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Portfolio Improvement and Asset Allocation

by

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A thesis submitted in partial fulfillment of

the requirements for the Degree of Doctor of Philosophy

January 2008

CERTIFICATION OF ORIGINALITY

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Abstract

This thesis studies the application of the Sharpe rule and Value-at-Risk in dealing with the portfolio improvement problem. It proposes that a portion of the portfolio value should be invested in some other assets for portfolio improvement. The generalized Sharpe rule is first used to assess the performances of assets or portfolios. Analytic results are derived to show that some assets with better performance are selected for portfolio improvement. By applying the Sharpe rule, it can be determined that new stocks are worthy of adding to the old portfolio if the average return rate of these stocks is greater than the return rate of the old portfolio multiplied by the sum of the elasticity of the Value-at-Risk (VaR) and 1. One attraction of our approach is diversification. Consideration is also given to the 'optimal' number of new assets to be added in two specific cases (i.e., arithmetic series and geometric series regarding the sequences of expected returns and standard deviations). Some interesting simulation results show that a new portfolio with the 'highest' Sharpe ratio can be obtained by adding only a few new assets.

Motivated from the simulations that a few new assets need to be added for portfolio improvement, we also formulate the portfolio improvement problem using the mean-variance approach with equality cardinality constraint. In the formulation, variance is regarded as the risk. The equality cardinality constraint restricts that a given number of new stocks are selected for portfolio improvement. Under the assumptions that all the stocks are uncorrelated, analytical solutions to the formulated problem are derived for two specific cases: the expected returns of stocks are all equal to the desired return, and the expected

returns of stocks are not all equal. The problem is also formulated with inequality cardinality constraint. Comparison is conducted to the problems formulated with equality cardinality constraint and with inequality cardinality constraint. Though the inequality cardinality constraint is set, numerical results show that in most of our simulated cases, the inequality cardinality constraint becomes equality at the optimal solution.

The need of innovation and progress in risk management leads to the popularity of VaR. In another formulation of the portfolio improvement problem, we propose to use VaR instead of variance as a risk measure. Due to some desirable properties of Conditional VaR (CVaR), it makes CVaR much easier to be handled than VaR. The portfolio improvement problem is formulated into a mean-CVaR problem. The problem is then solved under the normality and non-normality assumptions about the portfolio returns. Experimental results show that as the number of scenarios increases, the loss random variable approaches normality under the former assumption; however, such convergence is not observed under the latter assumption.

Acknowledgements

I am deeply indebted to my supervisors, Professor Xiaoqi Yang and Dr. Heung Wong, for their enthusiastic support and encouragement throughout the study of my PhD degree. I benefit greatly from countless insightful discussions with them. They helped me develop idea of writing this thesis as well as the challenging research that lies behind it. Their valuable advice, constructive criticisms and inspiring guidance are greatly appreciated. It has been an invaluable and unforgettable experience for me to do research under their supervision. I would also like to express my gratitude to the Board of Examiners, Dr. Wai-Cheung Ip as the chairman, Prof. Yanping Lin and Prof. Lizhi Liao as the external examiners, for their valuable comments and suggestions.

Sincere thanks go to the staff in the Department of Applied Mathematics at The Hong Kong Polytechnic University for their kind assistance in administrative support and computer-related problem solving.

I also greatly appreciate the financial support from The Hong Kong Polytechnic University, whose research studentship supported me throughout my study.

I would like to give my special thanks to Benny Fu who is always there to share the joys and frustrations I felt and support me during the entire period of this study.

Last, but not least, I wish to thank my family for their encouragement to pursue my study. Especially, I thank my parents for giving me life in the first place, for their unconditional love and support.

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Chapter 1

Introduction

1.1 Motivation and Objectives

Portfolio selection is a complex and challenging problem in financial management. The earliest approach to solving the portfolio selection problem is the mean-variance approach which is proposed by Markowitz (1952). In general, the portfolio selection problem treats the construction of efficient portfolios. The idea of Markowitz's model is that investor should hold mean-variance efficient portfolios.

The resulting efficient portfolios from solving the portfolio optimization problem may satisfy some investors with a specific risk tolerant at the moment. However, after some time, due to the uncertainty in the stock market, the selected portfolio may not best fit some of the investors. An investor may make a request for an improvement in the return on the portfolio. Of course, it is possible to sell the existing portfolio and buy another one with different combination of stocks that satisfies the investor. This will totally change the investment strategy and transaction cost is involved. On the other hand, the existing portfolio may still gain profit but it is under the expectation of the investor as s/he found some other stocks with better performance in the market. In practice, the investor would rather enhance the existing portfolio than trade it altogether. The motivation of this project is basically driven by the need of improving an existing portfolio in portfolio management.

Enhancing an existing portfolio is the objective of this thesis. One approach is to improve an existing portfolio by investing in some new assets. In this process, we need to select a few attractive assets from those in the market. As return and risk are two important quantities in measuring the performance of an investment, it is crucial to consider both return and risk in the selection of assets. Sharpe ratio is one of the most popular performance measures. It is defined as the ratio of the expected return to the standard deviation of the returns. It captures both return and risk (Sharpe (1966, 1975, 1994), Dowd (1998, 1999, 2000), Hodges (1998) and Amin and Kat (2002)). By the generalized Sharpe rule, a new asset with higher Sharpe ratio has higher priority to be selected.

Theoretically, a portfolio can consist of a large number of assets. However, some empirical results show that an efficient portfolio may constitute a small number of assets. Motivating from observations, we investigate into the determination of the 'optimal' number of new assets to be invested in a portfolio. Here, 'optimal' means that the minimum number of new stocks is selected to form a new portfolio with the highest Sharpe ratio.

In a general view, the portfolio improvement problem can be formulated into a mean-variance problem for analysis. Here, variance is referred to as the risk of the portfolio. In a standard mean-variance model, the number of assets in a portfolio is not restricted. However, it is not practical to involve too many assets in a portfolio. For controlling the number of assets to be invested in a portfolio, a cardinality constraint is introduced. With an inequality cardinality constraint, the number of selected assets is set in a range; comparatively, with an equality cardinality constraint, the number of selected assets is fixed. Our goal is to solve the mean-variance problem with an equality cardinality constraint for portfolio improvement and compare it with the formulation with an inequality cardinality constraint correspondingly. Taking cardinality constraint into consideration makes the problem more difficult to be solved than the standard mean-variance problem. Our approach is to solve the problem by the Xpress Solver in which the Interior Point method and cutting-plane strategies are applied. Discussions on heuristics or exact solution methodologies for the cardinality constrained mean-variance model can also be found in Chang et al. (2000), Crama and Schyns (2003), Jobst et al. (2001), Bienstock (1996) and Li et al. (2006).

By definition, variance is a measure of the dispersion or spread of a distribution. Unfortunately it cannot tell how much market risk the portfolio is carrying. In contrast with variance, Value-at-Risk (VaR), a single statistical measure of possible portfolio losses, is gaining its popularity as it can quantify market risk. Due to some desirable properties, e.g. sub-additivity, conditional Value-at-Risk (CVaR), which is the expected loss exceeding VaR, is more attractive than VaR (Uryasev and Rockafellar (2000)). Hence, a new approach is to formulate the portfolio improvement problem into a mean-CVaR problem with a cardinality constraint.

To solve the mean-CVaR problem, it is crucial to make an assumption about the expected returns of assets. It is usually assumed that the returns have a multivariate normal distribution. This assumption is popular and in widespread use. For example, as stated in the Technical Document provided by JPMorgan (1996, p. 13), when computing a portfolio's VaR using RiskMetrics, it is assumed that the portfolio return is normally distributed. RiskMetrics provided by JPMorgan is a set of techniques and data to measure market risks in portfolios of fixed income instruments, equities, foreign exchange, commodities, and their derivatives issued in over 30 countries. When the expected returns of assets have a multivariate normal distribution, the portfolio VaR and CVaR can be expressed in terms of expected return and variance of the portfolio returns. As pointed out in Dowd (1998), normality gives us a simple and tractable expression for VaR.

However, it is observable that the true underlying distribution may not be normal (fat tails, etc.). The VaR computed under the assumption of normality may be underestimated. To compare with the results obtained under the assumption of normality, it is possible to solve the mean-CVaR problem under non-normality assumption. It is convenient to assume that the expected returns of stocks have a multivariate T distribution. A T distribution has a fatter tail than a normal one. Moreover, it provides an easy and intuitively plausible way to estimate VaR. Hence, we intend to solve the mean-CVaR problem under this assumption and compare the result with that under the normality assumption.

1.2 Literature Review

1.2.1 Modern Portfolio Theory (MPT)

The work done by Markowitz (1952) is contributed to the basis of the Modern Portfolio Theory (MPT). In principle, MPT is about the optimization of portfolios for rational investors and the pricing of risky assets. It is to deliver solution methods to the portfolio selection problem. See Ingersoll (1987) for a rigorous and comprehensive representation of MPT. Also, an intuitive introduction and insights of MPT can be found in Harrington (1987), Copeland, Weston and Shastri (2005), Rasmussen (2003), Elton et al. (2003) and Litterman (2003).

Starting from the pioneering work of portfolio selection by Markowitz (1952), the portfolio optimization problem has long been investigated by practitioners and researchers. In the 50's and 60's of the twentieth century, the mean-variance analysis was developed by Tobin (1958), Sharpe (1963, 1964), and Lintner (1965), among others. With a heuristic introduction to the basic portfolio selection problem, Martin (1955) analyzed and explained, by reference to empirical data, some of the work done by Markowitz (1952). More discussions on investment balance and portfolio decision is carried out in Tobin (1958). He considered the liquidity preference theory which takes as given the choices determining how much wealth is to be invested in monetary assets and concerns itself with the allocation of these amounts among cash and alternative monetary assets. The concerns in the theory are also treated in the portfolio optimization problem.

The mean-variance approach has a great regard to the tradeoff between return and risk. A general belief is that a higher risk needs to be borne in order to acquire a higher return in an investment. In the portfolio theory, it is assumed that investors always prefer higher returns to lower returns for a given level of risk; likewise for a given level of return, one prefers lower risk to higher risk. Risk can be measured in terms of variance or standard deviation of the return (Markowitz (1952, 1959)). Hence, when analyzing and solving the portfolio selection problem, one popular approach is so-called mean-variance analysis. In this approach, the portfolio selection problem can be formulated to minimize the variance of the portfolio subject to a prospective level of return; or equivalently, maximize the return on the portfolio subject to a tolerant level of risk. Consequently, mean-variance efficient portfolios result from the optimization problem. Luenberger (1997, p. 157) illustrates that the efficient portfolios provide the best mean-variance combinations for most investors.

The original Markowitz mean-variance portfolio selection problem is treated a single period investment. See, for example, Levy and Markowitz (1979), Pulley (1981, 1985) and Kroll et al. (1984). They all favour the mean-variance model. By applying Lagrange multipliers, Merton (1972) derived an analytical solution to the Markowitz problem under the assumption that short selling is allowed. It leads to some insightful implications about the characteristics of the efficient portfolio frontiers. See also Markowitz (2000) for a detailed discussion on the solution to the general portfolio selection model.

Later, the mean-variance approach is in widespread application. It is applied for portfolio optimization and developed to analyze problems from conventional single period to multi-period. The Markowitz mean-variance portfolio optimization problem is extendable to be considered in multiperiod and for the continuous-time case, which can be derived by different approaches. Recently, Li and Ng (2000) derived explicit solutions to the discrete-time, multiperiod mean-variance problem. For more discussions on the mean-variance problem in the multiperiod case, see also Mossin (1968), Samuelson (1969), Hakansson (1971), Francis (1976), Campbell et al. (1997), and Çelikyurt and Özekici (2007).

For the continuous-time, dynamic mean-variance problem in a complete market, Zhou and Li (2000) solved it by diffusion process with deterministic coefficients. Li, Zhou and Lim (2002) considered the mean-variance portfolio selection problem in continuous-time under the constraint that short-selling of stocks is prohibited. More about the mean-variance portfolio selection problem in the continuous-time case can be found in Merton (1969, 1971), Karatzas et al.

(1987), Cox and Huang (1989), Duffie and Richardson (1991), Dumas and Luciano (1991), Grossman and Zhou (1996), Zhou and Yin (2003), and Bielecki et al. (2005).

Except portfolio optimization, the basic concepts of MPT include the two-fund theorem, the Capital Asset Pricing Model (CAPM), the capital market line and the security market line (Luenberger (1997), Panjer et al. (1998)). Müller (1989) summarized some main results in MPT. Specifically, he presented the Markowitz approach and discussed CAPM.

Luenberger (1997) illustrated the two-fund theorem. He stated that according to the two-fund theorem, two mutual funds could provide a complete investment service for everyone. However, this conclusion is based on some assumptions, e.g. investors are only concerned about mean and variance.

An equilibrium model for asset pricing, CAPM, was developed by Sharpe (1964) and Lintner (1965). Harrington (1983, p. 29) discussed some assumptions behind CAPM. For example, it is assumed that there is a risk-free asset, and investors can borrow and lend at the risk-free rate. This assumption is crucial. The risk-free asset simplifies that curved efficient frontier of MPT to the linear efficient frontier of the CAPM.

Sharpe (2000, p. 84) stated that the slope of the capital market line indicates the trade-off between expected return and risk (uncertainty). Sharpe (2000, p. 95) summarized that the security market line indicates the relationship between expected return and volatility and thus indicates the manner in which characteristic lines are related.

1.2.2 Portfolio Improvement

One of the financial planning services provided by financial institutions is portfolio management. It helps clients to construct strategies in balancing return and risk. Of course, the task is to maximize the return and minimize the risk. Due to the uncertainty of the equity market, an existing/old portfolio may perform worse as time went by. There is a need to improve the existing portfolio. Portfolio improvement is an important task in portfolio management. However, there are limited literatures on this issue (Hodges and Schaefer (1977), Sharpe (1987), Dowd (1998, 1999), Fabozzi (1999), Larsen and Resnick (2001), Spice and Hogan (2002), Bowden (2003), and Liu and Pan (2003)).

In an early literature, Hodges and Schaefer (1977) described a simple linear programming model for improving an initial bond portfolio. Their goal is to minimize the cost of achieving a given maturity profile of portfolio cash flows at a given tax rate. In the improvement, the yield on bond portfolio is increased without reducing any future after-tax cash flows.

In a close view with the problems faced by portfolio managers, Sharpe (1987) presented an algorithm for portfolio improvement. In the implementation of his approach, each iteration selects the 'best' security for purchase and the 'worst' for sale. Hence, an initial feasible portfolio is improved. Finally, the maximum improvement will be obtained.

Dowd (1998, 1999) applied the generalized Sharpe rule on the derivation of criteria to check the worthiness of adding a specific new stock into the old portfolio. In this process, a necessary condition is that the Sharpe ratio of the new portfolio must be greater than that of the old portfolio. In other words, the new portfolio performs better than the old portfolio which has been improved.

Fabozzi (1999, p. 261) illustrated two methods to improve risk-adjusted portfolio return: creating a 'tiled' portfolio and utilizing the future markets. The former constructed portfolio can be designed to maintain a strong relationship with a benchmark by minimizing the variance of the tracking error. The latter method involves the use of stock index futures. The strategy can be referred to as indexing enhancement and its focus is on risk control.

By applying modern portfolio theory (MPT), Larsen and Resnick (2001) demonstrated the potential for various ex ante portfolio parameter estimation techniques and optimization/holding-period frequency intervals to enhance managed portfolio returns relative to a benchmark.

With empirical evidence, Spice and Hogan (2002) showed the wise use of venture investing for improving overall portfolio performance. Moreover, they suggested several points that financial advisors new to venture investing can help clients who wish to participate in venture investing.

More recently, several approaches for portfolio improvement are proposed. Bowden (2003) suggested two approaches to portfolio enhancement. The first is based on traditional beta analysis. The second is non-parametric in nature and plots ordered mean difference schedules for the enhancement against the base portfolio. Liu and Pan (2003) proposed dynamic derivative strategies for asset allocation, and found that improving the portfolio efficiency is done from derivative investing. Winkelmann (2004) concluded that the portfolio efficiency can be improved by introducing a portable alpha program, introducing an active overlay program, diversifying the private equity portfolio more and increasing active risk.

1.2.3 Risk Measures

Li et al. (2006) pointed out that construction of a suitable risk measure plays an essential role in portfolio selection. Szegö (2002) presented the definition of risk measure and the main recently proposed risk measures. Variance is one of the most popular risk measures. Unfortunately, there are several conceptual difficulties with using variance/standard deviation as a measure of risk. As stated in Bertsimas et al. (2004), quadratic utility displays the undesirable properties of satiation and of increasing absolute risk aversion; see also Huang and Litzenerger (1988); moreover, the assumption of elliptically symmetric return distributions rules out possible asymmetry in the return distribution of assets, which commonly occurs in practice. Furthermore, asymmetric return distributions make standard deviation an intuitively inadequate risk measure. It is demanding to devise an alternative risk measure.

VaR is one of the alternative risk measures. A formal definition of VaR in Dowd (1998) expresses it as the maximum expected loss over a given horizon period at a given level of confidence. See also Linsmeier and Pearson (2000) and Jorion (2001) for an introduction to the concept and methodology of VaR. Recently, VaR is one of the most popular tools in risk management. It is widely used by practitioners, such as fund managers, dealers, corporate treasurers, and regulators. VaR is also in widespread use in banks, since the Basel Committee on Banking Supervision (1996, 2003) allows banks to use VaR when determining their capital-adequacy requirements arising from their exposure to market risk.

Hence, VaR is proposed to be used instead of variance in the mean-variance analysis for portfolio selection by some researchers. Alexander and Baptista (2001) examined the economic and equilibrium implications arising from a mean-VaR model for portfolio selection. They also compared the model with a mean-variance model and observed that the mean-VaR efficient set converges to the mean-variance set as the confidence level at which VaR is computed increases.

However, Artzner et al. (1999) shows that VaR is not a coherent risk measure since it lacks of sub-additivity property which makes it difficult to be handled. See also Acerbi and Tasche (2002). Another risk measure, Conditional Value-at-Risk (CVaR), is introduced. CVaR is defined as the expected loss exceeding VaR. So, CVaR is closely related to VaR, but has more attractive properties such as sub-additivity, convexity and coherence. The proof of convexity and coherence of CVaR can be found in Rockafellar and Uryasev (2002). Due to these desirable properties, CVaR is more attractive than VaR.

By proposing CVaR as a risk measure, Uryasev and Rockafellar (2000) solved a mean-CVaR portfolio optimization problem under the normality assumption. Uryasev (2000) outlined the approach for simultaneous minimization of CVaR and calculation of VaR. Alternatively, Palmquist et al. (1999) investigated the model of constraining CVaR in order to find a portfolio with maximal return. With regards to the VaR and CVaR constraints, Alexander and Baptista (2001) made a comparison between the mean-CVaR with the mean-VaR models and hence showed some implications. They concluded that a CVaR constraint dominates a VaR constraint as a risk management tool when the CVaR bound is set between two specific levels. Regarding a portfolio of derivatives, Alexander et al. (2004) proposed to include cost as an additional preference criterion for the CVaR optimization problem. They demonstrated that it is possible to compute an optimal CVaR derivative portfolio with significantly fewer instruments.

1.3 Outline of the Thesis

The rest of the thesis is organized as follows. Chapter 2 presents the basic concepts on asset allocation, portfolio optimization and probability theory.

Chapter 3 discusses in details on how to improve an existing portfolio by applying the Sharpe rule and Value-at-Risk. Some criteria are derived to check whether it is worthwhile investing in some new assets for constructing a new portfolio with better performance (Yu et al. (2007)). It can be observed from the numerical examples that only a few new assets are selected to be invested for a 'best' performed new portfolio. The 'best' performed new portfolio is the one with the 'highest' Sharpe ratio among the others by applying the Sharpe rule. The generalized Sharpe rule states that the higher the Sharpe ratio, the better the performance of the portfolio. Sharpe ratio is defined as the ratio of the expected return to the standard deviation of the portfolio returns. It captures both return and risk. Motivating from observations, we try to determine the 'optimal' number of new stocks to be invested in Section 3.2. Here, 'optimal' means that the minimum number of new stocks is selected to form a new portfolio with the highest Sharpe ratio. In the derivation, it is assumed that both the expected returns of stocks and the standard deviations are in arithmetic sequences or in geometric sequences, respectively.

More generally, our portfolio improvement problem can be formulated into a mean-variance problem. Chapter 4 investigates into the formulation with an equality cardinality constraint. Under the assumptions that all the assets are uncorrelated, the expected returns on assets are all equal to the desired return, but the variances are different, we derive some analytical solutions to the problem in Section 4.2. Moreover, we also derive analytical solutions to the problem without the assumption of equal expected return for each asset. As the cardinality constraint restricts the number of new stocks to be invested, it is required to select a specific number of new stocks from those given ones in solving our problem. Section 4.3 demonstrates our proposed stock picking strategy for solving our problem in the cases of picking 2 and 3 stocks respectively. It can be shown that, under some assumptions, the variance of the portfolio returns in the list of possible combinations constructed by our stock picking strategy is monotonically increasing. Actually, our formulated mean-variance problem is a Mixed Binary Quadratic Programming (MBQP) problem. Section 4.4 presents some procedures to solve the (MBQP) problem by using the Xpress Solver. Furthermore, Subsection 4.4.1 illustrates the application of the Xpress Solver in solving our (MBQP) problem with numerical examples. On the other hand, we also formulate our problem with inequality cardinality constraint. In contrast with equality cardinality constraint, inequality cardinality constraint does not restrict the number of selected new stocks to be fixed, but within a specific number. Comparison is carried out for the two formulated problems with some numerical examples in Section 4.5.

As a more comprehensive risk measure, CVaR is used instead of variance in the formulation of the portfolio improvement problem in Chapter 5. For the ease of implementation, the mean-CVaR problem is first solved by assuming that the expected returns of stocks have a multivariate normal distribution in Section 5.2. It is observed from the numerical results that as the number of scenarios increases, the values of VaR and CVaR are closer to their ‘true’ values. In contrast, the mean-CVaR problem is also solved under the non-normality assumption. Due to the fact that it is plausible to estimate VaR with a T distribution, it is assumed that the expected returns of stocks have a multivariate Student’s T distribution. Section 5.3 discusses on the solving of the mean-CVaR problem under the non-normality assumption and concludes with experimental results.

Finally, Chapter 6 contains discussions and conclusions.

Chapter 2

Fundamentals of Asset Allocation and Portfolio Management

2.1 Asset Allocation

In the role of financial planning, portfolio managers always seek a suitable investment opportunity to fulfill the financial needs of a particular investor. Many investment opportunities compose of different kinds of assets. Assets can be categorized into several classes according to return and risk. Some examples of major asset classes are cash, bonds, stocks and real estate. Among these asset classes, stocks have the highest returns and investing in cash offers the lowest return. Due to the tradeoff between return and risk, an asset class with higher return will bear higher risk. Though stocks have the highest return, they are most volatile and thus have the highest risk; in contrast with stocks, investing in cash is much safer and its concern is inflation risk.

The idea behind asset allocation is to divide your investment amount or portfolio into different asset classes. In this process, it will provide you with an investment strategy to achieve the highest expected return in a tolerant level of portfolio risk for a specific time horizon. This is a goal of many investment managers and investors, and hence gains the popularity of asset allocation in financial planning. Since different investors possess different levels of risk tolerance, determination of one's asset allocation is personal. In other words, it is different for person with different financial needs.

There are several factors affecting the determination of asset allocation. One of the most important factors is risk tolerance. Risk tolerance expresses one's ability and willingness to expose to loss in an investment for higher portfolio returns. It is hard to quantify one's risk tolerance as it is subjective. Many financial service companies provide investors with a risk tolerance questionnaire. The questionnaire is designed with several questions which provide some indication of the general attribute toward risk of a typical investor. Risk tolerance of an investor is then determined after s/he completed the questionnaire. Though it may not match his/her actual attitude toward investment risk, it indicates the profile of risk that fits him/her. Another important factor is investment time horizon. Asset price varying from time to time leads to an uncertainty in gain and loss of an investment. As pointed out in Frush (2007), time horizon greatly impacts expectations for asset class returns, asset class volatility, and correlations among assets classes. Long-term and short-term investments should have different strategies. It is general to invest in more equities, e.g. stocks, and fewer fixed-income assets, e.g. bonds and cash, for a longer time horizon. Thus, time horizon plays an important role in asset allocation. It is helpful to determine portfolio balance between equities and fixed-income assets.

Frush (2007) summarized some key benefits of asset allocation and gave detailed commentaries on each benefit. The key benefits are listed in the following. Some more detailed discussions on asset allocation can be found in this book.

- Minimizes retirement plan losses
- Promotes an optimal portfolio
- Eliminates what does not work

- Supports quick and easy reoptimization
- Maximizes portfolio risk-adjusted return
- Promotes simple portfolio design and construction
- Allows for easy contribution decisions
- Minimizes portfolio volatility
- Minimizes investor time and effort
- Promotes a more diversified portfolio
- Provides maximum avoidance of market weakness
- Delivers the highest impact value
- Reduces trading costs

2.2 Portfolio Management

From the financial point of view, a portfolio is a collection of investments in certain assets such as cash, stocks, bonds, real estate, options and future contracts. Different asset is possessing different level of risk. Different investor has his/her own attitude toward the risk. There is a need for every investor to make a choice from among an enormous number of assets. Selection of several assets for a desired portfolio is not an easy job for many investors. Some of them may ask for help from portfolio managers in various financial service companies. The most important issue for portfolio managers is to determine the risk tolerance of their clients. Investors always prefer higher return and lower risk. However, a general belief is that an investment with a higher return bears higher risk. Portfolio management plays an important role in balancing return and risk of investments.

Besides determining clients' risk tolerance to manage clients' portfolio, portfolio manager must take into account other considerations, such as the amount of resources available for investing, tax status of the investor, liquidity needs and time horizon of investment. More details of individual discussion on these considerations can be found in Brentani (2004).

The goal of portfolio management is to construct portfolios comprising various assets and securities that satisfy investors financial needs, and hence to manage the portfolios in order to achieve investment objectives. Portfolio construction can be done by return-risk analysis in two ways:

- Minimize the risk for a given expected return
- Maximize the expected return for a given level of risk

In the modern portfolio theory model, it interprets risk in terms of the standard deviation of the portfolio returns.

Apart from construction of an efficient portfolio, portfolio management involves the evaluation of portfolio performance. Return and risk of an investment are two main considerations that investors take into account to evaluate the performance of a portfolio. Every investor prefers a portfolio with the highest return and the lowest risk. However, there is a general belief that 'No Pain No Gain'. This belief is also true in investments, that is, higher risk needs to be bore in order to achieve higher return. In other words, there is a tradeoff between return and risk.

To fairly compare the performances among portfolios, the easiest way is to compare the rates of returns amongst portfolios with similar risk level. However, this process may be misleading, in which some portfolio managers may concentrate on particular subgroups and the portfolio profile may not be actually comparable. There is a need to devise a single measure for comparing portfolio performance. The measure must take both return and risk into account. Risk-adjusted returns are introduced as portfolio performance measures, in which portfolio returns must be adjusted for risk to compare portfolio performance meaningfully. They may not be perfect measurements, but they do provide useful information about portfolios. Some popular risk-adjusted measures calculate risk-adjusted returns using mean-variance criteria and measure both return and risk. Sharpe ratio is one of the most popular risk-adjusted measures.

Under the generalized Sharpe rule, Sharpe ratio can be expressed as the ratio of the expected rate of return to the standard deviation of the portfolio returns, i.e.

$$\text{Sharpe ratio} = \frac{R_P}{\sigma_P}, \tag{2.1}$$

where R_P is the expected rate of return to a portfolio and σ_P is the standard deviation of the portfolio returns. Note that σ_P is referred to as the risk of the portfolio. Obviously, a rising of the return or a falling of the standard deviation leads to an increase in the Sharpe ratio. It implies that a portfolio with a higher Sharpe ratio is preferable.

As shown in expression (2.1), the expected rate of return R_P and the standard deviation σ_P must be known in advance for obtaining the value of the Sharpe ratio. Note that we would like to shorten the name 'expected rate of return' to 'expected return' for simplicity in the rest of this thesis.

Suppose that a portfolio consists of n assets with weights x_i, i = 1, 2, …, n. The expected returns of the individual assets are $R_1, R_2, \ldots, R_n$ respectively. It can be shown that the expected return on the portfolio in terms of the expected returns of the individual assets is

$$R_P = x_1 R_1 + x_2 R_2 + \cdots + x_n R_n . \tag{2.2}$$

This expression can be interpreted as the weighted average of the expected returns of n assets.

Suppose that the standard deviation of the return on asset i is σ_i, and the covariance of the returns on asset i with asset j is σ_{ij}, i, j = 1, 2, …, n. The variance, i.e. square of the standard deviation, of the portfolio return can be expressed as

$$\sigma_P^2 = \sum_{i,j=1}^{n} x_i x_j \sigma_{ij} , \tag{2.3}$$

which is equivalent to

$$\sigma_P^2 = \sum_{i=1}^{n} x_i^2 \sigma_i^2 + 2 \sum_{\substack{i,j=1 \\ i \neq j}}^{n} x_i x_j \rho_{i,j} \sigma_i \sigma_j , \tag{2.4}$$

by recalling $\sigma_{ii} = \sigma_i^2$, $\sigma_{ij} = \sigma_{ji} = \rho_{i,j}\, \sigma_i\, \sigma_j$ for $i \neq j$, and $\rho_{i,j}$ is the correlation coefficient of the returns on asset i with asset j.

In order to obtain the expected return, the standard deviation and covariance of an asset, say stock, it is required to specify the length of the investment period, such as a day, a week, a month, a quarter or a year. For example, the daily expected return on stock i can be estimated roughly as

$$R_i = \frac{1}{T} \sum_{t=1}^{T} \frac{p_i^{t+1} - p_i^t}{p_i^t} , \tag{2.5}$$

where p_i^t is the closing price of stock i on day t and p_i^{t+1} is the closing price of stock i on day $t+1$. For the closing prices, one obvious source is historical data. Note that T is the number of days over a long period of time, e.g. $T = 250$ if we calculated the daily expected return over a year and assume 250 trading days per year.

Likewise, the variance of stock i can be estimated by averaging the square of the day's deviations from the expected value. That is, the variance of stock i, σ_i^2, can be estimated by

$$\sigma_i^2 = \frac{1}{T-1}\sum_{t=1}^{T}\left(\frac{p_i^{t+1} - p_i^t}{p_i^t} - R_i\right)^2. \tag{2.6}$$

In a similar manner, the covariance between stocks i and j, for $i \neq j$, can be estimated as follows

$$\sigma_{ij} = \frac{1}{T-1}\sum_{t=1}^{T}\left(\frac{p_i^{t+1} - p_i^t}{p_i^t} - R_i\right)\left(\frac{p_j^{t+1} - p_j^t}{p_j^t} - R_j\right), \tag{2.7}$$

where R_i and R_j are the daily expected returns on stocks i and j respectively. Notice that the correlation coefficient of the returns on stocks i and j is defined as

$$\rho_{i,j} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}. \tag{2.8}$$

It can be shown that $-1 \leq \rho_{i,j} \leq 1$. If $\rho_{i,j} = 0$, then the returns on stocks i and j are said to be uncorrelated. In this case, the returns on stocks i and j are independent. If $\rho_{i,j} > 0$, then the returns on stocks i and j are said to be positively correlated. This is the case that they have an increasing linear relationship. On the other hand, if $\rho_{i,j} < 0$, then the returns on stocks i and j are said to be negatively correlated and they have a decreasing linear relationship.

As mentioned before, modern portfolio theory interprets risk in terms of the standard deviation of returns. However, one key insight of the portfolio theory is that the risk of any individual asset is not the standard deviation, but rather the extent to the contributions of that asset to the overall portfolio risk. It is also observable that the standard deviation only measures the spread of the portfolio values but does not quantify the losses involved in the portfolio. Value-at-Risk (VaR) is one of the most popular quantitative measures of losses due to market movements. Comparatively, VaR is a single and comprehensive risk measure.

The history of VaR started in the late 1980s. With the introduction of derivative instruments to offset the risks in existing instruments, positions and portfolios, the inclusion of large numbers of cash and derivative instruments in many portfolios made the magnitudes of the risks in portfolios not obvious. This led to a demand for quantitative measures of market risk in portfolios. VaR was one of such measures and was first used to measure the risks of trading portfolios by major financial firms. It was then developed accompanying the development of risk management guidelines in the early 1990s. Currently, VaR is used by most major derivatives dealers to measure and manage market risk. It also gains popularity in use by banks to calculate their capital requirements for market risk. With the allowance of VaR models to be used to calculate capital requirements for foreign exchange positions by the European Union's Capital Adequacy Directive in 1996, it is made to move toward allowing VaR to calculate capital requirements for other market risks. The developments of VaR and risk management are elaborated in Linsmeier and Pearson (1996), Dowd (1998) and Holton (2003).

In the definition, VaR is the maximum amount likely to be lost over some period at some specific confidence level. It usually refers to a particular amount of money. In estimating the value of VaR, two parameters, i.e. holding period and confidence level, need to be specified. The holding period is usually set to be one day or one month. The possible confidence levels for VaR are 99%, 95% and 90%. The choice of these two parameters was discussed in Dowd (1998). He pointed out that there are four factors that affect the choice of holding period. Except the first factor, i.e. the liquidity of the markets in which the institution operates, the other three factors all suggests a very short holding period. One reason is to justify a normal approximation. A shorter holding period helps make the normal approximation more defensible. A second reason is to accommodate changes in the portfolio itself. The longer the holding period, the more likely portfolio managers are to change the portfolio, particularly if it is making losses.

For the choice of confidence level, Dowd (1998, p. 53) concluded as follows: Different VaR confidence levels are appropriate for different purposes: a low one for validation, a high one for risk management and capital requirements, and perhaps a medium or high one for accounting/comparison purposes. However, there is no compelling reason for an institution to work with on confidence level alone: there is no need for an institution to choose a low confidence level when assessing its capital requirements, say, just because model validation requires a VaR based on a low confidence level. Within reason, the institution could use a high confidence level when determining capital requirements and a low one when conducting validation exercises. In short, an institution should generally use whatever confidence level is appropriate to the task at hand.

2.3 Some Common Assumptions about the Probability Density Function of the Portfolio Return

Let us discuss the definition of the probability density function (p.d.f.). Hogg and Tanis (1993, p.192) depicted that the p.d.f. of a random variable X of the continuous type, with space S that is an interval or union of intervals, is an integrable function $f(x)$ satisfying the following conditions:

(a) $f(x) > 0, \quad x \in S$.

(b) $\int_S f(x)\, dx = 1$.

(c) The probability of the event $X \in A$ is

$$P(X \in A) = \int_A f(x)\, dx .$$

As pointed out in Jorion (2001), VaR describes the probability boundary of potential loss. The value of VaR is closely related to the probability of a return less than the cut-off returns. In general, VaR can be derived from the probability distribution of the portfolio return. Suppose the daily return of a portfolio is a continuous-type random variable, R, and the p.d.f. of the portfolio return is $f(R)$. At a level of confidence of $1-c$, the probability of a return less than the cut-off return is

$$\Pr[R < R^*] = \int_{-\infty}^{R^*} f(R)\, dR = c, \tag{2.9}$$

where R^* is called the quantile of the distribution, which is the cut-off return with a fixed probability of being exceeded.

Dowd (1998) showed that VaR can be represented in terms of absolute dollar loss, or in terms of loss relative to the mean. See also Jorion (2001). The former is simply the maximum expected loss amount with a given level of confidence, measured from the current level of wealth. With this definition, the VaR in absolute dollar terms can be expressed in terms of the cut-off return R^* as

$$\text{VaR(absolute)} = -R^*W, \tag{2.10}$$

in which W is the initial portfolio value. The latter VaR is defined in terms of the maximum expected loss amount with a given level of confidence, measured relative to the mean expected return over the period. With a given mean return μ, the VaR relative to the mean is

$$\text{VaR(relative)} = -R^*W + \mu W. \tag{2.11}$$

Note that by using a parametric approach to VaR, we can work with either absolute VaR or relative VaR. Dowd (1998, p. 43) states: In any case, if we are dealing with a short time period, the difference between absolute and relative VaRs will be fairly small anyway, so we may as well use whichever VaR is more convenient.

To compute VaR, assume that daily returns are identically and independently distributed. When estimating VaR, it is critical to make some assumptions about the p.d.f. of the portfolio return. The specification in the previous paragraph is valid for any distribution, discrete or continuous, fat- or thin-tailed.

One of the most common assumptions about the portfolio return is the normality assumption. That is, the portfolio return is normally distributed. Under the normality assumption, the random variable R has a normal distribution and its p.d.f. is defined by

$$f(R) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(R-\mu)^2}{2\sigma^2}\right], \qquad -\infty < R < \infty, \tag{2.12}$$

where μ and σ are parameters satisfying $-\infty < \mu < \infty$ and $0 < \sigma < \infty$. Briefly speaking, R is $N(\mu, \sigma^2)$ in which μ and σ are the mean and standard deviation of R.

If a random variable Z is $N(0,1)$ with mean 0 and standard deviation 1, then Z has a standard normal distribution. The p.d.f of Z is

$$f(z) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2}\right], \qquad -\infty < z < \infty. \tag{2.13}$$

Moreover, the cumulative distribution function (CDF) of Z is

$$\Phi(z) = \Pr\left[Z \le z\right] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{u^2}{2}\right] du. \tag{2.14}$$

Notice that values of $\Phi(z)$ for $z \ge 0$ are given in the normal probability table. Since the standard normal p.d.f. is symmetric about its mean, it is true that $\Phi(-z) = 1 - \Phi(z)$ for all real z.

Under the normality assumption, deriving from (2.9) shows that

$$R^* = \mu + \Phi^{-1}(c)\sigma, \tag{2.15}$$

where $\Phi^{-1}(c)$ is the inverse standard normal cumulative distribution function and can be expressed as

$$\Phi^{-1}(c) = \sqrt{2}\, \mathrm{erf}^{-1}(2c - 1), \qquad c \in (0\,,1)\,. \tag{2.16}$$

Here, $\mathrm{erf}^{-1}(s)$ denotes the inverse function of the error function

$$\mathrm{erf}(s) = \frac{2}{\sqrt{\pi}} \int_{0}^{s} e^{-t^2} dt.$$

It follows from (2.10) and (2.15) that

$$\mathrm{VaR(absolute)} = -\mu W - \Phi^{-1}(c)\ \sigma W. \tag{2.17}$$

Similarly, substituting (2.15) into (2.11) yields

$$\text{VaR(relative)} = -\,\Phi^{-1}(c)\ \sigma W. \qquad (2.18)$$

As observing from some evidence that not all underlying returns are normally distributed, it is plausible to make a non-normality assumption about the portfolio return. It is observable that some distributions of the portfolio return have fatter tails than a normal one. One of the most popular distributions that have fatter tails than a normal one is the Student's T distribution.

Theorem 5.7-1 of Hogg and Tanis (1993, p. 299) states that:

If Z is a random variable that is $N(0,1)$, if U is a random variable that is $\chi^2(v)$, and if Z and U are independent, then

$$T = \frac{Z}{\sqrt{U/v}}$$

has a T distribution with v degrees of freedom. Its p.d.f. is

$$g(t) = \frac{\Gamma\left[(v+1)/2\right]}{\sqrt{\pi v}\,\Gamma\left(v/2\right)\left(1+t^2/v\right)^{(v+1)/2}}, \quad -\infty < t < \infty. \qquad (2.19)$$

Note that the distribution of T is completely determined by the number v. The graph of the p.d.f. of T is symmetric about the vertical axis $t = 0$. Moreover, the graph of the p.d.f. of T is similar to the graph of the p.d.f. of a standard normal distribution but has heavier tails. It implies that there is more extreme probability in the T distribution than in the standard normal one. Because of the symmetry of the T distribution, it can be shown that the mean of T is

$$\mu = E(T) = 0, \qquad \text{for } v \ge 2,$$

and the variance of T is

$$\sigma^2 = \text{Var}(T) = E(T^2) = \frac{v}{v-2}, \qquad \text{for } v \ge 3.$$

Some values of $\Pr(T \le t)$ can be found in the Student's T distribution table for $v = 1, 2, \ldots, 30$. Under the assumptions that the portfolio return has a T distribution, the VaR in absolute dollar terms can be expressed as

$$\text{VaR(absolute)} = -\mu W - F^{-1}(c;v)\ \sigma W, \tag{2.20}$$

where $F^{-1}(c;v)$ is the inverse of standard Student's T cumulative distribution function with v degrees of freedom. In a similar manner, we can show that the relative VaR is

$$\text{VaR(relative)} = -\ F^{-1}(c;v)\ \sigma W, \tag{2.21}$$

which does not depend on the mean return μ.

Chapter 3

Incorporating Sharpe Ratio and Value-at-Risk into Asset Allocation

The Sharpe ratio was introduced and proposed to be used as a measure for the performance of mutual funds in Sharpe (1966). The Sharpe rule states that the higher the Sharpe ratio, the better the performance of a mutual fund. It has become more and more popular and is extended for use by investors in decision making. For example, to compare the performances of two portfolios, the Sharpe ratio can be defined as the ratio of the expected return on the corresponding portfolio to the standard deviation of the portfolio returns. In the definition of the Sharpe ratio, standard deviation can be referred to as the risk of the portfolio. Thus the Sharpe ratio captures both risk and return in a single measure for comparison between two portfolios. According to the Sharpe rule, one portfolio is preferred to another if it has a higher Sharpe ratio. A falling of the risk or a rising of the return leads to a rise in the Sharpe ratio.

The generalized Sharpe rule has been discussed thoroughly in Dowd (1998, 1999). Under some assumptions, the Sharpe ratio can be expressed in terms of Value-at-risk (VaR). A formal definition of VaR presented in Dowd (1998) expresses it as the maximum expected loss over a given horizon period at a given level of confidence. VaR is accepted as one of the most popular and useful tools in risk measurement and management. Most risk managers and derivative dealers use VaR to measure risks in both local and global markets. In the field of financial planning, VaR is used to measure risk exposures to clients so as to assist in devising suitable investments and hedging strategies. VaR is also shown in the financial reports of most corporations. So, the expression of the Sharpe ratio in terms of VaR gives us an insight into the problems of risk measurement and management. Dowd (1999) also discussed the uses of the Sharpe rule in making investment decisions, hedging and managing portfolios.

As shown in Dowd (1998, 1999), the generalized Sharpe rule can be used to determine a bound for the expected return on a new asset for assessing whether it is worthwhile purchasing it and putting it into an existing portfolio. The case considers an old portfolio first, then a new asset, called asset A, which is added in to form a new portfolio. The desired new portfolio consists of an amount, denoted by a, invested in asset A and an amount $(1-a)$ invested in the old portfolio. It is assumed that the overall portfolio value does not change. By applying the Sharpe rule, the new portfolio is preferable if the expected return on asset A satisfies the following inequality presented in Dowd (1999),

$$R_A \geq R_{old} + (\sigma_{new}/\sigma_{old} - 1)R_{old}/a \tag{3.1}$$

where R_A and R_{old} are the expected returns on asset A and the old portfolio, and σ_{new} and σ_{old} are the standard deviations of returns on the new and old portfolios. The observation is that it is not worthwhile purchasing the new asset unless its expected return is not less than the value on the right-hand side of (3.1). By assuming normality of the distribution of returns on portfolios, Dowd (1999) showed that (3.1) could be written in an equivalent VaR form as

$$R_A \geq R_{old} + \left(\frac{\text{VAR}_{new}}{\text{VAR}_{old}} - 1\right) R_{old} \Big/ a$$

or

$$R_A \geq R_{old}\left[\left(\frac{\text{VAR}_{new}}{\text{VAR}_{old}}-1\right)\Big/a+1\right]. \tag{3.2}$$

Criterion (3.2) is used to justify the worthiness of adding the new asset into the old portfolio. It implies that the new asset is worthwhile adding to an old portfolio if its expected return is greater than the expected return on the old portfolio multiplied by the sum of the elasticity of the VaR and 1. Notice that Dowd (1998, 1999) considered the case of adding only one new asset into the old portfolio to obtain a new portfolio. From another point of view, adding a favorable new asset would improve the performance of an existing portfolio. In other words, portfolio improvement could be done by adding a new asset to an existing portfolio.

In a broader view, Dowd's approach is extendable to consider adding more than one new asset. In this chapter, we propose to add a number of new assets to an existing portfolio in order to form a new portfolio with better performance. Sharpe ratio also acts as a performance measure in our approach. One objective of this chapter is to derive some criteria to judge the worthiness of adding a number of new assets to an old portfolio for portfolio improvement. We assume that the overall portfolio value remains the same before and after adding some new assets. Hence, in the resulting new portfolio, a certain amount of portfolio value is invested in some new assets and the remaining amount is invested in the old portfolio. To avoid solving another portfolio selection problem, it is suggested to add new assets with higher Sharpe ratios first.

Before adding a specific asset to a portfolio, one should be aware of a certain amount of accompanying risk. Due to the fact that prices of some kinds of assets (e.g., stocks) are uncertain, one may bear a certain amount of risk when purchasing a stock. Though the stock may gain a profit, there is always a concern about the risk carried by the stock. Risk reduction is one of the goals for all investment managers. They usually prefer to manage the risks by diversification. Diversification is one of the risk reduction methods. The exposure to risk is reduced by investing in a number of stocks that are fundamentally different from one another. With this concern, our approach to adding a number of new assets to an existing portfolio should benefit from the effect on diversification. Nevertheless, it will be shown in the following section that diversification is carried out in our approach.

