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DIGITAL SIGNAL PROCESSING FOR DISCRETE
NONLINEAR FREQUENCY DIVISION
MULTIPLEXING TRANSMISSION

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PhD

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Digital Signal Processing for Discrete Nonlinear
Frequency Division Multiplexing Transmission

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requirements for the degree of Doctor of
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Abstract

Nonlinear Frequency Division Multiplexing (NFDM) has attracted significant attention as a potential solution to overcome fiber nonlinearity in optical communication systems. Despite its theoretical elegance, the practical implementation of dual-polarization (DP) discrete NFDM has been hindered by the lack of a complete digital signal processing (DSP) framework—particularly one that eliminates the need for training sequences and operates entirely within the nonlinear Fourier domain. This thesis presents two key advancements to address these limitations. Both are developed analytically and validated through numerical simulations as well as experimental demonstrations.

First, the half back rotation algorithm is proposed as a simple yet effective equalization method for discrete NFDM systems. This algorithm restores the simplicity inherent to NFDM theory, eliminates the need for training sequences, and is shown to be optimal in performance. The theoretical analysis also provides an alternative proof of the Gordon-Haus effect in the language of NFT, and extends it to a phase-domain counterpart.

Second, a blind polarization demultiplexing algorithm is developed specifically for discrete NFDM signals, enabling the realization of a full DSP flow for DP discrete NFDM systems. The DSP flow operates entirely in the nonlinear frequency domain and integrates the half back rotation algorithm with other standard DSP techniques.

Furthermore, based on the refined noise model that accounts for both transfer noise and direct noise, the optimal symbol rate that minimizes the overall noise variance of a discrete NFDM system is identified. This finding offers a possible explanation for the historically low information rates observed in discrete NFDM systems.

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The journey of pursuing knowledge is not always smooth or joyful. Instead, it is common to be in darkness and fog, occasionally illuminated by flashes of insight. I remember I had a few of my own small “Eureka!” moments, usually late at night or before dawn, when I suddenly understood an equation that had long confused me, or when my tentative, probing derivation finally led me to meaningful results. However, slight frustration may follow soon since it had taken me quite a long time to figure it out. My main research object turned out to be a randomly walking soliton, which is interestingly analogous to my journey. Conserving, persevering, resilient, localized, obsolete...a few soliton words come to my mind...

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I cherish the summer when this journey began, enjoying the learning, accompanied by rock guitar solos in my earphones.

List of Publications

Journal Papers:

- [1] Chuang Xu, Gai Zhou, and Alan Pak Tao Lau, “A Full Digital Signal Processing Flow for Dual-Polarization Discrete Nonlinear Frequency Division Multiplexing Systems,” *Journal of Lightwave Technology*, Under review (2025).
- [2] Chuang Xu and Alan Pak Tao Lau, “Optimal Nonlinear Spectral Back Rotation for Discrete Eigenvalue Transmission Systems,” *Journal of Lightwave Technology*, **43**(11), 5129-5139 (2025).
- [3] Chuang Xu, Gai Zhou, Chao Lu, and Alan Pak Tao Lau, “81 Gb/s Dual-Polarization Discrete Eigenvalue Transmission Based on Nonlinear Fourier Transform,” *IEEE Photonics Technology Letters*, **35**(11), pp. 617-620 (2023).

Conference Papers:

- [1] Chuang Xu and Alan Pak Tao Lau, “Feed-forward Perturbation-Based Compensation for Nonlinear Optical Transmission,” in *Optical Fiber Communications Conference and Exhibition (OFC)*, pp. 1-3, Los Angeles, USA, 2026. (Submitted)
- [2] Chuang Xu, Jingchuan Wang, and Alan Pak Tao Lau, “Optimal Symbol Rate for Discrete Nonlinear Frequency Division Multiplexing Transmissions,” in *51st European Conference on Optical Communication (ECOC)*, pp. 1-3, Copenhagen, Denmark, 2025.
- [3] Chuang Xu, Gai Zhou, and Alan Pak Tao Lau, “A Full DSP Flow for Dual-Polarization Discrete NFDm Systems,” in *30th OptoElectronics and Communications Conference (OECC)*, pp. 1-3, Sapporo, Japan, 2025.
- [4] Chuang Xu and Alan Pak Tao Lau, “New Perspectives on Perturbation-based Pre-Distortion and Post-Compensation for Nonlinear Optical Transmission,” in *30th OptoElectronics and Communications Conference (OECC)*, pp. 1-3, Sapporo, Japan, 2025.
- [5] Chuang Xu and Alan Pak Tao Lau, “Optimal Nonlinear Spectral Back-Rotation for Discrete Eigenvalue NFT Transmission Systems,” in *Optical Fiber Communications Conference and Exhibition (OFC)*, pp. 1-3, San Diego, USA, 2024.

- [6] Chuang Xu, Yikun Chen, Kangping Zhong, Ke Zhang, Chao Lu, Cheng Wang, and Alan Pak Tao Lau, “6.48 Tb/s Transmissions Using 50 GHz Integrated Lithium Niobate Flat-Top Electro-Optic Combs,” in *Optical Fiber Communications Conference and Exhibition (OFC)*, pp. 1-3, San Diego, USA, 2024.
- [7] Chuang Xu and Alan Pak Tao Lau, “A Method of Generating Second-Order Soliton with Specified Time Positions,” in *Asia Communications and Photonics Conference (ACP)*, pp. 1-3, Wuhan, China, 2023.
- [8] Yan He, Zhiqun Zhai, Sai Chen, Huan Zhang, Chuang Xu, Liang Dou, Yuanchao Su, Chao Lu, Alan Pak Tao Lau, “Transceiver Penalty and Amplifier Noise Figure Characterization for Accurate QoT Estimation in Hyperscale Disaggregated DCI Networks,” in *Optical Fiber Communications Conference and Exhibition (OFC)*, pp. 1-3, San Francisco, USA, 2025.
- [9] Liwang Lu, Chuang Xu, Jingchun Wang, Cheng Chen, Chao Lu, Yaxi Yan, and Alan Pak Tao Lau, “Integrated sensing and communications using external cavity laser with laser frequency comb,” in *Optical Fiber Communications Conference and Exhibition (OFC)*, pp. 1-3, San Francisco, USA, 2025.

List of Acronyms

AOM	Acousto-Optic Modulator
APSK	Amplitude Phase Shift Keying
ASE	Amplified Spontaneous Emission
BER	Bit Error Rate
BPS	Blind Phase Search
BR	Back Rotation
CD	Chromatic Dispersion
CMA	Constant Modulus Algorithm
CFR	Carrier Frequency Recovery
CPR	Carrier Phase Recovery
DAC	Digital-to-Analog Converter
DBP	Digital Back-Propagation
DCF	Dispersion Compensating Fiber
DP	Dual-Polarization
DSP	Digital Signal Processing
EDFA	Erbium-Doped Fiber Amplifier
EVM	Error Vector Magnitude
FO	Frequency Offset
INFT	Inverse Nonlinear Fourier Transform
IST	Inverse Scattering Transform
IQ	In-phase and Quadrature
LMMSE	Linear Minimum Mean Square Error
LO	Local Oscillator
ME	Manakov Equation
MIMO	Multiple Input Multiple Output

ML	Machine Learning
MZ	Mach-Zehnder
NFT	Nonlinear Fourier Transform
NFDM	Nonlinear Frequency Division Multiplexing
NLSE	Nonlinear Schrodinger Equation
NZ-DSF	Non-Zero Dispersion-Shifted Fiber
OOK	On-Off Keying
PBC	Polarization Beam Combiner
PBS	Polarization Beam Splitter
PDE	Partial Differential Equation
PDL	Polarization Dependent Loss
PMD	Polarization Mode Dispersion
PSK	Phase Shift Keying
QAM	Quadrature Amplitude Modulation
RDE	Radius-Directed Equalization
SMF	Single-Mode Fiber
SNR	Signal-to-Noise Ratio
SOP	State of Polarization
SP	Single-Polarization
SpS	Samples per Symbol

List of Symbols

Physical Items

α	fiber power loss coefficient
$\Delta\beta_0$	propagation constant mismatch between X and Y polarization
$\Delta\beta_1$	group-velocity mismatch between X and Y polarization
β_2	group velocity dispersion coefficient
$\hat{\boldsymbol{\beta}}$	birefringence vector in the Stokes space
$E(l, \tau)$	complex envelope of SP signal
$\boldsymbol{E}(l, \tau)$	complex envelope of DP signal
l, τ	physical spatial, temporal variable
L_A	span length
L_D	dispersion length
T_0	time normalization parameter
$f(z)$	signal amplitude profile along transmission
R_s	symbol rate
$\boldsymbol{R}(l)$	continuous polarization rotation profile of signal along l
$\boldsymbol{R}_i$	abrupt polarization rotation of signal after the i^{th} span
$\mathbf{I}$	identity matrix
γ	Kerr nonlinearity coefficient
γ'	PDL in linear unit
ω	angular frequency
$\boldsymbol{\sigma}$	logarithm of PDL matrix
$\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3$	Pauli matrices
$\hat{\boldsymbol{\sigma}}$	$[\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3]^T$, “Pauli matrices” vector in form

NFT Items

α	real part of the eigenvalue
β	imaginary part of the eigenvalue
$\Delta\beta_0$	split in the initial β
$a(\lambda_i), a(\lambda)$	discrete, continuous a -coefficient
$b(\lambda_i), b(\lambda)$	discrete, continuous b -coefficient
b_i	discrete b -coefficient after the i^{th} EDFA
$\mathbf{b}(\lambda_i)$	discrete b -coefficient vector for DP signal
B_i	logarithm of the discrete b -coefficient after the i^{th} EDFA
ϵ_i	direct noise of B contributed by the i^{th} EDFA
L	normalized span length
Γ	normalized fiber power loss coefficient
Γ'	extra normalized fiber power loss coefficient
$\mathbf{L}, \mathbf{A}$	Lax pair for NLSE
$q(z, t)$	normalized complex envelope of SP signal
$\mathbf{Q}(z, t)$	normalized complex envelope of DP signal
z, t	normalized spatial, temporal variable
$Q_d(\lambda_i), Q_c(\lambda)$	discrete, continuous nonlinear spectrum
Q_{PA}	power-normalized path-averaged soliton
λ_i, λ	discrete, continuous eigenvalue
$\mathbf{v}$	eigenvector of $\mathbf{L}$
σ_α^2	variance of noise in α by each EDFA
σ_β^2	variance of noise in β by each EDFA
$\sigma_{\beta_0}^2$	variance of the split of initial β

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Chapter 1

Introduction

1.1 Background

Since the seminal proposal of Charles Kao [1] and the early commercial deployment of fiber-optic systems in the late 1980s, optical fiber communication has become the backbone of global data infrastructure, offering bandwidths orders of magnitude greater than alternative media such as copper wire or wireless links. Over the past three decades, both laboratory demonstrations and commercial systems have witnessed a dramatic increase in transmission capacity as illustrated in Fig. 1.1. However, sustaining such an exponential growth is facing fundamental challenges, primarily due to the nonlinear nature of optical fiber. State-of-the-art experiments are approaching the theoretical capacity limit of standard Single-Mode fiber (SMF), with only marginal growing space estimated at 20-30% [2, 3].

In its early days, fiber-optic communication was more a subject of applied physics than information theory. Data were transmitted using a simple On–Off Keying (OOK) scheme: a “light-on” pulse represented a binary “1” and “light-off” represented “0”. Such systems, known as Intensity Modulation and Direct Detection (IM/DD), dominated the first generation of optical links. The available bandwidth of fiber communication was effectively infinite compared with the data traffic needs back then. A transformative advance came with the invention of the Erbium-Doped Fiber Amplifier (EDFA), which, along with the development of in-line dispersion compensation techniques, effectively extends the unregenerated (no electro-optical regeneration) transmission distance. Remarkably, the gain bandwidth of EDFA aligns almost perfectly with the minimum-loss window of silica fiber—a fortunate coincidence that catalyzed the rapid scaling of Wavelength Division Multiplexing (WDM) systems, allowing nearly a hundred optical channels to co-propagate in a single fiber, thereby multiplying the aggregate capacity by orders of magnitude.

Another leap occurred around 2010 with the adoption of coherent detection and digital signal processing (DSP) in fiber-optic communication. These innovations unlocked

full control over the optical field, introducing new degrees of freedom in modulation, and virtually all long-haul fiber systems rely on digital coherent technology today. Crucially, coherent DSP eliminates the need for in-line dispersion-compensating modules; instead, chromatic dispersion accumulated over thousands of kilometers can be digitally reversed at the receiver, simplifying system design and extending reach. In addition, such dispersion-unmanaged transmission turned out to be more tolerant to fiber nonlinear impairments. Together with rapid advances in high-speed electronics, these innovations have driven per-wavelength data rates from 10 Gbps to 100 Gbps and beyond; the latest commercial deployments have achieved 1.6 Tbps per wavelength [4]. However, further increasing the system capacity is becoming progressively more difficult, primarily because of the nonlinear effects present in optical fibers.

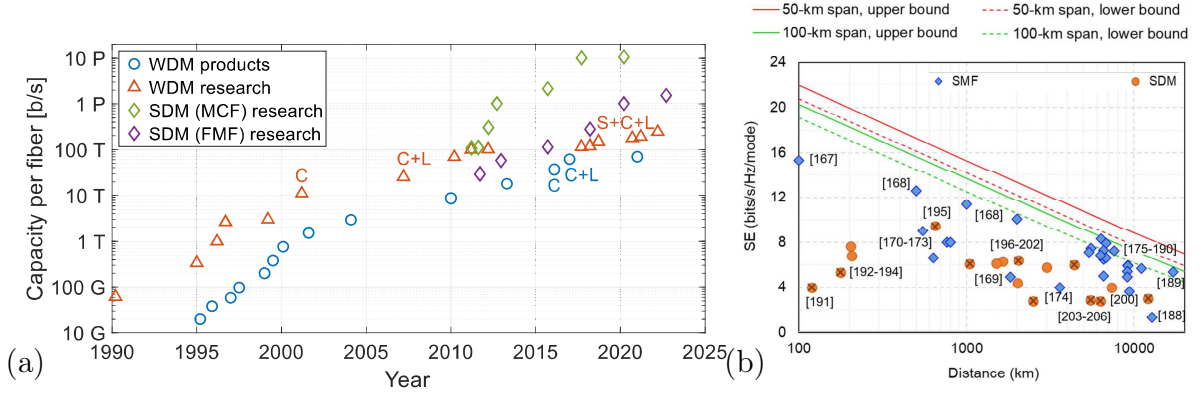


Figure 1.1: (a) Evolution of fiber communication capacity. Reproduced from [5]; (b) State-of-the-art experiments are approaching the theoretical capacity limit of standard SMF. Reproduced from [3]

When a light pulse propagates in an optical fiber, it will be jointly affected by three main factors: fiber attenuation, fiber dispersion, and fiber nonlinearity. Thanks to EDFA and digital coherent transmission technologies, the attenuation and dispersion problems have become trivial. However, the fiber nonlinearity effect is in a cubic proportion to the signal power, which means the Signal-to-Noise Ratio (SNR) will not monotonically increase with increasing signal power, but turn down after a certain threshold, leading to the “nonlinear Shannon limit” (Fig. 1.2(a)). Moreover, nonlinearity-caused signal distortions are hard and expensive to remove, and this issue has long been the central research problem in the fiber optic communication field.

Various schemes have been investigated to alleviate the detrimental effects of fiber nonlinearity, such as Digital Back-Propagation (DBP), Machine Learning (ML), and Anti-Resonant Fiber (ARF).

- DBP was proposed to recover the signal by letting it transmit through a “virtual fiber system” in the digital domain [7]. The virtual system has parameters exactly

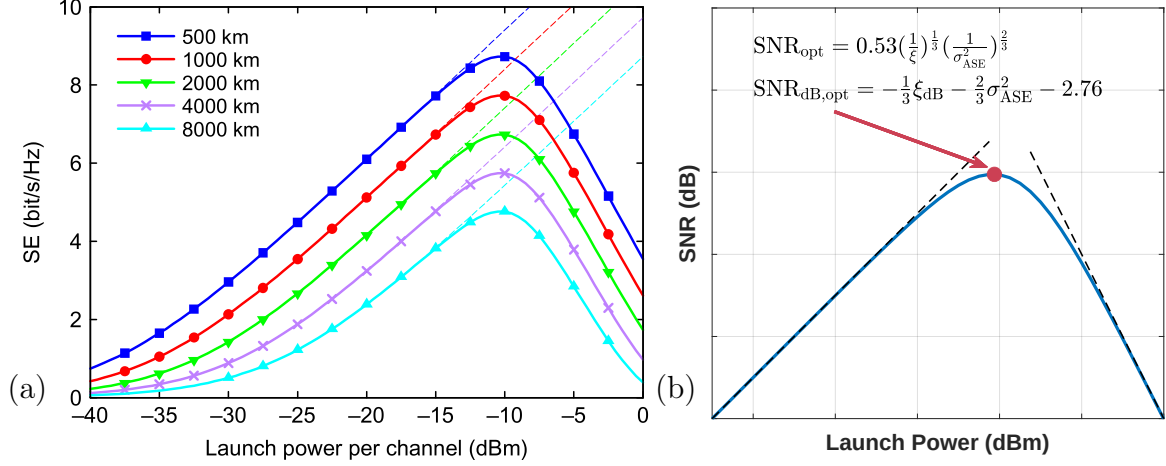


Figure 1.2: (a) Nonlinear Shannon limit to the fiber-optic channel capacity. Reproduced from [6, Chap. 20]; (b) Optimal SNR vs. launch power, showing 1/3 dB SNR gain per 1 dB reduction in nonlinear noise. ξ is the nonlinear noise coefficient.

opposite to the real system, so hopefully the nonlinear noise generated by the real and virtual link will cancel each other out. This method is straightforward but imposes a heavy calculation burden, impeding its implementation. Instead, it became a benchmark for evaluating other fiber nonlinear compensation algorithms.

- ML has received a tremendous boost in recent years across multiple disciplines, and fiber optic communication research has increasingly embraced its methodologies. In a sense, this situation in turn reflects that fiber nonlinearity compensation is indeed a tough task. The combination of ML and DBP is investigated, and it's found that ML can provide a Q-factor gain of up to 1.5 dB [8].
- ARF [9], a particular type of hollow-core fiber that emerged in recent years, offers a completely different view angle to this problem. The hollow-core characteristic gives it the inherent super-low nonlinearity (almost zero), and by continuous improvement on both design and manufacturing ends, the attenuation of ARF has been reduced to < 0.11 dB/km, even lower than the best record (< 0.14 dB/km) of conventional standard SMF, and moreover, ARF can support a very wide and tunable spectral range. However, this attractive fiber requires very precise manufacturing crafts, and the yield is still quite low (hence high cost), which hinders its application by far.

Besides, the Gaussian Noise (GN) model and its variants [10–12] are well-established analytical tools for estimating the power of nonlinear noise in current “pseudo-linear” coherent transmission systems. These systems operate primarily in the linear regime, so fiber nonlinearity can be treated as a perturbation. By accounting for both amplified spontaneous emission noise from optical amplifiers and nonlinear distortions, an effective

SNR can be derived. The analytical expression predicts an optimal launch power and the corresponding maximum SNR. It shows that reducing the strength of nonlinear noise by 1 dB yields only about a 1/3 dB improvement in overall SNR (Fig. 1.2(b)) [3], highlighting that fiber nonlinearity compensation is challenging and offers diminishing returns.

While current systems are mostly designed for linear transmission, equipped with compensation techniques aiming at suppressing the negative impact of fiber nonlinearity, a novel method named Nonlinear Frequency Division Multiplexing (NFDM) in optical communications has been growing in recent years. It embraces fiber nonlinearity as the inseparable intrinsic property of the fiber channel, rather than regarding it as an undesirable effect, and thus is expected to be paradigm-shifting for the next generation of fiber optic communication.

1.2 Soliton and discrete NFDM

Soliton, a classical and fancy object in mathematics and physics, is a special nonlinear waveform that can preserve its shape during propagation. Being highly nonlinear, it can even recover shape and velocity after collision with other solitons, showing particle-like properties. Naturally, it attracted enormous attention from fiber optic communication researchers in the 1980-90s [13] due to the intrinsic “unchanged” property, which is very suitable for communication applications—of course, we wish the pulses at the receiver are unchanged compared to those at the transmitter, so the information will not be distorted. People developed beautiful fiber soliton theories and demonstrated soliton experiments with incredible transmission distances (up to tens of millions of kilometers!) in that era [14]. However, the information rate became a major limit for soliton transmission due to its incompatibility with advanced modulation formats back then. The traditional soliton transmission scheme has become obsolete with the rise of the digital coherent transmission scheme widely deployed today, and the research interest in soliton transmission has also dropped sharply.

This situation started to change in recent years—soliton communication regained some sight, and it was absorbed into a more generalized concept—NFDM communication [15–17]. Soliton transmission turns out to be one branch of it, known as the discrete NFDM transmission. Researchers realized that the information can be embedded in the nonlinear spectrum (which can generally be discrete or continuous) rather than in the time-domain waveform or linear spectrum. By doing so, the information carried by light will not be distorted by the fiber’s nonlinearity but will just go through linear evolution, if the fiber has no attenuation. This innovation was inspired by an elegant and ingenious analytic method for solving the NLSE, namely the Nonlinear Fourier

Transform (NFT), developed by mathematicians [18]. Being special in the time domain, a soliton is also special in the nonlinear frequency domain—it only has a discrete nonlinear spectrum and no continuous nonlinear spectrum at all. Thanks to the development of new theories (NFDM, fast NFT algorithm [19]) and the mature digital coherent transmission technology, information can be encoded on the discrete eigenvalue and discrete spectrum of soliton in practice, which means, if seeing in the time domain, not only the amplitude but also the phase, time position, and velocity of soliton can be modulated, leading to essentially a modern version of soliton transmission.

The prototype of NFDM transmission was first proposed in 1993 [20], i.e., the golden era of soliton transmission. Different from the mainstream, the authors used the soliton's eigenvalue to carry the information because, according to the NFDM theorem, the eigenvalue is invariant throughout the whole transmission path. However, the first experimental demonstration had not been done until 2015 [21], because in the early years, numerical tools for eigenvalue calculation of a signal were not readily available. Since then, more and more works have been done to explore the potentials of NFDM, ranging from using discrete spectrum only [21], continuous spectrum only [22] to discrete-continuous-joint spectrum [23], from Single Polarization (SP) to Dual Polarization (DP) [24, 25], from EDFA-based systems [24] to Raman amplification-based systems [25].

The transmission rates of those works, however, were relatively low, partly due to the inconsistency between reality and the precondition for the NFDM theorem, which requires the optical fiber to be lossless. The path-averaged model [26] was proposed to reduce this mismatch, but NFDM still has not fulfilled its all potential. This thesis focuses on investigating the mechanism of discrete NFDM systems and develops the field by proposing algorithms for its practical implementation.

1.3 Research status and gaps

Research status

In 2014, Yousefi and Kschischang formalized the theory of NFDM. The mathematical foundations, numerical methods, and modulation schemes of NFDM were systematically proposed in their trilogy papers [15–17], in which they proposed that both discrete and continuous spectrum can be information carriers. The first experimental demonstration of NFDM was soon conducted in 2015, where four discrete eigenvalues were modulated, achieving 1.5 Gbps [27]. This is a relatively simple version of NFDM, as each subcarrier (in this case, the discrete eigenvalue) is On-Off Keying (OOK)-modulated, hence giving 4 bits/symbol in total. Alternatively, the discrete spectrum can also be modulated but usually with a single deterministic discrete eigenvalue, which corresponds to a 1st-order time-domain soliton that is more stable in transmission and occupies less

bandwidth¹. Amplitude and Phase Shift Keying (APSK) were found to be a better choice of modulation for discrete spectrum over Quadrature Amplitude Modulation (QAM) [28], and Linear Minimum Mean Square Error (LMMSE) filter was used to denoise the discrete spectrum as the noise on eigenvalue and that on discrete spectrum was found to be highly correlated [29]. Besides, b -coefficient was found to be a better information carrier than the discrete spectrum [29]. In [30], both eigenvalue and b -coefficient were modulated by 16QAM and 2A8SK, respectively, which gives 8 bits/symbol, and a record of 54 Gbps over 1200 km transmission was achieved. The adjacent soliton pulses were set with eigenvalue that has opposite real values to bring in designed symbol interactions, which turns out to be beneficial to the signal quality [31]. Thanks to the fact that the Manakov Equation (ME) governing the Dual-Polarization (DP) signal propagation is also integrable by the NFT method, DP NFDM system theory [32] can be developed. The DP NFDM was first experimentally demonstrated with discrete spectrum modulation, reaching 8 Gbps and proving its feasibility [25].

Continuous NFDM generally has a higher information rate as it can contain many more subcarriers, ranging from single-wavelength 64 Gbps in the first experiment [33] to the latest 25-wavelength 433.3 Gbps [34] (and see the refs in [34]). However, a continuous NFDM system suffers from dispersion-induced pulse broadening, which requires adding a guard interval between signal bursts. Full-spectrum NFDM has also been proposed, where information is encoded on both the discrete and continuous spectrum. These studies show the possibility of fully utilizing the available degrees of freedom in the nonlinear spectrum, but the improvement of information rate is usually limited by adding the discrete part [23, 33, 35, 36].

Research gaps.

Firstly, the simple equalization method promised by the NFDM theory becomes problematic for discrete NFDM systems because the discrete eigenvalues are perturbed by noise loading during transmission. Instead, LMMSE is widely used as an effective equalization method in discrete NFDM systems. However, this method is rather phenomenological without probing into the underlying mechanism. Besides, LMMSE involves using a training sequence, which reduces the information rate of discrete NFDM systems.

Secondly, as a novel communication scheme, a lot of work on NFT and NFDM theories themselves needed to be done in the early development stage. Thus, investigation on the DSP for practical NFDM transmission has long been shelved, especially for the polarization demultiplexing algorithm. NFDM studies are usually conducted under

¹ Modulation only on the discrete eigenvalue or discrete spectrum are both special case of discrete NFDM—the former has multiple OOK-modulated subcarriers, while the latter has only one subcarrier that is high-order-modulated.

relatively ideal transmission conditions, or through workarounds (e.g., polarization recovery by manual polarization controller or training sequence), which hinder the application of NFDM in realistic fiber optic systems. Hence, a full DSP flow that allows for various practical impairments is needed for (especially, DP) discrete NFDM transmission.

Thirdly, after a decade of development, the information rate of discrete NFDM is still low, although advanced modulation formats have been enabled in modern soliton transmission with the help of NFDM theory and digital coherent techniques. While the technical craft consistently needs improvement, there may be fundamental restrictions underlying that need to be investigated.

1.4 Outline and contributions

The structure of this thesis is as follows:

- In **Chapter 2**, necessary theoretical foundations on digital coherent transmission, NFT, and NFDM, especially the connection between optical soliton and discrete NFDM, are introduced.
- In **Chapter 3**, the performance gain by deploying a DP discrete NFDM system with misalignment on the orthogonal polarizations is numerically and experimentally investigated.
- In **Chapter 4**, based on proper modeling of the noise accumulation in a discrete NFDM system, half back rotation algorithm is proposed as a simple yet effective equalization method, which achieves the same level of performance as the conventionally-used LMMSE filter. Numerical and experimental verification with conducted and practical non-ideal factors analysis.
- In **Chapter 5**, a full DSP flow that completely works in the nonlinear frequency domain is proposed to enable the DP discrete NFDM system to work in a practical situation. Numerical and experimental verification are conducted with practical non-ideal factors analysis.
- In **Chapter 6**, the noise evolution model for discrete NFDM is further discussed, and the optimal symbol rate of the discrete NFDM system is derived and numerically obtained, which reaches a balance between “direct noise ”and “transfer noise”.
- In **Chapter 7**, conclusions and outlooks for future research are given.

The principal contributions of this thesis are:

- Proposed half back rotation as a conceptually concise equalization algorithm for an SP discrete NFDM system, which retrieves the promised simplicity of the NFDM scheme. It achieves the same performance as the conventional method but eliminates the need for training sequences, and therefore increases the information rate. Besides, the analysis methodology provides another proof of the well-known Gordon-Haus effect and shows that a similar effect also exists on the soliton phase evolution. The mechanism of correlation between noise on the discrete eigenvalue and b -coefficient is elucidated.
- Proposed a polarization demultiplexing algorithm and extended the back rotation algorithm for DP discrete NFDM systems. With these two critical steps accomplished, a full DSP flow for the DP discrete NFDM system is proposed, which paves the way for systematic investigation and practical implementation of discrete NFDM transmission.
- Pointing out the fundamental limitation on increasing symbol rate in discrete NFDM systems, revealing why the symbol rates of discrete NFDM systems have been low over the years.
- Analyzed the influence of non-ideal factors, including bias control, polarization-dependent loss, and gain-loss imbalance, on the discrete NFDM system, which reveals its sensitivity to the practical environment.

Chapter 2

Preliminaries

This chapter provides the necessary preliminaries for understanding the mechanism of NFT and NFDM. Fundamentals of digital coherent transmission systems are included as well because NFDM is realized based on coherent detection that recovers the full electrical field of the signal.

2.1 Digital coherent transmission system

Digital coherent transmission technologies enable transmission of high-volume data over long distances, with the merits of high spectral efficiency and resilience to channel impairments. As shown in Fig. 2.1, the typical structure of a digital coherent transmission system can be divided into four parts—the transmitter (for modulation and launching signal), the fiber channel (nonlinear propagation), the receiver front-end (for coherent detection), and finally the DSP unit (for channel impairments compensation and signal recovery).

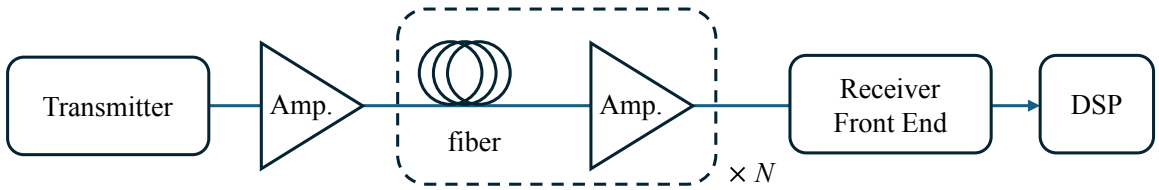


Figure 2.1: Digital coherent transmission system consists of a transmitter, a fiber channel, a receiver front-end, and DSP. Amp. means amplifier, which can be an EDFA or a Raman amplifier.

2.1.1 Transmitter

At the transmitter, information is digitally encoded in the waveform directly or in the (linear or nonlinear) spectrum first, which is then transformed into the corresponding

waveform. These digital signals are then converted into baseband signals via digital-to-analog converters (DACs). The baseband signal subsequently modulates an optical carrier generated by a laser.

The modulation is usually achieved with a Mach–Zehnder (MZ) modulator, which consists of two phase modulators in its up and down arms. As shown in Fig. 2.2, the optical signal is first equally split into two arms and modulated by different phase shifts, and equally recombined at the output. The transfer function of the MZ modulator is given by the ratio of the output electrical field to the input one:

$$\frac{E_{\text{out}}(t)}{E_{\text{in}}(t)} = \frac{1}{2} \left(e^{j\varphi_{\text{PM}_1}(t)} + e^{j\varphi_{\text{PM}_2}(t)} \right), \quad (2.1)$$

where $\varphi_{\text{PM}_1}(t)$ and $\varphi_{\text{PM}_2}(t)$ denote the phase shifts applied to the optical signals by the upper and lower arms of the modulator, respectively.

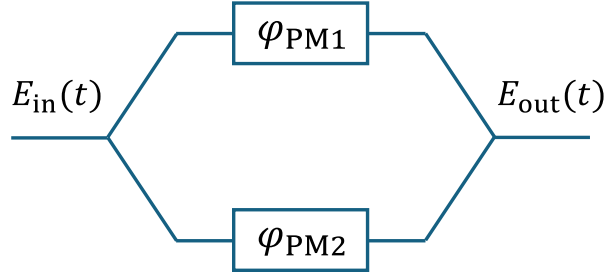


Figure 2.2: Structure of an MZ modulator.

The overall structure becomes an amplitude modulator if the phase shift in the two arms is opposite to each other (namely “push-pull” mode), $\varphi_{\text{PM}_1}(t) = -\varphi_{\text{PM}_2}(t) = \varphi(t)$,

$$\begin{aligned} \frac{E_{\text{out}}(t)}{E_{\text{in}}(t)} &= \frac{1}{2} \left(e^{j\varphi(t)} + e^{-j\varphi(t)} \right) \\ &= \cos[\varphi(t)] \\ &= \cos \left[\frac{V_{\text{bias}} + m(t)}{2V_{\pi}} \pi \right], \end{aligned} \quad (2.2)$$

where $m(t)$ is the baseband signal, V_{bias} is the bias voltage level, and V_{π} corresponds to the voltage range that the output optical power goes through from its maximum to minimum when no modulation signal is applied. If let $V_{\text{bias}} = -V_{\pi} + 2kV_{\pi}$, $k \in \mathbb{Z}$,

$$\frac{E_{\text{out}}(t)}{E_{\text{in}}(t)} = \sin \left[\frac{m(t)}{2V_{\pi}} \pi \right] \approx \frac{m(t)}{2V_{\pi}} \pi. \quad (2.3)$$

Then, the optical field is linearly modulated by $m(t)$ in amplitude, given that V_{bias} is properly set at $-V_{\pi} + 2kV_{\pi}$ and swing range of $m(t)$ is moderate, as shown in Fig. 2.4. The bias offset point is called “Null Point” since the output will be zero if no modulation signal is fed.

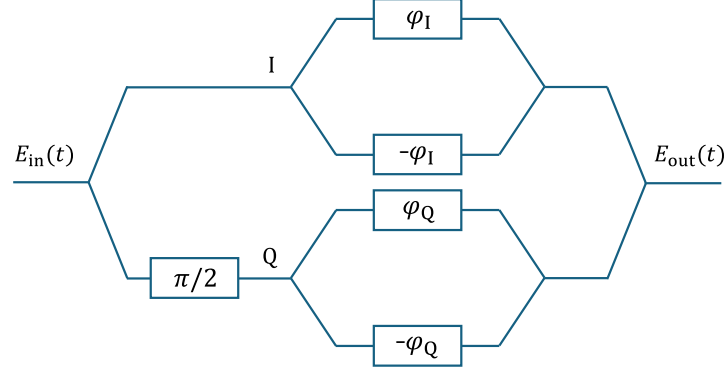


Figure 2.3: Structure of an IQ modulator.

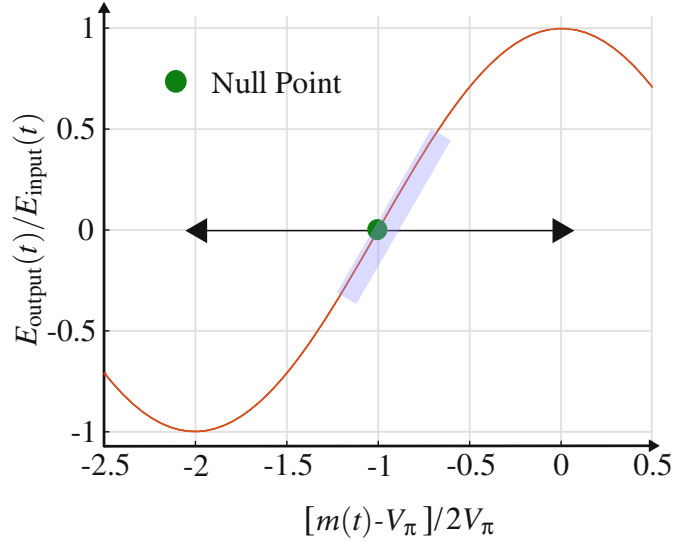


Figure 2.4: Transfer function of MZ modulator is approximately linear when the bias voltage is offset at $-V_\pi$ and $m(t)$ amplitude is moderate. Adapted from [37, Chap. 2.3] with permission from Springer Nature.

To realize an in-phase and quadrature (IQ) modulator, two MZ modulators work in parallel except that one of them is concatenated with a $\pi/2$ phase shifter, as shown in Fig. 2.3. Each of them serves as an amplitude modulator. The transfer function becomes

$$\frac{E_{\text{out}}(t)}{E_{\text{in}}(t)} \approx \frac{m_I(t)}{2V_{\pi,I}}\pi + j\frac{m_Q(t)}{2V_{\pi,Q}}\pi. \quad (2.4)$$

Furthermore, if we set two IQ modulators for the X and Y polarizations of the carrier, respectively, a DP-IQ modulator can be realized. As shown in Fig. 2.5, the optical carrier is first split into X and Y polarizations by a polarization beam splitter (PBS), and independently modulated by two IQ modulators before being recombined by a polarization beam combiner (PBC), forming a DP-IQ modulated signal. By doing so, the spectral efficiency can be doubled, as the X and Y polarizations of the optical carrier carry independent information simultaneously, a technique known as polarization multiplexing, which has become a standard practice in digital coherent transmission

systems.

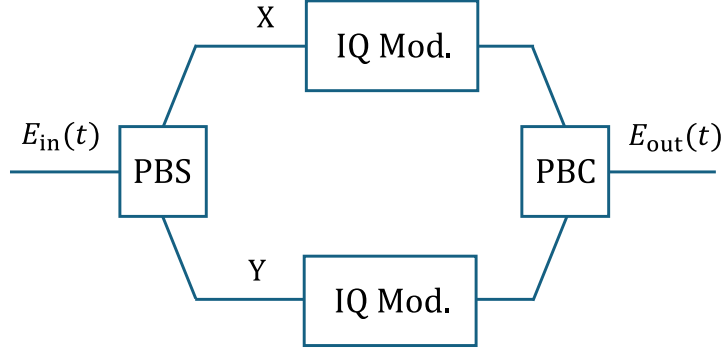


Figure 2.5: Structure of a DP-IQ modulator.

2.1.2 Optical fiber channel

Scalar case

The governing equation of propagation of a scalar signal in an SMF is the Nonlinear Schrodinger's Equation (NLSE), given by:

$$j\frac{\partial E(l, \tau)}{\partial l} - \frac{\beta_2}{2}\frac{\partial^2 E(l, \tau)}{\partial \tau^2} + \gamma|E(l, \tau)|^2 E(l, \tau) = 0, \quad (2.5)$$

where fiber loss, polarization effect, higher-order dispersion, Raman effect, and noise are neglected. $E(l, \tau)$ is the complex envelope of the optical signal, τ (s) is the temporal variable in the time-retarded frame, l (m) is the spatial variable, β_2 (s²/m) is the group velocity dispersion coefficient¹, and γ (1/W/m) is the Kerr fiber nonlinearity coefficient.

The second term describes the effect of fiber (group velocity) dispersion. When neglecting the effect of Kerr nonlinearity, fiber dispersion results in pulse broadening during propagation:

$$j\frac{\partial E(l, \tau)}{\partial l} - \frac{\beta_2}{2}\frac{\partial^2 E(l, \tau)}{\partial \tau^2} = 0, \quad (2.6)$$

which is readily solved in the frequency domain. Taking Fourier transform of E with respect to τ , $\tilde{E}(l, \omega) = \int_{-\infty}^{\infty} E(l, \tau)e^{-j\omega\tau} d\tau$, we have

$$j\frac{\partial \tilde{E}(l, \omega)}{\partial l} + \frac{\beta_2}{2}\omega^2 \tilde{E}(l, \omega) = 0, \quad (2.7)$$

$$\tilde{E}(l, \omega) = \tilde{E}(0, \omega)e^{j\frac{\beta_2}{2}\omega^2 l}, \quad (2.8)$$

¹ β_2 can be positive or negative, corresponding to normal dispersion and anomalous dispersion, respectively. In this thesis, we only discuss the case of $\beta_2 < 0$.

and the time-domain solution can be obtained by taking the inverse Fourier transform,

$$\begin{aligned} E(l, \tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{E}(l, \omega) e^{j\omega\tau} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{E}(0, \omega) e^{j\frac{\beta_2}{2}\omega^2 l} e^{j\omega\tau} d\omega. \end{aligned} \quad (2.9)$$

This integral can be expressed explicitly when the transmitted signal is a Gaussian pulse, $E(0, \tau) = \exp\left(-\frac{\tau^2}{2\tau_0^2}\right)$, where τ_0 is the pulse half-width at 1/e-power point.

$$E(l, \tau) = \frac{\tau_0}{(\tau_0^2 - j\beta_2 l)^{1/2}} \exp\left[-\frac{\tau^2}{2(\tau_0^2 - j\beta_2 l)}\right]. \quad (2.10)$$

Thus, during transmission, the pulse shape is still Gaussian, but the pulse half-width grows to

$$\begin{aligned} \tau(l) &= |\tau_0^2 - j\beta_2 l| \\ &= \tau_0 \left[1 + \left(\frac{l}{L_D}\right)^2\right]^{1/2}, \end{aligned} \quad (2.11)$$

where $L_D = \tau_0^2/|\beta_2|$ is defined as the dispersion length. After propagation of L_D , the pulse width is increased by a factor of $\sqrt{2}$.

The third term describes the effect of Kerr nonlinearity. When neglecting the effect of dispersion:

$$j \frac{\partial E(l, \tau)}{\partial l} + \gamma |E(l, \tau)|^2 E(l, \tau) = 0, \quad (2.12)$$

which is solved by $E(l, \tau) = E(0, \tau) e^{j\gamma |E(0, \tau)|^2 l}$, indicating Kerr nonlinearity introduces a nonlinear phase shift to the signal during propagation. The nonlinear phase shift is proportional to the pulse intensity in the time domain, and thus introduces instantaneous frequency change (frequency chirp) to the pulse, hence broadening of the pulse spectrum.

When both fiber dispersion and Kerr nonlinearity are taken into account, the evolution of the optical field $E(l, \tau)$ becomes highly complicated. Except for special cases such as soliton solutions, the NLSE is generally not analytically solvable. In practice, its solution is either obtained numerically—typically using the Split-Step Fourier Method (SSFM) [38, Chap. 2.4]—or semi-analytically via perturbation method [39].

Vector case

A single-mode optical fiber supports two orthogonally polarized modes that, in an ideal circularly symmetric fiber, are degenerate—i.e., they share the same propagation constant and thus the same phase and group velocities. However, real-world fibers always exhibit slight deviations from perfect circular symmetry due to imperfections introduced during manufacturing and installation. It breaks the degeneracy and induces birefringence, which is defined as a difference in the propagation constants of the two

polarization modes. To fully describe the signal field, a complex 2×1 vector is needed,

$$\mathbf{E}(l, \tau) = \begin{bmatrix} E_x(l, \tau) \\ E_y(l, \tau) \end{bmatrix}, \quad (2.13)$$

and the propagation of such a vectorial field in a constant linearly birefringent fiber is described by [40–42]

$$\begin{aligned} j \frac{\partial E_x}{\partial l} + \Delta\beta_0 E_x + j\beta_{1x} \frac{\partial E_x}{\partial \tau} - \frac{\beta_2}{2} \frac{\partial^2 E_x}{\partial \tau^2} + \gamma \left(|E_x|^2 + \frac{2}{3} |E_y|^2 \right) E_x + \frac{1}{3} \gamma E_y^2 E_x^* &= 0, \\ j \frac{\partial E_y}{\partial l} - \Delta\beta_0 E_y + j\beta_{1y} \frac{\partial E_y}{\partial \tau} - \frac{\beta_2}{2} \frac{\partial^2 E_y}{\partial \tau^2} + \gamma \left(\frac{2}{3} |E_x|^2 + |E_y|^2 \right) E_y + \frac{1}{3} \gamma E_x^2 E_y^* &= 0, \end{aligned} \quad (2.14)$$

where the birefringent axis are chosen to be aligned with X and Y axis, $\Delta\beta_0 = (\beta_{0,x} - \beta_{0,y})/2$ captures the difference of propagation constant between X and Y polarization, which results in polarization state evolution during propagation². $\beta_{1,x/y} = d\beta_{x,y}/d\omega$ is the reciprocal of the group velocity of X and Y polarization, and β_2 is assumed to be the same for both X and Y polarization. The 2/3 scaling factor of cross-phase modulation (XPM) from the opposite polarization originates from the symmetry in the 3rd-order susceptibility of silica [38, Chap. 6.1], [43, Chap. 14.2].

