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**JOINT OPTIMIZATION OF
TRUNK-FEEDER URBAN TRANSIT
NETWORKS**

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Joint Optimization of Trunk-Feeder Urban
Transit Networks

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A thesis submitted in partial fulfilment of the requirements for
the degree of Doctor of Philosophy

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Abstract

In the context of rapidly sprawling urban areas and expanding intercity express rail networks, efficient public transportation systems are increasingly vital for sustainable urban mobility. This thesis addresses the critical challenges of designing and optimizing transit networks through a comprehensive examination of feeder service design and trunk-feeder system integration. By developing continuum approximation models and optimization approaches, this research contributes to the evolution of transit network planning, offering solutions that can significantly enhance service quality while reducing generalized costs. The research explores the optimization of feeder transit services across two complementary categories: fixed-route services and demand-responsive services, each suitable for different operational contexts and demand patterns, before culminating in an integrated trunk-feeder system design that synthesizes these approaches.

For fixed-route feeder services, continuum approximation models are developed to optimize heterogeneous networks in rectangular service regions adjacent to rail terminals. This work extends previous research by: (i) simultaneously optimizing heterogeneous stop spacings along with line spacings and headways to account for various demand patterns; (ii) incorporating the effects of passenger boarding and alighting numbers on bus dwell times and patron transfer delays at rail terminals; and (iii) examining the advantages of asymmetric coordination between trunk and feeder schedules in both service directions. To address the increased modeling complexity, a novel semi-analytical method is introduced that combines analytically derived properties of the optimal solution with an iterative search algorithm. The research reveals several important insights, including that integrating heterogeneous stop spacing optimization reduces system costs by approximately 4% under specific operating conditions, with cost savings increasing with demand heterogeneity but decreasing with demand rate and service region size. Additionally, designing feeder lines where buses pick up and drop off passengers along the service region's shorter side significantly lowers system costs (by 6% when the region's aspect ratio is 0.5), while coordinating trunk and feeder schedules in both service directions yields additional cost savings of up to 20%.

Complementing this analysis, the research also examines flex-route feeder (or in some studies termed demand-responsive connector, DRC), which provide seamless connections between travelers' origins/destinations and major transportation hubs under conditions

where fixed-route services may be less efficient. This phrase distinctly identifies and analyzes two commonly used DRC operating strategies: (i) the "fully-flexible routing" strategy, where vehicles serve only requests received before dispatch through optimal tours, and (ii) the "semi-flexible routing" strategy, where vehicles follow predefined paths through swaths to serve requests received en route. Unlike previous studies that often adopted oversimplified approaches, this work develops refined analytical models for both strategies that accurately incorporate the second-order effects of stochastic demand and utilize improved local tour length formulas. Numerical experiments demonstrate that these enhanced models significantly reduce cost estimation errors—to within 2% for fully-flexible routing and 0.25% for semi-flexible routing—compared to previous models with errors of 8-12% and 6.3%, respectively. The research provides precise determination of critical demand densities for selecting between the two DRC strategies and fixed-route feeder service, revealing that as demand density and region size increase, the optimal strategy transitions from fully-flexible to semi-flexible routing before ultimately shifting to fixed-route service. Furthermore, the study identifies zoning as pivotal in DRC service design, with fully-flexible routing favoring square-shaped zones and semi-flexible routing preferring elongated rectangular zones.

Culminating this comprehensive investigation, the thesis presents a continuous model for optimizing integrated trunk-feeder transit networks under spatially heterogeneous demand. This model accounts for the integration of trunkline and feeder transit modes, offering significantly greater flexibility and applicability compared to prior studies that focused primarily on uniform networks or single service types. To address the heightened complexity in modelling, a sophisticated semi-analytical approach is developed that merges key analytical characteristics of the optimal solution with an iterative algorithm. Numerical results demonstrate that this integrated approach achieves approximately 21% cost savings compared to previous models, highlighting the substantial benefits of considering heterogeneous demand distributions and multi-modal integration in transit network design.

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Chapter 1

Introduction

Section 1.1 introduces the background for this thesis. Section 1.2 reviews the relevant studies. Section 1.3 summarizes the contributions of the thesis. The outline of this thesis is furnished in Section 1.4.

1.1 Background

The global trend of urbanization has accelerated significantly in recent decades (Mor et al., 2020), leading to expanding metropolitan areas, sprawling suburban developments, and the emergence of new mega-cities and urban clusters (Yu et al., 2019). This rapid urban growth has increased long-distance commuting demands between suburban regions, satellite towns, and city centers. To accommodate these growing travel needs, efficient and high-capacity trunk transit systems—such as rapid rail transit and commuter rail lines—have been widely developed. These transit lines typically feature widely spaced stations, sometimes several kilometers apart, designed to maintain high travel speeds and operational efficiency.

For instance, the Bay Area Rapid Transit (BART) system in California connects numerous small cities in the East Bay and San Francisco Peninsula with the downtown areas of San Francisco and Oakland, with station spacings of up to 8 km (<https://www.bart.gov/>; <https://www.google.com/maps>). Similarly, China’s booming high-speed rail network exemplifies this trend, supporting the rise of satellite cities such as Langfang, located approximately 60 km from Beijing yet accessible within only 20 minutes by high-speed rail. The extensive spacing between stations creates large catchment areas, where many travelers reside beyond comfortable walking distances, resulting in spatially heterogeneous and often sparse local demand distributions. Thus, designing efficient feeder service networks to accommodate these first- and last-mile trips has become crucial for ensuring seamless connectivity, accessibility, and enhancing the overall attractiveness of mass transit systems (Montenegro et al., 2021).

Feeder transit services are typically categorized into two primary forms: fixed-route and flex-route services (Sangveraphunsiri et al., 2022). Fixed-route feeders operate according to predetermined schedules and routes, offering a higher level of reliability, energy efficiency, and affordability, especially in densely populated areas with substantial and stable demand (Chang and Schonfeld, 1991; Petit et al., 2021; Wong et al., 2023). Such fixed-route systems provide consistent and predictable services, maintaining dependable performance.

Conversely, flex-route feeder services, also termed demand-responsive connectors (DRCs), offer enhanced convenience by providing near door-to-door service. These services dynamically respond to passenger requests and are particularly suitable for lower-demand areas, irregularly shaped zones, or user groups requiring additional assistance, such as elderly or disabled passengers (Kersting et al., 2021; Wang et al., 2023). Flex-route feeders are increasingly supported by advances in information and communication technologies, including user-friendly mobile applications, real-time vehicle tracking, and efficient dispatch systems, significantly improving user experience and operational responsiveness (Liu and Ouyang, 2021; Ma et al., 2019). Nevertheless, these benefits typically incur higher operating costs compared to traditional fixed-route services, presenting a trade-off between operational efficiency and passenger convenience (Wang et al., 2014, 2023).

Given these trade-offs and heterogeneous demand characteristics in urbanized regions, the design and optimization of integrated trunk-feeder transit systems have emerged as a critical research area. An integrated trunk-feeder system combines the advantages of trunk systems, offering high-capacity, high-speed transit connections between major urban nodes, with feeder services that provide local accessibility and flexible service coverage. Such integrated systems, when optimized to cater spatial heterogeneity, have the potential to substantially improve network-wide connectivity, passenger convenience, and overall system efficiency by effectively bridging the accessibility gaps created by widely spaced trunk line stations.

1.2 Literature Review

Section 1.2.1 discusses existing studies on the design and optimization of fixed-route feeder services. Section 1.2.2 reviews the growing body of research on demand-responsive connector services. Lastly, Section 1.2.3 surveys the literature on large-scale trunk-feeder transit network design.

1.2.1 Fixed-route feeder service design

Two types of models have been commonly employed to address the feeder service optimization problem: discrete models and continuous models. Discrete models incorporate numerous details representing specific physical and geographical constraints in the operating environment, such as the exact locations of candidate stops, specific road networks, and traffic conditions (Martins and Pato, 1998; Shrivastava and O’Mahony, 2006; Ciaffi et al., 2012; Lee et al., 2021). In contrast, continuous approximation (CA) models are more parsimonious, utilizing only a few parameters and design variables to represent the feeder system by approximating heterogeneous discrete details with continuous functions for demand density, line and stop spacings, and so on (Ouyang et al., 2014; Chen et al., 2015, 2018; Luo et al., 2021). The choice between these two modeling paradigms involves a fundamental trade-off between realism and tractability. The granular details captured by discrete models come at a computational cost: due to nonlinearity and the curse of dimensionality, their complexity grows exponentially with problem size, necessitating the use of heuristic methods (Fan and Ran, 2021) that often limit their application to small-scale networks and make it challenging to guarantee solution quality. Continuous models, by abstracting away such details, sacrifice some representational accuracy but gain substantial analytical advantages. They can be solved to global or near-global optimality using efficient analytical or numerical methods, scale effectively to large networks, reveal structural insights into optimal design principles.

Most works in this field assumed spatially uniform demands (Chang and Schonfeld, 1991; Kim and Schonfeld, 2012, 2013, 2014; Sivakumaran et al., 2014; Kim and Schonfeld, 2015; Su and Fan, 2019; Badia and Jenelius, 2020, 2021). Thus, they cannot be directly applied to real-world feeder system designs under spatially heterogeneous demand. In certain scenarios, spatial heterogeneity in demand cannot be overlooked, as exemplified by the satellite cities mentioned earlier. Most of the cited works also assumed uniform line spacing and headway, while largely neglecting stop spacing optimization. Moreover, they typically assumed fixed dwell time and transfer time between trunk and feeder services, disregarding the relationship between dwell and transfer times and the number of boarding and alighting patrons. Notably, Chang and Schonfeld (1991) and Kim and Schonfeld (2012, 2013, 2014, 2015) considered vehicle size optimization.

Only a few studies have developed feeder network design models for the more realistic, spatially heterogeneous demand. Among them, Wirasinghe (1980) and Sivakumaran et al. (2012) assumed demand is heterogeneous in only one direction (either along the rail line or perpendicular to it), while Kuah and Perl (1988) and Yang et al. (2020b) considered demand heterogeneity in both directions of the service region. Most of these studies did not optimize stop spacings, likely due to mathematical complexity. For instance, Quadrifoglio and Li (2009) stated that deriving the optimal stop spacing was “often

quite hard.” To the best of our knowledge, [Kuah and Perl \(1988\)](#) is the only study that optimized stop spacings under heterogeneous demand. However, their model relied on oversimplified and somewhat unrealistic assumptions (probably necessary to make the problem mathematically tractable), such as very small rail station spacing and ignoring the operating cost associated with bus dwell times. In addition, nearly all these works assumed fixed dwell and transfer times and unlimited feeder bus capacity.

To summarize, despite the extensive history of literature on fixed-route feeder service design, the following research gaps have not yet been adequately addressed:

(i) The optimization of heterogeneous stop spacings has been largely neglected in the literature. Most studies assume that the stop spacing is a constant or do not explicitly model stop locations and bus dwell times. [Kuah and Perl \(1988\)](#) is the only work that optimized heterogeneous stop spacings under heterogeneous demand, but their model also has its drawbacks, as discussed earlier in this paper. Note that real-world stop spacings vary significantly; for example, [Percus and Martin \(1996\)](#) states that real-world stop spacings range from 200 m to 800 m. And they substantially affect passengers’ access distances and in-vehicle travel times ([Daganzo and Ouyang, 2019](#)). Therefore, stop spacings should be jointly optimized with line spacings and headways in a heterogeneous manner.

(ii) Most studies assume constant dwell times and transfer times for feeder bus services, oversimplifying the impact of passenger boarding and alighting on dwell times. [Su and Fan \(2019\)](#) is the only exception, but it assumes uniform demand. Unlike rail transit, bus dwell time mainly depends on the number of passengers getting on and off at a stop ([Daganzo and Ouyang, 2019](#)). Ignoring this relationship can render errors in estimating bus roundtrip times and in-vehicle travel times for passengers. Particularly, when bus size is jointly optimized, employing a fleet of smaller vehicles can result in lower operating costs, increased service frequencies, and reduced travel times and roundtrip durations due to the fewer passengers each vehicle carries. This trade-off and its impact on the overall system cost cannot be accurately represented if the relationship between the number of boarding and alighting passengers and the dwell times is not taken into account.

(iii) While previous studies (e.g., [Sivakumaran et al., 2012](#); [Kim and Schonfeld, 2012, 2014](#); [Yang et al., 2020b](#)) have investigated the advantages of coordinating schedules between trunk and feeder services, their conclusions are hindered by modeling limitations. These works, for instance, overlooked stop spacing and variable dwell times, assumed uniform line spacing and headway, or generated suboptimal solutions. In addition, these studies only focused on coordination in one direction (the patron collection direction), while the coordination effects in the patron distribution direction were ignored. As we will see soon, schedule coordination in the patron distribution direction provides a smaller yet noteworthy benefit. The overall benefit of schedule coordination is significantly greater

than previously predicted in the literature.

Besides, a simple but important question seems to be neglected in literature. Specifically, for a rectangular or approximately rectangular feeder service region (which is often found in grid trunk-line networks; see [Chien and Schonfeld \(1997\)](#)), which feeder line layout is more cost-effective: each line making stops along the shorter side of the rectangle and traveling nonstop along the longer side, or vice versa? We will demonstrate that the difference between the two layout options is sizeable.

1.2.2 Demand-responsive connector service design

1.2.2.1 Optimal DRC design

The study of DRC design has seen the application of various methodologies, each with its strengths and limitations. Discrete mathematical programming models have been notably used to tailor feeder system designs to specific real-world networks, as seen in the work of [Pan et al. \(2015\)](#). These models are detailed and precise (e.g., [Yu et al., 2015](#)) but are constrained by substantial computational complexity, often deemed NP-hard problems. Another approach is the use of simulation, as explored by [Li and Quadrifoglio \(2010\)](#), which offers a dynamic representation of DRC operations and can yield a realistic portrayal of their performance. Despite its practicality, the simulation approach is computationally intensive—perhaps even more so than discrete modeling methods—and may not always provide clear analytical insights.

To address the limitations of the above approaches, the so-called “analytical models” (e.g., [Li and Quadrifoglio, 2010](#)) have gained traction for their ability to distill complex systems into essential operating variables. These models stand out for their generality, offering insights into the intricate cause-and-effect relationships between key operating features and the optimal DRC design. Moreover, these models can often be solved efficiently, facilitating the extraction of operational insights that are crucial for enhancing our comprehension and implementation of DRCs in practical scenarios. For example, [Quadrifoglio and Li \(2009\)](#) employed analytical models to derive approximate closed-form expressions for the critical demand density for choosing between fixed-route and flex-route services. Their models, while insightful, were limited to single and two-zone scenarios. [Qiu et al. \(2014\)](#), [Zheng et al. \(2018b\)](#), and [Zheng et al. \(2019\)](#) investigated feeder services where vehicles deviate from a set route on an ad-hoc basis to accommodate real-time demand requests. [Zheng et al. \(2018a\)](#) examined the implications of route deviation, enabling buses to alter their fixed routes for more direct passenger service. [Sangveraphunsiri et al. \(2022\)](#) proposed an innovative hybrid concept known as the “Jitney-lite” strategy, which employs a fixed-route service for the outbound leg (from passengers’ origins to a mass transit terminal) and a flex-route service for the inbound leg (from the terminal to passengers’ destinations).

The aforementioned studies presuppose that a single vehicle serves the entire service region or that the region is divided into a fixed number of zones with one vehicle allocated per zone. These approaches may lead to impractically long tour lengths and travel times, particularly in larger regions. To address this issue, [Chang and Schonfeld \(1991\)](#) developed continuum approximation models that optimize zone partitioning and vehicle sizes for flex-route feeder services. They also pinpointed the critical demand threshold that necessitates a switch between fixed and flex-route services. Subsequent research conducted by Kim and Schonfeld from 2012 to 2015 introduced a series of improvements to the model: optimizing service headways ([Kim and Schonfeld, 2012](#)), integrating mixed fleets with varying vehicle sizes ([Kim and Schonfeld, 2013](#)), examining multi-region service coordination ([Kim and Schonfeld, 2014](#)), and incorporating elastic demand ([Kim and Schonfeld, 2015](#)). More recent follow-up works focused on the stochastic dynamic switching between fixed and flex-route services ([Guo et al., 2017](#)), and the derivation of closed-form relationship between optimal headway and zone size ([Kim et al., 2019](#)).

1.2.2.2 Tour Length Estimation and Operational Setup Under Distinct Routing Strategies

A fundamental issue in optimizing DRC systems is the estimation of flex-route vehicle tour lengths. This problem aligns with the classic Traveling Salesman Problem (TSP)—to find the shortest tour length that visits a specified number of locations. While established algorithms or commercial solvers are adequate for resolving specific TSP instances (e.g., the one depicted in [Fig. 1.1a](#)), analytical modeling of DRC services demands a general closed-form expression for the expected TSP tour length, rather than case-specific TSP solutions. To this end, an asymptotic tour length formula was provided by [Beardwood et al. \(1959\)](#) and later extended by [Stein \(1978a,b\)](#). This foundational result asserts that the expected tour length for visiting a sufficiently large number of points denoted by q can be estimated as $k^*\sqrt{qA}$, where A is the area of the bounded zone containing the q points, and k^* is a constant. This formula has been extensively adopted in the literature of DRC design.

Determining the value of k^* is essential yet challenging. Unfortunately, the existing research has not yielded satisfactory results for DRC model applications. Most relevant studies reported k^* values ranging from 0.712 to 0.765, obtained under the Euclidean distance metric ([Bonomi and Lutton, 1984](#); [Lee and Choi, 1994](#); [Percus and Martin, 1996](#); [Stein, 1978a,b](#)). In the context of transportation, however, the Manhattan metric is more commonly used due to the grid-like nature of urban road networks (e.g., [Daganzo, 2010](#); [Fan et al., 2018](#); [Nourbakhsh and Ouyang, 2012](#); [Wu et al., 2020](#)). Regrettably, there are scant estimates of k^* that consider Manhattan geometry. [Chakraborti and Chakrabarti \(2000\)](#) proposed a k^* value of 0.93 for cases where $q \rightarrow \infty$. For smaller values of q , [Yang](#)

et al. (2020a) demonstrated that k^* is not a constant but varies depending on q and the service zone’s geometry. They offered a regression-based formula $k^* = 1.1055 - 0.008q + 1.0297S/q$, where S stands for the aspect ratio of a rectangular zone. Although their formula exhibits commendable precision for $10 \leq q \leq 100$, our numerical tests indicate that its accuracy decreases significantly for even smaller q values ($2 \leq q \leq 15$), which are typical for DRC vehicles serving low-demand areas. The observed error in these instances can be over 12%.

Additionally, Daganzo (1984) proposed an efficient heuristic method for tour construction, where a service zone is cut into a swath of width w_0 , as exemplified in Fig. 1.1b. Vehicles navigate along the swath, making lateral detours to pick up passengers, a strategy known in some DRC literature as “no-backtracking routing” (e.g., Qiu et al., 2014). We label it semi-flexible routing in this paper, as the vehicle’s route is fixed barring those lateral detours. This routing heuristic allows for an analytical expression for the expected tour length, dependent on swath width w_0 . Daganzo demonstrated that the minimum expected tour length is $1.15\sqrt{qA}$ when w_0 is set to $w_0^* = \sqrt{3A/q}$. This k^* value of 1.15 has since been widely used in DRC literature to estimate tour lengths (e.g., Chang and Schonfeld, 1991). Furthermore, Daganzo (1984) explicitly addressed cases where w_0^* exceeds half the shorter side of the service area (i.e., $\min(l, w) < 2w_0^*$). In such scenarios, he proposed setting $w_0^* = \min(l/2, w/2)$ to construct a single loop swath. However, it doesn’t account for the fact that w_0^* is often unachievable in practice. For rectangular zones that are typically assumed in DRC models, the zone length or width should be an integer multiple of w_0 , resulting in a k^* value that is necessarily higher than 1.15 and dependent on the zone’s dimensions.

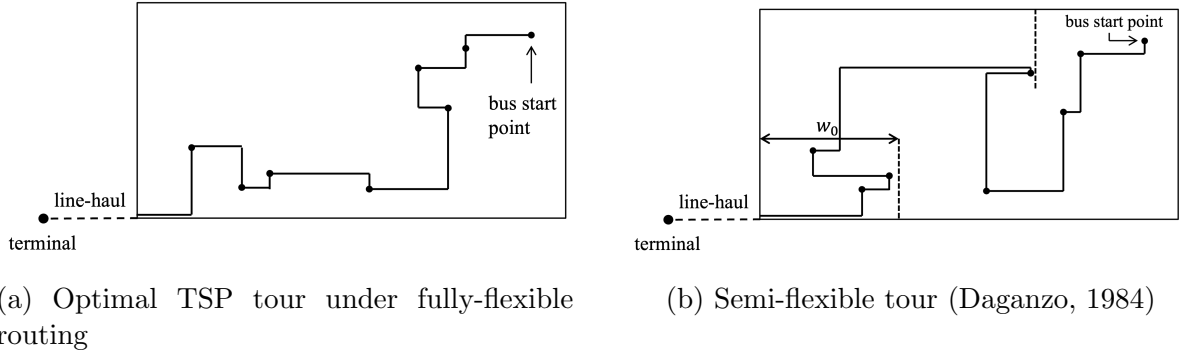


Figure 1.1: Two routing strategies

Despite the imprecision of k^* values reported in the literature, e.g., 0.93 for the optimal TSP routing strategy and 1.15 for semi-flexible routing, a rough comparison still suggests that an optimized TSP tour is on average 19% shorter than a semi-flexible heuristic, implying the great potential of using the former strategy.

Moreover, the two strategies differ not only on tour lengths but also on the operational setups. The optimized TSP routing, termed “fully-flexible routing strategy” in

this paper, follows a dynamic offline approach, where all service requests assigned to a vehicle must be known before its dispatch, and the vehicle’s route remains fixed during its trip. This structure leads to a passenger waiting time (at home) that includes both (i) the time from making the service request to the vehicle’s dispatch, and (ii) the time for the dispatched vehicle to reach the pick-up point. In contrast, the semi-flexible routing strategy relies on dynamic online routing for deviations from a “base route” determined by the predefined swath, allowing vehicles to pick up new passengers en route. Therefore, the average waiting time is significantly reduced by eliminating the wait for vehicle dispatch. The distinct operational mechanisms of each strategy represent an element that is often overlooked in DRC literature.

In summary, fully-flexible routing achieves reduced vehicle tour lengths but extends passenger wait times, whereas semi-flexible routing typically shortens wait times at the expense of longer tours. Therefore, determining which strategy optimally balances passenger experience and operational costs remains unresolved, motivating the present study.

1.2.3 Trunk-feeder network design

In transit network design using continuous models, many studies consider uniform demand assumptions due to their analytical tractability and computational simplicity (e.g., [Daganzo, 2010](#); [Chen and Nie, 2017](#)). Among these, [Sivakumaran et al. \(2014\)](#) analyzed trunk-feeder systems with uniform line and station distribution, providing insights into the coordination benefits between trunk and feeder services under idealized conditions. [Fan et al. \(2018\)](#) advanced this field by developing models for intersecting bimodal transit networks, where express and local lines intersect to provide multiple route options with 1-3 transfers, showing that such networks can outperform single-mode designs under moderate to high demand levels with cost savings up to 6%. More recently, [Luo and Nie \(2023\)](#) developed an integrated bus-bike system design model under uniform demand, demonstrating that bike-sharing services can enhance system flexibility by allowing travelers to choose non-closest bus stops, with over 20% of travelers making such choices to minimize transfer penalties.

However, due to the significant methodological complexity involved in handling spatial heterogeneity, only a few studies focus on optimizing heterogeneous design under heterogeneous demand, which aligns more closely with real-world conditions where demand varies substantially across different districts of a metropolitan area. [Chien and Schonfeld \(1997\)](#) were pioneers in designing heterogeneous transit networks, achieving varied line spacing under heterogeneous demand conditions and demonstrating the potential benefits of adapting network configuration to demand patterns. Building upon this foundational work, [Ouyang et al. \(2014\)](#) proposed a more sophisticated transit network design approach under heterogeneous demand, where the number of parallel network lines

varies spatially at a power-of-two rate with the demand rate, establishing important theoretical relationships between demand intensity and optimal network density. Extending this methodology further, [Petit and Ouyang \(2022\)](#) developed an optimization method for flexible transit systems, with flexible transit tubes increasing at a power-of-two rate with demand, bridging the gap between fixed-route and demand-responsive service designs.

Despite advances in both uniform and heterogeneous network design, existing studies have primarily focused on either trunk line optimization or feeder service design in isolation, with limited attention to the integrated optimization of trunk-feeder systems under spatially heterogeneous demand. Existing trunk-feeder models developed under uniform demand assumptions cannot adequately capture the performance benefits and cost savings that may arise from implementing heterogeneous designs under spatially varying demand densities. Current research also lacks comprehensive modeling approaches for simultaneously optimizing heterogeneous trunk and feeder service design variables under spatially heterogeneous demand. This necessitates the development of new methodological frameworks to account for the superior performance of such integrated heterogeneous systems and quantify the potential cost advantages compared to previous models.

1.3 Major Contributions

This thesis addresses trunk-feeder transit network optimization by systematically building from individual feeder service components to integrated trunk-feeder system design. The research first develops models for fixed-route feeder services, which are optimal for high-demand corridors and large service areas where operational efficiency is paramount. The study then addresses demand-responsive connector (DRC) services, which serve as a complementary solution for lower-demand areas and specialized access requirements where the flexibility of door-to-door service is advantageous. Finally, the research integrates feeder service designs with trunk line optimization to develop comprehensive trunk-feeder network design methodologies. Each component of this hierarchical approach yields both methodological advances that enhance the cost-effectiveness of optimal designs and novel numerical insights that provide practical guidance for transit system design and operation. The contributions across these three interconnected domains are introduced in Sections [1.3.1](#) to [1.3.3](#).

1.3.1 Fixed-route feeder service design

For the fixed-route feeder service design, our method stand out by considering the effects of passenger boarding and alighting on bus dwell times and transfer times, while also optimizing heterogeneous stop spacings alongside line spacings and headways. This added complexity not only enhances model accuracy of our model in reflecting real-world

scenarios but also yields considerable cost-saving benefits. For example, refining models of dwell and transfer times improves the fidelity of associated cost items by up to 5%. Furthermore, optimizing heterogeneous stop spacings boosts system performance by up to 4% compared to uniform stop spacing, with potential cost savings exceeding 10% when compared to models that only optimize line spacings and headways. Additionally, our model that integrates schedule coordination reveals even greater benefits—cost savings up to 20%—when trunk and feeder services are coordinated in both feeder travel directions. This suggests that coordinating schedules in both directions is advisable whenever feasible.

Other important findings and their practical implications demonstrate that the benefit of the heterogeneous design largely depends on the demand’s spatial pattern, with greater benefits observed for more heterogeneous demand and when the spatial concentration of demand shifted away from the rail terminal, while higher demand rates and larger service region sizes tend to reduce the benefit. Compared to homogeneous or partially-homogeneous feeder network designs, an optimal heterogeneous feeder network uses smaller buses because the heterogeneous design better accommodates heterogeneous demand, resulting in more evenly distributed passenger loading across lines and vehicles. Furthermore, feeder lines should be designed to enable buses to travel along the shorter side of the service region for pick-up and drop-off, while traveling nonstop along the longer side to the rail station, as selecting the appropriate layout can further reduce system costs by 6% or more, depending on the service region’s shape.

1.3.2 DRC service design

Our work on DRC service design addresses critical gaps in the literature through three major methodological contributions that enhance both modeling accuracy and practical applicability.

First of all, we have precisely defined the fully-flexible and semi-flexible routing strategies for DRC operations and formulated respective optimization models, addressing previous literature’s tendency to conflate these distinct operational approaches. Our numerical analyses reveal the significant difference between the two strategies’ performance under various operating conditions, yielding key insights instrumental for optimal DRC service designs and operations.

Second, to improve the modeling accuracy, we have recalibrated the estimates of the k^* values for fully-flexible routing in realistic DRC scenarios, which involve smaller occupancies and diverse service zone dimensions, in the context of the Manhattan distance metric. We have also rectified the previously incorrect use of $k^* = 1.15$ for semi-flexible routing by incorporating the constrained relationship between zone dimensions and the swath width.

Third, our research refines and advances the modeling of stochastic demand in DRC design optimization, accounting for the variability in vehicle occupancy due to random demand and its impacts on key cost terms. Prior models often relied solely on expected vehicle occupancy, neglecting the variation of occupancy, which might be adequate for cost functions linear in vehicle occupancy but falls short for nonlinear cost components, such as cumulative passenger wait times and in-vehicle travel times. This discrepancy arises because the expected value of a nonlinear function of occupancy is typically not equal to the function evaluated at the expected occupancy. To address this, our methodology employs a Taylor series approximation, capturing higher-order effects of demand variability, and integrates the vehicle occupancy distribution into our models to ensure more nuanced and accurate optimization of DRC design.

These methodological advancements enable the derivation of insights that were previously unattainable with less precise models. By reducing cost estimation errors from 8-12% to within 2% for fully-flexible routing and from 6.3% to 0.25% for semi-flexible routing, we correct prior underestimations of DRC’s benefits and provide more reliable thresholds for choosing between the two DRC strategies and fixed-route services. Zoning plays a pivotal role in DRC service design, with fully-flexible routing performing optimally with square-shaped zones, while semi-flexible routing demonstrates enhanced efficiency in elongated rectangular zones. The fully-flexible routing strategy is more effective in regions characterized by lower demand densities and smaller service areas, making it particularly suitable for last-mile operations, while the semi-flexible routing strategy is better suited for regions with relatively higher demand densities and larger service areas, such as those surrounding transportation hubs and major public venues.

1.3.3 Trunk-feeder network design

Building upon the foundation established by our work on feeder service design, we finally address the critical gap in integrated trunk-feeder system design under spatially heterogeneous demand. While existing studies have primarily focused on either trunk line optimization or feeder service design in isolation, the interaction between trunk and feeder services becomes increasingly complex when both demand patterns and service characteristics vary spatially, necessitating more comprehensive modeling approaches that can capture these interdependencies.

This research presents a continuum approximation model to optimize the integrated design of trunk-feeder transit networks, considering spatially heterogeneous demands and heterogeneous designs. The optimization framework jointly determines key decision variables including trunk line spacings and headways alongside feeder line spacings, stop spacings, and headways, with the objective of minimizing total system costs encompassing both agency operating costs and patrons’ travel time costs. A heuristic iterative algorithm

is developed to solve the complex optimization model, enabling practical implementation for real-world transit system design. Due to the limited time, only fixed-route feeder is incorporated in this joint optimization effort, while potential extensions to including flex-route feeder services can be conducted in the future using a similar approach.

Numerical analysis demonstrates significant performance improvements, with the integrated optimization approach achieving up to 24% cost savings compared to previous models that optimize trunk and feeder components separately. The results also offer strategic insights into mode selection: rail-bus systems are shown to be particularly advantageous in high-demand settings, while bus-bus systems are more cost-efficient in lower-demand environments as a result of their lower infrastructure and operational costs. Furthermore, feeder service configurations are more sensitive to spatial demand variations than to the type of trunk mode employed, highlighting the importance of localized design responsiveness in achieving overall system efficiency.

1.4 Organization of the Thesis

This thesis is organized into five chapters.

Chapter 2 focuses on heterogeneous fixed-route feeder service design under spatially heterogeneous demand. The chapter begins by introducing the development of continuum approximation models for optimizing heterogeneous fixed-route feeder transit networks. A two-stage iterative algorithm is then developed that combines analytically derived optimality properties with numerical search procedures to solve the optimization model. Numerical analysis is conducted to demonstrate the cost savings achieved by the proposed heterogeneous design compared to previous uniform approaches, followed by sensitivity analysis that yields practical insights for transit system design and operation.

Chapter 3 addresses demand-responsive connector (DRC) service design, developing analytical models for both fully-flexible routing and semi-flexible routing strategies. The chapter details the problem formulation with rigorous approaches to accommodating stochastic demand and more accurate estimations of tour lengths. The solution method is elaborated upon, followed by thorough numerical case studies that compare the efficacy of the two routing strategies across various operating scenarios. Additionally, the optimally designed DRC services are compared with optimal fixed-route feeder services within different operational contexts to establish service selection guidelines.

Chapter 4 presents the integrated trunk-feeder network design methodology that builds upon the foundation established in the previous fixed-route feeder service design. The chapter describes the modeling framework that jointly optimizes both trunk and feeder service design variables. An iterative solution algorithm is then introduced, followed by numerical analysis that includes comparisons with previous models and sensitivity analysis to demonstrate the substantial benefits of integrated system design.

Chapter 5 concludes the thesis by summarizing the key contributions across three core chapters. This chapter also discusses the limitations of the current work and outlines promising directions for future research, including potential extensions to incorporate additional service modes and operational considerations.

Chapter 2

Fixed-Route Feeder Service Design

2.1 Problem Description

Figure 2.1 depicts a schematic representation of a feeder network. A feeder bus terminal, marked by the black dot, is located at the lower-left corner of the figure. This terminal can be a trunk-line (e.g., rail transit) station in a city's suburban area or an intercity rail station¹. It is assumed that the terminal's catchment area, which represents the region served by the feeder network, is a rectangular zone centered at the terminal, encompassing a dense street grid. A quarter of this catchment area is defined as $\Omega \equiv \{(x, y) \mid 0 \leq x \leq L, 0 \leq y \leq W\}$; see Figure 2.1.

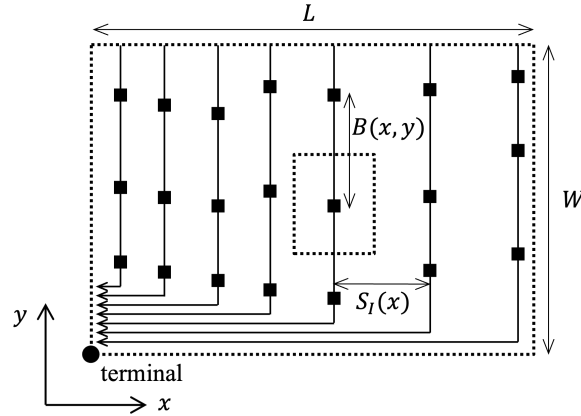


Figure 2.1: The feeder system layout.

Owing to the symmetric nature of the area, it is only necessary to identify the optimal design for feeder lines carrying patrons within Ω , both to and from the terminal. We presume that feeder buses collect or distribute passengers along the y -axis, while they undertake nonstop travel along the x -axis either towards or away from the terminal, as

¹The terminal may also represent a central business district in a medium or small-sized city, a shopping center within an urban district, or an airport. In such instances, the feeder service effectively functions as a shuttle service. Nevertheless, our models remain applicable to these cases, requiring only minor modifications.

illustrated by the thin solid lines in the figure. The positions of feeder bus stops are indicated by black squares.

Travel demands in this service area are expressed as continuous, time-invariant density functions². Specifically, the demand density from location $(x, y) \in \Omega$ to the terminal (in the patron-collection direction) is denoted by $\lambda_p(x, y)$, while the demand density from the terminal to $(x, y) \in \Omega$ (in the patron-distribution direction) is denoted by $\lambda_d(x, y)$. The line spacing $S_I(x)$ and the headways in the patron-collection and distribution directions, $H_{I_p}(x)$ and $H_{I_d}(x)$ respectively, are functions of x . The stop spacing $B(x, y)$ is a function of both x and y . In the context of the continuum approximation (CA) framework, these decision functions are assumed to be integrable on $0 \leq x \leq L$ and Ω , respectively.

We further assume that: (i) each patron chooses the nearest feeder stop on the closest line for boarding or alighting; (ii) feeder buses on any specific line maintain regular headways; (iii) patrons arrive at the feeder stops randomly without consulting the service timetable; and (iv) all patrons waiting at the terminal or a feeder bus stop will be picked up by the next arriving bus (i.e., no patron will be left behind). These assumptions are commonly employed in the literature (e.g., Daganzo, 2010; Chen et al., 2015; Fan et al., 2018).

The problem aims to minimize the generalized cost of the feeder system, consisting of the patrons' travel cost and the agency's operating cost. All cost components are expressed in units of hours; in other words, the agency cost items originally expressed in monetary units are converted to hours to ensure they can be added together with the patron costs. The following section develops these cost terms incurred during one hour of bus operations.

2.2 Mathematical Formulation

The cost model of patrons' travel time cost is formulated in Sections 2.2.1. Sections 2.2.2 gives the model of agency cost. Section 2.2.3 furnishes the optimization model without schedule coordination. Schedule coordination between the trunk line service and the feeders is modeled in Section 2.2.4. The notations used in this section and their definitions are listed in Appendix A.

²The models can readily be adapted to real-world situations where demand densities fluctuate over time, e.g., between peak and off-peak periods. To account for these variations, one can model the costs associated with different time periods and aggregate them accordingly; see Wu et al. (2020) for an example.

