The Hong Kong Polytechnic University

Department of Applied Mathematics

Model Risks in the Valuation of Equity Indexed Annuities

by

Fung Hon Kwok

A thesis submitted in partial fulfillment of

the requirements for the Degree of Doctor of Philosophy

July 2007

CERTIFICATION OF ORIGINALITY

I hereby declare that this thesis is my own work and that, to the best of my knowledge and belief, it reproduces no material previously published or written, nor material that has been accepted for the award of any other degree or diploma, except where due acknowledgment has been made in the text.

____________________ (Signed)

Fung, Hon Kwok (Name of student)

Abstract

Over last several decades, a lot of work has been done to incorporate various features of asset price movements into stochastic models, including (but not limited to) stochastic volatility of stock price, randomness of interest rate and the correlation between stock return and return volatility (i.e. the leverage effect). This thesis presents an empirical study in the pricing errors for equity indexed annuities (EIAs) arising from the use of different interest rate models and volatility models.

For the interest rate risk, I will inspect the pricing differences between the use of the stochastic volatility model of Heston (1993) and the use a combination of the Heston model with the stochastic interest rate model of Cox, Ingersoll and Ross (1985). The classical Black-Scholes (1973) model will also be compared against its combination with the extended Vasicek model due to Hull and White (1990). Unlike most equity options in the literature, EIAs typically have moderate to long maturities. While in the valuing the former one may consider the interest rate as deterministic, this may not be the case for pricing EIAs. This part of the thesis is a partial attempt to answer this question.

For volatility risks, although the number of existing volatility models is vast, most of them are impractical for EIA valuation. This is because the EIAs are Bermudan options (owing to the presence of surrender terms) but most popular volatility models simply do not admit closed form or semi-closed form formulas for the densities or characteristic functions of the stock return conditional on the initial and final volatilities. To solve this problem, I adopt the finite-state, continuous-time regime switching Levy stock return model proposed by Chourdakis (2004). This model also has the merit that the leverage effect can be built in easily. However, the model was initially intended to be used as an approximation to Heston's (1993) model. As such, it is not risk-neutral. In addition, for general time-changed Levy processes, that how to find an equivalent martingale measure is currently a very confusing issue in the literature. In this thesis I modify Chourdakis' model so that it is used directly as the model of, but not an approximation to, a physical process. A way to obtain a structure-preserving equivalent martingale measure is also proposed.

Given the conditional density or characteristic function for a regime switching model, one can compute Bermudan option prices by using the sequential quadrature method developed by Sullivan (2000a,b). We will explain the idea of this method and give a convergence proof to it in this thesis.

Acknowledgments

I would like to express my deep gratitude to my thesis supervisors, Dr Leong Kwan Li and Prof. Elias S.W. Shiu, for giving me enormous freedom in doing my research work and allowing me to pace myself, despite of my somewhat slow progress. I would also like to thank Ms Christine Yeung and Dr Leong Fu Cheung for providing me useful advices and also interesting conversation pieces during many dinners.

In the course of my study, Prof. Shiu had asked me to review a number of papers that were submitted for publication in a few academic journals. When I was pursuing a M.Phil. degree in Mathematics at the University of Hong Kong many years ago, all the papers I reviewed were about pure mathematics. Judging the values of this kind of papers was relatively easy. Usually, a technically correct paper would be accepted for publication as long as the problems it considered or the proofs it provided were interesting and novel. The papers that Prof. Shiu asked me to review, however, were about actuarial science and economics. While these papers still contained many mathematical details, their values must also be judged from context. Is a method for solving a problem *pragmatic*? Is the problem considered in this paper no longer *relevant*? Although the idea of this computational method is interesting and highly innovative, can it outperform other recently developed numerical procedures? Questions like these had made judgments difficult. Yet they did broaden my horizons. So, thank you, Prof. Shiu, for these interesting experiences.

Finally, I would also like to thank the Canadian Institute of Actuaries for giving me an Outstanding Paper Award (and a cash prize) at the *2006 Stochastic Modeling Symposium & Investment Seminar* held in Toronto. Having the recognition of a professional body is really very encouraging.

Contents

I Model risks for equity-indexed annuities **1**

1 Introduction **3**

2 Preliminaries **9**
2.1 Equity-indexed annuities and other exotic options 9
2.1.1 Equity-indexed annuities 9
2.1.2 Other exotic options 12
2.2 Research methodology 13
2.2.1 Different sides of model risk 13
2.2.2 Basic testing procedure 14

3 Interest rate risk for equity-indexed annuities **17**
3.1 Overview 17
3.2 The SVSI model and the BSHW model 17
3.2.1 The SVSI model 17
3.2.2 The BSHW model 19
3.3 Pricing ratchets and cliquets 22
3.4 Some qualitative analysis 24
3.5 Numerical experiments 27
3.5.1 Experimental procedure 30
3.5.2 EIA prices: SV model vs SVSI model 31
3.5.3 EIA prices: BS model vs BSHW model 33
3.5.4 Cliquet price differences 36
3.6 Conclusion 38

4 Volatility risk for equity-indexed annuities **41**
4.1 Overview 41
4.2 Economic models 42
4.2.1 Black-Scholes model 42
4.2.2 Levy models 43
4.2.3 Heston (1993) model and its extensions 49
4.2.4 Γ-OU model 51
4.2.5 Continuous-time regime-switching (CTRS) models 53
4.3 Risk-neutral evaluation of Levy and CTRS models 61
4.3.1 Levy models 62
4.3.2 CTRS models 65
4.4 Model estimation and calibration 68
4.4.1 Estimation 68
4.4.2 Calibration 70
4.5 Option pricing 77
4.6 Numerical experiments 79
4.6.1 Experimental set-up 79
4.6.2 Term-end CARs 80
4.6.3 Term-end PTPs, Asian-end CARs and reverse HWM PTPs 83

4.6.4 Cliquets and barrier options . . . 87
4.7 Conclusion . . . 90

II Implementation and application of the sequential quadrature method 125

5 The sequential quadrature method 127
5.1 Three variants of the sequential quadrature method . . . 127
5.1.1 The pure interpolation approach . . . 130
5.1.2 The pure quadrature approach . . . 131
5.1.3 The mixed approach . . . 131
5.2 Convergence of the mixed approach . . . 133

6 Example: pricing discrete dynamic fund protections 137
6.1 Overview . . . 137
6.2 Risk neutral valuation under the lognormal process . . . 139
6.3 Pricing dynamic fund protections under the CEV process . . . 140
6.4 Numerical experiments . . . 142
6.5 Concluding remarks . . . 144

A The BSHW model 147

B Proofs for propositions in chapter 4 155

Part I

Model risks for equity-indexed annuities

Chapter 1

Introduction

Over last few decades, a lot of work has been done to extend the Black-Scholes (1973) model so that various features of the financial market can be incorporated. Nowadays, different people may evaluate the price of the same financial derivative using vastly different financial models. Naturally, the prices of financial derivatives has become dependent on numerous model assumptions.

In finance, it is now widely accepted that stock price volatilities should be considered stochastic. In contrast, in pricing stock options, interest rates are often considered deterministic. While the reasons are seldom discussed, this is probably due to the fact that most traded stock options have short lifetimes. Since interest rates are empirically much less volatile than stock prices, when an option will expire shortly, incorporating stochastic interest rate into a stock price model may induce only technical or programming difficulties but not any noticeable improvement in pricing performance. In fact, it is reported (in, e.g. Bakshi *et al.* 1997 or Jiang 2002) that the randomness of interest rates has only minimal impact on the pricing of plain vanilla European options. In practice, interest rates are considered stochastic usually only when interest rate derivatives (such as interest rate swaps or guaranteed annuity options) are considered.

Interestingly, although equity-indexed annuities (EIAs) are clearly stock options with long maturities, they are predominantly priced with deterministic interest rate term structures in the literature. For example, among Buetow Jr. (1999), Tiong (2000), Wilmott (2002), Hardy (2003 and 2004), Lin and Tan (2003), Eales and Tunaru (2004), Jaimungal (2004), Schoutens *et al.* (2003, 2005) and Windcliff *et al.* (2004), stochastic interest rate is considered only in Lin and Tan (2003). Again, the reasons behind this are seldom stated explicitly, but the comment of Jaimungal (2004) probably explains what most researchers and practitioners have in their mind:

> "... when the short rate and asset price process are uncorrelated, the interest rate dynamics can be factored from the equity dynamics, and the pricing formulae which arise are those of Tiong (2000) with r replaced by the average rate over the term of the option, $r_{avg} = \frac{1}{T-t} \ln E^Q \left[\exp\{- \int_t^T r_s ds\} \right]$. In a market calibrated model, this average corresponds to the prevailing spot interest rate of the appropriate maturity. Furthermore, although EIA products typically have long terms (around 10 years), the embedded reset features are typically applied over $\frac{1}{2} \sim 1$ year periods. This implies that forward prices spanning a single year or less are the most relevant, and such interest rates have significantly lower volatility than, for example, a 10-year spot rate."

In contrast, not only are discounted stock prices much more volatile than spot interest rates, the former also exhibit a number of features that have greatly influenced the development of equity price models. These features include the fat-tailed distributions of stock returns, volatility clustering and the leverage effect, among others. As more and more models have been developed to incorporate these features, researchers and practitioners have become aware of the fact that even if the distributions of stock return over a fixed period under different models are almost identical, these models can still have very different finite-dimensional distributions. Consequently, there is a widespread belief that in pricing most path-dependent exotic options (including EIAs), information about the micro-structure of the stock price process is very important. Borrowing the words of Heyer (2004, p.15),

> "... you absolutely must be very careful about the instrument you choose to calibrate to. For example, if I calibrate to vanilla puts and then try to price ratchet structures, I absolutely guarantee you you'll wind up with a nonsense set of prices. The reason being that the put only cares about the distribution of returns at a specific time, but the ratchet actually cares about how you got there as it meandered through different periods."

The first half of Heyer's argument (that calibrating a model to plain-vanilla option prices alone does not guarantee small pricing errors in exotic options) is in fact supported by some striking research results. In a well-known paper by Schoutens *et al.* (2003, the paper is also slightly modified and published as Schoutens *et al.* 2005), seven stochastic volatility models are calibrated to fit the prices of 144 European call options with maturities ranging from two weeks to five years. Upon calibration, the model prices almost fit the market prices very well — the largest relative pricing error is only about 2.2%. However, when these calibrated models are applied to evaluate the prices of some exotic options (including barrier options, barrier digital options, lookback options and cliquets), the relative price differences between them can be as high as[1] 2333%! The price differences they find for cliquets are also up to about[2] 56%. Schoutens *et al.* conclude that, although different price dynamics may give almost identical marginal distributions, their *path structures* can be radically different. Since many exotic options have path-dependent payoffs, their option prices can be susceptible to these structural differences, no matter how well the models are calibrated to vanilla option prices.

To sum up, there are a few common beliefs about pricing a path-dependent option:

(a) Calibrating a model only to market prices of plain-vanilla options can result in large pricing errors.

(b) Therefore, one should calibrate the pricing model to the market prices of some path-

[1] The highest price they obtain for a down-and-in barrier option is 2.19, while the lowest price is 0.09. So the relative error is $(2.19/0.09 - 1) \times 100\% = 2333\%$.

[2] Computed as $(0.1131/0.0724 - 1) \times 100\%$.

dependent options.

(c) For EIAs, the interest rate risk is small, as EIAs typically have payoffs that depend on stock returns over a number of consecutive, non-overlapping but short time periods, and spot short-term interest rates are empirically quite non-volatile.

Note that, while assumption (a) in the above is supported by the results of Schoutens *et al.* (and other results that we have not reported here) for a few types of options, in general no evidence has been given to support or disprove it for other financial derivatives. Likewise, although assumptions (b) and (c) seem to be commonplace, they are, to this date and to our knowledge, only folklore assumptions that have not been backed by any empirical evidence. It is, however, very difficult for academics to verify assumption (b) because path-dependent options are typically traded over the counter and their price history is hard to obtain. In this thesis, we will focus on (a) and (c) in the above in the context of EIAs.

Ideally, if we want to study the effects of interest rate risk and volatility risk on the valuation of EIAs, we should consider a number of stock price models that feature both stochastic interest rate and stochastic (stock price) volatility. However, the number of such models are scarce in the literature. Since many stock options have very short maturities (insurance products such as EIAs are notable exceptions in this regard), in financial literature most researchers simply do not feel the need to make interest rates stochastic in building a stock price model. Actually, incorporating both stochastic interest rate and stochastic volatility in a single model can be technically challenging, especially when the number of stochastic factors in any modern interest rate model is overwhelmingly large. Even when the amalgam of interest rate and stock price volatility can be made realistic and meaningful, reliable ways to estimate and calibrate the combined economic model and efficient methods to calculate option prices are still needed for a pricing model to become practical.

A few models that incorporate both stochastic volatility and stochastic interest rate are better known to practitioners. The first one is an extension of the stochastic volatility model of Heston (1993) by Bakshi *et al.* (1997). The second one, developed by Duffie *et al.* (2000), extends the Heston model in a way that the log equity price and the interest rate must be affine functions of a common (but perhaps multidimensional) source of randomness. The third model, or strictly speaking a third approach, is the uncertain parameter model developed independently by Avellaneda *et al.* (1995) and Lyons (1995). This approach is interesting in that it requires only the upper and lower bounds of the volatility and the spot interest rate but not their exact dynamics to be specified. The model gives rise to an upper bound and a lower bound of the option price.

Unfortunately, it is difficult for us to apply these models in our study. With the models of Bakshi *et al.* (1997) and Duffie *et al.* (2000), the problems are mainly computational. For instance, some EIAs have structures similar to Bermudan style Asian or lookback options, which we find difficult to price under the Heston model or its extensions. For the uncertain parameter model, it is not known how conservative the bounds of the option prices are. Also, the stock price process is assumed to be a diffusion. Whether the model can be extended to include jumps in the stock price remains unclear.

In view of the difficulty in building or applying an equity price model with stochastic volatility and stochastic interest rate, we consider the two sources of randomness separately. In next chapter, we will first introduce the EIAs that are of interest in our study and explain our basic research methodology. Then we will investigate the interest rate risk for EIAs in chapter 3 using a few relatively simple models. For ease of computation, we assume that surrender is not allowed, so that every EIA becomes a European option which we can price using Monte Carlo simulation. We will change our focus to volatility risk in chapter 4, in which interest rates are assumed to be deterministic. Inspired by Schoutens *et al.* (2003), we will consider a number of Levy models and time-changed Levy models and also in the no-surrender case the model of Bakshi *et al.* (1997). However, owing to a technical flaw in Schoutens *et al.* (2003), the time-changed Levy models we use here are not those they consider but the continuous-time regime switching (CTRS) models developed by Choundakis (2002). Under these CTRS models, the finite dimensional distributions of stock price processes are available in semi-closed forms. Hence in theory we can evaluate the price of a Bermudan option by numerical integration. The computational complexity involved, however, can be prohibitive if the numerical procedure is not carefully devised. In our study, we will apply the *sequential quadrature method* developed independently by Hunt and Kennedy (2000), Sullivan (2000a,b) and Tse *et al.* (2001) to compute the prices of Bermudan options. In part II of this thesis, we will introduce the idea of sequential quadrature in chapter 5 and present an example application of this numerical procedure in the subsequent chapter. Technical proofs for chapters 3 and 4 are deferred to the appendices.

Throughout this thesis, we will denote by ϖ the imaginary number $\sqrt{-1}$. The commonly used notations i and j for $\sqrt{-1}$ are reserved for array indices.

Chapter 2

Preliminaries

2.1 Equity-indexed annuities and other exotic options

In this section we will briefly describe the payoff structures of the EIAs and exotic options concerned in our study. Most of the EIAs below have been considered by Lin and Tan (2003), which to our knowledge is the first paper to address the issue of interest rate risk in EIA valuation. For a comprehensive and broader discussion of other EIAs (and investment guarantees in general), the monograph by Hardy (2003) is highly recommended.

2.1.1 Equity-indexed annuities

In our study, we restrict our attention to two most common forms of equity-indexed annuities, namely point-to-point contracts and ratchets.

Point-to-point contract (PTP)

Suppose the contract is sold at time 0 and it will expire at the end of year T. Let $\tau \in \{1, 2, \ldots, T\}$ be the year when one or more of the following events occur: the contract expires, the policy holder dies or surrenders. Then the insurance company is liable to pay at the end of year τ the following amount of money:

$$H(\tau) = P \text{ mid} \left[1 + F_\tau,\ 1 + \alpha(R(\tau) - 1),\ 1 + C_\tau\right],$$

where P is the contract's premium, $\alpha > 0$ is the participation rate (which is typically smaller than one), $R(\tau)$ is some formula for calculating the equity return, F_τ and C_τ are respectively the floor and cap rates to the contract's gain, and $\mathrm{mid}(x, y, z)$ denotes the middle value or the median of three numbers x, y, z. The formula for $R(\tau)$ is responsible for different contract designs. Three popular designs are

$$\text{(Term-end)} \qquad R(\tau) = \frac{S(\tau)}{S(0)},$$

$$\text{(Asian-end)} \qquad R(\tau) = \frac{1}{12} \sum_{m=1}^{12} \frac{S\left(\tau - 1 + \frac{m}{12}\right)}{S(0)},$$

$$\text{(High-watermark)} \qquad R(\tau) = \max_{1 \le m \le 12\tau} \frac{S(m/12)}{S(0)},$$

were $S(t)$ represents the level of a certain stock index[1] at time t. In the term-end design, $R(\tau)$ represents the usual equity return $S(\tau)/S(0)$ over the period $[0, \tau]$, while in the Asian-end design, $R(\tau)$ is the simple average of the equity returns evaluated at the end of each of the twelve months in year τ with respect to the index level at the beginning of the year. In the high-watermark (HWM) design, $R(\tau)$ is the highest value of $S(t)/S(0)$ observed at the month ends throughout the lifetime of the contract. As pointed out by Lin and Tan (2003), when there is no cap ($C_\tau = \infty$), the Asian-end contract and the high-watermark PTP essentially reduce to an Asian option and a lookback option respectively.

When surrender is allowed, computing the prices of Asian-end and HWM PTPs is a difficult task. Yet in next chapter this poses no problem because we will assume for simplicity that surrender is not allowed, and hence we can evaluate the price of these PTPs by Monte Carlo simulation. In chapter 4, however, we will consider the possibility of surrender. In this case, owing to computational difficulties, we exclude Asian-end and HWM PTPs from our experiments. Instead, we will consider the following variant of HWM PTP:

$$\text{(Reverse high-watermark)} \qquad R(\tau) = \max_{0 \le m \le (12\tau - 1)} \frac{S(\tau)}{S(m/12)}.$$

[1] For ease of presentation, in the rest of this thesis, we will simply call a stock index a 'stock'. Thus 'stock price' means the level of a stock index, such as the S&P 500 index, and 'stock return' or 'equity return' mean the return of an index.

Despite the apparent similarity in their payoff structures, it is easier to price a reverse high-watermark (RHWM) PTP than to price an ordinary HWM PTP. Roughly speaking, since

$$R(t) = \max\left\{R\left(t - \frac{1}{12}\right), \frac{S\left(t-\frac{1}{12}\right)}{S(0)}\frac{S(t)}{S\left(t-\frac{1}{12}\right)}\right\}$$

in a HWM design and

$$R(t) = R\left(t - \frac{1}{12}\right)\max\left\{1, \frac{S(t)}{S\left(t-\frac{1}{12}\right)}\right\}.$$

in a RHWM design, more information is needed to compute the time-t price of an ordinary HWM PTP.

Compound annual ratchet (CAR)

The term *'ratchet'* is borrowed from the mechanical tool that bears the same name. In essence, it refers to a discretely monitored option written on a certain underlying asset such that over each monitoring period, the value of an investment before possible deduction of a management fee is guaranteed to increase (i.e. ratcheted up) if the price of the underlying asset rise, and never fall below a prespecified level if the market falls. In our experiments, we will consider a popular type of ratchet, which is called *compound annual ratchet* (CAR). The payoff of a CAR can be viewed as the product of the payoffs of a number of PTPs. It is given by

$$H(\tau) = P \text{ mid}\left\{1 + F_\tau,\ \prod_{t=1}^{\tau} \text{mid}\left[1 + f,\ 1 + \alpha(R(t) - 1) - \gamma,\ 1 + c\right],\ 1 + C_\tau\right\},$$

where P, F_τ, C_τ have the same meanings as they have before, $\gamma \geq 0$ represents a yield spread (which is a form of management fee) and f, c represent the local floor rate and local cap rate respectively. The symbol $R(t)$ still represents some kind of equity return, but with different formulas:

(Term-end) $$R(t) = \frac{S(t)}{S(t-1)},$$

(Asian-end) $$R(t) = \frac{1}{12}\sum_{m=1}^{12}\frac{S\left(t-1+\frac{m}{12}\right)}{S(t-1)}.$$

For both PTPs and CARs, when the parameters are chosen such that the fair prices of an EIA is equal to its premium P, we call α the *critical participation rate*.

2.1.2 Other exotic options

We are interested to know whether the severity of model risk is affected by the structure of the option contract, so we will also compute the prices of several European path-dependent options. These exotic options are essentially part of those considered in Schoutens *et al.* (2003), except that the contract parameters may be different.

Cliquet

Although the term 'ratchet' can be used interchangeably with *'cliquet'* (the French word for the same mechanical tool) to mean the same kind of investment guarantee, a more commonplace use of the latter is to describe a European option whose payoff is of the form

$$H(T) = P \text{ mid} \left[\sum_{i=1}^{N} \text{mid} \left(\frac{S_{t_i}}{S_{t_{i-1}}} - 1,\ f,\ c \right),\ F,\ C \right]$$

where $0 < t_1 < t_2 < \ldots < t_N = T$ are a series of time epochs and f, c, F, C represent the local floor rate, local cap rate, global floor rate and global cap rate respectively. For instance, this is the cliquet option that is found to be susceptible to volatility risk in Schoutens *et al.* (2003). Throughout this thesis we will maintain the difference in usage between 'ratchet' and 'cliquet'.

From the pricing aspect, there are two important differences between a cliquet and a ratchet. First, the P in a cliquet is only notional, but it is paid upfront if one buys a ratchet. Consequently the premium for a cliquet includes only the price of protection but not the (notional) amount of investment P, while the premium for a ratchet includes only P but not the price of the protection embedded in the contract (the latter is charged by the use of the yield spread γ and by giving only α of the excess equity return $R(t)$ without dividends). Second, the payoff of a ratchet is contingent on the time when the policyholder dies or surrenders, but a cliquet is strictly a European option without any exposure to mortality risks. Actually, the term 'ratchet' is more popular in the insurance literature but not in the general financial literature. In contrast, the term 'cliquet' is often used outside insurance context and in many cases the underlyings are not equities.

Barrier option

The payoffs of the barrier options we concern are given by

$$\text{(Down-and-out)} \qquad \mathrm{DOB} = (S(T)-K)^+ \,\mathbb{1}\left(\min_{0\le t\le T} S(t) > H\right),$$

$$\text{(Down-and-in)} \qquad \mathrm{DIB} = (S(T)-K)^+ \,\mathbb{1}\left(\min_{0\le t\le T} S(t) \le H\right),$$

$$\text{(Up-and-out)} \qquad \mathrm{UOB} = (S(T)-K)^+ \,\mathbb{1}\left(\max_{0\le t\le T} S(t) < H\right),$$

$$\text{(Up-and-in)} \qquad \mathrm{UIB} = (S(T)-K)^+ \,\mathbb{1}\left(\max_{0\le t\le T} S(t) \ge H\right).$$

For convenience of computation, instead of monitoring the barrier condition continuously, we will consider instead the daily monitored versions of these options. So, assuming that there are 252 trading days in each year, the payoff of a DOB is given by

$$(S(T)-K)^+ \,\mathbb{1}\left(\min_{0\le i\le 252T} S\left(\frac{i}{252}\right) > H\right)$$

and vice versa.

2.2 Research methodology

2.2.1 Different sides of model risk

Traders nowadays rely heavily on mathematical models to price and construct hedges for financial derivatives. Consequently, they are exposed to the risk of using a deficient model, which is usually called 'model risk'. There are many causes for model risk, including model misspecification, incorrect estimation and/or improper calibration of model parameters and difficulties in implementing numerical procedures for estimation, calibration, pricing, hedging or risk management purposes. In this thesis, we focus on the misspecification of models.

Misspecifying a model may lead to a number of problems. Most notably, it may result in unreasonable prices of contracts, poor hedging performance or severe errors in some risk measures such as value-at-risk (VaR) or conditional tail expectation (CTE). To study the severity of different problems, different data or different amount of data are required. For instance, owing to the existing practice to recalibrate models regularly, if we want to study the effect of model risk on hedging, historical time-series data for plain-vanilla option prices are required. In our study, due to severe limitation in obtaining historical data, we will only focus on the pricing errors arising from model risk.

Pricing errors are usually measured in terms of relative errors. If the price of an EIA is p according to some 'true' model but we sell this EIA at price p', the relative pricing error is given by $\frac{p'}{p} - 1$. Although one can measure the pricing error using the plain difference $p' - p$ between the two prices, this error measure is less reasonable as losing one cent in every ten dollars is clearly less disastrous than losing one cent in every dollar. We remark, however, that large pricing errors do not necessarily entail large hedging errors, because the error in the value of a hedging portfolio depends not only on the relative pricing errors in the EIA and the hedging instruments, but also on the sensitivities of them to various quantities. Nevertheless, in the rest of this thesis, unless otherwise specified, the term 'model risk' and 'pricing errors' always refer to the *relative pricing errors* between competing pricing models.

2.2.2 Basic testing procedure

Since pricing error is only about whether an EIA is sold at a wrong price at the inception of the EIA contract, its computation involves much fewer technical details than the computation of, say, hedging errors. Still, if we want to obtain a complete picture of the severity of pricing errors, we should determine them not only using many different competing models but also by considering a large number of economic scenarios. Therefore, in an ideal procedure of testing the size of model risk, one may first estimate the parameters for a set of competing models using a time series data sample. Then for each day in a subsequent time series, the models are recalibrated to the market prices of vanilla options and the pricing errors for the EIA are calculated.

In practice, however, there may be difficulties in carrying out such a testing procedure. In particular, we have not enough option price data for model calibration[2]. There are a number of approaches to get around this problem.

The first approach is to simply use the estimation and calibration results reported by the researchers who possess enough data, but in this case we can only use the models they use to perform our experiments. We will take this approach in some of our experiments when we study interest rate risk.

In the second approach, we may use synthetic data instead of historical data, i.e. we assume that one of our competing models is the 'true' one and then we consider a range of parameters for this model. We will also take this approach when we study interest rate risk. This completely removes our data availability problem, but two new issues arise. First, we must decide what ranges of parameters are reasonable. If the parameter domain is unbounded, any conclusion drawn from our experimental results may be based on unrealistic economic scenarios. Second, we want to calibrate our models to true option prices but our data are only synthetic. While we can assign the most complicated model as the true one when we have a nested chain of competing models, how to pick a true model when our models are not nested? In chapter 3, we will discuss these issues in more details.

A third approach to solve our problem is to use only the cross-sectional option price data on a single day, instead of a series of cross-sectional data. This is essentially the approach adopted by Hirsa *et al.* (2003) and Schoutens*et al.* (2003, 2005) and we will take this approach in chapter 4, where we study the volatility risk for EIAs. It has the advantages that the option data we need are readily available from published research papers and there will be no need to assign any one of the models as 'true', but the downside is that

[2] I have access to the Datastream financial database at this university. However, owing to perhaps licensing issues or a recent change in the software interface, S&P 500 option data, for instance, are no longer available from the option catalogue or the query interface.

any conclusion we draw from our experimental results may be applicable only to some particular economic states. Nonetheless, we will discuss later in this thesis why some of our experimental results should still hold in general economic scenarios.

Chapter 3

Interest rate risk for equity-indexed annuities

3.1 Overview

In this chapter, we will inspect the interest rate risk for EIAs. We will first introduce our equity price models in next section. Then we shall discuss various aspects on the pricing of EIAs (sec. 3.3) and explain why it is difficult to make any prediction about the size of model risk when the model parameters or contract parameters change (sec. 3.4). After that we will explain our experimental procedure and present our results (sec. 3.5) and conclude this chapter in sec. 3.6. Most mathematical details for this chapter are deferred to appendix A.

3.2 The SVSI model and the BSHW model

3.2.1 The SVSI model

The first model we consider is commonly referred to as the SVSI model. Let $S(t)$ denotes the price of the underlying asset (a portfolio of stocks or a stock index in our case) and $r(t)$ denotes the force of interest at time t. The SVSI model assumes that under the risk-neutral measure (i.e. the so-called $\mathbb{Q}$-measure), $S(t)$ and $r(t)$ are governed by the following stochastic differential equations (SDEs):

$$\frac{dS(t)}{S(t)} = (r(t) - q(t))\,dt + \sqrt{v(t)}\; dW_S^*(t), \tag{3.1}$$

$$dv(t) = (\theta_v^* - \kappa_v^* v(t))dt + \sigma_v \sqrt{v(t)}\; dW_v^*(t), \tag{3.2}$$

$$dr(t) = (\theta_r^* - \kappa_r^* r(t))dt + \sigma_r \sqrt{r(t)}\; dW_r^*(t), \tag{3.3}$$

where $q(t)$ represents the (deterministic) continuous dividend yield, $\theta_v^*, \theta_r^*, \kappa_v^*, \kappa_r^*, \sigma_v, \sigma_r$ are some nonnegative constants and $W_S^*(t), W_v^*(t)$ and $W_r^*(t)$ are standard Brownian motions such that the latter is uncorrelated with the former two but $dW_S^*(t)\; dW_v^*(t) = \rho dt$ for some constant $\rho \in [-1, 1]$. Typically, due to the stylised leverage effect for asset returns, we have $\rho < 0$. Equations (3.1) and (3.2) are actually the defining equations of the stochastic volatility (SV) model of Heston (1993) and eq. (3.3) is the stochastic interest rate model by Cox, Ingersoll and Ross (1985, CIR hereafter). Under the $\mathbb{Q}$-measure, $r(t)$ is a scalar multiple of a noncentral chi-square random variable (see Cox *et al.* 1985 for details). Its variance under the $\mathbb{Q}$-measure is given by:

$$\mathrm{var}^{\mathbb{Q}}(r(t)) = \frac{r(0)\sigma_r^2}{\kappa_r^*}(e^{-\kappa_r^* t} - e^{-2\kappa_r^* t}) + \frac{\theta_r^* \sigma_r^2}{2\kappa_r^*}(1 - e^{-\kappa_r^* t})^2. \tag{3.4}$$

Let $D(t, T)$ denote the time-t price of a zero coupon bond with maturity $T \geq t$. Under the SVSI model (actually under the CIR model), we have

$$D(t, T) = D\left(t, T; r(t)\right) = \exp\left\{A(t, T) - B(t, T)r(t)\right\}, \tag{3.5}$$

where (see appendix A for proof)

$$B(t, T) = \frac{2(1 - e^{-h(T-t)})}{2h - (h - \kappa_r^*)(1 - e^{-h(T-t)})}, \quad h = \sqrt{\kappa_r^{*2} + 2\sigma_r^2}, \tag{3.6}$$

$$A(t, T) = -\frac{\theta_r^*}{\sigma_r^2}\left\{(h - \kappa_r^*)(T - t) + 2\ln\left[1 - \frac{(h - \kappa_r^*)(1 - e^{-h(T-t)})}{2h}\right]\right\}. \tag{3.7}$$

There is a semi-analytic formula for the time-t price C of a European call with strike K

and maturity T. It is of the form

$$C = S(t)\Pi_1(t,T,S(t),r(t),v(t)) - KD(t,T)\Pi_2(t,T,S(t),r(t),v(t)), \tag{3.8}$$

$$\Pi_j(t,T,S(t),r(t),v(t)) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-\varpi u \ln K} f_j(t,T,S(t),r(t),v(t),u)}{iu}\right] du \tag{3.9}$$

where f_j $(j = 1,2)$ is the characteristic functions for Π_j (hence the second formula is just a Fourier inversion formula) and $\varpi = \sqrt{-1}$, as we have mentioned at the end of Chapter 1. The reader may refer to formulas (A10) and (A11) of Bakshi *et al.* (1997) for details (but replace the term $-\lambda i\phi\mu_J\tau$ by $-\lambda i\phi(\mu_J + q)\tau$ in both formulae). The idea of using characteristic functions and Fourier transforms to find European option prices is due to Heston (1993) and it has become a standard technique today. The derivations of f_j are very similar in different extensions of the Heston model. For instances, see Heston (1993), Scott (1997), Bates (1996) and Bakshi *et al.* (1997).

3.2.2 The BSHW model

The BSHW model specifies that in the risk-neutral world, $S(t)$ and $r(t)$ are governed by the following SDEs:

$$\frac{dS(t)}{S(t)} = (r(t) - q(t))\,dt + \sigma_S(t)\ dW_S^*(t), \tag{3.10}$$

$$dr(t) = (\theta_r^*(t) - \kappa_r^*\ r(t))dt + \sigma_r\ dW_r^*(t) \tag{3.11}$$

Here $q(t), \sigma_S(t)$ and $\theta_r^*(t)$ are functions of time, σ_r is a constant and $W_S^*(t), W_r^*(t)$ are two standard Brownian motions that have constant correlation: $dW_S^*(t)\ dW_r^*(t) = \rho dt$. This model has been considered by Lin and Tan (2003) with constant θ_r^*. One can see that when $r(t)$ is deterministic, the stock price dynamics (3.10) is just the classical Black-Scholes (BS) model. The interest rate dynamics (3.11), which was proposed by Hull and White (1990), is commonly referred to as the *Hull-White model* or the *extended Vasicek model* (it reduces to the model of Vasicek 1977 when θ_r^* is constant, hence the latter name).

Let $f(t,T)$ denotes the forward rate that one can contract for at time t to invest in the money market at time T, i.e.

$$f(t,T) = -\frac{\partial}{\partial T} \log D(t,T).$$

Under the BSHW model, the price of a bond is still given by the formula (3.5), but with $A(t,T)$ and $B(t,T)$ given by (again, see appendix A for proof)

$$\begin{aligned} B(t,T) &= \frac{1}{\kappa_r^*}(1 - e^{-\kappa_r^*(T-t)}), & (3.12) \\ A(t,T) &= B(t,T)m(t) - \int_t^T f(0,u)du - \frac{\sigma_r^2}{2}\int_t^T B(0,u)^2 du \\ &\quad + \frac{\sigma_r^2}{2}\int_t^T B(u,T)^2 du, & (3.13) \\ m(t) &= f(0,t) + \frac{\sigma_r^2}{2}B(0,t)^2. & (3.14) \end{aligned}$$

Furthermore, if $f(0,t)$ is differentiable w.r.t. t, the function $\theta_r^*(t)$ is related to $m(t)$ by

$$\theta_r^*(t) = m'(t) + \kappa_r^* m(t). \tag{3.15}$$

The significance of equations (3.14) and (3.15) is that the Hull-White model is a *market model* (it predates modern market models like the LIBOR market model and the swap market model). That is, we can always choose $\theta_r^*(t)$ such that the theoretical forward curve $f(0,t)$ matches the one observed in the market, provided the latter satisfies a certain regularity conditions (continuously differentiable in our case). Conversely, specifying $\theta_r^*(t)$ is equivalent to specifying the theoretical initial forward curve. For example, in the original Vasicek model, θ_r^* is taken to be $\kappa_r^*\theta$ for some constant θ. By equations (3.14) and (3.15), this means we *model* the theoretical forward curve as

$$f(0,t) = r(0)e^{-\kappa_r^* t} + \theta B(0,t) - \frac{\sigma_r^2}{2\kappa_r^{*2}} B(0,t)^2.$$

In our numerical experiments, we will consider the Hull-White model instead of the Vasicek model and we shall fit $f(0,t)$ to the market's forward curves.

Under the BSHW model, the variance of $r(t)$ is identical under the physical measure and the $\mathbb{Q}$-measure. It is given by

$$\mathrm{var}^{\mathbb{Q}}(r(t)) = \frac{\sigma_r^2}{2\kappa_r^*}(1 - e^{-2\kappa_r^* t}). \tag{3.16}$$

In this chapter, however, we will mostly work under another pricing measure, namely the T-forward measure $\mathbb{Q}^T$, in order to ease computations. A T-forward measure is the measure under which the price process of every tradable financial instrument, when denominated by the price $D(0,T)$ of the discount bond with maturity T (T is fixed), is a martingale. It will be shown in the appendix that for any $s \le t \le T$, the conditional distribution of $(r(t), \ln S(t))$ given $(r(s), \ln S(s))$ is bivariate normal under $\mathbb{Q}^T$:

$$\begin{aligned} r(t) - m(t) &= e^{-\kappa_r^*(t-s)}(r(s) - m(s)) - \frac{\sigma_r^2}{\kappa_r^*}B(s,t) + \frac{\sigma_r^2}{2\kappa_r^*}B(2s,2t)e^{-\kappa_r^*(T-t)} \\ &\quad + Y(s,t), \qquad (3.17) \\ \frac{S(t)}{S(s)} &= \frac{D(t,T;r(t))}{D(s,T,r(s))}\exp\left[-\int_s^t q(u)du - \frac{1}{2}\text{var}(X(s,t)) + X(s,t)\right] \qquad (3.18) \end{aligned}$$

where $(X(s,t), Y(s,t))$ is a bivariate normal random variable such that $E^{\mathbb{Q}^T}(X(s,t)) = E^{\mathbb{Q}^T}(Y(s,t)) = 0$,

$$\begin{aligned} \text{var}(X(s,t)) &= \int_s^t \left|\sigma_S(u)(\rho, \sqrt{1-\rho^2}) + (\sigma_r B(u,T),\ 0)\right|^2 du, \qquad (3.19) \\ \text{var}(Y(s,t)) &= \frac{\sigma_r^2}{2}B(s,t), \qquad (3.20) \\ \text{cov}(X(s,t), Y(s,t)) &= \rho\sigma_r\sigma_S B(s,t) + \frac{\sigma_r^2}{\kappa_r^*}B(s,t) - \frac{\sigma_r^2}{2\kappa_r^*}B(2s,2t)e^{-\kappa_r^*(T-t)} \qquad (3.21) \end{aligned}$$

and $(X(s_1,t_1), Y(s_1,t_1))$ and $(X(s_2,t_2), Y(s_2,t_2))$ are independent for any two nonoverlapping intervals $(s_1,t_1), (s_2,t_2) \subset [0,T]$.

Let $C(S(t), K, t, T, D(t,T), \sigma^2)$ denotes the time-t price of a European call option with maturity T and strike price K according to the Black-Scholes formula, i.e.

$$\begin{aligned} C(S(t), K, t, T, D(t,T), \sigma^2) &= S(t)e^{-\int_t^T q(u)du}N(d_1) - KD(t,T)N(d_2), \qquad (3.22) \\ d_1 &= \frac{\ln\left(S(t)e^{-\int_t^T q(u)du}\right) - \ln(KD(t,T)) + \frac{1}{2}\sigma^2(T-t)}{\sqrt{\sigma^2(T-t)}}, \\ d_2 &= d_1 - \sqrt{\sigma^2(T-t)} \end{aligned}$$

where $N(z)$ is the standard normal distribution function. Under the BSHW model we can calculate the call option price as[1]

$$C\left(S(t), K, t, T, D(t,T), \frac{1}{T}\mathrm{var}(X(t,T))\right).$$

3.3 Pricing ratchets and cliquets

The price of a European option that pays off $H(\tau)$ at time τ is given by

$$E^{\mathbb{Q}}\left[e^{-\int_0^\tau r(u)du} H(\tau)\right]$$

if the $\mathbb{Q}$-measure is used, or

$$D(0,T)E^{\mathbb{Q}^T}\left[\frac{H(\tau)}{D(\tau,T)}\right]$$

when the T-forward measure ($T \geq \tau$) is used. Since a cliquet is a European option, its price can be evaluated using these two formulas. When we ignore surrenders and mortality risk, a ratchet is a European option that pays off $H(T)$ at time T. So the above pricing formulas also apply.

We can account for mortality risks in these formulas easily when the survival probabilities are deterministic. Roughly speaking, the standard approach is to assume that mortality risks and financial risks are independent under both the physical measure and the pricing measure (which is the $\mathbb{Q}$-measure or T-forward measure in our case). So the physical mortality law remains unchanged under the pricing measure and mortality risks can be diversified by selling enough policies. (See Lin and Tan (2003) for using the percentile premium principle to deal with the case where only a small number of policies are sold.) Let ${}_tp_x$ be the probability that a person at age x can survive more than t years, ${}_tq_x = 1 - {}_tp_x$ and $q_x = {}_1q_x$. Let also $w_t = {}_{t-1}p_x \; q_{x+t-1}$ for $t = 1, 2, \ldots, T-1$ (i.e. w_t is the probability that the person at age x at time 0 will die in year t), and $w_T = 1 - \sum_{t=1}^{T-1} w_t$. Then the pricing formula becomes

$$\sum_{t=1}^{T} w_t \; E^{\mathbb{Q}}\left[e^{-\int_0^t r(u)du} H(t)\right].$$

[1] When the BS model is mixed with other interest rate models than the Hull-White model, similar closed-form solutions exist. See, e.g. Merton (1973), Rabinovitch (1989), Hull (1993) or Musiela and Rutkowski (2005, p.399).

When the T-forward measure $\mathbb{Q}^T$ is used instead of the risk-neutral measure $\mathbb{Q}$, the formula becomes

$$\sum_{t=0}^{T} w_t \; D(0,T) E^{\mathbb{Q}^T}\left[\frac{H(t)}{D(t,T)}\right].$$

In either case, the value of a ratchet is the weighted sum of the values of a number of European options of different maturities.

Incorporating a *stochastic mortality law* into our pricing formulas, however, is a difficult issue. When the survival probabilities themselves are also stochastic, it is unclear how to hedge against the longevity risk. Therefore, in the sequel, we assume a deterministic mortality law. In addition, although it is easy to include surrender behaviour into our pricing formulas by replacing each European option component by its Bermudan (i.e. discretely monitored American) counterpart, in order to ease our programming and computational burden, we also assume that there are no surrenders.

In some simple cases, we can calculate the price of a ratchet efficiently. For instance, given the lifetime τ, the payoff of a PTP with term-end design is a constant plus the difference of the payoffs of two European calls:

$$\begin{aligned} & P \times \text{mid}\left[1+\alpha\left(\frac{S(\tau)}{S(0)}-1\right),\ 1+F_\tau,\ 1+C_\tau\right] \\ = \quad & P\left\{(1+F_\tau)+\max\left[\alpha\frac{S(\tau)}{S(0)}-(\alpha+F_\tau),\ 0\right]-\max\left[\alpha\frac{S(\tau)}{S(0)}-(\alpha+C_\tau),\ 0\right]\right\}. \end{aligned}$$

Consequently the price of a term-end point-to-point contract is just a linear combination of the prices of discount bonds and call spreads:

$$\text{PTP price} = P\sum_{t=1}^{T} w_t\left[(1+F_t)D(0,t)+C(\alpha,\alpha+F_t,0,t)-C(\alpha,\alpha+C_t,0,t)\right].$$

Since a European call admits the semi-analytic formula (3.8) under the SVSI model and a Black-Scholes formula (but with total variance of the form (3.19)) under the BSHW model, we can calculate the PTP price easily. It is also possible to compute the price of a ratchet efficiently when no simple pricing formula exists. For example, we can use the sequential quadrature method described in part II of this thesis to evaluate efficiently the price of a term-end CAR that has global floor and cap.

For more complicated ratchets, we rely on Monte Carlo simulation to obtain their prices. In our study, we follow Schoutens *et al.* (2003) and simulate the SVSI model dynamics (3.1)–(3.3) using Euler discretisation[2], except that we discretise each year into 252 instead of 250 days (the former is divisible by 12). For example, given $S(t)$ and $v(t)$, we can simulate $S(t+\Delta t)$ and $v(t+\Delta t)$ ($\Delta t = 1/252$) by putting

$$\begin{aligned}
\ln \frac{S(t+\Delta t)}{S(t)} &= \left(r(t) - q(t) - \frac{1}{2} v(t) \right) \Delta t + \sqrt{v(t)\Delta t} \; \left(\rho Z_1 + \sqrt{1-\rho^2} Z_2 \right), \\
v(t+\Delta t) &= \max \left[v(t) + (\theta_v^* - \kappa_v^* v(t))\Delta t + \sigma_v \sqrt{v(t)\Delta t} \; Z_1, \; 0 \right]
\end{aligned}$$

where Z_1, Z_2 are two independent standard normal random numbers. For the BSHW model, one can again simulate (3.10)–(3.11) using Euler discretisation, as Lin and Tan (2003) did. Alternatively one can simulate the price dynamics under the T-forward measure, where the formulas for exact simulation are given by (3.17)–(3.21). As I have very limited computational resources, I have chosen the exact simulation method[3].

3.4 Some qualitative analysis

Later on, we will carry out some numerical experiments to examine how large can the pricing errors due to interest rate risk be. In this section, we will justify the necessity of this kind of empirical research by illustrating the limitations in qualitative analysis.

Suppose the equity price $S(t)$ and short rate $r(t)$ are governed by the BSHW model, but we price a ratchet according to the following risk-neutral dynamics:

$$\begin{aligned}
\frac{dS(t)}{S(t)} &= (r(t) - q(t))dt + \sigma_S(t) \; dW_S^*(t), \\
r(t) &= f(0,t),
\end{aligned}$$

[2] Broadie and Kaya (2004) have devised an exact simulation method for the SVSI model, but we find it too difficult to implement correctly.

[3] Note that the stock price and the interest rate are simulated on a daily basis in Euler discretisation, as opposed to a monthly/yearly basis in exact simulation. So the exact simulation method is clearly more efficient. In theory, exact simulation is also more accurate, but our experience shows that when the BSHW model is discretised on a daily basis, there is no noticeable discretisation error. So the use of exact simulation here only gives us an advantage in speed but not in accuracy. If computation time is not a great concern, we advise the reader to follow Lin and Tan (2003) and use Euler discretisation because its implementation is less error prone.

i.e. we price the EIA as if the stock price follow the BS model with the deterministic interest rate term given by the initial forward curve. How will this mistake affect the ratchet's price?

If the option in question is a vanilla European call, its price under the BSHW model is given by Black-Scholes formula (3.22), with $\sigma^2 T = \mathrm{var}(X(0,T))$. Under the BS model, the same formula applies, but $\sigma^2 T = \int_0^T \sigma_S^2(u)du$. So, if one mistakes the BS model as the true economic model, then the option will be underpriced if and only if $\mathrm{var}(X(0,T))$ is smaller than $\sigma_S^2 T$. It is quite easy to analyse the behaviour of $\mathrm{var}(X(0,T))$. In fact, from (3.19) we see that $\mathrm{var}(X(0,T))$ is monotonic increasing in κ_r^* and ρ and also quadratic in σ_r. This is well documented in the literature (see, e.g. Hull 1993 or Rabinovitch 1989), but even so, the behaviour of the pricing error is complicated enough to render some analysis impractical.

For example, if we believe that the option is currently underpriced and we anticipate a decrease in, say, σ_r, then we can be assured that the option will remain underpriced (because $\mathrm{var}(X(0,T)) - \sigma_S^2 T$ is convex and $\sigma_r = 0$ is a root), but unless we know the true model and the exact values of the model parameters, we could never know how the size of the pricing error will change. Similarly, consider a term-end PTP. Recall that this contract can be replicated as a portfolio of discount bonds and call spreads. Unless the annuitant is very old at inception of the contract, the probability that he/she can survive at least up to the PTP's expiration day is very high. So pricing error in the PTP is mainly contributed by the pricing error in the T-year call spread. Now, the price of this call spread is given by

$$CS(\alpha, F_T, C_T, T, \sigma^2) := C(\alpha, \alpha + F_T, 0, T, \sigma^2) - C(\alpha, \alpha + C_T, 0, T, \sigma^2),$$

where $\sigma^2 T = \mathrm{var}(X(0,T))$ under the BSHW model and $\sigma^2 T = \int_0^T \sigma_S^2(u)du$ under the BS model. When $F_T = C_T$, this call spread has zero pricing error, regardless of the model parameters. When C_T increases slightly, pricing error due to model risk arises. So the size of the pricing error should initially increase with C_T, but it may not remain monotonic when C_T continues to grow (see fig. 3.1 for example). This causes difficulties in product design, because there is no safe way to choose a floor rate or a cap rate so that the model risk is guaranteed to be small.

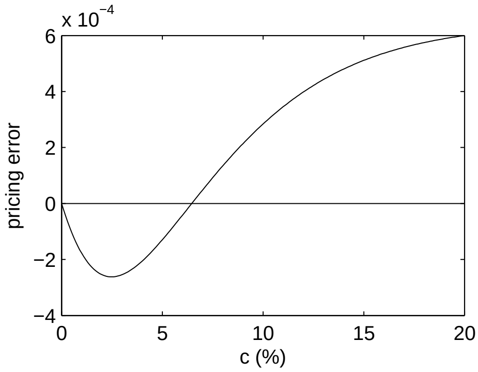

Figure 3.1. Plain pricing error in a call spread. Here the BSHW model is the true model and the BS model is the wrong one. The contract parameters are $T = 7,\ \alpha = 0.2,\ F_T = 0,\ C_T = (1+c)^T - 1$, where c ranges from 0 to 20%. The model parameters are $q(t) \equiv 0,\ \sigma_S(t) \equiv 0.3,\ \rho = 0,\ \sigma_r = 0.02,\ \kappa_r^* = 0.2,\ D(0,T) = 1.05^{-T}$.

Analysing the effects of σ_r, ρ and κ_r^* is even harder. The problem is that the price of the call spread is not necessarily a monotonic function of σ^2. This can be seen from the fact that the vega of the call spread,

$$\frac{d\ CS(\alpha, F_T, C_T, T; \sigma^2)}{d\sigma} = \phi\left(\frac{\log \dfrac{\alpha e^{-\int_0^T q(u)du}}{(\alpha + F_T)D(0,T)} + \dfrac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - \phi\left(\frac{\log \dfrac{\alpha e^{-\int_0^T q(u)du}}{(\alpha + C_T)D(0,T)} + \dfrac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right),$$

(ϕ is the standard normal density function), can be zero for some positive σ. Consequently, when σ increases, the value of CS (as a function of σ^2) may first increases and then decreases, or equivalently, the pricing difference in the call spread may first be negative, then becomes positive and then becomes negative again. Since $\text{var}(X(0,T))$ is monotonic in ρ, κ_r^* and quadratic in σ_r, this in turn means the pricing error can change sign more than once when ρ or κ_r^* increases or even change sign up to four times when σ_r increases. For PTPs of other designs or for CARs, the situation may be even more complex.

Despite it is difficult to analyse the behaviours of pricing errors, in view of equations (3.17)–(3.21), it is still reasonable to say that in these examples, the pricing errors in ratchets are related to $\text{var}(Y(s,t))$ and the discrepancy between $\text{var}(X(s,t))$ and $\int_s^t \sigma_S^2(u)du$ (where the length of the period $[s,t]$ may span from a month to several years, depending on the ratchet type). When these terms are small, we can still hope that the pricing errors are small. However, as qualitative analysis is impossible, we must rely on numerical experiments to study the impacts of stochastic interest rates.

3.5 Numerical experiments

Choices of parameters

Owing to the problems in data solicitation we mentioned in chapter 2, we will use parameters from Bakshi *et al.* (1997) for the SV and SVSI models. Although Bakshi *et al.* . (1997) have not estimated these parameters first under the statistical measure, they have calibrated the two models using a large set of time series option price data. For the BS and BSHW models, we will use synthetic parameters. Since the parameters for our four models come from two different sources, we will consider the two pairs of models separately. So, we will compute the pricing errors for EIAs between the SV model and the SVSI model and between the BS model and the BSHW model, but not pricing errors between, e.g., the BS model and the SVSI model.

The followings are the model parameters for the SV and SVSI models. The parameters in (a) and (b) below are taken from Bakshi *et al.* (1997), in which the SV and SVSI models are calibrated using U.S. Treasury bills and S&P 500 option prices sampled from mid-1988 to mid-1991:

(a) SV: $\kappa_v^* = 1.15,\ \theta_v^* = 0.04,\ \sigma_v^* = 0.39,\ \rho = -0.64,\ \sqrt{v(0)} = 0.1866.$

(b) SVSI: $\kappa_v^* = 0.98,\ \theta_v^* = 0.04,\ \sigma_v^* = 0.42,\ \rho = -0.76,\ \sqrt{v(0)} = 0.1865,\ \kappa_r^* = 0.58,\ \theta_r^* = 0.02,\ \sigma_r^* = 0.03.$

(c) $r(0) \in \{0.01, 0.02, \ldots, 0.08\}$. In the SV model, the interest rate term structure is fixed and is determined by the forward curve observed at time 0, i.e. we set $r(t) = f(0,t) = -\frac{\partial}{\partial t}D(0,t)$ (or equivalently $\exp(-\int_0^t r(u)du) = D(0,t)$), where $D(0,t)$ is calculated under the SVSI model.

(d) The dividend yield $q(t)$ is taken to be 3.5%, which is about the average dividend yield of the S&P 500 index over the sample period.

Note that by the formula (3.4), the long-term risk-neutral standard deviation of the short rate is given by

$$\sigma_\infty := \sqrt{\lim_{t\to\infty} \operatorname{var}^{\mathbb{Q}}(r(t))} = \sqrt{\frac{\theta_r^* \sigma_r^2}{2\kappa_r^*}} \approx 5.1 \times 10^{-3}.$$

So in the long run, the short rate is very involatile — its standard deviation is only about 51 basis points.

For the BSHW model, we consider the following combinations of parameters:

(e) $\sigma_S = 0.3$,

(f) $\sigma_D = \frac{\sigma_r}{\kappa_r^*}(1 - e^{-\kappa_r^*}) = 0.01, 0.02, 0.03, 0.04$,

(g) $\kappa_r^* = 0.9376, 0.6936, 0.3747, 0.1140$,

(h) $\rho \in \{-0.3, -0.15, 0, 0.15, 0.3\}$,

(i) Initial interest rate term structure (required in the computations of (3.13) and hence (3.18)): inferred from the 60 U.S. Treasury yield curves sampled on every first non-holiday of every January, April, July and December, from 1991-2005.

(j) $q(t) = 3.5\%$ per annum.

Note that we do not control σ_r directly, because it does not make any sense to say that a certain value of σ_r is reasonable without taking the value of κ_r^* into account. What we actually control are σ_D and σ_∞, where the formula for σ_D is given in (f) above and

$$\sigma_\infty := \sqrt{\lim_{t\to\infty} \operatorname{var}(r(t))} = \sqrt{\lim_{t\to\infty} \operatorname{var}^{\mathbb{Q}}(r(t))} = \frac{\sigma_r}{\sqrt{2\kappa_r^*}}$$

according to formula (3.16). Actually σ_D is the volatility coefficient of the one-year discount bond price (see appendix for proof). According to Hull (1993, p.436), a high estimate for σ_D is about 0.02. So the highest value 0.04 of σ_D we specify in the above should be high enough. Now, if we interpret σ_∞ as an unconditional standard deviation, an 8%

annualised standard deviation of $r(t)$ should be large enough. So we set the maximum of σ_∞ to be about 0.08. However, given σ_D, the value of σ_∞ cannot be too low, because $\frac{\sigma_\infty}{\sigma_D} = \sqrt{\frac{\kappa_r^*}{2} \frac{1}{1-e^{-\kappa_r^*}}}$ is mathematically bounded below by some constant $\omega_0 \approx 1.108$. Therefore we set $\sigma_\infty = \omega_0 \sigma_D \left\{ \frac{65}{64}, \frac{17}{16}, \frac{5}{4}, 2 \right\}$. The values of (κ_r^*, σ_r) can now be inferred, and they are plotted in fig. 3.2. The four ratios $\frac{65}{64}, \frac{17}{16}, \frac{5}{4}, 2$ of $\frac{\sigma_\infty}{\omega_0 \sigma_D}$ are chosen in a way that the combinations of (κ_r^*, σ_r) are visually more or less evenly spaced.

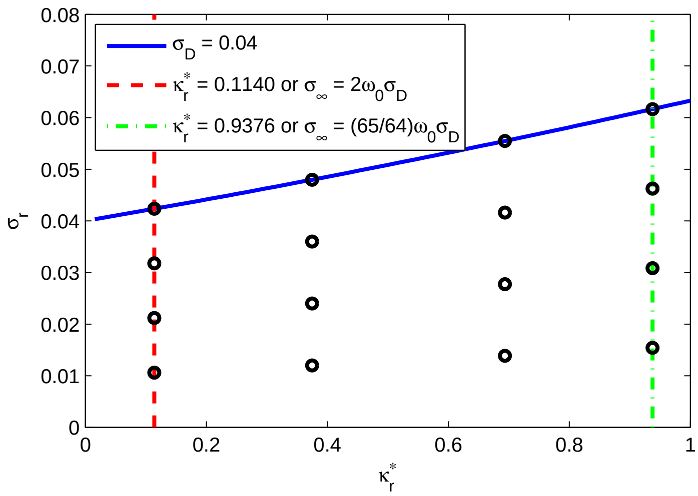

Figure 3.2. Sample values (circled spots) of σ_r and κ_r^* we use in the BSHW model.

We will compute the prices of several CARs, PTPs and cliquets in our experiments. For the former two types of contracts, we consider the following parameters:

- $T = 7, P = 1$ and $f = 0.03$,
- $c \in \{0.15, 0.2, 0.25, 0.3, \infty\}$ when SV or SVSI models are concerned, and $c \in \{0.2, 0.25, 0.3, \infty\}$ when we use the BS or BSHW models,
- $1 + F_\tau = (1 + f)^\tau$ and $1 + C_\tau = (1 + c)^\tau$ (so that our PTPs and CARs have identical minimum or maximum possible payoffs).

We also assume that the annuitant is of age 58 and his/her survival probabilities are taken from the 1980-94 U.S. Life Table.

We take the cliquet parameters from Schoutens *et al.* (2003): $T = 3$, $N = 6$, $t_i = \frac{iT}{N}$, $f = -0.03$, $c = 0.05$, $F = -0.05$, $C = \infty$. Since both f and F are negative, the cliquet price can be negative.

3.5.1 Experimental procedure

We assume that, for all EIAs considered in our experiments, no surrender is allowed. We will compute, under the SV and SVSI models, the prices of PTPs with term-end, Asian-end and high-watermark designs and also the prices of CARs with term-end and Asian-end designs. For the BS and BSHW models, since the U.S. Treasury website does not provide one-month yield values before mid-2001, we calculate the prices of PTPs and CARs of the term-end design only. Note that, since a term-end PTP is just a portfolio of bonds and calls, if the BS model is calibrated using vanilla option prices, there will be no pricing errors. Therefore, in calculating PTP prices, the BS model is not calibrated. For CARs, we calculate the BS prices both with and without model calibration. We remark that when the BS model is calibrated, the stock price volatility is no longer a constant but a function of time $\sigma_S(t)$. We can set it to the step function that satisfies

$$\sigma_S^2(t) = \mathrm{var}(X(0,t)) - \mathrm{var}(X(0,t-1))$$

and $\sigma(u) = \sigma(t)$ for all $u \in (t-1, t]$ and $t = 1, 2, \ldots, T$.

For each type of ratchet and each combination of contract and model parameters, we first find the critical participation rate α under the 'wrong' model (i.e. the SV or BS model). This can be evaluated by using the bisection method or the secant method. Given α, the ratchet price is calculated again under the price model with the 'true' model (the SVSI or BSHW model). The percentage pricing error (i.e. relative pricing error multiplied by 100%) is then measured by

$$\left(\frac{\text{SV price}}{\text{SVSI price}} - 1\right) \times 100\% \quad \text{or} \quad \left(\frac{\text{BS price}}{\text{BSHW price}} - 1\right) \times 100\%,$$

depending on the asset price models. This error is negative when the ratchet is underpriced under the wrong model and positive otherwise.

After computing the pricing errors for our EIAs, we will also compute the prices of our cliquets for the two pairs of models. We are interested to know if the striking results of Schoutens (2003) also occur in our experiment, although folklore assumption says that they should not.

3.5.2 EIA prices: SV model vs SVSI model

The pricing biases of the SV model for all five different types of ratchets are listed in table 3.3 and a few summary statistics are given in table 3.4. Except for term-end PTPs, the prices of all contracts here as well as the sensitivities of the SVSI prices to the critical α are calculated using simulation. The maximum standard errors of the simulated prices and the sensitivities are shown in the first and the sixth rows of table 3.4. They are so small that the calculated prices and sensitivities should be precise enough. In table 3.3, the pricing errors in the term-end PTPs (panel A) are nonzero because instead of calibrating the SV model to the SVSI prices, Bakshi *et al.* (1997) calibrate both models using market prices of vanilla options.

