EFFECTIVE MEDIUM THEORY OF ELASTIC AND THERMOELASTIC PROPERTIES OF FIBER COMPOSITES

A THESIS

SUBMITTED TO

THE HONG KONG POLYTECHNIC UNIVERSITY

BY

CHEN LAI TAK

FOR THE DEGREE OF

MASTER OF PHILOSOPHY

IN PHSICS

AUGUST 1998

Table of Contents

Acknowledgments

I would like to thank my supervisor, Prof. F.G. Shin, who gives me many invaluable ideas and suggestions to my research. Under his supervision, I can learn not only the fundamentals of the subject but also how to tackle problems in a logical and effective way so as to progress step by step towards the solution of the problems. Also thanks are due to my co-supervisor, Dr. Y.W. Wong, who has alerted me of many of the experimental data that I used in my research.

Many thanks to members of my family, my friends from the PolyU volleyball club, my former classmates in the technical institute, and my office-mate Mr. Jack Leung and his friend Ms. Pauline Yeung who give me advice when I was in trouble. Special thanks to Ms. Tong Man who has given me much encouragement and moral support during the time of writing this thesis.

The support of an MPhil studentship stipend is also here gratefully acknowledged.

Abstract

Fiber composite materials are investigated for many years and a wealth of theories and equations for the elastic and thermoelastic properties of composites have been accumulated.. The existing equations show good agreement with experimental data in the regime of low fiber volume concentration, but normally show much larger discrepancy at the high fiber volume concentration. We are proposing a new method to study the composite properties at high fiber volume concentration regime. Our work is based on a recent effective medium theory (EMT).

Numerical calculation has been adopted to determine the elastic and thermoelastic properties of unidirectional fiber composites with anisotropic constituents and of composites with short randomly oriented fibers. Moreover, the coupled partial differential equations of EMT relating the five elastic moduli of the unidirectional fiber composites, which are based on the Hashin bounds, are solved analytically. In our project, we also use other equations, such as those of Chamis and Hashin, for the purpose of comparing with EMT results. The Chamis equations are for unidirectional composites with anisotropic fibers but isotropic matrix, while the Hashin bounds are good for composites with anisotropic constituents. For the EMT calculations of the thermoelastic properties of unidirectional composites with anisotropic constituents, we base on the Hashin bounds for calculating elastic properties and then use the equations of thermal expansion coefficients derived by Rosen and Hashin. Results computed by other equations, e.g. the equations by Chamis, Chamberlain, and Rojstaczer *et al.* are used for comparison. For short

randomly oriented fiber composites, the Tandon and Weng equations are adopted for the EMT calculation. Results are compared with the Tandon and Weng equations, and also with the Halpin-Tsai equations for aligned fibers followed by randomizing through averaging.

Measured values of some actual systems are reported in the literature are used to illustrate the ability of EMT on the determination of elastic and thermoelastic properties. They are the composite systems of polyethylene fibers in polyethylene matrix, liquid crystalline polymer fibers in polycarbonate matrix, graphite fibers in epoxy matrix and Kevlar fibers in epoxy matrix. For short fiber composites, we use systems of steel fibers in concrete cement, glass fibers in polyester matrix, whisker SiC fibers in Al_2O_3 and Si_3N_4 matrix as well as SiO_2 spheres, Al_2O_3 fibers and Si_3N_4 fibers in Kerimid 601 matrix (K601). Furthermore, the difference in the predicted values of the effective elastic moduli of typical randomly oriented glass fiber/ epoxy composites between the EMT and Tandon and Weng equations have been investigated.

The EMT results have quite good agreement with the experimental values in the predictions of elastic moduli and thermal expansion coefficients of unidirectional fiber composites. Also the EMT results have excellent agreement with measured values for composites with spherical fillers especially at high volume concentration. Moreover, in the prediction for elastic moduli of three dimensional randomly oriented short fiber composites, the EMT results are close to the experimental data and good results are obtained in the prediction of thermal expansion coefficient. In addition, we have demonstrated th edifference at the high concentration regime between the EMT and the Tandon and Weng calculations in the prediction of elastic moduli of a three dimensional randomly oriented glass fiber/ epoxy composite as a function of fiber aspect ratio from 0.0001 to 10000. In conclusion, this work shows that the EMT is able to give adequate predictions of the fiber composite properties investigated.

Chapter 1 Introduction

1.1 Background

Many theoretical and experimental works on composite materials have been reported in periodicals and journals. In composite materials science, composites can be divided into two main categories: the long fiber and short fiber reinforced composite materials. The long fiber composites have high fiber aspect ratio which may be taken to be infinite, while the short fiber composites will have relatively low fiber aspect ratio, which is often less than one hundred. Composites containing spherical inclusions are known as particulate composites. Designers and engineers are most concerned with the mechanical properties of composite materials in structural design. Fewer works in the literature deal with other composite properties such as thermal properties, expansivity due to water absorption, and thermal conductivity. However, such properties are also dependent on the mechanical properties of the composite.

In the early investigations, bounds for Young's modulus are given by Voigt and Reuss. The "rule of mixtures", the Voigt result, is widely adopted to calculate the Young's modulus of a unidirectional fiber composite with isotropic constituents. However, it is found that the prediction by the rule of mixtures is not good enough and thus other equations and theories have emerged. These include theories and formulas given by investigators like Whitney, Behrens, Halpin and Tsai, and Chamis. Particular mention should be given to Hill's self consistent model to determine the effective elastic moduli of composite systems with isotropic constituents as well as Hashin's bounds which are tighter bounds compared with those of Voigt and Reuss. In Hashin's work, we can find the equations for elastic moduli of unidirectional fiber reinforced composites with anisotropic constituents. in which a replacement scheme is used to convert equations for properties of composites with isotropic constituents to those with anisotropic constituents. Theories and equations for the thermal expansion coefficients of unidirectional fiber composites have also been proposed by many other investigators, such as Yates *et al.*, Rosen and Hashin, Chamis, Schapery, and Chamberlain. Yates *et al.* also provide experimental data on the thermal expansion coefficients as well as elastic moduli of unidirectional fiber composites.

The short fiber composites may be classified according to fiber orientation: fibers may be aligned, random-in-plane, randomly oriented in 3-D, or more generally the fiber orientation is prescribed by a distribution function. Short fiber composites are usually fabricated by injection molding, thus the orientation of the fibers inside the composite are not easy to control. Therefore, the injected short fiber composites may exhibit skin-core structure in which the properties of the skin layer is different to that of the core.

Eshelby and Russel have tackled the elasticity problem of aligned short fiber composites. Chou and Nomura use the self consistent approach to derive equations for the effective elastic moduli, thermal expansion coefficients and thermal conductivity. Laws and McLaughlin have studied the fiber orientation effect on the elastic moduli of the composites. Berthelot has derived elasticity equations by assuming the fibers are in rectangular form and regularly arranged. Halpin and Tsai have derived semi-empirical equations which are widely applied by engineers. Based on the equations for the moduli of aligned short fiber composites, the elastic properties of random-in-plane and random-in-3D composites can be calculated through an averaging procedure. Such is done by Halpin and Kardos, Halpin and Pagano, and Lim and Han. The elasticity and thermal properties of two and three dimensional random fiber composites with isotropic constituents are recently discussed by Tandon and Weng.

In fact, most equations have similar predictions in the limit of small volume concentration of embedded fibers. However, large discrepancy is found in high fiber volume concentration. It is difficult to judge which theory is better for the estimation of the composite properties. Recently an effective medium theory (EMT) is proposed to tackle the problem of calculating dielectric properties of binary composites. This theory is then extended to calculate the effective elastic moduli and thermal expansion coefficients of unidirectional fiber composites with isotropic constituents. The predicted results have good agreement with experimental data in all cases examined. In this thesis, we intend to further extend the EMT to other cases. Our aim is to apply the EMT to predict the thermoelastic properties of unidirectional fiber composites with anisotropic constituents and also to isotropic composites with randomly oriented short fiber reinforcement.

Chapter 2 of this thesis gives an introduction of Hashin's equations and the EMT formulation, both numerical and analytical, of the elasticity problem of unidirectional fiber composites with anisotropic constituents. Chapter 3 gives the EMT prediction of thermal expansion coefficients of composites with anisotropic constituents by the numerical approach. Chapter 4 introduces the calculation of moduli and the thermal expansion coefficient of composites with randomly oriented short fibers. Finally some conclusions will be made in Chapter 5.

1.2 Literature Review

1.2.1 Unidirectional fiber composites

The simple and inverse rules of mixtures are two of the earliest results in the science of composite materials which are equations used to predict the bounds of effective elastic moduli of composites with isotropic constituents [Voigt, 1910; Reuss, 1929]. Much later, a self-consistent model is introduced for the study of the effective elastic constants of aggregates of crystals [Hershey, 1954; Kröner, 1958]. Then, Budiansky [Budiansky, 1965] and Hill [Hill, 1965] have extended the idea to solve the problems of multiphase media. Hill has developed equations for overall elastic moduli of the fiber composites with transversely isotropic phases in terms of properties of the phases and the fiber volume concentrations [Hill, 1964a; 1964b; 1964c]. Somewhat different works on the determination of effective elastic moduli of composite materials have been published by other authors [Kerner, 1956; Hashin and Rosen, 1964; Hermans, 1967; Chen and Cheng, 1967; Behrens, 1969; Chen and Cheng 1970].

Hashin has been able to establish bounds for the overall elastic properties of a multiphase composite material by a variational approach [Hashin, 1962; Hashin, 1963; Hashin, 1974]. In a further development, Hashin and Rosen have derived bounds for effective elastic moduli of a composite with parallel circular fibers by the variational approach [Hashin and Rosen, 1964]. Finally Hashin is able to obtain the elastic moduli of unidirectional composites with anisotropic constituents by introducing a replacement scheme which allows the isotropic constituents to be replaced by anisotropic constituents [Hashin, 1979].

The well-known equations of Halpin and Tsai are based on the work of Hermans [Hermans, 1967] whose results make use of Hill's self-consistent approach [Hill, 1963; Hill, 1964a]. Halpin and Kardos have reviewed the formulation of the Halpin-Tsai equations and discussed the parameters in the equations [Halpin and Kardos, 1976]. Chamis has also studied and developed micromechanics equations for hygral, thermal and mechanical properties of composite materials [Chamis, 1984]. Before this publication, Chamis and Sendeckyi have given a critique on the theories predicting elastic and thermal properties of unidirectional fiber composite materials due to many investigators[Chamis and Sendeckyi, 1968]. More recently, some investigators has extended previous theories to calculate the overall elastic moduli of unidirectional composites [Siboni, 1994; Low *et al.*, 1994; Darras *et al.*, 1995; Wilczynski and Lewinski, 1995].

The foregoing is a review of the literature on the effective elástic moduli of unidirectional composites. We are going to review the results on the effective thermal expansion coefficients (ETECs) of this kind of composites in the following paragraphs.

Wang has derived the equations for the prediction of ETECs of composites with isotropic phases, the embedded phase is assumed to be hollow cylinders with different diameters and have no interaction with each other [Wang, 1966]. The theories due to Turner [Turner, 1946], Kerner [Kerner, 1956] and Arthur and Coulson [Arthur and Coulson, 1964] are good for dilute suspensions of inclusions. Levin is concerned with the determination of the macroscopic ETECs as a function of the mechanical properties and the thermal expansion coefficients of the phases [Levin, 1967]. An energy approach has been employed by Schapery and bounds are obtained for ETECs of isotropic and anisotropic composite materials with isotropic phases. The bounds are actually derived by employing extremum principles of thermoelasticity. This model is extensively used by investigators in predicting the ETECs of unidirectional fiber reinforced composite materials [Schapery, 1968].

Rosen and Hashin have later extended the work by Levin [Levin, 1967] and derived bounds for the ETECs of composite materials with anisotropic constituents using the known bounds are given for the elastic moduli. The bounds they derived for the ETECs of composites with isotropic constituents as well as the expressions for anisotropic composite materials with isotropic constituents are similar to the results by Levin, and the equations reduce to Schapery's results when the phases are isotropic [Rosen and Hashin, 1970].

The equations due to Chamberlain are derived for the case of a fiber embedded in a cylindrical matrix in which the radial displacements on the surface of the cylinder are related to the transverse thermal expansion coefficient (TEC). In the transverse TEC equation, a packing factor is involved so as to identify the fiber packing geometry in the composite [Chamberlain, 1968]. On the other hand, Chamis used a simple force-balance approach to derive formulas for the mechanical and thermal properties of a unidirectional fiber reinforced composite with transversely isotropic fibers. Again the axial TEC formula is identical to Schapery's. However, the effect of Poisson's ratio has not been considered [Chamis, 1984].

Besides theoretical studies, experimental work has been carried out by Yates *et al.* [Yates *et al.*, 1977]. The ETECs of high tensile strength carbon fiber in epoxy at different fiber volume concentrations are interferometrically measured. The formulas due to Schapery [Schapery, 1968], Chamberlain [Chamberlain, 1968] and Schneider [Schneider, 1971] are used to compare with measured data. Strife and Prewo [Strife and Prewo, 1979] have studied the ETECs of unidirectionally and bi-directionally reinforced Kevlar/ epoxy composites prepared by conventional wet winding procedures. Moreover, Bowles and Tompkins have compared the equations for predicting the ETECs of unidirectional fiber reinforced composites with different graphite fibers in metal and ceramic matrices [Bowles and Tompkins, 1989].

1.2.2 Short fiber composites

Short fiber composite materials have been studied by many investigators due to their wide engineering application. An attraction is the high flexibility in fabrication. They can be produced by either injection or sheet molding process so that the production cost is lower as compared with fabricating long fiber composite materials. The fibers in short fiber composites can be unidirectional by aligned, random-in-plane and randomly-oriented. We will briefly give a review on these three different forms.

The importance in the application of short fiber composites are reviewed by Chou and Kelly [Chou and Kelly, 1976]. A number of theoretical works have been devoted to predict the effective properties of aligned short fiber composites with isotropic constituents, or with transversely isotropic fibers in an isotropic matrix. The Hill's self-consistent approach [Hill, 1965], the Russel's results [Russel, 1972; Russel, 1973] and Hahn's model [Tsai and Hahn, 1980; Hahn *et al.*, 1986] are all derived on the basis of micromechanics and are frequently used for the prediction of effective elastic moduli. However, in engineering, the Halpin-Tsai equations are frequently used instead [Halpin and Tsai, 1967].

The self-consistent approach is used by several investigators. Laws and McLaughlin [Laws and McLaughlin, 1978] assume the fibers are various-sized spheroids but have the same aspect ratio. In their work, fiber misalignment is not considered. Chou *et al.* model the composite with distributed ellipsoidal inclusions.

By calculating the stress-strain fields, the effective elastic moduli can then be calculated [Chou *et al.*, 1980]. Chou and Nomura examine the fiber orientation effect on the elastic moduli, thermal conductivity and thermal expansion coefficients of aligned short fiber composites [Chou and Nomura, 1980]. In the theory, bounds are used to calculate the effective elastic moduli for short fiber composites with isotropic constituents and with fiber misorientation [Nomura, 1980].

Berthelot works on the misalignment effects on the elastic properties of aligned fiber composites with misoriented short fibers [Berthelot, 1982a; 1982b]. In his model, an averaging procedure is applied to the stiffness and compliance constants of an perfect aligned fiber composite to determine the misoriented stiffness and compliance constants. The upper bound is obtained in terms of stiffness constants while the lower bound is given by the compliance constants. Similar works and theories are derived by Aboudi [Aboudi, 1983], Halpin and Tsai [Halpin, 1969; Whitney *et al.*, 1984]. Christensen [Christensen, 1991] has reviewed the work in the determination of the elastic moduli of composites at dilute fiber volume concentration by Russel [Russel, 1972; Russel, 1973] who makes use of Eshelby's the results on the elastic fields associated with an ellipsoidal inclusion [Eshelby, 1957]. Equations for the five independent elastic moduli have been derived and explicitly determined for low fiber volume concentration. The inclusion is assumed to take the form of a slender prolate ellipsoid. The effective elastic moduli are expressed in terms of the aspect ratio of the inclusions and the properties of the phases.

For composites with randomly oriented fibers, Halpin and Pagano [Halpin and Pagano, 1969] and Halpin *et al.* [Halpin *et al.*, 1971] have developed the laminate analogy for two- and three dimensional randomly oriented composites. Also, Lim and Han have derived equations for the determination of the effective elastic moduli for randomly oriented composites [Lim and Han, 1986]. Tandon and Weng have given explicit equations for such composites [Tandon and Weng, 1986]. They start with the Eshelby and Mori-Taka theories for aligned fiber composites and, for the three dimensional random fiber case, the composite is considered to be macroscopically isotropic and the elastic properties can be decomposed into hydrostatic and deviatoric parts leading to the effective bulk and shear moduli. These results will become useful in Chapter 4. Corresponding experimental investigations are carried out and can be found in the literature [Nishimatsu and Gurland, 1960; Richard, 1975; Fishers *et al.*, 1992].

We now take a look at the thermal expansion coefficients of short fiber composites. Nomura and Chou have developed equations for the thermal stress coefficients and thermal expansion coefficients for aligned short fiber composites [Nomura and Chou, 1981]. These results are then extended by Takao and Taya [Takao and Taya, 1985] who use the calculation on carbon fiber reinforced aluminum composites. In the reference [Christensen, 1991], the ETEC equations for composites with isotropic constituents are derived by adopting the work due to Rosen and Hashin [Rosen and Hashin, 1970]. Moreover, Craft and Christensen [Craft and Christensen, 1981] have derived equations for the thermal expansion coefficients of composites with two- and three dimensional randomly oriented fibers.

1.2.3 Effective Medium Theory (EMT)

The concept of an effective medium theory [Shin *et al.*, 1989] has been firstly applied to binary mixtures, in which a single algebraic functional equation for the mixture dielectric constant is formulated. Based on this approach, the formulation for a symmetric dielectric binary mixture gives a simple symmetric formula for the effective dielectric constant [Shin *et al.*, 1990]. Then the idea is extended to deal with the effective elastic moduli of isotropic composites with isotropic constituents. The EMT formulation is adopted to determine the binary mixture properties involving two substrate variables, the results from EMT shows an improvement on the prediction of the shear modulus of composites with spherical inclusions [Shin *et al.*, 1993a] as well as the elastic properties of a solid with a dispersion of soft inclusions or voids [Shin *et al.*, 1993b; Au *et al.*, 1994a]. In a further development, the EMT is extended to calculate the elastic moduli of unidirectional fiber composites with isotropic constituents [Au *et al.*, 1994b]. Also the thermal expansion coefficients of unidirectional fiber reinforced composites with isotropic constituents are calculated by the EMT numerical approach and better agreement with experimental data is obtained when compared with the results calculated from Kerner and Schapery [Au and Shin, 1995]. Furthermore, new dielectric mixture equation are derived for binary mixtures when interaction between the constituents phases are considered [Leung, 1996]. Currently, the EMT is applied to the determination of thermoelastic properties of unidirectional fiber composites with anisotropic constituents [Chen *et al.*., 1997]. In the last paper, we have illustrated the use of EMT numerical computation on the effective elastic moduli and thermal expansion coefficients, with good agreement with experimental data.

It should be mentioned that, implicit in the use of EMT, the fillers are assumed to be evenly distributed in the composite.

1.3 Aim and scope of this work

Most of existing elastic and thermoelastic equations have similar predictions at low fiber volume concentrations while at high fiber volume concentrations very different predictions may be obtained. Prompted by its previous success, this work aims to further make use of EMT to composite materials science. The scope of our investigation is to apply the EMT to calculate the elastic and thermoelastic properties of composites with anisotropic constituents and with three dimensional randomly oriented short fibers. We are interested in a comparison of the EMT results with results from other well-known equations, especially in the high fiber volume concentration regime. Also results will be compared with experimental values available from the literature so as to indicate the relative merits of the theories.

Chapter 2 Effective elastic moduli of unidirectional fiber composites with anisotropic constituents

2.1 Introduction to Hashin's expressions

In this Chapter, we are going to estimate the effective elastic moduli of a unidirectional fiber composite with anisotropic constituents based on the effective medium theory (EMT) by numerical and also analytical approaches. Comparison will be made with experimental data and with other theories. Good agreement of EMT results with experimental data is obtained.

We shall start with Hashin's expressions for the effective elastic moduli of a unidirectional fiber composite with anisotropic constituents, which are derived from the Composite Cylinder Assemblage (CCA) model [Hashin, 1974; Hashin, 1979]. The reason for choosing Hashin's expressions rather than many others is that the model is based on micromechanical studies of elasticity and has rigorous form which also can determine elastic moduli of a composite with anisotropic properties in both phases. In addition, these equations are widely adopted to solve related problems and known to have good agreement with experimental data, at least for low fiber fraction. Here we will start with this CCA model and outline the derivation of the effective moduli expressions. We use the same symbols as in Hashin's work [Hashin, 1979].

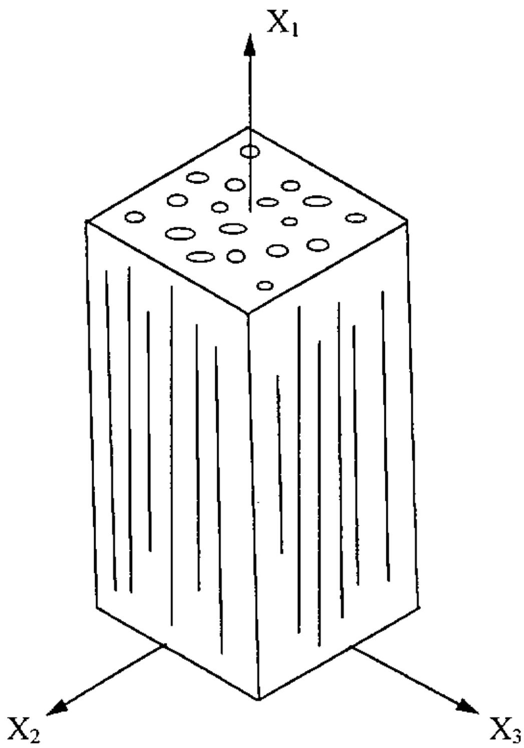


Figure 2.1.1 Unidirectional fiber composite model.

Consider a unidirectional long cylindrical fiber composite as shown in Figure 2.1.1, in which the phases are both transversely isotropic materials. X_1 is the designated fiber direction and X_2 and X_3 are in the transverse plane normal to the fibers, and average stress-strain relations are as follows:

$$\begin{aligned}
\bar{\sigma}_{11} &= n'\bar{\varepsilon}_{11} + l'(\bar{\varepsilon}_{22} + \bar{\varepsilon}_{33}) \\
\bar{\sigma}_{22} &= l'\bar{\varepsilon}_{11} + (k'+G_T')\bar{\varepsilon}_{22} + (k'-G_T')\bar{\varepsilon}_{33} \qquad \text{......(2.1.1a)} \\
\bar{\sigma}_{33} &= l'\bar{\varepsilon}_{11} + (k'-G_T')\bar{\varepsilon}_{22} + (k'+G_T')\bar{\varepsilon}_{33}
\end{aligned}$$

$$\bar{\sigma}_{12} = 2G_A'\bar{\varepsilon}_{12}$$

$$\bar{\sigma}_{23} = 2G_T'\bar{\varepsilon}_{23} \qquad \text{......(2.1.1b)}$$

$$\bar{\sigma}_{13} = 2G_A'\bar{\varepsilon}_{13}$$

The definition of homogeneous boundary conditions is through either

$$u_i(S) = \varepsilon_{ij}^0 x_j \qquad \text{......(2.1.2)}$$

or

$$T_i(S) = \sigma_{ij}^0 n_j \qquad \text{......(2.1.3)}$$

where ε_{ij}^0 is the constant strains and σ_{ij}^0 is the constant stresses, $u_i(S)$ is the displacement fields on the composite surface while $T_i(S)$ is the traction fields on the composite surface.

Here we will outline the derivation of the resulting effective elastic moduli solving the boundary-value problems. To obtain the transverse bulk modulus k', a state of plane strain is imposed on a fiber reinforced cylinder with $\varepsilon_{22}^0 = \varepsilon_{33}^0 = \varepsilon^0$ and others vanish. According to equation (2.1.1) we get $\bar{\sigma}_{22} = \bar{\sigma}_{33} = 2k'\varepsilon^0$. From this, it is easy to obtain the expression of k'. The axial Young's modulus E_A' and the axial Poisson's ratio ν_A' are borrowed from Hill's results [Hill, 1964a], then l' and n' are calculated from the moduli interrelations. The transverse shear modulus G_T' is computed from the equation (2.1.1b) by imposing $\varepsilon_{23}^{0} \neq 0$ and others vanish. This defines the transverse shear modulus by $\overline{\sigma}_{23} = 2G_T^{'}\varepsilon_{23}^{0}$, which may be solved from the corresponding boundary value problem. Lastly, The axial shear modulus $G_A^{'}$ is defined by $\overline{\sigma}_{12} = 2G_A^{'}\varepsilon_{12}^{0}$ if the only if ε_{12}^{0} is the nonvanishing average strain. Furthermore, by applying a replacement scheme, the resulting expressions are applicable to a unidirectional fiber composite with anisotropic phases. The case of isotropic k and G are to be replaced by the case of transversely isotropic K and G_T phase moduli for all the results of $k^{'}$ and $G_T^{'}$ for a composite with isotropic constituents. Similarly, all bounds for $k^{'}$ and $G_T^{'}$ for isotropic phases transform into the counterpart bounds for the case of transversely isotropic phases. Other moduli have to be expressed in terms of k and G and the replacement can then be carried out.

The replacement of isotropic moduli by transversely isotropic moduli are summarized in Table 2.1.1 .

Isotropic phase moduli	Transversely isotropic phase moduli
k	K
G (in transverse shear modulus expression)	G_T
E	$\frac{(3K - G_T)G_T}{K}$
ν	$\frac{1}{2}(1 - \frac{G_T}{K})$
G (in axial shear modulus expression)	G_A

Table 2.1.1 The replacement scheme for expressions of unidirectional fiber composite materials [Hashin, 1979].

Finally the resulting expressions for the elastic moduli are as follows:

$$K' = \frac{K_1(K_2 + G_{T1})(1-\phi) + K_2(K_1 + G_{T1})\phi}{(K_2 + G_{T1})(1-\phi) + (K_1 + G_{T1})\phi} \qquad \text{......(2.1.4a)}$$

$$E_A' = E_{A1}(1-\phi) + E_{A2}\phi + \frac{4(\nu_{A2} - \nu_{A1})^2\phi(1-\phi)}{\frac{1-\phi}{K_2} + \frac{\phi}{K_1} + \frac{1}{G_{T1}}} \qquad \text{......(2.1.4b)}$$

$$\nu_A' = \nu_{A1}(1-\phi) + \nu_{A2}\phi + \frac{(\nu_{A2} - \nu_{A1})(\frac{1}{K_1} - \frac{1}{K_2})\phi(1-\phi)}{\frac{1-\phi}{K_2} + \frac{\phi}{K_1} + \frac{1}{G_{T1}}} \qquad \text{......(2.1.4c)}$$

$$G_A' = G_{A1}\frac{G_{A1}(1-\phi) + G_{A2}(1+\phi)}{G_{A1}(1+\phi) + G_{A2}(1-\phi)} \qquad \text{......(2.1.4d)}$$

Only bounds are obtained for transverse shear modulus: $G'_{T(+)} > G'_T > G'_{T(-)}$.

When the fibers are stiffer than the matrix, i.e. $G_{T2} > G_{T1}$ and $K_2 > K_1$,

$$G'_{T(+)} = G_{T2} + \frac{(1-\phi)}{\frac{1}{G_{T1} - G_{T2}} + \frac{K_2 + 2G_{T2}}{2G_{T2}(K_2 + G_{T2})}\phi} \qquad \text{......(2.1.4e)}$$

$$G'_{T(-)} = G_{T1} + \frac{\phi}{\dfrac{1}{G_{T2} - G_{T1}} + \dfrac{K_1 + 2G_{T1}}{2G_{T1}(K_1 + G_{T1})}(1 - \phi)} \qquad \text{......(2.1.4f)}$$

Also, $E'_{T(\pm)}$ and $\nu'_{T(\pm)}$ are given by

$$E'_{T(\pm)} = \frac{4K'G'_{T(\pm)}}{K' + \psi G'_{T(\pm)}}$$

$$\text{......(2.1.5)}$$

$$\nu'_{T(\pm)} = \frac{K' - \psi G'_{T(\mp)}}{K' + \psi G'_{T(\mp)}}$$

where

$$\psi = 1 + \frac{4K'\nu'_A}{E'_A}$$

$E_A^{'}$, E_{A1}, E_{A2} = axial Young's modulus of composite, matrix, fiber, respectively

$E_T^{'}$, E_{T1}, E_{T2} = transverse Young's modulus of composite, matrix, fiber, respectively

$G_A^{'}$, G_{A1}, G_{A2} = axial shear modulus of composite, matrix, fiber, respectively

$G_T^{'}$, G_{T1}, G_{T2} = transverse shear modulus of composite, matrix, fiber, respectively

$\nu_A^{'}$, ν_{A1}, ν_{A2} = axial Poisson's ratio of composite, matrix, fiber, respectively

$\nu_T^{'}$, ν_{T1}, ν_{T2} = transverse Poisson's ratio of composite, matrix, fiber, respectively

K , K_1, K_2 = plain strain bulk modulus of composite, matrix, fiber, respectively

k' , k_1, k_2 = transverse bulk modulus of composite, matrix, fiber, respectively

ϕ = volume concentration of fiber

2.2 Numerical calculations based on EMT

We shall first introduce the term "increment function" to facilitate the discussion to follow. An increment function associated with any property P of a composite is defined as $(\frac{\partial P}{\partial \phi})_{\phi=0}$ i.e. the derivative of P with respect to the fiber volume concentration ϕ evaluated at $\phi = 0$. Five increment functions can be determined in our present problem corresponding to the five elastic moduli expressions of Hashin. In the case of transverse shear modulus, the upper and lower bounds give the same increment function.

The five increment functions calculated from Hashin's expressions are as follows:

$$K'_{\phi}\Big|_{\phi=0} = \frac{1}{\frac{1}{K_2 - K_1} + \frac{1}{K_1 + G_{T1}}} \qquad \text{......(2.2.1a)}$$

$$E'_{A\phi}\Big|_{\phi=0} = E_{A2} - E_{A1} + \frac{4(\nu_{A2} - \nu_{A1})^2}{\frac{1}{K_2} + \frac{1}{G_{T1}}} \qquad \text{......(2.2.1b)}$$

$$\nu'_{A\phi}\Big|_{\phi=0} = \nu_{A2} - \nu_{A1} + \frac{(\nu_{A2} - \nu_{A1})(\frac{1}{K_1} - \frac{1}{K_2})}{\frac{1}{K_2} + \frac{1}{G_{T1}}} \qquad \text{......(2.2.1c)}$$

$$G'_{A\phi}\Big|_{\phi=0} = \frac{1}{\dfrac{1}{G_{A2}-G_{A1}} + \dfrac{1}{2G_{A1}}} \qquad \text{......(2.2.1d)}$$

$$G'_{T(+)\phi}\Big|_{\phi=0} = G'_{T(-)\phi}\Big|_{\phi=0} = \frac{2(G_{T2}-G_{T1})G_{T1}(K_1+G_{T1})}{K_1(G_{T1}+G_{T2})+2G_{T1}G_{T2}} \qquad \text{......(2.2.1e)}$$

Because of equation (2.2.1e),

$$G'_{T\phi}\Big|_{\phi=0} = \frac{2(G_{T2}-G_{T1})G_{T1}(K_1+G_{T1})}{K_1(G_{T1}+G_{T2})+2G_{T1}G_{T2}} \qquad \text{......(2.2.1f)}$$

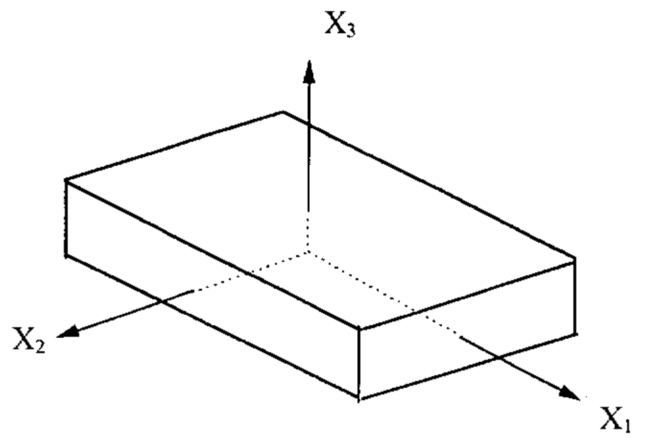
The procedures of numerical calculation for the determination of the effective elastic moduli of a unidirectional fiber composite with anisotropic constituents at an arbitrary fiber volume concentration is discussed below.

Without any reinforcement, the fiber volume concentration in the composite is $\phi_0 = 0$. First, we suppose a very small volume δ of anisotropic fibers, say $\delta \approx 0.001$ with respect to unit volume of composite, are embedded in a homogeneous and anisotropic matrix medium which has elastic moduli K_1, E_{A1}, ν_{A1}, G_{A1} and G_{T1} or in brief $m_{1\text{-}5}$. Fiber elastic moduli are K_2, E_{A2}, ν_{A2}, G_{A2} and G_{T2} or in brief $i_{1\text{-}5}$. Therefore, the corresponding effective elastic moduli are functions $f_{1\text{-}5}$ of $\phi_1 = \delta$ and the elastic properties of the phases, which can be determined by expressions valid at small fiber volume concentration:

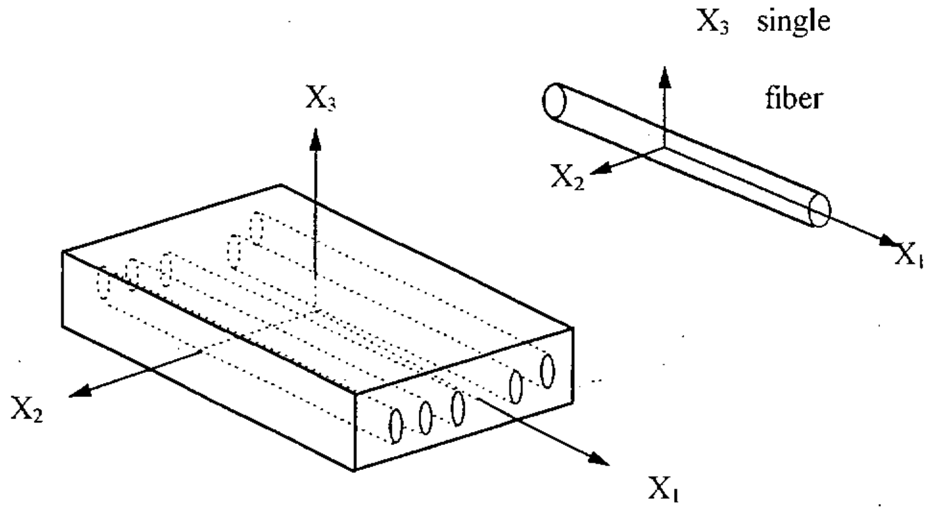
Effective Elastic Moduli = Corresponding Matrix Property + Corresponding Increment Function × Small Fiber Volume Concentration δ

Then take the calculated elastic moduli as new initial values of the medium. Same fiber volume concentration is then embedded in that new medium and the fiber volume in the composite ϕ_2 now is equal to $\phi_1 + \delta - \phi_1\delta$. Following this procedure, the next new fiber volume concentration is $\phi_2 + \delta - \phi_2\delta$. We can then generalize the formula for fiber volume concentration in each step to get $\phi_{n+1} = \phi_n + \delta - \phi_n\delta$, where n is from zero to any positive integer, and thus it is possible to calculate the effective elastic moduli of a fiber composite at arbitrary fiber volume concentration.

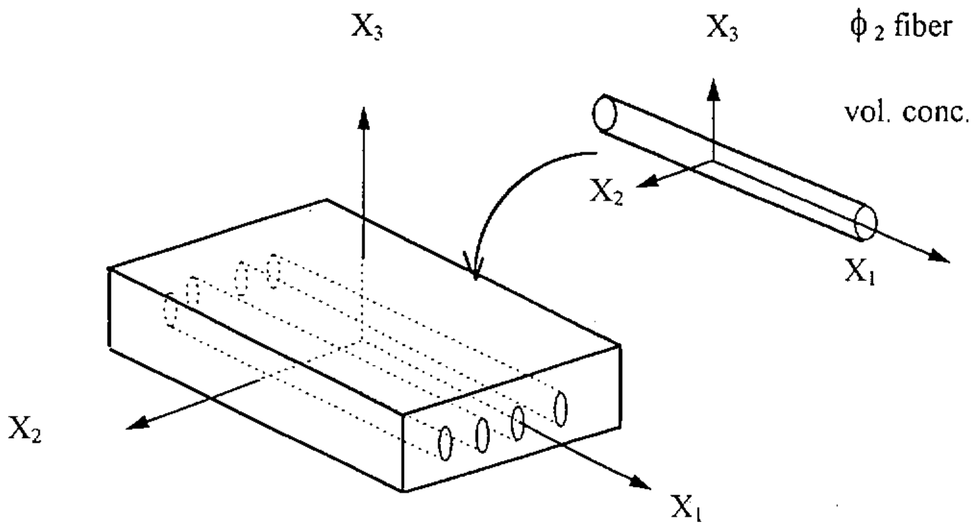
These steps can be illustrated by the following pictures in Figure 2.2.1 .



