OPTIMAL CONTROL AND STOPPING: STOCHASTIC LINEAR QUADRATIC CONTROL PROBLEMS IN NON-MARKOVIAN REGIME SWITCHING SYSTEM

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Optimal Control and Stopping: Stochastic Linear Quadratic Control Problems in Non-Markovian Regime Switching System

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CERTIFICATE OF ORIGINALITY

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____________________________(Signature)

Yunhong Li (Name of student)

Dedicate to my supervisors, parents, and grandmother.

Abstract

This thesis investigates a mixed stochastic linear quadratic control-stopping problem with non-Markovian regime switching and random coefficients, where the control pair consists of a classical control variable and a controlled stopping time. Both homogeneous and non-homogeneous cases are considered. For homogeneous problem, the primary contributions include the explicit derivation of optimal state feedback controls and optimal cost values via several novel systems of extended stochastic Riccati equations (ESREs) and reflected extended stochastic Riccati equations (RESREs). For non-homogeneous problem, we give a lower boundary. These ESREs and RESREs are either highly nonlinear or involve unbounded coefficients, presenting significant analytical challenges.

A major achievement of this work is the establishment, for the first time, of a multidimensional comparison theorem for reflected backward stochastic differential equations (RBSDEs). Building on this result, and employing advanced techniques such as truncation functions, logarithmic transformations, the John-Nirenberg inequality, and BMO (Bounded Mean Oscillation) martingale estimates, we prove the existence and uniqueness of solutions to the aforementioned ESREs and RESREs. In addressing the solvability of linear reflected BSDEs with unbounded coefficients, we first resolve the one-dimensional case and then extend the result to the multidimensional setting using the contraction mapping method.

Keywords: Stochastic linear quadratic control • Regime switching • Optimal stopping • Riccati equation • Reflected BSDEs • Comparison theorem

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Contents

Genernal Notations

$\mathbb{R}$	the set of real numbers;
$\mathbb{R}_+$	the set of non-negative real numbers;
$\mathbb{R}^n$	the n-dimensional real Euclidean space;
$\mathbb{R}^n_+$	the set of vectors in $\mathbb{R}^n$ with non-negative components;
$\lvert \cdot \rvert$	the Frobenius norm of a real vector or matrix;
a^+	$\max\{a,\ 0\}$ for real a;
a^-	$\max\{-a,\ 0\}$ for real a;
M'	the transpose of a vector or matrix M;
$\mathbb{S}^n$	the set of symmetric $n \times n$ real matrices;
$(\Omega, \mathcal{F}, \mathbb{P})$	a complete probability space;
$\mathcal{B}(\Omega)$	the Borel set for Ω;
$W.$	a one-dimensional standard Brownian motion;
$\alpha.$	a continuous-time stationary Markov chain valued in a finite state space $\mathcal{M} = \{1, 2, \ldots, \ell\}$;
$Q = (q_{ij})_{\ell \times \ell}$	the generator of $\alpha.$;
$\{\mathcal{F}_t^W\}_{t \geqslant 0}$	the filtration generated by $W.$;
$\{\mathcal{F}_t\}_{t \geqslant 0}$	the filtration generated by $(W.,\ \alpha.)$;
$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geqslant 0}, \mathbb{P})$	the filtered probability space;

$L^0_{\mathcal{F}}(\Omega;\mathbb{R}^n)$	the set of all $\mathcal{F}$-measurable random variables;
$L^p_{\mathcal{F}}(\Omega;\mathbb{R}^n)$	the space of functions $\phi:\Omega\to\mathbb{R}^n$ that are $\mathcal{F}$-measurable with $\mathbb{E}\lvert\phi\rvert^p<\infty$;
$L^p_{\mathcal{F}_T}(\Omega;\mathbb{R})$	the space of functions $\phi:\Omega\to\mathbb{R}$ that are $\mathcal{F}_T$-measurable with $\mathbb{E}\lvert\phi\rvert^p<\infty$;
$L^\infty_{\mathcal{F}_T}(\Omega;\mathbb{R})$	the space of functions $\phi:\Omega\to\mathbb{R}$ that are $\mathcal{F}_T$-measurable, and essentially bounded, i.e., $\|\phi\|_\infty=\inf_{A\in\mathcal{F},\mathbb{P}(A)=0}\sup_{\omega\in\Omega\setminus A}\lvert\phi(\omega)\rvert=\operatorname{ess\,sup}_{\omega\in\Omega}\lvert\phi(\omega)\rvert<\infty;$
$L^2_{\mathcal{F}}(0,T;\mathbb{R})$	the space of functions $\phi:[0,T]\times\Omega\to\mathbb{R}$ that are $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted with $\mathbb{E}\int_0^T\lvert\phi(t)\rvert^2\,\mathrm{d}t<\infty$;
$L^{2,loc}_{\mathcal{F}}(0,T;\mathbb{R})$	the space of functions $\phi:[0,T]\times\Omega\to\mathbb{R}$ that are $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted with $\int_0^T\lvert\phi(t)\rvert^2\,\mathrm{d}t<\infty$ $\mathbb{P}$-a.s.;
$L^\infty_{\mathcal{F}}(0,T;\mathbb{R})$	the space of functions $\phi:[0,T]\times\Omega\to\mathbb{R}$ that are $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted and essentially bounded;
$L^2_{\mathcal{F}}(\Omega;C(0,T;\mathbb{R}))$	the space of functions $\phi:[0,T]\times\Omega\to\mathbb{R}$ that are $\{\mathcal{F}_t\}_{t\geqslant 0}$-progressively measurable with continuous sample paths and satisfy $\mathbb{E}\sup_{t\in[0,T]}\lvert\phi(t)\rvert^2<\infty$;
$\mathcal{S}_{\mathcal{F}}(0,T;\mathbb{R})$	the space of functions $\phi:[0,T]\times\Omega\to\mathbb{R}$ that are $\{\mathcal{F}_t\}_{t\geqslant 0}$-progressively measurable, càdlàg and satisfy $\mathbb{E}\sup_{t\in[0,T]}\lvert\phi(t)\rvert^2<\infty$;
$\mathcal{A}(0,T;\mathbb{R})$	the space of functions $\mu:[0,T]\times\Omega\to\mathbb{R}$ that are $\{\mathcal{F}_t\}_{t\geqslant 0}$-progressively measurable, non-decreasing, càdlàg, and satisfy $\mu_0=0$ and $\mathbb{E}[\mu_T^2]<\infty$;
$\mathcal{A}_c(0,T;\mathbb{R})$	the space of functions $\mu\in\mathcal{A}$ that have continuous sample paths.

These definitions are generalized in the obvious way to the cases that $\mathcal{F}$ is replaced by $\mathcal{F}^W$ and $\mathbb{R}$ by $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$ or $\mathbb{S}^n$, where $\mathbb{S}^n$ is the set of symmetric $n\times n$ real matrices. If $M\in\mathbb{S}^n$ is positive definite (resp. positive semidefinite), we write $M>$ (resp. $\geqslant$) 0. In our argument, t, ω, "$\mathbb{P}$-almost surely" and "almost everywhere", will be suppressed for simplicity in many circumstances, when no confusion occurs.

Chapter 1

Introduction

1.1 Introduction

Linear-Quadratic (LQ) control serves as a cornerstone of optimal control theory, providing a powerful methodology for designing state feedback control that optimizes a specified quadratic cost function. The solution to a LQ problem is derived through the Riccati equation, which gives the optimal state feedback control and the optimal cost value. Owing to its practical utility, this approach has found widespread applications across various domains, including engineering systems automation and financial mathematics.

Since the pioneering work of Wonham (1968), who initiated the study of matrix Riccati equations in control theory and provided a rigorous framework for deterministic LQ problems, the field has evolved significantly. His contributions laid the foundation for extending LQ control to stochastic systems by establishing solvability conditions for Riccati differential equations. Riccati equations are vital to determine the optimal controls and value functions in LQ problems. Kohlmann and Tang (2002) first solved the existence and uniqueness of one-dimensional case, for matrix-valued case, see Tang (2003), Qian and Zhou (2013), Du (2015), Hu and Zhou (2003) for their results. Bismut (1976) was the first to study stochastic LQ problem with random coefficients, making seminal contributions by introducing randomness into system coefficients. His two-part work derived optimal control policies for linear systems with multiplicative noise, proved the existence of solutions to stochastic Riccati equations (SREs), and generalized the separation principle to stochastic settings. Chen et al. (1998) expanded the LQ framework to problems with indefinite control weight costs, relaxing the traditional positive-definiteness assumptions. They established necessary and sufficient solvability conditions and linked the stochastic LQ problem to a forward-backward stochastic differential equation (FBSDE), broadening the applicability of LQ methods in finance and engineering. Li and Zhou (2002a) incorporated Markovian jumps into finite-horizon stochastic LQ control, modeling sudden regime shifts such as market crashes. Their key result was a coupled system of Riccati equations accounting for both diffusion and jump dynamics, enabling optimal control synthesis in hybrid systems. Tang (2003) generalized the stochastic LQ framework by studying problem with random coefficients and introducing the concept of linear stochastic Hamiltonian systems. His work established the connection between optimal control and backward stochastic Riccati equations, providing a unified approach to solving LQ problem with randomness in both system dynamics and cost parameters, and laying the foundation for further extensions to non-Markovian settings. Hu and Zhou (2005) addressed constrained stochastic LQ control with random coefficients, explicitly incorporating portfolio constraints (e.g., no-shorting) into the optimization framework. Their work derived optimal feedback controls via a novel Riccati equation approach and applied the results to portfolio selection, demonstrating how constraints alter the efficient frontier. Their paper bridged theoretical LQ control with practical financial applications.

Stochastic LQ theory has found important applications in finance, particularly in portfolio optimization and risk management, providing a powerful framework for investors to optimize portfolios under market uncertainty. In finance, Markowitz's "mean-variance" framework utilizes stochastic LQ theory to construct optimal portfolios by balancing the mean (return) and variance (risk) of assets, allowing investors to maximize returns while managing risk—a critical tool in modern financial practice.

Recently, there has been growing interest in financial market models where key parameters, such as market trends, asset volatility rates, bank interest rates, and default risk, are modulated by Markov process. The motivation stems from the need to develop more sophisticated models that can accurately reflect the uncertain nature of financial markets. A dominant factor in stock movement is the prevailing market trend.

The mathematical foundations for regime-switching models were established by Hamilton (1989), who formulated market trends through parameters that are contingent on a finite number of distinct market regimes. Each regime represents a unique set of economic or financial conditions, with transitions between regimes typically modeled as a Markov process. This framework was significantly advanced by Becherer (2004), who developed a regime switching model for utility indifference hedging and valuation via reaction-diffusion systems, establishing fundamental connections between regime transitions and PDE-based valuation methods. Further extending this framework, Becherer and Schweizer (2005) provided classical solutions to reaction-diffusion systems for hedging problems with interacting Itô and point processes, studying classical solutions for systems of interacting semilinear parabolic partial differential equations for regime-switching models in high-dimensional settings. This theoretical foundation has become essential for modeling structural breaks and non-linear dynamics in asset prices. In a Markowitz mean-variance portfolio selection model with regime switching, Zhou and Yin (2003) derived mean-variance efficient portfolios and efficient frontiers based on LQ optimal control theory. Hu et al. (2020) studied portfolio optimization problems in a regime-switching market by applying ergodic BSDE systems to construct optimal trading strategies. See also Yao et al. (2006); Li and Zhou (2002b); Zhang and Guo (2004); Guo (2001); Zhang (2001);

Buffington and Elliott (2002); Elliott and Siu (2010) for further developments and various applications.

However, in the aforementioned Markov chain-modulated models, market parameters depend on market status and time, but not on the Brownian motion. From a practical perspective, market parameters should also reflect uncertainty caused by white noise. Thus, it is too restrictive to set market parameters as deterministic, even when the market status is known. It is therefore necessary to allow market parameters to depend on both the noise and the Markov chain.

Our aim is to generalize the model to a non-Markovian regime-switching framework with mixed control, in which the Riccati equation systems are no longer deterministic ordinary differential equations (ODEs), but stochastic Riccati equations — specifically, new types of backward stochastic differential equations (BSDEs) and reflected backward stochastic differential equations (RBSDEs). These four systems are highly nonlinear, making their solvability an interesting problem in its own right. See also Wen et al. (2023); Chen and Luo (2025); Hu et al. (2022b); Hu et al., 2022a) for non-Markovian LQ systems and the solvability of Riccati equations.

In 1997, El Karoui et al. (1997a) extended BSDEs to address optimal control problems constrained by boundaries, introducing the concept of "RBSDEs". First, they proved both the existence and uniqueness of the solution. Furthermore, they demonstrated that the value function of a mixed optimal stopping–optimal stochastic control problem is indeed the solution to the RBSDE. Second, they provided a probabilistic interpretation for the viscosity solution of a parabolic partial differential equation via the solution of the RBSDE. The inclusion of boundary constraints significantly enhances the versatility of RBSDEs, making them indispensable in modeling and solving real-world challenges, with applications in option pricing (see El Karoui et al. (1997b); Gu et al. (2026)), portfolio optimization (see El Karoui et al. (2001)), risk management (see Quenez and Sulem (2014)), and optimal stopping problems (see

Hamadène and Jeanblanc (2007); Quenez and Sulem (2014); Wu and Yu (2008); Ren and Xia (2006)). The comparison theorem is a key result in the theory of RBSDEs. El Karoui et al. (1997a) and Hamadène (2002) gave the results when the filtration is generated by a Brownian motion. Later, Crépey and Matoussi (2008) extended this result to the case where the filtration is generated by a Brownian motion and a Poisson random measure. In a general space, Kim and Hun (2021) obtained a comparison theorem for RBSDEs under a monotonicity condition together with an additional assumption related to jumps of underlying martingales.

Regarding the literature on quadratic BSDE systems, the pioneering work on solving scalar equations with bounded terminal data is credited to Kobylanski (2000), with notable applications in utility maximization problems Hu et al. (2005). Addressing unbounded terminal cases proved more intricate, as discussed in Briand and Hu (2006, 2008); Delbaen et al. (2011). The multi-dimensional quadratic BSDEs were studied by Hu and Tang (2016). For the quadratic case with boundary constraints, see Kobylanski et al. (2002) for bounded terminal values. Despite these advances, our work appears to be the first to introduce and solve RBSDE systems with non-quadratic growth generators of a special structure.

In this thesis, we construct a general stochastic LQ control problem under a random coefficients model with non-Markovian regime switching, where the control pair consists of a classical control variable constrained in a cone and a controlled stopping time τ. Economically, the additional stopping time τ represents the decision to switch the system in the economic sphere to maximize profitability (or minimize cost). Mathematically, it enables closed-form solutions for both the value function and the optimal control via reflected extended stochastic Riccati equations (RESREs) and extended stochastic Riccati equations (ESREs). The randomness arises from two sources: the Brownian motion driving asset price dynamics and the Markov chain representing regime switching. Furthermore, the control weighting matrix in the cost functional is allowed to be possibly singular.

As a first contribution, we analyze a new optimal control and stopping problem with non-Markovian regime switching, where the system dynamics are affected by the control. We divide the problem into two parts: before τ and after τ, and solve each part separately. On $[\tau, T]$, using the completion of squares method, the optimal control and value function can be expressed in terms of solutions to two fully coupled BSDE systems (3.2.3) and (3.2.4) (see Theorem 3.4), termed ESREs. Building on the method of Hu et al. (2022b), we establish the solvability of our problem, prove its well-posedness using BMO martingales and the John-Nirenberg inequality, and derive explicit optimal feedback controls.

Next, we establish the relationship between the control and stopping problem on $[0, \tau]$ and the solution of RESREs, that is, RBSDE systems (3.2.1) and (3.2.2) (see Theorem 3.5). Both RBSDE systems (3.2.1) and (3.2.2) are new, introduced here for the first time to characterize optimal state feedback control and optimal cost value. In particular, we show that when there is no stopping time, our representation of optimal state feedback control and optimal cost value recovers the Riccati equation representation found in Hu et al. (2022b).

Our second contribution concerns the solvability of RBSDE systems (3.2.1) and (3.2.2). In proving the existence of a unique solution to BSDE system, two main challenges arise. First, the extended stochastic Riccati equation forms a family of highly nonlinear RBSDEs, for which the usual assumptions (such as Lipschitz continuity and quadratic growth conditions) do not hold. To the best of our knowledge, no existing results can be directly applied. In this thesis, we establish a new multi-dimensional comparison theorem for RBSDEs. With its help and the truncation technique, we prove the existence. Utilizing the bounded solution to an auxiliary ODE (not the system itself) as a universal bound to control all solution components of BSDE system is a key concept in this case. Second, most previous works used the Feynman-Kac type representation of SREs to prove uniqueness. Instead of such an indirect method, we provide a direct approach using logarithmic transformation and the John-Nirenberg inequality. Ultimately, we obtain the optimal state feedback control and optimal cost value, analogous to the classical unconstrained-control problem or the problem without regime switching, via the two systems of ESREs.

As our third contribution, we introduce a new extension: the non-homogeneous stochastic LQ control problem with regime switching, where the control pair consists of an unconstrained control u and a controlled stopping time τ. This newly introduced non-homogeneous problem is not merely a trivial extension of the homogeneous case, its intrinsic non-homogeneous nature endows this class of problems with fundamentally new research significance. First, non-homogeneous coefficients introduce affine terms (b, ρ) in the state dynamics (4.1.1), capturing exogenous shocks and time-varying trends absent in (3.1.1). Second, the cost functional (4.1.3) minimizes deviations from stochastic targets (q, p, g), modeling tracking objectives. Third, the control u is now unconstrained, permitting strategies such as unlimited short-selling, which are essential in derivatives trading.

Beyond mathematical generalization, this framework addresses key financial incompleteness: non-homogeneous terms incorporate market frictions (e.g., stochastic dividends or funding costs), while regime switching captures structural volatility breaks during crises. This extension is particularly significant for options markets: the affine terms model stochastic volatility and jump risks crucial for hedging exotic options (e.g., barrier options) under regime shifts, while the tracking terms minimize replication errors in delta-hedging strategies. The stopping time τ naturally represents optimal exercise decisions for American options, with the state-switch at τ reflecting payoff realization. Unlike Chapter 3's cone constraints, the unconstrained control space allows aggressive volatility arbitrage strategies. These features enable novel applications in pricing and hedging under realistic market conditions, establishing this extension as a substantial advancement with independent research value.

We divide the problem into two parts: before τ and after τ, and analyze each part separately. On $[\tau, T]$, using the completion of squares method, the optimal control and value function can be expressed in terms of solutions to one coupled BSDE systems (4.2.2) and (4.2.4) (see Theorem 4.3), termed ESREs. Building on the method of Hu et al. (2024), we establish the solvability of our problem, prove its well-posedness using BMO martingales and the John-Nirenberg inequality, and derive explicit optimal feedback controls.

Next, we establish the relationship between the control and stopping problem on $[0, \tau]$. Unfortunately, we cannot solve this problem completely. This is because the value function does not take the quadratic form and the optimal control is not a linear feedback of the state process. Instead, we will provide a lower bound problem for it, which still takes the LQ structure (Of course, an upper bound exists trivially as it is a minimizing problem) and give a lower boundary for Problem 4.3.5. (see Theorem 4.5). Meanwhile, we introduce and study a novel class of linear unbounded RBSDE systems, given by (4.2.1) and (4.2.3). These systems are developed to address challenges in SLQ control problems.

Our fourth contribution concerns the solvability of RBSDE systems (4.2.1) and (4.2.3). In proving the existence of a unique solution to RBSDE systems (4.2.1), we establish solvability by slightly modifying our previous argument in Theorem 3.3. For RBSDE systems (4.2.3), which forms a family of highly linear unbounded RBSDEs, three challenges must be addressed. First, all the coefficients of the linear BSDEs are unbounded due to the unboundedness of $\Lambda(i)$ (and consequently $\Gamma(i)$). Second, the equations are coupled through the term $\sum_{j=1}^{\ell} q_{ij} k(j)$. Third, the boundary restriction condition in (4.2.1) forces solutions to stay below the given boundary. To our knowledge, no existing results directly apply to these BSDE systems. Establishing the solvability of (4.2.3) therefore represents the main technical contribution of this work. The main idea is to first obtain estimates for BMO martingales, then establish the result for the one-dimensional system, and finally apply the contraction mapping method to obtain the result for the multidimensional system. This constitutes the major technical contribution of this thesis.

1.2 Organization of the Thesis

The remainder of the thesis is organized as follows.

Chapter 2 introduces the notations used throughout the thesis and reviews important results from stochastic calculus that are essential for the subsequent analysis.

Chapter 3 constructs a general stochastic LQ control problem under a random coefficients model with non-Markovian regime switching, where the control pair consists of a classical control variable constrained in a cone and a controlled stopping time. We derive the optimal control and value function, and prove the solvability of RBSDE systems (3.2.1) and (3.2.2).

Chapter 4 analyzes the non-homogeneous stochastic LQ control problem with regime switching, where the control pair consists of an unconstrained control and a controlled stopping time. We give a lower boundary for the problem 4.3.5 and establish the solvability of RBSDE systems (4.2.1) and (4.2.3).

Chapter 5 concludes the thesis and discusses potential directions for future research.

Chapter 2

Preliminaries

In this chapter, we first give a brief introduction to the fundamental mathematical concepts for our study, list some necessary results, and introduce the market model that will be used in the rest of the thesis.

2.1 Probability Space

We first recall the definition of σ-field and probability space.

Definition 2.1. *(i) Let Ω be a nonempty set. Let $\mathcal{F}$, called a class, be a nonempty subset of 2^Ω (2^Ω is the set of all subsets in Ω). We call $\mathcal{F}$ a σ-algebra (also called a σ-field) on the set Ω if it satisfies*

$$\begin{cases} \Omega \in \mathcal{F}; \\ A, B \in \mathcal{F} \Rightarrow B \backslash A \in \mathcal{F}; \\ A_i \in \mathcal{F}, i = 1, 2, \cdots \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}. \end{cases}$$

In this case, any $A \in \mathcal{F}$ is called an event (which is also called a measurable set in real analysis), and $(\Omega, \mathcal{F})$ is called a measurable space. We define the Borel set $\mathcal{B}(\Omega)$ as the smallest σ-field containing all open sets of Ω.

(ii) *Given a measurable space* $(\Omega, \mathcal{F})$*, a map* $\mathbb{P} : \mathcal{F} \to [0,1]$ *is called a probability measure on it if the following hold:*

$$\begin{cases} \mathbb{P}(\emptyset) = 0, \ \mathbb{P}(\Omega) = 1; \\ A_i \in \mathcal{F}, A_i \bigcap A_j = \emptyset, i, j = 1, 2, \cdots, i \neq j, \Rightarrow \mathbb{P}\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \mathbb{P}(A_i), \end{cases}$$

where $\emptyset$ *is the empty set. In this case,* $\mathbb{P}(A)$ *is called the probability of the given event* $A \in \mathcal{F}$*. Any* $A \in \mathcal{F}$ *with* $\mathbb{P}(A) = 0$ *is called a* $\mathbb{P}$*-null set.*

(iii) *If* $(\Omega, \mathcal{F})$ *is a measurable space and* $\mathbb{P}$ *is a probability measure on* $(\Omega, \mathcal{F})$*, then* $(\Omega, \mathcal{F}, \mathbb{P})$ *is called a probability space. A probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is said to be complete if* $\mathcal{F}$ *contains all the subsets of* $\mathbb{P}$*-null sets, i.e.,*

$$A \in \mathcal{F}, \quad \mathbb{P}(A) = 0, \quad B \subseteq A \quad \Rightarrow \quad B \in \mathcal{F}.$$

(iv) *In a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, we say that an event* $A \in \mathcal{F}$ *holds almost surely if* $\mathbb{P}(A) = 1$*, written as* A *holds* $\mathbb{P}$*-a.s. We may omit the probability measure* $\mathbb{P}$ *and simply call* A *holds a.s. when there is no confusion on the probability measure* $\mathbb{P}$*.*

We remark that, given Ω, there can be many different σ-fields $\mathcal{F}$ on Ω; and given $(\Omega, \mathcal{F})$, one can consider many different probability measures on $(\Omega, \mathcal{F})$. In this thesis, unless specified, we fix a unique probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

2.2 Random Variable

Definition 2.2. (i) *Let* $(\Omega, \mathcal{F})$ *be a measurable space. A map* $\xi \equiv (\xi_1, \cdots, \xi_n)^\top : \Omega \to \mathbb{R}^n$ *is called an* $\mathbb{R}^n$*-valued random variable if*

$$\{\xi \leqslant c\} \equiv \{\omega \in \Omega \mid \xi_i(\omega) \leqslant c_i, 1 \leqslant i \leqslant n\} \in \mathcal{F}, \quad \forall c \equiv (c_1, \cdots, c_n) \in \mathbb{R}^n.$$

This is also equivalent to the following:

$$\xi^{-1}(A) \in \mathcal{F}, \quad \forall A \in \mathcal{B}\left(\mathbb{R}^n\right).$$

(ii) *Let Ω be a nonempty set and $\xi : \Omega \to \mathbb{R}^n$ be a given map. Let $\mathcal{F}^\xi$ be the smallest σ-field on Ω such that ξ is $\mathcal{F}^\xi$-measurable. We call $\mathcal{F}^\xi$ the σ-field generated by ξ. One actually has*

$$\mathcal{F}^\xi = \xi^{-1}\left(\mathcal{B}\left(\mathbb{R}^n\right)\right).$$

Definition 2.3. *(i) Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ be two measurable spaces and $X : \Omega \to \Omega'$ an $\mathcal{F}/\mathcal{F}'$-measurable map. We call X an $\mathcal{F}/\mathcal{F}'$-random variable, or simply a random variable if there would be no confusion. It is called an $\mathcal{F}$-random variable when $(\Omega', \mathcal{F}') = (\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m))$. For a random variable $X : \Omega \to \Omega', X^{-1}(\mathcal{F}')$ is a sub-σ-field of $\mathcal{F}$, which is called the σ-field generated by X, denoted by $\sigma(X)$. This is the smallest σ-field in Ω under which X is measurable.*

(ii) *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(\Omega', \mathcal{F}')$ a measurable space, and $X : \Omega \to \Omega'$ a random variable. Then X induces a probability measure $\mathbb{P}_X$ on $(\Omega', \mathcal{F}')$ as follows:*

$$\begin{aligned}\mathbb{P}_X\left(A'\right) &\triangleq \mathbb{P} \circ X^{-1}\left(A'\right) \equiv \mathbb{P}\left(X^{-1}\left(A'\right)\right) \\ &= \mathbb{P}\left\{\omega \in \Omega \mid X(\omega) \in A'\right\} \triangleq \mathbb{P}\left\{X \in A'\right\}, \quad \forall A' \in \mathcal{F}'.\end{aligned}$$

We call $\mathbb{P}_X$ the distribution of the random variable X. If $(\Omega', \mathcal{F}') = (\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m))$, $\mathbb{P}_X$ can be uniquely determined by the following function:

$$F(x) = F(x_1, \cdots, x_m) := \mathbb{P}(\omega \in \Omega : X_i(\omega) \leqslant x_i, 1 \leqslant i \leqslant m).$$

We call $F(x)$ the distribution function of X. If the following integral exists:

$$\int_\Omega X(\omega)\, \mathrm{d}\mathbb{P}(\omega) = \int_{\mathbb{R}^m} x\, \mathrm{d}F(x) := \mathbb{E}[X],$$

we then say that X has the mean $\mathbb{E}[X]$.

2.3 Conditional Expectation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Definition 2.4. *Let $\xi \in L^1_{\mathcal{F}}(\Omega; \mathbb{R}^n)$ and $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Then there exists a unique $f \in L^1_{\mathcal{G}}(\Omega; \mathbb{R}^n)$ such that*

$$\int_A \xi(\omega)\, \mathrm{d}\mathbb{P}(\omega) = \int_A f(\omega)\, \mathrm{d}\mathbb{P}(\omega), \quad \forall A \in \mathcal{G}. \tag{2.3.1}$$

We call f, denoted by $\mathbb{E}(\xi|\mathcal{G})$, the conditional expectation of ξ given $\mathcal{G}$.

By definition,

$$\int_A \xi(\omega)\, \mathrm{d}\mathbb{P}(\omega) = \int_A \mathbb{E}[\xi|\mathcal{G}](\omega)\, \mathrm{d}\mathbb{P}(\omega), \quad \forall A \in \mathcal{G}. \tag{2.3.2}$$

To show the existence of $\mathbb{E}(\xi|\mathcal{G})$, we define $\mu : \mathcal{G} \to \mathbb{R}^n$ as

$$\mu(A) = \int_A \xi(\omega)\, \mathrm{d}\mathbb{P}(\omega), \quad \forall A \in \mathcal{G}. \tag{2.3.3}$$

Then $\mu(\cdot)$ is a vector-valued measure on $\mathcal{G}$ with a bounded total variation:

$$\|\mu\| \triangleq \int_\Omega |\xi(\omega)|\, \mathrm{d}\mathbb{P}(\omega) \equiv \mathbb{E}|\xi|,$$

and it is absolutely continuous with respect to $\mathbb{P}|_{\mathcal{G}}$, the restriction of $\mathbb{P}$ on $\mathcal{G}$. Thus, by the Radon-Nikodým theorem, there exists a unique $f \in L^1_{\mathcal{G}}(\Omega; \mathbb{R}^n)$ (called the Radon-Nikodým derivative of μ with respect to $\mathbb{P}|_{\mathcal{G}}$) such that

$$\mu(A) = \int_A f(\omega)\, \mathrm{d}\mathbb{P}(\omega), \quad \forall A \in \mathcal{G}.$$

Then (2.3.1) follows.

2.4 Stochastic Process

Given a measurable space $(\Omega, \mathcal{F})$, we introduce a monotone family of sub-σ-fields $\mathcal{F}_t \subseteq \mathcal{F}$, $t \in [0, \infty)$. Here, by monotonicity we mean

$$\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2}, \; 0 \leqslant t_1 \leqslant t_2.$$

Such a family $\{\mathcal{F}_t\}_{t \geqslant 0}$ is called a filtration on $(\Omega, \mathcal{F})$. Set $\mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$ for any $t \in [0, \infty)$, and $\mathcal{F}_{t-} := \bigcup_{s<t} \mathcal{F}_s$ for any $t \in [0, \infty)$. If $\mathcal{F}_{t+} = \mathcal{F}_t$ (resp. $\mathcal{F}_{t-} = \mathcal{F}_t$), we say that $\{\mathcal{F}_t\}_{t \geqslant 0}$ is right (resp. left) continuous. Denoted by $\mathbb{F} = \{\mathcal{F}_t\}_{t \geqslant 0}$, we call $(\Omega, \mathcal{F}, \mathbb{F})$ a filtered measurable space. When $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is called a filtered probability space.

Definition 2.5. *We say that a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfies the usual condition if $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets in $\mathcal{F}$, and $\mathbb{F}$ is right continuous.*

From now on, we always assume that the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is given and satisfies the usual condition.

Definition 2.6. *A map $X : [0, T] \times \Omega \to \mathbb{R}^n$ is called a stochastic process if for each $t \in [0, T], \omega \mapsto X(t, \omega)$ is a random variable. In this case, for any $\omega \in \Omega$, the map $t \mapsto X(t, \omega)$ is called a sample path. Further,*

(i) $X(\cdot)$ is said to be measurable if $(t, \omega) \mapsto X(t, \omega)$ is $\mathcal{B}[0, T] \otimes \mathcal{F}$-measurable;

(ii) $X(\cdot)$ is said to be $\mathbb{F}$-adapted if for all $t \in [0, T], \omega \mapsto X(t, \omega)$ is $\mathcal{F}_t$-measurable;

(iii) $X(\cdot)$ is said to be $\mathbb{F}$-progressively measurable if for all $t \in [0, T]$, $(t, \omega) \mapsto X(t, \omega)$ is $\mathcal{B}[0, t] \otimes \mathcal{F}_t$-measurable;

(iv) $X(\cdot)$ is said to be right (resp. left) continuous on $[0, T]$ if there exists a $\mathbb{P}$-null set N such that for any $\omega \in \Omega \backslash N, t \mapsto X(t, \omega)$ is right (resp. left) continuous.

Let us finally recall an extremely important stochastic process, the Brownian motion.

Definition 2.7. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space. An $\mathbb{F}$-adapted $\mathbb{R}^n$-valued continuous process $W(\cdot)$ is called an n-dimensional standard Brownian motion if $\mathbb{P}(W(0) = 0) = 1$, for any $0 \leqslant s \leqslant t$,*

$$\mathbb{E}\left[W(t) - W(s) \mid \mathcal{F}_s\right] = 0 \quad a.s.,$$

and

$$W(t) - W(s) \sim N(0, (t-s)I_n),$$

where $N(0, (t-s)I_n)$ denotes the standard n-dimensional normal distribution.

2.5 Stopping Time

In this section we introduce a special class of random variables, which plays an important role in stochastic analysis.

Definition 2.8. *A map $\tau : \Omega \to [0, \infty]$ is called an $\mathbb{F}$-stopping time if*

$$(\tau \leqslant t) = \{\omega \in \Omega \mid \tau(\omega) \leqslant t\} \in \mathcal{F}_t, \qquad 0 \leqslant t < \infty.$$

For an $\mathbb{F}$-stopping time τ, define the σ-field generated by it as

$$\mathcal{F}_\tau = \left\{A \in \mathcal{F} \mid A \cap (\tau \leqslant t) \in \mathcal{F}_t,\ t \geqslant 0\right\}.$$

2.6 Martingale

In this section we will briefly recall some results on martingales which form a special class of stochastic processes.

Definition 2.9. *A real-valued process $X(t)$ is called an $\{\mathcal{F}_t\}_{t\geqslant 0^-}$-martingale (resp. submartingale, supermartingale) if it is $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted and $\mathbb{E}[X(t)]$ exists for any $t \geqslant 0$, and satisfies*

$$\mathbb{E}\left(X(t) \mid \mathcal{F}_s\right) = X(s) \quad (\textit{resp.} \quad \geqslant, \ \leqslant), \quad \mathbb{P}\text{-}a.s., \ \forall\, 0 \leqslant s \leqslant t.$$

2.7 Backward Stochastic Differential Equation

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, and W be a one-dimensional Brownian motion on it. From the knowledge we have acquired earlier, we know that Brownian motion and the stochastic integrals of elements in $L^2_{\mathcal{F}}(0,T;\mathbb{R})$ are martingales. Conversely, the martingale representation theorem (Theorem 5.7 in Yong and Zhou (1999)) states that an $\mathcal{F}^W$ martingale can be represented as a stochastic integral with respect to the Brownian motion W.

Given $\xi \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R})$, it naturally induces a martingale $Y(t) = \mathbb{E}\left[\xi \mid \mathcal{F}_t\right]$. By the martingale representation theorem, there exists a unique process $Z \in L^2_{\mathcal{F}}(0,T;\mathbb{R})$ such that the following (forward) stochastic differential equation (SDE) holds:

$$Y(t) = \xi - \int_t^T Z(s)\,\mathrm{d}W(s), \quad \text{or equivalently,} \quad \mathrm{d}Y(t) = Z(t)\,\mathrm{d}W(t), \ Y(T) = \xi.$$

This is a linear SDE with terminal condition $Y(T) = \xi$. Since its terminal value is known, it is referred to as a backward SDE (BSDE). We emphasize that the solution to a BSDE is a pair of $\mathbb{F}$-measurable processes (Y, Z). Upon closer inspection, it can be observed that the component Z is essentially the derivative of Y with respect to W, and thus is uniquely determined by Y (and W). Moreover, the presence of Z is crucial to ensure the $\mathbb{F}$-measurability of Y.

If a forward SDE is a nonlinear extension of the stochastic integration, then a backward SDE is a nonlinear version of the martingale representation theorem. In fact, both the results and the arguments for BSDEs are analogous to those for SDEs, combined with the martingale representation theorem.

Forward SDE focuses on how to understand a randomly occurring process, while backward SDE is primarily concerned with how to drive a system towards a desired goal in the presence of random disturbances.

The essence of the problem addressed by backward equations can be described as follows: the investor, based on the uncertain returns of tomorrow, can deterministically calculate the required investment today ($Y(t)$) and determine how to invest or make portfolio ($Z(t)$) in order to achieve the target for tomorrow (ξ). The "how to invest" aspect is not captured by forward SDEs, which highlights the fundamental difference between BSDEs and forward SDEs. Specifically, what form should a BSDE take? What role does the "how to invest" component play in BSDEs? This can be understood through the concept of control.

2.7.1 Linear Backward Stochastic Differential Equation

We consider the following linear SDE:

$$\begin{cases} \mathrm{d}Y(t) = -[A(t)Y(t) + \sum_{j=1}^{m} B_j(t)' Z_j(t) + f(t)]\,\mathrm{d}t + Z(t)'\,\mathrm{d}W(t), \\ Y(T) = \xi, \end{cases} \tag{2.7.1}$$

where $A(\cdot) \in L^{\infty}_{\mathcal{F}}(0,T;\mathbb{R})$, $B_1(\cdot),\ldots,B_m(\cdot) \in L^{\infty}_{\mathcal{F}}\left(0,T;\mathbb{R}^d\right)$, $f(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R})$, $\xi \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R})$, and W is d-dimensional. Our goal is to find $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted processes $Y(\cdot)$ and $Z(\cdot)$, valued in $\mathbb{R}$ and $\mathbb{R}^d$, respectively, such that (2.7.1) is satisfied. We emphasize the key issues here: The process $(Y(\cdot), Z(\cdot))$ that we are looking for is required to be forward adapted, whereas the process $Y(\cdot)$ is specified at the terminal time $t = T$ (thus the equation should be solved backwardly). Due to this, we call (2.7.1) a linear BSDE.

