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DEPARTMENT OF INDUSTRIAL AND SYSTEMS ENGINEERING

Cognitive Network Process with Fuzzy Soft Computing Technique in Collective Decision Aiding

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Certificate of Originality

I hereby declare that this thesis is my own work and that, to the best of my knowledge and belief, it produces no material previously published or written, nor material which has been accepted for the award of any other degree or diploma, except where due acknowledge has been made in the text.

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Abstract

The multi-criteria and multi-expert decision aiding models investigate the problems of identifying candidates, analyzing the criteria, and selecting the best alternative(s) based on the aggregation of the perceptions and preferences of the group decision makers. Although many studies have investigated these problems, there are no conclusions as to a single decision model that can dominate others. Among the various well-known models, the Analytic Hierarchy Process (AHP) /Analytic Network Process (ANP) is popular, and is applied in various domains, although there are some limitations. The Cognitive Network Process (CNP) is developed on the improvement of AHP/ANP with the cognitive decision process.

The CNP model is one of the models of the multi-criteria and multi-experts decision aiding. It applies the interdisciplinary techniques of decision sciences, cognitive sciences and fuzzy soft computing, on the basis of the mathematical modeling development.

The cognitive architecture of the CNP is mainly comprised of five processes: Problem Cognition Process (PGP), Cognitive Assessment Process (CAP), Cognitive Prioritization Process (CPP), Multiple Information Fusion Process (MIP), and Decisional Volition Process (DVP). In PGP, decision problems are formed as a Structural Assessment Network (SAN). In CAP, a Compound Linguistic Ordinal Scale (CLOS) model is proposed for the improvement of rating activities of the assessment. In CPP, a Cognitive

Prioritization Operator (CPO) of a Pairwise Opposite Matrix (POM) is proposed to derive the utility set from the POM. In MIP, a Cognitive Style and Aggregation Operator (CSAO) model is proposed for selection of aggregation operators to aggregate the utility sets with respect to the attitudes or cognitive styles of the decision makers. In DVP, a valuation function of the utility sets is used to provide the decision solution.

The framework of CNP includes primitive and extent types. The primitive type is a individual decision making model using linguistic variables represented by crisp numbers. The extent types include the notions of the collective judgments and fuzzy linguistic variables.

The main contribution of the CNP includes the mathematical developments of CLOS, POM, CPO, CSAO, fuzzy POM, and fuzzy CPO. The numerical analyses with the discussions of these concepts are performed respectively. Five cases selected from other publications illustrate the usability and validity of the CNP, with comparisons with the (fuzzy) AHP/ANP, and complementation with other decision models.

Like the impacts of AHP/ANP, the proposed CNP can be applied in many domains such as material management, transportation management, psychometrics, social sciences, business research, decision sciences, computer sciences, and engineering management. The CNP is the ideal alternative of the AHP/ANP.

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Chapter 1 Introduction

1.1 Research background and motivation

In modern scientific decision making, there is a need to weigh, rate and quantify uncertain attributes with numerical determination as there is insufficient information to understand the attributes of the decision problem. These activities of weighing, rating and quantifying are subjective measures. A common finding is that subjective measures have been extremely popular in operational settings due to their high face validity and the ease of data collection (Yeh and Wickens, 1988). Subjective measures are relatively inexpensive to obtain, non-intrusive, convenient, and easy to analyze (Liou and Wang, 1994). Subjective measures are essential for the input of the decision system. The decision systems have been studied by many authors, most of which are listed in the web site of International Society on Multiple Criteria Decision Making (2009). The significant models of decision making, rating scales, and the techniques including soft computing and cognition are reviewed in chapter 2.

1.2 Research problems and gaps

The design of suitable linguistic terms for subjective measures is essential to the accuracy of the evaluation. In the cases of the general evaluation processes, the experts design the criteria with numerical grades from 1 to 5 in a questionnaire. Usually, they just provide the numbers with linguistic terms, as in the Likert Scale (Likert, 1932), such as 5 for strongly agree, 4 for agree, 3 for neutral, 2 for disagree, and 1 for strongly disagree. When the raters observe the possible choices, the brain will process the external information, and then choose one from the predefined options to describe their perceptions. This induces seven fundamental questions:

a) whether numbers are appropriate to represent the actual measurement of human thinking - as human judgment unlikely provides accurate numerical determination. In the real-world, the uncertainty, constraints, and even the vague knowledge of the experts imply that decision makers cannot provide exact numbers to express their opinions (Ben-Arieh and Chen, 2006). The use of linguistic labels makes expert judgment more reliable and consistent (Ben-Arieh and Chen, 2006).

b) whether numbers are appropriate to represent these linguistic terms. In fact, better designed survey forms apply adjectives associated with numbers. In the above example, why are all the intervals between the adjacent terms equal to 1? What are the arguments that the interval between neutral and agree is equal to the interval between agree and disagree? The commonly used Likert categories are not necessarily evenly spaced along this level of agreement continuum, although researchers frequently assume that they are (Blaikie, 2003). In fact, the mapping from the linguistic domain to the numerical domain seems not to be defined in any theoretical mathematical models. The questions of the semantics of a language for the representation of measurement results are not yet clear (Muravyov and Savolainen, 1997).

c) whether there is a sufficient number of options for the rating to distinguish the difference. Is 5 enough? Or 9? Or even more? Miller (1956) indicated that an expert could manage a set with (7 ± 2) terms while Bonissine et al (1986) pointed out that one could manage up to 11 or 13 terms. This is open to discussion as 11 or 13 terms are excessive for raters to make decisions. Chapter 4 proposes a Compound Linguistic Ordinal Scale (CLOS) model which can handle (7±2)((7±2)-1)+1=[21,73] linguistic terms and more.

d) whether the decision makers or auditors rethink their choices. Using precise values to rate the fuzzy environment is a single thinking step, which is regarded as hasty decision making, and usually induces excessively subjective benchmarking.

e) whether the same rating categories can be applied to every question. In the above example, can the five linguistic terms apply to every question in the whole questionnaire?

f) whether each question is of the same weight if there are multi-criteria decisions. In other words, is each question of the same importance?

g) whether people's votes all carry the same weight if there are multi-expert decisions.

