The Augmented Lagrangian Methods and Applications

by

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Certificate of Originality

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Contents

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Abstract

The purpose of this thesis is to study a general augmented Lagrangian scheme for optimization and optimal control problems. We establish zero duality gap and exact penalty properties between a primal optimization problem and its augmented Lagrangian dual problem, and characterize local and global solutions for a class of non-Lipschitz penalty problems. We also obtain the existence of an optimal control for an optimal control problem governed by a variational inequality with monotone type mappings, and establish zero duality gap between this optimal control problem and its nonlinear Lagrangian dual problem.

Under the assumptions that the augmenting function satisfies the level-coercive condition and the perturbation function satisfies a growth condition, a necessary and sufficient condition for a vector to support an exact penalty representation of the problem of minimizing an extended real function is established. Moreover, in general Banach spaces, under the assumption that the augmenting function satisfies a valley at zero condition and the perturbation function satisfies a growth condition, a necessary and sufficient condition for a zero duality gap property between the primal problem and its augmented Lagrangian dual problem is established.

We show that under some conditions the inequality and equality constrained optimization problem and its augmented Lagrangian problem both have optimal solutions. On the other hand it is shown that every weak limit point of a sequence of optimal solutions generated by its augmented Lagrangian problem is a solution of the original constrained optimization problem. Sufficient conditions for the existence of an exact penalization representation and an asymptotically minimizing sequence for a constrained optimization problem are established.

It is shown that the second order sufficient condition implies a strict local minimum of a class of non-Lipschitz penalty problems with any positive penalty parameter. The generalized representation condition and the second order sufficient

condition imply a global minimum of these penalty problems. We apply our results to quadratic programming and linear fractional programming problems.

We study an optimal control problem where the state system is defined by a variational inequality problem with monotone type mappings. We first study a variational inequality problem for monotone type mappings. Under some general coercive assumption, we establish existence results of a solution of variational inequality problems with generalized pseudomonotone mappings, generalized pseudo-monotone perturbation and T-pseudomonotone perturbation of maximal monotone mappings respectively. We obtain several existence results of an optimal control of the optimal control problem governed by a variational inequality with monotone type mappings. Moreover, as an application, we get several existence results of an optimal control for the optimal control problem where the system is defined by a quasilinear elliptic variational inequality problem with an obstacle. By using nonlinear Lagrangian methods, we obtain one necessary condition and several sufficient conditions for the zero duality gap property between the optimal control problem where the state of the system is defined by a variational inequality problem for monotone type mappings and its nonlinear Lagrangian dual problem. We also apply our results to an example where the variational inequality problem leads to a linear elliptic obstacle problem.

The study of this thesis has used tools from nonlinear functional analysis, nonlinear programming, nonsmooth analysis and numerical linear algebra.

Chapter 1

Introduction

1.1 Augmented Lagrangian methods

The theory of augmented Lagrangian has many applications in the study of optimization problems. It is so attractive and powerful that a wealth of papers has been published, e.g., [10], [22], [29], [56], [62], [63], [64], [90], [95], [97], [100], [105] and [117]. Constrained optimization problems can be studied by using augmented Lagrangian functions, as such a constrained optimization problem can be solved by solving one or a sequence of unconstrained optimization problems. The augmented Lagrangian method can be viewed as a combination of penalty function method and Lagrangian multiplier method, which is able to eliminate many of the disadvantages associated with either method alone.

Actually, a constrained optimization problem is determined by the interaction of two distinct subproblems: the feasibility subproblem and the subproblem of minimizing the objective function. Therefore a constrained optimization problem can be solved by the unconstrained minimization of a merit function only if this function is able to represent well the combination of the two preceding subproblems. The initial idea in defining merit functions was to add the original objective function penalty terms which weigh the violation of the constraints. Since the penalty approach attempts to solve a constrained optimization problem by the minimization of an unconstrained function, the main motivation for the use of penalty methods is that of solving the constrained optimization problem by employing some unconstrained minimization algorithm. Penalty methods include interior point (barrier) and exterior point methods. The approximation is accomplished in the case of exterior point penalty methods by adding to the objective function a term that prescribes a high cost for violation of the constraints, and in the case of interior point methods by adding a term that favors points interior to the feasible region over those near the boundary.

Using Hahn-Banach separation theorem for convex sets or Ky Fan and Sion minimax theorem, the duality theory for convex programming via an ordinary Lagrangian has been well established (see [20] and [80]). However, for nonconvex optimization problems, a nonzero duality gap may exist when an ordinary Lagrangian is used. In order to overcome this drawback, quadratic augmented Lagrangians were introduced for constrained optimization problems. This method was first proposed by Hestenes [54] and Powell [94] to solve a mathematical program with only equality constraints. It was later extended by Rockafellar to solve optimization problems with both equality and inequality constraints (see [95, 96]). As noted in Bertsekas [15], in comparison with the traditional (quadratic) penalty method for constrained optimization problems, convergence of augmented Lagrangian method usually does not require that the penalty parameter tends to infinity. This important advantage results in elimination or at least moderation of the ill conditioning problem in the traditional penalty method. Another important advantage of augmented Lagrangian method is that its convergence rate is considerably better than that of the traditional penalty method.

In recent years, the theory of augmented Lagrangian has been widely studied by many authors in different aspects. In [97], a corner-stone book in optimization, an augmented Lagrangian with a convex augmenting function was introduced and the corresponding zero duality property was obtained. A level-bounded augmenting function was given by Huang and Yang in [56] where the convexity of augmenting functions in [97] was replaced by a level-boundedness condition. The level-bounded augmented Lagrangian scheme includes nonconvex and nonsmooth penalty functions in [81] and Lagrangian-type functions in [101] as special cases. Furthermore, a peak at 0 augmenting function was introduced in [100, 101] and applied to establish an equivalence of the zero duality gap properties of a corresponding augmented Lagrangian dual problem and a Lagrangian-type dual problem. Under a growth condition, one can relate the value of the dual problem to the behavior of the perturbation function. In [95], Rockafellar has given the definition of a quadratic growth condition in a finite-dimensional space and obtained a necessary and sufficient condition for a zero duality gap between a constrained problem and its augmented

Lagrangian dual problem. In [90], Penot has introduced the definition of a general growth condition in an infinite-dimensional space. In [103], Rubinov *et al.* gave a detailed and excellent survey on various Lagrange-type functions for nonconvex constrained scalar optimization problems. Necessary and sufficient conditions are given for the zero duality gap property and exact penalization. In [117], Yang and Huang established an equivalence between two types of zero duality gap properties, which are described using augmented Lagrangian dual functions and nonlinear Lagrangian dual functions, respectively. Furthermore, they introduced the concept of partially strictly monotone functions and applied it to construct a class of nonlinear penalty functions for a constrained optimization problem in [118]. This class of nonlinear penalty functions includes some (nonlinear) penalty functions currently used in the literature as special cases. They proved that each limit point of the second-order stationary points of the nonlinear penalty problems is a second-order stationary point of the original constrained optimization problem. In [43], Gasimov and Rubinov examined augmented Lagrangians for optimization problems with a single (either inequality or equality) constraint. In [42], Gasimov presented augmented Lagrangians for nonconvex minimization problems with equality constraints. He obtained the saddle point optimality conditions and some strong duality results. In [105], Shapiro and Sun considered a minimization problem where the constraints are given in a form of set inclusion in Hilbert spaces and introduced the augmented Lagrangian dual of this problem by applying methods developed by Rockafellar and Wets [97]. The existence of augmented Lagrange multipliers is studied, and especially second-order necessary and sufficient conditions for the existence of an augmented Lagrange multiplier are established. Augmented Lagrangian methods can also be used in solving variational inequality problems, see [6] and [107]. In [6], Auslender and Teboulle considered a new class of multiplier interior point methods for solving variational inequality problems.

The existence of an exact penalty function is important as such the optimal solution of the original constrained optimization problem can be found by solving only one unconstrained optimization problem [15, 19, 24]. Necessary and sufficient conditions for the existence of exact penalty parameters have been established for different situations (see [19, 97, 99]). It was established by Burke [19] that the existence of an exact penalty function is equivalent to the calmness condition first introduced by Clarke [23]. Lower-order penalty functions have been investigated in [81], [99] and [119]. In [119], Yang and Huang considered a smooth mathematical program with complementarity constraints (MPCC), and they applied a lower-order penalty method to transform MPCC into a unconstrained optimization problem.

They also derived optimality conditions for the penalty problems using a smooth approximate variational principle, and established that the limit point of a sequence of points that satisfy the second-order necessary optimality conditions of penalty problems is a strongly stationary point of the original MPCC if the limit point is feasible to MPCC. Numerical examples were presented to demonstrate and compare the effectiveness of the proposed method over existing methods in the literature. In [101], some numerical examples were presented to demonstrate that a non-Lipschitz penalty function can be successfully applied to solve a class of concave programming problems, where the classical penalty method has failed. It is worth noting that these penalty functions have some theoretical advantages. Firstly the nonconvex and nonsmooth penalty functions in [81] require weaker conditions to guarantee the existence of exact penalty functions than the classical l_1 penalty functions. Secondly nonlinear penalty functions in [99] and [101] admit a smaller least exact penalty parameter than that of the l_1 penalty function.

Exact penalty functions can be subdivided into two main classes: nondifferentiable exact penalty functions and continuously differentiable exact penalty functions. Nondifferentiable exact penalty functions were introduced for the first time by Zangwill in [125], and have been widely investigated in recent years ([8, 25, 26, 34, 52]). Continuously differentiable exact penalty functions were introduced by Fletcher [36] for equality constrained problems and by Glad and Polak [45] for inequality constrained problems, and were further investigated by [27, 30]. In particular, in [27], Di Pillo and Facchinei introduced a new continuously differentiable exact penalty function for the solution of nonlinear programming problems with a compact feasible set. The approach proposed in [27] has been further investigated by Lucidi in [79] and [30]. In [28], Di Pillo and Facchinei introduced formal definitions of exactness for penalty functions and state sufficient conditions for a penalty function to be exact according to these definitions. They dealt with a unified framework which applied to both the nondifferentiable and the continuously differentiable case.

It is worth noting that existence and convergence of an optimal path generated by penalty/dual problems toward the optimal set is important for numerical solution methods as shown in [5, 56, 117, 134]. Therefore, in this thesis, we are interested in exploring the convergence of optimal solutions for the augmented Lagrangian problem, discuss some necessary and sufficient conditions for the zero duality gap property between the primal problem and its augmented Lagrangian dual problem, and apply these results to solve some variational inequality problems in Sobolev spaces.

We note that augmented Lagrangian methods have been applied to optimization problems in infinite-dimensional Banach spaces. For example, quadratic augmented Lagrangian methods have been used in optimal control problems for nonlinear differential equation in Sobolev space, e.g., [22, 63, 64, 109]. Actually, the optimal control problem governed by a quasilinear elliptic variational inequality with an obstacle (see [77, 124]) is a nonconvex optimization problem in Sobolev spaces. It needs to use the theory of augmented Lagrangian in infinite-dimensional Banach spaces. Therefore, one of my interests is to obtain some new results about augmented Lagrangian in infinite-dimensional Banach spaces in this thesis.

In Chapter 2, we introduce the concept of a valley at 0 augmenting function and apply it to construct a class of valley at 0 augmented Lagrangian functions. In this chapter, we explore a general augmented Lagrangian scheme with a valley at 0 augmenting function in reflexive Banach spaces. Under the assumption that the perturbation function satisfies the growth condition and the augmenting function satisfies a valley at 0 condition, we establish a necessary and sufficient condition for a zero duality gap property between the primal problem and its augmented Lagrangian dual problem in general Banach spaces, which includes Theorem 2.9 and Corollary 2.10 in [90] as special cases. We apply it to variational problems in Sobolev spaces.

In Chapter 3, we obtain some exact augmented Lagrangian representation results in the framework of new augmented Lagrangian under weaker conditions than the ones in [56, 97]. In infinite dimensional Banach spaces, we obtain that the inequality and equality constrained optimization problem and its augmented Lagrangian problem both have optimal solutions under some conditions. We establish sufficient conditions of an exact penalization representation for the constrained problems. Furthermore, we obtain a sufficient condition of an asymptotically minimizing sequence for a constrained problem, which generalizes Theorem 3 in [95] to the non-quadratic case. Without any coercive assumption on the objective function and constraint functions, we obtain a sufficient condition of an exact penalization representation for the constrained problem in finite dimensional spaces.

In Chapter 4, we introduce a class of penalty functions which is more general than the ones in [52], [81], [88] and [113]. We prove that any strict local minimum satisfying a second-order sufficient condition for an inequality and equality constrained optimization problem is a strict local minimum of this penalty function with any positive penalty parameter, and that any global minimum satisfying a second-order global sufficient condition and a generalized representation condition for the original problem is a global minimum of this penalty function with some positive penalty parameter. We apply our results to quadratic and linear fractional programming problems.

1.2 Optimal control problems governed by a variational inequality

The optimal control problem for an elliptic variational inequality proposed by Lions ([73, 74, 75]) is the following minimization problem:

$$\begin{aligned}&\min\ \ G(u)+L(w)\\&\text{subject to } (w,u)\in U_{ad}\times K, \text{and } u\in S(w),\end{aligned}\tag{1.1}$$

where, for each $w \in U_{ad}$, $S(w)$ is the set of solutions of the following variational inequality problem:

$$\langle A(u), v-u\rangle \geq \langle f-B(w), v-u\rangle, \forall v\in K,\tag{1.2}$$

and K is a closed and convex cone of a Hilbert space $\mathbf{V}$, U_{ad} is a nonempty closed set of a Hilbert space $\mathbf{U}$, $G : K \to \mathbf{R}^+$, $L : U_{ad} \to \mathbf{R}^+$, $A \in L(\mathbf{V}, \mathbf{V}^*)$, $B \in L(U_{ad}, \mathbf{V}^*)$ and $f \in \mathbf{V}^*$. If there exist $(w_0, u_0) \in U_{ad} \times K,$ and $u_0 \in S(w_0)$, such that

$$G(u_0)+L(w_0)=\min_{(w,u)\in U_{ad}\times K, u\in S(w)} G(u)+L(w),$$

then w_0 is called an optimal control for minimization problem (1.1).

As Lions [74] pointed out, finding necessary and sufficient conditions for the optimal control and constructing algorithms amenable to numerical computation for the approximation of the optimal control are two important objectives of the optimal control theory. Necessary and sufficient conditions for optimal control problems governed by variational inequalities have been investigated by a number of authors (see Lions [73, 74, 75], Adams and Lenhart [2], Barbu [7], Mignot and Puel [87], He [53], Bergounioux [10] and Ye [122, 123]).

The theory of the optimal control problem (1.1) has been widely studied by many authors using different methods. One of the methods is the approximation of the variational inequality by an equation where the maximal monotone operator is approached by a differentiable single-value mapping with Moreau-Yosida approximation techniques. This method, mainly due to Barbu [7], leads to several existence results and to first-order optimality systems. It is worth noting that most results in these papers are obtained in Hilbert spaces, variational inequalities are of linear or semilinear elliptic type and the objectives (cost) are quadratic ones of the state and control.

Recently, Lou [77], Ye and Chen [124] considered the existence, regularity and necessary condition of the optimal control problem governed by a quasilinear elliptic variational inequality respectively. In [77], Lou introduced an approximate problem and gave estimates of optimal pairs for the approximate problem. By using the obtained results, he got the existence and regularity of the solution to the original problem. In [124], Ye and Chen approximated the variational inequality by a family of quasilinear elliptic equations, and proved that the optimal pairs for the approximate problem converges to the solution of the original problem. Using the weak convergence methods, they established some optimality conditions.

In order to study the optimal control problem governed by an obstacle problem, we investigate the obstacle problem first. Obstacle problems are actually special variational inequalities. These problems, especially monotone variational inequality problems, have attracted much attention in recent years. There are many interesting theoretical questions arising from these problems and many applications in mechanics, applied mathematics, social science, industry, and differential equations which can be cast as such problems. For example, see [7, 53, 55, 67, 71, 104, 108, 126, 131, 133] and the references cited therein. Recent results involving maximal monotone mappings and their perturbations can be found in [49, 55, 65, 66]. By using existence results of variational inequalities for monotone type mappings, we can establish surjective results for the corresponding monotone type mappings. For example, Huang and Zhou [60] studied the variational inequalities with a sum of a maximal monotone mapping T and a T-pseudomonotone mapping T_0, and, under some coercive conditions, obtained the surjective results of perturbed maximal monotone mappings, which extend and improve the corresponding results of [106, 127, 128]. Recently, by using some coercive condition which is weaker than the one in Browder's result (see [17]), Guan et al [48, 49] obtained some results about ranges of generalized pseudo-monotone perturbations of maximal monotone operators.

Penalty function methods have often been used in the study of optimization theory and methodology for constrained mathematical programs and optimal control problems (see [10], [11], [84], [85] and [97]). Bergounioux used these methods to approximate the optimal control problem of obstacle problems in [10] and [11]. In [11], he interpreted the variational inequality as a state equation, introducing another control function as in [87], and obtained first order necessary optimality conditions by using classical penalty methods. In [10], he introduced an associated relaxed problem, and devoted to a saddle-point formulation of the optimality system.

Quadratic augmented Lagrangian methods have been used in optimal control problems for nonlinear differential equation or elliptic variational inequality in Sobolev space, e.g., [22, 63, 64, 109]. In this method, differential equation or elliptic variational inequality was treated as an equality constraint or inequality constraint which is realized by a Lagrangian term together with a penalty functional. Chen and Zou [22], and Guo and Zou [50] investigated elliptic and parabolic systems by using augmented Lagrangian methods. The identification of parameters in elliptic and parabolic systems was formulated as a constrained minimization problem combining the output least squares and the equation error method. The minimization problem is then proved to be equivalent to the saddle-point problem of an augmented Lagrangian.

In Chapter 5, enlightened by [48, 49, 60], we consider variational inequality problems for generalized pseudo-monotone mappings and perturbed maximal monotone mappings. Under a more general coercive condition than the one used by [48, 49], we establish some existence results for a solution of variational inequality problems for generalized pseudo-monotone mappings and generalized pseudo-monotone perturbations of maximal monotone mappings respectively. Moreover, we obtain an existence result of a solution of a variational inequality problem for T-pseudomonotone perturbations of maximal monotone mapping T. The optimal control problem considered in this chapter is one with a monotone type variational inequality constraint in Banach spaces. This is actually a nonsmooth and nonconvex infinite-dimensional optimization problem. We obtain several existence results of an optimal control of the optimal control problem governed by a quasilinear elliptic variational inequality.

In Chapter 6, motivated by the idea presented in [100], we introduce the concept of a modified nonlinear Lagrangian function and obtain a necessary condition and sufficient condition for the zero duality gap property between the optimal control problem and its nonlinear Lagrangian dual problem. We apply a power penalty method to the optimal control problem, and obtain approximate optimal solutions of the penalty function that converges weakly to the optimal solution of the original optimal control problem.

We end this chapter by mentioning that the thesis is based on the following papers written by the author and her cooperators during the period of stay in the Department of Applied Mathematics, The Hong Kong Polytechnic University as a graduate student:

[1] Zhou, Y.Y. and Yang, X. Q., *Some results about duality and exact penalization,* Journal of Global Optimization, Vol. 29, pp. 497-509, 2004.

[2] Zhou, Y.Y., Yang, X. Q. and Teo, K. L., *Optimal control problems governed by a variational inequality and nonlinear Lagrangian methods*, Optimization, to appear.

[3] Zhou, Y.Y. and Yang, X. Q., *Lagrangian function, non-quadratic growth condition and exact penalization*, Operations Research Letter, to appear.

[4] Zhou, Y.Y., Yang, X. Q. and Teo, K. L., *The existence results for optimal control problems governed by a variational inequality,* Journal of Mathematical analysis and Application, to appear.

[5] Zhou, Y.Y. and Yang, X. Q., *Duality and penalization in optimization via an augmented Lagrangian function with applications*, Journal of Optimization Theory and Applications (revised).

[6] Zhou, Y.Y., *Some results about perturbed maximal monotone mappings*, Computers and Mathematics with Applications (revised).

[7] Rubinov, A.M., Yang, X. Q. and Zhou, Y.Y., *A Lagrange penalty reformulation method for a constrained optimization*, in preparation.

[8] Yang, X. Q. and Zhou, Y.Y., *Second-order sufficient optimality conditions for lower order exact penalty functions*, in preparation.

Chapter 2

Duality via an Augmented Lagrangian Function with Applications

2.1 Introduction

If a nonlinear programming problem is analyzed in terms of its ordinary Lagrangian function, there is usually a duality gap, unless the objective and constraint functions are convex. It is shown by Rockafellar [95] that the gap can be removed by using a quadratic augmented Lagrangian function. In [97], an augmented Lagrangian with a convex augmenting function was introduced and the corresponding zero duality property was obtained. In [56], Huang and Yang considered a augmented Lagrangian with a level-bounded augmenting function for the problem of minimizing a nonconvex extended-real-valued function defined over $\mathbf{R}^n$. Furthermore, a valley at 0 augmenting function was given in [134] where the level-boundedness condition of augmenting functions in [56] is replaced by the valley at 0 property. Such a formalism allows one to introduce a suitable dual problem having the same optimal value as the original one. It is worth noting that all these studies are carried out in finite dimension spaces.

In summary, there are four types of augmenting functions in the literature: (i) a quadratic augmenting function; (ii) a convex augmenting function; (iii) a level-bounded augmenting function; (iv) a valley at 0 augmenting function. Under some conditions, their implication relations are (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

In this chapter, we explore an augmented Lagrangian scheme with a valley at 0 augmenting function. It is shown that every weak limit point of a sequence of optimal solutions generated by augmented Lagrangian problems is a solution of the original problem. A zero duality gap property between the primal problem and the augmented Lagrangian dual problem is obtained. Under the assumption that the augmenting function satisfies a valley at zero condition and the perturbation function satisfies a growth condition, a necessary and sufficient condition for a zero duality gap property between the primal problem and its augmented Lagrangian dual problem is established. We apply it to variational problems in Sobolev spaces.

The outline of this chapter is as follows:

In Section 2.2, we present basic definitions, notations and some preliminary results. In Section 2.3, in general Banach spaces, we obtain a zero duality gap property between the primal problem and its augmented Lagrangian dual problem by assuming that the perturbation function satisfies a growth condition and by replacing the level coercivity of the augmenting function in [90] by a valley at 0 property. In Section 2.4, we establish the existence result of the solutions of a primal problem (a minimization problem) and its augmented Lagrangian problem in reflexive Banach spaces. We obtain that every weak limit point of a path of optimal solutions generated by the augmented Lagrangian problems is the solution of its primal problem. We obtain the zero duality gap property between the primal problem and the augmented Lagrangian dual problem. In Section 2.5, as an application, we discuss the relationship between the primal problem and its augmented Lagrangian problem about two kinds of variational inequality problems in Sobolev spaces and get several results.

2.2 Augmented Lagrangians

In the following, let $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{E}$ be three Banach spaces, B_V be the closed unit ball of $\mathbf{V}$ and $\overline{\mathbf{R}} = \mathbf{R}\bigcup\{-\infty, +\infty\}$. Let $\mathbf{U}^*$ denote the dual space of $\mathbf{U}$, $\langle f, u\rangle$ be the value of $f \in \mathbf{U}^*$ at $u \in \mathbf{U}$. We use the standard notation "$u_n \to u_0$" to denote strong convergence of a sequence u_n in $\mathbf{U}$ to u, i.e., $\|u_n - u\| \to 0$ as $n \to +\infty$, where $\|\cdot\|$ is a norm in $\mathbf{U}$, and "$u_n \rightharpoonup u_0$" to denote weak convergence of a sequence in $\mathbf{U}$, i.e., for any $f \in \mathbf{U}^*$, we have $\langle f, u_n\rangle \to \langle f, u\rangle$, as $n \to +\infty$.

Definition 2.2.1 *A subset $C \subset \mathbf{V}$ is said to be convex, if for every choice of $x_1, x_2 \in C$ one has*

$$(1-t)x_1 + tx_2 \in C, \quad \textit{for} \quad \textit{all}\ t \in (0,1).$$

A function f on a convex set C is said to be convex if for every choice of $x_1, x_2 \in C$ one has

$$f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2), \quad \textit{for} \quad \textit{all}\ t \in (0,1).$$

Definition 2.2.2 *Let $X \subset \mathbf{V}$ be a closed subset and $f : X \to \overline{\mathbf{R}}$ be an extended real-valued function. The function f is said to be level-bounded on X if, for any $\alpha \in \mathbf{R}$, the level set $\{v \in X : f(v) \leq \alpha\}$ is bounded.*

Definition 2.2.3 *Let $X \subset \mathbf{V}$ be a closed subset and $f : X \to \overline{\mathbf{R}}$ be an extended real-valued function. The function f is said to have a valley at 0 in X if, $f(0) = 0$, $f(v) > 0$, for all $v \neq 0$, and $c_\delta = \inf_{\|v\| \geq \delta} f(v) > 0$, for each $\delta > 0$.*

Definition 2.2.4 *A continuous function $f : \mathbf{V} \to \mathbf{R}$ is called a peak at zero if $f(v) < 0 = f(0)$ for all $v \neq 0$ and $\sup_{\|z\| \geq \delta} f(v) < 0$ for all $\delta > 0$.*

Definition 2.2.5 *Let $X \subset \mathbf{V}$ be a closed subset and $f : X \to \overline{\mathbf{R}}$ be an extended real-valued function. f is said to satisfy the $0-$coercive condition if*

$$\lim_{\|v\| \to +\infty} f(v) = +\infty,$$

it is said to satisfy the level-coercive condition if it is bounded below on bounded sets and satisfies

$$\liminf_{\|v\| \to +\infty} \frac{f(v)}{\|v\|} > 0,$$

whereas it is coercive if it is bounded below on bounded sets and

$$\liminf_{\|v\| \to +\infty} \frac{f(v)}{\|v\|} = +\infty.$$

Let us compare the definitions above by the following example.

Example 2.2.1 (a) Let $v \in \mathbf{V}$, $f(v) = \|v\|^\gamma$ for an exponent $\gamma \in (1, \infty)$, then f is coercive, but for $\gamma = 1$, it is merely level-coercive. For $\gamma \in (0,1)$, f is level-bounded, but is not level-coercive. It is clear that when $\gamma \in (0,1)$, f is a level-bounded function, but is not a convex one.

(b) Let $v \in \mathbf{V}$,

$$f_1(v) = \begin{cases} \|v\|^\gamma + \|v\|^p, & \text{if } \ \|v\| < 1 \\ |\sin \frac{\pi\|v\|^\gamma}{2}| + 1, & \text{if } \ 1 \leq \|v\| < +\infty, \end{cases}$$

$$f_2(v) = \begin{cases} \|v\|^\gamma, & \text{if } \|v\| \leq 1 \\ 1, & \text{if } \|v\| > 1, \end{cases}$$

where $0 < \gamma < 1$, $p > 0$. Then, they merely have a valley at 0, none of them is level-bounded or convex.

Definition 2.2.6 *A function $\bar{f} : \mathbf{U} \times \mathbf{V} \to \overline{\mathbf{R}}$ with value $\bar{f}(u, v)$ is said to be level-bounded in u locally uniform in v if, for each $\bar{v} \in \mathbf{V}$ and $\alpha \in \mathbf{R}$, there exists a neighborhood $U(\bar{v})$ of $\bar{v}$ along with a bounded set $D \subset \mathbf{U}$, such that $\{u \in \mathbf{U} : \bar{f}(u, v) \leq \alpha\} \subset D$ for any $v \in U(\bar{v})$.*

Let $X \subset \mathbf{V}$ be a closed subset and $\sigma : X \to \overline{\mathbf{R}}$. Let

$$\text{argmin}_v \sigma = \{v' \in X : \sigma(v') = \min_{v \in X} \sigma(v)\}.$$

By Proposition 3.23 and Corollary 3.27 in [97], we have

Proposition 2.2.1 For any proper, lsc function σ on $\mathbf{R}^m$, level coercivity implies level boundedness. When σ is convex the two properties are equivalent.

Proposition 2.2.2 For any proper, lsc convex function σ on $\mathbf{R}^m$, if, some set $\{v \in X : \sigma(v) \leq \alpha\}$ (α is a constant) is both nonempty and bounded, for instance the level set $\text{argmin}_v \sigma$, then σ must be level bounded.

Definition 2.2.7 *Let $\sigma : \mathbf{V} \to \overline{\mathbf{R}}$ be an extended real-valued function. The function σ is said to a convex augmenting function if it is proper, weakly lower semicontinuous and convex in $\mathbf{V}$, $\text{argmin}_v \sigma = \{0\}$ and $\sigma(0) = 0$.*

Definition 2.2.8 *Let $\sigma : \mathbf{V} \to \overline{\mathbf{R}}$ be an extended real-valued function. The function σ is said to a level-bounded augmenting function if it is proper, weakly lower semicontinuous and level bounded in $\mathbf{V}$, $\text{argmin}_v \sigma = \{0\}$ and $\sigma(0) = 0$.*

Definition 2.2.9 *A function $\sigma : \mathbf{V} \to \overline{\mathbf{R}}$ is said to be a valley at 0 augmenting function if it is proper, weakly lower semicontinuous and has a valley at 0 in $\mathbf{V}$.*

When $\mathbf{U}$ and $\mathbf{V}$ are finite dimensional spaces, these definitions on augmenting functions can be found in [56], [97], [100] and [134] respectively. Now we give an example of a valley at 0 augmenting function $\sigma(v)$. Let $v \in \mathbf{V}$, define

$$\sigma(v) := \begin{cases} \|v\|^\gamma, & \text{if } \|v\| < 1, \\ 1, & \text{if } \|v\| \geq 1, \end{cases} \tag{2.1}$$

where $\gamma > 0$. We have the following result.

Lemma 2.2.1 *Let $\sigma(v)$ be defined by (2.1). Then $\sigma(v)$ is a valley at 0 augmenting function.*

Proof: It is clear that $\sigma(v)$ has a valley at 0. We only need to prove that $\sigma(v)$ is weakly lower semicontinuous for each $v \in \mathbf{V}$. Arguing by contradiction that if $\sigma(v)$ is not weakly lower semicontinuous at some $v_0 \in \mathbf{V}$, then there exists a sequence $\{v_n\} \subset \mathbf{V}$ with $v_n \rightharpoonup v_0$ as $n \to \infty$, but $\liminf\limits_{n\to\infty} \sigma(v_n) < \sigma(v_0)$. Then there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that

$$\lim_{k\to\infty} \sigma(v_{n_k}) = \liminf_{n\to\infty} \sigma(v_n) < \sigma(v_0). \tag{2.2}$$

We consider the following two cases:

(i) if $\|v_0\| \geq 1$, it implies that $\sigma(v_0) = 1$. Hence from (2.2), there exists a sufficiently large K such that $\sigma(v_{n_k}) < 1$ for $k > K$. Thus $\|v_{n_k}\| < 1$ for $k > K$. It follows from the weak lower semi-continuity of the norm that

$$\liminf_{k\to\infty} \|v_{n_k}\| \geq \|v_0\|. \tag{2.3}$$

Noticing that $\|v_{n_k}\| < 1$ for $k > K$ and $\|v_0\| \geq 1$, it implies from (2.3) that $\lim_{k\to\infty} \|v_{n_k}\| = \|v_0\| = 1$. Therefore $\lim_{k\to\infty} \sigma(v_{n_k}) = \lim_{k\to\infty} \|v_{n_k}\|^\gamma = 1$, which contradicts (2.2).

(ii) if $\|v_0\| < 1$, then $\sigma(v_0) = \|v_0\|^\gamma$. By (2.2), $\|v_{n_k}\| < 1$, $\sigma(v_{n_k}) = \|v_{n_k}\|^\gamma$ and

$$\liminf_{k\to\infty} \|v_{n_k}\|^\gamma < \|v_0\|^\gamma. \tag{2.4}$$

Since $v_{n_k} \rightharpoonup v_0$, it follows from the same reason as in (2.3) that $\liminf_{k\to\infty} \|v_{n_k}\| \geq \|v_0\|$. Without loss of generality, we can assume that $\lim\limits_{k\to\infty} \|v_{n_k}\| \geq \|v_0\|$. Consequently, $\forall\ \varepsilon > 0$, there exists K large enough such that $\|v_{n_k}\| \geq \|v_0\| - \varepsilon$ for all $k > K$, and

$$\|v_{n_k}\|^\gamma \geq (\|v_0\| - \varepsilon)^\gamma \quad \forall\ k > K,$$

thus

$$\liminf_{k\to\infty} \|v_{n_k}\|^\gamma \geq (\|v_0\| - \varepsilon)^\gamma,$$

and then, by the arbitrariness of ε,

$$\liminf_{k\to\infty} \|v_{n_k}\|^\gamma \geq \|v_0\|^\gamma,$$

which contradicts (2.4). Thus, by (i) and (ii), we have shown that $\sigma(v)$ is weakly lower semicontinuous at each $v \in \mathbf{V}$.

Therefore, $\sigma(v)$ is a valley at 0 augmenting function. ▮

Proposition 2.2.3 Let $\mathbf{V}$ be a Banach space. (i) If $\sigma : \mathbf{V} \to \overline{\mathbf{R}}$ is a level-coercive function, then it is a level-bounded function. (ii) If the space $\mathbf{V}$ is reflexive, $\sigma : \mathbf{V} \to \overline{\mathbf{R}}_+$ is a level-bounded augmenting function, then it is a valley at 0 augmenting function.

Proof: (i) If σ is not a level-bounded function, then there exists a constant a, such that $\{v \in \mathbf{V} : \sigma(v) \leq a\}$ is unbounded. There exist $v_k \in \{v \in X : \sigma(v) \leq a\}$, such that $\|v_k\| \to \infty$ as $k \to \infty$. Thus, $\liminf_{k\to+\infty} \frac{\sigma(v_k)}{\|v_k\|} = 0$, which is impossible since σ is level-coercive.

(ii) If σ does not have a valley at 0 in $\mathbf{V}$, then there exists $\delta > 0$ such that $c_\delta = \inf_{\|v\|\geq\delta}\sigma(v) = 0$, hence there exists $\{v_j\} \subseteq \mathbf{V}$,$\|v_j\| \geq \delta$ such that $\sigma(v_j) \to c_\delta$. If $\{v_j\}$ is unbounded, by the definition of level-bounded function, we get $\{\sigma(v_j)\}$ is unbounded. Thus there exists a subsequence of $\{\sigma(v_j)\}$ converging to infinity. This contradicts to $c_\delta = 0$. Hence $\{v_j\}$ is bounded. Since $\mathbf{V}$ is reflexive, there exists v_0, such that $v_j \rightharpoonup v_0$ as $j \to \infty$. The weak lower semi-continuity of σ and $\mathrm{argmin}_v\sigma(v) = \{0\}$ imply $\sigma(v_0) = 0$, hence $v_0 = 0$, but $\|v_0\| \geq \delta$, which implies a contradiction. ▮

Remark 2.2.2 (i) It is clear that if $\sigma : \mathbf{V} \to \mathbf{R}$ is a continuous function, then σ has a valley at 0 if and only if $-\sigma$ has a peak at 0, see [100]. If σ is continuous at 0, then the concepts of σ having a valley at 0 and σ being a potential are the same, see [90].

(ii) If σ is a convex augmenting function in space $\mathbf{R}^m$ introduced by [97], that is, σ is a convex, proper, lower semicontinuous function, $\mathrm{argmin}_v\sigma = \{0\}$ and $\sigma(0) = 0$, then by Proposition 2.2.1, σ satisfies the level bounded condition. Thus, by Proposition 2.2.2, σ satisfies the level-coercive condition.

(iii) If the space $\mathbf{V}$ is reflexive, $\sigma : \mathbf{V} \to \overline{\mathbf{R}}_+$ is a level-bounded augmenting function, then, by Proposition 2.2.3, it is a valley at 0 augmenting function. On the other hand, if σ is a valley at 0 augmenting function, then it may not be a level-bounded function. For example, let us consider function $\sigma(v)$, where $\sigma(v)$ is defined by (2.1). By Lemma 2.2.1, $\sigma(v)$ is a valley at 0 augmenting function. But $\{v \in \mathbf{V} : \sigma(v) \leq 2\} = \mathbf{V}$ is unbounded. Thus $\sigma(v)$ is not a level-bounded function.

Hence the concept of a valley at 0 augmenting function is weaker than that of a level-bounded augmenting function which was introduced by [56].

Let $\varphi : \mathbf{U} \to \overline{\mathbf{R}}$ be an extended real-valued function and $\bar{f} : \mathbf{U} \times \mathbf{V} \to \overline{\mathbf{R}}$ be a dualizing parametrization function for φ, i.e., $\bar{f}(u, 0) = \varphi(u), \quad \forall u \in \mathbf{U}$. Let $\sigma : \mathbf{V} \to \mathbf{R}_+ \bigcup\{+\infty\}$ and $g : \mathbf{E} \times \mathbf{V} \to \mathbf{R}$. Let $(y, r) \in \mathbf{E} \times (0, +\infty)$. We will consider the primal problem

$$(P) \qquad \inf_{u \in \mathbf{U}} \varphi(u). \tag{2.5}$$

Definition 2.2.10 *Consider the primal problem (P). Let $\bar{f} : \mathbf{U} \times \mathbf{V} \to \overline{\mathbf{R}}$ be a dualizing parametrization function for φ, $\sigma : \mathbf{V} \to \mathbf{R}_+ \bigcup\{+\infty\}$ and $g : \mathbf{E} \times \mathbf{V} \to \mathbf{R}$.*

(i) The augmented Lagrangian (with parameter $r > 0$) $\bar{l} : \mathbf{U} \times \mathbf{E} \times (0, +\infty) \to \overline{\mathbf{R}}$ is defined by

$$\bar{l}(u, y, r) = \inf\{\bar{f}(u, v) - g(y, v) + r\sigma(v) : v \in \mathbf{V}\}, \quad u \in \mathbf{U}, y \in \mathbf{E}, r > 0.$$

(ii) The augmented Lagrangian dual function is defined by

$$\bar{\psi}(y, r) = \inf\{\bar{l}(u, y, r) : u \in \mathbf{U}\}, \quad y \in \mathbf{E}, r > 0. \tag{2.6}$$

(iii) The augmented Lagrangian dual problem is defined as

$$(D) \qquad \sup_{(y,r) \in \mathbf{E} \times (0,+\infty)} \bar{\psi}(y, r). \tag{2.7}$$

Remark 2.2.3 If $\mathbf{U}$ and $\mathbf{V}$ are finite dimension spaces, $g(y, v) = \langle y, v \rangle,\ y, v \in \mathbf{V}$, where $\langle y, v \rangle$ is the inner product in $\mathbf{V}$, then Definition 2.2.10 reduces to the one in [97]. If we only consider a constrained optimization problem in finite dimensional spaces, it is unnecessary to extend the inner product $\langle y, u \rangle$ to a general function $g(y, u)$. But in some infinite dimensional spaces, for example, in Sobolev space $W_0^{k,p}(\Omega)$ $(p \neq 2)$, there is no inner product (see p.147 in [44]). If we consider a constrained optimization problem in such spaces by using the augmented Lagrangian method, it is necessary to introduce a general bifunction to replace the inner product.

In the following, denote $\inf(P) = \inf\limits_{u\in\mathbf{U}} \varphi(u)$ and $\sup(D) = \sup\limits_{(y,r)\in\mathbf{E}\times(0,+\infty)} \bar{\psi}(y,r)$. We will consider the problem (P) and its augmented Lagrangian problem:

$$P(y,r) \qquad \inf_{(u,v)\in\mathbf{U}\times\mathbf{V}} \{\bar{f}(u,v) - g(y,v) + r\sigma(v)\}.$$

Let us look at an example to see how the simple primal problem relates with its augmented Lagrangian problem $P(y,r)$:

Example 2.2.2 Consider the following constrained program

$$(\mathrm{P}_1) \qquad \begin{aligned} &\inf\ f(x) \\ &\text{s.t. } x \in X, \\ &\qquad g_j(x) \leq 0, \quad j = 1,\cdots,m_1 \\ &\qquad g_j(x) = 0, \quad j = m_1+1,\cdots,m, \end{aligned}$$

where $X \subset \mathbf{R}^n$ is a nonempty and closed set, $f, g_j : X \to \mathbf{R}$, $j = 1,\cdots,m$. Denote by X_0 the set of feasible solutions of (P_1), i.e.,

$$X_0 = \{x \in X : g_j(x) \leq 0, j = 1,\cdots,m_1; g_j(x) = 0, j = m_1+1,\cdots,m\}.$$

Let

$$\varphi(x) = \begin{cases} f(x), & \text{if } x \in X_0, \\ +\infty, & \text{if } x \in \mathbf{R}^n \backslash X_0. \end{cases}$$

Then (P_1) is equivalent to the following problem (P_1') in the sense that the two problems have the same set of (locally) optimal solutions and the same optimal value

$$(\mathrm{P}_1') \qquad \inf_{x\in\mathbf{R}^n} \varphi(x).$$

Define the dualizing parametrization function:

$$\bar{f}_P(x,u) = f(x) + \delta_{\mathbf{R}_-^{m_1}\times\{0_{m-m_1}\}}(G(x)+u) + \delta_X(x),\ x \in \mathbf{R}^n,\ u \in \mathbf{R}^m,$$

where 0_{m-m_1} is the origin of $\mathbf{R}^{m-m_1}$, $G(x) = (g_1(x), \cdots, g_m(x))$ and δ_D is the indicator function of the set D, i.e.,

$$\delta_D(x) = \begin{cases} 0, & \text{if } x \in D \\ +\infty, & \text{otherwise.} \end{cases}$$

The augmented Lagrangian for (P_1') is

$$\bar{l}_P(x,y,r) = \inf\{\bar{f}_P(x,u) - \langle y,u\rangle + r\sigma(u) : u \in \mathbf{R}^m\},$$

where $y \in \mathbf{R}^m$, σ is an augmenting function. The above Lagrangian can be expressed as

$$\bar{l}_P(x,y,r) = \begin{cases} f(x) + \sum_{j=1}^{m} y_j g_j(x) + \inf_{v\geq 0}\{\sum_{j=1}^{m_1} y_j v_j + r\sigma(-g_1(x) - v_1, \\ \cdots, -g_{m_1}(x) - v_{m_1}, -g_{m_1+1}(x), \cdots, -g_m(x))\}, \text{if } x \in X, \\ +\infty, \quad \text{otherwise} \end{cases}$$

where $v = (v_1, \cdots, v_{m_1})$.

Define the perturbation function $p : \mathbf{V} \to \overline{\mathbf{R}}$ by

$$p(v) = \inf\{\bar{f}(u,v) : u \in \mathbf{U}\}. \tag{2.8}$$

Note that the augmented Lagrangian problem $P(y,r)$ is the same as the problem of evaluating the augmented Lagrangian dual function $\bar{\psi}(y,r)$. Then, from the definition, $p(0)$ and $\bar{\psi}(y,r)$ are the optimal values of the problems (P) and $P(y,r)$, respectively.

Definition 2.2.11 *Let the function p be defined by (2.8), and $\sigma : \mathbf{V} \to \overline{\mathbf{R}}$ be an extended real-valued function. The function p is said to satisfy the growth condition if, for any $\rho > 0$, there exist $a, b \in \mathbf{R}$ such that*

$$p(v) \geq b - a\sigma(v), \quad \forall v \in \mathbf{V} \setminus \rho B_V, \tag{2.9}$$

where $\rho B_V = \{v : v \in \mathbf{V}, \|v\| \leq \rho\}$.

Let us give an example to see how the definitions above are related to a nonlinear programming problem.

Example 2.2.3 Let $f_0, f_1, \cdots f_m$ be real-valued functions defined on a set $X \subset \mathbf{R}^n$. Consider the nonlinear programming problem:

$$(P_2) \qquad \begin{aligned} &\inf\ f_0(u) \\ &\text{s.t. } u \in X, \\ &\quad f_j(u) \leq 0, \quad j = 1, \cdots, m. \end{aligned}$$

In [95], the augmented Lagrangian function associated with problem (P_2) is

$$L(u,y,r) = f_0(u) + \sum_{j=1}^{m}[y_j\max\{f_j(u), \frac{-y_j}{2r}\} + r\max^2\{f_j(u), \frac{-y_j}{2r}\}],$$

for $u \in X$, $(y,r) \in T$, where $T = \mathbf{R}^m \times (0,+\infty)$. The augmented Lagrangian dual problem for (P_2) is:

$$(D_2) \qquad \sup_{(y,r)\in T} \bar{\psi}(y,r),$$

where $\bar{\psi}(y,r) = \inf_{u\in X} L(u,y,r) < +\infty$. The optimal value in (D_2) is by definition

$$\sup(D_2) = \sup_{(y,r)\in T}\inf_{u\in X} L(u,y,r).$$

For each $(u,v) \in \mathbf{R}^n \times \mathbf{R}^m$, define

$$\bar{f}(u,v) = \begin{cases} f_0(u), & \text{if } u \in X, f_j(u) \le v_j \text{ for } j = 1,\cdots,m; \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$\varphi(u) = \begin{cases} f_0(u), & \text{if } u \in X, f_j(u) \le 0 \text{ for } j = 1,\cdots,m; \\ +\infty, & \text{otherwise.} \end{cases}$$

Then, $\bar{f}(u,0) = \varphi(u)$ and $\inf_{u\in\mathbf{R}^n}\varphi(u) = \inf(P_2)$. Let $g(y,v) = \langle y,v\rangle$, $\sigma(v) = \|v\|^2$. It is elementary to calculate that

$$L(u,y,r) = \inf_{v\in\mathbf{R}^m}\{\bar{f}(u,v) + \langle y,v\rangle + r\|v\|^2\} = \inf_{v\in\mathbf{R}^m}\{\bar{f}(u,v) - g(y,v) + r\sigma(v)\}.$$

Therefore, the optimal value in (D_2) is the same as the one in Definition 2.2.10. Besides, Rockafellar has given the definition of quadratic growth condition in a finite-dimensional space in [95] and obtained the following result.