2.2.1 Patrons' travel time cost

The patrons' travel time cost comprises three components: (i) the time spent accessing (for the patron-collection direction) and egressing (for the distribution direction) feeder stops; (ii) the waiting time at feeder stops and the transfer time at the terminal; and (iii) the in-vehicle travel time on feeder buses. They are formulated in Sections 2.3.1 to 2.3.3, respectively³. Note that while bus fare is another component of patron cost, it is excluded from our generalized cost model. This exclusion is due to the fare constituting a monetary transfer from patrons to the transit agency, thereby representing a cost for patrons and a revenue source for the agency.

2.2.1.1 Access and egress cost

The total access and egress time per hour of bus operations, denoted by C_A , is formulated as follows:

$$C_A = \int_{x=0}^L \int_{y=0}^W \frac{S_I(x) + B(x, y)}{4v_W} [\lambda_p(x, y) + \lambda_d(x, y)] dy dx \quad (2.1)$$

where $\frac{S_I(x) + B(x, y)}{4}$ represents the average access or egress distance per passenger originating from or destined for location (x, y) , and v_W denotes the walking speed. Note that the double integral part related to $B(x, y)$ would not appear (i.e., the RHS of (2.1) would reduce to a single integral of decision function $S_I(x)$) if stop spacings are not jointly optimized.

2.2.1.2 Waiting and transfer cost

The total waiting and transfer time per hour of bus operations, denoted by C_W , is formulated as:

$$C_W = C_{Wp} + C_{Wd} \quad (2.2)$$

$$C_{Wp} = \int_{x=0}^L \int_{y=0}^W \frac{H_{Ip}(x)}{2} \lambda_p(x, y) dy dx + \int_{x=0}^L \int_{y=0}^W \left(\frac{\tau_a}{2} S_I(x) H_{Ip}(x) \int_{y=0}^W \lambda_p(x, y) dy + t_{f-t} + \frac{H_t}{2} \right) \lambda_p(x, y) dy dx \quad (2.3)$$

$$C_{Wd} = \int_{x=0}^L \int_{y=0}^W \left(t_{t-f} + \frac{H_{Id}(x)}{2} + \frac{\tau_b}{2} S_I(x) H_t \int_{y=0}^W \lambda_d(x, y) dy \right) \lambda_d(x, y) dy dx \quad (2.4)$$

where C_{Wp} and C_{Wd} denote the total waiting and transfer times for patrons traveling to and from the terminal, respectively. The first term on the right-hand side of Eq. (2.3)

³In this paper, the total patron cost is formulated by summing the three patron travel time components. Alternatively, different weighting coefficients could be applied to these components to account for the varying values of time during different stages of travel.

corresponds to the total waiting time at the feeder stops, with $H_{Ip}(x)/2$ being the average waiting time per patron for the bus line located at x . The second term on the RHS of Eq. (2.3) accounts for the total patron delay at the terminal, which comprises three components: the time lost due to patrons' alighting process from feeder buses, the transfer delay from feeder to trunk-line transit, and the waiting time for a trunk-line vehicle. Here, τ_a denotes the alighting time per patron, t_{f-t} the transfer delay per patron (including walking time), and H_t the trunk-line service headway at the terminal. These components are illustrated in Figure 2.2.

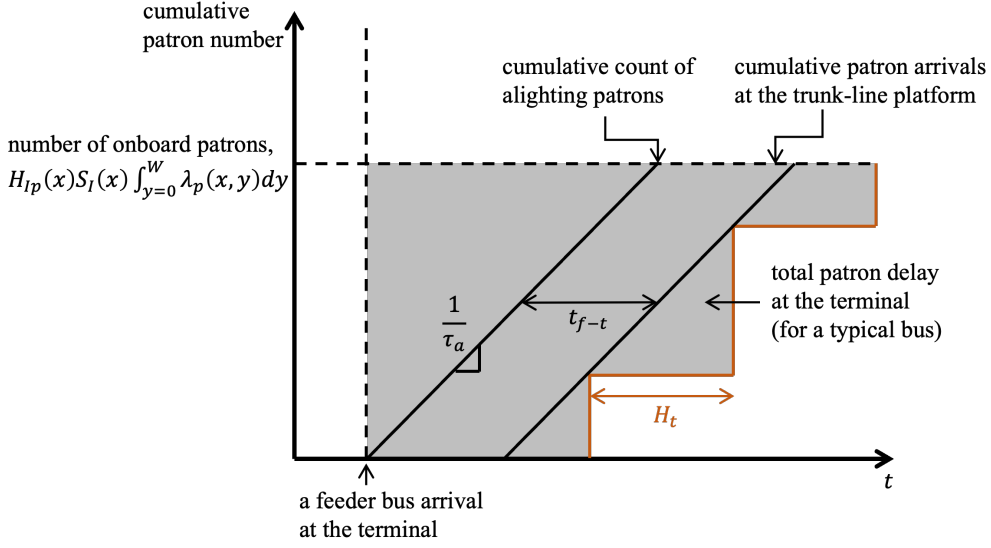


Figure 2.2: Patron delays at the terminal (in the patron-collection direction)

Note in particular that $S_I(x)H_{Ip}(x) \int_{y=0}^W \lambda_p(x, y) dy$ represents the number of onboard patrons per bus on a line located at x , while $\frac{\tau_a}{2}S_I(x)H_{Ip}(x) \int_{y=0}^W \lambda_p(x, y) dy$ corresponds to the average time loss per patron during the alighting process at the terminal. Similarly, the three components on the right-hand side of Eq. (2.4) are: the transfer delay from trunk-line to feeder, the waiting time for the feeder service, and the boarding time loss, where t_{t-f} denotes the transfer delay per patron and τ_b the boarding time per patron.

The boarding and alighting time losses in Eqs. (2.3) and (2.4) are significant, as all patrons will board or alight at the terminal, resulting in the longest bus dwell time at this location. Regrettably, these factors were overlooked in previous feeder design models (e.g., Chang and Schonfeld, 1991; Kim and Schonfeld, 2012; Su and Fan, 2019; Badia and Jenelius, 2020; Yang et al., 2020b).

Additionally, note that the formulas for C_{Wp} and C_{Wd} are asymmetrical. Thus, modeling a single direction of travel or two symmetrical directions (e.g., Chang and Schonfeld, 1991; Kim and Schonfeld, 2012; Sivakumaran et al., 2012) is insufficient for optimal feeder system design.

2.2.1.3 In-vehicle travel cost

The total in-vehicle travel time per operation hour, denoted by C_T , is composed of three parts: (i) the time to overcome the distance, C_{T1} ; (ii) the time lost to bus deceleration, acceleration, and door operations at stops, C_{T2} ; and (iii) the time lost to patrons' boarding and alighting behavior, C_{T3} . Formulations of these cost items are given in Eqs. (2.5)–(2.8).

$$C_T = C_{T1} + C_{T2} + C_{T3} \quad (2.5)$$

$$C_{T1} = \int_{x=0}^L \int_{y=0}^W \frac{x+y}{v_I} [\lambda_p(x, y) + \lambda_d(x, y)] dy dx \quad (2.6)$$

$$C_{T2} = \tau_0 \int_{x=0}^L \int_{y=0}^W \frac{\int_{z=y}^W (\lambda_p(x, z) + \lambda_d(x, z)) dz}{B(x, y)} dy dx \quad (2.7)$$

$$C_{T3} = \int_{x=0}^L \int_{y=0}^W \left[\tau_b S_I(x) H_{Ip}(x) \lambda_p(x, y) \int_{z=y}^W \lambda_p(x, z) dz \right. \\ \left. + \tau_a S_I(x) H_{Id}(x) \lambda_d(x, y) \int_{z=y}^W \lambda_d(x, z) dz \right] dy dx \quad (2.8)$$

where v_I denotes the cruise speed of feeder buses, and τ_0 is the fixed time loss per bus at a stop due to deceleration, acceleration, and door operations. In Eq. (2.7), the terms $\int_{z=y}^W \lambda_p(x, z) dz$ and $\int_{z=y}^W \lambda_d(x, z) dz$ represent the onboard patron flows passing y within a swath of width dx for the collection and distribution directions, respectively.

The first term on the right-hand side of Eq. (2.8) is derived as follows. The term $S_I(x) H_{Ip}(x) \int_{z=y}^W \lambda_p(x, z) dz$ represents the number of patrons onboard a bus on the line located at x when the bus passes y in the collection direction. Meanwhile, $\tau_b \lambda_p(x, y) dy dx$ corresponds to the boarding time loss generated by patrons originating from an area of size $dx \times dy$ at (x, y) . This time loss is then multiplied by $S_I(x) H_{Ip}(x) \int_{z=y}^W \lambda_p(x, z) dz$ since it affects all onboard patrons of a bus passing (x, y) . The second term on the RHS of Eq. (2.8) is developed in a similar manner. Note again that Eq. (2.7) would not involve a double integral of decision function $B(x, y)$ if stop spacings were not jointly optimized. Moreover, Eq. (2.8) would not exist if the dependency of bus dwell times on patron boarding and alighting behavior were disregarded.

2.2.2 Agency cost

During each hour of operation, the feeder service operator incurs three types of costs: (i) the stop infrastructure cost, denoted by C_s ; (ii) the operating cost associated with bus-km traveled (e.g., fuel cost), C_{vk} ; and (iii) the cost related to the bus fleet, including amortized bus purchase and maintenance costs, as well as driver wages, C_{vh} . It is worth mentioning that, since asymmetric headways in the two directions are allowed, some buses would experience layovers. For simplicity, however, the costs associated with layovers and

deadheading trips are disregarded (e.g., [Kim and Schonfeld, 2014](#); [Mei et al., 2021](#)). Note that these costs are converted to hours using the average value of time for patrons, θ . The formulations for these costs are as follows:

$$C_s = \frac{\pi_s}{\theta} \int_{x=0}^L \int_{y=0}^W \frac{1}{S_I(x)B(x,y)} dy dx \quad (2.9)$$

$$C_{vk} = \frac{\pi_v}{\theta} \int_{x=0}^L \frac{W+x}{S_I(x)} \left(\frac{1}{H_{Ip}(x)} + \frac{1}{H_{Id}(x)} \right) dx \quad (2.10)$$

$$C_{vh} = \frac{\pi_m}{\theta} V_h \quad (2.11)$$

where π_s , π_v , and π_m denote the unit costs per feeder stop (\$/stop), per bus-km traveled (\$/bus-km), and per bus-hour (\$/bus-hour), respectively. Note that $\frac{1}{S_I(x)B(x,y)} dy dx$ is the number of stops in the area of size $dx \times dy$ at (x, y) , and $\frac{W+x}{S_I(x)} dx$ is the bus line length contributed by a “vertical” swath of width dx at x . The value V_h in Eq. (2.11) denotes the required bus fleet size, which includes: (i) the number of buses actively traveling between stops, V_{h1} ; and (ii) the number of buses dwelling at stops. The latter can be divided into the number of buses needed due to deceleration, acceleration, and door operations, V_{h2} , as well as those required for passenger boarding and alighting, V_{h3} . These three components are formulated in Eqs. (2.12)–(2.13), respectively:

$$V_{h1} = \int_{x=0}^L \frac{W+x}{S_I(x)v_I} \left(\frac{1}{H_{Ip}(x)} + \frac{1}{H_{Id}(x)} \right) dx \quad (2.12)$$

$$V_{h2} = \int_{x=0}^L \int_{y=0}^W \frac{\tau_0}{S_I(x)B(x,y)} \left(\frac{1}{H_{Ip}(x)} + \frac{1}{H_{Id}(x)} \right) dy dx \quad (2.13)$$

$$V_{h3} = (\tau_a + \tau_b) \int_{x=0}^L \int_{y=0}^W [\lambda_p(x,y) + \lambda_d(x,y)] dy dx \quad (2.14)$$

Derivation of (2.12)–(2.14) is similar to those presented earlier in this section.

2.2.3 Optimization model

The generalized cost can be expressed as the sum of all the above cost components:

$$GC = C_s + C_{vk} + C_{vh} + C_A + C_W + C_T \quad (2.15)$$

The optimization problem is formulated as follows, with decision variables including the stop spacing, $B(x, y)$, the line spacing, $S_I(x)$, and the headways in the patron-collection and distribution directions, $H_{Ip}(x)$ and $H_{Id}(x)$, respectively:

$$\min \quad GC \quad (2.16)$$

$$\text{subject to: } \int_{y=0}^W \lambda_p(x, y) dy S_I(x) H_{Ip}(x) \leq K \quad (2.17)$$

$$\int_{y=0}^W \lambda_d(x, y) dy S_I(x) H_{Id}(x) \leq K \quad (2.18)$$

$$H_{\min} \leq H_{Ip}(x) \leq H_{\max} \quad (2.19)$$

$$\max\{H_{\min}, H_t\} \leq H_{Id}(x) \leq H_{\max} \quad (2.20)$$

where K denotes a feeder bus's patron-carrying capacity; and $H_{\min}$, $H_{\max}$ the minimum and maximum allowable headways, respectively. Constraints (2.17) and (2.18) ensure that the feeder bus capacity is sufficient for carrying the patrons. Constraints (2.19) and (2.20) are boundary constraints. Note that the feeder headway in the distribution direction, $H_{Id}(x)$, must be greater than or equal to the trunk-line headway, H_t . Otherwise, some feeder buses would not have any passengers to collect at the terminal.

2.2.4 Modeling schedule coordination

Feeder and trunk-line schedules can be coordinated to minimize waiting times during transfers (Sivakumaran et al., 2012). Coordination in both patron-collection and distribution directions can effectively eliminate the waiting time components, $H_t/2$ in Eq. (2.3) and $H_{Id}(x)/2$ in Eq. (2.4). However, the coordination schemes in the two directions are different. In the patron-collection direction, coordination necessitates that a trunk-line vehicle is dwelling at the terminal when patrons alighting from a feeder bus arrive at the trunk platform. This is only feasible if trunk-line vehicles bound for all destinations arrive at the terminal simultaneously. This may occur when a single trunk line serves the terminal and most transferring patrons travel in the same direction (e.g., during rush hour commuting). Meanwhile, coordination in the distribution direction merely requires a feeder bus to be ready for departure when patrons disembarking a trunk-line vehicle arrive at the feeder stop. In this case, coordination can be accomplished even when the terminal serves multiple trunk lines operating on distinct schedules.

For simplicity, we assume that the terminal serves only one trunk line, with patrons traveling to and from a city center using this line. This assumption facilitates coordination between feeder and trunk-line services. Specifically, to achieve coordination in the collection direction, a feeder line's headway must be an integer multiple of H_t , i.e.,

$$H_{Ip}(x) = kH_t \quad (k = 1, 2, 3, \dots) \quad (2.21)$$

while in the distribution direction, all feeder-line headways must equal H_t ⁴, i.e.,

$$H_{Id}(x) = H_t, \quad \forall x \in [0, L] \quad (2.22)$$

Under schedule coordination, the total waiting and transfer times in both travel directions, originally represented by Eq. (2.3) and Eq. (2.4), are replaced by the following equations:

$$C'_{Wp} = \int_{x=0}^L \int_{y=0}^W \frac{H_{Ip}(x)}{2} \lambda_p(x, y) dy dx + \int_{x=0}^L \int_{y=0}^W \left(\tau_a S_I(x) H_{Ip}(x) \int_{y=0}^W \lambda_p(x, y) dy + t_{f-t} \right) \lambda_p(x, y) dy dx \quad (2.23)$$

$$C'_{Wd} = \int_{x=0}^L \int_{y=0}^W \left(t_{t-f} + \tau_b S_I(x) H_t \int_{y=0}^W \lambda_d(x, y) dy \right) \lambda_d(x, y) dy dx \quad (2.24)$$

In Eq. (2.23), the second term consists of the total patron alighting time loss and the total transfer delay. Note that the waiting time for the trunk-line vehicle, $H_t/2$, is eliminated due to the coordination, and the alighting time loss term in Eq. (2.23) is doubled. The derivation is illustrated in Figure 2.3 for a case of $k = 1$. Eq. (2.24) is developed in a similar manner.

The remaining part of the optimization model involving schedule coordination is identical to (2.15) – (2.20).

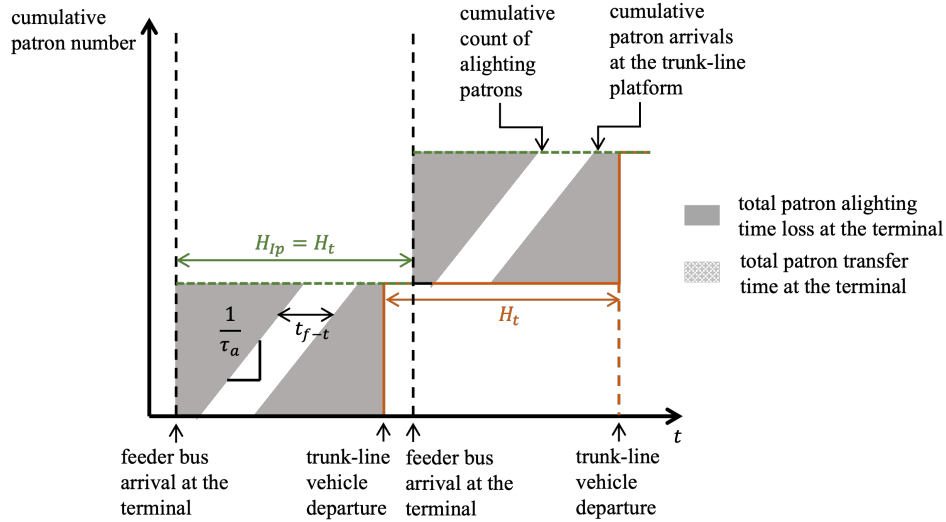


Figure 2.3: Patron delays at the terminal under schedule coordination with $k = 1$ (the collection direction)

⁴Eq.s (2.21) and Eq.s (2.22) suggest that the headway for collection, $H_{Ip}(x)$, is always greater than or equal to the headway for distribution, $H_{Id}(x)$. This setup might not align well with scenarios where distribution-direction demand is lower, such as during a morning peak. In such cases, it may be more efficient to operate the distribution service without coordination, allowing for larger headways to better match the reduced demand.

2.3 Solution Method

For brevity, this section only focuses on the solution approach for Eqs. (2.15)–(2.20). The model involving schedule coordination (i.e., Eqs. (2.23) and (2.24)) can be solved in a highly similar fashion. Section 2.3.1 derives the properties characterizing an optimal solution. An iterative heuristic algorithm is developed to solve the optimization problem, with details provided in Appendix C.

2.3.1 Optimality properties

To simplify the presentation, we begin by defining the following aggregate demand quantities and functions:

$$\Lambda = \Lambda_p + \Lambda_d \quad (2.25a)$$

$$\Lambda_p = \int_{x=0}^L \int_{y=0}^W \lambda_p(x, y) dy dx \quad (2.25b)$$

$$\Lambda_d = \int_{x=0}^L \int_{y=0}^W \lambda_d(x, y) dy dx \quad (2.25c)$$

$$\Lambda_{px}(x) = \int_{y=0}^W \lambda_p(x, y) dy, \quad 0 \leq x \leq L \quad (2.25d)$$

$$\Lambda_{dx}(x) = \int_{y=0}^W \lambda_d(x, y) dy, \quad 0 \leq x \leq L \quad (2.25e)$$

$$\Lambda_{pxy}(x, y) = \int_{z=y}^W \lambda_p(x, z) dz, \quad 0 \leq x \leq L, \quad 0 \leq y \leq W \quad (2.25f)$$

$$\Lambda_{dxy}(x, y) = \int_{z=y}^W \lambda_d(x, z) dz, \quad 0 \leq x \leq L, \quad 0 \leq y \leq W \quad (2.25g)$$

$$\mathcal{M}_{px}(x) = \int_{y=0}^W \lambda_p(x, y) \Lambda_{pxy}(x, y) dy, \quad 0 \leq x \leq L \quad (2.25h)$$

$$\mathcal{M}_{dx}(x) = \int_{y=0}^W \lambda_d(x, y) \Lambda_{dxy}(x, y) dy, \quad 0 \leq x \leq L \quad (2.25i)$$

where Λ , Λ_p , and Λ_d represent the total hourly demand for both directions combined, the collection direction, and the distribution direction, respectively; $\Lambda_{px}(x)$ and $\Lambda_{dx}(x)$ the demand densities integrated over the y-direction at location x for collection and distribution travels, respectively; and $\Lambda_{pxy}(x, y)$ and $\Lambda_{dxy}(x, y)$ the demand densities at location x , integrated from y to the top edge of Ω , for collection and distribution travels, respectively (note that $\Lambda_{pxy}(x, 0) = \Lambda_{px}(x)$ and $\Lambda_{dxy}(x, 0) = \Lambda_{dx}(x)$). $\Lambda_{pxy}(x, y)$ and $\Lambda_{dxy}(x, y)$ are utilized to calculate the onboard patron numbers at y for a bus on a line located at x . Lastly, $\mathcal{M}_{px}(x)$ and $\mathcal{M}_{dx}(x)$ are employed to calculate the total dwell time loss for all patrons combined.

The following functions are defined for further simplification of the optimality properties:

$$\alpha(x) = \frac{1}{\theta} \left(\pi_v(W+x) + \frac{\pi_m(W+x)}{v_I} + \pi_m \tau_0 \int_{y=0}^W \frac{1}{B(x,y)} dy \right), \quad 0 \leq x \leq L \quad (2.26a)$$

$$\beta_p(x) = \tau_a[\Lambda_{px}(x)]^2 + 2\tau_b\mathcal{M}_{px}(x), \quad 0 \leq x \leq L \quad (2.26b)$$

$$\beta_d(x) = 2\tau_a\mathcal{M}_{dx}(x), \quad 0 \leq x \leq L \quad (2.26c)$$

$$\gamma(x) = \frac{1}{\theta} \pi_s \int_{y=0}^W \frac{1}{B(x,y)} dy, \quad 0 \leq x \leq L \quad (2.26d)$$

Here $\alpha(x)$ denotes the agency cost rate per bus on a line located at x , excluding the agency cost associated with boarding and alighting patrons; $\beta_p(x)$ and $\beta_d(x)$ are related to the boarding and alighting time loss in the collection and distribution travel directions, respectively; and $\gamma(x)$ represents the stop infrastructure cost rate for a line located at x .

Note that $\Lambda_{pxy}(x,y)$, $\Lambda_{dxy}(x,y)$, $\mathcal{M}_{px}(x)$, $\mathcal{M}_{dx}(x)$, $\beta_p(x)$, and $\beta_d(x)$ are needed because the impacts of boarding and alighting patrons on bus dwell times and transfer delays are explicitly modeled.

The optimal solution to (2.15)–(2.20) must satisfy the following analytical properties for any $x \in [0, L]$ and $y \in [0, W]$. The derivation of these properties can be found in Appendix B.

$$H_{Ip}^*(x) = \text{mid} \left\{ H_{\min}, \min \left\{ \frac{K}{\Lambda_{px}(x)S_I^*(x)}, H_{\max} \right\}, \sqrt{\frac{2\alpha(x)/(S_I^*(x))^{-1}}{\Lambda_{px}(x) + \beta_p(x)S_I^*(x)}} \right\} \quad (2.27a)$$

$$H_{Id}^*(x) = \text{mid} \left\{ \max \{H_{\min}, H_t\}, \min \left\{ \frac{K}{\Lambda_{dx}(x)S_I^*(x)}, H_{\max} \right\}, \sqrt{\frac{2\alpha(x)/(S_I^*(x))^{-1}}{\Lambda_{dx}(x) + \beta_d(x)S_I^*(x)}} \right\} \quad (2.27b)$$

$$S_I^*(x) = \min \left\{ \sqrt{\frac{2\alpha(x) (H_{Ip}^*(x)^{-1} + H_{Id}^*(x)^{-1}) + 2\gamma(x)}{\frac{\Lambda_{px}(x) + \Lambda_{dx}(x)}{2v_W} + \tau_b[\Lambda_{dx}(x)]^2 H_t + \beta_p(x)H_{Ip}^*(x) + \beta_d(x)H_{Id}^*(x)}}, \frac{K}{\Lambda_{px}(x)H_{Ip}^*(x)}, \frac{K}{\Lambda_{dx}(x)H_{Id}^*(x)} \right\} \quad (2.27c)$$

$$B^*(x,y) = \sqrt{\frac{4v_W \left[\frac{1}{\mu S_I^*(x)} \left(\pi_s + \frac{\pi_m \tau_0}{H_{Ip}^*(x)} + \frac{\pi_m \tau_0}{H_{Id}^*(x)} \right) + \tau_0(\Lambda_{pxy}(x,y) + \Lambda_{dxy}(x,y)) \right]}{\lambda_p(x,y) + \lambda_d(x,y)}} \quad (2.27d)$$

Eqs. (2.27a)–(2.27c) show that the optimal headways and line spacings are negatively correlated with demand and positively correlated with the agency cost rate, $\alpha(x)$. In other words, lower demand (or higher agency cost rates) leads to larger bus service headways and line spacings. Note if the relationship between boarding and alighting patrons and bus dwell times is ignored, τ_0 and $\alpha(x)$ would be larger, resulting in an overestimation of

optimal headways and line spacings.

Eq. (2.27d) reveals that the optimal stop spacing is positively correlated with the agency cost rates and the number of onboard patrons (represented by $\Lambda_{pxy}(x, y) + \Lambda_{dxy}(x, y)$). But it is negatively correlated with the local demand density $\lambda_p(x, y) + \lambda_d(x, y)$ since shorter stop spacings can reduce access and egress costs. Notably, the optimal heterogeneous stop spacing is directly related to the local line spacing and headway, i.e., longer line spacing and headways result in shorter stop spacings. This highlights the importance of jointly and properly optimizing heterogeneous stop spacing. Eq. (2.27d) also shows that stop spacing would be overestimated if the impacts of boarding and alighting patrons on bus dwell times are ignored, thanks to the larger τ_0 that would occur.

Based on the above properties, we propose an iterative heuristic algorithm to solve the optimization problem. The detail of this algorithm is relegated to Appendix C.

2.4 Numerical Case Studies

2.4.1 Experimental setup

Our models are compatible with any continuous demand functions. In scenarios where real-world demand data come in zonal or raster formats, we employ interpolation techniques to create a continuous demand function (e.g., Fan and Ran, 2021). One effective method is using the “Gridfit” tool from Matlab (<https://ww2.mathworks.cn/matlabcentral/fileexchange/8998-surface-fitting-using-gridfit>), which constructs a smooth surface from scattered data points. For illustrative purposes, our numerical case studies utilize the following demand functions:

$$\lambda_p(x, y) = \text{TrN}(x \mid \mu_{xp}, \sigma_{xp}, 0, L) \cdot \text{TrN}(y \mid \mu_{yp}, \sigma_{yp}, 0, W) \cdot \Lambda_p \quad (2.28)$$

$$\lambda_d(x, y) = \text{TrN}(x \mid \mu_{xd}, \sigma_{xd}, 0, L) \cdot \text{TrN}(y \mid \mu_{yd}, \sigma_{yd}, 0, W) \cdot \Lambda_d \quad (2.29)$$

where $\text{TrN}(x \mid \mu, \sigma, a, b)$ indicates the probability density function of a truncated normal distribution with mean μ , standard deviation σ , and support $[a, b]$. Note that σ_{xp} , σ_{yp} , σ_{xd} , and σ_{yd} can be viewed as proxies for demand heterogeneity. A larger value of σ_{xp} , σ_{yp} , σ_{xd} , or σ_{yd} signifies that the spatial distribution of trip origins or destinations is “flatter.” Particularly, if $\sigma_{xp} = \sigma_{yp} = \sigma_{xd} = \sigma_{yd} = \infty$, the demand is uniformly distributed over the service region.

Additionally, note that patrons in close proximity to the terminal may prefer to walk rather than take feeder buses. To account for this preference, we assume that patrons with an origin or destination within 300 meters from the terminal, using the Manhattan distance measure, will not take the feeder buses.

Numerical instances in this section use the parameter values displayed in Table 2.1

(Jaiswal, 2010; Gu et al., 2016; Chen et al., 2018; Mei et al., 2021; Sangveraphunsiri et al., 2022) unless otherwise specified. The service region size ($2 \times 3 \text{ km}^2$) is suitable for representing a quarter of a small town or satellite city connected to a trunk transit system (e.g., the city of Langfang connected to the intercity high-speed rail network, and cities in the San Francisco Bay Area connected to the BART system). The relationships between π_v , π_m and K , θ are obtained using empirical data; see Appendix D for details.

Table 2.1: Parameter values

Notation	Description	Value	Unit
L	Length of the service region	3	km
W	Width of the service region	2	km
θ	Value of time	20	\$/h
π_s	Amortized unit cost per feeder stop infrastructure per operation hour	0	\$/stop/h
π_v	Unit cost per bus-km traveled	$0.0314 + 0.0039K$	\$/vehicle·km
π_m	Unit cost per bus hour	$2.068 + 0.108K + 2\theta$	\$/vehicle·h
τ_0	Bus dwell time per stop	12/3600	h/stop
τ_a	Alighting time per patron	2/3600	h/patron
τ_b	Boarding time per patron	4/3600	h/patron
v_W	Walking speed	2	km/h
v_I	Bus cruise speed	25	km/h
t_{f-t}	Transfer delay from feeder transit to trunk per patron	3/60	h/patron
t_{t-f}	Transfer delay from trunk to feeder transit per patron	3/60	h/patron
$H_{\min}$	Minimum feeder headway	3/60	h
$H_{\max}$	Maximum feeder headway	30/60	h
H_t	Trunk line headway	5/60	h
ε	Error tolerance	0.0001	—

Regarding the hyperparameters of the solution algorithm, we find that $m = 30$ and $n = 20$ yield sufficiently accurate numerical solutions. Further increasing m or n renders a relative percentage change in the generalized cost of less than 0.5%. Thus, the values of m and n are set to 30 and 20, respectively. The error tolerance ε is set to 0.0001. Each numerical instance is solved using various combinations of initial values, consistently converging to the same optimal solution. A typical instance’s solution is found within 30 seconds.

2.4.2 Validating the accuracy of the CA cost functions

To examine the accuracy of the CA cost functions, we first convert the optimal CA solution, which comprises continuous functions for headways, line and stop spacings, into a specific feeder system design. This is done using the method described in Appendix E. Following this, we recalculate the generalized cost and each cost component for compar-

ison. We apply a demand pattern with $\Lambda_p = \Lambda_d = 1200$ patrons/h, $\mu_{xp} = \mu_{xd} = 0$, $\mu_{yp} = \mu_{yd} = 0$, $\sigma_{xp} = \sigma_{xd} = L/4$, and $\sigma_{yp} = \sigma_{yd} = W/4$. These parameter values suggest that the demand is concentrated near the terminal, a common scenario where the trunk transit station is located in an area with high neighboring demand. The spatial distribution of the demand is illustrated in Fig. 2.4. We set the bus capacity at $K = 10$ (e.g., a 10-passenger Ford Transit Van). The converted feeder network, comprising the layout of feeder lines and the locations of stops (denoted by square markers), is illustrated in Fig. 2.5. The figure reveals that the bus lines and stops are more densely clustered near the terminal, reflecting the demand distribution shown in Fig. 2.4. It is worth noting that the line and stop locations can be fine-tuned to align with the local street network. For a case study on adapting a conceptual design to a local city street network, please refer to Estrada et al. (2011).

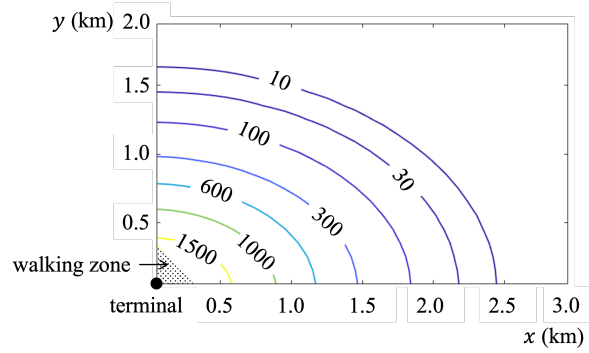


Figure 2.4: The demand densities (patron/km²/h) in the collection direction for a terminal-centered demand

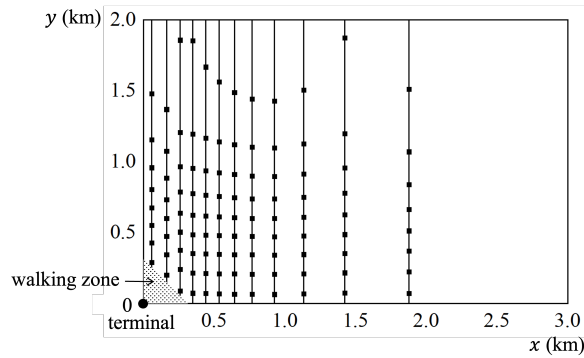


Figure 2.5: The specific feeder system layout

Percentage errors are computed between the recalculated cost terms and those of the CA cost functions. The results are displayed in Table 2.2. Note that the error in generalized cost is only 0.73%, and the error in each cost component remains consistently below 2%. This demonstrates the accuracy of our CA cost functions.

Table 2.2: Percentage errors in cost terms

Cost Term	CA (h)	Specific Plan (h)	Percentage Error
AC	98.58	97.85	0.75%
C_A	85.96	85.38	0.68%
C_W	254.80	254.05	0.30%
C_T	125.33	123.29	1.65%
UC	466.10	462.73	0.73%
GC	564.68	560.58	0.73%

2.4.3 Optimal feeder system designs without schedule coordination

We use the same terminal-centered demand pattern as described in Section 2.4.2 and analyze two scenarios with distinct time values: $\theta = 5$ \$/h for a low-wage city and $\theta = 20$ \$/h for a high-wage city.

First, we examine how the optimal generalized cost varies with bus capacity K in the two scenarios, as depicted in Fig. 2.6. These figures reveal that the generalized cost is sensitive to K when K is small, but becomes less sensitive for larger K values. Overall, the optimal bus capacity is not significantly affected by the value of time. The high-wage city exhibits lower sensitivity since the agency cost is associated with a smaller weight. For the same reason, the high-wage city favors a slightly smaller vehicle.

Throughout the remainder of Section 2.4, we will focus on the optimal feeder designs using the optimal K obtained through exhaustive search.

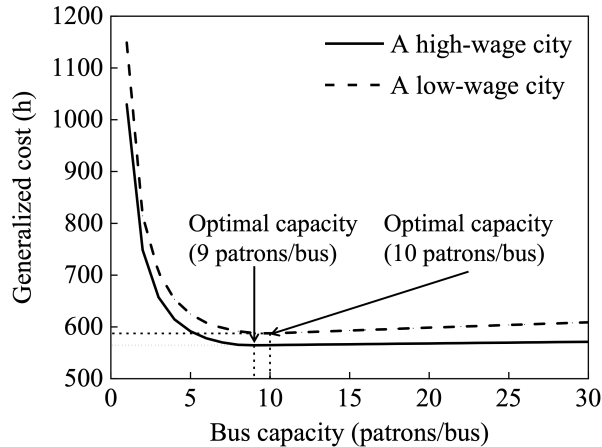


Figure 2.6: Generalized cost versus bus capacity

Figs. 2.7a and 2.7b plot the optimal line spacings and collection-direction headways against the location of lines (x) under the two scenarios, respectively. In each scenario, the distributions of spatially-heterogeneous line spacings and headways match the demand distribution (see Fig. 2.4); that is, greater transit service frequencies and line densities

are observed in areas with higher demand densities near the terminal. The high-wage city features higher line densities and service frequencies, as patron costs carry a larger weight. Nonetheless, the discrepancy between low- and high-wage cities remains relatively small. This can be attributed to the fact that 90% of the agency cost, specifically the cost component associated with fleet size, C_{vh} , is proportional to the value of time θ ; see the formula for π_m in Table 2.1 and its derivation in Appendix D. When C_{vh} is converted to hours by dividing by θ (see Eq. (2.11)), the impact of θ is mostly neutralized. In essence, since the relationship between the value of time and agency cost rates is factored into the model, the generalized cost, as well as the optimal system design, are only marginally influenced by the city's wealth level.