Over the range of parameters and yield curves we consider, the smallest pricing error is about -0.4% (first column in panel D, table 3.3). Even the largest pricing bias is fairly small (-1.26%, with $c = \infty$ and $r = 8\%$ in panel C). However, all pricing errors are negative, meaning that in the scenarios we consider, one is expected to lose money on selling all five kinds of ratchets.

By comparing the results in table 3.3, we see that the pricing errors are the largest in high-watermark ratchets, smaller in term-end ones and the smallest in ratchets with Asian-end design. This is consistent with the usual perceived sizes of fluctuation of the payoffs of these three types of instruments. However, although the paths of $(H(1), H(2), \ldots, H(7))$ are less fluctuated for CARs than for PTPs (because of the presence of local caps and floors in CARs), the pricing errors in Asian-end CARs are not necessarily smaller in size than those in Asian-end PTPs. See, e.g. the results in panels B and E when $c = 15\%$ and $r = 7\%$

or 8%. Similarly, although the paths of $(H(1), H(2), \ldots, H(7))$ are less fluctuated when the cap rate is small than when it is large, in panels A, B, D and E we see that the sizes of the pricing errors are actually smaller when c is larger. The reason was explained in section 4.

c	r							
	1%	2%	3%	4%	5%	6%	7%	8%
				(A) Term-end PTPs				
15%	−0.63%	−0.67%	−0.71%	−0.74%	−0.77%	−0.80%	−0.83%	−0.85%
20%	−0.63%	−0.66%	−0.69%	−0.72%	−0.74%	−0.76%	−0.77%	−0.79%
25%	−0.62%	−0.66%	−0.69%	−0.71%	−0.74%	−0.75%	−0.77%	−0.78%
30%	−0.62%	−0.66%	−0.69%	−0.71%	−0.73%	−0.75%	−0.77%	−0.78%
∞	−0.62%	−0.66%	−0.69%	−0.71%	−0.73%	−0.75%	−0.77%	−0.78%
				(B) Asian-end PTPs				
15%	−0.62%	−0.65%	−0.68%	−0.71%	−0.73%	−0.76%	−0.77%	−0.80%
20%	−0.60%	−0.63%	−0.66%	−0.69%	−0.71%	−0.73%	−0.75%	−0.76%
25%	−0.60%	−0.63%	−0.65%	−0.68%	−0.70%	−0.71%	−0.72%	−0.73%
30%	−0.59%	−0.62%	−0.65%	−0.68%	−0.71%	−0.72%	−0.74%	−0.74%
∞	−0.55%	−0.58%	−0.61%	−0.62%	−0.64%	−0.65%	−0.66%	−0.66%
				(C) High-watermark PTPs				
15%	−0.69%	−0.72%	−0.74%	−0.77%	−0.79%	−0.81%	−0.83%	−0.84%
20%	−0.73%	−0.77%	−0.81%	−0.85%	−0.87%	−0.90%	−0.92%	−0.94%
25%	−0.76%	−0.79%	−0.83%	−0.86%	−0.90%	−0.93%	−0.95%	−0.98%
30%	−0.78%	−0.83%	−0.87%	−0.90%	−0.93%	−0.97%	−0.99%	−1.01%
∞	−0.90%	−0.96%	−1.02%	−1.07%	−1.12%	−1.17%	−1.22%	−1.26%
				(D) Term-end CARs				
15%	−0.43%	−0.48%	−0.52%	−0.58%	−0.63%	−0.69%	−0.75%	−0.82%
20%	−0.42%	−0.45%	−0.48%	−0.51%	−0.54%	−0.58%	−0.62%	−0.66%
25%	−0.40%	−0.43%	−0.45%	−0.48%	−0.51%	−0.55%	−0.58%	−0.60%
30%	−0.40%	−0.42%	−0.44%	−0.46%	−0.48%	−0.50%	−0.53%	−0.55%
∞	−0.40%	−0.43%	−0.45%	−0.47%	−0.50%	−0.52%	−0.54%	−0.56%
				(E) Asian-end CARs				
15%	−0.50%	−0.53%	−0.56%	−0.60%	−0.65%	−0.71%	−0.78%	−0.86%
20%	−0.50%	−0.53%	−0.56%	−0.59%	−0.62%	−0.65%	−0.69%	−0.71%
25%	−0.53%	−0.56%	−0.58%	−0.60%	−0.62%	−0.65%	−0.68%	−0.70%
30%	−0.54%	−0.57%	−0.59%	−0.62%	−0.64%	−0.67%	−0.69%	−0.70%
∞	−0.45%	−0.47%	−0.48%	−0.50%	−0.52%	−0.53%	−0.55%	−0.55%

Table 3.3. Percentage pricing errors in the ratchets when the SVSI model is the true model and the SV model is the wrong one.

In practice, to protect the insurance company from model risks, one can specify in the contract a lower participation rate than the critical one. For any fixed participation rate α_0, let SV(α_0) denotes the SV price of the ratchet (i.e. the price as perceived by the insurance company) and SVSI(α_0) the SVSI price (the true price). Suppose α is the perceived critical participation rate according to the SV model and $(1-\Delta)\alpha$ is the participation rate specified in the contract ($\Delta \geq 0$). Then the policyholder would pay \$1 to the insurance company and the latter is expected to pay off SVSI$((1-\Delta)\alpha)$ to the former when the contract expires (despite the insurance company thinks that it is liable for an amount of SV$((1-\Delta)\alpha)$.

Therefore the pricing error is given by

$$1 - \text{SVSI}\left((1-\Delta)\alpha\right) \;\approx\; 1 - \text{SVSI}(\alpha) + \alpha\Delta \times \frac{d}{d\alpha}\text{SVSI}(\alpha).$$

If we want the pricing error to be nonnegative, then at least we should set

$$\Delta = \max\left\{0,\; \frac{\text{SVSI}(\alpha) - 1}{\alpha \times \frac{d}{d\alpha}\text{SVSI}(\alpha)}\right\}.$$

In table 3.4, the margin of safety is estimated by $\max\{\Delta\}$, where the maximum is taken over all combinations of parameters. In all the scenarios we consider, we can get rid of the underpricing errors by deducting 0.0567α from the perceived critical participation rate α.

	PTP			CAR	
	Term-end	Asian-end	High-watermark	Term-end	Asian-end
max. s.e. of prices	-	0.00020	0.00025	0.00023	0.00026
min. critical α	70.13%	73.97%	33.32%	35.93%	61.22%
max. critical α	84.72%	90.08%	46.43%	96.91%	131.03%
min. sensit.	0.2567	0.2384	0.4905	0.1502	0.1231
max. sensit.	0.3238	0.3107	0.6440	0.6774	0.3862
safety margin Δ	0.0354	0.0341	0.0494	0.0567	0.0536

Table 3.4. SV model vs SVSI model: a few summary statistics.

3.5.3 EIA prices: BS model vs BSHW model

The percentile plots of the pricing errors are given in figure 3.5 and some statistics are listed in table 3.6. Unlike those in last subsection, the pricing errors here are not always negative. In particular, when the BS model is calibrated, about 60% of the results are positive.

When the BS model is not calibrated, fig. 3.5 reveals that PTPs tend to have larger underpricing errors but smaller overpricing errors than CARs do. Once again, this shows that we cannot judge the size of pricing error based on the size of room that the payoffs $H(1), H(2), \ldots, H(7)$ can vary within. In addition, while we are tempted to think that a calibrated model should give smaller-sized pricing errors, the opposite is observed in fig. 3.5, regardless of whether underpricing or overpricing is concerned.

All these observations show how tricky it is to predict the signs and sizes of pricing biases. Of course, if the pricing errors are always small, our inability of predicting their

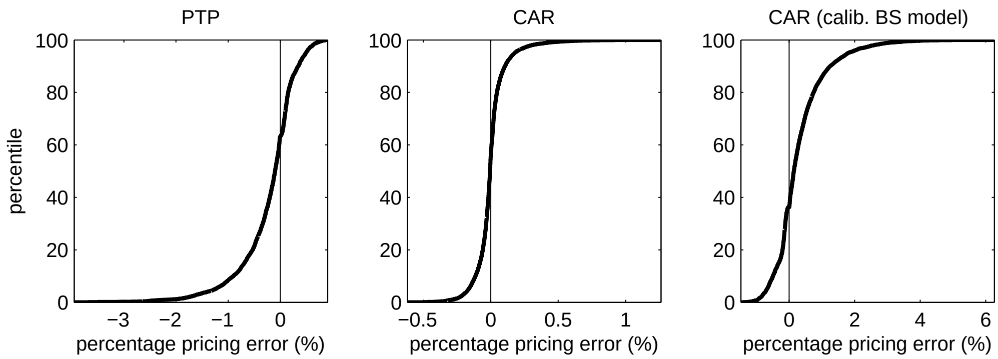

Figure 3.5. Plot of the percentage pricing errors in the ratchets.

	(A) non-restricted			(B) restricted		
	min.	median	max.	min.	median	max.
			PTP			
pricing error	−3.95%	−0.10%	0.91%	−1.08%	−0.05%	0.67%
critical α	12.17%	83.75%	151.29%	25.26%	77.18%	99.90%
sensitivity	0.0861	0.2253	0.3548	0.1605	0.2267	0.3121
			CAR			
pricing error	−0.62%	0.00%	1.26%	−0.27%	0.00%	0.27%
critical α	10.13%	32.55%	74.61%	20.03%	35.45%	74.61%
sensitivity	0.1979	0.6642	0.9208	0.2069	0.6757	0.9098
			CAR (calib. BS model)			
pricing error	−1.47%	0.15%	6.27%	−1.10%	0.06%	1.70%
critical α	8.30%	32.08%	75.68%	20.00%	35.40%	75.42%
sensitivity	0.2087	0.6637	0.9161	0.2096	0.6763	0.9095

Table 3.6. Statistics for the percentage pricing errors, critical participation rates and price sensitivities to participation rate evaluated at critical α. The numbers on panel B are calculated based on the restricted scenarios that satisfy $\sigma_D \leq 0.02$ and $0.2 \leq \alpha \leq 1$.

sizes or signs is less of a problem. However, as shown in figure 3.5 and table 3.6, the largest-sized underpricing errors in PTPs is −3.95%, which is not small. In terms of margin of safety, table 3.7 shows that the margin needed for a PTP needs is 0.2159α, which is much larger than the margin for CARs and also those observed in the SV vs SVSI case. However, if we restrict our attention to some less extreme scenarios where $\sigma_D \leq 0.02$ and neglect the cases $\alpha \notin [0.2, 1]$ (where α is too high or too low), the largest-sized underpricing error and the margin of safety become −1.08% and 0.0609α respectively, which are quite small.

	(A) non-restricted	(B) restricted
PTP	0.2159	0.0609
CAR	0.0228	0.0101
CAR (calib. BS model)	0.0414	0.0303

Table 3.7. Estimated safety margins when the BS model is used in a BSHW world. See the description of table 3.6 for the meaning of a restricted scenario.

For CARs, it is encouraging to see that both the underpricing errors and the margins of safety are still very small. However, it should be noted that when the BS model is calibrated, the maximum overpricing error is quite large (6.27%).

In fig. 3.8–3.11 we show the effects of changing parameter values. In some sense, these effects are not examined *ceteris paribus* here, because the pricing error is a function of the perceived critical participation rate α, which in turn is a nonlinear and implicit function of all parameters. However, in another experiment (of which the results are not reported here) we found that if we fix $\alpha = 0.25, 0.5$ or 0.75 and calculated the *plain* pricing errors (BS price minus BSHW price) instead of relative errors, the percentile plots thus produced are qualitatively no different from those in fig. 3.8–3.11. So, let us pretend that α is constant and the error measure in question is the plain price difference.

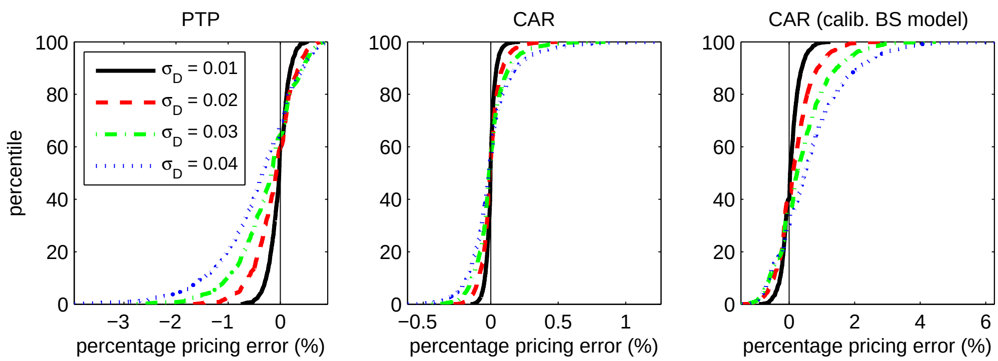

Figure 3.8. Percentile plot of the percentage pricing errors in the ratchets, according to different values of the 1-year discount bond volatility σ_D.

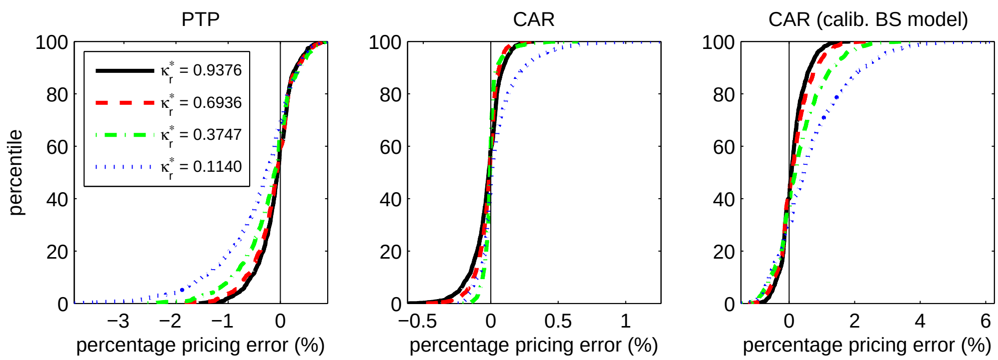

Figure 3.9. Percentile plot of the percentage pricing errors in the ratchets, according to different values of κ_r^*.

Our discussion at the end of section 4 shows that when the value of a parameter moves in one direction, the pricing error can change signs a few times. The behaviours of the pricing errors in fig. 3.8–3.11 are much nicer. In these figures, if we fix a percentile level

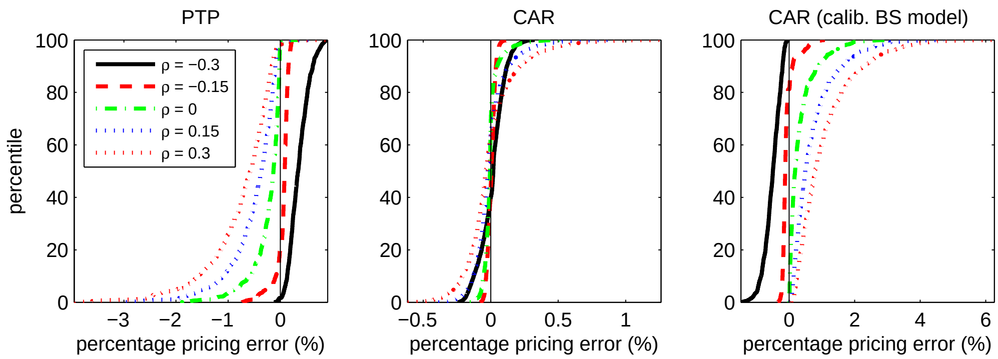

Figure 3.10. Percentile plot of the percentage pricing errors in the ratchets, according to different values of ρ.

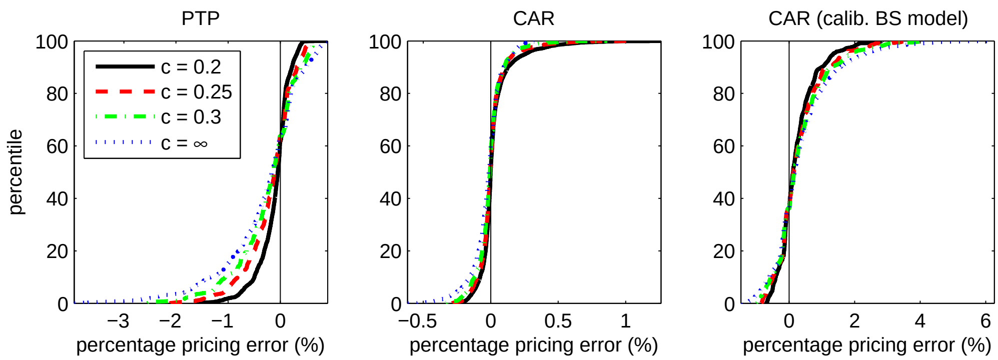

Figure 3.11. Percentile plot of the percentage pricing errors in the ratchets, according to different values of the local cap rate c.

and increase a parameter from a low value to a high value, the pricing error would either change monotonically or first move in one direction and then in another. Yet the patterns of change are inconsistent across different types of ratchets or different types of models. For instance, when the BS model is not calibrated, the movement of the pricing errors in CARs are very different from those in PTPs. Interestingly, when the pricing errors in CARs are calculated using a calibrated BS model, the effects of the parameters on the pricing errors in CARs are very similar to those on the *negative* of the pricing errors of PTPs (i.e. the percentile plots for PTPs and CARs look alike if the latter are rotated by 180°). Yet this is probably incidental.

3.5.4 Cliquet price differences

As we mentioned in the first chapter, Schoutens *et al.* (2003) show that even when a number of stock models are almost perfectly calibrated to vanilla call prices, the cliquet prices they give can differ by up to 56%. This striking result highlights the significance of volatility risk in the pricing of path dependent options. Here we repeat their experiment, but our purpose is to examine the relevance of interest rate risk. The results are given in table 3.12 and fig. 3.13. To much of our surprise, the largest-sized underpricing error is −87% in the SV vs SVSI case and the maximum overpricing error is 77% in the BS vs BSHW case. The latter number becomes 76% if we consider only the less extreme scenarios where $\sigma_D \leq 2\%$. These numbers are higher than the 56% reported by Schoutens *et al.*. Even though such direct comparison is not justified, our results clearly show that interest rate risks are hardly ignorable in cliquet pricing.

r_0	SVSI price	SV price	pricing error
1%	0.00304	0.00040	−87%
2%	0.00524	0.00247	−53%
3%	0.00751	0.00459	−39%
4%	0.00982	0.00676	−31%
5%	0.01217	0.00899	−26%
6%	0.01457	0.01127	−23%
7%	0.01703	0.01360	−20%
8%	0.01954	0.01599	−18%

Table 3.12. Percentage pricing errors in cliquets when the SVSI model is the true model and the SV model is the wrong one. The standard errors in the prices here range from 8.37×10^{-5} to 8.53×10^{-5}.

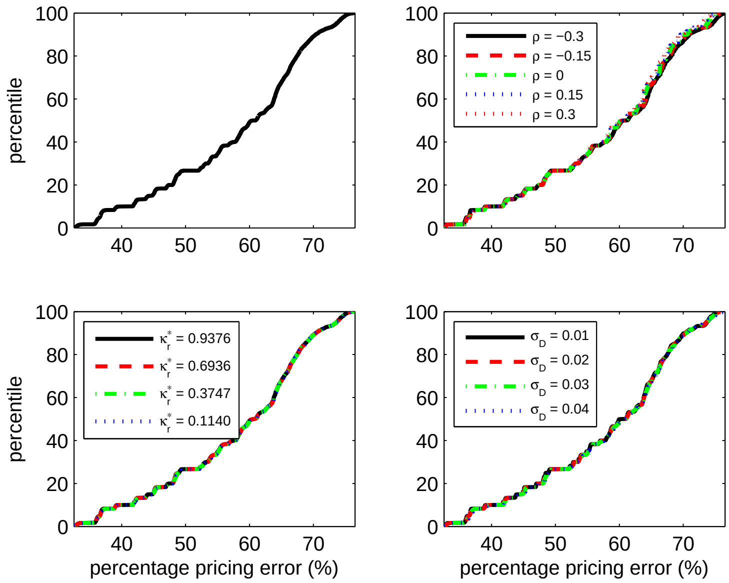

Figure 3.13. Percentile plot of the percentage pricing errors in cliquets (BS model vs BSHW model).

Fig. 3.13 also reveals a mysterious phenomenon. It seems that the pricing errors are insensitive to either of σ_D, κ_r^*, ρ or c. In other words, the only 'parameter' that is responsible for different pricing errors here is the initial yield curve. I am not sure if this is just a special instance or a general phenomenon.

Why are the pricing errors so large for cliquets (the largest-sized error being -87%) but not for CARs and PTPs (with errors up to 6.27%)? If we look at the plain price differences (table 3.14) instead of the percentage pricing errors, we find that the pricing errors in our EIAs and cliquets are actually of the same order of magnitude. So, cliquets apparently have very large *relative* pricing errors simply because the nominal principal P they seek to protect is not included in the premiums. We will go back to this point in next chapter.

	SV vs SVSI	BS vs BSHW
7-yr term-end CAR	$\$-0.0082$	\$0.059
3-yr cliquet	$\$-0.0035$	\$0.020

Table 3.14. The largest-sized plain price differences for term-end CARs and cliquets.

3.6 Conclusion

Is it necessary for risk managers of EIAs to model a stochastic interest rate environment? For term-end PTPs with no surrender, the answer in theory is negative because one can get rid of all of the interest rate risk by replicating such a PTP by a portfolio of discount bonds and call options. Certainly this is only theoretical because in practice holding long-term over-the-counter options to its expiry date may create liquidity and credit risk problems that are hard to manage, but the replication argument itself shows that the pricing errors in term-end PTPs should be small when the pricing models are calibrated. This is supported by the results in our comparison of the SV prices against the SVSI prices, where the 'wrong' model (the SV model) is calibrated, albeit non-perfectly. Indirect evidence can also be found in our comparison of the BS model against the BSHW model, where the BS model is not calibrated and the pricing errors and safety margins are found to be quite large. So, for term-end PTPs with no surrender, the key for managing model risk seems to be calibration but not stochastic interest rate modelling.

For the other three types of EIAs, it appears that the underpricing errors are fairly small and these errors can be easily absorbed by assigning a mild margin of safety. In particular, the pricing errors between the SV and SVSI models for HWM PTPs are surprisingly small (they can be underpriced by up to 1.26%), although they are — as expected — the largest among the pricing errors for all four types of EIAs we have considered. So, *solely based on our results* and purely from the pricing aspect, there seems to be no need to worry about the randomness of interest rates. However, in case the interest rate environment is very volatile or some less usual participation rates are considered, the overpricing errors for term-end CARs can be quite large (up to 6.27% in our experiments). While one will not lose money on selling an overpriced ratchet, such a large extent of overpricing may make the ratchet looks less competitive. More importantly, large pricing errors (regardless of their signs) can be detrimental to hedging, because the hedge positions may appear not self-financed.

For cliquets, we find that percentage pricing errors are huge and are much larger than those for ratchets. However, the plain pricing errors for both instruments are of the same order of magnitude. So, depending on one's error measure, the pricing errors in cliquets can be material or illusory. At any rate, our results suggest that the impact of interest rate risks on cliquet prices are not only significant, but also comparable to the impact of volatility risks. This should surprise many people, as the maturity of the cliquet we consider is just three years long and the stock returns in the payoff function have time spans of merely six months long.

Apart from the sizes of pricing errors, we also concern about how these errors change with the interest rate environment. In the second half of sec. 4, we argue that due to the nonlinear shape of the payoff function, it is very difficult to explain — even after the fact — the effects of changing different parameters on the pricing errors. Our experimental results show that pricing errors can behave very differently for different models or different instruments. In practice, the problem is further complicated by the presence of errors in model estimation and calibration. So, the authors believe that one should be very careful in accepting any presumption about the effects of changing interest rate environment on pricing errors.

Before ending this chapter, the reader should be cautioned that our results are merely preliminary, as we have examined only a limited number of asset price models under a limited number of scenarios. If more competing pricing models and more scenarios are included, one will likely see larger pricing errors than those we observe here, although the sizes of these errors may hopefully remain small. We hope we can do a more elaborate research in the future.

Chapter 4

Volatility risk for equity-indexed annuities

4.1 Overview

In this chapter we change our focus to volatility risk for EIAs. Although the term 'volatility risk' originates from the concern about the effect of the randomness of implied volatilities of plain-vanilla options on the pricing of exotic options, it should not be taken as the risk about the volatility of the underlying because such volatility is simply not defined under any model with random jumps in the equity price. It should be understood as the risk about the misspecification of the path structure of the discounted equity price.

The organisation of this chapter is similar to that in the former one. Essentially, we will first introduce the economic models we include in our experiments, then explain how to calibrate our models and how to set up our experiments and finally we will present and discuss our results. However, unlike the previous chapter, which is basically a piece of empirical research, there are a few mathematical contributions here. In particular, we will show that some existing 'risk-neutral' pricing models are not really risk-neutral and we will also explain how to obtain equivalent martingale measures for some Levy processes and a class of time-changed Levy processes called continuous-time regime switching (CTRS) models.

4.2 Economic models

The number of stock price models developed after the seminal work of Black and Scholes (1973) is so vast that even a casual overview on their development can take pages to complete. In this section we will only introduce a few of them. These selected models will be included in our experiments. While not all of them are genuine stochastic volatility models, they do reflect certain important aspects of a realistic stock price process.

Throughout, let us assume that the spot interest rate r is deterministic and our stock pays a constant and continuous dividend yield q. We shall denote the stock price by S and the log discounted total return of the stock price by X. By convention we assume that $X_0 = 0$ (we will use notations like X_t and $X(t)$ interchangeably throughout this chapter). So, we have

$$S_t = S_0 \exp\left[\int_0^t (r_s - q)ds + X_t\right].$$

In the sequel we shall focus on our models of X.

4.2.1 Black-Scholes model

In the classical Black-Scholes model, the stock price volatility σ is treated as a constant. However, owing to the presence of volatility smiles, if we calibrate this classical model to the market prices of vanilla options, we will get very poor results. This adverse effect can be somehow mitigated by considering σ as a deterministic function of time. So, in our experiments, the risk-neutral stochastic differential equation (SDE) for X is postulated as

$$dX_t = -\frac{1}{2}\sigma_t^2 dt + \sigma_t dW_t^{\mathbb{Q}},$$

where $W^{\mathbb{Q}}$ is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}$. With such extension, the increment $X_t - X_u$ for any $u < t$ is distributed under $\mathbb{Q}$ as a Normal distribution with mean $-\frac{1}{2}V(u,t)$ and variance $V(u,t)$, where

$$V(u,t) = \int_u^t \sigma_s^2 ds.$$

Simulation of this extended model is straightforward. For option pricing, to calculate the time-t price of a vanilla European option with expiry date T, one can simply apply the Black-Scholes formula, but replace the total variance term $(T-t)\sigma^2$ by $V(t,T)$.

In practice, σ_t is not known in advance, but obtained from model calibration. Given the market prices of a set of vanilla options with different maturities $T_1 < T_2 < \ldots < T_n$, we can find values of $V(0,T_1), \ldots, V(0,T_n)$ for which the errors between theoretical prices and market prices are the smallest with respect to some error metric. Any value of $V(0,t)$ for t between 0 and T_n can practically be obtained by function interpolation. For convenience, I use cubic spline interpolation in this chapter. For $t > T_n$, we put $\sigma_t^2 = \sigma_{T_n} = \frac{1}{T}V(0,T)$. So σ is a piecewise smooth curve and the stochastic integral of σ with respect to $W^{\mathbb{Q}}$ always exists. Note that although σ is nonconstant here, it is still a deterministic quantity. So this extended model cannot exhibit volatility smile across strikes.

4.2.2 Levy models

Another way to extend the classical Black-Scholes model is to use a non-Gaussian but stationary distribution for the increments of X. This can be achieved by using Levy processes. The flourishment of Levy processes (other than Brownian motions and Poisson processes) in option pricing theory can be attributed to Eberlein and Keller (1995), Barndorff-Nielsen (1995) and Madan *et al.* (1998), who independently developed various classes of Levy processes with their students and co-workers. Applications of Levy processes in fair valuation and hedging of insurance contracts have been studied by several researchers, e.g. Aase (2000), Ballotta (2005), Jaimungal and Young (2005) and Riesner (2006). Readers who are interested in the general theory of Levy processes can consult the two excellent treatises written by Bertoin (1996) and Sato (1999). Applebaum (2004) has also surveyed some important theoretical results and has included in his book a very well written overview on the subject as well as a section on mathematical finance. For financial applications, the monographs written by Schoutens (2003) and Cont and Tankov (2004) are very useful, and Kyprianou *et al.* (2005) contains many interesting results from recent research.

A d-dimensional stochastic process $X = (X_t, t \geq 0)$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *Levy process* if (a) $X_0 = 0$ almost surely (a.s.), (b) X has independent and stationary increments and (c) X is stochastically continuous, i.e. for any $\epsilon > 0$ and any $s \geq 0$,

$$\lim_{t \to s} \mathbb{P}(|X(t) - X(s)| > \epsilon) = 0.$$

Owing to property (b), if we model the log discounted total return as a non-Brownian Levy process, the model can produce volatility smiles across strikes (due to non-Gaussian increments) but cannot exhibit stochastic volatility (because of the stationarity of increments). Property (b) also makes the distribution of $X_t - X_s$ *infinitely divisible* for any $0 \leq s < t$. Recall that a random variable is called infinitely divisible if for any positive integer n, the distribution of is the sum of n i.i.d. random variables. For a Levy process increment, this is always satisfied as

$$X_t - X_s = (X_{s+n\Delta} - X_{s+(n-1)\Delta}) + (X_{s+(n-1)\Delta} - X_{s+(n-2)\Delta}) + \ldots + (X_{s+\Delta} - X_s) \quad (\Delta = (t-s)/n).$$

The famous Levy-Khintchine formula says that any probability measure μ on $\mathbb{R}^d$ is infinitely divisible if and only if there exists a vector $b \in \mathbb{R}^d$, a positive semidefinite symmetric matrix $A \in \mathbb{R}^{d \times d}$ and a Levy measure Π defined on $\mathbb{R}^d \setminus 0$ (i.e. a measure such that $\int_{\mathbb{R}^d \setminus 0} (1 \wedge ||x||^2) \Pi(dx)$ is finite) such that the characteristic function $\phi_\mu(u)$ of μ can be written as

$$\begin{aligned} \phi_\mu(u) &= \exp\{\zeta_\mu(u)\}, \\ \zeta_\mu(u) &= \varpi b^\top u - \frac{1}{2} u^\top A u + \int_{\mathbb{R}^d \setminus 0} \left(e^{\varpi u^\top x} - 1 - \varpi u^\top x \mathbb{1}\{||x|| < 1\} \right) \Pi(dx) \end{aligned}$$

In particular, since X_1 is infinitely divisible, for any rational number $t > 0$, the characteristic function of X_t is given by $\phi_{X_t}(u) = e^{t\zeta_{X_1}(u)}$ (we abuse the notations here so that we do not need to distinguish between X_t or X_1 from the probability measures they induce). Now condition (c) essentially guarantees that for any real (vector) u, the mapping $t \mapsto \phi_{X_t}(u)$ is continuous in t. So one can prove (see, e.g. Bingham *et al.* 1987, pp. 4-6) that for any $t \geq 0$,

$$\phi_{X_t}(u) = \phi^t(u) = e^{t\zeta(u)},$$

where $\phi(u)$ and $\zeta(u)$ are shorthands for $\phi_{X_1}(u)$ and $\zeta_{X_1}(u)$ respectively (we shall drop the subscript hereafter when there is no confusion). The function ζ is called the *Levy symbol* or *characteristic exponent* of X. The triple (b, A, Π) is usually called the *Levy triple* or *characteristic triple.* If X' is a Levy process with Levy triple (b', A', Π') defined on the same filtered probability space as X, then $X + X'$ is a Levy process with Levy triple $(b + b', A + A', \Pi + \Pi')$.

When Π is a discrete distribution, say $\Pi = \sum_{j=1}^{n} \lambda_j \delta_{x_j}$ where δ_x denotes a Dirac mass at x, we can rewrite the above characteristic function of X_t as

$$\begin{aligned}\phi_{X_t}(u) = \exp\left\{\varpi t u^\top b - \frac{t}{2} u^\top A u\right\} \times \prod_{||x_j||<1} \exp\left\{t\lambda_j\left(e^{\varpi u^\top x_j} - 1 - \varpi u^\top x_j\right)\right\} \\ \times \prod_{||x_j||\geq 1} \exp\left\{t\lambda_j\left(e^{\varpi u^\top x_j} - 1\right)\right\}.\end{aligned}$$

So such a Levy process is the sum of a Brownian motion with drift, a number of compensated Poisson processes with smaller jump sizes and a number of Poisson processes with larger jump sizes. Roughly speaking, we can interpret a Levy process similarly in general. Owing to the presence of jumps, realised paths of Levy processes (except when they are Brownian motions) are discontinuous. However, just like we can always choose a continuous version for a Brownian motion, we can always choose a version of Levy process that has a.s. cadlag (right continuous with left limits) paths.

In our numerical experiments, all the Levy processes we consider are based on the following three processes.

Meixner process (Schoutens and Teugels 1998, Schoutens, 2001)

The Meixner(α, β, δ) distribution is defined for parameters where $\alpha > 0$, $-\pi < \beta < \pi$ and $\delta > 0$. Its characteristic function is given by

$$\phi(u) = \left(\frac{\cos\left(\frac{\beta}{2}\right)}{\cosh\left(\frac{\alpha u - i\beta}{2}\right)} \right)^{2\delta}$$

and its mean is $\alpha\delta\tan(\beta/2)$. It has a Levy triple of the form $(b, 0, \Pi)$, where

$$b = \alpha\delta\tan\left(\frac{\beta}{2}\right) - 2\delta \int_1^\infty \frac{\sinh(\beta x/\alpha)}{\sinh(\pi x/\alpha)} dx,$$

$$\Pi(dx) = \delta \frac{\exp(\beta x/\alpha)}{x \sinh(\pi x/\alpha)} dx.$$

A Meixner(α, β, δ) process is defined as the Levy process L such that L_1 has the Meixner(α, β, δ) distribution. So, if L is such a process, then the law of the increment $L_t - L_s$ is identical to the law of L_{t-s}, which is a Meixner$(\alpha, \beta, (t-s)\delta)$ distribution.

At the time of writing, there is not any exact simulation method for the Meixner process. There are, however, two simulation methods for approximating a Meixner process. The first method, developed by Asmussen and Rosinski (2001) and also described in Schoutens (2003, section 8.2.1), replaces small jumps in the process by a Brownian motion, divides large jump sizes into intervals and lumps the jumps in each interval into a single Poisson process with a constant jump size. The method is generic in the sense that it can be used to approximate many other Levy processes as long as the Levy measure in the Levy-Khintchine formula satisfies a certain technical condition. However, the performance of this method can be susceptible to the way that the large jumps are discretised.

The second simulation method was developed by Madan and Yor (2005). It views a Meixner(α, β, δ) process L as a time-changed Brownian motion

$$L_t = \frac{\beta}{\alpha}\tau_t^* + W_{\tau_t^*}.$$

The stochastic time change τ_t^* is then simulated as an infinite sum

$$\tau_t^* = \alpha\delta\sqrt{\frac{2\varepsilon}{\pi}} + \sum_{j=1}^{\infty} \frac{\varepsilon}{u_j^2} \mathbb{1}\{g(\varepsilon/u_j^2) > w_j\} \tag{4.1}$$

where $\varepsilon > 0$ is a small number such that jump sizes smaller than ε are in principle lumped into a drift term that corresponds to the first summand in the above equation and $\{u_j\}$, $\{w_j\}$ are two independent sequences of i.i.d. uniform random variates. The function g is a certain distribution function that has an infinite series representation. Madan and Yor (2005) have not explained how to calculate g effectively and how to truncate the infinite series in (4.1). Due to the difficulties in implementing the above two simulation methods, we will not perform simulations for the Meixner process in our numerical experiments.

Normal inverse Gaussian (NIG) process (Barndorff-Nielsen 1995).

The NIG(α, β, δ) distribution is defined for $\alpha > 0$, $|\beta| < \alpha$ and $\delta > 0$. It is a special case of the generalised hyperbolic distribution GH$(-\frac{1}{2}, \alpha, \beta, \delta, 0)$ developed by Eberlein and Prause (1998). Its characteristic function is given by

$$\phi(u) = \exp\left\{-\delta\left[\sqrt{\alpha^2 - (\beta + \varpi u)^2} - \sqrt{\alpha^2 - \beta^2}\right]\right\}$$

and its mean is $\delta\beta/\sqrt{\alpha^2 - \beta^2}$. The NIG$(\alpha, \beta, \delta)$ process is defined as the Levy process L such that L_1 has a NIG(α, β, δ) distribution. So, for such a process, L_t follows a NIG$(\alpha, \beta, t\delta)$ distribution. The NIG process can be explicitly written as a time-changed Brownian motion:

$$L_t = \beta\delta^2 I_t + \delta W_{I_t}.$$

Here I is an Inverse Gaussian (IG) process with parameters $a = 1$ and $b = \delta\sqrt{\alpha^2 - \beta^2}$ (so that I_t follows the IG(at, b) distribution). As a result, we can simulate the NIG process exactly. See chapters 5 and 8 of Schoutens (2003) for a description of the IG process and an exact simulation method for the IG distribution.

Variance gamma (VG) process (Madan and Seneta 1987, Madan *et al.* 1998).

There are two conventions to specify a VG distribution. The first specifies the distribution using three parameters $\nu > 0$, $\sigma > 0$ and $\theta \in \mathbb{R}$, where θ is the mean of distribution. The functional form of the characteristic function under this convention is given by

$$\phi(u) = \left(1 - \varpi u\theta\nu + \frac{1}{2}\sigma^2\nu u^2\right)^{-1/\nu}.$$

The second convention uses three positive parameters C, G and M, where

$$\begin{aligned} C &= 1/\nu, \\ G &= 1/\left(\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} - \frac{1}{2}\theta\nu\right), \\ M &= 1/\left(\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} + \frac{1}{2}\theta\nu\right) \end{aligned}$$

or equivalently,

$$\begin{aligned} \nu &= 1/C, \\ \sigma &= \sqrt{\frac{2C}{GM}}, \\ \theta &= \frac{(G-M)\sigma^2}{2} = \frac{C(G-M)}{GM}. \end{aligned}$$

Under this convention, the VG(C, G, M) distribution is a special case of the CGMY(C, G, M, Y) distribution (Carr *et al.* 2002) with $Y = 0$. The characteristic function of the VG distribution can be written as

$$\phi(u) = \left[\frac{GM}{GM + (M-G)iu + u^2}\right]^C$$

and its Levy triple is in the form of $(b, 0, \Pi)$, where

$$\begin{aligned} b &= -C\left[\frac{e^{-M} - 1}{M} - \frac{e^{-G} - 1}{G}\right], \\ \Pi(dx) &= \left(-\frac{Ce^{Gx}}{x}\mathbb{1}\{x < 0\} + \frac{Ce^{-Mx}}{x}\mathbb{1}\{x > 0\}\right) dx. \end{aligned}$$

A VG(ν, σ, θ) (respectively VG(C, G, M)) process is defined as the Levy process L such that L_1 has the VG(ν, σ, θ) (respectively VG(C, G, M)) distribution. For such a process, L_t follows a VG$(\nu/t, \sigma\sqrt{t}, \theta t)$ (respectively VG(Ct, G, M)) distribution. Exact simulation method exists for the VG process because we can view a VG process as a time-changed Brownian motion of the form

$$L_t = \theta G_t + \sigma W_{G_t},$$

where G is a Gamma process with parameters $a = b = 1/\nu = C$ (and hence G_t has Gamma(at, b) distribution). See chapters 5 and 8 of Schoutens (2003) for a description of the Gamma process and an exact simulation method for the Gamma distribution.

Superposition of pure jump processes and diffusions. The three Levy processes above are pure jump processes, i.e. in each of their Levy triples there is no Brownian component ($A = 0$). Madan *et al.* (1998) argue that pure jump processes are adequate for modelling real world price processes, but for some reasons that we will explain later, we shall consider one-dimensional Levy processes with both diffusion parts and jump parts. These processes take the following form:

$$mt + W_{At} + L_t, \tag{4.2}$$

where $m \in \mathbb{R}$, $A > 0$, W is a standard Brownian motion and L is an independent Meixner, NIG or VG process. (The Levy triple for such a process is hence of the form (m, A, Π).)

4.2.3 Heston (1993) model and its extensions

The stochastic volatility model by Heston (1993, abbreviated as *SV model* hereafter) and its extensions (such as Bates 1996, Scott 1997, Bakshi *et al.* 1997, Duffie *et al.* 2000 or Kou 2002, 2004) are perhaps the most influential and widely used models in equity price modelling. In our experiments, we consider the extension developed by Bakshi *et al.* (1997)[1]. This extension, which we call the SVJ model, postulates that the risk-neutral dynamics of the stock price process is governed by the following SDEs:

$$dS_t = (r_t - q_t - \lambda_J \mu_J) S_{t-} dt + \sigma_t S_{t-} \left(\rho dW_t^{\mathbb{Q}(1)} + \sqrt{1-\rho^2} dW_t^{\mathbb{Q}(2)} \right) + J_t S_{t-} dN_t, \tag{4.3}$$

$$d\sigma_t^2 = \kappa(\eta - \sigma_t^2) dt + \lambda \sigma_t dW_t^{\mathbb{Q}(1)}, \tag{4.4}$$

where $W^{\mathbb{Q}(1)}$, $W^{\mathbb{Q}(2)}$ are two standard Brownian motions, μ_J is a constant, $\{J_t\}$ are i.i.d. jump sizes and N is a Poisson process with intensity parameter λ_J. The four processes

[1]This extended model encompasses jumps and stochastic volatility in the stock price and randomness in the interest rate and is sometimes referred to as the SVJ-SI model. When there are no jumps, it reduces to the SVSI model we considered in the previous chapter; when the interest rate is not stochastic, it reduces to the SVJ model we consider here.)

$W^{\mathbb{Q}(1)}$, $W^{\mathbb{Q}(2)}$, J and N are assumed to be independent. The SV model is the special case where there is no jump (i.e. $\lambda_J = 0$ and the dN_t term is void). In the most popular choice for J_t, the distribution of $\log(1+J_t)$ is taken to be normal:

$$\log(1+J_t) \sim N\left(\log(1+\mu_J) - \frac{1}{2}\sigma_J^2,\ \sigma_J^2\right).$$

Another choice for $\log(1+J_t)$, proposed by Kou and Wang (2004), is the doubly exponential distribution, but we opt to use the lognormal version in this thesis. The characteristic function of the discounted total return X_t in this case is given by

$$\begin{aligned}
\phi(u,t) &= E^Q\left[e^{\varpi u X_t}\middle| S_0, \sigma_0^2\right] \\
&= \exp\left\{\frac{\eta\kappa}{\lambda^2}\left[(\kappa - \rho\lambda\varpi u - d)t - 2\log\frac{1-ge^{-dt}}{1-g}\right] + \frac{\sigma_0^2}{\lambda^2}\left[\frac{(\kappa-\rho\lambda\varpi u - d)(1-ge^{-dt})}{1-g}\right]\right. \\
&\quad \left. -\lambda_J\mu_J\varpi u t + \lambda_J t\left[\exp\left(\left(\log(1+\mu_J) - \frac{\sigma_J^2}{2}\right)\varpi u - u^2\right) - 1\right]\right\}, \\
d &= \sqrt{(\kappa - \rho\lambda\varpi u)^2 + \lambda^2(\varpi u + u^2)}, \\
g &= \frac{\kappa - \rho\lambda\varpi u - d}{\kappa - \rho\lambda\varpi u + d}.
\end{aligned}$$

For the physical process, it is usually assumed there is a risk premium of the form $\gamma + \beta\sigma_t^2$ in the stock price and also a risk premium $\xi\sigma_t^2$ in the variance rate, so that under the physical measure $\mathbb{P}$ the dynamics of S and σ^2 takes the following form:

$$\begin{aligned}
dS_t &= (r_t - q_t - \lambda_J\mu_J + \gamma + \beta\sigma_t^2)S_{t-}dt + \sigma_t S_{t-}\left(\rho dW_t^{\mathbb{P}(1)} + \sqrt{1-\rho^2}dW_t^{\mathbb{P}(2)}\right) + J_t S_{t-} dN_t, \\
d\sigma_t^2 &= \left[\kappa(\eta - \sigma_t^2) + \xi\sigma_t^2\right]dt + \lambda\sigma_t dW_t^{\mathbb{P}(1)}.
\end{aligned}$$

In this thesis, however, we will only make use of the risk-neutral dynamics.

The SVJ process is usually simulated using Euler discretisation. That is, if we want to simulate S_t from time 0 to time T, we first discretise $[0,T]$ into, say, n subintervals of length $\Delta t = T/n$ and then we simulate $S_i := S_{i\Delta t}$ and $v_i := \sigma_{i\Delta t}^2$ (we define r_i and q_i analogously) as follows (here we simulate the risk-neutral process, the procedure for simulating the physical process is similar):

$$S_{i+1} = S_i + (r_i - q_i - \lambda_J \mu_J) S_i \Delta t + \sqrt{v_i \Delta t}\; S_i \left(\rho Z_i^{(1)} + \sqrt{1-\rho^2} Z_i^{(2)} \right) + S_i \sum_{k=1}^{\Delta N_i} J_i^{(k)},$$

$$v_{i+1} = v_i + \kappa(\eta - v_i)\Delta t + \lambda \sqrt{v_i \Delta t} Z_i^{(1)}$$

where $\Delta N_1, \Delta N_2, \ldots$ are independent Poisson random variates with intensity parameter $\lambda_J \Delta t$, each $J_i^{(k)}$ is a random variate of the form

$$J_i^{(k)} = \exp\left\{ \left[\log(1+\mu_J) - \frac{\sigma_J^2}{2} \right] + \sigma_J Z_i^{(3,k)} \right\} - 1$$

and $Z_i^{(1)}, Z_i^{(2)}, Z_i^{(3,1)}, Z_i^{(3,2)}, \ldots$ are independent standard normal random variates. To ensure that v_{i+1} is nonnegative, a further modification of either the form $v_{i+1} \leftarrow \max(v_{i+1}, 0)$ or the form $v_{i+1} \leftarrow |v_{i+1}|$ is usually taken after v_{i+1} is generated.

The Euler discretisation method is inexact. In fact, until recently, no exact simulation method for the SVJ process was known to exist. By using some results of Pitman and Yor (1982) on squared Bessel processes, Broadie and Kaya (2004) obtain a formula for the conditional characteristic function of $\int_u^t \sigma_s^2 ds$ given σ_u^2 and σ_t^2. In turn, they also obtain an exact simulation method for (S_t, σ_t^2). Since the implementation of this method involves many technical details, in this paper we still use Euler discretisation for simulation purpose.

4.2.4 Γ-OU model

Barndorff-Nielsen and Shephard (2001) have proposed a class of models (called BN-S models hereafter) in which the variance rate of the diffusion part of X is an Ornstein-Uhlenbeck (OU) type process driven by a Levy subordinator:

$$dX_t = (\mu + \beta \sigma_t^2)dt + \sigma_t dW_t + \rho dZ_{\lambda t},$$

$$d\sigma_t^2 = -\lambda \sigma_t^2 dt + dZ_{\lambda t}.$$

Here W is a standard Brownian motion and Z is an independent Levy process, called a background driving Levy process (BDLP), such that its Levy symbol is of the form $\zeta(u) = \int_{\mathbb{R}+} (e^{iux} - 1) w(x) dx$ (hence Z has no continuous component and no negative jumps but its Levy measure Π has a density) and $\zeta(u)$ satisfies several technical conditions. Given the information up to time t, it can be shown that the conditional Laplace transform of the increment $X_T - X_t$ is given by

$$E\left[e^{u(X_T-X_t)}\middle|\mathcal{F}_t\right] = \exp\left[\frac{1-e^{-\lambda(T-t)}}{2\lambda}(u^2+2\beta u)\sigma_t^2 + (T-t)\mu u + \lambda\int_t^T k\left(f(s,u)\right)ds\right],$$

$$k(u) = \log E[e^{uZ_1}] = \zeta(-iu),$$

$$f(s,u) = \rho u + \frac{1-e^{-\lambda(T-s)}}{2\lambda}(u^2+2\beta u).$$

Barndorff-Nielsen and Shephard (2001) have obtained two concrete BN-S models via the use of self-decomposable laws[2]. Let D be a self-decomposable law. Then (see Sato 1999, sec. 17) there exists a BDLP Z that has D as its invariant distribution (i.e. if $Y \sim D$, then $(Y + Z_t) \sim D$ regardless of t). Moreover, the cumulant function $k^D(u)$ of D is related to the cumulant function $k(u)$ of Z_1 through the formula

$$k(u) = u\frac{dk^D(u)}{du}.$$

Processes built from such invariant self-decomposable laws are called D-OU processes. Barndorff-Nielsen *et al.* (2002) have considered the cases where D is a Gamma distribution or an IG distribution. In each of both cases, $\int_t^T k\left(f(s,u)\right)ds$ has a closed-form formula, so one can obtain the conditional Laplace transform of $X_T - X_t$ easily. Nicolato and Venardos (2003) have intensively studied the structures of equivalent martingale measures in these models. In particular, there exists an equivalent martingale measure (EMM) $\mathbb{Q}$ under which the BN-S model structure is preserved, and under such $\mathbb{Q}$, we must have $\beta = -\frac{1}{2}$ and $\mu = -\lambda k(\rho)$ (note: the functional form of k here under $\mathbb{Q}$ may be different from the one under the physical measure $\mathbb{P}$). Furthermore, if X is a Γ-OU (resp. IG-OU process), then one can choose an EMM under which X is still a Γ-OU (resp. IG-OU) process.

In our numerical experiments, we will consider Γ-OU processes in the risk-neutral world

[2] A distribution D is called self-decomposable if there exists a family of characteristic functions $\{\phi_c : c \in (0,1)\}$ such that $\phi_D(u) = \phi_D(cu)\phi_c(u)$ for all $c \in (0,1)$

only. In this case, the integral of $k(f(s))$ is given as follows[3]:

$$
\begin{aligned}
k(u) &= \frac{au}{b-u}, \\
\int k\left(f(s,u)\right) ds &= \frac{a}{\lambda(b-f_2)}\left[\lambda f_2 s - b\log\left(b - f(s,u)\right)\right], \\
\lambda \int_t^T k\left(f(s,u)\right) ds &= \frac{a}{b-f_2}\left[\lambda(T-t)f_2 - b\log\frac{b-f_1}{b-\rho u}\right], \\
f_1 &= \rho u + \frac{1-e^{-\lambda(T-t)}}{2\lambda}(u^2-u), \\
f_2 &= \rho u + \frac{1}{2\lambda}(u^2-u).
\end{aligned}
$$

Furthermore, since Z_t is a Levy process and $k(u) = au/(b-u)$, we have

$$
E(e^{uZ_{\lambda t}}) = e^{\lambda t k(u)} = \exp\left[\lambda a t\left(\frac{b}{b-u} - 1\right)\right].
$$

In other words, $Z_{\lambda t}$ is a compound Poisson process with intensity λa and jump size distribution $\mathrm{Exp}(b) \overset{L}{=} \mathrm{Gamma}(1,b)$. So the simulation of a Γ-OU process using Euler discretisation is straightforward.

4.2.5 Continuous-time regime-switching (CTRS) models

In modern option pricing theory, the discounted total return process X is usually modelled as a positive semimartingale (this is not necessarily the case; the most notable exception is to model X as a fractional Brownian motion). Loosely speaking, this ensures that small changes (or errors) in a nonanticipating hedging strategy will only result in small changes in the value of the hedging portfolio. Since every semimartingale — including all Levy processes — can be written as a time-changed Brownian motion (Monroe 1978)[4], every semimartinagle can in turn be written as a time-changed Levy process.

Certainly, such a representation may not be easily made explicit, but in many models we can explicitly write X as the sum of time-changed Levy processes, i.e.

$$
X_t = \sum_{i=1}^{n} Y^{(i)}_{T^{(i)}_t},
$$

[3]Note that in the expressions for f_1, f_2, there is a denominator λ. Many different accounts, such as Barndorff-Nielsen *et al.* (2002, table 3.1), Nicolato and Venardos (2003, table 2.1) and Schoutens (2003, p.87), contain typos such that this denominator is missing.

[4]The stochastic time change is a positive and increasing semimartingale that may be correlated with the Brownian motion.

where each $Y^{(i)}$ is a Levy process and each $T^{(i)}$ is a certain stochastic time change. All models we consider in this chapter can be reformulated in this manner. For instances, in the extended Black-Scholes model, we can reformulate the risk-neutral dynamics of X_t as follows:

$$X_t = Y_{T_t},$$
$$Y_t = -\frac{1}{2}t + W_t,$$
$$T_t = \int_0^t \sigma_s^2 ds,$$

where W is a standard Brownian motion under $\mathbb{Q}$; in the SVJ model, using Ito's formula, we can write the SDE for X in the following form:

$$dX_t = (a + b\sigma_t^2)dt + \sigma_t dW_t + \log(1 + J_t)dN_t,$$

where a, b are constants and W_t is a standard Brownian motion. So

$$X_t = at + \widetilde{W}_{T_t} + Z_t,$$

where T_t is defined as in the above, $\widetilde{W}_t$ is a Brownian motion with unit volatility and drift b, and $Z_t = \int_0^t \log(1 + J_t)dN_t$ is a compound Poisson process. The quantity T_t is usually interpreted as the *business time*, whereas t is interpreted as the *calendar time*.

Except the Black-Scholes model, each of the models we have mentioned so far has an infinite number of economic states. Chourdakis (2002, 2004) has proposed a finite-state Markovian model. More specifically, suppose ι is a time-homogeneous continuous-time Markov chain with n states (say, $\iota(t) = 1, 2, \dots, n$), $\mu_1, \dots, \mu_n$ are some real numbers, $\{Y^{(i)}\}_{i=1,\dots,n}$ are n independent Levy processes and $\{J^{(i,j)}\}_{i,j=1,\dots,n;\ i\neq j}$ are $n^2 - n$ families of stochastic processes such that each $J^{(i,j)} = \left(J_t^{(i,j)} : t \geq 0\right)$ is by itself a family of i.i.d. random variables. It is assumed that ι and the two families of processes $\{Y^{(i)}\}$ and $\{J^{(i,j)}\}$ are independent of each other. Chourdakis models the log-price X as follows:

$$X_t = \sum_{i=1}^{n} \int_0^t \mathbb{1}(\iota_{s-} = i)dY_s^{(i)} + \sum_{0\leq s\leq t} \mathbb{1}(\iota_{s-} \neq \iota_s)J_s^{(\iota_{s-},\iota_s)}. \tag{4.5}$$

In other words, if ι remains in state i on some interval $[t_1, t_2)$ and changes to state j at time t_2, then increments of X over the time period $[t_1, t_2]$ are given by the increments of the Levy process $Y^{(i)}$, except that X will experience at time t_2 an additional jump whose size is distributed according to the law of $J_{t_2}^{(i,j)}$. Some models, such as those considered in Edwards (2005), Elliott *et al.* (2005) and Boyle *et al.* (2007), are special cases of Chourdakis' model when state transitions do not induce any jump.

Chourdakis' model has the merit that given any $t > s$, the characteristic function of the increment $X_t - X_s$ given (ι_s, ι_t) is readily available. Let $\mathbf{A} = (a_{ij})$ be the intensity matrix of the Markov chain ι (do not confuse this $\mathbf{A}$ with the A in the Levy triple) and $P(t) = (P_{ij}(t)) = \exp(t\mathbf{A})$, so that $P_{ij}(t)$ is the transition probability

$$P_{ij}(t) = \mathbb{P}(\iota_t = j | \iota_0 = i) = \mathbb{P}(\iota_{s+t} = j | \iota_s = i).$$

Let $\Phi(u; s, t)$ be the $n \times n$ matrix whose (i, j)-th entry is $P_{ij}(t - s)$ times the characteristic function of $X_t - X_s$ conditional on (ι_s, ι_t), i.e.,

$$\Phi_{ij}(u; s, t) = P_{ij}(t - s) \; E\left(e^{\varpi u(X_t - X_s)} | \iota_s = i, \iota_t = j\right).$$

Chourdakis (2002) shows that this weighted characteristic function matrix is time-homogeneous and can be written explicitly as

$$\Phi(u; s, t) = \Phi(u; t - s) = (\Phi_{ij}(u; t - s)) := \exp\left[(t - s)B(u)\right] \tag{4.6}$$

where $B(u) = (b_{ij}(u))$ is the square matrix such that

$$b_{ii}(u) = a_{ii} + \zeta_i(u),$$

$$b_{ij}(u) = a_{ij}\phi_{ij}(u) \text{ for } i \neq j.$$

Here ζ_i denotes the Levy symbol of $Y^{(i)}$ and ϕ_{ij} denotes the characteristic function of $J_t^{(i,j)}$ for any t.

CTRS process as Markov chain approximation

In principle, to specify a CTRS process, we need to specify n Levy processes, n^2-n transition intensities (a_{ij} for all $i \neq j$) and n^2 jump size distributions. Chourdakis has explained how to use his model as an approximation to existing stochastic volatility models. In this regard the amount of details to be specified is vastly reduced. To illustrate, consider a diffusion process:

$$dy_t = \mu(y_{t-})dt + \Sigma(y_{t-})dW_t.$$

Let $y'_1 < y'_2 < \ldots < y'_n$ denote a uniformly spaced grid of nodes with spacing $y'_i - y'_{i-1} = h$, and ι be a continuous-time Markov chain with n states and the following intensities (where μ^+ and μ^- denote the positive and negative parts of μ):

$$a_{i,i-1} = \frac{\mu^-(y'_i)}{h} + \frac{\Sigma^2(y'_i)}{2h^2}, \tag{4.7}$$

$$a_{i,i+1} = \frac{\mu^+(y'_i)}{h} + \frac{\Sigma^2(y'_i)}{2h^2}. \tag{4.8}$$

Here the diagonal entries $a_{i,i}$ are by definition equal to $-\sum_{i\neq j} a_{ij}$ and by convention, all other unspecified off-diagonal entries are equal to zero. Kushner (1990) proves that, when the spacing h is small enough and the range that the grid of nodes covers is large enough, the stochastic process y'_ι can approximate y arbitrarily well provided that μ and Σ satisfy some mild technical conditions.

Now, suppose we want to approximate the SV model by a CTRS process. Using Ito's formula, if we put[5] $y_t = \log \sigma_t^2$, then (4.3) and (4.4) give

$$dX_t = -\frac{1}{2}e^{y_t}dt + e^{y_t/2}\left(\rho dW_t^{(1)} + \sqrt{1-\rho^2}dW_t^{(2)}\right), \tag{4.9}$$

$$dy_t = \kappa\left[\left(\eta - \frac{\lambda^2}{2\kappa}\right)e^{-y_t} - 1\right]dt + \lambda e^{-y_t/2}dW_t^{(1)}. \tag{4.10}$$

[5] We may put $y_t = \sigma_t^2$ as well, as we shall do later in specifying the Levy/Γ-OU model. However, in practice, it seems that a finer spacing is needed for a low variance rate than for a high variance rate. So I follow Chourdakis (2002, 2004) and try to use the log variance rate wherever possible.