Proposition 2.2.4 Consider problems (P_2) and (D_2). If $\sigma(v) = \|u\|^2$ and p satisfies the quadratic growth condition, i.e., there exist real numbers $r \ge 0$ and q such that

$$p(u) \ge q - r\|u\|^2 \quad \forall u \in \mathbf{R}^m,$$

then $\sup(D_2) = \inf(P_2) > -\infty$ iff $p(0)$ is finite and p is lsc at 0.

In the next section, we will generalize this result to the non-quadratic case in general infinite dimensional Banach spaces.

Remark 2.2.4 If, for $y' = \theta$ (the origin of $\mathbf{E}$), there exists $r' > 0$, such that $\bar{\psi}(\theta,r')$ is finite, then, for $\alpha = \bar{\psi}(\theta,r')$, we have

$$p(v) \ge \alpha - r'\sigma(v), \quad \forall v \in \mathbf{V}.$$

That is, p satisfies the growth condition.

We discuss the property of the augmented Lagrangian for the primal problem (P) in the follows.

The usual duality theory involves the generalized Fenchel conjugate p^* of p, which is given by

$$p^*(y) = \sup_{v\in\mathbf{V}}(g(y,v) - p(v)).$$

Using the coupling function $c : \mathbf{V} \times \mathbf{E} \times (0, +\infty) \to \mathbf{R}$ given by

$$c(v, y, r) = g(y, v) - r\sigma(v),$$

we define the conjugate of p as

$$p^c(y, r) := \sup_{v\in\mathbf{V}}(c(v, y, r) - p(v)).$$

Thus the conjugate p^c of p can be computed with the help of the generalized Fenchel conjugacy,

$$p^c(y, r) = (p + r\sigma)^*(y).$$

It is easy to see that the augmented Lagrangian $\bar{l}(u, y, r)$ is concave and upper semicontinuous in (y, r) and nondecreasing in r. It can be expressed as

$$\begin{aligned}
\bar{l}(u, y, r) &= -\sup_{v\in\mathbf{V}}(g(y, v) - r\sigma(v) - \bar{f}(u, v)) \\
&= -\sup_{v\in\mathbf{V}}(c(v, y, r) - \bar{f}(u, v)) \\
&= -(\bar{f}_u)^c(y, r) \\
&= -(\bar{f}_u + r\sigma)^*(y),
\end{aligned}$$

where $\bar{f}_u(v) = \bar{f}(u, v)$. We define the conjugate of function $q : \mathbf{E} \times (0, +\infty) \to \mathbf{R}\bigcup\{-\infty, +\infty\}$ as

$$q^c(v) := \sup_{(y,r)\in\mathbf{E}\times(0,+\infty)}(c(v, y, r) - q(y, r)).$$

The biconjugate of p is $p^{cc} := (p^c)^c$. Then

$$\begin{aligned}
p^{cc}(0) &= \sup_{(y,r)\in\mathbf{E}\times(0,+\infty)}(-(p + r\sigma)^*(y)) \\
&= \sup_{(y,r)\in\mathbf{E}\times(0,+\infty)}(-\sup_{v\in\mathbf{V}}(c(v, y, r) - p(v))) \\
&= \sup_{(y,r)\in\mathbf{E}\times(0,+\infty)}(\inf_{v\in\mathbf{V}}(p(v) - g(y, v) + r\sigma(v))) \\
&= \sup_{(y,r)\in\mathbf{E}\times(0,+\infty)}\bar{\psi}(y, r).
\end{aligned} \tag{2.10}$$

For each $(y, r) \in \mathbf{E} \times (0, +\infty)$, using Definition 3.2.1, we have

$$\bar{l}(u, y, r) \leq \bar{f}(u, 0) = \varphi(u).$$

Thus

$$\inf_{u \in \mathbf{U}} \bar{l}(u, y, r) \leq \inf_{u \in \mathbf{U}} \varphi(u).$$

That is

$$\bar{\psi}(y, r) \leq p(0), \quad \forall (y, r) \in \mathbf{E} \times (0, +\infty). \tag{2.11}$$

Hence, (2.10) and (2.11) imply that the following weak duality holds:

$$p^{cc}(0) \leq p(0). \tag{2.12}$$

2.3 Non-quadratic growth condition and zero duality gap

Suppose that the perturbation function $p : \mathbf{V} \to \overline{\mathbf{R}}$ is defined by (2.8). As shown in the last section, under a growth condition, one can relate the value of the dual problem to the behavior of the perturbation function. In this section, under the assumption that the perturbation function satisfies the growth condition (2.9) and the augmenting function satisfies a valley at 0 condition, we establish that the lower semi-continuity of the perturbation function at 0 is a necessary and sufficient condition for a zero duality gap property between the primal problem and its augmented Lagrangian dual problem in general Banach spaces.

Lemma 2.3.1 *Let $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{E}$ be three Banach spaces. Suppose the function $p : \mathbf{V} \to \overline{\mathbf{R}}$ satisfies the growth condition (2.9), the function $\sigma : \mathbf{V} \to \overline{\mathbf{R}}$ has a valley at 0 and $g : \mathbf{E} \times \mathbf{V} \to \mathbf{R}$ satisfies $|g(y, v)| \leq d(y)\|v\|^{\mu}, \forall (y, v) \in \mathbf{E} \times \mathbf{V}$, where $d(y) \geq 0, d(0) = 0$ and $0 < \mu \leq 1$. Then,*

$$\sup(D) = \sup_{r>0} \bar{\psi}(0, r) = \liminf_{v \to 0} p(v) \leq \inf(P). \tag{2.13}$$

Furthermore, if there exists a neighborhood O of 0, such that p is bounded below on O, then

$$-\infty < \sup(D) \leq \inf(P). \tag{2.14}$$

Proof: By Definition 2.2.10, for each $\rho > 0$, and each $(y, r) \in \mathbf{E} \times \mathbf{R}_+$, we have

$$\bar{\psi}(y, r) \leq \inf\{p(v), v \in \rho B_V\} + \rho^\mu d(y) + r \sup \sigma(\rho B_V), \tag{2.15}$$

where $\rho B_V = \{v : v \in \mathbf{V}, \|v\| \leq \rho\}$. Note that σ has a valley at 0 in X, $\sup \sigma(\rho B_V) \to 0$ as $\rho \to 0$. Thus, by (2.15),

$$\bar{\psi}(y, r) \leq \liminf_{v \to 0} p(v).$$

This implies

$$\sup_{(y,r) \in \mathbf{E} \times \mathbf{R}_+} \bar{\psi}(y, r) \leq \liminf_{v \to 0} p(v). \tag{2.16}$$

On the other hand, let $s < \liminf_{v \to 0} p(v)$. There exists a $\rho > 0$, such that

$$p(v) + r\sigma(v) \geq s, \forall v \in \rho B_V. \tag{2.17}$$

Again since σ have a valley at 0 in X, $\inf_{v \in \mathbf{V} \setminus \rho B_V} \sigma(v) > 0$. There exists a $t_0 > 0$, such that

$$(t_0 - a)\sigma(v) > s - b, \forall v \in \mathbf{V} \setminus \rho B_V. \tag{2.18}$$

It follows from the growth condition of p and (2.18) that, for each $r > t_0$,

$$p(v) + r\sigma(v) \geq b + (r - a)\sigma(v) \geq s, \forall v \in \mathbf{V} \setminus \rho B_V. \tag{2.19}$$

Hence, combining (2.17) and (2.19), we get

$$\sup_{r > 0} \bar{\psi}(0, r) \geq \liminf_{v \to 0} p(v). \tag{2.20}$$

Therefore, it follows from (2.16) and (2.20) that (2.13) holds. If there exists a neighborhood O of 0, such that p is bounded below on O, $\liminf_{v \to 0} p(v) > -\infty$. Thus (2.14) holds. ∎

Remark 2.3.1 It is noted that if $\mathbf{E} = \mathbf{V}$ is a finite dimensional space, then $g(y, v) = \langle y, v \rangle$ satisfies $|g(y, v)| \leq d(y)\|v\|^\mu, \forall (y, v) \in \mathbf{E} \times \mathbf{V}$, where $d(y) \geq 0, d(0) = 0$ and $0 < \mu \leq 1$.

Theorem 2.3.1 *Let* $\mathbf{U}$, $\mathbf{V}$ *and* $\mathbf{E}$ *be three Banach spaces. Suppose that the function* $p : \mathbf{V} \to \overline{\mathbf{R}}$ *satisfies the growth condition (2.9), the function* $\sigma : \mathbf{V} \to \overline{\mathbf{R}}$ *has a valley at* 0 *and* $g : \mathbf{E} \times \mathbf{V} \to \mathbf{R}$ *satisfies* $|g(y, v)| \leq d(y)\|v\|^\mu, \forall (y, v) \in \mathbf{E} \times \mathbf{V}$, *where* $d(y) \geq 0, d(0) = 0$ *and* $0 < \mu \leq 1$. *Then,* $\sup(D) = \inf(P) > -\infty$ *if and only if* $p(0)$ *is finite and* p *is lsc at* 0.

Proof: If $\sup(D) = \inf(P) > -\infty$, then, using Lemma 2.3.1, we have

$$\liminf_{v \to 0} p(v) = \inf(P).$$

Note that $\inf(P) = p(0)$. Hence, p is lsc at 0 and $p(0)$ is finite.

If p is lsc at 0 and $p(0)$ is finite, then

$$\liminf_{v \to 0} p(v) \geq p(0) = \inf(P).$$

Again using Lemma 2.3.1,

$$\sup(D) = \liminf_{v \to 0} p(v) = \inf(P) > -\infty.$$

▮

2.4 Optimal path and zero duality gap

In this section, in the framework of augmented Lagrangians with a valley at zero augmenting function, we establish the existence result of a solution of the primal problem (P) and its augmented Lagrangian problem $P(y, r)$ in a reflexive Banach space, obtain that every weak limit point of a path of optimal solutions generated by the problem $P(y, r)$ is a solution of problem (P), and get a zero duality gap property between the primal problem (P) and its augmented Lagrangian dual problem (2.7).

We have the following Lemma.

Lemma 2.4.1 *Suppose that* $\{(u_n, v_n)\}$ *is a sequence of* $\mathbf{U} \times \mathbf{V}$ *with* $(u_n, v_n) \rightharpoonup (u_0, v_0)$ *as* $n \to \infty$*, then there exist* $\{u_{n_k}\} \subset \{u_n\}$ *and* $\{v_{n_k}\} \subset \{v_n\}$ *such that* $u_{n_k} \rightharpoonup u_0$ *and* $v_{n_k} \rightharpoonup v_0$ *as* $k \to \infty$*.*

Proof: The proof is elementary and is omitted. ▮

Then we have

Theorem 2.4.1 *Let* $\mathbf{U}$ *and* $\mathbf{V}$ *be reflexive Banach spaces,* $\mathbf{E}$ *be a Banach space,* $\bar{f} : \mathbf{U} \times \mathbf{V} \to \overline{\mathbf{R}}$ *be a dualizing parametrization function for* φ*,* $\sigma : \mathbf{V} \to \mathbf{R}_+ \bigcup \{+\infty\}$ *be a* *valley at 0 augmenting function and $g : \mathbf{E} \times \mathbf{V} \to \mathbf{R}$ be a weakly continuous function with $g(0,0) = 0$. Assume that $\bar{f}(u,v)$ is proper, weakly lower semicontinuous, and level-bounded in u locally uniform in v. Then the primal problem (P) has at least one solution. Furthermore suppose that there exists $(\bar{y}, \bar{r}) \in \mathbf{E} \times (0, +\infty)$ such that*

$$\inf\{\bar{l}(u, \bar{y}, \bar{r}) : u \in \mathbf{U}\} > -\infty. \tag{2.21}$$

Then

(i) There exists $r_0 > \bar{r}$, such that, for any $r \geq r_0$, the augmented Lagrangian problem $P(\bar{y}, r)$ has at least one solution.

(ii) Every weak limit point of the sequence $\{u_r\}$ is the solution of the primal problem (P), where (u_r, v_r) with $r \geq r_0$ is a solution of the augmented Lagrangian problem $P(\bar{y}, r)$.

Proof: Since the function $\bar{f}(u,v)$ is the dualizing parametrization function for φ, $\bar{f}(u,v)$ is proper, weakly lower semicontinuous, and level-bounded in u locally uniform in v, φ is proper, weakly lower semicontinuous and level-bounded. It is easy to prove that there exists $u_0 \in \mathbf{U}$, such that $\varphi(u_0) = \inf\limits_{u \in \mathbf{U}} \varphi(u)$. So we only need to prove (i) and (ii).

(i) Since $\sigma(v) \geq 0$, we have

$$\inf\{\bar{l}(u, \bar{y}, r) : u \in \mathbf{U}\} \geq \inf\{\bar{l}(u, \bar{y}, \bar{r}) : u \in \mathbf{U}\} \quad \forall r \geq \bar{r}.$$

Let $m_r{}^* = \inf\{\bar{l}(u, \bar{y}, r) : u \in \mathbf{U}\}$, $m^* = \inf\{\bar{l}(u, \bar{y}, \bar{r}) : u \in \mathbf{U}\}$. It is obvious that $m_r{}^* \leq p(0)$.

By the definition of $m_r{}^*$, there exists a minimizing sequence $(u_j, v_j) \in \mathbf{U} \times \mathbf{V}$, satisfying

$$\bar{f}(u_j, v_j) + r\sigma(v_j) - g(\bar{y}, v_j) \to m_r{}^*. \tag{2.22}$$

From (2.21), we have

$$\bar{f}(u_j, v_j) + r\sigma(v_j) - g(\bar{y}, v_j) \geq m^* + (r - \bar{r})\sigma(v_j). \tag{2.23}$$

It follows from (2.22) and (2.23) that, for some $\epsilon_0 \geq 0$, there exists an integer $N > 0$, such that

$$m^* + (r - \bar{r})\sigma(v_j) \leq m_r{}^* + \epsilon_0 \leq p(0) + \epsilon_0, \forall j > N.$$

That is

$$\sigma(v_j) \le \frac{p(0)+\epsilon_0 - m^*}{r-\bar{r}}, \forall j > N. \tag{2.24}$$

Since σ has a valley at 0, $c_\delta = \inf_{\|v\|\ge\delta}\sigma(v) > 0$, for each $\delta > 0$. Denote $r_0 = \frac{p(0)+\epsilon_0-m^*}{c_\delta} + \bar{r}$ and let $r > r_0$. From (2.24), we have $\sigma(v_j) < c_\delta, \forall j > N$. This implies that $v_j \in \{v \in \mathbf{V} : \|v\| \le \delta\}, \forall j > N$. Because $\bar{f}(u,v)$ is level-bounded in u locally uniform in v, it follows from (2.22) that $\{u_j\}$ is bounded, so $\{(u_j,v_j)\}$ is bounded. Because $\mathbf{U}$ and $\mathbf{V}$ are reflexive Banach spaces, the product space $\mathbf{U}\times\mathbf{V}$ is a reflexive Banach space. Thus there exists a weakly convergent subsequence of $\{(u_j,v_j)\}$. Without loss of generality, we may assume that $(u_j,v_j) \rightharpoonup (u_0,v_0)$. By Lemma 2.4.1, $u_{j_k} \rightharpoonup u_0$ and $v_{j_k} \rightharpoonup v_0$ as $k \to \infty$. The weak lower semicontinuity of f and σ, together with (2.22), implies

$$\begin{aligned}\bar{f}(u_0,v_0) + r\sigma(v_0) - g(\bar{y},v_0) &\le \liminf_{k\to+\infty} \bar{f}(u_{j_k},v_{j_k}) + r\liminf_{k\to+\infty}\sigma(v_{j_k}) - \lim_{k\to+\infty} g(\bar{y},v_{j_k})\\ &\le {m_r}^*. \end{aligned} \tag{2.25}$$

Hence

$$\bar{f}(u_0,v_0) + r\sigma(v_0) - g(\bar{y},v_0) = \inf_{(u,v)\in\mathbf{U}\times\mathbf{V}}\{\bar{f}(u,v)+r\sigma(v)-g(\bar{y},v)\}.$$

(ii) Let (u_r,v_r) with $r \ge r_0$ be the solution of the problem $P(\bar{y},r)$. Arbitrarily fix $u' \in \mathbf{U}$ such that $-\infty < \varphi(u') < +\infty$. We have

$$\varphi(u') \ge \bar{f}(u_r,v_r) + r\sigma(v_r) - g(\bar{y},v_r). \tag{2.26}$$

It follows from (2.21) that

$$\varphi(u') \ge m^* + (r-\bar{r})\sigma(v_r).$$

Thus,

$$\sigma(v_r) \le \frac{\varphi(u')-m^*}{r-\bar{r}}. \tag{2.27}$$

Let $r_1 = \frac{\varphi(u')-m^*}{c_\delta} + \bar{r}$. Then $\sigma(v_r) \le c_\delta$, so $\{v_r\}$ is bounded. Because $\bar{f}(u,v)$ is level-bounded in u locally uniform in v, it follows from (2.26) that $\{u_r\}$ is bounded, so $\{(u_r,v_r)\}$ is bounded. Then, the reflexivity of $\mathbf{U}\times\mathbf{V}$ implies that there exist $r_0 < r_j \to +\infty$ and $(u^*,v^*) \in \mathbf{U}\times\mathbf{V}$ such that $(u_{r_j},v_{r_j}) \rightharpoonup (u^*,v^*)$. The inequality (2.27), together with weak lower semicontinuity of σ, gives

$$\sigma(v^*) \le \liminf_{j\to+\infty}\sigma(v_{r_j}) = 0.$$

Therefore, $v^* = 0$.

Since (u_r, v_r) with $r \geq r_0$ is the solution of the problem $P(\bar{y}, r)$,

$$\bar{f}(u_r, v_r) - g(\bar{y}, v_r) \leq \bar{f}(u, 0) = \varphi(u), \quad \forall u \in \mathbf{U}.$$

Using the weak lower semicontinuity of $\bar{f}(u, v)$, we get

$$\varphi(u^*) \leq \varphi(u), \quad \forall u \in \mathbf{U}.$$

So u^* is the solution of primal problem (P). ∎

Theorem 2.4.2 *Let* $\mathbf{U}$ *and* $\mathbf{V}$ *be reflexive Banach spaces,* $\mathbf{E}$ *be a Banach space. Suppose that* $\bar{f}(u, v)$ *and* $\sigma(v)$ *satisfy the same conditions as in Theorem 2.4.1. Then the zero duality gap holds:*

$$p^{cc}(0) = p(0). \tag{2.28}$$

Proof: Since the weak duality (2.12) holds, we only need to prove

$$p^{cc}(0) \geq p(0).$$

From (i) of Theorem 2.4.1, there exists $r_0 > \bar{r}$, such that for any $r \geq r_0$, the augmented Lagrangian problem $P(\bar{y}, r)$ has at least one solution. Then, for any $r \geq r_0$, there exists (u_r, v_r), such that

$$\bar{f}(u_r, v_r) - g(\bar{y}, v_r) + r\sigma(v_r) = \bar{\psi}(\bar{y}, r). \tag{2.29}$$

It follows from the proof of (ii) in Theorem 2.4.1 that there exist $r_0 < r_j \to +\infty$ and $(u^*, 0) \in \mathbf{U} \times \mathbf{V}$ such that $(u_{r_j}, v_{r_j}) \rightharpoonup (u^*, 0)$, where $\varphi(u^*) = p(0)$. Since $\bar{f}(u, v)$ is weakly lower semicontinuous, we have

$$\liminf_{j \to +\infty} \bar{f}(u_{r_j}, v_{r_j}) \geq \bar{f}(u^*, 0).$$

This, combined with (2.29), yields

$$\liminf_{j \to +\infty} \bar{\psi}(\bar{y}, r_j) \geq p(0).$$

It is obvious that $\bar{\psi}(\bar{y}, r)$ is increasing in r, thus, for $\forall \varepsilon > 0$, there exists a $K > 0$, such that

$$\bar{\psi}(\bar{y}, r_j) \geq p(0) - \varepsilon, \quad \forall j > K.$$

Therefore

$$\sup_{(y,r)\in\mathbf{E}\times(0,+\infty)} \bar{\psi}(y,r) \geq p(0) - \varepsilon.$$

By the arbitrariness of ε, we conclude that

$$\sup_{(y,r)\in\mathbf{E}\times(0,+\infty)} \bar{\psi}(y,r) \geq p(0).$$

From (2.10), we have

$$p^{cc}(0) \geq p(0).$$

So zero duality gap (2.28) holds. ▮

2.5 Application to a variational inequality problem

Let Ω be a bounded domain of $\mathbf{R}^N$ with the smooth boundary and $H_0^1(\Omega) = W_0^{1,2}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ in the norm $\|u\| = \{\int_\Omega |\nabla v|^2\, dx\}^{1/2}$. Let $K = \{u \in H_0^1(\Omega) : u \leq d^*\ a.e. in\ \Omega\}$, where $d^* \in H_0^1(\Omega)$.

Assumption 1. $a_{i,j}, a_0, \frac{\partial b_i}{\partial x_i} \in L^\infty(\Omega)$ (the Banach space of essential bounded measurable functions on Ω), and, for some constant $c_1 > 0$,

$$a_0 - \frac{1}{2}\sum_{i=1}^{N} \frac{\partial b_i}{\partial x_i} \geq c_1. \tag{2.30}$$

Assumption 2. For some constant $c_2 > 0$,

$$\sum_{i,j=1}^{N} a_{i,j}\xi_i\xi_j \geq c_2\|\xi\|^2, \quad a.e.\ in\ \Omega, \quad \forall \xi = (\xi_1, ..., \xi_N) \in \mathbf{R}^N. \tag{2.31}$$

Define

$$a(u,v) = \sum_{i,j=1}^{N} \int_\Omega a_{i,j} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\, dx + \int_\Omega \Big(a_0 - \frac{1}{2}\sum_{i=1}^{N}\frac{\partial b_i}{\partial x_i}\Big) uv\, dx, \quad u, v \in K. \tag{2.32}$$

Let $f \in H^{-1}(\Omega)$ (the dual space of $H_0^1(\Omega)$). The following three problems are equivalent (see, Theorem 3.1 in [20]):

Problem 1: Find $u_0 \in K$, such that

$$a(u_0, u - u_0) \geq \langle f, u - u_0 \rangle \quad \forall u \in K.$$

Problem 2: Find $u_0 \in K$, such that

$$\frac{1}{2}a(u_0, u_0) - \langle f, u_0 \rangle = \inf_{u \in K}[\frac{1}{2}a(u, u) - \langle f, u \rangle].$$

Problem 3: Find $u_0 \in H_0^1(\Omega)$, such that

$$\frac{1}{2}a(u_0, u_0) - \langle f, u_0 \rangle + \delta_{\mathbf{R}^-}(u_0 - d^*) = \inf_{u \in H_0^1(\Omega)} (\frac{1}{2}a(u, u) - \langle f, u \rangle + \delta_{\mathbf{R}^-}(u - d^*)),$$

where

$$\delta_{\mathbf{R}^-}(u - d^*) = \begin{cases} 0, & \text{if } u \leq d^*, \ a.e. \ in \ \Omega; \\ +\infty, & \text{otherwise.} \end{cases}$$

Define

$$\begin{aligned} J(u) &:= \frac{1}{2}a(u, u) - \langle f, u \rangle, \\ \psi_1(u) &:= J(u) + \delta_{\mathbf{R}^-}(u - d^*), \\ \bar{f}_1(u, v) &:= J(u) + \delta_{\mathbf{R}^-}(u - d^* + v), \end{aligned} \tag{2.33}$$

and

$$\sigma(v) := \begin{cases} \left(\int_\Omega |\nabla v|^2 \, dx\right)^{\gamma/2}, & \text{if } \|v\| < 1, \\ 1, & \text{if } \|v\| \geq 1, \end{cases} \tag{2.34}$$

where $0 < \gamma < 1$. It follows from Lemma 2.2.1 that $\sigma(v)$ has a valley at 0. Moreover, $\bar{f}_1(u, v)$ is a dualizing parametrization function for $\psi_1(u)$. In the following, we are concerned with the primal problem

$$(P_3) \qquad \inf_{u \in H_0^1(\Omega)} \psi_1(u)$$

and the augmented Lagrangian problem

$$Q_1(y, r) \qquad \inf_{(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)} \{\bar{f}_1(u, v) + r\sigma(v) - \langle y, v \rangle\},$$

where $(y, r) \in H^{-1}(\Omega) \times (0, +\infty)$. Actually the primal problem (P_3) is the Problem 3. Now we have the expression of the augmented Lagrangian $\bar{l}(u, y, r)$ for the primal problem (P_3) as follows,

$$\begin{aligned} \bar{l}(u, y, r) &= \inf\{\bar{f}_1(u, v) - \langle y, v \rangle + r\sigma(v) : v \in H_0^1(\Omega)\} \\ &= J(u) + \inf\{(r\sigma(d^* - u + v) - \langle y, d^* - u + v \rangle) : v \in \mathbf{E}_0\}, \end{aligned} \tag{2.35}$$

where $\mathbf{E}_0 = \{u \in H_0^1(\Omega) : u \le 0,\ a.e. in\ \Omega\}$. Let $y = \theta$ (the zero element of $H^{-1}(\Omega)$). Then the augmented Lagrangian problem $Q_1(y, r)$ is turned into the problem

$$Q_1(r) \qquad \inf_{(u,v)\in H_0^1(\Omega)\times \mathbf{E}_0} \{J(u) + r\sigma(d^* - u + v)\}.$$

Lemma 2.5.1 *Let Assumptions 1 and 2 hold and $a(u, v)$ be defined by (2.32). Then Problem 2 has a unique solution.*

Proof: Since

$$J(u) = \frac{1}{2}a(u, u) - \langle f, u\rangle,$$

J is twice Gâteaux differentiable in $H_0^1(\Omega)$. We have

$$J'(u, \omega) = a(u, \omega) - \langle f, \omega\rangle,$$

and

$$J''(u, \omega, \varpi) = a(\omega, \varpi) = a(\varpi, \omega) \quad \forall u, \omega, \varpi \in H_0^1(\Omega).$$

It follows from (2.30) and (2.31) that there exist some positive constants c_3 , c_4 and c_5, such that

$$a(u, u) \ge c_2 \int_\Omega \sum_{i=1}^{N} (\frac{\partial u}{\partial x_i})^2\, dx + c_1 \int_\Omega u^2\, dx \ge c_3\|u\|^2,$$

and

$$a(u, v) \le c_4 (\int_\Omega \sum_{i=1}^{N} (\frac{\partial u}{\partial x_i})^2\, dx)^{\frac{1}{2}}\ (\int_\Omega \sum_{i=1}^{N} (\frac{\partial v}{\partial x_i})^2\, dx)^{\frac{1}{2}} + c_5(\int_\Omega u^2\, dx)^{\frac{1}{2}}\ (\int_\Omega v^2\, dx)^{\frac{1}{2}}$$
$$\le c_6\|u\|\|v\|.$$

Hence

$$J(u) \ge \frac{1}{2}c_6\|u\|^2 - \|f\|\|u\|, \tag{2.36}$$

and

$$J''(u, \omega, \omega) = a(\omega, \omega) \ge c_3\|\omega\|^2.$$

Thus, $J(u)$ is strictly convex, $J'(u, \omega)$ is linear and continuous for ω. Therefore $J(u)$ is weakly lower semicontinuous. There exists a minimizing sequence $u_j \in K$, satisfying

$$\lim_{j\to\infty} J(u_j) = \inf_{u\in K} J(u).$$

From (2.36), the sequence $\{u_j\}$ is bounded. By the reflexivity of $H_0^1(\Omega)$, there exists $\{u_{j_k}\} \subseteq \{u_j\}$, such that $u_{j_k} \rightharpoonup u_0$. Thus, from the weak lower semicontinuity of $J(u)$, we have

$$J(u_0) \leq \liminf_{k\to\infty} J(u_{j_k}) = \inf_{u\in K} J(u).$$

Because $u_j \in K$ and K is a closed and convex set, K is a weakly closed set. Thus $u_0 \in K$. So $J(u_0) = \inf_{u\in K} J(u)$. Noticing that $J(u)$ is strictly convex, u_0 is the unique solution of Problem 2. ▮

Theorem 2.5.1 *Let Assumptions 1 and 2 hold. For the primal problem (P_3) and the augmented Lagrangian problem $Q_1(r)$, we have the following results:*

(i) The primal problem (P_3) has a unique solution u_0.

(ii) There exists $r_0 > 0$, such that for any $r \geq r_0$, the augmented Lagrangian problem $Q_1(r)$ has a solution (u_r, v_r).

(iii) Every weak limit point of the sequence $\{u_r\}$ is the solution of primal problem (P_3).

Proof: Since Problem 2 and Problem 3 are equivalent, the conclusion of (i) follows from Lemma 2.5.1.

Let $y = \theta \in H^{-1}(\Omega)$. Then (2.35) and (2.36) imply that

$$\bar{l}(u,\theta,r) = J(u) + r\mathrm{inf}_{v\in\mathbf{E}_0}\{\sigma(d^* - u + v)\} \geq \frac{1}{2}c_6\|u\|^2 - \|f\|\|u\|,$$

and that there exists $M > 0$ large enough such that

$$\bar{l}(u,\theta,r) \geq J(0), \quad \forall\, u \in H_0^1(\Omega)\backslash B_M, \tag{2.37}$$

where $B_M = \{u \in H_0^1(\Omega) : \|u\| \leq M\}$. It is obvious that

$$\inf_{u\in B_M} \bar{l}(u,\theta,r) \leq J(0).$$

It follows from the proof of Lemma 2.5.1 that $J(u)$ is weakly lower semicontinuous. Then there exists $u_0 \in H_0^1(\Omega)$, such that $J(u_0) = \inf_{u\in B_M} J(u)$. Hence

$$J(u_0) \leq \inf_{u\in B_M} \bar{l}(u,\theta,r) \leq J(0).$$

This and (2.37) imply that

$$\inf_{u\in H_0^1(\Omega)} \bar{l}(u,\theta,r) \geq J(u_0) > -\infty.$$

Thus, according to Lemma 2.2.1 and using the same argument as in Theorem 2.4.1, there exists $r_0 > 0$, such that for any $r \geq r_0$, the augmented Lagrangian problem $Q_1(r)$ has a solution. This proves (ii).

Let (u_r, v_r) be the solution of the augmented Lagrangian problem $Q_1(r)$. Then

$$J(u_r) + r\sigma(d^* - u_r + v_r) \leq J(u) + r\sigma(d^* - u + v), \quad \forall (u,v) \in H_0^1(\Omega) \times \mathbf{E}_0.$$

Let $u = u_0, v = u - d^*$, where u_0 is the unique solution of the primal problem (P_3). Then

$$J(u_r) + r\sigma(d^* - u_r + v_r) \leq J(u_0).$$

Hence

$$J(u_r) \leq J(u_0), \tag{2.38}$$

and

$$\sigma(d^* - u_r + v_r) \leq \frac{1}{r}(J(u_0) - J(u_r)). \tag{2.39}$$

Again from (2.36), $J(u_r) \geq \frac{1}{2}c_6\|u_r\|^2 - \|f\|\|u_r\|$. From this and (2.38), we get that $\{u_r\}$ is bounded, i.e., there exists C such that $\|u_r\| \leq C$. From (2.39), we have

$$\sigma(d^* - u_r + v_r) \leq \frac{1}{r}(J(u_0) - \frac{1}{2}c_6\|u_r\|^2 + \|f\|\|u_r\|) \leq \frac{1}{r}(J(u_0) + C\|f\|). \tag{2.40}$$

Hence $\{v_r\}$ is bounded. There exists a subsequence $\{(u_{r_j}, v_{r_j})\}$ such that $u_{r_j} \rightharpoonup u', v_{r_j} \rightharpoonup v'$. By the weak lower semicontinuity of $J(u)$ and (2.38), we have

$$J(u') \leq \liminf_{j\to\infty} J(u_{r_j}) \leq J(u_0).$$

It follows from (2.40) that $d^* - u_{r_j} + v_{r_j} \to 0$. Let $\forall q \in H^{-1}(\Omega)$, $\langle q, d^* - u_{r_j} + v_{r_j}\rangle \to \langle q, d^* - u' + v'\rangle = 0$, hence $d^* - u' + v' = 0$. Since $\mathbf{E}_0$ is a weakly closed set, $\{v_{r_j}\} \in \mathbf{E}_0$ and $v_{r_j} \rightharpoonup v'$, it follows that $v' \in \mathbf{E}_0$ and $u' - d^* \in \mathbf{E}_0$, that is, $u' \leq d^*, a.e. in\ \Omega$. So u' is the solution of the primal problem (P_3) and $u' = u_0$.

∎

Theorem 2.5.2 *Let Assumptions 1 and 2 hold. For the primal problem* (P_3) *and the augmented Lagrangian problem* $Q_1(y,r)$*, the zero duality gap holds:*

$$\inf_{u\in H_0^1(\Omega)} \psi_1(u) = \sup_{(y,r)\in H^{-1}(\Omega)\times(0,+\infty)} \inf_{(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)} \{\bar{f}_1(u,v) + r\sigma(v) - \langle y, v\rangle\}.$$

Proof: According to Theorem 2.5.1, there exists $r_0 > 0$, such that for any $r \geq r_0$, the augmented Lagrangian problem $Q_1(r)$ has a solution. Let (u_r, v_r) be the solution of the problem $Q_1(r)$. Then

$$\begin{aligned} J(u_r) + r\sigma(d^* - u_r + v_r) &= \inf_{(u,v)\in H_0^1(\Omega)\times \mathbf{E}_0} \{J(u) + r\sigma(d^* - u + v)\} \\ &= \inf_{u\in H_0^1(\Omega)} \bar{l}(u, \theta, r). \end{aligned} \tag{2.41}$$

Let l be the limit point of the sequence $\{J(u_r)\}$. Then there exists a subsequence $\{u_{r_j}\}$, such that

$$\lim_{j\to\infty} J(u_{r_j}) = l. \tag{2.42}$$

Let $u = u_0$ and $v = u_0 - d^*$ in (2.41), where u_0 is the unique solution of the primal problem (P_3). Then

$$J(u_{r_j}) \leq J(u_0) = \inf_{u\in K} J(u), \tag{2.43}$$

and

$$\sigma(d^* - u_{r_j} + v_{r_j}) \leq \frac{1}{r_j}(J(u_0) - J(u_{r_j})). \tag{2.44}$$

Using (2.43), (2.44) and the same arguments as in the proof of Theorem 2.5.1, we have that $\{u_{r_j}\}$ and $\{v_{r_j}\}$ are bounded and $u_{r_j} \rightharpoonup u' \in K, v_{r_j} \rightharpoonup v' \in \mathbf{E}_0$. By (2.42), (2.43) and the weak lower semicontinuity of $J(u)$, we get

$$J(u_0) \leq J(u') \leq \liminf_{j\to\infty} J(u_{r_j}) = l \leq J(u_0),$$

that is, $\lim\limits_{j\to\infty} J(u_{r_j}) = J(u_0)$. Thus it follows from (2.41) that

$$J(u_0) = \lim_{j\to\infty} J(u_{r_j}) \leq \liminf_{j\to\infty} \inf_{u\in H_0^1(\Omega)} \bar{l}(u, \theta, r_j).$$

So

$$J(u_0) \leq \sup_{(y,r)\in H^{-1}(\Omega)\times(0,+\infty)} \inf_{u\in H_0^1(\Omega)} \bar{l}(u, y, r).$$

Since $J(u_0) = \inf\limits_{u\in H_0^1(\Omega)} \psi_1(u)$, it follows from (2.35) that

$$\inf_{u\in H_0^1(\Omega)} \psi_1(u) \leq \sup_{(y,r)\in H^{-1}(\Omega)\times(0,+\infty)} \inf_{(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)} \{\bar{f}_1(u,v) + r\sigma(v) - \langle y, v\rangle\}. \tag{2.45}$$

On the other hand, for each $(y, r) \in \mathbf{E} \times (0, +\infty)$, we have

$$\bar{l}(u, y, r) \leq \bar{f}_1(u, 0) = \psi_1(u),$$

thus

$$\sup_{(y,r)\in H^{-1}(\Omega)\times(0,+\infty)} \inf_{u\in H_0^1(\Omega)} \bar{l}(u, y, r) \leq \inf_{u\in H_0^1(\Omega)} \psi_1(u).$$

Therefore, this and (2.45) imply that the zero duality gap holds. ▮

Function $\psi_1(u)$ defined by (2.33) involves a bilinear function $a(u,v)$. Now we consider another function $\psi_2(u)$, which involves a nonlinear bi-function.

Define

$$\psi_2(u) = \Big(\int_\Omega |\nabla u|^2\,dx\Big)^{\alpha_1/2} + \int_\Omega \Big(a_0 - \frac{1}{2}\sum_{i=1}^N \frac{\partial b_i}{\partial x_i}\Big) u^{\alpha_2}\,dx \tag{2.46}$$

and

$$\begin{aligned}\bar{f}_2(u,v) = &\Big(\int_\Omega |\nabla u|^2\,dx\Big)^{\alpha_1/2} + \Big(\int_\Omega |\nabla v|^2\,dx\Big)^{\beta_1/2} + \int_\Omega \Big(a_0 - \frac{1}{2}\sum_{i=1}^N \frac{\partial b_i}{\partial x_i}\Big) u^{\alpha_2}\,dx \\ &+ \int_\Omega \Big(a_0 - \frac{1}{2}\sum_{i=1}^N \frac{\partial b_i}{\partial x_i}\Big) v^{\beta_2}\,dx,\end{aligned} \tag{2.47}$$

where $a_0, \frac{\partial b_i}{\partial x_i} \in L^\infty(\Omega)$ and satisfy (2.30), $\alpha_1, \alpha_2, \beta_1$ and β_a are some constants satisfying

$$1 \le \alpha_2 < \alpha_1 < 2^* := \frac{2N}{N-2}, \qquad 1 \le \beta_2 < \beta_1 < 2^*.$$

It is clear that $\psi_2(u) = \bar{f}_2(u,0)$.

The following Sobolev imbedding theorem is useful in this part.

Theorem 2.5.3 *([44] Sobolev imbedding theorem) The spaces $W_0^{1,p}(\Omega)$ are compactly imbedded in the spaces $L^q(\Omega)$ (i.e., the imbedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact) for any $q < np/(n-p) = p^*$, if $p < n$. That is, $W_0^{1,p}(\Omega) \subset L^q(\Omega)$, a bounded set in $W_0^{1,p}(\Omega)$ must be precompact in $L^q(\Omega)$, and there exists a constant $C = c(n,p)$ such that, for any $u \in W_0^{1,p}(\Omega)$,*

$$\|u\|_q \le C\|Du\|_p.$$

We have the following results.

Lemma 2.5.2 *$\bar{f}_2(u,v)$ is weakly lower semicontinuous.*

Proof: Let $(u_n, v_n) \rightharpoonup (u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$ as $n \to \infty$. By Lemma 2.4.1, there exist subsequences of $\{u_n\}$ and $\{v_n\}$ (without loss of generality, we still denote as $\{u_n\}$ and $\{v_n\}$), such that $u_n \rightharpoonup u_0$ and $v_n \rightharpoonup v_0$ as $n \to \infty$. Then

$$\begin{aligned}\liminf_{n\to\infty} \bar{f}_2(u_n, v_n) \geq & \liminf_{n\to\infty} \left(\int_\Omega |\nabla u_n|^2\,dx\right)^{\alpha_1/2} + \liminf_{n\to\infty} \left(\int_\Omega |\nabla v_n|^2\right)^{\beta_1/2} \\ & + \liminf_{n\to\infty} \int_\Omega \Big(a_0 - \frac{1}{2}\sum_{i=1}^N \frac{\partial b_i}{\partial x_i}\Big) u_n^{\alpha_2}\,dx \\ & + \liminf_{n\to\infty} \int_\Omega \Big(a_0 - \frac{1}{2}\sum_{i=1}^N \frac{\partial b_i}{\partial x_i}\Big) v_n^{\beta_2}\,dx.\end{aligned} \tag{2.48}$$

Since Ω is bounded, it follows from Theorem 2.5.3 that the embedding $H_0^1(\Omega) \hookrightarrow L^t(\Omega)$ is compact for $1 < t < 2^*$. Going if necessary, we may assume that $u_n \to u_0$ in $L^{\alpha_2}(\Omega)$ and $v_n \to v_0$ in $L^{\beta_2}(\Omega)$ as $n \to \infty$. (2.48) and the weak lower semicontinuity of the norm imply that

$$\liminf_{n\to\infty} \bar{f}_2(u_n, v_n) \geq \bar{f}_2(u_0, v_0),$$

that is, $\bar{f}_2(u, v)$ is weakly lower semicontinuous for each $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$. ▮

Lemma 2.5.3 *$\bar{f}_2(u, v)$ is level-bounded in u locally uniformly in v.*

Proof: For all $\bar{v} \in H_0^1(\Omega)$, denote $U(\bar{v}) = \{v \in H_0^1(\Omega) : \ \|v - \bar{v}\| \leq 1\}$. There exists a constant $C_1 > 0$ such that $\|v\| \leq C_1$ for all $v \in U(\bar{v})$. We have

$$\begin{aligned}\bar{f}_2(u, v) \geq & \left(\int_\Omega |\nabla u|^2\,dx\right)^{\alpha_1/2} + \left(\int_\Omega |\nabla v|^2\right)^{\beta_1/2} + \int_\Omega \Big(a_0 - \frac{1}{2}\sum_{i=1}^N \frac{\partial b_i}{\partial x_i}\Big)\big(u^{\alpha_2} + v^{\beta_2}\big)\,dx \\ \geq & \|u\|^{\alpha_1} + \|v\|^{\beta_1} - C_2\|u\|^{\alpha_2} - C_3\|v\|^{\beta_2} \\ \geq & \|u\|^{\alpha_1} - C_2\|u\|^{\alpha_2} - C_4,\end{aligned}$$

for some constants $C_2, C_3, C_4 > 0$.

If, for all $a \in \mathbf{R}$, $\bar{f}_2(u, v) \leq a$, then, from the preceding inequality, there exists a constant $C_5 > 0$ such that $\|u\| \leq C_5$. This yields that $\bar{f}_2(u, v)$ is level-bounded in u locally uniformly in v. ▮

Lemma 2.5.4 *For each $(\bar{y}, \bar{r}) \in H^{-1}(\Omega) \times (0, +\infty)$, we have*

$$\inf_{(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)} \{\bar{f}_2(u, v) + \bar{r}\sigma(v) - \langle \bar{y}, v\rangle\} > -\infty. \tag{2.49}$$

Proof: It follows from (2.30) that

$$\bar{f}_2(u,v) + \bar{r}\sigma(v) - \langle \bar{y}, v\rangle \geq \|u\|^{\alpha_1} + \|v\|^{\beta_1} - C_6\|u\|^{\alpha_2} - C_7\|v\|^{\beta_2} + \bar{r} - \|\bar{y}\|\|v\|,$$

for some constants $C_6, C_7 > 0$. Then, there exists $M \geq 0$ large enough such that

$$\bar{f}_2(u,v) + \bar{r}\sigma(v) - \langle \bar{y}, v\rangle\} \geq \bar{f}_2(0,0), \quad \forall\, u, v \in H_0^1(\Omega)\backslash B_M, \tag{2.50}$$

where $B_M = \{u \in H_0^1(\Omega) : \|u\| \leq M\}$. Noting that

$$\inf_{(u,v)\in B_M\times B_M} \{\bar{f}_2(u,v) + \bar{r}\sigma(v) - \langle \bar{y}, v\rangle\} \leq \bar{f}_2(0,0),$$

and that $B_M \times B_M$ is weakly compact, it follows from Lemma 2.2.1 and Lemma 2.5.2 that $\bar{f}_2(u,v) + \bar{r}\sigma(v)$ is weakly lower semicontinuous. Thus there exists $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$ such that

$$\bar{f}_2(u_0,v_0) + \bar{r}\sigma(v_0) - \langle \bar{y}, v_0\rangle = \inf_{(u,v)\in B_M\times B_M} \{\bar{f}_2(u,v) + \bar{r}\sigma(v) - \langle \bar{y}, v\rangle\} \leq \bar{f}_2(0,0).$$

This and (2.50) imply that

$$\bar{f}_2(u_0,v_0) + \bar{r}\sigma(v_0) - \langle \bar{y}, v_0\rangle = \inf_{(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)} \{\bar{f}_2(u,v) + \bar{r}\sigma(v) - \langle \bar{y}, v\rangle\}.$$

Therefore (2.49) holds. ∎

Theorem 2.5.4 *For the primal problem*

$$(P_4) \qquad \inf_{u\in H_0^1(\Omega)} \psi_2(u)$$

and the augmented Lagrangian problem

$$Q_2(y,r) \qquad \inf_{(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)} \{\bar{f}_2(u,v) + r\sigma(v) - \langle y, v\rangle\},$$

where $(y,r) \in H^{-1}(\Omega) \times (0,+\infty)$, we have the following results:

(i) For each $(\bar{y},\bar{r}) \in H^{-1}(\Omega)\times(0,+\infty)$, there exists $\hat{r} > \bar{r}$, such that, for any $r \geq \hat{r}$, the optimal solution set $S(\bar{y}, r)$ of the augmented Lagrangian problem $Q_2(\bar{y}, r)$ is nonempty and weakly compact.