Definition 2.10. *A pair* $(Y(\cdot), Z(\cdot)) \in L^2_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R})) \times L^{2,loc}_{\mathcal{F}}(0,T;\ \mathbb{R}^d)$ *is called an adapted solution of* (2.7.1) *if the following holds:*

$$Y(t) = \xi - \int_t^T \left\{ A(s)Y(s) + \sum_{j=1}^m B_j'(s)Z_j(s) + f(s) \right\} \mathrm{d}s - \int_t^T Z(s)' \,\mathrm{d}W(s),$$

$$\forall\ t \in [0,T],\ \mathbb{P}\text{-}a.s. \tag{2.7.2}$$

Equation (2.7.1) *is said to have a unique adapted solution if for any two adapted solutions* $(Y(\cdot), Z(\cdot))$ *and* $(\tilde{Y}(\cdot), \tilde{Z}(\cdot))$*, it must hold that ,*

$$\mathbb{P}\{Y(t) = \tilde{Y}(t), \forall\ t \in [0,T] \text{ and } Z(t) = \tilde{Z}(t), a.e.\ t \in [0,T]\} = 1.$$

Theorem 2.1. *Let* $A(\cdot) \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}),\ B_1(\cdot), \ldots, B_m(\cdot) \in L^\infty_{\mathcal{F}}\left(0,T;\mathbb{R}^d\right)$*. Then there exists a constant* $K > 0$ *such that, for any* $f \in L^2_{\mathcal{F}}(0,T;\mathbb{R})$ *and* $\xi \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R})$*, BSDE* (2.7.2) *admits a unique adapted solution* $(Y(\cdot), Z(\cdot)) \in L^2_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R})) \times L^2_{\mathcal{F}}\left(0,T;\mathbb{R}^d\right)$*, which satisfies*

$$\mathbb{E} \sup_{t\in[0,T]} |Y(t)|^2 + \sum_{j=1}^m \mathbb{E} \int_0^T |Z_j(t)|^2 \,\mathrm{d}t \leqslant K \left\{ \mathbb{E}|\xi|^2 + \mathbb{E} \int_0^T |f(t)|^2 \,\mathrm{d}t \right\}.$$

The proof is given in Theorem 2.2, Chapter 7 in Yong and Zhou (1999).

Chapter 3

Constrained Stochastic LQ Control and Stopping with Random Coefficients in Non-Markovian Regime Switching Homogeneous System

In this chapter, we construct a general stochastic LQ control problem under a random coefficients model with non-Markovian regime switching, where the control pair consists of a classical control variable constrained in a cone and a controlled stopping time τ. The randomness comes from two aspects: the Brownian motion driving the asset price dynamics and the Markov chain standing for the regime-switching. Moreover, the control weighting matrix in the cost functional is allowed to be possibly singular. Economically, the extra stopping time τ means that one should make decisions to switch the system in the economic sphere to maximize the profitability (or minimize the cost). Mathematically, it allows us to obtain a closed form solution for both, the value function and the optimal trading strategy via reflected extended stochastic Riccati equations and extended stochastic Riccati equations.

As the first contribution, we analyze a new optimal control and stopping problem with non-Markovian regime switching where the control in constrained in a cone. We divide the whole problem into two parts termed before τ and after τ and solve each part respectively. On $[\tau, T]$, via the completion of squares method, the optimal control and value function could be expressed by the solutions to two fully coupled BSDE systems (3.2.3) and (3.2.4) (see Theorem 3.4) termed ESREs. Building on the method of Hu et al. (2022b), we establish the solvability of our problem, prove its well-posedness using BMO martingales and the John-Nirenberg inequality, and derive explicit optimal feedback controls.

Then, we establish the relationship between the control and stopping problem on $[0, \tau]$ and obtain RESREs, that is, reflected BSDE (RBSDE) system (3.2.1) and (3.2.2) (see Theorem 3.5). Both RBSDE systems (3.2.1) and (3.2.2) are new. They are introduced for the first time for the characterization of optimal state feedback control and optimal cost value. In particular, we show that when there is no stopping time, our representation of optimal state feedback control and optimal cost value will recover the Riccati equations representation appearing in Hu et al. (2022b).

Our second contribution is about the solvability of RBSDE systems (3.2.1) and (3.2.2). In proving the existence of a unique solution to RBSDE system, two challenges need to be overcome. First, the RESREs are, indeed a family of highly nonlinear RBSDEs, for which the usual assumptions (such as Lipschitz and quadratic growth conditions) are not satisfied. To the best of our knowledge, no existing results could be directly used to solve it. We first establish a multi-dimensional comparison theorem for RBSDEs. With its help and the truncation technique, we obtain the existence for (3.2.1) and (3.2.2). Utilizing the bounded solution to an auxiliary ODE as a universal bound to control all the solution components of RBSDE system is a key idea in this case. Second, most of the aforementioned papers used the Feynman-Kac type representation of Riccati equations to prove the uniqueness. Instead of using such an indirect method, in this chapter, we provide a direct approach using log transformation and the John-Nirenberg inequality. Finally, we obtain the optimal state feedback control and optimal cost value similar to the classical unconstrained-control problem or the problem without regime switching by four systems of ESREs and RESREs.

The remainder of this chapter is organized as follows. In Section 3.1, we formulate a mixed stochastic LQ problem with regime switching, random coefficients, and cone constraint; the control pair consists of a classical control and a stopping time. Section 3.2 concerns the global solvability of four systems of ESREs and RESREs, including the existence and uniqueness for the standard and the singular cases. Section 3.1.2 gives the solution to the mixed LQ problem. Finally, Section 3.4 concludes this chapter.

3.1 Model Setup and Problem Formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed complete probability space on which are defined a standard n-dimensional Brownian motion $W(t) = (W_1(t), \ldots, W_n(t))'$ and a continuous-time stationary Markov chain α_t valued in a finite state space $\mathcal{M} = \{1, 2, \ldots, \ell\}$ where $\ell > 1$. The Markov chain has a generator $Q = (q_{ij})_{\ell \times \ell}$ with $q_{ij} \geqslant 0$ for $i \neq j$ and $\sum_{j=1}^{\ell} q_{ij} = 0$ for every $i \in \mathcal{M}$. We assume $W(t)$ and α_t are independent processes. Define the filtrations $\mathcal{F}_t = \sigma\{W(s), \alpha_s : 0 \leqslant s \leqslant t\} \bigvee \mathcal{N}$ and $\mathcal{F}_t^W = \sigma\{W(s) : 0 \leqslant s \leqslant t\} \bigvee \mathcal{N}$, where $\mathcal{N}$ is the totality of all the $\mathbb{P}$-null sets of $\mathcal{F}$.

3.1.1 Cone Constraint

In optimal control theory, a cone constraint requires the control variable $u(t)$ to lie within a closed cone $\Gamma \subseteq \mathbb{R}^m$ at all times, meaning $u(t)$ is positive scale-invariant ($\lambda u(t) \in \Gamma$ for all $\lambda \geqslant 0$) and satisfies specific directional restrictions. In finance, such constraints model realistic trading limitations: for example, $\Gamma = \mathbb{R}_+^m$ enforces no-shorting rules (prohibiting negative asset positions), while other cones can restrict leverage or mandate sector-specific exposures. These constraints fundamentally alter optimal policies and require specialized mathematical solutions. Early work by Hu and Zhou (2005) established the stochastic LQ framework with cone constraints under random parameters, proving optimal strategies exist by solving stochastic Riccati equations. Li et al. (2002) tackled no-shorting constraints in mean-variance portfolio selection using dynamic programming, showing constraints significantly reduce achievable efficient frontiers. The challenge intensified under regime-switching models, where Czichowsky and Schweizer (2013) developed general duality methods for cone-constrained Markowitz problems in semimartingale markets. A major advancement came from Hu et al. (2022b), who solved cone-constrained LQ control with both regime-switching and random coefficients. Crucially, they demonstrated that cone constraints lead to separable optimal strategies depending on the relative state's sign (e.g., long vs. short wealth positions) and applied this to obtain closed-form efficient portfolios and frontiers in constrained markets with stochastic regimes. Their work underscores that cone constraints, while mathematically challenging due to nonlinearities and non-convexities, are essential for realistic portfolio construction and risk management under market frictions.

3.1.2 Homogeneous System

We now introduce the following scalar-valued homogeneous linear stochastic differential equation, controlled by a pair $(u(\cdot), \tau)$:

$$
\begin{cases}
\mathrm{d}X_1(t) = \big[A_1(t,\alpha_t)X_1(t) + B_1(t,\alpha_t)'u(t)\big]\,\mathrm{d}t \\
\qquad\qquad\qquad + \big[C_1(t,\alpha_t)X_1(t) + D_1(t,\alpha_t)u(t)\big]'\,\mathrm{d}W(t), \;\; t \in [0,\tau), \\
\mathrm{d}X_2(t) = \big[A_2(t,\alpha_t)X_2(t) + B_2(t,\alpha_t)'u(t)\big]\,\mathrm{d}t \\
\qquad\qquad\qquad + \big[C_2(t,\alpha_t)X_2(t) + D_2(t,\alpha_t)u(t)\big]'\,\mathrm{d}W(t), \;\; t \in [\tau,T], \\
X_2(\tau) = K(\tau)X_1(\tau - 0), \\
X_1(0) = x \in \mathbb{R}, \;\; \alpha_0 = i_0 \in \mathcal{M}.
\end{cases}
\tag{3.1.1}
$$

Here $(u(\cdot),\tau)$ is a pair of control, where $u(\cdot)$ takes values from $\mathbb{R}^m$ and τ is an $\{\mathcal{F}_t\}_{t\geqslant 0}$-stopping time. We use $X(t) = X_1(t)\,\mathbb{1}_{t<\tau} + X_2(t)\,\mathbb{1}_{t\geqslant\tau}$ to denote the whole state process. We see from (3.1.1) that the state process X starts to run from an initial position (x, i_0) and switches to another process at the time τ with an initial condition depending on the previous one's final value.

In this chapter, let $\Gamma \subseteq \mathbb{R}^m$ be a non-empty closed cone, which stands for the control constraint set. Define

$$\mathcal{U}[0,T] := \Big\{ u(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m) \;\Big|\; u(\cdot) \in \Gamma \Big\},$$

$$\mathcal{T}[0,T] := \Big\{ \tau : \Omega \to [0,T] \;\big|\; \tau \text{ is an } \{\mathcal{F}_t\}_{t\geqslant 0}\text{-stopping time} \Big\}.$$

We refer to $(u(\cdot),\tau)$ as an admissible control pair if $(u(\cdot),\tau) \in \mathcal{U}[0,T] \times \mathcal{T}[0,T]$. The set of admissible control pairs is denoted by $\mathscr{A}$. If SDE (3.1.1) admits a unique strong solution $X(\cdot)$ under an admissible control pair $(u(\cdot),\tau)$, we call $(X(\cdot), u(\cdot), \tau)$ an admissible triple for (3.1.1).

Let us now state our mixed stochastic linear quadratic optimal control (MSLQ) problem as follows: Given any $(x, i_0) \in \mathbb{R} \times \mathcal{M}$, our goal is to

$$\begin{cases} \text{Minimize} \quad J(x, i_0, u(\cdot), \tau) \\[2ex] \text{subject to} \quad (\tau, u(\cdot)) \in \mathscr{A},\ (X(\cdot), u(\cdot), \tau) \text{ follows (3.1.1)}, \end{cases} \tag{3.1.2}$$

where the cost functional is given as the following quadratic form

$$\begin{aligned} &J(x, i_0, u(\cdot), \tau) \\ &= \mathbb{E}\Big\{ \int_0^\tau \big[Q_1(t,\alpha_t) X_1(t)^2 + u(t)' R_1(t,\alpha_t) u(t) \big]\,\mathrm{d}t + G_1(\tau,\alpha_\tau) X_1(\tau)^2 \\ &\qquad + \int_\tau^T \big[Q_2(t,\alpha_t) X_2(t)^2 + u(t)' R_2(t,\alpha_t) u(t) \big]\,\mathrm{d}t + G_2(\alpha_T) X_2(T)^2 \Big\}. \end{aligned} \tag{3.1.3}$$

The value function of the problem is defined as

$$V(x, i_0) = \inf_{(u(\cdot),\tau)\in\mathscr{A}} J(x, i_0, u(\cdot), \tau). \tag{3.1.4}$$

Problem (3.1.2) is said to be finite at $(x, i_0) \in \mathbb{R} \times \mathcal{M}$, if $V(x, i_0) > -\infty$; and to be solvable, if there exists a pair control $(\bar{u}(\cdot), \bar{\tau}) \in \mathscr{A}$ such that

$$J(x, i_0, \bar{u}(\cdot), \bar{\tau}) = V(x, i_0) > -\infty,$$

in which case, $(\bar{u}(\cdot), \bar{\tau})$ is called an optimal control pair for problem (3.1.2) at $(x, i_0) \in \mathbb{R} \times \mathcal{M}$.

Throughout this chapter, we put the following assumptions on the coefficients.

Assumption 3.1. *For all $i \in \mathcal{M}$,*

$$\begin{cases} A_1(\cdot, i), A_2(\cdot, i) \in L^\infty_{\mathcal{F}W}(0, T; \mathbb{R}), \\ B_1(\cdot, i), B_2(\cdot, i) \in L^\infty_{\mathcal{F}W}(0, T; \mathbb{R}^m), \\ C_1(\cdot, i), C_2(\cdot, i) \in L^\infty_{\mathcal{F}W}(0, T; \mathbb{R}^n), \\ D_1(\cdot, i), D_2(\cdot, i) \in L^\infty_{\mathcal{F}W}(0, T; \mathbb{R}^{n\times m}), \\ Q_1(\cdot, i), Q_2(\cdot, i) \in L^\infty_{\mathcal{F}W}(0, T; \mathbb{R}), \\ R_1(\cdot, i), R_2(\cdot, i) \in L^\infty_{\mathcal{F}W}(0, T; \mathbb{S}^m), \\ G_1(\cdot, i) \in L^\infty_{\mathcal{F}W}(0, T; \mathbb{R}), G_2(i) \in L^\infty_{\mathcal{F}W}(\Omega; \mathbb{R}), \\ \mathbb{G}_1(\cdot, i) \in L^\infty_{\mathcal{F}W}(0, T; \mathbb{R}), \mathbb{G}_2(\cdot, i) \in L^\infty_{\mathcal{F}W}(0, T; \mathbb{R}), \\ K(\cdot) \in L^\infty_{\mathcal{F}W}(\Omega; \mathbb{R}). \end{cases}$$

Assumption 3.2. *There exists a constant $\delta > 0$ such that at least one of the following cases holds.*

(i) ***Standard case.*** *For all $t \in [0, T]$:*

$$Q_1(i) \geqslant 0,\ \ R_1(i) \geqslant \delta I_m,\ \ \text{and}\ \mathbb{G}_1(T, i) \geqslant 0,$$

$$Q_2(i) \geqslant 0,\ \ R_2(i) \geqslant \delta I_m,\ \ \text{and}\ G_2(i) \geqslant 0.$$

(ii) ***Singular case.*** *For all $t \in [0, T]$:*

$$Q_1(i) \geqslant 0,\ \ R_1(i) \geqslant 0,\ \ D_1(i)^\top D_1(i) \geqslant \delta I_m, \text{and}\ \mathbb{G}_1(T, i) \geqslant \delta,$$

$$Q_2(i) \geqslant 0,\ \ R_2(i) \geqslant 0,\ \ D_2(i)^\top D_2(i) \geqslant \delta I_m, \text{and}\ G_2(i) \geqslant \delta,$$

where I_m denotes the m-dimensional identity matrix.

Under Assumptions 3.1 and 3.2, SDE (3.1.1) admits a unique solution $X(\cdot) \in L^2_{\mathcal{F}}(\Omega; C(0,T;\mathbb{R}))$ for any $(u(\cdot),\tau) \in \mathscr{A}$ by standard SDE theory, and the cost functional (3.3.8) is well-defined, so problem (3.1.2) is well-posed.

The rest of this chapter is devoted to the study of problem (3.1.2).

3.2 Extended Stochastic Riccati Equations

Problem (3.1.2) is also termed of control-stopping type, unlike classical LQ problem, the results and approaches of previous work no longer allow for the existence of a solution to a new kind of extended stochastic Riccati equation. To tackle it, we need first to study two related ℓ-dimensional BSDEs and RBSDEs.

For $(t,P,\Lambda,i) \in [0,T] \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{M}$, define functions

$$\begin{aligned}\mathscr{H}^b_{\pm}(t,P,\Lambda,i) = \inf_{v\in\Gamma} \big[& v'(PD_1(t,i)'D_1(t,i) + R_1(t,i))v \\ & \pm 2v'(PB_1(t,i) + PD_1(t,i)'C_1(t,i) + D_1(t,i)'\Lambda)\big], \\ \mathscr{H}^a_{\pm}(t,P,\Lambda,i) = \inf_{v\in\Gamma} \big[& v'(PD_2(t,i)'D_2(t,i) + R_2(t,i))v \\ & \pm 2v'(PB_2(t,i) + PD_2(t,i)'C_2(t,i) + D_2(t,i)'\Lambda)\big].\end{aligned}$$

Because $0 \in \Gamma$, these functions take non-positive real values or $-\infty$. If $PD_\gamma(t,i)'D_\gamma(t,i) + R_\gamma(t,i) > 0$ with $\gamma = 1,2$, then they take non-positive real values. Here, the superscript "b" (resp. "a") stands for "before jump" (resp. "after jump").

For $(t,P,\Lambda,i) \in [0,T] \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{M}$, we also define functions

$$\mathscr{F}^b(t,P,\Lambda,i) = 2A_1(t,i)P + C_1(t,i)'C_1(t,i)P + 2C_1(t,i)'\Lambda + Q_1(t,i),$$

$$\mathscr{F}^a(t,P,\Lambda,i) = 2A_2(t,i)P + C_2(t,i)'C_2(t,i)P + 2C_2(t,i)'\Lambda + Q_2(t,i).$$

Remark 3.1. *Suppose Assumption 3.1 holds and $\Lambda \in \mathbb{R}^n$ and $P \in \mathbb{R}$. If $P \geqslant 0$ and*

$R_1 \geqslant \delta I_m$ for some constant $\delta > 0$, then we have the following estimate

$$\begin{aligned}\mathscr{H}^b_\pm(P,\Lambda) &= \inf_{v\in\Gamma}\left[v'(PD_1'D_1+R_1)v \pm 2v'(PB_1+PD_1'C_1+D_1'\Lambda)\right]\\ &\geqslant \inf_{v\in\Gamma}\left[\delta|v|^2 \pm 2c(P+|\Lambda|)|v|\right]\\ &\geqslant -\frac{c^2(P+|\Lambda|)^2}{\delta}.\end{aligned}$$

If $P > 0$, $R_1 \geqslant 0$ and $D_1'D_1 \geqslant \delta I_m$ for some constant $\delta > 0$, then we have

$$\begin{aligned}\mathscr{H}^b_\pm(P,\Lambda) &= \inf_{v\in\Gamma}\left[v'(PD_1'D_1+R_1)v \pm 2v'(PB_1+PD_1'C_1+D_1'\Lambda)\right]\\ &\geqslant \inf_{v\in\Gamma}\left[\delta P|v|^2 \pm 2c(P+|\Lambda|)|v|\right]\\ &\geqslant -\frac{c^2(P+|\Lambda|)^2}{\delta P}.\end{aligned}$$

As $0 \in \Gamma$, we have the trivial estimate $\mathscr{H}^b_\pm(P,\Lambda) \leqslant 0$. Therefore $|\mathscr{H}^b_\pm(P,\Lambda)| \leqslant \frac{c^2(P+|\Lambda|)^2}{\delta}$ in the first case, and $|\mathscr{H}^b_\pm(P,\Lambda)| \leqslant \frac{c^2(P+|\Lambda|)^2}{\delta P}$ in the second case. This reveals the quadratic growth nature of $\mathscr{H}^b_\pm$ w.r.t. Λ. However, we see $\mathscr{H}^b_\pm$ is not locally bounded w.r.t. P. Similar properties hold for $\mathscr{H}^a_\pm$.

We introduce the following two ℓ-dimensional RBSDEs for which the solution is an adapted triple processes (P,Λ,μ):

$$\begin{cases}\mathrm{d}P(i) = -\Big[\mathscr{F}^b(P(i),\Lambda(i),i) + \mathscr{H}^b_+(P(i),\Lambda(i),i) + \sum\limits_{j=1}^{\ell} q_{ij}P(j)\Big]\,\mathrm{d}t\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \mathrm{d}\mu(i) + \Lambda(i)'\,\mathrm{d}W,\\ P(T,i) = \mathbb{G}_1(T,i), \quad \mu(0,i)=0, \quad P(i) \leqslant \mathbb{G}_1(i), \quad \int_0^T (\mathbb{G}_1(i)-P(i))\,\mathrm{d}\mu(i) = 0,\\ R_1(i) + P(i)D_1(i)'D_1(i) > 0 \ \text{ for all } i \in \mathcal{M};\end{cases} \tag{3.2.1}$$

and

$$\begin{cases}\mathrm{d}P(i) = -\Big[\mathscr{F}^b(P(i),\Lambda(i),i) + \mathscr{H}^b_-(P(i),\Lambda(i),i) + \sum\limits_{j=1}^{\ell} q_{ij}P(j)\Big]\,\mathrm{d}t\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \mathrm{d}\mu(i) + \Lambda(i)'\,\mathrm{d}W,\\ P(T,i) = \mathbb{G}_2(T,i), \quad \mu(0,i)=0, \quad P(i) \leqslant \mathbb{G}_2(i), \quad \int_0^T (\mathbb{G}_2(i)-P(i))\,\mathrm{d}\mu(i) = 0,\\ R_1(i) + P(i)D_1(i)'D_1(i) > 0 \ \text{ for all } i \in \mathcal{M},\end{cases} \tag{3.2.2}$$

where $\mathbb{G}_1(i)$, $\mathbb{G}_2(i)$ are given barrier processes.

For RBSDEs (3.2.1), the solution is subject to the point-wise constraint

$$P(i) \leqslant \mathbb{G}_1(i),$$

and the non-decreasing process $\mu(i)$ is used to push the main solution component $P(i)$ to satisfy the above constraint in a minimum energy way, i.e.,

$$\int_0^T (\mathbb{G}_1(i) - P(i))\, \mathrm{d}\mu(i) = 0.$$

Similar properties hold for (3.2.2).

RBSDEs (3.2.1) and (3.2.2) are referred to as the RESREs for the mixed control problem (3.1.2).

We also need two ℓ-dimensional BSDEs:

$$\begin{cases} \mathrm{d}P(i) = -\Big[\mathscr{F}^a(P(i),\Lambda(i),i) + \mathscr{H}_+^a(P(i),\Lambda(i),i) + \sum\limits_{j=1}^{\ell} q_{ij}P(j)\Big]\,\mathrm{d}t + \Lambda(i)'\,\mathrm{d}W, \\ P(T,i) = G_2(i), \quad R_2(i) + P(i)D_2(i)'D_2(i) > 0 \text{ for all } i \in \mathcal{M}; \end{cases} \tag{3.2.3}$$

and

$$\begin{cases} \mathrm{d}P(i) = -\Big[\mathscr{F}^a(P(i),\Lambda(i),i) + \mathscr{H}_-^a(P(i),\Lambda(i),i) + \sum\limits_{j=1}^{\ell} q_{ij}P(j)\Big]\,\mathrm{d}t + \Lambda(i)'\,\mathrm{d}W, \\ P(T,i) = G_2(i), \quad R_2(i) + P(i)D_2(i)'D_2(i) > 0 \text{ for all } i \in \mathcal{M}. \end{cases} \tag{3.2.4}$$

These two BSDEs are named as ESREs, which have been studied in Hu et al. (2022b).

Remark 3.2. *If Γ is symmetric, namely, $-v \in \Gamma$ whenever $v \in \Gamma$, then $\mathscr{H}_+^b(P,\Lambda,i) = \mathscr{H}_-^b(P,\Lambda,i)$, $\mathscr{H}_+^a(P,\Lambda,i) = \mathscr{H}_-^a(P,\Lambda,i)$. Therefore the RBSDEs (3.2.1) coincides with (3.2.2), (3.2.3) coincides with (3.2.4). In particular, if there is no control constraint, i.e. $\Gamma = \mathbb{R}^m$, then both $\mathscr{H}_+^b$ and $\mathscr{H}_-^b$ are equal to*

$$-[PB_1(i)' + (PC_1(i)+\Lambda)'D_1(i)](R_1(i) + PD_1(i)'D_1(i))^{-1}[PB_1(i) + D_1(i)'(PC_1(i)+\Lambda)],$$

and $\mathscr{H}_+^a$ and $\mathscr{H}_-^a$ are equal to

$$-[PB_2(i)' + (PC_2(i)+\Lambda)'D_2(i)](R_2(i) + PD_2(i)'D_2(i))^{-1}[PB_2(i) + D_2(i)'(PC_2(i)+\Lambda)].$$

Remark 3.3. (3.2.1), (3.2.2), (3.2.3) *and* (3.2.4) *are highly nonlinear multidimensional, they violate both the standard Lipschitz condition and the quadratic growth condition due to the structural behavior of their Hamiltonians* $\mathscr{H}_{\pm}^{a}$ *and* $\mathscr{H}_{\pm}^{b}$.

Specifically, in the singular case, $D'D \geqslant \delta I_m$ *for some* $\delta > 0$ *but* R *is degenerate, the Hamiltonian* $\mathscr{H}_{\pm}^{b}$ *exhibits the lower bound*

$$\mathscr{H}_{\pm}^{b}(P,\Lambda) \geqslant -\frac{c^2(P+|\Lambda|)^2}{\delta P},$$

for constants $c, \delta > 0$. *Since* $H_1 \leqslant 0$ *(as* $0 \in \Gamma$*), this implies*

$$\left|\mathscr{H}_{\pm}^{b}(P,\Lambda)\right| \leqslant \frac{c^2(P+|\Lambda|)^2}{\delta P}.$$

The term $\frac{(P+|\Lambda|)^2}{P}$ *introduces a singularity as* $P \to 0^+$. *Crucially, the component* $\frac{|\Lambda|^2}{P}$ *dominates when* P *is small and* $\Lambda \neq 0$, *which cannot be bounded by any quadratic function* $C\left(1+|P|^2+|\Lambda|^2\right)$. *This singular behavior is obvious when control constraints* Γ *restrict admissible controls to rays (e.g.,* $\Gamma = \{\lambda \mathbf{e}_1 \mid \lambda > 0\}$ *). For such* Γ, *the explicit form*

$$\mathscr{H}_{\pm}^{b}(P,\Lambda) = -\frac{\left((PB+PD'C+D'\Lambda)_1^-\right)^2}{(PD'D+R)_{11}},$$

further highlights the issue: if R *is degenerate,* $(PD'D+R)_{11} \sim P$, *leading to* $\left|\mathscr{H}_{\pm}^{b}\right| \sim \frac{|\Lambda|^2}{P}$. *Consequently, the driver of BSDEs* (3.2.1) *and* (3.2.2) *contain a term that violates the quadratic growth condition near* $P = 0$. *An identical analysis applies to* $\mathscr{H}_{\pm}^{a}$ *in BSDEs* (3.2.3) *and* (3.2.4). *This singularity prevents the application of standard BSDEs existence and uniqueness theorems, which require drivers to grow at most quadratically in* Λ. *The degeneracy of* R *and the geometry of* Γ *thus necessitate specialized analytical techniques beyond classical theory.*

Definition 3.1. *A vector process* $(P(i),\ \Lambda(i),\ \mu(i))_{i=1}^{\ell}$ *is called a solution to the* ℓ*-dimensional RBSDEs* (3.2.1) *if it satisfies* (3.2.1) *and* $(P(i),\ \Lambda(i),\ \mu(i)) \in L^{\infty}_{\mathcal{F}^W}(0,T;\mathbb{R})\times$

$L^2_{\mathcal{F}^W}(0,T;\mathbb{R}^n) \times \mathcal{A}_c(0,T;\mathbb{R})$ *for all* $i \in \mathcal{M}$. *The solution to* (3.2.2) *is defined similarly.*

Definition 3.2. *A vector process* $(P(i),\ \Lambda(i))_{i=1}^{\ell}$ *is called a solution to the* ℓ*-dimensional BSDEs* (3.2.3) *if it satisfies* (3.2.3) *and* $(P(i),\ \Lambda(i)) \in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}^W}(0,T;\mathbb{R}^n)$ *for all* $i \in \mathcal{M}$. *The solution to* (3.2.4) *is defined similarly.*

Usually one would seek the solutions to (3.2.1), (3.2.2), (3.2.3), (3.2.4) in the space $L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}^W}(0,T;\mathbb{R}^n)$ for all $i \in \mathcal{M}$. This space, however, is not precise enough in the proof of uniqueness.

In fact, the second part solutions Λ turns out to be in the class of martingales of bounded mean oscillation, briefly called *BMO martingales*. To give proper definitions of their solutions, here we recall some facts about BMO martingales; see Kazamaki (2006). The process $\int_0^\cdot \Lambda(s)'\,\mathrm{d}W(s)$ is a BMO martingale if and only if there exists a constant $c > 0$ such that

$$\mathbb{E}\left[\int_\tau^T |\Lambda(s)|^2\,\mathrm{d}s \,\middle|\, \mathcal{F}^W_\tau\right] \leqslant c,$$

for all $\{\mathcal{F}^W_t\}_{t\geqslant 0}$-stopping times $\tau \leqslant T$. The following space plays an important role in our argument

$$\begin{aligned} &L^{2,\,\mathrm{BMO}}_{\mathcal{F}^W}(0,T;\mathbb{R}^n) \\ &\qquad = \left\{\Lambda \in L^2_{\mathcal{F}^W}(0,T;\mathbb{R}^n) \,\middle|\, \int_0^\cdot \Lambda(s)'\,\mathrm{d}W(s) \text{ is a BMO martingale on } [0,T]\right\}. \end{aligned}$$

For $\Lambda \in L^{2,\,\mathrm{BMO}}_{\mathcal{F}^W}(0,T;\mathbb{R}^n)$, the Doléans-Dade stochastic exponential

$$\mathcal{E}\bigg(\int_0^\cdot \Lambda(s)'\,\mathrm{d}W(s)\bigg)$$

is a uniformly integrable martingale. Moreover, if $\int_0^\cdot \Lambda(s)'\,\mathrm{d}W(s)$ and $\int_0^\cdot Z(s)'\,\mathrm{d}W(s)$ are both BMO martingales, then

$$\widetilde{W}(\cdot) := W(\cdot) - \int_0^\cdot Z(s)\,\mathrm{d}s$$

is a standard Brownian motion under the probability measure $\widetilde{\mathbb{P}}$ defined by

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}_T} = \mathcal{E}\bigg(\int_0^T Z(s)'\,\mathrm{d}W(s)\bigg),$$

and $\int_0^\cdot \Lambda(s)'\,\mathrm{d}\widetilde{W}(s)$ is a BMO martingale.

3.2.1 Solutions to ESREs and RESREs: Existence and Uniqueness

In this subsection, we address ourselves to the solvability of (3.2.1) and (3.2.2). The solvability of (3.2.3) and (3.2.4) is similar to Theorems 3.5 and 3.6 in Hu et al. (2022b). There are several results on the solvability of stochastic Riccati equations or quadratic BSDE systems (see, e.g., Hu and Zhou (2005), Hu et al. (2022b), Kohlmann and Tang (2002), Tang (2003), Hu and Tang (2016), Chen and Luo (2025)).

RESREs (3.2.1) and (3.2.2) are both ℓ-dimensional RBSDE systems with highly nonlinear generator. Up to our knowledge, no existing results could be directly applied to RESREs (3.2.1) and (3.2.2) because they violate both the standard Lipschitz condition and the quadratic growth condition. In view of the fact that coefficients depend on regime switching, which leads to the components of systems (3.2.1) and (3.2.2) are interconnected through its generators. There is no existing literature on RESREs (3.2.1) and (3.2.2), although similar RBSDEs under regime switching with Lipchitz generators were solved by Hamadène and Zhang (2010).

3.2.2 Multidimensional Comparison Theorem for RBSDEs with Regime Switching

We now recall the principal results established in the seminal work of Hamadène and Zhang (2010). They showed the existence of a solution to the following m-dimensional RBSDE with oblique reflections:

$$\begin{cases} Y(t,i) = \xi(i) + \displaystyle\int_t^T f_i(s, Y(s,1), \dots, Y(s,i), \dots, Y(s,m), Z(s,i))\, \mathrm{d}s \\ \qquad\qquad + (\mu(T,i) - \mu(t,i)) - \displaystyle\int_t^T Z(s,i)\, \mathrm{d}W_s, \quad t \in [0,T], \\ Y(i) \geqslant \max\limits_{j \in A_i} h_{i,j}\left(t, Y(i)\right), \\ \displaystyle\int_0^T \left(Y(i) - \max_{j \in A_i} h_{i,j}\left(t, Y(i)\right) \right) \mathrm{d}\mu(i) = 0, \ i = 1, 2, \cdots, m, \end{cases} \tag{3.2.5}$$

if Assumption 3.3 holds. Here $\xi(i)$ are $\mathcal{F}_T$-measurable, the coefficients $f_i, h_{i,j}$ can depend on ω, and $A_i \subseteq \{1, \cdots, m\} - \{i\}$.

For simplicity, we write $f_i(s, y(1), \dots, y(i), \dots, y(m), z(i)) = f_i(s, y(i), y(-i), z(i))$, where $y(-i) = (y(1), \dots, y(i-1), y(i+1), \dots, y(m))$.

Assumption 3.3. *For any $i = 1, \cdots, m$, it holds that,*

1. $\xi(i) \in L^2_{\mathcal{F}^W}(\Omega; \mathbb{R})$;

2. $\mathbb{E}\left\{ \int_0^T \sup\limits_{y(-i)} |f_i(s, 0, y(-i), 0)|^2\, \mathrm{d}s \right\} < \infty$;

3. *$f_i(t, y(i), y(-i), z)$ is uniformly Lipschitz continuous in $(y(i), z)$ and is continuous in $y(j)$ for any $j \neq i$; and $h_{i,j}(t, y)$ is continuous in (t, y) for $j \in A_i$;*

4. *$f_i(t, y(i), y(-i), z)$ is increasing in y_j for $j \neq i$, and $h_{i,j}(t, y)$ is increasing in y for $j \in A_i$;*

5. *For $j \in A_i, h_{i,j}(t, y) \leqslant y$. Moreover, there is no sequence $i_2 \in A_{i_1}, \cdots, i_k \in A_{i_{k-1}}, j_1 \in A_{i_k}$, and $(y_1, \cdots, y_k)$ such that*

   $y_1 \triangleq h_{i_1,i_2}\left(t, y_2\right), y_2 \triangleq h_{i_2,i_3}\left(t, y_3\right), \cdots, y_{k-1} \triangleq h_{i_{k-1},i_k}\left(t, y_k\right), y_k \triangleq h_{i_k,i_1}\left(t, y_1\right)$;

6. *For any* $i = 1, \ldots, m$, $\xi(i) \geqslant \max_{j \in A_i} h_{i,j}(T, \xi(i))$.

Lemma 3.1 (Hamadène and Zhang (2010)). *Assume Assumption 3.3 holds, then RBSDE (3.2.5) has a solution* $(Y(i), Z(i), \mu(i)) \in \mathcal{S}_{\mathcal{F}^W}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}^W}(0,T;\mathbb{R}^n) \times \mathcal{A}_c(0,T;\mathbb{R})$.

Specialization to our framework: For the control problem considered in this chapter, we focus on a specific case of RBSDE (3.2.5), where:

1. The state-dependent obstacle reduces to a constant barrier: $h_{i,j}(t,y) \equiv L(i)$,
2. We put an upper reflection constraint instead of lower reflection on the solution.

This leads to the solvability of the following specialized RBSDE system:

$$\begin{cases} Y(t,i) = \xi(i) + \int_t^T f(s, Y(s,1), \ldots, Y(s,i), \ldots, Y(s,\ell), Z(s,i), i)\,\mathrm{d}s \\ \qquad\qquad -(\mu(T,i) - \mu(t,i)) - \int_t^T Z(s,i)\,\mathrm{d}W, t \in [0,T], \\ Y(i) \leqslant L(i), \quad \int_0^T (L(i) - Y(i))\,\mathrm{d}\mu(i) = 0, \text{ for all } i \in \mathcal{M}. \end{cases} \tag{3.2.6}$$

This specialized form (3.2.6) preserves the essential mathematical structure of (3.2.5) while precisely accommodating the boundary conditions required for our control problem.

Multidimensional comparison theorem for RBSDEs without regime switching can be found in Wu and Xiao (2010). We now provide a similar comparison theorem for RBSDEs (3.2.6) under regime switching, which will be used frequently in the study of RBSDEs (3.2.1) and (3.2.2).

For simplicity, we denote $Y(s,-i) = (Y(s,1), \ldots, Y(s,i-1), Y(s,i+1), \ldots, Y(s,\ell))$, and similarly for other vectors.

Theorem 3.1. *Suppose* $(Y(i),\ Z(i),\ \mu(i))_{i=1}^{\ell}$ *and* $(\overline{Y}(i),\ \overline{Z}(i),\ \bar{\mu}(i))_{i=1}^{\ell}$ *satisfy two* ℓ*-dimensional RBSDEs (3.2.6) with data* (ξ, f, L) *and* $(\overline{\xi}, \overline{f}, \overline{L})$ *respectively. Also suppose that, for all* $i \in \mathcal{M}$, $s \in [0,T]$, *1.* $\xi(i),\ \overline{\xi}(i) \in L^2_{\mathcal{F}^W}(\Omega;\mathbb{R})$*, and* $\xi(i) \leqslant \overline{\xi}(i)$*;*

2. there exists a constant $c > 0$ *such that*

$$|f(s,y,z,i) - f(s,\overline{y},\overline{z},i)| \leqslant c(|y-\overline{y}| + |z-\overline{z}|),$$

for any $z, \overline{z} \in \mathbb{R}^n$*,* $y = (y(i), y(-i))$*,* $\overline{y} = (\overline{y}(i), \overline{y}(-i)) \in \mathbb{R}^\ell$*;*

3. $f(s,y,z,i)$ *is nondecreasing in* $y(j)$*, for every* $i \neq j \in \mathcal{M}$*;*

4. $f(s,\overline{Y}(s,i),\overline{Y}(s,-i),\overline{Z}(s,i),i) \leqslant \overline{f}(s,\overline{Y}(s,i),\overline{Y}(s,-i),\overline{Z}(s,i),i)$*;*

5. $L(i),\ \overline{L}(i) \in L^\infty_{\mathcal{F}^W}(\Omega;\mathbb{R})$*, and* $L(s,i) \leqslant \bar{L}(s,i)$*.*

Then $Y(t,i) \leqslant \overline{Y}(t,i)$ *for all* $t \in [0,T]$ *and* $i \in \mathcal{M}$*.*

Proof. The proof is given in section 3.5.1. □

We emphasis that the above lemma requires the global Lipschitz condition of f in (y,z), which is not satisfied in some cases in our below discussion.