We can switch into the moving frame moving with the mean velocity of X and Y polarization by letting $\hat{E}_{x/y}(l, T) = E_{x/y}(l, \tau)$ with $T = l - \bar{\beta}_1 \tau$, $\bar{\beta}_1 = (\beta_{1x} + \beta_{1y})/2$, $\Delta\beta_1 = (\beta_{1x} - \beta_{1y})/2 > 0$,

$$\begin{aligned} j \frac{\partial \hat{E}_x}{\partial l} + \Delta\beta_0 \hat{E}_x + j\Delta\beta_1 \frac{\partial \hat{E}_x}{\partial T} - \frac{\beta_2}{2} \frac{\partial^2 \hat{E}_x}{\partial T^2} + \gamma \left(|\hat{E}_x|^2 + \frac{2}{3} |\hat{E}_y|^2 \right) \hat{E}_x + \frac{1}{3} \gamma \hat{E}_y^2 \hat{E}_x^* &= 0, \\ j \frac{\partial \hat{E}_y}{\partial l} - \Delta\beta_0 \hat{E}_y - j\Delta\beta_1 \frac{\partial \hat{E}_y}{\partial T} - \frac{\beta_2}{2} \frac{\partial^2 \hat{E}_y}{\partial T^2} + \gamma \left(\frac{2}{3} |\hat{E}_x|^2 + |\hat{E}_y|^2 \right) \hat{E}_y + \frac{1}{3} \gamma \hat{E}_x^2 \hat{E}_y^* &= 0, \end{aligned} \quad (2.15)$$

which can be put in a compact vectorial form (we switch T back to τ to release notation burden):

$$j \frac{\partial \mathbf{E}}{\partial l} + \Delta\beta_0 \boldsymbol{\sigma}_1 \mathbf{E} + j\Delta\beta_1 \boldsymbol{\sigma}_1 \frac{\partial \mathbf{E}}{\partial \tau} - \frac{\beta_2}{2} \frac{\partial^2 \mathbf{E}}{\partial \tau^2} + \gamma \left(|\mathbf{E}|^2 \mathbf{E} - \frac{1}{3} s_3 \boldsymbol{\sigma}_3 \mathbf{E} \right) = 0, \quad (2.16)$$

where $\mathbf{E} = [\hat{E}_x, \hat{E}_y]^T$, $s_3 = \mathbf{E}^\dagger \boldsymbol{\sigma}_3 \mathbf{E}$ and $\boldsymbol{\sigma}_i$ are the Pauli matrices,³

$$\boldsymbol{\sigma}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix}. \quad (2.17)$$

² It is common in the literature that 2.14 is commonly written without the second term with $\Delta\beta_0$, which is absorbed into the final term as a fast-changing contribution and then neglected. However, as we are concerned about the polarization state, the $\Delta\beta_0$ term should be kept.

³ This definition is usually used in polarization optics, which is an index-shifted version of the standard one proposed by Wolfgang Pauli for quantum mechanics.

In a long fiber, the birefringence axis is randomly changed along the transmission direction. This can be numerically modeled by concatenating short birefringent fiber sections, where the birefringence is linear and constant within each section, but randomly distributed in strength and orientation angle among sections. Analytically, the random birefringence change can be included by introducing l -dependence of $\Delta\beta_0$, $\Delta\beta_1$, and varying azimuth angle θ of the birefringence axis:

$$j\frac{\partial \mathbf{E}}{\partial l} + \Delta\beta_0(l)\boldsymbol{\sigma}_\theta \mathbf{E} + j\Delta\beta_1(l)\boldsymbol{\sigma}_\theta \frac{\partial \mathbf{E}}{\partial \tau} - \frac{\beta_2}{2} \frac{\partial^2 \mathbf{E}}{\partial \tau^2} + \gamma \left(|\mathbf{E}|^2 \mathbf{E} - \frac{1}{3} s_3 \boldsymbol{\sigma}_3 \mathbf{E} \right) = 0, \quad (2.18)$$

where $\boldsymbol{\sigma}_\theta = \cos[\theta(l)]\boldsymbol{\sigma}_1 + \sin[\theta(l)]\boldsymbol{\sigma}_2$ resulting from the linear birefringence with azimuth angle of $\theta/2$.⁴

If we assume $\mathbf{E}$ is a single-frequency continuous wave and ignore the nonlinear terms, then 2.18 reduces to the equation that describes the polarization evolution of $\mathbf{E}$,

$$j\frac{\partial \mathbf{E}}{\partial l} + \Delta\beta_0(l)\boldsymbol{\sigma}_\theta \mathbf{E} = 0, \quad (2.19)$$

which can be derived as follows. The Jones matrix of a short fiber section with length of δ , linear birefringence of $\Delta\beta_0$, azimuth angle of $\theta/2$ is

$$\mathbf{r} \begin{bmatrix} e^{+j\Delta\beta_0\delta} & 0 \\ 0 & e^{-j\Delta\beta_0\delta} \end{bmatrix} \mathbf{r}^{-1} \quad \text{with} \quad \mathbf{r} = \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix}. \quad (2.20)$$

Let $\mathbf{E}$ go through it, we obtain $\mathbf{E}(l + \delta) = \mathbf{r} \begin{bmatrix} e^{+j\Delta\beta_0\delta} & 0 \\ 0 & e^{-j\Delta\beta_0\delta} \end{bmatrix} \mathbf{r}^{-1} \mathbf{E}(l)$. Hence, the differential equation of $\mathbf{E}(l)$ is given by ($\mathbf{I}$ is for the identity matrix)

$$\begin{aligned} \frac{\partial \mathbf{E}(l)}{\partial l} &= \lim_{\delta \rightarrow 0} \frac{\mathbf{E}(l + \delta) - \mathbf{E}(l)}{\delta} = \left[\mathbf{r} \begin{bmatrix} e^{+j\Delta\beta_0\delta} & 0 \\ 0 & e^{-j\Delta\beta_0\delta} \end{bmatrix} \mathbf{r}^{-1} - \mathbf{I} \right] \frac{\mathbf{E}(l)}{\delta} \\ &= \left[\mathbf{r} \begin{bmatrix} 1 + j\Delta\beta_0\delta & 0 \\ 0 & 1 - j\Delta\beta_0\delta \end{bmatrix} \mathbf{r}^{-1} - \mathbf{I} \right] \frac{\mathbf{E}(l)}{\delta} \\ &= j\Delta\beta_0 \mathbf{r} \boldsymbol{\sigma}_1 \mathbf{r}^{-1} \mathbf{E}(l) \\ &= j\Delta\beta_0 \boldsymbol{\sigma}_\theta \mathbf{E}(l), \end{aligned} \quad (2.21)$$

which is for the linear birefringence case. Generally, the polarization evolution for arbitrary birefringence is described by [43, Chap. 7.5]

$$j\frac{\partial \mathbf{E}}{\partial l} + \frac{1}{2} \hat{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{\sigma}} \mathbf{E} = 0, \quad (2.22)$$

where $\hat{\boldsymbol{\beta}}$ is the 3×1 birefringence vector in Stoke space, and $\hat{\boldsymbol{\sigma}} = [\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3]^T$ is a 3×1 vector in form. In our linear birefringence case, $\hat{\boldsymbol{\beta}} = 2\Delta\beta_0[\cos \theta, \sin \theta, 0]^T$, $\hat{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{\sigma}} = 2\sigma_\theta$.

⁴ The azimuth angle θ lives in the Stokes space, so the corresponding orientation angle in Jones space is $\theta/2$.

Therefore, we may separate the fast polarization change induced by the random birefringence from the slow-varying effect induced by dispersion and nonlinearity by letting $\mathbf{E}(l, \tau) = \mathbf{R}(l)\mathbf{Q}(l, \tau)$, with $\mathbf{R}(l)$ satisfying $d\mathbf{R}/dl = j\Delta\beta_0\boldsymbol{\sigma}_\theta\mathbf{R}$, $\mathbf{R}(0) = \mathbf{I}$, which translates 2.18 into the evolution of $\mathbf{Q}(l, \tau)$

$$j\frac{\partial\mathbf{Q}}{\partial l} + j\Delta\beta_1(l)\bar{\boldsymbol{\sigma}}_\theta(l)\frac{\partial\mathbf{Q}}{\partial\tau} - \frac{\beta_2}{2}\frac{\partial^2\mathbf{Q}}{\partial\tau^2} + \gamma\left(|\mathbf{Q}|^2\mathbf{Q} - \frac{1}{3}\bar{s}_3\bar{\boldsymbol{\sigma}}_3(l)\mathbf{Q}\right) = 0 \quad (2.23)$$

with $\bar{\boldsymbol{\sigma}}_\theta(l) = \mathbf{R}^{-1}\boldsymbol{\sigma}_\theta\mathbf{R}$, $\bar{\boldsymbol{\sigma}}_3(l) = \mathbf{R}^{-1}\boldsymbol{\sigma}_3\mathbf{R}$, $\bar{s}_3 = \mathbf{Q}^\dagger\boldsymbol{\sigma}_3\mathbf{Q}$, and rewrite it into

$$j\frac{\partial\mathbf{Q}}{\partial l} - \frac{\beta_2}{2}\frac{\partial^2\mathbf{Q}}{\partial\tau^2} + \frac{8}{9}\gamma|\mathbf{Q}|^2\mathbf{Q} = -j\Delta\beta_1(l)\bar{\boldsymbol{\sigma}}_\theta(l)\frac{\partial\mathbf{Q}}{\partial\tau} + \frac{1}{3}\gamma\left(\bar{s}_3\bar{\boldsymbol{\sigma}}_3(l)\mathbf{Q} - \frac{1}{3}|\mathbf{Q}|^2\mathbf{Q}\right). \quad (2.24)$$

Due to fast polarization change along transmission, we have $\langle s_1 \rangle = \langle s_2 \rangle = \langle s_3 \rangle = \frac{1}{3}|\mathbf{E}|^2$, where $\langle \cdot \rangle$ means taking average. Besides, $\mathbf{R}$ keeps unitary during propagation⁵, so $|\mathbf{Q}|^2 = |\mathbf{E}|^2$, and note that $\bar{s}_3 = s_3$. Therefore, $\langle \bar{s}_3 \rangle = \frac{1}{3}|\mathbf{Q}|^2$, indicating that the second term of the RHS may be ignored. If further neglecting the first term of the RHS, which is responsible for Polarization Mode Dispersion (PMD), we arrive at the Manakov Equation (ME) [41, 42]:

$$j\frac{\partial\mathbf{Q}(l, \tau)}{\partial l} - \frac{\beta_2}{2}\frac{\partial^2\mathbf{Q}(l, \tau)}{\partial\tau^2} + \frac{8}{9}\gamma|\mathbf{Q}(l, \tau)|^2\mathbf{Q}(l, \tau) = 0. \quad (2.25)$$

The ME separates the fast polarization state evolution and the slow evolution due to fiber dispersion and nonlinearity, and is widely used in modeling dual-polarization optical fiber systems. In 1974, S. V. Manakov [44] solved ME by means of NFT, and showed the existence of soliton solutions for ME.

It is conventional and useful to transform NLSE (2.5) and ME (2.25) into dimensionless form, which will also facilitate the establishment of their corresponding NFT:

$$j\frac{\partial q(z, t)}{\partial z} + \frac{1}{2}\frac{\partial^2 q(z, t)}{\partial t^2} + |q(z, t)|^2 q(z, t) = 0 \quad (2.26)$$

$$j\frac{\partial \mathbf{q}(z, t)}{\partial z} + \frac{1}{2}\frac{\partial^2 \mathbf{q}(z, t)}{\partial t^2} + |\mathbf{q}(z, t)|^2 \mathbf{q}(z, t) = 0 \quad (2.27)$$

by variable substitution:

$$q(z, t) = \frac{E(l, \tau)}{\sqrt{P_{sp}}}, \quad \mathbf{q}(z, t) = \frac{\mathbf{Q}(l, \tau)}{\sqrt{P_{dp}}}, \quad z = \frac{l}{L_D}, \quad t = \frac{\tau}{T_0} \quad (2.28)$$

where $P_{sp} = 1/(\gamma L_D)$, $P_{dp} = 1/(\frac{8}{9}\gamma L_D)$, $L_D = T_0^2/|\beta_2|$ (L_D is known as the dispersion length) and T_0 is a freely chosen time normalization parameter.

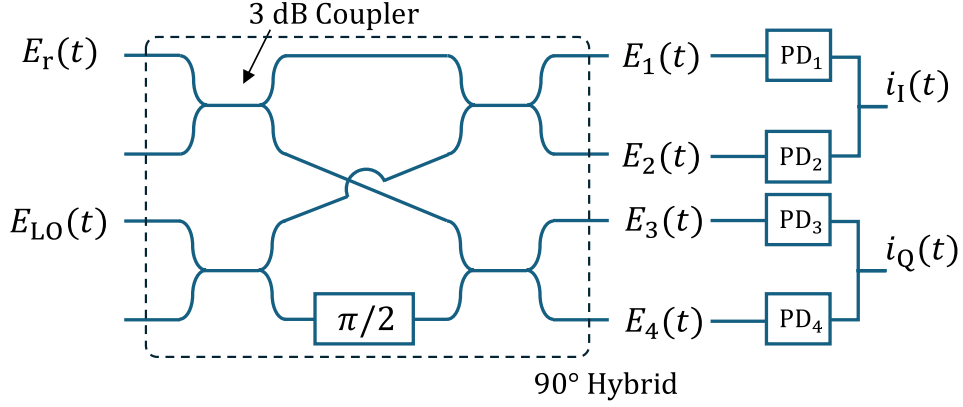


Figure 2.6: Structure of an SP IQ receiver.

2.1.3 Receiver

At the receiver, coherent detection is employed—the incoming signal is mixed with a local oscillator (LO) laser in a 90° hybrid, allowing for simultaneous recovery of both the amplitude and phase of the optical field.

A 90° hybrid consists of four 3-dB couplers and a $\pi/2$ phase shifter, as shown in Fig. 2.6. The transfer function of a 3-dB coupler is $H_c = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Then, the electrical field at points 1, 2, 3, and 4 is given by

$$\begin{aligned} E_1(t) &= \frac{1}{2}E_r(t) + \frac{1}{2}E_{LO}(t), \\ E_2(t) &= \frac{1}{2}E_r(t) - \frac{1}{2}E_{LO}(t), \\ E_3(t) &= \frac{1}{2}E_r(t) + \frac{1}{2}jE_{LO}(t), \\ E_4(t) &= \frac{1}{2}E_r(t) - \frac{1}{2}jE_{LO}(t), \end{aligned} \tag{2.29}$$

and they are detected by the two sets of balanced photodetectors (BPD) with photodetector responsivity of R (A/W), to recover the baseband signal:

$$\begin{aligned} i_I(t) &= R|E_1(t)|^2 - R|E_2(t)|^2 \\ &= R \cdot \mathbf{Re} \{ E_r(t) E_{LO}^*(t) \}, \end{aligned} \tag{2.30}$$

$$\begin{aligned} i_Q(t) &= R|E_3(t)|^2 - R|E_4(t)|^2 \\ &= R \cdot \mathbf{Im} \{ E_r(t) E_{LO}^*(t) \}, \end{aligned} \tag{2.31}$$

$$\begin{aligned} i_C(t) &= i_I(t) + j i_Q(t) \\ &= R \cdot E_r(t) E_{LO}^*. \end{aligned} \tag{2.32}$$

In coherent optical receivers employing polarization multiplexing, the dual-polarization optical front-end is typically realized by integrating two single-polarization receiver

⁵ $\frac{d}{dt} \mathbf{R}^\dagger \mathbf{R} = j\Delta\beta_0 \mathbf{R}^\dagger (\boldsymbol{\sigma}_\theta - \boldsymbol{\sigma}_\theta^\dagger) \mathbf{R} = 0$, so $\mathbf{R}^\dagger(z) \mathbf{R}(z) = \mathbf{R}^\dagger(0) \mathbf{R}(0) = \mathbf{I}$.

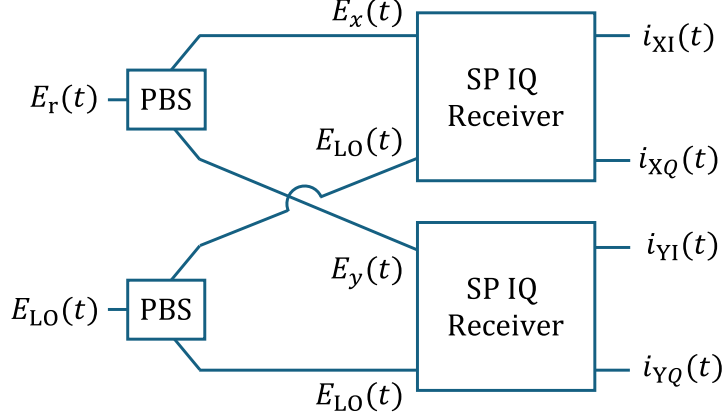


Figure 2.7: Structure of a DP coherent receiver.

branches. Both the received signal and LO laser signal are split into two orthogonal polarization states by polarization beam splitters and mixed in two 90° hybrids, respectively, as shown in Fig. 2.7. The four sets of BPDs output are

$$\begin{aligned}
 i_{XI}(t) &= R \cdot \mathbf{Re} \{ E_x(t) E_{LO}^*(t) \}, \\
 i_{XQ}(t) &= R \cdot \mathbf{Im} \{ E_x(t) E_{LO}^*(t) \}, \\
 i_{YI}(t) &= R \cdot \mathbf{Re} \{ E_y(t) E_{LO}^*(t) \}, \\
 i_{YQ}(t) &= R \cdot \mathbf{Im} \{ E_y(t) E_{LO}^*(t) \}.
 \end{aligned} \tag{2.33}$$

The coherently detected complex baseband signal is then digitized by high-speed analog-to-digital converters (ADCs). The digitized signal is subsequently processed by a DSP flow comprising specialized modules for chromatic dispersion compensation, polarization demultiplexing, carrier phase recovery, and equalization of nonlinear distortions, etc.

2.1.4 Digital signal processing

Although coherent detection was first developed as early as the 1980s, its potential was limited at that time because fiber optic communication systems were still based on on-off keying (OOK) and did not support advanced modulation formats. Moreover, the advent of EDFA and WDM significantly increased system capacity, rendering coherent detection largely unnecessary. Chromatic dispersion (CD) was compensated using physical means such as inline dispersion-compensating fiber (DCF), and the dispersion map required careful design.

In contrast, modern coherent optical systems exploit powerful DSP to overcome these historical limitations. DSP enables the use of complex modulation formats—such as quadrature amplitude modulation (QAM)—which substantially improve spectral efficiency and data rates. It also facilitates electronic CD compensation, eliminating

the need for DCF and shifting system design from dispersion-managed to dispersion-uncompensated links. The latter turns out to be more tolerant to fiber nonlinearity in addition to offering advantages such as lowered loss, reduced cost, and simplified network management.

Furthermore, DSP provides essential functionalities such as carrier phase recovery, adaptive equalization, and forward error correction (FEC). Beyond these core capabilities, DSP seamlessly integrates with emerging digital communication techniques—including probabilistic constellation shaping (PCS) for enhanced spectral efficiency, digital sub-carriers, and digital backpropagation (DBP) to further approach the capacity limit of the fiber channel. The synergy between coherent detection and advanced DSP not only mitigates the impairments imposed by the optical fiber channel but also enables real-time, adaptive optimization of system performance—establishing digital coherent transmission as indispensable for high-speed, long-haul, and flexible optical networks. Here, we briefly introduce the typical flow of DSP in modern coherent transmission systems.

CD Compensation As given by 2.8, the effect of fiber dispersion is an all-pass filter in the frequency domain. Thus, it can be readily compensated for in the digital domain. The electrical signals after BPD are first sampled by ADC, usually in 2 samples per symbol (SpS). Consider an N -length sample block with a sampling rate of f_s , the digital spectrum can be readily obtained by the Discrete Fourier Transform (DFT) algorithm, and CD compensation is done by multiplying the spectrum with the digital inverse CD transfer function 2.34, followed by inverse DFT to bring the signal back to the time domain. In a practical scenario, the sampled data are continuously incoming, and they are segmented into a proper length for CD compensation. An “overlap-save” method is used to avoid artifacts at the edge of the segments from circular convolution and enabling proper real-time processing.

$$H_{-CD}[n] = \exp \left[-\frac{j\beta_2 L}{2} \left(2\pi f_s \frac{n}{N} \right)^2 \right], \quad -\frac{N}{2} \leq n \leq \frac{N}{2} - 1. \quad (2.34)$$

Adaptive Equalization Due to random change of birefringence, the polarization state of the received signal is randomly rotated and time-varying. This is analogous to the wireless communications systems using Multi-Input-Multi-Output (MIMO), and a 2×2 MIMO was shown to be capable of recovering the original polarization effectively. Note that a 1-tap filter is sufficient to recover polarization rotation, since it is sufficient to form the inverse Jones matrix of the transmission system. However, considering there are PMD, residual CD, and other possible band limit effects, the adaptive filter is extended to a multi-tap form.

The 2×2 MIMO structure is shown in Fig. 2.8, where the inputs are CD-compensated 2-SpS signal x_{in} , y_{in} , and the output are 1-SpS polarization-recovered signal x_{out} , y_{out} ,

given by

$$\begin{aligned}
x_{\text{out}}^k &= \mathbf{x}_{\text{in}}^n * \bar{\mathbf{h}}_{\text{xx}}^k + \mathbf{y}_{\text{in}}^n * \bar{\mathbf{h}}_{\text{xy}}^k \\
&= (x_{\text{in}}^n, x_{\text{in}}^{n-1}, \dots, x_{\text{in}}^{n-N_T+1}) \cdot (\bar{h}_{\text{xx}}^1, \bar{h}_{\text{xx}}^2, \dots, \bar{h}_{\text{xx}}^{N_T})^T + (y_{\text{in}}^n, y_{\text{in}}^{n-1}, \dots, y_{\text{in}}^{n-N_T+1}) \cdot (\bar{h}_{\text{xy}}^1, \bar{h}_{\text{xy}}^2, \dots, \bar{h}_{\text{xy}}^{N_T})^T, \\
y_{\text{out}}^k &= \mathbf{x}_{\text{in}}^n * \bar{\mathbf{h}}_{\text{yx}}^k + \mathbf{y}_{\text{in}}^n * \bar{\mathbf{h}}_{\text{yy}}^k \\
&= (x_{\text{in}}^n, x_{\text{in}}^{n-1}, \dots, x_{\text{in}}^{n-N_T+1}) \cdot (\bar{h}_{\text{yx}}^1, \bar{h}_{\text{yx}}^2, \dots, \bar{h}_{\text{yx}}^{N_T})^T + (y_{\text{in}}^n, y_{\text{in}}^{n-1}, \dots, y_{\text{in}}^{n-N_T+1}) \cdot (\bar{h}_{\text{yy}}^1, \bar{h}_{\text{yy}}^2, \dots, \bar{h}_{\text{yy}}^{N_T})^T,
\end{aligned} \tag{2.35}$$

where N_T means the length of filter taps, $*$ means convolution, $n = 2k$, the bar denotes complex conjugate, and $\mathbf{h}_{\text{xx,xy,yx,yy}}^k$ are the N_T -tap filter coefficients that are adaptively updated by

$$\begin{aligned}
\mathbf{h}_{\text{xx}}^k &= \mathbf{h}_{\text{xx}}^{k-1} + \mu \mathbf{x}_{\text{in}}^n (\varepsilon_x^k)^*, \\
\mathbf{h}_{\text{xy}}^k &= \mathbf{h}_{\text{xy}}^{k-1} + \mu \mathbf{y}_{\text{in}}^n (\varepsilon_x^k)^*, \\
\mathbf{h}_{\text{yx}}^k &= \mathbf{h}_{\text{yx}}^{k-1} + \mu \mathbf{x}_{\text{in}}^n (\varepsilon_y^k)^*, \\
\mathbf{h}_{\text{yy}}^k &= \mathbf{h}_{\text{yy}}^{k-1} + \mu \mathbf{y}_{\text{in}}^n (\varepsilon_y^k)^*,
\end{aligned} \tag{2.36}$$

where the error terms (ε) are defined differently depending on the adopted specific algorithm as follows.

- Constant-Modulus Algorithm (CMA)

$$\begin{aligned}
\varepsilon_x^k &= \left(R_2 - |x_{\text{out}}^k|^2 \right) x_{\text{out}}^k, \\
\varepsilon_y^k &= \left(R_2 - |y_{\text{out}}^k|^2 \right) y_{\text{out}}^k,
\end{aligned} \tag{2.37}$$

where R_2 is a real-valued constant radius and given by $R_2 = E\{|x_0|^4\}/E\{|x_0|^2\}$, and x_0 means the clean constellation points.

- Radius-Directed Equalizer (RDE)

$$\begin{aligned}
\varepsilon_x^k &= \left[Q_r^2 \left(|x_{\text{out}}^k|^2 \right) - |x_{\text{out}}^k|^2 \right] x_{\text{out}}^k, \\
\varepsilon_y^k &= \left[Q_r^2 \left(|y_{\text{out}}^k|^2 \right) - |y_{\text{out}}^k|^2 \right] y_{\text{out}}^k,
\end{aligned} \tag{2.38}$$

where Q_r is a radius-decision function that returns the closest radius to the output symbol.

- Decision-Directed Least-Mean-Square (DD-LMS)

$$\begin{aligned}
\varepsilon_x^k &= \mathbf{d} \{ x_{\text{out}}^k \} - x_{\text{out}}^k, \\
\varepsilon_y^k &= \mathbf{d} \{ y_{\text{out}}^k \} - y_{\text{out}}^k,
\end{aligned} \tag{2.39}$$

where d denotes the final symbol decision function after the CFR and CFR stage described below.

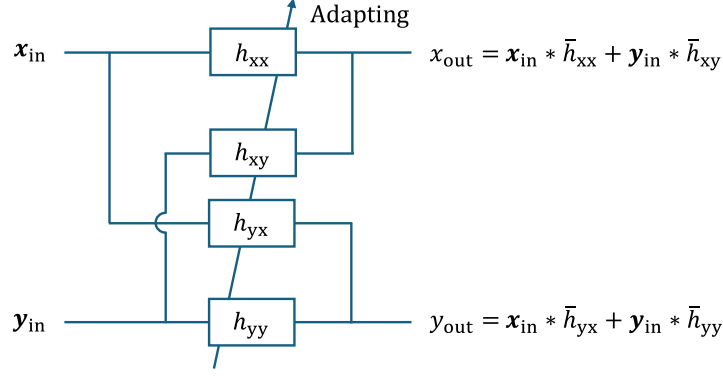


Figure 2.8: Structure of the MIMO filter for adaptive equalization.

Carrier Frequency Recovery (CFR) M^{th} -power algorithm is widely used for CFR. By taking DFT to the M^{th} -powered signal, there will be a spectral spike indicating (M -folds of) the frequency offset between LO and the optical carrier (2.40). The estimation range is restricted in $[-\frac{R_s}{2M}, \frac{R_s}{2M}]$, so signal with higher symbol rate (R_s) is more tolerant to frequency offset. Although this algorithm is originally designed for M -PSK signals, it also works well with $M = 4$ for QAM signals, as there are implicit structures of 4PSK in a QAM constellation.

$$\Delta f_{\text{est}} = \frac{1}{M} \arg \max_f |\text{DFT}\{y^M[k]\}|. \quad (2.40)$$

Carrier Phase Recovery (CPR) For Phase Shift Keying (PSK) modulation, Viterbi-Viterbi algorithm [45] can be used, while for QAM or other constellation modulation, Blind Phase Searching (BPS) algorithm [46] is commonly used, which is realized by calculating the error distance between a small block of symbols rotated by tentative phases and a standard clean constellation, and choose the tentative phase that minimized the average error among the block of symbols. These algorithms premise that the phase noise is relatively slow-varying compared with the symbol period, which is commonly valid for fiber optic communication, as the symbol rate is typically up to several tens of gigahertz. Thus, a higher symbol rate signal is also more resilient to phase noise.

2.2 NFT and NFDM

2.2.1 NFT

It is found that the normalized NLSE is “integrable” by the method called “Nonlinear Fourier Transform” (NFT)⁶, although it is a nonlinear Partial Differential Equation (PDE). The outline of the solving process highly resembles that of a linear PDE—the

⁶ Also called Inverse Scattering Transform (IST), which is more commonly used in the field of physics.

Fourier Transform (FT) method, which is the reason why the method is termed NFT⁷. By NFT, the nonlinear evolution of $q(z, t)$ becomes a linear evolution of its “nonlinear spectrum”, which inspired researchers to encode information in this domain to obtain an underlying linear channel and avoid nonlinear interference for optical fiber transmission.

As mentioned, NFT was established during the process of seeking for analytical solution to nonlinear PDEs. In 1968, Lax [47] published a general method that turned out to be suitable for solving several nonlinear PDEs, including the KdV equation, the Sine-Gordon equation, and the NLSE. The process is as follows [15, 48]:

Find two operators L and A , which are dependent on the signal $q(z, t)$, and L has eigenvalue λ and eigenfunction $\mathbf{v}$:

$$L\mathbf{v} = \lambda\mathbf{v}, \quad (2.41)$$

then it can be proved that if L and A satisfy 2.42 (if so, they are called a Lax pair), then λ is invariant to z , although L itself depends on z for its dependence on $q(z, t)$,

$$\frac{\partial L}{\partial z} = AL - LA. \quad (2.42)$$

Take derivative of 2.41 and use 2.42, we get

$$(L - \lambda) \left(\frac{\partial \mathbf{v}}{\partial z} - A\mathbf{v} \right) = \frac{\partial \lambda}{\partial z} \mathbf{v} = 0, \quad (2.43)$$

which indicates that $\left(\frac{\partial \mathbf{v}}{\partial z} - A\mathbf{v} \right)$ is also an eigenfunction of L with eigenvalue λ , so

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial z} - A\mathbf{v} &= d\mathbf{v}, \\ \frac{\partial \mathbf{v}}{\partial z} &= M\mathbf{v}, \end{aligned} \quad (2.44)$$

where we write $M = A + dI$, d is a scalar and I is the identity operator. Thus, M determines a linear evolution of the eigenvector $\mathbf{v}$ along z . By doing so, the evolution of a nonlinear PDE is translated into the linear evolution of $\mathbf{v}$, and the PDE can be solved analytically. A key step of this methodology is to find suitable Lax pairs for the nonlinear PDE to be solved. Zakharov and Shabat [49] found the suitable Lax pairs for the normalized NLSE,

$$L = \begin{bmatrix} j\frac{\partial}{\partial t} & q \\ -q^* & -j\frac{\partial}{\partial t} \end{bmatrix}, \quad A = \begin{bmatrix} j\frac{\partial^2}{\partial t^2} + \frac{j}{2}|q|^2 & q\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial q}{\partial t} \\ -q^*\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial q^*}{\partial t} & -j\frac{\partial^2}{\partial t^2} - \frac{j}{2}|q|^2 \end{bmatrix}. \quad (2.45)$$

⁷ NFT and FT are only similar in the general working flow of solving PDE, while their implementations are very different and NFT is specific to the type of the nonlinear PDE that is to be solved.

Substituting L and A into 2.42 returns the [normalized NLSE](#) (4.3)⁸:

$$\begin{bmatrix} 0 & \frac{\partial q}{\partial z} \\ -\frac{\partial q^*}{\partial z} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{j}{2} \frac{\partial^2 q}{\partial t^2} + j|q|^2 q \\ \frac{j}{2} \frac{\partial^2 q^*}{\partial t^2} + j|q|^2 q^* & 0 \end{bmatrix}. \quad (2.46)$$

In other words, if q satisfies the NLSE, then L and A defined above form a Lax pair, and thus the eigenvalue of L is invariant to z , and the eigenfunction of L evolves linearly along the z direction. These two properties turn out to be useful (but not straightforward) in determining the evolution of the nonlinear spectrum of $q(z, t)$, which will be defined below.

An equivalent statement of 2.41 is:

$$\frac{\partial \mathbf{v}}{\partial t} = \begin{bmatrix} -j\lambda & jq(t) \\ jq^*(t) & j\lambda \end{bmatrix} \mathbf{v}. \quad (2.47)$$

For an arbitrary time domain waveform $q(z = z_0, t)$, which satisfies $q(z = z_0, t) \rightarrow 0$ when $|t| \rightarrow \infty$, we plug $q(z = z_0, t)$ into 2.47 and omit the z variable hereafter for simplicity (because the following calculation is irrelevant to z coordinate).

The basis of the solution 2.47 consists of two linearly independent functions. Define the adjoint of a vector $\mathbf{v}(t) = [v_1(t), v_2(t)]^T$ to be $\bar{\mathbf{v}}(t) = [v_2^*(t), -v_1^*(t)]^T$, where $[\cdot]^T$ means matrix transpose. Then it can be shown that if $\mathbf{v}(t; \lambda)$ is the solution of 2.47 parameterized by λ , and let $\mathbf{u}(t; \lambda^*)$ denote the solution of 2.47 parameterized by λ^* , then the adjoint of $\mathbf{u}(t; \lambda^*)$, denoted by $\bar{\mathbf{u}}(t; \lambda^*)$ is also a solution of 2.47 parameterized by λ . What's more, $\mathbf{v}(t; \lambda)$ and $\bar{\mathbf{u}}(t; \lambda^*)$ are shown to be linearly independent, so they can form a basis for solutions of 2.47.

Let $\bar{\psi}, \psi$ be a set of basis of solutions of 2.47, which asymptotically approach to 2.48a, respectively, and let $\varphi, -\bar{\varphi}$ be another set of basis of solutions of 2.47, which asymptotically approach 2.48b, respectively, as shown in Fig. 2.9.

$$\bar{\psi}(t, \lambda) \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-j\lambda t}; \quad \psi(t, \lambda) \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{+j\lambda t}; \quad \text{as } t \rightarrow +\infty, \quad (2.48a)$$

$$\varphi(t, \lambda) \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-j\lambda t}; \quad -\bar{\varphi}(t, \lambda) \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{+j\lambda t}; \quad \text{as } t \rightarrow -\infty. \quad (2.48b)$$

As both pairs of them form a set of solution basis, we can use $\bar{\psi}, \psi$ to express φ by a linear combination

$$\varphi(t, \lambda) = a(\lambda) \bar{\psi}(t, \lambda) + b(\lambda) \psi(t, \lambda), \quad (2.49)$$

⁸ Note the calculateion rules of differential operators, $\partial_t f = f_t + f \partial_t$, $\partial_{tt} f = f_{tt} + 2f_t \partial_t + \partial_{tt}$, $\partial_t f \partial_t = f_t \partial_t + f \partial_{tt}$, where f is a multiplication operator.

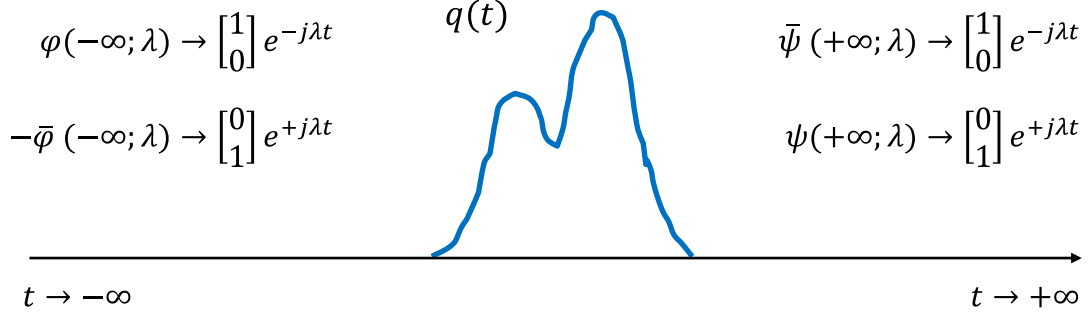


Figure 2.9: The asymptotical behavior of the two sets of solution basis $\bar{\psi}$, ψ and φ , $-\bar{\varphi}$.

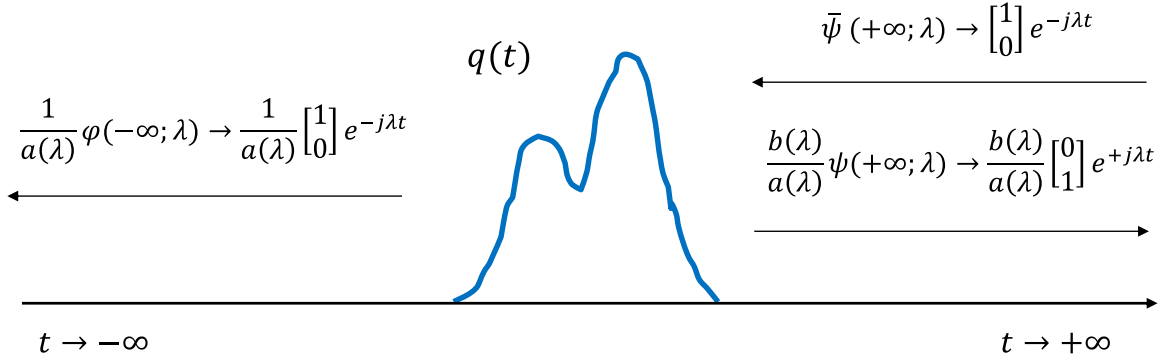


Figure 2.10: The analogy of NFT to the scattering problem in quantum mechanics.

where $a(\lambda)$, $b(\lambda)$ are called the scattering coefficients and can be shown to be invariant to t . $a(\lambda)$, $b(\lambda)$ are obtained by

$$\begin{aligned} a(\lambda) &= \lim_{t \rightarrow +\infty} \varphi_1 e^{+j\lambda t}, \\ b(\lambda) &= \lim_{t \rightarrow +\infty} \varphi_2 e^{-j\lambda t}. \end{aligned} \quad (2.50)$$

It is worth noting that 2.49 can be rewritten into

$$\frac{1}{a(\lambda)} \varphi(t, \lambda) = \bar{\psi}(t, \lambda) + \frac{b(\lambda)}{a(\lambda)} \psi(t, \lambda), \quad (2.51)$$

and a non-rigorous interpretation of it is to make an analogy to the scattering problem in quantum mechanics, regarding the RHS as an incident wave from $t \rightarrow +\infty$ and a reflected wave back to $t \rightarrow +\infty$, and the LHS is the transmitted wave to $t \rightarrow -\infty$ as shown in Fig. 2.10, then $1/a(\lambda)$ and $b(\lambda)/a(\lambda)$ are the transmission and reflection coefficients, respectively.

By far, it has been assumed that $\lambda \in \mathbb{R}$, but it can be proved that λ can be extended to the upper complex plane, $\lambda \in \mathbb{C}^+$. To make 2.49 still hold, $a(\lambda)$ needs to be zero, and this is only satisfied for some discrete eigenvalue, denoted by λ_i . The continuous and discrete spectrum is then defined as

$$Q_c(\lambda) = \frac{b(\lambda)}{a(\lambda)}, \quad Q_d(\lambda_i) = \frac{b(\lambda_i)}{a'(\lambda_i)}, \quad \text{with } a'(\lambda_i) = \left. \frac{da(\lambda)}{d\lambda} \right|_{\lambda=\lambda_i}. \quad (2.52)$$

The collection of $\{\lambda, Q_c(\lambda), \lambda_i, Q_d(\lambda_i)\}$ is called the nonlinear spectrum of signal $q(t)$. The inverse NFT (INFT) given by 2.53 involves reconstructing the time-domain signal from its nonlinear spectrum, which explicitly shows that the signal consists of the dispersive and solitonic components determined by the continuous and discrete spectrum, respectively. The detailed process of INFT is attached in Appendix A. Numerical examples of NFT of rectangular and sech functions are given below to provide a visual perception.

$$q(t) = \underbrace{\frac{j}{\pi} \int_{-\infty}^{+\infty} Q_c^*(\xi) \psi_2^*(t, \xi) e^{-j\xi t} d\xi}_{\text{dispersive component}} - 2 \underbrace{\sum_{i=1}^n Q_d^*(\lambda_i) \psi_2^*(t, \lambda_i) e^{-j\lambda_i^* t}}_{\text{solitonic component}}, \quad (2.53)$$

Some useful properties of NFT are listed below [15]:

- Phase shift: $e^{j\phi} q(t) \leftrightarrow e^{-j\phi} Q_c(\lambda), e^{-j\phi} Q_d(\lambda_i)$.
- Frequency shift: $e^{-2j\omega t} q(t) \leftrightarrow Q_c(\lambda - \omega), Q_d(\lambda_i - \omega)$.
- Time shift: $q(t - t_0) \leftrightarrow e^{-2j\lambda t_0} Q_c(\lambda), e^{-2j\lambda_i t_0} Q_d(\lambda_i)$.
- Parseval's identity: $\int_{-\infty}^{\infty} |q(t)|^2 dt = \frac{1}{\pi} \int_{-\infty}^{\infty} \ln(1 + |Q_c(\lambda)|^2) d\lambda + 4 \sum_{i=1}^N \text{Im}\{\lambda_i\}$.

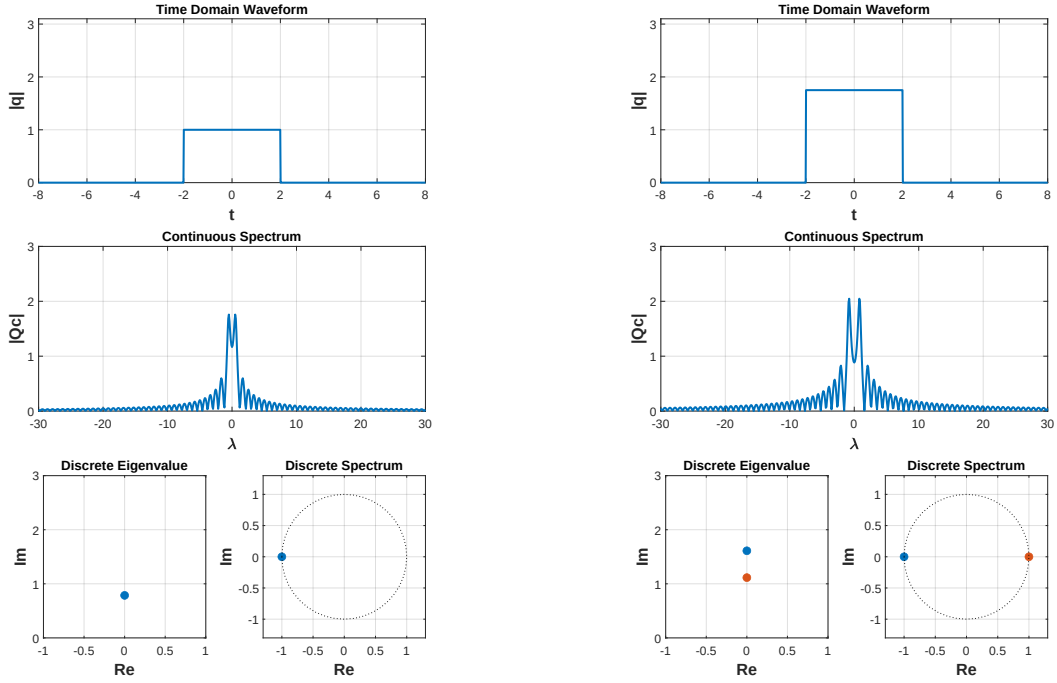
Another important issue is how the nonlinear spectrum evolves when the envelope $q(z, t)$ propagates. It can be shown that their evolution is in a rather simple linear manner—phase rotation without interference between different eigenvalues just like the evolution of linear Fourier modes for a linear PDE ⁹:

$$\begin{aligned} \lambda_i(z = L) &= \lambda_i(z = 0), \\ Q_d(z = L, \lambda_i) &= Q_d(z = 0, \lambda_i) e^{-2j\lambda_i^2 L}, \\ Q_c(z = L, \lambda) &= Q_c(z = 0, \lambda) e^{-2j\lambda^2 L}. \end{aligned} \quad (2.54)$$

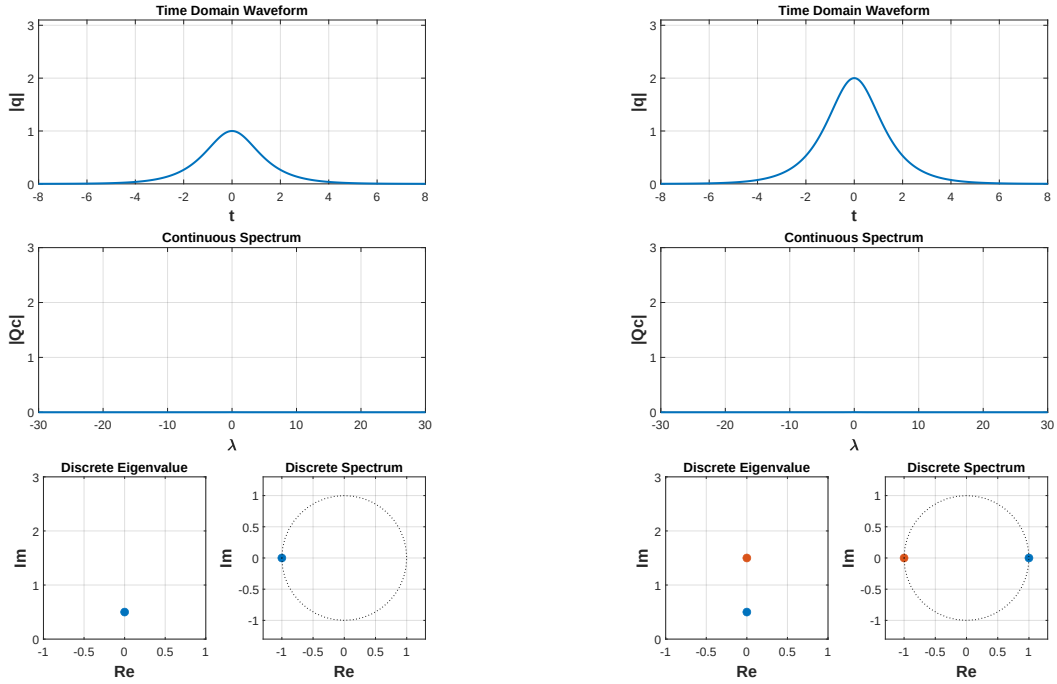
This property reveals the hidden linear structure of NLSE and makes the NFT a powerful tool to solve it. With the help of NFT/INFT and the evolution rule of the nonlinear spectrum, we can solve the NLSE with an initial condition by the following steps, illustrated in Fig. 2.11:

1. Conduct NFT to the initial signal $q(z = 0, t)$ and obtain the initial nonlinear spectrum;
2. Evolve the nonlinear spectrum for a certain distance L according to 2.54;
3. Conduct INFT to the evolved spectrum data to obtain the signal at L , $q(z = L, t)$.