Step 1. A homogeneous and anisotropic matrix with elastic moduli $m_{1\text{-}5}$.



Step 2. Anisotropic fibers, with elastic moduli $i_{1\text{-}5}$, are embedded in the anisotropic matrix along the matrix axis at very small volume concentration δ. The elastic properties of the composite is $f_{1\text{-}5}$ ($m_{1\text{-}5}$, $i_{1\text{-}5}$, ϕ_1) where $\phi_1 = \delta$.



Step 3. Then take this composite as a new homogeneous and anisotropic matrix, with elastic moduli $f_{1\text{-}5}$ ($m_{1\text{-}5}$, $i_{1\text{-}5}$, ϕ_1). δ volume concentration of fiber is embedded in this new medium. Now the fiber volume concentration is $\phi_2 = \phi_1 + \delta - \phi_1\delta$ and the effective elastic moduli of the composite is $f_{1\text{-}5}$ ($f_{1\text{-}5}$ ($m_{1\text{-}5}$, $i_{1\text{-}5}$, ϕ_1), $i_{1\text{-}5}$, δ).

Figure 2.2.1 The iterative procedures of EMT in computation of effective elastic moduli of a unidirectional fiber composite material with anisotropic constituents.

By repeating the last step, the effective elastic moduli of this composite can be calculated at arbitrary fiber volume concentration ϕ_{n-1}, from the EMT equation f_j ($m_{1\text{-}5}$, $i_{1\text{-}5}$, ϕ_{n-1}) $= f_j$ ($f_{1\text{-}5}$ ($m_{1\text{-}5}$, $i_{1\text{-}5}$, ϕ_n), $i_{1\text{-}5}$, δ). Actually, this is an iteration, or say continuous replacement, of composite properties from the old medium to a new medium and it is assumed the fibers are uniformly distributed throughout the composite at each step.

2.3 Analytical calculations based on EMT

In this section, five coupled expressions for effective elastic moduli of a unidirectional fiber composite with anisotropic constituents are obtained by analytical EMT calculations. The analytical results will be compared with the numerical results.

In the last section, five increment functions, equations (2.2.1a - 2.2.1d, 2.2.1f), are obtained. These are used as the starting point in the following calculation.

Recalling that the notation for matrix and fiber elastic moduli are $m_{1\text{-}5}$ and $i_{1\text{-}5}$, and ϕ is the volume concentration of fiber in the composite. After embedding reinforcing fibers in the matrix, the composite elastic moduli K, $E_A^{'}$, $\nu_A^{'}$, $G_A^{'}$ and $G_T^{'}$ are functions of $m_{1\text{-}5}$, $i_{1\text{-}5}$ and ϕ which we denoted by $f_j\,(m_{1\text{-}5}, i_{1\text{-}5}, \phi)$. Consider a composite with fiber concentration ϕ_1. The effective elastic moduli of this composite material are thus represented by $f_j\,(m_{1\text{-}5}, i_{1\text{-}5}, \phi_1)$. Then use this material as a new matrix material which we embed a volume concentration ϕ_2 of fiber. The fiber volume concentration in the new composite is $\phi = \phi_1 + \phi_2 - \phi_1\phi_2$ and its the effective elastic moduli are given by

$$K(m_{1\text{-}5},\ i_{1\text{-}5},\ \phi_1+\phi_2-\phi_1\phi_2) = K\ \ (K(m_{1\text{-}5},\ i_{1\text{-}5},\ \phi_1),\ E'_A(m_{1\text{-}5},\ i_{1\text{-}5},\ \phi_1),$$
$$\nu'_A(m_{1\text{-}5},\ i_{1\text{-}5},\ \phi_1), G'_A(m_{1\text{-}5},\ i_{1\text{-}5}, \phi_1),\ G'_T(m_{1\text{-}5},\ i_{1\text{-}5}, \phi_1),\ i_{1\text{-}5},\ \phi_2) \qquad \text{......(2.3.1a)}$$

$$E'_A(m_{1\text{-}5},\ i_{1\text{-}5},\ \phi_1+\phi_2-\phi_1\phi_2) = E'_A\ \ (K(m_{1\text{-}5},\ i_{1\text{-}5},\ \phi_1),\ E'_A(m_{1\text{-}5},\ i_{1\text{-}5},\ \phi_1),$$
$$\nu'_A(m_{1\text{-}5},\ i_{1\text{-}5},\ \phi_1), G'_A(m_{1\text{-}5},\ i_{1\text{-}5}, \phi_1),\ G'_T(m_{1\text{-}5},\ i_{1\text{-}5}, \phi_1),\ i_{1\text{-}5},\ \phi_2) \qquad \text{......(2.3.1b)}$$

$$\nu'_A(m_{1\text{-}5},\ i_{1\text{-}5},\ \phi_1+\phi_2-\phi_1\phi_2) = \nu'_A\ \ (K(m_{1\text{-}5},\ i_{1\text{-}5},\ \phi_1),\ E'_A(m_{1\text{-}5},\ i_{1\text{-}5},\ \phi_1),$$
$$\nu'_A(m_{1\text{-}5},\ i_{1\text{-}5},\ \phi_1), G'_A(m_{1\text{-}5},\ i_{1\text{-}5}, \phi_1),\ G'_T(m_{1\text{-}5},\ i_{1\text{-}5}, \phi_1),\ i_{1\text{-}5},\ \phi_2) \qquad \text{......(2.3.1c)}$$

$$G'_A(m_{1\text{-}5},\ i_{1\text{-}5},\ \phi_1+\phi_2-\phi_1\phi_2) = G'_A\ \ (K(m_{1\text{-}5},\ i_{1\text{-}5},\ \phi_1),\ E'_A(m_{1\text{-}5},\ i_{1\text{-}5},\ \phi_1),$$
$$\nu'_A(m_{1\text{-}5},\ i_{1\text{-}5},\ \phi_1), G'_A(m_{1\text{-}5},\ i_{1\text{-}5}, \phi_1),\ G'_T(m_{1\text{-}5},\ i_{1\text{-}5}, \phi_1),\ i_{1\text{-}5},\ \phi_2) \qquad \text{......(2.3.1d)}$$

$$G'_T(m_{1\text{-}5},\ i_{1\text{-}5},\ \phi_1+\phi_2-\phi_1\phi_2) = G'_T\ \ (K(m_{1\text{-}5},\ i_{1\text{-}5},\ \phi_1),\ E'_A(m_{1\text{-}5},\ i_{1\text{-}5},\ \phi_1),$$
$$\nu'_A(m_{1\text{-}5},\ i_{1\text{-}5},\ \phi_1), G'_A(m_{1\text{-}5},\ i_{1\text{-}5}, \phi_1),\ G'_T(m_{1\text{-}5},\ i_{1\text{-}5}, \phi_1),\ i_{1\text{-}5},\ \phi_2) \qquad \text{......(2.3.1e)}$$

We then partially differentiate equations (2.3.1a) to (2.3.1e) with respect to the ϕ_1 and set ϕ_1 equal to 0. Then by renaming ϕ_2 as ϕ and the matrix properties K_1, E_{A1}, ν_{A1}, G_{A1} and G_{T1} as K, E_A, ν_A, G_A and G_T, a set of partial differential equations is obtained:

$$(1-\phi)\frac{\partial K'}{\partial \phi} = K'_{\phi}\Big|_{\phi=0}\frac{\partial K'}{\partial K} + E'_{A\phi}\Big|_{\phi=0}\frac{\partial K'}{\partial E_A} + \nu'_{A\phi}\Big|_{\phi=0}\frac{\partial K'}{\partial \nu_A} + G'_{A\phi}\Big|_{\phi=0}\frac{\partial K'}{\partial G_A} + G'_{T\phi}\Big|_{\phi=0}\frac{\partial K'}{\partial G_T}$$

......(2.3.2a)

$$(1-\phi)\frac{\partial E'_A}{\partial \phi} = K'_{\phi}\Big|_{\phi=0}\frac{\partial E'_A}{\partial K} + E'_{A\phi}\Big|_{\phi=0}\frac{\partial E'_A}{\partial E_A} + \nu'_{A\phi}\Big|_{\phi=0}\frac{\partial E'_A}{\partial \nu_A} + G'_{A\phi}\Big|_{\phi=0}\frac{\partial E'_A}{\partial G_A} + G'_{T\phi}\Big|_{\phi=0}\frac{\partial E'_A}{\partial G_T}$$

......(2.3.2b)

$$(1-\phi)\frac{\partial \nu'_A}{\partial \phi} = K'_{\phi}\Big|_{\phi=0}\frac{\partial \nu'_A}{\partial K} + E'_{A\phi}\Big|_{\phi=0}\frac{\partial \nu'_A}{\partial E_A} + \nu'_{A\phi}\Big|_{\phi=0}\frac{\partial \nu'_A}{\partial \nu_A} + G'_{A\phi}\Big|_{\phi=0}\frac{\partial \nu'_A}{\partial G_A} + G'_{T\phi}\Big|_{\phi=0}\frac{\partial \nu'_A}{\partial G_T}$$

......(2.3.2c)

$$(1-\phi)\frac{\partial G'_A}{\partial \phi} = K'_{\phi}\Big|_{\phi=0}\frac{\partial G'_A}{\partial K} + E'_{A\phi}\Big|_{\phi=0}\frac{\partial G'_A}{\partial E_A} + \nu'_{A\phi}\Big|_{\phi=0}\frac{\partial G'_A}{\partial \nu_A} + G'_{A\phi}\Big|_{\phi=0}\frac{\partial G'_A}{\partial G_A} + G'_{T\phi}\Big|_{\phi=0}\frac{\partial G'_A}{\partial G_T}$$

......(2.3.2d)

$$(1-\phi)\frac{\partial G'_T}{\partial \phi} = K'_{\phi}\Big|_{\phi=0}\frac{\partial G'_T}{\partial K} + E'_{A\phi}\Big|_{\phi=0}\frac{\partial G'_T}{\partial E_A} + \nu'_{A\phi}\Big|_{\phi=0}\frac{\partial G'_T}{\partial \nu_A} + G'_{A\phi}\Big|_{\phi=0}\frac{\partial G'_T}{\partial G_A} + G'_{T\phi}\Big|_{\phi=0}\frac{\partial G'_T}{\partial G_T}$$

......(2.3.2e)

These partial differential equations (PDEs) have the same coefficients, and thus the same general solution. The method of characteristics may be used to give

$$-\frac{d\phi}{(1-\phi)}=\frac{dK}{\left.\frac{\partial K'}{\partial \phi}\right|_{\phi=0}}=\frac{dE_A}{\left.\frac{\partial E'_A}{\partial \phi}\right|_{\phi=0}}=\frac{d\nu_A}{\left.\frac{\partial \nu'_A}{\partial \phi}\right|_{\phi=0}}=\frac{dG_A}{\left.\frac{\partial G'_A}{\partial \phi}\right|_{\phi=0}}=\frac{dG_T}{\left.\frac{\partial G'_T}{\partial \phi}\right|_{\phi=0}} \qquad \text{......(2.3.3)}$$

2.3.1 Solution of first-order linear partial differential equations: Cauchy problem and method of characteristics

Before going to solve the five PDEs it is relevant to give an introduction to linear equations and their solution. The Cauchy problem is presented because it is directly applicable to our problem. In this part, the Cauchy problem and its solution by the method of characteristics will be discussed.

For a given first-order partial linear differential equation,

$$a_1(x_1,x_2,\ldots,x_n)\frac{\partial u}{\partial x_1}+\ldots+a_n(x_1,x_2,\ldots,x_n)\frac{\partial u}{\partial x_n}=0 \qquad \text{......(2.3.4)}$$

where a_i are real valued functions of the n real variables, the method of characteristics consists in solving the following equation(s):

$$\frac{dx_1}{a_1(x_1,x_2,\ldots,x_n)} = \frac{dx_2}{a_2(x_1,x_2,\ldots,x_n)} = \ldots = \frac{dx_n}{a_n(x_1,x_2,\ldots,x_n)} \qquad \ldots\ldots(2.3.5)$$

for $i = 1,2,\ldots,n$

This is known as the characteristic equation of equation (2.3.4). A function

$$\psi(x_1, x_2, \ldots, x_n) = \text{constant} \qquad \ldots\ldots(2.3.6)$$

which satisfies (2.3.5) is known as a first integral. There are altogether n - 1 independent first integrals from equation (2.3.5).

Now we consider the Cauchy (initial value) problem:

$$\begin{cases} \sum_{i=1}^{n} a_i(x_1,x_2,\ldots,x_n)\dfrac{\partial u}{\partial x_i} = 0 \\ u\Big|_{x_1=x_1^0} = \varphi(x_2,\ldots,x_n) \end{cases} \qquad \ldots\ldots(2.3.7)$$

In the equation (2.3.7), the function $\varphi(x_2,\ldots, x_n)$ is known and is continuous and differentiable. Now assume that the functions $\psi_i(x_1, x_2,\ldots, x_n)$, where $i = 1, 2,\ldots, n - 1$, are n - 1 independent first integrals. Putting $x_1 = x_1^0$ into ψ_i defines $\overline{\psi}_i$:

$$\begin{cases} \psi_1(x_1^0, x_2, \ldots, x_n) = \overline{\psi_1} \\ \psi_2(x_1^0, x_2, \ldots, x_n) = \overline{\psi_2} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \psi_{n-1}(x_1^0, x_2, \ldots, x_n) = \overline{\psi}_{n-1} \end{cases} \qquad \ldots\ldots(2.3.8)$$

from which solutions of x_2, ..., x_n may be obtained,

$$\begin{cases} x_2 = \omega_2(\overline{\psi}_1, \overline{\psi}_2, \ldots, \overline{\psi}_{n-1}) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n = \omega_n(\overline{\psi}_1, \overline{\psi}_2, \ldots, \overline{\psi}_{n-1}) \end{cases} \qquad \ldots\ldots(2.3.9)$$

As a result, the solution of the Cauchy problem is as follow:

$$u = \varphi(\omega_2(\psi_1, \psi_2, \ldots, \psi_{n-1}), \ldots\ldots, \omega_n(\psi_1, \psi_2, \ldots, \psi_{n-1})) \qquad \ldots\ldots(2.3.10)$$

2.3.2 Calculation of G_A' for unidirectional fiber composites

In this and the following sections, we will present the calculation of the effective moduli of unidirectional fiber composites with anisotropic phases by integrating equation (2.3.3).

The expression for G_A' is now derived first, from equation (2.3.3), by considering

$$-\frac{d\phi}{(1-\phi)} = \frac{dG_A}{G_{A\phi}\big|_{\phi=0}} \qquad \text{......(2.3.11)}$$

On the right hand side, the denominator is the increment function of G_A' which is given explicitly by equation (2.2.1d), thus,

$$\frac{d\phi}{(1-\phi)} = (\frac{1}{G_{A2}-G_A} + \frac{1}{2G_A})dG_A \qquad \text{......(2.3.12)}$$

By integrating equation (2.3.12),

$$\ln(1-\phi) = -\ln(G_{A2}-G_A) + \frac{1}{2}\ln G_A + const. \qquad \text{......(2.3.13)}$$

from which a first integral $\Phi_{G_A}(\phi)$ is obtained:

$$\Phi_{G_A}(\phi) = \frac{G_{A2} - G_A}{\sqrt{G_A}}(1-\phi) \qquad \text{......(2.3.14)}$$

Then evaluate this first integral at $\phi = 0$ to get

$$\Phi_{G_A}(0) = \frac{G_{A2} - G_{A1}}{\sqrt{G_{A1}}} \qquad \text{......(2.3.15)}$$

Following the prescription of the Cauchy problem, the resulting expression for the shear modulus G_A' of the composite is, finally,

$$\frac{G_{A2} - G_A'}{\sqrt{G_A'}} = \frac{G_{A2} - G_{A1}}{\sqrt{G_{A1}}}(1-\phi) \qquad \text{......(2.3.16)}$$

2.3.3 Calculation of G_T' and K' for unidirectional fiber composites

For the evaluation of G_T', again a pair of equations are picked up from equation (2.3.3) with explicit increment functions from (2.2.1a) and (2.2.1f):

$$-\frac{d\phi}{(1-\phi)}=\left(\frac{1}{K_2-K}+\frac{1}{K+G_T}\right)dK \qquad \text{......(2.3.17)}$$

$$-\frac{d\phi}{(1-\phi)}=\left(\frac{K(G_T+G_{T2})+2G_{T2}G_T}{2G_T(G_{T2}-G_T)(K+G_T)}\right)dG_T \qquad \text{......(2.3.18)}$$

However, it is necessary to simplify the denominator in equation (2.3.18) by use of partial fractions before integration and this results in:

$$-\frac{2d\phi}{(1-\phi)}=\left(\frac{1}{K+G_T}+\frac{2}{G_{T2}-G_T}+\frac{1}{G_T}\right)dG_T \qquad \text{......(2.3.19)}$$

We now aim to relate the composite elastic moduli G_T' and K'. This can be done by adding equation (2.3.17) to (2.3.19) multiplied by two, thus

$$-\frac{3d\phi}{(1-\phi)}=\left(\frac{1}{K_2-K}+\frac{1}{K+G_T}\right)dK+\left(\frac{1}{K+G_T}+\frac{2}{G_{T2}-G_T}+\frac{1}{G_T}\right)dG_T \qquad \text{.....(2.3.20)}$$

which may be put as

$$-\frac{3d\phi}{(1-\phi)}=\frac{-d(K_2-K)}{K_2-K}+\frac{d(K+G_T)}{K+G_T}-\frac{2d(G_{T2}-G_T)}{G_{T2}-G_T}+\frac{dG_T}{G_T} \qquad \text{......(2.3.21)}$$

By integrating equation (2.3.21), a first integral is obtained. As in section 2.3.1, we can evaluate this function at $\phi = 0$, which then gives

$$\frac{(K_2-K^{'})(G_{T2}-G_T^{'})^2}{(K^{'}+G_T^{'})G_T}=\frac{(K_2-K_1)(G_{T2}-G_{T1})^2}{(K_1+G_{T1})G_{T1}}(1-\phi)^3 \qquad \text{.......(2.3.22)}$$

However, equation (2.3.22) involves two composite elastic moduli at the same time. We shall have to establish one more equation for further evaluation. From equations (2.3.17) and (2.3.18), we get

$$\left(\frac{1}{K_2-K}+\frac{1}{K+G_T}\right)dK=\left(\frac{K(G_T+G_{T2})+2G_{T2}G_T}{2G_T(G_{T2}-G_T)(K+G_T)}\right)dG_T \qquad \text{......(2.3.23)}$$

This implies

$$\frac{dK}{dG_T}=(K_2-K)\left(\frac{K(G_T+G_{T2})+2G_{T2}G_T}{2G_T(G_{T2}-G_T)(K+G_T)}\right) \qquad \text{......(2.3.24)}$$

We use the substitution $\mu \equiv \dfrac{1}{K - K_2}$ to cast equation (2.3.24) into the following form:

$$\frac{d\mu}{dG_T} - \frac{K_2(G_T + G_{T2}) + 2G_T G_{T2}}{2(K_2 + G_T)G_T(G_{T2} - G_T)}\mu = \frac{G_T + G_{T2}}{2(K_2 + G_T)G_T(G_{T2} - G_T)} \qquad \text{......(2.3.25)}$$

Equation (2.3.25) is then a linear ordinary differential equation and may be integrated to give

$$\mu\frac{G_{T2} - G_T}{\sqrt{G_T(K_2 + G_T)}} = \frac{\sqrt{G_T}}{K_2\sqrt{K_2 + G_T}} + \frac{G_{T2}}{K_2\sqrt{G_T(K_2 + G_T)}} - \frac{2G_{T2}\sqrt{K_2 + G_T}}{K_2^2\sqrt{G_T}} + const.$$

......(2.3.26)

The integration constant may be evaluated by considering $\phi = 0$ where $G_T = G_{T1}$ and $K = K_1$. Solving equations (2.3.22) and (2.3.26) results in:

$$\frac{\dfrac{\Psi}{K_2 G_{T2}^2}(1-\phi)^3\left(\dfrac{3}{G_T'} + \dfrac{2}{K_2} - \dfrac{1}{G_{T2}}\right) - \left(\dfrac{1}{G_T'} - \dfrac{1}{G_{T2}}\right)^3}{(1-\phi)^3\left(\dfrac{1}{G_T'} + \dfrac{1}{K_2}\right)^{\frac{3}{2}}} = \frac{\dfrac{\Psi}{K_2 G_{T2}^2}\left(\dfrac{3}{G_{T1}} + \dfrac{2}{K_2} - \dfrac{1}{G_{T2}}\right) - \left(\dfrac{1}{G_{T1}} - \dfrac{1}{G_{T2}}\right)^3}{\left(\dfrac{1}{G_{T1}} + \dfrac{1}{K_2}\right)^{\frac{3}{2}}}$$

......(2.3.27)

where

$$\Psi = \frac{(K_2 - K_1)(G_{T2} - G_{T1})^2}{(K_1 + G_{T1})G_{T1}}$$

Indeed, equation (2.3.27) is the $G_T^{'}$ expression at arbitrary fiber volume concentration. Thus once the value of $G_T^{'}$ is calculated from given phase moduli and fiber volume concentration, $K^{'}$ can also be calculated from equation (2.3.22).

2.3.4 Calculation of ν_A' for unidirectional fiber composites

From equations (2.3.3), (2.2.1a) and (2.2.1c) ,the following is obtained:

$$\frac{dK}{\dfrac{1}{\left(\dfrac{1}{K_2-K}+\dfrac{1}{K+G_T}\right)}}=\frac{d\nu_A}{(\nu_{A2}-\nu_A)\left(1+\dfrac{\dfrac{1}{K}-\dfrac{1}{K_2}}{\dfrac{1}{K_2}+\dfrac{1}{G_T}}\right)} \qquad \text{......(2.3.28)}$$

Upon rearrangement,

$$\frac{(K_2+G_T)}{(K_2-K)(K+G_T)}dK=\frac{1}{(\nu_{A2}-\nu_A)}\frac{K(K_2+G_T)}{K_2(K+G_T)}d\nu_A \qquad \text{......(2.3.29)}$$

which can be simplified to

$$\frac{K_2}{(K_2-K)K}dK=\frac{1}{\nu_{A2}-\nu_A}d\nu_A \qquad \text{......(2.3.30)}$$

This can be integrated to give a first integral from

$$-\ln(K_2-K)+\ln K=-\ln(\nu_{A2}-\nu_A)+const. \qquad \text{......(2.3.31)}$$

and the solution of the Cauchy problem is

$$\frac{K}{K_2 - K'}(\nu_{A2} - \nu'_A) = \frac{K_1}{K_2 - K_1}(\nu_{A2} - \nu_{A1})$$

or

$$\frac{\nu_{A2} - \nu'_A}{\dfrac{1}{K'} - \dfrac{1}{K_2}} = \frac{\nu_{A2} - \nu_{A1}}{\dfrac{1}{K_1} - \dfrac{1}{K_2}} \qquad \text{......(2.3.32)}$$

This equation can be arranged in the form below and the ν'_A value can be calculated in terms of K', the phase moduli, and the fiber volume concentration:

$$\nu'_A = \nu_{A1}(1-\phi) + \nu_{A2}\phi - \frac{\nu_{A2} - \nu_{A1}}{\dfrac{1}{K_2} - \dfrac{1}{K_1}}\left(\frac{1-\phi}{K_1} + \frac{\phi}{K_2} - \frac{1}{K'}\right) \qquad \text{......(2.3.33)}$$

2.3.5 Calculation of $E_A^{'}$ for unidirectional fiber composites

The equations to be used are as shown below from equations (2.3.3), (2.2.1b) and (2.2.1c):

$$-\frac{d\phi}{(1-\phi)}=\frac{dE_A}{E_{A2}-E_A+\frac{4(\nu_{A2}-\nu_A)^2}{\frac{1}{K_2}+\frac{1}{G_T}}}=\frac{d\nu_A}{\nu_{A2}-\nu_A+\frac{(\nu_{A2}-\nu_A)(\frac{1}{K}-\frac{1}{K_2})}{\frac{1}{K_2}+\frac{1}{G_T}}}$$

......(2.3.34)

From equation (2.3.32), we get

$$\frac{\nu_{A2}-\nu_A^{'}}{\frac{1}{K^{'}}-\frac{1}{K_2}}\equiv B=const. \qquad \text{or} \qquad \frac{1}{K^{'}}-\frac{1}{K_2}=\frac{1}{B}(\nu_{A2}-\nu_A^{'})$$

Thus,

$$-\frac{d\phi}{(1-\phi)}=\frac{dE_A}{E_{A2}-E_A+\frac{4(\nu_{A2}-\nu_A)^2}{\frac{1}{K_2}+\frac{1}{G_T}}}=\frac{d\nu_A}{\nu_{A2}-\nu_A+\frac{1}{B}\frac{(\nu_{A2}-\nu_A)^2}{\frac{1}{K_2}+\frac{1}{G_T}}} \quad ......(2.3.35)$$

from which it follows that

$$-\frac{d\phi}{(1-\phi)}=\frac{dE_A-4Bdv_A}{E_{A2}-E_A+\dfrac{4(v_{A2}-v_A)^2}{\dfrac{1}{K_2}+\dfrac{1}{G_T}}-4B[v_{A2}-v_A+\dfrac{1}{B}\dfrac{(v_{A2}-v_A)^2}{\dfrac{1}{K_2}+\dfrac{1}{G_T}}]}$$

......(2.3.36)

Simplifying, we get

$$\frac{-d(E_{A2}-E_A)+4Bd(v_{A2}-v_A)}{(E_{A2}-E_A)-4B(v_{A2}-v_A)}=\frac{-d\phi}{(1-\phi)}$$

or

$$\frac{-d[E_{A2}-E_A-4B(v_{A2}-v_A)]}{E_{A2}-E_A-4B(v_{A2}-v_A)}=\frac{-d\phi}{(1-\phi)} \qquad \text{......(2.3.37)}$$

Integration then gives a first integral $\Phi_{E_A}(\phi)$,

$$\ln(1-\phi)=-\ln[E_{A2}-E_A-4B(v_{A2}-v_A)]+\ln\Phi_{E_A}(\phi) \qquad \text{......(2.3.38)}$$

Upon evaluation at $\phi = 0$, it gives

$$\Phi_{E_A}(0)=E_{A2}-E_{A1}-\frac{4(v_{A2}-v_{A1})^2}{\dfrac{1}{K_1}-\dfrac{1}{K_2}} \qquad \text{......(2.3.39)}$$

resulting in an expression for $E_A^{'}$ as follows:

$$E_{A2} - E_A^{'} - \frac{4(\nu_{A2} - \nu_A^{'})^2}{\frac{1}{K^{'}} - \frac{1}{K_2}} = [E_{A2} - E_{A1} - \frac{4(\nu_{A2} - \nu_{A1})^2}{\frac{1}{K_1} - \frac{1}{K_2}}](1-\phi) \quad ....(2.3.40)$$

This is still too complicated for it involves three composite elastic moduli. From equation (2.3.40), we can solve for $E_A^{'}$,

$$E_A^{'} = E_{A2}\phi + E_{A1}(1-\phi) + \frac{4(\nu_{A2} - \nu_{A1})^2}{\frac{1}{K_1} - \frac{1}{K_2}}(1-\phi) - \frac{4(\nu_{A2} - \nu_A^{'})^2}{\frac{1}{K^{'}} - \frac{1}{K_2}} \quad ......(2.3.41)$$

and since $\frac{\nu_{A2} - \nu_A^{'}}{\frac{1}{K^{'}} - \frac{1}{K_2}} = \frac{\nu_{A2} - \nu_{A1}}{\frac{1}{K_1} - \frac{1}{K_2}}$, we can write

$$E_A^{'} = E_{A2}\phi + E_{A1}(1-\phi) + \frac{4(\nu_{A2} - \nu_{A1})^2}{\frac{1}{K_1} - \frac{1}{K_2}}(1-\phi) - \frac{4(\nu_{A2} - \nu_{A1})^2}{(\frac{1}{K_1} - \frac{1}{K_2})^2}(\frac{1}{K^{'}} - \frac{1}{K_2})$$

......(2.3.42)

Hence

$$E_A^{'} = E_{A2}\phi + E_{A1}(1-\phi) + \frac{4(\nu_{A2} - \nu_{A1})^2}{(\frac{1}{K_1} - \frac{1}{K_2})^2}(\frac{1-\phi}{K_1} + \frac{\phi}{K_2} - \frac{1}{K^{'}})$$

......(2.3.43)

This $E_A^{'}$ expression involves only the $K^{'}$ value which can be calculated from equation (2.3.22) and equation (2.3.27).

2.4 Results and Discussion

In the previous sections, we have discussed a numerical approach and an analytical approach for calculating the elastic moduli of unidirectional fiber composites by EMT. We first make a comparison between these two calculations to cross-check their validity by using as an example the composite system of liquid crystalline polymer and polycarbonate (LCP/ PC). The results are tabulated in Table 2.4.1. It is observed that the difference is found only at the fourth digit which shows the two calculations give almost the same results.

ϕ	E_A' (N)	E_A' (A)	G_A' (N)	G_A' (A)	ν_A' (N)	ν_A' (A)	K' (N)	K' (A)
0	1.24	1.24	0.43	0.43	0.441	0.441	3.65	3.65
0.1	13.492	13.491	0.464	0.465	0.4457	0.4455	3.6827	3.6825
0.2	25.687	25.685	0.5102	0.5101	0.4504	0.4503	3.7315	3.7316
0.3	37.811	37.813	0.5565	0.5563	0.4551	0.4550	3.7492	3.7493
0.4	49.974	49.974	0.6078	0.6077	0.4597	0.4599	3.7833	3.7834
0.5	62.133	62.132	0.6645	0.6646	0.4644	0.4643	3.8180	3.8179
0.6	74.305	74.306	0.7274	0.7273	0.4690	0.4691	3.8533	3.8535
0.7	86.496	89.498	0.7968	0.7967	0.4736	0.4738	3.8891	3.8890
0.8	98.658	98.657	0.8732	0.8733	0.4783	0.4784	3.9254	3.9255
0.9	110.831	110.830	0.9575	0.9574	0.4829	0.4828	3.9624	3.9625

Table 2.4.1 Comparison of elastic constants of the LCP/ PC composite system obtained by numerical and analytical calculations. (N) and (A) denote the numerical and analytical results.

Next, the EMT results are here compared with experimental data reported in the literature as well as with Hashin bounds and the equation of Chamis in Figures 2.4.1a to 2.4.1g, 2.4.2a to 2.4.2g and 2.4.3a to 2.4.3g for three composite systems. The effective elastic moduli are plotted against the fiber volume concentration, ϕ. The equations due to Hashin and Chamis are listed in Appendix A and the phase properties in Appendix C. Since the Chamis equations do not include an equation for plane strain bulk modulus $K^{'}$, only Hashin and EMT calculations are presented in Figures 2.4.1c, 2.4.2c and 2.4.3c. In the following, we are going to discuss the relative merits of the different formulas in these three composite systems.

In one system, the gel-spun polyethylene fibers (PE) are embedded in low-density polyethylene matrix (PE) to form a PE/ PE composite. From the reference [Choy *et al.*, 1995], the polyethylene fibers are highly anisotropic with axial Young's modulus about 40 times higher than the transverse Young's modulus. However, the transverse shear modulus is only 5% smaller than the axial shear modulus. In addition, the axial Young's modulus ratio of fiber to matrix is about 99 times. The five stiffness constants are ultrasonically measured by the contact method and determined from propagation velocities. The effective elastic moduli are calculated from the resulting stiffness constants. The reason for selecting this system for comparison with theory is due to its high anisotropy.

From Figures 2.4.1a to 2.4.1g, the experimental and theoretical results of the effective elastic moduli for the PE/ PE composite are depicted. The predicted $G_A^{'}$ and $G_T^{'}$ results by Chamis shows over-prediction throughout the whole range of fiber volume concentration. The worst Chamis prediction is found in the transverse Poisson's ratio ν_T', since the experimental ν_T' shows a maximum at fiber volume concentration of about 0.2 and then gradually descent .

The Hashin bounds are quite narrow for this system and show generally good agreement with experimental data. It is found that the upper bound has better estimation for overall elastic moduli, except for ν_T', which is nicely predicted by the lower bound. Similar predictions have been obtained by EMT, and they falls within the Hashin bounds. Almost the same predictions are obtained at fiber volume concentration below 0.2 where the EMT results are usually close to the lower bounds.

Another system is a composite prepared from blending polycarbonate (PC) with different weight percentages of a thermotropic liquid crystalline polymer (LCP) [Lau, 1995], and the composite is in the form of drawn strands. It is revealed that the LCP domains in the blends become more elongated in high-drawn strands and in blends with high LCP volume concentration. The data of effective elastic moduli are plotted in Figures 2.4.2a to 2.4.2g.

The Chamis equation over-predicts G_A'. On the other hand, under-predictions are found in the case of G_T', E_T' and ν_T'. Large discrepancies between the equation and the data have been revealed in G_T' and especially in E_T'. However, the ν_T' has better agreement with the experimental data at low fiber volume concentration. The predictions of $E_A^{'}$ and $\nu_A^{'}$ are close to the rule of mixtures for all the models and the data lie on the predicted curves. The Hashin bounds and EMT calculations give almost the same estimations on the effective elastic moduli. It can be observed from the figures that the EMT results and the Hashin bounds often collapse onto the same lines in this composite system.

The third composite system is graphite fibers (ModmorII) in epoxy matrix. The original work [Kriz and Stinchcomb, 1979] aims to extrapolate the complete set of elastic moduli of transversely isotropic fibers from experimental unidirectional fiber reinforced materials data measured by the ultrasonic technique. But the $\nu_A^{'}$ data show fluctuation, contributing to error in the determination.

Almost the same predictions in $E_A^{'}$ are made by all models. Reasonable results are obtained by Chamis in $G_A^{'}$ as shown in Figure 2.4.3a, while in Figure 2.4.3g, it shows under-prediction for $\nu_T^{'}$. On the other hand, it is found that over-predictions are obtained in $G_T^{'}$ and $E_T^{'}$. All elastic moduli predicted by EMT fall within the Hashin bounds. Wide bounds are obtained by Hashin in $G_A^{'}$ and $\nu_T^{'}$. Experimental data are often closer to the lower bounds. On the contrary, the EMT results show good agreement with the data, since the predicted curves are close to the lower bounds of Hashin. Exception is observed in $\nu_T^{'}$ where the data fall within the Hashin bounds.