As some criteria have been established to check whether it is worthwhile using a portion of the portfolio amount to invest in some new assets, it is possible that different numbers of assets are worthwhile adding to an old portfolio. Of course, if the number of assets to be invested is not specified, then one can add as many new assets as possible in order to reduce portfolio risk. However, adding too many assets to a portfolio is not practical. Moreover, transaction cost is another concern. As more assets are added to the portfolio, the transaction cost will increase as well. We intend to show that achieving the goal of portfolio improvement can be done by adding a small number of new assets.

Regarding the Sharpe ratio as a performance measure of portfolios, investors always prefer a portfolio with the highest Sharpe ratio. The other objective of this chapter is to determine the 'optimal' number of new assets to be added that maximizes the Sharpe ratio of the new portfolio. It is shown that the Sharpe ratio can be expressed in terms of the number of new assets. By applying the mathematical software, Mathematica, we simplify the expression of the Sharpe ratio and hence determine the 'optimal' number of new assets.

This chapter is organized as follows. The next section looks at applying the Sharpe rule to derive some criteria, which can be used to judge whether it is worthwhile investing in some new assets. It then discusses the justification for considering the general case (i.e., investing in n new assets), and compares it with the specific case proposed by Dowd (1998, 1999). The main advantage of our case is that diversification is carried out in our approach. Subsection 3.1.1 presents a numerical example in the Hong Kong stock market to illustrate the applications of the Sharpe rule. It discusses the numerical results and draws some conclusions. Section 3.2 concentrates on the derivation of the 'optimal' number of new assets to be added in the old portfolio with regard to two specific cases: the sequences of the means and the standard derivations of the portfolios are in arithmetic progression or geometric progression. Some experimental results show that only a few new assets are selected to be added in the old portfolio to obtain the 'optimal' new portfolio.

3.1 Methodology and Derivation

Consider an old portfolio whose expected return and standard deviation are denoted by R_{old} and σ_{old}. Let n be the number of new assets added to the old portfolio, and a_i be the weight of asset i in the new portfolio for $i = 1, 2, \ldots, n$. Assume that the overall portfolio value, W, does not change after adding n new assets to the old portfolio. This implies that a portion of the investable amount in the old portfolio is transferred and invested in the additional assets. Thus, the weight of the old portfolio in the new portfolio is $\left[1-\left(a_1+a_2+\cdots+a_n\right)\right]$, or $\left[1-\sum_{i=1}^{n} a_i\right]$. The expected return on the new portfolio, R_{new}, can be expressed as

$$R_{new} = \left(1 - \sum_{i=1}^{n} a_i\right) R_{old} + \sum_{i=1}^{n} a_i R_i \,, \tag{3.3}$$

where R_i is the expected return on asset i for $i = 1, 2, \ldots, n$. Also, an expression for the standard deviation of the new portfolio can be obtained as

$$\sigma_{new} = \left[\left(1 - \sum_{i=1}^{n} a_i\right)^2 \sigma_{old}^2 + \sum_{i=1}^{n} a_i^2 \sigma_i^2 + 2\sum_{i=1}^{n} a_i \left(1 - \sum_{i=1}^{n} a_i\right) \rho_{i,old} \sigma_i \sigma_{old} + 2 \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_i a_j \rho_{i,j} \sigma_i \sigma_j \right]^{1/2} \tag{3.4}$$

where σ_i is the standard deviation of new asset *i*, $\rho_{i,old}$ is the correlation coefficient between the returns on new asset i and the old portfolio, and $\rho_{i,j}$ is the correlation coefficient between the returns on new assets *i* and *j*, for $i, j = 1, 2, \ldots n;\ i \neq j.$

According to the Sharpe rule, if a new portfolio is preferred to the old one, then the Sharpe ratio of the new portfolio should be greater than or equal to the Sharpe ratio of the old portfolio. That is, the new portfolio is preferred if it satisfies

$$\frac{R_{new}}{\sigma_{new}} \geq \frac{R_{old}}{\sigma_{old}}$$

or

$$R_{new} \geq R_{old} \left(\frac{\sigma_{new}}{\sigma_{old}} \right).$$

Replace R_{new} by the expression on the right-hand side of (3.3) to obtain

$$\left(1 - \sum_{i=1}^{n} a_i\right) R_{old} + \sum_{i=1}^{n} a_i R_i \geq R_{old} \left(\frac{\sigma_{new}}{\sigma_{old}} \right), \tag{3.5}$$

which can be rearranged to

$$\sum_{i=1}^{n} a_i R_i \geq R_{old}\left[\left(\frac{\sigma_{new}}{\sigma_{old}}\right)-1+\sum_{i=1}^{n} a_i\right] \tag{3.6}$$

or

$$\sum_{i=1}^{n} a_i R_i \geq R_{old}\left[\left(\frac{\sigma_{new}-\sigma_{old}}{\sigma_{old}}\right)+\sum_{i=1}^{n} a_i\right]. \tag{3.7}$$

Under the assumption of $0<\sum_{i=1}^{n} a_i<1$, dividing both sides of (3.7) by $\sum_{i=1}^{n} a_i$ yields

$$\frac{1}{\sum_{i=1}^{n} a_i}\sum_{i=1}^{n} a_i R_i \geq R_{old}\left[\left(\frac{\sigma_{new}-\sigma_{old}}{\sigma_{old}}\right)\Big/\sum_{i=1}^{n} a_i+1\right]. \tag{3.8}$$

The expression on the left-hand side of (3.8) is the weighted average of the returns on n new assets. Notice that the term, $\left(\frac{\sigma_{new}-\sigma_{old}}{\sigma_{old}}\right)\Big/\sum_{i=1}^{n} a_i$, on the right-hand side of (3.8) is the elasticity of the standard deviation with respect to $\sum_{i=1}^{n} a_i$. Inequality (3.8) indicates that it is worthwhile acquiring n new assets if their weighted average of returns is not less than the value of the expression on the right-hand side of (3.8). It can also be observed from (3.8) that the greater the standard deviation of the new portfolio, the greater the required weighted average of the returns on n assets, and so the greater the expected return on the new portfolio. This bears out the general belief in investment science. A higher risk needs to be borne in order to acquire a higher return in an investment.

Recalling from Section 2.3, under the normality assumption about the probability density function of a portfolio return, the VaR relative to the mean in (2.18) is given as

$$\text{VaR} = -\alpha\sigma W,$$

where $\alpha = \Phi^{-1}(c)$ is the value reflecting the specific confidence level c (e.g. $\alpha = -1.645$ for a confidence level $c = 95\%$), σ is the standard deviation of the portfolio return, and W is the portfolio value. The above definition of VaR can be used to derive expressions of VaRs for the old and new portfolios, denoted by VaR_{old} and VaR_{new}, respectively. As the overall portfolio value is supposed to be fixed during the acquisition of a new portfolio, as a combination of the old portfolio and n new assets, the ratio of VaR_{new} to VaR_{old} in the same confidence interval is equivalent to the ratio of σ_{new} to σ_{old}:

$$\frac{\text{VaR}_{new}}{\text{VaR}_{old}} = \frac{\sigma_{new}}{\sigma_{old}}.$$

The incremental VaR, denoted by IVaR, is defined as the difference between VaR_{old} and VaR_{new} (i.e., $\text{IVaR} = \text{VaR}_{new} - \text{VaR}_{old}$). Thus, the expression on the right-hand side of (3.8) can be presented in terms of VaR as

$$\frac{1}{\sum_{i=1}^{n} a_i} \sum_{i=1}^{n} a_i R_i \geq R_{old} \left[\left(\frac{\text{VaR}_{new} - \text{VaR}_{old}}{\text{VaR}_{old}} \right) \Big/ \sum_{i=1}^{n} a_i + 1 \right]$$

or

$$\frac{1}{\sum_{i=1}^{n} a_i} \sum_{i=1}^{n} a_i R_i \geq R_{old} \left[\left(\frac{\text{IVaR}}{\text{VaR}_{old}} \right) \Big/ \sum_{i=1}^{n} a_i + 1 \right]. \tag{3.9}$$

The criterion depicted in (3.9) shows that as IVaR increases, a greater weighted average of the returns on n new assets is required. The term $\left(\text{IVaR}/\text{VaR}_{old}\right) \big/ \sum_{i=1}^{n} a_i$ on the right-hand side of (3.9) can be considered the elasticity of the VaR with respect to $\sum_{i=1}^{n} a_i$. The implication that can be made from (3.9) is: the greater the elasticity, the greater the risk in the new portfolio, the greater the required weighted average of returns on n new assets, and the greater the returns on the new portfolio. Details of the VaR elasticity can be found in Dowd (2000), where examples were used to illustrate the relationship between the required returns and VaR elasticities.

The advantage of inequality (3.9) will emerge when comparing it with (3.2) derived in Dowd (1999). Some assumptions are made to compare these two inequalities. The old portfolios in this chapter and in Dowd (1999) are assumed to be the same (i.e., they have the same portfolio value W, expected return R_{old}, and standard deviation σ_{old}). The amount invested in a single new asset in Dowd's case is the same as the total amount invested in n new assets (i.e., $a = \sum_{i=1}^{n} a_i$). For the ease of comparison, identical symbols are used for common terms. Inequality (3.2) can be rewritten as

$$R_A \geq R_{old} \left[\left(\frac{\mathrm{IVaR}_D}{\mathrm{VaR}_{old}} \right) \Big/ a + 1 \right], \tag{3.10}$$

where IVaR_D is the incremental VaR in Dowd's case. Expressions on the right-hand side of (3.9) and (3.10) are lower bounds of the returns on new assets. The comparison between these two lower bounds can be replaced by comparing the values of IVaR with IVaR_D. Recalling the definition of VaR, the IVaR in (3.9) can be expressed as

$$\begin{aligned} \mathrm{IVaR} &= \mathrm{VaR}_{new} - \mathrm{VaR}_{old} \\ &= -\alpha W \left(\sigma_{new} - \sigma_{old} \right). \end{aligned} \tag{3.11}$$

To distinguish the standard deviation of the new portfolio in Dowd's case from σ_{new} in this chapter, σ_{new}^{D} is used to denote the standard deviation in Dowd's case. Thus, an expression for IVaR_D is

$$\text{IVaR}_D = -\alpha W\left(\sigma_{new}^D - \sigma_{old}\right). \tag{3.12}$$

Following from (3.11) and (3.12), comparing σ_{new} with σ_{new}^D becomes more straightforward than comparing IVaR with IVaR_D. According to Dowd (1999), σ_{new}^D has the expression

$$\sigma_{new}^D = \left[(1-a)^2 \sigma_{old}^2 + a^2\sigma_A^2 + 2a(1-a)\rho_{A,old}\,\sigma_A\,\sigma_{old}\right]^{1/2}, \tag{3.13}$$

where σ_A is the standard deviation of the returns on asset A, and $\rho_{A,old}$ is the correlation coefficient between the returns on asset A and the old portfolio. The expression for σ_{new} in (3.4) can be rewritten as

$$\sigma_{new} = \left[(1-a)^2 \sigma_{old}^2 + \sum_{i=1}^{n} a_i^2\sigma_i^2 + 2(1-a)\sum_{i=1}^{n} a_i\rho_{i,old}\sigma_i\sigma_{old} + 2\sum_{\substack{i,j=1 \\ i\neq j}}^{n} a_i a_j \rho_{i,j}\sigma_i\sigma_j\right]^{1/2}. \tag{3.14}$$

Common term $(1-a)^2\sigma_{old}^2$ is observed in (3.13) and (3.14). Comparison between (3.13) and (3.14) cannot be conducted. The first terms on the right-hand side are the same, but the remaining terms are different. It is helpful to consider a special case where all the correlation coefficients are equal to zero (i.e., $\rho_{A,old} = 0$, $\rho_{i,old} = 0$, and $\rho_{i,j}$, for $i, j = 1, 2, \ldots, n$, $i \neq j$). In other words, the new assets are uncorrelated with each other as well as uncorrelated with the old portfolio. Hence, (3.13) and (3.14) become

$$\sigma_{new}^D = \left[(1-a)^2 \sigma_{old}^2 + a^2\sigma_A^2\right]^{1/2} \tag{3.15}$$

and

$$\sigma_{new} = \left[(1-a)^2 \sigma_{old}^2 + \sum_{i=1}^{n} a_i^2\sigma_i^2\right]^{1/2}. \tag{3.16}$$

The only difference between the expressions on the right-hand sides of (3.15) and (3.16) is the second term.

For simplicity, the standard deviations are assumed to be the same for all new assets, say $\sigma_i^2 = \sigma^2$. Suppose n new assets are equally weighted in the new portfolio, that is, a_i is the same for all i. Since $a = \sum_{i=1}^{n} a_i$ then $a_i = a/n$ for all i, and therefore

$$\sum_{i=1}^{n} a_i^2 \sigma_i^2 = \sum_{i=1}^{n} \left(\frac{a}{n}\right)^2 \sigma^2 = \frac{1}{n} a^2 \sigma^2. \tag{3.17}$$

Hence as $n \to \infty$ then $\sum_{i=1}^{n} a_i^2 \sigma_i^2 \to 0.$ This shows the effect of diversification.

In the well-diversified case, σ_{new} is less than σ_{new}^D in Dowd's case. This implies that IVaR in (3.11) is less than IVaR_D in (3.12), and so the value on the right-hand side of (3.9) is less than that on the right-hand side of (3.10). In the general case considered in this chapter, a smaller lower bound for the acquiring expected return on the new assets can be obtained by diversification.

In the case of $\sum_{i=1}^{n} a_i^2 \sigma_i^2 \to 0,$ it can be shown further that

$$\sigma_{new} \to (1-a)\sigma_{old}$$

and

$$\text{IVaR} \to \alpha W\left(a\sigma_{old}\right). \tag{3.18}$$

Thus, $\text{IVaR} < 0$ is observed as $\alpha < 0$, which implies that the overall risk of the portfolio is reduced. This objective is similar to that of the hedging decision. The hedging decision is a popular but complicated problem in the field of risk management. Risk managers always try to identify risk exposure and seek a way to reduce risk in investments. One of their goals is to make a suitable hedging decision. Diversification and hedging have the same intention; that is, to reduce risk of a portfolio. However, they are different approaches. Diversification assumes that the new assets are uncorrelated with each other and the portfolio. When addressing the hedging decision, some assets which are negative correlated with the portfolio are sought. Litterman (1996) explored identifying and reducing risk further.

3.1.1 Numerical Examples

In this subsection, a numerical example is presented for illustration. Concentration is given to the stock market of Hong Kong. Since the Hang Seng Index (HSI) is the main index in the Hong Kong stock market, the 33 constituent stocks[1] in the HSI are considered. The constituent stocks can be classified into four market sectors: Commerce and Industry, Financials, Properties, and Utilities. Here, the daily closing prices of stocks from August 1996 to July 1997 are used for analysis. Based on the collected data, the daily expected return and standard deviation can be calculated for each stock. Hence, the Sharpe ratios of the stocks are obtained.

Consider an existing portfolio consisting of 10 stocks in the Properties sector. The stocks are selected by the Principal Component Analysis (PCA) which is a useful technique for data analysis and helps to discover the patterns in data of high dimension in order to reduce the number of dimensions of the data set. Components are extracted for the data set by PCA. The data set's variation can largely be explained by several principal components. The underlying

[1] Starting from July 1964 (the Hang Seng Index was first published) till June 2007, the number of constituent stocks in HSI varies from 30 to 39. The list of constituent stocks is changed from time to time. The 33 constituent stocks considered in this thesis are from the constituent list after change in August 1996.

dominator(s) can be identified in the principal components. Some introductory information of PCA can be found in Johnson and Wichern (1992) and Smith (2002). PCA can be carried out by using the statistical software SPSS. Here, ignoring the presentation of analytical steps, observations on the analysis are discussed. The result shows that the data set of the returns on 33 stocks can be represented by seven principal components. The first principal component is dominated by 10 stocks in the properties sector. This conforms to a great event of the downturn of the property market during the period 1996/97.

For the ease of manipulation, the 10 stocks are assumed to be equally weighted in the existing (old) portfolio. Let the portfolio value $W = 1$. The confidence interval is set to be 95% and the value reflected the confidence interval is $\alpha = -1.645$. The daily expected return, standard deviation, and VaR of the old portfolio are calculated as follows:

Daily Expected Return	**Standard Deviation**	**VaR**
0.0009772	0.0056477	0.0092904

Now, the remaining 23 constituent stocks in the HSI are considered being added to the old portfolio. The selection criterion is the Sharpe ratio. In Table 3.1, the 23 stocks are ranked by the Sharpe ratio in descending order. PCCW is observed to have the greatest expected return but with the highest standard deviation (risk). So, it is not the one with the highest Sharpe ratio. HSBC Holdings has the highest Sharpe ratio, as shown in Table 3.1. According to the Sharpe rule, the first stock to be added to the old portfolio is HSBC Holdings.

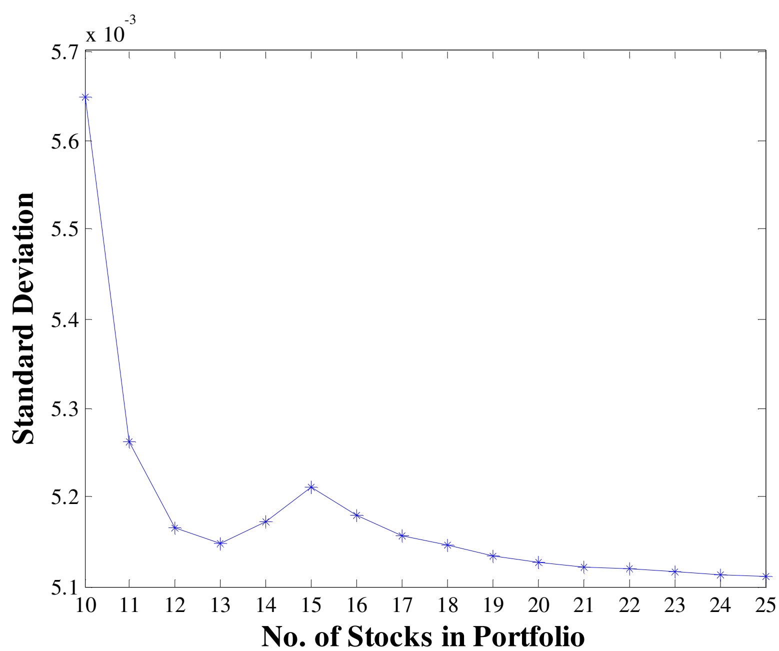
Table 3.1 The 23 Stocks Sorted by Sharpe Ratio in a Descending Order

No.	Stock Name	Daily Expected Return	Standard Deviation	Sharpe Ratio (%)
1	HSBC HOLDINGS	0.003250	0.013593	23.912
2	HONG KONG AND CHINA GAS	0.002087	0.012395	16.834
3	HUTCHISON WHAMPOA	0.002113	0.016216	13.031
4	GUANGDONG INVESTMENT	0.003849	0.029640	12.987
5	PCCW	0.005095	0.042908	11.875
6	CITIC PACIFIC	0.001871	0.016139	11.590
7	BANK OF EAST ASIA	0.001434	0.013001	11.030
8	NEW WORLD DEV.	0.002042	0.019360	10.545
9	CLP HOLDINGS	0.001414	0.013636	10.366
10	HONG KONG ELECTRIC	0.001437	0.015007	9.574
11	HANG SENG BANK	0.001720	0.018019	9.544
12	SCMP GROUP	0.002117	0.024608	8.604
13	JOHNSON ELECTRIC HDG.	0.001521	0.019491	7.803
14	CATHAY PACIFIC AIRWAYS	0.000847	0.016554	5.119
15	WHARF HOLDINGS	0.000994	0.019536	5.090
16	SWIRE PACIFIC 'A'	0.000548	0.016159	3.393
17	HOPEWELL HOLDINGS	0.000712	0.023106	3.082
18	TELEVISION BROADCASTS	0.000414	0.015956	2.595
19	SHUN TAK HOLDINGS	0.000298	0.018373	1.619
20	ORIENTAL PRESS GROUP	-0.000700	0.028379	-2.466
21	SHANGRI - LA ASIA	-0.000447	0.015355	-2.913
22	FIRST PACIFIC	-0.000767	0.019908	-3.851
23	HONGKONG & SHAI.HTLS.	-0.000689	0.016019	-4.302

Here, the effect of the number of stocks on the standard deviation of the portfolio is studied. The cases to be considered are n = 1, 2, …, 15. In these cases, the corresponding number of stocks in the new portfolios are 11, 12, … , 25. Assume that $\rho_{i,old} = 0$ and $\rho_{i,j} = 0$, for $i, j = 1, 2, \ldots, n$, $i \neq j$. Suppose n new stocks are equally weighted and $\sum_{i=1}^{n} a_i = 10\%$. Under these assumptions, the values of the standard deviation of the new portfolios can be calculated by (3.16).

Figure 3.1 shows the standard deviations of new portfolios vary against the number of stocks in the portfolios. The standard deviation of the portfolio drops dramatically first, then increases a bit before finally leveling out. Even though the standard deviation increases when adding four or five stocks to the old portfolio, the effect of the diversification can be seen when the standard deviation decreases eventually. All the values of the standard deviation of the new portfolios are less than that of the old portfolio.

Figure 3.1 Standard Deviations of New Portfolios against Number of Stocks



After calculating the values of standard deviations for the new portfolios, the values of IVaR can also be calculated by applying (3.11). Table 3.2 presents the values of standard deviation and IVaR for some portfolios. The standard deviation of the portfolio is getting increasingly smaller as many more stocks are added to the old portfolio. For the four specific new portfolios in Table 3.2, the values of IVaR are less than zero which implies that the overall risk of the portfolio is reduced.

Table 3.2 Standard Deviation and IVaR

Portfolios	Standard Deviation	IVaR
Old (10 stocks)	0.0056477	
New (11 stocks)	0.0052615	-0.0006352
New (15 stocks)	0.0052119	-0.0007168
New (20 stocks)	0.0051273	-0.0008560
New (25 stocks)	0.0051112	-0.0008824

Representatively, the worthiness of adding new stocks to the old portfolio is discussed for the cases of n = 1, 5, 10 and 15. The second column of Table 3.3 presents the values of the weighted average of returns on n new stocks, which are the values on the left-hand side of (3.9). The third column of Table 3.3 shows the values on the right-hand side of (3.9). Inequality (3.9) expresses that if the value on the left-hand side is not less than that on the right, then it is worthwhile adding n new assets to the old portfolio. As shown in Table 3.3, for the four specific cases, the values in Column 2 are greater than those in Column 3. Thus, it is worthwhile adding new stocks to the old portfolio in these four cases.

Table 3.3 Comparison of Values on both Sides of (3.9)

New Stocks Added	Value on the Left-hand Side of (3.9)	Value on the Right-hand Side of (3.9)
1 stock	0.0032504	0.0003091
5 stocks	0.0032790	0.0002233
10 stocks	0.0024591	0.0000769
15 stocks	0.0002119	0.0000491

Similarly, by comparing the values on the left- and right-hand sides of (3.9), it can be shown that new assets are worthwhile adding to the old portfolio for $n = 1, 2, \ldots, 15$. However, some questions must be addressed here. Due to diversification, the standard deviation (risk) gets lower when more stocks are added. It is reasonable to consider whether investing in as many stocks as possible is justified. Of course it is possible to invest in many stocks if it satisfies inequality (3.9) in theory. Therefore, one can consider whether the portfolio with more stocks is better. Though the risk is reduced by adding noticeably more stocks, the return may be reduced as well. Thus, consideration must also be given to the return on the portfolio and using the Sharpe ratio to determine a better portfolio. A reduction in return may be due to an increase in transaction cost which is another important factor, but it is not considered here for simplicity. It is shown in our example that though the transaction cost is neglected, the daily expected return of the new portfolio is reduced as more stocks are added, see Table 3.4. Moreover, it can be observed that the Sharpe ratio of the new portfolio is reduced as more than 5 stocks are added. As both the daily expected return and standard deviation are reduced, the reduction in Sharpe ratio is consequent on that the rate of change of the expected return is much faster than that of standard deviation.

Table 3.4 Return, Standard Deviation and Sharpe Ratio

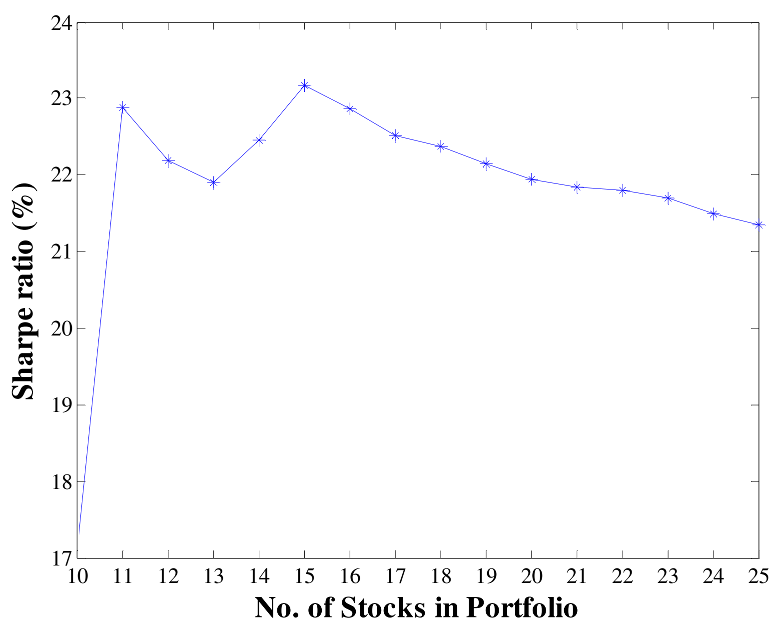
Portfolios	Daily Expected Return	Standard Deviation	Sharpe Ratio (%)
Old (10 stocks)	0.0009772	0.0056477	17.3029
New (11 stocks)	0.0012045	0.0052615	22.8932
New (12 stocks)	0.0011463	0.0051655	22.1924
New (13 stocks)	0.0011278	0.0051482	21.9072
New (14 stocks)	0.0011620	0.0051731	22.4620
New (15 stocks)	**0.0012074**	0.0052119	**23.1658**
New (16 stocks)	0.0011839	0.0051798	22.8562
New (17 stocks)	0.0011609	0.0051576	22.5085
New (18 stocks)	0.0011513	0.0051459	22.3721
New (19 stocks)	0.0011368	0.0051350	22.1376
New (20 stocks)	0.0011254	0.0051273	21.9491
New (21 stocks)	0.0011187	0.0051223	21.8396
New (22 stocks)	0.0011164	0.0051201	21.8041
New (23 stocks)	0.0011099	0.0051168	21.6906
New (24 stocks)	0.0010995	0.0051135	21.5011
New (25 stocks)	0.0010914	**0.0051112**	21.3535

Table 3.4 presents the values of daily expected return, standard d eviation, and Sharpe ratio of the old and new portfolios. Investors generally prefer the portfolio with higher return and lower risk. As shown in Table 3.4, the highest daily expected return, Sharpe ratio and the lowest standard deviation are bold for reference. Among these portfolios, the portfolio with 25 stocks is the one with the largest number of stocks and the lowest standard deviation. However, its

daily expected return is comparatively low. The most efficient portfolio is not the one with the largest number of stocks.

In Table 3.4, the most efficient portfolio is the one with 15 stocks since it has the highest Sharpe ratio and highest daily expected return. This can also be observed from Figure 3.2. By the Sharpe rule, the portfolio with the highest Sharpe ratio is the best one. This implies that one can earn more from the best portfolio than from the others for the same level of risk. Thus, one can earn more from the portfolio with 15 stocks than from the others. Though it has a high risk, it is still the best one if its risk is in the range of our risk tolerance.

Figure 3.2 Sharpe Ratios of New Portfolios against Number of Stocks



3.2 Determination of the 'Optimal' Number of New Assets

When the formulae derived in the previous section are used for decision making, investors may face the following problem. Suppose many new assets are available and only a small number of assets are demanded to be added to an old portfolio. Investors may encounter difficulties in choosing among these new assets. Under the assumptions that all the new assets are uncorrelated with each other and with the old portfolio, it is possible to make comparison among assets by applying some rules. As the Sharpe rule is fair to compare the performances of portfolios or assets, the rule can be applied to choose the assets with better performances. Accordingly, it is reasonable to sort the new assets by the Sharpe ratio in descending order. The assets with higher Sharpe ratios are chosen first.

If the number of assets to be added is not specified, another problem regarding the optimal number of assets added to the old portfolio arises. Before answering this question, some assumptions are made for the expected returns and standard deviations of the assets. There are n available new assets which are already sorted by the Sharpe ratio in descending order:

$$\frac{R_1}{\sigma_1} \geq \frac{R_2}{\sigma_2} \geq \cdots \geq \frac{R_{n-1}}{\sigma_{n-1}} \geq \frac{R_n}{\sigma_n} . \tag{3.19}$$

The first m new assets are chosen to be added to the old portfolio, for m = 1, 2, …, n. Thus, it is possible to obtain n new portfolios. The m new assets are assumed to be uncorrelated with each other and with the old portfolio, and equally weighted in the new portfolio. Under these assumptions, the expressions of expected return and standard deviation for a new portfolio in (3.3) and (3.4) can be simplified to

$$R_{new} = (1-a) R_{old} + \frac{a}{m} \sum_{i=1}^{m} R_i \tag{3.20}$$

and

$$\sigma_{new} = \left[(1-a)^2 \sigma_{old}^2 + \left(\frac{a}{m} \right)^2 \sum_{i=1}^{m} \sigma_i^2 \right]^{1/2} . \tag{3.21}$$

The following two subsections will consider two specific cases for the ease of analysis. The two specific cases are: arithmetic series and geometric series. These deal with the sequences of the expected returns and standard deviations. Though some closed-form formulae cannot be obtained to determine the optimal value of *m*, some interesting results can be observed from the simulated examples. The simulation results show that a new portfolio with the highest Sharpe ratio can be obtained by adding a few new assets under some assumptions.

3.2.1 Arithmetic Series Case

The first case to be considered is that both the expected returns and the standard deviations of the assets are arithmetic series, respectively. That is, $\{R_1, R_2, \dots, R_n\}$ and $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$ are assumed to be arithmetic sequences. Thus, the expected return and standard deviation of asset i can be expressed as

$$R_i = R_1 + (i-1)\gamma \tag{3.22}$$

and

$$\sigma_i = \sigma_1 + (i-1)\beta \tag{3.23}$$

for $i = 1, 2, \dots, n$ and $n \geq 1$, where γ and β are constants. Since standard deviations must be positive, $\beta > 0$ is assumed to avoid the existence of negative standard deviation. Following from (3.19), (3.22) and (3.23),

$$\frac{R_{n-1}}{\sigma_{n-1}} = \frac{R_n - \gamma}{\sigma_n - \beta} \geq \frac{R_n}{\sigma_n} \tag{3.24}$$

which can be simplified to

$$\frac{\gamma}{\beta} \leq \frac{R_n}{\sigma_n}. \tag{3.25}$$

This shows that the ratio of γ to β is bounded above by the Sharpe ratios for all assets.

It follows from (3.22) and (3.23) that the sums on the right-hand side of (3.20) and (3.21) can be simplified by applying the formula for an arithmetic series. Consequently, (3.20) and (3.21) can be reduced to

$$R_{new} = (1-a)\, R_{old} + a\, R_1 - \frac{a\gamma}{2} + \frac{a\gamma}{2} m \tag{3.26}$$

and

$$\sigma_{new} = \left[(1-a)^2 \sigma_{old}^2 + a^2 \left(\sigma_1^2 - \beta\sigma_1 + \frac{\beta^2}{6} \right) \frac{1}{m} + a^2 \left(\beta\sigma_1 - \frac{\beta^2}{2} \right) + \frac{a^2\beta^2}{3} m \right]^{1/2}, \tag{3.27}$$

where $1 \leq m \leq n$.

If the values of all the variables are assumed to be given except *m*, it is observed from (3.26) and (3.27) that R_{new} and σ_{new} can be considered functions of *m*. Hence, the Sharpe ratio for the new portfolio

$$\mathrm{SR}_{new} = \frac{R_{new}}{\sigma_{new}}$$

can be expressed by (3.25) and (3.26) in terms of the variable *m*. The goal is to find the optimal value of m that maximizes SR_{new}.

Since the Sharpe ratio can be considered a function of *m*, the optimal value of m can be obtained by differentiating SR_{new} with respect to m and setting the derivative equal to zero. The first derivative of SR_{new} can be derived by applying the quotient rule for derivatives. Setting $\frac{\mathrm{d}}{\mathrm{d}m}\mathrm{SR}_{new}=0$ implies that

$$\frac{\sigma_{new}\frac{\mathrm{d}}{\mathrm{d}m}(R_{new})-R_{new}\frac{\mathrm{d}}{\mathrm{d}m}(\sigma_{new})}{\sigma_{new}^2}=0\,. \tag{3.28}$$

The derivatives of R_{new} and σ_{new} with respect to m can be derived from (3.26) and (3.27) to give

$$\frac{\mathrm{d}}{\mathrm{d}m}(R_{new})=\frac{a\gamma}{2}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}m}(\sigma_{new})=\frac{1}{2}\sigma_{new}^{-1}\left[-a^2\left(\sigma_1^2-\beta\sigma_1+\frac{\beta^2}{6}\right)\frac{1}{m^2}+\frac{a^2\beta^2}{3}\right].$$

After substituting the expressions of R_{new}, σ_{new}, $\frac{\mathrm{d}}{\mathrm{d}m}(R_{new})$, and $\frac{\mathrm{d}}{\mathrm{d}m}(\sigma_{new})$ into the left-hand side of (3.28), a complicated equation is obtained. Nowadays, some mathematical software can assist in simplifying complicated equations. One of the powerful software is Mathematica.

By using Mathematica, we can obtain the expression for the first term in equation (3.28) as

$$\sigma_{new}\frac{\mathrm{d}}{\mathrm{d}m}(R_{new})=\frac{a\gamma}{2}\left[(1-a)^2\sigma_{old}^2+a^2\left(\sigma_1^2-\beta\sigma_1+\frac{\beta^2}{6}\right)\frac{1}{m}+a^2\left(\beta\sigma_1-\frac{\beta^2}{2}\right)+\frac{a^2\beta^2}{3}m\right]^{1/2}. \tag{3.29}$$

Similarly, the second term on the left-hand side of equation (3.28) can be expressed as

$$R_{new}\frac{\mathrm{d}}{\mathrm{d}m}(\sigma_{new})=\frac{a^2}{24m^2}\sigma_{new}^{-1}\left[a(m-1)\gamma+2aR_1+2(1-a)R_{old}\right]\left[(2m^2-1)\beta^2+6\beta\sigma_1-6\sigma_1^2\right]. \tag{3.30}$$

Hence, the expression on the left-hand side of (3.28) is simplified to

$$\frac{\sigma_{new}\frac{\mathrm{d}}{\mathrm{d}m}\left(R_{new}\right)-R_{new}\frac{\mathrm{d}}{\mathrm{d}m}\left(\sigma_{new}\right)}{\sigma_{new}^2}$$

$$=\Bigg[\sqrt{\frac{3}{2}}\,a\Big(-2a^2R_1\left(\left(2m^2-1\right)\beta^2+6\beta\sigma_1^2\right)+2\left(a-1\right)a\,R_{old}\left(\left(2m^2-1\right)\beta^2+6\beta\sigma_1-6\sigma_1^2\right)+$$
$$\gamma\Big(a^2\left(-1-3m-4m^2+2m^3\right)\beta^2+6a^2\left(1-3m+2m^2\right)\beta\sigma_1+6a^2\left(3m-1\right)\sigma_1^2+$$
$$12\left(1-a\right)^2m^2\sigma_{old}^2\Big)\Big)\Bigg]\Big/$$
$$\left[2m^2\left(\frac{a^2\left(1-3m+2m^2\right)\beta^2+6a^2\left(m-1\right)\beta\sigma_1+6\left(a-1\right)^2m\,\sigma_{old}^2}{m}\right)^{3/2}\right]$$

(3.31)

Let

$$f\left(m\right)=\sqrt{\frac{3}{2}}\,a\Big(-2a^2R_1\left(\left(2m^2-1\right)\beta^2+6\beta\sigma_1^2\right)+2\left(a-1\right)a\,R_{old}\left(\left(2m^2-1\right)\beta^2+6\beta\sigma_1-6\sigma_1^2\right)+$$
$$\gamma\Big(a^2\left(-1-3m-4m^2+2m^3\right)\beta^2+6a^2\left(1-3m+2m^2\right)\beta\sigma_1+6a^2\left(3m-1\right)\sigma_1^2+$$
$$12\left(1-a\right)^2m^2\sigma_{old}^2\Big)\Big).$$

(3.32)

Note that $f\left(m\right)$ is the numerator of the expression on the right-hand side of (3.31). Obviously, with given values of a, R_1, R_{old}, σ_1, σ_{old}, β and γ, it is a function of *m*. It follows that equation (3.28) is equivalent to

$$f\left(m\right)=0. \tag{3.33}$$

For the ease of observation, (3.32) can be rewritten as

$$f\left(m\right)=\sqrt{\frac{3}{2}}\,a\Big[-2a^2R_1\left(6\beta\sigma_1^2-\beta^2\right)+2\left(a-1\right)a\,R_{old}\left(-\beta^2+6\beta\sigma_1-6\sigma_1^2\right)+$$
$$\gamma\left(-a^2\beta^2+6a^2\beta\sigma_1-6a^2\sigma_1^2\right)\Big]+\sqrt{\frac{3}{2}}\,a^3\,\gamma\left(-3\beta^2-18\beta\sigma_1+18\sigma_1^2\right)m+$$
$$2\sqrt{6}\,a\left[\beta^2\left(\left(a-1\right)a\,R_{old}-a^2R_1m^2\right)+\gamma\left(-a^2\beta^2+3a^2\beta\sigma_1+3\left(1-a\right)^2\sigma_{old}^2\right)\right]m^2+$$
$$\sqrt{6}\;a^3\beta^2\,\gamma\,m^3.$$

(3.34)

It is obvious that the largest power of m in function $f(m)$ is three. There are at most three roots for the polynomial cubic equation (3.33). Mathematica can help determine the roots, i.e. the values of m, for this polynomial cubic equation. Since the expressions of the roots are rather lengthy, they are not presented here, but in the Appendix to this chapter.

After obtaining the values of the three roots, we can apply the second derivative test to determine whether the function SR_{new} is a maximum or minimum at the point of these three roots. A necessary condition for applying the test is that the given function is twice differentiable. Obviously, the second derivative of the function SR_{new} with respect to m can be obtained by differentiating (3.31) one more time. After some simplification, we get the following expression for the second derivative:

$$\frac{\mathrm{d}^2}{\mathrm{d}m^2}(\mathrm{SR}_{new})$$

$$=\Bigg[\sqrt{\frac{3}{2}}a^2\Big(-4am\gamma\left(\left(2m^2-1\right)\beta^2+6\beta\sigma_1\right)-4\left(a(m-1)\gamma+2aR_1+2(1-a)R_{old}\right)\bullet$$
$$\left(\beta^2-6\beta\sigma_1+6\sigma_1^2\right)+\frac{3a^2\left(a(m-1)\gamma+2aR_1+2(1-a)R_{old}\right)\left(\left(2m^2-1\right)\beta^2+6\beta\sigma_1-6\sigma_1^2\right)^2}{a^2\left(1-3m+2m^2\right)\beta^2+6a^2(m-1)\beta\sigma_1+6(a-1)^2m\sigma_{old}^2}\Big)\Bigg]\Big/$$
$$\left[4m^2\left(\frac{a^2\left(1-3m+2m^2\right)\beta^2+6a^2(m-1)\beta\sigma_1+6(a-1)^2m\sigma_{old}^2}{m}\right)^{3/2}\right] \tag{3.35}$$

By substituting the values of three roots into (3.35), we can determine the optimal value of m that maximizes SR_{new}. In the following, some numerical examples are implemented to show some interesting results.

The expected returns and standard deviations of new assets are assumed in arithmetic sequences which can be constructed for some given values of R_1, σ_1, γ, and β. Here are the given values of some parameters:

a	R_{old}	σ_{old}	R_1	σ_1
10%	0.002177	0.004598	0.00264	0.00466

Hence, the values of Sharpe ratios can be calculated for the old portfolio and the first new asset (i.e., SR_{old} = 0.473467 and SR_1 = 0.566524). Under the assumptions stated in this section, all the given new assets are sorted by the Sharpe ratio in descending order. This indicates that the first new asset is the one with the highest Sharpe ratio.

Now, the values of γ and β for constructing the sequences $\{R_1, R_2, \ldots, R_n\}$ and $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ are required. However, constructing a very long sequence (with very large n) may not be realistic. Here, n is set as 100. Thus, at most 100 new assets from the sequence can be invested and the possible values of m are 1, 2, …, 100. Since the values of γ and β affect the features of the sequences and hence the optimal value of *m*, studying the effect should be significant. The assumption of $\beta > 0$ implies that the standard deviations in the sequence are monotonically increasing. It is possible to consider a suitable value range of γ and β, respectively, in order to get different combinations of these two variables. One acceptable setting is to allow γ range between –0.001 and 0.001 and β range between 0 and 0.002. When the values of these two variables are obtained, it must be ensured that they satisfy inequality (3.25).

Let us consider the case with $\beta = 0.0004$ and $\gamma = -\ 0.0001$ first. After substituting these given values of the parameters into the expression on the right-hand side of (3.34), we can solve equation (3.33) by Mathematica to obtain three roots as $m_1 = 1.59328$, $m_2 = -1.61059$, and $m_3 = -64730.2$. With the values of three roots, we can apply the second derivative test to determine the whether the function has a maximum or minimum at these three point. Since it is assumed that $1 \le m \le n$, m_2 and m_3 are not practical. They are neglected. Consequently, substituting the values of the parameters and the first root, $m_1 = 1.59328$, into the second derivative (3.35) yields

$$\frac{\mathrm{d}^2}{\mathrm{d}m^2}\left(\mathrm{SR}_{new}\right)\Big|_{m=1.59328} = -0.00165505 < 0.$$

Hence, SR_{new} has a local maximum at the point. We conclude that $m = 1.59328$ is the optimal value that maximizes the Sharpe ratio SR_{new}. Since m is the number of new assets to be invested, it should be an integer number. We suppose to set $m^* = 2$, which is the nearest integer to $m = 1.59328$. In case of obtaining an optimal m which is not an integer, no doubt we can check for both two nearest integers to get the optimal integer value of m that maximizes SR_{new}. For example, in our previous case with $m = 1.59328$, we shall check for $m = 1$ and $m = 2$. For $m = 1$, we get $\mathrm{SR}_{new}\big|_{m=1} = 0.53389$; for $m = 2$, the Sharpe ratio of the new portfolio is $\mathrm{SR}_{new}\big|_{m=2} = 0.53421$. Thus, we can conclude that $m^* = 2$ is the optimal value that maximizes SR_{new}.

After analyzing different combinations of γ and β, the following observations are reached and categorized:

- For $\gamma < 0$ and $\beta > 0$, a new portfolio with the highest Sharpe ratio is reached by adding a few new assets. For example, the optimal value

$m = 2$ is obtained for $\gamma = -0.0001$ and $\beta = 0.0004$, see Figure 3.3. As $\gamma < 0$, the expected returns in the sequence are decreasing. And $\beta > 0$ implies that the standard deviations are increasing. Hence, the Sharpe ratios are decreasing, which satisfies condition (3.19).

- For $0 < \gamma < \beta$, but β is much greater than γ, a new portfolio with the highest Sharpe ratio can also be obtained by adding only a small number of new assets. One example is setting $\gamma = 0.00001$ and $\beta = 0.002$, see Figure 3.4. The optimal number of new assets added is 5.
- For $0 < \gamma < \beta$, and β is close to γ, it can be observed that as the number of new assets added is increasing, the Sharpe ratio is increasing as well. Therefore, a new portfolio with the highest Sharpe ratio cannot be reached by adding a small number of new assets. See Figure 3.5.

Though a small optimal value of m cannot be obtained for all the cases, it can be obtained in most of the cases. This interesting result is useful to investors because once it is found that γ and β satisfy the conditions in the first or second cases, a small number of new assets can be added to obtain a new portfolio with better performance.