Example 1: the NFT of a rectangle function of different heights.



Example 2: the NFT of a sech function of different heights, note this example only has the discrete spectrum, and the time domain waveform is a soliton.



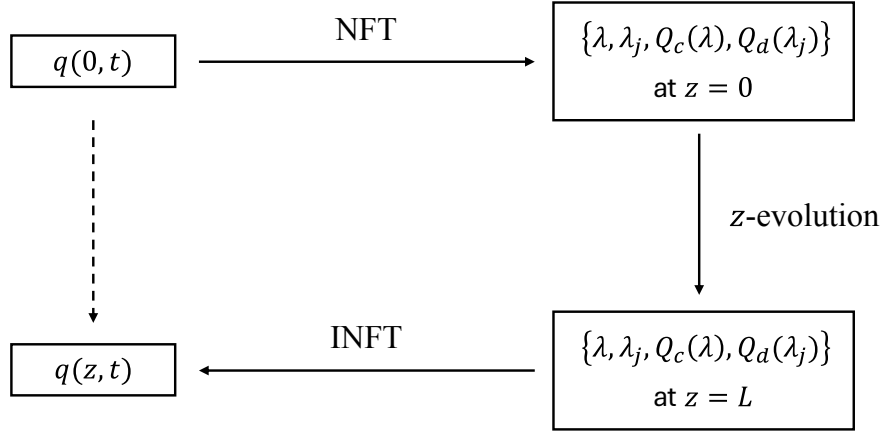


Figure 2.11: The procedure of solving the NLSE by NIFT/INFT.

Particularly, when the nonlinear spectrum only contains the discrete spectrum, the corresponding time domain signal is a soliton. Considering the 1st-order soliton, whose nonlinear spectrum only contains one discrete eigenvalue, let the discrete eigenvalue be $\lambda = \alpha + j\beta$, and the initial discrete spectrum be $Q_d(z=0) = |Q_d(z=0)|e^{j\angle Q_d(z=0)}$. Then, the corresponding soliton can be explicitly given by

$$q(z, t) = 2\beta \operatorname{sech} [2\beta (t - t_0 + 2\alpha z)] e^{j2(\beta^2 - \alpha^2)z - j2\alpha t + j\phi_0}, \quad (2.55)$$

where

$$2\beta t_0 = \ln \left(\frac{|Q_d(z=0)|}{2\beta} \right) = \ln |b(z=0)|, \quad (2.56)$$

$$\phi_0 = \pi - \angle Q_d(z=0) = \pi/2 - \angle b(z=0). \quad (2.57)$$

Therefore, the real part of λ determines the frequency and 1/group velocity of the soliton, and the imaginary part of λ determines the height and width of the soliton. The magnitude and phase of the discrete spectrum (or b -coefficient) determine the time position and phase of the soliton, respectively. With NIFT theory and coherent detection ready, it becomes possible to apply advanced modulation formats to soliton transmission. The modulation can be made on the discrete eigenvalue and/or discrete nonlinear spectrum, as shown in Fig. 2.12.

For a DP signal transmission in a long fiber with fast random polarization evolution, ME (2.27) should be used, and it also has a corresponding NIFT, given by S. V. Manakov [44]. The nonlinear spectrum of a DP signal governed by ME is 2-dimensional, $k = 1, 2$

⁹ Another common version of the evolution equation writes with $e^{-4j\lambda_i^2 L}$ and $e^{-4j\lambda^2 L}$, which results from another choice of normalizing NLSE, see [50, Table 2.1].

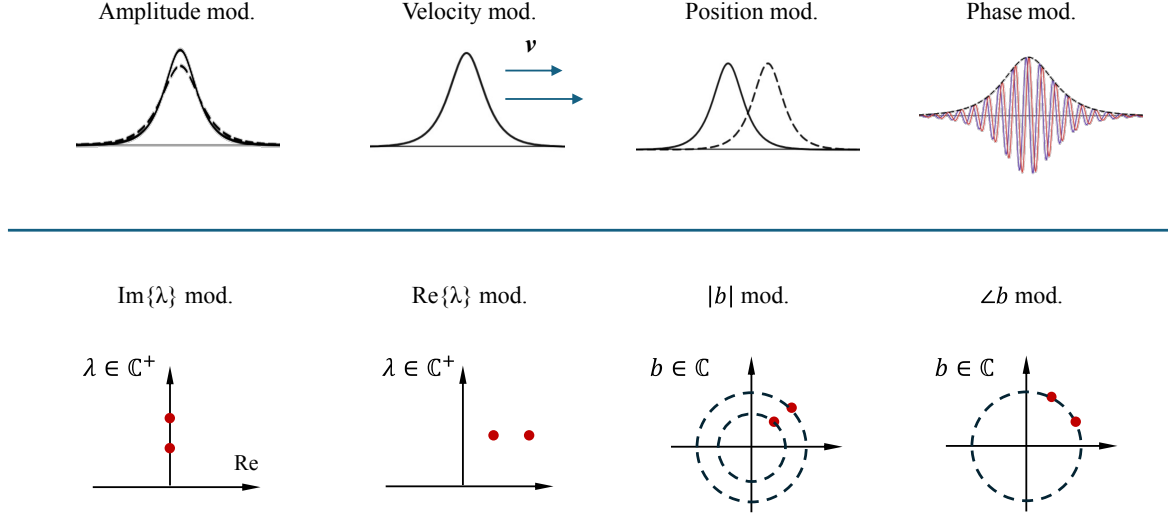


Figure 2.12: Modulation of the imaginary/real part of λ corresponds to modulation of the height/velocity of soliton; modulation of the magnitude/phase of b corresponds to modulation of the time-shift/phase of soliton.

¹⁰:

$$\begin{cases} Q_c^k(\lambda) = \frac{b_k(\lambda)}{a(\lambda)}, & \lambda \in \mathbb{R}, \\ Q_d^k(\lambda_i) = \frac{b_k(\lambda_i)}{a'(\lambda_i)}, & \lambda_i \in \mathbb{C}^+, \end{cases} \quad (2.58)$$

where each polarization has its own b -coefficients, but shares the same a -coefficient and eigenvalue. Similar to the SP case, for a single eigenvalue DP soliton, the discrete spectrum of it is explicitly related to its time domain parameters, which is presented in Chap. 5.2.

2.2.2 NFDM

The original purpose of NFT is to solve nonlinear PDEs, while it inspires the optical communication community to encode information onto the nonlinear spectrum due to its linear evolution, which is favorable because linear equalization theory and techniques are mature. Such a transmission scheme is called NFDM, where information can be encoded either in the discrete spectrum (called discrete NFDM) or continuous spectrum (called continuous NFDM), or both (called full-spectrum NFDM). In contrast, conventionally, information is encoded in the time domain waveform or the linear Fourier spectrum, and thus, they will suffer from fiber-nonlinearity-induced distortion and interference. NFDM is thought to have the potential to break the nonlinearity bottleneck of fiber nonlinearity, and the architecture of NFDM is shown in Fig. 2.13:

1. *INFT at the transmitter.* The nonlinear spectrum is modulated by information to

¹⁰ It is conventional to use 1, 2 instead of x, y for the nonlinear spectrum of a DP signal.

be sent, then the time domain signal is generated by taking INFT of the modulated nonlinear spectrum. Note that we can modulate the discrete eigenvalue, discrete spectrum, continuous spectrum, or a combination of them. Besides, the $b(\lambda_i)$ is more commonly modulated than $Q_d(\lambda_i)$ as the former brings less noise [29];

2. *Transmission in the SMF.* The generated time domain signal is transmitted through the SMF, which will be nonlinearly distorted by fiber dispersion and nonlinearity in a complicated way, but the information coded in the nonlinear spectrum evolves simply and linearly according to 2.54;
3. *NFT at the receiver.* At the receiver, the received signal is processed by NFT, and the nonlinear spectrum is obtained. The information sent from the transmitter side can be easily recovered by back rotating the received nonlinear spectrum (inverse of 2.54).

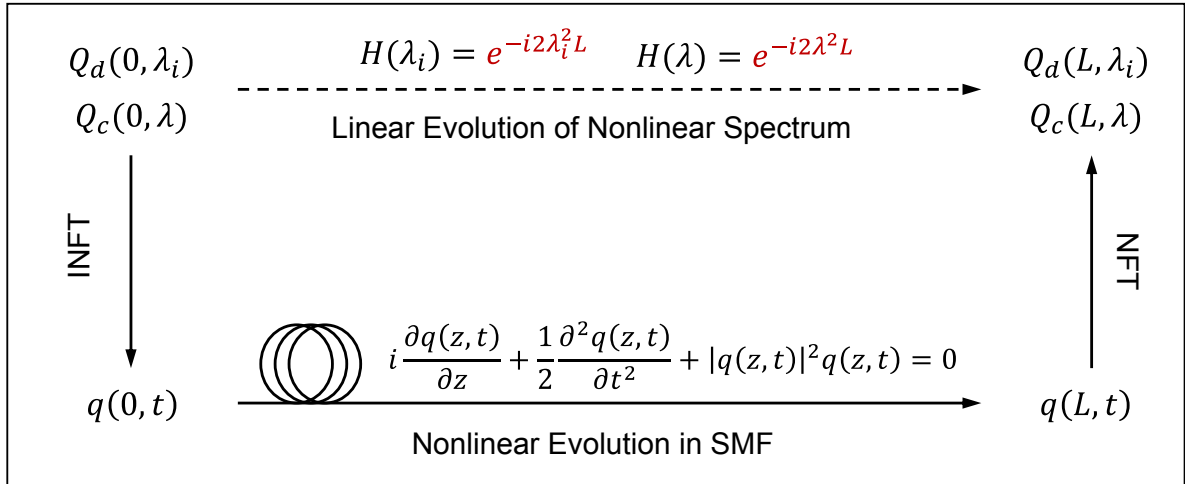


Figure 2.13: The architecture of NFD. Information is encoded in the linearly evolving nonlinear spectrum, without suffering from nonlinear distortion.

2.3 Path-averaged soliton

When $\beta_2 < 0$, the effect of fiber dispersion and Kerr nonlinearity can exactly cancel each other out if the pulse is in the shape of a sech function with a proper power, i.e., a 1st-order optical temporal soliton is formed, which will not change its envelope during propagation. However, the real-world fiber has loss, which will reduce the pulse power and hence the stimulated fiber nonlinearity effect, and the balance with dispersion will be broken. EDFA is widely used to periodically amplify the attenuated signal during transmission. A practical issue is whether an optical soliton can survive in the presence of fiber loss and EDFA gain. Intuitively, it would be reasonable to increase the soliton

power higher than the theoretical value to stimulate more fiber nonlinearity, which can balance out the shortage of fiber nonlinearity toward the end of a span. This intuitive idea was first validated experimentally [51] and later justified by theoretical analysis that introduced the concept of “path-averaged” soliton (or “guiding-center” soliton) [52].

The NLSE that includes fiber loss and EDFA gain is given by

$$j\frac{\partial q}{\partial z} + \frac{1}{2}\frac{\partial^2 q}{\partial t^2} + |q|^2 q = j\left[-\frac{\Gamma}{2} + \frac{\Gamma L}{2}\sum_{n=1}^N \delta(z - nL)\right]q, \quad (2.59)$$

where $\Gamma = \alpha L_D$, $L = L_A/L_D$ denote the normalized fiber power loss coefficient and normalized span length, respectively. Logarithm-scaled δ -function is used to model the lumped gain, and the logarithmic scaling of the δ -function is explained in Appendix C. We may separate the periodic amplitude variation due to fiber loss and periodic EDFA gain by writing $q(z, t) = f(z)Q(z, t)$ with $f(z)$ satisfying

$$\frac{df(z)}{dz} = \left[-\frac{\Gamma}{2} + \frac{\Gamma L}{2}\sum_{n=1}^N \delta(z - nL)\right]f(z) \quad (2.60)$$

to obtain

$$j\frac{\partial Q}{\partial z} + \frac{1}{2}\frac{\partial^2 Q}{\partial t^2} + |Q|^2 Q = [1 - a^2(z)]|Q|^2 Q. \quad (2.61)$$

If the span length is much shorter than the dispersion length, the power scaling factor $a^2(z)$ is fast-oscillating, and if $a^2(z)$ has a mean value of 1, namely $\frac{1}{L}\int_0^L a^2(z)dz = 1$, the RHS of 2.61 is zero-mean and fast-oscillating, which makes its contribution negligible. Since $a^2(z) = a^2(0)e^{-\Gamma z}$, to make its mean value 1, we need $a^2(0) = G \ln G / (G - 1)$, where $G = e^{\Gamma L}$ is the EDFA gain (equal to span loss). Going back to the physical world, that means amplifying the theoretical soliton launching power P_{Sol} by $G \ln G / (G - 1)$. By doing so, the pulse goes through dramatic power evolution during propagation, but retains its shape very well. This property was used in the “eigenvalue communication” [20], by which the eigenvalues remain almost unchanged during transmission. Fig. 2.14 shows an example of a path-averaged soliton that keeps its normalized envelope invariant during transmission, where $G = 10$, $L_A = 50$ km.

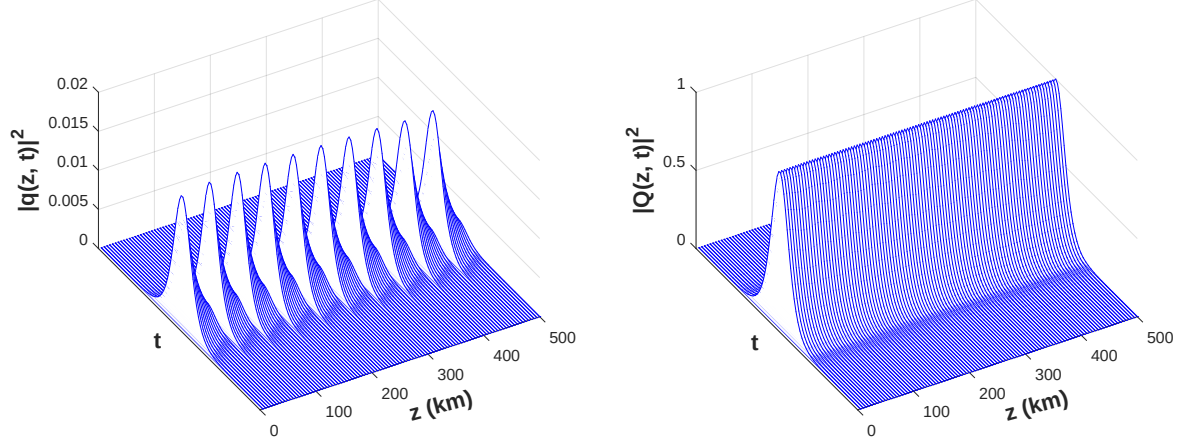


Figure 2.14: With launching power $\frac{G \ln G}{G-1} P_{\text{Sol}}$, the path-averaged soliton $q(z, t)$ keeps its normalized envelope $Q(z, t)$ almost invariant during transmission.

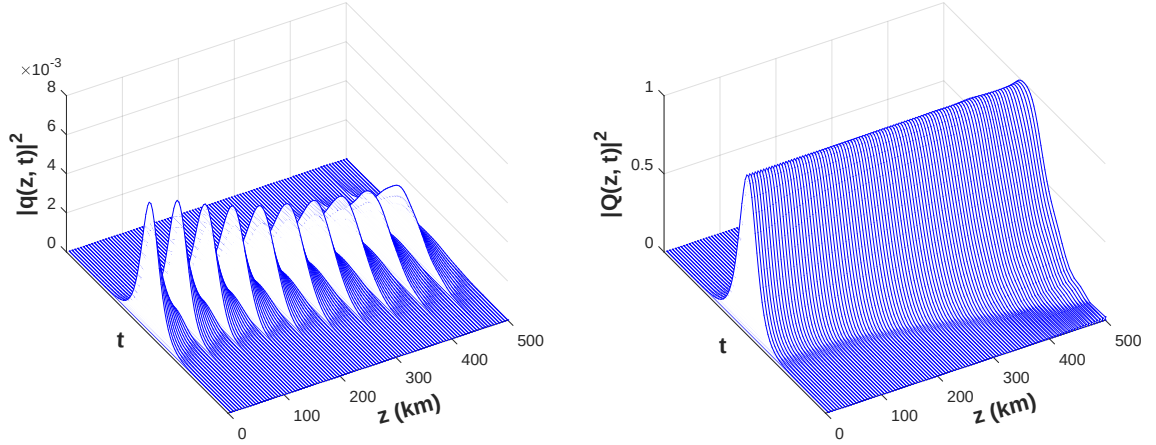


Figure 2.15: Launching power P_{Sol} , envelope of $q(z, t)$ envelope cannot be maintained.

Chapter 3

81 Gbps Dual-Polarization Discrete Eigenvalue Transmissions based on Nonlinear Fourier Transform

The DP NFDM system can be implemented either in its original form or in a simplified version, where the X and Y polarizations are separately equipped with two SP NFDM signals. This chapter investigates the gain brought by signal misalignment between polarizations in a simplified DP NFDM system.

3.1 Introduction

Nonlinear frequency division multiplexing (NFDM) is being investigated as a new approach that incorporates Kerr nonlinearity arising in signal propagation over the optical fibers [15, 53] through Nonlinear Fourier transform (NFT). In NFDM systems, information is encoded on the nonlinear discrete and continuous spectrum of signals, which propagate along the fiber without mutual interference. In recent years, digital signal processing (DSP) techniques such as linear minimum mean square error (LMMSE) estimator [29], overlapping-tolerant soliton design [31] and multi-symbol (MS) maximum likelihood (ML) detection [30] improved the performance of single-polarization (SP) discrete eigenvalue transmissions up to 54 Gbps over 1200 km. Dual-polarization (DP) discrete eigenvalue transmissions were also reported [25, 54, 55] and different DSP schemes [56], including nonlinear MMSE and neural networks, were applied to mitigate transmission impairments, but the demonstrated bit rates remain relatively low.

In this chapter, we extend our previous DSP work on SP discrete eigenvalue transmissions [30] and demonstrate DP discrete eigenvalue transmissions with joint eigenvalue and b -coefficient modulation. To reduce the computational complexity, we independently encode/decode soliton trains on each polarization using SP NFT instead of the gen-

eral DP NFT based on the Manakov equations [57]. We further propose shifting the soliton train in one polarization by half a symbol period relative to the signal in the other polarization to reduce the cross-polarization distortions and generalize the MS LMMSE and ML detection to incorporate neighboring signals in other polarizations. A 96 Gbps (net 81 Gbps) DP discrete eigenvalue transmission over 650 km is experimentally demonstrated using the aforementioned signal design and receiver DSP.

3.2 Signal design and receiver DSP for dual-polarization discrete eigenvalue transmission

In our polarization multiplexed discrete-eigenvalue transmissions, we will separately encode/decode two independent soliton pulse trains for each polarization using SP NFT instead of the full DP NFT by solving the Manakov–Zakharov–Shabat spectral problem associated with the Manakov equations due to its computational simplicity. For each soliton pulse train in the $x(y)$ -polarization, the eigenvalue and b -coefficient, $\lambda_{i,x(y)}$ and $b_{i,x(y)}(\lambda_{i,x(y)})$ of the i^{th} information symbol are modulated by a QAM and APSK constellation, respectively. We will add symmetrical real parts, α and $-\alpha$, to $\lambda_{i,x(y)}$ of neighboring pulses so that they will interact and pass through each other along propagation [30]¹. The i^{th} soliton pulse with support $[(i-1)T, iT]$, where T is the symbol period in normalized time, is then calculated through the inverse NFT (INFT) operation. We also propose shifting the soliton pulse train of one polarization by half a symbol period (Fig. 3.1), so that the cross-polarization interference can be reduced to some extent. We note that while shifting the solitons in one polarization by half a symbol period was proposed in traditional soliton literature in the 1990s [58], current research in discrete eigenvalue transmissions is based on digital coherent detection and modulation on multiple degrees of freedom of a soliton, so they are considerably different. Therefore, half-symbol shifting may induce very different impacts on transmission quality and is worth a new investigation.

At the receiver, we will segment the received waveforms into intervals of $2T$ followed by the NFT operation. The obtained pairs of eigenvalue and b -coefficients $\{\tilde{\lambda}_{i,x(y)}, \tilde{b}_{i,x(y)}(\tilde{\lambda}_{i,x(y)})\}$, $\{\tilde{\lambda}_{i+1,x(y)}, \tilde{b}_{i+1,x(y)}(\tilde{\lambda}_{i+1,x(y)})\}$ will undergo the following sequence of DSP described in [30]: 1) LMMSE filter to mitigate discrete eigenvalue noise $\Delta\tilde{\lambda}_{i,x(y)}$, using a -coefficients of the continuous spectrum (CS); 2) MS-ML detection on eigenvalue; 3) MS-LMMSE filter to mitigate noise in $\tilde{b}_{i,x(y)}(\tilde{\lambda}_{i,x(y)})$ using discrete eigenvalue noise $\Delta\tilde{\lambda}_{i,x(y)}$; 4) MS-ML detection on b -coefficients. Note that in previous SP discrete eigen-

¹ In this work, we used a pair of two adjacent 1st-order solitons to approximate a 2nd-order soliton. We published a method of generating a 2nd-order soliton with specified peak position later, and it is attached in Appendix D.

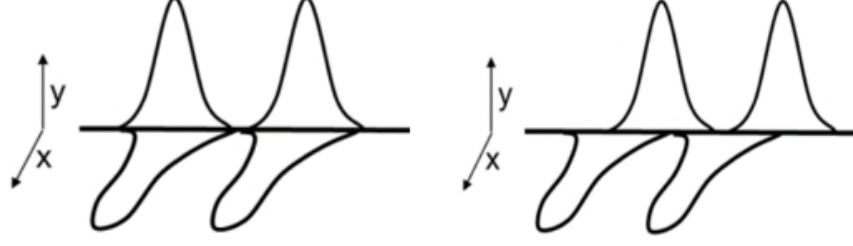


Figure 3.1: Shifting a soliton train by half a symbol period in one polarization to reduce polarization crosstalk.

value transmissions, the MS-LMMSE filter for b -coefficients incorporate the eigenvalue noise of neighboring pulses but in the proposed polarization multiplexed scenario, the MS-LMMSE filter can also include neighboring pulses in the other polarization so that the noise of $\tilde{b}_{i,x}(\tilde{\lambda}_{i,x})$ is also mitigated by a linear combination of $\Delta\tilde{\lambda}_{i-1,y}$, $\Delta\tilde{\lambda}_{i,y}$, $\Delta\tilde{\lambda}_{i+1,y}$ in addition to the eigenvalue noise of its own polarization $\Delta\tilde{\lambda}_{i-1,x}$, $\Delta\tilde{\lambda}_{i,x}$, $\Delta\tilde{\lambda}_{i+1,x}$. We will refer to this as the DP-MS-LMMSE filter and the former filters in SP transmissions as the SP-MS-LMMSE filter.

3.3 Simulation results

To investigate the benefits of the proposed shifting scheme at the transmitter (Tx) and the DP-MS-LMMSE filter at the receiver (Rx), we conducted simulations of DP 6 GBaud soliton signals with 16QAM-modulated λ and 8PSK-modulated $b(\lambda)$. The DP soliton trains were transmitted in multiple spans of 50-km non-zero dispersion-shifted fiber (NZ-DSF). The PMD is simulated by the “coarse-step” method, by which the fiber is represented as a concatenation of many small sections of length 1 km. The polarization state of the optical field at the end of each section is randomly rotated to a new point on the Poincaré sphere, and the overall differential group delay (DGD) follows Maxwellian’s distribution with polarization mode dispersion (PMD) coefficients of 0, 0.05, 0.1, or 0.15 ps/km^{1/2}. For each transmission distance, the value of $\mathbf{Re}\{\lambda\}$ is chosen such that each soliton will pass through 1 neighboring soliton and separate again at the receiver. The received signal was first PMD-compensated by the inverse Jones matrix of the fiber link, followed by fast NFT [59] and the proposed DSP algorithms. The launch power was swept to find the best signal-to-noise ratio (SNR) performance at each transmission distance. The SNR is calculated from Error Vector Magnitude (EVM) estimation from the received signal distribution by $\text{SNR} = 1/\text{EVM}^2$. The SNR results of λ and $b(\lambda)$ modulation for different Tx/Rx schemes are shown in Fig. 3.2. Shifting one of the polarizations by half a symbol period provides a clear gain both in the SNR of 16-QAM modulated λ and 8-PSK modulated $b(\lambda)$ of more than 5 dB.

In addition, in the PMD-free case, DP-MS-LMMSE also slightly improves SNR

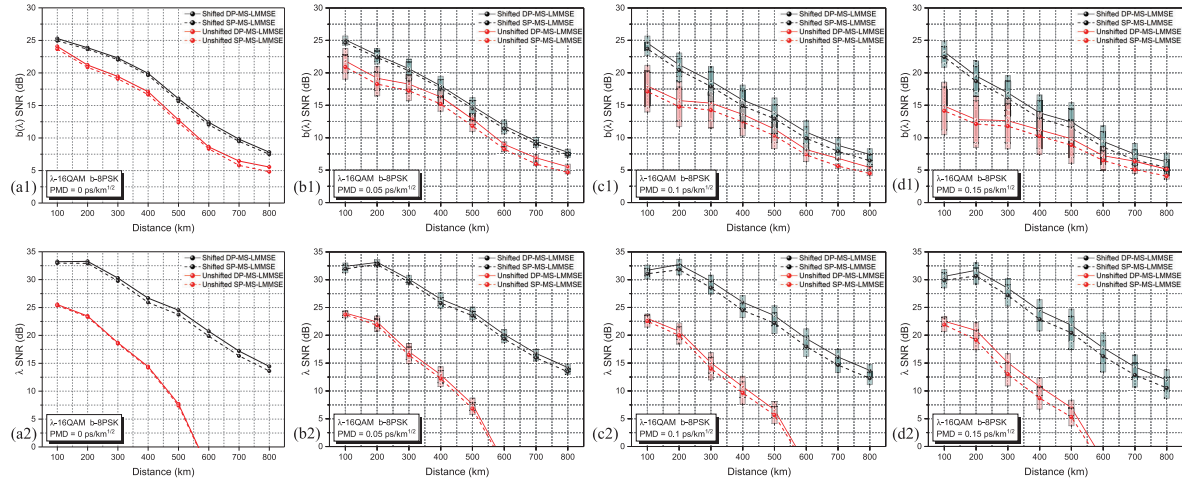


Figure 3.2: Simulation results for SNR of λ and $b(\lambda)$ of 6 GBaud DP λ 16QAM-b8PSK signals without PMD (a), with PMD of $0.05 \text{ ps/km}^{1/2}$ (b), $0.10 \text{ ps/km}^{1/2}$ (c), and $0.15 \text{ ps/km}^{1/2}$ (d). The box's vertical range and sphere markers represent the ± 1 standard deviation and mean value, respectively.

(less than 0.5 dB) in both shifting and non-shifting schemes. We attribute this to the little soliton interaction of X- and Y-polarization when PMD is absent. When PMD is introduced, the SNR gain by DP-MS-LMMSE becomes more significant, while the SNR gain by half-symbol shifting is generally maintained for different transmission distances. Thus, both the shifting scheme and the DP-MS-LMMSE filter are shown to be effective in improving transmission performance in DP discrete eigenvalue transmission systems.

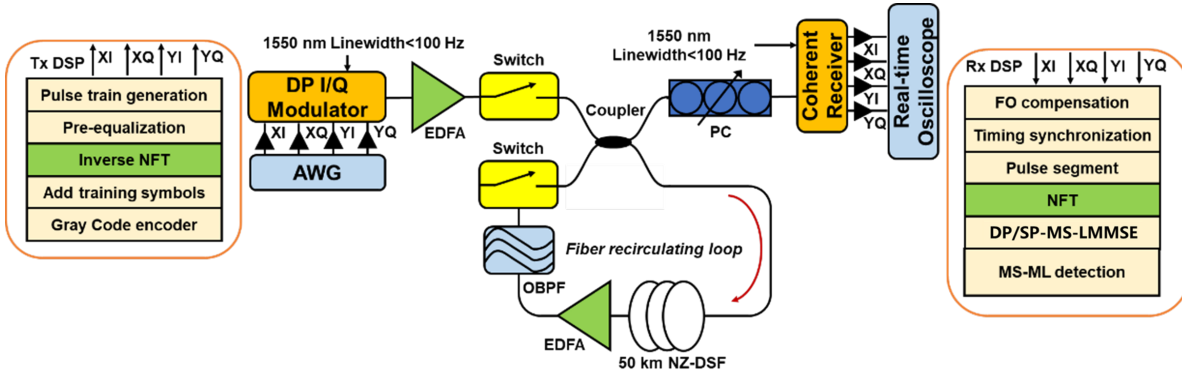


Figure 3.3: Experimental setup for polarization multiplexed discrete-eigenvalue transmission.

3.4 Experimental setup and results

We conducted experiments with offline signal processing to investigate the practical performance of the proposed polarization shifting and DP-MS-LMMSE filter in discrete eigenvalue transmissions. Fig. 3.3 shows the experimental setup and the offline DSP structure. Gray coding was separately applied to the constellation points in λ and $b(\lambda)$.

For the b -coefficient constellation design, we set $|b(\lambda)|$ in the two rings of 16-APSK to be $\exp(\pm \text{Im}\{\lambda\} \Delta T)$ so that their corresponding time-domain waveforms were shifted by $\pm \Delta T/2$ from the center of the time window regardless of λ and we experimentally optimized ΔT to be 0.5. The horizontal and vertical distances of the λ constellation were optimized to be 0.03 and 0.07, which is a trade-off between the separation of signal points in the λ constellation plane and inducing signal bandwidth expansion and nonlinear distortions to neighboring pulses. For each λ - $b(\lambda)$ -jointly-modulated symbol, we performed the INFT to obtain the corresponding time-domain waveform and then pre-compensated it to reduce the effect of the transmitter component imperfections, and then the symbols were loaded onto the arbitrary waveform generator (Keysight M8194A, 120 GSa/s, 45 GHz). The waveform then went into the DP-IQ modulator (Fujitsu FTM7992HM, 3 dB bandwidth of 35 GHz) to modulate the optical carrier, before being amplified by the Tx Erbium-doped fiber amplifier (EDFA) and launched into the fiber recirculating loop. The loop consisted of 1 span of 50-km NZ-DSF (Dispersion: 4 ps/km/nm, loss: 0.2 dB/km) and an EDFA to fully compensate for the loss of the NZ-DSF, optical switch, and coupler, which is about 19 dB in total. A flat-top optical filter (3-dB bandwidth of 1 nm) was used at the loop EDFA output to suppress out-of-band amplified spontaneous emission (ASE) noise. Both the transmitter laser and local oscillator were from the same fiber laser source (NKT Koheras BASIK Fiber laser, linewidth < 100 Hz). After alignment by a polarization controller, the received signal was coherently detected and then sampled by a digital oscilloscope (Keysight, UXR-0804A, 256 GSa/s, 80 GHz) and processed offline. The sampled signal was passed through a series of DSP units as shown in Fig. 3.3. After frequency offset (FO) compensation² and timing synchronization, the received signal was sliced into $2T$ -long blocks. NFFT was then applied to obtain the nonlinear spectrum, followed by a series of LMMSE and ML detection in [30]. Since fiber loss and other unaccounted experimental factors broke the integrability of the NLSE, the constellations of the received signal for both λ and $b(\lambda)$ were distorted, and training symbols were used for estimating the appropriate constellation points for noise calculations and applications of LMMSE and MS-ML algorithms.

The post-DSP constellations of signals (6 GBaud DP λ 16QAM- b 16APSK, making a total bit rate of 96 Gbit/s) received at 500 km with and without half-symbol shifting, and processed by DP-MS-LMMSE vs SP-MS-LMMSE are shown in Fig. 3.4. Clearly, shifting one of the soliton trains by half a symbol period brings significant performance improvements, thus verifying that mutual distortions between the two polarizations can be effectively reduced by time shifting. The use of DP-MS-LMMSE over SP-MS-LMMSE

² Although Tx and Rx shared the same laser, the Acousto-Optic Modulator (AOM) that constitutes the fiber recirculating loop introduced 100 MHz FO.

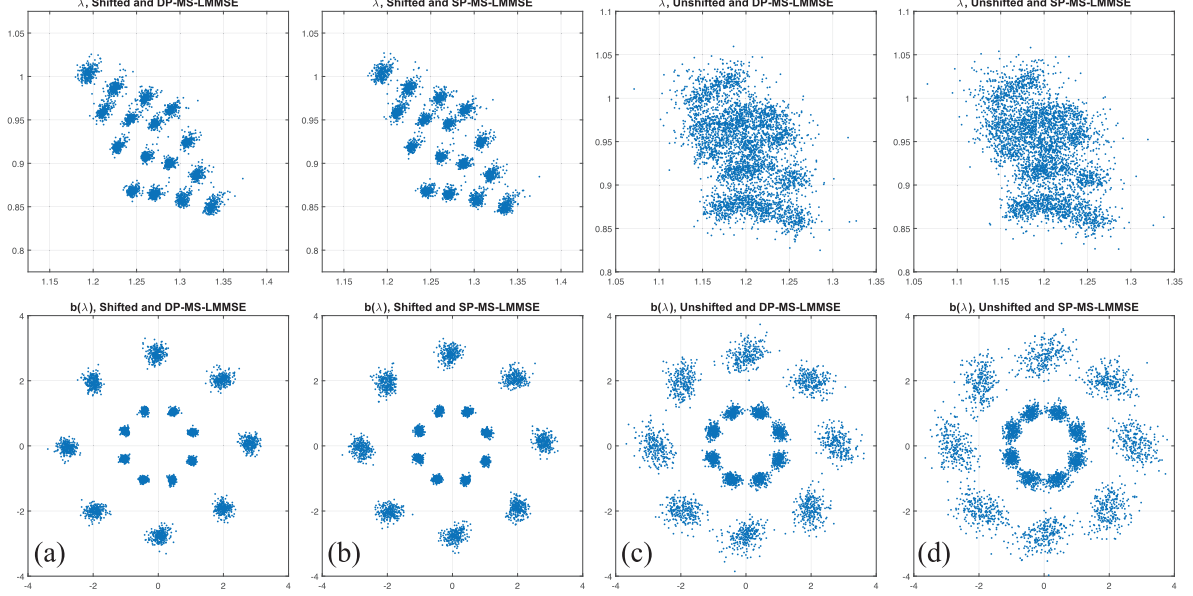


Figure 3.4: Received λ and $b(\lambda)$ distributions for 6 GBaud DP discrete eigenvalue transmissions over 500 km. (a) with time shifting and using DP-MS-LMMSE; (b) with time shifting and using SP-MS-LMMSE; (c) without time shifting and using DP-MS-LMMSE; (d) without time shifting and using SP-MS-LMMSE.

further reduces the noise in λ and $b(\lambda)$ slightly.

It should be noted that in our setup, the noise in $b(\lambda)$ is better mitigated compared with that in λ , and almost all the bit detection errors originate from λ detection. Fig. 3.5(a) compares the final BER with and without half-symbol shifting and using DP-MS-LMMSE vs SP-MS-LMMSE at 500 km. It can be seen that the shifting can reduce the optimal BER by more than an order of magnitude. Using the optimal power for each distance, the BER as a function of distance is shown in Fig. 3.5(b), indicating that the DP-MS-LMMSE filter can slightly improve the transmission distance compared with SP-MS-LMMSE. For an SD-FEC threshold of $2.4\text{E-}2$ (assuming an inner LDPC code with a rate of 9/10 and an outer HD-FEC staircase code with 6.25% overhead), a total of 96 Gbps (net 81 Gbps) transmitted over 650 km can be achieved.

3.5 Conclusion

We studied dual-polarization discrete eigenvalue transmissions by separately encoding and decoding soliton trains on each polarization using SP NFT. We proposed shifting the soliton train of one polarization by half a symbol and using DP-MS-LMMSE filters to improve transmission performance, and a 96 Gbps (net 81 Gbps) transmission over 650 km is experimentally demonstrated. It should be noted that while doubling the bit rate for the same distance is a reasonable first assessment/expectation for DP compared to SP systems, DP NFT systems impose more stringent constraints on the accuracy

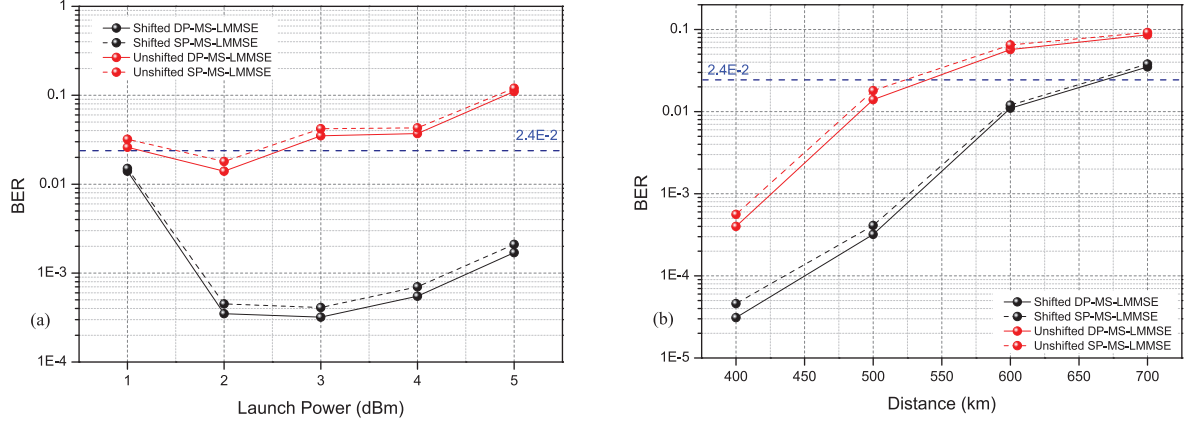


Figure 3.5: (a) BER vs launched power for 6 GBaud DP discrete eigenvalue transmission over 500 km with 16QAM on λ and 16APSK on $b(\lambda)$; (b) BER vs distance. The launched powers are optimized for each distance.

and precision of the experimental setup and are more difficult to realize at present. In addition, DP NFT transmissions and their experimental demonstrations are a relatively new area of research, and numerous new issues remain to be addressed, including polarization crosstalk and other effects, and related DP signal processing techniques in the NFT context. These are important future research topics for DP NFT systems.

Chapter 4

Optimal Nonlinear Spectral Back Rotation for Discrete Eigenvalue Transmission Systems

The primary advantage of NFDM is that it translates the nonlinear fiber channel into a linear one and hence promises a simple signal recovery algorithm. However, this becomes problematic for discrete NFDM systems due to noise-perturbed eigenvalues, and thus the linear minimum mean square error (LMMSE) filter is widely used as an alternative method. This chapter investigates the evolution of noise in the discrete NFDM system and proposes half back rotation as a simple yet effective algorithm for signal recovery, which retrieves the simplicity and elegance promised by the NFDM principle.

4.1 Introduction

Nonlinear Frequency Division Multiplexing (NFDM) is a promising scheme for addressing the fiber nonlinearity-induced capacity crunch. Using the nonlinear Fourier transform (NFT) corresponding to the nonlinear Schrodinger equation (NLSE) that governs signal propagation in an SMF, information can be encoded into the signal's nonlinear spectrum, and each nonlinear spectral component evolves linearly as a spectral phase rotation without mutual interference during propagation in a lossless fiber. Therefore, in principle, transmission impairment compensation and signal recovery can be achieved by back rotating the phase of the nonlinear spectrum.

As an extension of soliton communication, discrete eigenvalue communication was first proposed by Hasegawa and Nyu [20] in 1993, followed by a more general transmission scheme—NFDM proposed by Yousefi and Kschischang [15] in 2014—where information can be encoded in discrete and/or continuous nonlinear spectrum. Ordinary digital coherent transmission waveform combined with NFT detection was used in early experimental

works for back-to-back [60] and long-haul [61, 62] transmission, while Dong et al [27] conducted the first experimental demonstration of discrete NFDM signal transmission containing up to four eigenvalues. Later, advanced modulation formats were applied to the discrete spectrum to carry information, but significant noise on eigenvalues was observed, so the authors used ideal noiseless eigenvalues in back rotation for recovery [63]. Besides, encoding information on b -coefficient rather than discrete spectrum was found to be beneficial [29]. Furthermore, joint modulation of discrete and continuous spectrum [23] and joint modulation of eigenvalue and discrete b -coefficient [30] were investigated.

Although the integrability of NLSE is broken by fiber loss, the NFDM scheme is still feasible with distributed amplification [24], or with discrete lumped amplification by adopting the path-averaging model [26, 52]. However, amplified spontaneous emission (ASE) noise from optical amplifiers distorts the nonlinear spectrum and discrete eigenvalues, and therefore discrete NFDM signal cannot be recovered by simply back rotating the nonlinear spectrum with noise-corrupted eigenvalues. To address the problem, the LMMSE filter was proposed as a high correlation between the received eigenvalue and b -coefficient was observed [29]. Nonlinear filters and machine learning [56] have also been studied to reduce noise in b -coefficients, but these schemes have higher computational complexity and require training symbols to learn the filter coefficients.

In our previous work [64], we built a simple yet effective model of ASE noise accumulation on discrete eigenvalues and b -coefficients during propagation in a practical lossy discrete eigenvalue transmission system, and proposed back rotating the received nonlinear spectrum by half of the full propagation distance to minimize the b -coefficient noise, achieving nearly the same level of performance as LMMSE filter. We hereby extend the work with detailed analytical derivations on the optimality of the half back rotation, performance comparisons between optimal half back rotation and LMMSE filter, and detailed analysis of non-ideal factors, as well as providing experimental verifications of the proposed signal processing algorithm.

This chapter is organized as follows. In Section 4.2, we briefly review the basics of NFT and NFDM, and then introduce the noise accumulation model and prove the optimality of half back rotation. In Section 4.3, numerical results are presented, followed by discussions of practical non-ideal factors that may affect the effectiveness of the half back rotation algorithm in Section 4.4. Section 4.5 contains experimental verification, and Section 4.6 concludes the chapter.

4.2 Principle

4.2.1 NFT and NFDM

The evolution of the signal's complex envelope $E(l, \tau)$ in an SMF is described by the NLSE if fiber loss, polarization effect, higher-order dispersion, Raman effect, and noise are neglected:

$$j \frac{\partial E(l, \tau)}{\partial l} - \frac{\beta_2}{2} \frac{\partial^2 E(l, \tau)}{\partial \tau^2} + \gamma |E(l, \tau)|^2 E(l, \tau) = 0, \quad (4.1)$$

where τ (s) is the temporal variable in the time-retarded frame, l (m) is the spatial variable, β_2 (s²/m) is the group velocity dispersion coefficient, and γ (1/W/m) is the fiber nonlinearity coefficient.

It is conventional to convert 4.1 into a dimensionless form by changing variables:

$$q(z, t) = \frac{E(l, \tau)}{\sqrt{P}}, z = \frac{l}{L_D}, t = \frac{\tau}{T_0}, \quad (4.2)$$

where $P = 1/(\gamma L_D)$, $L_D = T_0^2/|\beta_2|$ (L_D is known as the dispersion length) and T_0 is a freely chosen time normalization parameter. For the anomalous dispersion situation ($\beta_2 < 0$), the normalized NLSE is given by

$$j \frac{\partial q(z, t)}{\partial z} + \frac{1}{2} \frac{\partial^2 q(z, t)}{\partial t^2} + |q(z, t)|^2 q(z, t) = 0. \quad (4.3)$$

The nonlinear spectrum of $q(z_0, t)$ is obtained by solving the eigenvalue problem of the Lax operator associated with the NLSE,

$$\frac{\partial \mathbf{v}}{\partial t} = \begin{bmatrix} -j\lambda & jq(z_0, t) \\ jq^*(z_0, t) & j\lambda \end{bmatrix} \mathbf{v}, \quad (4.4)$$

where $\mathbf{v} = [v_1, v_2]^T$ ($[\cdot]^T$ denotes transpose) is a 2×1 vector that solves (4.4). The scattering coefficients $a(z_0, \lambda)$ and $b(z_0, \lambda)$ are given by (the spatial variable z_0 is omitted for clear notation)

$$\begin{bmatrix} a(\lambda) \\ b(\lambda) \end{bmatrix} = \lim_{t \rightarrow +\infty} \begin{bmatrix} v_1 e^{+j\lambda t} \\ v_2 e^{-j\lambda t} \end{bmatrix} \text{ with } \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \xrightarrow{t \rightarrow -\infty} \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-j\lambda t}. \quad (4.5)$$

The continuous and discrete spectral coefficient is defined as $Q_c(\lambda) = b(\lambda)/a(\lambda)$, $Q_d(\lambda_i) = b(\lambda_i)/a'(\lambda_i)$ where $\lambda_i \in \mathbb{C}^+$ is the discrete eigenvalue satisfying $a(\lambda_i) = 0$, and $\lambda \in \mathbb{R}$ is the continuous eigenvalue. During propagation, the eigenvalue stays unchanged, while the time-domain nonlinear signal evolution is translated into linear (forward) evolution of b -coefficient, given by $b(z_0 + z, \lambda) = b(z_0, \lambda) e^{-2j\lambda^2 z}$ and $b(z_0 + z, \lambda_i) = b(z_0, \lambda_i) e^{-2j\lambda_i^2 z}$.