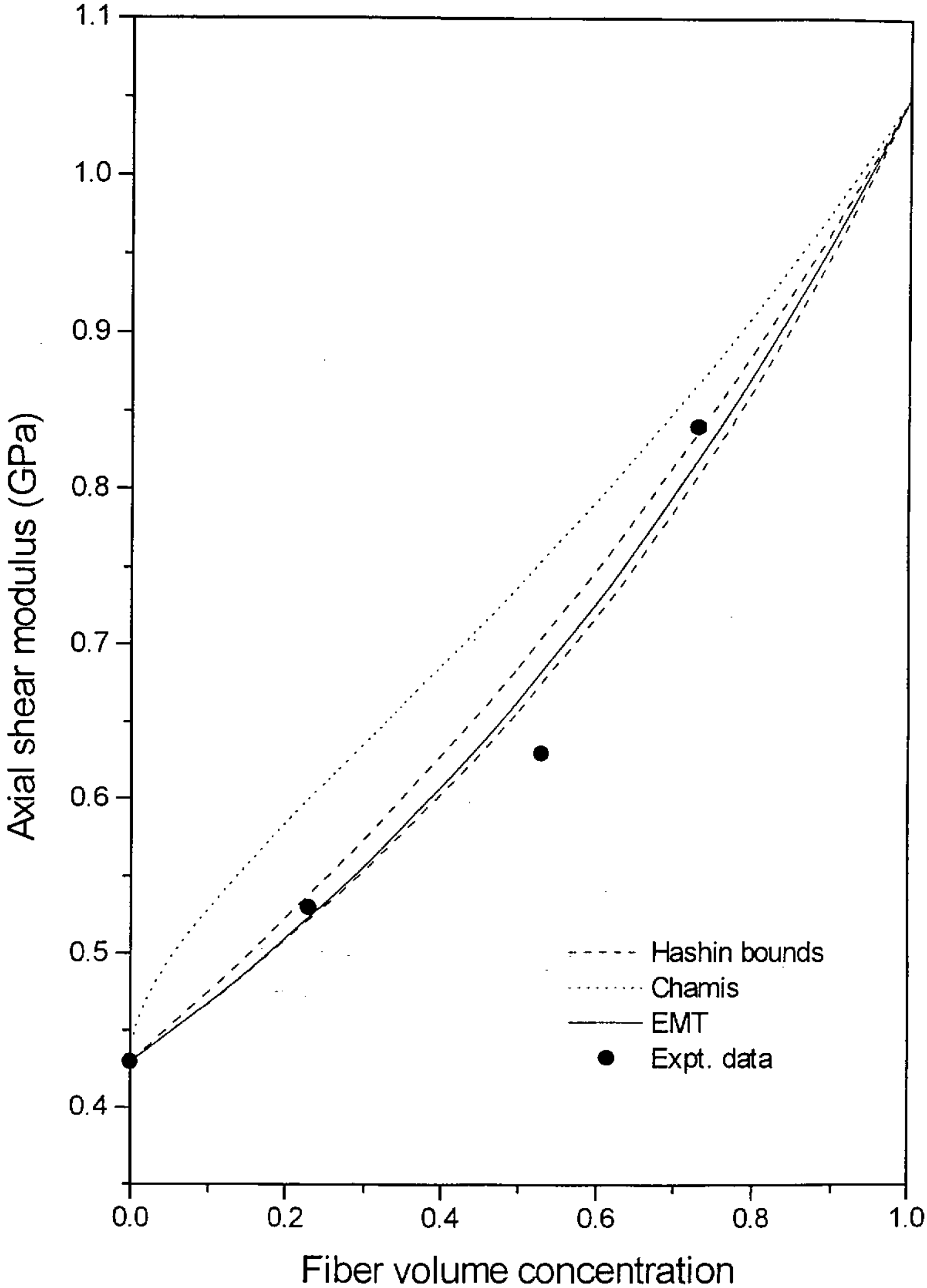


Figure 2.4.1a $G_A^{'}$ versus ϕ of PE/ PE composite.

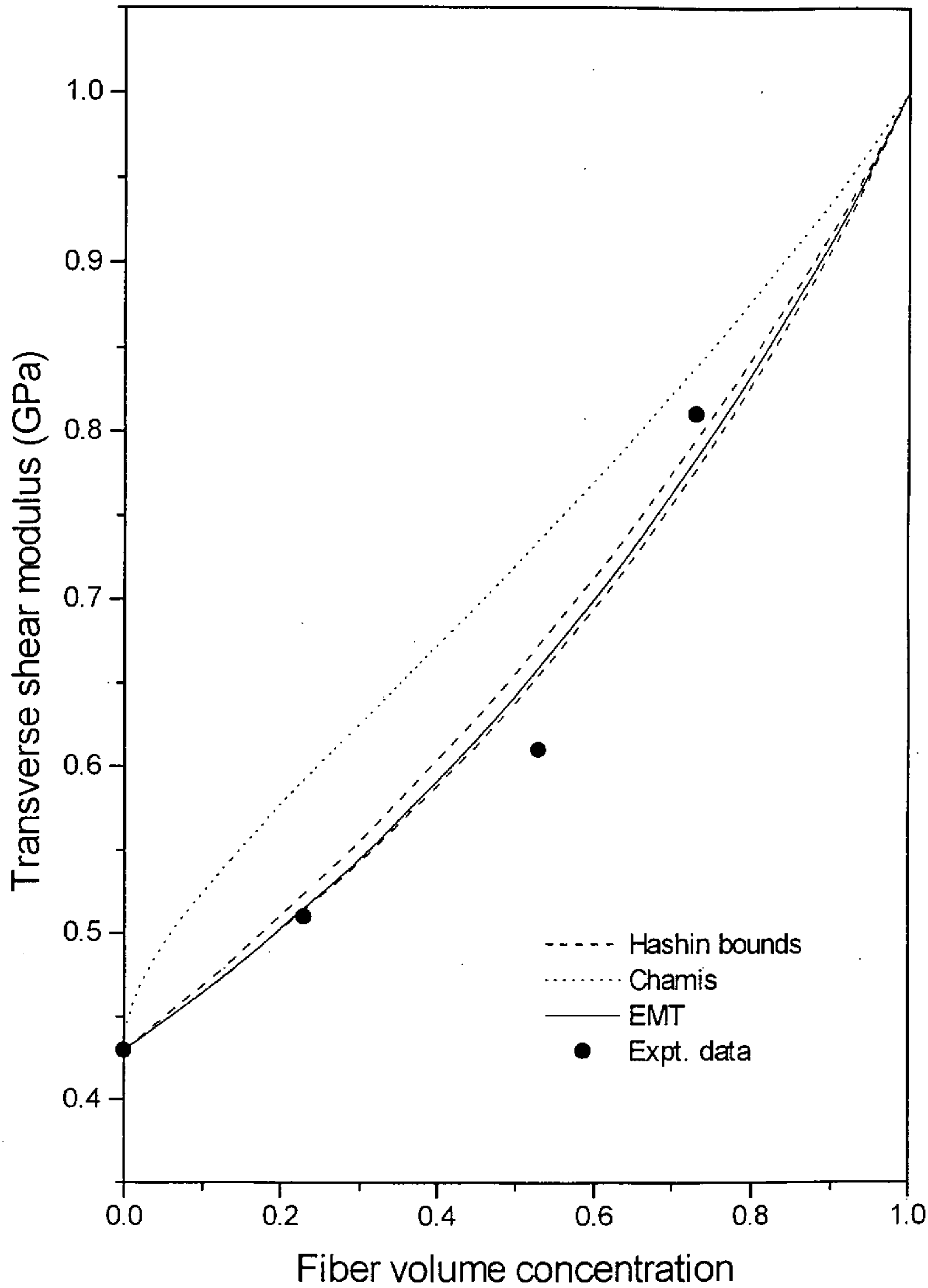


Figure 2.4.1b G_T' versus ϕ of PE/ PE composite.

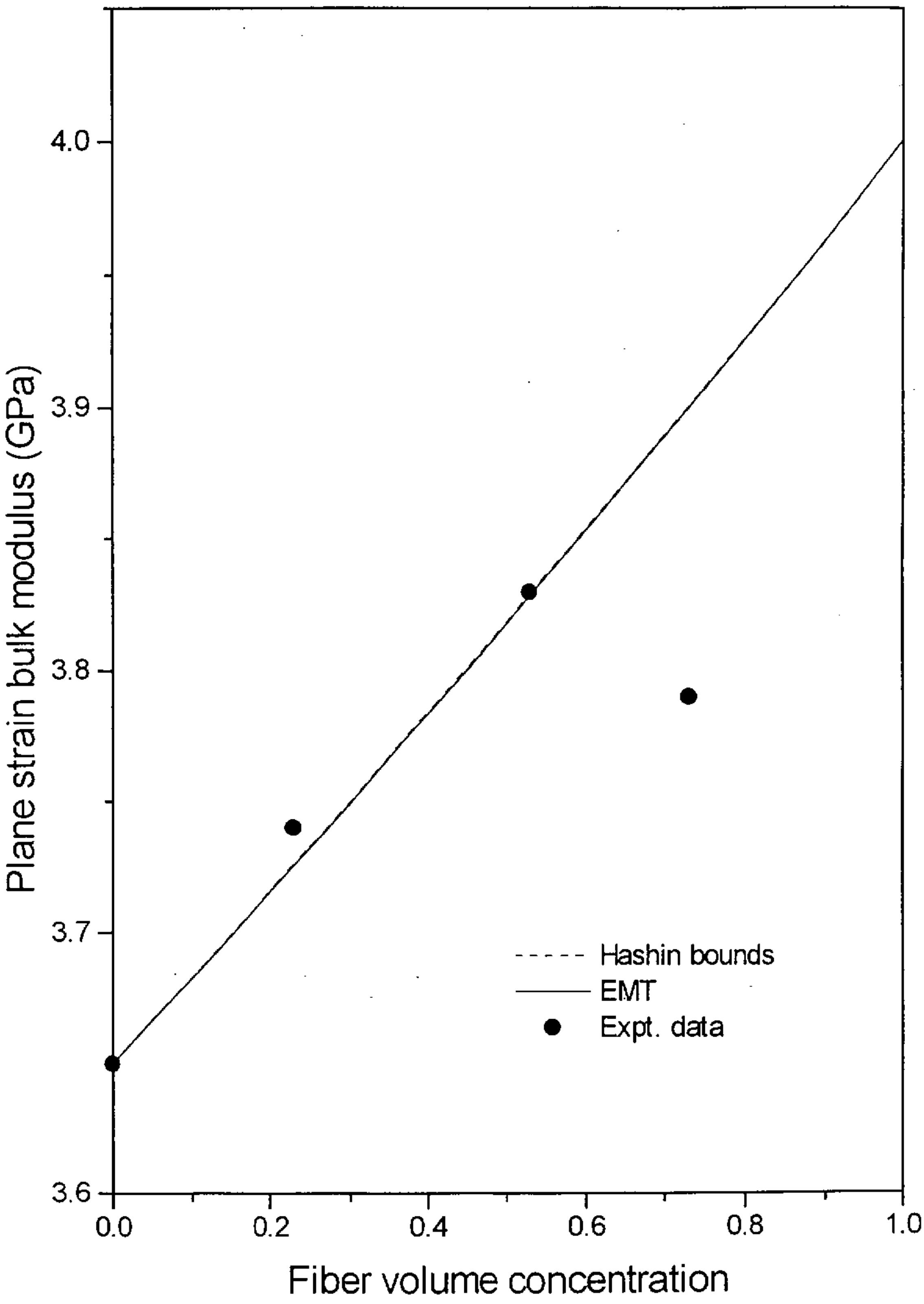


Figure 2.4.1c K' versus ϕ of PE/ PE composite.

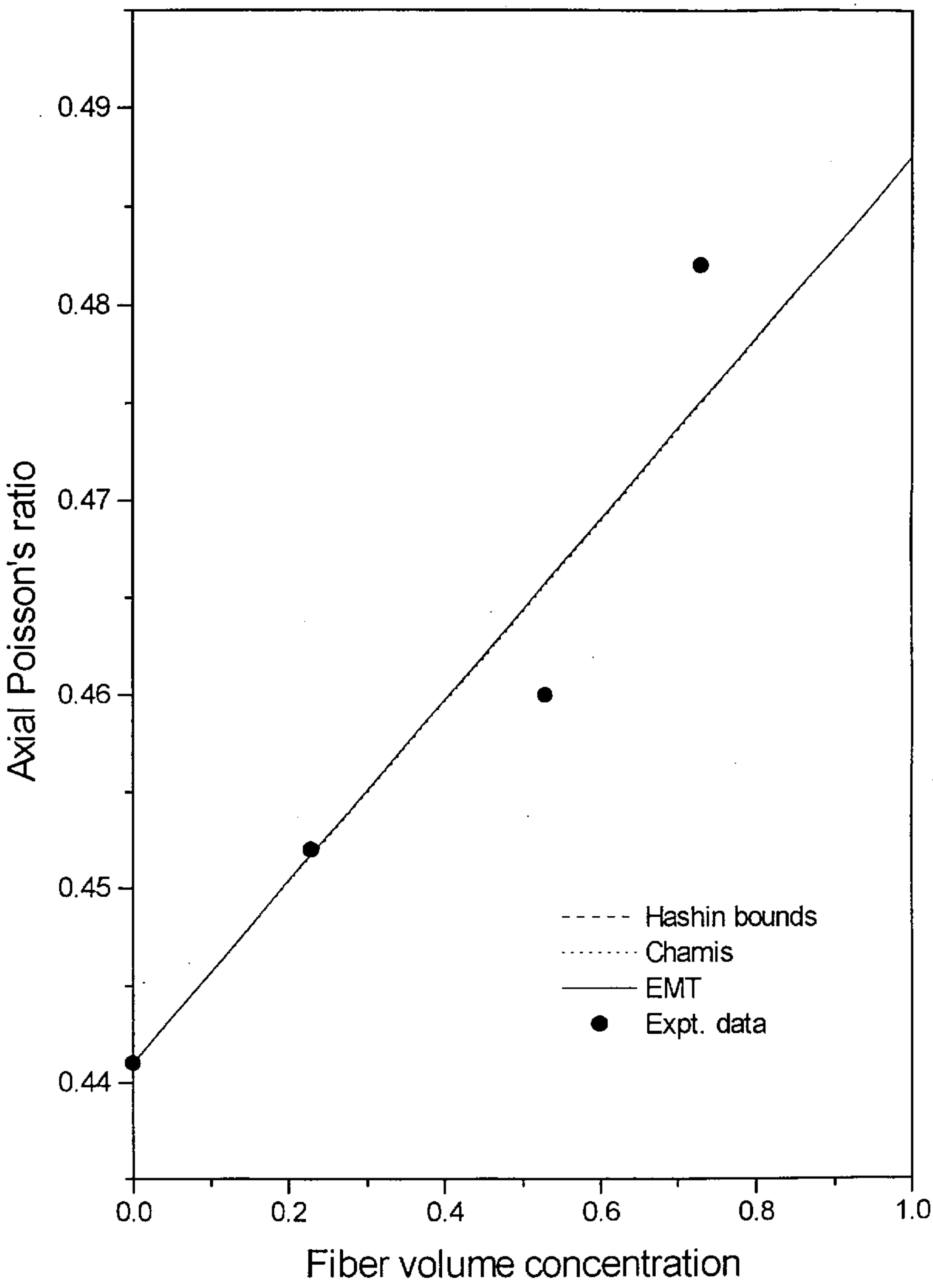


Figure 2.4.1d $\nu_A^{'}$ versus ϕ of PE/ PE composite.

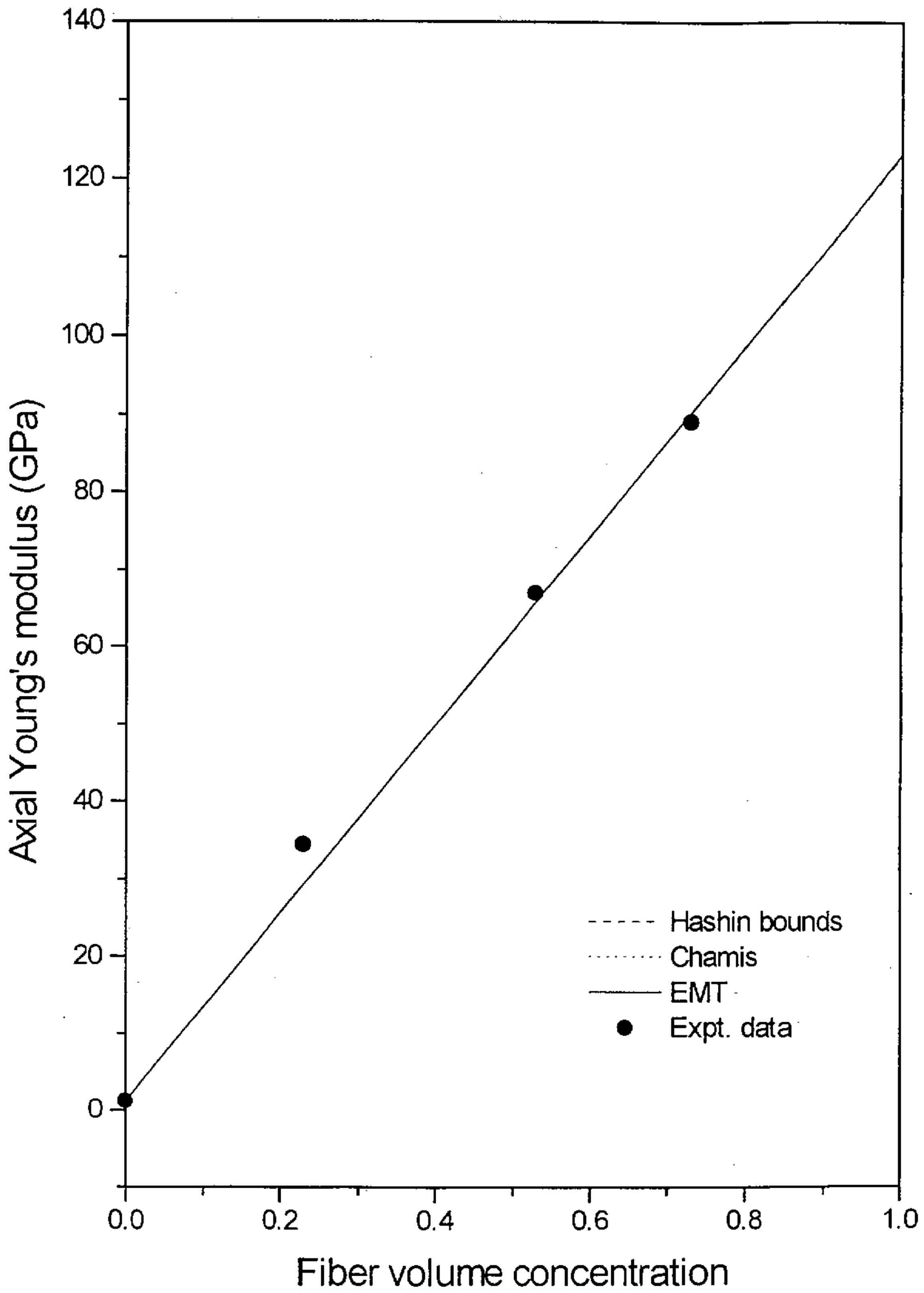


Figure 2.4.1e E_A' versus ϕ of PE/ PE composite.

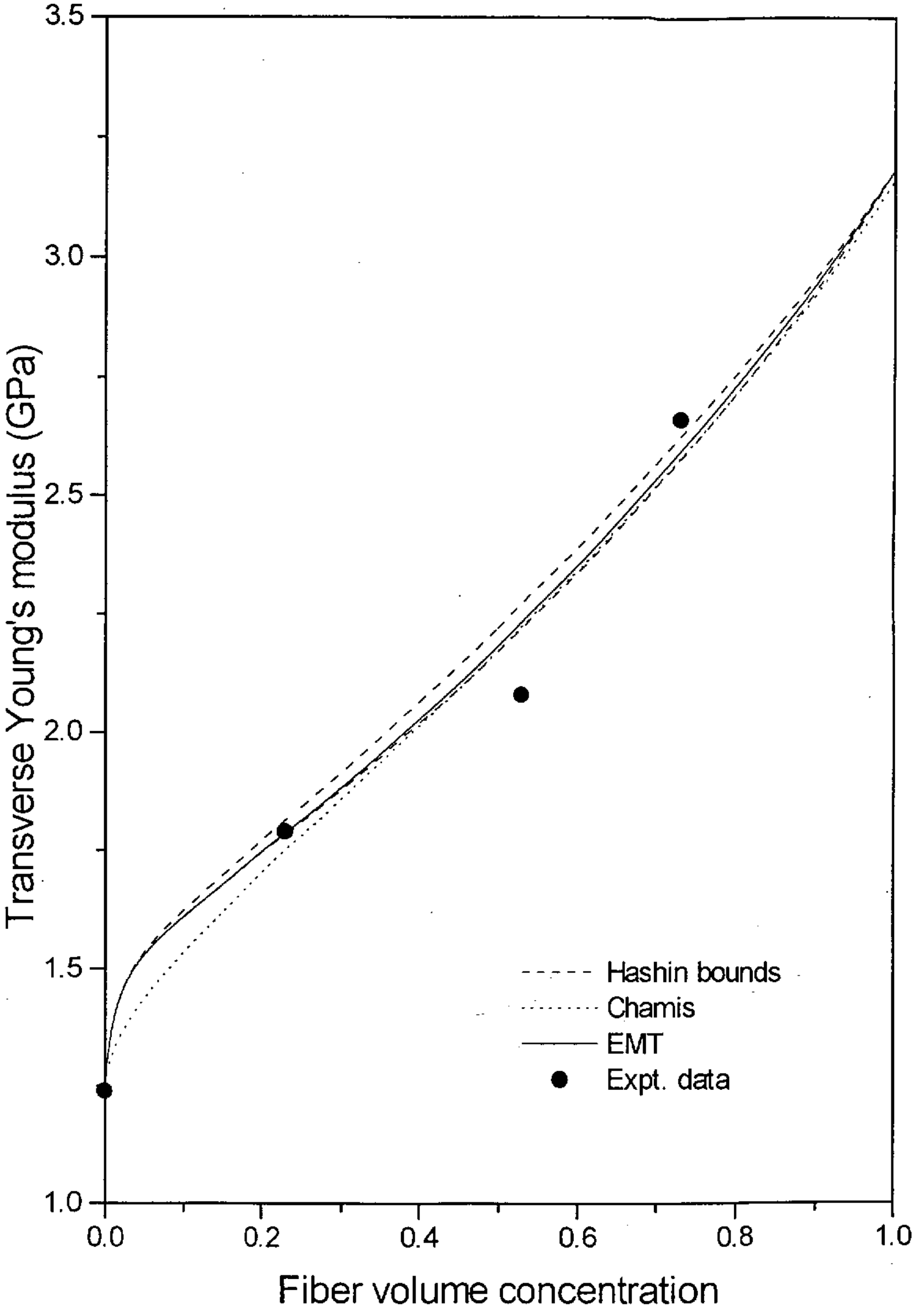


Figure 2.4.1f $E_T^{'}$ versus ϕ of PE/ PE composite.

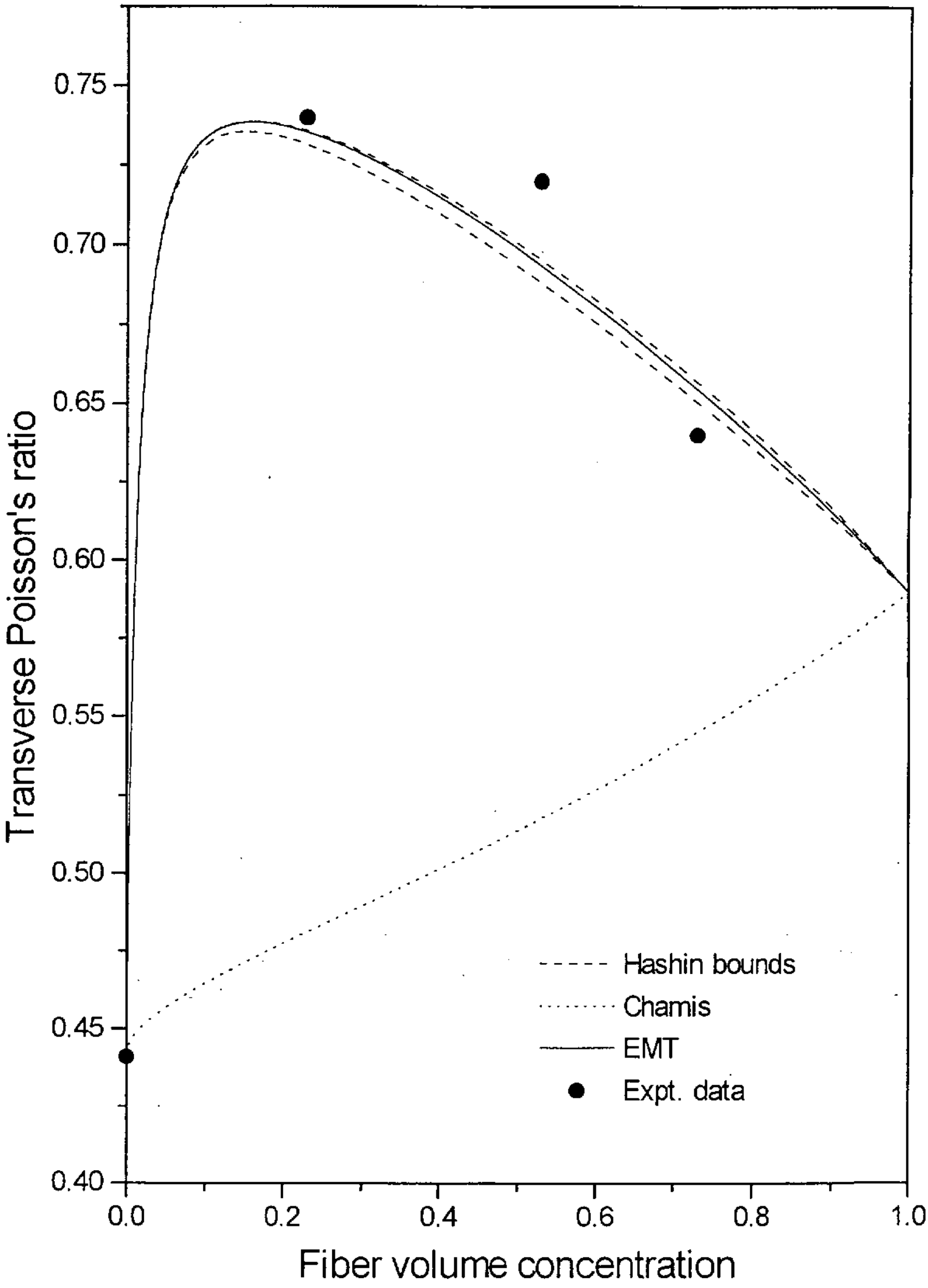


Figure 2.4.1g $\nu_T^{'}$ versus ϕ of PE/ PE composite.

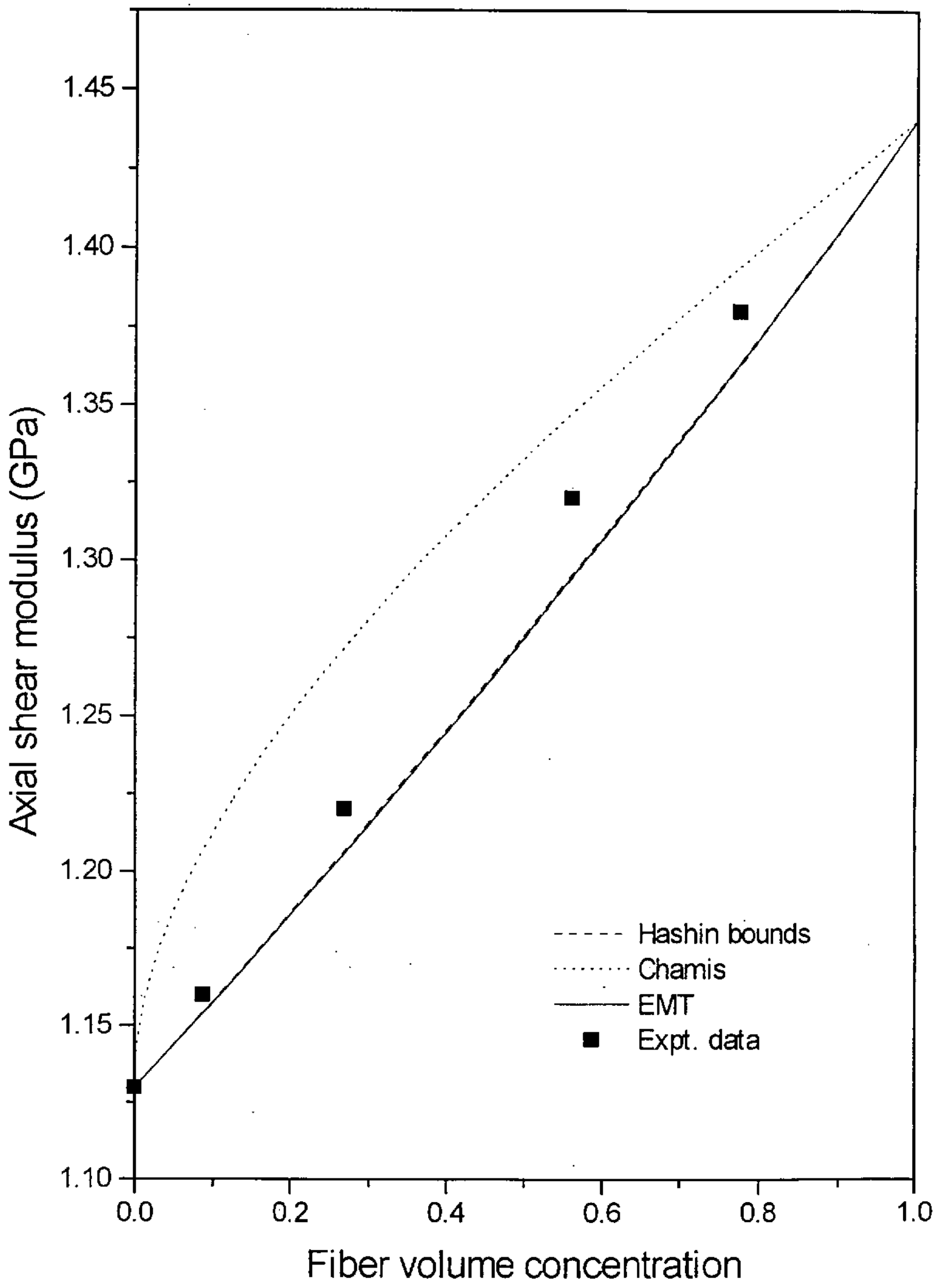


Figure 2.4.2a $G_A^{'}$ versus ϕ of LCP/ PC composite.

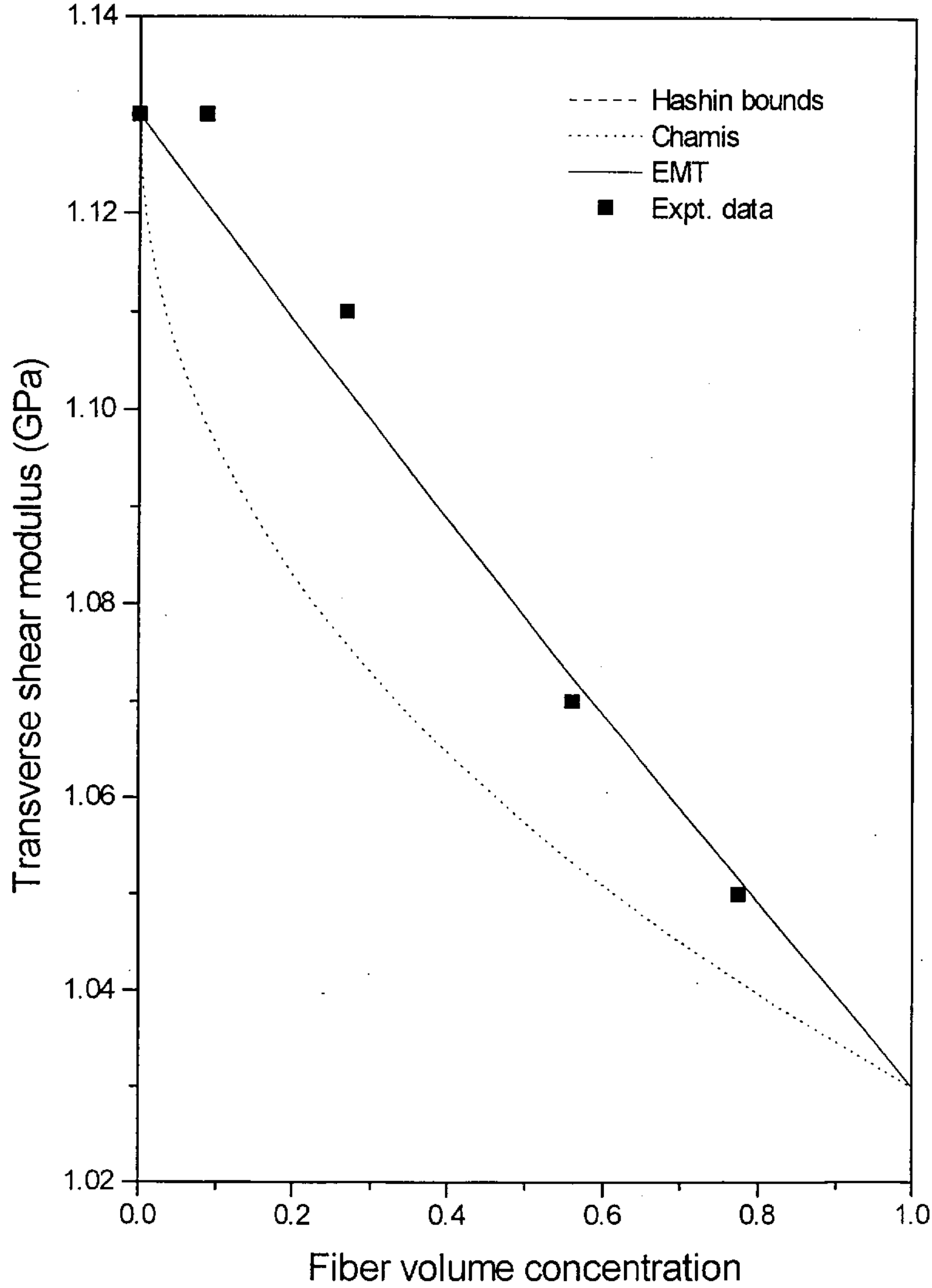


Figure 2.4.2b $G_T^{'}$ versus ϕ of LCP/ PC composite.

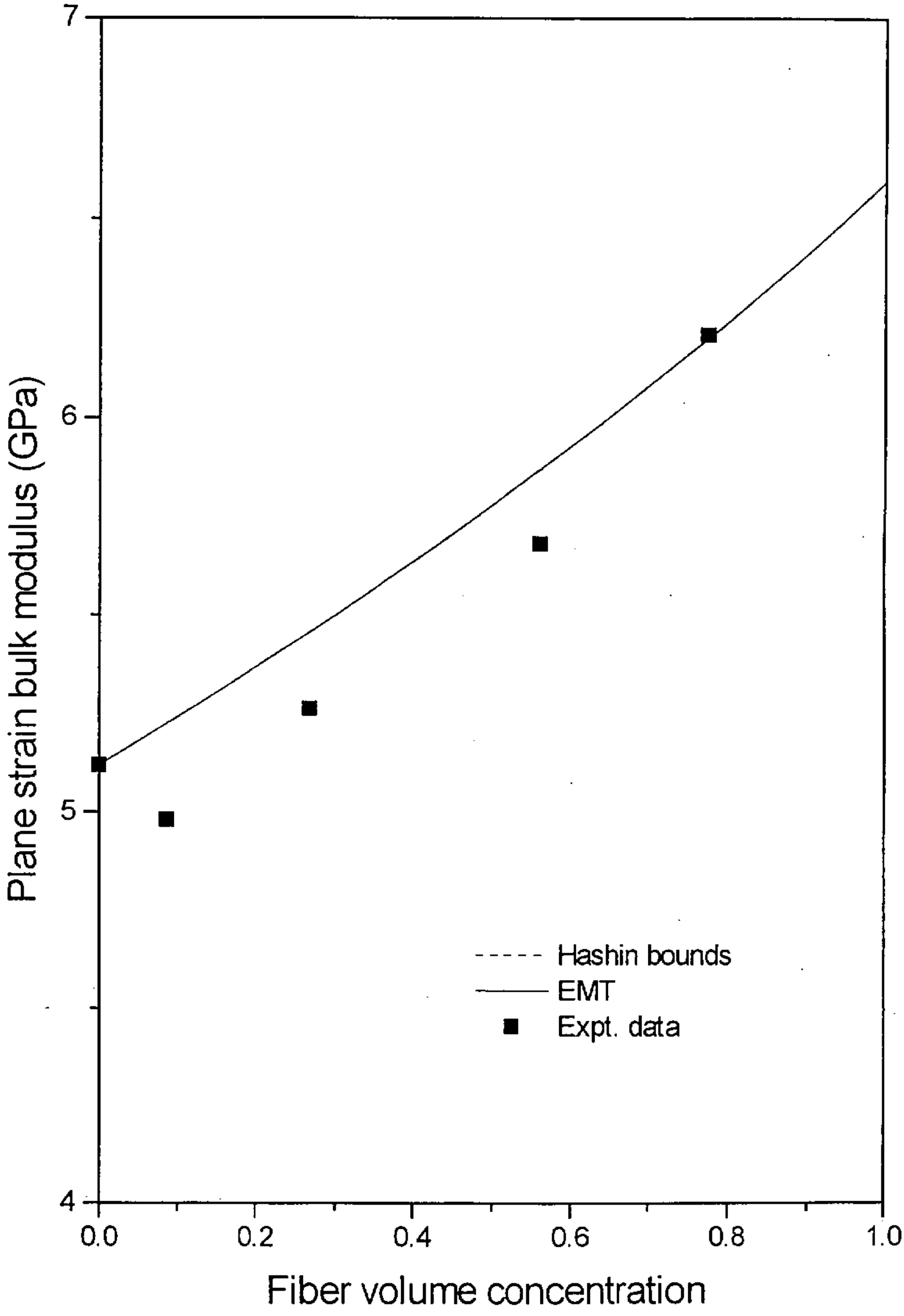


Figure 2.4.2c K' versus ϕ of LCP/ PC composite.