For example, is there any difference in the opinion of someone with five-years' working experience and the vote of someone without any experience, or just one year's experience, when it comes to evaluating work-related objects?

h) whether a pairwise rating method is superior to a direct rating method?

i) which aggregation method is the most preferable as different aggregation operators produce different results.

It seems there is a lack of theoretical mathematical models (as the literature does not contain any related mathematical model for modeling the distribution of the linguistic terms for the terms set in a matrix) associated with an appropriate management framework for addressing the questions (a) to (e).

To address the problems (a) to (e), and typically (b), adopting a statistical model may be the correct approach. However, to allocate the right number to each linguistic term is time consuming. Additionally it is expensive to acquire the sample data, and the result is usually not universally applied. This can be seen in the research from Hakel (1968) and a comparison with Simpson (1944) who investigated twenty modifying words such as usually, often, sometimes, occasionally, seldom, and rarely and commonly.

A probability model, such as the Gaussian normal distribution, may help in finding the solution for the linguistic terms in discrete order such as "…, little below good, absolutely good, little above good, …". A fundamental assumption of probability is entailed in the axiom of additivity where all events that satisfy specific properties must add up to one. This assumption forces the conclusion that the probability of an event occurring necessarily entails knowledge of the remaining events. Therefore the boundary of each event is crisp, and cannot be fuzzy. This articulates the challenge of measuring the fuzziness associated with an expert judgment, as a probability model is not appropriate for finding the distribution of linguistic terms**.** However, no research is found to deal with this case using fuzzy theory. Chapter 4 proposes the Compound Linguistic Ordinal Scale model to address these issues.

For the question (f) to (h), the popular tool to produce the weights of the criteria or experts is the pairwise comparison of the AHP (Saaty, 1980). Yuen and Lau (2009) have compared the direct rating method with AHP, and concluded that the approximated value of the Pairwise Reciprocal Matrix (PRM) of AHP is questionable. On the other hand, the direct rating method is excessively subjective. Chapter 5 proposes the Cognitive Prioritization Operator (CPO) and the Pairwise Opposite Matrix (POM) to address these issues

For question (i), the weighted arithmetic mean is the most popular operator (e.g.

AHP, SMART, ELECTRE, DEA and PROMETHEE in chapter 2.2). The alternative is the geometric mean (e.g. generalized means in chapter 2.2). However, they may produce different results. Chapter 6 proposes the Cognitive Style and Aggregation Operator (CSAO) model to address this issue.

1.3 Research objectives

To fill the above gaps, this research proposes the Cognitive Network Process model comprising the key concepts of the Compound Linguistic Ordinal Scale (CLOS), the Cognitive Prioritization Operator (CPO) , the Pairwise Opposite Matrix (POM), and the Cognitive Style and Aggregation Operator (CSAO).

The framework of CNP includes the primitive and extent types. The primitive type is the individual decision making model using the linguistic variable represented by a crisp number. The extent type is the collective decision making model using the linguistic variable represented by a fuzzy number.

The Cognitive Network Process concerns the improvement of the analytical network process in view of the novel definitions for the cognitive process, including the sensation and perception of the problems from the experts by using CLOS, CPO, POM, and CSAO. The CNP should be an ideal framework to improve the complex decision making process. For the impacts, like the impacts of AHP/ANP, the proposed CNP can be applied in many domains such as material management, inventory management, transportation management, psychometrics, social sciences, business research, decision sciences, computer sciences, and engineering management. The CNP is the ideal alternative for AHP/ANP.

1.4 Organization of the research

The thesis is divided into nine chapters. The outlines of the remaining chapters are as follows:

i. **Chapter 2** reviews the fundamental concepts as the basis for the development of the Cognitive Network Process (CNP) model. The CNP is the model which mainly intends to address the limitations of the Analytic Hierarchy Process (AHP) /Analytic Network Process (ANP). Thus two sections for AHP/ANP and fuzzy AHP are presented in detail in this chapter after some key concepts and other well-known decision models are reviewed: Multi-attribute Utility Theory (MAUT), Simple Multi-attribute Rating Technique (SMART), generalized means, preference relation, ELECTRE, PROMETHEE, DEA, and TOPSIS. The CNP is the broad concept, rather than the primitive CNP, on the development of the improvement of ANP. The concepts of soft computing and cognitive sciences are also reviewed to extend the notion of the primitive CNP. The reviews of soft computing include fuzzy linguistic variables, membership functions, basic operations of fuzzy sets, and aggregation operators, whilst the notions of cognitive sciences includes the topics of cognitive psychology, cognition and decision making, perception and computation intelligence, cognitive style and decision making, and cognitive architecture and intelligent decision agent systems. The topics of the rating scale and measurement are also reviewed: the definitions of measurement and scale, Likert-like scales, syntactic forms, and computation rules.

ii. **Chapter 3** discusses the cognitive architecture of the cognitive network process, which includes definitions, algorithms and formulations of the general notations of the process algebra of CNP. It states various high motivations for further discussion in the following chapters 4-8.