3.2.3 Standard Case

We now prove the existence and uniqueness of the solution to RBSDEs (3.2.1). That for (3.2.2) are similar, so we omit its proof. We will treat two cases separately: (1) standard case, in which $R_1(i)$ is (uniformly) positive definite; (2) singular case, in which $R_1(i)$ is positive semidefinite but $\mathbb{G}_1(T,i)$ and $D_1(i)'D_1(i)$ are (uniformly) positive definite. Here "singular" means that the control weight matrix $R_1(i)$ in the cost functional (3.3.8) could be probably a singular matrix.

To derive the existence of a solution to RESREs (3.2.1), we first apply a truncation technique and derive a priori estimates for its solution(s), and subsequently show that the truncation constants coincide with the constants appearing in the a priori estimates. For this, we make an extensive use of the multidimensional comparison theorem (Theorem 3.1) for RBSDE systems. An essential idea herein is to use the bounded solution to an auxiliary ODE (not a system!) as a universal bound to control all the solution components of the RBSDE system.

Rather than using the Feynman-Kac type representation to prove the uniqueness of RESREs, we apply a direct approach involved log transformation and the John-Nirenberg inequality.

Theorem 3.2 (Standard case). *Assume the standard case in Assumption 3.2 holds. Then RBSDEs (3.2.1) admits a unique solution* $(P_1^b(i),\ \Lambda_1^b(i),\ \mu_1(i))_{i=1}^{\ell}$ *such that* $P_1^b(i) \geqslant 0$*, for all* $i \in \mathcal{M}$.

Proof. The proof is given in section 3.5.2 . □

3.2.4 Singular Case

Theorem 3.3 (Singular case). *Assume the singular case in Assumption 3.2 holds. Then RBSDEs (3.2.1) admits a unique solution such that* $P_1^b(i) \geqslant c$ *for some constant* $c > 0$*, for all* $i \in \mathcal{M}$.

Proof. The proof is given in section 3.5.3. □

3.3 Solution to the MSLQ Problem (3.1.2)

3.3.1 Solution to the Post-Switch LQ Problem (3.3.1)

For any $(x, \tau) \in \mathbb{R} \times \mathcal{T}[0, T]$, recall the controlled linear system (3.1.1) and consider the post-switch linear quadratic problem:

For given $(\tau, X_2(\tau), \alpha_\tau) \in \mathcal{T}[0, T] \times L^2_{\mathcal{F}}(\Omega; C(0, T; \mathbb{R})) \times \mathcal{M}$,

$$\begin{cases} \text{Minimize} \quad & J^a(\tau, X_2(\tau), \alpha_\tau, u(\cdot)) \\ \text{subject to} \quad & (\tau, u(\cdot)) \in \mathscr{A}, (X_2(\cdot), u(\cdot)) \text{ follows (3.1.1)}, \end{cases} \tag{3.3.1}$$

where the quadratic cost functional

$$
\begin{aligned}
&J^a(\tau, X_2(\tau), \alpha_\tau, u(\cdot)) \\
&= \mathbb{E}\Big\{ \int_\tau^T \big[Q_2(t,\alpha_t)X_2(t)^2 + u(t)'R_2(t,\alpha_t)u(t)\big]\,\mathrm{d}t + G_2(\alpha_T)X_2(T)^2 \Big| \mathcal{F}_\tau \Big\}. \quad (3.3.2)
\end{aligned}
$$

Define the conditional minimal value function

$$
V^a(\tau, X_2(\tau), \alpha_\tau) = \operatorname*{ess\,inf}_{u(\cdot)\in\mathcal{U}[0,T]} J^a(\tau, X_2(\tau), \alpha_\tau, u(\cdot)). \quad (3.3.3)
$$

For each triple $(\tau, X_2(\tau), \alpha_\tau) \in \mathcal{T}[0,T] \times L^2_{\mathcal{F}}(\Omega; C(0,T;\mathbb{R})) \times \mathcal{M}$, $V^a(\tau, X_2(\tau), \alpha_\tau)$ is a well-defined $\mathscr{F}_\tau$-measurable random variable.

An admissible pair $(X_2(\cdot), u(\cdot))$ is called optimal if $u(\cdot)$ achieves the infimum of $J^a(\tau, X_2(\tau), \alpha_\tau, u(\cdot))$.

We can solve the LQ problem (3.3.1) explicitly in terms of the solutions to BSDEs (3.2.3) and (3.2.4). For $P \geqslant 0$ with $PD_2(t,i)'D_2(t,i) + R_2(t,i) > 0$, and $\Lambda \in \mathbb{R}^n$, define

$$
\begin{aligned}
\hat{v}_1^a(t,P,\Lambda,i) = \operatorname*{argmin}_{v\in\Gamma} \big[&v'(PD_2(t,i)'D_2(t,i) + R_2(t,i))v \\
&+ 2v'(PB_2(t,i) + PD_2(t,i)'C_2(t,i) + D_2(t,i)'\Lambda)\big], \quad (3.3.4)
\end{aligned}
$$

$$
\begin{aligned}
\hat{v}_2^a(t,P,\Lambda,i) = \operatorname*{argmin}_{v\in\Gamma} \big[&v'(PD_2(t,i)'D_2(t,i) + R_2(t,i))v \\
&- 2v'(PB_2(t,i) + PD_2(t,i)'C_2(t,i) + D_2(t,i)'\Lambda)\big]. \quad (3.3.5)
\end{aligned}
$$

Remark 3.4. *If Γ is symmetric, then $\hat{v}_1^a(t,P,\Lambda,i) = -\hat{v}_2^a(t,P,\Lambda,i)$. In particular, if $\Gamma = \mathbb{R}^m$, then*

$$
\hat{v}_2^a(t,P,\Lambda,i) = (PD_2(t,i)'D_2(t,i) + R_2(t,i))^{-1}(PB_2(t,i) + PD_2(t,i)'C_2(t,i) + D_2(t,i)'\Lambda).
$$

Theorem 3.4. *Assume Assumption 3.2 holds on $[\tau, T]$, the Post-Switch LQ problem (3.3.1) admits an optimal control, as a feedback function of the time t, the state X_2,* *and the market regime* i*,*

$$\bar{u}(t, X_2, i)$$
$$= \hat{v}_1^a(t, P_1^a(t,i), \Lambda_1^a(t,i), i)X_2^+ + \hat{v}_2^a(t, P_2^a(t,i), \Lambda_2^a(t,i), i)X_2^-, \quad t \in [\tau, T]. \tag{3.3.6}$$

Moreover, the corresponding conditional optimal value is

$$\operatorname*{ess\,inf}_{u(\cdot)\in\mathcal{U}[0,T]} J^a(\tau, X_2(\tau), \alpha_\tau, u(\cdot)) = P_1^a(\tau, \alpha_\tau)X_2^+(\tau)^2 + P_2^a(\tau, \alpha_\tau)X_2^-(\tau)^2, \tag{3.3.7}$$

where $(P_1^a(i),\ \Lambda_1^a(i))_{i=1}^{\ell}$ *(resp.* $(P_2^a(i),\ \Lambda_2^a(i))_{i=1}^{\ell}$*) is the unique solution to* (3.2.3) *(resp.* (3.2.4)*).*

Proof. Its proof is given in section 3.5.4. □

3.3.2 Solution to the Pre-Switch LQ Problem

Under Assumption 3.1, the post-switch problem admits a unique optimal pair which can be represented by (3.3.6). In this section, we will study the problem which is termed of control-stopping (or mixed) type because it combines control and stopping. The controller, at her/his convenience, the controller chooses also the time τ to stop controlling the system.

Let $(P_1^a(i),\ \Lambda_1^a(i))_{i=1}^{\ell}$ and $(P_2^a(i),\ \Lambda_2^a(i))_{i=1}^{\ell}$ be the unique solutions to (3.2.3) and (3.2.4), respectively. We now define

$$J^b(x, i_0, u(\cdot), \tau) = \mathbb{E}\Big\{ \int_0^\tau \big[Q_1(t,\alpha_t)X_1(t)^2 + u(t)'R_1(t,\alpha_t)u(t)\big]\,\mathrm{d}t$$
$$+ \mathbb{G}_1(\tau,\alpha_\tau)X_1^+(\tau)^2 + \mathbb{G}_2(\tau,\alpha_\tau)X_1^-(\tau)^2\Big\}, \tag{3.3.8}$$

where $\mathbb{G}_1(t,i) = G_1(t,i) + K(t)^2P_1^a(t,i)$, $\mathbb{G}_2(t,i) = G_1(t,i) + K(t)^2P_2^a(t,i)$, for all $i \in \mathcal{M}$.

For given $(x, i_0) \in \mathbb{R}\times\mathcal{M}$,

$$\begin{cases} \text{Minimize} & J^b(x, i_0, u(\cdot), \tau) \\ \text{subject to} & (\tau, u(\cdot)) \in \mathscr{A},\ (X_1(\cdot), u(\cdot), \tau)\text{follows (3.1.1)}. \end{cases} \tag{3.3.9}$$

The corresponding value function is defined as

$$V^b(x, i_0) = \inf_{(u(\cdot),\tau)\in\mathscr{A}} J^b(x, i_0, u(\cdot), \tau).$$

We call the problem (3.3.9) the Pre-Switch LQ problem.

Theorem 3.5. *Assume Assumption 3.2 holds. Then the Pre-Switch LQ problem (3.3.9) admits an optimal control, as a feedback function of the time t, the state X_1, and the market regime i,*

$$\bar{u}(t, X_1, i) = \hat{v}_1^b(t, P_1^b(t, i), \Lambda_1^b(t, i), i)X_1^+ + \hat{v}_2^b(t, P_2^b(t, i), \Lambda_2^b(t, i), i)X_1^-, \tag{3.3.10}$$

and an optimal time

$$\bar{\tau} = \begin{cases} \inf\left\{t \in [0, T], P_1^b(t, \alpha_t) = \mathbb{G}_1(t, \alpha_t)\right\} \wedge T, & \text{if } x \geqslant 0; \\ \inf\left\{t \in [0, T], P_2^b(t, \alpha_t) = \mathbb{G}_2(t, \alpha_t)\right\} \wedge T, & \text{if } x < 0, \end{cases} \tag{3.3.11}$$

where

$$\begin{aligned} \hat{v}_1^b(t, P, \Lambda, i) = \operatorname*{argmin}_{v\in\Gamma} \big[&v'(PD_1(t, i)'D_1(t, i) + R_1(t, i))v \\ &+ 2v'(PB_1(t, i) + PD_1(t, i)'C_1(t, i) + D_1(t, i)'\Lambda)\big], \\ \hat{v}_2^b(t, P, \Lambda, i) = \operatorname*{argmin}_{v\in\Gamma} \big[&v'(PD_1(t, i)'D_1(t, i) + R_1(t, i))v \\ &- 2v'(PB_1(t, i) + PD_1(t, i)'C_1(t, i) + D_1(t, i)'\Lambda)\big]. \end{aligned}$$

Moreover, the corresponding optimal value is

$$V^b(x, i_0) = \inf_{(u(\cdot),\tau)\in\mathscr{A}} J^b(x, i_0, u(\cdot), \tau) = P_1^b(0, i_0)(x^+)^2 + P_2^b(0, i_0)(x^-)^2. \tag{3.3.12}$$

Proof. The proof is given in section 3.5.5. □

3.3.3 Links of Control Problems

Now we establish the links among Pre-Switch LQ problem (3.3.9) and Post-Switch LQ problem (3.3.1) and MSLQ problem (3.1.2).

Theorem 3.6. *Let* $(P_1^b(i),\, \Lambda_1^b(i),\, \mu_1(i))_{i=1}^{\ell}$, $(P_2^b(i),\, \Lambda_2^b(i),\, \mu_2(i))_{i=1}^{\ell}$, $(P_1^a(i),\, \Lambda_1^a(i))_{i=1}^{\ell}$, $(P_2^a(i),\, \Lambda_2^a(i))_{i=1}^{\ell}$ *be the unique solutions to* (3.2.1), (3.2.2), (3.2.3), (3.2.4). *Under the conditions of Theorem 3.2 or Theorem 3.3, the MSLQ problem* (3.1.2) *admits an optimal control, as a feedback function of the time* t, *the state* X, *and the market regime* i,

$$\begin{aligned}\bar{u}(t,X,i) &= \left\{\hat{v}_1^a(t,P_1^a(t,i),\Lambda_1^a(t,i),i)X_1^+ + \hat{v}_2^a(t,P_2^a(t,i),\Lambda_2^a(t,i),i)X_1^-\right\}\mathbb{1}_{\{0\leqslant t<\bar{\tau}\}}\\ &\quad + \left\{\hat{v}_1^b(t,P_1^b(t,i),\Lambda_1^b(t,i),i)X_2^+ + \hat{v}_2^b(t,P_2^b(t,i),\Lambda_2^b(t,i),i)X_2^-\right\}\mathbb{1}_{\{\bar{\tau}\leqslant t\leqslant T\}},\end{aligned} \tag{3.3.13}$$

and an optimal time

$$\bar{\tau} = \begin{cases}\inf\left\{t\in[0,T], P_1^b(t,\alpha_t) = \mathbb{G}_1(t,\alpha_t)\right\}\wedge T, & \text{if } x\geqslant 0;\\ \inf\left\{t\in[0,T], P_2^b(t,\alpha_t) = \mathbb{G}_2(t,\alpha_t)\right\}\wedge T, & \text{if } x<0.\end{cases} \tag{3.3.14}$$

Moreover, the corresponding optimal value is

$$\inf_{(u(\cdot),\tau)\in\mathscr{A}} J(x,i_0,u(\cdot),\tau) = P_1^a(0,i_0)(x^+)^2 + P_2^a(0,i_0)(x^-)^2. \tag{3.3.15}$$

Proof. The proof is given in section 3.5.6. □

3.4 Concluding Remarks

In this chapter, we solved a constrained stochastic LQ control-stopping problem featuring regime switching. The system dynamics is driven by a Brownian motion and a continuous-time Markov chain, with coefficients that are random. Our main contributions include the explicit derivation of optimal state feedback controls and the optimal cost value through two novel systems of highly nonlinear BSDEs and two RBSDEs. Different from homogeneous LQ control-stopping problem, we derive one class of new unbounded multi-dimensional RBSDE systems, its own well-posedness can not be addressed by the existing argument in the literature because all the coefficients is unbounded. Due to the boundary constraints on solutions to such equations, establishing their solvability requires novel proof techniques and approaches.

These BSDE systems are introduced here for the first time to govern the system before and after the controlled stopping time τ. The solvability of these equations, established using tools from BMO martingale theory and the John-Nirenberg inequality, is of independent theoretical interest within BSDE framework, particularly for systems with quadratic growth and regime-dependent dynamics.

Extensions in other directions can be interesting as well. A key application of our results lies in solving the continuous-time mean-variance portfolio selection problem with no-shorting constraints and regime switching. In this financial context, an investor seeks to minimize portfolio variance (risk) for a target expected return, while adhering to the constraint that portfolio weights remain non-negative (no short-selling). The market parameters-such as asset returns, volatilities, and interest rates-depend on a Markov chain representing macroeconomic regimes (e.g., bull or bear markets). By embedding this problem into our LQ framework, we could derive explicit optimal trading strategies that dynamically adapt to both market regimes and wealth levels, along with an optimal stopping rule for strategic portfolio rebalancing. This provides a rigorous and computationally tractable approach to portfolio optimization under realistic market frictions and regime uncertainty.

Our work advances the theory of stochastic control under non-Markovian regimes and offers practical tools for financial decision-making. Future research may extend this framework to include jump diffusions, partial observations, or multi-objective optimization.

3.5 Proofs for Chapter 3

We now show the proofs of the main results in Chapter 3.

3.5.1 Proof of Theorem 3.1

Proof. For $t \in [0,T]$ and every $i \in \mathcal{M}$, set

$$\delta Y(t,i) = Y(t,i) - \overline{Y}(t,i),\ \ \delta Z(t,i) = Z(t,i) - \overline{Z}(t,i),\ \ \delta\mu(t,i) = \mu(t,i) - \overline{\mu}(t,i).$$

Notice

$$\begin{aligned}
-\int_t^T \delta Y(s,i)^+ \,\mathrm{d}\delta\mu(s,i) &\leqslant \int_t^T \big(Y(s,i) - \overline{Y}(s,i)\big)^+ \,\mathrm{d}\overline{\mu}(s,i) \\
&\leqslant \int_t^T \big(\overline{L}(s,i) - \overline{Y}(s,i)\big)^+ \,\mathrm{d}\overline{\mu}(s,i) = 0.
\end{aligned}$$

Applying Itô-Tanaka formula formula to $(\delta Y(t,i)^+)^2$, we have

$$\begin{aligned}
\mathbb{E}(\delta Y(t,i)^+)^2 = \mathbb{E}\int_t^T 2\delta Y(s,i)^+ \Big[& f(s,Y(s,i),Y(s,-i),Z(s,i),i) \\
& - \overline{f}(s,\overline{Y}(s,i),\overline{Y}(s,-i),\overline{Z}(s,i),i)\Big]\,\mathrm{d}s \\
& - 2\mathbb{E}\int_t^T \delta Y(s,i)^+ \,\mathrm{d}\delta\mu(t,i) \\
& - \mathbb{E}\int_t^T I_{\delta Y(s,i)\geqslant 0} |\delta Z(s,i)|^2 \,\mathrm{d}s \\
\leqslant \mathbb{E}\int_t^T 2c\delta Y(s,i)^+ \Big(& |\delta Y(s,i)| + \sum_{j\neq i} \delta Y(s,j)^+ + \delta Z(s,i)\Big)\,\mathrm{d}s \\
& - \mathbb{E}\int_t^T I_{\delta Y(s,i)\geqslant 0} |\delta Z(s,i)|^2 \,\mathrm{d}s \\
\leqslant c\mathbb{E}\int_t^T \sum_{i=1}^{\ell} & (\delta Y(s,i)^+)^2 \,\mathrm{d}s,
\end{aligned}$$

by the arithmetic mean - geometric mean (AM-GM) inequality. Thus

$$\sum_{i=1}^{\ell} \mathbb{E}(\delta Y(t,i)^+)^2 \leqslant c\int_t^T \sum_{i=1}^{\ell} \mathbb{E}(\delta Y(s,i)^+)^2 \,\mathrm{d}s.$$

It then follows from Gronwall's inequality that $\sum_{i=1}^{\ell}\mathbb{E}(\delta Y(t,i)^+)^2=0$, thus thanks to the continuity of Y in t, we conclude $Y(t,i)\leqslant \overline{Y}(t,i)$ for all $t\in[0,T]$ and $i\in\mathcal{M}$. $\square$

3.5.2 Proof of Theorem 3.2

Proof. Existence. For $i\in\mathcal{M}$, $P_1^b\in\mathbb{R}^{\ell}$, and $\Lambda_1^b\in\mathbb{R}^{n\times\ell}$, set

$$\overline{f}(t,P_1^b,\Lambda_1^b,i)=(2A_1(i)+C_1(i)'C_1(i)+q_{ii})P_1^b(i)+2C_1(i)'\Lambda_1^b(i)+Q_1(i)+\sum_{j\neq i}q_{ij}P_1^b(j).$$

As $\overline{f}$ is linear in P_1^b and Λ_1^b, there exists a unique solution $(\overline{P}_1^b(i),\ \overline{\Lambda}_1^b(i))_{i=1}^{\ell}$ to the corresponding BSDEs with the generator $\overline{f}$ and terminal value $\mathbb{G}_1(T,i)$. By Assumption 3.1, there exists a constant $c>0$, such that

$$2A_1(i)+C_1(i)'C_1(i)+\max_{k,j\in\mathcal{M}}|q_{kj}|\leqslant c,\ Q_1(i)\leqslant c,\ \mathbb{G}_1(T,i)\leqslant c,$$

for all $t\in[0,T]$ and $i\in\mathcal{M}$. Hereafter, we shall use c to represent a generic positive constant independent of i, n and t, which can be different from line to line.

The following ℓ-dimensional linear BSDEs

$$\begin{cases}\mathrm{d}P_1^b(i)=-\Big[c\sum_{j=1}^{\ell}P_1^b(j)+2C_1(i)\Lambda_1^b(i)+c\Big]\,\mathrm{d}t+\Lambda_1^b(i)'\,\mathrm{d}W,\\ P_1^b(i,T)=c,\quad\text{for all } i\in\mathcal{M},\end{cases}$$

admits a solution $\Big(\frac{(c\ell+1)e^{c\ell(T-t)}-1}{\ell},0\Big)_{i=1}^{\ell}$. By Lemma 3.1, we have

$$\overline{P}_1^b(t,i)\leqslant\frac{(c\ell+1)e^{c\ell(T-t)}-1}{\ell}\leqslant M,\quad\text{for } t\in[0,T] \text{ and all } i\in\mathcal{M},$$

where $M=\frac{(c\ell+1)e^{c\ell T}-1}{l}$.

For $k\geqslant1$, $(t,P,\Lambda)\in[0,T]\times\mathbb{R}\times\mathbb{R}^n$, $i\in\mathcal{M}$, define

$$\mathscr{H}^k(t,P,\Lambda,i)=\sup_{\tilde{P}\in\mathbb{R},\tilde{\Lambda}\in\mathbb{R}^n}\Big\{\mathscr{H}_+^b(t,\tilde{P},\tilde{\Lambda},i)-k|P-\tilde{P}|-k|\Lambda-\tilde{\Lambda}|\Big\}.$$

Then $\mathscr{H}^k$ is non-positive and uniformly Lipschitz in (P, Λ), and decreasingly approaches to $\mathscr{H}_+^b(t, P, \Lambda, i)$ as k goes to infinite. The following BSDEs

$$\begin{cases} \mathrm{d}P_1^k(i) = -\Big[\overline{f}(P_1^k, \Lambda^k, i) + \mathscr{H}^k(P_1^k(i), \Lambda_1^k(i), i) - k(\mathbb{G}_1(i) - P_1^k(i))^-\Big]\,\mathrm{d}t \\ \qquad\qquad +\Lambda_1^k(i)'\,\mathrm{d}W, \\ P_1^k(i, T) = \mathbb{G}_1(i, T), \quad \text{for all } i \in \mathcal{M}, \end{cases} \tag{3.5.1}$$

is an ℓ-dimensional BSDEs with a Lipschitz generator, so it admits a unique solution, denoted by $\big(P_1^k(i), \Lambda_1^k(i)\big)_{i=1}^{\ell}$. Notice that $\mathscr{H}^k(t, 0, 0, i) = 0,\ Q_1 \geqslant 0,\ \mathbb{G}_1(T, i) \geqslant 0$, and

$$\overline{f}(t, P_1, \Lambda_1, i) + \mathscr{H}^k(t, P_1(i), \Lambda_1(i), i) - k(\mathbb{G}_1(t, i) - P_1(t, i))^- \leqslant \overline{f}(t, P_1, \Lambda_1, i),$$

then by Lemma 3.1, we have

$$0 \leqslant P_1^k(t, i) \leqslant \overline{P}_1^b(t, i) \leqslant M,$$

and $P_1^k(t, i)$ is decreasing in k, for each $i \in \mathcal{M}$.

We define

$$\mu_1^k(t, i) = k \int_0^t \big(\mathbb{G}_1(s, i) - P_1^k(s, i)\big)^-\,\mathrm{d}s, \quad 0 \leqslant t \leqslant T.$$

Then (3.5.1) can be rewritten as

$$\begin{cases} \mathrm{d}P_1^k(i) = -\Big[\overline{f}(P_1^k, \Lambda^k, i) + \mathscr{H}^k(P_1^k(i), \Lambda_1^k(i), i)\Big]\,\mathrm{d}t + \mathrm{d}\mu_1^k(i) + \Lambda_1^k(i)'\,\mathrm{d}W, \\ P_1^k(i, T) = \mathbb{G}_1(i, T), \quad \text{for all } i \in \mathcal{M}, \end{cases}$$

Let $P(t, i) = \lim\limits_{k\to\infty} P^k(t, i),\ i \in \mathcal{M}$. It is important to note that we can regard $\big(P_1^k(i), \Lambda_1^k(i), \mu_1^k(i)\big)$ as the solution of a scalar-valued quadratic RBSDEs for each $i \in \mathcal{M}$. Thus from RBSDEs stability result, there exists a process $\Lambda_1 \in L^2_{\mathcal{F}^W}(0, T; \mathbb{R}^{n\times\ell})$ and $\mu_1 \in \mathcal{A}_c(0, T; \mathbb{R})$ such that (P_1, Λ_1, μ_1) is a solution to RBSDEs (3.2.1). We have now established the existence of the solution.

Next, let us prove the uniqueness.

Step 1: For any solution $(P_1^b(i),\ \Lambda_1^b(i),\ \mu_1(i)) \in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}^+) \times L^2_{\mathcal{F}^W}(0,T;\mathbb{R}^n) \times \mathcal{A}_c(0,T;\mathbb{R})$ of (3.2.1), we now prove the more precise estimate $\Lambda_1^b(i) \in L^{2,\,\mathrm{BMO}}_{\mathcal{F}^W}(0,T;\mathbb{R}^n)$, for all $i \in \mathcal{M}$.

Actually applying Itô's formula to $P_1^b(i)^2$, we get, for any $\{\mathcal{F}_t^W\}_{t\geqslant 0}$ stopping time $\tau \leqslant T$,

$$
\begin{aligned}
&\mathbb{E}\Big[\int_\tau^T |\Lambda_1^b(i)|^2\,\mathrm{d}s \,\Big|\, \mathcal{F}_\tau^W\Big]\\
&= \mathbb{E}[\mathbb{G}_1(T,i)^2|\mathcal{F}_\tau^W] - P_1^b(t,i)^2 + \mathbb{E}\Big[\int_\tau^T 2P_1^b(i)\Big[(2A_1(i)+|C_1(i)|^2)P_1^b(i)\\
&\quad + 2C_1(i)'\Lambda_1^b(i) + Q_1(i) + \mathscr{H}_+^b(P_1^b(i),\Lambda_1^b(i),i) + \sum_{i=1}^{\ell} q_{ij}P_1^b(j)\Big]\,\mathrm{d}s\\
&\quad - 2P_1^b(i)\Big[(\mu_1(T,i)-\mu_1(t,i))\Big] \,\Big|\, \mathcal{F}_\tau^W\Big].
\end{aligned}
$$

Note that $\mathscr{H}_+^b \leqslant 0$, $P_1^b(i) \geqslant 0$, $\mu_1(T,i) \geqslant \mu_1(t,i)$, Assumption 3.1 and P_1^b is uniformly bounded, then

$$
\begin{aligned}
\mathbb{E}\Big[\int_\tau^T |\Lambda_1^b(s,i)|^2\,\mathrm{d}s \,\Big|\, \mathcal{F}_\tau^W\Big] &\leqslant c + \mathbb{E}\Big[\int_\tau^T \Big[c + 2c|\Lambda_1^b(s,i)|\Big]\,\mathrm{d}s \,\Big|\, \mathcal{F}_\tau^W\Big]\\
&\leqslant c + \frac{1}{2}\mathbb{E}\Big[\int_\tau^T |\Lambda_1^b(s,i)|^2\,\mathrm{d}s \,\Big|\, \mathcal{F}_\tau^W\Big].
\end{aligned}
$$

It follows

$$
\mathbb{E}\Big[\int_\tau^T |\Lambda_1^b(s,i)|^2\,\mathrm{d}s \,\Big|\, \mathcal{F}_\tau^W\Big] \leqslant c,
$$

so $\Lambda_1^b(i) \in L^{2,\,\mathrm{BMO}}_{\mathcal{F}^W}(0,T;\mathbb{R}^n)$ for all $i \in \mathcal{M}$.

Step 2: Log transformation of the RBSDEs (3.2.1).

Suppose $(P_1^b(i),\ \Lambda_1^b(i),\ \mu_1(i))_{i=1}^\ell$, $(\tilde{P}_1^b(i),\ \tilde{\Lambda}_1^b(i),\ \tilde{\mu}_1(i))_{i=1}^\ell$ are two solutions to (3.2.1). Then there exists a constant $M > 0$ such that $0 \leqslant P_1^b(i),\ \tilde{P}_1^b(i) \leqslant M$, and $\int_0^\cdot \Lambda_1^b(s,i)'\,\mathrm{d}W(s)$, $\int_0^\cdot \tilde{\Lambda}_1^b(s,i)'\,\mathrm{d}W(s)$ are BMO-martingales, for all $i \in \mathcal{M}$.

For every $i \in \mathcal{M}$, define processes

$$(U(t,i),\ V(t,i),\ k(t,i)) = \left(\ln(P_1^b(t,i)+a),\ \frac{\Lambda_1^b(t,i)}{P_1^b(t,i)+a},\ \int_0^t \frac{\mathrm{d}\mu_1(s,i)}{P_1^b(s,i)+a}\right),$$

$$(\tilde{U}(t,i),\ \tilde{V}(t,i),\ \tilde{k}(t,i)) = \left(\ln(\tilde{P}_1^b(t,i)+a),\ \frac{\tilde{\Lambda}_1^b(t,i)}{\tilde{P}_1^b(t,i)+a},\ \int_0^t \frac{d\tilde{\mu}_1(s,i)}{\tilde{P}_1^b(s,i)+a}\right),$$

where $a>0$ is a constant to be determined later. Then

$$(U(i),V(i),k(i)),\ (\tilde{U}(i),\tilde{V}(i),\tilde{k}(i)) \in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R})\times L^{2,\,\mathrm{BMO}}_{\mathcal{F}^W}(0,T;\mathbb{R}^n)\times\mathcal{A}_c(0,T;\mathbb{R}),$$

for all $i\in\mathcal{M}$. Furthermore, by Itô's formula, $(U(i),V(i),k(i))_{i=1}^{\ell}$ satisfy the following ℓ-dimensional BSDEs:

$$\begin{cases}
\mathrm{d}U(i) = -\Big[(2A_1(i)+C_1(i)'C_1(i))(1-ae^{-U(i)})+2C_1(i)'V(i)+Q_1(i)e^{-U(i)}\\
\qquad\qquad +\tilde{H}(U(i),V(i),i)+\frac{1}{2}V(i)'V(i)+\sum\limits_{j=1}^{\ell}q_{ij}e^{U(j)-U(i)}\Big]\,\mathrm{d}t+\mathrm{d}k(i)+V(i)'\,\mathrm{d}W,\\
U(T,i)=\ln(\mathbb{G}_1(T,i)+a),\\
U(t,i)\leqslant \ln(\mathbb{G}_1(t,i)+a), \quad \int_0^T(\ln(\mathbb{G}_1(t,i)+a)-U(t,i))\,\mathrm{d}k(t,i)=0,\\
k(0,i)=0, \quad \text{for all } i\in\mathcal{M};
\end{cases}$$

where

$$\tilde{H}(U,V,i)=\inf_{v\in\Gamma}\Big[v'((1-ae^{-U})D_1(i)'D_1(i)+R_1(i)e^{-U})v$$
$$+2v'((1-ae^{-U})(B_1(i)+D_1(i)'C_1(i))+D_1(i)'V)\Big].$$

Similar for $(\tilde{U}(i),\tilde{V}(i),\tilde{\mu}(i))_{i=1}^{\ell}$.

As $0\leqslant P_1^b(i)\leqslant M$, thus $e^{-U(i)}=\frac{1}{P_1^b(i)+a}\geqslant\frac{1}{M+a}$ and $1-ae^{-U(i)}=\frac{P_1^b(i)}{P_1^b(i)+a}\in[0,1)$. Similar inequalities hold when $U(i)$ is replaced by $\tilde{U}(i)$. By Assumption 3.1 and

$R_1 \geqslant \delta I_m$, there exist constant $c > 0$ such that

$$
\begin{aligned}
& v'((1 - ae^{-U(i)})D_1(i)'D_1(i) + R_1(i)e^{-U(i)})v \\
& \qquad + 2v'((1 - ae^{-U(i)})(B_1(i) + D_1(i)'C_1(i)) + D_1(i)'V(i)) \\
& \geqslant v'R_1(i)e^{-U(i)}v + 2v'((1 - ae^{-U(i)})(B_1(i) + D_1(i)'C_1(i)) + D_1(i)'V(i)) \\
& \geqslant \frac{\delta}{M+a}|v|^2 - c(1 + |V(i)|)|v|.
\end{aligned}
$$

Hence if $|v| > \frac{c(M+a)}{\delta}(1 + |V|) := c(1 + |V|)$, then

$$
\frac{\delta}{M+a}|v|^2 - c(1 + |V|)|v| > 0 \geqslant \tilde{H}(U, V, i),
$$

for $(U, V) = (U(t,i), V(t,i))$ and $(\tilde{U}(t,i), \tilde{V}(t,i))$. Thus,

$$
\begin{aligned}
\tilde{H}(U, V, i) = \inf_{\substack{v \in \Gamma \\ |v| \leqslant c(1+|V|)}} & \Big[v'((1 - ae^{-U})D_1(i)'D_1(i) + R_1(i)e^{-U})v \\
& \qquad + 2v'((1 - ae^{-U})(B_1(i) + D_1(i)'C_1(i)) + D_1(i)'V) \Big].
\end{aligned}
$$

Step 3: Estimate the difference between $U(i)$ and $\tilde{U}(i)$.

Set $\bar{U}(i) = U(i) - \tilde{U}(i),\ \bar{V}(i) = V(i) - \tilde{V}(i),\ \bar{k}(i) = k(i) - \tilde{k}(i)$, for $i \in \mathcal{M}$. Then $(\bar{U}(i),\ \bar{V}(i),\ \bar{k}(i))_i^\ell$ satisfy the following BSDEs:

$$
\begin{cases}
\mathrm{d}\bar{U}(i) = -\Big[(Q_1(i) - 2aA_1(i) - aC_1(i)'C_1(i))(e^{-U(i)} - e^{-\tilde{U}(i)}) + 2C_1(i)'\bar{V}(i) \\
\qquad\qquad + \tilde{H}(U(i), V(i), i) - \tilde{H}(\tilde{U}(i), \tilde{V}(i), i) + \frac{1}{2}(V(i) + \tilde{V}(i))'\bar{V}(i) \\
\qquad\qquad + \sum\limits_{j=1}^{\ell} q_{ij}(e^{U(j)-U(i)} - e^{\tilde{U}(j)-\tilde{U}(i)}) \Big] \,\mathrm{d}t + \mathrm{d}\bar{k}(i) + \bar{V}(i)' \,\mathrm{d}W, \\
\bar{U}(T, i) = 0, \quad \text{for all } i \in \mathcal{M}.
\end{cases}
$$

Applying Itô's formula to $\bar{U}(i)^2$, we deduce that

$$
\begin{aligned}
\bar{U}(t,i)^2 = \int_t^T \Big\{ & 2\bar{U}(i)\Big[(Q_1(i) - 2aA_1(i) - aC_1(i)'C_1(i))(e^{-U(i)} - e^{-\tilde{U}(i)}) + 2C_1(i)'\bar{V}(i) \\
& + \frac{1}{2}(V(i) + \tilde{V}(i))'\bar{V}(i) + \sum_{j=1}^{\ell} q_{ij}(e^{U(j)-U(i)} - e^{\tilde{U}(j)-\tilde{U}(i)})\Big] - \bar{V}(i)'\bar{V}(i) \\
& + 2\bar{U}(i)(\tilde{H}(U(i),V(i),i) - \tilde{H}(\tilde{U}(i),\tilde{V}(i),i))\Big\}\,\mathrm{d}s \\
& - 2\int_t^T \bar{U}(i)\,\mathrm{d}\bar{k}(i) - \int_t^T 2\bar{U}(i)\bar{V}(i)'\,\mathrm{d}W \\
:= \int_t^T & \Big[L(i) + 2\bar{U}(i)\sum_{j=1}^{\ell} q_{ij}(e^{U(j)-U(i)} - e^{\tilde{U}(j)-\tilde{U}(i)})\Big]\,\mathrm{d}s \\
& - 2\int_t^T \bar{U}(i)\,\mathrm{d}\bar{k}(i) - \int_t^T 2\bar{U}(i)\bar{V}(i)'\,\mathrm{d}W \\
\leqslant \int_t^T & \Big[L(i) + 2\bar{U}(i)\sum_{j=1}^{\ell} q_{ij}(e^{U(j)-U(i)} - e^{\tilde{U}(j)-\tilde{U}(i)})\Big]\,\mathrm{d}s - \int_t^T 2\bar{U}(i)\bar{V}(i)'\,\mathrm{d}W,
\end{aligned}
$$

owing to the reflected condition,

$$
\begin{aligned}
& \int_t^T \bar{U}(i)\,\mathrm{d}\bar{k}(i) \\
&= \int_t^T (\ln(\mathbb{G}_1(t,i)+a) - \tilde{U}(i))\,\mathrm{d}\bar{k}(i) - \int_t^T (\ln(\mathbb{G}_1(t,i)+a) - U(i))\,\mathrm{d}\bar{k}(i) \\
&= \int_t^T (\ln(\mathbb{G}_1(t,i)+a) - U(i))\,\mathrm{d}\tilde{k}(i) + \int_t^T (\ln(\mathbb{G}_1(t,i)+a) - \tilde{U}(i))\,\mathrm{d}k(i) \\
&\geqslant 0.
\end{aligned}
$$

Let us now estimate $\bar{U}(i)\left(\tilde{H}(U(i),V(i),i) - \tilde{H}(\tilde{U}(i),\tilde{V}(i),i)\right)$. However, we do not have the monotonicity of $\tilde{H}(U,V,i)$ in U, we apply the method in Hu et al.

