

which along with (5.107) implies

$$\frac{1}{2}\frac{d}{dt}\int_\Omega W^2 + \frac{\alpha}{2}\int_\Omega W^2 \le \frac{\beta^2c_4^2}{2\alpha}\int_\Omega u^2 \le c_5 e^{-\frac{(mm_{M_0}-r)}{3}}(t-t_2), \quad \forall t>t_2. \tag{5.109}$$

From (5.109), we have

$$\|w(\cdot,t)-\hat{w}(\cdot,t)\|_{L^2} = \|W(\cdot,t)\|_{L^2} \le c_6 e^{-c_7(t-t_2)}, \quad \forall t>t_2, \tag{5.110}$$

where $c_7 := \min\{\frac{mm_{M_0}-r}{6}, \frac{\alpha}{4}\}$. Applying Gagliardo-Nirenberg inequality in two dimensional space and the $W^{1,\infty}$-boundedness of w and $\hat{w}$ (see (5.3) and (5.46), respectively) and (5.110), one has

$$\|w(\cdot,t)-\hat{w}(\cdot,t)\|_{L^\infty} \le \|w(\cdot,t)-\hat{w}(\cdot,t)\|_{W^{1,\infty}}^{\frac{1}{2}}\|w(\cdot,t)-\hat{w}(\cdot,t)\|_{L^2}^{\frac{1}{2}} \le c_8 e^{-\frac{c_7}{2}}(t-t_2), \quad \forall t>t_2,$$

which combined with (5.107) gives (5.7) by letting $C_1 := c_8 e^{\frac{c_7t_2}{2}}$ and $\theta_1 := \frac{c_7}{2}$. $\square$

In the following, we assume $h(x,t) \equiv h(t)$ and shall establish the global stability and uniqueness of the positive T-periodic solution to (5.1). When $m < m^*$, we deduce from Theorem 5.2 that the system (5.1) admits at least one positive T-periodic solution $(u^*, w^*)(x,t) =: (u^*, w^*)$. On the other hand, note that the positive T-periodic solution $(u_*, w_*)(t) =: (u_*, w_*)$ satisfies the following ordinary differential equation (ODE) system

$$\begin{cases} (u_*)_t = u_*(r - u_* - mw_*), & t > 0, \\ (w_*)_t = h(t) - \alpha w_* - \beta u_* w_*, & t > 0, \\ u_*(t) = u_*(t+T) > 0, \quad w_*(t) = w_*(t+T) > 0, & t \geq 0. \end{cases} \tag{5.111}$$

It is also the positive T-periodic solution of (5.1). We shall show that the positive T-periodic solution $(u_*, w_*)(t)$ is globally asymptotically stable, and hence it is the unique positive T-periodic solution.

To this end, let (u, w) be a solution obtained in Theorem 5.1 and denote $V := w - w_*$, one deduces from (5.1) and (5.111) that

$$\begin{cases} u_t = d_1 \Delta u + \chi \nabla \cdot (u \nabla V) + u(r - mV - mw_*) - u^2, & x \in \Omega, \ t > 0, \\ V_t = d_2 \Delta V - \alpha V - \beta u V - \beta w_*(u - u_*), & x \in \Omega, \ t > 0, \\ \partial_\nu u = \partial_\nu V = 0, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x), V(x,0) = w_0(x) - w_*(x,0), & x \in \Omega, \end{cases} \tag{5.112}$$

where we have used the fact that (u_*, w_*) is independent of x. Then we introduce the following entropy functional:

$$\mathcal{F}(t) := \frac{A_2 \beta}{m} \int_\Omega \frac{1}{u_*} \left(u - u_* - u_* \ln \frac{u}{u_*} \right) + \frac{1}{2A_1} \int_\Omega V^2,$$

where $A_1 := \max\limits_{t \in [0,T]} \{w_*(t)\}$ and $A_2 := \max\limits_{t \in [0,T]} \{u_*(t)\}$.

Proof of Theorem 5.4(ii). Our proof is divided into two steps:

Step 1. In this step, we show the global stability and uniqueness of (u_*, w_*). Some direct calculations yield that

$$\begin{aligned} \frac{d}{dt}\mathcal{F}(t) =& \frac{A_2\beta}{m} \frac{d}{dt} \int_\Omega \frac{1}{u_*} \left(u - u_* - u_* \ln \frac{u}{u_*} \right) + \frac{1}{2A_1} \frac{d}{dt} \left(\int_\Omega V^2 \right) \\ =& - \int_\Omega (\nabla \frac{u}{u_*}, \nabla V) I_1 (\nabla \frac{u}{u_*}, \nabla V)^{\mathcal{T}} - \int_\Omega \frac{A_2\beta}{m u_*}(u - u_*)^2 - \int_\Omega \frac{A_2 \beta}{u_*}(u - u_*)V \\ & - \int_\Omega \frac{\beta w_*}{A_1} V (u - u_*) - \int_\Omega \frac{\alpha + \beta u}{A_1} V^2 \\ =& - \int_\Omega (\nabla \frac{u}{u_*}, \nabla V) I_1 (\nabla \frac{u}{u_*}, \nabla V)^{\mathcal{T}} - \int_\Omega (u - u_*, V)\, I_2\, (u - u_*, V)^{\mathcal{T}} \end{aligned} \tag{5.113}$$

where

$$I_1 = \begin{pmatrix} \frac{A_2\beta d_1 G''\left(\frac{u}{u_*}\right)}{m} & \frac{A_2\beta\chi u G''\left(\frac{u}{u_*}\right)}{2mu_*} \\ \frac{A_2\beta\chi u G''\left(\frac{u}{u_*}\right)}{2mu_*} & \frac{d_2}{A_1} \end{pmatrix}, \quad I_2 = \begin{pmatrix} \frac{A_2\beta}{mu_*} & \frac{\beta(\frac{A_2}{u_*}+\frac{w_*}{A_1})}{2} \\ \frac{\beta(\frac{A_2}{u_*}+\frac{w_*}{A_1})}{2} & \frac{\alpha+\beta u}{A_1} \end{pmatrix}$$

with the function $G(s) := s - 1 - \ln s$. Then I_1 is positive definite iff

$$\frac{A_2\beta d_1 d_2}{mA_1} G''\left(\frac{u}{u_*}\right) > \frac{A_2^2\beta^2\chi^2 u^2}{4m^2u_*^2} G''\left(\frac{u}{u_*}\right)^2,$$

which is equivalent to $\frac{4d_1d_2m}{A_2A_1\beta} > \chi^2$. This can be ensured by (5.8).