iii. **Chapter 4** proposes the Compound Linguistic Ordinal Scale (CLOS) Model. Rating Scale Models (RSMs) have been applied in survey or questionnaire applications in various research areas. Their rating interfaces, however, possibly lead to problems concerning the choices of linguistic terms, accuracy of linguistic representation of numbers and decisions in rating dilemmas. To address the above problems, this chapter proposes a CLOS Model, which is an ordinal-in-ordinal scale model, as a promised alternative for the classic RSMs, which provide usually 7±2 options. CLOS, which provides (7±2)((7±2)-1)+1=[21,73] options or more, is a

Deductive Rating Strategy (DRS) of the Hedge-Direction-Atom Linguistic Representation Model (HAD LPM), with a cross reference relationship. The simulation result indicates that the proposed model helps to reduce the bias of the rating dilemma for a single rater and more accurately reflects consistency among raters. The contribution is that the model can be applied in large scale systems, surveys and questionnaires, psychometrics and collective multi-criteria decision problems of various fields.

iv. **Chapter 5** proposes the cognitive prioritization operators (CPOs) of the pairwise opposite matrices (POMs). The pairwise reciprocal matrix of AHP has been studied by many scholars. However, there are significant queries about the appropriateness of using the pairwise reciprocal matrix (PRM) to represent the pairwise comparison. This research proposes the POM as the ideal alternative with respect to the human linguistic cognition of the rating scales of the comparison. Several cognitive prioritization operators (CPOs) are proposed to derive the individual utility vector (or priority vector) of the pairwise opposite matrix. Not only are the rigorous mathematical proofs of the new models demonstrated, but solutions of the CPOs are also illustrated by the presentation of graph theory. The measurement models of the CPOs for the POM are also developed. The comprehensive numerical analyses show the validity of CPOs and how the PRM is superior to the POM. POM and

CPOs, which correct the fallacy of the PRM associated with the prioritization operators, should be the ideal models for multi-criteria decision making problems in various fields.

v. The selection of the aggregation operators can be determined by the cognitive style. **Chapter 6** proposes a Cognitive Style and Aggregation Operator (CSAO) model to analyze the mapping relationship between aggregation operators and cognitive styles represented by the decision attitudes. The numerical examples illustrate how decision attitudes of the aggregation operators can be determined by the selection strategy of CSAOs I and II. The CSAO model can be applied in decision making systems with the selection problems of the appropriate aggregation operators with considerations of decision attitudes.

vi. The narrow definition of the CNP is of a single decision maker, and the compound linguistic variable in the crisp value. This is called primitive CNP. **Chapter 7** extends the concept of CNP, and proposes a broader definition of CNP, which is named the fuzzy collective cognitive network process (FCCNP). FCCNP is of multiple decision makers with fuzzy inputs. FCCNP can be divided as the collective CNP (CCNP) and fuzzy CNP (FCNP). CCNP is of multiple decision makers with crisp inputs whilst FCNP is of a single decision maker with fuzzy inputs. The numerical analysis is performed to validate the essential functions, i.e. the fuzzy prioritization operators.

vii. In **chapter 8**, five cases are illustrated and discussed. Case 1 presents the high school section (Saaty, 1980, p26-28) with comparisons of primitive CNP and AHP models. Case 2 presents the transportation company selection problem (Kulak and Kahramna, 2005) with comparisons of the primitive CNP and AHP, and both prioritization measurement models are also used. Case 3 compares the CNP and the improved ANP models for the R&D project selection problem (Yuen and Lau, 2009). Case 4 compares the fuzzy CNP and Fuzzy AHP models for the software product selection problem (Yuen and Lau, 2008c). Case 5 illustrates the use of the fuzzy collective CNP model as the evaluation model for the problem of supplier number optimization (Berger et al, 2004).

viii. Finally, **chapter 9** concludes the work undertaken. The contributions and the motivations of this research are also presented.

Chapter 2 Literature Review

2.1 Introduction

This chapter reviews the fundamental concepts as the basis for the development of the Cognitive Network Process (CNP) model. The CNP is the model which mainly intends to address the limitations of the Analytic Hierarchy Process (AHP) /Analytic Network Process (ANP). Thus two sections for AHP/ANP and fuzzy AHP are presented in detail in this chapter after some key concepts and other well-known decision models are reviewed. Unless specified, in this chapter, the prioritization operator of AHP means analytic prioritization operator, and pairwise comparison or pairwise matrix means the analytic pairwise comparison and analytic pairwise matrix. However, these names are not for other chapters which imply cognitive ones of the CNP. The CNP is the broad concept rather than the primitive CNP on the development of the improvement of ANP. The concepts of soft computing and cognitive sciences are also reviewed to extend the notion of the primitive CNP. The framework CNP is an interdisciplinary notion. It can be incorporated with other decision models. Thus some decision models, which potentially can be fused with the CNP, are also reviewed.

In some sections, some new equations are proposed by the author to bridge the gap for some concepts, especially for the use of the comparison with CNP model, after the key concepts are reviewed. Particularly in the AHP and Fuzzy AHP sections, the new functions are used for the comparisons in Chapter 8.

