The Hong Kong Polytechnic University

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Computational Optimization of Mutual Insurance Systems: A Quasi-Variational Inequality Approach

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A thesis submitted in partial fulfilment of the requirements for the degree of
Doctor of Philosophy

May 2008

Certificate of Originality

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Abstract

It is well known that the optimal control of a stochastic system represents a general problem which can be found in many areas such as inventory control, financial engineering, and lately federal (national) reserve management, and so on. If the underlying system involves with some fixed transaction costs the problem will turn out to be known as an *impulse control problem.* The framework of solving this type of problem is initially developed by Bensoussan [**10**] and Aubin [**5**]. They proved that the optimal solution to an impulse control problem can be sufficiently characterized by Quasi-Variational Inequality (QVI). With these profound findings and fundamental developments in impulse control theory, a mathematically rigid HJB-QVI system (deterministic), which is formulated in the form of a functional boundary-value problem of non-linear Hamilton-Jacob-Bellman (HJB) equations, has been established as a general methodology for solving impulse control problems (stochastic). In theory, optimal solution to a stochastic impulse control problem can be determined by solving a corresponding deterministic HJB-QVI system. However, in reality, HJB-QVI system of a practical impulse control problem is often too complicated to have an analytical solution in closed forms. As far as we can ascertain from the literature, apart from very few extremely simplified problems [**34, 11, 28**], closed-form analytical solution to an HJB-QVI system is seldom attainable. In this study, we obtain computational properties of the aforementioned QVI systems associated with impulse control

problems, and provide computational methods for solving the QVI systems, which we categorize into two major classes: 1) QVI systems with analytically solvable HJB equations; 2) QVI systems with analytically unsolvable HJB equations. We begin with the study on the class-1 QVI systems. Although general solutions to underlying HJB equation of a class-1 QVI systems are obtainable, the associated QVI system may still need to be solved in non-closed forms. We present the solution for two class-1 type QVI systems in Chapter 2 and Chapter 3. For class-2 QVI systems, to obtain numerical solutions a computational optimization algorithm presented in Chapter 4. In the last chapter we again consider a class-1 QVI system. It has a non-symmetric cost structure, which has particular application in mutual insurance reserve control problem. A novel computation algorithm is developed to determine numerically an optimal $(a, A; B, b)$ policy.

Acknowledgements

I would like to thank my supervisor Prof. John J. Liu from the Department of Logistics and Maritime Studies. I am very grateful for his continuous guidance and support throughout my study period.

I would like to thank The Hong Kong Polytechnic University for the financial support, which makes this study possible. I would also thank the Department of Logistics and Maritime Studies for the wonderful seminar series which gave me a lot of inspiration of my research.

My thanks also go to many of my friends and colleagues: Jiejian Feng, Vivek Kanhangad, Chris Lo, Jinwen Ou. Jie Pan, Huajun Tang, Xiangzhen Kong, Jun Wang, Haisha Zhen. It has been my greatest time to study, discuss and play with them all.

Last, but not least, I would like thank asll my family. Their continuous unconditional support gave me the strength and courage to accomplish this study.

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CHAPTER 1

Introduction

1.1. Impulse Control and Quasi-Variational Inequality

The theory of impulse control is founded upon the well-established theory of stochastic control which is concerned with seeking optimal classical controls (as opposed to impulse controls) for a general class of stochastic diffusion systems. In concrete terms, a classical control variable is of the nature of *speed* and *rate* (e.g. speed of a rocket and rate of lending interest); while an impulse control variable is of the nature of *location* and *position* (e.g. initial inventory positioning and rescue injection to central reserve.)

Classical Stochastic Control. Consider a stochastic diffusion system with the following dynamics:

$$dx = \mu(x,u)dt + \sigma(x,u)dw$$

where $x \in R^n$ is the state, $u \in R^m$ is classical control, and dw is an n-dimension Brownian motion. The diffusion system contains a drift term $\mu(x,u)$ and a disturbance term $\sigma(x,u)$. A classic stochastic control system is presented in a standard form as following:

$$\begin{cases} V = \inf\limits_{u} \int\limits_{0}^{T} L(x,u)dt + \psi(x_T) \\ \text{s.t.} \\ \; dx = \mu(x,u)dt + \sigma(x,u)dw_t, \text{ given } x_0 \end{cases}$$

It is well established through dynamic programming that an optimal solution of a stochastic control problem can be characterized by a deterministic second-order partial differential equation (PDE), termed Hamilton-Jacob-Bellman equation. Without loss of generality, the study herein is confined to the 1-dimension diffusion systems, i.e. the state variable x is of one dimension. It shall be noted that the control u can still be multi-dimensional.

Impulse Control. Imposed upon an aforementioned diffusion system, an impulse control ξ_k can be exerted at a discrete point in time t_k $(k = 1, 2, \cdots)$ to shift the position of system state x, incurring an impulse control cost $K(\xi_k)$. Under impulse control, the state of diffusion system at time t, $x(t)$, is now expressed as:

$$x(t) = x_0 + \sum_{k=1} \xi_k + \int_0^t \mu(x, u)d\tau + \int_0^t \sigma(x, u)dw_\tau$$

The value function of an impulse control problem can be then be constructed as follows:

$$V(x) = \inf_{x(\cdot),u(\cdot)} \left(\sum_{k=1}^{\infty} e^{-rt_k} K(\xi_k) + \int_0^{\infty} e^{-r\tau} L(x(\tau), u(\tau))d\tau \right)$$

Due to Bensoussan and Lions (1984)[**10**], the value function under impulse control can be characterized as a solution to QVI (quasi-variational inequalities):

$$\begin{cases} (V - \mathcal{M} * V)(\mathcal{L}(V) - f(x)) = 0 \\ V < \mathcal{M} * V \\ \mathcal{L}(V) < f(x) \end{cases}$$

where $\mathcal{L}$ is a second order linear differential operator and $\mathcal{M}*$ represents an inf-convolution on V. Condition $V < \mathcal{M} * V$ always takes effect on the intervention region of V and the criteria $\mathcal{M} * V$ is not a fix one as well, which is not the case for an ordinary VI (variational inequality) problem.

1.2. Literature Review

There are a substantial volume of literature on deterministic and stochastic formulations of inventory problem [**8, 9**]. Bensoussan [**10**] first established the bridge between value functions and quasi-variational inequalities. This gave a theoretical foundation for subsequent researches in a deterministic framework allowed for both ordinary and impulse control. The main goal was to characterize a unique continuous, uniformly bounded viscosity solutions of Bensoussan – Lions type QVI. Bensoussan and Menaldi [**12**] provided a general framework for studying hybrid control systems. However, in many cases the optimal value functions are not necessarily smooth (e.g. non twice differentiable). An obvious example can be found in Berovic [**13**]. It is shown in Frankowska [**26**], viability theory can be used to link value functions and generalized lower semi-continuous solutions to QVIs. General analytical tools for dealing with this class of problems have been developed by Aubin [**7**]. The implications of viability theory regarding properties of hybrid systems are shown in Aubin, Lygeros et al. [**6**], with specific characterization of lower semi-continuous value functions for impulse control problem in infinite horizon.

Most inventory system actions, such as order goods with ordering cost, have discrete impulse characteristics. There is a large volume of literature on applying QVIs and viability theory to analyze inventory system. For example, Sulem [**34**] studied an inventory control problem under uncontrolled diffusion (demand) over an infinite horizon. In this model, the change of inventory level is classified into two types: one is driven by an uncontrollable Ito process of customers' demand, comprising of a deterministic drift term (i.e, the mean demand), as well as a stochastic disturbance term (i.e. the variability of the demand). Another type of change in inventory level is discrete type, that is, at some discrete points in time inventory orders are made which create jumps in the system states (e.g. inventory levels.) Associated with the two types of state transitions (namely continuous and discrete), there are two types of associated cost in the objective function as well. Under a differentiable Lagrange cost as objective function, the problem is treated as an impulse control under uncontrollable diffusion, and is solvable in principle via the associated QVI's of Bensoussan – Lions type. Our study will be helpful for the proposed study of a more general model with hybrid control. Moreover, two extensions are reported in two other papers: one to a two-dimension case by Sulem [**34**], and the other to deteriorating goods by Lakdere, Amin et al. [**32**].

A recent result was obtained by Berovic [**14**] and the model discussed in his paper is mostly related to our work. He also discussed a hybrid control of an inventory system (it was called "generalized inventory") in his model. A comparison between differential inclusion version QVIs and Bensoussan – Lion type QVI was shown in this paper. Characteristics of value function are reviewed. An algorithm to compute the optimal control of a hybrid system with the data of value function was proposed. Finally, an example of a simplified model is discussed. It is claimed that it was the first time an example was shown with specific inventory system for the using of analytic machinery, which has been thoroughly discussed in previous literature. However, we still find there are several limitations in this paper which left enhancement space for us to study. Though this theoretical study of the "generalized inventory" system is elaborated, the example he showed is too simple to characterize the system. The continuous control part of the hybrid concept was not considered in this model. This reduces the "generalized inventory" system (supposed to a hybrid system) to an ordinary impulse control system of Bensoussan [**10**]. Another limitation of the paper by Berovic and Vinter is that the algorithm used to calculate the trajectory requires a known "terminal" value of the value function at each impulse instant. However, the terminal values at impulse instants are unknown before the value function is solved from the corresponding HJB-QVI's, which alone is known to be more complicated than finding the optimal trajectory.

The mathematical framework to the impulse control problems in the study is from Bensoussan and Lions [**10**]. Since impulse control characterizes a class of optimal control problem, it have wide use in many area. Impulse control has been studied in inventory control [**27**], exchange rate problem [**22**], dividend policy for insurance [**20**], and portfolio optimization with transaction costs [**29**]. In many economic and financial applications where the controlled process is described as an Itō diffusion, the cost structure involves a setup cost, so the solution to the problem is always related to Hamilton-Jacobi-Bellman equation and quasi-variational inequalities.

Upon the property of quasi-variational inequalities conditions the state space can be split into intervention and continuous (no intervention) regions. We can characterize the dynamic behavior for each region and give conditions for the relation between then. Base on these observations, the common analytical way to solve the QVI follows these steps: First, by analyzing the characteristics of the QVI, conjecture the continuous region and intervention region. Then find the associated policy. Find the value function for the policy and prove the optimality of the value function. After this procedure we come to the conclusion that the impulse control associated with the QVI is an optimal control. These steps are often very difficult in real application and the success mainly depends on the form of the controlled process, reward and cost functions. But for some classes of problems it is still possible to solve them by using analytical method. Examples in the chapter 2 and chapter 3 will apply this method to investigate several problems originate from mutual insurance control problem.

For impulse control problem, besides the variable cost, we usually consider a positive set up cost associated with the impulse control, so the optimal strategy is always is to apply the control on a discrete way. The control is directly applied to the state of the system. Besides impulse control, there is another type of control known as singular control, which also gives a possibility of direct control on the system's state. Since the control cost in the case of singular control does not include set up cost, the optimal control could appear continuous in time. The behavior a singular control looks different from that of impulse control, but mathematically singular control is a special case of general impulse control problem with zero setup cost. Motivated by this, we extend the work in Kummar and Muthuraman [**30**], in which an algorithm is given to solve a singular control problem. Inspired by the fact that singular control and impulse control have an internal relation, we developed an algorithm to solve a class of impulse control problems. In Chapter 4 we provide this result together with numerical examples. Those examples is thought to be difficult or impossible to be solved before.

Another possible way to solve impulse control problem is to decompose the original problem into sequential optimal stopping time sub-problems. Impulse control problem usually involves solving a stochastic differential equations with an in-explicit boundary condition, which usual expressed as a convolution on the solution itself. For optimal stopping time problem, there is also a stochastic differential equations to be solved, but with an explicit boundary, which is always specified beforehand. However the boundary condition for the optimal stopping problem usually is a fixed boundary, which means we know exactly where the boundary condition is applied on. So we can see there is a natural connection between the optimal impulse control problem and optimal stopping time problem. For a general optimal stopping problem, finite element method can be developed to solve it numerically. In Chapter 5, we clarify this relation and show the uniqueness and existence of the solution under this approach. Numerical example will also be given to demonstration the effectiveness of this method.

Another thing need to be pointed out here is that Costa and Davis [**25**] take the value improvement approach while the others take the quasi-variational inequality approach. This study is mainly based on the latter approach.

CHAPTER 2

Contingent Impulse Control

2.1. Introduction

A mutual insurance organization, compared to a normal commercial insurance firm, adopts a none principal-agent type of risk-pooling mechanism comprising two contingent options (impulse control), namely, contingent calls and refunds. We develop a band-type contingent options (BTCO) model for mutual insurance, and derive QVI (quasi-variational inequality) characteristics for the optimality of a BTCO policy which was introduced for cash management by Constantinides and Richard [**24**]. Based on the QVI-characteristics system, we show that an optimal BTCO policy can be determined by solving a boundary-value problem that is constructed with the QVI-characteristics. Finally, the QVI-characteristics based solution method is tested with numerical examples of mutual insurance management. With these findings, we argue that contingent options constitute an alternative incentive scheme that preserves the revelation principle under a non-principal-agent setting.

2.2. Two-way Contingent Control with One End Set to Zero

Mutual insurance organization (such as a marine mutual insurance club) as compared to a commercial insurance firm that is founded upon an asymmetric information structure of principal-agent type [**31**], differs fundamentally in a none principal-agent type of risk-pooling mechanism. Unlike commercial insurance which requires non-linear premium pricing (e.g., basic premium, plus claims-dependent deductible price breaks, etc.), mutual insurance adopts a linear premium pricing scheme. For example of marine insurance, the premium for a vessel is determined by a fixed rate proportional to the tonnage of the vessel, that is, the regular premium charge of a vessel is calculated by multiplying the tonnage of the vessel with a vessel-independent premium rate. This can be further evidenced from the unique pricing scheme of mutual reserve options: A linear claims-independent premium is charged at a predetermined rate p (i.e., dollars per unit time) so as to build up a mutual reserve; while contingent options of a "call" (an impulse control to increase the reserve) or a "refund" (an impulse control to reduce the reserve) can be exercised with certain costs (fixed costs plus variable costs), when the reserve runs "low" or "high," respectively. In general, a real option that is defined over an interval domain is termed a band type of option. For the case of mutual insurance, the contingent option is limited within a "band" defined by bounds, namely, a call-threshold (lower bound) and a refund-bound (upper bound). As a special case of the two-boundary band type of contingent options, inventory ordering policies (e.g., $(s,\ S)$ ordering) only consist of a single lower-bound threshold (i.e., re-order point s), prohibiting any returned inventory. That is, inventory ordering is typically of a single-boundary band type, with an unbounded upper bound (i.e., refund-bound.)

We study in this chapter optimal contingent option policies for reserve management in mutual insurance under a continuous-time reserve diffusion process. We shall note that the result of the study is applicable to a range of mutual risk problems, including cash reserve, dividend, and inventory management, as the ones studied by Constantinides and Richard [**24**], Cadenillas, Choulli, Taksar, and Zhang [**20**], and Bensoussan, Liu, and Sethi [**11**]. The study of contingent options in mutual insurance is motivated by the ongoing search for the secret of why the P&I Club - a marine mutual insurance - has been predominating in the marine insurance market for over 150 years. Based on the findings in this study, we argue it is the unique contingent options that have secured the success of a P&I Club without subscribing to the typical non-linear pricing scheme under a principal-agent setting.

Assuming an exogenous diffusion claim process, we formulate the above mutual insurance problem as a band-type contingent option (BTCO) model which seeks cost-minimizing contingent option policies under a linear reserve holding cost structure and a fixed regular premium rate. We show that an optimal contingent option policy exists in the form of a stationary *band-type* policy as obtained for cash management by Constantinides and Richard [**24**]. Without using QVI characteristics, Constantinides and Richard proved the existence of optimal band-type policies in the form of two $(s,\ S)$ policies constructed at the two boundaries, that is, the system state is regulated within an interval $[a,\ b]$ by applying an $(a,\ A)$ policy at the lower boundary $a \in (-\infty, \infty)$, and a $(B,\ b)$ policy at the upper boundary b, with $a \leq A \leq B < b$. In general, a closed-form analytical solution for the optimal options as characterized by the four boundary parameters is still unobtainable. Only for a special case with $a = 0$ (i.e., requiring non-negative reserve), Harrison, Sellke and Taylor [**27**] obtained a closed-form solution to the associated HJB equation, and then developed an iterative method to computationally calculate the optimal BTCO parameters, $0 = a \leq A \leq B < b$. We note that the model developed in this study is of a stationary band with $a = 0$. Also, we note that most impulse control models for cash and inventory management are focused on single-boundary band option, for example, either with a single lower-bound option for contingent calls (as in an inventory system), or a single upper-bound option for contingent refunds (as in dividend management). Recent research on single-boundary band-type option models can be found in Bensoussan, Liu, and Sethi [**11**] who proved optimality of $(s,\ S)$ policy in an inventory system facing compound Poisson and diffusion demands, and in Cadenillas, Choulli, Taksar, and Zhang [**20**] who studied dividend policies in a general insurance firm.

Differing from the non-QVI approaches as typically taken in cash and inventory research, we start with deriving QVI (quasi-variational inequalities) characteristics of BTCO model, consisting of HJB optimality conditions for the continuation region and inf-convolution constraints on the boundaries. Based on the QVI characteristics, we then construct a boundary-value problem of the corresponding HJB equation, termed a QVI-characteristics system, for the solution of an optimal BTCO policy. An optimal BTCO policy for reserve management in mutual insurance can be determined and computed, by solving the boundary-value problem embedded in the QVI-characteristics system. So far in QVI research literature, closed-form analytical solutions, either for HJB equations or for inf-convolution constraints, are unattainable, except for the case of $a = 0$ as studied by Harrison, Sellke and Taylor [**27**], where HJB equation can be solved in closed-form but inf-convolution conditions remain unsolvable analytically. In sum, the QVI-characteristics system developed in this study gives a unified solution method for solving contingent-option problems in mutual insurance, as well as for solving cash and inventory management problems.

2.3. A band-type contingent option Model

First, we define the insurance claim process in a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. The cumulative insurance claim $d(t)$ over the interval $[0, t]$ is a stochastic diffusion process

$$d(t) = \eta t + \sigma w(t), \tag{2.3.1}$$

where η is a constant drift, σ is a non-negative constant disturbance term, and $w(t)$ denotes a Wiener process with $w(0) = 0$. Let $y(t)$ be the insurance reserve at time $t \geq 0$ with

$y(0) = y$, and $p \in (0, \infty)$ be a constant premium rate. Without contingent options, the cumulative reserve $y(t)$ over the interval $[0, t]$ follows a diffusion process:

$$y(t) = y + pt - d(t) = y + \mu t - \sigma w(t) \tag{2.3.2}$$

where $\mu = p - \eta$. Next, we construct a stationary band-type contingent option (BTCO) policy, denoted as $\Theta\{(a, A); (B, b)\}$, that is associated with the reserve diffusion process (2.3.2), including a contingent call option and a contingent refund option. Similar to Constantinides and Richard [**24**], the stationary BTCO policy, $\Theta\{(a, A); (B, b)\}$, is defined as:

Definition 1. **(BTCO Policy Parameters)** For two pairs of parameters (a, A) and (b, B) such that $a \leq A \leq B < b$ with $a = 0$ and A, b and B all controllable, we define:

- Contingent calls $\{(\tau_i^a, \xi_i^a) : i = 1, 2, \cdots\}$: Given a call-threshold $a = 0$, $\tau_i^a = \inf\left\{t > \tau_{i-1}^a : \ y(t) = a\right\}$ (with $\tau_0^a = 0$) is defined as the ith *call time* (Markov), and $\xi_i^a = A - a$ is the amount (controllable) of the ith contingent call so as to increase the reserve up to the level of $A = a + \xi_i^a$. It shall be noted that the call option defined herein is stationary, i.e., $\xi_i^a = \xi^a = A - a$ for all $i = 1, 2, \cdots$.
- Contingent refunds $\left\{(\tau_j^b, \xi_j^b) : j = 1, 2, \cdots\right\}$: Given a refund-threshold $b > a$ (controllable), $\tau_j^b = \inf\left\{t > \tau_{j-1}^b : \ y(t) = b\right\}$ (with $\tau_0^b = 0$) is defined as the jth *refund time* (Markov), and $\xi_j^b = b - B$ is the amount (controllable) of the jth contingent refund so as to decrease the reserve down to the level of $B = b - \xi_j^B$). It shall be

noted that the refund option defined herein is stationary, i.e., $\xi_j^b = \xi^b = b - B$ for all $j = 1, 2, \cdots$.

Definition 2. **(BTCO Policy)** Let $x(t)$ denote the cumulative reserve at time t. A stationary band-type contingent option (BTCO) policy, denoted as$\Theta\left\{(a, A); (b, B)\right\}$(or simply $\Theta(a; B)$), regulates a reserve diffusion process according to the following rules:

$$x(t^+) = \begin{cases} x(t) + (A - x(t)) & \text{if } x(t) \leq a \\ x(t) & \text{if } a < x(t) < b \\ x(t) - (x(t) - b) & \text{if } b \leq x(t) \end{cases} \tag{2.3.3}$$

Now, we can write the cumulative reserve under a BTCO policy, denoted by $x^{\Theta(a,b)}(t)$ with a given initial state $x = x(0)$, as following:

$$x^{\Theta(a,b)}(t) = x + \mu t - \sigma w(t) + Q(t) - R(t), \tag{2.3.4}$$

where $Q(t) = \sum_{\left\{i:\tau_i^a<t\right\}} \xi_i^a$ represents cumulative amount of calls over interval $[0, t)$, and $R(t) = \sum_{\left\{j:\tau_j^b<t\right\}} \xi_j^b$ represents cumulative amount of refunds over $[0, t)$. The cost associated with holding reserve at time t is given as $\rho x(t)$ with $\rho \geq 0$. The costs of the i-th contingent call and j-th contingent refund, denoted respectively as $g^+(\xi)$ and $g^-(\xi)$, are given as:

$$g^+(\xi) = \begin{cases} K^+ + c^+\xi & \xi > 0 \\ 0, & \xi = 0 \end{cases}, \text{ and } g^-(\xi) = \begin{cases} K^- + c^-\xi & \xi > 0 \\ 0, & \xi = 0 \end{cases}, \tag{2.3.5}$$

where $K^+, K^-, c^+,$ and c^- are all non-negative. c^- here represent the difference of opportunity cost and the unit dividend benefits. This representation is suggest by Harrison [**27**].

Here we consider the case that opportunity cost is larger than the dividend benefits and it is positive. The cost objective function, $C(y, \Theta)$, given initial reserve $y = y(0)$ under BTCO policy $\Theta(a; b)$, can be written as:

$$C(x, \Theta) = E\left\{\int_0^\infty \rho x^\Theta(t) e^{-rt} dt + G^+(\Theta) + G^-(\Theta)\right\} \tag{2.3.6}$$

It includes both holding cost which is continuously charged in time and the costs incurred by the contingent options. The total costs of contingent options are given by

$$G^+(\Theta) = \sum_{i=1}^{\infty} g^+(\xi^a) e^{-r\tau_i^a}; \text{ and } G^-(\Theta) = \sum_{j=1}^{\infty} g^-(\xi^B) e^{-r\tau^b} \tag{2.3.7}$$

Let $\mathcal{X}$ denote the set of all admissible option policies, and $\mathcal{X}^\Theta$ denote the subset of $\mathcal{X}$ containing all $\Theta(a, b)$ policies. Then, the associated value function can be defined from (2.3.6) as:

$$V(x) = \inf_{\Theta \in \mathcal{X}^\Theta} C(x; \Theta) \tag{2.3.8}$$

To this end, a band-type contingent options (BTCO) model can be formulated as:

$$\begin{cases} V(x) = \inf\limits_{\Theta \in \mathcal{X}^\Theta} C(x; \Theta) = \inf\limits_{\Theta \in \chi^\Theta} E\left\{\int_0^\infty \rho x^\Theta(t) e^{-rt} dt + G^+(\Theta) + G^-(\Theta)\right\} \\ \text{s.t.} \\ \quad x^{\Theta(a,b)}(t) = x + \mu t - \sigma w(t) + Q(t) - R(t) \\ \quad G^+(\Theta), G^-(\Theta): \text{ as given in (2.3.7)} \end{cases} \tag{2.3.9}$$

The optimality of a band-type impulse control policy for cash and inventory management has been obtained by Constantinides and Richard [**24**]. Now, we show the optimality of BTCO policy for mutual insurance. To this end, we obtain the following:

THEOREM 3. *For BTCO Model (3.3.5), there exists an optimal band-type contingent policy,* $\psi^\star = \{\Theta^\star\{(a^\star,\ A^\star);(b^\star,\ B^\star)\}\}$*, with* $0 = a \le A^\star \le B^\star < b^\star$*, under which there exists a twice-differentiable* $\pi(x)$ *such that the following equality is satisfied for almost everywhere (a.e.) in* $x \in (a,\ b)$*:*

$$\pi(x){=}_{a.e.}V(x) = C\left(x, \psi^*\right)$$

PROOF. By virtual of Theorem 1 together with Corollary 1 and 2 of Constantinides and Richard [**24**], there exists a band-type contingent policy $\Theta(p) = \{(a,\ A);(b,\ B)|p\}$ with $a \le A \le B < b$ for $a \in (-\infty, \infty)$, under which there exists a twice-differentiable function $\pi(x;\Theta)$ such that the following partial differential equation is satisfied over the interval $x \in (a,\ b)$:

$$-\frac{\sigma^2}{2}\frac{\partial^2\pi(x;\Theta)}{\partial x^2} - \mu\frac{\partial\pi(x;\Theta)}{\partial x} + r\pi(x;\Theta) - \rho x = 0 \tag{2.3.10}$$

Letting $\psi^* = \{\Theta^*\}$with $a = 0$, we then obtain $\pi(x) = C\left(x, \psi^*\right)$, *a.e.* for $x \in (0, b)$, with which we conclude the proof.

□

2.4. Characteristics of Optimal BTCO Policy

In this section, we characterize the optimal band-type contingent option (BTCO) policy using quasi-variational inequalities (QVI). First, we obtain a QVI-characteristics system associated with BTCO model (3.3.5):

Proposition 4. *Let* $\pi(x) =_{a.e.} \inf_{\Theta \in \mathcal{X}^{\Theta}} C(x, \Theta)$*, almost everywhere (a.e.) in* $x \in (0, b)$*, be a twice-differentiable function as defined in Theorem 3, where* $C(x, \Theta)$ *is as given in (2.3.6) for every* $x \geq 0$*. Then,* $\pi(x)$ *satisfies the following QVI characteristics system of BTCO model for every* $x \geq 0$*:*

$$\begin{cases} \mathcal{L}^p \pi(x) \leq \rho x, \\ \pi(x) \leq \left(g^- \otimes \pi\right)(x), \\ \left\{\pi(x) - \left(g^- \otimes \pi\right)(x)\right\}\left\{\mathcal{L}^p \pi(x) - \rho x\right\} = 0 \\ \pi(0) = \left(g^+ \otimes \pi\right)(0), \end{cases} \tag{2.4.1}$$

where the differential operator $\mathcal{L}^p \pi(x)$*, and inf-convolutions* $\left(g^+ \otimes \pi\right)(x)$ *and* $\left(g^- \otimes \pi\right)(x)$ *are defined as following:*

$$\mathcal{L}^p \pi(x) = -\frac{\sigma^2}{2}\frac{d^2\pi(x)}{dx^2} - \mu\frac{d\pi(x)}{dx} + r\pi(x)$$

$$\begin{aligned} \left(g^- \otimes \pi\right)(x) &= \inf\nolimits_{0<\xi<x}\left(g^-(\xi) + \pi(x - \xi)\right) \\ &= \inf\nolimits_{0<\xi<x}\left(K^- + c^-\xi + \pi(x - \xi)\right) \end{aligned}$$

$$\begin{aligned} \left(g^+ \otimes \pi\right)(x) &= \inf\nolimits_{\xi>0}\left(g^+(\xi) + \pi(x + \xi)\right) \\ &= \inf\nolimits_{\xi>0}\left(K^+ + c^+\xi + \pi(x + \xi)\right) \end{aligned}$$

Proof. Letting $\mathcal{L}^p\pi(y) = -\frac{\sigma^2}{2}\frac{d^2\pi(y)}{dy^2} - \mu\frac{d\pi(y)}{dy} + r\pi(y)$, we can write the HJB equation for the BTCO model as following,

$$\mathcal{L}^p\pi(y) - \rho y = 0$$

The rest of the Proof is a standard application of impulse control and QVI analysis, as rigorously presented in Bensoussan and Lions [**10**].