Next, we show that the matrix I_2 is positive definite under some conditions. It is easy to obtain

$$0 < r - \frac{mh^*}{\alpha} \le u_* \le r, \quad \frac{h_*}{\alpha+\beta r} < w_* \le \frac{h^*}{\alpha}. \tag{5.114}$$

One can check that I_2 is positive definite iff $\mathrm{Det}(I_2) > 0$, where

$$\mathrm{Det}(I_2) > \frac{\alpha A_2\beta}{u_*A_1m} - \frac{\beta^2\left(1+\frac{A_2}{u_*}\right)^2}{4} \ge \frac{\alpha\beta}{A_1m} - \frac{\beta^2\left(1+\frac{A_2}{\min_{t\in[0,T]}\{u_*(t)\}}\right)^2}{4} =: J.$$

Applying (5.114) implies

$$0 < r - \frac{mh^*}{\alpha} \le \min_{t\in(0,T)}\{u_*(t)\} \le A_2 \le r, \quad \frac{h_*}{\alpha+\beta r} < A_1 \le \frac{h^*}{\alpha}, \tag{5.115}$$

then

$$\mathrm{Det}(I_2) > J \ge \frac{\alpha^2\beta}{h^*m} - \frac{\beta^2\left(1+\frac{r}{r-mh^*/\alpha}\right)^2}{4} > 0,$$

which is guaranteed by (5.9). Hence, we deduce from (5.113) that there is a constant $c_1 > 0$ such that

$$\mathcal{F}'(t) \le -c_1\int_\Omega \left[(u-u_*)^2 + (w-w_*)^2\right] =: -c_1\mathcal{G}(t). \tag{5.116}$$

Denote $l(s) := s - b\ln s$, we apply the fact $l'(b) = 1 - \frac{b}{b} = 0$ and Taylor's expansion to obtain that for all $b > 0,\ s > 0$

$$s - b - b\ln\frac{s}{b} = l(s) - l(b) = \frac{l''(\tilde{b})}{2}(s-b)^2 = \frac{b}{2\tilde{b}^2}(s-b)^2 \ge 0, \tag{5.117}$$

where $\tilde{b}$ is between s and b. Taking $s = u$ and $b = u_*$ in (5.117) gives

$$u - u_* - u_*\ln\frac{u}{u_*} = \frac{u_*}{2b_1^2}(u-u_*)^2 \ge 0, \tag{5.118}$$

where b_1 is between u and u_*. On the other hand, by Theorem 5.2, when $m < \frac{r\alpha}{h^*}(\leq \frac{r}{\hat{w}^*} \leq m^*)$, there exist some constants $c_2 > 0$ and $t_1 > 0$ such that $u \geq c_2$ for all $t > t_1$. This along with (5.118), (5.114) and Theorem 5.1 enable us to find constants $c_3 > 0,\ c_4 > 0$ such that

$$\frac{1}{c_3}\int_\Omega (u-u_*)^2\,dx \leq \frac{A_2\beta}{m}\int_\Omega \frac{1}{u_*}\left(u-u_*-u_*\ln\frac{u}{u_*}\right)dx \leq c_4\int_\Omega (u-u_*)^2\,dx, \quad \forall t > t_1. \tag{5.119}$$

And (5.115) implies

$$\frac{1}{c_5}\int_\Omega (w-w_*)^2dx \leq \frac{1}{2A_2}\int_\Omega (w-w_*)^2 \leq c_6\int_\Omega (w-w_*)^2dx, \quad \forall t > 0. \tag{5.120}$$

Then (5.119) along with (5.120) updates (5.116) as

$$\frac{d}{dt}\mathcal{F}(t) \leq -c_7\mathcal{F}(t), \quad \forall t > t_1. \tag{5.121}$$

Solving (5.121) directly, and applying (5.119) and (5.120) again, we can find some positive constants $c_8, c_9, t_2 > t_1$ such that $\mathcal{G}(t) \leq c_8 e^{-c_9(t-t_2)}$ for all $t > t_2$. Using (5.119), (5.120) and $V := w - w_*$, one has

$$\|u-u_*\|_{L^2} + \|w-w_*\|_{L^2} \leq c_{10}e^{-c_{11}(t-t_2)}, \quad \forall t > t_2. \tag{5.122}$$

Applying the Gagliardo-Nirenberg inequality in two dimensional space derives

$$\|u-u_*\|_{L^\infty} \leq c_{12}\|u-u_*\|_{W^{1,\infty}}^{\frac{1}{2}}\|u-u_*\|_{L^2}^{\frac{1}{2}}, \quad \|w-w_*\|_{L^\infty} \leq c_{12}\|w-w_*\|_{W^{1,\infty}}^{\frac{1}{2}}\|w-w_*\|_{L^2}^{\frac{1}{2}}, \tag{5.123}$$

which along with (5.122), uniform boundeness (see (5.52)) and $(u_*, w_*) \in [C^1([0,T])]^2$ gives (5.10). $\square$

Chapter 6

Conclusions and Future Works

6.1 Conclusions

In this thesis, we studied in Chapter 2 an indirect predator-prey model with anti-predation, which describing by a density-dependent type cross-diffusion. We established the existence, uniqueness, and uniform-in-time boundedness and global stability of positive classical solutions in any dimensional bounded domain. Furthermore, we proved the existence of non-constant positive steady-state solution and non-existence of Hopf bifurcation when the prey takes the anti-predation strategy. These results show that the anti-predation helps create spatial heterogeneity (steady state patterns), which is sharply different from the density-dependent type indirect prey-taxis (which exhibits Hopf bifurcation) and the case without cross-diffusion (where no patterns emerge).