Figure 3.3 Sharpe Ratios of New Portfolios against Number of Stocks added in Arithmetic Series Case ($\gamma < 0$, $\beta > 0$)

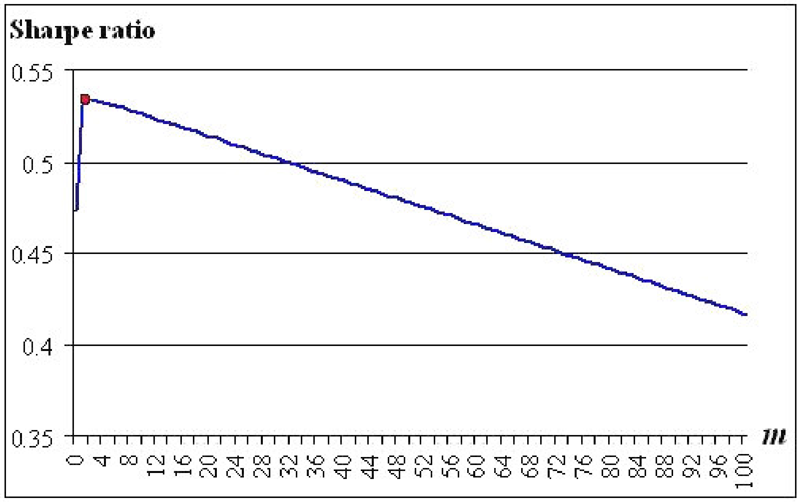


Figure 3.4 Sharpe Ratios of New Portfolios against Number of Stocks added in Arithmetic Series Case ($0 < \gamma << \beta$)

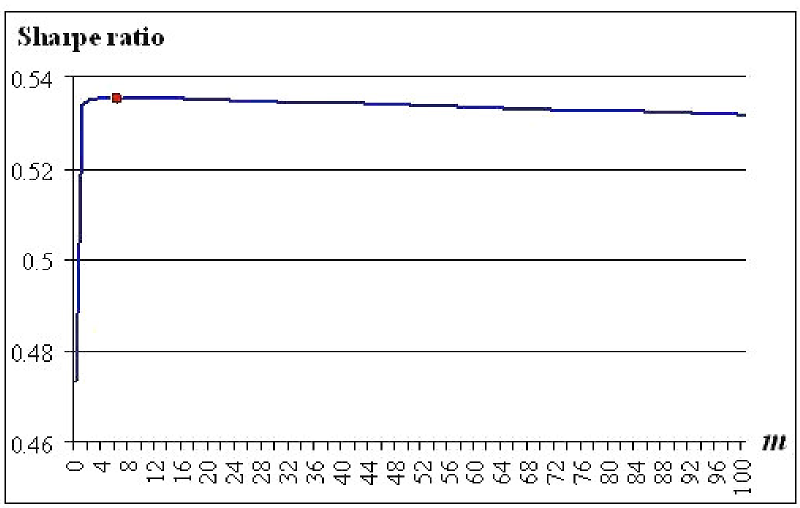
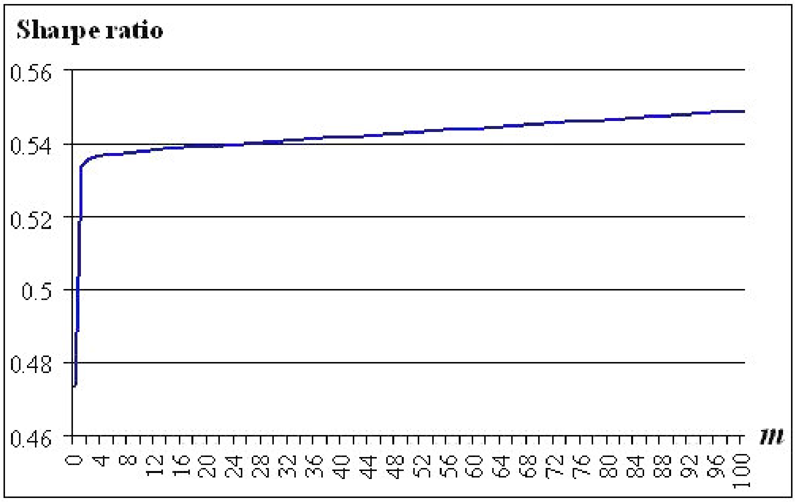


Figure 3.5 Sharpe Ratios of New Portfolios against Number of Stocks added in Arithmetic Series Case ($0 < \gamma < \beta$)



3.2.2 Geometric Series Case

This subsection considers the case that $\{R_1, R_2, \ldots, R_n\}$ and $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ are geometric sequences. Accordingly, for $i = 1, 2, \ldots, n$ and $n \geq 1$, the expected return and standard deviation of asset i are given as

$$R_i = \delta^{i-1} R_1 \tag{3.36}$$

and

$$\sigma_i = \theta^{i-1} \sigma_1, \tag{3.37}$$

where δ and θ are constants. Since a negative return is unfavorable and standard deviations must be positive, both δ and θ are assumed to be positive. Applying (3.19), (3.36), and (3.37) shows that

$$\frac{\delta}{\theta} \le 1, \tag{3.38}$$

in which the ratio of δ to θ is bounded above by one. In other words, the ratio of the returns for two successive assets, δ, is always less than or equal to the ratio of the standard deviations, θ.

The sums on the right-hand sides of (3.20) and (3.21) can be simplified by applying the formula for geometric series. Hence, expressions of the expected return and the standard deviation for the new portfolio are obtained as

$$R_{new} = (1-a)\,R_{old} + \frac{a}{m}\left(\frac{1-\delta^m}{1-\delta}\right)R_1 \tag{3.39}$$

and

$$\sigma_{new} = \left[(1-a)^2\,\sigma_{old}^2 + \left(\frac{a}{m}\right)^2\left(\frac{1-\theta^{2m}}{1-\theta^2}\right)\sigma_1^2\right]^{1/2}. \tag{3.40}$$

Notice that $\delta \ne 1$ and $\theta \ne 1$, otherwise (3.39) and (3.40) become meaningless. For the given values of a, R_{old}, R_1, σ_{old}, σ_1, δ, and θ, the Sharpe ratio of the new portfolio can be expressed by (3.39) and (3.40) as a function of m, i.e.

$$\mathrm{SR}_{new} = \frac{(1-a)\,R_{old} + \dfrac{a}{m}\left(\dfrac{1-\delta^m}{1-\delta}\right)R_1}{\left[(1-a)^2\,\sigma_{old}^2 + \left(\dfrac{a}{m}\right)^2\left(\dfrac{1-\theta^{2m}}{1-\theta^2}\right)\sigma_1^2\right]^{1/2}}. \tag{3.41}$$

The first derivatives of R_{new} and σ_{new} can be derived and substituted into the left-hand side of (3.28) to obtain

$$0 = \Big[a\Big(a\left(1-\theta^{2m} + m\theta^{2m}\ln\theta\right)\left(a(1-\delta^m)R_1 + (1-a)(1-\delta)R_{old}\, m\right)\sigma_1^2 + \left(1-\delta^m + m\delta^m \ln\delta\right)R_1\left(a^2(\theta^{2m}-1)\sigma_1^2 + (1-a)^2 m^2(\theta^2-1)\sigma_{old}^2\right)\Big)\Big] \Big/ \left[m^4(1-\delta)(1-\theta^2)\left((1-a)^2\sigma_{old}^2 + \left(\frac{a}{m}\right)^2\left(\frac{1-\theta^{2m}}{1-\theta^2}\right)\sigma_1^2 \right)^{3/2} \right], \tag{3.42}$$

which is equivalent to

$$0 = a\left(1-\theta^{2m} + m\theta^{2m}\ln\theta\right)\left(a(1-\delta^m)R_1 + (1-a)(1-\delta)R_{old}\, m\right)\sigma_1^2 + \left(1-\delta^m + m\delta^m\ln\delta\right)R_1\left(a^2(\theta^{2m}-1)\sigma_1^2 + (1-a)^2 m^2(\theta^2-1)\sigma_{old}^2\right). \tag{3.43}$$

Since equation (3.42) is even more complicated than that in the arithmetic series case, it cannot be solved directly for the optimal value of *m*. Here, some numerical results are shown for analysis.

Now, $\{R_1, R_2, \ldots, R_n\}$ and $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ are regarded as geometric sequences. For n = 100, m can be any integers between 1 and 100. Suppose with $a = 30\%$ and the values of R_{old}, σ_{old}, R_1, and σ_1 remain the same as stated in the previous subsection. To study how the values of δ and θ affect the value of the optimal *m*, both δ and θ are set ranging between 0 and 2, but $\delta \neq 1$ and $\theta \neq 1$. It should also be ensured that θ must always be greater than or equal to δ following from (3.38). It can be concluded from some simulation results that:

- Once δ and θ satisfy condition (3.38), a small optimal value of m is obtained. That is, a new portfolio with the highest Sharpe ratio is reached by adding only a few new assets. For example, when $\delta = 1.2$ and $\theta = 1.3$, the optimal value of m is 11, see Figure 3.6.

- More specifically, for $\frac{\delta}{\theta} = 1$, i.e. $\delta = \theta$, the Sharpe ratios for all the new assets are the same.
    - If $0 < \delta = \theta < 1$, we observe that both $\{R_1, R_2, \ldots, R_n\}$ and $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ are decreasing sequences. As m increases, both R_{new} and σ_{new} decrease as well; moreover, the rates of change drop and become steady. The rate of change for σ_{new} may be greater than that for R_{new} at the beginning, but it will be smaller eventually. The optimal m is obtained at this turning point. For example, when $\delta = \theta = 0.8$, the optimal value of m is 2. See Table 3.5. It can be observed in this case that the rate of change for σ_{new} is greater than that for R_{new} when $m \leq 2$; but the rate of change for σ_{new} is smaller than that for R_{new} when $m > 2$. Thus, the optimal m is at the turning point.
    - If $1 < \delta = \theta < 2$, then both $\{R_1, R_2, \ldots, R_n\}$ and $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ are increasing sequences. We have similar observations as those stated above. The rate of change for σ_{new} is smaller than that for R_{new} at the beginning, but it will be greater eventually. The optimal m is also obtained at the turning point. For example, when $\delta = \theta = 1.3$, the rate of change for σ_{new} is smaller than that for R_{new} when $m \leq 19$; but the rate of change for σ_{new} is greater than that for R_{new} when $m > 19$. The optimal value of m is 19. Moreover, as both δ and θ increase, the optimal value of m

decreases. For instance, when $\delta = \theta = 1.6$, the optimal value of m becomes to be 9. See Table 3.5.

- For $0 < \frac{\delta}{\theta} \ll 1$, i.e. $\delta \ll \theta$, it is usually required to add a few new stocks to obtain a portfolio with the highest Sharpe ratio. One of the examples is: $\delta = 0.1$ and $\theta = 1.2$, i.e. $\frac{\delta}{\theta} = \frac{0.1}{1.2} = 0.0833 \ll 1$, the optimal value of m is 1. In this case, $\{R_1, R_2, \ldots, R_n\}$ is a decreasing sequence, but $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ is an increasing sequence. As more new stocks are added, i.e. m increases, a falling of R_{new} and a rising of σ_{new} leads to a decrease in the Sharpe ratio.

Consequently, in the geometric series case, the optimal value of m is usually small if δ and θ satisfy the condition (3.38) and both δ and θ are strictly less than 1.

Figure 3.6 Sharpe Ratios of New Portfolios against Number of Stocks added in Geometric Series Case

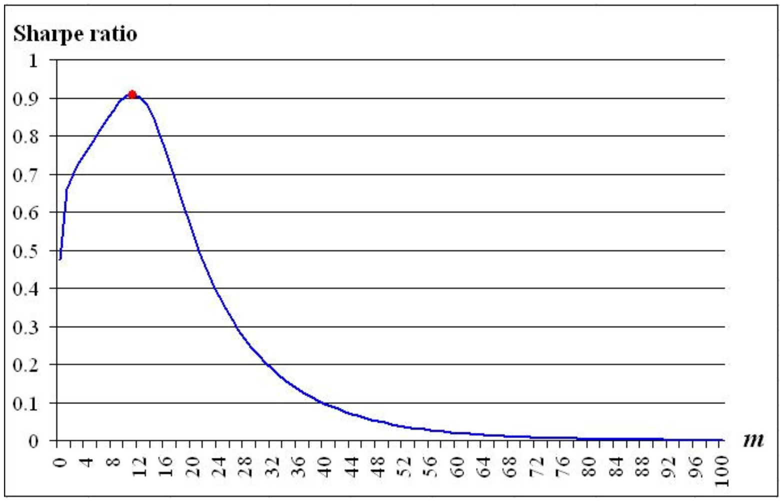


Table 3.5 The Optimal No. of New Stocks Added (m) for Different Values of δ and θ (δ and θ)

δ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
θ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
No. of New Stocks	1	1	1	1	1	1	1	2	2
δ	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
θ	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
No. of New Stocks	63	29	19	14	11	9	8	7	6

Appendix to Chapter 3: Roots for Equation (3.33)

A3.1 Expression of the First Root for Equation (3.33)

```
Solve[SR_D1 == 0, m]
```

$$\Big\{\Big\{m \to$$

$$\frac{2\left(a^2\beta^2\gamma + a^2\beta^2 R_1 + a\beta^2 R_{old} - a^2\beta^2 R_{old} - 3a^2\beta\gamma\sigma_1 - 3\gamma\sigma_{old}^2 + 6a\gamma\sigma_{old}^2 - 3a^2\gamma\sigma_{old}^2\right)}{3a^2\beta^2\gamma} -$$

$$\Big(18a^2\beta^2\gamma\left(a^2\beta^2\gamma - 6a^2\beta\gamma\sigma_1 + 6a^2\gamma\sigma_1^2\right) -$$

$$16$$

$$\big(a^2\beta^2\gamma + a^2\beta^2 R_1 + a\beta^2 R_{old} - a^2\beta^2 R_{old} - 3a^2\beta\gamma\sigma_1 - 3\gamma\sigma_{old}^2 + 6a\gamma\sigma_{old}^2 -$$

$$3a^2\gamma\sigma_{old}^2\big)^2\Big) \Big/$$

$$\Big(3\, 2^{2/3} a^2\beta^2\gamma$$

$$\Big(20a^6\beta^6\gamma^3 - 48a^6\beta^6\gamma^2 R_1 + 384a^6\beta^6\gamma R_1^2 + 128a^6\beta^6 R_1^3 - 48a^5\beta^6\gamma^2 R_{old} +$$

$$48a^6\beta^6\gamma^2 R_{old} + 768a^5\beta^6\gamma R_1 R_{old} - 768a^6\beta^6\gamma R_1 R_{old} + 384a^5\beta^6 R_1^2 R_{old} -$$

$$384a^6\beta^6 R_1^2 R_{old} + 384a^4\beta^6\gamma R_{old}^2 - 768a^5\beta^6\gamma R_{old}^2 + 384a^6\beta^6\gamma R_{old}^2 +$$

$$384a^4\beta^6 R_1 R_{old}^2 - 768a^5\beta^6 R_1 R_{old}^2 + 384a^6\beta^6 R_1 R_{old}^2 + 128a^3\beta^6 R_{old}^3 -$$

$$384a^4\beta^6 R_{old}^3 + 384a^5\beta^6 R_{old}^3 - 128a^6\beta^6 R_{old}^3 + 144a^6\beta^5\gamma^3\sigma_1 +$$

$$288a^6\beta^5\gamma^2 R_1\sigma_1 - 1152a^6\beta^5\gamma R_1^2\sigma_1 + 288a^5\beta^5\gamma^2 R_{old}\sigma_1 -$$

$$288a^6\beta^5\gamma^2 R_{old}\sigma_1 - 2304a^5\beta^5\gamma R_1 R_{old}\sigma_1 + 2304a^6\beta^5\gamma R_1 R_{old}\sigma_1 -$$

$$1152a^4\beta^5\gamma R_{old}^2\sigma_1 + 2304a^5\beta^5\gamma R_{old}^2\sigma_1 - 1152a^6\beta^5\gamma R_{old}^2\sigma_1 -$$

$$1080a^6\beta^4\gamma^3\sigma_1^2 + 864a^6\beta^4\gamma^2 R_1\sigma_1^2 + 864a^5\beta^4\gamma^2 R_{old}\sigma_1^2 -$$

$$864a^6\beta^4\gamma^2 R_{old}\sigma_1^2 + 432a^6\beta^3\gamma^3\sigma_1^3 - 504a^4\beta^4\gamma^3\sigma_{old}^2 + 1008a^5\beta^4\gamma^3\sigma_{old}^2 -$$

$$504a^6\beta^4\gamma^3\sigma_{old}^2 - 2304a^4\beta^4\gamma^2 R_1\sigma_{old}^2 + 4608a^5\beta^4\gamma^2 R_1\sigma_{old}^2 -$$

$$2304a^6\beta^4\gamma^2 R_1\sigma_{old}^2 - 1152a^4\beta^4\gamma R_1^2\sigma_{old}^2 + 2304a^5\beta^4\gamma R_1^2\sigma_{old}^2 -$$

$$1152a^6\beta^4\gamma R_1^2\sigma_{old}^2 - 2304a^3\beta^4\gamma^2 R_{old}\sigma_{old}^2 + 6912a^4\beta^4\gamma^2 R_{old}\sigma_{old}^2 -$$

$$6912a^5\beta^4\gamma^2 R_{old}\sigma_{old}^2 + 2304a^6\beta^4\gamma^2 R_{old}\sigma_{old}^2 - 2304a^3\beta^4\gamma R_1 R_{old}\sigma_{old}^2 +$$

$$6912a^4\beta^4\gamma R_1 R_{old}\sigma_{old}^2 - 6912a^5\beta^4\gamma R_1 R_{old}\sigma_{old}^2 + 2304a^6\beta^4\gamma R_1 R_{old}\sigma_{old}^2 -$$

$$1152a^2\beta^4\gamma R_{old}^2\sigma_{old}^2 + 4608a^3\beta^4\gamma R_{old}^2\sigma_{old}^2 - 6912a^4\beta^4\gamma R_{old}^2\sigma_{old}^2 +$$

$$4608a^5\beta^4\gamma R_{old}^2\sigma_{old}^2 - 1152a^6\beta^4\gamma R_{old}^2\sigma_{old}^2 + 3024a^4\beta^3\gamma^3\sigma_1\sigma_{old}^2 -$$

$$6048a^5\beta^3\gamma^3\sigma_1\sigma_{old}^2 + 3024a^6\beta^3\gamma^3\sigma_1\sigma_{old}^2 + 6912a^4\beta^3\gamma^2 R_1\sigma_1\sigma_{old}^2 -$$

$$13\,824a^5\beta^3\gamma^2 R_1\sigma_1\sigma_{old}^2 + 6912a^6\beta^3\gamma^2 R_1\sigma_1\sigma_{old}^2 + 6912a^3\beta^3\gamma^2 R_{old}\sigma_1\sigma_{old}^2 -$$

$$20\,736a^4\beta^3\gamma^2 R_{old}\sigma_1\sigma_{old}^2 + 20\,736a^5\beta^3\gamma^2 R_{old}\sigma_1\sigma_{old}^2 -$$

$$6912a^6\beta^3\gamma^2 R_{old}\sigma_1\sigma_{old}^2 - 6480a^4\beta^2\gamma^3\sigma_1^2\sigma_{old}^2 + 12\,960a^5\beta^2\gamma^3\sigma_1^2\sigma_{old}^2 -$$

$$6480a^6\beta^2\gamma^3\sigma_1^2\sigma_{old}^2 + 3456a^2\beta^2\gamma^3\sigma_{old}^4 - 13\,824a^3\beta^2\gamma^3\sigma_{old}^4 +$$

$$20\,736a^4\beta^2\gamma^3\sigma_{old}^4 - 13\,824a^5\beta^2\gamma^3\sigma_{old}^4 + 3456a^6\beta^2\gamma^3\sigma_{old}^4 +$$

$$3456a^2\beta^2\gamma^2 R_1\sigma_{old}^4 - 13\,824a^3\beta^2\gamma^2 R_1\sigma_{old}^4 + 20\,736a^4\beta^2\gamma^2 R_1\sigma_{old}^4 -$$

$$13\,824a^5\beta^2\gamma^2 R_1\sigma_{old}^4 + 3456a^6\beta^2\gamma^2 R_1\sigma_{old}^4 + 3456a\beta^2\gamma^2 R_{old}\sigma_{old}^4 -$$

$$17\,280a^2\beta^2\gamma^2 R_{old}\sigma_{old}^4 + 34\,560a^3\beta^2\gamma^2 R_{old}\sigma_{old}^4 - 34\,560a^4\beta^2\gamma^2 R_{old}\sigma_{old}^4 +$$

$$
\begin{aligned}
&17280\,a^5\beta^2\gamma^2 R_{old}\,\sigma_{old}^4 - 3456\,a^6\beta^2\gamma^2 R_{old}\,\sigma_{old}^4 - 10368\,a^2\beta\gamma^3\sigma_1\sigma_{old}^4 + \\
&41472\,a^3\beta\gamma^3\sigma_1\sigma_{old}^4 - 62208\,a^4\beta\gamma^3\sigma_1\sigma_{old}^4 + 41472\,a^5\beta\gamma^3\sigma_1\sigma_{old}^4 - \\
&10368\,a^6\beta\gamma^3\sigma_1\sigma_{old}^4 - 3456\,\gamma^3\sigma_{old}^6 + 20736\,a\,\gamma^3\sigma_{old}^6 - 51840\,a^2\gamma^3\sigma_{old}^6 + \\
&69120\,a^3\gamma^3\sigma_{old}^6 - 51840\,a^4\gamma^3\sigma_{old}^6 + 20736\,a^5\gamma^3\sigma_{old}^6 - 3456\,a^6\gamma^3\sigma_{old}^6 + \\
&\sqrt{\ }\Big(\Big(20\,a^6\beta^6\gamma^3 - 48\,a^6\beta^6\gamma^2 R_1 + 384\,a^6\beta^6\gamma R_1^2 + 128\,a^6\beta^6 R_1^3 - \\
&\quad 48\,a^5\beta^6\gamma^2 R_{old} + 48\,a^6\beta^6\gamma^2 R_{old} + 768\,a^5\beta^6\gamma R_1 R_{old} - \\
&\quad 768\,a^6\beta^6\gamma R_1 R_{old} + 384\,a^5\beta^6 R_1^2 R_{old} - 384\,a^6\beta^6 R_1^2 R_{old} + \\
&\quad 384\,a^4\beta^6\gamma R_{old}^2 - 768\,a^5\beta^6\gamma R_{old}^2 + 384\,a^6\beta^6\gamma R_{old}^2 + \\
&\quad 384\,a^4\beta^6 R_1 R_{old}^2 - 768\,a^5\beta^6 R_1 R_{old}^2 + 384\,a^6\beta^6 R_1 R_{old}^2 + \\
&\quad 128\,a^3\beta^6 R_{old}^3 - 384\,a^4\beta^6 R_{old}^3 + 384\,a^5\beta^6 R_{old}^3 - 128\,a^6\beta^6 R_{old}^3 + \\
&\quad 144\,a^6\beta^5\gamma^3\sigma_1 + 288\,a^6\beta^5\gamma^2 R_1\sigma_1 - 1152\,a^6\beta^5\gamma R_1^2\sigma_1 + \\
&\quad 288\,a^5\beta^5\gamma^2 R_{old}\sigma_1 - 288\,a^6\beta^5\gamma^2 R_{old}\sigma_1 - 2304\,a^5\beta^5\gamma R_1 R_{old}\sigma_1 + \\
&\quad 2304\,a^6\beta^5\gamma R_1 R_{old}\sigma_1 - 1152\,a^4\beta^5\gamma R_{old}^2\sigma_1 + 2304\,a^5\beta^5\gamma R_{old}^2\sigma_1 - \\
&\quad 1152\,a^6\beta^5\gamma R_{old}^2\sigma_1 - 1080\,a^6\beta^4\gamma^3\sigma_1^2 + 864\,a^6\beta^4\gamma^2 R_1\sigma_1^2 + \\
&\quad 864\,a^5\beta^4\gamma^2 R_{old}\sigma_1^2 - 864\,a^6\beta^4\gamma^2 R_{old}\sigma_1^2 + 432\,a^6\beta^3\gamma^3\sigma_1^3 - \\
&\quad 504\,a^4\beta^4\gamma^3\sigma_{old}^2 + 1008\,a^5\beta^4\gamma^3\sigma_{old}^2 - 504\,a^6\beta^4\gamma^3\sigma_{old}^2 - \\
&\quad 2304\,a^4\beta^4\gamma^2 R_1\sigma_{old}^2 + 4608\,a^5\beta^4\gamma^2 R_1\sigma_{old}^2 - 2304\,a^6\beta^4\gamma^2 R_1\sigma_{old}^2 - \\
&\quad 1152\,a^4\beta^4\gamma R_1^2\sigma_{old}^2 + 2304\,a^5\beta^4\gamma R_1^2\sigma_{old}^2 - 1152\,a^6\beta^4\gamma R_1^2\sigma_{old}^2 - \\
&\quad 2304\,a^3\beta^4\gamma^2 R_{old}\sigma_{old}^2 + 6912\,a^4\beta^4\gamma^2 R_{old}\sigma_{old}^2 - \\
&\quad 6912\,a^5\beta^4\gamma^2 R_{old}\sigma_{old}^2 + 2304\,a^6\beta^4\gamma^2 R_{old}\sigma_{old}^2 - \\
&\quad 2304\,a^3\beta^4\gamma R_1 R_{old}\sigma_{old}^2 + 6912\,a^4\beta^4\gamma R_1 R_{old}\sigma_{old}^2 - \\
&\quad 6912\,a^5\beta^4\gamma R_1 R_{old}\sigma_{old}^2 + 2304\,a^6\beta^4\gamma R_1 R_{old}\sigma_{old}^2 - \\
&\quad 1152\,a^2\beta^4\gamma R_{old}^2\sigma_{old}^2 + 4608\,a^3\beta^4\gamma R_{old}^2\sigma_{old}^2 - 6912\,a^4\beta^4\gamma R_{old}^2\sigma_{old}^2 + \\
&\quad 4608\,a^5\beta^4\gamma R_{old}^2\sigma_{old}^2 - 1152\,a^6\beta^4\gamma R_{old}^2\sigma_{old}^2 + 3024\,a^4\beta^3\gamma^3\sigma_1\sigma_{old}^2 - \\
&\quad 6048\,a^5\beta^3\gamma^3\sigma_1\sigma_{old}^2 + 3024\,a^6\beta^3\gamma^3\sigma_1\sigma_{old}^2 + 6912\,a^4\beta^3\gamma^2 R_1\sigma_1\sigma_{old}^2 - \\
&\quad 13824\,a^5\beta^3\gamma^2 R_1\sigma_1\sigma_{old}^2 + 6912\,a^6\beta^3\gamma^2 R_1\sigma_1\sigma_{old}^2 + \\
&\quad 6912\,a^3\beta^3\gamma^2 R_{old}\sigma_1\sigma_{old}^2 - 20736\,a^4\beta^3\gamma^2 R_{old}\sigma_1\sigma_{old}^2 + \\
&\quad 20736\,a^5\beta^3\gamma^2 R_{old}\sigma_1\sigma_{old}^2 - 6912\,a^6\beta^3\gamma^2 R_{old}\sigma_1\sigma_{old}^2 - \\
&\quad 6480\,a^4\beta^2\gamma^3\sigma_1^2\sigma_{old}^2 + 12960\,a^5\beta^2\gamma^3\sigma_1^2\sigma_{old}^2 - 6480\,a^6\beta^2\gamma^3\sigma_1^2\sigma_{old}^2 + \\
&\quad 3456\,a^2\beta^2\gamma^3\sigma_{old}^4 - 13824\,a^3\beta^2\gamma^3\sigma_{old}^4 + 20736\,a^4\beta^2\gamma^3\sigma_{old}^4 - \\
&\quad 13824\,a^5\beta^2\gamma^3\sigma_{old}^4 + 3456\,a^6\beta^2\gamma^3\sigma_{old}^4 + 3456\,a^2\beta^2\gamma^2 R_1\sigma_{old}^4 - \\
&\quad 13824\,a^3\beta^2\gamma^2 R_1\sigma_{old}^4 + 20736\,a^4\beta^2\gamma^2 R_1\sigma_{old}^4 - \\
&\quad 13824\,a^5\beta^2\gamma^2 R_1\sigma_{old}^4 + 3456\,a^6\beta^2\gamma^2 R_1\sigma_{old}^4 + 3456\,a\,\beta^2\gamma^2 R_{old}\sigma_{old}^4 - \\
&\quad 17280\,a^2\beta^2\gamma^2 R_{old}\sigma_{old}^4 + 34560\,a^3\beta^2\gamma^2 R_{old}\sigma_{old}^4 - \\
&\quad 34560\,a^4\beta^2\gamma^2 R_{old}\sigma_{old}^4 + 17280\,a^5\beta^2\gamma^2 R_{old}\sigma_{old}^4 - \\
&\quad 3456\,a^6\beta^2\gamma^2 R_{old}\sigma_{old}^4 - 10368\,a^2\beta\gamma^3\sigma_1\sigma_{old}^4 + 41472\,a^3\beta\gamma^3\sigma_1\sigma_{old}^4 - \\
&\quad 62208\,a^4\beta\gamma^3\sigma_1\sigma_{old}^4 + 41472\,a^5\beta\gamma^3\sigma_1\sigma_{old}^4 - 10368\,a^6\beta\gamma^3\sigma_1\sigma_{old}^4 - \\
&\quad 3456\,\gamma^3\sigma_{old}^6 + 20736\,a\,\gamma^3\sigma_{old}^6 - 51840\,a^2\gamma^3\sigma_{old}^6 + \\
&\quad 69120\,a^3\gamma^3\sigma_{old}^6 - 51840\,a^4\gamma^3\sigma_{old}^6 + 20736\,a^5\gamma^3\sigma_{old}^6 - \\
&\quad 3456\,a^6\gamma^3\sigma_{old}^6\Big)^2 + \\
&\ 4\,\Big(18\,a^2\beta^2\gamma\,\big(a^2\beta^2\gamma - 6\,a^2\beta\gamma\sigma_1 + 6\,a^2\gamma\sigma_1^2\big) - \\
&\qquad 16\,\big(a^2\beta^2\gamma + a^2\beta^2 R_1 + a\,\beta^2 R_{old} - a^2\beta^2 R_{old} - 3\,a^2\beta\gamma\sigma_1 - \\
&\qquad\quad 3\,\gamma\sigma_{old}^2 + 6\,a\,\gamma\sigma_{old}^2 - 3\,a^2\gamma\sigma_{old}^2\big)^2\Big)^3\Big)\Big)^{1/3}\Big) + \\
&\frac{1}{6\,2^{1/3}\,a^2\beta^2\gamma} \\
&\Big(20\,a^6\beta^6\gamma^3 - 48\,a^6\beta^6\gamma^2 R_1 + 384\,a^6\beta^6\gamma R_1^2 + 128\,a^6\beta^6 R_1^3 - 48\,a^5\beta^6\gamma^2 R_{old} +
\end{aligned}
$$

$$48 a^6 \beta^6 \gamma^2 R_{old} + 768 a^5 \beta^6 \gamma R_1 R_{old} - 768 a^6 \beta^6 \gamma R_1 R_{old} + 384 a^5 \beta^6 R_1^2 R_{old} -$$
$$384 a^6 \beta^6 R_1^2 R_{old} + 384 a^4 \beta^6 \gamma R_{old}^2 - 768 a^5 \beta^6 \gamma R_{old}^2 + 384 a^6 \beta^6 \gamma R_{old}^2 +$$
$$384 a^4 \beta^6 R_1 R_{old}^2 - 768 a^5 \beta^6 R_1 R_{old}^2 + 384 a^6 \beta^6 R_1 R_{old}^2 + 128 a^3 \beta^6 R_{old}^3 -$$
$$384 a^4 \beta^6 R_{old}^3 + 384 a^5 \beta^6 R_{old}^3 - 128 a^6 \beta^6 R_{old}^3 + 144 a^6 \beta^5 \gamma^3 \sigma_1 +$$
$$288 a^6 \beta^5 \gamma^2 R_1 \sigma_1 - 1152 a^6 \beta^5 \gamma R_1^2 \sigma_1 + 288 a^5 \beta^5 \gamma^2 R_{old} \sigma_1 -$$
$$288 a^6 \beta^5 \gamma^2 R_{old} \sigma_1 - 2304 a^5 \beta^5 \gamma R_1 R_{old} \sigma_1 + 2304 a^6 \beta^5 \gamma R_1 R_{old} \sigma_1 -$$
$$1152 a^4 \beta^5 \gamma R_{old}^2 \sigma_1 + 2304 a^5 \beta^5 \gamma R_{old}^2 \sigma_1 - 1152 a^6 \beta^5 \gamma R_{old}^2 \sigma_1 -$$
$$1080 a^6 \beta^4 \gamma^3 \sigma_1^2 + 864 a^6 \beta^4 \gamma^2 R_1 \sigma_1^2 + 864 a^5 \beta^4 \gamma^2 R_{old} \sigma_1^2 - 864 a^6 \beta^4 \gamma^2 R_{old} \sigma_1^2 +$$
$$432 a^6 \beta^3 \gamma^3 \sigma_1^3 - 504 a^4 \beta^4 \gamma^3 \sigma_{old}^2 + 1008 a^5 \beta^4 \gamma^3 \sigma_{old}^2 - 504 a^6 \beta^4 \gamma^3 \sigma_{old}^2 -$$
$$2304 a^4 \beta^4 \gamma^2 R_1 \sigma_{old}^2 + 4608 a^5 \beta^4 \gamma^2 R_1 \sigma_{old}^2 - 2304 a^6 \beta^4 \gamma^2 R_1 \sigma_{old}^2 -$$
$$1152 a^4 \beta^4 \gamma R_1^2 \sigma_{old}^2 + 2304 a^5 \beta^4 \gamma R_1^2 \sigma_{old}^2 - 1152 a^6 \beta^4 \gamma R_1^2 \sigma_{old}^2 -$$
$$2304 a^3 \beta^4 \gamma^2 R_{old} \sigma_{old}^2 + 6912 a^4 \beta^4 \gamma^2 R_{old} \sigma_{old}^2 - 6912 a^5 \beta^4 \gamma^2 R_{old} \sigma_{old}^2 +$$
$$2304 a^6 \beta^4 \gamma^2 R_{old} \sigma_{old}^2 - 2304 a^3 \beta^4 \gamma R_1 R_{old} \sigma_{old}^2 + 6912 a^4 \beta^4 \gamma R_1 R_{old} \sigma_{old}^2 -$$
$$6912 a^5 \beta^4 \gamma R_1 R_{old} \sigma_{old}^2 + 2304 a^6 \beta^4 \gamma R_1 R_{old} \sigma_{old}^2 - 1152 a^2 \beta^4 \gamma R_{old}^2 \sigma_{old}^2 +$$
$$4608 a^3 \beta^4 \gamma R_{old}^2 \sigma_{old}^2 - 6912 a^4 \beta^4 \gamma R_{old}^2 \sigma_{old}^2 + 4608 a^5 \beta^4 \gamma R_{old}^2 \sigma_{old}^2 -$$
$$1152 a^6 \beta^4 \gamma R_{old}^2 \sigma_{old}^2 + 3024 a^4 \beta^3 \gamma^3 \sigma_1 \sigma_{old}^2 - 6048 a^5 \beta^3 \gamma^3 \sigma_1 \sigma_{old}^2 +$$
$$3024 a^6 \beta^3 \gamma^3 \sigma_1 \sigma_{old}^2 + 6912 a^4 \beta^3 \gamma^2 R_1 \sigma_1 \sigma_{old}^2 - 13\,824 a^5 \beta^3 \gamma^2 R_1 \sigma_1 \sigma_{old}^2 +$$
$$6912 a^6 \beta^3 \gamma^2 R_1 \sigma_1 \sigma_{old}^2 + 6912 a^3 \beta^3 \gamma^2 R_{old} \sigma_1 \sigma_{old}^2 - 20\,736 a^4 \beta^3 \gamma^2 R_{old} \sigma_1 \sigma_{old}^2 +$$
$$20\,736 a^5 \beta^3 \gamma^2 R_{old} \sigma_1 \sigma_{old}^2 - 6912 a^6 \beta^3 \gamma^2 R_{old} \sigma_1 \sigma_{old}^2 - 6480 a^4 \beta^2 \gamma^3 \sigma_1^2 \sigma_{old}^2 +$$
$$12\,960 a^5 \beta^2 \gamma^3 \sigma_1^2 \sigma_{old}^2 - 6480 a^6 \beta^2 \gamma^3 \sigma_1^2 \sigma_{old}^2 + 3456 a^2 \beta^2 \gamma^3 \sigma_{old}^4 -$$
$$13\,824 a^3 \beta^2 \gamma^3 \sigma_{old}^4 + 20\,736 a^4 \beta^2 \gamma^3 \sigma_{old}^4 - 13\,824 a^5 \beta^2 \gamma^3 \sigma_{old}^4 +$$
$$3456 a^6 \beta^2 \gamma^3 \sigma_{old}^4 + 3456 a^2 \beta^2 \gamma^2 R_1 \sigma_{old}^4 - 13\,824 a^3 \beta^2 \gamma^2 R_1 \sigma_{old}^4 +$$
$$20\,736 a^4 \beta^2 \gamma^2 R_1 \sigma_{old}^4 - 13\,824 a^5 \beta^2 \gamma^2 R_1 \sigma_{old}^4 + 3456 a^6 \beta^2 \gamma^2 R_1 \sigma_{old}^4 +$$
$$3456 a \beta^2 \gamma^2 R_{old} \sigma_{old}^4 - 17\,280 a^2 \beta^2 \gamma^2 R_{old} \sigma_{old}^4 + 34\,560 a^3 \beta^2 \gamma^2 R_{old} \sigma_{old}^4 -$$
$$34\,560 a^4 \beta^2 \gamma^2 R_{old} \sigma_{old}^4 + 17\,280 a^5 \beta^2 \gamma^2 R_{old} \sigma_{old}^4 - 3456 a^6 \beta^2 \gamma^2 R_{old} \sigma_{old}^4 -$$
$$10\,368 a^2 \beta \gamma^3 \sigma_1 \sigma_{old}^4 + 41\,472 a^3 \beta \gamma^3 \sigma_1 \sigma_{old}^4 - 62\,208 a^4 \beta \gamma^3 \sigma_1 \sigma_{old}^4 +$$
$$41\,472 a^5 \beta \gamma^3 \sigma_1 \sigma_{old}^4 - 10\,368 a^6 \beta \gamma^3 \sigma_1 \sigma_{old}^4 - 3456 \gamma^3 \sigma_{old}^6 + 20\,736 a \gamma^3 \sigma_{old}^6 -$$
$$51\,840 a^2 \gamma^3 \sigma_{old}^6 + 69\,120 a^3 \gamma^3 \sigma_{old}^6 - 51\,840 a^4 \gamma^3 \sigma_{old}^6 + 20\,736 a^5 \gamma^3 \sigma_{old}^6 -$$
$$3456 a^6 \gamma^3 \sigma_{old}^6 +$$
$$\sqrt{\Big(\Big(20 a^6 \beta^6 \gamma^3 - 48 a^6 \beta^6 \gamma^2 R_1 + 384 a^6 \beta^6 \gamma R_1^2 + 128 a^6 \beta^6 R_1^3 -}$$
$$48 a^5 \beta^6 \gamma^2 R_{old} + 48 a^6 \beta^6 \gamma^2 R_{old} + 768 a^5 \beta^6 \gamma R_1 R_{old} -$$
$$768 a^6 \beta^6 \gamma R_1 R_{old} + 384 a^5 \beta^6 R_1^2 R_{old} - 384 a^6 \beta^6 R_1^2 R_{old} +$$
$$384 a^4 \beta^6 \gamma R_{old}^2 - 768 a^5 \beta^6 \gamma R_{old}^2 + 384 a^6 \beta^6 \gamma R_{old}^2 +$$
$$384 a^4 \beta^6 R_1 R_{old}^2 - 768 a^5 \beta^6 R_1 R_{old}^2 + 384 a^6 \beta^6 R_1 R_{old}^2 +$$
$$128 a^3 \beta^6 R_{old}^3 - 384 a^4 \beta^6 R_{old}^3 + 384 a^5 \beta^6 R_{old}^3 - 128 a^6 \beta^6 R_{old}^3 +$$
$$144 a^6 \beta^5 \gamma^3 \sigma_1 + 288 a^6 \beta^5 \gamma^2 R_1 \sigma_1 - 1152 a^6 \beta^5 \gamma R_1^2 \sigma_1 +$$
$$288 a^5 \beta^5 \gamma^2 R_{old} \sigma_1 - 288 a^6 \beta^5 \gamma^2 R_{old} \sigma_1 - 2304 a^5 \beta^5 \gamma R_1 R_{old} \sigma_1 +$$
$$2304 a^6 \beta^5 \gamma R_1 R_{old} \sigma_1 - 1152 a^4 \beta^5 \gamma R_{old}^2 \sigma_1 + 2304 a^5 \beta^5 \gamma R_{old}^2 \sigma_1 -$$
$$1152 a^6 \beta^5 \gamma R_{old}^2 \sigma_1 - 1080 a^6 \beta^4 \gamma^3 \sigma_1^2 + 864 a^6 \beta^4 \gamma^2 R_1 \sigma_1^2 +$$
$$864 a^5 \beta^4 \gamma^2 R_{old} \sigma_1^2 - 864 a^6 \beta^4 \gamma^2 R_{old} \sigma_1^2 + 432 a^6 \beta^3 \gamma^3 \sigma_1^3 -$$
$$504 a^4 \beta^4 \gamma^3 \sigma_{old}^2 + 1008 a^5 \beta^4 \gamma^3 \sigma_{old}^2 - 504 a^6 \beta^4 \gamma^3 \sigma_{old}^2 -$$
$$2304 a^4 \beta^4 \gamma^2 R_1 \sigma_{old}^2 + 4608 a^5 \beta^4 \gamma^2 R_1 \sigma_{old}^2 - 2304 a^6 \beta^4 \gamma^2 R_1 \sigma_{old}^2 -$$
$$1152 a^4 \beta^4 \gamma R_1^2 \sigma_{old}^2 + 2304 a^5 \beta^4 \gamma R_1^2 \sigma_{old}^2 - 1152 a^6 \beta^4 \gamma R_1^2 \sigma_{old}^2 -$$
$$2304 a^3 \beta^4 \gamma^2 R_{old} \sigma_{old}^2 + 6912 a^4 \beta^4 \gamma^2 R_{old} \sigma_{old}^2 - 6912 a^5 \beta^4 \gamma^2 R_{old} \sigma_{old}^2 +$$
$$2304 a^6 \beta^4 \gamma^2 R_{old} \sigma_{old}^2 - 2304 a^3 \beta^4 \gamma R_1 R_{old} \sigma_{old}^2 +$$
$$6912 a^4 \beta^4 \gamma R_1 R_{old} \sigma_{old}^2 - 6912 a^5 \beta^4 \gamma R_1 R_{old} \sigma_{old}^2 +$$
$$2304 a^6 \beta^4 \gamma R_1 R_{old} \sigma_{old}^2 - 1152 a^2 \beta^4 \gamma R_{old}^2 \sigma_{old}^2 + 4608 a^3 \beta^4 \gamma R_{old}^2 \sigma_{old}^2 -$$

$$6912\, a^4 \beta^4 \gamma R_{old}^2 \sigma_{old}^2 + 4608\, a^5 \beta^4 \gamma R_{old}^2 \sigma_{old}^2 - 1152\, a^6 \beta^4 \gamma R_{old}^2 \sigma_{old}^2 +$$
$$3024\, a^4 \beta^3 \gamma^3 \sigma_1 \sigma_{old}^2 - 6048\, a^5 \beta^3 \gamma^3 \sigma_1 \sigma_{old}^2 + 3024\, a^6 \beta^3 \gamma^3 \sigma_1 \sigma_{old}^2 +$$
$$6912\, a^4 \beta^3 \gamma^2 R_1 \sigma_1 \sigma_{old}^2 - 13\,824\, a^5 \beta^3 \gamma^2 R_1 \sigma_1 \sigma_{old}^2 +$$
$$6912\, a^6 \beta^3 \gamma^2 R_1 \sigma_1 \sigma_{old}^2 + 6912\, a^3 \beta^3 \gamma^2 R_{old} \sigma_1 \sigma_{old}^2 -$$
$$20\,736\, a^4 \beta^3 \gamma^2 R_{old} \sigma_1 \sigma_{old}^2 + 20\,736\, a^5 \beta^3 \gamma^2 R_{old} \sigma_1 \sigma_{old}^2 -$$
$$6912\, a^6 \beta^3 \gamma^2 R_{old} \sigma_1 \sigma_{old}^2 - 6480\, a^4 \beta^2 \gamma^3 \sigma_1^2 \sigma_{old}^2 + 12\,960\, a^5 \beta^2 \gamma^3 \sigma_1^2 \sigma_{old}^2 -$$
$$6480\, a^6 \beta^2 \gamma^3 \sigma_1^2 \sigma_{old}^2 + 3456\, a^2 \beta^2 \gamma^3 \sigma_{old}^4 - 13\,824\, a^3 \beta^2 \gamma^3 \sigma_{old}^4 +$$
$$20\,736\, a^4 \beta^2 \gamma^3 \sigma_{old}^4 - 13\,824\, a^5 \beta^2 \gamma^3 \sigma_{old}^4 + 3456\, a^6 \beta^2 \gamma^3 \sigma_{old}^4 +$$
$$3456\, a^2 \beta^2 \gamma^2 R_1 \sigma_{old}^4 - 13\,824\, a^3 \beta^2 \gamma^2 R_1 \sigma_{old}^4 + 20\,736\, a^4 \beta^2 \gamma^2 R_1 \sigma_{old}^4 -$$
$$13\,824\, a^5 \beta^2 \gamma^2 R_1 \sigma_{old}^4 + 3456\, a^6 \beta^2 \gamma^2 R_1 \sigma_{old}^4 + 3456\, a \beta^2 \gamma^2 R_{old} \sigma_{old}^4 -$$
$$17\,280\, a^2 \beta^2 \gamma^2 R_{old} \sigma_{old}^4 + 34\,560\, a^3 \beta^2 \gamma^2 R_{old} \sigma_{old}^4 -$$
$$34\,560\, a^4 \beta^2 \gamma^2 R_{old} \sigma_{old}^4 + 17\,280\, a^5 \beta^2 \gamma^2 R_{old} \sigma_{old}^4 -$$
$$3456\, a^6 \beta^2 \gamma^2 R_{old} \sigma_{old}^4 - 10\,368\, a^2 \beta \gamma^3 \sigma_1 \sigma_{old}^4 + 41\,472\, a^3 \beta \gamma^3 \sigma_1 \sigma_{old}^4 -$$
$$62\,208\, a^4 \beta \gamma^3 \sigma_1 \sigma_{old}^4 + 41\,472\, a^5 \beta \gamma^3 \sigma_1 \sigma_{old}^4 - 10\,368\, a^6 \beta \gamma^3 \sigma_1 \sigma_{old}^4 -$$
$$3456\, \gamma^3 \sigma_{old}^6 + 20\,736\, a \gamma^3 \sigma_{old}^6 - 51\,840\, a^2 \gamma^3 \sigma_{old}^6 + 69\,120\, a^3 \gamma^3 \sigma_{old}^6 -$$
$$51\,840\, a^4 \gamma^3 \sigma_{old}^6 + 20\,736\, a^5 \gamma^3 \sigma_{old}^6 - 3456\, a^6 \gamma^3 \sigma_{old}^6\Big)^2 +$$
$$4\,\Big(18\, a^2 \beta^2 \gamma \left(a^2 \beta^2 \gamma - 6\, a^2 \beta \gamma \sigma_1 + 6\, a^2 \gamma \sigma_1^2\right) -$$
$$16\,\big(a^2 \beta^2 \gamma + a^2 \beta^2 R_1 + a \beta^2 R_{old} - a^2 \beta^2 R_{old} - 3\, a^2 \beta \gamma \sigma_1 -$$
$$3\, \gamma \sigma_{old}^2 + 6\, a \gamma \sigma_{old}^2 - 3\, a^2 \gamma \sigma_{old}^2\big)^2\Big)^3\Big)\Big)^{1/3}\Big\},$$