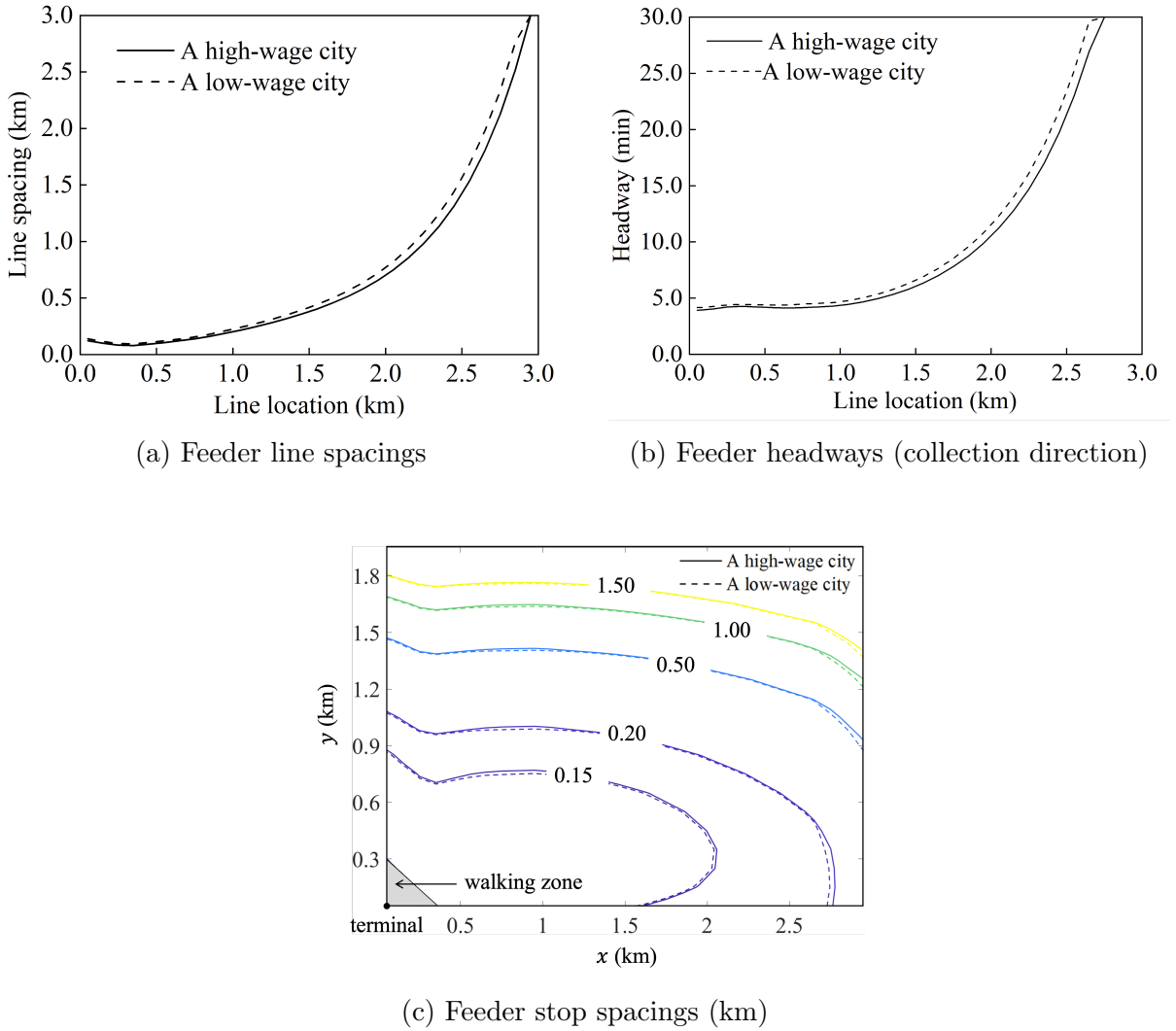


Figure 2.7: Optimal design variables under the terminal-centered demand

Fig. 2.7c displays the optimal stop spacings under the two scenarios. As expected, the stop spacing also decreases as the demand density increases and appears to be insensitive to the value of time. In both scenarios, *the optimal stop spacing varies in a rather wide range*, highlighting the need for heterogeneous stop spacing planning un-

der the spatially heterogeneous demand. Next, we compare our heterogeneous design against three benchmark designs, including: (i) an optimal design with heterogeneous line spacings and headways but uniform stop spacing; (ii) an optimal design featuring spatially-heterogeneous headways with uniform line and stop spacings; and (iii) an optimal design with a uniform line spacing, stop spacing, and headway. All four optimal designs use the same demand pattern and a value of time set at 20 \$/h. Results are presented in Table 2.3, where the range of optimal values for each heterogeneous decision variable is shown in brackets, followed by the average value in parentheses. The last row

Table 2.3: Optimal heterogeneous, partially-homogeneous, and homogeneous designs under terminal-centered demand

Variable	Heterogeneous design	Uniform stop spacing	Uniform line and stop spacings	Uniform line, stop spacings, and headway
K^*	9	9	12	22
$B^*(x, y)$ (km)	[0.12, 2.00] (0.23)	0.16	0.15	0.16
$S_I^*(x)$ (km)	[0.08, 3.00] (0.24)	[0.09, 3.00] (0.25)	0.20	0.22
$H_{Ip}^*(x)$ (min)	[3.9, 30.0] (10.7)	[4.0, 30.0] (11.0)	[3.0, 30.0] (14.2)	5.2
$H_{Id}^*(x)$ (min)	[5.0, 30.0] (11.1)	[5.0, 30.0] (11.3)	[5.0, 30.0] (14.8)	5.2
AC (h)	98.51	102.02	94.23	139.89
UC (h)	465.89	470.19	487.80	507.19
GC (h)	564.39	572.20	582.04	647.09
GC saving	1.36% (compared to design with uniform stop spacing)	1.68% (compared to design with uniform line and stop spacings)	10.05% (compared to the homogeneous design)	—

of Table 2.3 demonstrates that, compared to the homogeneous design, heterogenizing the headway results in a cost saving of 10.05%. Additionally, further heterogenizing the line spacing yields an extra saving of 1.68%, while heterogenizing the stop spacing contributes an added saving of 1.36%. Note that each number is computed by comparing the GC against a “more homogeneous” design, which is the one in the column immediately to the right of the current design in Table 2.3. The last number is a conservative estimate, as it is derived from a design with optimized (though uniform) stop spacing. Should stop spacing be disregarded or assumed constant in the model, as in previous studies such as Chang and Schonfeld (1991) and Yang et al. (2020b), even greater savings would be realized. For example, compared to using a fixed stop spacing of 500 m, the cost saving stemmed from our heterogeneous design amounts to 11.37%. Furthermore, larger savings are also observed under alternative demand patterns, as will be discussed shortly.

Lastly, we compare the generalized cost of the heterogeneous design to the cost of the optimal design without accounting for the impacts of boarding and alighting passengers on dwell times and transfer times (details are omitted in the interest of brevity). In the

latter design, the bus dwell time is set at 30 seconds, as in Gu et al. (2016). Numerical results across various demand levels reveal that the relative percentage error lies between 2–5%. This underscores the importance of incorporating passenger boarding and alighting processes in the model.

2.4.4 Sensitivity analysis

In this section, we examine the sensitivity of the optimal heterogeneous design and cost savings in comparison to the benchmark designs. Sensitivity analyses concerning the demand patterns, demand rates, service region sizes, and feeder line layouts are furnished in subsections (i)–(iv), respectively.

2.4.4.1 Sensitivity to the demand pattern

We investigate the benefit of heterogenous design under two distinct demand patterns for a high-wage city ($\theta = 20$ \$/h). The first pattern remains terminal-centered but exhibits a more concentrated spatial distribution with $\sigma_{xp} = \sigma_{xd} = L/8$ and $\sigma_{yp} = \sigma_{yd} = W/8$. The other parameter values are consistent with those in Section 2.4.3. The outcomes of the optimal heterogeneous design and the three benchmark designs are presented in Table 2.4. The last row of this table is derived in a similar fashion to Table 2.3.

Table 2.4: Optimal designs when demand is more concentrated near the terminal

Variable	Heterogeneous design	Uniform stop spacing	Uniform line and stop spacing	Uniform line, stop spacing, and headway
K (pax)	8	13	13	33
$B^*(x, y)$ (km)	[0.09, 2.00] (0.34)	0.13	0.13	0.14
$S_I(x)$ (km)	[0.03, 3.00] (0.23)	[0.04, 3.00] (0.26)	0.13	0.20
$H_{Ip}(x)$ (min)	[3.9, 30.0] (19.4)	[4.1, 30.0] (19.7)	[3.0, 30.0] (21.6)	5.9
$H_{Id}(x)$ (min)	[5.0, 30.0] (19.6)	[5.0, 30.0] (19.8)	[5.0, 30.0] (21.9)	5.9
AC (h)	83.42	88.21	96.09	142.72
UC (h)	313.50	322.88	330.84	384.98
GC (h)	396.92	411.09	426.93	527.70
GC gap	3.45%	3.71%	19.1%	–

Table 2.4 shows significantly larger percentage cost savings under the more concentrated demand pattern compared to Table 2.3. For example, heterogenizing the stop spacings now produces a 3.45% cost saving. This suggests that *the heterogeneous design is more advantageous when demand is more spatially heterogeneous*. Also, designs with higher degrees of heterogeneity tend to favor smaller feeder buses, as the optimal spatially-heterogeneous line and stop spacings align more closely with the heterogeneous demand (see Fig. 2.7), rendering the patrons more evenly distributed across the lines and vehicles.

The second demand pattern employs the same standard deviation parameters as in Section 2.4.3, but with demand centered at the far end of the service region, i.e., $\mu_{xp} = \mu_{xd} = L$ and $\mu_{yp} = \mu_{yd} = W$. This remotely-centered pattern also has real-world examples. Other parameters remain the same as before. The results under this demand pattern are provided in Table 2.5.

Table 2.5: Optimal designs when demand is centered at the remote end

Variable	Heterogeneous design	Uniform stop spacing	Uniform line and stop spacing	Uniform line, stop spacing, and headway
K (pax)	11	11	12	22
$B^*(x, y)$ (km)	[0.08, 2.00] (0.28)	0.24	0.24	0.25
$S_I(x)$ (km)	[0.10, 2.29] (0.19)	[0.10, 2.41] (0.19)	0.18	0.20
$H_{Ip}(x)$ (min)	[4.4, 30.0] (10.1)	[4.5, 30.0] (10.2)	[3.1, 30.0] (14.1)	5.2
$H_{Id}(x)$ (min)	[5.0, 30.0] (10.3)	[5.0, 30.0] (10.5)	[5.0, 30.0] (14.6)	5.3
AC (h)	122.23	122.72	117.34	145.66
UC (h)	813.40	853.58	862.96	877.85
GC (h)	935.62	976.30	980.30	1023.50
GC gap	4.17%	0.41%	4.22%	—

Table 2.5 demonstrates that merely heterogenizing stop spacings results in an even greater cost saving of 4.17%. This is likely due to the better alignment of optimal stop spacings with this demand pattern. Specifically, $B^*(x, y)$ is small near the service region’s top edge due to high demand density and low onboard patron flow, while it is large near the bottom owing to low demand density and high onboard patron flow; see Eq. (2.25g). Recall that the advantages of our design would be even more pronounced when compared to previous feeder optimization models that utilize a constant, non-optimized stop spacing.

2.4.4.2 Sensitivity to the demand rate

In this subsection, we examine the terminal-centered demand pattern with $\Lambda_p = \Lambda_d$ taking a smaller value (400 patrons/h). The results are provided in Table 2.6. Comparing the cost savings with those in Table 2.3 reveals that the percentage cost savings of the optimal heterogeneous design increase moderately as demand diminishes. In addition, lower demand entails fewer lines served by smaller vehicles operating at lower frequencies, which is a logical outcome. However, the stop spacings remain largely unaffected by the demand rate.

2.4.4.3 Sensitivity to the service region’s size

In this analysis, we continue to use the terminal-centered demand pattern but examine a smaller service region with $L = 1.5$ km and $W = 1$ km. To ensure a fair comparison,

Table 2.6: Optimal designs under terminal-centered demand with lower rates

Variable	Heterogeneous design	Uniform stop spacing	Uniform line and stop spacing	Uniform line, stop spacing, and headway
K (pax)	7	7	8	18
$B^*(x, y)$ (km)	[0.13, 2.00] (0.25)	0.17	0.16	0.17
$S_I(x)$ (km)	[0.22, 3.00] (0.49)	[0.22, 3.00] (0.49)	0.40	0.42
$H_{Ip}(x)$ (min)	[4.4, 30.0] (13.3)	[4.6, 30.0] (13.5)	[3.2, 30.0] (15.4)	6.7
$H_{Id}(x)$ (min)	[5.0, 30.0] (13.5)	[5.0, 30.0] (13.7)	[5.0, 30.0] (15.8)	6.7
AC (h)	43.77	40.62	40.10	54.89
UC (h)	173.70	175.65	182.58	195.01
GC (h)	212.74	216.27	222.68	249.90
GC gap	1.63%	2.88%	10.89%	—

we keep the average demand density (i.e., Λ_p/LW and Λ_d/LW) consistent with the instance in Table 2.3. The results are summarized in Table 2.7. The table indicates that the percentage cost savings of the heterogeneous design increase as the service region shrinks, especially the cost saving resulting from spatially-heterogenizing headways. This time, the headways and stop spacings demonstrate relative insensitivity to changes in service region size, while the average line spacing increases significantly as the service region contracts. The optimal K also decreases, since the demand carried by each line is lower. The results in Tables 2.3–2.7 reveal that spatially-heterogenizing headways always yields the largest benefit. Nevertheless, *heterogenizing stop spacings also contributes to moderate yet consistent improvements*.

Table 2.7: Optimal designs with a 1×1.5 km² service region

Variable	Heterogeneous design	Uniform stop spacing	Uniform line and stop spacing	Uniform line, stop spacing, and headway
K (pax)	6	6	9	12
$B^*(x, y)$ (km)	[0.10, 1.00] (0.20)	0.12	0.11	0.13
$S_I(x)$ (km)	[0.17, 1.50] (0.35)	[0.17, 1.50] (0.36)	0.37	0.29
$H_{Ip}(x)$ (min)	[3.6, 30.0] (10.7)	[3.9, 30.0] (11.0)	[3.0, 30.0] (12.8)	4.4
$H_{Id}(x)$ (min)	[5.0, 30.0] (11.2)	[5.0, 30.0] (11.4)	[5.0, 30.0] (13.6)	5.8
AC (h)	17.78	18.90	15.68	28.18
UC (h)	85.23	86.35	93.91	108.09
GC (h)	103.01	105.25	109.60	136.27
GC gap	2.13%	3.97%	19.57%	—

2.4.4.4 Sensitivity to the feeder line layout

In this subsection, we compare two alternative feeder line layouts: (i) feeder buses travel along the y -direction for patron pick-up or drop-off and move along the x -direction to

and from the terminal without stopping; and (ii) feeder buses load and unload passengers along the x -direction and perform nonstop travel along the y -direction. We let the rectangular service region’s aspect ratio, W/L , vary from 0.1 to 2, and plot the generalized costs per patron for the two layouts in Fig. 2.8. The service region’s area $L \times W$ and other parameter values are kept consistent with those in Section 2.4.3 (with $\theta = 20$ \$/h). The figure reveals that the generalized cost is consistently lower when buses make stops along the shorter side of the service region. In this configuration, more bus lines are deployed along the longer side, resulting in each bus carrying fewer patrons and reduced boarding delays. The cost gap between the two layouts is substantial when the service region is “slim.” For example, with $W/L = 2$ or 0.5, the more efficient layout outperforms the alternative by 6.49% in generalized cost. This finding highlights the importance of selecting the appropriate layout, particularly when the service region is more oblong than square. Notably, this straightforward result has not been previously documented in the literature; for example, [Sivakumaran et al. \(2012\)](#) arranged the feeder lines along the longer side of the service region, which our analysis indicates is a suboptimal design.

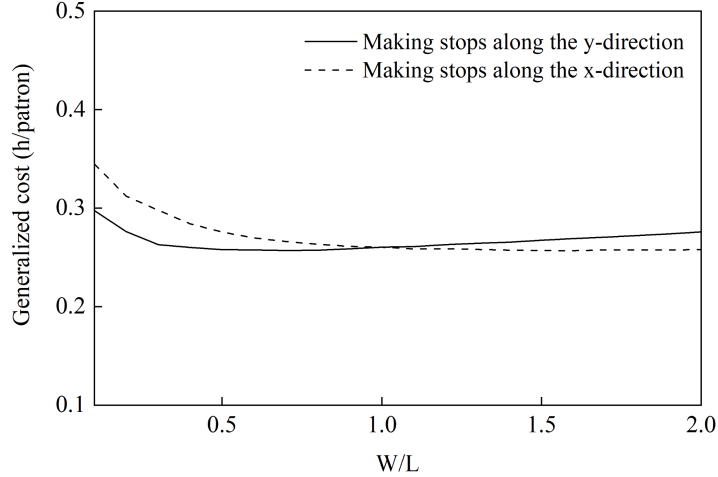


Figure 2.8: Comparison between two alternative feeder line layouts

2.4.5 Benefits of schedule coordination

We now explore the benefits of schedule coordination. The feeder network is reoptimized considering schedule coordination in the collection direction only, in the distribution direction only, and in both directions simultaneously. The percentage of generalized cost savings compared to the uncoordinated case is plotted against $H_t \in \{5, 7.5, 10\}$ minutes in Fig. 2.9. Other parameters remain the same as those in Section 2.4.3 (with $\theta = 20$ \$/h).

Fig. 2.9 demonstrates the sizeable benefits of schedule coordination. Coordination in the collection direction brings cost savings ranging from 7.42% to 10.95%. Coordination in the distribution direction yields a smaller but still noteworthy benefit (−0.04% to 7.35%),

particularly when H_t is large. Moreover, coordinating in both directions achieves savings of up to 20.10%. Intriguingly, this value slightly exceeds the combined percentage cost savings of coordinating in individual directions. This can be attributed to two possible reasons: (i) waiting time constitutes a large portion of the generalized cost; and (ii) simultaneously coordinating the schedules in both directions may facilitate more efficient optimization of line spacings, stop spacings, and the vehicle size (factors that concurrently impact the generalized costs in both directions).

Additionally, the figure indicates that coordination becomes increasingly advantageous as trunk-line headway grows. This is owing to two reasons: (i) the waiting time saved by coordination rises with trunk-line headway (see Section 2.2.4); and (ii) the agency cost savings also grow as larger feeder bus headways (in multiples of H_t) are enforced.

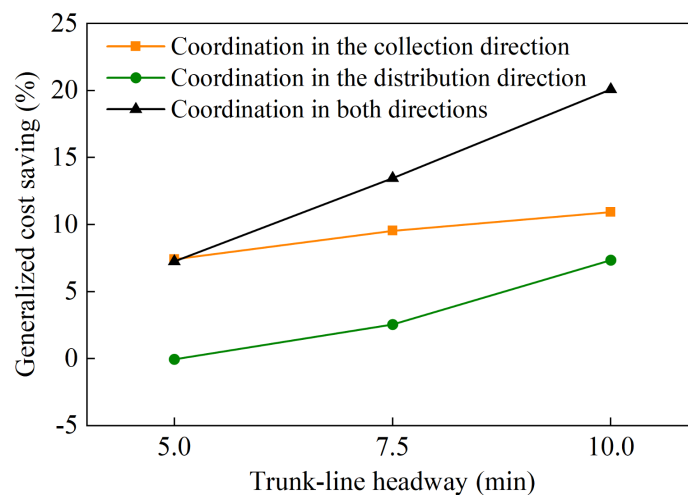


Figure 2.9: Comparison between two alternative feeder line layouts

Chapter 3

Demand Responsive Connector Service Design

3.1 Problem Description

Fig. 3.1 illustrates the idealized network for the DRC service. A DRC bus terminal, marked by a black dot, is depicted at the lower-left corner. This terminal acts as a nexus for transfers, connecting passengers to a transportation hub such as an intercity rail station. The terminal's catchment area is assumed to be a rectangle centered around it. Without loss of generality, we focus our design efforts on a quarter of this rectangular area, i.e., a service region denoted by $\Omega \equiv \{(x, y) \mid 0 \leq x \leq L, 0 \leq y \leq W\}$; see Fig. 3.1. Outbound demand requests are modeled as a spatial Poisson process (Sangveraphunsiri et al., 2022) with a uniform intensity of λ_p patrons/km²/h. In the inbound direction, the demand requests follow a similar process but at a distinct intensity of λ_d patrons/km²/h. Unlike Chapter 2, this chapter assumes the demand is uniformly distributed. The primary reason is that we incorporate stochastic passenger demand in our modeling framework, and uniform demand distribution facilitates the analytical tractability of the stochastic demand modeling.

For both routing strategies, the catchment area is partitioned into $M \times N$ identical rectangular zones, each measuring $l \times w$, where $l = L/N$ and $w = W/M$. For ease of reference, zones are identified by coordinates (m, n) , where m increments from bottom to top and n from left to right. It is assumed that transit vehicles follow a fixed-headway schedule for both collecting and distributing passengers, denoted as $H_{p(m,n)}$ and $H_{d(m,n)}$, respectively, for each zone (m, n) . Note that fixed-headway dispatching, in contrast to fixed-occupancy, tends to be more favorable among DRC patrons and is more commonly implemented in real-world operations (Dzisi et al., 2022; Wong et al., 2023). Further note that optimal headways may vary between zones as a result of the differences in line-haul distances associated with each zone. Zones requiring longer line-hauls will generally

need longer round-trip times and incur higher costs per trip, which necessitates larger headways to balance those costs.

We assume each service zone has a designated layover point where vehicles temporarily dwell between trips. A delivery vehicle travels from the terminal to a service zone and stops at the layover point after delivering passengers. Conversely, a pickup vehicle departs from the layover point, picks up passengers, and then travels to the terminal. Therefore, the same vehicle may sequentially serve both delivery and pickup trips within the same zone. Since the two directions may have unequal headways, deadheading trips may be needed. For simplicity, however, the costs associated with layovers and deadheading trips are disregarded. The same treatment was adopted in other planning-level studies on transit system design (e.g., [Chang and Schonfeld, 1991](#); [Li and Quadrioglio, 2010](#); [Kim and Schonfeld, 2012, 2013, 2015](#)).

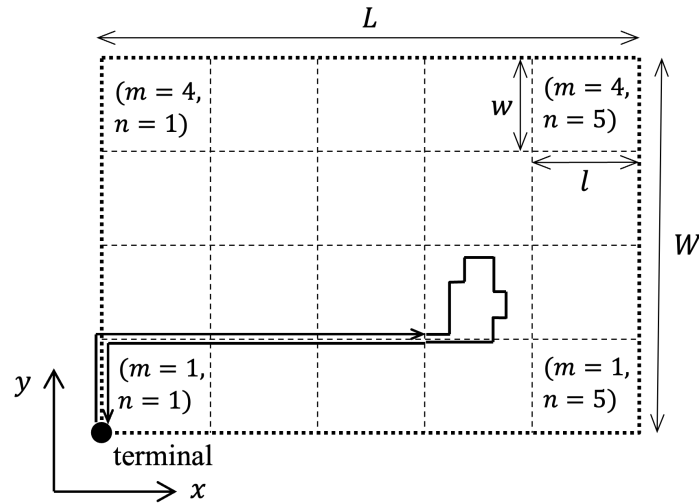


Figure 3.1: DRC service network

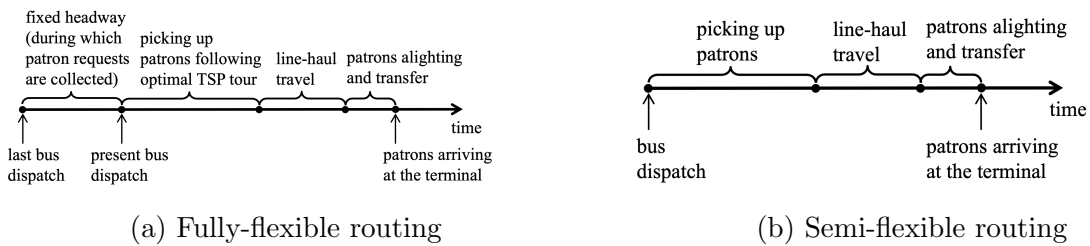


Figure 3.2: Optimal design variables under the terminal-centered demand

Fig. 3.2a demonstrates the service process for the fully-flexible routing strategy, focusing on the outbound direction as an example. Under this strategy, a patron submits a request, perhaps through a mobile phone app, before the next bus dispatch. The bus scheduled to depart will serve all requests received within the headway, following an optimal TSP tour calculated using commercial solvers or established algorithms, such as

the Concorde TSP Solver. Subsequently, the bus proceeds on its line-haul journey to the terminal.

Under the fixed-headway setting, the number of patrons served by a DRC bus follows a Poisson distribution with respective means $\lambda_p H_{p(m,n)} lw$ for the outbound direction and $\lambda_d H_{d(m,n)} lw$ for the inbound direction, respectively. The variables to be optimized in this strategy include the zoning dimensions, l and w (or, equivalently, M and N), the headways, $H_{p(m,n)}$ and $H_{d(m,n)}$, and the bus capacity, K .

Contrastingly, the semi-flexible routing strategy depicted in Fig. 3.2b does not necessitate patron pre-registration prior to the dispatch of buses. Under this scheme, patrons can request trips spontaneously, and they are subsequently picked up by the next bus that passes by, which could already be in the midst of its tour at the time the request is made. The optimization of this strategy revolves around the following decision variables: the zoning dimensions, l and w (or, equivalently, M and N), the headways, $H_{p(m,n)}$ and $H_{d(m,n)}$, the bus capacity, K , and the swath width, w_0 . We assume that buses maintain roughly regular headways at each longitudinal position along the swath. Thus, the number of patrons carried by a bus under this scheme also follows a Poisson distribution with mean $\lambda_p H_{p(m,n)} lw$ for outbound and $\lambda_d H_{d(m,n)} lw$ for inbound.

We also adopt the simplifying assumption that only one patron is picked up or dropped off per local tour stop. This assumption is commonly utilized in transit research (e.g., [Quadrifoglio and Li, 2009](#); [Sangveraphunsiri et al., 2022](#)).

In line with established practices in the field ([Ouyang et al., 2014](#); [Chen and Nie, 2017](#)), our objective is to minimize the total system cost of the DRC service under each routing strategy. This total system cost, sometimes referred to as "generalized cost" in the literature ([Daganzo, 2010](#); [Gu et al., 2016](#); [Mei et al., 2021](#)), encapsulates both the travel time experienced by patrons and the operating costs incurred by the agency. Note that patrons' travel time is measured in hours, while agency costs are in monetary terms. To add up these cost terms with disparate units, we convert the agency's costs into equivalent temporal units using the average patron's value of time, expressed in \$/hour ([Daganzo, 2010](#)). In addition, bus fares, which represent a monetary transfer from patrons to the transit agency, are omitted from the total cost calculation. While fares are a cost to patrons, they are concurrently revenue for the agency, thereby netting out in the context of system cost assessment.

Subsequent sections will elaborate on the cost models for one hour of DRC operations under the fully-flexible and semi-flexible routing strategies.

3.2 Mathematical Formulation

Section 3.2.1 formulates the cost components associated with the fully-flexible routing strategy. Section 3.2.3 develops models for the semi-flexible routing strategy. The opti-

mization models are described in Section 3.2.3. The notations used in this section and their definitions are listed in Appendix F.

3.2.1 Cost models of the fully-flexible routing strategy

3.2.1.1 Estimation of expected TSP tour length

Recall that the expected optimal TSP tour length within a designated area A asymptotically scales with $\sqrt{qA}$ as q approaches infinity (Beardwood et al., 1959), where q represents the number of stops. However, when q is small, as in our feeder service design study, the applicability of this asymptotic relationship requires further investigation. Drawing on the dimensional analysis introduced by Daganzo (2005), the expected tour length must adhere to a form that ensures dimensional consistency. Specifically, under the assumption of uniform demand and a rectangular service zone, the tour length should take the form $f(q, S)\sqrt{A}$, where $\sqrt{A}$ must be included for dimensional consistency, and the dimensionless coefficient $f(q, S)$ is a function of q and the length-to-width ratio S . To be consistent with the literature, we define a dimensionless k^* as $k^* = \frac{f(q, S)}{\sqrt{q}}$, leading to an expected tour length of the form $k^*\sqrt{qA}$. Precisely estimating the scaling factor k^* for fewer (≤ 15) stops in Manhattan geometry remains unexplored.

Given that the passenger load for a DRC bus is commonly lower than 15, we aim to establish an empirical model for k^* grounded in Monte Carlo simulation outcomes in this section. We execute a series of simulation scenarios that encompass a range of stops from 2 to 15 and various aspect ratios from 1 to 3. (In practice, the number of stops and aspect ratios are most likely within the ranges specified above. Numerical results (Figs. 6b–c) confirm that the optimal service zone aspect ratio and the number of patrons per vehicle under fully-flexible routing consistently fall within the selected ranges.) For each scenario, we randomly generate and optimally solve 500 TSP instances using an exact algorithm implemented in MATLAB R2020a. The experimental setup is detailed in Appendix G. Following these simulations, we compute the average k^* for every scenario, which are used to calibrate the model.

Through regression analysis, the following model, inspired by the Weibull probability density function, emerges as the best fit for the simulation data:

$$k^* = (\beta_1 S + \beta_2) q^{\beta_3} \exp(\beta_4 q^{\beta_5}) \quad (3.1)$$

where q denotes the number of stops and $S \geq 1$ the aspect ratio. The calibrated coefficient values are tabulated as follows:

Eq. (3.1) provides a refined estimation of the optimal TSP tour lengths pertinent to our feeder system. The mean absolute percentage errors (MAPE) across the specified ranges of q and S are showcased in Fig. 3.3. For comparison, Fig. 3.3 also includes the

Table 3.1: Values of coefficients in Eq. (3.1)

β_1	β_2	β_3	β_4	β_5
0.1102	1.4569	-0.1472	-2.5508	-2.6396

MAPE values based on the model from Yang et al. (2020b) and the k^* value proposed by Chakraborti and Chakrabarti (2000); see Section 1.2 for the details of those benchmarks. The depicted results indicate that *the error margins of the models from earlier studies are considerably high, with some scenarios yielding errors surpassing 40%. In stark contrast, our method provides a markedly closer estimate to the true values, maintaining a maximum MAPE under 5%.*

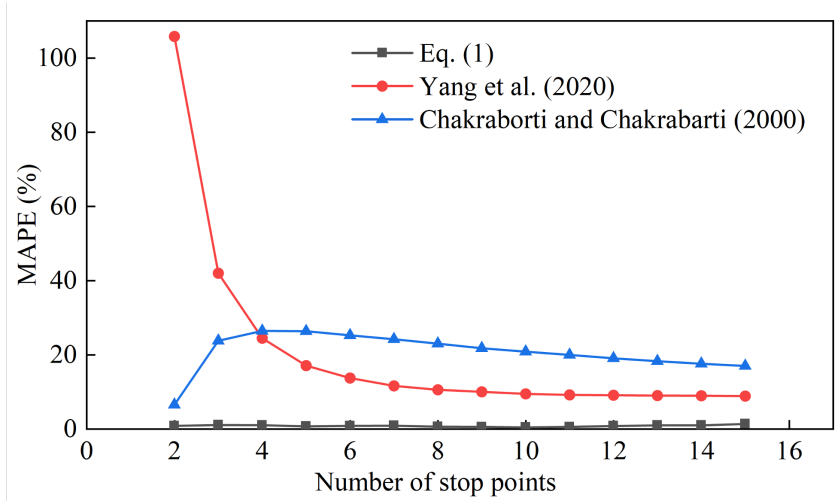


Figure 3.3: The MAPEs of k^* under fully-flexible routing

3.2.1.2 Patrons' travel time cost

The travel time cost for patrons is broken down into four elements: (i) waiting time at home; (ii) travel time during the local tour; (iii) travel time on the main line-haul; and (iv) transfer time at the terminal. Each of these components is detailed as follows:

(i) Waiting time at home

This element applies exclusively to the outbound leg where patrons are being picked up from home. The aggregate waiting time at home per operational hour of the service, represented by C_W , is calculated by summing the total waiting times for each zone, $C_{W(m,n)}$, where $1 \leq m \leq M$, $1 \leq n \leq N$. A patron's waiting time is comprised of: (a) the interval preceding the bus dispatch; and (b) the duration from bus dispatch until its arrival. The expected value for the first part is $H_p(m,n)/2$, since the service requests following a Poisson process are uniformly distributed over time. The second part is, on average, half of the local TSP tour's travel time. Consequently, the total waiting time is

expressed as follows:

$$\begin{aligned}
C_{W(m,n)} &= \alpha \cdot \left\{ \frac{1}{H_{p(m,n)}} \cdot \mathbb{E} \left[Q_{p(m,n)} \cdot \frac{H_{p(m,n)}}{2} \right] + \frac{1}{2H_{p(m,n)}} \cdot \mathbb{E} \left[Q_{p(m,n)} \cdot \left(\frac{1}{v_I} (\beta_1 S + \beta_2) \right. \right. \right. \\
&\quad \left. \left. \cdot (Q_{p(m,n)} + 1)^{\beta_3} \cdot e^{\beta_4(Q_{p(m,n)}+1)^{\beta_5}} \cdot \sqrt{(Q_{p(m,n)} + 1)lw + Q_{p(m,n)}\tau_p} \right) \right] \left. \right\} \\
&= \frac{\alpha}{2} \cdot \mathbb{E}[Q_{p(m,n)}] + \frac{\alpha(\beta_1 S + \beta_2)\sqrt{lw}}{2H_{p(m,n)}v_I} \cdot \mathbb{E} \left[(Q_{p(m,n)} + 1)^{\beta_3+3/2} - (Q_{p(m,n)} + 1)^{\beta_3+1/2} \right. \\
&\quad \left. \cdot e^{\beta_4(Q_{p(m,n)}+1)^{\beta_5}} \right] + \frac{\alpha\tau_p}{2H_{p(m,n)}} \cdot \mathbb{E}[Q_{p(m,n)}^2] \tag{3.2}
\end{aligned}$$

$$C_W = \sum_{n=1}^N \sum_{m=1}^M C_{W(m,n)} \tag{3.3}$$

where $\mathbb{E}[\cdot]$ denotes the expected value operator; $Q_p(m, n)$ the random number of patrons collected by a bus serving zone (m, n) ; v_I the bus cruising speed; $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ represent the corresponding coefficients used in Eq. (3.1); τ_p denotes the time loss per stop during the outbound tour; and $\alpha \in [0, 1]$ is the discount factor applied to the time value of waiting at home. This discount factor is introduced to reflect the reduced disutility of waiting at home as opposed to traveling in a vehicle, acknowledging the possibility for patrons to engage in other activities in a more comfortable and convenient environment (Algers et al., 1975).

The total number of stops in a TSP tour includes the designated dispatch point of the bus and is given by $Q_p(m, n) + 1$, where $Q_p(m, n)$ is Poisson-distributed with a mean of $\lambda_p H_p(m, n)lw$. Utilizing this information along with the Taylor expansion method, we derive an approximation for the expected values appearing in the right-hand-side (RHS) of Eq. (3.2), i.e., $\mathbb{E} \left[(Q_p(m, n) + 1)^{\beta_3+3/2} - (Q_p(m, n) + 1)^{\beta_3+1/2} \right] e^{\beta_4(Q_p(m, n)+1)^{\beta_5}}$ and $\mathbb{E}[Q_p(m, n)^2]$. Note that these terms are the expected values of nonlinear functions of $Q_p(m, n)$, and relying on the simple expected value $\mathbb{E}[Q_p(m, n)]$ for their calculation could result in notable errors. The details of the derivation of these expected values are relegated to Appendix H.

(ii) In-vehicle travel time of local tours

The patrons' total in-vehicle travel times in the local tour, for the outbound leg (C_{Tp}) and the inbound leg (C_{Td}) per service hour, are sums of the corresponding in-vehicle travel times for each zone, denoted $C_{Tp(m,n)}$ and $C_{Td(m,n)}$, respectively. A patron's in-vehicle travel time consists of two components: (i) the time for vehicle travel to overcome the distance between stops, and (ii) the dwell time at each stop. The formulas for these components are as follows:

$$\begin{aligned}
C_{Tp(m,n)} &= \frac{1}{2H_{p(m,n)}} \cdot \mathbb{E} \left[Q_{p(m,n)} \cdot \left(\frac{1}{v_I} (\beta_1 S + \beta_2) (Q_{p(m,n)} + 1)^{\beta_3} e^{\beta_4 (Q_{p(m,n)} + 1)^{\beta_5}} \right. \right. \\
&\quad \left. \left. \cdot \sqrt{(Q_{p(m,n)} + 1)lw} + Q_{p(m,n)} \tau_p \right) \right] \\
&= \frac{(\beta_1 S + \beta_2) \sqrt{lw}}{2H_{p(m,n)} v_I} \cdot \mathbb{E} \left[(Q_{p(m,n)} + 1)^{\beta_3 + 3/2} - (Q_{p(m,n)} + 1)^{\beta_3 + 1/2} \cdot e^{\beta_4 (Q_{p(m,n)} + 1)^{\beta_5}} \right] \\
&\quad + \frac{\tau_p}{2H_{p(m,n)}} \cdot \mathbb{E}[Q_{p(m,n)}^2] \tag{3.4}
\end{aligned}$$

$$\begin{aligned}
C_{Td(m,n)} &= \frac{1}{2H_{d(m,n)}} \cdot \mathbb{E} \left[Q_{d(m,n)} \cdot \left(\frac{1}{v_I} (\beta_1 S + \beta_2) (Q_{d(m,n)} + 1)^{\beta_3} e^{\beta_4 (Q_{d(m,n)} + 1)^{\beta_5}} \right. \right. \\
&\quad \left. \left. \cdot \sqrt{(Q_{d(m,n)} + 1)lw} + Q_{d(m,n)} \tau_d \right) \right] \\
&= \frac{(\beta_1 S + \beta_2) \sqrt{lw}}{2H_{d(m,n)} v_I} \cdot \mathbb{E} \left[(Q_{d(m,n)} + 1)^{\beta_3 + 3/2} - (Q_{d(m,n)} + 1)^{\beta_3 + 1/2} \cdot e^{\beta_4 (Q_{d(m,n)} + 1)^{\beta_5}} \right] \\
&\quad + \frac{\tau_d}{2H_{d(m,n)}} \cdot \mathbb{E}[Q_{d(m,n)}^2] \tag{3.5}
\end{aligned}$$

$$C_{Tp} = \sum_{n=1}^N \sum_{m=1}^M C_{Tp(m,n)} \tag{3.6}$$

$$C_{Td} = \sum_{n=1}^N \sum_{m=1}^M C_{Td(m,n)} \tag{3.7}$$

where τ_d denotes the time penalty incurred at each stop in the inbound leg of service. It is important to note that, on average, a patron's in-vehicle travel time is equivalent to half of the total duration of the vehicle's local tour, regardless of travel direction. Accordingly, in Eqs. (3.4) and (3.5), this time is calculated as half the sum of the expected travel time over the tour length and the total expected dwell time. The expected values on the RHS of (3.4) and (3.5) are computed using the same method as the one employed for calculating those expected values in Eq. (3.2).