□

We shall note that the HJB equation for the continuous region, i.e., $\mathcal{L}^p\pi(x) - \rho x = 0$, can be solved analytically by the general solution has a complex form and not easy to be analyzed when considering with the free boundary constrains. Next, we show that an analytical solution for the HJB equation can be obtained via transformation of value function. Letting $\pi(x) = w(x) + \dfrac{\rho x}{r} + \dfrac{\mu\rho}{r^2}$ and denoting $\pi'(x) = \dfrac{d\pi(x)}{dx}$, we can write:

$$\pi'(x) = w'(x) + \frac{\rho}{r}, \text{ and } \pi''(x) = w''$$

Then, we can write the HJB equation $\mathcal{L}^p\pi(x) - \rho x = 0$ in terms of $w(x)$ as following:

$$-\frac{\sigma^2}{2}w''(x) - \mu\left(w'(x) + \frac{\rho}{r}\right) + r\left(w(x) + \frac{\rho x}{r} + \frac{\mu\rho}{r^2}\right) - \rho x = 0 \tag{2.4.2}$$

The above equation (2.4.2) can be easily reduced to the following:

$$-\frac{\sigma^2}{2}w''(x) - \mu w'(x) + rw(x) = 0 \tag{2.4.3}$$

The transformed HJB equation (2.4.3) is a homogeneous ODE, which can be solved in the closed form as

$$w(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

where λ_1, λ_2 are the roots of the corresponding characteristic equation, and $C_1, C_2$are coefficients of specific solution determined from terminal conditions. This development is significant to the tractability of a BTCO problem. In particular, it is straight forward to show the transferability of solutions to ODE (2.4.3), that is, for any constant c, if $w(x)$ solves ODE (2.4.3), then so does $w(x+c)$. This transferability ensures some simplified analytical results of the inf-convolution. To see this, we write the inf-convolutions of QVI system (2.4.1) in terms of $w(x)$ using $\pi(x) = w(x) + \frac{\rho x}{r} + \frac{\mu\rho}{r^2}$. With the inf-convolution operators as defined in QVI characteristics system (2.4.1), we can write:

$$\begin{aligned}
\left(g^{+} \otimes \pi\right)(x) &= g^{+}(x) + \pi(0) \\
&= K^{+} + c^{+}x + w(0) + \frac{\mu\rho}{r^2}
\end{aligned}$$

$$\begin{aligned}
\left(g^{-} \otimes \pi\right)(x) &= \inf_{\xi>0}\left(g^{-}(\xi) + \pi(x-\xi)\right) \\
&= \inf_{\xi>0}\left(K^{-} + c^{-}\xi + w(x-\xi) + \frac{\rho(x-\xi)}{r} + \frac{\mu\rho}{r^2}\right)
\end{aligned}$$

Using impulse costs given in (2.3.5), it is immediate to derive:

$$\begin{aligned}
\left(g^{+} \otimes \pi\right)(x) &= K^{+} + \overline{c}^{+}(x) - \frac{\rho}{r}x + \frac{\mu\rho}{r^2} + w(0) \\
&= \left(\overline{g}^{+} \otimes w\right)(x) - \frac{\rho}{r}x + \frac{\mu\rho}{r^2}
\end{aligned}$$

$$\begin{aligned}
\left(g^{-} \otimes \pi\right)(x) &= \inf_{\xi>0}\left(K^{-} + \frac{\mu\rho}{r^2} + \overline{c}^{-}(\xi) + w(x-\xi)\right) \\
&= \left(\overline{g}^{-} \otimes w\right)(x) + \frac{\rho}{r}x + \frac{\mu\rho}{r^2}
\end{aligned}$$

where $\overline{c}^{+} = c^{+} + \frac{\rho}{r}$, $\overline{c}^{-} = c^{-} - \frac{\rho}{r}$, and

$$\overline{g}^{+}(\xi) = K^{+} + \overline{c}^{+}\xi, \text{ and } \overline{g}^{-} = K^{-} + \overline{c}^{-}\xi. \tag{2.4.4}$$

With the equalities (2.4.4) above, we obtain the following equivalences of inf-convolution conditions:

$$\left\{ \begin{array}{lll} \pi(y) - \left(g^{+} \otimes \pi\right)(x) = 0 & \Longleftrightarrow & w(y) - \left(\overline{g}^{+} \otimes w\right)(x) = 0 \\ \pi(y) - \left(g^{-} \otimes \pi\right)(x) = 0 & \Longleftrightarrow & w(y) - \left(\overline{g}^{-} \otimes w\right)(x) = 0 \end{array} \right. \tag{2.4.5}$$

Then the QVI characteristics system can be expressed in terms of $w(x)$ under the transformation $w(x) = \pi(x) - \left(\frac{\rho x}{r} + \frac{\mu \rho}{r^2}\right)$, via the following theorem:

THEOREM 5. *Let $\pi(x) \in C^1(R_+)$ be a solution of QVI characteristics system (2.4.1), and let $w(x) = \pi(x) - \left(\frac{\rho x}{r} + \frac{\mu \rho}{r^2}\right)$. Then, there exists an optimal stationary (time invariant) policy $\psi = \{\Theta\{(0,\ A);(b,\ B)\}\}$,* such that *the QVI characteristics system of BTCO can be equivalently expressed in terms of $w(x)$ as follows:*

$$\left\{ \begin{array}{ll} -\frac{1}{2}\sigma^2 w'' - \mu w' + rw = 0 & \textit{for } 0 \leq x \leq b \\ w(0) = w(A) + K^{+} + \overline{c}^{+} A & \\ w(b) = w(B) + K^{-} + \overline{c}^{-}(b - B) & \\ w'(A) = -\overline{c}^{+} & \\ w'(B) = \overline{c}^{-} & \\ w'(b) = \overline{c}^{-} & \end{array} \right. \tag{2.4.6}$$

where $\overline{c}^{+}$ and $\overline{c}^{-}$ are as defined via equalities in (2.4.4).

PROOF. Applying inf-convolutions in (2.4.5) for $x = 0$ and $x = b$, we obtain

$$w(0) = w(A) + K^{+} + \overline{c}^{+} A, \text{ and } w(b) = w(B) + K^{-} + \overline{c}^{-}(B - b)$$

respectively. The differentiability of $\pi(x)$ ensures the first-order condition of the inf-convolutions, that is,

$$\frac{\partial\left(K^{+}+\frac{\mu\rho}{r^{2}}+\overline{c}^{+}(A)+w(x+A)\right)}{\partial A}=0, \text{ and}$$

$$\frac{\partial\left(K^{-}+\frac{\mu\rho}{r^{2}}+\overline{c}^{-}(\xi^{b})+w(x-\xi^{b})\right)}{\partial \xi^{b}}=0$$

Immediately, we have $w'(A)=-\overline{c}^{+}$and $w'(b)=\overline{c}^{-}$. Denoting $w'(x)=\dfrac{dw(x)}{dx}$

$$\frac{\partial w(x+\xi)}{\partial \xi}=\frac{dw(x+\xi)}{dx}\cdot\frac{d\xi}{d\xi}=w'(x+\xi)=w'(x),$$

we obtain for the first-order inf-convolution condition the following:

$$w'(A)=-\overline{c}^{+}, \text{ and } w'(b)=w'(B)=\overline{c}^{-}$$

Together with the transformed HJB characteristic equation (2.4.3), the proof of Theorem is completed.

□

We can conclude from Theorem 5 that the optimal BTCO policy $\Theta\{(a,\ A);(b,\ B)\}$ is of a coupled $(s,\ S)$ policy, specifically a policy of combining an $(a,\ A)$ and $(b,\ B)$ policy, where $A=a+\xi^{a}$ and $B=b-\xi^{B}$. Also, it can be verified that the QVI-characteristics system of BTCO (2.4.6) poses as a two-point boundary value problem of ODE $-\frac{1}{2}\sigma^{2}w''-\mu w'+rw=0$, that is, the solution of BTCO model comprises a value function w which solves ODE $-\frac{1}{2}\sigma^{2}w''-\mu w'+rw=0$ and the boundary parameters $0=a\leq A\leq B<b$ which satisfy the inf-convolution conditions, all as specified in the

BTCO QVI-characteristics system (2.4.6). Then, the solution function $\pi(x)$to the original BTCO model can be immediately obtained as $\pi(x) = w(x) + \dfrac{\rho x}{r} + \dfrac{\mu\rho}{r^2}$.

2.5. Numerical Experiments on BTCO Model with $a = 0$

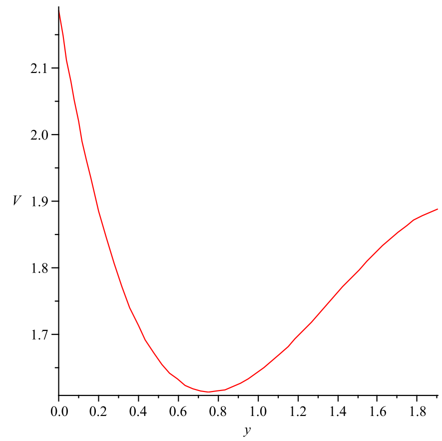
In this section, we conduct numerical experiments by using the QVI-characteristics system (2.4.6) to solve the BTCO problem for a mutual insurance club. In this computational experiment, we test how the change of system parameters, such as drift μ, disturbance σ, and option costs, will affect the optimal BTCO policies in terms of the policy parameters such as A, B, and b. The tests results are presented in a series of figures and plots, with A shown in "black" dotted line, B in "blue" dotted line, and b in "red" dotted line. With $a = 0$, we shall note that each call amount is calculated as $\xi^a = A - a = A$, and each refund amount is calculated as $\xi^b = b - B$.

First, we obtain sample values of the objective cost function $V(x)$ of BTCO model, as illustrated in Figure 2.5.1, using the following set of basic parameters:

$\mu = 0.01; r := 0.08; \sigma = 0.3; \rho = 0.1; K^+ = 0.5; K^- = 0.7; c^+ = 0.1; c^- = 0.1$

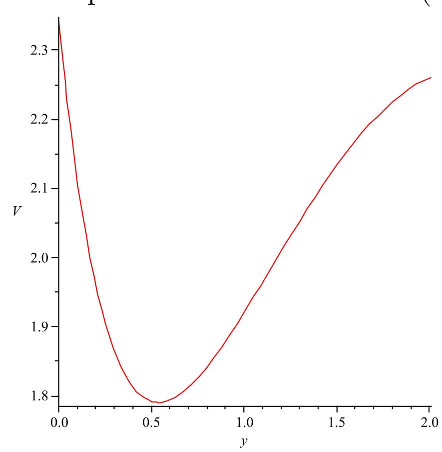
As we can see from Figure 2.5.1, there is an optimal reserve level $x \simeq$ 0.760 with $\mu = 0.01$, at which the minimized objective cost of BTCO model attains a single minimum of $V(0.760) \simeq 1.615$. The optimal policy is $\Theta\{(0, 0.682); (0.853, 1.903)\}$ (i.e., $A = 0.682$, $B = 0.853$, and $b = 1.903$; $\xi^b = b - B = 1.05$). We then increase μ by ten times, from 0.01 to 0.1, and computed another trajectory of objective value function for the system, see Figure 2.5.2. In this case the optimal policy is $\Theta\{(0, 0.492); (0.594, 2.013)\}$ (i.e., $A = 0.682$, $B = 0.594$, and $b = 2.013$; $\xi^b = 1.509$), and the optimal reserve level

FIGURE 2.5.1. Optimal Cost Function $V(x)$ with $\mu = 0.01$)



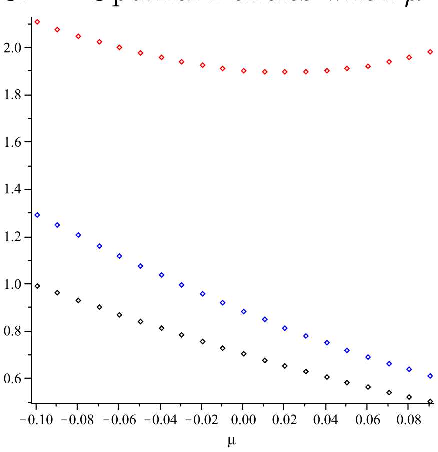
reduced slightly to $x \simeq 0.539$, with minimum of the optimal objective value function attained at $V(0.539) \simeq 1.790$.

FIGURE 2.5.2. Optimal Cost Function $V(x)$ with $\mu = 0.1$



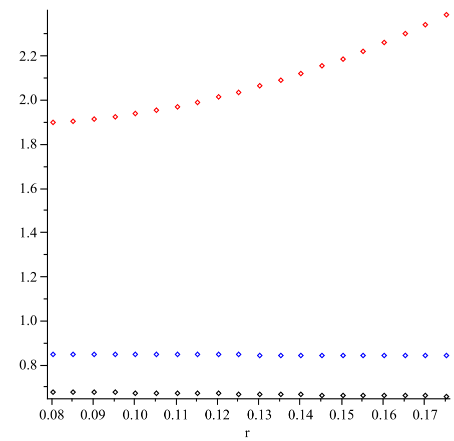
Next, we test on the impact of regular reserve build-up rate $\mu = p - \eta$, where p is the regular premium charge and η is average amount of claims. For a fixed η, drift μ represents the regular premium charge, with $\mu < 0$ representing the case where regular premium rate is less than that of claims. Changing the value of μ over the interval $[-0.1,\ 0.1]$ with all the rest parameters fixed, we computed the corresponding BTCO policy parameters A, B, and b, as illustrated in Figure 2.5.3. We can see from Figure 2.5.3 that an increased regular premium rate μ will result in a decreased A (i.e., the amount of each contingent call), as shown in black dotted line in Figure 2.5.3. This finding is consistent with the intuition that higher premium contribution on a regular basis from individual members will reduce the needs to call for contingent supplemental contributions from all the members in case of insufficient reserves. As to refund upper-boundary b shown in red dotted line in Figure 2.5.3, the mutual insurance club tends to reduce the frequency of refunds (by increasing refund threshold b), while at the same time the club will increase the amount of refund $\xi^b = b - B$, noting that the refund lower-boundary B is decreasing along with μ. It is interesting to note that the refund $\xi^b = b - B$ is a non-linear and increasing function of premium price represented by μ. It can be summarized from Figure 2.5.3 that even under a linear regular pricing policy, the revelation of asymmetric information can be still achieved by allowing claim-independent contingent options, especially with a non-linear refund scheme.

FIGURE 2.5.3. Optimal Policies when $\mu = -0.1 \sim 0.1$



Next, fixing other parameters while only letting the discount rate r to vary between $r = 0.08 \sim 0.18$, we test the impact of exogenous finance market condition on the refund parameter b, as illustrated in Figure 2.5.4. The mutual insurance club would adjust the refunding threshold to a higher level when the interest rate r becomes higher, indicating an increased value of future cash flow and therefore an increased magnitude of risk in the future. In other word, the Club should keep more reserve so as to prepare for the increased risk in the future. Compared to the refund threshold, the other policy variables A and B are almost insensitive to the change in interest rate r.

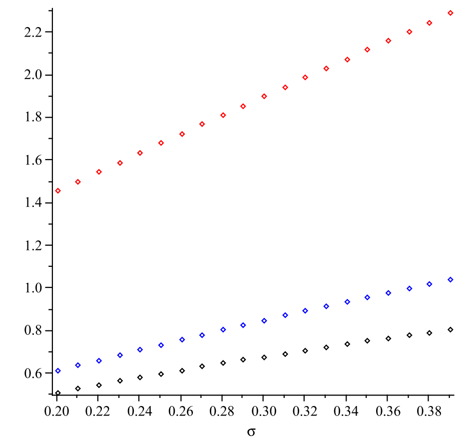
FIGURE 2.5.4. Optimal Policies when $r = 0.08 \sim 0.18$



As σ is defined as the volatility of the claim process, so it directly reflects the intrinsic demand risk of insurance market. From Figure 2.5.5, it shows that all three decision variable, A, B, and b will increase noticeably along with σ. The mutual insurance Club will become more conservative, by increasing all the thresholds for both contingent calls (A) and contingent refunds (B and b).

A higher value in ρ gives a higher time-continuous cost of holding reserve, which will encourage lower thresholds for both call and refund options, as shown in Figure 2.5.6. It

FIGURE 2.5.5. Optimal Policies when $\sigma = 0.2 \sim 0.4$



is interesting to note that if the holding cost becomes negligible ($\rho \to 0$), an unlimited reserve can be afforded by raising the refund level b (in red) to infinity.

FIGURE 2.5.6. Optimal Policies when $\rho = 0.01 \sim 0.2$

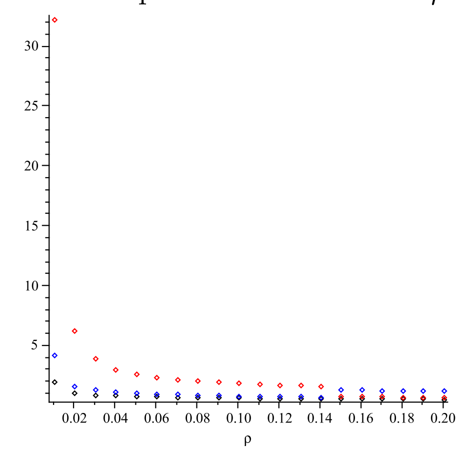
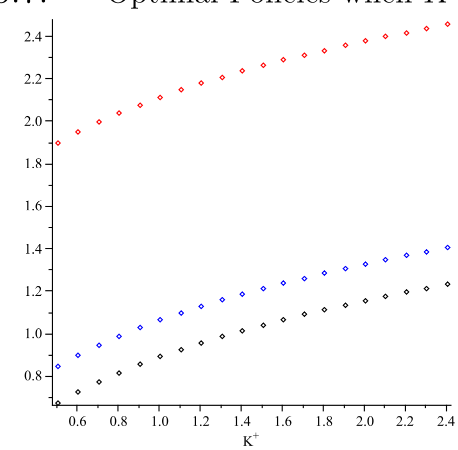


Figure 2.5.7 and 2.5.8 display the relationship between impulse control setup costs and the three parameters of a BTCO policy. In figure 2.5.7, we can see that all three policy parameters, A, B, and b, are increasing along with the setup cost of a contingent call (K^+). With an increasing A and fixed $a = 0$ the amount of call $\xi^a = A - a$ will increase as

K^+ increases, which results in a decreased frequency of contingent calls for a higher setup cost of each call. Also, we can see from Figure 2.5.7 that both B and b are increasing along with K^+, but the amount of refund $\xi^b = b - B$ largely remains unchanged as K^+ increases. On the other hand, the situation seems different when the cost of refund K^- is changed (see figure 2.5.8). In this case, the call-amount A is insensitive to the change in refund setup cost K^-, but refund parameters B and b are going opposite ways. The refund lower-bound B is decreasing with K^-, and the refund upper-bound b increases in an accelerated manner along with K^-, which resulted in accelerated increase in the refund amount $b - B$. In sum, the optimal refund policy seems to be more sensitive to the setup costs than the optimal call option does.

FIGURE 2.5.7. Optimal Policies when $K^+ = 0.5 \sim 2.5$



In a similar manner, the variable costs $c^+$and $c^-$of contingent calls and refunds are related to the BTCO policy parameters, as shown in Figure 2.5.9 and 2.5.10.

FIGURE 2.5.8. Optimal Policies when $K^- = 0.5 \sim 2.5$

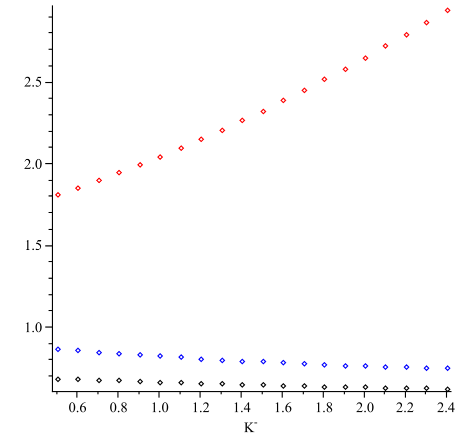
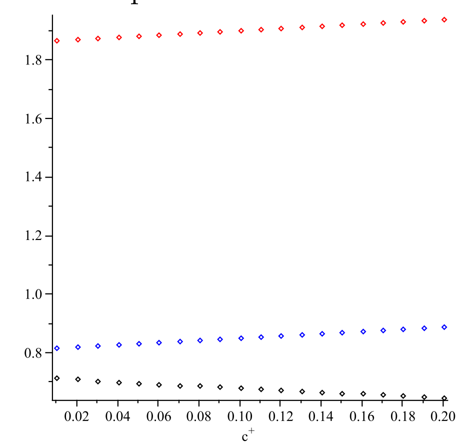


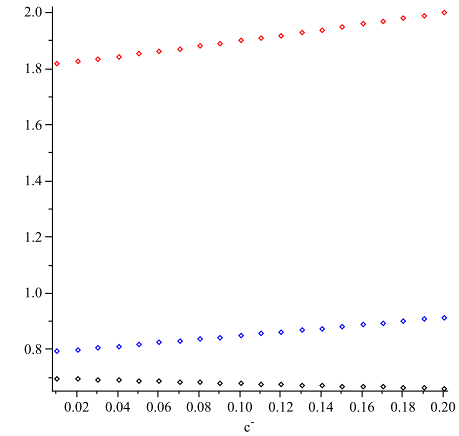
FIGURE 2.5.9. Optimal Policies when $c^+ = 0.01 \sim 0.2$



2.6. Conclusion

In this study, we obtained QVI characteristics of optimal band-type contingent potions (BTCO) in mutual insurance, and developed an explicit solution method for the BTCO model, by solving a two-point boundary problem of HJB equation subject to a set of inf-convolution boundary conditions. This solution method is tested with numerical sensitivity analysis of BTCO parameters pertinent to mutual insurance. We argue that contingent

FIGURE 2.5.10. Optimal Policies when $c^- = 0.01 \sim 0.2$



options form an alternative incentive scheme that preserves the revelation principle under a non-principal-agent setting.

CHAPTER 3

Hybrid Control

3.1. Introduction

We study a stochastic control problem with both classical and impulse control. The model is inspired from the operation of mutual insurance firms. The mutual insurance firm are allowed to adjust the reinsurance rate in the continuous time horizon. The firm can also execute both refund option and recall option which will increase and decrease the reserve position instantaneously.

We fund a band type policy together with a bang-bang control policy is the optimal policy for the firm to achieve a lowest maintenance cost. Constantinides[**24**] has studied this problem for the Cash Management Problem, with linear holding and penalty holding cost. The control can be applied in both ways of upward and downward. He proved that there exists one and only one two bands type optimal control policy for this system. But he did not provide an explicit way to find this solutions.

Many relevant studies are focused on finding the solution to a specific set of differential equations which satisfies the underlie QVI condition naturally. This strategy can be seen in literature [**11, 19, 22, 23, 28, 1, 15, 16**].

To solve the optimal impulse control problem, Costa and Davis [**25**] take the value improvement approach while the others take the quasi-variational inequality approach. The study in this chapter is based on the latter approach.

3.2. Feasible Control(Model)

we define the insurance claim process on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. The cumulative insurance claim $D(t)$ over the interval $[0, t]$ is a stochastic diffusion process.

$$D(t) = \eta t + \sigma w(t), \tag{3.2.1}$$

where η is a positive constant number which represent the drift part the diffusion process, σ is a non-negative constant disturbance term which represent the magnitude of the random noise. W_t denotes a Wiener process with $W_0 = 0$. We denote the premium rate collecting continuously by the mutual firm by p. We assume $p > \eta$. If without any control the total reserve process is still a diffusion process. We use μ to denote the drift: $p - \eta$.

Admissible hybrid control:

$$U = \left\{u(t), \left(\xi_1^+, \xi_2^+ ...\right), \left(\tau_1^-, \xi_1^-, \tau_2^-, \xi_2^- ...\right)\right\}$$

This notation means the mutual insurance firm can choose the reinsurance factor $u(t) \in [0,1]$ at time t continuously. When the reserve reaches zero at time τ_i^+, the firm should make a call from its shareholders. ξ_i^+ denotes the amount of the call. The firm can also make a decision on the time τ_j^- and amount of dividend ξ_j^-, it will refund to its shareholders. Please note that τ_j^- is a decision variable, whereas τ_i^+ is not. We denote by $\mathcal{U}$ the set of all admissible controls.

The cumulative amount of calls over interval $[0,t)$, and the cumulative amount of refunds over $[0,t)$ then can be denoted as:

$$Q(t) = \sum_{\left\{i:\tau_i^+ < t\right\}} \xi_i^+$$

$$R(t) = \sum_{\left\{j:\tau_j^- < t\right\}} \xi_j^-$$

The dynamics of the state process X under an admissible control is given by:

$$X(t) = x + \int_0^t \mu u(s)\, ds + \int_0^t \sigma u(s)\, dW_t + Q(t) - R(t), \tag{3.2.2}$$

Since the firm can take a call option, it will never confront the situation of bankruptcy, which a normal insurance company will meet in general.

3.3. Cost Structure and Value Function

The costs of the i-th contingent call and j-th contingent refund, denoted respectively as $g^{+}(\xi_i^a)$ and $g^{-}(\xi_j^B)$, are given as:

$$g^{+}\left(\xi_i^{+}\right) = \begin{cases} K^{+} + c^{+}\xi_i^{+} & \xi_i^{+} > 0 \\ 0, & \xi_i^{-} = 0 \end{cases} \tag{3.3.1}$$

$$g^{-}\left(\xi_j^{-}\right) = \begin{cases} K^{-} - c^{-}\xi_j^{-} & \xi_j^{-} > 0 \\ 0, & \xi_j^{B} = 0 \end{cases} \tag{3.3.2}$$

where $K^{+}, K^{-}, c^{+},$ and c^{-} are all non-negative. c^{-} here represents the unit benefits caused by the dividend from the mutual insurance firm, which will reduce the total costs in the object function. ξ_j^{-} is the dividend paid back to the shareholders, so it should be introduced as a reducing factor to the cost functions. Let r be a given discount rate. We define the discounted cost function for any admissible control $U \in \mathcal{U}$ with initial state $x \in [0, \infty)$:

$$C(x; U) = E_x \left\{ \sum_{i=1}^{\infty} g^{+}\left(\xi_i^{+}\right) e^{-r\tau_i^{+}} + \sum_{j=1}^{\infty} g^{-}\left(\xi_j^{-}\right) e^{-r\tau_j^{-}} \right\} \tag{3.3.3}$$

The objective control is to find the control policy to minimize the associated cost function. We define the value function according to the cost function (3.3.3) as:

$$V(x) = \inf_{U \in \mathcal{U}} C(x; U) \tag{3.3.4}$$

To this end, our model can be summarized as:

$$\begin{cases} V(x) = \inf\limits_{U \in \mathcal{U}} C(x;U) = \inf\limits_{U \in \mathcal{U}} E_x \left\{ \sum_{i=1}^{\infty} g^+ \left(\xi_i^+\right) e^{-r\tau_i^+} + \sum_{j=1}^{\infty} g^- \left(\xi_j^-\right) e^{-r\tau_j^-} \right\} \\ \text{s.t.} \\ \quad X(t) = x + \int_0^t \mu u(s)\, ds + \int_0^t \sigma u(s)\, dW_t + Q(t) - R(t), \\ \quad g^+, g^- : \text{ as given in (3.3.1) and (3.3.2)} \end{cases} \tag{3.3.5}$$

3.4. Optimal Control and Characteristic of the Value Function

We define the infinitesimal generator $\mathcal{L}$ for function $\phi : [0,\infty) \mapsto \mathbb{R}$

$$(\mathcal{L}^u \phi)(x) = -\frac{1}{2} u^2 \sigma^2 \frac{d^2 \phi(x)}{dx^2} - u\mu \frac{d\phi(x)}{dx}$$

$$\begin{aligned} (M\phi)(x) &= \inf_{0<\xi<x} \left(g^-(\xi) + \phi(x-\xi) \right) \\ &= \inf_{0<\xi<x} \left(K^- - c^-\xi + \phi(y-\xi) \right) \end{aligned}$$

The value function should satisfy following inequalities with an implicit boundary condition:

$$\mathcal{L}^u V + rV \leq 0 \tag{3.4.1}$$

$$V \leq MV \tag{3.4.2}$$

$$(V - MV)\left(\min_{u\in[0,1]} \mathcal{L}^u V + rV\right) = 0 \tag{3.4.3}$$

$$V(0) = \inf_{\xi>0}\left(g^+(\xi) + V(\xi)\right) \tag{3.4.4}$$

These set of equations can be proved as the sufficient condition for the value function.