A3.2 Expression of the Second Root for Equation (3.33)

$$
\Big\{m \to
\frac{2\left(a^2\beta^2\gamma + a^2\beta^2 R_1 + a\beta^2 R_{old} - a^2\beta^2 R_{old} - 3a^2\beta\gamma\sigma_1 - 3\gamma\sigma_{old}^2 + 6a\gamma\sigma_{old}^2 - 3a^2\gamma\sigma_{old}^2\right)}{3a^2\beta^2\gamma} +
$$

$$
\begin{aligned}
&\Big(\left(1+i\sqrt{3}\right)\Big(18a^2\beta^2\gamma\left(a^2\beta^2\gamma - 6a^2\beta\gamma\sigma_1 + 6a^2\gamma\sigma_1^2\right) - \\
&\quad 16\left(a^2\beta^2\gamma + a^2\beta^2R_1 + a\beta^2R_{old} - a^2\beta^2R_{old} - 3a^2\beta\gamma\sigma_1 - 3\gamma\sigma_{old}^2 + \right.\\
&\qquad \left. 6a\gamma\sigma_{old}^2 - 3a^2\gamma\sigma_{old}^2\right)^2\Big)\Big) \Big/ \\
&\Big(6\,2^{2/3}a^2\beta^2\gamma \\
&\quad \Big(20a^6\beta^6\gamma^3 - 48a^6\beta^6\gamma^2R_1 + 384a^6\beta^6\gamma R_1^2 + 128a^6\beta^6R_1^3 - 48a^5\beta^6\gamma^2R_{old} + \\
&\qquad 48a^6\beta^6\gamma^2R_{old} + 768a^5\beta^6\gamma R_1R_{old} - 768a^6\beta^6\gamma R_1R_{old} + 384a^5\beta^6R_1^2R_{old} - \\
&\qquad 384a^6\beta^6R_1^2R_{old} + 384a^4\beta^6\gamma R_{old}^2 - 768a^5\beta^6\gamma R_{old}^2 + 384a^6\beta^6\gamma R_{old}^2 + \\
&\qquad 384a^4\beta^6R_1R_{old}^2 - 768a^5\beta^6R_1R_{old}^2 + 384a^6\beta^6R_1R_{old}^2 + 128a^3\beta^6R_{old}^3 - \\
&\qquad 384a^4\beta^6R_{old}^3 + 384a^5\beta^6R_{old}^3 - 128a^6\beta^6R_{old}^3 + 144a^6\beta^5\gamma^3\sigma_1 + \\
&\qquad 288a^6\beta^5\gamma^2R_1\sigma_1 - 1152a^6\beta^5\gamma R_1^2\sigma_1 + 288a^5\beta^5\gamma^2R_{old}\sigma_1 - \\
&\qquad 288a^6\beta^5\gamma^2R_{old}\sigma_1 - 2304a^5\beta^5\gamma R_1R_{old}\sigma_1 + 2304a^6\beta^5\gamma R_1R_{old}\sigma_1 - \\
&\qquad 1152a^4\beta^5\gamma R_{old}^2\sigma_1 + 2304a^5\beta^5\gamma R_{old}^2\sigma_1 - 1152a^6\beta^5\gamma R_{old}^2\sigma_1 - \\
&\qquad 1080a^6\beta^4\gamma^3\sigma_1^2 + 864a^6\beta^4\gamma^2R_1\sigma_1^2 + 864a^5\beta^4\gamma^2R_{old}\sigma_1^2 - \\
&\qquad 864a^6\beta^4\gamma^2R_{old}\sigma_1^2 + 432a^6\beta^3\gamma^3\sigma_1^3 - 504a^4\beta^4\gamma^3\sigma_{old}^2 + 1008a^5\beta^4\gamma^3\sigma_{old}^2 - \\
&\qquad 504a^6\beta^4\gamma^3\sigma_{old}^2 - 2304a^4\beta^4\gamma^2R_1\sigma_{old}^2 + 4608a^5\beta^4\gamma^2R_1\sigma_{old}^2 - \\
&\qquad 2304a^6\beta^4\gamma^2R_1\sigma_{old}^2 - 1152a^4\beta^4\gamma R_1^2\sigma_{old}^2 + 2304a^5\beta^4\gamma R_1^2\sigma_{old}^2 - \\
&\qquad 1152a^6\beta^4\gamma R_1^2\sigma_{old}^2 - 2304a^3\beta^4\gamma^2R_{old}\sigma_{old}^2 + 6912a^4\beta^4\gamma^2R_{old}\sigma_{old}^2 - \\
&\qquad 6912a^5\beta^4\gamma^2R_{old}\sigma_{old}^2 + 2304a^6\beta^4\gamma^2R_{old}\sigma_{old}^2 - 2304a^3\beta^4\gamma R_1R_{old}\sigma_{old}^2 + \\
&\qquad 6912a^4\beta^4\gamma R_1R_{old}\sigma_{old}^2 - 6912a^5\beta^4\gamma R_1R_{old}\sigma_{old}^2 + 2304a^6\beta^4\gamma R_1R_{old}\sigma_{old}^2 - \\
&\qquad 1152a^2\beta^4\gamma R_{old}^2\sigma_{old}^2 + 4608a^3\beta^4\gamma R_{old}^2\sigma_{old}^2 - 6912a^4\beta^4\gamma R_{old}^2\sigma_{old}^2 + \\
&\qquad 4608a^5\beta^4\gamma R_{old}^2\sigma_{old}^2 - 1152a^6\beta^4\gamma R_{old}^2\sigma_{old}^2 + 3024a^4\beta^3\gamma^3\sigma_1\sigma_{old}^2 - \\
&\qquad 6048a^5\beta^3\gamma^3\sigma_1\sigma_{old}^2 + 3024a^6\beta^3\gamma^3\sigma_1\sigma_{old}^2 + 6912a^4\beta^3\gamma^2R_1\sigma_1\sigma_{old}^2 - \\
&\qquad 13\,824a^5\beta^3\gamma^2R_1\sigma_1\sigma_{old}^2 + 6912a^6\beta^3\gamma^2R_1\sigma_1\sigma_{old}^2 + 6912a^3\beta^3\gamma^2R_{old}\sigma_1\sigma_{old}^2 - \\
&\qquad 20\,736a^4\beta^3\gamma^2R_{old}\sigma_1\sigma_{old}^2 + 20\,736a^5\beta^3\gamma^2R_{old}\sigma_1\sigma_{old}^2 - \\
&\qquad 6912a^6\beta^3\gamma^2R_{old}\sigma_1\sigma_{old}^2 - 6480a^4\beta^2\gamma^3\sigma_1^2\sigma_{old}^2 + 12\,960a^5\beta^2\gamma^3\sigma_1^2\sigma_{old}^2 - \\
&\qquad 6480a^6\beta^2\gamma^3\sigma_1^2\sigma_{old}^2 + 3456a^2\beta^2\gamma^3\sigma_{old}^4 - 13\,824a^3\beta^2\gamma^3\sigma_{old}^4 + \\
&\qquad 20\,736a^4\beta^2\gamma^3\sigma_{old}^4 - 13\,824a^5\beta^2\gamma^3\sigma_{old}^4 + 3456a^6\beta^2\gamma^3\sigma_{old}^4 + \\
&\qquad 3456a^2\beta^2\gamma^2R_1\sigma_{old}^4 - 13\,824a^3\beta^2\gamma^2R_1\sigma_{old}^4 + 20\,736a^4\beta^2\gamma^2R_1\sigma_{old}^4 - \\
&\qquad 13\,824a^5\beta^2\gamma^2R_1\sigma_{old}^4 + 3456a^6\beta^2\gamma^2R_1\sigma_{old}^4 + 3456a\beta^2\gamma^2R_{old}\sigma_{old}^4 - \\
&\qquad 17\,280a^2\beta^2\gamma^2R_{old}\sigma_{old}^4 + 34\,560a^3\beta^2\gamma^2R_{old}\sigma_{old}^4 - 34\,560a^4\beta^2\gamma^2R_{old}\sigma_{old}^4 + \\
&\qquad 17\,280a^5\beta^2\gamma^2R_{old}\sigma_{old}^4 - 3456a^6\beta^2\gamma^2R_{old}\sigma_{old}^4 - 10\,368a^2\beta\gamma^3\sigma_1\sigma_{old}^4 + \\
&\qquad 41\,472a^3\beta\gamma^3\sigma_1\sigma_{old}^4 - 62\,208a^4\beta\gamma^3\sigma_1\sigma_{old}^4 + 41\,472a^5\beta\gamma^3\sigma_1\sigma_{old}^4 - \\
&\qquad 10\,368a^6\beta\gamma^3\sigma_1\sigma_{old}^4 - 3456\gamma^3\sigma_{old}^6 + 20\,736a\gamma^3\sigma_{old}^6 - 51\,840a^2\gamma^3\sigma_{old}^6 + \\
&\qquad 69\,120a^3\gamma^3\sigma_{old}^6 - 51\,840a^4\gamma^3\sigma_{old}^6 + 20\,736a^5\gamma^3\sigma_{old}^6 - 3456a^6\gamma^3\sigma_{old}^6 + \\
&\qquad \sqrt{\Big(\Big(20a^6\beta^6\gamma^3 - 48a^6\beta^6\gamma^2R_1 + 384a^6\beta^6\gamma R_1^2 + 128a^6\beta^6R_1^3 - } \\
&\qquad\qquad 48a^5\beta^6\gamma^2R_{old} + 48a^6\beta^6\gamma^2R_{old} + 768a^5\beta^6\gamma R_1R_{old} - \\
&\qquad\qquad 768a^6\beta^6\gamma R_1R_{old} + 384a^5\beta^6R_1^2R_{old} - 384a^6\beta^6R_1^2R_{old} +
\end{aligned}
$$

$$384 a^4 \beta^6 \gamma R_{\mathrm{old}}^2 - 768 a^5 \beta^6 \gamma R_{\mathrm{old}}^2 + 384 a^6 \beta^6 \gamma R_{\mathrm{old}}^2 +$$
$$384 a^4 \beta^6 R_1 R_{\mathrm{old}}^2 - 768 a^5 \beta^6 R_1 R_{\mathrm{old}}^2 + 384 a^6 \beta^6 R_1 R_{\mathrm{old}}^2 +$$
$$128 a^3 \beta^6 R_{\mathrm{old}}^3 - 384 a^4 \beta^6 R_{\mathrm{old}}^3 + 384 a^5 \beta^6 R_{\mathrm{old}}^3 - 128 a^6 \beta^6 R_{\mathrm{old}}^3 +$$
$$144 a^6 \beta^5 \gamma^3 \sigma_1 + 288 a^6 \beta^5 \gamma^2 R_1 \sigma_1 - 1152 a^6 \beta^5 \gamma R_1^2 \sigma_1 +$$
$$288 a^5 \beta^5 \gamma^2 R_{\mathrm{old}} \sigma_1 - 288 a^6 \beta^5 \gamma^2 R_{\mathrm{old}} \sigma_1 - 2304 a^5 \beta^5 \gamma R_1 R_{\mathrm{old}} \sigma_1 +$$
$$2304 a^6 \beta^5 \gamma R_1 R_{\mathrm{old}} \sigma_1 - 1152 a^4 \beta^5 \gamma R_{\mathrm{old}}^2 \sigma_1 + 2304 a^5 \beta^5 \gamma R_{\mathrm{old}}^2 \sigma_1 -$$
$$1152 a^6 \beta^5 \gamma R_{\mathrm{old}}^2 \sigma_1 - 1080 a^6 \beta^4 \gamma^3 \sigma_1^2 + 864 a^6 \beta^4 \gamma^2 R_1 \sigma_1^2 +$$
$$864 a^5 \beta^4 \gamma^2 R_{\mathrm{old}} \sigma_1^2 - 864 a^6 \beta^4 \gamma^2 R_{\mathrm{old}} \sigma_1^2 + 432 a^6 \beta^3 \gamma^3 \sigma_1^3 -$$
$$504 a^4 \beta^4 \gamma^3 \sigma_{\mathrm{old}}^2 + 1008 a^5 \beta^4 \gamma^3 \sigma_{\mathrm{old}}^2 - 504 a^6 \beta^4 \gamma^3 \sigma_{\mathrm{old}}^2 -$$
$$2304 a^4 \beta^4 \gamma^2 R_1 \sigma_{\mathrm{old}}^2 + 4608 a^5 \beta^4 \gamma^2 R_1 \sigma_{\mathrm{old}}^2 - 2304 a^6 \beta^4 \gamma^2 R_1 \sigma_{\mathrm{old}}^2 -$$
$$1152 a^4 \beta^4 \gamma R_1^2 \sigma_{\mathrm{old}}^2 + 2304 a^5 \beta^4 \gamma R_1^2 \sigma_{\mathrm{old}}^2 - 1152 a^6 \beta^4 \gamma R_1^2 \sigma_{\mathrm{old}}^2 -$$
$$2304 a^3 \beta^4 \gamma^2 R_{\mathrm{old}} \sigma_{\mathrm{old}}^2 + 6912 a^4 \beta^4 \gamma^2 R_{\mathrm{old}} \sigma_{\mathrm{old}}^2 -$$
$$6912 a^5 \beta^4 \gamma^2 R_{\mathrm{old}} \sigma_{\mathrm{old}}^2 + 2304 a^6 \beta^4 \gamma^2 R_{\mathrm{old}} \sigma_{\mathrm{old}}^2 -$$
$$2304 a^3 \beta^4 \gamma R_1 R_{\mathrm{old}} \sigma_{\mathrm{old}}^2 + 6912 a^4 \beta^4 \gamma R_1 R_{\mathrm{old}} \sigma_{\mathrm{old}}^2 -$$
$$6912 a^5 \beta^4 \gamma R_1 R_{\mathrm{old}} \sigma_{\mathrm{old}}^2 + 2304 a^6 \beta^4 \gamma R_1 R_{\mathrm{old}} \sigma_{\mathrm{old}}^2 -$$
$$1152 a^2 \beta^4 \gamma R_{\mathrm{old}}^2 \sigma_{\mathrm{old}}^2 + 4608 a^3 \beta^4 \gamma R_{\mathrm{old}}^2 \sigma_{\mathrm{old}}^2 - 6912 a^4 \beta^4 \gamma R_{\mathrm{old}}^2 \sigma_{\mathrm{old}}^2 +$$
$$4608 a^5 \beta^4 \gamma R_{\mathrm{old}}^2 \sigma_{\mathrm{old}}^2 - 1152 a^6 \beta^4 \gamma R_{\mathrm{old}}^2 \sigma_{\mathrm{old}}^2 + 3024 a^4 \beta^3 \gamma^3 \sigma_1 \sigma_{\mathrm{old}}^2 -$$
$$6048 a^5 \beta^3 \gamma^3 \sigma_1 \sigma_{\mathrm{old}}^2 + 3024 a^6 \beta^3 \gamma^3 \sigma_1 \sigma_{\mathrm{old}}^2 + 6912 a^4 \beta^3 \gamma^2 R_1 \sigma_1 \sigma_{\mathrm{old}}^2 -$$
$$13\,824 a^5 \beta^3 \gamma^2 R_1 \sigma_1 \sigma_{\mathrm{old}}^2 + 6912 a^6 \beta^3 \gamma^2 R_1 \sigma_1 \sigma_{\mathrm{old}}^2 +$$
$$6912 a^3 \beta^3 \gamma^2 R_{\mathrm{old}} \sigma_1 \sigma_{\mathrm{old}}^2 - 20\,736 a^4 \beta^3 \gamma^2 R_{\mathrm{old}} \sigma_1 \sigma_{\mathrm{old}}^2 +$$
$$20\,736 a^5 \beta^3 \gamma^2 R_{\mathrm{old}} \sigma_1 \sigma_{\mathrm{old}}^2 - 6912 a^6 \beta^3 \gamma^2 R_{\mathrm{old}} \sigma_1 \sigma_{\mathrm{old}}^2 -$$
$$6480 a^4 \beta^2 \gamma^3 \sigma_1^2 \sigma_{\mathrm{old}}^2 + 12\,960 a^5 \beta^2 \gamma^3 \sigma_1^2 \sigma_{\mathrm{old}}^2 - 6480 a^6 \beta^2 \gamma^3 \sigma_1^2 \sigma_{\mathrm{old}}^2 +$$
$$3456 a^2 \beta^2 \gamma^3 \sigma_{\mathrm{old}}^4 - 13\,824 a^3 \beta^2 \gamma^3 \sigma_{\mathrm{old}}^4 + 20\,736 a^4 \beta^2 \gamma^3 \sigma_{\mathrm{old}}^4 -$$
$$13\,824 a^5 \beta^2 \gamma^3 \sigma_{\mathrm{old}}^4 + 3456 a^6 \beta^2 \gamma^3 \sigma_{\mathrm{old}}^4 + 3456 a^2 \beta^2 \gamma^2 R_1 \sigma_{\mathrm{old}}^4 -$$
$$13\,824 a^3 \beta^2 \gamma^2 R_1 \sigma_{\mathrm{old}}^4 + 20\,736 a^4 \beta^2 \gamma^2 R_1 \sigma_{\mathrm{old}}^4 -$$
$$13\,824 a^5 \beta^2 \gamma^2 R_1 \sigma_{\mathrm{old}}^4 + 3456 a^6 \beta^2 \gamma^2 R_1 \sigma_{\mathrm{old}}^4 + 3456 a \beta^2 \gamma^2 R_{\mathrm{old}} \sigma_{\mathrm{old}}^4 -$$
$$17\,280 a^2 \beta^2 \gamma^2 R_{\mathrm{old}} \sigma_{\mathrm{old}}^4 + 34\,560 a^3 \beta^2 \gamma^2 R_{\mathrm{old}} \sigma_{\mathrm{old}}^4 -$$
$$34\,560 a^4 \beta^2 \gamma^2 R_{\mathrm{old}} \sigma_{\mathrm{old}}^4 + 17\,280 a^5 \beta^2 \gamma^2 R_{\mathrm{old}} \sigma_{\mathrm{old}}^4 -$$
$$3456 a^6 \beta^2 \gamma^2 R_{\mathrm{old}} \sigma_{\mathrm{old}}^4 - 10\,368 a^2 \beta \gamma^3 \sigma_1 \sigma_{\mathrm{old}}^4 + 41\,472 a^3 \beta \gamma^3 \sigma_1 \sigma_{\mathrm{old}}^4 -$$
$$62\,208 a^4 \beta \gamma^3 \sigma_1 \sigma_{\mathrm{old}}^4 + 41\,472 a^5 \beta \gamma^3 \sigma_1 \sigma_{\mathrm{old}}^4 - 10\,368 a^6 \beta \gamma^3 \sigma_1 \sigma_{\mathrm{old}}^4 -$$
$$3456 \gamma^3 \sigma_{\mathrm{old}}^6 + 20\,736 a \gamma^3 \sigma_{\mathrm{old}}^6 - 51\,840 a^2 \gamma^3 \sigma_{\mathrm{old}}^6 +$$
$$69\,120 a^3 \gamma^3 \sigma_{\mathrm{old}}^6 - 51\,840 a^4 \gamma^3 \sigma_{\mathrm{old}}^6 + 20\,736 a^5 \gamma^3 \sigma_{\mathrm{old}}^6 -$$
$$3456 a^6 \gamma^3 \sigma_{\mathrm{old}}^6)^2 +$$
$$4 \Big(18 a^2 \beta^2 \gamma \left(a^2 \beta^2 \gamma - 6 a^2 \beta \gamma \sigma_1 + 6 a^2 \gamma \sigma_1^2\right) -$$
$$16 \big(a^2 \beta^2 \gamma + a^2 \beta^2 R_1 + a \beta^2 R_{\mathrm{old}} - a^2 \beta^2 R_{\mathrm{old}} - 3 a^2 \beta \gamma \sigma_1 -$$
$$3 \gamma \sigma_{\mathrm{old}}^2 + 6 a \gamma \sigma_{\mathrm{old}}^2 - 3 a^2 \gamma \sigma_{\mathrm{old}}^2\big)^2\Big)^3\Big)\Big)^{1/3}\Big) -$$
$$\frac{1}{12\, 2^{1/3} a^2 \beta^2 \gamma}$$
$$\left(1 - i \sqrt{3}\right)$$
$$\Big(20 a^6 \beta^6 \gamma^3 - 48 a^6 \beta^6 \gamma^2 R_1 + 384 a^6 \beta^6 \gamma R_1^2 + 128 a^6 \beta^6 R_1^3 - 48 a^5 \beta^6 \gamma^2 R_{\mathrm{old}} +$$
$$48 a^6 \beta^6 \gamma^2 R_{\mathrm{old}} + 768 a^5 \beta^6 \gamma R_1 R_{\mathrm{old}} - 768 a^6 \beta^6 \gamma R_1 R_{\mathrm{old}} + 384 a^5 \beta^6 R_1^2 R_{\mathrm{old}} -$$
$$384 a^6 \beta^6 R_1^2 R_{\mathrm{old}} + 384 a^4 \beta^6 \gamma R_{\mathrm{old}}^2 - 768 a^5 \beta^6 \gamma R_{\mathrm{old}}^2 + 384 a^6 \beta^6 \gamma R_{\mathrm{old}}^2 +$$
$$384 a^4 \beta^6 R_1 R_{\mathrm{old}}^2 - 768 a^5 \beta^6 R_1 R_{\mathrm{old}}^2 + 384 a^6 \beta^6 R_1 R_{\mathrm{old}}^2 + 128 a^3 \beta^6 R_{\mathrm{old}}^3 -$$
$$384 a^4 \beta^6 R_{\mathrm{old}}^3 + 384 a^5 \beta^6 R_{\mathrm{old}}^3 - 128 a^6 \beta^6 R_{\mathrm{old}}^3 + 144 a^6 \beta^5 \gamma^3 \sigma_1 +$$

$$288\,a^6\,\beta^5\,\gamma^2\,R_1\,\sigma_1 - 1152\,a^6\,\beta^5\,\gamma\,R_1^2\,\sigma_1 + 288\,a^5\,\beta^5\,\gamma^2\,R_{old}\,\sigma_1 -$$
$$288\,a^6\,\beta^5\,\gamma^2\,R_{old}\,\sigma_1 - 2304\,a^5\,\beta^5\,\gamma\,R_1\,R_{old}\,\sigma_1 + 2304\,a^6\,\beta^5\,\gamma\,R_1\,R_{old}\,\sigma_1 -$$
$$1152\,a^4\,\beta^5\,\gamma\,R_{old}^2\,\sigma_1 + 2304\,a^5\,\beta^5\,\gamma\,R_{old}^2\,\sigma_1 - 1152\,a^6\,\beta^5\,\gamma\,R_{old}^2\,\sigma_1 -$$
$$1080\,a^6\,\beta^4\,\gamma^3\,\sigma_1^2 + 864\,a^6\,\beta^4\,\gamma^2\,R_1\,\sigma_1^2 + 864\,a^5\,\beta^4\,\gamma^2\,R_{old}\,\sigma_1^2 -$$
$$864\,a^6\,\beta^4\,\gamma^2\,R_{old}\,\sigma_1^2 + 432\,a^6\,\beta^3\,\gamma^3\,\sigma_1^3 - 504\,a^4\,\beta^4\,\gamma^3\,\sigma_{old}^2 + 1008\,a^5\,\beta^4\,\gamma^3\,\sigma_{old}^2 -$$
$$504\,a^6\,\beta^4\,\gamma^3\,\sigma_{old}^2 - 2304\,a^4\,\beta^4\,\gamma^2\,R_1\,\sigma_{old}^2 + 4608\,a^5\,\beta^4\,\gamma^2\,R_1\,\sigma_{old}^2 -$$
$$2304\,a^6\,\beta^4\,\gamma^2\,R_1\,\sigma_{old}^2 - 1152\,a^4\,\beta^4\,\gamma\,R_1^2\,\sigma_{old}^2 + 2304\,a^5\,\beta^4\,\gamma\,R_1^2\,\sigma_{old}^2 -$$
$$1152\,a^6\,\beta^4\,\gamma\,R_1^2\,\sigma_{old}^2 - 2304\,a^3\,\beta^4\,\gamma^2\,R_{old}\,\sigma_{old}^2 + 6912\,a^4\,\beta^4\,\gamma^2\,R_{old}\,\sigma_{old}^2 -$$
$$6912\,a^5\,\beta^4\,\gamma^2\,R_{old}\,\sigma_{old}^2 + 2304\,a^6\,\beta^4\,\gamma^2\,R_{old}\,\sigma_{old}^2 - 2304\,a^3\,\beta^4\,\gamma\,R_1\,R_{old}\,\sigma_{old}^2 +$$
$$6912\,a^4\,\beta^4\,\gamma\,R_1\,R_{old}\,\sigma_{old}^2 - 6912\,a^5\,\beta^4\,\gamma\,R_1\,R_{old}\,\sigma_{old}^2 + 2304\,a^6\,\beta^4\,\gamma\,R_1\,R_{old}\,\sigma_{old}^2 -$$
$$1152\,a^2\,\beta^4\,\gamma\,R_{old}^2\,\sigma_{old}^2 + 4608\,a^3\,\beta^4\,\gamma\,R_{old}^2\,\sigma_{old}^2 - 6912\,a^4\,\beta^4\,\gamma\,R_{old}^2\,\sigma_{old}^2 +$$
$$4608\,a^5\,\beta^4\,\gamma\,R_{old}^2\,\sigma_{old}^2 - 1152\,a^6\,\beta^4\,\gamma\,R_{old}^2\,\sigma_{old}^2 + 3024\,a^4\,\beta^3\,\gamma^3\,\sigma_1\,\sigma_{old}^2 -$$
$$6048\,a^5\,\beta^3\,\gamma^3\,\sigma_1\,\sigma_{old}^2 + 3024\,a^6\,\beta^3\,\gamma^3\,\sigma_1\,\sigma_{old}^2 + 6912\,a^4\,\beta^3\,\gamma^2\,R_1\,\sigma_1\,\sigma_{old}^2 -$$
$$13\,824\,a^5\,\beta^3\,\gamma^2\,R_1\,\sigma_1\,\sigma_{old}^2 + 6912\,a^6\,\beta^3\,\gamma^2\,R_1\,\sigma_1\,\sigma_{old}^2 + 6912\,a^3\,\beta^3\,\gamma^2\,R_{old}\,\sigma_1\,\sigma_{old}^2 -$$
$$20\,736\,a^4\,\beta^3\,\gamma^2\,R_{old}\,\sigma_1\,\sigma_{old}^2 + 20\,736\,a^5\,\beta^3\,\gamma^2\,R_{old}\,\sigma_1\,\sigma_{old}^2 -$$
$$6912\,a^6\,\beta^3\,\gamma^2\,R_{old}\,\sigma_1\,\sigma_{old}^2 - 6480\,a^4\,\beta^2\,\gamma^3\,\sigma_1^2\,\sigma_{old}^2 + 12\,960\,a^5\,\beta^2\,\gamma^3\,\sigma_1^2\,\sigma_{old}^2 -$$
$$6480\,a^6\,\beta^2\,\gamma^3\,\sigma_1^2\,\sigma_{old}^2 + 3456\,a^2\,\beta^2\,\gamma^3\,\sigma_{old}^4 - 13\,824\,a^3\,\beta^2\,\gamma^3\,\sigma_{old}^4 +$$
$$20\,736\,a^4\,\beta^2\,\gamma^3\,\sigma_{old}^4 - 13\,824\,a^5\,\beta^2\,\gamma^3\,\sigma_{old}^4 + 3456\,a^6\,\beta^2\,\gamma^3\,\sigma_{old}^4 +$$
$$3456\,a^2\,\beta^2\,\gamma^2\,R_1\,\sigma_{old}^4 - 13\,824\,a^3\,\beta^2\,\gamma^2\,R_1\,\sigma_{old}^4 + 20\,736\,a^4\,\beta^2\,\gamma^2\,R_1\,\sigma_{old}^4 -$$
$$13\,824\,a^5\,\beta^2\,\gamma^2\,R_1\,\sigma_{old}^4 + 3456\,a^6\,\beta^2\,\gamma^2\,R_1\,\sigma_{old}^4 + 3456\,a\,\beta^2\,\gamma^2\,R_{old}\,\sigma_{old}^4 -$$
$$17\,280\,a^2\,\beta^2\,\gamma^2\,R_{old}\,\sigma_{old}^4 + 34\,560\,a^3\,\beta^2\,\gamma^2\,R_{old}\,\sigma_{old}^4 - 34\,560\,a^4\,\beta^2\,\gamma^2\,R_{old}\,\sigma_{old}^4 +$$
$$17\,280\,a^5\,\beta^2\,\gamma^2\,R_{old}\,\sigma_{old}^4 - 3456\,a^6\,\beta^2\,\gamma^2\,R_{old}\,\sigma_{old}^4 - 10\,368\,a^2\,\beta\,\gamma^3\,\sigma_1\,\sigma_{old}^4 +$$
$$41\,472\,a^3\,\beta\,\gamma^3\,\sigma_1\,\sigma_{old}^4 - 62\,208\,a^4\,\beta\,\gamma^3\,\sigma_1\,\sigma_{old}^4 + 41\,472\,a^5\,\beta\,\gamma^3\,\sigma_1\,\sigma_{old}^4 -$$
$$10\,368\,a^6\,\beta\,\gamma^3\,\sigma_1\,\sigma_{old}^4 - 3456\,\gamma^3\,\sigma_{old}^6 + 20\,736\,a\,\gamma^3\,\sigma_{old}^6 - 51\,840\,a^2\,\gamma^3\,\sigma_{old}^6 +$$
$$69\,120\,a^3\,\gamma^3\,\sigma_{old}^6 - 51\,840\,a^4\,\gamma^3\,\sigma_{old}^6 + 20\,736\,a^5\,\gamma^3\,\sigma_{old}^6 - 3456\,a^6\,\gamma^3\,\sigma_{old}^6 +$$
$$\sqrt{\Big(\Big(20\,a^6\,\beta^6\,\gamma^3 - 48\,a^6\,\beta^6\,\gamma^2\,R_1 + 384\,a^6\,\beta^6\,\gamma\,R_1^2 + 128\,a^6\,\beta^6\,R_1^3 -}$$
$$48\,a^5\,\beta^6\,\gamma^2\,R_{old} + 48\,a^6\,\beta^6\,\gamma^2\,R_{old} + 768\,a^5\,\beta^6\,\gamma\,R_1\,R_{old} -$$
$$768\,a^6\,\beta^6\,\gamma\,R_1\,R_{old} + 384\,a^5\,\beta^6\,R_1^2\,R_{old} - 384\,a^6\,\beta^6\,R_1^2\,R_{old} +$$
$$384\,a^4\,\beta^6\,\gamma\,R_{old}^2 - 768\,a^5\,\beta^6\,\gamma\,R_{old}^2 + 384\,a^6\,\beta^6\,\gamma\,R_{old}^2 +$$
$$384\,a^4\,\beta^6\,R_1\,R_{old}^2 - 768\,a^5\,\beta^6\,R_1\,R_{old}^2 + 384\,a^6\,\beta^6\,R_1\,R_{old}^2 +$$
$$128\,a^3\,\beta^6\,R_{old}^3 - 384\,a^4\,\beta^6\,R_{old}^3 + 384\,a^5\,\beta^6\,R_{old}^3 - 128\,a^6\,\beta^6\,R_{old}^3 +$$
$$144\,a^6\,\beta^5\,\gamma^3\,\sigma_1 + 288\,a^6\,\beta^5\,\gamma^2\,R_1\,\sigma_1 - 1152\,a^6\,\beta^5\,\gamma\,R_1^2\,\sigma_1 +$$
$$288\,a^5\,\beta^5\,\gamma^2\,R_{old}\,\sigma_1 - 288\,a^6\,\beta^5\,\gamma^2\,R_{old}\,\sigma_1 - 2304\,a^5\,\beta^5\,\gamma\,R_1\,R_{old}\,\sigma_1 +$$
$$2304\,a^6\,\beta^5\,\gamma\,R_1\,R_{old}\,\sigma_1 - 1152\,a^4\,\beta^5\,\gamma\,R_{old}^2\,\sigma_1 + 2304\,a^5\,\beta^5\,\gamma\,R_{old}^2\,\sigma_1 -$$
$$1152\,a^6\,\beta^5\,\gamma\,R_{old}^2\,\sigma_1 - 1080\,a^6\,\beta^4\,\gamma^3\,\sigma_1^2 + 864\,a^6\,\beta^4\,\gamma^2\,R_1\,\sigma_1^2 +$$
$$864\,a^5\,\beta^4\,\gamma^2\,R_{old}\,\sigma_1^2 - 864\,a^6\,\beta^4\,\gamma^2\,R_{old}\,\sigma_1^2 + 432\,a^6\,\beta^3\,\gamma^3\,\sigma_1^3 -$$
$$504\,a^4\,\beta^4\,\gamma^3\,\sigma_{old}^2 + 1008\,a^5\,\beta^4\,\gamma^3\,\sigma_{old}^2 - 504\,a^6\,\beta^4\,\gamma^3\,\sigma_{old}^2 -$$
$$2304\,a^4\,\beta^4\,\gamma^2\,R_1\,\sigma_{old}^2 + 4608\,a^5\,\beta^4\,\gamma^2\,R_1\,\sigma_{old}^2 - 2304\,a^6\,\beta^4\,\gamma^2\,R_1\,\sigma_{old}^2 -$$
$$1152\,a^4\,\beta^4\,\gamma\,R_1^2\,\sigma_{old}^2 + 2304\,a^5\,\beta^4\,\gamma\,R_1^2\,\sigma_{old}^2 - 1152\,a^6\,\beta^4\,\gamma\,R_1^2\,\sigma_{old}^2 -$$
$$2304\,a^3\,\beta^4\,\gamma^2\,R_{old}\,\sigma_{old}^2 + 6912\,a^4\,\beta^4\,\gamma^2\,R_{old}\,\sigma_{old}^2 - 6912\,a^5\,\beta^4\,\gamma^2\,R_{old}\,\sigma_{old}^2 +$$
$$2304\,a^6\,\beta^4\,\gamma^2\,R_{old}\,\sigma_{old}^2 - 2304\,a^3\,\beta^4\,\gamma\,R_1\,R_{old}\,\sigma_{old}^2 +$$
$$6912\,a^4\,\beta^4\,\gamma\,R_1\,R_{old}\,\sigma_{old}^2 - 6912\,a^5\,\beta^4\,\gamma\,R_1\,R_{old}\,\sigma_{old}^2 +$$
$$2304\,a^6\,\beta^4\,\gamma\,R_1\,R_{old}\,\sigma_{old}^2 - 1152\,a^2\,\beta^4\,\gamma\,R_{old}^2\,\sigma_{old}^2 +$$
$$4608\,a^3\,\beta^4\,\gamma\,R_{old}^2\,\sigma_{old}^2 - 6912\,a^4\,\beta^4\,\gamma\,R_{old}^2\,\sigma_{old}^2 + 4608\,a^5\,\beta^4\,\gamma\,R_{old}^2\,\sigma_{old}^2 -$$

$$
\begin{aligned}
&1152\,a^6\,\beta^4\,\gamma\,R_{old}^2\,\sigma_{old}^2 + 3024\,a^4\,\beta^3\,\gamma^3\,\sigma_1\,\sigma_{old}^2 - 6048\,a^5\,\beta^3\,\gamma^3\,\sigma_1\,\sigma_{old}^2 + \\
&3024\,a^6\,\beta^3\,\gamma^3\,\sigma_1\,\sigma_{old}^2 + 6912\,a^4\,\beta^3\,\gamma^2\,R_1\,\sigma_1\,\sigma_{old}^2 - \\
&13\,824\,a^5\,\beta^3\,\gamma^2\,R_1\,\sigma_1\,\sigma_{old}^2 + 6912\,a^6\,\beta^3\,\gamma^2\,R_1\,\sigma_1\,\sigma_{old}^2 + \\
&6912\,a^3\,\beta^3\,\gamma^2\,R_{old}\,\sigma_1\,\sigma_{old}^2 - 20\,736\,a^4\,\beta^3\,\gamma^2\,R_{old}\,\sigma_1\,\sigma_{old}^2 + \\
&20\,736\,a^5\,\beta^3\,\gamma^2\,R_{old}\,\sigma_1\,\sigma_{old}^2 - 6912\,a^6\,\beta^3\,\gamma^2\,R_{old}\,\sigma_1\,\sigma_{old}^2 - \\
&6480\,a^4\,\beta^2\,\gamma^3\,\sigma_1^2\,\sigma_{old}^2 + 12\,960\,a^5\,\beta^2\,\gamma^3\,\sigma_1^2\,\sigma_{old}^2 - 6480\,a^6\,\beta^2\,\gamma^3\,\sigma_1^2\,\sigma_{old}^2 + \\
&3456\,a^2\,\beta^2\,\gamma^3\,\sigma_{old}^4 - 13\,824\,a^3\,\beta^2\,\gamma^3\,\sigma_{old}^4 + 20\,736\,a^4\,\beta^2\,\gamma^3\,\sigma_{old}^4 - \\
&13\,824\,a^5\,\beta^2\,\gamma^3\,\sigma_{old}^4 + 3456\,a^6\,\beta^2\,\gamma^3\,\sigma_{old}^4 + 3456\,a^2\,\beta^2\,\gamma^2\,R_1\,\sigma_{old}^4 - \\
&13\,824\,a^3\,\beta^2\,\gamma^2\,R_1\,\sigma_{old}^4 + 20\,736\,a^4\,\beta^2\,\gamma^2\,R_1\,\sigma_{old}^4 - 13\,824\,a^5\,\beta^2\,\gamma^2\,R_1\,\sigma_{old}^4 + \\
&3456\,a^6\,\beta^2\,\gamma^2\,R_1\,\sigma_{old}^4 + 3456\,a\,\beta^2\,\gamma^2\,R_{old}\,\sigma_{old}^4 - 17\,280\,a^2\,\beta^2\,\gamma^2\,R_{old}\,\sigma_{old}^4 + \\
&34\,560\,a^3\,\beta^2\,\gamma^2\,R_{old}\,\sigma_{old}^4 - 34\,560\,a^4\,\beta^2\,\gamma^2\,R_{old}\,\sigma_{old}^4 + \\
&17\,280\,a^5\,\beta^2\,\gamma^2\,R_{old}\,\sigma_{old}^4 - 3456\,a^6\,\beta^2\,\gamma^2\,R_{old}\,\sigma_{old}^4 - \\
&10\,368\,a^2\,\beta\,\gamma^3\,\sigma_1\,\sigma_{old}^4 + 41\,472\,a^3\,\beta\,\gamma^3\,\sigma_1\,\sigma_{old}^4 - 62\,208\,a^4\,\beta\,\gamma^3\,\sigma_1\,\sigma_{old}^4 + \\
&41\,472\,a^5\,\beta\,\gamma^3\,\sigma_1\,\sigma_{old}^4 - 10\,368\,a^6\,\beta\,\gamma^3\,\sigma_1\,\sigma_{old}^4 - 3456\,\gamma^3\,\sigma_{old}^6 + \\
&20\,736\,a\,\gamma^3\,\sigma_{old}^6 - 51\,840\,a^2\,\gamma^3\,\sigma_{old}^6 + 69\,120\,a^3\,\gamma^3\,\sigma_{old}^6 - \\
&51\,840\,a^4\,\gamma^3\,\sigma_{old}^6 + 20\,736\,a^5\,\gamma^3\,\sigma_{old}^6 - 3456\,a^6\,\gamma^3\,\sigma_{old}^6\Big)^2 + \\
&4\,\Big(18\,a^2\,\beta^2\,\gamma\,\big(a^2\,\beta^2\,\gamma - 6\,a^2\,\beta\,\gamma\,\sigma_1 + 6\,a^2\,\gamma\,\sigma_1^2\big) - \\
&\quad 16\,\big(a^2\,\beta^2\,\gamma + a^2\,\beta^2\,R_1 + a\,\beta^2\,R_{old} - a^2\,\beta^2\,R_{old} - 3\,a^2\,\beta\,\gamma\,\sigma_1 - \\
&\qquad 3\,\gamma\,\sigma_{old}^2 + 6\,a\,\gamma\,\sigma_{old}^2 - 3\,a^2\,\gamma\,\sigma_{old}^2\big)^2\Big)^3\Big)\Big)^{1/3}\Big\},
\end{aligned}
$$