(ii) Every weak limit point of the sequence $\{u_r\}$ is the solution of primal problem (P_4), where (u_r, v_r) with $r \geq \hat{r}$ is the solution of the augmented Lagrangian problem $Q_2(\bar{y}, r)$.

(iii)

$$\inf_{u\in H_0^1(\Omega)} \psi_2(u) = \sup_{(y,r)\in H^{-1}(\Omega)\times(0,+\infty)} \inf_{(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)} \{\bar{f}_2(u,v) + r\sigma(v) - \langle y, v\rangle\}.$$

Proof: (i) According to Lemma 2.2.1, Lemmas 2.5.2-2.5.4 and Theorem 2.4.1, there exists $r_0 > \bar{r}$, such that for any $r \geq r_0$, the optimal solution set $S(\bar{y}, r)$ is nonempty.

Let

$$\bar{t} = \inf_{(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)} \{\bar{f}_2(u,v) + \bar{r}\sigma(v) - \langle \bar{y}, v\rangle\}.$$

Then $\bar{t} > -\infty$. Fixing a $u_0 \in H_0^1(\Omega)$, let $\hat{r} > \psi_2(u_0) - \bar{t} + \bar{r}$. For any $r \geq \hat{r}$, we will verify that $S(\bar{y}, r)$ is weakly compact.

Let $r \geq \hat{r}$, $(u_{r_j}, v_{r_j}) \in S(\bar{y}, r)$. Then

$$\bar{f}_2(u_{r_j}, v_{r_j}) + r\sigma(v_{r_j}) - \langle \bar{y}, v_{r_j}\rangle \leq \psi_2(u_0). \tag{2.51}$$

Thus

$$\sigma(v_{r_j}) \leq \frac{\psi_2(u_0) - \bar{t}}{r - \bar{r}} < 1.$$

By the definition of $\sigma(v)$, we obtain $\|v_{r_j}\| < 1$. Hence it follows from (2.51) that

$$\bar{f}_2(u_{r_j}, v_{r_j}) \leq \psi_2(u_0) + \|\bar{y}\|.$$

Using Lemma 2.5.3, we get that $\{u_{r_j}\}$ is bounded. Therefore $S(\bar{y}, r)$ is bounded.

Let $(u_j, v_j) \in S(\bar{y}, r)$, $(u_j, v_j) \rightharpoonup (u_0, v_0)$ as $j \to \infty$. Then

$$\bar{f}_2(u_j, v_j) - \langle y, v_j\rangle + r\sigma(v_j) \leq \inf_{(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)} \{\bar{f}_2(u,v) + r\sigma(v) - \langle \bar{y}, v\rangle\}.$$

The weak lower semicontinuities of $\bar{f}_2(u,v)$ and $\sigma(v)$ imply

$$\bar{f}_2(u_0, v_0) + r\sigma(v_0) - \langle \bar{y}, v_0\rangle \leq \liminf_{j\to+\infty} \{\bar{f}_2(u_j, v_j) - \langle y, v_j\rangle + r\sigma(v_j)\}.$$

So

$$\bar{f}_2(u_0, v_0) + r\sigma(v_0) - \langle \bar{y}, v_0\rangle = \inf_{(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)} \{\bar{f}_2(u,v) + r\sigma(v) - \langle \bar{y}, v\rangle\}.$$

That is, $(u_0, v_0) \in S(\bar{y}, r)$. Therefore, $S(\bar{y}, r)$ is a bounded and weakly closed set, and, so is weakly compact.

(ii) and (iii) follow from Theorems 2.4.1-2.4.2. ▮

Chapter 3

Penalization via an Augmented Lagrangian Function

3.1 Introduction

It is possible to construct an exact penalty representation for a prime problem or a constrained optimization problem, that is, the solution of an augmented Lagrangian function yields an exact solution to the original problem for a finite value of the penalty parameter. With these functions it is not necessary to solve an infinite sequence of augmented Lagrangian problems to obtain the correct solution of the original problem.

This chapter is organized as follows:

In Section 3.2, suppose that the perturbation function satisfies a growth condition and the augmenting function satisfies the level-coercivity condition or has a valley at zero, we establish exact penalization results for a minimization problem of an extended real valued function, which includes Theorem 3.2 in [56] as a special case. We also obtain necessary and sufficient conditions for an exact penalty representation in the framework of augmented Lagrangians with a valley at zero augmenting function. In Section 3.3, we discuss the relationship between the solutions of a constrained optimization problem and that of its augmented Lagrangian problem and get some convergence and exact penalty results for a constrained optimization problem in infinite dimensional Banach spaces. We get the existence of an asymptotically minimizing sequence for a constrained optimization problem, which generalizes Theorem 3 in [95] to the non-quadratic case. In Section 3.4, the augmented penalty function method is applied to a reformulated constrained optimization problem. Exact penalty results without any coercive assumption on the objective function and constraint functions are obtained.

3.2 The exact penalty representation in Banach spaces

Let us consider the primal problem (2.5) again. Recall that

$$(P) \qquad \inf_{u\in\mathbf{U}} \varphi(u),$$

where $\varphi : \mathbf{U} \to \overline{\mathbf{R}}$. Let $\text{argmin}_u \varphi(u) = \{u' : \varphi(u') = \min_{u\in\mathbf{U}} \varphi(u)\}$. Suppose that the perturbation function $p : \mathbf{V} \to \overline{\mathbf{R}}$ is given by (2.8) again. In this section, we obtain necessary and sufficient conditions for an exact penalty representation in the framework of augmented Lagrangians under different conditions.

Definition 3.2.1 *(exact penalty representation) Consider the problem (P). Let the augmented Lagrangian $\bar{l}$ be defined as in Definition 2.2.10. A vector $\bar{y} \in \mathbf{E}$ is said to support an exact penalty representation for the problem (P) if there exists $\bar{r} > 0$ such that*

$$p(0) = \inf_{u\in\mathbf{U}} \bar{l}(u, \bar{y}, r), \quad \forall r \geq \bar{r} \tag{3.1}$$

and

$$\mathit{argmin}_u \varphi(u) = \mathit{argmin}_u \bar{l}(u, \bar{y}, r), \quad \forall r \geq \bar{r}. \tag{3.2}$$

Lemma 3.2.1 *Let $\mathbf{U}$ and $\mathbf{E}$ be two Banach spaces, $\mathbf{V}$ be a reflexive Banach space and $\bar{y} \in \mathbf{E}$. Assume that the function $p : \mathbf{V} \to \overline{\mathbf{R}}$ satisfies the growth condition (2.9), the function $\sigma : \mathbf{V} \to \overline{\mathbf{R}}$ satisfies the level-coercive condition and $g : \mathbf{E} \times \mathbf{V} \to \mathbf{R}$ satisfies $|g(y, v)| \leq d(y)\|v\|^\mu, \forall (y, v) \in \mathbf{E} \times \mathbf{V}$, where $d(y) \geq 0, d(0) = 0$ and $0 < \mu \leq 1$. Functions $p, g(\bar{y}, \cdot)$ and σ are proper and weakly lower semicontinuous in $\mathbf{V}$. Then, there exists a $r_0 > 0$ such that, for each $r \geq r_0$, there exists a $v_r \in \mathbf{V}$, such that*

$$p(v_r) - g(\bar{y}, v_r) + r\sigma(v_r) = \inf_{v\in\mathbf{V}} \big(p(v) - g(\bar{y}, v) + r\sigma(v)\big) = \bar{\psi}(\bar{y}, r). \tag{3.3}$$

Proof: Since σ satisfies the level-coercive condition, there exist $\varepsilon > 0$ and $\tau > 0$, such that

$$\sigma(v) \geq \varepsilon\|v\|, \quad \forall v \in \mathbf{V}\backslash\tau B_V, \tag{3.4}$$

where $\tau B_V = \{v : v \in \mathbf{V}, \|v\| \leq \tau\}$. Since $p : \mathbf{V} \to \overline{\mathbf{R}}$ satisfies the growth condition (2.9), there exist $a, b \in \mathbf{R}$ such that

$$p(v) \geq b - a\sigma(v), \quad \forall v \in \mathbf{V} \setminus \tau B_V. \tag{3.5}$$

Let $r_0 > \frac{d(\bar{y})}{\varepsilon\tau^{1-\mu}} + a$ and $r > r_0$. (3.4) and (3.5) imply that,

$$\begin{aligned} p(v) - g(\bar{y}, v) + r\sigma(v) &\geq b - a\sigma(v) - d(\bar{y})\|v\|^\mu + r\sigma(v) \\ &\geq b - d(\bar{y})\|v\|^\mu + (r - a)\varepsilon\|v\| \\ &\geq b + ((r_0 - a)\varepsilon - d(\bar{y})/\tau^{1-\mu})\|v\|, \quad \forall v \in \mathbf{V} \setminus \tau B_V. \end{aligned} \tag{3.6}$$

Let $r > r_0$ and

$$C_r = \inf_{v\in\mathbf{V}} \big(p(v) - g(\bar{y}, v) + r\sigma(v)\big).$$

Then, there exists a minimizing sequence $\{v_j\} \subset \mathbf{V}$, satisfying

$$p(v_j) + r\sigma(v_j) - g(\bar{y}, v_j) \leq C_r + \frac{1}{j}. \tag{3.7}$$

Hence, from (3.6) and (3.7), $\{v_j\}$ is bounded. Because $\mathbf{V}$ is a reflexive Banach space, there exists a weakly convergent subsequence of $\{v_j\}$. Without loss of generality, we may assume that $v_j \rightharpoonup v_r$. Therefore,

$$C_r \leq p(v_r) - g(\bar{y}, v_r) + r\sigma(v_r) \leq \liminf_{j\to\infty}(p(v_j) + r\sigma(v_j) - g(\bar{y}, v_j)) \leq C_r.$$

Thus, (3.3) holds. ∎

Lemma 3.2.2 *Let* $\mathbf{U}$ *and* $\mathbf{E}$ *be two Banach spaces,* $\mathbf{V}$ *be a reflexive Banach space and* $\bar{y} \in \mathbf{E}$*. Suppose that the functions* p*,* g *and* σ *satisfy the conditions given in Lemma 3.2.1. Moreover,* $argmin_y\sigma(y) = \{0\}$*,* $\sigma(0) = 0$*,* $v_r \in \mathbf{V}(r \geq r_0)$ *satisfy (3.3). Then, there exists a weakly convergent subsequence* $\{v_{r_j}\}$ *such that* $v_{r_j} \rightharpoonup 0$*, as* $r_j \to \infty$*.*

Proof: Denote

$$C_0 = \inf_{v\in\mathbf{V}} \big(p(v) - g(\bar{y}, v) + r_0\sigma(v)\big).$$

By Lemma 3.2.1, $C_0 > -\infty$. Since, for each $r > r_0$, $v_r (r \geq r_0)$ satisfies (3.3), and

$$\bar{\psi}(\bar{y}, r) = p(v_r) - g(\bar{y}, v_r) + r\sigma(v_r) \geq C_0 + (r - r_0)\sigma(v_r). \tag{3.8}$$

It is clear that

$$\bar{\psi}(\bar{y}, r) = \inf\{\bar{l}(u, \bar{y}, r) : u \in \mathbf{U}\} \leq \inf\{\bar{f}(u, 0) : u \in \mathbf{U}\} = p(0). \tag{3.9}$$

(3.8) and (3.9) imply

$$\sigma(v_r) \leq \frac{p(0) - C_0}{r - r_0}.$$

Thus, $\sigma(v_r) \to 0$ as $r \to \infty$. By using (3.4) given in the proof of Lemma 3.2.1, $\{v_r\}$ is bounded. Thus there exists a weakly convergent subsequence $\{v_{r_j}\}$ such that $v_{r_j} \rightharpoonup v_0$, as $r_j \to \infty$. The weak lower semicontinuity of σ implies $\sigma(v_0) = 0$. Hence, $v_0 = 0$. ▮

Theorem 3.2.1 *Let* $\mathbf{U}$ *and* $\mathbf{E}$ *be two Banach spaces,* $\mathbf{V}$ *be a reflexive Banach space. Assume that the function* $p : \mathbf{V} \to \overline{\mathbf{R}}$ *satisfies the growth condition (2.9),* $\sigma : \mathbf{V} \to \overline{\mathbf{R}}$ *satisfies the level-coercive condition,* $\text{argmin}_y \sigma(y) = \{0\}$, $\sigma(0) = 0$, $g : \mathbf{E} \times \mathbf{V} \to \mathbf{R}$ *satisfies* $|g(y, v)| \leq d(y)\|v\|^{\mu}, \forall (y, v) \in \mathbf{E} \times \mathbf{V}$, *where* $d(y) \geq 0, d(0) = 0$ *and* $0 < \mu \leq 1$. *Functions* $p, g(\bar{y}, \cdot)$ *and* σ *are proper, weakly lower semicontinuous in* $\mathbf{V}$. *Then a vector* $\bar{y}$ *supports an exact penalty representation for the primal problem* (P) *if and only if there exist* $r' > 0$ *and a neighborhood* U *of* $0 \in \mathbf{V}$ *such that*

$$p(0) \leq p(v) - g(\bar{y}, v) + r'\sigma(v), \quad \forall v \in U. \tag{3.10}$$

Proof: The necessity is clear. We only need to prove the sufficiency.

It follows from Lemma 3.2.1 that there exists a $r_0 > 0$ and, for each $r \geq r_0$, there exists a $v_r \in \mathbf{V}$, such that (3.3) holds. By Lemma 3.2.2, there exists a weakly convergent subsequence $\{v_{r_j}\}$ such that $v_{r_j} \rightharpoonup 0$, as $r_j \to \infty$. Since $p(v)$ and $g(\bar{y}, \cdot)$ are weakly lower semicontinuous, we have

$$\liminf_{j \to +\infty} p(v_{r_j}) \geq p(0),$$

and

$$\liminf_{j \to +\infty} g(\bar{y}, v_{r_j}) \geq g(\bar{y}, 0) = 0.$$

Hence

$$\liminf_{j \to +\infty} \bar{\psi}(\bar{y}, r_j) \geq \liminf_{j \to +\infty} \{p(v_{r_j}) - g(\bar{y}, v_{r_j}) + r_j\sigma(v_{r_j})\} \geq p(0). \tag{3.11}$$

By Definition 2.2.10, we have $\bar{l}(u,\bar{y},r_j)\leq\varphi(u)$, and

$$\bar{\psi}(y,r)=\inf\{\bar{l}(u,y,r):u\in\mathbf{U}\}\leq\inf_{u\in\mathbf{U}}\varphi(u)=p(0).$$

Thus,

$$\lim_{j\to+\infty}\bar{\psi}(\bar{y},r_j)=p(0).$$

Therefore, for $\forall\epsilon>0$, there exists $j_*>0$, such that

$$|\bar{\psi}(\bar{y},r_{j_*})-p(0)|<\epsilon.$$

Thus

$$p(0)-\epsilon<p(v)-g(\bar{y},v)+r_{j_*}\sigma(v),\quad\forall v\in\mathbf{V}.\tag{3.12}$$

In assuming (3.10), there is no loss of generality in taking U to be a ball $\tau B_V=\{u\in\mathbf{V},\|u\|\ \leq\tau\},\ \tau>0$.

Since σ has a valley at 0, there exists $\delta_0>0$, such that

$$\sigma(v)\geq\delta_0,\quad\forall v\in\mathbf{V}\backslash\tau B_V.\tag{3.13}$$

Letting $r^*>\max\{\bar{r},\frac{\epsilon}{\delta_0}+r_{j_*}\}$, we have

$$(r^*-r_{j_*})\sigma(v)\geq(r^*-r_{j_*})\delta_0>\epsilon,\quad\forall v\in\mathbf{V}\backslash\tau B_V.$$

This and (3.12) imply

$$p(0)\leq p(v)-g(\bar{y},v)+r^*\sigma(v),\quad\forall v\in\mathbf{V}\backslash\tau B_V.$$

Thus the above inequality combined with (3.10), yields that

$$p(0)\leq p(v)-g(\bar{y},v)+r^*\sigma(v),\quad\forall v\in\mathbf{V}.\tag{3.14}$$

By Definition 2.2.10, we have

$$\bar{l}(u,\bar{y},r)=\inf\{\bar{f}(u,v)-g(\bar{y},v)+r\sigma(v):v\in\mathbf{V}\}$$

and

$$\inf\{\bar{l}(u,\bar{y},r):u\in\mathbf{U}\}=\inf\{p(v)+r\sigma(v)-g(\bar{y},v):v\in\mathbf{V}\}.\tag{3.15}$$

Hence, by (3.14), we get

$$p(0)=\inf_{u\in\mathbf{U}}\bar{l}(u,\bar{y},r),\quad\forall r>r^*.\tag{3.16}$$

Let $r > r^*$ and $\bar{u} \in \text{argmin}_u \varphi(u)$. $\varphi(\bar{u}) = \inf_{u \in \mathbf{U}} \varphi(u) = p(0)$. Hence,

$$\bar{u} \in \text{argmin}_{u \in \mathbf{U}} \{f(u, 0) + r\sigma(0) - g(\bar{y}, 0)\}. \tag{3.17}$$

Since $\sigma(0) = 0$, $\sigma(u) > 0, \forall u \neq 0$, by using (3.14), we have

$$\text{argmin}_{v \in \mathbf{V}} \{p(v) + r\sigma(v) - g(\bar{y}, v)\} = \{0\} \quad \forall r > r^*. \tag{3.18}$$

(3.17) and (3.18) imply

$$(\bar{u}, 0) \in \text{argmin}_{(u,v) \in \mathbf{U} \times \mathbf{V}} \{f(u, v) + r\sigma(v) - g(\bar{y}, v)\}. \tag{3.19}$$

Thus, from (3.15) and (3.19), we get

$$\bar{u} \in \text{argmin}_u \bar{l}(u, \bar{y}, r), \quad \forall r > r^*,$$

i.e.

$$\text{argmin}_u \varphi(u) \subset \text{argmin}_u \bar{l}(u, \bar{y}, r), \quad \forall r > r^*. \tag{3.20}$$

Similarly, we have

$$\text{argmin}_u \bar{l}(u, \bar{y}, r) \subset \text{argmin}_u \varphi(u), \quad \forall r > r^*. \tag{3.21}$$

Therefore, from (3.20) and (3.21), we have proved that

$$\text{argmin}_u \varphi(u) = \text{argmin}_u \bar{l}(u, \bar{y}, r), \quad \forall r > r^*. \tag{3.22}$$

(3.16) and (3.22) imply that $\bar{y}$ supports an exact penalty representation for the problem (P). ∎

Let $\bar{y} = \theta$ (the zero element of $\mathbf{E}$). Similar to Lemmas 3.2.1 and 3.2.2, we have

Lemma 3.2.3 *Let $\mathbf{U}$ and $\mathbf{E}$ be two Banach spaces, $\mathbf{V}$ be a reflexive Banach space. Assume that the function $p : \mathbf{V} \to \overline{\mathbf{R}}$ satisfies the growth condition (2.9), the function $\sigma : \mathbf{V} \to \overline{\mathbf{R}}$ is a valley at 0 augmenting function. p is proper, weakly lower semicontinuous in $\mathbf{V}$. Then,*

(i) There exists a $r_0 > 0$ and, for each $r \geq r_0$, there exists a $v_r \in \mathbf{V}$, such that

$$p(v_r) + r\sigma(v_r) = \inf_{v \in \mathbf{V}} \big(p(v) + r\sigma(v)\big) = \bar{\psi}(\theta, r). \tag{3.23}$$

(ii) There exists a weakly convergent subsequence $\{v_{r_j}\}$ of the sequence $\{v_r\}$ obtained in (3.23), such that $v_{r_j} \rightharpoonup 0$, as $r_j \to \infty$.

Proof: Let $\tau > 0$. Since σ has a valley at 0, there exists $\varepsilon > 0$, such that

$$\sigma(v) \geq \varepsilon, \quad \forall v \in \mathbf{V} \backslash \tau B_V, \tag{3.24}$$

where $\tau B_V = \{v : v \in \mathbf{V}, \|v\| \leq \tau\}$. Since $p : \mathbf{V} \to \overline{\mathbf{R}}$ satisfies the growth condition, there exist $a, b \in \mathbf{R}$ such that

$$p(v) \geq b - a\sigma(v), \quad \forall v \in \mathbf{V} \setminus \tau B_V. \tag{3.25}$$

Let $r_0 > a$ and $r > r_0$. (3.24) and (3.25) imply,

$$\begin{aligned} p(v) + r\sigma(v) &\geq b - a\sigma(v) + r\sigma(v) \\ &\geq b + (r-a)\varepsilon, \quad \forall v \in \mathbf{V} \setminus \tau B_V. \end{aligned} \tag{3.26}$$

It follows from the proof of Lemmas 3.2.1 and 3.2.2 that the conclusions (i) and (ii) hold. ▮

By using Lemma 3.2.3, similar to the proof of Theorem 3.2.1, we have

Theorem 3.2.2 *Let* $\mathbf{U}$ *and* $\mathbf{E}$ *be two Banach spaces,* $\mathbf{V}$ *be a reflexive Banach space. If the function* $p : \mathbf{V} \to \overline{\mathbf{R}}$ *satisfies the growth condition (2.9), the function* $\sigma : \mathbf{V} \to \overline{\mathbf{R}}$ *is a valley at* 0 *augmenting function and p is proper, weakly lower semicontinuous in* $\mathbf{V}$*, then a vector* $\bar{y} = \theta$ *supports an exact penalty representation for the primal problem* (P) *if and only if there exist* $r' > 0$ *and a neighborhood* U *of* $0 \in \mathbf{V}$ *such that*

$$p(0) \leq p(v) + r'\sigma(v), \quad \forall v \in U. \tag{3.27}$$

Remark 3.2.1 In the case where $\bar{y} \neq \theta$ supports an exact penalty representation, it is need that σ satisfies the level-coercive condition, $\text{argmin}_y \sigma(y) = \{0\}$ and $\sigma(0) = 0$. In the case where $\bar{y} = \theta$ supports an exact penalty representation, it is only need that σ is a valley at 0 augmenting function.

Let $\mathbf{U} = \mathbf{R}^n, \mathbf{V} = \mathbf{R}^m, \bar{y} = 0 \in \mathbf{R}^m$. From Theorem 3.2.2, we have

Corollary 3.2.1 *(Theorem 3.2 in [56]) Let* $\sigma : X \to \overline{\mathbf{R}}$ *be a level-bounded augmenting function, where* $X \subset \mathbf{R}^m$ *be a closed subset. In the framework of the augmented Lagrangian* $\bar{l}$ *defined in Definition 2.2.10. The following statements are true:*

(i) If $\bar{y} = 0$ *supports an exact penalty representation, then there exist* $\bar{r} > 0$ *and a neighborhood* W *of* $0 \in \mathbf{R}^m$ *such that*

$$p(v) \geq p(0) - \bar{r}\sigma(v), \quad \forall v \in W.$$

(ii) The converse of (i) is true if $p(0)$ is finite, there exist $\bar{r}' > 0$ and $m^* \in \mathbf{R}$ such that $\bar{f}(u,v) + \bar{r}'\sigma(v) \geq m^*, \forall u \in \mathbf{R}^n, v \in \mathbf{R}^m$.

Proof: Because σ is a level-bounded augmenting function, that is, σ is proper, lower semicontinuous, level-bounded on $\mathbf{R}^m$, $\text{argmin}_y \sigma(y) = \{0\}$, $\sigma(0) = 0$, it is a valley at 0 augmenting function. From condition (ii), we have

$$p(v) \geq m^* - \bar{r}'\sigma(u), \quad \forall v \in R^m.$$

That is, p satisfies the growth condition. Hence, by Theorem 3.2.2, the conclusions of this corollary hold. ▮

Theorem 3.2.3 *Let $\mathbf{U}$ and $\mathbf{V}$ be reflexive Banach spaces, $\mathbf{E}$ be a Banach space, $\bar{f} : \mathbf{U} \times \mathbf{V} \to \overline{\mathbf{R}}$ be a dualizing parameterization function for φ, $\sigma : \mathbf{V} \to \mathbf{R}_+ \bigcup \{+\infty\}$ be a valley at 0 augmenting function and $g : \mathbf{E} \times \mathbf{V} \to \mathbf{R}$ be a weakly continuous function with $g(0,0) = 0$. Assume that $\bar{f}(u,v)$ is proper, weakly lower semicontinuous, and level-bounded in u locally uniformly in v. Furthermore suppose that there exists $(\bar{y}, \bar{r}) \in \mathbf{E} \times (0, +\infty)$ such that*

$$\inf\{\bar{l}(u, \bar{y}, \bar{r}) : u \in \mathbf{U}\} > -\infty. \tag{3.28}$$

Then a vector $\bar{y}$ supports an exact penalty representation for the primal problem (P) if and only if there exist $r' > 0$ and a neighborhood U of $0 \in \mathbf{V}$ such that

$$p(0) \leq p(v) - g(\bar{y}, v) + r'\sigma(v), \quad \forall v \in U. \tag{3.29}$$

Proof: Necessity. It follows from Theorem 2.4.1 that there exists $u_0 \in \mathbf{U}$, such that $\varphi(u_0) = \inf_{u \in \mathbf{U}} \varphi(u)$. Thus, $p(0)$ is finite. Since $\bar{y}$ supports an exact penalty representation, there exists $\bar{r} > 0$ such that (3.1) holds with $r = \bar{r}$, i.e.,

$$p(0) = \inf\{\bar{l}(u, \bar{y}, \bar{r}) : u \in \mathbf{U}\}.$$

Hence

$$p(0) = \inf\{\bar{f}(u,v) - g(\bar{y}, v) + \bar{r}\sigma(v) : (u,v) \in \mathbf{U} \times \mathbf{V}\}.$$

Consequently,

$$p(0) \leq p(v) - g(\bar{y}, v) + \bar{r}\sigma(v), \quad \forall v \in \mathbf{V}.$$

This proves the necessity.

Now we prove the sufficiency. We notice that, if there exists $r^* > r' > 0$, such that

$$p(0) \leq p(v) - g(\bar{y}, v) + r^*\sigma(v), \quad \forall v \in \mathbf{V}. \tag{3.30}$$

Then

$$p(0) \leq \bar{f}(u, v) - g(\bar{y}, v) + r^*\sigma(v), \quad \forall u \in \mathbf{U}, v \in \mathbf{V}.$$

Hence $p(0) \leq \bar{\psi}(\bar{y}, r^*)$. Since $\bar{\psi}(y, r)$ is increasing in r, $p(0) \leq \bar{\psi}(\bar{y}, r), \forall r \geq r^*$. Therefore, from (2.11), we have

$$p(0) = \bar{\psi}(\bar{y}, r), \forall r \geq r^*.$$

Thus

$$p(0) = \inf_{u \in \mathbf{U}} \bar{l}(u, \bar{y}, r), \quad \forall r \geq r^*. \tag{3.31}$$

That is, (3.1) in Definition 3.2.1 holds if (3.30) holds. In assuming (3.29), there is no loss of generality in taking U to be a ball $B_{\delta_0} = \{u \in \mathbf{V}, \|u\| \ \leq \delta_0\}$, $\delta_0 > 0$. So in order to prove (3.1) in Definition 3.2.1 holds, we only need to prove that (3.30) holds for all $v \in \mathbf{V} \backslash B_{\delta_0}$.

It follows from (i) of Theorem 2.4.1 that there exists $r_0 > \bar{r}$, such that for any $r \geq r_0$, the augmented Lagrangian problem $P(\bar{y}, r)$ has at least one solution. Let (u_r, v_r) with $r \geq r_0$ be the solution of the problem $P(\bar{y}, r)$. It follows from the proof of (ii) in Theorem 2.4.1 that there exist $r_0 < r_j \to +\infty$ and $(u^*, 0) \in \mathbf{U} \times \mathbf{V}$ such that $(u_{r_j}, v_{r_j}) \rightharpoonup (u^*, 0)$, where $\varphi(u^*) = p(0)$. Since $\bar{f}(u, v)$ is weakly lower semicontinuous,

$$p(0) = \bar{f}(u^*, 0) \leq \liminf_{j \to +\infty} \bar{f}(u_{r_j}, v_{r_j}).$$

Hence

$$\liminf_{j \to +\infty} \bar{\psi}(\bar{y}, r_j) \geq \liminf_{j \to +\infty} \{\bar{f}(u_{r_j}, v_{r_j}) - g(y, v_{r_j}) + r\sigma(v_{r_j})\} \geq p(0).$$

By (2.11), we have $\limsup_{j \to +\infty} \bar{\psi}(\bar{y}, r_j) \leq p(0)$. So

$$\lim_{j \to +\infty} \bar{\psi}(\bar{y}, r_j) = p(0).$$

Therefore, for $\forall \epsilon > 0$, there exists $j_0 > 0$, such that

$$|\bar{\psi}(\bar{y}, r_{j_0}) - p(0)| < \epsilon.$$

Thus

$$p(0) - \epsilon < \bar{l}(u, \bar{y}, r_{j_0}), \quad \forall u \in \mathbf{U},$$

i.e.,

$$p(0) - \epsilon < p(v) - g(\bar{y}, v) + r_{j_0}\sigma(v), \quad \forall v \in \mathbf{V}. \tag{3.32}$$

Since σ has a valley at 0 in $\mathbf{V}$, $c_{\delta_0} = \inf_{v \in \mathbf{V} \setminus B_{\delta_0}} \sigma(v) > 0$. Letting $r^* > \max\{\bar{r}, \frac{\epsilon}{c_{\delta_0}} + r_{j_0}\}$, we have

$$(r^* - r_{j_0})\sigma(u) \geq (r^* - r_{j_0})c_{\delta_0} > \epsilon, \quad \forall v \in \mathbf{V} \setminus B_{\delta_0}.$$

This and (3.32) imply

$$p(0) \leq p(v) - g(\bar{y}, v) + r^*\sigma(v), \quad \forall v \in \mathbf{V} \setminus B_{\delta_0}.$$

Thus this, combined with (3.29), yields that (3.30) holds. So we get (3.31).

Fix $r > r^*$, and define

$$h(u, v) := \bar{f}(u, v) + r\sigma(v) - g(\bar{y}, v).$$

It is obvious that

$$\inf_{u \in \mathbf{U}} h(u, v) = p(v) + r\sigma(v) - g(\bar{y}, v),$$
$$\inf_{v \in \mathbf{V}} h(u, v) = \bar{l}(u, \bar{y}, r),$$

and

$$\begin{aligned}
\mathrm{argmin}_{u,v} h(u, v) = \{(u', v') | u' &\in \mathrm{argmin}_u h(u, v'), \\
v' &\in \mathrm{argmin}_v p(v) + r\sigma(v) - g(\bar{y}, v)\} \\
= \{(u', v') | u' &\in \mathrm{argmin}_u \bar{l}(u, \bar{y}, r),\ v' \in \mathrm{argmin}_v h(u', v)\}.
\end{aligned} \tag{3.33}$$

For any $\bar{u} \in \mathrm{argmin}_u \varphi(u)$, we have

$$\bar{u} \in \mathrm{argmin}_u h(u, 0).$$

Since $\sigma(0) = 0$, using (3.30), we have

$$\mathrm{argmin}_{v \in \mathbf{V}} \{p(v) + r\sigma(v) - g(\bar{y}, v)\} = \{0\}. \tag{3.34}$$

It follow from (3.33) and (3.34) that

$$(u', 0) \in \mathrm{argmin}_{u,v} h(u, v).$$

So

$$u' \in \mathrm{argmin}_u \bar{l}(u, \bar{y}, r), \quad \forall r \geq r^*,$$

i.e.

$$\mathrm{argmin}_u \varphi(u) \subset \mathrm{argmin}_u \bar{l}(u, \bar{y}, r), \quad \forall r \geq r^*. \tag{3.35}$$

For any $u' \in \mathrm{argmin}_u \bar{l}(u, \bar{y}, r), \forall r \geq r^*$, we have

$$\inf_{v \in \mathbf{V}} h(u', v) = \bar{l}(u', \bar{y}, r) = \inf_{u \in \mathbf{U}} \bar{l}(u, \bar{y}, r) = \inf_{u \in \mathbf{U}} \inf_{v \in \mathbf{V}} h(u, v) = \inf_{v \in \mathbf{V}} (p(v) + r\sigma(v) - g(\bar{y}, v)).$$

This and (3.34) imply

$$\inf_{v \in \mathbf{V}} h(u', v) = p(0) = h(u', 0).$$

Thus, $0 \in \mathrm{argmin}_v h(u', v)$. Again from (3.33) that $u' \in \mathrm{argmin}_u h(u, 0)$, i.e.

$$u' \in \mathrm{argmin}_u \varphi(u).$$

So

$$\mathrm{argmin}_u \bar{l}(u, \bar{y}, r) \subset \mathrm{argmin}_u \varphi(u), \quad \forall r \geq r^*. \tag{3.36}$$

Therefore, from (3.35) and (3.36), we have proved that

$$\mathrm{argmin}_u \varphi(u) = \mathrm{argmin}_u \bar{l}(u, \bar{y}, r), \quad \forall r \geq r^*. \tag{3.37}$$

(3.31) and (3.37) imply that $\bar{y}$ supports an exact penalty representation for the problem (P). ▮

In the following, we apply the exact penalization representation results of the augmented Lagrangian scheme in Theorem 3.2.3 to the case of finite dimensional spaces and show that conditions required are weaker than the ones in [56] and [97].

In Definition 2.2.10, let $\mathbf{U} = \mathbf{R}^n$, $\mathbf{V} = \mathbf{R}^m$, $g(y, v) = \langle y, v \rangle$, where $\langle y, v \rangle$ denotes the inner product in $\mathbf{R}^m$. The primal problem (2.5) turns out to be:

$$\inf_{u \in \mathbf{R}^n} \varphi(u), \tag{3.38}$$

where $\varphi : \mathbf{R}^n \to \overline{\mathbf{R}}$ is an extended real-valued function. Let $\bar{f}$ be any dualizing parameterization function for φ, and σ be a valley at 0 augmenting function. The augmented Lagrangian (with parameter $r > 0$) $\bar{l} : \mathbf{R}^n \times \mathbf{R}^m \times (0, +\infty) \to \overline{\mathbf{R}}$ is defined by

$$\bar{l}(u, y, r) = \inf\{\bar{f}(u, v) - \langle y, v \rangle + r\sigma(v) : v \in \mathbf{R}^m\}, \quad u \in \mathbf{R}^n, y \in \mathbf{R}^m, r > 0.$$

We have the following result.

Lemma 3.2.4 *Assume that $\bar{f}(u,v)$ is proper, lower semicontinuous, and level-bounded in u locally uniformly in v. Suppose that one of following conditions is satisfied:*
(i) $\bar{y}$ supports an exact penalty representation for the problem (3.38). That is, there exists $\bar{r} > 0$ such that

$$p(0) = \inf_{x \in \mathbf{R}^n} \bar{l}(x, \bar{y}, r), \quad \forall r \geq \bar{r} \tag{3.39}$$

and

$$argmin_x \varphi(x) = argmin_x \bar{l}(x, \bar{y}, r), \quad \forall r \geq \bar{r}. \tag{3.40}$$

(ii) $p(0)$ is finite and there exists $\bar{r}' > 0$ such that

$$\bar{m}' = inf\{\bar{f}(u,v) - \langle \bar{y}, v \rangle + \bar{r}'\sigma(v) : (u,v) \in \mathbf{R}^n \times \mathbf{R}^m\} > -\infty. \tag{3.41}$$

Then, we have

$$p(0) = sup_{r \geq r_*} inf_{u \in \mathbf{R}^n} \bar{l}(u, \bar{y}, r) \tag{3.42}$$

where $r_* = max\{\bar{r}, \bar{r}'\}$.

Proof: If the condition (i) is satisfied, it is easy to see (3.42) holds.
If the condition (ii) is satisfied, we prove (3.42) by contradiction. It is clearly that the weak duality holds:

$$\bar{\psi}(y,r) \leq p(0), \quad \forall (y,r) \in \mathbf{R}^m \times (0, +\infty).$$

If (3.42) doesn't hold, then there exists $\epsilon_0 > 0$, such that

$$p(0) > \sup_{r \geq r_*} \inf_{u \in \mathbf{R}^n} \bar{l}(u, \bar{y}, r) + \epsilon_0.$$

Then there exist $u^k \in \mathbf{R}^n$ and $v^k \in \mathbf{R}^m$ such that

$$p(0) \geq \bar{f}(u^k, v^k) - \langle \bar{y}, v^k \rangle + r\sigma(v^k) + \epsilon_0, \quad \forall r \geq r_*. \tag{3.43}$$

Since σ has a valley at 0, $c_\delta = \inf_{\|v\| \geq \delta} \sigma(v) > 0$, for each $\delta > 0$. Denote $r_0 = \frac{p(0) - \epsilon_0 - \bar{m}'}{c_\delta} + r_*$. From (3.41) and (3.43), we have

$$\sigma(v^k) \leq \frac{p(0) - \epsilon_0 - \bar{m}'}{r - \bar{r}'} < c_\delta, \quad \forall r > r_0. \tag{3.44}$$

This implies $v^k \in \{v \in \mathbf{R}^m : \|v\| \leq \delta\}$. Because $\bar{f}(u,v)$ is level-bounded in u locally uniform in v, it follows from (3.43) that $\{u^k\}$ is bounded. So $\{(u^k, v^k)\}$ is bounded. Assume, without loss of generality, that $(u^k, v^k) \to (\bar{u}, \bar{v})$. Let $r_0 < r^k \to +\infty$, (3.44) implies $\bar{v} = 0$. The lsc of f and σ combined with (3.43) yields $p(0) \geq p(0) + \epsilon_0$. This is a contradiction. So, (3.42) holds. ∎

By using Lemma 3.2.4, we have

Theorem 3.2.4 *Assume that $\bar{f}(u, v)$ is proper, lower semicontinuous, and level-bounded in u locally uniformly in v. The following statements are true:*

(i) If $\bar{y}$ supports an exact penalty representation for the problem (3.38), then there exist $\bar{r} > 0$ and a neighborhood W of $0 \in \mathbf{R}^m$ such that

$$p(v) \geq p(0) + \langle \bar{y}, v \rangle - \bar{r}\sigma(v), \quad \forall v \in W. \tag{3.45}$$

(ii) The converse of (i) is true if the condition (ii) of Lemma 3.2.4 is satisfied.

Remark 3.2.2 In Theorem 3.2.4, we don't need to assume either that σ is a convex augmenting function ([97]) or there exist $\tau > 0$ and $N > 0$ such that $\sigma(v) \geq \tau \|v\|$ when $\|v\| \geq N$ ([56]). Thus Theorem 3.2.4 improves the corresponding results in [56] and [97].

Example 3.2.1 Consider the following simple problem:

$$\begin{aligned} &\inf\ u^2 \\ &\text{s.t. } u \in \mathbf{R}, u \leq 0. \end{aligned} \tag{3.46}$$

It is easy to see that $u = 0$ is the minimum point of (3.46). Define

$$\bar{f}(u, v) = \begin{cases} u^2, & \text{if } u \leq v\ ; \\ +\infty, & \text{otherwise.} \end{cases}$$

Then,

$$p(v) = \begin{cases} v^2, & \text{if } v \leq 0\ ; \\ 0, & \text{if } v > 0. \end{cases}$$

Let $\varphi(u) = \bar{f}(u, 0)$. Then $\inf_{u \leq 0} \varphi(u) = \inf_{u \leq 0} u^2 = p(0) = 0$. Let $g(y, v) = yv^\gamma$, $\sigma(v) = |v|^\gamma$, $\gamma > 0$.

$$\begin{aligned} \bar{l}(u, y, r) &= \inf\{\bar{f}(u, v) - g(y, v) + r\sigma(v) : v \in \mathbf{R}\} \\ &\geq \inf\{u^2 - y|v|^\gamma + r|v|^\gamma : u \leq v, v \in \mathbf{R}\}. \end{aligned}$$

Thus, there exist $\bar{y} \in \mathbf{R}$ and $\bar{r} > 0$, such that $u^2 - y|v|^\gamma + r|v|^\gamma > -\infty$, i.e., $\bar{l}(u, \bar{y}, \bar{r}) > -\infty$. For $\bar{y} \in \mathbf{R}$, there exists a $r' > 0$, such that

$$p(0) \leq p(v) - g(\bar{y}, v) + r'\sigma(v), \quad \forall v \in \mathbf{R}.$$

That is, the conditions (3.28) and (3.29) in Theorem 3.2.3 are satisfied. It is clear that there exists a $\bar{r} > r'$, such that

$$p(0) = \inf_{u \in \mathbf{U}} \bar{l}(u, \bar{y}, r) = 0, \quad \forall r \geq \bar{r},$$

and

$$\text{argmin}_u \varphi(u) = \text{argmin}_u \bar{l}(u, \bar{y}, r) = \{0\}, \quad \forall r \geq \bar{r}.$$

3.3 Inequality and equality constrained optimization problem

In this section, we will consider the following constrained optimization problem in infinite dimensional spaces:

$$(P_1)\qquad \begin{aligned} &\inf\ f(u)\\ &\text{s.t. } u \in X,\\ &\qquad g_j(u) \le 0, \quad j = 1, \cdots, m_1\\ &\qquad g_j(u) = 0, \quad j = m_1+1, \cdots, m, \end{aligned}$$

where $\mathbf{U}$ and $\mathbf{W}$ are two Banach spaces, $X \subset \mathbf{U}$ is a nonempty and closed set, $f, g_j(j = 1, \cdots, m_1) : X \to \mathbf{R}$, $g_j(j = m_1+1, \cdots, m) : X \to \mathbf{W}$. The optimal value of (P_1) is denoted by M_{P_1}. Denote by X_0 the set of feasible solutions of (P_1), i.e.,

$$X_0 = \{u \in X : g_j(u) \le 0, j = 1, \cdots, m_1; g_j(u) = 0, j = m_1+1, \cdots, m\}.$$

For the constrained problem (P_1), let

$$\varphi(u) = \begin{cases} f(u), & \text{if } u \in X_0,\\ +\infty, & \text{if } u \in \mathbf{U} \backslash X_0. \end{cases}$$

Then (P_1) is actually equivalent to the primal problem (P) in the sense that the two problems have the same set of (locally) optimal solutions and the same optimal value.

Define the dualizing parameterization function:

$$\bar{f}(u, v) = f(u) + \delta_{\mathbf{R}_-^{m_1} \times \{0_{m-m_1}\}}(G(u) + v) + \delta_X(u),\ u \in \mathbf{U},\ v \in \mathbf{R}^m, \tag{3.47}$$

where 0_{m-m_1} is the origin of $\mathbf{R}^{m-m_1}$, $G(u) = (g_1(u), \cdots, g_{m_1}(u), \|g_{m_1+1}(u)\|, \cdots, \|g_m(u)\|)$ and δ_D is the indicator function of the set D, i.e.,

$$\delta_D(u) = \begin{cases} 0, & \text{if } u \in D,\\ +\infty, & \text{otherwise.} \end{cases}$$

Let $\mathbf{V} = \mathbf{R}^m$, $y = (y_1, y_2, \cdots, y_m) \in \mathbf{R}^m, r > 0$, the augmented Lagrangian for (P) be

$$\bar{l}(u, y, r) = \inf\{\bar{f}(u, v) - \langle y, v\rangle + r\sigma(v) : v \in \mathbf{R}^m\},\ u \in \mathbf{U},$$

where $v = (v_1, v_2, \cdots, v_m)$. Let $g_j(u) + v_j = -w_j \in \mathbf{R}_-, j = 1, \cdots, m_1$, and $\|g_j(u)\| + v_j = 0, j = m_1 + 1, \cdots, m$. By some calculations, the above Lagrangian can be expressed as

$$\bar{l}(u, y, r) = \begin{cases} f(u) + \sum_{j=1}^{m_1} y_j g_j(u) + \sum_{j=m_1+1}^{m} y_j \|g_j(u)\| + \inf_{w \geq 0_{m_1}} \{\sum_{j=1}^{m_1} y_j w_j + r\sigma(-g_1(u) \\ -w_1, \cdots, -g_{m_1}(u) - w_{m_1}, -\|g_{m_1+1}(u)\|, \cdots, -\|g_m(u)\|)\}, \text{if } u \in X, \\ +\infty, \quad \text{otherwise} \end{cases} \tag{3.48}$$

where $w = (w_1, \cdots, w_{m_1})$, 0_{m_1} is the origin of $\mathbf{R}^{m_1}$.