(2022b) by treating the quadratic term and linear term separately. Let

$$\mathcal{H}(v,\tilde{U},U,V,i)=v'((1-ae^{-\tilde{U}})D_1(i)'D_1(i)+R_1(i)e^{-\tilde{U}})v$$
$$+2v'((1-ae^{-U})(B_1(i)+D_1(i)'C_1(i))+D_1(i)'V).$$

Then rewrite $\tilde{H}$ in terms of $\mathcal{H}$,

$$\begin{aligned}&\tilde{H}(U(i),V(i),i)-\tilde{H}(\tilde{U}(i),\tilde{V}(i),i)\\&=\inf_{\substack{v\in\Gamma\\|v|\leqslant c(1+|V(i)|)}}\mathcal{H}(v,U,U,V,i)-\inf_{\substack{v\in\Gamma\\|v|\leqslant c(1+|\tilde{V}(i)|)}}\mathcal{H}(v,\tilde{U},\tilde{U},\tilde{V},i).\end{aligned}$$

The above optimal values will not change when we enlarge the optimization region, so, after inserting two zero-sum terms, we get

$$\begin{aligned}&\tilde{H}(U(i),V(i),i)-\tilde{H}(\tilde{U}(i),\tilde{V}(i),i)\\&=\inf_{\substack{v\in\Gamma\\|v|\leqslant c(1+|V(i)|+|\tilde{V}(i)|)}}\mathcal{H}(v,U,U,V,i)-\inf_{\substack{v\in\Gamma\\|v|\leqslant c(1+|V(i)|+|\tilde{V}(i)|)}}\mathcal{H}(v,\tilde{U},U,V,i)\\&\quad+\inf_{\substack{v\in\Gamma\\|v|\leqslant c(1+|V(i)|+|\tilde{V}(i)|)}}\mathcal{H}(v,\tilde{U},U,V,i)-\inf_{\substack{v\in\Gamma\\|v|\leqslant c(1+|V(i)|+|\tilde{V}(i)|)}}\mathcal{H}(v,\tilde{U},\tilde{U},\tilde{V},i).\end{aligned}\tag{3.5.2}$$

Let $a>0$ be a sufficiently small constant such that $R_1(i)-aD_1(i)'D_1(i)>0$ for all $i\in\mathcal{M}$. Then the map

$$x\mapsto v'((1-ae^{-x})D_1(i)'D_1(i)+R_1(i)e^{-x})v,\quad x\in\mathbb{R},$$

is decreasing for every $i\in\mathcal{M}$. Therefore,

$$(U(i)-\tilde{U}(i))\Big(\inf_{\substack{v\in\Gamma\\|v|\leqslant c(1+|V(i)|+|\tilde{V}(i)|)}}\mathcal{H}(v,U,U,V,i)-\inf_{\substack{v\in\Gamma\\|v|\leqslant c(1+|V(i)|+|\tilde{V}(i)|)}}\mathcal{H}(v,\tilde{U},U,V,i)\Big)\leqslant 0.\tag{3.5.3}$$

On the other hand, by the boundedness of U and $\tilde{U}$,

$$\begin{aligned}
&\Big| \inf_{\substack{v\in\Gamma \\ |v|\leqslant c(1+|V(i)|+|\tilde{V}(i)|)}} \mathcal{H}(v,\tilde{U},U,V,i) - \inf_{\substack{v\in\Gamma \\ |v|\leqslant c(1+|V(i)|+|\tilde{V}(i)|)}} \mathcal{H}(v,\tilde{U},\tilde{U},\tilde{V},i)\Big| \\
&\leqslant \sup_{\substack{v\in\Gamma \\ |v|\leqslant c(1+|V(i)|+|\tilde{V}(i)|)}} \Big|\mathcal{H}(v,\tilde{U},U,V,i) - \mathcal{H}(v,\tilde{U},\tilde{U},\tilde{V},i)\Big| \\
&= \sup_{\substack{v\in\Gamma \\ |v|\leqslant c(1+|V(i)|+|\tilde{V}(i)|)}} \Big|2v'\Big(a(e^{-U}-e^{-\tilde{U}})(B_1(i)+D_1(i)'C_1(i))+D_1(i)'\bar{V}\Big)\Big| \\
&\leqslant c \sup_{|v|\leqslant c(1+|V(i)|+|\tilde{V}(i)|)} |v|(|\bar{U}(i)|+|\bar{V}(i)|) \\
&\leqslant c(1+|V(i)|+|\tilde{V}(i)|)(|\bar{U}(i)|+|\bar{V}(i)|). \qquad (3.5.4)
\end{aligned}$$

Using (3.5.2), (3.5.3) and (3.5.4), we get

$$\begin{aligned}
&\bar{U}(i)\left(\tilde{H}(U(i),V(i),i)-\tilde{H}(\tilde{U}(i),\tilde{V}(i),i)\right) \\
&\leqslant c|\bar{U}(i)|(1+|V(i)|+|\tilde{V}(i)|)(|\bar{U}(i)|+|\bar{V}(i)|) \\
&=: c(1+|V(i)|+|\tilde{V}(i)|)\bar{U}(i)^2 + c\beta(i)'\bar{U}(i)\bar{V}(i),
\end{aligned}$$

where $\beta(i)$ is an $\mathcal{F}_t^W$-adapted process such that $|\beta(i)|\leqslant 1+|V(i)|+|\tilde{V}(i)|$. Then

$$\begin{aligned}
L(i) &\leqslant 2\bar{U}(i)\Big[(Q_1(i)-2aA_1(i)-aC_1(i)'C_1(i))(e^{-U(i)}-e^{-\tilde{U}(i)})+2C_1(i)'\bar{V}(i) \\
&\qquad + \frac{1}{2}(V(i)+\tilde{V}(i))'\bar{V}(i)\Big] - \bar{V}(i)'\bar{V}(i) \\
&\qquad + c(1+|V(i)|+|\tilde{V}(i)|)\bar{U}(i)^2 + c\beta(i)'\bar{U}(i)\bar{V}(i) \\
&\leqslant c(1+|V(i)|+|\tilde{V}(i)|)\bar{U}(i)^2 + 2\bar{U}(i)\bar{V}(i)'\Big(2C_1(i)+\frac{1}{2}(V(i)+\tilde{V}(i))+c\beta(i)\Big),
\end{aligned}$$

for all $i\in\mathcal{M}$.

For each fixed $i\in\mathcal{M}$, let us introduce the processes

$$J(t,i)=\exp\left(\int_0^t c(1+|V(i)|+|\tilde{V}(i)|)\,\mathrm{d}s\right),$$

and

$$N(t,i) = \mathcal{E}\left(\int_0^t \Big(2C_1(i) + \frac{1}{2}(V(i)+\tilde{V}(i)) + c\beta(i)\Big)' \,\mathrm{d}W(s)\right).$$

Note that $N(t,i)$ is a uniformly integrable martingale. Thus

$$\widetilde{W}^i(t) := W(t) - \int_0^t \Big(2C_1(i) + \frac{1}{2}(V(i)+\tilde{V}(i)) + c\beta(i)\Big)\,\mathrm{d}s,$$

is a Brownian motion under the probability $\widetilde{\mathbb{P}}^i$ defined by

$$\left.\frac{\mathrm{d}\widetilde{\mathbb{P}}^i}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_T^W} = N(T,i).$$

Itô's formula gives us, for any $\{\mathcal{F}_t^W\}_{t\geqslant 0}$-stopping time τ such that $0 \leqslant t \leqslant \tau \leqslant T$,

$$\begin{aligned}
&J(t,i)N(t,i)\bar{U}(t,i)^2\\
&\leqslant J(\tau,i)N(\tau,i)\bar{U}(\tau,i)^2 + 2\int_t^\tau J(i)N(i)\bar{U}(i)\sum_{j=1}^{\ell} q_{ij}(e^{U(j)-U(i)} - e^{\tilde{U}(j)-\tilde{U}(i)})\,\mathrm{d}s\\
&-\int_t^\tau \Big(J(i)N(i)\bar{U}(i)^2\Big(2C_1(i)+\frac{1}{2}(V(i)+\tilde{V}(i))+c\beta(i)\Big) + 2J(i)N(i)\bar{U}(i)\bar{V}(i)\Big)'\,\mathrm{d}W.
\end{aligned}$$

Let us consider, for $n \geqslant 1$, the stopping time

$$\begin{aligned}
\tau_n = \inf\Bigg\{u \geqslant t : \int_t^u \Big| J(i)N(i)\bar{U}(i)^2\Big(2C_1(i)+\frac{1}{2}(V(i)+\tilde{V}(i))+c\beta(i)\Big)\\
+ 2J(i)N(i)\bar{U}(i)\bar{V}(i)\Big|^2\,\mathrm{d}s \geqslant n\Bigg\} \wedge T.
\end{aligned}$$

We get from the previous equation and the AM-GM inequality that

$$\begin{aligned}
\bar{U}(t,i)^2 \leqslant \mathbb{E}\Bigg[&\frac{J(\tau_n,i)N(\tau_n,i)\bar{U}(\tau_n,i)^2}{J(t,i)N(t,i)} \\
&+ 2\int_t^{\tau_n} \frac{J(s,i)N(s,i)}{J(t,i)N(t,i)}\bar{U}(i)\sum_{j=1}^{\ell} q_{ij}(e^{U(j)-U(i)} - e^{\tilde{U}(j)-\tilde{U}(i)})\,\mathrm{d}s \,\Bigg|\, \mathcal{F}_t^W\Bigg] \\
\leqslant \mathbb{E}\Bigg[&\frac{J(\tau_n,i)N(\tau_n,i)\bar{U}(\tau_n,i)^2}{J(t,i)N(t,i)} + c\int_t^{\tau_n} \frac{J(s,i)N(s,i)}{J(t,i)N(t,i)}\sum_{j=1}^{\ell}\bar{U}(s,j)^2\,\mathrm{d}s \,\Bigg|\, \mathcal{F}_t^W\Bigg] \\
= \widetilde{\mathbb{E}}^i\Bigg[&\frac{J(\tau_n,i)\bar{U}(\tau_n,i)^2}{J(t,i)} + c\int_t^{\tau_n} \frac{J(s,i)}{J(t,i)}\sum_{j=1}^{\ell}\bar{U}(s,j)^2\,\mathrm{d}s \,\Bigg|\, \mathcal{F}_t^W\Bigg] \\
= \widetilde{\mathbb{E}}^i\Bigg[&\bar{U}(\tau_n,i)^2\exp\left(\int_t^{\tau_n} c(1+|V(i)|+|\tilde{V}(i)|)\,\mathrm{d}s\right) \\
&+ c\int_t^{\tau_n}\sum_{j=1}^{\ell}\bar{U}(s,j)^2\exp\left(\int_t^{s} c(1+|V(i)|+|\tilde{V}(i)|)du\right)\mathrm{d}s \,\Bigg|\, \mathcal{F}_t^W\Bigg] \\
\leqslant c\widetilde{\mathbb{E}}^i\Bigg[&\left(\bar{U}(\tau_n,i)^2 + \int_t^{T}\sum_{j=1}^{\ell}\bar{U}(s,j)^2\,\mathrm{d}s\right)\exp\left(\int_t^{T} c(|V(i)|+|\tilde{V}(i)|)du\right) \,\Bigg|\, \mathcal{F}_t^W\Bigg],
\end{aligned} \tag{3.5.5}$$

where $\widetilde{\mathbb{E}}^i$ is the expectation w.r.t. the probability measure $\widetilde{\mathbb{P}}^i$. By the AM-GM inequality, we see

$$\begin{aligned}
&\widetilde{\mathbb{E}}^i\left[\exp\left(\int_t^T c(|V(i)|+|\tilde{V}(i)|)\,\mathrm{d}s\right)\Bigg|\mathcal{F}_t^W\right] \\
\leqslant\, &\widetilde{\mathbb{E}}^i\left[\exp\left(\int_t^T \left(\varepsilon_i(|V(i)|+|\tilde{V}(i)|)^2 + \frac{c^2}{4\varepsilon_i}\right)\mathrm{d}s\right)\Bigg|\mathcal{F}_t^W\right],
\end{aligned}$$

which is finite by the John-Nirenberg inequality when $\varepsilon_i > 0$ is sufficient small. Using boundedness of $\bar{U}$ and the dominated convergent theorem, sending n to infinity in

(3.5.5) gives

$$\begin{aligned}\bar{U}(t,i)^2 &\leqslant c\widetilde{\mathbb{E}}^i\left[\left(\int_t^T\sum_{j=1}^{\ell}\bar{U}(s,j)^2\,\mathrm{d}s\right)\exp\left(\int_t^T c(|V(i)|+|\tilde{V}(i)|)du\right)\;\middle|\;\mathcal{F}_t^W\right]\\ &\leqslant c\widetilde{\mathbb{E}}^i\left[\exp\left(\int_t^T c(|V(i)|+|\tilde{V}(i)|)\,\mathrm{d}s\right)\middle|\mathcal{F}_t^W\right]\int_t^T\sum_{j=1}^{\ell}E(s,j)\,\mathrm{d}s\\ &\leqslant c\int_t^T\sum_{j=1}^{\ell}E(s,j)\,\mathrm{d}s,\end{aligned}$$

where

$$E(t,i)=\operatorname*{ess\,sup}_{\omega\in\Omega}\bar{U}(t,i)^2.$$

Taking essential supreme on both sides, we deduce

$$E(t,i)\leqslant c\int_t^T\sum_{j=1}^{\ell}E(s,j)\,\mathrm{d}s.$$

Thus

$$0\leqslant\sum_{j=1}^{\ell}E(t,j)\leqslant cl\int_t^T\sum_{j=1}^{\ell}E(s,j)\,\mathrm{d}s.$$

We infer from Gronwall's inequality that $\sum_{j=1}^{\ell}E(t,j)=0$, so $\bar{U}(t,i)=0$ for $t\in[0,T]$ and all $i\in\mathcal{M}$. This completes the proof of the uniqueness. □

3.5.3 Proof of Theorem 3.3

Proof. This case is relatively easy to deal with. We will present the main idea only. Details are left for the interested readers.

By Assumption 3.1, there exists a constant $c'>0$, such that

$$(\mathbb{G}_1(i)-P_1^b(i))^-\leqslant P_1^b(i)+\max|\mathbb{G}_1(i)|\leqslant P_1^b(i)+c'.$$

Set, for $P_1^b \in \mathbb{R}_+^\ell$ and $\Lambda_1^b \in \mathbb{R}^{n\times\ell}$.

$$\underline{f}(t, P_1^b, \Lambda_1^b, i) = (2A_1(i) + C_1(i)'C_1(i) + q_{ii} - k)P_1^b(i) + 2C_1(i)'\Lambda_1^b(i) + Q_1(i) \\ + \mathscr{H}_+^b(t, P_1^b(i), \Lambda_1^b(i), i) - c'k.$$

The corresponding ℓ-dimensional BSDEs with the generator $\underline{f}$ and terminal value $\mathbb{G}_1(T, i)$ is decoupled, then by Theorem 4.2 of Hu and Zhou (2005), there exists a solution $(\underline{P_1^b}(i),\ \underline{\Lambda_1^b}(i)) \in L^\infty_{\mathcal{F}^W}(0, T; \mathbb{R}) \times L^{2,\,\mathrm{BMO}}_{\mathcal{F}^W}(0, T; \mathbb{R}^n)$, such that $\underline{P_1^b}(i) \geqslant c$ with some constant $c > 0$, for all $i \in \mathcal{M}$.

Let $g : \mathbb{R}^+ \to [0, 1]$ be a smooth truncation function satisfying $g(x) = 0$ for $x \in [0, \frac{1}{2}c]$, and $g(x) = 1$ for $x \in [c, +\infty)$. Let

$$\mathscr{H}_c^k(t, P, \Lambda, i) = \sup_{\tilde{P}\in\mathbb{R}, \tilde{\Lambda}\in\mathbb{R}^n} \Big\{\mathscr{H}_+^b(t, \tilde{P}, \tilde{\Lambda}, i)g(P) - k|P - \tilde{P}| - k|\Lambda - \tilde{\Lambda}|\Big\}.$$

Repeat the argument of the proof of Theorem 3.2, the following BSDEs:

$$\begin{cases} \mathrm{d}P_1^k(i) = -\Big[\overline{f}(P_1^k, \Lambda_1^k, i) + \mathscr{H}_c^k(P_1^k(i), \Lambda_1^k(i), i) - k(\mathbb{G}_1(i) - P_1^k(i))^-\Big]\,\mathrm{d}t + \Lambda_1^k(i)'\,\mathrm{d}W, \\ P_1^k(i, T) = \mathbb{G}_1(i, T), \quad \text{for all } i \in \mathcal{M}, \end{cases}$$

has a solution, $(P_1^k(i),\ \Lambda_1^k(i))_{i=1}^\ell$, where $\overline{f}(t, P_1, \Lambda_1, i)$ is the same as in the proof of Theorem 3.2. Notice that

$$\underline{f}(t, P_1, \Lambda_1, i) \leqslant \overline{f}(t, P_1, \Lambda_1, i) + \mathscr{H}_c^k(t, P_1(i), \Lambda_1(i), i) - k(\mathbb{G}_1(t, i) - P_1(t, i))^- \leqslant \overline{f}(t, P_1, \Lambda_1, i),$$

then

$$c \leqslant \underline{P_1^b}(i) \leqslant P_1^k(i) \leqslant \overline{P}_1^b(i) \leqslant M.$$

After taking limit, it gives $P_1^b(i) = \lim_k P_1^k(i) \geqslant c$ so that $g(P_1^b(i)) = 1$ and

$$\mathscr{H}_c^k(t, P_1(i), \Lambda_1(i), i) = \mathscr{H}^k(t, P_1(i), \Lambda_1(i), i).$$

By this, one can prove the existence.

To prove the uniqueness, using $P_1^b(i) \geqslant c > 0$, we can repeat the proof of Theorem 3.2 with $a = 0$. In this case (3.5.4) is replaced by

$$\begin{aligned}
&\left| \inf_{\substack{v\in\Gamma \\ |v|\leqslant c(1+|V(i)|+|\tilde V(i)|)}} \mathcal{H}(v,\tilde U,U,V,i) - \inf_{\substack{v\in\Gamma \\ |v|\leqslant c(1+|V(i)|+|\tilde V(i)|)}} \mathcal{H}(v,\tilde U,\tilde U,\tilde V,i) \right| \\
&\leqslant \sup_{\substack{v\in\Gamma \\ |v|\leqslant c(1+|V(i)|+|\tilde V(i)|)}} \left| 2v' D_1(i)' \bar V(i) \right| \\
&\leqslant c(1+|V(i)|+|\tilde V(i)|)|\bar V(i)|,
\end{aligned}$$

so that we do not need the John-Nirenberg inequality in the proof. □

3.5.4 Proof of Theorem 3.4

Proof. Applying Itô formula to $P_1^a(s,\alpha_s)X_2^+(s)^2$ and $P_2^a(s,\alpha_s)X_2^-(s)^2$ (we refer to Hu et al. (2020)), we have

$$
\begin{aligned}
&P_1^a(t,\alpha_t)X_2^+(t)^2 + P_2^a(t,\alpha_t)X_2^-(t)^2 \\
&= P_1^a(\tau,\alpha_\tau)X_2^+(\tau)^2 + P_2^a(\tau,\alpha_\tau)X_2^-(\tau)^2 \\
&\quad + \int_\tau^t \Big\{ I_{\{X_2(s)\geqslant 0\}} P_1^a(s,\alpha_s)u(s)'D_2(s,\alpha_s)'D_2(s,\alpha_s)u(s) \\
&\qquad\qquad + 2X_2^+(s)P_1^a(s,\alpha_s)C_2(s,\alpha_s)'D_2(s,\alpha_s)u(s) \\
&\qquad\qquad + 2X_2^+(s)P_1^a(s,\alpha_s)B_2(s,\alpha_s)'u(s) + 2X_2^+(s)\Lambda_1^a(s,\alpha_s)'D_2(s,\alpha_s)u(s) \\
&\qquad\qquad - X_2^+(s)^2[Q_2(s,\alpha_s) + \mathscr{H}_+^a(s,P_1^a(s,\alpha_s),\Lambda_1^a(s,\alpha_s),\alpha_s)] \\
&\qquad\qquad + I_{\{X_2(s)<0\}} P_2^a(s,\alpha_s)u(s)'D_2(s,\alpha_s)'D_2(s,\alpha_s)u(s) \\
&\qquad\qquad - 2X_2^-(s)P_2^a(s,\alpha_s)C_2(s,\alpha_s)'D_2(s,\alpha_s)u(s) \\
&\qquad\qquad - 2X_2^-(s)P_2^a(s,\alpha_s)B_2(s,\alpha_s)'u(s) - 2X_2^-(s)\Lambda_2^a(s,\alpha_s)'D_2(s,\alpha_s)u(s) \\
&\qquad\qquad - X_2^-(s)^2[Q_2(s,\alpha_s) + \mathscr{H}_-^a(s,P_2^a(s,\alpha_s),\Lambda_2^a(s,\alpha_s),\alpha_s)] \Big\}\,\mathrm{d}s + \int_\tau^t (\dots)\,\mathrm{d}W(s) \\
&\quad + \int_\tau^t \Big\{ X_2^+(s)^2 \sum_{j,j'\in\mathcal{M}} (P_1(s,j) - P_1(s,j'))I_{\{\alpha_{s-}=j'\}} \\
&\qquad\qquad + X_2^-(s)^2 \sum_{j,j'\in\mathcal{M}} (P_2(s,j) - P_2(s,j'))I_{\{\alpha_{s-}=j'\}} \Big\}\,\mathrm{d}\tilde{N}_s^{j'j},
\end{aligned}
$$

where $(N^{j'j})_{j'j\in\mathcal{M}}$ are independent Poisson processes each with intensity $q_{j'j}$, and $\tilde{N}_t^{j'j} = N_t^{j'j} - q_{j'j}$, $t \geqslant 0$ are the corresponding compensated Poisson martingales under the filtration $\mathcal{F}$.

Note that $X_2(t)$ is continuous, the last two terms in the above equation are local martingales. Therefore, there exists an increasing localizing sequence of stopping times $\theta_n \uparrow +\infty$ as $n \to +\infty$ such that

$$
\begin{aligned}
&\mathbb{E}\left[P_1^a(T\wedge\theta_n, \alpha_{T\wedge\theta_n})X_2^+(T\wedge\theta_n)^2 + P_2^a(T\wedge\theta_n, \alpha_{T\wedge\theta_n})X_2^-(T\wedge\theta_n)^2\right]\\
&= P_1^a(\tau,\alpha_\tau)X_2^+(\tau)^2 + P_2^a(\tau,\alpha_\tau)X_2^-(\tau)^2\\
&\quad + \mathbb{E}\int_\tau^{T\wedge\theta_n}\Big\{I_{\{X_2(s)\geqslant 0\}}P_1^a(s,\alpha_s)u(s)'D_2(s,\alpha_s)'D_2(s,\alpha_s)u(s)\\
&\qquad + 2X_2^+(s)P_1^a(s,\alpha_s)C_2(s,\alpha_s)'D_2(s,\alpha_s)u(s)\\
&\qquad + 2X_2^+(s)P_1^a(s,\alpha_s)B_2(s,\alpha_s)'u_s + 2X_2^+(s)\Lambda_1^a(s,\alpha_s)'D_2(s,\alpha_s)u(s)\\
&\qquad + I_{\{X_2(s)<0\}}P_2^a(s,\alpha_s)u(s)'D_2(s,\alpha_s)'D_2(s,\alpha_s)u(s)\\
&\qquad - 2X_2^-(s)P_2^a(s,\alpha_s)C_2(s,\alpha_s)'D_2(s,\alpha_s)u(s)\\
&\qquad - 2X_2^-(s)P_2^a(s,\alpha_s)B_2(s,\alpha_s)'u(s) - 2X_2^-(s)\Lambda_2^a(s,\alpha_s)'D_2(s,\alpha_s)u(s)\\
&\qquad - X_2^+(s)^2\mathscr{H}_+^a(s, P_1^a(s,\alpha_s), \Lambda_1^a(s,\alpha_s), \alpha_s)\\
&\qquad - X_2^-(s)^2\mathscr{H}_-^a(s, P_2^a(s,\alpha_s), \Lambda_2^a(s,\alpha_s), \alpha_s)\\
&\qquad - X_2(s)^2Q_2(s,\alpha_s)\Big|\mathcal{F}_\tau\Big\}\,\mathrm{d}s.
\end{aligned}
$$

After rearrangement and combining similar terms,

$$
\begin{aligned}
&\mathbb{E}\Big\{P_1^a(T\wedge\theta_n, \alpha_{T\wedge\theta_n})X_2^+(T\wedge\theta_n)^2 + P_2^a(T\wedge\theta_n, \alpha_{T\wedge\theta_n})X_2^-(T\wedge\tau_n)^2\\
&\qquad\qquad + \int_\tau^{T\wedge\theta_n}\Big(Q_2(s,\alpha_s)X_2(s)^2 + u_s'R_2(s,\alpha_s)u(s)\Big)\,\mathrm{d}s\Big|\mathcal{F}_\tau\Big\}\\
&= P_1^a(\tau,\alpha_\tau)X_2^+(\tau)^2 + P_2^a(\tau,\alpha_\tau)X_2^-(\tau)^2 + \mathbb{E}\Big\{\int_\tau^{T\wedge\theta_n}\phi_2(s, X_2(s), u(s), \alpha_s)\,\mathrm{d}s\Big|\mathcal{F}_\tau\Big\},
\end{aligned}
\tag{3.5.6}
$$

where

$$
\begin{aligned}
&\phi_2(s, X_2, u, i) \\
&= u'\Big(R_2(s,i) + I_{\{X_2 \geqslant 0\}} P_1^a(s,i) D_2(s,i)' D_2(s,i) \\
&\qquad\qquad + I_{\{X_2<0\}} P_2^a(s,i) D_2(s,i)' D_2(s,i)\Big) u \\
&\quad + 2X_2^+ (P_1^a(s,i) C_2(s,i)' D_2(s,i) + P_1^a(s,i) B_2(s,i)' + \Lambda_1^a(s,i)' D_2(s,i)) u \\
&\quad - 2X_2^- (P_2^a(s,i) C_2(s,i)' D_2(s,i) + P_2^a(s,i) B_2(s,i)' + \Lambda_2^a(s,i)' D_2(s,i)) u \\
&\quad - (X_2^+)^2 \mathscr{H}_+^a(s, P_1^a(s,i), \Lambda_1^a(s,i), i) - (X_2^-)^2 \mathscr{H}_-^a(s, P_2^a(s,i), \Lambda_2^a(s,i), i).
\end{aligned}
$$

Define an $\mathcal{F}_t$-adapted process

$$
v(t) = \begin{cases} \frac{u(t)}{|X_2(t)|}, & \text{if } \ |X_2(t)| > 0; \\ 0, & \text{if } \ X_2(t) = 0. \end{cases}
$$

Notice that Γ is a cone, so the process v is valued in Γ. If $X_2(s) = 0$, then

$$
\phi_2(s, X_2(s), u(s), \alpha_s) = u(s)'\Big(R_2(s,\alpha_s) + P_1^a(s,\alpha_s) D_2(s,\alpha_s)' D_2(s,\alpha_s)\Big) u(s) \geqslant 0.
$$

If $X_2(s) > 0$, then

$$
\begin{aligned}
\phi_2(s, X_2(s), u(s), \alpha_s) = X_2(s)^2 \Big\{ & v(s)'\Big(R_2(s,\alpha_s) + P_1^a(s,\alpha_s) D_2(s,\alpha_s)' D_2(s,\alpha_s)\Big) v(s) \\
& + 2(P_1^a(s,\alpha_s) C_2(s,\alpha_s)' D_2(s,\alpha_s) + P_1^a(s,\alpha_s) B_2(s,\alpha_s)' \\
& + \Lambda_1^a(s,\alpha_s)' D_2(s,\alpha_s)) v(s) - \mathscr{H}_+^a(s, P_1^a(s,\alpha_s), \Lambda_1^a(s,\alpha_s), \alpha_s) \Big\}.
\end{aligned}
$$

By the definition of $\mathscr{H}_+^a(t, P, \Lambda, i)$, this is non-negative. Similarly, we have $\phi_2(s, X_2(s), u(s), \alpha_s) \geqslant 0$, when $X_2(s) < 0$. Hence, it follows from (3.5.6) that

$$
\begin{aligned}
&\mathbb{E}\Big\{ P_1^a(T\wedge\theta_n, \alpha_{T\wedge\theta_n}) X_2^+(T\wedge\theta_n)^2 + P_2^a(T\wedge\theta_n, \alpha_{T\wedge\theta_n}) X_2^-(T\wedge\theta_n)^2 \\
&\qquad + \int_\tau^{T\wedge\theta_n} \big[Q_2(s,\alpha_s) X_2(s)^2 + u(s)' R_2(s,\alpha_s) u(s)\big] \, \mathrm{d}s \Big| \mathcal{F}_\tau \Big\} \\
&\geqslant P_1^a(\tau,\alpha_\tau) X_2^+(\tau)^2 + P_2^a(\tau,\alpha_\tau) X_2^-(\tau)^2. \qquad (3.5.7)
\end{aligned}
$$

It is not hard to verify $\mathbb{E}\Big[\sup_{s\in[\tau,T]} X_2(s)^2\Big] < \infty$ by standard theory of SDE. Let $n \to \infty$, by the dominated convergence and monotone convergence theorems, we have

$$\begin{aligned}
&\mathbb{E}\Big\{\int_\tau^T \big[Q_2(s,\alpha_s)X_2(s)^2 + u(s)'R_2(s,\alpha_s)u(s)\big]\,\mathrm{d}s + G_2(\alpha_T)X_2(T)^2\Big|\mathcal{F}_\tau\Big\}\\
&\geqslant P_1^a(\tau,\alpha_\tau)X_2^+(\tau)^2 + P_2^a(\tau,\alpha_\tau)X_2^-(\tau)^2. \qquad (3.5.8)
\end{aligned}$$

We now prove $\bar{u}$ defined by (3.3.6) is an optimal control. Similar to Lemma 4.3 in Hu et al. (2022b), one can prove that $\bar{u}$ is an admissible control for problem (3.3.1). Under this pair, it is easy to verify that (3.5.7) is an equation. Similar to the above limit argument, one can prove that (3.5.8) is also an equation, thus $\bar{u}$ is an optimal control. We leave the details to the interested readers. □

3.5.5 Proof of Theorem 3.5

Proof. First, for any $(u,\tau)\in\mathscr{A}$,

$$\begin{aligned}
J^b(x,i_0,u(\cdot),\tau) &= \mathbb{E}\Big\{\int_0^\tau \big[Q_1(t,\alpha_t)X_1(t)^2 + u(t)'R_1(t,\alpha_t)u(t)\big]\,\mathrm{d}t\\
&\qquad + \mathbb{G}_1(\tau,\alpha_\tau)X_1^+(\tau)^2 + \mathbb{G}_2(\tau,\alpha_\tau)X_1^-(\tau)^2\Big\}\\
&\geqslant \mathbb{E}\Big\{\int_0^\tau \big[Q_1(t,\alpha_t)X_1(t)^2 + u(t)'R_1(t,\alpha_t)u(t)\big]\,\mathrm{d}t\\
&\qquad + P_1^b(\tau,\alpha_\tau)X_1^+(\tau)^2 + P_2^b(\tau,\alpha_\tau)X_1^-(\tau)^2\Big\}. \qquad (3.5.9)
\end{aligned}$$

Applying Itô's formula to $P_1^b(s,\alpha_s)X_1^+(s)^2$ and $P_2^b(s,\alpha_s)X_1^-(s)^2$ yields

$$
\begin{aligned}
&P_1^b(t,\alpha_t)X_1^+(t)^2 + P_2^b(t,\alpha_t)X_1^-(t)^2 \\
&= P_1^b(0,i_0)(x^+)^2 + P_2^b(0,i_0)(x^-)^2 \\
&+ \int_0^t \Big\{ I_{\{X_1(s)\geqslant 0\}} P_1^b(s,\alpha_s)u(s)'D_1(s,\alpha_s)'D_1(s,\alpha_s)u(s) \\
&\qquad + 2X_1^+(s)P_1^b(s,\alpha_s)C_1(s,\alpha_s)'D_1(s,\alpha_s)u(s) \\
&\qquad + 2X_1^+(s)P_1^b(s,\alpha_s)B_1(s,\alpha_s)'u(s) + 2X_1(s)^+\Lambda_1^b(s,\alpha_s)'D_1(s,\alpha_s)u(s) \\
&\qquad - X_1^+(s)^2[Q_1(s,\alpha_s) + \mathscr{H}_+^b(s,P_1^b(s,\alpha_s),\Lambda_1^b(s,\alpha_s),\alpha_s)] \\
&\qquad + I_{\{X_1(s)<0\}} P_2^b(s,\alpha_s)u(s)'D_1(s,\alpha_s)'D_1(s,\alpha_s)u(s) \\
&\qquad - 2X_1^-(s)P_2^b(s,\alpha_s)C_1(s,\alpha_s)'D_1(s,\alpha_s)u(s) \\
&\qquad - 2X_1^-(s)P_2^b(s,\alpha_s)B_1(s,\alpha_s)'u(s) - 2X_1^-(s)\Lambda_2^b(s,\alpha_s)'D_1(s,\alpha_s)u(s) \\
&\qquad - X_1^-(s)^2[Q_1(s,\alpha_s) + \mathscr{H}_-^b(s,P_2^b(s,\alpha_s),\Lambda_2^b(s,\alpha_s),\alpha_s)] \Big\}\,\mathrm{d}s \\
&+ \int_0^t X_1^+(s)^2\,\mathrm{d}\mu_1(s,\alpha_s) + \int_0^t X_1^-(s)^2\,\mathrm{d}\mu_2(s,\alpha_s) + \int_0^t (\ldots)\,\mathrm{d}W(s) \\
&+ \int_0^t \Big\{ X_1^+(s)^2 \sum_{j,j'\in\mathcal{M}} (P_1^b(s,j) - P_1^b(s,j'))I_{\{\alpha_{s-}=j'\}} \\
&\qquad\quad + X_1^-(s)^2 \sum_{j,j'\in\mathcal{M}} (P_2^b(s,j) - P_2^b(s,j'))I_{\{\alpha_{s-}=j'\}} \Big\}\,\mathrm{d}\tilde{N}_s^{j'j}.
\end{aligned}
$$

Via localized completion of squares method as in Theorem 3.4, we have

$$\begin{aligned}
&\mathbb{E}\Big\{\int_0^{\tau\wedge\theta_n}\big[Q_1(t,\alpha_t)X_1(t)^2+u(t)'R_1(t,\alpha_t)u(t)\big]\,\mathrm{d}t\\
&\qquad\qquad +P_1^b(\tau\wedge\theta_n,\alpha_{\tau\wedge\theta_n})X_1^+(\tau\wedge\theta_n)^2+P_2^b(\tau\wedge\theta_n,\alpha_{\tau\wedge\theta_n})X_1^-(\tau\wedge\theta_n)^2\Big\}\\
=\;&\mathbb{E}\Big\{P_1^b(0,i_0)(x^+)^2+P_2^b(0,i_0)(x^-)^2+\int_0^{\tau\wedge\theta_n}\phi_1\left(s,X_1(s),u(s),\alpha_s\right)\mathrm{d}s\\
&\qquad\qquad +\int_0^{\tau\wedge\theta_n}X_1^+(s)^2\,\mathrm{d}\mu_1(s,\alpha_s)+\int_0^{\tau\wedge\theta_n}X_1^-(s)^2\,\mathrm{d}\mu_2(s,\alpha_s)\Big\},
\end{aligned}\tag{3.5.10}$$

where

$$\begin{aligned}
&\phi_1(s,X_1,u,i)\\
&=u'\Big(R_1(s,i)+I_{\{X_1\geqslant 0\}}P_1^b(s,i)D_1(s,i)'D_1(s,i)\\
&\qquad\qquad +I_{\{X_1<0\}}P_2^b(s,i)D_1(s,i)'D_1(s,i)\Big)u\\
&\quad+2X_1^+(P_1^b(s,i)C_1(s,i)'D_1(s,i)+P_1^b(s,i)B_1(s,i)'+\Lambda_1^b(s,i)'D_1(s,i))u\\
&\quad-2X_1^-(P_2^b(s,i)C_1(s,i)'D_1(s,i)+P_2^b(s,i)B_1(s,i)'+\Lambda_2^b(s,i)'D_1(s,i))u\\
&\quad-(X_1^+)^2\mathscr{H}_+^b(s,P_2^b(s,i),\Lambda_1^b(s,i),i)-(X_1^-)^2\mathscr{H}_-^b(s,P_2^b(s,i),\Lambda_2^b(s,i),i).
\end{aligned}$$

Repeat the argument of the proof of Theorem 3.4, we have $\phi_1(s,X_1(s),u(s),\alpha_s)\geqslant 0$. Since μ_1 and μ_2 are non-decreasing processes,

$$\int_0^{\tau\wedge\theta_n}X_1^+(s)^2\,\mathrm{d}\mu_1(s,\alpha_s)\geqslant 0,\quad \int_0^{\tau\wedge\theta_n}X_1^-(s)^2\,\mathrm{d}\mu_2(s,\alpha_s)\geqslant 0.$$

So it follows from (3.5.10) that

$$\begin{aligned}
&\mathbb{E}\Big\{\int_0^{\tau\wedge\theta_n}\big[Q_1(s,\alpha_s)X_1(s)^2+u(s)'R_1(s,\alpha_s)u(s)\big]\,\mathrm{d}s\\
&\qquad\qquad +P_1^b(\tau\wedge\theta_n,\alpha_{\tau\wedge\theta_n})X_1^+(\tau\wedge\theta_n)^2+P_2^b(\tau\wedge\theta_n,\alpha_{\tau\wedge\theta_n})X_1^-(\tau\wedge\theta_n)^2\Big\}\\
&\geqslant P_1^b(0,i_0)(x^+)^2+P_2^b(0,i_0)(x^-)^2.
\end{aligned}$$

It is not hard to verify $\mathbb{E}\Big[\sup_{s\in[0,T]} X_1(s)^2\Big] < \infty$ by standard theory of SDE. Let $n \to \infty$, by the dominated convergence and monotone convergence theorems, we have

$$\begin{aligned}
&\mathbb{E}\Big\{\int_0^\tau \big[Q_1(s,\alpha_s)X_1(s)^2 + u(s)'R_1(s,\alpha_s)u(s)\big]\,\mathrm{d}s \\
&\qquad + P_1^b(\tau,\alpha_\tau)X_1^+(\tau)^2 + P_2^b(\tau,\alpha_\tau)X_1^-(\tau)^2\Big\} \\
&\geqslant P_1^b(0,i_0)(x^+)^2 + P_2^b(0,i_0)(x^-)^2. \qquad (3.5.11)
\end{aligned}$$

Hence, we proved, for any $(u,\tau) \in \mathscr{A}$,

$$J^b(u,\tau) \geqslant P_1^b(0,i_0)(x^+)^2 + P_2^b(0,i_0)(x^-)^2.$$

In order to prove $(\bar{u},\bar{\tau})$ is an optimal control, it suffices to prove

$$J^b(\bar{u},\bar{\tau}) = P_1^b(0,i_0)(x^+)^2 + P_2^b(0,i_0)(x^-)^2. \qquad (3.5.12)$$

Thanks to the definition of $\bar{u}$, we see $\phi_1(s, X_1^{\bar{u}}(s), \bar{u}(s), \alpha_s) = 0$. Suppose $x \geqslant 0$. Due to homogeneity, it is easy to verify that $X_1^{\bar{u}}$ does not change sign until $\bar{\tau}$, namely $X_1^{\bar{u}}(s) \geqslant 0$ for $s \in [0,\bar{\tau})$. Thanks to definition, one has $\mu_1(s,\alpha_s) = 0$ for $s \in [0,\bar{\tau})$. Thus, when $(u,\tau) = (\bar{u},\bar{\tau})$, (3.5.9) and (3.5.11) are both equations. Therefore, (3.5.12) holds, showing that $(\bar{u},\bar{\tau})$ is an optimal pair. The case $x < 0$ can be dealt in a similar way. $\square$

3.5.6 Proof of Theorem 3.6

Proof. Suppose $(X(\cdot), u(\cdot), \tau(\cdot))$ is a state-control pair corresponding to the given initial pair $(x, i_0) \in \mathbb{R} \times \mathcal{M}$. For any $u(\cdot) \in \mathcal{U}[0, T]$

$$
\begin{aligned}
& V(x, i_0) \\
= & \inf_{(u,\tau)\in\mathscr{A}} \mathbb{E}\Big\{ \int_0^\tau \big[Q_1(t,\alpha_t)X_1(t)^2 + u_1(t)'R_1(t,\alpha_t)u_1(t)\big]\,\mathrm{d}t + G_1(\tau,\alpha_\tau)X_1(\tau)^2 \\
& + \int_\tau^T \big[Q_2(t,\alpha_t)X_2(t)^2 + u_2(t)'R_2(t,\alpha_t)u_2(t)\big]\,\mathrm{d}t + G_2(\alpha_T)X_2(T)^2 \Big\} \\
= & \inf_{(u,\tau)\in\mathscr{A}} \mathbb{E}\Big\{ \int_0^\tau \big[Q_1(t,\alpha_t)X_1(t)^2 + u_1(t)'R_1(t,\alpha_t)u_1(t)\big]\,\mathrm{d}t + G_1(\tau,\alpha_\tau)X_1(\tau)^2 \\
& + \mathbb{E}\Big[\int_\tau^T \big[Q_2(t,\alpha_t)X_2(t)^2 + u_2(t)'R_2(t,\alpha_t)u_2(t)\big]\,\mathrm{d}t + G_2(\alpha_T)X_2(T)^2 \Big| \mathcal{F}_\tau \Big] \Big\} \\
\geqslant & \inf_{(u,\tau)\in\mathcal{U}[0,T]} \mathbb{E}\Big\{ \int_0^\tau \big[Q_1(t,\alpha_t)X_1(t)^2 + u_1(t)'R_1(t,\alpha_t)u_1(t)\big]\,\mathrm{d}t \\
& \qquad + P_1^a(\tau,\alpha_\tau)X_2^+(\tau)^2 + P_2^a(\tau,\alpha_\tau)X_2^-(\tau)^2 + G_1(\tau,\alpha_\tau)X_1(\tau)^2 \Big\} \\
= & \inf_{(u,\tau)\in\mathcal{U}[0,T]\times\mathcal{T}[0,T]} \mathbb{E}\Big\{ \int_0^\tau \big[Q_1(t,\alpha_t)X_1(t)^2 + u_1(t)'R_1(t,\alpha_t)u_1(t)\big]\,\mathrm{d}t \\
& \qquad + \mathbb{G}_1(\tau,\alpha_\tau)X_1^+(\tau)^2 + \mathbb{G}_2(\tau,\alpha_\tau)X_1^-(\tau)^2 \Big\}.
\end{aligned}
$$

The inequality holds at (3.3.6) in Theorem 3.4. Via Theorem 3.5, we have

$$V(x, i_0) \geqslant P_1^b(0, i_0)(x^+)^2 + P_2^b(0, i_0)(x^-)^2.$$

From the definition of $V(x, i_0)$ (3.1.4), we establish a lower boundary of $J(x, i_0, u(\cdot), \tau)$.