THEOREM 6. *If there is a function $v \in C^1$ on $[0,\infty)$, satisfy condition* (3.4.1),(3.4.2),(3.4.3) *and* (3.4.4), *then for every $x \in [0,\infty)$, we have:*

$$v(x) \leq V(x)$$

Next, we will get the optimal policy associated with the QVI conditions.

3.5. Hybrid Control and its Optimality

From 4.1.7 we have:

$$\min \mathcal{L}^u V + rV = 0 \tag{3.5.1}$$

differentiate $\mathcal{L}^u V$ with respect to u, we have the first order condition:

$$u = -\frac{\mu V_1'}{\sigma^2 V_1''} \tag{3.5.2}$$

Substitute (3.5.2) to (2.4.2) and suppose $V'' \neq 0$ on the continuous control area

$$2r\sigma^2 V_1 V_1'' + \mu^2 \left(V_1'\right)^2 = 0 \tag{3.5.3}$$

For $x_0 < x < A$, $u(x) = \max[0,1] = 1$, then the HJB becomes:

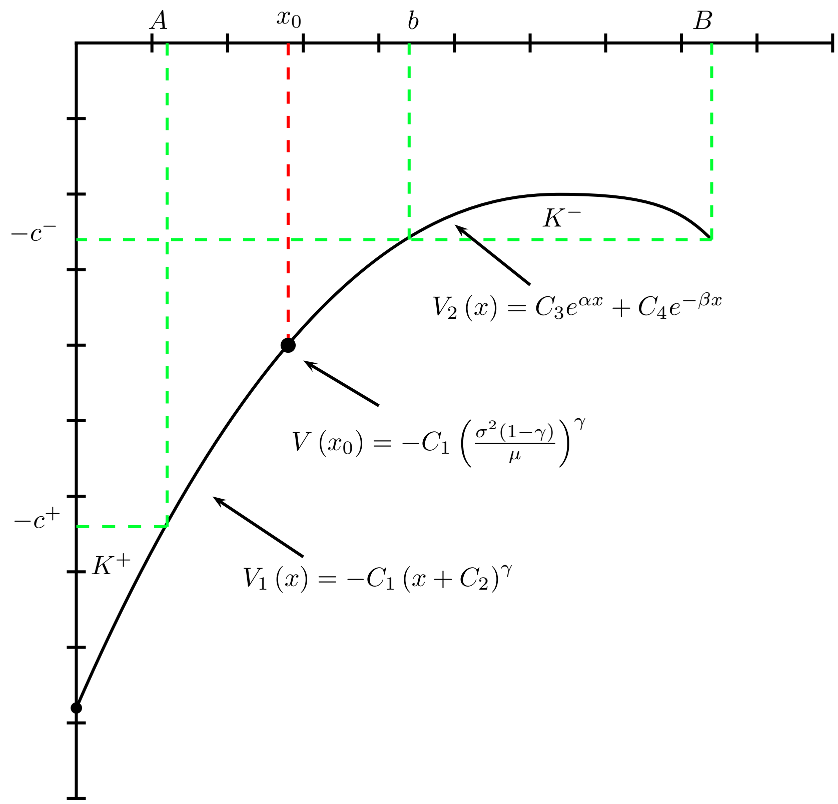
$$\frac{1}{2}\sigma^2 V_2'' + \mu V_2' - rV_2 = 0 \tag{3.5.4}$$

All we need to do next is to solve following set of equations,

$$\begin{cases} 2r\sigma^2 V(x) V''(x) + \mu^2 \left(V'(x)\right)^2 = 0 & \text{, for } 0 < x < x_0 \\ \frac{1}{2}\sigma^2 V''(x) + \mu V'(x) - rV(x) = 0 & \text{, for } x_0 < x < b \\ V(0) = V(A) + K^+ + c^+ A \\ V(b) = V(B) + K^- - c^-(B - b) \\ V'(A) = -c^+ \\ V'(b) = -c^- \\ V'(B) = -c^- \\ V_{x \to x_0^-}(x) = V_{x_0^+ \leftarrow x}(x) \\ V'_{x \to x_0^-}(x) = V'_{x_0^+ \leftarrow x}(x) \\ V''_{x \to x_0^-}(x) = V''_{x_0^+ \leftarrow x}(x) \end{cases} \tag{3.5.5}$$

Next, we will give the explicit expression for A,b,B, $x_0$and the value function $V(x)$.

Figure 3.5.1. Optimal Policy Parameters



The general solution of (3.5.3) is:

$$V_1(x) = -C_1 (x + C_2)^\gamma \tag{3.5.6}$$

Where

$$\gamma = \frac{1}{1 + \frac{\mu^2}{2r\sigma^2}}$$

Notice that

$$0 < \gamma < 1$$

From (3.5.2), we get the expression for $u(x)$ in term of C_2.

$$u(x) = \frac{\mu(x + C_2)}{\sigma^2(1-\gamma)} \tag{3.5.7}$$

Solve for x_0 by $u(x_0) = 1$:

$$x_0 = \frac{\sigma^2(1-\gamma)}{\mu} - C_2 \tag{3.5.8}$$

Due to the facts that $u(x) \in (0,1)$ and $u(x)$ is an increasing function, when $x > x_0$ $u(x) = 1$. When $0 < x < x_0$, the equation (3.5.7) is satisfied.

Substitute (3.5.8) into (3.5.6), we get

$$V(x_0) = V_1(x_0) = -C_1\left(\frac{\sigma^2(1-\gamma)}{\mu}\right)^{\gamma} \tag{3.5.9}$$

The general solution for 3.5.4 is:

$$V_2(x) = C_3 e^{\alpha x} + C_4 e^{-\beta x} \tag{3.5.10}$$

where,

$$\alpha = \frac{\sqrt{\mu^2 + 2r\sigma^2} - \mu}{\sigma^2}$$
$$\beta = \frac{\sqrt{\mu^2 + 2r\sigma^2} + \mu}{\sigma^2}$$

with $0 < \alpha < \beta$. Based on the ODE and boundary conditions given above, we can analyze the properties of the solution.

$$V_1'(x) = -\gamma C_1 (x + C_2)^{\gamma - 1}$$

$$-1 < \gamma - 1 < 0$$

$$V_1'(0) = -\gamma C_1 C_2^{\gamma-1}$$

3.5.1. Case 1: $A < x_0 < B$**.** This case is shown in figure 3.5.1. Suppose that V is twice continuous differentiable on x_0, $V_1'(x_0) = V_2'(x_0)$,$V_1''(x_0) = V_2''(x_0)$. from (3.5.2), we get:

$$V_2''(x_0) = \frac{\mu}{\sigma^2} V_2'(x_0), \tag{3.5.11}$$

Substitute above equation into the second equation of (3.5.5), we also get equation: $V_2'(x_0) = \frac{2r}{3\mu} V_2(x_0)$. Let $W(x)$ be the solution for the following ODE with boundary condition:

$$\begin{cases} \frac{1}{2}\sigma^2 W'' + \mu W' - rW = 0 \\ W'(0) = -k^- \\ W'\left(\xi^b\right) = -k^- \\ W(0) - W\left(\xi^b\right) = k^- \xi^b - K^- \end{cases}$$

It can be easily verified that the general solution for $W(x)$ is same as (3.5.10), and there exists one and only one ξ^b satisfy above equations. And we denote the corresponding solution as:

$$W^b(x) = C_1^b e^{\alpha x} + C_2^b e^{-\beta x}$$

From figure (3.5.1), we see that $V_2(x) = W^b(x - B)$. Now we can express $V_2(x)$ in term of B.

$$V_2(x) = C_1^b e^{\alpha(x-B)} + C_2^b e^{-\beta(x-B)}$$

From (3.5.11) we get:

$$\alpha C_1^b e^{\alpha(x_0-B)} - \beta C_2^b e^{-\beta(x_0-B)} = \frac{2r}{3\mu}\left(C_1^b e^{\alpha(x_0-B)} + C_2^b e^{-\beta(x_0-B)}\right)$$

From this equation we can solve for $\delta := x_0 - B$, then we can express x_0 by B: $x_0 = B + \delta$. Therefore,

$$V_1(x_0) = V_2(x_0) = C_1^b e^{\alpha\delta} + C_2^b e^{-\beta\delta} \tag{3.5.12}$$

together with (3.5.9), we now can solve C_1 explicitly as follow:

$$C_1 = -\left(C_1^b e^{\alpha\delta} + C_2^B e^{-\beta\delta}\right)\left(\frac{\sigma^2(1-\gamma)}{\mu}\right)^{-\gamma} \tag{3.5.13}$$

use condition given by (3.5.5) and (3.5.6)

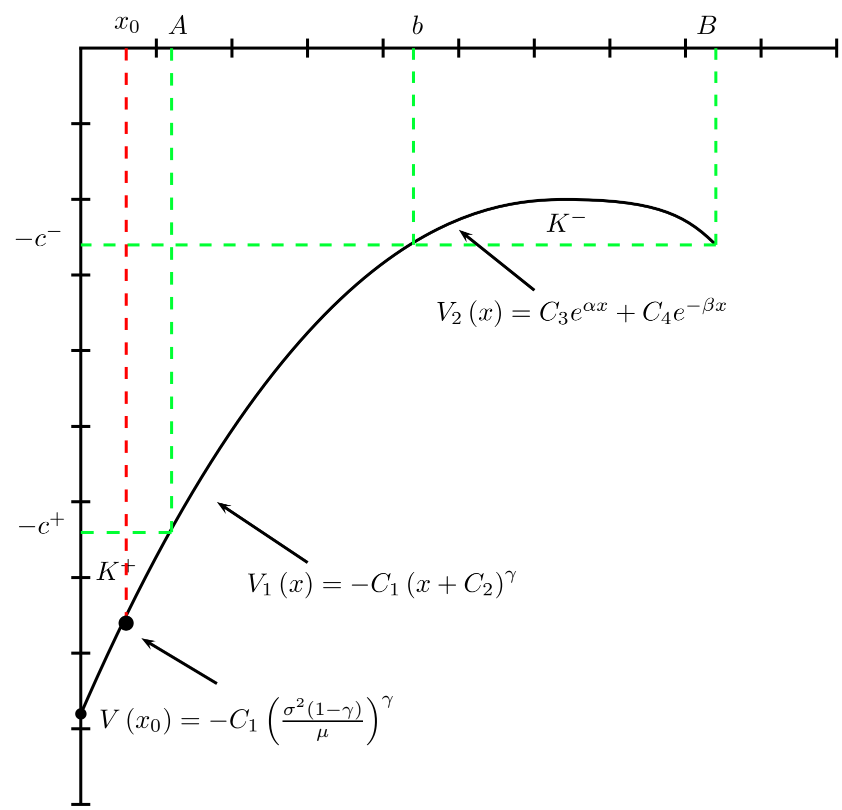
$$\begin{cases} -C_1\gamma(A + C_2)^{\gamma-1} = -c^+ \\ -C_1 C_2^\gamma + C_1(A + C_2)^\gamma = K^+ + c^+ A \end{cases}$$

The argument A can be reduced from above equations, and we get the equation :

$$-C_1 C_2^\gamma + C_1\left(\frac{c^+}{C_1\gamma}\right)^{\gamma-\gamma^2} = K^+ + c^+\left(\left(\frac{c^+}{C_1\gamma}\right)^{1-\gamma} - C_2\right)$$

The only unknown in above is C_2, which can be easily solved by Newton method.

FIGURE 3.5.2. Optimal Policy Parameters in the Case 2



3.5.2. Case 2: $0 < x_0 < A$**.** The solution for C_1 is the same as case 1, and from (3.5.13) we have

$$V(0) = V_1(0) = -C_1 C_2^\gamma$$

By using the same trick in case 1, we can express $V_2(x)$ as:

$$V_2(x) = C_1^b e^{\alpha(x-B)} + C_2^b e^{-\beta(x-B)}$$

Since $V_2'(A) = -c^+$, we get following equation

$$\alpha C_1^b e^{\alpha(A-B)} - \beta C_2^b e^{-\beta(A-B)} = -c^+$$

The quantity of $A-B$ can be solved, then we can calculate the value for $V(A)$:

$$\begin{aligned} V(A) &= V_2(A) \\ &= C_1^b e^{\alpha(A-B)} + C_2^b e^{-\beta(A-B)} \end{aligned}$$

and from (3.5.12)

$$\begin{aligned} V_2(x_0) &= C_1^b e^{\alpha\delta} + C_2^b e^{-\beta\delta} \\ &= -C_1 \left(\frac{\sigma^2 (1-\gamma)}{\mu} \right)^{\gamma} \end{aligned}$$

Form (3.5.5), we know that $V(0) - V(A) = c^+ (A - x_0) + c^+ (x_0 - 0) + K^+$, together with (3.5.8):

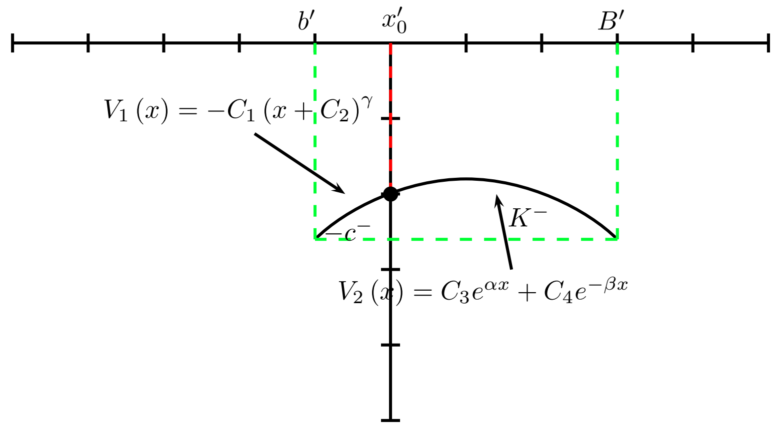
$$\begin{aligned} -C_1 C_2^{\gamma} - C_1^b e^{\alpha(A-B)} + C_2^b e^{-\beta(A-B)} &= c^+ ((A-B) - (x_0 - B)) + c^+ x_0 + K^+ \\ -C_1 C_2^{\gamma} &= c^+ ((A-B) - (x_0 - B)) + c^+ \left(\frac{\sigma^2 (1-\gamma)}{\mu} - C_2 \right) + K^+ \end{aligned}$$

where $A-B$ and $x_0 - B$ is already known. To this end, the only unknown in above equation is C_2

3.5.3. Case 3: $B < x_0 < b$**.** We shift the value function $V(x)$ to the left by amount of x_0, and denote the new function with $x_0' = 0$ as $W(x) = V(x + x_0)$. Correspondingly, $W_1(x) = V_1(x + x_0)$, and $W_2(x) = V_2(x + x_0)$. From (3.5.6) and (3.5.10) we get the general function for $W_1(x)$ and $W_2(x)$:

$$\begin{cases} W_1(x) = -C_1 (x + C_2')^{\gamma} \\ W_2(x) = C_3' e^{\alpha x} + C_4' e^{-\beta x} \end{cases} \tag{3.5.14}$$

FIGURE 3.5.3. Optimal Policy Parameters in the Case 3



, where $C_2' = C_2 + x_0$. From (3.5.8), we get C_2'

$$C_2' = \frac{\sigma^2(1-\gamma)}{\mu}$$

Since $W(x)$ is continuously differentiable at x_0',

$$W_2(0) = W_1(0) = -C_2'^{\gamma} C_1$$

$$W_2'(0) = W_1'(0) = -\gamma C_2'^{\gamma-1} C_1$$

from (3.5.14),

$$\begin{cases} C_3' + C_4' = -C_2'^{\gamma} C_1 \\ \alpha C_3' - \beta C_4' = -\gamma C_2'^{\gamma-1} C_1 \end{cases}$$

To this end, we can get the expression for $C_3'$and C_4' with only one unknown argument C_1 in it:

$$
\begin{cases}
C_3' = -\frac{\frac{\gamma}{C_2'}+\beta}{\alpha+\beta} C_2'^{\gamma} C_1 \\
C_4' = \frac{\frac{\gamma}{C_2'}-\alpha}{\alpha+\beta} C_2'^{\gamma} C_1
\end{cases}
\tag{3.5.15}
$$

Let $\lambda_1 = -\frac{\frac{\gamma}{C_2'}+\beta}{\alpha+\beta} C_2'^{\gamma}$, $\lambda_2 = \frac{\frac{\gamma}{C_2'}-\alpha}{\alpha+\beta} C_2'^{\gamma}$

$$
\lambda_1 + \lambda_2 = -C_2'^{\gamma}
$$

By imply the seventh condition in (3.5.5), we know that $W_2'(b') = -c^-$. Equivalently,

$$
\alpha\lambda_1 C_1 e^{\alpha b'} - \beta\lambda_2 C_1 e^{-\beta b'} = -c^-
$$

we now can solve for C_1, with the only unknown argument b' in the expression:

$$
C_1 = \frac{c^-}{\beta\lambda_2 e^{-\beta b'} - \alpha\lambda_1 e^{\alpha b'}}
\tag{3.5.16}
$$

We also know that $W_2(b') = \lambda_1 C_1 e^{\alpha b'} + \lambda_2 C_1 e^{-\beta b'} = \frac{c^-\left(\lambda_1 e^{\alpha b'} + \lambda_2 e^{-\beta b'}\right)}{\beta\lambda_2 e^{-\beta b'} - \alpha\lambda_1 e^{\alpha b'}}$

From the sixth condition in (3.5.5), we know that $W_1'(B') = -c^-$. By also using the first equation in (3.5.14), we get

$$
B' + C_2' = \left(\frac{c^-}{\gamma C_1}\right)^{\frac{1}{\gamma-1}}
$$

so

$$
W_1(B') = -C_1\left(B' + C_2'\right)^{\gamma} = -C_1\left(\frac{c^-}{\gamma C_1}\right)^{\frac{\gamma}{\gamma-1}}
$$

by using the forth condition in (3.5.5), we get $W_1(b') - W_2(B') = K^- - c^-(b' - B')$

$$
\begin{aligned}
-\left(\frac{c^-}{\gamma C_1}\right)^{\frac{\gamma}{\gamma-1}} - \lambda_1 e^{\alpha b'} - \lambda_2 e^{-\beta b'} &= \frac{K^- - c^-(b' - B')}{C_1} \\
-\left(\frac{\beta\lambda_2 e^{-\beta b'} - \alpha\lambda_1 e^{\alpha b'}}{\gamma}\right)^{\frac{\gamma}{\gamma-1}} - \lambda_1 e^{\alpha b'} - \lambda_2 e^{-\beta b'} &= \left(\frac{K^-}{c^-} - b' + B'\right)\left(\beta\lambda_2 e^{-\beta b'} - \alpha\lambda_1 e^{\alpha b'}\right)
\end{aligned}
$$

the only unknown in above equation is b', which can be easily solved numerically. We can now solve for C_1 in (3.5.16). Since we've got C_1 , C_2' The C_3' and C_4' are also clear to us by using (3.5.15). Therefore the expressions for both $W_1(x)$ and $W_2(x)$(3.5.14) are known to us. To this end, we solve the last case explicitly.

CHAPTER 4

Algorithms for Solving Impulse Control Problem

4.1. The Model

We define the process on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with right continuous filtration $\mathcal{F}_t$.Let W_t denotes a Wiener process with respect to this filtration and $W_0 = 0$. We consider control process that are defined by $\{(\tau_1, \xi_1), (\tau_2, \xi_2) ...\}$, where $\xi_i \neq 0$ denotes the amount of ith impulse control at time $\tau_i > 0$. We denote the cumulative amount of impulse over interval $[0, t)$, as:$Q(t) = \sum_{\{i:\tau_i < t\}} \xi_i$. Given the control process and the initial state x. The state process X is expressed as:

$$X(t) = x + \int_0^t \mu(X(s))\, ds + \int_0^t \sigma(X(s))\, dW_s + Q(t), \tag{4.1.1}$$

The state process is governed by the impulse control and a Brownian motion with drift μ and σ. The control process $\{(\tau_1, \xi_1), (\tau_2, \xi_2) ...\}$ are to be thought of pushing X in positive or negative direction at Markov stopping time τ_i.

Given a initial state x and a fixed discount factor $r > 0$ a control process $U = \{(\tau_1, \xi_1), (\tau_2, \xi_2) ...\}$ is said to be *admissible* if

$$\forall T \in [0, \infty): \qquad P\left\{\lim_{n\to\infty} \tau_n \leqslant T\right\}$$

$$\lim_{T\to\infty} E\left[e^{-rT} X(T+)\right] = 0$$

Above condition is necessary for the total cost of impulse control to be finite. We use $\mathcal{U}$ denotes the set that contains all admissible controls.

Since the firm can take a call option, it will never confront bankruptcy, which is a situation a normal insurance company will always meet in general.

The contingent options will incur a setup cost together with a variable cost proportional to the amount of control. Let g^+be the cost for making a call option and let g^- be the cost for make a refund option. Suppose $\xi_i^+ > 0$is the amount of money that the firm makes in the ith call, and $\xi_j^- > 0$ is the amount of dividend that the firm makes in the jth refund option. Then we have:

$$g(\xi) = \begin{cases} K^+ + c^+\xi_i^+ & \text{if } \xi > 0 \\ K^- - c^-\xi_j^- & \text{if } \xi < 0 \end{cases}$$

where $K^+, K^-, c^+,$ and c^- are non-negative constant, $K^+$and K^- denote the set up cost for the two-way impulse control. c^+ denote the proportional cost associate with the impulse control that shift the state positively. Respectively, c^- denote the proportional cost that associate the negative impulse control. Let r be a given discount factor. We now can define the discounted cost function for any admissible control $U \in \mathcal{U}$ with the initial state $x \in [0, \infty)$:

$$C(x; U) = E_x \left\{ \sum_{i=1}^{\infty} g(\xi_i) e^{-r\tau_i} + \int_0^{\infty} e^{-rs} \rho X(s)\, ds \right\} \tag{4.1.2}$$

The objective is to find a control to minimize the cost that is associated with. We define the value function according to the cost function (4.1.2) as:

$$V(x) = \inf_{U \in \mathcal{U}} C(x; U) \tag{4.1.3}$$

the corresponding control U^* is the optimal control.

For any twice differentiable function $\phi(x)$, we define the infinitesimal generator $\mathcal{L}$:

$$(4.1.4)\quad (\mathcal{L}\phi)(x) = \min_{u\in[0,1]}\left(\lim_{t\to 0}\frac{E_x\left[\phi\left(X_t\right)\right]-\phi\left(x\right)}{t}\right) = \min_{u\in[0,1]}\left(u\mu\frac{d\phi\left(x\right)}{dx}+\frac{1}{2}u^2\sigma^2\frac{d^2\phi\left(x\right)}{dx^2}\right)$$

Then we define the QVI formulation similar to preceding sections:

$$\mathcal{L}V + rV \leq 0 \tag{4.1.5}$$

$$V \leq MV \tag{4.1.6}$$

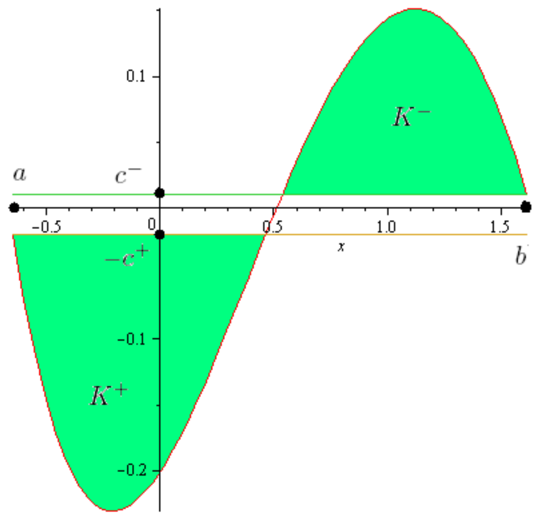
$$\left(V - MV\right)\left(\mathcal{L}V + rV\right) = 0 \tag{4.1.7}$$

These conditions will serve as sufficient condition for the optimal value function.

4.2. The Iterative Method

Because of the complexity of the differential equation with free boundary condition's, there is few stochastic control problems can be solved analytically [**4**]. Kumar [**30**] provide a numerical method to solve singular control problem. The basic idea behind this method is to solve sequential fixed boundary differential equations to approach the original problem of solving differential equations with free boundary condition. Inspired by this method, we provide a method in this section such that if we choose a boundary with a range wide enough (actually the position is also need to be considered, which is a limitation that Kumar's method also has, but he did not point it out in the paper), then we solve sequential differential equations with fixed boundary, then we will always get a convergence

FIGURE 4.2.1. A Particular First Order Derivative of the Value Function



solution to the optimal solution of the original problem with free boundary condition. It will be showed by examples. And This method can also be implemented to Geometric Brownian Motion, which is the basic model for many financial engineering problem.

Our object is to solve the following algebraic equation.

$$\begin{cases} -\frac{1}{2}\sigma^2 V'' - \mu V' + rV = f \\ V(a) = V(A) + K^+ + c^+(A-a) \\ V(b) = V(B) + K^- + c^-(b-B) \\ V'(a) = V'(A) = -c^+ \\ V'(b) = V'(B) = c^- \end{cases} \tag{4.2.1}$$

which means we need to find a solution, whose first derivative function appears as following graph:

In other words, we need to find a boundary (a, b), and solve an ordinary differential equation (ODE) in this band. Also a set of conditions needs to be satisfied. The condition says that the value of the first derivative at a and b must equal to $-c^+$ and c^- respectively. And the area above the line $y = c^-$ should equal to K^-; the area bellow the line $y = -c^+$ must equal to K^+. Note that if the boundary value is given, and consider the conditions on the boundary, this problem is a typical problem of solving ordinary differential equation with Neumann boundary condition. We have efficient way to solve this type of problem either analytically or numerically.