In fiber optics, a signal that only has a discrete nonlinear spectrum is known as an optical soliton in the time domain, and a 1st-order soliton's temporal position and phase are determined by $\ln |b(\lambda_1)|$ and $\angle b(\lambda_1)$, respectively. For transmission in the real-world lossy fiber with lumped amplification by Erbium-doped fiber amplifier (EDFA), path-averaged soliton should be used, which means that if span length $L_A \ll L_D$, increasing the theoretical soliton launch power by $G \ln G / (G - 1)$ can make the normalized pulse envelope well retained during propagation even though the pulse power varies dramatically across spans, where G is the EDFA gain (also span loss).

4.2.2 Nonlinear spectral back rotation

Consider an N -span EDFA transmission system with each span of equal normalized length L , i.e., $L = L_A / L_D$ as shown in Fig. 4.1. At each EDFA, circularly symmetric complex white Gaussian ASE noise is added to the signal in the time domain. The ASE-induced noise to the discrete eigenvalues is generally not Gaussian but can be approximated by a conditional Gaussian distribution in the small noise limit [17]. We consider a signal with only one discrete eigenvalue, $\lambda_0 = \alpha_0 + j\beta_0 = 0 + 0.5j$, i.e., a 1st-order soliton in the time domain, and assume that at each EDFA, independent Gaussian noise is added to the eigenvalues and b -coefficients, and they are free from ASE noise during propagation between EDFAs. Their values right after the i^{th} EDFA are denoted by α_i , β_i , and b_i , respectively (We will use the subscript “ i ” to indicate span number hereafter, rather than the ordinal of eigenvalue, as there is only one eigenvalue for each pulse).



Figure 4.1: Schematic diagram of the transmission link. Gaussian noise is independently added to λ and b at each EDFA.

During propagation, instead of staying unchanged, the eigenvalue evolves in a random-walk manner and gradually deviates from its original value due to ASE noise loading, and the b -coefficient also deviates from its ideal evolution trace due to noise accumulation on itself, and more importantly, the deviated eigenvalue. Mathematically, $b_{i+1} = b_i e^{-2j(\alpha_i + j\beta_i)^2 L} + n_{i+1}$, where n_i stands for the noise added to the b -coefficient by the i^{th}

EDFA. The received b -coefficient can be written as:

$$\begin{aligned} b_{N+1} &= \left((b_0 + n_1) e^{-2j\lambda_1^2 L} + n_2 \right) e^{-2j\lambda_2^2 L} \dots + n_{N+1} \\ &= b_0 e^{-2jL \sum_{i=1}^N \lambda_i^2} + \sum_{i=1}^N n_i e^{-2jL \sum_{k=i}^N \lambda_k^2} + n_{N+1}. \end{aligned} \quad (4.6)$$

In practice, we noticed that $|n_i| \ll |b_0|$, which makes the first term dominant, so the following terms can be safely dropped. In other words, the deviation of the b -coefficient is mostly contributed by the deviated eigenvalue, rather than the noise on the b -coefficient itself¹. Besides, note that during propagation, $\alpha_i \approx 0$ and $\beta_i \approx 0.5$, which makes the deviation of α and β contribute much more significantly to the deviation of $|b_i|$ and $\angle b_i$, respectively. Thus, we can obtain $|b_{i+1}| \approx |b_i| e^{4\alpha_i \beta_i L} \approx |b_i| e^{2\alpha_i L}$ and $\angle b_{i+1} \approx \angle b_i - 2(\alpha_i^2 - \beta_i^2)L \approx \angle b_i + 2\beta_i^2 L$ (see Appendix B), and it is more convenient to work with the logarithm of $|b|$:

$$\ln |b_{N+1}| = \ln |b_0| + 2L \sum_{i=1}^N \alpha_i, \quad (4.7)$$

$$\angle b_{N+1} = \angle b_0 + 2L \sum_{i=1}^N \beta_i^2. \quad (4.8)$$

The random evolution traces of λ_i and b_i are schematically drawn in Fig. 4.2. We noticed that within each span, the evolution of $\angle b$ follows a curved trace as shown in Fig. 4.9, rather than the straight line given by the NFT evolution equation (while we plot it by straight line segments for simplicity). However, the trace curves back and coincides with the straight line at the end of each span, making the forward evolution model still work well. The curved evolution is due to the unique property of the path-averaged soliton, which will be discussed more in Section 4.4.

Without loss of generality, we set the initial b -coefficient $b_0 = 1$ so $\ln |b_0| = \angle b_0 = 0$. We also assume the noise added to the eigenvalues at each EDFA is identically and independently Gaussian distributed. That is to say, α_i and β_i can be understood as a “discrete” Wiener process. More complex models can be found in [65–67], which argued that the noise in the discrete eigenvalues is generally not Gaussian, but in the following derivation, only independence is needed.

After transmission over the N -span system with $N + 1$ EDFAs, the received signal is processed by NFT. The noisy eigenvalue $\lambda_{N+1} = \alpha_{N+1} + j\beta_{N+1}$, and b -coefficient b_{N+1} are obtained. We then back rotate b_{N+1} by $e^{2j\lambda_{N+1}^2 z}$ with z ranging from 0 to NL , correspondingly referred to as zero to full back rotation. This is equivalent to

¹ We later found that this will not hold if the symbol rate is too low, which makes the noise on b itself dominant and half back rotation or LMMSE less useful, but the optimality of half back rotation is unchanged. This topic will be discussed in Chap. 6

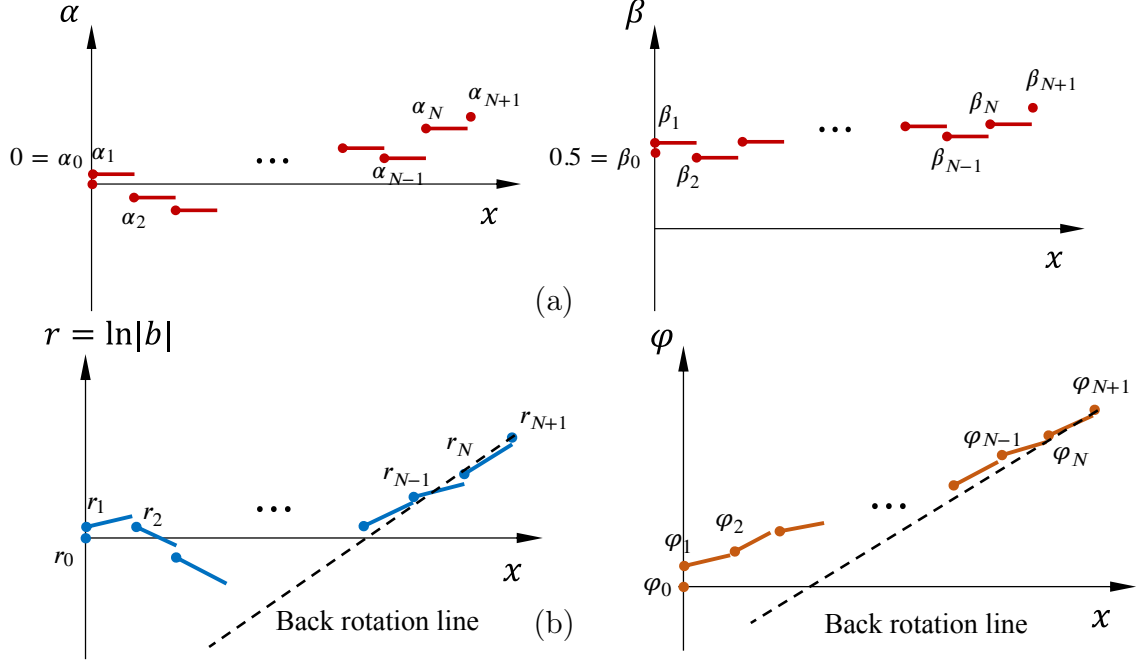


Figure 4.2: Random evolution of λ_i and b_i due to the ASE noise added by each EDFA, and the x-axis is normalized to L , i.e., $x = z/L$.

moving back along the straight dashed line in Fig. 4.2 that starts from point (N, b_{N+1}) with a slope of $2L\alpha_{N+1}$ or $2L\beta_{N+1}^2$, respectively. Note that in a noiseless case, the evolution and back rotation traces of the b -coefficient will exactly coincide, so a perfect recovery of the b -coefficient can be achieved. However, due to ASE noise loading, the eigenvalue evolves in a random-walk manner, and the b -coefficient evolution trace is approximately the cumulative summation of the random walk of the eigenvalue. Therefore, the evolution trace and back rotation line will never be the same but only tangent to each other at point (N, b_{N+1}) . More specifically, the back rotation line for $\ln|b_{N+1}|$ is $r(x) = 2L\alpha_{N+1}(x - N) + r_{N+1}$, where we used $r(x) = \ln|b(x)|$, $r_{N+1} = \ln|b_{N+1}|$ for clear notation, $x = z/L$ is the position coordinate normalized to L , and $b(x)$ denotes the value of b -coefficient that back rotated to the position x . Note that $r(x)$ is a random variable since its coefficients α_{N+1} and r_{N+1} are random variables. The variance of $r(x)$ is derived (in Appendix B) to be

$$\text{Var}(r(x)) = 4L^2\sigma_\alpha^2(N+1) \left[x(x-N) + \frac{N(2N+1)}{6} \right], \quad (4.9)$$

where σ_α^2 is the variance of the noise added to α by each EDFA, which can be obtained by simulation or experiment. The $N(N+1)(2N+1) \propto N^3$ term results from the variance of r_{N+1} , and this cubic dependence is consistent with the well-known Gordon-Haus effect [68, 69]. To minimize $\text{Var}(r(x))$, we set $\frac{d}{dx} \text{Var}(r(x)) = 0$ and obtain $x_{\text{opt}} = N/2$.

Similarly, the back rotation line for $\angle b_N$ is $\varphi(x) = 2L\beta_{N+1}^2(x - N) + \varphi_{N+1}$, where $\varphi(x)$ represents $\angle b(x)$ for clear notation. The variance of $\varphi(x)$ is derived (in Appendix B) to be

$$\text{Var}(\varphi(x)) \approx 4L^2\sigma_\beta^2(N+1) \left[x(x-N) + \frac{N(2N+1)}{6} \right], \quad (4.10)$$

where σ_β^2 is the variance of the noise added to β at each EDFA, and we also obtain $x_{\text{opt}} = N/2$ that minimizes the variance. In short, both the variance of $\ln|b|$ and $\angle b$ are minimized when the back rotation length is half of the full transmission distance. Note that the noise variance will be reduced by a factor of four with half back rotation, hence a 6 dB gain in SNR. Such a result has some resemblance to the optimal scale factor for nonlinear phase noise compensation in the chromatic-dispersion-free scenario [70]. The theoretical analysis provides an intuitive interpretation and effectively offers an alternative proof of the Gordon-Haus effect in the language of NFT, and extends it to a phase-domain counterpart. Note that when a full back rotation of the nonlinear spectral phase is applied ($x = 0$), both the variance of $\ln|b|$ and $\angle b$ become the same as those at the receiver, so a full back rotation does not help to reduce the noise on b -coefficients at all. Besides, back rotating with the noiseless eigenvalue ($0 + 0.5j$) at the transmitter is effectively the same operation across different symbols (back rotation lines are all parallel to each other) and hence does not reduce the variance of $\ln|b|$ and $\angle b$ too.

Theoretically, to make a perfect back rotation, we need all noisy eigenvalues at each EDFA and back rotate with them for each span in sequence, while in practice, we only have access to the noisy eigenvalue at the receiver. However, the received noisy eigenvalue results from ASE noise accumulation along transmission, while the received noisy b -coefficient results from the accumulation of the accumulated ASE noise at each EDFA. Thus, the received noisy eigenvalue still contains considerable information about all the noise introduced along transmission and helps denoise the noisy b -coefficient to a certain extent. This fact is also reflected in their high correlation that has been experimentally reported [29], and it will be theoretically analyzed in Section 4.2.3. The optimality of half back rotation intuitively indicates that we should make some use of the received eigenvalue, but not fully trust it.

4.2.3 Relation to LMMSE filter

We calculated the signal-to-noise ratio (SNR) of the signal after back rotation by $\text{SNR} = 1/\text{EVM}_{\text{RMS}}^2$, where EVM_{RMS} stands for root mean square error vector magnitude (EVM) commonly used in conventional coherent communications. It is noted from Fig. 4.3(b) that the signal SNR after half back rotation is close to that by LMMSE filter and applying an LMMSE filter to the half back rotated signal barely further improves the SNR². This motivates us to investigate the correlation between the b -coefficient and the

² Thus, if a “wrong” distance is used for back rotation, it will still be okay as long as LMMSE is used in the end. I learned from a fellow at a conference that he uses the effective fiber length for back rotation, and zero or full transmission distance is also used by people, as far as I know.

eigenvalue during the back rotation process since the LMMSE filter essentially denoises the b -coefficients by making use of their correlations with eigenvalues. With the noise model described above, the correlation coefficients between $r(x)$ and α_{N+1} , $\varphi(x)$ and β_{N+1} are derived (in Appendix B) to be

$$\rho_{\varphi(x), \beta_{N+1}} \approx \rho_{r(x), \alpha_{N+1}} = \frac{x - N/2}{\sqrt{x(x - N) + N(2N + 1)/6}}. \quad (4.11)$$

Clearly, the correlation coefficient reaches zero right at half back rotation, which is exactly what an LMMSE filter achieves (fully removing the correlation), and hence leads to the same SNR performance.

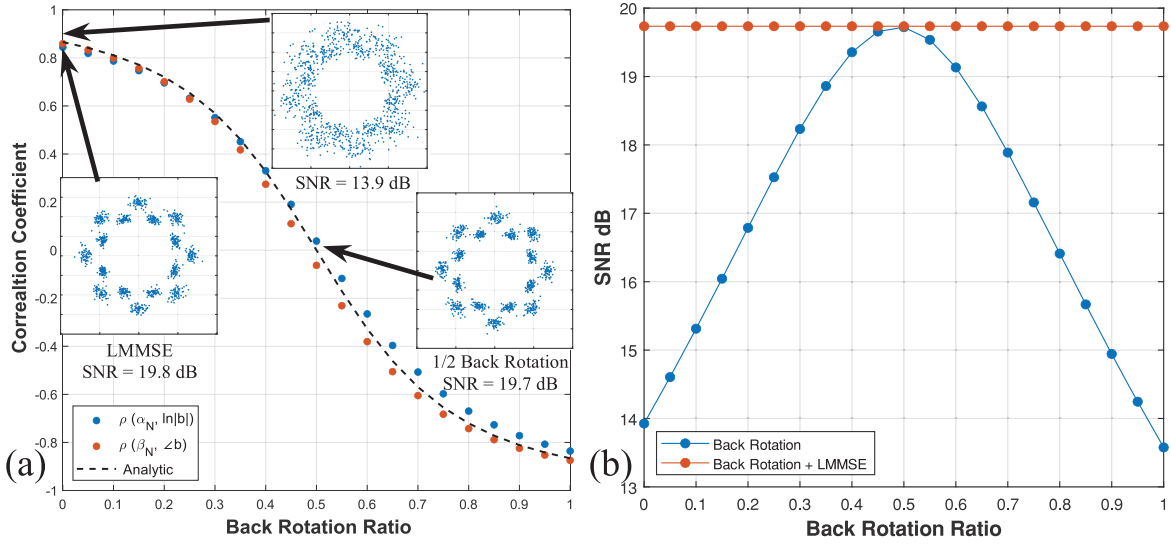


Figure 4.3: (a) Simulated (dots) and analytic (dashed curve) correlation coefficients between the b and λ during back rotation. Constellations of the received signal, received signal processed by LMMSE filter or half back rotation are inset; (b) SNR of b -coefficient with back rotation only and with back rotation plus LMMSE filter.

4.3 Numerical results

We conducted simulations on a 40×50 km EDFA system with non-zero dispersion-shifted fiber (NZ-DSF, loss coefficient: 0.2 dB/km, dispersion coefficient: 4 ps/km/nm). Ideal transmitters, EDFA, and receivers were used in the simulation, while the possible non-ideal factors' effect will be discussed in Section 4.4. 1024 path-averaged 1st-order soliton pulses with 2A8PSK-modulated b -coefficients were used to estimate the variance of $\ln |b|$, $\angle b$, and SNR during the forward evolution and back rotation process. The ratio of symbol interval to soliton's full width at half maximum (FWHM) was set to 6 to avoid interaction between the adjacent solitons, to isolate the impact of ASE noise. The symbol rate was set to 3 GBaud, leading to a dispersion length, L_D of 194.6 km. NFT was applied to the power-normalized signal every 10 km to study the evolution of eigenvalues and b -coefficients of the soliton pulses. For back rotation, we evaluated the b -coefficients every 100 km backward from the receiver. Fig. 4.4 shows the forward evolution and back rotation traces of $\ln |b|$ and $\angle b$, where the red curve and black dots represent the deviations $\ln |b_i| - \ln |b_0|$ and $\angle b_i - \angle b_0$ of each random pulse during forward evolution and back rotation, respectively. Interestingly, the evolution of $\angle b$ is curved in each span and forms a stair-like trace (curved in each span) over the full link.

Fig. 4.5 shows that the variances of $\ln |b|$ and $\angle b$ indeed reach their minimum with a half back rotation, and the SNR of b -coefficient reaches the maximum, with an SNR gain of about 6 dB. The SNR for full back rotation is close to that at Rx, which is consistent with the theoretical analysis. Signal distributions at different back rotation ratios are inset for visual illustration of the optimality of the proposed half back rotation algorithm. In addition, the analytical expression of $\rho_{\varphi(x), \beta_{N+1}}$ and $\rho_{r(x), \alpha_{N+1}}$ match the simulated correlation coefficients quite well as shown in Fig. 4.3(a), and clearly, the correlation coefficient reduces to zero with half back rotation as expected and reaches its maximum of around $\pm\sqrt{3}/2 \approx \pm 0.87$ (given N is not too small) with zero or full back rotation.

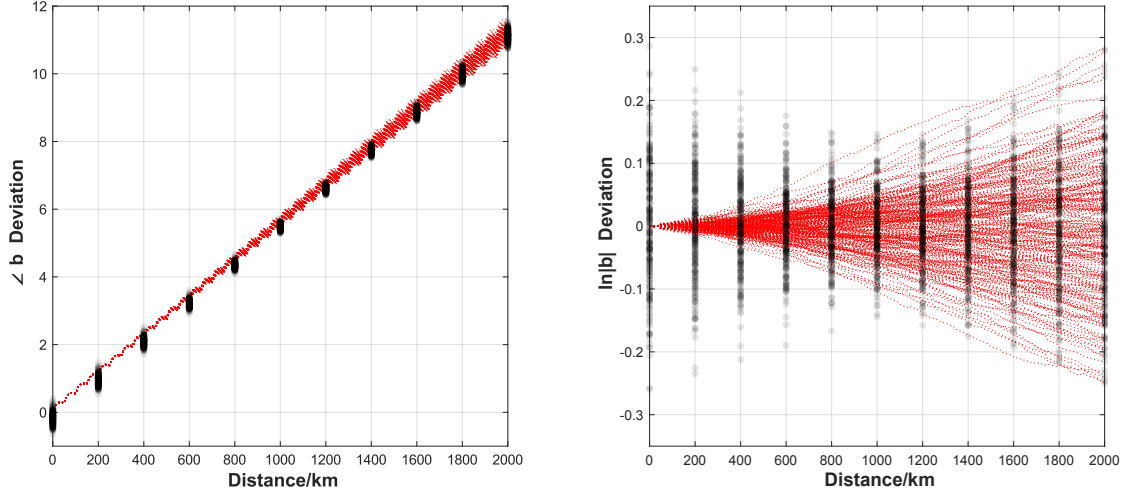


Figure 4.4: Forward evolution (red curves) and back rotation (black dots) of $\angle b$ and $\ln|b|$. Semi-transparent dots are used to better show their dispersity during back rotation, and back rotation dots are plotted every 200 km for a clearer view.

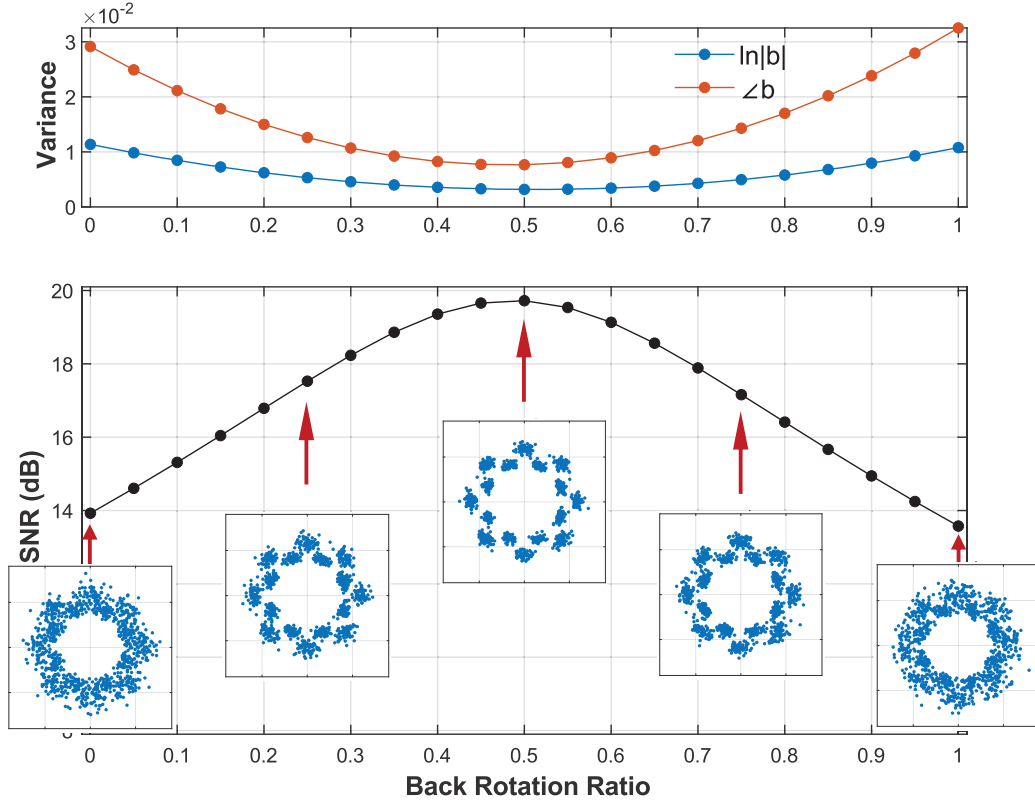


Figure 4.5: Variance of $\ln|b|$, $\angle b$ and the SNR of b -coefficient for different ratios of back rotation. Signal distributions at different back rotation ratios are inset for visual illustration.

4.4 Effect of non-ideal factors

The analytical noise model and the corresponding numerical simulation are established based on ideal transceivers and a channel without other sources of noise and distortions. Here, we discuss the impact of possible non-ideal factors in practice on the performance of the proposed half back rotation algorithm.

4.4.1 Non-ideal bias control

In a digital coherent system, the soliton signal was generated by an IQ modulator, which ideally should be biased at the null point ($V_{\text{bias}} = -V_\pi + 2kV_\pi, k \in \mathbb{Z}$ in Chap. 2). We noticed that non-ideal bias control can cause fluctuation in the soliton's amplitude right after modulation, which translates into fluctuation of β_0 . Moreover, the amplitude fluctuation depends on the soliton's phase, hence the phase of b -coefficient, so the constellation clusters of b will rotate with different angular velocities that are determined by their corresponding β^2 , resulting in an asymmetric constellation at the receiver side, which will also be experimentally shown in Section 4.5. Specifically, let $s(t) = \text{sech}(t)e^{j\theta}$ denote an ideal dimensionless 1st-order soliton, and $\delta_{IQ} = \delta_I + j\delta_Q$ denote the bias on I and Q. Then the soliton shifted by IQ bias is $s'(t) = s(t) + \delta_{IQ}$, which has instantaneous power of $P'(t) = |s(t) + \delta_{IQ}|^2 \approx \text{sech}^2(t) + 2\text{sech}(t)R\cos(\theta - \phi)$, where $R = |\delta_{IQ}|$, $\phi = \arctan(\delta_Q/\delta_I)$. We may use Parseval's identity of nonlinear Fourier Transform [15] to obtain the β after the soliton is perturbed by the IQ bias,

$$\int_{-\infty}^{\infty} P'(t)dt = \frac{1}{\pi} \int_{-\infty}^{\infty} \log(1 + |\tilde{q}(\lambda)|^2) d\lambda + 4\beta, \quad (4.12)$$

which gives

$$\beta \approx 0.5 + \frac{\pi}{2}R\cos(\theta - \phi), \quad (4.13)$$

where we have assumed the number of eigenvalues is still 1, and the continuous nonlinear spectrum $\tilde{q}(\lambda)$ can be ignored, given that the non-ideal IQ bias is typically small in practice. Then, for a specific δ_{IQ} , β is determined by the soliton's phase θ , and so as $\angle b$ ($\angle b = \pi/2 - \theta$). Thus, different b -coefficient phases will lead to different initial values of β when IQ bias exists. This can be illustrated by the following simulation example. 28 ideal 1st-order solitons are b -modulated by 14 different phases and 2 amplitudes, and $\delta_{IQ} = 0.01 + 0.02j$ are applied to them. Fig. 4.6(a) shows the changes in amplitude due to the non-zero IQ bias, which are dependent on $\angle b$. Note that in Fig. 4.6(b), the IQ-bias-perturbed β follows the sinusoidal curve given by 4.13 quite well.

Let $\Delta\beta_0$ denote the initial errors in β_0 . The back rotation line becomes $\tilde{\varphi}(x) = 2L\tilde{\beta}_{N+1}^2(x - N) + \tilde{\varphi}_{N+1}$, where $\tilde{\beta}_{N+1} = \beta_{N+1} + \Delta\beta_0$ and $\tilde{\varphi}_{N+1} = 2L\sum_{i=1}^N(\beta_i + \Delta\beta_0)^2$ are

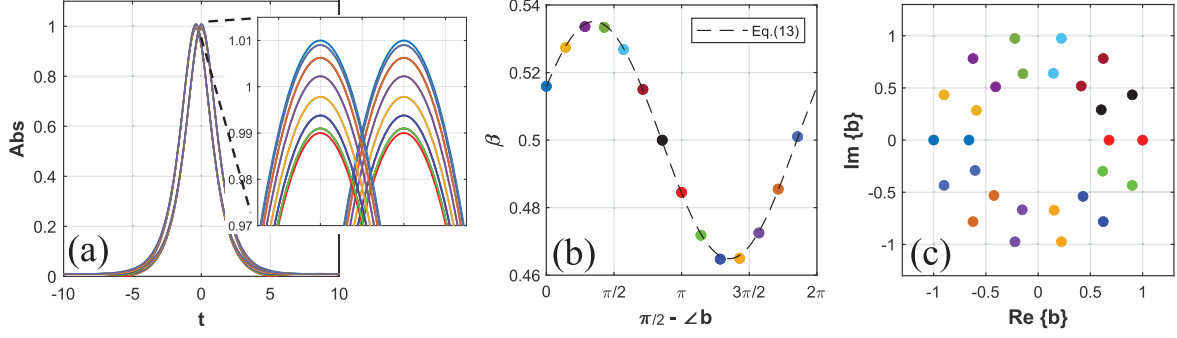


Figure 4.6: (a) The amplitude of 2A14PSK-modulated solitons perturbed by a given IQ bias; (b) the IQ-bias-perturbed β ; (c) IQ bias does not corrupt b directly at Tx, but induces β -dependent phase noise during propagation.

the counterparts to β_{N+1} and φ_{N+1} , respectively. The optimal back rotation location for $\angle b$ will be at (see Appendix B)

$$x_{\text{opt}} = \frac{N(N+1)\sigma_{\beta}^2}{2[(N+1)\sigma_{\beta}^2 + \sigma_{\beta_0}^2]}. \quad (4.14)$$

Thus, if the initial noise on β_0 is significant enough such that $\sigma_{\beta_0}^2$ cannot be neglected (such as non-ideal bias control or low SNR at Tx EDFA), the optimal back rotation for phase will be greater than a half, shifting toward full back rotation, and it is reduced to the ideal case when $\sigma_{\beta_0}^2 = 0$. An intuitive interpretation of this phenomenon is that the initial error introduced by the shifted bias affects the whole transmission process, which is rather “deterministic” compared with the ASE noise gradually added along the transmission line, so a full back rotation is necessary to undo the effect. Overall, the optimal back rotation ratio for $\angle b$ stands between 1/2 and 1. The back rotation for $\ln |b|$ is much less affected, and the optimal ratio is still 1/2 as α is not affected by bias offset.

As an illustrative example, we offset a transmitted signal $s(t) = \sum_{k=-\infty}^{\infty} \text{sech}(t - T_k)e^{j\theta_k}$ by $\delta_I = 1.5\text{E-}3$ (non-ideal bias only happens on I, and here the value of δ_I is in the physical domain) in the dimensionless domain. Fig. 4.7 shows the corresponding forward evolution traces of b -coefficient and eigenvalue. The initial imaginary part β_0 splits into various values at the transmitter (Fig. 4.7 (c)) due to the phase-dependent power fluctuation mentioned above, resulting in different phase evolution velocities indicated by the different trace slopes in Fig. 4.7(a) and thus an asymmetric constellation at the receiver. Fig. 4.8(a) and (c) show the back rotation of $\angle b$ and the corresponding variance versus different back rotation ratios, respectively. As predicted by 4.14, the lowest variance of $\angle b$ happens with a back rotation ratio greater than 1/2, while that for $\ln |b|$ is still 1/2.

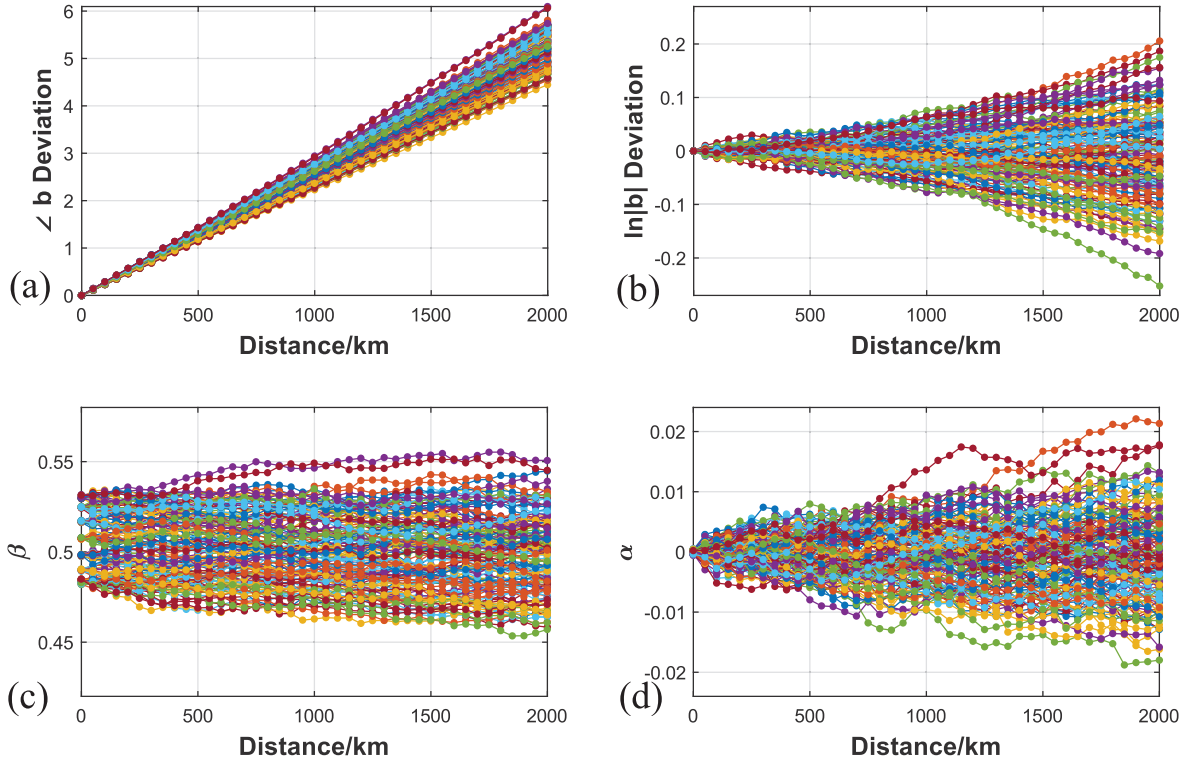


Figure 4.7: Evolution of (a) $\angle b$, (b) $|b|$, (c) β and (d) α when $\delta_I = 1.5\text{E-}3$ is added on $s_I(t)$ at transmitter.

4.4.2 Gain-loss imbalance in fiber loop

In a strict sense, the path-averaged soliton is not a “true” soliton but just a soliton-like pulse. Although the path-averaged soliton’s power evolves periodically due to fiber loss and periodical discrete amplification, its envelope can remain almost unchanged in shape during propagation. The analytical expression of the path-averaged soliton up to first-order correction [52] is

$$q\left(nL_a + \hat{L}, t\right) = a_0 e^{-\Gamma \hat{L}/2} \left(Q_0 + \tilde{Q}\right), \quad (4.15)$$

where $\tilde{Q} = j\tilde{A}_1(\hat{L})|Q_0|^2 Q_0$, $\hat{L}$ is the space coordinates between the i^{th} and $(i+1)^{\text{th}}$ EDFA, $0 < \hat{L} < L$, and $\tilde{A}_1(\hat{L})$ is a coefficient determined by fiber loss and span length. Γ is the normalized power loss coefficient given by $\Gamma = \alpha L_D$, a_0 the initial amplitude, and Q_0 the ‘true’ soliton solution of NLSE.

Let Q_{PA} denotes $(Q_0 + \tilde{Q})$, which is a true soliton Q_0 perturbed by $\tilde{Q}$, and NFT was essentially performed on Q_{PA} . As shown in Fig. 4.9, we noticed that the evolution of $\angle b(Q_{\text{PA}})$ does not follow the straight line of $2(\beta^2 - \alpha^2)z$ given by the NFT theory but evolves along a curved line. However, it eventually curves back and crosses the $2(\beta^2 - \alpha^2)z$ straight line at the end of a span, which makes our forward evolution model and hence back rotation scheme still work.

If the EDFA gain is slightly lower than the span loss, the pulse amplitude after each

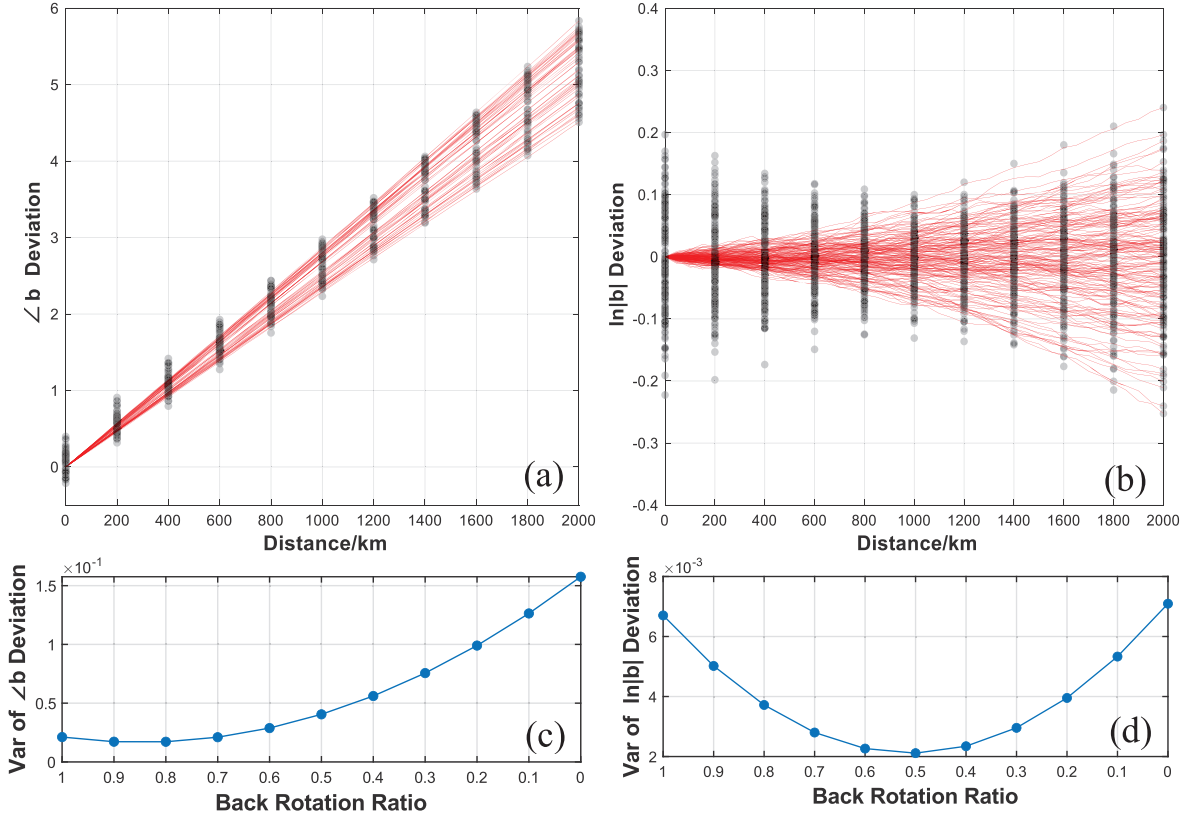


Figure 4.8: (a) Forward evolution and (b) back rotation of $\angle b$ and $\ln|b|$ with $\delta_I = 1.5\text{E-}3$. (c) and (d) show the corresponding variance versus the back rotation ratio.

EDFA will gradually decrease, and we noticed the rotation of $\angle b$ slows down during propagation, while β is still around 0.5 because we always conduct NFT to the power-normalized pulse, Q_{PA} . In such a case, we may consider the pulses as path-averaged solitons perturbed by (tiny) fiber loss. Note that the normalized power loss coefficient $\Gamma = \alpha L_D = 0.2 \times 194.6/4.343 = 8.96 > 1$, which actually does not satisfy the loss perturbation condition of $\Gamma \ll 1$ [38]. However, the periodic amplification essentially extends the distance over which the effect of fiber loss becomes significant to the path-averaged soliton. The gain-loss difference over a span may be interpreted as an “effective” perturbative loss (or gain) under the framework of the path-averaged soliton. With the contribution of fiber loss and periodic EDFA amplification included, the normalized NLSE is given by

$$j \frac{\partial q(z, t)}{\partial z} + \frac{1}{2} \frac{\partial^2 q(z, t)}{\partial t^2} + |q(z, t)|^2 q(z, t) = j \left[-\frac{\Gamma + \Gamma'}{2} + \frac{\Gamma}{2} L \sum_{n=1}^N \delta(z - nL) \right] q(z, t), \quad (4.16)$$

where δ is the Dirac delta function modeling the EDFA amplification, Γ' represents the extra normalized loss/gain term originating from the loss-gain imbalance in the loop.

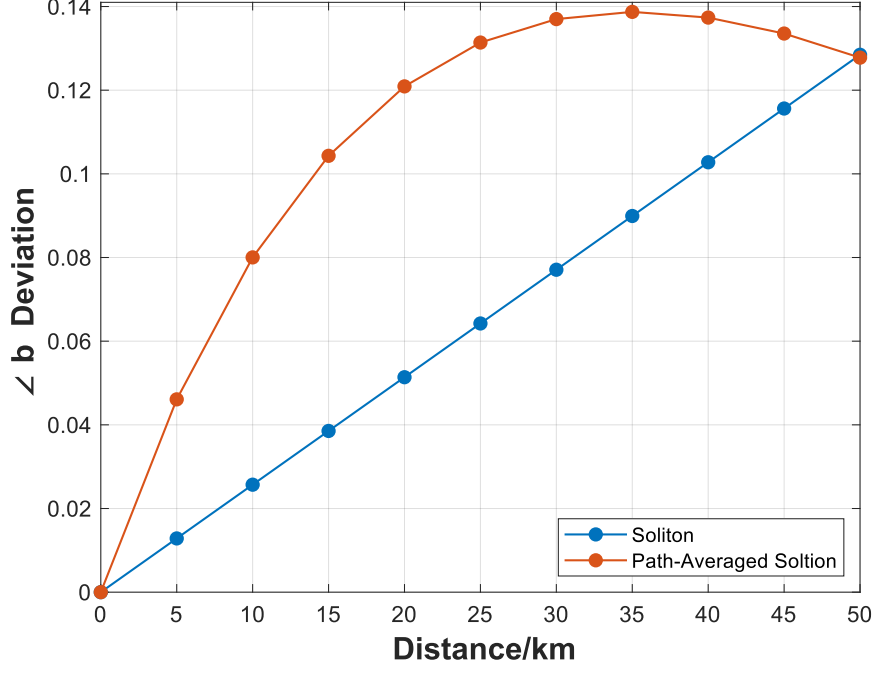


Figure 4.9: Evolution of $\angle b(Q_{PA})$ and $\angle b(Q_0)$ within a span.

Let $q(z, t) = f(z) Q(z, t)$, where $f(z)$ satisfies

$$\frac{df(z)}{dz} = \left[-\frac{\Gamma}{2} + \frac{\Gamma}{2} L \sum_{n=1}^N \delta(z - nL) \right] f(z), \quad (4.17)$$

and then we have

$$j \frac{\partial Q(z, t)}{\partial z} + \frac{1}{2} \frac{\partial^2 Q(z, t)}{\partial t^2} + a^2(z) |Q(z, t)|^2 Q(z, t) = -j \frac{\Gamma'}{2} Q(z, t). \quad (4.18)$$

$a^2(z)$ is fast-varying with respect to the dispersion length and may be replaced by its average value $\langle a^2(z) \rangle$, and for a path-average soliton, the launch power is increased such that $\langle a^2(z) \rangle = \frac{1}{L} \int_0^L a^2(z) dz = 1$, then 4.18 may be effectively interpreted as a path-averaged soliton transmission under perturbative fiber loss Γ' . Suppose the EDFA gain is, for example, 0.03 dB less than the span loss in the fiber loop, then $\Gamma' = 0.03 \text{ (dB)} L_D/L_A = 0.03/4.343 \times 194.6/50 = 0.027 \ll 1$, which satisfies the criterion of perturbation. The phase evolution of a soliton under perturbative attenuation is given by [71]

$$\varphi(z) = \varphi(0) + \frac{1 - \exp(-2\Gamma'z)}{4\Gamma'}, \quad (4.19)$$

which exhibits a slowing-down trend, and it matches quite well with the simulated phase evolution trace (ASE is turned off) as shown in Fig. 4.10. As an illustrative example, Fig. 4.11(a) and (b) show the evolution and back rotation of $\angle b$ and $\ln |b|$, respectively, with ASE turned on and loop gain set to be 0.03 dB less than loop loss. The evolution of $\angle b$ slows down as predicted. However, the received eigenvalue is still around $0 + 0.5j$,

which makes it “larger” than what is needed for back rotation and the back rotation line non-tangent to the forward evolution of $\angle b$ anymore. Thus, the optimal back rotation ratio for it will be less than $1/2$, while the back rotation of $\ln |b|$ is less affected because the gain-loss imbalance does not affect the temporal position of a soliton. Similarly, if the gain is slightly higher than the loss in each span, the optimal back rotation ratio for $\angle b$ will be more than $1/2$.

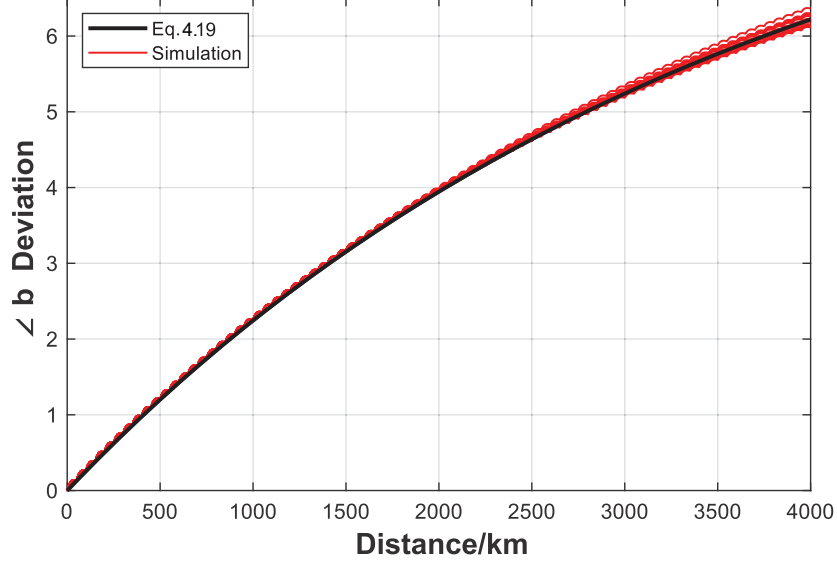


Figure 4.10: Comparison of 4.19 and simulated $\angle b(Q_{PA})$ trace with loop gain less than loop loss by 0.03 dB, ASE free.

