AMERICAN OPTION PRICING AND PENALTY METHODS

by

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CERTIFICATE OF ORIGINALITY

I hereby declare that this submission is my own work and that, to the best of my knowledge and belief, it contains no material previously published or written by another person nor material which to a substantial extent has been accepted for the award of any other degree or diploma of a university or other institute of higher learning, except where due acknowledgment is made in the text.

KAI ZHANG

Abstract

The main purpose of this thesis is to study penalty approaches to American option pricing problems. We consider penalty approaches to pricing plain American options, American options with jump diffusion processes and two-asset American options. Convergence properties of these methods are investigated. Also, the numerical schemes – finite element method and fitted finite volume method – for solving the penalized PDE are developed. Finally, an augmented Lagrangian method is applied to solving the plain American option pricing. Empirical tests are carried out to illustrate the effectiveness and usefulness of our methods.

For plain American option pricing, based on the theory of variational inequalities, a monotonic penalty approach is developed and its convergence properties are established in some appropriate infinite dimensional spaces. We derive the convergence rate of the combination of two power penalty functions. This convergence rate gives a unified result on that of higher and lower order penalty functions. After that, a fitted finite volume method is applied to finding the numerical solution of the penalized nonlinear PDE. We then test this method empirically, and compare it with projected successive over relaxation method (PSOR for short). We conclude that the monotonic penalty method is roughly comparable with the PSOR method, but is more desirable for its robustness under changes in market parameters, and furthermore the effect of the time reserving of the monotonic penalty method becomes significantly enhanced as the number of space steps increases.

Pricing American options with jump diffusion processes can be formulated as a partial integro-differential complementarity problem. We propose a power penalty approach for solving this complementarity problem. The convergence analysis of this method is established in some appropriate infinite dimensional spaces. Then, using the

finite element method, we propose a numerical scheme to solve the penalized problem and carry out the numerical tests to illustrate the efficiency of our method.

The two-asset American option pricing problem is formulated as a continuous complementarity problem involving a two dimensional Black-Scholes operator. By using a power penalty method, the two-asset American option model is reformulated as a two dimensional nonlinear parabolic PDE. By introducing a weighted Sobolev space and the corresponding norm, the coerciveness and continuity of the bilinear operator in the variational problem are derived. Hence, the unique solvability of the original and penalized problems is established. The convergence rate of the power penalty method is obtained in some appropriate infinite dimensional spaces. Moreover, to overcome the computational difficulty of the convection-dominated Black-Scholes operator, a novel fitted finite volume method is proposed to solve the penalized nonlinear two dimensional PDE. We perform numerical tests empirically to illustrate the efficiency of our new method.

Finally, based on the fitted finite volume discretization, an algorithm is developed by applying an augmented Lagrangian method (ALM for short) to pricing the plain American option. Convergence properties of ALM are considered. By empirical numerical experiments, we conclude that ALM is more effective than penalty method and Lagrangian method, and comparable with the PSOR method. Furthermore, numerical results show that ALM is more robust in terms of computation time under the changes in market parameters: interest rate and volatility.

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Notations

μ, μ_1, μ_2	drift rates
r	risk-free interest rate
$\sigma, \sigma_1, \sigma_2$	volatilities
S, S_1, S_2, x, y	prices of underlying assets
t, τ	time variables
w_1, w_2	weights of underlying assets x, y (or S_1, S_2), respectively
ρ	correlation of underlying assets x, y (or S_1, S_2)
V	option value
T	expiry date
K, K_1, K_2	striking prices
$V^*, \mathbf{\Lambda}$	payoff functions
W	Brownian process
N	Possion process
ν	the mean arrival rate of jumps of the Possion process
η	an impulse function producing a jump from S to $(1+\eta)\,S$
κ	expectation $E\,(\eta)$
μ_η	mean of the jumps
δ	variance of the jumps
L	original Black-Scholes operator
$\mathcal{L}$	Black-Scholes operator after variable changes

l_1	linear penalty function
l_2	quadratic penalty function
l_k	kth power penalty function
ξ	penalty parameter
λ	Lagrangian multiplier vector with component λ_i
ζ	damping parameter of damped Newton method

$\mathbb{R}^n$	n dimensional space
$\mathbf{S}, \mathbf{I}$	subsets of $\mathbb{R}$
Ω	subset of $\mathbb{R}^2$
Γ	boundaries of Ω
$L^p(\mathbf{S})$	denote the space of all p-integrable functions on $\mathbf{S}$
$H^{m.p}(\mathbf{S})$	Sobolev space over $\mathbf{S}$: $H^{m,p}(\mathbf{S}) = \{v : v \in L^p(\mathbf{S}), \frac{d^\alpha v}{dx^\alpha} \in L^p(\mathbf{S}),\ \forall\ 0 \leq \lvert\alpha\rvert \leq m\}$,
$H^m_\varpi, H^m_{0,\varpi}$	weighted Sobolev spaces
$\mathcal{K}$	closed convex set

PSOR	projected successive over relaxation method
ALM	augmented Lagrangian method
PDE	partial differential equation
PIDE	partial integro-differential equation
DCP	differential complementarity problem
PIDC	partial integro-differential complementarity problem
VI	variational inequality problem
FEM	finite element method
FVM	finite volume method

Chapter 1

Introduction

1.1 Financial Models

A derivative security is a financial asset whose payoff depends on the value of some underlying variable. The underlying variable can be traded asset, such as a stock; an index portfolio; a future's price; a currency; or some location or the volatility of an index. The payoff can involve various patterns of cash flows. Payments can be spread evenly through time, occur at specific dates, or a combination of the two. Derivatives are also referred to as contingent claims.

An option is a derivative security that gives the right to buy or sell the underlying asset, on or before some maturity date T, for a prespecified price K, called the striking price or exercise price. A call (put) option is a right to buy (sell). Because exercise is a right and not an obligation, the exercise payoff is $V^* = \max\{S - K, 0\}$ for a call option and $V^* = \max\{K - S, 0\}$ for a put option, where S denotes the price of the underlying asset. Option can be European style, which can only be exercised at the maturity date, or American style, where exercise is at the discretion of the holder, at any time before or at the maturity date.

Plain options, such as those described above, were introduced on organized option exchanges such as the Chicago Board of Option Exchange (CBOE), which was established in 1973. Since then, different types of options have led to the creation of numerous products designed to fill the needs of various types of investors. Path-dependent options, such as barrier options, Asian options, and lookbacks are examples of contractual forms that have emerged since and are now routinely traded in markets or quoted by financial institutions, or both. Even more exotic types of contracts, whose payoffs depend on multiple underlying assets or on occupation times of predetermined regions, have emerged in recent years. Because options are highly complex financial instruments, pricing option correctly is crucial in financial management. The option pricing problem is both computationally challenging as well as practically significant. It has attracted great interests amongst researchers, since the publication of the pioneer work by Black and Scholes [10] and Merton [92].

1.1.1 Pricing American Vanilla Option

In 1973, one of the most important insights of the seminal papers by Black and Scholes [10] was to show how to derive an analytic formula to pricing options, which is based on the no-arbitrage valuation theory. The significance of the Black-Scholes option pricing model and its extensions are far beyond the theoretical framework. They have been widely used by practitioners to price a variety of options. The basic Black-Scholes analysis starts from the premise that the underlying asset price S follows a geometric Brownian motion process

$$\frac{dS}{S} = \mu dt + \sigma dW, \tag{1.1.1}$$

where μ and σ are constants, which represent the expected (total) return on the asset, and the return volatility, respectively. The process W is a standard Brownian motion that captures the underlying uncertainty in the market. Trading in this asset is assumed to be unrestricted, i.e., no taxes, transactions costs, constraints, or other frictions. Likewise, investors can invest without restrictions, at the constant risk-free rate r. By using Ito's lemma [64, 7] and riskless hedging principle [82, 17], it can be shown that the European vanilla option price $V(S,t)$ is governed by the famous Black-Scholes equation

$$-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV = 0 \tag{1.1.2}$$

on the domain $R^+ \times [0, T)$, along with the boundary and terminal conditions

$$\begin{aligned} V(S,T) &= V^*(S), \\ V(0,t) &= V^*(0)e^{-r(T-t)}, \\ \lim_{S\to\infty} V(S,t) &= V^*(\infty)e^{-r(T-t)}. \end{aligned}$$

Explicit formulas for the fair price of European vanilla options are available in the literature, see, for example, [85]. As mentioned above, American vanilla options differ from European agreements in that they can be exercised at any time up to and including the expiry date of the contract. Most exchange-traded options are American-style, and they potentially have a higher value than European-style due to the extra early exercise feature. In [127, 126], the American vanilla option price V is formulated as the solution of a complementarity problem as follows.

$$\begin{cases} LV \geq 0, \\ V - V^* \geq 0, \\ LV \cdot (V - V^*) = 0, \end{cases} \tag{1.1.3}$$

where L represents the Black-Scholes operator. For plain American vanilla options,

$$LV = -\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV.$$

Terminal and boundary conditions need to be specified for a particular option just as in the European case. It is well known that for an American call option without dividends it is never optimal to exercise the option before maturity [76, 13]. Hence the problem reduces to pricing an European option. However, for an American put option, there may exist an optimal stopping time before maturity. Hence, in this thesis we will restrict our attention to the pricing of American-style put options.

1.1.2 Pricing American Option with Jump Diffusion Processes

The standard Black-Scholes equation (1.1.2) is one of the most successfully and widely used tools in financial economics. The underlying model (1.1.1) and its related assumptions are simple and elegant. But as a tool for real-world applications the model must be comparable to financial data. Deviations between the model and empirical evidence offer opportunities for the development of more realistic models, which, in turn, create new computational challenges for the pricing and hedging of derivative securities. Recently, empirical findings have shown that the standard Black-Scholes assumption of lognormal stock diffusion with constant volatility is not consistent with the market price. This phenomenon is often referred to as the volatility skew or smile [3, 34]. It exists in all the major stock index markets today. In order to capture the existence of volatility smiles, various extensions of the Black-Scholes model have been proposed. Generally speaking, three approaches are being studied: the stochastic volatility approach [65, 56, 84, 26, 47, 117, 80], the jump diffusion model approach [93, 135, 35, 137, 14, 23, 58] and the deterministic volatility function approach [34]. In [3], the advantages and disadvantages of these three approaches have been carefully studied and the jump diffusion model is being identified to be a more adaptable approach. The jump diffusion model is introduced by Merton in [93]. Contrary to the Black-Scholes model [10], the stock price in jump diffusion model is not a continuous function of time. This allows to account for large changes in market prices due to rare events. More importantly, the jump diffusion model yields implied volatility curves similar to volatility smiles observed on markets.

For the underlying price, Merton [93] proposed the following jump diffusion model:

$$\frac{dS}{S} = \mu dt + \sigma dW + \eta dN,$$

where N is a Poisson process with intensity ν, η is an impulse function producing a jump from S to $(1+\eta)S$, which is taken to be a lognormally distributed jump amplitude with probability density

$$G(\eta) = \frac{1}{\sqrt{2\pi}\delta\eta} \exp\left(\frac{(\log\eta - \mu_\eta)^2}{2\delta^2}\right),$$

The process N is independent of W. This model has three additional parameters: ν determines the arrival rate of jumps; and μ_η and δ determine the mean and variance of the jumps in return respectively. In [93, 94], Merton derived that the price $V(S,t)$ of European option with jump diffusion processes is governed by the following partial integro-differential equation

$$-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r - \nu\kappa)\, S\frac{\partial V}{\partial S} + (r+\nu)V - \left(\int_0^\infty V(S\eta)G(\eta)d\eta\right) = 0 \quad (1.1.4)$$

on the domain $R^+ \times [0, T)$, where κ represents the expectation $E(\eta)$, along with the boundary and terminal conditions

$$\begin{aligned} V(S, T) &= V^*(S), \\ V(0, t) &= V^*(0)e^{-r(T-t)}, \\ \lim_{S \to \infty} V(S, t) &= V^*(\infty)e^{-r(T-t)}. \end{aligned}$$

Due to the feature of early exercising, the price V of an American option with jump diffusion processes is formulated as the solution to a complementarity problem as follows.

$$\begin{cases} LV \geq 0, \\ V - V^* \geq 0, \\ LV \cdot (V - V^*) = 0, \end{cases}$$

where

$$LV = -\frac{\partial V}{\partial t} - \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} - (r - \nu\kappa)S\frac{\partial V}{\partial S} + (r + \nu)V - \left(\int_0^\infty V(S\eta)G(\eta)d\eta \right).$$

Terminal and boundary conditions need to be specified for a particular option just as in the case of European option.

1.1.3 Pricing Two-Asset American Option

While many single-asset models work well in practice, such as (1.1.2) and (1.1.4), they are not satisfactory for models that have payoff dependent on two or more correlated underlying assets [55, 123, 16, 33, 49, 71]. Examples of these multi-state options include index options, basket options, cross-currency options, exchange options, options on the extremum of several assets, etc. Also, it is common for corporate security to contain embedded options whose payoff depends on several state variables. Thus, if the payoff depends on two underlying assets that are correlated in terminal conditions, then their joint distribution depends on their correlation – a consideration that a single-factor model cannot capture. In order to capture realistic correlation patterns, and thus covariance structures, multi-factor models, such as two-asset models, are needed.

Pricing two-asset option starts from the premise that the two underlying assets S_1

and S_2 follow the following geometric Brownian motion processes equations.

$$\frac{dS_1}{S_1} = \mu_1 \, dt + \sigma_1 \, dW_1,$$
$$\frac{dS_2}{S_2} = \mu_2 \, dt + \sigma_2 \, dW_2,$$

where μ_1 and μ_2 are the drift rates, σ_1 and σ_2 are the deterministic local volatility of the assets S_1 and S_2, respectively, W_1 and W_2 are the standard Brownian motions followed by the assets S_1 and S_2, respectively. For the two assets, they are assumed to be correlated by $\rho \in [-1, 1]$. By using the no-arbitrage theory and Ito's formula, it is derived that the two-asset European option price $V(S_1, S_2, t)$ is governed by the following two dimensional Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\left[\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + 2\rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2}\right] + r\left[S_1 \frac{\partial V}{\partial S_1} + S_2 \frac{\partial V}{\partial S_2}\right] - rV = 0,$$

where r is the risk free interest rate, with some appropriate boundary and terminal conditions. For example, for European basket option, the boundary conditions can be taken as:

$$V(0, S_2, t) = g_1(S_2, t), \quad V(S_1, 0, t) = g_2(S_1, t),$$
$$\lim_{S_1 \to \infty} V(S_1, S_2, t) = 0, \quad \lim_{S_2 \to \infty} V(S_1, S_2, t) = 0,$$

and terminal condition:

$$V(S_1, S_2, t = T) = V^*(S_1, S_2),$$

Here, g_1 and g_2 are given functions providing suitable boundary conditions, $V^*(S_1, S_2)$ is the payoff function. Typically, g_1 and g_2 are determined by solving the associated one dimensional European option problem, see [121]. For American two-asset option, its price is formulated as the solution to a complementarity problem as follows.

$$\begin{cases} LV \geq 0, \\ V - V^* \geq 0, \\ LV \cdot (V - V^*) = 0, \end{cases} \tag{1.1.5}$$

where L represents the Black-Scholes operator:

$$LV = -\frac{\partial V}{\partial t} - \frac{1}{2}\left[\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + 2\rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2}\right] - r\left[S_1 \frac{\partial V}{\partial S_1} + S_2 \frac{\partial V}{\partial S_2}\right] + rV.$$

Terminal and boundary conditions need to be specified for a particular option just as in the case of European option.

There are also some other kinds of American-style options, such as uncertain volatility model, transaction model [83], Asian option model, barrier option model, stochastic model, passport option model, and so on. All of them can be cast into the framework of (1.1.5) with specified operator L. See [127, 126, 19, 43, 41] for detailed analysis.

1.2 American Option Pricing and Numerical Methods

Unlike European options, closed-form solutions are not available for pricing American options. So the popular American options must be priced numerically, even for the simplest case of constant coefficients. Two broad classes of numerical methods have been developed to pricing American options: stochastic and PDE approaches. The Monte-Carlo simulation method [12, 82, 50, 64], and tree or lattice method [105, 25, 82, 15] are examples of stochastic approach. Explicit method [44], projected successive over-relaxation method [127, 62, 64, 118], linear programming method [31, 32], Lagrangian method [122], l_1 and l_2 penalty method [44, 35, 45, 37, 43, 41, 137], and barrier penalty method [98, 97, 96], are examples of PDE approach.

Monte-Carlo simulation method [95] has been proved to be a powerful and versatile technique [12, 82, 50]. It is basically a numerical procedure for estimating the expected value of a random variable, and so it is suitable to derivative pricing problems represented as expectations. Pricing a financial instrument using Monte-Carlo simulation involves three steps: (i) simulate the sample paths for the underlying asset in the derivative model over the life of the derivatives, according to the risk neutral probability distributions; (ii) for each simulated sample path, evaluate the discounted cash flow of the derivatives; (iii) take the sample average of the discounted cash flow over all sample paths. One of the advantages of the Monte-Carlo simulation method is the simplicity of the algorithm. The implementation of this method is straight forward. Another major advantage of this method is that it is easy to accommodate terminal payoff function in an option model [82]. The main drawback of the Monte-Carlo simulation method is the demand for a large number of simulation trials in order to achieve a high level of accuracy. Furthermore, unlike European options, American options are more difficult to price by using the Monte-Carlo simulation method, since they can be exercised at any time before the expiry date. Recently, several papers have attempted to estimate the American option price using the Monte-Carlo simulation method [120, 6, 18, 109]. Generally speaking, they attempted to estimate the expected return at exercise conditional only on known information using sophisticated algorithms.

The lattice method for pricing derivative securities was suggested by Parkinson [105, 107] and Cox, Ross, Rubinstein [25]. This scheme is most widely used in the finance community for the valuation of a wide variety of option models, due primarily to its ease of implementation and pedagogical appeal. Unlike the continuous Black-Scholes framework of analysis, the lattice method bypasses the derivation of partial differential equations and so the comprehension of the method is accessible to a much wider audience in the finance community. In the lattice model, the asset price movement is simulated by a discrete random walk model, which converges to a continuous lognormal diffusion as the time interval between successive steps tends to zero. The lattice model is consistent with the risk neutrality argument where the option price obtained from the model depends only on the growth rate of a riskless bond but independent of the expected rate of return of the asset price. The lattice scheme can be easily modified to incorporate added features in an American option contract. For example, with a slight modification of the dynamic programming procedure [82, 127, 66], we can incorporate the effects of callable features in an American option. Some enhanced forms of the lattice scheme have been proposed in the literature. For instance, a good survey of these enhanced schemes can be found in Broadie and Demples' paper [15]. The lattice method is simple and computationally inexpensive. But some disadvantages are also obvious, such as the lack of accuracy of the results obtained [46]. Thus, the use of these results in the real financial market could have great adverse consequences.

Currently, the PDE approach is the most popular one. As a result of the possibility of early exercising in American-style options, part of the valuation problem consists of identifying the optimal exercise policy, i.e. the exercise time that maximizes value for the holder of the security. Pricing an American option is a free boundary problem (which is a PDE problem) [113, 91, 119, 92, 118], i.e. at any time t, there exists a value of S that marks the boundary of two regions: one in which early exercise is preferred and one in which the option should be held. An arbitrage-free argument shows that the valuation of an American option is always greater than or equal to the payoff at the time of exercise. The return on the portfolio containing the option should be no more than the return on a riskless asset. Further, the holder of the option would choose to exercise early when holding the option is less valuable than exercising it immediately and reinvesting the funds in a riskless asset. See, for example [127], for a detailed explanation. Thus, the problem can be formulated more formally as a partial differential complementarity problem given in the previous subsection. The resulting partial differential complementarity problem can be equivalently regarded as a variational inequality problem [70, 118, 17]. By using the finite difference, finite volume or finite element discretizations, explicit method [44], projected successive over-relaxation method (PSOR) [127, 62, 64, 118], linear programming method [31, 32], Lagrangian method [122], l_1 and l_2 penalty methods [44, 35, 45, 37, 43, 41, 137], barrier penalty method [98, 97, 96], etc, have been developed to solve this partial differential complementarity problem or variational inequality problem.

In the current practice, the most common method of handling the early exercise condition is simply to advance the discrete solution over a timestep ignoring the constraint, and then to apply the constraint explicitly, see, for example, [14, 70]. The main disadvantage of this approach is that the solution is inconsistent at the beginning of each timestep (i.e. the discrete form of the LCP is not approximately satisfied). Thus, this approach can only be regarded as a first order approximation in time. On the other hand, the explicit application of the constraint is computationally very inexpensive.

So far, the most popular algorithm has been the PSOR method. It is easy to implement. The PSOR method was proposed by Cryer [27] to solve the linear complementarity problem. This method, which is based on the usual SOR method for solving linear system, is modified to update only non-negative SOR solutions. In general, this method is fast and robust for many kinds of American option pricing problems. See, for example, [127, 126, 11, 112]. However, its convergence rate depends crucially on the choice of the relaxation parameter and it exhibits exponential solution-time behavior as the number of space discretization points increases, see [44, 31, 32, 11, 131, 133]. A multigrid method has been suggested in [21] to accelerate convergence of the basic relaxation method. Although this is a promising technique, multigrid methods are usually strongly coupled to the type of discretization used, and hence they are complicated to implement.

Recently, Dempster, Hutton and Richards [31, 32] proposed the linear programming method for pricing American options with an aim to overcome the problems encountered with the PSOR technique. By showing that the complementarity problem is equivalent to an abstract linear programming problem, the pricing problem is transformed into a linear programming problem. This method evaluates the American option in times that are varied linearly with respect to the space discretization. This feature is very suitable for the use of the linear programming method to solve a sequence of one space dimension problems. However, it is not well equipped to handle sparse matrix systems, especially in the case of multi-asset options. Moreover, it was shown [11] that this method is invalid for the implicit difference scheme.

Besides those methods we mentioned above, there are several other methods to price American options, such as Lagrangian method, barrier penalty method and the nonlocal boundary condition method in [1, 2]. Lagrangian method for the valuation of American options was used by Vàzque in [122]. This method is used to solve an equivalent quadratic programming problem. By applying the Uzawa's duality method [53], an algorithm was developed in [122] for solving the American option pricing problem. For this method, the optimal exercise boundary can be easily obtained from Kuhn-Tucker multipliers by taking into account the fact that the exercise boundary vanishes only outside the active set. However, the unsatisfactory convergence rate and low accuracy are the two major disadvantages. A barrier penalty method was first proposed by Nielson [98, 97, 96] to solve the American vanilla option pricing problem. For the complementarity problem resulting from the American option pricing problem, a barrier term was added in the partial differential inequality. Some convergence properties of this method were shown in [98, 97, 96]. However, its convergence rate has not been obtained. For this reason, it is less popular than other methods. In [96], this method was used to price multi-asset American options. In [60, 59], Hon applied this method to pricing both single-asset and multi-asset American option pricing problems with a specific numerical scheme – radial basis function method (RBF). Allegretto, Lin and Yang in [1, 2] proposed an exact nonlocal boundary condition method, and presented the finite element error estimates for this method. Essentially, this method introduces a new nonlocal boundary condition, which is mathematically exact. Then the problem is reformulated as a variational inequality problem on a very narrow region, without changing the solution.

There is a large number of general methods for solving linear complementarity problems [79, 100, 24]. We can divide these methods into essentially two categories: direct methods, such as pivoting techniques [24, 78], and iterative methods, such as Newton's method [100, 24, 103, 22] and interior point algorithms [79]. Some of these methods, which have been applied specifically to American option pricing, include pivoting methods [62], and interior point methods [63, 102]. As pointed out in [63], pivoting methods (such as Lemke's algorithm [24]) is not well equipped to handle sparse systems, especially for problems with more than one dimension (multi-factor options).

As well known, complementarity problems (both linear and nonlinear) can be posed in the form of a set of nonlinear equations, see [103, 102, 104]. Various nonsmooth Newton methods have been suggested for these types of problems [72, 74, 39, 101, 104]. More recently, combinations of nonsmooth Newton and smoothing methods have been proposed [75].

It is well known that a complementarity problem (or, equivalently, a variational inequality problem) can be solved by a penalty method [30, 51, 77, 48, 99, 114, 38]. In [8, 77, 51], the quadratic (l_2) and linear (l_1) penalty methods were used to solve a variational inequality problem. With the rigorous variational analysis, it has been shown that the solution to the penalized problem converges to that of the original variational inequality problem. Moreover, for quadratic and l_1 penalty methods, their convergence rates have been shown to be of order $\mathcal{O}(\xi^{-1/4})$ and $\mathcal{O}(\xi^{-1/2})$ respectively in [8, 77, 51], where ξ is the penalty parameter. Since the American option pricing problem can be cast into the framework of complementarity problem or variational inequality problem, quadratic and l_1 penalty methods were used to price American options. In the context of American option pricing, Forsyth and Zvan [44, 137] first developed the penalty approach to this problem. In [44], l_1 penalty method was used to price an American vanilla option, where quadratic convergence rate was obtained in terms of selected time steps. The quadratic and l_1 penalty methods were used to price American options with stochastic volatility in [137]. In [37, 35, 89, 88, 90], l_1 penalty method was applied to valuating American options with jump diffusion processes. Moreover, for other American-style exotic options, such as shout options, American Asian option under jump diffusion, uncertain volatility models, quadratic and l_1 penalty methods were widely used, see [128], [41] and [36], respectively. In these papers, quadratic and l_1 penalty methods were used to solve sequential linear complementary problems, which are derived from the original differential complementarity problem by the discretization scheme. An advantage of these methods is that it is simple to implement and can make full use of the existing softwares to handle the sparse matrix structure. This technique is suitable for any type of discretization, for any dimension, and for any unstructured meshes. It also works for multiple-connected problems, and problems with nonlinearity, such as uncertain volatility models, drift-dominated problems, transaction cost models and jump diffusion models [127]. A major disadvantage is that the solution obtained by the penalty methods only satisfies approximately the complementarity conditions. However, the error can be controlled by adjusting the penalty parameter.

Although the error of the quadratic and l_1 penalty methods can be controlled by adjusting the penalty parameter, the convergence rates of these two penalty methods ($\mathcal{O}(\xi^{-1/4})$ and $\mathcal{O}(\xi^{-1/2})$, respectively) imply that much larger penalty parameters should be attained in order to obtain a desirable accuracy level of the solutions. It is widely acknowledged [40] that, too large penalty parameters can cause computational difficulties. Thus, in order to overcome this difficulty, a more general power penalty method ($l_k, k > 0$), especially the lower order penalty method ($[\cdot]_+^k, 0 < k < 1$), was proposed in [125] to solve the differential complementarity problems, based on the theories of variational inequality and complementarity problems.

Methods based on lower order penalty functions for mathematical programs with equilibrium constraints and nonlinear optimization problems have been under investigation in the last three decades, see Luo and Pang [86], Pang [102], Luo et al. [87], Rubinov [110], and Yang and Huang [130]. In [87], Luo et al gave a global exact penalty function result for a lower order penalty function. Rubinov, Yang and Bagirov [111] showed that the existence of an exact lower order penalty function requires much weaker conditions than that of the classical l_1 penalty function. Moreover, the least exact penalty parameter of the lower order penalty function is smaller than that of the l_1 penalty function. Because of these features, the study of lower order penalty functions has attracted extensive attention in recent years. The applications of lower order penalty functions can be found in [129, 125, 130, 131, 132, 134]. It is worth pointing out that a lower order penalty function is in general nonsmooth and non-Lipschitz. Most numerical methods in nonlinear programming require differentiable functions. Thus, lower order penalty functions may cause some difficulties in numerical implementation. However, with the help of smoothing techniques (see [129, 125]), we are able to compute the solution of nonlinear PDE equations with lower order penalty functions.

Due to the advantages of the lower order penalty method, Wang, Yang and Teo [125] proposed a power penalty method to price American vanilla options. They reformulated the American option pricing problem as a variational inequality problem and the resulting variational inequality problem was then transformed into a nonlinear parabolic partial differential equation by adding a power penalty term. In some appropriate infinite dimensional spaces, they also obtained the rate of convergence of the power penalty approach. It has been shown that the rate of convergence of the l_k penalty method is of order $\mathcal{O}(\xi^{-1/2k})$. This is a much faster convergence rate than that of l_1 penalty method when $0 < k < 1$.

1.3 Contributions and Outlines

In this thesis, we aim to further explore the penalty approach to valuating a broad class of problems related to American options, including the standard American option. The fitted finite volume method and finite element method are developed to solve the penalized problem. We propose a monotonic penalty method for plain American option pricing problem, and develop the fitted finite volume method to solve it. We develop a power penalty method for pricing American option with jump diffusion processes, and propose the finite element method to solve it. We also develop a power penalty method for two-asset American option pricing problem, and propose the two-dimensional version of the fitted finite volume method to solve it. Finally, we propose the augmented Lagrangian method for American option pricing problem. For each method, its convergence analysis is carried out. We also conducted some empirical numerical tests to verify the effectiveness of our methods.

In Chapter 2, we first introduce the American option pricing model and its formulations: differential complementarity problem and variational inequality problem. Then, we consider the equivalence between the complementarity problem and variational inequality problem. Within the framework of complementarity problem and variational inequalities theories, a monotonic penalty method is proposed, which produces a nonlinear parabolic PDE. Based on the theory of variational inequalities, the convergence properties of the monotonic penalty approach are established in some appropriate infinite dimensional spaces. The solvability of the penalized problem is investigated. We also point out that the quadratic penalty method, linear penalty method, lower order penalty method, 'value at zero' penalty method, and the combination of two power penalty methods are all special cases of monotonic penalty methods. We derive the convergence rate of the combination of two power penalty methods in some appropriate infinite dimensional spaces. After that, the fitted finite volume method is applied to obtaining the numerical solution to the penalized nonlinear PDE. We then test this method empirically, and compare it with the PSOR (projected successive over relaxation) method. First, we compare the solution times of penalty methods and PSOR method under different time-space discretizations. We find that the monotonic penalty method is roughly comparable with the PSOR method, and the saving in computational time of the monotonic penalty method becomes more significant as the number of space steps increases. Second, we compare the solution times of these two methods under changes in market parameters. we conclude that the monotonic penalty method is more robust with respect to the changes in market parameters, such as interest rate and volatility.

In Chapter 3, pricing of American options with jump diffusion processes is studied. First, we present the mathematical model of the problem, which is a partial integro-differential complementarity problem. Then, we present its equivalent form: variational inequality problem. The equivalence between the complementarity problem and variational inequality problem is shown. With the help of the complementarity problem and variational inequality problem, we propose a power penalty approach to this partial integro-differential complementarity problem. The solvability of the penalized problem is considered. After that, the convergence analysis of the power penalty approach, in some appropriate infinite dimensional spaces, is established. We show that the convergence rate of $\mathcal{O}(\xi^{-1/2k})$ can be achieved by the power penalty approach (l_k, $k > 0$), where ξ is the penalty parameter and k is the order of the power penalty function. Therefore, when k is small, we need only a very small ξ to achieve a given accuracy.

This is significantly superior over the existing theoretical results for the quadratic and linear penalty methods. Finally, a numerical scheme is developed to solve the penalized nonlinear equation. In this scheme, the finite element method is used in space and θ-scheme is used in time. More specifically, a θ_J-scheme time-stepping method is applied to the jump integral term, which can overcome the difficulty caused by the convolution term in the penalized partial integro-differential equation. To this end, two numerical examples, i.e. an American vanilla option with jump diffusion and an American butterfly option with jump diffusion, are given to illustrate the efficiency and usefulness of the power penalty method.

In Chapter 4, a two-asset American option is investigated. We first present a mathematical model of the two-asset American option, which is described as a partial differential complementarity problem involving a two-dimensional Black-Scholes operator. The divergence form of this operator is derived. For the convenience of theoretical analysis, an equivalent standard form satisfying homogenous Dirichlet boundary conditions is derived from the original form. Then, we show that this reformulated problem is equivalent to a variational inequality problem. For the variational inequality problem, we introduce a weighted Sobolev space and its norm to prove the coerciveness and continuity of the corresponding bilinear operator. With the coerciveness and continuity of the corresponding bilinear operator, the unique solvability of the variational problem is established. On the basis of equivalence between variational inequalities of first kind and those of second kind, we develop a power penalty method for pricing the two-asset American option model, which is described by a high dimensional nonlinear parabolic PDE. In some appropriate infinite dimensional spaces, the convergence rate of the power penalty method is given. We show that the convergence rate of the l_k penalty method is of order $\mathcal{O}(\xi^{-1/2k})$, where ξ is the penalty parameter and k is the order of the power penalty function (l_k, $k > 0$). After that, a two-dimensional version of the fitted finite volume method is proposed to solve the penalized nonlinear multi-dimensional PDE. The two-dimensional version of the fitted finite volume method is parallelized to the one-dimensional version as stated in Chapter 2, but its deduction is much more complicated and nontrivial. The computational formulas for the two-dimensional version is carefully deduced. We also develop the solution method for the resulting nonlinear system from the fitted finite volume discretization. Finally, numerical examples are designed to illustrate the usefulness of the power penalty methods for pricing two-asset

American option. These power penalty methods include quadratic penalty method, linear penalty method, and lower order penalty method.

In Chapter 5, by combining the advantages of both Lagrangian method and penalty method, we introduce the augmented Lagrangian method (ALM) for American option pricing. First, the continuous models of American option pricing, which are described by a partial differential complementarity problem and a corresponding variational inequality problem, are discretized by using the fitted finite volume method. This leads to a large scale finite dimensional variational inequality problem or linear complementarity problem. Due to the fitted finite volume discretization, the obtained variational inequalities can be efficiently solved by the augmented Lagrangian method [57, 67, 68, 73]. Thus, we adapt the augmented Lagrangian method to pricing an American vanilla put option. We present the explicit augmented Lagrangian formulation and design the corresponding augmented Lagrangian algorithm. The existence of the Lagrangian multiplier is characterized. For a penalty parameter, ξ, a linear convergence rate, i.e. of order $\mathcal{O}(1/\xi)$, of the augmented Lagrangian method is given. Moreover, for fixed penalty parameter ξ, a superlinear convergence rate is shown. To explore the advantages of the ALM over penalty methods and the Lagrangian method, empirical tests are carried out with two sets of problems. Furthermore, to compare the ALM method with the PSOR method, empirical experiments with different market parameters σ and r and different step sizes of space variable and time variable are implemented. It is observed that the ALM method is more robust in terms of changes in market parameters than the PSOR method.

Finally, concluding remarks are presented in Chapter 6, where some future problems and research directions are also included.

1.4 Preliminaries

In this section, we introduce some functional spaces and notations, which will be used later in this thesis.

For an open set $\mathbf{S} \subset \mathbb{R}$ and $1 \leq p \leq \infty$, let $H^{m,p}(\mathbf{S})$ denote the Sobolev space over the domain S defined by

$$H^{m,p}(\mathbf{S}) = \left\{ v : v \in L^p(\mathbf{S}), \frac{d^\alpha v}{dx^\alpha} \in L^p(\mathbf{S}),\ \forall\ 0 \le |\alpha| \le m \right\},$$

where its norm $\|\cdot\|_{H^{m,p}(\mathbf{S})}$ or $\|\cdot\|_{m,p,\mathbf{S}}$ is defined by

$$\|v\|_{m,p,\mathbf{S}} = \|v\|_{H^{m,p}(\mathbf{S})} = \left(\sum_{|\alpha|\le m} \int_{\mathbf{S}} \left| \frac{d^\alpha v}{dx^\alpha} \right|^p dx \right)^{1/p},$$

where α is a positive integer. We simply use $H^m(\mathbf{S})$ with the norm $\|\cdot\|_{H^m(\mathbf{S})}$ or $\|\cdot\|_{m,\mathbf{S}}$ to denote $H^{m,2}(\mathbf{S})$ with the norm $\|\cdot\|_{H^{m,2}(\mathbf{S})}$ or $\|\cdot\|_{m,2,\mathbf{S}}$.

Let $L^p(\mathbf{S})$ denote the space of all p-integrable functions on $\mathbf{S}$ with the norm $\|\cdot\|_{L^p(\mathbf{S})}$. We put

$$H_0^m(\mathbf{I}) = \{ v(\cdot,t) : v(\cdot,t) \in H^m(\mathbf{I}),\ v(0,t) = v(X,t) = 0 \}.$$

Finally, for any Hilbert space $H(\mathbf{I})$, the norm of $L^p(0,T;H(\mathbf{I}))$ is denoted by

$$\|v(\cdot,t)\|_{L^p(0,T;H(\mathbf{I}))} = \left(\int_0^T \|v(\cdot,t)\|_{H(\mathbf{I})}^p\, dt \right)^{1/p}.$$

Obviously, $L^p(0,T;L^p(\mathbf{I})) = L^p(\mathbf{I}\times(0,T)) = L^p(\Omega)$.

In the case of one dimensional space, we define the weighted Sobolev space $H^1_{0,\varpi}(\mathbf{I})$ as

$$H^1_{0,\varpi}(\mathbf{I}) = \left\{ v(\cdot,t) : v, xv_x \in H_0^1(\mathbf{I}), \forall x \in \mathbf{I} \right\}.$$

We put

$$\mathcal{K} = \left\{ v(\cdot,t) : v(t) \in H^1_{0,\varpi}(\mathbf{I}),\ v(\cdot,t) \le u^*(\cdot,t) \right\},$$

where $u^*(\cdot,t)$ is a given function. It is easy to verify that $\mathcal{K}$ is a convex and closed subset of $H^1_{0,\varpi}(\mathbf{I})$.

In the case of two dimensional space, let $\Omega = (0,X)\times(0,Y) \subset \mathbb{R}^2$ and Γ denote the boundaries of Ω. Clearly,

$$\Gamma = \{x=0, 0\le y\le Y\} \cup \{y=0, 0\le x\le X\} \cup \{x=X, 0\le y\le Y\} \cup \{y=Y, 0\le x\le X\}.$$

We can define the weighted Sobolev space $H^1_\varpi(\Omega)$ as

$$H^1_\varpi(\Omega) = \left\{ v(\cdot,\cdot,t) : v, xv_x, yv_y \in L^2(\Omega),\quad \forall x,y \in \Omega \right\}$$

with its norm denoted by $\|\cdot\|_{1,\varpi}$. We put

$$H^1_{0,\varpi}(\Omega) = \left\{ v(\cdot,\cdot,t) : v(\cdot,\cdot,t) \in H^1_{\varpi}(\Omega),\ v|_{\Gamma} = 0 \right\},$$

$$\mathcal{K} = \left\{ v(\cdot,\cdot,t) : v(t) \in H^1_{0,\varpi}(\Omega),\ v(\cdot,\cdot,t) \leq u^*(\cdot,\cdot,t) \right\},$$

where $u^*(\cdot,\cdot,t)$ is a given function. It is clear that $\mathcal{K}$ is a convex and closed subset of $H^1_{0,\varpi}(\Omega)$.