The algorithm is as following. Start with a_0,b_0:

(1) For given boundaries a_k b_k solve the ODE with the boundary condition $a_k = -c^+$ $b_k = c^-$

(2) For upper boundary b_k check if the area above the line $y = c^-$ is bigger than K^- (impulse cost), if not go to step 5 (also applied to the boundary a_k). Set $b_k := b_{k+1}$

(3) For upper boundary b_k, set the new upper boundary to the point b_{k+1}.(see the following picture for more details)(also applied to the lower boundary a_k)

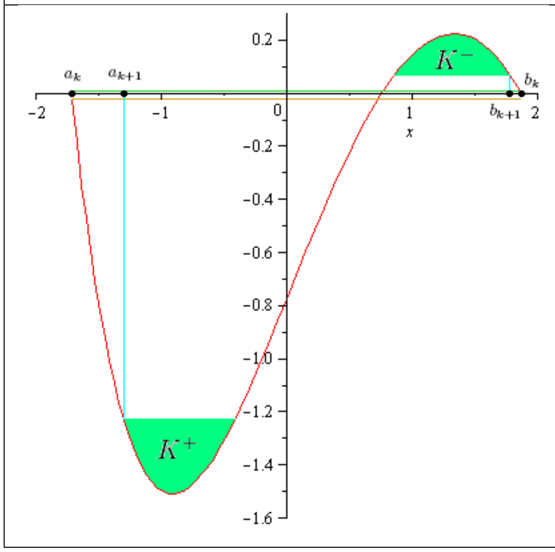
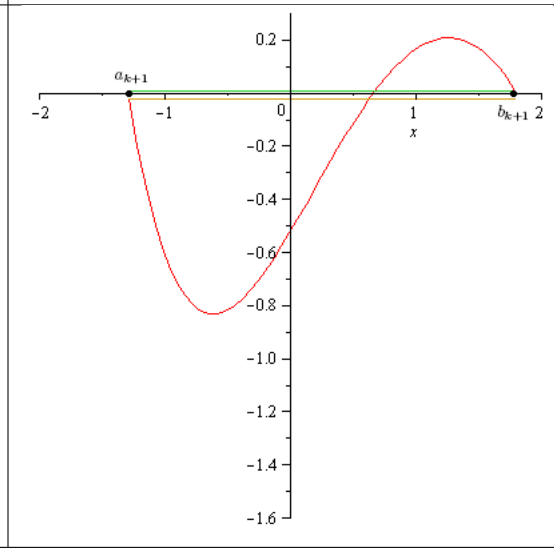
(4) If convergence condition is note satisfied, go to step 1. Otherwise, stop the program, show the result.

(5) Set the new boundary b_k as $\frac{b_k+b_{k-1}}{2}$(*backward motion*) (also applied to the boundary a_k)

(6) go to step 1

Following process is picked from the round 12 of the example showed in the last part of this mail. It looks different from the last version, as the graph is displayed in fixed coordinate.

Table 1. The First Derivative of the Value Function Before and After the Transformation

before the transformation	after the transformation
	

As mentioned above, in each iterative procedure, we solve an ODE with Neumann boundary condition. This result is corresponding to another impulse control problem with different boundary conditions. We call this problem as intermediate problem, which is similar to the original one. The only difference between these problems is on the conditions related with the setup costs. So the basic idea of the algorithm is to reduce this difference by changing the boundary. The next step is to determine where to set the new boundary. Looking at the value function displayed in the left figure of Table 1, we tell see that the only conditions on which the v' violates is the constrains containing integrals, which represents the amount of areas above $y = -c^+$ and bellow $y = c^-$.

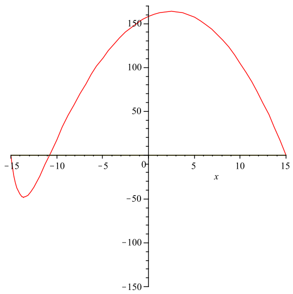
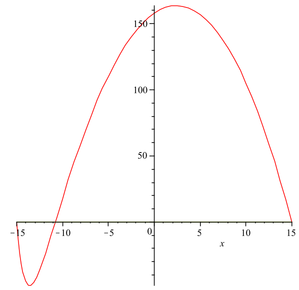
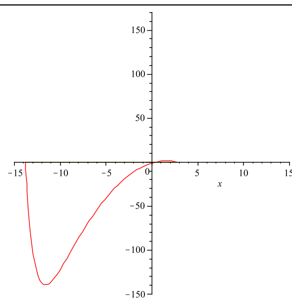
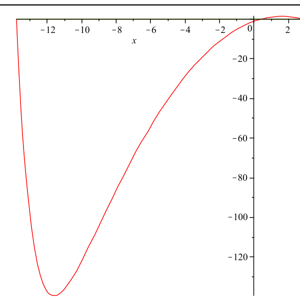
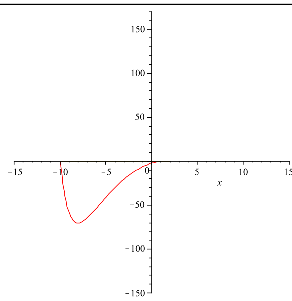
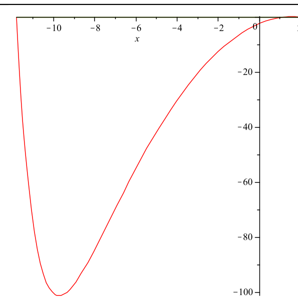
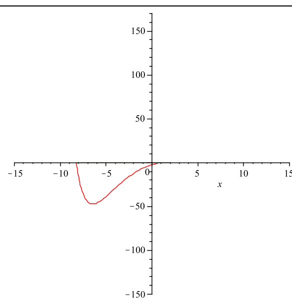
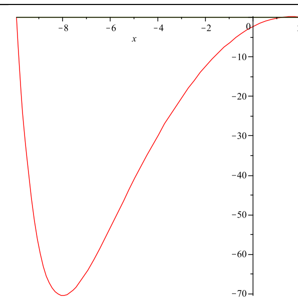
Here is an experiment for the problem with the same parameters as those in Kummar [**30**]. The difference is that we have setup costs $K^+$and K^-in addition. If one sets $K^+ = 0$ and $K^- = 0$, the problem is exactly the same as Kumar's. And my algorithm will also behave the same as Kumar's. So his case is a special case of ours in this sense. The parameters of the system in this test are as follows:

$$\mu = 1; \sigma = \sqrt{2}; r = 0.01; f = (x - 0.6)^2; K^+ = 0.15; K^- = 0.1; c^+ = 0.02; c^- = 0.01$$

We used Maple to do this experiment, since this ODE is quite simple and can be solved symbolically. If this method is proved to be able to work in a general situation, we can use Matlab instead, which provides more powerful finite element method package to solve ODEs with fixed boundary condition [**2, 3, 17, 18**].

We started with initial lower boundary -15,upper boundary 15. The graphs bellow show the results of the first 9 iterations. We display the experiment results in following table. To make it easy to do comparison between each round of iterations. V's are plotted by using both same coordinate units and adapted coordinate units.

Table 2: Illustration for the Searching Process

Round	Same coordinate units	Adapted coordinate units
1		
2		
3		
4		

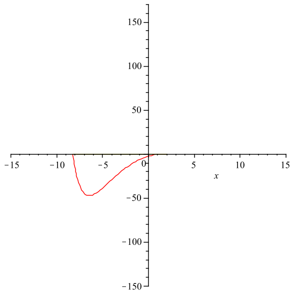
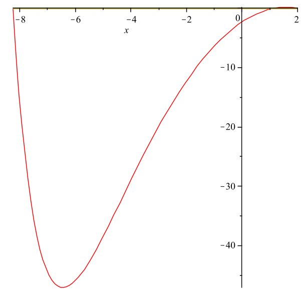
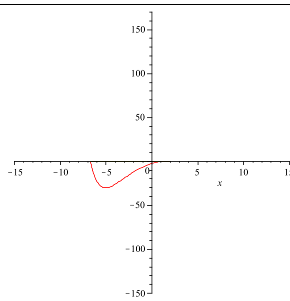
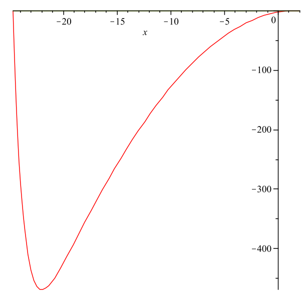
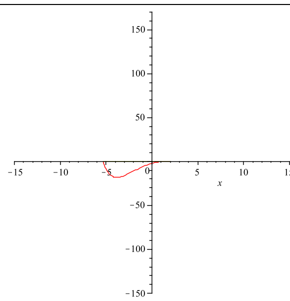
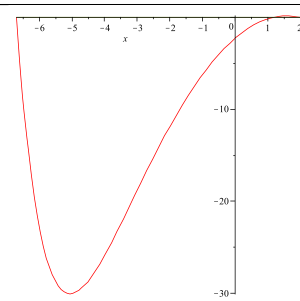
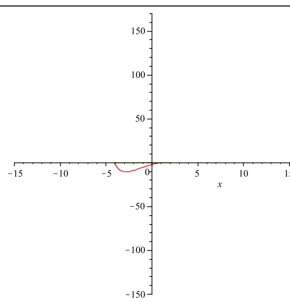
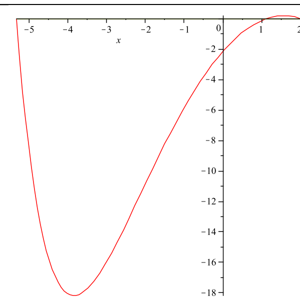
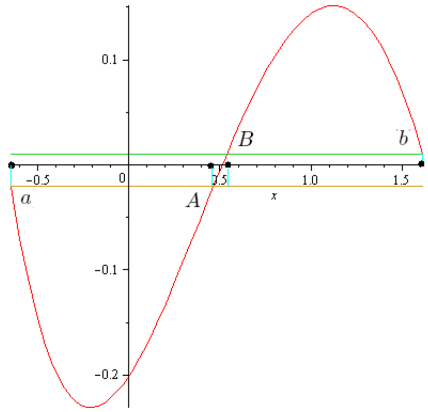
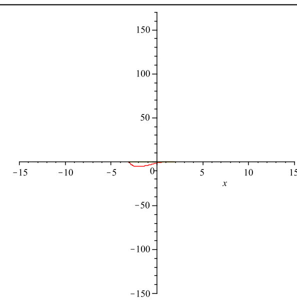
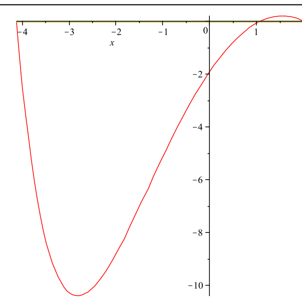
5		
6		
7		
8		

FIGURE 4.2.2. First Derivative of Optimal Value Function (round 38)



9		

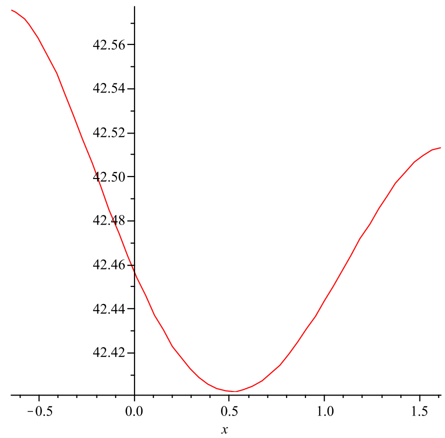
The table bellow shows parameters in each round iteration. The upper boundary a, the lower boundary b. The size of area bellow $y = -c^+$ denoted by K^+, respectively the size of the area above $y = c^-$ is denoted by K^-. Note that the object value of $K^+$is 0.15, and the object value of $K^-$is 0.1. Bold font means the *backward motion* will happen in the next round.

We apply this method to problem studied in [**21**], where the dynamic part is driven by geometric Brownian motion. The ODE is more complicated. The author get the solution by a Newton method, which mainly depends on a good guess of the optimal solution. If we started with a rather wide coupled boundaries, then the method can still converge to

Table 3. Evolution of the Key Parameters in the Iterations

round	a	b	K^+	K^-
1	-15.00000000	15.00000000	127.8006811	2809.222413
2	-13.79006963	2.833602333	916.3861385	2.503136830
3	-11.80542829	2.019206247	596.9333575	.1033454120
4	-9.952180513	2.011191078	365.5977234	.1006641309
5	**-8.245445357**	**2.009137254**	**213.0740609**	**0.9881751749e-1**
6	-8.245445357	2.010164166	213.0660675	.1008244991
7	-6.697500009	2.009756361	117.2683297	.1004972753
8	-5.317689805	2.005149880	60.35214109	.1097531784
9	-4.120416978	1.995252967	28.82829437	.1215396882
10	-3.116853829	1.967232622	12.76872536	.1367392000
11	-2.314152770	1.922359409	5.316041410	.1509999270
12	-1.710982106	1.859186936	2.188781759	.1551931728
13	**-1.288341844**	**1.789800887**	**.959290824**	**.146252981**
14	-1.016215904	1.729258873	.490343968	.137847891
15	-.8530880828	1.684170384	.305080196	.121753584
16	-.7590157520	1.654706038	.223421743	.113720866
17	-.7084508295	1.633742211	.187590327	.106120824
18	-.6792108264	1.624636765	.168458110	.103163311
19	-.6629750727	1.619788346	.162116156	.100008726
20	-.6567113469	1.616346790	.160000270	.101758375
21	-.6519996112	1.615211451	.154340860	.100256005
22	**-.6480652449**	**1.610853010**	**.153129500**	**0.99288121e-1**
23	**-.6480652449**	**1.613032230**	**.152762136**	**0.99950337e-1**
24	-.6480652449	1.614121840	.150580289	.101782534
25	**-.6471350521**	**1.611600857**	**.150819407**	**0.99821809e-1**
26	**-.6471350521**	**1.612861348**	**.151919588**	**0.98938880e-1**
27	-.6471350521	1.613491594	.150971471	.103998873
28	**-.6469536576**	**1.609611163**	**.151777255**	**0.95719986e-1**
29	**-.6469536576**	**1.611551378**	**.150009370**	**0.97979208e-1**
30	**-.6469536576**	**1.612521486**	**.152625446**	**0.98608344e-1**
31	**-.6469536576**	**1.613006540**	**.149440186**	**0.99429022e-1**
32	**-.6470443548**	**1.613249067**	**.151703376**	**0.99711676e-1**
33	-.6470443548	1.613370330	.150906114	.100915941
34	-.6468584565	1.612230890	.151288507	.100250484
35	**-.6462966202**	**1.611551921**	**.149021874**	**0.98874363e-1**
36	**-.6465775383**	**1.611891406**	**.149156200**	**.100063384**
37	**-.6467179974**	**1.611891406**	**.149943409**	**0.98875446e-1**
38	**-.6467882269**	**1.611891406**	**.150832090**	**0.98776894e-1**

FIGURE 4.2.3. Optimal Value Function (round 38):



the solution he got in about thirty rounds. And the method could also be extended to solve system with two controllable state variables.

This method has more advantage in the performance on the convergence and the speed speed aspects. It has a more complex behavior than the one in Kumar's case, since the backward motion may happen. However, the backward motion has potential to speed up the convergence of the process.

4.3. Limitation and Improvement

Our method is an extension of Kumar's method. If we set the setup costs equal zero, then the problem will degenerate to the problem of a singular control which is studied in Kumar's paper. The corresponding algorithm introduced in this study will perform exactly the same way as Kumar's algorithm. By doing the experiments we found a limitation which is faced by both of the methods. In some situation, the peak point in the first derivative function is very close to the boundary point. The new boundary points are then very likely to the close to the old ones. As a result, the range of the inactive area shrinks very slow.

In this situation we propose a greedy search strategy. For the right hand side of $V'(x)$, we choose a vertical line $x = b_{k+1}$ such that the area within $V'(x)$, $x = b_{k+1}$, and $y = c^-$ equals K^-. A similar method can be applied to get a_{k+1}. By using this search strategy, we can avoid the situation described above. In some cases this search strategy has better performance, but more back motions are involved. We postpone the convergence analysis of this search method to future study.

CHAPTER 5

The Impulse Control Model for Mutual Insurance Optimization

Two-band Impulse Control for Mutual Insurance OptimizationThis research is support in part by GRF grant PolyU5230/06E

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Abstract. Motivated by our ongoing study of marine mutual insurance, we develop in this paper a two-band impulse control model for optimal regulation of mutual reserve, which has profound implications to a wide range of applications such as optimal regulation of reserves of central banks. The Dynamic Programming characteristics of this two-band impulse control leads to a Quasi-Variational Inequality (QVI) with two sides, one at a lower boundary a and the other at an upper boundary b. It is analogous to the QVI of inventory control, except that the (s, S) solution is replaced by a couple of (s, S), which we call an (a, A,B, b) two-band policy with $a < A \leq B < b$. After the proof of existence of an optimal two-band policy by Constantinides and Richard [**24**] (1978), there has been little advancement in either solution method or application of a two-band control problem, except for two simplified problems: one of an inventory problem with $a = 0$ (i.e., no shortage allowable) by Harrison, Selke and Taylor [**27**] (1983); and the other of exchange rate control in central banks with quadratic cost objective (i.e., symmetrical holding and shortage costs) by Cadenillas [**21**] (1999). Under a general inventory cost structure in a real-world mutual insurance setting, we obtain in this paper new findings in analytical characteristics of an optimal $(a, A; B, b)$ two-band policy, including: 1) Some useful analytical properties of an optimal two-band policy are obtained, based on which an optimal solution for mutual insurance is found to contain a combination of both cash and credit reserves (i.e., $a < 0$ and $b > 0$.) 2) A novel computational optimization algorithm is then developed to solve general non-symmetric reserve regulation problems, which enable us to conduct, for the first time, comprehensive numerical sensitivity analysis on two-band problems.

5.1. Introduction

This article is motivated by our ongoing study of marine mutual insurance which operates on a two-way contingent option scheme, specifically a *contingent refund option* to reduce the mutual reserve and a *contingent call option* to increase the reserve. Although this problem formulation with two-way contingent options had been considered in cash management, see Constantinides-Richard [**24**] (1978) and in intervention of central bank in exchange rate control, see Cadenillas-Zapatero [**21**] (1999), practical adoption of a two-way option has not been identified, until a recent study of marine mutual insurance as reported in Liu and Yuan [**33**] (2006). The aforementioned previous studies follow the general methodology of Impulse Control introduced in the context of Inventory Control, see Bensoussan-Lions [**10**] (1984), where only one-way option is found practically necessary and the well known (s, S) policy (referred in this study as a *single-band* policy) is justified completely by Impulse Control theory and Q.V.I. methodology. In the case of cash management, two-way options (i.e., contingent call and refund) are theoretically adoptable although not practically tractable, and the optimal policy can potentially take the form of an $(a, A; B, b)$ two-band policy with $a < A \leq B < b$, where the system state is regulated within an interval $[a, b]$ with $a < b$ by applying an (a, A) single-band policy at the lower boundary a with A as *order-up-to* level, and applying a (B, b) policy at the upper boundary b with B as *refund-down-to* level. It is obviously more complex to solve for an optimal $(a, A; B, b)$ two-band policy than a single-band one, and as well as to develop solution methods for a two-band impulse control problem. As a result, determination of optimal parameters a, A, B and b is difficult and remains as a challenging research area. This level of sophistication, plus the lack of practical interests found in real-world applications, are believed to be the main reasons why the two-band impulse control problem has generated little research interest after the initial works mentioned above.

It is until Liu and Yuan [**33**] (2006) that the two-band impulse control is found to have long been adopted in marine mutual insurance which boasts a sustaining success for over 150 years. Under an asymmetric cost structure with both holding and shortage costs, a mutual insurance adopts two-way contingent options, namely, options of *contingent call* and *contingent refund*, which are generally unenviable in a non-mutual insurance. In mutual insurance practice, a *contingent call* collects cash contribution from mutual members, on a contingent basis, to raise the mutual reserve when the reserve runs low; while a *contingent refund* is devised to distribute excessive funds back to mutual members when mutual reserve runs high. In other words, *contingent options* adopted in mutual insurance are of a two-band impulse control. Based on our field study of P&I clubs, which constitute the key form of marine mutual insurance around the world, the two-way contingent options are so far implemented on an intuitive basis without a rigid theoretical and computational framework, and so are the implementation of two-band policies, lacking of a systematic and effective method of optimal solutions for them. On the other hand, the two-band control is further found to entail profound implications to a wide range of applications such as regulation of reserves of central banks, which further intensified the need for development of attainable solution methods for optimal two-band control. Motivated as such, this study is devoted to novel and effective solution methods of two-band impulse control problems.

The key contribution of this study relates to obtaining much simplified QVI characteristics of a two-band policy, which then enable us to develop effective computational QVI solution algorithms for mutual insurance optimization in a real-world setting. The reason why our approach developed in this study is so much simplified that can effectively solve realistic problems lies in the fact that we decompose the problem and deal only with one threshold problem at a time, whereas the initial approach has to consider the system globally. Our method enables derivation of many interesting properties, which simplify considerably the analytical as well as the numerical treatment. For example, our solution method for an optimal two-band policy needs to solve at most a system of 3 nonlinear equations, 50% reduced from a system of 6 nonlinear equations as required in the work of Cadenillas-Zapatero [**21**] (1999), of which a general solution remains intractable, either analytically or computationally. The QVI characteristics and properties obtained in this study lead to some major developments in optimization of mutual insurance. Under a general inventory cost structure in a real-world mutual insurance setting, we obtain in this study new findings in analytical characteristics of an optimal $(a, A; B, b)$ two-band policy, including: 1) An optimal policy solution for mutual insurance is found to contain a combination of both cash and credit reserves (i.e., $a < 0$ and $b > 0$.) This finding can be witnessed in the national stimulus measure by US Federal Reserve in facing the global financial crisis of 2008, namely, a combination of cash injection of amount of $b > 0$ and issuance of government bonds ($a < 0$). 2) A novel computational optimization algorithm is then developed to solve general non-symmetric reserve regulation problems which are unattainable in the previous studies. The techniques presented in this study are efficient and practical, so as to provide solutions to real-world problems of mutual insurance and reserve regulation as well.

We propose as future work to extend the methodology to approach more realistic situations, such as under a multi-dimension diffusion state and treatment of jumps in claims.

5.2. Impulse Control Model of Mutual Insurance Optimization

5.2.1. Formulation of the problem. Stemmed from mutual insurance, a mutual we are considering here is generally referred to as a mutual fund reserve $x(t)$ against collective claims of a Wiener disturbance $w(t)$. Consider a probability space $(\Omega, a, \mathcal{P})$ on which a standard Wiener process $w(t)$ is defined. When there is no control we consider the reserve process with a net premium rate μ as its drift:

$$x(t) = x + \mu t + \sigma w(t) \tag{5.2.1}$$

We define $\mathcal{F}^t$ to be the filtration generated by $w(t)$. An impulse control in terms of two-way contingent option is an increasing sequence of $\mathcal{F}^t$-adapted stopping times τ_i and contingent-option variables $\xi_i \neq 0$. We modify the trajectory of reserve (5.2.1) to get an option-regulated reserve process by setting

$$x(\tau_i) = x(\tau_i - 0) + \xi_i$$

$$x(t) = x(\tau_i) + \mu(t - \tau_i) + \sigma(w(t) - w(\tau_i)) \tag{5.2.2}$$

for $\tau_i \leq t < \tau_{i+1}$

$$x(0) = x$$

If we define

$$\nu(t) = \sum_i \xi_i \cdot \mathbb{I}_{\tau_i \leq t} \tag{5.2.3}$$

than we can write for $t \geq 0$

$$x(t) = x + \mu t + \sigma w(t) + \nu(t) \tag{5.2.4}$$

We next define a cost function to be optimized. We set

$$f(x) = hx^{+} + px^{-} \tag{5.2.5}$$

$$g(\xi) = K^{+}\mathbb{1}_{\xi>0} + K^{-}\mathbb{1}_{\xi<0} + c^{+}\xi^{+} + c^{-}\xi^{-} \tag{5.2.6}$$

where $K^{+}, K^{-} > 0$; $\xi^{+} = \xi \cdot \mathbb{1}_{\xi>0}$ denotes *a contingent call*, and $\xi^{-} = \xi \cdot \mathbb{1}_{\xi<0}$ denotes a *contingent refund.*

We next set

$$J_x(\nu) = E\left[\int_0^\infty f(x(t))\, e^{-rt}\mathrm{d}t + \sum_i g(\xi_i)\, e^{-r\tau_i}\right] \tag{5.2.7}$$

and we consider the value function

$$V(x) = \inf_{\nu} J_x(\nu) \tag{5.2.8}$$

This is the formulation of two-band impulse control problem.

5.2.2. Dynamic programming under Impulse Control. If the value function $V(x)$ is C^1 then the classical optimality principle approach leads to the following analogue of Bellman equation

$$\begin{aligned} \Gamma V(x) \leq f(x) = hx^{+} + px^{-} \qquad \forall x \\ V(x) \leq MV(x) \\ (\Gamma V(x) - f(x))(V(x) - MV(x)) = 0 \end{aligned} \tag{5.2.9}$$

where

$$\Gamma V(x) = -\frac{1}{2}\sigma^2 V''(x) - \mu V'(x) + rV(x) \tag{5.2.10}$$

$$MV(x) = \inf_{\xi \neq 0} \left[g(\xi) + V(x + \xi) \right]$$

This problem is called a Q.V.I. (Quasi Variational Inequality). We solve (5.2.9) and find that it has a two band solution. Namely there exists $a < 0$, $b > 0$, such that

$$\begin{aligned} \Gamma V(x) &= hx^+ + px^- && a < x < b \\ V(x) &= -c^+x + K^+ + \inf_{a \leq y \leq b} \left(c^+y + V(y) \right) && x \leq a \\ V(x) &= c^-x + K^- + \inf_{a \leq y \leq b} \left(-c^-y + V(y) \right) && x \geq b \\ V(x) &\in C^1(R) \end{aligned} \tag{5.2.11}$$

Moreover, there exist two numbers A, B with

$$\begin{aligned} c^+A + V(A) &= \inf_{a \leq y \leq b} \left(c^+y + V(y) \right) \\ -c^-B + V(B) &= \inf_{a \leq y \leq b} \left(-c^-y + V(y) \right) \end{aligned} \tag{5.2.12}$$

We have $a < A < B < b$, but A can be positive or negative, as well as B.

We derive from these numbers a feedback $\hat{\xi}(x)$ such that

$$\hat{\xi}(x) = \left| \begin{array}{ll} A - x & \text{if } x \leq a \\ 0 & \text{if } a \leq x \leq b \\ B - x & \text{if } x \geq b \end{array} \right. \tag{5.2.13}$$

and from this feedback we attain an impulse control called two-band impulse control. Namely

$$\begin{aligned}
&\hat{\tau}_1 = \inf_{t \geq 0} \{x(t) \,|x(t) < a \text{ or } x(t) > b\} \\
&x(\hat{\tau}_1) = x(\hat{\tau}_1 - 0) + \hat{\xi}(x(\hat{\tau}_1 - 0)) \\
&\hat{\tau}_2 = \inf_{t \geq \hat{\tau}_1} \{x(t) \,|x(t) < a \text{ or } x(t) > b\} \\
&x(\hat{\tau}_2) = x(\hat{\tau}_2 - 0) + \hat{\xi}(x(\hat{\tau}_2 - 0)) \\
&\ldots
\end{aligned} \tag{5.2.14}$$

Let $\hat{\nu}_x$ be this impulse control (depends on the initial condition x). We note that $x(\hat{\tau}_i) = A$ or B, $\forall i$. A standard argument (verification argument) shows that

$$J_x(\hat{\nu}_x) = V(x)$$

and then $\hat{\nu}_x$ is an optimal impulse control.