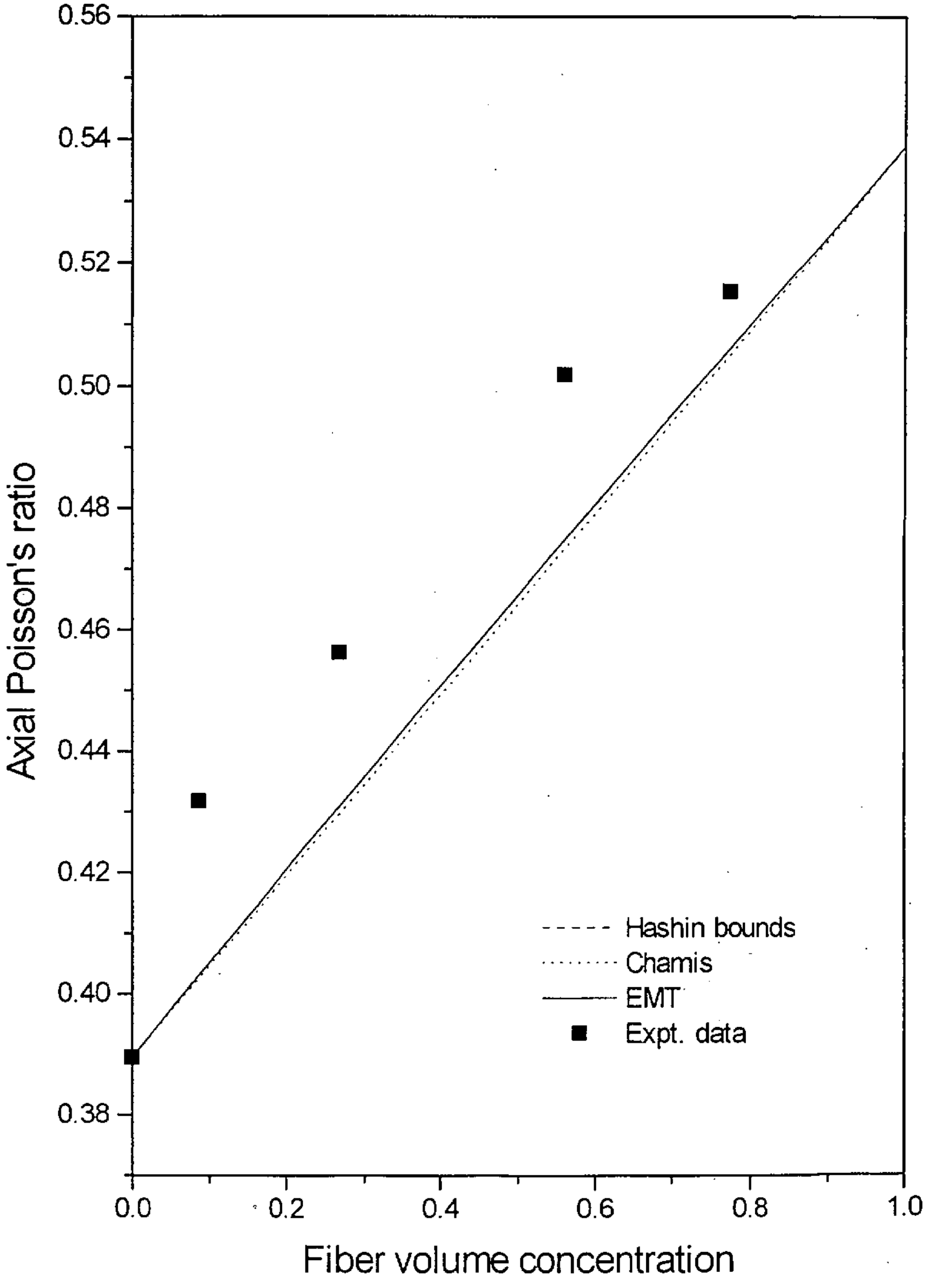


Figure 2.4.2d ν'_A versus ϕ of LCP/ PC composite.

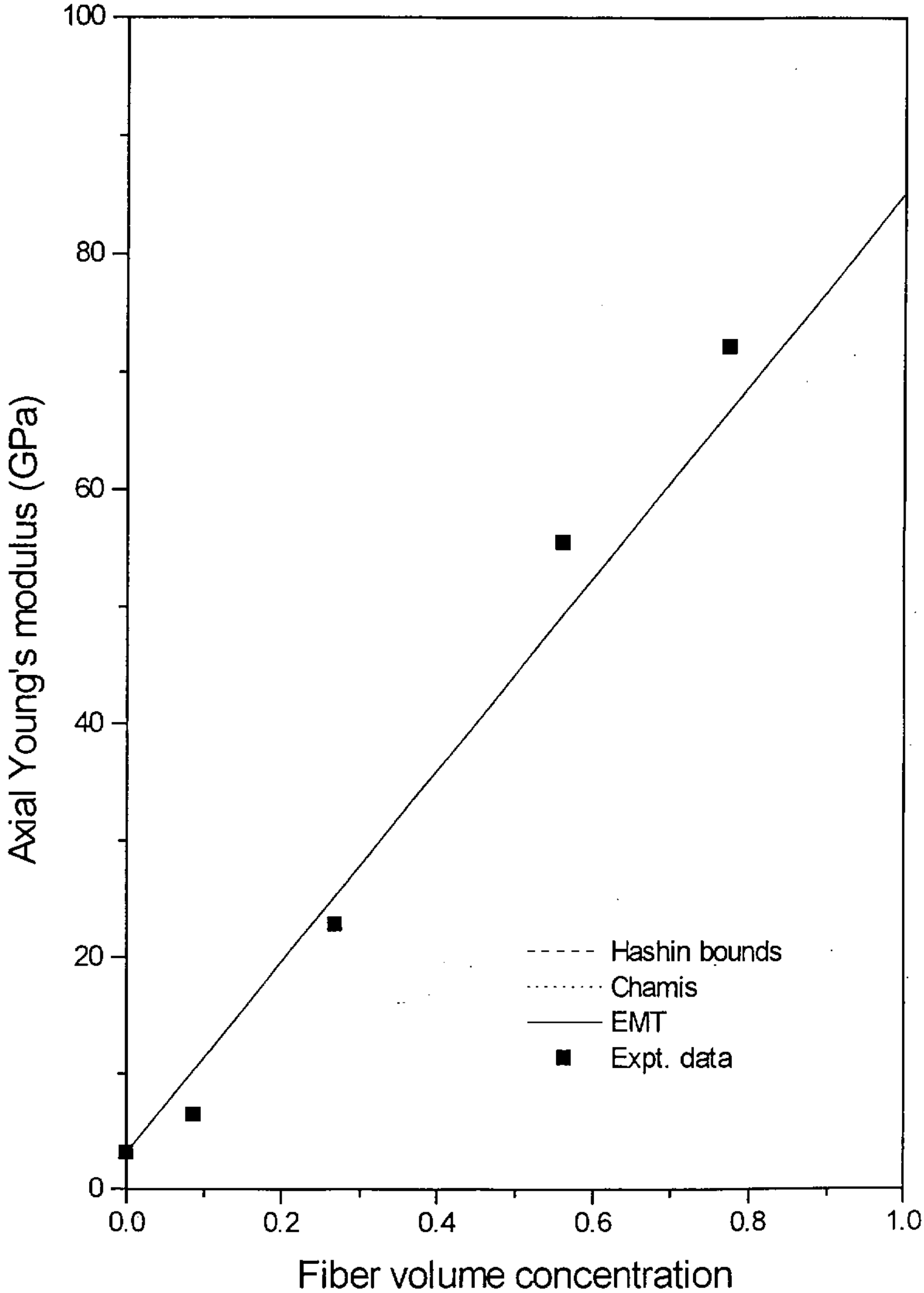


Figure 2.4.2e $E_A^{'}$ versus ϕ of LCP/ PC composite.

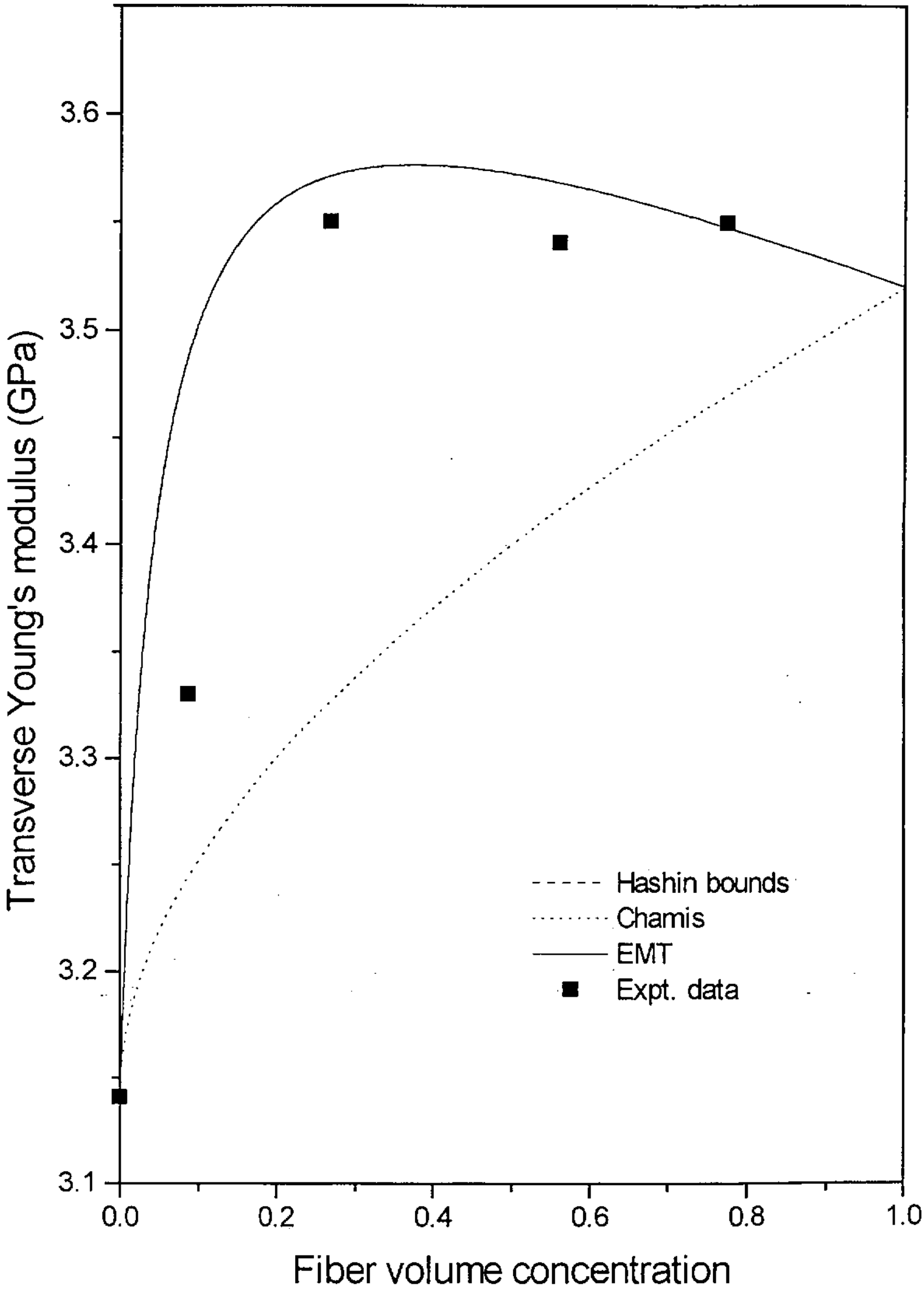


Figure 2.4.2f $E_T^{'}$ versus ϕ of LCP/ PC composite.

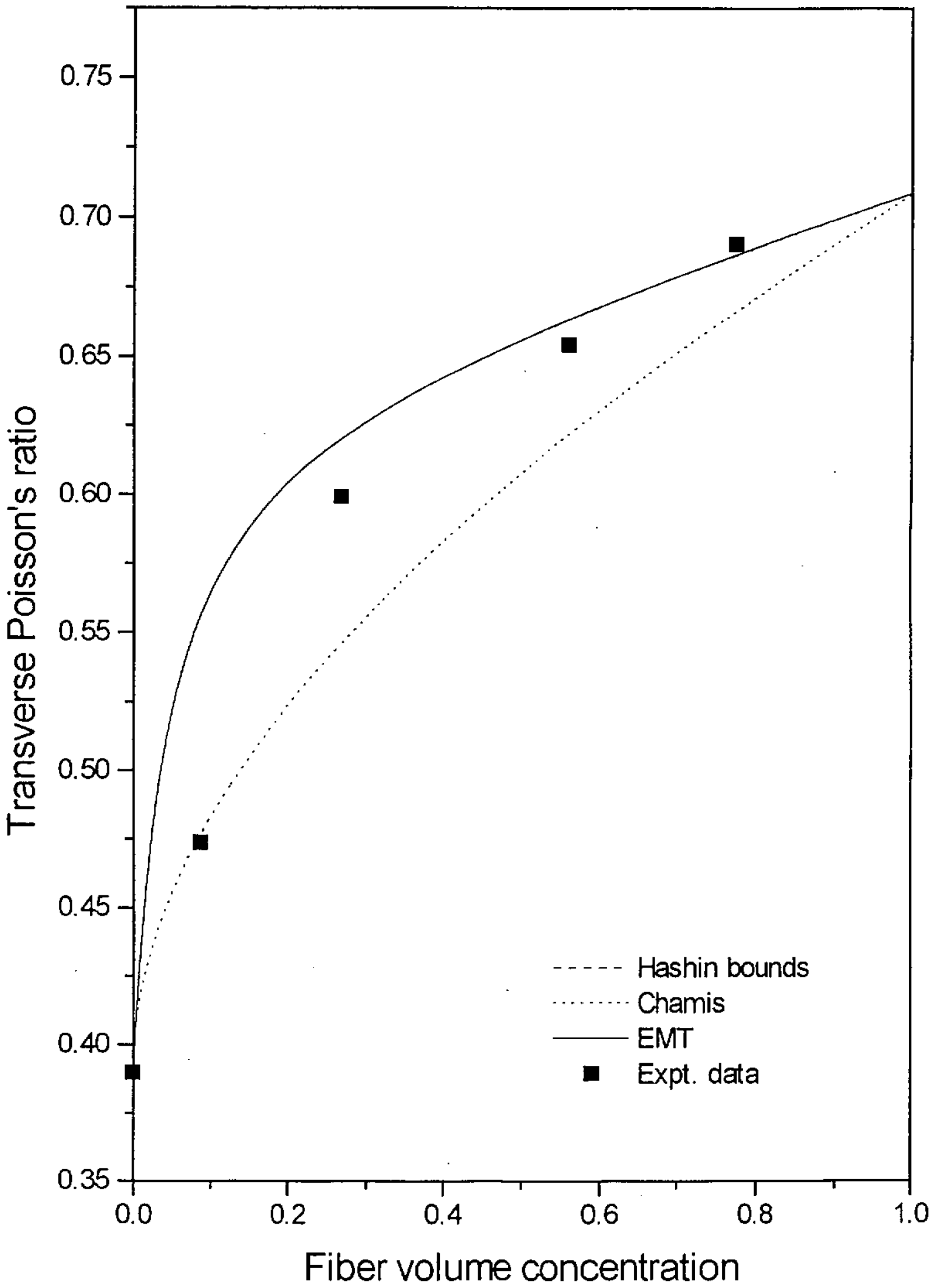


Figure 2.4.2g ν_T' versus ϕ of LCP/ PC composite.

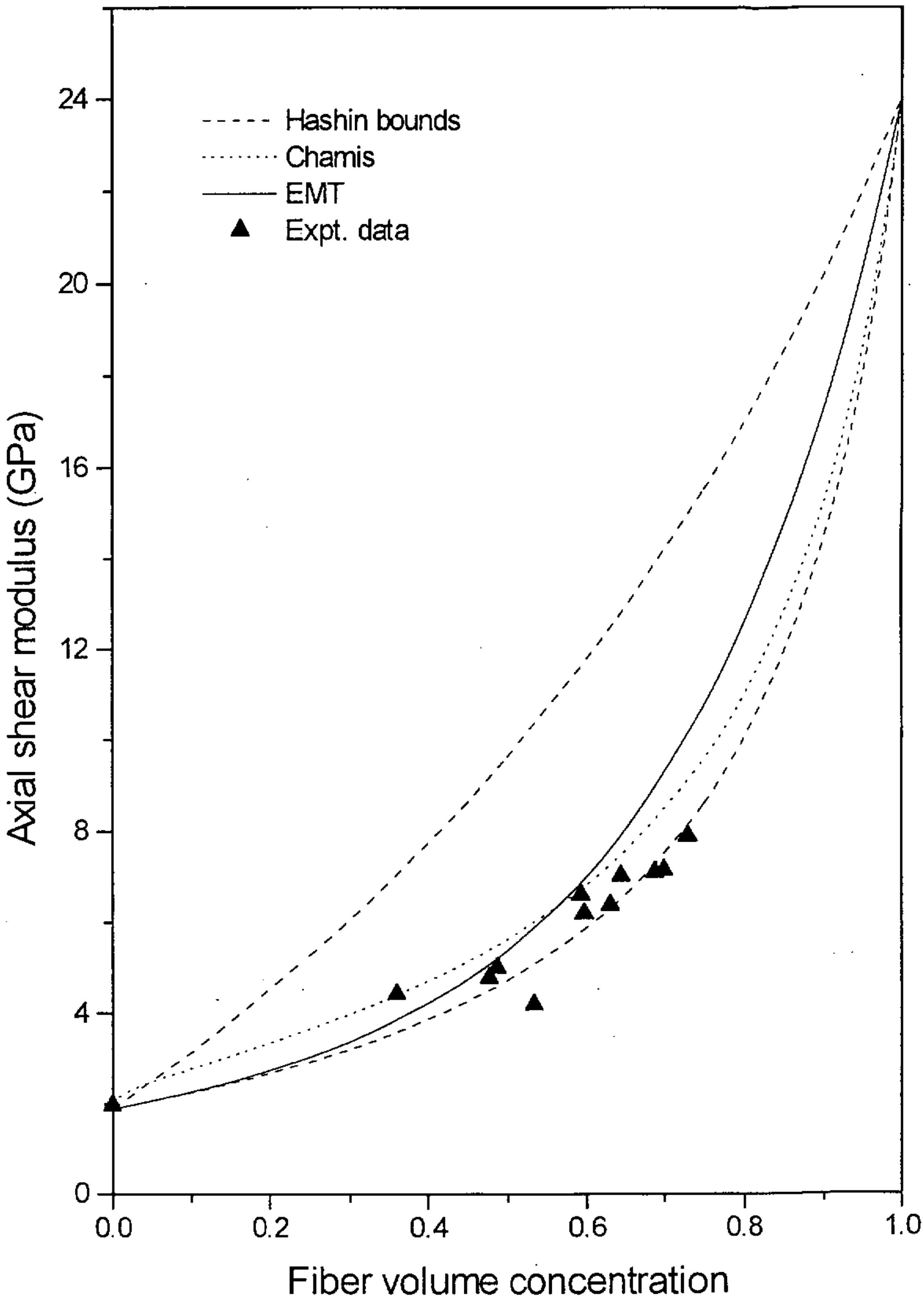


Figure 2.4.3a G_A' versus ϕ of Modmor II/ Epoxy composite.

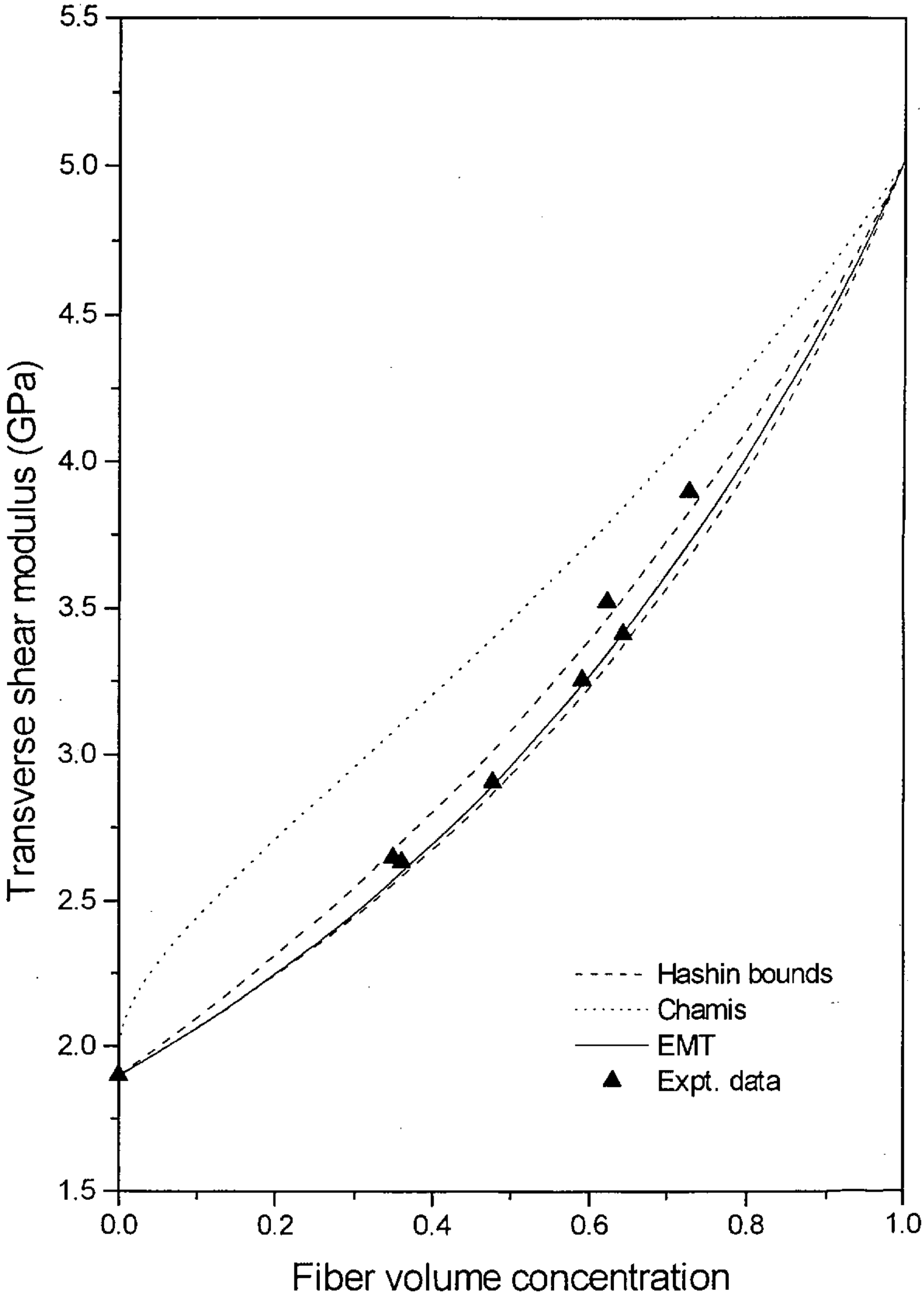


Figure 2.4.3b $G_T^{'}$ versus ϕ of Modmor II/ Epoxy composite.

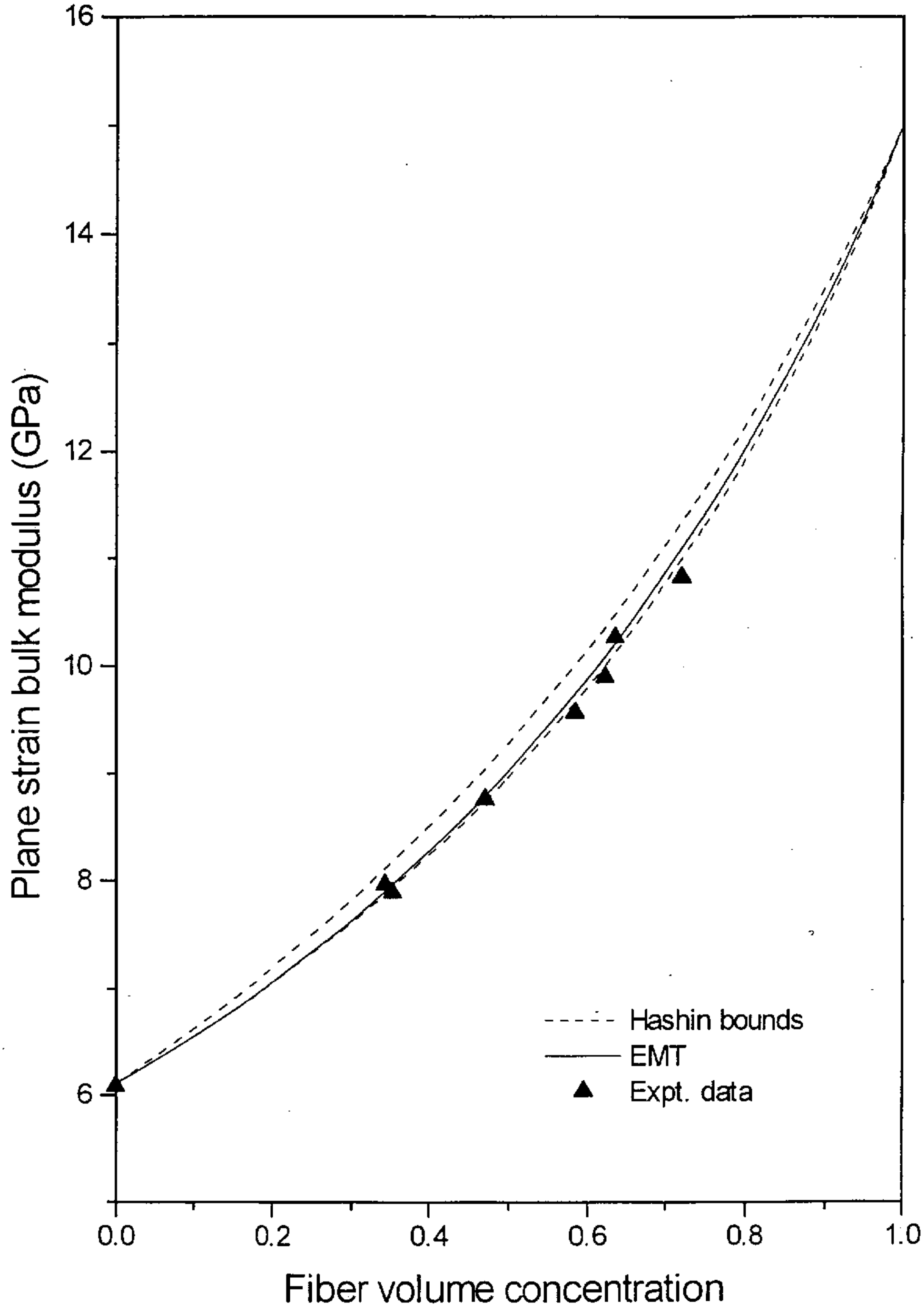


Figure 2.4.3c K versus ϕ of Modmor II/ Epoxy composite.

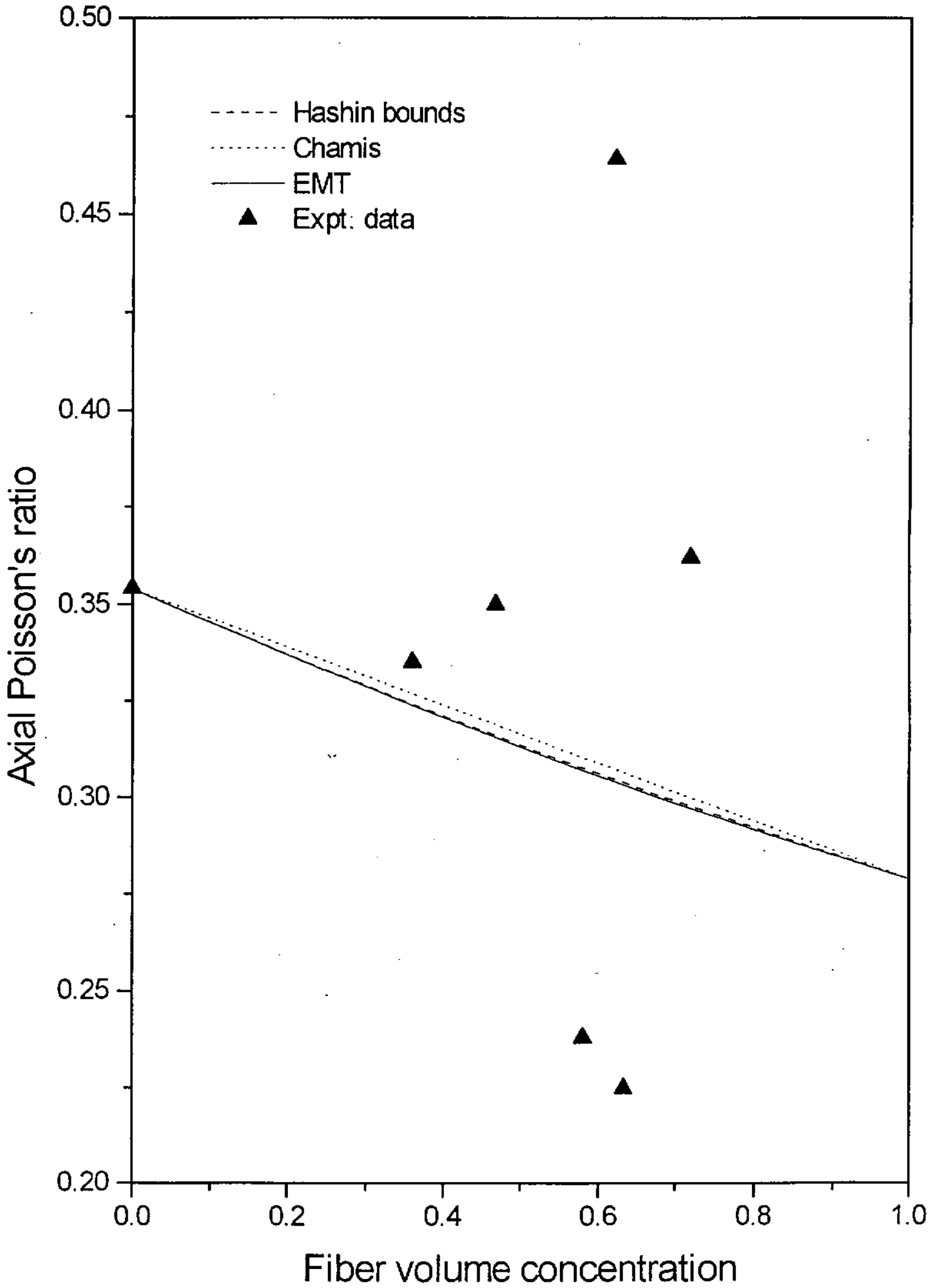


Figure 2.4.3d ν'_A versus ϕ of Modmor II/ Epoxy composite.

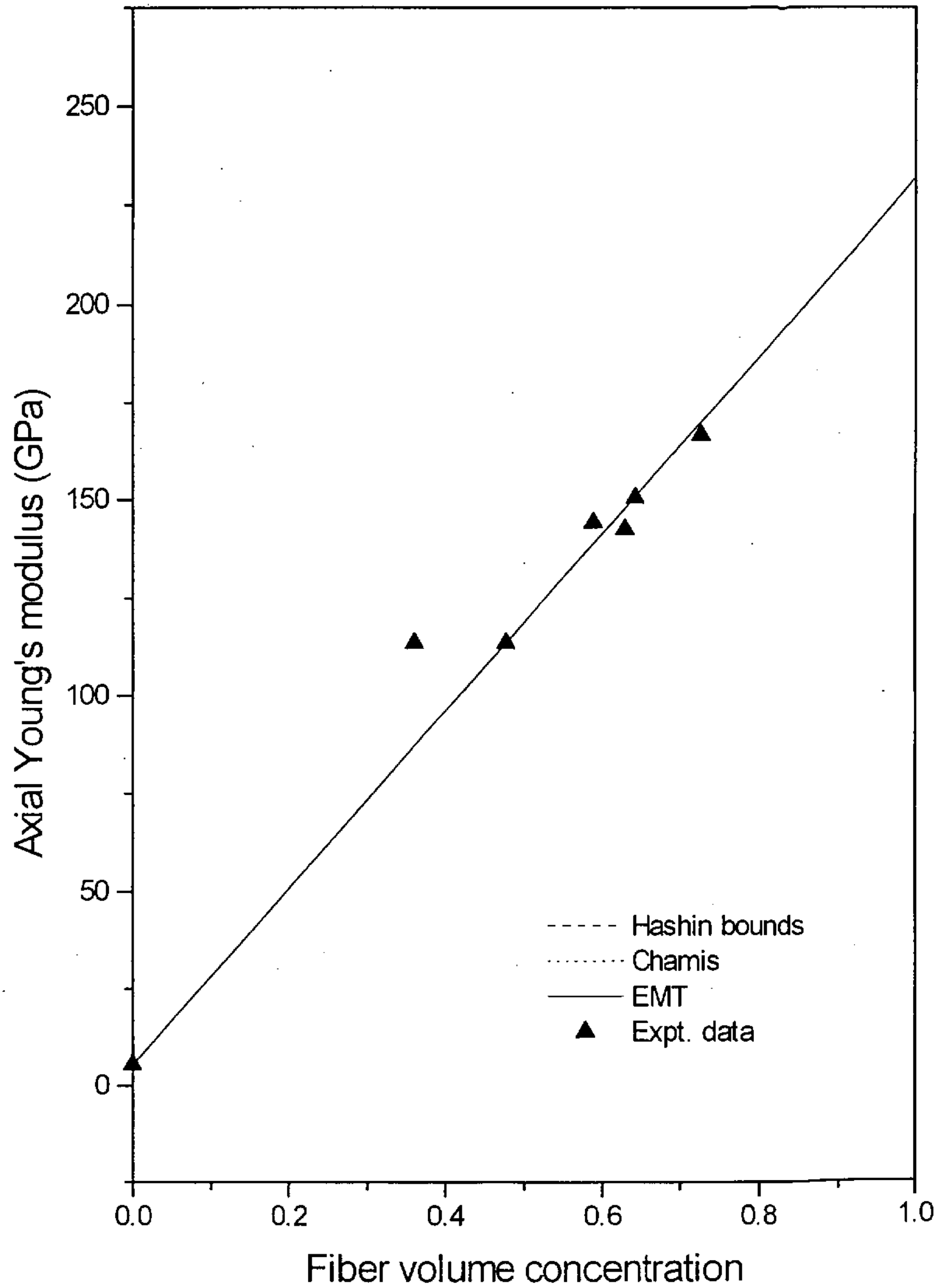


Figure 2.4.3e $E_A^{'}$ versus ϕ of Modmor II/ Epoxy composite.

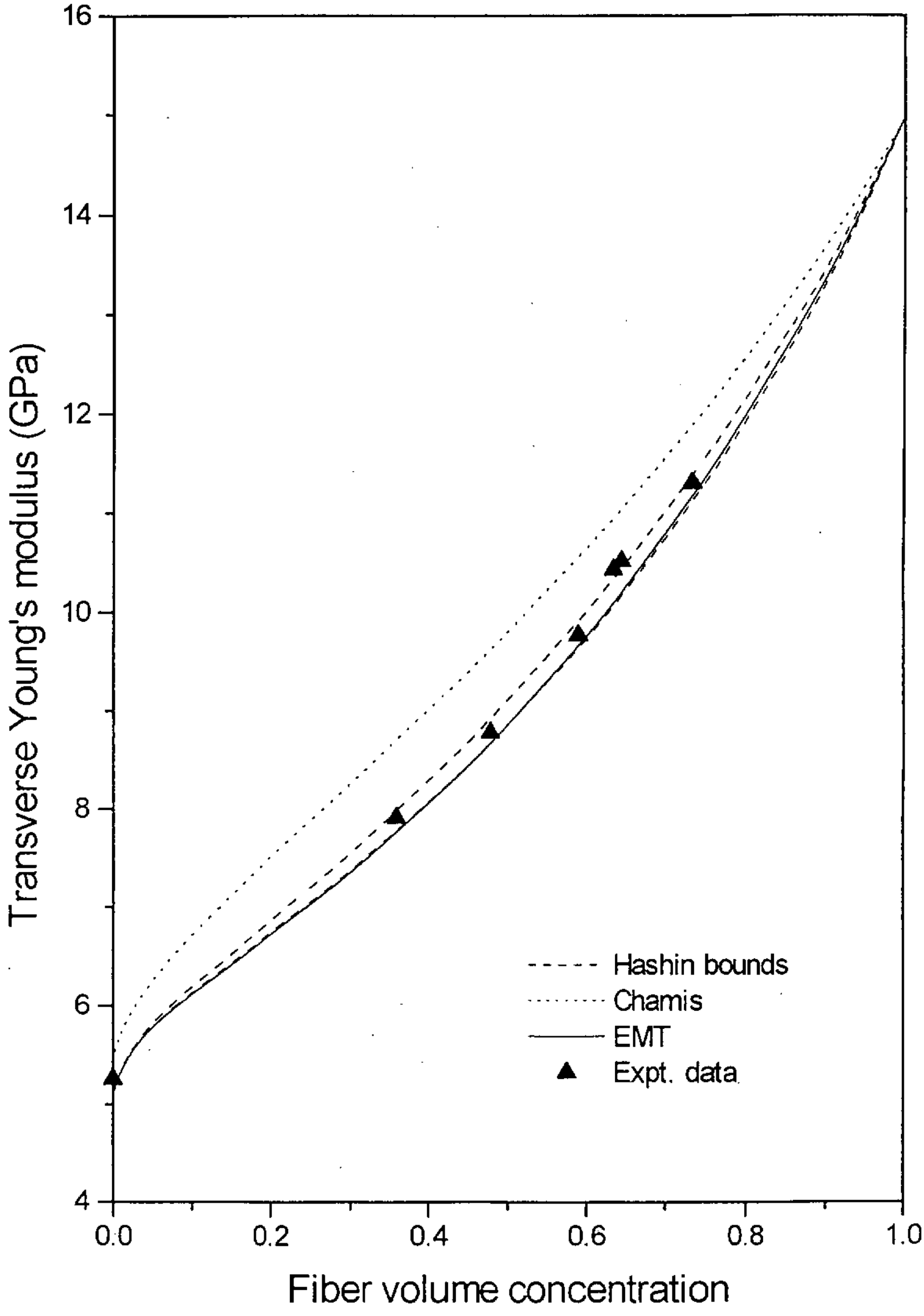


Figure 2.4.3f $E_T^{'}$ versus ϕ of Modmor II/ Epoxy composite.