Chapter 2

A Monotonic Penalty Method for American Option Pricing

In this chapter, we propose a monotonic penalty method to price American vanilla put option. The monotonic penalty function includes l_2, l_1, l_k $(0 < k < 1)$, and their linear combinations as special cases. By adding a monotonic penalty term, the original differential complementarity problem is converted into a nonlinear Black-Scholes equation with a strong monotonic operator. We establish the convergence of the monotonic penalty method within the framework of the variational inequalities under the general situation. For a general monotonic penalty method, it is impossible to give its explicit convergence rate. However, we are able to derive the convergence rate for the linear combination of two different l_k penalty functions $(0 < k < \infty)$. This result unifies several convergence rates given in [38] and [125]. We shall study and compare the performance of the higher order, l_1, l_k, valley at zero and the combination of power penalty functions in solving the American option pricing problem with different market parameters. Numerical tests show that some of the monotonic penalty methods are particularly fast-convergent and accurate for our proposed option pricing models. From numerical experiments, we find that solution times of the monotonic penalty methods increase in a linear rate with respect to space steps and that these solution times do not depend on the changes in the interest rate and volatility. This is an obvious advantage over the PSOR method, where the exponential solution time behavior is observed when the number of the space steps gets large.

2.1 The Monotonic Penalty Approach

Consider an asset with price x which satisfies the following stochastic differential equation

$$dx = rx\,dt + \sigma x\,dW,$$

where W is a standard Brownian motion, r is the risk-free interest rate, σ denotes a deterministic local volatility. Let $V(x,t)$ denote the value of a standard American put option, T the *expiry time*, and K the *striking price*. It is well known, under the non-arbitrage assumption, that the American option pricing problem can be formally stated as a linear differential complementarity problem [127] as follows.

(DCP)

$$\begin{cases} LV \geq 0, \\ V - V^* \geq 0, \\ LV \cdot (V - V^*) = 0, \\ \text{a.e. in the region } \ \Omega = I \times (0,T), \end{cases} \tag{2.1.1}$$

where

$$L\colon = -\frac{\partial}{\partial t} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} - rx\frac{\partial}{\partial x} + r$$

denotes the Black-Scholes differential operator, $V^*(x)$ is the *payoff function* defined by $V^*(x) = \max\{K - x, 0\}$. $V(x,t)$ at the final time T is given by

$$V(x,T) = V^*(x), \tag{2.1.2}$$

and $I = (0, X) \subset \mathbb{R}$ is the variable range of the underlying asset price. Realistically, we should choose $X \gg K$. Additionally, the boundary conditions are:

$$\begin{aligned} V(0,t) &= K, \\ V(X,t) &= 0. \end{aligned} \tag{2.1.3}$$

The system of (2.1.1)–(2.1.3) is the original American option pricing model.

For convenience, we adopt the reformulation technique in [125] to transform (2.1.1)–(2.1.3) into an equivalent standard form satisfying homogeneous Dirichlet boundary conditions. By introducing a new variable

$$u(x,t) = e^{\beta t}(V_0(x) - V(x,t))$$

with

$$\beta = \sigma^2,\ V_0(x) = \left(1 - \frac{x}{X}\right) K,$$

we obtain the following (LCP) which is equivalent to (DCP).

Problem 2.1.1 *(LCP)*

$$\begin{cases} \mathcal{L}u \le f, \\ u - u^* \le 0, \\ (\mathcal{L}u - f) \cdot (u - u^*) = 0, \\ a.e. \ \textit{in the region} \ \ \Omega = I \times (0,T), \end{cases} \tag{2.1.4}$$

where

$$\mathcal{L}\colon = -\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\left[ax^2 \frac{\partial}{\partial x} + bx\right] + c$$

is in the self-adjoint form with

$$a = \frac{1}{2}\sigma^2, \quad b = r - \sigma^2, \quad c = r + b + \beta, \ \textit{and} \ f(x,t) = e^{\beta t} L V_0(x),$$

the payoff function becomes

$$u^*(x,t) = e^{\beta t}(V^*(x) - V_0(x,t)), \tag{2.1.5}$$

and the new boundary conditions are

$$u(0,t) = u(X,t) = 0, \quad t \in [0,T). \tag{2.1.6}$$

In what follows, we will simply write $v(t)$ when we regard $v(\cdot,t)$ as an element of $H^1_{0,\varpi}(I)$. We will also suppress the independent time variable t (or τ), when it causes no confusion in doing so.

In fact, the linear complementarity problem (2.1.4)–(2.1.6) can also be reformulated as the following variational inequality problem (2.1.7).

Problem 2.1.2 *Find $u(t) \in \mathcal{K}$, such that, for all $v(t) \in \mathcal{K}$,*

$$\left(-\frac{\partial u}{\partial t}, v - u\right) + A(u, v - u; t) \ge (f, v - u), \quad a.e. \ \textit{in} \ (0,T) \tag{2.1.7}$$

where

$$A(u,v;t) = (ax^2 u' + bxu, v') + (cu, v)$$

is a bilinear form, and

$$\mathcal{K} = \left\{ v(t) \in H^1_{0,\varpi}(I) : v(t) \leq u^*(t) \right\}$$

is a convex and closed subset of $H^1_{0,\varpi}(I)$.

Now, we prove the coerciveness and continuity of the operator $A(u, v; t)$. Let $\| \cdot \|_A$ be a functional on $H^1_{0,\varpi}(I)$ defined by

$$\|v\|_A^2 = \left(x^2 v', v'\right) + (v, v)$$

for any $v \in H^1_{0,\varpi}(I)$. It is easy to see that $\| \cdot \|_A$ is a weighted energy norm on $H^1_{0,\varpi}(I)$.

Lemma 2.1.1 *There exist positive constants* C *and* M*, independent of* v*, such that for any* $v, w \in H^1_{0,\varpi}(I)$*,*

$$A(v, v; t) \geq C \, \|v\|_A^2 \,, \tag{2.1.8}$$

$$|A(v, w; t)| \leq M \, \|v\|_A \, \|w\|_A \,, \tag{2.1.9}$$

for $t \in (0, T)$.

Proof. For any $v \in H^1_{0,\varpi}(I)$,

$$\int_0^X bxvv'dx = \int_0^X bxvdv = bxv^2|_0^X - \int_0^X bvd\,(xv) = -\int_0^X bxvv'dx - \int_0^X bv^2dx.$$

Thus,

$$\int_0^X bxvv'dx = -\frac{1}{2}\int_0^X bv^2dx.$$

Therefore, it follows from the above relationship that

$$\begin{aligned} A(v, v; t) &= (ax^2v' + bxv, v') + (cv, v) \\ &= (ax^2v', v') + ((r + b + \beta - b/2)v, v) \\ &= (ax^2v', v') + \frac{1}{2}((3r + 2\beta - \sigma^2)v, v) \\ &\geq C[(x^2v', v') + (v, v)]. \end{aligned}$$

This proves (2.1.8).

The proof of (2.1.9) is standard and is hence omitted here. ■

From Lemma 2.1.1, we have

Lemma 2.1.2 *Problem 2.1.2 is the variational form of Problem 2.1.1.*

Proof. For any $w \in \mathcal{K}$, it follows from the definition of $\mathcal{K}$ that

$$w - u^* \le 0, \quad \text{a.e. on } I.$$

Multiplying both sides of the first inequality of (2.1.4) by $w - u^*$ for an arbitrary $w \in \mathcal{K}$ and integrating the second term by parts, we obtain

$$\left(-\frac{\partial u}{\partial t}, w - u^*\right) + A(u, w - u^*; t) \ge (f, w - u^*), \quad \text{a.e. in } (0, T). \tag{2.1.10}$$

Since $\mathcal{K}$ is a convex and closed subset of $H^1_{0,\varpi}(I)$, we may write w as $w = \theta v + (1-\theta)u$, where $u, v \in \mathcal{K}$ and $\theta \in [0, 1]$. Therefore, (2.1.10) becomes

$$\begin{aligned}&\left(-\frac{\partial u}{\partial t}, \theta\,(v-u)\right) + \left(-\frac{\partial u}{\partial t}, u - u^*\right) + A(u, \theta\,(v-u)\,; t) + A(u, u - u^*; t)\\ &\ge (f, \theta\,(v-u)) + (f, u-u^*)\,, \quad \text{a.e. in } (0, T).\end{aligned} \tag{2.1.11}$$

On the other hand, from the third relationship of (2.1.4), we have

$$(\mathcal{L}u - f, u - u^*) = 0,$$

i.e.

$$\left(-\frac{\partial u}{\partial t}, u - u^*\right) + A(u, u - u^*; t) - (f, u - u^*) = 0.$$

Therefore, (2.1.11) reduces to

$$\left(-\frac{\partial u}{\partial t}, \theta\,(v-u)\right) + A\left(u, \theta\,(v-u)\,; t\right) \ge \left(f, \theta\,(v-u)\right), \quad \text{a.e. in } (0, T). \tag{2.1.12}$$

Since $\theta \in [0, 1]$, we see that (2.1.12) leads to

$$\left(-\frac{\partial u}{\partial t}, v - u\right) + A(u, v - u; t) \ge (f, v - u), \quad \text{a.e. in } (0, T).$$

■

Lemma 2.1.3 *Variational inequality (2.1.7) has a unique solution.*

Proof. In fact, by virtue of the coerciveness of the operator $A(u; v; t)$, the conclusion is a consequence of Theorem 2.3 in [8], in which the unique solvability of a parabolic variational inequality problem is established. ■

In order to introduce the monotonic penalty approach, we first give the definition of a monotonic operator. A function $\rho : L^2(\Omega) \to L^2(\Omega)$ is called monotonic, if for any $u, v \in L^2(\Omega)$, it holds

$$(\rho(u) - \rho(v), u - v) \geq 0.$$

Now, a monotonic penalty approach to Problem 2.1.1 is stated as follows.

Problem 2.1.3

$$-\frac{\partial u_\xi}{\partial t} - \frac{\partial}{\partial x}\left[ax^2 \frac{\partial u_\xi}{\partial x} + bx\right] + cu_\xi + \xi\, \rho(u_\xi(x,t)) = f(x,t), \quad (x,t) \in \Omega \tag{2.1.13}$$

with the given boundary and final conditions

$$u_\xi(0,t) = 0 = u_\xi(X,t),$$
$$u_\xi(x,T) = u^*(x,T),$$

where $\xi > 1$ is the penalty parameter and ρ is a continuous, monotonic penalty function subject to

$$\begin{cases} \rho(u) > 0, & if \quad u(\cdot,t) \notin \mathcal{K}, \\ \rho(u) = 0, & if \quad u(\cdot,t) \in \mathcal{K}. \end{cases}$$

In the next section, we will present rigorous mathematical convergence analysis, that is, solutions $u_\xi(x,t)$ of (2.1.13) tend to that of (2.1.4) as $\xi \to +\infty$. Before proceeding, we first give the variational form of Problem 2.1.3 as below.

Problem 2.1.4 *Find $u_\xi(t) \in H^1_{0,\varpi}(I)$ such that, for all $v(t) \in H^1_{0,\varpi}(I)$,*

$$\left(-\frac{\partial u_\xi}{\partial t}, v\right) + A(u_\xi, v; t) + \xi\,(\rho(u_\xi), v) = (f, v), \quad a.e. \ in \ (0,T). \tag{2.1.14}$$

For Problem 2.1.3, we have the following unique solvability result.

Theorem 2.1.1 *Suppose that $\rho : L^2(\Omega) \to L^2(\Omega)$ satisfies the following conditions*

1. *ρ is monotonic in $L^2(\Omega)$; and*

2. *ρ is continuous.*

Then, Problem 2.1.3 has a unique solution.

Proof. First, note that $f(x,t) = e^{\beta t} L V_0$, where V_0 is sufficiently smooth in (x,t). We now prove this theorem by showing that the variational form of the nonlinear operator on the left-hand side of (2.1.13) is strictly monotone and continuous. In fact, for any $v_1, v_2 \in H^1_{0,\varpi}(I)$ a.e. in $(0,T)$ with the final condition being equal to $u^*(x,T)$ at $t = T$, it follows from the integration by parts that

$$\begin{aligned}
& \left(\mathcal{L}\left(v_1 - v_2\right), v_1 - v_2\right) + \xi\left(\rho(v_1) - \rho(v_2), v_1 - v_2\right) \\
= & \left(-\frac{\partial\left(v_1 - v_2\right)}{\partial \tau}, v_1 - v_2\right) + A\left(v_1 - v_2, v_1 - v_2; t\right) + \xi\left(\rho(v_1) - \rho(v_2), v_1 - v_2\right).
\end{aligned} \tag{2.1.15}$$

First, consider the function $\rho(v_1)$. By definition, it is a monotonic function. Obviously, this function is non-decreasing in v. Thus

$$\xi\left(\rho(v_1) - \rho(v_2), v_1 - v_2\right) = \xi \int_{-R}^{R} \left(\rho(v_1) - \rho(v_2)\right)\left(v_1 - v_2\right) dx \geq 0.$$

Denote $e(\tau) = v_1(\tau) - v_2(\tau)$. Integrating both sides of (2.1.15) from 0 to T, and using the above inequality and (2.1.8), we have

$$\begin{aligned}
& \int_0^T \left[(\mathcal{L}e(\tau), e(\tau)) + \xi\left(\rho(v_1) - \rho(v_2), v_1 - v_2\right)\right] d\tau \\
= & \int_0^T \left[\left(-\frac{\partial e(\tau)}{\partial \tau}, e(\tau)\right) + A\left(e(\tau), e(\tau)\right) + \xi\left(\rho(v_1) - \rho(v_2), v_1 - v_2\right)\right] d\tau \\
\geq & \int_0^T \left(-\frac{\partial e(\tau)}{\partial \tau}, e(\tau)\right) d\tau + C\int_0^T \|e(\tau)\|_A \, d\tau.
\end{aligned} \tag{2.1.16}$$

However, for any $t \in (0,T)$, integrating by parts gives

$$\int_t^T \left(-\frac{\partial e(\tau)}{\partial \tau}, e(\tau)\right) d\tau = (e(\tau), e(\tau)) - \int_t^T \left(-\frac{\partial e\left(\tau\right)}{\partial \tau}, e(\tau)\right) d\tau,$$

because $e(T) = 0$. From this, it follows that

$$\int_t^T \left(-\frac{\partial e(\tau)}{\partial \tau}, e(\tau)\right) d\tau = \frac{1}{2}\left(e(\tau), e(\tau)\right) \geq 0. \tag{2.1.17}$$

Therefore, from (2.1.16), (2.1.17), we get

$$\int_0^T \left[\left(\mathcal{L} \left(v_1 - v_2 \right), v_1 - v_2 \right) + \xi \left(\rho(v_1) - \rho(v_2), v_1 - v_2 \right) \right] d\tau \geq C \left\| v_1 - v_2 \right\|^2_{L^2(0,T;H^1_{0,\varpi}(I))}.$$

This implies that the operator on the left-hand side of (2.1.13) is strictly monotone.

Moreover, for any $v, w \in L^2(0, T; H^1_{0,\varpi}(I))$, it is easy to show by a standard argument that

$$\int_0^T \left| A \left(v, w; t \right) \right| dt \leq C \left\| v \right\|_{L^2(0,T;H^1_{0,\varpi}(I))} \left\| w \right\|_{L^2(0,T;H^1_{0,\varpi}(I))},$$

which means that the operator $A(v, w; t)$ is continuous in both v and w. Also, it is obvious that $(\rho(v), w)$ is continuous in both v and w. Therefore, using a standard result (see, for example, page 37 in [54]), we can conclude that Problem 2.1.3 is uniquely solvable. ∎

2.2 Convergence Analysis of The Monotonic Penalization

Many regularity results on the solution of the penalty problem can be found in several monographs, such as [28], [8] and [106]. In brief, under the assumption that u_ξ and $f(t)$ are sufficiently smooth, we have the following regularity results.

$$\frac{\partial u_\xi(x,t)}{\partial t}, \; u_\xi(x,t) \in L^2(0, T; H^1_{0,\varpi}(I)) \cap L^\infty(0, T; L^2(I)),$$

and

$$\rho(u_\xi(x,t)), f(x,t) \in L^\infty(0, T; L^2(I)).$$

On this basis, we have the following convergence result.

Theorem 2.2.1 *Let u and u_ξ be the solutions to Problem 2.1.2 and Problem 2.1.3, respectively. Then,*

$$\lim_{\xi \to \infty} \left(\| u_\xi(x,t) - u(x,t) \|_{L^\infty(0,T;L^2(I))} + \| u_\xi(x,t) - u(x,t) \|_{L^2(0,T;H^1_{0,\varpi}(I))} \right) = 0.$$

Proof. The proof is divided into three parts. First, we obtain *a prior estimates* for $\{u_\xi\}$. Then *the weak convergence* of $\{u_\xi\}$, and finally *the strong convergence* of $\{u_\xi\}$.

(I) *A prior estimate* for $\{u_\xi\}$.

We have the estimates

$$\|u_\xi(x,t)\|_{L^\infty(0,T;L^2(I))} + \|u_\xi(x,t)\|_{L^2(0,T;H^1_{0,\varpi}(I))} \le C, \tag{2.2.18}$$

where C is independent of ξ and u_ξ.

Let $v_0(t) \in K$. Then $\rho(v_0) = 0$. Setting $v(t) = u_\xi(t) - v_0(t)$, we have

$$\left(-\frac{\partial u_\xi}{\partial t}, u_\xi - v_0\right) + A(u_\xi, u_\xi - v_0; t) + \xi\left(\rho(u_\xi) - \rho(v_0), u_\xi - v_0\right) = (f(t), u_\xi - v_0).$$

Since ρ is monotonic, it follows that $(\rho(u_\xi) - \rho(v_0), u_\xi - v_0) \ge 0$. Thus, we get

$$\left(-\frac{\partial u_\xi}{\partial t}, u_\xi - v_0\right) + A(u_\xi, u_\xi - v_0); t) \le \left(f, u_\xi - v_0\right),$$

and hence

$$-\frac{1}{2}\frac{d}{dt}\left|u_\xi - v_0\right|^2 + A(u_\xi, u_\xi; t) \le (f, u_\xi - v_0) + A(u_\xi, v_0; t).$$

Therefore,

$$-\frac{1}{2}\frac{d}{dt}\left|u_\xi - v_0\right|^2 + \alpha\left\|u_\xi\right\|^2 \le c\left|u_\xi\right|^2 + c\left\|u_\xi\right\|\left\|v_0\right\| + c\left\|f\right\|\left\|u_\xi - v_0\right\|.$$

Since v_0 is a bounded element of $L^2(\Omega)$, we have

$$-\frac{d}{dt}\left|u_\xi - v_0\right|^2 + 2\alpha\left\|u_\xi\right\|^2 \le c\left|u_\xi - v_0\right|^2 + \alpha\left\|u_\xi\right\|^2 + c(1 + \|f\|^2).$$

Integrating both sides of the above inequality from t to T, we obtain

$$\begin{aligned}&\left|u_\xi - v_0\right|^2 + \alpha\int_t^T \left\|u_\xi\right\|^2 d\tau \\ &\le c\int_t^T \left|u_\xi - v_0\right|^2 d\tau + c\,(T-t) + \left(\int_t^T \left\|f\right\|^2 d\tau\right) + \left|u_\xi(T) - v_0(T)\right|^2.\end{aligned} \tag{2.2.19}$$

Now, let $\eta(t) = |u_\xi(t) - v_0(t)|^2$. Then, the above inequality can be expressed as:

$$\begin{cases} \eta(t) \le c\displaystyle\int_t^T \eta(\tau)d\tau + d; \\ d \le cT + (\displaystyle\int_0^T \|f\|^2 d\tau) + |u_\xi(T) - v_0(T)|^2. \end{cases}$$

By virtue of Gronwall's inequality, the above inequalities imply $\eta(t) \leq d\exp(ct)$, i.e.

$$|u_\xi(t) - v_0(t)|^2 \leq d\exp(ct).$$

On this basis, we deduce that

$$\|u_\xi(x,t)\|_{L^\infty(0,T;L^2(I))} \leq C, \quad (C \textit{ independent of } \xi \textit{ and } u_\xi) \tag{2.2.20}$$

Thus, from (2.2.19) and (2.2.20), it follows that (2.2.18) is valid.

(II) *Weak convergence* of $\{u_\xi\}$.

(2.2.18) implies that $\{u_\xi\}$ is uniformly bounded in the space $L^2(0,T;H^1_{0,\varpi}(I)) \cap L^\infty(0,T;L^2(I))$. Therefore, there exists a subsequence of $\{u_\xi\}$, still denoting it by $\{u_\xi\}$, such that

$$\lim_{\xi\to\infty} u_\xi = \bar{u}, \textit{ weakly in } L^2(0,T;H^1_{0,\varpi}(I)) \cap L^\infty(0,T;L^2(I)).$$

Our next task is to show that $\bar{u}$ is a solution to Problem 2.1.2.

From (2.1.14), we have

$$\int_t^T (\rho(u_\xi), v)\, d\tau = \frac{1}{\xi}\left[\int_t^T (f,v)d\tau - \int_t^T \left(\frac{\partial u_\xi}{\partial \tau}, v\right) d\tau - \int_t^T A(u_\xi, v; \tau) d\tau\right].$$

Thus,

$$\|\rho(u_\xi)\|_{L^\infty(0,T;L^2(I))} = O(\frac{1}{\xi}).$$

Therefore, when $\xi \to \infty$, we have $u_\xi \to \bar{u}$, and $\|\rho(u_\xi)\|_{L^\infty(0,T;L^2(I))} \to 0$. Hence,

$$\|\rho(\bar{u})\|_{L^\infty(0,T;L^2(I))} = 0.$$

So, we obtain $\rho(\bar{u}) = 0$. By the definition of $\rho(\bar{u})$, we see that $\bar{u}(x,t) \in K$. In addition, $\bar{u} \in L^2(0,T;L^2(I))$.

For any $v \in K$, we have $\rho(v) = 0$. Replacing v with $v - u_\xi$ in (2.1.14), it follows that

$$\left(-\frac{\partial u_\xi}{\partial t}, v - u_\xi\right) + A\left(u_\xi, v - u_\xi; t)\right) + \xi\left(\rho(u_\xi) - \rho(v), v - u_\xi\right) = (f, v - u_\xi), \quad a.e. \text{ in } (0,T).$$

By the monotonicity of ρ, we get, for $v \in K$,

$$(\rho(u_\xi) - \rho(v), v - u_\xi) \leq 0.$$

Thus,

$$\left(-\frac{\partial u_\xi}{\partial t}, v-u_\xi\right)+A(u_\xi, v-u_\xi;t)-(f,v-u_\xi)\geq 0, \quad a.e. \text{ in } (0,T).$$

Therefore,

$$\left(-\frac{\partial u_\xi}{\partial t}, v\right)+A(u_\xi, v;t)-(f,v)\geq \left(-\frac{\partial u_\xi}{\partial t}, u_\xi\right)+A(u_\xi,u_\xi;t), \ a.e. \text{ in } (0,T). \quad (2.2.21)$$

Integrating both sides of (2.2.21) from 0 to T, we obtain

$$\begin{aligned}
&\int_t^T\left(-\frac{\partial u_\xi}{\partial \tau}, v\right)d\tau+\int_t^T A(u_\xi,v;\tau)d\tau-\int_t^T(f,v)d\tau\\
\geq &\int_t^T\left(-\frac{\partial u_\xi}{\partial \tau}, u_\xi\right)d\tau+\int_t^T A(u_\xi,u_\xi;\tau)d\tau\\
= &(u_\xi,u_\xi)+\int_t^T A(u_\xi,u_\xi;\tau)d\tau. \qquad (2.2.22)
\end{aligned}$$

We deduce from the properties of weak convergence that

$$\liminf_{\xi\to\infty}\left((u_\xi,u_\xi)+\int_t^T A(u_\xi,u_\xi;\tau)d\tau\right)\geq(\bar{u},\bar{u})+\int_t^T A(\bar{u},\bar{u};\tau)d\tau. \quad (2.2.23)$$

From (2.2.22) and (2.2.23), we have

$$\begin{aligned}
&\liminf_{\xi\to\infty}\left[\int_t^T\left(-\frac{\partial u_\xi}{\partial \tau}, v\right)d\tau+\int_t^T A(u_\xi,v;\tau)d\tau-\int_t^T(f,v)d\tau\right]\\
\geq &\int_t^T\left(-\frac{\partial \bar{u}}{\partial \tau}, \bar{u}\right)d\tau+\int_t^T A(\bar{u},\bar{u};\tau)d\tau.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_t^T\left(-\frac{\partial \bar{u}}{\partial \tau}, v\right)d\tau+\int_t^T A(\bar{u},v;\tau)d\tau-\int_t^T(f,v)d\tau\\
\geq &\int_t^T\left(-\frac{\partial \bar{u}}{\partial \tau}, \overline{u}\right)d\tau+\int_t^T A(\bar{u},\bar{u};\tau)d\tau.
\end{aligned}$$

Thus,

$$\int_t^T\left(-\frac{\partial \bar{u}}{\partial \tau}, v-\bar{u}\right)d\tau+\int_t^T A(\bar{u},v-u;\tau)d\tau\geq\int_t^T(f,v-\bar{u})d\tau,$$

i.e.

$$\left(-\frac{\partial \bar{u}}{\partial t}, v-\bar{u}\right)+A(\bar{u},v-\bar{u};t)\geq(f,v-\bar{u}), \quad \forall v\in K, \quad a.e. \text{ in } (0,T).$$

This shows that $\bar{u}=u$, and that the whole sequence $\{u_\xi\}$ converges weakly to u.

(III) *Strong convergence* of $\{u_\xi\}$.

We need to prove

$$\lim_{\xi\to\infty}\left(\|u_\xi(x,t)-u(x,t)\|_{L^\infty(0,T;L^2(I))}+\|u_\xi(x,t)-u(x,t)\|_{L^2(0,T;H^1_{0,\varpi}(I))}\right)=0.$$

Setting $v(t)=u_\xi(t)-u(t)$ in (2.1.14), we have

$$\left(-\frac{\partial u_\xi}{\partial t},u_\xi-u\right)+A(u_\xi,u_\xi-u;t)+\xi\left(\rho(u_\xi)-\rho(u),u_\xi-u\right)=(f,u_\xi-u),\ a.e.\ \text{in } (0,T).$$

Using the monotonic property of ρ, we have

$$\left(-\frac{\partial u_\xi}{\partial t},u_\xi-u\right)+A(u_\xi,u_\xi-u;t)\le(f,u_\xi-u),\ \ a.e.\ \text{in } (0,T). \tag{2.2.24}$$

Reformulating (2.2.24) yields

$$\begin{aligned}&\left(-\frac{\partial\left(u_\xi-u\right)}{\partial t},u_\xi-u\right)+A(u_\xi-u,u_\xi-u;t)\\ \le\ &(f,u_\xi-u)+\left(-\frac{\partial u}{\partial t},u_\xi-u\right)+A(u,u_\xi-u;t).\end{aligned} \tag{2.2.25}$$

Integrating both sides of (2.2.25) from 0 to T and then using the coerciveness property of the operator A, we obtain

$$\begin{aligned}&|u_\xi-u|^2+\alpha\int_t^T\|u_\xi-u\|^2_{H^1_{0,\varpi}(I)}\,d\tau\\ \le\ &c\left[\int_t^T\|f\|^2_{L^2(0,T;L^2(I))}\,d\tau\right]^{\frac12}\left[\int_t^T\|u_\xi-u\|^2_{L^2(0,T;L^2(I))}\,d\tau\right]^{\frac12}\\ &+c\left[\int_t^T\|u\|^2_{L^2(0,T;L^2(I))}\,d\tau\right]^{\frac12}\left[\int_t^T\|u_\xi-u\|^2_{L^2(0,T;L^2(I))}\,d\tau\right]^{\frac12}\\ &+c\left[\int_t^T\left\|\frac{\partial u}{\partial\tau}\right\|^2_{L^2(0,T;L^2(I))}\,d\tau\right]^{\frac12}\left[\int_t^T\|u_\xi-u\|^2_{L^2(0,T;L^2(I))}\,d\tau\right]^{\frac12}\\ \triangleq\ &C\left[\int_t^T\|u_\xi-u\|^2_{L^2(0,T;L^2(I))}\,d\tau\right]^{\frac12}.\end{aligned} \tag{2.2.26}$$

Thus, we deduce from (2.2.26) that

$$\lim_{\xi\to\infty}\left(\|u_\xi(x,t)-u(x,t)\|_{L^\infty(0,T;L^2(I))}+\|u_\xi(x,t)-u(x,t)\|_{L^2(0,T;H^1_{0,\varpi}(I))}\right)=0.$$

■

2.3 Some Special Monotonic Penalty Functions

The class of monotonic penalty functions is very rich. The l_2 penalty function, l_1 penalty function and lower order penalty function l_k $(0 < k < 1)$ are the most frequently used monotonic penalty functions. These penalty methods for American option pricing have been extensively studied in [125, 44, 137]. In this section, we shall propose two new monotonic penalty functions, i.e. 'valley at zero' penalty function and a combination of two power penalty functions.

1. *'Valley at zero' penalty function*

$$\rho(u) = \begin{cases} 1, & if \quad u \geq 1, \\ u, & if \quad 0 \leq u \leq 1, \\ 0, & if \quad u \leq 0. \end{cases} \tag{2.3.27}$$

For a detailed study of 'valley at zero' penalty function, see [136]. It is easy to see that (2.3.27) given above is a monotonic penalty function. When $u \leq 1$ this penalty function is identical to the l_1 penalty function. From our numerical experiments, we find that its convergence rate is just the same as that of the l_1 penalty function. The reason for this phenomenon is that the option pricing problem is a well-behaved initial value problem [137]. The error at every iteration is far below one, so $\rho(u) = u$, as $0 \leq u \leq 1$, plays the dominating role. Thus, for the American option pricing problem, we conclude that the convergence rate of the 'valley at zero' penalty function (2.3.27) is the same as that of the l_1 penalty function.

2. *A combination of two power penalty functions*

$$c_{k,m}(u) = [\max\{u, 0\}]^{1/k} + [\max\{u, 0\}]^m, \quad 0 < k, m < \infty. \tag{2.3.28}$$

It is clear that the combined function $c_{k,m}(u)$ given by (2.3.28) is also monotonic. The motivation for this new kind of penalty functions is that for problems that u is not a 'good' initial guess, a higher order penalty term (e.g. $m > 1$) plays a dominating role in the behavior of u, controlling it to converge to near zero as quickly as possible. Then, the problem will behave as a well defined initial value problem, hence asymptotically the lower order penalty term (e.g. $k > 1$) will play the dominating role. The combination of these two power penalty functions possesses a good convergence behavior with a desirable convergence rate. These advantages are clearly seen from our numerical experiments.

In the following, we will establish a convergence rate of the penalty function (2.3.28).

Lemma 2.3.1 *Let the combined penalty function $c_{k,m}(u)$ be used and $u_\xi \in L_p(\Omega)$ be the solution to Problem 2.1.4. Then there exists a positive constant C, independent of u_ξ and ξ, such that*

$$\|[u_\xi - u^*]_+\|_{L_p(\Omega)} \leq \frac{C}{\xi^{1/(p-1)}}, \tag{2.3.29}$$

$$\|[u_\xi - u^*]_+\|_{L^\infty(0,T;L^2(I))} + \|[u_\xi - u^*]_+\|_{L^\infty(0,T;H^1_{0,\varpi}(I))} \leq \frac{C}{\xi^{1/(2p-2)}}, \tag{2.3.30}$$

where $p = 1 + \frac{m+1/k}{2}$.

Proof. Assume that C is a generic positive constant, independent of u_ξ and ξ. To simplify the notation, we let $\varphi(\cdot,t) = [u_\xi(\cdot,t) - u^*]_+ \in H^1_{0,\varpi}(I)$ for almost all $t \in (0,T)$, where $[u_\xi(\cdot,t) - u^*]_+ = \max\{u_\xi(\cdot,t) - u^*, 0\}$.

Now, setting $v(t) = \varphi(\cdot,t)$ in (2.1.14), and replacing $\rho(u)$ with

$$c_{k,m}(u) = [\max\{u,0\}]^{1/k} + [\max\{u,0\}]^m,$$

we have

$$\left(\frac{-\partial u_\xi}{\partial t}, \varphi\right) + A(u_\xi, \varphi; t) + \xi(\varphi^m + \varphi^{1/k}, \varphi) = (f, \varphi), \text{ a.e. in } (0,T). \tag{2.3.31}$$

Taking $-(\dfrac{\partial u^*}{\partial t}, \varphi) + A(u^*, \varphi; t)$ away from both sides of (2.3.31) gives

$$\begin{aligned}
&-\left(\frac{\partial(u_\xi - u^*)}{\partial t}, \varphi\right) + A(u_\xi - u^*, \varphi; t) + \xi(\varphi^m + \varphi^{1/k}, \varphi) \\
&= (f, \varphi) + \left(\frac{\partial u^*}{\partial t}, \varphi\right) - A(u^*, \varphi; t).
\end{aligned} \tag{2.3.32}$$

Integrating both sides of (2.3.32) from t to T and using the coerciveness property of the operator A and Hölder's inequality, we get

$$\begin{aligned}
&\frac{1}{2}(\varphi, \varphi) + \int_t^T ||\varphi||_A^2 d\tau + \xi \int_t^T \left(\varphi^m + \varphi^{1/k}, \varphi\right) d\tau \\
&\leq \int_t^T (f, \varphi) d\tau - \beta \int_t^T e^{\beta\tau}(V_0 - V^*, \varphi) d\tau - \int_t^T A(u^*, \varphi; \tau) d\tau \\
&\leq C \left(\int_t^T ||\varphi||^p_{L^p(I)} d\tau\right)^{1/p} + \beta \int_t^T e^{\beta\tau}(V_0 - V^*, \varphi) d\tau - \int_t^T A(u^*, \varphi; \tau) d\tau.
\end{aligned} \tag{2.3.33}$$

Noting that $a+b \geq 2\sqrt{ab}$, if $a, b \geq 0$, we have

$$\begin{aligned} &\frac{1}{2}(\varphi,\varphi) + \int_t^T ||\varphi||_A^2 d\tau + \xi \int_t^T ||\varphi||_{L^p(I)}^p d\tau \\ \leq &\frac{1}{2}(\varphi,\varphi) + \int_t^T ||\varphi||_A^2 d\tau + \xi \int_t^T \left(\varphi^m + \varphi^{1/k}, \varphi\right) d\tau \end{aligned} \tag{2.3.34}$$

where $p = \frac{m+1/k}{2}$.

From (2.3.33) and (2.3.34), we have

$$\frac{1}{2}(\varphi,\varphi) + \int_t^T ||\varphi||_A^2 d\tau + \xi \int_t^T ||\varphi||_{L^p(I)}^p d\tau \leq C \left(\int_t^T ||\varphi||_{L^p(I)}^p d\tau \right)^{1/p}, \text{ for all } t \in (0,T). \tag{2.3.35}$$

This implies that

$$\xi \int_t^T ||\varphi||_{L^p(I)}^p d\tau \leq C \left(\int_t^T ||\varphi||_{L^p(I)}^p d\tau \right)^{1/p}.$$

From this, it follows that

$$\left(\int_t^T ||\varphi||_{L^p(I)}^p d\tau \right)^{1/p} \leq \frac{C}{\xi^{1/(p-1)}}, \quad where \;\; p = 1 + \frac{m+1/k}{2}. \tag{2.3.36}$$

Now, from (2.3.35) and (2.3.36), we have

$$\frac{1}{2}(\varphi,\varphi) + \int_t^T ||\varphi||_A^2 d\tau \leq C \left(\int_t^T ||\varphi||_{L^p(I)}^p d\tau \right)^{1/p} \leq \frac{C}{\xi^{1/(p-1)}},$$

from which, it follows that

$$(\varphi,\varphi)^{\frac{1}{2}} + \left(\int_t^T ||\varphi||_A^2 d\tau \right)^{1/2} \leq \frac{C}{\xi^{1/(2p-2)}}$$

for all $t \in (0,T)$. Clearly, by replacing $\varphi(\cdot,t)$ with $[u_\xi(\cdot,t) - u^*]_+$, we obtain readily (2.3.29) and (2.3.30). ■

Theorem 2.3.1 *Assume that the assumptions of Theorem 2.2.1 are satisfied. Then, for the combined penalty function $c_{k,m}(u)$ defined by (2.3.28), it holds that*

$$\|u_\xi(x,t) - u(x,t)\|_{L^\infty(0,T;L^2(I))} + \|u_\xi(x,t) - u(x,t)\|_{L^2(0,T;H^1_{0,\varpi}(I))} \leq \frac{C}{\xi^{1/(m+1/k)}}, \tag{2.3.37}$$

as $\xi \to \infty$.

In particular, when $k = 2$ and $m = 2$,

$$\|u_\xi(x,t) - u(x,t)\|_{L^\infty(0,T;L^2(I))} + \|u_\xi(x,t) - u(x,t)\|_{L^2(0,T;H^1_{0,\varpi}(I))} \leq \frac{C}{\xi^{2/5}},$$

as $\xi \to \infty$.