5.2.3. QVI Characteristics of Mutual Insurance Optimization. Recall that the functions $f(x)$ and $g(x)$ are given respectively as,

$$f(x) = hx^+ + px^- \tag{5.2.15}$$

$$g(\xi) = K^+ 1\!\!1_{\xi>0} + K^- 1\!\!1_{\xi<0} + c^+\xi^+ + c^-\xi^- \tag{5.2.16}$$

and that the M operator is defined as

$$MV(x) = \inf_{\xi \neq 0} [g(\xi) + V(x+\xi)] \tag{5.2.17}$$

Under a two-band impulse control (5.2.14), for $a \leq 0$ and $b \geq 0$ we set

$$P(a,b)\varphi = \inf_{a \leq x \leq b} \left(c^{+}x + \varphi(x) \right) \tag{5.2.18}$$

$$Q(a,b)\varphi = \inf_{a \leq x \leq b} \left(-c^{-}x + \varphi(x) \right) \tag{5.2.19}$$

Then, the mutual insurance problem becomes to find $V(x) \in C^{1}$ such that

$$\begin{aligned} &\Gamma V(x) = hx^{+} + px^{-}, \text{ a.e. } , a < x < b \\ &V(x) = -c^{+}x + K^{+} + P(a,b)V, x \leq a \\ &V(x) = c^{-}x + K^{-} + Q(a,b)V, x \geq b \end{aligned} \tag{5.2.20}$$

together with the continuity condition

$$\begin{aligned} &V(a) = -c^{+}a + K^{+} + P(a,b)V \\ &V(b) = c^{-}b + K^{-} + Q(a,b)V \\ &V'(a) = -c^{+} \\ &V'(b) = c^{-} \end{aligned} \tag{5.2.21}$$

The equalities in (5.2.20) and (5.2.21) are referred to as *QVI characteristics* associated with two-band impulse control problem, including mutual insurance optimization and general reserve regulation.

5.2.4. Solution of the QVI. We now give the conditions under which a solution of the two band control problem solves the Q.V.I. We state

PROPOSITION 7. *If $V(x) \in C^1$ is a solution of (5.2.20) and (5.2.21), such that*

$$V''(a+0) \leq 0 \qquad , V''(b-0) \leq 0 \tag{5.2.22}$$

$$V(x) \leq -c^+x + K^+ + \inf_{x<y<b}\Big(V(y) + c^+y\Big) \qquad \forall x \in [a,b] \tag{5.2.23}$$

$$V(x) \leq c^-x + K^- + \inf_{a<y<x}\Big(V(y) - c^-y\Big) \qquad \forall x \in [a,b] \tag{5.2.24}$$

then it is a solution of Q.V.I. (5.2.9).

PROOF. We prove that

$$V(x) = MV(x) \qquad \forall x \leq a \tag{5.2.25}$$

$$V(x) = MV(x) \qquad \forall x \geq b \tag{5.2.26}$$

Let us prove the first relation. Take $x \leq a$ then we first have

$$\inf_{y \geq x}(c^+y + V(y)) = P(a,b)V$$

Indeed we notice that

$$\inf_{y \geq x}(c^+y + V(y)) = \inf_{x \leq y \leq b}(c^+y + V(y))$$

since for $y > b$, $V(y) + c^+y$ is affine increasing. Next for $x \leq y \leq a$, one has

$$c^+y + V(y) = K^+ + P(a,b)V > P(a,b)V$$

Therefore

$$\inf_{x\le y\le b}(c^+y+V(y))=\inf_{a\le y\le b}(c^+y+V(y))=P(a,b)V$$

Hence from (5.2.23) for $x\le a$ we have

$$V(x)=-c^+x+K^++\inf_{y\ge x}(c^+y+V(y)) \tag{5.2.27}$$

But +3pt

$$\begin{aligned} MV\left(x\right)= &\ \min\left[K^++\inf_{y>x}(c^+y+V(y))-c^+x,K^-+\inf_{y<x}(-c^-y+V(y))+c^-x\right] \\ = &\ \min\left[K^++\inf_{y\ge x}(c^+y+V(y))-c^+x,K^-+\inf_{y<x}(-c^-y+V(y))+c^-x\right] \end{aligned}$$

and from (5.2.27)

$$MV\left(x\right)=\min\left[V\left(x\right),K^-+\inf_{y<x}(-c^-y+V(y))+c^-x\right]$$

But

$$\begin{aligned} K^-+\inf_{y<x}(-c^-y+V(y))+c^-x &= K^-+\inf_{y<x}\left(-c^-y-c^+y+K^++P\left(a,b\right)V\right)+c^-x \\ &= K^-+K^+-c^+x+P\left(a,b\right)V \\ &= K^-+V\left(x\right)>V\left(x\right) \end{aligned}$$

and the first relation (5.2.25) is proved.

A similar proof is done for the second relation (5.2.26). Therefore the product condition in (5.2.9) is satisfied.

We next have to prove

$$\Gamma V\left(x\right)\le f\left(x\right)\qquad\forall x\le a\text{ or }x\ge b$$

Consider $x \leq a$, we have to prove

$$\mu c^{+} + r\left(-c^{+}x + K^{+} + P\left(a,b\right)V\right) \leq -px$$

or

$$\mu c^{+} + pa + rV\left(a\right) \leq -\left(p - rc^{+}\right)\left(x-a\right)$$

Now

$$-\frac{1}{2}\sigma^{2}V''\left(a\right) - \mu c^{+} + rV\left(a\right) + pa = 0$$

and by notice of the of the first condition (5.2.22)

$$\mu c^{+} + rV\left(a\right) + pa = \frac{1}{2}\sigma^{2}V''\left(a\right) \leq 0 \leq -\left(p - rc^{+}\right)\left(x-a\right)$$

For $x \geq b$, we have to prove

$$-\mu c^{-} + r\left(-c^{-}x + K^{-} + Q\left(a,b\right)V\right) \leq hx$$

or

$$-\mu c^{-} + r\left(V\left(b\right) + c^{-}x - c^{-}b\right) \leq hx$$

or

$$-\mu c^{-} + rV\left(b\right) - hb \leq \left(h - rc^{-}\right)\left(x-b\right)$$

or

$$\frac{1}{2}\sigma^{2}V''\left(b\right) \leq \left(h - rc^{-}\right)\left(x-b\right)$$

which is satisfied thanks to the second condition (5.2.22).

We finally have to check that

$$V\left(x\right) \leq MV\left(x\right) \qquad \forall x \in \left(a,b\right)$$

which is equivalent to check

$$V\left(x\right) \leq -c^{+}x + K^{+} + \inf_{x<y<b}\Big(V\left(y\right) + c^{+}y\Big)$$

$$V\left(x\right) \leq c^{-}x + K^{-} + \inf_{a<y<x}\Big(V\left(y\right) - c^{-}y\Big)$$

which follows directly from the assumption (5.2.23), (5.2.24).

The proof has been completed □

5.2.5. Functions v, v^{+}, v^{-}. Consider $v\left(x\right) = V'\left(x\right)$. We derive from (5.2.20), (5.2.21) the problem for v, namely

(5.2.28)
$$\begin{aligned}&\Gamma v\left(x\right) = f'\left(x\right) a < x < b\\&v\left(x\right) = -c^{+}x \leq a\\&v\left(x\right) = c^{-}x \geq b\end{aligned}$$

and the continuity condition

(5.2.29)
$$0 = K^{+} + \inf_{a\leq x\leq b}\int_{a}^{x}\Big(v\left(\xi\right) + c^{+}\Big)\,\mathrm{d}\xi$$

(5.2.30)
$$0 = -K^{-} + \sup_{a\leq x\leq b}\int_{x}^{b}\Big(v\left(\xi\right) - c^{-}\Big)\,\mathrm{d}\xi$$

We next define

$$v^{+}\left(x\right) = v\left(x\right) + c^{+}$$

$$v^{-}\left(x\right) = v\left(x\right) - c^{-}$$

Then we derive the problem for v^+, v^-

(5.2.31)
$$\begin{aligned} &\Gamma v^+(x) = f'(x) + rc^+ \\ &v^+(a) = 0 \qquad v^+(b) = c^+ + c^- \end{aligned}$$

(5.2.32)
$$\begin{aligned} &\Gamma v^-(x) = f'(x) - rc^- \\ &v^-(a) = -\left(c^+ + c^-\right) \qquad v^-(b) = 0 \end{aligned}$$

and (5.2.29), (5.2.30) become

(5.2.33)
$$0 = K^+ + \inf_{a \le x \le b} \int_a^x v^+(\xi)\, \mathrm{d}\xi$$

(5.2.34)
$$0 = -K^- + \sup_{a \le x \le b} \int_x^b v^-(\xi)\, \mathrm{d}\xi$$

The method is to solve (5.2.31), (5.2.32) for any fixed values of a, b and look for a, b solutions of the algebraic equations

(5.2.35)
$$F(a,b) = 0$$

(5.2.36)
$$G(a,b) = 0$$

with

(5.2.37)
$$F(a,b) = K^+ + \inf_{a \le x \le b} \int_a^x v^+(\xi; a, b)\, \mathrm{d}\xi$$

$$G\left(a,b\right)-K^{-}+\sup_{a\leq x\leq b}\int_{x}^{b}v^{-}\left(\xi;a,b\right)\mathrm{d}\xi \tag{5.2.38}$$

We shall make the assumptions

$$p-rc^{+}>0 \qquad h-rc^{-}>0 \tag{5.2.39}$$

Note that in order to obtain a solution of the Q.V.I. ((5.2.9)), we need to check that the condition of Proposition 7 are satisfied, namely

$$\left(v^{+}\right)'\left(a;a,b\right)\leq 0 \qquad \left(v^{-}\right)'\left(b;a,b\right)\leq 0 \tag{5.2.40}$$

$$0\leq K^{+}+\inf_{x<y<b}\int_{x}^{y}v^{+}\left(\xi;a,b\right)\mathrm{d}\xi, \qquad \forall x\in[a,b] \tag{5.2.41}$$

$$0\geq K^{-}+\sup_{a<y<x}\int_{y}^{x}v^{-}\left(\xi;a,b\right)\mathrm{d}\xi, \qquad \forall x\in[a,b] \tag{5.2.42}$$

5.2.6. The Functions $v^{+}(x;a,b)$. Let ρ_1, ρ_2 be the roots of the second order equation

$$-\frac{1}{2}\sigma^2\rho^2-\mu\rho+r=0$$

namely

$$\rho_1=-\frac{\mu+\sqrt{\mu^2+2r\sigma^2}}{\sigma^2}$$
$$\rho_2=-\frac{\mu-\sqrt{\mu^2+2r\sigma^2}}{\sigma^2}$$

We note that

$$\rho_1<0, \quad \rho_2>0$$

and

$$\rho_1 + \rho_2 = -\frac{2\mu}{\sigma^2}$$
$$\rho_1 \rho_2 = -\frac{2r}{\sigma^2}$$

We have an analytic formula for the function $v^+(x; a, b)$, namely+3pt

$$\begin{aligned} v^+(x;a,b) &= \Gamma^+(a,b)(\exp\rho_1(x-a) - \exp\rho_2(x-a)) \\ &- \frac{2}{\sigma^2(\rho_1-\rho_2)}\int_a^x (f' + rc^+)(\xi)(\exp\rho_1(x-\xi) - \exp\rho_2(x-\xi))d\xi \end{aligned} \tag{5.2.43}$$

with

$$\Gamma^+(a,b) = \frac{Z(a,b)}{\exp\rho_1(b-a) - \exp\rho_2(b-a)} \tag{5.2.44}$$

$$Z(a,b) = c^+ + c^- + \frac{2}{\sigma^2(\rho_1-\rho_2)}\int_a^b (f' + rc^+)(\xi)(\exp\rho_1(b-\xi) - \exp\rho_2(b-\xi))d\xi \tag{5.2.45}$$

Lemma 8. *Under the first assumption (5.2.39), the function $Z(a,b)$is increasing in a. Moreover $Z(0,b) > 0$, $Z(-\infty, b) = -\infty$.*

Proof. Note that

$$\frac{\partial Z}{\partial a} = \frac{2}{\sigma^2(\rho_1-\rho_2)}(p - rc^+)(\exp\rho_1(b-a) - \exp\rho_2(b-a)) > 0$$

Clearly $Z(0,b) > 0$ and for $x < 0$

$$\begin{aligned} (f' + rc^+)(x)\frac{\exp\rho_1(b-x) - \exp\rho_2(b-x)}{\rho_1-\rho_2} &= -\left(p - rc^+\right)\frac{\exp\rho_1(b-x) - \exp\rho_2(b-x)}{\rho_1-\rho_2} \\ &\to -\infty \quad as \quad x \to -\infty \end{aligned}$$

hence $Z(-\infty, b) = -\infty$ □

It follows that $Z(a,b)$ (and thus $\Gamma^{+}(a,b)$) vanish at a unique point $a_0\,(b) < 0$, $\forall b$. We have $\Gamma^{+}(a,b) < 0$ for $a > a_0\,(b)$, $\Gamma^{+}(a,b) > 0$ for $a < a_0\,(b)$ ($\Gamma^{+}(a,b)$ and $Z(a,b)$ have opposite signs).

Note that

$$(5.2.46)\qquad \begin{aligned} \left(v^{+}\right)'(a;a,b) = \Gamma^{+}(a,b)(\rho_1-\rho_2) &\leq 0 \quad \text{if } a \leq a_0\,(b) \\ \text{and } &> 0 \quad \text{if } a > a_0\,(b) \end{aligned}$$

Therefore the first condition (5.2.40) is satisfied if $a \leq a_0\,(b)$, and not satisfied otherwise. This is a first restriction on the choice of the pair a, b.

We next state the following essential result:

Proposition 9. *Under the first assumption (5.2.39), we have*

$$a \geq a_0\,(b) \Rightarrow v^{+}\,(x;a,b) > 0 \quad \forall\, x \in (a,b]$$

If $a \leq a_0\,(b)$, there exists a unique $A\,(a,b)$ such that $v^{+}\,(x;a,b) < 0 \quad \forall\, x \in (a,A)$ and $v^{+}\,(x;a,b) > 0 \quad \forall\, x \in (A,b]$, with $v^{+}\,(A;a,b) = 0$.

Proof. Suppose first $a \geq a_0\,(b)$, then $(v^{+})'\,(a;a,b) \geq 0$. Then

$$-\frac{1}{2}\sigma^2\left(v^{+}\right)''(a;a,b) = \mu\left(v^{+}\right)'(a;a,b) - \left(p - rc^{+}\right)$$

hence $(v^{+})''\,(a;a,b) > 0$, Therefore there exists a small interval $a,\ a+\epsilon$, on which $(v^{+})'\,(x;a,b) > 0$. We claim there can not be a value $\bar{x} \in (a,b)$ such that $v^{+}\,(x;a,b) = 0$. If such points exist, consider the smallest one called $\bar{x}$. Then $v^{+}\,(x;a,b) > 0$ for $x \in (a,\bar{x})$. Necessarily $\bar{x} > 0$, otherwise there will be a local positive maximum in $(a,\bar{x})$ hence in $(a,0)$. This is impossible by maximum principle considerations. Indeed, for such a point

x^* we shall have $\Gamma v^+(x^*; a, b) > 0$ which contradicts $\Gamma v^+(x^*; a, b) = -p + rc^- < 0$. Therefore $\bar{x} > 0$. Moreover $(v^+)'(\bar{x}; a, b) < 0$. Since $v^+(b; a, b) > 0$ the function $v^+(x; a, b)$ admits a negative minimum in $(0, b)$ which is impossible by maximum principle considerations. Indeed, in such a point, called also x^* to save notation, we have $\Gamma v^+(x^*; a, b) < 0$ which is a contradiction with $\Gamma v^+(x^*; a, b) = h + rc^+ > 0$.

We have then proved that $v^+(x; a, b) > 0 \quad \forall\, x \in (a, b]$. Hence the first part of the proposition has been proved. Suppose now that $a < a_0(b)$. Then $(v^+)'(a; a, b) < 0$. It follows that at least on a small interval $(a, a + \epsilon)$, $v^+(x; a, b) < 0$, $\forall x \in (a, a + \epsilon)$. Since $v^+(b; a, b) > 0$ the function must vanish in (a, b). Let us call $A(a, b)$ the first value in (a, b) for which $v^+(b; a, b) > 0$. In fact, we are going to show that A is unique and that $v^+(x; a, b) > 0, \forall x \in (A, b]$. The fact that $v^+(x; a, b) \leq 0, \forall x \in (a, A)$ follows from the definition of A. Suppose that the statement is not true, then there will be points $x \in (A, b)$ for which $v^+(x; a, b) < 0$. There cannot be a situation where $v^+(x; a, b) = 0$ while the function does not become strictly negative. This is because the derivative is continuous. Therefore the function must have a strictly negative minimum, denoted by $\hat{x}$ in the interval (A, b). This minimum cannot lie on $(0, b)$. This would contradict again the maximum principle. Indeed $\Gamma v^+(\hat{x}; a, b) < 0$ and thus cannot be equal to $h + rc^+$. This situation implies that A must be strictly negative. In addition, note that $(v^+)'(A; a, b) > 0$. Therefore, there must be also a positive maximum on $(A, 0)$. This is impossible, by the same reasoning done at the beginning of the proof. □

Proposition 9 implies that the first condition (5.2.40) is satisfied whenever $a \geq a_0(b)$ and not satisfied otherwise. In view of (5.2.46), it follows that having simultaneously the first condition (5.2.40) and (5.2.41) on $v^+(x;a,b)$ to be satisfied impose the unique choice

$$a = a_0\left(b\right) \tag{5.2.47}$$

We proceed by stating properties of the curve $a_0\left(b\right)$. We first perform a complete calculation of $Z\left(a,b\right)$ defined by (5.2.45). We have first+3pt

$$\begin{aligned} Z(a,b) &= -\frac{h-rc^-}{r} + \frac{1}{r(\rho_2-\rho_1)}\{(p+h)\left(\rho_2 \exp \rho_1 b - \rho_1 \exp \rho_2 b\right) \\ &- \left(p-rc^+\right)\left(\rho_2 \exp \rho_1(b-a) - \rho_1 \exp \rho_2(b-a)\right)\Big\} \end{aligned} \tag{5.2.48}$$

and thus $a_0\left(b\right)$ is the unique solution of

(5.2.49)
$$\rho_2 \exp \rho_1(b{-}a_0\left(b\right)){-}\rho_1 \exp \rho_2(b{-}a_0\left(b\right)) = -\frac{\left(h-rc^-\right)\left(\rho_2-\rho_1\right)}{p-rc^+}{+}\frac{p+h}{p-rc^+}\left(\rho_2 \exp \rho_1 b - \rho_1 \exp \rho_2 b\right)$$

This equation defies uniquely $a_0\left(b\right) < 0 \;\forall b \in [0,\infty)$. We have in particular for $b=0$

$$\rho_2 \exp -\rho_1 a_0\left(0\right) - \rho_1 \exp -\rho_2 a_0\left(0\right) = \frac{p+rc^-}{p-rc^+}\left(\rho_2-\rho_1\right) \tag{5.2.50}$$

To get the value $a_0\left(\infty\right)$we perform a limit argument. We can write from (5.2.60) +3pt

$$\begin{aligned} &\rho_2 \exp\left(\left(\rho_1-\rho_2\right)b - \rho_1 a_0\left(b\right)\right) - \rho_1 \exp -\rho_2 a_0\left(b\right) = -\frac{h-rc^-}{p-rc^+}\left(\rho_2-\rho_1\right)\exp -\rho_1 b \\ &+ \frac{p+h}{p-rc^-}\left(\rho_2 \exp\left(\rho_1-\rho_2\right)b - \rho_1\right) \end{aligned}$$

and letting $b \to \infty$ we obtain

$$\exp -\rho_2 a_0\left(\infty\right) = \frac{p+h}{p-rc^+} \tag{5.2.51}$$

The function $a_0(b)$ is not monotone. Using (5.2.60) we can compute

$$1 - a_0'(b) = \frac{p+h}{p-rc^+} \frac{\exp \rho_1 b - \exp \rho_2 b}{\exp \rho_1 (b - a_0(b)) - \exp \rho_2 (b - a_0(b))} \tag{5.2.52}$$

The right hand side is positive, so we have

$$a_0'(b) \leq 1, \quad a_0'(0) = 1, \quad a_0'(\infty) = 0 \tag{5.2.53}$$

We are going to check the following property

PROPOSITION 10. *Under the first assumption (5.2.39) the function $a_0(b)$ is uniquely defined on $[0, \infty)$ by equation (5.2.49). We have $a_0(b) < 0$ $\forall b \in [0, \infty)$. The values and $a_0(0)$ and $a_0(\infty)$ are given by (5.2.50), (5.2.51). We also have (5.2.53). There exists a unique value b^* such that $a_0'(b^*) = 0$ $a_0'(b) > 0$ $0 \leq b < b^*$ and $a_0'(b) < 0$ for $b > b^*$. The function $a_0(b)$ increases on $(0, b^*)$ from $a_0(0)$ to $a_0(b^*)$ then decreases for $b > b^*$ from $a_0(b^*)$ to $a_0(\infty)$.*

PROOF. From formula (5.2.52) the points where $a_0'(b) = 0$ must satisfy

$$1 = \frac{p+h}{p-rc^+} \frac{\exp \rho_1 b - \exp \rho_2 b}{\exp \rho_1 (b - a_0(b)) - \exp \rho_2 (b - a_0(b))} \tag{5.2.54}$$

So we get, combining (5.2.54) and (5.2.60)

$$\exp -\rho_2 a_0(b) = \frac{p + h - (h - rc^-) \exp -\rho_2 b}{p - rc^+}$$

$$\exp -\rho_1 a_0(b) = \frac{p + h - (h - rc^-) \exp -\rho_1 b}{p - rc^+}$$

Eliminating $a_0(b)$, we obtain the equation for b, namely

(5.2.55)
$$\left(p+h-\left(h-rc^-\right)\exp -\rho_2 b\right)^{-\frac{\rho_1}{\rho_2}}\left(p+h-\left(h-rc^-\right)\exp -\rho_1 b\right)=\left(p-rc^+\right)^{1-\frac{\rho_1}{\rho_2}}$$

Let $\psi(b)$ be the function to the left hand side of (5.2.55). We have

$$\psi'(b)=\rho_1\left(h-rc^-\right)(p+h)\left(p+h-\left(h-rc^-\right)\exp -\rho_2 b\right)^{-\frac{\rho_1}{\rho_2}-1}(\exp -\rho_1 b-\exp -\rho_2 b)<0$$

hence $\psi(b)$ decreases from $(p+rc^-)^{1-\frac{\rho_1}{\rho_2}}$ to $-\infty$. So there exists a unique value $b^*>0$ such that (5.2.55) holds. This is the unique point with finite value such that $a_0'(b)=0$. Since $a_0'(0)=1$, it follows that $a_0'(b)>0$ for $b\in[0,b^*)$. It is negative for $b\in(b^*,\infty)$. Indeed we can compute the derivative $a_0''(b^*)$. We have +3pt

$$-a_0''(b)=\frac{p+h}{p-rc^+}\{(\rho_1\exp\rho_1 b-\rho_2\exp\rho_2 b)(\exp\rho_1(b-a_0(b))-\exp\rho_2(b-a_0(b)))$$
$$-(\exp\rho_1 b-\exp\rho_2 b)(\rho_1\exp\rho_1(b-a_0(b))-\rho_2\exp\rho_2(b-a_0(b)))(1-a_0'(b))\}$$
$$\cdot\frac{1}{(\exp\rho_1(b-a_0(b))-\exp\rho_2(b-a_0(b)))^2}$$

For $b=b^*$, we have, since $a_0'(b^*)=0$

(5.2.56)
$$-a_0''(b^*)=\frac{p+h}{p-rc^+}(\rho_1-\rho_2)\exp(\rho_1+\rho_2)b^*\frac{\exp -\rho_1 a_0(b^*)-\exp -\rho_2 a_0(b^*)}{(\exp\rho_1(b^*-a_0(b^*))-\exp\rho_2(b^*-a_0(b^*)))^2}$$

hence $a_0''(b^*)<0$. Therefore for $b\in(b^*+\epsilon)$ we have $a_0'(b)<0$. But it cannot have a 0 on (b^*,∞). So $a_0'(b)$ remain negative on (b^*,∞). This completes the proof. $\square$

5.2.7. The function $v^-(x;a,b)$. Although $v^-(x;a,b) = v^+(x;a,b) - (c^+ + c^-)$, it is preferable to use (5.2.31). We get in this way the formulas

$$\begin{aligned} v^-(x;a,b) =& \Gamma^-(a,b)(\exp\rho_1(x-b) - \exp\rho_2(x-b)) \\ &+ \frac{2}{\sigma^2(\rho_1-\rho_2)}\int_x^b (f'-rc^-)(\xi)(\exp\rho_1(x-\xi) - \exp\rho_2(x-\xi))d\xi \end{aligned} \tag{5.2.57}$$

with

$$\Gamma^-(a,b) = \frac{\tilde{Z}(a,b)}{\exp\rho_1(a-b) - \exp\rho_2(a-b)}$$

$$\tilde{Z}(a,b) = -(c^+ + c^-) + \frac{2}{\sigma^2(\rho_1-\rho_2)}\int_a^b (f'-rc^-)(\xi)(\exp\rho_1(a-\xi) - \exp\rho_2(a-\xi))d\xi \tag{5.2.58}$$

We have the equivalent of Lemma 8

LEMMA 11. *Under the second assumption (5.2.39), the function $\tilde{Z}(a,b)$ is increasing in b. Moreover $\tilde{Z}(a,+\infty) = +\infty$ and $\tilde{Z}(a,0) < 0$.*

PROOF. We compute

$$\frac{\partial \tilde{Z}}{\partial b}(a,b) = -\frac{2}{\sigma^2(\rho_1-\rho_2)}(h-rc^-)(\exp\rho_1(a-b) - \exp\rho_2(a-b)) > 0$$

□

It follows that $\Gamma^-(a,b)$ vanishes in a unique point $b_0(a) > 0\ \forall a$. We have $\Gamma^-(a,b) < 0$ for $b < b_0(a)$ and $\Gamma^-(a,b) > 0$ for $b > b_0(a)$ ($\Gamma^-(a,b)$ and $\tilde{Z}(a,b)$ have the same sign). We then note that

$$\begin{aligned} \left(v^-\right)'(b;a,b) = (\rho_1-\rho_2)\Gamma^-(a,b) &\leq 0 \quad \text{if } b \geq b_0(a) \\ \text{and } &> 0 \quad \text{if } b < b_0(a) \end{aligned} \tag{5.2.59}$$

Therefore the second condition 5.2.40 is satisfied if $b \geq b_0(a)$, and not satisfied otherwise.

We proceed with the equivalent of Proposition 9, stated without proof

Proposition 12. *Under the second assumption (5.2.39), we have*

$$b \leq b_0(a) \Rightarrow v^-(x; a, b) < 0 \quad \forall\, x \in [a, b)$$

If $b > b_0(a)$, there exists a unique $B(a,b)$ such that $v^-(x; a, b) > 0 \quad \forall\, x \in (B, b)$ and $v^-(x; a, b) < 0 \quad \forall\, x \in [a, B)$, with $v^-(B; a, b) = 0$.