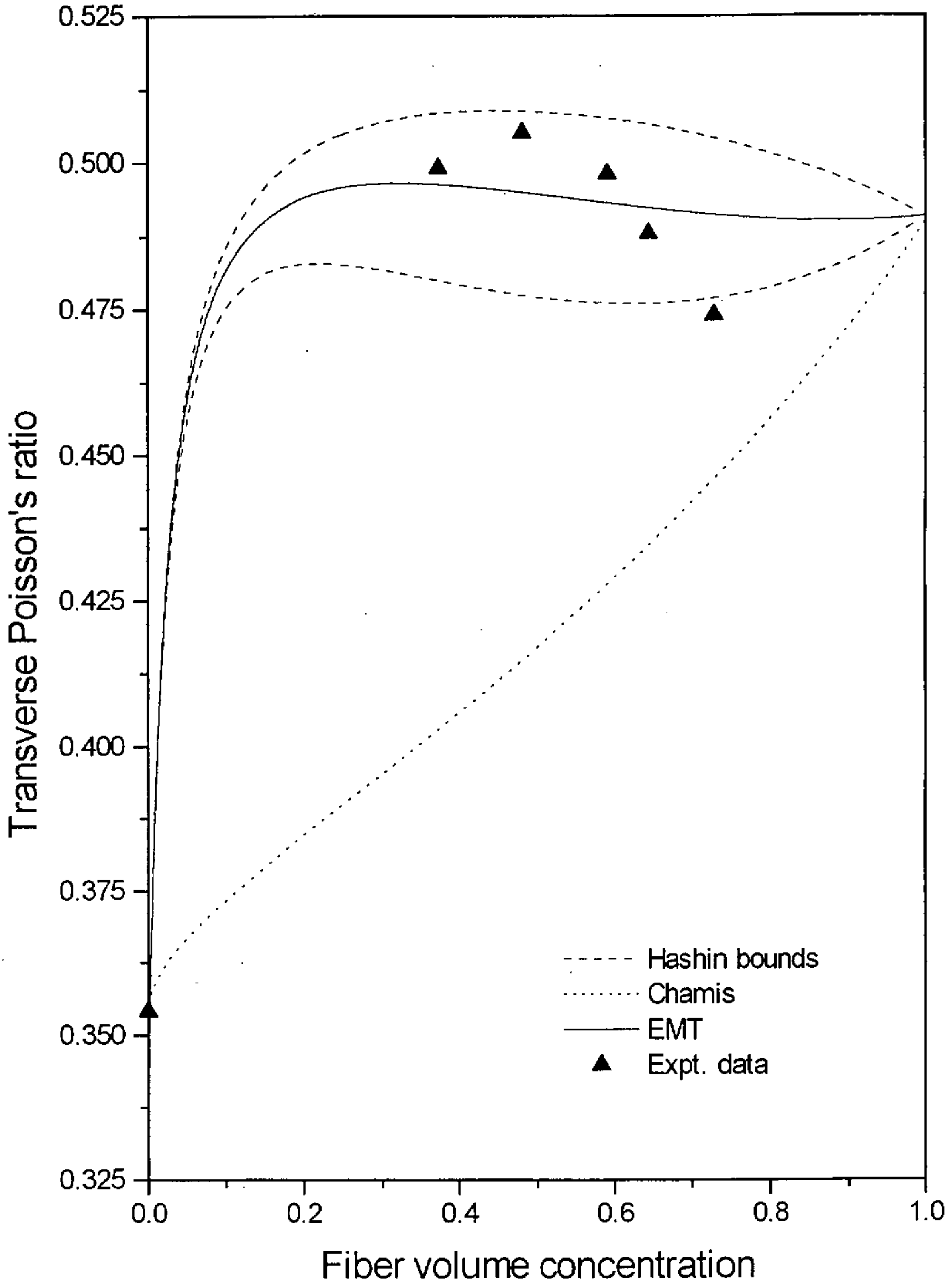


Figure 2.4.3f ν_T versus ϕ of Modmor II/ Epoxy composite.

Chapter 3 Effective thermal expansion coefficients of unidirectional fiber composites with anisotropic constituents

3.1 Effective thermal expansion coefficients (ETECs) of unidirectional fiber composites

Referring to the calculation on effective elastic moduli of a unidirectional fiber reinforced composite material with anisotropic constituents in Chapter 2, five independent elastic moduli can be obtained at arbitrary fiber volume concentrations. From these results, the effective thermal expansion coefficients (ETECs) can then be calculated, since the expressions of the ETECs involve the effective elastic moduli and the thermal expansion coefficients of the phases.

The EMT calculation of ETECs will be based on the theory by Rosen and Hashin [Rosen and Hashin, 1970]. The following will outline the theory leading to the computational algorithms.

Elegant expressions of ETECs of a unidirectional fiber composite material with isotropic constituents are developed by Levin [Levin, 1967] based on thermoelasticity. Rosen and Hashin [Rosen and Hashin, 1970] have later extended Levin's method to anisotropic composites with anisotropic constituents.

In their work, they consider (i) a prescribed stress and (ii) a prescribed temperature on a representative volume element with surface S. Thus in the case of prescribed stress,

$$\left.\begin{aligned} \sigma_k &= \sigma^0_{kl} n_l \\ T &= 0 \end{aligned}\right\} \quad \text{on S} \qquad \text{......(3.1.1)}$$

Under the condition of uniform stress $\overline{\sigma}^*_{kl}$, the volumetrically averaged stresses are given by

$$\begin{aligned} \overline{\sigma}^*_{kl} &= \sigma^0_{kl} \\ \overline{T} &= 0 \end{aligned} \qquad \text{......(3.1.2)}$$

The averaged strain and stress can be related as follows:

$$\overline{\varepsilon}^*_{kl} = S_{klmn} \sigma^0_{mn} \qquad \text{......(3.1.3)}$$

where S_{klmn} are the effective compliances for the heterogeneous medium.

For the problem of prescribed temperature:

$$\left.\begin{array}{l}\sigma_k = 0 \\ T = T_o\end{array}\right\} \quad \text{on S} \qquad \text{......(3.1.4)}$$

From equation (3.1.4) the volumetric averages are given by

$$\begin{array}{l}\overline{\sigma}_{kl} = 0 \\ \overline{T} = T_o\end{array} \qquad \text{......(3.1.5)}$$

The corresponding average strains are well expressed as

$$\overline{\varepsilon}_{kl} = \alpha_{kl} T_o \qquad \text{......(3.1.6)}$$

where α_{kl} are defined as the unknown ETECs.

If we multiple the σ^*_{kl} from the stress problem by the ε_{kl} from the temperature problem and integrate over volume, we get

$$\int_V \sigma^*_{kl} \varepsilon_{kl} dv = \int_V (\sigma^*_{kl} \varpi_k)_{,l} dv \qquad \text{......(3.1.7)}$$

where V is the volume and ϖ_k is the displacement field.

By putting $\sigma^*_{kl,l} = 0$ and using the divergence theorem we have

$$\int_V \sigma^*_{kl}\varepsilon_{kl}dv = \int_S \sigma^*_{kl}\varpi_k n_l ds \qquad \text{......(3.1.8)}$$

where n_l is the unit vector normal to the surface. However, we have $\sigma^*_{kl} = \sigma^0_{kl}$ on the surface, and this yields

$$\int_V \sigma^*_{kl}\varepsilon_{kl}dv = \sigma^0_{kl}\bar{\varepsilon}_{kl}V \qquad \text{......(3.1.9)}$$

By use of equation (3.1.6), equation (3.1.9) becomes

$$\int_V \sigma^*_{kl}\varepsilon_{kl}dv = \sigma^0_{kl}\alpha_{kl}T_o V \qquad \text{......(3.1.10)}$$

and by definition, we have

$$\overline{\sigma}^*_{kl} = \sum_{r=1}^{R} v_r \overline{\sigma}^{*(r)}_{kl} \qquad \text{......(3.1.11)}$$

where v_r is the phase volume concentration of the rth phase of R phases.

Write the linear transformation between $\overline{\sigma}_{kl}^{*(r)}$ and σ_{kl}^{0} as

$$\overline{\sigma}_{kl}^{*(r)} = H_{klmn}^{(r)} \sigma_{mn}^{0} \qquad \text{......(3.1.12)}$$

where $H_{klmn}^{(r)}$ are regarded as influence coefficients and can be determined from the solution of the stress problem. Combining equations (3.1.11) and (3.1.12) yields

$$\overline{\sigma}_{kl}^{*} = \sum_{r=1}^{R} v_r H_{klmn}^{(r)} \sigma_{mn}^{0} \qquad \text{......(3.1.13)}$$

Since $\overline{\sigma}_{kl}^{*} = \sigma_{kl}^{0}$, this results in

$$\sum_{r=1}^{R} v_r H_{klmn}^{(r)} = I_{klmn} = \frac{\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm}}{2} \qquad \text{.......(3.1.14)}$$

where I_{klmn} is the fourth-rank symmetric unit tensor.

The unknown ETECs can be determined by making use of equations (3.1.13) and (3.1.6) in (3.1.10), thus

$$\alpha_{kl} = \sum_{r=1}^{R} v_r \alpha_{mn}^{(r)} H_{mnkl}^{(r)} \qquad \text{......(3.1.15)}$$

Noting that the average strain is defined as

$$\overline{\varepsilon}^{*}_{kl} = \sum_{r=1}^{R} v_r \overline{\varepsilon}^{*(r)}_{kl} \qquad ......(3.1.16)$$

and putting this into equation (3.1.3), and then making use of (3.1.12), we get

$$S_{klmn} = \sum_{r=1}^{R} v_r S^{(r)}_{klpq} H^{(r)}_{pqmn} \qquad ......(3.1.17)$$

Equation (3.1.15) together with (3.1.14) and (3.1.17) determines the ETECs, α_{kl}, in terms of the phase properties and the effective compliances, S_{klmn}, of the composite by eliminating the influence coefficients, $H^{(r)}_{klmn}$. Thus for a two-phase composite the resulting equations for ETECs are as follows:

$$\alpha_{kl} = \overline{\alpha}_{kl} + (\alpha^{(2)}_{mn} - \alpha^{(1)}_{mn}) P_{kmns} (S_{rskl} - \overline{S}_{rskl})$$

or

$$\alpha_{kl} = \alpha^{(1)}_{kl} + (\alpha^{(2)}_{mn} - \alpha^{(1)}_{mn}) P_{mnrs} (S_{rskl} - S^{(1)}_{rskl}) \qquad ......(3.1.18)$$

where $$P_{rskl} (S^{(2)}_{rskl} - S^{(1)}_{rskl}) = I_{klmn}$$

Here $\overline{\alpha}_{kl}$ and $\overline{S}_{klmn}$ are the average of the thermal expansion coefficients and compliances of the composite. In equation (3.1.18), the terms $S^{(2)}_{rskl}$ and $S^{(1)}_{rskl}$ are the compliances of fiber and matrix, respectively.

For a transversely isotropic composite, there are only two ETECs: $\alpha_{22} = \alpha_{33} = \alpha_T$ and $\alpha_{11} = \alpha_A$. The equation for the axial ETEC is

$$\alpha_A = \alpha_{A1} + (\alpha_{A2} - \alpha_{A1})\sum_{j=1}^{3} P_{11jj}(S_{jj11} - S^{(1)}_{jj11}) + (\alpha_{T2} - \alpha_{T1})\sum_{j=1}^{3}(P_{22jj} + P_{33jj})(S_{jj11} - S^{(1)}_{jj11})$$

......(3.1.19)

and for the transverse ETEC

$$\alpha_T = \alpha_{T1} + (\alpha_{T2} - \alpha_{T1})\sum_{j=1}^{3} P_{11jj}(S_{jj22} - S^{(1)}_{jj22}) + (\alpha_{T2} - \alpha_{T1})\sum_{j=1}^{3}(P_{22jj} + P_{33jj})(S_{jj22} - S^{(1)}_{jj22})$$

......(3.1.20)

For a transversely isotropic composite material, the overall compliances are given in terms of effective elastic moduli as

$$S = \begin{bmatrix} \dfrac{1}{E_A} & \dfrac{-\nu_A}{E_A} & \dfrac{-\nu_A}{E_A} \\ \dfrac{-\nu_A}{E_A} & \dfrac{1}{E_T} & \dfrac{-\nu_T}{E_T} \\ \dfrac{-\nu_A}{E_A} & \dfrac{-\nu_T}{E_T} & \dfrac{1}{E_T} \end{bmatrix}, \qquad \text{......(3.1.21)}$$

Then the values of P_{rskl} can be calculated from equation (3.1.18) by using equation (3.1.21) and the results are

$$P_{1111} = \frac{(\frac{1}{E_{A2}} - \frac{1}{E_{A1}})^2 - (-\frac{\nu_{T2}}{E_{T2}} + \frac{-\nu_{T1}}{E_{T1}})^2}{\Delta}$$

$$P_{2222} = P_{3333} = \frac{(\frac{1}{E_{A2}} - \frac{1}{E_{A1}})(\frac{1}{E_{T2}} - \frac{1}{E_{T1}}) - (-\frac{\nu_{A2}}{E_{A2}} + \frac{\nu_{A1}}{E_{A1}})^2}{\Delta}$$

$$P_{1122} = P_{2211} = \frac{(-\frac{\nu_{A2}}{E_{A2}} + \frac{\nu_{A1}}{E_{A1}})[-\frac{(1+\nu_{T2})}{E_{T2}} + \frac{(1+\nu_{T1})}{E_{T1}}]}{\Delta}$$

$$P_{2233} = P_{3322} = \frac{(-\frac{\nu_{A2}}{E_{A2}} + \frac{\nu_{A1}}{E_{A1}})^2 - (\frac{1}{E_{A2}} - \frac{1}{E_{A1}})(-\frac{\nu_{T2}}{E_{T2}} + \frac{\nu_{T1}}{E_{T1}})}{\Delta}$$

......(3.1.22)

where

$$\Delta = (\frac{1}{E_{T2}} - \frac{1}{E_{T1}} + \frac{\nu_{T2}}{E_{T2}} + \frac{\nu_{T1}}{E_{T1}})[(\frac{1}{E_{A2}} - \frac{1}{E_{A1}})(\frac{1}{E_{T2}} - \frac{1}{E_{T1}} + \frac{\nu_{T2}}{E_{T2}} + \frac{\nu_{T1}}{E_{T1}}) - 2(\frac{-\nu_{A2}}{E_{A2}} + \frac{-\nu_{A1}}{E_{A1}})^2]$$

3.2 Numerical calculations of ETECs by EMT

A straight forward way to calculate the ETECs is first to determine the five effective elastic moduli of the composite. Accordingly, we can calculate the axial shear modulus from equation (2.3.16), the transverse shear modulus from equation (2.3.27) and then obtain the value of the plane strain bulk modulus $K^{'}$ by use of equation (2.3.22). Once the value of $K^{'}$ is calculated, the values of $\nu_{A}^{'}$ and $E_{A}^{'}$ are then determined from equations (2.3.33) and (2.3.43). Hence we can calculate the values of the transverse Young's modulus $E_{T}^{'}$ and transverse Poisson's ratio $\nu_{T}^{'}$, with equation (2.1.5). These moduli results then give the overall compliances by equation (3.1.21) and the values of P_{rskl} by use of equation (3.1.22). Thus the ETECs are determined in accordance with the results of the overall compliances and P_{rskl} values.

An alternative is to do the calculations following EMT which will be discussed below. Major results will be compared with experimental data as well as with other model theories.

In Chapter 2, we have shown the algorithm of EMT calculations in computing the effective elastic moduli of unidirectional fiber composite materials with anisotropic constituents. Suppose fibers of volume fraction $\delta = 0.001$ is embedded in the anisotropic matrix. Based on equations (2.3.16), (2.3.22), (2.3.27), (2.3.33) and (2.3.43), the five effective elastic moduli are then calculated and the transverse Young's moduli and transverse Poisson's ratio will be obtained by equation (2.1.5).

Then we can calculate the term $(S_{rskl}^{(2)} - S_{rskl}^{(1)})$ and the overall compliances from equations (3.1.17) and (3.1.18), The overall P_{rskl} values are also calculated according to equation (3.1.22). By substituting the results of overall compliances, the P_{rskl} values and the ETECs of the phases into equations (3.1.19) and (3.1.20), the ETECs of a unidirectional fiber composite with δ fiber volume concentration are calculated. Based on the concept of EMT, this composite is treated as a new anisotropic matrix with computed effective moduli. Again, fibers of volume concentration δ is embedded into this new matrix and the previous steps repeated, but this time the moduli with subscripts 1, in equations (3.1.19), (3.1.20), (3.1.21) and (3.1.22), and with superscript (1) in equation (3.1.18), are correspondingly replaced by the new matrix moduli. For each iteration the fiber volume concentration changes to V + δ - Vδ, where V is the previous fiber volume concentration in the composite; similarly the ETECs are calculated at V + δ - Vδ fiber volume concentration.

By such an iteration, the ETECs of a unidirectional fiber composite material with anisotropic constituents can be calculated at arbitrary fiber volume concentration.

Experimental data are again taken from Choy *et al.* [Choy *et al.*, 1985], Lau[Lau, 1995] as well as Rojstaczer, Cohn and Marom (RCM) [Rojstaczer *et al.*, 1985]. The reason for choosing these composite systems is that anisotropic fibers are used to reinforce an isotropic matrix, which is the case dealt with in the equations from other theories, thus making comparison possible. In addition, two of the systems are used previously in Chapter 2. Therefore, we can give a fuller discussion on the systems in this Chapter. The experimental data and the theoretical predictions are depicted in Figures 3.3.1, 3.3.2 and 3.3.3.

3.3 Results and discussion

The formulas due to RCM, Chamis, Chamberlain and EMT calculations are used to predict the ETECs of unidirectional fiber reinforced composite materials with anisotropic constituents. We choose these models to compare with EMT calculations because they are relatively well known and applicable to composites with transversely isotropic fibers in an isotropic matrix. It is worth to state that the equations of Chamberlain involve a packing factor which indicates the fiber packing geometry. Its value for square packing of fibers is 0.7854 and 0.9069 for hexagonal packing. In the figures, curves for both cases are produced. The values of the ETECs of the phases of the composite systems used are tabulated in Appendix C.

The data for the ETECs of PE/ PE and LCP/ PC are determined from thermal mechanical analysis by Choy *et al.* [Choy *et al.*, 1995] and Lau [Lau, 1995]. From the PE/ PE data, the transverse ETEC of fiber is calculated as 1.17×10^{-4} K^{-1} by the authors using a modification of the Schapery equations to accommodate anisotropic constituents. Indeed, such a result is only an approximation since when $\phi = 1$, the transverse ETEC of the composite is not equal to that of fibers in their equation. The EMT calculation based on a fiber transverse ETEC of 1.799×10^{-4} K^{-1} [Rojstaczer *et al.* 1985] seem to fit the data very well.

For the PE/ PE composite system shown in Figure 3.3.1, the same axial α_A predictions have been made by the models and are close to the experimental data. In the transverse direction, the Chamis results show under-prediction as compared with the experimental data but the trend seems correct: the ETEC increases from the matrix value to a maximum value of 2.9163×10^{-4} K^{-1} and then descents to the fiber property. The two curves of Chamberlain both seem to under-predict. RCM and EMT calculations have similar results and the maximum values reached are 3.204×10^{-4} K^{-1} and 3.178×10^{-4} K^{-1}, respectively. Nevertheless, the experimental data lie on the predicted curves.

Figure 3.3.2 shows the ETECs of LCP/ PC composites as a function of volume concentration. In the axial direction, similarly, the models have approximately the same α_A values and are close to the measured data. It is found that the α_T experimental data are close to the Chamis formula at high volume concentration but it is not the case at low volume concentration. Chamberlain's formula has good prediction for this composite system, since it is evident from the figure that the predicted curves are closer to the experimental data at low as well as high volume concentration. Over-predictions are obtained by the RCM formula and EMT calculation at low volume concentration but fairly good predictions at high volume concentration.

For the Kevlar 49/ Epoxy composite depicted in Figure 3.3.3, RCM, Chamis and Chamberlain have roughly the same predictions in the axial direction. However, the prediction of EMT is closer to the experimental data for the whole range of fiber volume concentration. In the transverse direction, Chamis and Chamberlain both show under-estimation. The predictions of both RCM and EMT are close to the experimental data.

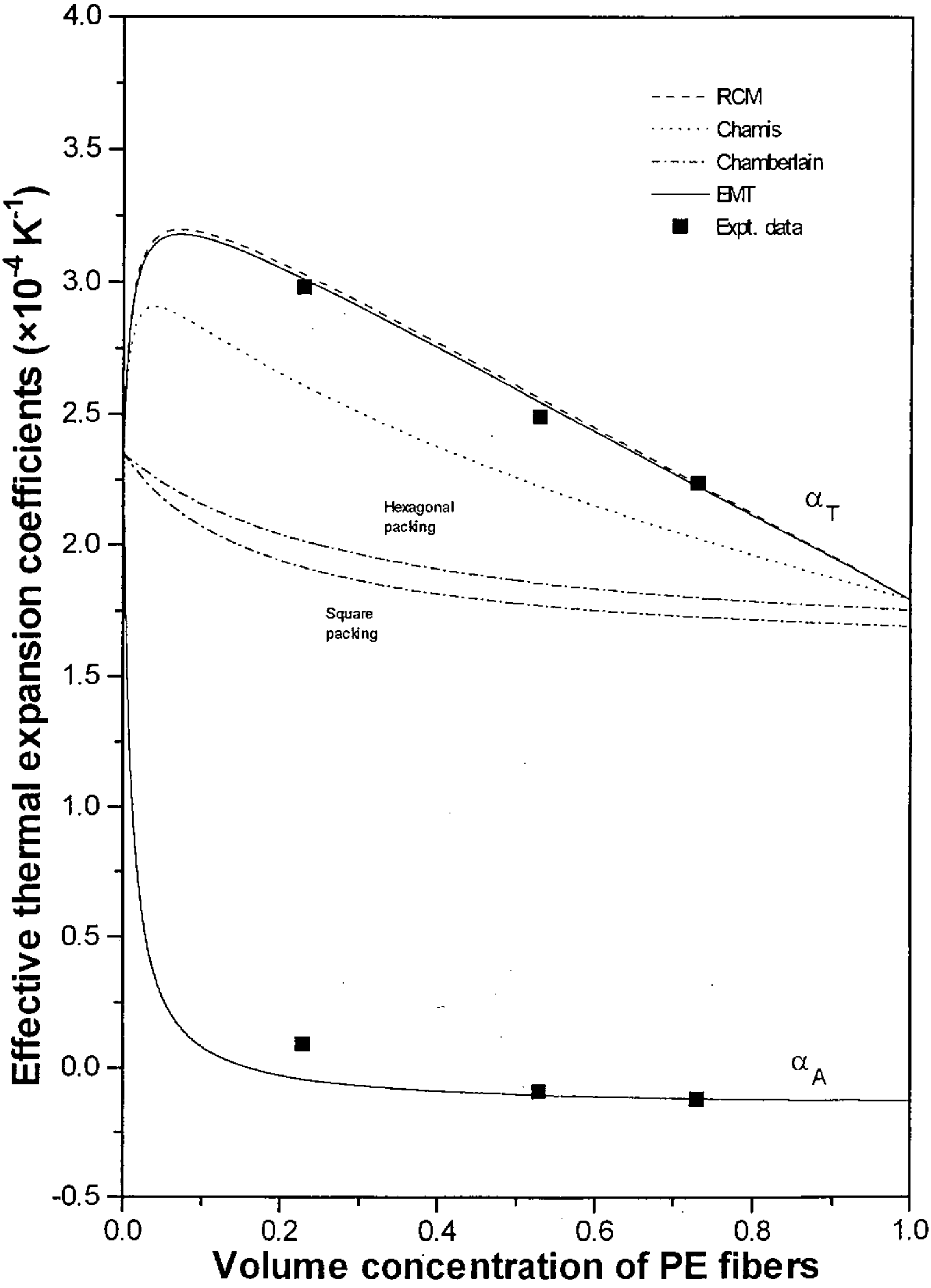


Figure 3.3.1 The effective thermal expansion coefficients of PE/ PE composites as a function of fiber volume concentration.

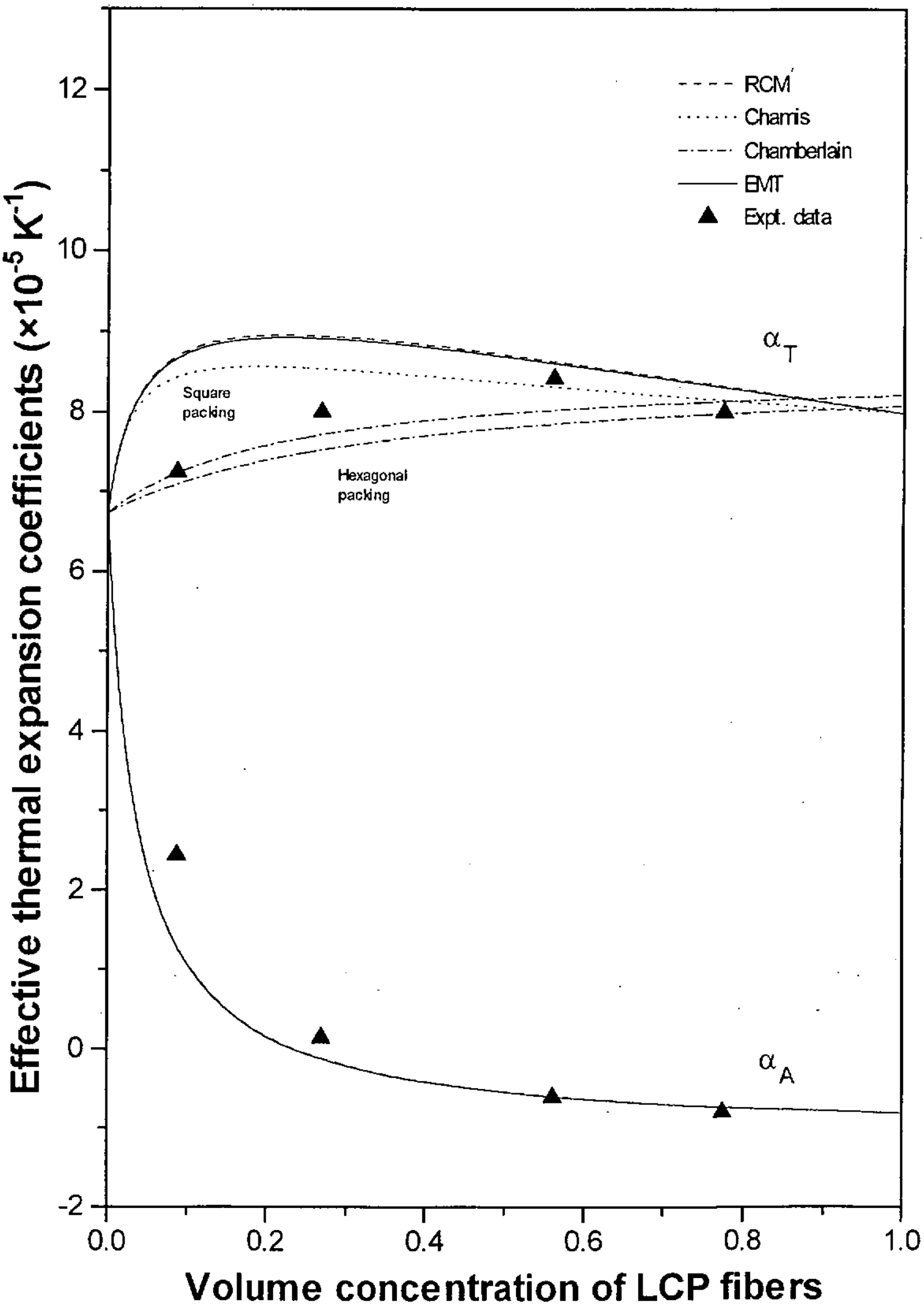


Figure 3.3.2 The effective thermal expansion coefficients of LCP/ PC composites as a function of fiber volume concentration.

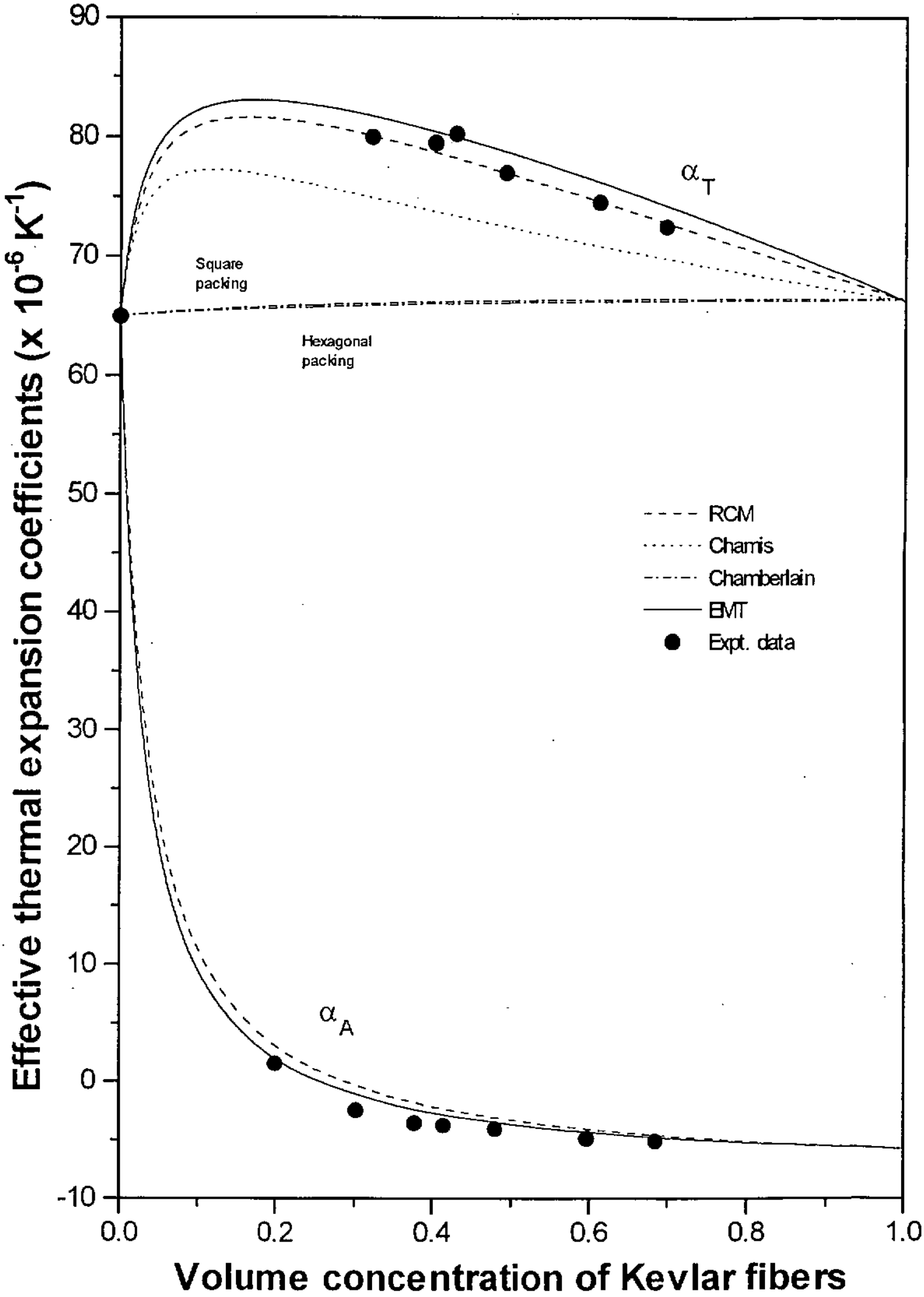


Figure 3.3.3 The effective thermal expansion coefficients of Kelvar 49/ epoxy composites as a function of fiber volume concentration.

Chapter 4 Three dimensional randomly oriented short fiber composites

In this chapter, we will focus attention on three-dimensional randomly oriented short fiber composites and base our EMT treatment on the work by Tandon and Weng [Tandon and Weng, 1986b]. The formulation of this problem will now be introduced and the resulting calculation will be compared with experimental data reported in the literature.

4.1 Elastic moduli and ETEC of three dimensional randomly oriented short fiber composites

General principles involving Eshelby's "stress-free" transformation strain and Hill's "strain-free" stress have been discussed by Weng [Weng, 1984] under traction- and displacement-prescribed boundary conditions. Later Tandon and Weng [Tandon and Weng, 1986a] have used the same idea to study the stress distribution in and around spherical inclusions. With their method, the distinct roles played by the matrix and the inclusions are fully accounted for, even at finite filler concentration. The Tandon and Weng [Tandon and Weng, 1986b] equations for the effective elastic moduli of three dimensional randomly oriented fiber composites are derived by calculating the average stress in the matrix following the above approach, together with a standard method for averaging over fiber orientations. With the inclusions oriented randomly in 3-D, allows the decomposition of the average stress or strain into hydrostatic and deviatoric parts, ultimately leading to the following equations for bulk K and shear G moduli as follows:

$$K = \frac{K_m}{1+\phi T} \quad \text{and} \quad G = \frac{G_m}{1+\phi U} \qquad \text{......(4.1.1)}$$

where

$$T = \frac{T_2}{T_1} \quad \text{and} \quad U = \frac{U_2}{U_1} \qquad \text{......(4.1.2)}$$

$$T_1 = 1 + \phi \frac{[2(S_{1122} + S_{2222} + S_{2233} - 1)(p_3 + p_4) + (S_{1111} + 2S_{2211} - 1)(p_1 - 2p_2)]}{3p}$$

......(4.1.3a)

$$T_2 = \frac{p_1 - 2(p_2 - p_3 - p_4)}{3p} \qquad \text{......(4.1.3b)}$$

$$U_1 = 1 - \phi[\frac{2(2S_{1212} - 1)}{5(2S_{1212} + \frac{G_m}{G_i - G_m})} + \frac{2S_{2323} - 1}{3(2S_{2323} + \frac{G_m}{G_i - G_m})}$$

$$- \frac{(S_{1122} - S_{2233})(2p_3 - p_4 + p_5 p) + 2(S_{1111} - S_{2211} - 1)(p_1 + p_2)}{15p} \qquad \text{......(4.1.4a)}$$

$$- \frac{(S_{1122} - S_{2222} + 1)(2p_3 - p_4 - p_5 p)}{15p}]$$

$$U_2 = \frac{2(p_1 + p_2 - p_3) + p_4 + p_5 p}{15p} - \frac{2}{5(2S_{1212} + \frac{G_m}{G_i - G_m})} - \frac{1}{3(2S_{2323} + \frac{G_m}{G_i - G_m})} \qquad \text{......(4.1.4b)}$$

Here ϕ, G_m and G_i are the inclusion volume concentration, the shear moduli of the matrix and the inclusion, respectively, and the values for the constants p and the Eshelby S_{ijkl} tensor are given in Appendix D in terms of matrix properties and the fiber aspect ratio, AR.