Proof. We still use the notation of Lemma 2.3.1. Setting $v_- = \min(v, 0)$ and $R_\xi = u - u^* + [u_\xi - u^*]_-$, it follows that

$$u - u_\xi = R_\xi - \varphi, \text{ and } (\varphi^\alpha, [u_\xi - u^*]_-) = [u_\xi - u^*]_+^\alpha [u_\xi - u^*]_- \equiv 0, \ \alpha > 0. \tag{2.3.38}$$

Set $v = u - R_\xi$ in (2.1.7) and $v = R_\xi$ in (2.1.14). Then, by replacing $\rho(u)$ with $c_{k,m}(u) = [\max\{u, 0\}]^{1/k} + [\max\{u, 0\}]^m$, we obtain

$$\left(-\frac{\partial u}{\partial t}, -R_\xi\right) + A(u, -R_\xi; t) \geq (f, -R_\xi), \tag{2.3.39}$$

$$\left(-\frac{\partial u_\xi}{\partial t}, R_\xi\right) + A(u_\xi, R_\xi; t) + \xi\left(\varphi^m + \varphi^{1/k}, R_\xi\right) = (f, R_\xi). \tag{2.3.40}$$

Combining (2.3.39) and (2.3.40) gives

$$\left(-\frac{\partial(u_\xi - u)}{\partial t}, R_\xi\right) + A(u_\xi - u, R_\xi; t) + \xi(\varphi^m + \varphi^{1/k}, R_\xi) \geq 0.$$

It follows from $u \leq u^*$ that

$$(\varphi^m + \varphi^{1/k}, R_\xi) = (\varphi^m + \varphi^{1/k}, u - u^*) + (\varphi^m + \varphi^{1/k}, [u_\xi - u^*]_-) = (\varphi^m + \varphi^{1/k}, u - u^*) \leq 0.$$

Thus,

$$\left(-\frac{\partial(u - u_\xi)}{\partial t}, R_\xi\right) + A(u - u_\xi, R_\xi; t) \leq 0.$$

From (2.3.38), we get

$$\left(-\frac{\partial R_\xi}{\partial t}, R_\xi\right) + A(R_\xi, R_\xi; t) \leq \left(-\frac{\partial \varphi}{\partial t}, R_\xi\right) + A(\varphi, R_\xi; t).$$

Integrating both sides of the above from t to T and then using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&(R_\xi, R_\xi) + \int_t^T A(R_\xi, R_\xi; \tau) d\tau \\
&\leq \int_t^T \left(-\frac{\partial \varphi}{\partial \tau}, R_\xi\right) d\tau + \int_t^T A(\varphi, R_\xi; \tau) d\tau \\
&\leq (\varphi, R_\xi) + \int_t^T \left(\varphi, \frac{\partial R_\xi}{\partial \tau}\right) d\tau + \int_t^T A(\varphi, R_\xi; \tau) d\tau \\
&\leq (\varphi, R_\xi) + \int_t^T \left(\varphi, \frac{\partial R_\xi}{\partial \tau}\right) d\tau + \int_t^T A(\varphi, R_\xi; \tau) d\tau + \int_t^T \left(\varphi, \frac{\partial u}{\partial \tau}\right) d\tau \\
&\leq ||\varphi||_{L^\infty(0,T;L^2(I))} ||R_\xi||_{L^\infty(0,T;L^2(I))} + C_1 ||\varphi||_{L^2(0,T;H^1_{0,\varpi}(I))} ||R_\xi||_{L^2(0,T;H^1_{0,\varpi}(I))} \\
&\quad + C_2 ||\varphi||_{L^p(\Omega)} \left\|\frac{\partial u}{\partial t}\right\|_{L^q(\Omega)} + ||V_0 - V^*||_{L^q(\Omega)} \\
&\leq ||\varphi||_{L^\infty(0,T;L^2(I))} ||R_\xi||_{L^\infty(0,T;L^2(I))} + C_1 ||\varphi||_{L^2(0,T;H^1_{0,\varpi}(I))} ||R_\xi||_{L^2(0,T;H^1_{0,\varpi}(I))} + \frac{C_2}{\xi^{1/(p-1)}},
\end{aligned}$$

where $p = 1 + \frac{m+1/k}{2}$, and $1/p + 1/q = 1$.

Using the coerciveness property of the operator A and (2.3.30), we obtain

$$\left(||R_\xi||_{L^\infty(0,T;L^2(I))} + ||R_\xi||_{L^2(0,T;H^1_{0,\varpi}(I))}\right)^2 \le \frac{C}{\xi^{1/(p-1)}},$$

and hence,

$$||R_\xi||_{L^\infty(0,T;L^2(I))} + ||R_\xi||_{L^2(0,T;H^1_{0,\varpi}(I))} \le \frac{C}{\xi^{1/(2p-2)}}.$$

By the triangle inequality, we finally have

$$\begin{aligned}
&||u - u_\xi||_{L^\infty(0,T;L^2(I))} + ||u - u_\xi||_{L^2(0,T;H^1_{0,\varpi}(I))} \\
\le\ & ||R_\xi||_{L^\infty(0,T;L^2(I))} + ||R_\xi||_{L^2(0,T;H^1_{0,\varpi}(I))} + ||\varphi||_{L^\infty(0,T;L^2(I))} + ||\varphi||_{L^2(0,T;H^1_{0,\varpi}(I))} \\
\le\ & \frac{C}{\xi^{1/(2p-2)}},
\end{aligned}$$

This is the estimate (2.3.37).

Setting $k = 2$ and $m = 2$, and then $p = 9/4$, we have

$$\|u_\xi(x,t) - u(x,t)\|_{L^\infty(0,T;L^2(I))} + \|u_\xi(x,t) - u(x,t)\|_{L^2(0,T;H^1_{0,\varpi}(I))} \le \frac{C}{\xi^{2/5}}.$$

■

Remark 2.3.1 *From the proof of Theorem 2.3.1, it can be seen that if we set $m = 1/k$, where $k > 1$, then we get the convergence rate, of order $\mathcal{O}(\xi^{-1/2m})$, of the lower order penalty function $l_{1/k}$ or l_m. Likewise, we obtain the convergence rates of l_2 and l_1 penalty functions by setting $1/k = m = 2$ and $m = k = 1$, respectively. Therefore, Theorem 2.3.1 actually presents a unified convergence rate for all the penalty functions considered in [125] for American option pricing and in [38] for variational inequalities.*

2.4 The Fitted Finite Volume Method

The standard finite difference method is widely applied to valuating the option pricing problems, see [115, 118, 127]. However, it is well known [115] that when using the standard finite difference method to solve those problems involving the convection-diffusion operator, such as the Black-Scholes operator, some numerical difficulties may be encountered. The main reason is that when the volatility or the asset price is small, the Black-Scholes operator becomes a convection-dominated operator. Hence, the standard finite difference method will produce numerical oscillation, which will significantly affect the accuracy of the hedging parameters. The detailed analysis of this phenomenon can be found in [40, 115].

To overcome this numerical difficulty, the fitted finite volume discretization scheme was proposed in [125, 124] to solve the variational problem obtained as an equivalent formulation from the option pricing. The fitted finite volume method is based on the idea proposed by Allen and Southwell for convection-diffusion problems [29]. It has been shown in [125, 124] that the fitted finite volume method is numerically stable. Essentially, this method is based on finite volume formulation with a fitted local approximation to the solution. The local approximation is determined by a set of two-point boundary value problems defined on the element edges. This fitting technique effectively eliminates the oscillation phenomenon.

In what follows, we give a brief account of the fitted finite volume method applied to (2.1.13). To avoid overloading of symbols, in the rest of this section, we suppress the subscript ξ of u_ξ used in the previous part.

Let the interval $I = (0, X)$ be divided into N sub-intervals

$$I_i := (x_i, x_{i+1}), \quad i = 0, \dots, N-1$$

with $0 = x_0 < x_1 < \cdots < x_N = X$. For each $i = 0, 1, ..., N-1$, we put $h_i = x_{i+1} - x_i$ and $h = \max_{0 \le i \le N-1} h_i$. We also let $x_{i-1/2} = (x_{i-1} + x_i)/2$ and $x_{i+1/2} = (x_i + x_{i+1})/2$ for each $i = 1, 2, \dots, N-1$. These mid-points form a second partition of $(0, X)$ if we define $x_{-1/2} = x_0$ and $x_{N+1/2} = x_N$. Finally, we set $l_i := x_{i+1/2} - x_{i-1/2}$ for $i = 1, \dots, N-1$.

Integrating (2.1.13) over $(x_{i-1/2}, x_{i+1/2})$ and applying the mid-point quadrature rule to the first, third and last terms, we obtain $N-1$ 'balance equations'

$$-\frac{\partial u_i}{\partial t} l_i - \left[x_{i+1/2}\, \psi(u)|_{x_{i+1/2}} - x_{i-1/2}\, \psi(u)|_{x-1/2} \right] + \left[c_i u_i - \xi \rho(u_i) \right] l_i = 0, \qquad (2.4.41)$$

for $i = 1, \dots, N-1$, where $c_i = c(x_i)$, u_i is the nodal approximation to $u(x_i, t)$, which is to be determined, and $\psi(u)$ is a *flux* associated with u defined by

$$\psi(u) := ax \frac{\partial u}{\partial x} + bu. \qquad (2.4.42)$$

The key idea of the fitted volume method is to derive an approximation to the flux at the two end points $x_{i-1/2}$ and $x_{i+1/2}$. When $i \neq 0$, the flux (2.4.42) can be approximated by the following two-point boundary value problem

$$\begin{cases} \left(aux' + b_{i+1/2}u\right)' = 0, \quad x \in I_i \\ \; u(x_i) = u_i, \quad u(x_{i+1}) = u_{i+1}, \end{cases} \tag{2.4.43}$$

where $b_{i+1/2} = b(x_{i+1/2})$. When $i = 0$, $I_0 = (0, x_1)$, (2.4.43) is degenerated, so we cannot use the above approximation. Instead, we consider another approximation by adding an extra degree of freedom in the following form

$$\begin{cases} \left(aux' + b_{1/2}u\right)' = C_1, \quad \text{in} \quad (0, x_1) \\ u(0) = u_0, \qquad\quad u(x_1) = u_1. \end{cases} \tag{2.4.44}$$

Now, using (2.4.43) and (2.4.44), we can define a global piecewise constant approximation to the flux $\psi(u)$ by $\psi_h(u)$ satisfying

$$\psi_h(u) = \begin{cases} b_{i+1/2}\dfrac{x_{i+1}^{\alpha_i}u_{i+1} - x_i^{\alpha_i}u_i}{x_{i+1}^{\alpha_i} - x_i^{\alpha_i}}, \quad x \in I_i, \; i \neq 0, \\ \frac{1}{2}\left[\left(a + b_{1/2}u_1\right) - \left(a - b_0\right)u_0\right], \; x \in (0, x_1). \end{cases} \tag{2.4.45}$$

Substituting (2.4.45) into (2.4.41), we obtain

$$-\frac{\partial u_i}{\partial t}l_i + e_{i-1}^i u_{i-1} + e_i^i u_i + e_{i+1}^i u_{i+1} + d(u_i) = 0, \tag{2.4.46}$$

where

$$\begin{cases} e_0^1 &=& -\dfrac{x_1(a - b_{1/2})}{4}, \\ e_1^1 &=& \dfrac{x_1(a - b_{1/2})}{4} + \dfrac{b_1\, x_{1+1/2}\, x_1^{\alpha_1}}{x_2^{\alpha_1} - x_1^{\alpha_1}} + c_1 l_1, \\ e_2^1 &=& \dfrac{b_{i+1/2}\, x_{i+1/2}\, x_{i+1}^{\alpha_i}}{x_{i+1}^{\alpha_i} - x_i^{\alpha^i}}, \end{cases}$$

and

$$\begin{cases} e_{i-1}^i &=& -\dfrac{b_{i-1}\, x_{i-1/2}\, x_{i-1}^{\alpha_i - 1}}{x_i^{\alpha_i - 1} - x_{i-1}^{\alpha^{i-1}}}, \\ e_i^i &=& \dfrac{b_{i-1}\, x_{i-1/2}\, x_i^{\alpha_i - 1}}{x_i^{\alpha_i - 1} - x_{i-1}^{\alpha^{i-1}}} + \dfrac{b_{i+1/2}\, x_{i+1/2}\, x_i^{\alpha_i}}{x_{i+1}^{\alpha_i} - x_i^{\alpha_i}} + c_i l_i, \\ e_{i+1}^i &=& \dfrac{b_{i+1/2}\, x_{i+1/2}\, x_{i+1}^{\alpha_i}}{x_{i+1}^{\alpha_i} - x_i^{\alpha^i}}, \end{cases}$$

for $i = 2, \ldots N - 1$. The nonlinear term

$$d(u_i) = -\xi l_i \rho(u_i).$$

We now discretize the time horizon $[0, T]$ by letting t_i $(i = 0, 1, \ldots K)$ be a set of partition points in $[0, T]$ satisfying $T = t_0 > t_1, \ldots, > t_K = 0$. Then, we apply the two-level implicit time-stepping method with a splitting parameter $\theta \in [0, 1/2]$ to (2.4.46). We can write down a compact form of the discrete system by defining

$$\begin{cases} \mathbf{u}^k &= (u_1^k, \ldots, u_{N-1}^k)^T, \\ E_i^k &= (0, \ldots, 0, e_{i-1}^i, e_i^i, e_{i+1}^i, 0, \ldots, 0), \\ E^k &= (E_1^k, \ldots, E_{N-1}^k)^T, \\ G^k &= diag(-l_1/\triangle t_k, \ldots, -l_{N-1}/\triangle t_k), \\ D(\mathbf{u}^k) &= (d(u_1), \ldots, d(u_{N-1}))^T, \end{cases}$$

for $k = 0, 1, \ldots, K-1$, where $\triangle t_k = t_{k+1} - t_k < 0$, $\mathbf{U}^k$ denotes the approximation of u at $t = t_k$ and $E_i^k = E_i(t_k)$. Then, the final full discrete system can be written as:

$$\left(\theta E^{k+1} + G^k\right) \mathbf{u}^{k+1} + \theta D(\mathbf{u}^{k+1}) = \left(G^k - (1-\theta)\, E^k\right) \mathbf{u}^k - (1-\theta)\, D(\mathbf{u}^k), \qquad (2.4.47)$$

with u_0 and u_N satisfying the given boundary conditions (2.1.6).

2.5 Numerical Experiments

Now, we consider the numerical solution to the penalized equation (2.1.13), where the fitted finite volume method [124] is used in the space discretization. Details of the fitted finite volume method for the Black-Scholes differential equation can be found in [125], [124]. Crank-Nicolson method proposed in [108] and [44] is used for the discretization of time. It is worth noting that the lower order penalty function and the combined penalty function are nonsmooth. Moreover, they are not even locally Lipschitz continuous at $u = 0$. This, in turn, implies that the lower order penalty function is not semismooth. Hence, the generalized Newton method [104] cannot be applied to this kind of situations. Nonsmooth penalized equation (2.1.13) is smoothed by a polynomial function as shown in Appendix 7.2. We will use a test problem that is the same as the one in [44]. All the numerical results were computed in the double precision on a Pentium IV 2.8 GHz, 512M memory PC under the Visual C++.net environment.

2.5.1 Comparison of Different Penalty Methods

The test problem is a standard American vanilla put option with the following two sets of parameters

$$\text{(i) } T = 0.25,\ K = 100,\ r = 0.10,\ \sigma = 0.2.$$
$$\text{(ii) } T = 0.25,\ K = 100,\ r = 0.10,\ \sigma = 0.8.$$

We take the asset pricing (S) space as $[0, 1000]$. We divide it into 1000 sub-intervals, that is, $\Delta S = 1$, and divide the time interval $[0, 0.25]$ into 100 sub-intervals, that is, $\Delta t = 0.0025$. The value at time $t = 0$, and $S = K = 100$ are computed and compared. Take $V = 14.6747$, when $\sigma = 0.8$, and $V = 3.06444$, when $\sigma = 0.2$ as the respective 'reference solutions'. These reference solutions were computed in [44].

For clarity, $\{\xi_n\}$ is denoted as a sequence of penalty parameters. For each ξ_n, a solution to Problem 2.1.3 is found, denoted by V_n. Let

$$\Delta V_n = V_n - V_{n-1}, \quad R_n = \frac{\Delta V_{n-1}}{\Delta V_n}$$

be the difference of the solutions and the ratio of changes corresponding to the two successive penalty parameters, respectively. Tables 2.1, 2.2, 2.3, 2.4 and 2.5 show the details of the numerical results of the five monotonic penalty methods, where $l_1, l_{1/2}$ and l_2 stand for the linear penalty method, $\frac{1}{2}$th order penalty method and quadratic penalty method, respectively.

From Tables 2.1, 2.2, 2.3, 2.4 and 2.5, some desirable conclusions can be drawn.

1. Among the $l_2, l_1, l_{1/2}$, 'valley at zero' and $l_2 + l_{1/2}$ penalty methods, with the same level of accuracy, the $l_2 + l_{1/2}$ and $l_{1/2}$ penalty methods need smaller penalty parameters, while the l_2 penalty method needs the largest. For the 'valley at zero' penalty method, the results obtained are similar to those of the l_1 penalty method. These findings suggest that the combined penalty function $l_2 + l_{1/2}$ is preferable to the others.

2. From the columns ξ_n and R_n of every table, we can observe a rough convergence speed. For the l_2 penalty method, when the penalty parameter is increased 4 times, the relative error $R_n = \frac{\Delta V_{n-1}}{\Delta V_n}$ is reduced to half, see Table 2.1. For the

Table 2.1: l_2 penalty method.

$\sigma = 0.2$				$\sigma = 0.8$			
ξ_n	V_n	ΔV_n	R_n	ξ_n	V_n	ΔV_n	R_n
250	3.02733			250	14.6310		
1000	3.04530	0.01797		1000	14.6515	0.0205	
4000	3.05481	0.00951	1.9	4000	14.6628	0.0113	1.8
16000	3.05969	0.00488	1.9	16000	14.6688	0.0060	1.9
64000	3.06215	0.00246	2.0	64000	14.6718	0.0030	2.0
256000	3.06337	0.00122	2.0	256000	14.6734	0.0016	1.9
1024000	3.06398	0.00061	2.0	1024000	14.6742	0.0008	2.0
4096000	3.06428	0.00030	2.0	4096000	14.6746	0.0004	2.0
16384000	*Failed*	*Failed*	*Failed*	16384000	*Failed*	*Failed*	*Failed*

l_1 penalty method, when the penalty parameter is increased 2 times, the relative error R_n is reduced to half, see Table 2.2. However, for the $l_{1/2}$ penalty method, when the penalty parameter is increased 2 times, the relative error $R_n = \frac{\Delta V_{n-1}}{\Delta V_n}$ is reduced to one fourth, see Table 2.3 – approximately a second order convergence rate with respect to penalty parameter.

3. From Tables 2.5 and 2.6, we also find that there is little difference between the convergence rates of $l_{1/2}$ and $l_2 + l_{1/2}$. However, with the same level of penalty parameter, the results obtained by $l_2 + l_{1/2}$ penalty method are more accurate than those of $l_{1/2}$, when the penalty parameters are small. However, when the penalty parameters get larger, the results become nearly the same. In fact, when penalty parameters are small, the situation is corresponding to a less accuracy requirement. Thus, the initial value problem is not so well behaved. Hence, the l_2 penalty function plays the role of controlling the error of the solution obtained. However, as the penalty parameter gets larger, a higher accuracy is required. When u is large, the values of the functions are mainly from the contribution of the l_2 penalty function. On the other hand, when u is small, it is the $l_{1/2}$ penalty function which plays the dominating role.

Table 2.2: l_1 penalty method with smoothing interval $(0, 10^{-3})$.

$\sigma = 0.2$				$\sigma = 0.8$			
ξ_n	V_n	ΔV_n	R_n	ξ_n	V_n	ΔV_n	R_n
125	3.04804			125	14.6541		
250	3.05622	0.01797		250	14.6641	0.0100	
500	3.06039	0.00951	1.9	500	14.6695	0.0054	1.9
1000	3.06250	0.00488	1.9	1000	14.6722	0.0027	2.0
2000	3.06354	0.00246	2.0	2000	14.6736	0.0014	2.0
4000	3.06406	0.00122	2.0	4000	14.6743	0.0007	2.0
8000	3.06432	0.00061	2.0	8000	14.6746	0.0003	2.3
16000	*Failed*	*Failed*	*Failed*	16000	*Failed*	*Failed*	*Failed*

Table 2.3: Lower penalty method ($l_{1/2}$) with smoothing interval $(0, 10^{-3})$.

$\sigma = 0.2$				$\sigma = 0.8$			
ξ_n	V_n	ΔV_n	R_n	ξ_n	V_n	ΔV_n	R_n
10	2.98285			10	14.5838		
20	3.03279	0.04994		20	14.6352	0.0792	
40	3.05569	0.02290	2.2	40	14.6630	0.0278	2.8
80	3.06229	0.00660	3.5	80	14.6719	0.0089	3.1
160	3.06400	0.00171	3.8	160	14.6742	0.0023	3.9
320	3.06443	0.00043	4.0	320	14.6747	0.0005	4.6
640	*Failed*	*Failed*	*Failed*	640	*Failed*	*Failed*	*Failed*

Table 2.4: 'Valley at zero' penalty function with smoothing interval $(0, 10^{-3})$.

$\sigma = 0.2$				$\sigma = 0.8$			
ξ_n	V_n	ΔV_n	R_n	ξ_n	V_n	ΔV_n	R_n
125	3.04804			125	14.6541		
250	3.05622	0.01797		250	14.6641	0.0100	
500	3.06039	0.00951	1.9	500	14.6695	0.0054	1.9
1000	3.06250	0.00488	1.9	1000	14.6722	0.0027	2.0
2000	3.06354	0.00246	2.0	2000	14.6736	0.0014	2.0
4000	3.06406	0.00122	2.0	4000	14.6743	0.0007	2.0
8000	3.06432	0.00061	2.0	8000	14.6746	0.0003	2.3
16000	$Failed$	$Failed$	$Failed$	16000	$Failed$	$Failed$	$Failed$

Table 2.5: Comparison of l_2 and $l_2 + l_{1/2}$ penalty methods with $\sigma = 0.2$ and smoothing interval $(0, 10^{-3})$.

$l_{1/2}$				$l_{1/2} + l_2$			
ξ_n	V_n	ΔV_n	R_n	ξ_n	V_n	ΔV_n	R_n
5	2.92458			5	2.94239		
10	2.98285	0.05827		10	2.99335	0.05096	
20	3.03279	0.04994	1.2	20	3.03498	0.04163	1.2
40	3.05569	0.02290	2.2	40	3.05582	0.02084	2.0
80	3.06229	0.00660	3.5	80	3.06229	0.00647	3.2
160	3.06400	0.00171	3.8	160	3.06400	0.00171	3.9
320	3.06443	0.00043	4.0	320	3.06433	0.00043	4.0
640	$Failed$	$Failed$	$Failed$	640	$Failed$	$Failed$	$Failed$

Table 2.6: Comparison of l_2 and $l_2 + l_{1/2}$ penalty methods with $\sigma = 0.8$ and smoothing interval $(0, 10^{-3})$.

$l_{1/2}$				$l_{1/2} + l_2$			
ξ_n	V_n	ΔV_n	R_n	ξ_n	V_n	ΔV_n	R_n
5	14.5216			5	14.5457		
10	14.5839	0.0623		10	14.5935	0.0478	
20	14.6352	0.0514	1.2	20	14.6378	0.0443	1.0
40	14.6630	0.0278	2.8	40	14.6632	0.0254	1.7
80	14.6719	0.0089	3.1	80	14.6719	0.0087	2.9
160	14.6742	0.0023	3.9	160	14.6742	0.0023	3.9
320	14.6747	0.0005	4.6	320	14.6747	0.0005	4.6
640	*Failed*	*Failed*	*Failed*	640	*Failed*	*Failed*	*Failed*

2.5.2 Comparison of Solution Times for The PSOR and Monotonic Penalty Methods

In this subsection, comparison of solution times for the PSOR method and monotonic penalty method is presented. As well, under various market parameters, the validity of the monotonic penalty method and the PSOR method is studied via numerical experiments.

Comparison of Solution Time

As we pointed out in the introduction, the PSOR method is commonly used in American option pricing because of its simplicity and robustness. In this section, from the numerical experiments, we can give a rough comparison of solution times of the monotonic penalty method and the PSOR method.

Example 2.5.1 *An American put option with the parameters: striking price $K = 100$, the interest rate $r = 0.1$, expiry date $T = 0.25$ and two different volatilities $\sigma = 0.2$ and* 0.8.

Table 2.7: Solution times of monotonic penalty method and PSOR method with different space steps

		$\sigma = 0.2$		$\sigma = 0.8$	
N_s	N_t	$T_{l_{1/2}+l_2}$	T_{PSOR}	$T_{l_{1/2}+l_2}$	T_{PSOR}
200	200	0.07005	0.01043	0.08750	0.26300
400	200	0.13047	0.01978	0.14765	0.13152
800	200	0.23255	0.05573	0.27995	0.86223
1600	200	0.46933	0.55182	0.56301	6.01483

Table 2.8: Solution times of monotonic penalty method and PSOR method with different time steps

		$\sigma = 0.2$		$\sigma = 0.8$	
N_s	N_t	$T_{l_{1/2}+l_2}$	T_{PSOR}	$T_{l_{1/2}+l_2}$	T_{PSOR}
800	100	0.16380	0.04167	0.15858	0.83438
800	200	0.26198	0.05598	0.27995	0.86233
800	400	0.42343	0.08412	0.51983	0.90027
800	800	0.72187	0.13802	0.94115	0.96355

In this example, the security maximum price is chosen to be $S_{\max} = 1000 = 10K$. Table 2.7 shows solution times of the PSOR method and monotonic penalty method under different space steps N_s and the same time steps N_t. Figure 2.1 presents the corresponding plots of solution times for these two methods as a function of space steps N_s.

Similarly, Table 2.8 and Figure 2.2 show results and graphs of the PSOR method and monotonic penalty method as the number of time steps N_t increases.

From Tables 2.7, 2.8 and Figures 2.1 and 2.2, we can draw some conclusions as below.

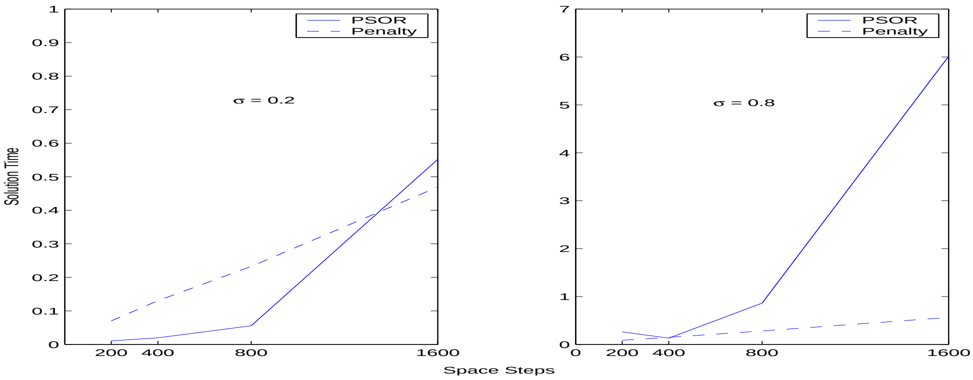

Figure 2.1: Solution times of PSOR and monotonic penalty method for different space steps

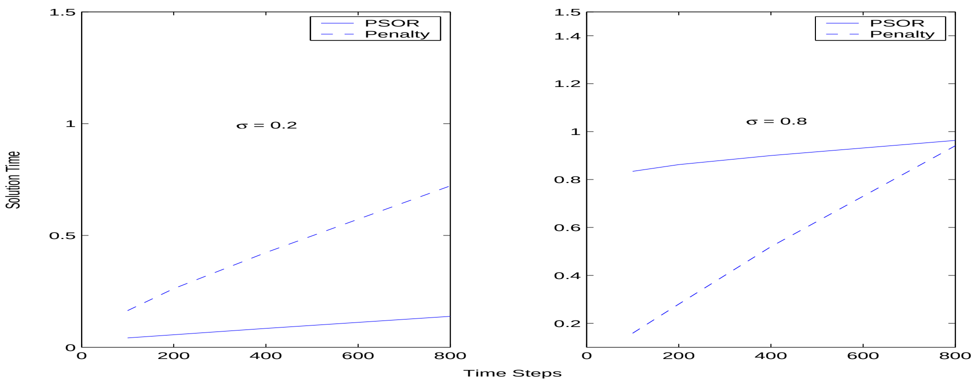

Figure 2.2: Solution times of PSOR and monotonic penalty method for different time steps

1. The monotonic penalty method $l_{1/2} + l_2$ gives almost linear computational time as a function of space steps N_s. On the other hand, the PSOR method exhibits exponential solution-time behavior as the number, N_s, of the space steps increases. For a smaller value of N_s, the PSOR method performs faster than the monotonic penalty method. However, as the number of the space steps increases, the effect on the time reserving of the monotonic penalty method becomes significantly enhanced.

2. For the monotonic penalty method, the computational time is linearly dependent on the number, N_t, of time steps. On the other hand, the PSOR method possesses an interesting property that the computational time is very flat as a function of N_t.

Validity of The Monotonic Penalty Method Under Different Market Parameters

For a standard American vanilla option, there are two parameters to be considered – the interest rate r and volatility σ. Here, we compare the solution times used for the PSOR and $l_2 + l_{1/2}$ penalty methods when the risk-free interest rate r is a constant taking, respectively, the value of $0.05, 0.10, 0.20, 0.40$ and 0.80 for each case, while the volatility σ is also a constant, taking, respectively, $0.10, 0.20, 0.40$ and 0.80. All other parameters are the same as those in Example 2.5.1.

Tables 2.9, 2.10 and Figure 2.3 show results and graphs of the PSOR method and monotonic penalty method as functions of market parameters σ and r.

From Tables 2.9, 2.10 and Figure 2.3, we can observe that

1. The monotonic penalty method is much more stable and robust when compared with the PSOR method. That is, the solution time of the monotonic penalty method appears to be independent of the market parameters σ and r.

2. For small σ and large r, the PSOR method needs less solution time than the monotonic penalty method. In contrast, the monotonic penalty method needs much less solution time than the PSOR method when σ is large and r is small.

Table 2.9: Solution times of PSOR method with different market parameters

	r			
σ	0.10	0.20	0.40	0.80
0.10	0.10052	0.09895	0.93483	0.97642
0.20	0.25157	0.24662	0.23438	0.90339
0.40	1.12005	1.07238	1.03125	0.92473
0.80	6.05442	6.02578	5.91875	5.65860

Table 2.10: Solution times of monotonic penalty method with different market parameters

	r			
σ	0.10	0.20	0.40	0.80
0.10	0.38750	0.34113	0.25417	0.28658
0.20	0.46442	0.41407	0.38098	0.34505
0.40	0.51772	0.47890	0.47448	0.42473
0.80	0.58152	0.55572	0.54272	0.51900

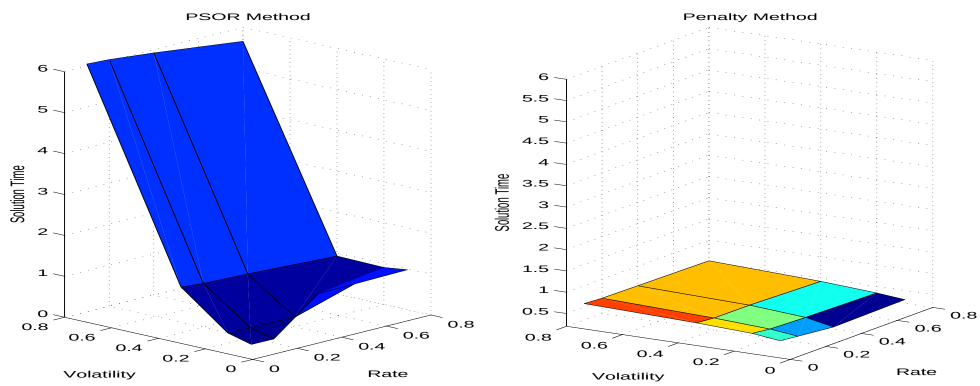

Figure 2.3: Solution times of PSOR and monotonic penalty methods for different market parameters

2.6 Summary

We have studied the monotonic penalty method for pricing American options. By using the equivalence between LCP and variational inequalities, the solvability and convergence properties of the monotonic penalty method were proposed. We have shown that the solution to the monotonic penalized nonlinear equation converges to that of the original LCP. A unified convergence rate of several monotonic penalty methods was established. Numerical experiments clearly confirmed the theoretical results obtained. The numerical results showed that the monotonic penalty method worked better than the PSOR method when the number of space steps is large or the parameter σ is small or r is large.

Chapter 3

A Power Penalty Method for American Option Pricing with Jump Diffusion Processes

In the original Black-Scholes analysis of the valuation of a plain option, constant volatility is assumed. Recently, empirical findings have shown that the standard Black-Scholes assumption of lognormal stock diffusion with constant volatility is not consistent with the market price. This phenomenon is often referred to as the volatility skew or smile [3] and exists in all the major stock index markets today. In order to capture the existence of volatility smiles, several extensions of the Black-Scholes model have been proposed. Generally speaking, three approaches are being studied: the stochastic volatility approach [64], the jump diffusion model approach [93, 135, 35] and the deterministic volatility function approach [34]. In [3], the advantages and disadvantages of these three approaches have been carefully studied and the jump diffusion model is being identified to be a more adaptable approach.

The jump diffusion model was introduced by Merton in [93]. Contrary to the Black-Scholes model [10], the stock price in jump diffusion model is not a continuous function of time. This allows to account for large changes in market prices due to rare events. More importantly, volatility curves obtained from the jump diffusion model exhibit behaviors similar to volatility smiles observed in markets.

In this chapter, we aim to develop a power penalty method to price American options arising from a jump diffusion model. Originally, this model is formulated as a partial integro-differential complementarity problem (PIDCP). With the help of the variational inequalities, we can transform the PIDCP into an equivalent variational form. Then, by applying the power penalty approach, it gives rise to a penalized nonlinear equation. We also study the convergence of the power penalty approach in some appropriate infinite dimensional spaces and show that a rate of convergence of $\mathcal{O}(\xi^{-1/2k})$ can be achieved, where ξ is the penalty parameter and k is the order of the power penalty function (l_k, $k > 0$). We also propose a numerical scheme to solving the penalized nonlinear equation, in which the finite element method is used in space and θ-scheme is used in time. More specifically, a θ_J-scheme time-stepping method is applied to the jump integral term. Finally, numerical examples are presented to illustrate the efficiency of the power penalty method.

3.1 Mathematical Model

Let S denote the underlying stock price. In the Merton model, the stock price is governed by the following stochastic differential equation (SDE)

$$\frac{dS}{S} = \mu dt + \sigma dW + \eta dN,$$

where

μ is the drift rate,

σ is the deterministic local volatility,

W is a standard Brownian motion,

N is a Possion process with intensity ν, $dN = \begin{cases} 0 & \text{with probability } 1 - \nu dt, \\ 1 & \text{with probability } \nu dt, \end{cases}$

ν is the mean arrival rate of jumps of the Possion process,

η is an impulse function producing a jump from S to $(1 + \eta)\, S$.

The Possion and Brownian motion processes are assumed to be uncorrelated. In the Merton model [93], η is taken to be a lognormally distributed jump amplitude with probability density

$$G(\eta) = \frac{1}{\sqrt{2\pi}\delta\eta} \exp\left(\frac{(\log\eta - \mu_\eta)^2}{2\delta^2}\right),$$

where μ_η and δ denote, respectively, the mean and variance of the jumps.

Let $V(S,t)$ represent the value of an American option with striking price K. If we define

$$LV = -\frac{\partial V}{\partial t} - \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} - (r - \nu\kappa)\, S\frac{\partial V}{\partial S} + (r+\xi)V - \left(\int_0^\infty V(S\eta)G(\eta)d\eta\right),$$

then the following partial integro-differential complementarity problem for the value of $V(S,t)$ is found in [35, 45, 118, 62]:

$$\begin{cases} LV(S,t) \geq 0, \\ V(S,t) - V^*(S) \geq 0, \\ LV(S,t)\cdot(V(S,t) - V^*(S)) = 0, \end{cases} \tag{3.1.1}$$

where

T	is the expiry date,
r	is the risk free rate,
t	is the current time with $0 \leq t \leq T$,
κ	is the expectation $E(\eta)$ which is given by

$$\kappa = E(\eta) = \int_0^\infty yG(y)dy = \exp\left(\mu_\eta + \delta^2/2\right) - 1.$$

In the case of a put option, the boundary conditions are:

$$V(0,t) = K, \text{ and } V(X,t) = 0, \tag{3.1.2}$$

where X is chosen so that $X >> K$, and the *payoff function* $V^*(S)$ is defined by

$$V^*(S) = V(S,T) = \max\{K - S, 0\}. \tag{3.1.3}$$

(3.1.1) – (3.1.3) is the original partial integro-differential complementarity problem. We define log-price of the underlying asset $x = \log(S)$, and use the changes of variables:

$$\tau = T - t,\ y = \log(\eta),\ \eta = e^y \text{ and } d\eta = e^y dy.$$

Also, for convenience, we transform (3.1.1) – (3.1.3) into an equivalent standard form satisfying homogeneous Dirichlet boundary conditions. For this, we define the following operator [89]

$$\begin{cases} A_{BS}(u) = -\dfrac{\sigma^2}{2}\dfrac{\partial^2 u}{\partial x^2} + \left(\dfrac{\sigma^2}{2} - r\right)\dfrac{\partial u}{\partial x} + ru, \\ A_j(u) = \nu\left(\kappa\dfrac{\partial u}{\partial x} + u - \displaystyle\int_{-\infty}^{\infty} u\,(x+y)\,g\,(y)\,dy\right) \\ \mathcal{L}(u) = \dfrac{\partial u}{\partial \tau} + A_{BS}(u) + A_j(u), \end{cases}$$

where

$$g\,(\eta) = \frac{1}{\sqrt{2\pi}\delta}\exp\left(\left[(y-\mu_\eta)\,/2\delta\right]^2\right).$$

Now, by introducing a new variable

$$U(x,\tau) = V(x,\tau) - \psi(x),$$

where $\psi(x) = V^*(e^x)$, we obtain the following complementarity problem which is equivalent to (3.1.1).