The structure of this chapter is as follows. Chapter 2.2 reviews the concepts of Multi-Criteria Decision Aiding (MCDA) including Multi-attribute Utility Theory (MAUT), Simple Multi-attribute Rating Technique (SMART), generalized means, preference relation, ELECTRE, PROMETHEE, TOPSIS and Data Envelopment Analysis (DEA). Chapter 2.3 presents the essential reviews of soft computing including fuzzy linguistic variables, membership functions, basic operations of fuzzy sets, and aggregation operators. Chapter 2.4 presents the reviews of the rating scales and measurement including the definitions of measurement and scale, Likert-like scales, syntactic forms, and computation rules. Chapter 2.5 presents the concepts of AHP/ANP in depth including the pairwise (reciprocal) matrix, (analytical) prioritization operators (AOs), synthesis methods, and measurement models for POs. Chapter 2.6 reviews the two types of Fuzzy Analytic Hierarchy Process, Extent Analysis Method (EAM) and modified fuzzy Logarithmic Least Squares Method (mf-LLSM), and the fuzzy measurement models are also proposed for comparing these two methods, as the literature review did not reveal any measurement models of the fuzzy POs. Finally, chapter 2.7 briefly reviews the notions of cognitive sciences including the topics of cognitive psychology, cognition and decision making, perception and computation intelligence, cognitive style and decision making, and cognitive architecture and intelligent decision agent system.

2.2 Multi criteria decision aiding

The Multi-Criteria Decision Aiding (MCDA) or Multi-Criteria Decision Making (MCDM) models have been studied extensively. The huge bibliography of MCDA research can be found in the web site of the International Society on Multiple Criteria Decision Making (2009). The recommended textbooks are in (Ozturk et al., 2005). This section reviews and presents only the essential concepts and selected models related to the CNP framework.

Prior to introduction of MCDA models, a decision matrix $\bar{O}$ is defined in the following form:

$$\bar{O} = \begin{array}{c c} & \begin{array}{ccccc} (w_1 & \cdots & w_j & \cdots & w_n) \\ c_1 & \cdots & c_j & \cdots & c_n \end{array} \\ \begin{array}{c} T_1 \\ \vdots \\ T_i \\ \vdots \\ T_m \end{array} & \left(\begin{array}{ccccc} & & & & \\ & & & & \\ & & r_{ij} & & \\ & & & & \\ & & & & \end{array} \right) \end{array} \tag{2.2.1}$$

$c_j \in C$ is the criterion. $r_{ij} \in r$ is the rating score or utility values from the rating. $w_j \in W$ is the weight of the criterion c_j, and usually is normalized, ie. $\sum_{j=1}^{n} w_j = 1$. $T_i \in T$ is the alternative. $\bar{O}$ can be written in its transposition form in other study. Both styles have appeared in the literature.

2.2.1 MAUT

Most of the MCDA models apply Multi-attribute Utility Theory (MAUT). The basis of MAUT is the use of utility functions. Utility functions can be applied to transform the raw performance values of the alternatives against diver criteria, both factual (objective, quantitative) and judgmental (subjective, qualitative), to a common, dimensionless scale (Fulop, 2005). The CNP is developed on the basis of MAUT.

2.2.2 SMART

The Simple Multi-attribute Rating Technique (SMART) is the weight arithmetic mean of the set of rating values $\{r_{ij}\}$ of the alternatives in a general decision matrix. It has the form:

$$T_i = \frac{\sum_{j=1}^{m} w_j r_{ij}}{\sum_{j=1}^{m} w_j}, \quad i = 1, \ldots, n \qquad (2.2.2)$$

, where r_{ij} is the utility value or performance value.

2.2.3 Generalized means

The generalized means model is the weighted geometric mean of the ranking values of the alternatives in a general decision matrix. It has the form:

$$`T_i = \prod_{j=1}^{m} r_{ij}^{\frac{w_j}{\sum_{j=1}^{m} w_j}} \quad , \quad i = 1, \ldots, n \tag{2.2.3}$$

The difference of SMART and generalized means is only in the aggregation operator definition. If each aggregation operator (AO) produces one type of decision model, there are many MCDA models, as Chapter 2.3.4 reviews a number of AOs. Chapter 6 proposes the Cognitive Style of Aggregation Operator (CSAO) model to utilize the AOs. In the default setting, like AHP, the weighted arithmetic mean (*wam*) is applied.

2.2.4 Axioms of preference relation

According to the review of Ozturk et al. (2005), the notion of binary relations appears for the first time in De Morgan's study (1864). It is defined as a set of ordered pairs in Peirce's works (1880, 1881, 1883). Some of the first work dedicated to the study of preference relations can be found in (Dushinik and Miller, 1941; Sott and Suppes, 1958). More general concepts for models of arbitrary relations will be introduced in (Tarski, 1954, 1955).The following notations adopts Rouben and Vincke's work (1985) with a modification that fits for the notation system of CNP (Chapter 3).

Considering one cluster $Clst\left(nd, \{gn_i\}\right)$, *nd* is the node, $gn = \{gn_i\}$ is the set of granular data of *nd*. The following definition holds.

Definition 2.1 (Binary Relation): Let gn be a finite set of elements ($gn_1, gn_2, \ldots gn_z$), a binary relation R on the set gn is a subset of the Cartesian product $gn \times gn$, that is, a set of order pairs, i.e. $\{(gn_j, gn_k)\}$ such that gn_j and gn_k are in $gn : R \subseteq gn \times gn$.