A3.3 Expression of the Third Root for Equation (3.33)

$$
\begin{gathered}
\{m \to \\
\frac{2\,(a^2 \beta^2 \gamma + a^2 \beta^2 R_1 + a \beta^2 R_{old} - a^2 \beta^2 R_{old} - 3 a^2 \beta \gamma \sigma_1 - 3 \gamma \sigma_{old}^2 + 6 a \gamma \sigma_{old}^2 - 3 a^2 \gamma \sigma_{old}^2)}{3 a^2 \beta^2 \gamma} + \\
((1 - i \sqrt{3})\,(18 a^2 \beta^2 \gamma\,(a^2 \beta^2 \gamma - 6 a^2 \beta \gamma \sigma_1 + 6 a^2 \gamma \sigma_1^2) - \\
16\,(a^2 \beta^2 \gamma + a^2 \beta^2 R_1 + a \beta^2 R_{old} - a^2 \beta^2 R_{old} - 3 a^2 \beta \gamma \sigma_1 - 3 \gamma \sigma_{old}^2 + \\
6 a \gamma \sigma_{old}^2 - 3 a^2 \gamma \sigma_{old}^2)^2)) / \\
(6\, 2^{2/3} a^2 \beta^2 \gamma \\
(20 a^6 \beta^6 \gamma^3 - 48 a^6 \beta^6 \gamma^2 R_1 + 384 a^6 \beta^6 \gamma R_1^2 + 128 a^6 \beta^6 R_1^3 - 48 a^5 \beta^6 \gamma^2 R_{old} + \\
48 a^6 \beta^6 \gamma^2 R_{old} + 768 a^5 \beta^6 \gamma R_1 R_{old} - 768 a^6 \beta^6 \gamma R_1 R_{old} + 384 a^5 \beta^6 R_1^2 R_{old} - \\
384 a^6 \beta^6 R_1^2 R_{old} + 384 a^4 \beta^6 \gamma R_{old}^2 - 768 a^5 \beta^6 \gamma R_{old}^2 + 384 a^6 \beta^6 \gamma R_{old}^2 + \\
384 a^4 \beta^6 R_1 R_{old}^2 - 768 a^5 \beta^6 R_1 R_{old}^2 + 384 a^6 \beta^6 R_1 R_{old}^2 + 128 a^3 \beta^6 R_{old}^3 - \\
384 a^4 \beta^6 R_{old}^3 + 384 a^5 \beta^6 R_{old}^3 - 128 a^6 \beta^6 R_{old}^3 + 144 a^6 \beta^5 \gamma^3 \sigma_1 + \\
288 a^6 \beta^5 \gamma^2 R_1 \sigma_1 - 1152 a^6 \beta^5 \gamma R_1^2 \sigma_1 + 288 a^5 \beta^5 \gamma^2 R_{old} \sigma_1 - \\
288 a^6 \beta^5 \gamma^2 R_{old} \sigma_1 - 2304 a^5 \beta^5 \gamma R_1 R_{old} \sigma_1 + 2304 a^6 \beta^5 \gamma R_1 R_{old} \sigma_1 - \\
1152 a^4 \beta^5 \gamma R_{old}^2 \sigma_1 + 2304 a^5 \beta^5 \gamma R_{old}^2 \sigma_1 - 1152 a^6 \beta^5 \gamma R_{old}^2 \sigma_1 - \\
1080 a^6 \beta^4 \gamma^3 \sigma_1^2 + 864 a^6 \beta^4 \gamma^2 R_1 \sigma_1^2 + 864 a^5 \beta^4 \gamma^2 R_{old} \sigma_1^2 - \\
864 a^6 \beta^4 \gamma^2 R_{old} \sigma_1^2 + 432 a^6 \beta^3 \gamma^3 \sigma_1^3 - 504 a^4 \beta^4 \gamma^3 \sigma_{old}^2 + 1008 a^5 \beta^4 \gamma^3 \sigma_{old}^2 - \\
504 a^6 \beta^4 \gamma^3 \sigma_{old}^2 - 2304 a^4 \beta^4 \gamma^2 R_1 \sigma_{old}^2 + 4608 a^5 \beta^4 \gamma^2 R_1 \sigma_{old}^2 - \\
2304 a^6 \beta^4 \gamma^2 R_1 \sigma_{old}^2 - 1152 a^4 \beta^4 \gamma R_1^2 \sigma_{old}^2 + 2304 a^5 \beta^4 \gamma R_1^2 \sigma_{old}^2 - \\
1152 a^6 \beta^4 \gamma R_1^2 \sigma_{old}^2 - 2304 a^3 \beta^4 \gamma^2 R_{old} \sigma_{old}^2 + 6912 a^4 \beta^4 \gamma^2 R_{old} \sigma_{old}^2 - \\
6912 a^5 \beta^4 \gamma^2 R_{old} \sigma_{old}^2 + 2304 a^6 \beta^4 \gamma^2 R_{old} \sigma_{old}^2 - 2304 a^3 \beta^4 \gamma R_1 R_{old} \sigma_{old}^2 + \\
6912 a^4 \beta^4 \gamma R_1 R_{old} \sigma_{old}^2 - 6912 a^5 \beta^4 \gamma R_1 R_{old} \sigma_{old}^2 + 2304 a^6 \beta^4 \gamma R_1 R_{old} \sigma_{old}^2 - \\
1152 a^2 \beta^4 \gamma R_{old}^2 \sigma_{old}^2 + 4608 a^3 \beta^4 \gamma R_{old}^2 \sigma_{old}^2 - 6912 a^4 \beta^4 \gamma R_{old}^2 \sigma_{old}^2 + \\
4608 a^5 \beta^4 \gamma R_{old}^2 \sigma_{old}^2 - 1152 a^6 \beta^4 \gamma R_{old}^2 \sigma_{old}^2 + 3024 a^4 \beta^3 \gamma^3 \sigma_1 \sigma_{old}^2 - \\
6048 a^5 \beta^3 \gamma^3 \sigma_1 \sigma_{old}^2 + 3024 a^6 \beta^3 \gamma^3 \sigma_1 \sigma_{old}^2 + 6912 a^4 \beta^3 \gamma^2 R_1 \sigma_1 \sigma_{old}^2 - \\
13\,824 a^5 \beta^3 \gamma^2 R_1 \sigma_1 \sigma_{old}^2 + 6912 a^6 \beta^3 \gamma^2 R_1 \sigma_1 \sigma_{old}^2 + 6912 a^3 \beta^3 \gamma^2 R_{old} \sigma_1 \sigma_{old}^2 - \\
20\,736 a^4 \beta^3 \gamma^2 R_{old} \sigma_1 \sigma_{old}^2 + 20\,736 a^5 \beta^3 \gamma^2 R_{old} \sigma_1 \sigma_{old}^2 - \\
6912 a^6 \beta^3 \gamma^2 R_{old} \sigma_1 \sigma_{old}^2 - 6480 a^4 \beta^2 \gamma^3 \sigma_1^2 \sigma_{old}^2 + 12\,960 a^5 \beta^2 \gamma^3 \sigma_1^2 \sigma_{old}^2 - \\
6480 a^6 \beta^2 \gamma^3 \sigma_1^2 \sigma_{old}^2 + 3456 a^2 \beta^2 \gamma^3 \sigma_{old}^4 - 13\,824 a^3 \beta^2 \gamma^3 \sigma_{old}^4 + \\
20\,736 a^4 \beta^2 \gamma^3 \sigma_{old}^4 - 13\,824 a^5 \beta^2 \gamma^3 \sigma_{old}^4 + 3456 a^6 \beta^2 \gamma^3 \sigma_{old}^4 + \\
3456 a^2 \beta^2 \gamma^2 R_1 \sigma_{old}^4 - 13\,824 a^3 \beta^2 \gamma^2 R_1 \sigma_{old}^4 + 20\,736 a^4 \beta^2 \gamma^2 R_1 \sigma_{old}^4 - \\
13\,824 a^5 \beta^2 \gamma^2 R_1 \sigma_{old}^4 + 3456 a^6 \beta^2 \gamma^2 R_1 \sigma_{old}^4 + 3456 a \beta^2 \gamma^2 R_{old} \sigma_{old}^4 - \\
17\,280 a^2 \beta^2 \gamma^2 R_{old} \sigma_{old}^4 + 34\,560 a^3 \beta^2 \gamma^2 R_{old} \sigma_{old}^4 - 34\,560 a^4 \beta^2 \gamma^2 R_{old} \sigma_{old}^4 + \\
17\,280 a^5 \beta^2 \gamma^2 R_{old} \sigma_{old}^4 - 3456 a^6 \beta^2 \gamma^2 R_{old} \sigma_{old}^4 - 10\,368 a^2 \beta \gamma^3 \sigma_1 \sigma_{old}^4 + \\
41\,472 a^3 \beta \gamma^3 \sigma_1 \sigma_{old}^4 - 62\,208 a^4 \beta \gamma^3 \sigma_1 \sigma_{old}^4 + 41\,472 a^5 \beta \gamma^3 \sigma_1 \sigma_{old}^4 - \\
10\,368 a^6 \beta \gamma^3 \sigma_1 \sigma_{old}^4 - 3456 \gamma^3 \sigma_{old}^6 + 20\,736 a \gamma^3 \sigma_{old}^6 - 51\,840 a^2 \gamma^3 \sigma_{old}^6 + \\
69\,120 a^3 \gamma^3 \sigma_{old}^6 - 51\,840 a^4 \gamma^3 \sigma_{old}^6 + 20\,736 a^5 \gamma^3 \sigma_{old}^6 - 3456 a^6 \gamma^3 \sigma_{old}^6 + \\
\sqrt{}((20 a^6 \beta^6 \gamma^3 - 48 a^6 \beta^6 \gamma^2 R_1 + 384 a^6 \beta^6 \gamma R_1^2 + 128 a^6 \beta^6 R_1^3 - \\
48 a^5 \beta^6 \gamma^2 R_{old} + 48 a^6 \beta^6 \gamma^2 R_{old} + 768 a^5 \beta^6 \gamma R_1 R_{old} - \\
768 a^6 \beta^6 \gamma R_1 R_{old} + 384 a^5 \beta^6 R_1^2 R_{old} - 384 a^6 \beta^6 R_1^2 R_{old} +
\end{gathered}
$$

$$384 a^4 \beta^6 \gamma R_{\text{old}}^2 - 768 a^5 \beta^6 \gamma R_{\text{old}}^2 + 384 a^6 \beta^6 \gamma R_{\text{old}}^2 +$$
$$384 a^4 \beta^6 R_1 R_{\text{old}}^2 - 768 a^5 \beta^6 R_1 R_{\text{old}}^2 + 384 a^6 \beta^6 R_1 R_{\text{old}}^2 +$$
$$128 a^3 \beta^6 R_{\text{old}}^3 - 384 a^4 \beta^6 R_{\text{old}}^3 + 384 a^5 \beta^6 R_{\text{old}}^3 - 128 a^6 \beta^6 R_{\text{old}}^3 +$$
$$144 a^6 \beta^5 \gamma^3 \sigma_1 + 288 a^6 \beta^5 \gamma^2 R_1 \sigma_1 - 1152 a^6 \beta^5 \gamma R_1^2 \sigma_1 +$$
$$288 a^5 \beta^5 \gamma^2 R_{\text{old}} \sigma_1 - 288 a^6 \beta^5 \gamma^2 R_{\text{old}} \sigma_1 - 2304 a^5 \beta^5 \gamma R_1 R_{\text{old}} \sigma_1 +$$
$$2304 a^6 \beta^5 \gamma R_1 R_{\text{old}} \sigma_1 - 1152 a^4 \beta^5 \gamma R_{\text{old}}^2 \sigma_1 + 2304 a^5 \beta^5 \gamma R_{\text{old}}^2 \sigma_1 -$$
$$1152 a^6 \beta^5 \gamma R_{\text{old}}^2 \sigma_1 - 1080 a^6 \beta^4 \gamma^3 \sigma_1^2 + 864 a^6 \beta^4 \gamma^2 R_1 \sigma_1^2 +$$
$$864 a^5 \beta^4 \gamma^2 R_{\text{old}} \sigma_1^2 - 864 a^6 \beta^4 \gamma^2 R_{\text{old}} \sigma_1^2 + 432 a^6 \beta^3 \gamma^3 \sigma_1^3 -$$
$$504 a^4 \beta^4 \gamma^3 \sigma_{\text{old}}^2 + 1008 a^5 \beta^4 \gamma^3 \sigma_{\text{old}}^2 - 504 a^6 \beta^4 \gamma^3 \sigma_{\text{old}}^2 -$$
$$2304 a^4 \beta^4 \gamma^2 R_1 \sigma_{\text{old}}^2 + 4608 a^5 \beta^4 \gamma^2 R_1 \sigma_{\text{old}}^2 - 2304 a^6 \beta^4 \gamma^2 R_1 \sigma_{\text{old}}^2 -$$
$$1152 a^4 \beta^4 \gamma R_1^2 \sigma_{\text{old}}^2 + 2304 a^5 \beta^4 \gamma R_1^2 \sigma_{\text{old}}^2 - 1152 a^6 \beta^4 \gamma R_1^2 \sigma_{\text{old}}^2 -$$
$$2304 a^3 \beta^4 \gamma^2 R_{\text{old}} \sigma_{\text{old}}^2 + 6912 a^4 \beta^4 \gamma^2 R_{\text{old}} \sigma_{\text{old}}^2 -$$
$$6912 a^5 \beta^4 \gamma^2 R_{\text{old}} \sigma_{\text{old}}^2 + 2304 a^6 \beta^4 \gamma^2 R_{\text{old}} \sigma_{\text{old}}^2 -$$
$$2304 a^3 \beta^4 \gamma R_1 R_{\text{old}} \sigma_{\text{old}}^2 + 6912 a^4 \beta^4 \gamma R_1 R_{\text{old}} \sigma_{\text{old}}^2 -$$
$$6912 a^5 \beta^4 \gamma R_1 R_{\text{old}} \sigma_{\text{old}}^2 + 2304 a^6 \beta^4 \gamma R_1 R_{\text{old}} \sigma_{\text{old}}^2 -$$
$$1152 a^2 \beta^4 \gamma R_{\text{old}}^2 \sigma_{\text{old}}^2 + 4608 a^3 \beta^4 \gamma R_{\text{old}}^2 \sigma_{\text{old}}^2 - 6912 a^4 \beta^4 \gamma R_{\text{old}}^2 \sigma_{\text{old}}^2 +$$
$$4608 a^5 \beta^4 \gamma R_{\text{old}}^2 \sigma_{\text{old}}^2 - 1152 a^6 \beta^4 \gamma R_{\text{old}}^2 \sigma_{\text{old}}^2 + 3024 a^4 \beta^3 \gamma^3 \sigma_1 \sigma_{\text{old}}^2 -$$
$$6048 a^5 \beta^3 \gamma^3 \sigma_1 \sigma_{\text{old}}^2 + 3024 a^6 \beta^3 \gamma^3 \sigma_1 \sigma_{\text{old}}^2 + 6912 a^4 \beta^3 \gamma^2 R_1 \sigma_1 \sigma_{\text{old}}^2 -$$
$$13\,824 a^5 \beta^3 \gamma^2 R_1 \sigma_1 \sigma_{\text{old}}^2 + 6912 a^6 \beta^3 \gamma^2 R_1 \sigma_1 \sigma_{\text{old}}^2 +$$
$$6912 a^3 \beta^3 \gamma^2 R_{\text{old}} \sigma_1 \sigma_{\text{old}}^2 - 20\,736 a^4 \beta^3 \gamma^2 R_{\text{old}} \sigma_1 \sigma_{\text{old}}^2 +$$
$$20\,736 a^5 \beta^3 \gamma^2 R_{\text{old}} \sigma_1 \sigma_{\text{old}}^2 - 6912 a^6 \beta^3 \gamma^2 R_{\text{old}} \sigma_1 \sigma_{\text{old}}^2 -$$
$$6480 a^4 \beta^2 \gamma^3 \sigma_1^2 \sigma_{\text{old}}^2 + 12\,960 a^5 \beta^2 \gamma^3 \sigma_1^2 \sigma_{\text{old}}^2 - 6480 a^6 \beta^2 \gamma^3 \sigma_1^2 \sigma_{\text{old}}^2 +$$
$$3456 a^2 \beta^2 \gamma^3 \sigma_{\text{old}}^4 - 13\,824 a^3 \beta^2 \gamma^3 \sigma_{\text{old}}^4 + 20\,736 a^4 \beta^2 \gamma^3 \sigma_{\text{old}}^4 -$$
$$13\,824 a^5 \beta^2 \gamma^3 \sigma_{\text{old}}^4 + 3456 a^6 \beta^2 \gamma^3 \sigma_{\text{old}}^4 + 3456 a^2 \beta^2 \gamma^2 R_1 \sigma_{\text{old}}^4 -$$
$$13\,824 a^3 \beta^2 \gamma^2 R_1 \sigma_{\text{old}}^4 + 20\,736 a^4 \beta^2 \gamma^2 R_1 \sigma_{\text{old}}^4 -$$
$$13\,824 a^5 \beta^2 \gamma^2 R_1 \sigma_{\text{old}}^4 + 3456 a^6 \beta^2 \gamma^2 R_1 \sigma_{\text{old}}^4 + 3456 a \beta^2 \gamma^2 R_{\text{old}} \sigma_{\text{old}}^4 -$$
$$17\,280 a^2 \beta^2 \gamma^2 R_{\text{old}} \sigma_{\text{old}}^4 + 34\,560 a^3 \beta^2 \gamma^2 R_{\text{old}} \sigma_{\text{old}}^4 -$$
$$34\,560 a^4 \beta^2 \gamma^2 R_{\text{old}} \sigma_{\text{old}}^4 + 17\,280 a^5 \beta^2 \gamma^2 R_{\text{old}} \sigma_{\text{old}}^4 -$$
$$3456 a^6 \beta^2 \gamma^2 R_{\text{old}} \sigma_{\text{old}}^4 - 10\,368 a^2 \beta \gamma^3 \sigma_1 \sigma_{\text{old}}^4 + 41\,472 a^3 \beta \gamma^3 \sigma_1 \sigma_{\text{old}}^4 -$$
$$62\,208 a^4 \beta \gamma^3 \sigma_1 \sigma_{\text{old}}^4 + 41\,472 a^5 \beta \gamma^3 \sigma_1 \sigma_{\text{old}}^4 - 10\,368 a^6 \beta \gamma^3 \sigma_1 \sigma_{\text{old}}^4 -$$
$$3456 \gamma^3 \sigma_{\text{old}}^6 + 20\,736 a \gamma^3 \sigma_{\text{old}}^6 - 51\,840 a^2 \gamma^3 \sigma_{\text{old}}^6 +$$
$$69\,120 a^3 \gamma^3 \sigma_{\text{old}}^6 - 51\,840 a^4 \gamma^3 \sigma_{\text{old}}^6 + 20\,736 a^5 \gamma^3 \sigma_{\text{old}}^6 -$$
$$3456 a^6 \gamma^3 \sigma_{\text{old}}^6 \Big)^2 +$$
$$4 \Big(18 a^2 \beta^2 \gamma \left(a^2 \beta^2 \gamma - 6 a^2 \beta \gamma \sigma_1 + 6 a^2 \gamma \sigma_1^2\right) -$$
$$16 \Big(a^2 \beta^2 \gamma + a^2 \beta^2 R_1 + a \beta^2 R_{\text{old}} - a^2 \beta^2 R_{\text{old}} - 3 a^2 \beta \gamma \sigma_1 -$$
$$3 \gamma \sigma_{\text{old}}^2 + 6 a \gamma \sigma_{\text{old}}^2 - 3 a^2 \gamma \sigma_{\text{old}}^2 \Big)^2 \Big)^3 \Big)\Big)\Big)^{1/3} \Big) -$$
$$\frac{1}{12\, 2^{1/3} a^2 \beta^2 \gamma}$$
$$\left(1 + i \sqrt{3}\right)$$
$$\Big(20 a^6 \beta^6 \gamma^3 - 48 a^6 \beta^6 \gamma^2 R_1 + 384 a^6 \beta^6 \gamma R_1^2 + 128 a^6 \beta^6 R_1^3 - 48 a^5 \beta^6 \gamma^2 R_{\text{old}} +$$
$$48 a^6 \beta^6 \gamma^2 R_{\text{old}} + 768 a^5 \beta^6 \gamma R_1 R_{\text{old}} - 768 a^6 \beta^6 \gamma R_1 R_{\text{old}} + 384 a^5 \beta^6 R_1^2 R_{\text{old}} -$$
$$384 a^6 \beta^6 R_1^2 R_{\text{old}} + 384 a^4 \beta^6 \gamma R_{\text{old}}^2 - 768 a^5 \beta^6 \gamma R_{\text{old}}^2 + 384 a^6 \beta^6 \gamma R_{\text{old}}^2 +$$
$$384 a^4 \beta^6 R_{\text{old}}^3 + 384 a^5 \beta^6 R_{\text{old}}^3 - 128 a^6 \beta^6 R_{\text{old}}^3 + 144 a^6 \beta^5 \gamma^3 \sigma_1 +$$
$$288 a^6 \beta^5 \gamma^2 R_1 \sigma_1 - 1152 a^6 \beta^5 \gamma R_1^2 \sigma_1 + 288 a^5 \beta^5 \gamma^2 R_{\text{old}} \sigma_1 -$$

$$288\, a^6 \beta^5 \gamma^2 R_{old}\, \sigma_1 - 2304\, a^5 \beta^5 \gamma R_1 R_{old}\, \sigma_1 + 2304\, a^6 \beta^5 \gamma R_1 R_{old}\, \sigma_1 -$$

$$1152\, a^4 \beta^5 \gamma R_{old}^2 \sigma_1 + 2304\, a^5 \beta^5 \gamma R_{old}^2 \sigma_1 - 1152\, a^6 \beta^5 \gamma R_{old}^2 \sigma_1 -$$

$$1080\, a^6 \beta^4 \gamma^3 \sigma_1^2 + 864\, a^6 \beta^4 \gamma^2 R_1 \sigma_1^2 + 864\, a^5 \beta^4 \gamma^2 R_{old}\, \sigma_1^2 -$$

$$864\, a^6 \beta^4 \gamma^2 R_{old}\, \sigma_1^2 + 432\, a^6 \beta^3 \gamma^3 \sigma_1^3 - 504\, a^4 \beta^4 \gamma^3 \sigma_{old}^2 + 1008\, a^5 \beta^4 \gamma^3 \sigma_{old}^2 -$$

$$504\, a^6 \beta^4 \gamma^3 \sigma_{old}^2 - 2304\, a^4 \beta^4 \gamma^2 R_1 \sigma_{old}^2 + 4608\, a^5 \beta^4 \gamma^2 R_1 \sigma_{old}^2 -$$

$$2304\, a^6 \beta^4 \gamma^2 R_1 \sigma_{old}^2 - 1152\, a^4 \beta^4 \gamma R_1^2 \sigma_{old}^2 + 2304\, a^5 \beta^4 \gamma R_1^2 \sigma_{old}^2 -$$

$$1152\, a^6 \beta^4 \gamma R_1^2 \sigma_{old}^2 - 2304\, a^3 \beta^4 \gamma^2 R_{old}\, \sigma_{old}^2 + 6912\, a^4 \beta^4 \gamma^2 R_{old}\, \sigma_{old}^2 -$$

$$6912\, a^5 \beta^4 \gamma^2 R_{old}\, \sigma_{old}^2 + 2304\, a^6 \beta^4 \gamma^2 R_{old}\, \sigma_{old}^2 - 2304\, a^3 \beta^4 \gamma R_1 R_{old}\, \sigma_{old}^2 +$$

$$6912\, a^4 \beta^4 \gamma R_1 R_{old}\, \sigma_{old}^2 - 6912\, a^5 \beta^4 \gamma R_1 R_{old}\, \sigma_{old}^2 + 2304\, a^6 \beta^4 \gamma R_1 R_{old}\, \sigma_{old}^2 -$$

$$1152\, a^2 \beta^4 \gamma R_{old}^2 \sigma_{old}^2 + 4608\, a^3 \beta^4 \gamma R_{old}^2 \sigma_{old}^2 - 6912\, a^4 \beta^4 \gamma R_{old}^2 \sigma_{old}^2 +$$

$$4608\, a^5 \beta^4 \gamma R_{old}^2 \sigma_{old}^2 - 1152\, a^6 \beta^4 \gamma R_{old}^2 \sigma_{old}^2 + 3024\, a^4 \beta^3 \gamma^3 \sigma_1 \sigma_{old}^2 -$$

$$6048\, a^5 \beta^3 \gamma^3 \sigma_1 \sigma_{old}^2 + 3024\, a^6 \beta^3 \gamma^3 \sigma_1 \sigma_{old}^2 + 6912\, a^4 \beta^3 \gamma^2 R_1 \sigma_1 \sigma_{old}^2 -$$

$$13\,824\, a^5 \beta^3 \gamma^2 R_1 \sigma_1 \sigma_{old}^2 + 6912\, a^6 \beta^3 \gamma^2 R_1 \sigma_1 \sigma_{old}^2 + 6912\, a^3 \beta^3 \gamma^2 R_{old}\, \sigma_1 \sigma_{old}^2 -$$

$$20\,736\, a^4 \beta^3 \gamma^2 R_{old}\, \sigma_1 \sigma_{old}^2 + 20\,736\, a^5 \beta^3 \gamma^2 R_{old}\, \sigma_1 \sigma_{old}^2 -$$

$$6912\, a^6 \beta^3 \gamma^2 R_{old}\, \sigma_1 \sigma_{old}^2 - 6480\, a^4 \beta^2 \gamma^3 \sigma_1^2 \sigma_{old}^2 + 12\,960\, a^5 \beta^2 \gamma^3 \sigma_1^2 \sigma_{old}^2 -$$

$$6480\, a^6 \beta^2 \gamma^3 \sigma_1^2 \sigma_{old}^2 + 3456\, a^2 \beta^2 \gamma^3 \sigma_{old}^4 - 13\,824\, a^3 \beta^2 \gamma^3 \sigma_{old}^4 +$$

$$20\,736\, a^4 \beta^2 \gamma^3 \sigma_{old}^4 - 13\,824\, a^5 \beta^2 \gamma^3 \sigma_{old}^4 + 3456\, a^6 \beta^2 \gamma^3 \sigma_{old}^4 +$$

$$3456\, a^2 \beta^2 \gamma^2 R_1 \sigma_{old}^4 - 13\,824\, a^3 \beta^2 \gamma^2 R_1 \sigma_{old}^4 + 20\,736\, a^4 \beta^2 \gamma^2 R_1 \sigma_{old}^4 -$$

$$13\,824\, a^5 \beta^2 \gamma^2 R_1 \sigma_{old}^4 + 3456\, a^6 \beta^2 \gamma^2 R_1 \sigma_{old}^4 + 3456\, a\, \beta^2 \gamma^2 R_{old}\, \sigma_{old}^4 -$$

$$17\,280\, a^2 \beta^2 \gamma^2 R_{old}\, \sigma_{old}^4 + 34\,560\, a^3 \beta^2 \gamma^2 R_{old}\, \sigma_{old}^4 - 34\,560\, a^4 \beta^2 \gamma^2 R_{old}\, \sigma_{old}^4 +$$

$$17\,280\, a^5 \beta^2 \gamma^2 R_{old}\, \sigma_{old}^4 - 3456\, a^6 \beta^2 \gamma^2 R_{old}\, \sigma_{old}^4 - 10\,368\, a^2 \beta \gamma^3 \sigma_1 \sigma_{old}^4 +$$

$$41\,472\, a^3 \beta \gamma^3 \sigma_1 \sigma_{old}^4 - 62\,208\, a^4 \beta \gamma^3 \sigma_1 \sigma_{old}^4 + 41\,472\, a^5 \beta \gamma^3 \sigma_1 \sigma_{old}^4 -$$

$$10\,368\, a^6 \beta \gamma^3 \sigma_1 \sigma_{old}^4 - 3456\, \gamma^3 \sigma_{old}^6 + 20\,736\, a\, \gamma^3 \sigma_{old}^6 - 51\,840\, a^2 \gamma^3 \sigma_{old}^6 +$$

$$69\,120\, a^3 \gamma^3 \sigma_{old}^6 - 51\,840\, a^4 \gamma^3 \sigma_{old}^6 + 20\,736\, a^5 \gamma^3 \sigma_{old}^6 - 3456\, a^6 \gamma^3 \sigma_{old}^6 +$$

$$\sqrt{}\Big(\Big(20\, a^6 \beta^6 \gamma^3 - 48\, a^6 \beta^6 \gamma^2 R_1 + 384\, a^6 \beta^6 \gamma R_1^2 + 128\, a^6 \beta^6 R_1^3 -$$

$$48\, a^5 \beta^6 \gamma^2 R_{old} + 48\, a^6 \beta^6 \gamma^2 R_{old} + 768\, a^5 \beta^6 \gamma R_1 R_{old} -$$

$$768\, a^6 \beta^6 \gamma R_1 R_{old} + 384\, a^5 \beta^6 R_1^2 R_{old} - 384\, a^6 \beta^6 R_1^2 R_{old} +$$

$$384\, a^4 \beta^6 \gamma R_{old}^2 - 768\, a^5 \beta^6 \gamma R_{old}^2 + 384\, a^6 \beta^6 \gamma R_{old}^2 +$$

$$384\, a^4 \beta^6 R_1 R_{old}^2 - 768\, a^5 \beta^6 R_1 R_{old}^2 + 384\, a^6 \beta^6 R_1 R_{old}^2 +$$

$$128\, a^3 \beta^6 R_{old}^3 - 384\, a^4 \beta^6 R_{old}^3 + 384\, a^5 \beta^6 R_{old}^3 - 128\, a^6 \beta^6 R_{old}^3 +$$

$$144\, a^6 \beta^5 \gamma^3 \sigma_1 + 288\, a^6 \beta^5 \gamma^2 R_1 \sigma_1 - 1152\, a^6 \beta^5 \gamma R_1^2 \sigma_1 +$$

$$288\, a^5 \beta^5 \gamma^2 R_{old}\, \sigma_1 - 288\, a^6 \beta^5 \gamma^2 R_{old}\, \sigma_1 - 2304\, a^5 \beta^5 \gamma R_1 R_{old}\, \sigma_1 +$$

$$2304\, a^6 \beta^5 \gamma R_1 R_{old}\, \sigma_1 - 1152\, a^4 \beta^5 \gamma R_{old}^2 \sigma_1 + 2304\, a^5 \beta^5 \gamma R_{old}^2 \sigma_1 -$$

$$1152\, a^6 \beta^5 \gamma R_{old}^2 \sigma_1 - 1080\, a^6 \beta^4 \gamma^3 \sigma_1^2 + 864\, a^6 \beta^4 \gamma^2 R_1 \sigma_1^2 +$$

$$864\, a^5 \beta^4 \gamma^2 R_{old}\, \sigma_1^2 - 864\, a^6 \beta^4 \gamma^2 R_{old}\, \sigma_1^2 + 432\, a^6 \beta^3 \gamma^3 \sigma_1^3 -$$

$$504\, a^4 \beta^4 \gamma^3 \sigma_{old}^2 + 1008\, a^5 \beta^4 \gamma^3 \sigma_{old}^2 - 504\, a^6 \beta^4 \gamma^3 \sigma_{old}^2 -$$

$$2304\, a^4 \beta^4 \gamma^2 R_1 \sigma_{old}^2 + 4608\, a^5 \beta^4 \gamma^2 R_1 \sigma_{old}^2 - 2304\, a^6 \beta^4 \gamma^2 R_1 \sigma_{old}^2 -$$

$$1152\, a^4 \beta^4 \gamma R_1^2 \sigma_{old}^2 + 2304\, a^5 \beta^4 \gamma R_1^2 \sigma_{old}^2 - 1152\, a^6 \beta^4 \gamma R_1^2 \sigma_{old}^2 -$$

$$2304\, a^3 \beta^4 \gamma^2 R_{old}\, \sigma_{old}^2 + 6912\, a^4 \beta^4 \gamma^2 R_{old}\, \sigma_{old}^2 - 6912\, a^5 \beta^4 \gamma^2 R_{old}\, \sigma_{old}^2 +$$

$$2304\, a^6 \beta^4 \gamma^2 R_{old}\, \sigma_{old}^2 - 2304\, a^3 \beta^4 \gamma R_1 R_{old}\, \sigma_{old}^2 +$$

$$6912\, a^4 \beta^4 \gamma R_1 R_{old}\, \sigma_{old}^2 - 6912\, a^5 \beta^4 \gamma R_1 R_{old}\, \sigma_{old}^2 +$$

$$2304\, a^6 \beta^4 \gamma R_1 R_{old}\, \sigma_{old}^2 - 1152\, a^2 \beta^4 \gamma R_{old}^2 \sigma_{old}^2 +$$

$$4608\, a^3 \beta^4 \gamma R_{old}^2 \sigma_{old}^2 - 6912\, a^4 \beta^4 \gamma R_{old}^2 \sigma_{old}^2 + 4608\, a^5 \beta^4 \gamma R_{old}^2 \sigma_{old}^2 -$$

$$1152\, a^6 \beta^4 \gamma R_{old}^2 \sigma_{old}^2 + 3024\, a^4 \beta^3 \gamma^3 \sigma_1 \sigma_{old}^2 - 6048\, a^5 \beta^3 \gamma^3 \sigma_1 \sigma_{old}^2 +$$

$$
\begin{aligned}
&3024\, a^6 \beta^3 \gamma^3 \sigma_1 \sigma_{old}^2 + 6912\, a^4 \beta^3 \gamma^2 R_1 \sigma_1 \sigma_{old}^2 - \\
&13\,824\, a^5 \beta^3 \gamma^2 R_1 \sigma_1 \sigma_{old}^2 + 6912\, a^6 \beta^3 \gamma^2 R_1 \sigma_1 \sigma_{old}^2 + \\
&6912\, a^3 \beta^3 \gamma^2 R_{old} \sigma_1 \sigma_{old}^2 - 20\,736\, a^4 \beta^3 \gamma^2 R_{old} \sigma_1 \sigma_{old}^2 + \\
&20\,736\, a^5 \beta^3 \gamma^2 R_{old} \sigma_1 \sigma_{old}^2 - 6912\, a^6 \beta^3 \gamma^2 R_{old} \sigma_1 \sigma_{old}^2 - \\
&6480\, a^4 \beta^2 \gamma^3 \sigma_1^2 \sigma_{old}^2 + 12\,960\, a^5 \beta^2 \gamma^3 \sigma_1^2 \sigma_{old}^2 - 6480\, a^6 \beta^2 \gamma^3 \sigma_1^2 \sigma_{old}^2 + \\
&3456\, a^2 \beta^2 \gamma^3 \sigma_{old}^4 - 13\,824\, a^3 \beta^2 \gamma^3 \sigma_{old}^4 + 20\,736\, a^4 \beta^2 \gamma^3 \sigma_{old}^4 - \\
&13\,824\, a^5 \beta^2 \gamma^3 \sigma_{old}^4 + 3456\, a^6 \beta^2 \gamma^3 \sigma_{old}^4 + 3456\, a^2 \beta^2 \gamma^2 R_1 \sigma_{old}^4 - \\
&13\,824\, a^3 \beta^2 \gamma^2 R_1 \sigma_{old}^4 + 20\,736\, a^4 \beta^2 \gamma^2 R_1 \sigma_{old}^4 - 13\,824\, a^5 \beta^2 \gamma^2 R_1 \sigma_{old}^4 + \\
&3456\, a^6 \beta^2 \gamma^2 R_1 \sigma_{old}^4 + 3456\, a \beta^2 \gamma^2 R_{old} \sigma_{old}^4 - 17\,280\, a^2 \beta^2 \gamma^2 R_{old} \sigma_{old}^4 + \\
&34\,560\, a^3 \beta^2 \gamma^2 R_{old} \sigma_{old}^4 - 34\,560\, a^4 \beta^2 \gamma^2 R_{old} \sigma_{old}^4 + \\
&17\,280\, a^5 \beta^2 \gamma^2 R_{old} \sigma_{old}^4 - 3456\, a^6 \beta^2 \gamma^2 R_{old} \sigma_{old}^4 - \\
&10\,368\, a^2 \beta \gamma^3 \sigma_1 \sigma_{old}^4 + 41\,472\, a^3 \beta \gamma^3 \sigma_1 \sigma_{old}^4 - 62\,208\, a^4 \beta \gamma^3 \sigma_1 \sigma_{old}^4 + \\
&41\,472\, a^5 \beta \gamma^3 \sigma_1 \sigma_{old}^4 - 10\,368\, a^6 \beta \gamma^3 \sigma_1 \sigma_{old}^4 - 3456\, \gamma^3 \sigma_{old}^6 + \\
&20\,736\, a \gamma^3 \sigma_{old}^6 - 51\,840\, a^2 \gamma^3 \sigma_{old}^6 + 69\,120\, a^3 \gamma^3 \sigma_{old}^6 - \\
&51\,840\, a^4 \gamma^3 \sigma_{old}^6 + 20\,736\, a^5 \gamma^3 \sigma_{old}^6 - 3456\, a^6 \gamma^3 \sigma_{old}^6 \Big)^2 + \\
&4 \Big(18\, a^2 \beta^2 \gamma \left(a^2 \beta^2 \gamma - 6\, a^2 \beta \gamma \sigma_1 + 6\, a^2 \gamma \sigma_1^2\right) - \\
&\quad 16 \big(a^2 \beta^2 \gamma + a^2 \beta^2 R_1 + a \beta^2 R_{old} - a^2 \beta^2 R_{old} - 3\, a^2 \beta \gamma \sigma_1 - \\
&\qquad 3\, \gamma \sigma_{old}^2 + 6\, a \gamma \sigma_{old}^2 - 3\, a^2 \gamma \sigma_{old}^2\big)^2\Big)^3\Big)\Big)^{1/3}\Big\}\Big\}
\end{aligned}
$$

Chapter 4

The Mean-Variance Approach to Portfolio Improvement

So far, we have developed some criteria to judge the worthiness of adding some new assets to an existing portfolio. As a result, a new portfolio with better performance is obtained. Portfolio improvement is accomplished by acquiring some new assets. In the process of acquiring some new assets, it is necessary to compare the performances of a new constructed portfolio with that of a benchmark, i.e. the existing portfolio. Because people are usually concerned about return and risk when comparing the performances of portfolios and risk can be measured by the variance of the portfolio returns, we take both return and variance into account in solving the portfolio improvement problem. This chapter looks into developing a model to solve the portfolio improvement problem in a single period by applying the mean-variance analysis. More specifically, the portfolio improvement problem is formulated into an optimization problem, in which the variance of the new portfolio is minimized subject to some constraints including a return constraint.

In the settlement of stock trading, transaction fees are charged. Hence, transaction cost is a concern when acquiring some new assets. Moreover, adding too many assets to a portfolio is not practical in view of manageability. Due to the facts, a cardinality constraint is introduced to restrict the number of selected new assets not to be too large in our model.

Before developing models to solve the problem, let us make some assumptions and introduce some notations. Assume that the overall portfolio value does not change, i.e. the portfolio values of the old and new portfolios are the same. The expected return and standard deviation of stock i are denoted by R_i and σ_i, for $i = 1, 2, \ldots, n$. Let x_{old} be the weight of the old portfolio and x_i be the weight of stock i in the new portfolio, for $i = 1, 2, \ldots, n$. For convenience, the weights is represented by a $(n+1)$-dimensional vector, denoted $\mathbf{x}$, where the transpose of $\mathbf{x}$ is

$$\mathbf{x}^T = [x_{old}, x_1, x_2, \cdots, x_n]. \tag{4.1}$$

Similarly, a $(n+1)$-dimensional vector of returns is given by

$$\mathbf{R} = [R_{old}, R_1, R_2, \cdots, R_n]. \tag{4.2}$$

Thus, the expected return on the new portfolio can be expressed as

$$\mu(\mathbf{x}) = R_{old} x_{old} + \sum_{i=1}^{n} R_i x_i = \mathbf{R}\mathbf{x}. \tag{4.3}$$

The old portfolio is treated as a whole and consideration is given to the correlations of the returns on new stocks and the old portfolio. The variance-covariance matrix, denoted by $\mathbf{\Sigma}$, is

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_{old}^2 & \sigma_{old,1} & \sigma_{old,2} & \cdots & \sigma_{old,n} \\ \sigma_{1,old} & \sigma_1^2 & \sigma_{1,2} & \cdots & \sigma_{1,n} \\ \sigma_{2,old} & \sigma_{2,1} & \sigma_2^2 & \cdots & \sigma_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{n,old} & \sigma_{n,1} & \sigma_{n,2} & \cdots & \sigma_n^2 \end{bmatrix}, \tag{4.4}$$

where $\sigma_{i,old} = \sigma_{old,i}$ is the covariance of the returns on new stock i and the old portfolio and $\sigma_{i,j} = \sigma_{j,i}$ is the covariance of the returns on new stocks i and j, for $i, j = 1, 2, \ldots, n;\ i \neq j$. Hence, the variance of the new portfolio return can be expressed as

$$v(\mathbf{x}) = x_{old}^2 \sigma_{old}^2 + \sum_{i=1}^{n} x_i^2 \sigma_i^2 + 2\sum_{i=1}^{n} x_i x_{old} \sigma_{i,old} + 2\sum_{\substack{i,j=1 \\ i \neq j}}^{n} x_i x_j \sigma_{i,j} = \mathbf{x}^T \mathbf{\Sigma} \mathbf{x}, \tag{4.5}$$

and the standard deviation is

$$\sigma(\mathbf{x}) = \sqrt{\mathbf{x}^T \mathbf{\Sigma} \mathbf{x}}. \tag{4.6}$$

By introducing the cardinality constraint for the restriction on the number of selected new stocks, Section 4.1 formulates our problem with an equality cardinality constraint. It discusses the formulation and derives analytical solutions to the problem under some assumptions in Section 4.2. Though we cannot obtain a closed-form solution in the general case, the analytical solutions in some special cases provide some insight for solving the problem. In solving our problem, it is required to pick some stocks from n given stocks. A good stock picking strategy makes the solution time shorter. With this concern, Section 4.3 illustrates a stock picking strategy for picking 2 and 3 stocks. As our problem can be categorized as a Mixed Binary Quadratic Programming (MBQP) problem, some developed optimization solvers can solve such a problem. Section 4.4 shows some procedures to solve our problem by using the Xpress Solver. A numerical example is presented for illustration. Section 4.5 considers another model with an inequality cardinality constraint. Comparison with the model with an equality cardinality constraint is carried out by using a numerical example.

4.1 Problem Formulation with Equality Cardinality Constraint

Investors always prefer a portfolio with higher return or lower risk. Risk is usually measured by the standard deviation or variance. The classical portfolio optimization problem is to minimize the risk of achieving a given level of return. The objective of our formulation is to minimize the variance of the new portfolio subject to some constraints, e.g. the return on the new portfolio must be greater than a desired return, $\bar{R}$. Here, it is recommended to set $\bar{R} \geq R_{old}$ because the spirit of our problem is to improve the return on the existing portfolio.

Under the mean-variance theory, our problem can be formulated as follows:

(MBQP) Minimize $$\frac{1}{2}v(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \mathbf{\Sigma x} \tag{4.7a}$$

Subject to $$\mu(\mathbf{x}) = \mathbf{Rx} \geq \bar{R}, \tag{4.7b}$$

$$x_{old} + \sum_{i=1}^{n} x_i = 1, \tag{4.7c}$$

$$x_{old} \geq a, \tag{4.7d}$$

$$0 \leq x_i \leq y_i, \quad \text{for} \quad i = 1, 2, \ldots, n, \tag{4.7e}$$

$$\sum_{i=1}^{n} y_i = m, \tag{4.7f}$$

$$y_i \in \{0,1\}, \quad \text{for} \quad i = 1, 2, \ldots, n, \tag{4.7g}$$

$$y_i - x_i \leq u_i, \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{4.7h}$$

Here, a is the minimum portion of portfolio value invested in the old portfolio and m is a positive integer number. It is only meaningful to set $m \leq n$. We presume that $0 < a < 1$. Note that y_i is an indicator variable for x_i; if new stock i is selected to be invested in the new portfolio, i.e. $x_i > 0$, then $y_i = 1$; else new stock i is not selected, then $x_i = 0$ and $y_i = 0$. Thus, for all i, y_i is a binary variable; it is equal to either 0 or 1. The constraint $x_{old} + \sum_{i=1}^{n} x_i = 1$ restricts that a portion of portfolio value is invested in the old portfolio and the remaining portions are invested in some new stocks. It follows from (4.7c) and (4.7d) that $\sum_{i=1}^{n} x_i \le 1 - a$, i.e. the total portions invested in the new stocks do not exceed $1 - a$. By setting $x_i \ge 0$ for all i, short selling is not allowed in the problem (MBQP). With the restrictions $x_i \le y_i$ for all i, x_i is either equal to 0 or a percentage (between 0 and 1). Moreover, we introduce linear constraints (4.7h) to restrict x_i not to be too small. When x_i is too small, we treat it as zero. Notice that u_i is a constant and $0 < u_i < 1$. For example, when $u_i = 0.99$, x_i is set to be not less than 1% for those selected stocks.

In problem (MBQP), we introduce an equality cardinality constraint $\sum_{i=1}^{n} y_i = m$. It specifies that the total number of new stocks invested in the new portfolio must be exactly equal to m. In other words, it is required to choose exactly m stocks from n given new stocks. There are ${}_nC_m$ combinations for selection. For a fixed value of m, as the value of n increases, the number of combinations increases exponentially. Thus, the computational time required to solve the problem will increase exponentially. For a fixed value of n, the value of m also affects the computational time required to solve the problem. Moreover, transaction costs increase as more stocks are traded. Because of the concern in computational time and cost, m is requested to be a small number in practice.

In the formulation of problem (MBQP), we intend to minimize the variance of the portfolio return subject to some equality and inequality constraints including a return constraint, i.e. $\mathbf{Rx} \geq \bar{R}$. For an optimization problem, it is crucial to analyze the characteristics of the objective function and hence identify the problem type in order to seek a suitable solution method. Here, the objective function $v(\mathbf{x})$ is an expression of the variance which is a quadratic function of the decision variables $\mathbf{x}$. Without constraints (4.7d), (4.7e), (4.7f) and (4.7g), our problem can be reduced to the classical Markowitz mean-variance problem which is a convex quadratic programming (QP) problem. As pointed out in Nocedal and Wright (1999), quadratic programs can always be solved (or can be shown to be infeasible) in a finite number of iterations, but the effort required to find a solution depends strongly on the characteristics of the objective function and the number of inequality constraints. It is obvious that the variance-covariance matrix $\mathbf{\Sigma}$ in our objective function is symmetric. It can be shown that $\mathbf{\Sigma}$ is positive semidefinite. In this case, the Markowitz mean-variance problem is a convex QP. Solving such a problem is no much difficult than a linear programming (LP) problem.

However, regarding the constraints (4.7g), the indicator variables y_i, for all i, are restricted to 0-1 values. Moreover, the inclusion of constraints (4.7d), (4.7e) and (4.7f) makes the problem (MBQP) more difficult to be solved than the standard Markowitz mean-variance problem. Actually, our problem is a Mixed Binary Quadratic Programming which can be abbreviated to MBQP, i.e. the name of our problem. For an integer nonlinear programming problem, it can be solved by a branch-and-bound approach that applies a nonlinear solver to successive sub-problems.

4.2 Analytical Solutions to the Problem in Some Special Cases

Before solving our problem (MBQP), it is interesting to discuss the solution in some special cases. Obviously, problem (MBQP) can be reduced to a model similar to the Markowitz model by ignoring some constraints, say (4.7d), (4.7e), (4.7f), (4.7g) and (4.7h), and setting (4.7b) with equality constraint. As stated and demonstrated in Luenberger (1997), for n assets, a system of $n+2$ linear equations can be obtained for efficient set. These equations can be solved with standard methods for $n+2$ unknowns including two Lagrange multipliers and n portfolio weights.

Suppose that there are n assets which are uncorrelated. You may invest in any one, or in any combination of them. The mean rate return $\bar{R}$ is the same for each asset, but the variances are different. The return on asset i has a variance of σ_i^2 for $i = 1, 2, \ldots, n.$ This is the problem stated in Chapter 6 of Luenberger (1997). It is required to express the minimum-variance point in terms of

$$\bar{\sigma}^2 = \left(\sum_{i=1}^{n} \frac{1}{\sigma_i^2} \right)^{-1}. \tag{4.8}$$

It can be shown that the n portfolio weights x_i for $i = 1, 2, \ldots, n$ have the following expression

$$x_i = \frac{\bar{\sigma}^2}{\sigma_i^2}. \tag{4.9}$$

Hence, the minimum variance is equal to $\bar{\sigma}^2$.