Problem 3.1.1

$$\begin{cases} U(x,\tau) \geq 0, \\ \mathcal{L}U(x,\tau) \geq f(x,\tau), \\ (\mathcal{L}U(x,\tau) - f(x,\tau)) \cdot U(x,\tau) = 0, \end{cases} \tag{3.1.4}$$

with the initial condition

$$U(x,0) = 0, \tag{3.1.5}$$

and the boundary conditions

$$U(-R,\tau) = 0 = U(R,\tau), \tag{3.1.6}$$

where the domain of U *is* $(-R,R)\times(0,T)$ *and*

$$f(x,\tau) = -A_{BS}(\psi) - A_j(\psi).$$

Remark 3.1.1 *It is worth noting that after the change of variables mentioned above, the solution region defined by (3.1.1) – (3.1.3) is changed as well. From the computational viewpoint, we should choose a large enough* R *so that the result obtained by solving Problem 3.1.1 in the region* $I = (-R,R)$ *is equivalent to that of (3.1.1) – (3.1.3) in* $(0,S)$.

In fact, the complementarity problem (3.1.1) – (3.1.3) can also be reformulated as an equivalent variational inequality problem. By introducing the following bilinear form

$$a_{BS}(u,v;\tau) = \frac{\sigma^2}{2}\int_I \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}dx + \left(\frac{\sigma^2}{2} - r\right)\int_I \frac{\partial u}{\partial x}vdx + r\int_I uvdx,$$

$$a_j\left(u,v;\tau\right) = \nu\kappa\int_I \frac{\partial u}{\partial x}vdx + \nu\int_I uvdx - \nu\int_I\int_{-\infty}^{\infty} u(x+y)g(y)v(x)dydx,$$

we obtain the equivalent variational inequality problem given by

Problem 3.1.2 *Find $U \in L^2((0,T);H_0^1(I))$, $U_\tau \in L^2(\Omega)$, such that $U(\cdot,\tau) \in \mathcal{K}$, and for all $v \in \mathcal{K}$,*

$$\left(\frac{\partial U}{\partial \tau}, v - U\right) + a_{BS}(U, v-U)) + a_j(U, v-U) \geq (f, v-U), \;\; a.e. \;\; in \;\; (0,T), \quad (3.1.7)$$

where

$$\mathcal{K} = \left\{v(\tau) \in H_0^1(I) : v(\tau) \geq 0\right\}.$$

Theorem 3.1.1 *Problem 3.1.2 is the variational form of Problem 3.1.1.*

The proof of Theorem 3.1.1 is similar to that given for Lemma 2.1.2, and hence is omitted here.

The solvability of Problem 3.1.2 lies on the properties of the operator $a_{BS}(u,v;\tau)$ and $a_j(u,v;\tau)$. In [89], the properties of the operators $a_{BS}(u,v;\tau)$ and $a_j(u,v;\tau)$ have been extensively studied. The following lemma presents some useful properties which are required later in this chapter.

Lemma 3.1.1 *[89] There exist positive constants C, M_1, and M_2, such that, for all ϕ, $\psi \in H_0^1(I)$,*

$$a_j(\phi,\phi;\tau) \geq 0, \quad (3.1.8)$$

$$a_{BS}(\phi,\phi;\tau) \geq C\sigma^2 \left\|\phi\right\|^2_{H_0^1(I)}, \quad (3.1.9)$$

and

$$|a_j(\phi,\psi;\tau)| \leq M_1 \left\|\phi\right\|_{H_0^1(I)} \left\|\psi\right\|_{H_0^1(I)},$$

$$|a_{BS}(\phi,\psi;\tau)| \leq M_2 \left\|\phi\right\|_{H_0^1(I)} \left\|\psi\right\|_{H_0^1(I)}.$$

The proof of Lemma 3.1.1 can be found in [90]. Based on this lemma, the following theorem is easily obtained.

Theorem 3.1.2 *Variational inequality (3.1.7) has a unique solution, and it holds that*

$$U, U_\tau \in L^2(0,T;H_0^1(I)) \cap L^\infty(0,T;L^2(I)).$$

For the proof of Theorem 3.1.2, see [53].

Remark 3.1.2 *If we set $\xi = 0$ in the operator $\mathcal{L}$, then $\mathcal{L}$ is degenerated to the standard Black-Scholes model with constant volatility assumption. Obviously, the analysis given above can be carried over to this case.*

In the next section, we will develop a power penalty approach to solving Problem 3.1.1.

3.2 The Power Penalty Approach

The power penalty approach to Problem 3.1.1 is stated as follows:

Problem 3.2.1

$$\mathcal{L}U_\xi(x,t) - \xi\,[-U_\xi]_+^k = f(x,t), \quad (x,t) \in \Omega \tag{3.2.10}$$

with the initial condition and boundary conditions

$$U_\xi(x,0) = 0,$$
$$U_\xi(-R,\tau) = 0 = U_\xi(R,\tau),$$

where $\xi > 0$ is the penalty parameter and $[-U_\xi]_+^k$ is the kth ($k > 0$) order penalty function defined by

$$[-U_\xi]_+^k = [\max\{-U_\xi(x,t), 0\}]^k .$$

Remark 3.2.1 *Corresponding to $k = 1$, $0 < k < 1$ and $k > 1$, are, respectively, the l_1, lower order and higher order penalty approaches. For example, $k = 2$ is the quadratic penalty method, $k = 1$ is the linear penalty method and $k = 1/2$ is the $1/2$th order penalty method.*

The variational form of Problem 3.2.1 is given below.

Problem 3.2.2 *Find $U_\xi \in L^2((0,T);H_0^1(I))$, $\frac{\partial U_\xi}{\partial \tau} \in L^2(\Omega)$, such that, for all $v \in \mathcal{K}$,*

$$\left(\frac{\partial U_\xi}{\partial \tau}, v\right) + a_{BS}(U_\xi, v;\tau) + a_j(U_\xi, v;\tau) - \xi\left([-U_\xi]_+^k, v;\tau\right) = (f, v), \quad \text{a.e. in } (0,T). \tag{3.2.11}$$

For Problem 3.2.1 (or Problem 3.2.2), we have the following unique solvability result.

Theorem 3.2.1 *Problem 3.2.1 (or Problem 3.2.2) has a unique solution.*

Proof. For any $v_1, v_2 \in H_0^1(I)$ a.e. in $(0,T)$ with the initial conditions $v_1(x,0) = 0$ and $v_2(x,0) = 0$, it follows from the integration by parts that

$$\begin{aligned}
&(\mathcal{L}(v_1 - v_2), v_1 - v_2) - \xi\left([-v_1]_+^k - [-v_2]_+^k, v_1 - v_2\right) \\
=& \left(\frac{\partial (v_1 - v_2)}{\partial \tau}, v_1 - v_2\right) + a_{BS}(v_1 - v_2, v_1 - v_2) \\
&+ a_j(v_1 - v_2, v_1 - v_2) + \xi\left([-v_2]_+^k - [-v_1]_+^k, v_1 - v_2\right).
\end{aligned} \tag{3.2.12}$$

First, consider the function $[-v]_+^k$. By definition, it is equal to $[\max\{-v, 0\}]^k$. Obviously, this function is non-decreasing in v, thus

$$\xi\left([-v_2]_+^k - [-v_1]_+^k, v_1 - v_2\right) = \xi\int_{-R}^{R}\left([-v_2]_+^k - [-v_1]_+^k\right)(v_1 - v_2)\,dx \geq 0. \tag{3.2.13}$$

Denote $e(\tau) = v_1(\tau) - v_2(\tau)$. Integrating both sides of (3.2.12) from 0 to T, and using the inequalities (3.1.8), (3.1.9) and (3.2.13), we have

$$\begin{aligned}
&\int_0^T \left[(\mathcal{L}e(\tau), e(\tau)) - \xi\left([-v_1(\tau)]_+^k - [-v_2(\tau)]_+^k, e(\tau)\right)\right] d\tau \\
=& \int_0^T \left(\frac{\partial e(\tau)}{\partial \tau}, e(\tau)\right) + a_{BS}(e, e;\tau) + a_j(e, e;\tau) + \xi\left([-v_2]_+^k - [-v_1]_+^k, e(\tau)\right) d\tau \\
\geq& \int_0^T \left(\frac{\partial e(\tau)}{\partial \tau}, e(\tau)\right) d\tau + C\int_0^T \|e(\tau)\|_{H^1(I)}\,d\tau.
\end{aligned} \tag{3.2.14}$$

However, for any $t \in (0,T)$, integrating by parts gives

$$\int_0^t \left(\frac{\partial e(\tau)}{\partial \tau}, e(\tau) \right) d\tau = (e(\tau), e(\tau)) - \int_0^t \left(\frac{\partial e(\tau)}{\partial \tau}, e(\tau) \right) d\tau.$$

Taking $e(0) = v_1(x,0) - v_2(x,0) = 0$, we have

$$\int_0^t \left(\frac{\partial e(\tau)}{\partial \tau}, e(\tau) \right) d\tau = \frac{1}{2} (e(\tau), e(\tau)) \geq 0. \tag{3.2.15}$$

Therefore, from (3.2.14), (3.2.15) and (3.2.12), we get

$$\int_0^T \left[\left(\mathcal{L} \left(v_1 - v_2 \right), v_1 - v_2 \right) - \xi \left([-v_1]_+^k - [-v_2]_+^k, v_1 - v_2 \right) \right] d\tau$$
$$\geq C \left\| v_1 - v_2 \right\|_{L^2(0,T;H_0^1(I))}^2 .$$

This implies that the operator on the left-hand side of (3.2.10) is strictly monotone.

Moreover, for any v, $w \in L^2(0,T;H_0^1(I))$, it is easy to show that

$$\int_0^T \left(a_{BS}(v,w;\tau) + a_j(v,w;\tau) \right) d\tau \leq C \left\| v \right\|_{L^2(0,T;H_0^1(I))} \left\| w \right\|_{L^2(0,T;H_0^1(I))},$$

which means that the operator $a_{BS}(v,w;\tau) + a_j(v,w;\tau)$ is continuous in both v and w. Also, it is obvious that $([v]_+^k, w)$ is continuous in both v and w. Furthermore, it can be easily seen that the function $f(x,t)$ in (3.2.10) belongs to the space $L^2(\Omega)$. Therefore, using a standard result in [54], we can conclude that Problem 3.2.1 (or Problem 3.2.2) is uniquely solvable. ■

Many results on the regularity of the solution to Problem 3.2.1 are now available in the literature, see, for example [54, 28, 8]. In particular, the solution U_ξ to Problem 3.2.1 is such that

$$U, \frac{\partial U}{\partial \tau} \in L^2 \left(0, T; H_0^1 \left(I \right) \right) \cap L^\infty \left(0, T; L^2 \left(I \right) \right).$$

3.3 Convergence Analysis of The Power Penalization

In this section, we will show that the solution to Problem 3.2.1 converges to that of Problem 3.1.2 as $\xi \to \infty$ with the rate of order $\mathcal{O}(\xi^{-1/2k})$ in some appropriate norms. We begin this discussion with the following lemma.

Lemma 3.3.1 *Let U_ξ be the solution to Problem (3.2.1). If $U_\xi \in L^p(\Omega)$, then there exists a positive constant $C > 0$, independent of U_ξ and ξ, such that*

$$\left\|[-U_\xi]_+\right\|_{L^p(\Omega)} \leq \frac{C}{\xi^{1/k}}, \tag{3.3.16}$$

$$\left\|[-U_\xi]_+\right\|_{L^\infty(0,T;L^2(I))} + \left\|[-U_\xi]_+\right\|_{L^2(0,T;H_0^1(I))} \leq \frac{C}{\xi^{1/2k}}, \tag{3.3.17}$$

where k is the order of the power penalty function used in (3.2.10) and $p = 1 + k$.

Proof. Assume that C is a generic positive constant, independent of U_ξ and ξ. Let $\phi(\tau) = [-U_\xi(\tau)]_+ \in H_0^1(I)$ for almost all $\tau \in (0, T)$.

Now, setting $v = \phi$ in (3.2.11) , we have

$$\left(\frac{\partial U_\xi}{\partial \tau}, \phi\right) + a_{BS}(U_\xi, \phi; \tau) + a_j(U_\xi, v; \tau) - \xi\left([-U_\xi]_+^k, \phi\right) = (f, \phi), \quad \text{a.e. in } (0, T). \tag{3.3.18}$$

As $\phi(\tau) = [-U_\xi(\tau)]_+$, (3.3.18) can be reformulated as:

$$\left(\frac{\partial \phi}{\partial \tau}, \phi\right) + a_{BS}(\phi, \phi; \tau) + a_j(\phi, \phi; \tau) + \xi\left(\phi^k, \phi\right) = (-f, \phi), \quad \text{a.e. in } (0, T).$$

Integrating from 0 to t, we get

$$\frac{1}{2}(\phi, \phi) + \int_0^t a_{BS}(\phi, \phi; \tau) d\tau + \int_0^t a_j(\phi, \phi; \tau) d\tau + \xi \int_0^t \left(\phi^k, \phi\right) d\tau = \int_0^t (-f, \phi) d\tau.$$

Using (3.1.8), (3.1.9), the fact that $f \in L^2(\Omega)$ and Hölder's inequality, we obtain

$$\frac{1}{2}(\phi, \phi) + C \int_0^t \|\phi\|_{H_0^1(I)}^2 d\tau + \xi \int_0^t \|\phi\|_{L^p(I)}^p d\tau \leq C \int_0^t \|\phi\|_{L^p(I)}^p d\tau. \tag{3.3.19}$$

From the above inequality, we can easily see that

$$\xi \int_0^t \|\phi\|_{L^p(I)}^p d\tau \leq C \int_0^t \|\phi\|_{L^p(I)}^p d\tau,$$

or

$$\left(\int_0^t \|\phi\|_{L^p(I)}^p d\tau\right)^{1-1/p} \leq C\xi^{-1}.$$

From this, it follows that

$$\left(\int_0^t \|\phi\|_{L^p(I)}^p d\tau\right)^{1/p} \leq C\xi^{\frac{-1}{p(1-1/p)}} = \frac{C}{\xi^{1/k}}, \tag{3.3.20}$$

since $p = 1 + k$. This proves (3.3.16).

Now, from (3.3.19) and (3.3.20), we have

$$\frac{1}{2}(\phi,\phi) + \int_0^t \|\phi\|^2_{H_0^1(I)}\,d\tau \le C\int_0^t \|\phi\|^p_{L^p(I)}\,d\tau \le \frac{C}{\xi^{1/k}},$$

from which, it follows that

$$(\phi,\phi)^{1/2} + \left(\int_0^t \|\phi\|^2_{H_0^1(I)}\,d\tau\right)^{1/2} \le \frac{C}{\xi^{1/2k}},$$

for all $t \in (0,T)$. This inequality, in turn, implies that

$$\left\|[-U_\xi]_+\right\|_{L^\infty(0,T;L^2(I))} + \|[-U_\xi]_+\|_{L^2(0,T;H_0^1(I))} \le \frac{C}{\xi^{1/2k}}.$$

■

On the basis of Lemma 3.3.1, we next show that the solution to Problem 3.2.1 converges to that of Problem 3.1.2 as $\xi \to \infty$ with the convergence rate of order $O(\xi^{-1/2k})$ in some appropriate norms.

Theorem 3.3.1 *Suppose that the assumptions in Lemma 3.3.1 are fulfilled, and let U and U_ξ be the solutions to Problem 3.1.2 and Problem 3.2.1, respectively. If $\frac{\partial U}{\partial \tau} \in L^{1/k+1}(\Omega)$, then there exists a constant $C > 0$, independent of ξ, U, U_ξ, such that*

$$\|U - U_\xi\|_{L^\infty(0,T;L^2(I))} + \|U - U_\xi\|_{L^2(0,T;H_0^1(I))} \le \frac{C}{\xi^{1/2k}}, \tag{3.3.21}$$

where k is the order of the power penalty function used in (3.2.10).

Proof. We follow the notation used in Lemma 3.3.1. Now, we decompose $U - U_\xi$ as:

$$U - U_\xi = U + [-U_\xi]_+ - [-U_\xi]_- = \phi + U - [-U_\xi]_- \triangleq \phi + R_\xi, \tag{3.3.22}$$

where

$$\begin{aligned} \phi &= [-U_\xi]_+\,, \\ [-U_\xi]_- &= -\min\{-U_\xi, 0\}, \\ R_\xi &= U - [-U_\xi]_-\,. \end{aligned}$$

Setting $v = U - R_\xi$ in (3.1.7) and $v = R_\xi$ in (3.2.11), we have, respectively,

$$\left(\tfrac{\partial U}{\partial \tau}, -R_\xi\right) + a_{BS}(U, -R_\xi;\tau)) + a_j(U, -R_\xi;\tau) \ge (f, -R_\xi), \tag{3.3.23}$$

$$\left(\tfrac{\partial U_\xi}{\partial \tau}, R_\xi\right) + a_{BS}(U_\xi, R_\xi;\tau) + a_j(U_\xi, R_\xi;\tau) - \xi\left(\phi^k, R_\xi\right) = (f, R_\xi). \tag{3.3.24}$$

Adding up (3.3.23) and (3.3.24) gives

$$\left(\frac{\partial\left(U_\xi-U\right)}{\partial\tau}, R_\xi\right) + a_{BS}(U_\xi - U, R_\xi;\tau) + a_j(U_\xi - U, R_\xi;\tau) - \xi\left(\phi^k, R_\xi\right) \geq 0. \quad (3.3.25)$$

However,

$$\left(\phi^k, R_\xi\right) = \left(\phi^k, U - [-U_\xi]_-\right) = \left(\phi^k, U\right) - \left(\phi^k, [-U_\xi]_-\right) = \left(\phi^k, U\right) \geq 0, \quad (3.3.26)$$

since $U \geq 0$ ($U \in \mathcal{K}$) and $\phi^k \geq 0$. Thus, from (3.3.25) and (3.3.26), we obtain

$$\left(\frac{\partial\left(U_\xi-U\right)}{\partial\tau}, R_\xi\right) + a_{BS}(U_\xi - U, R_\xi;\tau) + a_j(U_\xi - U, R_\xi;\tau) \geq 0,$$

or

$$\left(\frac{\partial\left(U-U_\xi\right)}{\partial\tau}, R_\xi\right) + a_{BS}(U - U_\xi, R_\xi;\tau) + a_j(U - U_\xi, R_\xi;\tau) \leq 0. \quad (3.3.27)$$

Using (3.3.22), (3.3.27) becomes

$$\left(\frac{\partial R_\xi}{\partial\tau}, R_\xi\right) + a_{BS}(R_\xi, R_\xi;\tau) + a_j(R_\xi, R_\xi;\tau) \leq \left(-\frac{\partial\phi}{\partial\tau}, R_\xi\right) - a_{BS}(\phi, R_\xi;\tau) - a_j(\phi, R_\xi;\tau).$$

Integrating both sides of the above inequality from 0 to t, and using Cauchy-Schwartz's inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}(R_\xi, R_\xi) + \int_0^t a_{BS}(R_\xi, R_\xi;\tau)d\tau + \int_0^t a_j(R_\xi, R_\xi;\tau)d\tau \\
&\leq (-\phi, R_\xi) + \int_0^t \left(-\phi, \frac{\partial R_\xi}{\partial\tau}\right)d\tau + \int_0^t \left(a_{BS}(-\phi, R_\xi;\tau) + a_j(-\phi, R_\xi;\tau)\right)d\tau \\
&\leq \|\phi\|_{L^\infty(0,T;L^2(I))}\|R_\xi\|_{L^\infty(0,T;L^2(I))} + C\|\phi\|_{L^2(0,T;H_0^1(I))}\|r_\xi\|_{L^2(0,T;H_0^1(I))} \\
&\quad + \int_0^t \left(-\phi, \frac{\partial R_\xi}{\partial\tau}\right)d\tau, \quad (3.3.28)
\end{aligned}$$

for all $t \in (0,T)$. Since $(\phi, [-U_\xi]_-) = 0$ for almost all $t \in (0,T)$, it follows from (3.3.16) that

$$\int_0^t \left(-\phi, \frac{\partial R_\xi}{\partial\tau}\right)d\tau = \int_0^t \left(-\phi, \frac{\partial U}{\partial\tau}\right)d\tau \leq C\|\phi\|_{L^p(\Omega)}\left\|\frac{\partial U}{\partial t}\right\|_{L^q(\Omega)} \leq \frac{C}{\xi^{1/k}}, \quad (3.3.29)$$

where $p = 1+k$ and $q = 1/k+1$. Substituting (3.3.29) into (3.3.28), and using (3.1.8), (3.1.9) and (3.3.17), we get

$$\begin{aligned}
&\left(\|R_\xi\|_{L^\infty(0,T;L^2(I))} + \|R_\xi\|_{L^2(0,T;H_0^1(I))}\right)^2 \\
&\leq C\left(\frac{1}{2}\|R_\xi\|^2_{L^\infty(0,T;L^2(I))} + \|r_\xi\|^2_{L^2(0,T;H_0^1(I))}\right) \\
&\leq C\left(\|\phi\|^2_{L^\infty(0,T;L^2(I))} + \|\phi\|^2_{L^2(0,T;H_0^1(I))}\right)\cdot\left(\|R_\xi\|^2_{L^\infty(0,T;L^2(I))} + \|R_\xi\|^2_{L^2(0,T;H_0^1(I))} + \xi^{-1/k}\right) \\
&\leq C\left[\xi^{-1/2k}\left(\|R_\xi\|_{L^\infty(0,T;L^2(I))} + \|R_\xi\|_{L^2(0,T;H_0^1(I))}\right) + \xi^{-1/k}\right]. \quad (3.3.30)
\end{aligned}$$

This is in the form of

$$y^2 \leq C\rho^{1/2} y + C\rho,$$

which can be rewritten as:

$$\left(y - \frac{1}{2}C\rho^{1/2}\right)^2 \leq \left(C + \frac{C^2}{4}\right)\rho.$$

Clearly, it implies that

$$y \leq C\rho^{1/2}.$$

Replacing y with $\|R_\xi\|_{L^\infty(0,T;L^2(I))} + \|R_\xi\|_{L^2(0,T;H_0^1;(I))}$ and ρ with $\xi^{-1/k}$, it follows from (3.3.30) that

$$\|R_\xi\|_{L^\infty(0,T;L^2(I))} + \|R_\xi\|_{L^2(0,T;H_0^1(I))} \leq \frac{C}{\xi^{\frac{1}{2k}}}.$$

Using the triangle inequality, the above inequality and (3.3.17), we obtain

$$\begin{aligned}
&\|U - U_\xi\|_{L^\infty(0,T;L^2(I))} + \|U - U_\xi\|_{L^2(0,T;H_0^1(I))} \\
\leq &\left(\|R_\xi\|_{L^\infty(0,T;L^2(I))} + \|R_\xi\|_{L^2(0,T;H_0^1(I))}\right) + \left(\|\phi\|_{L^\infty(0,T;L^2(I))} + \|\phi\|_{L^2(0,T;H_0^1(I))}\right) \\
\leq &\frac{C}{\xi^{1/2k}}.
\end{aligned}$$

■

Remark 3.3.1 *It is worth noting that if we set $k = 1$ in (3.3.21), then we can show that the convergent rate of the l_1 penalty method is $O(\xi^{-1/2})$. This result is consistent with that obtained in [8]. When $0 < k < 1$, a faster convergent rate of order $O(\xi^{-1/2k})$ is achieved. This is consistent with the conclusion made in the study of minimization problem studied in [111]. With these, we can conclude that with the same level of accuracy, the lower order power penalty approach $l_k\,(0 < k < 1)$ needs much less penalty parameter than the l_1 penalty approach [35, 45, 89]. In other words, by using the $l_k\,(0 < k < 1)$ penalty method, we can achieve a much higher level of accuracy with a small penalty parameter.*

3.4 Numerical Scheme for The Penalized Equation

In this section, we concentrate on the numerical solution to the penalized equation (3.2.10). This is a nonlinear parabolic partial integro-differential equation. For its numerical solution, we first discretize it in time using the so-called θ-scheme and in space by a standard linear finite element method. This yields a nonlinear algebraic system. Then, we will give a solution algorithm for nonlinear algebraic system.

Let the space interval $I = (-R, R)$ be divided into $N+1$ sub-intervals

$$I_i = (x_i, x_{i+1}), \quad i = 1, 2, \ldots, N+1$$

with $-R = x_1 < x_2 < \ldots < x_{N+1} = R$. We denote this partition as $\mathcal{T}$ and $h_i = x_{i+1} - x_i$. Let V_h be the space of continuous piecewise linear functions with respect to the partition $\mathcal{T}$. In the following, we omit the subscript ξ of U_ξ when it causes no confusion in doing so. The semi-discretization form of (3.2.10) reads:

$$\begin{cases} \text{Find } U_h \in L^2(0,T;V_h), \text{ such that for all } v_h \in V_h, \\ \dfrac{d}{d\tau}(U_h, v_h) + a_{BS}(U_h, v_h;\tau) + a_j(U_h, v_h;\tau) - \xi([-U_h]_+^k, v_h) = (f, v_h), \\ U_1 = 0 = U_{N+1} \quad \text{and} \quad U_h(x,0) = 0, \ a.e. \text{ in } (0,T), \end{cases}$$

Moreover, for computational convenience, we apply the piecewise constant element for the nonlinear penalty term.

Let $s = T/M$ with $M \in \mathbb{N}$ be the number of time steps. Let us denote by $U^m, m = 0, 1, \ldots, M$, the solution to the following backward Euler discretization of (3.2.11):

$$\begin{cases} \text{Find } U_h^{m+1} \in V_h, \ m = 0, 1, \ \ldots, \ M, \text{ such that for all } v_h \in V_h, \\ (\frac{U_h^{m+1} - U_h^m}{s}, v_h) + a_{BS}(U_h^{m+\theta}, v_h;\tau) + a_j(U_h^{m+\theta_j}, v_h;\tau) - \xi([-U_h^{m+\theta}]_+^k, v_h) = (f^m, v_h), \\ U_1^0 = 0 = U_{N+1}^0(R,\tau) \quad \text{and} \quad U_h^0(x,0) = 0, \ a.e. \text{ in } (0,T). \end{cases} \tag{3.4.31}$$

Here,

$$U^{m+\theta} := \theta U^{m+1} + (1-\theta) U^m,$$

and similarly

$$U^{m+\theta_J} := \theta_J U^{m+1} + (1-\theta_J) U^m,$$

with $0 \leq \theta, \ \theta_J \leq 1$.

The sequence of finite dimensional variational equations (3.4.31) corresponds to a sequence of matrix form systems. Specifically, let $\mathcal{B} = \{b_j\}_{1 \leq j \leq N}$ be a basis for V_h, i.e. $V_h = span(\mathcal{B})$. Let $\mathbf{M}$ denote the mass matrix with respect to $\mathcal{B}$, and let $\mathbf{A}_{BS}$ and $\mathbf{A}_j$

denote the stiffness matrices of $a_{BS}(\cdot,\cdot)$ and $a_j(\cdot,\cdot)$ with respect to $\mathcal{B}$, respectively, i.e.

$$\begin{cases} \mathbf{M}_{i,j} = (b_i, b_j)_{L^2(I)}, \\ \mathbf{A}_{BS} = a_{BS}(b_i, b_j), \\ \mathbf{A}_{BS} = a_j(b_i, b_j). \end{cases}$$

Also, let $\mathbf{F}^m$ be the load vector defined by

$$\mathbf{F}_j^m = (f^m, b_j)_{L^2(I)}\,.$$

Therefore, the matrix form of (3.4.31) is written as:

$$\begin{cases} \text{Find } \underline{U}^{m+1} \in V_h,\ m = 0, 1,\ \ldots,\ M, \text{ such that} \\ \frac{1}{s}\mathbf{M}\left(\underline{U}^{m+1} - \underline{U}^m\right) + \mathbf{A}_{BS}\underline{U}^{m+\theta} + \mathbf{A}_j\underline{U}^{m+\theta_j} - \xi\mathbf{D}(\underline{U}^{m+\theta}) = \mathbf{F}^m, \\ \underline{U}_1^0 = 0 = \underline{U}_{N+1}^0(R,\tau), \quad \text{and} \quad \underline{U}_h^0(x,0) = 0, \end{cases}$$

where $\underline{U}^m$ is the coefficient vector of U_h^m with respect to $\mathcal{B}$ and

$$\mathbf{D}(\underline{U}^{m+\theta}) = (h_1[d_1]_+^k, \cdots,\ h_i[d_i]_+^k, \cdots,\ h_{N+1}[d_{N+1}]_+^k)^T, \tag{3.4.32}$$

with $d_i = -(\underline{U}^{m+\theta})_i$.

Remark 3.4.1 *Due to the non-local property of the operator A_j, the matrix $\mathbf{A}_j$ is fully filled. In fact, it is a Toeplitz matrix. Hence, we can use the fast Fourier transform method to calculate it. In this way, the computation time will be significantly reduced [45, 118].*

If we set $\theta = \theta_j = \frac{1}{2}$ in the time stepping scheme, we obtain the Crank-Nicolson method. Noting that the operator

$$a_j(u,v) = \nu k \int_I u_x v dx + \nu \int_I uv dx - \nu \int_I \int_{-\infty}^{\infty} u(x+y)g(y)v(x)dydx$$

is not a local operator, we can see that the stiffness matrix $\mathbf{A}_j(\underline{U}^{m+\theta_j})$ is a dense one. The computation of the non-local operator is time-consuming. So, we use the so-called 'lagging' technique which is recommended by Tavella in [118]. That is, we can use the Crank-Nicolson time differential in the PDE part of the equation $a(u,v)$, and just evaluate the integral term at the 'old' (lagged) time step:

$$\frac{1}{s}\mathbf{M}\left(\underline{U}^{m+1} - \underline{U}^m\right) + \mathbf{A}_{BS}\underline{U}^{m+\theta} - \xi\mathbf{D}(\underline{U}^{m+\theta}) = \mathbf{F}^m - \mathbf{A}_j\underline{U}^m. \tag{3.4.33}$$

Now the unknown values at the new time step $\underline{U}^{m+1}$ are coupled only by the partial differential operators, leading to a sparse system of discrete equations. The convolution term couples $\underline{U}$ over a wide range of x, but these are all known values.

Now, we can use the classical Newton method to solve the nonlinear equation (3.4.33). First, we reformulate (3.4.33) as:

$$\left[\frac{1}{s}\mathbf{M}+\theta\mathbf{A}_{BS}\right]\underline{U}^{m+1}-\xi\mathbf{D}(\theta\underline{U}^{m+1}+(1-\theta)\underline{U}^{m}) = \mathbf{F}^{m}-\left[\mathbf{A}_j+(1-\theta)\mathbf{A}_{BS}\right]\underline{U}^{m}. \quad (3.4.34)$$

Applying the damped Newton method to (3.4.34) gives

$$\begin{aligned}\left[\frac{1}{s}\mathbf{M}+\theta\mathbf{A}_{BS}-\theta\xi\mathbf{J}_D(\varpi^{l-1})\right]\delta\varpi^{l} &= \mathbf{F}^{m}-\left[\mathbf{A}_j+(1-\theta)\,\mathbf{A}_{BS}\right]\underline{U}^{m}+(1-\theta)\,\xi\mathbf{D}(\underline{U}^{m})\\ &\quad -\left[s\mathbf{M}+\theta\mathbf{A}_{BS}\right]\varpi^{l-1}+\theta\xi\mathbf{D}(\varpi^{l-1}),\\ \varpi^{l} &= \varpi^{l-1}+\zeta\cdot\delta\varpi^{l},\end{aligned}$$

for $l=1,2,\ldots$, with ϖ^0 a given initial guess, where $\mathbf{J}_D(\varpi^{l-1})$ denotes the Jacobian of the column vector $\mathbf{D}(\varpi)$, and $\zeta\in(0,1]$ denotes a damping parameter. We then choose

$$\underline{U}^{m+1}=\lim_{l\to\infty}\varpi^{l}.$$

It is worth noting that when taking $0<k\leq 1$ with the l_k penalty function, we can see from (3.4.32) that $([-d_i]_+^k)'\to\infty$ as $(-d)_i\to 0^+$. Hence, the Jacobian $\mathbf{J}_D(\varpi^{l-1})$ is singular. In order to overcome this difficulty, we use the smoothing technique, as given in Appendix 7.2, to smooth out the term $([-d_i]_+^k)$.

Now, we can give the solution algorithm as follows:

Penalty Iteration

Let $(\underline{U}^{m+1})^0=\underline{U}^m$

Let $(\widetilde{\underline{U}})^j=(\underline{U}^{m+1})^j$

Let $(\widetilde{D})^j=D((\underline{U}^{m+1})^j)$

Loop:

For $j=0,1,2,\cdots$ until convergence

Solve

$$\frac{1}{s}\mathbf{M}\left(\underline{U}^{m+1}-\underline{U}^{m}\right)+\mathbf{A}_{BS}\left(\underline{U}^{m+\theta}\right)+\mathbf{A}_j\left(\underline{U}^{m+\theta_j}\right)-\xi\mathbf{I}\left[-\underline{U}\right]_+^k=\mathbf{F},$$
$$\underline{U}^0=0,$$

End

3.5 Numerical Experiments

In this section, we present two examples to illustrate the performance and convergence of the power penalty approach. The first example is the American put option, and the second is the American butterfly option. For these two types of options, we compare the penalty parameters of the l_1 penalty method and the lower order penalty method ($l_{1/2}$) under several different discretization schemes.

3.5.1 American Put Option Example

Put options are contingent contracts with payoff at maturity

$$V^*(S) = (K - S)_+ = \max\{K - S, 0\},$$

where K and S are striking price and asset price, respectively.

Table 3.1: Market Parameters used in American put option examples.

T	0.25
ν	0.10
δ	0.45
μ_η	-0.9
σ	0.15
r	0.05
K	100
B-S Implied Volatility	0.1886

We take the example which is used in [35], where the market parameters are listed in Table 3.1. The computational results are listed in Tables 3.2, 3.3 and 3.4. In these tables, 'Pen. Param.' means the penalty parameters. 'Aver. Iterations' means the average number of iterations required in the penalty method during every time step. 'Comp. Time' is the CPU time of the computation. All the numerical results were computed with the Matlab code on a Pentium IV PC. Finally, we depict the option value in Figure 3.1, which is obtained by $l_{1/2}$ penalty method with the discretization: the number of time steps = 100 and the number of space steps = 256.

Table 3.2: Computed results of American put option in the Merton model with $S = 1.0$ and the discretization: the number of time steps = 50 and the number of space steps = 128.

	l_1 penalty method				$l_{1/2}$ penalty method			
Pen. Param.	10^3	10^6	10^8	10^{13}	20	200	2000	20000
Option Value	0.03436	0.03435	0.03435	0.03435	0.03435	0.03435	0.03435	0.03435
Free Boundary	0.91051	0.91051	0.91051	0.91051	0.91051	0.91051	0.91051	0.91051
Aver. Iteration	$13-15$	$12-13$	$12-13$	$12-13$	$12-13$	$13-15$	$12-13$	$12-13$
Comp. Time	13	10	10	10	12	10	10	10

Table 3.3: Computed results of American put option in the Merton model with $S = 1.0$ and the discretization: the number of time steps = 50 and the number of space steps = 256.

	l_1 penalty method				$l_{1/2}$ penalty method			
Pen. Param.	10^4	10^6	10^{10}	10^{14}	80	8000	80000	800000
Option Value	0.03226	0.03225	0.03225	0.03225	0.03226	0.03225	0.03225	0.03225
Free Boundary	0.94678	0.94678	0.94678	0.94678	0.94678	0.94678	0.94678	0.94678
Aver. Iteration	$15-17$	$12-13$	$12-13$	$12-13$	$13-14$	$12-13$	$12-13$	$12-13$
Comp. Time	24	21	21	21	26	21	21	21

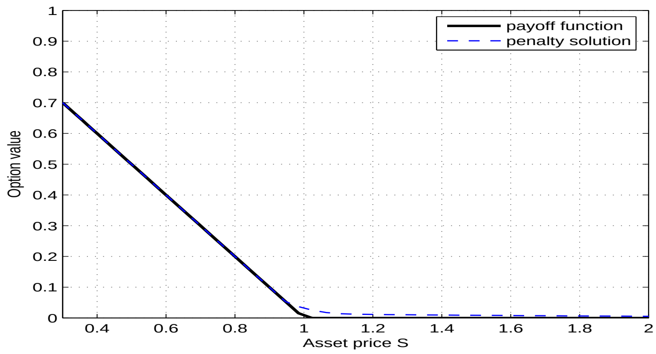

Figure 3.1: American put option value at time $t = 0$, obtained by $l_{1/2}$ penalty method with the discretization: the number of time steps = 100 and the number of space steps = 256.

Table 3.4: Computed results of American put option in the Merton model with $S = 1.0$ and the discretization: the number of time steps = 100 and the number of space steps = 256.

	l_1 penalty method				$l_{1/2}$ penalty method			
Pen. Param.	10^4	10^7	10^{10}	10^{13}	80	8000	80000	800000
Option Value	0.03238	0.03238	0.03238	0.03238	0.03438	0.03238	0.03238	0.03238
Free Boundary	0.94678	0.94678	0.94678	0.94678	0.94678	0.94678	0.94678	0.94678
Aver. Iteration	$12-13$	$11-12$	$11-12$	$11-12$	$12-13$	$11-12$	$11-12$	$11-12$
Comp. Time	40	38	38	38	40	38	38	38

3.5.2 American Butterfly Option Example

In this example, American butterfly option [127] is considered. This model is more challenging. A butterfly option has the payoff

$$V^*(S) = (S - K_1)_+ - 2\left(S - \frac{(K_1 + K_2)}{2}\right)_+ + (S - K_2)_+ ,$$

where K_1 and K_2 ($K_1 < K_2$) are striking prices specified by the contract. We can see that the butterfly option is a combination of three put options, which can be easily seen if we arrange the payoff in the form

$$V^*(S) = (K_1 - S)_+ - 2\left(\frac{(K_1 + K_2)}{2} - S\right)_+ + (K_2 - S)_+ .$$

It is clear that American butterfly option possesses two free boundaries (left one - L. Boundary and right one - R. Boundary).

For our example, all the parameters are the same as those listed in Table 3.1, except $K_1 = 90$ and $K_2 = 110$ for the butterfly option payoff. The numerical results are listed in Table 3.5. We depict the option value in Figure 3.2, which is obtained by $l_{1/2}$ penalty method with the discretization: the number of time steps = 100 and the number of space steps = 256.