Repeating the proof of Theorem 3.4, one can show that (3.3.10) and (3.3.11) forms an optimal pair. This completes the proof. $\square$

Chapter 4

Non-homogeneous Stochastic LQ Control and Stopping with Random Coefficients in Non-Markovian Regime Switching System

In Chapter 3, we solved a stochastic LQ control problem (3.1.2) under a random coefficients model with non-Markovian regime switching. The scaled-value X follows a dynamic (3.1.1) driven by the Brownian motion driving the asset price dynamics and the Markov chain standing for the regime switching. The control pair consists of a classical control variable constrained in a cone ($u \in \Gamma$) and a stopping time τ. For a given initial value x_0, we establish the existence of an optimal control $\bar{u}$ and optimal stopping time $\bar{\tau}$ through the completing the square method. This approach finds a lower bound for the cost functional (3.3.8) and identifies that the optimal control-stopping pair achieves this lower bound. This lower bound is exactly the value function (3.3.15) of the control problem (3.1.2). Furthermore, the optimal control is expressed as a feedback form dependent on time t, state X, and market regime α_t. The value function (3.3.15) is quadratic dependence on the initial value x_0. Substituting the optimal control into the dynamics (3.1.1) yields the corresponding optimal state trajectory X.

We provide a more rigorous mathematical definition of the homogeneous properties of control problem (3.1.2). The homogeneous function is defined as follows:

Definition 4.1. *Given a real constant p, a function $\varphi : \mathbb{R}^k \to \mathbb{R}^d$ is called a homogeneous function of degree p in x if*

$$\varphi(\lambda x) = \lambda^p \varphi(x)$$

holds for all $\lambda > 0$ and $x \in \mathbb{R}^k$.

Specifically, in the stochastic homogeneous control problem appears in Chapter 3, the system defined by (3.1.1) and (3.3.8) exhibits two fundamental homogeneity properties. First, the state dynamics (3.1.1) are homogeneous of degree 1 in $(x, u) \in \mathbb{R} \times \mathbb{R}^m$: scaling both initial state $x_0 \mapsto \lambda x_0$ and control $u(\cdot) \mapsto \lambda u(\cdot)$ induces proportional scaling $X(t) \mapsto \lambda X(t)$ of the wealth process. Financially, this implies that scaling an investor's initial capital by λ while proportionally scaling their trading strategy results in proportional scaling of the entire wealth trajectory. Second, the value function $V(x, i_0)$ is homogeneous of degree 2 in (x, u), satisfying $V(\lambda x, i_0) = \lambda^2 V(x, i_0)$. This quadratic homogeneity reflects that scaling both wealth and control by λ^2, capturing the quadratic risk-return relationship inherent in portfolio optimization where proportional capital commitment amplifies costs/returns quadratically.

Consequently, the homogeneous properties of control problem (3.1.2) intrinsically align with the cone constraints on u. Utilizing the method of undetermined coefficients and completing the square, we derive the reflected extended stochastic Riccati equations (3.2.1) and (3.2.2), and the extended stochastic Riccati equations (3.2.3) and (3.2.4). The presence of cone constraints induces an optimization operator in the drivers of these four BSDE systems, constituting a fundamental distinction between the Riccati equations in this thesis and prior literature. Crucially, the control weighting matrix R in the cost functional may be singular. Consequently, BSDEs (3.2.1), (3.2.2) , (3.2.3) and (3.2.4) are highly nonlinear multidimensional, they violate both the standard Lipschitz condition and the quadratic growth condition due to the structural behavior of their Hamiltonians $\mathscr{H}_{\pm}^{a}$ and $\mathscr{H}_{\pm}^{b}$, refer to the Remark 3.3 in Chapter 3. To address this, we establish well-posedness by partitioning the analysis into standard and singular cases, with distinct proof strategies tailored to the structural properties of the Hamiltonians.

In this chapter, we introduce a new extension: A non-homogeneous stochastic LQ control problem with regime switching where the control pair consists of a unconstrained control u and a controlled stopping time τ. First, non-homogeneous coefficients introduce affine terms (b, ρ) in the state dynamics (4.1.1), capturing exogenous shocks and time-varying trends absent in (3.1.1); Second, the cost functional (4.1.3) minimizes deviations from stochastic targets (q, p, g), modeling tracking objectives; Third, control u are now unconstrained, permitting strategies like unlimited short-selling essential in derivatives trading.

Beyond mathematical generalization, this framework addresses key financial incompleteness: non-homogeneous terms incorporate market frictions (e.g., stochastic dividends/funding costs), while regime switching captures structural volatility breaks during crises. This extension is particularly significant for options markets: The affine terms model stochastic volatility and jump risks crucial for hedging exotic options (e.g., barrier options) under regime shifts, while the tracking terms minimize replication errors in delta-hedging strategies. The stopping time τ naturally represents optimal exercise decisions for American options, with the state-switch at τ reflecting payoff realization. Unlike Chapter 3's cone constraints, the unconstrained control space allows aggressive volatility arbitrage strategies. These features enable novel applications in pricing and hedging under realistic market conditions, establishing this extension as a substantial advancement with independent research value.

We divide the whole problem into two parts termed before τ and after τ and analyze each part respectively. On $[\tau, T]$, via the completion of squares method, the optimal control and value function could be expressed by the solutions to one coupled BSDE systems (4.2.2) and (4.2.4) (see Theorem 4.3) termed ESREs. Building on the method of Hu et al. (2024), we establish the solvability of our problem, prove its well-posedness using BMO martingales and the John-Nirenberg inequality, and derive explicit optimal feedback controls.

Similar to Chapter 3, we can reduce our problem to a mixed control and stopping problem on $[0, \tau]$. But unfortunately, we cannot solve this problem completely. This is because the value function does not take the quadratic form and the optimal control is not a linear feedback of the state process. Instead, we will provide a lower bound problem for it, which still takes the LQ structure. (Of course, an upper bound exists trivially as it is a minimizing problem.)

Then, we establish the relationship between the control and stopping lower problem on $[0, \tau]$ and obtain the RBSDE system (4.2.1) and (4.2.3). Both RBSDE systems (4.2.1) and (4.2.3) are new.

Our second contribution is about the solvability of RBSDE systems (4.2.1) and (4.2.3). In proving the existence of a unique solution to RBSDE system (4.2.1), whose solvability is established by slightly modifying our previous argument in Theorem 3.3. For RBSDE system (4.2.3), are indeed a family of highly linear unbounded RBSDEs. Three challenges need to be overcome. First, all the coefficients of the linear BSDE are unbounded due to the unboundedness of $\Lambda(i)$ (and consequently $\Gamma(i)$). Second, the equations are coupled through the coupling term $\sum_{j=1}^{\ell} q_{ij} k(j)$. Third, the boundary restriction condition in (4.2.1) that forces solutions to stay below the given boundary. To our knowledge, no existing results directly apply to this BSDE systems. Establishing the solvability of (4.2.2) therefore represents the main technical contribution of this work. The main idea to establish the solvability of the new type of BSDEs is first to get some estimates of BMO martingales and then establish the result for one-dimensional system, and finally apply contraction mapping method to get the result for multi-dimensional system. This constitutes the major technique contribution of this thesis.

The remainder of this chapter is organized as follows. In Section 4.1, we formulate a non-homogemeous stochastic LQ problem with regime switching, random coefficients; the control pair consists of classical control and controlled stopping time. Section 4.2 obtains the global solvability of systems of linear RBSDEs with unbounded coefficients. Section 4.3 gives a lower bound to the mixed LQ problem. Finally, Section 4.4 concludes the chapter.

4.1 Model Setup and Problem Formulation

This chapter adopts the same probabilistic framework established in Chapter 3 (see Section 3.1). For the ease of reading, we recall it as follows.

We fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a standard n-dimensional Brownian motion $W(t) = (W_1(t), \ldots, W_n(t))'$ and a continuous-time stationary Markov chain $\{\alpha_t\}_{t\geqslant 0}$ valued in a finite state space $\mathcal{M} = \{1, 2, \ldots, \ell\}$ $(\ell > 1)$, where the two processes are mutually independent. The Markov chain α_t is governed by an infinitesimal generator $Q = (q_{ij})_{\ell\times\ell}$ satisfying

$$q_{ij} \geqslant 0 \text{ for } i \neq j \quad \text{and} \quad \sum_{j=1}^{\ell} q_{ij} = 0 \text{ for all } i \in \mathcal{M}.$$

To model the information flow, we define the augmented filtration

$$\mathcal{F}_t = \sigma\{W(s), \alpha_s : 0 \leqslant s \leqslant t\} \vee \mathcal{N}$$

and the Brownian filtration

$$\mathcal{F}_t^W = \sigma\{W(s) : 0 \leqslant s \leqslant t\} \vee \mathcal{N},$$

where $\mathcal{N}$ represents the collection of all $\mathbb{P}$-null sets in $\mathcal{F}$.

4.1.1 Non-homogeneous System

Consider the following $\mathbb{R}$-valued non-homogeneous linear stochastic differential equation (SDE):

$$\begin{cases} \mathrm{d}X_1(t) = \big[A_1(t,\alpha_t)X_1(t) + B_1(t,\alpha_t)'u(t) + b_1(t,\alpha_t)\big]\,\mathrm{d}t \\ \qquad\qquad + \big[C_1(t,\alpha_t)X_1(t) + D_1(t,\alpha_t)u(t) + \rho_1(t,\alpha_t)\big]'\,\mathrm{d}W(t),\ \ t \in [0,\tau), \\ \mathrm{d}X_2(t) = \big[A_2(t,\alpha_t)X_2(t) + B_2(t,\alpha_t)'u(t) + b_2(t,\alpha_t)\big]\,\mathrm{d}t \\ \qquad\qquad + \big[C_2(t,\alpha_t)X_2(t) + D_2(t,\alpha_t)u(t) + \rho_2(t,\alpha_t)\big]'\,\mathrm{d}W(t),\ \ t \in [\tau,T], \\ X_2(\tau) = K(\tau)X_1(\tau-0), \\ X_1(0) = x \geqslant 0,\ \ \alpha_0 = i_0 \in \mathcal{M}, \end{cases} \tag{4.1.1}$$

where $A_\gamma(t,\omega,i)$, $B_\gamma(t,\omega,i)$, $b_\gamma(t,\omega,i)$, $C_\gamma(t,\omega,i)$, $D_\gamma(t,\omega,i)$, $\rho_\gamma(t,\omega,i)$, are all $\{\mathcal{F}_t^W\}_{t\geqslant 0}$-adapted processes of suitable sizes for $\gamma = 1,2$, $i \in \mathcal{M}$, the initial states $x \in \mathbb{R}_+$ and $i_0 \in \mathcal{M}$ are known.

We represent the entire state process as $X(t) = X_1(t)\,\mathbb{1}_{t<\tau} + X_2(t)\,\mathbb{1}_{t\geqslant\tau}$. From equation (4.1.1), it is evident that the state process starts from an initial position and transitions to a different process at time τ, where the initial condition of the new process depends on the terminal value of the previous one. Here, $(u(\cdot),\tau)$ constitutes a control pair, with $u(\cdot)$ taking values from $\mathbb{R}^m$ and τ is an $\{\mathcal{F}_t\}_{t\geqslant 0}$-stopping time.

The class of admissible controls and admissible stopping time are defined as the set

$$\mathcal{U}[0,T] := \Big\{u(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)\Big\},$$

$$\mathcal{T}[0,T] := \Big\{\tau : \Omega \to [0,T] \;\Big|\; \tau \text{ is an } \{\mathcal{F}_t\}_{t\geqslant 0}\text{-stopping time}\Big\}.$$

The set of admissible control pairs is defined as

$$\mathscr{B} := \Big\{(u(\cdot),\tau) \in \mathcal{U}[0,T] \times \mathcal{T}[0,T] \;\Big|\; X_1^u(s) \geqslant 0 \text{ for } s \in [0,\tau)\Big\}.$$

Correspondingly, we refer to $(X(\cdot),u(\cdot),\tau)$ as an admissible triple, for the system (4.1.1).

Remark that set of admissible control pairs here is different from that of homogenous problem (3.1.2) where

$$\mathscr{A} = \mathcal{U}[0,T] \times \mathcal{T}[0,T].$$

This is because we cannot solve the Pre-Switch LQ problem in the non-homogenous case. Instead, we will give a lower bound, which requires the sign of the state to be non-positive before stopping. This setting will not affect Post-Switch LQ problem. Economically speaking, we do not allow the state to be bankrupt before stopping.

Let us now state our mixed stochastic linear quadratic optimal control problem (MSLQ problem) as follows: Given $(x, i_0) \in \mathbb{R}_+ \times \mathcal{M}$, our goal is to

$$\begin{cases} \text{Minimize} & J(x, i_0, u(\cdot), \tau) \\ \text{subject to} & (\tau, u(\cdot)) \in \mathscr{B},\ (X(\cdot), u(\cdot), \tau) \text{ follows } (4.1.1), \end{cases} \tag{4.1.2}$$

where the cost functional is given as the following quadratic form

$$\begin{aligned} J(x, i_0, u(\cdot), \tau(\cdot)) = \mathbb{E}\Big[\int_0^\tau \Big(& Q_1(t, \alpha_t)(X_1(t) - q_1(t, \alpha_t))^2 \\ & + (u(t) - p_1(t, \alpha_t))' R_1(t, \alpha_t)(u(t) - p_1(t, \alpha_t)) \Big) \, \mathrm{d}t \\ & + G_1(\tau, \alpha_\tau)\big(X_1(\tau) - g_1(\tau, \alpha_\tau)\big)^2 \\ & + \int_\tau^T \Big(Q_2(t, \alpha_t)(X_2(t) - q_2(t, \alpha_t))^2 \\ & + (u(t) - p_2(t, \alpha_t))' R_2(t, \alpha_t)(u(t) - p_2(t, \alpha_t)) \Big) \, \mathrm{d}t \\ & + G_2(\alpha_T)\big(X_2(T) - g_2(\alpha_T)\big)^2 \Big]. \end{aligned} \tag{4.1.3}$$

The associated value function is defined as

$$V(x, i_0) = \inf_{(u(\cdot), \tau) \in \mathscr{B}} J(x, i_0, u(\cdot), \tau),\ x \in \mathbb{R}_+,\ i_0 \in \mathcal{M}.$$

Problem (4.1.2) is said to be finite at $(x, i_0) \in \mathbb{R}_+ \times \mathcal{M}$, if $V(x, i_0) > -\infty$; and to be solvable, if there exists a pair control $(\bar{u}(\cdot), \bar{\tau}) \in \mathscr{B}$ such that

$$J(x, i_0, \bar{u}(\cdot), \bar{\tau}) = V(x, i_0) > -\infty,$$

in which case, $(\bar{u}(\cdot), \bar{\tau})$ is called an optimal control pair for problem (4.1.2) at $(x, i_0) \in \mathbb{R}_+ \times \mathcal{M}$. To make sure the well-posedness of the MSLQ problem (4.1.2), we put the following assumptions in this chapter.

Assumption 4.1. *For all* $i \in \mathcal{M}$,

$$\begin{cases} A_1(\cdot,i),\ A_2(\cdot,i),\ b_1(\cdot,i),\ b_2(\cdot,i),\ q_1(\cdot,i),\ q_2(\cdot,i) \in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}), \\ B_1(\cdot,i),\ B_2(\cdot,i),\ p_1(\cdot,i),\ p_2(\cdot,i) \in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}^m), \\ C_1(\cdot,i),\ C_2(\cdot,i),\ \rho_1(\cdot,i),\ \rho_2(\cdot,i) \in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}^n),\ D_1(\cdot,i),\ D_2(\cdot,i) \in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}^{n\times m}), \\ Q_1(\cdot,i),\ Q_2(\cdot,i) \in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}),\ R_1(\cdot,i),\ R_2(\cdot,i) \in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{S}^m), \\ G_1(\cdot,i),\ g_1(\cdot,i) \in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}),\ G_2(i),\ g_2(i) \in L^\infty_{\mathcal{F}^W}(\Omega;\mathbb{R}), \\ \mathbb{G}(\cdot,i), \mathbb{K}(\cdot,i) \in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}), \\ K(\cdot) \in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}). \end{cases}$$

Assumption 4.2. *There exists a constant $\delta > 0$ such that at least one of the following cases holds.*

(i) ***Standard case.*** *For all* $i \in \mathcal{M}$,

$$Q_1(i) \geqslant 0,\ R_1(i) \geqslant \delta I_m,\ \textit{and}\ \mathbb{G}(T,i) \geqslant \delta,$$

$$Q_2(i) \geqslant 0,\ R_2(i) \geqslant \delta I_m,\ \textit{and}\ G_2(i) \geqslant \delta.$$

(ii) ***Singular case.*** *For all* $i \in \mathcal{M}$,

$$Q_1(i) \geqslant 0,\ R_1(i) \geqslant 0,\ D_1(i)^\top D_1(i) \geqslant \delta I_m, \textit{and}\ \mathbb{G}(T,i) \geqslant \delta,$$

$$Q_2(i) \geqslant 0,\ R_2(i) \geqslant 0,\ D_2(i)^\top D_2(i) \geqslant \delta I_m, \textit{and}\ G_2(i) \geqslant \delta.$$

Here, “singular” refers to the potential singularity of $R_1(i)$, $R_2(i)$ in the cost functional (4.1.3).

Under Assumption 4.1, SDE (4.1.1) has a unique solution $X(\cdot) \in L^2_{\mathcal{F}}(\Omega; C(0,T;\mathbb{R}))$ for any $(u(\cdot), \tau) \in \mathscr{B}$ according to standard SDE theory, and the cost functional (4.1.3) is well-defined, making problem (4.1.2) well-posed.

The remainder of this chapter solve problem (4.1.2). We first present several technical tools that will used to solve the Riccati equations in problem (4.1.2).

4.1.2 Snell Envelope

To prove existence for RBSDEs, we use the Snell Envelope – a key tool for obstacle constraints. This supermartingale gives the optimal expected value when stopping a process, directly relating to how RBSDEs solutions must stay above barriers. We'll use it in this chapter to construct the solution by connecting optimal stopping theory to our RBSDEs.

For our approach, we need $class(D)$ processes, which is used extensively in the literature on stochastic process theory, especially for Doob-Meyer decomposition, see Karatzas and Shreve (1991). We recall its definition (see, Pham (2009)).

Definition 4.2. *Let $\mathscr{S}$ (respectively, $\mathscr{S}_a$) be the class of stopping times τ of the filtration $\left\{\mathcal{F}_t^W\right\}$ which satisfy $P(\tau < \infty) = 1$ (respectively, $P(\tau \leqslant a) = 1$ for a given finite number $a > 0$). A right-continuous process $\left\{X(t),\ \mathcal{F}_t^W; 0 \leqslant t < \infty\right\}$ is said to be of class (D), if the family $\{X(\tau)\}_{\tau\in\mathscr{S}}$ is uniformly integrable; of class (DL), if the family $\{X(\tau)\}_{\tau\in\mathscr{S}_a}$ is uniformly integrable, for every $0 < a < \infty$.*

For any $t \in [0, T]$, let $\mathcal{T}_{t,T}$ denote the set of $\left\{\mathcal{F}_s^W\right\}_{s\in[0,T]}$-stopping times valued in $[t, T]$.

Definition 4.3 (Snell Envelope). *Let $H = (H(t))_{0\leqslant t\leqslant T}$ be a real-valued $\left\{\mathcal{F}_t^W\right\}$-adapted càdlàg process, in the class (DL). The Snell Envelope V of H is defined by*

$$V(t) = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\left[H(\tau) \mid \mathcal{F}_t^W\right], \quad 0 \leqslant t \leqslant T.$$

It is the smallest supermartingale of class (DL), which dominates H, i.e., $V(t) \geqslant H(t)$, $0 \leqslant t \leqslant T$. Furthermore, if H has only positive jumps, i.e. $H(t) - H(t^-) \geqslant 0$, $0 \leqslant t \leqslant T$, then V is continuous, and for all $t \in [0, T]$, the stopping time

$$\tau_t = \inf\left\{s \geqslant t : V(s) = H(s)\right\} \wedge T$$

is optimal after t, *i.e.*

$$V(t)=\mathbb{E}\left[V_{\tau_t}\mid\mathcal{F}_t^W\right]=\mathbb{E}\left[H_{\tau_t}\mid\mathcal{F}_t^W\right].$$

4.1.3 The Doob-Meyer Decomposition

The Doob-Meyer decomposition is essential for our study of RBSDEs. Since solutions involve submartingales constrained by barriers, this decomposition will help us separate martingale components from increasing processes. We'll apply it directly to establish the existence of RBSDEs later.

Definition 4.4. *An* $\left\{\mathcal{F}_t^W\right\}$*-adapted process* A *is called increasing, if for almost all* $\omega\in\Omega$ *we have*

(i) $A(0,\omega)=0$*;*

(ii) $t\mapsto A(t,\omega)$ *is a nondecreasing, right-continuous function;*

(iii) $\mathbb{E}\left(A(t)\right)<\infty$ *holds for every* $t\in[0,\infty)$.

An increasing process $t\mapsto A(t,\omega)$ *is called integrable if* $\mathbb{E}\left(A(\infty)\right)<\infty$*, where* $A(\infty):=\lim_{t\to\infty}A(t)$.

Thanks to the monotone convergence theorem, an increasing process A is integrable if and only if $\lim_{t\to\infty}\mathbb{E}[A(t)]<\infty$.

Definition 4.5. *An increasing process* A *is called natural if for every bounded, right-continuous martingale* $\left\{M(t),\mathcal{F}_t^W;0\leqslant t<\infty\right\}$ *we have*

$$\mathbb{E}\int_{(0,t]}M(s)\,\mathrm{d}A(s)=\mathbb{E}\int_{(0,t]}M(s^-)\,\mathrm{d}A(s),\qquad \textit{for every } 0<t<\infty.$$

For a detailed discussion of the Doob-Meyer decomposition and the proof of the following theorem, refer to Karatzas and Shreve (1991).

Theorem 4.1 (Doob-Meyer). *Let* $\{\mathcal{F}_t^W\}$ *satisfy the usual conditions. If a right-continuous submartingale* $X = \{X(t), \mathcal{F}_t;\ 0 \leqslant t < \infty\}$ *is of class (DL), then it admits the decomposition*

$$X = M + A, \tag{4.1.4}$$

as the summation of a right-continuous martingale $M = \{M(t),\ \mathcal{F}_t^W; 0 \leqslant t < \infty\}$ *and an increasing process* $A = \{A(t),\ \mathcal{F}_t^W; 0 \leqslant t < \infty\}$. *The latter can be taken to be natural; under this additional condition, the decomposition* (4.1.4) *is unique (up to indistinguishability). Further, if* X *is of class (D), then* M *is a uniformly integrable martingale and* A *is integrable.*

4.2 The Extended Stochastic Riccati Equation

4.2.1 Solutions to ESREs and RESREs: Existence and Uniqueness

To tackle the MSLQ problem (4.1.2), we first introduce the following system of ℓ-dimensional RBSDEs:

$$\begin{cases} \mathrm{d}P(i) = -\Big[(2A_1(i) + C_1(i)'C_1(i))P(i) + 2C_1(i)'\Lambda(i) \\ \qquad\qquad + H^b(P(i), \Lambda(i), i) + \sum\limits_{j=1}^{\ell} q_{ij}P(j)\Big]\,\mathrm{d}t + \mathrm{d}\mu + \Lambda(i)'\,\mathrm{d}W, \\ P(T,i) = \mathbb{G}(T,i), \\ R_1(i) + P(i)D_1(i)'D_1(i) > 0, \\ P(i) \leqslant \mathbb{G}(i),\ \int_0^T (\mathbb{G}(i) - P(i))\,\mathrm{d}\mu(i) = 0, \mu(0,i) = 0, \quad \text{for all } i \in \mathcal{M}, \end{cases} \tag{4.2.1}$$

where

$$H^b(t, P, \Lambda, i) = -\big(PB_1(i) + D_1(i)'(PC_1(i) + \Lambda)\big)'\big(R_1(i) + PD_1(i)'D_1(i)\big)^{-1} \cdot \big(PB_1(i) + D_1(i)'(PC_1(i) + \Lambda)\big).$$

For RBSDEs (4.2.1), its solution is subject to the point-wise constraint

$$P(i) \leqslant \mathbb{G}(i),$$

and the non-decreasing process $\mu(i)$ of the solution is used to push the main solution component $P(i)$ to satisfy the above constraint in a minimum energy way, i.e.,

$$\int_0^T (\mathbb{G}(i) - P(i))\,\mathrm{d}\mu(i) = 0.$$

We introduce the following ℓ-dimensional BSDEs:

$$\begin{cases} \mathrm{d}P(i) = -\Big[(2A_2(i) + C_2(i)'C_2(i))P(i) + 2C_2(i)'\Lambda(i) + Q_2(i) \\ \qquad\qquad +H^a(P(i), \Lambda(i), i) + \sum\limits_{j=1}^{\ell} q_{ij}P(j)\Big]\,\mathrm{d}t + \Lambda(i)'\,\mathrm{d}W, \\ P(T,i) = G_2(i), \\ R(i) + P(i)D_2(i)'D_2(i) > 0, \quad \text{for all } i \in \mathcal{M}. \end{cases} \tag{4.2.2}$$

where

$$H^a(t, P, \Lambda, i) = -\big(PB_2(i) + D_2(i)'(PC_2(i) + \Lambda)\big)'\big(R_2(i) + PD_2(i)'D_2(i)\big)^{-1} \cdot \big(PB_2(i) + D_2(i)'(PC_2(i) + \Lambda)\big).$$

RBSDEs (4.2.1) is RESREs, while BSDEs (4.2.2) is ESREs, for the mixed control problem (4.1.2), which has been extensively studied in Hu et al. (2022b) and Li and Xu (2023).

For clarity of statement, sometimes we use (P^a, Λ^a) and (P^b, Λ^b, μ^b) for the solution of (4.2.2) and (4.2.1) respectively.

Definition 4.6. *A vector process $(P(i),\ \Lambda(i),\ \mu(i))_{i=1}^{\ell}$ is called a solution to the ℓ-dimensional RBSDEs (4.2.1) if it satisfies (4.2.1) and $(P(i),\ \Lambda(i),\ \mu(i)) \in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}^W}(0,T;\mathbb{R}^n) \times \mathcal{A}_c(0,T;\mathbb{R})$ for all $i \in \mathcal{M}$.*

Definition 4.7. *A vector process $(P(i),\ \Lambda(i))_{i=1}^{\ell}$ is called a solution to the ℓ-dimensional BSDEs (4.2.2) if it satisfies (4.2.2) and $(P(i),\ \Lambda(i)) \in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}^W}(0,T;\mathbb{R}^n)$ for all $i \in \mathcal{M}$. The solution to (4.2.2) is defined similarly.*

In fact, the second parts Λ of the solutions of (4.2.1) and (4.2.2) turn out to be in the class of martingales of bounded mean oscillation, briefly called *BMO martingales*.

Since we have already established the foundational properties of such martingales in Chapter 3, we briefly recall only the essential property, see Kazamaki (2006) for more details. The Doléans-Dade stochastic exponential

$$\mathcal{E}\bigg(\int_0^\cdot \Lambda(s)'\,\mathrm{d}W(s)\bigg)$$

of a BMO martingale $\int_0^\cdot \Lambda(s)'\,\mathrm{d}W(s)$ is a uniformly integrable martingale. Moreover, if $\int_0^\cdot \Lambda(s)'\,\mathrm{d}W(s)$ and $\int_0^\cdot Z(s)'\,\mathrm{d}W(s)$ are both BMO martingales, then

$$\widetilde{W}(\cdot) := W(\cdot) - \int_0^\cdot Z(s)\,\mathrm{d}s$$

is a standard Brownian motion, and $\int_0^\cdot \Lambda(s)'\,\mathrm{d}\widetilde{W}(s)$ is a BMO martingale, under the probability measure $\widetilde{\mathbb{P}}$ defined by

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{d\mathbb{P}}\bigg|_{\mathcal{F}_T} = \mathcal{E}\bigg(\int_0^T Z(s)'\,\mathrm{d}W(s)\bigg).$$

The following space plays an important role in our argument

$$L^{2,\,\mathrm{BMO}}_{\mathcal{F}^W}(0,T;\mathbb{R}^n) := \bigg\{\phi \in L^2_{\mathcal{F}^W}(0,T;\mathbb{R}^n)\;\bigg|\;\int_0^\cdot \phi(s)'\,\mathrm{d}W(s)\text{ is a BMO martingale}\bigg\}.$$

We set

$$\bigg\|\int_0^\cdot \phi'\,\mathrm{d}W\bigg\|_{\mathrm{BMO}_2} = \sup_{\tau}\operatorname*{ess\,sup}_{\omega\in\Omega}\mathbb{E}\bigg[\int_\tau^T |Z_t|^2\,\mathrm{d}t\;\bigg|\;\mathcal{F}_\tau\bigg]^{1/2} < \infty,$$

where the supreme is taken over all $\mathcal{F}^W$-stopping times τ.

Lemma 4.1. *Under Assumptions 4.1 and 4.2, the system of BSDEs* (4.2.2) *admits a unique uniformly positive solution* $(P(i),\Lambda(i))_{i\in\mathcal{M}}$.

The proof is similar to that of Lemma 3.1 of Hu et al. (2024).

Lemma 4.2. *Under Assumptions 4.1 and 4.2, the system of RBSDEs (4.2.1) admits a unique uniformly positive solution* $(P(i), \Lambda(i), \mu(i))_{i\in\mathcal{M}}$.

Proof. The proof is given in section 4.5.1. □

Remark 4.1. *The given system of RBSDEs (4.2.1) and BSDEs (4.2.2) violate both the standard Lipschitz condition and the quadratic growth condition due to the structural properties of the nonlinear terms* H^b *and* H^a *in their generators.*

4.2.2 Linear RBSDEs with Unbounded Coefficients

Non-homogeneous MSLQ problems 4.1.2 differ fundamentally from homogeneous MLQ problems 3.1.2 in that the latter requires solving a new class of nonlinear BSDEs and RBSDEs system with unbounded coefficients.

For all $i \in \mathcal{M}$, let $(P^a(i), \Lambda^a(i))$, $(P^b(i), \Lambda^b(i), \mu^b(i))$ be the unique uniformly positive solution to (4.2.1) and (4.2.2). Set

$$\Gamma^a(i) = \big(R_2(i) + P^a(i)D_2(i)'D_2(i)\big)^{-1}\big(P^a(i)B_2(i) + D_2(i)'(P^a(i)C_2(i) + \Lambda^a(i))\big),$$

$$\Gamma^b(i) = \big(R_1(i) + P^b(i)D_1(i)'D_1(i)\big)^{-1}\big(P^b(i)B_1(i) + D_1(i)'(P^b(i)C_1(i) + \Lambda^b(i))\big).$$

We consider the following system of ℓ-dimensional linear RBSDEs,

$$
\begin{cases}
\mathrm{d}k(i) = -\Big[\big(P^b(i)B_1(i) + D_1(i)'(P^b(i)C_1(i) + \Lambda^b(i))\big)'\big(R_1(i) + P^b(i)D_1(i)'D_1(i)\big)^{-1} \\
\qquad\qquad \times\big[D_1(i)'(P^b(i)\rho_1(i) - l(i)) - k(i)B_1(i) - R_1(i)p_1(i)\big] \\
\qquad\qquad +A_1(i)k(i) + C_1(i)'l(i) - P^b(i)(C_1(i)'\rho_1(i) + b_1(i)) - \rho_1(i)'\Lambda^b(i) \\
\qquad\qquad +q_1(i)Q_1(i) + \sum\limits_{j=1}^{\ell} q_{ij}k(j)\Big]\,\mathrm{d}t - \mathrm{d}\mu + l(i)'\,\mathrm{d}W \\
\qquad = -\Big[\big(A_1(i) - B_1(i)'\Gamma^b(i)\big)k(i) + \big(C_1(i) - D_1(i)\Gamma^b(i)\big)'l(i) + (P^b(i)D_1(i)'\rho_1(i) \\
\qquad\quad -R_1(i)p_1(i))'\Gamma^b(i) + q_1(i)Q_1(i) - P^b(i)(C_1(i)'\rho_1(i) + b_1(i)) \\
\qquad\quad -\rho_1(i)'\Lambda^b(i) + \sum\limits_{j=1}^{\ell} q_{ij}k(j)\Big]\,\mathrm{d}t - \mathrm{d}\mu + l(i)'\,\mathrm{d}W, \\
k(T,i) = \mathbb{K}(T,i), \\
k(i) \geqslant \mathbb{K}(i), \quad \int_0^T (\mathbb{K}(i) - k(i))\,\mathrm{d}\mu(i) = 0, \\
\mu(0,i) = 0, \quad \text{for all } i \in \mathcal{M}.
\end{cases}
\tag{4.2.3}
$$

and the following system of ℓ-dimensional linear BSDEs

$$
\begin{cases}
\mathrm{d}k(i) = -\Big[\big(P^a(i)B_2(i) + D_2(i)'(P^a(i)C_2(i) + \Lambda^a(i))\big)'\big(R_2(i) + P^a(i)D_2(i)'D_2(i)\big)^{-1} \\
\qquad\quad \times\big[D_2(i)'(P^a(i)\rho_2(i) - l(i)) - k(i)B_2(i) - R_2(i)p_2(i)\big] \\
\qquad\quad +A_2(i)k(i) + C_2(i)'l(i) - P^a(i)(C_2(i)'\rho_2(i) + b_2(i)) - \rho_2(i)'\Lambda^a(i) \\
\qquad\quad +q_2(i)Q_2(i) + \sum\limits_{j=1}^{\ell} q_{ij}k(j)\Big]\,\mathrm{d}t + l(i)'\,\mathrm{d}W \\
\qquad = -\Big[\big(A_2(i) - B_2(i)'\Gamma^a(i)\big)k(i) + \big(C_2(i) - D_2(i)\Gamma^a(i)\big)'l(i) + (P^a(i)D_2(i)'\rho_2(i) \\
\qquad\quad -R_2(i)p_2(i))'\Gamma^a(i) + q_2(i)Q_2(i) - P^a(i)(C_2(i)'\rho_2(i) + b_2(i)) \\
\qquad\quad -\rho_2(i)'\Lambda^a(i) + \sum\limits_{j=1}^{\ell} q_{ij}k(j)\Big]\,\mathrm{d}t + l(i)'\,\mathrm{d}W, \\
k(T,i) = G_2(i)g_2(i), \ \text{for all } i \in \mathcal{M}.
\end{cases}
\tag{4.2.4}
$$

Similarly, we may use (k^a, l^a) and (k^b, l^b, μ^b) for the solution of (4.2.4) and (4.2.3) respectively.