(iii) Line-haul travel time

Let C_{Lp} and C_{Ld} represent the total line-haul travel times for patrons in the outbound and inbound legs, respectively, per service hour. Since delivery vehicles operate from the terminal to service zones, while pickup vehicles travel in the opposite direction, each line-haul travel time accounts only for one-way travel. Therefore, the line-haul travel times are calculated by summing up the respective line-haul travel times for each zone (m, n) ,

denoted by $C_{Lp(m,n)}$ and $C_{Ld(m,n)}$. These values are formulated as follows:

$$\begin{aligned} C_{Lp(m,n)} &= \frac{1}{H_{p(m,n)}} \cdot \mathbb{E} \left[Q_{p(m,n)} \cdot \frac{(m-1)w + (n-1)l}{v_I} \right] \\ &= \frac{(m-1)w + (n-1)l}{H_{p(m,n)}v_I} \cdot \mathbb{E} [Q_{p(m,n)}] \end{aligned} \quad (3.8)$$

$$\begin{aligned} C_{Ld(m,n)} &= \frac{1}{H_{d(m,n)}} \cdot \mathbb{E} \left[Q_{d(m,n)} \cdot \frac{(m-1)w + (n-1)l}{v_I} \right] \\ &= \frac{(m-1)w + (n-1)l}{H_{d(m,n)}v_I} \cdot \mathbb{E} [Q_{d(m,n)}] \end{aligned} \quad (3.9)$$

$$C_{Lp} = \sum_{n=1}^N \sum_{m=1}^M C_{Lp(m,n)} \quad (3.10)$$

$$C_{Ld} = \sum_{n=1}^N \sum_{m=1}^M C_{Ld(m,n)} \quad (3.11)$$

where the term $(m-1)w + (n-1)l$ specifies the line-haul distance for a vehicle traveling between zone (m, n) and the terminal.

(iv) Transfer time at the terminal

Let C_{Rp} and C_{Rd} denote the total transfer times per service hour for patrons in the outbound and inbound directions, respectively, which are calculated as the sums of the zone-specific transfer times, denoted by $C_{Rp(m,n)}$ and $C_{Rd(m,n)}$. The formulas for these times are as follows:

$$\begin{aligned} C_{Rp(m,n)} &= \frac{1}{H_{p(m,n)}} \cdot \mathbb{E} \left[Q_{p(m,n)} \cdot \left(\frac{\tau_a Q_{p(m,n)}}{2} + t_{f-t} + \frac{H_t}{2} \right) \right] \\ &= \frac{\mathbb{E}[Q_{p(m,n)}]}{H_{p(m,n)}} \left(t_{f-t} + \frac{H_t}{2} \right) + \frac{\tau_a}{2H_{p(m,n)}} \cdot \mathbb{E}[Q_{p(m,n)}^2] \end{aligned} \quad (3.12)$$

$$\begin{aligned} C_{Rd(m,n)} &= \frac{1}{H_{d(m,n)}} \cdot \mathbb{E} \left[Q_{d(m,n)} \cdot \left(\frac{\tau_b Q_{d(m,n)}}{2} + t_{t-f} + \frac{(\gamma_{(m,n)} - 1)H_{d(m,n)}}{2\gamma_{(m,n)}} \right) \right] \\ &= \frac{\mathbb{E}[Q_{d(m,n)}]}{H_{d(m,n)}} \left(t_{t-f} + \frac{(\gamma_{(m,n)} - 1)H_{d(m,n)}}{2\gamma_{(m,n)}} \right) + \frac{\tau_b}{2H_{d(m,n)}} \cdot \mathbb{E}[Q_{d(m,n)}^2] \end{aligned} \quad (3.13)$$

$$C_{Rp} = \sum_{n=1}^N \sum_{m=1}^M C_{Rp(m,n)} \quad (3.14)$$

$$C_{Rd} = \sum_{n=1}^N \sum_{m=1}^M C_{Rd(m,n)} \quad (3.15)$$

where H_t represents the trunk-line headway; t_{f-t} and t_{t-f} the outbound and inbound transfer delays per patron (including walking time), respectively; τ_a and τ_b the alighting and boarding times per patron, respectively; and $\gamma_{(m,n)}$ is an integer representing the ratio

between $H_{d(m,n)}$ and H_t . The RHS of (3.12) indicates that an average patron's transfer time consists of three components: (a) the time spent in the alighting process at the rail terminal; (b) the transfer delay; and (c) the waiting time for boarding a trunk-line vehicle (e.g., a train). Eq. (3.13) is derived in a similar manner, with the distinction that the third component—here, the waiting time for boarding a feeder bus—is $\frac{(\gamma(m,n)-1)H_{d(m,n)}}{2\gamma(m,n)}$ instead of $H_t/2$. This difference arises from the assumptions that the feeder headway in the inbound direction, $H_{d(m,n)}$, is an integer multiple of the trunk-line headway, H_t , and that the feeder bus dispatches are synchronized with the trunk vehicle arrivals. This synchronization ensures that the expected load of patrons is consistent across all feeder buses.

3.2.1.3 Agency cost

The DRC service operator bears two types of costs per service hour: (i) fuel and vehicle maintenance costs, C_{vk} , which are proportional to the distance traveled in vehicle-kilometers; and (ii) costs associated with fleet purchase (amortized over time) and driver wages, C_{vh} , which are proportional to the vehicle-hours traveled. These two costs are the sums of the corresponding cost components for each zone (m, n) , denoted by $C_{vkp(m,n)}$ and $C_{vhp(m,n)}$ in the outbound leg, and $C_{vkd(m,n)}$ and $C_{vhd(m,n)}$ in the inbound leg, respectively. These costs are calculated as follows:

$$\begin{aligned}
C_{vkp(m,n)} &= \frac{\pi_v}{\theta H_{p(m,n)}} \cdot \left((m-1)w + (n-1)l + \mathbb{E} \left[(\beta_1 S + \beta_2)(Q_{p(m,n)} + 1)^{\beta_3} e^{\beta_4(Q_{p(m,n)} + 1)^{\beta_5}} \right. \right. \\
&\quad \left. \left. + \sqrt{(Q_{p(m,n)} + 1)lw} \right] \right) \\
&= \frac{\pi_v}{\theta H_{p(m,n)}} ((m-1)w + (n-1)l) \\
&\quad + \frac{\pi_v(\beta_1 S + \beta_2)\sqrt{lw}}{\theta H_{p(m,n)}} \cdot \mathbb{E} \left[(Q_{p(m,n)} + 1)^{\beta_3 + \frac{1}{2}} e^{\beta_4(Q_{p(m,n)} + 1)^{\beta_5}} \right]
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
C_{vkd(m,n)} &= \frac{\pi_v}{\theta H_{d(m,n)}} ((m-1)w + (n-1)l) \\
&\quad + \frac{\pi_v(\beta_1 S + \beta_2)\sqrt{lw}}{\theta H_{d(m,n)}} \cdot \mathbb{E} \left[(Q_{d(m,n)} + 1)^{\beta_3 + \frac{1}{2}} e^{\beta_4(Q_{d(m,n)} + 1)^{\beta_5}} \right]
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
C_{vhp(m,n)} &= \frac{\pi_m}{\theta H_{p(m,n)} v_I} ((m-1)w + (n-1)l) \\
&\quad + \frac{\pi_m(\beta_1 S + \beta_2)\sqrt{lw}}{\theta H_{p(m,n)} v_I} \cdot \mathbb{E} \left[(Q_{p(m,n)} + 1)^{\beta_3 + \frac{1}{2}} e^{\beta_4(Q_{p(m,n)} + 1)^{\beta_5}} \right] \\
&\quad + \frac{\pi_m \tau_p}{\theta H_{p(m,n)}} \cdot \mathbb{E}[Q_{p(m,n)}]
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
C_{vhd(m,n)} &= \frac{\pi_m}{\theta H_{d(m,n)} v_I} ((m-1)w + (n-1)l) \\
&+ \frac{\pi_m(\beta_1 S + \beta_2)\sqrt{lw}}{\theta H_{d(m,n)} v_I} \cdot \mathbb{E} \left[(Q_{d(m,n)} + 1)^{\beta_3 + \frac{1}{2}} e^{\beta_4(Q_{d(m,n)} + 1)^{\beta_5}} \right] \\
&+ \frac{\pi_m \tau_d}{\theta H_{d(m,n)}} \cdot \mathbb{E}[Q_{d(m,n)}]
\end{aligned} \tag{3.19}$$

$$C_{vk} = \sum_{n=1}^N \sum_{m=1}^M (C_{vkp(m,n)} + C_{vkd(m,n)}) \tag{3.20}$$

$$C_{vh} = \sum_{n=1}^N \sum_{m=1}^M (C_{vhp(m,n)} + C_{vhd(m,n)}) \tag{3.21}$$

where π_v (\$/vehicle-km) and π_m (\$/vehicle-hour) denote the unit costs per vehicle-km and vehicle-hour traveled, respectively; and θ (\$/hour) the value of time used to convert monetary costs into equivalent temporal costs. π_v and π_m are expressed as linear functions of the vehicle size, K ; see Section 3.4.1. Specifically, the average distance traveled by each bus, as included in the parentheses in the RHS of Eq. (3.16) and (3.17), accounts for both line-haul and local tour distances. Similarly, the average travel time per bus, factored into the RHS of Eq. (3.18) and (3.29), encompasses the cruise times for line-haul and local tours, as well as the additional time incurred at stops.

3.2.2 Cost models of the semi-flexible routing strategy

For ease of presentation, we will slightly abuse the notation in this section. Unless indicated otherwise, the same notations employed in the subsequent subsections retain the definitions provided in 3.2.1, with the understanding that they now refer to the context of the semi-flexible routing strategy.

3.2.2.1 Estimation of expected TSP tour length

Consider a rectangular zone with an area A , in which a swath of width w_0 has been cut. Under the semi-flexible routing strategy, where a bus is to visit q stops to pick up patrons, Daganzo (1984) provides the formula for calculating the bus's tour length using the Manhattan metric:

$$L_s = \frac{qw_0}{3} + \frac{A}{w_0} \tag{3.22}$$

The RHS of Eq. (22) reaches its minimum, $L_s^* = 2\sqrt{qA/3} \approx 1.15\sqrt{qA}$, when $w_0 = w_0^* = \sqrt{3A/q}$. However, this minimum is unattainable if the dimensions of the zone are not integer multiples of w_0^* . In general, to determine the optimal swath width, the

following two constraints must be met:

$$w_0 \leq \min\{l, w\} \quad (3.23)$$

$$w_0 \in \left\{ l, w, \frac{l}{2}, \frac{w}{2}, \frac{l}{3}, \frac{w}{3}, \frac{l}{4}, \frac{w}{4}, \dots \right\} \quad (3.24)$$

Constraint (3.23) guarantees that w_0 is not too large compared to the zone size. Constraint (3.24) ensures that the swath can be feasibly cut.

We perform numerical experiments across a range of scenarios characterized by varying numbers of stops, q (ranging from 2 to 15), and aspect ratios, S (ranging from 1 to 3). For each scenario, we calculate the MAPE between the minimum optimal tour length, given by $L_s^* = 1.15\sqrt{qA}$, and the tour length optimized using Eq.(3.22) while adhering to constraints (3.23) and (3.24). As depicted in 3.4, the MAPE is substantial, reaching over 12% in scenarios with a smaller number of stops. The results underscore the importance of considering the zone's dimensional constraints within the modeling process for semi-flexible routing.

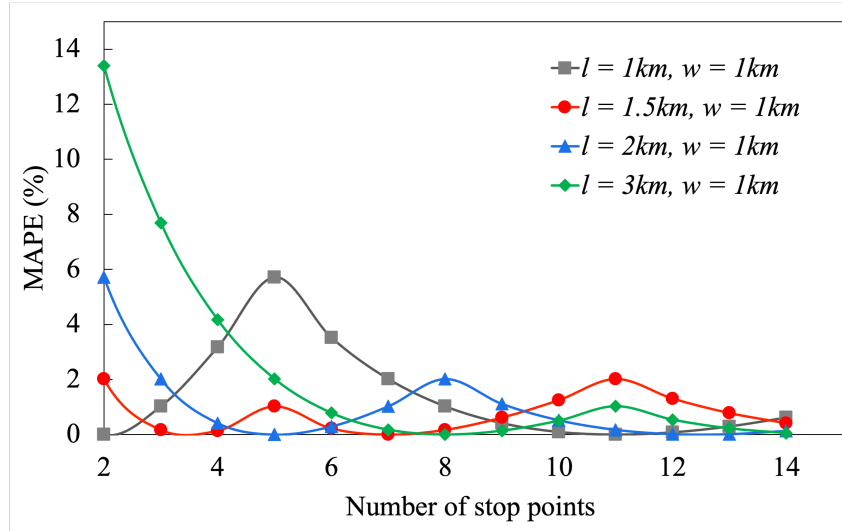


Figure 3.4: The MAPEs of tour length under semi-flexible routing

3.2.2.2 Patrons' travel time cost

The formulations of patrons' total line-haul travel time and transfer time under semi-flexible routing are identical to those under the fully-flexible routing strategy, so we will not repeat them here. Instead, we focus on presenting the formulations for the total waiting time at home and the in-vehicle travel time during the local tour.

(i) Waiting time at home

$$C_{W(m,n)} = \frac{\alpha}{H_p(m,n)} \mathbb{E}[Q_{p(m,n)}] \left(\frac{H_p(m,n)}{2} + \frac{w_0}{3v_I} \right) \quad (3.25)$$

$$C_W = \sum_{n=1}^N \sum_{m=1}^M C_{W(m,n)} \quad (3.26)$$

where $w_0/(3v_I)$ represents the average travel time for a bus to cover the lateral displacement necessary to pick up a patron, as provided by [Daganzo \(1984\)](#). In contrast to Eq.(3.2), it is evident that the waiting time under semi-flexible routing is reduced by an amount equivalent to half the local-tour travel time when compared to fully-flexible routing. This reduction arises because the waiting time with fully-flexible routing includes both the interval from the request being received to the vehicle's dispatch (which is on average half of the service headway) and the subsequent time taken for the vehicle to reach the pick-up point (the average of which is half of the local-tour travel time). In semi-flexible routing, however, the waiting time is solely composed of the period until the arrival of the next bus, which, on average, is half the service headway.

(ii) In-vehicle travel time of local tours

$$\begin{aligned} C_{Tp(m,n)} &= \frac{1}{2H_{p(m,n)}} \mathbb{E} \left[Q_{p(m,n)} \left(\frac{1}{v_I} \left(\frac{Q_{p(m,n)}w_0}{3} + \frac{lw}{w_0} + \frac{w_0}{2} \right) + Q_{p(m,n)}\tau_p \right) \right] \\ &= \frac{1}{2H_{p(m,n)}} \left(\left(\frac{lw}{v_Iw_0} + \frac{w_0}{2v_I} \right) \mathbb{E}[Q_{p(m,n)}] + \left(\frac{w_0}{3v_I} + \tau_p \right) \mathbb{E}[Q_{p(m,n)}^2] \right) \end{aligned} \quad (3.27)$$

$$\begin{aligned} C_{Td(m,n)} &= \frac{1}{2H_{d(m,n)}} \mathbb{E} \left[Q_{d(m,n)} \left(\frac{1}{v_I} \left(\frac{Q_{d(m,n)}w_0}{3} + \frac{lw}{w_0} + \frac{w_0}{2} \right) + Q_{d(m,n)}\tau_d \right) \right] \\ &= \frac{1}{2H_{d(m,n)}} \left(\left(\frac{lw}{v_Iw_0} + \frac{w_0}{2v_I} \right) \mathbb{E}[Q_{d(m,n)}] + \left(\frac{w_0}{3v_I} + \tau_d \right) \mathbb{E}[Q_{d(m,n)}^2] \right) \end{aligned} \quad (3.28)$$

$$C_{Tp} = \sum_{n=1}^N \sum_{m=1}^M C_{Tp(m,n)} \quad (3.29)$$

$$C_{Td} = \sum_{n=1}^N \sum_{m=1}^M C_{Td(m,n)} \quad (3.30)$$

In Eq.(3.27) and (3.28), a bus's local tour travel time is composed of cruising time and the time spent at stops. On average, a patron's in-vehicle travel time for the local tour is half the total travel time of the bus's local tour. Note that an adjustment of $w_0/2$ is incorporated to account for the average distance from the end of the local tour, situated (on average) along the swath's midline, to the start of the line-haul journey at the zone's bottom left corner.

3.2.2.3 Agency cost

The agency costs associated with vehicle-kms and vehicle-hours traveled under the semi-flexible routing strategy are formulated as follows:

$$\begin{aligned}
C_{vk(m,n)} &= \frac{\pi_v}{\theta} \left(\frac{1}{H_{p(m,n)}} \mathbb{E} \left[\frac{Q_{p(m,n)} w_0}{3} + \frac{lw}{w_0} + \frac{w_0}{2} + (m-1)w + (n-1)l \right] \right. \\
&\quad \left. + \frac{1}{H_{d(m,n)}} \mathbb{E} \left[\frac{Q_{d(m,n)} w_0}{3} + \frac{lw}{w_0} + (m-1)w + (n-1)l \right] \right) \\
&= \frac{\pi_v}{\theta} \left(\left(\frac{1}{H_{p(m,n)}} + \frac{1}{H_{d(m,n)}} \right) \left(\frac{lw}{w_0} + \frac{w_0}{2} + (m-1)w + (n-1)l \right) \right. \\
&\quad \left. + \left(\frac{\mathbb{E}[Q_{p(m,n)}]}{H_{p(m,n)}} + \frac{\mathbb{E}[Q_{d(m,n)}]}{H_{d(m,n)}} \right) \frac{w_0}{3} \right) \tag{3.31}
\end{aligned}$$

$$\begin{aligned}
C_{vh(m,n)} &= \frac{\pi_m}{\theta} \left(\frac{1}{H_{p(m,n)}} \left(\mathbb{E} \left[\frac{1}{v_I} \left(\frac{Q_{p(m,n)} w_0}{3} + \frac{lw}{w_0} + \frac{w_0}{2} + (m-1)w + (n-1)l \right) \right] \right. \right. \\
&\quad \left. \left. + \mathbb{E}[Q_{p(m,n)}] \tau_p \right) + \frac{1}{H_{d(m,n)}} \left(\mathbb{E} \left[\frac{1}{v_I} \left(\frac{Q_{d(m,n)} w_0}{3} + \frac{lw}{w_0} + \frac{w_0}{2} + (m-1)w + (n-1)l \right) \right] \right. \right. \\
&\quad \left. \left. + \mathbb{E}[Q_{d(m,n)}] \tau_d \right) \right) \\
&= \frac{\pi_m}{\theta} \left(\frac{1}{v_I} \left(\left(\frac{1}{H_{p(m,n)}} + \frac{1}{H_{d(m,n)}} \right) \left(\frac{lw}{w_0} + \frac{w_0}{2} + (m-1)w + (n-1)l \right) \right. \right. \\
&\quad \left. \left. + \left(\frac{\mathbb{E}[Q_{p(m,n)}]}{H_{p(m,n)}} + \frac{\mathbb{E}[Q_{d(m,n)}]}{H_{d(m,n)}} \right) \frac{w_0}{3} \right) + \left(\frac{\mathbb{E}[Q_{p(m,n)}] \tau_p}{H_{p(m,n)}} + \frac{\mathbb{E}[Q_{d(m,n)}] \tau_d}{H_{d(m,n)}} \right) \right) \tag{3.32}
\end{aligned}$$

$$C_{vk} = \sum_{n=1}^N \sum_{m=1}^M (C_{vk(m,n)} + C_{vk(m,n)}) \tag{3.33}$$

$$C_{vh} = \sum_{n=1}^N \sum_{m=1}^M (C_{vh(m,n)} + C_{vh(m,n)}) \tag{3.34}$$

3.2.3 Optimization model

For the fully-flexible routing strategy, the objective function, i.e., the total societal cost of the DRC system, can be written as the sum of all the cost components presented in Section 3.2.2:

$$GC = C_W + C_{Tp} + C_{Td} + C_{Lp} + C_{Ld} + C_{Rp} + C_{Rd} + C_{vk} + C_{vh} \tag{3.35a}$$

The optimization problem is formulated as follows:

$$\min GC \tag{3.35b}$$

subject to:

$$\lambda_p H_{p(m,n)} l w + (2\lambda_p H_{p(m,n)} l w)^{1/2} \leq K \quad (3.35c)$$

$$\lambda_d H_{d(m,n)} l w + (2\lambda_d H_{d(m,n)} l w)^{1/2} \leq K \quad (3.35d)$$

$$H_{d(m,n)} = \gamma_{(m,n)} H_t \quad (3.35e)$$

$$H_{\min} \leq H_{p(m,n)} \leq H_{\max} \quad (3.35f)$$

$$\max\{H_{\min}, H_t\} \leq H_{d(m,n)} \leq H_{\max} \quad (3.35g)$$

where K denotes a DRC bus's patron-carrying capacity; $H_{\min}$ and $H_{\max}$ the minimum and maximum headways, respectively; and $\gamma_{(m,n)}$ is an integer. Constraints (3.35c) and (3.35d) ensure that the DRC bus's capacity is no less than the average number of patrons onboard plus two standard deviations. This statistically covers approximately 98% of all instances, thereby minimizing the risk of capacity violations. Constraint (3.35e) requires that the DRC headway in the inbound direction is an integer multiple of the trunk vehicle headway.

For the semi-flexible routing strategy, the optimization formulation remains consistent with the one used here, though the cost models differ, and it includes additional constraints (3.23) and (3.24).

3.3 Solution Method

First, note that the four integer variables: M , N , K and $\gamma_{(m,n)}$, as well as w_0 in the semi-flexible routing strategy can take values only from limited ranges. These constraints make it feasible to employ an exhaustive search to determine their optimal values. Specifically, in our approach, we define the ranges as $K \in \{1, 2, \dots, 20\}$, $M \in \{1, 2, \dots, 6\}$, $N \in \{1, 2, \dots, 6\}$, and $\gamma_{(m,n)} \in \{1, 2, \dots, 5\}$ where $m \in \{1, 2, \dots, M\}$, $n \in \{1, 2, \dots, N\}$, and $w_0 \in \{l, w, l/2, w/2, l/3, w/3, l/4, w/4\}$, subject to the condition that $w_0 \leq \min\{l, w\}$ (the values of l and w can be determined when M and N are known).

For each combination of M , N , K and $\gamma_{(m,n)}$ (along with w_0 for semi-flexible routing), we aim to find the optimal value for $H_{p(m,n)}$ where $m \in \{1, 2, \dots, M\}$ and $n \in \{1, 2, \dots, N\}$ (note that $H_{d(m,n)}$ is determined by $\gamma_{(m,n)} H_t$, as shown in Eq. (3.35e). The optimization problem for $H_{p(m,n)}$ is a nonlinear and non-convex mathematical program. We solve this problem using the sequential quadratic programming method, implemented via the `fmincon` function in MATLAB R2020a. Given the non-convex nature of the problem, we execute multiple runs with different initial values to achieve a high-quality, near-optimal solution. Ultimately, we select the solution with the lowest cost from all evaluated combinations of the discrete variables.

The detailed solution algorithm under the fully-flexible routing strategy is provided

below:

Step 1: Iterate over each combination of $\{M, N, K, \gamma_{(m,n)}\} \in \{1, 2, \dots, 6\} \times \{1, 2, \dots, 6\} \times \{1, 2, \dots, 20\} \times \{1, 2, \dots, 5\}$, where “ $\times$ ” denotes the Cartesian product:

- 1.1 Assign random initial values to $H_{p(m,n)}$ within the range of $[H_{\min}, H_{\max}]$ for $m \in \{1, 2, \dots, M\}$ and $n \in \{1, 2, \dots, N\}$.
- 1.2 Use `fmincon` to find the optimal solution for $H_{p(m,n)}^*$. Record the solution and the associated total cost.
- 1.3 Repeat sub-steps 1.1 and 1.2 for 20 times and record the lowest-cost solution for the current set of $\{M, N, K, \gamma_{(m,n)}\}$.

Step 2: From all combinations of $\{M, N, K, \gamma_{(m,n)}\}$, select the solution with the lowest cost and output it.

Under the semi-flexible routing strategy, Step 1 is adjusted by including the feasible set of w_0 in the Cartesian product.

3.4 Numerical Case Studies

Section 3.4.1 delineates the setup of the numerical case studies. A comparative analysis of the performance of our models against those using the k^* values/formulas from preceding research is presented in Section 3.4.2. The two DRC routing strategies, fully-flexible routing and semi-flexible routing, are compared in Section 3.4.3. Section 3.4.4 delves into the performance sensitivity of both DRC strategies and a fixed-route feeder design concerning various key operational parameters, including the demand density, the service region dimensions, the value of time, and the discount factor for the home-waiting time value.

3.4.1 Experimental setup

Our numerical instances utilize the parameter values outlined in Table 3.2, except where noted otherwise. These values are borrowed from literatures (Daganzo, 2010; Li and Quadrifoglio, 2010; Kim and Schonfeld, 2015; Sangveraphunsiri et al., 2022; Zhen and Gu, 2024). We chose a service region size of 2×2 km², which represents a quarter of a suburban center or a satellite town linked to a trunk transit line. The demand density is set at 40 patrons/h/km² for both inbound and outbound directions. A sensitivity analysis regarding these parameter values will be explored in Section 3.4.4.

In our models, coefficient α denotes the discount factor for wait time value at home. Following earlier studies (Nourbakhsh and Ouyang, 2012; Qiu et al., 2014; Zheng et al.,

Table 3.2: Parameter values

Notation	Description	Value	Unit
L	Length of the service region	2	km
W	Width of the service region	2	km
λ_p	Outbound demand density	40	patrons/h/km ²
λ_d	Inbound demand density	40	patrons/h/km ²
θ	Value of time	20	\$/h
α	Discount factor of waiting time value at home	0.3	—
π_v	Unit cost per bus-km traveled	$0.0314 + 0.0039K$	\$/vehiclekm
π_m	Unit cost per bus-hour traveled	$2.068 + 0.108K + 2\theta$	\$/vehicleh
τ_0	Dwell time loss per stop due to bus acceleration and deceleration	26/3600	h/stop
τ_p	Dwell time per stop in the outbound leg	30/3600	h/patron
τ_d	Dwell time per stop in the inbound leg	28/3600	h/patron
τ_a	Alighting time per patron	2/3600	h/patron
τ_b	Boarding time per patron	4/3600	h/patron
v_I	Bus cruise speed	25	km/h
t_{f-t}	Transfer delay from feeder to trunk transit per patron	3/60	h/patron
t_{t-f}	Transfer delay from trunk transit to feeder per patron	3/60	h/patron
$H_{\min}$	Minimum feeder headway	3/60	h
$H_{\max}$	Maximum feeder headway	1	h
H_t	Headway of the trunk line	5/60	h
β_1	Coefficient of Eq. (1)	0.1102	—
β_2	Coefficient of Eq. (1)	1.4569	—
β_3	Coefficient of Eq. (1)	-0.1472	—
β_4	Coefficient of Eq. (1)	-2.5508	—
β_5	Coefficient of Eq. (1)	-2.6396	—

2018a; Petit et al., 2021), α varies between 0 and 1, with higher values indicating a more significant wait time impact on the overall cost. For our DRC service, α is set below 1, as DRC bus dispatch times are known to the patrons. This permits patrons to plan other activities during their wait, leading to a more agreeable waiting experience compared to being in transit (Ohmori et al., 2004).

Additionally, the operating cost parameters π_v and π_m are modeled as linear functions of vehicle capacity K , supported by empirical validation from transit operator data (Oldfield and Bly, 1988). This form has been widely adopted in transit optimization studies (e.g., Chang and Schonfeld, 1991; Kim and Schonfeld, 2014) to effectively balance model simplicity and cost estimation accuracy.

The optimization problem was solved in a PC powered by an Intel Core i7-1165G7 processor with 16.0 GB of RAM, clocked at 2.80 GHz. The solution algorithm was implemented using MATLAB 2020a. On average, solving a case required about 20 seconds of computational time.

3.4.2 Benefits of accurate modeling

Compared to previous DRC studies, an important improvement in our work is the integration of more precise models for estimating tour lengths and the inclusion of second-order effects of stochastic demand within the framework of optimal DRC design. This section is dedicated to evaluating the impacts of these methodological enhancements.

Specifically, we conduct comparisons with select metrics against designs based on previous tour length estimates without considering the second-order effects of demand stochasticity (e.g., Chang and Schonfeld, 1991; Kim and Schonfeld, 2014). To this end, we implement Monte Carlo simulation to emulate actual costs for certain designs and contrast them with costs of our models and previous studies' models. We evaluate 32 scenarios with varying parameters: $\lambda_p = \lambda_d \in \{10, 40\}$ patrons/h/km², $f \in \{0.3, 0.9\}$, $\theta \in \{5, 20\}$ \$/h, $L \in \{2, 3\}$ km, and $W \in \{2, 3\}$ km. For each scenario, we use our models to optimize the DRC design under both routing strategies, then freeze the design variables (M , N , K , $\gamma_{(m,n)}$, w_0 , $H_{p(m,n)}$, and $H_{d(m,n)}$) and simulate the costs. Each simulation run handles the demand randomly generated within one hour using the spatial Poisson process. To find the expected generalized cost, we run at least 1000 simulations for each scenario and strategy. If the expected generalized cost does not converge after 1000 simulations, the number of simulation runs will be increased until convergence is achieved. We stipulate that convergence is attained when the expected generalized cost's standard deviation is less than 0.05. Then, we compare simulation results with our theoretical estimates and those using previous models. The percentage errors in total cost using various tour length estimates for fully-flexible and semi-flexible routing strategies are presented in Tables 3.3 and 3.4, respectively. Also shown are percentages of: (i) overall tour length estimation

errors; (ii) errors in the cumulative passenger time loss due to pick-ups and drop-offs at stops, i.e., the aggregated values of $\sum_{m=1}^M \sum_{n=1}^N \left(\frac{\tau_p}{H_{p(m,n)}} \mathbb{E}[Q_{p(m,n)}^2] + \frac{\tau_d}{H_{d(m,n)}} \mathbb{E}[Q_{d(m,n)}^2] \right)$ in Eq.(3.4)–(3.5) across all zones; and (iii) instances where demand exceeds vehicle capacity. We include the item (ii) to illustrate the impacts of accounting for the second-order stochastic effects of demand. Table 3.3 indicates that with fully-flexible routing, the

Table 3.3: Modeling errors for the fully-flexible routing strategy

Metric	Our model		Yang et al. (2020)		Chakraborti and Chakrabarti (2000)	
	Average	Maximum	Average	Maximum	Average	Maximum
Errors in GC	1.97%	4.74%	11.73%	23.63%	8.05%	10.50%
Errors in out-bound tour length	1.32%	4.71%	33.05%	68.99%	18.19%	25.24%
Errors in in-bound tour length	1.38%	3.93%	28.60%	61.55%	19.54%	26.88%
Errors in cumulative pick-up time loss	0.50%	1.14%	22.37%	39.94%	15.79%	20.56%
Errors in cumulative drop-off time loss	0.64%	1.29%	16.30%	24.42%	19.42%	29.49%
Overcapacity	0.45%	0.57%	13.46%	16.93%	12.73%	18.02%

Table 3.4: Modeling errors for the semi-flexible routing strategy

Metric	Our model		Tour length estimation using $k^{star} = 1.15$ and excluding second-order stochastic effect	
	Average	Maximum	Average	Maximum
Errors in GC	0.25%	0.53%	6.32%	10.57%
Errors in outbound tour length	0.43%	0.72%	36.83%	42.81%
Errors in inbound tour length	0.40%	0.81%	35.70%	44.52%
Errors in cumulative pick-up time loss	0.71%	1.02%	43.95%	45.99%
Errors in cumulative drop-off time loss	0.65%	1.24%	38.63%	48.72%
Overcapacity	0.43%	0.51%	14.45%	17.50%

average percentage error in total cost incorporating our k^* formula and the second-order stochastic effects is under 2%, which clearly demonstrates the precision of our cost approximations. In contrast, errors exceed 8% and can reach up to 24% when using previous k^* values or formulas and neglecting second-order stochastic effects. Although some readers

might still deem these improvements moderate, it is because the cost components related to our model enhancements constitute a limited portion of the overall cost in the present case.

A closer look at tour length estimates reveals even greater discrepancies. Our average errors remain below 2%, whereas previous k^* values or formulas result in average errors between 18–33%, with extremes up to 69%. This indicates that *for scenarios where local tour lengths significantly impact overall cost (e.g., shorter line-haul distances and larger zones, potentially due to a limited fleet size), our model enhancements will provide a substantial increase in estimation precision.*

Regarding cumulative pick-up/drop-off time losses, our model maintains average errors under 1%, while previous models that ignore second-order stochastic effects show average errors between 15–22%, with extremes reaching 40%. This suggests that *more significant improvements in generalized cost estimation are expected when pick-up/drop-off times are a larger portion of the total cost, such as in systems designed for transporting people with disabilities, who have considerably longer boarding (τ_p) and alighting (τ_d) times.*

Finally, our model keeps the chance of overcapacity as low as 0.45%. In comparison, overlooking second-order stochastic effects results in an overcapacity probability of 12–13%. This further underscores the necessity of considering second-order stochastic demand effects.

For the semi-flexible routing, the average total cost error of our refined model is 0.25%, marking a sizeable improvement over the previous models (6.3%). Errors in tour length estimates and cumulative pick-up/drop-off time losses are kept below 1% with our model, in contrast to the more than 30% errors produced by earlier models. Additionally, our model yields a probability of overcapacity of merely 0.43%, compared to 14% under previous models. These findings highlight the critical importance of the adjustments and improvements implemented in our modeling approach.

3.4.3 Performance comparison between two routing strategies

We conduct a head-to-head comparison between the optimized fully-flexible and semi-flexible routing strategies for the parameter values provided in Table 3.2. Key design variables, costs, and other metrics are presented in Table 3.5. Variable notations adorned with top bars represent mean values, averaged over either zones or individual patrons.

The first row of Table 5 shows a 3.1% generalized cost saving when opting for semi-flexible routing over fully-flexible routing. Upon closer examination, it becomes evident that semi-flexible routing offers both lower patron costs (row 2) and lower agency costs (row 3).

Further comparison of the patron cost components (rows 4–7) indicates that the

Table 3.5: Optimal designs, costs, and other metrics under the two routing strategies

Row number	Parameters	Fully-flexible routing	Semi-flexible routing
1	$\overline{GC}$, mean total cost, min/patron	18.29	17.73
2	$\overline{UC}$, mean user cost, min/patron	11.96	11.62
3	$\overline{AC}$, mean agency cost, min/patron	6.33	6.11
4	$\overline{C_W}$, mean waiting time at home per patron, min/patron	1.01	0.57
5	$\overline{C_T}$, mean local-tour in-vehicle time, min/patron	2.10	2.45
6	$\overline{C_L}$, mean line-haul travel time, min/patron	1.2	0.9
7	$\overline{C_R}$, mean transfer time, min/patron	2.18	2.19
8	K , bus capacity, patrons/bus	8	9
9	$M \times N$, number of service zones partitioned	2×2	1×4
10	w_0 , swath width under semi-flexible routing, km	–	0.5
11	$\overline{H_{p(m,n)}}$, mean headway in outbound direction, min	4.98	6.80
12	$\overline{H_{d(m,n)}}$, mean headway in inbound direction, min	5.00	5.00
13	$\overline{Q_{p(m,n)}}$, mean occupancy outbound, patrons/bus	3.32	4.54
14	$\overline{Q_{d(m,n)}}$, mean occupancy inbound, patrons/bus	3.33	3.33
15	$\overline{k_p}$, mean k^* in outbound direction	1.19	1.41
16	$\overline{k_d}$, mean k^* in inbound direction	1.20	1.54
17	Mean tour length in outbound direction, km	2.46	3.01
18	Mean tour length in inbound direction, km	2.49	2.81

greatest portion of savings comes from reduced waiting times for patrons (row 4). This reduction is attributed to *the increased waiting times associated with the fully-flexible routing strategy* (refer to Fig.3.2a). On the other hand, *fully-flexible routing results in considerably shorter local-tour times* (row 5), primarily due to two factors: (i) the enhanced efficiency of the optimal TSP tours, evident from substantially lower k^* values (rows 15–16); and (ii) each vehicle in the fully-flexible routing strategy transports fewer passengers compared to the semi-flexible strategy in the outbound direction (see row 13). The latter is caused by the shorter headway in fully-flexible routing (see row 11), a result of (i) shorter headways helping to counterbalance the negative effects of longer waiting times, and (ii) reduced round-trip times under fully-flexible routing (see row 17) enabling a higher frequency of service. Despite the efficiency gains from shorter local tours in fully-flexible routing, these do not compensate for the increased waiting times, thus leading to higher overall patron costs for this strategy in the present case study.