We now study the properties of the curve $b = b_0(a)$ as we did for the curve $a = a_0(b)$. We compute completely $\tilde{Z}(a,b)$. We get from (5.2.58)

$$\tilde{Z}(a,b) = \frac{p - rc^+}{r} + \frac{1}{r(\rho_1 - \rho_2)}\{(p+h)(\rho_2 \exp \rho_1 a - \rho_1 \exp \rho_2 a)$$
$$- \left(h - rc^-\right)(\rho_2 \exp \rho_1 (a-b) - \rho_1 \exp \rho_2 (a-b))\Big\}$$

and thus $b_0(a)$ is the unique solution of

(5.2.60)
$$\rho_2 \exp \rho_1 (a - b_0(a)) - \rho_1 \exp \rho_2 (a - b_0(a)) = -\frac{(p - rc^+)(\rho_2 - \rho_1)}{h - rc^-} + \frac{p+h}{h - rc^-}(\rho_2 \exp \rho_1 a - \rho_1 \exp \rho_2 a)$$

This equation defines uniquely $b_0(a) > 0\ \forall a \in (\infty, 0]$. In particular for $a = 0$, we get

$$\rho_2 \exp -\rho_1 b_0(0) - \rho_1 \exp -\rho_2 b_0(0) = (\rho_2 - \rho_1)\frac{h + rc^+}{h - rc^-} \tag{5.2.61}$$

The value $b_0(-\infty)$ is obtained from+3pt

$$\rho_2 \exp -\rho_1 b_0(a) - \rho_1 \exp((\rho_2 - \rho_1) a - \rho_2 b_0(a)) = -\frac{p - rc^+}{h - rc^-}(\rho_2 - \rho_1) \exp -\rho_1 a$$
$$+ \frac{p+h}{h - rc^-}(\rho_2 - \rho_1 \exp(\rho_2 - \rho_1) a)$$

and letting a tend to $-\infty$ which yields

$$\exp -\rho_1 b_0(-\infty) = \frac{p+h}{h-rc^-} \tag{5.2.62}$$

The derivative $b_0'(a)$ is given by

$$1-b_0'(a) = \frac{p+h}{h-rc^-}\frac{\exp\rho_1 a - \exp\rho_2 a}{\exp\rho_1(a-b_0(a)) - \exp\rho_2(a-b_0(a))} \tag{5.2.63}$$

We have

$$b_0'(a) \leq 1, \quad b_0'(0) = 1, \quad b_0'(-\infty) = 0 \tag{5.2.64}$$

We next state the analogue of Proposition 10

Proposition 13. *Under the second assumption (5.2.39) the function $b_0(a)$ is uniquely defined on $(-\infty, 0]$ by equation (5.2.60). We have $b_0(a) > 0$ $\forall a \in (-\infty, 0]$. The values $b_0(0)$ and $b_0(-\infty)$are given by (5.2.61), (5.2.62). We have also (5.2.64). There exists a unique a^* such that $b_0'(a^*) = 0$ $b_0'(a) > 0$ for $a^* < a \leq 0$. and $b_0'(a) < 0$ for $a < a^*$. The function $b_0(a)$ decreases on $(-\infty, a^*)$ from $b_0(-\infty)$ to $b_0(b^*)$ then increases on $(a^*, 0)$ from $b_0(b^*)$ to $b_0(0)$.*

Proof. Similar to that of Proposition 10, we only give the relation defining a^*. The points such that $b_0'(a) = 0$ satisfy (from (5.2.63))

$$1 = \frac{p+h}{h-rc^-}\frac{\exp\rho_1 a - \exp\rho_2 a}{\exp\rho_1(a-b_0(a)) - \exp\rho_2(a-b_0(a))} \tag{5.2.65}$$

We combine (5.2.65) and (5.2.60) which yields to

$$\left(h-rc^-\right)^{1-\frac{\rho_1}{\rho_2}} = \left(p+h-\left(p-rc^+\right)\exp -\rho_1 a\right)^{-\frac{\rho_1}{\rho_2}}\left(p+h-\left(p-rc^+\right)\exp -\rho_2 a\right)$$

which is the analogue of (5.2.55). This equation has a unique solution a^*. We can also compute

(5.2.66)

$$-b_0''(a^*) = \frac{p+h}{h-rc^-}(\rho_1-\rho_2)\exp(\rho_1+\rho_2)a^*\frac{\exp-\rho_1 b_0(a^*)-\exp-\rho_2 b_0(a^*)}{(\exp\rho_1(a^*-b_0(a^*))-\exp\rho_2(a^*-b_0(a^*)))^2}<0$$

hence $b_0''(a^*)>0$. Therefore on $(a^*-\epsilon, a^*)$ we have $b_0'(a)<0$ if ϵ is sufficiently small. Since $b_0'(a)$ cannot vanish on $(-\infty, a^*)$, then it remains negative, and the proof has been completed. □

5.3. Choice of a, b

5.3.1. Functions $A(a,b)$, $B(a,b)$. The functions $v^+(x;a,b)$, $v^-(x;a,b)$ are as those defined previously in sections 5.2.6 and 5.2.7. Using Proposition 9, we have defined, when $a<a_0(b)$, a unique $A(a,b)$ such that

(5.3.1) $$v^+(A;a,b)=0 \qquad \text{if } a<a_0(b)$$

Since $v^+(x;a,b)>0$ for $a<x\le b$, in case $a\ge a_0(b)$ and $v^+(a;a,b)=0$, it is convenient to set

(5.3.2) $$A(a,b)=a \qquad \text{for } a\ge a_0(b)$$

Therefore $A(a,b)$ is uniquely defined by formulas (5.3.1), (5.3.2), $\forall a,b$. Similarly, using Proposition (12), we define a unique $B(a,b)$ by formulas

(5.3.3) $$v^-(B;a,b)=0 \qquad \text{if } b>b_0(a)$$

and

$$B\,(a,b) = b \qquad \text{for } b \leq b_0(a) \tag{5.3.4}$$

5.3.2. Verification of Conditions. We know from (5.2.46) and (5.2.59) that we must have

$$a \leq a_0\,(b) \qquad b \geq b_0(a) \tag{5.3.5}$$

in order to verify (5.2.40). Next we note that

$$F\,(a,b) = K^+ + \int_a^{A(a,b)} v^+\,(\xi; a, b)\,\mathrm{d}\xi \tag{5.3.6}$$

$$G\,(a,b) = -K^- + \int_{B(a,b)}^{b} v^-\,(\xi; a, b)\,\mathrm{d}\xi \tag{5.3.7}$$

Next for $x \in [a, A]$

$$\begin{aligned} K^+ + \inf_{x<y<b} \int_x^y v^+\,(\xi; a, b)\,\mathrm{d}\xi &= \int_x^A v^+\,(\xi; a, b)\,\mathrm{d}\xi + K^+ \\ &\geq \int_a^b v^+\,(\xi; a, b)\,\mathrm{d}\xi + K^+ = 0 \end{aligned}$$

and for $x \in [A, b]$,

$$K^+ + \inf_{x<y<b} \int_x^y v^+\,(\xi; a, b)\,\mathrm{d}\xi = K^+ > 0$$

Therefore (5.2.41) is satisfied. Similarly for $x \in [B, b]$

$$\begin{aligned} -K^- + \sup_{a<y<x} \int_y^x v^-\,(\xi; a, b)\,\mathrm{d}\xi &= -K^- + \int_B^x v^-\,(\xi; a, b)\,\mathrm{d}\xi \\ &\leq -K^- + \int_B^b v^-\,(\xi; a, b)\,\mathrm{d}\xi = 0 \end{aligned}$$

and for $x \in [a, B]$

$$-K^- + \sup_{a<y<x} \int_y^x v^- (\xi; a, b)\, \mathrm{d}\xi = -K^- < 0$$

Therefore conditions (5.2.41), (5.2.42) are satisfied.

So if we find a, b verifying (5.3.5) and (5.2.35), (5.2.36) and consider $v^+ (x; a, b)$, $v^- (x; a, b)$ and

$$\begin{aligned} v (x; a, b) &= v^+ (x; a, b) - c^+ \\ &= v^- (x; a, b) + c^- \end{aligned}$$

then the function $V (x)$ defined by +3pt

$$\begin{aligned} &V (x) = f (x) + \mu v (x) + \frac{1}{2}\sigma^2 v' (x) \quad x \in (a, b) \\ &V (a) = f (a) - \mu c^+ + \frac{1}{2}\sigma^2 v' (a) \\ &V (b) = f (b) + \mu c^- + \frac{1}{2}\sigma^2 v' (b) \\ &V (x) = -c^+ x + V (a) + c^+ a \qquad x \leq a \\ &V (x) = c^- + V (b) - c^- b \qquad x \geq b \end{aligned}$$

is the solution of the QVI ((5.2.9)).

5.3.3. Functions $a (b)$ and $b (a)$. We first need to study how the function $v (x; a, b)$ behaves in a, b, for x fixed (so for a in $(-\infty, x)$ and b in (∞, x) when x is fixed).

We have+3pt

$$\frac{\partial v}{\partial a}(x; a, b) = \frac{\partial v^+}{\partial a}(x; a, b) = \frac{\partial v^-}{\partial a}(x; a, b)$$
$$\frac{\partial v}{\partial b}(x; a, b) = \frac{\partial v^+}{\partial b}(x; a, b) = \frac{\partial v^-}{\partial b}(x; a, b)$$

Using (5.2.57) we have

$$\frac{\partial v^+}{\partial a}(x;a,b) = \frac{\partial \Gamma^-}{\partial a}(a,b)(\exp \rho_1(x-b) - \exp \rho_2(x-b))$$

and

$$\frac{\partial v^+}{\partial a}(a;a,b) = \frac{\partial \Gamma^-}{\partial a}(a,b)(\exp \rho_1(a-b) - \exp \rho_2(a-b))$$

On the other hand since $v^+(a;a,b) = 0$, we have also

$$\frac{\partial v^+}{\partial x}(a;a,b) + \frac{\partial v^+}{\partial a}(a;a,b) = 0$$

From (5.2.46) we have

$$\frac{\partial v^+}{\partial x}(a;a,b) = \Gamma^+(a,b)(\rho_1 - \rho_2)$$

Therefore

$$\frac{\partial v^+}{\partial a}(a;a,b) = \Gamma^+(a,b)(\rho_2 - \rho_1)$$

hence

$$\frac{\partial \Gamma^-}{\partial a}(a,b) = \frac{\Gamma^+(a,b)(\rho_2 - \rho_1)}{\exp \rho_1(a-b) - \exp \rho_2(a-b)}$$

and thus we have attained

$$\frac{\partial v^+}{\partial a}(x;a,b) = \frac{\Gamma^+(a,b)(\rho_2 - \rho_1)(\exp \rho_1(x-b) - \exp \rho_2(x-b))}{\exp \rho_1(a-b) - \exp \rho_2(a-b)}$$

So we can state that+3pt

$$\begin{aligned} &\frac{\partial v^+}{\partial a}(x;a,b) > 0 \qquad \text{if } a < a_0(b) \\ &\frac{\partial v^+}{\partial a}(x;a,b) < 0 \qquad \text{if } a > a_0(b) \end{aligned} \tag{5.3.8}$$

We can obtain a simpler formula for $\frac{\partial v^-}{\partial b}(x;a,b)$

$$\begin{aligned}\frac{\partial v^-}{\partial b}(x;a,b) &= \frac{\partial v^+}{\partial b}(x;a,b)\\ &= \frac{\partial \Gamma^+}{\partial b}(a,b)(\exp\rho_1(x-a)-\exp\rho_2(x-a))\end{aligned}$$

Using

$$v^-(b;a,b)=0$$

We can write

$$\frac{\partial v^-}{\partial x}(b;a,b)+\frac{\partial v^-}{\partial b}(b;a,b)=0$$

hence

$$\frac{\partial v^-}{\partial b}(b;a,b)=\Gamma^-(a,b)(\rho_2-\rho_1)$$

and

$$\frac{\partial \Gamma^+}{\partial b}(a,b)=\frac{\Gamma^-(a,b)(\rho_2-\rho_1)}{\exp\rho_1(b-a)-\exp\rho_2(b-a)}$$

Finally

$$\frac{\partial v^-}{\partial b}(x;a,b)=\frac{\Gamma^-(a,b)(\rho_2-\rho_1)(\exp\rho_1(x-a)-\exp\rho_2(x-a))}{\exp\rho_1(b-a)-\exp\rho_2(b-a)}$$

So we can state that+3pt

$$\begin{aligned}&\frac{\partial v^-}{\partial b}(x;a,b)>0 \qquad \text{for } b>b_0(a)\\ &\frac{\partial v^-}{\partial b}(x;a,b)<0 \qquad \text{for } b<b_0(a)\end{aligned} \tag{5.3.9}$$

We can now state

Proposition 14. *The function* $F(a,b)$, $G(a,b)$ *defined by* (5.2.37), (5.2.38) *satisfy+3pt*

$$\begin{aligned} &\frac{\partial F}{\partial a}(a,b)=0 \qquad \text{if } 0\geq a\geq a_0(b) \\ &\frac{\partial F}{\partial a}(a,b)>0 \qquad \text{if } 0\leq a< a_0(b) \end{aligned} \tag{5.3.10}$$

$$\begin{aligned} &\frac{\partial G}{\partial b}(a,b)=0 \qquad \text{if } b\leq b_0(a) \\ &\frac{\partial G}{\partial b}(a,b)>0 \qquad \text{if } b> b_0(a) \end{aligned} \tag{5.3.11}$$

Proof. We first note that $F(a,b)=K^+$ if $a\geq a_0(b)$ and $G(a,b)=-K^-$ if $b\leq b_0(a)$ Moreover from formulas (5.3.6), (5.3.7) we have for $a<a_0(b)$ and $b>b_0(a)$

$$\frac{\partial F}{\partial a}(a,b)=\int_a^{A(a,b)}\frac{\partial v^+}{\partial a}(\xi;a,b)\mathrm{d}\xi>0$$

from (5.3.8). Similarly for $b>b_0(a)$

$$\frac{\partial G}{\partial b}(a,b)=\int_{B(a,b)}^{b}\frac{\partial v^-}{\partial b}(\xi;a,b)\mathrm{d}\xi>0$$

from (5.3.9). □

Proposition 15. *There exists a unique* $a(b)<a_0(b)$ *such that*

$$F(a(b),b)=0 \tag{5.3.12}$$

and a unique $b(a)>b_0(a)$ *such that*

$$G(a,b(a))=0 \tag{5.3.13}$$

Proof. We have $F(a_0(b), b) = K^+$. We shall check that $F(-\infty, b) = -\infty$. Since $F(a,b)$ is monotone increasing in a for fixed b then the point $a(b)$ is uniquely defined. Similarly $G(a, b_0(a)) = -K^-$. We shall check that $G(a, +\infty) = +\infty$. Since $G(a,b)$ is monotone increasing in b for fixed a the point $b(a)$ is uniquely defined.

we now prove that

$$F(-\infty, b) = -\infty \tag{5.3.14}$$

the proof of $G(a, +\infty) = +\infty$ is similar. We can compute explicitly $v^+(x; -\infty, b)$ and obtain+3pt

$$v^+(x; -\infty, b) = -\frac{p - rc^+}{r} + \left(c^+ + c^-\right) \exp \rho_2 (x - b) - \frac{p - rc^+}{r(\rho_2 - \rho_1)} \exp \rho_2 (x - b) (\rho_1 \exp \rho_2 b - \rho_2 \text{e}$$

$$+ \frac{h + rc^+}{r(\rho_2 - \rho_1)} \exp \rho_2 (x - b) (\rho_1 (1 - \exp \rho_2 b) - \rho_2 (1 - \exp \rho_1 b))$$

for $x \leq 0$

$$v^+(x; -\infty, b) = \left(c^+ + c^-\right) \exp \rho_2 (x - b) - \frac{p - rc^+}{r(\rho_2 - \rho_1)} [\exp \rho_2 (x - b) (\rho_1 \exp \rho_2 b - \rho_2 \exp \rho_1 b)$$

$$+ \rho_2 \exp \rho_1 x - \rho_1 \exp \rho_2 x] + \frac{h + rc^+}{r(\rho_2 - \rho_1)} [\exp \rho_2 (x - b) (\rho_1 (1 - \exp \rho_2 b) - \rho_2 (1 - \exp \rho_1 b))$$

$$+ (\rho_2 (1 - \exp \rho_1 x) - \rho_1 (1 - \exp \rho_2 x))]$$

Therefore $v^+(x; -\infty, b) \to -\frac{p - rc^+}{r}$ as $x \to -\infty$ and $v^+(b; -\infty, b) = c^+ + c^-$. Therefore the point $A(-\infty, b)$ is well defined (as we know) and finite. It follows that

$$\int_{-\infty}^{A(-\infty,b)} v^+(\xi; -\infty, b) \mathrm{d}\xi = -\infty$$

and thus (5.3.14) holds. $\square$

5.3.4. Finding a, b. As we have seen in section 5.3.2, the choice of a, b boils down to solving (5.2.35) and (5.2.36) with $a \leq a_0(b)$ and $b \geq b_0(a)$. This is equivalent to finding a crossing point of the curves $a(b)$ and $b(a)$.

So we must find $\hat{a}$, $\hat{b}$ such that

$$\hat{a} = a\left(\hat{b}\right) \quad ; \quad \hat{b} = b(\hat{a}) \tag{5.3.15}$$

The existence of such a pair is a consequence of the property

$$b(-\infty) > 0 \text{ finite} \tag{5.3.16}$$

Indeed if (5.3.16) is true the function

$$u(a) = a(b(a)) - a$$

is $+\infty$ for $a = -\infty$ and $u(0) = a(b(0)) < 0$. Since it is a continuous function of a it crosses the axis $u(a) = 0$.

Let us check (5.3.16). Assume that $b(-\infty) = +\infty$, then we must have $G(-\infty, +\infty) = 0$.

We are going to prove that

$$G(-\infty, +\infty) = +\infty \tag{5.3.17}$$

and thus we obtain a contradiction.

To prove (5.3.17) we first note that+3pt

$$v^{-}(x;-\infty,+\infty)=\frac{h-rc^{-}}{r}-\frac{(p+h)\,\rho_2}{r\,(\rho_2-\rho_1)}\exp\rho_1 x \quad x>0$$

$$v^{-}(x;-\infty,+\infty)=-\frac{p+rc^{-}}{r}-\frac{(p+h)\,\rho_1}{r\,(\rho_2-\rho_1)}\exp\rho_2 x \quad x<0$$

This function if monotone increasing from $-\frac{p+rc^{-}}{r}$ to $\frac{h-rc^{-}}{r}$, so the value $B(-\infty,+\infty)$ is finite. It follows that

$$G(-\infty,+\infty)=-K^{-}+\int_{B(-\infty,+\infty)}^{+\infty}v^{-}(\xi;-\infty,+\infty)\mathrm{d}\xi=+\infty$$

hence (5.3.17) is verified.

□

The pair $\hat{a}$, $\hat{b}$ solution of (5.3.15) and the corresponding function $V\left(x;\hat{a},\hat{b}\right)$ constitute the solution referred to in Theorem.

5.4. Properties of $a(b)$ and $b(a)$

5.4.1. Monotonicity properties. The curves $a(b)$ and $b(a)$ are not monotone, but we can derive some properties. From (5.3.12) we can obtain

$$\frac{\partial F}{\partial a}(a(b),b)a'(b)+\frac{\partial F}{\partial b}(a(b),b)=0$$

Since $a(b)<a_0(b)$, we have from (5.3.10)

$$\frac{\partial F}{\partial a}(a(b),b)>0$$

Next

$$\begin{aligned}\frac{\partial F}{\partial b}(a,b) = & \int_a^{A(a,b)} \frac{\partial v^+}{\partial b}(\xi;a,b)\mathrm{d}\xi \\ = & \int_a^{A(a,b)} \frac{\partial v^-}{\partial b}(\xi;a,b)\mathrm{d}\xi\end{aligned}$$

From (5.3.9) it follows that

$$\begin{aligned}\frac{\partial F}{\partial b}(a,b) > 0 \quad \text{for} \quad b > b_0\left(a\right) \text{ and } a < a_0\left(b\right) \\ \frac{\partial F}{\partial b}(a,b) < 0 \quad \text{for} \quad b < b_0\left(a\right) \text{ and } a < a_0\left(b\right)\end{aligned}$$

Therefore we can state

$$\begin{aligned}&\text{If } b > b_0\left(a\left(b\right)\right) \text{ then } a'\left(b\right) < 0 \\ &\text{If } b < b_0\left(a\left(b\right)\right) \text{ then } a'\left(b\right) > 0\end{aligned} \tag{5.4.1}$$

+3ptA similar analysis holds for $b\left(a\right)$.

We have

$$\frac{\partial G}{\partial a}(a,b\left(a\right)) + \frac{\partial G}{\partial b}(a,b\left(a\right))b'(a) = 0$$

Since $b\left(a\right) > b_0\left(a\right)$, we have from (5.3.11)

$$\frac{\partial G}{\partial b}(a,b\left(a\right)) > 0$$

Next

$$\begin{aligned}\frac{\partial G}{\partial a}(a,b) = & \int_{B(a,b)}^{b} \frac{\partial v^-}{\partial a}(\xi;a,b)\mathrm{d}\xi \\ = & \int_{B(a,b)}^{b} \frac{\partial v^+}{\partial a}(\xi;a,b)\mathrm{d}\xi\end{aligned}$$

and from (5.3.8) it follows that

$$\frac{\partial G}{\partial a}(a,b)>0 \quad \text{if} \quad a<a_0\left(b\right) \text{ and } b>b_0\left(a\right)$$
$$\frac{\partial G}{\partial a}(a,b)<0 \quad \text{if} \quad a>a_0\left(b\right) \text{ and } b>b_0\left(a\right)$$

Therefore we can assert

$$\begin{aligned}&\text{If } a<a_0\left(b\left(a\right)\right) \text{ then } b'\left(a\right)<0\\ &\text{If } a>a_0\left(b\left(a\right)\right) \text{ then } b'\left(a\right)>0\end{aligned} \tag{5.4.2}$$

It follows in particular from (5.4.1) and (5.4.2) that

$$b'\left(0\right)>0, \quad a'\left(0\right)>0$$

We will show later on that these values equal 1.

5.4.2. Expressions for $F\left(a,b\right)$ and $G\left(a,b\right)$. We can compute $F\left(a,b\right)$ and $G\left(a,b\right)$ by formula (5.3.10), (5.3.11) in terms of a, b, A and a, b, B respectively. One of the difficulties is that A and B can be positive or negative Note that

$$A\left(a,b\right)<B\left(a,b\right)$$

This is clear if $A\left(a,b\right)=a$ or $B\left(a,b\right)=b$. Otherwise we have

$$v^{+}\left(A;a,b\right)=0 \qquad , \; v^{+}\left(B;a,b\right)=c^{+}+c^{-}$$

Since $v^{+}\left(x;a,b\right)<0$ for $x\in\left(a,A\right)$, necessarily $B>A$. We begin by giving the expressions of $v^{+}\left(x;a,b\right)$ and $v^{-}\left(x;a,b\right)$

We have

for $x > 0$

$$v^+(x;a,b) = \left(c^+ + c^-\right) \frac{\exp \rho_1 (x-a) - \exp \rho_2 (x-a)}{\exp \rho_1 (b-a) - \exp \rho_2 (b-a)}$$

$$- \frac{h + rc^+}{r} \frac{(\exp \rho_1 (x-a) - \exp \rho_2 (x-a) - \exp \rho_1 (b-a) + \exp \rho_2 (b-a))}{\exp \rho_1 (b-a) - \exp \rho_2 (b-a)}$$

$$+ \frac{(p+h)(\exp(\rho_2 x + \rho_1 b) - \exp(\rho_1 x + \rho_2 b))(\rho_1 \exp -\rho_1 a - \rho_2 \exp -\rho_2 a)}{r(\rho_2 - \rho_1)(\exp \rho_1 (b-a) - \exp \rho_2 (b-a))}$$

$$+ \frac{p - rc^+}{r} \frac{\exp -(\rho_1 + \rho_2) a (\exp(\rho_2 x + \rho_1 b) - \exp(\rho_1 x + \rho_2 b))}{\exp \rho_1 (b-a) - \exp \rho_2 (b-a)}$$

for $x < 0$

$$v^+(x;a,b) = \frac{\exp \rho_1 (x-a) - \exp \rho_2 (x-a)}{\exp \rho_1 (b-a) - \exp \rho_2 (b-a)} \left[-\frac{h - rc^-}{r} \right.$$

$$\left. + \frac{(p+h)}{r(\rho_2 - \rho_1)} (\rho_2 \exp \rho_1 b - \rho_1 \exp \rho_2 b) \right] \tag{5.4.3}$$

$$- \frac{p - rc^+}{r(\exp \rho_1 (b-a) - \exp \rho_2 (b-a))} [\exp \rho_2 (b-a) (\exp \rho_1 (x-a) - 1)$$

$$- \exp \rho_1 (b-a) (\exp \rho_2 (x-a) - 1)]$$

To compute $F(a,b)$ for $a < a_0(b)$, we note that from the differential equation (5.2.31) we have

$$-\frac{\sigma^2}{2} \left(\left(v^+\right)' (A) - \left(v^+\right)' (a) \right) + r \int_a^{A(a,b)} v^+(\xi;a,b) \, \mathrm{d}\xi = \int_a^{A(a,b)} \left(f'(\xi) + rc^+ \right) \mathrm{d}\xi$$

Recalling that $\rho_1\rho_2 = -\frac{2r}{\sigma^2}$, we get

$$\int_a^{A(a,b)} v^+(\xi; a, b) = -\frac{1}{\rho_1\rho_2}\left(\left(v^+\right)'(A) - \left(v^+\right)'(a)\right) + \frac{1}{r}\int_a^{A(a,b)} \left(f'(\xi) + rc^+\right) \mathrm{d}\xi$$

Now from (5.2.43)

$$\left(v^+\right)'(x; a, b) = \Gamma^+(a, b)\left(\rho_1 \exp\rho_1 (x-a) - \rho_2 \exp\rho_2 (x-a)\right)$$
$$-\frac{2}{\sigma^2(\rho_1-\rho_2)}\int_a^x (f' + rc^+)(\xi)(\rho_1 \exp\rho_1(x-\xi) - \rho_2\exp\rho_2(x-\xi))d\xi$$

hence

$$\left(v^+\right)'(A; a, b) - \left(v^+\right)'(a; a, b) = \Gamma^+(a, b)\left(\rho_1\left(\exp\rho_1(A-a) - 1\right) - \rho_2\left(\exp\rho_2(A-a) - 1\right)\right)$$
$$-\frac{2}{\sigma^2(\rho_1-\rho_2)}\int_a^A (f' + rc^+)(\xi)(\rho_1\exp\rho_1(A-\xi) - \rho_2\exp\rho_2(A-\xi))d\xi$$

But since $v^+(A; a, b = 0)$we can write

$$\Gamma^+(a, b) = \frac{2}{\sigma^2(\rho_1-\rho_2)}\frac{\int_a^A (f' + rc^+)(\xi)(\exp\rho_1(A-\xi) - \exp\rho_2(A-\xi))d\xi}{\exp\rho_1(A-a) - \exp\rho_2(A-a)}$$

Collecting results we can assert that

$$F(a, b) = K^+ + \frac{1}{r}\int_a^A \left(f' + rc^+\right)(\xi)\,\mathrm{d}\xi$$
$$-\frac{1}{r}\int_a^A \left(f' + rc^+\right)(\xi)\,\frac{\exp\rho_2(A-\xi)\left(\exp\rho_1(A-a) - 1\right) - \exp\rho_1(A-\xi)\left(\exp\rho_2(A-a) - 1\right)}{\exp\rho_1(A-a) - \exp\rho_2(A-a)}\mathrm{d}\xi$$

and we obtain the formula

(5.4.4)

$$F(a,b) = K^{+} + \frac{1}{r}\left[-\left(p - rc^{+}\right)(A-a) + (h+p)A^{+}\right]$$

$$+\frac{1}{r}\frac{p-rc^{+}}{\rho_1\rho_2}(\rho_1-\rho_2)\frac{(\exp\rho_1(A-a)-1)(\exp\rho_2(A-a)-1)}{\exp\rho_1(A-a)-\exp\rho_2(A-a)}$$

$$+\frac{1}{r}\frac{p+h}{\rho_1\rho_2}\frac{[\rho_1(\exp\rho_1(A-a)-1)(1-\exp\rho_2A^{+})-\rho_2(\exp\rho_2(A-a)-1)(1-\exp\rho_1A^{+})]}{\exp\rho_1(A-a)-\exp\rho_2(A-a)}$$

Let us give equivalent formulas for $v^{-}(x;a,b)$ and $G(a,b)$. We have

(5.4.5)

for $x > 0$

$$v^{-}(x;a,b) = \frac{\exp\rho_1(x-b)-\exp\rho_2(x-b)}{\exp\rho_1(a-b)-\exp\rho_2(a-b)}\left\{\frac{p-rc^{+}}{r}+\frac{(p+h)}{r(\rho_2-\rho_1)}(\rho_1\exp\rho_2 a-\rho_2\exp\rho_1 a)\right.$$

$$\left.+\frac{h-rc^{-}}{r}\frac{\exp\rho_2(a-b)(\exp\rho_1(x-b)-1)-\exp\rho_1(a-b)(\exp\rho_2(x-b)-1)}{\exp\rho_1(a-b)-\exp\rho_2(a-b)}\right\}$$

for $x < 0$

$$v^{-}(x;a,b) = -\left(c^{+}+c^{-}\right)\frac{\exp\rho_1(x-b)-\exp\rho_2(x-b)}{\exp\rho_1(a-b)-\exp\rho_2(a-b)}$$

$$+\frac{(h-rc^{-})\exp-(\rho_1+\rho_2)b}{r(\exp\rho_1(a-b)-\exp\rho_2(a-b))}(\exp(\rho_1x+\rho_2a)-\exp(\rho_2x+\rho_1a))$$

$$+\frac{(p+h)(\exp(\rho_1x+\rho_2a)-\exp(\rho_2x+\rho_1a))(\rho_2\exp-\rho_2b-\rho_1\exp-\rho_1b)}{r(\rho_1-\rho_2)(\exp\rho_1(a-b)-\exp\rho_2(a-b))}$$

$$+\frac{p+rc^{-}}{r}\frac{(\exp\rho_1(x-b)-\exp\rho_2(x-b)-\exp\rho_1(a-b)+\exp\rho_2(a-b))}{\exp\rho_1(a-b)-\exp\rho_2(a-b)}$$

and then

(5.4.6)

$$G(a,b) = -K^- + \frac{1}{r}\left[\left(h - rc^-\right)(b-B) - (h+p)\,B^-\right]$$

$$+\frac{p+h}{\rho_1\rho_2}\frac{\left[\rho_1\left(\exp\rho_1(B-b)-1\right)\left(1-\exp-\rho_2B^-\right)-\rho_2\left(\exp\rho_2(B-b)-1\right)\left(1-\exp-\rho_1B^-\right)\right]}{\exp\rho_1(B-b)-\exp\rho_2(B-b)}$$

$$+\frac{(\rho_1-\rho_2)}{\rho_1\rho_2}\left(h-rc^-\right)\frac{\left(\exp\rho_1(B-b)-1\right)\left(\exp\rho_2(B-b)-1\right)}{\exp\rho_1(B-b)-\exp\rho_2(B-b)}$$

5.4.3. How to compute $a(b)$ and $b(a)$. $a(b)$ is the solution of $F(a(b),b) = 0$. However looking at (5.4.4) the expression depends on A. So in fact we define $a(b)$ and $A(a(b),b)$ jointly by solving the system

$$F(a,b) = 0 \qquad v^+(A;a,b) = 0 \tag{5.4.7}$$

with unknowns a, A and b fixed. In solving (5.4.7) in a, A for b fixed, the value of A might be positive or negative.