Consider the augmented Lagrangian problem of the primal problem (P):

$$(Q_{y,r}) \qquad \inf_{(u,w) \in X \times \mathbf{R}_+^{m_1}} \{f(u) + \textstyle\sum_{j=1}^{m_1} y_j (g_j(u) + w_j) + \sum_{j=m_1+1}^{m} y_j \|g_j(u)\| \\ + r\sigma(-g_1(u) - w_1, \cdots, -g_{m_1}(u) - w_{m_1}, \\ -\|g_{m_1+1}(u)\|, \cdots, -\|g_m(u)\|)\}.$$

Actually, $(Q_{y,r})$ is the same as the problem of evaluating the augmented Lagrangian dual function $\bar{\psi}(y, r)$. If $u \in X_0$, and $w_j = -g_j(u) \geq 0 (j = 1, \cdots, m_1)$, then

$$\sigma(-g_1(u) - w_1, \cdots, -g_{m_1}(u) - w_{m_1}, -\|g_{m_1+1}(u)\|, \cdots, -\|g_m(u)\|) = 0.$$

Thus, we have

$$\bar{\psi}(y, r) \leq M_{P_1}. \tag{3.49}$$

Theorem 3.3.1 *Let* $\mathbf{U}$ *and* $\mathbf{W}$ *be two reflexive Banach spaces,* $X \subset \mathbf{U}$ *be a nonempty and closed set,* $\sigma : X \to \mathbf{R}^m$ *be a valley at* 0 *augmenting function, and let* $f : X \to \mathbf{R}$ *be a weakly lower semi-continuous function,* $g_j (j = 1, \cdots, m_1) : X \to \mathbf{R}$ *be bounded below and weakly continuous functions,* $g_j (j = m_1 + 1, \cdots, m) : X \to \mathbf{W}$ *be continuous operators from the weak topology of* $\mathbf{U}$ *to the topology of* $\mathbf{W}$*. Suppose* f *is level-bounded on* X*,* $y \in \mathbf{R}^m$*. Then*

(i) the problem (P_1) *has at least one solution* u_0*.*

(ii) there exists $r_0 > 0$*, such that for any* $r \geq r_0$*, the problem* $(Q_{y,r})$ *has at least one solution* $(u_r, w_r) \in X \times \mathbf{R}_+^{m_1}$*.*

(iii) every weak limit point of the sequence $\{u_r\}$ *is a solution of problem* (P_1)*.*

(iv)

$$\lim_{r\to\infty} f(u_r) = f(u_0) = M_{P_1}.$$

Proof: (i) There exists a minimizing sequence $u_k \in X_0$, satisfying

$$\lim_{k\to\infty} f(u_k) = \inf_{u\in X_0} f(u) = M_{P_1}.$$

Since f is level-bounded on X, $\{u_k\}$ is bounded. Note that $\mathbf{U}$ is a reflexive Banach space, there exists a weakly convergent subsequence of $\{u_k\}$. Without loss of generality, we may assume that $u_k \rightharpoonup u_0$. Thus,

$$f(u_0) \leq \liminf_{k\to\infty} f(u_k) = \inf_{u\in X_0} f(u),$$

$$g_j(u_0) = \lim_{k\to\infty} g_j(u_k) \leq 0 \quad (j = 1, \cdots, m_1),$$

and

$$g_j(u_0) = \lim_{k\to\infty} g_j(u_k) = 0 \quad (j = m_1+1, \cdots, m).$$

We conclude that u_0 is a solution of the problem (P_1).

(ii) We claim that f is bounded below on X. If not, there exist $u_k \in X$, such that $f(u_k) \to -\infty$. Since f is level-bounded on X, for any $\alpha \in R$, the set $\{u \in X : f(u) \leq \alpha\}$ is bounded. When k is large enough, $u_k \in \{u \in X : f(u) \leq \alpha\}$, thus $\{u_k\}$ is bounded. Because $\mathbf{U}$ is a reflexive Banach space and f is a weakly lower semicontinuous function, without loss of generality, we may assume that $u_k \rightharpoonup u_* \in X$ and

$$\liminf_{k\to\infty} f(u_k) \geq f(u_*).$$

This is impossible since $f(u_k) \to -\infty$. Hence f is bounded below on X, that is, there exists $d_0 \in \mathbf{R}$, such that

$$f(u) \geq d_0, \quad \forall u \in X. \tag{3.50}$$

Since σ has a valley at 0, $c_1 = \inf_{\|v\|\geq 1} \sigma(v) > 0$. Define $r_0 = \frac{M_{P_1} - d_0}{c_1}$, and let $r > r_0$. Suppose that $(u_k, w_k) \in X \times \mathbf{R}_+^{m_1}$ is a minimizing sequence such that

$$\begin{aligned}\lim\nolimits_{k\to\infty} &(f(u_k) + \textstyle\sum_{j=1}^{m_1} y_j(g_j(u_k) + w_j^k) + \sum_{j=m_1+1}^{m} y_j\|g_j(u_k)\| \\ &+ r\sigma(-g_1(u_k) - w_1^k, \cdots, -g_{m_1}(u_k) - w_{m_1}^k, \\ &-\|g_{m_1+1}(u_k)\|, \cdots, -\|g_m(u_k)\|)) = \bar{\psi}(y, r),\end{aligned} \tag{3.51}$$

where $w_k = (w_1^k, w_2^k, \cdots, w_{m_1}^k)$. Since $g_j (j = 1, \cdots, m_1) : X \to \mathbf{R}$ is bounded below and f is level-bounded on X, by (3.49) and (3.51), $\{u_k\}$ is bounded. We may assume that $u_k \rightharpoonup u_r$ as $k \to \infty$ and

$$\liminf_{k\to\infty} f(u_k) \geq f(u_r) \geq d_0. \tag{3.52}$$

We will prove that $\{w_k\} \subset \mathbf{R}_+^{m_1}$ is bounded. If $\{w_k\}$ is unbounded, by the weak continuity of $g_j (j = 1, \cdots, m_1)$, we have

$$\lim_{k\to\infty} (g_j(u_k) + w_j^k) = +\infty, \quad \text{as } k \to \infty.$$

Let k be large enough such that $g_j(u_k) + w_j^k \geq 1$. It follows from (3.49) and (3.51) that

$$\begin{aligned} M_{P_1} + \tfrac{1}{k} &\geq \bar{\psi}(y, r) + \tfrac{1}{k} \\ &\geq f(u_k) + r\sigma(-g_1(u_k) - w_1^k, \cdots, -g_{m_1}(u_k) - w_{m_1}^k, \\ &\quad -\|g_{m_1+1}(u_k)\|, \cdots, -\|g_m(u_k)\|) \\ &\geq f(u_k) + rc_1, \end{aligned} \tag{3.53}$$

where $c_1 = \inf_{\|v\|\geq 1} \sigma(v) > 0$. Thus, by (3.52) and (3.53), we get

$$M_{P_1} \geq d_0 + rc_1,$$

which is impossible since $r > r_0 = \frac{M_{P_1} - d_0}{c_1}$. Hence, $\{w_k\}$ is bounded. We may assume that $w_k \to w_r$ as $k \to \infty$. Therefore,

$$\begin{aligned} \bar{\psi}(y, r) &\leq f(u_r) + \textstyle\sum_{j=1}^{m_1} y_j (g_j(u_r) + w_j^r) + \sum_{j=m_1+1}^{m} y_j \|g_j(u_r)\| \\ &\quad + r\sigma(-g_1(u_r) - w_1^r, \cdots, -g_{m_1}(u_r) - w_{m_1}^r, -\|g_{m_1+1}(u_r)\|, \cdots, -\|g_m(u_r)\|) \\ &\leq \liminf_{k\to\infty} (f(u_k) + \textstyle\sum_{j=1}^{m_1} y_j (g_j(u_k) + w_j^k) + \sum_{j=m_1+1}^{m} y_j \|g_j(u_k)\| \\ &\quad + r\sigma(-g_1(u_k) - w_1^k, \cdots, -g_{m_1}(u_k) - w_{m_1}^k, -\|g_{m_1+1}(u_k)\|, \cdots, -\|g_m(u_k)\|)) \\ &= \bar{\psi}(y, r). \end{aligned}$$

Thus, the problem $(Q_{y,r})$ has at least one solution (u_r, w_r) as $r > r_0$.

(iii) Noting that u_0 is the solution of the problem (P_1) and (u_r, w_r) is the solution of the problem $(Q_{y,r})$, we have

$$\begin{aligned} &f(u_r) + \textstyle\sum_{j=1}^{m_1} y_j (g_j(u_r) + w_j^r) + \sum_{j=m_1+1}^{m} y_j \|g_j(u_r)\| + r\sigma(-g_1(u_r) - w_1^r, \cdots, \\ &-g_{m_1}(u_r) - w_{m_1}^r, -\|g_{m_1+1}(u_r)\|, \cdots, -\|g_m(u_r)\|) \leq f(u_0). \end{aligned} \tag{3.54}$$

Similar to the proof of (ii), by using (3.54), we can prove that $\{(u_r, w_r)\}$ is bounded in $X \times \mathbf{R}_+^{m_1}$. Thus we may assume that $u_r \rightharpoonup u^*$ and $w_r \to w^*$ as $r \to +\infty$. From (3.54), we have

$$\lim_{r\to\infty} \sigma(-g_1(u_r) - w_1^r, \cdots, -g_{m_1}(u_r) - w_{m_1}^r, -\|g_{m_1+1}(u_r)\|, \cdots, -\|g_m(u_r)\|) = 0.$$

Noting that σ is a valley at 0 augmenting function, we have

$$\begin{array}{l} \lim_{r\to\infty}(g_j(u_r) + w_j^r) = 0, \quad j = 1, 2, \cdots, m_1, \\ \lim_{r\to\infty} \|g_j(u_r)\| = 0, \quad j = m_1 + 1, \cdots, m. \end{array} \tag{3.55}$$

Since f is a weakly lower semi-continuous function, $g_j (j = 1, \cdots, m_1)$ are weakly continuous functions, $g_j (j = m_1 + 1, \cdots, m)$ are continuous operators from the weak topology of $\mathbf{U}$ to the topology of $\mathbf{W}$, by (3.54) and (3.55), we get

$$f(u^*) \le \liminf_{r\to+\infty} f(u_r) \le f(u_0),$$

$$g_j(u^*) = \lim_{r\to\infty} g_j(u_j^r) = -w_j^* \le \lim_{r\to+\infty} -w_j^r \le 0 \quad (j = 1, \cdots, m_1),$$

and

$$g_j(u^*) = \lim_{r\to\infty} g_j(u_j^r) = 0 \quad (j = m_1 + 1, \cdots, m).$$

Since u_0 is a solution of (P_1), these inequalities above imply that u^* is a solution of (P_1).

(iv) Suppose that $l \in \mathbf{R}$ is a limit point of sequence $\{f(u_r)\}$. That is, there exists a subsequence $\{u_{r_k}\}$ of $\{u_r\}$ such that

$$\lim_{k\to\infty} f(u_{r_k}) = l.$$

By using (3.54), $\{u_{r_k}\}$ and $\{w_{r_k}\}$ are bounded. Without loss of generality, we assume that $u_{r_k} \rightharpoonup u'$. Similar to the proof of (iii), we have

$$f(u') \le f(u_0),$$

$$g_j(u') \le 0 \quad (j = 1, \cdots, m_1),$$

and

$$g_j(u') = 0 \quad (j = m_1 + 1, \cdots, m).$$

Since u_0 is a solution of (P_1), u' is a solution of (P_1). Hence, $f(u') = f(u_0) = M_{P_1}$. The weak lower semicontinuity of f implies $f(u') \le \lim_{k\to\infty} f(u_{r_k}) = l$. Again by (3.54), we have $\limsup_{k\to\infty} f(u_{r_k}) \le f(u_0)$. Thus,

$$\lim_{k\to\infty} f(u_{r_k}) = l = f(u_0) = f(u') = M_{P_1}.$$

Therefore,

$$\lim_{r\to+\infty} f(u_r) = f(u_0) = M_{P_1}.$$

∎

Theorem 3.3.2 *Let* $\mathbf{U}$ *and* $\mathbf{W}$ *be two reflexive Banach spaces,* $X \subset \mathbf{U}$ *be a nonempty and closed set,* $\sigma : X \to \mathbf{R}^m$ *be a valley at* 0 *augmenting function, and let* $f : X \to \mathbf{R}$ *be a level-bounded and weakly lower semi-continuous function,* $g_j(j = 1, \cdots, m_1) : X \to \mathbf{R}$ *be bounded below and weakly continuous functions,* $g_j(j = m_1 + 1, \cdots, m) : X \to \mathbf{W}$ *be continuous operators from the weak topology of* $\mathbf{U}$ *to the topology of* $\mathbf{W}$*. Suppose that* $\bar{y} \in \mathbf{R}^m$*, and that there exists a* $r_* > 0$ *and a neighborhood* B *of* $0 \in \mathbf{R}^m$ *such that*

$$M_{P_1} \le \inf_{v\in B}(p(v) - \langle \bar{y}, v\rangle + r_*\sigma(v)). \tag{3.56}$$

Then there exists a $r^* > 0$ *such that*

$$M_{P_1} = \inf\{L(u, \bar{y}, r) : u \in X\}, \quad \forall r > r^*,$$

and

$$argmin_{u\in X_0} f(u) = argmin_{u\in X} L(u, \bar{y}, r^*), \quad \forall r > r^*.$$

Proof: By Theorem 3.3.1, there exists $r_0 > \bar{r}$, such that for any $r \ge r_0$, the problem $(Q_{\bar{y},r})$ has at least one solution $(u_r, w_r) \in X \times \mathbf{R}_+^{m_1}$, and

$$\lim_{r\to\infty} f(u_r) = M_{P_1}.$$

Thus

$$\begin{aligned}\bar{\psi}(\bar{y}, r) = {} & f(u_r) + \textstyle\sum_{j=1}^{m_1} y_j(g_j(u_r) + w_j^r) + \sum_{j=m_1+1}^{m} y_j\|g_j(u_r)\| \\ & + r\sigma(-g_1(u_r) - w_1^r, \cdots, -g_{m_1}(u_r) - w_{m_1}^r, -\|g_{m_1+1}(u_r)\|, \cdots, -\|g_m(u_r)\|).\end{aligned}$$

Denote

$$v_j^r = -(g_j(u_r) + w_j^r), j = 1, \cdots, m_1,$$

and

$$v_j^r = -\|g_j(u_r)\|, j = m_1 + 1, \cdots, m.$$

Then

$$G(u_r) + v_r \in \mathbf{R}_-^{m_1} \times \{0_{m-m_1}\},$$

where 0_{m-m_1} is the origin of $\mathbf{R}^{m-m_1}$, $G(u_r) = (g_1(u_r), \cdots, g_{m_1}(u_r), \|g_{m_1+1}(u_r)\|, \cdots, \|g_m(u_r)\|)$, $v_r = (v_1^r, v_2^r, \cdots, v_m^r)$. By (3.47), we have $\bar{f}(u_r, v_r) = f(u_r)$ and

$$\bar{\psi}(\bar{y}, r) = \bar{f}(u_r, v_r) - \langle \bar{y}, v_r \rangle + r\sigma(v_r).$$

It follows from (3.55) that $\lim_{r\to\infty} v_r = 0$. Hence

$$\limsup_{r\to\infty} \bar{\psi}(\bar{y}, r) \geq \lim_{r\to\infty} f(u_r) = M_{P_1}.$$

Therefore, for $\forall \epsilon > 0$, there exists $r_0 > 0$, such that

$$M_{P_1} \leq \bar{\psi}(\bar{y}, r_0) + \epsilon.$$

That is,

$$M_{P_1} \leq p(v) + \langle \bar{y}, v \rangle + r_0\sigma(v) + \epsilon, \quad \forall v \in \mathbf{R}^m.$$

In condition (3.56), without loss of generality, we may take $B = B_\delta = \{u \in \mathbf{R}^m, \|u\| \leq \delta\}$, $\delta > 0$. Since σ has a valley at 0 in $\mathbf{R}^m$, $c_\delta = \inf_{v\in\mathbf{R}^m\setminus B_\delta} \sigma(v) > 0$. Let $r^* > \max\{r_0, r_*\}$ large enough such that

$$\epsilon + r_0\sigma(v) \leq r\sigma(v), \quad \forall v \in \mathbf{R}^m\setminus B_\delta, \forall r > r^*.$$

Thus, for each $r > r^*$,

$$M_{P_1} \leq p(v) + \langle \bar{y}, v \rangle + r\sigma(v). \quad \forall v \in \mathbf{R}^m\setminus B_\delta.$$

This and (3.56) imply

$$M_{P_1} \leq p(v) + \langle \bar{y}, v \rangle + r\sigma(v), \quad \forall v \in \mathbf{R}^m.$$

Hence,

$$M_{P_1} = \inf_{u\in X} L(u, \bar{y}, r), \quad \forall r > r^*.$$

If $u^* \in \operatorname{argmin}_{u\in X_0} f(u)$, then, for each $r > r^*$, $f(u^*) = M_{P_1} = \inf\{L(u, \bar{y}, r) : u \in X\}$. Since $u^* \in X_0$, $G(u^*) \in \mathbf{R}_-^{m_1} \times \{0_{m-m_1}\}$,

$$\begin{aligned} \bar{f}(u^*, 0) &= f(u^*) \\ &= \inf\{L(u, \bar{y}, r) : u \in X\} \\ &= \inf\{\bar{f}(u, v) - \langle \bar{y}, v \rangle + r\sigma(v) : u \in \mathbf{U}, v \in \mathbf{R}^m\}. \end{aligned}$$

Thus,

$$(u^*, 0) \in \operatorname{argmin}_{(u,v)\in\mathbf{U}\times\mathbf{R}^m}(\bar{f}(u, v) - \langle \bar{y}, v \rangle + r\sigma(v)).$$

This implies

$$\bar{f}(u^*,0)=\inf\{\bar{f}(u^*,v)-\langle \bar{y},v\rangle+r\sigma(v):v\in\mathbf{R}^m\}=L(u^*,\bar{y},r).$$

Therefore,

$$L(u^*,\bar{y},r)=\inf_{u\in X}L(u,\bar{y},r).$$

That is, $u^*\in \mathrm{argmin}_{u\in X}L(u,\bar{y},r)$.

On the other hand, if $u^*\in \mathrm{argmin}_{u\in X}L(u,\bar{y},r)$, $r>r^*$, then $L(u^*,\bar{y},r)=\inf_{u\in X}L(u,\bar{y},r)=M_{P_1}$, i.e.,

$$\begin{aligned}&\inf\{\bar{f}(u^*,v)-\langle \bar{y},v\rangle+r\sigma(v):v\in\mathbf{R}^m\}\\&=\inf\{\bar{f}(u,v)-\langle \bar{y},v\rangle+r\sigma(v):u\in\mathbf{U},v\in\mathbf{R}^m\}\\&=\inf\nolimits_{u\in X_0}f(u).\end{aligned}$$

Noting that $\inf_{u\in X_0}f(u)=\inf_{u\in X}\bar{f}(u,0)$, we get

$$(u^*,0)\in \mathrm{argmin}_{(u,v)\in\mathbf{U}\times\mathbf{R}^m}(\bar{f}(u,v)-\langle \bar{y},v\rangle+r\sigma(v)).$$

Hence, $u^*\in X_0$ and $f(u^*)=M_{P_1}$. That is, $u^*\in \mathrm{argmin}_{u\in X_0}f(u)$. Therefore,

$$\mathrm{argmin}_{u\in X_0}f(u)=\mathrm{argmin}_{u\in X}L(u,\bar{y},r),\quad \forall r>r^*.$$

∎

Let the function p be defined by (2.8). The quantity

$$\liminf_{v\to 0}p(v)\tag{3.57}$$

is called the asymptotic optimal value for the primal problem (P). It can also be described as the minimum of

$$\limsup_{k\to\infty}f(u_k)\tag{3.58}$$

over all asymptotically feasible sequences $\{u_k\}$ for the constrained problem (P_1): that is, sequences in X satisfying

$$\limsup_{k\to\infty}g_j(u_k)\le 0,\quad \text{for } j=1,\cdots,m_1,\tag{3.59}$$

and

$$\lim_{k\to\infty}g_j(u_k)=0,\quad \text{for } j=m_1+1,\cdots,m.\tag{3.60}$$

Similar to [95], a sequence $\{u_k\}$ is called asymptotically minimizing for the constrained problem (P_1) if it is asymptotically feasible and yields the minimum possible value for (3.58).

Theorem 3.3.3 *Let* $\mathbf{U}$ *and* $\mathbf{W}$ *be two Banach spaces,* $X \subset \mathbf{U}$ *be a nonempty and closed set,* σ *be a valley at* 0 *augmenting function,* $\{(y_k, r_k)\}$ *be a sequence such that, for some* $\delta > 0$,

$$\lim_{k\to\infty} \bar{\psi}(y_k, r_k - \delta) = \sup_{(y,r)\in\mathbf{V}^*\times(0,+\infty)} \bar{\psi}(y,r) < +\infty. \tag{3.61}$$

Let $u_k \in X$ *satisfy*

$$\bar{l}(u_k, y_k, r_k) \le \inf_{u\in X} \bar{l}(u, y_k, r_k) + \alpha_k, \tag{3.62}$$

where $\alpha_k \to 0$. *Then,* $\{u_k\}$ *is asymptotically feasible. Moreover, if* $\{y_k\}$ *is bounded, then* $\{u_k\}$ *is an asymptotically minimizing sequence for* (P_1).

Proof: From (3.61) and (3.62), we have

$$\bar{l}(u_k, y_k, r_k) \le \bar{\psi}(y_k, r_k) + \alpha_k \le \sup_{(y,r)\in\mathbf{V}^*\times(0,+\infty)} \bar{\psi}(y,r) + \alpha_k < +\infty. \tag{3.63}$$

This and (3.48) imply $u_k \in X$ and

$$\begin{aligned}\bar{l}(u_k, y_k, r_k) = {} & f(u_k) + \textstyle\sum_{j=1}^{m_1} y_j^k g_j(u_k) + \sum_{j=m_1+1}^{m} y_j^k \|g_j(u_k)\| + \inf_{w\ge 0}\{\sum_{j=1}^{m_1} y_j^k w_j + \\ & r_k\sigma(-g_1(u_k) - w_1, \cdots, -g_{m_1}(u_k) - w_{m_1}, -\|g_{m_1+1}(u_k)\|, \cdots, -\|g_m(u_k)\|)\} \\ & \ge \bar{\psi}(y_k, r_k - \delta) + \delta \inf_{w\ge 0}\{\sigma(-g_1(u_k) - w_1, \cdots, \\ & -g_{m_1}(u_k) - w_{m_1}, -\|g_{m_1+1}(u_k)\|, \cdots, -\|g_m(u_k)\|)\}.\end{aligned} \tag{3.64}$$

(3.64), combined with (3.61) and (3.63), yields that

$$\begin{aligned}& \inf_{w\ge 0}\{\sigma(-g_1(u_k) - w_1, \cdots, -g_{m_1}(u_k) - w_{m_1}, -\|g_{m_1+1}(u_k)\|, \cdots, -\|g_m(u_k)\|)\} \\ & \le \tfrac{1}{\delta}\Big(\sup_{(y,r)\in\mathbf{V}^*\times(0,+\infty)} \bar{\psi}(y,r) - \bar{\psi}(y_k, r_k - \delta) + \alpha_k\Big) \to 0, \quad \text{as } k \to \infty,\end{aligned} \tag{3.65}$$

and

$$\lim_{k\to 0} \bar{l}(u_k, y_k, r_k) = \sup_{(y,r)\in\mathbf{V}^*\times(0,+\infty)} \bar{\psi}(y,r). \tag{3.66}$$

Since σ has a valley at 0 in $\mathbf{R}^m$, from (3.65), there exist $w_k = (w_1^k, \cdots, w_{m_1}^k) \ge 0$, such that

$$g_j(u_k) + w_j^k \to 0, \quad j = 1, \cdots, m_1,$$

and

$$\|g_j(u_k)\| \to 0, \quad j = m_1 + 1, \cdots, m.$$

Thus

$$\limsup_{k\to 0} g_j(u_k) \le -\liminf_{k\to 0} w_j^k \le 0, \quad j = 1, \cdots, m_1,$$

and

$$\lim_{k\to 0} g_j(u_k) = 0, \quad j = m_1+1, \cdots, m.$$

That is, $\{u_k\}$ satisfies (3.59) and (3.60). Therefore, $\{u_k\}$ is asymptotically feasible. (3.63) and (3.64) imply

$$\begin{aligned}\liminf_{k\to\infty} f(u_k) &\le \liminf_{k\to\infty} \bar{l}(u_k, y_k, r_k)\\ &\le \limsup_{k\to\infty}\Big(\sup_{(y,r)\in\mathbf{V}^*\times(0,+\infty)} \bar{\psi}(y,r) + \alpha_k\Big)\\ &= \sup_{(y,r)\in\mathbf{V}^*\times(0,+\infty)} \bar{\psi}(y,r).\end{aligned}$$

On the other hand,

$$\begin{aligned}\bar{l}(u_k, y, r) &= \inf\{\bar{f}(u_k, v) - \langle y, v\rangle + r\sigma(v) : v \in \mathbf{R}^m\}\\ &\le \bar{f}(u_k, 0) - \langle y, 0\rangle + r\sigma(0)\\ &= f(u_k).\end{aligned}$$

Thus,

$$\limsup_{k\to\infty} f(u_k) \ge \sup_{(y,r)\in\mathbf{V}^*\times(0,+\infty)} \bar{\psi}(y,r).$$

Therefore, $\lim_{k\to\infty} f(u_k) = \sup\limits_{(y,r)\in\mathbf{V}^*\times(0,+\infty)} \bar{\psi}(y,r)$. Moreover, if $\{y_k\}$ is bounded, then (3.64) implies

$$\lim_{k\to 0} f(u_k) = \sup_{(y,r)\in\mathbf{V}^*\times(0,+\infty)} \bar{\psi}(y,r).$$

Hence, $\{u_k\}$ is an asymptotically minimizing sequence for (P_1). ∎

Theorem 3.3.3 generalizes Theorem 3 of [95] to the non-quadratic case in infinite dimensional Banach spaces.

3.4 Exact penalty representation for the constrained problem in the finite dimensional spaces

In this section, we will consider the following constrained optimization problem in finite dimensional spaces:

$$(P_1) \qquad \begin{aligned}&\inf\ f(x)\\ &\text{s.t. } x \in X,\\ &\qquad g_j(x) \le 0, \quad j = 1, \cdots, m_1\\ &\qquad g_j(x) = 0, \quad j = m_1+1, \cdots, m,\end{aligned}$$

where $X \subset \mathbf{R}^n$ is a nonempty and closed set, $f, g_j(j = 1, \cdots, m_1) : X \to \mathbf{R}$ are lsc real valued functions, $g_j(j = m_1 + 1, \cdots, m) : X \to \mathbf{R}$ are real valued continuous functions. In this section, we reformulate the problem (P_1) into an optimization problem with a single constraint and a modified objective function. We then introduce a linear Lagrangian function for the reformulated optimization problem and establish a sufficient condition of an exact penalization representation for the reformulated constrained optimization problem.

The optimal value of (P_1) is denoted by M_{P_1}. Denote by X_0 the set of feasible solutions of (P_1), i.e.,

$$X_0 = \{x \in X : g_j(x) \leq 0, j = 1, \cdots, m_1; g_j(x) = 0, j = m_1 + 1, \cdots, m\}.$$

Consider the absolute-value penalty function

$$g(x) = \sum_{j=1}^{m_1} g_j^+(x) + \sum_{j=m_1+1}^{m} |g_j(x)|, \tag{3.67}$$

where $g_j^+(x) = \max(0, g_j(x))$. It is clear that $g(x) = 0$ if and only if $x \in X_0$. Denote $f_0(x) = f(x) + \rho(x)g(x)$, where $\rho : X \to \mathbf{R}^+$ is a lsc real valued function. Consider the following reformulated optimization problem with a single constraint:

$$(P_1^*) \qquad \begin{array}{c} \inf f_0(x) \\ \text{s.t } x \in X,\ g(x) \leq 0, \end{array}$$

We consider the classical linear penalty function for this problem,

$$L(x, r) = f_0(x) + rg(x) = f(x) + (r + \rho(x))g(x),$$

and the dual function

$$\psi(r) = \inf\{L(x, r) : x \in X\},\ r > 0.$$

Then,

$$\begin{aligned} \psi(r) &= \inf\{L(x, r) : x \in X\} \\ &= \inf\{f(x) + (r + \rho(x))g(x) : x \in X\} \\ &= \inf\{f(x) + (r + \rho(x))\sum_{j=1}^{m} |u_j| : x \in X, u \in \mathbf{R}^m \\ &\qquad g_j^+(x) = u_j, j = 1, \cdots, m_1; g_j(x) = u_j, j = m_1 + 1, \cdots, m\}. \end{aligned} \tag{3.68}$$

We have $\psi(r) \le \inf\{L(x,r) : x \in X_0\} = M_{P_1}$. Denote by S_r the set of optimal solutions of problem $\inf\{L(x,r) : x \in X\}$. Let $u = (u_1, u_2, \cdots, u_m) \in \mathbf{R}^m$. Consider also the perturbed problem of the original constrained optimization problem (P_1):

$$(P_u) \qquad \inf_{x \in Z(u)} f(x),$$

where $Z(u) = \{x \in X : g_j(x) \le u_j, j = 1, \cdots, m_1; g_j(x) = u_j, j = m_1 + 1, \cdots, m\}$. Denote by $\beta(u)$ the optimal value of problem (P_u) and $\beta_0(u) = \inf_{x \in Z(u)} f_0(x)$.

Lemma 3.4.1 *Suppose that*

1). $f(x) \ge c > -\infty, \quad \forall x \in X.$

2). $\lim\limits_{\|x\| \to \infty} g(x) > 0.$

3). $\lim\limits_{\|x\| \to \infty} \rho(x) = +\infty.$

4). $X_0 \ne \phi$ *and* M_{P_1} *is finite.*

Then, S_r *is nonempty and compact whenever* $r > 0$*, and*

$$\lim_{r \to +\infty} \psi(r) = M_{P_1}.$$

Proof: By the definition, $\psi(r) < +\infty$. Fix $r > 0$, there exists a minimizing sequence $\{x_j\} \subset X$, satisfying

$$f(x_j) + (r + \rho(x_j))g(x_j) \le \psi(r) + \frac{1}{j}.$$

We claim that $\{x_j\}$ is bounded. Otherwise, we may assume $\|x_j\| \to \infty$ as $j \to \infty$. By conditions 1), 2) and 3), we have $\lim_{j \to \infty} g(x_j) > 0$, $\lim_{j \to \infty} \rho(x_j) = +\infty$. Thus

$$f(x_j) + (r + \rho(x_j)g(x_j)) \to +\infty.$$

This is impossible since $\psi(r) < +\infty$. Therefore $\{x_j\}$ is bounded. Without loss of generality, we may assume that $x_j \to x_r$. Therefore,

$$\psi(r) \le f(x_r) + (r + \rho(x_r))g(x_r) \le \liminf_{j \to \infty}(f(x_j) + (r + \rho(x_j))g(x_j)) \le \psi(r).$$

Thus,

$$\psi(r) = f(x_r) + (r + \rho(x_r))g(x_r), \tag{3.69}$$

that is, $x_r \in S_r$. Let $x_0 \in X_0$. Denote

$$M_r = \{x \in X : L(x, r) \le L(x_0, r) = f(x_0)\}.$$

Clearly, $S_r \subset M_r$. We show that M_r is compact for any $r > 0$. Suppose to the contrary that there exist $0 < r_k \to +\infty$ and $x_k \in M_r$ such that $\|x_k\| \to +\infty$. By conditions 1), 2) and 3), we have

$$f(x_0) \ge L(x_k, r) = f(x_k) + (r + \rho(x_k) g(x_k)) \to +\infty,$$

which is impossible. Thus, S_r is nonempty and compact for $r > 0$.

Ler $r_0 > 0$, $r > r_0$. By (3.69),

$$\psi(r_0) \le f(x_r) + (r_0 + \rho(x_r)) g(x_r) = \psi(r) + (r_0 - r) g(x_r) \le M_{P_1} + (r_0 - r) g(x_r).$$

Hence

$$g(x_r) \le \frac{M_{P_1} - \psi(r_0)}{r - r_0}.$$

Therefore, $g(x_r) \to 0$ as $r \to +\infty$. Set

$$u_j^r = g_j^+(x_r), j = 1, \cdots, m_1; u_j^r = g_j(x_r), j = m_1 + 1, \cdots, m.$$

Let $u_r = (u_1^r, \cdots, u_m^r)$. Then we have

$$\|u_r\| = \sum_{j=1}^{m} |u_j^r| = g(x_r) \to 0, \tag{3.70}$$

as $r \to +\infty$. Clearly, $x_r \in Z(u_r)$. Thus,

$$\beta(u_r) \le f(x_r). \tag{3.71}$$

We claim that

$$\liminf_{r \to +\infty} \beta(u_r) \ge M_{P_1}. \tag{3.72}$$

On the contrary, suppose that there exists an $\epsilon_0 > 0$ such that

$$\liminf_{r \to +\infty} \beta(u_r) \le M_{P_1} - \epsilon_0.$$

Then, there exists a subsequence $\{x'_{r_k}\} \subset Z(u_{r_k})$ such that

$$f(x'_{r_k}) \le M_{P_1} - \frac{\epsilon_0}{2}.$$

It follows from condition 2) and (3.70) that $\{x'_{r_k}\}$ is bounded. We may assume that $x'_{r_k} \to x_0$ as $k \to \infty$. Then $x_0 \in X_0$. Hence,

$$M_{P_1} \leq f(x_0) \leq \liminf_{k\to\infty} f(x'_{r_k}) \leq M_{P_1} - \frac{\epsilon_0}{2}.$$

It is a contradiction. By (3.69), (3.71) and (3.72), we get

$$\liminf_{r\to+\infty} \psi(r) \geq \liminf_{r\to+\infty} f(x_r) \geq M_{P_1}.$$

Since $\psi(r) \leq M_{P_1}$,

$$\lim_{r\to+\infty} \psi(r) = M_{P_1}.$$

∎

Theorem 3.4.1 *Suppose that all conditions in Lemma 3.4.1 are satisfied. Then the following two statements are equivalent:*

(i) there exists a $\bar{r} > 0$ such that

$$M_{P_1} = \inf\{L(x,r) : x \in X\}, \quad \forall r > \bar{r},$$

and

$$argmin_{x\in X_0} f(x) = argmin_{x\in X} L(x,r), \quad \forall r > \bar{r},$$

(ii) there exist a $r' > 0$ and a neighborhood U of $0 \in \mathbf{R}^m$ such that

$$\begin{aligned} M_{P_1} \leq \inf\{f(x) + (r+\rho(x))\sum_{j=1}^{m} |u_j| : x \in X, u \in U \\ g_j^+(x) = u_j, j = 1,\cdots,m_1; g_j(x) = u_j, j = m_1+1,\cdots,m\}, \end{aligned} \tag{3.73}$$

where $r > r'$.

Proof: Suppose (i) holds. From (3.68), we have,

$$\begin{aligned} M_{P_1} &= \inf\{L(x,\bar{r}) : x \in X\} \\ &\leq \inf\{f(x) + (r+\rho(x))\sum_{j=1}^{m} |u_j| : x \in X, u \in \mathbf{R}^m \\ &\quad g_j^+(x) = u_j, j = 1,\cdots,m_1; g_j(x) = u_j, j = m_1+1,\cdots,m\}, \end{aligned}$$

where $r > \bar{r}$. That is, (ii) holds.

Suppose (ii) holds. We prove that (i) holds in the following. By Lemma 3.4.1, for $\forall \epsilon > 0$, there exists $r_* > 0$, such that

$$|\psi(r_*) - M_{P_1}| < \epsilon.$$

By (3.68), we have

$$M_{P_1} \leq f(x) + (r_* + \rho(x)) \sum_{j=1}^{m} |u_j| + \epsilon, \quad \forall x \in X, \ \forall u \in \mathbf{R}^m, \tag{3.74}$$

where $u_j = g_j^+(x), j = 1, \cdots, m_1; u_j = g_j(x), j = m_1 + 1, \cdots, m$. In assuming (3.73), without loss of generality, we may take $U = B_s = \{u \in \mathbf{R}^m, \|u\| \ \leq s\}, \ s > 0$. There exists a $\bar{r} > \max(r_*, r')$, such that

$$\epsilon \leq (\bar{r} - r_*)\|u\|, \quad \forall u \in \mathbf{R}^m \backslash B_s.$$

Hence, from (3.73) and (3.74), we have

$$M_{P_1} \leq f(x) + (\bar{r} + \rho(x)) \sum_{j=1}^{m} |u_j|, \quad \forall x \in X, \ \forall u \in \mathbf{R}^m, \tag{3.75}$$

where $u_j = g_j^+(x), j = 1, \cdots, m_1; u_j = g_j(x), j = m_1 + 1, \cdots, m$. Consequently,

$$M_{P_1} = \inf\{L(x, r) : x \in X\}, \quad \forall r > \bar{r}.$$

If $x^* \in \mathrm{argmin}_{x \in X_0} f(x)$, then $f(x^*) = M_{P_1} = \inf\{L(x, r) : x \in X\}, \ \forall r > \bar{r}$. Since $x^* \in X_0$, $g(x^*) = 0$, $L(x^*, r) = f(x^*) = \inf\{L(x, r) : x \in X\}, \ \forall r > \bar{r}$. Thus, $x^* \in \mathrm{argmin}_{x \in X} L(x, r), \ \forall r > \bar{r}$. On the other hand, if $x^* \in \mathrm{argmin}_{x \in X} L(x, r), \ \forall r > \bar{r}$, then $L(x^*, r) = \inf\{L(x, r) : x \in X\} = M_{P_1}$, i.e., $f(x^*) + (r + \rho(x^*))g(x^*) = M_{P_1}, \ \forall r > \bar{r}$. We will show by contradiction that x^* must be feasible for problem (P). If x^* is infeasible then $g(x^*) > 0$. Choose a $x_0 \in X_0$ and let

$$r > \max\{\frac{f(x_0) - f(x^*)}{g(x^*)}, \bar{r}\}.$$

We then have

$$f(x_0) = f(x_0) + (r + \rho(x_0))g(x_0) \geq f(x^*) + (r + \rho(x^*))g(x^*) > f(x_0).$$

This gives a contradiction and hence $x^* \in X_0$. Thus, $M_{P_1} \leq f(x^*) = f(x^*) + (r + \rho(x^*))g(x^*) = M_{P_1}$ $(\forall r > \bar{r})$, that is, $f(x^*) = M_{P_1}$. Therefore, $x^* \in \mathrm{argmin}_{x \in X_0} f(x)$. Therefore,

$$\mathrm{argmin}_{x \in X_0} f(x) = \mathrm{argmin}_{x \in X} L(x, r). \quad \forall r > \bar{r}.$$

∎

It is noted that $\beta_0(u) = \inf_{x \in Z(u)} f_0(x)$, $f_0(x) = f(x) + \rho(x)g(x)$. Then we have

Corollary 3.4.1 *Suppose that all conditions in Lemma 3.4.1 are satisfied. If there exist a $r' > 0$ and a neighborhood U of $0 \in \mathbf{R}^m$ such that*

$$M_{P_1} \leq \beta_0(u) + r \sum_{j=1}^{m} |u_j|, \quad \forall u \in U, \forall r \geq r'. \tag{3.76}$$

Then there exists a $\bar{r} > 0$ such that

$$M_{P_1} = \inf\{L(x, r) : x \in X\}, \quad \forall r > \bar{r},$$

and

$$argmin_{x \in X_0} f(x) = argmin_{x \in X} L(x, r), \quad \forall r > \bar{r}.$$

Corollary 3.4.2 *Suppose that all conditions in Lemma 3.4.1 are satisfied. If there exist a $r' > 0$ and a neighborhood U of $0 \in \mathbf{R}^m$ such that*

$$M_{P_1} \leq \beta(u) + (r + \bar{\rho}) \sum_{j=1}^{m} |u_j|, \quad \forall u \in U, \forall r \geq r', \tag{3.77}$$

where $\bar{\rho} = \inf_{x \in X} \rho(x)$. Then there exists a $\bar{r} > 0$ such that

$$M_{P_1} = \inf\{L(x, r) : x \in X\}, \quad \forall r > \bar{r},$$

and

$$argmin_{x \in X_0} f(x) = argmin_{x \in X} L(x, r), \quad \forall r > \bar{r}.$$

Proof: By (3.77), we know that

$$\begin{aligned} M_{P_1} &\leq \inf_{x \in Z(u)} f(x) + (r + \inf_{x \in X} \rho(x)) \sum_{j=1}^{m} |u_j|, \quad \forall u \in U, \forall r \geq r', \\ &\leq \inf\{f(x) + (r + \rho(x)) \sum_{j=1}^{m} |u_j| : x \in X\}, \quad \forall u \in U, \forall r \geq r', \end{aligned}$$

where $u_j = g_j^+(x), j = 1, \cdots, m_1; u_j = g_j(x), j = m_1 + 1, \cdots, m$. Hence, (3.73) given in Theorem 3.4.1 holds. Therefore, the conclusion holds by virtue of Theorem 3.4.1.

∎

Remark 3.4.1 It is noted that, in our results above, we don't need any coercive assumption on the objective function and constraint functions. If we give some coercive assumption on the objective function or constraint functions, we can obtain some exact penalty results similar to ones in Theorem 3.4.1 and Corollary 3.4.2. The proof is similar and thus omitted.

Lemma 3.4.2 *Suppose that*

1). $\lim_{\|x\|\to\infty} \max\{f(x), g(x)\} = +\infty.$

2). $X_0 \neq \phi$ *and* M_{P_1} *is finite.*

Then, S_r *is nonempty and compact whenever* $r > 0$, *and*

$$\lim_{r\to+\infty} \psi(r) = M_{P_1}.$$

Theorem 3.4.2 *Suppose that all conditions in Lemma 3.4.2 are satisfied. Then the following two statements are equivalent:*

(i) there exists a $\bar{r} > 0$ *such that*

$$M_{P_1} = \inf\{L(x, \bar{r}) : x \in X\},$$

and

$$argmin_{x\in X_0} f(x) = argmin_{x\in X} L(x, \bar{r}),$$

(ii) there exist a $r' > 0$ *and a neighborhood* U *of* $0 \in \mathbf{R}^m$ *such that*

$$\begin{aligned} M_{P_1} \leq \inf\{f(x) + (r + \rho(x)) \sum_{j=1}^{m} |u_j| : x \in X, u \in U \\ g_j^+(x) = u_j, j = 1, \cdots, m_1; g_j(x) = u_j, j = m_1 + 1, \cdots, m\}, \end{aligned} \tag{3.78}$$

where $r > r'$.

Corollary 3.4.3 *Suppose that all conditions in Lemma 3.4.2 are satisfied. Then the following two statements are equivalent:*

(i) there exists a $\bar{r} > 0$ *such that*

$$M_{P_1} = \inf\{L(x, \bar{r}) : x \in X\},$$

and

$$argmin_{x\in X_0} f(x) = argmin_{x\in X} L(x,\bar{r}),$$

(ii) there exist a $r' > 0$ *and a neighborhood* U *of* $0 \in \mathbf{R}^m$ *such that*

$$M_{P_1} \leq \beta(u) + r\sum_{j=1}^{m} |u_j|, \quad \forall u \in U, \forall r \geq r'. \tag{3.79}$$

Chapter 4

Second-Order Sufficient Optimality Conditions for Lower Order Exact Penalty Functions

4.1 Introduction

Consider the following constrained optimization problem:

$$(P_1) \qquad \begin{aligned} \inf\ & f(x) \\ \text{s.t. } & x \in \mathbf{R}^n, \\ & g_j(x) \leq 0, \quad j = 1, \cdots, m_1 \\ & g_j(x) = 0, \quad j = m_1 + 1, \cdots, m, \end{aligned}$$

where $f, g_j (j = 1, \cdots, m) : \mathbf{R}^n \to \mathbf{R}$ are real valued twice continuously differentiable functions. Denote by X_0 the set of feasible solutions of (P_1), i.e.,

$$X_0 = \{x \in \mathbf{R}^n : g_j(x) \leq 0, j = 1, \cdots, m_1; g_j(x) = 0, j = m_1 + 1, \cdots, m\}.$$

Second-order sufficient optimality conditions play an important role in establishing the existence of exact penalty functions. For example, see [52, 80, 113]. In [52], in order to solve a nonlinear programming problem with equality and inequality constraints, Han and Mangasarian introduced a class of exact penalty functions depending on a fixed vector norm related with the constraints. For a sufficiently large but finite value of the penalty parameter, the penalty function has a local minimum point at any strict local minimum point of the constrained nonlinear programming problem satisfying a second-order sufficient optimality condition. It is shown in [80] that any strict local minimum satisfying a second-order sufficient condition for the original problem is a strict local minimum of the classical l_1 penalty function with a large enough penalty parameter. In [113], Wu *et al*. considered a lower order penalty function and its ε-smoothing for an inequality constrained nonlinear programming problem. It is shown that any strict local minimum satisfying a second-order sufficient condition for the original problem is a strict local minimum of the lower order penalty function with any positive penalty parameter.