In Chapter 3, a three-species Lotka-Volterra food chain model with intraguild predation and taxis mechanisms (prey-taxis and alarm-taxis) was studied. We established the existence, uniqueness, and uniform-in-time boundedness and global stability of positive classical solutions in one dimensional bounded interval. Furthermore, we focused on exploring the effects of intraguild predation and taxis mechanisms (prey-taxis and alarm-taxis). Our numerical simulations demonstrate the following points:

(a) Even in the absence of prey-taxis, as long as alarm-taxis is sufficiently strong, pattern formation will occur regardless of whether intraguild predation is included. Hence, the signal taxis mechanism plays an indispensable and essential role in promoting spatially inhomogeneous patterns.

(b) The prey-taxis plays very different effects for the system (3.3) between the cases that without intraguild predation (i.e., $\gamma_1 = \gamma_2 = 0$) and with intraguild predation (i.e., $\gamma_1, \gamma_2 > 0$). When $\gamma_1 = \gamma_2 = 0$, $\phi(u, v) = uv$, prey-taxis has a stabilization effect on the homogeneous steady state (see Figure 3.2) while it has a destabilization effect in the intraguild predation model with prey-taxis (i.e., $\gamma_1, \gamma_2 > 0,\ \phi(u,v) = v$) (see Figure 3.3), which contrasts with the well-known results that the prey-taxis serves to enhance the stability of the spatially homogeneous steady state in two-species predator-prey systems. As for intraguild predation model with alarm-taxis (i.e., $\gamma_1, \gamma_2 > 0,\ \phi(u,v) = uv$), the effects of prey-taxi ξ on pattern formations are more complicated. The system may subsequently undergo steady state bifurcations, no pattern formations and Hopf bifurcations as ξ increasing from 0 to 4 and then to 45, see Figure 3.4.

In Chapter 4, we explored an SIS model with a cross-diffusion dispersal strategy for the infected individuals. The existence, uniqueness, and uniform-in-time boundedness of positive classical solutions in two dimensional bounded domaine were proven. In addition, we defined the basic reproduction number R_0 and established the threshold dynamics in terms of R_0 as well as the global stability of constant steady states. Finally, we gave some numerical simulations. Our results demonstrate that the cross-diffusion dispersal strategy can reduce R_0 and help eradicate the diseases even if the habitat is high-risk in contrast to the situation without cross-diffusion.

In Chapter 5, we proved the existence, uniqueness, uniform-in-time boundedness and *ultimately uniform boundedness* of positive classical solutions to a population-toxicant model in time-periodic environment with toxicant-taxis in two dimensional bounded domain. Furthermore, we demonstrated the uniform persistence for any cross-diffusion coefficients $\chi \geq 0$ and examined the effects of cross-diffusion on uniform persistence for special form of $h(x,t)$. Additionally, we established the global stability of the non-constant semi-trivial T-periodic solution $(0, \hat{w})$ for general case of $h(x,t)$ and the positive T-periodic solution for the special case where $h(x,t) = h(t)$. Our results show that the strong toxicant-taxis (i.e., cross-diffusion) destabilizes the semi-trivial T-periodic solution $(0, \hat{w})$, and helps aquatic species to survive in a polluted environment.

Our thesis develops some new ideas/methods to overcome the difficulties caused by the inapplicability of the comparison principle in cross-diffusion models. For example, the proof ideas and outlines developed in Chapter 5 can be applied to prove the existence of time-periodic solutions or non-constant steady-state solutions, as well as uniform persistence for general cross-diffusion models.

6.2 Future Works

Except for the problems addressed in our thesis, several other pertinent questions remain open for further investigation:

(1) Note that the assumption (H_0) in Chapter 2 implies $d_P > 0$. Therefore, exploring the solution behavior of $d_P = 0$, which may involve potential degeneracy, is also worth considering in the future (e.g., [149], [150]). Additionally, to study the effects of density-dependent diffusion, our study specifically focuses on the Holling type I functional response function, and hence other types of response functions would also be worthwhile to further investigate.

(2) In Chapter 4, we introduced the expression for R_0 and found it to be related to the density-dependent rate function. Although we numerically demonstrated that this function can reduce R_0 and help eradicate the diseases, proving $R_0 < \hat{R}_0$ analytically (where $\hat{R}_0$ is the basic reproduction number when the density-dependent rate function is constant) remains an intriguing and challenging task. Additionally, while we proved the existence of an EE when $R_0 > 1$ in this chapter, the uniqueness of non-trivial EE when $R_0 > 1$ and the existence/uniqueness of non-trivial EE when $R_0 \leq 1$ remain unresolved. Moreover, the conditions for the global stability of constant DFE and EE are currently stringent, necessitating further work to relax these conditions.

(3) For the population-toxicant model in time-periodic environment with toxicant-taxis, we only proved that the effects of cross-diffusion for special $h(x,t) = a(x) + b(x,t)$ with $0 \leq |b(x,t)| \ll 1$, and the global stability of positive T-periodic solution for $h(x,t) = h(t)$. Proving these results for general $h(x,t)$ remains open. Additionally, according to the results in Theorem 5.4, the global dynamics for the system (5.1) are still unclear when m is at an intermediate and warrant further exploration.

Drawing from aforementioned works and ideas presented in Chapter 2 - 5, aside from the framework developed in Chapter 5 which can be used to demonstrate uniform persistence and the existence of periodic solutions or non-constant steady-state solutions for general cross-diffusion systems, it is necessary and challenging to develop frameworks that clarify the global stability and examine the effects of cross-diffusion on the principal eigenvalue. This topic may warrant further investigation.