Substitute the second SDE into the first, we get

$$
\begin{aligned}
dX_t &= -\frac{1}{2}e^{y_t}dt + \frac{\rho}{\lambda}\left\{e^{y_t}dy_t - \kappa\left[\left(\eta - \frac{\lambda^2}{2\kappa}\right) - e^{y_t}\right]dt\right\} + e^{y_t/2}\sqrt{1-\rho^2}dW_t^{(2)} \\
&= \left[\left(\frac{\rho\kappa}{\lambda} - \frac{1}{2}\right)e^{y_t} - \frac{\rho\kappa}{\lambda}\left(\eta - \frac{\lambda^2}{2\kappa}\right)\right]dt + e^{y_t/2}\sqrt{1-\rho^2}dW_t^{(2)} + \frac{\rho}{\lambda}e^{y_t}dy_t.
\end{aligned}
$$

So, if we approximate y by y'_ι, then X can be approximated by the CTRS model where

$$
\begin{aligned}
Y_t^{(i)} &= \left[\left(\frac{\rho\kappa}{\lambda} - \frac{1}{2}\right)e^{y'_i} - \frac{\rho\kappa}{\lambda}\left(\eta - \frac{\lambda^2}{2\kappa}\right)\right]t + \int_0^t e^{y'_i/2}\sqrt{1-\rho^2}W_s^{(2)}, \\
J_t^{(i,j)} &= \begin{cases} \frac{\rho}{\lambda}e^{y'_i}h & \text{if } j = i+1, \\ -\frac{\rho}{\lambda}e^{y'_i}h & \text{if } j = i-1, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

In our experiments, instead of using CTRS models as approximations of existing stochastic volatility models, we use existing models as guidelines to specify our CTRS models. Whether the CTRS models would converge to these existing models is not our concern. In the following we will give the concrete classes of CTRS models that we use in our experiments.

Levy/CIR models

We consider the Markov chain described by (4.7), (4.8) together with the square-root process (4.10):

$$
a_{i,i-1} = \frac{\kappa}{h}\left[\left(\eta - \frac{\lambda^2}{2\kappa}\right)\frac{1}{c_i} - 1\right]^- + \frac{\lambda^2}{2c_ih^2}, \tag{4.11}
$$

$$
a_{i,i+1} = \frac{\kappa}{h}\left[\left(\eta - \frac{\lambda^2}{2\kappa}\right)\frac{1}{c_i} - 1\right]^+ + \frac{\lambda^2}{2c_ih^2}. \tag{4.12}
$$

where $c_i = e^{y'_i}$. Let Y be a Levy process of the form (4.2). We define X as in (4.5), where

$$
J_t^{(i,j)} = h_{ij} := \begin{cases} \rho c_i h & \text{if } j = i+1, \\ -\rho c_i h & \text{if } j = i-1, \\ 0 & \text{otherwise.} \end{cases}
$$

$$
Y_t^{(i)} \stackrel{L}{=} -\sum_{j\neq i} a_{ij}\left(e^{h_{ij}} - 1\right)t + Y_{c_it},
$$

where Y is a Levy process of the form (4.2). So the transition-induced jumps $(J_t^{(i,j)})$ are basically those in the previous approximation of the SV process, except that the constant λ is absorbed into ρ. We call the process X a Meix/CIR process if the L in (4.2) is a Meixner process, and vice versa.

Levy/Γ-OU models

We can also use a Markov chain to mimic a Gamma-OU(λ, ρ, a, b) process. Conceptually, we think of y_t as a surrogate for σ_t^2 and let $y_1' < y_2' < \ldots < y_n'$ be n evenly spaced *positive* numbers. When the transition intensities for ι are given by

$$a_{i,i-1} = \lambda y_i'/h, \tag{4.13}$$

$$a_{i,j} = \lambda ab(j-i)he^{-b(j-i)h} \quad (j > i), \tag{4.14}$$

the process y_ι' can be used to mimic a Gamma-OU(λ, ρ, a, b) process. Let $c_i = y_i'$ and Y be a Levy process of the form (4.2). We define X as in (4.5), where

$$J_t^{(i,j)} = h_{ij} := \begin{cases} \rho(j-i)h & \text{if } j-i \geq -1, \\ 0 & \text{otherwise.} \end{cases}$$

$$Y_t^{(i)} \stackrel{L}{=} -\sum_{j \neq i} a_{ij} \left(e^{h_{ij}} - 1\right) t + Y_{c_i t},$$

where Y is a Levy process of the form (4.2). We call X a Meix/Γ-OU process if the L in (4.2) is a Meixner process, and vice versa.

Relationship between CTRS models and RSLN models

A discrete-time regime-switching model, called regime-switching lognormal (RSLN) model, was introduced by Naik (1993) to the option pricing community and by Hardy (2001) to actuaries. It has been the focus of numerous research works. See, e.g. Bollen (1998), Kolkiewicz *et al.* (2001) and Edwards (2005). In this model, $\iota = (\iota(t_0), \iota(t_1), \ldots)$ is a discrete-time Markov chain and $X = (X(t_0), X(t_1), \ldots)$ is such that:

$$X(t_k) - X(t_{k-1}) = \mu_{\iota(t_k)} + \sigma_{\iota(t_k)} Z(t_k),$$

where μ_i and σ_i $(i = 1, 2, \ldots, n)$ are constants and $(Z(t_k) : k = 1, 2 \ldots)$ is a family of i.i.d. standard normal random variables.

Despite the apparent similarity between the CTRS model and the RSLN model, there are subtle differences. In a RSLN process, the distribution of the increment $X(t_k) - X(t_{k-1})$ depends only on $\iota(t_k)$. For the purpose of our discussion, let us call a process like this *vertex-dependent*. For a discretely sampled CTRS process, of which the incremental distribution depends not only on $\iota(t_k)$ but also on $\iota(t_{k-1})$, we say it is *edge-dependent*. Certainly we can always reformulate an edge-dependent model into a vertex-dependent one by redefining the state variable as $\widetilde{\iota}(t_k) = (\iota(t_{k-1}), \iota(t_k))$, but then the intensity matrix of $\widetilde{\iota}$ has a special sparseness pattern that is not necessarily present in the intensity matrix of a RSLN process. So in the sequel we refrain from using this trick.

Another difference between a RSLN-like vertex-dependent model and a CTRS model is that in the former, some time scale is more special than the others. Suppose X^{half} and X^{ann} are respectively the half-yearly sampled and annually sampled time series of a certain log-price process X, where X^{half} is a general RSLN-like regime switching process such that its transition probabilities may be time-dependent and the distributions of its increments can be non-normal. For ease of discussion, suppose ι has only two states. In the following proposition, we show that X^{half} and X^{ann} cannot both be vertex-dependent regime switching processes unless some unusual conditions hold.

Proposition 4.1. *Suppose for all $u \in \mathbb{R}$, we have*

$$\begin{pmatrix} \alpha\varphi_1(u) & (1-\alpha)\varphi_2(u) \\ (1-\beta)\varphi_1(u) & \beta\varphi_2(u) \end{pmatrix} = \begin{pmatrix} p\phi_1(u) & (1-p)\phi_2(u) \\ (1-q)\phi_1(u) & q\phi_2(u) \end{pmatrix} \begin{pmatrix} \tilde{p}\widetilde{\phi}_1(u) & (1-\tilde{p})\widetilde{\phi}_2(u) \\ (1-\tilde{q})\widetilde{\phi}_1(u) & \tilde{q}\widetilde{\phi}_2(u) \end{pmatrix}.$$

Then either (a) $p + q = 1$, or $p + q \neq 1$ but at least one of the followings hold: (b) $(\tilde{p}, \tilde{q}) = (0, 0)$ or $(1, 1)$, (c) $\phi_1 = \phi_2$, or (d) both $\widetilde{\phi}_1$ and $\widetilde{\phi}_2$ vanish on some nonempty open intervals.

Here we allow the Markov chain to be time-inhomogeneous and the distributions of the increments of X^{half} to be time-dependent. The symbols $\alpha, \beta, p, q, \tilde{p}, \tilde{q}$ are transition probabilities and $\varphi_1, \varphi_2, \phi_1, \phi_2, \widetilde{\phi}_1, \widetilde{\phi}_2$ are characteristic functions for appropriate time periods. Note that both conditions (a) and (c) imply that the increment of X^{half} over the first half of the year has a state-independent distribution and condition (b) implies that state transitions in the second half of the year are certain. Thus X^{half} is not a genuine stochastic volatility model if any one of these three conditions hold. For condition (d), although there do exist some probability distributions whose characteristic functions vanish on some nonempty open intervals (e.g. as pointed out by Feller 1968, p.503, the characteristic function corresponding to the probability density function $f(x) = \frac{1-\cos(ax)}{\pi a x^2}$ is given by $\phi(u) = 1 - \frac{|u|}{a}$ for $|u| \leq a$ and $\phi(u) = 0$ otherwise), to our knowledge none of them has been used in option pricing literature.

In short, if an annually sampled log-price process is a RSLN-like regime switching model, then unless we enlarge the space of state variables or impose some strange conditions, vertex dependency will be lost if we double or halve the sampling frequency. So, in such a model, some time scale is more special than the others. Whether this is financially unconvincing is controversial (see e.g. the interesting discussion between Klein and Hardy 2001). In contrast, there is no special time scale in CTRS models.

Number of states in CTRS models

How many states of ι are needed in building a useful CTRS model? The work of Hardy (2001) shows that for fitting the physical process of X with the RSLN model, two states are enough. So a small number of states are probably also enough for CTRS models.

The situation is very different if we want to calibrate our asset price model to market option prices. An issue in RLSN or CTRS models that has been overlooked by most researchers in the literature is that when the Markov chain has n states, RSLN or CTRS models can exhibit only n different kinds of volatility smiles. However, the number of volatility smiles we can observe in an option market is virtually infinite. So, even if we can calibrate a RSLN or CTRS model to the today's market prices nicely, it is unlikely that the calibrated model can fit tomorrow's prices well. In other words, if the number of states is small, we would have to recalibrate our model frequently and the model parameters may change drastically from day to day. While this may not be a real issue in some applications (e.g. when calibration is done only because we want to create an initial static hedge), in general such instability of model parameters is undesirable.

In our numerical experiments, since we only calibrate our models once using a cross section of option price data, model recalibration is not an issue. Therefore, for computational efficiency, we consider only a small number of states.

4.3 Risk-neutral evaluation of Levy and CTRS models

In virtually all modern stock price models, equivalent martingale measures (EMMs) are no longer unique (and hence the market is incomplete). That how to choose an EMM has become an issue in option pricing theory. There are essentially two lines of approaches. In the first one, one picks a measure change that has some economic underpinnings or technical advantages. For instance, one approach that actuaries may find familiar is to use an Esscher transform (Gerber and Shiu 1994, 1996), i.e. a Radon-Nikodym derivative of the form

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \frac{e^{\theta X_t}}{E^{\mathbb{P}}(e^{\theta X_t})}$$

to obtain an equivalent pricing measure. This approach has a number of merits. Most importantly, the characteristic function of the discounted stock price under the pricing measure,

$$\phi^{\mathbb{Q}}(u) = \frac{\phi^{\mathbb{P}}(u - i\theta)}{\phi^{\mathbb{P}}(-i\theta)},$$

is readily available. In addition, in many cases the use of Esscher transform is theoretically justified because it can be derived from equilibrium arguments (e.g. Keller 1997) or the EMM it gives is a minimal martingale measures (e.g. Chan 1999) in the sense of Follmer and Schweizer (1991) or a minimal entropy martingale measure (see, e.g. Chan 1999 or Miyahara and Fujiwara 2003). Unfortunately, this kind of measure changes do not always exist and option prices obtained from these measure changes may not agree with market prices.

Another approach is to calibrate the asset price model to market prices of options. Suppose we have estimated the model parameters under the physical measure and we are given the prices of a set of liquidly traded options. If we can specify the set of EMMs that we are interested in, we can use an optimisation routine to find the EMM that gives rise to the least price discrepancies (w.r.t. some error metric).

Despite the ill-posedness of the calibration itself (the price discrepancies may remain small for two very different sets of model parameters), this approach is popular among practitioners and is of particular importance when one wants to create hedges using the kind of options that the asset price model is calibrated to. Following Hirsa *et al.* (2003) and Schoutens *et al.* (2003, 2005), we also take this approach in our experiments. So, the main issue here is to specify the EMMs of interest. In the rest of this section, we will discuss how to obtain an EMM in each of the Levy models or CTRS models we consider in this paper.

4.3.1 Levy models

The characterisation of EMMs is in general a difficult problem. The usual practice is to study structure preserving EMMs. For Levy models, this means we opt to obtain an EMM so that under both the physical measure $\mathbb{P}$ and the EMM $\mathbb{Q}$, X remains a Levy process of the same class. But even so, the current literature on EMMs for Levy models can be confusing. For instance, Madan *et al.* (1998, p.87) commented that if $\mathbb{P}$ is a VG distribution with parameters $\nu_S, \sigma_S, \theta_S$ and $\mathbb{Q}$ is a VG distribution with parameters $\nu_{RN}, \sigma_{RN}, \theta_{RN}$, then $\mathbb{P}$ and $\mathbb{Q}$ are always equivalent and "there is no link between ... [the two sets of] ... parameters". While their comment is correct, some people may be misled to think that the parameters for any two VG *processes* (as opposed to *distributions*) are completely unrelated.

Structure preserving locally equivalent measures for Levy processes have actually been studied thoroughly. The followings are the main results.

Theorem 4.2. (See, e.g. theorems 33.1-2 of Sato 1999, proposition 9.8 of Cont and Tankov 2004 or proposition 2.19 of Raible 2000; detailed expressions of the U_t below can be found in the the former two references.) *Let X be a Levy process with Levy triple (b, A, Π) under the probability measure P. Then there is a probability measure $\mathbb{Q}$ such that it is locally equivalent to $\mathbb{P}$ and X is a $\mathbb{Q}$-Levy process with Levy triple (b', A', Π') if and only if the following conditions are satisfied:*

(a) $A = A'$.

(b) Π *and* Π' *are locally equivalent with*

$$\int_{\mathbb{R}} \left(\sqrt{\frac{d\Pi'}{d\Pi}} - 1 \right)^2 \Pi(dx) < \infty.$$

(c) If X has no Brownian component (i.e. $A = 0$), then in addition we must have

$$b' - b = \int_{-1}^{1} x \left(\Pi' - \Pi \right)(dx).$$

When the above conditions are satisfied, the Radon-Nikodym derivative takes the form of $\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{U_t}$, *where U_t is a $\mathbb{P}$-Levy process.*

With this theorem, we can deduce how the parameter values change when we model X as the superposition of a diffusion and a Meixner, NIG or VG process.

Proposition 4.3. *Suppose $X_t = \mu t + W_{At} + L_t$ is a $\mathbb{P}$-Levy process, where $\mu \in \mathbb{R}$, $A \geq 0$ and W, L are respectively a standard Brownian motion and a pure jump Levy process under $\mathbb{P}$. Suppose X is also a $\mathbb{Q}$-Levy process under some $\mathbb{Q} \overset{loc.}{\sim} \mathbb{P}$ (so that $A = A'$ by theorem 4.2).*

(i) If L is a $\mathbb{P}$-Meixner(α, β, δ) *process and a* $\mathbb{Q}$-Meixner$(\alpha', \beta', \delta')$ *process, then $\alpha\delta = \alpha'\delta'$. In addition, if $A = A' = 0$, we must have $\mu = \mu'$.*

(ii) If L is a $\mathbb{P}$-NIG(α, β, δ) *process and a* $\mathbb{Q}$-NIG$(\alpha', \beta', \delta')$ *process, then $\delta = \delta'$. In addition, if $A = A' = 0$, we must have $\mu = \mu'$.*

(iii) If L is a $\mathbb{P}$-VG(C,G,M) process and a $\mathbb{Q}$-VG(C',G',M') process, then $C=C'$. In addition, if $A=A'=0$, then

$$\mu' - C\int_0^1 \left(e^{-M'x} - e^{-G'x}\right) dx = \mu - C\int_0^1 \left(e^{-Mx} - e^{-Gx}\right) dx.$$

In view of theorem 4.2 or the above proposition, when X has no diffusion component, we cannot change its mean at will. This is quite different from the situation in the Black-Scholes world and is a point that the reader should take caution with. For instance, Schoutens (2003, pp.79-80) wrongly claims that if $X_t = L_t$ for some $\mathbb{P}$-Levy process L, then there always exists an EMM $\mathbb{Q}$ under which $X_t = -\zeta^{\mathbb{P}}_{L_1}(-\varpi)t + L'_t$, where L' is a $\mathbb{Q}$-Levy process with the same parameters as L. However, theorem 4.2 shows that this is impossible unless L has a Brownian component or $\mathbb{P}$ is already a martingale measure (so that $\zeta^{\mathbb{P}}_{L_1}(-\varpi)=0$).

At any rate, by proposition 4.3, if X has Brownian component and we want the jump component L to be, say, a NIG process under both $\mathbb{P}$ and a locally equivalent probability measure $\mathbb{Q}$, then there are two free parameters α' and β' to specify. However, for $\mathbb{Q}$ to be a martingale measure, α' and β' must further satisfy the constraint

$$\begin{aligned} 1 = E^{\mathbb{Q}}(e^{X_1}) = \phi^{\mathbb{Q}}(-\varpi) &= \exp\left[\mu - \delta\left(\sqrt{(\alpha')^2-(\beta'+1)^2} - \sqrt{(\alpha')^2-(\beta')^2}\right)\right] \\ \Leftrightarrow \sqrt{(\alpha')^2-(\beta'+1)^2} - \sqrt{(\alpha')^2-(\beta')^2} &= \frac{\mu}{\delta}. \qquad (4.15) \end{aligned}$$

Therefore, provided that μ and δ have been estimated using statistical data, there is only one free parameter to choose in $\mathbb{Q}$. Having fewer free parameters can be advantageous because we can avoid overfitting in model calibration, but the result of calibration can be poor. In our experiments, we opt to include adopt a nonzero A so that the model can be calibrated better. So, for each Levy model of X in the form of (4.2) under $\mathbb{P}$, we will consider local equivalent measures $\mathbb{Q}$ under which

$$X_t = -\left(\frac{A}{2} + \zeta^{\mathbb{Q}}_{L_1}(-\varpi)\right)t + \widetilde{W}_{At} + L_t,$$

where $\widetilde{W}$ is a standard $\mathbb{Q}$-Brownian motion and $\zeta^{\mathbb{Q}}_{L_1}$ is the Levy symbol of L under $\mathbb{Q}$. Since $A \neq 0$ in our Levy models, proposition 4.3 guarantees that such $\mathbb{Q}$ must exist although it may not be unique. Obviously X is a $\mathbb{Q}$-martingale in this case, so $\mathbb{Q}$ is an EMM.

4.3.2 CTRS models

Changing measures for time-changed Levy models is even more confusing. Consider a Levy process Y and a random time change T. Carr and Wu (2004, p.138) suggest that one can use transformations of the following form, which they also call Esscher transforms[6], to obtain a locally equivalent measure $\mathbb{Q}$:

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(\theta Y_{T_t} - T_t \phi_{Y_1}(-i\theta)\right)$$

They cite a result of Kushler and Sorensen (1997), which basically says that if T_t satisfies some technical conditions, then one can state the *local characteristics* of the semimartingale Y_T easily in terms of the Levy triple of Y. However, they did not explain how this result can be turned into an operational method for deriving the dynamics of Y_T in the risk-neutral world[7]. Moreover, despite the name "characteristics", local characteristics do not really characterise stochastic processes. In fact, Kushler and Sorensen (1997, p.293) has shown that two stochastic processes can have identical local characteristics but different distribution laws.

Another way to change measures has been adopted by Hirsa *et al.* (2003), Schoutens (2003) and Schoutens *et al.* (2003, 2005), who thought that when Y and T are independent and $T_t = \int_0^t \sigma_u^2 du$ for some Markov process σ, the following process would automatically be risk-neutral:

$$S_t = S_0\ e^{(r-q)t} \frac{e^{Y_{T_t}}}{E\left[e^{Y_{T_t}} \,\middle|\, \sigma_0\right]}.$$

Unfortunately, this is wrong. Although we do have $e^{-(r-q)t}E[S_t|\sigma_0] = S_0$, risk-neutrality requires that $X_t = e^{-(r-q)t}S_t$ is a martingale. But this means

$$E\left[e^{Y_{T_t}-Y_{T_u}} \,\middle|\, \sigma_u\right] = \frac{E\left[e^{Y_{T_t}} \,\middle|\, \sigma_0\right]}{E\left[e^{Y_{T_u}} \,\middle|\, \sigma_0\right]}.$$

Since the left hand side depends on σ_u but the right hand side does not, this equality does

[6]Note that as opposed to what Carr and Wu (2004, pp.122, 138) say, these Esscher transforms are not Dolean-Dade exponentials of θY_{T_t}.

[7]They did illustrate how to derive the risk-neutral dynamics for the SV model using Esscher transform, but this was done using Girsanov's theorem, not Kushler and Sorensen's result.

not hold unless $(\sigma_t : t \geq 0)$ is a family of independent random variables. But then S_t is no longer a genuine stochastic volatility model.

For CTRS models, Chourdakis (2002) has shown in principle how to use an equilibrium argument to derive risk-neutral parameters. However, he does not explain how to choose a utility function such that the derivation of risk-neutral parameters is mathematically tractable. In our study, we discard the utility maximisation approach and focus on Radon-Nikodym derivatives directly.

Recall that $\mathbf{A} = (a_{ij})$ is the intensity matrix for the Markov chain ι. For any $0 \leq s \leq t$, let $\iota(s,t) := (\iota_u : s \leq u \leq t)$ denote the path of ι on $[s,t]$ and $p(\iota(s,t))$ denote the likelihood for this segment of path given ι_s. That is, within the time period $(s,t]$, if the state of ι changes only at some m time points $t_1, \ldots, t_m$ and if we write $t_0 = s$ and $\iota_j = \iota(t_j)$, then

$$p(\iota(s,t)) = \left(\prod_{j=1}^{m} a_{\iota_{j-1}\iota_j} \exp\left((t_j - t_{j-1}) a_{\iota_{j-1},\iota_{j-1}} \right) \right) a_{\iota_m \iota_m} \exp\left((t - t_m) a_{\iota_m,\iota_m} \right).$$

Now, consider a CTRS process X of the form (4.5). The following proposition shows that any locally equivalent measure change for the Levy processes $Y^{(i)}$ and any change of positive transition intensities for ι can be separately carried over to the CTRS model[8].

Proposition 4.4. *Suppose X is a CTRS process of the form (4.5) and $U^{(1)}, U^{(2)}, \ldots, U^{(n)}$ are n Levy processes that take the role of U in theorem 4.2 for respectively the Levy processes $Y^{(1)}, Y^{(2)}, \ldots, Y^{(n)}$. Let $\widetilde{\mathbf{A}} = (\widetilde{a}_{ij})$ be an intensity matrix such that $a_{ij} = 0$ if and only if $\widetilde{a}_{ij} = 0$; let $q(s,t)$ denote the path likelihood of $\iota(s,t)$ when the intensity matrix of ι is replaced by $\widetilde{\mathbf{A}}$. When ι changes state exactly at times $t_1 < t_2 < \ldots < t_m$ within the time period $(0,t]$, we write $t_{m+1} = t$, $\iota_j = \iota(t_j)$ and define*

$$\eta_X(t) = \exp\left\{ \sum_{j=0}^{m} \left[U^{(\iota_j)}(t_{j+1}) - U^{(\iota_j)}(t_j) \right] \right\} \frac{q(\iota(0,t))}{p(\iota(0,t))}.$$

Then η_X defines a Radon-Nikodym derivative that transforms $\mathbb{P}$ into a locally equivalent measure $\mathbb{Q}$, under which the intensity matrix of ι is given by $\widetilde{\mathbf{A}}$ and each $Y^{(i)}$ is still a Levy

[8]Should the sizes of the transition-induced jumps be stochastic, we could also perform similar changes. Nevertheless we are content to leave those jumps constant here.

process. Moreover, $\mathbb{Q}$ *is a martingale measure if for all* i*, the Levy symbol of* $Y_1^{(i)}$ *under* $\mathbb{Q}$ *satisfies*

$$\zeta_i^{\mathbb{Q}}(-\varpi) = -\sum_{j \neq i} \widetilde{a}_{ij} \left(e^{h_{ij}} - 1 \right). \tag{4.16}$$

As this proposition indicates, the measure transform for CTRS models can be quite arbitrary. For simplicity, we only consider structure preserving measure transforms in our study. Recall that in our Levy/CIR or Levy/Γ-OU models, the Levy processes $Y^{(i)}$ are of the form

$$Y_t^{(i)} \stackrel{L}{=} -\sum_{j \neq i} a_{ij} \left(e^{h_{ij}} - 1 \right) t + Y_{c_i t},$$

where

$$Y_t = mt + W_{At} + L_t.$$

Therefore,

$$Y_t^{(i)} \stackrel{L}{=} \left[-\sum_{j \neq i} a_{ij} \left(e^{h_{ij}} - 1 \right) + mc_i \right] t + W_{c_i At} + L_{c_i t}. \tag{4.17}$$

Without loss of generality, suppose the intensity matrix A is constructed using formulae (4.11)–(4.12). Now, since $Y^{(i)}$ has a nonzero Brownian component, by proposition 4.3, we can change the drift of $Y^{(i)}$ at will via a change of measure. Consequently, if $\widetilde{\mathbf{A}}$ is another intensity matrix that is constructed using formulae (4.11)–(4.12), then $a_{ij} = 0$ if and only if $\widetilde{a}_{ij} = 0$ and hence by proposition 4.4, there exists an EMM $\mathbb{Q}$ under which

$$Y_t^{(i)} \stackrel{L}{=} \left[-\sum_{j \neq i} \widetilde{a}_{ij} \left(e^{h_{ij}} - 1 \right) - c_i \left(\frac{A}{2} + \zeta_{L_1}^{\mathbb{Q}}(-\varpi) \right) \right] t + \widetilde{W}_{c_i At} + L_{c_i t}, \tag{4.18}$$

where $\widetilde{W}$ is a standard $\mathbb{Q}$-Brownian motion, $\zeta_{L_1}^{\mathbb{Q}}$ is the Levy symbol of L under $\mathbb{Q}$ and the parameters of L under $\mathbb{P}$ and $\mathbb{Q}$ may be different (see proposition 4.3). Obviously, equality (4.16) holds in this case. So $\mathbb{Q}$ is a martingale measure. Furthermore, if we define

$$\widetilde{Y}_t = -\left(\frac{A}{2} + \zeta_{L_1}^{\mathbb{Q}}(-\varpi) \right) t + \widetilde{W}_{At} + L_t,$$

then $\widetilde{Y}$ is a $\mathbb{Q}$-martingale such that under $\mathbb{Q}$,

$$Y_t^{(i)} \stackrel{L}{=} -\sum_{j \neq i} \widetilde{a}_{ij} \left(e^{h_{ij}} - 1 \right) t + \widetilde{Y}_{c_i t}.$$

In short, proposition 4.4 says that if we want to perform a martingale transform under the Levy/CIR (resp. Levy/Γ-OU) model, we may do so *as if*[9] we can separately transform the exponential Levy process e^Y into a martingale and change the CIR (resp. Γ-OU) parameters arbitrarily.

This outlines an important difference between the SV model and its approximating CTRS model, namely, in CTRS models we can freely adjust the CIR parameter λ by equivalent measure transform but in the SV model we cannot (otherwise the quadratic variation of σ^2 will not be preserved). Therefore, although we can use a CTRS model to approximate a SV model by increasing the number of states of ι, we cannot pass the associated measure transform to the limit.

4.4 Model estimation and calibration

4.4.1 Estimation

We will estimate Levy models CTRS models by the maximum likelihood method. As the characteristic functions of the Meixner, NIG and VG distributions are known in closed forms, we can compute their likelihood functions by Fourier inversion. For CTRS models, since the transitional likelihood functions are state-dependent, we must filter the state variables out in order to obtain MLEs. The detailed procedure for doing this for RSLN models (or vertex-dependent models in general) is described in Hardy (2001, 2003). Extending this procedure to deal with discretely sampled CTRS processes is straightforward.

Since the observed time series data of X under regime switching models are serially correlated, it is reasonable to suspect (see, e.g. Hardy 2001, p.52) that asymptotic results of MLEs no longer hold. A pleasant result of Bickel *et al.* (1998), however, shows that under some mild conditions, MLEs of RSLN models and CTRS models are still asymptotically normal and the observed information matrices are still consistent estimators of the Fisher

[9] We say "as if" because we do not really perform a measure change to make e^Y an exponential martingale. Instead, we perform a measure change so that the dynamics of each $Y^{(i)}$ under $\mathbb{P}$, as described in (4.17), becomes (4.18) under $\mathbb{Q}$.

information matrices. Nevertheless, in our experiments, the primary reason for estimating our models is to see whether imposing an additional constraint (that the risk-neutral model parameters must keep their logical links with their physical counterparts) on the calibrated model would reduce price discrepancies between models. Therefore the precision of the MLEs is not a concern here.

Finding MLEs for SV, SVJ or BN-S models is tricky. These models are usually estimated by other methods (see, e.g. Barndorff-Nielsen and Shephard 2001 or Jiang 2002). For simplicity, in our experiments, we will not estimate these three models and we will consider them only in risk-neutral world.

The data for the estimation of X include historical data for the monthly returns and total returns of the S&P 500 index (SPX) and also historical data for the U.S. Treasury yield curves. The former two sets of data are downloaded from Standard & Poor's website (`www.standardandpoors.com`) and the latter one is obtained from the U.S. Treasury (`www.ustreas.gov`). Using these data, we can construct time series data for the log discounted total return X.

For all Levy models and CTRS Levy models, we set $A = 0.01 = 0.1^2$ in (4.2). We also set $n = 7,\ y'_1 = -1,\ y'_n = 1,\ h = 1/3$ with $c_i = e^{y'_i}$ in (4.11) and (4.12) for all Levy/CIR models and set $n = 7,\ y'_1 = 0.4,\ y'_n = 1.6,\ h = 0.2$ with $c_i = y'_i$ in (4.13), (4.14) for all Levy/Γ-OU models. Table 4.17 contains our estimation results. For convenience, we will call the estimated parameters the S&P 500 $\mathbb{P}$ parameters.

We have several interesting observations on the estimation results. First, the unconditional densities of X_1 for the estimated models are plotted in fig. 4.6(a). As shown in the figure, the density function for the VG/CIR model is quite different from the densities for the others. Although this may indicate that the VG/CIR model does not fit the data well, it is also possible that different models that fit the time series data well can exhibit different unconditional densities.

Second, the one-year transition probabilities for ι are listed in table 4.21. We see in the

NIG/Γ-OU model and the VG/Γ-OU model, ι almost always stay at state 1. This shows that in order to fit the physical unconditional distribution of equity returns well, the use of a genuine stochastic volatility model is not really necessary.

Third, although one may not find fig. 4.6(a) visually clear, the figure does show that that densities for the three Levy models are visually indistinguishable. So are the densities for the Meix/CIR and NIG/CIR models or the densities for the NIG/Γ-OU and VG/Γ-OU models. More interestingly, table 4.21 shows that the differences in the transition probabilities between the Meix/CIR and NIG/CIR models or between the NIG/Γ-OU and VG/Γ-OU models are almost zero. This suggests that in modelling the equity price, the choice in the Levy model is perhaps less important than the choice in the business time.

In hindsight, the value 0.1^2 we assign to A seems to be too large. In fact, for the Meixner, Meix/CIR, NIG/CIR, VG/Γ-OU models in table 4.17, the Levy jump components are almost non-existent (i.e., the Levy processes in these four estimated models are almost identical to a Brownian motion). After finishing all our experiments, we found that if we set $A = 0.0001 = 0.01^2$, then a log-likelihood value of about 270 can be obtained for the VG/CIR model. However, since it is very time consuming to do all our experiments again[10], we accept our results of estimation here.

4.4.2 Calibration

In equations (3.8) and (3.9), we have described the (Fourier) transform-based method that Heston (1993) developed for vanilla option pricing. For each evaluation of an option with maturity T, Heston's method requires the computation of two Fourier transforms and also the expectations of $\log S_T$ under two different probability measures. This method was then improved by Carr and Madan (1998), so that for each option evaluation, only one Fourier transform is required and the expectations of $\log S_T$ are not needed. In our study, we will

[10] If we use a smaller value for A, the characteristic functions would decay slower and hence more quadrature nodes are needed to calculate the density functions. Consequently the time to finish all our experiments would become longer.

apply Carr and Madan's method to calibrate our models.

Let φ denotes the risk-neutral characteristic function of $\log S_T$. For a Levy/CIR model, for instance, when the initial state is ι_0, we have

$$\varphi(u;\iota_0) = E(e^{\varpi u S_T}|\iota_0) = e^{\varpi u(\log S_0 + \int_0^T r_s ds - qT)} E(e^{\varpi u X_T};\iota_0).$$

Therefore, by formula (4.6), we have

$$\varphi(u;\iota_0) = e^{\varpi u(\log S_0 + \int_0^T r_s ds - qT)}\ \mathbf{e}_{\iota_0}^\top\ \Phi(u;T)\ \sum_{i=1}^{n} \mathbf{e}_i$$

where $\mathbf{e}_i$ denotes the i-th unit vector in the usual orthonormal basis of $\mathbb{R}^n$. Let α be a positive number such that $E(S_T^{1+\alpha})$ exists. Such an α can be obtained by testing whether $\varphi(-\varpi(1+\alpha);\iota_0) < \infty$. In our experiments, we follow Carr and Madan's suggestions to take $\alpha = 0.75$ and find no problems. Carr and Madan (1998) show that the price of a European call option with maturity T and strike price K is given by

$$C(K,T) = \frac{\exp(-\alpha \log K)}{\pi} \int_0^\infty \frac{e^{-\varpi u \log K - \int_0^T r_s ds}\varphi(u-(\alpha+1)\varpi;\iota_0)}{\alpha^2+\alpha-u^2+\varpi(2\alpha+1)u} du. \tag{4.19}$$

In our numerical experiments, we calibrate our models in three different scenarios. In the first two scenarios, the option data are taken from Schoutens (2003, fig. 4.5 and appendix C). They consist of the closing prices of 75 vanilla S&P 500 (SPX) call options observed on April 18, 2002. The maturities of these options range from 0.088 to 1.708 years. Since the maturities are fractional, the bond yields are obtained using cubic spline interpolation. The zero yields observed for that date are given in table 4.1. The interest rate r is related to the yield rate y as

$$\int_0^t r_s ds = t \log(1+y_t).$$

The SPX closed at 1124.47 on April 18, 2002. Here we ignore the synchronisation problem in recording the index value and option prices. We follow Schoutens and set the continuous dividend yield q is set to 1.2% per year. In the first scenario, whenever we have estimated the model parameters by maximum likelihood estimation, we will retain the links between statistical parameters and risk-neutral parameters. For example, for the Meixner process, the quantity $\alpha\delta$ must be preserved, by proposition 4.3. In the second scenario, we neglect all statistical parameters and calibrate our models directly using option data. We will call the set of data in the first scenario the S&P 500 $\mathbb{P}+\mathbb{Q}$ data and the data set in the second scenario the S&P 500 $\mathbb{Q}$ data. Parameters of our models derived in these two scenarios are called S&P 500 $\mathbb{P}+\mathbb{Q}$ parameters and S&P 500 $\mathbb{Q}$ parameters respectively.

The data set for the third scenario is taken from Schoutens *et al.* (2005, table 4.1). We will call this the Eurostoxx 50 data set and the parameters obtained from this data set the Eurostoxx 50 parameters. The data set comprises 144 plain vanilla Euro Stoxx 50 index (SX5E) call options prices inferred from the implied volatility surface observed on October 7, 2003. The index value was 2461.44 when the volatility surface was observed (it is not a closing price). The maturities of these options range from 0.0361 year to 5.1639 years. Judging from the longest maturity in this data set, these options are probably traded over the counter. While such long-dated options may be thinly traded and their market prices may not reflect their true prices, they are still the most liquid options among all long-dated equity derivatives. Hence we still include them for calibration purpose here.

We take the Euro Stoxx 50 index data (SX5E) and the total return index (SX5T) data from the Dow Jones EURO STOXX 50 website (`www.stoxx.com`). Using linear regression, the (continuous) dividend yield is found to be around 2.36% over the year prior to Oct 7, 2003 and we set this as the constant dividend yield. The interest rate data for the EU 15 zone are downloaded from Eurostat's website (`epp.eurostat.ec.europa.eu`) and are given in table 4.1. The yield curve for maturities from one and seven year is constructed using cubic spline interpolation, and zero yields for maturities less than one year are extrapolated[11] using the first piece of cubic polynomial in the spline curve. Since we have no detailed historical data for the zero yields, we cannot compile enough time series data for

[11] In hindsight, we can use prime rates or LIBOR rates to obtain the zero yields for shorter maturities, but this does not resolve the problem that the yields for very short maturities are either unobservable or subject to wild fluctuation. So, for technical convenience, we opt to extrapolate the yield curve here.

U.S. Treasury yield rates on April 18, 2002										
maturity T (years)	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3	5	7	10	20
yield rate y_T (%)	1.70	1.72	1.91	2.40	3.38	3.97	4.64	5.02	5.23	5.88
Yield rates for the EU 15 zone on Oct 7, 2003										
maturity T (years)	1	2	3	4	5	6	7			
yield rate y_T (%)	2.144	2.535	2.922	3.245	3.514	3.742	3.937			

Table 4.1. Some U.S. and European yield rates.

the discounted total return process. Therefore, in this scenario, we will not estimate our models but calibrate them directly.

In each of the three scenarios, the calibrated model parameters are obtained by minimising the root mean squared error (RMSE) between the model prices and market prices:

$$\text{RMSE} = \sqrt{\sum_{\text{all options}} \frac{(\text{model price} - \text{market price})^2}{\text{number of options}}}.$$

This error metric is used in a number of research works as well. See, e.g. the list of papers cited by Bakshi *et al.* (1997, p.2017) and also the works of Jiang (2002), Schoutens (2003) and Schoutens *et al.* (2003, 2005). Alternatively one can minimise the mean squared percentage error, but that would put too much emphasis on calculating the cheaper options accurately, especially for those deeply in-the-money or out-of-money options that are thinly traded. For purpose of comparison, we follow Schoutens (2003) and calculate also some other error measures. Here we include the average absolute error (AAE) and the average relative error (ARE):

$$\text{AAE} = \frac{1}{\text{no. of options}} \sum_{\text{options}} |\text{model price} - \text{market price}|,$$

$$\text{ARE} = \frac{1}{\text{no. of options}} \sum_{\text{options}} \frac{|\text{model price} - \text{market price}|}{\text{market price}}.$$

The optimisation process for the CTRS models is very time consuming. For each data set, the search of a minimum may involve the computation of 78 or 144 option prices for tens of thousand times in the first place. In addition, the time taken for computing an option price under a CTRS model is relatively much longer than that under a Levy model. Furthermore, since ι has seven states in our setting, finding each minimum of RMSE requires running the optimisation routine seven times, each for a different initial state of ι. So, in

	S&P 500 $\mathbb{P}+\mathbb{Q}$			S&P 500 $\mathbb{Q}$			Eurostoxx 50		
	RMSE	AAE	ARE	RMSE	AAE	ARE	RMSE	AAE	ARE
Meix	1.4101	1.1266	0.0312	1.2896	1.0335	0.0316	12.5554	10.0951	0.0922
NIG	1.3087	1.0645	0.0332	1.2076	0.9641	0.0294	12.2625	9.8523	0.0905
VG	4.0594	3.3768	0.1020	1.3577	1.0967	0.0324	12.7114	10.2943	0.0950
Meix/CIR	2.4737	1.9249	0.0515	1.6231	1.3417	0.0380	2.2033	1.7397	0.0159
NIG/CIR	2.4736	1.9248	0.0515	0.3588	0.2752	0.0078	2.2113	1.7533	0.0164
VG/CIR	2.5653	1.9320	0.0506	0.5584	0.4607	0.0136	3.1357	2.6038	0.0233
Meix/Γ-OU	1.2008	0.9979	0.0300	0.7239	0.5479	0.0128	5.4327	4.3730	0.0552
NIG/Γ-OU	1.0607	0.8562	0.0252	0.7809	0.5720	0.0123	3.6617	2.8328	0.0177
VG/Γ-OU	2.5066	1.8702	0.0393	0.7250	0.5500	0.0129	4.1229	3.1924	0.0210
BS				4.9258	3.9343	0.1127	29.4545	24.7259	0.1963
SV				0.5572	0.4531	0.0153	2.2308	1.8248	0.0162
SVJ				0.4365	0.3298	0.0095	1.8378	1.4727	0.0096
Γ-OU				0.5224	0.3818	0.0097	2.5928	2.1619	0.0176

Table 4.2. Results of model calibration.

order to save time, we do not really attempt to find the real minimum, but a local minimum with a fixed initial state. In our experiments, we arbitrarily choose $\iota_0 = 4$.

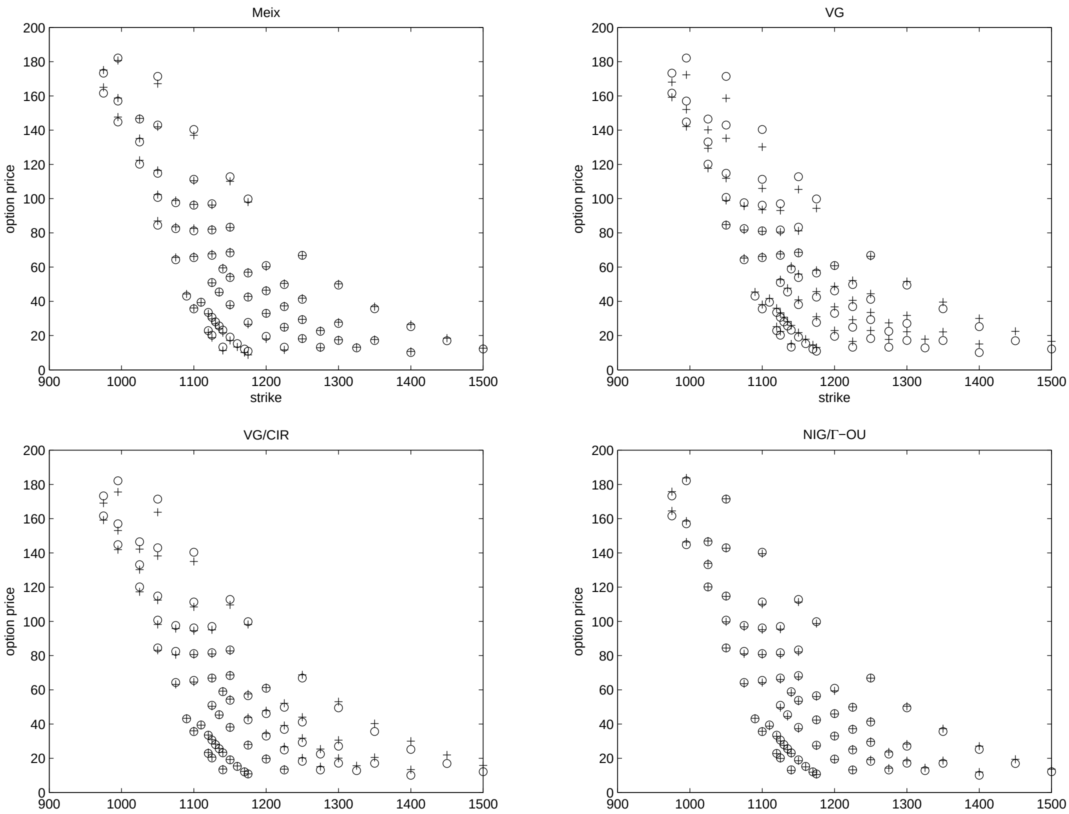

Figure 4.3. Some calibration results with S&P 500 $\mathbb{P}+\mathbb{Q}$ parameters. The circles are market prices and the crosshairs are model prices.

Parameters of the calibrated models are given in tables 4.18–4.20. Table 4.2 contains the values of the error measures. We see that when the maturities of the options are long (up to five years for the Eurostoxx 50 data set), Levy models do not calibrate well. This is probably due to the fact that volatility smiles do change over time. Moreover, since the Black-Scholes model cannot even generate a genuine volatility smile, its performance is the the worst among all models.

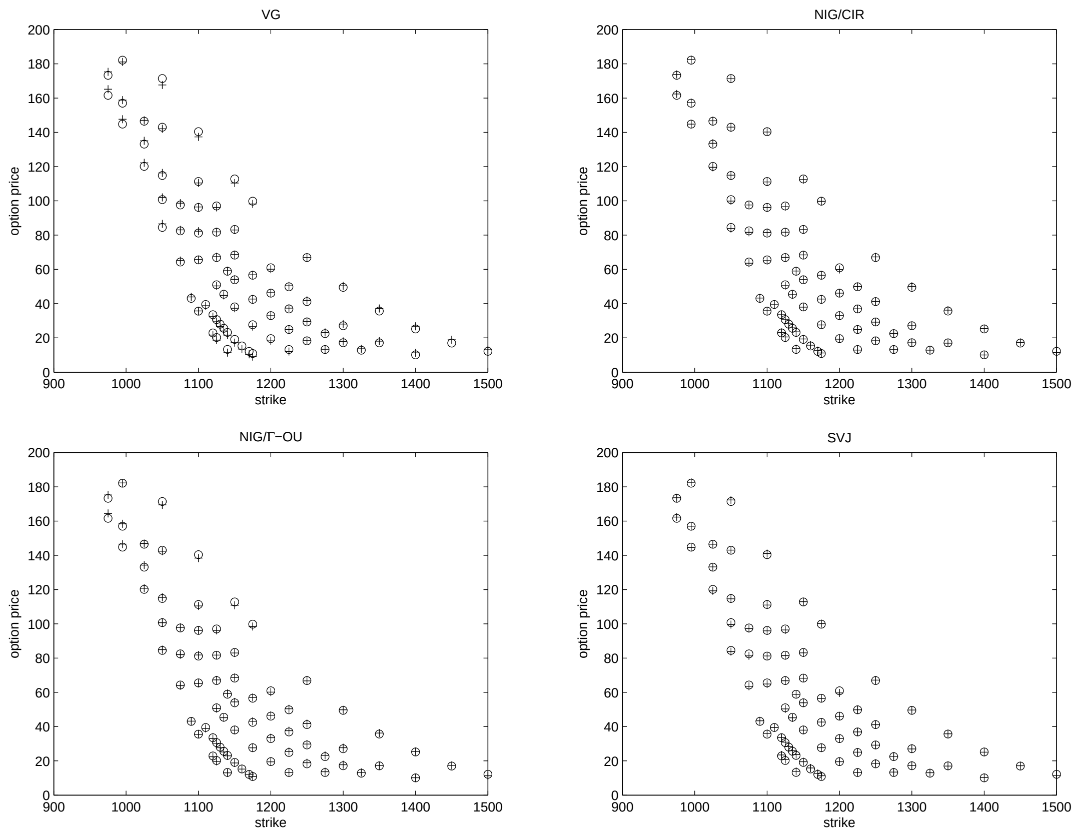

Figure 4.4. Some calibration results with S&P 500 $\mathbb{Q}$ parameters.

Overall speaking, when there is no need to maintain the links between statistical parameters and risk-neutral parameters (i.e. for S&P 500 $\mathbb{Q}$ and Eurostoxx 50 parameters), the calibration results for the three continuous-state models (SV, SVJ and Γ-OU) are always among the best. In particular, the average relative errors for the SVJ prices are less than 1%, which are practically almost perfect results. To visualise this, see figures 4.4 and 4.5. Nevertheless, CTRS models perform fairly well and in some cases, they even outperform the three continuous-state models. For instance, with S&P 500 $\mathbb{Q}$ parameters, the NIG/CIR model gives a nearly perfect calibration result: the average relative error is merely 0.78%, which is even better than the 0.95% given by the the SVJ model. See fig. 4.4 for a visualisation of some calibration results.

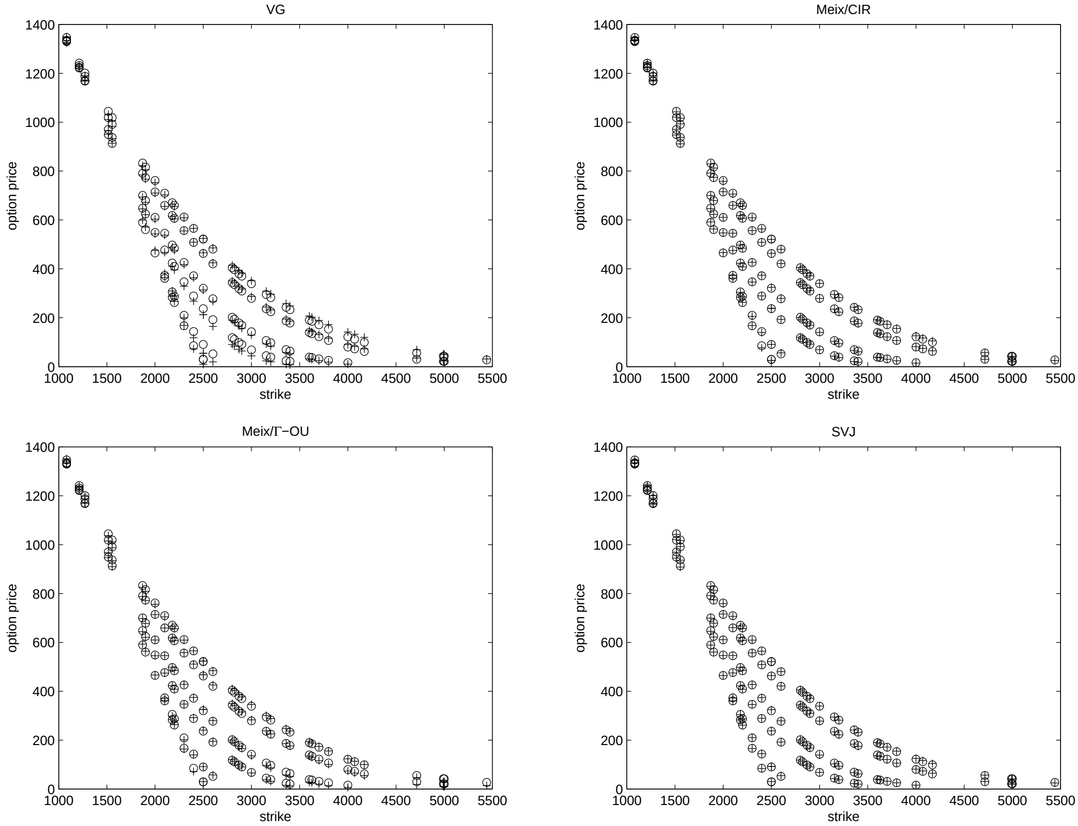

Figure 4.5. Some calibration results with Eurostoxx 50 parameters.

From table 4.2, we observe that the calibration results for S&P 500 $\mathbb{Q}$ parameters are better than those for those for S&P 500 $\mathbb{P}+\mathbb{Q}$ parameters. This is within our expectation, as the optimisation of RMSEs for S&P 500 $\mathbb{Q}$ parameters is less constrained. However, with S&P 500 $\mathbb{P}+\mathbb{Q}$ parameters, the calibration results are mediocre and the CTRS models perform worse than the three Levy models. Certainly this may indicate that our Levy models and CTRS models are not good candidates for describing the true dynamics of the S&P 500 index, but there may be other reasons behind this as well. First, this mediocrity may simply reflect the increased difficulties in doing nonlinear optimisation when there are more constraints. Second, while we view the performances of Levy and CTRS models as mediocre, it is possible that the SV, SVJ or Γ-OU may also perform not so well if the local equivalence of measures is maintained. Finally, we should not overlook the possibility that the risk-neutral dynamics in reality may be disconnected with its physical counterpart. At any rate, since the results for S&P 500 $\mathbb{P}+\mathbb{Q}$ parameters are decent enough, we will accept the use of these parameters in our experiments.

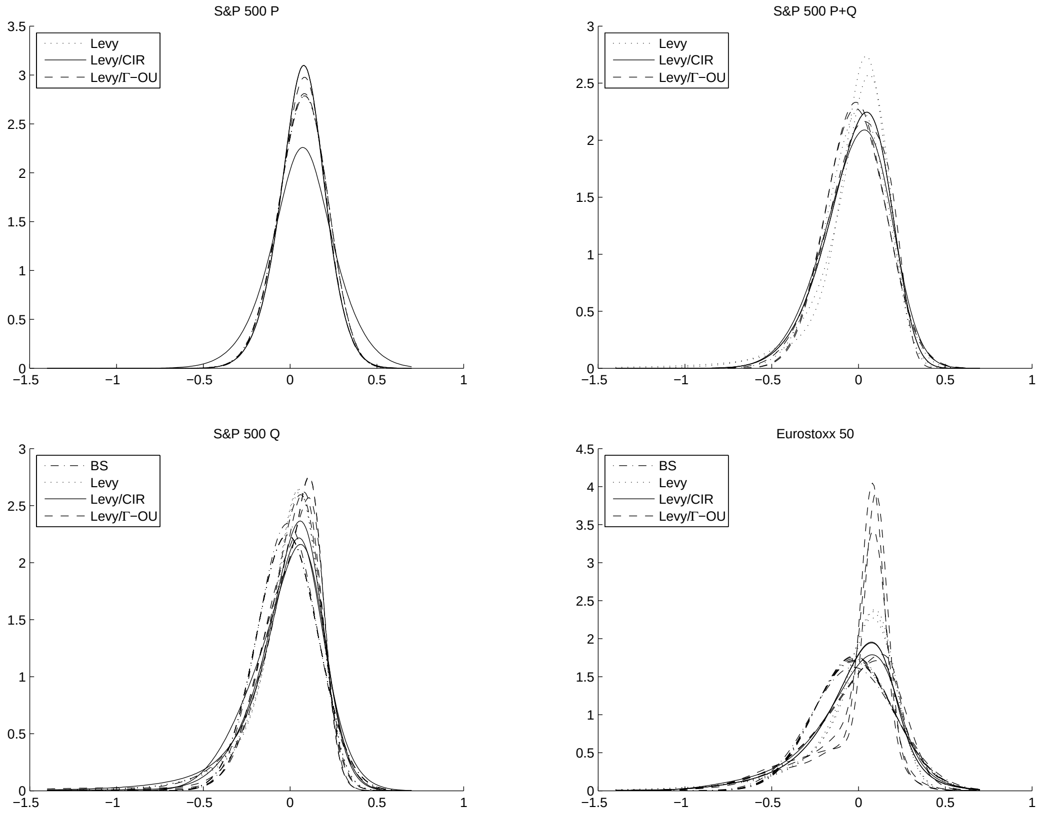

Figure 4.6. Unconditional densities of the one-year log discounted total return with (a) S&P 500 $\mathbb{P}$ parameters, (b) S&P 500 $\mathbb{P}+\mathbb{Q}$ parameters, (c) S&P 500 $\mathbb{Q}$ parameters and (d) Eurostoxx 50 parameters. For the Black-Scholes model, densities of the one-year log discounted total return for over each of the first seven years are plotted.

4.5 Option pricing

In our numerical experiments, we will calculate the prices of four kinds of EIAs (two types of PTPs and two types of CARs) under different models. We will deal with the mortality risk in the way we did in last chapter, i.e. we will calculate the price of an EIA with maturity T as a weighted sum of the prices of T Bermudan options with maturities ranging from 1 to T, where the weights are several survival probabilities.

In principle, under the SV, SVJ or BN-S-Gamma model we can evaluate the price of a Bermudan option by solving a partial (integro-)differential equation, but for simplicity, these models will be excluded from our experiment when surrenders are allowed. For the rest of our models, we will apply sequential quadrature to compute the Bermudan option prices. We shall discuss the details of this numerical procedure in part II. Here we remark that in order to use sequential quadrature, we need to know the risk-neutral densities of equity returns. In case of term-end EIAs or RHWM PTPs, the equity returns are the annual or monthly increments of X. So their densities can be obtained by applying Fourier inversion on the characteristic functions of X_1 or $X_{1/12}$. For Asian-end CARs, however, we need to know the density of the monthly average $R(t) = \frac{1}{12}\sum_{m=1}^{12} \frac{S(t-1+\frac{m}{12})}{S(t-1)}$ of stock returns in each year and so there is a slight complication.

Let $\mathbf{f}(x;t) = (f_{ij}(x;t))$ denote the $n \times n$ matrix whose (i,j)-th entry, $f_{ij}(x;t)$, is $P_{ij}(t)$ times the conditional density of $X_{s+t} - X_s$ given that $\iota_s = i$ and $\iota_{s+t} = j$. It can be computed using the Fourier inversion formula

$$\mathbf{f}(x;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\varpi ux}\Phi(u;t)du \tag{4.20}$$

(where integration is carried out entrywise) provided that the usual technical condition $\int_{-\infty}^{\infty} |\phi_{Y^{(i)}}(u)|du < \infty$ (which is satisfied in our Levy/CIR and Levy/Γ-OU models) is satisfied for all i.

Let $g_{ijk}(x)$ be the product of the transition probability $P_{ij}(1)$ with the conditional density of $R(k)$ given $\iota_{k-1} = i$ and $\iota_k = j$, and let

$$\begin{aligned}
\overline{X}_k^{(11)} &= \log \frac{S(k)}{S\left(k-1+\frac{11}{12}\right)}, \\
\overline{X}_k^{(m)} &= \log\left[\frac{11-m}{12-m}(1+\overline{X}_k^{(m+1)})\frac{S\left(k-1+\frac{m+1}{12}\right)}{S\left(k-1+\frac{m}{12}\right)}\right] \qquad (0 \le m < 11).
\end{aligned}$$

Then $\overline{X}_k^{(0)} = \log R(k)$ and the density $g_{ijk}^{(m)}(x)$ of $\overline{X}_k^{(m)}$, given $\iota(k+\frac{m}{12}) = i$ and $\iota(k+\frac{m+1}{12}) = j$, is given by the recursion formula

$$\begin{aligned}
g_{ijk}^{(11)}(x) &= f_{ij}\left(x - \int_{k-\frac{1}{12}}^{k}(r_s - q_s)ds,\ \frac{1}{12}\right), \\
g_{ijk}^{(m)}(x) &= \sum_{l=1}^{n}\int_{-\infty}^{\infty} g_{ilk}^{(m+1)}(y)f_{lj}\left(\log\left[\frac{(12-m)e^x}{(11-m)(1+e^y)}\right] - \int_{k-1+\frac{m}{12}}^{k-1+\frac{m+1}{12}}(r_s - q_s)ds,\ \frac{1}{12}\right)dy.
\end{aligned}$$

We can evaluate the density function $g_{ijk} = g_{ijk}^{(0)}$ by sequential quadrature. In actual computation, g_{ijk} is precomputed at a set of nodes. For any x that lies outside the convex hull of these nodes, we set $g_{ijk}(x) = 0$; for x inside the convex hull, we obtain $g_{ijk}(x)$ by function interpolation.

4.6 Numerical experiments

4.6.1 Experimental set-up

We will calculate the prices of those EIAs and options mentioned in sec. 4.2 under the following models:

Meix,	NIG,	VG,
Meix/CIR,	NIG/CIR,	VG/CIR,
Meix/Γ-OU,	NIG/Γ-OU,	VG/Γ-OU
BS,	SV,	SVJ,

where the model parameters were obtained in the previous section. Below are the contract parameters. We always normalise the initial stock index level so that $S_0 = 1$.

- Term-end CARs and Asian-end CARs. We set $T = 3, 7$, $f = 0$, $c = 0.1, 0.2, 0.3$, $F_\tau = 0.9(1.03)^\tau - 1$ ($\tau = 1, 2, \dots, T$) and $C_\tau = \infty$ (i.e. no global cap). The premium P_0 is assumed to be 1.

- Term-end and RHWM PTPs. We set $T = 3, 7$, $F_\tau = \max\{0, 0.9(1.03)^\tau - 1\}$ and $C_\tau = (1 + c)^\tau - 1$ for $c = 0.1, 0.2, 0.3$, so that these PTPs and the above two types of CARs have identical maximum and minimum payoffs when the contracts expire or when the policy holder surrenders. The premium P_0 is assumed to be 1.

- Cliquets. We set $T = 7$ and $t_i = i$. All other contract parameters are set to those of the PTPs.

- Barrier options. We set $T = 7$ and $K = 1$. For DIBs and DOBs, we set $H = 0.5, 0.55, 0.6, \dots, 0.95$; for UIBs and UOBs, we set $H = 1.05, 1.1, 1.15, \dots, 1.5$.

Both the Bermudan prices and no-surrender (i.e. European) prices of the CARs or PTPs will be calculated. However, for simplicity, in calculating the Bermudan prices the

SV and SVJ models will be excluded; the Bermudan EIA prices under the other models are calculated using sequential quadrature. The no-surrender prices of our EIAs and the prices of our exotic European options will be reckoned using Monte Carlo simulation. Unless otherwise specified, each Monte Carlo price is calculated using 200,000 simulation paths. Due to the difficulty in simulating the Meixner process, the Meixner model and the other two Meixner-based models CTRS models are removed from all simulation experiments. In general, regardless of the option type, when the S&P 500 $\mathbb{P}+\mathbb{Q}$ parameters are used, the BS, SV and SVJ models are excluded because we have not determined their model parameters.