Therefore using the relations for linear isotropic elasticity, we can calculate the Poisson's ratio ν and also Young's modulus E from the equations:

$$\nu = \frac{3K - 2G}{2(3K + G)}; \qquad E = \frac{9KG}{3K + G} \qquad \text{......(4.1.5)}$$

In EMT calculations, as before, a small amount δ of inclusions are put into the matrix such that the effective elastic moduli of the composites can be determined by equations (4.1.1) to (4.1.5) for the three dimensional randomly oriented short fiber composite system. The matrix properties are then replaced by these calculated composite properties. Again a small amount of inclusions are embedded to this new matrix, as so forth. The volume concentration variable ϕ is renewed in every iteration step, and is given by

$$\phi_{n+1} = \phi_n + \delta(1-\phi_n) = \phi_n + \delta - \phi_n\delta \qquad ......(4.1.6)$$

where δ is the volume concentration of inclusion in each iteration. This procedure thus determines the effective elastic moduli for a three dimensional randomly oriented short fiber composite system. In the next session, the numerical results are used to compare with other theories [Tandon and Weng, 1986; Christensen, 1991; Whiney *et al.*, 1984] as well as experimental data.

The above calculation is easily extended to determine the ETEC for the same composite system. Once the effective elastic moduli are calculated, the corresponding ETEC can be determined from the equation for macroscopically isotropic composites:

$$\alpha_{iso} = \alpha_i + \frac{(\alpha_m - \alpha_i)}{(\frac{1}{K_m} - \frac{1}{K_i})}\left[\frac{1}{K} - \frac{1}{K_i}\right] \qquad ......(4.1.7)$$

where the α_{iso}, α_m, α_i are the thermal expansion coefficient of the isotropic composite, the matrix and the inclusion. By the concept of EMT, the values of α_m and K_m are replaced by α_{iso} and K in each iteration. In this way the ETEC of a three dimensional randomly oriented short fiber composite can be numerically determined for the whole range of inclusion volume concentration. Computed results will be compared with other equations [Craft and Christensen, 1981; Christensen, 1991] for several sets of experimental data.

4.2 Results and discussion

In this section, the EMT computed results for the elastic and thermoelastic properties of three dimensional randomly oriented composites will be discussed and compared with other theories and experimental data. In the first part, we will present the results for effective elastic moduli of isotropic composites with spherical inclusions (AR = 1) and short fibers (AR ~ 1 -100), respectively. Moreover, we have studied the variation of effective elastic moduli with different fiber aspect ratio AR from 0.0001 to 10000; these results will be compared with Tandon and Weng results (T-W) [Tandon and Weng, 1986]. Secondly, we would like to display the results for the prediction of the thermal expansion coefficient of three dimensional randomly oriented composites. In this part, the equations for thermal expansion coefficient of Craft and Christensen (Craft-Chris) [Craft and Christensen, 1981] for long fiber composites and also equations for composites with spherical inclusions [Christensen, 1991] are used for comparison.

Prediction of Young's modulus of two composite systems embedded with spherical inclusions are examined. Figure 4.2.1 depicts the results of those theoretical models applicable to the glass spheres reinforced polyester composite system (E_m = 1.72 GPa, ν_m = 0.45, E_i = 70.3 GPa, ν_i = 0.21) [Richard, 1975]. We can observe that Hashin's lower bound (Hashin LB) [Hashin, 1967] has similar prediction as T-W and good fit with the data is obtained up to 35% volume concentration. On the other hand, EMT has better agreement with the data for a wider concentration range and is higher than both T-W and Hashin LB.

Figure 4.2.2 shows the Young's modulus results of the tungsten carbide spheres reinforced cobalt matrix composite system (E_m = 206.8 GPa, ν_m = 0.3, E_i = 703 GPa, ν_i = 0.22). The experimental data are obtained by Nishimatsu and Gurland [Nishimatsu and Gurland, 1960]. The predictions of the three models are reasonably good.

We then turn to the EMT calculations for composites with randomly oriented short fibers which have fiber aspect ratio higher than 1 but less than 100. Figure 4.2.3 displays the Young's modulus of a steel fiber reinforced concrete cement composite system (E_m = 20.802 GPa, ν_m = 0.2081, E_i = 200 GPa, ν_i = 0.3) with fiber aspect ratio 50 [Williamson, 1973]. As the aspect ratio is not 1, Hashin LB is no longer used for comparison. Instead, we choose the Halpin-Tsai equations [Whitney *et al.*, 1984] for aligned short fiber composites to calculate the effective elastic moduli first and then use the equations for isotropic Young's modulus and Poisson's ratio for 3-D randomized composite due to Christensen (H-T-C) [Christensen, 1991]. In Figure 4.2.3, the prediction of H-T-C shows over estimation, while the predictions by T-W and EMT coincide and have good agreement with experimental values.

Another set of experimental data is taken from Fishers *et al.* [Fishers *et al.*, 1992]. They have studied the elastic moduli of Al_2O_3 (E_{Al2O3} = 399.7 GPa, ν_{Al2O3} = 0.24) and Si_3N_4 (E_{Si3N4} = 316.5 GPa, ν_{Si3N4} = 0.255) ceramics reinforced with SiC whiskers (E_{SiC} = 548 GPa, ν_{SiC} = 0.14) with aspect ratio 30-50. Fishers *et al.*'s moduli values for SiC whiskers are determined from β-SiC whiskers and the shear modulus of SiC whiskers averages to 188 GPa. The

Young's and shear moduli results for the SiC/ Al_2O_3 composite are shown in Figures 4.2.4a and 4.2.4b. In both Figures 4.2.4a and 4.2.4b, good agreement with the data are obtained by T-W and EMT results but the H-T-C results show over-prediction. In Figure 4.2.4c, the H-T-C under-predicts as compared with the data and T-W and EMT results are close to the data. Similar results have been obtained for the SiC/ Si_3N_4 composite system shown in Figures 4.2.5a, 4.2.5b and 4.2.5c. We note that the data for both composite systems with 5%SiC show abnormal values; these are caused, according to Fishers *et al.*, by the formation of pores during processing and the nonhomogeneous mixing and distribution of whiskers which can influence the elastic properties of the composites. The above proves that both the T-W and EMT calculations are able to predict the elastic moduli of composites with randomly oriented short fibers. Furthermore, they seem to show no distinction.

We have examined the capability of both T-W and EMT calculations for composites with spherical inclusions as well as randomly oriented short fibers. Since they show almost no difference in predictions, we intend to find out whether the difference in calculated elastic moduli are negligible under other circumstances. An industrial glass fiber reinforced epoxy composite with randomly oriented short fibers is used for such comparison (E_m = 2.76 GPa, ν_m = 0.35, E_i = 72.4 GPa, ν_i = 0.2). The elastic moduli of composites with 30%, 60% and 90% of glass fibers have been calculated with aspect ratio ranging from 0.0001 to 10000. (Strictly speaking, the inclusions will normally not be called "fibers" if AR is less than 1). The isotropic Young's, shear and bulk moduli as well as Poisson's ratio are depicted in Figures 4.2.6a to 4.2.6d as a function of aspect ratio.

From Figures 4.2.6a to 4.2.6c, we can observe that the results of infinitely long geometry are smaller than that of platelet inclusions but greater than the results from spherical geometry. The Figures show that the curves are level-off at the infinitely long geometrical regime ($\log_{10}$ AR > 2), and at the other end, they come to have similar results and reach a plateau ($\log_{10}$ AR < -3). In Figure 4.2.6d, a peak is found at spherical geometry regime, the lowest values are obtained by platelet inclusions, while the infinitely long geometry results lies between the two cases. Clear discrepancies between the two models can be found and are largest in the 90% composite. These figures indicate that T-W and EMT do not always give close predictions, especially at high filler volume concentration.

We now come to EMT calculations of the thermal expansion coefficient of three dimensional randomly oriented short fiber composite systems. Experimental data can be found in [Takei *et al.*, 1991]. Three different kinds of fillers are embedded in Kerimid 601 matrix (K601): Al_2O_3 short fibers, Si_3N_4 whiskers and SiO_2 spheres. It is reported that the thermal expansion coefficient of Kerimid 601 matrix may vary between 50 - 80 $\times$ 10^{-6} K^{-1} and this is because the thermal expansion coefficient is affected by the volatile component content. In Figure 4.2.7, 4.2.8 and 4.2.9, different thermal expansion values have been marked for the matrix for different composites; these matrix thermal expansion coefficient values are as determined by Takei *et al.* by use of which they obtain fairly good agreement between the experimental and theoretical thermal expansion coefficient values of the composites.

Figure 4.2.7 shows the volumetric variations for thermal expansion coefficient of the SiO_2/ K601 composite system and the thermal expansion coefficient of K601 is taken to be 50×10^{-6} K^{-1}. All models have the correct trend in the prediction and the Christensen and T-W result seem to have better agreement with the data. The deviations between EMT and the other two models become larger as the volume concentration of SiO_2 spheres increases.

The results for the Al_2O_3/ K601 composite system is shown in Figure 4.2.8. The Al_2O_3 has aspect ratio 15 and the thermal expansion coefficient of K601 is taken to be 56×10^{-6} K^{-1}. Obviously all models do not have reasonable agreement with the data, although the Craft-Chris results are better. It has been explained [Takei *et al.*, 1991] that the Al_2O_3/ K601 composites were made by the paper-making method where the Al_2O_3 fibers will tend to form paper layers which are better described by a two dimensional randomly oriented short fiber composite model which show a laminated structure.

Lastly, the Si_3N_4/ K601 composite system is examined and the aspect ratio of the Si_3N_4 fibers is 17.5 and the thermal expansion coefficient of K601 is taken as 65×10^{-6} K^{-1}. The results are depicted in Figure 4.2.9, the data are denoted by squares and triangles for the Si_3N_4/ K601 composites made by premix and paper-making methods, respectively. The Si_3N_4/ K601 composites are made by the premix method at filler volume concentration from 10% to 30%, giving a three dimensional randomly oriented fiber composite. From 30% to 50%, the composites are made by the paper-making method which shall giving a two dimensional randomly oriented fiber composite. However, they mentioned that these data seems consistent with the prediction of three dimensional randomly oriented fiber composites. And thus the data are used in our examination. Figure 4.2.9 shows clearly that the prediction by Christensen is far away from the data points. T-W predicts fairly well at low volume concentration. On the contrary, Craft-Chris predicts quite poorly at low volume concentrations but has better estimation at high volume concentration. Among the models, the prediction due to EMT has excellent agreement with the experimental values for the whole range.

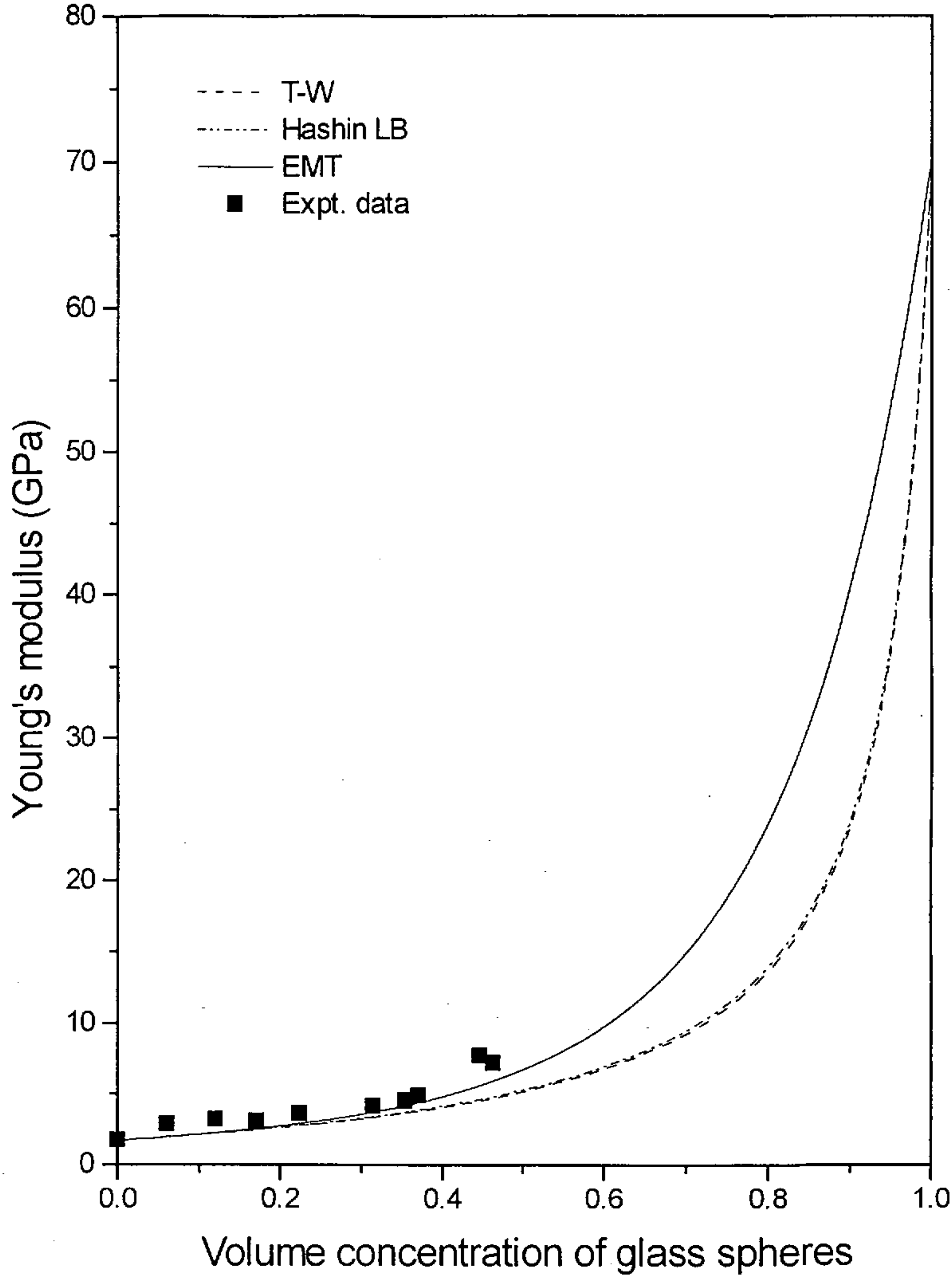


Figure 4.2.1 The Young's modulus of glass sphere/ polyester composites as a function of sphere volume concentration.

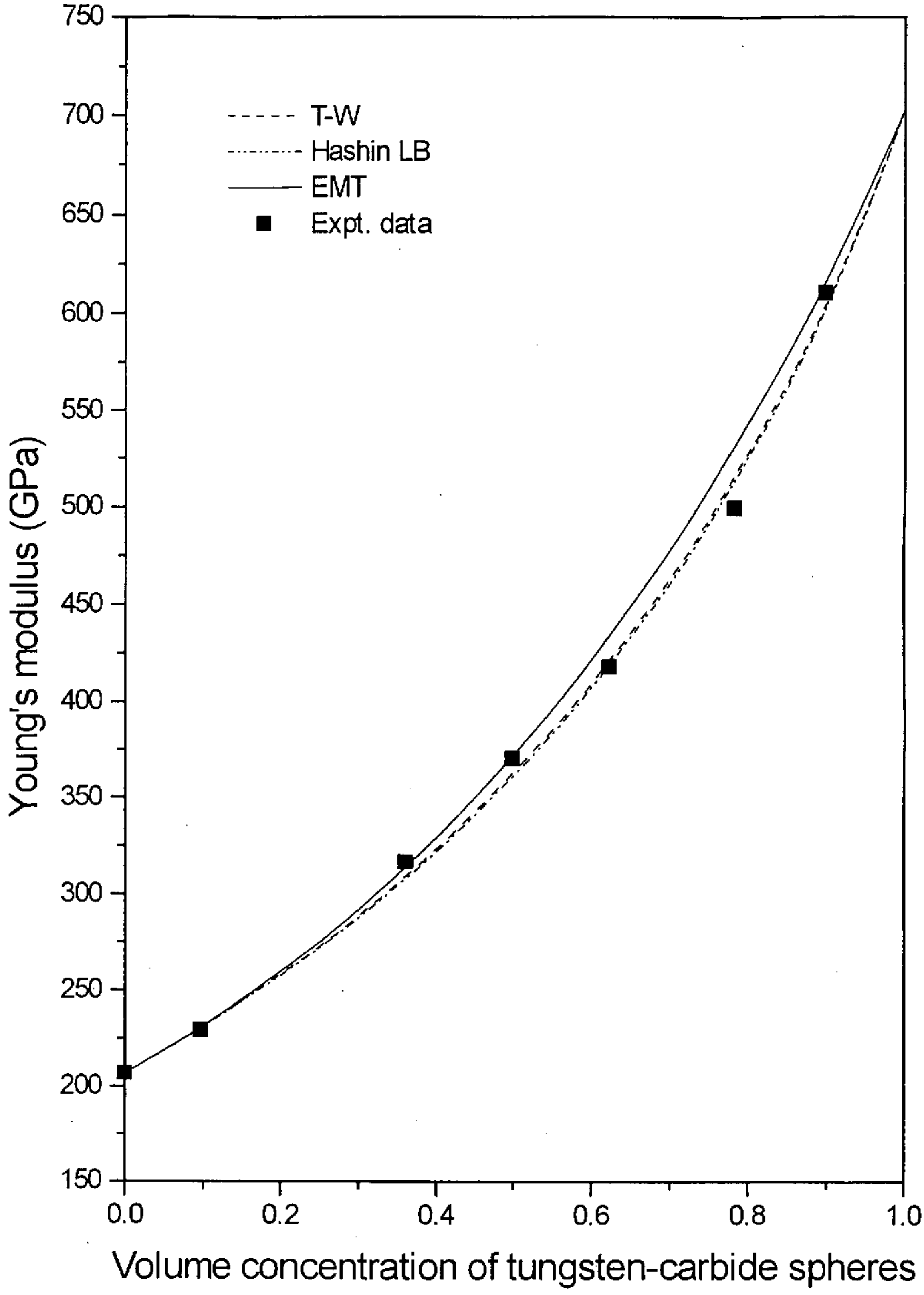


Figure 4.2.2 The Young's modulus of tungsten-carbide/ cobalt composites as a function of sphere volume concentration.

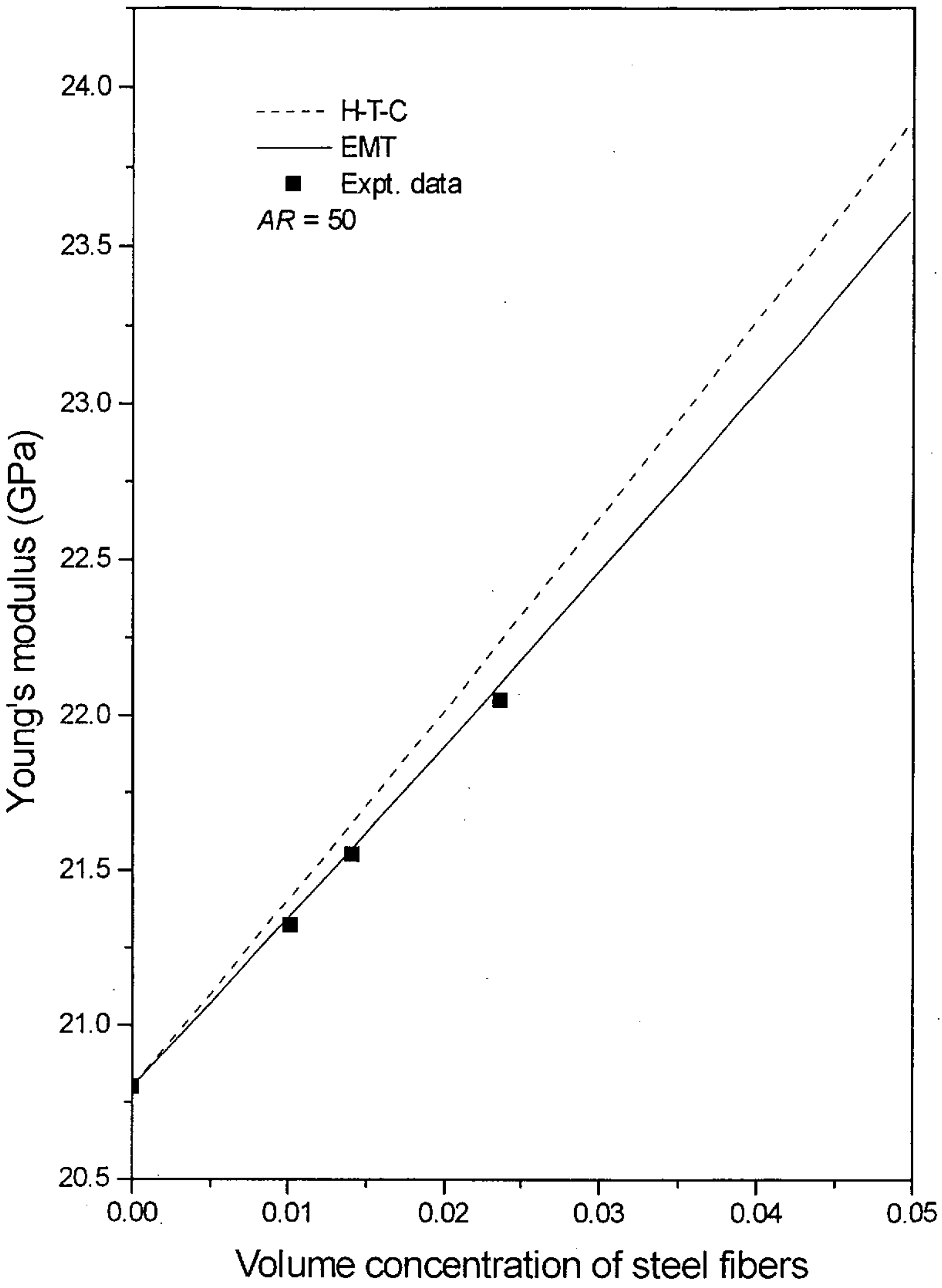


Figure 4.2.3 The Young's modulus of steel fiber/ concrete cement composites as a function of fiber volume concentration.

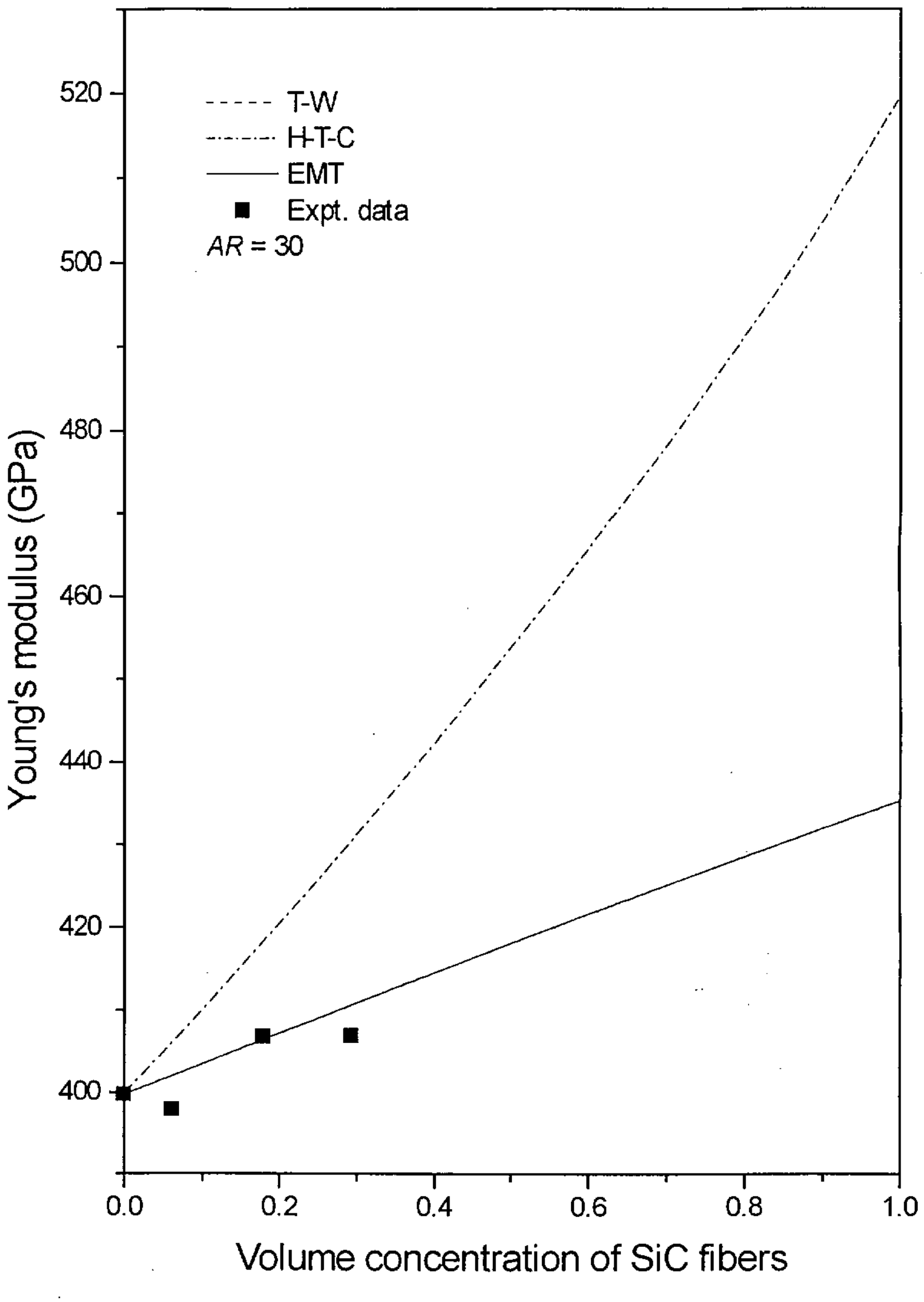


Figure 4.2.4a The Young's modulus of SiC/ Al_2O_3 composites as a function of fiber volume concentration.

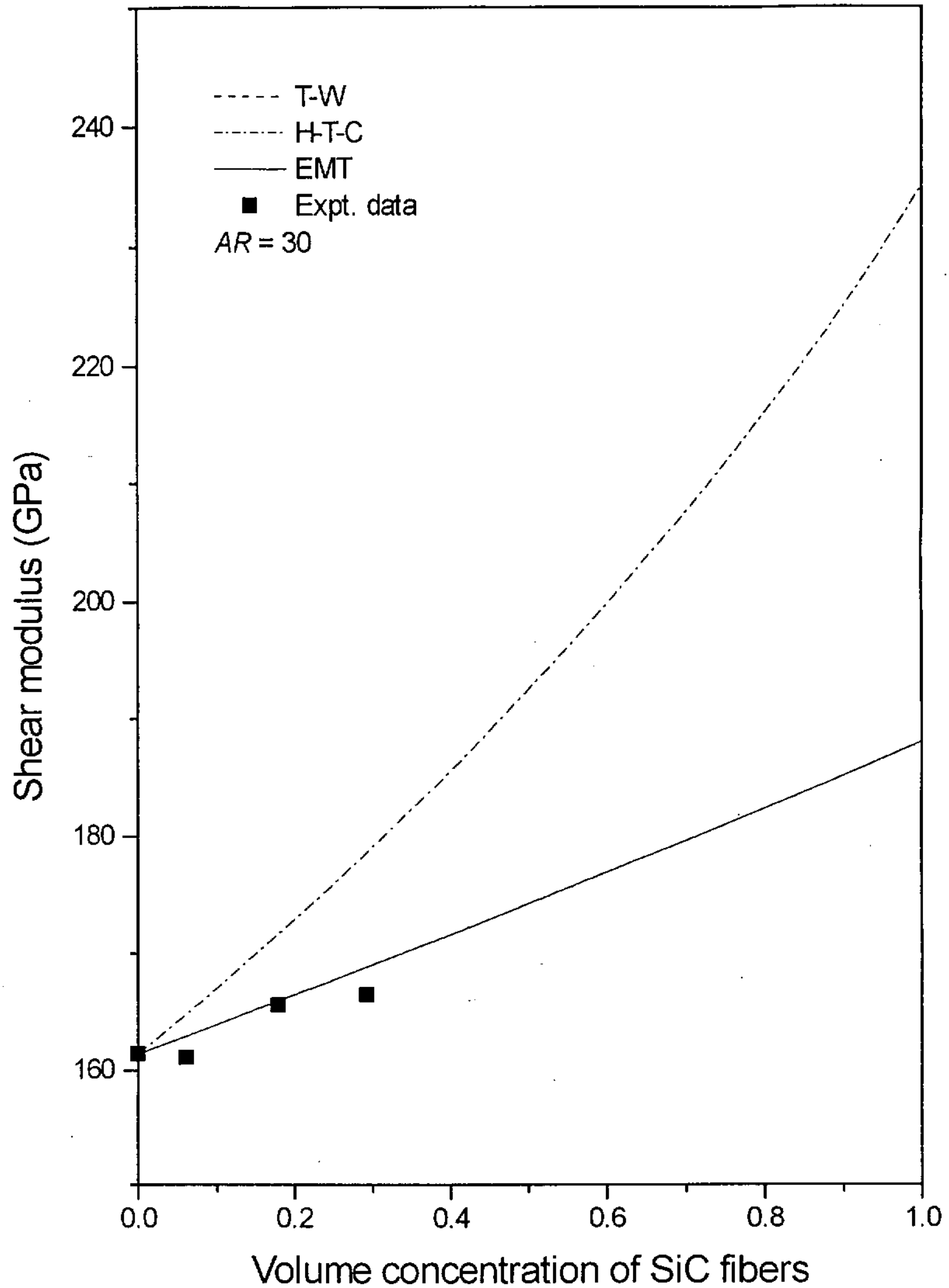


Figure 4.2.4b The shear modulus of $SiC/\ Al_2O_3$ composites as a function of fiber volume concentration.

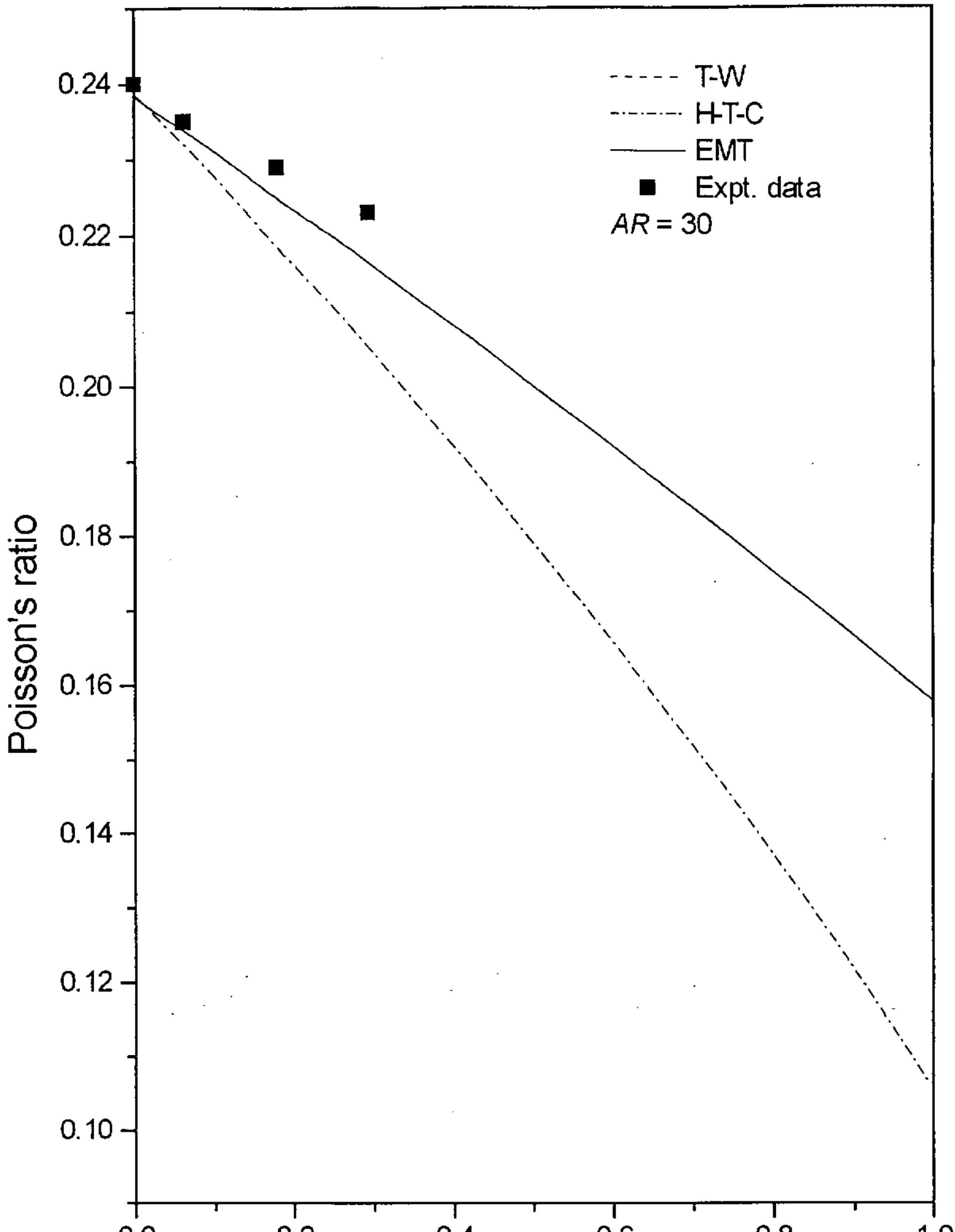


Figure 4.2.4c The Poisson's ratio of SiC/ Al_2O_3 composites as a function of fiber volume concentration.

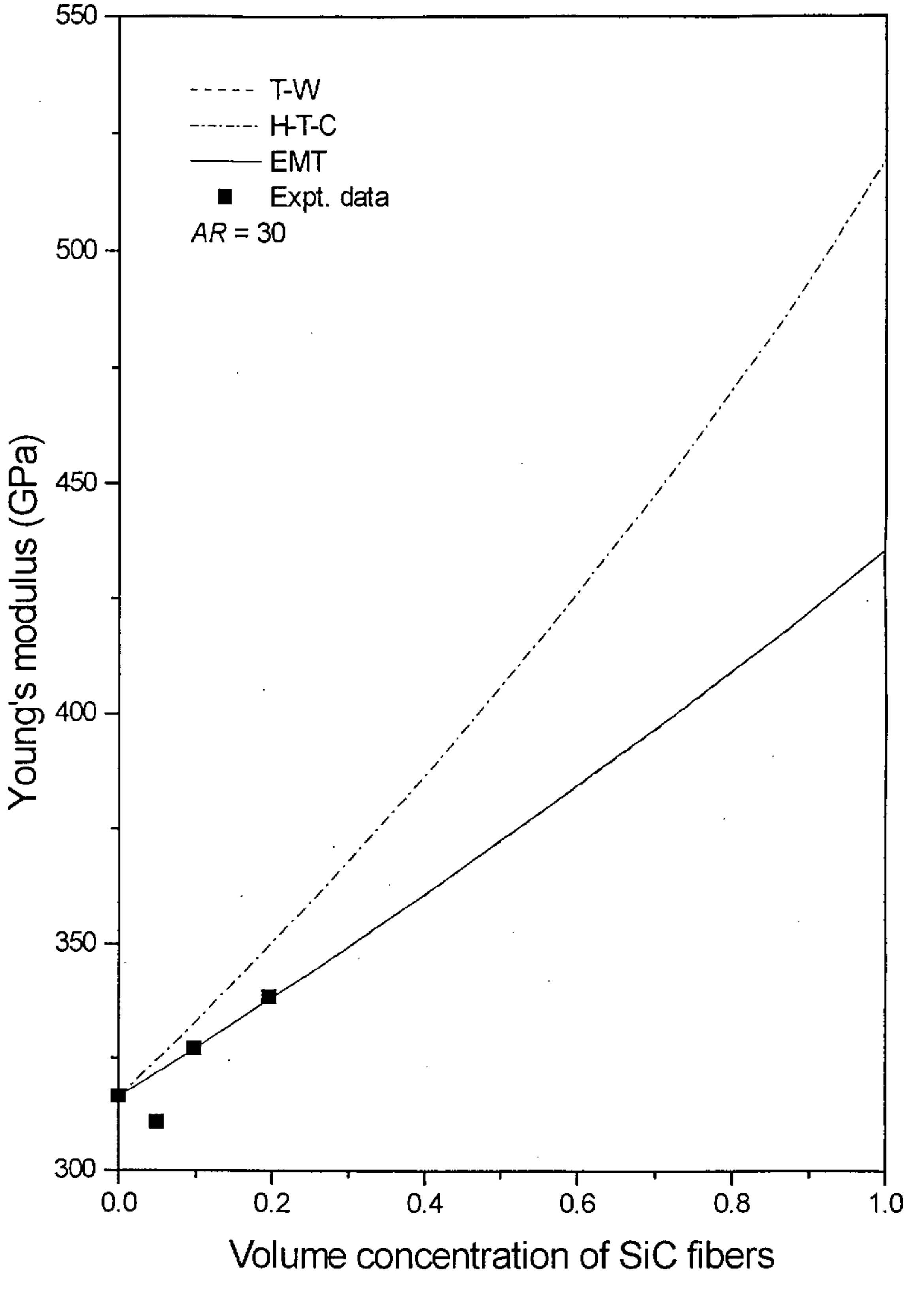


Figure 4.2.5a The Young's modulus of SiC/ Si_3N_4 composites as a function of fiber volume concentration.