4.5 Experimental setup and results

Experiments with offline digital signal processing (DSP) were conducted to verify the effectiveness of half back rotation. Fig. 4.12 shows the experimental setup and the offline DSP flow. 2A8PSK constellation was used for the b -coefficient modulation with ring radii of $\exp(\pm\beta_0 T_\Delta)$. Thus, the corresponding normalized time-domain waveforms were centered at $T_\Delta/2$ respectively, where $T_\Delta = 0.8$ was used in the experiment. At the transmitter, the corresponding time-domain waveform was obtained by INFT on the 2A8PSK symbols. After being added with a synchronization head, it was loaded into the arbitrary waveform generator (AWG, Keysight M8196A). The electrical waveform was then sent into an integrated coherent digital modulator (3 dB bandwidth of 35 GHz) to modulate the optical carrier, and the modulated signal was then amplified by the Tx-EDFA and launched into the fiber loop. To generate a path-averaged soliton, the launched power was increased by $G \ln G / (G - 1)$ with respect to the theoretical power for generating an optical soliton, where G is the total loss of the 50-km fiber. The loop consisted of one span of 50-km NZ-DSF with a dispersion coefficient of 4 ps/km/nm and

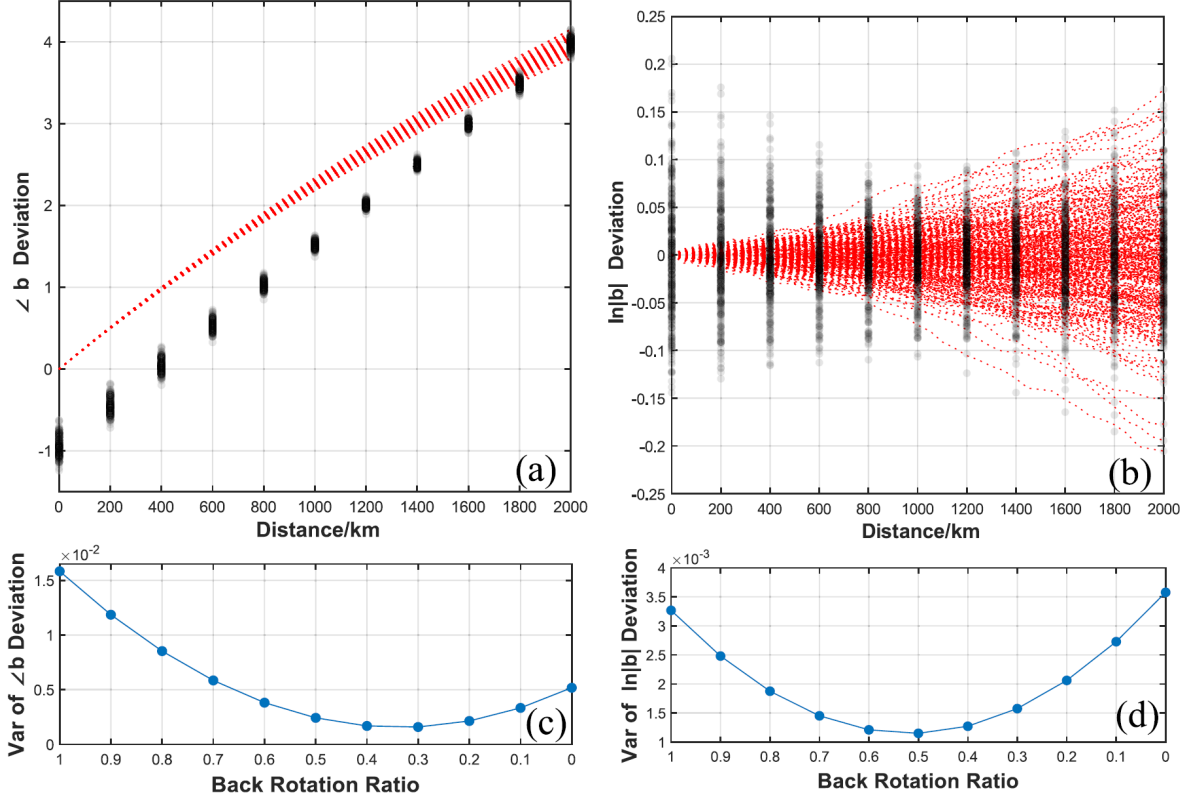


Figure 4.11: (a) Forward evolution and (b) back rotation of $\angle b$ and $\ln|b|$ with loop gain set to be 0.03 dB less than loop loss; (c) and (d) show the corresponding variance versus the back rotation ratio.

a power loss coefficient of 0.2 dB/km, and a loop-EDFA to compensate for the overall loss (about 19 dB) of the NZ-DSF, optical switch, and 50:50 coupler. A flat-top optical filter (3-dB bandwidth of 1 nm centered at ~ 1550 nm) was used after the loop-EDFA to suppress out-of-band ASE noise. Both the transmitter laser and local oscillator shared the same fiber laser source (NKT Koheras BASIK Fiber laser, linewidth < 100 Hz).

After polarization alignment by a manual polarization controller, the received signal was coherently detected and then sampled by a digital oscilloscope (Keysight, DSAZ594A) and processed offline. The sampled signal passed through a series of DSP stages as shown in the inset of Fig. 4.12. After frequency offset (FO) compensation for the AOM-induced 100 MHz frequency shift and timing synchronization, the received signal was sliced into individual pulses and power-normalized, and NFT was then applied to obtain the eigenvalue and b -coefficient for each pulse, followed by back rotation and/or LMMSE filter. The baud rate was 2 GBaud, and the pulse FWHM was set to be $1/8$ of the symbol period, leading to a dispersion length of 246 km that ensures the stable existence of a path-averaged soliton with the span length of 50 km. The normalization time parameter T_0 was calculated to be 35.45 ps with the designed eigenvalue $\lambda_0 = 0 + 0.5j$ and FWHM.

We conducted a soliton transmission experiment over 1500 km with the parameters described above and back-rotated the received noisy b -coefficients with the received

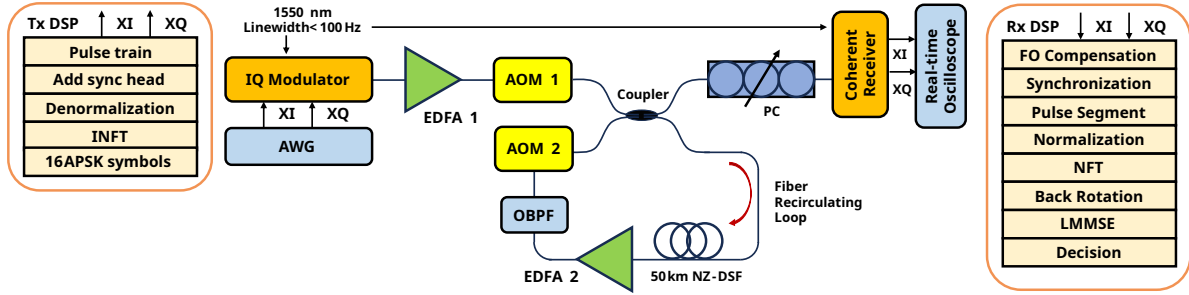


Figure 4.12: Experimental setup for discrete eigenvalue transmission. AOM: Acousto-Optic Modulator, OBPF: Optical Band-Pass Filter.

noisy eigenvalues. A fully trained LMMSE filter was also used after back rotation for performance comparison. Fig. 4.13(a) shows the BER at different ratios of back rotation with/without further processing by an LMMSE filter. The lowest BER happened at around 0.6 back rotation, but the LMMSE filter further decreased the BER after back rotation and returned the lowest BER at a back rotation ratio of 1. This is because in the experiment, the eigenvalue was perturbed by non-ideal factors at the transmitter, which results in asymmetric constellation distributions during the process of forward evolution and also back rotation as discussed in Section 4.4.1. Note that in such a situation, back rotation has double effects: one is reducing the noise variance in each constellation cluster, for which a half back rotation will be optimal, while the other is bringing the mean of each constellation cluster back to the original symmetrically distributed position, for which a full back rotation will be optimal. These double effects make the overall optimal back rotation ratio greater than $1/2$, as theoretically analyzed before, which is a result of the trade-off between the variance within each cluster and the variance among clusters. Besides, we noticed that in the experiment, $|b|$ also became asymmetry, which indicates a split of α at the transmitter. Although the cause needs further investigation, the back rotation effect on $\ln|b|$ is similar to that on $\angle b$, so a full back rotation restores circular symmetry to the constellation both in radial and angular direction, while the noise in each cluster is raised again due to the “excessive” back rotation. However, since the LMMSE filter can remove the noise within each cluster resulting from the “excessive” back rotation, the BER can be further reduced when the back rotation ratio is over $1/2$. To isolate the influence of constellation asymmetry, we calculated the signal SNR with respect to each cluster’s mean point and obtained a flat SNR curve versus different back rotation ratios when further applying an LMMSE filter, as shown in Fig. 4.13(b), which is consistent with the simulation results in Section 4.3.

4.6 Conclusion

We propose back rotating the received noisy b -coefficient by half of the transmission distance with the received noisy eigenvalues for discrete eigenvalue transmission. The effectiveness of half back rotation was derived analytically and verified by simulation and experiment. It gives the same level of performance compared to the LMMSE filter, but with less calculation complexity and no need for training data. We also derive and discuss the sensitivity of the proposed back rotation algorithm to non-ideal practical factors such as non-ideal bias control and gain-loss imbalance. Extensions to distributed amplification systems and dual-polarization systems are expected to be natural and straightforward.

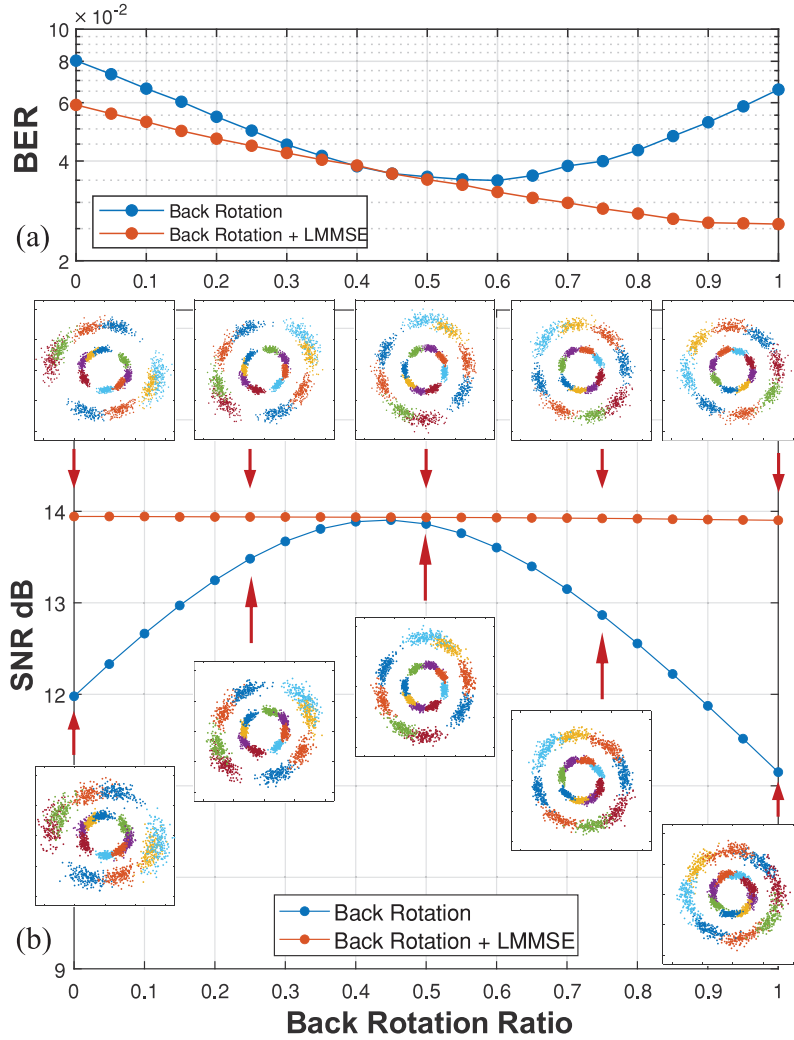


Figure 4.13: (a) BER at different ratios of back rotation with/without further applying an LMMSE filter; (b) SNR (within cluster) at different ratios of back rotation with/without further applying an LMMSE filter. Insets are corresponding signal distributions.

Chapter 5

A Full Digital Signal Processing Flow for Dual-Polarization Discrete Nonlinear Frequency Division Multiplexing Systems

In NFDM studies, due to its infancy, practical impairments are usually circumvented through temporary solutions. Meanwhile, the development of DSP algorithms for NFDM transmission has been slow and unsystematic, which hinders the application of NFDM in practical fiber optic systems. With half back rotation ready and noting that it is the counterpart of dispersion compensation in ordinary coherent systems, a full DSP flow for DP discrete NFDM transmission, which allows for various practical impairments, is close at hand as long as a proper polarization recovery algorithm can be proposed. This chapter explores this issue.

5.1 Introduction

Nonlinear Frequency Division Multiplexing (NFDM) is a strategy to address the capacity limitations caused by fiber nonlinearity. By utilizing the nonlinear Fourier transform (NFT) associated with the Manakov Equation that describes the propagation of a Dual-Polarization (DP) signal in a Single-Mode Fiber (SMF), information can be encoded on the signal's nonlinear spectrum. In an ideal lossless fiber, the nonlinear spectrum evolves linearly without mutual interference, thereby creating a set of parallel communication channels that are theoretically optimal for nonlinear fiber transmission [15, 44].

In real-world applications, however, the integrability of the Manakov Equation is broken due to fiber loss and Amplified Spontaneous Emission (ASE) noise. This ASE noise translates into noise in the nonlinear frequency domain, making both the nonlinear

spectrum and discrete eigenvalues noisy. The noise in the received discrete eigenvalues prevents signal recovery by simply applying a (full) Back Rotation (BR) to the nonlinear spectrum, but leads to significant errors instead. To address this issue, various Digital Signal Processing (DSP) algorithms have been investigated, including linear minimum mean square error (LMMSE) filters [29, 30], nonlinear filters, and machine learning methods [56]. This became the main research direction in the early days of discrete NFDM transmission, which investigates the emerging innovative scheme under relatively ideal transmission conditions, either by simulations, or by experiments with homodyne detection using extremely-narrow-linewidth lasers and polarization controllers to skip over polarization and carrier recovery. Without these impairments, those studies could focus on reducing the noise on the received nonlinear spectrum, which was found to be mainly contributed by the noisy discrete eigenvalues afterwards. Besides, some studies jumped out of the framework of NFDM by using a neural network receiver [54] to seek performance improvements. Although these approaches can be effective in denoising the nonlinear spectrum, they often involve considerable complexity and need a training sequence. In contrast, we proposed half nonlinear spectral BR [64] as a simple yet effective algorithm for single-polarization (SP) discrete NFDM systems. This approach was shown to be optimal in terms of reducing noise power on b -coefficients, and moreover, it retrieves the simplicity and elegance of signal processing promised by the NFDM principle. In [72], a similar idea called “mean back rotation” equalizer was heuristically proposed, but the mechanism was not illuminated. We note that it is effectively equivalent to half BR except for including the initial eigenvalue, which is a constant and does not affect the performance of BR.

In recent years, researchers started to address the practical impairments, including polarization rotation, polarization mode dispersion, carrier Frequency Offset (FO), and phase noise that normally exist in any coherent transmission systems (and a discrete NFDM system is no exception). First, training sequences were widely used for those impairments (e.g., polarization demultiplexing [73], FO estimation [74]). The training sequence usually operates separately in the time domain and is irrelevant to the NFDM mechanism, while the estimated channel impairments can be used in equalizing the successive NFDM symbols since the channel impairments are usually slow-varying compared with the symbol rate. Alternatively, DSP algorithms that directly work in the nonlinear frequency domain have also been investigated, which can avoid the information rate loss due to the training sequence (see [75] for polarization demultiplexing, [76–78] for FO estimation and [74, 79–82] for carrier phase recovery), and neural network has also been used for carrier recovery [21, 83]. These advancements in DSP techniques paved the way for establishing a more integral and efficient discrete NFDM transmission system, but they were typically specific to only one or two individual impairments, while

a full DSP flow working in the nonlinear domain, especially for DP discrete NFDM systems, has not been proposed yet.

In our previous work [84], we extended the half BR algorithm to DP discrete NFDM transmission and realized polarization recovery by using a 1 Sample-per-Symbol (SpS) Multiple-Input-Multiple-Output (MIMO) filter. We then proposed a full DP discrete NFDM DSP flow that completely works in the nonlinear frequency domain for the first time. We hereby extend the work with a detailed investigation of the DSP flow in combination with the LMMSE filter, and a detailed analysis of non-ideal factors, as well as experimental verifications.

This chapter is organized as follows. In Section 5.2, we briefly review the basics of DP discrete NFDM transmission. In Section 5.3, the proposed DSP flow for the DP discrete NFDM system is presented, followed by numerical verifications in Section 5.4. Sections 5.5 and 5.6 pay special attention to the influence of PDL and non-ideal IQ bias on DP discrete NFDM transmission, respectively. Section 5.7 contains experimental verification of the full DSP flow, and Section 5.8 concludes the chapter. We use $[\cdot]^T$, $[\cdot]^*$, and $[\cdot]^\dagger$ to denote transpose, complex conjugate, and Hermitian transpose, respectively. Vectors and matrices are indicated by bold fonts.

5.2 DP-NFT and NFDM

The propagation of a DP signal $\mathbf{Q}(l, \tau) = [Q_1(l, \tau), Q_2(l, \tau)]^T$ in an SMF is governed by the Manakov Equation, where the fiber loss, polarization mode dispersion (PMD), and noise are ignored, and the fast evolution of State of Polarization (SOP) during propagation has been averaged out. The Manakov Equation is given by

$$j \frac{\partial \mathbf{Q}(l, \tau)}{\partial l} - \frac{\beta_2}{2} \frac{\partial^2 \mathbf{Q}(l, \tau)}{\partial \tau^2} + \frac{8}{9} \gamma |\mathbf{Q}(l, \tau)|^2 \mathbf{Q}(l, \tau) = 0, \quad (5.1)$$

which can be normalized into the dimensionless form

$$j \frac{\partial \mathbf{q}(z, t)}{\partial z} + \frac{1}{2} \frac{\partial^2 \mathbf{q}(z, t)}{\partial t^2} + |\mathbf{q}(z, t)|^2 \mathbf{q}(z, t) = 0 \quad (5.2)$$

by variable substitution

$$\mathbf{q}(z, t) = \frac{\mathbf{Q}(l, \tau)}{\sqrt{P}}, \quad t = \frac{\tau}{T_0}, \quad z = \frac{l}{L_D}. \quad (5.3)$$

where $j = \sqrt{-1}$, τ (s) is the temporal variable in the time-retarded frame, l (m) is the spatial variable, β_2 (s²/m) is the group velocity dispersion coefficient, and γ (1/W/m) is the fiber nonlinearity parameter. $P = |\beta_2|/(\frac{8}{9}\gamma T_0^2)$, $L_D = T_0^2/|\beta_2|$ (L_D is known as the dispersion length), and T_0 is a free normalization coefficient.

The NFT of $\mathbf{q}(z_0, t) = [q_1(z_0, t), q_2(z_0, t)]^T$ is conducted by solving the eigenvalue problem [44]

$$\begin{bmatrix} j\partial_t & q_1(z_0, t) & q_2(z_0, t) \\ q_1^*(z_0, t) & j\partial_t & 0 \\ q_2^*(z_0, t) & 0 & j\partial_t \end{bmatrix} \mathbf{v} = \lambda \mathbf{v} \quad (5.4)$$

with the boundary condition $\lim_{t \rightarrow -\infty} \mathbf{v}(t) = [1, 0, 0]^T e^{-j\lambda t}$, and $\partial_t \equiv \frac{\partial}{\partial t}$. The discrete scattering coefficients $a(z_0, \lambda_i)$, $b_1(z_0, \lambda_i)$ and $b_2(z_0, \lambda_i)$ are given by

$$\begin{aligned} a(z_0, \lambda_i) &= \lim_{t \rightarrow \infty} v_1(t) e^{+j\lambda_i t}, \\ b_1(z_0, \lambda_i) &= \lim_{t \rightarrow \infty} v_2(t) e^{-j\lambda_i t}, \\ b_2(z_0, \lambda_i) &= \lim_{t \rightarrow \infty} v_3(t) e^{-j\lambda_i t}, \end{aligned} \quad (5.5)$$

where $\lambda_i \in \mathbb{C}^+$ and satisfies $a(\lambda_i) = 0$.

By NFT, the nonlinear evolution of the time-domain signal is transformed into the linear evolution of b -coefficients without mutual interference:

$$\begin{aligned} b_1(z_0 + z, \lambda_i) &= b_1(z_0, \lambda_i) e^{-2j\lambda_i^2 z}, \\ b_2(z_0 + z, \lambda_i) &= b_2(z_0, \lambda_i) e^{-2j\lambda_i^2 z}. \end{aligned} \quad (5.6)$$

A 1st-order DP soliton has a single discrete eigenvalue $\lambda = \alpha + j\beta$ (subscript omitted), whose time-domain waveform and b -coefficient are explicitly related by [44, 72]:

$$\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = -2\beta \operatorname{sech}(2\beta(t - \Delta t)) e^{-2j\alpha t} \begin{bmatrix} \cos(\theta) e^{-j\phi_1} \\ \sin(\theta) e^{-j\phi_2} \end{bmatrix}, \quad (5.7)$$

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = e^{2\beta\Delta t} \begin{bmatrix} \cos(\theta) e^{j\phi_1} \\ \sin(\theta) e^{j\phi_2} \end{bmatrix}, \quad (5.8)$$

and the Jones vector of $\mathbf{q}(t)$ is related to $\mathbf{b}$ by

$$\mathbf{J}(\mathbf{q}) = \begin{bmatrix} \cos(\theta) e^{-j\phi_1} \\ \sin(\theta) e^{-j\phi_2} \end{bmatrix} = \frac{\mathbf{b}^*}{|\mathbf{b}|}. \quad (5.9)$$

Thus, $|b_1|/|b_2|$ determines the magnitude ratio between X and Y polarization, the phase difference between b_1 and b_2 determines the (minus) phase difference between X and Y polarization, and $\ln \|\mathbf{b}\|/2\beta$ determines the initial time position of the soliton pulse.

In this work, we use the bidirectional algorithm for computing the discrete b -coefficients [85].

5.3 DSP flow for DP discrete NFDm transmission

The DSP flow is shown in Fig. 5.1. First, the common DSP stages may include synchronization, power normalization, and waveform segmenting to properly allocate the received pulses into the processing time window, followed by DP-NFT to correctly get the eigenvalue and b -coefficients, which is critical for the following DSP modules to remove channel impairments and demodulate the signal, as described in the following.

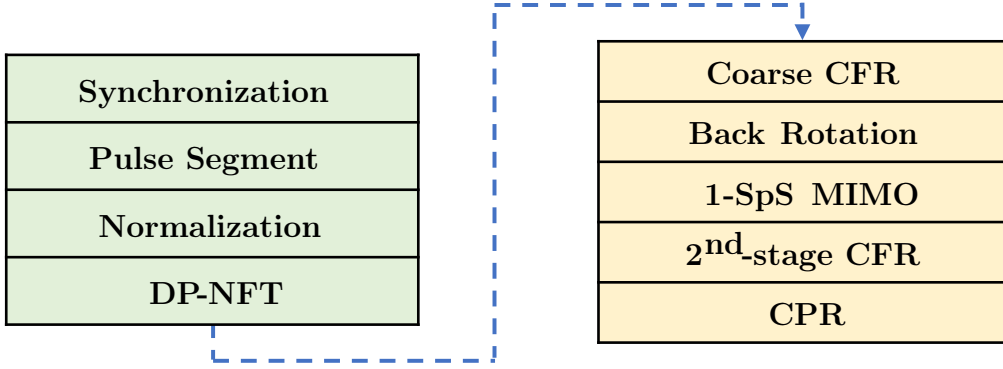


Figure 5.1: The proposed full DSP flow for DP discrete NFDm transmission.

5.3.1 Back Rotation (BR)

The evolution rule of $\mathbf{b}$ (5.6) for a discrete NFDm system has the same structure as the transfer function of chromatic dispersion in an ordinary coherent transmission system, and so it would be reasonable to set the DSP flow similar to that in an ordinary coherent transmission system. Therefore, we set BR as the first stage of the DSP flow, which benefits the following stages since it produces a less noisy $\mathbf{b}$. However, due to the noisy nature of the discrete eigenvalue during transmission, the BR should be done by only half of the whole transmission distance to minimize the noise on b -coefficients, as we have shown in the SP case [64]. Since b_1 and b_2 share the same evolution equation, the optimality of half BR can be directly extended to the DP case. If ignoring the noise directly added on $\mathbf{b}$, which is reasonable for symbol rates greater than 2 GBaud (see Chap. 6), the noise variance of $\ln |b_{1,2}|$ and $\angle b_{1,2}$ are given by

$$\text{Var}(r_{1,2}(x)) = 4L^2\sigma_\alpha^2(N+1)\left[x(x-N) + \frac{N(2N+1)}{6}\right], \quad (5.10a)$$

$$\text{Var}(\varphi_{1,2}(x)) = 4L^2\sigma_\beta^2(N+1)\left[x(x-N) + \frac{N(2N+1)}{6}\right], \quad (5.10b)$$

where r and φ denote $\ln |b|$ and $\angle b$, and σ_α^2 , σ_β^2 denote the noise variance on α and β due to i.i.d. ASE loading at each EDFA, respectively. Ideally, when a half BR is done, the

noise variance of $\ln |b_{1,2}|$ and $\angle b_{1,2}$ will be reduced by four times, equivalently, a gain of 6 dB in Signal-to-Noise Ratio (SNR). We have shown in [86] that BR and LMMSE return the same performance in an ideal SP transmission situation. However, when practical impairments such as polarization rotation, FO, and phase noise exist, the correlation between the eigenvalue and b -coefficient can be completely broken, making LMMSE incapable of equalization. In contrast, BR is not affected by those impairments.

5.3.2 Polarization Demultiplexing

The Manakov Equation is obtained by averaging out the fast SOP change of the signal during propagation, while for practical DP signal transmission, polarization demultiplexing must be conducted. Conventionally, polarization demultiplexing is usually done (in the time domain) before conducting NFT, for example, by using a training sequence [87, 88] or by maximizing the polarization extinction ratio of the soliton pulse sequence [75]. However, a blind polarization recovery algorithm that directly works on the nonlinear frequency domain has not been proposed yet. Here, we explain that polarization demultiplexing can be conducted directly in the nonlinear frequency domain, which reduces the complexity and overhead of the overall DSP flow.

Polarization rotation and the 1st-order PMD of SMF can be modeled by the (frequency-dependent) Jones matrix $\mathbf{R}(\omega)$ in the ordinary frequency domain [37, Chap. 3.3]. However, as the symbol rate of discrete NFDM transmission is typically low (up to several GBaud), we may neglect the PMD effect [89] and the frequency dependence of $\mathbf{R}(\omega)$. In such a way, the polarization effect in the time or frequency domain is described by the same Jones matrix $\mathbf{R}$ (frequency-independent), i.e., $\tilde{\mathbf{q}}(L, \omega) = \mathbf{R}\tilde{\mathbf{q}}(0, \omega) \Leftrightarrow \mathbf{q}(L, t) = \mathbf{R}\mathbf{q}(0, t)$, where $\tilde{\mathbf{q}}$ denotes the ordinary Fourier transform of $\mathbf{q}$. Since $\mathbf{R}$ is unitary, its corresponding effect in the nonlinear frequency domain will simply be its complex conjugate [57, 90]: $\mathbf{q}(L, t) = \mathbf{R}\mathbf{q}(0, t) \Leftrightarrow \mathbf{b}(L, \lambda_i) = \mathbf{R}^*\mathbf{b}(0, \lambda_i)$. In other words, $\mathbf{b}$ also goes through a polarization rotation, and hence it is feasible to recover the original SOP by directly applying the well-known adaptive equalization algorithms based on MIMO structure, such as Constant Modulus Algorithm (CMA) and Radius-Directed Equalization (RDE) on $\mathbf{b}$.

Note that in ordinary coherent systems, the MIMO stage is usually 2-SpS conducted to self-adaptively adjust the sampling instants to the optimal ones, while for discrete NFDM systems, the NFT result λ_i and $\mathbf{b}(\lambda_i)$ is one-to-one mapped from the time-domain symbol (a soliton pulse), which effectively means 1-SpS. However, there is a synchronization stage to the received signal for segmenting the consecutive soliton pulses into proper separate time windows, which ensures obtaining correct NFT results for those segments. During such a synchronization process, the “optimal sampling instant” has been implicitly achieved. Thus, a 1-SpS MIMO filter can be directly applied to the

$\mathbf{b}$ for polarization recovery. Besides, we set the MIMO filter tap number to one since it is enough to realize polarization demultiplexing, and also due to the fact that multi-tap means linear combination of adjacent b -coefficients, which makes no physical sense.

We set the 1-SpS MIMO after BR, as a counterpart of the 2 SpS MIMO following chromatic dispersion compensation in ordinary coherent systems. Moreover, once polarization is properly demultiplexed, off-the-shelf Carrier Frequency Recovery (CFR) and Carrier Phase Recovery (CPR) algorithms can be directly concatenated after the 1-SpS MIMO, forming a complete DSP flow fully working in the nonlinear frequency domain.

5.3.3 Carrier Frequency Recovery

As a traditional blind CFR method, the M^{th} -power algorithm [91] has a FO estimation range limited within $[-\frac{R_s}{2M}, \frac{R_s}{2M}]$, where R_s denotes the symbol rate, and M comes from the M -fold phase-symmetry in the constellation. Therefore, it is hard to be directly applied since the symbol rate in discrete NFDM systems is relatively low, compared to the FO that can be up to 5 GHz for commercially applied lasers [92]. However, a favorable property of the discrete NFDM signal is that the FO between the carrier and Local Oscillator (LO) directly translates into the shift in the real part of the eigenvalue, $\Delta\alpha$. Thus, a coarse CFR can be easily done by rotating the k^{th} b -vector by $e^{-2jk\Delta\alpha_k T}$, where $T = T_s/T_0$ is the normalized symbol period and $\Delta\alpha_k$ can be replaced by $\mathbf{E}[\alpha_k]$ ($\alpha = 0$ at Tx). Then, the residual FO will be small enough to be further recovered by the M^{th} -power CFR algorithm. We set the coarse CFR before BR, and back rotate using the eigenvalue with its mean real part removed, $\lambda_k - \mathbf{E}[\alpha_k]$.

5.3.4 Carrier Phase Recovery

Amplitude Phase Shift Keying (APSK) is commonly used in discrete spectrum modulation as it is found to outperform Quadrature Amplitude Modulation (QAM) [28], and it is a more natural choice for the logarithmic transform. Therefore, we adopted APSK modulation, and we applied the Blind Phase Search (BPS) algorithm [46] for CPR because it can be applied to arbitrary constellations. Other algorithms widely used in ordinary coherent systems are also expected to be effective, while verification of those algorithms is beyond the scope of this research. The whole DSP flow at the receiver is shown in Fig. 5.1.

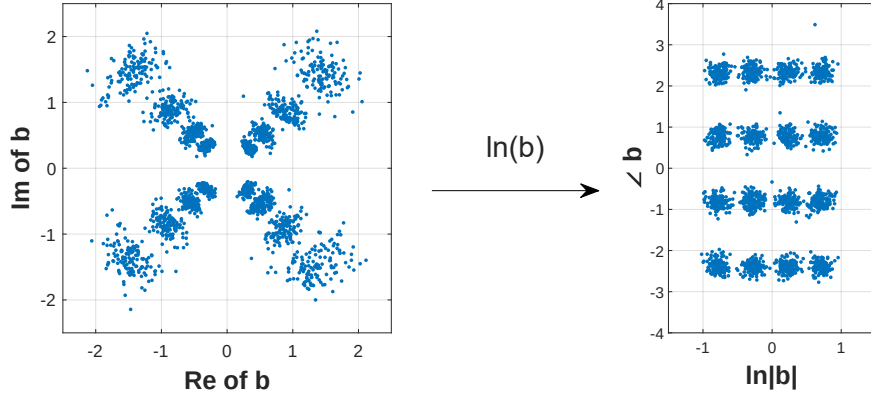


Figure 5.2: Logarithmic transform of a mAn PSK signal used in discrete NFDm transmission. Here $m = n = 4$ is shown as an example.

5.4 Simulation results

5.4.1 EDFA System

We conducted simulations on a 40×50 km fiber recirculating loop system, where Non-Zero Dispersion-Shifted Fiber (NZ-DSF, power loss coefficient: 0.2 dB/km, dispersion coefficient: 4 ps/km/nm), ideal model of transmitters, EDFA and receivers were used in the simulation except for setting a FO of 6 GHz and laser linewidth of 100 kHz for both the Tx and Rx laser. The Noise Figure (NF) of the EDFA was set to 8 dB to accelerate noise accumulation. 2048 path-averaged 1st-order DP soliton pulses (eigenvalue $\lambda_0 = 0 + j\beta_0 = 0 + 0.5j$) were used to estimate $\text{Var}(\ln |b_{1,2}|)$, $\text{Var}(\angle b_{1,2})$, and SNR during the forward evolution and BR process, where b_1 and b_2 are independently modulated with a 4A8PSK constellation. The ring radii of $b_{1,2}$ is set to $\exp(\pm\beta_0 T)/\sqrt{2}$, where $T = 0.5$. The scaling factor of $1/\sqrt{2}$ is to make the mean of $|b| \approx 1$, so the corresponding time-domain soliton pulse is located at the center of the time window.

The ratio of symbol interval to the Full Width at Half Maximum (FWHM) of the soliton was set to 6 to avoid interaction between the adjacent solitons. The symbol rate was set to 3 GBaud, leading to a dispersion length L_D of 194.6 km. DP-NFT was applied to the signal after power normalization. We varied the BR ratio from 0 to 1 to verify the optimality of half BR when embedded in the full DSP flow. In this chapter, SNR is calculated based on the logarithm of $b_{1,2}$, which eliminates the dependence of noise variance on the signal magnitude, as shown in Fig. 5.2, and such a transform is also consistent with the logic of analyzing $\text{Var}(\ln |b_{1,2}|)$ and $\text{Var}(\angle b_{1,2})$.

The signal distribution at different stages of the proposed DSP flow is shown in Fig. 5.3, which verifies the effectiveness of the proposed DSP flow. It is worth noting that the FO (6 GHz) is twice the value of the symbol rate (3 GBaud), but the 2nd-stage CFR still successfully works thanks to the coarse CFR at the beginning of the DSP flow. Fig. 5.4

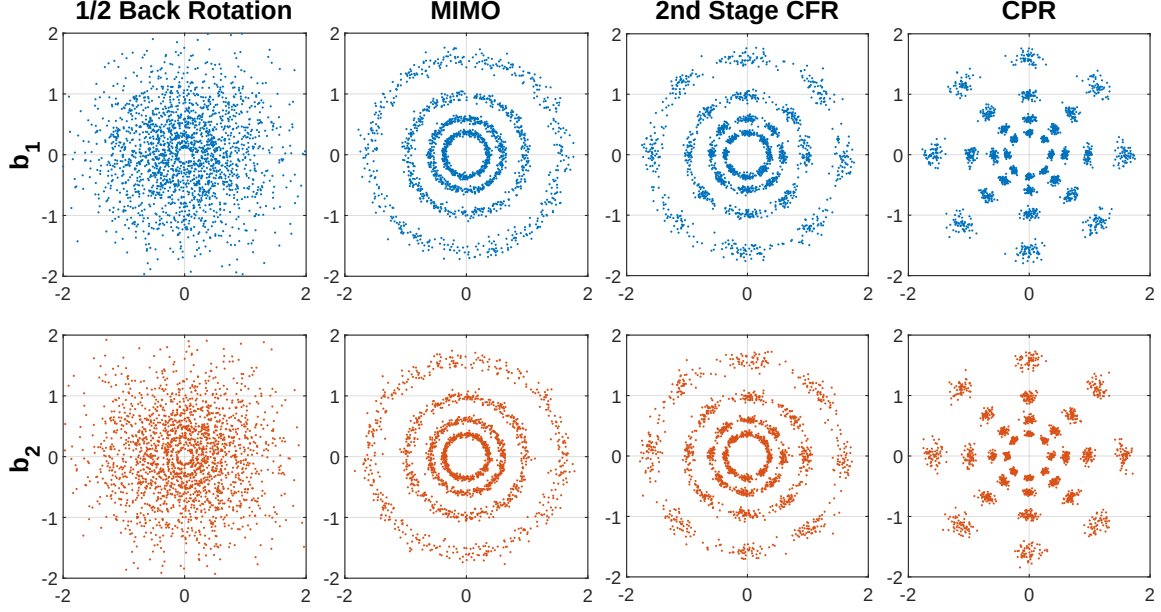


Figure 5.3: Signal distribution at different stages of the DSP flow.

shows the variance and SNR of the DP signal after the DSP flow. An ideal transmission result (only ASE is included as impairment, and the DSP flow is reduced to BR) is added for comparison. We can see a gap of ~ 0.9 dB between them at half BR, which may be regarded as the penalty due to the practical transmission situation with DSP.

5.4.2 Raman System

It is worth noting that the BR algorithm also works on ideal distributed amplification systems. In such a case, strictly, the $b_{1,2}$ evolution should be described by a stochastic integral over the random-walk trace of the eigenvalue. However, the same result can also be obtained by simply considering the ideal distributed amplification as the continuous limit of discrete amplification with the amplifier distance approaching zero.

The gain of real-world Raman amplification, however, is not uniformly distributed along the fiber—it is higher at the location closer to the pump light source, resulting in a curved power profile within a span. In simulation, we used a 24.6 dBm single pump laser with a frequency higher than the signal carrier frequency by 13.15 THz, which counter-propagates from the end of each span, providing an on-off gain of 10 dB (equal to the span loss) in simulation. The resultant signal power profile in each span is shown in Fig. 5.5(a). By setting the launch power such that $\frac{1}{L} \int_0^L a^2(z) dz = 1$, where $a^2(z)$ describes the power profile of the signal along the fiber, a path-averaged soliton can well exist in the Raman amplification system. Without a strict argument, we simply apply the BR algorithm to a practical Raman system and verify its effectiveness by the following simulation results. FO and laser linewidth are the same as in the EDFA case.

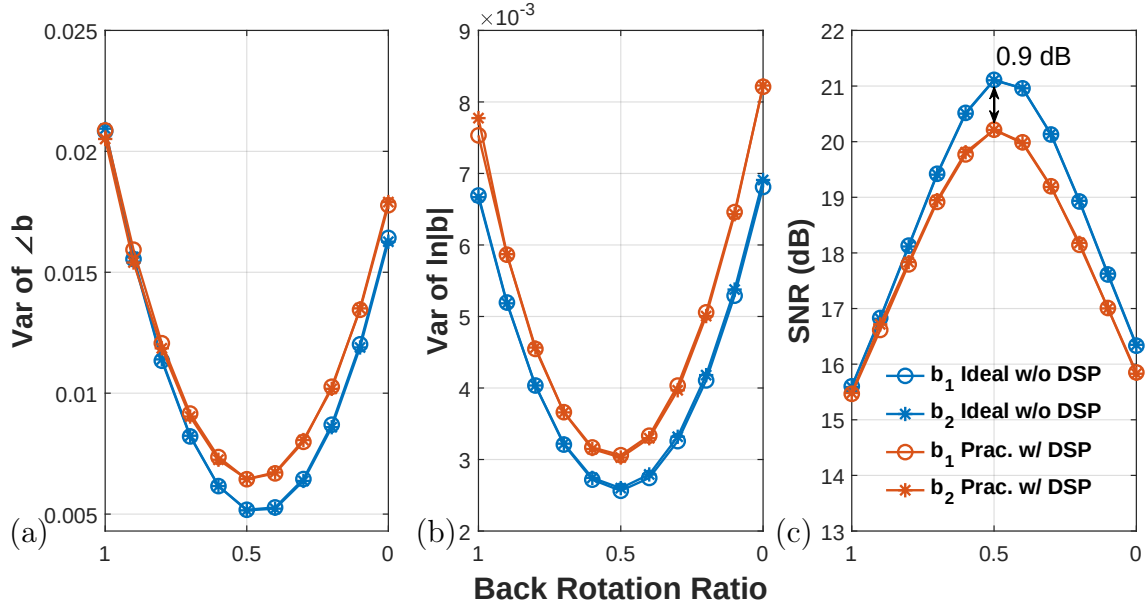


Figure 5.4: Variance in phase (a), log-magnitude (b), and SNR (c) of $b_{1,2}$ versus different BR ratios used in the DSP flow. Ideal transmission with BR only is added for comparison.

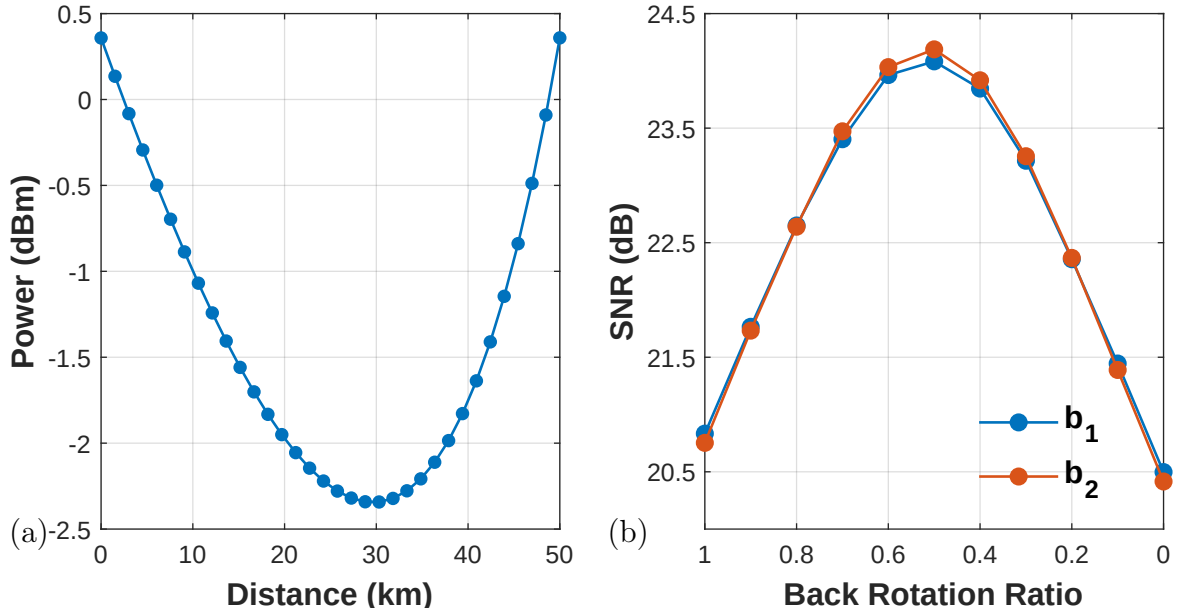


Figure 5.5: (a) Signal power profile in each span with backward Raman amplification used; (b) SNR of the b -coefficients after DSP processing with different BR ratio.

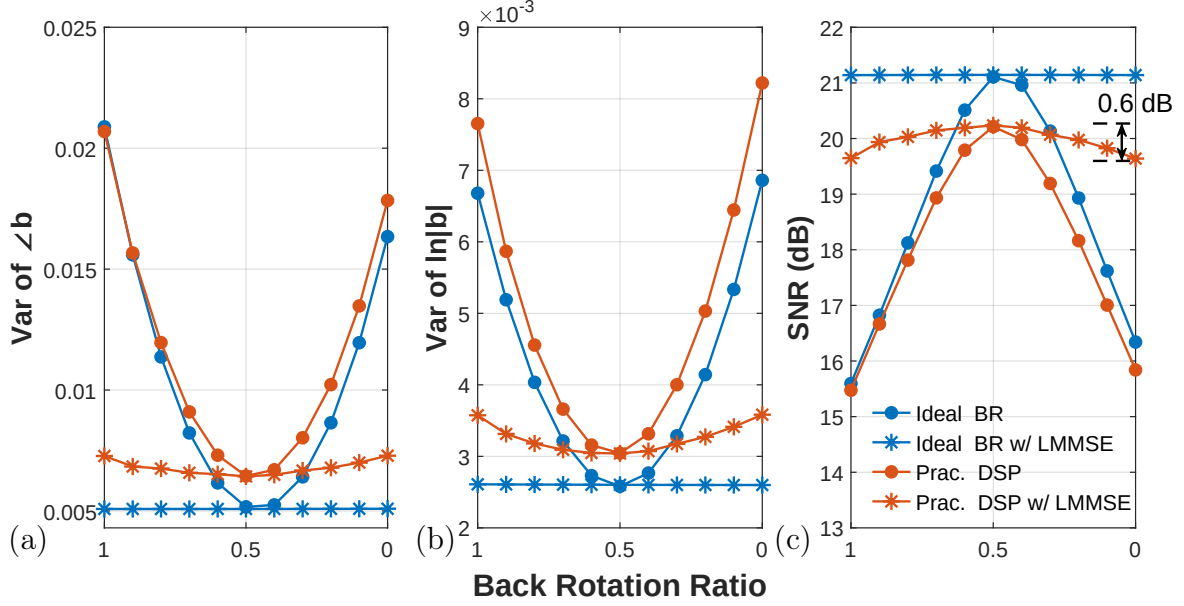


Figure 5.6: Variance in phase (a), log-magnitude (b), and SNR (c) of the DP signal after DSP flow processing with or without a following LMMSE. Ideal transmission without polarization rotation, FO, or phase noise is added for comparison.