For an ordered pair (gn_j, gn_k) which belongs to *R*, the notation is indifferently of the form as follows:

$$(gn_j, gn_k) \in R \quad \text{or} \quad gn_j\ R\ gn_k \quad \text{or} \quad R(gn_j, gn_k) \tag{2.2.4}$$

Let *R* and *T* be two binary relations on the same set gn. The set operations are of the form:

Inclusion: $R \subseteq T$ iff $gn_j\ R\ gn_k \rightarrow gn_j\ \mathrm{T}\ gn_k$; (2.2.5)

Union: $gn_j\ (R \cup T)\ gn_k$ iff $gn_j\ R\ gn_k$ or $gn_j\ \mathrm{T}\ gn_k$; (2.2.6)

Intersection: $gn_j\ (R \cap T)\ gn_k$ iff $gn_j\ R\ gn_k$ and $gn_j\ \mathrm{T}\ gn_k$; (2.2.7)

Relative Product: $gn_j\ (R.T)\ gn_k$ iff $\exists gn_i \in$ gn: $(gn_j\ R\ gn_i)$ and $(gn_i\ \mathrm{T}\ gn_k)$;

(2.2.8)

The properties of the asymmetric R^a, the symmetric R^s and the complementary part $R^{'}$ of the binary relation *R* are shown as follows:

$$gn_j\ R^a\ gn_k \text{ iff } gn_j\ R\ gn_k \text{ and } \neg(gn_k\ R\ gn_j); \tag{2.2.9}$$

$$gn_j\ R^s\ gn_k \text{ iff } gn_j\ R\ gn_k \text{ and } gn_k\ R\ gn_j; \tag{2.2.10}$$

$$gn_j\ R^{'}\ gn_k \text{ iff } \neg(gn_j\ R\ gn_k) \text{ and } \neg(gn_k\ R\ gn_j) \tag{2.2.11}$$

The complement R^c, the converse (the dual) R^d, and the co-dual R^{cd} of R are the forms respectively:

$$gn_j \ R^c \ gn_k \ \text{ iff } \ \neg\left(gn_j \ R \ gn_k\right); \tag{2.2.12}$$

$$gn_j \ R^d \ gn_k \ \text{ iff } \ gn_k \ R \ gn_j; \tag{2.2.13}$$

$$gn_j \ R^{cd} \ gn_k \ \text{ iff } \ \neg\left(gn_k \ R \ gn_j\right); \tag{2.2.14}$$

More relations of R are illustrated as follows:

Reflexive, if $gn_j \ R \ gn_k, \ \forall gn_j \ \in gn$; (2.2.15)

Irreflexive, if $gn_j \ R^c \ gn_j, \ \forall gn_j \ \in gn$; (2.2.16)

Symmetric, if $gn_j \ R \ gn_k \rightarrow gn_k \ R \ gn_j, \forall gn_j, gn_k \ \in gn$; (2.2.17)

Antisymmetric, if $\left(gn_j \ R \ gn_k, gn_k \ R \ gn_j\right) \rightarrow gn_j = gn_k, \ \forall gn_j, gn_k \ \in gn$; (2.2.18)

Asymmetric, if $gn_j \ R \ gn_k \rightarrow gn_k \ R^c \ gn_j, \ \forall gn_j, gn_k \ \in gn$; (2.2.19)

Complete, if $\left(gn_j \ R \ gn_k \ \ or \ \ gn_k \ R \ gn_j\right), \ \forall gn_j \neq gn_k \ \in gn$; (2.2.20)

Strongly complete, if $gn_j \ R \ gn_k \ \ or \ \ gn_k \ R \ gn_j, \ \forall gn_j, gn_k \ \in gn$; (2.2.21)

Transitive, if, $\left(gn_j \ R \ gn_i, gn_i \ R \ gn_k\right) \rightarrow gn_j \ R \ gn_k, \ \forall gn_i, gn_j, gn_k \ \in gn$; (2.2.22)

Negatively transitive $\left(gn_j \ R^c \ gn_i, gn_i \ R^c \ gn_k\right) \rightarrow gn_j \ R^c \ gn_k, \forall gn_i, gn_j, gn_k \ \in gn$;

(2.2.23)

Negative transitive, if $gn_j \ R \ gn_k \rightarrow \left(gn_j \ R \ gn_i, gn_i \ R \ gn_k\right), \ \forall gn_i, gn_j, gn_k \ \in gn$;

(2.2.24)

Semitransitive, if $\left(gn_j \ R \ gn_i, gn_i \ R \ gn_k\right) \rightarrow \left(gn_j \ R \ gn_{i'}, gn_{i'} \ R \ gn_k\right)$,

$$\forall gn_i, gn_{i'}, gn_j, gn_k \in gn; \tag{2.2.25}$$

Ferrers relation, if $\left(gn_j\ R\ gn_i, gn_{i'}\ R\ gn_k\right) \rightarrow \left(gn_j\ R\ gn_{i'}, gn_i\ R\ gn_k\right)$,

$$\forall gn_i, gn_{i'}, gn_j, gn_k \in gn. \tag{2.2.26}$$

Iff R is reflexive, symmetric and transitive, R is equivalent relation E of the form:

$$gn_j \ \mathrm{E}\ gn_k \ \text{ iff } \ \forall gn_j \in gn \begin{cases} gn_j \ \mathrm{R}\ gn_i \rightarrow gn_k \ \mathrm{R}\ gn_i \\ gn_i \ \mathrm{R}\ gn_j \rightarrow gn_i \ \mathrm{R}\ gn_k \end{cases}; \tag{2.2.27}$$

The binary preference relation with utility theory is of the form:

$$u\left(gn_j\ R\ gn_k\right) = \begin{cases} 1 & \mathit{iff} \ \ gn_j\ R\ gn_k = \mathit{true} \\ 0 & \mathit{iff} \ \ gn_j\ R\ gn_k = \mathit{false} \end{cases}; \tag{2.2.28}$$

, where $\{1,0\}$ is the set of the utility values.