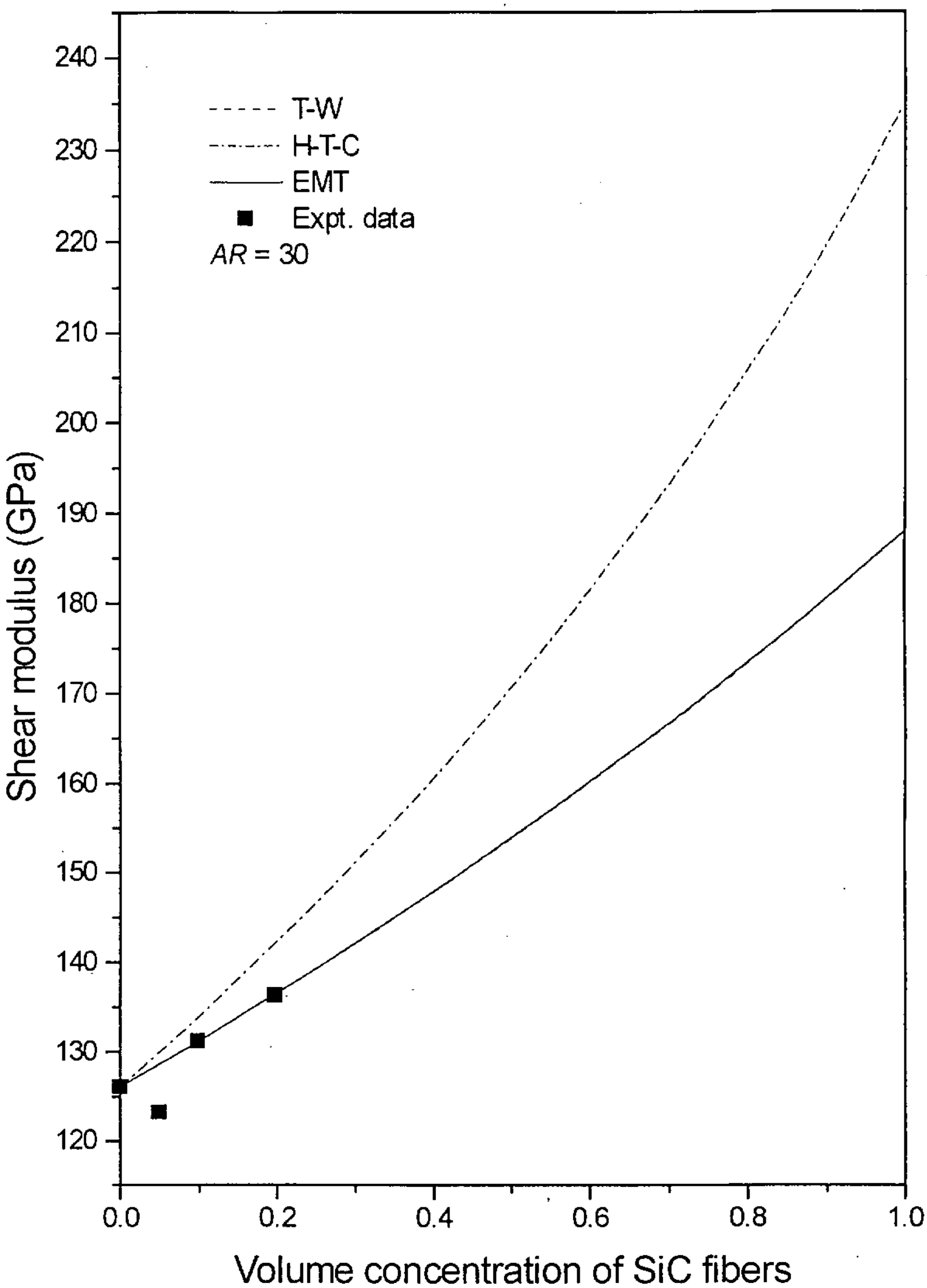


Figure 4.2.5b The shear modulus of SiC/ Si_3N_4 composites as a function of fiber volume concentration.

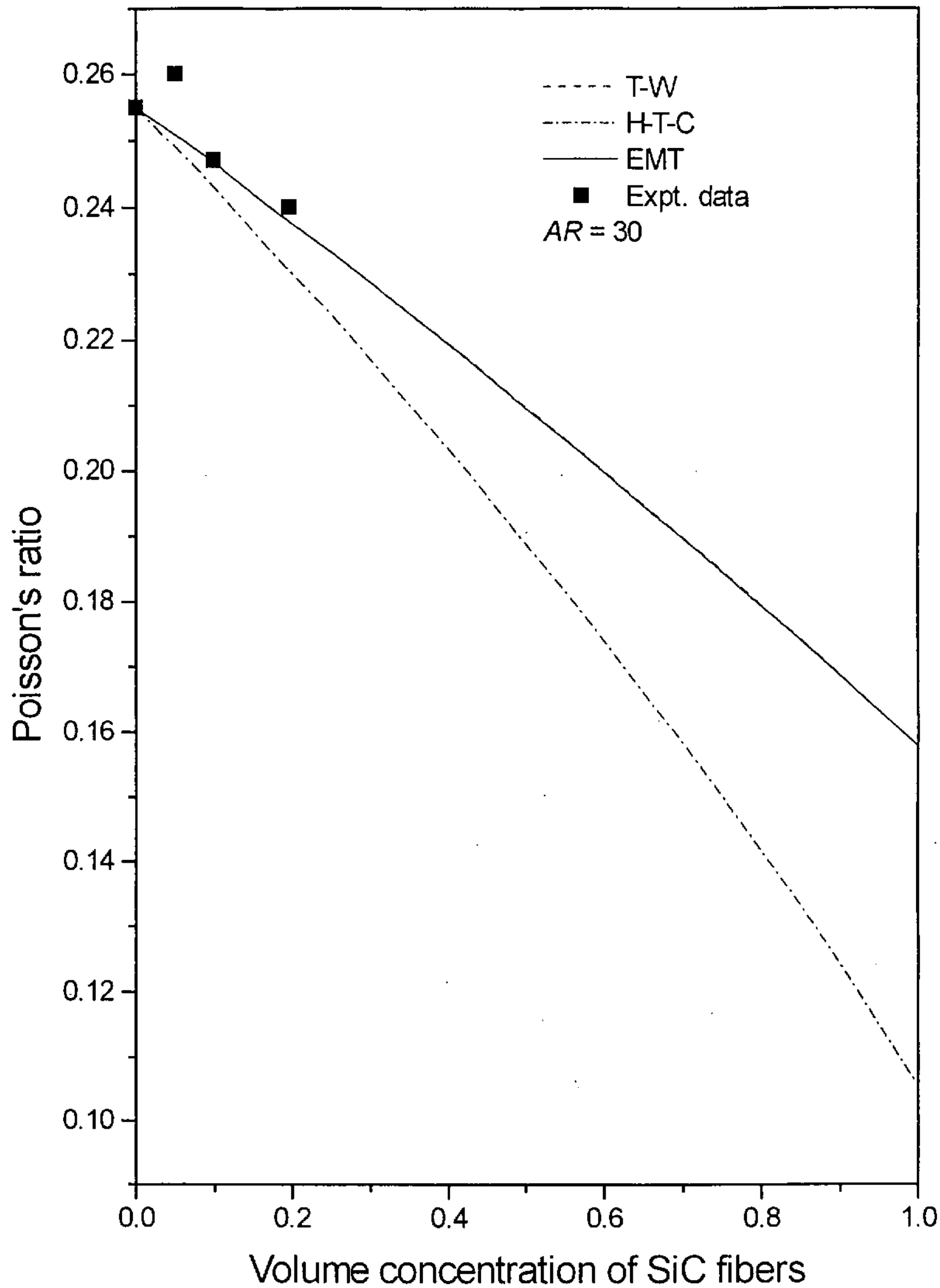


Figure 4.2.5c The Poisson's ratio of SiC/ Si_3N_4 composites as a function of fiber volume concentration.

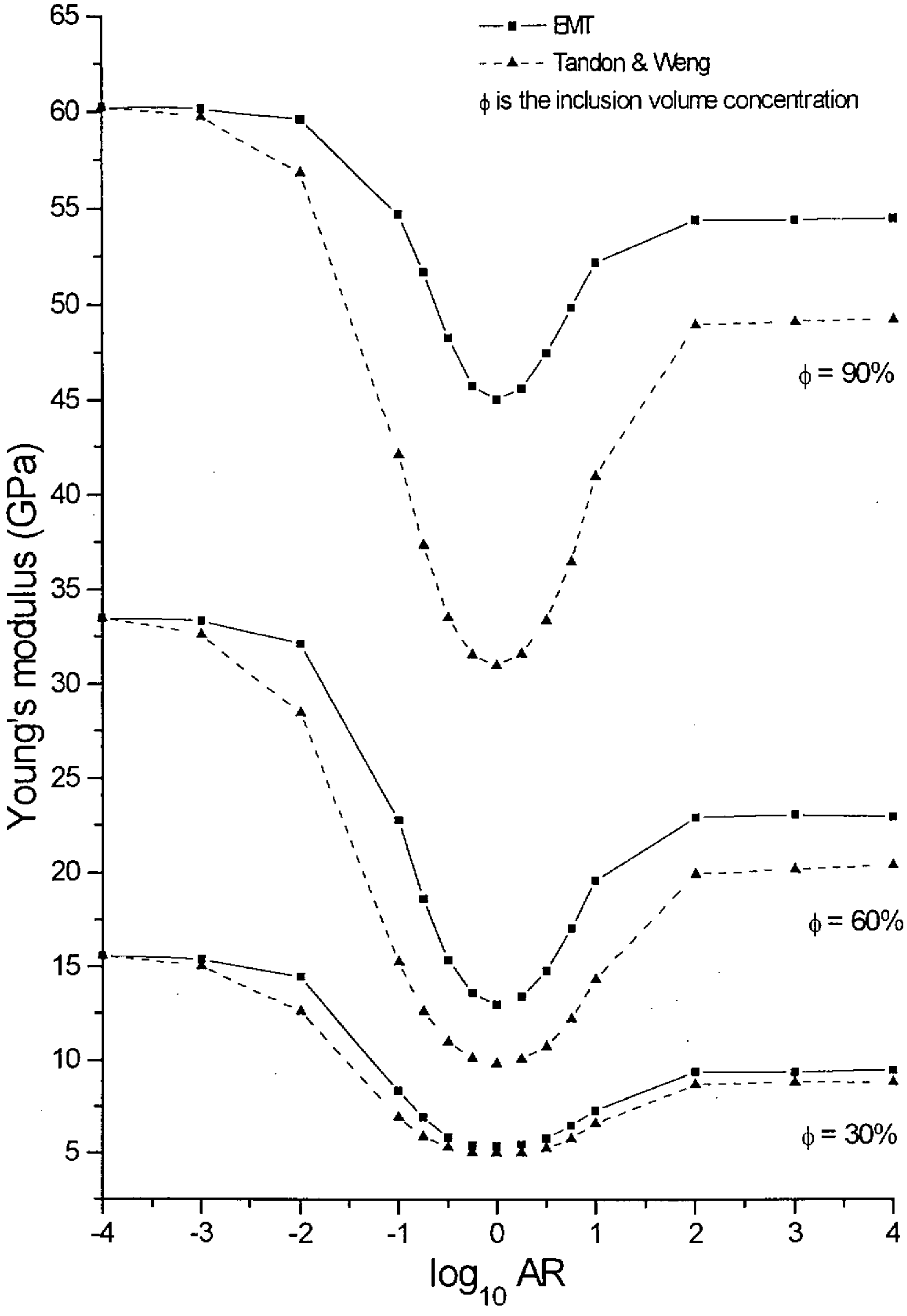


Figure 4.2.6a The Young's modulus of glass/ epoxy composites as a function of aspect ratio.

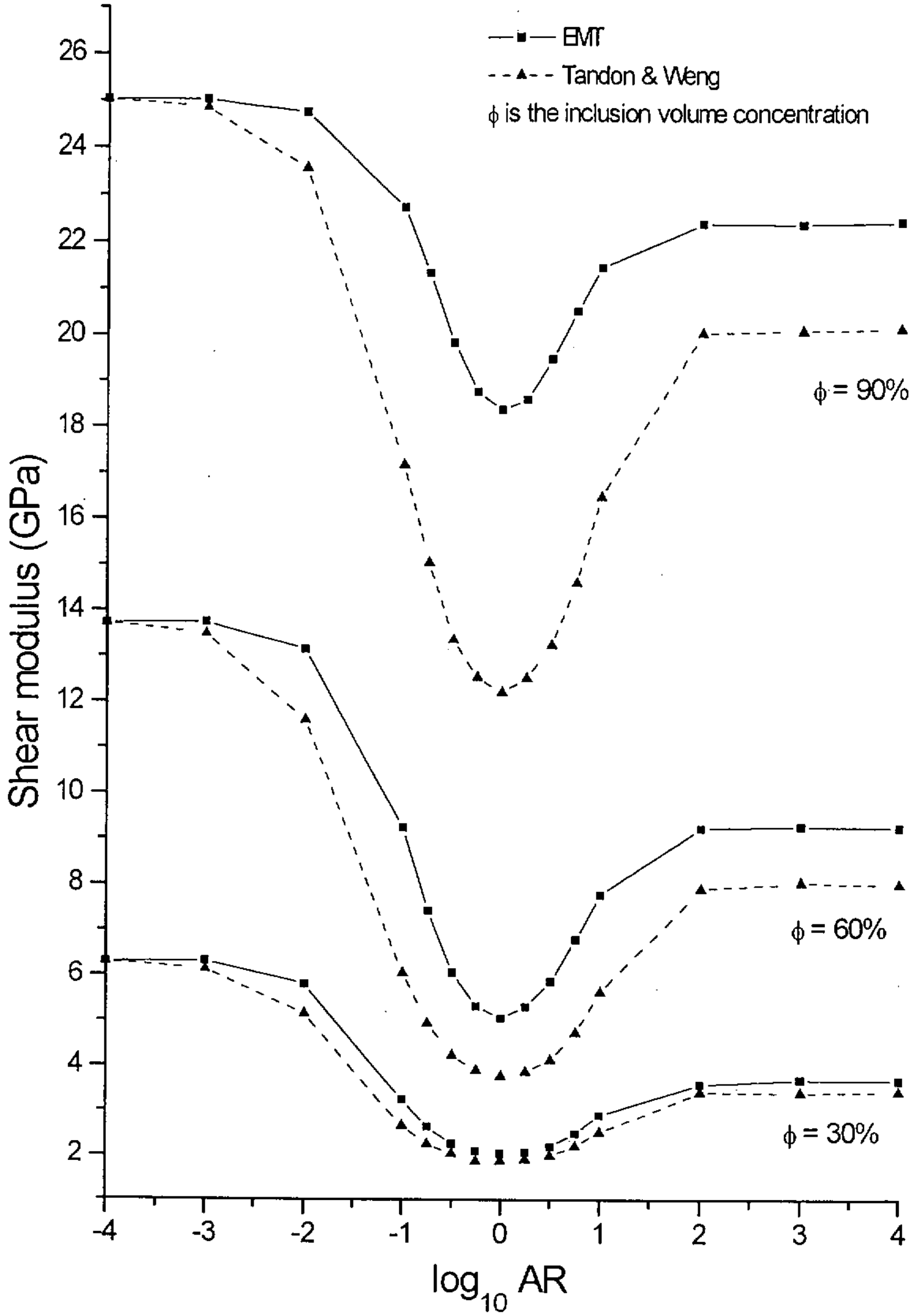


Figure 4.2.6b The shear modulus of glass/ epoxy composites as a function of aspect ratio.

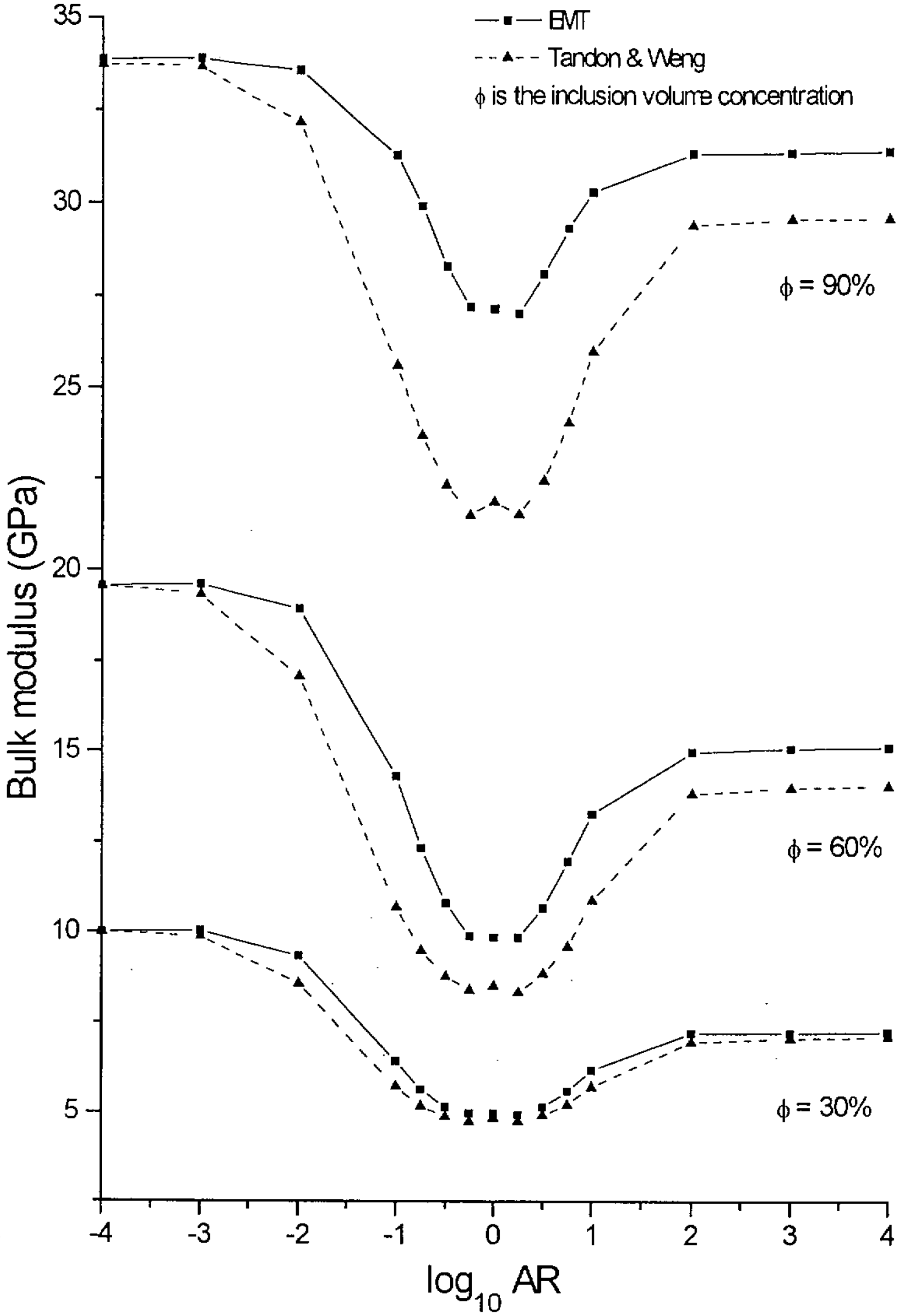


Figure 4.2.6c The bulk modulus of glass/ epoxy composites as a function of aspect ratio.

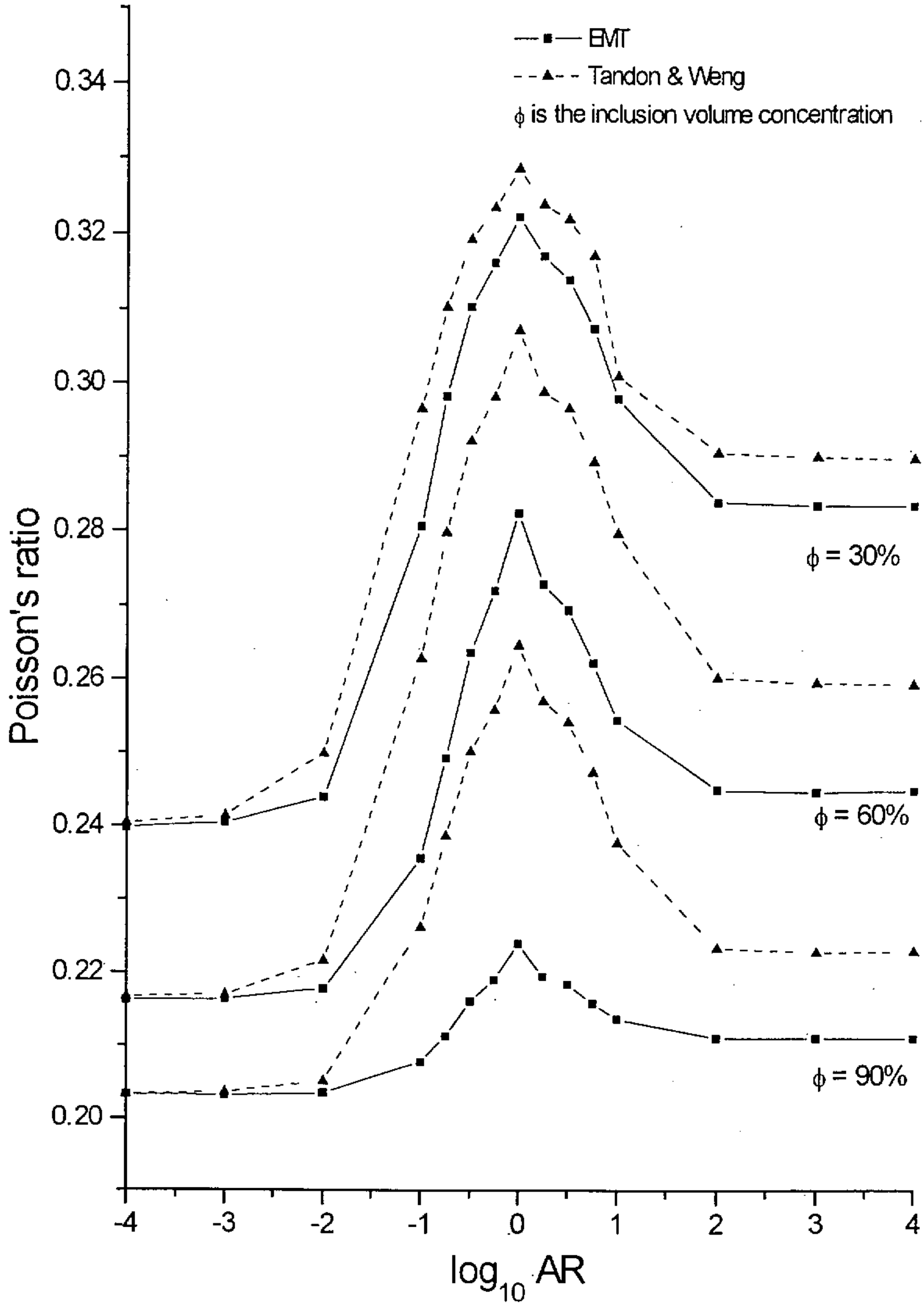


Figure 4.2.6d The Poisson's ratio of glass/ epoxy composites as a function of aspect ratio.

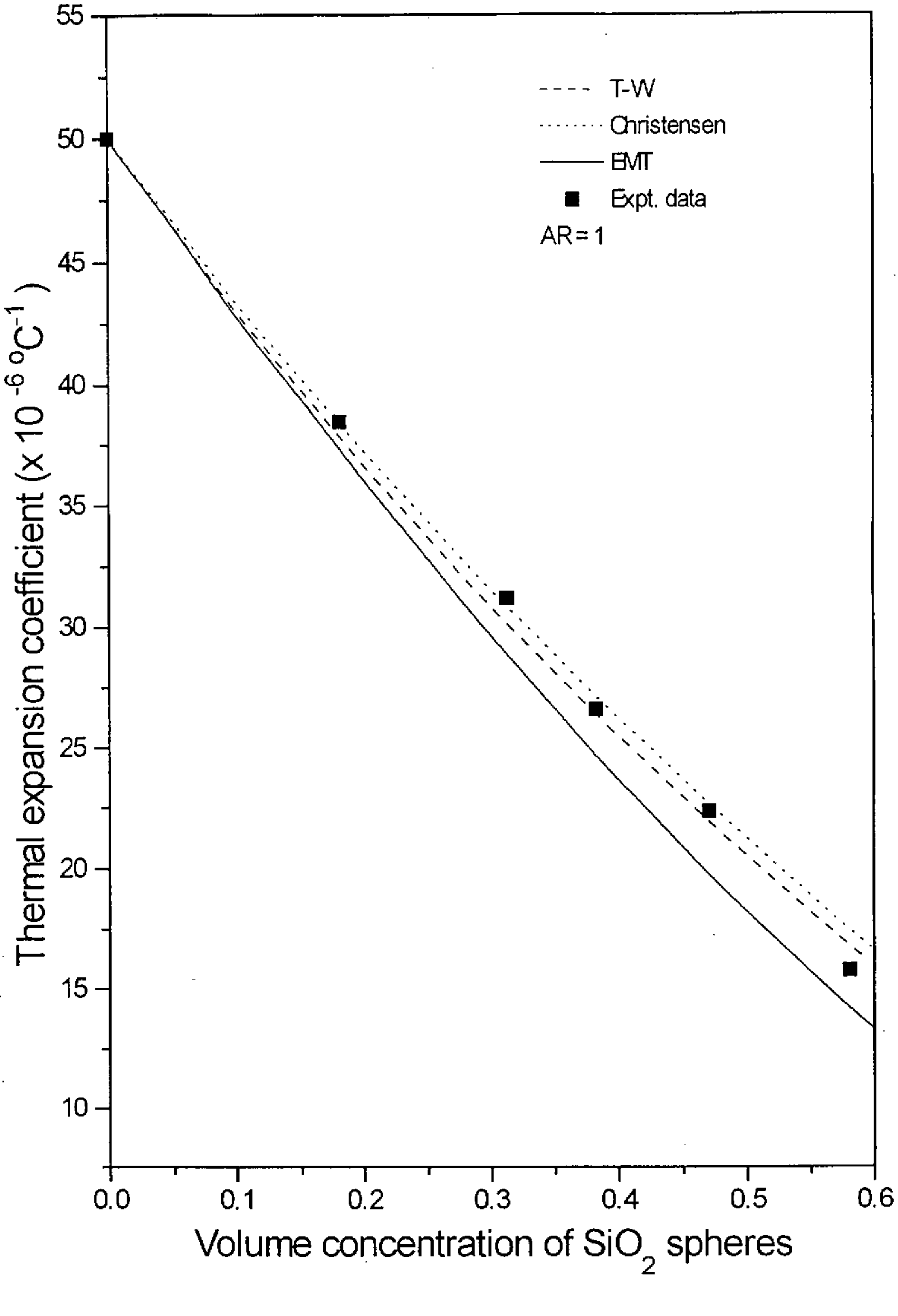


Figure 4.2.7 The thermal expansion coefficient of SiO_2/ K601 composite as a function of sphere volume concentration.

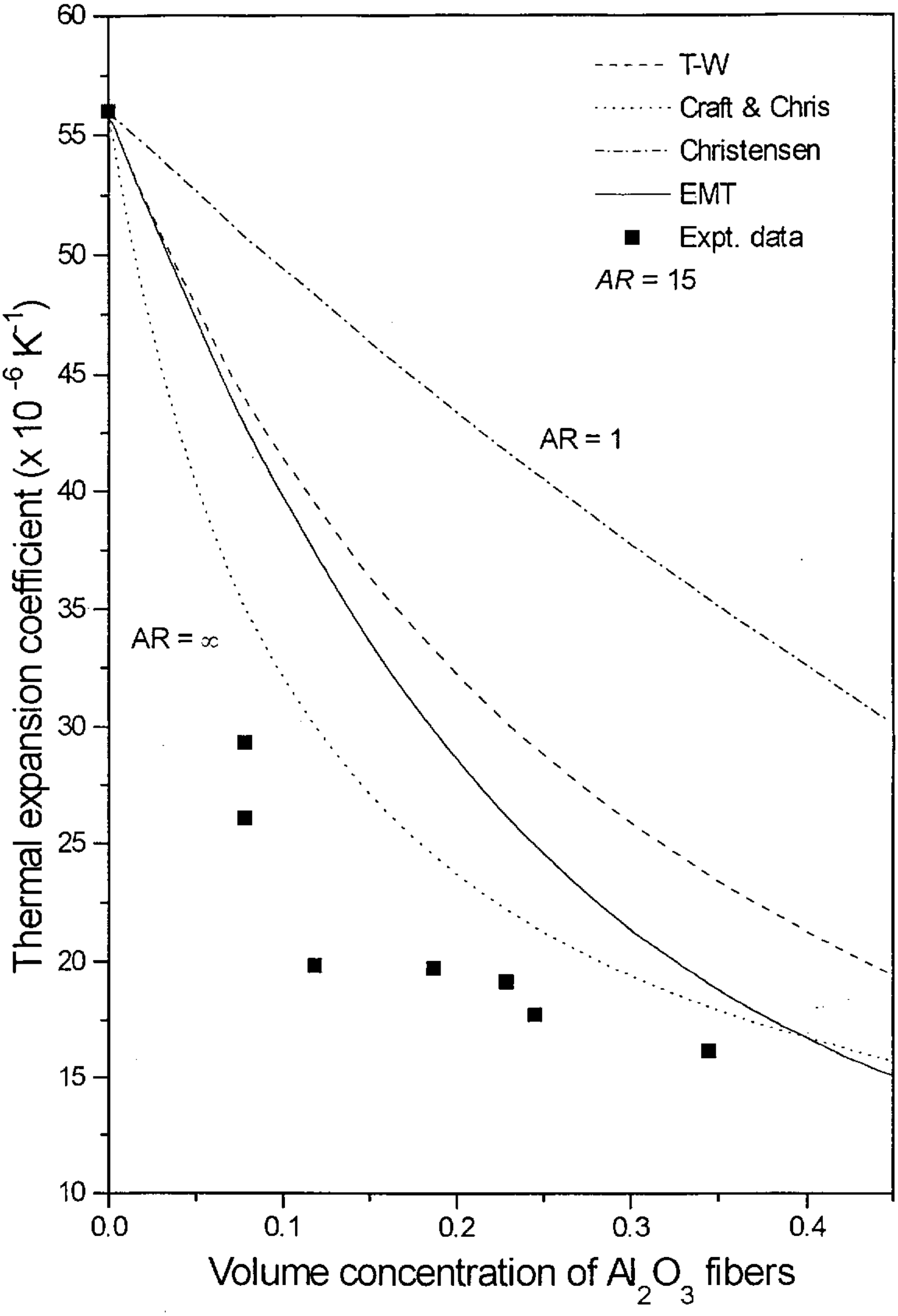


Figure 4.2.8 The thermal expansion coefficient of Al_2O_3/ K601 composite as a function of sphere volume concentration.

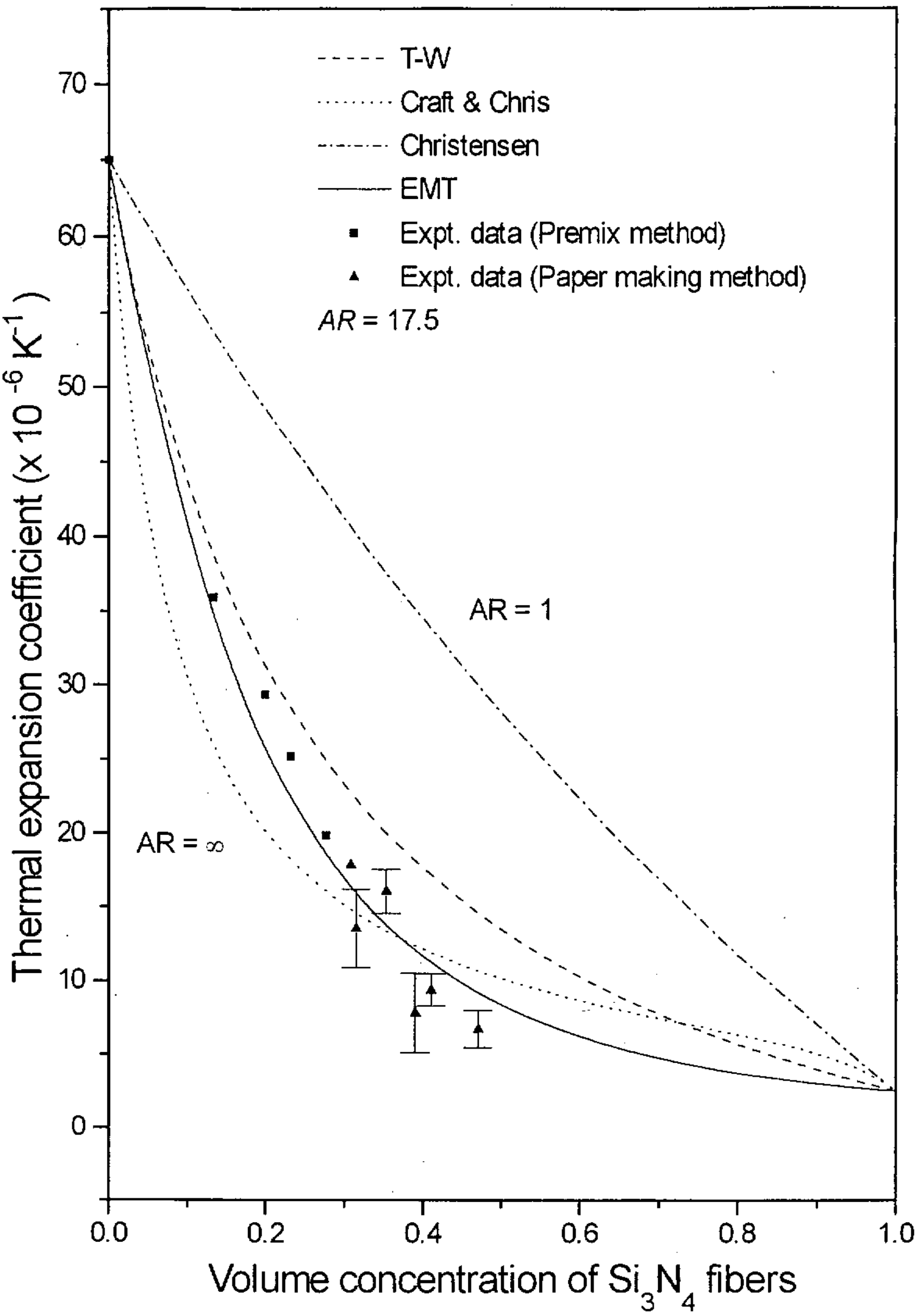


Figure 4.2.9 The thermal expansion coefficient of Si_3N_4/ K601 composite as a function of sphere volume concentration.

Chapter 5 Conclusions

We have proposed a mathematical approach to deal with the problems of calculating the effective elastic and thermoelastic properties of composite materials. The problems tackled are the determination of effective elastic moduli and thermal expansion coefficients of unidirectional fiber reinforced composites with anisotropic constituents, and of composites with short randomly oriented isotropic short fibers. Also, we have been able to give analytical expressions for the prediction of the effective elastic moduli of unidirectional fiber composites with anisotropic constituents.

By use of the concept of Effective Medium Theory (EMT), the effective elastic moduli of unidirectional fiber composites can be numerically calculated according to what we have described, and it is found that the results compare favorably with experimental data. We have also evaluated the five coupled EMT equations for the effective elastic moduli, as a Cauchy problem, by analytical approach. The resulting equations are then compared with some existing equations and bounds of Hashin.

On the prediction of thermal expansion coefficients by EMT, the results show very good agreement with the experimental data. Since for thermal expansion coefficient in the axial direction, the prediction of all equations are similar and thus the difference can only be found in the transverse direction. The EMT results are close to the results by RCM as well as close to experimental values.

The overall elastic moduli of composites with 3-D randomly oriented short fibers or spherical inclusions are investigated. For the case of composites with spherical inclusions, the results show excellent agreement with the measured data especially at high volume concentration. It is observed that both T-W and Hashin LB have lower predictions compared with the data. Furthermore, we have also estimated the thermal expansion coefficient of isotropic composites with isotropic constituents. Three cases have been examined and the composites are reinforced with fibers which have different aspect ratios. However, it is worth to point out that the matrix has no exact thermal expansion coefficient and this relates to the volatile component content in the composite. Therefore, in the calculation, the thermal expansion coefficient of the matrix is different in each composite system. The results of EMT fall in between the spherical particulate and long fiber composites predictions. It shows that the calculation is reasonably good for fibers with aspect ratio between one and infinity.