Bibliography

[1] T. Ahmed, M. Zounemat-Kermani, and M. Scholz. Climate change, water quality and water-related challenges: a review with focus on Pakistan. *Int. J. Environ. Res. Public. Health*, 17, 2020.

[2] I. Ahn and C. Yoon. Global well-posedness and stability analysis of prey-predator model with indirect prey-taxis. *J. Differential Equations*, 268(8):4222–4255, 2020.

[3] N.D. Alikakos. L^p bounds of solutions of reaction-diffusion equations. *Comm. Partial Differential Equations*, 4(8):827–868, 1979.

[4] L.J.S. Allen, B.M. Bolker, Y. Lou, and A.L. Nevai. Asymptotic profiles of the steady states for an SIS epidemic reaction-diffusion model. *Discrete Contin. Dyn. Syst.*, 21(1):1–20, 2008.

[5] H. Amann. Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. *SIAM Rev.*, 18(4):620–709, 1976.

[6] H. Amann. Dynamic theory of quasilinear parabolic equations. III. Global existence. *Math. Z.*, 202(2):219–250, 1989.

[7] H. Amann. Dynamic theory of quasilinear parabolic equations. II. Reaction-diffusion systems. *Differential Integral Equations*, 3(1):13–75, 1990.

[8] H. Amann. Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems. In *Function spaces, Differential Operators and Nonlinear Analysis (Friedrichroda, 1992)*, volume 133 of *Teubner-Texte Math.*, pages 9–126. Teubner, Stuttgart, 1993.

[9] C.V.M. Araújo, M. Moreira-Santos, and R. Ribeiro. Active and passive spatial avoidance by aquatic organisms from environmental stressors: a complementary perspective and a critical review. *Envir. Inter.*, 92-93:405–415, 2016.

[10] D. Balcan et al. Multiscale mobility networks and the spatial spreading of infectious diseases. *Proc. Nat. Acad. Sci.*, 106(51):21484–21489, 2009.

[11] I. Barbălat. Systèmes d'équations différentielles d'oscillations non linéaires. *Rev. Math. Pures Appl.*, 4:267–270, 1959.

[12] J.P Bourguignon and H. Brezis. Remarks on the euler equation. *J.Funct.Anal.*, 15(4):341–363, 1974.

[13] D. Brockmann and D. Helbing. The hidden geometry of complex, network-driven contagion phenomena. *Science*, 342(6164):1337–1342, 2013.

[14] L.M.T. Broyles, E.L. Pakhtigian, A.Y. Rosinger, and A. Mejia. Climate and hydrological seasonal effects on household water insecurity: a systematic review. *WIREs Water*, 9, 2022.

[15] M.D. Burkenroad. A possible function of bioluminescence. *J. Mar. Res.*, 5:161–164, 1943.

[16] Y.L. Cai, Q. Cao, and Z.-A. Wang. Asymptotic dynamics and spatial patterns of a ratio-dependent predator-prey system with prey-taxis. *Appl. Anal.*, 101(1):81–99, 2022.

[17] R.S. Cantrell and C. Cosner. *Spatial Ecology via Reaction-Diffusion Equations*. John Wiley and Sons Ltd., Chichester, UK, 2003.

[18] X.R. Cao. Global bounded solutions of the higher-dimensional Keller-Segel system under smallness conditions in optimal spaces. *Discrete Contin. Dyn. Syst.*, 35(5):1891–1904, 2015.

[19] K. Castellano and R.B. Salako. On the effect of lowering population's movement to control the spread of an infectious disease. *J. Differential Equations*, 316:1–27, 2022.

[20] X.F. Chen and Y. Lou. Principal eigenvalue and eigenfunctions of an elliptic operator with large advection and its application to a competition model. *Indiana Univ. Math. J.*, 57(2):627–658, 2008.

[21] J.W. Chu. Steady states of a diffusive population-toxicant model with negative toxicant-taxis. *Acta Appl. Math.*, 190:Paper No. 13, 23, 2024.

[22] J.W. Chu and H.-Y. Jin. Global dynamics of a three-species Lotka-Volterra food chain model with intraguild predation and taxis mechanisms. *J. Nonlinear Sci.*, 35(3):Paper No. 56, 47, 2025.

[23] J.W. Chu, H.-Y. Jin, and F. Liu. Global boundedness and stabilization for a three species food chain model with density-dependent motion. *submitted*, 2024.

[24] J.W. Chu and S.B. Li. Global dynamics of an indirect prey-taxis system with an anti-predation mechanism. *J. Differential Equations*, 385:424–462, 2024.

[25] J.W. Chu and Z.-A. Wang. Global dynamics of an SIS epidemic model with cross-diffusion: applications to quarantine measures. *Nonlinearity*, 38(5):Paper No. 055010, 27, 2025.

[26] B. Clucas. Defensive chemicals. In *Encyclopedia of Animal Behavior*, pages 481–486. Academic Press, Oxford, 2010.

[27] R. Courant and D. Hilbert. *Methods of Mathematical Physics, Vol. I.* Interscience, New York, 1993.

[28] E.N. Dancer. On the indices of fixed points of mappings in cones and applications. *J. Math. Anal. Appl.*, 91:131–151, 1983.

[29] E.N. Dancer. On positive solutions of some pairs of differential equations. *Trans. Amer. Math. Soc.*, 284:729–743, 1984.

[30] M.C.M. de Jong, O. Diekmann, and J.A.P. Heesterbeek. *How does transmission of infection depend on population size?*, pages 84–94. Cambridge University Press, United Kingdom, 1995.

[31] K. Deng and Y.X. Wu. Dynamics of a susceptible-infected-susceptible epidemic reaction-diffusion model. *Proc. Roy. Soc. Edinburgh Sect. A*, 146(5):929–946, 2016.

[32] X.M. Deng, Q.H. Huang, and Z.-A. Wang. Global dynamics and pattern formation in a diffusive population-toxicant model with toxicant-taxis. *SIAM J. Appl. Math.*, 83(6):2212–2236, 2023.