Fig. 5.5(b) shows the DSP demodulation result for the Raman amplification system. The signal distribution is omitted here because it is similar to the case of EDFA, but just with less noise distribution. The SNR curve peaking at 0.5 indicates the effectiveness of BR and also the whole DSP flow.

5.4.3 Combination with LMMSE

We note that the DSP flow can be further combined with an LMMSE filter. After CPR, the signal distribution is clear enough, which makes it possible to be further processed by LMMSE. As shown in [86], at half BR, LMMSE does not provide improvement in the case of SP homodyne detection. However, when practical impairments (SOP rotation, FO, and phase noise) and the corresponding DSP modules are included, a processing penalty may be introduced when the initial noise is large. We apply an LMMSE filter (training ratio, 0.1) at the end of the DSP flow with data from Section 5.4.1 to investigate the influence. Since the performance of b_1 and b_2 is very close, the SNR curves are replaced by their mean value for clear visualization.

As shown in Fig. 5.6, in the ideal transmission case (only ASE is included as impairment, and the DSP flow is reduced to BR), half BR shows the same performance as 0 BR + LMMSE. However, when practical impairments and the full DSP flow are included, half BR shows an SNR gain of ~ 0.6 dB over 0 BR + LMMSE. Again, at the optimal BR ratio, further applying LMMSE shows little gain. Thus, it is favorable to use BR instead of LMMSE in a practical transmission situation, which not only avoids

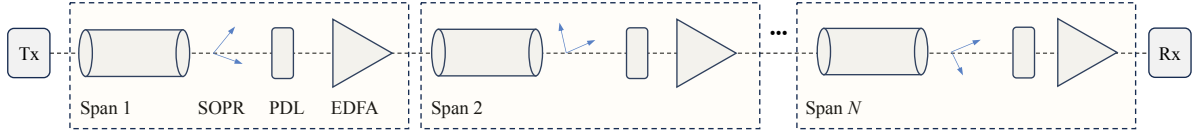


Figure 5.7: DP transmission model. Each span consists of a span of lossy fiber, a lumped SOP rotation (SOPR), a lumped PDL, and an EDFA.

using a training sequence but also provides a performance gain.

5.5 The impact of PDL

A feature of discrete NFDM systems is that the eigenvalue can be disturbed by certain channel impairments, which in turn translates into extra noise on $\mathbf{b}$, and it also affects the BR process because it can feed BR with “improper” eigenvalues. We found that PDL can significantly affect a DP discrete NFDM system when conducting experiments, and we analyze it in the following.

5.5.1 PDL perturbation on eigenvalue

The eigenvalue of $\mathbf{q}(t)$ is invariant under polarization rotation, while it will be perturbed by PDL and then heavily distorts the evolution of $\mathbf{b}$. This can be first intuitively understood by the fact that a DP soliton pulse will go through SOP change and hence different power distribution in the X and Y polarization during propagation, which results in different overall power weakening of the pulse when it passes through a PDL device, thus causing the fluctuation of β . We analyze it mathematically in the following.

The fluctuation in β due to PDL can be understood in two ways. First, we may infer the fluctuation of β from the fluctuation of the soliton pulse amplitude caused by PDL. We found this method works fine when PDL is small, but it gets less accurate with large PDL or longer transmission distance. Alternatively, we applied perturbation analysis by adding the SOP rotation and PDL as perturbative terms into the Manakov Equation and analyzing their effect on the evolution of β , by which we obtained more accurate results.

The transmission system may be modeled as shown in Fig. 5.7. In each span, the signal propagation in the fiber is governed by the loss-included Manakov Equation, and at the end of each span, a lumped SOP rotation that represents the overall SOP rotation of a span, a lumped PDL, and an EDFA are added successively. The SOP rotation can be the same or randomly different from span to span, which corresponds to a fiber loop without or with SOP scrambling (or a real fiber link), respectively.

We start with an SOP-scrambling-free fiber loop case and use 8PSK modulation for

clear visualization. A lumped PDL of 0.05 dB in the X direction is inserted before the EDFA after each span. Due to the periodic nature of the fiber loop, the signal polarization evolves periodically as it recirculates in the loop, which can be clearly illustrated in the Stokes space, as shown in Fig. 5.8. In the Stokes space, the SOP rotation manifests itself as a rotation on the surface of the Poincaré sphere along a rotation axis, where the axis direction and rotation angle are determined by the birefringence of the fiber span. Since $|b_1| = |b_2|$ for 8PSK modulation, the initial SOP lies on the unit circle on the S_2OS_3 plane ($S_1 = 0$). The rotation axis of a single span is indicated as the red arrow in Fig. 5.8, then the initial SOP (different color indicates different initial SOP) rotates around the birefringent axis and forms a circle on the surface of the Poincaré sphere. Due to the symmetric arrangement of the 8PSK constellation and hence the initial SOP, the SOP rotation traces in the Stokes space are also in symmetric form. Note that when PDL exists, the rotation axis will not be a perfect circle but tends to fall towards the rotational axis [93, Chap. 14]. The periodical change of power distribution between X and Y polarization can be easily visualized by the periodical evolution of S_1 , by viewing on the 2D S_1OS_2 plane (Fig. 5.8(b)). However, different initial SOPs result in different evolution traces, which may evolve with $S_1 > 0$ or $S_1 < 0$ most of the time, or in a half-and-half manner. For those mostly with $S_1 > 0$, the power of soliton pulse is always mainly distributed in the X polarization direction, so PDL in X direction decreases the pulse power most severely, thus reducing β most severely throughout transmission, and vice versa, while for the SOP trace in a half-and-half manner with S_1 , their β fluctuates up and down. In contrast, when SOP scrambling is added, the evolution trace of SOP becomes much more random, making $S_1 > 0$ and $S_1 < 0$ evenly distributed (as shown in Fig. 5.9(a) and 5.9(b), where only 2 SOP traces are shown for a clear view), so the amplitude of soliton and β fluctuates up and down from span to span regardless of the initial SOP.

To keep the power balance, we compensate for the PDL-caused extra loss by increasing the EDFA gain (dB) by $\text{PDL}(\text{dB})/2$, which works reasonably fine for a small value of PDL. Let the initial power ratio on X polarization be P_x , then after the first PDL element and EDFA gain, the power distribution between X and Y polarization changes from $[P_x^0, P_y^0]^T$ to $[P_x^1, P_y^1]^T$, given by

$$\begin{bmatrix} P_x^0 \\ P_y^0 \end{bmatrix} = \begin{bmatrix} P_x \\ 1 - P_x \end{bmatrix} \xrightarrow{\text{PDL}} \begin{bmatrix} P_x \gamma' \\ 1 - P_x \end{bmatrix} \xrightarrow{\text{EDFA}} \begin{bmatrix} P_x \sqrt{\gamma'} \\ (1 - P_x)/\sqrt{\gamma'} \end{bmatrix} = \begin{bmatrix} P_x^1 \\ P_y^1 \end{bmatrix}, \quad (5.11)$$

and the soliton height A jumps from $A_0 = 1$ to

$$\begin{aligned} A_1 &= \gamma'^{(-1/4)} \sqrt{1 + (\gamma' - 1)P_x} \\ &\approx \gamma'^{(-1/4)} (1 + (\gamma' - 1)P_x/2) \\ &= \gamma'^{(-1/4)} (1 + (\gamma' - 1)(S_1^0 + 1)/4), \end{aligned} \quad (5.12)$$

where $\gamma' = 10^{PDL(dB)/10}$. After recirculating for N loops, the soliton height becomes

$$A_N \approx \gamma'^{(-N/4)} \prod_{n=1}^N (1 + (\gamma' - 1)(S_1^n + 1)/4), \quad (5.13)$$

which reveals that the fluctuation of soliton pulse amplitude (and hence β , $\beta_N = A_N - 0.5$) is determined by the evolution history of S_1 .

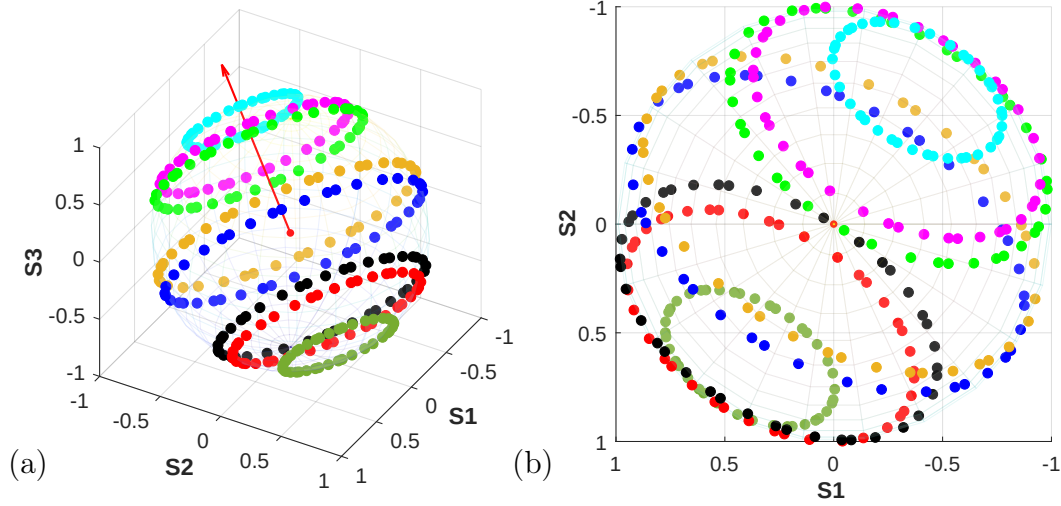


Figure 5.8: (a) SOP evolution in the Stokes space of loop transmission without SOP scrambling. Eight different initial SOP result in eight evolution traces, indicated by different colors; (b) 2D view of (a) from the S_1OS_2 plane above.

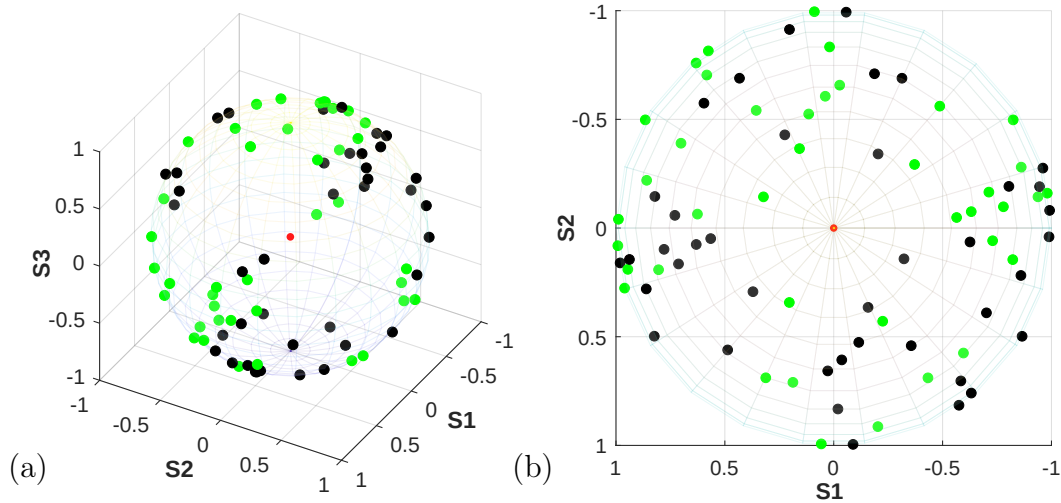


Figure 5.9: (a) SOP evolution in the Stokes space of loop transmission with SOP scrambling; (b) 2D view of (a) from the S_1OS_2 plane above.

The impact of PDL and SOP on λ can also be addressed by strict perturbative

analysis. The evolution equation of $\mathbf{q} = \mathbf{q}(z, t)$ may be written as

$$j \frac{\partial \mathbf{q}}{\partial z} + \frac{1}{2} \frac{\partial^2 \mathbf{q}}{\partial t^2} + |\mathbf{q}|^2 \mathbf{q} = j \left[-\frac{\Gamma}{2} + \frac{\Gamma L}{2} \sum_{n=1}^N \delta(z - nL) \right. \\ \left. + \frac{\boldsymbol{\sigma}}{4} \sum_{n=1}^N \delta(z - nL) + \sum_{n=1}^N \ln(\mathbf{R}_n) \delta(z - nL) \right] \mathbf{q}, \quad (5.14)$$

where $\Delta = \text{PDL}(\text{dB})/4.343$, $\boldsymbol{\sigma} = \begin{bmatrix} \Delta & 0 \\ 0 & -\Delta \end{bmatrix}$, and terms on the right-hand side of 5.14 represent the fiber loss, lumped EDFA gain, lumped PDL, and SOP rotation, respectively. δ -function is used to model the abrupt change of SOP or amplitude, and the logarithmic scaling of the δ -function is explained in the Appendix C. We may separate the periodic amplitude variation due to fiber loss and periodic EDFA gain by writing $\mathbf{q}(z, t) = f(z)\mathbf{p}(z, t)$ with $f(z)$ satisfying

$$\frac{df(z)}{dz} = \left[-\frac{\Gamma}{2} + \frac{\Gamma L}{2} \sum_{n=1}^N \delta(z - nL) \right] f(z) \quad (5.15)$$

to obtain

$$j \frac{\partial \mathbf{p}}{\partial z} + \frac{1}{2} \frac{\partial^2 \mathbf{p}}{\partial t^2} + f^2(z) |\mathbf{p}|^2 \mathbf{p} = j \left[\frac{\boldsymbol{\sigma}}{4} \sum_{n=1}^N \delta(z - nL) + \sum_{n=1}^N \ln(\mathbf{R}_n) \delta(z - nL) \right] \mathbf{p}. \quad (5.16)$$

Similarly, we can write $\mathbf{p}(z, t) = \mathbf{T}(z)\mathbf{s}(z, t)$ to peel off the SOP rotation effect, where $\mathbf{T}(z)$ is a “staircase” function along z whose k^{th} “stair height” is the overall SOP rotation matrix $\mathbf{R}_k \mathbf{R}_{k-1} \dots \mathbf{R}_2 \mathbf{R}_1$ that indicates the accumulated SOP rotation effect up to z ($z \geq kL$). $\mathbf{R}_k$ denotes the SOP rotation matrix induced by the k^{th} span. Thus, $\mathbf{T}(z)$ satisfies

$$\frac{d\mathbf{T}(z)}{dz} = \left[\sum_{n=1}^N \ln(\mathbf{R}_n) \delta(z - nL) \right] \mathbf{T}(z), \quad (5.17)$$

and we will have

$$j \frac{\partial \mathbf{s}}{\partial z} + \frac{1}{2} \frac{\partial^2 \mathbf{s}}{\partial t^2} + a^2(z) |\mathbf{s}|^2 \mathbf{s} = j \boldsymbol{\epsilon}(\mathbf{s}), \quad (5.18)$$

where

$$\boldsymbol{\epsilon}(\mathbf{s}) = \left[\mathbf{T}^\dagger(z) \left(\frac{\boldsymbol{\sigma}}{4} \sum_{n=1}^N \delta(z - nL) \right) \mathbf{T}(z) \right] \mathbf{s} \quad (5.19)$$

is the perturbation to the Manakov Equation, and the evolution of β under such a perturbation may be given by [38, 94]

$$\frac{d\beta}{dz} = \frac{1}{2} \text{Re} \int_{-\infty}^{\infty} \mathbf{s}^\dagger \boldsymbol{\epsilon}(\mathbf{s}) dt \quad (5.20)$$

with (actually, the integral is purely real)

$$\int_{-\infty}^{\infty} \mathbf{s}^\dagger \boldsymbol{\epsilon}(\mathbf{s}) dt = \beta \left[\mathbf{b}^T \mathbf{T}^\dagger(z) \left(\boldsymbol{\sigma} \sum_{n=1}^N \delta(z - nL) \right) \mathbf{T}(z) \mathbf{b}^* \right], \quad (5.21)$$

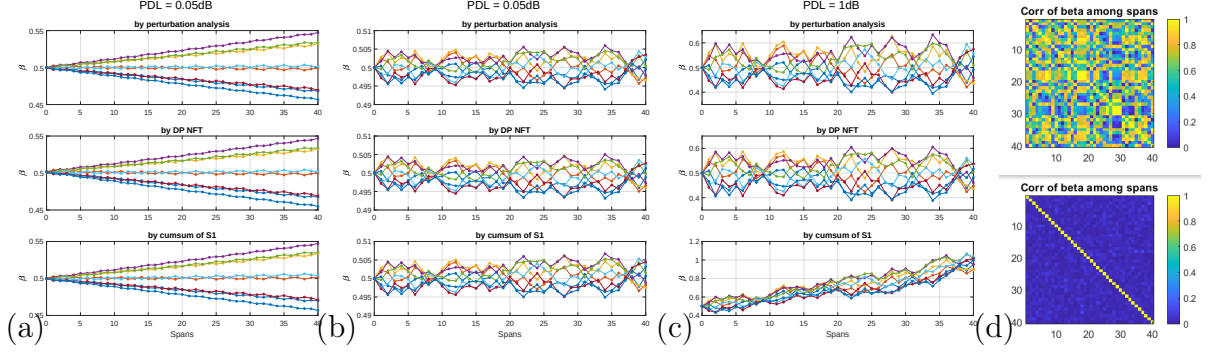


Figure 5.10: Comparison of the evolution of β obtained by Eq. 5.13, 5.24 and DP-NFT with (a) PDL = 0.05 dB, no SOP scrambling; (b) PDL = 0.05 dB, SOP scrambling; (c) PDL = 1 dB, SOP scrambling; (d) Correlation of β among different spans due to PDL = 0.1 dB (up) and ASE (down).

where we assumed that under such perturbation, the soliton retains the sech pulse shape but with distance-varying β , and we have used the fact of

$$\mathbf{s} = -2\beta(z) \operatorname{sech}(2\beta(z)t) e^{j\phi(z)} \mathbf{b}^*, \quad (5.22)$$

and

$$\int_{-\infty}^{\infty} \left(2\beta \operatorname{sech}(2\beta t) \right)^2 dt = 4\beta. \quad (5.23)$$

Thus,

$$\beta(z) = \beta(0) \exp \left[\int_0^z \left(\mathbf{b}^\dagger \mathbf{T}^\dagger(z') \left(\frac{\boldsymbol{\sigma}}{2} \sum_{n=1}^N \delta(z' - nL) \right) \mathbf{T}(z') \mathbf{b} \right) dz' \right]. \quad (5.24)$$

If PDL = 0 dB, i.e., $\boldsymbol{\sigma} = \mathbf{0}$, 5.24 indicates that β keeps invariant during transmission.

We conducted a simulation of 8PSK-modulated DP solitons transmission over a 40×50 km ASE-free system. A fixed or random SOP rotation is included after each span, followed by PDL = 0.05 dB or 1 dB. We extracted the signal after each span and conducted DP-NFT to obtain the eigenvalues for comparison with the theoretical analysis above. Fig. 5.10(a), (b), and (c) compare the β obtained by 5.13, 5.24, and DP-NFT under different amounts of PDL, without and with SOP scrambling, respectively. It can be seen that both methods successfully predict the evolution of β with PDL = 0.05 dB, while only 5.24 still shows great agreement with the result of DP-NFT in the case of high PDL of 1 dB. Besides, if no SOP scrambling is added, the split of β becomes more and more severe along propagation due to the periodic evolution of SOP (Fig. 5.10(a)). In contrast, the evolution of β becomes fluctuating when SOP scrambling is added (Fig. 5.10(c)).

5.5.2 Impact on BR and DSP

Although the split of eigenvalue can be greatly suppressed and transformed into a fluctuating evolution by adding SOP scrambling in the fiber loop. However, such a

fluctuation is very different from the fluctuation induced by the independent ASE noise loading at each EDFA, because the PDL-induced fluctuations from each span are certainly dependent through SOP evolution from span to span, which is verified by the numerical result in Fig. 5.10(d), where the zero correlation between $\beta_{m,n}$ ($m \neq n$) in PDL-free case is shown for comparison. The optimality of half BR can be lost due to the breakdown of the evolution model of λ and $\mathbf{b}$. The same conclusion also applies to a non-loop fiber transmission system where the SOP rotation and the direction of PDL are different from span to span.

We conducted simulations with 4A8PSK in three cases to show the impact of PDL on BR and DSP. The first one is an ideal transmission case with BR at Rx only, the second one is the same as the first except for introducing PDL of 0.1 dB, and the third one contains PDL of 0.1 dB and other practical impairments (random SOP rotation, FO, phase noise) with the whole DSP flow at Rx. As discussed above, the PDL-induced fluctuation is highly dependent from span to span, and the theoretical foundation of half BR fails. Such a fluctuation can make the final β unsuitable for BR, either “larger” or “smaller” than the proper value for different transmitted span numbers, and the DSP may fail to converge. As a result, the BR curve at different span numbers can be quite different, as shown in Fig. 5.11(b) and (c), not guaranteed to be optimal at half, while the optimal BR ratio is always 0.5 for any span numbers in the PDL-free case (Fig. 5.11(a)).

5.6 The impact of IQ bias

Ideally, the bias of XI, XQ, YI, and YQ should all be offset at the null points of a DP-IQ modulator. Similar to the SP case [86], we found that non-ideal IQ bias causes the fluctuation of the soliton pulse height and hence β . However, the bias that exists in a certain polarization will affect the evolution of $\angle b$ of the same and orthogonal polarizations very differently.

We may first consider that the non-ideal IQ bias only exists in X polarization. In the normalized time domain, let $\mathbf{q}(t) = \text{sech}(t)[\cos(\theta)e^{-j\phi_1}, \sin(\theta)e^{-j\phi_2}]^T$ denote an ideal dimensionless DP 1st-order soliton, and $\delta_X \in \mathbb{C}$ denote a small non-zero IQ bias offset on X polarization. Then the soliton shifted by δ_X is $\mathbf{q}'(t) = \mathbf{q}(t) + [\delta_X, 0]^T$, whose instantaneous power will be $P'(t) \approx \text{sech}^2(t) + 2 \text{sech}(t) \cos(\theta) |\delta_X| \cos(\phi_1 + \angle \delta_X)$. We may use Parseval's identity of nonlinear Fourier Transform [15] to obtain the δ_X -perturbed β :

$$\int_{-\infty}^{\infty} P'(t) dt \approx 4\beta, \quad (5.25)$$

which gives

$$\beta \approx 0.5 + \frac{\pi}{2} \cos(\theta) |\delta_X| \cos(\phi_1 + \angle \delta_X), \quad (5.26)$$

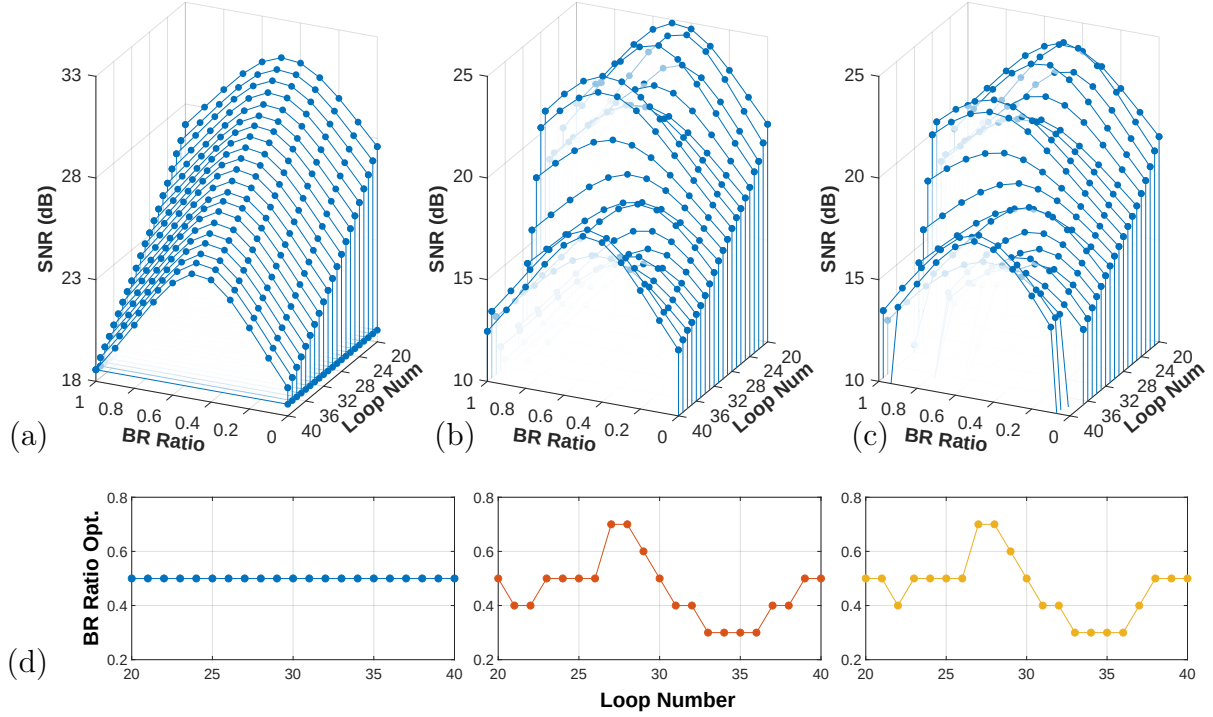


Figure 5.11: (a) Ideal transmission, PDL = 0 dB, BR at Rx; (b) Ideal transmission except for PDL = 0.1 dB, BR at Rx; (c) Practical transmission with PDL = 0.1 dB, DSP flow at Rx; (d) The optimal BR ratio vs loop number corresponds to (a), (b) and (c).

where we have assumed the number of discrete eigenvalues remains 1 and the continuous nonlinear spectrum is negligible, given that the IQ bias offset is typically small in practice. Then, for a specific δ_X , β is determined by the soliton phase in X polarization, and so is $\angle b_1$. Thus, δ_X leads to $\angle b_1$ -dependent perturbed β .

Therefore, as shown in Fig. 5.12, the constellation points in the same cluster will result in the same value of perturbed β , and so they share the same phase velocity during transmission, while the phase velocity varies among clusters. Consequently, the signal distribution gradually becomes asymmetric during propagation. However, b_2 will evolve in a very different way. Due to the independent modulation between b_1 and b_2 , the points in each cluster of b_2 evolve with all possible phase velocities, which results in strong phase noise within each cluster of b_2 . However, the means of those clusters will remain in the symmetric arrangement. Although $\angle b_1$ and $\angle b_2$ evolve very differently, the overall phase noise power (variance with respect to the ideal constellation) remains the same between them during propagation due to the law of total variance—the phase variance is distributed completely among clusters for b_1 and completely within clusters for b_2 , respectively.

Note that for an SP discrete NFDm signal, a non-ideal bias only leads to asymmetric signal distribution. However, in the DP case, the orthogonal polarization gains extra phase noise at the same time. Generally, non-ideal bias exists both in X and Y polarization

in practice, and their effects superimpose, making the cluster of b_1 and b_2 both spread and asymmetric, as shown in Fig. 5.12. This makes the DP discrete NFDM system seem to be more sensitive to non-ideal bias, but we found it has the same sensitivity as an SP discrete NFDM system by the following analysis and numerical verification.

The perturbed β of a DP and SP discrete NFDM signal (suppose the SP signal is on the X polarization) is given by

$$\beta_D \approx 0.5 + \frac{\pi}{2} \left(\cos(\theta) |\delta_X| \cos(\phi_1 + \angle \delta_X) + \sin(\theta) |\delta_Y| \cos(\phi_2 + \angle \delta_Y) \right), \quad (5.27a)$$

$$\beta_S \approx 0.5 + \frac{\pi}{2} |\delta_X| \cos(\phi_1 + \angle \delta_X), \quad (5.27b)$$

respectively. By the independence between b_1 and b_2 , we can obtain

$$\text{Var}(\beta_S) = \frac{\pi^2}{8} |\delta_X|^2, \quad (5.28a)$$

$$\text{Var}(\beta_D) = \frac{\pi^2}{16} (|\delta_X|^2 + |\delta_Y|^2). \quad (5.28b)$$

Suppose the intensity of non-ideal IQ bias is on the same level for X and Y polarization, i.e., $|\delta_X|^2 \approx |\delta_Y|^2$, we obtain $\text{Var}(\beta_S) \approx \text{Var}(\beta_D)$, which will lead to the same level of phase noise to b -coefficient through 5.10(b).

Besides, note that a DP and an SP discrete NFDM signal also have the same SNR on b over the same transmission parameters. This is very different from the ordinary coherent signal, in which an SP signal has a 3 dB higher SNR than a DP one when they have the same signal power. For example, consider a DP soliton that is time-shifted by noise loading. The time jitter will then shift the X and Y polarization components by the same amount, regardless of the power ratio between them (in the case of power fully distributed in one polarization, this reduces to the case of SP). In the language of NFT, the noise on $\ln |b_1|$ and $\ln |b_2|$ is the same. Besides, the noise of b is reflected in time and phase jitter, which is acquired through transmission. Even though half of the ASE power can be filtered out at Rx for an SP signal, it may only reduce the noise on λ and the “direct noise” on b , while the “transfer noise” on b (induced by noisy λ) is “historical” will not be affected. Therefore, the DP discrete signal has the same sensitivity to both the non-ideal IQ bias and ASE noise, as verified by the following simulation example.

As pointed out by [86], the non-ideal IQ bias induced split of β serves as an initial distortion which affects the phase evolution of $b_{1,2}$ throughout the whole transmission line, and thus a full BR is needed to undo its effect. Fig. 5.13 shows the BR curve for an SP and DP discrete NFDM signal, respectively, where the optimal BR ratio shifts to 0.7. Both of them are ideally transmitted through a 40×50 km system with EDFA NF of 8 dB and IQ bias (in the physical domain) of $\delta_X = 8 \times 10^{-4} - 6.4 \times 10^{-4}j$, $\delta_Y = -8 \times 10^{-4} + 6.4 \times 10^{-4}j$.

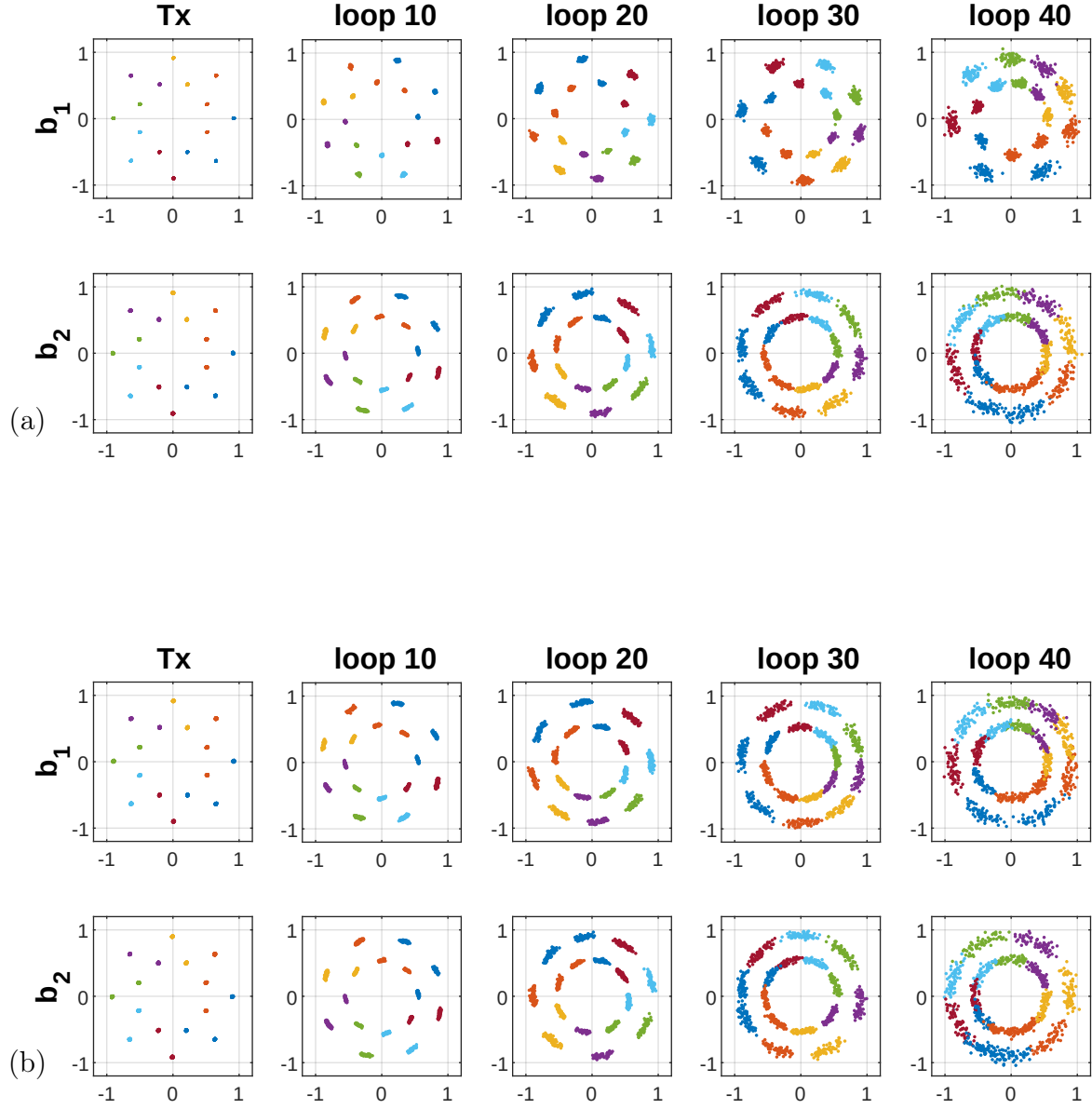


Figure 5.12: Evolution of $b_{1,2}$ when non-ideal IQ bias is present on (a) X polarization only and (b) both X and Y polarization.

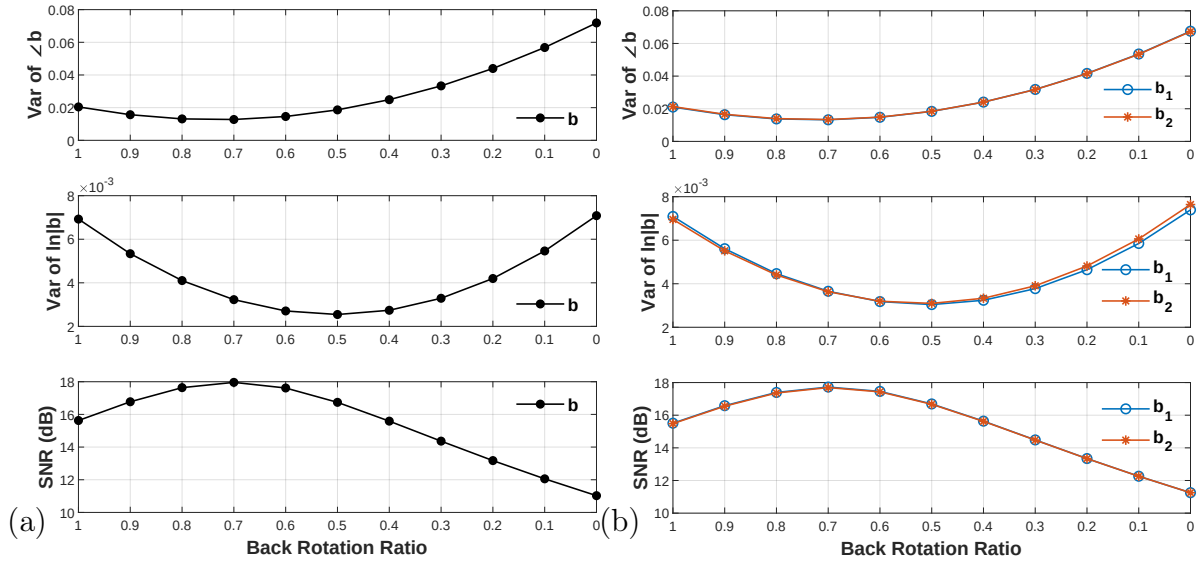


Figure 5.13: BR curve of (a) SP discrete NFDMA signal with IQ bias of $\delta_X = 8 \times 10^{-3} - 6.4 \times 10^{-3}j$ and (b) DP discrete NFDMA signal with IQ bias of $\delta_X = 8 \times 10^{-3} - 6.4 \times 10^{-3}j$, $\delta_Y = -8 \times 10^{-3} + 6.4 \times 10^{-3}j$. NF = 8 dB in both cases.

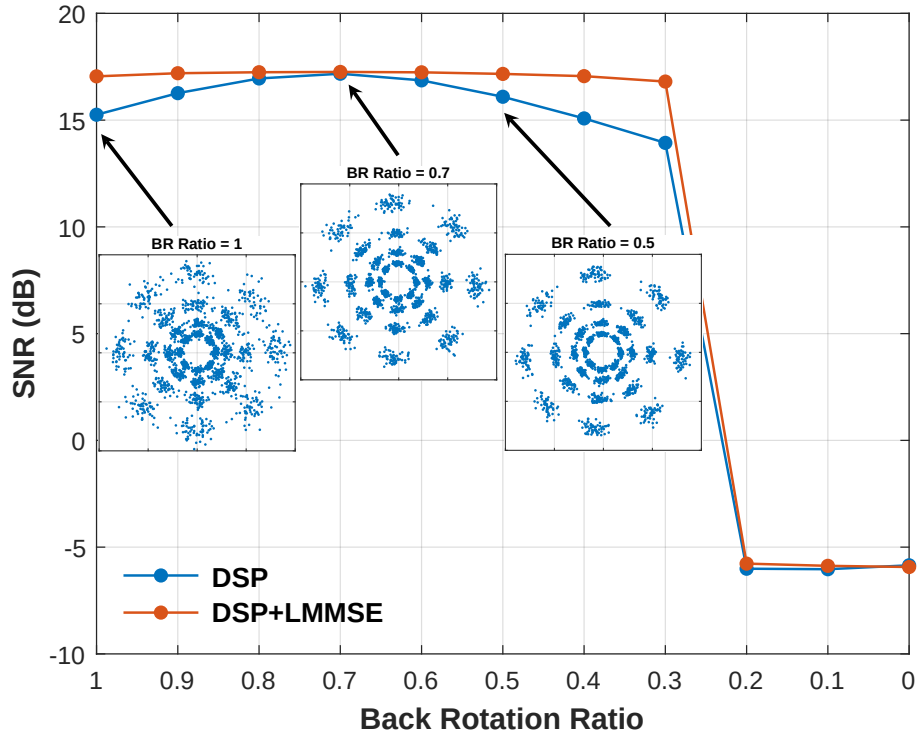


Figure 5.14: SNR versus BR ratio used in the DSP flow when non-ideal IQ bias exists. Signal distributions of b_1 processed by DSP are shown in insets for BR ratios of 0.5, 0.7, and 1, respectively.

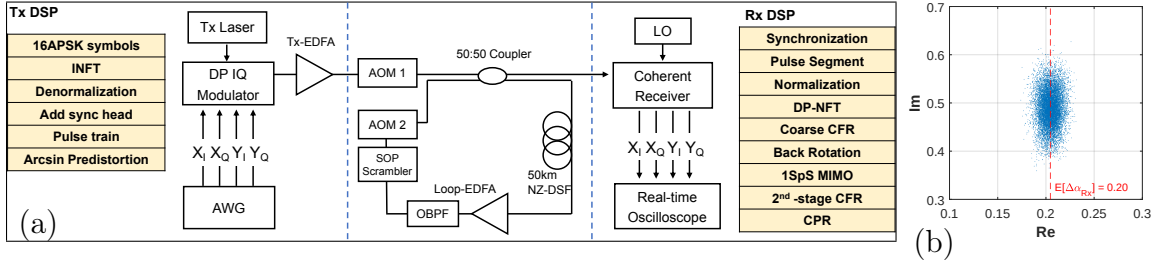


Figure 5.15: (a) Experimental setup and DSP flow in Tx and Rx; (b) The received eigenvalue, whose real part is shifted by 0.21 on average due to the FO between the signal carrier and LO.

Besides, it should be noted that in a practical transmission situation involving CFR and CPR, the demodulation process can fail if non-ideal IQ bias exists. This is because the M^{th} -power CFR requires the phase-symmetry in the constellation, by which it can eliminate the phase modulation of the signal and reveal the FO information. Also, a CPR algorithm requires constellation symmetry to work properly. In such a case, BR (with a large enough BR ratio) becomes indispensable to restore the signal distribution to a sufficiently symmetric form to make the following CFR and CPR work normally. To verify this, we add SOP rotation, $\text{FO} = 6 \text{ GHz}$, and $\text{Tx} + \text{Rx}$ linewidth of $100 + 100 \text{ kHz}$ to the system in Fig. 5.13 and demodulate with the DSP flow plus an optional LMMSE. The result is shown in Fig. 5.14, we can see that when the BR ratio is low, the DSP fails, and so as the following LMMSE. As the BR ratio gets higher, the constellation symmetry is more and more restored (see the inset signal distribution examples), and the DSP flow starts to survive. As discussed above, a full BR completely restores the symmetry signal distribution and the IQ bias-induced phase noise, while the ASE-induced noise is raised again. Overall, the DSP flow returns a maximum SNR with BR ratio = 0.7, which is the same as in the ideal transmission case. Once DSP survives and returns a recognizable signal distribution, LMMSE can function normally, returning a flat SNR curve for different BR ratios because LMMSE can further kill the remaining correlation between back-rotated $\mathbf{b}$ and λ .

5.7 Experiment

Experiments with offline DSP were conducted to verify the effectiveness of the proposed DSP flow for DP discrete NFDm transmission. Fig. 5.15 shows the experimental setup and the offline DSP flow. 2A8PSK constellation with ring radii of $\exp(\pm\beta_0 T)/\sqrt{2}$, $T = 0.5$ and eigenvalue of $\lambda_0 = 0 + j\beta_0 = 0 + 0.5j$ were used for the b -coefficient modulation. At the transmitter, the corresponding time-domain waveform was obtained by DP inverse-NFT on the 2A8PSK symbols. The baud rate was 3 GBaud, and the soliton pulse FWHM was set to be 1/6 of the symbol period. The normalization time parameter

T_0 was calculated to be 31.5 ps.

After a synchronization header was added, the DP discrete NFDM signal was loaded into the arbitrary waveform generator (AWG, Keysight M8194A). The generated baseband waveform was then sent into a DP-IQ modulator (3 dB bandwidth of 35 GHz) to modulate the optical carrier at 1550 nm, and the modulated signal was then amplified by the Tx-EDFA and launched into the fiber loop. The loop consisted of a span of 50 km NZ-DSF (power loss coefficient: 0.2 dB/km, dispersion coefficient: 4 ps/km/nm), an SOP scrambler, and a loop-EDFA to compensate for the overall loss (about 22 dB per loop). The SOP scrambling (in discrete SOP switching mode) rate is set to over 4000 jumps per second to match the fiber loop recirculating period. To generate path-averaged solitons, the launched power was increased by $G \ln(G)/(G - 1) = 2.558$ with respect to the theoretical power for generating an optical soliton, where $G = 10$ dB, equal to the loss of the 50 km NZ-DSF. A flat-top optical filter (3-dB bandwidth of 1 nm centered at 1550 nm) was used after the loop-EDFA to suppress the out-of-band ASE noise. Two separate ordinary laser sources (linewidth < 100 kHz) served as the transmitter laser and LO, respectively.

After transmission of 1000 km (20 loops), the signal was coherently detected and then sampled by a digital oscilloscope (Keysight, UXR0804A) and processed offline by the proposed DSP flow as shown in Fig. 5.15(a). First, timing synchronization, power normalization, and waveform segmenting are successively done. DP-NFT was then conducted to obtain the eigenvalue and b -coefficient for each soliton pulse. The mean shift in the real part of the received eigenvalue is 0.21, which translates into a physical FO of 2.07 GHz, as shown in Fig. 5.15(b). A coarse CFR is done accordingly, followed by BR, 1-SpS MIMO, 2nd-stage CFR, CPR, and an optional LMMSE.

Fig. 5.16 shows the experimental results of the noise variance and SNR of $b_{1,2}$, processed by the proposed DSP flow with and without an LMMSE (training ratio = 0.1) filter appended at the end. We observed stronger phase noise than amplitude noise in $b_{1,2}$, which can also be inferred from the received eigenvalue distribution that has stronger noise in the imaginary part than real part, and this makes the SNR performance dominated by phase noise. We attribute this to non-ideal IQ bias-induced initial split of β and extra fluctuation of β during transmission due to PDL. This is also reflected in Fig. 5.16 (d), where the signal distribution symmetry is gradually restored as the BR ratio goes to 1. As discussed in Chap. 4, the optimal BR for $\text{Var}(\angle b_{1,2})$ is greater than 0.5 while that of $\ln|b_{1,2}|$ remains 0.5.

Fig. 5.17 shows another set of experimental results, in which the DSP fails when no BR is applied to the received signal. The optimal BR ratio shifted to 0.8, and we attributed it to the time-varying IQ bias during the experiment. These results indicate that the optimality of half BR can be fragile in practice, and thus, it is recommended to

demodulate the signal using DSP flow with a large BR followed by LMMSE to stably recover the signal. Finally, we note that the noise level in $b_{1,2}$ can be different in the experiment, which may be caused by other non-ideal factors and will be the topic of further investigation.