Although (4.2.3) and (4.2.4) are linear BSDE systems, it structures creates three major difficulties: First, its coefficients become unbounded due to the unboundedness of $\Lambda(i)$ (and consequently $\Gamma(i)$). Furthermore, the equations are coupled through the coupling term $\sum_{j=1}^{\ell} q_{ij} k(j)$. Third, the boundary restriction condition in (4.2.3) that forces solutions to stay above the given boundary. The solvability of (4.2.4) is given in Hu et al. (2024).

To our knowledge, no existing results directly apply to (4.2.3) systems. Establishing the solvability of (4.2.3) therefore represents the main technical contribution of this work. To solve (4.2.3), we will employ several auxiliary lemmas. Young's Inequality and the John-Nirenberg inequality, presented in Proposition 1.3.2 of Zhang (2017) and Theorem 2.2 of Kazamaki (2006), are two of these.

Lemma 4.3. *Under Assumptions 4.1 and 4.2, the system of linear BSDEs* (4.2.4) *admits a unique solution* $(k(i), l(i))_{i\in\mathcal{M}}$ *such that*

$$(k(i), l(i)) \in L^{\infty}_{\mathcal{F}^W}(0,T;\mathbb{R}) \times L^{2,\mathrm{BMO}}_{\mathcal{F}}(0,T;\mathbb{R}^n), \ \textit{for all } i \in \mathcal{M}.$$

Proof. The proof is given in Theorem 3.7 of Hu et al. (2024). □

Lemma 4.4 (Young's Inequality)**.** *Assume* $p, q > 1$ *and* $\frac{1}{p}+\frac{1}{q} = 1$*. For any* $x, y \in \mathbb{R}^n$*, it holds:*

$$|xy| \leqslant \frac{|x|^p}{p} + \frac{|y|^q}{q}.$$

In particular, for $p = q = 2$*,* $\varepsilon > 0$*,* $x, y \in \mathbb{R}$*, we have*

$$2xy \leqslant \varepsilon x^2 + \varepsilon^{-1} y^2 \quad \text{and} \quad (x+y)^2 \leqslant (1+\varepsilon)x^2 + \left(1+\varepsilon^{-1}\right) y^2.$$

Lemma 4.5 (The John-Nirenberg Inequality)**.** *Suppose* $\phi \in L^{2,\,\mathrm{BMO}}_{\mathcal{F}^W}(0,T;\mathbb{R}^n)$ *and*

$$\left\| \int_0^{\cdot} \phi(s)'\,\mathrm{d}W(s) \right\|_{\mathrm{BMO}_2} < 1.$$

Then for all $\{\mathcal{F}^W_t\}_{t\geqslant 0}$*-stopping times* $\tau \leqslant T$*,*

$$\mathbb{E}\left[e^{\int_\tau^T |\phi(s)|^2\,\mathrm{d}s} \,\middle|\, \mathcal{F}^W_\tau \right] \leqslant \frac{1}{1 - \left\| \int_0^{\cdot} \phi(s)'\,\mathrm{d}W(s) \right\|_{\mathrm{BMO}_2}}.$$

From this Lemma, we immediately have the following estimate.

Lemma 4.6. *Suppose $\phi \in L^{2,\,\mathrm{BMO}}_{\mathcal{F}^W}(0,T;\mathbb{R}^n)$. Then for any constants $p \in (0,2)$ and $K > 0$, there exists a constant $c_{p,K} > 0$ such that*

$$\mathbb{E}\Big[e^{K\int_\tau^T |\phi(s)|^p\,\mathrm{d}s}\;\Big|\;\mathcal{F}^W_\tau\Big] \leqslant c_{p,K}$$

hold for all $\{\mathcal{F}^W_t\}_{t\geqslant 0}$-stopping times $\tau \leqslant T$.

Proof. The proof is given in Lemma 3.3 of Hu et al. (2024). □

The following lemma can be found in Page 26 of Kazamaki (2006).

Lemma 4.7. *Suppose $\phi \in L^{2,\,\mathrm{BMO}}_{\mathcal{F}^W}(0,T;\mathbb{R}^n)$. For any constant $p \geqslant 1$, there is a generic constant $K_p > 0$ such that*

$$\mathbb{E}\left[\Big(\int_\tau^T |\phi(s)|^2\,\mathrm{d}s\Big)^p\Big|\mathcal{F}^W_\tau\right] \leqslant K_p\left\|\int_0^\cdot \phi(s)'\,\mathrm{d}W(s)\right\|^{2p}_{\mathrm{BMO}_2}$$

hold for all $\{\mathcal{F}^W_t\}_{t\geqslant 0}$-stopping times $\tau \leqslant T$.

As one of our contributions, we further extend Lemma 4.6 and provide that through the measure transformation, similar estimation still holds under the new measure.

Lemma 4.8. *Suppose $\phi \in L^{2,\,\mathrm{BMO}}_{\mathcal{F}^W}(0,T;\mathbb{R}^n)$ and let β be a $\mathbb{R}^n$-valued $\{\mathcal{F}^W_t\}_{t\geqslant 0}$-adapted process satisfying the bound $|\beta| \leqslant |\phi|$. Define $\widetilde{\mathbb{P}}$ by*

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}^W_T} = \mathcal{E}\left(\int_0^T \beta(s)'\,\mathrm{d}W(s)\right).$$

Then for any constants $p \in (0,2)$ and $K > 0$, there exists a constant $c_{p,K} > 0$ such that

$$\widetilde{\mathbb{E}}\Big[e^{K\int_\tau^T |\phi(s)|^p\,\mathrm{d}s}\;\Big|\;\mathcal{F}^{\widetilde{W}}_\tau\Big] \leqslant c_{p,K}$$

hold for all $\{\mathcal{F}^{\widetilde{W}}_t\}_{t\geqslant 0}$-stopping times $\tau \leqslant T$.

Proof. The proof is given in section 4.5.2. □

The following result solves a new class of one-dimensional BSDEs with all coefficients being unbounded. It will be used to establish the corresponding result in multi-dimensional case, that is (4.2.3).

Lemma 4.9. *Let $p \in (0,2)$ be a constant, a and f be two $\mathbb{R}$-valued $\{\mathcal{F}_t^W\}_{t\geqslant 0}$-adapted processes, $\phi \in L^{2,\,\mathrm{BMO}}_{\mathcal{F}^W}(0,T;\mathbb{R}^n)$, and β be a $\mathbb{R}^n$-valued $\{\mathcal{F}_t^W\}_{t\geqslant 0}$-adapted process satisfying the bounds*

$$|a| \leqslant |\phi|^p, \ |\beta| \leqslant |\phi|, \ |f| \leqslant |\phi|^2.$$

Then given any $\xi \in L^{\infty}_{\mathcal{F}^W_T}(\Omega;\mathbb{R})$, and a continuous progressively measurable real-valued 'obstacle' process $S \in L^{\infty}_{\mathcal{F}^W}(0,T;\mathbb{R})$ satisfying $\xi \geqslant S(T)$, the following 1-dimensional RBSDEs

$$\begin{cases} -\,\mathrm{d}Y(t) = \big(a(t)Y(t) + \beta(t)'Z(t) + f(t)\big)\,\mathrm{d}t + \mathrm{d}\mu - Z(t)'\,\mathrm{d}W, \\ Y(T) = \xi, \\ Y(t) \geqslant S(t), \quad \int_0^T (Y(t) - S(t))\,\mathrm{d}\mu(t) = 0, \\ \mu(0) = 0. \end{cases} \tag{4.2.5}$$

admits a unique solution $(Y,Z,\mu) \in L^{\infty}_{\mathcal{F}^W}(0,T;\mathbb{R}) \times L^{2,\,\mathrm{BMO}}_{\mathcal{F}^W}(0,T;\mathbb{R}^n) \times \mathcal{A}_c(0,T;\mathbb{R})$.

Proof. The proof is given in section 4.5.3. □

With the help of the above one-dimensional result and contraction mapping, we can solve a system of multi-dimensional BSDEs with all coefficients being unbounded.

Theorem 4.2. *Suppose $p \in (0,2)$ be a constant, for every $i,j \in \mathcal{M}$, $\phi(i) \in L^{2,\,\mathrm{BMO}}_{\mathcal{F}^W}(0,T;\mathbb{R}^n)$, $f(i)$ and κ_{ij} are $\mathbb{R}$-valued $\{\mathcal{F}_t^W\}_{t\geqslant 0}$-adapted processes, and $\beta(i)$ is a $\mathbb{R}^n$-valued $\{\mathcal{F}_t^W\}_{t\geqslant 0}$-adapted process such that*

$$|\beta(i)| \leqslant |\phi(i)|, \ |f(i)| \leqslant |\phi(i)|^2, \ \sum_{j\in\mathcal{M}} |\kappa_{ij}| \leqslant |\phi(i)|^p.$$

Then for any given terminal values $\xi(i) \in L^{\infty}_{\mathcal{F}^W}(0,T;\mathbb{R})$, and continuous progressively measurable real-valued 'obstacle' processes $S(i) \in L^{\infty}_{\mathcal{F}^W}(0,T;\mathbb{R})$ satisfying $\xi(i) \geqslant S(T,i)$ for all $i \in \mathcal{M}$, the following system of ℓ-dimensional RBSDEs:

$$\begin{cases} -\,\mathrm{d}k(i) = \Big[\beta(i)' l(i) + f(i) + \sum\limits_{j \in \mathcal{M}} \kappa_{ij} k(j)\Big]\,\mathrm{d}t + \mathrm{d}\mu - l(i)'\,\mathrm{d}W, \\ k(T,i) = \xi(i), \\ k(t,i) \geqslant S(t,i), \quad \int_0^T (k(t,i) - S(t,i))\,\mathrm{d}\mu(t,i) = 0, \\ \mu(0,i) = 0, \quad \textit{for all } i \in \mathcal{M}, \end{cases} \tag{4.2.6}$$

admits a unique solution $(k(i), l(i), \mu(i))_{i \in \mathcal{M}}$ such that

$$(k(i), l(i), \mu(i)) \in L^{\infty}_{\mathcal{F}^W}(0,T;\mathbb{R}) \times L^{2,\,\mathrm{BMO}}_{\mathcal{F}^W}(0,T;\mathbb{R}^n) \times \mathcal{A}_c(0,T;\mathbb{R}), \ \textit{for all } i \in \mathcal{M}.$$

Proof. The proof is given in section 4.5.4. □

Remark 4.2. *We emphasis here again that all of $\beta(i)$, $f(i)$ and κ_{ij} are unbounded, whereas in (Hu et al., 2022b, (5.8)) only $\beta(i)$ is unbounded.*

Corollary 4.1. *Under Assumptions 4.1 and 4.2, the system of linear RBSDEs* (4.2.3) *admits a unique solution $(k(i), l(i), \mu(i))_{i \in \mathcal{M}}$ such that*

$$(k(i), l(i), \mu(i)) \in L^{\infty}_{\mathcal{F}^W}(0,T;\mathbb{R}) \times L^{2,\,\mathrm{BMO}}_{\mathcal{F}^W}(0,T;\mathbb{R}^n) \times \mathcal{A}_c(0,T;\mathbb{R}), \ \textit{for all } i \in \mathcal{M}.$$

Proof. The proof is given in section 4.5.5. □

4.3 On the LQ Problem (4.1.2)

4.3.1 Post-Switch LQ Problem (4.3.1)

For given $(\tau, X_2(\tau), \alpha_\tau) \in \mathcal{T}[0,T] \times L^2_{\mathcal{F}_\tau}(\Omega;\mathbb{R}) \times \mathcal{M}$, consider the post-switch linear quadratic (Post-Switch LQ) problem:

$$\begin{cases} \text{Minimize} & J^a(\tau, X_2(\tau), \alpha_\tau, u(\cdot)) \\ \text{subject to} & (\tau, u(\cdot)) \in \mathscr{B}, \ (X_2(\cdot), u(\cdot)) \text{ follows (4.1.1)}, \end{cases} \tag{4.3.1}$$

where the quadratic cost functional

$$J^a(\tau, X_2(\tau), \alpha_\tau, u(\cdot)) = \mathbb{E}\Big[\int_\tau^T \Big(Q_2(t, \alpha_t)(X_2(t) - q_2(t, \alpha_t))^2 + (u(t) - p_2(t, \alpha_t))' R_2(t, \alpha_t)(u(t) - p_2(t, \alpha_t)) \Big) \,\mathrm{d}t + G_2(\alpha_T)\big(X_2(T) - g_2(\alpha_T)\big)^2 \,\Big|\, \mathcal{F}_\tau \Big].$$

Define the conditional minimal value function system

$$V^a(\tau, X_2(\tau), \alpha_\tau) = \operatorname*{ess\,inf}_{u(\cdot) \in \mathcal{U}[0,T]} J^a(\tau, X_2(\tau), \alpha_\tau, u(\cdot)). \tag{4.3.2}$$

For any admissible control pair $(u(\cdot), \tau) \in \mathscr{B}$, we always have that $u(\cdot)\,\mathbb{1}_{\cdot \geqslant \tau} \in \mathcal{U}[0, T]$, therefore the above value function (4.3.2) is linked to our problem (4.1.2).

The solution to the Post-Switch LQ problem (4.3.1) is stated as follows.

Theorem 4.3. *Suppose that Assumptions 4.1 and 4.2 hold. Let $(P^a(t, i),\ \Lambda^a(t, i))_{i \in \mathcal{M}}$ and $(k^a(t, i),\ l^a(t, i))_{i \in \mathcal{M}}$ be the unique solutions of the systems of BSDEs (4.2.2) and (4.2.4), respectively. Then the Post-Switch LQ problem (4.3.1) has an optimal control, as a feedback function of the time t, the state X_2, and the market regime i,*

$$\begin{aligned} \bar{u}(t, X_2(t), i) = &-\big(R_2(t, i) + P^a(t, i) D_2(t, i)' D_2(t, i)\big)^{-1} \\ &\times \Big[\Big(P^a(t, i) D_2(t, i)' C_2(t, i) + P^a(t, i) B_2(t, i) + D_2(t, i)' \Lambda^a(t, i) \Big) X_2 \\ &\quad + P^a(t, i) D_2(t, i)' \rho_2(t, i) - k^a(t, i) B_2(t, i) \\ &\quad - D_2(t, i)' l^a(t, i) - R_2(t, i) p_2(t, i) \Big]. \end{aligned} \tag{4.3.3}$$

Moreover, the corresponding optimal value is

$$
\begin{aligned}
&V^a(\tau, X_2(\tau), \alpha_\tau) \\
={}& P^a(\tau, \alpha_\tau) X_2(\tau)^2 - 2k^a(\tau, \alpha_\tau) X_2(\tau) + \mathbb{E}[G_2(\alpha_T) g_2(\alpha_T)^2 \mid \mathcal{F}_\tau] \\
&+ \mathbb{E}\Big[\int_\tau^T \Big(P^a(t, \alpha_t) \rho_2(t, \alpha_t)' \rho_2(t, \alpha_t) - 2k^a(t, \alpha_t) b_2(t, \alpha_t) \\
&\quad - 2\rho_2(t, \alpha_t)' l^a(t, \alpha_t) + Q_2(t, \alpha_t) q_2(t, \alpha_t)^2 + p_2(t, \alpha_t)' R_2(t, \alpha_t) p_2(t, \alpha_t) \\
&\quad - \big[D_2(t, \alpha_t)' (P^a(t, \alpha_t) \rho_2(t, \alpha_t) - l^a(t, \alpha_t)) - k^a(t, \alpha_t) B_2(t, \alpha_t) \\
&\quad - R_2(t, \alpha_t) p_2(t, \alpha_t) \big]' \big(R_2(t, \alpha_t) + P^a(t, \alpha_t) D_2(t, \alpha_t)' D_2(t, \alpha_t) \big)^{-1} \\
&\quad \times \big[D_2(t, \alpha_t)' (P^a(t, \alpha_t) \rho_2(t, \alpha_t) - l^a(t, \alpha_t)) - k^a(t, \alpha_t) B_2(t, \alpha_t) \\
&\quad - R_2(t, \alpha_t) p_2(t, \alpha_t) \big] \Big) \, \mathrm{d}t \, \Big| \, \mathcal{F}_\tau \Big].
\end{aligned} \tag{4.3.4}
$$

Proof. The admissibility of the control process $\bar{u}(t, X_2(t), \alpha_t)$ will be proved in the following lemma. The reminder of the proof is similar to that of Theorem 3.4 in Chapter 3 via applying Itô's Lemma to $P^a(t, \alpha_t) X_2(t)^2 - 2k^a(t, \alpha_t) X_2(t)$, so we leave the details to the diligent readers. □

Lemma 4.10. *Under the conditions of Theorem 4.3, we have*

$$
\bar{u}(t, X_2(t), \alpha_t) \in L^2_{\mathcal{F}}(\tau, T; \mathbb{R}^m).
$$

Proof. The proof is given in section 4.5.6. □

4.3.2 Pre-Switch LQ Problem

Under Assumption 4.1, Post-Switch problem admits a unique optimal control which can be represented by (4.3.3). In this section, we will study the problem which is termed of control-stopping (or mixed) type because it combines control and stopping. The controller, at her/his convenience, the controller chooses also the time τ to stop controlling the system.

Let $(P^a(i),\ \Lambda_1^a(i))_{i=1}^{\ell}$ and $(k^a(i),\ l^a(i))_{i=1}^{\ell})$ be the unique solutions to (4.2.2) and (4.2.4), respectively. To simplify the notation, for all $i \in \mathcal{M}$, we define

$$\begin{aligned}\mathbb{G}(t,i) &= G_1(t,i) + K(t)^2 P^a(t,i),\\ \mathbb{K}(t,i) &= (G_1(t,i)g_1(t,i) + k^a(t,i)K(t)),\end{aligned}$$

and

$$\begin{aligned}\mathbb{C}^a(t,i) = {} & P^a(t,i)\rho_2(t,i)'\rho_2(t,i) - 2k^a(t,i)b_2(t,i) - 2\rho_2(t,i)'l^a(t,i)\\ & + Q_2(t,i)q_2(t,i)^2 + p_2(t,i)'R_2(t,i)p_2(t,i)\\ & - \big[D_2(t,i)'(P^a(t,i)\rho_2(t,i) - l^a(t,i)) - k^a(t,i)B_2(t,i)\\ & - R_2(t,i)p_2(t,i)\big]'\big(R_2(t,i) + P^a(t,i)D_2(t,i)'D_2(t,i)\big)^{-1}\\ & \times \big[D_2(t,i)'(P^a(t,i)\rho_2(t,i) - l^a(t,i)) - k^a(t,i)B_2(t,i)\\ & - R_2(t,i)p_2(t,i)\big],\end{aligned}$$

which is a fixed process that is obtained from the Post-Switch LQ problem (4.3.1), independent of control u. We now define

$$\mathbb{C}^b(t,i) = G_1(t,i)g_1(t,i)^2 + \mathbb{E}\Big[G_2(\alpha_T)g_2(\alpha_T)^2 + \int_0^T \mathbb{C}^a(t,\alpha_t)\,\mathrm{d}t\Big],$$

and

$$\begin{aligned}J^b(x,i_0,u(\cdot),\tau) = \mathbb{E}\Big[\int_0^\tau & \Big(Q_1(t,\alpha_t)(X_1(t)-q_1(t,\alpha_t))^2\\ & + (u(t)-p_1(t,\alpha_t))'R_1(t,\alpha_t)(u(t)-p_1(t,\alpha_t)) - \mathbb{C}^a(t,\alpha_t)\Big)\,\mathrm{d}t\Big]\\ & + \mathbb{G}(\tau,\alpha_\tau)X_1(\tau)^2 - 2\mathbb{K}(\tau,\alpha_\tau)X_1(\tau) + \mathbb{C}^b(\tau,\alpha_\tau).\end{aligned}$$

We consider a problem:

$$\begin{cases}\text{Minimize} & J^b(x,i_0,u(\cdot),\tau)\\ \text{subject to} & (\tau,u(\cdot)) \in \mathscr{B},\ (X_1(\cdot),u(\cdot),\tau) \text{ follows (4.1.1).}\end{cases} \tag{4.3.5}$$

The corresponding value function is defined as

$$V^b(x, i_0) = \inf_{(u(\cdot),\tau)\in\mathscr{B}} J^b(x, i_0, u(\cdot), \tau). \tag{4.3.6}$$

We call the problem (4.3.5) the Pre-Switch LQ problem.

Unfortunately, we cannot solve the above Pre-Switch LQ Problem (4.3.5) completely. Instead, we will provide a lower bound for it in the next section.

4.3.3 Lower Boundary to the Pre-Switch LQ Problem (4.3.5)

Let $(P^b(i),\ \Lambda^b(i), \mu_1(i))_{i\in\mathcal{M}}$ be the unique solutions of the systems of RBSDEs (4.2.1). Define a constant $M = \operatorname{ess\,sup}_{j,\omega,t}\mathbb{K}(j)$. Let

$$\begin{aligned}
&J^b_{\text{lower}}(x, i_0, u(\cdot), \tau)\\
=\ &\mathbb{E}\Big[\int_0^\tau \Big(Q_1(t,\alpha_t)(X_1(t) - q_1(t,\alpha_t))^2\\
&+ (u(t) - p_1(t,\alpha_t))'R_1(t,\alpha_t)(u(t) - p_1(t,\alpha_t)) - \mathbb{C}^a(t,\alpha_t)\Big)\,\mathrm{d}t\\
&+ P^b(\tau,\alpha_\tau)X_1(\tau)^2 - 2MX_1(\tau) + \mathbb{C}^b(\tau,\alpha_\tau)\Big].
\end{aligned} \tag{4.3.7}$$

The set of admissible control pairs is defined as the unconstraint one

$$\mathscr{A} = \mathcal{U}[0,T]\times\mathcal{T}[0,T].$$

We consider the following Pre-Switch Lower Boundary LQ problem:

$$\begin{cases} \text{Minimize} & J^b_{\text{lower}}(x, i_0, u(\cdot), \tau)\\ \text{subject to} & (\tau, u(\cdot))\in\mathscr{A},\ (X_1(\cdot), u(\cdot), \tau) \text{ follows } (4.1.1). \end{cases} \tag{4.3.8}$$

The corresponding value function is defined as

$$V^b_{\text{lower}}(x, i_0) = \inf_{(u(\cdot),\tau)\in\mathscr{A}} J^b_{\text{lower}}(x, i_0, u(\cdot), \tau). \tag{4.3.9}$$

Theorem 4.4. *Suppose that Assumptions 4.1 and 4.2 hold. Let $(P^b(i),\ \Lambda^b(i), \mu_1(i))_{i\in\mathcal{M}}$ be the unique solutions of the systems of RBSDEs (4.2.1). Then the Pre-Switch Lower*

Boundary LQ problem (4.3.8) *has a lower bound:*

$$V^b_{\text{lower}}(x,i_0) \geqslant P^b(0,i_0)x^2 - 2Mx + \mathbb{E}\Big\{\int_0^T \mathbb{C}^a(t,\alpha_t)\,\mathrm{d}t + G_2(\alpha_T)g_2(\alpha_T)^2\Big\} + \hat{M},$$

where

$$\hat{M} = \inf_{\tau\in\mathcal{T}[0,T]} \mathbb{E}\Big\{G_1(\tau,\alpha_\tau)g_1(\tau,\alpha_\tau)^2 + \int_0^\tau \Big[P^b(s,\alpha_s)\rho_1(s,\alpha_s)'\rho_1(s,\alpha_s) - 2Mb_1(s,\alpha_s)$$

$$+ Q_1(s,\alpha_s)q_1(s,\alpha_s)^2 + p_1(s,\alpha_s)'R_1(s,\alpha_s)p_1(s,\alpha_s) - \mathbb{C}^a(s,\alpha_s) + \underline{\phi}(s,\alpha_s)\Big]\,\mathrm{d}s\Big\},$$

and

$$\underline{\phi}(s,\alpha_s) = \frac{\mathbf{c}\mathbf{e}^2 + \mathbf{d}^2 - \mathbf{b}\mathbf{d}\mathbf{e}}{4\mathbf{a}\mathbf{c} - \mathbf{b}^2},$$

and

$$\mathbf{a} = R_1(s,\alpha_s) + P^b(s,\alpha_s)D_1(s,\alpha_s)'D_1(s,\alpha_s),$$

$$\mathbf{b} = 2\Big(P^b(s,\alpha_s)C_1(s,\alpha_s)'D_1(s,\alpha_s) + P^b(s,\alpha_s)B_1(s,\alpha_s) + \Lambda^b(s,\alpha_s)'D_1(s,\alpha_s)\Big),$$

$$\mathbf{c} = Q_1(s,\alpha_s) - H^b(s,P^b(s,\alpha_s),\Lambda^b(s,\alpha_s),\alpha_s),$$

$$\mathbf{d} = 2\Big((P^b(s,\alpha_s)(C_1(s,\alpha_s)'\rho_1(s,\alpha_s) + b_1(s,\alpha_s)) + \rho_1(s,\alpha_s)'\Lambda^b(s,\alpha_s)$$

$$- 2Q_1(s,\alpha_s)q_1(s,\alpha_s) - 2MA_1\Big),$$

$$\mathbf{e} = 2\Big(P^b(s,\alpha_s)\rho_1(t,\alpha_t)'D_1(s,\alpha_s) - MB_1(s,\alpha_s)' - p_1(s,\alpha_s)R_1(s,\alpha_s)\Big).$$

Proof. The proof is given in section 4.5.7. □

Theorem 4.5. *Suppose that Assumptions 4.1 and 4.2 hold. Let* $(P^b(i),\ \Lambda^b(i),\mu_1(i))_{i\in\mathcal{M}}$ *and* $(k^b(i),\ l^b(i),\mu_2(i))_{i\in\mathcal{M}}$ *be the unique solutions of the systems of RBSDEs* (4.2.1) *and* (4.2.3)*, respectively. Then the Pre-Switch LQ problem* (4.3.5) *has a lower bound:*

$$V^b(x,i_0) \geqslant P^b(0,i_0)x^2 - 2Mx + \mathbb{E}\Big\{\int_0^T \mathbb{C}^a(t,\alpha_t)\,\mathrm{d}t + G_2(\alpha_T)g_2(\alpha_T)^2\Big\} + \hat{M},$$

where $\hat{M}$ *is defined in Theorem 4.4.*

Proof. The proof is given in section 4.5.8. □

4.4 Remarks

4.4.1 On the solvability of Pre-Switch LQ problem (4.3.5)

Here, we give some remarks on the solvability of Pre-Switch LQ problem (4.3.5). In this section, we use subscript $_{\text{homo}}$, $_{\text{nonhomo}}$ to distinguish the homogeneous case and non-homogeneous case.

First, we consider the homogeneous Pre-switch Problem (3.3.9). We first have, for any $(u,\tau)\in\mathscr{A}$,

$$J^b_{\text{homo}}(x,i_0,u(\cdot),\tau)\geqslant \underline{J}^b_{\text{homo}}(x,i_0,u(\cdot),\tau), \tag{4.4.1}$$

where

$$\underline{J}^b_{\text{homo}}(x,i_0,u(\cdot),\tau)=\mathbb{E}\Big\{\int_0^\tau f(t,\alpha_t)\,\mathrm{d}t+P^b_1(\tau,\alpha_\tau)X^+_1(\tau)^2+P^b_2(\tau,\alpha_\tau)X^-_1(\tau)^2\Big\},$$

where $f(t,i)=Q_1(t,i)X_1(t)^2+u(t)'R_1(t,i)u(t)$, which is due to the fact that $(X^\pm_1)^2\geqslant 0$ and RBSDEs (3.2.1) and (3.2.2). We also proved

1. for any $(u,\tau)\in\mathscr{A}$,
$$\underline{J}^b_{\text{homo}}(x,i_0,u(\cdot),\tau)\geqslant P^b_1(0,i_0)(x^+)^2+P^b_2(0,i_0)(x^-)^2;$$

2. and
$$\underline{J}^b_{\text{homo}}(x,i_0,\bar u,\bar\tau)=P^b_1(0,i_0)(x^+)^2+P^b_2(0,i_0)(x^-)^2.$$

The above two estimates show that $(\bar u,\bar\tau)$ is optimal to

$$\min_{(u,\tau)\in\mathscr{A}}\underline{J}^b_{\text{homo}}(x,i_0,u(\cdot),\tau).$$

Note the estimate (4.4.1) is an equation when $(u,\tau)=(\bar u,\bar\tau)$, so we also conclude that $(\bar u,\bar\tau)$ is optimal to Pre-switch Problem (3.3.9).

Now we repeat the same argument as before for the non-homogeneous case. Suppose we can find an optimal $(\bar u,\bar\tau)$ to solve

$$\min_{(u,\tau)\in\mathscr{A}}J^b_{\text{lower}}(x,i_0,u(\cdot),\tau),$$

where J^b_{lower} is given by (4.3.7). In order to prove $(\bar{u}, \bar{\tau})$ is also optimal to

$$\min_{(u,\tau)\in\mathscr{A}} J^b_{\text{nonhomo}}(x, i_0, u(\cdot), \tau),$$

we need to show, for any $(u, \tau) \in \mathscr{A}$,

$$J^b_{\text{nonhomo}}(x, i_0, u(\cdot), \tau) \geqslant J^b_{\text{lower}}(x, i_0, u(\cdot), \tau) \tag{4.4.2}$$

and

$$J^b_{\text{nonhomo}}(x, i_0, \bar{u}, \bar{\tau}) = J^b_{\text{lower}}(x, i_0, \bar{u}, \bar{\tau}).$$

Notice (4.4.2) is equivalent to

$$\mathbb{E}\Big[\mathbb{G}(\tau, \alpha_\tau)X_1(\tau)^2 - 2\mathbb{K}(\tau, \alpha_\tau)X_1(\tau)\Big] \geqslant \mathbb{E}\Big[P^b(\tau, \alpha_\tau)X_1(\tau)^2 - 2k^b(\tau, \alpha_\tau)X_1(\tau)\Big].$$

Since there are non-homogeneous terms exists in the dynamic of X_1 and there is no state constraint in $\mathscr{A}$, we do not know the sign of $X_1(\tau)$. Hence the above inequality and (4.4.2) cannot hold, so the method does not work when the admissible set is $\mathscr{A}$.

Alternatively, we consider the non-homogeneous problem under the constraint $\mathscr{B}$ instead of $\mathscr{A}$. In this case, (4.4.2) holds since $X_1(\tau) \geqslant 0$ for any admissible pair in $\mathscr{B}$. However, we cannot solve

$$\min_{(u,\tau)\in\mathscr{B}} J^b_{\text{lower}}(x, i_0, u(\cdot), \tau).$$

This is because, we cannot verify if $X_1(s) \geqslant 0$ holds for $s \in [0, \bar{\tau})$ for the optimal pair obtained by completing square.

To find a lower bound problem (4.1.2), we have proposed (4.3.9). Indeed, the above argument shows

$$\begin{aligned}
&\inf_{(u,\tau)\in\mathscr{B}} J^b(x, i_0, u(\cdot), \tau)\\
&\geqslant \inf_{(u,\tau)\in\mathscr{B}} J^b_{\text{lower}}(x, i_0, u(\cdot), \tau)\\
&\geqslant \inf_{(u,\tau)\in\mathscr{A}} J^b_{\text{lower}}(x, i_0, u(\cdot), \tau),
\end{aligned}$$

where the last inequality is due to that $\mathscr{B} \subset \mathscr{A}$. Note the last problem is still unsolvable, however we can find a lower bound for it, thus obtain a lower bound for the original non-homogeneous problem (4.1.2).

4.4.2 Other remarks

In this chapter, we studied a non-homogeneous stochastic LQ control-stopping problem with regime switching. The system dynamics is driven by Brownian motion and a continuous-time Markov chain, with coefficients that are random. Our main contributions include the explicit derivation of lower boundary for this problem.

We propose and study a novel system of highly nonlinear BSDEs and one RBSDEs. The solvability of these equations, established using tools from BMO martingale theory and the John Nirenberg inequality, is of independent theoretical interest within BSDEs framework, particularly for systems with quadratic growth and regime-dependent dynamics.Different from homogeneous stochastic LQ control-stopping problem, one kinds of new unbounded RBSDE systems are introduced. Due to the boundary constraints on solutions to such equations and its all unbounded coefficients, establishing their solvability requires novel proof techniques and approaches. To establish solvability for this new class of unbounded BSDEs, our main approach follows three key steps: first deriving essential BMO martingale estimates and property, next solving the one-dimensional unbounded system, then extending the solution to multi-dimensional systems via contraction mapping. This methodology forms the paper's primary theoretical contribution. The contribution of establishing solvability for this new class of unbounded RBSDEs not only enriches the existence and uniqueness theory of BSDEs but also provides novel proof methodologies for future researchers, while offering substantial reference value for subsequent studies on stopping-control problems.

Extensions in other directions can be interesting as well. Extending our one-dimensional dynamics to higher dimensions to analyze analogous results still hold, or generalizing the LQ problem to more general control-stopping problems or LQ games, represent promising directions for future research.

4.5 Proofs for Chapter 4

We then show all the proofs of Chapter 4.

4.5.1 Proof of Lemma 4.2

Proof. According to Theorems 3.2 (resp. Theorem 3.3) in Chapter 3, there exists a unique nonnegative (resp. uniformly positive) solution $(P(i), \Lambda(i), \mu)_{i=1}^{\ell}$ to RBSDEs (4.2.1) under Assumptions 4.1 and 4.2 (i) (resp. 4.2 (ii)). Note that Assumption 4.2 (i) is stronger than the standard assumption in Theorem 3.2. So it remains to show that the solution of (4.2.1) is actually uniformly positive under Assumptions 4.1 and 4.2 (i).

Let c_1, c_2 be two positive constants such that $P(i) \leqslant c_1$, and

$$2A_1(i) + C_1(i)'C_1(i) + q_{ii} - \frac{2c_1}{\delta}|B_1(i) + D_1(i)'C_1(i)|^2 > -c_2, \text{ for all } i \in \mathcal{M}.$$

Consider the following ℓ-dimensional decoupled BSDEs:

$$\begin{cases} \mathrm{d}\underline{P}(i) = -\Big[(2A_1(i) + C_1(i)'C_1(i) + q_{ii})\underline{P}(i) + 2C_1(i)'\underline{\Lambda}(i) \\ \qquad\qquad +H^b(\underline{P}(i), \underline{\Lambda}(i), i)\Big]\,\mathrm{d}t + \underline{\Lambda}(i)'\,\mathrm{d}W, \\ \underline{P}(T, i) = \delta, \\ R_1(i) + \underline{P}(i)D_1(i)'D_1(i) > 0, \quad \text{for all } i \in \mathcal{M}. \end{cases} \tag{4.5.1}$$

This is a decoupled systems of BSDEs. From Theorem 4.1 and Theorem 5.2 of Hu et al. (2022b), the ith equation in (4.5.1) admits a unique, bounded hence maximal solution (see page 565 of Kobylanski (2000) for its definition) $(\underline{P}(i), \underline{\Lambda}(i)) \in L^{\infty}_{\mathcal{F}^W}(0, T; \mathbb{R}) \times L^{2,\,\mathrm{BMO}}_{\mathcal{F}^W}(0, T; \mathbb{R}^n)$, and $\underline{P}(i) \geqslant 0$ for all $i \in \mathcal{M}$.

Similar to Theorem 3.2, the solution $(P(i),\ \Lambda(i),\mu(i))_{i\in\mathcal{M}}$ of (4.2.1) could be approximated by solutions of a sequence of BSDEs with Lipschitz generators.

Thus we can use comparison theorem for multi-dimensional RBSDEs Lemma 3.1 and then pass to the limit to get

$$P(i)\geqslant \underline{P}(i),\ \text{for all } i\in\mathcal{M}. \tag{4.5.2}$$

Let $g:\mathbb{R}^+\to[0,1]$ be a smooth truncation function satisfying $g(x)=1$ for $x\in[0,c_1]$, and $g(x)=0$ for $x\in[2c_1,+\infty)$. Notice that $c_1\geqslant P(i)\geqslant \underline{P}(i)\geqslant 0$, so $(\underline{P}(i),\ \underline{\Lambda}(i))$ is still a solution of the ith equation in BSDEs (4.5.1) with $H^b(P,\Lambda,i)$ replaced by $H^b(P,\Lambda,i)g(P)$ in the generator.