[33] O. Diekmann and J.A.P. Heesterbeek. *Mathematical Epidemiology of Infectious Diseases: Model Building, Analysis and Interpretation*, volume 5. John Wiley & Sons, 2000.

[34] W. Ding, W.Z. Huang, and S. Kansakar. Traveling wave solutions for a diffusive SIS epidemic model. *Discrete Contin. Dyn. Syst. Ser. B*, 18(5):1291–1304, 2013.

[35] N. Erbilgin and K.F. Raffa. Modulation of predator attraction to pheromones of two prey species by stereochemistry of plant volatiles. *Oecologia*, 127(3):444–453, 2001.

[36] R.A. Erickson, S.B. Cox, J.L. Oates, T.A. Anderson, C.J. Salice, and K.R. Long. A daphnia population model that considers pesticide exposure and demographic stochasticity. *Ecol. Model.*, 275:37–47, 2014.

[37] M.C.O. Ferrari, B.D. Wisenden, and D.P. Chivers. Chemical ecology of predator-prey interactions in aquatic ecosystems: a review and prospectus. *Canadian Journal of Zoology*, 88(7):698–724, 2010.

[38] R.A. Fisher. The wave of advance of advantageous genes. *Annals of Eugenics*, 7:355–

369, 1937.

[39] X.F. Fu, L.H. Tang, C.L. Liu, J.D. Huang, T. Hwa, and P. Lenz. Stripe formation in bacterial systems with density-suppressed motility. *Phys. Rev. Lett.*, 108(19):198102, 2012.

[40] K. Fujie and J. Jiang. Global existence for a kinetic model of pattern formation with density-suppressed motilities. *J. Differential Equations*, 269(6):5338–5378, 2020.

[41] D. Gilbarg and N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, Berlin, 2nd edition, 1983.

[42] J.K. Hale. *Asymptotic Behavior of Dissipative Systems*, volume 25 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1988.

[43] J.K. Hale and O. Lopes. Fixed point theorems and dissipative processes. *J. Differential Equations*, 13:391–402, 1973.

[44] J.K. Hale and P. Waltman. Persistence in infinite-dimensional systems. *SIAM J. Math. Anal.*, 20(2):388–395, 1989.

[45] T.G. Hallam, C.E. Clark, and R.R. Lassiter. Effects of toxicants on populations: A qualitative: Approach i. equilibrium environmental exposure. *Ecol. Model.*, 18:291–304, 1983.

[46] E.C. Haskell and J. Bell. A model of the burglar alarm hypothesis of prey alarm calls. *Theor. Popul. Biol.*, 141:1–13, 2021.

[47] A. Hasting and T. Powell. Chaos in a three-species food chain. *Ecology*, 72(3):896–903, 1991.

[48] T.I. Hayashi, M. Kamo, and Y. Tanaka. Population-level ecological effect assessment: estimating the effect of toxic chemicals on density-dependent populations. *Ecological Research*, 24:945–954, 2009.

[49] X. He and S.N. Zheng. Global boundedness of solutions in a reaction-diffusion system of predator-prey model with prey-taxis. *Appl. Math. Lett.*, 49:73–77, 2015.

[50] J.A.P. Heesterbeek. A brief history of R_0 and a recipe for its calculation. *Acta biotheoretica*, 50(3):189–204, 2002.

[51] H.W. Hethcote. The mathematics of infectious diseases. *SIAM Rev.*, 42(4):599–653, 2000.

[52] C.D. Hoefler, M. Taylor, and E.M. Jakob. Chemosensory response to prey in Phidippus audax and Pardosa milvina. *J. Arachnol.*, 30(1):155–158, 2002.

[53] R.D. Holt and G.A. Polis. A theoretical framework for intraguild predation. *The American Naturalist*, 149:745–764, 1997.

[54] D. Horstmann and M. Winkler. Boundedness vs. blow-up in a chemotaxis system. *J. Differential Equations*, 215(1):52–107, 2005.

[55] S.B. Hsu, S.G. Ruan, and T.H. Yang. Analysis of three species Lotka-Volterra food web models with omnivory. *J. Math. Anal. Appl.*, 426:659–687, 2015.

[56] Q.H. Huang, G. Seo, and C. Shan. Bifurcation and global dynamics in a toxin-dependent aquatic population model. *Math. Biosci.*, 296:26–35, 2018.

[57] Q.H. Huang, H. Wang, and M.A. Lewis. The impact of environmental toxins on predator-prey dynamics. *J. Theor. Biol.*, 378:2015, 12-30.

[58] J. Húska. Harnack inequality and exponential separation for oblique derivative problems on Lipschitz domains. *J. Differential Equations*, 226(2):541–557, 2006.

[59] V. Hutson, K. Mischaikow, and P. Poláčik. The evolution of dispersal rates in a heterogeneous time-periodic environment. *J. Math. Biol.*, 43(6):501–533, 2001.

[60] J.S. Jia, X. Lu, Y. Yuan, G. Xu, J. Jia, and N.A. Christakis. Population flow drives spatio-temporal distribution of COVID-19 in China. *Nature*, 582(7812):389–394, 2020.

[61] C.H. Jin. Global solvability and boundedness to a coupled chemotaxis-fluid model with arbitrary porous medium diffusion. *J. Differential Equations*, 265(1):332–353, 2018.

[62] C.H. Jin and X.P. Zhang. Chemotaxis driven formation of time periodic pattern and steady state pattern. *J. Differential Equations*, 370:498–536, 2023.

[63] H.-Y. Jin, Y.-J. Kim, and Z.-A. Wang. Boundedness, stabilization, and pattern formation driven by density-suppressed motility. *SIAM J. Appl. Math.*, 78(3):1632–1657, 2018.

[64] H.-Y. Jin and Z.-A. Wang. Global stability of prey-taxis systems. *J. Differential Equations*, 262(3):1257–1290, 2017.