Furthermore, fully-flexible routing incurs higher agency costs than semi-flexible routing, largely due to its more frequent service.

An additional notable observation is the difference in zone partitioning between the two routing strategies: fully-flexible routing divides the service region into four square zones of $1 \times 1 \text{ km}^2$, whereas semi-flexible routing uses four elongated zones of $0.5 \times 2 \text{ km}^2$; see row 9. *The square zoning in fully-flexible routing, which maintains a minimal aspect ratio of 1, evidently reduces the length of local tours*, as indicated by the positive coefficient β_1 in Eq.(3.1). Conversely, *semi-flexible routing prefers elongated zones, which naturally form narrow swaths with a small w_0 , minimizing lateral detours for buses*.

3.4.4 Sensitivity analysis

3.4.4.1 Sensitivity to the demand density

In this analysis, we explore demand densities ranging from 2 to 210 patrons/h/km². The values for other parameters remain consistent with those used in Section 3.4.1. The outcomes are visualized in Fig. 3.5.

The key insight from Fig. 3.5 is that the optimal service design transitions from fully-flexible routing to semi-flexible routing, and ultimately to fixed-route service as demand density increases. At lower demand densities (below 21 patrons/h/km²), fully-flexible routing is the most cost-effective, with savings of up to 18% compared to semi-flexible routing. As demand increases, semi-flexible routing becomes the preferable option, predominantly due to its headways being more responsive to changes in demand, thereby reducing patrons' waiting times by a greater amount. Fixed-route service becomes the most advantageous option only when demand density exceeds 199 patrons/h/km², a shift attributable to its higher commercial speeds and the elimination of detours compared to DRC.

Another interesting observation comes from comparing the cost reduction rates of the three designs in Fig. 3.5a as demand density grows from 2 to 210 patrons/h/km², the generalized cost per patron for the fixed-route service decreases by 52%, for the semi-flexible routing by 41%, and for the fully-flexible routing by 26%. These results indicate that the fixed-route service demonstrates the most substantial economies of scale, followed by semi-flexible routing, and then fully-flexible routing. The trend suggests that increased routing flexibility tends to reduce economies of scale. This finding is consistent with observations in other transport modes such as taxis and private vehicles, which offer the highest flexibility but the least economies of scale compared to structured transit systems.

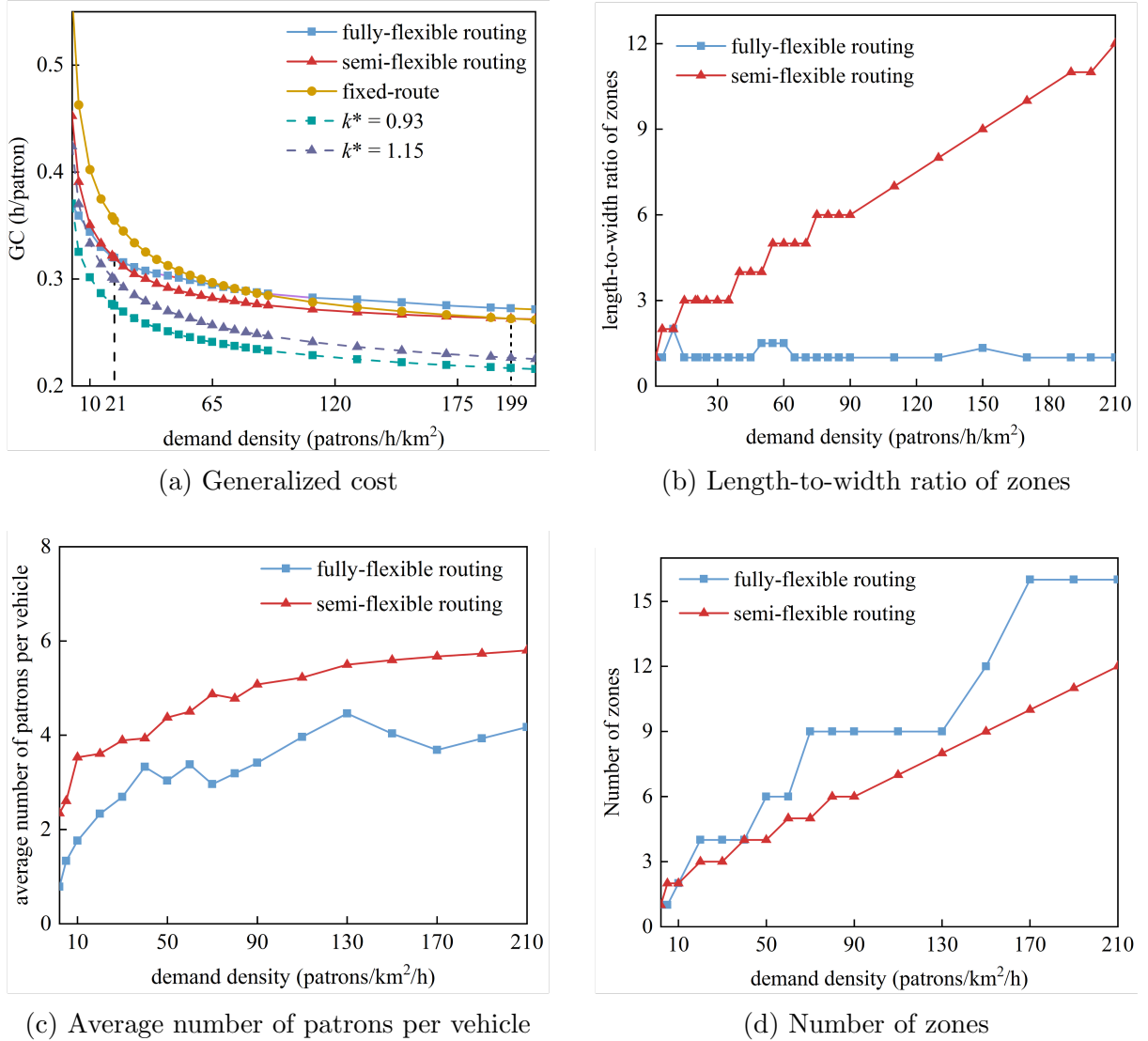


Figure 3.5: Effects of the demand density

For further comparison, Fig. 3.5a also includes the optimal DRC costs using k^* values from previous studies (0.93 and 1.15). Notably, employing these historical k^* values tends to significantly overestimate the benefits of DRC services and the critical demand density

between DRC and fixed-route service (which is beyond the range of demand density investigated here). This underscores the importance of accurate parameter estimation in optimal service designs.

Fig. 3.5b highlights a notable difference in the zonal configurations adopted by fully-flexible and semi-flexible routing strategies. For fully-flexible routing, the zonal aspect ratio (l/w) remains between 1 and 2, showing minimal sensitivity to changes in demand density. This indicates a preference for square zones, which help minimize local tour lengths. In contrast, semi-flexible routing increasingly favors more elongated zones as demand density rises.

Figs. 3.5c and 3.5d show that for both routing strategies, as demand increases, there is a significant increase in the number of zones, while the number of patrons per vehicle only grows moderately. This pattern underscores the effectiveness of strategic zonal division in keeping tour lengths short, thereby minimizing generalized costs. Additionally, as shown in Fig. 3.5c, fully-flexible routing consistently accommodates fewer patrons per vehicle due to its smaller zones (for most cases) and shorter headways. In rare cases where demand density falls below 10 patrons/km²/h, zones under the fully-flexible strategy can be larger than those under the semi-flexible strategy. However, the former's shorter headways (as explained in Section 3.4.3) still lead to fewer patrons per vehicle.

3.4.4.2 Sensitivity to region size and shape

First, we explore the effects of different region sizes, ranging from 2 to 32 km², with a fixed aspect ratio of 1. Other parameters are again set to the values presented in Section 3.4.1. The results under the three service designs are presented in Fig. 3.6.

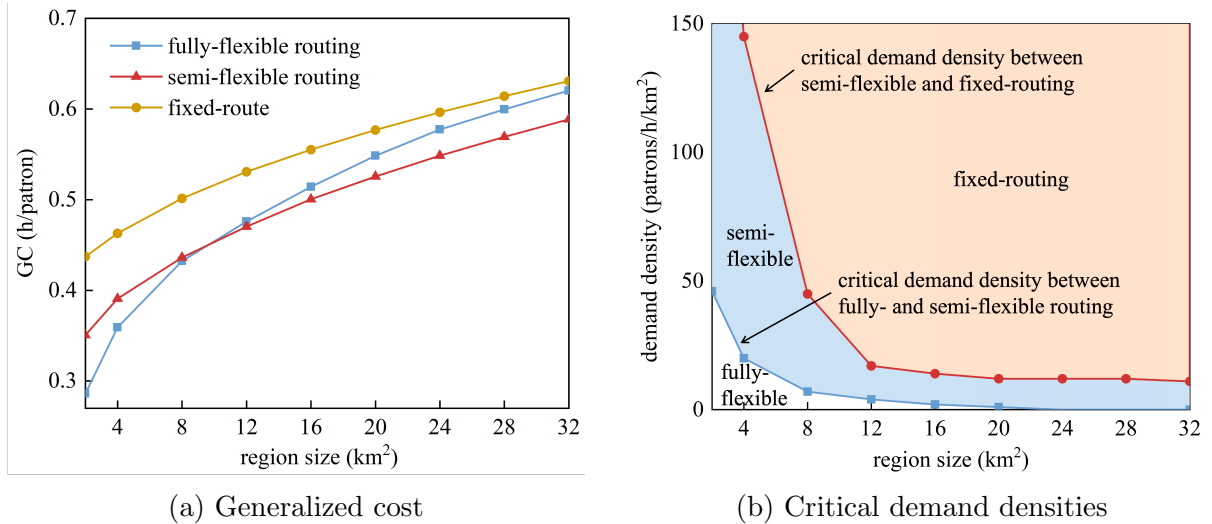


Figure 3.6: Effects of the region size

Fig. 3.6a shows that fully-flexible routing offers considerable advantages in smaller service regions, achieving cost savings of up to 18% compared to semi-flexible routing and

35% compared to fixed-route service. As the size of the region increases, semi-flexible routing becomes the preferred choice, primarily because its headways and agency costs are more sensitive to the region size, leading to significantly reduced waiting times and lower agency costs per patron. Should the region size increase beyond the scope depicted in Fig. 3.6a, fixed-route service is likely to become the most effective design, attributed to its higher commercial speed and reduced operating costs in larger service areas.

Fig. 3.5a identifies the critical demand densities at which the optimal choice of feeder service transitions among the three considered designs. Fig. 3.6b illustrates how these critical demand densities shift in response to changes in region size. Remarkably, these critical demand densities decrease significantly as the region size grows. Fig. 3.6b can be used to determine the most suitable feeder service configuration under varying operational conditions.

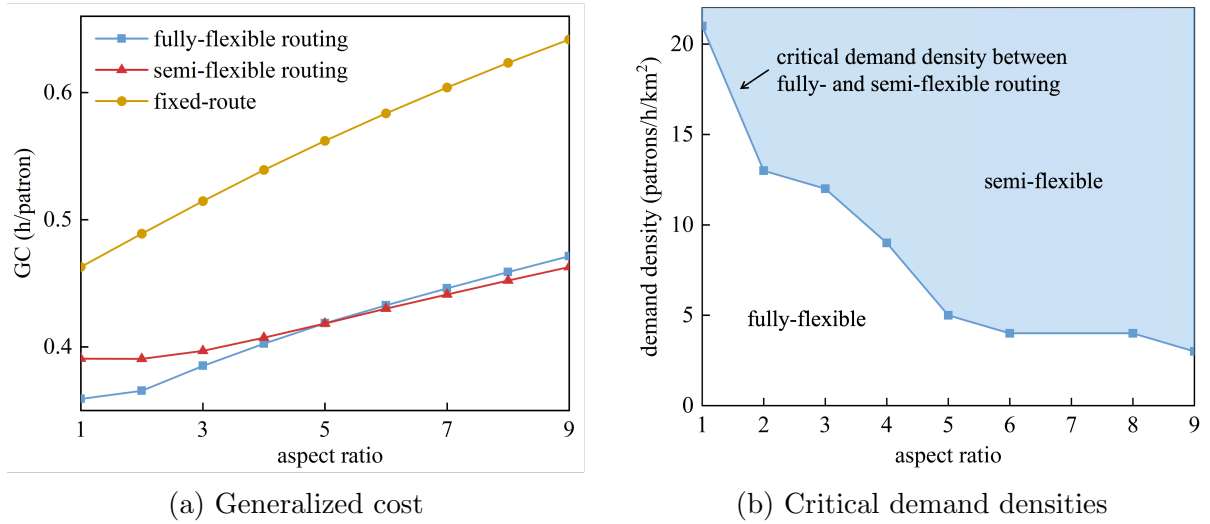


Figure 3.7: Effects of the region shape

We also examine the impact of the service region's aspect ratio, allowing it to vary from 1 to 10 while maintaining a constant region size of 4 km². Fig. 3.7a shows a rising trend in generalized costs as the service region becomes more elongated, a consequence of increased tour lengths. Among the three feeder service schemes, semi-flexible routing exhibits the least sensitivity to changes in the aspect ratio, demonstrating its superior adaptability to variations in shape. Fig. 3.7b illustrates how the critical demand density threshold between fully-flexible and semi-flexible routing decreases as the aspect ratio increases, further highlighting the adaptability of semi-flexible routing to different service region shapes. The critical density distinguishing fixed-route service from DRC shows minimal sensitivity to aspect ratio changes. This curve is not depicted in Fig. 3.7b, as it falls outside the scope of this figure.

3.4.4.3 Sensitivity to the value of time

Fig. 3.8 explores how the value of time affects the generalized costs of the three strategies, with all other parameter values held consistent with those outlined in Section 3.4.1. The results show that the value of time has an insignificant impact on the choice between strategies. However, as the value of time increases, the benefits of fully-flexible routing become slightly more pronounced, primarily due to a greater reduction in agency costs compared to the other alternatives.

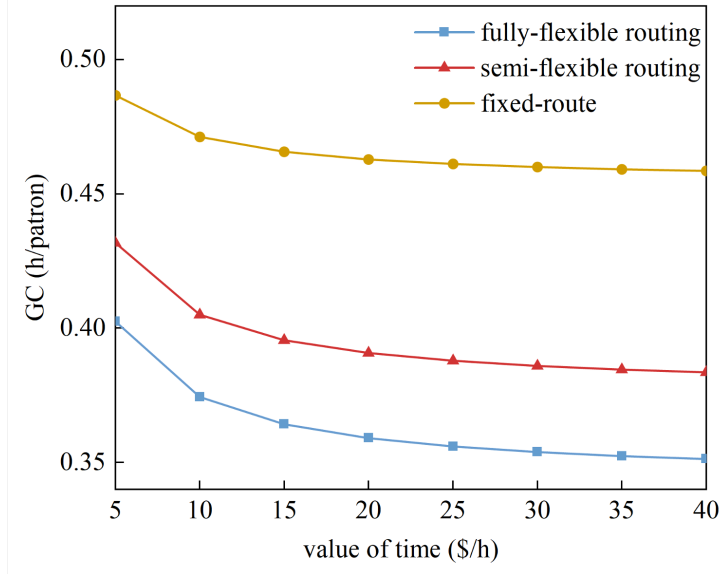


Figure 3.8: Effects of the value of time

3.4.4.4 Sensitivity to the discount factor of the waiting time value at home

Recall that the value patrons place on waiting time at home can vary significantly across different scenarios. Understanding this parameter's impact is crucial for designing effective feeder systems. Flex-route services often attribute a lower value to waiting time because they allow passengers to wait at home, offering enhanced comfort and convenience (Algers et al., 1975). This setup enables passengers to engage in other activities, thus making waiting time less burdensome and more tolerable.

In this section, we examine how the two flexible routing strategies perform under varying values of α , which ranges from 0 to 1. Fig. 3.9 presents these results. For comparative purposes, results from the fixed-route service are also included, which do not depend on α . Here we assume the value of waiting time at bus stops (for fixed-route service) is equal to that of in-vehicle travel time (Guo et al., 2011). The inbound and outbound demand densities are set at 15 patrons/km²/h. Fig. 3.9 illustrates how the effectiveness of the two strategies diverges depending on the value of α . For $\alpha \leq 0.48$, fully-flexible routing is more advantageous, offering potential cost savings of up to

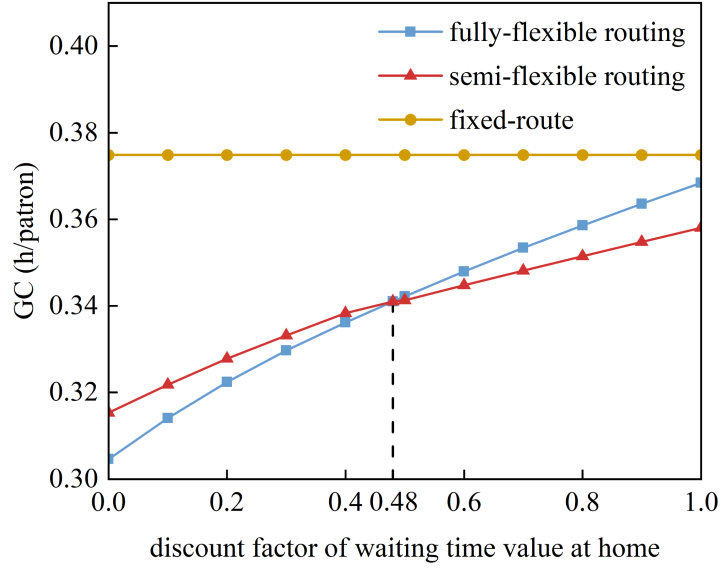


Figure 3.9: Effects of the discount factor of home-waiting time

3.4%. Within this α range, the lower weighting of waiting time in the generalized costs compensates for the longer waiting times associated with fully-flexible routing. However, as α increases beyond this point, the significance of waiting time in the generalized costs grows, diminishing the benefits of fully-flexible routing due to its longer waiting times. Consequently, semi-flexible routing begins to show greater advantages, consistently outperforming fully-flexible routing by maintaining shorter waiting times for passengers.

It is important to note that under the current parameter settings, fixed-route service is never the optimal choice, even with $\alpha = 1$. However, under other operating scenarios with high values of α (not detailed in this paper for brevity), fixed-route service could outperform the DRC alternatives.

Chapter 4

Trunk-Feeder Transit Network Design under Heterogeneous Demand

4.1 Problem Description

Fig. 4.1 illustrates the idealized network for the trunk-feeder system. The service region is assumed to be a rectangle with length L and width W . Fig. 4.1(a) depicts the trunk-line network, where black solid lines represent the trunk transit lines with spatially heterogeneous spacings. Trunk-line stations are positioned at the line intersections. These stations are designed to facilitate passenger transfers between trunk lines and between feeder services and trunk lines. Note that, unlike many real-world metro systems, our trunk-line network does not include non-transfer stations situated between these transfer points. This design choice is based on the premise that efficient feeder services provide rapid and convenient access to the transfer stations. By eliminating non-transfer trunk-line stations, we enhanced trunk-line transit speed without compromising passenger access convenience.

The horizontal and vertical lines are termed as *X-lines* and *Y-lines*, respectively. The spacings between adjacent Y-lines, denoted by $S_Y(x)$, and between X-lines, denoted by $S_X(y)$, are functions of the coordinates x and y , respectively. Note that these also represent the stations spacings between X-lines and Y-lines, respectively. Trunk-line vehicles operate on distinct headways, with $H_X(y)$ for X-lines and $H_Y(x)$ for Y-lines, respectively. We assume that the headways in both travel directions on the same line are identical, ensuring balanced vehicle flows.

Admittedly, our trunk-line network setup is idealized and may appear less flexible than some other network models found in the literature (e.g., the power-of-two bifurcation network in Ouyang et al., 2014). However, this idealization is crucial for reducing the

modeling complexity of the integrated network design problem, while still effectively addressing spatial heterogeneity. By incorporating distinctly designed feeder line that serve each transfer station, the integrated network gains greater flexibility, allowing it to more efficiently accommodate spatially-varied demand patterns.

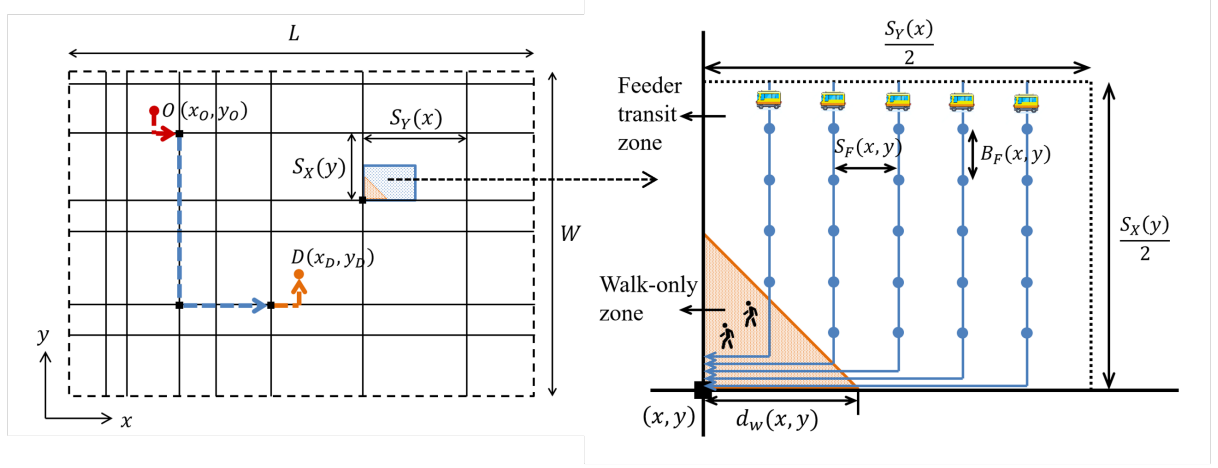


Figure 4.1: A trunk-feeder transit system network

Feeder services are designed for all trunk-line stations, unless walking offers greater passenger convenience. Fig. 4.1 shows the layout of a feeder system serving the trunk-line station located at coordinate (x, y) , covering a quarter of the station's catchment area, which is a rectangle with a length of $S_Y(x)/2$ and a width of $S_X(y)/2$ ¹. Patrons can choose between walking and taking feeder transit based on their distance from the trunk-line station. Those closer to the station may opt to walk. For simplicity, we define a walk-only zone around the trunk-line station, bounded by an equidistant contour line at a “critical distance” from the station, denoted by $d_w(x, y)$. Assuming a rectilinear metric, the contour line forms a 45-degree angle, as depicted in Fig. 4.1, with the walk-only zone shaded in orange. The whole walk-only zone around the transfer station is therefore a square. Patrons whose origin or destination is located within the walk-only zone will choose to walk, while others will opt to take feeder buses. A similar analysis of the walk-only zone can be found in Wu et al. (2020).

The blue lines and dots in Fig. 4.1 represent feeder routes and stops, respectively. Feeder lines are laid out along the Y-direction. The lines and stops distributed uniformly within the feeder zone, with spacings of $S_F(x, y)$ and $B_F(x, y)$, respectively. The same headway, denoted by $H_F(x, y)$, is assumed for all lines in both outbound and inbound directions, which helps reduce deadheading costs for feeder vehicles. We acknowledge

¹This represents an approximation of the actual catchment area, which performs well when $S_Y(x)$ and $S_X(y)$ vary slowly. This is a common assumption in the literature on continuum approximation (e.g., Wirasinghe and Ghoneim, 1981). Under this assumption, the trunk-line station's catchment area is approximately symmetric around the two intersecting trunk lines at the station. This symmetry allows the feeder system design from one quarter of the catchment area to be replicated across the remaining three quarters. For further details, please refer to Chapter 2.

that some assumptions employed here, e.g., the uniform settings for feeder line and stop spacings, as well as headways, are a simplification of the more general heterogeneous feeder service design detailed in Chapter 2. This simplification is again a deliberate choice to reduce modeling complexity, making the optimization of the heterogeneous trunk-feeder network tractable. Once the trunk-line network is optimized, the local feeder systems can be re-optimized using the approach presented in Chapter 2, allowing for the incorporation of heterogeneous layouts and demands. This will ultimately lead to improved systemic performance.

Due to the time limit, flex-route feeder service was not included in this chapter but will be explored in our future work. This decision was made because the primary goal of this chapter is to demonstrate the cost-effectiveness of the jointly optimized heterogeneous trunk-feeder network, which is already substantial even without considering flex-route feeders.

The demand density over the city is modeled as a spatially continuous function, $\lambda(x_O, y_O, x_D, y_D)$ (patrons/km⁴/h), where (x_O, y_O) and (x_D, y_D) represent the demand origin and destination, respectively. We assume that a patron accesses and egresses from the trunk system through the stations that are closest to his origin and destination (termed the origin and destination stations in this chapter), respectively. Similarly, if the patron utilizes a feeder service, he will access or egress the feeder system through the nearest feeder stop; see Chapter 2. Fig. 4.1 illustrates a route of a patron traveling from (x_O, y_O) to (x_D, y_D) . The patron first walks or takes feeder transit to reach his origin station, indicated by the red dashed line in the figure. He then boards the trunk-line transit and travels along the blue dashed line. Finally, the patron arrives at his destination by walking or feeder; see the orange dashed line. Thus, each patron experiences one to three transfers, depending on the selected modes of access and egress.

In summary, the trunk-feeder network design optimization problem involves the following decision functions and variables: line spacings of trunk lines, $S_X(y)$ and $S_Y(x)$; headways of trunk lines, $H_X(y)$ and $H_Y(x)$; line spacings of feeder lines, $S_F(x, y)$; stop spacings of feeder lines, $B_F(x, y)$; headways of feeder lines, $H_F(x, y)$; and feeder bus capacity, K_F . Following established practices (Ouyang et al., 2014; Chen and Nie, 2017), we define our optimization objective as minimizing the total system cost.

4.2 Mathematical Formulation

The cost models for the trunk-line and feeder subsystems are developed in Sections 4.2.1 and 4.2.2, respectively, while the overall optimization model is introduced in Section 4.2.3. The notations used in this section and their definitions are listed in Appendix I.

4.2.1 Trunk-line service model

4.2.1.1 Patrons' travel time

The travel time cost for patrons on trunk lines is further divided into three components: (i) waiting time at the origin station, T_W ; (ii) transfer time at the transfer station, T_R ; and (iii) in-vehicle travel time on the trunk line, T_T . Each of these components is detailed as follows.

(i) Waiting time at origin stations

The derivation of this cost term hinges on the widely accepted assumption that patrons arrive at their origin stations randomly, irrespective of their mode of access. This implies that patrons who walk do not check the service timetable, a common behavior associated with high-frequency trunk-line services (Daganzo, 2010; Chen et al., 2015; Fan et al., 2018). Similarly, for those taking feeder buses, their arrivals are not coordinated with the trunk-line schedule.² Following this assumption, the average waiting time for a patron is half of the trunk-line headway. Note that this assumption is conservative, as more sophisticated trip planning and schedule coordination (Sivakumaran et al., 2012; Zhen and Gu, 2024) could potentially reduce costs. The formulation of T_W is as follows:

$$\begin{aligned} T_W &= \int_{x=0}^L \frac{\Lambda_{OY}(x)}{2} \cdot \frac{H_Y(x)}{2} dx + \int_{y=0}^W \frac{\Lambda_{OX}(y)}{2} \cdot \frac{H_X(y)}{2} dy \\ &= \frac{1}{4} \left(\int_{x=0}^L \Lambda_{OY}(x) H_Y(x) dx + \int_{y=0}^W \Lambda_{OX}(y) H_X(y) dy \right) \end{aligned} \quad (4.1)$$

$$\Lambda_{OY}(x) = \int_{y_O=0}^W \int_{x_D=0}^L \int_{y_D=0}^W \lambda(x_O, y_O, x_D, y_D) dy_D dx_D dy_O \quad (4.2)$$

$$\Lambda_{OX}(y) = \int_{x_O=0}^L \int_{x_D=0}^L \int_{y_D=0}^W \lambda(x_O, y_O, x_D, y_D) dy_D dx_D dx_O \quad (4.3)$$

where $\Lambda_{OY}(x)dx$ represents the cumulative demand *originating in* a $W \times dx$ vertical slice at x , regardless of the destination. Here we assume that patrons accessing any trunk-line station are equally likely to board a Y-line or an X-line. Therefore, only half of $\Lambda_{OY}(x)dx$ will board a Y-line at x . Consequently, the total waiting time for patrons boarding Y-lines is calculated by the integral $\int_{x=0}^L \frac{\Lambda_{OY}(x)}{2} \cdot \frac{H_Y(x)}{2} dx$. Similarly, the total waiting time for those boarding X-lines is given by $\int_{y=0}^W \frac{\Lambda_{OX}(y)}{2} \cdot \frac{H_X(y)}{2} dy$, where $\Lambda_{OX}(y)dy$ denotes the cumulative demand *originating in* a $L \times dy$ horizontal slice located at y .

²This assumption is made because coordinating feeder bus arrivals at a transfer station with trunk-line schedules across all four directions (positive and negative X- and Y-directions) is infeasible. However, during real commuting periods, it is often possible to coordinate a local feeder service's schedule with the trunk-line direction that serves the majority of commuters, potentially reducing overall system costs. Please refer to Chapter 2 for further discussion on this matter.

(ii) Transfer time at trunk-line transfer stations

The derivation of T_R is similar to that of the patron waiting time T_W since a significant portion of T_R constitutes the waiting time at transfer stations. The key difference between that portion of T_R and T_W lies in their dependency on different lines of the trunk-line network. Specifically, T_W depends on the headway of the first trunk line that each patron boards, which correlates with the patron's origin (x_O, y_O) . Conversely, T_R is based upon the headway of the second trunk line that a patron takes, which is associated with his destination (x_D, y_D) .

For the sake of simplicity, we assume that every trip within the trunk-line network involves exactly one transfer. Thus, we disregard the minor possibility that a patron might reach their destination using a single trunk line directly, a simplification also adopted in other studies (Daganzo, 2010; Ouyang et al., 2014). The formulation of T_R is given by:

$$T_R = \frac{1}{4} \left(\int_{x=0}^L \Lambda_{DY}(x) H_Y(x) dx + \int_{y=0}^W \Lambda_{DX}(y) H_X(y) dy \right) + \Lambda \delta_T \quad (4.4)$$

$$\Lambda = \int_{x_O=0}^L \int_{y_O=0}^W \int_{x_D=0}^L \int_{y_D=0}^W \lambda(x_O, y_O, x_D, y_D) dy_D dx_D dy_O dx_O \quad (4.5)$$

$$\Lambda_{DY}(x) = \int_{y_D=0}^W \int_{x_O=0}^L \int_{y_O=0}^W \lambda(x_O, y_O, x_D, y_D) dy_O dx_O dy_D \quad (4.6)$$

$$\Lambda_{DX}(y) = \int_{x_D=0}^L \int_{x_O=0}^L \int_{y_O=0}^W \lambda(x_O, y_O, x_D, y_D) dy_O dx_O dx_D \quad (4.7)$$

where Λ denotes the total demand; $\Lambda_{DY}(x)dx$ and $\Lambda_{DX}(y)dy$ denote the cumulative demands destined for a $W \times dx$ vertical slice at x and for a $L \times dy$ horizontal slice located at y , respectively; and δ_T denotes the transfer penalty.

(iii) In-vehicle travel time on trunk lines

The in-vehicle travel time (T_T) is derived as follows:

$$T_T = \int_{x=0}^L \frac{Q_X(x)}{v_X(x)} dx + \int_{y=0}^W \frac{Q_Y(y)}{v_Y(y)} dy \quad (4.8)$$

$$\begin{aligned} Q_X(x) = & \int_{y_D=0}^W \int_{x_D=x}^L \int_{y_O=0}^W \int_{x_O=0}^x \lambda(x_O, y_O, x_D, y_D) dx_O dy_O dx_D dy_D \\ & + \int_{y_D=0}^W \int_{x_D=0}^x \int_{y_O=0}^W \int_{x_O=x}^L \lambda(x_O, y_O, x_D, y_D) dx_O dy_O dx_D dy_D \end{aligned} \quad (4.9)$$

$$\begin{aligned} Q_Y(y) = & \int_{y_D=y}^W \int_{x_D=0}^L \int_{y_O=0}^y \int_{x_O=0}^L \lambda(x_O, y_O, x_D, y_D) dx_O dy_O dx_D dy_D \\ & + \int_{y_D=0}^y \int_{x_D=0}^L \int_{y_O=y}^W \int_{x_O=0}^L \lambda(x_O, y_O, x_D, y_D) dx_O dy_O dx_D dy_D \end{aligned} \quad (4.10)$$

$$(4.1)$$

$$\frac{1}{v_X(x)} = \frac{1}{v_T} + \frac{\tau_T}{S_Y(x)} \quad (4.11)$$

$$\frac{1}{v_Y(y)} = \frac{1}{v_T} + \frac{\tau_T}{S_X(y)} \quad (4.12)$$

where $Q_X(x)$ and $Q_Y(y)$ denote the cross-sectional patron flows for the entire city width W at x and for the entire city length L at y , respectively; $v_X(x)$ and $v_Y(y)$ the commercial speeds for X-lines at x and for Y-lines at y , respectively; and τ_T the dwell time per station, including vehicle acceleration, deceleration, door operations, and patron boarding and alighting. The first part of Eq. (4.9) represents the patron flow passing the cross section at x , originating from the left side of the catchment area and destined for the right side, while the second part represents the flow in the opposite direction. Eq. (4.10) is derived similarly.

4.2.1.2 Agency cost

During each hour of operation, the operator incurs four types of costs: (i) line infrastructure costs, T_L ; (ii) station infrastructure costs, T_S ; (iii) operating costs associated with vehicle-km traveled (e.g., fuel cost), T_K ; and (iv) fleet-related costs, including amortized vehicle purchase and maintenance costs, and driver wages, T_H .

(i) Line infrastructure cost

$$T_L = \frac{2\pi_{lT}}{\mu} \left(W \int_{x=0}^L \frac{dx}{S_Y(x)} + L \int_{y=0}^W \frac{dy}{S_X(y)} \right) \quad (4.13)$$

where π_{lT} (\$/km/h) denotes the amortized trunk-line infrastructure cost per hour; μ the value of time; $\int_{x=0}^L dx/S_Y(x)$ and $\int_{y=0}^W dy/S_X(y)$ represent the numbers of Y-lines and X-lines, respectively. The coefficient 2 accounts for both directions of the lines.

(ii) Station infrastructure cost

$$T_S = \frac{\pi_{sT}}{\mu} \int_{x=0}^L \frac{dx}{S_Y(x)} \int_{y=0}^W \frac{dy}{S_X(y)} \quad (4.14)$$

where π_{sT} (\$/station/h) denotes the amortized trunk-line station infrastructure cost per hour.

(iii) Distance-based operating cost

$$T_K = \frac{2\pi_{kT}}{\mu} \left(W \int_{x=0}^L \frac{dx}{S_Y(x)H_Y(x)} + L \int_{y=0}^W \frac{dy}{S_X(y)H_X(y)} \right) \quad (4.15)$$

where π_{kT} (\$/km/h) denotes the unit cost per vehicle-km traveled on the trunk-line. The

total vehicle-kms traveled per hour of transit operations is calculated as the product of the line infrastructure length and the service frequency.

(iv) Time-based operating cost

$$T_H = \frac{2\pi_{hT}}{\mu} \left[\left(\frac{W}{v_T} + \tau_T \int_{y=0}^W \frac{dy}{S_X(y)} \right) \int_{x=0}^L \frac{dx}{S_Y(x)H_Y(x)} + \left(\frac{L}{v_T} + \tau_T \int_{x=0}^L \frac{dx}{S_Y(x)} \right) \int_{y=0}^W \frac{dy}{S_X(y)H_X(y)} \right] \quad (4.16)$$

where π_{hT} (\$/hour/h) denotes the unit cost per vehicle-hour traveled on the trunk line. The total vehicle hours traveled is the total vehicle kms traveled divided by the commercial speed. The average commercial speed on a Y-line and X-line are $1 / \left(\frac{W}{v_T} + \tau_T \int_{y=0}^W \frac{dy}{S_X(y)} \right)$ and $1 / \left(\frac{L}{v_T} + \tau_T \int_{x=0}^L \frac{dx}{S_Y(x)} \right)$, respectively.