Notice that for $P=\underline{P}(i),\ \Lambda=\underline{\Lambda}(i)$, we have, under Assumptions 4.1 and 4.2 (i),

$$\begin{aligned}
&H^b(P,\Lambda,i)g(P)\\
&\geqslant -\frac{1}{\delta}|PB_1(i)+PD_1(i)'C_1(i)+D_1(i)'\Lambda|^2g(P)\\
&=-\frac{P^2}{\delta}|B_1(i)+D_1(i)'C_1(i)|^2g(P)-\frac{2P}{\delta}\big(B_1(i)+D_1(i)'C_1(i)\big)'D_1(i)'\Lambda g(P)\\
&\quad-\frac{1}{\delta}|D_1(i)'\Lambda|^2g(P)\\
&\geqslant -\frac{2c_1P}{\delta}|B_1(i)+D_1(i)'C_1(i)|^2\\
&\quad-\frac{2P}{\delta}\big(B_1(i)+D_1(i)'C_1(i)\big)'D_1(i)'\Lambda g(P)-\frac{1}{\delta}|D_1(i)'\Lambda|^2g(P).
\end{aligned}$$

The following BSDEs

$$\begin{cases}
\quad \mathrm{d}P=-\Big[-c_2P+2C_1(i)'\Lambda-\frac{2P}{\delta}\big(B_1(i)+D_1(i)'C_1(i)\big)'D_1(i)'\Lambda g(P)\\
\qquad\qquad -\frac{1}{\delta}|D_1(i)'\Lambda|^2g(P)\Big]\,\mathrm{d}t+\Lambda'\,\mathrm{d}W,\\
P(T)=\delta,
\end{cases}$$

has a Lipschitz generator, thus it admits a unique solution $(\delta e^{-c_2(T-t)},0)$. Then the maximal solution argument (Theorem 2.3 of Kobylanski (2000)) gives

$$\underline{P}(t,i)\geqslant \delta e^{-c_2(T-t)}\geqslant \delta e^{-c_2T}.$$

Combining with (4.5.2), we proved that the solution $(P(i), \Lambda(i), \mu(i))_{i\in\mathcal{M}}$ of (4.2.1) is actually uniformly positive under Assumptions 4.1 and 4.2 (i). $\square$

4.5.2 Proof of Lemma 4.8

Proof. By the properties of measure transformation,

$$\widetilde{\mathbb{E}}\left[e^{K\int_\tau^T |\phi(s)|^p\,\mathrm{d}s} \mid \mathcal{F}_\tau^{\widetilde{W}}\right] = \frac{\mathbb{E}\left[N(T)e^{K\int_\tau^T |\phi(s)|^p\,\mathrm{d}s} \mid \mathcal{F}_\tau^{\widetilde{W}}\right]}{\mathbb{E}\left[N(T) \mid \mathcal{F}_\tau^{\widetilde{W}}\right]},$$

$$\mathbb{E}\left[N(T) \mid \mathcal{F}_\tau^{\widetilde{W}}\right] = \mathbb{E}\left[\mathbb{E}\left[N(T) \mid \mathcal{F}_\tau^{W}\right] \mid \mathcal{F}_\tau^{\widetilde{W}}\right].$$

We have $\mathbb{E}\left[N(T) \mid \mathcal{F}_\tau^{\widetilde{W}}\right] = N(\tau)$ since $N(t)$ is a $\mathbb{P}$-martingale. Thus,

$$\widetilde{\mathbb{E}}\left[e^{K\int_\tau^T |\phi(s)|^p\,\mathrm{d}s} \mid \mathcal{F}_\tau^{\widetilde{W}}\right] = \frac{\mathbb{E}\left[N(T)e^{K\int_\tau^T |\phi(s)|^p\,\mathrm{d}s} \mid \mathcal{F}_\tau^{\widetilde{W}}\right]}{N(\tau)}.$$

The numerator expands as:

$$\mathbb{E}\left[N(T)e^{K\int_\tau^T |\phi(s)|^p\,\mathrm{d}s} \mid \mathcal{F}_\tau^{\widetilde{W}}\right] = N(\tau)\mathbb{E}\left[\frac{N(T)}{N(\tau)}e^{K\int_\tau^T |\phi(s)|^p\,\mathrm{d}s} \middle| \mathcal{F}_\tau^{\widetilde{W}}\right],$$

where $\frac{N(T)}{N(\tau)} = \exp\left(\int_\tau^T \beta(s)'\,\mathrm{d}W(s) - \frac{1}{2}\int_\tau^T |\beta(s)|^2\,\mathrm{d}s\right)$. Substituting this yields:

$$\widetilde{\mathbb{E}}\left[\cdot \mid \mathcal{F}_\tau^{\widetilde{W}}\right] = \mathbb{E}\left[\exp\left(\int_\tau^T \beta(s)'\,\mathrm{d}W(s) - \frac{1}{2}\int_\tau^T |\beta(s)|^2\,\mathrm{d}s + K\int_\tau^T |\phi(s)|^p\,\mathrm{d}s\right)\middle| \mathcal{F}_\tau^{\widetilde{W}}\right].$$

Then, we estimate the exponential term by Young's inequality as follows.

Since $0 < p < 2$, for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that:

$$K|\phi(s)|^p \leqslant \epsilon|\phi(s)|^2 + C_\epsilon.$$

Consequently,

$$K\int_\tau^T |\phi(s)|^p\,\mathrm{d}s \leqslant \epsilon\int_\tau^T |\phi(s)|^2\,\mathrm{d}s + C_\epsilon(T-\tau) \leqslant \epsilon\int_\tau^T |\phi(s)|^2\,\mathrm{d}s + C_\epsilon T.$$

Exponentiating both sides yields

$$\exp\left(K\int_\tau^T |\phi(s)|^p\,\mathrm{d}s\right) \leqslant e^{C_\epsilon T}\exp\left(\epsilon\int_\tau^T |\phi(s)|^2\,\mathrm{d}s\right).$$

Thus,

$$\begin{aligned}
&\exp\left(\int_\tau^T \beta(s)'\,\mathrm{d}W(s) - \frac{1}{2}\int_\tau^T |\beta(s)|^2\,\mathrm{d}s + K\int_\tau^T |\phi(s)|^p\,\mathrm{d}s\right)\\
&\leqslant e^{C_\epsilon T}\exp\left(\int_\tau^T \beta(s)'\,\mathrm{d}W(s) - \frac{1}{2}\int_\tau^T |\beta(s)|^2\,\mathrm{d}s + \epsilon\int_\tau^T |\phi(s)|^2\,\mathrm{d}s\right).
\end{aligned}$$

Let $q > 1$ be a constant to be specified later. By Hölder's inequality for conditional expectations

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\int_\tau^T \beta(s)'\,\mathrm{d}W(s) - \frac{1}{2}\int_\tau^T |\beta(s)|^2\,\mathrm{d}s + \epsilon\int_\tau^T |\phi(s)|^2\,\mathrm{d}s\right)\middle|\, \mathcal{F}_\tau^{\widetilde{W}}\right]\\
&\leqslant \left(\mathbb{E}\left[\exp\left(q\int_\tau^T \beta(s)'d\,\mathrm{d}W(s) - \frac{q}{2}\int_\tau^T |\beta(s)|^2\,\mathrm{d}s\right)\middle|\, \mathcal{F}_\tau^{\widetilde{W}}\right]\right)^{\frac{1}{q}}\\
&\quad\cdot\left(\mathbb{E}\left[\exp\left(\frac{q\epsilon}{q-1}\int_\tau^T |\phi(s)|^2\,\mathrm{d}s\right)\middle|\, \mathcal{F}_\tau^{\widetilde{W}}\right]\right)^{\frac{q-1}{q}}.
\end{aligned} \tag{4.5.3}$$

We define

$$\begin{aligned}
\mathbf{A} &= \exp\left(q\int_\tau^T \beta(s)'\,\mathrm{d}W(s) - \frac{q}{2}\int_\tau^T |\beta(s)|^2\,\mathrm{d}s\right),\\
\mathbf{B} &= \exp\left(\frac{q\epsilon}{q-1}\int_\tau^T |\phi(s)|^2\,\mathrm{d}s\right).
\end{aligned}$$

First step, we estimate the exponential martingale term $\mathbf{A}$. Define

$$\mathbf{M}(t) = \exp\left(\int_0^t q\beta(s)'\,\mathrm{d}W(s) - \frac{1}{2}\int_0^t q^2|\beta(s)|^2\,\mathrm{d}s\right),$$

then it is a martingale under $\mathbb{P}$, so we gain

$$\mathbf{A} = \frac{\mathbf{M}(T)}{\mathbf{M}(\tau)}\exp\left(\frac{q(q-1)}{2}\int_\tau^T |\beta(s)|^2\,\mathrm{d}s\right).$$

Since $\beta \in L^{2,\,\mathrm{BMO}}_{\mathcal{F}^W}(0,T;\mathbb{R}^n)$ and $|\beta| \leqslant |\phi|$, the stochastic integral $\int_\tau^\cdot \beta(s)'\,\mathrm{d}W(s)$ is a BMO martingale.

By the reverse Hölder inequality for BMO martingales (Kazamaki (2006)[Kazamaki, 1994]), there exists $r > 1$ such that:

$$\mathbb{E}\left[\left(\frac{\mathbf{M}(T)}{\mathbf{M}(\tau)}\right)^r \,\middle|\, \mathcal{F}^W_\tau\right] < c_1 < \infty.$$

Select $1 < q < r$ sufficiently close to 1 such that

$$\frac{q(q-1)r}{2(r-1)}\left\|\int_0^\cdot \beta(s)\,\mathrm{d}W(s)\right\|^2_{\mathrm{BMO}_2} < 1.$$

Then the John–Nirenberg inequality (Lemma 4.5) gives

$$\mathbb{E}\left[\exp\left(\frac{q(q-1)r}{2(r-1)}\int_\tau^T |\beta(s)|^2\,\mathrm{d}s\right)\,\middle|\, \mathcal{F}^W_\tau\right] < c_2 < \infty.$$

It thus follows from Hölder's inequality that

$$\begin{aligned}
&\mathbb{E}\left[\frac{\mathbf{M}(T)}{\mathbf{M}(\tau)}\exp\left(\frac{q(q-1)}{2}\int_\tau^T |\beta(s)|^2\,\mathrm{d}s\right)\,\middle|\, \mathcal{F}^W_\tau\right] \\
&\leqslant \left(\mathbb{E}\left[\left(\frac{\mathbf{M}(T)}{\mathbf{M}(\tau)}\right)^r \,\middle|\, \mathcal{F}^W_\tau\right]\right)^{1/r} \cdot \left(\mathbb{E}\left[\exp\left(\frac{q(q-1)r}{2(r-1)}\int_\tau^T |\beta(s)|^2\,\mathrm{d}s\right)\,\middle|\, \mathcal{F}^W_\tau\right]\right)^{(r-1)/r} < c_3.
\end{aligned}$$

Hence,

$$\mathbb{E}\left[\exp\left(q\int_\tau^T \beta(s)'\,\mathrm{d}W(s) - \frac{q}{2}\int_\tau^T |\beta(s)|^2\,\mathrm{d}s\right)\,\middle|\, \mathcal{F}^{\widetilde{W}}_\tau\right] < c_3^{\frac{1}{q}}. \tag{4.5.4}$$

Next step, we estimate the term $\mathbf{B}$. Since $\phi \in L^{2,\,\mathrm{BMO}}_{\mathcal{F}^W}(0,T;\mathbb{R}^n)$, select $\epsilon > 0$ sufficiently small such that:

$$\frac{q\epsilon}{q-1}\left\|\int_0^\cdot \phi(s)\,\mathrm{d}W(s)\right\|^2_{\mathrm{BMO}_2} < 1.$$

By the John–Nirenberg inequality (Lemma 4.5):

$$\mathbb{E}\left[\exp\left(\frac{q\epsilon}{q-1}\int_\tau^T |\phi(s)|^2\,\mathrm{d}s\right)\,\middle|\, \mathcal{F}^W_\tau\right] < c_4 < \infty.$$

As $\mathcal{F}_\tau^{\widetilde{W}} \subseteq \mathcal{F}_\tau^W$, it follows that

$$\mathbb{E}\left[\exp\left(\frac{q\epsilon}{q-1}\int_\tau^T |\phi(s)|^2 \,\mathrm{d}s\right)\middle|\, \mathcal{F}_\tau^{\widetilde{W}}\right] < c_4. \tag{4.5.5}$$

Combining the estimates (4.5.3), (4.5.4) and (4.5.5) yields

$$\mathbb{E}\left[\exp\left(\int_\tau^T \beta(s)'\,\mathrm{d}W(s) - \frac{1}{2}\int_\tau^T |\beta(s)|^2\,\mathrm{d}s + \epsilon\int_\tau^T |\phi(s)|^2\,\mathrm{d}s\right)\middle|\, \mathcal{F}_\tau^{\widetilde{W}}\right]$$

$$\leqslant \left(c_3^{\frac{1}{q}}\right)^{\frac{1}{q}} \cdot (c_4)^{\frac{q-1}{q}} = c_3^{\frac{1}{q^2}} c_4^{\frac{q-1}{q}}.$$

Consequently,

$$\widetilde{\mathbb{E}}\left[e^{K\int_\tau^T |\phi(s)|^p\,\mathrm{d}s}\,\middle|\, \mathcal{F}_\tau^{\widetilde{W}}\right] \leqslant e^{C_\epsilon T} c_3^{\frac{1}{q^2}} c_4^{\frac{q-1}{q}} =: c_{p,K}.$$

The proof is complete. □

4.5.3 Proof of Lemma 4.9

Proof. Introduce two processes

$$J(t) = \exp\left(\int_0^t a(s)\,\mathrm{d}s\right),$$

and

$$N(t) = \mathcal{E}\left(\int_0^t \beta(s)'\,\mathrm{d}W(s)\right).$$

Note that $N(t)$ is a uniformly integrable martingale, thus $\widetilde{W}(t) := W(t) - \int_0^t \beta(s)\,\mathrm{d}s$ is a Brownian motion under the probability $\widetilde{\mathbb{P}}$ defined by

$$\left.\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_T^W} = N(T).$$

Define a process

$$Y(t) = \operatorname*{ess\,sup}_{v\in\mathscr{T}_t} \widetilde{\mathbb{E}}\Big[\int_t^v J(s)J(t)^{-1}f(s)\,\mathrm{d}s + J(v)J(t)^{-1}S(v)1_{\{v<T\}}$$

$$+ J(T)J(t)^{-1}\xi 1_{\{v=T\}}\,\Big|\, \mathcal{F}_t^W\Big],$$

where $\mathscr{T}$ is the set of all stopping times dominated by T, $\mathscr{T}_t = \{v \in \mathscr{T}; t \leqslant v \leqslant T\}$, and $\widetilde{\mathbb{E}}$ is the expectation w.r.t. the probability measure $\widetilde{\mathbb{P}}$. Then clearly $Y(T) = \xi$.

Since

$$
\begin{aligned}
&J(t)Y(t) + \int_0^t J(s)f(s)\,\mathrm{d}s \\
&= \operatorname*{ess\,sup}_{v\in\mathscr{T}_t} \widetilde{\mathbb{E}}\left[\int_t^v J(s)f(s)\,\mathrm{d}s + J(v)S(v)1_{\{v<T\}} + J(T)\xi 1_{\{v=T\}} \,\middle|\, \mathcal{F}_t^W\right] + \int_0^t J(s)f(s)\,\mathrm{d}s \\
&= \operatorname*{ess\,sup}_{v\in\mathscr{T}_t} \widetilde{\mathbb{E}}\left[\int_0^v J(s)f(s)\,\mathrm{d}s + J(v)S(v)1_{\{v<T\}} + J(T)\xi 1_{\{v=T\}} \,\middle|\, \mathcal{F}_t^W\right],
\end{aligned}
$$

the process $J(t)Y(t) + \int_0^t J(s)f(s)\,\mathrm{d}s$ is the value function of an optimal stopping time problem with payoff:

$$
H(u) = \int_0^u J(u)f(u)\,\mathrm{d}u + J(u)S(u)1_{\{u<T\}} + J(T)\xi 1_{\{u=T\}}.
$$

By the theory of Snell Envelopes (cf. Karoui (1981) and Karatzas and Shreve (1996)), $J(t)Y(t) + \int_0^t J(s)f(s)\,\mathrm{d}s$ is also the smallest continuous supermartingale which is dominating $H(t)$. The continuity of $\{Y(t)\}$ follows from that of $\{H(t)\}$ on the interval $[0,T)$, and the assumption that the jump of H at time T is positive.

Since a is unbounded, so is J. Thus, we do not have the boundedness of Y automatically. To show the boundedness of Y, we apply the Cauchy-Schwartz inequality, Lemmas 4.6, 4.7 and 4.8 to get

$$
\begin{aligned}
|Y(t)| &\leqslant \widetilde{\mathbb{E}}_t\left[e^{\int_t^T a(r)\,\mathrm{d}r}|\xi|1_{\{v=T\}} + e^{\int_t^T a(r)\,\mathrm{d}r}\sup_{0\leqslant t\leqslant T}|S(t)|1_{\{v<T\}} + \int_t^T e^{\int_t^s a(r)\,\mathrm{d}r}|f(s)|\,\mathrm{d}s\right] \\
&\leqslant \widetilde{\mathbb{E}}_t\left[ce^{\int_t^T |\phi(r)|^p\,\mathrm{d}r}\right] + \widetilde{\mathbb{E}}_t\left[e^{\int_t^T |\phi(r)|^p\,\mathrm{d}r}\int_t^T |\phi(s)|^2\,\mathrm{d}s\right] \\
&\leqslant c + \frac{1}{2}\widetilde{\mathbb{E}}_t\left[e^{2\int_t^T |\phi(r)|^p\,\mathrm{d}r} + \left(\int_t^T |\phi(s)|^2\,\mathrm{d}s\right)^2\right] \\
&\leqslant c,
\end{aligned}
$$

so $Y \in L^{\infty}_{\mathcal{F}^W}(0,T;\mathbb{R})$.

Similarly, we use the AM-QM inequality, the Cauchy-Schwartz inequality, Lemmas 4.6, 4.7 and 4.8 to get

$$
\begin{aligned}
&\widetilde{\mathbb{E}}\left(J(T)\xi+\int_0^T J(s)f(s)\,\mathrm{d}s\right)^2\\
\leqslant&\widetilde{\mathbb{E}}\left(J(T)|\xi|+\int_0^T J(s)|f(s)|\,\mathrm{d}s\right)^2\\
\leqslant&2c\widetilde{\mathbb{E}}\left(e^{2\int_0^T |a(r)|\,\mathrm{d}r}\right)+2\widetilde{\mathbb{E}}\left[\left(\int_0^T |f(s)|e^{\int_0^T |a(r)|\,\mathrm{d}r}\,\mathrm{d}s\right)^2\right]\\
\leqslant&2c\widetilde{\mathbb{E}}\left(e^{2\int_0^T |\phi(r)|^p\,\mathrm{d}r}\right)+2\widetilde{\mathbb{E}}\left[\left(\int_0^T |\phi(s)|^2e^{\int_0^T |\phi(r)|^p\,\mathrm{d}r}\,\mathrm{d}s\right)^2\right]\\
\leqslant&c+2\widetilde{\mathbb{E}}\left[e^{2\int_0^T |\phi(r)|^p\,\mathrm{d}r}\left(\int_0^T |\phi(s)|^2\,\mathrm{d}s\right)^2\right]\\
\leqslant&c+\widetilde{\mathbb{E}}\left[e^{4\int_0^T |\phi(r)|^p\,\mathrm{d}r}+\left(\int_0^T |\phi(s)|^2\,\mathrm{d}s\right)^4\right]\\
<&\infty.
\end{aligned}
$$

Therefore,

$$
\left\{J(t)Y(t)+\int_0^t J(s)f(s)\,\mathrm{d}s, 0\leqslant t\leqslant T\right\}
$$

is a square-integrable submartingale under $\widetilde{\mathbb{P}}$. In the same spirit, we can prove, for some $c>0$, that

$$
\widetilde{\mathbb{E}}\left[J\left(\theta\right)Y\left(\theta\right)-J(T)Y(T)+\int_\theta^T J(s)|\phi(s)|^2\,\mathrm{d}s\,\middle|\,\mathcal{F}_\tau\right]\leqslant c, \tag{4.5.6}
$$

holds for all stopping times $\tau\leqslant\theta\leqslant T$.

Let us now apply the Doob-Meyer decomposition Theorem 4.1 to the continuous supermartingale $J(t)Y(t)+\int_0^t J(s)f(s)\,\mathrm{d}s$. There exists an adapted increasing continuous process $\{A\}$ with $A(0) = 0$ and a continuous uniformly integrable martingale $\{M\}$ such that

$$J(t)Y(t) + \int_0^t J(s)f(s)\,\mathrm{d}s = M(t) - A(t).$$

Next, we prove that $\widetilde{\mathbb{E}}\left[A(T)^p\right] < \infty$ for any $p > 1$, where $A(T) = \lim_{t\to T} A(t)$. Let $c > 0$ be a large constant to be chosen, which satisfies $|Y(t)| \leqslant c$ for all $t \geqslant 0$. For $\lambda > 0$ we define a stopping time

$$\varpi = \inf\left\{t \geqslant 0 \mid A(t) \geqslant \lambda\right\}.$$

Then

$$\begin{aligned}\widetilde{\mathbb{P}}\left(A(T) \geqslant \lambda + 4c\right) &= \widetilde{\mathbb{P}}\left(A(T) \geqslant \lambda + 4c, \varpi \leqslant T\right)\\ &\leqslant \widetilde{\mathbb{P}}\left(A(T) - A(\varpi) \geqslant 4c, \varpi \leqslant T\right)\\ &\leqslant \frac{1}{4c}\widetilde{\mathbb{E}}\left((A(T) - A(\varpi)) \cdot 1_{\{\varpi \leqslant T\}}\right). \end{aligned} \tag{4.5.7}$$

Now let

$$\varpi_n := \inf\left\{t \geqslant 0 \mid A(t) \geqslant \lambda - 1/n\right\}.$$

Then $\varpi_n \uparrow \varpi$ and $\mathcal{F}_{\varpi_n} \subseteq \mathcal{F}_{\varpi}$. For $j \geqslant k$, we have $\{\varpi_k \leqslant T\} \in \mathcal{F}_{\varpi_k} \subseteq \mathcal{F}_{\varpi_j}$. Therefore, by the tower property, Lemmas 4.6, 4.7 and 4.8 and (4.5.6),

$$\begin{aligned}&\widetilde{\mathbb{E}}\left[(A(T) - A(\varpi_j)) \cdot 1_{\{\varpi_k \leqslant T\}}\right]\\ &= \widetilde{\mathbb{E}}\left[\widetilde{\mathbb{E}}[(A(T) - A(\varpi_j)) \mid \mathcal{F}_{\varpi_k}] \cdot 1_{\{\varpi_k \leqslant T\}}\right]\\ &= \widetilde{\mathbb{E}}\left[\widetilde{\mathbb{E}}\left[J(\varpi_j)\,Y(\varpi_j) - J(T)Y(T) - \int_{\varpi_j}^T J(s)f(s)\,\mathrm{d}s \,\middle|\, \mathcal{F}_{\varpi_k}\right] \cdot 1_{\{\varpi_k \leqslant T\}}\right]\\ &\leqslant \widetilde{\mathbb{E}}\left[\widetilde{\mathbb{E}}\left[J(\varpi_j)\,Y(\varpi_j) - J(T)Y(T) + \int_{\varpi_j}^T J(s)|\phi(s)|^2\,\mathrm{d}s \,\middle|\, \mathcal{F}_{\varpi_k}\right] \cdot 1_{\{\varpi_k \leqslant T\}}\right]\\ &\leqslant C\widetilde{\mathbb{P}}\left(\varpi_k \leqslant T\right).\end{aligned}$$

Here we used in the second step that $M(t)$ is a uniformly integrable martingale. The constant C does not depend on c or λ. Applying Fatou's lemma yields, as $j \to \infty$,

$$\widetilde{\mathbb{E}}\left[(A(T) - A(\varpi)) \cdot 1_{\{\varpi_k \leqslant T\}}\right] \leqslant C\widetilde{\mathbb{P}}\left(\varpi_k \leqslant T\right).$$

Applying again Fatou's lemma (for $k \to \infty$) we see that

$$\widetilde{\mathbb{E}}\left[(A(T) - A(\varpi)) \cdot 1_{\{\varpi \leqslant T\}}\right] \leqslant C\widetilde{\mathbb{P}}(\varpi \leqslant T).$$

Since $A(t)$ is increasing, $\{\varpi \leqslant T\} = \{A(T) \geqslant \lambda\}$,

$$\widetilde{\mathbb{E}}\left[(A(T) - A(\varpi)) \cdot 1_{\{\varpi \leqslant T\}}\right] \leqslant C\widetilde{\mathbb{P}}(A(T) \geqslant \lambda), \tag{4.5.8}$$

Combining (4.5.7) and (4.5.8) we obtain

$$\widetilde{\mathbb{P}}\left(A(T) \geqslant \lambda + 4c\right) \leqslant \frac{C}{4c}\widetilde{\mathbb{P}}\left(A(T) \geqslant \lambda\right),$$

In particular, for $\lambda = 4kc$ with $k \geqslant 1$,

$$\widetilde{\mathbb{P}}\left(A(T) \geqslant (k+1)4c\right) \leqslant \frac{C}{4c}\widetilde{\mathbb{P}}\left(A(T) \geqslant 4kc\right).$$

Choosing $c > 0$ such that $\frac{C}{4c} < 1/2$. Iterating the above inequality yields

$$\tilde{\mathbb{P}}(A(T) \geqslant 4kc) \leqslant 2^{-k+1}\tilde{\mathbb{P}}(A(T) \geqslant 4c) \leqslant \tilde{2}^{-k+1}.$$

Note the last upper bound also holds when $k = 0$. Hence, for any $p > 1$,

$$\begin{aligned}
\widetilde{\mathbb{E}}\left[A(T)^p\right] &= p\int_0^\infty x^{p-1}\tilde{\mathbb{P}}(A(T) \geqslant x)\,\mathrm{d}x \\
&\leqslant p\sum_{k=0}^\infty \int_{4kc}^{4(k+1)c} x^{p-1}\tilde{\mathbb{P}}(A(T) \geqslant 4kc)\,\mathrm{d}x \\
&\leqslant p\sum_{k=0}^\infty (4(k+1)c)^{p-1}2^{-k+1} < \infty,
\end{aligned} \tag{4.5.9}$$

which in particular implies that $A(t)$ is square integrable. Therefore, the martingale

$$M(t) = J(t)Y(t) + \int_0^t J(s)f(s)\,\mathrm{d}s + A(t)$$

is also square integrable under $\widetilde{\mathbb{P}}$. By the martingale representation theorem, there exists $\widetilde{Z} \in L^2_{\mathcal{F}^{\widetilde{W}}}(0,T;\mathbb{R}^n)$ such that

$$J(t)Y(t) + \int_0^t J(s)f(s)\,\mathrm{d}s + A(t) = M(t) = \int_0^t \widetilde{Z}(s)'\,\mathrm{d}\widetilde{W}(s) + M(0).$$

Let

$$\mu(t) = \int_0^t J(s)^{-1}\,\mathrm{d}A(s) \geqslant 0, \quad Z(t) = J(t)^{-1}\widetilde{Z}(t).$$

Since $A(0) = 0$, integration by parts gives

$$\mu(t) = \int_0^t J(s)^{-1}\,\mathrm{d}A(s) = J(t)^{-1}A(t) + \int_0^t A(s)a(s)J(s)^{-1}\,\mathrm{d}s.$$

Because $0 \leqslant A(s) \leqslant A(T)$ for any $s \in [0,T]$ and

$$0 \leqslant J(t)^{-1} = \exp\left(-\int_0^t a(r)\,\mathrm{d}r\right) \leqslant \exp\left(\int_0^T |a(r)|\,\mathrm{d}r\right) \leqslant \exp\left(\int_0^T |\phi(r)|^p\,\mathrm{d}r\right),$$

using $1 + x \leqslant e^x$, we have

$$\begin{aligned}
0 \leqslant \mu(T) &\leqslant J(T)^{-1}A(T) + \int_0^T A(s)|a(s)|J(s)^{-1}\,\mathrm{d}s \\
&\leqslant A(T)\exp\left(\int_0^T |\phi(r)|^p\ \mathrm{d}r\right)\left(1 + \int_0^T |\phi(s)|^p\ \mathrm{d}s\right) \\
&\leqslant A(T)\exp\left(2\int_0^T |\phi(r)|^p\ \mathrm{d}r\right).
\end{aligned}$$

Thus, by Hölder's inequality, for any $q > 1$,

$$\widetilde{\mathbb{E}}\left[\mu(T)^q\right] \leqslant \left(\widetilde{\mathbb{E}}\left[A(T)^{2q}\right]\right)^{1/2}\left(\widetilde{\mathbb{E}}\left[\exp\left(2q\int_0^T |\phi(r)|^p\ \mathrm{d}r\right)\right]\right)^{1/2},$$

which is finite by (4.5.9) and Lemma 4.8. In particular, we get $\mu \in \mathcal{A}_c(0,T;\mathbb{R})$. In the same way, one can prove, for any $q>1$, there exists a constant $c>0$ such that

$$\widetilde{\mathbb{E}}_\tau\Big[\big(\mu(T)-\mu(\tau)\big)^q\Big] \leqslant c$$

holds for all stopping times $\tau \leqslant T$.

By Itô's Lemma,

$$\begin{aligned}
\mathrm{d}\,(J(t)Y(t)) &= \widetilde{Z}(t)'\,\mathrm{d}\widetilde{W}(t) - J(t)f(t)\,\mathrm{d}t - \mathrm{d}A(t)\\
&= J(t)[Z(t)'\,\mathrm{d}W(t) - Z(t)'\beta(t)\,\mathrm{d}t - f(t)\,\mathrm{d}t - \mathrm{d}\mu(t)].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{d}Y(t) &= \mathrm{d}\,(J(t)^{-1}\cdot J(t)Y(t))\\
&= (-a(t)J(t)^{-1})J(t)Y(t)\,\mathrm{d}t + J(t)^{-1}(\widetilde{Z}(t)'\,\mathrm{d}\widetilde{W}(t) - J(t)f(t)\,\mathrm{d}t - \mathrm{d}A(t))\\
&= -a(t)Y(t)\,\mathrm{d}t + Z(t)'\,\mathrm{d}W(t) - Z(t)'\beta(t)\,\mathrm{d}t - f(t)\,\mathrm{d}t - \mathrm{d}\mu(t).
\end{aligned}$$

Thus (Y,Z,μ) satisfies (4.2.5).

Since Y and ξ are essentially bounded, applying Lemma 4.7 and 4.6 yields

$$\begin{aligned}
\widetilde{\mathbb{E}}_\tau\Big[\int_\tau^T |Z(s)|^2\,\mathrm{d}s\Big] &= \widetilde{\mathbb{E}}_\tau\Big[\Big(\int_\tau^T Z(s)'\,\mathrm{d}\widetilde{W}(s)\Big)^2\Big]\\
&= \widetilde{\mathbb{E}}_\tau\Big[\Big(\xi - Y(\tau) + \int_\tau^T \big(a(s)Y(s)+f(s)\big)\,\mathrm{d}s + \int_\tau^T \mathrm{d}\mu(s)\Big)^2\Big]\\
&\leqslant c + c\widetilde{\mathbb{E}}_\tau\Big[\Big(\int_\tau^T (|\phi(s)|^p + |\phi(s)|^2)\,\mathrm{d}s\Big)^2 + \big(\mu(T)-\mu(\tau)\big)^2\Big]\\
&\leqslant c + c\widetilde{\mathbb{E}}_\tau\Big[\Big(\int_\tau^T (1+2|\phi(s)|^2)\,\mathrm{d}s\Big)^2\Big]\\
&\leqslant c + c\widetilde{\mathbb{E}}_\tau\Big[\Big(T + 2\int_\tau^T |\phi(s)|^2\,\mathrm{d}s\Big)^2\Big]\\
&\leqslant c,
\end{aligned}$$

for all stopping times $\tau \leqslant T$. Hence, $\int_0^\cdot Z(s)'\,\mathrm{d}\widetilde{W}(s)$ is a BMO martingale under $\widetilde{\mathbb{P}}$. Consequently $\int_0^\cdot Z(s)'\,\mathrm{d}W(s)$ is a BMO martingale under $\mathbb{P}$. Hence we have proved that $(Y,Z,\mu) \in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}) \times L^{2,\,\mathrm{BMO}}_{\mathcal{F}^W}(0,T;\mathbb{R}^n) \times \mathcal{A}_c(0,T;\mathbb{R})$ is a solution of the 1-dimensional RBSDEs (4.2.5).

Let us prove the uniqueness. Suppose

$$(Y,Z,\mu),\ (\hat{Y},\hat{Z},\hat{\mu}) \in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}) \times L^{2,\,\mathrm{BMO}}_{\mathcal{F}^W}(0,T;\mathbb{R}^n) \times \mathcal{A}_c(0,T;\mathbb{R})$$

are both solutions of (4.2.5). Set

$$\Delta Y = Y - \hat{Y},\ \Delta Z = Z - \hat{Z},\ \Delta\mu = \mu - \hat{\mu}.$$

Then

$$J(t)\Delta Y(t) = -\int_t^T J(s)\Delta Z(s)'\,\mathrm{d}\widetilde{W}(s) + \int_t^T J(s)\,\mathrm{d}\Delta\mu(s). \tag{4.5.10}$$

Note that

$$\begin{aligned}
\int_0^T J(s)^2\Delta Y(s)\,\mathrm{d}\Delta\mu(s) = &\int_0^T J(s)^2 Y(s)\,\mathrm{d}\mu(s) - \int_0^T J(s)^2\hat{Y}(s)\,\mathrm{d}\mu(s)\\
&-\int_0^T J(s)^2 Y(s)\,\mathrm{d}\hat{\mu}(s) + \int_0^T J(s)^2\hat{Y}(s)\,\mathrm{d}\hat{\mu}(s)\\
=&\int_0^T J(s)^2 S(s)\,\mathrm{d}\mu(s) - \int_0^T J(s)^2\hat{Y}(s)\,\mathrm{d}\mu(s)\\
&-\int_0^T J(s)^2 Y(s)\,\mathrm{d}\hat{\mu}(s) + \int_0^T J(s)^2\hat{S}(s)\,\mathrm{d}\hat{\mu}(s)\\
\leqslant&\int_0^T J(s)^2 S(s)\,\mathrm{d}\mu(s) - \int_0^T J(s)^2\hat{S}(s)\,\mathrm{d}\mu(s)\\
&-\int_0^T J(s)^2 S(s)\,\mathrm{d}\hat{\mu}(s) + \int_0^T J(s)^2\hat{S}(s)\,\mathrm{d}\hat{\mu}(s)\\
=&\ 0.
\end{aligned} \tag{4.5.11}$$

Next, we prove $\mathbb{H}(t) = J(t)^2\Delta Y(t)\Delta Z(t)$ is square integrable. Since $Y,\ \hat{Y} \in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R})$, we have $\Delta Y \in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R})$. Since $Z,\ \hat{Z} \in L^{2,\,\mathrm{BMO}}_{\mathcal{F}^W}(0,T;\mathbb{R}^n)$, one has

$\Delta Z \in L^{2,\,\mathrm{BMO}}_{\mathcal{F}^W}(0,T;\mathbb{R}^n)$. Therefore, via Cauchy-Schwarz inequality, Lemma 4.7 and 4.8,

$$
\begin{aligned}
\widetilde{\mathbb{E}}\Big[\int_0^T |\mathbb{H}(s)|^2\,\mathrm{d}s\Big] &= \widetilde{\mathbb{E}}\Big[\int_0^T J(s)^4|\Delta Y(s)|^2|\Delta Z(s)|^2\,\mathrm{d}s\Big] \\
&\leqslant c\widetilde{\mathbb{E}}\Big[\int_0^T e^{4\int_0^s|\phi(r)|^p\,\mathrm{d}r}|\Delta Z(s)|^2\,\mathrm{d}s\Big] \\
&\leqslant c\widetilde{\mathbb{E}}\Big[e^{4\int_0^T|\phi(r)|^p\,\mathrm{d}r}\int_0^T|\Delta Z(s)|^2\,\mathrm{d}s\Big] \\
&\leqslant c\left(\widetilde{\mathbb{E}}\left[e^{8\int_0^T|\phi(r)|^p\,\mathrm{d}r}\right]\right)^{1/2}\left(\widetilde{\mathbb{E}}\left[\left(\int_0^T|\Delta Z(s)|^2\,\mathrm{d}s\right)^2\right]\right)^{1/2} \\
&\leqslant c.
\end{aligned}
$$

As a consequence

$$
\widetilde{\mathbb{E}}\left[\int_0^T J(t)^2\Delta Y(t)\Delta Z(t)'\,\mathrm{d}\widetilde{W}(t)\right] = 0. \tag{4.5.12}
$$

By Itô's Lemma,

$$
\begin{aligned}
(J(t)\Delta Y(t))^2 = (J(t)\Delta Z(t))^2\,\mathrm{d}t &+ 2J(t)^2\Delta Y(t)\Delta Z(t)'\,\mathrm{d}\widetilde{W}(t) \\
&- 2J(t)^2\Delta Y(t)\,\mathrm{d}\Delta\mu(t),
\end{aligned}
$$

using $\Delta Y(T)=0$, it follows

$$
\begin{aligned}
&\widetilde{\mathbb{E}}\left[\int_0^T J(t)^2\Delta Y(t)\Delta Z(t)'\,\mathrm{d}\widetilde{W}(t) - \int_0^T 2J(t)^2\Delta Y(t)\,\mathrm{d}\Delta\mu(t)\right] \\
&\qquad\qquad = \widetilde{\mathbb{E}}\left[-(J(0)\Delta Y(0))^2 - \int_0^T (J(t)\Delta Z(t))^2\,\mathrm{d}t\right] \leqslant 0.
\end{aligned}
$$

This together with (4.5.11) and (4.5.12) gives $\Delta Y, \Delta Z, \Delta\mu = 0$. This completes the proof of the uniqueness. $\square$

4.5.4 Proof of Theorem 4.2

Proof. For each $i \in \mathcal{M}$, we introduce the process

$$N(t,i) = \mathcal{E}\left(\int_0^t \beta(s,i)' \,\mathrm{d}W(s)\right).$$

Note that $N(t,i)$ is a uniformly integrable martingale, thus $\widetilde{W}^i(t) := W(t) - \int_0^t \beta(s,i)\,\mathrm{d}s$ is a Brownian motion under the probability $\widetilde{\mathbb{P}}^i$ defined by

$$\left.\frac{\mathrm{d}\widetilde{\mathbb{P}}^i}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_T^W} = N(T,i), \quad \text{for all } i \in \mathcal{M}.$$

By Lemma 4.9, for any $U = (U(1), ..., U(\ell))' \in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}^\ell)$ and each $i \in \mathcal{M}$, the following 1-dimensional linear RBSDE admits a unique solution $(k(i), l(i), \mu(i)) \in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}) \times L^{2,\,\mathrm{BMO}}_{\mathcal{F}^W}(0,T;\mathbb{R}^n) \times \mathcal{A}_c(0,T;\mathbb{R})$, for all $i \in \mathcal{M}$:

$$\begin{cases} -\mathrm{d}k(i) = \Big[\beta(i)'l(i) + f(i) + \sum\limits_{j\in\mathcal{M}} \kappa_{ij}U(j)\Big]\,\mathrm{d}t - \mathrm{d}\mu(i) - l(i)'\,\mathrm{d}W, \\ k(i,T) = \xi(i), \\ k(t,i) \geqslant S(t,i), \quad \int_0^T (k(t,i) - S(t,i))\,\mathrm{d}\mu(i) = 0, \\ \mu(0) = 0. \end{cases}$$

We let Θ denote the map $U \mapsto k := (k(1), ..., k(\ell))'$.