Let us consider the following model reduced from problem (MBQP):

(MBQP_R) Minimize $$\frac{1}{2} \sum_{i,j=1}^{n} \sigma_{ij} x_i x_j \tag{4.10a}$$

Subject to $$\sum_{i=1}^{n} R_i x_i = \overline{R}, \tag{4.10b}$$

$$\sum_{i=1}^{n} x_i = 1, \tag{4.10c}$$

$$0 \le x_i \le y_i, \quad \text{for} \quad i = 1, 2, \ldots, n, \tag{4.10d}$$

$$\sum_{i=1}^{n} y_i = m, \tag{4.10e}$$

$$y_i\left(1 - y_i\right) = 0, \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{4.10f}$$

$$y_i - x_i \le u_i, \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{4.10g}$$

Notice that constraint (4.10f) is used instead of constraint (4.7g) to present an algebraic expression of (4.7g) for the ease of derivation. Regarding problem (MBQP_R), it is complicated to determine an analytical solution. For simplicity, some assumptions are addressed here. Suppose the n assets are uncorrelated. The expected returns R_is are all equal to the desired return $\overline{R}$, but the variances are different. Both m and n are fixed. In this case, the number of combinations for portfolio selection, i.e. ${}_nC_m$, is fixed. With the uncorrelated assumption on assets, the variance of the portfolio in the objective function can be simplified and expressed as

$$\sum_{i,j=1}^{n} \sigma_{ij} x_i x_j = \sum_{i=1}^{n} x_i^2 \sigma_i^2. \tag{4.11}$$

Note that under the assumption of equal expected return for all assets, constraint (4.10b) can be simplified and is equivalent to constraint (4.10c). In a manner similar to the solution to the problem in Luenberger (1997), we can also show that

$$x_i = \frac{\overline{\sigma}_I^2}{\sigma_i^2}, \qquad \text{for } i \in I \text{ and } I \subseteq N, \tag{4.12}$$

where $\overline{\sigma}_I^2 = \left(\sum_{i \in I} \frac{1}{\sigma_i^2} \right)^{-1}$ and the set I is the subset of the set $N = \{1, 2, \ldots, n\}$. Note that the cardinality of a set, e.g. S, measures the number of elements of the set and is denoted by $|S|$. It is obvious that $|N| = n$ and $|I| \leq n$. Actually, $|I| = m$ and this specifies that only m assets are chosen from n assets. However, the elements in the set I are still not determined. Suppose the variances are ranked in an ascending order as

$$\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_n.$$

Obviously, the optimal solution is

$$\begin{cases} x_i^* = \dfrac{\overline{\sigma}_I^2}{\sigma_i^2}, & i \in I, I = \{1, 2, \ldots, m\}, \\ x_i^* = 0, & i \in N \setminus I. \end{cases} \tag{4.13}$$

The corresponding minimum variance is equal to

$$\sum_{i=1}^{n} x_i^{*2} \sigma_i^2 = \sum_{i=1}^{m} x_i^{*2} \sigma_i^2 = \overline{\sigma}_I^4 \left(\sum_{i=1}^{m} \frac{1}{\sigma_i^2} \right)^{-1} = \overline{\sigma}_I^2. \tag{4.14}$$

The result is straightforward in common sense that investors always prefer assets with lower risk, say standard deviation, for the same level of return.

By ignoring the assumption of equal mean rate of return for each asset, we consider problem (MBQP_R) without constraints (4.10d) – (4.10g):

(MV_L) Minimize $$\frac{1}{2} \sum_{i,j=1}^{n} \sigma_{ij} x_i x_j \tag{4.15a}$$

Subject to $$\sum_{i=1}^{n} x_i R_i = \overline{R} \tag{4.15b}$$

$$\sum_{i=1}^{n} x_i = 1 \tag{4.15c}$$

Under the uncorrelated assumption on assets, the objective function (4.15a) can be simplified to the function on the right-hand side of (4.11). By introducing two Lagrange multipliers λ and μ, we can obtain

$$L = \frac{1}{2}\sum_{i=1}^{n} x_i^2 \, \sigma_i^2 - \lambda\left(\sum_{i=1}^{n} x_i \, R_i - \overline{R}\right) - \mu\left(\sum_{i=1}^{n} x_i - 1\right). \tag{4.16}$$

Set the derivatives $\partial L = 0,$ which yields a system of equations:

$$\begin{cases} \dfrac{\partial L}{\partial x_i} = x_i \, \sigma_i^2 - \lambda R_i - \mu = 0, \qquad i = 1, \ldots, n, \\ \dfrac{\partial L}{\partial \lambda} = \sum_{i=1}^{n} x_i \, R_i - \overline{R} = 0, \\ \dfrac{\partial L}{\partial \mu} = \sum_{i=1}^{n} x_i - 1 = 0. \end{cases}$$

Solving the system of equations above yields

$$x_i = \frac{\lambda R_i + \mu}{\sigma_i^2}, \tag{4.17}$$

where

$$\lambda = \frac{\overline{R}\left(\sum_{i=1}^{n} \frac{1}{\sigma_i^2}\right) - \left(\sum_{i=1}^{n} \frac{R_i}{\sigma_i^2}\right)}{\left(\sum_{i=1}^{n} \frac{R_i^2}{\sigma_i^2}\right)\left(\sum_{i=1}^{n} \frac{1}{\sigma_i^2}\right) - \left(\sum_{i=1}^{n} \frac{R_i}{\sigma_i^2}\right)^2}, \tag{4.18}$$

and

$$\mu = \frac{\left(\sum_{i=1}^{n} \frac{R_i^2}{\sigma_i^2}\right) - \overline{R}\left(\sum_{i=1}^{n} \frac{R_i}{\sigma_i^2}\right)}{\left(\sum_{i=1}^{n} \frac{R_i^2}{\sigma_i^2}\right)\left(\sum_{i=1}^{n} \frac{1}{\sigma_i^2}\right) - \left(\sum_{i=1}^{n} \frac{R_i}{\sigma_i^2}\right)^2}. \tag{4.19}$$

With regard to the expressions of λ and μ in (4.18) and (4.19) respectively, we can show that they are expressed meaningfully only when the return rates of assets are not all equal. It can be observed explicitly that as all the assets have the same return rate, the same denominator in both (4.18) and (4.19) becomes zero.

The expressions of λ and μ can be simplified by introducing some new notation. Let $A=\sum_{i=1}^{n}\frac{1}{\sigma_i^2}$, $B=\sum_{i=1}^{n}\frac{R_i}{\sigma_i^2}$ and $C=\sum_{i=1}^{n}\frac{R_i^2}{\sigma_i^2}$. It follows from (4.18) and (4.19) that λ and μ can be expressed in terms of A, B and C as

$$\lambda=\frac{\bar{R}A-B}{AC-B^2} \tag{4.20}$$

and

$$\mu=\frac{C-\bar{R}B}{AC-B^2}. \tag{4.21}$$

Recall that the variance of the portfolio returns can be expressed as

$$\begin{aligned}\sum_{i=1}^{n}x_i^2\,\sigma_i^2 &= \sum_{i=1}^{n}\frac{\left(\lambda R_i+\mu\right)^2}{\sigma_i^2}\\ &=\lambda^2\left(\sum_{i=1}^{n}\frac{R_i^2}{\sigma_i^2}\right)+2\lambda\mu\left(\sum_{i=1}^{n}\frac{R_i}{\sigma_i^2}\right)+\mu^2\left(\sum_{i=1}^{n}\frac{1}{\sigma_i^2}\right),\end{aligned} \tag{4.22}$$

which is equivalent to

$$\sum_{i=1}^{n}x_i^2\,\sigma_i^2=C\lambda^2+2B\lambda\mu+A\mu^2. \tag{4.23}$$

The constraint (4.15b) can be rewritten as

$$\sum_{i=1}^{n}x_i\,R_i=\lambda\left(\sum_{i=1}^{n}\frac{R_i^2}{\sigma_i^2}\right)+\mu\left(\sum_{i=1}^{n}\frac{R_i}{\sigma_i^2}\right)=\bar{R},$$

or

$$C\lambda+B\mu=\bar{R}. \tag{4.24}$$

Similarly, constraint (4.15c) can be simplified to

$$B\lambda+A\mu=1. \tag{4.25}$$

It follows from (4.23), (4.24) and (4.25) that

$$\sum_{i=1}^{n} x_i^2 \sigma_i^2 = \lambda \bar{R} + \mu . \tag{4.26}$$

By substituting (4.20) and (4.21) into (4.26), we can obtain

$$\sum_{i=1}^{n} x_i^2 \sigma_i^2 = \frac{A\bar{R}^2 - 2B\bar{R} + C}{AC - B^2}, \tag{4.27}$$

which can be considered as a function of $\bar{R}$ as A, B and C are all constant with given values of R_i s and σ_i^2 s. For different values of $\bar{R}$, formula (4.27) gives different values for minimum variance of the portfolio returns. It can also be shown that without the assumption of no correlation between assets, problem (MV_L) can be solved and the variance of the portfolio returns can be expressed as a function of the desired return $\bar{R}$. See Merton (1972) for a more detailed derivation of the variance of a frontier portfolio in the general case.

4.3 Stock Picking Strategy

The previous subsection has derived some analytical solutions to the problem in some special cases. Intuitively, to solve our problem (MBQP_R) under the uncorrelated assumption on assets, we can construct ${}_nC_m$ combinations of assets, i.e. set I, and then calculate the variance for each portfolio for comparison. Finally, the optimal portfolio is the one with the smallest variance. However, as m and n become large, the number of combinations increases and the computation time for the problem becomes longer. Moreover, one may encounter difficulties in the construction of a well-structured sequence of combinations of assets, e.g. stocks, which can increase the search efficiency for a solution. Since solving our problem gives a portfolio with minimum variance, this section will try to construct a sequence of combinations of stocks in portfolios for efficient search. In general, it is not easy to construct a structured sequence of combinations, in which the variances of the portfolios are in an ascending order. Some assumptions are made in this section to simplify the expression of variance in order to construct an approximately increasing sequence of combinations. Our approach is to consider the unconstrained problem, i.e. the problem (MBQP_R), without the return constraint for a simpler solution first. Then we will check whether the possible portfolios in the sequence of combinations satisfy the return constraint. For those feasible portfolios, the portfolio with the smallest variance is the optimal one.

Let us consider problem (MV_L) again. Suppose all the stocks are uncorrelated. We assume that the return constraint (4.15b) is released and the problem becomes an unconstrained problem. Solving the problem by introducing a Lagrange multiplier μ yields

$$x_i = \frac{\mu}{\sigma_i^2}, \tag{4.28}$$

for $i = 1, 2, \ldots, n,$ where

$$\mu = \left(\sum_{i=1}^{n} \frac{1}{\sigma_i^2} \right)^{-1}. \tag{4.29}$$

Hence, the variance of the portfolio is

$$\sum_{i=1}^{n} x_i^2 \sigma_i^2 = \left(\sum_{i=1}^{n} \frac{1}{\sigma_i^2} \right)^{-1}. \tag{4.30}$$

Under the same assumptions, we can show that for problem (MBQP_R) without constraint (4.10b), the portfolio weights x_i is equivalent to (4.12) and zero otherwise. Hence, the variance of the portfolio can be expressed as $\overline{\sigma}_I^2$. It has been shown in Section 4.2 that in the solution, the first m stocks with smaller standard deviations are selected if all the stocks have the same expected return. However, the solution is not determined without the assumption of equal expected return of stocks.

The following paragraphs concentrate on constructing a structured sequence of combinations. One more assumption is addressed here. Assume that $\sigma_i^2 \propto R_i$, i.e. $\sigma_i^2 = cR_i$, for all i, where c is a constant. The idea behind this assumption is the tradeoff between return and risk, i.e. R_i and σ_i^2, in which a higher risk must be borne for a higher return. Replacing σ_i^2 by cR_i in the expression of the variance of the portfolio, $\overline{\sigma}_I^2$, gives

$$\sum_{i\in I} x_i^2\,\sigma_i^2 = \overline{\sigma}_I^2 = \left(\sum_{i\in I}\frac{1}{cR_i}\right)^{-1} = c\left(\sum_{i\in I}\frac{1}{R_i}\right)^{-1}, \tag{4.31}$$

where $I \subseteq N$ and $N = \{1, 2, \ldots, n\}$. Since c is a constant, to minimize $\sum_{i\in I} x_i^2\,\sigma_i^2$ is equivalent to minimize $\left(\sum_{i\in I}\frac{1}{R_i}\right)^{-1}$.

For specific values of m and n, there are ${}_nC_m$ possible portfolios for comparison. Intuitively, for an efficient portfolio that satisfies the return constraint, at least one of the expected returns of the selected stocks should be greater than the desired return $\overline{R}$. For n given stocks, their expected returns can be calculated and ranked in an ascending order. Suppose $\overline{R}$ is ranked between R_k and R_{k+1}. Now, we have

$$R_1 < R_2 < \cdots < R_k < \overline{R} < R_{k+1} < \cdots < R_n. \tag{4.32}$$

Since smaller values of R_i s give a smaller value of $\left(\sum_{i \in I} \frac{1}{R_i}\right)^{-1}$, we propose to pick $m-1$ stocks from the first k ranked stocks and 1 stock from the last $n-k$ stocks in (4.32), with $m-1 \le k \le n-1$. In this stock picking strategy, the number of combinations becomes ${}_kC_{m-1} \bullet {}_{n-k}C_1 = (n-k) \bullet {}_kC_{m-1}$ which should be less than ${}_nC_m$. For the ease of tracking and comparison, we suppose that the expected returns of stocks, $\{R_1, R_2, \dots, R_n\}$, is an arithmetic sequence. Thus, the expected return of stock i can be expressed in terms of R_1 as in (3.22) for $i = 1, 2, \dots, n$, and $n \ge 1$.

Let us consider the simplest case $m = 2$ first. With given values of n and k, we should pick 1 stock from the first k given stocks and another from the last $n-k$ stocks. It turns out that the number of combinations, i.e. portfolios, is $k \bullet (n-k)$.

The following algorithm demonstrates our stock picking strategy for the case of $m = 2$.

Algorithm 4.1 (Stock Picking Strategy for $m = 2$)

Step 1. (Initialization) k value (must be $m-1 \le k \le n-1$); $i = 1$, $j = k+1$; $\text{exit\_value} = k + n + 1$; $\text{sum\_value} = i + j$.

Step 2. (Selection) If $i \le k$ and $j \le n$, then enter the list and go to Step 3; otherwise, go to Step 3.

Step 3. (Partition) If $j = k+1$, then go to Step 4; else, go to Step 5.

Step 4. (Update_1) $i = i+1$, $\text{sum\_value} = \text{sum\_value} + 1$, $j = \text{sum\_value} - i$. If $\text{sum\_value} = \text{exit\_value}$, then stop; else, go to Step 2.

Step 5. (Update_2) $j = j-1$, $i = \text{sum\_value} - j$; go to Step 2.

The goal of Algorithm 4.1 is to construct a sequential list of possible sets of stock indexes in a two-stock portfolio. In Algorithm 4.1, the sum_value is corresponding to the sum of the indexes of the two constituent stocks, i.e. i and j, in the portfolio. The list is started from the set with the smallest sum_value and ended at the set with the largest sum_value. Since we want to pick 1 stock from the first k given stocks and another from the last $n-k$ stocks, we have $1 \le i \le k$ and $k+1 \le j \le n$. Hence, $k+2 \le \text{sum\_value} \le k+n$.

As an illustration, suppose $n=19$, $m=2$ and $k=7$. The first set to be considered in the list is (9;1,8) where $\text{sum\_value}=9$, $i=1$ and $j=8$. We also have $1 \le i \le 7$, $8 \le j \le 19$ and $9 \le \text{sum\_value} \le 26$. Table 4.1 displays a sequential list of possible sets of stock indexes for the case of $m=2$. The total number of possible sets is $k \bullet (n-k) = 7 \times 12 = 84$. We observe that the sum_value is in a non-descending order. Each set refers to a possible portfolio, e.g. (9;1,8) refers to a portfolio consisting of two stocks with expected returns R_1 and R_8. For the portfolios with the same sum_value in Table 4.1, we suppose that the portfolio in a more 'extreme' set has a smaller variance, so the more 'extreme' set is selected first. Here, 'extreme' means that one of the two constituent stocks in the 'extreme' set has the smallest expected return and the other has the largest expected return among the stocks in the sets with equal sum of expected returns of stocks. For example, for $\text{sum\_value}=12$, the set (12;1,11) is more 'extreme' than the set (12;2,10), (12;2,10) is more 'extreme' than (12;3,9), and so on. It is presumed that the portfolios in Table 4.1 with the same sum_value are sorted by variance in an ascending order. This will be proved in Proposition 4.1.

Table 4.1 List of Possible Sets of Stock Indexes for m = 2

sum_value	i	j	sum_value	i	j	sum_value	i	j
9	1	8	16	1	15	20	1	19
10	1	9	16	2	14	20	2	18
10	2	8	16	3	13	20	3	17
11	1	10	16	4	12	20	4	16
11	2	9	16	5	11	20	5	15
11	3	8	16	6	10	20	6	14
12	1	11	16	7	9	20	7	13
12	2	10	17	1	16	21	2	19
12	3	9	17	2	15	21	3	18
12	4	8	17	3	14	21	4	17
13	1	12	17	4	13	21	5	16
13	2	11	17	5	12	21	6	15
13	3	10	17	6	11	21	7	14
13	4	9	17	7	10	22	3	19
13	5	8	18	1	17	22	4	18
14	1	13	18	2	16	22	5	17
14	2	12	18	3	15	22	6	16
14	3	11	18	4	14	22	7	15
14	4	10	18	5	13	23	4	19
14	5	9	18	6	12	23	5	18
14	6	8	18	7	11	23	6	17
15	1	14	19	1	18	23	7	16
15	2	13	19	2	17	24	5	19
15	3	12	19	3	16	24	6	18
15	4	11	19	4	15	24	7	17
15	5	10	19	5	14	25	6	19
15	6	9	19	6	13	25	7	18
15	7	8	19	7	12	26	7	19

Obviously, if only one of the stock indexes in two sets are different, then the portfolio with smaller sum_value has smaller variance; For instance, the variance of the portfolio in set $(9;1,8)$ is smaller than that in set $(10;1,9)$, and the variance of the portfolio in set $(10;1,9)$ is smaller than that in set $(11;1,10)$. However, for two portfolios with different stock indexes and sum_value s, it is still uncertain which one has smaller variance. In Table 4.1, it is required to check for some pairs of sets, e.g. $(10;2,8)$ and $(11;1,10)$, $(11;3,8)$ and $(12;1,11)$, and so on.

Proposition 4.1 Suppose (R_i, R_j) and $(R_{i'}, R_{j'})$ are two portfolios with two different stocks respectively. Here, $R_i, R_j, R_{i'}, R_{j'}$ are the expected returns of stocks i, j, i', j'. It is given that $0 < R_1 \le R_i < R_{i'} \le R_k < R_{k+1} \le R_{j'} < R_j \le R_n$.

(i) If $R_i + R_j = R_{i'} + R_{j'}$, then

$$\left(\frac{1}{R_{i'}} + \frac{1}{R_{j'}}\right)^{-1} > \left(\frac{1}{R_i} + \frac{1}{R_j}\right)^{-1}. \tag{4.33}$$

(ii) It is given that $R_i, R_j, R_{i'}, R_{j'}$ are in an arithmetic sequence, i.e. $R_i = R_1 + (i-1)\gamma = R_1 + a_i\gamma$, where $a_i = i-1$ and $\gamma > 0$ is the common difference between two terms. If $R_{i'} + R_{j'} < R_i + R_j$ and γ satisfies

$\gamma_2 < 0 < \gamma < \gamma_1$

or

$0 < \gamma_2 < \gamma < \gamma_1$,

where

$$\gamma_1 = \left[\frac{\left(a_i a_j - a_{i'} a_{j'}\right) + \sqrt{\left(a_i - a_{i'}\right)\left(a_j - a_{i'}\right)\left(a_i - a_{j'}\right)\left(a_j - a_{j'}\right)}}{a_{i'} a_{j'}\left(a_i + a_j\right) - a_i a_j\left(a_{i'} + a_{j'}\right)}\right] R_1 \tag{4.34}$$

and

$$\gamma_2 = \left[\frac{\left(a_i a_j - a_{i'} a_{j'}\right) - \sqrt{\left(a_i - a_{i'}\right)\left(a_j - a_{i'}\right)\left(a_i - a_{j'}\right)\left(a_j - a_{j'}\right)}}{a_{i'} a_{j'}\left(a_i + a_j\right) - a_i a_j\left(a_{i'} + a_{j'}\right)}\right] R_1, \tag{4.35}$$

then

$$\left(\frac{1}{R_{i'}}+\frac{1}{R_{j'}}\right)^{-1}<\left(\frac{1}{R_i}+\frac{1}{R_j}\right)^{-1}. \tag{4.36}$$

Proof:

(i) Let $R_{i'}=R_i+d$ and $d>0$. Since $R_i+R_j=R_{i'}+R_{j'}$, it follows that $R_{j'}=R_j-d$. As $0<R_i<R_{i'}<R_{j'}<R_j$, to show (4.33) is equivalent to show

$$\frac{1}{R_{i'}}+\frac{1}{R_{j'}}<\frac{1}{R_i}+\frac{1}{R_j} \tag{4.37}$$

$$\Rightarrow \frac{1}{R_{i'}R_{j'}}<\frac{1}{R_iR_j} \tag{4.38}$$

$$\Rightarrow R_iR_j<(R_i+d)(R_j-d) \tag{4.39}$$

$$\Rightarrow 0<-R_id+R_jd-d^2 \tag{4.40}$$

$$\Rightarrow R_i<R_j-d \tag{4.41}$$

It is given from the assumption that $R_j-d=R_{j'}$ and $R_i<R_{j'}$ is true. So Proposition 4.1(i) is proved.

(ii) As given, we can express $R_i, R_j, R_{i'}, R_{j'}$ in terms of R_1 and γ: $R_i=R_1+a_i\,\gamma$, $R_j=R_1+a_j\,\gamma$, $R_{i'}=R_1+a_{i'}\,\gamma$ and $R_{j'}=R_1+a_{j'}\,\gamma$, where $a_i=i-1$, $a_j=j-1$, $a_{i'}=i'-1$ and $a_{j'}=j'-1$. From the given information, we have $0<a_i<a_{i'}<a_{j'}<a_j$ and $a_{i'}+a_{j'}<a_i+a_j$. To show inequality (4.36) is equivalent to show

$$\frac{R_{i'}R_{j'}}{R_{i'}+R_{j'}}<\frac{R_iR_j}{R_i+R_j} \tag{4.42}$$

$$\Rightarrow R_i R_j \left(R_{i'} + R_{j'}\right) > R_{i'} R_{j'} \left(R_i + R_j\right) \tag{4.43}$$

$$\Rightarrow \left(R_1 + a_i\,\gamma\right)\left(R_1 + a_j\,\gamma\right)\left(2R_1 + \left(a_{i'} + a_{j'}\right)\gamma\right) > \left(R_1 + a_{i'}\,\gamma\right)\left(R_1 + a_{j'}\,\gamma\right)\left(2R_1 + \left(a_i + a_j\right)\gamma\right) \tag{4.44}$$

$$\Rightarrow \left[\left(a_i + a_j\right) - \left(a_{i'} + a_{j'}\right)\right] R_1^2\,\gamma + 2\left(a_i a_j - a_{i'} a_{j'}\right) R_1\,\gamma^2 + \left[a_i a_j \left(a_{i'} + a_{j'}\right) - a_{i'} a_{j'} \left(a_i + a_j\right)\right]\gamma^3 > 0 \tag{4.45}$$

For $\gamma > 0$, dividing inequality (4.45) by $(-\gamma)$ yields

$$\left[\left(a_{i'} + a_{j'}\right) - \left(a_i + a_j\right)\right] R_1^2 + 2\left(a_{i'} a_{j'} - a_i a_j\right) R_1\,\gamma + \left[a_{i'} a_{j'} \left(a_i + a_j\right) - a_i a_j \left(a_{i'} + a_{j'}\right)\right]\gamma^2 < 0$$

or

$$\left(\gamma - \gamma_1\right)\left(\gamma - \gamma_2\right) < 0, \tag{4.46}$$

where γ_1 and γ_2 are expressed as in (4.34) and (4.35). It is obvious that $\gamma_2 < \gamma_1$. To satisfy inequality (4.46), we must have

$$\gamma_2 < \gamma < \gamma_1$$

and

$$\gamma > 0.$$

We have the following outcomes:

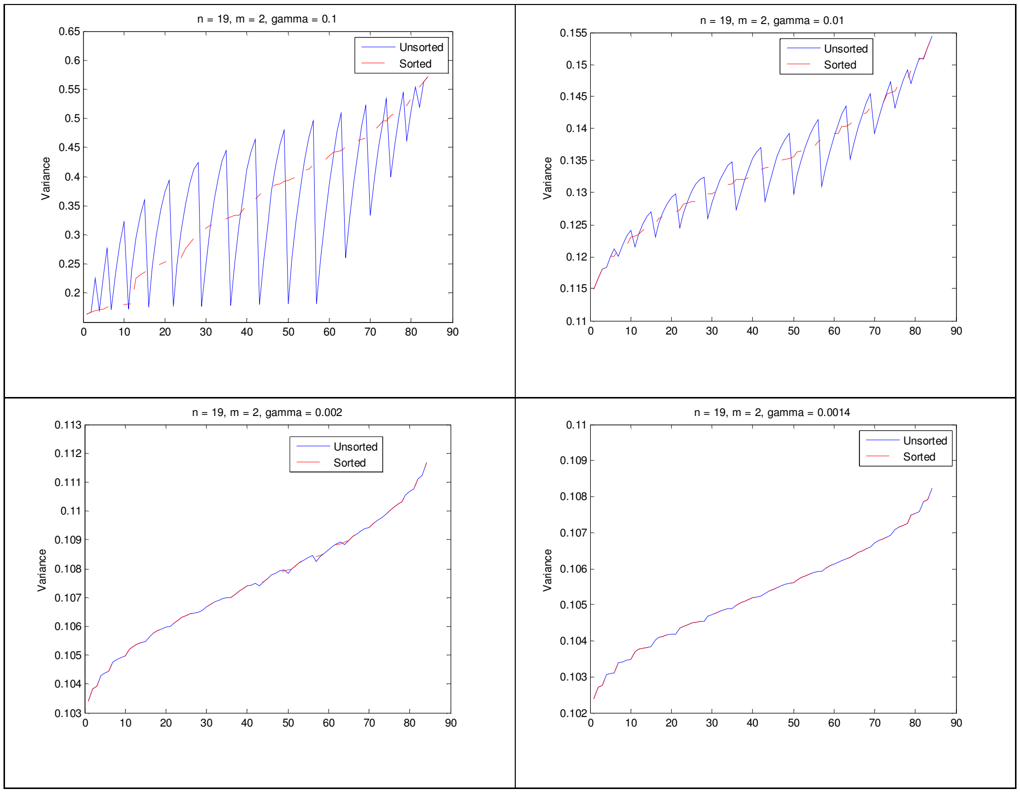
- If $\gamma_2 < \gamma_1 < 0$, then it is impossible to get a value of γ that leads to inequality (4.36).
- If $\gamma_2 < 0 < \gamma_1$, it follows that inequality (4.36) is true when $0 < \gamma < \gamma_1$.
- If $0 < \gamma_2 < \gamma_1$, then $0 < \gamma_2 < \gamma < \gamma_1$ and inequality (4.36) is obtained.

Thus, the previous results show that Proposition 4.1(ii) is proved.■

Note that the expressions on the left- and right-hand sides of inequality (4.33) are main factors in the variances of the two corresponding portfolios as in (4.31). Proposition 4.1(i) implies that for two given portfolios, if each consists of two stocks and the sums of the expected returns of their constituent stocks are equal, then the portfolio with the more ‘extreme’ set of stocks has smaller variance. By assuming that $\{R_1, R_2, \ldots, R_n\}$ is an arithmetic sequence, we can observe from (3.22) that the expected return is closely related to the index of the stock, i.e. i in (3.22). A stock with larger index has larger expected return. The sets with equal sum of expected returns of stocks also have equal sum of indexes of stocks. So in the ‘extreme’ set, the indexes of the constituent stocks are ‘extreme’. In our list of possible sets of stock indexes, we presume that the portfolio with smaller sum_value has smaller variance. This may not be true for all the cases. But it has been proved in Proposition 4.1(ii) that it is true when γ satisfies some conditions.

The following numerical example illustrates our stock picking strategy for $m = 2$. In the numerical example, $R_1 = 0.2$ and γ is ranged from 0 to 0.1. By (3.22), the expected return of stock i can be obtained, for $i = 2, \ldots, 19$. Hence, the variances of portfolios can be calculated for all the sets in Table 4.1. The trend of the variances of portfolios is affected by the values of γ. Figure 4.1 shows the trends of the variance for four cases, i.e. γ = 0.1, 0.01, 0.002 and 0.0014 respectively.

Figure 4.1 The Trend of Variances of Portfolios (n = 19, m = 2)



The ‘unsorted’ curve refers to the graph of the variances of portfolios according to the sequencing list of sets in Table 4.1; and the ‘sorted’ curve is a graph of variances of portfolios which is ranked in an ascending order. Notice that our goal is to apply our stock picking strategy to construct a list of possible sets, in which each set is corresponding to a portfolio and the variances of portfolios are in a non-decreasing order. Here, the ‘sorted’ curve is used as a reference. Once the ‘unsorted’ and ‘sorted’ curves are the same, it implies that the variances of portfolios in our list of sets are in an ascending order. We can observe from Figure 4.1 that as γ is getting smaller, the ‘unsorted’ curve is getting closer to the ‘sorted’ one. In the case of $\gamma = 0.0014$, the ‘unsorted’ one is the same as the ‘sorted’ one.

More specifically, it has been shown in Proposition 4.1(ii) that the portfolio with smaller sum_value has smaller variance if γ satisfies some conditions depending on γ_1 and γ_2. In the following the range of γ is determined for our example. Some pairs of sets obtained from Table 4.1 with different sum_values are required to be compared, see Table 4.2. By applying the formulas (4.34) and (4.35) in Proposition 4.1(ii), we can determine the values of γ_1 and γ_2 for different pairs of sets with different sum_values . It can be observed from Table 4.2 that for all the pairs of sets, $\gamma_1 > 0$ and $\gamma_2 < 0$. Hence, in order to obtain inequality (4.36), γ must satisfy $0 < \gamma < \gamma_1$ for a corresponding pair of sets. For instance, γ must satisfy $0 < \gamma < 0.002632$ for the pair of sets, i.e. (14;6,8) and (15;1,14). It is observable from Table 4.2 that the smallest value of γ_1 is 0.001424 for the pair of sets, (19;7,12) and (20;1,19). If γ satisfies $0 < \gamma < 0.001424$, inequality (4.36) is obtained for all the pairs of sets listed in Table 4.2. Notice that this range of γ is for a particular value of R_1, i.e. $R_1 = 0.2$. As shown in expressions (4.34) and (4.35), γ_1 and γ_2 depend on R_1. We can determine a range of the ratio, γ/R_1, that leads to inequality (4.36). The smallest value of γ_1/R_1 is 0.007120. It implies that once γ/R_1 is in the range $(0,\ 0.007120)$, the variances of portfolio returns with the sets in Table 4.1 are in an ascending order.

Table 4.2 Values of γ_1 and γ_2 for Pairs of Sets with Different Sum_Values

sum_value			i	j	i'	j'	γ_1	γ_2	γ_1 / R_1	γ_2 / R_1
10 11	2 1	8 10	1	10	2	8	0.011375	-0.055819	0.056873	-0.279095
11 12	3 1	8 11	1	11	3	8	0.006186	-0.046186	0.030931	-0.230931
12 13	4 1	8 12	1	12	4	8	0.004262	-0.040626	0.021312	-0.203130
13 14	5 1	8 13	1	13	5	8	0.003254	-0.036587	0.016269	-0.182936
14 15	6 1	8 14	1	14	6	8	0.002632	-0.033401	0.013160	-0.167006
15 16	7 1	8 15	1	15	7	8	0.002210	-0.030781	0.011050	-0.153907
16 17	7 1	9 16	1	16	7	9	0.001942	-0.028609	0.009710	-0.143043
17 18	7 1	10 17	1	17	7	10	0.001732	-0.026732	0.008659	-0.133659
18 19	7 1	11 18	1	18	7	11	0.001563	-0.025092	0.007814	-0.125461
19 20	7 1	12 19	1	19	7	12	**0.001424**	-0.023646	**0.007120**	-0.118231
20 21	7 2	13 19	2	19	7	13	0.001710	-0.022400	0.008552	-0.112001
21 22	7 3	14 19	3	19	7	14	0.002142	-0.021320	0.010709	-0.106599
22 23	7 4	15 19	4	19	7	15	0.002865	-0.020409	0.014327	-0.102046
23 24	7 5	16 19	5	19	7	16	0.004334	-0.019719	0.021672	-0.098595
24 25	7 6	17 19	6	19	7	17	0.008990	-0.019516	0.044948	-0.097579

Now, let us consider a more complicated case, i.e. $m = 3$. In this case, 3 stocks are required to pick from n given stocks. Suppose all the expected returns of stocks are ranked in an ascending order and the desired return $\bar{R}$ is lined between R_k and R_{k+1}. In a similar manner to the case of m = 2, we propose to pick 2 stocks from the first k and one stock from the last $n-k$ stocks. Algorithm 4.2 demonstrates the stock picking strategy for the case of $m=3$. In this case, the sum_value is ranged from $k+4$ to $2k+n-1$. The strategy is to pick stocks forming a smaller sum_value first.

Algorithm 4.2 (Stock Picking Strategy for $m=3$)

Step 1. (Initialization) k value (must be $m-1 \le k \le n-1$); $i=1$, $h=2$ and $j=k+1$, $\text{exit\_value} = (k-1)+k+n+1 = 2k+n$; $\text{sum\_value} = i+h+j$.

Step 2. (Selection) If $i \le k-1$, $h \le k$ and $j \le n$, then enter the list and go to Step 3; otherwise, go to Step 3.

Step 3. (Partition) If $j=k+1$, then go to Step 4; else, go to Step 5.

Step 4. (Update_1) $i=i+1$, $h=i+1$, $j=\text{sum\_value}-i-h$. If $j<k+1$, then go to Step 6; else, go to Step 2.

Step 5. (Update_2) $j=j-1$, $i=\text{sum\_value}-j$; go to Step 2.

Step 6. (Exit) $i=1$, $h=2$, $\text{sum\_value}=\text{sum\_value}+1$, $j=\text{sum\_value}-i-h$. If $\text{sum\_value}=\text{exit\_value}$, then stop; else, go to Step 2.

A numerical example is presented here to illustrate our stock picking strategy for the case of $m=3$. Suppose $n=19$ and k = 7. As from the initialization step, the first set to be consider is $(11;1,2,8)$ where $\text{sum\_value}=11$, $i=1$, $h=2$ and $j=8$. The search will be ended at the set $(32;6,7,19)$ with $\text{sum\_value}=32$.

The total number of possible sets is $(n-k)\bullet{}_{k}C_{m-1} = (19-7)\bullet{}_{7}C_{2} = 252$. As the size of the list of all possible sets is large, Table 4.3 displays the first 50 sets for discussion.

Table 4.3 List of Some Possible Sets of Stock Indexes for *m* = 3

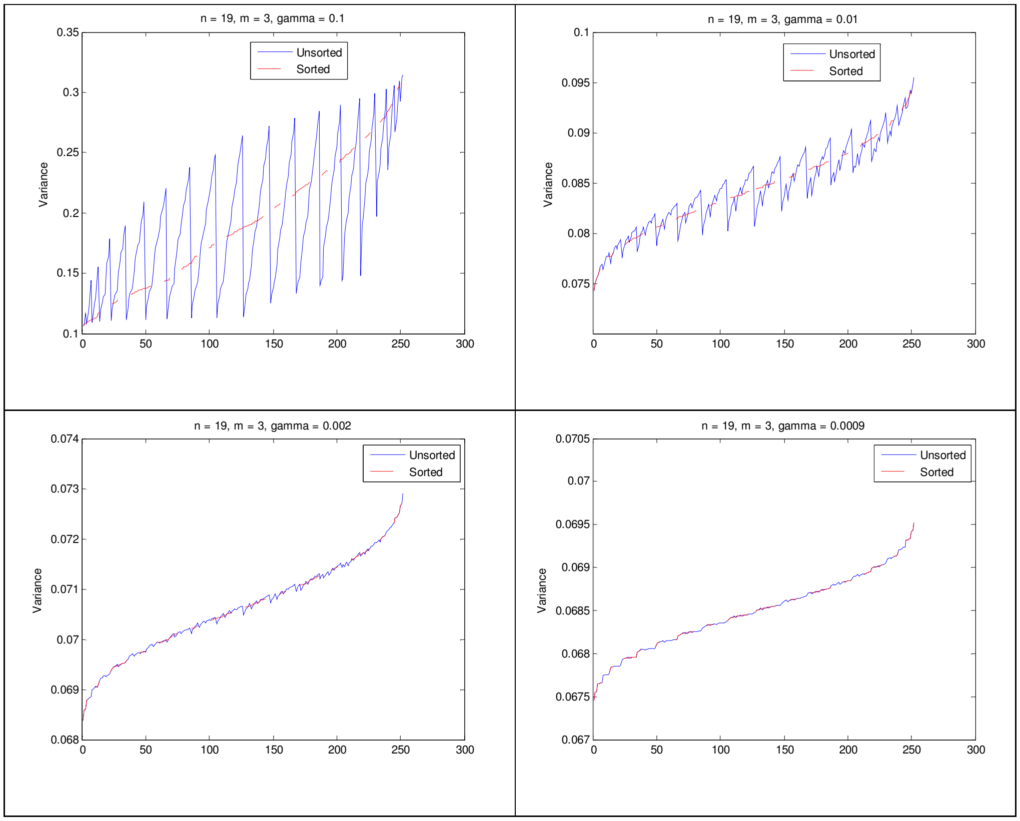
sum_value	*i*	*h*	*j*		sum_value	*i*	*h*	*j*
11	1	2	8		16	1	5	10
12	1	2	9		16	1	6	9
12	1	3	8		16	1	7	8
13	1	2	10		16	2	3	11
13	1	3	9		16	2	4	10
13	1	4	8		16	2	5	9
13	2	3	8		16	2	6	8
14	1	2	11		16	3	4	9
14	1	3	10		16	3	5	8
14	1	4	9		17	1	2	14
14	1	5	8		17	1	3	13
14	2	3	9		17	1	4	12
14	2	4	8		17	1	5	11
15	1	2	12		17	1	6	10
15	1	3	11		17	1	7	9
15	1	4	10		17	2	3	12
15	1	5	9		17	2	4	11
15	1	6	8		17	2	5	10
15	2	3	10		17	2	6	9
15	2	4	9		17	2	7	8
15	2	5	8		17	3	4	10
15	3	4	8		17	3	5	9
16	1	2	13		17	3	6	8
16	1	3	12		17	4	5	8
16	1	4	11		18	1	2	15

It is obvious that if two of the stocks indexes in two sets are the same, the portfolio formed by the set with smaller sum_value has smaller variance. One example in Table 4.3 is the variance of the portfolio in set $(11; 1, 2, 8)$ is smaller than that in the set $(12; 1, 2, 9)$. If only one of the stock indexes in two sets is the same and the sum_value is the same, then we can easily show that the portfolio in a more 'extreme' set has smaller variance. Actually, this result is extended from Proposition 4.1. Considering two sets $(12;1,2,9)$ and $(12;1,3,8)$, the portfolio in set $(12;1,2,9)$ has smaller variance since the set $(12;1,2,9)$ is more 'extreme' than $(12;1,3,8)$. Moreover, as an extension from Proposition 4.2, we can compare two sets with one stock index being the same and different sum_value, e.g. $(12;1,3,8)$ and $(13;1,2,10)$. In such a case, we can show that the portfolio in the set with smaller sum_value has smaller variance as γ is small enough. Here, we assume that $\{R_1, R_2, \ldots, R_n\}$ is an arithmetic sequence and $\gamma > 0$ is the common difference of successive terms.

However, if all the stocks indexes in two sets are different, it is hard to determine the one has smaller variance in general even though these two sets has the same sum_value. We can consider a graphical approach and observe the trend of variances of portfolio returns in the list of possible sets.

We suppose that $R_1 = 0.2$ and γ is ranged from 0.0014 to 0.1. We have similar observations as for the case of m = 2. As the value of γ is getting smaller, the graph of the 'unsorted' one is getting closer to that of the 'sorted' one. See Figure 4.2. The variances of the portfolio returns are approximately in an ascending order in the list of sets constructed by our stock picking strategy.

Figure 4.2 The Trend of Variances of Portfolios (n = 19, m = 3)



So far, we have discussed our stock picking strategy for $m = 2$ and $m = 3$. We have shown that the variances of portfolio returns in our constructed stock picking list are in an ascending order under some assumptions for $m = 2$. Though we cannot proved that the variances are in an ascending order in all the cases for $m = 3$, we show by graph that the variance are approximately in an ascending order. It is even more complicated to show the trend of variances for larger values of *m*. Our discussion about the cases of $m = 2$ and $m = 3$ gives an insight into the stock picking strategy. It may not be perfect, but may be extendable to cases with larger values of *m*.

4.4 Problem Solving by the Xpress Solver

Though we cannot derive an analytical solution to our problem without any assumptions, our problem can be solved by using an optimization solver. This section presents some procedures for solving our problems by the Xpress Solver.

Nowadays, various optimization solvers are developed to help people solve different kinds of optimization problems efficiently. To deal with the Mixed Integer Programming (MIP) or QP problems, one possible solution is to use the Xpress Solver developed by Dash Optimization Limited. The Xpress Solver engineer uses the natural extension of the Interior Point or Newton-Barrier method to solve QP problems. Provided that sufficient memory is available, this solver is able to solve extremely large QP problems. However, it is appropriate only for positive definite quadratic objectives (when minimizing; negative definite when maximizing). More examples of applications are discussed in Guéret et al. (2002).

As discussed above, our problem (MBQP) is formulated into a minimization problem. The model is built with a collection of defined variables, constraints and objective function. By using the Xpress Solver to solve our problem, the first stage is to get the model and develop it into the syntax of the Mosel language. Mosel is an advanced programming language and environment. It has some favorable features, such as easy to use, easily extended and supporting dynamic objects. More features of Mosel can be found in Xpress-Mosel User Guide provided by Dash Optimization. In developing the model into a Mosel file, every decision variable must be declared. In our problem, we have two sets of decision variables, i.e. x_i's and y_i's. Notice that MIP and QP variables are of type 'mpvar'. By default, Mosel assumes that all 'mpvar' variables are constrained to be non-negative unless it is informed otherwise. So our decision variables x_i's and y_i's are of type 'mpvar' and non-negative by default.

Then, the next stage is to solve the model developed in Mosel. It can be specified to Mosel that the problem is to be solved by using the Xpress-Optimizer. The Xpress-Optimizer algorithms enable us to solve LP, MIP, QP and MIQP (Mixed Integer Quadratic Programming) problems. It uses a sophisticated branch and bound algorithm to solve MIP and MIQP problems. To reduce problem size and solution time, MIP pre-solve algorithm pre-processes the problem. In order to improve the quality of bounds and reduce the size of the global search, it also use some advanced cutting-plane strategies, such as Flow covers, GUB covers, Lift and Project, Clique cuts, Flow paths, Mixed integer rounding and Gomory fractional cuts. More information on the Xpress-Optimizer and related topic can be found in URL http://www.dashoptimization.com/home/products/products_optimizer.html.

So far, we have developed a model and specified the optimizer to solve it. The final stage is to obtain a solution to our problem. Under Microsoft Windows, Xpress-IVE, the Xpress Interactive Visual Environment, is used to work with Mosel models. When a model is run, the program output is displayed in the output window pane. In the following subsection, a numerical example of our problem solved by the Xpress-Optimizer is illustrated.

4.4.1 Numerical Examples

In this subsection, consideration is given to the stock market in Hong Kong. More specifically, some constituent stocks in the Hang Seng Index (HSI) are considered. Here, we used historical daily closing prices of stocks from August 1996 to July 1997. Suppose that an investor is holding an existing portfolio, called old portfolio. The historical daily expected return and the standard deviation of the return are given as follows:

Daily Expected Return, R_{old}	Standard Deviation, σ_{old}
0.0009772	0.013279

S/he requests to improve the performance of the existing portfolio. There are 19 new stocks that are attractive to him/her in the stock market. In this situation, $n = 19$. The historical daily expected returns of the stocks, standard deviations of the returns and Sharpe ratios are given in Table 4.4. The Sharpe ratio is defined as the ratio of the daily expected return to the standard deviation of the returns. A falling of the standard deviation or a rising of the return leads to a rise in the Sharpe ratio. It can be used to measure the performance of a portfolio or a stock. The higher the Sharpe ratio, the better the performance of the stock. Among 19 new stocks in Table 4.4, stock 4 has the highest Sharpe ratio.

Table 4.4 List of 19 New Stocks with Daily Expected Return, Standard Deviation and Sharpe Ratio

No.	Stock Name	Daily Expected Return	Standard Deviation	Sharpe Ratio (%)
1	CLP HOLDINGS	0.001414	0.013636	10.3658
2	HONG KONG AND CHINA GAS	0.002087	**0.012395**	16.8335
3	WHARF HOLDINGS	0.000994	0.019536	5.0905
4	HSBC HOLDINGS	0.003250	0.013593	**23.9118**
5	HONG KONG ELECTRIC	0.001437	0.015007	9.5743
6	PCCW	**0.005095**	0.042908	11.8752
7	HANG SENG BANK	0.001720	0.018019	9.5442
8	HUTCHISON WHAMPOA	0.002113	0.016216	13.0310
9	NEW WORLD DEV.	0.002042	0.019360	10.5453
10	SWIRE PACIFIC 'A'	0.000548	0.016159	3.3925
11	BANK OF EAST ASIA	0.001434	0.013001	11.0296
12	HOPEWELL HOLDINGS	0.000712	0.023106	3.0817
13	JOHNSON ELECTRIC HDG.	0.001521	0.019491	7.8029
14	SHUN TAK HOLDINGS	0.000298	0.018373	1.6192
15	CITIC PACIFIC	0.001871	0.016139	11.5901
16	GUANGDONG INVESTMENT	0.003849	0.029640	12.9869
17	CATHAY PACIFIC AIRWAYS	0.000847	0.016554	5.1195
18	TELEVISION BROADCASTS	0.000414	0.015956	2.5953
19	SCMP GROUP	0.002117	0.024608	8.6037

By analyzing the historical data, the variance-covariance matrix, $\mathbf{\Sigma}$, can be also obtained. Suppose 3 new stocks are requested to be chosen from 19 stocks for investing in the new portfolio, i.e. $m = 3$. We set $a = 70\%$, i.e. only at most 30% of the portfolio value is used to invest in those 3 selected new stocks. The desired return is $\overline{R} = 0.0016$ which is greater than the expected return on the old portfolio. Under the return constraint, the return on the new portfolio is restricted to be greater than or equal to the desired return. To restrict the weight of selected stock i, x_i, not to be too small, we set u_i = 0.999. Hence, we have $x_i \geq 0.1\%$ for m selected stocks.