4.2.2 Feeder-line service model

The feeder service model is derived under the assumption that the demand density is uniformly distributed within each feeder zone, equating to the demand density of the connected trunk-line station at (x, y) . This simplification is necessary for building an integrated trunk-feeder network design model that is continuous at every spatial coordinate, and can be decomposed locally for solving it efficiently. It incurs only a slight error when spatial variations in demand are not drastic.

4.2.2.1 Patrons' travel time

The travel time cost for patrons on feeder lines is divided into three components: (i) walking time, F_A , which includes the time for patrons within the walk-only zone to walk to and from trunk-line stations, and for patrons outside the zone to walk to and from feeder stops; (ii) waiting and transfer time, F_W , which includes the time for patrons waiting for feeder transit and for transferring from the feeder stop to the trunk-line station; and (iii) in-vehicle travel time on feeder lines, F_T , for patrons traveling on feeder buses. Each of these components is detailed as follows:

(i) Walking time

$$F_A = \int_{x=0}^L \int_{y=0}^W \frac{1}{S_X(y)S_Y(x)} (F_{AW} + F_{AF}) dy dx \quad (4.17)$$

$$F_{AW} = \frac{D_W A_W}{v_W} (\lambda_p(x, y) + \lambda_d(x, y)) \quad (4.18)$$

$$F_{AF} = \frac{S_F(x, y) + B_F(x, y)}{4v_W} (S_Y(x)S_X(y) - A_W) (\lambda_p(x, y) + \lambda_d(x, y)) \quad (4.19)$$

$$\lambda_p(x, y) = \int_{x_D=0}^L \int_{y_D=0}^W \lambda(x, y, x_D, y_D) dy_D dx_D \quad (4.20)$$

$$\lambda_d(x, y) = \int_{x_O=0}^L \int_{y_O=0}^W \lambda(x_O, y_O, x, y) dy_O dx_O \quad (4.21)$$

where F_{AW} denotes the total walking time for patrons within the squared walk-only zone in the catchment area of a transfer station located at (x, y) ; v_W the walking speed; D_W the average walking distance for patrons within the walk-only zone; A_W the size of the squared walk-only zone; and F_{AF} represents the total walking time of patrons in the catchment area but outside the walk-only zone, covering the distance to and from feeder stops. In Eq. (4.17), the sum of F_{AW} and F_{AF} is divided by the catchment area size $S_X(y)S_Y(x)$ to obtain the "density" of walking time at (x, y) , and then integrated over the city to compute the city-wide walking cost. Note that all the following feeder-service cost components are derived in similar manners. In Eq. (4.19), $\frac{S_F(x, y) + B_F(x, y)}{4v_W}$ represents the average walking time per patron to or from the nearest feeder bus stop in the feeder zone at (x, y) ; $(S_Y(x)S_X(y) - A_W) (\lambda_p(x, y) + \lambda_d(x, y))$ indicates the number of patrons choosing feeder transit in the feeder zone at (x, y) . In Eq. (4.20), $\lambda_p(x, y)$ represents the pickup demand density at location (x, y) , indicating the total demand for trips originating from this location to all possible destinations within the service region. In Eq. (4.21), $\lambda_d(x, y)$ represents the drop-off demand density at location (x, y) , indicating the total demand for trips destined to this location from all possible origins within the service region.

(ii) Waiting and transfer time

$$F_W = \int_{x=0}^L \int_{y=0}^W \frac{1}{S_X(y)S_Y(x)} (F_{Wp} + F_{Wd}) dy dx \quad (4.22)$$

$$F_{Wp} = \left(\frac{H_F(x, y)}{2} + t_{f-t} \right) (S_Y(x)S_X(y) - A_W) \lambda_p(x, y) \quad (4.23)$$

$$F_{Wd} = \left(\frac{H_F(x, y)}{2} + t_{t-f} \right) (S_Y(x)S_X(y) - A_W) \lambda_d(x, y) \quad (4.24)$$

where F_{Wp} and F_{Wd} denote the total waiting and transfer times for outbound and inbound service in the catchment area of a transfer station located at (x, y) , respectively; t_{f-t} and t_{t-f} represent the transfer penalties per patron from feeder stops to trunk-line stations and vice versa, respectively. The average time for each patron waiting for a feeder vehicle is given by $H_F(x, y)/2$.

(iii) In-vehicle travel time on feeder lines

$$F_T = \int_{x=0}^L \int_{y=0}^W \frac{1}{S_X(y)S_Y(x)} (F_{Tt} + F_{Td}) dy dx \quad (4.25)$$

$$F_{Tt} = \frac{1}{v_F} \left(\frac{S_X(y) + S_Y(x)}{4} \cdot S_Y(x)S_X(y) - A_W D_W \right) (\lambda_p(x, y) + \lambda_d(x, y)) \quad (4.26)$$

$$F_{Td} = \frac{\tau_F S_X(y)}{4B_F(x, y)} (S_Y(x)S_X(y) - A_W) (\lambda_p(x, y) + \lambda_d(x, y)) \quad (4.27)$$

where F_{Tt} denotes the total in-vehicle cruise time required to overcome the travel distance; F_{Td} the total dwell time incurred by feeder buses; τ_F the feeder transit dwell time per stop; and v_F the cruising speed of feeder transit. Eq. (4.26) is derived as follows: the total travel distance for patrons in the feeder zone at (x, y) is given by $\frac{S_X(y) + S_Y(x)}{4} \cdot S_X(y)S_Y(x) (\lambda_p(x, y) + \lambda_d(x, y))$. For patrons located within the walk-only zone, the total access distance is $A_W D_W (\lambda_p(x, y) + \lambda_d(x, y))$. Thus, the in-vehicle travel distance in the feeder zone is $\left(\frac{S_X(y) + S_Y(x)}{4} \cdot \frac{S_X(y)S_Y(x)}{4} - A_W D_W \right) \cdot (\lambda_p(x, y) + \lambda_d(x, y))$.

In Eq. (4.27), the term $\frac{S_X(y)}{4B_F(x, y)}$ represents the average number of feeder stops visited per patron in the feeder zone at (x, y) . This estimate may slightly overstate the true number of visited stops, as no bus stops are required within the walk-only zone.

4.2.2.2 Agency cost

In the case of the feeder transit system, the agency incurs similar types of costs as the trunk line: (i) line infrastructure costs, F_L ; (ii) stop infrastructure costs, F_S ; (iii) operating costs associated with vehicle-km traveled, F_K ; and (iv) costs related to the bus fleet, F_H . The formulations for these costs are as follows.

(i) Line infrastructure cost

$$F_L = \frac{4\pi_{lF}}{\mu} \int_{x=0}^L \int_{y=0}^W \frac{1}{S_X(y)S_Y(x)} \left(\frac{S_X(y)}{2} + \frac{S_Y(x)}{4} \right) \left(\frac{2S_Y(x)}{2S_F(x, y)} \right) dy dx \quad (4.28)$$

where π_{lF} (\$/km/h) denotes the amortized feeder-line infrastructure cost per hour. Here the term $\left(\frac{S_X(y)}{2} + \frac{S_Y(x)}{4} \right)$ represents average length of a feeder line in the feeder zone at (x, y) , and $\left(\frac{2S_Y(x)}{2S_F(x, y)} \right)$ represents the number of feeder lines serving both directions in a quarter feeder zone at (x, y) . Note that the integral is multiplied by 4 to account for all four quarters.

(ii) Stop infrastructure cost

$$F_S = \frac{4\pi_{sF}}{\mu} \int_{x=0}^L \int_{y=0}^W \frac{1}{S_X(y)S_Y(x)} \left(\frac{S_Y(x)}{2S_F(x, y)} \right) \left(\frac{S_X(y)}{2B_F(x, y)} \right) dy dx \quad (4.29)$$

where π_{sF} (\$/stop/h) denotes the amortized feeder stop infrastructure cost per hour.

(iii) Distance-based operating cost

$$\begin{aligned}
F_K &= \frac{4\pi_{kF}}{\mu} \int_{x=0}^L \int_{y=0}^W \frac{1}{S_X(y)S_Y(x)} \left(\frac{S_X(y)}{2} + \frac{S_Y(x)}{4} \right) \\
&\quad \cdot \frac{S_Y(x)}{2S_F(x,y)} \left(\frac{1}{H_F(x,y)} + \frac{1}{H_F(x,y)} \right) dy dx \\
&= \frac{\pi_{kF}}{\mu} \int_{x=0}^L \int_{y=0}^W \frac{2S_X(y) + S_Y(x)}{S_X(y)S_F(x,y)H_F(x,y)} dy dx
\end{aligned} \tag{4.30}$$

where π_{kF} (\$/km/h) denotes the unit cost per vehicle-km traveled on the feeder-line. The derivation follows the same logic as Eq. (4.25).

(iv) Time-based operating cost

$$F_H = \frac{4\pi_{hF}}{\mu} \int_{x=0}^L \int_{y=0}^W \frac{1}{S_X(y)S_Y(x)} (F_{Ht} + F_{Hd}) dy dx \tag{4.31}$$

$$\begin{aligned}
F_{Ht} &= \left(\frac{S_X(y)}{2} + \frac{S_Y(x)}{4} \right) \cdot \frac{S_Y(x)}{2S_F(x,y)} \cdot \frac{2}{H_F(x,y)} \cdot \frac{1}{v_F} \\
&= \frac{S_Y(x)(2S_X(y) + S_Y(x))}{4v_F S_F(x,y)H_F(x,y)}
\end{aligned} \tag{4.32}$$

$$\begin{aligned}
F_{Hd} &= \frac{\tau_F S_Y(x)}{2S_F(x,y)} \cdot \frac{S_X(y)}{2B_F(x,y)} \cdot \frac{2}{H_F(x,y)} \\
&= \frac{\tau_F S_Y(x)S_X(y)}{2S_F(x,y)B_F(x,y)H_F(x,y)}
\end{aligned} \tag{4.33}$$

where π_{hF} (\$/vehicle·h) denotes the unit cost per vehicle-hour traveled on feeder lines; F_{Ht} is the total bus cruising time in a quarter feeder zone per hour; and F_{Hd} represents the total bus dwell time in a quarter feeder zone per hour. The term $\frac{S_X(y)}{2B_F(x,y)}$ indicates the number of stops per feeder line though it may be slightly overestimated due to the absence of stops within the walk-only zone.

4.2.3 Optimization model

The generalized cost can be expressed as the sum of all the above cost components:

$$GC = T_W + T_R + T_T + T_L + T_S + T_K + T_H + F_A + F_W + F_T + F_K + F_H \tag{4.34}$$

The optimization problem is formulated as follows:

$$\min \quad \text{GC} \tag{4.35a}$$

$$\text{s.t.} \quad Q_{X+}(y) \cdot S_X(y) \cdot H_X(y) \leq K_T \tag{4.35b}$$

$$Q_{X-}(y) \cdot S_X(y) \cdot H_X(y) \leq K_T \tag{4.35c}$$

$$Q_{Y+}(x) \cdot S_Y(x) \cdot H_Y(x) \leq K_T \tag{4.35d}$$

$$Q_{Y-}(x) \cdot S_Y(x) \cdot H_Y(x) \leq K_T \tag{4.35e}$$

$$\frac{\lambda_p(x, y) \cdot S_F(x, y) \cdot H_F(x, y) \cdot S_X(y)}{2} \leq K_F \tag{4.35f}$$

$$\frac{\lambda_d(x, y) \cdot S_F(x, y) \cdot H_F(x, y) \cdot S_X(y)}{2} \leq K_F \tag{4.35g}$$

$$H_{T \min} \leq H_X(y) \leq H_{T \max} \tag{4.35h}$$

$$H_{T \min} \leq H_Y(x) \leq H_{T \max} \tag{4.35i}$$

$$H_{F \min} \leq H_F(x, y) \leq H_{F \max} \tag{4.35j}$$

$$0 \leq S_X(y) \leq W \tag{4.35k}$$

$$0 \leq S_Y(x) \leq L \tag{4.35l}$$

$$0 \leq S_F(x, y) \leq \frac{S_Y(x)}{2} \tag{4.35m}$$

$$0 \leq B_F(x, y) \leq \frac{S_X(y)}{2} \tag{4.35n}$$

where $Q_{X+}(y)$, $Q_{X-}(y)$, $Q_{Y+}(x)$, and $Q_{Y-}(x)$ represent the patron flows in the positive x -direction, negative x -direction, positive y -direction, and negative y -direction, respectively. Detailed formulations are provided in Appendix J. K_T and K_F denote the capacities of trunk-line and feeder-line vehicles. $H_{T \min}$, $H_{T \max}$, $H_{F \min}$, and $H_{F \max}$ represent the minimum and maximum allowable headways for trunk and feeder services, respective. Constraints (4.35b)–(4.35e) enforce the trunk-line vehicle capacity, (4.35f)–(4.35g) enforce the feeder-line capacity, (4.35h)–(4.35j) set bounds on the headways, and (4.35k)–(4.35n) constrain the line and stop spacings.

4.3 Solution Method

Section 4.3.1 derives the properties characterizing an optimal solution. Section 4.3.2 presents a heuristic solution algorithm built upon these properties.

4.3.1 Optimality properties

Combining all the cost metrics, we can rewrite the generalized cost as:

$$\text{GC} = \Lambda \delta_T$$

$$\begin{aligned}
& + \int_{x=0}^L \left[\frac{1}{4} \Lambda_{OY}(x) H_Y(x) + \frac{1}{4} \Lambda_{YD}(x) H_Y(x) + \frac{Q_X(x)}{v_T} + \frac{Q_X(x) \tau_T}{S_Y(x)} \right. \\
& + \left. \frac{2\pi_{lT} W}{\mu S_Y(x)} + \frac{2\pi_{kT} W}{\mu S_Y(x) H_Y(x)} + \frac{2\pi_{hT} W}{\mu v_T S_Y(x) H_Y(x)} \right] dx \\
& + \int_{y=0}^W \left[\frac{1}{4} \Lambda_{OX}(y) H_X(y) + \frac{1}{4} \Lambda_{XD}(y) H_X(y) + \frac{Q_Y(y)}{v_T} + \frac{Q_Y(y) \tau_T}{S_X(y)} \right. \\
& + \left. \frac{2\pi_{lT} L}{\mu S_X(y)} + \frac{2\pi_{kT} L}{\mu S_X(y) H_X(y)} + \frac{2\pi_{hT} L}{\mu v_T S_X(y) H_X(y)} \right] dy \\
& + \int_{x=0}^L \int_{y=0}^W \left[\frac{\pi_{sT}}{\mu S_Y(x) S_X(y)} + \frac{2\pi_{hT} \tau_T}{\mu S_Y(x) H_Y(x) S_X(y)} + \frac{2\pi_{hT} \tau_T}{\mu S_Y(x) S_X(y) H_X(y)} \right. \\
& + \left. \frac{4D_W A_W (\lambda_p(x, y) + \lambda_d(x, y))}{v_W S_X(y) S_Y(x)} \right. \\
& + \left. \left(\frac{S_F(x, y) + B_F(x, y)}{4v_W} - \frac{(S_F(x, y) + B_F(x, y)) A_W}{v_W S_X(y) S_Y(x)} \right) (\lambda_p(x, y) + \lambda_d(x, y)) \right. \\
& + \left(\frac{H_F(x, y)}{2} + t_{f-t} \right) \left(1 - \frac{4A_W}{S_X(y) S_Y(x)} \right) \lambda_p(x, y) \\
& + \left(\frac{H_F(x, y)}{2} + t_{t-f} \right) \left(1 - \frac{4A_W}{S_X(y) S_Y(x)} \right) \lambda_d(x, y) \\
& + \frac{\lambda_p(x, y)}{4} (H_Y(x) + H_X(y)) \\
& + \left(\frac{S_X(y) + S_Y(x)}{4v_F} - \frac{4A_W D_W}{S_X(y) S_Y(x) v_F} \right) (\lambda_p(x, y) + \lambda_d(x, y)) \\
& + \left(\frac{\tau_F S_X(y)}{4B_F(x, y)} - \frac{A_W \tau_F}{S_Y(x) B_F(x, y)} \right) (\lambda_p(x, y) + \lambda_d(x, y)) \\
& + \frac{\pi_{sF}}{\mu S_F(x, y) B_F(x, y)} + \frac{2\pi_{kF}}{\mu S_F(x, y) H_F(x, y)} + \frac{\pi_{kF} S_Y(x)}{\mu S_X(y) S_F(x, y) H_F(x, y)} \\
& + \frac{2\pi_{hF}}{\mu v_F S_F(x, y) H_F(x, y)} + \frac{\pi_{hF} S_Y(x)}{\mu S_X(y) v_F S_F(x, y) H_F(x, y)} \\
& + \left. \frac{2\pi_{hF} \tau_F}{\mu S_F(x, y) B_F(x, y) H_F(x, y)} \right] dy dx
\end{aligned} \tag{4.36}$$

The cost terms can be grouped into three parts. The first part of the objective function is constant. The second part contains single integrals relating to either x or y . The third part contains double integrals related to both x and y .

By taking first order derivatives with respect to individual decision functions and considering the boundary constraints (4.35b)–(4.35n), the optimal solution to our GC minimization problem must satisfy the following analytical properties for any $x \in [0, L]$ and $y \in [0, W]$.

$$S_X^*(y) = \min \left\{ \hat{S}_X(y), \frac{K_T}{Q_{WE}(y)H_X(y)}, \frac{K_T}{Q_{EW}(y)H_X(y)}, \frac{2K_F}{\lambda_p(x, y)S_F(x, y)H_F(x, y)}, \frac{2K_F}{\lambda_d(x, y)S_F(x, y)H_F(x, y)}, W \right\} \quad (4.37a)$$

$$S_Y^*(x) = \min \left\{ \hat{S}_Y(x), \frac{K_T}{Q_{SN}(x)H_Y(x)}, \frac{K_T}{Q_{NS}(x)H_Y(x)}, L \right\} \quad (4.37b)$$

$$H_X^*(y) = \text{mid} \left\{ H_{T \min}, \hat{H}_X(y), \min \left\{ \frac{K_T}{Q_{WE}(y)S_X(y)}, \frac{K_T}{Q_{EW}(y)S_X(y)}, H_{T \max} \right\} \right\} \quad (4.37c)$$

$$H_Y^*(x) = \text{mid} \left\{ H_{T \min}, \hat{H}_Y(x), \min \left\{ \frac{K_T}{Q_{SN}(x)S_Y(x)}, \frac{K_T}{Q_{NS}(x)S_Y(x)}, H_{T \max} \right\} \right\} \quad (4.37d)$$

$$S_F^*(x, y) = \min \left\{ \hat{S}_F(x, y), \frac{2K_F}{\lambda_p(x, y)H_F(x, y)S_X(y)}, \frac{2K_F}{\lambda_d(x, y)H_F(x, y)S_X(y)}, \frac{S_Y(x)}{2} \right\} \quad (4.37e)$$

$$H_F^*(x, y) = \text{mid} \left\{ H_{F \min}, \hat{H}_F(x, y), \min \left\{ \frac{2K_F}{\lambda_p(x, y)S_F(x, y)S_X(y)}, \frac{2K_F}{\lambda_d(x, y)S_F(x, y)S_X(y)}, H_{F \max} \right\} \right\} \quad (4.37f)$$

$$B_F^*(x, y) = \min \left\{ \hat{B}_F(x, y), \frac{S_X(y)}{2} \right\} \quad (4.37g)$$

where notations with hats denote solutions that satisfy the first-order optimality condition, as shown below.

$$\begin{aligned}
\hat{S}_X(y) = & \sqrt{ \frac{Q_Y(y)\tau_T + \frac{2L}{\mu} \left(\pi_{lT} + \frac{\pi_{kT}}{H_X(y)} + \frac{\pi_{hT}}{v_T H_X(y)} \right) + \left(\frac{\pi_{sT}}{\mu} + \frac{2\pi_{hT}\tau_T}{\mu H_X(y)} \right) \int_{x=0}^L \frac{1}{S_Y(x)} dx + \frac{2\pi_{hT}\tau_T}{\mu} \int_{x=0}^L \frac{1}{S_Y(x)H_Y(x)} dx + \left(\frac{1}{v_W} - \frac{1}{v_F} \right) 4D_W A_W \int_{x=0}^L \frac{\lambda_p(x,y) + \lambda_d(x,y)}{S_Y(x)} dx - \frac{A_W}{v_W} \int_{x=0}^L \frac{(S_F(x,y) + B_F(x,y))(\lambda_p(x,y) + \lambda_d(x,y))}{S_Y(x)} dx - 4A_W \left(\int_{x=0}^L \frac{(H_F(x,y)/2 + t_{f-t})\lambda_p(x,y)}{S_Y(x)} dx + \int_{x=0}^L \frac{(H_F(x,y)/2 + t_{t-f})\lambda_d(x,y)}{S_Y(x)} dx \right) + \left(\frac{\pi_{kF}}{\mu} + \frac{\pi_{hF}}{\mu v_F} \right) \int_{x=0}^L \frac{S_Y(x)}{S_F(x,y)H_F(x,y)} dx }{ \frac{1}{4v_F} \int_{x=0}^L (\lambda_p(x,y) + \lambda_d(x,y)) dx + \frac{\tau_F}{4} \int_{x=0}^L \frac{\lambda_p(x,y) + \lambda_d(x,y)}{B_F(x,y)} dx } } \quad (4.38a)
\end{aligned}$$

$$\begin{aligned}
\hat{S}_Y(x) = & \sqrt{ \frac{Q_X(x)\tau_T + \frac{2W}{\mu} \left(\pi_{lT} + \frac{\pi_{kT}}{H_Y(x)} + \frac{\pi_{hT}}{v_T H_Y(x)} \right) + \left(\frac{\pi_{sT}}{\mu} + \frac{2\pi_{hT}\tau_T}{\mu H_Y(x)} \right) \int_{y=0}^W \frac{1}{S_X(y)} dy + \frac{2\pi_{hT}\tau_T}{\mu} \int_{y=0}^W \frac{1}{S_X(y)H_X(y)} dy + \left(\frac{1}{v_W} - \frac{1}{v_F} \right) 4D_W A_W \int_{y=0}^W \frac{\lambda_p(x,y) + \lambda_d(x,y)}{S_X(y)} dy - \frac{A_W}{v_W} \int_{y=0}^W \frac{(S_F(x,y) + B_F(x,y))(\lambda_p(x,y) + \lambda_d(x,y))}{S_X(y)} dy - 4A_W \left(\int_{y=0}^W \frac{(H_F(x,y)/2 + t_{f-t})\lambda_p(x,y)}{S_X(y)} dy + \int_{y=0}^W \frac{(H_F(x,y)/2 + t_{t-f})\lambda_d(x,y)}{S_X(y)} dy \right) - A_W \tau_F \int_{y=0}^W \frac{\lambda_p(x,y) + \lambda_d(x,y)}{B_F(x,y)} dy }{ \frac{1}{4v_F} \int_{y=0}^W (\lambda_p(x,y) + \lambda_d(x,y)) dy + \left(\frac{\pi_{kF}}{\mu} + \frac{\pi_{hF}}{\mu v_F} \right) \int_{y=0}^W \frac{1}{S_X(y)S_F(x,y)H_F(x,y)} dy } } \quad (4.38b)
\end{aligned}$$

$$\hat{H}_X(y) = \sqrt{\frac{\frac{2}{\mu S_X(y)} \left(\pi_{kT} L + \frac{\pi_{hT} L}{v_T} + \pi_{hT} \tau_T \int_{x=0}^L \frac{dx}{S_Y(x)} \right)}{\frac{1}{4} \left(\Lambda_{OX}(y) + \Lambda_{DX}(y) + \int_{x=0}^L \lambda_p(x, y) dx \right)}} \quad (4.38c)$$

$$\hat{H}_Y(x) = \sqrt{\frac{\frac{2}{\mu S_Y(x)} \left(\pi_{kT} W + \frac{\pi_{hT} W}{v_T} + \pi_{hT} \tau_T \int_{y=0}^W \frac{dy}{S_X(y)} \right)}{\frac{1}{4} \left(\Lambda_{OY}(x) + \Lambda_{DY}(x) + \int_{y=0}^W \lambda_p(x, y) dy \right)}} \quad (4.38d)$$

$$\hat{S}_F(x, y) = \sqrt{\frac{\frac{1}{\mu} \left(\frac{\pi_{sF}}{B_F(x, y)} + \frac{2\pi_{kF}}{H_F(x, y)} + \frac{\pi_{kF} S_Y(x)}{S_X(y) H_F(x, y)} + \frac{2\pi_{hF}}{v_F H_F(x, y)} + \frac{\pi_{hF} S_Y(x)}{S_X(y) v_F H_F(x, y)} + \frac{2\pi_{hF} \tau_F}{B_F(x, y) H_F(x, y)} \right)}{\frac{\lambda_p(x, y) + \lambda_d(x, y)}{v_W} \left(\frac{1}{4} - \frac{A_W}{S_X(y) S_Y(x)} \right)}} \quad (4.38e)$$

$$\hat{H}_F(x, y) = \sqrt{\frac{\frac{1}{\mu S_F(x, y)} \left(2\pi_{kF} + \frac{\pi_{kF} S_Y(x)}{S_X(y)} + \frac{2\pi_{hF}}{v_F} + \frac{\pi_{hF} S_Y(x)}{S_X(y) v_F} + \frac{2\pi_{hF} \tau_F}{B_F(x, y)} \right)}{\frac{\lambda_p(x, y) + \lambda_d(x, y)}{2} \left(1 - \frac{4A_W}{S_X(y) S_Y(x)} \right)}} \quad (4.38f)$$

$$\hat{B}_F(x, y) = \sqrt{\frac{\left(\frac{\tau_F S_X(y)}{4} - \frac{A_W \tau_F}{S_Y(x)} \right) (\lambda_p(x, y) + \lambda_d(x, y)) + \frac{1}{\mu S_F(x, y)} \left(\pi_{sF} + \frac{2\pi_{hF} \tau_F}{H_F(x, y)} \right)}{\frac{(\lambda_p(x, y) + \lambda_d(x, y))}{v_W} \left(\frac{1}{4} - \frac{A_W}{S_X(y) S_Y(x)} \right)}} \quad (4.38g)$$

Based on the above analytical properties, several key insights emerge regarding the interdependencies between trunk and feeder system design variables. Firstly, Eq. (4.38g) reveals that when feeder lines are sparse (large $S_F(x, y)$) or service frequency is low (large $H_F(x, y)$), the model prescribes denser stops to maintain service quality. Additionally, stop spacing increases with the spacings of X-lines, $S_X(y)$, which aligns with our design where feeder lines are laid out along the Y-direction, allowing better control of dwell time costs across larger catchment areas.

Secondly, the frequent occurrence of the ratio $\frac{S_Y(x)}{S_X(y)}$ in Eqs.(4.38e) and(4.38f) highlights the cost sensitivity associated with spatial asymmetry in service regions. Since

$S_Y(x)$ appears in the numerator, whereas $S_X(y)$ occupies the denominator, elongated service regions in the x -direction necessitate sparser feeder line configurations and longer headways to mitigate distance-related operational costs. Furthermore, $S_Y(x)$ plays a direct role in determining feeder line lengths, thereby exerting a more pronounced influence on overall system cost. In contrast, $S_X(y)$ primarily governs stop density, with comparatively milder cost implications. Moreover, the expansion of walk-only zones leads to increased feeder line spacing and headways, as reduced feeder ridership makes it more cost-effective to operate services with larger spacing and lower frequencies in areas with lower demand.

In addition, examining Eqs. (4.37c) and (4.37d) reveals that trunk headways remain largely independent of feeder system design, being primarily determined by trunk line spacing—larger spacing induces higher service frequencies to maintain capacity, while denser lines allows lower frequencies. However, the denominators in Eqs. (4.38a) and (4.38b) expose that $S_Y(x)$ is influenced by feeder line spacing, while $S_X(y)$ responds more to feeder stop spacing. This directional specificity stems from the geometric structure of our model, where feeder line spacings influence the spacings of Y-lines, and feeder stop spacings affects the spacings of X-lines.

4.3.2 A heuristic iterative solution method

The optimization problem presented above is solved using a heuristic iterative algorithm. The detailed solution procedure is outlined in the following steps:

Stage 1: Initialization.

Discretize the service region $L \times W$ into an $m \times n$ lattice of points, setting $x_i = (i - 0.5)L/n$, $i = 1, 2, \dots, n$, and $y_j = (j - 0.5)W/m$, $j = 1, 2, \dots, m$.

Initialize $S_Y^{(0)}(x_i)$, $H_Y^{(0)}(x_i)$, $S_X^{(0)}(y_j)$, $H_X^{(0)}(y_j)$, $S_F^{(0)}(x_i, y_j)$, and $H_F^{(0)}(x_i, y_j)$ to satisfy the boundary constraints for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

Stage 2: Compute $B_F(x, y)$, $H_F(x, y)$, and $S_F(x, y)$.

For $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$: Calculate $B_F^{(k)}(x_i, y_j)$ using Eq. (4.37g), taking $S_Y^{(k-1)}(x_i)$, $H_Y^{(k-1)}(x_i)$, $S_X^{(k-1)}(y_j)$, $H_X^{(k-1)}(y_j)$, $S_F^{(k-1)}(x_i, y_j)$, and $H_F^{(k-1)}(x_i, y_j)$ as input. Calculate $S_F^{(k)}(x_i, y_j)$ using Eq. (4.37e). Calculate $H_F^{(k)}(x_i, y_j)$ using Eq. (4.37f).

Stage 3: Compute $H_X(y)$, $H_Y(x)$, $S_X(y)$, and $S_Y(x)$.

For $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$: Calculate $H_X^{(k)}(y_j)$ using Eq. (4.37c). Calculate $H_Y^{(k)}(x_i)$ using Eq. (4.37d). Calculate $S_X^{(k)}(y_j)$ using Eq. (4.37a). Calculate $S_Y^{(k)}(x_i)$ using Eq. (4.37b).

Stage 4: Check convergence.

If the following conditions are all satisfied:

$$\begin{aligned}
|B_F^{(k+1)}(x_i, y_j) - B_F^{(k)}(x_i, y_j)| &\leq \varepsilon, & |S_F^{(k+1)}(x_i, y_j) - S_F^{(k)}(x_i, y_j)| &\leq \varepsilon, \\
|H_F^{(k+1)}(x_i, y_j) - H_F^{(k)}(x_i, y_j)| &\leq \varepsilon, & |H_Y^{(k+1)}(x_i) - H_Y^{(k)}(x_i)| &\leq \varepsilon, \\
|H_X^{(k+1)}(y_j) - H_X^{(k)}(y_j)| &\leq \varepsilon, & |S_Y^{(k+1)}(x_i) - S_Y^{(k)}(x_i)| &\leq \varepsilon, \\
|S_X^{(k+1)}(y_j) - S_X^{(k)}(y_j)| &\leq \varepsilon,
\end{aligned}$$

then the solution has converged. Otherwise, set $k \leftarrow k + 1$ and return to Stage 2.

4.4 Numerical Case Studies

Section 4.4.1 presents the setup of the numerical case studies. Section 4.4.2 discusses the optimal system designs under various demand patterns and service modes. Section 4.4.3 compares the performance of our proposed model with those of previous benchmark studies.

4.4.1 Experimental setup

Our models are compatible with arbitrary continuous demand functions. For illustrative purposes, the numerical case studies in this part adopt a specific class of demand functions, as defined in Eq. (4.39), which follows the same functional form as that used in Ouyang et al. (2014). Two representative demand patterns are tested: a mono-centric city and a commuter city, as illustrated in Fig. 4.2a and 4.2b. The corresponding parameter settings for each demand scenario are summarized in 4.1.

$$\lambda(x_O, y_O, x_D, y_D) = \prod_{i=1}^2 \left(a_1 + a_2 \sum_{j=1}^2 \exp \left[-(a_{3j}x_i - b_{ij})^2 - (a_{4j}y_i - \bar{b}_{ij})^2 \right] \right) \quad (4.39)$$

Table 4.1: Parameter values for different demand patterns

Case	a_1	a_2	a_{31}	a_{32}	a_{41}	a_{42}	b_{11}	b_{12}	b_{21}	b_{22}	$\bar{b}_{11}$	$\bar{b}_{12}$	$\bar{b}_{21}$	$\bar{b}_{22}$
Mono-centric city	0.0016	1.4	0.25	0	0.25	0	2.5	0	2.5	0	2.5	0	2.5	0
Commuter city	0.44	1.783	0.5	0	0.5	0	3	0.5	6	0.5	6	0.5	3	0.5

Numerical instances in this section use the parameter values listed in Tables 4.2 and 4.3 (Gu et al., 2016; Chen and Nie, 2017; Mei et al., 2021; Zhen et al., 2025) unless otherwise specified. The service region is set to $20 \times 20 \text{ km}^2$, which is representative of

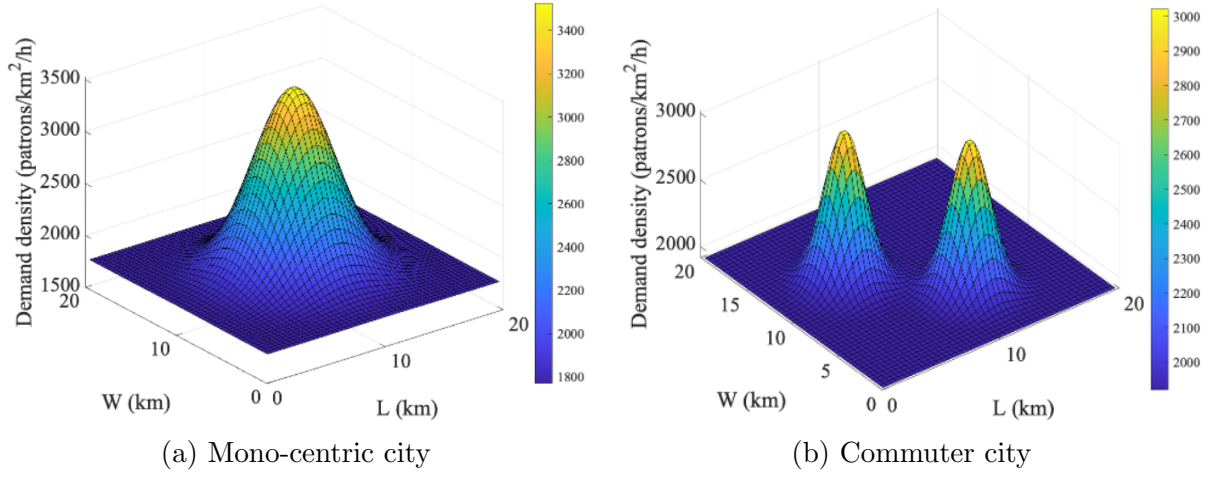


Figure 4.2: Demand distributions

typical urban areas where trunk-feeder transit systems are applicable. The length of the walking zone is set to 500 m, reflecting common practice in real-world transit planning. The value of time is assumed to be \$20/h, and the average walking speed is set to 2 km/h, following [Daganzo \(2010\)](#).

Table 4.2: Parameter values of trunk services

Mode	K_T (patrons/veh)	π_{lT} (\$/km)	π_{sT} (\$/station)	π_{kT} (\$/veh·km)	π_{hT} (\$/veh·h)
Rail	3000	$594 + 19.8\mu$	$294 + 9.8\mu$	2.2	$101 + 5\mu$
Bus	80	$6 + 0.2\mu$	$0.42 + 0.014\mu$	0.59	$2.66 + 3\mu$
	v_T (km/h)	τ_T (s)	$H_{\min}$ (h)	$H_{\max}$ (h)	δ_T (s)
Rail	60	45	1.5/60	1	60
Bus	25	30	1/60	1	60

Table 4.3: Parameter values of feeder service

K_T (patrons/veh)	π_{lT} (\$/km)	π_{sT} (\$/station)	π_{kT} (\$/veh·km)	π_{hT} (\$/veh·h)
10	0	0	$0.0314 + 0.0039K$	$2.068 + 0.108K + 2\mu$
v_T (km/h)	τ_F (s)	$H_{\min}$ (h)	$H_{\max}$ (h)	t_{f-t}/t_{t-f} (min)
20	15	3/60	1	3

Regarding the hyperparameters of the solution algorithm, we find that using a discretization resolution of $m = 100$ and $n = 100$ provides sufficiently accurate numerical solutions. The error tolerance ε is set to 0.0001. Each numerical instance is solved using multiple combinations of initial values, all of which consistently converge to the same optimal solution. A typical instance is solved within 90 seconds.