[65] H.-Y. Jin and Z.-A. Wang. Global dynamics and spatio-temporal patterns of predator-prey systems with density-dependent motion. *European J. Appl. Math.*, 32(4):652–682, 2021.

[66] H.-Y. Jin, Z.-A. Wang, and L. Wu. Global dynamics of a three-species spatial food chain model. *J. Differential Equations*, 333:144–183, 2022.

[67] H.-Y. Jin, Z.-A. Wang, and L. Wu. Global solvability and stability of an alarm-taxis system. *SIAM J. Math. Anal.*, 55(4):2838–2876, 2023.

[68] P. Kareiva and G. Odell. Swarms of predators exhibit "preytaxis" if individual predators use area-restricted search. *Am. Nat.*, 130(2):233–270, 1987.

[69] E.F. Keller and L.A. Segel. Initiation of slime mold aggregation viewed as an instability. *J. Theoret. Biol.*, 26(3):399–415, 1970.

[70] W.O. Kermack and À.G. Mckendrick. A contribution to the mathematical theory of epidemics. *Proc. R. Soc. A*, 115:700–721, 1927.

[71] Hansjörg Kielhöfer. *Bifurcation Theory: An Introduction with Applications to PDEs*, volume 156. Springer Science & Business Media, 2006.

[72] H. Kierstead and Slobodkin L.B. The size of water masses containing plankton bloom. *J. Marine Research*, 12:141–147, 1953.

[73] A. Klebanoff and A. Hastings. Chaos in three-species food chains. *J. Math. Biol.*, 32:427–451, 1994.

[74] M.U.G. Kraemer et al. The effect of human mobility and control measures on the covid-19 epidemic in China. *Science*, 368(6490):493–497, 2020.

[75] N. Krikorian. The Volterra model for three species predator-prey systems: boundedness and stability. *J. Math. Biol.*, 7(2):117–132, 1979.

[76] O. Ladyženskaja, V. Solonnikov, and N. Ural′ceva. *Linear and Quasi-linear Equations of Parabolic Type*, volume 23. American Mathematical Society, 1968.

[77] K.-Y. Lam and Y. Lou. *Introduction to Reaction-Diffusion Equations: Theory and Applications to Spatial Ecology and Evolutionary Biology*. Springer Cham, 2022.

[78] D. Le. Cross diffusion systems on n spatial dimensional domains. *Indiana University Mathematics Journal*, 51:625–643, 2002.

[79] J.M. Lee, T. Hillen, and M.A. Lewis. Continuous traveling waves for prey-taxis. *Bull. Math. Biol.*, 70(8):654–676, 2008.

[80] J.M. Lee, T. Hillen, and M.A. Lewis. Pattern formation in prey-taxis systems. *J. Biol. Dyn.*, 3(6):551–573, 2009.

[81] H.C. Li, R. Peng, and F.-B. Wang. Varying total population enhances disease persistence: Qualitative analysis on a diffusive SIS epidemic model. *J. Differential Equations*, 262(2):885–913, 2017.

[82] H.C. Li, R. Peng, and Z.-A. Wang. On a diffusive susceptible-infected-susceptible

epidemic model with mass action mechanism and birth-death effect: analysis, simulations, and comparison with other mechanisms. *SIAM J. Appl. Math.*, 78(4):2129–2153, 2018.

[83] H.C. Li, R. Peng, and T. Xiang. Dynamics and asymptotic profiles of endemic equilibrium for two frequency-dependent SIS epidemic models with cross-diffusion. *European J. Appl. Math.*, 31(1):26–56, 2020.

[84] H.C. Li and T. Xiang. On an SIS epidemic model with power-like nonlinear incidence and with/without cross-diffusion. *Stud. Appl. Math.*, 153(1):Paper No. e12683, 46, 2024.

[85] L.G. Li. Coexistence theorems of steady states for predator-prey interacting systems. *Trans. Amer. Math. Soc.*, 305:143–166, 1988.

[86] Z.Z. Li and Z.-A Wang. Population dynamics in closed polluted aquatic ecosystems with time-periodic input of toxicants. *J. Differential Equations*, 442:Paper No. 113502, 36, 2025.

[87] Gary M. Lieberman. *Second Order Parabolic Differential Equations*. World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[88] C.-S. Lin, W.-M. Ni, and I. Takagi. Large amplitude stationary solutions to a chemotaxis system. *J. Differential Equations*, 72(1):1–27, 1988.

[89] C.L. Liu, X.F. Fu, and L.Z. Liu et al. Sequential establishment of stripe patterns in an expanding cell population. *Science*, 334(6053):238–241, 2011.

[90] S. Liu, Y. Lou, R. Peng, and M.L. Zhou. Asymptotics of the principal eigenvalue for a linear time-periodic parabolic operator I: large advection. *SIAM J. Math. Anal.*, 53(5):5243–5277, 2021.

[91] J.E. Losey and R.F. Denno. The escape response of pea aphids to foliar-foraging predators: factors affecting dropping behaviour. *Ecol. Entomol.*, 23(1):53–61, 1998.

[92] Y. Lou and T. Nagylaki. Evolution of a semilinear parabolic system for migration and selection without dominance. *J. Differential Equations*, 225(2):624–665, 2006.

[93] Y. Lou and W.-M. Ni. Diffusion, self-diffusion and cross-diffusion. *J. Differential Equations*, 131(1):79–131, 1996.

[94] Y. Lou and W.-M. Ni. Diffusion vs cross-diffusion: an elliptic approach. *J. Differential Equations*, 154(1):157–190, 1999.

[95] A. Lunardi. Analytic semigroups and optimal regularity in parabolic problems. *Progr. Nonlinear Differential Equations Appl.*, page 16, 1995.

[96] Y. Luo. Global existence and stability of the classical solution to a density-dependent prey-predator model with indirect prey-taxis. *Math. Biosci. Eng.*, 18(5):6672–6699, 2021.