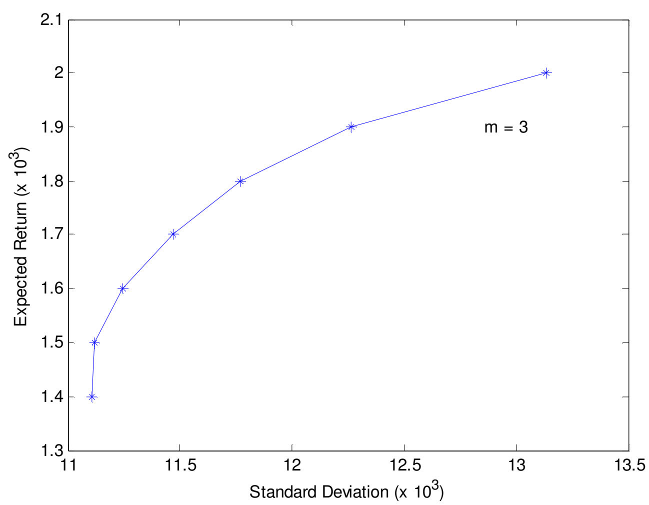
With given values of some parameters, the problem (MBQP) can be translated to the Xpress-Mosel language and saved as a model file. Hence, the decision variables x_i s and y_i s are determined from solving the problem by the Xpress Optimizer.

The output displays the new portfolio with minimum variance that satisfied all the constraints in problem (MBQP). The selected stocks are the ones labeling No. 2, 4 and 6. The corresponding portions invested in these 3 stocks are 11.8635%, 13.8565% and 4.2797% respectively. These percentages are summed up to 29.9997%. Thus, 70.0003% of the portfolio value is still invested in the old portfolio. Note that Stocks 2, 4 and 6 have higher Sharpe ratios among the other stocks. Stock 4 with the highest Sharpe ratio is selected to be invested with the greatest weights among the other two selected stocks. It shows that the new stocks with higher Sharpe ratios will be selected to form a new portfolio with better performance. The following table displays the return and the standard deviation of the return for the new portfolio.

Daily Expected Return, R_{new}	Standard Deviation, σ_{new}
0.0016	0.0112453

It is obvious that the daily expected return on the new portfolio is greater than that on the old portfolio. And the standard deviation (referring to risk) of the return on the new portfolio is lower than that of the old portfolio. Hence, the overall performance of the new portfolio is better than the old portfolio. It achieves the goal of the problem. Figure 4.3 illustrates the efficient frontier of problem (MBQP) with $n = 19$ and $m = 3$. An observation from analyzing the efficient frontier is that Stock 4 is always selected for the new portfolios lying on the efficient frontier in Figure 4.3. This may be due to the fact that Stock 4 has the highest Sharpe ratio among the new stocks.

Figure 4.3 The Efficient Frontier (n = 19, m = 3)



4.5 Comparison with the Problem Formulated with Inequality Cardinality Constraint

As mention in Section 4.1, the motivation of setting an equality constraint comes from the analytical results of the portfolio improvement problem in which only a few new assets are required to be invested. With the equality cardinality constraint, the number of new stocks selected for the new portfolio is fixed to a desired small number. The problem formulated in this style satisfies investors with clear ideas on his/her desired number of selected stocks. However, some investors may be uncertain about the desired number. To favor such investors, we can formulate our problem in another way with an inequality cardinality constraint $\sum_{i=1}^{n} y_i \leq m$. In this consideration, the number of selected new stocks is not fixed to a specific number; however, it is set to range from 1 to *m*.

Regarding a portfolio selection problem, an inequality cardinality constrained problem formulation was discussed in Li, Sun and Wang (2006). They considered the optimal lot solution to the cardinality constrained mean-variance formulation for portfolio selection under concave transaction costs. They incorporated two important discrete features — round lots and cardinality constraint — into their model. It only allows trade of integer lots of stocks and restrains the portfolio from too widely spread. Since the main purpose of this section is to make a comparison between the models with equality and inequality cardinality constraints, we consider only one discrete feature, i.e. cardinality constraint, in our models.

Having solved a portfolio improvement problem formulated with an equality cardinality constraint in the previous section, we investigate the following problem formulation with an inequality cardinality constraint in this section.

(MBQP_IN) Minimize $v(\mathbf{x}) = \mathbf{x}^T \boldsymbol{\Sigma} \mathbf{x}$ (4.47a)

Subject to $\mu(\mathbf{x}) = \mathbf{R}\mathbf{x} \geq \overline{R},$ (4.47b)

$$x_{old} + \sum_{i=1}^{n} x_i = 1, \tag{4.47c}$$

$$x_{old} \geq a, \tag{4.47d}$$

$$0 \leq x_i \leq y_i, \quad \text{for} \quad i = 1, 2, \ldots, n, \tag{4.47e}$$

$$\sum_{i=1}^{n} y_i \leq m, \tag{4.47f}$$

$$y_i \in \{0,1\}, \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{4.47g}$$

$$y_i - x_i \leq u_i, \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{4.47h}$$

By comparing problems (MBQP) with (MBQP_IN), we observe that the only difference is the setting on cardinality constraint. It is obvious that the equality constraint (4.7f) is a special case of constraint (4.47f). The set $\left\{ y_1, \ldots, y_n : \sum_{i=1}^{n} y_i = m \right\}$ is a subset of $\left\{ y_1, \ldots, y_n : \sum_{i=1}^{n} y_i \leq m \right\}$. The number of combinations regarding the inequality constraint becomes much greater than that regarding the equality constraint if m and n are large integers. Hence, the computational time required to solve problem (MBQP_IN) is longer than that to solve problem (MBQP). Though these two problem formulations have the same intention to improve the existing portfolio by investing in a few new assets, it is interesting to compare the results obtained from these two models. The next subsection illustrates the comparison between the problems formulated with an equality cardinality constraint and with an inequality cardinality constraint.

4.5.1 Numerical Example

The numerical example in the previous subsection is considered again. Now, two formulations are implemented. The first one is our problem (MBQP). The other one is constructed by replacing the constraint $\sum_{i=1}^{n} y_i = m$ by $\sum_{i=1}^{n} y_i \leq m$ and denoted by (MBQP_IN). Table 4.4 presents 19 available new stocks for selection.

With n = 19, we consider the cases of m = 1, 2, …, 7. For problem (MBQP), exact m stocks need to be identified from n available stocks for investment. In contrast with problem (MBQP), the possible number of selected stocks for problem (MBQP_IN) can be 1, 2, 3, 4, 5, 6 or 7, if m = 7. Both problems (MBQP) and (MBQP_IN) are solved by the Xpress Solver following the procedures shown in Section 4.4.

Table 4.5 and Table 4.6 display the weights of old portfolio and new stocks in the new portfolios, minimum variances, expected returns and Sharpe ratios (SR) in each case for two problems (MBQP) and (MBQP_IN). Sharpe ratio can be used to measure the performance of portfolios. It suggests that the higher the Sharpe ratio, the better the performance of the portfolio. The highest Sharpe ratio is bold in both Table 4.5 and Table 4.6. In Table 4.5, the new portfolio in the case of $m = 4$ has the highest Sharpe ratio among the others for problem (MBQP). However, in Table 4.6, the new portfolio in the case of $m = 7$ has the highest Sharpe ratio for problem (MBQP_IN). Thus, setting the equality constraint in the problem will require adding a small number of new stocks to construct a new portfolio with higher Sharpe ratio.

One argument is that the highest Sharpe ratio in Table 4.6 is greater than that in Table 4.5. However, by comparing these two highest Sharpe ratios, we notice that their difference is small. Moreover, the transaction cost is not encountered in these two problems. Transaction cost will increase as more stocks are purchased. So from the point of view of cost, it is desirable to purchase a small number of new stocks to construct a new portfolio with better performance. It can also be observed that though the inequality constraint is set to allow investors investing into less than m stocks in problem (MBQP_IN), some of the cases in Table 4.6

require investing in exact m stocks to obtain a new portfolio with minimum variance, e.g. when m = 1, 2, 3 or 6.

Table 4.5 Output for Problem (MBQP)

Cases	m = 1	m = 2	m = 3	m = 4	m = 5	m = 6	m = 7
Weight of old portfolio	70.0009%	70%	70%	70%	70.0001%	70.0002%	70%
Stock 1	0	0	0	0	0	0	0
Stock 2	0	5.0588%	11.8635%	12.7198%	8.2292%	8.2542%	11.5966%
Stock 3	0	0	0	0	0	0	0
Stock 4	29.9991%	24.9411%	13.8565%	10.5573%	11.0203%	12.0081%	10.8647%
Stock 5	0	0	0	0	0	0	0.4755%
Stock 6	0	0	4.2797%	5.7396%	3.8684%	4.1154%	5.8960%
Stock 7	0	0	0	0	0	0	0
Stock 8	0	0	0	0	0	0	0
Stock 9	0	0	0	0	0	0	0
Stock 10	0	0	0	0	0	0	0
Stock 11	0	0	0	0	0	0	0
Stock 12	0	0	0	0	0	0	0
Stock 13	0	0	0	0	0	1.2376%	0.4815%
Stock 14	0	0	0	0	0	0	0
Stock 15	0	0	0	0	0	0	0
Stock 16	0	0	0	0.9833%	4.6719%	3.1089%	0.1006%
Stock 17	0	0	0	0	0	0	0
Stock 18	0	0	0	0	2.2101%	1.2756%	0.5850%
Stock 19	0	0	0	0	0	0	0
Sum	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%
Minimum Variance ($\times 10^4$)	1.4215	1.3591	1.2646	1.2585	1.3141	1.2933	1.2591
Expected Return ($\times 10^4$)	16.5915	16.0029	16.0004	16.0001	16.0005	16.0006	16.0003
Sharpe ratio (%)	13.9157	13.7271	14.2286	**14.2625**	13.9577	14.0697	14.2593

Table 4.6 Output for Problem (MBQP_IN)

Cases	**m = 1**	**m = 2**	**m = 3**	**m = 4**	**m = 5**	**m = 6**	**m = 7**
Weight of old portfolio	72.5977%	70.0001%	70.0003%	70.0003%	70.0001%	70.0002%	70.0003%
Stock 1	0	0	0	0	0	0	0
Stock 2	0	5.0589%	11.8634%	11.8630%	10.0958%	6.2088%	11.6554%
Stock 3	0	0	0	0	0	0	0
Stock 4	27.4023%	24.9411%	13.8566%	13.8572%	14.0490%	10.3679%	10.9461%
Stock 5	0	0	0	0	0	3.4200%	0.4241%
Stock 6	0	0	4.2796%	4.2794%	4.4661%	3.7830%	5.9132%
Stock 7	0	0	0	0	0	0	0
Stock 8	0	0	0	0	0	0	0
Stock 9	0	0	0	0	0	0	0
Stock 10	0	0	0	0	0	0	0
Stock 11	0	0	0	0	0	0	0
Stock 12	0	0	0	0	0	0	0
Stock 13	0	0	0	0	1.3890%	1.3688%	0.4714%
Stock 14	0	0	0	0	0	0	0
Stock 15	0	0	0	0	0	0	0
Stock 16	0	0	0	0	0	4.8513%	0
Stock 17	0	0	0	0	0	0	0
Stock 18	0	0	0	0	0	0	0.5895%
Stock 19	0	0	0	0	0	0	0
Sum	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%
Minimum Variance ($\times 10^4$)	1.4387	1.3591	1.2646	1.2646	1.2643	1.3171	1.2584
Expected Return ($\times 10^4$)	16.0012	16.0029	16.0004	16.0004	16.0004	16.0005	16.0004
Sharpe ratio (%)	13.3404	13.7271	14.2286	14.2286	14.2301	13.9419	**14.2633**

Chapter 5

Portfolio Optimization by Solving the Mean-CVaR Problem

The previous chapter has investigated a portfolio improvement problem by formulating it into a mean-risk optimization problem where the risk is measured by the standard deviation or variance. By using such a risk measure, one can only tell how the portfolio value/return varies but does not know how much market risk the portfolio is taking. Currently, in the risk measurement and management framework, Value-at-Risk (VaR) is one of the most popular tools to measure market risk. It is a more comprehensive risk measure than standard deviation or variance. VaR is preferable to measure risk in portfolio optimization problem because it is a single statistical measure to quantify market risk encountered in a portfolio. However, VaR is not easy to handle because of its undesirable properties such as non-subadditivity and non-convexity. Uryasev and Rockafellar (2000) proposed to use Conditional VaR (CVaR) instead of VaR for optimizing a portfolio of financial instruments. CVaR is defined as the expected loss exceeding VaR. It is similar to VaR but has more attractive properties than VaR such as sub-additivity, convexity and coherency; see Pflug (2000), Rockafellar and Uryasev (2000, 2001).

Due to the favorable properties of CVaR, it gains popularity to be a risk measure in practical applications to risk management and portfolio optimization. This chapter regards CVaR as a risk measure instead of variance and develops a mean-risk model to solve the portfolio improvement problem. It first considers the problem and discusses on its formulation with CVaR as the objective function. Then, further consideration is given to the problem solving under the normality and non-normality assumptions about the loss function. Numerical examples are presented for both assumptions.

5.1 A General Problem and Its Formulation

Consider again the problem described in Chapter 4 that an investor wants to improve an existing portfolio by investing a portion of portfolio value in some new stocks in the stock market. The assumptions made in Chapter 4 are still valid in this chapter. Because of the consideration of CVaR as a risk measure, some more notations need to be introduced. Referring to a definition in Dowd (1998), VaR is the maximum expected loss over a given horizon period at a given level of confidence. Hence, to calculate VaR, a horizon period such as a day, a week, a month or a year, and a level of confidence such as 90%, 95% or 99% confidence need to be presumed.

Furthermore, the expressions of VaR and CVaR are closely related to the loss function. It is necessary to define the loss function in order to establish functions for VaR and CVaR. Let $f(\mathbf{x},\mathbf{z})$ be the loss function which depends on the $(n+1)$-dimensional decision vector $\mathbf{x}$ defined in (4.1) and the random vector $\mathbf{z}\in\mathbb{R}^k$. The vector $\mathbf{z}$ represents the uncertainties that can affect the loss. Note that $f:\mathbb{R}^{n+1}\times\mathbb{R}^k\rightarrow\mathbb{R}$. The value of the loss function can be positive or negative. If it is negative, it turns out to be a gain. Consider a probability function

$$\Psi(\mathbf{x},\zeta)=\int_{f(\mathbf{x},\mathbf{z})\leq\zeta} p(\mathbf{x},\mathbf{z})\,d\mathbf{z}, \tag{5.1}$$

which is the probability that the loss function does not exceed some threshold value ζ. Here, $p(\mathbf{x},\mathbf{z})$ is the probability density function of the random vector $\mathbf{z}$, which also depends on the parameter vector $\mathbf{x}$. Suppose c is a constant that defines a level of confidence and $0 < c < 1$. The quantile function

$$\zeta(\mathbf{x},c) = \min\left\{\zeta \in \mathbb{R} : \Psi\left(\mathbf{x},\zeta\right) \geq c\right\} \tag{5.2}$$

is called Value-at-Risk (VaR). For the definition of CVaR, it is the conditional expected value of the loss $f(\mathbf{x},\mathbf{z})$ given that the loss exceed the quantile $\zeta(\mathbf{x},c)$, and can be expressed as

$$\frac{\mathrm{K}(\mathbf{x},\zeta)}{1-c},$$

where

$$\mathrm{K}(\mathbf{x},\zeta) = \int_{f(\mathbf{x},\mathbf{z}) \geq \zeta(\mathbf{x},c)} f\left(\mathbf{x},\mathbf{z}\right) p(\mathbf{x},\mathbf{z})\, d\mathbf{z}. \tag{5.3}$$

Hence, if CVaR is regarded as a risk measure, the objective function in problem (MBQP) can be replaced by $\frac{\mathrm{K}(\mathbf{x},\zeta)}{1-c}$ or equivalently the function $\mathrm{K}(\mathbf{x},\zeta)$ because $1-c$ is a positive constant. Now, the following formulation of the problem (MCVaR) is considered:

(MCVaR) Minimize $\mathrm{K}(\mathbf{x},\zeta)$ (5.4a)

Subject to $\mu(\mathbf{x}) = \mathbf{R}\mathbf{x} \geq \overline{R}$, (5.4b)

$x_{old} + \sum_{i=1}^{n} x_i = 1,$ (5.4c)

$x_{old} \geq a,$ (5.4d)

$0 \leq x_i \leq y_i, \quad \text{for } i = 1, 2, \ldots, n,$ (5.4e)

$$\sum_{i=1}^{n} y_i = m, \tag{5.4f}$$

$$y_i \in \{0,1\}, \quad \text{for } i = 1,2,\ldots,n. \tag{5.4g}$$

$$y_i - x_i \le u_i, \quad \text{for} \quad i = 1,2,\ldots,n. \tag{5.4h}$$

5.2 Problem Solving under Normality Assumption

One approach to solve the problem (MCVaR) is to approximate the objective function $\mathrm{K}(\mathbf{x},\zeta)$ by discretization. This process makes the function suitable for numerical evaluation. According to Rockafellar and Uryasev (2000), the function $\mathrm{K}(\mathbf{x},\zeta)$ can be reduced to

$$\mathrm{K}(\mathbf{x},\zeta) = (1-c)\zeta + \int_{f(\mathbf{x},\mathbf{z}) \ge \zeta} \left(f(\mathbf{x},\mathbf{z}) - \zeta\right) p(\mathbf{x},\mathbf{z})\, d\mathbf{z}$$

$$= (1-c)\zeta + \int_{\mathbf{z}\in\mathbb{R}^k} \left(f(\mathbf{x},\mathbf{z}) - \zeta\right)^+ p(\mathbf{x},\mathbf{z})\, d\mathbf{z} \tag{5.5}$$

where $\left(f(\mathbf{x},\mathbf{z}) - \zeta\right)^+ = \max\left\{0, f(\mathbf{x},\mathbf{z}) - \zeta\right\}$. Consider a simple case that the density function $p(\mathbf{x},\mathbf{z})$ does not depend on the decision vector $\mathbf{x}$ and hence it reduces to $p(\mathbf{z})$. If the function $f(\mathbf{x},\mathbf{z})$ is convex with respect to $\mathbf{x}$, then the function $\mathrm{K}(\mathbf{x},\zeta)$ is also convex with respect to $\mathbf{x}$. By discretization, the integral in the second term of $\mathrm{K}(\mathbf{x},\zeta)$ in (5.3) can be approximated using scenarios $\mathbf{z}_j$, $j = 1,2,\ldots,J$, which are sampled with the density function $p(\mathbf{x},\mathbf{z}) = p(\mathbf{z})$. That is

$$\int_{\mathbf{z}\in\mathbb{R}^k} \left(f(\mathbf{x},\mathbf{z}) - \zeta\right)^+ p(\mathbf{z})\, d\mathbf{z} \approx J^{-1} \sum_{j=1}^{J} \left(f(\mathbf{x},\mathbf{z}_j) - \zeta\right)^+.$$

Hence, the objective function of the problem can be reduced and approximated by

$$\tilde{\mathrm{K}}(\mathbf{x},\zeta)=(1-c)\zeta+J^{-1}\sum_{j=1}^{J}\left(f\left(\mathbf{x},\mathbf{z}_j\right)-\zeta\right)^{+}. \qquad (5.6)$$

Consider a simple case that the random vector $\mathbf{z}$ is normally distributed. The scenarios $\mathbf{z}_j$, $j=1,2,\ldots,J$, can be generated from a multivariate normal distribution with mean $\mathbf{R}$ and variance-covariance $\mathbf{\Sigma}$ by MATLAB. In the Statistics Toolbox of MATLAB, the function '$\mathrm{mvnrnd}(\mathbf{R},\mathbf{\Sigma},J)$' returns a J-by-$(n+1)$ matrix of random vectors chosen from the multivariate normal distribution with a common 1-by-$(n+1)$ mean vector $\mathbf{R}$, and a common $(n+1)$-by-$(n+1)$ covariance matrix $\mathbf{\Sigma}$. A detailed illustration of the MATLAB function 'mvnrnd' can be found in the Statistics Toolbox User's Guild in the MATLAB webpage (URL: http://www.mathworks.com/access/helpdesk/help/pdf_doc/stats/stats.pdf). Notice that for each scenario, $\mathbf{z}_j\in\mathbb{R}^{n+1}$, to approximate the objective function with the scenarios, the loss function $f(\mathbf{x},\mathbf{z}_j)$ needs to be calculated first. The sample loss of the portfolio can be defined by

$$f(\mathbf{x},\mathbf{z}_j)=-\mathbf{z}_j^T\mathbf{x}, \qquad (5.7)$$

which is a linear function of the control vector $\mathbf{x}$. To make the objective function more apparent for linear programming, dummy variables λ_j s are introduced. Hence, the approximated objective function $\tilde{\mathrm{K}}(\mathbf{x},\zeta)$ is reduced to

$$\tilde{\mathrm{K}}(\mathbf{x},\zeta)=(1-c)\zeta+J^{-1}\sum_{j=1}^{J}\lambda_j. \qquad (5.8)$$

Two more constraints about the dummy variables are added to the problem (MCVaR). They are

$$\lambda_j \geq -\mathbf{z}_j^T \mathbf{x} - \zeta, \quad j = 1, \ldots, J, \tag{5.9}$$

and

$$\lambda_j \geq 0, \quad j = 1, \ldots, J. \tag{5.10}$$

Constraints (5.9) and (5.10) restrict the sum in (5.8) to involve scenarios with loss exceeding the threshold only. Thus, the objective function in problem (MCVaR) becomes a linear function.

Since investing a very small portion of portfolio value into a new stock is not meaningful and increases the transaction cost, some more constraints are added to avoid the weights of new stocks to be too small. Consider constraints (5.4h)

$$y_i - x_i \leq u_i \quad \text{for} \quad i = 1, 2, \ldots, n,$$

where $0 < u_i < 1$, for $i = 1, 2, \ldots, n$. The constraints assign a tighter lower bound to each x_i.

By solving problem (MCVaR), we can obtain an optimal value of CVaR. At the same time, the value of VaR can also be determined. Recalling from Dowd (1998), VaR can be calculated parametrically. A critical issue for the calculation is the assumptions about the probability density function of the portfolio return, i.e. $p(\mathbf{x}, \mathbf{z})$. In practice, it is often assumed that $p(\mathbf{x}, \mathbf{z})$ represents a normal distribution. If $f(\mathbf{x}, \mathbf{z})$ is normally distributed, it can be shown that the absolute VaR can be calculated by the following expression derived from (2.17) with $W = 1$:

$$\text{Cal\_VaR}(\mathbf{x}) = -\mu(\mathbf{x}) - \Phi^{-1}(c)\sigma(\mathbf{x}) \tag{5.11}$$

where $\Phi^{-1}(c)$ is the inverse standard normal cumulative distribution function which can be obtained from (2.13), $\mu(\mathbf{x})$ and $\sigma(\mathbf{x})$ are the mean and standard deviation of the portfolio return.

Notice that the notation, Cal_VaR, is used to distinguish the VaR calculated by formula (5.11) from that obtained by solving problem (MCVaR). It is interesting to compare these two VaR values. Let

$$\text{diff\_VaR}(\mathbf{x}^*) = \frac{\left|\zeta(\mathbf{x}^*, c) - \text{Cal\_VaR}(\mathbf{x}^*)\right|}{\text{Cal\_VaR}(\mathbf{x}^*)} \times 100\%. \tag{5.12}$$

Here, $\mathbf{x}^*$ is the optimal solution to problem (MCVaR) and $\zeta(\mathbf{x}^*, c)$ is the VaR value of the optimal portfolio. In (5.12), $\text{diff\_VaR}(\mathbf{x}^*)$ is equivalent to the absolute percentage difference between $\zeta(\mathbf{x}^*, c)$ and $\text{Cal\_VaR}(\mathbf{x}^*)$. The loss random variable is supposed to approach normality as the value of $\text{diff\_VaR}(\mathbf{x}^*)$ approaches zero.

Similarly, the value of the Conditional VaR, denoted by Cal_CVaR, can be calculated by

$$\text{Cal\_CVaR}(\mathbf{x}) = -\mu(\mathbf{x}) - \alpha(c)\sigma(\mathbf{x}), \tag{5.13}$$

and

$$\alpha(c) = \left(\sqrt{2\pi}\exp\left(\text{erf}^{-1}(2c-1)\right)^2 (1-c)\right)^{-1}, \tag{5.14}$$

where $\exp(s)$ denotes the exponential function of s. Suppose

$$\text{diff\_CVaR}(\mathbf{x}^*) = \frac{\left|\frac{\tilde{K}(\mathbf{x}^*, \zeta)}{1-c} - \text{Cal\_CVaR}(\mathbf{x}^*)\right|}{\text{Cal\_CVaR}(\mathbf{x}^*)} \times 100\%, \tag{5.15}$$

where $\frac{\tilde{K}(\mathbf{x}^*, \zeta)}{1-c}$ is the CVaR value of the optimal portfolio. Obviously, $\text{diff\_CVaR}(\mathbf{x}^*)$ is equivalent to the absolute percentage difference between $\frac{\tilde{K}(\mathbf{x}^*, \zeta)}{1-c}$ and $\text{Cal\_CVaR}(\mathbf{x}^*)$.

In the following subsection, a numerical example is illustrated for different sample sizes. The problem is implemented with Xpress-Mosel and solved with the Xpress-Optimizer. The VaR and CVaR obtained by solving problem (MCVaR) in different sample sizes are compared with Cal_VaR(x) and Cal_CVaR(x) calculated numerically.

5.2.1 Numerical Examples

Suppose an investor is holding an old portfolio with historical daily expected return, 0.0009772, and the standard deviation of the returns, 0.013279. Again the 19 new stocks listed in Table 4.1 are considered to be selected for the new portfolio. Note that $n = 19$. The daily expected returns and variance-covariance matrix for the old portfolio and 19 new stocks are obtained by analyzing the historical data in the period from August 1996 to July 1997.

Under the assumption that the random vector $\mathbf{z}$ is normally distributed, the scenarios $\mathbf{z}_j$, $j = 1, 2, \ldots, J$, can be generated from a multivariate normal distribution with known daily expected returns and variance-covariance matrix by using MATLAB. The number of scenarios is set to be $J = 1000, 3000, 5000, 10000$ and 20000. For example, if J = 1000, the MATLAB program will output a 20-by-1000 matrix displaying the scenarios. Note that $\mathbf{z}_j \in \mathbb{R}^{20}$ for $j = 1, 2, \ldots, 1000$. The following table displays some possible values of the main parameters.

Parameters	Values
desired return, $\overline{R}$	0.0016
level of confidence, c	90%
portion invested in the old portfolio, a	60%, 70% or 80%
number of selected new stocks, m	1, 2, 3, 4, 5, 6 or 7

We suppose $u_i = 99\%$ for $i = 1, 2, \ldots, 19$, which means that $x_i \geq 1\%$ for some selected stock i. Apparently, with a given value of a, x_i must not exceed the weight $(1-a)$ in the new portfolio.

Solving the problem (MCVaR), we can obtain the values of CVaR, VaR and the corresponding values of the decision variables, x_i s. Hence, the values of expected return $\mu(\mathbf{x})$ and variance $v(\mathbf{x})$ of the optimal new portfolio can be calculated by formulas (4.3) and (4.5). Moreover, the Sharpe ratio of the optimal new portfolio can be also obtained. If several portfolios are compared by the Sharpe ratio, the portfolio with the highest Sharpe ratio is preferable to the others. Corresponding to the definition, the Sharpe ratio captures two quantities, say return and standard deviation. Standard deviation is considered as the measure of risk. Comparatively, VaR or CVaR is a more comprehensive and popular risk measure. By considering CVaR as a risk measure, we introduce a risk-adjusted quantity for measuring portfolio performance, called RC ratio, which is a ratio of return to CVaR. The RC ratio is defined as

$$\text{RC ratio} = \frac{\mu(\mathbf{x})}{\frac{\mathrm{K}(\mathbf{x},\zeta)}{1-c}}, \tag{5.16}$$

where $\mu(\mathbf{x})$ is the expected return on portfolio and $\frac{\mathrm{K}(\mathbf{x},\zeta)}{1-c}$ is the CVaR value of a portfolio in the confidence level c. As investors always favor portfolio with higher return and lower risk, RC ratio is the higher the better.

Table 5.1 CVaR Value, Expected Return and RC Ratio for c = 90% and a = 60%

No. of Scenarios	Cases	m = 1	m = 2	m = 3	m = 4	m = 5	m = 6	m = 7
1000	CVaR	0.017296	0.015720	0.015397	0.015182	0.015111	0.015107	0.015113
	Expected Return	0.0019	0.0016	0.0017	0.0016	0.0016	0.0016	0.0016
	RC ratio (%)	**10.9074**	10.1781	10.7518	10.5386	10.5885	10.5911	10.5871
3000	CVaR	0.019446	0.017327	0.017198	0.017011	0.016969	0.016967	0.016971
	Expected Return	0.0019	0.0016	0.0016	0.0016	0.0016	0.0016	0.0016
	RC ratio (%)	**9.7014**	9.2333	9.3315	9.4055	9.4290	9.4299	9.4279
5000	CVaR	0.018751	0.017120	0.016912	0.016636	0.016591	0.016576	0.016578
	Expected Return	0.0016	0.0016	0.0016	0.0016	0.0016	0.0016	0.0016
	RC ratio (%)	8.5333	9.3461	9.4611	9.6180	9.6445	**9.6528**	9.6518
10000	CVaR	0.018741	0.017109	0.016868	0.016650	0.016611	0.016607	0.016609
	Expected Return	0.0019	0.0016	0.0016	0.0016	0.0016	0.0016	0.0016
	RC ratio (%)	**10.0663**	9.3514	9.6135	9.6096	9.6322	9.6343	9.6330
20000	CVaR	0.018783	0.017020	0.016802	0.016552	0.016518	0.016512	0.016509
	Expected Return	0.0019	0.0016	0.0016	0.0016	0.0016	0.0016	0.0016
	RC ratio (%)	**10.0435**	9.4000	9.7691	9.6663	9.6863	9.6897	9.6917

Tables 5.1, 5.2 and 5.3 show the CVaR value, expected return, and RC ratio in the confidence level $c = 90\%$ for a = 60%, 70% and 80% respectively. In these tables, the highest RC ratios for different number of scenarios are put in bold font for reference. It can be observed from these tables that among the portfolios with the same number of scenarios, the m values of the portfolios with the highest RC ratio do not exceed 6. In other words, the new portfolios in the cases of $m \leq 6$ are preferred to the others if we consider the RC ratio as the selection criteria. It implies that in our model, only a few new stocks are required

to be invested into the old portfolio to construct a new portfolio with better performance.

Table 5.2 CVaR Value, Expected Return and RC Ratio for $c = 90\%$ and $a = 70\%$

No. of Scenarios	Cases	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$
1000	CVaR	0.017899	0.017041	0.016544	0.016523	0.016524	0.016559	0.016594
	Expected Return	0.0017	0.0016	0.0016	0.0016	0.0016	0.0016	0.0016
	RC ratio (%)	**9.2698**	9.3889	9.6712	9.6839	9.6830	9.6625	9.6421
3000	CVaR	0.019894	0.018741	0.018478	0.018476	0.018483	0.018503	0.018536
	Expected Return	0.0017	0.0016	0.0016	0.0016	0.0016	0.0016	0.0016
	RC ratio (%)	**8.3402**	8.5372	8.6588	8.6600	8.6567	8.6471	8.6318
5000	CVaR	0.019282	0.018651	0.018151	0.018142	0.018143	0.018156	0.018175
	Expected Return	0.0017	0.0016	0.0016	0.0016	0.0016	0.0016	0.0016
	RC ratio (%)	8.6048	8.5785	8.8152	8.8193	8.8191	**8.8125**	8.8035
10000	CVaR	0.019291	0.018573	0.018133	0.018130	0.018140	0.018164	0.018198
	Expected Return	0.0017	0.0016	0.0016	0.0016	0.0016	0.0016	0.0016
	RC ratio (%)	8.6009	**8.6145**	8.8238	8.8251	8.8202	8.8085	8.7925
20000	CVaR	0.019324	0.018369	0.018031	0.018028	0.018034	0.018046	0.018065
	Expected Return	0.0017	0.0016	0.0016	0.0016	0.0016	0.0016	0.0016
	RC ratio (%)	8.5863	**8.7101**	8.8738	8.8749	8.8724	8.8663	8.8571

Table 5.3 CVaR Value, Expected Return and RC Ratio for c = 90% and a = 80%

No. of Scenarios	Cases	m = 1	m = 2	m = 3	m = 4	m = 5	m = 6	m = 7
1000	CVaR	0.020391	0.018518	0.018549	0.018594	0.018694	0.018832	0.018990
	Expected Return	0.0016	0.0016	0.0016	0.0016	0.0016	0.0016	0.0016
	RC ratio (%)	**7.8469**	8.6406	8.6261	8.6053	8.5595	8.4966	8.4258
3000	CVaR	0.021909	0.020378	0.020389	0.020400	0.020482	0.020582	0.020698
	Expected Return	0.0016	0.0016	0.0016	0.0016	0.0016	0.0016	0.0016
	RC ratio (%)	**7.3031**	7.8522	7.8478	7.8435	7.8122	7.7739	7.7306
5000	CVaR	0.021933	0.020058	0.020056	0.020107	0.020230	0.020378	0.020551
	Expected Return	0.0016	0.0016	0.0016	0.0016	0.0016	0.0016	0.0016
	RC ratio (%)	7.2952	7.9773	7.9780	7.9578	7.9094	**7.8518**	7.7857
10000	CVaR	0.021948	0.020103	0.020121	0.020153	0.020265	0.020410	0.020577
	Expected Return	0.0016	0.0016	0.0016	0.0016	0.0016	0.0016	0.0016
	RC ratio (%)	7.2903	**7.9595**	7.9523	7.9396	7.8957	7.8395	7.7760
20000	CVaR	0.021575	0.019911	0.019929	0.019950	0.020033	0.020158	0.020293
	Expected Return	0.0016	0.0016	0.0016	0.0016	0.0016	0.0016	0.0016
	RC ratio (%)	7.4162	**8.0361**	8.0287	8.0206	7.9871	7.9376	7.8849

When scenarios are generated for solving problem (MCVaR), the assumption of normal distribution is made. Historical data in the period form August 1996 to July 1997 are available for analysis. The data can be examined graphically by histograms. Interesting features can be observed from the constructed diagrams and hence provide information to identify the type of probability distribution. Observation from the histograms for the available stocks shows that the returns on stocks are not all perfectly normally distributed. In the following, the problem (MCVaR) will be solved by using several sets of generated scenarios. Hence, by comparing the VaR and CVaR obtained from problem (MCVaR) and the ones calculated by (5.12) and (5.15), we may show whether the assumption of normality will affect the result.

The following observations are concluded from the output of problem (MCVaR):

- As more scenarios are generated for the problem, most of the values of $\text{diff\_VaR}(\mathbf{x}^*)$ and $\text{diff\_CVaR}(\mathbf{x}^*)$ for VaR and CVaR respectively become smaller. It implies that the loss random variable will approach normality as more scenarios are generated. When the number of scenarios is $J = 10000$, all the absolute percentage differences for VaR and CVaR are less than or around 1% by comparing with the Cal_VaR and Cal_CVaR calculated by (5.12) and (5.15) respectively. See Tables 5.4, 5.5 and 5.6 for $\text{diff\_VaR}(\mathbf{x}^*)$ and Tables 5.7, 5.8 and 5.9 for $\text{diff\_CVaR}(\mathbf{x}^*)$ in the confidence levels c = 90%, 95% and 99% correspondingly.
- Furthermore, in the cases $m = 1, 2, \ldots, 7$, Stock 4 is always selected to be invested into the portfolio with the highest weight. Observe from Table 4.1 that Stock 4 has the highest Sharpe ratio among 19 new stocks. It shows that the stocks with higher Sharpe ratio will be selected with higher priority.
- Apparently, all other things being equal, the result shows that a rise in the confidence level c leads to a rise in both the VaR and CVaR values. This can be shown by expressions (5.11) and (5.13) under the normality assumption. Due to the fact that a larger value of c leads to larger

values of $-\Phi^{-1}(c)$ and $-\alpha_1(c)$, it also leads to larger values of Cal_VaR and Cal_CVaR. Moreover, it can be observed that the CVaR value is always greater than the VaR value.

Table 5.4 Cal_VaR, VaR and Diff_VaR for *c* = 90%

a	No. of Scenarios	Cases	*m* = 1	*m* = 2	*m* = 3	*m* = 4	*m* = 5	*m* = 6	*m* = 7
60%	1000	Cal_VaR	0.01315	0.01210	0.01190	0.01178	0.01169	0.01169	0.01173
		VaR	0.01239	0.01087	0.01089	0.01107	0.01093	0.01089	0.01104
		Diff_VaR	**5.82%**	**10.19%**	**8.52%**	**6.07%**	**6.53%**	**6.88%**	**5.96%**
	5000	Cal_VaR	0.01315	0.01210	0.01191	0.01171	0.01168	0.01168	0.01168
		VaR	0.01330	0.01235	0.01214	0.01198	0.01194	0.01200	0.01193
		Diff_VaR	**1.07%**	**2.02%**	**1.89%**	**2.31%**	**2.17%**	**2.81%**	**2.20%**
	10000	Cal_VaR	0.01315	0.01210	0.01191	0.01171	0.01168	0.01168	0.01168
		VaR	0.01322	0.01217	0.01193	0.01162	0.01170	0.01170	0.01172
		Diff_VaR	**0.48%**	**0.55%**	**0.15%**	**0.77%**	**0.13%**	**0.15%**	**0.35%**
	20000	Cal_VaR	0.01315	0.01210	0.01190	0.01171	0.01168	0.01168	0.01168
		VaR	0.01313	0.01193	0.01174	0.01155	0.01153	0.01153	0.01151
		Diff_VaR	**0.16%**	**1.42%**	**1.41%**	**1.29%**	**1.29%**	**1.33%**	**1.39%**
70%	1000	Cal_VaR	0.01362	0.01307	0.01278	0.01278	0.01279	0.01281	0.01284
		VaR	0.01272	0.01201	0.01198	0.01210	0.01209	0.01218	0.01203
		Diff_VaR	**6.58%**	**8.17%**	**6.25%**	**5.36%**	**5.52%**	**4.98%**	**6.31%**
	5000	Cal_VaR	0.01362	0.01307	0.01278	0.01278	0.01278	0.01279	0.01280
		VaR	0.01380	0.01332	0.01303	0.01306	0.01306	0.01307	0.01316
		Diff_VaR	**1.30%**	**1.86%**	**1.99%**	**2.19%**	**2.19%**	**2.18%**	**2.80%**
	10000	Cal_VaR	0.01362	0.01307	0.01278	0.01278	0.01278	0.01279	0.01280
		VaR	0.01367	0.01311	0.01283	0.01281	0.01282	0.01281	0.01285
		Diff_VaR	**0.38%**	**0.29%**	**0.41%**	**0.28%**	**0.31%**	**0.22%**	**0.43%**
	20000	Cal_VaR	0.01362	0.01307	0.01278	0.01278	0.01278	0.01279	0.01280
		VaR	0.01365	0.01279	0.01271	0.01268	0.01269	0.01265	0.01270
		Diff_VaR	**0.21%**	**2.14%**	**0.57%**	**0.79%**	**0.73%**	**1.07%**	**0.78%**
80%	1000	Cal_VaR	0.01540	0.01413	0.01415	0.01417	0.01425	0.01434	0.01445
		VaR	0.01442	0.01349	0.01344	0.01344	0.01347	0.01342	0.01345
		Diff_VaR	**6.39%**	**4.55%**	**5.04%**	**5.17%**	**5.48%**	**6.44%**	**6.97%**
	5000	Cal_VaR	0.01540	0.01413	0.01415	0.01417	0.01425	0.01435	0.01446
		VaR	0.01592	0.01448	0.01442	0.01446	0.01458	0.01466	0.01486
		Diff_VaR	**3.37%**	**2.51%**	**1.89%**	**2.01%**	**2.32%**	**2.20%**	**2.76%**
	10000	Cal_VaR	0.01540	0.01413	0.01415	0.01417	0.01425	0.01434	0.01445
		VaR	0.01544	0.01415	0.01421	0.01419	0.01421	0.01429	0.01445
		Diff_VaR	**0.25%**	**0.18%**	**0.42%**	**0.12%**	**0.28%**	**0.33%**	**0.05%**
	20000	Cal_VaR	0.01540	0.01413	0.01415	0.01417	0.01425	0.01435	0.01445
		VaR	0.01520	0.01409	0.01408	0.01409	0.01416	0.01419	0.01430
		Diff_VaR	**1.34%**	**0.28%**	**0.51%**	**0.60%**	**0.63%**	**1.07%**	**1.08%**

Table 5.5 Cal_VaR, VaR and Diff_VaR for c = 95%

a	No. of Scenarios	Cases	*m* = 1	*m* = 2	*m* = 3	*m* = 4	*m* = 5	*m* = 6	*m* = 7
60%	1000	Cal_VaR	0.01742	0.01599	0.01574	0.01558	0.01546	0.01546	0.01551
		VaR	0.01583	0.01473	0.01421	0.01383	0.01371	0.01370	0.01366
		Diff_VaR	**9.12%**	**7.88%**	**9.72%**	**11.23%**	**11.34%**	**11.42%**	**11.95%**
	5000	Cal_VaR	0.01742	0.01599	0.01575	0.01548	0.01545	0.01544	0.01544
		VaR	0.01762	0.01584	0.01557	0.01556	0.01544	0.01557	0.01550
		Diff_VaR	**1.14%**	**0.92%**	**1.16%**	**0.51%**	**0.03%**	**0.86%**	**0.36%**
	10000	Cal_VaR	0.01742	0.01599	0.01575	0.01548	0.01545	0.01544	0.01544
		VaR	0.01740	0.01607	0.01582	0.01562	0.01572	0.01558	0.01556
		Diff_VaR	**0.08%**	**0.52%**	**0.43%**	**0.89%**	**1.76%**	**0.87%**	**0.78%**
	20000	Cal_VaR	0.01742	0.01599	0.01574	0.01548	0.01545	0.01545	0.01544
		VaR	0.01741	0.01573	0.01551	0.01530	0.01529	0.01529	0.01526
		Diff_VaR	**0.06%**	**1.59%**	**1.46%**	**1.14%**	**1.02%**	**1.04%**	**1.19%**
70%	1000	Cal_VaR	0.01795	0.01723	0.01686	0.01686	0.01687	0.01690	0.01694
		VaR	0.01646	0.01576	0.01520	0.01535	0.01531	0.01535	0.01554
		Diff_VaR	**8.30%**	**8.55%**	**9.84%**	**8.96%**	**9.28%**	**9.19%**	**8.28%**
	5000	Cal_VaR	0.01795	0.01723	0.01686	0.01685	0.01685	0.01687	0.01688
		VaR	0.01812	0.01744	0.01697	0.01700	0.01693	0.01704	0.01713
		Diff_VaR	**0.96%**	**1.22%**	**0.70%**	**0.89%**	**0.46%**	**1.02%**	**1.50%**
	10000	Cal_VaR	0.01795	0.01723	0.01686	0.01685	0.01685	0.01687	0.01688
		VaR	0.01803	0.01736	0.01700	0.01703	0.01698	0.01702	0.01705
		Diff_VaR	**0.46%**	**0.72%**	**0.83%**	**1.05%**	**0.78%**	**0.89%**	**1.04%**
	20000	Cal_VaR	0.01795	0.01723	0.01686	0.01686	0.01686	0.01687	0.01688
		VaR	0.01797	0.01694	0.01671	0.01669	0.01670	0.01674	0.01673
		Diff_VaR	**0.10%**	**1.70%**	**0.90%**	**1.01%**	**0.93%**	**0.80%**	**0.90%**
80%	1000	Cal_VaR	0.02022	0.01859	0.01862	0.01864	0.01874	0.01886	0.01901
		VaR	0.01907	0.01717	0.01735	0.01730	0.01751	0.01772	0.01785
		Diff_VaR	**5.71%**	**7.63%**	**6.83%**	**7.20%**	**6.55%**	**6.05%**	**6.08%**
	5000	Cal_VaR	0.02022	0.01859	0.01861	0.01864	0.01874	0.01887	0.01901
		VaR	0.02101	0.01903	0.01902	0.01902	0.01918	0.01937	0.01972
		Diff_VaR	**3.88%**	**2.36%**	**2.19%**	**2.02%**	**2.32%**	**2.67%**	**3.71%**
	10000	Cal_VaR	0.02022	0.01859	0.01861	0.01864	0.01874	0.01886	0.01901
		VaR	0.02050	0.01883	0.01885	0.01887	0.01889	0.01906	0.01923
		Diff_VaR	**1.41%**	**1.29%**	**1.26%**	**1.20%**	**0.78%**	**1.04%**	**1.20%**
	20000	Cal_VaR	0.02022	0.01859	0.01862	0.01864	0.01874	0.01887	0.01901
		VaR	0.02003	0.01852	0.01854	0.01855	0.01863	0.01880	0.01890
		Diff_VaR	**0.94%**	**0.38%**	**0.45%**	**0.48%**	**0.57%**	**0.38%**	**0.57%**