Also in our work, we have demonstrated the difference between T-W and EMT calculations in the prediction of effective elastic moduli of composites with randomly oriented isotropic short fibers as a function of "fiber" aspect ratio from 0.0001 to 10000. The calculation gives more information about the difference between the predicted overall moduli of composites by EMT and other theories.

In some practical cases, a thrid phase may exist between the filler and the matrix. One common example is that of a composite with coated filler. In such a case, this composite can be treated as a sequence of two binary composite problems. The filler together with the coating agent can be treated as a new filler. The first binary composite problem is the calculation of the properties of the new filler. The second is one of the new filler in the matrix. This approach allows the use of EMT to calculate the overall properties of the original composite.

Essentially when choosing to use EMT, no additional assumptions are imposed on the material system apart from that the fillers are uniformly distributed in the composite. Assumptions and restrictions that are inherent in the equations from which the increment functions are derived will still be prevalent.

In conclusion, the EMT has been proposed to determine the elastic and thermoelastic properties of unidirectional fiber composites with anisotropic constituents and applied to determine the thermoelastic properties of composites with randomly oriented fibers and spherical inclusions. The accuracy of the predictions have been examined against experimental values and assessed in relation to other models. The EMT model is found to give reasonable results in all the cases investigated. We suggest that the EMT could be applied to determine many other properties of composite systems, such as hygrothermal properties and thermal conductivity.

References

Aboudi J.
"The Effective Moduli of Short-Fiber Composites"
Int. J. Solids Structures, Vol. 19, No. 8, pp.693-707 (1983)

Arthur G. and Coulson J.A.
J. Nucl. Mater., Vol. 13, p.242 (1964)

Au W.M., Tsui W.L. and Shin F.G.
"Effective Medium Theory in Elastic Properties of Isotropic Composite Materials"
ACTA Materiae Compositae Sinica, Vol. 11, No. 1, pp.49-58 (1994a)

Au W.M., Tsui W.L. and Shin F.G.
"Numerical Calculations of Elastic Properties of Unidirectional Fiber Reinforced Composites Based on Effective Medium Thoery"
ACTA Materiae Compositae Sinica, Vol. 11, No. 3, pp.50-55 (1994b)

Au W.M. and Shin F.G.
"Calculations of Expansion Coefficient of Composite Materials by Numerical Method Based on Effective Medium Theory"
ACTA Materiae Compositae Sinica, Vol. 12, No. 2, pp.52-58 (1995)

Behrens E.
"Elastic Constants of Filamentary Composites with Haxegonal Symmetry"
J. Acou. Soc. Am., Vol. 45, No. 6, pp.1567-1570 (1969)

Berthelot J.M.
"Effect of Fibre Misalignment on the Elastic Properties of Oriented Discontinuous Fibre Composites"
Fibre Sci. Techn., Vol. 17, pp.25-39 (1982a)

Berthelot J.M.
"Molding Influence on the Mechanical Properties of Sheet Molding Compounds Part I-Elastic Properties"
Fibre Sci. Techn., Vol. 17, pp.235-244 (1982b)

Bowles D.E. and Tompkins S.S.
"Prediction of Coefficients of Thermal Expansion for Unidirectional Composites"
J. Comp. Mater., Vol. 23, pp.370-388 (1989)

Budiansky B.
"On the Elastic Moduli of Some Heterogeneous Materials"
J. Mech. Phys. Solids Vol. 13, pp.223-227 (1965)

Chamberlain N.J.
"Deviation of Expansion Coefficients for a Fibre Reinforced Composite"
BAC Report SON(P), Vol. 33 (1968)

Chamis C.C. and Sendeckyi G.P.
"Critique on Theories Predicting Thermoelastic Properties of Fibrous Composites"
J. Comp. Mater. Vol. 2, No. 3, pp.332-358 (1968)

Chamis C.C.
"Simplified Composite Micromechanics Equations for Hygral, Thermal, and Mechanical Properties"
SAMPE Quarterly, pp.14-23 (1984)

Chen C.H. and Cheng S.
"Mechanical Properties of Fiber Reinforced Composites"
J. Comp. Mater., Vol. 1, pp.30-40 (1967)

Chen C.H. and Cheng S.
"Mechanical Properties of Anisotropic Fiber-Reinforced Composites"
J. Appl. Mech., Vol. 37, pp.186-189 (1970)

Chen L.T., Shin F.G. and Wong Y.W.
"Numerical Calculation of Thermoelastic Properties of Unidirectional Fiber Composites with Anisotropic Constituents"
4th ICCE, Ed. by David H, pp.119-120 (July 1997)

Chou T.W. and Kelly A.
"Fiber Composites" in Challenges and Opportunities in Materials Science and Engineering"
Mater. Sci. Engng., Vol. 25, p.35 (1976)

Chou T.W. and Nomura S.
"Fibre Orientation Effects on the Thermoelastic Properties of Short-Fibre Composites"
Fibre Sci. Techn., Vol. 14, pp.279-291 (1980)

Chou T.W., Nomura S. and Taya M.
"A Self-Consistent Approach to the Elastic Stiffness of Short-Fiber Composites"
J. Comp. Mater., Vol. 14, pp.178-188 (1980)

Choy C.L., Kwok K.W. and Ma H.M.
"Elastic Constants and Thermal Expansivity of Gel-Spun Polyethylene Fiber and Its Composites"
Polym. Comp., Vol. 16, No. 5, pp.357-362 (1995)

Choy C.L., Lau K.W., Wong Y.W., Ma H.M. and Yee A.F.
"Thermal Conductivity and Thermal Expansivity of in Situ Composites of a Liquid Crystalline Polymer and Polycarbonate"
Polym. Engng. Sci., Vol. 36, No. 6, pp.827-834 (1996)

Christensen R.M.
"Mechanics of Composite Materials"
Kreger Publishing Company, Malabar, Florida. (1991)

Craft W.J. and Christensen R.M.
"Coefficients of Thermal Expansion for Composites with Randomly Oriented Fibers"
J. Comp. Mater., Vol. 15, pp.2-20 (1981)

Darras O., Duckett R.A., Hine P.J. and Ward I.M.
"Anisotropic Elasticity of Oriented Polyethylene Materials"
Comp. Sci. Techn., Vol. 51, pp.131-138, (1995)

Eshelby J.D.
"The Determination of the Elastic Field of an Ellipsoidal Inclusion, and Related Problems"
Proc. R. Soc. Lond., Vol. A241, pp.376-396 (1957)

Fishers E.S., Manghnani M.H., Wang J.F. and Rourbort J.L.
"Elastic Properties of Al_2O_3 and Si_3N_4 Matrix Composites with SiC Whiskers Reinforcement"
J. Am. Ceram. Soc., Vol. 75, pp.908-914 (1992)

Hahn H.T., Jernia K.L. and Chiou W.
"Thermoelastic Behaviour of Injection-Molded Thermoplastic Composites"
Proc. Inter. Symp. Comp. Mater. Struc., Beijing, China, pp.68-74 (1986)

Halpin J.C. and Kardos J.L.
"The Halpin-Tsai Equations-A Review"
Polym. Engng. Sci., Vol. 16, No. 5, pp.344-352 (1976)

Halpin J.C. and Pagano N.J.
"The Laminate Approximation for Randomly Oriented Fibrous Composites"
J. Comp. Mater., Vol. 3, pp.720-724 (1969)

Halpin J.C. and Tsai S.W.
"Environmental Factors in Composite Materials Design"
AFML TR 67-423, (1967)

Halpin J.C.
"Stiffness and Expansion Estimations for Oriented Short Fiber Composites"
J. Comp. Mater., Vol. 3, pp.732-734 (1969)

Halpin J.C., Jerine K. and Whitney J.M.
"The Laminate Analogy for 2 and 3 Dimensional Composite Materials"
J. Comp. Mater., Vol. 5, pp.36-44 (1971)

Hashin Z.
"A Variational Approach to the Theory of the Elastic Behaviour of Multiphase Materials"
J. Mech. Phys. Solid, Vol. 11, pp.127-140 (1963)

Hashin Z.
"Theory of Fiber Reinforced Materials"
Pennsylvania University, (NASA-CR-1974) (Mar. 1974)

Hashin Z.
"Analysis of Properties of Fiber Composites with Anisotropic Constituents"
J. Appl. Mech. Vol. 46, pp.543-550 (1979)

Hashin Z. and Rosen B.W.
"The Elastic Moduli of Fiber-Reinforced Materials"
J. Appl. Mech. Vol. 31, pp.223-232 (1964)

Hermans J.J.
"The Elastic Properties of Fiber Reinforced Materials when the Fibers are Aligned"
Proc. R. Academy, Amsterdam, B70, pp.1-9 (1967)

Hershey A.V.
"The Elasticity of an Isotropic Aggregate of Anisotropic Cubic Crystals"
J. Appl. Mech., Vol. 21, pp.236-240 (1954)

Hill R.
"Elastic Properties of Reinforced Solids : Some Theoretical Principles"
J. Mech. Phys. Solids, Vol. 11, pp.357-372 (1963)

Hill R.
"Theory of Mechanical Properties of Fibre-Strengthened Materials : I. Elastic Behaviour"
J. Mech. Phys. Solids, Vol. 12, pp.199-212 (1964a)

Hill R.
"Theory of Mechanical Properties of Fibre-Strengthened Materials : II. Inelastic Behaviour"
J. Mech. Phys. Solids, Vol.12, pp.213-218 (1964b)

Hill R.
"Theory of Mechanical Properties of Fibre-Strengthened Materials : III. Self-Consistent Model"
J. Mech. Phys. Solids, Vol. 13, pp.189-198 (1965a)

Hill R.
"A Self-Consistent Mechanics of Composite Materials"
J. Mech. Phys. Solids, Vol. 13, pp.213-225 (1965b)

Kerner E.H.
"The Elastic and Thermo-elastic Properties of Composite Media"
Proc. Phys. Soc., B69, pp.808-813 (1956)

Kröner E.
"Berechnung der elastichen Konstanten des Vielkristalls aus den Konstanten des Einkristalls"
Zeitschrift fur Physik, Vol. 151, pp.504-518 (1958)

Kriz R.D. and Stinchcomb W.W.
"Elastic Moduli of Transversely Isotropic Graphite Fibers and Their Composites"
Exptl. Mech., p.41 (1979)

Lau K.W. Eddy
"Mechanical and Thermal Properties of Extruded Liquid Crystalline Polymer and Blends of Liquid Crystalline Polymer and Polycarbonate"
Dissertation of the Department of Applied Physics of The Hong Kong Polytechnic University, (1995)

Laws N. and McLaughlin R.
"The Effect of Fibre Length on the Overall Moduli of Composite Materials"
J. Mech. Phys. Solids, Vol. 27, pp.1-13 (1978)

Leung C.W.
"Study of Dielectric Formulation of Binary Mixture with Interaction Terms by Effective Medium Theory"
Dissertation of the Department of Applied Physics of The Hong Kong Polytechnic University, (1996)

Levin V.M.
"Thermal Expansion Coefficients of Heterogeneous Materials"
Mekhanika Tverdogo Tela, Vol. 2, No. 1, pp.88-94 (1967)

Lim T. and Han K.S.
"Prediction of Effective Stiffness on Short Fiber Composite Materials"
Proc. 7th ICCM, Vol. 1, pp.296-303 (1989)

Low B.Y., Gaardner S.D., Pittman Jr. C.U. and Hackett R.M.
"A Micromechanical Characterization of Graphite-Fiber/ Epoxy Composites Containing a Heterogeneous Interphase Region"
Comp. Sci. Techn., Vol. 50, pp.589-606 (1994)

Nishimatsu C. and Gurland
J. Trans. ASM, Vol.5 p.469 (1960)

Nomura S.
"Micromechanics of Short-Fiber Composite Materials"
Dissertation of the Faculty of the University of Delaware (1980)

Nomura S. and Chou T.W.
"Effective Thermoelastic Constants of Short Fiber Composites"
Inter. J. Engng. Sci., Vol. 19, pp.1-9 (1981)

Reuss A.
"Berechnung der Fliessgrenze von Mischkristallen auf Grund der Plastiziatsbedingung fur Einkristalle"
Z. Angew. Math. Mech. (in German), Vol. 9, p.49 (1929)

Richard T.G.
"The Mechanical Behaviour of a Solid Microsphere Filled Composite"
J. Comp. Mater., Vol. 9 pp.108-113 (1975)

Rojstaczer S., Cohn D. and Marom G.
"Thermal Expansion of Kevlar Fibres and Composites"
J. Mater. Sci. lett., Vol. 4, pp.1233-1236 (1985)

Rosen B.W. and Hashin Z.
"Effective Thermal Expansion Coefficients and Specific Heats of Composites Materials"
Inter. J. Engng. Sci., Vol. 8, pp.157-173 (1970)

Russel W.B. and Acrivos A.
"On the Effective Moduli of Composite Materials: Slender Rigid Inclusions at Dilute Concentrations"
J. Appl. Math. Phys. (ZAMP), Vol. 23, pp.433-64 (1972)

Russel W.B.
"On the Effective Moduli of Composite Materials: Effect of Fiber Length and Geometry at Dilute Concentrations"
J. Appl. Math. Phys..(ZAMP), Vol. 24, pp.581-600 (1973)

Schapery R.A.
"Thermal Expansion Coefficients of Composite Materials Based on Energy Principles"
J. Comp. Mater., Vol. 2, No. 3, pp.380-404 (1968)

Schneider W.
Kunststoffe Vol. 61, p.23 (1971)

Shin F.G., Tsui W.L. and Yeung Y.Y.
"Dielectric Constant of Binary Mixtures"
J. Mater. Sci. Lett., Vol. 8, pp.1383-1385 (1989)

Shin F.G., Tsui W.L. and Yeung Y.Y.
"Effective Medium Formalism of Binary Mixture Properties Involving Two Substrate Variables"
J. Mater. Sci. Lett., Vol. 12, pp.1163-1165 (1993a)

Shin F.G., Tsui W.L., Yeung Y.Y. and Au W.M.
"Elastic Properties of a Solid with a Dispersion of Soft Inclusions or Voids"
J. Mater. Sci. Lett., Vol. 12, pp.1632-1634 (1993b)

Shin F.G., Yeung Y.Y. and Tsui W.L.
"On Symmetrical Dielectric Binary Mixture Formulas"
J. Mater. Sci. Lett., Vol. 9, pp.948-950 (1990)

Siboni G.
"Prediction of the Effective Elastic Properties of Multiphase Composite Media with Orthotropic Constituents"
Comp. Sci. and Techn., Vol. 50, pp.293-298, (1994)

Strifė J.R. and Prewo K.M.
"The Thermal Expansion Behaviour of Unidirectional and Bidirectional Kevlar/ Epoxy Composites"
J. Comp. Mater., Vol. 13, pp.264-277 (1979)

Takahashi K. and Harakawa K.
"Analysis of the Thermal Expansion Coefficients of Particular-Filled Polymers"
Environmental Effects on Composite Materials, Springer G.S., Vol. 12, pp.384-362 (1984)

Takao Y. and Taya M.
"Thermal Expansion Coefficients and Thermal Stress in an Aligned Short Fiber Composite with Application to a Short Carbon Fiber/ Aluminum"
J. Appl. Mech., Vol. 52, pp.806-810 (1985)

Takei T., Hatta H. and Taya M.
"Thermal Expansion Behaviour of Particular-filled Composites I: Single Reinforcing Phase"
Mater. Sci. and Engng., Vol. A131, pp.133-143 (1991)

Tandon G.P. and Weng G.J.
"Stress Distribution in and around Spheroidal Inclusions and Voids at Finite Concentration"
J. Appl. Mech., Vol. 53, pp.511-518, (1986a)

Tandon G.P. and Weng G.J.
"Average Stress in the Matrix and Effective Moduli of Randomly Oriented Composites"
Comp. Sci. and Techn., Vol. 27, pp.111-132, (1986b)

Tsai S.W. and Hahn H.T.
"Introduction to Composite Materials"
Technomic Publishing Co., p.398 (1980)

Turner P.S.
J. Res. NBS, Vol.37, p.239 (1946)

Voigt W.
Lehrfbuch der Kristallphysik (in German) Teubner, Leipzig (1910)

Wang G.A.
"Elastic Constants and Thermal Expansion of Certain Bodies with Inhomogeneous Regular Structure"
Soviet Physics-Doklady, Vol. 11, No. 2, pp.176-178 (1966)

Weng G.J.
"Some Elastic Properties of Reinforced Solids, With Special Reference to Isotropic Ones Containing Spherical Inclusions"
Inter. J. Engng. Sci., Vol. 22, pp.845-856 (1984)

Whitney J.M., Daniel I.M. and Pipes R.B.
"Experimental Mechanics of Fiber Reinforced Composite Materials"
The Society for Experimental Mechanics, pp.10-12 (1984)

Wilczynski A. and Lewinski J.
"Predicting the Properties of Unidirectional Fibrous Composites with Monotropic Reinforcement"
Comp. Sci. and Techn., Vol. 51, pp.139-143 (1995)

Williamson G.R.
"The Effect of Steel Fibers on the Compressive Strength of Concrete"
International Symposium on Fiber Reinforced Concrete.
American Concrete Institute. (1973)

Yates B., Overy M.J., Sargent J.P. McCalla B.A. and Kingston-L D.M., Philips L.N., Rogers K.F.
"The Thermal Expansion of Carbon Fibre-Reinforced Plastics Part 2 The Influence of Fibre Volume Fraction"
J. Mater. Sci., Vol. 12, pp.718-724 (1977)

Appendix A

The Hashin bounds are given here; (-) and (+) signs represent the lower and upper bounds, respectively.

The bounds of plane strain bulk modulus

$$K^{(-)} = K_1 + \frac{\phi}{\dfrac{1}{K_2 - K_1} + \dfrac{(1-\phi)}{K_1 - G_{T1}}} \qquad \text{......(A1a)}$$

$$K^{(+)} = K_2 + \frac{(1-\phi)}{\dfrac{1}{K_1 - K_2} + \dfrac{\phi}{K_2 - G_{T2}}} \qquad \text{......(A1b)}$$

The bounds of transverse shear modulus

$$G_T^{(-)} = G_{T1} + \frac{\phi}{\dfrac{1}{G_{T2} - G_{T1}} + \dfrac{K_1 + 2G_{T1}}{2G_{T1}(K_1 + G_{T1})}(1-\phi)} \qquad \text{......(A2a)}$$

$$G_T^{(+)} = G_{T2} \frac{(1-\phi)}{\dfrac{1}{G_{T1} - G_{T2}} + \dfrac{K_2 + 2G_{T2}}{2G_{T2}(K_2 + G_{T2})}\phi} \qquad \text{......(A2b)}$$

The bounds of axial shear modulus

$$G_A^{(-)} = G_{A1} + \frac{\phi}{\dfrac{1}{G_{A2} - G_{A1}} + \dfrac{(1-\phi)}{2G_{A1}}} \qquad \text{......(A3a)}$$

$$G_A^{(+)} = G_{A2} + \frac{(1-\phi)}{\dfrac{1}{G_{A1} - G_{A2}} + \dfrac{\phi}{2G_{A2}}} \qquad \text{......(A3b)}$$

The bounds of axial Young's modulus

$$E_A^{(-)} = E_{A1}(1-\phi) + E_{A2}\phi + \frac{4\phi(1-\phi)(\nu_{A1} - \nu_{A2})^2}{\dfrac{(1-\phi)}{K_2} + \dfrac{\phi}{K_1} + \dfrac{1}{G_{T1}}} \qquad \text{......(A4a)}$$

$$E_A^{(+)} = E_{A1}(1-\phi) + E_{A2}\phi + \frac{4\phi(1-\phi)(\nu_{A1} - \nu_{A2})^2}{\dfrac{(1-\phi)}{K_2} + \dfrac{\phi}{K_1} + \dfrac{1}{G_{T2}}} \qquad \text{......(A4b)}$$

The bounds of axial Poisson's ratio

$$\nu_A^{(-)} = \nu_{A1}(1-\phi) + \nu_{A2}\phi + \frac{\phi(1-\phi)(\nu_{A2} - \nu_{A1})(\dfrac{1}{K_1} - \dfrac{1}{K_2})}{\dfrac{(1-\phi)}{K_2} + \dfrac{\phi}{K_1} + \dfrac{1}{G_{T1}}} \qquad \text{......(A5a)}$$

$$\nu_A^{(+)} = \nu_{A1}(1-\phi) + \nu_{A2}\phi + \frac{\phi(1-\phi)(\nu_{A2} - \nu_{A1})(\dfrac{1}{K_1} - \dfrac{1}{K_2})}{\dfrac{(1-\phi)}{K_2} + \dfrac{\phi}{K_1} + \dfrac{1}{G_{T2}}} \qquad \text{......(A5b)}$$

The bounds of transverse Young's modulus

$$E_T^{(\pm)} = \frac{4K^{(\pm)}G_T^{(\pm)}}{K^{(\pm)} + w^{(\pm)}G_T^{(\pm)}} \qquad \text{......(A6a)}$$

where $$w^{(\pm)} = 1 + \frac{4K^{(\pm)}\nu_A^{(\mp)}}{E_A^{(\pm)}}$$

The bounds of transverse Poisson's ratio

$$\nu_T^{(\pm)} = \frac{K^{(\pm)} - w^{(\pm)}G_T^{(\mp)}}{K^{(\pm)} + w^{(\pm)}G_T^{(\mp)}} \qquad \text{......(A6b)}$$

The equations for effective elastic moduli by Chamis are as follows:

The transverse Young's modulus

$$E_T = \frac{E_1}{1-\sqrt{\phi}(1-\frac{E_1}{E_{T2}})} \qquad \text{......(A7)}$$

The axial shear modulus

$$G_A = \frac{G_1}{1-\sqrt{\phi}(1-\frac{G_1}{G_{A2}})} \qquad \text{......(A8)}$$

The transverse shear modulus

$$G_T = \frac{G_1}{1-\sqrt{\phi}(1-\frac{G_1}{G_{T2}})} \qquad \text{......(A9)}$$

The axial Poisson's ratio

$$\nu_A = \phi\nu_{A2} + (1-\phi)\nu_1 \qquad \text{......(A10)}$$

The transverse Poisson's ratio

$$\nu_T = \frac{E_T}{2G_T} - 1 \qquad \text{......(A11)}$$

Appendix B

Some equations are given below for the effective thermal expansion coefficients of unidirectional fiber composites with isotropic matrix and anisotropic fibers, and of three dimensional randomly oriented fiber composites with isotropic phases.

Chamis equations

Axial thermal expansion coefficient

$$\alpha_A = \frac{\phi\alpha_{fA}E_{fA} + (1-\phi)\alpha_m E_m}{E_A} \qquad \text{......(B1)}$$

Transverse thermal expansion coefficient

$$\alpha_T = \alpha_{fT}\sqrt{\phi} + (1-\sqrt{\phi})(1+\phi\nu_m\frac{E_{fA}}{E_A})\alpha_m \qquad \text{......(B2)}$$

where ν_m, E_m, E_{fA} and E_A are the Poisson's ratio and Young's modulus of the isotropic matrix, axial Young's modulus of the fiber and the effective Young's modulus of a unidirectional fiber composite. α_m , α_{fA} , and α_{fT} are the thermal expansion coefficients of the matrix, the fiber in axial and transverse directions, and ϕ is the fiber volume concentration.

Chamberlain equations

Axial thermal expansion coefficient

$$\alpha_A = \frac{\phi\alpha_{fA}E_{fA} + (1-\phi)\alpha_m E_m}{E_A} \qquad \text{......(B3)}$$

Transverse thermal expansion coefficient

$$\alpha_T = \alpha_m + \frac{2(\alpha_{fT} - \alpha_m)\phi}{\nu_m[F - 1 + (1-\phi)] + (F + \phi) + \frac{E_m}{E_{fA}}(1 - \nu_{fA})[F - 1 + (1-\phi)]} \qquad \text{......(B4)}$$

where F is a packing factor, equal to 0.9096 and 0.7854 for hexagonal and square packing. ν_{fA} is the axial Poisson's ratio of fiber.

Appendix C

Table of moduli values for Section 2.4

Constituents	E_A (GPa)	ν_A	G_A (GPa)	G_T (GPa)	K (GPa)	E_T (GPa)	ν_T
PE matrix	1.24	0.4410	0.43	0.43	3.646	1.24	0.441
PE fiber	122.9	0.4875	1.05	1.00	4.00	3.18	0.5901
PC matrix	3.141	0.3896	1.13	1.13	5.119	3.141	0.3896
LCP fiber	85.25	0.5387	1.44	1.03	6.59	3.52	0.7087
Epoxy matrix (for ModmorII/ Ep)	5.35	0.354	1.976	1.976	6.766	5.34	0.354
ModmorII fiber	232	0.279	24.0	5.02	15.0	15.0	0.49
Epoxy matrix (for Kelvar/ Ep)	3.5	0.35	1.296	1.296	4.321	3.5	0.35
Kevlar fiber	113	0.63	2.01	0.95	1.816	2.49	0.31

Table of thermal expansion coefficients for Section 3.2

Constituents	α_A (K^{-1})	α_T (K^{-1})
PE matrix	2.35×10^{-4}	2.35×10^{-4}
PE fiber	-0.125×10^{-4}	1.7956×10^{-4}
PC matrix	0.673×10^{-4}	0.673×10^{-4}
LCP fiber	-0.0795×10^{-4}	0.7999×10^{-4}
Epoxy matrix	65×10^{-6}	65×10^{-6}
Kevlar fiber	-5.7×10^{-6}	66.3×10^{-6}

Table of moduli values for Section 4.2

Constituents	Young's modulus	Poisson's ratio
Polyester matrix	2.5×10^5 psi	0.45
Glass sphere	102×10^5 psi	0.21
Cobalt matrix	206.8 GPa	0.3
Tungsten carbide sphere	703 GPa	0.22
Concrete cement	20.802 GPa	0.2081
Steel fiber (AR = 50)	200 GPa	0.3
Al_2O_3 matrix	399.7 GPa	0.24
Si_3N_4 matrix	316.5 GPa	0.255
SiC whisker (AR = 30)	548 GPa	0.14
Epoxy matrix	2.76 GPa	0.35
Glass fiber	72.4 GPa	0.2

Table of elastic moduli and thermal expansion coefficient for Section 4.2

Constituents	Young's modulus	Poisson's ratio	Thermal expansion coefficient ($\times 10^{-6}$ K^{-1})
Kerimid 601 matrix	3.5 GPa	0.35	50-80
SiO_2 sphere	70 GPa	0.22	0.5
Al_2O_3 short fiber (AR = 15)	300 GPa	0.22	8
Si_3N_4 whisker (AR = 17.5)	385 GPa	0..27	2.5

Appendix D

The components of Eshelby's S_{ijkl} tensor

$$S_{1111}=\frac{1}{2(1-\nu_m)}\left[1-2\nu_m+\frac{3AR^2-1}{AR^2-1}(1-2\nu_m+\frac{3AR^2}{AR^2-1})Z\right]$$

$$S_{2222}=S_{3333}=\frac{1}{8(1-\nu_m)}\frac{3AR^2}{AR^2-1}+\frac{1}{4(1-\nu_m)}\left[1-2\nu_m-\frac{9}{4(AR^2-1)}\right]Z$$

$$S_{2233}=S_{3322}=\frac{1}{4(1-\nu_m)}\left\{\frac{AR^2}{2(AR^2-1)}-\left[1-2\nu_m+\frac{3}{4(AR^2-1)}\right]Z\right\}$$

$$S_{2211}=S_{3311}=-\frac{1}{2(1-\nu_m)}\frac{AR^2}{AR^2-1}+\frac{1}{4(1-\nu_m)}\left[\frac{3AR^2}{AR^2-1}-(1-2\nu_m)\right]Z$$

$$S_{1122}=S_{1133}=-\frac{1}{2(1-\nu_m)}(1-2\nu_m+\frac{1}{AR^2-1})+\frac{1}{2(1-\nu_m)}\left[1-2\nu_m+\frac{3}{2(AR^2-1)}\right]Z$$

$$S_{2323}=S_{3232}=\frac{1}{4(1-\nu_m)}\left\{\frac{AR^2}{2(AR^2-1)}+\left[1-2\nu_m-\frac{3}{2(AR^2-1)}\right]Z\right\}$$

$$S_{1212}=S_{1313}=\frac{1}{4(1-\nu_m)}\left\{1-2\nu_m-\frac{AR^2+1}{AR^2-1}-\frac{1}{2}\left[1-2\nu_m-\frac{3(AR^2+1)}{AR^2-1}\right]Z\right\}$$

......(D1)

where ν_m is the Poisson's ratio of the matrix and AR is the fiber aspect ratio and Z, for prolate shape, is given by

$$Z = \frac{AR}{(AR^2 - 1)^{\frac{3}{2}}}\left[AR(AR^2 - 1)^{\frac{1}{2}} - \cosh^{-1} AR\right]$$

and for oblate shape inclusion, it is

$$Z = \frac{AR}{(1 - AR^2)^{\frac{3}{2}}}\left[\cos^{-1} AR - AR(1 - AR^2)^{\frac{1}{2}}\right]$$

The values for the p and q constants are given by

$$p_1 = 6(K_i - K_m)(G_i - G_m)(S_{2222} + S_{2233} - 1) - 2(K_m G_i - K_i G_m) + 6K_i(G_i - G_m)$$

$$p_2 = 6(K_i - K_m)(G_i - G_m)S_{1133} + 2(K_m G_i - K_i G_m)$$

$$p_3 = -6(K_i - K_m)(G_i - G_m)S_{3311} + 2(K_m G_i - K_i G_m)$$

$$p_4 = 6(K_i - K_m)(G_i - G_m)(S_{1111} - 1) + 2(K_m G_i - K_i G_m) + 6G_i(K_i - K_m)$$

$$p_5 = \frac{1}{S_{3322} - S_{3333} + 1 - \dfrac{G_m}{G_i - G_m}}$$

$$p = 6(K_i - K_m)(G_i - G_m)\left[2S_{1133}S_{3311} - (S_{1111} - 1)(S_{3322} + S_{3333} - 1)\right]$$

$$+2(K_m G_i - K_i G_m)\left[2(S_{1133} + S_{3311}) + (S_{1111} - S_{3322} - S_{3333})\right] - 6K_i(G_i - G_m)(S_{1111} - 1)$$

$$-6G_i(K_i - K_m)(S_{2222} + S_{2233} - 1) - 6K_i G_i$$

......(D2)

$$q_1 = \frac{1}{16p}\{2p_3(6S_{1122} + S_{2222} + S_{2233} - 1) + p_4[3(S_{2222} + S_{2233} - 1) + 2S_{1122}]$$

$$+3p_5p(S_{2222} - S_{2233} - 1) + 2p_1[3(S_{1111} - 1) + S_{2211}] - 2p_2(S_{1111} + 3S_{2211} - 1)$$

$$-\frac{4p(2S_{1212} - 1)}{2S_{1212} + \dfrac{G_m}{G_i - G_m}}\}$$

$$q_2 = \frac{1}{16p}\{2p_3[2S_{1122} + 3(S_{2222} + S_{2233} - 1)] + p_4(6S_{1122} + S_{2222} + S_{2233} - 1)$$

$$+p_5p(S_{2222} - S_{2233} - 1) + 2p_1(S_{1111} + 3S_{2211} - 1) - 2p_2[S_{2211} + 3(S_{1111} - 1)]$$

$$+\frac{4p(2S_{1212} - 1)}{2S_{1212} + \dfrac{G_m}{G_i - G_m}}\}$$

$$q_3 = \frac{1}{4p}\{-2p_2(S_{1111} + S_{2211} - 1) + p_4(2S_{1122} + S_{2222} + S_{2233} - 1) - p_5p(S_{2222} - S_{2233} - 1)\}$$

$$q_4 = \frac{1}{4p}\{2(p_1 - p_2)S_{2211} + (2p_3 + p_4)(S_{2222} + S_{2233} - 1) - p_5p(S_{2222} - S_{2233} - 1)\}$$

$$q_5 = \frac{1}{4p}\{-2p_2S_{2211} + p_4(S_{2222} + S_{2233} - 1) + p_5p(S_{2222} - S_{2233} - 1)\}$$

......(D3)

where G_m, G_i, K_m and K_i are the shear moduli of matrix and fiber and bulk moduli of matrix and fiber.

Halpin and Tsai equations

The equations for the effective elastic moduli of aligned short fiber composites are:

The axial Young's modulus

$$E_A = \frac{E_{mA}[1 + 2AR\eta_{E_A}\phi]}{(1 - \eta_{E_A}\phi)} \qquad \text{......(D4)}$$

The transverse Young's modulus

$$E_T = \frac{E_{mT}[1 + 2AR\eta_{E_T}\phi]}{(1 - \eta_{E_T}\phi)} \qquad \text{......(D5)}$$

The axial Poisson's ratio

$$\nu_A = \nu_{fA}\phi + \nu_{mA}(1 - \phi) \qquad \text{......(D6)}$$

The plane strain bulk modulus

$$K = \frac{(K_f + G_{mT})K_m + (K_f - K_m)G_{mT}\phi}{(K_f + G_{mT}) - (K_f - K_m)\phi} \qquad \text{......(D7)}$$

The axial shear modulus

$$G_A = \frac{G_{mA}[G_{fA}(1 + \phi) + G_{mA}(1 - \phi)]}{G_{fA}(1 - \phi) + G_{mA}(1 + \phi)} \qquad \text{......(D8)}$$

The transverse Poisson's ratio

$$\nu_T = 1 - \frac{E_T(E_A + 4\nu_A^2 K)}{2E_A K} \qquad \text{......(D9)}$$

The transverse shear modulus

$$G_T = \frac{E_T}{2(1+\nu_T)} \qquad \text{......(D10)}$$

where

$$\eta_{E_A} = \frac{(\frac{E_{fA}}{E_{mA}} - 1)}{[\frac{E_{fA}}{E_{mA}} + 2AR]} \qquad \text{and} \qquad \eta_{E_T} = \frac{(\frac{E_{fT}}{E_{mT}} - 1)}{(\frac{E_{fT}}{E_{mT}} + 2)}$$

and m, f, A and T denote the matrix, fiber, axial and transverse directions.

Christensen

The equations for Young's modulus and Poisson's ratio of three dimensional randomly oriented fiber composites are:

$$E_{random} = \frac{[E_A + 4(1+\nu_A)^2 K][E_A + (1-2\nu_A)^2 K + 6(G_A + G_T)]}{3[2E_A + (8\nu_A + 12\nu_A + 7)K + 2(G_A + G_T)]} \qquad \text{......(D11)}$$

$$\nu_{random} = \frac{[E_A + 2(2\nu_A^2 + 8\nu_A + 3)K - 4(G_A + G_T)]}{3[2E_A + (8\nu_A + 12\nu_A + 7)K + 2(G_A + G_T)]} \qquad \text{......(D12)}$$

Craft and Christensen

The equation for the thermal expansion coefficient of a three dimensional randomly oriented fiber composite is

$$\alpha_{random} = \frac{[E_A + 4\nu_A(1+\nu_A)K]\alpha_P + 4(1+\nu_A)K\alpha_Q}{E_A + 4(1+\nu_A)^2 K} \qquad \text{......(D13)}$$

where

$$\alpha_P = \overline{\alpha} + \frac{(\alpha_f - \alpha_m)}{(\frac{1}{k_m} - \frac{1}{k_f})}[\frac{3(1-2\nu_A)}{E_A} - (\overline{\frac{1}{k}})] \qquad \text{......(D14a)}$$

$$\alpha_Q = \overline{\alpha} + \frac{(\alpha_f - \alpha_m)}{(\frac{1}{k_m} - \frac{1}{k_f})}[\frac{3}{2K} - \frac{3\nu_A(1-2\nu_A)}{E_A} - (\overline{\frac{1}{k}})] \qquad \text{......(D14b)}$$

and

$$\overline{\alpha} = \phi\alpha_f + (1-\phi)\alpha_m \quad \text{and} \quad (\overline{\frac{1}{k}}) = \frac{\phi}{k_f} + \frac{(1-\phi)}{k_m} \qquad \text{......(D14c)}$$

Hashin LB

Hashin's lower bounds of the effective moduli for an isotropic composite with spherical filler:

$$k_s = k_m + \frac{(k_i - k_m)\phi}{1 + \dfrac{(1-\phi)(k_i - k_m)}{(k_m + \frac{4}{3}G_m)}} \qquad \text{......(D15)}$$

$$G_s = G_m + \frac{(G_i - G_m)\phi}{1 + \dfrac{(1-\phi)(G_i - G_m)}{(G_m + G_L)}} \qquad \text{......(D16)}$$

where $$G_L = \frac{3}{2(\frac{1}{G_m} + \frac{10}{9k_m + 8G_m})}$$

The expressions above assume $(G_i - G_m)(k_i - k_m) \geq 0$.