The reader may note that the Γ-OU model is not included in our experiments, despite the fact that it is the inspiration of our Levy/Γ-OU models. This is because the discretisation errors in simulating the Γ-OU process are too high. Table 4.25 lists the exact no-surrender prices (calculated using the Carr-Madan formula (4.19)) of the term-end PTPs under the SV, SVJ and Γ-OU models using S&P 500 $\mathbb{Q}$ parameters. The participation rates are taken from panel B of table 4.35 and the simulation errors are listed in panel B of table 4.25. One can see that while most simulation errors under the SV or SVJ models are less than one percent of the true prices, the simulation errors for the Γ-OU model can be as large as 10%.

We will report our experimental results immediately. In order to save space and computation time, we shall will consider all three parameter sets S&P 500 $\mathbb{P}+\mathbb{Q}$, S&P 500 $\mathbb{Q}$ and Eurostoxx 50 only when term-end CARs are concerned. For each of the other contingent claims, only one set of parameters will be used.

4.6.2 Term-end CARs

Table 4.7 lists the critical participation rates for the term-end CARs under different models. We see that the critical participation rates for $T = 3$ and $T = 7$ are almost identical. This indicates that with the initial states of the economy we consider in our experiment, the policy holder is likely to surrender the ratchet within three years. We also see that the critical α remains about the same when the local cap rate c is increased from 0.2 to 0.3.

This would hardly surprise a practitioner, as many CARs in reality have local cap rates lower than 0.2.

Now, suppose $T = 3$ and $c = 0.1$. According to table 4.7, the critical participation rate of the CAR under the Meixner model with S&P 500 $\mathbb{P} + \mathbb{Q}$ parameters is 0.3137. However, if the true model is the NIG model, the true critical participation rate should be 0.3194. So, if one specifies $\alpha = 0.3137$ in the CAR contract, the ratchet would be overpriced. In general, if α is the participation rate specified in the contract and CAR(α) denotes the CAR price under the true model when the participation rate is α, the profit or loss of the issuer of the CAR, or equivalently the pricing error, is given by $1 - \text{CAR}(\alpha)$ (because the ratchet is sold at one dollar).

	$c = 0.1$		$c = 0.2$		$c = 0.3$	
	$T = 3$	$T = 7$	$T = 3$	$T = 7$	$T = 3$	$T = 7$
S&P 500 $\mathbb{P} + \mathbb{Q}$ parameters						
Meix	0.3137	0.3137	0.3062	0.3062	0.3062	0.3062
NIG	0.3194	0.3194	0.3075	0.3075	0.3066	0.3065
VG	0.3379	0.3379	0.3142	0.3142	0.3140	0.3140
Meix/CIR	0.3207	0.3207	0.3098	0.3097	0.3097	0.3097
NIG/CIR	0.3207	0.3207	0.3097	0.3097	0.3097	0.3097
VG/CIR	0.3250	0.3250	0.3106	0.3106	0.3105	0.3105
Meix/Γ-OU	0.3246	0.3246	0.3111	0.3111	0.3109	0.3109
NIG/Γ-OU	0.3245	0.3245	0.3084	0.3084	0.3062	0.3062
VG/Γ-OU	0.3203	0.3203	0.3111	0.3111	0.3111	0.3111
S&P 500 $\mathbb{Q}$ parameters						
Meix	0.3177	0.3177	0.3074	0.3074	0.3072	0.3072
NIG	0.3174	0.3174	0.3072	0.3072	0.3068	0.3068
VG	0.3164	0.3164	0.3069	0.3069	0.3068	0.3068
Meix/CIR	0.3151	0.3151	0.3076	0.3076	0.3076	0.3076
NIG/CIR	0.3093	0.3093	0.2995	0.2995	0.2956	0.2956
VG/CIR	0.3157	0.3157	0.3078	0.3078	0.3078	0.3078
Meix/Γ-OU	0.3023	0.3023	0.2883	0.2883	0.2789	0.2789
NIG/Γ-OU	0.3058	0.3058	0.2941	0.2941	0.2874	0.2874
VG/Γ-OU	0.3025	0.3025	0.2886	0.2886	0.2793	0.2793
BS	0.3591	0.3591	0.3237	0.3237	0.3229	0.3229
Eurostoxx 50 parameters						
Meix	0.2528	0.2528	0.2479	0.2479	0.2476	0.2476
NIG	0.2517	0.2517	0.2470	0.2470	0.2466	0.2466
VG	0.2494	0.2494	0.2460	0.2459	0.2458	0.2458
Meix/CIR	0.2235	0.2235	0.2145	0.2144	0.2134	0.2131
NIG/CIR	0.2235	0.2235	0.2145	0.2144	0.2134	0.2131
VG/CIR	0.2194	0.2194	0.2118	0.2118	0.2112	0.2109
Meix/Γ-OU	0.2131	0.2131	0.2077	0.2077	0.2041	0.2041
NIG/Γ-OU	0.2274	0.2274	0.2173	0.2173	0.2144	0.2144
VG/Γ-OU	0.2257	0.2257	0.2150	0.2150	0.2097	0.2097
BS	0.2502	0.2502	0.2224	0.2224	0.2214	0.2214

Table 4.7. Critical participation rates of term-end CARs under different models.

In tables 4.26, 4.27, 4.28, we give the pricing errors for the ratchets under each model but using the other models' critical participation rates. For instance, in table 4.26, we see that among the nine models we consider, the Meixner and NIG models tend to underprice our CARs, while the three Levy/CIR models and the VG/Γ-OU models are less susceptible to model risk.

As the critical participation rates subject to different models in table 4.7 are more diverse when the S&P 500 $\mathbb{Q}$ parameters are used than when the S&P 500 $\mathbb{P}+\mathbb{Q}$ parameters are used, the pricing errors become larger when the S&P 500 $\mathbb{Q}$ parameters are used. This is evident if we compare table 4.26 with table 4.27. However, since the pricing errors in term-end CARs for both parameter sets are very small, the slightly inferior performance of the S&P 500 $\mathbb{Q}$ parameters here is unimportant and coincidental.

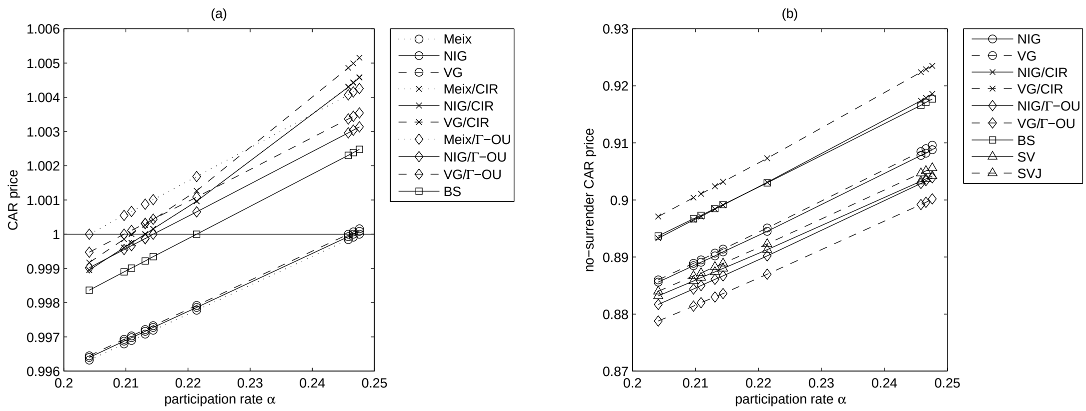

Figure 4.8. (a) Values of $100 \times (1 - \text{term-end CAR price})$ when surrender is allowed. Here $T = 7, c = 0.3$ and Eurostoxx 50 parameters are used. (b) The corresponding no-surrender prices.

In our experiments, relatively larger pricing errors are observed when $T = 7, c = 0.3$ and the Eurostoxx 50 parameters are used (see fig. 4.8(a) for a plot of them), but still the largest one of them represents just a loss of 0.52 cents for each dollar. So, apparently the pricing errors in term-end CARs are small. However, we have not yet taken into account any computation error in the CAR price. Unfortunately, since all Bermudan prices here are evaluated using a single method (sequential quadrature), we cannot verify their accuracies. Therefore, we also compute the no-surrender prices of our ratchets using both sequential quadrature and Monte Carlo simulation. These Monte Carlo no-surrender prices are given in tables 4.29, 4.30, 4.31 and the errors of the sequential quadrature prices by the Monte

Carlo prices as true prices are given in tables 4.32, 4.33, 4.34. We see that the errors in the sequential quadrature prices are fairly small — the largest of them is about 1.1 cents (under the VG/Γ-OU model with S&P 500 $\mathbb{Q}$ parameters, $T = 7$ and $c = 0.3$; see table 4.33). If we believe that the inaccuracies in Bermudan CAR prices are also about of this size, then the largest-sized pricing error for the Bermudan CARs should be at most two to three percent. So our previous conclusion that the model risk for term-end CARs is small is still justified.

In fig. 4.8(b) we plot the no-surrender prices corresponding to the CARs and parameters we have considered in fig. 4.8. The SV and SVJ models are also included in our simulation. As seen from the figure, the discrepancies between no-surrender prices are within about 2.5 cents and the inclusion of the two additional models does not bring any fundamental change to the size of the model risk.

4.6.3 Term-end PTPs, Asian-end CARs and reverse HWM PTPs

Tables 4.35 reports the critical participation rates for our term-end PTPs under each model with each set of parameters. The pricing errors obtained with S&P 500 $\mathbb{P} + \mathbb{Q}$ parameters are listed in table 4.36. The no-surrender prices are given in table 4.37. When surrender is not allowed, every European term-end PTP can be replicated by a cash amount plus a long position in a plain vanilla call option and a short position in another call. Therefore in table 4.37, the no-surrender prices are obtained by the Carr-Madan formula (4.19) but not by simulation. The inaccuracies in these no-surrender prices if sequential quadrature is used are reported in table 4.38. As seen from the table, these inaccuracies are tiny. In fact, the largest of them is merely 0.11 cent. So, we are convinced that the sequential quadrature also evaluates the Bermudan prices accurately. In view of the results in table 4.36, we conclude that the model risk for term-end PTPs are small. This is within our expectation. As the value of a Bermudan term-end PTP is largely contributed by its no-surrender price and the latter can be calculated from vanilla option prices, any equity price model that is well-calibrated to market prices of plain vanilla options should be able to generate the no-surrender price and in turn the Bermudan price accurately.

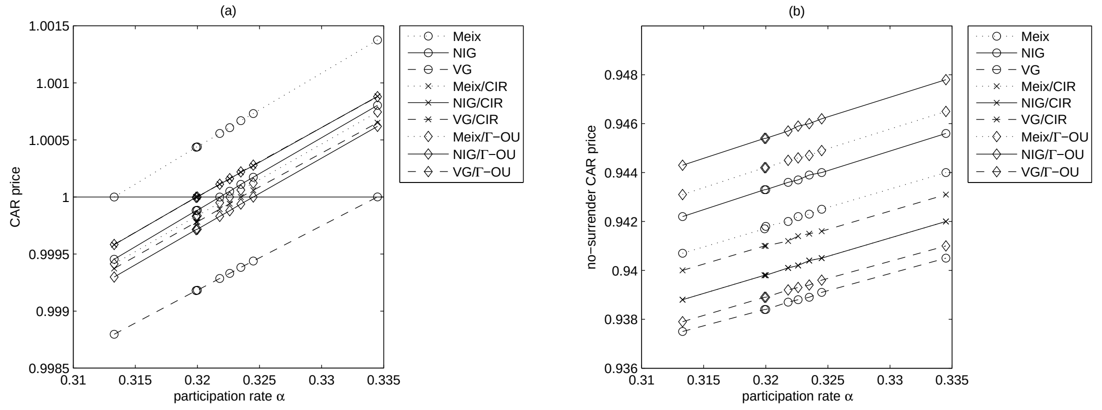

Figure 4.9. (a) Values of 100 × (1 − term-end PTP price) when surrender is allowed. Here $T = 3, c = 0.1$ and S&P 500 $\mathbb{P} + \mathbb{Q}$ parameters are used. (b) The corresponding no-surrender prices.

The critical participation rates for Asian-end CARs and RHWM PTPs and the associated experimental results for are contained in tables 4.39–4.46. The Asian-end CAR prices are reckoned using the S&P 500 $\mathbb{Q}$ parameters and the reverse HWM PTPs are calculated using the Eurostoxx 50 parameters. For Asian-end CARs, the largest pricing error is about 0.3 cent and it occurs when $c = 0.3$ and $T = 7$. Fig. 4.10 plots the discrepancies between the prices calculated with different models in this case. Since table 4.42 (with $c = 0.3$ and $T = 7$) shows that the sequential quadrature errors for the no-surrender prices are at most 1.35 cents[12], we believe that the largest sequential quadrature errors and in turn the maximum pricing errors for the Asian-end CARs are about one to two cents.

For RHWM PTPs, we met a computational problem. Although in principle we can evaluate the price of such an EIA using sequential quadrature, in reality, since the variance of the equity return $R(t)$ is much higher in the case of reverse HWM PTP than in the case of a term-end PTP, the quadrature domains for the former EIA are also much larger than those for the latter one. In effect, the valuation of reverse HWM PTP is computationally intensive and the results are usually less accurate. Therefore, in our experiment, we do

[12] One may observe that the quadrature errors here are larger than those in the term-end CAR prices. In general, although the quadrature domain for an Asian-end CAR is usually smaller than that for a term-end CAR (because average return have smaller variance than yearly return), the distribution of the average return is more sharp-peaked than that of the yearly return. Therefore more quadrature nodes are needed to evaluate the Asian-end CAR price accurately.

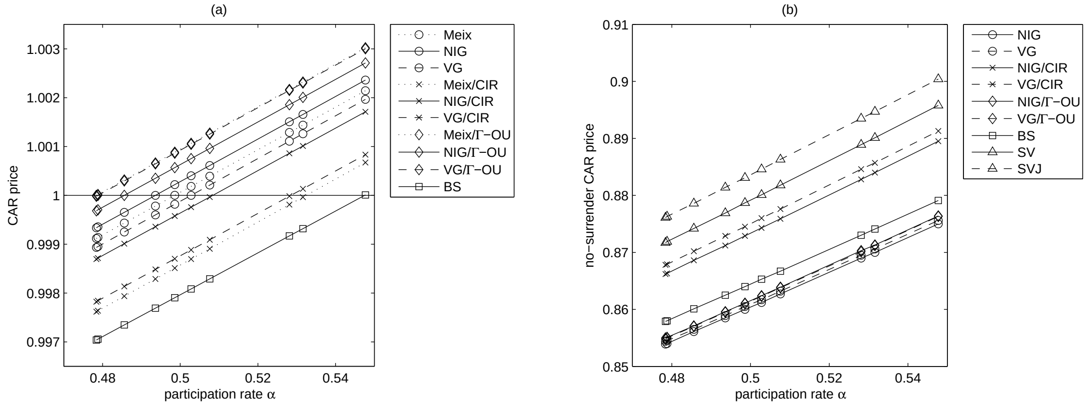

Figure 4.10. (a) Values of $100 \times (1 -$ Asian-end CAR price) when surrender is allowed. Here $T = 7, c = 0.3$ and S&P 500 $\mathbb{Q}$ parameters are used. (b) The corresponding no-surrender prices.

not calculate the critical participation rate to a high precision. As a consequence, in table 4.44, one can see that the diagonal of the pricing error matrix is nonzero. This fact is also reflected in fig. 4.11(a), where none of the nodes of the polygonal graphs lie on the horizontal (α-)axis. In all similar figures that we have presented earlier, for each graph there is exactly one node lying on the horizontal axis.

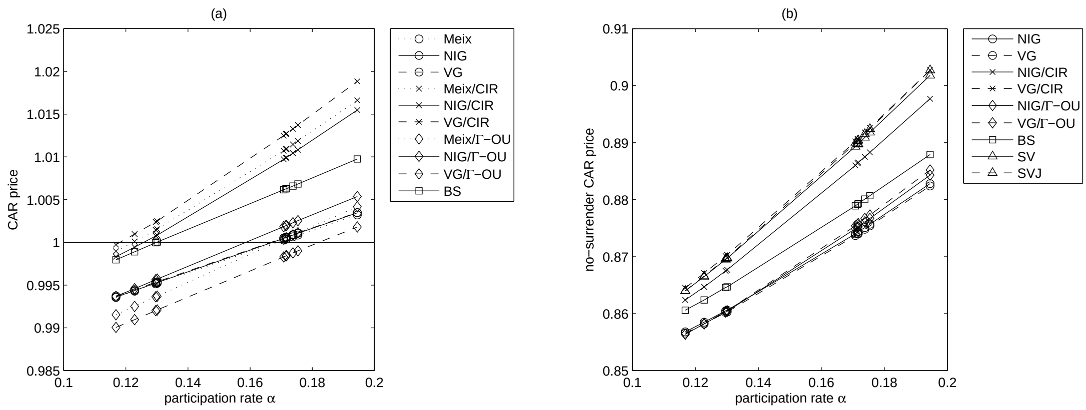

Figure 4.11. (a) Values of $100 \times (1-$reverse HWM PTP price) when surrender is allowed. Here $T = 7, c = 0.1$ and Eurostoxx 50 parameters are used. (b) The corresponding no-surrender prices.

From table 4.46, we also see that the numerical errors for the no-surrender prices for the VG/Γ-OU model are quite large (up to about 3.2 cents). In addition, except for the Black-Scholes model (whose numerical errors are negligible), the numerical errors for all other models are negative. This suggests that for those models that have the quadrature domains (not reported here) we used in our computation are perhaps not large enough. Therefore it is reasonable to expect the numerical errors in the Bermudan prices to be also negative.

If we believe that the numerical errors in Bermudan prices are of about the same sizes as the errors in the no-surrender prices, then in fig. 4.11(a), we should move the graph for the VG/Γ-OU model down by about 3.2 cents and also other graphs by different magnitudes (as dictated by the errors in table 4.46. After such adjustment, the maximum pricing error in fig. 4.11(a) would become a bit more than 3 cents. This certainly is not a very precise approximation, but it seems plausible to claim that the pricing errors for Bermudan RHWM PTPs are far below 10 cents.

Note that in our previous approximations of pricing errors for Bermudan EIAs, we have not included the SV and SVJ models (because Bermudan prices for these models have not been computed). However, since in tables 4.33, 4.34, 4.42, 4.46 and fig. 4.12(a), 4.10(b), 4.11(b) we do not observe any dramatic differences between the no-surrender prices computed by the SV or SVJ models and the prices computed by other models, we maintain our claims about the sizes of the pricing errors for each type of EIA.

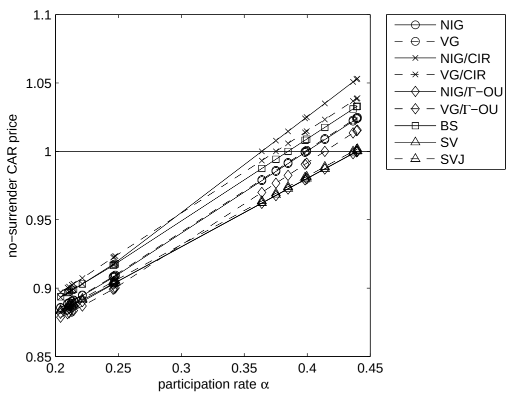

Figure 4.12. Linear extrapolation of the no-surrender CAR prices in 4.8.

Roughly speaking, at the same participation rate, the discrepancies no-surrender prices of our EIAs are larger than or only slightly smaller than those between Bermudan prices. Compare, e.g., subplots (a) and (b) in fig. 4.8, 4.9, 4.10 or 4.11. Note that the values of α at which we evaluate the no-surrender prices in these figures are the critical participation rates of the Bermudan CARs, not of the European CARs. In order to save time, we have not computed the critical participation rates for the European CARs. However, since the no-surrender prices appears to vary linearly with α, we may extrapolate each nearly straight line in the figure to obtain an approximation of the critical participation rate for the European CARs. The result of extrapolation for the term-end CAR prices, for instance, is shown in fig. 4.12(b). One can see that the pricing errors we obtain from such extrapolation are up to about five cents, which is larger than the discrepancies in the Bermudan prices. The same holds if we also extrapolate the prices in fig. 4.9(b), 4.10(b) or 4.11(b). Certainly one shall not put too much trust in such extrapolation, but it seems reasonable to believe that discrepancies between no-surrender prices of our EIAs are in general larger than those between Bermudan prices. This is because the possible payoffs in the presence of the surrender option are less diverse and the exercise prices of Bermudan CARs are model-independent.

Fig. 4.8–4.11 and tables 4.26–4.45 also show that the discrepancies between prices obtained by the three Levy models (Meixner, NIG and VG) are very small. In fact, since increments of Levy processes have stationary, independent and infinitely divisible distributions, if two Levy models have identical T-year equity return distributions at time 0, their associated Levy processes must have identical finite dimensional distributions and hence they must give identical (exotic or plain vanilla) option prices. So, if we can calibrate our Levy models to market prices of options well enough, we should expect these models to generate similar option prices. This is not necessarily true for stochastic volatility models. When two stochastic volatility models have identical plain vanilla option prices at time 0, this at best means they agree on the current distribution of equity returns. There is no reason to believe that their future equity return distributions should appear similar.

4.6.4 Cliquets and barrier options

We have seen that the pricing errors for our EIAs are small. This leads us to ask why our results have a sharp contrast with the results of Schoutens *et al.* (2003, 2005). In last chapter, we argue that the answer lies in the fact that *the nominal asset that an EIA seeks* *to protect is included in the payoff*, but this is not true for the options (barrier options, lookbacks and cliquets) considered in Schoutens *et al.* (2003, 2005).

To illustrate, consider a term-end CAR with no surrender, no global cap, no global floor and no yield spread. Again, assume the premium is one dollar. Then the payoff of this term-end CAR is given by

$$H(T) = \prod_{t=1}^{T} \text{mid}\left[1+f, 1+\alpha\left(R(t)-1\right), 1+c\right]$$

where α is the (perceived) critical participation rate. Let

$$h(t) = \text{mid}\left[0, R(t) - \left(1+\frac{f}{\alpha}\right), \frac{c-f}{\alpha}\right] = \left[R(t) - \left(1+\frac{f}{\alpha}\right)\right]^{+} - \left[R(t) - \left(1+\frac{c}{\alpha}\right)\right]^{+},$$

i.e. $h(t)$ is the payoff of a forward-start one-year call spread. Then

$$\begin{aligned} H(T) &= \prod_{t=1}^{T}[(1+f)+\alpha h(t)] \\ &= (1+f)^T \left[1 + \left(\frac{\alpha}{1+f}\right)\sum_{1\le t\le T} h(t) + \left(\frac{\alpha}{1+f}\right)^2 \sum_{1\le t_1<t_2\le T} h(t_1)h(t_2) \right. \\ &\quad \left. + \ldots + \left(\frac{\alpha}{1+f}\right)^T \prod_{t=1}^{T} h(t)\right]. \end{aligned} \tag{4.21}$$

Since $(1+f)^T$ is a constant, the pricing error for it must be zero. Hence the presence of this guaranteed minimum payoff can help lowering the overall *relative error* in pricing this ratchet.

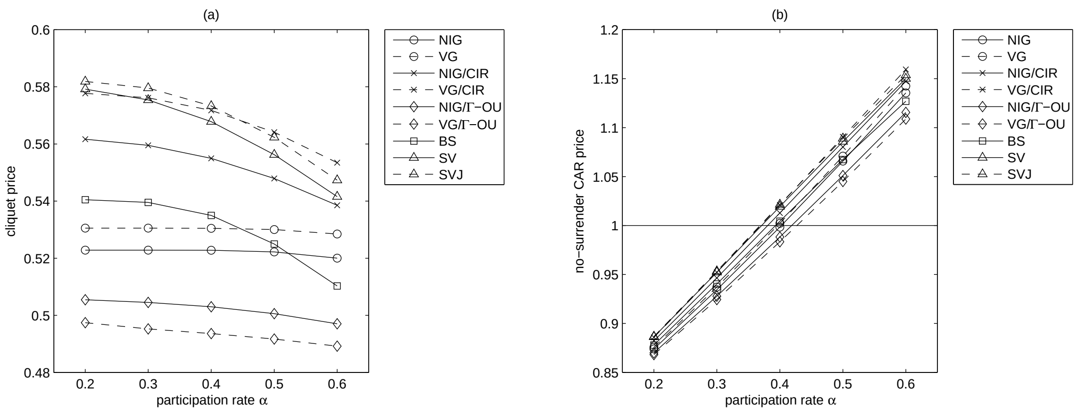

Figure 4.13. Some cliquet and term-end CAR prices calculated with Eurostoxx 50 parameters and $T = 7, f = 0, c = 0.3, F = 0, C = +\infty$. We assume there is no mortality risk.

Fig. 4.13 plots the cliquet prices $\exp(-\int_0^T r_s ds)E^{\mathbb{Q}}\left[\sum_{1\le t\le T} h(t)\right]$ and the CAR prices over a range of different participation rates and under different models when $T = 7$, $f = 0$

and $c = 0.3$. For simplicity, we assume there is no mortality risk here and the prices are calculated using only Eurostoxx 50 50 parameters with 10000 simulation paths. The detailed prices are given in table 4.14. Despite errors in the CAR prices comprise not only from errors in cliquet prices but also errors from the second and higher-order terms in (4.21), table 4.14 shows that in absolute terms, the pricing errors for cliquets (6 to 8 cents) are only slightly larger than those for CARs (2 to 5 cents). This is because each t-th order term in (4.21) is weighted by the factor $\left(\frac{\alpha}{1+f}\right)^t$, which is smaller than one. In relative terms, since the CAR price contains a component $\exp(-\int_0^T r_s ds)(1+f)^T$ that is invariant across models and large in size, the relative errors in the CAR prices (2 to 5 percent) are significantly lower than that in the cliquet prices (13 to 17 percent).

T	α	NIG	VG	NIG/CIR	VG/CIR	NIG/Γ-OU	VG/Γ-OU	BS	SV	SVJ
cliquet	0.2	0.5228	0.5305	0.5617	0.5777	0.5055	0.4974	0.5405	0.5792	0.5819
	0.3	0.5228	0.5305	0.5595	0.5761	0.5045	0.4953	0.5395	0.5754	0.5796
	0.4	0.5228	0.5305	0.5549	0.5718	0.5030	0.4936	0.5350	0.5678	0.5733
	0.5	0.5222	0.5300	0.5479	0.5641	0.5006	0.4917	0.5249	0.5563	0.5623
	0.6	0.5200	0.5285	0.5385	0.5535	0.4970	0.4892	0.5103	0.5416	0.5473
CAR	0.2	0.8741	0.8758	0.8823	0.8859	0.8701	0.8683	0.8780	0.8863	0.8867
	0.3	0.9345	0.9372	0.9466	0.9523	0.9278	0.9245	0.9404	0.9523	0.9534
	0.4	0.9985	1.0023	1.0128	1.0210	0.9883	0.9835	1.0045	1.0194	1.0215
	0.5	1.0658	1.0709	1.0802	1.0905	1.0510	1.0450	1.0674	1.0858	1.0889
	0.6	1.1354	1.1424	1.1474	1.1594	1.1156	1.1088	1.1269	1.1501	1.1537

Table 4.14. Monte Carlo prices of some cliquets and term-end CARs. Here S&P 500 $\mathbb{Q}$ parameters are used and we do not consider mortality risk. Percentagewise, the standard errors of simulation for the cliquet prices and CAR prices are respectively less than 0.61% and 0.37% (e.g. when $T = 1$ and $\alpha = 0.2$, the standard error for the cliquet under the NIG model is less than 0.0061×0.5228).

Another example for this 'principal-not-included-in-premium' effect is given table 4.47, in which we replicate the experiment of Schoutens (2005) to determine the prices of several barrier options under, except that the maturity is seven years here and we use our models with S&P 500 $\mathbb{Q}$ parameters. The 'principal' that these options seek to protect is the strike price ($K = S_0 = 1$ in our experiment), which is not included in the option premium. For each barrier level, the maximum-to-minimum ratios of the option prices obtained with our models are given in table 4.48. These ratios and the prices for DIBs are plotted in fig. 4.15. Although the maximum difference between the DIB prices here is only about 7.6 cents (which is the difference between the SVJ price and VG price for $H = 0.95$ in table 4.47), in relative term (which the pricing error is defined), the highest DIB price obtained with one model can be almost 54 times the lowest one (see table 4.48; the ratio of SVJ DIB price to VG/Γ-OU price is 54.29 when $H = 0.5$)! This is even more dramatic than the result of Schoutens *et al.* (2003, 2005), in which the DIB price obtained with one model is 'just' 22 times the price obtained with another. Note that such a high ratio occurs when the barrier level is very low ($H = 0.5$) and the option prices across all models are cheap. In this case, the standard errors of simulation are very high — in our experiments they can be as high as 10% of the option price. But even if both the highest price and the lowest price are, say, two standard errors from the true prices, the ratio between them is still $54.29 \times (1 - 0.2)/(1 + 0.2) \approx 36.2$, which is by no means a small number. However, should the 'principal' be included in the option premium (so that the option payoff is $\max(K, S(T))$ instead of $(S(T) - K)^+$ if the lower barrier H has been reached), the relative errors in all barrier option prices in table 4.47 would become only a few percent.

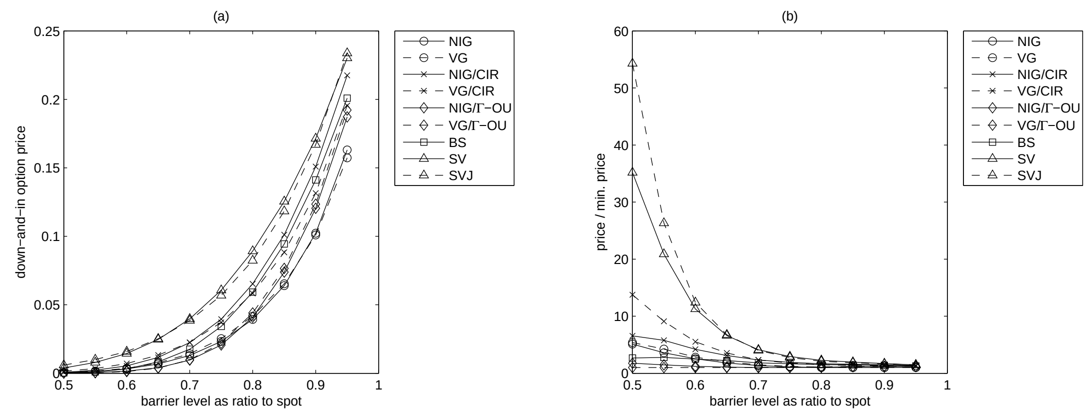

Figure 4.15. (a) DIB prices (with S&P 500 $\mathbb{Q}$ parameters). (b) Ratios of DIB prices evaluated using all models to the minimum prices among all models.

4.7 Conclusion

We have experimentally shown that the pricing errors for EIAs due to model misspecification are small. We have also explained that such model risk should not be substantial (at least far lower than that for other exotic options) because the principal of an EIA is included in the premium of the EIA. The reader, however, should bear in mind that we have adopted only one single measure of model risk here, namely, the relative error in the option price. This risk measure makes sense here because our major concern is the consequence of mispricing. Yet, in other aspects of risk management or product design, we need to consider other risk measures and we may arrive at different conclusions.

For instance, in reality it is important to determine the value-at-risk or conditional tail expectation of the liability distribution, but in our experiments we have not computed these quantities so as to save time. Also, we have not touched upon hedging errors in our experiments. Since the stock market is in practice incomplete, the hedging cost for an exotic option always carries a non-diversifiable component. The variation in such component, however, can be different for any two equity price models even if they agree on the prices of EIAs and/or the prices of plain-vanilla options. In reality, this issue is further complicated by the practice of model recalibration. Since we have not enough option price data, we do not pursue this issue in our study.

Nonetheless, the reader should be reminded that hedging errors are very different in nature from (relative) pricing errors. For example, fig. 4.16 plots the prices and maximum-to-minimum price ratios of our DOBs. As shown in this figure and fig. 4.15, the maximum pricing error for DOBs is far lower than that in DIBs. But that does not make DIBs riskier hedging instruments than DOBs are when all models are calibrated perfectly to plain-vanilla option prices. This is because the sum of a DIB and a DOB with identical maturity, strike price and barrier level is equal to the price of a plain-vanilla option. So, every hedging portfolio containing the underlying, a money market account and some DIBs will have the same pricing errors as the equivalent portfolio containing the underlying, a money market account, some plain-vanilla options and some DOBs. Roughly speaking, the hedging errors in general involves both the *plain* (as opposed to relative) pricing errors in the hedging instruments and the sensitivities of the EIA prices to the prices of the hedging instruments over the EIA's lifetime. They bear little (or no) relationship to the relative pricing errors in the hedging instruments.

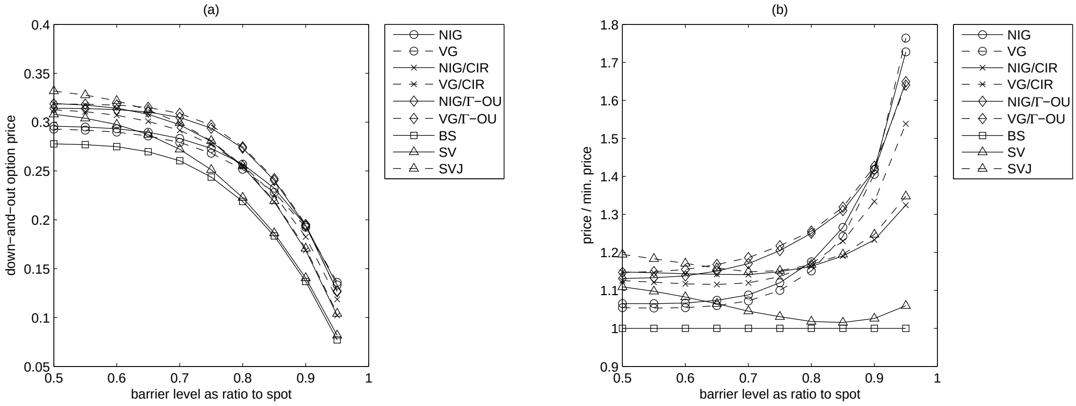

Figure 4.16. (a) DOB prices (with S&P 500 $\mathbb{Q}$ parameters). (b) Ratios of DOB prices evaluated using all models to the minimum prices among all models.

Finally, in relation to product design, it is also important to consider the discrepancies in the critical participation rates given by different models. Although the pricing errors in EIA prices are small, they form a systematic risk that cannot be diversified. Therefore, to alleviate the pricing error, one can lower the participation rate specified in the EIA contract. Unfortunately, as shown in tables 4.7, 4.35, 4.39 and 4.43, the differences between the highest value of α and the lowest one can be very large. With S&P 500 $\mathbb{Q}$ parameters, for instance, they can be up to 0.05 for term-end CARs, 0.25 for term-end PTPs, 0.07 for Asian-end CARs and 0.16 for reverse high-watermark PTPs. As a result, if we rely on adjusting the participation rate as a major tactic to reduce the volatility risk, we may have to sell the EIAs at a very unpopular low participation rate. From this perspective one can claim that the volatility risk (as measured by discrepancy in critical participation rate) for EIAs is profound.

Meixner								
log-likelihood	m	α	β	δ				
262.6508	0.4996775	3.515842×10^{-4}	-3.127717	8.4576112				
NIG								
log-likelihood	m	α	β	δ				
262.6467	0.6960995	4654.506	-4625.685	0.06988474				
VG								
log-likelihood	m	ν	σ	θ				
262.6508	0.4996519	0.05911537	1.537686×10^{-3}	-0.4285767				
Meix/CIR								
log-likelihood	m	α	β	δ	κ	η	λ	ρ
266.6849	0.04555943	7.171381×10^{-5}	7.992495×10^{-5}	1.053654×10^{-3}	0.1496271	4.973732	1.220004	-0.05049887
NIG/CIR								
log-likelihood	m	α	β	δ	κ	η	λ	ρ
266.6844	0.04495629	0.6383216	0.4819588	1.096622×10^{-6}	0.1490792	4.975700	1.218009	-0.05049782
VG/CIR								
log-likelihood	m	ν	σ	θ	κ	η	λ	ρ
265.1812	0.5000209	0.05020042	5.412863×10^{-5}	-0.4580512	1.011410×10^{-3}	287.48164	0.7625771	-0.05895189
Meix/Γ-OU								
log-likelihood	m	α	β	δ	λ	a	b	ρ
262.4856	2.174371	0.01975670	-1.833048	77.40086	26.89863	5.201181	8.474000×10^{-8}	-2.149814
NIG/Γ-OU								
log-likelihood	m	α	β	δ	λ	a	b	ρ
262.6412	1.560547	85.42745	-50.79530	1.869106	14.48220	1.65030×10^{-7}	2.344002	-6.085683
VG/Γ-OU								
log-likelihood	m	ν	σ	θ	λ	a	b	ρ
267.8209	-94.27025	1.003258×10^{-6}	0.01893145	94.35367	0.4380703	3.281281	1.211525	-0.1399573

Table 4.17. Nine estimated models for the log discounted total return process of the S&P 500 index.

Meixner							
RMSE	α	β	δ				
1.4101	6.810816×10^{-3}	-3.105724	0.4365941				
NIG							
RMSE	α	β	δ				
1.3087	7.407577	-6.082560	0.06988474				
VG							
RMSE	ν	σ	θ				
4.0594	0.05911537	1.671151×10^{-4}	-0.6044945				
Meix/CIR							
RMSE	α	β	δ	κ	η	λ	ρ
2.4737	1.379583×10^{-4}	-2.248244	5.477129×10^{-4}	2.227036	1.323325	2.427794	-0.05049887
NIG/CIR							
RMSE	α	β	δ	κ	η	λ	ρ
2.4736	0.02682220	0.02674482	1.096622×10^{-6}	2.227040	1.323280	2.427754	-0.05049782
VG/CIR							
RMSE	ν	σ	θ	κ	η	λ	ρ
2.5653	0.05020042	3.253157×10^{-5}	-0.3078786	8.187479×10^{-3}	159.4297	1.615752	-0.05895189
Meix/Γ-OU							
RMSE	α	β	δ	λ	a	b	ρ
1.1659	0.01068587	0.2326769	143.1035	6.004864×10^{-5}	2040.428	2.521318	-2.149814
NIG/Γ-OU							
RMSE	α	β	δ	λ	a	b	ρ
1.2522	6241.186	5914.039	1.869106	7.172200×10^{-6}	10136.07	5.590844	-6.085683
VG/Γ-OU							
RMSE	ν	σ	θ	λ	a	b	ρ
3.3963	1.003258×10^{-6}	0.002226980	-1.482618	4.449971	13.93517	0.03127421	-0.1399573

Table 4.18. Calibration of nine models to the S&P 500 $\mathbb{P}+\mathbb{Q}$ data set. For Levy/CIR and Levy/Γ-OU models, we set $\iota(0)=4$.

Meixner	RMSE	α	β	δ					
	1.2896	0.2542345	-2.173540	0.2361260					
NIG	RMSE	α	β	δ					
	1.2076	12.93220	-10.83855	0.08312326					
VG	RMSE	ν	σ	θ					
	1.3577	1.512398	0.1112353	-0.1170521					
Meix/CIR	RMSE	α	β	δ	κ	η	λ	ρ	
	1.6231	2.357486×10^{-4}	1.318927	181.1492	0.3898270	2.278931	1.221890	-0.1062445	
NIG/CIR	RMSE	α	β	δ	κ	η	λ	ρ	
	0.3588	9999.380	-9999.376	1.294448×10^{-4}	0.8362507	1.451250	0.9807027	-0.1125412	
VG/CIR	RMSE	ν	σ	θ	κ	η	λ	ρ	
	0.5584	6.272471	0.06621677	-0.05768710	0.2576452	1.667539	0.5051902	-0.1768512	
Meix/Γ-OU	RMSE	α	β	δ	λ	a	b	ρ	
	0.7239	0.02743529	-3.141207	3.279825×10^{-3}	0.7713004	10.11672	0.1029443	-0.2150597	
NIG/Γ-OU	RMSE	α	β	δ	λ	a	b	ρ	
	0.7809	17.38878	-17.38501	7.220126×10^{-3}	0.8218414	11.20389	0.1023007	-0.1931833	
VG/Γ-OU	RMSE	ν	σ	θ	λ	a	b	ρ	
	0.7250	147.5511	0.1932517	-0.4195801	0.7624066	10.13572	0.1029173	-0.2162819	
SV	RMSE	σ_0	κ	η	λ	ρ			
	0.5572	0.1585912	1.116216	0.05444639	0.3417730	-0.6945387			
SVJ	RMSE	σ_0	κ	η	λ	ρ	μ_J	σ_J	λ_J
	0.4365	0.1499472	0.01459178	0.5356103	0.1250240	-0.6142367	-0.6233325	0.3745147	0.02587005
BN-S-Gamma	RMSE	σ_0	λ	a	b	ρ			
	0.5224	0.1307365	1.260329	0.9817727	30.24491	-2.406227			
Black-Scholes	RMSE	$T=0.088$	$T=0.1840$	$T=0.4360$	$T=0.6920$	$T=0.9360$	$T=1.1920$	$T=1.7080$	
	4.9258	0.1542737	0.1580514	0.1708684	0.1723069	0.1692376	0.1760581	0.1777992	

Table 4.19. Calibration of thirteen models to the S&P 500 $\mathbb{Q}$ data set. For all Levy/CIR and Levy/Γ-OU models, we set $\iota(0)=4$. The numbers shown here for the Black-Scholes model are $\sqrt{\frac{1}{T}\int_0^T \sigma^2(s)ds}$ for the seven maturities T.

Meixner	RMSE	α	β	δ					
	12.5554	0.3381139	-2.332713	0.2053300					
NIG	RMSE	α	β	δ					
	12.2625	12.40196	-10.93724	0.1054963					
VG	RMSE	ν	σ	θ					
	12.7114	1.923651	0.1409525	-0.1667489					
Meix/CIR	RMSE	α	β	δ	κ	η	λ	ρ	
	2.2033	1.345887	-1.924540	0.01065796	0.3546080	0.5961431	0.6254714	-0.3135845	
NIG/CIR	RMSE	α	β	δ	κ	η	λ	ρ	
	2.2113	2.193882	-1.652319	0.02239679	0.3496299	0.5792671	0.6227605	-0.3092245	
VG/CIR	RMSE	ν	σ	θ	κ	η	λ	ρ	
	3.1357	1.076692	8.625504×10^{-4}	9.772339×10^{-4}	0.3724380	0.7624565	0.6524768	-0.3315771	
Meix/Γ-OU	RMSE	α	β	δ	λ	a	b	ρ	
	5.4327	0.01331816	-3.141492	1.448651×10^{-3}	0.8501681	17.36256	0.01933556	-0.5065685	
NIG/Γ-OU	RMSE	α	β	δ	λ	a	b	ρ	
	3.6617	1.735778	-1.553340	0.04026096	0.5781245	1.210263	0.4815865	-0.4815522	
VG/Γ-OU	RMSE	ν	σ	θ	λ	a	b	ρ	
	4.1229	98.38164	0.3565857	-0.1004087	0.5718345	9.819974	0.03772315	-0.5549661	
SV	RMSE	σ_0	κ	η	λ	ρ			
	2.2308	0.2593130	0.4750641	0.07844849	0.2730130	-0.7441461			
SVJ	RMSE	σ_0	κ	η	λ	ρ	μ_J	σ_J	λ_J
	1.8378	0.2447375	0.4842404	0.07140327	0.2629690	-0.9723238	0.1430381	3.000357×10^{-4}	0.4489377
BN-S-Gamma	RMSE	σ_0	λ	a	b	ρ			
	2.5928	0.2287317	1.130768	0.4749992	12.26941	-2.274977			
Black-Scholes	RMSE	$T = 0.0361$	$T = 0.2000$	$T = 1.1944$	$T = 2.1916$	$T = 4.2056$	$T = 5.1639$		
	29.4545	0.2539519	0.2570405	0.2434300	0.2352761	0.2319372	0.2327002		

Table 4.20. Calibration of thirteen models to the S&P 500 $\mathbb{Q}$ data set. For all Levy/CIR and Levy/Γ-OU models, we set $\iota(0) = 4$. The numbers shown here for the Black-Scholes model are $\sqrt{\frac{1}{T}\int_0^T \sigma^2(s)ds}$ for the six maturities T.

	Destination state						
	1	2	3	4	5	6	7
				Meix/CIR			
1	0.0956	0.1248	0.1532	0.1728	0.1736	0.1527	0.1274
2	0.0925	0.1209	0.1493	0.1701	0.1740	0.1574	0.1358
3	0.0853	0.1121	0.1403	0.1638	0.1746	0.1682	0.1558
4	0.0735	0.0977	0.1252	0.1529	0.1749	0.1858	0.1900
5	0.0579	0.0783	0.1046	0.1370	0.1736	0.2093	0.2393
6	0.0412	0.0574	0.0816	0.1179	0.1696	0.2343	0.2979
7	0.0291	0.0419	0.0640	0.1021	0.1642	0.2524	0.3462
equ.	0.0569	0.0768	0.1023	0.1338	0.1707	0.2107	0.2488
				NIG/CIR			
1	0.0958	0.1250	0.1534	0.1729	0.1736	0.1524	0.1269
2	0.0926	0.1211	0.1495	0.1702	0.1739	0.1572	0.1354
3	0.0854	0.1123	0.1404	0.1639	0.1746	0.1680	0.1554
4	0.0736	0.0977	0.1253	0.1530	0.1749	0.1857	0.1897
5	0.0579	0.0782	0.1046	0.1371	0.1737	0.2093	0.2392
6	0.0412	0.0573	0.0815	0.1179	0.1696	0.2344	0.2982
7	0.0290	0.0418	0.0639	0.1020	0.1641	0.2525	0.3467
equ.	0.0569	0.0768	0.1023	0.1338	0.1707	0.2107	0.2488
				VG/CIR			
1	0.1588	0.2013	0.2221	0.1996	0.1343	0.0623	0.0217
2	0.1444	0.1856	0.2117	0.2021	0.1486	0.0771	0.0305
3	0.1142	0.1518	0.1873	0.2035	0.1777	0.1116	0.0539
4	0.0736	0.1040	0.1460	0.1918	0.2111	0.1695	0.1041
5	0.0355	0.0549	0.0915	0.1515	0.2208	0.2428	0.2030
6	0.0118	0.0204	0.0413	0.0873	0.1744	0.2932	0.3714
7	0.0030	0.0058	0.0143	0.0386	0.1048	0.2670	0.5666
equ.	0.0427	0.0596	0.0831	0.1158	0.1614	0.2247	0.3127
				Meix/Γ-OU			
1	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
7	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
equ.	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
				NIG/Γ-OU			
1	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
7	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
equ.	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
				VG/Γ-OU			
1	0.4546	0.1289	0.1207	0.1091	0.0909	0.0641	0.0316
2	0.3351	0.2157	0.1301	0.1181	0.0983	0.0690	0.0337
3	0.2093	0.2379	0.1996	0.1312	0.1091	0.0761	0.0368
4	0.1186	0.2003	0.2447	0.1884	0.1225	0.0849	0.0407
5	0.0631	0.1435	0.2373	0.2454	0.1699	0.0956	0.0452
6	0.0322	0.0926	0.1949	0.2641	0.2319	0.1335	0.0508
7	0.0159	0.0553	0.1418	0.2410	0.2715	0.1955	0.0790
equ.	0.2306	0.1708	0.1791	0.1639	0.1301	0.0858	0.0397

Table 4.21. One-year transition probabilities and equilibrium distributions for ι with S&P 500 $\mathbb{P}$ parameters.

	Destination state						
	1	2	3	4	5	6	7
				Meix/CIR			
1	0.1014	0.1253	0.1480	0.1649	0.1700	0.1590	0.1315
2	0.1014	0.1252	0.1480	0.1649	0.1700	0.1590	0.1315
3	0.1013	0.1252	0.1479	0.1648	0.1701	0.1590	0.1316
4	0.1012	0.1251	0.1479	0.1648	0.1701	0.1592	0.1318
5	0.1011	0.1249	0.1477	0.1647	0.1701	0.1593	0.1320
6	0.1009	0.1248	0.1476	0.1646	0.1702	0.1596	0.1323
7	0.1008	0.1246	0.1474	0.1646	0.1702	0.1597	0.1326
equ.	0.1011	0.1250	0.1478	0.1647	0.1701	0.1593	0.1319
				NIG/CIR			
1	0.1014	0.1253	0.1480	0.1649	0.1700	0.1589	0.1315
2	0.1014	0.1252	0.1480	0.1649	0.1700	0.1590	0.1315
3	0.1013	0.1252	0.1479	0.1648	0.1701	0.1590	0.1316
4	0.1012	0.1251	0.1479	0.1648	0.1701	0.1592	0.1318
5	0.1011	0.1249	0.1477	0.1647	0.1701	0.1593	0.1320
6	0.1009	0.1248	0.1476	0.1646	0.1702	0.1596	0.1323
7	0.1008	0.1246	0.1474	0.1646	0.1702	0.1597	0.1326
equ.	0.1012	0.1250	0.1478	0.1647	0.1701	0.1593	0.1319
				VG/CIR			
1	0.0580	0.0793	0.1057	0.1361	0.1687	0.2030	0.2492
2	0.0569	0.0779	0.1041	0.1347	0.1682	0.2045	0.2537
3	0.0544	0.0747	0.1004	0.1315	0.1671	0.2081	0.2638
4	0.0503	0.0694	0.0944	0.1261	0.1652	0.2140	0.2806
5	0.0448	0.0623	0.0862	0.1187	0.1626	0.2219	0.3035
6	0.0388	0.0545	0.0773	0.1106	0.1596	0.2305	0.3287
7	0.0343	0.0487	0.0706	0.1045	0.1573	0.2369	0.3477
equ.	0.0429	0.0598	0.0834	0.1161	0.1616	0.2246	0.3116
				Meix/Γ-OU			
1	0.9143	0.0342	0.0213	0.0133	0.0083	0.0053	0.0033
2	0.0002	0.9168	0.0343	0.0214	0.0134	0.0085	0.0054
3	0.0000	0.0002	0.9214	0.0345	0.0216	0.0136	0.0086
4	0.0000	0.0000	0.0003	0.9289	0.0349	0.0220	0.0139
5	0.0000	0.0000	0.0000	0.0003	0.9416	0.0355	0.0225
6	0.0000	0.0000	0.0000	0.0000	0.0004	0.9630	0.0366
7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0005	0.9995
equ.	0.0000	0.0000	0.0000	0.0000	0.0001	0.0127	0.9873
				NIG/Γ-OU			
1	0.9613	0.0255	0.0087	0.0030	0.0010	0.0003	0.0001
2	0.0000	0.9614	0.0256	0.0087	0.0030	0.0010	0.0003
3	0.0000	0.0000	0.9617	0.0256	0.0087	0.0030	0.0010
4	0.0000	0.0000	0.0000	0.9626	0.0256	0.0088	0.0030
5	0.0000	0.0000	0.0000	0.0000	0.9653	0.0258	0.0089
6	0.0000	0.0000	0.0000	0.0000	0.0000	0.9737	0.0262
7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.9999
equ.	0.0000	0.0000	0.0000	0.0000	0.0000	0.0022	0.9978
				VG/Γ-OU			
1	0.6702	0.1143	0.0834	0.0590	0.0393	0.0233	0.0104
2	0.6702	0.1143	0.0834	0.0590	0.0393	0.0233	0.0104
3	0.6702	0.1143	0.0834	0.0590	0.0393	0.0233	0.0104
4	0.6702	0.1143	0.0834	0.0590	0.0393	0.0233	0.0104
5	0.6702	0.1143	0.0834	0.0590	0.0393	0.0233	0.0104
6	0.6702	0.1143	0.0834	0.0590	0.0393	0.0233	0.0104
7	0.6702	0.1143	0.0834	0.0590	0.0393	0.0233	0.0104
equ.	0.6702	0.1143	0.0834	0.0590	0.0393	0.0233	0.0104

Table 4.22. One-year transition probabilities and equilibrium distributions for ι with S&P 500 $\mathbb{P}+\mathbb{Q}$ parameters.

	Destination state						
	1	2	3	4	5	6	7
				Meix/CIR			
1	0.0999	0.1320	0.1619	0.1794	0.1740	0.1445	0.1082
2	0.0971	0.1285	0.1585	0.1774	0.1749	0.1489	0.1149
3	0.0905	0.1205	0.1505	0.1724	0.1766	0.1590	0.1303
4	0.0798	0.1073	0.1372	0.1639	0.1791	0.1758	0.1568
5	0.0654	0.0895	0.1188	0.1513	0.1811	0.1986	0.1952
6	0.0497	0.0696	0.0978	0.1358	0.1816	0.2241	0.2415
7	0.0376	0.0543	0.0810	0.1224	0.1802	0.2439	0.2807
equ.	0.0674	0.0916	0.1205	0.1514	0.1791	0.1960	0.1940
				NIG/CIR			
1	0.0728	0.1263	0.1947	0.2357	0.1964	0.1190	0.0551
2	0.0699	0.1218	0.1895	0.2333	0.1995	0.1254	0.0605
3	0.0638	0.1122	0.1783	0.2278	0.2058	0.1393	0.0728
4	0.0543	0.0971	0.1601	0.2174	0.2144	0.1621	0.0946
5	0.0422	0.0774	0.1349	0.1999	0.2224	0.1940	0.1292
6	0.0297	0.0565	0.1060	0.1755	0.2254	0.2308	0.1761
7	0.0204	0.0404	0.0821	0.1518	0.2224	0.2609	0.2219
equ.	0.0464	0.0838	0.1415	0.2013	0.2159	0.1859	0.1254
				VG/CIR			
1	0.1332	0.2327	0.2917	0.2299	0.0917	0.0185	0.0023
2	0.1082	0.1988	0.2786	0.2588	0.1227	0.0289	0.0042
3	0.0673	0.1383	0.2413	0.3007	0.1876	0.0551	0.0096
4	0.0291	0.0705	0.1649	0.3100	0.2865	0.1138	0.0253
5	0.0086	0.0247	0.0760	0.2116	0.3691	0.2353	0.0747
6	0.0019	0.0063	0.0244	0.0916	0.2566	0.3973	0.2219
7	0.0003	0.0013	0.0062	0.0298	0.1192	0.3244	0.5187
equ.	0.0235	0.0505	0.1017	0.1855	0.2511	0.2303	0.1575
				Meix/Γ-OU			
1	0.5175	0.1159	0.1152	0.1013	0.0774	0.0497	0.0230
2	0.4759	0.1561	0.1158	0.1018	0.0777	0.0498	0.0230
3	0.4053	0.2033	0.1378	0.1024	0.0781	0.0500	0.0231
4	0.3249	0.2326	0.1758	0.1146	0.0787	0.0503	0.0232
5	0.2484	0.2370	0.2134	0.1419	0.0854	0.0506	0.0233
6	0.1827	0.2206	0.2374	0.1777	0.1038	0.0543	0.0235
7	0.1299	0.1910	0.2425	0.2113	0.1336	0.0662	0.0254
equ.	0.4253	0.1649	0.1424	0.1127	0.0810	0.0505	0.0231
				NIG/Γ-OU			
1	0.4727	0.1257	0.1259	0.1109	0.0849	0.0546	0.0253
2	0.4418	0.1553	0.1264	0.1113	0.0851	0.0547	0.0254
3	0.3868	0.1931	0.1424	0.1120	0.0855	0.0549	0.0254
4	0.3209	0.2200	0.1718	0.1206	0.0860	0.0551	0.0255
5	0.2549	0.2283	0.2037	0.1412	0.0907	0.0554	0.0256
6	0.1952	0.2188	0.2275	0.1704	0.1043	0.0580	0.0258
7	0.1448	0.1963	0.2370	0.2005	0.1277	0.0665	0.0271
equ.	0.3936	0.1681	0.1493	0.1206	0.0878	0.0553	0.0255
				VG/Γ-OU			
1	0.5193	0.1150	0.1146	0.1010	0.0774	0.0497	0.0230
2	0.4762	0.1566	0.1152	0.1015	0.0777	0.0498	0.0231
3	0.4037	0.2048	0.1381	0.1022	0.0781	0.0500	0.0231
4	0.3217	0.2339	0.1772	0.1149	0.0786	0.0503	0.0232
5	0.2444	0.2373	0.2153	0.1433	0.0857	0.0507	0.0234
6	0.1785	0.2195	0.2389	0.1800	0.1050	0.0546	0.0235
7	0.1261	0.1888	0.2428	0.2137	0.1358	0.0671	0.0256
equ.	0.4247	0.1650	0.1425	0.1129	0.0811	0.0506	0.0232

Table 4.23. One-year transition probabilities and equilibrium distributions for ι with S&P 500 $\mathbb{Q}$ parameters.

	Destination state						
	1	2	3	4	5	6	7
				Meix/CIR			
1	0.2781	0.2794	0.2269	0.1381	0.0583	0.0161	0.0031
2	0.2569	0.2642	0.2269	0.1516	0.0724	0.0230	0.0050
3	0.2102	0.2286	0.2226	0.1793	0.1063	0.0418	0.0111
4	0.1446	0.1726	0.2026	0.2089	0.1618	0.0821	0.0274
5	0.0795	0.1073	0.1564	0.2106	0.2223	0.1552	0.0688
6	0.0339	0.0525	0.0947	0.1645	0.2390	0.2525	0.1628
7	0.0121	0.0215	0.0473	0.1030	0.1988	0.3052	0.3122
equ.	0.1790	0.1947	0.1932	0.1710	0.1314	0.0853	0.0455
				NIG/CIR			
1	0.2812	0.2803	0.2260	0.1366	0.0572	0.0157	0.0030
2	0.2598	0.2651	0.2262	0.1502	0.0713	0.0225	0.0049
3	0.2126	0.2295	0.2223	0.1783	0.1053	0.0412	0.0109
4	0.1460	0.1732	0.2027	0.2086	0.1612	0.0814	0.0270
5	0.0800	0.1074	0.1564	0.2107	0.2224	0.1548	0.0684
6	0.0339	0.0524	0.0945	0.1643	0.2392	0.2530	0.1628
7	0.0120	0.0213	0.0469	0.1023	0.1981	0.3055	0.3139
equ.	0.1817	0.1960	0.1932	0.1700	0.1301	0.0842	0.0449
				VG/CIR			
1	0.2478	0.2683	0.2344	0.1537	0.0702	0.0212	0.0044
2	0.2284	0.2527	0.2323	0.1658	0.0848	0.0291	0.0069
3	0.1868	0.2175	0.2238	0.1893	0.1185	0.0497	0.0143
4	0.1292	0.1637	0.1997	0.2121	0.1707	0.0916	0.0330
5	0.0720	0.1022	0.1524	0.2083	0.2244	0.1633	0.0774
6	0.0315	0.0508	0.0928	0.1621	0.2369	0.2542	0.1718
7	0.0116	0.0215	0.0475	0.1034	0.1990	0.3045	0.3125
equ.	0.1505	0.1768	0.1888	0.1790	0.1467	0.1012	0.0571
				Meix/Γ-OU			
1	0.7839	0.0620	0.0546	0.0430	0.0301	0.0182	0.0081
2	0.7281	0.1177	0.0546	0.0431	0.0301	0.0182	0.0081
3	0.6300	0.1905	0.0799	0.0431	0.0302	0.0182	0.0081
4	0.5150	0.2463	0.1276	0.0546	0.0302	0.0182	0.0082
5	0.4025	0.2721	0.1808	0.0828	0.0354	0.0183	0.0082
6	0.3035	0.2693	0.2237	0.1235	0.0512	0.0206	0.0082
7	0.2222	0.2462	0.2476	0.1667	0.0789	0.0291	0.0092
equ.	0.7301	0.0967	0.0684	0.0473	0.0311	0.0183	0.0082
				NIG/Γ-OU			
1	0.7925	0.0572	0.0507	0.0417	0.0306	0.0189	0.0084
2	0.6567	0.1875	0.0527	0.0433	0.0316	0.0195	0.0086
3	0.4767	0.2846	0.1305	0.0457	0.0332	0.0203	0.0089
4	0.3166	0.3025	0.2226	0.0924	0.0352	0.0214	0.0094
5	0.1976	0.2644	0.2748	0.1675	0.0633	0.0226	0.0098
6	0.1177	0.2043	0.2762	0.2320	0.1199	0.0394	0.0104
7	0.0677	0.1449	0.2417	0.2620	0.1834	0.0801	0.0201
equ.	0.6837	0.1153	0.0811	0.0553	0.0355	0.0203	0.0087
				VG/Γ-OU			
1	0.8158	0.0453	0.0436	0.0382	0.0294	0.0189	0.0087
2	0.6751	0.1857	0.0437	0.0384	0.0295	0.0190	0.0087
3	0.4879	0.2898	0.1263	0.0385	0.0296	0.0190	0.0088
4	0.3220	0.3089	0.2242	0.0872	0.0298	0.0191	0.0088
5	0.1993	0.2689	0.2790	0.1664	0.0584	0.0192	0.0088
6	0.1177	0.2062	0.2798	0.2337	0.1177	0.0361	0.0089
7	0.0669	0.1449	0.2432	0.2641	0.1837	0.0784	0.0188
equ.	0.7111	0.1026	0.0732	0.0509	0.0336	0.0198	0.0088

Table 4.24. One-year transition probabilities and equilibrium distributions for ι with Eurostoxx 50 parameters.