Similarly, the discrete preference with the discrete utility is of the form:

$$u\left(gn_j\ R_i\ gn_k\right) = v_i, \text{ where } \ v_i \in v : R_i \in \left\{R_i\right\}, v = \left(v_1, v_2, \ldots, v_n\right) \tag{2.2.29}$$

The continuous preference with the continue utility of the form:

$$u\left(gn_j\ R_i\ gn_k\right) = v', \quad \text{where } \ v' = f\left(R_i\right) \in \left[0,1\right]. \tag{2.2.30}$$

The continue interval $[0,1]$ can be other intervals. This is only the scaling problem. Out ranking methods, including ELECRE, AHP and CNP, are based on this combination of MAUT and preference model. The difference is the definition of the form, $u\left(gn_j\ R\ gn_k\right)$, including the scale values and operations.

The preference model can be shown in a matrix which is shown in following example.

Example 2.1

Let R be a binary relation defined in a set of granular data $gn = [gn_1, gn_2, gn_3, gn_4]$, which is from nd. If the set of the relation, R'={(gn_1, gn_2), (gn_2, gn_3), (gn_2, gn_4), (gn_3, gn_1), (gn_4, gn_2)}, is true and others pairs are false, the matrix representation of $R(Clst(nd, gn))$ is illustrated as follows:

$$R(Clst(nd,gn)) = \begin{array}{c} \\ gn_1 \\ gn_2 \\ gn_3 \\ gn_4 \end{array} \begin{array}{c} \begin{array}{cccc} gn_1 & gn_2 & gn_3 & gn_4 \end{array} \\ \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{24} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} \end{array} = \begin{array}{c} \\ gn_1 \\ gn_2 \\ gn_3 \\ gn_4 \end{array} \begin{array}{c} \begin{array}{cccc} gn_1 & gn_2 & gn_3 & gn_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

In CNP, the utility value (or performance value) is rated from the matrix of Compound Linguistic Ordinal Scale (CLOS) (chapter 4). The CNP applies the pairwise opposite matrix which is the preference function of the following definition using preference model notation:

$$u(gn_j \text{ R } gn_k) = \begin{cases} b^+ & , R = \succ \\ 0 & , R = \sim ; \\ b^- & , R = \prec \end{cases} \tag{2.2.31}$$

$\succ$ means the positive preference, $\prec$ means the negative preference, and $\sim$ means the equal preference. The level of the preference is rated from a score. b^+ is the positive number from the discrete interval scale derived from CLOS. b^{-} is the negative number from the opposite of the above CLOS. Details are in chapters 3, 4,5, and 7 respectively.

2.2.5 ELECTRE

The ELECTRE, ELimination Et Choix Traduisant la REalité, (ELimination and Choice Expressing REality), initially appeared in a French operations research journal by Roy (1968). The development of ELECRE methods lasts about four decades. The key version is shown in table 2.1. This section does not intend to review all versions of ELECRE, but only the most fundamental one, ELECRE I. The details of the ELECRE family can be referred to in the reviewed references (Pomerol and Barba-Romero, 2000; Figueira et al., 2005b; Bouyssou, 2006). The presentation of ELECTRE of Figueira et al. (2005b) is applied, but is modified to fit for the notation system of CNP. In fact, the Pairwise Opposite Matrix of CNP can be applied for determination of the weights of ELECTRE.

Considering $Clst\left(nd,\{gn_i\}\right)$, let the preference relation R be $\succsim$, where $gn_j \succsim gn_k$ means gn_j outranks gn_k (that is , "gn_j is at least as good as gn_k"). The Concordance index β and the discordance index δ are two essential concepts in ELECTRE.

Table 2.1: The various version of ELECTRE (Pomerol and Barba-Romero, 2000, p184)

Version	First reference	Type of criterion	Weights required	Fuzzy	Type of problem
I	Roy (1968)	simple	yes	No	selection
II	Roy and Bertier (1973)	simple	yes	little	ranking
III	Roy (1978)	Psudo	yes	yes	ranking
IV	Roy & Hugonnard (1982)	Psudo	no	No	ranking
IS	Roy and Skalla (1985)	psedo	yes	No	selection

For an ordered pair of granular information $\left(gn_j, gn_k\right)$, the concordance index is of the form:

$$\beta_{jk} = \beta\left(gn_j \succsim gn_k\right) = \sum_{\{i:g_i(gn_j)\geq g_i(gn_k)\}} w_i \tag{2.2.32}$$

, where $\sum_{i=1}^{z} w_i = 1$, and $\left\{i : g_i\left(gn_j\right) \geq g_i\left(gn_k\right)\right\}$ is the set of indices for all criteria belonging to the concordant coalition with the outranking relation, i.e. $gn_j \succeq gn_k$.

The value of the concordance index β_{jk} must be greater than or equal to a given concordance level, s, whose value generally falls within the range $\left[0.5, 1-\min\left(\left\{w_j\right\}\right)\right]$, i.e., $\beta_{jk} \geq s$

On the other hand, the discordance is measured by a discordance level defined as follows:

$$\delta_{jk} = \delta\left(gn_j \succsim gn_k\right) = \max_{\{i:g_i(gn_j)<g_i(gn_k)\}} \left\{g_i\left(gn_k\right) - g_i\left(gn_j\right)\right\} \tag{2.2.33}$$

The power of the discordance means that if its value surpasses a given level, v, the assertion is no longer valid. Discordant coalition expects no power whenever $\delta_{jk} \le \nu$.