4.4.2 Optimal System Designs

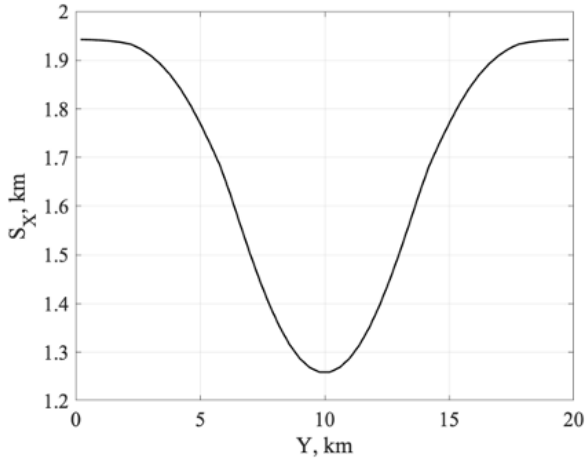
4.4.2.1 Optimal system design under different demand distributions

To illustrate the performance and applicability of our trunk-feeder transit network model under varying spatial demand distributions, this section compares the optimal system configurations under two distinct scenarios: mono-centric city and commuter city, which are shown in Section 4.1. The results are shown in Fig. 4.3.

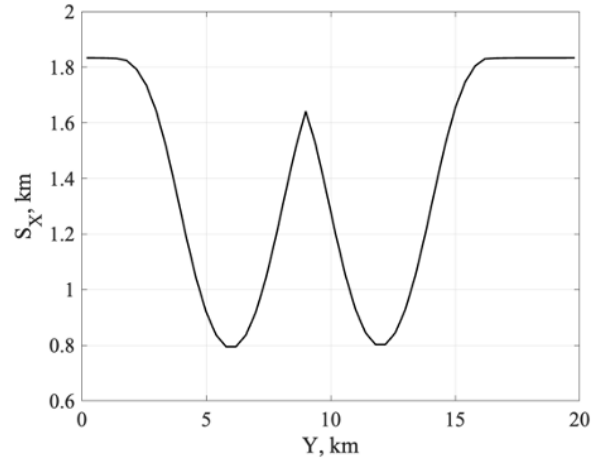
Comparing the results under two types of demand distribution in Fig. 4.5, we observe that the optimal system designs consistently and directly respond to the underlying spatial patterns of patron demand. Specifically, both scenarios demonstrate a clear trend: areas characterized by high travel demand are provided with denser services, as indicated by smaller spacings of X-lines (i.e., $S_X(y)$), higher service frequencies, and more closely spaced feeder lines and stops, accompanied by shorter feeder headways. Conversely, lower-demand regions are served by comparatively sparser services. These findings underscore the robustness of our optimization model and suggest its considerable potential for application across diverse urban contexts with varying demand patterns.

Interestingly, our numerical results reveal an insightful asymmetry in the optimal design of trunk-line spacings and headways. Specifically, as demonstrated in Fig. 4.3a and 4.3c, while the spacings of X-lines ($S_X(y)$) notably decrease in areas of higher demand, indicating denser line arrangements, the spacings of Y-lines ($S_Y(x)$) unexpectedly increase. This seemingly counterintuitive phenomenon arises from the directional coordination intrinsic to our trunk-feeder network design. Feeder line spacings oriented negative X-direction effectively fulfill the demand coverage requirements along the Y-direction through strategically placed, densely spaced feeder stops, thereby allowing the trunk lines in this direction to remain relatively sparse. A similar asymmetry is observed in trunk-line headways (see Figs. 4.3e and 4.4g). This coordinated trunk-feeder configuration underscores our model’s capability to achieve cost-efficient service coverage in high-demand urban areas through deliberate spatial and operational coordination.

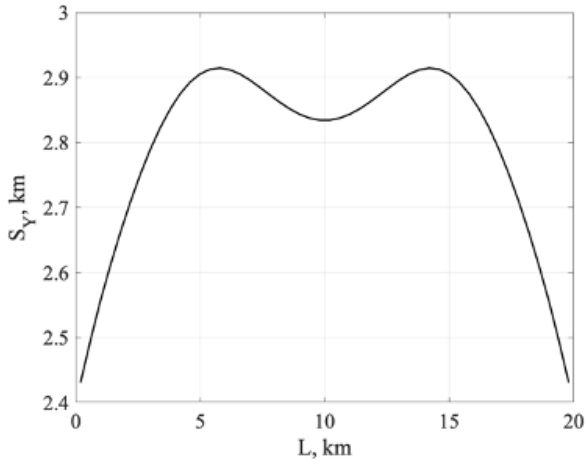
A notable observation from our results is that the spacings of Y-lines ($S_Y(x)$) consistently exceed those of X-lines ($S_X(y)$), resulting in feeder zones that are elongated in the X-direction. This finding aligns with our theoretical insights from Chapter 2, which demonstrated that optimal feeder system design favors operating lines along the shorter side of rectangular service zones to minimize generalized costs. Given our model’s setting that feeder lines operate exclusively in the Y-direction, the optimization naturally produces $S_Y(x) > S_X(y)$ to create service zones with shorter Y-dimensions. This represents the system’s adaptive response to achieve the best cost trade-off for feeder zone shapes under the fixed directional setting.



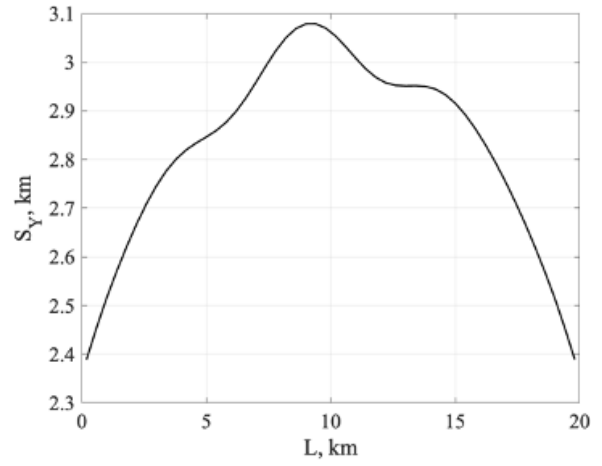
(a) Line spacings between two adjacent X-lines (mono-centric)



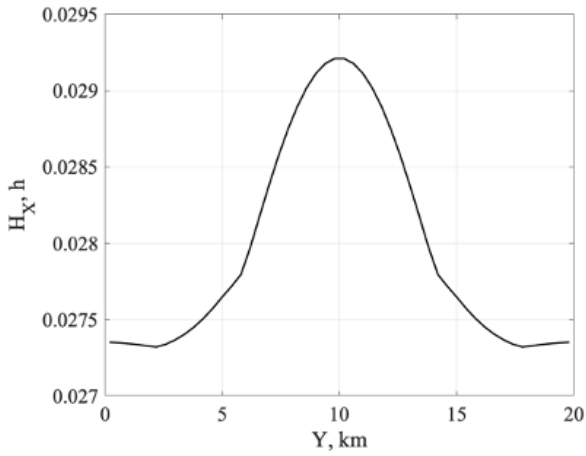
(b) Line spacings between two adjacent X-lines (commuter)



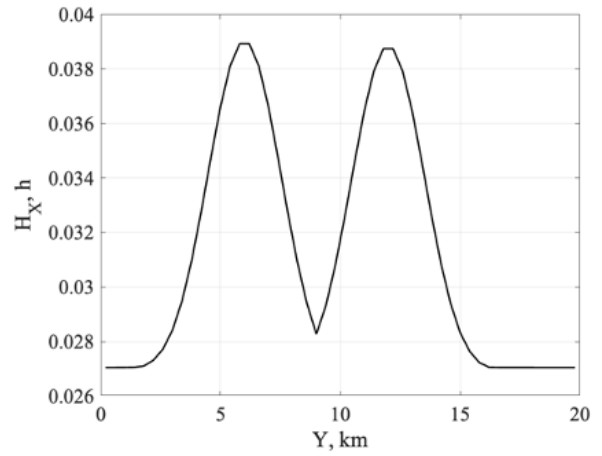
(c) Line spacings between two adjacent Y-lines (mono-centric)



(d) Line spacings between two adjacent Y-lines (commuter)

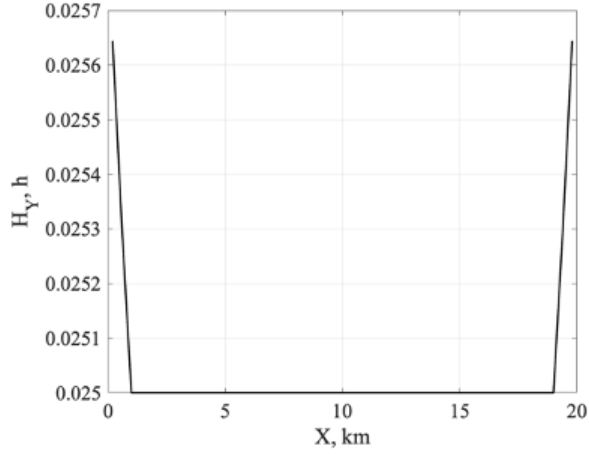


(e) Headways between two adjacent X-lines (mono-centric)

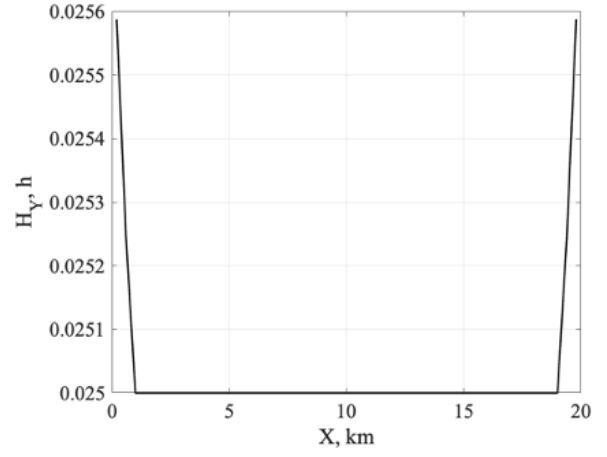


(f) Headways between two adjacent X-lines (commuter)

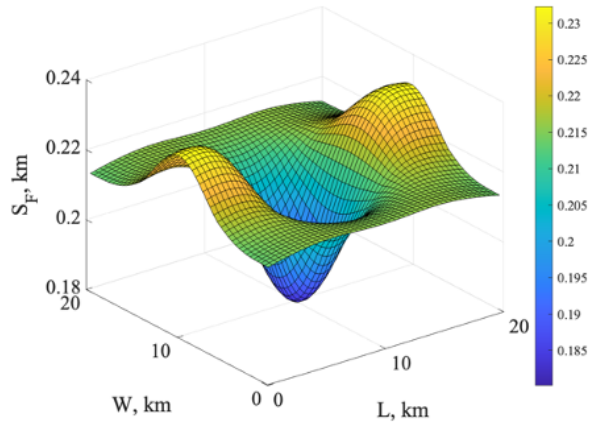
Figure 4.3: Optimal system design parameters (1/3)



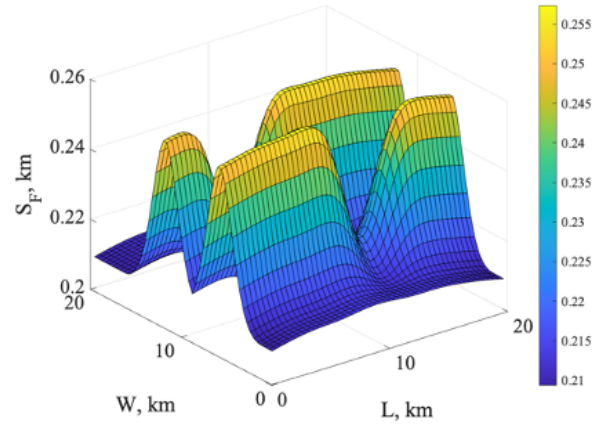
(g) Headways between two adjacent Y-lines (mono-centric)



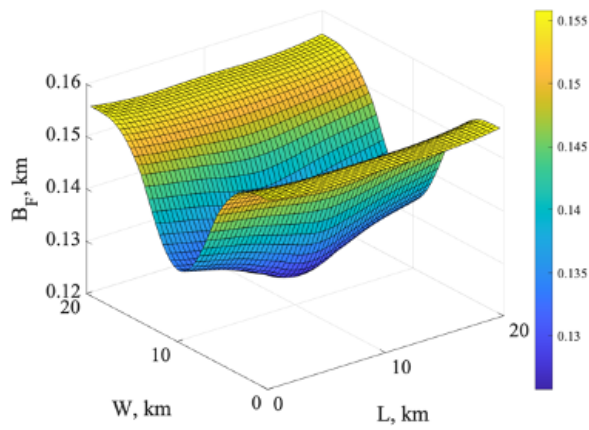
(h) Headways between two adjacent Y-lines (commuter)



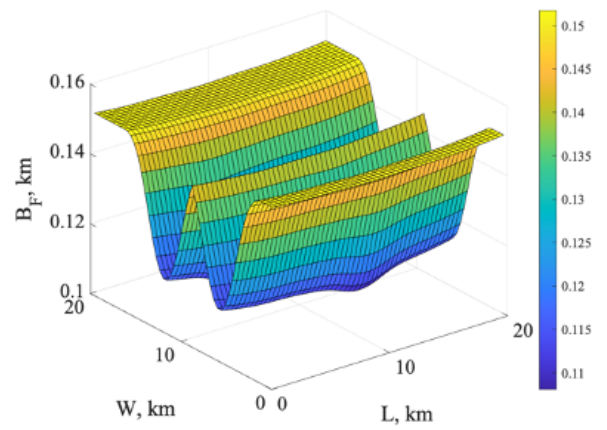
(i) Feeder line spacings (mono-centric)



(j) Feeder line spacings (commuter)



(k) Feeder stop spacings (mono-centric)



(l) Feeder stop spacings (commuter)

Figure 4.4: Optimal system design parameters (2/3)

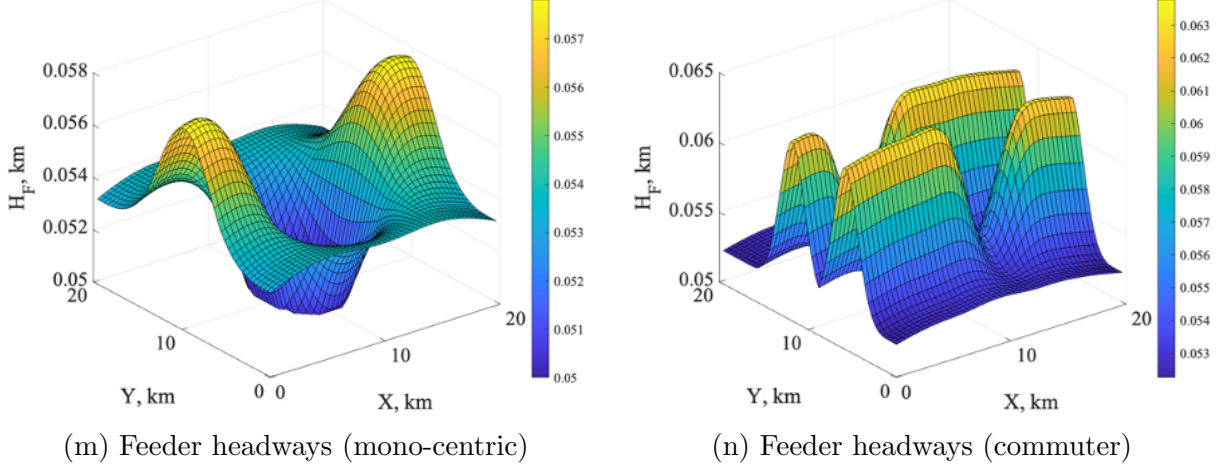


Figure 4.5: Optimal system design parameters (3/3)

4.4.2.2 Optimal Mode Selection Under Different demand levels

To further illustrate the effectiveness of our trunk-feeder transit network model, this section compares two distinct trunk-feeder mode combinations—namely, the rail-feeder bus and bus-feeder bus modes—across varying demand levels. Specifically, we examine a higher-demand scenario (identical to the mono-centric demand distribution presented in Section 4.1 and a lower-demand scenario, with parameters defined in Table 4.4. The optimal system design outcomes under these different scenarios are summarized in Table 4.5.

Table 4.4: Values of demand function parameters (lower demand density)

a_1	a_2	a_{31}	a_{32}	a_{41}	a_{42}	b_{11}	b_{12}	b_{21}	b_{22}	$\bar{b}_{11}$	$\bar{b}_{12}$	$\bar{b}_{21}$	$\bar{b}_{22}$
0.0016	1.4	0.5	0	0.25	0	2.5	0	2.5	0	2.5	0	2.5	0

Table 4.5 illustrates that in the higher-demand scenario, the rail-bus combination achieves a lower generalized cost (0.92 h/patron) compared to the bus-bus system (1.07 h/patron). This advantage primarily results from a substantial reduction in patrons' travel time cost (0.74 h/patron for rail-bus versus 1.00 h/patron for bus-bus). Despite the slightly higher agency cost associated with the rail-bus system (0.17 h/patron compared to 0.06 h/patron for bus-bus), the significantly reduced user cost ensures overall superior performance. The considerable savings in user costs under the rail-bus mode are attributed mainly to larger trunk-line spacings and higher operational speeds of rail services, both of which significantly reduce patrons' in-vehicle travel time. Moreover, our model explicitly incorporates feeder services, enabling patrons to access rail stations despite the greater spacing between rail trunk lines. Although the generalized cost of feeder services in the rail-bus scenario (0.44 h/patron) slightly exceeds that of the bus-bus scenario (0.40 h/patron), this additional feeder cost remains substantially lower than

Table 4.5: Optimal design parameters and cost components of different trunk-feeder service combinations under varying demand levels

Notation	Rail-bus (High)	Bus-bus (High)	Rail-bus (Low)	Bus-bus (Low)
Ave $S_X(y)$, average trunk-line spacings between X-lines (km)	1.68	1.53	3.93	1.96
Ave $S_Y(x)$, average trunk-line spacings between Y-lines (km)	2.80	1.63	5.79	1.61
Ave $H_X(y)$, average trunk-line headways on X-lines (min)	1.68	1.28	2.95	3.17
Ave $H_Y(x)$, average trunk-line headways on Y-lines (min)	1.50	1.25	2.47	3.49
Ave $S_F(x, y)$, average feeder line spacings (km)	0.21	0.21	0.40	0.40
Ave $B_F(x, y)$, average feeder stop spacings (km)	0.15	0.14	0.22	0.18
Ave $H_F(x, y)$, average feeder headways (min)	3.22	3.22	5.92	5.98
GC, generalized cost (h/patron)	0.92	1.07	1.44	1.27
AC, agency cost (h/patron)	0.17	0.06	0.49	0.15
UC, user travel time cost (h/patron)	0.74	1.00	0.95	1.12
GC_T , trunk system generalized cost (h/patron)	0.48	0.67	0.71	0.74
AC_T , trunk system agency cost (h/patron)	0.13	0.02	0.39	0.06
UC_T , trunk system user cost (h/patron)	0.35	0.65	0.33	0.68
GC_F , feeder system generalized cost (h/patron)	0.44	0.40	0.73	0.53
AC_F , feeder system agency cost (h/patron)	0.05	0.04	0.10	0.08
UC_F , feeder system user cost (h/patron)	0.39	0.35	0.63	0.45

the travel time savings realized by utilizing rail trunk lines.

Conversely, in the lower-demand scenario, the bus-bus combination proves to be more cost-effective, achieving a lower generalized cost (1.27 h/patron versus 1.44 h/patron for rail-bus). This improvement is largely attributed to the significantly lower agency cost of bus-bus systems (0.15 h/patron compared to 0.49 h/patron for rail-bus), underscoring the economic advantage of bus mode under reduced demand conditions. Although the patrons' travel time cost is slightly higher in the bus-bus scenario (1.12 h/patron compared to 0.95 h/patron for rail-bus), the substantial savings in agency cost more than compensate for this trade-off.

An examination of feeder service parameters further supports these conclusions. Interestingly, the optimal feeder line spacings and headways remain consistent across both service modes under each demand scenario, indicating that feeder design decisions are primarily responsive to local demand characteristics rather than trunk service type. This finding suggests a robustness and adaptability inherent in feeder design optimization, reinforcing the versatility of feeder systems as complementary components in trunk-feeder network design.

4.4.3 Comparisons with previous designs

In this section, we compare the system cost and its components in our model with those from similar studies by [Ouyang et al. \(2014\)](#) and [Chien and Schonfeld \(1997\)](#). Since both previous studies considered only a single service mode, we focus on the scenario where both trunk and feeder services are bus-based. The parameters chosen are the same as [Ouyang et al. \(2014\)](#). The results are presented in Table 4.6.

Table 4.6: Cost comparison with previous models under the mono-centric city scenario

Cost components	Our model	Ouyang et al. (2014)	Chien and Schonfeld (1997)
Agency cost (h/patron)	0.22	0.21	0.40
User cost (h/patron)	0.96	1.29	1.13
Generalized cost (h/patron)	1.18	1.50	1.53
Average number of transfers	3	1.0009	1

As shown in Table 4.6, our model achieves a 21% cost reduction compared to [Ouyang et al. \(2014\)](#) and a 24% reduction compared to [Chien and Schonfeld \(1997\)](#). By incorporating feeder buses, our model achieves considerably lower access and egress times, as the previous models rely solely on walking for access, and allows for larger trunk-line spacings, leading to notable reductions in in-vehicle travel time. The substantial performance gaps can be attributed to key limitations in the previous approaches: [Chien and Schonfeld \(1997\)](#) lacks a feeder system, forcing all passengers to walk for access and egress,

thereby decreasing system efficiency. While [Ouyang et al. \(2014\)](#) shows improvement, the power-of-two constraint in line spacing design still yields suboptimal solutions. Although our model introduces additional transfers compared to previous models (in some cases nearly three times as many), the system maintains a stable cost advantage, yielding an average saving of 20 minutes per patron. Moreover, these benefits will be further enhanced by opportunities for schedule coordination during peak hours, where feeder services can be aligned with the dominant trunk-line direction to mitigate transfer-related inconvenience.

Chapter 5

Conclusions

Section 5.1 summarizes the main contributions of this thesis. Section 5.2 discusses potential future research directions.

5.1 Contributions

The first main contribution is that we propose continuous approximation models for designing an optimal heterogeneous fixed-route feeder transit network. These models identify the optimal feeder service headways, line spacings, stop spacings, and vehicle size that minimize the generalized system cost. The models are solved by an iterative method that leverages the analytical properties of the optimal solution. Our models stand out by considering the effects of passenger boarding and alighting on bus dwell times and transfer times, while also optimizing heterogeneous stop spacings alongside line spacings and headways. This added complexity not only enhances model accuracy in reflecting real-world scenarios but also yields considerable cost-saving benefits. Refined models of dwell and transfer times improves the fidelity of associated cost items by up to 5%, while optimizing heterogeneous stop spacings boosts system performance by up to 4% compared to uniform stop spacing, with potential cost savings exceeding 10% when compared to models that only optimize line spacings and headways. Additionally, integrating schedule coordination reveals even greater benefits—cost savings up to 20%—when trunk and feeder services are coordinated in both feeder travel directions.

The research yields several important practical implications for transit system design. The benefits of heterogeneous design largely depend on demand spatial patterns, with greater advantages observed for more heterogeneous demand and when spatial concentration shifts away from rail terminals, while higher demand rates and larger service regions tend to reduce these benefits. Optimal heterogeneous feeder networks utilize smaller buses compared to homogeneous designs because they better accommodate heterogeneous demand, resulting in more evenly distributed passenger loading across lines

and vehicles. Furthermore, feeder lines should be designed to enable buses to travel along the shorter side of service regions for passenger operations while traveling nonstop along the longer side to rail stations, potentially reducing system costs by 6% or more depending on regional geometry. The developed models and algorithms can be readily implemented by local transit agencies to design fixed-route feeder networks within rectangular service regions, with generated plans capable of being fine-tuned to align with local street networks.

The second main contribution is that we have developed analytical models to design efficient DRC networks, focusing on two operational strategies: fully-flexible routing and semi-flexible routing. These models enable the determination of optimal zone dimensions, heterogeneous headways, and vehicle sizes, all aimed at minimizing overall system costs. To our knowledge, this is the first study to clearly differentiate between two commonly used DRC routing strategies that previous studies often conflated. We identify two main distinctions: whether exact optimal local tours are attained, and whether patrons are required to submit trip requests before vehicle dispatch. Consequently, while fully-flexible routing provides shorter local-tour times, this advantage comes at the expense of longer waiting times. Our models significantly improve the handling of stochastic demand and precision of tour length estimations by incorporating second-order effects of random passenger numbers and employing more accurate tour length estimates. For fully-flexible routing, average cost estimation errors are reduced from 8-12% to within 2%, while the semi-flexible routing model reduces errors from 6.3% to 0.25%. These enhancements help rectify previous underestimations of DRC advantages relative to fixed-route feeder services and provide more accurate assessments of critical demand density thresholds for service type selection.

The research establishes important operational guidelines for DRC service implementation. Fully-flexible routing strategies prove more effective in regions with lower demand densities and smaller areas, making them particularly suitable for last-mile services, while semi-flexible routing strategies excel in areas with higher demand densities and larger service areas, such as those serving airports, high-speed rail stations, and major public venues. Zoning emerges as a critical component in DRC service design, with the number of zones closely aligned with demand to ensure limited vehicle occupancy and local tour lengths. Fully-flexible routing favors square-shaped zones while semi-flexible routing tends toward elongated rectangular zone configurations. As demand increases, the optimal feeder service design evolves from fully-flexible routing to semi-flexible routing, and eventually to fixed-route services, indicating that greater routing flexibility tends to diminish economies of scale for feeder services.

The third main contribution is that we developed an optimization model for designing trunk-feeder transit networks under spatially heterogeneous demand. The model jointly determines line spacings, stop spacings, and headways across both hierarchical lay-

ers. The derived analytical solutions reveal fundamental system behaviors: sparse feeder lines require denser stops as a compensatory mechanism to maintain accessibility. Our results consistently show that optimal trunk-line spacings exhibit directional asymmetry, with $S_Y(x)$ exceeding $S_X(y)$, leading to feeder zones elongated in the X-direction. This supports our Chapter 2 insights that cost-efficient feeder systems tend to operate along the shorter side of service zones to collect passengers.

The practical implications are illustrated through numerical experiments, which demonstrate up to a 24% cost reduction compared to conventional grid networks. These improvements are primarily achieved by integrating feeder and trunk services that significantly reduce access times, while also benefiting from the implementation of heterogeneous designs. The framework also offers guidance on mode selection: rail-bus systems provide considerable benefits in high-demand contexts by reducing travel time, whereas bus-bus systems emerge as more cost-effective options in low-demand scenarios due to their lower agency costs. Additionally, feeder design decisions are found to be primarily responsive to local demand intensity rather than the type of trunk service employed.

5.2 Future Work

Future research could extend the current modeling of fully-flexible routing strategies by incorporating dynamic request insertion mechanisms. In the present study, fully-flexible tours are assumed to be determined prior to vehicle dispatch based on known passenger locations, following a static traveling salesman problem (TSP) structure. A dynamic extension would allow new requests to be inserted into an ongoing tour, enabling more responsive and potentially more efficient operations. This direction introduces new challenges such as designing real-time request acceptance rules, developing computationally tractable re-routing algorithms, and managing the trade-off between service responsiveness and operational fairness. Addressing these challenges could lead to hybrid routing strategies that outperform both fully-flexible and semi-flexible approaches under stochastic demand environments.

Another promising direction involves expanding the current framework to accommodate multiple access modes and fully integrate trunk and feeder system design. While this study focuses on fixed-route feeder services coordinated with heterogeneous trunk-line structures, future work may consider multiple feeder service modes such as DRC, shared bikes and ride-hailing. Incorporating such multimodal interactions would enable a more comprehensive assessment of access strategies, accounting for behavioral mode choice and urban network constraints. Such extensions would enhance the model’s generalizability and support more context-sensitive transit planning in diverse urban settings.

Appendix A

List of notations for Chapter 2

Table A.1: List of notations for Chapter 2

Notation	Description	Unit
Decision variables		
$B(x, y)$	Stop spacing at location (x, y)	km
$S_I(x)$	Line spacing at location x	km
$H_{Ip}(x)$	Feeder bus headway in the patron-collection direction at location x	h
$H_{Id}(x)$	Feeder bus headway in the patron-distribution direction at location x	h
K	Patron-carrying capacity of a feeder bus	patrons/vehicle
Cost terms and parameters (all cost terms are per hour of operations)		
AC	Total agency cost	h
UC	Total user cost	h
GC	Generalized cost	h
C_A	Total access and egress time	h
C_W	Total wait and transfer time	h
C_{Wp}	Wait and transfer time in the collection direction	h
C_{Wd}	Wait and transfer time in the distribution direction	h
C_T	Total in-vehicle travel time	h
C_{T1}	In-vehicle travel time to overcome the distance	h
C_{T2}	Time loss due to deceleration, acceleration, door operations	h
C_{T3}	Time loss due to boarding and alighting behavior	h
C_s	Stop infrastructure cost	h
C_{vk}	Operating cost associated with bus-km traveled	h
C_{vh}	Operating cost related to bus fleet size	h

Continued on next page

Table A.1 – continued from previous page

Notation	Description	Unit
V_h	Bus fleet size	–
V_{h1}	Buses actively traveling between stops	–
V_{h2}	Buses needed for deceleration/acceleration/door operation	–
V_{h3}	Buses required for boarding and alighting	–
π_s	Amortized cost of feeder stop infrastructure per hour	\$/stop/h
π_v	Cost per bus-km traveled	\$/vehicle·km
π_m	Cost per bus hour	\$/vehicle·h
Other parameters and variables		
$\lambda_p(x, y)$	Demand density in patron-collection direction at (x, y)	patrons/km ² /h
$\lambda_d(x, y)$	Demand density in patron-distribution direction at (x, y)	patrons/km ² /h
Λ_p	Total demand in the patron-collection direction	patrons/h
Λ_d	Total demand in the patron-distribution direction	patrons/h
Λ'_p	Collection demand after removing walking patrons	patrons/h
Λ'_d	Distribution demand after removing walking patrons	patrons/h
θ	Value of time	\$/h
W	Width of the service region	km
L	Length of the service region	km
v_W	Walking speed	km/h
v_I	Feeder bus cruise speed	km/h
τ_0	Time loss per stop due to door and speed change	h/stop
τ_a	Alighting time per patron	h/patron
τ_b	Boarding time per patron	h/patron
t_{f-t}	Transfer delay from feeder to trunk platform	h/patron
t_{t-f}	Transfer delay from trunk to feeder platform	h/patron
H_t	Headway of the trunk line	h
$H_{\min}$	Minimum feeder headway	h
$H_{\max}$	Maximum feeder headway	h

Appendix B

Derivation of the Optimality Properties

Combining all the cost metrics, we can rewrite the generalized cost (15a) as:

$$\begin{aligned}
GC = & \left\{ \frac{\pi_m}{\mu}(\tau_a + \tau_b)\Lambda + \int_{x=0}^L \left(\frac{x}{v_I}(\Lambda_{px}(x) + \Lambda_{dx}(x)) + \left(t_{f-t} + \frac{H_t}{2} \right) \Lambda_{px}(x) + t_{t-f} \Lambda_{dx}(x) \right) dx \right. \\
& + \left. \int_{y=0}^W \left(\frac{y}{v_I}(\Lambda_{py}(y) + \Lambda_{dy}(y)) \right) dy \right\} \\
& + \left\{ \int_{x=0}^L \left[\frac{\pi_v(W+x)}{\theta S_I(x)} \left(\frac{1}{H_{Ip}(x)} + \frac{1}{H_{Id}(x)} \right) + \frac{\pi_m(W+x)}{\theta S_I(x) v_I} \left(\frac{1}{H_{Ip}(x)} + \frac{1}{H_{Id}(x)} \right) \right. \right. \\
& + \frac{S_I(x)}{4v_W}(\Lambda_{px}(x) + \Lambda_{dx}(x)) + \frac{H_{Ip}(x)}{2}\Lambda_{px}(x) + \frac{H_{Id}(x)}{2}\Lambda_{dx}(x) + \frac{\tau_a}{2}S_I(x)H_{Ip}(x)[\Lambda_{px}(x)]^2 \\
& + \left. \left. \frac{\tau_b}{2}S_I(x)H_t[\Lambda_{dx}(x)]^2 + \tau_b S_I(x)H_{Ip}(x)\mathcal{M}_{px}(x) + \tau_a S_I(x)H_{Id}(x)\mathcal{M}_{dx}(x) \right] dx \right\} \\
& + \left\{ \int_{x=0}^L \int_{y=0}^W \left[\frac{\pi_s}{\theta} \cdot \frac{1}{S_I(x)B(x,y)} + \frac{\pi_m \tau_0}{\theta S_I(x)B(x,y)} \left(\frac{1}{H_{Ip}(x)} + \frac{1}{H_{Id}(x)} \right) \right. \right. \\
& + \left. \left. \frac{B(x,y)}{4v_W}(\lambda_p(x,y) + \lambda_d(x,y)) + \frac{\tau_0 \Lambda_{pxy}(x,y)}{B(x,y)} + \frac{\tau_0 \Lambda_{dxy}(x,y)}{B(x,y)} \right] dy dx \right\} \tag{B.1}
\end{aligned}$$

The part of the objective function in the first pair of braces is constant. The part in the second pair of braces contains single integrals relating to either x or y (involving decision functions $S_I(x)$, $H_{Ip}(x)$, $H_{Id}(x)$). Meanwhile, the part in the third pair of braces contains double integrals related to both x and y (involving the decision function $B(x,y)$).

Note that [B.1](#) is a convex function when only one decision variable ($S_I(x)$, $H_{Ip}(x)$, $H_{Id}(x)$, or $B(x,y)$) is considered, and the other variables are fixed. (This type of functions is known as the coordinate-wise convex functions, which are not necessarily convex themselves.) Further note that constraints [\(2.17\)](#)–[\(2.18\)](#) act as boundary constraints when only one decision variable is considered. As a result, the optimal solution for each decision variable can be developed analytically from first-order derivatives, assuming the other variables are fixed. These optimal solutions are:

$$H_{Ip}^*(x) = \text{mid} \left\{ H_{\min}, \min \left\{ \frac{K}{\Lambda_{px}(x)S_I^*(x)}, H_{\max} \right\}, \sqrt{\frac{2\alpha(x)S_I^*(x)^{-1}}{\Lambda_{px}(x) + \beta_p(x)S_I^*(x)}} \right\} \quad (\text{B.2})$$

$$H_{Id}^*(x) = \text{mid} \left\{ \max\{H_{\min}, H_t\}, \min \left\{ \frac{K}{\Lambda_{dx}(x)S_I^*(x)}, H_{\max} \right\}, \sqrt{\frac{2\alpha(x)S_I^*(x)^{-1}}{\Lambda_{dx}(x) + \beta_d(x)S_I^*(x)}} \right\} \quad (\text{B.3})$$

$$S_I^*(x) = \min \left\{ \sqrt{\frac{2\alpha(x) \left(H_{Ip}^*(x)^{-1} + H_{Id}^*(x)^{-1} \right) + 2\gamma(x)}{\frac{\Lambda_{px}(x) + \Lambda_{dx}(x)}{2v_W} + \tau_b[\Lambda_{dx}(x)]^2 H_t + \beta_p(x)H_{Ip}^*(x) + \beta_d(x)H_{Id}^*(x)}}, \frac{K}{\Lambda_{px}(x)H_{Ip}^*(x)}, \frac{K}{\Lambda_{dx}(x)H_{Id}^*(x)} \right\} \quad (\text{B.4})$$

$$B^*(x, y) = \sqrt{\frac{4v_W \left[\frac{1}{\mu S_I^*(x)} \left(\pi_s + \frac{\pi_m \tau_0}{H_{Ip}^*(x)} + \frac{\pi_m \tau_0}{H_{Id}^*(x)} \right) + \tau_0 (\Lambda_{pxy}(x, y) + \Lambda_{dxy}(x, y)) \right]}{\lambda_p(x, y) + \lambda_d(x, y)}} \quad (\text{B.5})$$

where function $\text{mid}\{z_1, z_2, z_3\}$ returns the middle value among the three arguments.

Appendix C

A heuristic iterative solution method

The algorithm consists of two main stages (i.e., Stage 1 and 2 in the following algorithm) that iterate until convergence is reached. In the first stage, we develop $B^*(x, y)$ using (2.27d) and given values of $H_{Ip}(x)$, $H_{Id}(x)$, and $S_I(x)$ for a lattice of discrete points defined on Ω . In the second stage, $S_I^*(x)$, $H_{Ip}^*(x)$, and $H_{Id}^*(x)$ are derived using (2.27a)–(2.27c) given the $B^*(x, y)$ obtained in the first stage.

Stage 0: Initialization.

- Discretize the service region Ω into a $m \times n$ lattice of points, setting $x_i = (i - 0.5)\frac{L}{n}$, $i = 1, 2, \dots, n$, and $y_j = (j - 0.5)\frac{W}{m}$, $j = 1, 2, \dots, m$.
- Initialize $H_{Ip}^{(0)}(x_i)$, $H_{Id}^{(0)}(x_i)$, and $S_I^{(0)}(x_i)$ to satisfy the boundary constraints for $i = 1, 2, \dots, n$. Set the outer-loop iteration count $k = 1$ and proceed to Stage 1.

Stage 1: Compute $B(x, y)$.