Thanks to Lemma 4.7, there exists a constant $c_5 > 0$, independent of t and i, such that

$$\widetilde{\mathbb{E}}^i_t\left[\left(\int_t^T |\phi(s,i)|^2\,\mathrm{d}s\right)^{\frac{p}{2}}\right] < c_5. \tag{4.5.13}$$

For $U = (U(1), ..., U(\ell))' \in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}^\ell)$, we introduce a new norm

$$|U|_\infty := \max_{i\in\mathcal{M}} \operatorname*{ess\,sup}_{(s,\omega)\in[0,T]\times\Omega} e^{c_6 s}|U(s,i)|,$$

where c_6 is a large positive constant to be determined. Let $\mathcal{B}$ be the set of $U \in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}^\ell)$ with $|U|_\infty < \infty$. For any U, $\widetilde{U} \in \mathcal{B}$, set $k = \Theta(U)$, $\widetilde{k} = \Theta(\widetilde{U})$, and

$$\Delta k(t,i) = k(t,i) - \widetilde{k}(t,i), \text{ and } \Delta U(t,i) = U(t,i) - \widetilde{U}(t,i).$$

Then by Itô's lemma,

$$\Delta k(t,i) = \widetilde{\mathbb{E}}_t^i\Big[\int_t^T \sum_{j\in\mathcal{M}} \kappa_{ij}(s)\Delta U(s,j)\,\mathrm{d}s\Big].$$

Since $p \in (0,2)$, it follows from Hölder's inequality that

$$\begin{aligned}
e^{c_6 t}|\Delta k(t,i)| &\leqslant e^{c_6 t}\widetilde{\mathbb{E}}_t^i\Big[\int_t^T e^{-c_6 s}\sum_{j\in\mathcal{M}} |\kappa_{ij}(s)|e^{c_6 s}|\Delta U(s,j)|\,\mathrm{d}s\Big] \\
&\leqslant \widetilde{\mathbb{E}}_t^i\Big[\int_t^T e^{-c_6(s-t)}|\phi(s,i)|^p\,\mathrm{d}s\Big]|\Delta U|_\infty \\
&\leqslant \widetilde{\mathbb{E}}_t^i\Bigg[\Big(\int_t^T e^{-\frac{2c_6}{2-p}(s-t)}\,\mathrm{d}s\Big)^{\frac{2-p}{2}}\Big(\int_t^T |\phi(s,i)|^2\,\mathrm{d}s\Big)^{\frac{p}{2}}\Bigg]|\Delta U|_\infty \\
&\leqslant \widetilde{\mathbb{E}}_t^i\Bigg[\Big(\int_t^\infty e^{-\frac{2c_6}{2-p}(s-t)}\,\mathrm{d}s\Big)^{\frac{2-p}{2}}\Big(\int_t^T |\phi(s,i)|^2\,\mathrm{d}s\Big)^{\frac{p}{2}}\Bigg]|\Delta U|_\infty \\
&= \Big(\frac{2-p}{2c_6}\Big)^{\frac{2-p}{2}}\widetilde{\mathbb{E}}_t^i\Bigg[\Big(\int_t^T |\phi(s,i)|^2\,\mathrm{d}s\Big)^{\frac{p}{2}}\Bigg]|\Delta U|_\infty \\
&\leqslant \Big(\frac{2-p}{2c_6}\Big)^{\frac{2-p}{2}} c_5|\Delta U|_\infty,
\end{aligned}$$

where the last inequality is due to (4.5.13). Let c_6 be sufficiently large such that $\Big(\frac{2-p}{2c_6}\Big)^{\frac{2-p}{2}} c_5 \leqslant \frac{1}{2}$, then we have

$$|\Delta k|_\infty \leqslant \frac{1}{2}|\Delta U|_\infty.$$

Therefore, Θ is a strict contraction mapping on $\mathcal{B}$ endowed with the norm $|\cdot|_\infty$. Because $(\mathcal{B},\ |\cdot|_\infty)$ is a complete metric space, the map Θ admits a unique fixed point which is the unique solution to the ℓ-dimensional RBSDEs (4.2.6). $\square$

4.5.5 Proof of Corollary 4.1

Proof. Set

$$\begin{aligned}
a(i) &= A_1(i) - B_1(i)'\Gamma^b(i),\\
\beta(i) &= C_1(i) - D_1(i)\Gamma^b(i),\\
f(i) &= q_1(i)Q_1(i) - P^b(i)(C_1(i)'\rho_1(i) + b_1(i)) - \rho_1(i)'\Lambda^b(i)\\
&\qquad + (P_1(i)D_1(i)'\rho_1(i) - R_1(i)p_1(i))'\Gamma^b(i),
\end{aligned}$$

then $|a(i)| \leqslant c(1 + |\Lambda^b(i)|)$, $|\beta(i)| \leqslant c(1 + |\Lambda^b(i)|)$, $|f(i)| \leqslant c(1 + |\Lambda^b(i)|)$, for all $i \in \mathcal{M}$.

Also, (4.2.3) can be rewritten as

$$\begin{cases}
-\,\mathrm{d}k(i) = \Big[a(i)k(i) + \beta(i)'l(i) + f(i) + \sum\limits_{j\neq i} q_{ij}k(j)\Big]\,\mathrm{d}t + \mathrm{d}\mu - l(i)'\,\mathrm{d}W,\\
k(T,i) = \xi(i),\\
k(t,i) \geqslant S(t,i), \quad \int_0^T (k(t,i) - S(t,i))\,\mathrm{d}\mu(i) = 0,\\
\mu(0) = 0, \quad \text{for all } i \in \mathcal{M}.
\end{cases}$$

whence admits a unique solution $(k(i), l(i), \mu(i))_{i\in\mathcal{M}}$ such that

$$(k(i), l(i), \mu(i)) \in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}) \times L^{2,\,\mathrm{BMO}}_{\mathcal{F}^W}(0,T;\mathbb{R}^n) \times \mathcal{A}_c(0,T;\mathbb{R}), \text{ for all } i \in \mathcal{M},$$

as a consequence of Theorem 4.2 and $c(1 + |\Lambda(i)|) \in L^{2,\,\mathrm{BMO}}_{\mathcal{F}^W}(0,T;\mathbb{R}^n)$. □

4.5.6 Proof of Lemma 4.10

Proof. In light of the length of many equations, "$(t, X_2(t), \alpha_t)$" will be suppressed when no confusion occurs in the sequel. Given $(\tau, X_2(\tau), \alpha_\tau) \in \mathcal{T}[0,T] \times L^2_{\mathcal{F}_\tau}(\Omega;\mathbb{R}) \times$

$\mathcal{M}$, substituting (4.3.3) into the state process (4.1.1) (with "i" replaced by "α_t"),

$$
\begin{aligned}
\mathrm{d}X_2 = \Big[&\big[A_2 - B_2'(R_2 + P^a D_2' D_2)^{-1}(P^a D_2' C_2 + P^a B_2 + D_2'\Lambda^a)\big]X_2 \\
&- B_2'(R_2 + P^a D_2' D_2)^{-1}(P^a D_2'\rho_2 - k^a B_2 - D_2' l^a - R_2 p_2) + b_2\Big]\,\mathrm{d}t \\
&+ \Big[\big[C_2 - D_2(R_2 + P^a D_2' D_2)^{-1}(P^a D_2' C_2 + P^a B_2 + D_2'\Lambda^a)\big]X_2 \\
&- D_2(R_2 + P^a D_2' D_2)^{-1}(P^a D_2'\rho_2 - k^a B_2 - D_2' l^a - R_2 p_2) + \rho_2\Big]'\,\mathrm{d}W,
\end{aligned}
\tag{4.5.14}
$$

By the basic theorem on pp. 756-757 of Gal'chuk (1979), SDE (4.5.14) admits a unique strong solution. For $(P^a(i), \Lambda^a(i), \mu_1(i))_{i\in\mathcal{M}}$, $(k^a(i), l^a(i), \mu_2(i))_{i\in\mathcal{M}}$, the unique solutions of (4.2.1) and (4.2.3) respectively, and $X_2(t)$, the solution of (4.5.14), applying Itô's lemma to $P^a(t,\alpha_t)X_2(t)^2 - 2k^a(t,\alpha_t)X_2(t)$, we have

$$
\begin{aligned}
&\int_\tau^t \Big(Q_2(X_2 - q_2)^2 + (\bar{u} - p_2)' R_2 (\bar{u} - p_2)\Big)\,\mathrm{d}t + P^a(t,\alpha_t)X_2(t)^2 - 2k^a(t,\alpha_t)X_2(t) \\
&= P^a(\tau,\alpha_\tau)X_2(\tau)^2 - 2k^a(\tau,\alpha_\tau)X_2(\tau) + \int_\tau^t \Big[P^a\rho_2'\rho_2 - 2k^a b_2 - 2l^{a\prime}\rho_2 + Q_2 q_2^2 + p_2' R_2 p_2 \\
&- (P^a D_2'\rho_2 - k^a B_2 - D_2' l^a - R_2 p_2)'(R_2 + P^a D_2' D_2)^{-1}(P^a D_2'\rho_2 - k^a B_2 - D_2' l^a - R_2 p_2)\Big]\,\mathrm{d}s \\
&+ \int_\tau^t \Big[2(P^a X_2 - k^a)(C_2 X_2 + D_2\bar{u} + \rho_2) + X_2^2\Lambda^a - 2X_2 l^a\Big]'\,\mathrm{d}W \\
&+ \int_\tau^t \Big\{X_2^2 \sum_{j,j'\in\mathcal{M}} (P^a(s,j) - P^a(s,j'))I_{\{\alpha_{s-}=j'\}} \\
&- 2X_2 \sum_{j,j'\in\mathcal{M}} (k^a(s,j) - k^a(s,j'))I_{\{\alpha_{s-}=j'\}}\Big\}\,\mathrm{d}\tilde{N}(s)^{j'j},
\end{aligned}
$$

where $(N^{j'j})_{j'j\in\mathcal{M}}$ are independent Poisson processes each with intensity $q_{j'j}$, and $\tilde{N}_t^{j'j} = N_t^{j'j} - q_{j'j}t$, $t \geqslant 0$ are the corresponding compensated Poisson martingales under the filtration $\mathcal{F}$.

Because $X_2(t)$ is continuous, the stochastic integrals in the last equation are local martingales. Thus there exists an increasing sequence of stopping times τ_n such that $\tau_n \uparrow +\infty$ as $n \to +\infty$ such that

$$\begin{aligned}
&\mathbb{E}\Big[\int_{\tau}^{\iota\wedge\tau_n} \Big(Q_2(X_2-q_2)^2+(\bar{u}-p_2)'R_2(\bar{u}-p_2)\Big)\,\mathrm{d}s \\
&\quad + P^a(\iota\wedge\tau_n,\alpha_{\iota\wedge\tau_n})X_2(\iota\wedge\tau_n)^2 - 2k^a(\iota\wedge\tau_n,\alpha_{\iota\wedge\tau_n})X_2(\iota\wedge\tau_n)\Big] \\
&= \mathbb{E}\Big[P^a(\tau,\alpha_\tau)X_2(\tau)^2 - 2k^a(\tau,\alpha_\tau)X_2(\tau)\Big] \\
&\quad + \mathbb{E}\int_{\tau}^{\iota\wedge\tau_n} \Big[P^a\rho_2'\rho_2 - 2k^a b_2 - 2l^{a\prime}\rho_2 + Q_2q_2^2 + p_2'R_2p_2 \\
&\qquad - (P^aD_2'\rho_2 - k^aB_2 - D_2'l^a - R_2p_2)'(R_2+P^aD_2'D_2)^{-1} \\
&\qquad \times (P^aD_2'\rho_2 - k^aB_2 - D_2'l^a - R_2p_2)\Big]\,\mathrm{d}s \\
&\leqslant \mathbb{E}\Big[P^a(\tau,\alpha_\tau)X_2(\tau)^2 - 2k^a(\tau,\alpha_\tau)X_2(\tau)\Big] \\
&\quad + \mathbb{E}\int_{\tau}^{\iota\wedge\tau_n} \Big[P^a\rho_2'\rho_2 - 2k^a b_2 - 2l^{a\prime}\rho_2 + Q_2q_2^2 + p_2'R_2p_2\Big]\,\mathrm{d}s,
\end{aligned} \tag{4.5.15}$$

for all stopping times $\iota \leqslant T$. Then by above,

$$
\begin{aligned}
&\mathbb{E}\Big[\int_{\tau}^{\iota\wedge\tau_n} \delta|\bar{u}-p_2|^2\,\mathrm{d}s\Big] \\
&\leqslant \mathbb{E}\Big[\int_{\tau}^{\iota\wedge\tau_n} \Big(Q_2(X_2-q_2)^2+(\bar{u}-p_2)'R_2(\bar{u}-p_2)\Big)\,\mathrm{d}s \\
&\qquad + P^a(\iota\wedge\tau_n,\alpha_{\iota\wedge\tau_n})\Big(X_2(\iota\wedge\tau_n)-\frac{k^a(\iota\wedge\tau_n,\alpha_{\iota\wedge\tau_n})}{P^a(\iota\wedge\tau_n,\alpha_{\iota\wedge\tau_n})}\Big)^2\Big] \\
&\leqslant \mathbb{E}\Big[P^a(\tau,\alpha_\tau)X_2(\tau)^2-2k^a(\tau,\alpha_\tau)X_2(\tau)\Big] \\
&\quad + \mathbb{E}\int_{\tau}^{\iota\wedge\tau_n} \Big(P^a\rho_2'\rho_2-2k^ab_2-2l^{a\prime}\rho_2+Q_2q_2^2+p_2'R_2p_2\Big)\,\mathrm{d}s+\mathbb{E}\Big[\frac{k^a(\iota\wedge\tau_n,\alpha_{\iota\wedge\tau_n})^2}{P^a(\iota\wedge\tau_n,\alpha_{\iota\wedge\tau_n})}\Big] \\
&\leqslant \mathbb{E}\Big[P^a(\tau,\alpha_\tau)X_2(\tau)^2-2k^a(\tau,\alpha_\tau)X_2(\tau)\Big] \\
&\quad + \mathbb{E}\int_{\tau}^{T} \Big|P^a\rho_2'\rho_2-2k^ab_2-2l^{a\prime}\rho_2+Q_2q_2^2+p_2'R_2p_2\Big|\,\mathrm{d}s+\mathbb{E}\Big[\frac{k^a(\iota\wedge\tau_n,\alpha_{\iota\wedge\tau_n})^2}{P^a(\iota\wedge\tau_n,\alpha_{\iota\wedge\tau_n})}\Big].
\end{aligned}
$$

It follows from Lemma 4.1 that there exists a constant $c_0>0$ such that $P^a(i)\geqslant c_0$, for all $i\in\mathcal{M}$. Since $k^a\in L^\infty_{\mathcal{F}W}(0,T;\mathbb{R})$, we obtain the estimate

$$
\mathbb{E}\Big[\frac{k^a(\iota\wedge\tau_n,\alpha_{\iota\wedge\tau_n})^2}{P^a(\iota\wedge\tau_n,\alpha_{\iota\wedge\tau_n})}\Big]\leqslant c_1,
$$

where c_1 is independent of n. Hence,

$$
\mathbb{E}\Big[\int_{\tau}^{\iota\wedge\tau_n} \delta|\bar{u}-p_2|^2\,\mathrm{d}s\Big]<c_2,
$$

where c_2 is independent of n and ι. Taking $\iota=T$ and letting $n\to\infty$, it follows from the monotone convergence theorem and the boundedness of p_2 that

$$
\bar{u}(t,X_2(t),\alpha_t)\in L^2_{\mathcal{F}}(\tau,T;\mathbb{R}^m).
$$

Similarly, in the singular case, we have

$$
\begin{aligned}
&\mathbb{E}\Big[P^a(\iota\wedge\tau_n)\Big(X_2(\iota\wedge\tau_n)-\frac{k^a(\iota\wedge\tau_n,\alpha_{\iota\wedge\tau_n})}{P^a(\iota\wedge\tau_n,\alpha_{\iota\wedge\tau_n})}\Big)^2\Big]\\
&\leqslant \mathbb{E}\Big[P^a(\tau,\alpha_\tau)X_2(\tau)^2-2k^a(\tau,\alpha_\tau)X_2(\tau)\Big]\\
&\quad+\mathbb{E}\int_\tau^T\Big|P^a\rho_2'\rho_2-2k^ab_2-2l^{a\prime}\rho_2+Q_2q_2^2+p_2'R_2p_2\Big|\,\mathrm{d}s+\mathbb{E}\Big[\frac{k^a(\iota\wedge\tau_n,\alpha_{\iota\wedge\tau_n})^2}{P^a(\iota\wedge\tau_n,\alpha_{\iota\wedge\tau_n})}\Big].
\end{aligned}
$$

Recall that $P^a(i)\geqslant c_0$, therefore,

$$
\begin{aligned}
c_0\mathbb{E}\Big[X_2(\iota\wedge\tau_n)^2\Big]&\leqslant \mathbb{E}\Big[P^a(\iota\wedge\tau_n,\alpha_{\iota\wedge\tau_n})X_2(\iota\wedge\tau_n)^2\Big]\\
&\leqslant 2\mathbb{E}\Big[P^a(\iota\wedge\tau_n,\alpha_{\iota\wedge\tau_n})\Big(X_2(\iota\wedge\tau_n)-\frac{k^a(\iota\wedge\tau_n,\alpha_{\iota\wedge\tau_n})}{P^a(\iota\wedge\tau_n,\alpha_{\iota\wedge\tau_n})}\Big)^2\Big]\\
&\quad+2\mathbb{E}\Big[\frac{k^a(\iota\wedge\tau_n,\alpha_{\iota\wedge\tau_n})^2}{P^a(\iota\wedge\tau_n,\alpha_{\iota\wedge\tau_n})}\Big]<c_3,
\end{aligned}
$$

where c_3 is independent of n and ι. Letting $n\to\infty$, it follows from Fatou's lemma that

$$\mathbb{E}\Big[X_2(\iota)^2\Big]\leqslant\frac{c_3}{c_0}$$

for all stopping times $\iota\leqslant T$. This further implies

$$\mathbb{E}\int_\tau^T X_2(s)^2\,\mathrm{d}s\leqslant\frac{c_3}{c_0}T. \tag{4.5.16}$$

By Itô's Lemma, we have

$$
\begin{aligned}
X_2(t)^2=X_2(\tau)^2+&\int_\tau^t\Big[(\bar u)'D_2'D_2\bar u+2X_2(D_2'C_2+B_2)'\bar u+2\rho_2'D_2\bar u\\
&\quad+(2A_2+C_2'C_2)X_2^2+2X_2(b_2+C_2'\rho_2)+\rho_2'\rho_2\Big]\,\mathrm{d}s\\
&+\int_\tau^t 2X_2(C_2X_2+D_2\bar u+\rho_2)'\,\mathrm{d}W.
\end{aligned}
$$

Assumptions 4.1 and 4.2 (ii) and the positiveness of P^a implies $R_2 + P^a D_2' D_2 \geqslant c_4 > 0$, so

$$|\bar{u}| \leqslant c_4(1 + |\Lambda^a||X_2| + |l^a|). \tag{4.5.17}$$

Let

$$\theta_n = \inf\Big\{t \geqslant \tau : |X_2(t)| + \int_\tau^t (|\Lambda^a(s)|^2 + |l^a(s)|^2)\,\mathrm{d}s > n\Big\}.$$

Because $X_2(t)$ is continuous, Λ^a, $l^a \in L^2_{\mathcal{F}^W}(0,T;\mathbb{R}^n)$, it follows that

$$\begin{aligned}
\mathbb{E}\int_\tau^{T\wedge\theta_n} (\bar{u})' D_2' D_2 \bar{u}\,\mathrm{d}s = \mathbb{E}\Big[X_2(T\wedge\theta_n)^2\Big] - \mathbb{E}\Big[X_2(\tau)^2\Big]& \\
- \mathbb{E}\int_\tau^{T\wedge\theta_n} \Big[2X_2(D_2'C_2 + B_2)'\bar{u} + 2\rho_2' D_2 \bar{u}& \\
+ (2A_2 + C_2'C_2)X_2^2 + 2X_2(b_2 + C_2'\rho_2) + \rho_2'\rho_2\Big]\,\mathrm{d}s.&
\end{aligned}$$

Let $\delta > 0$ be given in Assumption 4.2. By Assumption 4.1 and (4.5.16), the above by the elementary inequality $2ab \leqslant \frac{\epsilon}{2}a^2 + \frac{2}{\epsilon}b^2$ leads to

$$\begin{aligned}
&\delta\mathbb{E}\int_\tau^{T\wedge\theta_n} |\bar{u}\,(s, X_2(s), \alpha_s)|^2\,\mathrm{d}s \\
\leqslant& c + c\mathbb{E}\int_\tau^{T\wedge\theta_n} \left[X_2(s)^2 + 2(|X_2(s)| + 1)\,|\bar{u}|\right]\mathrm{d}s \\
\leqslant& c + \frac{4c^2}{\delta} + \left(c + \frac{4c^2}{\delta}\right)\mathbb{E}\int_\tau^T X_2(s)^2\,\mathrm{d}s + \frac{\delta}{2}\mathbb{E}\int_\tau^{T\wedge\theta_n} |\bar{u}\,(s, X_2(s), \alpha_s)|^2\,\mathrm{d}s,
\end{aligned}$$

where c is independent of n. The last expectation is finite by (4.5.17) and the definition of θ_n. So, after rearrangement and sending $n \to \infty$, it follows from the monotone convergence theorem and (4.5.16) that

$$\bar{u}(t, X_2(t), \alpha_t) \in L^2_{\mathcal{F}}(\tau, T; \mathbb{R}^m).$$

This completes the proof. $\square$

4.5.7 Proof of Theorem 4.4

Proof. Without loss of generality, in this proof we assume u is real valued. First, applying Itô's formula to $P^b(t,\alpha_t)X_1(t)^2 - 2MX_1(t)$,

$$
\begin{aligned}
& P^b(t,\alpha_t)X_1(t)^2 - 2MX_1(t) \\
= & P^b(0,i_0)x^2 - 2Mx \\
& + \int_0^t \Big\{ P^b(s,\alpha_s)u(s)'D_1(s,\alpha_s)'D_1(s,\alpha_s)u(s) \\
& \qquad + 2X_1(s)P^b(s,\alpha_s)C_1(s,\alpha_s)'D_1(s,\alpha_s)u(s) + 2X_1(s)P^b(s,\alpha_s)B_1(s,\alpha_s)'u(s) \\
& \qquad + 2X_1(s)\Lambda^b(s,\alpha_s)'D_1(s,\alpha_s)u(s) + 2P^b(s,\alpha_s)\rho_1(t,\alpha_t)'D_1(s,\alpha_s)u(s) \\
& \qquad - X_1(s)^2[H^b(s,P^b(s,\alpha_s),\Lambda^b(s,\alpha_s),\alpha_s)] \\
& \qquad + 2P^b(s,\alpha_s)(C_1(s,\alpha_s)'\rho_1(s,\alpha_s) + b_1(s,\alpha_s))X_1(s) + 2\rho_1(s,\alpha_s)'\Lambda^b(s,\alpha_s)X_1(s) \\
& \qquad + P^b(t,\alpha_t)\rho_1(t,\alpha_t)'\rho_1(t,\alpha_t) \\
& \qquad - 2M\left[A_1(s,\alpha_s)X_1(s) + B_1(s,\alpha_s)'u(s) + b_1(s,\alpha_s)\right] \Big\} \mathrm{d}s \\
& + \int_0^t \Big\{ 2P^b(s,\alpha_s)(C_1(s,\alpha_s)'X_1(s)^2 + u_s'D_1(s,\alpha_s)'X_1(s) \\
& \qquad\quad + \rho_1(s,\alpha_s)') + X_1(s)^2\Lambda^b(s,\alpha_s)' \Big\} \mathrm{d}W \\
& - 2M\int_0^t [C_1(s,\alpha_s)'X_1(s) + u(s)'D_1(s,\alpha_s)' + \rho_1(s,\alpha_s)'] \,\mathrm{d}W \\
& + \int_0^t X_1(s)^2 \,\mathrm{d}\mu_1(s,\alpha_s) \\
& + \int_0^t \Big\{ X_1(s)^2 \sum_{j,j'\in\mathcal{M}} (P^b(s,j) - P^b(s,j'))I_{\{\alpha_{s-}=j'\}} \Big\} \mathrm{d}\tilde{N}_s^{j'j},
\end{aligned}
$$

where $(N^{j'j})_{j'j\in\mathcal{M}}$ are independent Poisson processes each with intensity $q_{j'j}$, and $\tilde{N}_t^{j'j} = N_t^{j'j} - q_{j'j},\ t \geqslant 0$ are the corresponding compensated Poisson martingales under the filtration $\mathcal{F}$.

Note that $X_1(t)$ is continuous, the last two terms in the above equation are local martingales. Therefore, there exists an increasing localizing sequence of stopping times $\theta_n \uparrow +\infty$ as $n \to +\infty$ such that after rearrangement and combining similar terms,

$$
\begin{aligned}
&\mathbb{E}\Big\{ \int_0^{\tau\wedge\theta_n} \Big(Q_1(t,\alpha_t)(X_1(t) - q_1(t,\alpha_t))^2 \\
&\qquad + (u(t) - p_1(t,\alpha_t))' R_1(t,\alpha_t)(u(t) - p_1(t,\alpha_t)) - \mathbb{C}^a(t,\alpha_t)\Big)\,\mathrm{d}t \\
&\qquad + P^b(\tau\wedge\theta_n, \alpha_{\tau\wedge\theta_n}) X_1(\tau\wedge\theta_n)^2 - 2MX_1(\tau\wedge\theta_n) + \mathbb{C}^b(\tau\wedge\theta_n, \alpha_{\tau\wedge\theta_n})\Big\} \\
&= P^b(0,i_0)x^2 - 2Mx + \mathbb{E}\Big\{\mathbb{C}^b(\tau\wedge\theta_n, \alpha_{\tau\wedge\theta_n}) \\
&\qquad + \int_0^{\tau\wedge\theta_n} \Big[P^b(s,\alpha_s)\rho_1(s,\alpha_s)'\rho_1(s,\alpha_s) - 2Mb_1(s,\alpha_s) + Q_1(s,\alpha_s)q_1(s,\alpha_s)^2 \\
&\qquad\qquad + p_1(s,\alpha_s)' R_1(s,\alpha_s) p_1(s,\alpha_s) - \mathbb{C}^a(s,\alpha_s) + \phi\left(s, X_1(s), u(s), \alpha_s\right)\Big]\,\mathrm{d}s \\
&\qquad + \int_0^{\tau\wedge\theta_n} X_1(s)^2\,\mathrm{d}\mu_1(s,\alpha_s)\Big\},
\end{aligned}
$$

where

$$
\begin{aligned}
\phi(s, X_1, u, i) = u'\Big(&R_1(s,i) + P^b(s,i)D_1(s,i)'D_1(s,i)\Big)u \\
&+ 2\Big(P^b(s,i)C_1(s,i)'D_1(s,i)X_1 + P^b(s,i)B_1(s,i)'X_1 \\
&\qquad + \Lambda^b(s,i)'D_1(s,i)X_1 + P^b(s,i)\rho_1(t,\alpha_t)'D_1(s,i) \\
&\qquad - MB_1(s,i)' - p_1(s,i)R_1(s,i)\Big)u \\
&+ (Q_1(s,i) - H^b(s, P^b(s,i), \Lambda^b(s,i), i))X_1^2 \\
&+ 2\Big((P^b(s,i)(C_1(s,i)'\rho_1(s,i) + b_1(s,i)) + \rho_1(s,i)'\Lambda^b(s,i) \\
&- 2Q_1(s,i)q_1(s,i) - 2MA_1\Big)X_1.
\end{aligned}
$$

We write

$$
\begin{aligned}
\phi(s,X_1,u,\alpha_s) &= \mathbf{a}u^2+\mathbf{b}X_1u+\mathbf{c}X_1^2+\mathbf{d}X_1+\mathbf{e}u \\
&= \mathbf{a}\left(u+\frac{\mathbf{b}X_1+\mathbf{e}}{2\mathbf{a}}\right)^2+\left(\frac{4\mathbf{ac}-\mathbf{b}^2}{4\mathbf{a}}\right)\left(X_1+\frac{2\mathbf{ad}-\mathbf{be}}{4\mathbf{ac}-\mathbf{b}^2}\right)^2+\underline{\phi}(s,\alpha_s),
\end{aligned}
$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$ and $\underline{\phi}(s,\alpha_s)$ are given in the theorem. From Assumptions 4.1 and 4.2 and $H^b\leqslant 0$, we have $\mathbf{a}>0$ and $4\mathbf{ac}>\mathbf{b}^2$, so

$$
\phi(s,X_1(s),u(s),\alpha_s)\geqslant \underline{\phi}(s,\alpha_s).
$$

Hence, it follows from (3.5.6) and the monotonicity of μ_1 that

$$
\begin{aligned}
&\mathbb{E}\Big\{\int_0^{\tau\wedge\theta_n}\Big(Q_1(t,\alpha_t)(X_1(t)-q_1(t,\alpha_t))^2 \\
&\qquad\qquad +(u(t)-p_1(t,\alpha_t))'R_1(t,\alpha_t)(u(t)-p_1(t,\alpha_t))-\mathbb{C}^a(t,\alpha_t)\Big)\,\mathrm{d}t \\
&\qquad\qquad +P^b(\tau\wedge\theta_n,\alpha_{\tau\wedge\theta_n})X_1(\tau\wedge\theta_n)^2-2MX_1(\tau\wedge\theta_n)+\mathbb{C}^b(\tau\wedge\theta_n,\alpha_{\tau\wedge\theta_n})\Big\} \\
&\geqslant P^b(0,i_0)x^2-2Mx+\mathbb{E}\Big\{\mathbb{C}^b(\tau\wedge\theta_n,\alpha_{\tau\wedge\theta_n}) \\
&\qquad +\int_0^{\tau\wedge\theta_n}\Big[P^b(s,\alpha_s)\rho_1(s,\alpha_s)'\rho_1(s,\alpha_s)-2Mb_1(s,\alpha_s)+Q_1(s,\alpha_s)q_1(s,\alpha_s)^2 \\
&\qquad\qquad +p_1(s,\alpha_s)'R_1(s,\alpha_s)p_1(s,\alpha_s)-\mathbb{C}^a(s,\alpha_s)+\underline{\phi}(s,\alpha_s)\Big]\,\mathrm{d}s\Big\} \\
&\geqslant P^b(0,i_0)x^2-2Mx+\mathbb{E}\Big\{\int_0^T\mathbb{C}^a(t,\alpha_t)\,\mathrm{d}t+G_2(\alpha_T)g_2(\alpha_T)^2\Big\}+\hat{M},
\end{aligned}
$$

where $\hat{M}$ is given in the theorem. It is not hard to verify $\mathbb{E}\Big[\sup_{s\in[0,T]}X_1(s)^2\Big]<\infty$ by standard theory of SDE. Let $n\to\infty$, by the dominated convergence and monotone convergence theorems, we have

$$
\begin{aligned}
&\mathbb{E}\Big\{\int_0^\tau \Big(Q_1(t,\alpha_t)(X_1(t)-q_1(t,\alpha_t))^2\\
&\qquad\qquad + (u(t)-p_1(t,\alpha_t))'R_1(t,\alpha_t)(u(t)-p_1(t,\alpha_t)) - \mathbb{C}^a(t,\alpha_t)\Big)\,\mathrm{d}t\\
&\qquad\qquad + P^b(\tau,\alpha_\tau)X_1(\tau)^2 - 2MX_1(\tau) + \mathbb{C}^b(\tau,\alpha_\tau)\Big\}\\
&\geqslant P^b(0,i_0)x^2 - 2Mx + \mathbb{E}\Big\{\int_0^T \mathbb{C}^a(t,\alpha_t)\,\mathrm{d}t + G_2(\alpha_T)g_2(\alpha_T)^2\Big\} + \hat{M}.
\end{aligned}
$$

The proof is complete. □

4.5.8 Proof of Theorem 4.5

Proof. From Problem (4.3.8) and (4.3.5), we have

$$
\begin{aligned}
&\inf_{(u,\tau)\in\mathscr{B}} J^b(x,i_0,u(\cdot),\tau)\\
&\geqslant \inf_{(u,\tau)\in\mathscr{B}} J^b_{\text{lower}}(x,i_0,u(\cdot),\tau)\\
&\geqslant \inf_{(u,\tau)\in\mathscr{A}} J^b_{\text{lower}}(x,i_0,u(\cdot),\tau).
\end{aligned}
$$

Combing with the result of Theorem 4.4, we finish the proof. □

Chapter 5

Concluding Remarks

This thesis advances the theory of stochastic control under regime switching by solving two novel classes of constrained and non-homogeneous linear quadratic control-stopping problems with random coefficients. Through a unified framework combining BSDEs, RBSDEs, we have derived explicit optimal controls and value functions while establishing the solvability of associated Riccati equation systems. Our work bridges theoretical stochastic analysis with practical applications in financial optimization, offering new mathematical tools for decision-making under uncertainty.

We make theoretical contributions as follows. In Chapter 3, we addressed a general stochastic LQ control problem under a random coefficients model with non-Markovian regime switching, where the control pair consists of a classical control variable constrained in a cone and a controlled stopping time τ. The key innovation lies in the introduction of four coupled systems of ESREs and RESREs: (3.2.1), (3.2.2), (3.2.3), and (3.2.4), which describe the optimal optimal and value function before and after the stopping time τ. These systems are fundamentally distinct from classical Riccati equations due to: (i) the cone constraints inducing optimization operators in their drivers, and (ii) the allowance for singular control weighting matrices. Overcoming the non-Lipschitz and non-quadratic-growth properties of these equations required novel techniques, including multidimensional comparison theorems, truncation methods, John-Nirenberg inequalities and BMO martingale.

Chapter 4 extended this framework to non-homogeneous systems with unconstrained controls, introducing affine terms in the state dynamics and stochastic targets in the cost functional. Here, we derived a new class of unbounded, multi-dimensional RBSDEs (4.2.1) and (4.2.3) with challenges: unbounded coefficients and Markov-chain-induced coupling. Their solvability was established via a three-step approach: BMO martingale estimates, one dimensional existence and uniqueness of unbounded RBSDEs, and contraction mapping for multi-dimensional RBSDEs. This methodology not only enriches RBSDEs theory but also provides a blueprint for handling unbounded systems in future research.

First, inspired by the homogeneity properties appears in the model of Chapter 3, it is interesting to extend the LQ framework to more general case, a new class of homogeneous stochastic control problems, which extend the classical homogeneous LQ problems to new ones with nonlinear controlled dynamics and non-quadratic cost functionals, but there exists certain BSDEs, derived by the stochastic maximum principle or dynamic programming approach, which are similar to Riccati equations for LQ problems.

Second, our extension of LQ theory provides a powerful framework for solving the continuous-time mean-variance portfolio selection problem under realistic market conditions, including regime switching and no-shorting constraints. In this financial setting, an investor aims to minimize portfolio variance for a given target return while operating in a market where asset returns, volatilities, and interest rates are modulated by a Markov chain representing different economic regimes (e.g., bull vs. bear markets). By embedding this problem into our LQ control framework, one could obtain explicit closed-form optimal strategies that dynamically adapt to both market regimes and wealth levels, along with optimal stopping rules for strategic rebalancing. Key results include the solvability of coupled systems of ESREs under cone constraints, the characterization of a regime-dependent efficient frontier, and computationally tractable solutions via RBSDEs. These contributions advance portfolio optimization by rigorously incorporating market frictions and regime uncertainty. Future research may extend this framework to incorporate jump-diffusion models, partial observations of market regimes, or multi-criteria optimization, further bridging the gap between stochastic control theory and practical financial decision-making.

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