Both concordance and discordance indices have to be computed for every pair of actions $\left(gn_j, gn_k\right)$ in the set $gn = \left[gn_1, gn_2, \ldots gn_z\right]$ such that $gn_j \ne gn_k$. Such a computing procedure leads to a binary relation in comprehensive terms on the set gn. For each $\left(gn_j, gn_k\right)$, only one of the following four situation occurs:

1. $\left(gn_j \succsim gn_k\right)$ & $\neg\left(gn_k \succsim gn_j\right) \rightarrow \left(gn_j \succsim^{+} gn_k\right)$ (gn_j is strictly preferred to gn_k);
2. $\left(gn_k \succsim gn_j\right)$ & $\neg\left(gn_j \succsim gn_k\right) \rightarrow \left(gn_k \succsim^{+} gn_j\right)$ (gn_k is strictly preferred to gn_j);
3. $\left(gn_j \succsim gn_k\right)$ & $\left(gn_k \succsim gn_j\right) \rightarrow \left(gn_j \sim gn_k\right)$ (gn_j is indifferent to gn_k);
4. $\neg\left(gn_k \succsim gn_j\right) \& \neg\left(gn_k \succsim gn_j\right) \rightarrow \left(gn_j \succsim^{-} gn_k\right)$ (gn_j is incomparable to gn_k);

From the above forms, $\neg$ is the negation operation, $\rightarrow$ means “imply”. $\succsim^{+}$ means “be strictly preferred to”, $\succsim^{-}$ means “be incomparable to”, and $\sim$ means “be indifferent to”.

The second procedure consists of exploiting this outranking relation to identify a small as possible subset of actions, from which the best compromise action could be selected. Let $\overline{gn}$ be the partition. Each class on $\overline{gn} = \left[\overline{gn}_1, \overline{gn}_2, \ldots\right]$ is composed of a set of (considerate) equivalent actions. The new preference relation, $\succ'$, is defined on $\overline{gn}$, which is of the form:

$$\overline{gn}_p \succ' \overline{gn}_q \Leftrightarrow \left\{ \exists gn_j \in \overline{gn}_p \quad \& \quad \exists gn_k \in \overline{gn}_q \mid gn_j \succsim gn_k,\ for\ \overline{gn}_p \neq' \overline{gn}_q \right\}; \quad (2.2.34)$$

2.2.6 PROMETHEE

According to Brans and Mareshal (2005), PROMETHEE (Preference Ranking Organization METHod for Enriching Evaluations) was firstly developed by Brans in 1982 at a conference. PROMETHEE I deals with a partial preorder, PROMETHEE II deals with a complete preorder, PROMETHEE III deals with an interval order emphasizing indifference, PROMETHEE IV deals with continuous set of possible alternatives, PROMETHEE V supports the optimization under constraints and PROMETHEE VI is a representation of the human brain. GAIA provides graphical representation supporting the PROMETHEE. The details are reviewed by Brans and Mareshal (2005). This section only investigates the foundations, PROMETHEEs I and II (Brans and Mareshal , 2005), with the modification notation that fits for the CNP model.

Consider a typical decision matrix $\bar{O}$ shown in Eq. 2.2.1. Let a multi-criteria problem be of the form:

$$Max\left\{ r_{i1}, r_{i2} \ldots, r_{ij}, \ldots, r_{i1} : r_{in} \in r \right\}, \forall i \qquad (2.2.35)$$

The natural dominance relation associated to a multi-criteria problem of the above form is defined as follows:

$$\begin{cases} \forall j: r_{ij} \geq r_{i'j} \\ \exists k: r_{ik} > r_{i'k} \end{cases} \Leftrightarrow T_i \succsim T_{i'} \tag{2.2.36}$$

$$\forall j: r_{ij} = r_{i'j} \Leftrightarrow T_i \sim T_{i'} \tag{2.2.37}$$

$$\begin{cases} \exists j: r_{ij} > r_{i'j} \\ \exists k: r_{ik} < r_{i'k} \end{cases} \Leftrightarrow T_i \otimes T_{i'} \tag{2.2.38}$$

,where $\succsim, \sim$ and $\otimes$ stand for preference, indifference and incomparability.

Table 2.2: Types of generalized criteria ($P(d)$: Preference function) (Brans and Mareshal , 2005)

Generalized criterion	Definition	Parameters to fix
Usual criterion	$P(d) = \begin{cases} 0 & d \leq 0 \\ 1 & d > 0 \end{cases}$	-
U-Shape criterion	$P(d) = \begin{cases} 0 & d \leq q \\ 1 & d > q \end{cases}$	q
V-Shape criterion	$P(d) = \begin{cases} 0 & d \leq 0 \\ d/p & 0 \leq d \leq p \\ 1 & d > p \end{cases}$	p
Level criterion	$P(d) = \begin{cases} 0 & d \leq q \\ 1/2 & q \leq d \leq p \\ 1 & d > p \end{cases}$	p,q
V-Shape with indifference criterion	$P(d) = \begin{cases} 0 & d \leq q \\ (d-q)/(p-q) & q \leq d \leq p \\ 1 & d > p \end{cases}$	p,q
Gaussian criterion	$P(d) = \begin{cases} 0 & d \leq 0 \\ 1 - e^{-\frac{d^2}{2s^2}} & d > 0 \end{cases}$	s

The information requested to run PROMETHEE consists of the information between the criteria, and the information within the criteria. Information between the criteria refers to the determination of the weights of the criteria. Usually the set of the normalized weights, $W=\{w_j\}$, is applied such that $\sum_{j=1}^{n} w_j = 1$. W can be derived by pairwise opposite comparison (Chapter 5) using CLOS (Chapter 4).

The information within the criteria refers to the performance values on the criteria by the preference function of the form:

$$P_j(T_i,T_k)=F_j\left(d_j(T_i,T_k)\right),\ \ \forall T_i,T_k \in T, \tag{2.2.39}$$

,where the deviation function is of the form:

$$d_j(T_i,T_k)=r_{ij}-r_{kj}, \tag{2.2.40}$$

$$\text{for which, } 0 \le P_j(T_i,T_k) \le 1 \tag{2.2.41}$$

The pair $\{c_j,P_j(T_i,T_k)\}$ is called the generalized criterion associated with the criterion *j*. Such a generalized criterion has to be defined for each criterion. Brans and Mareshal (2005) proposed six types of preference functions (table 2.2), the parameters are defined as follows:

q is a threshold of indifference;

p is a threshold of strict preference;

s is an intermediate value between q and p.