			(A) Exact price			
SV	0.9439	0.8446	0.9426	0.8422	0.9426	0.8422
	0.9440	0.8448	0.9427	0.8425	0.9427	0.8425
	0.9437	0.8442	0.9425	0.8420	0.9425	0.8420
	0.9435	0.8440	0.9426	0.8422	0.9426	0.8422
	0.9554	0.8647	0.9530	0.8598	0.9530	0.8598
	0.9435	0.8440	0.9426	0.8423	0.9426	0.8423
	0.9791	0.9091	0.9703	0.8898	0.9702	0.8895
	0.9650	0.8822	0.9602	0.8723	0.9602	0.8722
	0.9784	0.9076	0.9698	0.8889	0.9697	0.8887
	0.9496	0.8541	0.9453	0.8468	0.9452	0.8466
SVJ	0.9439	0.8446	0.9426	0.8422	0.9426	0.8422
	0.9440	0.8448	0.9427	0.8425	0.9427	0.8425
	0.9437	0.8442	0.9425	0.8420	0.9425	0.8420
	0.9435	0.8440	0.9426	0.8422	0.9426	0.8422
	0.9554	0.8647	0.9530	0.8598	0.9530	0.8598
	0.9435	0.8440	0.9426	0.8423	0.9426	0.8423
	0.9791	0.9091	0.9703	0.8898	0.9702	0.8895
	0.9650	0.8822	0.9602	0.8723	0.9602	0.8722
	0.9784	0.9076	0.9698	0.8889	0.9697	0.8887
	0.9496	0.8541	0.9453	0.8468	0.9452	0.8466
Γ-OU	0.9439	0.8446	0.9426	0.8422	0.9426	0.8422
	0.9440	0.8448	0.9427	0.8425	0.9427	0.8425
	0.9437	0.8442	0.9425	0.8420	0.9425	0.8420
	0.9435	0.8440	0.9426	0.8422	0.9426	0.8422
	0.9554	0.8647	0.9530	0.8598	0.9530	0.8598
	0.9435	0.8440	0.9426	0.8423	0.9426	0.8423
	0.9791	0.9091	0.9703	0.8898	0.9702	0.8895
	0.9650	0.8822	0.9602	0.8723	0.9602	0.8722
	0.9784	0.9076	0.9698	0.8889	0.9697	0.8887
	0.9496	0.8541	0.9453	0.8468	0.9452	0.8466
			(B) Monte Carlo price minus exact price			
SV	−0.0080	−0.0056	−0.0079	−0.0055	−0.0079	−0.0055
	−0.0080	−0.0056	−0.0079	−0.0055	−0.0079	−0.0055
	−0.0080	−0.0055	−0.0079	−0.0055	−0.0079	−0.0055
	−0.0080	−0.0055	−0.0079	−0.0055	−0.0079	−0.0055
	−0.0094	−0.0065	−0.0095	−0.0066	−0.0095	−0.0066
	−0.0080	−0.0055	−0.0079	−0.0055	−0.0079	−0.0055
	−0.0107	−0.0074	−0.0121	−0.0084	−0.0120	−0.0084
	−0.0103	−0.0071	−0.0106	−0.0074	−0.0105	−0.0074
	−0.0107	−0.0074	−0.0120	−0.0084	−0.0120	−0.0083
	−0.0088	−0.0061	−0.0083	−0.0058	−0.0083	−0.0058
SVJ	−0.0096	−0.0062	−0.0094	−0.0060	−0.0094	−0.0060
	−0.0096	−0.0062	−0.0094	−0.0060	−0.0094	−0.0060
	−0.0095	−0.0062	−0.0094	−0.0060	−0.0094	−0.0060
	−0.0095	−0.0062	−0.0094	−0.0060	−0.0094	−0.0060
	−0.0115	−0.0076	−0.0112	−0.0072	−0.0112	−0.0072
	−0.0095	−0.0062	−0.0094	−0.0060	−0.0094	−0.0060
	−0.0145	−0.0099	−0.0143	−0.0092	−0.0143	−0.0092
	−0.0129	−0.0087	−0.0125	−0.0081	−0.0125	−0.0080
	−0.0145	−0.0098	−0.0142	−0.0092	−0.0142	−0.0091
	−0.0106	−0.0069	−0.0099	−0.0063	−0.0098	−0.0063
Γ-OU	0.0134	0.0144	0.0405	0.0594	0.0508	0.0821
	0.0134	0.0143	0.0405	0.0594	0.0509	0.0822
	0.0135	0.0146	0.0404	0.0594	0.0507	0.0820
	0.0136	0.0147	0.0405	0.0594	0.0508	0.0821
	0.0078	0.0031	0.0422	0.0593	0.0572	0.0894
	0.0136	0.0147	0.0405	0.0594	0.0509	0.0821
	−0.0068	−0.0259	0.0414	0.0549	0.0647	0.0975
	0.0023	−0.0078	0.0423	0.0580	0.0608	0.0934
	−0.0063	−0.0249	0.0415	0.0551	0.0645	0.0974
	0.0108	0.0092	0.0411	0.0596	0.0526	0.0841

Table 4.25. Exact term-end PTP prices (calculated using the Carr-Madan formula (4.19)) and their simulation errors for the SV, SVJ and Γ-OU models using S&P 500 $\mathbb{Q}$ parameters.

	α	Meix	NIG	VG	$\frac{\text{Meix}}{\text{CIR}}$	$\frac{\text{NIG}}{\text{CIR}}$	$\frac{\text{VG}}{\text{CIR}}$	$\frac{\text{Meix}}{\Gamma\text{-OU}}$	$\frac{\text{NIG}}{\Gamma\text{-OU}}$	$\frac{\text{VG}}{\Gamma\text{-OU}}$
$c = 0.1$	0.3137	0.00	0.04	0.13	0.04	0.04	0.07	0.07	0.06	0.04
$T = 3$	0.3194	−0.04	0.00	0.10	0.01	0.01	0.03	0.03	0.03	0.01
	0.3379	−0.15	−0.11	0.00	−0.10	−0.10	−0.07	−0.08	−0.08	−0.11
	0.3207	−0.05	−0.01	0.09	0.00	0.00	0.02	0.02	0.02	0.00
	0.3207	−0.05	−0.01	0.09	0.00	0.00	0.02	0.02	0.02	0.00
	0.3250	−0.07	−0.03	0.07	−0.03	−0.03	0.00	0.00	0.00	−0.03
	0.3246	−0.07	−0.03	0.07	−0.02	−0.02	0.00	0.00	0.00	−0.03
	0.3245	−0.07	−0.03	0.07	−0.02	−0.02	0.00	0.00	0.00	−0.03
	0.3203	−0.04	−0.01	0.09	0.00	0.00	0.03	0.03	0.03	0.00
$c = 0.1$	0.3137	0.00	0.04	0.13	0.04	0.04	0.07	0.07	0.06	0.04
$T = 7$	0.3194	−0.04	0.00	0.10	0.01	0.01	0.03	0.03	0.03	0.01
	0.3379	−0.15	−0.11	0.00	−0.10	−0.10	−0.07	−0.08	−0.08	−0.11
	0.3207	−0.05	−0.01	0.09	0.00	0.00	0.02	0.02	0.02	0.00
	0.3207	−0.05	−0.01	0.09	0.00	0.00	0.02	0.02	0.02	0.00
	0.3250	−0.07	−0.03	0.07	−0.03	−0.03	0.00	0.00	0.00	−0.03
	0.3246	−0.07	−0.03	0.07	−0.02	−0.02	0.00	0.00	0.00	−0.03
	0.3245	−0.07	−0.03	0.07	−0.02	−0.02	0.00	0.00	0.00	−0.03
	0.3203	−0.04	−0.01	0.09	0.00	0.00	0.03	0.03	0.03	0.00
$c = 0.2$	0.3062	0.00	0.01	0.06	0.03	0.03	0.03	0.04	0.02	0.04
$T = 3$	0.3075	−0.01	0.00	0.05	0.02	0.02	0.02	0.03	0.01	0.03
	0.3142	−0.06	−0.05	0.00	−0.03	−0.03	−0.03	−0.02	−0.04	−0.02
	0.3098	−0.03	−0.02	0.03	0.00	0.00	0.01	0.01	−0.01	0.01
	0.3097	−0.03	−0.02	0.03	0.00	0.00	0.01	0.01	−0.01	0.01
	0.3106	−0.03	−0.02	0.03	−0.01	−0.01	0.00	0.00	−0.02	0.00
	0.3111	−0.04	−0.03	0.02	−0.01	−0.01	0.00	0.00	−0.02	0.00
	0.3084	−0.02	−0.01	0.04	0.01	0.01	0.02	0.02	0.00	0.02
	0.3111	−0.04	−0.03	0.02	−0.01	−0.01	0.00	0.00	−0.02	0.00
$c = 0.2$	0.3062	0.00	0.01	0.06	0.03	0.03	0.03	0.04	0.02	0.04
$T = 7$	0.3075	−0.01	0.00	0.05	0.02	0.02	0.02	0.03	0.01	0.03
	0.3142	−0.06	−0.05	0.00	−0.03	−0.03	−0.03	−0.02	−0.04	−0.02
	0.3097	−0.03	−0.02	0.03	0.00	0.00	0.01	0.01	−0.01	0.01
	0.3097	−0.03	−0.02	0.03	0.00	0.00	0.01	0.01	−0.01	0.01
	0.3106	−0.03	−0.02	0.03	−0.01	−0.01	0.00	0.00	−0.02	0.00
	0.3111	−0.04	−0.03	0.02	−0.01	−0.01	0.00	0.00	−0.02	0.00
	0.3084	−0.02	−0.01	0.04	0.01	0.01	0.02	0.02	0.00	0.02
	0.3111	−0.04	−0.03	0.02	−0.01	−0.01	0.00	0.00	−0.02	0.00
$c = 0.3$	0.3062	0.00	0.00	0.06	0.03	0.03	0.03	0.04	0.00	0.04
$T = 3$	0.3066	0.00	0.00	0.06	0.02	0.02	0.03	0.03	0.00	0.03
	0.3140	−0.06	−0.06	0.00	−0.03	−0.03	−0.03	−0.02	−0.06	−0.02
	0.3097	−0.03	−0.02	0.03	0.00	0.00	0.01	0.01	−0.03	0.01
	0.3097	−0.03	−0.02	0.03	0.00	0.00	0.01	0.01	−0.03	0.01
	0.3105	−0.03	−0.03	0.03	−0.01	−0.01	0.00	0.00	−0.03	0.00
	0.3109	−0.04	−0.03	0.02	−0.01	−0.01	0.00	0.00	−0.04	0.00
	0.3062	0.00	0.00	0.06	0.03	0.03	0.03	0.04	0.00	0.04
	0.3111	−0.04	−0.03	0.02	−0.01	−0.01	0.00	0.00	−0.04	0.00
$c = 0.3$	0.3062	0.00	0.00	0.06	0.03	0.03	0.03	0.04	0.00	0.04
$T = 7$	0.3065	0.00	0.00	0.06	0.02	0.02	0.03	0.03	0.00	0.03
	0.3140	−0.06	−0.06	0.00	−0.03	−0.03	−0.03	−0.02	−0.06	−0.02
	0.3097	−0.03	−0.02	0.03	0.00	0.00	0.01	0.01	−0.03	0.01
	0.3097	−0.03	−0.02	0.03	0.00	0.00	0.01	0.01	−0.03	0.01
	0.3105	−0.03	−0.03	0.03	−0.01	−0.01	0.00	0.00	−0.03	0.00
	0.3109	−0.04	−0.03	0.02	−0.01	−0.01	0.00	0.00	−0.04	0.00
	0.3062	0.00	0.00	0.06	0.03	0.03	0.03	0.04	0.00	0.04
	0.3111	−0.04	−0.03	0.02	−0.01	−0.01	0.00	0.00	−0.04	0.00

Table 4.26. Values of $100 \times (1 - \text{term-end CAR price})$ using S&P 500 $\mathbb{P} + \mathbb{Q}$ parameters.

	α	Meix	NIG	VG	$\frac{\text{Meix}}{\text{CIR}}$	$\frac{\text{NIG}}{\text{CIR}}$	$\frac{\text{VG}}{\text{CIR}}$	$\frac{\text{Meix}}{\Gamma\text{-OU}}$	$\frac{\text{NIG}}{\Gamma\text{-OU}}$	$\frac{\text{VG}}{\Gamma\text{-OU}}$	BS
$c = 0.1$	0.3177	0.00	0.00	−0.01	−0.02	−0.06	−0.01	−0.10	−0.08	−0.10	0.20
$T = 3$	0.3174	0.00	0.00	−0.01	−0.01	−0.05	−0.01	−0.10	−0.08	−0.10	0.20
	0.3164	0.01	0.01	0.00	−0.01	−0.05	0.00	−0.09	−0.07	−0.09	0.21
	0.3151	0.02	0.01	0.01	0.00	−0.04	0.00	−0.09	−0.06	−0.08	0.21
	0.3093	0.05	0.05	0.05	0.04	0.00	0.04	−0.05	−0.02	−0.05	0.24
	0.3157	0.01	0.01	0.00	0.00	−0.04	0.00	−0.09	−0.07	−0.09	0.21
	0.3023	0.10	0.10	0.09	0.08	0.05	0.09	0.00	0.02	0.00	0.28
	0.3058	0.08	0.07	0.07	0.06	0.02	0.06	−0.02	0.00	−0.02	0.26
	0.3025	0.10	0.10	0.09	0.08	0.05	0.09	0.00	0.02	0.00	0.28
	0.3591	−0.25	−0.25	−0.26	−0.27	−0.31	−0.27	−0.36	−0.33	−0.36	0.00
$c = 0.1$	0.3177	0.00	0.00	−0.01	−0.02	−0.06	−0.01	−0.10	−0.08	−0.10	0.20
$T = 7$	0.3174	0.00	0.00	−0.01	−0.01	−0.05	−0.01	−0.10	−0.08	−0.10	0.20
	0.3164	0.01	0.01	0.00	−0.01	−0.05	0.00	−0.09	−0.07	−0.09	0.21
	0.3151	0.02	0.01	0.01	0.00	−0.04	0.00	−0.09	−0.06	−0.08	0.21
	0.3093	0.05	0.05	0.05	0.04	0.00	0.04	−0.05	−0.02	−0.05	0.24
	0.3157	0.01	0.01	0.00	0.00	−0.04	0.00	−0.09	−0.07	−0.09	0.21
	0.3023	0.10	0.10	0.09	0.08	0.05	0.09	0.00	0.02	0.00	0.28
	0.3058	0.08	0.07	0.07	0.06	0.02	0.06	−0.02	0.00	−0.02	0.26
	0.3025	0.10	0.10	0.09	0.08	0.05	0.09	0.00	0.02	0.00	0.28
	0.3591	−0.25	−0.25	−0.26	−0.27	−0.31	−0.27	−0.36	−0.33	−0.36	0.00
$c = 0.2$	0.3074	0.00	0.00	0.00	0.00	−0.06	0.00	−0.15	−0.10	−0.14	0.12
$T = 3$	0.3072	0.00	0.00	0.00	0.00	−0.06	0.00	−0.14	−0.10	−0.14	0.12
	0.3069	0.00	0.00	0.00	0.01	−0.06	0.01	−0.14	−0.10	−0.14	0.12
	0.3076	0.00	0.00	−0.01	0.00	−0.06	0.00	−0.15	−0.10	−0.14	0.11
	0.2995	0.06	0.06	0.06	0.06	0.00	0.06	−0.09	−0.04	−0.08	0.17
	0.3078	0.00	0.00	−0.01	0.00	−0.06	0.00	−0.15	−0.10	−0.15	0.11
	0.2883	0.15	0.14	0.14	0.15	0.09	0.15	0.00	0.04	0.00	0.25
	0.2941	0.10	0.10	0.10	0.10	0.04	0.10	−0.04	0.00	−0.04	0.21
	0.2886	0.14	0.14	0.14	0.14	0.08	0.15	0.00	0.04	0.00	0.25
	0.3237	−0.12	−0.13	−0.13	−0.12	−0.18	−0.12	−0.27	−0.23	−0.27	0.00
$c = 0.2$	0.3074	0.00	0.00	0.00	0.00	−0.06	0.00	−0.15	−0.10	−0.14	0.12
$T = 7$	0.3072	0.00	0.00	0.00	0.00	−0.06	0.01	−0.14	−0.10	−0.14	0.12
	0.3069	0.00	0.00	0.00	0.01	−0.06	0.01	−0.14	−0.10	−0.14	0.12
	0.3076	0.00	0.00	−0.01	0.00	−0.06	0.00	−0.15	−0.10	−0.14	0.11
	0.2995	0.06	0.06	0.06	0.06	0.00	0.06	−0.08	−0.04	−0.08	0.17
	0.3078	0.00	0.00	−0.01	0.00	−0.06	0.00	−0.15	−0.10	−0.15	0.11
	0.2883	0.15	0.14	0.14	0.15	0.09	0.15	0.00	0.04	0.00	0.25
	0.2941	0.10	0.10	0.10	0.10	0.04	0.10	−0.04	0.00	−0.04	0.21
	0.2886	0.14	0.14	0.14	0.14	0.08	0.15	0.00	0.04	0.00	0.25
	0.3237	−0.12	−0.13	−0.13	−0.12	−0.18	−0.12	−0.27	−0.23	−0.27	0.00
$c = 0.3$	0.3072	0.00	0.00	0.00	0.00	−0.09	0.00	−0.22	−0.15	−0.21	0.11
$T = 3$	0.3068	0.00	0.00	0.00	0.01	−0.09	0.01	−0.21	−0.15	−0.21	0.12
	0.3068	0.00	0.00	0.00	0.01	−0.09	0.01	−0.21	−0.15	−0.21	0.12
	0.3076	0.00	−0.01	−0.01	0.00	−0.09	0.00	−0.22	−0.15	−0.22	0.11
	0.2956	0.09	0.09	0.09	0.09	0.00	0.09	−0.13	−0.06	−0.12	0.20
	0.3078	0.00	−0.01	−0.01	0.00	−0.09	0.00	−0.22	−0.16	−0.22	0.11
	0.2789	0.22	0.21	0.21	0.22	0.13	0.22	0.00	0.07	0.00	0.32
	0.2874	0.15	0.15	0.15	0.15	0.06	0.15	−0.07	0.00	−0.06	0.26
	0.2793	0.21	0.21	0.21	0.22	0.12	0.22	0.00	0.06	0.00	0.32
	0.3229	−0.12	−0.12	−0.12	−0.12	−0.21	−0.11	−0.34	−0.27	−0.33	0.00
$c = 0.3$	0.3072	0.00	0.00	0.00	0.00	−0.09	0.00	−0.22	−0.15	−0.21	0.11
$T = 7$	0.3068	0.00	0.00	0.00	0.01	−0.09	0.01	−0.21	−0.15	−0.21	0.12
	0.3068	0.00	0.00	0.00	0.01	−0.09	0.01	−0.21	−0.15	−0.21	0.12
	0.3076	0.00	−0.01	−0.01	0.00	−0.09	0.00	−0.22	−0.15	−0.22	0.11
	0.2956	0.09	0.09	0.09	0.09	0.00	0.09	−0.13	−0.06	−0.12	0.20
	0.3078	0.00	−0.01	−0.01	0.00	−0.09	0.00	−0.22	−0.16	−0.22	0.11
	0.2789	0.22	0.21	0.21	0.22	0.13	0.22	0.00	0.07	0.00	0.32
	0.2874	0.15	0.15	0.15	0.15	0.06	0.15	−0.07	0.00	−0.06	0.26
	0.2793	0.21	0.21	0.21	0.22	0.12	0.22	0.00	0.06	0.00	0.32
	0.3229	−0.12	−0.12	−0.12	−0.12	−0.21	−0.11	−0.34	−0.27	−0.33	0.00

Table 4.27. Values of $100 \times (1 -$ term-end CAR price) using S&P 500 $\mathbb{Q}$ parameters.

	α	Meix	NIG	VG	$\frac{\text{Meix}}{\text{CIR}}$	$\frac{\text{NIG}}{\text{CIR}}$	$\frac{\text{VG}}{\text{CIR}}$	$\frac{\text{Meix}}{\Gamma\text{-OU}}$	$\frac{\text{NIG}}{\Gamma\text{-OU}}$	$\frac{\text{VG}}{\Gamma\text{-OU}}$	BS
$c = 0.1$	0.2528	0.00	−0.01	−0.03	−0.24	−0.24	−0.27	−0.34	−0.19	−0.20	−0.01
$T = 3$	0.2517	0.01	0.00	−0.02	−0.23	−0.23	−0.26	−0.33	−0.18	−0.20	−0.01
	0.2494	0.03	0.02	0.00	−0.21	−0.21	−0.24	−0.31	−0.17	−0.18	0.00
	0.2235	0.23	0.22	0.21	0.00	0.00	−0.03	−0.09	0.03	0.02	0.16
	0.2235	0.23	0.22	0.21	0.00	0.00	−0.03	−0.09	0.03	0.02	0.16
	0.2194	0.26	0.25	0.24	0.03	0.03	0.00	−0.06	0.06	0.05	0.19
	0.2131	0.31	0.30	0.29	0.09	0.09	0.05	0.00	0.11	0.10	0.23
	0.2274	0.20	0.19	0.17	−0.03	−0.03	−0.07	−0.13	0.00	−0.01	0.14
	0.2257	0.21	0.20	0.19	−0.02	−0.02	−0.05	−0.11	0.01	0.00	0.15
	0.2502	0.02	0.01	−0.01	−0.22	−0.22	−0.25	−0.32	−0.17	−0.18	0.00
$c = 0.1$	0.2528	0.00	−0.01	−0.03	−0.24	−0.24	−0.27	−0.34	−0.19	−0.20	−0.01
$T = 7$	0.2517	0.01	0.00	−0.02	−0.23	−0.23	−0.26	−0.33	−0.18	−0.20	−0.01
	0.2494	0.03	0.02	0.00	−0.21	−0.21	−0.24	−0.31	−0.17	−0.18	0.00
	0.2235	0.23	0.22	0.21	0.00	0.00	−0.03	−0.09	0.03	0.02	0.16
	0.2235	0.23	0.22	0.21	0.00	0.00	−0.03	−0.09	0.03	0.02	0.16
	0.2194	0.26	0.25	0.24	0.03	0.03	0.00	−0.06	0.06	0.05	0.19
	0.2131	0.31	0.30	0.29	0.09	0.09	0.05	0.00	0.11	0.10	0.23
	0.2274	0.20	0.19	0.17	−0.03	−0.03	−0.07	−0.13	0.00	−0.01	0.14
	0.2257	0.21	0.20	0.19	−0.02	−0.02	−0.05	−0.11	0.01	0.00	0.15
	0.2502	0.02	0.01	−0.01	−0.22	−0.22	−0.25	−0.32	−0.17	−0.18	0.00
$c = 0.2$	0.2479	0.00	−0.01	−0.02	−0.40	−0.40	−0.45	−0.39	−0.28	−0.30	−0.23
$T = 3$	0.2470	0.01	0.00	−0.01	−0.38	−0.38	−0.44	−0.38	−0.28	−0.30	−0.22
	0.2460	0.02	0.01	0.00	−0.37	−0.37	−0.42	−0.37	−0.27	−0.29	−0.21
	0.2145	0.28	0.28	0.27	0.00	0.00	−0.03	−0.07	0.03	0.00	0.07
	0.2145	0.28	0.28	0.27	0.00	0.00	−0.03	−0.07	0.03	0.01	0.07
	0.2118	0.30	0.30	0.29	0.03	0.03	0.00	−0.04	0.05	0.03	0.10
	0.2077	0.34	0.33	0.33	0.08	0.07	0.05	0.00	0.09	0.07	0.14
	0.2173	0.26	0.25	0.24	−0.03	−0.03	−0.06	−0.09	0.00	−0.02	0.05
	0.2150	0.28	0.27	0.26	−0.01	−0.01	−0.04	−0.07	0.02	0.00	0.07
	0.2224	0.22	0.21	0.20	−0.09	−0.09	−0.12	−0.14	−0.05	−0.07	0.00
$c = 0.2$	0.2479	0.00	−0.01	−0.02	−0.42	−0.42	−0.48	−0.39	−0.28	−0.30	−0.23
$T = 7$	0.2470	0.01	0.00	−0.01	−0.41	−0.41	−0.47	−0.38	−0.28	−0.30	−0.22
	0.2459	0.02	0.01	0.00	−0.39	−0.39	−0.45	−0.37	−0.27	−0.29	−0.21
	0.2144	0.28	0.28	0.27	0.00	0.00	−0.03	−0.07	0.03	0.01	0.07
	0.2144	0.28	0.28	0.27	0.00	0.00	−0.03	−0.07	0.03	0.01	0.07
	0.2118	0.30	0.30	0.29	0.03	0.03	0.00	−0.04	0.05	0.03	0.10
	0.2077	0.34	0.33	0.33	0.08	0.07	0.05	0.00	0.09	0.07	0.14
	0.2173	0.26	0.25	0.24	−0.03	−0.03	−0.06	−0.09	0.00	−0.02	0.05
	0.2150	0.28	0.27	0.26	−0.01	−0.01	−0.04	−0.07	0.02	0.00	0.07
	0.2224	0.22	0.21	0.20	−0.09	−0.09	−0.12	−0.14	−0.05	−0.07	0.00
$c = 0.3$	0.2476	0.00	−0.01	−0.02	−0.43	−0.43	−0.47	−0.43	−0.31	−0.35	−0.25
$T = 3$	0.2466	0.01	0.00	−0.01	−0.41	−0.41	−0.46	−0.42	−0.30	−0.34	−0.24
	0.2458	0.02	0.01	0.00	−0.40	−0.40	−0.45	−0.41	−0.30	−0.34	−0.23
	0.2134	0.29	0.28	0.28	0.00	0.00	−0.03	−0.09	0.01	−0.03	0.08
	0.2134	0.29	0.28	0.28	0.00	0.00	−0.03	−0.09	0.01	−0.03	0.08
	0.2112	0.31	0.30	0.29	0.02	0.02	0.00	−0.07	0.03	−0.01	0.10
	0.2041	0.37	0.36	0.36	0.11	0.10	0.08	0.00	0.10	0.05	0.16
	0.2144	0.28	0.27	0.27	−0.01	−0.01	−0.04	−0.10	0.00	−0.04	0.07
	0.2097	0.32	0.31	0.31	0.04	0.04	0.02	−0.05	0.04	0.00	0.11
	0.2214	0.22	0.21	0.21	−0.09	−0.09	−0.12	−0.17	−0.07	−0.11	0.00
$c = 0.3$	0.2476	0.00	−0.01	−0.02	−0.46	−0.46	−0.52	−0.43	−0.31	−0.35	−0.25
$T = 7$	0.2466	0.01	0.00	−0.01	−0.44	−0.44	−0.50	−0.42	−0.30	−0.34	−0.24
	0.2458	0.02	0.01	0.00	−0.43	−0.43	−0.49	−0.41	−0.30	−0.34	−0.23
	0.2131	0.29	0.28	0.28	0.00	0.00	−0.03	−0.09	0.01	−0.03	0.08
	0.2131	0.29	0.28	0.28	0.00	0.00	−0.03	−0.09	0.01	−0.03	0.08
	0.2109	0.31	0.30	0.30	0.03	0.03	0.00	−0.07	0.03	−0.01	0.10
	0.2041	0.37	0.36	0.36	0.10	0.10	0.08	0.00	0.10	0.05	0.16
	0.2144	0.28	0.27	0.27	−0.02	−0.02	−0.04	−0.10	0.00	−0.04	0.07
	0.2097	0.32	0.31	0.31	0.04	0.04	0.01	−0.05	0.04	0.00	0.11
	0.2214	0.22	0.21	0.21	−0.10	−0.10	−0.13	−0.17	−0.07	−0.11	0.00

Table 4.28. Values of $100 \times (1 - \text{term-end CAR price})$ using Eurostoxx 50 parameters.

	α	NIG	VG	$\frac{\text{NIG}}{\text{CIR}}$	$\frac{\text{VG}}{\text{CIR}}$	$\frac{\text{NIG}}{\Gamma\text{-OU}}$	$\frac{\text{VG}}{\Gamma\text{-OU}}$
$c = 0.1$	0.3137	0.9628	0.9589	0.9621	0.9623	0.9613	0.9622
$T = 3$	0.3194	0.9639	0.9599	0.9632	0.9633	0.9624	0.9633
	0.3379	0.9675	0.9629	0.9666	0.9665	0.9658	0.9667
	0.3207	0.9642	0.9601	0.9634	0.9636	0.9626	0.9635
	0.3207	0.9642	0.9601	0.9634	0.9636	0.9626	0.9635
	0.3250	0.9650	0.9608	0.9642	0.9643	0.9634	0.9643
	0.3246	0.9649	0.9608	0.9641	0.9642	0.9633	0.9643
	0.3245	0.9649	0.9607	0.9641	0.9642	0.9633	0.9642
	0.3203	0.9641	0.9600	0.9633	0.9635	0.9625	0.9634
$c = 0.1$	0.3137	0.8679	0.8600	0.8666	0.8675	0.8653	0.8665
$T = 7$	0.3194	0.8703	0.8620	0.8688	0.8696	0.8675	0.8688
	0.3379	0.8779	0.8682	0.8758	0.8762	0.8747	0.8758
	0.3207	0.8708	0.8624	0.8693	0.8701	0.8680	0.8693
	0.3207	0.8708	0.8624	0.8693	0.8701	0.8680	0.8693
	0.3250	0.8726	0.8639	0.8709	0.8716	0.8697	0.8709
	0.3246	0.8724	0.8638	0.8708	0.8715	0.8696	0.8708
	0.3245	0.8724	0.8637	0.8708	0.8714	0.8695	0.8708
	0.3203	0.8706	0.8623	0.8691	0.8699	0.8679	0.8691
$c = 0.2$	0.3062	0.9639	0.9618	0.9635	0.9653	0.9629	0.9628
$T = 3$	0.3075	0.9642	0.9621	0.9638	0.9657	0.9632	0.9631
	0.3142	0.9659	0.9638	0.9654	0.9673	0.9649	0.9647
	0.3098	0.9648	0.9627	0.9643	0.9662	0.9638	0.9636
	0.3097	0.9648	0.9627	0.9643	0.9662	0.9638	0.9636
	0.3106	0.9650	0.9629	0.9645	0.9664	0.9640	0.9638
	0.3111	0.9651	0.9630	0.9647	0.9666	0.9641	0.9640
	0.3084	0.9645	0.9624	0.9640	0.9659	0.9634	0.9633
	0.3111	0.9651	0.9630	0.9647	0.9666	0.9641	0.9640
$c = 0.2$	0.3062	0.8712	0.8676	0.8709	0.8762	0.8701	0.8688
$T = 7$	0.3075	0.8719	0.8682	0.8715	0.8769	0.8708	0.8695
	0.3142	0.8755	0.8716	0.8751	0.8806	0.8743	0.8730
	0.3097	0.8731	0.8694	0.8728	0.8782	0.8720	0.8707
	0.3097	0.8731	0.8694	0.8728	0.8782	0.8720	0.8707
	0.3106	0.8736	0.8698	0.8732	0.8786	0.8724	0.8711
	0.3111	0.8739	0.8700	0.8735	0.8789	0.8727	0.8714
	0.3084	0.8724	0.8687	0.8720	0.8774	0.8713	0.8700
	0.3111	0.8739	0.8701	0.8735	0.8789	0.8727	0.8714
$c = 0.3$	0.3062	0.9640	0.9619	0.9635	0.9654	0.9629	0.9628
$T = 3$	0.3066	0.9641	0.9620	0.9635	0.9655	0.9630	0.9628
	0.3140	0.9659	0.9638	0.9654	0.9674	0.9649	0.9647
	0.3097	0.9649	0.9627	0.9643	0.9663	0.9638	0.9636
	0.3097	0.9649	0.9627	0.9643	0.9663	0.9638	0.9636
	0.3105	0.9650	0.9629	0.9645	0.9665	0.9640	0.9638
	0.3109	0.9651	0.9630	0.9646	0.9666	0.9641	0.9639
	0.3062	0.9640	0.9619	0.9635	0.9654	0.9630	0.9628
	0.3111	0.9652	0.9631	0.9647	0.9667	0.9641	0.9640
$c = 0.3$	0.3062	0.8713	0.8677	0.8709	0.8765	0.8702	0.8688
$T = 7$	0.3065	0.8716	0.8679	0.8711	0.8767	0.8704	0.8690
	0.3140	0.8756	0.8717	0.8750	0.8808	0.8744	0.8729
	0.3097	0.8733	0.8695	0.8728	0.8784	0.8721	0.8707
	0.3097	0.8733	0.8695	0.8728	0.8784	0.8721	0.8707
	0.3105	0.8737	0.8699	0.8732	0.8789	0.8725	0.8711
	0.3109	0.8739	0.8701	0.8734	0.8791	0.8727	0.8713
	0.3062	0.8714	0.8677	0.8709	0.8765	0.8702	0.8688
	0.3111	0.8740	0.8702	0.8735	0.8792	0.8728	0.8714

Table 4.29. Monte Carlo no-surrender prices for term-end CARs using S&P 500 $\mathbb{P} + \mathbb{Q}$ parameters. In percentage, the maximum standard error here is found to be about 0.02% of the price.

	α	NIG	VG	$\frac{\text{NIG}}{\text{CIR}}$	$\frac{\text{VG}}{\text{CIR}}$	$\frac{\text{NIG}}{\Gamma\text{-OU}}$	$\frac{\text{VG}}{\Gamma\text{-OU}}$	BS	SV	SVJ
$c = 0.1$	0.3177	0.9637	0.9641	0.9659	0.9653	0.9655	0.9659	0.9573	0.9440	0.9420
$T = 3$	0.3174	0.9637	0.9640	0.9659	0.9652	0.9655	0.9658	0.9573	0.9439	0.9419
	0.3164	0.9635	0.9638	0.9657	0.9650	0.9652	0.9656	0.9571	0.9438	0.9418
	0.3151	0.9632	0.9636	0.9654	0.9648	0.9650	0.9653	0.9569	0.9436	0.9417
	0.3093	0.9620	0.9624	0.9643	0.9636	0.9637	0.9640	0.9560	0.9429	0.9409
	0.3157	0.9633	0.9637	0.9655	0.9649	0.9651	0.9654	0.9570	0.9437	0.9417
	0.3023	0.9606	0.9610	0.9628	0.9622	0.9622	0.9625	0.9548	0.9419	0.9400
	0.3058	0.9613	0.9617	0.9635	0.9629	0.9629	0.9633	0.9554	0.9424	0.9405
	0.3025	0.9607	0.9610	0.9629	0.9623	0.9622	0.9625	0.9548	0.9420	0.9401
	0.3591	0.9715	0.9719	0.9737	0.9728	0.9738	0.9744	0.9635	0.9490	0.9468
$c = 0.1$	0.3177	0.8701	0.8708	0.8750	0.8762	0.8743	0.8753	0.8580	0.8591	0.8560
$T = 7$	0.3174	0.8700	0.8707	0.8749	0.8761	0.8742	0.8752	0.8579	0.8590	0.8560
	0.3164	0.8695	0.8702	0.8745	0.8757	0.8737	0.8747	0.8576	0.8587	0.8557
	0.3151	0.8690	0.8697	0.8739	0.8752	0.8731	0.8741	0.8571	0.8582	0.8553
	0.3093	0.8665	0.8672	0.8715	0.8729	0.8704	0.8713	0.8553	0.8563	0.8535
	0.3157	0.8692	0.8699	0.8742	0.8754	0.8734	0.8744	0.8573	0.8584	0.8554
	0.3023	0.8635	0.8642	0.8685	0.8700	0.8672	0.8680	0.8530	0.8539	0.8513
	0.3058	0.8650	0.8657	0.8700	0.8714	0.8688	0.8697	0.8541	0.8550	0.8524
	0.3025	0.8636	0.8643	0.8686	0.8701	0.8673	0.8681	0.8531	0.8539	0.8514
	0.3591	0.8866	0.8875	0.8915	0.8918	0.8924	0.8939	0.8704	0.8723	0.8679
$c = 0.2$	0.3074	0.9641	0.9645	0.9670	0.9675	0.9647	0.9649	0.9612	0.9470	0.9454
$T = 3$	0.3072	0.9641	0.9644	0.9670	0.9674	0.9647	0.9649	0.9612	0.9470	0.9454
	0.3069	0.9640	0.9643	0.9669	0.9673	0.9646	0.9648	0.9611	0.9470	0.9453
	0.3076	0.9642	0.9645	0.9671	0.9675	0.9648	0.9650	0.9613	0.9471	0.9455
	0.2995	0.9622	0.9625	0.9650	0.9654	0.9628	0.9629	0.9593	0.9456	0.9440
	0.3078	0.9642	0.9646	0.9672	0.9676	0.9648	0.9650	0.9613	0.9471	0.9455
	0.2883	0.9594	0.9597	0.9621	0.9626	0.9600	0.9601	0.9567	0.9436	0.9421
	0.2941	0.9608	0.9611	0.9636	0.9640	0.9614	0.9616	0.9581	0.9446	0.9431
	0.2886	0.9595	0.9598	0.9622	0.9626	0.9600	0.9602	0.9568	0.9436	0.9421
	0.3237	0.9682	0.9685	0.9712	0.9716	0.9688	0.9690	0.9650	0.9500	0.9483
$c = 0.2$	0.3074	0.8720	0.8726	0.8795	0.8862	0.8731	0.8736	0.8683	0.8696	0.8741
$T = 7$	0.3072	0.8719	0.8725	0.8794	0.8860	0.8730	0.8735	0.8682	0.8695	0.8740
	0.3069	0.8717	0.8723	0.8792	0.8859	0.8729	0.8734	0.8680	0.8693	0.8739
	0.3076	0.8721	0.8727	0.8796	0.8863	0.8732	0.8737	0.8684	0.8697	0.8742
	0.2995	0.8677	0.8683	0.8750	0.8816	0.8688	0.8693	0.8643	0.8656	0.8702
	0.3078	0.8722	0.8728	0.8797	0.8864	0.8733	0.8738	0.8685	0.8698	0.8743
	0.2883	0.8618	0.8623	0.8687	0.8751	0.8628	0.8632	0.8589	0.8601	0.8648
	0.2941	0.8649	0.8654	0.8719	0.8784	0.8659	0.8663	0.8617	0.8629	0.8676
	0.2886	0.8620	0.8625	0.8689	0.8753	0.8629	0.8634	0.8590	0.8602	0.8650
	0.3237	0.8808	0.8815	0.8888	0.8957	0.8821	0.8827	0.8764	0.8777	0.8820
$c = 0.3$	0.3072	0.9641	0.9644	0.9670	0.9676	0.9647	0.9649	0.9614	0.9474	0.9459
$T = 3$	0.3068	0.9640	0.9643	0.9669	0.9675	0.9646	0.9648	0.9613	0.9473	0.9458
	0.3068	0.9640	0.9643	0.9669	0.9675	0.9646	0.9648	0.9613	0.9473	0.9458
	0.3076	0.9642	0.9645	0.9672	0.9677	0.9648	0.9650	0.9615	0.9475	0.9460
	0.2956	0.9612	0.9615	0.9640	0.9645	0.9618	0.9619	0.9586	0.9452	0.9437
	0.3078	0.9642	0.9646	0.9672	0.9677	0.9648	0.9650	0.9615	0.9475	0.9460
	0.2789	0.9571	0.9574	0.9597	0.9602	0.9576	0.9578	0.9546	0.9420	0.9407
	0.2874	0.9592	0.9595	0.9619	0.9624	0.9597	0.9599	0.9566	0.9437	0.9423
	0.2793	0.9572	0.9575	0.9599	0.9603	0.9577	0.9579	0.9547	0.9421	0.9408
	0.3229	0.9680	0.9683	0.9711	0.9716	0.9686	0.9688	0.9651	0.9504	0.9488
$c = 0.3$	0.3072	0.8719	0.8725	0.8795	0.8868	0.8730	0.8735	0.8687	0.8709	0.8782
$T = 7$	0.3068	0.8717	0.8723	0.8793	0.8866	0.8728	0.8733	0.8685	0.8706	0.8780
	0.3068	0.8717	0.8723	0.8793	0.8866	0.8728	0.8733	0.8685	0.8706	0.8780
	0.3076	0.8722	0.8728	0.8798	0.8871	0.8732	0.8737	0.8689	0.8711	0.8784
	0.2956	0.8657	0.8662	0.8729	0.8799	0.8667	0.8671	0.8628	0.8648	0.8720
	0.3078	0.8723	0.8728	0.8798	0.8872	0.8733	0.8738	0.8690	0.8712	0.8785
	0.2789	0.8568	0.8574	0.8635	0.8701	0.8577	0.8581	0.8545	0.8563	0.8632
	0.2874	0.8613	0.8619	0.8683	0.8751	0.8623	0.8627	0.8587	0.8606	0.8677
	0.2793	0.8571	0.8576	0.8638	0.8703	0.8580	0.8584	0.8547	0.8565	0.8634
	0.3229	0.8805	0.8811	0.8886	0.8963	0.8817	0.8822	0.8768	0.8791	0.8866

Table 4.30. Monte Carlo no-surrender prices for term-end CARs using S&P 500 $\mathbb{Q}$ parameters. In percentage, the maximum standard error here is found to be about 0.02% of the price.

	α	NIG	VG	$\frac{\text{NIG}}{\text{CIR}}$	$\frac{\text{VG}}{\text{CIR}}$	$\frac{\text{NIG}}{\Gamma\text{-OU}}$	$\frac{\text{VG}}{\Gamma\text{-OU}}$	BS	SV	SVJ
$c = 0.1$	0.2528	0.9809	0.9819	0.9835	0.9850	0.9805	0.9795	0.9783	0.9607	0.9615
$T = 3$	0.2517	0.9807	0.9817	0.9833	0.9847	0.9803	0.9792	0.9781	0.9605	0.9614
	0.2494	0.9801	0.9811	0.9827	0.9842	0.9798	0.9787	0.9777	0.9602	0.9611
	0.2235	0.9740	0.9747	0.9769	0.9784	0.9738	0.9728	0.9729	0.9567	0.9575
	0.2235	0.9740	0.9747	0.9769	0.9784	0.9738	0.9728	0.9729	0.9567	0.9575
	0.2194	0.9730	0.9737	0.9760	0.9774	0.9728	0.9718	0.9721	0.9561	0.9569
	0.2131	0.9714	0.9721	0.9745	0.9759	0.9713	0.9704	0.9709	0.9551	0.9560
	0.2274	0.9749	0.9757	0.9778	0.9793	0.9747	0.9737	0.9737	0.9572	0.9581
	0.2257	0.9745	0.9753	0.9774	0.9789	0.9743	0.9733	0.9734	0.9570	0.9578
	0.2502	0.9803	0.9813	0.9829	0.9844	0.9799	0.9789	0.9779	0.9603	0.9612
$c = 0.1$	0.2528	0.9084	0.9097	0.9095	0.9133	0.9018	0.8989	0.9039	0.8919	0.8933
$T = 7$	0.2517	0.9079	0.9092	0.9091	0.9128	0.9013	0.8984	0.9035	0.8915	0.8930
	0.2494	0.9067	0.9080	0.9081	0.9118	0.9002	0.8973	0.9027	0.8908	0.8922
	0.2235	0.8942	0.8950	0.8966	0.9001	0.8883	0.8854	0.8936	0.8822	0.8836
	0.2235	0.8942	0.8950	0.8966	0.9001	0.8883	0.8854	0.8936	0.8822	0.8836
	0.2194	0.8923	0.8930	0.8947	0.8982	0.8864	0.8836	0.8922	0.8808	0.8822
	0.2131	0.8892	0.8898	0.8918	0.8953	0.8835	0.8807	0.8898	0.8787	0.8800
	0.2274	0.8961	0.8970	0.8983	0.9019	0.8901	0.8872	0.8950	0.8835	0.8849
	0.2257	0.8953	0.8961	0.8975	0.9011	0.8893	0.8864	0.8944	0.8829	0.8843
	0.2502	0.9072	0.9084	0.9084	0.9122	0.9006	0.8977	0.9030	0.8910	0.8925
$c = 0.2$	0.2479	0.9809	0.9817	0.9865	0.9883	0.9816	0.9802	0.9838	0.9639	0.9649
$T = 3$	0.2470	0.9807	0.9814	0.9862	0.9880	0.9814	0.9800	0.9835	0.9638	0.9647
	0.2460	0.9804	0.9811	0.9859	0.9877	0.9811	0.9797	0.9832	0.9636	0.9645
	0.2145	0.9722	0.9728	0.9772	0.9787	0.9729	0.9717	0.9748	0.9578	0.9586
	0.2145	0.9721	0.9728	0.9771	0.9787	0.9729	0.9717	0.9748	0.9578	0.9586
	0.2118	0.9715	0.9721	0.9764	0.9779	0.9722	0.9710	0.9741	0.9573	0.9581
	0.2077	0.9704	0.9710	0.9753	0.9767	0.9711	0.9700	0.9730	0.9565	0.9573
	0.2173	0.9729	0.9736	0.9780	0.9795	0.9736	0.9724	0.9756	0.9583	0.9591
	0.2150	0.9723	0.9729	0.9773	0.9788	0.9730	0.9718	0.9750	0.9579	0.9587
	0.2224	0.9742	0.9749	0.9794	0.9810	0.9750	0.9737	0.9769	0.9593	0.9601
$c = 0.2$	0.2479	0.9090	0.9097	0.9175	0.9225	0.9035	0.8997	0.9169	0.9025	0.9043
$T = 7$	0.2470	0.9085	0.9092	0.9170	0.9220	0.9030	0.8992	0.9164	0.9021	0.9039
	0.2459	0.9079	0.9086	0.9164	0.9213	0.9025	0.8987	0.9158	0.9016	0.9034
	0.2144	0.8909	0.8914	0.8985	0.9026	0.8864	0.8831	0.8988	0.8870	0.8883
	0.2144	0.8909	0.8913	0.8985	0.9026	0.8864	0.8831	0.8988	0.8870	0.8882
	0.2118	0.8896	0.8900	0.8971	0.9011	0.8851	0.8819	0.8975	0.8858	0.8870
	0.2077	0.8874	0.8878	0.8948	0.8987	0.8831	0.8799	0.8953	0.8840	0.8851
	0.2173	0.8924	0.8929	0.9001	0.9043	0.8878	0.8845	0.9004	0.8883	0.8896
	0.2150	0.8912	0.8917	0.8988	0.9030	0.8867	0.8834	0.8992	0.8873	0.8885
	0.2224	0.8951	0.8956	0.9029	0.9073	0.8903	0.8869	0.9031	0.8906	0.8920
$c = 0.3$	0.2476	0.9808	0.9816	0.9868	0.9886	0.9818	0.9805	0.9841	0.9644	0.9652
$T = 3$	0.2466	0.9806	0.9813	0.9865	0.9882	0.9816	0.9802	0.9838	0.9642	0.9650
	0.2458	0.9803	0.9811	0.9863	0.9880	0.9814	0.9800	0.9836	0.9640	0.9649
	0.2134	0.9719	0.9725	0.9771	0.9785	0.9728	0.9717	0.9747	0.9579	0.9586
	0.2134	0.9719	0.9725	0.9771	0.9785	0.9728	0.9717	0.9747	0.9579	0.9586
	0.2112	0.9713	0.9719	0.9764	0.9779	0.9722	0.9711	0.9741	0.9574	0.9581
	0.2041	0.9694	0.9701	0.9744	0.9758	0.9703	0.9693	0.9721	0.9561	0.9568
	0.2144	0.9721	0.9728	0.9774	0.9788	0.9731	0.9720	0.9750	0.9581	0.9588
	0.2097	0.9709	0.9716	0.9760	0.9775	0.9718	0.9707	0.9737	0.9572	0.9578
	0.2214	0.9740	0.9746	0.9793	0.9809	0.9749	0.9737	0.9769	0.9594	0.9601
$c = 0.3$	0.2476	0.9088	0.9096	0.9186	0.9235	0.9039	0.9002	0.9177	0.9042	0.9056
$T = 7$	0.2466	0.9082	0.9090	0.9179	0.9229	0.9034	0.8996	0.9171	0.9037	0.9051
	0.2458	0.9078	0.9085	0.9174	0.9224	0.9029	0.8992	0.9166	0.9033	0.9047
	0.2131	0.8902	0.8907	0.8984	0.9024	0.8861	0.8830	0.8985	0.8874	0.8883
	0.2131	0.8902	0.8907	0.8984	0.9024	0.8861	0.8830	0.8985	0.8874	0.8883
	0.2109	0.8891	0.8895	0.8971	0.9011	0.8850	0.8820	0.8973	0.8864	0.8872
	0.2041	0.8856	0.8860	0.8933	0.8971	0.8817	0.8788	0.8937	0.8832	0.8840
	0.2144	0.8909	0.8914	0.8991	0.9032	0.8867	0.8836	0.8992	0.8880	0.8889
	0.2097	0.8885	0.8889	0.8965	0.9004	0.8844	0.8814	0.8967	0.8858	0.8867
	0.2214	0.8945	0.8951	0.9031	0.9073	0.8902	0.8870	0.9030	0.8913	0.8923

Table 4.31. Monte Carlo no-surrender prices for term-end CARs using Eurostoxx 50 parameters. In percentage, the maximum standard error here is found to be about 0.02% of the price.

	α	NIG	VG	$\frac{\text{NIG}}{\text{CIR}}$	$\frac{\text{VG}}{\text{CIR}}$	$\frac{\text{NIG}}{\Gamma\text{-OU}}$	$\frac{\text{VG}}{\Gamma\text{-OU}}$
$c = 0.1$	0.3137	−0.0000	−0.0001	0.0001	0.0001	0.0005	0.0000
$T = 3$	0.3194	−0.0000	−0.0001	0.0001	0.0000	0.0005	0.0000
	0.3379	−0.0000	−0.0001	0.0000	0.0000	0.0005	0.0001
	0.3207	−0.0000	−0.0001	0.0001	0.0000	0.0005	0.0000
	0.3207	−0.0000	−0.0001	0.0001	0.0000	0.0005	0.0000
	0.3250	−0.0000	−0.0001	0.0001	0.0000	0.0005	0.0000
	0.3246	−0.0000	−0.0001	0.0001	0.0000	0.0005	0.0000
	0.3245	−0.0000	−0.0001	0.0001	0.0000	0.0005	0.0000
	0.3203	−0.0000	−0.0001	0.0001	0.0000	0.0005	0.0000
$c = 0.1$	0.3137	0.0001	−0.0002	0.0000	0.0001	0.0007	0.0003
$T = 7$	0.3194	0.0001	−0.0002	0.0000	0.0001	0.0007	0.0004
	0.3379	0.0001	−0.0002	−0.0000	0.0000	0.0007	0.0005
	0.3207	0.0001	−0.0002	0.0000	0.0001	0.0007	0.0004
	0.3207	0.0001	−0.0002	0.0000	0.0001	0.0007	0.0004
	0.3250	0.0001	−0.0002	0.0000	0.0001	0.0007	0.0004
	0.3246	0.0001	−0.0002	0.0000	0.0001	0.0007	0.0004
	0.3245	0.0001	−0.0002	0.0000	0.0001	0.0007	0.0004
	0.3203	0.0001	−0.0002	0.0000	0.0001	0.0007	0.0004
$c = 0.2$	0.3062	0.0001	−0.0001	0.0001	0.0003	0.0009	−0.0002
$T = 3$	0.3075	−0.0011	−0.0009	0.0001	−0.0000	−0.0013	−0.0003
	0.3142	−0.0012	−0.0011	0.0000	−0.0001	−0.0014	−0.0003
	0.3098	0.0001	−0.0001	0.0001	0.0003	0.0009	−0.0002
	0.3097	0.0001	−0.0001	0.0001	0.0003	0.0009	−0.0002
	0.3106	0.0001	−0.0001	0.0001	0.0003	0.0009	−0.0002
	0.3111	0.0001	−0.0001	0.0001	0.0003	0.0009	−0.0002
	0.3084	−0.0011	−0.0009	0.0001	−0.0000	−0.0013	−0.0003
	0.3111	−0.0011	−0.0010	0.0000	−0.0001	−0.0014	−0.0003
$c = 0.2$	0.3062	−0.0004	−0.0006	0.0004	0.0009	−0.0002	−0.0002
$T = 7$	0.3075	−0.0004	−0.0007	0.0004	0.0009	−0.0002	−0.0002
	0.3142	0.0007	0.0000	0.0004	0.0012	0.0018	−0.0002
	0.3097	0.0007	0.0000	0.0004	0.0012	0.0018	−0.0002
	0.3097	−0.0004	−0.0007	0.0004	0.0009	−0.0003	−0.0002
	0.3106	0.0007	0.0000	0.0004	0.0012	0.0018	−0.0002
	0.3111	−0.0004	−0.0008	0.0004	0.0009	−0.0003	−0.0002
	0.3084	−0.0004	−0.0007	0.0004	0.0009	−0.0002	−0.0002
	0.3111	−0.0004	−0.0008	0.0004	0.0009	−0.0003	−0.0002
$c = 0.3$	0.3062	0.0003	−0.0001	0.0001	0.0003	0.0013	−0.0002
$T = 3$	0.3066	0.0003	−0.0001	0.0001	0.0003	0.0013	−0.0002
	0.3140	0.0003	−0.0001	0.0001	0.0004	0.0013	−0.0002
	0.3097	0.0003	−0.0001	0.0001	0.0003	0.0013	−0.0002
	0.3097	0.0003	−0.0001	0.0001	0.0003	0.0013	−0.0002
	0.3105	0.0003	−0.0001	0.0001	0.0004	0.0013	−0.0002
	0.3109	0.0003	−0.0001	0.0001	0.0004	0.0013	−0.0002
	0.3062	0.0003	−0.0001	0.0001	0.0003	0.0013	−0.0002
	0.3111	0.0003	−0.0001	0.0001	0.0004	0.0013	−0.0002
$c = 0.3$	0.3062	0.0010	−0.0001	0.0004	0.0012	0.0025	−0.0003
$T = 7$	0.3065	0.0010	−0.0001	0.0004	0.0012	0.0025	−0.0003
	0.3140	0.0010	−0.0001	0.0004	0.0012	0.0025	−0.0003
	0.3097	0.0010	−0.0001	0.0004	0.0012	0.0025	−0.0003
	0.3097	0.0010	−0.0001	0.0004	0.0012	0.0025	−0.0003
	0.3105	0.0010	−0.0001	0.0004	0.0012	0.0025	−0.0003
	0.3109	0.0010	−0.0001	0.0004	0.0012	0.0025	−0.0003
	0.3062	0.0010	−0.0001	0.0004	0.0012	0.0025	−0.0003
	0.3111	0.0010	−0.0001	0.0004	0.0012	0.0025	−0.0003

Table 4.32. Sequential quadrature prices minus Monte Carlo prices for the EIAs in table 4.29.

	α	NIG	VG	$\frac{\text{NIG}}{\text{CIR}}$	$\frac{\text{VG}}{\text{CIR}}$	$\frac{\text{NIG}}{\Gamma\text{-OU}}$	$\frac{\text{VG}}{\Gamma\text{-OU}}$	BS
$c = 0.1$	0.3177	0.0001	−0.0001	0.0011	0.0001	0.0013	0.0019	0.0002
$T = 3$	0.3174	0.0001	−0.0001	0.0011	0.0001	0.0013	0.0019	0.0002
	0.3164	0.0001	−0.0001	0.0011	0.0001	0.0013	0.0019	0.0002
	0.3151	0.0001	−0.0001	0.0011	0.0001	0.0013	0.0019	0.0002
	0.3093	0.0001	−0.0001	0.0011	0.0001	0.0013	0.0019	0.0002
	0.3157	0.0001	−0.0001	0.0011	0.0001	0.0013	0.0019	0.0002
	0.3023	0.0001	−0.0001	0.0011	0.0001	0.0013	0.0019	0.0002
	0.3058	0.0001	−0.0001	0.0011	0.0001	0.0013	0.0019	0.0002
	0.3025	0.0001	−0.0001	0.0011	0.0001	0.0013	0.0019	0.0002
	0.3591	0.0001	−0.0001	0.0011	0.0002	0.0014	0.0020	0.0003
$c = 0.1$	0.3177	0.0001	−0.0003	0.0027	−0.0004	0.0025	0.0036	−0.0000
$T = 7$	0.3174	0.0001	−0.0003	0.0027	−0.0004	0.0025	0.0036	−0.0000
	0.3164	0.0001	−0.0003	0.0027	−0.0004	0.0025	0.0036	−0.0000
	0.3151	0.0001	−0.0003	0.0027	−0.0004	0.0025	0.0036	−0.0000
	0.3093	0.0000	−0.0003	0.0027	−0.0004	0.0025	0.0036	−0.0000
	0.3157	0.0001	−0.0003	0.0027	−0.0004	0.0025	0.0036	−0.0000
	0.3023	0.0000	−0.0003	0.0027	−0.0004	0.0025	0.0035	−0.0000
	0.3058	0.0000	−0.0003	0.0027	−0.0004	0.0025	0.0036	−0.0000
	0.3025	0.0000	−0.0003	0.0027	−0.0004	0.0025	0.0035	−0.0000
	0.3591	0.0001	−0.0003	0.0028	−0.0002	0.0028	0.0039	−0.0000
$c = 0.2$	0.3074	0.0002	−0.0001	0.0022	−0.0001	0.0026	0.0037	0.0003
$T = 3$	0.3072	−0.0003	−0.0004	−0.0013	−0.0002	−0.0033	−0.0046	−0.0014
	0.3069	−0.0003	−0.0004	−0.0013	−0.0002	−0.0033	−0.0046	−0.0014
	0.3076	−0.0003	−0.0004	−0.0013	−0.0002	−0.0033	−0.0046	−0.0014
	0.2995	0.0002	−0.0001	0.0022	−0.0001	0.0026	0.0037	0.0003
	0.3078	0.0002	−0.0001	0.0022	−0.0001	0.0026	0.0037	0.0003
	0.2883	0.0002	−0.0001	0.0022	−0.0001	0.0026	0.0037	0.0003
	0.2941	0.0002	−0.0001	0.0022	−0.0001	0.0026	0.0037	0.0003
	0.2886	0.0002	−0.0001	0.0022	−0.0001	0.0026	0.0037	0.0003
	0.3237	−0.0005	−0.0005	−0.0014	−0.0003	−0.0033	−0.0046	−0.0022
$c = 0.2$	0.3074	0.0000	−0.0003	0.0025	−0.0010	−0.0001	−0.0001	−0.0013
$T = 7$	0.3072	0.0000	−0.0003	0.0025	−0.0010	−0.0001	−0.0001	−0.0013
	0.3069	0.0005	−0.0001	0.0057	−0.0009	0.0052	0.0074	0.0003
	0.3076	0.0005	−0.0001	0.0057	−0.0009	0.0052	0.0074	0.0003
	0.2995	0.0005	−0.0001	0.0057	−0.0009	0.0052	0.0073	0.0003
	0.3078	0.0000	−0.0004	0.0025	−0.0010	−0.0001	−0.0001	−0.0013
	0.2883	0.0005	−0.0000	0.0056	−0.0008	0.0052	0.0073	0.0003
	0.2941	0.0005	−0.0001	0.0056	−0.0008	0.0052	0.0073	0.0003
	0.2886	0.0001	−0.0002	0.0024	−0.0009	−0.0001	−0.0001	−0.0007
	0.3237	−0.0001	−0.0005	0.0026	−0.0011	−0.0001	−0.0001	−0.0020
$c = 0.3$	0.3072	0.0003	−0.0001	0.0031	−0.0002	0.0039	0.0055	0.0003
$T = 3$	0.3068	0.0003	−0.0001	0.0031	−0.0002	0.0039	0.0055	0.0003
	0.3068	0.0003	−0.0001	0.0031	−0.0002	0.0039	0.0055	0.0003
	0.3076	0.0003	−0.0001	0.0031	−0.0002	0.0039	0.0055	0.0003
	0.2956	−0.0001	−0.0001	−0.0007	−0.0002	−0.0024	−0.0034	0.0003
	0.3078	0.0003	−0.0001	0.0031	−0.0002	0.0039	0.0055	0.0003
	0.2789	0.0003	−0.0001	0.0031	−0.0001	0.0039	0.0055	0.0003
	0.2874	0.0003	−0.0001	0.0031	−0.0001	0.0039	0.0055	0.0003
	0.2793	0.0003	−0.0001	0.0031	−0.0001	0.0039	0.0055	0.0003
	0.3229	0.0003	−0.0001	0.0032	−0.0002	0.0039	0.0055	0.0003
$c = 0.3$	0.3072	0.0006	−0.0001	0.0078	−0.0010	0.0079	0.0111	0.0002
$T = 7$	0.3068	0.0006	−0.0001	0.0078	−0.0010	0.0079	0.0111	0.0002
	0.3068	0.0006	−0.0001	0.0078	−0.0010	0.0079	0.0111	0.0002
	0.3076	0.0006	−0.0001	0.0078	−0.0010	0.0079	0.0111	0.0002
	0.2956	0.0006	−0.0001	0.0076	−0.0010	0.0078	0.0110	0.0002
	0.3078	0.0006	−0.0001	0.0078	−0.0010	0.0079	0.0111	0.0002
	0.2789	0.0006	−0.0001	0.0075	−0.0009	0.0078	0.0109	0.0002
	0.2874	0.0006	−0.0001	0.0075	−0.0009	0.0078	0.0110	0.0002
	0.2793	0.0006	−0.0001	0.0075	−0.0009	0.0078	0.0109	0.0002
	0.3229	0.0006	−0.0002	0.0079	−0.0011	0.0080	0.0112	0.0002

Table 4.33. Sequential quadrature prices minus Monte Carlo prices for the EIAs in table 4.30.