[97] M. Martcheva. *An Introduction to Mathematical Epidemiology*, volume 61. Springer, 2015.

[98] K. McCann and A. Hastings. Re-evaluating the omnivory-stability relationship in food webs. *Proceedings of the Royal Society of London, Series B, Biological Sciences*, 264:1249–1254, 1997.

[99] H.W. Mckenzie, Y. Jin, J. Jacobsen, and M.A. Lewis. R_0 analysis of a spatiotemporal model for a stream population. *SIAM J. Appl. Dyn. Syst.*, 11(2):567–596, 2012.

[100] J.D. Murray. *Mathematical Biology I: An Introduction*, volume 17 of *Interdisciplinary Applied Mathematics*. Springer-Verlag, New York, third edition, 2002.

[101] L. Nirenberg. On elliptic partial differential equations. *Ann. Sc. Norm. Super. Pisa*, 13(3):115–162, 1959.

[102] L. Nirenberg. *Topics in Nonlinear Functional Analysis*, volume 6 of *Courant Lecture Notes*. American Mathematical Society, 2001.

[103] R.A. Pastorok, S.M. Bartell, S. Ferson, and L.R. Ginzburg. *Ecological Modeling in Risk Assessment: Chemical Effects on Populations, Ecosystems, and Landscapes*. CRC Press, 2016.

[104] R. Peng and S.Q. Liu. Global stability of the steady states of an SIS epidemic reaction-diffusion model. *Nonlinear Anal.*, 71(1-2):239–247, 2009.

[105] R. Peng, Z.-A. Wang, G.H. Zhang, and M.L. Zhou. Novel spatial profiles of some diffusive SIS epidemic models. *J. Math. Biol.*, 87(6):Paper No. 81, 36, 2023.

[106] R. Peng and G. Zhang. Asymptotics of the principal eigenvalue of a linear elliptic operator with large advection. *preprint*, 2025.

[107] R. Peng and X.-Q. Zhao. A reaction-diffusion SIS epidemic model in a time-periodic environment. *Nonlinearity*, 25(5):1451–1471, 2012.

[108] K. Peters, M. Bundschuh, and R.B. Schäfer. Review on the effects of toxicants on freshwater ecosystem functions. *Environmental pollution*, 180:324–9, 2013.

[109] Sébastien Picard. Notes on Hölder estimates for parabolic PDE. *Notes*, 2019.

[110] G.A. Polis. Complex trophic interactions in deserts: an empirical critique of food-web theory. *Amer. Nat.*, 138(1):123–155, 1991.

[111] M.M. Porzio and V. Vespri. Hölder estimates for local solutions of some doubly nonlinear degenerate parabolic equations. *J. Differential Equations*, 103(1):146–178, 1993.

[112] Coffey R., M.J. Paul, J. Stamp, A.T. Hamilton, and T.E. Johnson. A review of water quality responses to air temperature and precipitation changes 2: Nutrients, algal blooms, sediment, pathogens. *JAWRA J. Amer. Water Resour. Assoc.*, 55:844 – 868, 2018.

[113] W.H. Ruan and W. Feng. On the fixed point index and multiple steady-state solutions of reaction-diffusion systems. *Differential Integral Equations*, 8(2):371–391, 1995.

[114] G.R. Scott and K.A. Sloman. The effects of environmental pollutants on complex fish behaviour: integrating behavioural and physiological indicators of toxicity. *Aquat. Toxicol.*, 68(4):369–92, 2004.

[115] S. Shankar. *Nonlinear Systems: Analysis, Stability, and Control.* Springer, 1999.

[116] J. Skellam. Random dispersal in theoretical populations. *Biometrika*, 38:196–218, 1951.

[117] H.L. Smith and H.R. Thieme. *Dynamical Systems and Population Persistence*, volume 118 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2011.

[118] J.A. Spromberg and W.J. Birge. Modeling the effects of chronic toxicity on fish populations: The influence of life-history strategies. *Environ. Toxicol. Chem.*, 24(6):1532–1540, 2005.

[119] C. Stinner, C. Surulescu, and M. Winkler. Global weak solutions in a PDE-ODE system modeling multiscale cancer cell invasion. *SIAM J. Math. Anal.*, 46:1969–2007, 2014.

[120] K. Tanabe and T. Namba. Omnivory creates chaos in simple food web models. *Ecology*, 86(12):3411–3414, 2005.

[121] D. Tang and Z.-A. Wang. Coexistence of heterogeneous predator-prey systems with prey-dependent dispersal. *J. Differential Equations*, 409:461–497, 2024.

[122] Y.S. Tao. Global existence of classical solutions to a predator-prey model with nonlinear prey-taxis. *Nonlinear Anal. Real World Appl.*, 11(3):2056–2064, 2010.

[123] Y.S. Tao and Z.-A. Wang. Competing effects of attraction vs. repulsion in chemotaxis. *Math. Models Methods Appl. Sci.*, 23(1):1–36, 2013.

[124] Y.S. Tao and M. Winkler. Boundedness in a quasilinear parabolic-parabolic Keller-

Segel system with subcritical sensitivity. *J. Differential Equations*, 252:692–715, 2012.

[125] Y.S. Tao and M. Winkler. Large time behavior in a forager-exploiter model with different taxis strategies for two groups in search of food. *Math Models Methods Appl. Sci.*, 29(11):2151–2182, 2019.

[126] Y.S. Tao and M. Winkler. Analysis of a chemotaxis-SIS epidemic model with unbounded infection force. *Nonlinear Anal. Real World Appl.*, 71:Paper No. 103820, 16, 2023.

[127] Y.S. Tao and M. Winkler. Global smooth solutions in a three-dimensional cross-diffusive SIS epidemic model with saturated taxis at large densities. *Evol. Equ. Control Theory*, 12(6):1676–1687, 2023.

[128] Y.S. Tao and M. Winkler. Global solutions to a Keller-Segel-consumption system involving singularly signal-dependent motilities in domains of arbitrary dimension. *J. Differential Equations*, 343:390–418, 2023.