In this chapter, we introduce a class of penalty functions which is more general than the penalty functions used in [52], [81], [88] and [113]. We prove that any strict local minimum satisfying a second-order sufficient condition for the original problem is a strict local minimum of this class of penalty functions with any positive penalty parameter, and that any global minimum satisfying a second-order global sufficient condition for the original problem is a global minimum of this class of penalty functions with some positive penalty parameter.

The outline of this chapter is as follows:

In Section 4.2, we present some preliminary results. In Section 4.3, under the assumption that a second order sufficient condition is satisfied, we obtain a strict local minimum of the penalty problem. In Section 4.4, under the assumptions that a second order global sufficient condition is satisfied, we obtain a global minimum of the penalty problem. We apply our results to quadratic programming and linear fractional programming problems.

4.2 Preliminaries

Let $Q : \mathbf{R}_+ \to \mathbf{R}_+$ be a continuous function. We shall associate nonlinear programming problem (P_1) with the following class of penalty functions:

$$P(x,q) := f(x) + qQ(\|(g_1^+(x), \cdots, g_{m_1}^+(x), g_{m_1+1}(x), \cdots, g_m(x))\|), \tag{4.1}$$

where q is a nonnegative real number, $\|\cdot\|$ is a vector norm in $\mathbf{R}^m$. Let $0 < \beta \leq 1$. Assume that the following properties hold

$$Q(0) = 0, \quad Q(t) > 0 \quad \text{for } t > 0, \tag{4.2}$$

$$\infty > \lim_{t \to 0+} \frac{Q(t) - Q(0)}{t^\beta} \geq \Lambda > 0, \tag{4.3}$$

$$\liminf_{t \to +\infty} Q(t) > 0. \tag{4.4}$$

Example 4.2.1 *There exist a lots of functions that satisfy (4.2) - (4.4). For example:*

(i) $Q_1(t) = \sqrt{t} + t^2 (t \geq 0)$,

(ii) let $\beta \in (0,1), \quad Q_2(t) = t^\beta (t \geq 0)$.

Corresponding to Q_2 *and* $\|y\| = \sum_{j=1}^m |y_j|$, *we have the following penalty function:*

$$f(x) + q \left(\sum_{j=1}^{m_1} g_j^+(x) + \sum_{j=m_1+1}^{m} |g_j(x)| \right)^\beta,$$

which has been investigated in the study of mathematical programs with equilibrium constraints, see [81]. Corresponding to Q_2 *and* $\|y\| = \max\{|y_1|, \cdots, |y_m|\}$, *we have the following penalty function:*

$$f(x) + q \left(\max\{g_1^+(x), \cdots, g_{m_1}^+(x), |g_{m_1+1}(x)|, \cdots, |g_m(x)|\right)^\beta,$$

which has been investigated in [88].

Consider the following penalty problem:

$$(P_Q) \quad \min_{x \in \mathbf{R}^n} P(x, q).$$

Let $x^* \in X_0$ and

$$A(x^*) = \{j \in \{1, \cdots, m_1\} | g_j(x^*) = 0\}. \tag{4.5}$$

Let

$$L(x, \lambda) = f(x) + \sum_{j=1}^{m} \lambda_j g_j(x)$$

be the Lagrangian of problem (P_1).

In this chapter, we assume that $f, g_j (j = 1, \cdots, m) : \mathbf{R}^n \to \mathbf{R}$ are real valued twice continuously differentiable functions.

4.3 Local exact penalty functions

Proposition 4.3.1 *[52] (Second order sufficiency of a strict local minimum). Let* $(x^*, \lambda^*) \in \mathbf{R}^{n+m}$ *satisfy the Karush-Kuhn-Tucker necessary optimality condition for problem* (P_1)*:*

$$\begin{aligned} \nabla_x L(x^*, \lambda^*) = 0,& \\ g_j(x^*) \le 0,\ j = 1, \cdots, m_1& \\ \lambda_j^* \ge 0,\ j = 1, \cdots m_1& \\ \lambda_j^* g_j(x^*) = 0,\ j = 1, \cdots m_1& \\ g_j(x^*) = 0, j = m_1 + 1, \cdots m.& \end{aligned} \tag{4.6}$$

Let

$$W(x^*) = \left\{ y \in \mathbf{R}^n \left| \begin{array}{ll} \nabla^\top f(x^*) y \le 0, & \\ \nabla^\top g_j(x^*) y \le 0, & j \in A(x^*) \\ \nabla^\top g_j(x^*) y = 0, & j = m_1 + 1, \cdots, m \\ y \ne 0 & \end{array} \right. \right\}.$$

If

$$y^\top \nabla^2 L(x^*, \lambda^*) y > 0,\ \forall\ y \in W(x^*), \tag{4.7}$$

then x^* *is a strict local minimum (of order 2) for* (P_1)*.*

Define

$$U(x^*) = \left\{ y \in \mathbf{R}^n \left| \begin{array}{ll} \nabla^T g_j(x^*) y = 0, & j \in J(x^*) \\ \nabla^T g_j(x^*) y \le 0, & j \in K(x^*) \\ \nabla^T g_j(x^*) y = 0, & j = m_1 + 1, \cdots, m \\ y \ne 0 & \end{array} \right. \right\},$$

where $J(x^*)$ and $K(x^*)$ are the following subsets of $A(x^*)$:

$$\begin{aligned} J(x^*) &= \{i \in \{1, \cdots, m\} \mid g_i(x^*) = 0,\ \lambda_i^* > 0\}, \\ K(x^*) &= \{i \in \{1, \cdots, m\} \mid g_i(x^*) = 0,\ \lambda_i^* = 0\}. \end{aligned}$$

By [52], we have

Proposition 4.3.2 *Under the assumption of Proposition 4.3.1, (4.7) is equivalent to*

$$y^T \nabla^2 L(x^*, \lambda^*) y > 0,\ \forall\ y \in U(x^*).$$

Definition 4.3.1 *(Mangasarian-Fromovitz constraint qualification (MFCQ)) Let* $g_j(x^*) \leq 0,\ j = 1, \cdots, m_1,\ g_j(x^*) = 0, j = m_1 + 1, \cdots m,$ *and* $A(x^*)$ *be defined by (4.5). The constraints* $g_j(x),\ j = 1, \cdots, m$ *are said to satisfy the constraint qualification condition of [83] at* x^* *if,* $g_j(x),\ j = 1, \cdots, m_1,$ *are differentiable at* x^*, $g_j(x), j = m_1 + 1, \cdots m,$ *are continuously differentiable at* x^*, *and*

$$\begin{aligned} &\nabla g_j(x^*), j = m_1 + 1, \cdots m \text{ are linearly independent and,} \\ &\qquad\qquad \text{there exists a } y \in \mathbf{R}^n \text{such that} \\ &\nabla^\top g_j(x^*) y < 0, j \in A(x^*) \\ &\nabla^\top g_j(x^*) y = 0, j = m_1 + 1, \cdot \cdot \cdot, m. \end{aligned} \tag{4.8}$$

It is noted that the more stringent constraint qualification condition used by Pietrzykowski in [91], namely that the gradients $\nabla g_j(x^*),\ j \in A(x^*) \bigcup \{m_1 + 1, \cdot \cdot \cdot, m\}$, are linearly independent, implies the constraint qualification condition (4.8).

When $\beta = 1$, Theorem 4.4 of [52] shows that the combination of a strict local minimum and the MFCQ implies a local minimum of exact penalty for a large penalty parameter while Theorem 4.6 of [52] states that the second order sufficiency implies a strict local minimum of exact penalty function for a large penalty parameter. Now we prove that the second order sufficiency implies a strict local minimum of exact penalty for any penalty parameter $q > 0$ if the nonconvex function Q satisfies (4.2) and (4.3).

Theorem 4.3.1 *Let* $0 < \beta < 1$ *and* Q *satisfy (4.2) and (4.3). Suppose that all assumptions in Proposition 4.3.1 hold. Then* x^* *is a strict local minimum of the penalty problem* (P_Q) *for any* $q > 0$.

Proof: By contradiction, suppose that there exist a $q > 0$ and a sequence $\{x_k\}$ converging to x^*, such that $x_k \neq x^*$ for $k = 1, 2, \cdots$, and

$$P(x_k, q) \leq P(x^*, q).$$

Clearly, we have

$$f(x_k) \leq f(x^*), k = 1, 2, \cdots,$$

and

$$\|(g_1^+(x_k), \cdots, g_{m_1}^+(x_k), g_{m_1+1}(x_k), \cdots, g_m(x_k))\| \to 0,$$

as $k \to \infty$. Then it follows from (4.3) that

$$\lim_{n\to\infty} \frac{Q(\|(g_1^+(x_k), \cdots, g_{m_1}^+(x_k), g_{m_1+1}(x_k), \cdots, g_m(x_k))\|)}{\|(g_1^+(x_k), \cdots, g_{m_1}^+(x_k), g_{m_1+1}(x_k), \cdots, g_m(x_k))\|^\beta} \geq \Lambda > 0. \tag{4.9}$$

Denote

$$s_k = \frac{x_k - x^*}{\|x_k - x^*\|}.$$

Obviously, there exists a subsequence $\{s_{k_l}\}$ such that $\{s_{k_l}\}$ converging to a vector s with $\|s\| = 1$. Without loss of generality, suppose that

$$s = \lim_{k\to+\infty} s_k.$$

Since f and $g_j, j = 1, 2, \cdots, m$, are continuously differentiable at x^*, we have

$$0 \geq \frac{f(x_k) - f(x^*)}{\|x_k - x^*\|} = \nabla^\top f(x^*) s_k + \frac{o(\|x_k - x^*\|)}{\|x_k - x^*\|},$$

$$\frac{g_j^+(x_k)}{\|x_k - x^*\|} = \max\left\{0, \nabla^\top g_j(x^*) s_k + \frac{o(\|x_k - x^*\|)}{\|x_k - x^*\|}\right\},$$

where $j \in A(x^*)$, and

$$\frac{|g_j(x_k)|}{\|x_k - x^*\|} = \left|\nabla^\top g_j(x^*) s_k + \frac{o(\|x_k - x^*\|)}{\|x_k - x^*\|}\right|,$$

where $j = m_1 + 1, \cdots, m$. Furthermore, we have

$$\lim_{k\to+\infty} \left(\nabla^\top f(x^*) s_k + \frac{o(\|x_k - x^*\|)}{\|x_k - x^*\|}\right) = \nabla^\top f(x^*) s, \tag{4.10}$$

$$\lim_{k\to+\infty} \left(\max\left\{0, \nabla^\top g_j(x^*) s_k + \frac{o(\|x_k - x^*\|)}{\|x_k - x^*\|}\right\}\right)^\beta = \left(\max\left\{0, \nabla^\top g_j(x^*) s\right\}\right)^\beta, \tag{4.11}$$

for each $j \in A(x^*)$, and

$$\left|\nabla^\top g_j(x^*) s_k + \frac{o(\|x_k - x^*\|)}{\|x_k - x^*\|}\right|^\beta = |\nabla^\top g_j(x^*) s|^\beta, \tag{4.12}$$

for each $j = m_1 + 1, \cdots, m$. We have

$$\begin{aligned}
0 &\geq \frac{P(x_k, q) - P(x^*, q)}{\|x_k - x^*\|} \\
&= \frac{f(x_k) + qQ(\|(g_1^+(x_k), \cdots, g_{m_1}^+(x_k), g_{m_1+1}(x_k), \cdots, g_m(x_k))\|) - f(x^*)}{\|x_k - x^*\|} \\
&= \nabla^\top f(x^*) s_k + \frac{o(\|x_k - x^*\|)}{\|x_k - x^*\|} \\
&\quad + q\frac{Q(\|(g_1^+(x_k), \cdots, g_{m_1}^+(x_k), g_{m_1+1}(x_k), \cdots, g_m(x_k))\|)}{\|(g_1^+(x_k), \cdots, g_{m_1}^+(x_k), g_{m_1+1}(x_k), \cdots, g_m(x_k))\|^\beta} \\
&\quad \cdot \frac{\|(g_1^+(x_k), \cdots, g_{m_1}^+(x_k), g_{m_1+1}(x_k), \cdots, g_m(x_k))\|^\beta}{\|x_k - x^*\|^\beta} \cdot \|x_k - x^*\|^{\beta-1}.
\end{aligned}$$

Since $0 < \beta < 1$, the above inequality, combined with (4.9) - (4.12), yields

$$\begin{aligned} \nabla^\top f(x^*)s &\le 0, \\ \nabla^\top g_j(x^*)s &\le 0, \qquad j \in A(x^*) \\ \nabla^\top g_j(x^*)s &= 0, \;\; j = m_1+1, \cdots, m \\ s &\ne 0. \end{aligned}$$

Thus $s \in W(x^*)$. By (4.7), we have

$$s^\top \nabla^2 L(x^*, \lambda^*) s > 0. \tag{4.13}$$

If $j \notin A(x^*)$, that is, $g_j(x^*) \ne 0$, it yields from $\lambda_j^* g_j(x^*) = 0$ that $\lambda_j^* = 0$. Making use of the twice differentiability property, we obtain

$$\begin{aligned} & f(x_k) - f(x^*) + \sum_{j=1}^m \lambda_j^* g_j(x_k) \\ & = f(x_k) - f(x^*) + \sum_{j=1}^m (\lambda_j^* g_j(x_k) - \lambda_j^* g_j(x^*)) \\ & = \Big(\nabla^\top f(x^*) + \sum_{j=1}^m \lambda_j^* \nabla^\top g_j(x^*)\Big) s_k \|x_k - x^*\| \\ & + \frac{1}{2} s_k^\top (\nabla^2 f(x^*) + \sum_{j=1}^m \lambda_j^* \nabla^2 g_j(x^*)) s_k \|x_k - x^*\|^2 + o(\|x_k - x^*\|^2) \\ & = \Big(\nabla^\top f(x^*) + \sum_{j=1}^m \lambda_j^* \nabla^\top g_j(x^*)\Big) s_k \|x_k - x^*\| \\ & + \|x_k - x^*\|^2 \big(\frac{1}{2} s_k^\top \nabla^2 L(x^*, \lambda^*) s_k + \frac{o(\|x_k - x^*\|^2}{\|x_k - x^*\|^2})\big). \end{aligned}$$

This, combined with (4.6) and (4.13), implies that,

$$f(x_k) - f(x^*) + \sum_{j=1}^m \lambda_j^* g_j(x_k) > 0, \tag{4.14}$$

for sufficiently large k. Let $\Lambda_0 = \max_{j=1,\cdots,m} \lambda_j^*$. Then

$$\begin{aligned} & \ge P(x_k, q) - P(x^*, q) \\ & \ge f(x_k) - f(x^*) + \textstyle\sum_{j=1}^m \lambda_j^* g_j(x_k) - \Lambda_0 (\sum_{j \in A(x^*)} g_j^+(x_k) + \sum_{j=m_1+1}^m |g_j(x_k)|) \\ & + qQ(\|(g_1^+(x_k), \cdots, g_{m_1}^+(x_k), g_{m_1+1}(x_k), \cdots, g_m(x_k))\|). \end{aligned} \tag{4.15}$$

Let $0 < \Lambda' < \Lambda$. By (4.9), we have

$$\frac{Q(\|(g_1^+(x_k), \cdots, g_{m_1}^+(x_k), g_{m_1+1}(x_k), \cdots, g_m(x_k))\|)}{\|(g_1^+(x_k), \cdots, g_{m_1}^+(x_k), g_{m_1+1}(x_k), \cdots, g_m(x_k))\|^\beta} \ge \Lambda', \tag{4.16}$$

for sufficiently large k. Note that the norms are equivalent to each other in a finite dimensional normed space, hence

$$\|(g_1^+(x_k),\cdots,g_{m_1}^+(x_k),g_{m_1+1}(x_k),\cdots,g_m(x_k))\| \ge c_0(\sum_{j=1}^{m_1} g_j^+(x_k)^2 + \sum_{j=m_1+1}^{m} |g_j(x_k)|^2)^{\frac{1}{2}},$$

for some constant $c_0 > 0$. Consequently

$$\begin{aligned}
&\frac{\sum_{j\in A(x^*)} g_j^+(x_k) + \sum_{j=m_1+1}^m |g_j(x_k)|}{\|(g_1^+(x_k),\cdots,g_{m_1}^+(x_k),g_{m_1+1}(x_k),\cdots,g_m(x_k))\|^\beta} \\
&\le \frac{1}{c_0^\beta}(\sum_{j\in A(x^*)} g_j^+(x_k)^{1-\beta} + \sum_{j=m_1+1}^{m} |g_j(x_k)|^{1-\beta}) \to 0
\end{aligned} \tag{4.17}$$

as $k\to\infty$. By using (4.16) and (4.17), we obtain

$$\begin{aligned}
&q\frac{Q(\|(g_1^+(x_k),\cdots,g_{m_1}^+(x_k),g_{m_1+1}(x_k),\cdots,g_m(x_k))\|)}{\|(g_1^+(x_k),\cdots,g_{m_1}^+(x_k),g_{m_1+1}(x_k),\cdots,g_m(x_k))\|^\beta} \\
&-\Lambda_0\frac{\sum_{j\in A(x^*)} g_j^+(x_k) + \sum_{j=m_1+1}^m |g_j(x_k)|}{\|(g_1^+(x_k),\cdots,g_{m_1}^+(x_k),g_{m_1+1}(x_k),\cdots,g_m(x_k))\|^\beta} > 0,
\end{aligned}$$

for k is sufficiently large. Therefore, this, combined with (4.14) and (4.15), yields that, when k is sufficiently large, we have

$$P(x_k,q) - P(x^*,q) > 0, \tag{4.18}$$

which is a contradiction. ▮

4.4 Global exact penalty functions

Definition 4.4.1 *Let $x^* \in X_0$ and $V(x^*) \subset \mathbf{R}^n$ be a closed subset and $f:\mathbf{R}^n\to\mathbf{R}$ be twice continuously differentiable at x^*. We say that a generalized representation condition holds for f at x^* with respect to $\eta(x,x^*)\in V(x^*)$ if, for every $x\in\mathbf{R}^n$,*

$$f(x) = f(x^*) + \nabla f(x^*)^\top(x-x^*) + \frac{1}{2}\eta(x,x^*)^\top\nabla^2 f(x^*)\eta(x,x^*). \tag{4.19}$$

Definition 4.4.2 *We say that a pair (x^*,λ^*) satisfies the second order global sufficient condition if,*

$$\begin{aligned}
\nabla_x L(x^*,\lambda^*) &= 0, \\
g_j(x^*) &\le 0,\ j=1,\cdots,m_1 \\
\lambda_j^* &\ge 0,\ j=1,\cdots m_1 \\
\lambda_j^* g_j(x^*) &= 0,\ j=1,\cdots m_1 \\
g_j(x^*) &= 0, j=m_1+1,\cdots m
\end{aligned} \tag{4.20}$$

and

$$y^\top \nabla^2 L(x^*,\lambda^*)y \geq 0,\ \forall y \in V(x^*), \tag{4.21}$$

where $V(x^*) \subset \mathbf{R}^n$ is a closed subset, and $f, g_j (j = 1, \cdots, m)$ satisfy (4.19) with the same $\eta(x,x^*)$, $\eta(\cdot,x^*) : \mathbf{R}^n \to V(x^*)$ is continuous.

Theorem 4.4.1 *If there exists $q_0 > 0$, such that x^* is a global solution of the penalty problem (P_Q) for any $q > q_0$, where $P(x,q)$ is defined by (4.1) with Q satisfying (4.2), then x^* is a global minimum of the constrained optimization problem (P_1).*

Proof: We first show by contradiction that $x^* \in X_0$. If x^* is infeasible, then $\|(g_1^+(x^*), \cdots, g_{m_1}^+(x^*), g_{m_1+1}(x^*), \cdots, g_m(x^*))\| > 0$. Since Q satisfy (4.2),

$$Q(\|(g_1^+(x^*), \cdots, g_{m_1}^+(x^*), g_{m_1+1}(x^*), \cdots, g_m(x^*))\|) > 0.$$

Choose a $x_* \in X_0$, then

$$Q(\|(g_1^+(x_*), \cdots, g_{m_1}^+(x_*), g_{m_1+1}(x_*), \cdots, g_m(x_*))\|) = 0.$$

Let

$$q > \max\Big\{\frac{f(x_*) - f(x^*)}{Q(\|(g_1^+(x^*), \cdots, g_{m_1}^+(x^*), g_{m_1+1}(x^*), \cdots, g_m(x^*))\|)}, q_0\Big\}.$$

We then conclude that

$$\begin{aligned} f(x_*) &= f(x_*) + qQ(\|(g_1^+(x_*), \cdots, g_{m_1}^+(x_*), g_{m_1+1}(x_*), \cdots, g_m(x_*))\|) \\ &\geq f(x^*) + qQ(\|(g_1^+(x^*), \cdots, g_{m_1}^+(x^*), g_{m_1+1}(x^*), \cdots, g_m(x^*))\|) \\ &> f(x_*), \end{aligned}$$

which is impossible. Thus, $x^* \in X_0$ and

$$Q(\|(g_1^+(x^*), \cdots, g_{m_1}^+(x^*), g_{m_1+1}(x^*), \cdots, g_m(x^*))\|) = 0.$$

Therefore, for each $x \in X_0$,

$$f(x^*) = p(x^*,q) \leq P(x,q) = f(x).$$

Hence, x^* is a global solution of the constrained optimization problem (P_1). ∎

Theorem 4.4.2 *Suppose that (x^*,λ^*) satisfies the second order global sufficient condition. Assume that one of the following conditions holds:*

(i) $\limsup_{\|x\|\to\infty} f(x) = +\infty$;

(ii) $\liminf_{\|x\|\to\infty} g_j(x) = a > 0$, *for some* $j \in \{1, \cdots, m\}$.

Then there exists $q_0 > 0$*, such that* x^* *is a global solution of the penalty problem* (P_Q) *for any* $q > q_0$*, where* $P(x, q)$ *is defined by (4.1) with* Q *satisfying (4.2)-(4.4),* $0 < \beta \leq 1$.

Proof: Arguing by the contradiction, we may assume that there exist $\{q_k\}$ and $\{x_k\}$ with $q_k > 0$, $x_k \neq x^*$, $q_k \to \infty$ such that

$$P(x_k, q_k) < P(x^*, q_k), \tag{4.22}$$

that is,

$$f(x_k) + q_k Q(\|(g_1^+(x_k), \cdots, g_{m_1}^+(x_k), g_{m_1+1}(x_k), \cdots, g_m(x_k))\|) < f(x^*). \tag{4.23}$$

Consequently

$$f(x_k) < f(x^*). \tag{4.24}$$

We claim that $\{x_k\}$ is bounded. Indeed, if $\{x_k\}$ is unbounded, we may assume that $\|x_k\| \to \infty$ as $k \to \infty$. By assumptions (i) and (ii), either $\limsup_{k\to\infty} f(x_k) = +\infty$, a contradiction with (4.23), or, for some j_0, $\liminf_{k\to\infty} g_{j_0}(x_k) = a > 0$, which implies that there exist a subsequence $\{x_{k_i}\} \subset \{x_k\}$ and $0 < l_0 \leq +\infty$ such that

$$\|(g_1^+(x_{k_i}), \cdots, g_{m_1}^+(x_{k_i}), g_{m_1+1}(x_{k_i}), \cdots, g_m(x_{k_i}))\| \to l_0.$$

Hence, by (4.4), we have

$$q_{k_i} Q(\|(g_1^+(x_{k_i}), \cdots, g_{m_1}^+(x_{k_i}), g_{m_1+1}(x_{k_i}), \cdots, g_m(x_{k_i}))\|) \to \infty \text{ as } i \to \infty,$$

a contradiction with (4.23). Therefore $\{x_k\}$ is bounded. We can assume, going if necessary to a subsequence, that $x_k \to x_0$. From (4.23), we have

$$\|(g_1^+(x_0), \cdots, g_{m_1}^+(x_0), g_{m_1+1}(x_0), \cdots, g_m(x_0))\| = 0. \tag{4.25}$$

That is, $x_0 \in X_0$.

Since $f(x)$, and $g_j(x)(j = 1, \cdots, m)$ satisfy the generalized representation condition at x^*, there exists $\eta(x, x^*) \in V(x^*)$ such that

$$0 \geq f(x_k) - f(x^*) = \nabla f(x^*)^\top (x_k - x^*) + \frac{1}{2}\eta(x_n, x^*)^\top \nabla^2 f(x^*) \eta(x_n, x^*), \tag{4.26}$$

$$g_j^+(x_k) = \max\big\{0, g_j(x^*) + \nabla g_j(x^*)^\top(x_k - x^*) + \frac{1}{2}\eta(x_n, x^*)^\top \nabla^2 g_j(x^*)\eta(x_n, x^*)\big\},$$

where $j \in A(x^*)$, and

$$|g_j(x_k)| = |g_j(x^*) + \nabla g_j(x^*)^\top(x_k - x^*) + \frac{1}{2}\eta(x_n, x^*)^\top \nabla^2 g_j(x^*)\eta(x_n, x^*)|,$$

where $j = m_1 + 1, \cdots, m$. Note that

$$\begin{aligned}&\lim_{k\to\infty} \max\big\{0, \nabla g_j(x^*)^\top(x_k - x^*) + \frac{1}{2}\eta(x_n, x^*)^\top \nabla^2 g_j(x^*)\eta(x_n, x^*)\big\}\\ =&\max\big\{0, \nabla g_j(x^*)^\top(x_0 - x^*) + \frac{1}{2}\eta(x_0, x^*)^\top \nabla^2 g_j(x^*)\eta(x_0, x^*)\big\},\end{aligned} \tag{4.27}$$

for each $j \in A(x^*)$, and

$$\begin{aligned}&\lim_{k\to\infty} |\nabla g_j(x^*)^\top(x_k - x^*) + \frac{1}{2}\eta(x_n, x^*)^\top \nabla^2 g_j(x^*)\eta(x_n, x^*)|\\ &= |\nabla g_j(x^*)^\top(x_0 - x^*) + \frac{1}{2}\eta(x_0, x^*)^\top \nabla^2 g_j(x^*)\eta(x_0, x^*)|.\end{aligned} \tag{4.28}$$

for each $j \in \{m_1 + 1, \cdots, m\}$. Since $q_k \to \infty$ as $k \to \infty$, it follows from (4.23), (4.26), (4.27) and (4.28) that

$$\nabla f(x^*)^\top(x_0 - x^*) + \frac{1}{2}\eta(x_0, x^*)^\top \nabla^2 f(x^*)\eta(x_0, x^*) \le 0, \tag{4.29}$$

$$\max\big\{0, \nabla g_j(x^*)^\top(x_0 - x^*) + \frac{1}{2}\eta(x_0, x^*)^\top \nabla^2 g_j(x^*)\eta(x_0, x^*)\big\} = 0,$$

for each $j \in A(x^*)$, and

$$|\nabla g_j(x^*)^\top(x_0 - x^*) + \frac{1}{2}\eta(x_0, x^*)^\top \nabla^2 g_j(x^*)\eta(x_0, x^*)| = 0.$$

for each $j \in \{m_1 + 1, \cdots, m\}$. Hence

$$\nabla^\top g_j(x^*)(x_0 - x^*) + \frac{1}{2}\eta(x_0, x^*)^\top \nabla^2 g_j(x^*)\eta(x_0, x^*) \le 0, \tag{4.30}$$

for each $j \in A(x^*)$, and

$$\nabla g_j(x^*)^\top(x_0 - x^*) + \frac{1}{2}\eta(x_0, x^*)^\top \nabla^2 g_j(x^*)\eta(x_0, x^*) = 0, \tag{4.31}$$

for each $j \in m_1 + 1, \cdots, m$. It implies from the second order sufficient condition that $\nabla_x L(x^*, \lambda^*) = 0$, i.e.,

$$(\nabla^\top f(x^*) + \sum_{j=1}^{m} \lambda_j^* \nabla^\top g_j(x^*))(x_k - x^*) = 0, \tag{4.32}$$

$$\lambda_j^* g_j(x^*) = 0,\ j = 1, \cdots m_1, g_j(x^*) = 0, j = m_1 + 1, \cdots, m, \tag{4.33}$$

and that

$$\eta(x_k, x^*)^\top \nabla^2 L(x^*, \lambda^*)\eta(x_k, x^*) \geq 0. \tag{4.34}$$

Thus, by the generalized representation condition, (4.32), (4.33) and (4.34), we have

$$f(x_k) + \sum_{j=1}^{m} \lambda_j^* g_j(x_k) \geq f(x^*). \tag{4.35}$$

By (4.3), there exist $t_0 > 0$, $0 < \Lambda_1 < \Lambda$ such that

$$Q(t) \geq \Lambda_1 t^\beta. \quad \forall 0 < t < t_0. \tag{4.36}$$

It follows from (4.25) and (4.36) that

$$\begin{aligned}
&Q(\|(g_1^+(x_k), \cdots, g_{m_1}^+(x_k), g_{m_1+1}(x_k), \cdots, g_m(x_k))\|) \\
&\geq \Lambda_1 \|(g_1^+(x_k), \cdots, g_{m_1}^+(x_k), g_{m_1+1}(x_k), \cdots, g_m(x_k))\|^\beta \\
&\geq \Lambda_2 (\sum_{j=1}^{m} g_j^+(x_k) + \sum_{j=m_1+1}^{m} |g_j(x_k)|)^\beta,
\end{aligned} \tag{4.37}$$

for k large enough, where Λ_2 is some positive constant.

Case 1. $\beta = 1$. For k large enough, by (4.37) we have

$$\begin{aligned}
&P(x_k, q_k) - P(x^*, q_k) \\
=& f(x_k) - f(x^*) + q_k Q(\|(g_1^+(x_k), \cdots, g_{m_1}^+(x_k), g_{m_1+1}(x_k), \cdots, g_m(x_k))\|) \\
\geq& f(x_k) - f(x^*) + \Big(\sum_{j=1}^{m_1} \lambda_j^* g_j^+(x_k) + \sum_{j=m_1+1}^{m} \lambda_j^* |g_j(x_k)|\Big) \\
\geq& f(x_k) - f(x^*) + \sum_{j=1}^{m} \lambda_j^* g_j(x_k) \geq 0,
\end{aligned}$$

which contradicts (4.22).

Case 2. $0 < \beta < 1$. (4.24) and (4.35) imply

$$\sum_{j=1}^{m} \lambda_j^* g_j(x_k) > 0, \ \forall k. \tag{4.38}$$

That is, by generalized representation condition,

$$\sum_{j=1}^{m} \lambda_j^* \Big(\nabla g_j(x^*)^\top (x_k - x^*) + \frac{1}{2}\eta(x_n, x^*)^\top \nabla^2 g_j(x^*)\eta(x_n, x^*)\Big) > 0.$$

Let $k \to \infty$. Then

$$\sum_{j=1}^{m} \lambda_j^* \Big(\nabla g_j(x^*)^\top (x_0 - x^*) + \frac{1}{2} \eta(x, x_0)^\top \nabla^2 g_j(x^*) \eta(x, x_0) \Big) \geq 0. \tag{4.39}$$

If $j \notin A(x^*)$, that is, $g_j(x^*) \neq 0$, it yields from $\lambda_j^* g_j(x^*) = 0$ that $\lambda_j^* = 0$. Thus by (4.30), (4.31) and (4.39) we have

$$\sum_{j=1}^{m} \lambda_j^* \Big(\nabla^\top g_j(x^*)(x_0 - x^*) + \frac{1}{2} \eta(x_0, x^*)^\top \nabla^2 g_j(x^*) \eta(x_0, x^*) \Big) = 0, \tag{4.40}$$

Let $\Lambda_3 \geq \sup_{j=1,\cdots,m} \lambda_j^* > 0$. Since also $0 < \beta < 1$, by (4.37), we have

$$\begin{aligned}
& P(x_k, q) - P(x^*, q) \\
=& f(x_k) - f(x^*) + q_k Q(\|(g_1^+(x_k), \cdots, g_{m_1}^+(x_k), g_{m_1+1}(x_k), \cdots, g_m(x_k))\|) \\
\geq& f(x_k) - f(x^*) + \frac{q_k \Lambda_2}{\Lambda_3^\beta} \left(\sum_{j=1}^{m} \lambda_j^* g_j(x_k) \right)^\beta \\
=& f(x_k) - f(x^*) + \sum_{j=1}^{m} \lambda_j^* g_j(x_k) \\
& + \left(\sum_{j=1}^{m} \lambda_j^* g_j(x_k) \right)^\beta \left(\frac{q_k \Lambda_2}{\Lambda_3^\beta} - \left(\sum_{j=1}^{m} \lambda_j^* g_j(x_k) \right)^{1-\beta} \right) \\
=& f(x_k) - f(x^*) + \sum_{j=1}^{m} \lambda_j^* g_j(x_k) \\
& + \left(\sum_{j=1}^{m} \lambda_j^* g_j(x_k) \right)^\beta \Big(\frac{q_k \Lambda_2}{\Lambda_3^\beta} - \Big(\sum_{j=1}^{m} \lambda_j^* \big(\nabla g_j(x^*)^\top (x_k - x^*) \\
& + \frac{1}{2} \eta(x_n, x^*)^\top \nabla^2 g_j(x^*) \eta(x_n, x^*) \big) \Big)^{1-\beta} \Big), \text{ for sufficiently large } k.
\end{aligned}$$

It implies by the above inequality, (4.35), (4.38) and (4.40) that, for k large enough,

$$P(x_k, q) - P(x^*, q) > 0,$$

which contradicts (4.22). ▮

Example 4.4.1 *Let $\beta = 1/2$. Consider the problem*

$$\min 1, \ \ s.t. \ \ -x^2 = 0.$$

The penalty function is

$$\phi_q(x) = 1 + q|-x^2|^p = 1 + q|x|.$$

$x^* = 0$ *is a global minimum of the penalty function. Let*

$$L(x, \lambda) = 1 + \lambda(-x^2).$$

Then

$$\nabla L(x^*, \lambda^*) = -2\lambda^* x^* = 0, \ \lambda^* = -1,$$

$$\nabla^2 L(x^*, \lambda^*) = -2\lambda^* = 2 > 0.$$

Second order global sufficient condition is satisfied. But conditions (i) and (ii) of Theorem 4.4.2 are not satisfied. Thus condition (i) or (ii) is only a sufficient condition.

Under further assumptions on the objective and constraint functions of the constrained optimization problem (P_1), we can obtain more specific representation of the set $V(x^*)$.

Assume that $f(x), g_j(x), j = 1, \cdots, m$ satisfy the following conditions at x^*, for every $x \in \mathbf{R}^n$, there exist $\alpha(x, x^*) > 0$:

$$f(x) = f(x^*) + \nabla f(x^*)^\top (x - x^*) + \frac{1}{2}\alpha(x, x^*)(x - x^*)^\top \nabla^2 f(x^*)(x - x^*) \tag{4.41}$$

$$g_j(x) = g_j(x^*) + \nabla g_j(x^*)^\top (x - x^*) + \frac{1}{2}\alpha(x, x^*)(x - x^*)^\top \nabla^2 g_j(x^*)(x - x^*) \tag{4.42}$$

where $j = 1, \cdots, m$, and $\lim_{x \to x^*} \alpha(x, x^*) = 1$.

It is clear that, for a quadratic function $f(x) = \frac{1}{2}x^\top Ax + b^\top x + c$, we have

$$f(x) = f(x^*) + \nabla f(x^*)^\top (x - x^*) + \frac{1}{2}(x - x^*)^\top \nabla^2 f(x^*)(x - x^*),$$

and, for a linear fractional function $f(x) = \frac{a^\top x + r}{b^\top x + s}$, we have

$$f(x) = f(x^*) + \nabla f(x^*)^\top (x - x^*) + \frac{1}{2}\frac{b^\top x + s}{b^\top x^* + s}(x - x^*)^\top \nabla^2 f(x^*)(x - x^*).$$

Therefore, quadratic functions and linear fractional functions satisfy (4.41) and (4.42) with $\alpha(x, x^*) = 1$ and $\alpha(x, x^*) = \frac{b^\top x + s}{b^\top x^* + s}$ respectively.

Let $x^* \in \mathbf{R}^n$, and $A(x^*) = \{j \in \{1, \cdots, m_1\} | g_j(x^*) = 0\}$. Define

$$V_1(x^*) = \left\{ 0 \neq y \in \mathbf{R}^n \middle| \begin{array}{ll} \nabla^\top f(x^*) y \leq 0, & \\ \nabla^\top g_j(x^*) y \leq 0, & j \in A(x^*) \\ \nabla^\top g_j(x^*) y = 0, & j = m_1 + 1, \cdots, m \end{array} \right\}, \tag{4.43}$$

and

$$V_2(x^*) = \left\{ 0 \neq y \in \mathbf{R}^n \middle| \begin{array}{ll} \langle \nabla f(x^*), y \rangle + \frac{1}{2}\alpha(y, x^*) y^\top \nabla^2 f(x^*) y \leq 0, & \\ \langle \nabla g_j(x^*), y \rangle + \frac{1}{2}\alpha(y, x^*) y^\top \nabla^2 g_j(x^*) y \leq 0, & j \in A(x^*) \\ \langle \nabla g_j(x^*), y \rangle + \frac{1}{2}\alpha(y, x^*) y^\top \nabla^2 g_j(x^*) y = 0, & j = m_1 + 1, \cdots, m \end{array} \right\}. \tag{4.44}$$

We have

Theorem 4.4.3 *Let $0 < \beta < 1$ and Q satisfy (4.2) - (4.4). Suppose that $f(x), g_j(x)$, $j = 1, \cdots, m$ satisfy (4.41) and (4.42), and that (x^*, λ^*) satisfies (4.20), and (4.21) with a strict inequality, where $V(x^*)$ replaced by $V_1(x^*) \cup V_2(x^*)$, $V_1(x^*)$ and $V_2(x^*)$ are defined by (4.43) and (4.44), respectively. Assume that one of the following conditions holds:*

(i) $\limsup_{\|x\| \to \infty} f(x) = +\infty$*;*

(ii) $\liminf_{\|x\| \to \infty} g_j(x) = a > 0$*, for some $j \in \{1, \cdots, m\}$.*

Then there exists $q_0 > 0$ such that x^* is a strict global solution of the penalty problem (P_Q) for any $q > q_0$, where $P(x, q)$ is defined by (4.1).

Proof: Arguing by the contradiction, we may assume that there exist $\{q_k\}$ and $\{x_k\}$ with $q_k > 0$, $x_k \neq x^*$, $q_k \to \infty$ such that

$$P(x_k, q_k) \leq P(x^*, q_k), \tag{4.45}$$

that is,

$$f(x_k) + q_k Q(\|(g_1^+(x_k), \cdots, g_{m_1}^+(x_k), g_{m_1+1}(x_k), \cdots, g_m(x_k))\|) \leq f(x^*). \tag{4.46}$$

Consequently

$$f(x_k) \leq f(x^*). \tag{4.47}$$

Similar to the proof of Theorem 2.5.1, we obtain that $\{x_k\}$ is bounded, and going if necessary to a subsequence, $x_k \to x_0 \in X_0$.

Case 1. $x_0 = x^*$. We have $\lim_{k\to\infty} g_j(x_k) = g_j(x^*)$. By (4.3), we have

$$\lim_{k\to\infty} \frac{Q(\|(g_1^+(x_k), \cdots, g_{m_1}^+(x_k), g_{m_1+1}(x_k), \cdots, g_m(x_k))\|)}{\|(g_1^+(x_k), \cdots, g_{m_1}^+(x_k), g_{m_1+1}(x_k), \cdots, g_m(x_k))\|^\beta} \geq \Lambda > 0. \tag{4.48}$$

Let $x_k = x^* + \delta_k s_k$, where $s_k \in \mathbf{R}^n$, $\|s_k\| = 1$ and $\delta_k > 0$ for each k. Clearly, $\delta_k \to 0$ and there exists a convergent subsequence of the bounded sequence $\{s_k\}$ converging to some s^*. Without loss of generality, we assume that the sequence $\{s_k\}$ itself is convergent to s^*. Now by (4.47), (4.41) and (4.42), we have

$$0 \geq f(x_k) - f(x^*) = \delta_k \langle \nabla f(x^*), s_k\rangle + \frac{\delta_k^2}{2}\alpha(x_k, x^*) s_k^\top \nabla^2 f(x^*) s_k, \tag{4.49}$$

$$g_j(x_k) - g_j(x^*) = \delta_k \langle \nabla g_j(x^*), s_k\rangle + \frac{\delta_k^2}{2}\alpha(x_k, x^*) s_k^\top \nabla^2 g_j(x^*) s_k, \tag{4.50}$$

for each $j \in A(x^*)$, and

$$g_j(x_k) - g_j(x^*) = \delta_k \langle \nabla g_j(x^*), s_k\rangle + \frac{\delta_k^2}{2}\alpha(x_k, x^*) s_k^\top \nabla^2 g_j(x^*) s_k, \tag{4.51}$$

for each $j \in \{m_1 + 1, \cdots, m\}$. Then

$$\begin{aligned}
0 &\geq \frac{P(x_k, q_k) - P(x^*, q_k)}{\delta_k} \\
&= \frac{f(x_k) + q_k Q(\|(g_1^+(x_k), \cdots, g_{m_1}^+(x_k), g_{m_1+1}(x_k), \cdots, g_m(x_k))\|) - f(x^*)}{\delta_k} \\
&= \langle \nabla f(x^*), s_k\rangle + \frac{\delta_k}{2}\alpha(x_k, x^*) s_k^\top \nabla^2 f(x^*) s_k \\
&+ q_k \frac{Q(\|(g_1^+(x_k), \cdots, g_{m_1}^+(x_k), g_{m_1+1}(x_k), \cdots, g_m(x_k))\|)}{\|(g_1^+(x_k), \cdots, g_{m_1}^+(x_k), g_{m_1+1}(x_k), \cdots, g_m(x_k))\|^\beta} \\
&\quad \cdot \frac{\|(g_1^+(x_k), \cdots, g_{m_1}^+(x_k), g_{m_1+1}(x_k), \cdots, g_m(x_k))\|^\beta}{\delta_k^\beta} \cdot \delta_k^{\beta-1}.
\end{aligned}$$

Since $0 < \beta < 1$, the above inequality, combined with (4.48), (4.49), (4.50) and (4.51), yields

$$\langle \nabla f(x^*), s^*\rangle \leq 0,$$

$$\langle \nabla g_j(x^*), s^*\rangle \leq 0, \quad j \in A(x^*),$$

and

$$\langle \nabla g_j(x^*), s^*\rangle = 0, \quad j = m_1 + 1, \cdots, m.$$

Thus, $s^* \in V_1(x^*)$. Since (x^*, λ^*) satisfies the second order sufficient condition (4.20),

$$s^{*\top} \nabla^2 L(x^*, \lambda^*) s^* > 0. \tag{4.52}$$

Multiplying (4.50) and (4.51) by λ_j^*, add these to (4.49), and use (4.52), we obtain

$$\begin{aligned} & f(x_k) - f(x^*) + \textstyle\sum_{j=1}^m (\lambda_j^* g_j(x_k) - \lambda_j^* g_j(x^*)) \\ &= \tfrac{\delta_k^2}{2} \big(\alpha(x_k, x^*) s_k^\top \nabla^2 f(x^*) s_k + \textstyle\sum_{j=1}^m \alpha(x_k, x^*) \lambda_j^* s_k^\top \nabla^2 g_j(x^*) s_k \big) > 0, \end{aligned}$$

for sufficiently large k. Similar to the proof of in (4.16), (4.17) and (4.18) in Theorem 4.3.1, we have

$$P(x_k, q_k) - P(x^*, q_k) > 0,$$

which contradicts (4.45).

Case 2. $x_0 \neq x^*$. Similar to the proof of (4.29), (4.30) and (4.31) in Theorem 4.4.2, we have

$$\langle \nabla f(x^*), x_0 - x^* \rangle + \frac{1}{2} \alpha(x_0, x^*)(x_0 - x^*)^\top \nabla^2 f(x^*)(x_0 - x^*) \leq 0,$$

$$\langle \nabla g_j(x^*), x_0 - x^* \rangle + \frac{1}{2} \alpha(x_0, x^*)(x_0 - x^*)^\top \nabla^2 g_j(x^*)(x_0 - x^*) \leq 0,\ j \in A(x^*),$$

$$\langle \nabla g_j(x^*), x_0 - x^* \rangle + \frac{1}{2} \alpha(x_0, x^*)(x_0 - x^*)^\top \nabla^2 g_j(x^*)(x_0 - x^*) = 0,\ j = m_1 + 1, \cdots, m.$$

Therefore, $x_0 - x^* \in V_2(x^*)$. By the assumption, we get

$$(x_0 - x^*)^\top \nabla^2 L(x^*, \lambda^*)(x_0 - x^*) > 0.$$

Similarly, we can obtain (4.35)-(4.40) given in Theorem 4.4.2, and then

$$P(x_k, q_k) - P(x^*, q_k) > 0,$$

for k large enough, which contradicts (4.45).