If $A < 0$, then from (5.4.4) we get

$$\begin{aligned} &K^+ - \frac{p-rc^+}{r}(A-a) \\ &+\frac{1}{r}\frac{p-rc^+}{\rho_1\rho_2}(\rho_1-\rho_2)\frac{\left(\exp\rho_1(A-a)-1\right)\left(\exp\rho_2(A-a)-1\right)}{\exp\rho_1(A-a)-\exp\rho_2(A-a)} = 0 \end{aligned} \tag{5.4.8}$$

and from (5.4.3) we have also

(5.4.9)

$$-\left(h-rc^-\right)+(p+h)\frac{\rho_2\exp\rho_1 b-\rho_1\exp\rho_2}{\rho_2-\rho_1}=$$

$$\left(p-rc^+\right)\frac{\left(\exp\rho_2(b-a)\left(\exp\rho_1(A-a)-1\right)-\exp\rho_1(b-a)\left(\exp\rho_2(A-a)-1\right)\right)}{\exp\rho_1(A-a)-\exp\rho_2(A-a)}$$

The system (5.4.8), (5.4.9) can be solved sequentially. Indeed (5.4.8) defines $A-a$, and (5.4.9) defines a $A-a$.

If we consider the function

$$Z\left(x\right)=-x+\frac{\rho_1-\rho_2}{\rho_1\rho_2}\frac{\left(\exp\rho_1x-1\right)\left(\exp\rho_2x-1\right)}{\exp\rho_1x-\exp\rho_2x} \tag{5.4.10}$$

then (5.4.8) is equivalent to

$$Z(A-a)=\frac{-K^+r}{p-rc^+} \tag{5.4.11}$$

The function $Z\left(x\right)$ decreases for $x>0$. Indeed

$$\begin{aligned}Z'\left(x\right) &= -1+\frac{\rho_1-\rho_2}{\rho_1\rho_2}\frac{\rho_2\exp\rho_2x\left(\exp\rho_1x-1\right)^2-\rho_1\exp\rho_1x\left(\exp\rho_2x-1\right)^2}{\left(\exp\rho_1x-\exp\rho_2x\right)^2}\\ &= \frac{\left[\rho_2\left(\exp\rho_1x-1\right)-\rho_1\left(\exp\rho_2x-1\right)\right]\left[\rho_1\exp\rho_1x\left(\exp\rho_2x-1\right)-\rho_2\exp\rho_2x\left(\exp\rho_1x-1\right)\right]}{\rho_1\rho_2\left(\exp\rho_1x-\exp\rho_2x\right)^2}\end{aligned}$$

We have

$$\rho_1\exp\rho_1x\left(\exp\rho_2x-1\right)-\rho_2\exp\rho_2x\left(\exp\rho_1x-1\right)>0\qquad\text{for } x>0$$

$$\rho_2\left(\exp\rho_1x-1\right)-\rho_1\left(\exp\rho_2x-1\right)>0$$

hence $Z'\left(x\right)\leq 0$.

Moreover $Z\left(0\right)=0$, and $Z\left(+\infty\right)=-\infty$. Hence there exists a unique $\alpha>0$ such that

$$Z(\alpha)=\frac{-K^+r}{p-rc^+}$$

From (5.4.11) we can deduce that

$$A\left(a\left(b\right),b\right)-a\left(b\right)=\alpha\qquad\text{if}\quad A\left(a\left(b\right),b\right)\leq 0$$

Going back to (5.4.9) we deduce also

$$
\begin{aligned}
(5.4.12)\qquad & -\frac{(h-rc^{-})(\rho_2-\rho_1)}{p-rc^{+}}+\frac{(p+h)}{p-rc^{+}}(\rho_2\exp\rho_1 b-\rho_1\exp\rho_2 b)= \\
& (\rho_2-\rho_1)\frac{[\exp\rho_2(b-a(b))(\exp\rho_1\alpha-1)-\exp\rho_1(b-a(b))(\exp\rho_2\alpha-1)]}{\exp\rho_1\alpha-\exp\rho_2\alpha}
\end{aligned}
$$

which is thus an equation for $a(b)$. Introduce

$$\varphi(\alpha)=\frac{\rho_2(\exp\rho_1\alpha-1)-\rho_1(\exp\rho_2\alpha-1)}{\exp\rho_2\alpha-\exp\rho_1\alpha}$$

then

$$\varphi(\alpha)+\rho_1=\frac{(\rho_2-\rho_1)(\exp\rho_1\alpha-1)}{\exp\rho_2\alpha-\exp\rho_1\alpha}$$

$$\varphi(\alpha)+\rho_2=\frac{(\rho_2-\rho_1)(\exp\rho_1\alpha-1)}{\exp\rho_2\alpha-\exp\rho_1\alpha}$$

hence (5.4.12) becomes

$$-\frac{(h-rc^{-})(\rho_2-\rho_1)}{p-rc^{+}}+\frac{p+h}{p-rc^{+}}(\rho_2\exp\rho_1 b-\rho_1\exp\rho_2 b)=$$

$$(\varphi(\alpha)+\rho_2)\exp\rho_1(b-a(b))-(\varphi(\alpha)+\rho_1)\exp\rho_2(b-a(b))$$

also

$$
\begin{aligned}
(5.4.13)\qquad & \frac{r(\rho_2-\rho_1)(c^{+}+c^{-})+(p+h)(\rho_2(\exp\rho_1 b-1)-\rho_1(\exp\rho_2 b-1))}{p-rc^{+}} \\
& =(\varphi(\alpha)+\rho_2)(\exp\rho_1(b-a(b))-1)-(\varphi(\alpha)+\rho_1)(\exp\rho_2(b-a(b))-1)
\end{aligned}
$$

Let us define

$$L_\alpha(a,b)=(\varphi(\alpha)+\rho_2)(\exp\rho_1(b-a)-1)-(\varphi(\alpha)+\rho_1)(\exp\rho_2(b-a)-1)$$

then

$$\frac{\partial L_\alpha}{\partial a}(a,b) = -\rho_1(\varphi(\alpha)+\rho_2)\exp\rho_1(b-a) + \rho_2(\varphi(\alpha)+\rho_1)\exp\rho_2(b-a)$$

$$\begin{aligned}\frac{\partial^2 L_\alpha}{\partial a^2}(a,b) &= \rho_1^2(\varphi(\alpha)+\rho_2)\exp\rho_1(b-a) - \rho_2^2(\varphi(\alpha)+\rho_1)\exp\rho_2(b-a) \\ &> 0 \qquad \text{since } \varphi(\alpha)+\rho_1 < 0\end{aligned}$$

The value

$$\frac{\partial L_\alpha}{\partial a}(-\infty,b) = -\infty \qquad L_\alpha(-\infty,b) = +\infty$$

$$\frac{\partial L_\alpha}{\partial a}(0,b) = \rho_2(\varphi(\alpha)+\rho_1)\exp\rho_2 b - \rho_1(\varphi(\alpha)+\rho_2)\exp\rho_1 b$$

$$L_\alpha(0,b) = (\varphi(\alpha)+\rho_2)(\exp\rho_1 b - 1) - (\varphi(\alpha)+\rho_1)(\exp\rho_2 b - 1)$$

The value $\frac{\partial L_\alpha}{\partial a}(0,b)$ decreases in b from $(\rho_2-\rho_1)\varphi(\alpha) > 0$ to $-\infty$. So we define uniquely $\bar{b}$ with

$$\frac{\partial L_\alpha}{\partial a}\left(0,\bar{b}\right) = 0$$

and

$$\begin{aligned}\frac{\partial L_\alpha}{\partial a}(0,b) < 0 &\qquad \text{if} \quad b > \bar{b} \\ \frac{\partial L_\alpha}{\partial a}(0,b) > 0 &\qquad \text{if} \quad b < \bar{b}\end{aligned}$$

Therefore if $b > \bar{b}$, $\frac{\partial L_\alpha}{\partial a}(a,b) < 0$ and if $b < \bar{b}$, $\frac{\partial L_\alpha}{\partial a}(a,b) < 0$ if $a < \bar{a}(b)$ and $\frac{\partial L_\alpha}{\partial a}(a,b) > 0$ if $a > \bar{a}(b)$.

However the value on the left hand side of (5.4.13) is larger than $L_\alpha(0,b)$, as easily seen. Therefore the solution of

$$\frac{r(\rho_2-\rho_1)(c^++c^-)+(p+h)(\rho_2(\exp\rho_1 b-1)-\rho_1(\exp\rho_2 b-1))}{p-rc^+}=L_\alpha(a,b)$$

exists and is uniquely defined. We denote the solution by $a_\alpha(b)$. This notation is coherent with the fact that whenever $\alpha=0$, we get indeed $a_0(b)$ defined in section 5.2.6 ($v^+=0$).

Recalling the values of $Z(a,b)$ and $\Gamma^+(a,b)$, see (5.2.48) and (5.2.44) we can check easily that (5.4.13) is equivalent to

$$\Gamma^+(a,b)=\frac{p-rc^+}{r(\rho_2-\rho_1)}\varphi(\alpha)$$

and $\varphi(\alpha)>0$, since $\alpha>0$. It follows that

$$Z(a_\alpha,b)<0$$

which implies since $Z(a_\alpha,b)$ and $Z(a,b)$ is increasing in a, that

$$a_\alpha(b)<a_0(b)$$

We can thus assert that

$$\text{if}\quad a_\alpha(b)+\alpha<0\quad\text{then}\quad a(b)=a_\alpha(b) \tag{5.4.14}$$

since in that case $a_\alpha(b)$, $A(a_\alpha(b),b)=a_\alpha(b)+\alpha$ is a solution of (5.4.7).

The value $a_\alpha(\infty)$ is defined by

$$\exp \rho_2 a_\alpha(\infty) = \frac{p - rc^+}{h+p}\left(1 + \frac{\varphi(\alpha)}{\rho_1}\right)$$

Similarly, we can consider $b(a)$ and claim

$$G(a,b) = 0 \qquad v^-(B;a,b) = 0$$

with unknowns b and B when a is fixed.

The case $B > 0$ leads to similar simplification as for $A < 0$. We have from (5.4.6) and (5.4.5),

$$0 = -K^- + \frac{1}{r}\left(h - rc^-\right)(b-B) + \frac{(\rho_1 - \rho_2)}{\rho_1\rho_2}\left(h - rc^-\right)\frac{(\exp\rho_1(B-b)-1)(\exp\rho_2(B-b)-1)}{\exp\rho_1(B-b) - \exp\rho_2(B-b)}$$

which leads to $b - B = \beta$ with $\beta > 0$ solution of

$$Z(\beta) = \frac{-K^- r}{h - rc^-}$$

From (5.4.5) we get a relation similar to (5.4.13)

$$\begin{aligned}&\frac{r(\rho_2-\rho_1)(c^+ + c^-) + (p+h)(\rho_2(\exp\rho_1 a - 1) - \rho_1(\exp\rho_2 a - 1))}{h - rc^-}\\ &= (\varphi(-\beta)+\rho_2)(\exp\rho_1(a - b(a)) - 1) - (\varphi(-\beta)+\rho_1)(\exp\rho_2(a-b(a)) - 1)\end{aligned} \tag{5.4.15}$$

and $\varphi(-\beta) + \rho_2 > 0$.

We can show that (5.4.15) defines a unique $b_\beta(a)$, with

$$b_\beta(a) > b_0(a)$$

We have the analogue of (5.4.14), namely

$$\text{if} \quad b_\beta(a) - \beta > 0 \quad \text{then} \quad b(a) = b_\beta(a) \tag{5.4.16}$$

and $B(a, b(a)) = b_\beta(a) - \beta$

Whenever (5.4.14) or (5.4.16) are not satisfied, then the pair a, A or b, B have to be obtained simultaneously by solving a system

(5.4.17)

$$0 = K^+ + \frac{1}{r}\left[-\left(p - rc^+\right)(A-a) + (h+p)A\right]$$

$$+ \frac{1}{r}\frac{(p - rc^+)}{\rho_1\rho_2}(\rho_1 - \rho_2)\frac{(\exp\rho_1(A-a) - 1)(\exp\rho_2(A-a) - 1)}{\exp\rho_1(A-a) - \exp\rho_2(A-a)}$$

$$+ \frac{1}{r}\frac{(p+h)}{\rho_1\rho_2}\frac{[\rho_1(\exp\rho_1(A-a) - 1)(1 - \exp\rho_2 A) - \rho_2(\exp\rho_2(A-a) - 1)(1 - \exp\rho_1 A)]}{\exp\rho_1(A-a) - \exp\rho_2(A-a)}$$

$$\begin{aligned}
0 &= \left(c^+ + c^-\right)(\exp\rho_1(A-a) - \exp\rho_2(A-a)) \\
&- \frac{h + rc^+}{r}(\exp\rho_1(A-a) - \exp\rho_2(A-a) - \exp\rho_1(b-a) + \exp\rho_2(b-a)) \\
&+ \frac{(p+h)}{r(\rho_2 - \rho_1)}(\exp(\rho_2 A + \rho_1 b) - \exp(\rho_1 A + \rho_2 b))(\rho_1\exp -\rho_1 a - \rho_2\exp -\rho_2 a) \\
&+ \frac{p - rc^+}{r}\exp -(\rho_1 + \rho_2)a(\exp(\rho_2 A + \rho_1 b) - \exp(\rho_1 A + \rho_2 b))
\end{aligned} \tag{5.4.18}$$

In (5.4.17), (5.4.18) b is fixed and we obtain $a(b)$ and $A(a(b), b)$.

Similarly for b, B when a is given

(5.4.19)

$$0=-K^{-}+\frac{1}{r}\left[\left(h-rc^{-}\right)b+\left(p+rc^{-}\right)B\right]$$

$$+\frac{1}{r}\frac{(p+h)}{\rho_1\rho_2}\frac{\left[\rho_1\left(\exp\rho_1\left(B-b\right)-1\right)\left(1-\exp\rho_2 B\right)-\rho_2\left(\exp\rho_2\left(B-b\right)-1\right)\left(1-\exp\rho_1 B\right)\right]}{\exp\rho_1\left(B-b\right)-\exp\rho_2\left(B-b\right)}$$

$$+\frac{\left(\rho_1-\rho_2\right)\left(h-rc^{-}\right)\left(\exp\rho_1\left(B-b\right)-1\right)\left(\exp\rho_2\left(B-b\right)-1\right)}{\rho_1\rho_2\left(\exp\rho_1\left(B-b\right)-\exp\rho_2\left(B-b\right)\right)}$$

(5.4.20)

$$\begin{aligned}0=&-\left(c^{+}+c^{-}\right)\left(\exp\rho_1\left(B-b\right)-\exp\rho_2\left(B-b\right)\right)\\&+\frac{h-rc^{-}}{r}\exp-\left(\rho_1+\rho_2\right)b\left(\exp\left(\rho_1 B+\rho_2 a\right)-\exp\left(\rho_2 B+\rho_1 a\right)\right)\\&+\frac{p+h}{r\left(\rho_1-\rho_2\right)}\left(\exp\left(\rho_1 B+\rho_2 a\right)-\exp\left(\rho_2 B+\rho_1 a\right)\right)\left(\rho_2\exp-\rho_2 b-\rho_1\exp-\rho_1 a\right)\\&+\frac{p+rc^{-}}{r}\left(\exp\rho_1\left(B-b\right)-\exp\rho_2\left(B-b\right)-\exp\rho_1\left(a-b\right)+\exp\rho_2\left(a-b\right)\right)\end{aligned}$$

In (5.4.19), (5.4.20) a is given and we obtain $b\left(a\right)$and $B\left(a,b\left(a\right)\right)$.

5.4.4. Properties of $a_\alpha\left(b\right)$ and $b_\beta\left(a\right)$. From (5.4.13) we can obtain $a_\alpha^{'}\left(b\right)$

(5.4.21)

$$\frac{p+h}{p-rc^{+}}\rho_1\rho_2\left(\exp\rho_1 b-\exp\rho_2 b\right)=$$

$$\left(1-a_\alpha^{'}\left(b\right)\right)\left[\left(\varphi\left(\alpha\right)+\rho_2\right)\rho_1\exp\rho_1\left(b-a_\alpha\left(b\right)\right)-\left(\varphi\left(\alpha\right)+\rho_1\right)\rho_2\exp\rho_2\left(b-a_\alpha\left(b\right)\right)\right]$$

From the preceding section

$$\frac{\partial L_\alpha}{\partial a}\left(a_\alpha\left(b\right),b\right)<0$$

hence

$$\left(\varphi\left(\alpha\right)+\rho_2\right)\rho_1\exp\rho_1\left(b-a_\alpha\left(b\right)\right)-\left(\varphi\left(\alpha\right)+\rho_1\right)\rho_2\exp\rho_2\left(b-a_\alpha\left(b\right)\right)>0$$

and therefore

$$
(5.4.22) \qquad a_{\alpha}^{'}(b) = (\rho_2 - \rho_1) \frac{\{\exp \rho_1 b \left[(p+h)\rho_2 - (\rho_2 + \varphi(\alpha))(p - rc^{+}) \exp -\rho_1 a_{\alpha}(b)\right]}{(p - rc^{+}) \left[(\varphi(\alpha) + \rho_2)\rho_1 \exp \rho_1 (b - a_{\alpha}(b))\right.} \frac{-\rho_2 (h - rc^{-})\}}{\left.-(\varphi(\alpha) + \rho_1)\rho_2 \exp \rho_2 (b - a_{\alpha}(b))\right]}
$$

From (5.4.21) and (5.4.22) we see that

$$
a_{\alpha}^{'}(0) = 1 \ ; \ a_{\alpha}^{'}(b) < 0 \quad \text{for } b \text{ sufficient large}
$$

Let us look at the points b such that $a_{\alpha}^{'}(b) = 0$. From (5.4.15) and (5.4.13) we deduce as for (5.4.10), (5.4.11) the relations

$$
(5.4.23) \qquad \exp -\rho_2 a_{\alpha}(b) = \frac{\rho_1}{\rho_1 + \varphi(\alpha)} \frac{p + h - (h - rc^{-}) \exp -\rho_2 b}{p - rc^{+}}
$$

$$
(5.4.24) \qquad \exp -\rho_1 a_{\alpha}(b) = \frac{\rho_2}{\rho_2 + \varphi(\alpha)} \frac{p + h - (h - rc^{-}) \exp -\rho_1 b}{p - rc^{+}}
$$

We can eliminate $a_{\alpha}(b)$ and get the relation for b, we obtain the equation

$$
(5.4.25) \qquad \begin{aligned} &\left(p + h - \left(h - rc^{-}\right) \exp -\rho_1 b\right) \left(p + h - \left(h - rc^{-}\right) \exp -\rho_2 b\right)^{-\frac{\rho_1}{\rho_2}} \\ &= \left(p - rc^{+}\right)^{1 - \frac{\rho_1}{\rho_2}} \frac{\rho_2 + \varphi(\alpha)}{\rho_2} \left(\frac{\rho_1 + \varphi(\alpha)}{\rho_1}\right)^{-\frac{\rho_1}{\rho_2}} \end{aligned}
$$

If we consider the function $\left(1 + \frac{x}{\rho_2}\right)\left(1 + \frac{x}{\rho_1}\right)^{-\frac{\rho_1}{\rho_2}}$ for $0 < x < -\rho_1$, then it is decreasing and equal to 1 for $x > 0$, therefore

$$
\frac{\rho_2 + \varphi(\alpha)}{\rho_2} \left(\frac{\rho_1 + \varphi(\alpha)}{\rho_1}\right)^{-\frac{\rho_1}{\rho_2}} < 1
$$

We know that the left hand side decreases on $[0,\infty)$ from $(p+rc^-)^{1-\frac{\rho_1}{\rho_2}}$ to $-\infty$. Since

$$\left(p+rc^-\right)^{1-\frac{\rho_1}{\rho_2}} > \left(p-rc^+\right)^{1-\frac{\rho_1}{\rho_2}} \frac{\rho_2+\varphi\left(\alpha\right)}{\rho_2}\left(\frac{\rho_1+\varphi\left(\alpha\right)}{\rho_1}\right)^{-\frac{\rho_1}{\rho_2}}$$

We obtain that (5.4.25) has a unique solution b_α^*. Again if $\alpha=0$, we recover b^* defined in Proposition 10. We can compute $a_\alpha''\left(b_\alpha^*\right)$ in a way similar to $a_0''\left(b^*\right)$ (see (5.2.66)), namely

$$\begin{aligned}
-\,a_\alpha''\left(b_\alpha^*\right) &= \frac{p+h}{p-rc^+}\exp\left(\rho_1+\rho_2\right)b_\alpha^*\left(\rho_1-\rho_2\right) \\
&\cdot \frac{\left[\frac{\varphi(\alpha)+\rho_2}{\rho_2}\exp-\rho_1 a_\alpha\left(b_\alpha^*\right)-\frac{\varphi(\alpha)+\rho_1}{\rho_1}\exp-\rho_2 a_\alpha\left(b_\alpha^*\right)\right]}{\left[\frac{\varphi(\alpha)+\rho_2}{\rho_2}\exp\rho_1\left(b_\alpha^*-a_\alpha\left(b_\alpha^*\right)\right)-\frac{\varphi(\alpha)+\rho_1}{\rho_1}\exp\rho_2\left(b_\alpha^*-a_\alpha\left(b_\alpha^*\right)\right)\right]}
\end{aligned} \tag{5.4.26}$$

Although we cannot immediately give the sign of the bracket in the numerator, as we did in the case of $a_0\left(b\right)$, we can still claim that $a_\alpha''\left(b_\alpha^*\right)<0$.

Indeed we know that $a_0'\left(0\right)=1$. Since b_α^* is the only point, it means that $a_\alpha'\left(b\right)>0$ $\forall$ $b<b_\alpha^*$. Therefore necessarily $a_\alpha''\left(b_\alpha^*\right)\leq 0$. The case $a_\alpha''\left(b_\alpha^*\right)=0$ would imply from formula (5.4.26)

$$\frac{\varphi\left(\alpha\right)+\rho_2}{\rho_2}\exp-\rho_1 a_\alpha\left(b_\alpha^*\right)=\frac{\varphi\left(\alpha\right)+\rho_1}{\rho_1}\exp-\rho_2 a_\alpha\left(b_\alpha^*\right)$$

and from (5.4.23), (5.4.24) it will follow that

$$\exp-\rho_2 b_\alpha^*=\exp-\rho_1 b_\alpha^*$$

which is impossible since $b_\alpha^*>0$. We can then state the analogue of Proposition (10) and Proposition (13)

Proposition 16. *Under the assumption* (5.2.39)*, the functions* $a_\alpha\left(b\right)$ *and* $b_\beta\left(a\right)$ *are uniquely defined by* (5.4.13) *and* (5.4.15) *respectively on* $(-\infty,0)$ *and* $(0,\infty)$.

We have $a'_\alpha(0) = b'_\beta(0) = 1$. *There exist unique points* $b^*_\alpha > 0$ *and* $a^*_\beta < 0$ *such that*

$$a'_\alpha(b^*_\alpha) = 0 \quad , \quad b'_\beta\left(a^*_\beta\right) = 0$$

The function $a_\alpha(b)$ *increases on* $(0, b^*_\alpha)$, *then decreases on* (b^*_α, ∞). *The function* $b_\beta(a)$ *decreases on* $\left(-\infty, a^*_\beta\right)$, *then increases on* $\left(a^*_\beta, 0\right)$ *.*

The properties of $a_\alpha(b)$ and $b_\beta(a)$ lead to important consequences for $a(b)$ and $b(a)$. Using (5.4.14) and (5.4.16) we can state the

THEOREM 17. *Continue* (5.4.16). *We have*

(5.4.27) $$\text{if} \quad a_\alpha(b^*_\alpha) + \alpha < 0 \quad \text{then} \quad a(b) = a_\alpha(b)$$

(5.4.28) $$\text{if} \quad b_\beta\left(a^*_\beta\right) - \beta > 0 \quad \text{then} \quad b(a) = b_\beta(a)$$

PROOF. We have

$$a_\alpha(0) + \alpha < 0 \quad , \quad b_\beta(0) - \beta > 0$$

Suppose (5.4.27) is not satisfied, then if $a_\alpha(+\infty) + \alpha < 0$, there exist two values b^1_α, b^2_α with $b^1_\alpha < b^*_\alpha < b^2_\alpha$ solutions of

$$a_\alpha(b) + \alpha = 0$$

and

$$a(b) = a_\alpha(b) \quad \text{for} \quad b < b^1_\alpha \text{ and } b > b^2_\alpha$$

If $a_\alpha(\infty) + \alpha \geq 0$, then $b^2_\alpha = +\infty$.