Table 3.5: Computed results of American butterfly option in the Merton model with $S = 1.0$ and the discretization: the number of time steps = 100 and the number of space steps = 256.

	l_1 penalty method				$l_{1/2}$ penalty method			
Pen. Param.	10^4	10^7	10^{10}	10^{13}	80	8000	80000	800000
Option Value	0.05263	0.05261	0.05261	0.05261	0.05263	0.05261	0.05261	0.05261
L. Boundary	0.9845	0.9845	0.9845	0.9845	0.9845	0.9845	0.9845	0.9845
R. Boundary	1.0813	1.0813	1.0813	1.0813	1.0813	1.0813	1.0813	1.0813
Aver. Iteration	$12 - 13$	$12 - 13$	$12 - 13$	$12 - 13$	$12 - 13$	$12 - 13$	$12 - 13$	$12 - 13$
Comp. Time	40	40	40	40	40	40	40	40

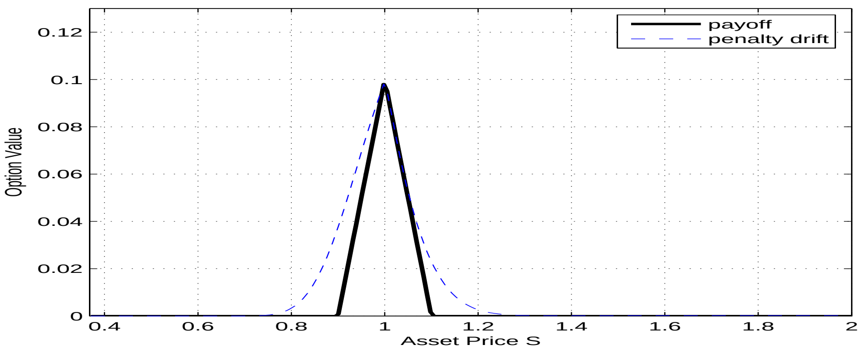

Figure 3.2: American Butterfly option value at time $t = 0$, obtained by $l_{1/2}$ penalty method with the discretization: the number of time steps = 100 and the number of space steps = 256.

3.5.3 Remarks

By comparing the results listed in the above two subsections, some interesting conclusions can be drawn. Also, several useful results are observed.

1. Both the l_1 and $l_{1/2}$ penalty methods are efficient to solve the American put and butterfly options. They are all robust with respect to the penalty parameters.

2. With the same level of accuracy, the $l_{1/2}$ penalty method needs much smaller penalty parameter than the l_1 penalty method, while their computational times are almost the same. Actually, a lower order penalty method has many advantages over a higher order penalty method. For details, see the study of this topic in the context of optimization theory reported in [111].

3. Exact penalty parameters are observed for both l_1 and $l_{1/2}$ penalty methods. These results are consistent with those for the classic optimization theory, see [111].

From the points 1, 2 and 3, we see that the power penalty method possesses several good properties. In particular, the power penalty method is a robust and efficient method to price the American option pricing with jump diffusion processes.

3.6 Summary

In this chapter, we proposed a power penalty approach to solving American option pricing with jump diffusion processes, which is a more practical model than the constant volatility Black-Scholes model. Within the framework of variational inequalities, we reformulated the PIDCP as a partial integro-differential equation. Then, we presented the unique solvability and convergence analysis of the resulting partial integro-differential equation. Moreover, a rate of convergence of this method was derived. By finite element discretization, a numerical scheme was proposed to solve the partial integro-differential equation. Finally, we demonstrated the efficiency of the power penalty method via solving an American put option example and an American butterfly option example. Desired results were obtained for both examples.

Chapter 4

A Power Penalty Method for Two-Asset American Option Pricing

For a single-asset American option pricing, it has been shown in [44, 125] and [131] that the solution obtained by the penalty approach converges to the original solution. In [137], the quadratic penalty method and l_1 penalty method were used to solve the American options with stochastic volatility, which are two-factor models and multi-dimensional complementarity problems. Also, in [137], a simple and intuitively equivalent relationship between the penalized problem and linear complementarity problem was given. However, there are few works that are devoted to the convergence analysis of the penalty approach to two-asset American option pricing. In the two previous chapters, the advantages of penalty method for American-style options over the PSOR method has been investigated: first, it is much more stable and computationally cheaper when compared with the PSOR method; second, it is more robust with respect to the changes of market parameters than the PSOR method. With these advantages, in this chapter, we develop a power penalty method for two-asset American options pricing. We first rewrite the continuous complementarity problem in a conservative form and give the corresponding variational form. Then, by applying variational inequalities theory, a power penalty approach to the complementarity problem is developed. We establish the unique solvability of the penalized nonlinear equation via the theory of abstract variational inequalities. After that, the convergence rate, of order $\mathcal{O}(\xi^{-1/2k})$, of the l_k penalty method is derived in some appropriate infinite dimensional spaces. To solve the penalized nonlinear equation effectively, the fitted finite volume method is proposed, which is known [125, 124, 131, 61] to be especially suitable for solving the Black-Scholes equation. Empirical numerical test is implemented to verify the effectiveness of our method.

4.1 Mathematical Model

Let x and y denote, respectively, the first and second underlying assets which follow the geometric Brownian motion processes

$$\begin{aligned} dx &= \mu_1 x dt + \sigma_1 x dW_1, \\ dy &= \mu_2 y dt + \sigma_2 y dW_2, \end{aligned}$$

where μ_1 and μ_2 are the drift rates of the assets x and y, σ_1 and σ_2 are the deterministic local volatilities of the assets x and y, W_1 and W_2 are the standard Brownian motions followed by the assets x and y, respectively. For the two assets, they are assumed to be positively correlated by $\rho \in [0,1]$ (see [116]) .

Let $V(x,y,t)$ represent the value of an European put option with expiry date T. We define

$$LV = -\frac{\partial V}{\partial t} - \frac{1}{2}\left[\sigma_1^2 x^2 \frac{\partial^2 V}{\partial x^2} + 2\rho\sigma_1\sigma_2 xy \frac{\partial^2 V}{\partial x \partial y} + \sigma_2^2 y^2 \frac{\partial^2 V}{\partial y^2}\right] - r\left[x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y}\right] + rV, \tag{4.1.1}$$

where r is the risk free interest rate. Then, by the no-arbitrage theory and Ito's formula, we can get the two-dimensional version of the Black-Scholes equation [121]

$$LV = 0,$$

with the *payoff function*

$$V^*(x,y) = \max\left(K - w_1 x - w_2 y, 0\right), \tag{4.1.2}$$

where $K > 0$ is the striking price, $w_1, w_2 \geq 0$ are the weights of the assets x and y, respectively.

For the case of American put options, there are possibilities of early exercise. Thus, an additional constraint is required

$$V - V^* \geq 0.$$

Consequently, the solution domain of the American option price can be divided into two parts: the hold regions and the exercise regions. In the hold regions, we have [127, 137]

$$LV = 0, \text{ and } V - V^* > 0,$$

while in the regions where it is optimal to exercise early, we have

$$LV > 0, \text{ and } V - V^* = 0.$$

In a nutshell, the value of the American put option is characterized by the following partial differential complementarity problem.

$$\begin{cases} LV \geq 0, \\ V - V^* \geq 0, \\ LV \cdot (V - V^*) = 0, \end{cases} \tag{4.1.3}$$

with the boundary conditions defined by

$$\begin{aligned} &V(0, y, t) = g_1(y, t), \quad &&V(x, 0, t) = g_2(x, t), \\ &V(X, y, t) = 0, &&V(x, Y, t) = 0, \end{aligned} \tag{4.1.4}$$

and terminal condition

$$V(x, y, t = T) = V^*(x, y), \tag{4.1.5}$$

where $X >> x$ and $Y >> y$. Here, g_1 and g_2 are given functions that provide suitable boundary conditions. Typically, g_1 and g_2 are determined via solving the associated one dimensional American put option problems, see [121]. We will give further details in Section 4.5.2.

For convenience of theoretical analysis, we first transform (4.1.1) into the conservative form

$$LV = -V_t - \nabla \cdot (\mathbf{D}\nabla V + \underline{b}V) + \overline{c}V, \tag{4.1.6}$$

where

$$\mathbf{D} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

It is not difficult to derive the following relationships by comparing (4.1.1) and (4.1.6).

$$\overline{c} = 3r - (\sigma_1^2 + \sigma_2^2 + \rho\sigma_1\sigma_2)\,,$$
$$\mathbf{D} = \begin{pmatrix} \frac{1}{2}\sigma_1^2 x^2 & \frac{1}{2}\rho\sigma_1\sigma_2 xy \\ \frac{1}{2}\rho\sigma_1\sigma_2 xy & \frac{1}{2}\sigma_2^2 y^2 \end{pmatrix},\ \underline{b} = \begin{pmatrix} rx - \sigma_1^2 x - \frac{1}{2}\rho\sigma_1\sigma_2 x \\ ry - \sigma_2^2 y - \frac{1}{2}\rho\sigma_1\sigma_2 y \end{pmatrix}. \tag{4.1.7}$$

Now, we aim to transform the complementarity problem (4.1.4)–(4.1.5) into an equivalent standard form satisfying homogeneous Dirichlet boundary conditions.

Let $V_0(x, y) \in H^2(\Omega)$ for every t such that V_0 satisfies the boundary conditions given in (4.1.4). This can be (theoretically) determined by, for example, the Laplace equation $\triangle V_0 = 0$ with the boundary condition (4.1.4). In this case, ∇V_0 is continuous on Ω. With the definition of V_0, we introduce a new function

$$u(x, y, t) = e^{\beta t}\,(V_0 - V)\,,$$

where $\beta = \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \rho\sigma_1\sigma_2)$. Taking LV_0 away from both sides of the first inequality of (4.1.3) and translating V in (4.1.3)–(4.1.5) into the new function u, it is easy to show that the complementarity problem (4.1.3)–(4.1.5) becomes

$$\begin{cases} \mathcal{L}u \leq f, \\ u - u^* \leq 0, \\ (\mathcal{L}u - f)\cdot(u - u^*) = 0, \end{cases} \tag{4.1.8}$$

where

$$\begin{aligned} \mathcal{L}u &= -u_t - \nabla\cdot(\mathbf{D}\nabla u + \underline{b}u) + \underline{c}V, \qquad \underline{c} = \overline{c} + \beta \\ f(x, y, t) &= e^{\beta t} L V_0, \qquad\qquad u^* = e^{\beta t}\,(V_0 - V^*)\,. \end{aligned} \tag{4.1.9}$$

The boundary conditions (4.1.4) now become

$$\begin{aligned} u(0, y, t) = u(X, y, t) = 0,\ \forall t \in [0, T] \text{ and } y \in [0, Y], \\ u(x, 0, t) = u(x, Y, t) = 0,\ \forall t \in [0, T] \text{ and } x \in [0, X], \end{aligned}$$

The terminal condition (4.1.4) becomes

$$u(x, y, t = T) = u^*(x, y).$$

4.2 Variational Analysis

In this section, we will give an equivalent variational inequality of (4.1.8). Before proceeding, let us first recall some standard notation, which can be found in Section 1.4. Let $\Omega = [0, X] \times [0, Y]$ and Γ denote the boundaries of Ω. Clearly,

$$\Gamma = \{x = 0, 0 \le y \le Y\} \cup \{y = 0, 0 \le x \le X\} \cup \{x = X, 0 \le y \le Y\} \cup \{y = Y, 0 \le x \le X\}.$$

We have defined the weighted Sobolev space $H^1_{\varpi}(\Omega)$ given by

$$H^1_{\varpi}(\Omega) = \left\{ v(\cdot,\cdot,t) : v, xv_x, yv_y \in L^2(\Omega), \forall x, y \in \Omega \right\}$$

with its norm denoted by $\|\cdot\|_{1,\varpi}$. We also have

$$H^1_{0,\varpi}(\Omega) = \left\{ v(\cdot,\cdot,t) : v(\cdot,\cdot,t) \in H^1_{\varpi}(\Omega),\ v|_{\Gamma} = 0 \right\},$$

$$\mathcal{K} = \left\{ v(\cdot,\cdot,t) : v(t) \in H^1_{0,\varpi}(\Omega),\ v(\cdot,\cdot,t) \le u^*(\cdot,\cdot,t) \right\},$$

where $u^*(\cdot,\cdot,t)$ is defined by (4.1.9). It is easy to verify that $\mathcal{K}$ is a convex and closed subset of $H^1_{0,\varpi}(\Omega)$. Finally, for any Hilbert space $H(\Omega)$, the norm of $L^p(0, T; H(\Omega))$ is denoted by

$$\|v(\cdot,t)\|_{L^p(0,T;H(\Omega))} = \left(\int_0^T \|v(\cdot,\cdot,t)\|^p_H \, dt \right)^{\frac{1}{p}}.$$

Obviously,

$$L^p(0, T; L^p(\Omega)) = L^p(\Omega \times (0, T)) = L^p(\Theta).$$

In what follows, we will simply write $v(t)$ when we regard $v(\cdot,\cdot,,t)$ as an element of $H^1_{0,\varpi}(\Omega)$. We will also suppress the independent time variable t (or τ), when it causes no confusion in doing so.

Now, we define the following variational inequality problem.

Problem 4.2.1 *Find $u(t) \in \mathcal{K}$ such that, for all $v \in \mathcal{K}$,*

$$\left(-\frac{\partial u(t)}{\partial t}, v - u(t) \right) + B(u(t), v - u(t); t) \ge (f, v - u(t)), \ \text{a.e. in } (0, T) \qquad (4.2.10)$$

where $B(u, v; t)$ is a bilinear form defined by

$$B(u, v; t) = (\mathbf{D}\nabla u + \underline{b}u, \nabla v) + (\underline{c}u, v), \quad u, v \in H^1_{0,\varpi}(\Omega). \qquad (4.2.11)$$

For this variational inequality problem, we have the following theorem.

Theorem 4.2.1 *Problem 4.2.1 is the variational form of the complementarity problem (4.1.8).*

Proof. For any $w \in \mathcal{K}$, it follows from the definition of $\mathcal{K}$ that

$$w - u^* \leq 0 \quad \text{a.e. on } \Theta.$$

Multiplying both sides of the first inequality of (4.1.8) by $w - u^*$, we obtain

$$\left(-\frac{\partial u}{\partial t}, w - u^*\right) - (\nabla \cdot (\mathbf{D}\nabla u + \underline{b}u) - \underline{c}u, w - u^*) \geq (f, w - u^*), \text{ a.e. in } (0, T).$$

Using the Gauss-divergence theory, we obtain

$$\left(-\frac{\partial u}{\partial t}, w - u^*\right) + B(u, w - u^*; t) \geq (f, w - u^*), \text{ a.e. in } (0, T). \tag{4.2.12}$$

Since $\mathcal{K}$ is a convex and closed subset of $H^1_{0,\varpi}(\Omega)$, we may write w as $w = \theta v + (1-\theta)u$, where $u, v \in \mathcal{K}$ and $\theta \in [0, 1]$. Therefore, (4.2.12) becomes

$$\begin{aligned} &\left(-\frac{\partial u}{\partial t}, \theta\,(v-u)\right) + B(u, \theta\,(v-u)\,; t) \\ &\geq (f, \theta\,(v-u)) - \left(\left(-\frac{\partial u}{\partial t}, u - u^*\right) + B(u, u - u^*; t) - (f, u - u^*)\right), \text{ a.e. in } (0, T). \end{aligned} \tag{4.2.13}$$

On the other hand, form the third relationship of (4.1.8), we have

$$(\mathcal{L}u - f, u - u^*) = 0,$$

i.e.

$$\left(-\frac{\partial u}{\partial t}, u - u^*\right) + B(u, u - u^*; t) - (f, u - u^*) = 0.$$

Therefore, (4.2.13) reduces to

$$\left(-\frac{\partial u}{\partial t}, \theta\,(v-u)\right) + B(u, \theta\,(v-u)\,; t) \geq (f, \theta\,(v-u)), \text{ a.e. in } (0, T). \tag{4.2.14}$$

Since $\theta \in [0, 1]$, we can see that (4.2.14) leads to

$$\left(-\frac{\partial u}{\partial t}, v - u\right) + B(u, v - u; t) \geq (f, v - u), \text{ a.e. in } (0, T).$$

■

In order to establish the unique solvability of Problem 4.2.1, we study the properties of the operator $B(u, v; t)$.

First, we define the semi-norm of B as follows:

$$|v|_{1,\varpi}^2 \triangleq \int_\Omega \left[x^2 v_x^2 + \rho\,(xv_x + yv_y)^2 + y^2 v_y^2\right] d\Omega, \tag{4.2.15}$$

for any $v \in H_{0,\varpi}^1(\Omega)$. It is easy to see that $|\cdot|_{1,\varpi}$ is a weighted semi-norm on $H_{0,\varpi}^1(\Omega)$.

Now, $\|v\|_{1,\varpi}$ is defined by

$$\|v\|_{1,\varpi}^2 \triangleq |v|_{1,\varpi}^2 + \|v\|_{0,\varpi}^2 .$$

Obviously, $\|\cdot\|_{1,\varpi}$ is a weighted energy norm on $H_{0,\varpi}^1(\Omega)$.

With the above definitions, we have the following lemma.

Lemma 4.2.1 *There exist positive constants C and M, independent of v and w, such that for any $v, w \in H_{0,\varpi}^1(\Omega)$,*

$$\begin{aligned} B(v, v; t) &\geq C\,\|v\|_{1,\varpi}^2, && (\textit{coerciveness}) \\ |B(v, w; t)| &\leq M\,\|v\|_{1,\varpi}\,\|w\|_{1,\varpi}. && (\textit{continuity}) \end{aligned}$$

Proof. Let C and M be two generic positive constants, independent of v and w. First, we note that for any $v \in H_{0,\varpi}^1(\Omega)$, we get by integrating by parts

$$\int_\Omega \underline{b}v \cdot \nabla v d\Omega = \int_{\partial\Omega} v^2 \underline{b} \cdot nds - \int_\Omega v\nabla \cdot (\underline{b}v)\, d\Omega = -\int_\Omega v\underline{b} \cdot \nabla v d\Omega - \int_\Omega v^2 \nabla \cdot \underline{b} d\Omega. \tag{4.2.16}$$

Hence, from (4.2.16), we have

$$\int_\Omega \underline{b}v \cdot \nabla v d\Omega = -\frac{1}{2}\int_\Omega \nabla \cdot \underline{b} v^2 d\Omega.$$

Therefore, using (4.2.11), (4.2.15) and (4.2.16), we obtain

$$\begin{aligned} B(v, v; t) = &\ (\mathbf{D}\nabla v + \underline{b}v, \nabla v) + \underline{c}(v, v) = (\mathbf{D}\nabla v, \nabla v) + (\underline{b}v, \nabla v) + \underline{c}\,(v, v) \\ = &\ \tfrac{1}{2}\textstyle\int_\Omega \left[(1-\rho)\,\sigma_1^2 x^2 v_x^2 + \rho\,(\sigma_1 x v_x + \sigma_2 y v_y)^2 + (1-\rho)\,\sigma_2^2 y^2 v_y^2\right] d\Omega \\ &+ \left(\underline{c} - \tfrac{1}{2}\nabla \cdot \underline{b}\right)(v, v) \\ \geq &\ C\,|v|_{1,\varpi}^2 + \left(\beta + 2r - \tfrac{1}{2}\left(\sigma_1^2 + \sigma_2^2 + \rho\sigma_1\sigma_2\right)\right)\|v\|_{0,\varpi}^2 \\ \geq &\ C\,\|v\|_{1,\varpi}^2, \end{aligned}$$

since $\beta = \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \rho\sigma_1\sigma_2)$.

Now, Let us show the continuity of B. For any $v, w \in H^1_{0,\varpi}(\Omega)$, we have

$$|B(v,w;t)| = |(\mathbf{D}\nabla v + \underline{b}v, \nabla w) + \underline{c}(v,w)| \leq |(\mathbf{D}\nabla v, \nabla w)| + |(\underline{b}v, \nabla w)| + |\underline{c}\,(v,w)| \,. \tag{4.2.17}$$

1. For $|(\mathbf{D}\nabla v, \nabla w)|$ in (4.2.17), by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
& |(\mathbf{D}\nabla v, \nabla w)| \\
= \ & \left|\int_\Omega \left(\tfrac{1}{2}\sigma_1^2 x^2 v_x w_x + \tfrac{1}{2}\sigma_2^2 y^2 v_y w_y + \tfrac{\varrho}{2}\sigma_1\sigma_2 xy\,(v_x w_y + v_y w_x)\right) d\Omega\right| \\
\leq \ & \tfrac{1}{2}\left[\int_\Omega \sigma_1^2 x^2 v_x^2 d\Omega\right]^{\frac{1}{2}} \cdot \left[\int_\Omega \sigma_1^2 x^2 w_x^2 d\Omega\right]^{\frac{1}{2}} + \tfrac{1}{2}\left[\int_\Omega \sigma_2^2 y^2 v_y^2 d\Omega\right]^{\frac{1}{2}} \cdot \left[\int_\Omega \sigma_2^2 y^2 w_y^2 d\Omega\right]^{\frac{1}{2}} + \\
& \tfrac{\varrho}{4}\left[\int_\Omega \sigma_1^2 x^2 v_x^2 d\Omega\right]^{\frac{1}{2}} \cdot \left[\int_\Omega \sigma_2^2 y^2 w_y^2 d\Omega\right]^{\frac{1}{2}} + \tfrac{\varrho}{4}\left[\int_\Omega \sigma_1^2 x^2 w_x^2 d\Omega\right]^{\frac{1}{2}} \cdot \left[\int_\Omega \sigma_2^2 y^2 v_y^2 d\Omega\right]^{\frac{1}{2}} \\
\leq \ & \left(\tfrac{1}{2} + \tfrac{\varrho}{4}\right)\left[\int_\Omega \left(\sigma_1^2 x^2 v_x^2 d\Omega + \sigma_2^2 y^2 v_y^2\right) d\Omega\right]^{\frac{1}{2}} \cdot \left[\int_\Omega \left(\sigma_1^2 x^2 w_x^2 d\Omega + \sigma_2^2 y^2 w_y^2\right) d\Omega\right]^{\frac{1}{2}} \\
\leq \ & M\,|v|_{1,\varpi}\,|w|_{1,\varpi}\,.
\end{aligned} \tag{4.2.18}$$

2. For $|(\underline{b}v, \nabla w)|$ in (4.2.17), by the expression of $\underline{b}$ in (4.1.7) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
& |(\underline{b}v, \nabla w)| \\
= \ & \left|\int_\Omega \underline{b}v \cdot \nabla w d\Omega\right| = \left|\int_{\partial\Omega} vw\underline{b} \cdot n ds - \int_\Omega w\nabla \cdot (\underline{b}v)\, d\Omega\right| \\
= \ & \left|-\int_\Omega w\underline{b} \cdot \nabla v d\Omega - \int_\Omega vw\nabla \cdot \underline{b} d\Omega\right| \\
\leq \ & \left|\int_\Omega w\underline{b} \cdot \nabla v d\Omega\right| + \left|\int_\Omega vw\nabla \cdot \underline{b} d\Omega\right| \\
\leq \ & \left|\int_\Omega w\left[\left(r - \sigma_1^2 - \tfrac{1}{2}\rho\sigma_1\sigma_2\right) xv_x + \left(r - \sigma_2^2 - \tfrac{1}{2}\rho\sigma_1\sigma_2\right) yv_y\right] d\Omega\right| + \left|\int_\Omega vw\nabla \cdot \underline{b} d\Omega\right| \\
\leq \ & M\,\|w\|_{0,\varpi}\,|v|_{1,\varpi} + M\,\|v\|_{0,\varpi}\,\|w\|_{0,\varpi}
\end{aligned} \tag{4.2.19}$$

3. For $|\underline{c}(v,w)|$ in (4.2.17), it is easy to see

$$|\underline{c}\,(v,w)| \leq M\,\|v\|_{0,\varpi}\,\|w\|_{0,\varpi}\,. \tag{4.2.20}$$

Summarizing (4.2.17)-(4.2.20), the continuity of B is obtained as follows:

$$|B\,(v,w;t)| \leq M\left(|v|_{1,\varpi}\,|w|_{1,\varpi} + \|w\|_{0,\varpi}\,|v|_{1,\varpi} + \|v\|_{0,\varpi}\,\|w\|_{0,\varpi}\right) \leq M\,\|v\|_{1,\varpi}\,\|w\|_{1,\varpi}\,.$$

■

Using Lemma 4.2.1 and the theory of abstract variational inequalities, the unique solvability of Problem 4.2.1 is established in the following theorem.

Theorem 4.2.2 *There exists a unique solution to Problem 4.2.1.*

4.3 The Power Penalty Approach

The underlying idea of the penalty approach to solving American option pricing is simple. Let $k > 0$ be a parameter. As for the single-asset option in [125], the power penalty approach to (4.1.8) is given as the following nonlinear equation.

$$\mathcal{L}u_\xi + \xi\,[u_\xi - u^*]_+^k = f, \qquad (x, y, t) \in \Theta \tag{4.3.21}$$

with the given boundary and final conditions

$$u_\xi(x, y, t)|_\Gamma = 0 \quad \text{and} \quad u_\xi(x, y, t = T) = u^*(x, y), \tag{4.3.22}$$

where $\xi > 1$ is the *penalty parameter.*

If $k = 2$, this penalty approach corresponds to the quadratic penalty approach. When $k = 1$, the typical l_1 penalty approach is obtained. When $0 < k < 1$, the so-called lower order penalty approach [111, 125] is achieved. For single-asset American option pricing, it has been shown [125, 131] that the solution to (4.3.21) converges to that of (4.1.8) at the rate of order $\mathcal{O}(\xi^{-1/2k})$. The same conclusion is valid for the multi-asset American option pricing. We will verify this conclusion in the following section.

Now, we give the rigorous mathematical derivation of the power penalty method for the complementarity problem (4.1.8). First, we give the penalty approach to the variational inequality (4.2.10).

Theorem 4.3.1 *The variational inequality (4.2.10) of the first kind is equivalent to the following nonlinear variational inequality of the second kind, i.e. the penalization of (4.2.10):*

Find $u_\xi(t) \in H^1_{0,\varpi}(\Omega)$ such that, for all $v \in H^1_{0,\varpi}(\Omega)$,

$$\left(-\frac{\partial u_\xi}{\partial t}, v - u_\xi\right) + B(u_\xi, v - u_\xi; t) + j(v) - j(u_\xi) \geq (f, v - u_\xi), \ \textit{a.e. in } (0, T) \tag{4.3.23}$$

where

$$j(v) = \frac{\xi}{k+1}\,[v - u^*]_+^{1+k}. \tag{4.3.24}$$

The above theorem presents a standard result on the equivalence between the variational inequalities of the first kind and those of the second kind, see [51]. The unique solvability is established by the coerciveness and continuity of the bilinear operator B and the lower semi-continuity of j, see [51]. Obviously, by virtue of Lemma 4.2.1 and (4.3.24), all the required conditions are satisfied. Hence, the penalization (4.3.23) is uniquely solvable.

From (4.3.24), we can see that $j(v)$ is differentiable. Hence, the penalization (4.3.23) is equivalent to the following problem.

Problem 4.3.1 *Find $u_\xi \in H^1_{0,\varpi}(\Omega)$ such that, for all $v \in H^1_{0,\varpi}(\Omega)$,*

$$\left(-\frac{\partial u_\xi}{\partial t}, v\right) + B(u_\xi, v; t) + (j'(u_\xi), v) = (f, v), \ \ a.e. \ in \ (0, T) \tag{4.3.25}$$

where

$$j'(v) = \xi \left[v - u^*\right]_+^k . \tag{4.3.26}$$

Clearly, Problem 4.3.1 is the variational form of the nonlinear equation (4.3.21). Hence, we obtain the power penalty approach to the complementarity problem (4.1.8). In the next section, we will study the convergence rate of the power penalty approach.

4.4 Convergence Analysis

We now show that the solution to Problem 4.3.1 converges to that of Problem 4.2.1 as the penalty parameter $\xi \to \infty$ with the convergence rate of order $\mathcal{O}(\xi^{-1/2k})$ in some proper norms. In doing so, we first give the following Lemma.

Lemma 4.4.1 *Let u_ξ be the solution to Problem 4.3.1. If $u_\xi \in L^p(\Theta)$, then there exists a positive constant C, independent of u_ξ and ξ, such that*

$$\begin{aligned} \left\|[u_\xi - u^*]_+\right\|_{L^p(\Theta)} &\leq \tfrac{C}{\xi^{1/k}} \\ \left\|[u_\xi - u^*]_+\right\|_{L^\infty(0,T;L^2(\Omega))} + \left\|[u_\xi - u^*]_+\right\|_{L^2(0,T;H^1_{0,\varpi}(\Omega))} &\leq \tfrac{C}{\xi^{1/2k}} \end{aligned} \tag{4.4.27}$$

where k is the power of the power penalty function and $p = 1 + k$.

Proof. Assume that C is a generic positive constant, independent of u_ξ and ξ. To simplify the notation, we let $\phi = [u_\xi - u^*]_+$. Obviously, $\phi \in H^1_{0.\varpi}(\Omega)$ a.e. in $(0, T)$.

Setting $v = \phi$ in (4.3.25) and (4.3.26), we have

$$\left(-\frac{\partial u_\xi}{\partial t}, \phi\right) + B(u_\xi, \phi; t) + \xi\left(\phi^k, \phi\right) = (f, \phi), \text{ a.e. in } (0, T)$$

$$\left(-\frac{\partial\left(u_\xi - u^*\right)}{\partial t}, \phi\right) + B(\left(u_\xi - u^*\right), \phi; t) + \xi\left(\phi^k, \phi\right) = (f, \phi) + \left(\frac{\partial u^*}{\partial t}, \phi\right) - B(u^*, \phi; t).$$

Integrating from t to T, we have

$$\int_t^T \left(-\frac{\partial\left(u_\xi - u^*\right)}{\partial t}, \phi\right) + B(\left(u_\xi - u^*\right), \phi; t) + \xi\left(\phi^k, \phi\right) = (f, \phi) + \left(\frac{\partial u^*}{\partial t}, \phi\right) - B(u^*, \phi; t) \tag{4.4.28}$$

Integrating both sides of (4.4.28) from t to T and using the coerciveness property of the operator B and Hölder's inequality, we get

$$\begin{aligned}
&\frac{1}{2}(\phi(t), \phi(t)) + \int_t^T ||\phi(\tau)||_B^2 d\tau + \xi\int_t^T \left(\phi^k, \phi\right) d\tau \\
&\leq \int_t^T (f(\tau), \phi(\tau))d\tau + \beta\int_t^T e^{\beta\tau}(V_0 - V^*, \phi(\tau))d\tau - \int_t^T B(u^*, \phi(\tau); \tau)d\tau \\
&\leq C\left(\int_t^T ||\phi(\tau)||^p_{L^p(\Omega)}d\tau\right)^{1/p} + \beta\int_t^T e^{\beta\tau}(V_0 - V^*, \phi(\tau))d\tau - \int_t^T B(u^*, \phi(\tau); \tau)d\tau.
\end{aligned} \tag{4.4.29}$$

Noting that $|V_0 - V^*|$ is uniformly bounded and $\beta = \sigma_1^2 + \sigma_2^2 + \frac{1}{2}\rho\sigma_1\sigma_2$, we have

$$\begin{aligned}
&\frac{1}{2}(\phi(t), \phi(t)) + \int_t^T ||\phi(\tau)||_B^2 d\tau + \xi\int_t^T ||\phi(\tau)||^p_{L^p(\Omega)}d\tau \\
&\leq C\left(\int_t^T ||\phi||^p_{L^p(\Omega)}d\tau\right)^{1/p} - \int_t^T B(u^*, \phi(\tau); \tau)d\tau.
\end{aligned} \tag{4.4.30}$$

Since $B(u, v; t) = (\mathbf{D}\nabla u + \underline{b}u, \nabla v) + (\underline{c}u, v)$, we have

$$-\int_t^T B(u^*, \phi(\tau); \tau)d\tau = -\int_t^T (\mathbf{D}\nabla u^* + \underline{b}u^*, \nabla\phi(\tau))d\tau - \int_t^T (\underline{c}u^*, \phi(\tau))d\tau. \tag{4.4.31}$$

Furthermore, by Green's theorem, we obtain

$$-\int_t^T \left(\underline{b}u^*, \nabla\phi(\tau)\right) d\tau \;=\; \int_t^T \int_\Omega \nabla\cdot\underline{b}u^*\phi(\tau)d\Omega d\tau - \int_t^T \int_\Gamma u^*\cdot n\phi(\tau)d\Gamma d\tau. \tag{4.4.32}$$

Let $\Omega_1 = \{0 < x < K/w_1, 0 < y < K/w_2, K - w_1x - w_2y > 0\}$ and $\Omega_2 = \Omega\backslash\overline{\Omega}_1$. We also let Γ_0 denote the interface of Ω_1 and Ω_2. Therefore, Γ_0 has two opposite orientations:

Γ_0^+ when it is oriented in the same direction as $\partial\Omega_1$, and Γ_0^- when it is oriented in the same direction as $\partial\Omega_2$. Consider only the integrant $(\mathbf{D}\nabla u^*, \nabla\phi)$ in (4.4.31). For $\phi \in H^1_{0,\varpi}(\Omega)$, note that $\phi = 0$ on Γ, we have

$$\begin{aligned}
&-(\mathbf{D}\nabla u^*, \nabla\phi) \\
&= -\int_\Omega (\mathbf{D}\nabla u^*)^T \nabla\phi d\Omega = -\int_{\Omega_1} (\mathbf{D}\nabla u^*)^T \nabla\phi d\Omega - \int_{\Omega_2} (\mathbf{D}\nabla u^*)^T \nabla\phi d\Omega \\
&= -\int_{\Gamma_0^+} \mathbf{D}\nabla u^* \cdot n\phi ds + \int_{\Omega_1} \nabla\cdot(\mathbf{D}\nabla u^*)\phi d\Omega - \int_{\Gamma_0^-} \mathbf{D}\nabla u^* \cdot n\phi ds + \int_{\Omega_2} \nabla\cdot(\mathbf{D}\nabla u^*)\phi d\Omega \\
&= -\int_{\Gamma_0^+} \left(\mathbf{D}\nabla u^*_- - \mathbf{D}\nabla u^*_+\right)\cdot n\phi ds + \int_\Omega \nabla\cdot(\mathbf{D}\nabla u^*)\phi d\Omega \qquad (4.4.33)
\end{aligned}$$

where n denotes the unit outward normal direction of the boundary segments and ∇u^*_- and ∇u^*_+ denote, respectively, the values of ∇u^* evaluated on the left and right sides of Γ_0^+. From $u^* = e^{\beta t}(V_0 - V^*)$ and (4.1.7), it is easy to see that

$$\nabla u^* = e^{\beta t}(\nabla V_0 - \nabla V^*).$$

Since $V_0 \in H^2(\Omega)$, ∇V_0 is continuous on Ω, as we mentioned before,

$$\begin{aligned}
\nabla u^*_- - \nabla u^*_+ &= e^{\beta t}\left[(\nabla V_0 - \nabla V^*)_- - (\nabla V_0 - \nabla V^*)_+\right] \\
&= e^{\beta t}\left(\nabla V^*_- - \nabla V^*_+\right) \\
&= e^{\beta t}(-w_1, -w_2)^T.
\end{aligned}$$

Furthermore, the unit outward-normal vector to Γ_0^+ is:

$$n = \nabla(K - w_1 x - w_2 y) / \|\nabla(K - w_1 x - w_2 y)\| = (-w_1, -w_2)^T / \left(w_1^2 + w_2^2\right)^{1/2}.$$

Therefore, estimate (4.4.33) becomes

$$\begin{aligned}
-(\mathbf{D}\nabla u^*, \nabla\phi) &= -\int_{\Gamma_0^+} e^{\beta t}\frac{(w_1, w_2)\,\mathbf{D}^T(w_1, w_2)^T}{(w_1^2 + w_2^2)^{1/2}}\phi ds + \int_\Omega \nabla\cdot(\mathbf{D}\nabla u^*)\phi d\Omega \\
&\le C\int_\Omega \phi(\tau) d\Omega,
\end{aligned}$$

because $\mathbf{D}$ is positive definite, ϕ is non-negative and $\nabla\cdot(\mathbf{D}\nabla u^*)$ is bounded above on Ω. Thus,

$$-\int_t^T (\mathbf{D}\nabla u, \nabla\phi(\tau))d\tau \le C\int_t^T\int_\Omega \phi(\tau)d\Omega d\tau \le C\left(\int_t^T ||\phi(\tau)||^p_{L^p(\Omega)}d\tau\right)^{1/p}. \quad (4.4.34)$$

Also, from (4.4.32), it follows that

$$-\int_t^T (\underline{b}u^*, \nabla\phi(\tau))\, d\tau \le C\int_t^T \int_\Omega \phi(\tau) d\Omega d\tau \le C\left(\int_t^T ||\phi(\tau)||^p_{L^p(\Omega)} d\tau\right)^{1/p}, \quad (4.4.35)$$

because $\nabla\cdot\underline{b}u^*$ is bounded above on Ω.

Thus, from (4.4.29) to (4.4.35), it follows that

$$\frac{1}{2}(\phi(t),\phi(t)) + \int_t^T ||\phi(\tau)||^2_B d\tau + \xi\int_t^T ||\phi(\tau)||^p_{L^p(\Omega)} d\tau$$
$$\le C\left(\int_t^T ||\phi(\tau)||^p_{L^p(\Omega)} d\tau\right)^{1/p}, \quad \text{a.e. in } (0,T) \quad (4.4.36)$$

This implies that

$$\xi\int_t^T ||\phi(\tau)||^p_{L^p(\Omega)} d\tau \le C\left(\int_t^T ||\phi(\tau)||^p_{L^p(\Omega)} d\tau\right)^{1/p}, \quad \text{a.e. in } (0,T).$$

From this, it follows that

$$\left(\int_t^T ||\phi(\tau)||^p_{L^p(\Omega)} d\tau\right)^{1/p} \le \frac{C}{\xi^{1/(p-1)}} = \frac{C}{\xi^{1/k}}, \text{ where } p = 1+k. \quad (4.4.37)$$

Now, from (4.4.36) and (4.4.37), we have

$$\frac{1}{2}(\phi(t),\phi(t)) + \int_t^T ||\phi(\tau)||^2_B d\tau \le C\left(\int_t^T ||\phi(\tau)||^p_{L^p(\Omega)} d\tau\right)^{1/p} \le \frac{C}{\xi^{1/k}},$$

from which, it follows that

$$(\phi(t),\phi(t))^{\frac{1}{2}} + \left(\int_t^T ||\phi(\tau)||^2_B d\tau\right)^{\frac{1}{2}} \le \frac{C}{\xi^{1/2k}}, \quad \text{a.e. in } (0,T).$$

Clearly, by replacing ϕ with $[u_\xi - u^*]_+$, we obtain readily (4.4.27). ■

Using the above Lemma, we are able to show that the solution to Problem 4.3.1 converges to that of Problem 4.2.1 at a rate of order $\xi^{-1/2k}$, as given in the next theorem.