- Calculate $B^{(k)}(x_i, y_j)$ using Eq. (2.27d) for $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, taking $H_{Ip}^{(k-1)}(x_i)$, $H_{Id}^{(k-1)}(x_i)$, and $S_I^{(k-1)}(x_i)$ as inputs. Note that the integrals are approximated by summations. Proceed to Stage 2.

Stage 2: Compute $H_{Ip}(x)$, $H_{Id}(x)$ and $S_I(x)$.

1. Set $\tilde{S}_I^{(k,0)}(x_i) = S_I^{(k-1)}(x_i)$, $i = 1, 2, \dots, n$. Initialize the inner-loop iteration count $k' = 1$.
2. Taking $B^{(k)}(x_i, y_j)$ and $\tilde{S}_I^{(k,k'-1)}(x_i)$ as inputs, calculate $\tilde{H}_{Ip}^{(k,k')}(x_i)$ and $\tilde{H}_{Id}^{(k,k')}(x_i)$ using Eqs. (2.27a)–(2.27b).
3. Taking $B^{(k)}(x_i, y_j)$, $\tilde{H}_{Ip}^{(k,k')}(x_i)$, and $\tilde{H}_{Id}^{(k,k')}(x_i)$ as inputs, calculate $\tilde{S}_I^{(k,k')}(x_i)$ by Eq. (2.27c).
4. Set $S_I^{(k)}(x_i) = \tilde{S}_I^{(k,k')}(x_i)$, $H_{Ip}^{(k)}(x_i) = \tilde{H}_{Ip}^{(k,k')}(x_i)$, and $H_{Id}^{(k)}(x_i) = \tilde{H}_{Id}^{(k,k')}(x_i)$. Check for convergence using the predefined error tolerance ϵ .

5. If the following convergence criteria are satisfied:

$$\sum_{i=1}^n \left| \tilde{S}_I^{(k,k')}(x_i) - \tilde{S}_I^{(k,k'-1)}(x_i) \right| \leq n\epsilon$$

$$\sum_{i=1}^n \left(\left| \tilde{H}_{Ip}^{(k,k')}(x_i) - \tilde{H}_{Ip}^{(k,k'-1)}(x_i) \right| + \left| \tilde{H}_{Id}^{(k,k')}(x_i) - \tilde{H}_{Id}^{(k,k'-1)}(x_i) \right| \right) \leq 2n\epsilon$$

then proceed to the next step. Otherwise, update $k' \leftarrow k' + 1$ and repeat steps (2) and (3).

6. Check convergence. If

$$\sum_{i=1}^n \left| S_I^{(k)}(x_i) - S_I^{(k-1)}(x_i) \right| \leq n\epsilon,$$

$$\sum_{i=1}^n \left(\left| H_{Ip}^{(k)}(x_i) - H_{Ip}^{(k-1)}(x_i) \right| + \left| H_{Id}^{(k)}(x_i) - H_{Id}^{(k-1)}(x_i) \right| \right) \leq 2n\epsilon,$$

$$\sum_{i=1}^n \sum_{j=1}^m \left| B^{(k)}(x_i, y_j) - B^{(k-1)}(x_i, y_j) \right| \leq mn\epsilon,$$

then set $S_I^*(x_i) = S_I^{(k)}(x_i)$, $H_{Ip}^*(x_i) = H_{Ip}^{(k)}(x_i)$, $H_{Id}^*(x_i) = H_{Id}^{(k)}(x_i)$, and $B^*(x_i, y_j) = B^{(k)}(x_i, y_j)$ for $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, and end the search. Otherwise, update $k \leftarrow k + 1$ and return to Stage 1.

The optimal CA solution, obtained on the point lattice, can be transformed into a specific layout of feeder lines. The methodology for this conversion is detailed in Appendix E.

Appendix D

Transit agency cost rates

The unit operating cost per bus-hour, π_m , is primarily determined by the amortized vehicle price, maintenance costs, and staff wages. The vehicle purchase and maintenance costs can be assumed to be an affine function of bus capacity (Oldfield and Bly, 1985), while the staff wages are proportional to the value of time. In other words, we can express π_m as:

$$\pi_m = a_m + b_m K + c_m \theta \quad (\text{D.1})$$

where a_m , b_m , and c_m are the cost coefficients.

The unit operating cost per bus-km traveled, π_v , is mainly determined by the fuel costs, which are related to K but not to θ . Thus, we can assume an affine relationship between π_v and K as follows:

$$\pi_v = a_v + b_v K \quad (\text{D.2})$$

where a_v and b_v are the cost coefficients.

To calibrate the cost coefficients, we utilized data from two popular bus models: the Volvo 8900 for a full-sized bus and the Ford Transit for a medium-sized one. Table D.1 displays the capacities, purchase costs, fuel consumption rates, and service lives of these vehicle models. The passenger-carrying capacity of a full-sized (single-deck, not articulated) bus like the Volvo 8900 may vary depending on the seat plan requested of the transit agency. For our analysis, we assume a capacity of 80 passengers, including standees. The purchase cost of Volvo 8900 was obtained from Horrox and Casale (2019). Its fuel consumption rate was extracted from the Volvo Buses Environmental Blog <https://volvobusesenvironmentblog.wordpress.com/>. As for the Ford Transit, we obtained its seat capacity and purchase cost from the Ford Transit price list https://www.ford.co.uk/content/dam/guxeu/uk/documents/pricelist/commercialvehicles/PL-Transit_Minibus.pdf, while its fuel consumption rate came from the Ford Media Center <https://media.ford.com/content/fordmedia/feu/nl/nl/news/2014/05/15/ford-launches-first-18-seat-transit-minibus--offers-improved-eff.html>. We learned the service lives of both models from Laver et al. (2007).

We assume two employees are assigned to each bus during operation hours, including one

Table D.1: Data of Volvo 8900 and Ford Transit

Models	Capacity	Price (\$)	Fuel consumption rate (liter/100 km)	Service life (year)
Volvo 8900	80	500,000	26	12
Ford Transit	17	76,000	7.4	5

driver and one manpower for maintenance, management, administration, and support services combined. Therefore, we set $c_m = 2$. To amortize the bus purchase costs, we assume that each bus operates for 16 hours a day. Additionally, we inflate the bus purchase cost by 50% to account for other cost items associated with the bus fleet, such as maintenance, insurance, and facility rent. Based on these assumptions, we determine that $a_m = 2.068\$/\text{vehicle} \cdot \text{h}$ and $b_m = 0.108\$/\text{h}$.

For the bus-km-related cost coefficients, we assume a prevailing gasoline price of \$1.1/liter. The bus-km-related costs are also inflated by 20% to cover the cost items other than fuel, such as mileage-related depreciation, vehicle deterioration, and toll. Based on these assumptions, we determine that $a_v = 0.0314\$/\text{vehicle} \cdot \text{km}$ and $b_v = 0.0039\$/\text{km}$.

Appendix E

Generating a specific feeder system design

Given $S_I^*(x_i)$ and $B^*(x_i, y_j)$ ($i = 1, 2, \dots, n$; $j = 1, 2, \dots, m$), we use the following three steps to generate line and stop locations. Step 1 generates line locations that closely match $S_I^*(x_i)$. Step 2 determines the continuous stop spacing function for each line generated in Step 1. Step 3 generates the stop locations for each line.

Step 1. Apply the spline curve fitting method to fit a cubic spline function $S_I(x)$ to $S_I^*(x_i)$ ($i = 1, 2, \dots, n$). Next, place one line at every x where: $\int_{z=0}^x \frac{dz}{S_I(x)} = k + 0.5$, $k = 1, 2, \dots$. Here, 0.5 is added to the RHS to ensure the first line's catchment zone resides on both sides of that line. The resulting line location set is denoted by $\Omega_S = \{x_p : p = 1, 2, \dots, N^S\}$, where N^S is the number of lines, and x_p^S the location of the p -th line satisfying $0 < x_1 < x_2 < \dots < x_{N^S} < L$.

Step 2. Use the gridded data interpolation method to fit a piecewise cubic function $B(x, y)$ to $B^*(x_i, y_j)$ ($i = 1, 2, \dots, n$; $j = 1, 2, \dots, m$). The continuous stop spacing function on the p -th line is $B(x_p, y)$.

Step 3. For each line $p \in \{1, 2, \dots, N^S\}$, place one stop at every y where: $\int_{z=0}^y \frac{dz}{B(x_p, y)} = k + 0.5$, $k = 1, 2, \dots$. The resulting stop location set is denoted by $\Omega_p^B = \{(x_p, y_q) : q = 1, 2, \dots, N_p^B\}$, where N_p^B is the number of stops on the p -th line. Coordinates y_q satisfy $0 < y_1 < y_2 < \dots < y_{N_p^B} < W$.

Finally, the headways of each line are obtained from $H_{Ip}^*(x)$ and $H_{Id}^*(x)$ at $x_p \in \Omega_S$.

Note that the CA solution renders $\int_0^L \frac{1}{S_I^*(x)} dx$ feeder lines, and this number is not always close to an integer. The above method guarantees finding the nearest integer number of lines. However, when that number is small, the percentage error could be relatively large (capped by $0.5 \left(\int_0^L \frac{1}{S_I^*(x)} dx \right)^{-1}$). To further reduce this error and improve the converted specific design, we can evenly allocate the fractional residual number of lines (positive or negative) to all lines by proportionally increasing or decreasing the line spacings. Similarly, stop spacings on each line can also be adjusted to eliminate the fractional residuals. In addition, one can reoptimize

stop spacings and headways using the CA model after line locations are determined by Step 1 of the above method. This will make the specific design more efficient. For simplicity, these fine-tuning steps are not included in our numerical case studies.

Appendix F

List of notations for Chapter 3

Table F.1: List of notations for Chapter 3

Notation	Description	Unit
Decision variables		
M	Number of service zones along the vertical direction	km
N	Number of service zones along the horizontal direction	km
w_0	Width of the swath in semi-flexible routing	km
$H_{p(m,n)}$	Headway in the outbound leg of zone (m, n)	h
$H_{d(m,n)}$	Headway in the inbound leg of zone (m, n)	h
K	The patron-carrying capacity of a bus	patrons/bus
Cost terms and parameters		
$C_W(m, n)$	Waiting time at home of zone (m, n)	h
C_W	Total waiting time at home	h
$C_{Tp(m,n)}$	Local tour travel time in the outbound leg of zone (m, n)	h
C_{Tp}	Total local tour travel time in the outbound leg	h
$C_{Td(m,n)}$	Local tour travel time in the inbound leg of zone (m, n)	h
C_{Td}	Total local tour travel time in the inbound leg	h
$C_{Lp(m,n)}$	Line-haul travel time in the outbound leg of zone (m, n)	h
C_{Lp}	Total line-haul travel time in the outbound leg	h
$C_{Ld(m,n)}$	Line-haul travel time in the inbound leg of zone (m, n)	h
C_{Ld}	Total line-haul travel time in the inbound leg direction	h
$C_{Rp(m,n)}$	Transfer time in the outbound leg of zone (m, n)	h
C_{Rp}	Total transfer time in the outbound leg	h
$C_{Rd(m,n)}$	Transfer time in the inbound leg of zone (m, n)	h
C_{Rd}	Total transfer time in the inbound leg	h

continued on next page

Notation	Description	Unit
$C_{vkp(m,n)}$	Distance-based agency cost in the outbound leg of zone (m, n)	h
$C_{vkd(m,n)}$	Distance-based agency cost in the inbound leg of zone (m, n)	h
C_{vk}	Total distance-based agency cost	h
$C_{vhp(m,n)}$	Fleet-based agency cost in the outbound leg of zone (m, n)	h
$C_{vhd(m,n)}$	Fleet-based agency cost in the inbound leg of zone (m, n)	h
C_{vh}	Total fleet-based agency cost	h
GC	Total societal cost (generalized cost)	h
π_v	The unit cost per bus-km traveled	\$/vehicle·km
π_m	The unit cost per bus hour	\$/vehicle·h
Other parameters and variables		
q	Number of stops in a TSP tour	-
A	Area of the service zone	km ²
k^*	Proportionality factor in the TSP tour length formula	-
S	Ratio of a service zone's length to width	patrons/h
W	Width of the service region	km
L	Length of the service region	km
w	Width of a service zone	km
l	Length of a service zone	km
m	The row number of a zone (from bottom to top)	-
n	The column number of a zone (from left to right)	-
λ_p	Demand density in the outbound direction	patrons/km ² /h
λ_d	Demand density in the inbound direction	patrons/km ² /h
$Q_{p(m,n)}$	Bus occupancy in zone (m, n) in the outbound leg	-
$Q_{d(m,n)}$	Bus occupancy in zone (m, n) in the inbound leg	-
$\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$	Calibrated coefficients in the optimal TSP tour length equation	-
v_I	Bus cruising speed	km/h
τ_p	Dwell time per stop in the outbound leg	h
τ_d	Dwell time per stop in the inbound leg	h
α	Discount factor of the time value of waiting at home	-
τ_a	Boarding time per patron at the terminal stop	h
τ_b	Alighting time per patron at the terminal stop	h
t_{f-t}	Transfer delay from the DRC stop to the rail-line platform	h/patron

continued on next page

Notation	Description	Unit
t_{t-f}	Transfer delay from the rail-line platform to the DRC stop	h/patron
H_t	Headway of the rail service	h
$\gamma_{(m,n)}$	Integer scale factor between $H_{d(m,n)}$ and H_t	-
θ	Value of time	\$/h
H_{min}	Minimum headway	h
H_{max}	Maximum headway	h

Appendix G

Estimating k^* for the Optimal TSP Tour Length

The dataset used for calibrating the formula of k^* includes 56 scenarios with various combinations of q and S values, where $q \in \{2, 3, \dots, 15\}$ and $S \in \{1, 1.5, 2, 3\}$. (Note that S is defined to be no less than 1.) In practical DRC services, scenarios where $q > 15$ or $S > 3$ are uncommon.

For each scenario, we generated and solved a sufficient number of TSP instances (at least 500) using an off-the-shelf exact solver to ensure the convergence of the average k^* value within a tolerance of 0.01. More information about the solver can be found at: <https://ww2.mathworks.cn/help/optim/ug/travelling-salesman-problem.html>.

Upon analyzing the results (which are provided in Table B.1 for reference), it became evident that k^* approximates a linear function of S . Additionally, the relationship between k^* and q resembles the probability density function of a Weibull distribution. Consequently, we calibrate the following formula for k^* :

$$k^* = (\beta_1 S + \beta_2) q^{\beta_3} e^{\beta_4 q^{\beta_5}} \quad (\text{G.1})$$

where β_1 , β_2 , β_3 , β_4 , and β_5 are coefficients obtained through the calibration of simulated data. Equation (G.1) represents the calibrated outcome. The simulated (average) k^* values are tabulated in a table below for reference.

Table G.1: Results of simulated k^*

S	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0.94	1.16	1.20	1.19	1.18	1.17	1.15	1.13	1.12	1.11	1.10	1.09	1.09	1.08
1.5	0.96	1.17	1.22	1.22	1.20	1.19	1.17	1.15	1.15	1.13	1.12	1.11	1.10	1.10
2	1.00	1.23	1.27	1.28	1.25	1.23	1.21	1.20	1.18	1.16	1.15	1.14	1.13	1.12
3	1.09	1.33	1.38	1.38	1.36	1.34	1.31	1.29	1.27	1.25	1.23	1.21	1.20	1.19

Appendix H

Approximations of expected values by Taylor expansion

For brevity, we use simplified notations Q , λ , and H to represent the number of passengers a vehicle needs to pick up in a service zone, the demand density, and the headway, respectively.

Due to the assumption of spatial Poisson demand, we have:

$$\mathbb{E}[Q] = \lambda H l w \quad (\text{H.1})$$

$$\text{Var}[Q] = \mathbb{E}[(Q - \mathbb{E}[Q])^2] = \lambda H l w \quad (\text{H.2})$$

$$\mathbb{E}[Q^2] = \mathbb{E}[Q]^2 + \text{Var}[Q] = (\lambda H l w)^2 + \lambda H l w \quad (\text{H.3})$$

We derive second-order approximations for $(Q+1)^{\beta_3+3/2}e^{\beta_4(Q+1)^{\beta_5}}$ and $(Q+1)^{\beta_3+1/2}e^{\beta_4(Q+1)^{\beta_5}}$, which are used in Eq. (2), (4), (5), and (16–19). Applying Taylor expansion at the point of the expected value $\mathbb{E}[Q]$, we have:

$$\begin{aligned} (Q+1)^{\beta_3+3/2}e^{\beta_4(Q+1)^{\beta_5}} &= e^{\beta_4(\mathbb{E}[Q]+1)^{\beta_5}} \left\{ (\mathbb{E}[Q]+1)^{\beta_3+3/2} \right. \\ &\quad + \left[(\beta_3+3/2)(\mathbb{E}[Q]+1)^{\beta_3+1/2} + \beta_4\beta_5(\mathbb{E}[Q]+1)^{\beta_3+\beta_5+1/2} \right] (Q - \mathbb{E}[Q]) \\ &\quad + \left[(\beta_3+3/2)(\beta_3+1/2)(\mathbb{E}[Q]+1)^{\beta_3-1/2} \right. \\ &\quad + \beta_4\beta_5(\beta_3+3/2)(\mathbb{E}[Q]+1)^{\beta_3+\beta_5-1/2} \\ &\quad + \beta_4\beta_5(\beta_3+\beta_5+1/2)(\mathbb{E}[Q]+1)^{\beta_3+\beta_5-1/2} \\ &\quad + \left. \beta_4^2\beta_5^2(\mathbb{E}[Q]+1)^{\beta_3+2\beta_5-1/2} \right] \frac{(Q - \mathbb{E}[Q])^2}{2} \\ &\quad \left. + \mathcal{O}((Q - \mathbb{E}[Q])^3) \right\} \quad (\text{H.4}) \end{aligned}$$

$$\begin{aligned}
(Q+1)^{\beta_3+1/2} e^{\beta_4(Q+1)^{\beta_5}} &= e^{\beta_4(\mathbb{E}[Q]+1)^{\beta_5}} \left\{ (\mathbb{E}[Q]+1)^{\beta_3+1/2} \right. \\
&+ \left[(\beta_3+1/2)(\mathbb{E}[Q]+1)^{\beta_3-1/2} + \beta_4\beta_5(\mathbb{E}[Q]+1)^{\beta_3+\beta_5-1/2} \right] (Q-\mathbb{E}[Q]) \\
&+ \left[(\beta_3+1/2)(\beta_3-1/2)(\mathbb{E}[Q]+1)^{\beta_3-3/2} \right. \\
&+ \beta_4\beta_5(\beta_3+1/2)(\mathbb{E}[Q]+1)^{\beta_3+\beta_5-3/2} \\
&+ \beta_4\beta_5(\beta_3+\beta_5-1/2)(\mathbb{E}[Q]+1)^{\beta_3+\beta_5-3/2} \\
&+ \left. \left. \beta_4^2\beta_5^2(\mathbb{E}[Q]+1)^{\beta_3+2\beta_5-3/2} \right] \frac{(Q-\mathbb{E}[Q])^2}{2} \right. \\
&+ \left. \mathcal{O}((Q-\mathbb{E}[Q])^3) \right\} \tag{H.5}
\end{aligned}$$

Apply the expected value operator to both sides and omit higher-order terms:

$$\begin{aligned}
\mathbb{E}[(Q+1)^{\beta_3+3/2} e^{\beta_4(Q+1)^{\beta_5}}] &\approx e^{\beta_4(\lambda Hlw+1)^{\beta_5}} \left\{ (\lambda Hlw+1)^{\beta_3+3/2} \right. \\
&+ \left[(\beta_3+3/2)(\beta_3+1/2)(\lambda Hlw+1)^{\beta_3-1/2} \right. \\
&+ \beta_4\beta_5(\beta_3+3/2)(\lambda Hlw+1)^{\beta_3+\beta_5-1/2} \\
&+ \beta_4\beta_5(\beta_3+\beta_5+1/2)(\lambda Hlw+1)^{\beta_3+\beta_5-1/2} \\
&+ \left. \left. \beta_4^2\beta_5^2(\lambda Hlw+1)^{\beta_3+2\beta_5-1/2} \right] \frac{\lambda Hlw}{2} \right\} \tag{H.6}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[(Q+1)^{\beta_3+1/2} e^{\beta_4(Q+1)^{\beta_5}}] &\approx e^{\beta_4(\lambda Hlw+1)^{\beta_5}} \left\{ (\lambda Hlw+1)^{\beta_3+1/2} \right. \\
&+ \left[(\beta_3+1/2)(\beta_3-1/2)(\lambda Hlw+1)^{\beta_3-3/2} \right. \\
&+ \beta_4\beta_5(\beta_3+1/2)(\lambda Hlw+1)^{\beta_3+\beta_5-3/2} \\
&+ \beta_4\beta_5(\beta_3+\beta_5-1/2)(\lambda Hlw+1)^{\beta_3+\beta_5-3/2} \\
&+ \left. \left. \beta_4^2\beta_5^2(\lambda Hlw+1)^{\beta_3+2\beta_5-3/2} \right] \frac{\lambda Hlw}{2} \right\} \tag{H.7}
\end{aligned}$$

Appendix I

List of notations for Chapter 4

Table I.1: List of notations for Chapter 4

Notation	Description	Unit
Decision variables		
$S_X(y)$	Trunk-line spacings between two adjacent X-lines	km
$S_Y(x)$	Trunk-line spacings between two adjacent Y-lines	km
$H_X(y)$	Trunk-line headways of vehicles travel on X-lines	h
$H_Y(x)$	Trunk-line headways of vehicles travel on Y-lines	h
$S_F(x, y)$	Feeder-line spacings in the feeder zone serving the trunk station at (x, y)	h
$B_F(x, y)$	Feeder-stop spacings in the feeder zone serving the trunk station at (x, y)	h
$H_F(x, y)$	Feeder-line headways in the feeder zone serving the trunk station at (x, y)	h
K_F	The patron-carrying capacity of feeder vehicle	patrons/bus
Cost terms and parameters		
T_W	Waiting time at the origin stations of trunk lines	h
T_R	Transfer time at trunk-line transfer stations	h
T_T	In-vehicle travel time on trunk lines	h
T_L	Trunk-line infrastructure cost	h
T_S	Trunk-station infrastructure cost	h
T_K	Trunk-line distance-based operating cost	h
T_H	Trunk-line time-based operating cost	h
F_A	Walking time	h
F_{AW}	Walking time of patrons within the walking zone	h
F_{AF}	Walking time of patrons outside the walking zone	h
F_W	Waiting time for feeder transit and transfer time between feeder stops and trunk-line stations	h

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Table I.1 – continued from previous page

Notation	Description	Unit
F_{Wp}	Waiting and transfer time for outbound service	h
F_{Wd}	Waiting and transfer time for inbound service	h
F_T	In-vehicle travel time on feeder lines	h
F_{Tt}	Travel time to overcome distance	h
F_{Td}	Travel time for bus dwell	h
F_L	Feeder-line infrastructure cost	h
F_S	Feeder-stop infrastructure cost	h
F_K	Feeder-line distance-based operating cost	h
F_H	Feeder-line time-based operating cost	h
F_{Ht}	The number of vehicles that are actively travelling between stops	h
F_{Hd}	The number of vehicles that are dwelling at stops	h
π_{lT}	The unit trunk-line infrastructure cost	\$/km
π_{sT}	The unit trunk-station infrastructure cost	\$/station
π_{kT}	The unit trunk-line vehicle-km traveled cost	\$/vehicle·km
π_{hT}	The unit trunk-line vehicle-hour traveled cost	\$/vehicle·h
π_{lF}	The unit feeder-line infrastructure cost	\$/km
π_{sF}	The unit feeder-stop infrastructure cost	\$/stop
π_{kF}	The unit feeder-line vehicle-km traveled cost	\$/vehicle·km
π_{hF}	The unit feeder-line vehicle-hour traveled cost	\$/vehicle·h
Other parameters and variables		
W	Width of the service region	–
L	Length of the service region	–
$\lambda(x_O, y_O, x_D, y_D)$	Demand density from the origin (x_O, y_O) to the destination (x_D, y_D)	patrons/km ⁴ /h
$\lambda_p(x, y)$	Demand density in the outbound direction	patrons/km ² /h
$\lambda_d(x, y)$	Demand density in the inbound direction	patrons/km ² /h
$\Lambda_{OY}(x)$	Cumulative boarding demand for Y-lines	patrons/km ² /h
$\Lambda_{OX}(y)$	Cumulative boarding demand for X-lines	patrons/km ² /h
$\Lambda_{DY}(x)$	Cumulative alighting demand for X-lines	patrons/km ² /h
$\Lambda_{DX}(y)$	Cumulative alighting demand for Y-lines	patrons/km ² /h
Λ	Total demand of the service region	patrons/km ² /h
$Q_X(x)$	Cross-sectional patron flows in X-lines	patrons/km/h
$Q_Y(y)$	Cross-sectional patron flows in Y-lines	patrons/km/h
$Q_{SN}(x)$	The patron flow in the south-north direction	patrons/km/h
$Q_{NS}(x)$	The patron flow in the north-south direction	patrons/km/h
$Q_{WE}(y)$	The patron flow in the west-east direction	patrons/km/h
$Q_{EW}(y)$	The patron flow in the east-west directions	patrons/km/h

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Table I.1 – continued from previous page

Notation	Description	Unit
δ_T	Transfer penalty of trunk lines	h
τ_T	Dwell time per trunk station	h
v_T	Cruising speed of trunk-line vehicles	km/h
K_T	Trunk-line vehicle capacity	patrons/veh
H_{Tmin}	Minimum trunk-line headway	h
H_{Tmax}	Maximum trunk-line headway	h
A_W	Size of the walking zone	km ²
D_W	Average walking distance per patron in the walking zone	km
v_W	Patrons' walking speed	km/h
t_{f-t}	Transfer penalty between feeder and trunk lines	h
t_{t-f}	Transfer penalty between trunk and feeder lines	h
τ_F	Dwell time per feeder stop	h
v_F	Cruising speed of feeder-line vehicles	km/h
K_F	Feeder-line vehicle capacity	patrons/veh
H_{Fmin}	Minimum feeder-line headway	h
H_{Fmax}	Maximum feeder-line headway	h

Appendix J

Derivation of passenger flows of different directions

The derivation of passenger flows in each direction is shown below.

$$Q_{X+}(y) = \frac{1}{2} \left(\int_{x_O=0}^x \int_{y_O=0}^W \int_{x_D=x}^L \lambda(x_O, y_O, x_D, y) dx_D dy_O dx_O \right. \\ \left. + \int_{x_O=0}^x \int_{y_D=0}^W \int_{x_D=x}^L \lambda(x_O, y, x_D, y_D) dx_D dy_D dx_O \right) \quad (\text{J.1})$$

$$Q_{X-}(y) = \frac{1}{2} \left(\int_{x_O=x}^L \int_{y_O=0}^W \int_{x_D=0}^x \lambda(x_O, y_O, x_D, y) dx_D dy_O dx_O \right. \\ \left. + \int_{x_O=x}^L \int_{y_D=0}^W \int_{x_D=0}^x \lambda(x_O, y, x_D, y_D) dx_D dy_D dx_O \right) \quad (\text{J.2})$$

$$Q_{Y+}(x) = \frac{1}{2} \left(\int_{y_O=0}^y \int_{x_O=0}^L \int_{y_D=y}^W \lambda(x_O, y_O, x, y_D) dy_D dx_O dy_O \right. \\ \left. + \int_{y_O=0}^y \int_{y_D=y}^W \int_{x_D=0}^L \lambda(x, y_O, x_D, y_D) dx_D dy_D dy_O \right) \quad (\text{J.3})$$

$$Q_{Y-}(x) = \frac{1}{2} \left(\int_{y_O=y}^W \int_{x_O=0}^L \int_{y_D=0}^y \lambda(x_O, y_O, x, y_D) dy_D dx_O dy_O \right. \\ \left. + \int_{y_O=y}^W \int_{y_D=0}^y \int_{x_D=0}^L \lambda(x, y_O, x_D, y_D) dx_D dy_D dy_O \right) \quad (\text{J.4})$$

Eq. (J.1) calculates the passenger flow $Q_{X+}(y)$ in the positive X-direction at location (x, y) by considering all trips that need to cross this point traveling in the positive X-direction. The first integral term represents passenger flow passing location (x, y) from origins located at smaller X-coordinates than position x , while the second term captures flows originating from a horizontal slice at y and heading to destinations at larger X-coordinates than position (x, y) . Both types of trips must utilize the trunk line at position (x, y) to travel in the positive X-direction. The factor of $1/2$ accounts for the assumption that trips are equally distributed between the two possible routing directions. Eqs. (J.2)–(J.4) are derived similarly.

Appendix K

Derivation of Walking Zone Parameters A_W and D_W

To determine the walking zone area A_W and the average walking distance within the walking zone D_W , we need to consider different geometric configurations based on the relationship between the walking threshold d_w and the trunk line spacings. The walking zone is defined as the area around a trunk station where patrons choose to walk directly to the terminal rather than use feeder bus services. The walking threshold d_w represents the maximum distance patrons are willing to walk, and the geometric configuration of the walking zone depends on the relative magnitudes of d_w and the trunk line spacings $S_X(y)$ and $S_Y(x)$.

Four distinct cases emerge based on the relationship between the walking threshold and the catchment area dimensions:

Case I: $d_w \leq \min\{\frac{S_Y(x)}{2}, \frac{S_X(y)}{2}\}$

In this case, the walking zone forms a complete circular sector within the rectangular catchment area. The area of the walking zone and the average walking distance are:

$$A_W(x, y) = \frac{d_w^2}{2} \quad (\text{K.1})$$

$$D_W(x, y) = \frac{2}{3}d_w \quad (\text{K.2})$$

Case II: $\min\{\frac{S_Y(x)}{2}, \frac{S_X(y)}{2}\} \leq d_w \leq \max\{\frac{S_Y(x)}{2}, \frac{S_X(y)}{2}\}$

In this case, the walking zone is truncated by one side of the rectangular catchment area:

$$A_W(x, y) = \frac{1}{2} \left[d_w - \min\left(\frac{S_Y(x)}{2}, \frac{S_X(y)}{2}\right) + d_w \right] \times \min\left(\frac{S_Y(x)}{2}, \frac{S_X(y)}{2}\right) \quad (\text{K.3})$$

$$D_W(x, y) = \frac{\frac{2d_w}{3} \cdot 2d_w^2 - 4 \int_0^{d_w - \min(\frac{S_Y(x)}{2}, \frac{S_X(y)}{2})} dx \int_{\min(\frac{S_Y(x)}{2}, \frac{S_X(y)}{2})}^{d_w - x} (x + y) dy}{A_W(x, y)} \quad (\text{K.4})$$

Case III: $\max\{\frac{S_Y(x)}{2}, \frac{S_X(y)}{2}\} \leq d_w \leq \frac{S_Y(x) + S_X(y)}{2}$

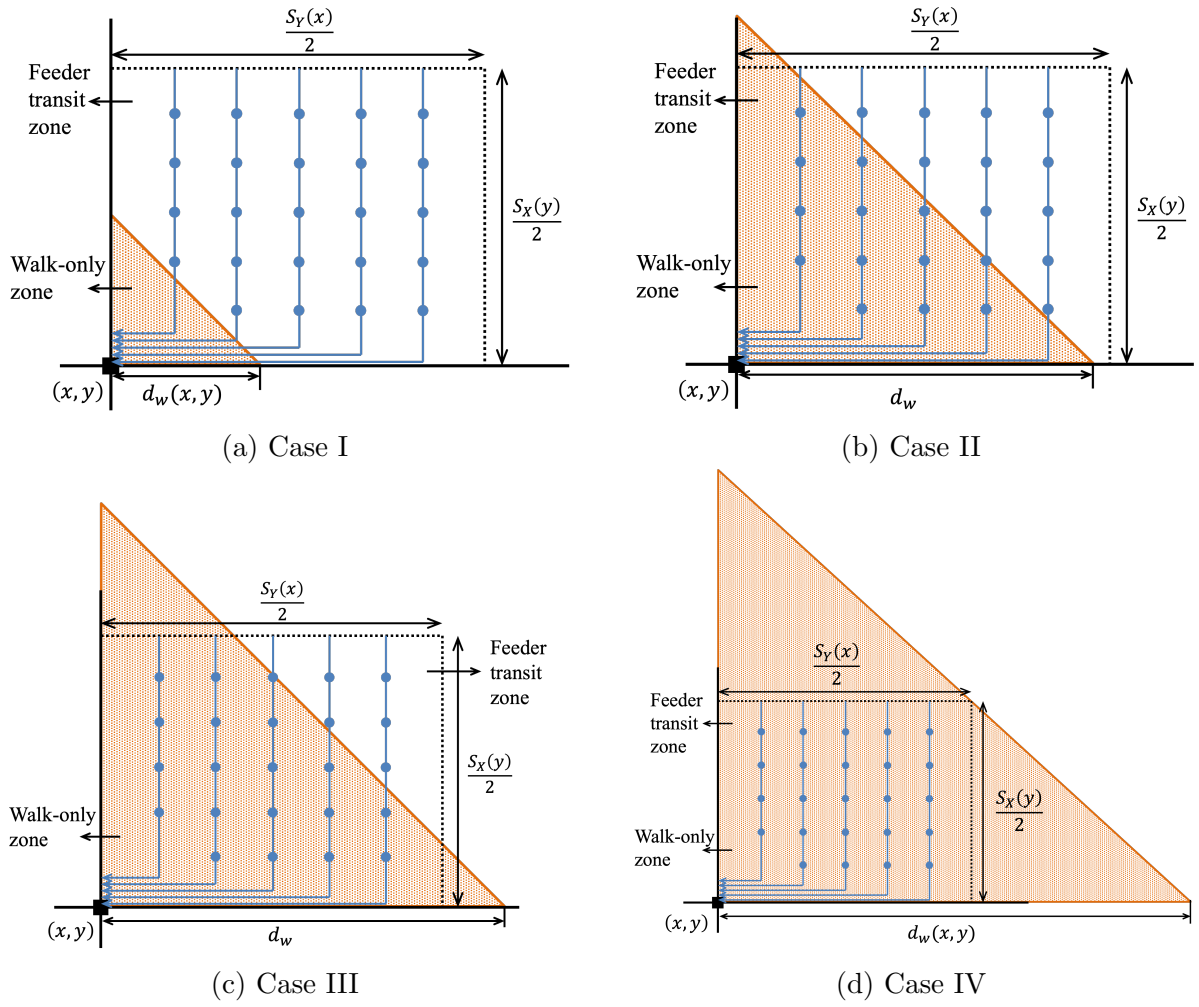


Figure K.1: The walk-only zone in a quarter catchment area of a trunk-line station

In this case, the walking zone is truncated by both sides of the rectangular catchment area:

$$A_W(x, y) = \frac{d_w^2}{2} - \frac{1}{2} \left(d_w - \frac{S_Y(x)}{2} \right)^2 - \frac{1}{2} \left(d_w - \frac{S_X(y)}{2} \right)^2 \quad (\text{K.5})$$

$$D_W(x, y) = \frac{\frac{2d_w}{3} \cdot 2d_w^2 - 4 \int_0^{d_w - \frac{S_X(y)}{2}} dx \int_{\frac{S_X(y)}{2}}^{d_w - x} (x + y) dy - 4 \int_{\frac{S_Y(x)}{2}}^{d_w} dx \int_0^{d_w - \frac{S_Y(x)}{2} - x} (x + y) dy}{A_W(x, y)} \quad (\text{K.6})$$

Case IV: $d_w \geq \frac{S_Y(x) + S_X(y)}{2}$

In this case, all patrons within the catchment area choose to walk to the terminal:

$$A_W(x, y) = \frac{S_X(y)S_Y(x)}{4} \quad (\text{K.7})$$

$$D_W(x, y) = \frac{S_X(y) + S_Y(x)}{4} \quad (\text{K.8})$$

While these four cases provide a complete mathematical description, implementing all cases would significantly increase computational complexity and prevent analytical solution derivation. To maintain tractability while preserving essential model behavior, we adopt a simplified formulation that considers only the extreme cases. This simplification approach enables derivation of analytical properties for the optimal solution, which would be impossible with the complete four-case formulation, while significantly reducing computational requirements compared to evaluating all cases at each iteration.

The simplified formulation assumes that when intermediate cases occur, the model provides feeder bus service instead of walking access. This represents a conservative overestimation of system costs, as operating feeder buses incurs higher costs than allowing walking access, but captures the essential trade-off between walking and feeder service while maintaining the core optimization insights. Therefore, the walking zone parameters are expressed as:

$$A_W = \begin{cases} 0, & \text{if } \frac{S_X(y)S_Y(x)}{4} \leq d_w^2 \\ \frac{1}{2}d_w^2, & \text{if } \frac{S_X(y)S_Y(x)}{4} > d_w^2 \end{cases} \quad (\text{K.9})$$

$$D_W = \begin{cases} 0, & \text{if } \frac{S_X(y)S_Y(x)}{4} \leq d_w^2 \\ \frac{2}{3}d_w, & \text{if } \frac{S_X(y)S_Y(x)}{4} > d_w^2 \end{cases} \quad (\text{K.10})$$

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