[129] J.I. Tello and D. Wrzosek. Predator-prey model with diffusion and indirect prey-taxis. *Math. Models Methods Appl. Sci.*, 26(11):2129–2162, 2016.

[130] R. Temam. *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, volume 68 of *Applied Mathematical Sciences*. Springer-Verlag, New York, second edition, 1997.

[131] H. Tian et al. An investigation of transmission control measures during the first 50 days of the COVID-19 epidemic in China. *Science*, 368(6491):638–642, 2020.

[132] K.B. Tierney. Chemical avoidance responses of fishes. *Aquat. Toxicol.*, 174:228–41, 2016.

[133] A.M. Turing. The chemical basis of morphogenesis. *Philos. Trans. Roy. Soc. London Ser. B*, 237(641):37–72, 1952.

[134] Y.V. Tyutyunov, D. Sen, L.I. Titova, and M. Banerjee. Predator overcomes the Allee effect due to indirect prey-taxis. *Ecol. Complex.*, 39:100772, 2019.

[135] Y.V. Tyutyunov, L.I. Titova, and I.N. Senina. Prey-taxis destabilizes homogeneous stationary state in spatial Gause-Kolmogorov-type model for predator-prey system. *Ecol. Complex.*, 31:170–180, 2017.

[136] P. Van den Driessche. Reproduction numbers of infectious disease models. *Infectious Disease Modelling*, 2(3):288–303, 2017.

[137] R.R. Vance. Predation and resource partitioning in one predator-two prey model

communities. *Amer. Nat.*, 112:797–813, 1978.

[138] J.P. Wang and M.X. Wang. Boundedness in the higher-dimensional Keller-Segel model with signal-dependent motility and logistic growth. *J. Math. Phys.*, 60(1):011507, 14, 2019.

[139] J.P. Wang and M.X. Wang. The dynamics of a predator-prey model with diffusion and indirect prey-taxis. *J. Dynam. Differential Equations*, 32(3):1291–1310, 2020.

[140] J.P. Wang and M.X. Wang. The dynamics of a predator-prey model with diffusion and indirect prey-taxis. *J. Dynam. Differential Equations*, 32(3):1291–1310, 2020.

[141] M.X. Wang. *Nonlinear Second Order Parabolic Equations.* CRC Press, Boca Raton, 2021.

[142] M.X. Wang and P.Y.H. Pang. *Nonlinear Second Order Elliptic Equations.* Springer, Singapore, [2024] ©2024.

[143] Q. Wang. Dynamics of a spatiotemporal model on populations in a polluted river. *Nonlinear Anal. Real World Appl.*, 71:103803, 2023.

[144] W. Wang and X.-Q. Zhao. Basic reproduction numbers for reaction-diffusion epidemic models. *SIAM J. Appl. Dyn. Syst.*, 11(4):1652–1673, 2012.

[145] Z.-A. Wang and L.Y. Wu. Lotka-volterra diffusion-advection competition system with dynamical resources. *Discrete Contin. Dyn. Syst. Ser. B*, 28(6):3322–3348, 2023.

[146] X.W. Wen, J.P. Ji, and B. Li. Asymptotic profiles of the endemic equilibrium to a diffusive SIS epidemic model with mass action infection mechanism. *J. Math. Anal. Appl.*, 458(1):715–729, 2018.

[147] M. Winkler. Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model. *J. Differential Equations*, 248(12):2889–2905, 2010.

[148] M. Winkler. Asymptotic homogenization in a three-dimensional nutrient taxis system involving food-supported proliferation. *J. Differential Equations*, 263(8):4826–4869, 2017.

[149] M. Winkler. Stabilization despite pervasive strong cross-degeneracies in a nonlinear diffusion model for migration-consumption interaction. *Nonlinearity*, 36:4438–4469, 2023.

[150] M. Winkler. A quantitative strong parabolic maximum principle and application to a taxis-type migration-consumption model involving signal-dependent degenerate diffusion. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 41(1):95–127, 2024.

[151] S. Wu, J. Shi, and B. Wu. Global existence of solutions and uniform persistence of a diffusive predator-prey model with prey-taxis. *J. Differential Equations*, 260(7):5847–5874, 2016.

[152] S.N. Wu, J.P. Shi, and B.Y. Wu. Global existence of solutions and uniform persistence of a diffusive predator-prey model with prey-taxis. *J. Differential Equations*, 260(7):5847–5874, 2016.

[153] Y.X. Wu and X.F. Zou. Asymptotic profiles of steady states for a diffusive SIS epidemic model with mass action infection mechanism. *J. Differential Equations*, 261(8):4424–4447, 2016.

[154] T.D. Wyatt. *Pheromones and Animal Behaviour*. Cambridge University Press, 2003.

[155] Y. Yu, Z. Ling, and Y. Zhou. Dynamical behavior of a spatiotemporal model in open advective environments. *Acta Appl. Math.*, 187:Paper No. 1, 16, 2023.

[156] X.Q. Zhao. Uniform persistence and periodic coexistence states in infinite-dimensional periodic semiflows with applications. *Canad. Appl. Math. Quart.*, 3:473–495, 1995.

[157] X.Q. Zhao. *Dynamical Systems in Population Biology*. Springer- Verlag, New York, 2003.

[158] P. Zhou and Q.H. Huang. A spatiotemporal model for the effects of toxicants on populations in a polluted river. *SIAM J. Appl. Math.*, 82:95–118, 2022.

[159] M. Zuk and G.R. Kolluru. Exploitation of sexual signals by predators and parasitoids. *The Quarterly Review of Biology*, 73(4):415–438, 1998.

[160] W.J. Zuo and Y.L. Song. Stability and double-Hopf bifurcations of a Gause-Kolmogorov-type predator-prey system with indirect prey-taxis. *J. Dynam. Differential Equations*, 33(4):1917–1957, 2021.