Theorem 4.4.1 *Let u and u_ξ be the solutions to Problem 4.2.1 and Problem 4.3.1, respectively. If $u_\xi \in L^p(\Theta)$ and $\frac{\partial u}{\partial t} \in L^{1/k+1}(\Theta)$, then there exists a positive constant C, independent of u_ξ and ξ, such that*

$$\|u-u_\xi\|_{L^\infty(0,T;L^2(\Omega))} + \|u-u_\xi\|_{L^2(0,T;H^1_{0,\varpi}(\Omega))} \le \frac{C}{\xi^{1/2k}} \quad (4.4.38)$$

where k is the power of the power penalty function.

Proof. We still use the notation of Lemma 4.4.1. Setting $v_- = -\min(v,0)$ and $R_\xi = u - u^* + [u_\xi - u^*]_-$, it follows that

$$\begin{aligned} u - u_\xi = R_\xi - \varphi, \\ (\varphi^\alpha, [u_\xi - u^*]_-) = [u_\xi - u^*]_+^\alpha [u_\xi - u^*]_- \equiv 0, \ \alpha > 0. \end{aligned} \tag{4.4.39}$$

Set $v = u - R_\xi$ in (4.2.10) and $v = R_\xi$ in (4.3.25), we obtain

$$\left(-\frac{\partial u}{\partial t}, -R_\xi\right) + B\left(u, -R_\xi; t\right) \geq \left(f, -R_\xi\right), \tag{4.4.40}$$

$$\left(-\frac{\partial u_\xi}{\partial t}, R_\xi\right) + B\left(u_\xi, R_\xi; t\right) + \xi\left(\phi^k, R_\xi\right) = \left(f, R_\xi\right). \tag{4.4.41}$$

Combining (4.4.40) and (4.4.41) gives

$$\left(-\frac{\partial(u_\xi - u)}{\partial t}, R_\xi\right) + B\left(u_\xi - u, R_\xi; t\right) + \xi\left(\phi^k, R_\xi\right) \geq 0.$$

It follows from $u \leq u^*$ and $\phi \geq 0$ that

$$\left(\phi^k, R_\xi\right) = \left(\phi^{1/k}, u - u^*\right) + \left(\phi^k, [u_\xi - u^*]_-\right) = \left(\phi^k, u - u^*\right) \leq 0.$$

Therefore,

$$\left(-\frac{\partial(u - u_\xi)}{\partial t}, R_\xi\right) + B(u - u_\xi, R_\xi; t) \leq 0.$$

From (4.4.39), it follows that

$$\left(-\frac{\partial R_\xi}{\partial t}, R_\xi\right) + B(R_\xi, R_\xi; t) \leq \left(-\frac{\partial \phi}{\partial t}, R_\xi\right) + B\left(\phi, R_\xi; t\right).$$

Integrating both sides of the above from $\tau = t$ to $\tau = T$ and then using Cauchy-Schwarz inequality and $(\varphi, [u_\xi - u^*]_-) = 0$, we obtain

$$\begin{aligned} &\frac{1}{2}\left(R_\xi, R_\xi\right) + \int_t^T B\left(R_\xi, R_\xi; \tau\right) d\tau \\ \leq &\int_t^T \left(-\frac{\partial \phi}{\partial \tau}, R_\xi\right) d\tau + \int_t^T B\left(\phi, R_\xi; \tau\right) d\tau \\ \leq &\left(\phi, R_\xi\right) + \int_t^T \left(\phi, \frac{\partial R_\xi}{\partial \tau}\right) d\tau + \int_t^T B(\phi, R_\xi; \tau) d\tau \\ \leq &\left(\phi, R_\xi\right) + \int_t^T B\left(\phi, R_\xi; \tau\right) d\tau + \int_t^T \left(\phi, \frac{\partial u}{\partial \tau}\right) d\tau \\ \leq &||\phi||_{L^\infty(0,T;L^2(\Omega))} ||R_\xi||_{L^\infty(0,T;L^2(\Omega))} + C||\phi||_{L^2(0,T;H^1_{0,\varpi}(\Omega))} ||R_\xi||_{L^2(0,T;H^1_{0,\varpi}(\Omega))} \\ &+ C||\phi||_{L^p(\Theta)} \left(\left\|\frac{\partial u}{\partial t}\right\|_{L^q(\Theta)} + ||V_0 - V^*||_{L^q(\Theta)}\right) \end{aligned} \tag{4.4.42}$$

where $p = 1 + k$, and $1/p + 1/q = 1$.

Since $u_\xi \in L^p(\Theta)$, and $\frac{\partial u}{\partial t} \in L^{1/k+1}(\Theta)$, it follows from (4.4.27) that

$$||\phi||_{L^p(\Theta)} \left(\left\| \frac{\partial u}{\partial t} \right\|_{L^q(\Theta)} + ||V_0 - V^*||_{L^q(\Theta)} \right) \leq \frac{C}{\xi^{1/k}} \tag{4.4.43}$$

Using the coerciveness property of the operator B, we have

$$\frac{1}{2}\left(R_\xi, R_\xi\right) + \int_t^T B\left(R_\xi, R_\xi; \tau\right) d\tau \geq \frac{1}{2}||R_\xi||_{L^\infty(0,T;L^2(\Omega))} + C||R_\xi||_{L^2(0,T;H^1_{0,\varpi}(\Omega))}. \tag{4.4.44}$$

Therefore, from (4.4.42) to (4.4.44), we have

$$\begin{aligned}
&\left(||R_\xi||_{L^\infty(0,T;L^2(\Omega))} + ||R_\xi||_{L^2(0,T;H^1_{0,\varpi}(\Omega))}\right)^2 \\
&\leq C\left(\frac{1}{2}||R_\xi||_{L^\infty(0,T;L^2(\Omega))} + ||R_\xi||_{L^2(0,T;H^1_{0,\varpi}(\Omega))}\right) \\
&\leq C\left[\left(||\phi||_{L^\infty(0,T;L^2(\Omega))} + ||\phi||_{L^2(0,T;H^1_{0,\varpi}(\Omega))}\right) \cdot \left(||R_\xi||_{L^\infty(0,T;L^2(\Omega))} + ||R_\xi||_{L^2(0,T;H^1_{0,\varpi}(\Omega))}\right)\right] \\
&\quad + C\xi^{-1/k} \\
&\leq C\left[\xi^{-1/2k}\left(||R_\xi||_{L^\infty(0,T;L^2(\Omega))} + ||R_\xi||_{L^2(0,T;H^1_{0,\varpi}(\Omega))}\right) + \xi^{-1/k}\right]
\end{aligned}$$

Clearly, the above inequalities imply that

$$||R_\xi||_{L^\infty(0,T;L^2(\Omega))} + ||R_\xi||_{L^2(0,T;H^1_{0,\varpi}(\Omega))} \leq \frac{C}{\xi^{1/2k}}.$$

Using the triangle inequality and (4.4.27), also noting that $u - u_\xi = R_\xi - \varphi$, we finally have

$$\begin{aligned}
&||u - u_\xi||_{L^\infty(0,T;L^2(\Omega))} + ||u - u_\xi||_{L^2(0,T;H^1_{0,\varpi}(\Omega))} \\
&\leq \left(||R_\xi||_{L^\infty(0,T;L^2(\Omega))} + ||R_\xi||_{L^2(0,T;H^1_{0,\varpi}(\Omega))}\right) + \left(||\phi||_{L^\infty(0,T;L^2(\Omega))} + ||\phi||_{L^2(0,T;H^1_{0,\varpi}(\Omega))}\right) \\
&\leq \frac{C}{\xi^{1/2k}}.
\end{aligned}$$

■

4.5 Discretization Scheme – The Fitted Finite Volume Method

The penalty approach to the complementarity problem (4.1.8) yields a nonlinear parabolic partial differential equation (4.3.21). In this section, we will present the fitted finite volume method for the *Eq. (4.3.21)*. This method is originally proposed in [125] and has been successfully used for solving the single-asset option pricing problem (both European and American options).

Based on the idea of the fitted finite volume method applied to the single-asset American option pricing, we develop the two-dimensional version of this method. The numerical method and analysis presented in this chapter can be easily extended to general n-dimensional American options.

For brevity, we will omit the subscript ξ in the discussion given below. But bear in mind that we refer to V as the solution to the penalized problem rather than the original complementarity problem. In what follows, we will give a detailed statement of the fitted finite volume method of a two-asset American option pricing.

Since the fitted finite volume method for two-asset American options is much more complicated, it will be described in several parts. First, we transform the penalized equation (4.3.21) into a form with the original variable V. Then the discretization of the space is described in the second part. In the third part, the boundary conditions are considered. Finally, the fitted finite volume method is given for the two-asset model.

It is easy to show that the penalized equation (4.3.21) with its final and boundary conditions (4.3.22) is equivalent to the following equation.

$$-V_t - \nabla \cdot (\mathbf{D}\nabla V + \underline{b}V) + \bar{c}V - \xi\,[V^* - V]_+^k = 0, \qquad (x, y, t) \in \Theta \tag{4.5.45}$$

with the boundary and final conditions

$$V(x, y, t)|_\Gamma = 0 \quad \text{and} \quad V(x, y, t = T) = V^*(x, y).$$

In what follows, we will give the fitted finite volume method for (4.5.45).

4.5.1 Discretization

First, we define the meshes for $(0, X)$. Let the interval $I_x := (0, X)$ be divided into N_x sub-intervals:

$$I_{x_i} := (x_i, x_{i+1}), \quad i = 0, \ldots, N_x - 1$$

with $0 = x_0 < x_1 < \cdots < x_{N_x} = X$. For each $i = 0, 1, ..., N_x - 1$, we put $h_{x_i} = x_{i+1} - x_i$ and $h_x = \max_{0 \le i \le N_x - 1} h_{x_i}$. We also let

$$x_{i-1/2} = \frac{(x_{i-1} + x_i)}{2} \text{ and } x_{i+1/2} = \frac{(x_i + x_{i+1})}{2},$$

for each $i = 1, 2, \ldots, N_x - 1$. These mid-points form a second partition of $(0, X)$ if we define $x_{-1/2} = x_0$ and $x_{N_x+1/2} = x_{N_x}$. Finally, we set $l_{x_i} := x_{i+1} - x_i$ for $i = 0, \ldots, N_x$.

Similarly, we define the meshes for $(0, Y)$. Let the interval $I_y := (0, Y)$ be divided into N_y sub-intervals:

$$I_{y_j} := (y_j, y_{j+1}), \quad j = 0, \ldots, N_y - 1$$

with $0 = y_0 < y_1 < \cdots < y_{N_y} = Y$. For each $j = 0, 1, ..., N_y - 1$, we put $h_{y_j} = y_{j+1} - y_j$ and $h_y = \max_{0 \le j \le N_y - 1} h_{y_j}$. We also let

$$y_{j-1/2} = \frac{(y_{j-1} + y_j)}{2} \text{ and } y_{j+1/2} = \frac{(y_j + y_{j+1})}{2},$$

for each $j = 1, 2, \ldots, N_y - 1$. These mid-points form a second partition of $(0, Y)$ if we define $y_{-1/2} = Y_0$ and $y_{N_y+1/2} = Y_{N_y}$. Finally, we set $l_{y_j} := y_{j+1} - y_j$ for $j = 0, \ldots, N_y$.

The above two meshes define a whole mesh on $\Omega := I_x \times I_y$, where all the mesh lines are perpendicular to one of the axes. Also, these mid-points form a second partition of Ω, which is called *boxes,* denoted by $\Omega_{i,j} = [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}]$, for $i = 0, 1, \ldots, N_x - 1$ and $j = 0, 1, \ldots, N_y - 1$. We call the boxes $\Omega_{i,j} = [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}]$ as the dual meshes of the original partition $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$.

4.5.2 Boundary Conditions

Before going on to derive the fitted finite volume method for (4.5.45), we first show how to determine the boundary condition functions $g_1(y, t)$ and $g_2(x, t)$.

1. On the boundary $x = 0$, the function $g_1(y, t)$ is to be determined. In this case, the following equation should be satisfied by $V(y, t) = w_2 g_2(y, t)$:

$$\begin{cases} -\dfrac{\partial V(y,t)}{\partial t} - \dfrac{1}{2}\sigma_2^2 y^2 \dfrac{\partial^2 V(y,t)}{\partial y^2} - ry\dfrac{\partial V(y,t)}{\partial y} + rV(y,t) - \xi\,[V^* - V]_+^k = 0, \\ V(0,t) = \frac{K}{w_2}, \qquad V(Y,t) = 0, \\ V(y,T) = V^*(0,y) = \max(\frac{K}{w_2} - y, 0). \end{cases} \tag{4.5.46}$$

2. On the boundary $y = 0$, the function $g_2(x,t)$ is to be determined. In this case, the following equation should be satisfied by $V(x,t) = w_1 g_1(x,t)$:

$$\begin{cases} -\dfrac{\partial V(x,t)}{\partial t} - \dfrac{1}{2}\sigma_1^2 x^2 \dfrac{\partial^2 V(x,t)}{\partial x^2} - rx\dfrac{\partial V(x,t)}{\partial x} + rV(x,t) - \xi\left[V^* - V\right]_+^k = 0, \\ V(0,t) = \frac{K}{w_1}, \quad V(X,t) = 0, \\ V(x,T) = V^*(x,0) = \max(\frac{K}{w_1} - x, 0). \end{cases} \tag{4.5.47}$$

All the above cases fall into the framework of the single-asset American option pricing problem with the discretization defined in Section 4.5.1. Thus, we can solve the equations (4.5.46) and (4.5.47) by the method for the single-asset American option problem.

4.5.3 The Fitted Finite Volume Method for Two-Asset Model

Now, we concentrate on deriving the fitted finite volume method for (4.5.45) based on the discretization defined in Section 4.5.1. Integrating (4.5.45) over the box $\Omega_{i,j} = [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}]$ and applying the mid-point quadrature rule to the first, third and last terms, we have $(N_x - 1) \times (N_y - 1)$ 'balance equations'

$$-\frac{\partial V_{i,j}}{\partial t} R_{i,j} - \int_{\Omega_{i,j}} \left(\nabla \cdot (\mathbf{D}\nabla V + \underline{b}V)\right) d\Omega_{ij} + \left[\overline{c}_{i,j} V_{i,j} - \xi\left[V_{i,j}^* - V_{i,j}\right]_+^{1/k}\right] R_{ij} = 0, \tag{4.5.48}$$

for $i = 1, \ldots, N_x - 1$ and $j = 1, \ldots, N_y - 1$, where $R_{i,j} = (x_{i+1/2} - x_{i-1/2}) \times (y_{j+1/2} - y_{j-1/2})$ is the area of the box $\Omega_{i,j}$, $V_{i,j} = V(x_i, y_j, t)$, $\underline{c}_{i,j} = \underline{c}(x_i, y_j)$ and $V_{i,j}^* = V^*(x_i, y_j)$.

Now, we consider the approximation of the second term in *Eq.(4.5.48)*. By Gauss's theorem, we have

$$-\int_{\Omega_{i,j}} \left(\nabla \cdot (\mathbf{D}\nabla V + \underline{b}V)\right) d\Omega_{ij} = -\int_{\partial\Omega_{i,j}} (\mathbf{D}\nabla V + \underline{b}V) \cdot n ds,$$

where $\partial\Omega_{i,j}$ are the boundaries of the domain $\Omega_{i,j}$ and n denotes the unit outward-normal vector to $\partial\Omega_{i,j}$. As the domain $\Omega_{i,j}$ is a rectangle, it is clear that $\partial\Omega_{i,j}$ is the four edges of the rectangle. Thus, we have

$$
\begin{aligned}
& -\int_{\Omega_{i,j}} (\nabla \cdot (\mathbf{D}\nabla V + \underline{b}V))\, d\Omega_{ij} \\
= & -\int_{\partial\Omega_{i,j}} (\mathbf{D}\nabla V + \underline{b}V) \cdot n ds \\
= & -\int_{(x_{i+1/2},y_{j-1/2})}^{(x_{i+1/2},y_{j+1/2})} (a_{11}V_x + a_{12}V_y + b_1 V)\, dy + \int_{(x_{i-1/2},y_{j-1/2})}^{(x_{i-1/2},y_{j+1/2})} (a_{11}V_x + a_{12}V_y + b_1 V)\, dy \\
& -\int_{(x_{i-1/2},y_{j+1/2})}^{(x_{i+1/2},y_{j+1/2})} (a_{21}V_x + a_{22}V_y + b_2 V)\, dx + \int_{(x_{i-1/2},y_{j-1/2})}^{(x_{i+1/2},y_{j-1/2})} (a_{21}V_x + a_{22}V_y + b_2 V)\, dx \\
\triangleq & -\mathbb{A} + \mathbb{B} - \mathbb{C} + \mathbb{D}
\end{aligned} \tag{4.5.49}
$$

We shall look at (4.5.49) term by term. First, for the term $\mathbb{A}$, we use the mid-point quadrature rule and (4.1.7). Then, we obtain

$$
\begin{aligned}
\mathbb{A} = & \ h_{y_j} \left[a_{11}V_x + a_{12}V_y + b_1 V\right]_{(x_{i+1/2},y_j)} \\
= & \ h_{y_j} \left[\tfrac{1}{2}\sigma_1^2 x^2 V_x + \tfrac{1}{2}\rho\sigma_1\sigma_2 xyV_y + \left(rx - \sigma_1^2 x - \tfrac{1}{2}\rho\sigma_1\sigma_2 x\right) V\right]_{(x_{i+1/2},y_j)} \\
= & \ h_{y_j} x_{{}_{i+1/2}} \left[\tfrac{1}{2}\sigma_1^2 xV_x + \left(r - \sigma_1^2 - \tfrac{1}{2}\rho\sigma_1\sigma_2\right) V\right]_{(x_{i+1/2},y_j)} + \left[\tfrac{1}{2}\rho\sigma_1\sigma_2 yV_y\right]_{(x_{i+1/2},y_j)} \\
\triangleq & \ h_{y_j} x_{{}_{i+1/2}} \left[\mathbb{I} + \mathbb{III}\right].
\end{aligned} \tag{4.5.50}
$$

We use the forward difference to approximate V_y, i.e.,

$$
V_y = \frac{V_{i,j+1} - V_{i,j}}{h_{y_j}}. \tag{4.5.51}
$$

Hence, by denoting $d = \frac{1}{2}\rho\sigma_1\sigma_2 y$, the term $\mathbb{III}$ is rewritten as:

$$
\left[\frac{1}{2}\rho\sigma_1\sigma_2 yV_y\right]_{(x_{i+1/2},y_j)} \simeq d_{i,j}\frac{V_{i,j+1} - V_{i,j}}{h_{y_j}}.
$$

where $d_{i,j} = d(x_i, y_j) = \frac{1}{2}\rho\sigma_1\sigma_2 y_j$. For the term $\mathbb{I}$, we find that it is essentially the same as (2.4.42) with the coefficients $a = \frac{1}{2}\sigma_1^2$ and $b = (r - \sigma_1^2 - \frac{1}{2}\rho\sigma_1\sigma_2)$. Therefore, following the discussion in Section 2.4.1, we have the following approximation to the term $\mathbb{I}$

$$
\begin{aligned}
& \left[\frac{1}{2}\sigma_1^2 xV_x + \left(r - \sigma_1^2 - \frac{1}{2}\rho\sigma_1\sigma_2\right) V\right]_{(x_{i+1/2},y_j)} \\
\simeq & \begin{cases} b_{i+1/2,j}\frac{x_{i+1}^{\alpha_{i,j}} V_{i+1,j} - x_i^{\alpha_{i,j}} V_{i,j}}{x_{i+1}^{\alpha_{i,j}} - x_i^{\alpha_{i,j}}}, & i \neq 0, \\ \frac{1}{2}\left[\left(a_j + b_{1/2,j} V_{1,j}\right) - \left(a_j - b_{0,j}\right) V_{0,j}\right], & i = 0. \end{cases}
\end{aligned} \tag{4.5.52}
$$

Here, $b_{i+1/2,j} = b(x_{i+1/2}, y_j)$ and $\alpha_{i,j} = b_{i+1/2,j}/a$.

Substituting (4.5.51) and (4.5.52) into (4.5.50), we finally obtain the approximation of the term $\mathbb{A}$ in (4.5.49), i.e.,

$$\begin{aligned}\mathbb{A} &= \int_{(x_{i+1/2},y_{j-1/2})}^{(x_{i+1/2},y_{j+1/2})} (a_{11}V_x + a_{12}V_y + b_1 V)\, dy \\ &\simeq \; h_{y_j} x_{i+1/2} (b_{i+1/2,j} \tfrac{x_{i+1}^{\alpha_{i,j}} V_{i+1,j} - x_i^{\alpha_{i,j}} V_{i,j}}{x_{i+1}^{\alpha_{i,j}} - x_i^{\alpha_{i,j}}} + d_{i,j} \tfrac{V_{i,j+1}-V_{i,j}}{h_{y_j}}), \qquad i \neq 0,\end{aligned} \tag{4.5.53}$$

Similarly, the approximation of the terms $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$ in (4.5.49) can be achieved as follows.

$$\begin{aligned}\mathbb{B} &= \int_{(x_{i-1/2},y_{j-1/2})}^{(x_{i-1/2},y_{j+1/2})} (a_{11}V_x + a_{12}V_y + b_1 V)\, dy \\ &\simeq \begin{cases} h_{y_j} x_{i-1/2} (b_{i-1/2,j} \tfrac{x_i^{\alpha_{i-1,j}} V_{i,j} - x_{i-1}^{\alpha_{i-1,j}} V_{i-1,j}}{x_i^{\alpha_{i-1,j}} - x_{i-1}^{\alpha_{i-1,j}}} + d_{i,j} \tfrac{V_{i,j+1}-V_{i,j}}{h_{y_j}}), \; i = 2, \dots N_x - 1, \\ h_{y_j} x_{1/2} (\tfrac{1}{2}[(a_j + b_{1/2,j})V_{1,j} - (a_j - b_{0,j})V_{0,j}] + d_{1,j} \tfrac{V_{1,j+1}-V_{1,j}}{h_{y_j}}), \; i = 1, \end{cases}\end{aligned} \tag{4.5.54}$$

$$\begin{aligned}\mathbb{C} &= \int_{(x_{i-1/2},y_{j+1/2})}^{(x_{i+1/2},y_{j+1/2})} (a_{21}V_x + a_{22}V_y + b_2 V)\, dx \\ &\simeq \; h_{x_i} y_{j+1/2} (\overline{b}_{i,j+1/2} \tfrac{y_{j+1}^{\overline{\alpha}_{i,j}} V_{i,j+1} - y_j^{\overline{\alpha}_{i,j}} V_{i,j}}{y_{j+1}^{\overline{\alpha}_{i,j}} - y_j^{\overline{\alpha}_{i,j}}} + \overline{d}_{i,j} \tfrac{V_{i+1,j}-V_{i,j}}{h_{x_i}}), \qquad j \neq 0,\end{aligned} \tag{4.5.55}$$

$$\begin{aligned}\mathbb{D} &= \int_{(x_{i+1/2},y_{j-1/2})}^{(x_{i+1/2},y_{j+1/2})} (a_{11}V_x + a_{12}V_y + b_1 V)\, dy \\ &\simeq \begin{cases} h_{x_i} y_{j-1/2} (\overline{b}_{i,j-1/2} \tfrac{y_j^{\overline{\alpha}_{i,j-1}} V_{i,j} - y_{j-1}^{\overline{\alpha}_{i,j-1}} V_{i,j-1}}{y_j^{\overline{\alpha}_{i,j-1}} - y_{j-1}^{\overline{\alpha}_{i,j-1}}} + \overline{d}_{i,j} \tfrac{V_{i+1,j}-V_{i,j}}{h_{x_i}}), \; j = 2, \dots N_y - 1, \\ h_{x_i} y_{1/2} (\tfrac{1}{2}[(\overline{a}_i + \overline{b}_{i,1/2})V_{i,1} - (\overline{a}_i - \overline{b}_{i,0})V_{i,0}] + \overline{d}_{i,1} \tfrac{V_{i+1,1}-V_{i,1}}{h_{x_i}}), \; j = 1, \end{cases}\end{aligned} \tag{4.5.56}$$

where

$$b = r - \sigma_1^2 - \tfrac{1}{2}\rho\sigma_1\sigma_2, \qquad a = \tfrac{1}{2}\sigma_1^2, \qquad d = \tfrac{1}{2}\rho\sigma_1\sigma_2 y,$$
$$b_{i,j+1/2} = b(x_i, y_{j+1/2}), \quad \alpha_{i,j} = b_{i,j+1/2}/a, \quad d_{i,j} = d(x_i, y_j),$$

and

$$\overline{b} = r - \sigma_2^2 - \tfrac{1}{2}\rho\sigma_1\sigma_2, \qquad \overline{a} = \tfrac{1}{2}\sigma_2^2, \qquad \overline{d} = \tfrac{1}{2}\rho\sigma_1\sigma_2 x,$$
$$\overline{b}_{i,j+1/2} = \overline{b}(x_i, y_{j+1/2}), \quad \overline{\alpha}_{i,j} = \overline{b}_{i,j+1/2}/\overline{a}, \quad \overline{d}_{i,j} = \overline{d}(x_i, y_j).$$

Using (4.5.53), (4.5.54), (4.5.55), (4.5.56) and (4.5.49), we obtain from (4.5.48) the following $(N_x - 1) \times (N_y - 1)$ equations.

$$
\begin{aligned}
-\frac{\partial V_{i,j}}{\partial t} R_{i,j} + e^{i,j}_{i-1,j} V_{i-1,j} + e^{i,j}_{i,j-1} V_{i,j-1} + e^{i,j}_{i,j} V_{i,j} + e^{i,j}_{i,j+1} V_{i,j+1} & \\
+ e^{i,j}_{i+1,j} V_{i+1,j} - \xi \left[V^*_{i,j} - V_{i,j} \right]_+^{1/k} R_{ij} & = 0
\end{aligned}
\tag{4.5.57}
$$

for $i = 1, \ldots, N_x - 1$ and $j = 1, \ldots, N_y - 1$. Here

$$
\left\{
\begin{array}{rcl}
e^{1,1}_{0,1} & = & -h_{y_1} \frac{x_{1/2}(a_1 - b_{0,1})}{2}, \\
e^{1,1}_{1,0} & = & -h_{x1} \frac{y_{1/2}(\overline{a}_1 - \overline{b}_{1,1/2})}{2}, \\
e^{1,1}_{1.1} & = & \begin{array}{l} h_{y_1}\left(\frac{b_{3/2,1} x_{3/2} x_1^{\alpha_{1,1}}}{x_2^{\alpha_{1,1}} - x_1^{\alpha_{1,1}}} + \frac{x_{1/2}(a_1 + b_{1/2,1})}{2} + \overline{d}_{1,1} \right) + \\ h_{x_1}\left(\frac{\overline{b}_{1,3/2} y_{3/2} y_1^{\overline{\alpha}_{1,1}}}{y_2^{\overline{\alpha}_{1,1}} - y_1^{\overline{\alpha}_{1,1}}} + \frac{(\overline{a}_1 + \overline{b}_{1,1/2}) y_{1/2}}{2} + d_{1,1} \right) + \overline{c}_{1,1} R_{1,1}, \end{array} \\
e^{1,1}_{1,2} & = & -h_{x_1}\left(\frac{\overline{b}_{1,3/2} y_{3/2} y_2^{\overline{\alpha}_{1,1}}}{y_2^{\overline{\alpha}_{1,1}} - y_1^{\overline{\alpha}_{1,1}}} + d_{1,1} \right), \\
e^{1,1}_{2,1} & = & -h_{y_1}\left(\frac{b_{3/2,1} x_{3/2} x_2^{\alpha_{1,1}}}{x_2^{\alpha_{1,1}} - x_1^{\alpha_{1,1}}} + \overline{d}_{1,1} \right),
\end{array}
\right.
\tag{4.5.58}
$$

and

$$
\left\{
\begin{array}{rcl}
e^{1,j}_{0,j} & = & -h_{y_j} \frac{x_{1/2}(a_j - b_{0,j})}{2}, \\
e^{1,j}_{1,j-1} & = & -h_{x_1} \frac{\overline{b}_{1,j-1/2} y_{j-1/2} y_{j-1}^{\overline{\alpha}_{1,j-1}}}{y_j^{\overline{\alpha}_{1,j-1}} - y_{j-1}^{\overline{\alpha}_{1,j-1}}}, \\
e^{1,j}_{1.j} & = & \begin{array}{l} h_{y_j}\left(\frac{b_{3/2,j} x_{3/2} x_1^{\alpha_{1,j}}}{x_2^{\alpha_{1,j}} - x_1^{\alpha_{1,j}}} + \frac{x_{1/2}(a_j + b_{1/2,j})}{2} + \overline{d}_{1,j} \right) + \\ h_{x_1}\left(\frac{\overline{b}_{1,j+1/2} y_{j+1/2} y_j^{\overline{\alpha}_{1,j}}}{y_{j+1}^{\overline{\alpha}_{1,j}} - y_j^{\overline{\alpha}_{1,j}}} + \frac{\overline{b}_{1,j-1/2} y_{j-1/2} y_j^{\overline{\alpha}_{1,j-1}}}{y_j^{\overline{\alpha}_{1,j-1}} - y_{j-1}^{\overline{\alpha}_{1,j-1}}} + d_{1,j} \right) + \underline{c}_{1,j} R_{1,j}, \end{array} \\
e^{1,j}_{1,j+1} & = & -h_{x_1}\left(\frac{\overline{b}_{1,j+1/2} y_{j+1/2} y_{j+1}^{\overline{\alpha}_{1,j}}}{y_{j+1}^{\overline{\alpha}_{1,j}} - y_j^{\overline{\alpha}_{1,j}}} \right) + d_{1,j}, \\
e^{1,j}_{2,j} & = & -h_{y_j}\left(\frac{b_{3/2,j} x_{3/2} x_2^{\alpha_{1,j}}}{x_2^{\alpha_{1,j}} - x_1^{\alpha_{1,j}}} + \overline{d}_{1,j} \right),
\end{array}
\right.
\tag{4.5.59}
$$

for $j = 2, \ldots, N_y - 1$,

$$
\left\{
\begin{array}{rcl}
e^{i,1}_{i-1,1} & = & -h_{y_1} \frac{b_{i-1/2,1} x_{i-1/2} x_{i-1}^{\alpha_{i-1,1}}}{x_i^{\alpha_{i-1,1}} - x_{i-1}^{\alpha_{i-1,1}}}, \\
e^{i,1}_{i,0} & = & -h_{x_i} \frac{y_{1/2}\left(\overline{a}_i - \overline{b}_{i,1/2}\right)}{2}, \\
e^{i,1}_{i,1} & = & \begin{array}{l} h_{y_1}\left(\frac{b_{i+1/2,1} x_{i+1/2} x_i^{\alpha_{i,1}}}{x_{i+1}^{\alpha_{i,1}} - x_i^{\alpha_{i,1}}} + \frac{b_{i-1/2,1} x_{i-1/2} x_i^{\alpha_{i-1,1}}}{x_i^{\alpha_{i-1,1}} - x_{i-1}^{\alpha_{i-1,1}}} + \overline{d}_{i,1} \right) + \\ h_{x_i}\left(\frac{\overline{b}_{i,3/2} y_{3/2} y_1^{\overline{\alpha}_{i,1}}}{y_2^{\overline{\alpha}_{i,1}} - y_1^{\overline{\alpha}_{i,1}}} + \frac{(\overline{a}_i + \overline{b}_{i,1/2}) y_{1/2}}{2} + d_{i,1} \right) + c_{i,1} R_{i,1}, \end{array} \\
e^{i,1}_{i,2} & = & -h_{x_i}\left(\frac{\overline{b}_{i,3/2} y_{3/2} y_2^{\overline{\alpha}_{i,1}}}{y_2^{\overline{\alpha}_{i,1}} - y_1^{\overline{\alpha}_{i,1}}} + d_{i,1} \right), \\
e^{i,1}_{i+1,1} & = & -h_{y_1}\left(\frac{b_{i+1/2,1} x_{i+1/2} x_{i+1}^{\alpha_{i,1}}}{x_{i+1}^{\alpha_{i,1}} - x_i^{\alpha_{i,1}}} + \overline{d}_{i,1} \right).
\end{array}
\right.
\tag{4.5.60}
$$

for $i = 2, \ldots, N_x - 1$, and

$$\left\{ \begin{array}{lcl} e^{i,j}_{i-1,j} & = & -h_{y_j} \frac{b_{i-1/2,j} x_{i-1/2} x_{i-1}^{\alpha_{i-1,j}}}{x_i^{\alpha_{i-1,j}} - x_{i-1}^{\alpha_{i-1,j}}}, \\ e^{i,j}_{i,j-1} & = & -h_{x_i} \frac{\overline{b}_{i,j-1/2} y_{j-1/2} y_{j-1}^{\overline{\alpha}_{i,j-1}}}{y_j^{\overline{\alpha}_{i,j-1}} - y_{j-1}^{\overline{\alpha}_{i,j-1}}}, \\ e^{i,j}_{i,j} & = & \begin{array}{l} h_{y_j} \left(\frac{b_{i+1/2,j} x_{i+1/2} x_i^{\alpha_{i,j}}}{x_{i+1}^{\alpha_{i,j}} - x_i^{\alpha_{i,j}}} + \frac{b_{i-1/2,j} x_{i-1/2} x_i^{\alpha_{i-1,j}}}{x_i^{\alpha_{i-1,j}} - x_{i-1}^{\alpha_{i-1,j}}} + \overline{d}_{i,j} \right) + \\ h_{x_i} \left(\frac{\overline{b}_{i,j+1/2} y_{j+1/2} y_j^{\overline{\alpha}_{i,j}}}{y_{j+1}^{\overline{\alpha}_{i,j}} - y_j^{\overline{\alpha}_{i,j}}} + \frac{\overline{b}_{i,j-1/2} y_{j-1/2} y_j^{\overline{\alpha}_{i,j-1}}}{y_j^{\overline{\alpha}_{i,j-1}} - y_{j-1}^{\overline{\alpha}_{i,j-1}}} + d_{i,j} \right) + \overline{c}_{i,j} R_{i,j}, \end{array} \\ e^{i,j}_{i,j+1} & = & -h_{x_i} \left(\frac{\overline{b}_{i,j+1/2} y_{j+1/2} y_{j+1}^{\overline{\alpha}_{i,j}}}{y_{j+1}^{\overline{\alpha}_{i,j}} - y_j^{\overline{\alpha}_{i,j}}} + d_{i,j} \right), \\ e^{i,j}_{i+1,j} & = & -h_{y_j} \left(\frac{b_{i+1/2,j} x_{i+1/2} x_{i+1}^{\alpha_{i,j}}}{x_{i+1}^{\alpha_{i,j}} - x_i^{\alpha_{i,j}}} + \overline{d}_{i,j} \right). \end{array} \right. \tag{4.5.61}$$

for $i = 2, \ldots, N_x - 1$, $j = 2, \ldots, N_y - 1$ and $e^{i,j}_{m,n} = 0$ if $m \neq i-1, i, i+1$ and $n \neq j-1, j, j+1$.

By defining

$$E_{i,j} = \left(0, \cdots, 0, e^{i,j}_{i-1,j}, 0, \cdots, 0, e^{i,j}_{i,j-1}, e^{i,j}_{i,j}, e^{i,j}_{i,j+1}, 0, \cdots, 0, e^{i,j}_{i+1,j}, 0, \cdots, 0\right)$$

for $i = 2, \ldots, N_x - 1$, $j = 2, \ldots, N_y - 1$, and

$$\mathbf{V} = \left(V_{1,1}, \cdots, V_{1,N_y-1}, V_{2,1}, \cdots, V_{2,N_y-1}, \cdots, V_{N_x-1,1}, \cdots, V_{N_x-1,N_y-1}\right)^T$$

with $V_{i,0}$, $i = 1, \ldots, N_x$ and $V_{0,j}$, $j = 1, \ldots, N_y$ in (4.5.57) being equal to the given boundary conditions, we can rewrite (4.5.57) as:

$$-\frac{\partial V_{i,j}}{\partial t} R_{i,j} + E_{i,j} \mathbf{V} + p(V_{i,j}) = 0, \tag{4.5.62}$$

where

$$p(V_{i,j}) = -\xi R_{i,j} [V^*_{i,j} - V_{i,j}]_+^{1/k}. \tag{4.5.63}$$

This is a system of $(N_x - 1)^2 \times (N_y - 1)^2$ linear equations with $(N_x - 1) \times (N_y - 1)$ unknown values.

Now, as we have done in the case of the single-asset American options, we discretize the time by letting t_i $(i = 0, 1, \ldots M)$ be a set of partition points in $[0, T]$ satisfying $T = t_0 > t_1, \ldots, > t_M = 0$. Then, we apply the two-level implicit time-stepping method with a splitting parameter $\theta \in [0, 1/2]$ to (4.5.62), we get the final full discrete system

$$\left(\theta E^{m+1} + G^m\right) \mathbf{V}^{m+1} + \theta D(\mathbf{V}^{m+1}) = \left(G^m - (1-\theta) E^m\right) \mathbf{V}^m - (1-\theta) D(\mathbf{V}^m), \tag{4.5.64}$$

where

$$\begin{aligned}
\mathbf{V}^m &= (V_{1,1}^m, \cdots, V_{1,N_y-1}^m, V_{2,1}^m, \cdots, V_{2,N_y-1}^m, \cdots, V_{N_x-1,1}^m, \cdots, V_{N_x-1,N_y-1}^m)^T, \\
E^m &= (E_{1,1}^m, \cdots, E_{1,N_y-1}^m, E_{2,1}^m, \cdots, E_{2,N_y-1}^m, \cdots, E_{N_x-1,1}^m, \cdots, E_{N_x-1,N_y-1}^m)^T, \\
G^m &= diag\left(-R_{1,1}/\triangle t_m, \ldots, -R_{N_x-1,N_y-1}/\triangle t_m\right), \\
D(V_{1,1}^m) &= (p(V_{1,1}^m), \cdots, p(V_{N_x-1,N_y-1}^m))^T,
\end{aligned} \tag{4.5.65}$$

for $m = 0, 1, \ldots, m-1$, where $\triangle t_m = t_{m+1} - t_m < 0$, $\mathbf{V}^m$ denotes the approximation of $\mathbf{V}$ at $t = t_m$ and $E_{i,j}^m = E_{i,j}(t_m)$.