	α	NIG	VG	$\frac{\text{NIG}}{\text{CIR}}$	$\frac{\text{VG}}{\text{CIR}}$	$\frac{\text{NIG}}{\Gamma\text{-OU}}$	$\frac{\text{VG}}{\Gamma\text{-OU}}$	BS
$c = 0.1$	0.2528	0.0002	−0.0001	0.0002	0.0003	0.0008	0.0015	−0.0000
$T = 3$	0.2517	0.0002	−0.0001	0.0002	0.0003	0.0008	0.0015	−0.0001
	0.2494	0.0002	−0.0001	0.0002	0.0003	0.0008	0.0015	−0.0001
	0.2235	0.0002	−0.0001	0.0002	0.0003	0.0008	0.0014	−0.0001
	0.2235	0.0002	−0.0001	0.0002	0.0003	0.0008	0.0014	−0.0001
	0.2194	0.0002	−0.0001	0.0002	0.0003	0.0008	0.0014	−0.0001
	0.2131	0.0002	−0.0001	0.0002	0.0003	0.0007	0.0014	−0.0001
	0.2274	0.0002	−0.0001	0.0002	0.0003	0.0008	0.0015	−0.0001
	0.2257	0.0002	−0.0001	0.0002	0.0003	0.0008	0.0014	−0.0001
	0.2502	0.0002	−0.0001	0.0002	0.0003	0.0008	0.0015	−0.0001
$c = 0.1$	0.2528	0.0001	−0.0001	0.0003	0.0004	0.0016	0.0027	−0.0001
$T = 7$	0.2517	0.0001	−0.0001	0.0003	0.0004	0.0016	0.0027	−0.0001
	0.2494	0.0001	−0.0001	0.0003	0.0004	0.0016	0.0027	−0.0001
	0.2235	0.0000	−0.0001	0.0002	0.0003	0.0014	0.0025	−0.0001
	0.2235	0.0000	−0.0001	0.0002	0.0003	0.0014	0.0025	−0.0001
	0.2194	0.0000	−0.0001	0.0002	0.0003	0.0014	0.0025	−0.0001
	0.2131	0.0000	−0.0002	0.0002	0.0003	0.0014	0.0024	−0.0001
	0.2274	0.0000	−0.0001	0.0002	0.0003	0.0015	0.0025	−0.0001
	0.2257	0.0000	−0.0001	0.0002	0.0003	0.0015	0.0025	−0.0001
	0.2502	0.0001	−0.0001	0.0003	0.0004	0.0016	0.0027	−0.0001
$c = 0.2$	0.2479	−0.0002	−0.0003	−0.0011	0.0003	−0.0036	−0.0043	−0.0065
$T = 3$	0.2470	−0.0002	−0.0003	−0.0011	0.0003	−0.0036	−0.0043	−0.0064
	0.2460	−0.0001	−0.0003	−0.0011	0.0003	−0.0036	−0.0043	−0.0062
	0.2145	0.0003	−0.0001	0.0003	0.0003	0.0012	0.0023	−0.0000
	0.2145	0.0003	−0.0001	0.0003	0.0003	0.0012	0.0023	−0.0000
	0.2118	0.0003	−0.0001	0.0003	0.0003	0.0012	0.0023	−0.0000
	0.2077	−0.0001	−0.0002	−0.0005	0.0003	−0.0026	−0.0039	−0.0024
	0.2173	0.0003	−0.0001	0.0003	0.0003	0.0012	0.0023	−0.0000
	0.2150	0.0003	−0.0001	0.0003	0.0003	0.0012	0.0023	−0.0000
	0.2224	0.0003	−0.0001	0.0003	0.0003	0.0012	0.0023	−0.0001
$c = 0.2$	0.2479	0.0002	0.0001	−0.0002	0.0012	−0.0015	−0.0012	−0.0057
$T = 7$	0.2470	0.0007	0.0003	0.0011	0.0012	0.0029	0.0049	0.0003
	0.2459	0.0007	0.0003	0.0011	0.0012	0.0029	0.0049	0.0003
	0.2144	0.0007	0.0004	0.0010	0.0010	0.0028	0.0047	0.0003
	0.2144	0.0007	0.0004	0.0010	0.0010	0.0028	0.0047	0.0003
	0.2118	0.0007	0.0004	0.0010	0.0010	0.0027	0.0046	0.0003
	0.2077	0.0007	0.0004	0.0009	0.0010	0.0027	0.0046	0.0003
	0.2173	0.0007	0.0004	0.0010	0.0010	0.0028	0.0047	0.0003
	0.2150	0.0007	0.0004	0.0010	0.0010	0.0028	0.0047	0.0003
	0.2224	0.0007	0.0004	0.0010	0.0011	0.0028	0.0047	0.0003
$c = 0.3$	0.2476	0.0004	−0.0000	0.0004	0.0004	0.0017	0.0033	−0.0001
$T = 3$	0.2466	0.0004	−0.0000	0.0004	0.0004	0.0017	0.0033	−0.0001
	0.2458	0.0004	−0.0000	0.0004	0.0004	0.0017	0.0033	−0.0001
	0.2134	0.0004	−0.0000	0.0004	0.0003	0.0016	0.0032	−0.0000
	0.2134	0.0004	−0.0000	0.0004	0.0003	0.0016	0.0032	−0.0000
	0.2112	0.0004	−0.0000	0.0004	0.0003	0.0016	0.0032	−0.0000
	0.2041	0.0004	−0.0000	0.0003	0.0003	0.0016	0.0032	−0.0000
	0.2144	0.0004	−0.0000	0.0004	0.0003	0.0016	0.0032	−0.0000
	0.2097	0.0004	−0.0000	0.0004	0.0003	0.0016	0.0032	−0.0000
	0.2214	0.0004	−0.0000	0.0004	0.0003	0.0017	0.0032	−0.0000
$c = 0.3$	0.2476	0.0007	0.0003	0.0011	0.0011	0.0037	0.0066	0.0002
$T = 7$	0.2466	0.0007	0.0003	0.0011	0.0011	0.0037	0.0066	0.0002
	0.2458	0.0007	0.0003	0.0011	0.0011	0.0037	0.0066	0.0002
	0.2131	0.0007	0.0003	0.0009	0.0009	0.0035	0.0063	0.0002
	0.2131	0.0007	0.0003	0.0009	0.0009	0.0035	0.0063	0.0002
	0.2109	0.0007	0.0003	0.0009	0.0009	0.0035	0.0063	0.0002
	0.2041	0.0007	0.0003	0.0009	0.0008	0.0035	0.0062	0.0002
	0.2144	0.0007	0.0003	0.0009	0.0009	0.0035	0.0063	0.0002
	0.2097	0.0007	0.0003	0.0009	0.0009	0.0035	0.0062	0.0002
	0.2214	0.0007	0.0003	0.0010	0.0009	0.0035	0.0063	0.0002

Table 4.34. Sequential quadrature prices minus Monte Carlo prices for the EIAs in table 4.31.

	$c = 0.1$		$c = 0.2$		$c = 0.3$	
	$T = 3$	$T = 7$	$T = 3$	$T = 7$	$T = 3$	$T = 7$
(A) S&P 500 $\mathbb{P} + \mathbb{Q}$ parameters						
Meix	0.3133	0.3133	0.3063	0.3063	0.3063	0.3063
NIG	0.3218	0.3216	0.3114	0.3114	0.3113	0.3113
VG	0.3345	0.3340	0.3141	0.3141	0.3139	0.3139
Meix/CIR	0.3199	0.3199	0.3097	0.3097	0.3097	0.3097
NIG/CIR	0.3199	0.3199	0.3097	0.3097	0.3097	0.3097
VG/CIR	0.3235	0.3234	0.3106	0.3106	0.3105	0.3105
Meix/Γ-OU	0.3226	0.3224	0.3110	0.3110	0.3109	0.3109
NIG/Γ-OU	0.3245	0.3242	0.3124	0.3124	0.3122	0.3122
VG/Γ-OU	0.3200	0.3199	0.3111	0.3111	0.3111	0.3111
(B) S&P 500 $\mathbb{Q}$ parameters						
Meix	0.3169	0.3168	0.3077	0.3077	0.3076	0.3076
NIG	0.3177	0.3176	0.3085	0.3085	0.3085	0.3085
VG	0.3156	0.3155	0.3070	0.3070	0.3070	0.3070
Meix/CIR	0.3146	0.3146	0.3076	0.3076	0.3076	0.3076
NIG/CIR	0.3887	0.3885	0.3683	0.3682	0.3682	0.3681
VG/CIR	0.3149	0.3149	0.3079	0.3079	0.3079	0.3079
Meix/Γ-OU	0.5637	0.5583	0.4693	0.4690	0.4684	0.4680
NIG/Γ-OU	0.4534	0.4524	0.4105	0.4104	0.4103	0.4101
VG/Γ-OU	0.5572	0.5521	0.4663	0.4662	0.4656	0.4654
BS	0.3520	0.3507	0.3235	0.3235	0.3229	0.3229
(C) Eurostoxx 50 parameters						
Meix	0.2524	0.2524	0.2485	0.2485	0.2485	0.2485
NIG	0.2525	0.2525	0.2487	0.2487	0.2487	0.2487
VG	0.2492	0.2492	0.2463	0.2463	0.2463	0.2463
Meix/CIR	0.2225	0.2223	0.2181	0.2180	0.2179	0.2178
NIG/CIR	0.2230	0.2228	0.2185	0.2185	0.2184	0.2183
VG/CIR	0.2190	0.2190	0.2157	0.2157	0.2157	0.2157
Meix/Γ-OU	0.2911	0.2910	0.2769	0.2766	0.2767	0.2762
NIG/Γ-OU	0.2510	0.2506	0.2410	0.2409	0.2404	0.2404
VG/Γ-OU	0.2998	0.2983	0.2784	0.2775	0.2769	0.2765
BS	0.2411	0.2402	0.2220	0.2220	0.2213	0.2213

Table 4.35. Critical participation rates of term-end PTPs under different models.

	α	Meix	NIG	VG	$\frac{\text{Meix}}{\text{CIR}}$	$\frac{\text{NIG}}{\text{CIR}}$	$\frac{\text{VG}}{\text{CIR}}$	$\frac{\text{Meix}}{\Gamma\text{-OU}}$	$\frac{\text{NIG}}{\Gamma\text{-OU}}$	$\frac{\text{VG}}{\Gamma\text{-OU}}$
$c = 0.1$	0.3133	0.00	0.05	0.12	0.04	0.04	0.06	0.06	0.07	0.04
$T = 3$	0.3218	−0.06	0.00	0.07	−0.01	−0.01	0.01	0.00	0.02	−0.01
	0.3345	−0.14	−0.08	0.00	−0.09	−0.09	−0.07	−0.07	−0.06	−0.09
	0.3199	−0.04	0.01	0.08	0.00	0.00	0.02	0.02	0.03	0.00
	0.3199	−0.04	0.01	0.08	0.00	0.00	0.02	0.02	0.03	0.00
	0.3235	−0.07	−0.01	0.06	−0.02	−0.02	0.00	−0.01	0.01	−0.02
	0.3226	−0.06	−0.01	0.07	−0.02	−0.02	0.01	0.00	0.01	−0.02
	0.3245	−0.07	−0.02	0.06	−0.03	−0.03	−0.01	−0.01	0.00	−0.03
	0.3200	−0.04	0.01	0.08	0.00	0.00	0.02	0.02	0.03	0.00
$c = 0.1$	0.3133	0.00	0.05	0.12	0.04	0.04	0.06	0.06	0.07	0.04
$T = 7$	0.3216	−0.06	0.00	0.07	−0.01	−0.01	0.01	0.00	0.02	−0.01
	0.3340	−0.13	−0.08	0.00	−0.09	−0.09	−0.06	−0.07	−0.06	−0.09
	0.3199	−0.04	0.01	0.08	0.00	0.00	0.02	0.02	0.03	0.00
	0.3199	−0.04	0.01	0.08	0.00	0.00	0.02	0.02	0.03	0.00
	0.3234	−0.07	−0.01	0.06	−0.02	−0.02	0.00	−0.01	0.01	−0.02
	0.3224	−0.06	0.00	0.07	−0.02	−0.02	0.01	0.00	0.01	−0.02
	0.3242	−0.07	−0.02	0.06	−0.03	−0.03	−0.01	−0.01	0.00	−0.03
	0.3199	−0.04	0.01	0.08	0.00	0.00	0.02	0.02	0.03	0.00
$c = 0.2$	0.3063	0.00	0.04	0.06	0.03	0.03	0.03	0.04	0.05	0.04
$T = 3$	0.3114	−0.04	0.00	0.02	−0.01	−0.01	−0.01	0.00	0.01	0.00
	0.3141	−0.06	−0.02	0.00	−0.03	−0.03	−0.03	−0.02	−0.01	−0.02
	0.3097	−0.03	0.01	0.03	0.00	0.00	0.01	0.01	0.02	0.01
	0.3097	−0.03	0.01	0.03	0.00	0.00	0.01	0.01	0.02	0.01
	0.3106	−0.03	0.01	0.03	−0.01	−0.01	0.00	0.00	0.01	0.00
	0.3110	−0.04	0.00	0.02	−0.01	−0.01	0.00	0.00	0.01	0.00
	0.3124	−0.05	−0.01	0.01	−0.02	−0.02	−0.01	−0.01	0.00	−0.01
	0.3111	−0.04	0.00	0.02	−0.01	−0.01	0.00	0.00	0.01	0.00
$c = 0.2$	0.3063	0.00	0.04	0.06	0.03	0.03	0.03	0.04	0.05	0.04
$T = 7$	0.3114	−0.04	0.00	0.02	−0.01	−0.01	−0.01	0.00	0.01	0.00
	0.3141	−0.06	−0.02	0.00	−0.03	−0.03	−0.03	−0.02	−0.01	−0.02
	0.3097	−0.03	0.01	0.03	0.00	0.00	0.01	0.01	0.02	0.01
	0.3097	−0.03	0.01	0.03	0.00	0.00	0.01	0.01	0.02	0.01
	0.3106	−0.03	0.01	0.03	−0.01	−0.01	0.00	0.00	0.01	0.00
	0.3110	−0.04	0.00	0.02	−0.01	−0.01	0.00	0.00	0.01	0.00
	0.3124	−0.05	−0.01	0.01	−0.02	−0.02	−0.01	−0.01	0.00	−0.01
	0.3111	−0.04	0.00	0.02	−0.01	−0.01	0.00	0.00	0.01	0.00
$c = 0.3$	0.3063	0.00	0.04	0.06	0.03	0.03	0.03	0.04	0.04	0.04
$T = 3$	0.3113	−0.04	0.00	0.02	−0.01	−0.01	−0.01	0.00	0.01	0.00
	0.3139	−0.06	−0.02	0.00	−0.03	−0.03	−0.03	−0.02	−0.01	−0.02
	0.3097	−0.03	0.01	0.03	0.00	0.00	0.01	0.01	0.02	0.01
	0.3097	−0.03	0.01	0.03	0.00	0.00	0.01	0.01	0.02	0.01
	0.3105	−0.03	0.01	0.03	−0.01	−0.01	0.00	0.00	0.01	0.00
	0.3109	−0.04	0.00	0.02	−0.01	−0.01	0.00	0.00	0.01	0.00
	0.3122	−0.05	−0.01	0.01	−0.02	−0.02	−0.01	−0.01	0.00	−0.01
	0.3111	−0.04	0.00	0.02	−0.01	−0.01	0.00	0.00	0.01	0.00
$c = 0.3$	0.3063	0.00	0.04	0.06	0.03	0.03	0.03	0.04	0.04	0.04
$T = 7$	0.3113	−0.04	0.00	0.02	−0.01	−0.01	−0.01	0.00	0.01	0.00
	0.3139	−0.06	−0.02	0.00	−0.03	−0.03	−0.03	−0.02	−0.01	−0.02
	0.3097	−0.03	0.01	0.03	0.00	0.00	0.01	0.01	0.02	0.01
	0.3097	−0.03	0.01	0.03	0.00	0.00	0.01	0.01	0.02	0.01
	0.3105	−0.03	0.01	0.03	−0.01	−0.01	0.00	0.00	0.01	0.00
	0.3109	−0.04	0.00	0.02	−0.01	−0.01	0.00	0.00	0.01	0.00
	0.3122	−0.05	−0.01	0.01	−0.02	−0.02	−0.01	−0.01	0.00	−0.01
	0.3111	−0.04	0.00	0.02	−0.01	−0.01	0.00	0.00	0.01	0.00

Table 4.36. Values of $100 \times (1 -$ term-end PTP price) using S&P 500 $\mathbb{P} + \mathbb{Q}$ parameters.

	α	Meix	NIG	VG	$\frac{\text{Meix}}{\text{CIR}}$	$\frac{\text{NIG}}{\text{CIR}}$	$\frac{\text{VG}}{\text{CIR}}$	$\frac{\text{Meix}}{\Gamma\text{-OU}}$	$\frac{\text{NIG}}{\Gamma\text{-OU}}$	$\frac{\text{VG}}{\Gamma\text{-OU}}$
$c = 0.1$	0.3133	0.9407	0.9422	0.9375	0.9388	0.9388	0.9400	0.9431	0.9443	0.9379
$T = 7$	0.3218	0.9420	0.9436	0.9387	0.9401	0.9401	0.9412	0.9445	0.9457	0.9392
	0.3345	0.9440	0.9456	0.9405	0.9420	0.9420	0.9431	0.9465	0.9478	0.9410
	0.3199	0.9417	0.9433	0.9384	0.9398	0.9398	0.9410	0.9442	0.9454	0.9389
	0.3199	0.9417	0.9433	0.9384	0.9398	0.9398	0.9410	0.9442	0.9454	0.9389
	0.3235	0.9423	0.9439	0.9389	0.9404	0.9404	0.9415	0.9447	0.9460	0.9394
	0.3226	0.9422	0.9437	0.9388	0.9402	0.9402	0.9414	0.9446	0.9459	0.9393
	0.3245	0.9425	0.9440	0.9391	0.9405	0.9405	0.9416	0.9449	0.9462	0.9396
	0.3200	0.9418	0.9433	0.9384	0.9398	0.9398	0.9410	0.9442	0.9454	0.9389
$c = 0.1$	0.3133	0.8379	0.8409	0.8340	0.8354	0.8354	0.8380	0.8428	0.8467	0.8338
$T = 3$	0.3216	0.8401	0.8432	0.8360	0.8375	0.8375	0.8401	0.8452	0.8492	0.8358
	0.3340	0.8433	0.8466	0.8390	0.8406	0.8406	0.8433	0.8487	0.8530	0.8389
	0.3199	0.8396	0.8427	0.8356	0.8371	0.8370	0.8397	0.8447	0.8487	0.8354
	0.3199	0.8396	0.8428	0.8356	0.8371	0.8371	0.8397	0.8447	0.8487	0.8354
	0.3234	0.8405	0.8437	0.8365	0.8379	0.8379	0.8406	0.8457	0.8498	0.8362
	0.3224	0.8403	0.8434	0.8362	0.8377	0.8377	0.8403	0.8454	0.8495	0.8360
	0.3242	0.8407	0.8439	0.8367	0.8381	0.8381	0.8408	0.8459	0.8500	0.8364
	0.3199	0.8396	0.8428	0.8356	0.8371	0.8371	0.8397	0.8447	0.8487	0.8354
$c = 0.2$	0.3063	0.9398	0.9414	0.9369	0.9380	0.9380	0.9393	0.9422	0.9435	0.9370
$T = 7$	0.3114	0.9406	0.9422	0.9377	0.9388	0.9388	0.9401	0.9431	0.9444	0.9378
	0.3141	0.9411	0.9427	0.9382	0.9392	0.9392	0.9406	0.9436	0.9448	0.9382
	0.3097	0.9403	0.9419	0.9375	0.9386	0.9386	0.9399	0.9428	0.9441	0.9376
	0.3097	0.9403	0.9419	0.9375	0.9386	0.9386	0.9399	0.9428	0.9441	0.9376
	0.3106	0.9405	0.9421	0.9376	0.9387	0.9387	0.9400	0.9430	0.9442	0.9377
	0.3110	0.9406	0.9422	0.9377	0.9388	0.9388	0.9401	0.9430	0.9443	0.9378
	0.3124	0.9408	0.9424	0.9379	0.9390	0.9390	0.9403	0.9433	0.9445	0.9380
	0.3111	0.9406	0.9422	0.9377	0.9388	0.9388	0.9401	0.9430	0.9443	0.9378
$c = 0.2$	0.3063	0.8362	0.8392	0.8327	0.8339	0.8339	0.8366	0.8411	0.8449	0.8322
$T = 3$	0.3114	0.8376	0.8407	0.8339	0.8352	0.8352	0.8380	0.8426	0.8465	0.8335
	0.3141	0.8383	0.8414	0.8346	0.8359	0.8359	0.8387	0.8434	0.8473	0.8342
	0.3097	0.8371	0.8402	0.8335	0.8347	0.8347	0.8375	0.8421	0.8460	0.8331
	0.3097	0.8371	0.8402	0.8335	0.8347	0.8347	0.8375	0.8421	0.8460	0.8331
	0.3106	0.8373	0.8404	0.8337	0.8350	0.8350	0.8377	0.8423	0.8462	0.8333
	0.3110	0.8375	0.8406	0.8338	0.8351	0.8351	0.8379	0.8425	0.8464	0.8334
	0.3124	0.8378	0.8409	0.8342	0.8354	0.8354	0.8382	0.8429	0.8468	0.8337
	0.3111	0.8375	0.8406	0.8339	0.8351	0.8351	0.8379	0.8425	0.8464	0.8334
$c = 0.3$	0.3063	0.9398	0.9414	0.9369	0.9380	0.9380	0.9393	0.9422	0.9435	0.9370
$T = 7$	0.3113	0.9406	0.9422	0.9377	0.9388	0.9388	0.9401	0.9431	0.9444	0.9378
	0.3139	0.9410	0.9426	0.9381	0.9392	0.9392	0.9405	0.9435	0.9448	0.9382
	0.3097	0.9403	0.9419	0.9375	0.9385	0.9385	0.9399	0.9428	0.9441	0.9376
	0.3097	0.9403	0.9419	0.9375	0.9385	0.9385	0.9399	0.9428	0.9441	0.9376
	0.3105	0.9405	0.9421	0.9376	0.9387	0.9387	0.9400	0.9429	0.9442	0.9377
	0.3109	0.9405	0.9421	0.9377	0.9387	0.9387	0.9401	0.9430	0.9443	0.9377
	0.3122	0.9408	0.9424	0.9379	0.9389	0.9389	0.9403	0.9432	0.9445	0.9379
	0.3111	0.9406	0.9422	0.9377	0.9388	0.9388	0.9401	0.9430	0.9443	0.9378
$c = 0.3$	0.3063	0.8362	0.8392	0.8327	0.8339	0.8339	0.8366	0.8411	0.8449	0.8322
$T = 3$	0.3113	0.8375	0.8406	0.8339	0.8351	0.8351	0.8379	0.8426	0.8464	0.8335
	0.3139	0.8382	0.8414	0.8346	0.8358	0.8358	0.8386	0.8433	0.8473	0.8341
	0.3097	0.8371	0.8402	0.8335	0.8347	0.8347	0.8375	0.8421	0.8460	0.8331
	0.3097	0.8371	0.8402	0.8335	0.8347	0.8347	0.8375	0.8421	0.8460	0.8331
	0.3105	0.8373	0.8404	0.8337	0.8349	0.8349	0.8377	0.8423	0.8462	0.8333
	0.3109	0.8374	0.8405	0.8338	0.8351	0.8351	0.8378	0.8424	0.8463	0.8334
	0.3122	0.8378	0.8409	0.8341	0.8354	0.8354	0.8382	0.8428	0.8467	0.8337
	0.3111	0.8375	0.8406	0.8339	0.8351	0.8351	0.8379	0.8425	0.8464	0.8334

Table 4.37. True prices of term-end PTPs using S&P 500 $\mathbb{P} + \mathbb{Q}$ parameters when surrender is not allowed.

	α	Meix	NIG	VG	$\frac{\text{Meix}}{\text{CIR}}$	$\frac{\text{NIG}}{\text{CIR}}$	$\frac{\text{VG}}{\text{CIR}}$	$\frac{\text{Meix}}{\Gamma\text{-OU}}$	$\frac{\text{NIG}}{\Gamma\text{-OU}}$	$\frac{\text{VG}}{\Gamma\text{-OU}}$
$c = 0.1$	0.3133	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	−0.0000
$T = 7$	0.3218	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
	0.3345	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
	0.3199	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	−0.0000
	0.3199	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	−0.0000
	0.3235	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
	0.3226	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
	0.3245	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	−0.0000
	0.3200	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	−0.0000
$c = 0.1$	0.3133	−0.0000	−0.0011	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
$T = 3$	0.3216	−0.0000	−0.0011	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
	0.3340	−0.0000	−0.0011	0.0000	0.0000	−0.0000	−0.0000	−0.0000	−0.0000	0.0000
	0.3199	−0.0000	−0.0011	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
	0.3199	−0.0000	−0.0011	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
	0.3234	−0.0000	−0.0011	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
	0.3224	−0.0000	−0.0011	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
	0.3242	−0.0000	−0.0011	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
	0.3199	−0.0000	−0.0011	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
$c = 0.2$	0.3063	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
$T = 7$	0.3114	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
	0.3141	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
	0.3097	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
	0.3097	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
	0.3106	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
	0.3110	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
	0.3124	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
	0.3111	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
$c = 0.2$	0.3063	−0.0000	−0.0011	0.0000	−0.0000	−0.0000	−0.0000	0.0000	−0.0000	0.0000
$T = 3$	0.3114	−0.0000	−0.0011	0.0000	0.0000	−0.0000	−0.0000	0.0000	−0.0000	0.0000
	0.3141	−0.0000	−0.0011	−0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	−0.0000
	0.3097	−0.0000	−0.0011	0.0000	0.0000	−0.0000	−0.0000	0.0000	−0.0000	0.0000
	0.3097	−0.0000	−0.0011	0.0000	0.0000	−0.0000	−0.0000	0.0000	−0.0000	0.0000
	0.3106	−0.0000	−0.0011	0.0000	0.0000	−0.0000	−0.0000	0.0000	−0.0000	0.0000
	0.3110	−0.0000	−0.0011	0.0000	0.0000	−0.0000	−0.0000	0.0000	−0.0000	0.0000
	0.3124	−0.0000	−0.0011	0.0000	−0.0000	−0.0000	−0.0000	0.0000	−0.0000	0.0000
	0.3111	−0.0000	−0.0011	0.0000	0.0000	−0.0000	−0.0000	0.0000	−0.0000	0.0000
$c = 0.3$	0.3063	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
$T = 7$	0.3113	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
	0.3139	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
	0.3097	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
	0.3097	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
	0.3105	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
	0.3109	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
	0.3122	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
	0.3111	−0.0000	−0.0005	0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	0.0000
$c = 0.3$	0.3063	−0.0000	−0.0011	0.0000	−0.0000	−0.0000	−0.0000	0.0000	−0.0000	0.0000
$T = 3$	0.3113	−0.0000	−0.0011	0.0000	0.0000	−0.0000	−0.0000	0.0000	−0.0000	0.0000
	0.3139	−0.0000	−0.0011	−0.0000	0.0000	−0.0000	0.0000	0.0000	−0.0000	−0.0000
	0.3097	−0.0000	−0.0011	0.0000	0.0000	−0.0000	−0.0000	0.0000	−0.0000	0.0000
	0.3097	−0.0000	−0.0011	0.0000	0.0000	−0.0000	−0.0000	0.0000	−0.0000	0.0000
	0.3105	−0.0000	−0.0011	0.0000	0.0000	−0.0000	−0.0000	0.0000	−0.0000	0.0000
	0.3109	−0.0000	−0.0011	0.0000	0.0000	−0.0000	−0.0000	0.0000	−0.0000	0.0000
	0.3122	−0.0000	−0.0011	0.0000	−0.0000	−0.0000	−0.0000	0.0000	−0.0000	0.0000
	0.3111	−0.0000	−0.0011	0.0000	0.0000	−0.0000	−0.0000	0.0000	−0.0000	0.0000

Table 4.38. Errors in the sequential quadrature prices of the EIAs mentioned in table 4.37.

	$c = 0.1$		$c = 0.2$		$c = 0.3$	
	$T = 3$	$T = 7$	$T = 3$	$T = 7$	$T = 3$	$T = 7$
S&P 500 $\mathbb{P} + \mathbb{Q}$ parameters						
Meix	0.5394	0.5394	0.5244	0.5244	0.5195	0.5195
NIG	0.5428	0.5428	0.5094	0.5094	0.4938	0.4938
VG	0.5597	0.5597	0.5259	0.5259	0.5238	0.5238
Meix/CIR	0.5463	0.5463	0.5301	0.5301	0.5290	0.5290
NIG/CIR	0.5461	0.5461	0.5298	0.5298	0.5285	0.5285
VG/CIR	0.5684	0.5684	0.5438	0.5438	0.5393	0.5393
Meix/Γ-OU	0.5211	0.5211	0.4672	0.4672	0.4290	0.4290
NIG/Γ-OU	0.5182	0.5182	0.4595	0.4595	0.4170	0.4170
VG/Γ-OU	0.5380	0.5380	0.5227	0.5227	0.5225	0.5225
S&P 500 $\mathbb{Q}$ parameters						
Meix	0.5399	0.5399	0.5127	0.5127	0.4986	0.4986
NIG	0.5371	0.5371	0.5090	0.5090	0.4936	0.4936
VG	0.5391	0.5391	0.5148	0.5148	0.5028	0.5028
Meix/CIR	0.5458	0.5458	0.5323	0.5323	0.5316	0.5316
NIG/CIR	0.5402	0.5402	0.5185	0.5185	0.5077	0.5076
VG/CIR	0.5518	0.5518	0.5318	0.5317	0.5283	0.5281
Meix/Γ-OU	0.5322	0.5322	0.4987	0.4987	0.4784	0.4784
NIG/Γ-OU	0.5331	0.5331	0.5031	0.5031	0.4856	0.4856
VG/Γ-OU	0.5325	0.5325	0.4990	0.4990	0.4788	0.4788
BS	0.6032	0.6032	0.5485	0.5485	0.5477	0.5477
Eurostoxx 50 parameters						
Meix	0.4207	0.4207	0.3991	0.3995	0.3842	0.3840
NIG	0.3705	0.3704	0.3115	0.3113	0.2555	0.2554
VG	0.3301	0.3300	0.2332	0.2330	0.1520	0.1517
Meix/CIR	0.2374	0.2283	0.2269	0.2144	0.2198	0.2055
NIG/CIR	0.3609	0.3608	0.3393	0.3393	0.3345	0.3341
VG/CIR	0.3572	0.3572	0.3395	0.3395	0.3378	0.3372
Meix/Γ-OU	0.3571	0.3571	0.3353	0.3357	0.3286	0.3284
NIG/Γ-OU	0.3765	0.3765	0.3459	0.3464	0.3358	0.3356
VG/Γ-OU	0.3830	0.3830	0.3465	0.3462	0.3346	0.3344
BS	0.3879	0.3879	0.3523	0.3527	0.3518	0.3519

Table 4.39. Critical participation rates of Asian-end CARs under different models.

	α	Meix	NIG	VG	$\frac{\text{Meix}}{\text{CIR}}$	$\frac{\text{NIG}}{\text{CIR}}$	$\frac{\text{VG}}{\text{CIR}}$	$\frac{\text{Meix}}{\Gamma\text{-OU}}$	$\frac{\text{NIG}}{\Gamma\text{-OU}}$	$\frac{\text{VG}}{\Gamma\text{-OU}}$	BS
$c = 0.1$	0.5399	0.00	−0.01	0.00	0.02	0.00	0.04	−0.03	−0.03	−0.03	0.18
$T = 3$	0.5371	0.01	0.00	0.01	0.03	0.01	0.05	−0.02	−0.02	−0.02	0.19
	0.5391	0.00	−0.01	0.00	0.02	0.00	0.04	−0.03	−0.02	−0.03	0.19
	0.5458	−0.02	−0.03	−0.03	0.00	−0.02	0.02	−0.05	−0.05	−0.05	0.16
	0.5402	0.00	−0.01	−0.01	0.02	0.00	0.04	−0.03	−0.03	−0.03	0.18
	0.5518	−0.05	−0.06	−0.05	−0.02	−0.04	0.00	−0.07	−0.07	−0.07	0.15
	0.5322	0.03	0.02	0.02	0.05	0.03	0.07	0.00	0.00	0.00	0.21
	0.5331	0.02	0.01	0.02	0.05	0.03	0.07	−0.01	0.00	0.00	0.20
	0.5325	0.03	0.02	0.02	0.05	0.03	0.07	0.00	0.00	0.00	0.21
	0.6032	−0.23	−0.24	−0.23	−0.21	−0.23	−0.18	−0.25	−0.25	−0.25	0.00
$c = 0.1$	0.5399	0.00	−0.01	0.00	0.02	0.00	0.04	−0.03	−0.03	−0.03	0.18
$T = 7$	0.5371	0.01	0.00	0.01	0.03	0.01	0.05	−0.02	−0.02	−0.02	0.19
	0.5391	0.00	−0.01	0.00	0.02	0.00	0.04	−0.03	−0.02	−0.03	0.19
	0.5458	−0.02	−0.03	−0.03	0.00	−0.02	0.02	−0.05	−0.05	−0.05	0.16
	0.5402	0.00	−0.01	−0.01	0.02	0.00	0.04	−0.03	−0.03	−0.03	0.18
	0.5518	−0.05	−0.06	−0.05	−0.02	−0.04	0.00	−0.07	−0.07	−0.07	0.15
	0.5322	0.03	0.02	0.02	0.05	0.03	0.07	0.00	0.00	0.00	0.21
	0.5331	0.02	0.01	0.02	0.05	0.03	0.07	−0.01	0.00	0.00	0.20
	0.5325	0.03	0.02	0.02	0.05	0.03	0.07	0.00	0.00	0.00	0.21
	0.6032	−0.23	−0.24	−0.23	−0.21	−0.23	−0.18	−0.25	−0.25	−0.25	0.00
$c = 0.2$	0.5127	0.00	−0.02	0.01	0.09	0.03	0.09	−0.06	−0.04	−0.06	0.15
$T = 3$	0.5090	0.01	0.00	0.02	0.10	0.04	0.10	−0.05	−0.03	−0.05	0.17
	0.5148	−0.01	−0.03	0.00	0.08	0.02	0.08	−0.07	−0.05	−0.07	0.14
	0.5323	−0.09	−0.10	−0.08	0.00	−0.06	0.00	−0.15	−0.13	−0.15	0.07
	0.5185	−0.03	−0.04	−0.02	0.06	0.00	0.06	−0.09	−0.07	−0.09	0.13
	0.5318	−0.09	−0.10	−0.08	0.00	−0.06	0.00	−0.15	−0.13	−0.14	0.07
	0.4987	0.06	0.04	0.07	0.15	0.09	0.15	0.00	0.02	0.00	0.21
	0.5031	0.04	0.02	0.05	0.13	0.07	0.13	−0.02	0.00	−0.02	0.19
	0.4990	0.06	0.04	0.07	0.15	0.09	0.15	0.00	0.02	0.00	0.21
	0.5485	−0.16	−0.17	−0.15	−0.07	−0.13	−0.07	−0.22	−0.20	−0.22	0.00
$c = 0.2$	0.5127	0.00	−0.02	0.01	0.09	0.03	0.09	−0.06	−0.04	−0.06	0.15
$T = 7$	0.5090	0.01	0.00	0.02	0.10	0.04	0.10	−0.05	−0.03	−0.05	0.17
	0.5148	−0.01	−0.03	0.00	0.08	0.02	0.08	−0.07	−0.05	−0.07	0.14
	0.5323	−0.09	−0.10	−0.08	0.00	−0.06	0.00	−0.15	−0.13	−0.15	0.07
	0.5185	−0.03	−0.04	−0.02	0.06	0.00	0.06	−0.09	−0.07	−0.09	0.13
	0.5317	−0.09	−0.10	−0.08	0.00	−0.06	0.00	−0.15	−0.13	−0.14	0.07
	0.4987	0.06	0.04	0.07	0.15	0.09	0.15	0.00	0.02	0.00	0.21
	0.5031	0.04	0.02	0.05	0.13	0.07	0.13	−0.02	0.00	−0.02	0.19
	0.4990	0.06	0.04	0.07	0.15	0.09	0.15	0.00	0.02	0.00	0.21
	0.5485	−0.16	−0.17	−0.15	−0.07	−0.13	−0.07	−0.22	−0.20	−0.22	0.00
$c = 0.3$	0.4986	0.00	−0.02	0.02	0.15	0.04	0.13	−0.09	−0.06	−0.09	0.21
$T = 3$	0.4936	0.02	0.00	0.04	0.17	0.06	0.15	−0.07	−0.03	−0.06	0.23
	0.5028	−0.02	−0.04	0.00	0.13	0.02	0.11	−0.11	−0.08	−0.10	0.19
	0.5316	−0.14	−0.17	−0.13	0.00	−0.10	−0.01	−0.23	−0.20	−0.23	0.07
	0.5077	−0.04	−0.06	−0.02	0.11	0.00	0.09	−0.13	−0.10	−0.13	0.17
	0.5283	−0.13	−0.15	−0.11	0.02	−0.09	0.00	−0.22	−0.19	−0.22	0.08
	0.4784	0.09	0.07	0.11	0.24	0.13	0.22	0.00	0.03	0.00	0.30
	0.4856	0.06	0.03	0.08	0.21	0.10	0.19	−0.03	0.00	−0.03	0.27
	0.4788	0.09	0.06	0.10	0.24	0.13	0.22	0.00	0.03	0.00	0.29
	0.5477	−0.21	−0.24	−0.20	−0.07	−0.17	−0.08	−0.30	−0.27	−0.30	0.00
$c = 0.3$	0.4986	0.00	−0.02	0.02	0.15	0.04	0.13	−0.09	−0.06	−0.09	0.21
$T = 7$	0.4936	0.02	0.00	0.04	0.17	0.06	0.15	−0.07	−0.03	−0.06	0.23
	0.5028	−0.02	−0.04	0.00	0.13	0.02	0.11	−0.11	−0.08	−0.10	0.19
	0.5316	−0.14	−0.17	−0.13	0.00	−0.10	−0.01	−0.23	−0.20	−0.23	0.07
	0.5076	−0.04	−0.06	−0.02	0.11	0.00	0.09	−0.13	−0.10	−0.13	0.17
	0.5281	−0.13	−0.15	−0.11	0.02	−0.09	0.00	−0.22	−0.19	−0.21	0.08
	0.4784	0.09	0.07	0.11	0.24	0.13	0.22	0.00	0.03	0.00	0.30
	0.4856	0.06	0.03	0.08	0.21	0.10	0.19	−0.03	0.00	−0.03	0.27
	0.4788	0.09	0.06	0.10	0.24	0.13	0.22	0.00	0.03	0.00	0.29
	0.5477	−0.21	−0.24	−0.20	−0.07	−0.17	−0.08	−0.30	−0.27	−0.30	0.00

Table 4.40. Values of $100 \times (1 - \text{Asian-end CAR price})$ using S&P 500 $\mathbb{Q}$ parameters.

	α	NIG	VG	$\frac{\text{NIG}}{\text{CIR}}$	$\frac{\text{VG}}{\text{CIR}}$	$\frac{\text{NIG}}{\Gamma\text{-OU}}$	$\frac{\text{VG}}{\Gamma\text{-OU}}$	BS	SV	SVJ
$c = 0.1$	0.5399	0.9619	0.9624	0.9648	0.9633	0.9636	0.9634	0.9596	0.9623	0.9601
$T = 3$	0.5371	0.9615	0.9621	0.9644	0.9630	0.9632	0.9631	0.9594	0.9620	0.9598
	0.5391	0.9618	0.9623	0.9647	0.9632	0.9635	0.9633	0.9596	0.9622	0.9600
	0.5458	0.9625	0.9631	0.9654	0.9640	0.9643	0.9641	0.9602	0.9629	0.9607
	0.5402	0.9619	0.9625	0.9648	0.9634	0.9636	0.9635	0.9597	0.9623	0.9601
	0.5518	0.9632	0.9638	0.9661	0.9646	0.9650	0.9649	0.9607	0.9635	0.9613
	0.5322	0.9610	0.9615	0.9639	0.9625	0.9626	0.9625	0.9589	0.9614	0.9593
	0.5331	0.9611	0.9616	0.9640	0.9626	0.9627	0.9626	0.9590	0.9615	0.9594
	0.5325	0.9610	0.9615	0.9639	0.9625	0.9626	0.9625	0.9589	0.9615	0.9593
	0.6032	0.9689	0.9696	0.9715	0.9700	0.9711	0.9709	0.9652	0.9688	0.9664
$c = 0.1$	0.5399	0.8665	0.8673	0.8728	0.8723	0.8702	0.8701	0.8604	0.8720	0.8692
$T = 7$	0.5371	0.8658	0.8666	0.8722	0.8717	0.8694	0.8694	0.8599	0.8714	0.8686
	0.5391	0.8663	0.8671	0.8726	0.8721	0.8700	0.8699	0.8602	0.8719	0.8690
	0.5458	0.8680	0.8687	0.8742	0.8736	0.8717	0.8717	0.8615	0.8734	0.8704
	0.5402	0.8666	0.8673	0.8729	0.8723	0.8702	0.8702	0.8605	0.8721	0.8693
	0.5518	0.8694	0.8702	0.8755	0.8750	0.8734	0.8733	0.8626	0.8747	0.8716
	0.5322	0.8646	0.8653	0.8710	0.8705	0.8681	0.8680	0.8589	0.8703	0.8676
	0.5331	0.8648	0.8656	0.8712	0.8707	0.8684	0.8683	0.8591	0.8705	0.8678
	0.5325	0.8647	0.8654	0.8711	0.8706	0.8682	0.8681	0.8590	0.8704	0.8677
	0.6032	0.8815	0.8825	0.8867	0.8860	0.8866	0.8868	0.8718	0.8857	0.8818
$c = 0.2$	0.5127	0.9604	0.9609	0.9655	0.9639	0.9613	0.9612	0.9628	0.9644	0.9619
$T = 3$	0.5090	0.9599	0.9604	0.9649	0.9634	0.9608	0.9607	0.9623	0.9639	0.9614
	0.5148	0.9607	0.9612	0.9658	0.9642	0.9616	0.9615	0.9631	0.9647	0.9622
	0.5323	0.9632	0.9637	0.9684	0.9668	0.9641	0.9640	0.9656	0.9672	0.9646
	0.5185	0.9612	0.9617	0.9664	0.9648	0.9621	0.9620	0.9636	0.9652	0.9627
	0.5318	0.9631	0.9636	0.9684	0.9667	0.9640	0.9639	0.9655	0.9672	0.9645
	0.4987	0.9584	0.9589	0.9634	0.9619	0.9593	0.9592	0.9608	0.9624	0.9599
	0.5031	0.9591	0.9595	0.9641	0.9625	0.9599	0.9598	0.9614	0.9630	0.9605
	0.4990	0.9585	0.9589	0.9634	0.9619	0.9593	0.9592	0.9608	0.9624	0.9600
	0.5485	0.9655	0.9660	0.9709	0.9692	0.9664	0.9663	0.9679	0.9696	0.9669
$c = 0.2$	0.5127	0.8642	0.8647	0.8773	0.8786	0.8655	0.8653	0.8679	0.8816	0.8843
$T = 7$	0.5090	0.8631	0.8636	0.8760	0.8774	0.8643	0.8642	0.8668	0.8804	0.8831
	0.5148	0.8649	0.8654	0.8780	0.8793	0.8661	0.8660	0.8685	0.8823	0.8850
	0.5323	0.8702	0.8708	0.8838	0.8851	0.8716	0.8714	0.8737	0.8881	0.8906
	0.5185	0.8660	0.8665	0.8792	0.8805	0.8673	0.8671	0.8696	0.8835	0.8862
	0.5317	0.8700	0.8706	0.8835	0.8849	0.8714	0.8712	0.8735	0.8879	0.8904
	0.4987	0.8600	0.8605	0.8726	0.8740	0.8611	0.8610	0.8637	0.8770	0.8798
	0.5031	0.8613	0.8618	0.8741	0.8754	0.8625	0.8624	0.8650	0.8785	0.8812
	0.4990	0.8601	0.8605	0.8727	0.8741	0.8612	0.8611	0.8638	0.8771	0.8799
	0.5485	0.8752	0.8758	0.8891	0.8904	0.8767	0.8765	0.8786	0.8934	0.8958
$c = 0.3$	0.4986	0.9584	0.9589	0.9635	0.9620	0.9593	0.9592	0.9609	0.9628	0.9603
$T = 3$	0.4936	0.9577	0.9582	0.9627	0.9612	0.9586	0.9585	0.9602	0.9621	0.9595
	0.5028	0.9590	0.9595	0.9641	0.9626	0.9599	0.9598	0.9616	0.9635	0.9609
	0.5316	0.9631	0.9636	0.9685	0.9669	0.9640	0.9639	0.9658	0.9678	0.9650
	0.5077	0.9597	0.9602	0.9648	0.9633	0.9606	0.9605	0.9623	0.9642	0.9616
	0.5283	0.9626	0.9631	0.9680	0.9664	0.9635	0.9634	0.9653	0.9673	0.9645
	0.4784	0.9556	0.9560	0.9604	0.9590	0.9564	0.9563	0.9580	0.9598	0.9574
	0.4856	0.9566	0.9570	0.9615	0.9600	0.9574	0.9573	0.9590	0.9609	0.9584
	0.4788	0.9556	0.9561	0.9605	0.9590	0.9564	0.9564	0.9581	0.9599	0.9574
	0.5477	0.9654	0.9659	0.9709	0.9693	0.9663	0.9662	0.9682	0.9702	0.9673
$c = 0.3$	0.4986	0.8600	0.8605	0.8729	0.8745	0.8611	0.8610	0.8640	0.8787	0.8831
$T = 7$	0.4936	0.8585	0.8589	0.8712	0.8729	0.8596	0.8595	0.8625	0.8769	0.8814
	0.5028	0.8612	0.8617	0.8743	0.8760	0.8624	0.8623	0.8653	0.8801	0.8846
	0.5316	0.8700	0.8706	0.8840	0.8857	0.8713	0.8712	0.8741	0.8901	0.8947
	0.5076	0.8627	0.8632	0.8759	0.8776	0.8639	0.8638	0.8667	0.8818	0.8863
	0.5281	0.8690	0.8695	0.8828	0.8846	0.8703	0.8701	0.8730	0.8889	0.8935
	0.4784	0.8539	0.8544	0.8662	0.8678	0.8550	0.8548	0.8579	0.8717	0.8761
	0.4856	0.8561	0.8565	0.8686	0.8702	0.8571	0.8570	0.8601	0.8742	0.8786
	0.4788	0.8540	0.8545	0.8663	0.8679	0.8551	0.8550	0.8580	0.8719	0.8763
	0.5477	0.8750	0.8756	0.8895	0.8913	0.8764	0.8763	0.8791	0.8958	0.9004

Table 4.41. Monte Carlo no-surrender prices for Asian-end CARs using S&P 500 $\mathbb{Q}$ parameters. In percentage, the maximum standard error here is found to be about 0.02% of the price.

	α	NIG	VG	$\frac{\text{NIG}}{\text{CIR}}$	$\frac{\text{VG}}{\text{CIR}}$	$\frac{\text{NIG}}{\Gamma\text{-OU}}$	$\frac{\text{VG}}{\Gamma\text{-OU}}$	BS
$c = 0.1$	0.5399	0.0020	0.0012	0.0005	0.0007	0.0017	0.0020	−0.0001
$T = 3$	0.5371	0.0020	0.0012	0.0005	0.0007	0.0017	0.0020	−0.0001
	0.5391	0.0020	0.0012	0.0005	0.0007	0.0017	0.0020	−0.0001
	0.5458	0.0020	0.0012	0.0005	0.0007	0.0017	0.0020	−0.0001
	0.5402	0.0020	0.0012	0.0005	0.0007	0.0017	0.0020	−0.0001
	0.5518	0.0020	0.0012	0.0006	0.0007	0.0017	0.0020	−0.0001
	0.5322	0.0020	0.0012	0.0005	0.0007	0.0017	0.0020	−0.0001
	0.5331	0.0020	0.0012	0.0005	0.0007	0.0017	0.0020	−0.0001
	0.5325	0.0020	0.0012	0.0005	0.0007	0.0017	0.0020	−0.0001
	0.6032	0.0020	0.0012	0.0006	0.0005	0.0016	0.0020	−0.0001
$c = 0.1$	0.5399	0.0034	0.0023	0.0015	0.0012	0.0034	0.0039	−0.0002
$T = 7$	0.5371	0.0034	0.0023	0.0015	0.0013	0.0034	0.0039	−0.0002
	0.5391	0.0034	0.0023	0.0015	0.0013	0.0034	0.0039	−0.0002
	0.5458	0.0034	0.0023	0.0015	0.0012	0.0034	0.0039	−0.0002
	0.5402	0.0034	0.0023	0.0015	0.0012	0.0034	0.0039	−0.0002
	0.5518	0.0034	0.0023	0.0016	0.0011	0.0034	0.0039	−0.0002
	0.5322	0.0034	0.0023	0.0015	0.0013	0.0034	0.0039	−0.0002
	0.5331	0.0034	0.0023	0.0015	0.0013	0.0034	0.0039	−0.0002
	0.5325	0.0034	0.0023	0.0015	0.0013	0.0034	0.0039	−0.0002
	0.6032	0.0035	0.0024	0.0018	0.0007	0.0034	0.0040	−0.0002
$c = 0.2$	0.5127	0.0038	0.0027	0.0015	0.0014	0.0037	0.0043	−0.0000
$T = 3$	0.5090	−0.0037	−0.0032	−0.0038	−0.0006	−0.0049	−0.0057	−0.0010
	0.5148	−0.0037	−0.0032	−0.0038	−0.0007	−0.0050	−0.0057	−0.0011
	0.5323	−0.0038	−0.0033	−0.0039	−0.0007	−0.0050	−0.0058	−0.0015
	0.5185	−0.0037	−0.0032	−0.0038	−0.0007	−0.0050	−0.0057	−0.0012
	0.5318	−0.0038	−0.0033	−0.0039	−0.0007	−0.0050	−0.0058	−0.0015
	0.4987	0.0038	0.0027	0.0015	0.0014	0.0038	0.0043	−0.0000
	0.5031	−0.0037	−0.0031	−0.0038	−0.0006	−0.0049	−0.0057	−0.0009
	0.4990	−0.0037	−0.0031	−0.0038	−0.0006	−0.0049	−0.0057	−0.0009
	0.5485	−0.0039	−0.0033	−0.0039	−0.0007	−0.0050	−0.0058	−0.0019
$c = 0.2$	0.5127	0.0075	0.0055	0.0037	0.0031	0.0078	0.0088	0.0001
$T = 7$	0.5090	0.0007	0.0003	−0.0011	0.0013	−0.0001	−0.0002	−0.0008
	0.5148	0.0007	0.0003	−0.0011	0.0013	−0.0001	−0.0002	−0.0009
	0.5323	0.0006	0.0002	−0.0012	0.0012	−0.0001	−0.0002	−0.0012
	0.5185	0.0075	0.0055	0.0037	0.0031	0.0078	0.0088	0.0001
	0.5317	0.0006	0.0002	−0.0012	0.0012	−0.0001	−0.0002	−0.0012
	0.4987	0.0074	0.0055	0.0037	0.0031	0.0078	0.0087	0.0001
	0.5031	0.0007	0.0003	−0.0010	0.0013	−0.0000	−0.0002	−0.0007
	0.4990	0.0007	0.0003	−0.0010	0.0013	−0.0000	−0.0002	−0.0006
	0.5485	0.0006	0.0001	−0.0012	0.0012	−0.0001	−0.0003	−0.0016
$c = 0.3$	0.4986	0.0056	0.0041	0.0029	0.0019	0.0057	0.0065	−0.0001
$T = 3$	0.4936	0.0056	0.0041	0.0029	0.0019	0.0057	0.0065	−0.0000
	0.5028	0.0056	0.0041	0.0029	0.0019	0.0057	0.0065	−0.0001
	0.5316	0.0057	0.0041	0.0028	0.0019	0.0057	0.0065	−0.0001
	0.5077	0.0056	0.0041	0.0029	0.0019	0.0057	0.0065	−0.0001
	0.5283	0.0057	0.0041	0.0028	0.0019	0.0057	0.0065	−0.0001
	0.4784	0.0056	0.0041	0.0029	0.0019	0.0057	0.0065	−0.0000
	0.4856	0.0056	0.0041	0.0029	0.0019	0.0057	0.0065	−0.0000
	0.4788	0.0056	0.0041	0.0029	0.0019	0.0057	0.0065	−0.0000
	0.5477	0.0057	0.0041	0.0028	0.0019	0.0058	0.0065	−0.0001
$c = 0.3$	0.4986	0.0112	0.0083	0.0067	0.0043	0.0117	0.0132	0.0000
$T = 7$	0.4936	0.0112	0.0083	0.0067	0.0043	0.0117	0.0132	0.0000
	0.5028	0.0112	0.0084	0.0067	0.0043	0.0118	0.0133	0.0000
	0.5316	0.0113	0.0084	0.0066	0.0042	0.0119	0.0134	0.0000
	0.5076	0.0112	0.0084	0.0067	0.0043	0.0118	0.0133	0.0000
	0.5281	0.0113	0.0084	0.0066	0.0042	0.0119	0.0134	0.0000
	0.4784	0.0111	0.0083	0.0067	0.0043	0.0117	0.0132	0.0000
	0.4856	0.0111	0.0083	0.0067	0.0043	0.0117	0.0132	0.0000
	0.4788	0.0111	0.0083	0.0067	0.0043	0.0117	0.0132	0.0000
	0.5477	0.0114	0.0085	0.0066	0.0042	0.0119	0.0135	−0.0000

Table 4.42. Sequential quadrature prices minus Monte Carlo prices for the EIAs mentioned in table 4.41.

	$c = 0.1$		$c = 0.2$		$c = 0.3$	
	$T = 3$	$T = 7$	$T = 3$	$T = 7$	$T = 3$	$T = 7$
S&P 500 $\mathbb{P} + \mathbb{Q}$ parameters						
Meix	0.2048	0.2048	0.2043	0.2042	0.2042	0.2046
NIG	0.2244	0.2244	0.2229	0.2228	0.2232	0.2232
VG	0.1941	0.1940	0.1932	0.1932	0.1932	0.1932
Meix/CIR	0.1804	0.1802	0.1802	0.1802	0.1802	0.1802
NIG/CIR	0.1800	0.1797	0.1798	0.1797	0.1798	0.1797
VG/CIR	0.1856	0.1855	0.1852	0.1851	0.1852	0.1851
Meix/Γ-OU	0.2057	0.2056	0.2047	0.2050	0.2050	0.2050
NIG/Γ-OU	0.3143	0.3143	0.3042	0.3042	0.3040	0.3040
VG/Γ-OU	0.1862	0.1862	0.1861	0.1861	0.1861	0.1861
S&P 500 $\mathbb{Q}$ parameters						
Meix	0.2101	0.2101	0.2092	0.2092	0.2092	0.2095
NIG	0.2118	0.2118	0.2110	0.2110	0.2110	0.2110
VG	0.2087	0.2083	0.2079	0.2079	0.2079	0.2079
Meix/CIR	0.1790	0.1791	0.1789	0.1788	0.1789	0.1787
NIG/CIR	0.2476	0.2476	0.2459	0.2459	0.2459	0.2459
VG/CIR	0.1966	0.1965	0.1964	0.1964	0.1964	0.1964
Meix/Γ-OU	0.3419	0.3418	0.3319	0.3318	0.3318	0.3318
NIG/Γ-OU	0.3015	0.3015	0.2972	0.2972	0.2972	0.2972
VG/Γ-OU	0.3402	0.3402	0.3305	0.3305	0.3304	0.3304
BS	0.1980	0.1977	0.1963	0.1962	0.1962	0.1961
Eurostoxx 50 parameters						
Meix	0.1716	0.1715	0.1712	0.1712	0.1711	0.1712
NIG	0.1738	0.1738	0.1735	0.1735	0.1735	0.1735
VG	0.1755	0.1754	0.1752	0.1752	0.1754	0.1752
Meix/CIR	0.1229	0.1228	0.1225	0.1224	0.1224	0.1224
NIG/CIR	0.1302	0.1301	0.1297	0.1296	0.1296	0.1296
VG/CIR	0.1167	0.1168	0.1167	0.1166	0.1167	0.1166
Meix/Γ-OU	0.1710	0.1708	0.1706	0.1707	0.1705	0.1706
NIG/Γ-OU	0.1720	0.1717	0.1701	0.1701	0.1698	0.1699
VG/Γ-OU	0.1951	0.1945	0.1919	0.1917	0.1912	0.1913
BS	0.1297	0.1296	0.1287	0.1286	0.1286	0.1286

Table 4.43. Critical participation rates of reverse HWM PTPs under different models.