Thus, there exists $q_0 > 0$ such that x^* is a strict global solution of the penalty problem (P_Q) for any $q > q_0$. ▮

If we only consider the following inequality constrained optimization problem

$$\begin{aligned} (P_0) \qquad & \inf\ f(x) \\ & \text{s.t. } x \in \mathbf{R}^n, \\ & \quad g_j(x) \leq 0, \quad j = 1, \cdots, m, \end{aligned}$$

where $f(x), g_j(x), j = 1, \cdots, m$ are convex quadratic functions. That is,

$$\begin{aligned} f(x) &= \tfrac{1}{2}x^\top A_0 x + \langle b_0, x\rangle + c_0, \\ g_j(x) &= \tfrac{1}{2}x^\top A_j x + \langle b_j, x\rangle + c_j, j = 1, \cdots, m, \end{aligned} \tag{4.53}$$

where A_j is an $l \times l$ positive semi-definite symmetric matrix, $b_j, x_j \in \mathbf{R}^n$ and c_j is a constant, $j = 0, 1, \cdots, m$. Let $x^* \in \mathbf{R}^n$, and $I(x^*) = \{j \in \{1, \cdots, m\} | g_j(x^*) = 0\}$.

Define

$$V_1'(x^*) = \left\{ y \in \mathbf{R}^n \middle| \begin{array}{c} \langle A_0 x^* + b_0, y\rangle \le 0, \\ \langle A_j x^* + b_j, y\rangle \le 0, \;\; j \in I(x^*) \\ y \neq 0, \end{array} \right\}, \tag{4.54}$$

and

$$V_2'(x^*) = \left\{ y \in \mathbf{R}^n \middle| \begin{array}{c} \langle A_0 x^* + b_0, y\rangle + \frac{1}{2}y^\top A_0 y \le 0, \\ \langle A_j x^* + b_j, y\rangle + \frac{1}{2}y^\top A_j y \le 0, \;\; j \in I(x^*) \\ y \neq 0, \end{array} \right\}. \tag{4.55}$$

Then, by Theorem 4.4.3, we have

Corollary 4.4.1 *Let Q satisfy (4.2) - (4.4) and $0 < \beta < 1$. Suppose that $f(x), g_j(x)$, $j = 1, \cdots, m$, are convex quadratic functions defined above, (x^*, λ^*) satisfies (4.20), and (4.21) with a strict inequality, where $V(x^*)$ is replaced by $V_1'(x^*)$. Assume that one of the following conditions holds:*

(i) $\limsup_{\|x\| \to \infty} f(x) = +\infty$*;*

(ii) $\liminf_{\|x\| \to \infty} g_j(x) = a > 0$*, for some* $j \in \{1, \cdots, m\}$.

Then there exists $q_0 > 0$ such that x^* is a strict global solution of the penalty problem (P_Q) for any $q > q_0$.

Proof: Since $f(x), g_j(x), j = 1, \cdots, m$, are convex quadratic functions, $V_2'(x^*) \subset V_1'(x^*)$. Therefore, the conclusion of this corollary follows from Theorem 4.4.3. ▌

Corollary 4.4.2 *Let $\beta = 1$ and $A_j x^* + b_j, j \in I(x^*)$, are linearly independent. Suppose that the other conditions in Corollary 4.4.1 are satisfied. Then there exists $q_0 > 0$ such that x^* is a strict global solution of the penalty problem (P_Q) for any $q > q_0$.*

Proof: It is easy to check, all conditions in Proposition 4.3.1 are satisfied. Thus, x^* is a strict local minimum for (P_0). Note that $f(x), g_j(x), j = 1, \cdots, m$, are convex, x^* is a strict global minimum for (P_0). Since $A_j x^* + b_j, j \in I(x^*)$, are linearly independent, the constraint qualification condition (4.8) hold at x^*. By Theorem 4.4 in [52], there exists $q_* > 0$ such that x^* is a local solution of the penalty problem (P_Q) for any $q > q_*$.

Now we claim that, there exists $q_0 > q_*$, such that x^* is a global solution of the penalty problem (P_Q) for any $q > q_0$. By contradiction, suppose that there exist $\{q_k\}$ and $\{x_k^*\}$ with $q_k > 0$, $x_k^* \neq x^*$, $q_k \to \infty$ such that

$$P(x_k^*, q_k) \leq P(x^*, q_k). \tag{4.56}$$

Similar to the proof of Theorem 4.4.2, we obtain that $\{x_k^*\}$ is bounded.

Let $q > q_*$, and let O be a compact set such that $x^*, x_k^* \in \mathrm{int}(O)$, where $k = 1, 2, \cdots$, and $\mathrm{int}(O)$ denotes the interior of O. Let

$$X = \{x \in O : g_j(x) \leq 0, j = 1, 2, \cdots, m\}.$$

Then X is a compact set. There exists a neighborhood $N(x^*, \delta_{x^*})$ of x^*, such that

$$P(x, q) > P(x^*, q), \quad \forall x \in N(x^*, \delta_{x^*}) \setminus \{x^*\}. \tag{4.57}$$

Furthermore, since $f(x) > f(x^*)$ for any $x \in X \setminus \{x^*\}$, there exists a neighborhood $N(x, \delta_x)$ of x, such that

$$f(y) > f(x^*), \quad \forall y \in N(x, \delta_x).$$

Consequently,

$$\begin{aligned} P(y, q) &= f(y) + qQ(\|(g_1^+(x), \cdots, g_m^+(x))\|) \\ &\geq f(y) > f(x^*) = P(x^*, q), \end{aligned} \tag{4.58}$$

for any

$$y \in \bigcup_{x \in X \setminus \{x^*\}} N(x, \delta_x).$$

By the compactness of X, there exist $x_i \in X, i = 1, 2, \cdots, K$, such that

$$X \subset \bigcup_{i=1,2,\cdots,K} N(x_i, \delta_{x_i}).$$

Therefore, it follows from (4.57) and (4.58) that

$$P(y, q) > P(x^*, q), \quad \forall y \in \bigcup_{i=1,2,\cdots,K} N(x_i, \delta_{x_i}) \setminus \{x^*\}. \tag{4.59}$$

Since $Q : \mathbf{R}_+ \to \mathbf{R}_+$ is a continuous function and satisfies (4.2), $O \setminus \bigcup_{i=1,2,\cdots,K} N(x_i, \delta_{x_i})$ is a compact set and

$$\|(g_1^+(y), \cdots, g_m^+(y))\| > 0, \forall y \in O \setminus \bigcup_{i=1,2,\cdots,K} N(x_i, \delta_{x_i}),$$

there exists a ρ_0, such that

$$Q(\|(g_1^+(y), \cdots, g_m^+(y))\|) > \rho_0, \forall y \in O \setminus \bigcup_{i=1,2,\cdots,K} N(x_i, \delta_{x_i}).$$

Let

$$q^* = \max\{q_*, \frac{a-b}{\rho_0}\},$$

where $a \geq \max_{y \in O} f(y), b \leq \min_{y \in O} f(y)$. Then when $q > q^*$,

$$\begin{aligned} P(y,q) &= f(y) + qQ(\|(g_1^+(x), \cdots, g_m^+(x))\|) \\ &> b + \tfrac{a-b}{\rho_0}\rho_0 + P(x^*, q) - f(x^*) \\ &\geq P(x^*, q). \end{aligned}$$

for any $O \setminus \bigcup_{i=1,2,\cdots,K} N(x_i, \delta_{x_i})$. This, combined with (4.59), yields that

$$P(y,q) > P(x^*, q), \tag{4.60}$$

for any $y \in O \setminus \{x^*\}$, when $q > q^*$. Note that $x_k^* \in \text{int}(O)$, $k = 1, 2, \cdots$, (4.56) contradicts (4.60). Therefore, the conclusion holds. ∎

Chapter 5

The Existence of a Solution for an Optimal Control Problem Governed by a Variational Inequality

5.1 Introduction

In this chapter, we study an optimal control problem where the state of the system is defined by a variational inequality problem. As an useful practical problem, the optimal control problem has attracted a lot of people to deal with it. In the literature, many authors have discussed similar problems concerning different aspects. See, [3, 7, 10, 11, 37, 38, 39, 76, 77, 124]. In [3], Adams et al. considered an optimal obstacle problem for an elliptic variational inequality for the homogeneous case, and established the existence and uniqueness of the optimal control problem. Chen studied an optimal control problem for a coupling system of a semilinear elliptic equation and an obstacle variational inequality in [21]. Lou considered the regularity of the obstacle control problem in [76, 77] for the homogeneous case with the major term being $p-$Laplacian. Bergounioux and Lenhart [12, 13] studied an obstacle optimal control problem for semilinear and bilateral obstacle problems. Recently, Ye and Chen [124] obtained the existence of an optimal control of the obstacle in a quasilinear elliptic variational inequality. Approximating the variational inequality by a family of quasilinear elliptic equations, and using the weak convergence methods, they establish some optimality conditions. In this chapter, we consider an optimal control problem governed by variational inequality with monotone type mappings in reflexive Banach spaces and apply our results to an optimal control problem governed by a quasilinear elliptic variational inequality.

The outline of this chapter is as follows. In Section 5.2, we present basic definitions, notations and some preliminary results. In Section 5.3, we establish some existence results of a solution of variational inequality problems for generalized pseudomonotone mappings and perturbed maximal monotone mappings respectively. In Section 5.4, we consider an optimal control problem governed by variational inequality with monotone type mappings in reflexive Banach spaces, which is more general than the problem (1.1) in Chapter 1. We obtain some existence results for this optimal control problem. In Section 5.5, as an application, we consider the optimal control problem governed by quasilinear elliptic variational inequality with an obstacle. By using the results obtained in Section 5.4, we obtain several existence results of optimal controls of this optimal control problem.

5.2 Preliminaries

In the following, let $\mathbf{X}$ be a real reflexive Banach space, $\mathbf{X}^*$ its dual space. We assume that the space $\mathbf{X}$ has been renormed so that $\mathbf{X}$ and its dual space are locally uniformly convex. Without confusion, the norms of $\mathbf{X}$ and $\mathbf{X}^*$ are denoted by the same notion $\|\cdot\|$. For $x \in \mathbf{X}$ and $x^* \in \mathbf{X}^*$, the symbol $\langle x^*, x\rangle$ stands for the value of x^* at x. Let $\mathbf{Y}$ be another real Banach space. For a multi-valued mapping $T : \mathbf{X} \to 2^{\mathbf{Y}}$, we set

$$D(T) = \{x \in \mathbf{X} : Tx \neq \emptyset\},$$
$$R(T) = \bigcup\{Tx : x \in D(T)\},$$

and

$$G(T) = \{[x, u] : x \in D(T), u \in R(T)\}.$$

$T : \mathbf{X} \to 2^{\mathbf{Y}}$ is said to be finitely continuous if T is upper semicontinuous from the topology of each finite-dimensional subspace F of $\mathbf{X}$ to the weak topology of $\mathbf{Y}$.

Definition 5.2.1 *Let $T : \mathbf{X} \to 2^{\mathbf{X}^*}$ be a multivalued mapping.*

(i) T is said to be monotone if, for any $x, y \in D(T)$, the inequality

$$\langle w - v, x - y\rangle \geq 0$$

holds for all $w \in T(x), v \in T(y)$*; A monotone mapping* T *is said to be maximal monotone, if* $\forall [y, v] \in G(T)$*, the inequality* $\langle w - v, x - y\rangle \geq 0$ *implies* $x \in D(T)$ *and* $w \in T(x)$*.*

(ii) T *is said to be pseudo-monotone if,* T *satisfies*

(m_1) $T(x)$ *is a nonempty, bounded, closed and convex subset of* $\mathbf{X}^*$ *for each* $x \in D(T)$*;*

(m_2) T *is finitely continuous;*

(m_3) *for each* $\{y_j\} \subset D(T)$*,* $w_j \in T(y_j)$ *satisfying* $y_j \rightharpoonup y_0 \in \mathbf{X}$ *and*

$$\limsup_{j\to\infty} \langle w_j, y_j - y_0\rangle \leq 0,$$

then, for every $x \in \mathbf{X}$*, there exists* $w(x) \in T(y_0)$ *such that*

$$\langle w(x), y_0 - x\rangle \leq \liminf_{j\to\infty} \langle w_j, y_j - x\rangle.$$

(iii) T *is said to be generalized pseudo-monotone if,* (m_1) *given above is satisfied and, for every pair of sequences* $\{y_j\}$ *and* $\{w_j\}$ *such that* $w_j \in T(y_j)$*,* $y_j \rightharpoonup y_0$*,* $w_j \rightharpoonup w_0$ *and*

$$\limsup_{j\to\infty} \langle w_j, y_j - y_0\rangle \leq 0,$$

then, we have $w_0 \in T(y_0)$ *and* $\langle w_j, y_j\rangle \to \langle w_0, y_0\rangle$*.*

(iv) T *is said to be of class* $(S)_+$ *if,* (m_1) *given above is satisfied and, for* $\{y_j\} \subset D(T)$*,* $y_j \rightharpoonup y_0 \in \mathbf{X}$*,* $w_j \in T(y_j)$ *satisfying* $\limsup_{j\to\infty} \langle w_j, y_j - y_0\rangle \leq 0$ *implies that* $y_j \to y_0$*.*

(v) T *is said to be quasi-bouneded if, for each* $M > 0$*, there exists a constant* $K(M) > 0$ *such that whenever* $[u, w]$ *lies in the* $G(T)$ *and* $\langle w, u\rangle \leq M\|u\|, \|u\| \leq M$*, then* $\|w\| \leq K(M)$*.*

(vi) T *is said to be demicontinuous if each* $y_n \to y_0$*,* $w_n \in T(y_n)$*, there exists a subsequence* $\{w_{n_k}\} \subset \{w_n\}$ *such that* $w_{n_k} \rightharpoonup w_0 \in T(y_0)$*.*

(vii) T *is said to be quasi-monotone if,* (m_1) *given above is satisfied and, for* $\{y_j\} \subset D(T)$*,* $y_j \rightharpoonup y_0 \in \mathbf{X}$*,* $w_j \in T(y_j)$*, we have*

$$\limsup_{j\to\infty} \langle w_j, y_j - y_0\rangle \geq 0.$$

Definition 5.2.2 *A mapping $T_0 : \mathbf{X} \to \mathbf{X}^*$ is said to be T-pseudo-monotone in Browder's sense ([16]), if for $\forall\{y_j\} \subset \mathbf{X}, y_j \rightharpoonup y \in \mathbf{X}$ and a bounded sequence $\{w_j\}$ with w_j in $T(y_j)$, suppose that $\limsup_{j\to\infty}\langle T_0(y_j), y_j - y\rangle \leq 0$. Then $T_0(y_j) \rightharpoonup T_0(y)$ and $\langle T_0(y_j), y_j - y\rangle \to 0$.*

By Pascali and Sburlan ([89] Proposition 1.3 on p.98 and Theorem 2.4 on p.106), we have

Proposition 5.2.1 A pseudo-monotone mapping is generalized pseudo-monotone. Moreover, a bounded generalized pseudo-monotone mapping which satisfies (m_1) is pseudo-monotone.

Proposition 5.2.2 Any maximal monotone mapping $A : \mathbf{X} \to 2^{\mathbf{X}^*}$ with $D(A) = \mathbf{X}$ is a pseudo-monotone,

For basic properties of monotone type mappings, we refer to [17, 18, 89, 126].

Let Φ be the set of all continuous and strictly increasing functions $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ with $\phi(0) = 0$ and $\lim_{r\to\infty}\phi(r) = \infty$. Let $\phi \in \Phi$ and define $J_\phi : \mathbf{X} \to 2^{\mathbf{X}^*}$ as follows:

$$J_\phi(x) = \{x^* \in \mathbf{X}^* : \langle x^*, x\rangle = \|x^*\|\ \|x\|, \|x^*\| = \phi(\|x\|)\}. \tag{5.1}$$

By the Hahn-Banach Theorem, $J_\phi(x) \neq \emptyset$ for any $x \in \mathbf{X}$. Since the space $\mathbf{X}$ and its dual are assumed to be locally uniformly convex, the duality mapping J_ϕ is single-valued and continuous. From [108], J_ϕ is monotone, and is a mapping of class $(S)_+$.

The following lemma can be found in Browder ([17] Lemma 1).

Lemma 5.2.1 *Let $\mathbf{E}$ be a Banach space, $\{x_n\}$ a sequence in $\mathbf{E}$, and $\{\alpha_n\}$ a sequence of positive constants with $\alpha_n \to 0$ as $n \to \infty$. Fix $r > 0$ and assume that for every $h \in \mathbf{E}^*$ with $\|h\| \leq r$ there exists a constant C_h such that $\langle h, x_n\rangle \leq \alpha_n\|x_n\| + C_h$, for all n. Then the sequence $\{x_n\}$ is bounded.*

Let $0 \in K$, and K be a closed subset of $\mathbf{X}$, $T : \mathbf{X} \to 2^{\mathbf{X}^*}$ with $D(T) = K$ which satisfies the following condition:

(a) there exists $x_* \in K$ such that

$$\lim_{y\in K, \|y\|\to\infty}\ \inf_{w\in T(y)}\langle w, y - x_*\rangle > 0. \tag{5.2}$$

If there exist $y_0 \in K$, and $w_0 \in T(y_0)$ such that

$$\langle w_0, x - y_0 \rangle \geq 0, \quad \forall x \in K,$$

we call that the variational inequality problem (denote as VIP(T, K)) has a solution.

We denote by Λ the family of all finite-dimensional subspaces F of $\mathbf{X}$, with F containing x_* of condition (a), ordered by inclusion. Let $K_F = K \bigcap F$. For each $F \in \Lambda$, we set

$$V_F = \{y \in K : \text{ there exists } w \in T(y) \text{ such that } \forall x \in K_F, \langle w, x - y \rangle \geq 0\}.$$

Let $\overline{V_F}^w$ be the weak closure of V_F.

Lemma 5.2.2 *([60] Lemma 2) Let K be a nonempty, closed and convex subset of $\mathbf{X}$ and let $T : \mathbf{X} \to 2^{\mathbf{X}^*}$ be a multivalued mapping with $D(T) = K$ and $T(x)$ a nonempty, bounded, closed and convex subset of $\mathbf{X}^*$ for each $x \in K$. Suppose that T satisfies condition (a) and, for each $F \in \Lambda$, $T : K_F \to 2^{\mathbf{X}^*}$ is upper semicontinuous from the topology of F to the weak topology of E^*. Then $\bigcap_{F\in\Lambda} \overline{V_F}^w \neq \emptyset$.*

Lemma 5.2.3 *Let K be a nonempty, closed and convex subset of $\mathbf{X}$ and let $T : \mathbf{X} \to 2^{\mathbf{X}^*}$ be a demicontinous mapping of $(S)_+$ with $D(T) = K$. Assume that T satisfies condition (a). Then the variational inequality problem VIP(T, K) has a solution.*

Proof: By Lemma 5.2.2, there exists a $y_0 \in \bigcap_{F\in\Lambda} \overline{V_F}^w$. For each $x \in K$, we can find a $F \in \Lambda$ such that $x \in F$, $y_0 \in F$. Since $y_0 \in \overline{V_F}^w$, there exists $\{y_n\} \subset V_F$ such that $y_n \rightharpoonup y_0$. By the definition of V_F, there exists $w_n \in T(y_n)$ such that

$$\langle w_n, x - y_n \rangle \geq 0, \tag{5.3}$$

and

$$\langle w_n, y_0 - y_n \rangle \geq 0.$$

Consequently,

$$\limsup_{n\to\infty} \langle w_n, y_n - y_0 \rangle \leq 0. \tag{5.4}$$

Since T is a demicontinous mapping of $(S)_+$, it follows from (5.4) that we have $\lim_{n\to\infty} y_n = y_0$, and there exists a subsequence $\{w_{n_k}\} \subset \{w_n\}$ such that $w_{n_k} \rightharpoonup w_0 \in T(y_0)$. Hence, by (5.3), we have $(w_0, x - y_0) \geq 0, \forall x \in K$. ∎

Lemma 5.2.4 *Let K be a nonempty, closed and convex subset of a reflexive Banach space $\mathbf{X}$ and let $T : K \to 2^{\mathbf{X}^*}$ be a bounded generalized pseudo-monotone mapping with conditions $(m_1), (m_2)$ and $D(T) = K$. Assume that T satisfies condition (a). Then the variational inequality problem VIP(T, K) has a solution.*

Proof: Similar to the proof of Lemma 5.2.3, again by Lemma 5.2.2, there exists a $y_0 \in \bigcap_{F\in\Lambda} \overline{V_F}^w$. For each $x \in K$, we can find a $F \in \Lambda$ such that $x \in F$, $y_0 \in F$. Since $y_0 \in \overline{V_F}^w$, there exists a sequence $\{y_n\} \subset V_F$ such that $y_n \to y_0$. By the definition of V_F, there exists $w_n \in T(y_n)$ such that (5.3) and (5.4) hold. Since $\{y_n\}$ is bounded and T is a bounded generalized pseudo-monotone mapping, $\{w_n\}$ is bounded. Thus, there exists a subsequence $\{w_{n_k}\} \subset \{w_n\}$ such that $w_{n_k} \rightharpoonup w_0$. From (5.4), we have $w_0 \in T(y_0)$ and

$$\langle w_{n_k}, y_{n_k}\rangle \to \langle w_0, y_0\rangle.$$

Hence by (5.3), we have

$$(w_0, x - y_0) \geq 0, \quad \forall x \in K.$$

∎

Lemma 5.2.5 *([60]) Let K be a nonempty, closed and convex subset of $\mathbf{X}$. Let $T : \mathbf{X} \to 2^{\mathbf{X}^*}$ is a bounded maximal monotone mapping such that $D(T) = \mathbf{X}$ and $T_0 : K \to \mathbf{X}^*$ is a finitely continuous T-pseudo-monotone mapping. Assume there exists $x_* \in K$ such that the following condition is satisfied,*

$$(d) \lim_{y\in K,\|y\|\to\infty} \inf_{w\in T(y)} \langle w + T_0(y), y - x_*\rangle > 0.$$

Then VIP($T + T_0, K$) has a solution.

5.3 Existence of variational inequalities for perturbed maximal monotone mappings

We denote by Γ the set of all functions $\beta : \mathbf{R}_+ \to \mathbf{R}_+$ such that $\beta(r) \to 0$ as $r \to \infty$.

Theorem 5.3.1 *Let K be a nonempty, closed and convex subset of $\mathbf{X}$, $0 \in K$ and let $T : \mathbf{X} \to 2^{\mathbf{X}^*}$ be a bounded finitely continuous generalized pseudo-monotone mapping with $D(T) = K$. Assume that T satisfies the following condition:*

(b) For every $s \in \mathbf{X}^*$ with $\|s\| \leq r$ there exist $x_s \in K$ and $\beta = \beta_s \in \Gamma$ such that

$$\inf_{w \in T(y)} \langle w - s, y - x_s \rangle \geq -\beta(\|y\|)\|y\|.$$

Then there exist $y_0 \in K$, and $w_0 \in T(y_0)$ such that

$$\langle w_0, x - y_0 \rangle \geq 0, \quad \forall x \in K.$$

Proof: First, we claim that, for any $[y_j, w_j] \in G(T)$ satisfying $y_j \rightharpoonup y_0 \in K$,

$$\limsup_{j \to \infty} \langle w_j, y_j - y_0 \rangle \geq 0. \tag{5.5}$$

In fact, if it is not so, there exist $y_j \in K, w_j \in T(y_j), y_j \rightharpoonup y_0 \in K$, such that

$$\limsup_{j \to \infty} \langle w_j, y_j - y_0 \rangle < 0. \tag{5.6}$$

From the boundedness and the generalized pseudo-monotonicity of T, we obtain that, there exists a subsequence $\{w_{j_k}\} \in \{w_j\}$ such that $w_{j_k} \rightharpoonup w_0 \in T(y_0)$, and

$$\lim_{k \to \infty} \langle w_{j_k}, y_{j_k} \rangle = \langle w_0, y_0 \rangle.$$

Therefore,

$$\lim_{k \to \infty} \langle w_{j_k}, y_{j_k} - y_0 \rangle = 0,$$

which contradicts (5.6).

Next, we claim that $T + \frac{1}{n} J_\phi$ is a mappings of $(S)_+$ for a fixed positive integer n and $\phi \in \Phi$. In fact, for any $[y_j, w_j] \in G(T)$ and $v_j^J \in J_\phi(y_j)$ satisfying $y_j \rightharpoonup y_0 \in K$,

$$\limsup_{j \to \infty} \langle w_j + \frac{1}{n} v_j^J, y_j - y_0 \rangle \leq 0.$$

By (5.5), we may assume without loss of generality that

$$\lim_{j \to \infty} \langle w_j, y_j - y_0 \rangle \geq 0.$$

Thus,

$$\limsup_{j \to \infty} \langle v_j^J, y_j - y_0 \rangle \leq n(\limsup_{j \to \infty} \langle w_j + \frac{1}{n} v_j^J, y_j - y_0 \rangle - \lim_{j \to \infty} \langle w_j, y_j - y_0 \rangle) \leq 0.$$

Since J_ϕ is a mapping of class $(S)_+$, $y_j \to y_0$. Therefore, $T + \frac{1}{n} J_\phi$ is a mappings of $(S)_+$.

Then, we claim that, for each positive integer n, $T + \frac{1}{n}J_\phi$ satisfies (5.2) in condition (a). Let $y \in K$, $y \neq 0$, $v_J \in J_\phi(y)$ and $w \in T(y)$. Since T is bounded, T is quasibounded. By Lemma 2.1 in [49], there exists a $\phi \in \Phi$ such that $\langle w, y\rangle \geq -\phi(\|y\|)$. We may choose the same $\phi \in \Phi$ such that

$$\langle w + \frac{1}{n}v_J, y\rangle \geq -\phi(\|y\|) + \frac{1}{n}\phi(\|y\|)\|y\| = (\frac{1}{n} - \frac{1}{\|y\|})\phi(\|y\|)\|y\|.$$

Thus, $T + \frac{1}{n}J_\phi$ satisfies (5.2) in condition (a).

Therefore, all conditions in Lemma 5.2.3 are satisfied. By Lemma 5.2.3, for each positive integer n, there exist $y_n \in K, w_n \in T(y_n)$ such that

$$\langle w_n + \frac{1}{n}Jy_n, x - y_n\rangle \geq 0, \quad \forall x \in K.$$

Consequently,

$$\langle w_n, y_n - x\rangle \leq -\frac{1}{n}\phi(\|y_n\|)(\|y_n\| - \|x\|), \quad \forall x \in K. \tag{5.7}$$

We claim that $\{y_n\}$ is bounded. If not so, then without loss of generality, we may assume that $\|y_n\| \to \infty$ as $n \to \infty$. Since T satisfies the condition (b), from (5.7) we have

$$\begin{aligned} -\beta(\|y_n\|)\|y_n\| &\leq \langle w_n - s, y_n - x_s\rangle \\ &\leq -\langle s, y_n - x_s\rangle - \tfrac{1}{n}\phi(\|y_n\|)(\|y_n\| - \|x_s\|) \\ &\leq -\langle s, y_n - x_s\rangle, \end{aligned}$$

for all large n, since $\|y_n\| \to \infty$ as $n \to \infty$. Therefore

$$\langle s, y_n\rangle \leq \langle s, x_s\rangle + \beta(\|y_n\|)\|y_n\|.$$

By Lemma 5.2.1, $\{y_n\}$ is bounded. This contradicts $\|y_n\| \to \infty$ as $n \to \infty$. It follows from the boundedness of T that $\{w_n\}$ is bounded. Without loss of generality, we may assume that $y_n \rightharpoonup y_0$ and $w_n \rightharpoonup w_0$. Again by (5.7), we get

$$\limsup_{n\to\infty}\langle w_n, y_n - x\rangle \leq 0, \quad \forall x \in K. \tag{5.8}$$

Thus, $\limsup_{n\to\infty}\langle w_n, y_n - y_0\rangle \leq 0$. Since T is a generalized pseudo-monotone mapping, $w_0 \in T(y_0)$ and $\lim_{n\to\infty}\langle w_n, y_n\rangle = \langle w_0, y_0\rangle$. Therefore, by (5.8), we have

$$\langle w_0, x - y_0\rangle \geq 0, \quad \forall x \in K.$$

∎

It is noted that the coercive condition (b) has been used in [48, 49], which is weaker than the one in Browder's result (see [17]).

Theorem 5.3.2 *Let K be a nonempty, closed and convex subset of $\mathbf{X}$, $0 \in K$, and let $A : \mathbf{X} \to 2^{\mathbf{X}^*}$ be a maximal monotone mapping (not necessarily bounded) with $D(A) = \mathbf{X}$ and $[0,0] \in G(A)$, $T : \mathbf{X} \to 2^{\mathbf{X}^*}$ be a quasi-bounded finitely continuous generalized pseudo-monotone mapping with $D(T) = K$. Assume that $A+T$ satisfies the condition (b) given in Theorem 5.3.1. Then VIP($A+T, K$) has a solution.*

Proof: By Pascali and Sburlan ([89] Lemma on p.142), monotone mapping $A : \mathbf{X} \to 2^{\mathbf{X}^*}$ with $0 \in IntD(A)$ is quasi-bounded. It is clear that, for any $[y_j, a_j] \in G(A)$ satisfying $y_j \rightharpoonup y_0 \in K$, we have

$$\limsup_{j \to \infty} \langle a_j, y_j - y_0 \rangle \geq 0. \tag{5.9}$$

Since T is a bounded generalized pseudo-monotone mapping, (5.5) given in the proof of Theorem 5.3.1 also holds. Since J_ϕ is a mapping of class $(S)_+$, by (5.5) and (5.9), we conclude that $A + T + \frac{1}{n} J_\phi$ is a mappings of $(S)_+$ for a positive integer n.

Let $y \in K$, $y \neq 0$, $v_J \in J_\phi(y)$ and $w \in T(y)$. The monotonicity of A and $[0,0] \in G(A)$ imply that $\forall a \in A(y), \langle a, y \rangle \geq 0$. By the boundedness of T and Lemma 2.1 in [49], there exists a $\phi \in \Phi$ such that $\langle w, y \rangle \geq -\phi(\|y\|)$. Since

$$\begin{aligned} \langle a + w + \tfrac{1}{n} v_J, y \rangle &\geq \langle w + \tfrac{1}{n} v_J, y \rangle \\ &\geq -\phi(\|y\|) + \tfrac{1}{n}\phi(\|y\|)\|y\| \\ &= (\tfrac{1}{n} - \tfrac{1}{\|y\|})\phi(\|y\|)\|y\|, \end{aligned} \tag{5.10}$$

$A + T + \frac{1}{n} J_\phi$ satisfies the condition (a). By Lemma 5.2.3, there exist $y_n \in K, a_n \in A(y_n), w_n \in T(y_n)$ and $v_n \in J_\phi(y_n)$ such that

$$\langle a_n + w_n + \frac{1}{n} v_n, x - y_n \rangle \geq 0, \quad \forall x \in K. \tag{5.11}$$

Thus,

$$\langle a_n + w_n, y_n - x \rangle \leq -\frac{1}{n}\phi(\|y_n\|)(\|y_n\| - \|x\|), \quad \forall x \in K. \tag{5.12}$$

We claim that $\{y_n\}$ is bounded. If it is not so, then without loss of generality, we may assume that $\|y_n\| \to \infty$ as $n \to \infty$. Thus, $\|y_n\| - \|x\| > 0$ for all large n. Since T satisfies the condition (b), For every $s \in \mathbf{X}^*$ with $\|s\| \leq r$ there exist $x_s \in K$ and $\beta = \beta_s \in \Gamma$ such that

$$\inf_{w \in T(y)} \langle w - s, y - x_s \rangle \geq -\beta(\|y\|)\|y\|.$$

Hence, from (5.12), we have

$$\begin{aligned}-\beta(\|y_n\|)\|y_n\| &\le \langle x_n + w_n - s, y_n - x_s\rangle \\ &\le \langle a_n + w_n, y_n - x_s\rangle - \langle s, y_n - x_s\rangle \\ &\le -\langle s, y_n - x_s\rangle,\end{aligned}$$

for all large n. Therefore

$$\langle s, y_n\rangle \le \langle s, x_s\rangle + \beta(\|y_n\|)\|y_n\|.$$

By Lemma 5.2.1, $\{y_n\}$ is bounded. This contradicts $\|y_n\| \to \infty$ as $n \to \infty$. Thus, $\{y_n\}$ is bounded. Then $\frac{1}{n}\phi(\|y_n\|) \to 0$ as $n \to \infty$. It follows from the boundedness of T that $\{w_n\}$ is bounded. By (5.12), we have $\|y_n\| \le M$ and

$$\langle a_n, y_n\rangle \le -\langle w_n, y_n\rangle - \frac{1}{n}\phi(\|y_n\|)\|y_n\| \le K(M)\|y_n\|,$$

where M and $K(M)$ are some positive constants. Recall that A is quasi-bounded. Then $\|a_n\| \le K(M)$. Without loss of generality, we may assume that $y_n \rightharpoonup y_0$, $a_n \rightharpoonup a_0$ and $w_n \rightharpoonup w_0$. Again by (5.11), we get

$$\limsup_{n\to\infty}\langle a_n + w_n, y_n - y_0\rangle \le 0. \tag{5.13}$$

Let $\bar{a} \in A(y_0)$, $\langle a_n - \bar{a}, y_n - y_0\rangle \ge 0$. Therefore,

$$\limsup_{n\to\infty}\langle w_n, y_n - y_0\rangle \le -\liminf_{n\to\infty}\langle a_n - \bar{a}, y_n - y_0\rangle + \lim_{n\to\infty}\langle \bar{a}, y_n - y_0\rangle \le 0.$$

Since T is a generalized pseudo-monotone mapping, $w_0 \in T(y_0)$ and $\lim_{n\to\infty}\langle w_n, y_n\rangle = \langle w_0, y_0\rangle$. This and (5.13) imply $\limsup_{n\to\infty}\langle a_n, y_n - y_0\rangle \le 0$. By Pascali and Sburlan ([89] Proposition 1.3 on p.98 and Theorem 2.4 on p.106), the maximal monotone mapping $A : \mathbf{X} \to 2^{\mathbf{X}^*}$ with $D(A) = \mathbf{X}$ is pseudo-monotone, and hence is generalized pseudo-monotone. So, $a_0 \in T(y_0)$ and $\lim_{n\to\infty}\langle a_n, y_n\rangle = \langle a_0, y_0\rangle$.

Therefore, by (5.11), we have $\langle a_0 + w_0, x - y_0\rangle \ge 0, \forall x \in K$. ∎

Theorem 5.3.3 *Let K be a nonempty, closed and convex subset of a reflexive Banach space $\mathbf{X}$, $0 \in K$, and let $T : \mathbf{X} \to 2^{\mathbf{X}^*}$ be a maximal monotone mapping (not necessarily bounded) with $D(T) = \mathbf{X}$ and $[0, 0] \in G(T)$, $T_0 : \mathbf{X} \to 2^{\mathbf{X}^*}$ be a bounded finitely continuous generalized pseudo-monotone mapping with $D(T_0) = K$. Assume T_0 satisfies the following condition:*

(c) there exists $x_* \in K$ such that

$$\lim_{y\in K, \|y\|\to\infty}\ \inf_{v\in T_0(y)} \frac{\langle v, y - x_*\rangle}{\|y\|} = +\infty.$$

Then VIP$(T + T_0, K)$ has a solution.

Proof: For $\forall \varepsilon > 0$, we consider the generalized *Yosida* approximations of T: $T_\varepsilon = (T^{-1} + \varepsilon J^{-1})^{-1}$, where J is the duality mapping, that is $J = J_\phi$, for $\phi(x) = x$, J_ϕ is defined by (5.1). By Proposition 12 in [18], $T_\varepsilon : \mathbf{X} \to 2^{\mathbf{X}^*}$ is a bounded maximal monotone mapping with $D(T_\varepsilon) = \mathbf{X}$ and $0 = T_\varepsilon(0)$, thus T_ε is a generalized pseudo-monotone mapping. Let $\hat{w} \in T_\varepsilon(x_*)$. The monotonicity of T_ε implies that $\forall w_\varepsilon \in T_\varepsilon(y), \langle w_\varepsilon - \hat{w}, y - x_* \rangle \geq 0$. Hence, from the condition (c), we have

$$\begin{aligned}
&\lim_{y \in \mathbf{X}, \|y\| \to \infty} \inf_{w_\varepsilon \in T_\varepsilon(y), v \in T_0(y)} \frac{\langle w_\varepsilon + v, y - x_* \rangle}{\|y\|} \\
&\geq \lim_{y \in E, \|y\| \to \infty} \inf_{v \in T_0(y)} \frac{\{\langle v, y - x_* \rangle + \langle \hat{w}, y - x_* \rangle\}}{\|y\|} \\
&= +\infty.
\end{aligned} \tag{5.14}$$

That is, $T_\varepsilon + T_0$ satisfies the condition (a). By Lemma 5.2.4, there exist $y_\varepsilon \in K, w_\varepsilon \in T_\varepsilon(y_\varepsilon)$ and $v_\varepsilon \in T_0(y_\varepsilon)$ such that

$$\langle w_\varepsilon + v_\varepsilon, x - y_\varepsilon \rangle \geq 0, \forall x \in K. \tag{5.15}$$

We claim that there exists $0 < \varepsilon < \varepsilon_0$, such that $\{y_\varepsilon\}$ $(\varepsilon > 0)$ is bounded. If it is not so, then without loss of generality, we may assume that $\|y_{\varepsilon_j}\| \to \infty$ as $j \to \infty$, where $\varepsilon_j \to 0$ as $j \to \infty$. It follows from (5.14) and (5.15) that

$$0 \geq \lim_{j \to \infty} \inf_{w_{\varepsilon_j} \in T_{\varepsilon_j}(y_{\varepsilon_j}), v_{\varepsilon_j} \in T_0(y_{\varepsilon_j})} \frac{\langle w_{\varepsilon_j} + v_{\varepsilon_j}, y_{\varepsilon_j} - x_* \rangle}{\|y_{\varepsilon_j}\|} = +\infty,$$

which is impossible. Thus, $\{y_\varepsilon\}$ $(\varepsilon > 0)$ is bounded. Consequently, $\{w_\varepsilon\}$ and $\{v_\varepsilon\}$ are bounded as well. There exists a $M > 0$ such that $\|y_\varepsilon\| \leq M, \|w_\varepsilon\| \leq M, \|v_\varepsilon\| \leq M (0 < \varepsilon < \varepsilon_0)$. It implies from the definition of T_ε that $y_\varepsilon \in (T^{-1} + \varepsilon J^{-1}) w_\varepsilon$. Hence there exists $x_\varepsilon \in D(T)$ such that $w_\varepsilon \in T(x_\varepsilon)$ and $\varepsilon w_\varepsilon \in J(y_\varepsilon - x_\varepsilon)$, that is

$$\|y_\varepsilon - x_\varepsilon\| = \varepsilon \|w_\varepsilon\| \leq \varepsilon M \leq \varepsilon_0 M. \tag{5.16}$$

We have $\|x_\varepsilon\| \leq (\varepsilon_0 + 1)M$. We may assume that $y_{\varepsilon_j} \rightharpoonup y_0, w_{\varepsilon_j} \rightharpoonup w_0, v_{\varepsilon_j} \rightharpoonup v_0$ and $x_{\varepsilon_j} \rightharpoonup x_0$, where $\varepsilon_j \to 0$ as $j \to \infty$. By (5.16), we get $x_0 = y_0$ and

$$\lim_{j \to \infty} \langle w_{\varepsilon_j}, x_{\varepsilon_j} - y_{\varepsilon_j} \rangle = 0. \tag{5.17}$$

Let $\bar{w} \in T(y_0)$. Since $w_{\varepsilon_j} \in T(x_{\varepsilon_j})$ and T is a monotone mapping,

$$\langle w_{\varepsilon_j}, x_{\varepsilon_j} - y_0 \rangle = \langle w_{\varepsilon_j} - \bar{w}, x_{\varepsilon_j} - y_0 \rangle + \langle \bar{w}, x_{\varepsilon_j} - y_0 \rangle \geq \langle \bar{w}, x_{\varepsilon_j} - y_0 \rangle.$$

Thus,

$$\liminf_{j \to \infty} \langle w_{\varepsilon_j}, x_{\varepsilon_j} - y_0 \rangle \geq 0. \tag{5.18}$$

It implies from (5.15), (5.17) and (5.18) that

$$\begin{aligned}\limsup_{j\to\infty}\langle v_{\varepsilon_j}, y_{\varepsilon_j}-y_0\rangle &\le \limsup_{j\to\infty}(\langle w_{\varepsilon_j}+v_{\varepsilon_j}, y_{\varepsilon_j}-y_0\rangle - \langle w_{\varepsilon_j}, y_{\varepsilon_j}-y_0\rangle)\\ &\le -\liminf_{j\to\infty}\langle w_{\varepsilon_j}, x_{\varepsilon_j}-y_0\rangle + \lim_{j\to\infty}\langle w_{\varepsilon_j}, x_{\varepsilon_j}-y_{\varepsilon_j}\rangle\\ &\le 0.\end{aligned}$$

It implies from the generalized pseudo-monotonicity of T_0 that $v_0 \in T_0(y_0)$ and

$$\lim_{j\to\infty}\langle v_{\varepsilon_j}, y_{\varepsilon_j}\rangle = \langle v_0, y_0\rangle. \tag{5.19}$$

This, combined with (5.15) and $w_{\varepsilon_j} \in T(x_{\varepsilon_j})$, yields

$$\limsup_{j\to\infty}\langle w_{\varepsilon_j}, x_{\varepsilon_j}-y_0\rangle \le \limsup_{j\to\infty}\langle w_{\varepsilon_j}, x_{\varepsilon_j}-y_{\varepsilon_j}\rangle + \limsup_{j\to\infty}\langle w_{\varepsilon_j}, y_{\varepsilon_j}-y_0\rangle \le 0.$$

Thus, by the generalized pseudo-monotonicity of T, we have $w_0 \in T(w_0)$ and

$$\lim_{j\to\infty}\langle w_{\varepsilon_j}, y_{\varepsilon_j}\rangle = \lim_{j\to\infty}\langle w_{\varepsilon_j}, x_{\varepsilon_j}\rangle + \lim_{j\to\infty}\langle w_{\varepsilon_j}, y_{\varepsilon_j}-x_{\varepsilon_j}\rangle = \langle w_0, y_0\rangle. \tag{5.20}$$

Again using (5.15), (5.19) and (5.20), we obtain $w_0 \in T(w_0)$, $v_0 \in T_0(y_0)$ and

$$(w_0 + v_0, x - y_0) \ge 0, \forall x \in K.$$

∎

Theorem 5.3.4 *Let $T : \mathbf{X} \to 2^{\mathbf{X}^*}$ be a maximal monotone mapping (not necessarily bounded) with $D(T) = \mathbf{X}$ and $T_0 : K \to \mathbf{X}^*$ be a quasi-bounded finitely continuous T-pseudo-monotone mapping. Assume $0 \in K$ and there exists $x_* \in K$ such that the condition (c) given in Theorem 5.3.3 is satisfied. Then VIP($T + T_0, K$) has a solution.*

Proof: We still consider the generalized *Yosida* approximations of T: $T_\varepsilon = (T^{-1} + \varepsilon J^{-1})^{-1}$. Then $T_\varepsilon : E \to 2^{E^*}$ is a bounded maximal monotone mapping with $D(T_\varepsilon) = E$ and $0 = T_\varepsilon(0)$. Similar to the proof of Theorem 5.3.3, we can prove that $T_\varepsilon + T_0$ satisfies the condition (a). Thus, by Lemma 5.2.5, there exist $y_\varepsilon \in K, w_\varepsilon \in T_\varepsilon(y_\varepsilon)$ such that

$$\langle w_\varepsilon + T_0(y_\varepsilon), x - y_\varepsilon\rangle \ge 0, \forall x \in K. \tag{5.21}$$

Similarly, $\{y_\varepsilon\}$, $\{T_0(y_\varepsilon)\}$ and $\{w_\varepsilon\}$ ($\varepsilon > 0$) are bounded. Since $w_\varepsilon \in T_\varepsilon(y_\varepsilon)$,

$$y_\varepsilon \in (T^{-1} + \varepsilon J^{-1})w_\varepsilon.$$

Hence there exists $x_\varepsilon \in D(T)$ such that $w_\varepsilon \in T(x_\varepsilon)$ and $\varepsilon w_\varepsilon \in J(y_\varepsilon - x_\varepsilon)$, that is

$$\|x_\varepsilon\| \leq \|y_\varepsilon - x_\varepsilon\| + \|y_\varepsilon\| = \varepsilon\|w_\varepsilon\| + \|y_\varepsilon\|.$$

Thus $\{x_\varepsilon\}$ $(\varepsilon > 0)$ is bounded. Passing if necessary to a subsequence, we may assume that $y_{\varepsilon_j} \rightharpoonup y_0, w_{\varepsilon_j} \rightharpoonup w_0, T_0(y_{\varepsilon_j}) \rightharpoonup v_0$ and $x_{\varepsilon_j} \rightharpoonup x_0$, where $\varepsilon_j \to 0$ as $j \to \infty$. The generalized pseudo-monotonicity of T_ε and Proposition 2 of [18] that

$$\liminf_{j\to\infty} \langle w_{\varepsilon_j}, x_{\varepsilon_j} - y_0\rangle \geq 0.$$

This and (5.21) imply,

$$\begin{aligned}
\limsup\nolimits_{j\to\infty}\langle T_0(y_{\varepsilon_j}), y_{\varepsilon_j} - y_0\rangle &\leq \limsup\nolimits_{j\to\infty}(\langle w_{\varepsilon_j} + T_0(y_{\varepsilon_j}), y_{\varepsilon_j} - y_0\rangle - \langle w_{\varepsilon_j}, y_{\varepsilon_j} - y_0\rangle) \\
&\leq -\liminf\nolimits_{j\to\infty}\langle w_{\varepsilon_j}, x_{\varepsilon_j} - y_0\rangle + \lim\nolimits_{j\to\infty}\langle w_{\varepsilon_j}, x_{\varepsilon_j} - y_{\varepsilon_j}\rangle \\
&\leq 0.
\end{aligned}$$

Since T_0 is T-pseudo-monotone, we have

$$\lim\langle T_0(y_{\varepsilon_j}), y_{\varepsilon_j}\rangle = \langle v_0, y_0\rangle,$$

and $v_0 = T_0(y_0)$. Again by (5.21), we obtain that

$$\limsup_{j\to\infty}\langle w_{\varepsilon_j}, y_{\varepsilon_j} - y_0\rangle \leq 0.$$

Hence, $\lim_{j\to\infty}\langle w_{\varepsilon_j}, y_{\varepsilon_j}\rangle = \langle w_0, y_0\rangle$, and $w_0 \in T(w_0)$. Therefore,

$$(w_0 + v_0, x - y_0) \geq 0, \forall x \in K.$$

∎

Remark 5.3.1 Since we establish some existence results of a solution of variational inequality problems for generalized pseudo-monotone or T-pseudo-monotone perturbations of maximal monotone mappings, by using the same method as the ones in [60], we can obtain the surjective results for generalized pseudo-monotone or T-pseudo-monotone perturbations of maximal monotone mappings.