5.8 Conclusion

We extend the half back rotation (BR) algorithm to DP discrete NFDM systems and propose the first set of complete DSP algorithms that fully operate in the nonlinear frequency domain. The proposed DSP flow is verified both numerically and experimentally with practical channel impairments, and the influence of PDL and non-ideal IQ bias on the evolution of eigenvalue and BR is characterized. In addition to the advantage of eliminating training sequences, BR is indispensable and provides better performance in practical transmission scenarios in the presence of non-ideal IQ bias.

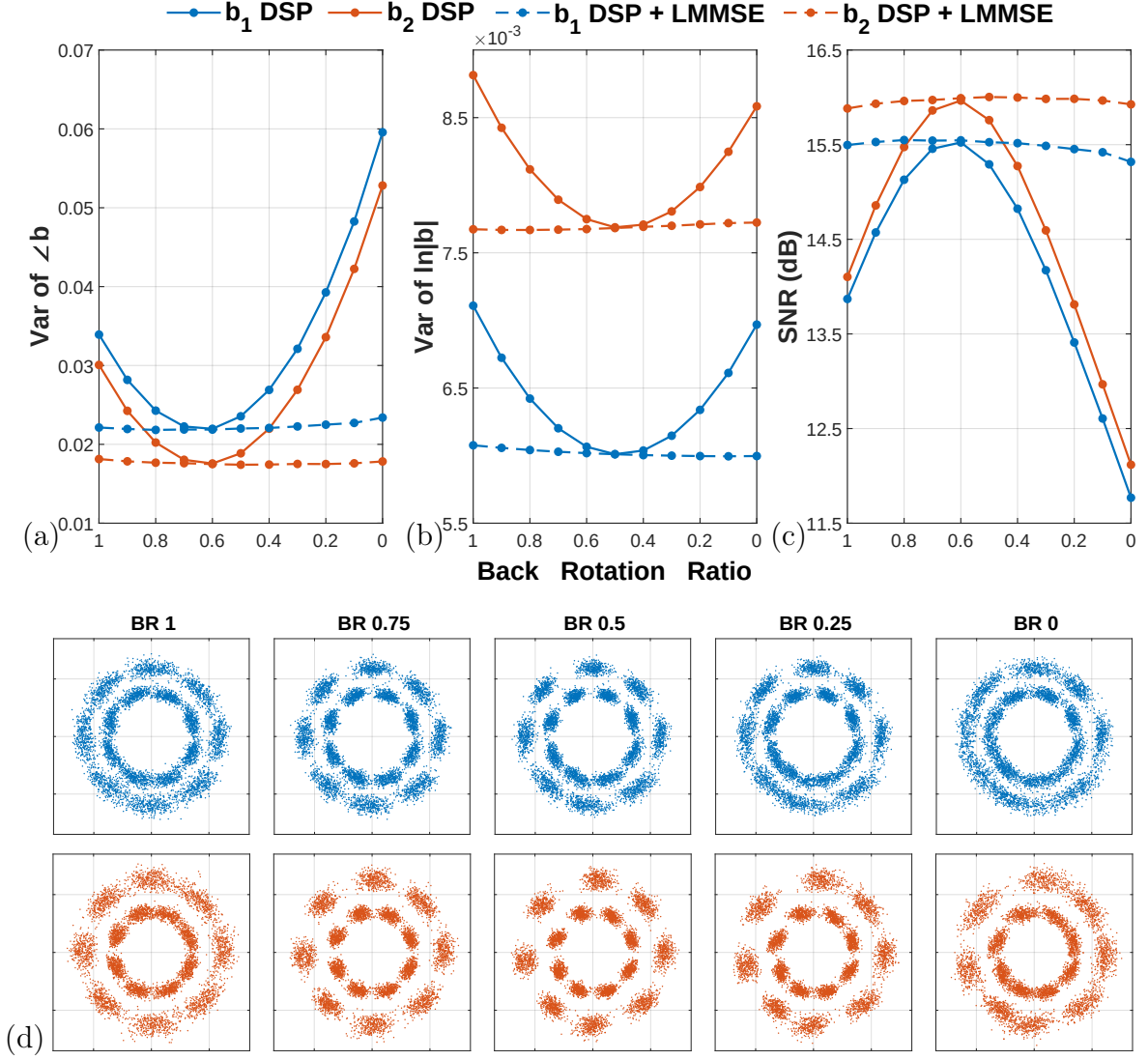


Figure 5.16: Experimental result of variance in phase (a), log-magnitude (b), and SNR (c) of $b_{1,2}$ versus different BR ratios used in the DSP flow; (d) Constellations of b_1 and b_2 at different BR ratios.

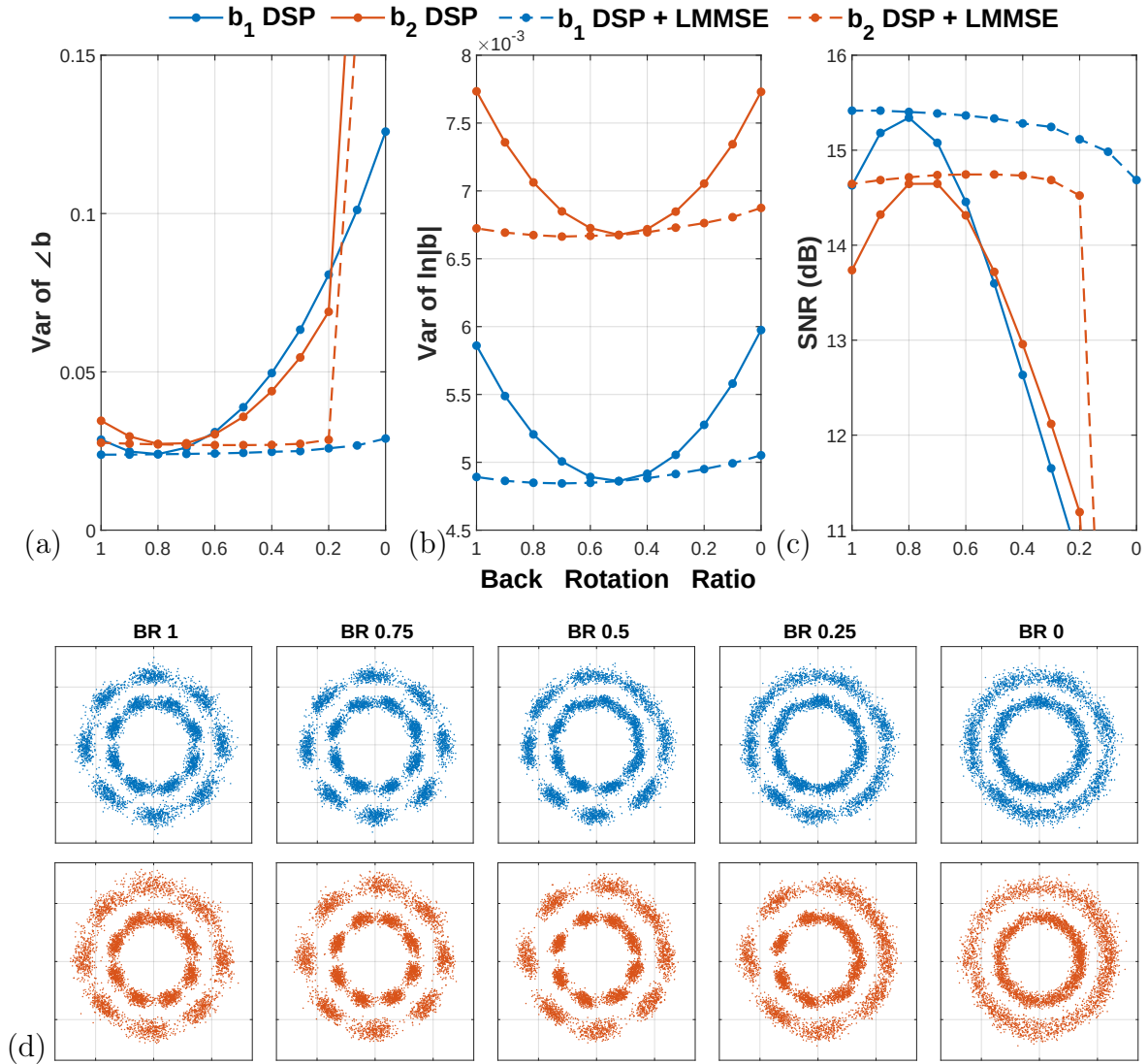


Figure 5.17: Another set of experimental results.

Chapter 6

Optimal Symbol Rate for Discrete Nonlinear Frequency Division Multiplexing Transmission

Although advanced modulation formats have been equipped in modern soliton transmission with the help of NFDM theory and digital coherent transmission techniques, the information rate of discrete NFDM is still low by far. Fig. 6.1 summarizes the symbol rates of experimental demonstration in the discrete NFDM system since 2014, the year when the NFDM theory was formally established [15–17]. It can be seen that after a decade of development, the symbol rate of the discrete NFDM system is only around the order of 1 Gbaud. In contrast, the ordinary signal carrier coherent optical transmission system reaches 200 Gbaud [4]. While the technical side, such as the DSP algorithm, needs developing, there may be fundamental limitations that need to be investigated. This chapter refines the noise evolution model and shows its dependence on the symbol rate of the discrete NFDM system, and finds that the symbol rate of the discrete NFDM system is strongly restricted.

6.1 Introduction

Nonlinear Frequency Division Multiplexing (NFDM) is a promising theory that transforms the nonlinear fiber channel into a linear evolution of the nonlinear spectrum, under the assumption that the fiber is lossless.

As real fiber is always lossy, the integrability of the NLSE is broken. Besides, to compensate for the loss, optical amplifiers are introduced, which also introduce noise during transmission. Consequently, the eigenvalue and the nonlinear spectrum become noisy, and the signal cannot be recovered by simply back rotating the discrete nonlinear spectrum by the full transmission distance, since the eigenvalues are contaminated with

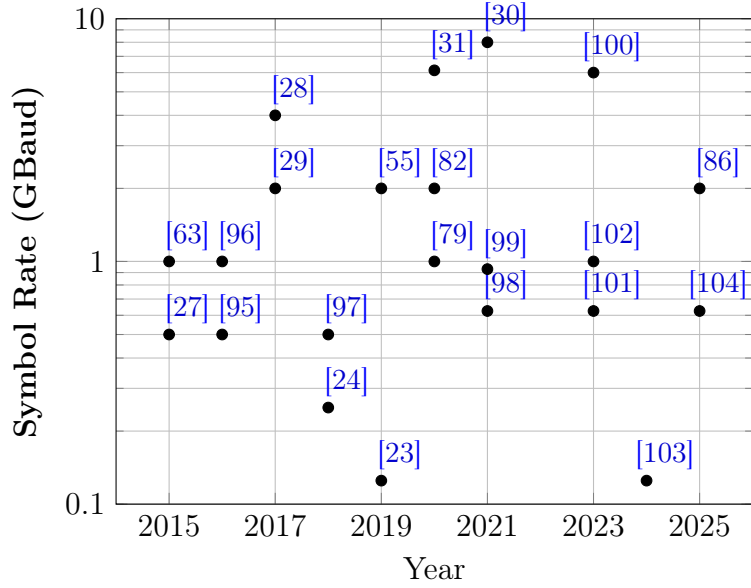


Figure 6.1: Symbol rate of discrete NFDN experimental demonstration reported since 2014.

noise.

Many efforts have been made to narrow the gap between the “lossless” assumption and the reality, including using Raman amplification [24] to reduce the power fluctuations along the fiber link, using linear minimum mean square error (LMMSE) estimator [29] for denoising on the nonlinear domain, nonlinear filters and machine learning [56] are also applied in this field. Recently, we proposed back rotating the noisy eigenvalue by half of the transmission distance to optimize the SNR of b -coefficient [64, 86].

In this paper, we further investigated the noise power accumulation in the b -coefficient and eigenvalue. We analytically and numerically study the dependence of overall noise on symbol rate and determine an optimal symbol rate for the discrete eigenvalue transmission system.

6.2 NFT and noise model

The analysis and result apply to a dual-polarization discrete system. However, we discuss in a single polarization condition for simplicity, where the signal propagation is described by the normalized NLSE, $jq_z + q_{tt}/2 + |q|^2q = 0$. The nonlinear spectrum of $q(t)$ is obtained by solving the eigenvalue problem, and the scattering coefficients $a(\lambda)$ and $b(\lambda)$ are given by [15]

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} &= \begin{bmatrix} -j\lambda & jq(t) \\ jq^*(t) & j\lambda \end{bmatrix} \mathbf{v}, \\ \begin{bmatrix} a(\lambda) \\ b(\lambda) \end{bmatrix} &= \lim_{t \rightarrow \infty} \begin{bmatrix} v_1 e^{+j\lambda t} \\ v_2 e^{-j\lambda t} \end{bmatrix} \text{ with } \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \xrightarrow{t \rightarrow -\infty} \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-j\lambda t}. \end{aligned} \quad (6.1)$$

The discrete spectrum is defined as $Q_a(\lambda_i) = b(\lambda_i)/a'(\lambda_i)$ where $\lambda_i \in \mathbb{C}^+$ is the solutions of $a(\lambda_i) = 0$. In this chapter, we restrict our discussion to a single discrete eigenvalue system. Information is encoded on the discrete b -coefficient $b(\lambda_i)$, whose evolution is governed by $b(z, \lambda_i) = b(0, \lambda_i) \exp(-2jL\lambda_i^2 z)$. (We will use i to denote the span number hereafter, rather than the eigenvalue ordinal, as there is only one eigenvalue.)

In an EDFA transmission system shown in Fig. 6.2 (same as the link diagram in Chap. 4), due to the perturbation of amplified spontaneous emission (ASE) noise from each EDFA, the evolution of λ_i may be modeled as a discrete random walk, and the evolution of b_i is an accumulation of the random walk of λ_i times a productive noise e^{ϵ_i} , given by $b_{i+1} = b_i e^{-2jL\lambda_i^2} e^{\epsilon_i}$, where $\epsilon_i \in \mathbb{C}$ [105]. It would be convenient to work with the logarithm of b , by which the productive noise is transferred into additive noise, and the noise variance dependence on the magnitude of the signal can be removed as well. Let B_i denote $\ln b_i$, so

$$B_{i+1} = B_i + \underbrace{(-2jL\lambda_i^2)}_{\Delta N_i^t} + \underbrace{\epsilon_i}_{\Delta N_i^d}, \quad (6.2)$$

$$B_{i+1} = B_0 + \underbrace{\left(-2jL \sum_{i=0}^N \lambda_i^2\right)}_{N_{i+1}^t} + \underbrace{\sum_{i=1}^{N+1} \epsilon_i}_{N_{i+1}^d}, \quad (6.3)$$

where $\lambda_i = \alpha_i + j\beta_i$ and b_i denote the eigenvalue and b -coefficient right after the i^{th} EDFA. $L = L_A/L_D$ is the normalized span length with respect to the dispersion length. By taking the logarithm, an APSK constellation commonly used in discrete NFDM transmission is transformed into a QAM constellation, as illustrated in Fig. 5.2.

We classify the noise terms as “Transfer Noise” (ΔN_i^t) and “Direct Noise” (ΔN_i^d), as indicated in 6.2, and the accumulated transfer noise and direct noise up to the i^{th} EDFA are denoted by N_i^t and N_i^d , respectively. It reduces to the model in Chap. 4 if the direct noise is ignored. To investigate the possible fundamental noise limitation on the performance of a discrete eigenvalue transmission system, we carried out ideal back rotation assuming λ_i is known for all i , given by $B_k^{\text{BRI}} = B_{N+1} - 2jL \sum_{i=N+1-k}^N \lambda_i^2$, $k = 0, 1, \dots, N$ (the larger k , the more back rotation). After a full ideal back rotation, transfer noise is fully removed, and only the accumulated direct noise remains. We also conduct practical back rotation given by $B_k^{\text{BRP}} = B_{N+1} - 2jL(N+1-k)\lambda_{N+1}^2$ since the receiver only has access to the received eigenvalue λ_{N+1} .

6.3 Impact of symbol rate on the overall noise

Given a fixed span length and duty cycle D_c (soliton FWHM/symbol period), we may vary the span number (N) and symbol rate (R_s) to investigate the dependence of B



Figure 6.2: Schematic diagram of the transmission link. ASE noise is independently added to λ and b at each EDFA.

noise on them.

In the time domain, SNR is given by

$$SNR = \frac{P_S}{P_N} \propto \frac{E_S R_S}{BWN_0} \propto \frac{E_S}{N_0} \propto \frac{R_S}{N_0}, \quad (6.4)$$

where P_S , P_N denote the power of the soliton signal and in-band ASE noise, respectively, and E_S , BW denote the energy, bandwidth of the soliton pulse, respectively. We used the fact that the energy of a soliton is inversely proportional to its pulse duration. N_0 represents the power spectral density of ASE noise.

Note that the time-domain signal has been power normalized before NFT operation to get λ and b , so the time-domain noise power can be represented by the NSR of the signal, and we assume that under a small noise condition, the time-domain noise power linearly translates into the NFT domain. Hence, we may assume

$$\text{Var}(\Delta\lambda_i) = p_1 R_s^{-1}, \quad \text{Var}(\Delta N_i^d) = p_2 R_s^{-1}, \quad (6.5)$$

where p_1 and p_2 are scaling constants. Physically, $\Delta\lambda_i$ and ΔN_i^d mean that the amplitude, frequency, phase, and time position of a soliton pulse are all slightly shifted due to ASE loading by each EDFA. By increasing the symbol rate, a higher SNR is obtained, and hence smaller $\Delta\lambda_i$ and ΔN_i^d . However, the dispersion length $L_D \propto R_s^{-2}$, and such a dependence will in turn dramatically amplify the R_s^{-1} dependent noise on λ_i and make the transfer noise R_s^3 dependent, which dominates the overall noise power and severely degrades the performance.

The noise variance of B_i is $\text{Var}(B_i) = \text{Var}(N_i^t + N_i^d) = \text{Var}(N_i^t) + \text{Var}(N_i^d)$, where we have assumed the transfer noise and direct noise are uncorrelated (which can be verified by simulations). Considering the perturbed eigenvalue is still around $0 + 0.5j$, i.e., $\alpha_i \approx 0$, $\beta_i \approx 0.5$, then $\lambda_i^2 \approx 0.25 - \beta_i + j\alpha_i$, thus $\text{Var}(\lambda_i^2) \approx \text{Var}(\lambda_i)$. According to 4.9 and 4.10 in Chap. 4, we obtain

$$\begin{aligned} \text{Var}(N_{N+1}^t) &= 4L^2 N^3 / 3 \cdot \text{Var}(\Delta\lambda_i) \\ &= 4(L_A/L_D)^2 N^3 / 3 \cdot p_1 R_s^{-1} \\ &= p_1 c N^3 R_s^3, \end{aligned} \quad (6.6)$$

$$\text{Var}(N_{N+1}^d) = N \text{Var}(\Delta N_i^d) = p_2 N R_s^{-1}, \quad (6.7)$$

where $c = \frac{4}{3}1.763^4 L_A^2 \beta_2^2 / D_c^4$, a constant independent on symbol rate. Therefore,

$$\text{Var}(B_{N+1}) = p_1 c N^3 R_s^3 + p_2 N R_s^{-1}, \quad (6.8)$$

and if half practical back rotation is used for denoising b -coefficient, which reduces the transfer noise by a factor of four [86] and does not affect the direct noise, the residual noise variance will be

$$\text{Var}(B_{\text{HBR}}) = p_1 c N^3 R_s^3 / 4 + p_2 N R_s^{-1}. \quad (6.9)$$

Eq. 6.8 and 6.9 indicate that the overall noise on B is dominated by the direct noise and transfer noise in the low and high symbol rate regions, respectively. Once p_1 and p_2 are obtained through simulation, the optimal symbol rate (in terms of minimizing signal noise) is obtained by letting $\frac{d}{dR_s} \text{Var}(B_{\text{HBR}}) = 0$,

$$\begin{aligned} R_s^{\text{opt}} &= \frac{1}{\sqrt{N}} \left(\frac{4p_2}{3p_1 c} \right)^{1/4} \\ &= \frac{D_c}{1.763 \sqrt{N L_A |\beta_2|}} \left(\frac{p_2}{p_1} \right)^{1/4}. \end{aligned} \quad (6.10)$$

6.4 Simulation verification

We conducted simulations on a 40×50 km EDFA system (fiber loss: 0.2 dB/km, dispersion: 4 ps/km/nm at 193.4 THz). Ideal transmitter, EDFA (NF = 5 dB) and receiver were used in the simulation to focus on ASE noise limitation on the system performance. 2048 path-averaged 1st-order soliton pulses were used to estimate the variance of B during forward propagation and back rotation, where b were modulated with a 4A4PSK constellation. The duty cycle was 1/8 to avoid interaction between adjacent pulses. The symbol rate ranges from 0.5 to 3.5 GBaud, leading to the dispersion length ranging from 3940.8 to 80.6 km. The symbol rate range was selected to ensure both the stable existence of the path-averaged soliton and sufficient SNR for the small noise assumption. NFT was applied to the coherently received signal after power normalization, and we back-rotated by rotation ratios from 0 to 1, using either ideal or practical back rotation.

Fig. 6.3(a) shows the correlation coefficient between α_i , $\ln |b_i|$, and β_i , $\angle b_i$, during forward propagation, and the analytic correlation coefficient 4.11 (ignoring the direct noise) is given for reference. The dominance of the transfer noise gradually increases during transmission, which is quite fast with a high symbol rate due to the correspondingly low dispersion length. For the 0.5 GBaud signal, the overall noise on b is basically uncorrelated with the noise on λ even at the end of transmission (hence half back

rotation or LMMSE will not have much improvement on it). In contrast, the correlation for the 3.5 GBaud signal quickly approaches the theoretical upper limit of 0.87 as the transfer noise quickly dominates.

Fig. 6.3(c) shows $\text{Var}(\Delta\lambda_i)$ and $\text{Var}(\Delta N_i^d)$ during propagation for different symbol rates. Here, each short curve is equivalent to an inset showing the noise variance vs span number. Similarly, the black and red curves in Fig. 6.4 are insets of back rotation for each symbol rate. The inset example shows $\Delta\alpha_i$ and $\Delta\beta_i$ during propagation in the 2 GBaud case, which is obtained by differentiating their evolution traces in Fig. 6.3(b), and then translated into $\text{Var}(\Delta\lambda_i)$. Both $\text{Var}(\Delta\lambda_i)$ and $\text{Var}(\Delta N_i^d)$ basically remain stable during transmission due to the i.i.d. ASE loading at each EDFA. The reciprocal dependence of them on R_s is clearly indicated by the black reciprocal fitting curve. From the fitting curve, we obtain $p_1 = 4.5 \times 10^3$, $p_2 = 4.8 \times 10^4$. In Fig. 6.3(d), the simulation results verified the assumption that ΔN_i^d are uncorrelated, so $\text{Var}(N_i^d)$ grows linearly with span number, where the gradient is $\text{Var}(\Delta N_i^d)$ as given by 6.7.

With p_1, p_2 obtained, we then verify 6.7 to 6.9. We plot the variance curve of B_k^{BRI} (red) and B_k^{BRP} (black) for each R_s in Fig. 6.4, where the end point of B_k^{BRI} is $\text{Var}(N_{N+1}^d)$, the start and middle point of B_k^{BRP} are $\text{Var}(B_{N+1})$ and $\text{Var}(B_{\text{HBR}})$, respectively. They are well matched with the values correspondingly predicted by 6.7-6.9, as shown in Fig. 6.4, and the optimal symbol rate R_s^{opt} is evaluated to be 1.27 GBaud. Such a low value may reveal why the symbol rate of the discrete NFDM system has long been low. Note that the result is obtained from the NZ-DSF transmission link, which has a lower value of $|\beta_2|$ (about a quarter of that of G. 652 fiber). Thus, the optimal symbol rate obtained here is even an optimistic estimation as indicated by 6.10. The NF of EDFA and constellation size are not shown in the analysis, so the result should be independent of them. To verify this, we conduct another simulation with 4A8PSK and NF = 3 dB, and we found the ratio of p_2/p_1 is almost the same, and hence the same optimal symbol rate, which implies a deeper relationship between the direct noise on λ and B .

The yellow curve (6.7) may be interpreted as the best performance, if possible, for a discrete NFDM system, and the purple curve (6.9) represents the optimal performance under linear equalization (e.g., half back rotation or LMMSE). Thus, the space between them may characterize the possible performance of advanced equalization schemes, such as using a nonlinear filter or machine learning.

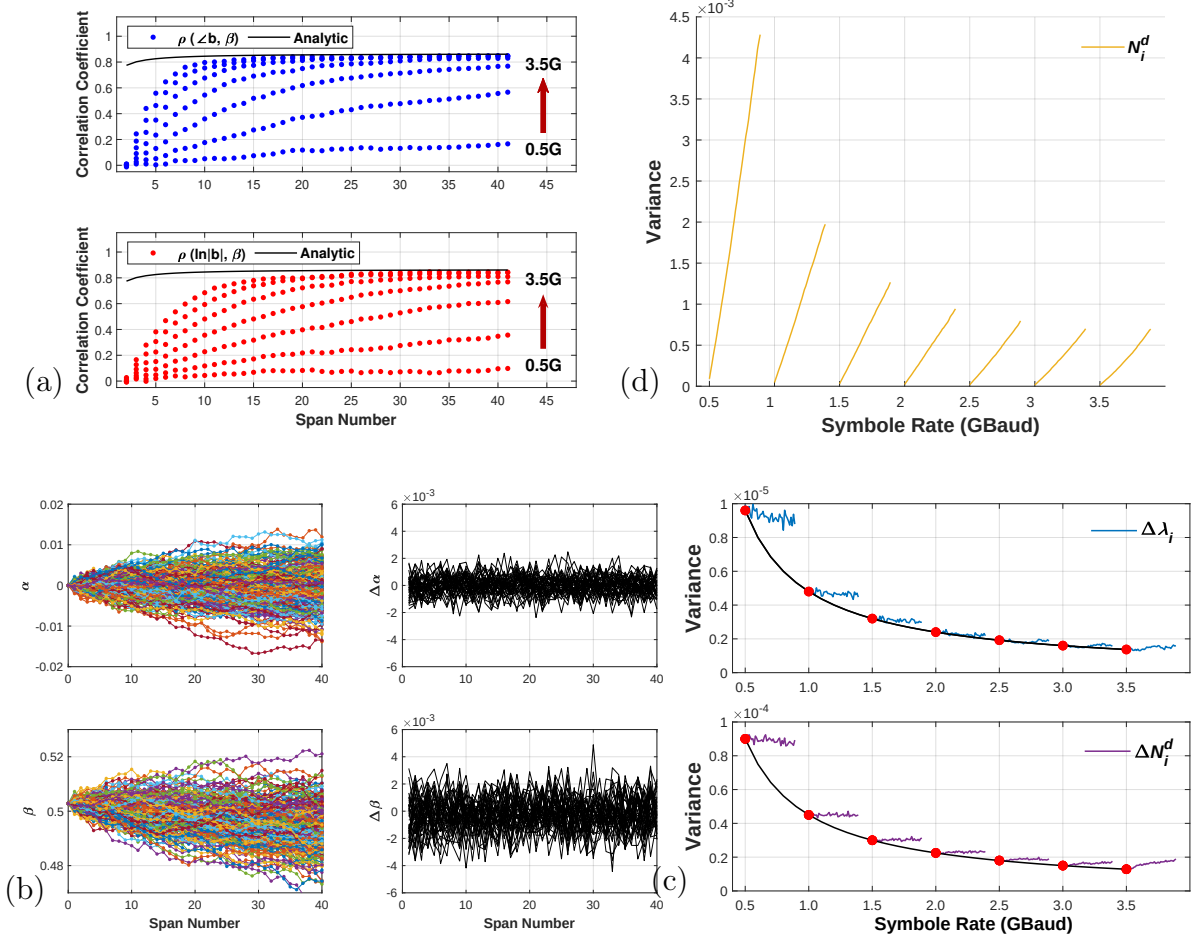


Figure 6.3: (a) Correlation coefficients between λ and b with different R_s ; (b) Illustration of the random walk of α_i and β_i with symbol rate of 2 GBaud, $\Delta\alpha_i = \alpha_i - \alpha_{i-1}$ and $\Delta\beta_i = \beta_i - \beta_{i-1}$; (c) Reciprocal dependence of $\text{Var}(\Delta\lambda_i)$ and $\text{Var}(\Delta N_i^d)$ on R_s ; (d) Linear dependence of $\text{Var}(N_i^d)$ on N .

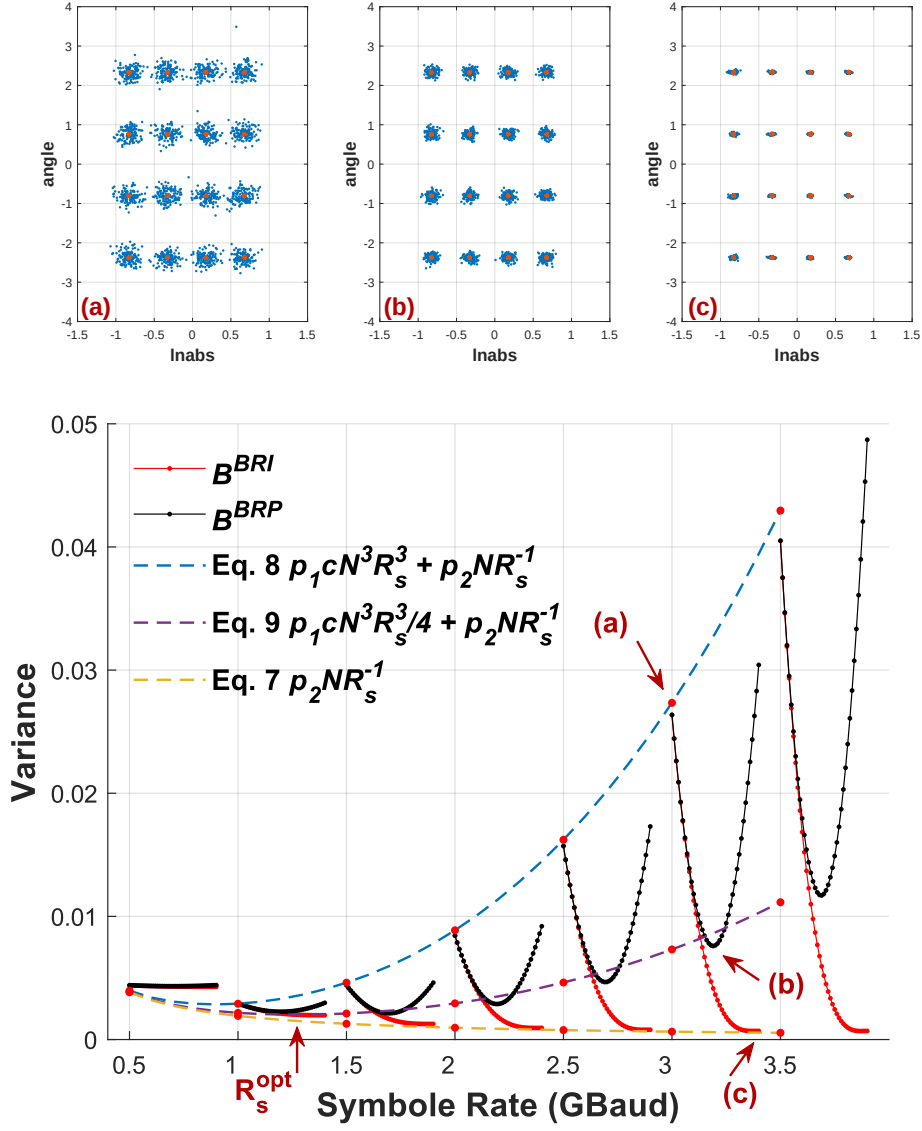


Figure 6.4: The noise variance of B with full ideal back rotation, zero back rotation, and half practical back rotation is well fitted by 6.7, 6.8, and 6.9, respectively. Insets: signal distribution of 3 GBaud.

6.5 Conclusion

We analyze the noise accumulation for a discrete NFDM system and show its dependence on symbol rate. The cubic dependence on symbol rate and transmission distance puts a strong limit on the performance of the discrete NFDM system. The optimal symbol rate is found to be low.

Chapter 7

Conclusion

7.1 Summary

In this thesis, we first demonstrated a record transmission rate for a dual-polarization (DP) discrete Nonlinear Fourier Division Multiplexing (NFDM) system by combining two symbol-misaligned single-polarization (SP) discrete NFDM signals and employing DP multi-symbol linear minimum mean-square error (LMMSE) detection.

Subsequently, we investigated the noise evolution mechanism in discrete NFDM systems. Building on this understanding, we proposed the half back rotation algorithm—a conceptually simple yet highly effective method that restores the intrinsic elegance of NFDM theory. Notably, this algorithm offers an alternative derivation of the Gordon–Haus effect within the nonlinear Fourier transform (NFT) framework and extends it to a phase-domain counterpart, revealing new insights into noise dynamics in the nonlinear spectral domain.

We then tackled a long-standing challenge: blind polarization demultiplexing for DP discrete NFDM in the nonlinear frequency domain. By integrating the half-back rotation with the blind polarization demultiplexing algorithm and other existing digital signal processing (DSP) techniques, we constructed the first complete DSP flow tailored specifically for DP discrete NFDM, which operates entirely in the nonlinear frequency domain and requires no training sequences. The proposed DSP flow was validated through both numerical simulations and experimental demonstrations, and we further analyzed in detail the impact of practical non-idealities such as non-ideal bias offset, loss-gain imbalance, and polarization-dependent loss on its performance.

Finally, we refined the noise model for discrete NFDM by explicitly accounting for direct noise contributions on the nonlinear spectrum. This refined model revealed the existence of an optimal symbol rate that minimizes total noise variance. Crucially, this optimum occurs at a relatively low symbol rate, which may explain the persistent difficulty of discrete NFDM systems in achieving high information rates despite their

theoretical promise.

Together, these contributions enhance the systematicity and feasibility of discrete NFDM systems, bridging gaps between nonlinear signal theory and real-world implementation. Furthermore, they also indicate that the limitations of discrete NFDM may be inherent, resulting from the interaction between nonlinear spectral evolution and stochastic noise.

7.2 Outlook

NFDM was proposed to harness, rather than mitigate, fiber nonlinearity—effectively transforming it from a detrimental impairment into a useful resource. In this context, discrete NFDM has demonstrated notable success, as it leverages high signal power to form optical solitons in the time domain, thereby exhibiting inherent resilience to nonlinear distortions.

However, this advantage comes at a cost—the vulnerability of the discrete eigenvalue to noise. The noise on the discrete eigenvalue will transfer into the b -coefficients in a magnified form through the NFDM transfer function, which turns out to be cubic-dependent on transmission distance and symbol rate, severely limiting the achievable symbol rate and resulting in a relatively low spectral efficiency for discrete NFDM systems. To address these limitations, future work may focus on the following directions:

1. Soliton control techniques developed in classical soliton communication can be investigated in discrete NFDM systems. The parabolic filter can reduce the Gordon-Haus jitter from z^3 -dependent to z -dependent, and the synchronous modulation can even fully suppress it [106, 107]. With an expectation that these techniques can also be successfully applied to the modern soliton system, the symbol rate of discrete NFDM is supposed to increase accordingly.
2. Since the Gordon-Haus jitter, i.e., the magnitude noise on b , can be greatly reduced in principle by applying the techniques mentioned above, reducing the phase noise on b (hence the noise on β) will become crucial for discrete NFDM to reach a higher performance. Methods of suppressing the noise on β can be explored in an analogous way to that on α .
3. The half back rotation and associated DSP flow proposed in this thesis are currently limited to single-eigenvalue systems. An important next step is to investigate whether these algorithms can be extended to multi-soliton (multi-eigenvalue) scenarios, which are essential for achieving higher data rates in NFDM transmission.

Appendix A

Inverse NFT

INFT is generally achieved by the ‘‘Gelfand-Levitan-Marchenko’’ (GLM) method [44, 108, 109]. However, we briefly attach the method given in [108]. For $\lambda \in \mathbb{C}^+$, 2.51 becomes

$$\frac{1}{a(\xi)}\varphi(t, \xi) = \bar{\psi}(t, \xi^*) + \frac{b(\xi)}{a(\xi)}\psi(t, \xi), \quad (\text{A.1})$$

and it is useful to multiply A.1 with $e^{j\xi t}/(\xi - \lambda)$ and turn it into an integral over the red contour C that goes from $-\infty$ to $+\infty$, and closely pass around the discrete eigenvalues, as shown in Fig. A.1,

$$\int_C \frac{\varphi(t, \xi)e^{j\xi t}}{a(\xi)(\xi - \lambda)}d\xi = \int_C \frac{\bar{\psi}(t, \xi^*)e^{j\xi t}}{(\xi - \lambda)}d\xi + \int_C \frac{b(\xi)\psi(t, \xi)e^{j\xi t}}{a(\xi)(\xi - \lambda)}d\xi. \quad (\text{A.2})$$

For the LHS of A.2, regard C as the difference between the closed counterclockwise contour $C + R_{+\infty}$ and $R_{+\infty}$, as shown in Fig. A.1(a), the integral becomes

$$\int_C \frac{\varphi(t, \xi)e^{j\xi t}}{a(\xi)(\xi - \lambda)}d\xi = \oint_{C+R_{+\infty}} \frac{\varphi(t, \xi)e^{j\xi t}}{a(\xi)(\xi - \lambda)}d\xi - \int_{R_{+\infty}} \frac{\varphi(t, \xi)e^{j\xi t}}{a(\xi)(\xi - \lambda)}d\xi. \quad (\text{A.3})$$

The first term on the RHS of A.3 is zero due to analyticity within the closed region. Besides, it can be shown that $\varphi(t, \xi)e^{j\xi t} \rightarrow [1, 0]^T$, $a(\xi) \rightarrow 1$ when $|\xi| \rightarrow \infty$, so A.3 becomes

$$\int_C \frac{\varphi(t, \xi)e^{j\xi t}}{a(\xi)(\xi - \lambda)}d\xi = - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \int_{R_{+\infty}} \frac{1}{(\xi - \lambda)}d\xi = -j\pi \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (\text{A.4})$$

For the first term of RHS of A.2, we regard C as the difference between the closed counterclockwise contour $-C + R_{-\infty}$ and $R_{-\infty}$, as shown in Fig. A.1(b), so the integral becomes

$$\begin{aligned} \int_C \frac{\bar{\psi}(t, \xi^*)e^{j\xi t}}{(\xi - \lambda)}d\xi &= - \oint_{-C+R_{-\infty}} \frac{\bar{\psi}(t, \xi^*)e^{j\xi t}}{(\xi - \lambda)}d\xi + \int_{R_{-\infty}} \frac{\bar{\psi}(t, \xi^*)e^{j\xi t}}{(\xi - \lambda)}d\xi \\ &= -2j\pi\bar{\psi}(t, \lambda^*)e^{j\lambda t} + j\pi \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned} \quad (\text{A.5})$$

For the second term of RHS of A.2, we regard C as the difference between the real axis and a closed counterclockwise contour ΔC , as shown in Fig. A.1(c), so the integral becomes

$$\begin{aligned} \int_C \frac{b(\xi)\psi(t, \xi)e^{j\xi t}}{a(\xi)(\xi - \lambda)} d\xi &= \int_{-\infty}^{+\infty} \frac{b(\xi)\psi(t, \xi)e^{j\xi t}}{a(\xi)(\xi - \lambda)} d\xi - \oint_{\Delta C} \frac{b(\xi)\psi(t, \xi)e^{j\xi t}}{a(\xi)(\xi - \lambda)} d\xi \\ &= \int_{-\infty}^{+\infty} \frac{b(\xi)\psi(t, \xi)e^{j\xi t}}{a(\xi)(\xi - \lambda)} d\xi - 2j\pi \sum_{i=1}^n \frac{b(\lambda_i)\psi(t, \lambda_i)e^{j\lambda_i t}}{a'(\lambda_i)(\lambda_i - \lambda)}. \end{aligned} \quad (\text{A.6})$$

With these results, A.2 becomes

$$\begin{aligned} \bar{\psi}(t, \lambda^*) e^{j\lambda t} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2j\pi} \int_{-\infty}^{+\infty} \frac{b(\xi)\psi(t, \xi)e^{j\xi t}}{a(\xi)(\xi - \lambda)} d\xi + \sum_{i=1}^n \frac{b(\lambda_i)\psi(t, \lambda_i)e^{j\lambda_i t}}{a'(\lambda_i)(\lambda - \lambda_i)} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2j\pi} \int_{-\infty}^{+\infty} Q_c(\xi) \frac{\psi(t, \xi)e^{j\xi t}}{(\xi - \lambda)} d\xi + \sum_{i=1}^n Q_d(\lambda_i) \frac{\psi(t, \lambda_i)e^{j\lambda_i t}}{(\lambda - \lambda_i)}, \end{aligned} \quad (\text{A.7})$$

and for $|\lambda| \rightarrow \infty$, A.7 approaches to

$$\bar{\psi}(t, \lambda^*) e^{j\lambda t} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{1}{2j\pi\lambda} \int_{-\infty}^{+\infty} Q_c(\xi)\psi(t, \xi)e^{j\xi t} d\xi + \frac{1}{\lambda} \sum_{i=1}^n Q_d(\lambda_i)\psi(t, \lambda_i)e^{j\lambda_i t}. \quad (\text{A.8})$$

Replace the λ^* with λ in A.8 and take the complex conjugate on both sides we get

$$\begin{bmatrix} \psi_2(t, \lambda) \\ -\psi_1(t, \lambda) \end{bmatrix} e^{-j\lambda t} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2j\pi\lambda} \int_{-\infty}^{+\infty} Q_c^*(\xi) \begin{bmatrix} \psi_1^*(t, \xi) \\ \psi_2^*(t, \xi) \end{bmatrix} e^{-j\xi t} d\xi + \frac{1}{\lambda} \sum_{i=1}^n Q_d^*(\lambda_i) \begin{bmatrix} \psi_1^*(t, \lambda_i) \\ \psi_2^*(t, \lambda_i) \end{bmatrix} e^{-j\lambda_i^* t}. \quad (\text{A.9})$$

Besides, for $|\lambda| \gg 1$ the asymptotic form of $\psi(t, \lambda)e^{-j\lambda t}$ can be proved to be

$$\psi(t, \lambda)e^{-j\lambda t} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{1}{2\lambda} \begin{bmatrix} q(t) \\ -j \int_t^{+\infty} |q(t')|^2 dt' \end{bmatrix}. \quad (\text{A.10})$$

Comparing A.9 and A.10, we get the formula for reconstructing $q(t)$ from the nonlinear spectrum, i.e., the inverse NFT:

$$q(t) = \frac{j}{\pi} \int_{-\infty}^{+\infty} Q_c^*(\xi)\psi_2^*(t, \xi)e^{-j\xi t} d\xi - 2 \sum_{i=1}^n Q_d^*(\lambda_i)\psi_2^*(t, \lambda_i)e^{-j\lambda_i^* t}, \quad (\text{A.11})$$

which shows the relationship between the time-domain signal and its nonlinear spectrum, where the first and second terms correspond to the dispersive and solitonic components of the signal, respectively. For a soliton, only the discrete spectrum exists, A.11 becomes

$$q(t) = -2 \sum_{i=1}^n Q_d^*(\lambda_i)\psi_2^*(t, \lambda_i)e^{-j\lambda_i^* t}, \quad (\text{A.12})$$

and A.7 becomes

$$\begin{bmatrix} \psi_2^*(t, \lambda^*) \\ -\psi_1^*(t, \lambda^*) \end{bmatrix} e^{j\lambda t} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sum_{i=1}^n Q_d(\lambda_i) \frac{\psi(t, \lambda_i)e^{j\lambda_i t}}{(\lambda - \lambda_i)}. \quad (\text{A.13})$$

Let $\lambda = \lambda_k^*$, it becomes

$$\begin{bmatrix} \psi_2^*(t, \lambda_k) \\ -\psi_1^*(t, \lambda_k) \end{bmatrix} e^{j\lambda_k^* t} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sum_{i=1}^n Q_d(\lambda_i) \frac{\psi(t, \lambda_i) e^{j\lambda_i t}}{(\lambda_k^* - \lambda_i)}. \quad (\text{A.14})$$

Consider the 1st-order soliton which only has one eigenvalue $\lambda_1 = \alpha + j\beta$,

$$\begin{cases} \psi_2^*(t, \lambda_1) e^{j\lambda_1^* t} = 1 + Q_d(\lambda_1) \frac{\psi_1(t, \lambda_1) e^{j\lambda_1 t}}{(\lambda_1^* - \lambda_1)} \\ \psi_1(t, \lambda_1) e^{-j\lambda_1 t} = Q_d^*(\lambda_1) \frac{\psi_2^*(t, \lambda_1) e^{-j\lambda_1^* t}}{(\lambda_1^* - \lambda_1)} \end{cases}. \quad (\text{A.15})$$

From A.15, we can get

$$\psi_2^*(t, \lambda_1) = e^{-j\lambda_1^* t} / \left(\frac{|Q_d(\lambda_1)|^2}{4\beta^2} e^{-4\beta t} + 1 \right). \quad (\text{A.16})$$

Thus,

$$q(t) = -2Q_d^*(\lambda_1) \psi_2^*(t, \lambda_1) e^{-j\lambda_1^* t} = -\frac{8\beta^2 Q_d^*(\lambda_1) e^{-j2\lambda_1^* t}}{4\beta^2 + |Q_d(\lambda_1)|^2 e^{-4\eta t}}, \quad (\text{A.17})$$

which can be rewritten into

$$q(t) = 2\beta \operatorname{sech}(2\beta + \theta) e^{-j2\alpha t + j\phi}, \quad (\text{A.18})$$

where

$$\begin{aligned} e^{-j\phi - \theta} &= \frac{Q_d(\lambda_1, z)}{-2\beta} = \frac{Q_d(\lambda_1, 0)}{-2\beta} e^{j2\lambda_1^2 z} \\ &= \frac{|Q_d(\lambda_1, 0)|}{2\beta} e^{-4\alpha\beta z} e^{2j(\alpha^2 - \beta^2)z + j\angle Q_d(\lambda_1, 0) - j\pi}. \end{aligned} \quad (\text{A.19})$$

Thus,

$$\begin{cases} \theta = 4\alpha\beta z - \ln(|Q_d(\lambda_1, 0)|/2\beta) \\ \phi = 2(\alpha^2 - \beta^2)z - \angle Q_d(\lambda_1, 0) + \pi \end{cases}, \quad (\text{A.20})$$

which is consistent with D.2. When only the discrete spectrum is present but the eigenvalues are multiple, the Darboux transform [17, Sect. III][110] is a very efficient way to generate the corresponding time domain waveform from an arbitrary number of eigenvalues. Therefore, it is very suitable for pure discrete NFD systems. Besides, the development of the fast NFT/INFT (FNFT/IFNFT) algorithm, which is an open-source project [59], greatly facilitates the NFT/INFT calculation.