	α	Meix	NIG	VG	$\frac{\text{Meix}}{\text{CIR}}$	$\frac{\text{NIG}}{\text{CIR}}$	$\frac{\text{VG}}{\text{CIR}}$	$\frac{\text{Meix}}{\Gamma\text{-OU}}$	$\frac{\text{NIG}}{\Gamma\text{-OU}}$	$\frac{\text{VG}}{\Gamma\text{-OU}}$	BS
$c = 0.1$	0.1716	−0.04	−0.06	−0.06	−1.06	−0.97	−1.23	−0.05	−0.20	0.16	−0.63
$T = 3$	0.1738	−0.07	−0.09	−0.09	−1.11	−1.02	−1.28	−0.09	−0.23	0.13	−0.66
	0.1755	−0.09	−0.11	−0.11	−1.15	−1.06	−1.32	−0.11	−0.25	0.10	−0.68
	0.1229	0.57	0.56	0.55	−0.01	0.05	−0.10	0.75	0.54	0.90	0.11
	0.1302	0.48	0.47	0.46	−0.16	−0.09	−0.25	0.63	0.43	0.79	−0.01
	0.1167	0.65	0.64	0.63	0.11	0.17	0.03	0.85	0.63	1.00	0.21
	0.1710	−0.03	−0.05	−0.06	−1.05	−0.95	−1.21	−0.04	−0.19	0.17	−0.62
	0.1720	−0.04	−0.06	−0.07	−1.07	−0.98	−1.24	−0.06	−0.20	0.15	−0.63
	0.1951	−0.33	−0.35	−0.36	−1.61	−1.51	−1.81	−0.43	−0.54	−0.18	−0.98
	0.1297	0.48	0.47	0.46	−0.15	−0.08	−0.24	0.63	0.43	0.80	0.00
$c = 0.1$	0.1715	−0.04	−0.06	−0.06	−1.09	−0.99	−1.27	−0.05	−0.20	0.16	−0.63
$T = 7$	0.1738	−0.07	−0.09	−0.09	−1.15	−1.04	−1.33	−0.09	−0.23	0.12	−0.66
	0.1754	−0.09	−0.11	−0.11	−1.19	−1.08	−1.37	−0.11	−0.25	0.10	−0.68
	0.1228	0.57	0.56	0.55	−0.01	0.06	−0.10	0.75	0.54	0.90	0.11
	0.1301	0.48	0.47	0.46	−0.15	−0.08	−0.25	0.63	0.43	0.79	0.00
	0.1168	0.64	0.64	0.63	0.10	0.17	0.03	0.85	0.63	0.99	0.20
	0.1708	−0.03	−0.05	−0.05	−1.07	−0.97	−1.25	−0.04	−0.18	0.17	−0.62
	0.1717	−0.04	−0.06	−0.06	−1.09	−0.99	−1.27	−0.05	−0.20	0.16	−0.63
	0.1945	−0.32	−0.35	−0.35	−1.66	−1.55	−1.88	−0.42	−0.54	−0.18	−0.98
	0.1296	0.49	0.48	0.47	−0.15	−0.08	−0.24	0.64	0.44	0.80	0.00
$c = 0.2$	0.1712	−0.04	−0.05	−0.06	−1.09	−0.99	−1.25	−0.05	−0.22	0.14	−0.69
$T = 3$	0.1735	−0.06	−0.08	−0.09	−1.15	−1.05	−1.31	−0.09	−0.25	0.10	−0.73
	0.1752	−0.09	−0.11	−0.11	−1.19	−1.09	−1.35	−0.11	−0.28	0.08	−0.75
	0.1225	0.57	0.57	0.56	−0.01	0.05	−0.09	0.75	0.54	0.90	0.10
	0.1297	0.48	0.47	0.46	−0.16	−0.09	−0.24	0.63	0.43	0.78	−0.01
	0.1167	0.65	0.64	0.63	0.10	0.16	0.03	0.85	0.63	0.99	0.20
	0.1706	−0.03	−0.05	−0.05	−1.08	−0.98	−1.23	−0.04	−0.21	0.15	−0.68
	0.1701	−0.02	−0.04	−0.05	−1.06	−0.97	−1.22	−0.03	−0.20	0.16	−0.67
	0.1919	−0.30	−0.32	−0.32	−1.61	−1.50	−1.80	−0.39	−0.54	−0.18	−1.04
	0.1287	0.50	0.49	0.48	−0.14	−0.07	−0.22	0.65	0.44	0.80	0.00
$c = 0.2$	0.1712	−0.04	−0.05	−0.06	−1.12	−1.02	−1.29	−0.05	−0.22	0.14	−0.69
$T = 7$	0.1735	−0.07	−0.08	−0.09	−1.18	−1.08	−1.36	−0.09	−0.25	0.10	−0.73
	0.1752	−0.09	−0.11	−0.11	−1.22	−1.12	−1.40	−0.12	−0.28	0.07	−0.75
	0.1224	0.57	0.57	0.56	−0.01	0.06	−0.09	0.75	0.54	0.89	0.10
	0.1296	0.48	0.48	0.47	−0.16	−0.09	−0.25	0.63	0.43	0.78	−0.01
	0.1166	0.65	0.64	0.63	0.10	0.17	0.03	0.85	0.63	0.99	0.20
	0.1707	−0.03	−0.05	−0.05	−1.11	−1.00	−1.28	−0.04	−0.21	0.14	−0.68
	0.1701	−0.02	−0.04	−0.05	−1.09	−0.99	−1.26	−0.03	−0.20	0.15	−0.67
	0.1917	−0.29	−0.32	−0.32	−1.66	−1.54	−1.87	−0.39	−0.54	−0.18	−1.04
	0.1286	0.50	0.49	0.48	−0.14	−0.07	−0.22	0.65	0.44	0.80	0.00
$c = 0.3$	0.1711	−0.04	−0.05	−0.06	−1.09	−0.99	−1.25	−0.05	−0.22	0.13	−0.69
$T = 3$	0.1735	−0.06	−0.08	−0.09	−1.15	−1.05	−1.31	−0.09	−0.25	0.10	−0.73
	0.1754	−0.09	−0.11	−0.11	−1.20	−1.10	−1.36	−0.12	−0.28	0.07	−0.76
	0.1224	0.57	0.57	0.56	−0.01	0.06	−0.09	0.75	0.54	0.89	0.10
	0.1296	0.48	0.48	0.47	−0.16	−0.08	−0.24	0.64	0.43	0.78	−0.01
	0.1167	0.65	0.64	0.63	0.10	0.16	0.03	0.85	0.63	0.98	0.20
	0.1705	−0.03	−0.05	−0.05	−1.08	−0.98	−1.23	−0.04	−0.21	0.14	−0.68
	0.1698	−0.02	−0.04	−0.04	−1.06	−0.96	−1.21	−0.03	−0.20	0.15	−0.67
	0.1912	−0.29	−0.31	−0.31	−1.59	−1.49	−1.78	−0.38	−0.53	−0.18	−1.03
	0.1286	0.50	0.49	0.48	−0.14	−0.07	−0.22	0.65	0.44	0.80	0.00
$c = 0.3$	0.1712	−0.04	−0.05	−0.06	−1.12	−1.02	−1.29	−0.05	−0.22	0.13	−0.69
$T = 7$	0.1735	−0.07	−0.08	−0.09	−1.18	−1.08	−1.36	−0.09	−0.25	0.09	−0.73
	0.1752	−0.09	−0.11	−0.11	−1.22	−1.12	−1.40	−0.12	−0.28	0.07	−0.76
	0.1224	0.58	0.57	0.56	−0.01	0.06	−0.09	0.75	0.54	0.89	0.11
	0.1296	0.48	0.48	0.47	−0.16	−0.08	−0.24	0.63	0.43	0.78	−0.01
	0.1166	0.65	0.64	0.63	0.10	0.17	0.03	0.85	0.63	0.98	0.20
	0.1706	−0.03	−0.05	−0.05	−1.11	−1.00	−1.28	−0.04	−0.21	0.14	−0.68
	0.1699	−0.02	−0.04	−0.04	−1.09	−0.99	−1.26	−0.03	−0.20	0.15	−0.67
	0.1913	−0.29	−0.31	−0.31	−1.65	−1.53	−1.86	−0.38	−0.53	−0.18	−1.03
	0.1286	0.50	0.49	0.48	−0.14	−0.06	−0.22	0.65	0.44	0.80	0.00

Table 4.44. Values of $100 \times (1 -$ reverse HWM PTP price$)$ using Eurostoxx 50 parameters.

	α	NIG	VG	NIG CIR	VG CIR	NIG Γ-OU	VG Γ-OU	BS	SV	SVJ
$c = 0.1$	0.1716	0.9645	0.9643	0.9771	0.9799	0.9697	0.9705	0.9691	0.9792	0.9803
$T = 3$	0.1738	0.9651	0.9649	0.9779	0.9807	0.9704	0.9712	0.9697	0.9800	0.9811
	0.1755	0.9656	0.9654	0.9785	0.9813	0.9709	0.9717	0.9702	0.9806	0.9817
	0.1229	0.9511	0.9509	0.9602	0.9622	0.9549	0.9556	0.9545	0.9617	0.9625
	0.1302	0.9531	0.9529	0.9627	0.9648	0.9571	0.9578	0.9567	0.9643	0.9651
	0.1167	0.9494	0.9492	0.9580	0.9599	0.9530	0.9536	0.9526	0.9594	0.9602
	0.1710	0.9643	0.9641	0.9769	0.9797	0.9695	0.9703	0.9689	0.9790	0.9801
	0.1720	0.9646	0.9644	0.9773	0.9801	0.9698	0.9707	0.9692	0.9793	0.9805
	0.1951	0.9709	0.9707	0.9852	0.9884	0.9768	0.9777	0.9760	0.9875	0.9889
	0.1297	0.9530	0.9528	0.9626	0.9647	0.9570	0.9577	0.9565	0.9641	0.9650
$c = 0.1$	0.1715	0.8743	0.8740	0.8864	0.8906	0.8752	0.8757	0.8792	0.8897	0.8903
$T = 7$	0.1738	0.8751	0.8748	0.8875	0.8918	0.8761	0.8767	0.8801	0.8909	0.8915
	0.1754	0.8757	0.8754	0.8883	0.8926	0.8767	0.8773	0.8807	0.8918	0.8924
	0.1228	0.8585	0.8582	0.8647	0.8671	0.8582	0.8582	0.8624	0.8665	0.8666
	0.1301	0.8606	0.8603	0.8677	0.8703	0.8604	0.8605	0.8647	0.8697	0.8698
	0.1168	0.8568	0.8566	0.8624	0.8645	0.8564	0.8564	0.8606	0.8640	0.8640
	0.1708	0.8740	0.8737	0.8860	0.8902	0.8749	0.8754	0.8789	0.8893	0.8899
	0.1717	0.8744	0.8740	0.8865	0.8906	0.8752	0.8758	0.8793	0.8898	0.8904
	0.1945	0.8828	0.8824	0.8977	0.9027	0.8843	0.8852	0.8879	0.9018	0.9027
	0.1296	0.8605	0.8602	0.8675	0.8701	0.8603	0.8604	0.8646	0.8695	0.8696
$c = 0.2$	0.1712	0.9644	0.9642	0.9771	0.9798	0.9697	0.9708	0.9692	0.9792	0.9802
$T = 3$	0.1735	0.9650	0.9648	0.9779	0.9807	0.9704	0.9715	0.9699	0.9800	0.9811
	0.1752	0.9655	0.9653	0.9785	0.9813	0.9709	0.9720	0.9704	0.9806	0.9817
	0.1225	0.9510	0.9508	0.9601	0.9620	0.9548	0.9556	0.9544	0.9615	0.9623
	0.1297	0.9529	0.9528	0.9626	0.9646	0.9570	0.9579	0.9566	0.9642	0.9650
	0.1167	0.9494	0.9492	0.9580	0.9599	0.9530	0.9538	0.9526	0.9595	0.9602
	0.1706	0.9642	0.9640	0.9769	0.9796	0.9695	0.9706	0.9690	0.9790	0.9800
	0.1701	0.9641	0.9639	0.9767	0.9794	0.9694	0.9704	0.9688	0.9788	0.9798
	0.1919	0.9701	0.9699	0.9843	0.9874	0.9761	0.9772	0.9754	0.9867	0.9879
	0.1287	0.9527	0.9525	0.9622	0.9643	0.9567	0.9576	0.9563	0.9638	0.9646
$c = 0.2$	0.1712	0.8742	0.8739	0.8863	0.8904	0.8751	0.8762	0.8793	0.8896	0.8902
$T = 7$	0.1735	0.8750	0.8747	0.8875	0.8916	0.8760	0.8772	0.8802	0.8908	0.8914
	0.1752	0.8756	0.8753	0.8883	0.8925	0.8767	0.8778	0.8808	0.8917	0.8923
	0.1224	0.8584	0.8581	0.8646	0.8669	0.8581	0.8584	0.8623	0.8664	0.8664
	0.1296	0.8605	0.8602	0.8676	0.8701	0.8603	0.8607	0.8646	0.8695	0.8696
	0.1166	0.8568	0.8565	0.8624	0.8644	0.8564	0.8566	0.8605	0.8639	0.8639
	0.1707	0.8740	0.8737	0.8861	0.8901	0.8749	0.8760	0.8791	0.8894	0.8899
	0.1701	0.8738	0.8735	0.8858	0.8899	0.8747	0.8758	0.8789	0.8891	0.8896
	0.1917	0.8817	0.8814	0.8964	0.9012	0.8833	0.8847	0.8872	0.9004	0.9012
	0.1286	0.8602	0.8599	0.8671	0.8697	0.8600	0.8604	0.8643	0.8691	0.8692
$c = 0.3$	0.1711	0.9644	0.9642	0.9771	0.9798	0.9697	0.9709	0.9691	0.9792	0.9802
$T = 3$	0.1735	0.9650	0.9648	0.9779	0.9806	0.9704	0.9716	0.9699	0.9800	0.9811
	0.1754	0.9655	0.9653	0.9786	0.9814	0.9710	0.9722	0.9705	0.9807	0.9818
	0.1224	0.9509	0.9508	0.9600	0.9620	0.9548	0.9557	0.9544	0.9615	0.9623
	0.1296	0.9529	0.9528	0.9626	0.9646	0.9570	0.9579	0.9566	0.9641	0.9649
	0.1167	0.9494	0.9492	0.9580	0.9599	0.9530	0.9539	0.9526	0.9594	0.9602
	0.1705	0.9642	0.9640	0.9769	0.9796	0.9695	0.9707	0.9690	0.9789	0.9800
	0.1698	0.9640	0.9638	0.9766	0.9793	0.9693	0.9705	0.9688	0.9787	0.9797
	0.1912	0.9699	0.9697	0.9841	0.9872	0.9759	0.9772	0.9753	0.9864	0.9876
	0.1286	0.9527	0.9525	0.9622	0.9643	0.9567	0.9576	0.9562	0.9638	0.9646
$c = 0.3$	0.1712	0.8742	0.8739	0.8863	0.8904	0.8751	0.8765	0.8793	0.8896	0.8902
$T = 7$	0.1735	0.8750	0.8747	0.8874	0.8916	0.8760	0.8774	0.8802	0.8908	0.8914
	0.1752	0.8756	0.8753	0.8883	0.8925	0.8767	0.8781	0.8808	0.8917	0.8923
	0.1224	0.8584	0.8581	0.8646	0.8669	0.8581	0.8585	0.8623	0.8663	0.8664
	0.1296	0.8605	0.8602	0.8675	0.8701	0.8603	0.8609	0.8646	0.8695	0.8696
	0.1166	0.8568	0.8565	0.8623	0.8644	0.8564	0.8567	0.8605	0.8639	0.8639
	0.1706	0.8740	0.8737	0.8861	0.8901	0.8749	0.8762	0.8791	0.8893	0.8899
	0.1699	0.8738	0.8734	0.8857	0.8898	0.8746	0.8760	0.8788	0.8890	0.8895
	0.1913	0.8816	0.8812	0.8962	0.9010	0.8831	0.8848	0.8870	0.9002	0.9010
	0.1286	0.8602	0.8599	0.8671	0.8696	0.8600	0.8605	0.8643	0.8690	0.8691

Table 4.45. Monte Carlo no-surrender prices for reverse HWM PTPs using Eurostoxx 50 parameters. In percentage, the maximum standard error here is found to be about 0.02% of the price.

	α	NIG	VG	$\frac{\text{NIG}}{\text{CIR}}$	$\frac{\text{VG}}{\text{CIR}}$	$\frac{\text{NIG}}{\Gamma\text{-OU}}$	$\frac{\text{VG}}{\Gamma\text{-OU}}$	BS
$c = 0.1$	0.1716	−0.0009	−0.0000	−0.0019	−0.0002	−0.0077	−0.0165	0.0003
$T = 3$	0.1738	−0.0009	−0.0000	−0.0018	−0.0002	−0.0077	−0.0165	0.0003
	0.1755	−0.0009	−0.0000	−0.0018	−0.0002	−0.0077	−0.0165	0.0003
	0.1229	−0.0008	−0.0001	−0.0019	−0.0001	−0.0078	−0.0164	0.0002
	0.1302	−0.0008	−0.0001	−0.0019	−0.0001	−0.0078	−0.0164	0.0002
	0.1167	−0.0008	−0.0001	−0.0019	−0.0001	−0.0078	−0.0164	0.0002
	0.1710	−0.0009	−0.0000	−0.0019	−0.0002	−0.0077	−0.0165	0.0003
	0.1720	−0.0009	−0.0000	−0.0019	−0.0002	−0.0077	−0.0165	0.0003
	0.1951	−0.0009	−0.0000	−0.0018	−0.0002	−0.0077	−0.0165	0.0003
	0.1297	−0.0008	−0.0001	−0.0019	−0.0001	−0.0078	−0.0164	0.0002
$c = 0.1$	0.1715	−0.0017	−0.0001	−0.0051	−0.0012	−0.0152	−0.0315	0.0003
$T = 7$	0.1738	−0.0017	−0.0001	−0.0051	−0.0012	−0.0152	−0.0315	0.0003
	0.1754	−0.0017	−0.0001	−0.0051	−0.0012	−0.0152	−0.0316	0.0003
	0.1228	−0.0017	−0.0002	−0.0050	−0.0010	−0.0153	−0.0314	0.0001
	0.1301	−0.0017	−0.0002	−0.0050	−0.0010	−0.0153	−0.0314	0.0002
	0.1168	−0.0017	−0.0002	−0.0050	−0.0010	−0.0153	−0.0314	0.0001
	0.1708	−0.0017	−0.0001	−0.0051	−0.0012	−0.0152	−0.0315	0.0003
	0.1717	−0.0017	−0.0001	−0.0051	−0.0012	−0.0152	−0.0315	0.0003
	0.1945	−0.0018	−0.0001	−0.0051	−0.0012	−0.0151	−0.0317	0.0003
	0.1296	−0.0017	−0.0002	−0.0050	−0.0010	−0.0153	−0.0314	0.0002
$c = 0.2$	0.1712	−0.0009	−0.0000	−0.0019	−0.0002	−0.0077	−0.0165	0.0003
$T = 3$	0.1735	−0.0009	−0.0000	−0.0019	−0.0002	−0.0077	−0.0165	0.0003
	0.1752	−0.0009	−0.0000	−0.0019	−0.0002	−0.0077	−0.0165	0.0003
	0.1225	−0.0008	−0.0001	−0.0019	−0.0001	−0.0078	−0.0164	0.0002
	0.1297	−0.0008	−0.0001	−0.0019	−0.0001	−0.0078	−0.0164	0.0002
	0.1167	−0.0008	−0.0001	−0.0019	−0.0001	−0.0078	−0.0164	0.0002
	0.1706	−0.0009	−0.0000	−0.0019	−0.0002	−0.0077	−0.0165	0.0003
	0.1701	−0.0009	−0.0000	−0.0019	−0.0002	−0.0077	−0.0165	0.0003
	0.1919	−0.0009	−0.0000	−0.0019	−0.0002	−0.0077	−0.0165	0.0003
	0.1287	−0.0008	−0.0001	−0.0019	−0.0001	−0.0078	−0.0164	0.0002
$c = 0.2$	0.1712	−0.0017	−0.0001	−0.0051	−0.0012	−0.0152	−0.0317	0.0003
$T = 7$	0.1735	−0.0017	−0.0001	−0.0051	−0.0012	−0.0152	−0.0317	0.0003
	0.1752	−0.0017	−0.0001	−0.0051	−0.0012	−0.0152	−0.0317	0.0003
	0.1224	−0.0017	−0.0002	−0.0050	−0.0010	−0.0153	−0.0315	0.0002
	0.1296	−0.0017	−0.0002	−0.0050	−0.0010	−0.0153	−0.0316	0.0002
	0.1166	−0.0017	−0.0002	−0.0050	−0.0010	−0.0153	−0.0315	0.0001
	0.1707	−0.0017	−0.0001	−0.0051	−0.0012	−0.0152	−0.0317	0.0003
	0.1701	−0.0017	−0.0001	−0.0051	−0.0012	−0.0152	−0.0317	0.0003
	0.1917	−0.0018	−0.0001	−0.0051	−0.0012	−0.0151	−0.0318	0.0003
	0.1286	−0.0017	−0.0002	−0.0050	−0.0010	−0.0153	−0.0315	0.0002
$c = 0.3$	0.1711	−0.0009	−0.0000	−0.0019	−0.0002	−0.0077	−0.0165	0.0003
$T = 3$	0.1735	−0.0009	−0.0000	−0.0019	−0.0002	−0.0077	−0.0165	0.0003
	0.1754	−0.0009	−0.0000	−0.0019	−0.0002	−0.0077	−0.0165	0.0003
	0.1224	−0.0008	−0.0001	−0.0019	−0.0001	−0.0078	−0.0164	0.0002
	0.1296	−0.0008	−0.0001	−0.0019	−0.0001	−0.0078	−0.0165	0.0002
	0.1167	−0.0008	−0.0001	−0.0019	−0.0001	−0.0078	−0.0164	0.0002
	0.1705	−0.0009	−0.0000	−0.0019	−0.0002	−0.0077	−0.0165	0.0003
	0.1698	−0.0009	−0.0000	−0.0019	−0.0002	−0.0077	−0.0165	0.0003
	0.1912	−0.0009	−0.0000	−0.0019	−0.0002	−0.0077	−0.0165	0.0003
	0.1286	−0.0008	−0.0001	−0.0019	−0.0001	−0.0078	−0.0164	0.0002
$c = 0.3$	0.1712	−0.0017	−0.0001	−0.0051	−0.0012	−0.0152	−0.0319	0.0003
$T = 7$	0.1735	−0.0017	−0.0001	−0.0051	−0.0012	−0.0152	−0.0319	0.0003
	0.1752	−0.0017	−0.0001	−0.0051	−0.0012	−0.0152	−0.0319	0.0003
	0.1224	−0.0017	−0.0002	−0.0050	−0.0010	−0.0153	−0.0317	0.0002
	0.1296	−0.0017	−0.0002	−0.0050	−0.0010	−0.0153	−0.0317	0.0002
	0.1166	−0.0017	−0.0002	−0.0050	−0.0010	−0.0154	−0.0316	0.0001
	0.1706	−0.0017	−0.0001	−0.0051	−0.0012	−0.0152	−0.0319	0.0003
	0.1699	−0.0017	−0.0001	−0.0051	−0.0012	−0.0152	−0.0319	0.0003
	0.1913	−0.0018	−0.0001	−0.0051	−0.0012	−0.0151	−0.0320	0.0003
	0.1286	−0.0017	−0.0002	−0.0050	−0.0010	−0.0153	−0.0317	0.0002

Table 4.46. Sequential quadrature prices minus Monte Carlo prices for the EIAs mentioned in table 4.45.

H	NIG	VG	$\frac{\text{NIG}}{\text{CIR}}$	$\frac{\text{VG}}{\text{CIR}}$	$\frac{\text{NIG}}{\Gamma\text{-OU}}$	$\frac{\text{VG}}{\Gamma\text{-OU}}$	BS	SV	SVJ	order
	Down-and-in barrier options (DIB)									
0.50	5.59	5.89	7.18	15.05	1.92	1.09	2.93	38.49	59.40	$\times 10^{-4}$
0.55	13.61	16.10	21.92	34.56	5.55	3.79	10.52	79.23	99.83	$\times 10^{-4}$
0.60	3.24	3.59	5.37	7.07	1.55	1.28	3.17	14.42	15.87	$\times 10^{-3}$
0.65	6.75	7.56	11.59	13.29	4.03	3.74	8.33	24.70	25.18	$\times 10^{-3}$
0.70	13.04	14.20	22.42	22.51	9.64	9.83	17.60	39.72	38.52	$\times 10^{-3}$
0.75	2.33	2.53	3.95	3.71	2.07	2.19	3.42	6.06	5.68	$\times 10^{-2}$
0.80	3.93	4.17	6.53	5.83	4.09	4.40	5.93	8.92	8.23	$\times 10^{-2}$
0.85	6.40	6.53	10.11	8.83	7.39	7.66	9.44	12.55	11.84	$\times 10^{-2}$
0.90	1.02	1.01	1.51	1.32	1.20	1.23	1.41	1.71	1.67	$\times 10^{-1}$
0.95	1.63	1.57	2.17	1.95	1.87	1.92	2.01	2.30	2.34	$\times 10^{-1}$
	Down-and-out barrier options (DOB)									
0.50	2.96	2.93	3.19	3.13	3.14	3.19	2.78	3.08	3.32	$\times 10^{-1}$
0.55	2.95	2.92	3.17	3.11	3.14	3.18	2.77	3.04	3.28	$\times 10^{-1}$
0.60	2.93	2.90	3.14	3.07	3.13	3.17	2.75	2.97	3.22	$\times 10^{-1}$
0.65	2.90	2.86	3.08	3.01	3.10	3.15	2.70	2.87	3.13	$\times 10^{-1}$
0.70	2.83	2.79	2.97	2.92	3.05	3.09	2.60	2.72	2.99	$\times 10^{-1}$
0.75	2.73	2.68	2.80	2.77	2.94	2.97	2.44	2.51	2.81	$\times 10^{-1}$
0.80	2.57	2.52	2.54	2.56	2.73	2.75	2.19	2.23	2.55	$\times 10^{-1}$
0.85	2.32	2.28	2.19	2.26	2.40	2.42	1.84	1.86	2.19	$\times 10^{-1}$
0.90	1.94	1.92	1.69	1.83	1.94	1.95	1.37	1.40	1.71	$\times 10^{-1}$
0.95	13.33	13.61	10.22	11.87	12.72	12.65	7.71	8.17	10.40	$\times 10^{-2}$
	Up-and-in barrier options (UIB)									
1.05	2.96	2.93	3.20	3.14	3.14	3.19	2.78	3.12	3.38	$\times 10^{-1}$
1.10	2.96	2.93	3.20	3.14	3.14	3.18	2.78	3.12	3.38	$\times 10^{-1}$
1.15	2.96	2.93	3.19	3.14	3.14	3.18	2.78	3.11	3.37	$\times 10^{-1}$
1.20	2.95	2.92	3.18	3.13	3.13	3.17	2.77	3.11	3.37	$\times 10^{-1}$
1.25	2.93	2.90	3.17	3.12	3.11	3.15	2.75	3.09	3.36	$\times 10^{-1}$
1.30	2.91	2.88	3.15	3.10	3.08	3.12	2.73	3.07	3.35	$\times 10^{-1}$
1.35	2.87	2.84	3.12	3.07	3.05	3.09	2.70	3.05	3.33	$\times 10^{-1}$
1.40	2.83	2.80	3.08	3.04	3.00	3.04	2.66	3.01	3.30	$\times 10^{-1}$
1.45	2.78	2.75	3.03	3.00	2.94	2.98	2.62	2.97	3.27	$\times 10^{-1}$
1.50	2.72	2.69	2.98	2.95	2.88	2.91	2.56	2.92	3.22	$\times 10^{-1}$
	Up-and-out barrier options (UOB)									
1.05	18.55	18.80	13.91	13.75	22.42	25.73	10.13	11.22	5.26	$\times 10^{-6}$
1.10	17.21	17.53	14.60	12.25	19.63	20.65	12.12	13.76	7.04	$\times 10^{-5}$
1.15	6.56	6.87	5.42	4.81	7.18	7.59	4.91	5.14	2.83	$\times 10^{-4}$
1.20	16.50	16.71	13.94	12.06	17.20	18.61	13.23	13.07	7.57	$\times 10^{-4}$
1.25	3.33	3.36	2.79	2.42	3.51	3.69	2.78	2.62	1.62	$\times 10^{-3}$
1.30	5.82	5.79	4.98	4.33	6.11	6.39	5.00	4.60	2.97	$\times 10^{-3}$
1.35	9.05	9.01	7.87	6.86	9.77	10.03	7.97	7.35	4.99	$\times 10^{-3}$
1.40	13.34	13.16	11.70	10.33	14.39	14.83	11.73	10.82	7.62	$\times 10^{-3}$
1.45	1.85	1.83	1.64	1.45	2.00	2.06	1.65	1.51	1.12	$\times 10^{-2}$
1.50	2.46	2.43	2.20	1.95	2.65	2.74	2.19	2.01	1.55	$\times 10^{-2}$

Table 4.47. Monte Carlo prices of several barrier options using S&P 500 $\mathbb{Q}$ parameters. The initial stock price is normalised to $S_0 = 1$. The price of the DIB option with $H = 0.5$ under the NIG/CIR model, e.g., is 7.18×10^{-4}. Here the prices of DIB options with $0.5 \leq H \leq 0.7$ and UOB options with $1.05 \leq H \leq 1.25$ are so cheap that the standard errors of these prices range from 1% to 10% of the simulated prices. For all other options, the standard errors are less than 1% of the simulated prices.

H	NIG	VG	$\frac{\text{NIG}}{\text{CIR}}$	$\frac{\text{VG}}{\text{CIR}}$	$\frac{\text{NIG}}{\Gamma\text{-OU}}$	$\frac{\text{VG}}{\Gamma\text{-OU}}$	BS	SV	SVJ
	Down-and-in barrier options (DIB)								
0.50	5.11	5.39	6.56	13.76	1.76	1.00	2.68	35.18	54.29
0.55	3.59	4.24	5.78	9.11	1.46	1.00	2.77	20.88	26.31
0.60	2.53	2.80	4.20	5.53	1.21	1.00	2.48	11.27	12.41
0.65	1.81	2.02	3.10	3.56	1.08	1.00	2.23	6.61	6.74
0.70	1.35	1.47	2.33	2.34	1.00	1.02	1.83	4.12	4.00
0.75	1.13	1.22	1.91	1.80	1.00	1.06	1.66	2.94	2.75
0.80	1.00	1.06	1.66	1.48	1.04	1.12	1.51	2.27	2.09
0.85	1.00	1.02	1.58	1.38	1.15	1.20	1.48	1.96	1.85
0.90	1.01	1.00	1.49	1.30	1.19	1.22	1.40	1.70	1.65
0.95	1.04	1.00	1.38	1.24	1.19	1.22	1.28	1.46	1.49
	Down-and-out barrier options (DOB)								
0.50	1.07	1.05	1.15	1.13	1.13	1.15	1.00	1.11	1.19
0.55	1.07	1.05	1.15	1.12	1.13	1.15	1.00	1.10	1.18
0.60	1.07	1.05	1.14	1.12	1.14	1.15	1.00	1.08	1.17
0.65	1.07	1.06	1.14	1.12	1.15	1.17	1.00	1.07	1.16
0.70	1.09	1.07	1.14	1.12	1.17	1.19	1.00	1.05	1.15
0.75	1.12	1.10	1.15	1.14	1.20	1.22	1.00	1.03	1.15
0.80	1.18	1.15	1.16	1.17	1.25	1.26	1.00	1.02	1.17
0.85	1.27	1.24	1.19	1.23	1.31	1.32	1.00	1.02	1.19
0.90	1.42	1.41	1.23	1.33	1.42	1.43	1.00	1.03	1.25
0.95	1.73	1.76	1.32	1.54	1.65	1.64	1.00	1.06	1.35
	Up-and-in barrier options (UIB)								
1.05	1.07	1.06	1.15	1.13	1.13	1.15	1.00	1.12	1.21
1.10	1.07	1.06	1.15	1.13	1.13	1.15	1.00	1.12	1.21
1.15	1.07	1.05	1.15	1.13	1.13	1.15	1.00	1.12	1.22
1.20	1.07	1.05	1.15	1.13	1.13	1.15	1.00	1.12	1.22
1.25	1.06	1.05	1.15	1.13	1.13	1.14	1.00	1.12	1.22
1.30	1.06	1.05	1.15	1.13	1.13	1.14	1.00	1.13	1.23
1.35	1.06	1.05	1.15	1.14	1.13	1.14	1.00	1.13	1.23
1.40	1.06	1.05	1.16	1.14	1.13	1.14	1.00	1.13	1.24
1.45	1.06	1.05	1.16	1.15	1.13	1.14	1.00	1.13	1.25
1.50	1.06	1.05	1.16	1.15	1.12	1.14	1.00	1.14	1.26
	Up-and-out barrier options (UOB)								
1.05	3.52	3.57	2.64	2.61	4.26	4.89	1.92	2.13	1.00
1.10	2.45	2.49	2.07	1.74	2.79	2.93	1.72	1.95	1.00
1.15	2.32	2.42	1.91	1.70	2.54	2.68	1.73	1.81	1.00
1.20	2.18	2.21	1.84	1.59	2.27	2.46	1.75	1.73	1.00
1.25	2.06	2.08	1.72	1.49	2.17	2.28	1.72	1.62	1.00
1.30	1.96	1.95	1.68	1.46	2.05	2.15	1.68	1.55	1.00
1.35	1.81	1.81	1.58	1.37	1.96	2.01	1.60	1.47	1.00
1.40	1.75	1.73	1.54	1.36	1.89	1.95	1.54	1.42	1.00
1.45	1.66	1.64	1.47	1.30	1.80	1.85	1.47	1.35	1.00
1.50	1.58	1.56	1.42	1.26	1.71	1.76	1.41	1.30	1.00

Table 4.48. Ratio of the simulation prices in table 4.47 from all models to the minimum one.

Part II

Implementation and application of the sequential quadrature method

Chapter 5

The sequential quadrature method

5.1 Three variants of the sequential quadrature method

Sequential quadrature refers to a numerical technique that combines the application of numerical integration in a dynamic programming fashion with approximation of integrands. The term was coined by Sullivan (2000a) and the method was proposed independently by Hunt and Kennedy (2000, pp.332-335) to evaluate the prices of Bermudan swaptions under the extended Vasicek model and by Sullivan (2000a,b) and Tse *et al.* (2001) to solve the pricing problems of American put options and barrier options under the Black-Scholes model. It is also used by Ben-Ameur *et al.* (2002) to evaluate the prices of some American style Asian options and by Fung and Li (2003, 2005) to determine the prices of dynamic fund protections under the CEV model as well as prices of barrier options under the GARCH model (we will report the results of Fung and Li 2003 in the next chapter). Fung and Li (2002) recognise that the method can in fact be applied to price any Bermudan option when the time-t price of the option is a deterministic function of the time-t value of a Markov process. Lord *et al.* (2007) recognises not only this fact, but also the fact that the price of a European option can be written as a convolution of two functions. Consequently he has found a way to apply fast Fourier transform (FFT) to the quadrature process and greatly reduces the computation time when a vast number of quadrature nodes are used.

We now illustrate the idea of sequential quadrature. Consider the valuation problem of a T-year term-end CAR where surrender is allowed and the log discounted total return X is a Levy process such that the probability density function of X_1 is known. For illustration purpose, suppose the participation rate α is equal to 1 and there are no yield spread and no local floor or cap. So the exercise value of the CAR at the end of year k is given by

$$P \text{ mid}\left\{1+F_k,\ \frac{S_k}{S_0},\ 1+C_k\right\}$$

and it can be written as a function of the form $V_k^{\text{exer}}(Z_k)$, where

$$Z_k = X_k = Z_{k-1} + X_k - X_{k-1}.$$

So Z_k is a continuous function of Z_{k-1} and $X_k - X_{k-1}$. Let $V_k(Z_k)$ denotes the time-k price of the ratchet and $V_k^{\text{cont}}(Z_k)$ denotes the *continuation value* of the ratchet, i.e. the value of the ratchet given that the policy holder continues to hold the ratchet at time k. By mathematical induction, if we set $V_T(Z_T) = V_T^{\text{exer}}(Z_T)$ and $V_T^{\text{cont}}(Z_T) = 0$, then we have

$$V_{k-1}^{\text{cont}}(Z_{k-1}) = e^{-\int_{k-1}^{k} r_s ds} E^Q\left[V_k(Z_k)\middle|\, Z_{k-1}\right],$$

$$V_{k-1}\left(Z_{k-1}\right) = \max\left\{V_{k-1}^{\text{cont}}(Z_{k-1}),\ V_{k-1}^{\text{exer}}(Z_{k-1})\right\}$$

for $k = T,\ T-1, \ldots, 1$. Substitute the second formula (with $k-1$ replaced by k) into the first, we get

$$V_{k-1}^{\text{cont}}(Z_{k-1}) = e^{-\int_{k-1}^{k} r_s ds} \int_{-\infty}^{\infty} \max\left\{V_k^{\text{cont}}\left(Z_{k-1}+x\right)\right),\ V_k^{\text{exer}}(Z_{k-1}+x)\right\} f(x)dx, \quad (5.1)$$

where $I_k = \mathbb{R}$ and $f(x)$ is the probability density function of $X_k - X_{k-1}$. Obvious, the initial price of the CAR, $V_0^{\text{cont}}(1)$ can be written as a T-dimensional integral. In numerical integration, we will truncate the integration domain, so that in practice the integration domains in both of the above formulations are closed intervals. For example, in (5.1) we may set the integration domain to $I_k = [\mu^{\mathbb{Q}} - n\sigma^{\mathbb{Q}}, \mu^{\mathbb{Q}} + n\sigma^{\mathbb{Q}}]$ where $\mu^{\mathbb{Q}} = E^{\mathbb{Q}}(X_1)$, $\sigma^{\mathbb{Q}} = \sqrt{\text{var}^{\mathbb{Q}}(X_1)}$ and n is a positive integer. We can make the truncation error arbitrarily small by increasing n. For normal distributions, setting $n = 6$ is usually enough; for other distributions, we may need a larger n.

Anyway, let us assume that I_k is a compact interval. Many valuation problems for Bermudan options, including all EIA valuation problems in part I, can be formulated in a way similar to (5.1):

$$V_{k-1}^{\text{cont}}(Z_{k-1}) = U_{k-1}(Z_{k-1}) + \rho_k \sum_{i=1}^{m_k} \int_{I_k^{(i)}(Z_{k-1})} \max\left\{V_k^{\text{cont}}\left(Z_k(Z_{k-1},x)\right),\ V_k^{\text{exer}}\left(Z_k(Z_{k-1},x)\right)\right\} f_k(x)dx, \tag{5.2}$$

where $U_{k-1}(Z_{k-1})$ is a computable function, ρ_k is the discount factor for the k-th year, $f_k(x)$ is a certain probability density function, $Z_k(Z_{k-1}, x)$ is a continuous function in Z_{k-1} and x, and each $I_k^{(i)}(Z_{k-1})$ is a compact subinterval of I_k (the location of this subinterval depends on Z_{k-1} and for each Z_{k-1} the number of these subintervals, m_k, depends on k). For example, when both the interest rate and dividend yield are zero, the time-$(k-1)$ price of an annually monitored down-and-out option is given by

$$\begin{aligned} V_{k-1}^{\text{cont}}(Z_{k-1}) &= \rho_k \int_{I_k} \max\left\{V_k^{\text{cont}}(Z_{k-1}+x),\ (Z_{k-1}+x-H_k)^+\right\} f_k(x)dx \\ &= \rho_k \int_{I_k \cap [H_k - Z_{k-1}, \infty)} \max\left\{V_k^{\text{cont}}(Z_{k-1}+x),\ (Z_{k-1}+x-H_k)\right\} f_k(x)dx \end{aligned}$$

where $Z_k = X_k$ and H_k is some barrier level. In this case, we can put $U_{k-1} = 0$, $m_k = 1$ and $I_k^{(1)}(z) = I_k \cap [H_k - z, \infty)$. In general, I_k may be broken into more than one subdomains and the purpose of doing so is to ensure that the integrand is continuous and piecewise differentiable function over each subdomain even when we are pricing some options with barrier or digital features. This will become important when we prove the convergence of sequential quadrature method. In practice, for many exercise functions V_k^{exer}, when each density function $f_k(x)$ is differentiable (which is usually the case), it can be shown by mathematical induction that V_k^{cont} is also continuously differentiable function for each k and the integrands in (5.2) are continuous and piecewise differentiable on their respective subdomains.

This formulation (5.2) can be easily extended to allow the uses of the extended Black-Scholes model or CTRS models , but for convenience of presentation, we only consider Levy models here. Also, without loss of generality, let us assume that $U_{k-1} = 0$ and $m_k = 1$ for all k. In principle, we can compute $V_{T-1}^{\text{cont}}(\cdot)$, $V_{T-2}^{\text{cont}}(\cdot), \ldots$ down to $V_0^{\text{cont}}(\cdot)$ recursively by numerical quadrature. So let us assume that we have computed in a previous recursion step the values of $V_k^{\text{cont}}(y)$ for a certain set of nodes $y \in \mathcal{N}_k$. In the current recursion step we want to evaluate $V_{k-1}^{\text{cont}}(z)$ for all z in some set $\mathcal{N}_{k-1}$. So we apply Simpson's rule on (5.2):

$$V_{k-1}^{\text{cont}}(z) \approx \rho_k \sum_{j=1}^{N} w_j \max\left\{V_k^{\text{cont}}\left(Z_k(z,x_j)\right),\ V_k^{\text{exer}}\left(Z_k(z,x_j)\right)\right\} f_k(x_j), \tag{5.3}$$

where $\{x_1, x_2, \ldots, x_N\}$ and $\{w_1, w_2, \ldots, w_N\}$ are respectively the quadrature nodes and weights when the Simpson's rule is applied to $I_k(z)$. Yet there is a caveat. In (5.3), we need to know the value of $V_k(z)$ at each $z = Z_k(y, x_j)$. However, in the previous recursion step, we have only computed $V_k(z)$ for all $z \in \mathcal{N}_k$. There are three approaches to resolve this problem.

5.1.1 The pure interpolation approach

The first one, developed by Hunt and Kennedy (2000), makes use of the fact that in many applications, the function Z_k takes the form of $Z_k(z,x) = z + x$ as in (5.1). So we can reformulate (5.2) as

$$V_{k-1}^{\text{cont}}(z) = \rho_k \int_{I_k(z)} V_k(z+x) f_k(x) dx.$$

Now the idea of Hunt and Kennedy (2000) is to completely abandon numerical quadrature. Instead, they use the set of values $V_k(\mathcal{N}_k)$ computed in the preceding recursion step to interpolate V_k as a polynomial. When f_k is a normal density function, there is an efficient algorithm for computing the exact values of integrals of the form $\int_a^b x^n f_k(x) dx$. Therefore the numerical integration of polynomials w.r.t. the normal density is not only fast, but also immune to quadrature error. However, since V_k is usually not smooth, one usually has to determine all discontinuities or non-differentiable points of V_k and break I_k into a number of subintervals on which V_k is smooth before employing this approach. Also, for other density functions than the normal density function, this approach may not apply.

5.1.2 The pure quadrature approach

The second approach, developed by Tse *et al.* (2001), reformulates (5.2) as

$$V_{k-1}^{\text{cont}}(Z_{k-1}) = \rho_k \int_{I_k} \max\left\{V_k^{\text{cont}}(y),\ V_k^{\text{exer}}(y)\right\} f_k(y - Z_{k-1})dy$$

(where the original integration domain $\mathbb{R}$ may be truncated into a different I_k). Now for each k, the set of nodes $\mathcal{N}_k$ at which V_k^{cont} is evaluated is simply taken to be the set of quadrature nodes for the interval I_k. So, in each recursion step, the values of V_k^{cont} needed in the quadrature process have already been calculated in the preceding step, and values of $f_k(y - Z_{k-1})$ are computed on the fly. Since this is a pure quadrature approach, when V_k is piecewise continuous for all k, most popular quadrature rules will guarantee the convergence of the computed option price to the true price. Furthermore, this approach has a wider field of applications than the first one has because it is applicable when f_k is not normal or when $Z_k(z, x)$ is of other form than $z + x$ (but it needs $Z_k(z, x)$ to be an invertible function of x, given z). However, its efficiency relies heavily on the efficiency in computing f_k. When no closed-form formula is available for f_k, performance of this approach may be poor.

Another pure quadrature approach is advocated by Lord *et al.* (2007), who employ FFT to do numerical integration. At present, their method is applicable only under Levy models, but it seems to be extensible to work under stochastic volatility models as well.

5.1.3 The mixed approach

The third approach, proposed by Sullivan (2000a,b), stands in the middle ground. Put it simply, in (5.3) the values of $V_k(Z_k(y, x_j))$ are interpolated from the outputs $V_k(\mathcal{N}_k)$ of the preceding recursion step. Like the pure interpolation approach, the aim of each recursion step in the mixed approach is to generate the values of V_{k-1} on a set of knots $\mathcal{N}_{k-1}$, where these outputs will be used for function interpolation in next recursion step. Although this approach involves both interpolation errors and quadrature errors, it is the most flexible one among the three approaches that we introduce in this chapter.

In (5.2), there are many ways to do numerical quadrature. In this thesis, we will consider only composite Newton-Cotes formulae (such as the composite trapezoidal rule and the composite Simpson's rule) and Gauss-Legendre quadrature. Rabinowitz (1987, lemma 1) shows that these quadrature rules are convergent for all Riemann-integrable functions. Abramowitz and Stegun (1972) has tabulated the nodes and weights for n-point Gauss-Legendre quadrature over the interval $[-1, 1]$ for some values of n. Computer packages for generating the Gauss-Legendre nodes and weights for a general n are also freely available on the internet. For small to moderate n (say, $n \leq 512$), the quadrature nodes and weights can be computed instantly.

Given $V_k(\mathcal{N}_k)$, there are also many ways to interpolate the curve $V_k(y)$. Two popular choices include cubic spline interpolation and Lagrange interpolation. However, in this chapter, we will only consider piecewise Lagrange interpolation and Chebyshev interpolation. The former refers to the partition of the domain of interpolation into subintervals and the application of Lagrange interpolation on every one of them. For instance, in (5.2), if $I_k = [a_k, b_k]$ and piecewise cubic interpolation is used, then for each k, we set $\mathcal{N}_k = \{y_i : y_i = a_k + i(b_k - a_k)/(3n),\ i = 0, 1, \ldots, 3n\}$ for some positive integer n. Whenever y lies inside some $[y_{3i}, y_{3i+3}]$, we approximate $V_k(y)$ by the cubic polynomial that passes through $(y_{3i}, V_k(y_{3i})), (y_{3i+1}, V_k(y_{3i+1})), (y_{3i+2}, V_k(y_{3i+2}))$ and $(y_{3i+3}, V_k(y_{3i+3}))$.

Sullivan (2000a,b) advocates the use of Chebyshev polynomials in sequential quadrature because empirically these polynomials are able to minimize the maximum approximation errors. In general, given a function $h(y)$ defined on an interval $[-1, 1]$ and a control parameter $p \in \{2, 3, \ldots, n\}$, we can approximate $h(y)$ by an order-n Chebyshev polynomial of the form

$$P(y) = \sum_{k=0}^{n-1} c_k \cos(k \cos^{-1}(y))$$

such that $h(y) = P(y)$ on the set of interpolation nodes

$$Y = \left\{ \cos\left[(k - \frac{1}{2})\frac{\pi}{p} \right] : k = 1, \ldots, n \right\}.$$

A computer program for determining the coefficients c_k (that are dependent on $h(y)$) is given by Press *et al.* (1992). Approximations of functions over other compact intervals can be achieved by a change of variables.

When h is a continuous function defined on a compact interval I, the $(3n + 1)$-point piecewise Lagrange polynomial that interpolates h will converge uniformly to h as $n \to \infty$. If h is also Dini-Lipschitz on I, then its order-n Chebyshev approximating polynomial will converge uniformly to h as $p \to \infty$ (see, e.g. Mason and Handscomb 2003, corollary 6.14A). Unlike cubic spline approximation, both Chebyshev and piecewise Lagrange interpolation are bounded linear operators with respect to the maximum norm.

5.2 Convergence of the mixed approach

In this section, we prove the uniform convergence property of the mixed approach. First, without loss of generality, let $U_{k-1} = 0$ in (5.2) be the zero function. Also, for ease of presentation, assume $m = 1$ in (5.2), so that we can reformulate the equation into a more abstract form:

$$C(z) = \int_{I^z} V(z, x) f(x) dx, \quad (z \in \widetilde{I}), \tag{5.4}$$

where $f(x)$ is a differentiable probability density function defined on $\mathbb{R}$, $I^z, \widetilde{I}$ are compact intervals such that for each $z \in \widetilde{I}$, $I^z \subseteq I$ and the end-points of I_z are continuous functions in z. The function $V(z, x)$ is assumed to be continuous and piecewise differentiable in x as well as continuous in z for all $z \in \widetilde{I}$.

Let us introduce some notations. We denote a quadrature rule with n nodes by Q_n. When we say that we evaluate $\int_I h(x)dx$ by Q_n, we mean the integral is approximated by a finite sum of the form

$$Q_n(h; I) = \mu(I) \sum_{i=1}^{n} w_{in}\, h(A_I(x_{in}))$$

where $-1 \le x_{1n} < x_{2n} < \ldots < x_{nn} \le 1$ and $w_{1n}, \ldots, w_{nn} \in \mathbb{R}$ are called the nodes and weights of Q_n respectively, $\mu(I)$ is half of the length of I and $A_I : [-1, 1] \to I$ is the affine function that takes -1 to a and 1 to b. For instance, the composite trapezoidal rule with m partitions is obtained when $n = m + 1$, $w_{1n} = w_{nn} = \frac{1}{m}$, $w_{in} = \frac{2}{m}$ for $i = 2, 3, \ldots, m$ and $x_{in} = -1 + \frac{2}{m}$ for each i. We will also denote the order-n Chebyshev approximation or the $(qn + 1)$-point piecewise Lagrange interpolation using degree-q polynomials by J_n. So the approximate function for $h(x)$ over I is written as $J_n(h; I)(x)$ and the application of sequential quadrature on the recursion step (5.2) can be expressed in the following form:

$$C_n(z) \approx U(z) + Q_n \left(J_n(V_n(z, \cdot); I_z) f; I_z \right),$$

where $C_n(z)$ corresponds to the value of $V_k^{\text{cont}}(z)$ obtained by sequential quadrature and $V_n(z, x)$ here represents the quadrature value of $V_k(Z_k(z, x))$ in (5.2) as if it has been computed in the preceding recursion step (it may not be computed if $Z_k(z, x) \notin \mathcal{N}_k$). By mathematical induction, we can prove the convergence of the option price obtained by sequential quadrature to the true price if we can show that $\lim_{n\to\infty} \|V_n - V\|_\infty = 0$ implies $\lim_{n\to\infty} \|C_n - C\|_\infty = 0$, where the first maximum norm is evaluated over $\mathcal{I}$ and the second one is evaluated over $\widetilde{I}$. Here is our result:

Proposition 5.1. *Consider equation (5.4). Suppose* $\{V_n(z, x) : n = 1, 2, \ldots\}$ *is a family of functions defined on* $\mathcal{I} = \{(z, x) : z \in \widetilde{I},\ x \in I^z\}$ *such that* $V_n(z, x)$ *is continuous and piecewise differentiable in* x*, continuous in* z *and* V_n *converges uniformly to* V *on* $\mathcal{I}$*. Let* Q_n *denotes a composite Newton-Cotes quadrature rule with n partitions or a n-point Gauss-Legendre quadrature rule, and* J_n *denotes an order-n Chebyshev interpolation operator or a* $(qn + 1)$*-point piecewise degree-q Lagrange interpolation operator. Then* $C_n(z) = U(z) + Q_n \left(J_n(V_n(z, \cdot); I^z) f; I^z \right)$ *converges uniformly to* C *on* $\widetilde{I}$*.*

Proof. We have

$$\begin{aligned} C(z) \equiv \int_{I^z} V(z,\cdot)f(x)dx &\approx A_n(z) = Q_n(V(z,\cdot)f;\ I^z) \\ &\approx B_n(z) = Q_n(V_n(z,\cdot)f;\ I^z) \\ &\approx C_n(z) = Q_n\left(J_n(V_n(z,\cdot); I^z)f;\ I^z\right). \end{aligned}$$

Since f, V, and V_n and the end-points of I^z are continuous functions and Q_n, J_n are bounded linear operators, the functions A_n, B_n, C_n and C are also continuous. Therefore, it suffices to prove pointwise convergence of A_n to C, B_n to A_n and C_n to B_n because $\widetilde{I}$ is compact. The convergence of $A_n(z)$ to $C(z)$ is a result of the aforementioned lemma of Rabinowitz (1987). B_n converges to A_n because V_n converges to V uniformly and $\|Q_n\left(V_n(z,\cdot)f - V(z,\cdot)f;\ I^z\right)\|_\infty \leq \|V_n - V\|_\infty \times \|f\|_\infty \times Q_n(1; I^z)$. So it remains to show that C_n converges to B_n. However, since Q_n is a bounded linear operator and f is bounded on I, it suffices to show that $J_n(V_n(z,\cdot); I^z)$ converges uniformly to $V_n(z,\cdot)$ on each I^z. Note that

$$\begin{aligned} &\|J_n(V_n(z,\cdot)) - V_n(z,\cdot)\|_\infty \\ &\quad \leq \|J_n(V_n(z,\cdot) - V(z,\cdot))\|_\infty + \|J_n(V(z,\cdot)) - V(z,\cdot)\|_\infty + \|V(z,\cdot) - V_n(z,\cdot)\|_\infty . \end{aligned}$$

Since J_n is a bounded linear operator and V_n converges to V uniformly, the first and the third summand on the right hand side converges to zero. As $V(z,x)$ is continuous and piecewise differentiable in x, it is Dini-Lipschitz continuous in x and hence $J_n(V(z,\cdot))$ converges uniformly to $V(z,\cdot)$ on I^z for each z. Therefore the second summand also converges to zero as $n \to \infty$. □

Although the previous proof is not difficult, its last step relies on the uniform approximation property of piecewise Lagrange or Chebyshev approximation. So our proof does not apply if we use some popular spline interpolation scheme such as cubic spline interpolation. Also, we have only proved the convergence of sequential quadrature but we have said nothing about the method's rate of convergence or numerical stability. Owing to the interplay between quadrature and interpolation, it is hard to perform any analysis on these two topics. However, I have not yet experienced any numerical instability issue in using sequential quadrature. Nor have I encountered any difficulty by taking cubic spline interpolation as the approximation scheme.

Chapter 6

Example: pricing discrete dynamic fund protections

6.1 Overview

Call and put options allow an investor to profit on upside or downside movements in the prices of underlying assets, while limiting the losses when the prices move adversely. Yet the protections these options offer are merely static. For example, if a put option is deeply out of the money, the buyer of this option can hardly get more than the strike price when the option expires.

Some path-dependent financial derivatives are designed to address the above issue. A floating-strike put option, for example, guarantees that the investor can exercise the option using the highest asset price realized throughout the option's lifetime. For investment funds, Gerber and Shiu (1998, 1999) and Gerber and Pafumi (2000) introduced a form of path-dependent derivative called *reset guarantee* or *dynamic fund protection*. The basic idea is to prevent a fund value from falling below a certain threshold level K over the fund's lifetime. Specifically, the basic fund unit is replaced by an upgraded fund unit, which begins with the same value as the basic one. During the protection's lifetime, whenever the value of the upgraded fund unit ever drops to K, just enough money will be endowed to it so that its value does not fall any further. On the other hand, if its value is above K in some time interval, it enjoys the same instantaneous rate of return as the original fund.

The relationship between the value of a basic fund unit, $\mathrm{s}(t)$, and the value of an upgraded fund unit, $S(t)$, can be formulated as follows:

$$S(t) = \begin{cases} \mathrm{s}(t) \max\left\{1, \max\limits_{0 \le t' \le t} \frac{K(t')}{\mathrm{s}(t')}\right\} & \text{if } \mathrm{s}(t') > 0 \text{ for all } t' \in [0, t], \\ 0 & \text{otherwise.} \end{cases}$$

Note that at inception $S(0) = \mathrm{s}(0)$ and we assume that both the basic fund and the upgraded fund default is $\mathrm{s}(t) = 0$.

When $\{\mathrm{s}(t)\}$ is modeled as a geometric Brownian motion, one can obtain explicit pricing formulas for perpetual protections (Gerber and Shiu 1998, 1999) as well as finite time protections (Gerber and Pafumi, 2000). The case where the basic fund price follows a constant elasticity of variance (CEV) process was also examined recently (Imai and Boyle, 2001).

In practice, it is difficult to monitor the fund price movement continuously, and discrete monitoring may be a more appropriate choice. Under discrete monitoring, instead of keeping $S(t) \ge K$ all the time, the upgraded fund will only be endowed at discrete time epochs $0 < t_1 < t_2 < \ldots < t_m = T$. Also, the protection thresholds $K_1, K_2, \ldots, K_m > 0$ at different time points may be independent. We can express the total asset value of the upgraded fund at time t_i as follows:

$$S(t_i) = \begin{cases} \mathrm{s}(t_i) \max\left\{1, \max\limits_{1 \le j \le i} \frac{K_j}{\mathrm{s}(t_j)}\right\} & \text{if } \mathrm{s}(t') > 0 \text{ for all } t' \in [0, t_i], \\ 0 & \text{otherwise} \end{cases} \tag{6.1}$$

$$= \begin{cases} \max\left\{\frac{\mathrm{s}(t_i)}{\mathrm{s}(t_{i-1})} S(t_{i-1}), K_i\right\} & \text{if } \mathrm{s}(t_i) > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6.2}$$

When the monitoring frequency is not high, the discrepancy between the prices of a continuous protection and a discrete protection can be substantial, meaning that discrete protections can be appealing to policy holders because they are cheaper. However, this also entails a difficulty in pricing discrete protections, because we cannot approximate their prices directly by the prices of their continuous counterparts. Although Imai and Boyle (2001) recently developed a fairly accurate continuity adjustment formula (in the spirit of Broadie *et al.* 1999) to approximate the prices of discretely monitored dynamic fund protections, this formula works only under the lognormal price process. For other price processes, such as the constant elasticity of variance (CEV) process, they had to resort to the Monte Carlo method, which is flexible but inefficient. In this chapter, we will use sequential quadrature to solve the pricing problem.

The layout of rest of this chapter is as follows. In next section, we formulate the valuation problem of a fund protection under the Black-Scholes model in form of (5.4). Then we will derive an analogous functional equation for the CEV model section 6.3. Numerical results obtained by our valuation method under the two price processes are compared to the results of Imai and Boyle (2001) and to the quasi-Monte Carlo prices in section 6.4. Section 6.5 concludes.

6.2 Risk neutral valuation under the lognormal process

In this section, we assume that we are living in a Black-Scholes world, such that the risk-neutral stochastic differential equation (SDE) for the price of a basic fund unit is given by

$$d\log \mathrm{s}(t) = \mu dt + \sigma dW(t) \quad (t \geq 0) \tag{6.3}$$

where r is the risk-free rate, δ is the continuous dividend yield of the investment fund, σ is the fund price's volatility, $W(t)$ is a standard Wiener process and

$$\mu = r - \delta - \frac{1}{2}\sigma^2.$$

Let $\Delta t_i = t_i - t_{i-1}$ and $\varphi_i(z)$ be the probability density function of the normal distribution $N(\mu\Delta t_i, \sigma\sqrt{\Delta t_i})$ for $i = 1, 2, \ldots, m$. Denote by $g_i(S)$ the risk-neutral price of the dynamic fund protection given that the asset value of the upgraded fund at time t_i is S. By (6.2), we have

$$
\begin{aligned}
g_m(S) &= \max(K_m - S, 0); && (6.4)\\
g_{i-1}(S) &= e^{-r\Delta t_i}\int_{-\infty}^{+\infty}\Big\{ I(Se^z \ge K_i)\ g_i(Se^z) + \\
&\qquad I(Se^z < K_i)\ [\underbrace{(K_i - Se^z)}_{\text{put}} + \underbrace{g_i(K_i)}_{\text{constant}}\ ]\Big\}\varphi_i(z)dz \\
&= e^{-r\Delta t_i}\int_{\kappa_i}^{\infty} g_i(Se^z)\varphi_i(z)dz + \\
&\qquad \Big[K_i e^{-r\Delta t_i}\Phi_i(d_2^{(i)}) - Se^{-\delta\Delta t_i}\Phi_i(d_1^{(i)})\Big] + e^{-r\Delta t_i} g_i(K_i)\Phi_i(d_2^{(i)}) && (6.5)
\end{aligned}
$$

where $I(\cdot)$ is the indicator function, $\kappa_i = \log(K_i/S)$, $d_2^{(i)} = (\log(K_i/S) - \mu\Delta t_i)/(\sigma\sqrt{\Delta t_i})$ and $d_1^{(i)} = d_2^{(i)} - \sigma\sqrt{\Delta t_i}$.

6.3 Pricing dynamic fund protections under the CEV process

In this section, we shall briefly describe how to evaluate dynamic fund protections under the CEV process. We assume that the underlying fund price satisfies the SDE

$$ds(t) = (r - \delta)s(t)dt + \sigma s(t)^{\alpha/2} dW(t) \tag{6.6}$$

in a risk-neutral world, where $0 \le \alpha \le 2$. When $\alpha = 2$, the price process reduces to a lognormal process.