5.4 Existence of an optimal control governed by a class of monotone type variational inequality

In this section, we assume that $\mathbf{W}, \mathbf{X}$ are two reflexive Banach spaces, U is a nonempty, closed set of $\mathbf{W}$ and K is a closed and convex cone of $\mathbf{X}$. Let J :

$U \times K \to \mathbf{R}$ be a real-valued function and $A : K \to \mathbf{X}^*$, $F : K \to \mathbf{X}^*$, $B : U \to \mathbf{X}^*$ be three given mappings. Consider the following optimal control problem governed by a variational inequality:

$$\begin{aligned} &\min \quad J(w, u) \\ &\text{subject to } (w, u) \in U \times K, \text{and } u \in S(w), \end{aligned} \tag{5.22}$$

where, for each $w \in U$, $S(w)$ is the solution set of the following abstract variational inequality problem:

$$\text{Find } u \in K : \langle A(u), v - u \rangle \geq \langle F(u) - B(w), v - u \rangle, \forall v \in K. \tag{5.23}$$

The optimal control problem (5.22) considered here is more general than (1.1) in the following aspects:

(i) A and B do not need to be linear, $F(u)$ does not need to be a constant function in $\mathbf{X}^*$ as in (1.1). In the following theorems, we will assume that A or $A - F$ is a monotone type mapping.

(ii) Hilbert spaces are replaced by reflexive Banach spaces.

In order to obtain the existence of an optimal control for problem (5.22), we should ensure the existence of a solution of the variational inequality (5.23). That is, we need to have $S(w) \neq \phi$. In the following we will give a sufficient condition to ensure that $S(w) \neq \phi$. Recall that the definitions of monotone type mappings have been introduced in Definition 5.2.1.

Lemma 5.4.1 *Assume that $\mathbf{W}, \mathbf{X}$ are two reflexive Banach spaces, U is a nonempty, closed set of $\mathbf{W}$ and K is a closed and convex cone of $\mathbf{X}$. Suppose that, for each $w \in U$, the following coercive condition is satisfied:*

$$\lim_{(w,u) \in U \times K, \|(w,u)\| \to +\infty} \langle (A - F)(u) + B(w), u \rangle = +\infty. \tag{5.24}$$

Moreover, assume that one of the following conditions is satisfied:

(i) A is a continuous mapping of class $(S)_+$ and F is a continuous and compact mapping,

(ii) $A - F$ is a continuous, bounded and generalized pseudo-monotone mapping.

Then, for each $w \in U$, $S(w) \neq \phi$, i.e., the variational inequality problem (5.23) *has a solution.*

Proof: Notice that K be unbounded. Taking $B_r = \{v \in \mathbf{X} : \ \|v\| \leq r\}$ and letting $K_r = B_r \cap K$, we get that K_r is a bounded, closed and convex subset of $\mathbf{X}$. We claim that, for each $w \in U$, there exists $u_r \in K_r$ such that

$$\langle A(u_r), v - u_r\rangle \geq \langle F(u_r) - B(w), v - u_r\rangle, \forall v \in K_r. \tag{5.25}$$

Indeed the arguments used in the proof of [130, Theorem 1] show that

$$\cap_{M\in\mathbf{\Sigma}}\mathrm{clw}(V_{r,M}) \neq \emptyset,$$

where $\mathbf{\Sigma}$ is the family of all finite-dimensional subspaces M of $\mathbf{X}$ with $K_{r,M} := K_r \cap M \neq \emptyset$,

$$V_{r,M} := \{u \in K_r : \ \langle A(u), v - u\rangle \geq \langle F(u) - B(w), v - u\rangle \ \ \forall v \in K_{r,M}\}, \forall M \in \mathbf{\Sigma},$$

$\mathrm{clw}(V_{r,M})$ is the weak closure of $V_{r,M}$. Therefore there exists $u_0 \in \mathrm{clw}(V_{r,M})$ for all $M \in \mathbf{\Sigma}$.

For each $v \in K_r$, we may find a finite-dimensional subspace M_0 of $\mathbf{X}$ such that $u_0 \in M_0$ and $v \in M_0$. It turns out $u_0 \in \mathrm{clw}(V_{r,M_0})$ since $M_0 \in \mathbf{\Sigma}$. Thus, there exists $\{u_j\} \subset V_{M_0}$ such that $u_j \rightharpoonup u_0$, by the definition of V_{r,M_0}, we have

$$\langle A(u_j), u_0 - u_j\rangle \geq \langle F(u_j) - B(w), u_0 - u_j\rangle. \tag{5.26}$$

(i) If A is a continuous mapping of class $(S)_+$ and F is compact, there exists $l_0 \in \mathbf{X}^*$ such that $F(u_j) \to l_0$. Therefore we have that

$$\begin{aligned}
&\limsup_{j\to\infty}\langle F(u_j) - B(w), u_j - u_0\rangle \\
&= \limsup_{j\to\infty}\langle F(u_j) - l_0, u_j - u_0\rangle + \lim_{j\to\infty}\langle l_0, u_j - u_0\rangle + \lim_{j\to\infty}\langle -B(w), u_j - u_0\rangle \\
&\leq 0.
\end{aligned} \tag{5.27}$$

Hence, from (5.26) and (5.27), we have

$$\limsup_{j\to\infty}\langle A(u_j), u_j - u_0\rangle \leq 0.$$

Since A is a continuous mapping of class $(S)_+$, $u_j \to u_0$, $A(u_j) \to A(u_0)$. Thus $F(u_j) \to F(u_0)$. Since $v \in K_{M_0}$, by using the definition of V_{r,M_0}, we obtain that

$$\langle A(u_j), v - u_j\rangle \geq \langle F(u_j) - B(w), v - u_j\rangle.$$

Consequently,

$$\langle A(u_0), v - u_0\rangle \geq \langle F(u_0) - B(w), v - u_0\rangle. \tag{5.28}$$

(ii) If $A - F$ is a continuous bounded generalized pseudo-monotone mapping, we may assume that $A(u_j) - F(u_j) \rightharpoonup t_0$. By (5.26), we get

$$\limsup_{j\to\infty}\langle A(u_j) - F(u_j), u_j - u_0\rangle \leq 0.$$

Therefore, $t_0 = A(u_0) - F(u_0)$ and $\langle A(u_j) - F(u_j), u_j\rangle \to \langle A(u_0) - F(u_0), u_0\rangle$. Hence (5.28) holds. The variational inequality (5.25) is solvable.

In particular, taking $v = 0$ in (5.25), we have

$$\langle A(u_r), u_r\rangle \leq \langle F(u_r) - B(w), u_r\rangle. \tag{5.29}$$

It follows from condition (5.24) that $\{u_r\}$ is bounded (Otherwise if $\|u_r\| \to \infty$, then by (5.24) we get that $\lim_{\|u_r\|\to\infty}\langle (A - F)(u_r) + B(w), u_r\rangle = +\infty$, which contradicts (5.29)), i.e., $\|u_r\| \leq M$ for some real number $M > 0$. Let $r = M + 1$. Then, for each $v \in K$, we can choose $t \in (0, 1)$ small enough such that $z = (1 - t)u_r + tv \in K_r$. Substituting z into (5.25), we obtain that u_r is a solution of the variational inequality (5.23).

∎

Theorem 5.4.1 *Assume that* $\mathbf{W}, \mathbf{X}$ *are two reflexive Banach spaces,* $J(w, u)$ *is a weakly lower semicontinuous function,* B *is continuous from the weak topology of* $\mathbf{W}$ *to the topology of* $\mathbf{X}^*$*. Assume that the conditions in Lemma 5.4.1 are satisfied. Then there exists an optimal control* $w_0 \in U$ *for problem (5.22).*

Proof: From Lemma 5.4.1, it follows that, for each $w \in U$, $S(w) \neq \phi$. Note that $S(w)$ is the solution set of the variational inequality (5.23) and $u \in S(w)$ is equivalent to $u \in S(w)$. Let $\{(w_n, u_n)\}_{n\in N}$ be a minimizing sequence for problem (5.22) such that

$$\lim_{n\to\infty} J(w_n, u_n) = \inf_{w\in U, u\in S(w)} J(w, u). \tag{5.30}$$

We claim that $\{(w_n, u_n)\}_{n\in N}$ is bounded. If not so, then there exists a subsequence $\{(w_{n_k}, u_{n_k})\}_{k\in N}$ such that $\|(w_{n_k}, u_{n_k})\| \to \infty$. It follows from the coercive condition (5.24) that

$$\lim_{k\to\infty}\langle (A - F)(u_{n_k}) + B(w_{n_k}), u_{n_k}\rangle = \infty. \tag{5.31}$$

By $u_n \in K_{w_n}(0)$, we have

$$\langle (A-F)u_n + B(w_n), u_n - v\rangle \leq 0, \forall v \in K. \tag{5.32}$$

Let $v = 0$. Then, we get $\langle (A-F)u_n + B(w_n), u_n\rangle \leq 0$, which contradicts (5.31). Hence, $\{(w_n, u_n)\}_{n\in N}$ is bounded.

By the reflexivity of $\mathbf{W}$ and $\mathbf{X}$, there exists a weakly convergent subsequence of $\{(w_n, u_n)\}$. Without loss of generality, we may assume that $(w_n, u_n) \rightharpoonup (w_0, u_0)$. Hence $w_n \rightharpoonup w_0 \in \mathbf{W}$ and $u_n \rightharpoonup u_0 \in \mathbf{X}$ as $n \to \infty$. Since U and K are weakly closed sets, $w_0 \in U$ and $u_0 \in K$. From (5.32), we have

$$\langle (A-F)u_n + B(w_n), u_n - u_0\rangle \leq 0. \tag{5.33}$$

Noting that B is continuous from the weak topology of $\mathbf{W}$ to the topology of $\mathbf{X}^*$, we have $B(w_n) \to B(w_0)$.

(i) If A is a continuous mapping of class $(S)_+$ and F is compact, then there exists an $l_0 \in \mathbf{X}^*$ such that $F(u_n) \to l_0$. Thus,

$$\begin{aligned}
&\limsup_{n\to\infty}\langle F(u_n) - B(w_n), u_n - u_0\rangle \\
&\leq \limsup_{n\to\infty}\langle B(w_0) - B(w_n), u_n - u_0\rangle + \limsup_{n\to\infty}\langle F(u_n) - l_0, u_n - u_0\rangle \\
&+ \lim_{n\to\infty}\langle -B(w_0), u_n - u_0\rangle + \lim_{n\to\infty}\langle l_0, u_n - u_0\rangle \\
&= 0.
\end{aligned} \tag{5.34}$$

(5.33) and (5.34) imply

$$\limsup_{n\to\infty}\langle A(u_n), u_n - u_0\rangle \leq 0. \tag{5.35}$$

Noting that $u_n \rightharpoonup u_0$ and A is a continuous mapping of class $(S)_+$, it implies that $u_n \to u_0$. Therefore, by (5.32), we have

$$\sup_{v\in K}\langle (A-F)(u_0) + B(w_0), u_0 - v\rangle \leq 0.$$

That is $u_0 \in K_{w_0}(0)$.

(ii) If $A - F$ is a continuous bounded generalized pseudo-monotone mapping, then we may assume that $A(u_n) - F(u_n) \rightharpoonup s_0$. Using (5.33), we have

$$\limsup_{n\to\infty}\langle (A-F)u_n, u_n - u_0\rangle \leq 0.$$

Thus $s_0 = (A-F)u_0$ and $\langle (A-F)u_n, u_n\rangle \to \langle (A-F)u_0, u_0\rangle$. It follows from (5.32) that $u_0 \in K_{w_0}(0)$.

Since $J(w,u)$ is a weakly lower semicontinuous function, it follows from (5.30) that

$$J(w_0,u_0) \le \liminf_{n\to\infty} J(w_n,u_n) = \inf_{w\in U, u\in S(w)} J(w,u).$$

So,

$$J(w_0,u_0) = \inf_{w\in U, u\in S(w)} J(w,u).$$

Then, $w_0 \in U$ is an optimal control for problem (5.22). ∎

Remark 5.4.1 If $\bar{A} : U \times K \to \mathbf{X}^*$ is strongly monotone in $u \in K$ and uniformly in $w \in U$, i.e., there exists a $\mu > 0$ such that

$$\begin{aligned}&\langle \bar{A}(w,u') - \bar{A}(w,u), u'-u\rangle \ge \mu||u'-u||^2,\\ &\forall (u',w),(u,w) \in U\times K,\end{aligned} \tag{5.36}$$

then $A(\bar{w},\cdot)$ is a continuous mapping of class $(S)_+$ for each $\bar{w} \in U$. In fact, for each $\bar{w} \in U$, if $u_n \rightharpoonup u_0 \in K$ and

$$\limsup_{n\to\infty} \langle \bar{A}(\bar{w},u_n), u_n-u_0\rangle \le 0,$$

we have $\lim\limits_{n\to\infty} \langle \bar{A}(\bar{w},u_0), u_n-u_0\rangle = 0$. Hence, it follows from (5.36) that $\|u_n-u_0\| \to 0$, i.e., $u_n \to u_0$. Therefore, $A(\bar{w},\cdot)$ is a continuous mapping of class $(S)_+$ for each $\bar{w} \in U$.

We consider the following optimal control problem of strongly monotone variational inequality (see [122]):

$$\begin{aligned}&\min\ \ J(w,u)\\ &\text{subject to } (w,u)\in U\times K,\ \langle \bar{A}(w,u), v-u\rangle \ge 0,\end{aligned} \tag{5.37}$$

where $\bar{A}$ is strongly monotone in $u \in K$ and uniformly in $w \in U$.

Now we will derive an existence result for the optimal control problem (5.37). It is clear that if $\bar{A} : U\times K \to \mathbf{X}^*$ is strongly monotone in $u \in K$ and uniformly in $w \in U$, then $\bar{A}$ satisfies the coercive condition (5.24). By Lemma 5.4.1 and Theorem 5.4.1, it follows that, for each $w \in U$, $S(w) \ne \phi$ and the following result holds.

Corollary 5.4.1 *Assume that* $\mathbf{W},\mathbf{X}$ *are two reflexive Banach spaces,* $J(w,u)$ *is a weakly lower semicontinuous function,* $\bar{A} : U\times K \to \mathbf{X}^*$ *is strongly monotone in* $u \in K$ *and uniformly in* $w \in U$. *Then, there exists an optimal control* $w_0 \in U$ *for problem (5.37).*

5.5 Optimal control problem governed by a quasilinear elliptic variational inequality

Let Ω be a bounded domain of $\mathbf{R}^N$ with smooth boundary, let $f : \Omega \times \mathbf{R} \to \mathbf{R}$ be a given function. Recall that $W_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\| = \left\{\int_\Omega |\nabla u|^p\right\}^{1/p}$ $(1 < p < N)$. Let U be a nonempty, closed and convex subset of the space $L^q(\Omega)$ $(1 < q < p^* = \frac{Np}{N-p})$, let $\tau : U \to L^{q'}(\Omega)(q' = \frac{p}{p-1})$ be a mapping. Denote the Sobolev space $W_0^{1,p}(\Omega)$ $(1 < p < N)$ and

$$\{\, u \in W_0^{1,p}(\Omega) : u(x) \geq 0 \ a.e. \ in \ \Omega\}$$

as $\mathbf{X}$ and K, respectively. It is clear that K is a closed and convex cone of $\mathbf{X}$. Denote the space $L^q(\Omega)$ $(1 < q < p^* = \frac{Np}{N-p})$ as $\mathbf{W}$. It is well-known that $\mathbf{X} = W_0^{1,p}(\Omega)$ is reflexive for $p > 1$.

In this section, we will consider the following optimal control problem. For each $w \in U$, we define $u \in K$ (the state of the system) as the solution of the following quasilinear elliptic variational inequality

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla(v-u) \geq \int_\Omega (f(x,u) - \tau(w))(v-u), \quad \forall v \in K, \tag{5.38}$$

where $1 < p < N$. We also denote the solution set of the variational inequality (5.38) as $S(w)$. We define the cost function J as

$$J(w,u) = \int_\Omega g(x,u) + \int_\Omega l(x,w), \tag{5.39}$$

where $g : \Omega \times K \to \mathbf{R}^+$, $l : \Omega \times U \to \mathbf{R}^+$. This section is concerned with the existence of $w_0 \in U$ (optimal control), $u_0 \in K$ and $u_0 \in S(w_0)$, such that

$$J(w_0, u_0) = \min_{(w,u)\in U\times K, u\in S(w)} J(w,u). \tag{5.40}$$

In case $p = 2$, (5.38) become a semilinear elliptic variational inequality. The optimal control problem governed by a semilinear elliptic variational inequality was studied by many authors in different aspects. For example, see [2], [10], [53], [76], [87] and the references cited therein. The optimal control problem governed by a quasilinear elliptic variational inequality was investigated by Lou, Ye and Chen in [77] and [124]. In [77] and [124], they introduced an approximate problem, gave estimates of optimal pairs for the approximate problem, and proved that the optimal pairs for the approximate problem convergence to the solution of the original problem. In this section, by using the results obtained in Sections 5.4, we obtain several existence results of optimal controls of the optimal control problem governed by the quasilinear elliptic variational inequality.

Recall that the p-Laplacian defined by

$$\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u),$$

$$\lambda_1 := \inf\{\int_\Omega |\nabla u|^p : \int_\Omega |u|^p = 1, u \in \mathbf{X}\} > 0$$

is simple and isolated, see for example [31] for details.

Define $A : K \to \mathbf{X}^*$, $F : K \times \mathbf{R} \to \mathbf{X}^*$ and $B : U \to \mathbf{X}^*$ as follows: for all $u, v \in K$ and $w \in U$,

$$\begin{array}{rl} \langle A(u), v\rangle : & = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v, \\ \langle F(u), v\rangle : & = \int_\Omega f(x,u)v, \\ \langle B(w), v\rangle : & = \int_\Omega \tau(w)v, \end{array} \tag{5.41}$$

where $\langle \cdot, \cdot \rangle$ stand for the duality pairing between $\mathbf{X}$ and $\mathbf{X}^*$.

Actually, solving (5.38) is equivalent to solving the abstract variational inequality (5.23).

In order to prove existence results for the quasilinear elliptic variational inequality and the optimal control problem governed by the quasilinear elliptic variational inequality, we prove the following lemma first.

Lemma 5.5.1 *Let A, F, B be defined by (5.41). Then*

(i) $A : K \to \mathbf{X}^*$ is a continuous mapping of class $(S)_+$.

(ii) If $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is a continuous function and satisfies

$$\lim_{|t|\to\infty} \frac{f(x,t)}{a(x)t^{s-1}} = \lambda_0, \tag{5.42}$$

uniformly a.e. *with respect to $x \in \Omega$, where $1 < s < p^*, 0 \le a(x) \in L^r(\Omega), r = p^*/(p^* - s)$. Then, $F : K \times \mathbf{R} \to \mathbf{X}^*$ is a compact and continuous mapping.*

(iii) If $\tau : U \to L^{q'}(\Omega)(q' = \frac{p}{p-1})$ is a weakly continuous mapping, then $B : U \to \mathbf{X}^*$ is continuous from the weak topology of $\mathbf{W}$ to the topology of $\mathbf{X}^*$.

Proof: (i) We prove that the mapping $A : K \to \mathbf{X}^*$ is of class $(S)_+$. Indeed, let $u_n \rightharpoonup u_0$ in $\mathbf{X}$ and $\limsup\limits_{n\to\infty}\langle A(u_n), u_n - u_0\rangle \le 0$. Then,

$$\lim_{n\to\infty}\int_\Omega |\nabla u_0|^{p-2}\nabla u_0(\nabla u_n - \nabla u_0) = 0, \tag{5.43}$$

and

$$\limsup_{n\to\infty}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n(\nabla u_n - \nabla u_0) \le 0. \tag{5.44}$$

Since

$$\begin{aligned}
&\langle A(u) - A(v), u - v\rangle \\
&= \textstyle\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)(\nabla u - \nabla v) \\
&= \textstyle\int_\Omega |\nabla u|^p + \int_\Omega |\nabla v|^p - \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v - \int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla u \\
&\ge (\|u\|^{p-1} - \|v\|^{p-1})(\|u\| - \|v\|),
\end{aligned} \tag{5.45}$$

it follows from (5.43), (5.44) and (5.45) that

$$\begin{aligned}
&\limsup_{n\to\infty}(\|u_n\|^{p-1} - \|u_0\|^{p-1})(\|u_n\| - \|u_0\|) \\
&\le \limsup_{n\to\infty}\langle A(u_n), u_n - u_0\rangle - \lim_{n\to\infty}\langle A(u), u_n - u_0\rangle \\
&\le 0.
\end{aligned}$$

Hence, $\lim\limits_{n\to\infty}\|u_n - u\| = 0$, and consequently $u_n \to u$ in $\mathbf{X}$, i.e., A is a mapping of class $(S)_+$. Similar to the result presented in Lemma 3.3 of [32], we note that A is continuous.

(ii) By condition (5.42), for $\varepsilon_0 > 0$, there exists an $M_0 > 0$, such that

$$\left|\frac{f(x,t)}{a(x)t^{s-1}} - \lambda_0\right| < \varepsilon_0, \quad \forall t(|t| > M_0).$$

That is,

$$|\, f(x,t)\, | < (\varepsilon_0 + \lambda_0)a(x)|t|^{s-1}, \quad \forall t(|t| > M_0).$$

This and the continuity of $f(x,t)$ imply that there exists a constant $c_0 > 0$ such that

$$|\, f(x,t)\, | \le c_0 + (\varepsilon_0 + \lambda_0)a(x)|t|^{s-1}, \quad \forall t(t \in \mathbf{R}). \tag{5.46}$$

Thus,

$$\langle F(u), v\rangle \le c_0\int_\Omega |v| + (\varepsilon_0 + \lambda_0)\int_\Omega a(x)|u|^{s-1}|v|. \tag{5.47}$$

Since $0 \le a(x) \in L^r(\Omega)$, and $r = p^*/(p^* - s)$, it follows from Hölder's inequality and Sobolev's inequality that

$$\int_\Omega |v| \le \Big(\int_\Omega 1\Big)^{1/p^{*\prime}}\Big(\int_\Omega |v|^{p^*}\Big)^{1/p^*} \le c_1\|v\|,$$

and

$$\begin{aligned}\textstyle\int_\Omega a(x)|u|^{s-1}|v| &\le \big(\textstyle\int_\Omega a^{(p^*)'}|u|^{(s-1)(p^*)'}\big)^{(s-1)/p^*}\big(\textstyle\int_\Omega |v|^{p^*}\big)^{1/p^*}\\ &\le \big(\textstyle\int_\Omega a^r\big)^{1/r}\big(\textstyle\int_\Omega |u|^{p^*}\big)^{(s-1)/p^*}\big(\textstyle\int_\Omega |v|^{p^*}\big)^{1/p^*}\\ &\le c_2\|u\|^{s-1}\|v\|,\end{aligned}$$

for some constants $c_1, c_2 > 0$. Therefore, by virtue of (5.47), F is well defined. We will show the compact continuity of F.

Let u_n be a bounded sequence in $\mathbf{X}$, where $\mathbf{X}$ is reflexive. If necessary, through a subsequence, we can assume that $u_n \rightharpoonup u_0$ in $\mathbf{X}$. Noting that $1 < s < p^*$, we have $\rho = (s-1)(p^*)' = \frac{s-1}{p^*-1}\, p^* < p^*$. By Theorem 2.5.3, without loss of generality, we may assume that $u_n \to u_0$ in $L^\rho(\Omega)$. Noting that $s - 1 = \rho/(p^*)'$, and using (5.46) and the continuity of the Nemytskii operator $u \to f(x,u)$ from $L^\rho(\Omega)$ to $L^{(p^*)'}(\Omega)$ ([68], Theorem 2.1), we get

$$\int_\Omega |f(x,u_n) - f(x,u_0)|^{(p^*)'} \to 0,$$

and hence

$$\sup_{\|v\|\le 1}\int_\Omega |(f(x,u_n) - f(x,u_0))v| \le c_3\Big(\int_\Omega |f(x,u_n) - f(x,u_0)|^{(p^*)'}\Big)^{1/(p^*)'} \to 0,$$

for some constant $c_3 > 0$. Thus F is a compact and continuous mapping.

(iii) Let $w_n \rightharpoonup w_0$ in U. Since $\tau : U \to L^{q'}(\Omega)$ is a weakly continuous mapping, we have $\tau(w_n) \to \tau(w_0)$ in $L^{q'}(\Omega)$. Hence,

$$\begin{aligned}\|B(w_n) - B(w_0)\|_* &= \sup_{\|v\|\le 1}|\langle B(w_n) - B(w_0), v\rangle|\\ &\le \sup_{\|v\|\le 1}\big(\textstyle\int_\Omega |\tau(w_n) - \tau(w_0)|^{q'}\big)^{1/q'}\big(\textstyle\int_\Omega |v|^q\big)^{1/q}\\ &\le c_4\big(\textstyle\int_\Omega |\tau(w_n) - \tau(w_0)|^{q'}\big)^{1/q'}\\ &\to 0,\end{aligned}$$

for some constant $c_4 > 0$. Therefore $\lim_{n\to\infty} B(w_n) = B(w_0)$ in $\mathbf{X}^*$. ▮

Remark 5.5.1 Note that the conditions that $1 < q' < p^*(q' = \frac{p}{p-1})$ and $\tau : U \to \mathbf{W}$ is an identity mapping are sufficient to guarantee that $B : U \to \mathbf{X}^*$ is continuous from the weak topology of $\mathbf{W}$ to the topology of $\mathbf{X}^*$. In fact, recall that $\mathbf{W} = L^q(\Omega)$, $U \in \mathbf{W}$ and $(L^q(\Omega))^* = L^{q'}(\Omega)$. If $1 < q' < p^*$ and $\|v\| \le 1$, then by using Sobolev embedding theorem, the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{q'}(\Omega)$ is compact and

$$\int_\Omega |v|^{q'} \le C_0||v|| \le C_0,$$

for some constant $C_0 > 0$. If $w_n \rightharpoonup w_0$ in U, the above inequality implies

$$\begin{aligned}\|B(w_n) - B(w_0)\|_* &= \sup_{\|v\|\le 1} \textstyle\int_\Omega (\tau(w_n) - \tau(w_0))v \\ &= \sup_{\|v\|\le 1} \textstyle\int_\Omega (w_n - w_0)v \\ &\to 0.\end{aligned}$$

Hence, we also have $\lim\limits_{n\to\infty} B(w_n) = B(w_0)$ in $\mathbf{X}^*$.

Theorem 5.5.1 *Let $1 < p < N$ and $1 < q < p^*$. Suppose that $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is a continuous function and satisfies*

$$\lim_{|t|\to\infty} \frac{f(x,t)}{a(x)t^{p-1}} = \lambda_0, \tag{5.48}$$

almost uniformly with respect to $x \in \Omega$, where $0 \le a(x) \in L^\infty(\Omega)$ and $0 \le \lambda_0\|a\|_\infty < \lambda_1$, λ_1 is the first eigenvalue of the p-Laplacian with the zero boundary value. If $\tau : U \to L^{q'}(\Omega)(q' = \frac{p}{p-1})$ is a mapping such that $\tau(U)$ is a bounded set, then the variational inequality problem (5.38) *has a solution.*

Proof: Notice that $K = \{ u \in W_0^{1,p}(\Omega) : u(x) \le \psi(x) \ a.e. \ in \ \Omega\}$ is a closed and convex subset of $\mathbf{X} = W_0^{1,p}(\Omega)$ with $0 \in K$. Since f satisfies (5.48), it satisfies (5.42) in Lemma 5.5.1 with $s = p$. By Lemma 5.5.1, we get that A is a continuous mapping of class $(S)_+$, and F is a compact continuous mapping, B is continuous from the topology of $\mathbf{W}$ to the weak topology of $\mathbf{X}^*$.

By Lemma 5.4.1, we will complete the proof by showing that A, F and B satisfy the coercive condition (5.24).

Let $I_1 := \langle A(u), u\rangle + \langle B(w), u\rangle$ and $I_2 := \langle F(u), u\rangle$. Then,

$$\langle (A - F)(u) + B(w), u\rangle = I_1 - I_2.$$

Since $\tau(U)$ is bounded, there exists a constant $\bar{C} > 0$ such that $\|\tau(w)\| \le \bar{C}, \forall w \in U$. Thus,

$$I_1 \ge \|u\|^p - \bar{C}\|u\|. \tag{5.49}$$

Let $\varepsilon_1 > 0$ satisfying $(\varepsilon_1 + \lambda_0)\|a\|_\infty < \lambda_1$. Since $0 \le \lambda_0\|a\|_\infty < \lambda_1$, it follows from (5.48) that, for this $\varepsilon_1 > 0$, there exists an $M_1 > 0$, such that

$$\left|\frac{f(x,t)}{a(x)t^{s-1}} - \lambda_0\right| < \varepsilon_1, \quad \forall t \ (|t| > M_1).$$

That is,

$$| f(x,t) |< (\varepsilon_0+\lambda_0)a(x)|t|^{p-1}, \quad \forall t \ (|t|>M_1).$$

Denote $\Omega_{M_1} = \{x \in \Omega : \ |u(x)| \leq M_1\}$. Then by the above inequality and the Sobolev imbedding theorem, we have

$$|I_2| \leq \int_{\Omega_{M_1}} |f(x,u,\lambda)||u| + \int_{\Omega-\Omega_{M_1}} |f(x,u,\lambda)||u| \leq c_5 + (\varepsilon_1+\lambda_0)\|a\|_\infty \int_\Omega |u|^{p-1} \tag{5.50}$$

for some positive constant c_5. Notice that $\lambda_1 = \inf\{\int_\Omega |\nabla u|^p : \ \int_\Omega |u|^p = 1, u \in \mathbf{X}\} > 0$, which implies that $\lambda_1 \int_\Omega |u|^p \leq \|u\|^p$. It follows from (5.49) and (5.50) that

$$\langle (A-F)(u)+B(w), u\rangle \geq |I_1|-|I_2| \geq \big(1-\frac{(\varepsilon_1+\lambda_0)\|a\|_\infty}{\lambda_1}\big)\|u\|^p - \bar{C}\|u\| - c_5. \tag{5.51}$$

Since $0<(\varepsilon_1+\lambda_0)\|a\|_\infty<\lambda_1$ and $1<p<N$, the right-hand side of (5.51) tends to $+\infty$ as $\|u\|\to\infty$. Therefore, the conclusion holds by virtue of Lemma 5.4.1. ▮

Theorem 5.5.2 *Suppose that $f : \Omega\times\mathbf{R}\to\mathbf{R}$ is a continuous function and satisfies condition (5.42) in Lemma 5.5.1 with $1<s<p$, let $\tau : U \to L^q(\Omega)$ is a weakly continuous mapping. Then, the variational inequality problem (5.38) has a solution.*

Proof: Similar to the proof as that of Theorem 5.5.1, we only need to show A, F and B satisfy the coercive condition (5.24). By condition (5.42), for each $\varepsilon>0$, there exists an $M_2>0$, such that

$$|\frac{f(x,t)}{a(x)t^{s-1}} - \lambda_0| < \varepsilon, \quad \forall t(|t|>M_2).$$

That is,

$$| f(x,t) |< (\varepsilon_0+\lambda_0)a(x)|t|^{s-1}, \quad \forall t(|t|>M_2).$$

Since $f(x,t)$ is bounded in $\Omega\times\{t\in\mathbf{R} : |t|\leq M_2\}$, there exists a constant $c_6>0$ such that

$$| f(x,t) |\leq c_6 + (\varepsilon+\lambda_0)a(x)|t|^{s-1}, \quad \forall t\in\mathbf{R}.$$

Thus,

$$\langle F(u), u\rangle \leq c_6\int_\Omega |u| + (\varepsilon+\lambda_0)\int_\Omega a(x)|u|^s.$$

Since $0\leq a(x)\in L^r(\Omega)$, and $r=p^*/(p^*-s)$, it follows from Hölder's inequality and Sobolev's inequality that

$$\begin{aligned}\langle F(u), u\rangle &\leq c_7\|u\| + (\varepsilon+\lambda_0)\big(\textstyle\int_\Omega a^r\big)^{1/r}\big(\int_\Omega |u|^{p^*}\big)^{s/p^*}\\ &\leq c_8\|u\| + c_9\|u\|^s,\end{aligned}$$

for some positive constants c_7, c_8 and c_9. Thus,

$$\langle (A-F)(u)+B(w),u\rangle \geq (1-c_9\|u\|^{s-p})\|u\|^p - \|\tau(w)\|\|u\| - c_8.$$

Since $1<s<p$ and $\tau(U)$ is bounded,

$$\lim_{\|u\|\to\infty} \langle (A-F)(u)+B(w),u\rangle = +\infty.$$

By Lemma 5.4.1, the variational inequality problem (5.38) has a solution. ∎

Lemma 5.5.2 *Let $\eta > 1$. Suppose that $g : \mathbf{R} \to \mathbf{R}$ is a C^1 convex function such that*

(i) $|g(t)| \leq C_1|t|^\eta$ $(\forall t \in \mathbf{R})$, *for some constant* C_1.

(ii) $|g'(t)| \leq C_2|t|^{\eta-1}$ $(\forall t \in \mathbf{R})$, *for some constant* C_2.

Then, the function Q defined by

$$Q(u) = \int_\Omega g(u), \quad u \in L^\eta(\Omega),$$

is weakly lower semicontinuous.

Proof: By Example 1.3 in [20], we know that the function Q is Gâteaux differentiable in the space $L^\eta(\Omega)$ and

$$Q'(u,\varphi) = Q'(u)\cdot\varphi = \int_\Omega g'(u)\varphi.$$

Since g is a convex function, $Q : L^\eta(\Omega) \to \mathbf{R}$ is convex. It is clear that $Q'(u) \in L^{\eta'}(\Omega)$ (the dual space of $L^\eta(\Omega)$). Thus, by Proposition 4.1 in [20], we conclude that Q is weakly lower semicontinuous. ∎

In the following theorems, we denote

$$J(w,u) = \int_\Omega g(u) + \int_\Omega l(w),$$

where $g, l : \mathbf{R} \to \mathbf{R}$ are two functions satisfying some conditions to be specified later.

Theorem 5.5.3 *Suppose that all the conditions given in Theorem 5.5.1 or Theorem 5.5.2 are satisfied. Let p, q defined as before, i.e., $1 < p < N, 1 < q < p^*$. Furthermore, assume that $g, l : \mathbf{R} \to \mathbf{R}$ are two C^1 convex functions such that*

(i) $|l(t)| \leq C_3|t|^p$ *and* $|g(t)| \leq C_3|t|^q$ $(\forall t \in \mathbf{R})$*, for some constant* C_3*;*

(ii) $|l'(t)| \leq C_4|t|^{p-1}$ *and* $|g'(t)| \leq C_4|t|^{q-1}$ $(\forall t \in \mathbf{R})$*, for some constant* C_4*.*

Then, there exists an optimal control $w_0 \in U$ for problem (5.40).

Proof: Since

$$J(w, u) = \int_\Omega g(u) + \int_\Omega l(w).$$

Let $(w_k, u_k) \rightharpoonup (w', u')$ in $L^q(\Omega) \times L^p(\Omega)$. Then, $w_k \rightharpoonup w'$ in $L^q(\Omega)$ and $u_k \rightharpoonup u'$ in $L^p(\Omega)$. By Lemma 5.5.2, we have

$$\int_\Omega g(u') \leq \liminf_{k\to\infty} \int_\Omega g(u_k)$$

and

$$\int_\Omega l(w') \leq \liminf_{k\to\infty} \int_\Omega l(w_k).$$

Hence, $J : L^q(\Omega) \times L^p(\Omega) \to \mathbf{R}$ is weakly lower semicontinuous.

Assume that $(w_n, u_n) \rightharpoonup (w_0, u_0)$ in $U \times K$. Then $w_n \rightharpoonup w_0$ in $U \subset L^q(\Omega)$ and $u_n \rightharpoonup u_0$ in K. By Theorem 2.5.3, when Ω is bounded, the imbedding $W_0^{1,p}(\Omega) \to L^p(\Omega)$ is a compact one. Hence, $u_n \to u_0$ in $L^p(\Omega)$ and $(w_n, u_n) \rightharpoonup (w_0, u_0)$ in $L^q(\Omega) \times L^p(\Omega)$. Note that $J : L^q(\Omega) \times L^p(\Omega) \to \mathbf{R}$ is weakly lower semicontinuous. Thus,

$$\liminf_{n\to\infty} J(w_n, u_n) \geq J(w_0, u_0).$$

Therefore, $J : U \times K \to \mathbf{R}$ is weakly lower semicontinuous.

Let A, B and F be defined as (5.41). Then, by Lemma 5.5.1, A is a continuous mapping of class $(S)_+$, F is a compact continuous mapping, B is continuous from the topology of $\mathbf{W}$ to the weak topology of $\mathbf{X}^*$. Since all the conditions in Theorem 5.5.1 or Theorem 5.5.2 are satisfied, it follows from Theorem 5.5.1 or Theorem 5.5.2 that the variational inequality (5.38) has a solution. That is, the variational inequality (5.23) has a solution. Furthermore, A, B and F satisfy all the conditions in Theorem 5.4.1, so from Theorem 5.4.1, there exists an optimal control $w_0 \in U$ for problem (5.40). ∎

Theorem 5.5.4 *Suppose that all the conditions given in Theorem 5.5.1 or Theorem 5.5.2 are satisfied. Let* $1 < \alpha, q < p^*$*,* $g(u) = C_5|u - u_d|^\alpha$ *and* $l(w) = C_6|w - w_d|^q$*, for some constants* C_5 *and* C_6*, where* $u_d \in L^\alpha(\Omega)$*,* $w_d \in L^q(\Omega)$*. Then, there exists an optimal control* $w_0 \in U$ *for problem (5.40).*

Proof: Note that

$$\begin{aligned} J(w,u) &= \int_\Omega g(u) + \int_\Omega l(w) \\ &= C_5 \int_\Omega |u - u_d|^\alpha + C_6 \int_\Omega |w - w_d|^q \\ &= C_5 \|u - u_d\|^\alpha_{L^\alpha} + C_6 \|w - w_d\|^q_{L^q}. \end{aligned}$$

By the weakly lower semi-continuity of the norm, $J : L^q(\Omega) \times L^\alpha(\Omega) \to \mathbf{R}$ is weakly lower semicontinuous.

Let $(w_n, u_n) \rightharpoonup (w_0, u_0) \in U \times K$. Then, $w_n \rightharpoonup w_0$ in U and $u_n \rightharpoonup u_0$ in K. Since $1 < \alpha < p^*$, it follows from Theorem 2.5.3 that the imbedding $W_0^{1,p}(\Omega) \to L^\alpha(\Omega)$ is a compact one. Hence, $u_n \to u_0$ in $L^\alpha(\Omega)$ and $(w_n, u_n) \rightharpoonup (w_0, u_0)$ in $L^q(\Omega) \times L^\alpha(\Omega)$. Thus, by using the weak lower-semi continuity of $J : L^q(\Omega) \times L^p(\Omega) \to \mathbf{R}$, $J : U \times K \to \mathbf{R}$ is weakly lower semicontinuous. Similar to the proof of Theorem 5.5.3, we conclude that there exists an optimal control $w_0 \in U$ for problem (5.40). ▮

Chapter 6

Optimal Control Problems Governed by a Variational Inequality via Nonlinear Lagrangian Methods

6.1 Introduction

The study of an optimal control problem where the state of the system is defined by a variational inequality problem has been widely investigated by many authors in different aspects. See Adams and Lenhart [2], Barbu [7], Bergounioux [10, 11], He [53], Lions [74], Mignot and Puel [87], Ye and Chen [124], for example. Lagrangian and penalty function methods have been used in the study of theory and methodology for optimal control problems (see [10], [84], [85] and [97]). These methods can be used to approximate the optimal control problem governed by a variational inequality. Recently, a class of nonlinear Lagrangian functions was introduced and applied to establish a zero duality gap result for a nonconvex optimization problem in [100] and [117]. The zero duality gap property for a nonconvex optimization problem is an important property to be utilized in the development of primal-dual methods, as the solution of the original constrained optimization problem can be obtained via solving its dual problem.

In this chapter, we consider optimal control problems governed by a variational inequality problem for monotone type mappings. We deal here with quite general variational inequalities, which include the ones studied in [10], [11], [74] and [85] as special cases. We propose a nonlinear Lagrangian approach for solving these problems. Similar to the results given in [100], we establish that the lower semicontinuous property of the perturbation function of the optimal control problem at 0 is an equivalent condition for the existence of the zero duality gap property between the optimal control problem and its nonlinear Lagrangian dual problem. But this lower semicontinuous property of the perturbation function at 0 is quite abstract and cannot be easily verified. To overcome this difficulty, we show that if the variational inequalities associated with optimal control problems are ones for some monotone type mappings, then the lower semicontinuous property of the perturbation function at 0 can be guaranteed. Therefore, the zero duality gap is obtained.

The outline of this chapter is as follows:

In Section 6.2, motivated by the idea presented in [100], we introduce the concept of a modified nonlinear Lagrangian function and obtain a necessary and sufficient condition for the zero duality gap property between the optimal control problem and its nonlinear Lagrangian dual problem. In Section 6.3, we obtain the zero duality gap property for the optimal control problem governed by variational inequalities involving monotone type mappings and its dual problem. In Section 6.4, we show that the optimal solution set of the power penalty problem is a nonempty bounded set and every weak limit point of a sequence of optimal solutions generated by the power penalty problem is a solution of the optimal control problem. In Section 6.5, we apply our results to an example where the variational inequality leads to a linear elliptic obstacle problem.

6.2 Lower semicontinuous property of the perturbation function

In this section, we assume that $\mathbf{W}$ and $\mathbf{X}$ are two Banach spaces, U is a nonempty closed set of $\mathbf{W}$ and K is a closed and convex cone of $\mathbf{X}$. Let $J : U \times K \to \mathbf{R}$ be a given function and $A : K \to \mathbf{X}^*$, $F : K \to \mathbf{X}^*$ and $B : U \to \mathbf{X}^*$ be three given mappings. The optimal control problem governed by a variational inequality is the following minimization problem:

$$\begin{aligned} &\min \;\; J(w,u) \\ &\text{s.t. } (w,u) \in U \times K, \text{and } u \in S(w), \end{aligned} \tag{6.1}$$

where, for each $w \in U$, $S(w)$ is the solution set of the following variational inequality problem:

$$\text{Find } u \in K : \langle A(u), v-u\rangle \geq \langle F(u) - B(w), v-u\rangle, \forall v \in K. \tag{6.2}$$

The optimal control problem (6.1) includes the ones studied in [10], [11], [74] and [85] as special cases.