4.5.4 Solution to Discrete System

After the lengthy description of the fitted finite volume method for the two-asset American option pricing, we finally obtain the nonlinear discrete system (4.5.64). In [125], the solution to the discrete system for the single-asset American options has been studied. In fact, system (4.5.64) has nearly the same structure but in a higher dimension. In the following, we will concentrate on the solution to system (4.5.64).

The Newton method will be applied to solving the nonlinear system (4.5.64). Note that when $0 < k < 1$, it follows from (4.5.63) that $p'(V_{i,j}^m) \to \infty$ as $V_{i,j}^* - V_{i,j} \to 0^+$. We apply the smoothing technique in Appendix 7.2 to overcoming this difficulty.

Now, applying Newton's method to (4.5.64) gives

$$\begin{aligned}
\left[\theta E^{m+1} + G^m + \theta J_D(\varpi^{l-1})\right] \delta\varpi^l &= \left[G^m - (1-\theta) E^m\right] \mathbf{V}^m - (1-\theta) D(\mathbf{V}^m) \\
&\quad - \left(\theta E^{m+1} + G^m\right) \varpi^l - \theta D(\varpi^{l-1}), \\
\varpi^l &= \varpi^{l-1} + \zeta \cdot \delta\varpi^l
\end{aligned} \tag{4.5.66}$$

for $l = 1, 2, \ldots$ with ϖ^0 being a given initial guess, where $J_D(\varpi)$ denotes the Jacobian of the column vector $D(\varpi)$ and $\zeta \in (0, 1]$ denotes a damping parameter. We then choose

$$\mathbf{V}^{m+1} = \lim_{l\to\infty} \varpi^l.$$

We will prove that the system matrix of (4.5.66) is an M-matrix.

Theorem 4.5.1 *For any given $m = 1, 2, \ldots M-1$, if $|\triangle t_m|$ is sufficiently small and $\bar{c} \geq 0$, then the system matrix of (4.5.66) is an M-matrix.*

Proof. From the definition of $D(\mathbf{V})$, it is easy to see that the Jacobian

$$J_D(\varpi^l) = diag(p'(V_{1,1}^m), \cdots, p'(V_{N_x-1,N_y-1}^m)),$$

i.e. it is a diagonal matrix. We also have $p'(V_{i,j}^m) \geq 0$ for all $i = 1, \ldots N_x - 1$ and $j = 1, \ldots N_y - 1$. Thus, to show that the system (4.5.66) is an M-matrix, it suffices to show that $\theta E^{m+1} + G^m$ is an M-matrix.

First, we note that $e_{m,n}^{i,j} \leq 0$ for all $m \neq i$, $n \neq j$ since

$$\frac{b_{i+1/2,j}}{x_{i+1}^{\alpha_{i,j}} - x_i^{\alpha_{i,j}}} = \frac{a\ \alpha_{i,j}}{x_{i+1}^{\alpha_{i,j}} - x_i^{\alpha_{i,j}}} > 0,\ \frac{\overline{b}_{i,j+1/2}}{y_{j+1}^{\overline{\alpha}_{i,j}} - y_j^{\overline{\alpha}_{i,j}}} = \frac{\overline{a}\ \overline{\alpha}_{i,j}}{y_{j+1}^{\overline{\alpha}_{i,j}} - y_j^{\overline{\alpha}_{i,j}}} > 0, \qquad (4.5.67)$$

for all $i = 1, \ldots, N_x - 1$, $j = 1, \ldots, N_y - 1$ and all $b_{i+1/2,j} \neq 0$, $\overline{b}_{i,j+1/2} \neq 0$. (4.5.67) also holds when $b_{i+1/2,j} \to 0$, $\overline{b}_{i,j+1/2} \to 0$. Furthermore, from (4.5.61), it follows that when $c_{i,j} \geq 0$, for all $i = 2, \ldots, N_x - 1$, $j = 2, \ldots, N_y - 1$,

$$\begin{aligned}\left(e_{i,j}^{i,j}\right)^{m+1} &\geq \left|(e_{i-1,j}^{i,j})^{m+1}\right| + \left|(e_{i,j-1}^{i,j})^{m+1}\right| + \left|(e_{i,j+1}^{i,j})^{m+1}\right| + \left|(e_{i+1,j}^{i,j})^{m+1}\right| + c_{i,j}^{m+1} R_{i,j} \\ &= \sum_{p=1}^{N_x-1} \sum_{q=1}^{N_y-1} \left|(e_{p,q}^{i,j})^{m+1}\right| + \underline{c}_{i,j}^{m+1} R_{i,j},\end{aligned}$$

since d and $\overline{d}$ are all non-negative. Therefore, E^{m+1} is diagonally dominant with respect to its columns. When $i = 1$ or $j = 1$ or $i = j = 1$, it follows from (4.5.58), (4.5.59) and (4.5.60) that

$$\begin{aligned}(e_{1,1}^{1,1})^{m+1} &\geq \sum_{p=1}^{N_x-1} \sum_{q=1}^{N_y-1} \left|(e_{p,q}^{1,1})^{m+1}\right|, \\ (e_{1,j}^{1,j})^{m+1} &\geq \sum_{p=1}^{N_x-1} \sum_{q=1}^{N_y-1} \left|(e_{p,q}^{1,j})^{m+1}\right|, \\ (e_{i,1}^{i,1})^{m+1} &\geq \sum_{p=1}^{N_x-1} \sum_{q=1}^{N_y-1} \left|(e_{p,q}^{i,1})^{m+1}\right|,\end{aligned}$$

if $c \geq 0$, $a_j^{m+1} + b_{1/2,j}^{m+1} \geq 0$ and $\overline{a}_i^{m+1} + \overline{b}_{i,1/2}^{m+1} \geq 0$. Hence, form the above analysis, we can see that for all i, j, E^{m+1} is diagonally dominant and satisfies that $e_{i,j}^{i,j} > 0$ and $e < 0$ for all $m \neq i$, $n \neq j$. This implies that E^{m+1} is an M-matrix.

For the second part, we first note that G^m of the system matrix (4.5.66) is a diagonal matrix with positive diagonal entries. When $|\triangle t_m|$ is sufficiently small, we also have

$$\theta \underline{c}_{i,j} + \frac{R_{i,j}}{-\triangle t_m} > 0.$$

So, $\theta E^{m+1} + G^m$ is an M-matrix. ∎

Theorem 4.5.1 implies that the fully discrete system (4.5.64) satisfies the discrete maximum principle and the discretization is monotone.

4.6 Numerical Experiments

In this section, we demonstrate the efficiency and usefulness of the above penalty and numerical method by solving the following test model problem with different values of the parameters k and ξ. Throughout this section, the following model parameters are used.

$$\begin{aligned} r &= 0.1, \\ \sigma_1 &= 0.2, \ \sigma_2 = 0.3, \\ \omega_1 &= 0.6, \ \omega_2 = 0.4, \\ K &= 1.0, \\ T &= 1.0. \end{aligned}$$

The correlation parameter ρ is chosen equal to 0 and 0.5. In order to perform simulations, we must choose an upper limit for the solution domain, that is a domain where option values outside are regarded worthless. For our model, we choose $X = Y = 4$.

It should be noted that in the special case when $k = 1$ and 2, the resulting penalty functions are corresponding to linear and quadratic penalty functions, which have been introduced to valuating the American option pricing problem, see [44, 35, 45, 131]. In the case of $k = 1/2$, the lower order penalty function $l_{1/2}$ is obtained. In [125, 131], the lower order penalty method l_k is carefully studied, where $k = 1/2$ is chosen. In this chapter, we choose $k = 2, 1$ and $1/2$ to implement the numerical tests. Some computational details are listed below.

1. When $k = 2$, the nonlinear system (4.5.64) is smooth. Therefore, the Newton's method is applied to solving (4.5.64).

2. When $k = 1$, the nonlinear system (4.5.64) is semismooth. We apply the semismooth Newton's method [104] to solving (4.5.64).

3. When $k = 1/2$, the nonlinear system (4.5.64) is nonsmooth. Hence, the smoothing technique in Appendix 7.2 is adapted so as to applying the Newton's method to solving (4.5.64). The choice of smoothing interval ε should be moderate. It should not be too large, otherwise the rate of convergence will be too low. On the other hand, ε should not be too small, otherwise the Jacobian will be highly singular. From experiments, in our numerical tests, ε is chosen moderately as 10^{-3}.

4. We choose the timestep splitting parameter $\theta = 1/2$ in (4.5.64) , i.e. the Crank-Nicolson scheme.

5. The choice of the penalty parameter should be moderate. It is well known that too large penalty parameter will lead to computational difficulty. In our numerical tests, the attainable maximum penalty parameters are reported. From experience, we choose $\xi = 100000$ when $k = 2$, $\xi = 1000$ when $k = 1$, $\xi = 10$ when $k = 1/2$, respectively.

All the implementations of our numerical method are conducted within the Matlab 7.0 framework. When $\rho = 0$, the option value and the value of the constraint $V - V^*$ at time to maturity are depicted in Figures 4.1-4.6. Figures 4.1 and 4.2 show the results computed by the quadratic penalty method with $k = 2, \xi = 100000$. Figures 4.3 and 4.4 show the results computed by the linear penalty method with $k = 1, \xi = 1000$. Figures 4.5 and 4.6 depict the results computed by the lower order penalty method (l_2) with $k = 1/2, \xi = 10$.

When $\rho = 0.5$, the option value and the value of the constraint $V - V^*$ at time to maturity are depicted in Figures 4.7-4.12. Figures 4.7 and 4.8 show the results computed by the quadratic penalty method with $k = 2, \xi = 100000$. Figures 4.9 and 4.10 depict the results computed by the linear penalty method with $k = 1, \xi = 1000$. Figures 4.11 and 4.12 show the results computed by the lower order penalty method (l_2) with $k = 1/2, \xi = 10$.

From our numerical tests, we can observe that the numerical results computed by the quadratic, l_1 and $l_{1/2}$ penalty methods are compatible. For quadratic penalty method,

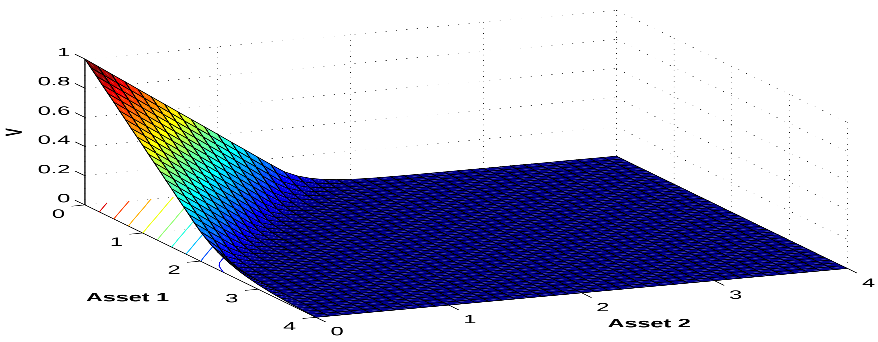

Figure 4.1: V at time to maturity computed by the quadratic penalty method ($\rho = 0$)

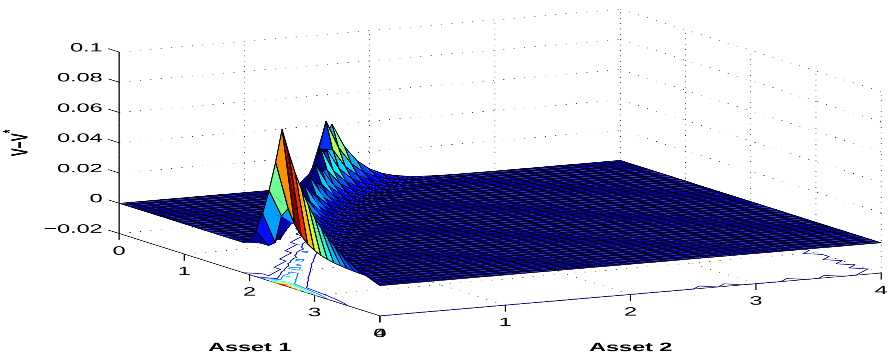

Figure 4.2: $V - V^*$ at time to maturity computed by the quadratic penalty method ($\rho = 0$)

it needs the largest penalty parameter and is computationally most consuming. For $l_{1/2}$ penalty methods, it needs the smallest penalty parameters and the least computational time. These observations are consistent with our theoretical result (4.4.38).

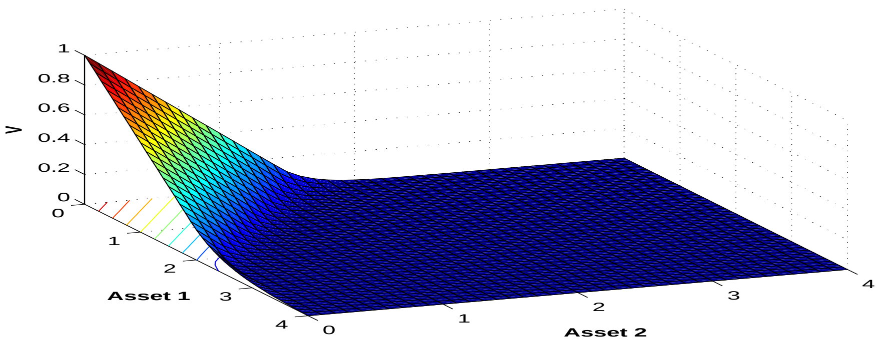

Figure 4.3: V at time to maturity computed by the linear penalty method ($\rho = 0$)

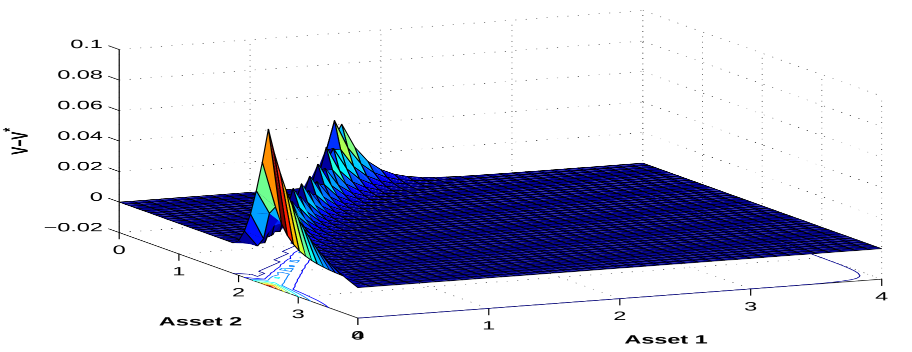

Figure 4.4: $V - V^*$ at time to maturity computed by the linear penalty method ($\rho = 0$)

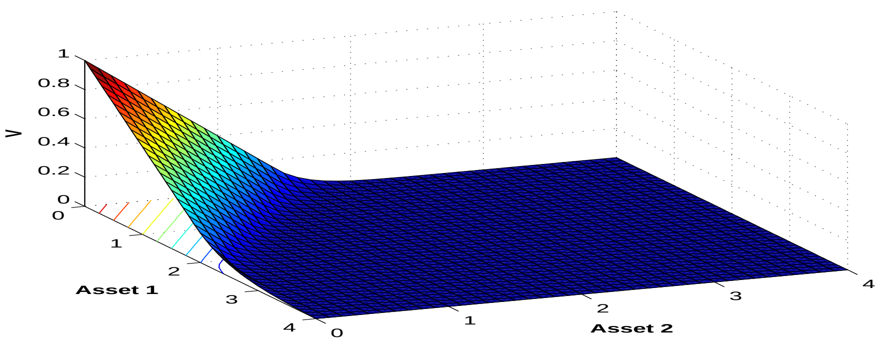

Figure 4.5: V at time to maturity computed by the lower order penalty method ($\rho = 0$)

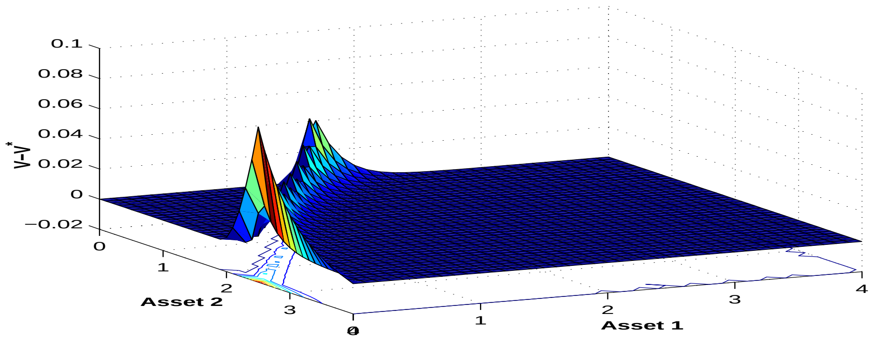

Figure 4.6: $V - V^*$ at time to maturity computed by the lower order penalty method ($\rho = 0$)

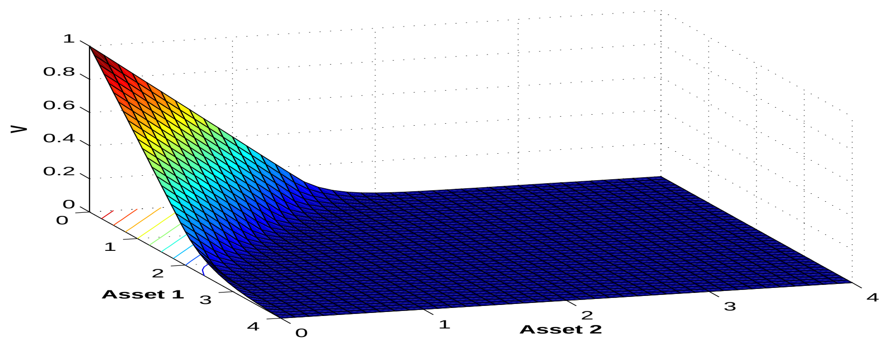

Figure 4.7: V at time to maturity computed by the quadratic penalty method ($\rho = 0.5$)

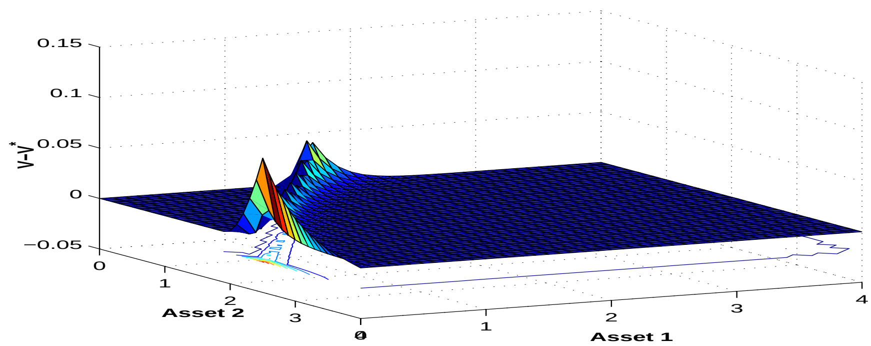

Figure 4.8: $V - V^*$ at time to maturity computed by the quadratic penalty method ($\rho = 0.5$)

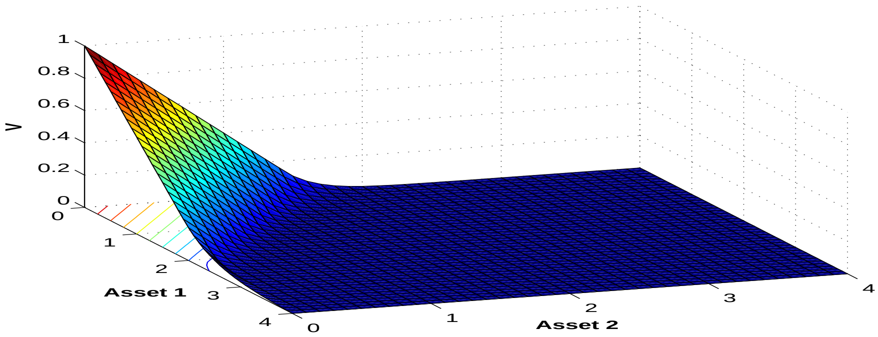

Figure 4.9: V at time to maturity computed by the linear penalty method ($\rho = 0.5$)

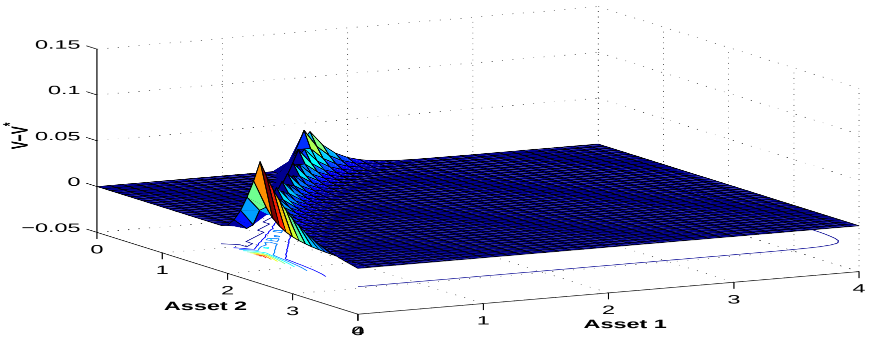

Figure 4.10: $V - V^*$ at time to maturity computed by the linear penalty method ($\rho = 0.5$)

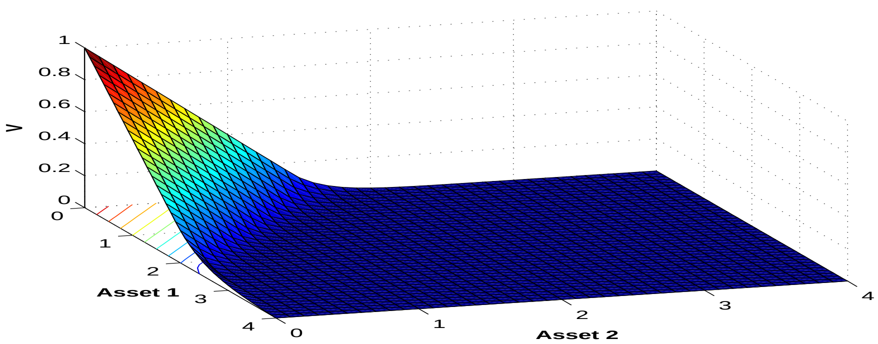

Figure 4.11: V at time to maturity computed by the lower order penalty method ($\rho = 0.5$)

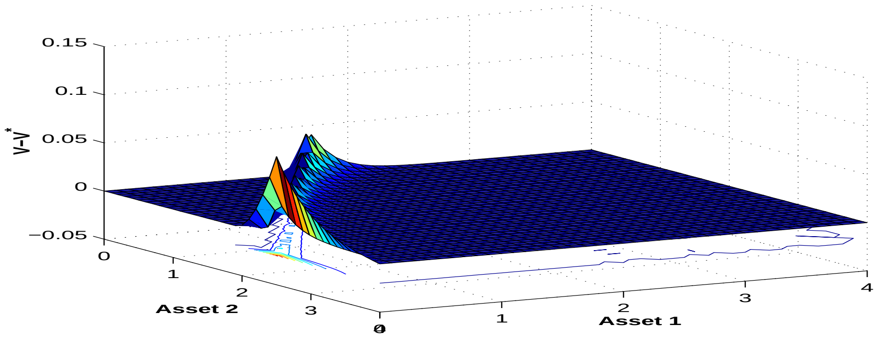

Figure 4.12: $V - V^*$ at time to maturity computed by the lower order penalty method ($\rho = 0.5$)

4.7 Summary

In this chapter, we have presented the power penalty method for the two-asset American option pricing problem and developed the fitted volume method to solve the nonlinear penalized parabolic PDE. By the variational analysis, we derived the penalty approach to the complementarity problem which is resulted from the two-asset American option pricing problem. The rate of convergence of the power penalty method was obtained in some infinite dimensional spaces. To solve the penalized PDE, we have proposed the two-dimensional version of the fitted finite volume method. Numerical procedure was developed carefully. Numerical examples were implemented to verify the theoretical results. The numerical results showed that the method performed very well for the test problems.

Chapter 5

Augmented Lagrangian Method for American Option Pricing

In the previous chapters, we focus on studying the penalty methods for several kinds of American-style option pricing. As we mentioned before, the convergence rates of l_1 and l_2 penalty methods are fast and their accuracy is high for valuating American option. However, to achieve a desirable accuracy, the l_2 and l_1 penalty methods require large enough penalty parameters. It is well known that a large penalty parameter could cause numerical difficulty. As for the power penalty method (l_k, $0 < k < 1$) for pricing American options, although a much smaller penalty parameter is required, the resulting nonlinear equation systems are nonsmooth and non-Lipschitz. Hence, much more computational time is required to solve the nonsmooth equation systems.

Lagrangian method for valuating American options was used in [122]. This method is to solve an equivalent quadratic programming problem. By using the Uzawa's duality method [53], an algorithm was developed in [122] for solving the American option pricing. For this method, the optimal exercise boundary can be easily obtained from Kuhn-Tucker multipliers by taking into account that exercise boundary vanishes only outside the active set. However, the unsatisfactory convergence rate and low accuracy are the two major disadvantages.

While Lagrangian method and penalty methods have been investigated to price American options, little attention is devoted to develop augmented Lagrangian method

(ALM) to solve those problems. It is widely known that ALM has several advantages over Lagrangian method and penalty method, see [9, 40, 51] and the references therein. First, to achieve the desired accuracy, the augmented Lagrangian method needs much less iterations than the Lagrangian method. Hence, the rate of convergence of ALM is faster than that of Lagrangian method. Second, much larger penalty parameters can be attained by ALM such that a more accurate solution can be obtained. Third, with the same level of accuracy, the augmented Lagrangian method requires a much smaller penalty parameter than the penalty method.

In this chapter, we formulate the continuous models resulted from the American put option pricing problem, that is, a partial differential complementarity problem and a variational inequality problem. The fitted finite volume method [125] is employed for the discretization of the variational inequality formulation. This leads to a sequence of large scale finite dimensional variational inequality problems. We adapt the augmented Lagrangian method to solving these variational inequalities. We present the explicit augmented Lagrangian formulation and design the corresponding augmented Lagrangian algorithm to solve the American put option pricing problem. The existence of the Lagrangian multiplier is established. Due to the elimination of oscillation in the fitted finite volume discretization, these resulting variational inequalities can be effectively solved by the augmented Lagrangian method [57, 67, 68, 73]. A convergence rate of order $\mathcal{O}(1/\xi)$ of the augmented Lagrangian method with respect to the penalty parameter ξ is obtained. Moreover, for fixed penalty parameter ξ, a superlinear convergence rate for each discrete variational inequality problem is established. To explore the advantages of the augmented Lagrangian method over penalty methods and the Lagrangian method, empirical tests are carried out with two sets of problems. Furthermore, to compare the augmented Lagrangian method with the PSOR method, empirical experiments with different market parameters σ and r and different step sizes of space variable and time variable are implemented. For the augmented Lagrangian method, including robustness with respect to changes of market parameters, the advantages over the PSOR method are revealed.

5.1 Continuous Models of American Option Pricing

First, let us recall the Black-Scholes operator

$$LV: \;= -\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV.$$

Then, we transform the Black-Scholes operator into the following self-adjoint form:

$$\mathcal{L}V: \;= -\frac{\partial V}{\partial t} - \frac{\partial}{\partial S}\left[aS^2\frac{\partial V}{\partial S} + bSV\right] + cV$$

with $a = \frac{1}{2}\sigma^2$, $b = r - \sigma^2$, $c = 2r - \sigma^2$, where $V(S,t)$ is the value of American put option. For convenience, we replace the asset price S with x. Then, the partial differential complementarity form of the American put option pricing problem can be stated as:

Problem 5.1.1

$$(DCP)\begin{cases} \mathcal{L}V(x,t) \geq 0, \\ V(x,t) - \Lambda(x,t) \geq 0, \\ \mathcal{L}V(x,t)\cdot(V(x,t) - \Lambda(x,t) = 0, \\ a.e.\ \Omega = I \times (0,T), \end{cases} \tag{5.1.1}$$

where $I = (0, X) \subset \mathbb{R}$ is the variable range of the underlying asset price, and the deterministic variable X is chosen to be large enough so as to reflect the practical reality.

The final condition and boundary conditions are:

$$V(x,T) = \Lambda(x,T),$$
$$V(0,t) = K, \quad \lim_{x\to\infty} V(x,t) = 0,$$

where $\Lambda(x,t)$ is the *payoff function* defined by $\Lambda(x,t) = max\{K - x, 0\}$.

It is easy to prove that the following variational inequality problem is equivalent to the partial differential complementarity problem (5.1.1).

Problem 5.1.2 *Find $V(t) \in \mathcal{K} = \{v \in H^1_{\varpi}(I) : v \geq \Lambda,\ V(0,t) = K,\ V(X,t) = 0\}$, such that, $\forall v \in \mathcal{K}$,*

$$\left(-\frac{\partial V(t)}{\partial t}, v - V(t)\right) + A\left(V\left(t\right), v - V(t); t\right) \geq 0, \quad a.e.\ in\ (0,T),$$

$$V(x,T) = \Lambda(x,T),$$

where

$$A(u,v;t) = \left(ax^2u' + bxu, v'\right) + (cu, v)$$

is a bilinear form, $u' \triangleq \partial u/\partial x$, $(\cdot,\cdot)$ *denotes the inner product on* $L^2(I)$, *and* $H^1_\varpi(I)$ *is the weighted Sobolev space defined by* $H^1_\varpi(I) = \{v(x) : v, x\partial v/\partial x \in L^2(I)\}$.

The unique solvability of Problem 5.1.2 lies on the coerciveness and continuity properties of the operator $A(u,v;t)$. See Lemma 2.1.1 and its proof.

The partial differential complementarity problem (5.1.1) implies that there exist an optimal exercise boundary $x^*(t)$ and an optimal exercise time $\rho(t)$ defined by

$$x^*(t) = \sup\{x : V(x,t) = \Lambda(x,t)\},$$
$$\rho(t) = \inf\{\tau \in [t,T] : V(x(\tau),\tau) = \Lambda(x(\tau),\tau)\},$$

respectively. The domain of the value function can thus be separated into a continuous region $\mathcal{C}$ (inactive region), on which the option has the value greater than the payoff for early exercise, and a stopping region $\mathcal{S}$ (active region), where the value of the option equals the payoff, see [69, 81]. Hence,

$$\mathcal{C} = \left\{(x,t) \in R^+ \times (0,T] : V(x,t) > \Lambda(x,t)\right\},$$
$$\mathcal{S} = \left\{(x,t) \in R^+ \times (0,T] : V(x,t) = \Lambda(x,t)\right\}.$$

In the next section, we propose a discrete form of system (5.1.1), which is based on the fitted finite volume method.

5.2 Fitted Finite Volume Discretization

We apply the finite difference method on time and the fitted finite volume method on space. Detailed formulation and analysis of the fitted finite volume method can be found in Section 2.4. In what follows, we give a brief account of the fitted finite volume discretization applied to (5.1.1).

Let the interval $I = (0, X)$ be divided into N sub-intervals

$$I_i := (x_i, x_{i+1}), \quad i = 0, \ldots, N-1,$$

with $0 = x_0 < x_1 < \cdots < x_N = X$. For each $i = 0, 1, ..., N-1$, we put $h_i = x_{i+1} - x_i$ and $h = \max_{0 \le i \le N-1} h_i$. We also let $x_{i-1/2} = (x_{i-1} + x_i)/2$ and $x_{i+1/2} = (x_i + x_{i+1})/2$ for each $i = 1, 2, \ldots, N-1$. These mid-points form a second partition of $(0, X)$ if we define $x_{-1/2} = x_0$ and $x_{N+1/2} = x_N$.

For time discretization, let t_m, $m = 0, 1, \ldots, M$, be a set of partition points in $[0, T]$ satisfying $T = t_0 > t_1 > \cdots > t_M = 0$. Then, we apply the two-level implicit time-stepping method with splitting parameter $\theta \in [1/2, 1]$.

Thus, by applying the fitted volume method to the variational problem 5.1.2, it leads to the following sequential finite dimensional variational inequality problems at times $t = t_m$ for $m = M-1, M-2, \ldots, 2, 1, 0$.

Problem 5.2.1 *Find* $\mathbf{V}^m \in \mathcal{K}^m = \{\mathbf{V}^m \in R^{N-1} : \mathbf{V}^m \ge \mathbf{\Lambda}^m\}$, *such that, for all* $\mathbf{U}^m \in \mathcal{K}^m$,

$$(\mathbf{M}_{fvm}\mathbf{V}^m, \mathbf{U}^m - \mathbf{V}^m) \ge \left(\mathbf{q}^{m+1}, \mathbf{U}^m - \mathbf{V}^m\right). \tag{5.2.2}$$

In Problem 5.2.1, $\mathbf{V}^m = (V_1^m, \cdots, V_{N-1}^m)^T$is the $(N-1)$-vector of variables, $\mathbf{\Lambda}^m = (\Lambda_1^m, \cdots, \Lambda_{N-1}^m)^T$ and $\mathbf{q}^m = (q_1^m, \cdots, q_{N-1}^m)^T$ are two $(N-1)$-vectors of constants.

The boundary and final conditions for Problem 5.2.1 are:

$$\begin{cases} V_0^m = K, \qquad V_N^m = 0, \quad m = 0, 1, \ldots, M \\ \mathbf{V}^0 = (\Lambda_1^0, \cdots, \Lambda_{N-1}^0)^T. \end{cases}$$

For the concise expression of the matrix $\mathbf{M}_{fvm}$, we define

$$\begin{aligned} \mathbf{G} &= diag(-l_1/\Delta t_m, -l_2/\Delta t_m, \cdots, -l_{N-1}/\Delta t_m), \\ \mathbf{E}_i &= (0, \cdots, 0, e_{i\,i-1}, e_{i\,i}, e_{i\,i+1}, 0, \cdots, 0), \\ \mathbf{E} &= (\mathbf{E}_1, \mathbf{E}_2, \cdots, \mathbf{E}_{N-1})^T, \end{aligned}$$

where, for $i = 1$,

$$\begin{cases} e_{10} &= -\dfrac{x_1(a - b_{1/2})}{4}, \\ e_{11} &= \dfrac{x_1(a - b_{1/2})}{4} + \dfrac{b_1\, x_{1+1/2}\, x_1^{\alpha_1}}{x_2^{\alpha_1} - x_1^{\alpha_1}} + c_1 l_1, \\ e_{12} &= \dfrac{b_{i+1/2}\, x_{i+1/2}\, x_{i+1}^{\alpha_i}}{x_{i+1}^{\alpha_i} - x_i^{\alpha^i}} \end{cases}$$

and, for $i = 2, \dots N-1$,

$$\begin{cases} e_{i\,i-1} &= -\dfrac{b_{i-1}\, x_{i-1/2}\, x_{i-1}^{\alpha_i - 1}}{x_i^{\alpha_i - 1} - x_{i-1}^{\alpha^{i-1}}}, \\ e_{i\,i} &= \dfrac{b_{i-1}\, x_{i-1/2}\, x_i^{\alpha_i - 1}}{x_i^{\alpha_i - 1} - x_{i-1}^{\alpha^{i-1}}} + \dfrac{b_{i+1/2}\, x_{i+1/2}\, x_i^{\alpha_i}}{x_{i+1}^{\alpha_i} - x_i^{\alpha_i}} + c_i l_i, \\ e_{i\,i+1} &= \dfrac{b_{i+1/2}\, x_{i+1/2}\, x_{i+1}^{\alpha_i}}{x_{i+1}^{\alpha_i} - x_i^{\alpha^i}}, \end{cases}$$

$\Delta t_m = t_{m+1} - t_m$ is defined as the time-step. Here, $\alpha_i = b_{i+1/2}/a$, and $l_i = x_{i+1/2} - x_{i-1/2}$.

Now the matrix $\mathbf{M}_{fvm}$ can be expressed as:

$$\mathbf{M}_{fvm} = \theta \mathbf{E} + \mathbf{G}.$$

By virtue of Theorem 5.2 in [125], we see that the matrix $\mathbf{M}_{fvm}$ is an M-matrix.

The constant vector $\mathbf{q}^{m+1}$ is defined by

$$\mathbf{q}^{m+1} = [\mathbf{G} - (1-\theta)\mathbf{E}]\, \mathbf{V}^{m+1} - \begin{bmatrix} e_{10} V_0^{m+1} \\ \vdots \\ e_{N-1\ N} V_N^{m+1} \end{bmatrix}_{N-1},$$

$m = 0, 1, \dots, M$, where $\mathbf{V}^{m+1}$ is computed from the previous time step.

Remark 5.2.1 *A matrix* $\mathbf{A}$ *is said to be an* M*-matrix if it is monotone and if, furthermore, all of its off-diagonal elements are non-positive, see [20]. Another equivalent definition of* M*-matrix is that* $\mathbf{A}$ *is nonsingular and all of its off-diagonal elements are non-positive and* $\mathbf{A}^{-1} \geq 0$*, see [68].*

For clarity and readability, we omit the superscript m and $m+1$ in the rest of this chapter.

5.3 Augmented Lagrangian Method for American Option Pricing

5.3.1 Augmented Lagrangian Algorithm

It is worth noting that with the number of the discrete points increases, the scale of Problem 5.2.1 gets larger. Thus, for solving this large scale problem, a fast and effective numerical optimization algorithm is in order. Consequently, we propose the augmented Lagrangian method for solving the variational inequality (5.2.2). As mentioned in Section 5.2, for time t, let t_m, $m = 0, 1, \dots, M$, be a set of partition points in $[0, T]$ satisfying $T = t_0 > t_1 > \cdots > t_M = 0$. This gives a sequence of variational inequalities (5.2.2). The augmented Lagrangian algorithm is applied to each discrete variational inequality (5.2.2).