Unlike under a lognormal process, the quantity $s(t_i)/s(t_{i-1})$ and in turn $S(t_i)$ in (6.2) are dependent on $s(t_{i-1})$ under a CEV process. Consequently, on each level of our backward recursion, the approximation space is no longer one-dimensional but two-dimensional. Yet, the basic idea remains unchanged and so we only derive the functional equation for recursion here. First, let $f(s_{t'}, s_t; t', t)$ denote the continuous part of the density $s_t = s(t) > 0$ conditional on $s_{t'} = s(t')$ $(t' < t)$ in a risk-neutral world. (The probability of $s_t = 0$ is nonzero when $\alpha < 2$.) Cox (1975, 1996) showed that

$$f(s_{t'}, s_t; t', t) = (2-\alpha)k^{\nu}\ (xz^{1-2\alpha})^{\frac{\nu}{2}}\ I_\nu(2\sqrt{xz})e^{-x-z}$$

where

$$
\begin{aligned}
k &= \frac{2(r-\delta)}{\sigma^2(2-\alpha)[e^{(r-\delta)(2-\alpha)(t-t')}-1]} \\
x &= k\mathrm{s}_{t'}^{2-\alpha}e^{(r-\delta)(2-\alpha)(t-t')} \\
z &= k\mathrm{s}_t^{2-\alpha} \\
\nu &= \frac{1}{2-\alpha}
\end{aligned}
$$

and $I_\nu(\cdot)$ is the modified Bessel function of the first kind of order ν. Now, let us define

$$
\lambda(t_i) = \begin{cases} \frac{\mathrm{s}(t_i)}{S(t_i)} & \text{if } \mathrm{s}(t_i) > 0 \\ 0 & \text{otherwise} \end{cases}
$$

Then $0 \leq \lambda(t_i) \leq 1$ and

$$
\begin{aligned}
S(t_i) &= \begin{cases} \max\left\{\frac{\mathrm{s}(t_i)}{\mathrm{s}(t_{i-1})}S(t_{i-1}), K_i\right\} & \text{if } \mathrm{s}(t_i) > 0 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \max\{\frac{\mathrm{s}(t_i)}{\lambda(t_{i-1})}, K_i\} & \text{if } \lambda(t_{i-1}) > 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Hence the price of the dynamic fund protection $g_{i-1}(S_{i-1}, \lambda_{i-1})$, given that $S(t_{i-1}) = S_{i-1} > 0$ and $\lambda(t_{i-1}) = \lambda_{i-1} > 0$, must satisfy:

$$
\begin{aligned}
& e^{r\Delta t_i} g_{i-1}(S_{i-1}, \lambda_{i-1}) \\
= & \int_{\lambda_{i-1}K_i}^{\infty} g_i(\frac{\mathrm{s}}{\lambda_{i-1}}, \lambda_{i-1}) f(\lambda_{i-1}S_{i-1},\ \mathrm{s};\ t_{i-1},\ t_i) d\mathrm{s} \\
& + \int_0^{\lambda_{i-1}K_i} \left[(K_i - \frac{\mathrm{s}}{\lambda_{i-1}}) + g_i(K_i, \frac{\mathrm{s}}{K_i})\right] f(\lambda_{i-1}S_{i-1},\ \mathrm{s};\ t_{i-1},\ t_i) d\mathrm{s}.
\end{aligned}
$$

Having this functional equation, we can use the sequential quadrature method to evaluate the dynamic fund protection. However, since we now have to approximate g in both the S-space and the λ-space, the total computational complexity will become higher.

6.4 Numerical experiments

In this section, we present results computed with the sequential quadrature method under daily (364 per year)[1], weekly (52 per year) and monthly (12 per year) monitoring, and compare them with those prices based on continuous monitoring or those obtained by other methods. Both the lognormal price process and the constant elasticity of variance (CEV) process were investigated. For convenience of presentation, we assume there is no dividend (hence $\delta = 0$). Gauss-Legendre quadrature and Chebyshev approximation are used in the sequential quadrature process. All numerical experiments were conducted with MATLAB 6.5 running on a Intel 1.9GHz processor with 512MB RAM.

Table 6.1 reports the Black-Scholes prices and CPU times obtained with different numbers (n) of quadrature/interpolation nodes. As the table shoes, the prices converge fairly quickly. In most cases, convergent prices are obtained as soon as $n = 24$. The table 6.1 also shows that the sequential quadrature method is very efficient. With $n = 40$ and 2184 monitorings (daily over 5 years), the valuation process takes less than a minute to complete.

To confirm that our prices are accurate, we compare our prices with those obtained by the Monte-Carlo (MC) method, the quasi-Monte Carlo (QMC) method and also Imai and Boyle's (2001) continuity adjustment formula. The MC prices are taken from Table 10 of Imai and Boyle (2001), in which the equation (6.6) with $\alpha = 2$ was discretised in 1092 steps, and 1092^2 sample values were averaged to obtain each price. Note that Imai and Boyle did not discretise (6.3), so that in calculating the MC prices, a uniform treatment could be given to both the lognormal process and the CEV process. However, in order to obtain accurate QMC prices, we shall use (6.3) for discretization.

Our simulation trials were generated with a Sobol sequence. The implementation of the sequence generator is based on the work of Joe and Kuo (2003), which extends the popular implementation by Bratley and Fox (1988) to 1111 dimensions. The zero vector

[1] We will compare our computational results with those obtained by Imai and Boyle (2001), who took the number of days in a year as $364 (= 52 \times 7)$.

was not included in the sequence. When we wanted to calculate a price using 2^n samples, the segment containing $(2^n + 1)$-th point to the (2^{n+1})-th point in the Sobol sequence were used.

The prices calculated with the four methods are listed in Table 6.2. Our prices clearly matches the QMC prices, and are quite close to the MC prices. Judging from the convergence patterns of the QMC prices, we are convinced that our prices are truly accurate.

For more elaborate comparisons, we compare all the MC prices listed in Table 10 of Imai and Boyle (2001) with our prices. See Table 6.3 here. Although each of the MC prices was calculated with over a million simulation trials, these MC prices are slightly inaccurate. Also, under daily monitoring, the MC prices are consistently worse than the results given by the continuity adjustment formula.

Table 6.3 also compares the prices of discrete protections with those of continuous protections. As expected, when the monitoring frequency increases, the price of a discrete protection becomes closer to the price of a continuous protection. In general, mispricing can be substantial when the price of a continuous protection is used as a surrogate price for its discrete counterpart: when $T = 5,\ K = 100$ and the protection is monitored daily, the price of the discrete protection differs from the price of the continuous protection by about 2.7%; when $T = 1,\ K = 80$ and the protection is monitored monthly, the difference is even as large as 36.8%. Although the table contains only a few experimental results, we believe the conclusion of the finding here is generally applicable.

Our final experiment investigates the pricing of discrete protections under the CEV process. We compare the MC prices listed in Table 11 of Imai and Boyle (2001) with our prices. See Table 6.4. We find that although the two set of prices agree fairly well, the MC prices are consistently larger than our prices by about one standard error. This may be due to the errors inherent in Euler discretisation.

6.5 Concluding remarks

We have derived functional equations for pricing discretely monitored dynamic fund protections under the Black-Scholes model and the CEV model, and demonstrated how to solve these equations efficiently using the sequential quadrature technique. Our numerical experiments confirm that the prices of continuous protections are not good approximations for those of discrete versions, especially when the number of monitorings is small. In general, we suggest the reader to treat discretely monitored financial derivatives and their continuous counterparts as two different classes of derivative securities. In fact, vast differences between them lie not only in the prices, but also in the ease of pricing. For example, our present pricing method for discrete dynamic fund protections allows the protection levels to be independent. For continuous protections, however, we are not aware of any pricing formula that allows deterministic but non-exponential protection levels, even under the lognormal price process.

Most discretely monitored derivative securities can only be evaluated numerically. Although it is possible to evaluate some of them accurately using various continuity adjustment formulas, no one has yet discovered such formulas for dynamic fund protections, except under the lognormal process. Therefore the pricing of discrete dynamic fund protections in general must rely on the use of efficient numerical methods.

In this study, we have essentially adopted Sullivan's (2000a, 2000b) variant of the sequential quadrature technique. Currently, sequential quadrature is not widely used in solving computational finance problems. However, it seems that the prices of many discretely monitored or time-discretised versions of financial derivatives can be easily formulated as the solutions of sets of functional equations. Sequential quadrature may therefore be useful in these cases.

K	No. of pts. (n)	Daily Price	Daily Time	Weekly Price	Weekly Time	Monthly Price	Monthly Time
$T = 1$							
	16	14.099	3.64	13.039	0.48	11.361	0.13
100	24	14.107	5.02	13.039	0.69	11.361	0.16
	32	14.106	6.52	13.039	0.91	11.361	0.19
	40	14.106	9.03	13.039	1.25	11.361	0.25
	16	5.675	3.55	5.180	0.50	4.445	0.11
90	24	5.680	4.88	5.180	0.67	4.445	0.16
	32	5.679	6.52	5.180	0.91	4.445	0.19
	40	5.679	9.00	5.180	1.25	4.445	0.25
	16	1.651	3.55	1.481	0.50	1.241	0.09
80	24	1.653	4.84	1.481	0.69	1.241	0.16
	32	1.653	6.55	1.481	0.91	1.241	0.19
	40	1.653	9.02	1.481	1.27	1.241	0.25
$T = 3$							
	16	23.084	11.36	21.942	1.59	20.009	0.38
100	24	23.126	15.38	21.943	2.20	20.009	0.47
	32	23.128	20.39	21.943	2.91	20.009	0.66
	40	23.128	28.13	21.943	3.95	20.009	0.89
	16	12.975	11.33	12.286	1.61	11.143	0.36
90	24	13.005	15.41	12.287	2.20	11.143	0.50
	32	13.005	20.45	12.287	2.92	11.143	0.66
	40	13.005	28.06	12.287	3.99	11.143	0.88
	16	6.377	11.34	6.005	1.63	5.397	0.36
80	24	6.394	15.41	6.005	2.17	5.397	0.48
	32	6.394	20.55	6.005	2.91	5.397	0.66
	40	6.394	28.06	6.005	3.95	5.397	0.88
$T = 5$							
	16	28.312	19.31	27.145	2.77	25.092	0.63
100	24	28.389	26.00	27.146	3.72	25.091	0.84
	32	28.391	34.50	27.146	4.92	25.091	1.13
	40	28.392	47.45	27.146	6.72	25.091	1.51
	16	17.451	19.28	16.705	2.76	15.396	0.64
90	24	17.513	26.01	16.706	3.73	15.396	0.84
	32	17.514	34.61	16.706	4.89	15.396	1.11
	40	17.514	47.33	16.706	6.81	15.396	1.50
	16	9.786	19.25	9.343	2.76	8.565	0.63
80	24	9.829	26.06	9.344	3.70	8.565	0.84
	32	9.829	34.51	9.344	4.92	8.565	1.13
	40	9.829	47.36	9.344	6.72	8.565	1.51

Table 6.1. Prices of discretely monitored dynamic fund protections under lognormal process. The prices were computed using n-point Chebyshev approximation and n-point Gauss-Legendre quadrature. The parameters for the basic contract are: $F_0 = 100,\ r = 0.04$ and $\sigma = 0.2$. CPU times are measured in seconds.

	Monthly, $T = 1$							Weekly, $T = 5$		
	$K = 100$		$K = 90$		$K = 80$			$K = 80$	
n	Price	Time	Price	Time	Price	Time	n	Price	Time
15	11.3490	25.70	4.4329	25.80	1.2499	25.63	17	9.3349	233.08
16	11.3735	51.23	4.4511	51.50	1.2469	51.23	18	9.3461	466.91
17	11.3609	102.28	4.4448	102.69	1.2409	102.41	19	9.3586	932.58
18	11.3619	204.66	4.4450	204.95	1.2427	204.78	20	9.3509	1859.67
19	11.3603	409.05	4.4431	410.45	1.2416	409.48	21	9.3461	3716.75
20	11.3611	816.70	4.4444	819.76	1.2410	818.78	22	9.3488	7430.36
21	11.3615	1631.17	4.4442	1640.47	1.2415	1637.06	23	9.3448	14856.80
22	11.3607	3259.38	4.4448	3283.01	1.2410	3276.02	24	9.3431	29752.00
23	11.3607	6521.41	4.4445	6565.06	1.2415	6549.38	25	9.3442	59770.75
(a)	11.3608	0.25	4.4446	0.25	1.2414	0.25	(a)	9.3441	6.72
(b)	11.375		4.461		1.254		(b)	9.340	

Table 6.2. Prices of monthly monitored dynamic fund protections obtained by using quasi-Monte Carlo method with 2^n Sobol points. Also listed are (a) the prices obtained with sequential quadrature and (b) the Monte-Carlo prices given by Imai and Boyle (2001) using 1092^2 samples. The parameters for the basic contract are: $F_0 = 100,\ r = 0.04$ and $\sigma = 0.2$. CPU times are measured in seconds.

		$T = 1$			$T = 3$			$T = 5$		
K		(a)	(b)	(c)	(a)	(b)	(c)	(a)	(b)	(c)
100	Monthly	11.361	11.375	11.096	20.009	20.060	19.890	25.091	25.097	25.021
	Weekly	13.039	13.053	12.977	21.943	21.993	21.915	27.146	27.097	27.130
	Daily	14.106	14.119	14.098	23.128	23.177	23.124	28.391	28.395	28.389
	∞	14.793	14.793	14.793	23.874	23.874	23.874	29.172	29.172	29.172
90	Monthly	4.445	4.461	4.197	11.143	11.194	10.999	15.396	15.395	15.294
	Weekly	5.180	5.196	5.121	12.287	12.338	12.253	16.706	16.709	16.682
	Daily	5.679	5.695	5.671	13.005	13.056	13.000	17.514	17.517	17.511
	∞	6.012	6.012	6.012	13.465	13.465	13.465	18.026	18.026	18.026
80	Monthly	1.241	1.254	1.119	5.397	5.357	5.295	8.565	8.559	8.487
	Weekly	1.481	1.494	1.451	6.005	6.054	5.981	9.344	9.340	9.326
	Daily	1.653	1.666	1.648	6.394	6.443	6.390	9.829	9.824	9.826
	∞	1.771	1.771	1.771	6.644	6.644	6.644	10.137	10.137	10.137

Table 6.3. Prices of discretely monitored dynamic fund protections under lognormal process. The prices in (a) are computed using sequential quadrature with 40 Chebyshev nodes and 40 Gauss-Legendre nodes, while the prices in (b) and (c) respectively refer to the Monte Carlo prices and the continuity adjustment prices obtained by Imai & Boyle (2001). The parameters for the basic contract are: $F_0 = 100,\ r = 0.04$ and $\sigma = 0.2$.

		$\alpha = 2$			$\alpha = 1$			$\alpha = 0$		
K		(a)	(b)	Std. err.	(a)	(b)	Std. err.	(a)	(b)	Std. err.
100	Monthly	11.361	11.375	(0.015)	11.640	11.653	(0.014)	12.002	12.014	(0.013)
	Weekly	13.039	13.053	(0.015)	13.439	13.452	(0.014)	13.958	13.971	(0.014)
	Daily	14.106	14.119	(0.015)	14.593	14.603	(0.014)	15.192	15.232	(0.014)
90	Monthly	4.445	4.461	(0.016)	4.809	4.824	(0.016)	5.246	5.260	(0.015)
	Weekly	5.180	5.196	(0.016)	5.631	5.647	(0.015)	6.188	6.203	(0.014)
	Daily	5.679	5.695	(0.016)	6.190	6.208	(0.015)	6.824	6.848	(0.014)
80	Monthly	1.241	1.254	(0.018)	1.557	1.570	(0.018)	1.951	1.963	(0.017)
	Weekly	1.481	1.494	(0.018)	1.863	1.876	(0.018)	2.350	2.361	(0.017)
	Daily	1.653	1.666	(0.018)	2.068	2.094	(0.017)	2.627	2.647	(0.016)

Table 6.4. Prices of discretely monitored dynamic fund protections under the CEV process. The prices in (a) are computed using sequential quadrature with 40 Chebyshev nodes and 40 Gauss-Legendre nodes, while the prices in (b) are the Monte Carlo prices obtained by Imai & Boyle (2001). The standard errors are listed in brackets. The parameters for the basic contract are: $F_0 = 100,\ r = 0.04, \sigma = 0.2$ and $T = 1$.

Appendix A

The BSHW model

The mathematical details for combining the extended Vasicek model with the Black-Scholes model are not difficult, but owing to the fact that interest rates in equity price models are usually considered deterministic in financial literature, the BSHW model and its derivation are not contained in many textbooks. For convenience, we summarise the details of the BSHW model below. These details are not our original work but written based on a concise discussion of of the Hull-White model in Pelsser (2000) and the material on a more general and sophisticated model in Musiela and Rutkowski (2005).

The CIR model and the Hull-White model

Despite EIA prices are evaluated under the pricing measure, it is helpful to begin our discussion with the physical probability measure. Suppose that in the physical world the force of interest $r(t)$ at time t is governed by the SDE

$$dr(t) = (\theta_r(t) - \kappa_r r(t))\, dt + \Sigma_r\,(r(t), t)\ \ dW_r(t), \tag{A.1}$$

where $\kappa_r \geq 0$ is a constant, θ_r and Σ_r are some functions and $W_r(t)$ is a standard Brownian motion. In a model of this kind, the (interest rate) market is in general incomplete. Intuitively this is because the short rate is not a tradable asset and there is no way to hedge away the interest rate risk. In fact, the market price of interest rate risk cannot be inferred from a no-arbitrage argument. More specifically, let $V(t, r(t))$ denote the time-t price of a financial instrument whose time-t value is determined by t and $r(t)$. Using Ito's lemma, we get

$$dV = \left[\frac{\partial V}{\partial t} + \frac{\Sigma_r^2}{2}\frac{\partial^2 V}{\partial r^2} + (\theta - \kappa_r r)\frac{\partial V}{\partial r}\right] dt + \Sigma_r \frac{\partial V}{\partial r} dW_r. \tag{A.2}$$

Now, consider a portfolio of two such financial instruments, $\Pi = V_1 - \Delta V_2$. We want to adjust the weight Δ dynamically so that the portfolio is both instantaneously riskless ($d\Pi = r\Pi dt$) and self-financed ($d\Pi = dV_1 - \Delta dV_2$). It is not difficult to see that the equation $dV_1 - \Delta dV_2 = r\Pi dt$ implies that $\Delta = \frac{\partial V_1}{\partial r} \Big/ \frac{\partial V_2}{\partial r}$ and

$$\frac{\frac{\partial V_1}{\partial t} + \frac{\Sigma_r^2}{2}\frac{\partial^2 V_1}{\partial r^2} + (\theta_r - \kappa_r r)\frac{\partial V_1}{\partial r} - rV_1}{\frac{\partial V_1}{\partial r}} = \frac{\frac{\partial V_2}{\partial t} + \frac{\Sigma_r^2}{2}\frac{\partial^2 V_2}{\partial r^2} + (\theta_r - \kappa_r r)\frac{\partial V_2}{\partial r} - rV_2}{\frac{\partial V_2}{\partial r}} \tag{A.3}$$

Since the left hand side of the equation contains terms of V_1 only and the right hand side contains V_2 only, the quotients on both sides must be a function that does not involve V_1 and V_2, e.g. a function of the form $\Lambda_r(r, t)$. Consequently, the price $V(t, r(t))$ of a discount bond must satisfy the partial differential equation (PDE)

$$\frac{\partial V}{\partial t} + \frac{\Sigma_r^2}{2}\frac{\partial^2 V}{\partial r^2} + (\theta_r - \kappa_r r - \Lambda_r)\frac{\partial V}{\partial r} - rV = 0. \tag{A.4}$$

However, the above no-arbitrage argument does not tell us the exact functional form of Λ_r, and certainly different specifications of Λ_r will give rise to different PDEs and in turn different solutions of V. Therefore the price of an interest rate derivative is not uniquely determined and the market is incomplete. Nevertheless, since there is only a single source of randomness in model (A.1), if derivatives with maturities only up to some T are concerned, the market can be completed (because Λ_r can be recovered) by including an interest rate derivative with maturity T (such as a discount bond) as a tradable asset. Formally, by the Feynman-Kac formula, once Λ_r is known, we can write $V(t, r) = E^{\mathbb{Q}}\left[\exp(-\int_t^T r(s)ds)V(T, r(T))\right]$ where $\mathbb{Q}$ is a probability measure under which

$$dr(t) = (\theta_r(t) - \kappa_r r(t) - \Lambda_r)dt + \Sigma_r\left(r(t), t\right)\ dW_r^*(t) \tag{A.5}$$

for some $\mathbb{Q}$-standard Brownian motion $W_r^*(t)$. The measure $\mathbb{Q}$ is usually called the risk-neutral measure or the $\mathbb{Q}$-measure, and $-\Lambda_r/\Sigma_r$ is referred to as the market price of risk.

Various well-known one-factor short rate models can be obtained by specifying different θ_r, Λ_r and Σ_r. The Vasicek (1977) model is obtained when θ_r, Σ_r and Λ_r are constants. The CIR model is obtained when θ_r is a constant, $\Sigma_r = \sigma_r \sqrt{r}$ and $\Lambda_r = \lambda_r r$ for some constants σ_r and λ_r. The model of Ho and Lee (1986) is obtained when $\kappa_r = 0$ and $\theta_r, \Sigma_r, \Lambda_r$ are functions of t. The Hull-White (1990) model, which includes the Vasicek model as a special case and the Ho-Lee model as a limiting case, is identical to the Ho-Lee model except that $\kappa_r > 0$. In this paper we write $\theta_r^* = \theta$ and $\kappa_r^* = \kappa_r - \lambda_r$ in the CIR model so that (3.3) is obtained. For the Hull-White model, we write $\theta_r^*(t) = \theta(t) - \Lambda_r(t)$ and $\kappa_r^* = \kappa_r$ and assume that $\Sigma_r(t)$ is equal to some constant σ_r, so that the risk-neutral SDEs for $r(t)$ is of the form (3.11).

From a broader perspective, all of the above-mentioned interest rate models fall into the class of *affine term structure models*, whose concept was popularised by Duffie and Kan (1996). Roughly speaking, an affine term structure model is one in which the time-t price $D(t,T)$ of a discount bond with maturity T is an exponential-affine function of some Markov diffusion process $z(t)$, i.e. $D(t,T) = e^{A(t,T) - B(t,T)z(t)}$ for some deterministic functions A and B that satisfy $A(T,T) = B(T,T) = 0$. We will not delve into the details of affine term structure models here, but note that in both the CIR model and the Hull-White model, $z(t)$ is just the short rate process $r(t)$ and hence the discount bond price can be expressed in the form of equation (3.5).

We now briefly explain how to determine $A(t,T)$ and $B(t,T)$ in the CIR model. Let A' denotes $\dfrac{\partial A(t,T)}{\partial t}$ and B' denotes $\dfrac{\partial B(t,T)}{\partial t}$. Recall that $\theta_r - \kappa_r r - \Lambda_r = \theta_r^* - \kappa_r^* r$. Substituting bond price formula (3.5) into the PDE (A.4), we get

$$(A' - B'r) + \frac{\sigma_r^2}{2} B^2 r - (\theta_r^* - \kappa_r^* r)B - r = 0.$$

Since the equation holds for all r, we must have

$$\begin{aligned} B' &= \frac{\sigma_r^2}{2} B^2 + \kappa_r^* B - 1, \\ A' &= \theta_r^* B. \end{aligned}$$

The first ordinary differential equation (ODE) can be written as $\int_t^T \frac{2dB}{\sigma_r^2(B-b_1)(B-b_2)} = T-t$, where b_1 and b_2 are the roots of the equation $\frac{\sigma_r^2}{2}B^2 + \kappa_r^* B - 1 = 0$. Using partial fraction to decompose the integrand and apply the terminal condition $B(T,T) = 0$, we can solve for $B(t,T)$. The function $A(t,T)$ is obtained easily by integrating the second ODE on both sides from 0 to t.

If we integrate (3.3) on both sides, take risk-neutral expectation and assume that we can change the order of integral signs, we obtain

$$E^{\mathbb{Q}}(r(t)) - r(0) = \theta_r^* t - \kappa_r^* \int_0^t E^{\mathbb{Q}}(r(u))du$$

Differentiate on both sides and solve the resulting ordinary differential equation (ODE), we see that

$$E^{\mathbb{Q}}(r(t)) = r(0)e^{-\kappa_r^* t} + \frac{\theta_r^*}{\kappa_r^*}(1 - e^{-\kappa_r^* t}).$$

We can apply the same technique to find $E^{\mathbb{Q}}(r(t)^2)$ (the SDE for $r(t)^2$ can be derived using Ito's formula) and obtain the variance formula (3.4).

For the Hull-White model, by substituting (3.5) into (A.4), we see that the volatility coefficient of $D(t,T)$ is $\sigma_r B(t,T)$ and hence $\sigma_D = \sigma_r B(0,1)$ is the initial volatility of a one-year discount bond. In addition, the previous substitution gives

$$(A' - B'r) + \frac{\sigma_r^2}{2}B^2 - (\theta_r^* - \kappa_r^* r)B - r = 0,$$

$$\Rightarrow \begin{cases} B' = \kappa_r^* B - 1, \\ A' = \theta_r^* B - \frac{\sigma_r^2}{2}B^2. \end{cases}$$

Solving these two ordinary differential equations, we get formula (3.12) and

$$A(t,T) = -\int_t^T \theta_r^*(u)B(u,T)du + \frac{\sigma_r^2}{2}\int_t^T B(u,T)^2 du. \tag{A.6}$$

Hence equation (3.5) gives

$$\begin{aligned} f(0,T) &= -\frac{\partial}{\partial T}\ln D(0,T) = -\frac{\partial A(0,T)}{\partial T} + \frac{\partial B(0,T)}{\partial T}r(0) \\ &= \int_0^T \theta_r^*(t)\left(1 - \kappa_r^* B(u,T)\right)du - \frac{\sigma_r^2}{2}\frac{\partial}{\partial T}\int_0^T B(u,T)^2 du + \left(1 - \kappa_r^* B(0,T)\right)r(0) \\ &= \int_0^T \theta_r^*(t)du + \kappa_r^* D(0,T) - \frac{\kappa_r^*\sigma_r^2}{2}\int_0^T B(u,T)^2 du - \frac{\sigma_r^2}{2}\frac{\partial}{\partial T}\int_0^T B(u,T)^2 du + r(0). \end{aligned}$$

Differentiate with respect to T once again, we obtain

$$\frac{\partial}{\partial T}f(0,T) = \theta_r^*(T) - \kappa_r^* f(0,T) - \frac{\kappa_r^*\sigma_r^2}{2}\frac{\partial}{\partial T}\int_0^T B(u,T)^2 du - \frac{\sigma_r^2}{2}\frac{\partial^2}{\partial T^2}\int_0^T B(u,T)^2 du$$

$$\Rightarrow \quad \theta_r^*(T) = \frac{\partial}{\partial T}\left(f(0,T) + \frac{\sigma_r^2}{2}\frac{\partial}{\partial T}\int_0^T B(u,T)^2 du\right) + \kappa_r^*\left(f(0,T) + \frac{\sigma_r^2}{2}\frac{\partial}{\partial T}\int_0^T B(u,T)^2 du\right).$$

Therefore we obtain equations (3.14) and (3.15). Also, by equation (A.6),

$$\begin{aligned}
A(t,T) &= -\int_t^T (m'(u) + \kappa_r^* m(u))B(u,T)du + \frac{\sigma_r^2}{2}\int_t^T B(u,T)^2 du \\
&= -\int_t^T e^{-\kappa_r^* u}B(u,T)\ d(e^{\kappa_r^* u}m(u)) + \frac{\sigma_r^2}{2}\int_t^T B(u,T)^2 du \\
&= B(t,T)m(t) + \int_t^T e^{\kappa_r^* u}m(u)\ d(e^{-\kappa_r^* u}B(u,T)) + \frac{\sigma_r^2}{2}\int_t^T B(u,T)^2 du \\
&= B(t,T)m(t) + \int_t^T m(u)(B' - \kappa_r^* B)du + \frac{\sigma_r^2}{2}\int_t^T B(u,T)^2 du \\
&= B(t,T)m(t) - \int_t^T m(u)du + \frac{\sigma_r^2}{2}\int_t^T B(u,T)^2 du \\
&= B(t,T)m(t) - \int_t^T f(0,u)du - \int_t^T \frac{\sigma_r^2}{2\kappa_r^{*2}}(1 - e^{-\kappa_r^* u})^2 du + \frac{\sigma_r^2}{2}\int_t^T B(u,T)^2 du.
\end{aligned}$$

Therefore we obtain formula (3.13). Finally, by equations (3.11) and (3.14), we have

$$\begin{aligned}
& dr(t) = (\theta_r^*(t) - \kappa_r^* r(t))dt + \sigma_r\ dW_r^*(t) \\
\Rightarrow\quad & d\left(r(t) - m(t)\right) + \kappa_r^*\left(r(t) - m(t)\right)dt + \sigma_r\ dW_r^*(t) \\
\Rightarrow\quad & d\left[e^{\kappa_r^* t}\left(r(t) - m(t)\right)\right] = \sigma_r e^{\kappa_r^* t}\ dW_r^*(t) \\
\Rightarrow\quad & e^{\kappa_r^* t}\left(r(t) - m(t)\right) = e^{\kappa_r^* s}\left(r(s) - m(s)\right) + \sigma_r \int_s^t e^{\kappa_r^* u}\ dW_r^*(u) \qquad \text{(A.7)}
\end{aligned}$$

for any $s \leq t$. Put $s = 0$ and take risk-neutral variance on both sides, formula (3.16) follows. The formula also holds under the physical measure because the drift change under the transformation $\sigma_r dW_r(t) = -\Lambda_r(t)dt + \sigma_r dW_r^*(t)$ is deterministic.

The BSHW model under the T-forward measure

Recall that $W_S^*(t)$ and $W_r^*(t)$ in the model dynamics (3.10) and (3.11) are two correlated standard Brownian motions. Without loss of generality, let $W_r^*(t) = W_1(t)$ and $W_S^*(t) = \rho W_1^*(t) + \sqrt{1-\rho^2}W_2^*(t)$ and write $\underline{W}^*(t) = (W_1^*(t), W_2^*(t))$, where $W_1^*(t),\ W_2^*(t)$ are two independent standard Brownian motions. Then we can rewrite equations (3.10), (A.2) and

(A.7) as

$$\frac{dS(t)}{S(t)} = (r(t) - q(t))\, dt + \sigma_S(t)(\rho, \sqrt{1-\rho^2}) \cdot d\underline{W}^*(t)$$

$$dD(t,T) = r(t)D(t,T)dt + \sigma_r \frac{\partial D}{\partial r} dW_r^*(t)$$

$$= r(t)D(t,T)dt - \sigma_r B(t,T)D(t,T)dW_r^*(t)$$

$$= r(t)D(t,T)dt + D(t,T)(-\sigma_r B(t,T), 0) \cdot d\underline{W}^*(t), \tag{A.8}$$

$$r(t) - m(t) = e^{-\kappa_r^*(t-s)}\,(r(s) - m(s)) + \sigma_r e^{-\kappa_r^* t} \int_s^t (e^{\kappa_r^* u}, 0) \cdot d\underline{W}^*(u) \tag{A.9}$$

Under a stochastic interest rate environment, it is usually easier to study the price dynamics under the so-called T-forward measure. This is a measure under which the price of all tradable assets denominated by the price of a discount bond is a martingale. Let T be a fixed date and let

$$Z(t,T) = \exp\left\{ -\int_0^t r(s)ds \right\} D(t,T) = \exp\left\{ -\int_0^t r(s)ds + A(t,T) - B(t,T)r(t) \right\}$$

denote the discounted price of the discount bond $D(t,T)$. By Ito's formula, equation (A.8) implies that

$$dZ(t,T) = Z(t,T)(-\sigma_r B(t,T), 0) \cdot d\underline{W}^*(t).$$

Hence

$$Z(t,T) = Z(0,T) \exp\left\{ -\frac{1}{2}\int_0^t \sigma_r^2 B^2(u,T)du + \int_0^t (-\sigma_r B(u,T),\ 0) \cdot d\underline{W}^*(u) \right\}.$$

Therefore, if we use the discount bond $D(t,T)$ with maturity T as the numeraire, and define the T-forward measure $\mathbb{Q}^T$ as a probability measure that has Radon-Nykodym derivative

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \frac{D(t,T)/D(0,T)}{B(t)/B(0)} = \frac{Z(t,T)}{Z(0,T)} = \exp\left\{ -\frac{1}{2}\int_0^t \sigma_r^2 B^2(u,T)du + \int_0^t (-\sigma_r B(u,T),\ 0) \cdot d\underline{W}^*(u) \right\},$$

then the stock and all discount bonds, when denominated by $D(t,T)$, are martingales under $\mathbb{Q}^T$. Also, by Girsanov Theorem, the process $\underline{W}^T(t)$ defined by

$$\underline{W}^T(t) = \underline{W}^*(t) - \int_0^t (-\sigma_r B(u,T),\ 0)du, \quad \forall t \in [0,T]$$

is a two-dimensional standard Brownian motion under $\mathbb{Q}^T$. Therefore, for all $0 \leq t \leq T$,

$$\begin{aligned}
\frac{d(S(t)/D(t,T))}{S(t)/D(t,T)} &= \frac{dS(t)}{S(t)} - \frac{dD(t,T)}{D(t,T)} - \frac{dS(t)}{S(t)}\frac{dD(t,T)}{D(t,T)} + \left(\frac{dD(t,T)}{D(t,T)}\right)^2 \\
&= -q(t)dt + \left[\sigma_S(t)(\rho, \sqrt{1-\rho^2}) + (\sigma_r B(u,T),\ 0)\right] \cdot (\sigma_r B(t,T)dt + d\underline{W}^*(t)) \\
&= -q(t)dt + \left[\sigma_S(t)(\rho, \sqrt{1-\rho^2}) - (-\sigma_r B(u,T),\ 0)\right] \cdot d\underline{W}^T(t).
\end{aligned}$$

Also, by equation (A.7),

$$\begin{aligned}
r(t) - m(t) &= e^{-\kappa_r^*(t-s)}\left(r(s) - m(s)\right) + \sigma_r e^{-\kappa_r^* t} \int_s^t (e^{\kappa_r^* u}, 0) \cdot (-\sigma_r B(u,T),\ 0)du \\
&\quad + \sigma_r e^{-\kappa_r^* t} \int_s^t (e^{\kappa_r^* u}, 0) \cdot d\underline{W}^T(u)
\end{aligned}$$

and equations (3.17)–(3.21) immediately follow.

Appendix B

Proofs for propositions in chapter 4

Proof of proposition 4.1

Suppose conditions (a), (b) and (d) do not hold. Then

$$\begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix}^{-1} \begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix} \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix} = \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix} \begin{pmatrix} \tilde{p} & 1-\tilde{p} \\ 1-\tilde{q} & \tilde{q} \end{pmatrix} \begin{pmatrix} \widetilde{\phi}_1 & 0 \\ 0 & \widetilde{\phi}_2 \end{pmatrix}.$$

That is,

$$\begin{pmatrix} \tilde{p} & 1-\tilde{p} \\ 1-\tilde{q} & \tilde{q} \end{pmatrix} \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix} = \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix} \begin{pmatrix} \tilde{p} & 1-\tilde{p} \\ 1-\tilde{q} & \tilde{q} \end{pmatrix} \begin{pmatrix} \widetilde{\phi}_1 & 0 \\ 0 & \widetilde{\phi}_2 \end{pmatrix}$$

and in turn

$$\begin{pmatrix} \tilde{p}\varphi_1 & (1-\tilde{p})\varphi_2 \\ (1-\tilde{q})\varphi_1 & \tilde{q}\varphi_2 \end{pmatrix} = \begin{pmatrix} \tilde{p}\phi_1\widetilde{\phi}_1 & (1-\tilde{p})\phi_1\widetilde{\phi}_2 \\ (1-\tilde{q})\phi_2\widetilde{\phi}_1 & \tilde{q}\phi_2\widetilde{\phi}_2 \end{pmatrix}.$$

Hence at least one of $\phi_1\widetilde{\phi}_1 = \phi_2\widetilde{\phi}_1$ or $\phi_1\widetilde{\phi}_2 = \phi_2\widetilde{\phi}_2$ is true. Now condition (c) follows because $\widetilde{\phi}_1, \widetilde{\phi}_2$ do not vanish on any nonempty open interval and ϕ_1, ϕ_2 are continuous.

Proof of proposition 4.3

For the proofs of (ii) and the first part of (iii), see Raible (2000, section 2.5). The second part of (iii) is simply a reformulation of condition (b) of theorem 4.2. It remains to prove statement (i). Recall that the Levy measure of a Meixner(α, β, δ) process is given by

$$\Pi(dx) = \frac{\delta \exp(\beta x/\alpha)}{x \sinh(\pi x/\alpha)} dx.$$

So Π and Π' have nonzero densities and hence they are locally equivalent. Now, by condition (b) of theorem 4.2,

$$\int_{\mathbb{R}} \left(\sqrt{\frac{d\Pi'}{d\Pi}} - 1 \right)^2 \Pi(dx) = \int_{\mathbb{R}} \left(\sqrt{\frac{\delta' \exp(\beta' x/\alpha') \sinh(\pi x/\alpha)}{\delta \exp(\beta x/\alpha) \sinh(\pi x/\alpha')}} - 1 \right)^2 \frac{\delta \exp(\beta x/\alpha)}{x \sinh(\pi x/\alpha)} dx < \infty.$$

Since the exponential terms approach 1 as $x \to 0$, the above is integrable if and only if

$$\left(\sqrt{\frac{\delta' \sinh(\pi x/\alpha)}{\delta \sinh(\pi x/\alpha')}} - 1 \right)^2 \frac{\delta}{x \sinh(\pi x/\alpha)}$$

is integrable near zero, but this means

$$\left(\sqrt{\frac{\delta'\alpha'}{\delta\alpha}} - 1 \right)^2 \frac{\alpha\delta}{\pi x^2}$$

is integrable near zero because $\sinh(y)$ can be expanded into $y + \frac{y^3}{3!} + \frac{y^5}{5!} + \ldots$. Therefore we must have $\alpha\delta = \alpha'\delta'$. Finally, if $A = A' = 0$, then by condition (b) of theorem 4.2, the quantity

$$B := \mu - \left(\alpha\delta \tan\frac{\beta}{2} - 2\delta \int_1^\infty \frac{\sinh(\beta x/\alpha)}{\sinh(\pi x/\alpha)} dx \right) - \int_{-1}^1 \frac{\delta \exp(\beta x/\alpha)}{\sinh(\pi x/\alpha)} dx \tag{B.1}$$

is preserved if we replace $(\alpha, \beta, \delta, \mu)$ by $(\alpha', \beta', \delta', \mu')$. Let $f(x) = \frac{\delta \exp(\beta x/\alpha)}{\sinh(\pi x/\alpha)}$. Then

$$\int_{-1}^1 f(x)dx = \int_0^1 f(x)dx + \int_0^1 f(-y)dy = \int_0^1 \frac{2\delta \sinh(\beta x/\alpha)}{\sinh(\pi x/\alpha)} dx.$$

Therefore

$$B = \mu - \alpha\delta \tan\frac{\beta}{2} + 2\delta \int_0^\infty \frac{\sinh(\beta x/\alpha)}{\sinh(\pi x/\alpha)} dx = \mu - \alpha\delta \tan\frac{\beta}{2} + 2\alpha\delta \int_0^\infty \frac{\sinh(\beta x)}{\sinh(\pi x)} dx.$$

However,

$$\begin{aligned}
&2\int_0^\infty \frac{\sinh(\beta x)}{\sinh(\pi x)} dx = 2\int_0^\infty \frac{e^{-\pi x}(e^{\beta x} - e^{-\beta x})}{1 - 2e^{-2\pi x}} dx \\
&= 2\int_0^\infty (e^{(\beta-\pi)x} - e^{-(\beta+\pi)x}) \sum_{n=0}^\infty e^{-2n\pi x} dx = 2\int_0^\infty \sum_{n=0}^\infty (e^{[\beta-(2n+1)\pi]x} - e^{-[\beta+(2n+1)\pi]x}) dx \\
&= 2\sum_{n=0}^\infty \left(\frac{-1}{\beta - (2n+1)\pi} - \frac{-1}{\beta + (2n+1)\pi} \right) = \sum_{n=0}^\infty \frac{-4\beta}{\beta^2 - [(2n+1)\pi]^2} \\
&= \sum_{n=0}^\infty \frac{-2(\beta/2)}{(\beta/2)^2 - \left(n + \frac{1}{2}\right)^2 \pi^2} = \tan\frac{\beta}{2},
\end{aligned}$$

where the last equality is due to $\tan z = \sum_{n=0}^\infty \frac{-2z}{z^2 - \left(n + \frac{1}{2}\right)^2 \pi^2}$. Therefore the preservation of B in (B.1) simply means $\mu = \mu'$.

Proof of proposition 4.4

Obviously, we have $\eta_X(t) > 0$ and $\eta_X(0) = 1$. Showing that η_X is a P-martingale is straightforward. First, it is clear that for any $0 \leq u_1 \leq u_2 \leq u_3$, we have $\eta_X(u_1, u_3) = \eta_X(u_1, u_2)\eta_X(u_2, u_3)$. Now, consider a time period $[s, t]$. Suppose we know that within this period, ι changes state exactly at times $t_1, t_2, \ldots, t_m$. Write $t_0 = s$ and $t_{m+1} = t$, then

$$\begin{aligned} E^{\mathbb{P}}\left(\left. \frac{\eta_X(t)}{\eta_X(s)} \right| \iota(s,t) \right) &= E^{\mathbb{P}}\left(\exp\left\{ \sum_{j=0}^{m} \left[U^{(\iota_j)}(t_{j+1}) - U^{(\iota_j)}(t_j) \right] \right\} \left. \frac{q(\iota(s,t))}{p(\iota(s,t))} \right| \iota(s,t) \right) \\ &= \frac{q(\iota(s,t))}{p(\iota(s,t))} \prod_{j=0}^{m} E^{\mathbb{P}}\left(\exp\left\{ U^{(\iota_j)}(t_{j+1}) - U^{(\iota_j)}(t_j) \right\} \Big| \iota(s,t) \right) \\ &= \frac{q(\iota(s,t))}{p(\iota(s,t))}. \end{aligned}$$

Therefore by the tower law,

$$\begin{aligned} E^{\mathbb{P}}\left(\eta_X(t) | \mathcal{F}_s \right) &= \eta_X(s) E^{\mathbb{P}}\left[E^{\mathbb{P}}\left(\left. \frac{\eta_X(t)}{\eta_X(s)} \right| \iota(s,t) \right) \Big| \mathcal{F}_s \right] \\ &= \eta_X(s) E^{\mathbb{P}}\left[\left. \frac{q(\iota(s,t))}{p(\iota(s,t))} \right| \mathcal{F}_s \right] \\ &= \eta_X(s). \end{aligned}$$

Finally, condition (4.16) is trivial because conditional on state i, the increment $X_{s+t} - X_s$ has the same law as the sum of a Levy process $Y_s^{(i)}$ and a compound Poisson process $\sum_{j \neq i} \int_0^t J_s^{(i,j)} dN_s^{(i,j)}$, where each $N^{(i,j)}$ is a Poisson process with intensity $\widetilde{a}_{ij}$.

Remark. Since the transition-induced jumps we consider in this paper are constant, if $\iota(s) = i$ on some interval $[u_1, u_2)$, we can determine the path of $Y^{(i)}(s) - Y^{(i)}(u_1)$ for $s \in [u_1, u_2]$ using the paths of X and ι:

$$Y^{(i)}(s) - Y^{(i)}(u_1) = X(s) - X(u_1) - \mathbb{1}\{s = u_2 \text{ and } \iota_s \neq i\} h_{i\iota_s}. \tag{B.2}$$

As $U^{(i)}$ is a Levy process, we can in turn back out $U^{(i)}(s) - U^{(i)}(u_1)$ using information of $Y^{(i)}(s) - Y^{(i)}(u_1)$ on $[u_1, u_2]$. Consequently, η_X is adapted to the information flow generated by X and ι. However, when the transition-induced have stochastic jump sizes, since there can be a nonzero probability that $Y^{(i)}$ and ι may jump together, we cannot almost surely distinguish jumps due to $Y^{(i)}$ from jumps due to a transition of states. So, the information about X and ι alone is not enough to make η_X an adapted process.

References

AASE, KNUT K 2000. "An equilibrium asset pricing model based on Levy processes: relations to stochastic volatility, and the survival hypothesis", *Insurance: Mathematics and Economics* 27(3): 345-363.

ABRAMOWITZ, MILTON, AND IRENE A. STEGUN 1972. *Handbook of Mathematical Functions.* Dover Publications, New York, USA.

APPLEBAUM, DAVID 2004. *Levy Processes and Stochastic Calculus.* Cambridge University Press, Cambridge, UK.

ASMUSSEN, S., AND J. ROSINSKI 2001. "Approximations of small jumps of Levy processes with a view towards simulation", *Journal of Applied Probability* 38(2): 482-493.

AVELLANEDA, M., A. LEVY, AND A. PARAS 1995. "Pricing and hedging derivative securities in markets with uncertain volatilities", *Applied Mathematical Finance* 2(2):73-88.

BALLOTTA, LAURA 2005. "A Levy process-based framework for the fair valuation of participating life insurance contracts", *Insurance: Mathematics and Economics* 37(2):173-196.

BARNDORFF-NIELSEN, OLE E. 1995. "Normal inverse Gaussian distributions and the modeling of stock returns", *Research Report no. 300*, Department of Theoretical Statistics, Aarhus University.

BARNDORFF-NIELSEN, OLE E., AND NEIL SHEPHARD 2001. "Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics", *Journal of the Royal Statistical Society, B* 63(2): 167-241.

BARNDORFF-NIELSEN, OLE E., ELISA NICOLATO, AND NEIL SHEPARD 2002. "Some recent developments in stochastic volatility modelling", *Quantitative Finance* 2(1): 11-23.

BAKSHI, GURDIP, CHARLES CAO AND ZHIWU CHEN 1997. "Empirical performance of alternative option pricing models", *Journal of Finance* 52(5): 2003-2049.

BATES, DAVID S. 1996. "Jumps and stochastic volatility: exchange rate processes implicit in PHLX Deutschmark options", *Review of Financial Studies* 9(1): 69-107.

BEN-AMEUR, HATEM, MICHELE BRETON, AND PIERRE L'ECUYER 2002. "A dynamic programming procedure for pricing American-Style Asian options", *Management Science* 48(5): 625-643.

BERTOIN, J. 1996. *Levy Processes*, Cambridge University Press, Cambridge, UK.

Bickel, Peter J., Ya'acov Ritov, and Tobias Ryden 1998. "Asymptotic normality of the maximum-likelihood estimator for general hidden Markov models", *Annals of Statistics* 26(4): 1614-1635.

Bingham, N.H., C.M. Goldie, and J.L. Teugels 1987. *Regular Variation*, Cambridge University Press, Cambridge, UK.

Bollen, Nicolas P.B. 1998. "Valuing options in regime-switching models", *Journal of Derivatives* 6(1): 38-49.

Boyle, Phelim P., Adam W. Kolkiewicz, and Ken Seng Tan 2001. "Valuation of the reset options embedded in some equity-linked insurance products", *North American Actuarial Journal* 5(3): 1-18.

Boyle, Phelim, and Thangaraj Draviam 2007. "Pricing exotic options under regime switching", *Insurance: Mathematics and Economics* 40(2): 267-282.

Bratley, Paul, and Bennett L. Fox 1988. "Algorithm 659: implementing Sobol's quasirandom sequence generator", *ACM Transactions on Mathematical Software* 14(1): 88-100.

Broadie, Mark, Paul Glasserman, and Steve Kou 1999. "Connecting discrete and continuous path-dependent options", *Finance and Stochastics* 3: 55-82.

Broadie, Mark, and Özgür Kaya 2004. "Exact simulation of stochastic volatility and other affine jump diffusion processes", Working Paper, Graduate School of Business, Columbia University.

Buetow Jr., Gerald 1999. "Ratchet options", *Journal of Financial and Strategic Decisions* 12(2): 17-30.

Carr, Peter and Dilip Madan 1998. "Option valuation using the fast Fourier transform", *Journal of Computational Finance* 2(4): 61-73.

Carr, Peter, and Liuren Wu 2004. "Time-changed Levy processes and option pricing", *Journal of Financial Economics* 71(1): 113-141.

Chan, T. 1999. "Pricing contingent claims on stocks driven by Levy processes", *Annals of Applied Probability* 9(2): 504-528.

Chourdakis, Kyriakos 2002. "Continuous time regime switching models and applications in estimating processes with stochastic volatility and jumps", *Working Paper*, Department of Economics, University of London, Queen Mary.

—— 2004. "Non-affine option pricing", *Journal of Derivatives* 11(3): 10-25.

Cont, Rama, and Peter Tankov 2004. *Financial Modelling with Jump Processes*, Chapman & Hall/CRC, London, UK.

Cox, John C. 1975. "Notes on option pricing I: constant elasticity of variance diffusions", working paper, Stanford University (reprinted in Cox 1996).

—— 1996. "The constant elasticity of variance option pricing model", *The Journal of Portfolio Management* 22(5): 15-17.

Cox, John, Jonathan Ingersoll, and Stephen Ross 1985. "A theory of the term structure of interest rates", *Econometrica* 53(2): 385-407.

Duffie, Darrell, Jun Pan, and K. Singleton 2000. "Transform analysis and asset pricing for affine jump-diffusions", *Econometrica* 68(6): 1343-1376.

EALES, BRIAN, AND RADU TUNARU 2004, "Financial engineering with reverse cliquet options", paper no. 81, Money Macro and Finance (MMF) Research Group Conference 2004, Money Macro and Finance Research Group.

EBERLEIN, E., AND U. KELLER 1995. "Hyperbolic distributions in finance", *Bernoulli* 1: 281-299.

EBERLEIN, E., AND K. PRAUSE 1998. "The generalized hyperbolic model: financial derivatives and risk measures", *FDM Preprint 56*, University of Freiburg.

EDWARDS, CRAIG 2005. "Derivative pricing models with regime switching: a general approach", *Journal of Derivatives* 13(1): 42-47.

ELLIOTT, ROBERT J., LEUNGLUNG CHAN, AND TAK KUEN SIU 2005. "Option pricing and Esscher transform under regime switching", *Annals of Finance* 1(4): 423-432.

FELLER, WILLIAM 1968. *An Introduction to Probability Theory and Its Applications.* John Wiley & Sons, Chichester, UK.

FOLLMER, H., AND M. SCHWEIZER 1991. "Hedging of contingent claims under incomplete information", in M.H.A. Davis and R.J. Elliotts (eds), *Applied Stochastic Analysis*, Gordon and Breach, New York, pp. 389-414.

FUNG, HON-KWOK, AND LEONG KWAN LI 2002. "Fast evaluations of some probability integrals arising from the valuations of discretely monitored derivative securities", *Working Paper*, Department of Applied Mathematics, Hong Kong Polytechnic University.

—— 2003. "Pricing discrete dynamic fund protections", *North American Actuarial Journal* 7(4):23-31.

—— 2005. "Pricing discretely monitored barrier options by a sequential quadrature method", working paper, Department of Applied Mathematics, Hong Kong Polytechnic University.

GERBER, HANS U., AND GERARD PAFUMI 2000. "Pricing dynamic investment fund protection", *North American Actuarial Journal* 4(2): 28-36.

GERBER, HANS U., AND ELIAS S.W. SHIU 1994. "Option pricing by Esscher-transforms", *Transactions of the Society of Actuaries* 46: 99-191.

—— 1996. "Actuarial bridges to dynamic hedging and option pricing", *Insurance: Mathematics and Economics* 18(3): 183-218.

—— 1998. "Pricing perpetual options for jump processes", *North American Actuarial Journal* 2(3): 101-112.

—— 1999. "From ruin theory to pricing reset guarantees and perpetual put options", *Insurance: Mathematics and Economics* 24(1): 3-14.

—— 2003a. "Pricing lookback options and dynamic guarantees", *North American Actuarial Journal* 7(1): 48-67.

—— 2003b. "Pricing perpetual fund protection with withdrawal option", To appear in *North American Actuarial Journal* 7(2).

HARDY, MARY 2001. "A regime-switching model of long-term stock returns", *North American Actuarial Journal* 5(2): 41-53.

—— 2003. *Investment Guarantees: Modeling and Risk Management for Equity-Linked Life Insurance*. Ontario, Canada: John Wiley & Sons, Inc.

—— 2004. "Ratchet Equity Indexed Annuities", the 14th Annual International AFIR Colloquium.

Heston, Steven L. 1993. "A closed-form solution for options with stochastic volatility with applications to bond and currency options", *Review of Financial Studies* 6(2): 327-343.

Heyer, Daniel 2004. "Hedging variable annuity guarantees: a practical discussion", in Francis Sabatini (mod.), session 31PD, *RECORD* (a publication of the Society of Actuaries) 30(2): 1-22.

Ho, Thomas, and Sang-Bin Lee 1986. "Term structure movements and pricing interest rate contingent claims", *Journal of Finance* 41: 1011-1029.

Hirsa, Ali, Georges Courtadon, and Dilip B. Madan 2003. "The effect of model risk on the valuation of barrier options", *Journal of Risk Finance* 4(2): 1-9.

Hull, John 1993. *Options, Futures, and Other Derivative Securities*, 2/e, Prentice-Hall Inc., New Jersey.

Hull, John, and Alan White 1990. "Pricing interest rate derivative securities", *Review of Financial Studies* 3: 573-92.

Hunt, P.J. ,and J.E. Kennedy 2000. *Financial Derivatives in Theory and Practice*, John Wiley & Sons, Chichester, UK.

Imai, Junichi, and Phelim P. Boyle 2001. "Dynamic fund protection", *North American Actuarial Journal* 5(3): 31-49.

Jaimungal, Sebastian 2004. "Pricing and hedging equity indexed annuities with variance-Gamma deviates", Working Paper, Department of Statistics, University of Toronto.

Jaimungal, Sebastian, and Virginia R. Young 2005. "Pricing equity-linked pure endowments with risky assets that follow Levy processes", *Insurance: Mathematics and Economics* 36(3): 329-346.

Jiang, George 2002. "Testing option pricing models with stochastic volatility, random jump and stochastic interest rate", *International Review of Finance* 3(3): 233-272.

Joe, Stephen, and Frances Y. Kuo 2003. "Remark on algorithm 659: implementing Sobol's quasirandom sequence generator", *ACM Transactions on Mathematical Software* 29(1):49-57.

Keller, U. 1997. "Realistic modelling of financial derivatives", *Dissertation*, University of Freiburg.

Klein, Gordon E., and Mary Hardy 2001. Discussion of the paper "A regime-switching model of long-term stock returns", *North American Actuarial Journal* 6(1): 171-176.

Kolkiewicz, Adam W., and Ken Seng Tan 2001. "Volatility risk for regime switching models", working paper, Department of Statistics and Actuarial Science, University of Waterloo.

Kou, S.G. 2002. "A jump diffusion model for option pricing", *Management Science* 48(8): 1086-1101.

KOU, S.G., AND HUI WANG 2004. "Option pricing under a double exponential jump diffusion model", *Management Science* 50(9): 1178-1192.

KUSHLER, U., AND M.SORENSEN 1997. *Exponential Families of Stochastic Processes*, Springer, New York.

KUSHNER, H.J 1990. "Numerical methods for stochastic control problems in continuous time." *SIAM Journal of Control and Optimization*, 28(5): 999-1048.

LIN, SHELDON, AND KEN SENG TAN 2003. "Valuation of equity-indexed annuities under stochastic interest rates", *North American Actuarial Journal* 7(4): 72-91.

LORD, R., F. FANG, F. BERVOETS, AND C.W. OOSTERLEE 2007. "A fast and accurate FFT-based method for pricing early-exercise options under Levy processes", *Working Paper*, Rabobank International, Delft University of Technology and Center for Mathematics and Computer Science (CWI).

LYONS, T.J. 1995. "Uncertain volatility and the risk-free synthesis of derivatives", *Applied Mathematical Finance* 2(2):117-133.

MADAN, D.B., AND E. SENETA 1990. "The VG model for share market returns", *Journal of Business* 63(): 511-524.

MADAN, D.B., P. CARR, AND E.C. CHANG 1998. "The variance Gamma process and option pricing", *European Finance Review* 2(): 79-105.

MASON, JOHN C., AND DAVID C. HANDSCOMB 2003. *Chebyshev Polynomials*, CRC Press, Florida, USA.

MERTON, ROBERT 1973. "Theory of rational option pricing", *Bell Journal of Economics and Management Science* 4: 141-83.

MIYAHARA Y., AND T. FUJIWARA 2003. "The minimal entropy martingale measures for geometric Levy processes", *Finance and Stochastics*, 7(4): 509-531.

MONROE, ITREL 1978. "Processes that can be embedded in Brownian motion", *Annals of Probability* 6(1): 42-56.

MUSIELA, MAREK, AND MAREK RUTKOWSKI 2005. *Martingale Methods in Financial Modelling*, 2/e, Springer-Verlag, Berlin.

NAIK, VASANTTILAK 1993. "Option valuation and hedging strategies with jumps in the volatility of asset returns", *Journal of Finance* 48(5): 1969-1984.

PELSSER, ANTOON 2000. *Efficient Methods for Valuing Interest Rate Derivatives*, Springer-Verlag, London, UK.

PRESS, WILLIAM H., BRIAN P. FLANNERY, SAUL A. TEUKOLSKY, AND WILLIAM T. VETTERLING 1993. *Numerical Recipes in C*, 2/e, Cambridge University Press, Cambridge, UK.

RABINOVITCH, RAMON 1989. "Pricing stock and bond options when the default-free rate is stochastic", *Journal of Financial and Quantitative Analysis* 24(4): 447-457.

RABINOWITZ, PHILIP 1987. "On the convergence of closed interpolatory integration rules based on the zeros of Gegenbauer polynomials", in M. Mori and R. Piessens (eds), *Numerical Quadrature*, Elsevier Science Publishers B.V., Amsterdam, The Netherlands.

RIESNER, MARTIN 2006. "Hedging life insurance contracts in a Levy process financial market", *Insurance: Mathematics and Economics* 38(3): 599-608.

Sato, K.I. 1999. *Levy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge, UK.

Schoutens, Wim 2001. "The Meixner process in finance", *EURANDOM Report 2001-2002*, EURANDOM, Eindhoven, Netherlands.

—— 2003. *Levy Processes in Finance*, John Wiley & Sons, Chichester, UK.

Schoutens, Wim, and J.L. Teugels 1998. "Levy processes, polynomials and martingales", *Communications in Statistics: Stochastic Models* 14(): 335-349.

Schoutens, Wim, Erwin Simons, and Jurgen Tistaert 2003. "A perfect calibration! Now what?" Frankfurt MathFinance Workshop.

—— 2005. "Model risk for exotic and moment derivatives", in Andreas Kyprianou, Wim Schoutens and Paul Wilmott (eds), *Exotic Option Pricing and Advanced Levy Models*, John Wiley & Sons, Chichester, UK.

Scott, Louis 1997. "Pricing stock options in a jump-diffusion model with stochastic volatility and interest rates: applications of Fourier inversion methods", *Mathematical Finance* 7(4): 413-426.

Sullivan, Michael A. 2000a. "Valuing American put options using Gaussian quadrature", *Review of Financial Studies* 13: 75-94.

—— 2000b. "Pricing discretely monitored barrier options", *Journal of Computational Finance* 3(4): 35-52.

Tiong, Serena 2000. "Valuing equity-indexed annuities", *North American Actuarial Journal* 4(4): 149-163.

Tse, Wai Man, Leong Kwan Li, and Kai Wang Ng 2001. "Pricing discrete barrier and hindsight options with the tridiagonal probability algorithm", *Management Science* 47(3): 383-93.

Vasicek, Oldrich 1977. "An equilibrium characterization of the term structure", *Journal of Financial Economics* 5(2): 177-88.

Wilmott, Paul 2002. "Cliquet options and volatility models", *Wilmott Magazine* Dec: 78-83.

Windcliff, Heath, Matin Le Roux, Peter Forsyth, and Kenneth Vetzal 2002. "Understanding the behavior and hedging of segregated funds offering the reset feature", *North American Actuarial Journal* 6(2): 107-124.

Windcliff, Heath, Peter Forsyth, and Kenneth Vetzal 2004. "Numerical methods and volatility models for valuing cliquet options", Technical Report, School of Computer Science, University of Waterloo.