Table 5.6 Cal_VaR, VaR and Diff_VaR for c = 99%

a	No. of Scenarios	Cases	m = 1	m = 2	m = 3	m = 4	m = 5	m = 6	m = 7
60%	1000	Cal_VaR	0.02542	0.02327	0.02295	0.02269	0.02253	0.02253	0.02260
		VaR	0.02479	0.02134	0.02078	0.02079	0.02046	0.02049	0.02039
		Diff_VaR	**2.49%**	**8.30%**	**9.45%**	**8.41%**	**9.21%**	**9.08%**	**9.80%**
	5000	Cal_VaR	0.02542	0.02327	0.02295	0.02255	0.02251	0.02250	0.02250
		VaR	0.02456	0.02300	0.02312	0.02250	0.02260	0.02219	0.02201
		Diff_VaR	**3.38%**	**1.17%**	**0.76%**	**0.23%**	**0.37%**	**1.40%**	**2.17%**
	10000	Cal_VaR	0.02542	0.02327	0.02295	0.02256	0.02251	0.02250	0.02251
		VaR	0.02561	0.02269	0.02252	0.02248	0.02238	0.02243	0.02255
		Diff_VaR	**0.76%**	**2.50%**	**1.88%**	**0.37%**	**0.58%**	**0.32%**	**0.22%**
	20000	Cal_VaR	0.02542	0.02328	0.02295	0.02255	0.02251	0.02251	0.02250
		VaR	0.02557	0.02325	0.02292	0.02280	0.02283	0.02275	0.02276
		Diff_VaR	**0.60%**	**0.12%**	**0.11%**	**1.08%**	**1.41%**	**1.05%**	**1.14%**
70%	1000	Cal_VaR	0.02608	0.02504	0.02450	0.02451	0.02453	0.02456	0.02462
		VaR	0.02583	0.02269	0.02194	0.02173	0.02170	0.02183	0.02173
		Diff_VaR	**0.93%**	**9.38%**	**10.46%**	**11.33%**	**11.50%**	**11.12%**	**11.72%**
	5000	Cal_VaR	0.02608	0.02504	0.02450	0.02450	0.02450	0.02452	0.02454
		VaR	0.02526	0.02492	0.02421	0.02412	0.02394	0.02408	0.02411
		Diff_VaR	**3.15%**	**0.44%**	**1.18%**	**1.55%**	**2.30%**	**1.78%**	**1.74%**
	10000	Cal_VaR	0.02608	0.02504	0.02450	0.02450	0.02450	0.02452	0.02454
		VaR	0.02641	0.02491	0.02445	0.02453	0.02449	0.02459	0.02458
		Diff_VaR	**1.29%**	**0.51%**	**0.22%**	**0.15%**	**0.04%**	**0.29%**	**0.16%**
	20000	Cal_VaR	0.02608	0.02504	0.02451	0.02450	0.02451	0.02452	0.02454
		VaR	0.02612	0.02514	0.02473	0.02469	0.02473	0.02467	0.02469
		Diff_VaR	**0.16%**	**0.42%**	**0.89%**	**0.75%**	**0.92%**	**0.60%**	**0.63%**
80%	1000	Cal_VaR	0.02926	0.02695	0.02700	0.02703	0.02717	0.02734	0.02754
		VaR	0.02539	0.02471	0.02396	0.02391	0.02384	0.02392	0.02404
		Diff_VaR	**13.24%**	**8.32%**	**11.24%**	**11.56%**	**12.24%**	**12.51%**	**12.72%**
	5000	Cal_VaR	0.02926	0.02695	0.02699	0.02703	0.02717	0.02735	0.02755
		VaR	0.02917	0.02631	0.02625	0.02624	0.02651	0.02680	0.02695
		Diff_VaR	**0.31%**	**2.39%**	**2.73%**	**2.92%**	**2.45%**	**2.00%**	**2.18%**
	10000	Cal_VaR	0.02926	0.02695	0.02699	0.02703	0.02717	0.02734	0.02754
		VaR	0.02913	0.02704	0.02707	0.02721	0.02721	0.02706	0.02728
		Diff_VaR	**0.44%**	**0.35%**	**0.31%**	**0.66%**	**0.13%**	**1.02%**	**0.97%**
	20000	Cal_VaR	0.02926	0.02695	0.02700	0.02703	0.02717	0.02735	0.02754
		VaR	0.02915	0.02711	0.02708	0.02710	0.02732	0.02752	0.02767
		Diff_VaR	**0.37%**	**0.57%**	**0.32%**	**0.24%**	**0.57%**	**0.62%**	**0.46%**

Table 5.7 Cal_CVaR, CVaR and Diff_CVaR for *c* = 90%

a	No. of Scenarios	Cases	*m* = 1	*m* = 2	*m* = 3	*m* = 4	*m* = 5	*m* = 6	*m* = 7
60%	1000	Cal_CVaR	0.01871	0.01716	0.01691	0.01673	0.01660	0.01660	0.01666
		CVaR	0.01730	0.01572	0.01540	0.01518	0.01511	0.01511	0.01511
		Diff_CVaR	**7.57%**	**8.41%**	**8.94%**	**9.24%**	**9.00%**	**9.02%**	**9.29%**
	5000	Cal_CVaR	0.01871	0.01716	0.01691	0.01662	0.01659	0.01658	0.01658
		CVaR	0.01875	0.01712	0.01691	0.01664	0.01659	0.01658	0.01658
		Diff_CVaR	**0.21%**	**0.25%**	**0.00%**	**0.08%**	**0.00%**	**0.03%**	**0.02%**
	10000	Cal_CVaR	0.01871	0.01716	0.01691	0.01663	0.01659	0.01658	0.01659
		CVaR	0.01874	0.01711	0.01687	0.01665	0.01661	0.01661	0.01661
		Diff_CVaR	**0.16%**	**0.31%**	**0.28%**	**0.13%**	**0.12%**	**0.15%**	**0.14%**
	20000	Cal_CVaR	0.01871	0.01717	0.01691	0.01662	0.01659	0.01659	0.01658
		CVaR	0.01878	0.01702	0.01680	0.01655	0.01652	0.01651	0.01651
		Diff_CVaR	**0.38%**	**0.85%**	**0.62%**	**0.41%**	**0.43%**	**0.46%**	**0.44%**
70%	1000	Cal_CVaR	0.01927	0.01849	0.01809	0.01810	0.01811	0.01814	0.01818
		CVaR	0.01790	0.01704	0.01654	0.01652	0.01652	0.01656	0.01659
		Diff_CVaR	**7.09%**	**7.86%**	**8.55%**	**8.71%**	**8.75%**	**8.71%**	**8.73%**
	5000	Cal_CVaR	0.01927	0.01849	0.01809	0.01809	0.01809	0.01810	0.01812
		CVaR	0.01928	0.01865	0.01815	0.01814	0.01814	0.01816	0.01818
		Diff_CVaR	**0.09%**	**0.85%**	**0.32%**	**0.29%**	**0.29%**	**0.29%**	**0.31%**
	10000	Cal_CVaR	0.01927	0.01849	0.01809	0.01809	0.01809	0.01810	0.01812
		CVaR	0.01929	0.01857	0.01813	0.01813	0.01814	0.01816	0.01820
		Diff_CVaR	**0.13%**	**0.43%**	**0.23%**	**0.23%**	**0.29%**	**0.34%**	**0.45%**
	20000	Cal_CVaR	0.01927	0.01849	0.01810	0.01809	0.01809	0.01811	0.01812
		CVaR	0.01932	0.01837	0.01803	0.01803	0.01803	0.01805	0.01806
		Diff_CVaR	**0.30%**	**0.68%**	**0.37%**	**0.35%**	**0.33%**	**0.34%**	**0.31%**
80%	1000	Cal_CVaR	0.02168	0.01994	0.01997	0.02000	0.02010	0.02023	0.02039
		CVaR	0.02039	0.01852	0.01855	0.01859	0.01869	0.01883	0.01899
		Diff_CVaR	**5.95%**	**7.13%**	**7.13%**	**7.02%**	**7.01%**	**6.92%**	**6.85%**
	5000	Cal_CVaR	0.02168	0.01994	0.01997	0.02000	0.02011	0.02024	0.02039
		CVaR	0.02193	0.02006	0.02006	0.02011	0.02023	0.02038	0.02055
		Diff_CVaR	**1.16%**	**0.59%**	**0.45%**	**0.54%**	**0.62%**	**0.68%**	**0.78%**
	10000	Cal_CVaR	0.02168	0.01994	0.01997	0.02000	0.02011	0.02023	0.02039
		CVaR	0.02195	0.02010	0.02012	0.02015	0.02027	0.02041	0.02058
		Diff_CVaR	**1.23%**	**0.82%**	**0.77%**	**0.77%**	**0.79%**	**0.88%**	**0.94%**
	20000	Cal_CVaR	0.02168	0.01994	0.01997	0.02000	0.02010	0.02024	0.02039
		CVaR	0.02158	0.01991	0.01993	0.01995	0.02003	0.02016	0.02029
		Diff_CVaR	**0.49%**	**0.14%**	**0.22%**	**0.25%**	**0.34%**	**0.41%**	**0.46%**

Table 5.8 Cal_CVaR, CVaR and Diff_CVaR for c = 95%

a	No. of Scenarios	Cases	m = 1	m = 2	m = 3	m = 4	m = 5	m = 6	m = 7
60%	1000	Cal_CVaR	0.02232	0.02045	0.02016	0.01994	0.01980	0.01980	0.01986
		CVaR	0.02089	0.01870	0.01830	0.01799	0.01793	0.01793	0.01793
		Diff_CVaR	**6.42%**	**8.57%**	**9.24%**	**9.78%**	**9.43%**	**9.43%**	**9.72%**
	5000	Cal_CVaR	0.02232	0.02045	0.02016	0.01982	0.01978	0.01977	0.01977
		CVaR	0.02234	0.02034	0.02002	0.01962	0.01955	0.01954	0.01953
		Diff_CVaR	**0.09%**	**0.53%**	**0.68%**	**0.99%**	**1.18%**	**1.16%**	**1.19%**
	10000	Cal_CVaR	0.02232	0.02045	0.02016	0.01982	0.01978	0.01977	0.01977
		CVaR	0.02230	0.02033	0.02008	0.01984	0.01981	0.01980	0.01980
		Diff_CVaR	**0.09%**	**0.62%**	**0.44%**	**0.10%**	**0.14%**	**0.13%**	**0.12%**
	20000	Cal_CVaR	0.02232	0.02046	0.02016	0.01982	0.01978	0.01978	0.01977
		CVaR	0.02244	0.02040	0.02015	0.01983	0.01980	0.01979	0.01978
		Diff_CVaR	**0.54%**	**0.27%**	**0.04%**	**0.06%**	**0.09%**	**0.06%**	**0.05%**
70%	1000	Cal_CVaR	0.02293	0.02202	0.02154	0.02155	0.02156	0.02160	0.02165
		CVaR	0.02159	0.02023	0.01963	0.01960	0.01962	0.01964	0.01968
		Diff_CVaR	**5.86%**	**8.13%**	**8.87%**	**9.06%**	**9.03%**	**9.07%**	**9.11%**
	5000	Cal_CVaR	0.02293	0.02202	0.02155	0.02154	0.02154	0.02156	0.02158
		CVaR	0.02289	0.02206	0.02140	0.02138	0.02138	0.02139	0.02142
		Diff_CVaR	**0.21%**	**0.21%**	**0.65%**	**0.75%**	**0.75%**	**0.77%**	**0.74%**
	10000	Cal_CVaR	0.02293	0.02202	0.02154	0.02154	0.02154	0.02156	0.02157
		CVaR	0.02292	0.02212	0.02160	0.02160	0.02161	0.02163	0.02168
		Diff_CVaR	**0.08%**	**0.48%**	**0.23%**	**0.27%**	**0.30%**	**0.36%**	**0.51%**
	20000	Cal_CVaR	0.02293	0.02202	0.02155	0.02154	0.02155	0.02156	0.02158
		CVaR	0.02304	0.02200	0.02157	0.02156	0.02157	0.02158	0.02160
		Diff_CVaR	**0.47%**	**0.10%**	**0.07%**	**0.08%**	**0.09%**	**0.07%**	**0.11%**
80%	1000	Cal_CVaR	0.02576	0.02372	0.02376	0.02379	0.02391	0.02406	0.02424
		CVaR	0.02425	0.02191	0.02192	0.02195	0.02205	0.02222	0.02243
		Diff_CVaR	**5.88%**	**7.61%**	**7.72%**	**7.70%**	**7.75%**	**7.63%**	**7.49%**
	5000	Cal_CVaR	0.02576	0.02372	0.02375	0.02379	0.02391	0.02407	0.02425
		CVaR	0.02568	0.02365	0.02367	0.02372	0.02384	0.02398	0.02414
		Diff_CVaR	**0.33%**	**0.30%**	**0.33%**	**0.28%**	**0.31%**	**0.38%**	**0.43%**
	10000	Cal_CVaR	0.02576	0.02372	0.02375	0.02379	0.02391	0.02406	0.02424
		CVaR	0.02616	0.02397	0.02400	0.02405	0.02420	0.02436	0.02456
		Diff_CVaR	**1.55%**	**1.06%**	**1.06%**	**1.13%**	**1.19%**	**1.26%**	**1.33%**
	20000	Cal_CVaR	0.02576	0.02372	0.02376	0.02379	0.02391	0.02407	0.02424
		CVaR	0.02578	0.02374	0.02377	0.02380	0.02391	0.02406	0.02424
		Diff_CVaR	**0.05%**	**0.11%**	**0.04%**	**0.05%**	**0.02%**	**0.03%**	**0.00%**

Table 5.9 Cal_CVaR, CVaR and Diff_CVaR for c = 99%

a	No. of Scenarios	Cases	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$
60%	1000	Cal_CVaR	0.02939	0.02689	0.02654	0.02623	0.02605	0.02605	0.02613
		CVaR	0.02740	0.02474	0.02381	0.02374	0.02371	0.02372	0.02373
		Diff_CVaR	**6.78%**	**8.01%**	**10.28%**	**9.51%**	**8.95%**	**8.94%**	**9.19%**
	5000	Cal_CVaR	0.02939	0.02689	0.02653	0.02607	0.02603	0.02601	0.02601
		CVaR	0.02871	0.02616	0.02578	0.02534	0.02509	0.02502	0.02500
		Diff_CVaR	**2.34%**	**2.72%**	**2.82%**	**2.80%**	**3.60%**	**3.82%**	**3.89%**
	10000	Cal_CVaR	0.02939	0.02689	0.02653	0.02608	0.02602	0.02601	0.02602
		CVaR	0.02915	0.02604	0.02583	0.02553	0.02552	0.02551	0.02552
		Diff_CVaR	**0.84%**	**3.17%**	**2.62%**	**2.12%**	**1.93%**	**1.93%**	**1.91%**
	20000	Cal_CVaR	0.02939	0.02690	0.02653	0.02607	0.02602	0.02602	0.02601
		CVaR	0.02997	0.02695	0.02670	0.02647	0.02644	0.02644	0.02644
		Diff_CVaR	**1.95%**	**0.19%**	**0.63%**	**1.55%**	**1.61%**	**1.60%**	**1.66%**
70%	1000	Cal_CVaR	0.03012	0.02892	0.02830	0.02831	0.02833	0.02838	0.02844
		CVaR	0.02875	0.02676	0.02638	0.02632	0.02634	0.02638	0.02642
		Diff_CVaR	**4.55%**	**7.46%**	**6.81%**	**7.06%**	**7.01%**	**7.04%**	**7.11%**
	5000	Cal_CVaR	0.03012	0.02892	0.02831	0.02830	0.02830	0.02832	0.02835
		CVaR	0.02927	0.02800	0.02736	0.02724	0.02720	0.02719	0.02720
		Diff_CVaR	**2.82%**	**3.17%**	**3.33%**	**3.75%**	**3.90%**	**4.00%**	**4.03%**
	10000	Cal_CVaR	0.03012	0.02892	0.02831	0.02830	0.02830	0.02832	0.02834
		CVaR	0.03000	0.02875	0.02801	0.02802	0.02806	0.02811	0.02817
		Diff_CVaR	**0.38%**	**0.58%**	**1.04%**	**1.00%**	**0.84%**	**0.76%**	**0.62%**
	20000	Cal_CVaR	0.03012	0.02892	0.02831	0.02831	0.02831	0.02833	0.02835
		CVaR	0.03077	0.02908	0.02876	0.02878	0.02880	0.02883	0.02887
		Diff_CVaR	**2.15%**	**0.56%**	**1.59%**	**1.69%**	**1.72%**	**1.76%**	**1.83%**
80%	1000	Cal_CVaR	0.03376	0.03111	0.03116	0.03120	0.03136	0.03156	0.03179
		CVaR	0.03268	0.02957	0.02948	0.02948	0.02948	0.02953	0.02983
		Diff_CVaR	**3.20%**	**4.96%**	**5.41%**	**5.52%**	**6.00%**	**6.41%**	**6.16%**
	5000	Cal_CVaR	0.03376	0.03111	0.03115	0.03120	0.03136	0.03157	0.03180
		CVaR	0.03224	0.02968	0.02971	0.02977	0.02987	0.03002	0.03025
		Diff_CVaR	**4.49%**	**4.60%**	**4.63%**	**4.60%**	**4.76%**	**4.89%**	**4.86%**
	10000	Cal_CVaR	0.03376	0.03111	0.03115	0.03120	0.03136	0.03156	0.03179
		CVaR	0.03408	0.03124	0.03130	0.03138	0.03155	0.03177	0.03204
		Diff_CVaR	**0.97%**	**0.43%**	**0.46%**	**0.58%**	**0.60%**	**0.68%**	**0.79%**
	20000	Cal_CVaR	0.03376	0.03111	0.03116	0.03120	0.03136	0.03157	0.03179
		CVaR	0.03390	0.03172	0.03171	0.03173	0.03180	0.03192	0.03206
		Diff_CVaR	**0.43%**	**1.95%**	**1.75%**	**1.70%**	**1.41%**	**1.11%**	**0.87%**

5.3 Problem Solving under Non-Normality Assumption

It is well known that there are several approaches to estimate the VaR value. The approach to calculate Cal_VaR in (5.11) can be classified into the parametric approach to VaR. In this approach, it is crucial to make some assumptions about the probability density function of the portfolio return. In practice, it is common to assume that the portfolio return is normally distributed. This is due to some advantages of normality. One of the most apparent attractions of normality is that it gives us a very simple and tractable expression for VaR, such as expression (5.11).

However, some researchers query whether it is reasonable to make the normality assumption about the portfolio returns. A large amount of empirical literature investigates this issue. See, for example, Butler and Schachter (1996), Dowd (1998), Jackson, Maude and Perraudin (1997), Longin (1994) and Venkataraman (1997). As portfolios usually consist of different instruments, the distribution of portfolio returns depends on the distributions of individual instruments. The return distributions vary from case to case. Many evidences show that the normality assumption is not too unreasonable and can be used as an approximation for the distribution of portfolio returns, but a huge amount of evidence shows that many individual return distributions are not normal. It is observable that many return distributions have fat tails. Fat tails imply that extraordinary losses will occur more frequently and be larger than that expected in the normal distribution. It is worried that the normality assumption leads to underestimate of the 'true' VaR.

To deal with fat tails in the distribution of portfolio returns, it is usual to treat it as a Student's T distribution. It is well known that the probability density function (p.d.f.) of the T distribution is symmetrical with respect to the vertical axis $t = 0$ and resembles the bell shape of the p.d.f. of the standard normal distribution, except that its tails are heavier than those of a normal one. Note that the T distribution depends on a single parameter v, i.e. the number of degrees of freedom, but not the mean or the standard deviation. It can be shown that as the number of degrees of freedom v increases, the T distribution converges to the standard normal distribution. Similar to the normal distribution, a T distribution provides an easy and intuitively plausible way to estimate VaR. We can conclude from Section 5.2 that under the normality assumption, the VaR and CVaR values solved from problem (MCVaR) differ decreasingly from the values of Cal_VaR and Cal_CVaR, as the number of scenarios increase. It is interesting to investigate the effect under the non-normality assumption, i.e. with a T distribution. This section will concentrate on this issue and illustrate with numerical examples.

As shown in Section 5.2, discretization transfers the objective function from a continuous form into its discrete counterpart for simplification. Hence, the objective function can be approximated and problem (MCVaR) can be solved by generating numbers of scenarios $\mathbf{z}_j$ for $j = 1, \ldots, J$. In this approach, it is crucial to make some reasonable assumptions about the distribution of the random vector $\mathbf{z}$. This section assumes that the distribution of $\mathbf{z}$ is a Student's T distribution. The Statistics Toolbox of MATLAB provides the function '$\text{mvtrnd}(\mathbf{C}, v, J)$' to returns a J-by-$(n+1)$ matrix of random numbers chosen from the multivariate T distribution. Here, $\mathbf{C}$ is a $(n+1)$-by-$(n+1)$ correlation matrix in which its diagonal elements are all 1 and other elements are the correlation coefficients, i.e.

$$\mathbf{C} = \begin{bmatrix} 1 & \rho_{old,1} & \rho_{old,2} & \cdots & \rho_{old,n} \\ \rho_{1,old} & 1 & \rho_{1,2} & \cdots & \rho_{1,n} \\ \rho_{2,old} & \rho_{2,1} & 1 & \cdots & \rho_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{n,old} & \rho_{n,1} & \rho_{n,2} & \cdots & 1 \end{bmatrix}. \tag{5.17}$$

Since $\rho_{i,j} = \rho_{j,i}$, $\mathbf{C}$ is a symmetric matrix. It is also assumed to be a positive definite matrix. Note that v is the number of degrees of freedom. For a T distribution, v is usually suggested ranging from 1 to 30. It is because when $v = 30$ or larger, the T distribution is approximately equivalent to a standard normal distribution.

Since the MATLAB function 'mvtrnd' does not take the mean and standard deviation into account, the J-by-$(n+1)$ matrix of random numbers cannot be used to calculate the sample loss and $\tilde{K}(\mathbf{x}, \zeta)$ at once. Let $\mathbf{q} = \text{mvtrnd}(\mathbf{C}, v, J)$ be the J-by-$(n+1)$ matrix. The scenarios $\mathbf{z}_j$ can be obtained by adjusting $\mathbf{q}_j$ as follows:

$$\mathbf{z}_j = \mathbf{R} + \mathbf{q}_j \bullet \mathbf{D}, \quad \text{for } j = 1, \ldots, J, \tag{5.18}$$

where $\mathbf{R}$ is the vector of returns given in (4.2), $\mathbf{q}_j$ is a 1-by-$(n+1)$ vector in the j^{th} row of the matrix $\mathbf{q}$, and $\mathbf{D}$ is defined by

$$\mathbf{D} = \begin{bmatrix} \sigma_{old} & 0 & 0 & \cdots & 0 \\ 0 & \sigma_1 & 0 & \cdots & 0 \\ 0 & 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_n \end{bmatrix}. \tag{5.19}$$

In the adjustment, the expected returns and standard deviations are considered. Hence, $\mathbf{z}_j$ can be substituted into (5.6), (5.7) for calculation. With the scenarios, problem (MCVaR) can be solved in the same manner as in the previous section.

In brief, it can be formulated in the Mosel language and implemented in the Xpress-IVE environment.

It has been shown that under normality assumption, the VaR can be estimated parametrically in a simple way. Due to the similarity of the normal distribution and the T distribution, it can be shown that with the T distribution, the VaR can also be estimated parametrically. With $W = 1$, the VaR in absolute dollar terms derived from (2.20) is

$$\text{Cal\_VaR}_{\text{T}}(\mathbf{x}) = -\mu(\mathbf{x}) - F^{-1}(c\,;v)\,\sigma(\mathbf{x}) \tag{5.20}$$

where c is the confidence level and $F^{-1}(c\,;v)$ is the inverse of standard Student's T cumulative distribution function with v degrees of freedom. For example, for a T distribution with v equal to 19 at the 95 percent confidence level, $F^{-1}(95\%;19)$ equals 1.729. Similarly, the values of $F^{-1}(c\,;v)$ with specific values of v and c can be obtained from the T distribution table.

With the assumption of T distributed, the absolute percentage difference between the VaR solved from problem (MCVaR), say $\zeta(\mathbf{x}^*, c)$, and $\text{Cal\_VaR}_{\text{T}}(\mathbf{x}^*)$ is

$$\text{diff\_VaR}_{\text{T}}(\mathbf{x}^*) = \frac{\left|\zeta(\mathbf{x}^*, c) - \text{Cal\_VaR}_{\text{T}}(\mathbf{x}^*)\right|}{\text{Cal\_VaR}_{\text{T}}(\mathbf{x}^*)} \times 100\%\,. \tag{5.21}$$

This percentage can be regarded as an indicator of the accuracy of the VaR value under T distributed assumption. Explicitly, a smaller percentage signifies higher accuracy.

5.3.1 Experimental Results

This subsection is dealing with the same problem as in Subsection 5.2.1 but with the assumption of T distribution. Under this assumption, the scenarios are generated from a multivariate T distribution and adjusted by formula (5.18). The values of the parameters are set as in Subsection 5.2.1.

Tables 5.10, 5.11 and 5.12 display the $\text{Cal\_VaR}_{\text{T}}$ solved from problem (MCVaR), the VaR_{T} values and the $\text{diff\_VaR}_{\text{T}}$ values for different values of a, m and different numbers of scenarios at 90%, 95% and 99% confidence levels respectively. The experimental results show that

- In these three tables, there is no evidence shown that as the number of scenarios increases, the percentage difference $\text{diff\_VaR}_{\text{T}}$ becomes smaller under the assumption of T-distribution. Some percentages may become even larger for a greater number of scenarios. This observation is a bit different from that under the normality assumption, which shows convergence.
- If we compare the VaR_{T} value with the VaR value and the $\text{Cal\_VaR}_{\text{T}}$ value with the Cal_VaR value, we observe that those values under normality assumption are always smaller than those under T distribution assumption. For example, Tables 5.4 and 5.10 display the VaR values in the same confidence level c = 90% for a = 70%, J = 5000 and $m = 2$, $\text{VaR} = 0.01332$ and $\text{Cal\_VaR} = 0.01307$ under normality assumption are smaller than $\text{VaR}_{\text{T}} = 0.01342$ and $\text{Cal\_VaR}_{\text{T}} = 0.01360$ under T distributed assumption, respectively. We

suppose that this may due to the fatter tail of a T distribution than a normal one.

Table 5.10 Cal_VaR$_T$, VaR$_T$ and Diff_VaR$_T$ for c = 90%

a	No. of Scenarios	Cases	m = 1	m = 2	m = 3	m = 4	m = 5	m = 6	m = 7
60%	1000	Cal_VaR$_T$	0.01370	0.01259	0.01259	0.01235	0.01226	0.01228	0.01227
		VaR$_T$	0.01400	0.01284	0.01269	0.01228	0.01229	0.01231	0.01227
		Diff_VaR$_T$	**2.16%**	**1.92%**	**0.78%**	**0.57%**	**0.26%**	**0.21%**	**0.05%**
	5000	Cal_VaR$_T$	0.01370	0.01259	0.01239	0.01219	0.01218	0.01217	0.01216
		VaR$_T$	0.01346	0.01254	0.01237	0.01183	0.01184	0.01182	0.01184
		Diff_VaR$_T$	**1.78%**	**0.41%**	**0.18%**	**2.88%**	**2.75%**	**2.87%**	**2.67%**
	10000	Cal_VaR$_T$	0.01370	0.01259	0.01239	0.01219	0.01217	0.01215	0.01215
		VaR$_T$	0.01420	0.01299	0.01287	0.01267	0.01265	0.01262	0.01261
		Diff_VaR$_T$	**3.67%**	**3.12%**	**3.84%**	**4.00%**	**3.96%**	**3.83%**	**3.77%**
	20000	Cal_VaR$_T$	0.01370	0.01336	0.01260	0.01251	0.01260	0.01267	0.01270
		VaR$_T$	0.01273	0.01174	0.01124	0.01109	0.01095	0.01091	0.01095
		Diff_VaR$_T$	**7.10%**	**12.13%**	**10.78%**	**11.40%**	**13.10%**	**13.92%**	**13.75%**
70%	1000	Cal_VaR$_T$	0.01417	0.01388	0.01332	0.01332	0.01335	0.01334	0.01339
		VaR$_T$	0.01407	0.01407	0.01346	0.01342	0.01337	0.01345	0.01337
		Diff_VaR$_T$	**0.76%**	**1.38%**	**1.10%**	**0.73%**	**0.15%**	**0.84%**	**0.17%**
	5000	Cal_VaR$_T$	0.01417	0.01360	0.01330	0.01330	0.01330	0.01331	0.01332
		VaR$_T$	0.01389	0.01342	0.01297	0.01294	0.01293	0.01294	0.01298
		Diff_VaR$_T$	**1.97%**	**1.36%**	**2.48%**	**2.71%**	**2.78%**	**2.81%**	**2.50%**
	10000	Cal_VaR$_T$	0.01417	0.01360	0.01330	0.01330	0.01330	0.01330	0.01332
		VaR$_T$	0.01459	0.01409	0.01370	0.01373	0.01376	0.01380	0.01374
		Diff_VaR$_T$	**2.91%**	**3.58%**	**3.06%**	**3.25%**	**3.47%**	**3.75%**	**3.20%**
	20000	Cal_VaR$_T$	0.01417	0.01455	0.01361	0.01364	0.01377	0.01378	0.01374
		VaR$_T$	0.01302	0.01264	0.01215	0.01203	0.01202	0.01204	0.01202
		Diff_VaR$_T$	**8.13%**	**13.16%**	**10.78%**	**11.79%**	**12.71%**	**12.63%**	**12.50%**
80%	1000	Cal_VaR$_T$	0.01601	0.01469	0.01472	0.01475	0.01484	0.01496	0.01507
		VaR$_T$	0.01585	0.01489	0.01470	0.01496	0.01484	0.01486	0.01493
		Diff_VaR$_T$	**1.02%**	**1.33%**	**0.14%**	**1.42%**	**0.03%**	**0.66%**	**0.94%**
	5000	Cal_VaR$_T$	0.01601	0.01469	0.01472	0.01474	0.01482	0.01492	0.01503
		VaR$_T$	0.01588	0.01444	0.01448	0.01449	0.01449	0.01466	0.01481
		Diff_VaR$_T$	**0.82%**	**1.71%**	**1.57%**	**1.69%**	**2.21%**	**1.78%**	**1.46%**
	10000	Cal_VaR$_T$	0.01601	0.01469	0.01472	0.01474	0.01482	0.01492	0.01503
		VaR$_T$	0.01658	0.01522	0.01526	0.01529	0.01528	0.01539	0.01553
		Diff_VaR$_T$	**3.54%**	**3.56%**	**3.69%**	**3.69%**	**3.15%**	**3.10%**	**3.31%**
	20000	Cal_VaR$_T$	0.01601	0.01469	0.01489	0.01494	0.01502	0.01512	0.01514
		VaR$_T$	0.01436	0.01339	0.01337	0.01340	0.01337	0.01345	0.01350
		Diff_VaR$_T$	**10.35%**	**8.87%**	**10.21%**	**10.28%**	**10.97%**	**11.04%**	**10.84%**

Table 5.11 Cal_VaR$_T$, VaR$_T$ and Diff_VaR$_T$ for c = 95%

a	No. of Scenarios	Cases	*m* = 1	*m* = 2	*m* = 3	*m* = 4	*m* = 5	*m* = 6	*m* = 7
60%	1000	Cal_VaR$_T$	0.01841	0.01689	0.01688	0.01657	0.01644	0.01648	0.01647
		VaR$_T$	0.01794	0.01659	0.01649	0.01645	0.01641	0.01641	0.01657
		Diff_VaR$_T$	**2.58%**	**1.78%**	**2.27%**	**0.74%**	**0.21%**	**0.41%**	**0.62%**
	5000	Cal_VaR$_T$	0.01841	0.01689	0.01663	0.01635	0.01634	0.01633	0.01632
		VaR$_T$	0.01837	0.01660	0.01649	0.01619	0.01622	0.01621	0.01621
		Diff_VaR$_T$	**0.24%**	**1.71%**	**0.84%**	**0.98%**	**0.79%**	**0.69%**	**0.69%**
	10000	Cal_VaR$_T$	0.01841	0.01689	0.01663	0.01635	0.01634	0.01631	0.01631
		VaR$_T$	0.01889	0.01728	0.01703	0.01688	0.01675	0.01658	0.01663
		Diff_VaR$_T$	**2.57%**	**2.36%**	**2.36%**	**3.23%**	**2.53%**	**1.64%**	**1.96%**
	20000	Cal_VaR$_T$	0.01841	0.01789	0.01689	0.01678	0.01689	0.01698	0.01702
		VaR$_T$	0.01776	0.01660	0.01615	0.01585	0.01724	0.01549	0.01545
		Diff_VaR$_T$	**3.57%**	**7.19%**	**4.38%**	**5.55%**	**2.03%**	**8.77%**	**9.23%**
70%	1000	Cal_VaR$_T$	0.01896	0.01856	0.01782	0.01783	0.01786	0.01786	0.01792
		VaR$_T$	0.01859	0.01879	0.01807	0.01798	0.01806	0.01799	0.01787
		Diff_VaR$_T$	**1.95%**	**1.24%**	**1.37%**	**0.82%**	**1.11%**	**0.74%**	**0.31%**
	5000	Cal_VaR$_T$	0.01896	0.01820	0.01780	0.01780	0.01780	0.01782	0.01782
		VaR$_T$	0.01861	0.01804	0.01785	0.01791	0.01789	0.01787	0.01787
		Diff_VaR$_T$	**1.85%**	**0.86%**	**0.29%**	**0.64%**	**0.48%**	**0.27%**	**0.27%**
	10000	Cal_VaR$_T$	0.01896	0.01820	0.01780	0.01780	0.01780	0.01781	0.01782
		VaR$_T$	0.01939	0.01850	0.01822	0.01819	0.01821	0.01831	0.01833
		Diff_VaR$_T$	**2.27%**	**1.64%**	**2.36%**	**2.19%**	**2.29%**	**2.80%**	**2.84%**
	20000	Cal_VaR$_T$	0.01896	0.01944	0.01821	0.01824	0.01842	0.01843	0.01838
		VaR$_T$	0.01813	0.01796	0.01749	0.01710	0.01711	0.01710	0.01711
		Diff_VaR$_T$	**4.38%**	**7.58%**	**3.98%**	**6.26%**	**7.07%**	**7.24%**	**6.92%**
80%	1000	Cal_VaR$_T$	0.02134	0.01962	0.01965	0.01969	0.01981	0.01997	0.02011
		VaR$_T$	0.02164	0.02000	0.01980	0.01981	0.01953	0.01979	0.01979
		Diff_VaR$_T$	**1.42%**	**1.93%**	**0.78%**	**0.62%**	**1.39%**	**0.89%**	**1.60%**
	5000	Cal_VaR$_T$	0.02134	0.01962	0.01965	0.01968	0.01978	0.01992	0.02006
		VaR$_T$	0.02084	0.01936	0.01947	0.01950	0.01965	0.01971	0.01990
		Diff_VaR$_T$	**2.32%**	**1.32%**	**0.91%**	**0.91%**	**0.66%**	**1.07%**	**0.78%**
	10000	Cal_VaR$_T$	0.02134	0.01962	0.01965	0.01968	0.01978	0.01992	0.02006
		VaR$_T$	0.02197	0.02034	0.02026	0.02036	0.02049	0.02061	0.02081
		Diff_VaR$_T$	**2.97%**	**3.69%**	**3.09%**	**3.44%**	**3.57%**	**3.48%**	**3.74%**
	20000	Cal_VaR$_T$	0.02134	0.01962	0.01987	0.01993	0.02005	0.02017	0.02020
		VaR$_T$	0.02029	0.01887	0.01885	0.01890	0.01890	0.01889	0.01900
		Diff_VaR$_T$	**4.94%**	**3.80%**	**5.15%**	**5.20%**	**5.72%**	**6.35%**	**5.92%**

Table 5.12 Cal_VaR$_T$, VaR$_T$ and Diff_VaR$_T$ for c = 99%

a	No. of Scenarios	Cases	m = 1	m = 2	m = 3	m = 4	m = 5	m = 6	m = 7
60%	1000	Cal_VaR$_T$	0.02792	0.02555	0.02554	0.02508	0.02490	0.02495	0.02494
		VaR$_T$	0.02630	0.02205	0.02229	0.02218	0.02213	0.02240	0.02243
		Diff_VaR$_T$	**5.83%**	**13.70%**	**12.73%**	**11.57%**	**11.14%**	**10.20%**	**10.05%**
	5000	Cal_VaR$_T$	0.02792	0.02555	0.02520	0.02477	0.02475	0.02473	0.02472
		VaR$_T$	0.02850	0.02524	0.02493	0.02447	0.02478	0.02450	0.02471
		Diff_VaR$_T$	**2.06%**	**1.21%**	**1.08%**	**1.20%**	**0.11%**	**0.92%**	**0.01%**
	10000	Cal_VaR$_T$	0.02792	0.02555	0.02520	0.02477	0.02474	0.02471	0.02471
		VaR$_T$	0.02811	0.02568	0.02512	0.02520	0.02525	0.02516	0.02511
		Diff_VaR$_T$	**0.67%**	**0.49%**	**0.32%**	**1.73%**	**2.04%**	**1.83%**	**1.63%**
	20000	Cal_VaR$_T$	0.02792	0.02702	0.02556	0.02539	0.02556	0.02569	0.02575
		VaR$_T$	0.02899	0.02608	0.02573	0.02511	0.02467	0.02520	0.02487
		Diff_VaR$_T$	**3.81%**	**3.47%**	**0.67%**	**1.12%**	**3.51%**	**1.90%**	**3.43%**
70%	1000	Cal_VaR$_T$	0.02862	0.02800	0.02693	0.02694	0.02698	0.02697	0.02707
		VaR$_T$	0.02630	0.02474	0.02469	0.02480	0.02484	0.02474	0.02494
		Diff_VaR$_T$	**8.14%**	**11.67%**	**8.30%**	**7.92%**	**7.96%**	**8.28%**	**7.90%**
	5000	Cal_VaR$_T$	0.02862	0.02748	0.02689	0.02689	0.02689	0.02692	0.02693
		VaR$_T$	0.02913	0.02718	0.02715	0.02696	0.02717	0.02726	0.02722
		Diff_VaR$_T$	**1.77%**	**1.08%**	**0.96%**	**0.23%**	**1.03%**	**1.28%**	**1.08%**
	10000	Cal_VaR$_T$	0.02862	0.02748	0.02689	0.02689	0.02689	0.02691	0.02693
		VaR$_T$	0.02870	0.02780	0.02703	0.02711	0.02694	0.02710	0.02705
		Diff_VaR$_T$	**0.28%**	**1.18%**	**0.49%**	**0.80%**	**0.17%**	**0.73%**	**0.45%**
	20000	Cal_VaR$_T$	0.02862	0.02929	0.02750	0.02754	0.02780	0.02782	0.02774
		VaR$_T$	0.02928	0.02780	0.02744	0.02749	0.02745	0.02743	0.02736
		Diff_VaR$_T$	**2.28%**	**5.12%**	**0.21%**	**0.19%**	**1.25%**	**1.40%**	**1.37%**
80%	1000	Cal_VaR$_T$	0.03209	0.02956	0.02961	0.02967	0.02984	0.03008	0.03029
		VaR$_T$	0.03168	0.02701	0.02718	0.02676	0.02660	0.02690	0.02747
		Diff_VaR$_T$	**1.27%**	**8.64%**	**8.18%**	**9.81%**	**10.86%**	**10.58%**	**9.32%**
	5000	Cal_VaR$_T$	0.03209	0.02956	0.02961	0.02966	0.02980	0.03001	0.03021
		VaR$_T$	0.03150	0.02968	0.02970	0.02997	0.02986	0.03027	0.03039
		Diff_VaR$_T$	**1.85%**	**0.39%**	**0.33%**	**1.07%**	**0.19%**	**0.87%**	**0.59%**
	10000	Cal_VaR$_T$	0.03209	0.02956	0.02961	0.02966	0.02980	0.03001	0.03021
		VaR$_T$	0.03264	0.03005	0.02987	0.02978	0.03025	0.03039	0.03035
		Diff_VaR$_T$	**1.71%**	**1.65%**	**0.91%**	**0.42%**	**1.52%**	**1.29%**	**0.47%**
	20000	Cal_VaR$_T$	0.03209	0.02956	0.02993	0.03003	0.03019	0.03037	0.03041
		VaR$_T$	0.03187	0.02985	0.03016	0.03010	0.03017	0.03001	0.03020
		Diff_VaR$_T$	**0.68%**	**0.96%**	**0.74%**	**0.26%**	**0.05%**	**1.19%**	**0.68%**

Chapter 6

Conclusions

This chapter gives some concluding remarks about this thesis and discusses some possible directions for further research.

The portfolio selection problem has long been investigated by researchers and practitioners since the pioneer work was done by Markowitz (1952). It deals with the construction of efficient portfolios. Due to the uncertainty of the financial market, it is emerging that an efficient portfolio may no longer perform well after some time. Improving an existing portfolio is an important issue. However, there are limited studies regarding this issue. This thesis has investigated the portfolio improvement problem in detail. It is proposed that an existing portfolio can be improved by adding some new attractive assets. The main contribution of this thesis is to derive some criteria to judge the worthiness of adding some new assets and to develop some solvable models to deal with the portfolio improvement problems.

In dealing with our portfolio improvement problem, we first carry out asset allocation by applying the Sharpe ratio and VaR in Chapter 3. Regarding our intention, after allocating some new assets into the old portfolio, the new portfolio obtained should have better performance than the old one. Since the Sharpe ratio captures both return and risk into a single measure, we use it as a performance measure for comparison between two portfolios. As a result, we derive a criterion (3.8) to judge whether it is worthwhile investing in some new assets. The criterion tells us a lower bound for the weighted average of the returns of the new assets. It can be applied easily in practice.

Moreover, as VaR has gained popularity in risk measurement and management, we regard VaR as a risk measure instead of standard deviation. With this regard, the Shape ratio can be expressed in terms of VaR under the assumption that the portfolio return is normally distributed. Hence, we derive a criterion in terms of VaR for broader applications in risk management. We have also shown that diversification is carried out in our approach.

Since adding in too many assets into a portfolio is not practical, we intend to determine the 'optimal' number of new assets to be added that maximizes the Sharpe ratio. For easy tracking, we assume that both the expected returns and standard deviations of the assets are arithmetic series and geometric series respectively. The simulation results show that the 'optimal' numbers of new assets are small in most of the cases under our assumptions.

In another view, we consider our portfolio improvement problem as an optimization problem seeking to minimize the portfolio risk subject to the return and other constraints. The number of stocks in a portfolio is not restricted in the Markowitz portfolio selection model. Taking the number of stocks into account is an important issue since it is impractical to add too many stocks in a portfolio. By introducing a cardinality constraint in our model, we restrict the number of stocks in a portfolio and consider a more complicated problem, i.e. Mixed Binary Quadratic Programming (MBQP) problem. To satisfy investors who want to invest in a small number of stocks for the ease of management, we formulate our problem with an equality cardinality constraint. We derive some analytical solutions to the problem under some assumptions. Moreover, stocking picking strategies for the case of picking 2 or 3 stocks are illustrated. It has been shown for the case of picking 2 stocks that the variances of the portfolios in the list of combinations constructed by our strategy are monotonically increasing in some cases. Without any assumptions, our formulated MBQP problem can be solved by the Xpress Solver. We present the procedure for solving the MBQP problem by the Xpress Solver and illustrate with numerical examples. Furthermore, our problem can be formulated with an inequality cardinality constraint. In this case, the number of stocks to be added is not fixed, but in a range. This formulation satisfies those investors who are concerned much more about the reward of the portfolio than the number of selected stocks. This formulated problem can also be solved by the Xpress Solver. Comparison between these two formulated problems is carried out with some numerical examples.

Due to the favorable properties of CVaR, it is much easier to be handled than VaR. By regarding CVaR as a risk measure, our portfolio improvement problem is formulated into a mean-CVaR problem. Our approach to solve the mean-CVaR problem is to approximate the objective function by discretization, which makes the function suitable for numerical evaluation. The problem is solved under both normality and non-normality assumptions about the loss random variables. In both cases, CVaR and VaR are obtained from solving the problem by Xpress Solver. By assuming that the loss random variables are multivariate normally distributed, a matrix of scenarios with respect to the returns is generated from a multivariate normal distribution with a given mean vector and a variance-covariance matrix by a MATLAB function. Simulation results show that as more scenarios are generated, the loss random variables approach normality. In a similar manner, our mean-CVaR problem is solved under non-normality assumption, i.e. with a Student's T distribution. The simulation results show that convergence is not achieved under T distribution assumption;

moreover, the VaR values obtained under the normality assumption are always smaller than those under T distributed assumption. May be this is due to the fatter tail of a T distribution than a normal one.

In the following, we present some possible directions for further research.

- In the real stock market, investors need to pay fees for every transaction. Actually, transaction cost is an important factor affecting the decision strategy in portfolio management. Though transaction cost may not be a large amount of money, ignoring the transaction cost in a portfolio selection model may lead to an inefficient portfolio in practice. From this point of view, our portfolio improvement problem can be formulated into a more complicated mean-variance and mean-CVaR models by taking transaction cost into account.
- In our stock picking strategy, we only consider the cases of picking 2 or 3 stocks. It is extendable to devise a more complete and systematic stock picking strategy for picking more stocks.
- In the determination of the 'optimal' number of new assets for portfolio improvement in Section 3.2, it is assumed that these new assets are equally weighted in the new portfolio. For further discussion, we can treat the problem without this assumption, in which new assets have different weightings in the new portfolio.

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