For each $w \in U$ and $y \in \mathbf{R}$, we define

$$g_w(u) = \sup_{v\in K}\langle (A-F)(u) + B(w), u-v\rangle, \tag{6.3}$$

$$K_w(y) = \{u \in K,\ g_w(u) \leq y\}. \tag{6.4}$$

Then, $g_w(u) \geq 0$ and $K_w(0) = \{u \in K :\ g_w(u) = 0\} = S(w)$.

Define the perturbation function β as

$$\beta(y) = \inf_{w\in U,\ u\in K_w(y)} J(w,u), y \in \mathbf{R}. \tag{6.5}$$

It is clear that $\beta(0)$ is the optimal value of problem (6.1).

Definition 6.2.1 *Let $U, K, \mathbf{W}$ and $\mathbf{X}$ be defined as before, and let Z be a subset in $\mathbf{R}$, $P : \mathbf{R} \times Z \to \mathbf{R}$ be a function. A nonlinear Lagrangian function L_P : $U \times K \times (0,+\infty) \to \mathbf{R}$ for problem (6.1) is defined as*

$$L_P(w,u,d) = P(J(w,u), dg_w(u)).$$

For each $d \in \mathbf{R}^+$, the function

$$F_P(d) = \inf_{(w,u)\in U\times K} L_P(w,u,d)$$

is called its nonlinear Lagrangian dual function. The equality

$$\beta(0) = \sup_{d\in\mathbf{R}^+} F_P(d) \tag{6.6}$$

is called the zero duality gap property.

Let us make the following assumptions on the function P:

(P_i) If $y_1 \le y_2$, then $P(y_1, z) \le P(y_2, z), \forall z \in Z$.

(P_{ii}) $P(y, 0) = y, \forall y \in \mathbf{R}^+$.

(P_{iii}) $P(y, z) \ge y, \forall (y, z) \in \mathbf{R}^+ \times Z$ and $\lim\limits_{z \in Z, |z| \to \infty} P(y, z) \ge \beta(0)$.

Remark 6.2.1 We notice that if a real-valued function P defined on a subset $\mathbf{R}^+ \times V$ of $\mathbf{R}^{1+m}$ satisfies (i) P is increasing, i.e., $y_1 \le y_2, z_1 \le z_2$ implies $P(y_1, z_1) \le P(y_2, z_2)$, where $(u_j, v_j) \in \mathbf{R}^+ \times V, j = 1, 2$; (ii) $P(y, 0_m) = y$ for all $y \in \mathbf{R}^+$, where 0_m is the origin of the space $\mathbf{R}^m$ and $0_m \in V$; (iii) there exist numbers $a_1 > 0, ..., a_m > 0$ such that $P(y_0, y_1, ..., y_m) \ge \max(y_0, a_1 y_1, ..., a_m y_m)$ for all $y_0 > 0$, $(y_0, y_1, ..., y_m) \in \mathbf{R}^m$, then it is not difficult to check that P satisfies assumptions (P_i), (P_{ii}) and (P_{iii}) with $Z = V \cap {\mathbf{R}_+}^m$. Conditions (i), (ii) and (iii) are the ones used in [100]. However, the converse is not true. The following example confirms this assertion.

Example 6.2.1. Let P be a real-valued function defined on $\mathbf{R}^2_+$ by

$$P(y, z) = \begin{cases} y, & \text{if } z = 0, \\ y + (yz)^{1/3} |\sin \frac{1}{z}|, & \text{if } 0 < z < \frac{2}{\pi}, \\ \quad y + (yz)^{(1/3)}, & \text{if } z \ge \frac{2}{\pi}. \end{cases}$$

It is easy to check that P satisfies assumptions (P_i), (P_{ii}) and (P_{iii}), but P is not increasing in both y and z, and there is no $a > 0$ such that $p(y, z) \ge \max(y, az)$.

In the following, we assume, without loss of generality, that, for some $m_0 \ge 0$,

$$J(w, u) \ge m_0, \forall (w, u) \in U \times K.$$

Motivated by the idea reported in [100], we derive a necessary condition for the zero duality gap property in the following lemma.

Lemma 6.2.1 *Assume that* $\mathbf{W}$ *and* $\mathbf{X}$ *are two Banach spaces, and* P *is a continuous function satisfying* (P_i) *and* (P_{ii}). *If the zero duality gap property (6.6) holds, then the perturbation function* β *is lower semicontinuous at the origin.*

Proof: On the contrary, suppose that there exist a $\delta > 0$ and a sequence $\{y_k\} \subset \mathbf{R}$ such that $y_k \to 0$ as $k \to \infty$ and

$$\beta(y_k) \le \beta(0) - \delta, k = 1, ... \tag{6.7}$$

Since the zero duality gap property (6.6) holds, there exists a $d \in \mathbf{R}^+$ such that

$$\begin{aligned}\beta(0) &< F_P(d) + \tfrac{\delta}{2} \\ &= \inf_{(w,u)\in U\times K} P(J(w,u), dg_w(u)) + \tfrac{\delta}{2} \\ &\leq \inf_{w\in U,\ u\in K_w(y_k)} P(J(w,u), dg_w(u)) + \tfrac{\delta}{2}.\end{aligned} \tag{6.8}$$

By (6.5), we have

$$\beta(y_k) = \inf_{w\in U, u\in K_w(y_k)} J(w,u).$$

Thus, there exist $w_k \in U,\ u_k \in K_{w_k}(y_k)$, such that

$$J(w_k, u_k) \leq \beta(y_k) + \frac{\delta}{4}.$$

Since P satisfies (P_i), the above inequality and (6.8) imply

$$\begin{aligned}\beta(0) &\leq P(J(w_k,u_k), dg_{w_k}(u_k)) + \tfrac{\delta}{2} \\ &\leq P(\beta(y_k) + \tfrac{\delta}{4}, dg_{w_k}(u_k)) + \tfrac{\delta}{2}.\end{aligned} \tag{6.9}$$

Because $w_k \in U,\ u_k \in K_{w_k}(y_k)$,

$$g_{w_k}(u_k) \leq y_k \to 0, \text{ as } k \to \infty. \tag{6.10}$$

Noting that P is continuous and satisfies (P_i) and (P_{ii}). Thus, (6.9), combined with (6.7) and (6.10), yields

$$\begin{aligned}\beta(0) &\leq \lim_{k\to\infty} P(\beta(0) - \tfrac{3\delta}{4}, h(dg_{w_k}(u_k))) + \tfrac{\delta}{2} \\ &= P(\beta(0) - \tfrac{3\delta}{4}, 0) + \tfrac{\delta}{2} \\ &= \beta(0) - \tfrac{\delta}{4},\end{aligned}$$

which is a contradiction. ∎

We will give a sufficient condition for the zero duality gap property in the following.

Lemma 6.2.2 *Assume that* $\mathbf{W}$ *and* $\mathbf{X}$ *are two Banach spaces, and* P *satisfies assumptions* (P_i)*,* (P_{ii}) *and* (P_{iii})*. Let* $-\infty < \beta(0) < +\infty$*. If the perturbation function* β *is lower semicontinuous at the origin, then the zero duality gap property (6.6) holds.*

Proof: Since $-\infty < \beta(0) < +\infty$,

$$\begin{aligned} F_P(d) &= \inf_{(w,u)\in U\times K} P(J(w,u), dg_w(u)) \\ &\le \inf_{(w,u)\in U\times K_w(0)} P(J(w,u), 0) \\ &= \inf_{w\in U,\ u\in K_w(0)} J(w,u) \quad \text{(using } (P_{ii})) \\ &= \beta(0). \end{aligned} \tag{6.11}$$

Assume that the zero duality gap property (6.6) is not valid. By (6.11), there exists a $\delta > 0$ such that

$$\beta(0) > \sup_{d\in\mathbf{R}^+} F_P(d) + \delta,$$

that is

$$\beta(0) \ge \inf_{(w,u)\in U\times K} P(J(w,u), dg_w(u)) + \delta, \quad \forall d \in \mathbf{R}^+.$$

Let $(w_k, u_k) \in U \times K$ satisfy

$$\beta(0) \ge P(J(w_k,u_k), kg_{w_k}(u_k)) + \frac{1}{2}\delta, \forall k. \tag{6.12}$$

We assert that $g_{w_k}(u_k) \to 0$ as $k \to \infty$. Otherwise, without loss of generality, we assume that

$$g_{w_k}(u_k) \ge a_0,$$

for some $a_0 > 0$. Then, $kg_{w_k}(u_k) \to \infty$. Therefore, this and condition (P_{iii}) imply

$$\lim_{k\to\infty} P(J(w_k,u_k), kg_{w_k}(u_k)) \ge \lim_{k\to\infty} P(m_0, kg_{w_k}(u_k)) \ge \beta(0),$$

which contradicts (6.12). Thus, $g_{w_k}(u_k) \to 0$ as $k \to \infty$. Let $y_k = g_{w_k}(u_k)$. Then, $y_k \to 0$, $u_k \in K_{w_k}(y_k)$. It follows from (6.5) that

$$\beta(y_k) \le J(w_k, u_k).$$

This, (6.12) and condition (P_{iii}) imply

$$\beta(0) \ge J(w_k,u_k) + \frac{1}{2}\delta \ge \beta(y_k) + \frac{1}{2}\delta. \tag{6.13}$$

The lower semicontinuous property of β, together with (6.13), gives

$$\beta(0) \ge \liminf_{k\to\infty} \beta(y_k) + \frac{1}{2}\delta \ge \beta(0) + \frac{1}{2}\delta,$$

which is a contradiction. ∎

6.3 Nonlinear Lagrangian duality theorems

In the following, we obtain the zero duality gap property for the optimal control problem governed by variational inequalities involving monotone type mappings and its nonlinear Lagrangian dual problem.

Lemma 6.3.1 *Assume that* $\mathbf{W}$ *and* $\mathbf{X}$ *are two reflexive Banach spaces,* $J : U \times K \to \mathbf{R}$ *is a weakly lower semicontinuous function,* $A : K \to \mathbf{X}^*$ *is a continuous mapping of class* $(S)_+$*,* $-F : K \to \mathbf{X}^*$ *is a continuous, bounded and generalized pseudo-monotone mapping, and* $B : U \to \mathbf{X}^*$ *is continuous from the weak topology of* $\mathbf{W}$ *to the topology of* $\mathbf{X}^*$*. Suppose that the following coercive condition is satisfied:*

$$\lim_{(w,u)\in U\times K, \|(w,u)\|\to+\infty} \langle (A-F)(u) + B(w), u\rangle = +\infty. \tag{6.14}$$

Then, $-\infty < \beta(0) < +\infty$*.*

Proof: First we shall show that $-F$ is a quasimonotone mapping. In fact, if it fails, then there exist $\{y_j\} \subset K$, $y_j \rightharpoonup y_0$, such that

$$\limsup_{j\to\infty} \langle -F(y_j), y_j - y_0\rangle < 0. \tag{6.15}$$

From the boundedness of $-F$, we may assume $-F(y_j) \rightharpoonup l_0 \in X^*$. By the generalized pseudo-monotonicity of $-F$, we obtain that $-F(y_0) = l_0$ and $\langle -F(y_j), y_j\rangle \to \langle l_0, y_0\rangle$. Thus

$$\lim_{j\to\infty} \langle -F(y_j), y_j - y_0\rangle = 0,$$

which contradict (6.15). By [14], $A - F$ is a continuous mapping of class $(S)_+$. It follows from Theorem 2.2 in [51] that the variational inequality problem (6.2) has a solution for each $w \in U$. That is, $S(w) \neq \emptyset$ for each $w \in U$.

Since $\beta(0) = \min_{(w,u)\in U\times K, u\in S(w)} J(w,u)$, there exists a minimizing sequence $\{(w_n, u_n)\}$ satisfying $u_n \in S(w_n)$, such that

$$J(w_n, u_n) \leq \beta(0) + \frac{1}{n}, n = 1, 2, \cdots. \tag{6.16}$$

By $u_n \in S(w_n)$, we have

$$\langle A(u_n), v - u_n\rangle \geq \langle F(u_n) - B(w_n), v - u_n\rangle, \quad \forall v \in K. \tag{6.17}$$

Let $v = 0$. Then,

$$\langle A(u_n) - F(u_n) + B(w_n), u_n \rangle \leq 0.$$

This and the coercive condition (6.14) imply that $\{(w_n, u_n)\}$ is bounded. Without loss of generality, we may assume that $w_n \rightharpoonup w_0$, $u_n \rightharpoonup u_0$. Since $B : U \to \mathbf{X}^*$ is continuous from the weak topology of $\mathbf{W}$ to the topology of $\mathbf{X}^*$, $B(w_n) \to B(w_0)$. By (6.17), we have

$$\limsup_{n \to \infty} \langle A(u_n) - F(u_n), u_n - u_0 \rangle \leq 0.$$

Noting that $A - F$ is a continuous mapping of class $(S)_+$, the inequality above implies that $u_n \to u_0$ and $(A - F)(u_n) \to (A - F)(u_0)$. It follows from (6.17) that

$$\langle A(u_0), v - u_0 \rangle \geq \langle F(u_0) - B(w_0), v - u_0 \rangle, \quad \forall v \in K.$$

That is, $u_0 \in S(w_0)$. The weak lower semicontinuous property of $J(u, v)$, together with (6.16), gives

$$J(w_0, u_0) \leq \liminf_{n \to \infty} J(w_n, u_n) \leq \beta(0).$$

Therefore, $J(w_0, u_0) = \beta(0)$.

∎

Lemma 6.3.2 *Assume that* $\mathbf{W}$ *and* $\mathbf{X}$ *are two reflexive Banach spaces,* $J : U \times K \to \mathbf{R}$ *is a weakly lower semicontinuous function,* $A - F : K \to \mathbf{X}^*$ *is a continuous, bounded and generalized pseudo-monotone mapping, and* $B : U \to \mathbf{X}^*$ *is continuous from the weak topology of* $\mathbf{W}$ *to the topology of* $\mathbf{X}^*$*. Suppose that the coercive condition (6.14) is satisfied. Then,* $-\infty < \beta(0) < +\infty$*.*

Proof: Define $J : \mathbf{X} \to 2^{\mathbf{X}^*}$ as follows:

$$J(x) = \{x^* \in \mathbf{X}^* : \langle x^*, x \rangle = \|x^*\| \, \|x\| = \|x\|^2\}.$$

It is easy to prove that J is a continuous mapping of class $(S)_+$. By [14], $A - F + \frac{1}{n}J$ is a continuous mapping of class $(S)_+$ for each $n, n = 1, 2, \cdots$. For each $w \in U$, it follows from Theorem 2.2 in [51] that there exists $y_n \in K$, such that

$$\langle A(y_n) + \frac{1}{n}J(y_n), v - y_n \rangle \geq \langle F(y_n) - B(w), v - y_n \rangle, \quad \forall v \in K. \tag{6.18}$$

Let $v = 0$. Then,

$$\langle A(y_n), y_n \rangle \leq \langle F(y_n) - B(w), y_n \rangle - \frac{1}{n}\langle J(y_n), y_n \rangle \leq \langle F(y_n) - B(w), y_n \rangle.$$

This and the coercive condition (6.14) imply that $\{y_n\}$ is bounded. Without loss of generality, we may assume that $y_n \rightharpoonup y_0$. By (6.18), we obtain

$$\limsup_{n\to\infty}\langle A(y_n) - F(y_n), y_n - y_0\rangle \le 0.$$

From the boundedness and the generalized pseudo-monotonicity of $A - F$, we have $A(y_n) - F(y_n) \rightharpoonup A(y_0) - F(y_0)$ and $\langle A(y_n) - F(y_n), y_n\rangle \to \langle A(y_0) - F(y_0), y_0\rangle$. Hence, by using (6.18), we get

$$\langle A(y_0), v - y_0\rangle \ge \langle F(y_0) - B(w), v - y_0\rangle, \quad \forall v \in K.$$

That is, $S(w) \ne \emptyset$. Similar to the proof of Lemma 6.3.1, we can show that $-\infty < \beta(0) < +\infty$. ∎

Theorem 6.3.1 *Suppose that* $\mathbf{W}$ *and* $\mathbf{X}$ *are two reflexive Banach spaces,* U *is a nonempty closed set of* $\mathbf{W}$ *and* K *is a closed and convex cone of* $\mathbf{X}$*. Let* P *satisfy assumptions* (P_i)*,* (P_{ii}) *and* (P_{iii})*, and let* $U \times K$ *be unbounded. Suppose that* $J : U \times K \to \mathbf{R}$ *is a weakly lower semicontinuous function,* $A : K \to \mathbf{X}^*$ *is a continuous mapping of class* $(S)_+$*,* $-F : K \to \mathbf{X}^*$ *is a continuous, bounded and generalized pseudo-monotone mapping, and* $B : U \to \mathbf{X}^*$ *is continuous from the weak topology of* $\mathbf{W}$ *to the topology of* $\mathbf{X}^*$*. Assume that the coercive condition (6.14) is satisfied. Then, the zero duality gap property (6.6) holds.*

Proof: From Lemma 6.3.1, $-\infty < \beta(0) < +\infty$. By Lemma 6.2.2, we only need to prove that the perturbation function β is lower semicontinuous at the origin.

On the contrary, suppose that there exists an $\epsilon_0 > 0$ such that

$$\liminf_{y\to 0}\beta(y) \le \beta(0) - \epsilon_0.$$

Then, there exist a sequence $\{y_k\} \subset \mathbf{R}$, $y_k \to 0$, $w_k \in U$ and $u_k \in K_{w_k}(y_k)$ such that

$$J(w_k, u_k) \le \beta(0) - \frac{\epsilon_0}{2}, k = 1, \cdots. \tag{6.19}$$

holds. By $u_k \in K_{w_k}(y_k)$, we have $g_{w_k}(u_k) \le y_k$, i.e.,

$$\sup\nolimits_{v\in K}\langle (A - F)(u_k) + B(w_k), u_k - v\rangle \le y_k. \tag{6.20}$$

Thus,

$$\langle (A - F)(u_k) + B(w_k), u_k\rangle \le y_k, k = 1, 2, \cdots.$$

This, combined with (6.19), implies that the sequence

$$\left\{\max\left(J(w_k,u_k),\langle (A-F)(u_k)+B(w_k),u_k\rangle\right)\right\}$$

is bounded. It follows from (6.14) that $\{(w_k,u_k)\}$ is bounded. Because $\mathbf{W}$ and $\mathbf{X}$ are reflexive Banach spaces, the product space $\mathbf{W}\times\mathbf{X}$ is a reflexive Banach space. Hence, there exist $\{w_{k_j}\}\subset\{w_k\}$ and $\{u_{k_j}\}\subset\{u_k\}$ such that $w_{k_j}\rightharpoonup w_0\in\mathbf{W}$ and $u_{k_j}\rightharpoonup u_0\in\mathbf{X}$. Since U and K are weakly closed sets, it is clear that $w_0\in U$ and $u_0\in K$. Using (6.20), we have

$$\langle (A-F)(u_{k_j})+B(w_{k_j}),u_{k_j}-u_0\rangle\le 0. \tag{6.21}$$

Since $-F:K\to\mathbf{X}^*$ is a continuous, bounded and generalized pseudo-monotone mapping, we claim that,

$$\limsup_{j\to\infty}\langle -F(u_{k_j}),u_{k_j}-u_0\rangle\ge 0. \tag{6.22}$$

In fact, if it is not so,

$$\limsup_{j\to\infty}\langle -F(u_{k_j}),u_{k_j}-u_0\rangle< 0. \tag{6.23}$$

From the boundedness and the generalized pseudomonotonicity of $-F$, we obtain $-F(u_{k_j})\rightharpoonup -F(u_0)$, and $\lim_{j\to\infty}\langle F(u_{k_j}),u_{k_j}\rangle=\langle F(u_0),u_0\rangle$. Therefore,

$$\lim_{j\to\infty}\langle F(u_{k_j}),u_{k_j}-u_0\rangle=0,$$

which contradicts (6.23). Since $B:U\to\mathbf{X}^*$ is continuous from the weak topology of $\mathbf{W}$ to the topology of $\mathbf{X}^*$, $B(w_{k_j})\to B(w_0)$. Without loss of generality, by (6.22), we may assume that

$$\lim_{j\to\infty}\langle F(u_{k_j}),u_{k_j}-u_0\rangle\le 0.$$

Therefore,

$$\begin{aligned}
&\limsup_{j\to\infty}\langle F(u_{k_j})-B(w_{k_j}),u_{k_j}-u_0\rangle\\
&\le\limsup_{j\to\infty}\langle B(w_0)-B(w_{k_j}),u_{k_j}-u_0\rangle+\lim_{j\to\infty}\langle F(u_{k_j}),u_{k_j}-u_0\rangle\\
&+\lim_{j\to\infty}\langle -B(w_0),u_{k_j}-u_0\rangle\\
&\le 0.
\end{aligned} \tag{6.24}$$

Combining (6.21) and (6.24), we get

$$\limsup_{j\to\infty}\langle A(u_{k_j}),u_{k_j}-u_0\rangle\le 0. \tag{6.25}$$

Noting that $u_{k_j} \rightharpoonup u_0$ and A is a continuous mapping of class $(S)_+$, it implies that $u_{k_j} \to u_0$ and $A(u_{k_j}) \to A(u_0)$. Therefore, by (6.20), we have

$$\sup_{v\in K}\langle (A-F)(u_0) + B(w_0), u_0 - v\rangle \le 0.$$

That is, $u_0 \in K_{w_0}(0)$.

By (6.19) and the weak lower semicontinuous property of $J(w,u)$, we have

$$\beta(0) \le J(w_0,u_0) \le \liminf_{j\to\infty} J(w_{k_j}, u_{k_j}) \le \beta(0) - \frac{\epsilon_0}{2},$$

which is impossible. Therefore, β is lower semicontinuous at the origin. ∎

Similar to the proof of Theorem 6.3.1, by using Lemma 6.3.2, we have the following theorem:

Theorem 6.3.2 *Suppose that* $\mathbf{W}$ *and* $\mathbf{X}$ *are two reflexive Banach spaces,* U *is a nonempty closed set of* $\mathbf{W}$ *and* K *is a closed and convex cone of* $\mathbf{X}$*. Let* P *satisfy assumptions* (P_i)*,* (P_{ii}) *and* (P_{iii})*, and let* $U \times K$ *be unbounded. Suppose that* $J : U \times K \to \mathbf{R}$ *is a weakly lower semicontinuous function,* $A - F : K \to \mathbf{X}^*$ *is a continuous, bounded and generalized pseudo-monotone mapping, and* $B : U \to \mathbf{X}^*$ *is continuous from the weak topology of* $\mathbf{W}$ *to the topology of* $\mathbf{X}^*$*. If (6.14) is satisfied, then the zero duality gap property (6.6) holds.*

In the following, we first give the concept of the upper semicontinuous set-valued mapping, and then obtain another sufficient condition for the zero duality gap property in Banach spaces.

Definition 6.3.1 *([4]) Let* $\mathbf{E}$ *and* $\mathbf{F}$ *be two topological spaces. Let* T*:* $\mathbf{E} \to 2^{\mathbf{F}}$ *be a set-valued mapping. Suppose that* $x_0 \in \mathbf{E}$*.* T *is said to be upper semicontinuous at* x_0 *if, for any open set* $O \subset \mathbf{F}$ *with* $T(x_0) \subset O$*, there exists an open set* $Q \subset \mathbf{E}$ *with* $x_0 \in Q$ *such that* $T(x) \subset O$*,* $\forall x \in Q$*.*

Theorem 6.3.3 *Assume that* $\mathbf{W}$ *and* $\mathbf{X}$ *are two Banach spaces,* P *satisfies assumptions* (P_i)*,* (P_{ii}) *and* (P_{iii})*,* U *is a compact set and* K *is a bounded set. Suppose that* $J(w,u)$ *is a lower semicontinuous function. If, for each* $w \in U$*,* $K_w(\cdot) : \mathbf{R}^+ \to 2^K$ *is upper semicontinuous at* $y = 0$*,* $K_w(0)$ *is a nonempty compact set. Then, the zero duality gap property (6.6) holds.*

Proof: By Lemma 6.2.2, we only need to prove that the perturbation function β is lower semicontinuous at the origin.

On the contrary, suppose that there exists an $\epsilon_0 > 0$ such that

$$\liminf_{y \to 0} \beta(y) \leq \beta(0) - \epsilon_0.$$

Then, there exist a sequence $\{y_k\} \subset \mathbf{R}$, $y_k \to 0$, $w_k \in U$ and $u_k \in K_{w_k}(y_k)$ such that

$$J(w_k, u_k) \leq \beta(0) - \frac{\epsilon_0}{2}, k = 1, \cdots. \tag{6.26}$$

Since U is a compact set, there exists a convergent subsequence of $\{w_k\}$. Without loss of generality, we may assume that $w_k \to \bar{w}$. By (6.3), we have

$$\begin{aligned} g_{\bar{w}}(u_k) &= \sup_{v \in K} \langle (A - F)(u_k) + B(\bar{w}), u_k - v \rangle \\ &\leq \sup_{v \in K} \langle (A - F)(u_k) + B(w_k), u_k - v \rangle + \sup_{v \in K} \langle B(\bar{w}) - B(w_k), u_k - v \rangle \\ &= g_{w_k}(u_k) + \sup_{v \in K} \langle B(\bar{w}) - B(w_k), u_k - v \rangle. \end{aligned} \tag{6.27}$$

Let $z_k = \sup_{v \in K} \langle B(\bar{w}) - B(w_k), u_k - v \rangle$. Since K is bounded and B is continuous, $z_k \to 0$.

It follows from $u_k \in K_{w_k}(y_k)$, (6.4) and (6.27) that

$$g_{\bar{w}}(u_k) \leq y_k + z_k, \quad \forall k.$$

That is, $u_k \in K_{\bar{w}}(y_k + z_k)$. Because $K_{\bar{w}}(y)$ is upper semicontinuous at $y = 0$, for any open set $O \subset K$ with $T(0) \subset O$, there exists an open set $Q \subset \mathbf{R}$ with $0 \in Q$ such that $K_{\bar{w}}(y) \subset O$, $\forall y \in Q$. Since $y_k \to 0$ and $z_k \to 0$, there exists an integer N large enough such that $y_k + z_k \in Q$ for all $k > N$. Thus, $K_{\bar{w}}(y_k + z_k) \subset O$. By $u_k \in K_{\bar{w}}(y_k + z_k)$, we get $u_k \in O$ ($\forall k \geq N$). Since $K_{\bar{w}}(0)$ is compact, and O is arbitrary, we conclude that $u_k \to \bar{u} \in K_{\bar{w}}(0)$.

By the lower semicontinuous property of $J(w, u)$ and (6.26), we have

$$J(\bar{w}, \bar{u}) \leq \liminf_{k \to \infty} J(w_k, u_k) \leq \beta(0) - \frac{\epsilon_0}{2}.$$

Noting that $\bar{u} \in K_{\bar{w}}(0)$, $\beta(0) \leq J(\bar{w}, \bar{u})$, we get

$$\beta(0) \leq \beta(0) - \frac{\epsilon_0}{2},$$

which is impossible.

∎

6.4 A convergence property for the power penalty problem

Let

$$L(w,u,d) = J(w,u) + d(g_w(u))^\mu,$$

where $0 < \mu$. This is called a $\mu-$power penalty function. It is easy to see that the power penalty function is a special nonlinear Lagrangian defined above. It is found that the power penalty method is effective and useful in application, see [111]. In this section, we will discuss existence results of solutions of the power penalty problem in reflexive Banach spaces. We show that every weak limit point of a sequence of optimal solutions generated by the power penalty problem is a solution for the optimal control problem.

Let $d > 0$. Consider the $\mu-$power penalty problem:

$$(PP_d) \quad \inf_{(w,u)\in U\times K} L(w,u,d).$$

Let S_d be the set of optimal solutions of problem (PP_d).

In order to establish the result on the existence of global solutions for the $\mu-$power penalty problem (PP_d) and their convergence property, we need some existence results of variational inequality problem (6.2). Similar to the proof of Lemma 6.3.2, we have the following lemma:

Lemma 6.4.1 *Suppose that* $\mathbf{W}$ *and* $\mathbf{X}$ *are two reflexive Banach spaces,* U *is a nonempty closed set of* $\mathbf{W}$ *and* K *is a closed and convex cone of* $\mathbf{X}$*. Assume that* $J(w,u) : U \times K \to \mathbf{R}$ *is a weakly lower semicontinuous function,* $A : K \to \mathbf{X}^*$ *and* $F : K \to \mathbf{X}^*$ *are continuous from the weak topology of* $\mathbf{X}$ *to the topology of* $\mathbf{X}^*$*. Suppose that, for each* $w \in U$*, the coercive condition (6.14) is satisfied. Then,* $-\infty < \beta(0) < +\infty$*.*

Theorem 6.4.1 *Suppose that* $\mathbf{W}$ *and* $\mathbf{X}$ *are two reflexive Banach spaces,* U *is a nonempty closed set of* $\mathbf{W}$ *and* K *is a closed and convex cone of* $\mathbf{X}$*. Assume that* $J(w,u) : U \times K \to \mathbf{R}$ *is a weakly lower semicontinuous function,* $A : K \to \mathbf{X}^*$ *and* $F : K \to \mathbf{X}^*$ *are continuous from the weak topology of* $\mathbf{X}$ *to the topology of* $\mathbf{X}^*$*,* $B : U \to \mathbf{X}^*$ *is continuous from the weak topology of* $\mathbf{W}$ *to the topology of* $\mathbf{X}^*$*. Assume that the coercive condition (6.14) is satisfied. Then,* *(i) There exists a $d_0 > 0$, such that, for any $d \geq d_0$, the optimal solution set S_d of problem (PP_d) is a nonempty bounded set.*

(ii) Every weak limit point of the sequence $\{w_d\}$ is an optimal control for the optimal control problem (6.1), where (w_d, u_d) is a solution of problem (PP_d) with $d \geq d_0$.

Proof: (i) By Lemma 6.4.1, $-\infty < \beta(0) < +\infty$. There exists $(w^*, u^*) \in U \times K, u^* \in K_{w^*}(0)$, such that $J(w^*, u^*) = \beta(0)$. Then, $g_{w^*}(u^*) = 0$ and $L(w^*, u^*, d) = J(w^*, u^*)$. Define

$$O_d = \{(w, u) : L(w, u, d) \leq L(w^*, u^*, d)\}.$$

We have

$$O_d = \{(w, u) : L(w, u, d) \leq J(w^*, u^*)\}.$$

Clearly,

$$(w^*, u^*) \in O_d, S_d \subset O_d, \quad \forall d > 0,$$

and

$$O_{d'} \subset O_d, \quad 0 < d < d'.$$

Now we show that there exists a $d_0 > 0$ such that O_d is bounded for any $d \geq d_0$.

On the contrary, suppose that there exist $0 < d_k \to +\infty$ and $(w_k, u_k) \in O_{d_k}$ such that $\|(w_k, u_k)\| \to +\infty$. Then, it follows from the coercive condition (6.14) that

$$\lim_{k \to +\infty} \langle (A - F)(u_k) + B(w_k), u_k \rangle = +\infty. \tag{6.28}$$

By $(w_k, u_k) \in O_{d_k}$, we have

$$L(w_k, u_k, d_k) \leq J(w^*, u^*).$$

That is,

$$J(w_k, u_k) + d_k(g_{w_k}(u_k))^\mu \leq J(w^*, u^*). \tag{6.29}$$

We claim that $g_{w_k}(u_k) \to 0$. In fact, if there exists a $\delta > 0$ such that $g_{w_k}(u_k) \geq \delta$, then

$$d_k(g_{w_k}(u_k))^\mu \to +\infty, \text{as } k \to \infty.$$

By (6.29), this is impossible. Hence, $g_{w_k}(u_k) \to 0$. Let $g_{w_k}(u_k) = y_k$. Then

$$\sup_{v \in K} \langle (A - F)(u_k) + B(w_k), u_k - v \rangle = y_k.$$

Thus,

$$\langle (A-F)(u_k) + B(w_k), u_k \rangle \leq y_k \to 0,$$

which contradicts (6.28). Thus, there exists a $d_0 > 0$ such that O_d is bounded for any $d \geq d_0$.

Let $d > 0$, and define $\inf_{w \in U, u \in K} L(w, u, d) = C_d$. Assume that $\{(w_n, u_n)\}$ is a minimizing sequence for the $\mu-$power penalty problem (PP_d), such that

$$L(w_n, u_n, d) \leq C_d + \frac{1}{n}, n = 1, 2, \cdots. \tag{6.30}$$

Thus,

$$(g_{w_n}(u_n))^{\mu} \leq \frac{C_d}{d} + \frac{1}{nd} \ .$$

That is,

$$\langle (A-F)(u_n) + B(w_n), u_n \rangle \leq \big(\frac{C_d}{d} + \frac{1}{nd}\big)^{\frac{1}{\mu}} \ .$$

It follows from the inequality above and (6.14) that $\{(w_n, u_n)\}$ is bounded. Without loss of generality, we assume that $(w_n, u_n) \rightharpoonup (w_0, u_0)$.

Since $J(w, u)$ is a weakly lower semicontinuous function, A and F is continuous from the weak topology of $\mathbf{X}$ to the topology of $\mathbf{X}^*$, B is continuous from the weak topology of $\mathbf{W}$ to the topology of $\mathbf{X}^*$, we have

$$J(w_0, u_0) \leq \liminf_{n \to \infty} J(w_n, u_n)$$

and

$$\lim_{n \to +\infty} \langle (A-F)(u_n) + B(w_n), u_n - v \rangle = \langle (A-F)(u_0) + B(w_0), u_0 - v \rangle, \forall v \in K.$$

Hence, from (6.30), we have

$$J(w_0, u_0) + d(g_{w_0}(u_0))^{\mu} \leq \liminf_{n \to \infty} (J(w_n, u_n) + d(g_{w_n}(u_n))^{\mu}) \leq C_d.$$

Therefore,

$$J(w_0, u_0) + d(g_{w_0}(u_0))^{\mu} = C_d,$$

i.e., S_d is a nonempty bounded set for any $d \geq d_0$.

(ii) Let $(w_d, u_d) \in S_d, \forall d > d_0$. Then, by an argument similar to that given for the proof of (i), we can show that O_{d_0} is bounded and $S_d \subset O_d \subset O_{d_0}$, $\forall d > d_0$. Thus, $\{(w_d, u_d)\}$ is bounded. Suppose that $(w_{d_k}, u_{d_k}) \rightharpoonup (\bar{w}, \bar{u})$ as $d_k \to +\infty$. We claim that $\bar{u} \in K_{\bar{w}}(0)$.

On the contrary, suppose that $\bar{u} \notin K_{\bar{w}}(0)$. Then, there exists a $\delta' > 0$ such that $g_{\bar{w}}(\bar{u}) \geq \delta'$. That is,

$$\sup_{v\in K}\langle (A-F)(\bar{u}) + B(\bar{w}), \bar{u} - v\rangle \geq \delta'.$$

Let $0 < \epsilon_0 < \delta'$. For this ϵ_0, there exists an integer $N_0 > 0$, such that

$$\sup_{v\in K}\langle (A-F)(u_{d_k}) + B(w_{d_k}), u_{d_k} - v\rangle \geq \delta' - \epsilon_0. \ \ \forall k \geq N_0,$$

i.e.,

$$g_{w_{d_k}}(u_{d_k}) \geq \delta' - \epsilon_0, \quad \forall k \geq N_0. \tag{6.31}$$

By $S_{d_k} \subset O_{d_k}$, we get

$$L(w_{d_k}, u_{d_k}, d_k) \leq J(w^*, u^*).$$

That is,

$$J(w_{d_k}, u_{d_k}) + d_k(g_{w_{d_k}}(u_{d_k}))^{\mu} \leq J(w^*, u^*).$$

Since $0 < d_k \to +\infty$ and $J(w_{d_k}, u_{d_k}) \leq 0$, from (6.31), the left side of the above inequality tends to $+\infty$, which is impossible. Therefore $\bar{u} \in K_{\bar{w}}(0)$, which, in turn, implies that $\bar{u} \in S(\bar{w})$.

By $(w_{d_k}, u_{d_k}) \in S_{d_k}$, for $\forall (w, u) \in U \times K, u \in S(w)$, we have

$$J(w_{d_k}, u_{d_k}) \leq L(w_{d_k}, u_{d_k}, d_k) \leq L(w, u, d_k) = J(w, u), \quad \forall d_k \geq d_0.$$

Since J is a weakly lower semicontinuous function,

$$J(\bar{w}, \bar{u}) \leq \liminf_{k\to+\infty} J(w_{d_k}, u_{d_k}) \leq J(w, u), \quad \forall (w, u) \in U \times K.$$

Noting that (w, u) is arbitrary, it follows that $\bar{w}$ is an optimal control for the optimal control problem (6.1). ▮

Remark 6.4.1 The results of optimal control governed by variational inequality in this chapter can be extended to the general form as in (5.37) (see p. 105).

6.5 Example of the obstacle problem

In this section, we study an example of an optimal control problem where the variational inequality constraint leads to an obstacle problem.

Let Ω be a bounded domain of $\mathbf{R}^N$ with the smooth boundary and $H_0^1(\Omega) = W_0^{1,2}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ in the norm $\|u\| = \{\int_\Omega |\nabla v|^2\, dx\}^{1/2}$. Let $K = \{u \in H_0^1(\Omega) : u \geq 0,\ a.e.\ in\ \Omega\}$. This set is a nonempty closed and convex cone of $H_0^1(\Omega)$. Define

$$a(u,v) = \sum_{i,j=1}^{N} \int_\Omega a_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx + \int_\Omega a_0 uv\, dx, \quad u, v \in K.$$

Assume $a_{i,j}, a_0 \in L^\infty(\Omega)$ (the Banach space of essential bounded measurable functions on Ω), $a_0(x) \geq c_1 > 0$, where c_1 is a constant. For some constant $c_2 > 0$,

$$\sum_{i,j=1}^{N} a_{i,j}\xi_i\xi_j \geq c_2\|\xi\|^2, \quad a.e.\ in\ \Omega, \quad \forall \xi = (\xi_1, ..., \xi_N) \in \mathbb{R}^N. \tag{6.32}$$

Let $f \in H^{-1}(\Omega)$ (the dual space of $H_0^1(\Omega)$) and let U be a nonempty closed and convex subset of $L^2(\Omega)$. For each $w \in U$, we define $u = u(w)$ (the state function of the system) as the solution of the variational inequality:

$$a(u, v-u) \geq \langle f + w, v - u\rangle, \quad v \in K. \tag{6.33}$$

It follows from [107] that (6.33) is an obstacle problem. Now, we consider the optimal control problem defined as follows:

$$\begin{array}{ll} (P_0) & \inf \left\{ J(u,w) = \frac{1}{2}\int_\Omega (u - z_d) + \frac{M}{2}\int_\Omega w^2 \right\} \\ & \text{s.t. } a(u, v-u) \geq \langle f + w, v-u\rangle, w \in U, v \in K, \end{array}$$

where $z_d \in L^2(\Omega)$ and $M > 0$. The optimal control problem (P_0) has been studied by Bergounioux [11], Mignot and Puel [87].

Theorem 6.5.1 *Let P satisfy assumptions (P_i), (P_{ii}) and (P_{iii}), and let the set $U \times K$ be unbounded. Suppose that condition (6.32) given above is satisfied. Then, the zero duality gap property*

$$\inf_{(w,u)\in U\times K, u\in S(w)} J(w,u) = \sup_{d\in\mathbf{R}^+} \inf_{(w,u)\in U\times K} P(J(w,u), dg_w(u)) \tag{6.34}$$

holds.

Proof: Denote $H_0^1(\Omega) = W_0^{1,2}(\Omega)$ and $L^2(\Omega)$ as $\mathbf{X}$ and $\mathbf{W}$, respectively. Define $\langle Au, v\rangle = a(u,v)$, $F(u) \equiv f$, $\langle B(w), v\rangle = \int_\Omega wv$. Then, $A \in L(H_0^1(\Omega), H^{-1}(\Omega))$. The coercive assumption (6.32) implies that

$$\langle Au, u\rangle \geq c_2\|u\|^2, \quad \forall u \in H_0^1(\Omega). \tag{6.35}$$

Suppose that $u_j \rightharpoonup u_0 \in \mathbf{X}$ satisfies

$$\limsup_{j\to\infty}\langle A(u_j), u_j - u_0\rangle \le 0.$$

It is clear that

$$\lim_{j\to\infty}\langle A(u_0), u_j - u_0\rangle = 0.$$

By (6.35), we have

$$\begin{aligned}\langle A(u_j), u_j - u_0\rangle &= \langle A(u_j - u_0), u_j - u_0\rangle + \langle A(u_0), u_j - u_0\rangle\\ &\ge c_2\|u_j - u_0\|^2 + \langle A(u_0), u_j - u_0\rangle.\end{aligned}$$

Thus $u_j \to u_0$. That is, A is a mapping of class $(S)_+$. It is easy to prove that A, F, B satisfy all the conditions in Theorem 6.3.2. Therefore, by Theorem 6.3.2, the zero duality gap property (6.34) holds.

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Chapter 7

Conclusions and Suggestions for Future Studies

In this thesis, we studied the theory of augmented Lagrangian and nonlinear Lagrangian scheme for constrained optimization problems and optimal control problems governed by a variational inequality.

In Chapter 2, we introduced the concept of a valley at 0 augmenting function, which includes a convex augmenting function and a level-bounded augmenting function as special cases, and applied it to construct a class of valley at 0 augmented Lagrangian functions. Under the assumption that the perturbation function satisfies the growth condition and the augmenting function satisfies a valley at 0 condition, we established a necessary and sufficient condition for a zero duality gap property between the primal problem and its augmented Lagrangian dual problem in general Banach spaces.

In Chapter 3, we obtained some exact penalty representation results in the framework of the new augmented Lagrangian. We established sufficient conditions of an exact penalization representation for constrained optimization problems. We obtained a sufficient condition of the existence of an asymptotically minimizing sequence for a constrained problem in infinite dimensional Banach spaces. Furthermore, without any coercive assumption on the objective function and constraint functions, we obtained a sufficient condition of an exact penalization representation for a constrained optimization problem in finite dimensional spaces.

In Chapter 4, we introduced a class of penalty functions. We proved that any strict local minimum satisfying a second-order sufficient condition for an inequality and equality constrained optimization problem is a strict local minimum of this penalty function with any positive penalty parameter, and that any global minimum satisfying a second-order global sufficient condition for the original problem is a global minimum of this penalty function with some positive penalty parameter. We applied our results to quadratic and linear fractional programming problems.

In Chapter 5, we established some existence results for a solution of variational inequality problems for generalized pseudo-monotone mappings and generalized pseudo-monotone perturbations of maximal monotone mappings respectively. We obtained several existence results of an optimal control of the optimal control problem governed by a quasilinear elliptic variational inequality.

In Chapter 6, we introduced a modified nonlinear Lagrangian function and obtained a necessary condition and sufficient condition for the zero duality gap property between the optimal control problem and its nonlinear Lagrangian dual problem. We applied a power penalty method to the optimal control problem, and obtained that a sequence of approximate optimal solutions of the penalty function converges weakly to the optimal solution of the original optimal control problem.

Overall, we obtained some new results and methods for the theory of augmented Lagrangian and nonlinear Lagrangian in Banach spaces. Some of our results can include the corresponding results studied by others as special cases, and some of our results are original. But some of our results are quite abstract. Thus it is difficult to apply these results to some practical problems. Moreover, in this thesis, we have established zero duality gap and exact penalty properties between a primal optimization problem and its augmented Lagrangian dual problem by using a weaker augmenting function in Banach spaces. Since this augmenting function is nonconvex and non-Lipschitz, it maybe difficult to apply it in the design of some effective and efficient optimization algorithms. Therefore, we should try to overcome these difficulties in the future research, and carry out some numerical computations for the approximation of the solution for constrained optimization problems and optimal control problems governed by a variational inequality.

The following is a list of some interesting problems for future research.

1 We will try to find more applications in practical problems such as American option price problem, infinite dimensional linear programming and transportation problems.

2 We will find more necessary and sufficient optimality conditions for optimal control problems that are amenable to numerical computation for the approximation of the optimal control for an optimal control problem governed by monotone type variational inequality.

3 We will establish second-order optimality conditions of augmented Lagrangian problems and characterize local and global solutions for augmented Lagrangian problems.

4 We will establish sufficient conditions for exact penalty by using the new results for the error bounds obtained by Wu and Ye [112].

In studying those problems mentioned above, we will obtain some new results by using the methods introduced by others, or find and introduce new methods to deal with these problems. We will be further concerned with these and related problems. We will try to get more useful results.

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