First, we define a vector $\lambda \in R_+^{N-1}$ and a penalty parameter $\xi \in R^+$. Let V_i^k and λ_j^k be the k^{th} estimate for V_i and λ_j, where $\mathbf{V}^k = (V_1^k, \cdots, V_{N-1}^k)^T$ and $\lambda^k = (\lambda_1^k, \cdots, \lambda_{N-1}^k)^T$. Now, as in [57], we introduce the augmented Lagrangian formulation for Problem 5.2.1 as follows.

$$\begin{cases} \mathbf{M}_{fvm}\mathbf{V}_q - \mathbf{q} - \lambda = \mathbf{0}, \\ \lambda = \max\left\{0, \lambda + \xi\left(\mathbf{\Lambda} - \mathbf{V}\right)\right\}, \quad \forall \xi \in R^+. \end{cases} \tag{5.3.3}$$

From Remark 5.2.1, it is obvious that $\mathbf{M}_{fvm}$ is monotone since it is an M-matrix. The max operator is also monotone. Thus, by virtue of the monotone operator theory, we see that (5.3.3) has a unique solution denoted by $(\mathbf{V}_\xi, \lambda_\xi)$, see [68].

Based on (5.3.3), the augmented Lagrangian algorithm [68] for pricing American options is given below.

Algorithm 5.3.1 *(ALM Algorithm) At each time level:*

1. *Let iteration step $k = 0$, and initialize $\xi > 0, \varepsilon > 0$, $\mathbf{V}^0 \geq \mathbf{\Lambda}$;*

2. *Set the inactive and active sets by*

$$\mathrm{I}^{k+1} = \left\{i \in \mathbb{N}\colon \lambda_i^k + \xi\left(\Lambda_i - V_i^k\right) \leq 0\right\},$$
$$\mathcal{A}^{k+1} = \left\{i \in \mathbb{N}\colon \lambda_i^k + \xi\left(\Lambda_i - V_i^k\right) > 0\right\},$$

where $\mathbb{N} = \{1, 2, \cdots, N-1\}$*;*

3. *Solve the following problem for* $\mathbf{V}^{k+1}$*:*

$$\mathbf{M}_{fvm}\mathbf{V}^{k+1} - \lambda^k = \mathbf{q};$$

4. *Update* λ_i^{k+1}*,*

$$\lambda_i^{k+1} = \begin{cases} 0 & on \quad \mathrm{I}^{k+1}, \\ \lambda_i^k + \xi(\Lambda_i - [V_i]^{k+1}) & on \quad \mathcal{A}^{k+1}; \end{cases}$$

5. *If* $\|\mathbf{V}^{k+1} - \mathbf{V}^k\| \leq \varepsilon$*, Stop; else, let* $k = k+1$*, and go to Step 2.*

It is worth noting that ALM is ideal for solving American option pricing, since at each time step this method only solves a sequence of linear algebraic equation system with sparse matrices. In the next subsection, we point out that the limiting points of the sequences $\{[\mathbf{V}]^k\}$ and $\{\lambda^k\}$ are, respectively, the option values $\mathbf{V}$ and the Lagrangian multiplier λ. Thus, by controlling the penalty parameter, we can obtain the result as accurately as required. Furthermore, by virtue of the Lagrangian multiplier λ, the optimal exercise boundary can be obtained directly. In fact, in this method, we can update Lagrangian multiplier λ automatically, which means that the optimal exercise boundary is obtained automatically, this is an additional major advantage over the penalty methods. On the other hand, in this method, the penalty parameter r is not required to take very large value.

Remark 5.3.1 *If we set the penalty parameter* $\xi = 0$ *in (5.3.3), then the Lagrangian method is obtained, which has been studied in [122]. It is worth noting that the Lagrangian multiplier method is strongly problem-dependent and does not usually give rise to an accurate solution. Moreover, the convergence of the Lagrangian method when applied to American options is very slow. All these shortcomings can be clearly observed in the numerical examples to be presented in Section 5.4.*

Remark 5.3.2 *If we substitute the second relation in (5.3.3) into the first one and then set* $\lambda = \mathbf{0}$*, then the linear penalty method is obtained, which has been studied in [44, 125]. It is worth noting that the penalty parameter is required to be chosen large enough if accurate results are to be obtained, see [40]. Hence, computational* *difficulty may be encountered. Furthermore, by the penalty method, the results obtained for the American option pricing only satisfy the complementarity constraint conditions approximately, and the optimal exercise boundary cannot be obtained directly. These points can be seen in the numerical examples to be presented in Section 5.4.*

5.3.2 Convergence Analysis of ALM

The convergence analysis of ALM applied to variational inequalities has been studied in [51, 53, 52, 68, 73]. In fact, due to the fact that the matrix $\mathbf{M}_{fvm}$ obtained by the fitted finite volume discretization is an M-matrix, some convergence properties of our method can be deduced from the results presented in [51, 53, 52, 68, 73]. In this subsection, we give the convergence results of the ALM Algorithm applied to American option pricing. The detailed proofs of these results are omitted, as they can be found in the relevant references.

Before going to discuss the convergence properties of the augmented Lagrangian algorithm, we characterize the solution to Problem 5.2.1 by an equivalent linear complementarity problem.

Lemma 5.3.1 *Problem 5.2.1 is characterized by the existence of $\lambda \in R_+^{N-1}$, such that the following linear complementarity problem is satisfied*

$$(LCP)\begin{cases} \mathbf{M}_{fvm}\mathbf{V} - \mathbf{q} - \lambda = \mathbf{0}, \\ \mathbf{V} - \mathbf{\Lambda} \geq \mathbf{0},\ \lambda \geq \mathbf{0}, (\lambda, \mathbf{V} - \mathbf{\Lambda}) = \mathbf{0}. \end{cases} \tag{5.3.4}$$

Proof. Since we omit the superscript m and $m+1$ in (5.2.2), Problem 5.2.1 can be rewritten as follows.

$$\begin{cases} \text{Find } \mathbf{V} \in \mathcal{K}^m = \{\mathbf{V} \in R^{N-1} : \mathbf{V} \geq \mathbf{\Lambda}\},\ \text{such that} \\ (\mathbf{M}_{fvm}\mathbf{V}, \mathbf{U} - \mathbf{V}) \geq (\mathbf{q}, \mathbf{U} - \mathbf{V})\,, \text{ for all } \mathbf{U} \in \mathcal{K}^m. \end{cases} \tag{5.3.5}$$

1. (5.3.5) implies (5.3.4).

For any $\mathbf{W} \in R_+^{N-1}$, we have

$$\mathbf{U} + \mathbf{W} \in \mathcal{K}^m.$$

Taking $\mathbf{V} = \mathbf{U} + \mathbf{W}$ in (5.3.5), it follows that $\forall\ \mathbf{W} \in R_+^{N-1}$

$$(\mathbf{M}_{fvm}\mathbf{V} - \mathbf{q}, \mathbf{W}) \geq \mathbf{0}, \tag{5.3.6}$$

and (5.3.6) clearly implies

$$\mathbf{M}_{fvm}\mathbf{V} - \mathbf{q} \geq \mathbf{0}.$$

Hence, there exists a vector $\lambda \in R_+^{N-1}$ such that

$$\lambda \triangleq \mathbf{M}_{fvm}\mathbf{V} - \mathbf{q} \geq \mathbf{0}. \tag{5.3.7}$$

Since $\mathbf{V} \in \mathcal{K}^m$, we have $\mathbf{V} \geq \mathbf{\Lambda}$, i.e., $\mathbf{V} - \mathbf{\Lambda} \geq \mathbf{0}$. From (5.3.7), we immediately obtain

$$(\mathbf{M}_{fvm}\mathbf{V} - \mathbf{q}, \mathbf{V} - \mathbf{\Lambda}) \geq \mathbf{0}. \tag{5.3.8}$$

On the other hand, it is obvious that $\mathbf{\Lambda} \in \mathcal{K}^m$. Hence, by taking $\mathbf{U} = \mathbf{\Lambda}$ in (5.3.5), we have

$$(\mathbf{M}_{fvm}\mathbf{V} - \mathbf{q}, \mathbf{\Lambda} - \mathbf{V}) \geq 0. \tag{5.3.9}$$

Thus, from (5.3.8), (5.3.9) and (5.3.7), we obtain

$$\begin{cases} \mathbf{M}_{fvm}\mathbf{V} - \mathbf{q} - \lambda = \mathbf{0}, \\ \mathbf{V} - \mathbf{\Lambda} \geq \mathbf{0},\ \lambda \geq \mathbf{0}, (\lambda, \mathbf{V} - \mathbf{\Lambda}) = \mathbf{0}. \end{cases}$$

2. (5.3.4) implies (5.3.5).

From (5.3.4), it is obvious that $\mathbf{V} \in \mathcal{K}$ and

$$(\mathbf{M}_{fvm}\mathbf{V} - \mathbf{q}, \mathbf{V} - \mathbf{\Lambda}) = \mathbf{0}. \tag{5.3.10}$$

Let $\mathbf{U} \in \mathcal{K}^m$. Then, $\mathbf{U} \geq \mathbf{\Lambda}$, i.e., $\mathbf{U} - \mathbf{\Lambda} \geq \mathbf{0}$. Since

$$\mathbf{M}_{fvm}\mathbf{V} - \mathbf{q} = \lambda \geq \mathbf{0},$$

we obtain

$$(\mathbf{M}_{fvm}\mathbf{V} - \mathbf{q}, \mathbf{U} - \mathbf{\Lambda}) \geq \mathbf{0}. \tag{5.3.11}$$

Subtracting (5.3.10) from (5.3.11), we have

$$(\mathbf{M}_{fvm}\mathbf{V} - \mathbf{q}, \mathbf{U} - \mathbf{V}) \geq \mathbf{0}, \quad \forall\ \mathbf{U} \in \mathcal{K}^m.$$

Thus, (5.3.4) implies (5.3.5). ∎

It is worth noting that, due to the fact that the matrix $\mathbf{M}_{fvm}$ is an M-matrix, (5.3.4) has a unique solution, which is denoted by $(\mathbf{V}, \lambda)$, see [57].

Remark 5.3.3 *From the optimization theory, we know that if* $\mathbf{M}_{fvm}$ *in (5.3.4) and (5.3.3) is symmetric and positive definite, then (ALM) is the KKT condition of*

$$(P) \qquad \begin{cases} \min F(\mathbf{V}) = \frac{1}{2}(\mathbf{V}, \mathbf{M}_{fvm}\mathbf{V}) - (\mathbf{q}, \mathbf{V}) \\ s.t.\ \mathbf{V} - \mathbf{\Lambda} \geq \mathbf{0}. \end{cases}$$

For Problem 5.2.1, the matrix $\mathbf{M}_{fvm}$ *is not symmetric. Hence, we cannot find the corresponding form of* (P)*. However, motivated by the relationship between* (P) *and (5.3.3) when* $\mathbf{M}_{fvm}$ *is symmetric, we propose the augmented Lagrangian formulation (5.3.3) of the complementarity system (5.3.4).*

The following lemma shows that the solution to (5.3.3) is convergent to that of (5.3.4), and hence, to that of Problem (5.2.1), when the penalty parameter tends to infinity.

Lemma 5.3.2 *Let* $(\mathbf{V}, \lambda)$ *and* $(\mathbf{V}_\xi, \lambda_\xi)$ *be the solutions to (5.3.4) and (5.3.3), respectively. Then,*

$$|\lambda_\xi - \lambda| = \mathcal{O}(1/\xi),$$
$$\|\mathbf{V}_\xi - \mathbf{V}\| = \mathcal{O}(1/\xi).$$

Lemma 5.3.2 presents a standard result on the ALM method applied to variational inequalities. Here, we omit the proof. For details, see Glowinsky's book [51], Chapter 2, Theorem 7.2. From this lemma, we can see that the error $\|\mathbf{V}_\xi - \mathbf{V}\|$ decreases linearly with the penalty parameter ξ. This result is verified by the numerical results in Section 5.4.

The convergence analysis of the augmented Lagrangian algorithm applied to (5.3.3) has been studied in [57, 67, 68, 73]. Due to the fact that the matrix $\mathbf{M}_{fvm}$ obtained by the fitted finite volume discretization is an M-matrix, the following conclusion is a consequence of those presented in [57, 67, 68, 73].

Theorem 5.3.1 *[57, Theorem 3.1] For a fixed* $\xi > 0$*, if* $\|(\mathbf{V}_\xi^0, \lambda_\xi^0) - (\mathbf{V}_\xi, \lambda_\xi)\|$ *is sufficient small, then*

$$(\mathbf{V}_\xi^k, \lambda_\xi^k) \to (\mathbf{V}_\xi, \lambda_\xi) \quad \textit{superlinearly.}$$

5.4 Empirical Tests

In this section, we perform some numerical tests to illustrate the performance of the augmented Lagrangian algorithm. First, we give a detailed comparison of ALM, the Lagrangian method and the penalty method. Then, by empirical tests, ALM is compared with the PSOR method in terms of step sizes of space and time. Finally, we show that ALM is robust with respect to changes of the market parameters: interest rate and volatility. All the numerical results were computed in the double precision on a Pentium IV 2.8 GHz, 512M memory PC under the Visual C++.net environment.

5.4.1 Comparison of ALM, Lagrangian Method and Penalty Method

The test problem is an American Vanilla Put option with the following two sets of parameters

$$\text{(i) } T = 0.25,\ K = 100,\ r = 0.10,\ \sigma = 0.2,$$
$$\text{(ii) } T = 0.25,\ K = 100,\ r = 0.10,\ \sigma = 0.8.$$

These test problems have been carefully studied in [44] and the numerical solutions are listed there. For financial derivatives markets, the model (i) is more realistic and standard. In order to show the strength of the ALM method, an unusually high volatility model (ii) is also considered. The security maximum price $S_{\max} = 1000$, i.e., the asset pricing (S) space is taken as $[0, 1000]$. We uniformly divide it into 1000 sub-intervals and uniformly divide the time interval $[0, 0.25]$ into 1000 sub-intervals. The values for time to maturity T, and $S = K$ are computed and compared.

Table 5.1 to Table 5.4 exhibit the comparison of the results by linear penalty method, power penalty method, ALM and Lagrangian method, respectively. Here, we choose the power penalty method with the lower order (1/2). In Table 5.1 to Table 5.3, the column 'V' indicates the option value, 'CPU' denotes the computational time (*minutes*), $\{\xi_n\}$ denotes a sequence of penalty parameters, '$*$' represents 'not available'. For each ξ_n, a solution to Problem (ALM) is found, denoted by V_n. Let

$$\Delta V_n = V_n - V_{n-1},\ R_n = \frac{\Delta V_{n-1}}{\Delta V_n}$$

be the difference of the solutions and the ratio of changes corresponding to the two successive penalty parameters, respectively. As the Lagrangian method does not possess penalty parameter, we only list the option value and computational time.

Table 5.1: Results by l_1 penalty method

$\sigma = 0.2$					$\sigma = 0.8$				
ξ_n	V_n	ΔV_n	R_n	CPU	ξ_n	V_n	ΔV_n	R_n	CPU
5	2.89463			0.206	5	14.5069			0.183
10	2.93705	0.04242		0.205	10	14.5427	0.0358		0.183
20	2.98162	0.04457	0.9	0.197	20	14.5841	0.0414	0.9	0.185
40	3.01690	0.03528	1.3	0.196	40	14.6207	0.0366	1.1	0.183
80	3.03900	0.02210	1.6	0.204	80	14.6458	0.0251	1.5	0.182
160	3.05136	0.01236	1.8	0.195	160	14.6606	0.0148	1.7	0.180
320	3.05780	0.00644	1.9	0.200	320	14.6687	0.0081	1.8	0.178
640	3.06112	0.00332	1.9	0.198	640	14.6730	0.0043	1.9	0.175
1280	3.06281	0.00169	2.0	0.168	1280	14.6751	0.0210	2.0	0.161
2560	*Failed*	*Failed*	*Failed*	$*$	2560	*Failed*	*Failed*	*Failed*	$*$
Max Tolerance $= 1.0 \times 10^{-5}$					Max Tolerance $= 1.0 \times 10^{-5}$				

In view of the results obtained, we can draw the following conclusions.

1. From the column 'R_n' of Table 5.3, the convergence rate of ALM to the penalty parameter is estimated. The rate is of order $\mathcal{O}(1/\xi)$, which is consistent with Lemma 5.3.2.

2. From the columns 'ξ_n' and 'V_n' of Tables 5.1 and 5.3, we observe that with the same level of accuracy, ALM is computationally more stable and cheaper than the linear penalty method. Moreover, ALM can attain much larger penalty parameters than the linear penalty method, and hence can achieve a better accuracy of the option values.

3. From the columns 'CPU' of Tables 5.2 and 5.3, we see that although a small penalty parameter is required and a much faster convergence rate is achieved by the power penalty method, the computational time is still greater than that of

Table 5.2: Results by power penalty method

$\sigma = 0.2$					$\sigma = 0.8$				
ξ_n	V_n	ΔV_n	R_n	CPU	ξ_n	V_n	ΔV_n	R_n	CPU
5	2.92437			0.296	10	14.5339			0.292
10	2.98262	0.05825		0.297	20	14.5861	0.0522		0.292
20	3.03255	0.04993	1.2	0.297	40	14.6375	0.0514	1.0	0.292
40	3.05548	0.02323	2.1	0.299	80	14.6653	0.0278	1.8	0.292
80	3.06206	0.00658	3.5	0.295	160	14.6741	0.0088	3.2	0.290
160	$Failed$	$Failed$	$Failed$	$*$	640	$Failed$	$Faild$	$Failed$	$*$
Max Tolerance $= 1.0 \times 10^{-5}$					Max Tolerance $= 1.0 \times 10^{-5}$				

Table 5.3: Results by ALM

$\sigma = 0.2$					$\sigma = 0.8$				
ξ_n	V_n	ΔV_n	R_n	CPU	ξ_n	V_n	ΔV_n	R_n	CPU
5	3.06688			0.190	5	14.6799			0.184
10	3.06596	0.00092		0.190	10	14.6788	0.0011	1.7	0.188
20	3.06532	0.00064	1.4	0.190	20	14.6783	0.0005	2.2	0.190
40	3.06495	0.00037	1.8	0.190	40	14.6779	0.0004	1.3	0.190
80	3.06476	0.00019	1.9	0.193	80	14.6777	0.0002	2.0	0.190
160	3.06467	0.00009	2.1	0.185	160	14.6776	0.0001	2.0	0.190
320	3.06462	0.00005	1.8	0.184	320	14.6774	0.0001	$*$	0.190
640	3.06459	0.00003	1.7	0.180	640	14.6774	0.0000	$*$	0.183
1280	3.06457	0.00002	1.5	0.182	1280	14.6774	0.0000	$*$	0.180
2560	3.06457	0.00001	$*$	0.180	2560	14.6774	0.0000	$*$	0.170
5120	3.06457	0.00001	$*$	0.175	5120	14.6774	0.0000	$*$	0.171
7000	3.06457	0.00001	$*$	0.203	7000	14.6774	0.0000	$*$	0.184
Max Tolerance $= 1.0 \times 10^{-5}$					Max Tolerance $= 1.0 \times 10^{-5}$				

Table 5.4: Results by Lagrangian method

$\sigma = 0.2$		$\sigma = 0.8$	
V_n	CPU	V_n	CPU
3.07869	0.447	14.7219	0.260
Max Tolerance $= 1.0 \times 10^{-2}$		Max Tolerance $= 1.0 \times 10^{-2}$	

ALM. This is due to the non-smooth nature of the lower order penalty term. It requires more time to solve the nonlinear and non-smooth systems.

4. From the columns 'V_n' and 'CPU' of Tables 5.3 and 5.4, we find that, compared with the Lagrangian method, ALM is more accurate and faster. This indicates that ALM possesses a faster rate of convergence than the Lagrangian method.

5. By using ALM, the early exercising boundary is automatically obtained according to the Lagrangian multiplier, during the process of computing the option value. However, the penalty method does not possess this advantage. This is another advantage of ALM over penalty methods.

Thus, we can conclude that ALM is the most effective method among all the four methods. It is especially fitted to solve the American option pricing problem.

By applying ALM, the option values for the two different volatility values are depicted in Figure 5.1. Also, the optimal exercise boundaries are revealed explicitly from Kuhn-Tucker multipliers obtained from the augmented Lagrangian method, which can be seen in Figure 5.2.

5.4.2 Comparison of ALM and PSOR

The PSOR method [62, 127] is a commonly used method both in practice and in research. To show that ALM is comparable with the PSOR method, we first compare ALM with the PSOR method in computational times under different space and time

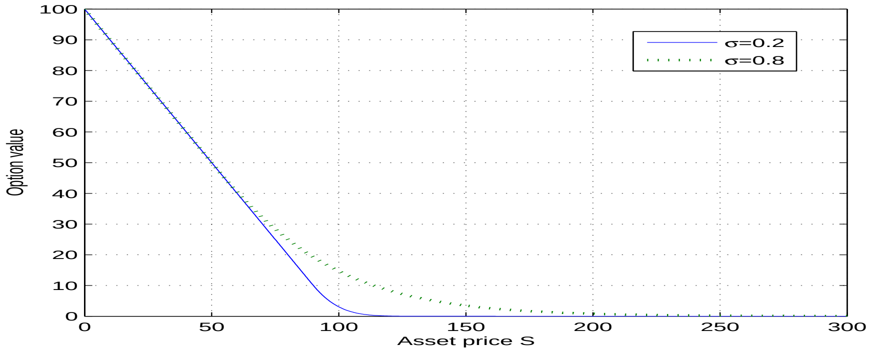

Figure 5.1: Option values at $t = 0$

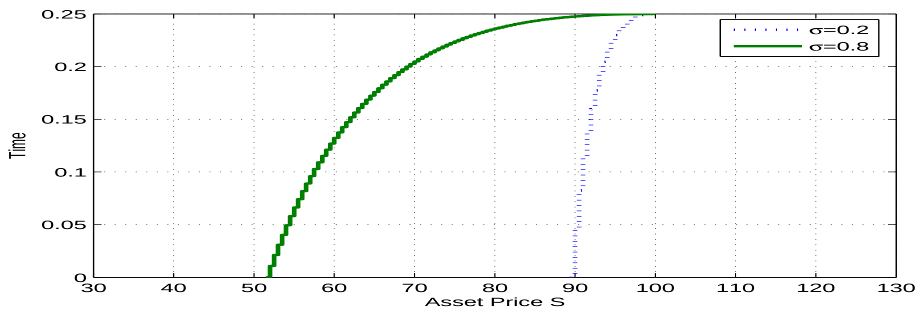

Figure 5.2: Optimal exercise boundaries

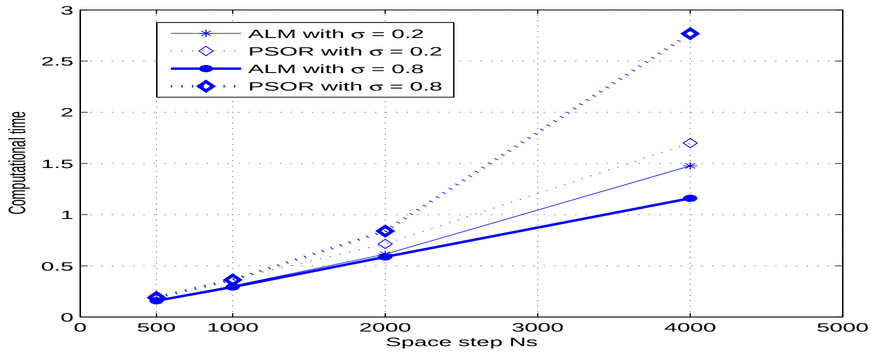

Figure 5.3: Solution times of ALM and PSOR with different space steps ($r = 0.10$)

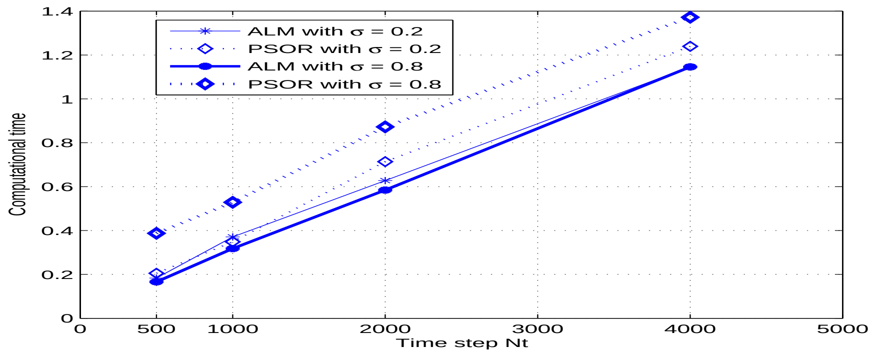

Figure 5.4: Solution times of ALM and PSOR with different time steps ($r = 0.10$)

discretizations. Then, the comparison of computational times of these two methods under different market parameters is investigated.

Figure 5.3 shows the computational times of ALM and PSOR under different space steps N_s and the same time steps N_t.

Figure 5.4 shows the computational times of ALM and PSOR under different time steps N_t and the same space steps N_s. Figure 4 gives the corresponding plots of computational times for these two methods as a function of time steps N_t.

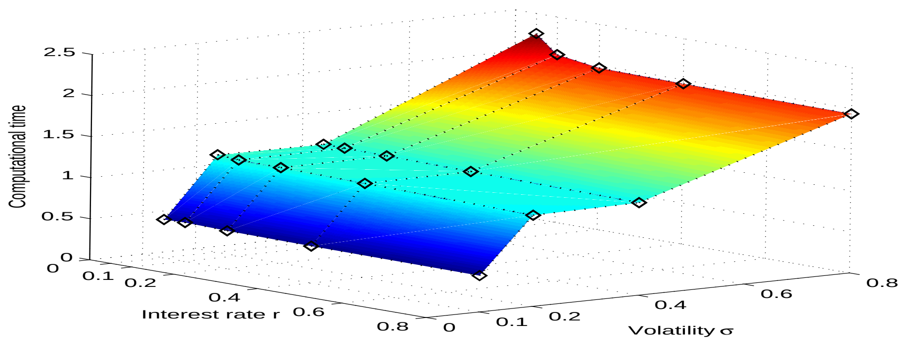

Figure 5.5: Solution times of PSOR method with different market parameters (time steps = 1000, space steps = 4000)

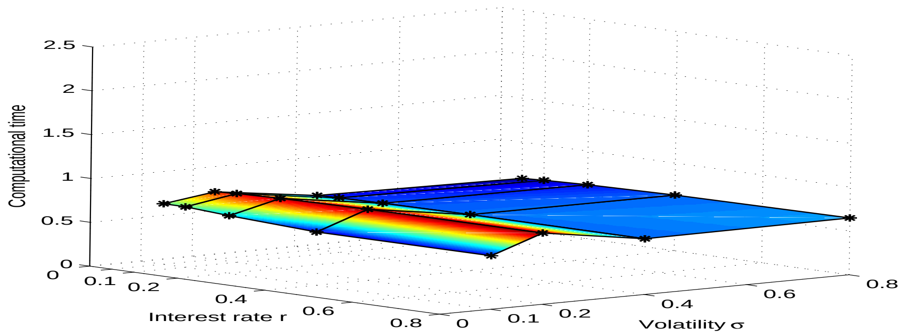

Figure 5.6: Solution times of ALM with different market parameters (time steps = 1000, space steps = 4000)

Figure 5.5 and Figure 5.6 show the computational times of ALM and PSOR under different market parameters: interest rate r and volatility σ. Figures 5.5 and 5.6 give the corresponding plots of solution times for these two methods as a function of r and σ.

In view of the comparison results, we can draw the following conclusions.

1. ALM gives almost linear computational time as a function of space steps N_s. However, PSOR exhibits exponential solution time behavior as the number, N_s,

of the space steps increases. As the space steps increase, the effect of the time reserving of ALM becomes more significant.

2. The computational times of ALM and PSOR depend linearly on the number, N_t, of time steps. The computational time of PSOR is a little bit greater than that of ALM.

3. ALM is much more stable and robust when compared with PSOR. That is, the computational time of ALM appears to be independent of the market parameters σ and r.

4. For small r and large σ, PSOR is faster than ALM. In contrast, ALM is much faster when r is large and σ is small.

From the numerical comparisons and empirical tests of the above two subsections, we conclude that ALM has several notable advantages over the penalty methods, Lagrangian method and the PSOR method: (i) it is computationally more stable and cheaper; (ii) it is much more stable and robust when compared with PSOR; (iii) ALM is robust with respect to the changes of market parameters; and (iv) early exercising boundary is automatically obtained during the process of computing the option value.

5.5 Summary

In this chapter, we proposed an augmented Lagrangian method in combination with the fitted finite volume discretization scheme for the solution to the variational inequality problem. This problem is an equivalent formulation of a stochastic optimal stopping problem arising in the study of the American option valuation. After the discretization, an equivalent linear complementarity problem was obtained. For the resulting LCP, the augmented Lagrangian method and the corresponding algorithm were proposed. Empirical tests were implemented. The numerical results obtained showed that the augmented Lagrangian method works very well for the American option valuation, and it is comparable with the PSOR method.

Chapter 6

Conclusions and Suggestions For Future Research

6.1 Conclusions

In this thesis, we have developed robust numerical methods for pricing American options under various generalizations, including American options with jump diffusion processes and two-asset American options. Each of these methods was developed for solving a penalized problem obtained by applying a penalty approach to a complementarity problem arising from the American option pricing. Effective algorithms for solving these penalized problems were presented and numerical tests were implemented. Some advantages of penalty methods over the PSOR method have been observed from empirical numerical experiments. In addition, we have also considered the augmented Lagrangian method for American option pricing.

Because of its early exercise feature, the pricing of an American option is a free boundary problem for which the holder can make a decision whether to exercise or hold it at any time during the lifetime of the option. This feature is naturally captured in the corresponding complementarity problem, which is subsequently transformed into a variational inequality problem. On this basis, we applied penalty approaches to the complementarity problem or the variational inequality problem. Based on the theories of variational inequalities, convergence properties of penalty approaches have been obtained in some appropriate infinite dimensional spaces. For some special penalty methods, e.g. the power penalty method, rates of convergence have been established.

Black-Scholes operators are parabolic partial differential ones. They are convection-diffusion operators, which can produce numerical oscillations if general numerical methods are to be used. Therefore, some effective numerical methods, including the fitted finite volume method and finite element method, have been proposed to solve the penalized nonlinear problems. These methods have overcome the difficulty of numerical oscillations. For the fitted finite volume method, a detailed description of the computational procedure was presented. For the penalized nonlinear problems, solution methods have been developed.

From numerical experiments, we also revealed some advantages of penalty methods over the PSOR method. We observed that penalty methods are much more stable and robust than the PSOR method. Penalty methods are computationally much less expensive than the PSOR method when the number of space steps increases. Moreover, penalty methods are more robust than the PSOR method with respect to the changes of the market parameters, such as interest rate and volatility.

We finally proposed the augmented Lagrangian method for American option pricing problem and developed the corresponding algorithm. In addition to having all the advantages of penalty method for American option pricing, the augmented Lagrangian method possesses another two major advantages: (i) rather than a nonlinear equation system obtained by penalty methods, a linear equation system is to be solved for the augmented Lagrangian method; and (ii) early exercising boundary is automatically obtained during the process of computing the option value.

We believe that the general penalty approach (including augmented Lagrangian method) can be successfully applied to pricing other more exotic options with the early exercise feature. As mentioned in the introduction, the study of the penalty methods and their application to option pricing are still very much in its infancy. Several further directions are discussed in the next section.

6.2 Future Research Directions

Several further research directions are listed in the following.

- To develop penalty approaches to a broader class of models, such as options with stochastic volatilities and transaction costs. It is of particular interest to develop penalty approaches to uncertain volatility models [5], where an optimization problem is involved in the penalized problem. Existing numerical procedures are computationally extremely expensive. There is thus an urgent need to develop efficient techniques for solving this problem effectively.

- In finance, a lot of problems can be formulated to optimal control problems under the Hamilton-Jacobi-Bellman framework [42]. Hence, to apply the penalty approach to the numerical solution of the optimal control model is both interesting and important.

- It is of importance to study the convergence rates of our methods with respect to both the penalty parameters (ξ) and discretization parameters (h). All the existing results on the convergence rate are either to the penalty parameters (ξ) or to the discretization parameters (h), see [4].

- To find the local volatility function implied by the American options via the theory of mathematical programming with equilibrium constraints. This is another important research area. It is an inverse problem where optimization methods are to be used to deal with the estimation and adjustment of implied volatilities for pricing American options on the Black-Scholes framework and its generalization. Applying penalty methods to solving the resulting mathematical programming with equilibrium constraints would appear to be of the right direction.

Chapter 7

Appendix

7.1 The Projected SOR Method

The Successive Over Relaxation (SOR) is a method frequently used to solve a certain class of matrix equations. The projected SOR method is discussed in detail in [38].

Consider a general linear complementarity problem (LCP):

$$\mathbf{Ax} \geq \mathbf{b}, \quad \mathbf{x} \geq \mathbf{c}, \quad (\mathbf{x} - \mathbf{c}) \cdot (\mathbf{Ax} - \mathbf{b}) = \mathbf{0}.$$

We assume only that the matrix $\mathbf{A}$ is invertible and positive definite (i.e. $\mathbf{x} \cdot (\mathbf{Ax}) > 0$ for any $\mathbf{x} \neq \mathbf{0}$). Then, it can be shown that there is a unique solution vector $\mathbf{x}$ for this LCP (see [38] for a proof).

The algorithm for finding the solution is an iterative procedure. Start with an initial guess $\mathbf{x}^0 \geq \mathbf{c}$ (the algorithm may not converge if $\mathbf{x}^0 < \mathbf{c}$). During each iteration, we form a new vector

$$\mathbf{x}^{k+1} = \left(x_1^{k+1}, x_2^{k+1}, \ldots, x_n^{k+1}\right),$$

from the current vector

$$\mathbf{x}^{k} = \left(x_1^{k}, x_2^{k}, \ldots, x_n^{k}\right),$$

by the following two-step process:

1. For each $i = 1, 2, \ldots, n$, we sequentially form the intermediate quantity y_i^{k+1},

given by

$$y_i^{k+1} = \frac{1}{A_{ii}} \left(b_i - \sum_{j=1}^{i-1} A_{i,j} x_j^{k+1} - \sum_{j=i-1}^{n} A_{i,j} x_j^k \right).$$

2. Form $\mathbf{x}^{k+1}$ by

$$x_i^{k+1} = \max \left(c_i, x_i^k + \varpi \left(y_i^{k+1} - x_i^k \right) \right).$$

Note that it is important to perform these two steps in sequence. We need the new value of x_{i-1}^{k+1} in order to find x_i^{k+1}. The only difference between this method and the classical SOR method is the test to make sure that $x_i^{k+1} \geq c_i$. The constant ϖ is called a relaxation parameter. Provided that $\mathbf{x}^0 \geq \mathbf{c}$ and $0 < \varpi < 2$, the method converges. It can be shown that the convergence can be optimized by choosing a particular value of $\varpi \in (1, 2)$, which depends on the matrix $\mathbf{A}$.

Each iteration defines a new vector $\mathbf{x}^{k+1} \geq \mathbf{c}$ such that as $k \to \infty$, $\mathbf{x}^{k+1} \to \mathbf{x}$, the solution of the problem. In practice, we stop the iteration once the following condition is satisfied

$$\left| \mathbf{x}^{k+1} - \mathbf{x}^k \right| < \varepsilon$$

where ε is some pre-chosen small tolerance. We then take $\mathbf{x}^{k+1}$ as the solution.

7.2 Smoothing Techniques

The penalty approach to American option pricing problem usually leads to a nonlinear system. In this thesis, we apply the classical Newton method to this system. When the lower order penalty method ($l_k, 0 < k < 1$) is used, a nonsmooth system is obtained, which should be smoothed out by some techniques. For example, for the term

$$d(u) = [u]_+^k,$$

we see that $d'(u) \to \infty$ as $u \to 0^+$. To overcome this difficulty, we smooth out $d(u)$ in the neighborhood of $[u]_+ = 0$ as follows:

$$d(u) = \begin{cases} u^k, & u \geq \varepsilon, \\ W([u]_+), & u < \varepsilon \end{cases}$$

for $0 < k < 1$, where $1 >> \varepsilon > 0$ is a transition parameter and $W(z)$ is a function which smoothes out the original $d(z)$ around $z = 0$. We choose

$$W(z) = c_1 + c_2 z + \cdots + c_n z^{n-1} + c_{n+1} z^n$$

for $n \geq 3$. To ensure that $d(z)$ is smooth, it is required that

$$W(0) = W'(0) = 0, \quad W(\varepsilon) = \varepsilon^k, \quad W'(\varepsilon) = \tfrac{1}{k}\varepsilon^{k-1}.$$

Using these four conditions and setting $c_3 = \cdots = c_{n-1} = 0$, we can easily find that

$$c_1 = c_2 = 0, \quad c_n = \varepsilon^{k-n+1}(n-k), \quad c_{n+1} = \varepsilon^{k-n}(k-n+1).$$

Thus,

$$d(u) = \begin{cases} u^k, & u \geq \varepsilon, \\ \varepsilon^{k-n+1}(n-k)[u]_+^{n-1} + \varepsilon^{k-n}(k-n+1)[u]_+^n, & u < \varepsilon. \end{cases}$$

The intuition of this choice of d is as follows. When $u \geq \varepsilon$, a given tolerance, u^k offers a convergence rate of order $\frac{1}{2k}$. When $u < \varepsilon$, we choose $d(u) = W([u]_+)$ to slow down the convergence. It has been proved by Wang [125] that the function $W([u]_+)$ is strictly increasing on $[0, \varepsilon]$ if $1/k \geq n$ for $n \geq 3$. Moreover, the nonlinear function $d(u)$ is smooth and increasing on $(-\infty, \infty)$ when $1/k \geq n$ for any integer $n \geq 3$.

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