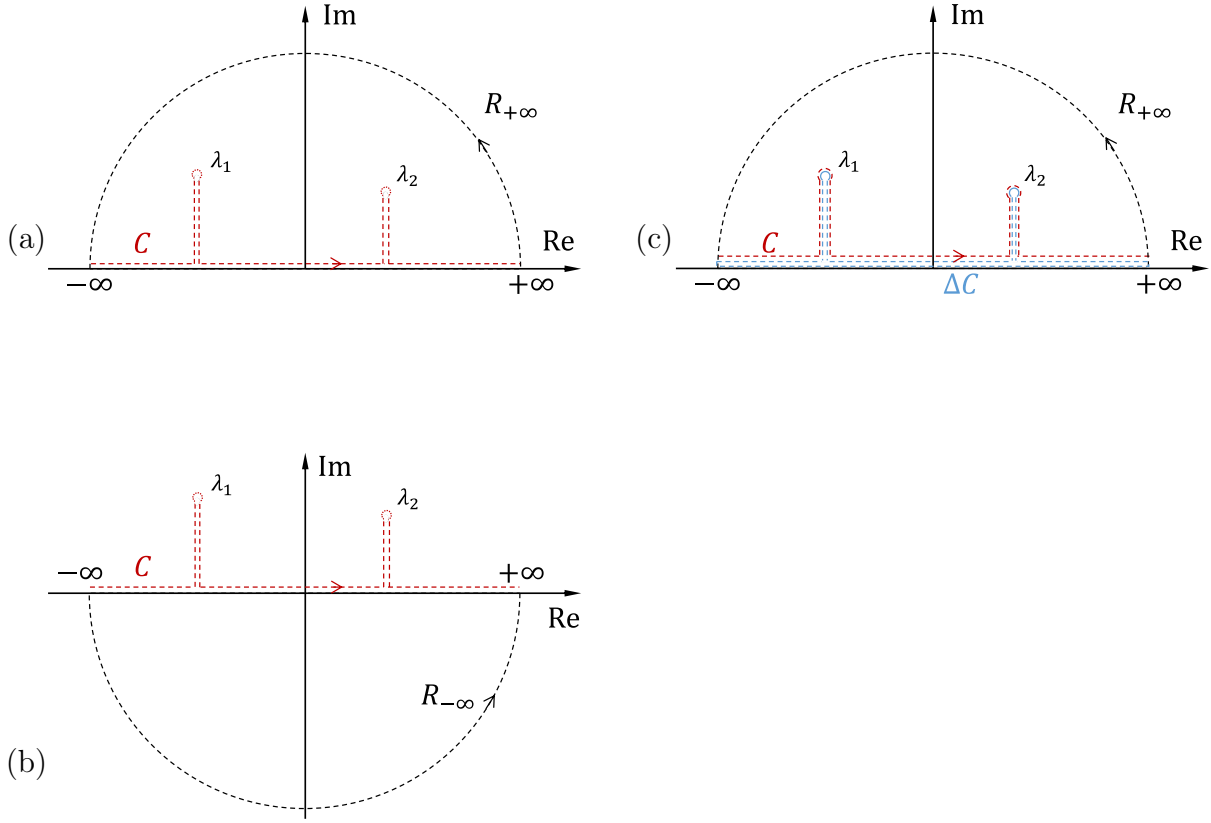


Figure A.1: Contour C is regarded as (a) $(C + R_{+\infty}) - R_{+\infty}$; (b) $R_{-\infty} - (-C + R_{-\infty})$; (c) Real axis $-\Delta C$. Adapted from [48].

Appendix B

Variance of b -coefficients

B.1 Approximation of $(\alpha_i + j\beta_i)^2$

Let $\tilde{\alpha}_i = \alpha_i - \alpha_0$, $\tilde{\beta}_i = \beta_i - \beta_0$ denote the accumulated noise in α and β at the i^{th} EDFA, respectively. We have

$$\begin{aligned} b_{i+1} &= b_i e^{-2j(\alpha_i + j\beta_i)^2 L} \\ &= b_i e^{4\alpha_i \beta_i L} e^{-2j(\alpha_i^2 - \beta_i^2)L}, \end{aligned} \tag{B.1}$$

and the exponent terms can be approximated as

$$\begin{aligned} 4\alpha_i \beta_i &= 4(\tilde{\alpha}_i + \alpha_0)(\tilde{\beta}_i + \beta_0) \\ &= 4(\tilde{\alpha}_i \tilde{\beta}_i + \alpha_0 \tilde{\beta}_i + \tilde{\alpha}_i \beta_0 + \alpha_0 \beta_0) \\ &\approx 4(\tilde{\alpha}_i \beta_0) \\ &= 2\tilde{\alpha}_i \\ &= 2\alpha_i, \end{aligned} \tag{B.2}$$

$$\begin{aligned} \alpha_i^2 - \beta_i^2 &= (\tilde{\alpha}_i + \alpha_0)^2 - \beta_i^2 \\ &= \tilde{\alpha}_i^2 - 2\tilde{\alpha}_i \alpha_0 + \alpha_0^2 - \beta_i^2 \\ &\approx -\beta_i^2, \end{aligned} \tag{B.3}$$

because $\alpha_0 = 0$, $\beta_0 = 0.5$, and $\tilde{\alpha}_i \tilde{\beta}_i$, $\tilde{\alpha}_i^2$ are negligible.

B.2 Variance of $\ln |b(x)|$

Let $\Delta\alpha_i = \alpha_i - \alpha_{i-1}$ denote the independent noise with a variance of σ_α^2 added to α by the i^{th} EDFA. We have

$$\begin{aligned}\text{Var}(r(x)) &= \text{Var}(2L\alpha_{N+1}(x - N) + r_{N+1}) \\ &= 4L^2(x - N)^2 \text{Var}(\alpha_{N+1}) + \text{Var}(r_{N+1}) \\ &\quad + 4L(x - N) \text{Cov}(r_{N+1}, \alpha_{N+1}),\end{aligned}\tag{B.4}$$

where

$$\begin{aligned}\text{Var}(\alpha_{N+1}) &= \text{Var}\left(\sum_{i=1}^{N+1} \Delta\alpha_i\right) \\ &= \sum_{i=1}^{N+1} \text{Var}(\Delta\alpha_i) + \sum_{i \neq j} \text{Cov}(\Delta\alpha_i, \Delta\alpha_j) \\ &= (N + 1)\sigma_\alpha^2,\end{aligned}\tag{B.5}$$

$$\begin{aligned}\text{Var}(r_{N+1}) &= \text{Var}\left(\ln |b_0| + 2L \sum_{i=1}^N \alpha_i\right) \\ &= 4L^2 \text{Var}\left(\sum_{i=1}^N \alpha_i\right) \\ &= 4L^2 \sum_{i,j=1}^N \text{Cov}(\alpha_i, \alpha_j) \\ &= 4L^2 \sigma_\alpha^2 \sum_{i,j=1}^N \min(i, j) \\ &= 4L^2 \sigma_\alpha^2 \frac{N(N+1)(2N+1)}{6},\end{aligned}\tag{B.6}$$

$$\begin{aligned}\text{Cov}(r_{N+1}, \alpha_{N+1}) &= \mathbf{E}[r_{N+1}\alpha_{N+1}] - \mathbf{E}[r_{N+1}] \mathbf{E}[\alpha_{N+1}] \\ &= \mathbf{E}[r_{N+1}\alpha_{N+1}] \\ &= \mathbf{E}\left[2L \sum_{i=1}^N \alpha_i \alpha_{N+1}\right] \\ &= 2L \sum_{i=1}^N \mathbf{E}[\alpha_i \alpha_{N+1}] \\ &= 2L \sigma_\alpha^2 \sum_{i=1}^N i \\ &= N(N+1)L\sigma_\alpha^2.\end{aligned}\tag{B.7}$$

Thus,

$$\text{Var}(r(x)) = 4L^2\sigma_\alpha^2(N+1) \left[x(x-N) + \frac{N(2N+1)}{6} \right]. \quad (\text{B.8})$$

B.3 Variance of $\angle \mathbf{b}(\mathbf{x})$

β_i may be regarded as $\beta_i = w_i + 0.5$, where w_i is a zero-mean random walk, and we obtain $\beta_i^2 = w_i^2 + w_i + 0.25 \approx \beta_i - 0.25$. Let $\Delta\beta_i = \beta_i - \beta_{i-1}$ denote the independent noise with a variance of σ_β^2 added to β by the i^{th} EDFA. We have

$$\begin{aligned} \text{Var}(\varphi(x)) &= \text{Var} \left(2L\beta_{N+1}^2(x-N) + \varphi_{N+1} \right) \\ &= 4L^2(x-N)^2 \text{Var}(\beta_{N+1}^2) + \text{Var}(\varphi_{N+1}) \\ &\quad + 4L(x-N) \text{Cov}(\varphi_{N+1}, \beta_{N+1}^2), \end{aligned} \quad (\text{B.9})$$

where

$$\begin{aligned} \text{Var}(\beta_{N+1}^2) &\approx \text{Var}(\beta_{N+1}) \\ &= \text{Var} \left(\sum_{i=1}^{N+1} \Delta\beta_i \right) \\ &= \sum_{i=1}^{N+1} \text{Var}(\Delta\beta_i) + \sum_{i \neq j} \text{Cov}(\Delta\beta_i, \Delta\beta_j) \\ &= (N+1)\sigma_\beta^2, \end{aligned} \quad (\text{B.10})$$

$$\begin{aligned} \text{Var}(\varphi_{N+1}) &= 4L^2 \text{Var} \left(\sum_{i=1}^N \beta_i^2 \right) \\ &\approx 4L^2 \sum_{i,j=1}^N \text{Cov}(\beta_i, \beta_j) \\ &= 4L^2\sigma_\beta^2 \sum_{i,j=1}^N \min(i, j) \\ &= 4L^2\sigma_\beta^2 \frac{N(N+1)(2N+1)}{6}, \end{aligned} \quad (\text{B.11})$$

$$\begin{aligned}
\text{Cov}(\varphi_{N+1}, \beta_{N+1}^2) &= \text{Cov}\left(2L \sum_{i=1}^N \beta_i^2, \beta_{N+1}^2\right) \\
&\approx 2L \sum_{i=1}^N \text{Cov}(\beta_i - 0.25, \beta_{N+1} - 0.25) \\
&= 2L\sigma_\beta^2 \sum_{i=1}^N i \\
&= N(N+1)L\sigma_\beta^2.
\end{aligned} \tag{B.12}$$

Thus,

$$\text{Var}(\varphi(x)) = 4L^2\sigma_\beta^2(N+1) \left[x(x-N) + \frac{N(2N+1)}{6} \right]. \tag{B.13}$$

B.4 Correlation coefficient of $\ln |\mathbf{b}(x)|$

$$\begin{aligned}
\rho(r(x), \alpha_{N+1}) &= \frac{\text{Cov}(r(x), \alpha_{N+1})}{\sqrt{\text{Var}(r(x)) \cdot \text{Var}(\alpha_{N+1})}} \\
&= \frac{\text{Cov}(2L\alpha_{N+1}(x-N) + r_{N+1}, \alpha_{N+1})}{\sqrt{\text{Var}(r(x)) \cdot \text{Var}(\alpha_{N+1})}} \\
&= \frac{2L(x-N) \text{Var}(\alpha_{N+1}) + \text{Cov}(r_{N+1}, \alpha_{N+1})}{\sqrt{\text{Var}(r(x)) \cdot \text{Var}(\alpha_{N+1})}} \\
&= \frac{x - N/2}{\sqrt{x(x-N) + N(2N+1)/6}},
\end{aligned} \tag{B.14}$$

$$\begin{aligned}
\rho(\varphi(x), \beta_{N+1}) &= \frac{\text{Cov}(\varphi(x), \beta_{N+1})}{\sqrt{\text{Var}(\varphi(x)) \cdot \text{Var}(\beta_{N+1})}} \\
&= \frac{\text{Cov}(2L\beta_{N+1}^2(x-N) + \varphi_{N+1}, \beta_{N+1})}{\sqrt{\text{Var}(\varphi(x)) \cdot \text{Var}(\beta_{N+1})}} \\
&= \frac{2L(x-N) \text{Cov}(\beta_{N+1}^2, \beta_{N+1}) + \text{Cov}(\varphi_{N+1}, \beta_{N+1})}{\sqrt{\text{Var}(\varphi(x)) \cdot \text{Var}(\beta_{N+1})}} \\
&\approx \frac{x - N/2}{\sqrt{x(x-N) + N(2N+1)/6}},
\end{aligned} \tag{B.15}$$

where we have used

$$\begin{aligned}
\text{Cov}(\beta_i^2, \beta_{N+1}) &\approx \text{Cov}(\beta_i - 0.25, \beta_{N+1} - 0.5) \\
&= \text{Cov}(\beta_i, \beta_{N+1}) \\
&= i\sigma_\beta^2,
\end{aligned} \tag{B.16}$$

and

$$\begin{aligned}
\text{Cov}(\varphi_{N+1}, \beta_{N+1}) &= \text{Cov}\left(2L \sum_{i=1}^N \beta_i^2, \beta_{N+1}\right) \\
&= 2L \sum_{i=1}^N \text{Cov}(\beta_i, \beta_{N+1}) \\
&= N(N+1)L\sigma_\beta^2.
\end{aligned} \tag{B.17}$$

B.5 Non-ideal bias control case

With split β_0 due to non-ideal bias control and/or low OSNR at the transmitter, the back rotation line becomes (with second-order terms neglected)

$$\tilde{\varphi}(x) = 2L\tilde{\beta}_{N+1}^2(x - N) + \tilde{\varphi}_{N+1}, \tag{B.18}$$

$$\begin{aligned}
\tilde{\beta}_{N+1}^2 &= (\beta_{N+1} + \Delta\beta_0)^2 \\
&\approx \beta_i + \Delta\beta_0 - 0.25,
\end{aligned}$$

$$\begin{aligned}
\tilde{\varphi}_{N+1} &= 2L \sum_{i=1}^N \tilde{\beta}_i^2 \\
&\approx 2L \sum_{i=1}^N (\beta_i + \Delta\beta_0 - 0.25).
\end{aligned} \tag{B.19}$$

Thus,

$$\begin{aligned}
\text{Var}(\tilde{\varphi}(x)) &= \text{Var}\left(2L\tilde{\beta}_{N+1}^2(x - N) + \tilde{\varphi}_{N+1}\right) \\
&= 4L^2(x - N)^2 \text{Var}\left(\tilde{\beta}_{N+1}^2\right) + \text{Var}\left(\tilde{\varphi}_{N+1}\right) \\
&\quad + 4L(x - N) \text{Cov}\left(\tilde{\varphi}_{N+1}, \tilde{\beta}_{N+1}^2\right),
\end{aligned} \tag{B.20}$$

where

$$\begin{aligned}
\text{Var}\left(\tilde{\beta}_{N+1}^2\right) &\approx \text{Var}(\beta_{N+1} + \Delta\beta_0 - 0.25) \\
&= (N+1)\sigma_\beta^2 + \sigma_{\beta_0}^2,
\end{aligned} \tag{B.21}$$

$$\text{Cov} \left(\tilde{\varphi}_{N+1}, \tilde{\beta}_{N+1}^2 \right) = 2L \sum_{i=1}^N \text{Cov} \left(\tilde{\beta}_i^2, \tilde{\beta}_{N+1}^2 \right) \quad (\text{B.22})$$

$$= 2L \sum_{i=1}^N \text{Cov} \left(\beta_i + \Delta\beta_0 - 0.25, \beta_{N+1} + \Delta\beta_0 - 0.25 \right) \quad (\text{B.23})$$

$$= 2L \sum_{i=1}^N \left(\text{Cov} \left(\beta_i, \beta_{N+1} \right) + \text{Var} \left(\Delta\beta_0 \right) \right) \\ = N(N+1)L\sigma_\beta^2 + 2NL\sigma_{\beta_0}^2. \quad (\text{B.24})$$

Let $\frac{d}{dx} \text{Var}(\tilde{\varphi}(x)) = 0$, we obtain

$$x_{\text{opt}} = N - \frac{\text{Cov} \left(\tilde{\varphi}_{N+1}, \tilde{\beta}_{N+1}^2 \right)}{\text{Var} \left(\tilde{\beta}_{N+1}^2 \right) 2L} \quad (\text{B.25})$$

$$= \frac{N(N+1)\sigma_\beta^2}{2 \left[(N+1)\sigma_\beta^2 + \sigma_{\beta_0}^2 \right]}. \quad (\text{B.26})$$

Appendix C

Abrupt Effects Modeled by Logarithm-Scaled δ -function

In [52, 111], the lumped gain of EDFA was included by scaling the δ -function by $e^{\Gamma L/2} - 1$:

$$\frac{\partial \mathbf{q}}{\partial z} = \left[-\frac{\Gamma}{2} + (e^{\frac{\Gamma L}{2}} - 1) \sum_{n=1}^N \delta(z - nL) \right] \mathbf{q}, \quad (\text{C.1})$$

which implies that $\mathbf{q}$ is amplified by $e^{\Gamma L/2}$ at $z_n = nL$, described by

$$\int_{z_n^-}^{z_n^+} \frac{\partial \mathbf{q}}{\partial z} dz = \int_{z_n^-}^{z_n^+} \left[-\frac{\Gamma}{2} + (e^{\frac{\Gamma L}{2}} - 1) \sum_{n=1}^N \delta(z - nL) \right] \mathbf{q} dz, \quad (\text{C.2})$$

$$\mathbf{q}(z_n^+) - \mathbf{q}(z_n^-) = (e^{\frac{\Gamma L}{2}} - 1) \mathbf{q}(z_n^-), \quad (\text{C.3})$$

$$\mathbf{q}(z_n^+) = e^{\frac{\Gamma L}{2}} \mathbf{q}(z_n^-). \quad (\text{C.4})$$

In this thesis, we instead used logarithmic scaling of the δ -function to include the abrupt changes caused by EDFA gain, SOP rotation, and PDL. For example, we can include the EDFA gain by

$$\frac{\partial \mathbf{q}}{\partial z} = \left[-\frac{\Gamma}{2} + \frac{\Gamma L}{2} \sum_{n=1}^N \delta(z - nL) \right] \mathbf{q}. \quad (\text{C.5})$$

This manipulation makes it more natural and convenient to directly solve $\partial \mathbf{q} / \partial z = g(z) \mathbf{q}$ in the canonical form

$$\mathbf{q}(z) = \exp \left(\int_0^z g(z') dz' \right) \mathbf{q}(0). \quad (\text{C.6})$$

In such a way, the integral in the exponential index becomes a “staircase” function with each “stair height” being the summation of the logarithm of each abrupt change. Let

$$g(z) = -\frac{\Gamma}{2} + \frac{\Gamma L}{2} \sum_{n=1}^N \delta(z - nL), \quad (\text{C.7})$$

it is convenient to obtain

$$\begin{aligned}\mathbf{q}(z) &= \exp\left(\int_0^z \left[-\frac{\Gamma}{2} + \frac{\Gamma L}{2} \sum_{n=1}^N \delta(z' - nL)\right] dz'\right) \mathbf{q}(0) \\ &= \exp\left(-\frac{\Gamma z}{2}\right) \exp\left(\frac{\Gamma L}{2} \left\lfloor \frac{z}{L} \right\rfloor\right) \mathbf{q}(0).\end{aligned}\tag{C.8}$$

Similarly, we scale the individual δ -functions by $\ln(\mathbf{R}_n)$ to model the abrupt SOP rotation after each span. Let

$$g(z) = \sum_{n=1}^N \ln(\mathbf{R}_n) \delta(z - nL),\tag{C.9}$$

then,

$$\begin{aligned}\mathbf{q}(z) &= \exp\left(\int_0^z \sum_{n=1}^N \ln(\mathbf{R}_n) \delta(z' - nL) dz'\right) \mathbf{q}(0) \\ &= \mathbf{T}(z) \mathbf{q}(0),\end{aligned}\tag{C.10}$$

where $\mathbf{T}(z)$ records the accumulate SOP rotation along z .

$$\mathbf{T}(z) = \begin{cases} \mathbf{I} & 0 \leq z < L \\ \mathbf{R}_1 & L \leq z < 2L \\ \mathbf{R}_2 \mathbf{R}_1 & 2L \leq z < 3L \\ \dots & \\ \mathbf{R}_N \dots \mathbf{R}_1 & NL \leq z \end{cases}\tag{C.11}$$

$$\frac{d\mathbf{T}(z)}{dz} = \left[\sum_{n=1}^N \ln(\mathbf{R}_n) \delta(z - nL) \right] \mathbf{T}(z),\tag{C.12}$$

or equivalently,

$$\frac{d \ln \mathbf{T}(z)}{dz} = \sum_{n=1}^N \ln(\mathbf{R}_n) \delta(z - nL),\tag{C.13}$$

which is illustrated in Fig. C.1 and C.2.

Finally, for PDL, let the gain of amplitude X and Y polarization be $e^{\Delta/4}$ and $e^{-\Delta/4}$, so the gain difference on power is e^{Δ} . We set

$$g(z) = \frac{\boldsymbol{\sigma}}{4} \sum_{n=1}^N \delta(z - nL) \quad \text{with} \quad \boldsymbol{\sigma} = \ln \begin{bmatrix} e^{\Delta} & 0 \\ 0 & e^{-\Delta} \end{bmatrix} = \begin{bmatrix} \Delta & 0 \\ 0 & -\Delta \end{bmatrix},\tag{C.14}$$

which gives

$$\begin{aligned}\mathbf{q}(z) &= \exp\left(\int_0^z \frac{\boldsymbol{\sigma}}{4} \sum_{n=1}^N \delta(z' - nL) dz'\right) \mathbf{q}(0) \\ &= \exp\left(\frac{\boldsymbol{\sigma}}{4} \left\lfloor \frac{z}{L} \right\rfloor\right) \mathbf{q}(0).\end{aligned}\tag{C.15}$$

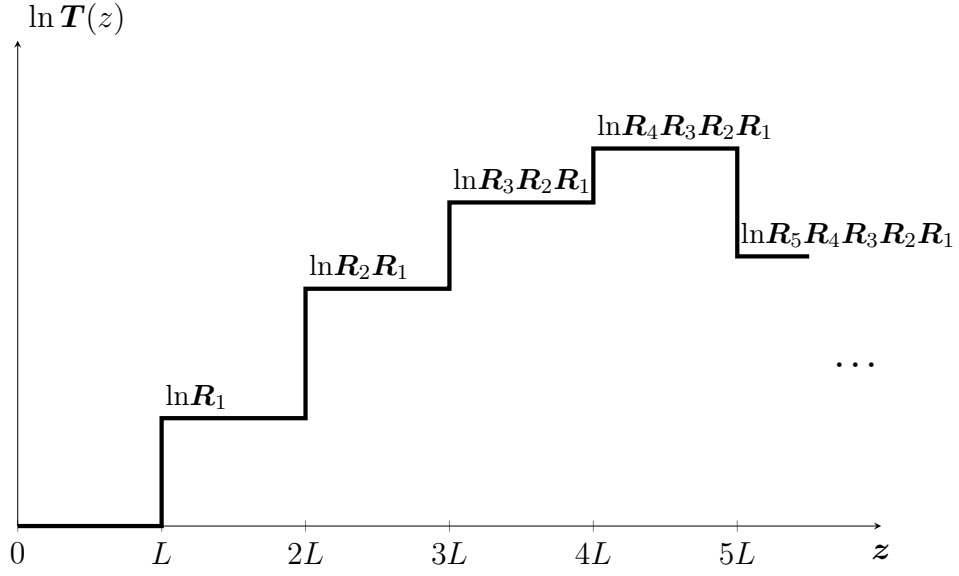


Figure C.1: Staircase function scaled by the logarithm of the overall SOP rotation until span n , $\ln \mathbf{R}_n \mathbf{R}_{n-1} \dots \mathbf{R}_1$.

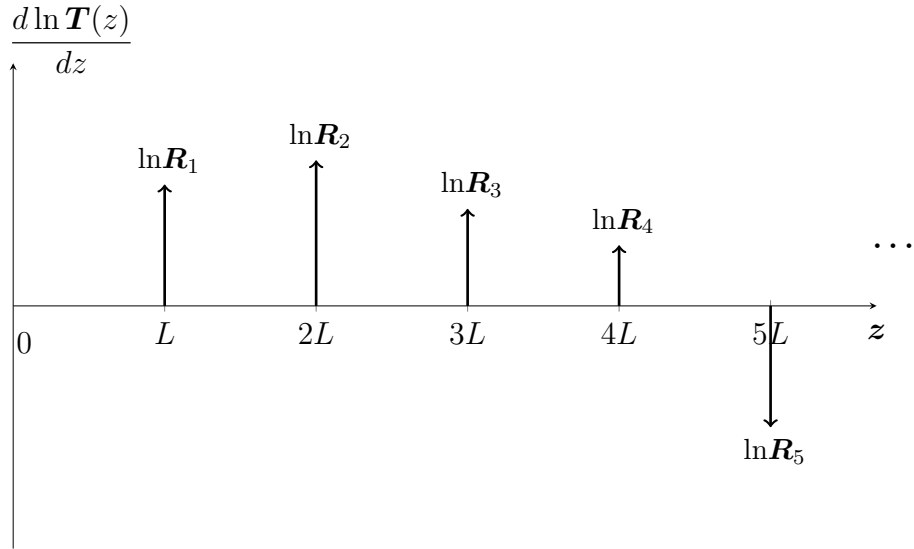


Figure C.2: δ -function scaled by the logarithm of SOP rotation matrix of the n^{th} span, $\ln \mathbf{R}_n$.

Appendix D

Generating Second-Order Soliton with Specified Time Positions

A 1st-order soliton (1-soliton), which has a single eigenvalue $\lambda = \alpha + j\beta$, can be described by 2.55

$$u(z, t) = 2\beta \operatorname{sech} [2\beta (t - t_0 + 2\alpha z)] e^{j2(\beta^2 - \alpha^2)z - j2\alpha t + j\phi_0}, \quad (\text{D.1})$$

where

$$2\beta t_0 = \ln \left(\frac{|Q_d(z=0)|}{2\beta} \right) = \ln |b(z=0)|, \quad (\text{D.2a})$$

$$\phi_0 = \pi - \angle u_d(z=0) = \pi/2 - \angle b(z=0). \quad (\text{D.2b})$$

Note the second equalities in D.2 (a) and (b) are obtained by the fact that in the 1-soliton case, $a'(\lambda)$ is shown to be $1/(2j\beta)$. The time position of the soliton peak and the phase of the 1-soliton are explicitly determined by the magnitude and phase of the b -coefficient, respectively. Therefore, a 1-soliton with arbitrary time position and phase can be readily generated by inverse NFT with λ and $b(\lambda)$.

A 2nd-order soliton (2-soliton) whose eigenvalues have different real parts can be asymptotically (i.e. when $z \rightarrow \pm\infty$) expressed as a sum of two 1-solitons, but $|b(\lambda_1)|$ and $|b(\lambda_2)|$ no longer correspond to the pulse peak positions in the time domain.

If two 1-solitons collide, they will shift in time and phase after collision, and the amount of shift is related to the height and velocity of these two solitons. Mollenauer and Gordon [112] analyzed the shift analytically by modeling the collision of two 1-solitons as the evolution of a 2-soliton. By comparing the time and phase terms in the asymptotical region at $z = \pm\infty$, the value of the shifts can be obtained. We found that by offsetting the time shift in the b -coefficients of the 2-soliton, the corresponding time domain waveform can be well located at the designed time position.

In [112], the amplitude and frequency of the two pulses depended on each other, while we extend the method to arbitrary amplitudes and frequencies by introducing a new variable, p , to the amplitude. Considering a 2-soliton, which can be expressed by a

sum of two components (which are not necessarily 1-soliton), $u(z, t) = u_1(z, t) + u_2(z, t)$ with amplitudes of $A_1 = p(1 + a)$, $A_2 = p(1 - a)$, and frequencies¹ of $\Omega_1 = \Omega(1 - a)$, $\Omega_2 = -\Omega(1 + a)$, where $p, \Omega > 0$, $|a| \leq 1$. u_1 and u_2 satisfy

$$\begin{bmatrix} M_{11} (\gamma_1^{-1} + \gamma_1^*) & M_{12} (\gamma_1^{-1} + \gamma_2^*) \\ M_{21} (\gamma_2^{-1} + \gamma_1^*) & M_{22} (\gamma_2^{-1} + \gamma_2^*) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (\text{D.3})$$

with $(i, k \in \{1, 2\})$

$$\begin{aligned} M_{ik} &= [A_i + A_k - j(\Omega_i - \Omega_k)]^{-1}, \\ \gamma_i &= e^{A_i(t-t_{i0}+\Omega_i z)+j(-\Omega_i t+(A_i^2-\Omega_i^2)z/2+\phi_{i0})}. \end{aligned} \quad (\text{D.4})$$

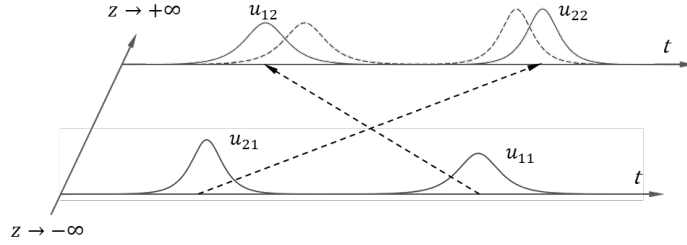


Figure D.1: A 2-soliton is asymptotically two 1-solitons at $z \rightarrow \pm\infty$. The dashed curve illustrates the case that u_{11}, u_{21} propagate without the existence of each other.

With those parameters, $u(z, t)$ can be analytically given, and when $z \rightarrow \pm\infty$, the analytical expression can be well approximated by a sum of two 1-solitons separated from each other, i.e., two humps, as illustrated in Fig. D.1. They are given by

$$\begin{aligned} u_{11}(z, t) &= p(1 + a) \operatorname{sech}(pt + py - \eta/2) e^{j(\theta+\phi-\delta\phi_1)}, \\ u_{12}(z, t) &= p(1 + a) \operatorname{sech}(pt + py + \eta/2) e^{j(\theta+\phi+\delta\phi_1)}, \\ u_{21}(z, t) &= p(1 - a) \operatorname{sech}(pt - py + \eta/2) e^{j(\theta+\phi-\delta\phi_2)}, \\ u_{22}(z, t) &= p(1 - a) \operatorname{sech}(pt - py - \eta/2) e^{j(\theta+\phi+\delta\phi_2)}, \end{aligned} \quad (\text{D.5})$$

where

$$\begin{aligned} y &= at + (1 - a^2)(\Omega z - t_0), \\ \theta &= a\Omega t + (1 + a^2)(p^2 - \Omega^2)z/2, \\ \phi &= -\Omega t + a(p^2 + \Omega^2)z + \phi_0, \\ \eta &= \ln[(p^2 + \Omega^2)/(p^2 a^2 + \Omega^2)], \\ \delta\phi_1 &= \operatorname{atan}(p/\Omega) - \operatorname{atan}(pa/\Omega), \\ \delta\phi_2 &= \operatorname{atan}(p/\Omega) + \operatorname{atan}(pa/\Omega). \end{aligned} \quad (\text{D.6})$$

¹ It may be better to express with “real parts” and “imaginary parts” for “amplitudes” and “frequencies”, but we follow the language in [112] first, then change it in the end.

Thus, the collision-induced time position shifts on u_{12} , u_{22} are given by (note the time position shift of the waveform at $z \rightarrow +\infty$ in Fig. D.1)

$$\begin{aligned}\delta t_1 &= \frac{\eta}{p(1+a)}, & \delta t_2 &= \frac{\eta}{p(1-a)}, \\ p &= \frac{A_1 + A_2}{2}, & a &= \frac{A_1 - A_2}{A_1 + A_2}.\end{aligned}\tag{D.7}$$

Note that the derivation above requires a mean frequency of $-a\Omega$, but it can be omitted as we are free to change the moving reference frame, by which the collision characteristics will not change. Thus, for simplicity, we may choose $\Omega_1 = -\Omega_2$ at first, and add a common frequency to the generated 2-soliton, if we want Ω_1 and Ω_2 to be asymmetric.

Suppose we want to generate a 2-soliton with two peaks located at t_1 and t_2 , with eigenvalues of $\lambda_1 = \alpha_1 + j\beta_1$ and $\lambda_2 = \alpha_2 + j\beta_2$. The method of offsetting the b -coefficients, b_1 and b_2 , is described as follows.

Step 1: rewrite D.7 in the language of NFT.

$$\begin{aligned}\delta t_1 &= \ln \left[\frac{(\beta_1 + \beta_2)^2 + (\alpha_1 - \alpha_2)^2}{(\beta_1 - \beta_2)^2 + (\alpha_1 - \alpha_2)^2} \right] / (2\beta_1), \\ \delta t_2 &= \ln \left[\frac{(\beta_1 + \beta_2)^2 + (\alpha_1 - \alpha_2)^2}{(\beta_1 - \beta_2)^2 + (\alpha_1 - \alpha_2)^2} \right] / (2\beta_2).\end{aligned}\tag{D.8}$$

Step 2: determine $|b_1|$ and $|b_2|$ by Eq. (3), but with offset time shifts $-\delta t_1/2$ and $+\delta t_2/2$, then add phase terms and φ_2 to obtain b_1 and b_2 , while $\bar{b}_1$ and $\bar{b}_2$ denote the b -coefficient without the time shift offset.

$$\begin{aligned}b_1 &= e^{2\beta_1(t_1 - \delta t_1/2) + j\varphi_1}, & b_2 &= e^{2\beta_2(t_2 + \delta t_2/2) + j\varphi_2}, \\ \bar{b}_1 &= e^{2\beta_1 t_1 + j\varphi_1}, & \bar{b}_2 &= e^{2\beta_2 t_2 + j\varphi_2}.\end{aligned}\tag{D.9}$$

Step 3: generate the 2-soliton waveform by INFT with b_1 , b_2 . Also, for comparison, the waveform of 2-soliton with $\bar{b}_1$, $\bar{b}_2$ and the waveform of a linear sum of two 1-solitons with $\bar{b}_1$, $\bar{b}_2$ are denoted by $\bar{u}(t)$ and $\tilde{u}(t)$, respectively.

$$\begin{aligned}u(t) &= \text{INFT} \{ \lambda_1, \lambda_2, b_1, b_2 \}, \\ \bar{u}(t) &= \text{INFT} \{ \lambda_1, \lambda_2, \bar{b}_1, \bar{b}_2 \}, \\ \tilde{u}(t) &= \text{INFT} \{ \lambda_1, \bar{b}_1 \} + \text{INFT} \{ \lambda_2, \bar{b}_2 \}.\end{aligned}\tag{D.10}$$

Three numerical examples are given in Fig. D.2 to demonstrate the effectiveness of the proposed method. The peak positions of the waveform are labeled with corresponding colors in the figure. In all cases, the peaks of $u(t)$ align well with the specified time positions, while that of $\bar{u}(t)$ shows unwanted mismatch.

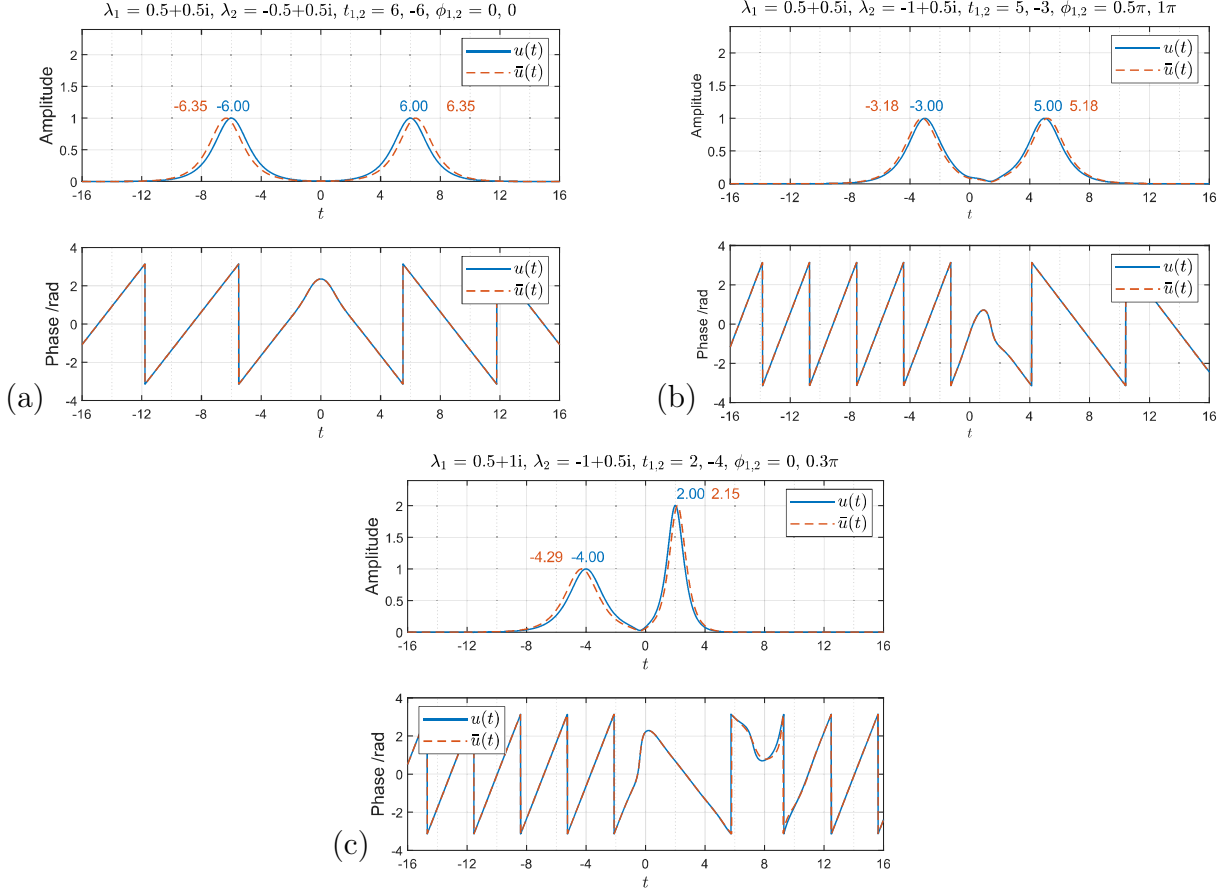


Figure D.2: Comparison of $u(t)$ and $\tilde{u}(t)$ for three different cases: (a) same amplitude, symmetric frequency; (b) same amplitude, asymmetric frequency; (c) different amplitude, asymmetric frequency. The waveforms of $u(t)$ match very well to the specified time positions.

We also compare the waveform of $u(t)$ and $\tilde{u}(t)$ and their eigenvalues, with fixed specified time positions $t_{1,2} = \pm 3$ and symmetric frequencies, whose absolute value varies from 0.05 to 1. Although their time domain waveforms match quite well, both the real and imaginary parts of the eigenvalue of $\tilde{u}(t)$ deviate from the designed value, especially when the real part is small, which will introduce eigenvalue noise at the transmitter in an NFDM system. This is because a sum of the time-domain waveform does not result in a sum of the nonlinear spectrum, and the asymptotic expression of a 2-soliton by the sum of two 1-solitons is only accurate when the 1-solitons are well separated from each other. This indicates that when the constituent pulses are close enough, it is preferable to generate a 2-soliton by our proposed method.

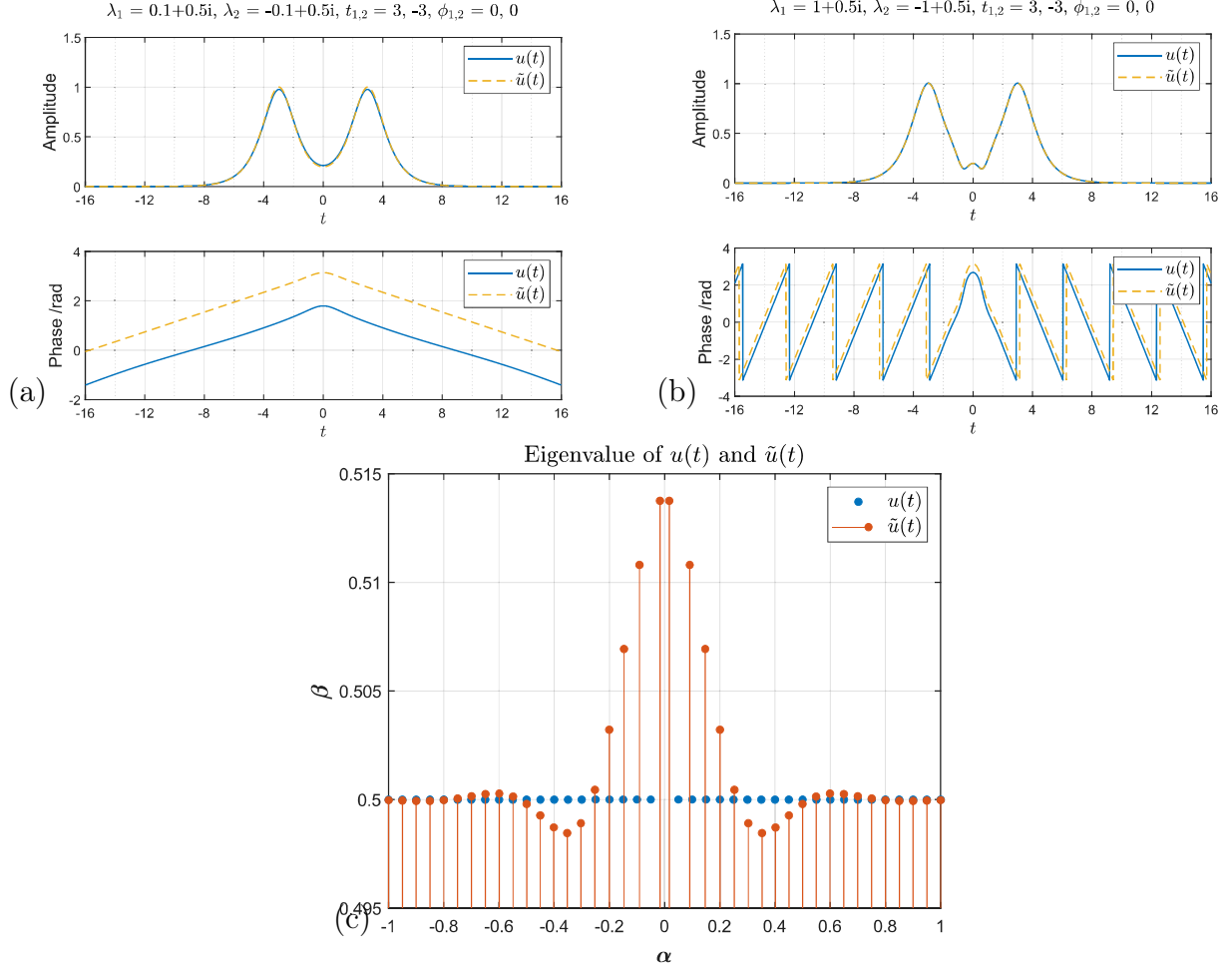


Figure D.3: Generating 2-soliton with eigenvalues of $\pm\alpha + 0.5j$ ($\alpha = 0.05, 0.1, \dots, 1$) by INFT with the proposed modified b -coefficient, $u(t)$, and by directly adding two corresponding 1-solitons, $\tilde{u}(t)$. (a) and (b) are time domain waveforms of $u(t)$ and $\tilde{u}(t)$ with small and big α , respectively; (c) eigenvalues of $u(t)$ and $\tilde{u}(t)$ with different α .

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