Similarly, suppose (5.4.28) is not satisfied, then if $b_\beta(-\infty) - \beta > 0$, then there exist two values a_β^1, a_β^2 with $a_\beta^1 < a_\beta^* < a_\beta^2$ solutions of

$$b_\beta(a) - \beta = 0$$

and

$$b(a) = b_\beta(a) \quad \text{for} \quad a > a_\beta^2 \text{ and } a < a_\beta^1$$

If $b_\beta(-\infty) - \beta \leq 0$, then $a_\beta^1 = -\infty$. □

5.4.5. More properties and search procedure. The only cases when $a(b) \neq a_\alpha(b)$ or $b(a) \neq b_\beta(a)$ is when one of the two conditions (5.4.27) and (5.4.28) is not satisfied, and

$$\text{if (5.4.27) is not satisfied, } b_\alpha^1 < b < b_\alpha^2$$

$$\text{if (5.4.28) is not satisfied, } a_\beta^1 < a < a_\beta^2$$

If we consider a point of intersection $\hat{a}, \hat{b}$ and if $\hat{A} = A\left(\hat{a}, \hat{b}\right)$, $\hat{B} = B\left(\hat{a}, \hat{b}\right)$, then we may have $\hat{A} < 0$ and $\hat{B} > 0$, in which case

$$\hat{A} = \hat{a} + \alpha \quad , \quad \hat{B} = \hat{b} - \beta$$

In such a case

$$\hat{a} = a\left(\hat{b}\right) = a_\alpha\left(\hat{b}\right)$$

$$\hat{b} = b(\hat{a}) = b_\beta(\hat{a})$$

and then $\hat{a}$, $\hat{b}$ is also a crossing point of the curves $a_\alpha(b)$ and $b_\beta(a)$.

If $\hat{A} > 0$ then $\hat{B} > 0$ so we have

$$\hat{B} = \hat{b} - \beta \quad , \quad b(\hat{a}) = b_\beta(\hat{a})$$

Moreover $\hat{a} = a\left(\hat{b}\right) \neq a_\alpha\left(\hat{b}\right)$, hence we must have $a_\alpha\left(b_\alpha^*\right) + \alpha > 0$ and the interval b_α^1, b_α^2 is defined (possibly $b_\alpha^2 = +\infty$). Necessarily

$$b_\alpha^1 < \hat{b} < b_\alpha^2$$

Since also $\hat{b} = b\left(\hat{a}\right) = b_\beta\left(\hat{a}\right)$, either $b_\beta\left(a_\beta^*\right) - \beta > 0$, or if $b_\beta\left(a_\beta^*\right) - \beta < 0$, the interval a_β^1, a_β^2 is well defined (possibly $a_\beta^1 = -\infty$) and

$$\hat{a} > a_\beta^2 \quad \text{or} \quad \hat{a} < a_\beta^1$$

The second possibility disappears if $a_\beta^1 + \alpha < 0$ (in particular if $a_\beta^1 = -\infty$).

Similarly if $\hat{B} < 0$ then $\hat{A} < 0$ so we have

$$\hat{A} = \hat{a} + \alpha \quad , \quad a\left(\hat{b}\right) = a_\beta\left(\hat{b}\right)$$

Moreover $\hat{b} = b\left(\hat{a}\right) \neq b_\beta\left(\hat{a}\right)$, hence we must have $b_\beta\left(a_\beta^*\right) - \beta < 0$ and the interval a_β^1, a_β^2 is well defined (possibly $a_\beta^1 = -\infty$). Necessarily

$$a_\beta^1 < \hat{a} < a_\beta^2$$

Since $\hat{a} = a\left(\hat{b}\right) = a_\beta\left(\hat{b}\right)$, then either $a_\alpha\left(b_\alpha^*\right) + \alpha < 0$, or if $a_\alpha\left(b_\alpha^*\right) + \alpha > 0$, then the values b_α^1, b_α^2 are well defined (possibly $b_\alpha^2 = +\infty$) and

$$\hat{b} < b_\alpha^1 \quad \text{or} \quad \hat{b} > b_\alpha^2$$

The second possibility disappears when $b_\alpha^2 > \beta$ (in particular when $b_\alpha^2 = +\infty$).

From this discussion we can define the following search procedure for the pair $\hat{a}$, $\hat{b}$. We consider the curves $a_\alpha\left(b\right)$ and $b_\beta\left(a\right)$.

(1) $a_\alpha\left(b_\alpha^*\right)+\alpha<0$, $b_\beta\left(a_\beta^*\right)-\beta>0$ then $\hat{a}$, $\hat{b}$ is a crossing point of $a_\alpha(b)$, $b_\beta(a)$

(2) Suppose

$$a_\alpha\left(b_\alpha^*\right)+\alpha>0 \quad \text{or} \quad b_\beta\left(a_\beta^*\right)-\beta<0$$

then consider the crossing points of $a_\alpha(b)$, $b_\beta(a)$. If there exists a crossing point $\hat{a}$, $\hat{b}$ such that $\hat{a}+\alpha<0$, $\hat{b}-\beta>0$ then this crossing point is a solution.

(3) Suppose

$$a_\alpha\left(b_\alpha^*\right)+\alpha>0 \quad \text{or} \quad b_\beta\left(a_\beta^*\right)-\beta<0$$

and the crossing points of $a_\alpha(b)$ and $b_\beta(a)$ do not satisfy the conditions of step 2. We consider alternative situations

▷ $a_\alpha\left(b_\alpha^*\right)+\alpha>0$, $b_\beta\left(a_\beta^*\right)-\beta>0$

then $b(a)=b_\beta(a)$.

The interval b_α^1, b_α^2 is well defined ($b_\alpha^2=+\infty$ if $a_\alpha(\infty)+\alpha\geq 0$). We look for a triple $\hat{a}$, $\hat{b}$, $\hat{A}$ such that $\hat{a}<0$, $\hat{A}>0$, $b_\alpha^1<\hat{b}<b_\alpha^2$, solution of a system of 3 nonlinear equations, namely (5.4.17), (5.4.18) and

$$\frac{r\left(\rho_2-\rho_1\right)\left(c^++c^-\right)+(p+h)\left(\rho_2\left(\exp\rho_1 a-1\right)-\rho_1\left(\exp\rho_2 a-1\right)\right)}{h-rc^-}= \tag{5.4.29}$$
$$\left(\varphi(-\beta)+\rho_2\right)\left(\exp\rho_1(a-b)-1\right)-\left(\varphi(-\beta)+\rho_1\right)\left(\exp\rho_2(a-b)-1\right)$$

Necessarily there exists a solution of this system of equations satisfying the constraints, and $\hat{B}=\hat{b}-\beta$.

▷ $a_\alpha\left(b_\alpha^*\right)+\alpha<0$, $b_\beta\left(a_\beta^*\right)-\beta<0$

then $a(b)=a_\alpha(b)$.

The interval $\left(a_\beta^1,a_\beta^2\right)$, is well defined ($a_\beta^1=-\infty$ if $b_\beta(-\infty)-\beta\leq 0$). We look for a triple $\hat{a}$, $\hat{b}$, $\hat{B}$ such that $a_\beta^1<\hat{a}<a_\beta^2$, $\hat{B}<0$, $\hat{b}>0$, solution of a system of 3 nonlinear equations, namely (5.4.19), (5.4.20) and

$$\frac{r\left(\rho_2-\rho_1\right)\left(c^{+}+c^{-}\right)+\left(p+h\right)\left(\rho_2\left(\exp\rho_1 b-1\right)-\rho_1\left(\exp\rho_2 b-1\right)\right)}{p-rc^{+}}= \tag{5.4.30}$$
$$\left(\varphi\left(\alpha\right)+\rho_2\right)\left(\exp\rho_1\left(b-a\right)-1\right)-\left(\varphi\left(\alpha\right)+\rho_1\right)\left(\exp\rho_2\left(b-a\right)-1\right)$$

Necessarily there exists a solution of this system of equations satisfying the constraints, and $\hat{A}=\hat{a}+\alpha$.

(4) Suppose

$$a_\alpha\left(b_\alpha^*\right)+\alpha>0 \quad \text{and} \quad b_\beta\left(a_\beta^*\right)-\beta<0$$

The intervals b_α^1, b_α^2 and a_β^1, a_β^2 are well defined ($b_\alpha^2=+\infty$ if $a_\alpha\left(\infty\right)+\alpha\geq 0$; $a_\beta^1=-\infty$ if $b_\beta\left(-\infty\right)-\beta\leq 0$). The solution $\hat{a}$, $\hat{b}$ belongs to one of two cases: Either $\hat{B}=\hat{b}-\beta$ and $\hat{a}$, $\hat{b}$, $\hat{A}$ are solutions of (5.4.17), (5.4.18), (5.4.29) with the constraints

$$a_\beta^2<\hat{a}<0 \quad \text{or} \quad \hat{a}<a_\beta^1 \quad \text{and} \quad \hat{A}>0 \quad b_\alpha^1<\hat{b}<b_\alpha^2$$

The possibility $\hat{a}<a_\beta^1$ disappears when $a_\beta^1+\alpha<0$. Or $\hat{A}=\hat{a}+\alpha$ and $\hat{a}$, $\hat{b}$, $\hat{B}$ are solutions of (5.4.19), (5.4.20), (5.4.30) with the constraints

$$0<\hat{b}<b_\alpha^1 \quad \text{or} \quad \hat{b}>b_\alpha^2 \quad \text{and} \quad \hat{B}<0 \quad a_\beta^1<\hat{a}<a_\beta^2$$

The above search procedure defines a crossing point $\hat{a}$, $\hat{b}$ of the two curves $a\left(b\right)$ and $b\left(a\right)$.

The solution is easily found if step 1 and 2 apply. In step 3 and 4 a system of 3 nonlinear algebraic equations has to be solved, with a minor limitation on the range of the solution, which the search.

5.5. Application to Mutual Insurance Optimization

Now we consider application of impulse control to mutual insurance optimization, in the context of real-world marine mutual insurance that we have been conducting field study on. Comparing to general (i.e., non-mutual) insurance, a mutual insurance differs in its unique two-way contingent options, namely, contingent calls $\nu^+(t)$ (i.e., contingent fund injection to mutual reserve $x(t)$), and contingent refunds $\nu^-(t)$ (i.e., contingent reduction from mutual reserve). A contingent option incurs a non-zero fixed cost (i.e., $K^+ > 0$ and $K^- > 0$). Using the same notation as above, a mutual insurance optimization can be defined as follows:

$$\begin{cases} V(x) = \inf_{\nu} J_x(\nu) \\ \\ x(t) = x + \mu t + \sigma w(t) + \nu^+(t) - \nu^-(t) \end{cases}$$

where $J_x(\nu)$ is as defined in (5.2.7), $\nu(t) = \nu^+(t) - \nu^-(t)$ is the two-way impulse control, and

$$\nu^+(t) = \sum_i \xi_i \cdot \mathbb{1}_{\xi_i > 0} \mathbb{1}_{\tau_i < t}$$

$$\nu^-(t) = -\sum_i \xi_i \cdot \mathbb{1}_{\xi_i < 0} \mathbb{1}_{\tau_i < t}$$

5.5.1. Numerical Test and Validation. With the numerical methods developed in the preceding section, we are able to conduct for the first time sensitivity analysis of an optimal two-band policy of real-world mutual insurance systems. The system parameters used in this experiment are representative of typical field data of P&I Clubs that we studied, and the basic set of data are selected as follows:

$\sigma = 0.3, \mu = 0, r = 0.06, p = 0.15, h = 0.1, c^+ = 0.2, c^- = 0.2, K^+ = 0.5, K^- = 0.5$

Using the numerical methods we developed in this study, we can compute the optimal two-band policy for the above basic case as $(a, A, B, b) = (-1.359984695, -0.016258323, 0.24848895$ and also the corresponding optimal $V'(x)$ function, as shown in Figure 5.5.1. For the purpose of validation and verification of properties we obtained in Section 4, we then computed for the above basic case the curves of $a_\alpha(b)$ and $b_\beta(a)$, as shown in figure 5.5.2, from which we can verify that the curves satisfy all the properties we derive in preceding sections. Specifically, the crossing point is $\left(\hat{a}, \hat{b}\right)$ with $\hat{a} < -\alpha$ and $\hat{b} > \beta$.

FIGURE 5.5.1. The solution$V'(x)$ for the basic case

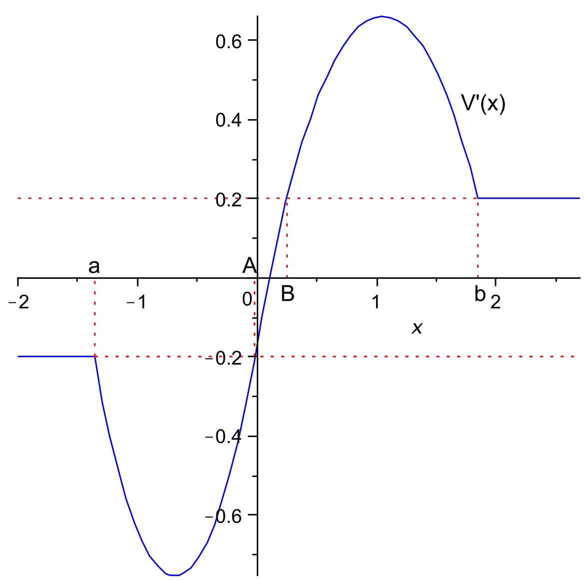
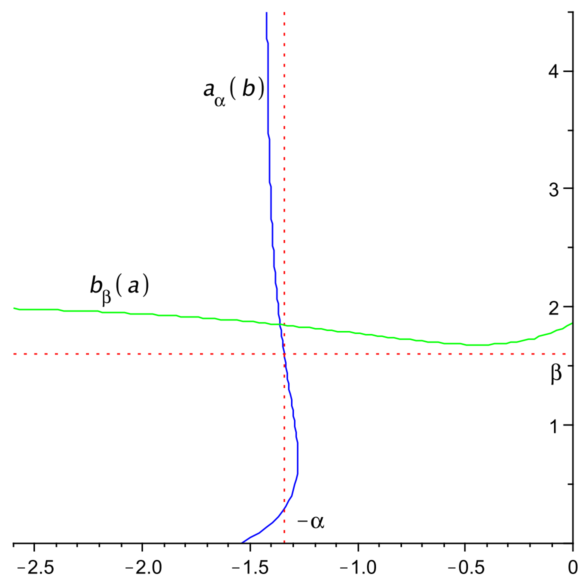


FIGURE 5.5.2. The crossing of $a_\alpha(b)$ and $b_\beta(a)$ gives the solution



For the sensitivity analysis on the system parameters, we first tested on the effect on optimal two-band policy under different values of drift μ. The numerical test results on the sensitivity of μ are summarized in Table 1.

TABLE 1. Sensitivity of Two-Band Policy with Respect to Drift μ

μ	-0.2	-0.1	0	0.1	0.2
b	3.467747478	2.563076507	1.850891187	1.585328554	1.658267126
B	1.569309222	0.875087411	0.248488952	-.1240502468	-.3892841203
A	0.7142289391	0.3748215795	-0.016258323	-.465740742	-.958488203
a	-1.178876904	-1.158125725	-1.359984695	-1.864303490	-2.500777449
$b-B$	1.898438	1.687989	1.602402	1.709379	2.047551
$A-a$	1.893106	1.532947	1.343726	1.398563	1.542289
$b-a$	4.646624	3.721202	3.210876	3.449632	4.159045
$\lvert\frac{b}{a}\rvert$	2.941569	2.213125	1.360965	0.850360	0.663101
$\lvert\frac{b}{b-a}\rvert$	0.746294	0.688776	0.576444	0.459565	0.398713
$\lvert\frac{a}{b-a}\rvert$	0.253706	0.311224	0.423556	0.540435	0.601287

The band width $(b - a\ )$ gives an optimal amount of total insurance reserve, which shall be maintained at minimum so long as to adequately cover the claims. From the Table 1 above, it is observed that the optimal total reserve $b - a$ reaches its minimum as drift μ approaches to zero. We notice that in the context of mutual insurance the drift μ represents the net income of reserve, i.e., premium minus claim. It suggests that an optimal premium policy is to set the premium rate equal to the average rate of claims.

We then test on sensitivity with respect to volatility factor (i.e., disturbance coefficient) σ, and obtained the results of sensitivity as summarized in Table 2 below:

TABLE 2. Sensitivity of Two-Band Policy with Respect to Volatility σ

σ	0.1	0.2	0.3	0.4	0.5
b	0.9502412424	1.429492145	1.850891187	2.239340879	2.605747546
B	0.0873056200	0.167030191	0.248488952	0.332075199	0.417647959
A	0.01201089395	0.001819393475	-0.016258323	-0.039649499	-0.066947517
a	-.6897591884	-1.047628992	-1.359984695	-1.646711267	-1.916331439
$b - B$	0.862936	1.262462	1.602402	1.907266	2.188100
$A - a$	0.701770	1.049448	1.343726	1.607062	1.849384
$b - a$	1.640000	2.477121	3.210876	3.886052	4.522079
$\lvert \frac{b}{a} \rvert$	1.377642	1.364502	1.360965	1.359887	1.359758
$\lvert \frac{b}{b-a} \rvert$	0.579415	0.577078	0.576444	0.576251	0.576228
$\lvert \frac{a}{b-a} \rvert$	0.420585	0.422922	0.423556	0.423749	0.423772

The numerical tests on the volatility factor σ conform to the intuition that the band width is increasing along with the volatility. That is, higher amount of total reserve is needed to face a more volatile insurance market. Using the concept of safety stock, the finding above conforms to the inventory theory that with the same average demand, higher volatility requires a higher level of safety stock in order to attain the same service level. To illustrate the impact of σ on optimal two-band policy, we compared the two curves of $V'(x)$ respectively for $\sigma = 0.1$ (as blue curve) and $\sigma = 0.5$ (as red curve), as shown in

Figure 5.5.3, and the corresponding two curves of value (cost) function $V(x)$, as shown in Figure 5.5.4. As we can see from Figure 5.5.3 that larger value of $\sigma = 0.5$ (i.e., red curves) results in larger total reserve $b - a$, and from Figure 5.5.4 that a larger σ requires a higher optimal cost function $V(x)$ which reaches its overall minimum at $x = \mu = 0$. We shall note that these numerical curves and results are now obtainable, only due to the properties obtained and then the numerical methods developed in this study, which have not been obtainable previously.

FIGURE 5.5.3. The solution $V'(x)$ for the case with $\sigma = 0.1$,$\sigma = 0.5$

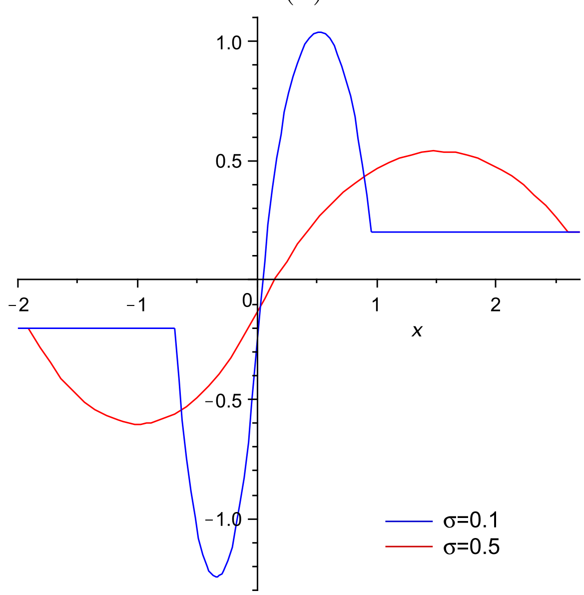
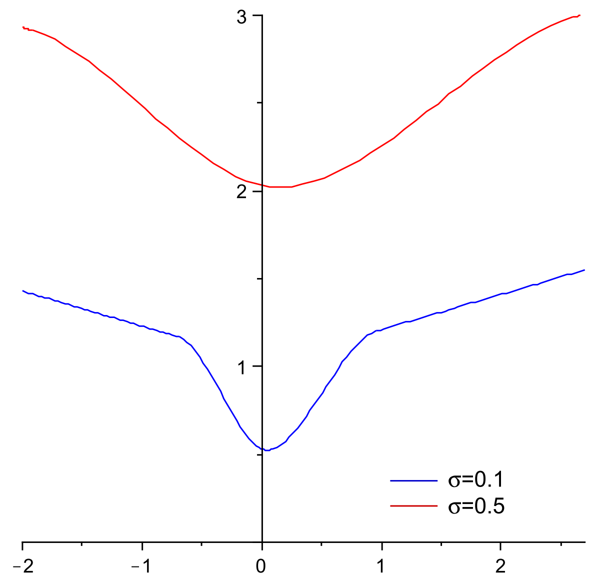


FIGURE 5.5.4. The solution $V(x)$ for the case with $\sigma = 0.1$,$\sigma = 0.5$



Numerical test result on holding cost h, and shortage penalty p are summarized, respectively, in Table 3 and Table 4 below.

TABLE 3. Sensitivity of Two-Band Policy with Respect to Holding Cost h

h	0.06	0.08	0.1	0.12	0.14
b	2.564163050	2.126935831	1.850891187	1.656443694	1.510105743
B	0.499453109	0.346914176	0.248488952	0.178973428	0.127008746
A	0.1033574044	0.03288433214	-0.016258323	-0.056178329	-0.090311753
a	-1.253941193	-1.312411254	-1.359984695	-1.399904701	-1.434038125
$b-B$	2.064710	1.780022	1.602402	1.477470	1.383097
$A-a$	1.357299	1.345296	1.343726	1.343726	1.343726
$b-a$	3.818104	3.439347	3.210876	3.056348	2.944144
$\lvert\frac{b}{a}\rvert$	2.044883	1.620632	1.360965	1.183255	1.053044
$\lvert\frac{b}{b-a}\rvert$	0.671580	0.618413	0.576444	0.541968	0.512918
$\lvert\frac{a}{b-a}\rvert$	0.328420	0.381587	0.423556	0.458032	0.487082

TABLE 4. Sensitivity of Two-Band Policy with Respect to Shortage Penalty p

p	0.11	0.13	0.15	0.17	0.19
b	1.771779794	1.814628628	1.850891187	1.882117240	1.909382930
B	0.169377559	0.212226393	0.248488952	0.279715005	0.306980695
A	-.110508333	-0.056613961	-0.016258323	0.01537187478	0.04263756519
a	-1.645629420	-1.483918570	-1.359984695	-1.261309449	-1.180438785
$b-B$	1.602402	1.602402	1.602402	1.602402	1.602402
$A-a$	1.535121	1.427305	1.343726	1.276681	1.223076
$b-a$	3.417409	3.298547	3.210876	3.143427	3.089822
$\lvert\frac{b}{a}\rvert$	1.076658	1.222863	1.360965	1.492193	1.617520
$\lvert\frac{b}{b-a}\rvert$	0.518457	0.550130	0.576444	0.598747	0.617959
$\lvert\frac{a}{b-a}\rvert$	0.481543	0.449870	0.423556	0.401253	0.382041

It is interesting to note from the numerical test results as presented in Table 3 and 4 that the total reserve $b-a$ decreases as either holding cost h increases or shortage penalty p increases; while an increase in holding cost h tends to decrease cash reserve b as compared to credit reserve $|a|$ (i.e., a decreased ratio $|\frac{b}{a}|$), and an increase in shortage penalty p tends to decrease credit reserve $|a|$ (i.e., an increased ratio $|\frac{b}{a}|$).

The test results on the costs related to impulse control, i.e., setup costs $K^+$and K^-, and proportional costs $c^+$and c^-, are presented in Table 5 and Table 6, respectively.

Table 5. Sensitivity of Two-Band Policy with Respect to Setup Costs $K^+$and K^-

	$K^+ = 0.5$		$K^+ = 0.7$	
	$K^- = 0.5$	$K^- = 0.7$	$K^- = 0.5$	$K^- = 0.7$
b	1.850891187	2.062172917	1.878632487	2.089339796
B	0.248488952	0.222786324	0.276230252	0.249953203
A	-0.016258323	-0.028899432	0.01188712155	-0.0005024186647
a	-1.359984695	-1.372625804	-1.519911452	-1.532094123
$b-B$	1.602402	1.839387	1.602402	1.839387
$A-a$	1.343726	1.343726	1.531799	1.531592
$b-a$	3.210876	3.434799	3.398544	3.621434
$\lvert\frac{b}{a}\rvert$	1.360965	1.502356	1.236014	1.363715
$\lvert\frac{b}{b-a}\rvert$	0.576444	0.600377	0.552776	0.576937
$\lvert\frac{a}{b-a}\rvert$	0.423556	0.399623	0.447224	0.423063

Table 6. Sensitivity of Two-Band Policy with Respect to Proportional Costs $c^+$and c^-

	$c^+ = 0.2$		$c^+ = 0.4$	
	$c^- = 0.2$	$c^- = 0.4$	$c^- = 0.2$	$c^- = 0.4$
b	1.850891187	2.067982222	1.890232222	2.106006479
B	0.248488952	0.367620930	0.287829987	0.405645187
A	-0.016258323	-0.037292575	-0.089842149	-.109820302
a	-1.359984695	-1.381018947	-1.481362467	-1.501340620
$b-B$	1.602402	1.700361	1.602402	1.700361
$A-a$	1.343726	1.343726	1.391520	1.391520
$b-a$	3.210876	3.449001	3.371595	3.607347
$\lvert\frac{b}{a}\rvert$	1.360965	1.497432	1.276009	1.402751
$\lvert\frac{b}{b-a}\rvert$	0.576444	0.599589	0.560634	0.583810
$\lvert\frac{a}{b-a}\rvert$	0.423556	0.400411	0.439366	0.416190

CHAPTER 6

Conclusion and Future Work

The impulse models developed in this study are either a pure impulse control system or a hybrid system (i.e. with a mix of impulse and continuous controls), and are associated with an HJB-QVI system. The value function of the these models is the solution to the associated HJB-QVIs with free boundary conditions, from which optimal controls in both continuous and impulse type, as well as optimal order release policy, can be determined. For mutual insurance control problems, Band-Type Contingent Option (BTCO) type policy is proved to be optimal policy, and we introduce efficient numerical algorithm to get the optimal solution. The efficiency is evidenced by experiment examples.

The results obtained in this study are applicable to a wide range of applications, including production inventory systems and cash reserve management. The key contribution of this study relates to derivation of much simplified QVI characteristics and development of computational QVI solution algorithms for mutual insurance optimization. The QVI characteristics and properties obtained in this study lead to some major developments in optimization of mutual insurance, such as: 1) an optimal insurance reserve policy consists of maintaining a combination of both cash reserve of amount $b > 0$ and credit reserve of amount $a < 0$ (e.g. reinsurance and credit loans). This finding can be lively witnessed in the national stimulus measure by US Federal Reserve, namely, a combination of cash injection and issuance of government bonds. 2) A novel computation algorithm is then developed to determine numerically an optimal $(a, A; B, b)$ policy, which is much simplified and effective for solving general reserve regulation problems under non-symmetric cost structure. The techniques presented in this study are particularly adequate to provide solutions to the problems of mutual insurance and general reserves regulation.

We propose in future work to extend the methodology to deal with more general situations, such as a system with a multi-dimension controllable diffusion state. Moreover, we would also investigate situations when there are exogenous impacts on the system. These applications will provide us more accurate formulation of the real world events and control systems. They are more complicated than those we have investigated in this study from mathematical aspect of view, however, we believe that the treatments introduced in this study would also be applicable to those applications.

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