As soon as evaluation table or the decision matrix $\bar{O}$, $\{c_j,P_j(T_i,T_k)\}$ and W are determined, the PROMETHEE procedure can be performed. Firstly, the aggregated preference indices and outranking flows are defined. The aggregated preference indices are of the form:

$$\pi\left(T_i,T_k\right)=\sum_{j=1}^{n}P_j\left(T_i,T_k\right)\cdot w_j\,,\quad \forall T_i,T_k\in T\,, \tag{2.2.42}$$

$$\pi\left(T_k,T_i\right)=\sum_{j=1}^{n}P_j\left(T_k,T_i\right)\cdot w_j\,,\quad \forall T_i,T_k\in T\,, \tag{2.2.43}$$

The aggregated preference index $\pi\left(T_i,T_k\right)$ expresses the degree of how T_i is preferred to T_k over all the criteria whilst $\pi\left(T_k,T_i\right)$ shows how T_k is preferred to T_i over all the criteria. The following properties hold.

$$\pi\left(T_i,T_i\right)=0\,; \tag{2.2.44}$$

$$0\le\pi\left(T_i,T_k\right)\le 1\,; \tag{2.2.45}$$

$$0\le\pi\left(T_k,T_i\right)\le 1\,; \tag{2.2.46}$$

$$0\le\left(\pi\left(T_i,T_k\right)+\pi\left(T_k,T_i\right)\right)\le 1\,. \tag{2.2.47}$$

If $\pi\left(T_i,T_k\right)$ approximate to 0, there is a weak global preference of T_i over T_k, and vice versa.

Each alternative T_i is facing (m-1) other alternatives in T. In order to rank the alternatives, the outranking flows are defined for the following two forms.

The positive outranking flow is of the form:

$$\phi^{+}\left(T_i\right)=\frac{1}{n-1}\sum_{k=1}^{m}\pi\left(T_i,T_k\right) \tag{2.2.48}$$

The negative outranking flow is of the form:

$$\phi^-(T_i) = \frac{1}{n-1}\sum_{k=1}^{m}\pi(T_k, T_i) \qquad (2.2.49)$$

The positive outranking flow expresses how an alternative T_i is outranking all the others. It is its power, and its outranking character. The higher $\phi^+(T_i)$ gives a better alternative. Conversely, the negative outranking flow expresses how an alternative T_i is outranked by all the others. It is its weakness, and its outranked character. The lower $\phi^-(T_i)$ gives a better alternative.

The PROMETHEE I partial ranking $(\succsim^I, \sim^I, \otimes^I)$ is obtained from both positive and negative outranking flows of the following three forms:

1. $T_i \succsim^I T_k$ iff ($\phi^+(T_i) > \phi^+(T_k)$ & $\phi^-(T_i) < \phi^-(T_k)$) or ($\phi^+(T_i) = \phi^+(T_k)$ & $\phi^-(T_i) = \phi^-(T_k)$) or ($\phi^+(T_i) > \phi^+(T_k)$ & $\phi^-(T_i) = \phi^-(T_k)$);
2. $T_i \sim^I T_k$ iff ($\phi^+(T_i) = \phi^+(T_k)$ & $\phi^-(T_i) = \phi^-(T_k)$);
3. $T_i \otimes^I T_k$ iff ($\phi^+(T_i) > \phi^+(T_k)$ & $\phi^-(T_i) > \phi^-(T_k)$) or ($\phi^+(T_i) < \phi^+(T_k)$ & $\phi^-(T_i) < \phi^-(T_k)$)

The PROMETHEE II consists of the $(\succsim^{II}, \sim^{II})$ complete ranking. The net outranking flow is applied and is of the form:

$$\phi(T_i) = \phi^+(T_i) - \phi^-(T_i), \quad \forall i \in \{1, \ldots, m\} \qquad (2.2.50)$$

The higher $\phi(T_i)$ follows the better alternative. Thus,

$$T_i \succsim^{II} T_k \text{ iff } \phi(T_i) > \phi(T_k); \qquad (2.2.51)$$

$$T_i \sim^{II} T_k \text{ iff } \phi(T_i) = \phi(T_k). \qquad (2.2.52)$$

The net outranking flow function can produce disputable results as more information gets lost by using this function ϕ. From the above forms, two properties hold.

$$-1 \le \phi(T_i) \le 1, \quad \forall i \in \{1,\dots,m\}; \tag{2.2.53}$$

$$\sum_{i=1}^{m} \phi(T_i) = 0. \tag{2.2.54}$$

As the net flow ϕ provides a complete ranking, it may be compared with a utility function (Brans and Mareshal, 2005). One advantage is that it is built on clear and simple preference information of weights and preferences functions, and that it does rely on comparative statements rather than on absolute statements (Brans and Mareshal, 2005). It is possible to integrate PROMETHEE into the CNP framework. However, this is beyond the scope of this research.

2.2.7 TOPSIS

TOPSIS stands for Technique for Order Preference by Similarity to Ideal Solution, which was initially proposed by Hwang and Yoon (1981). TOPSIS is of the notion that the chosen alternative should have the shortest distance from the positive-ideal solution (PIS) and the longest distance from the negative-ideal solution (NIS) (Hwang and Yoon, 1981). The presentation of TOPSIS (Yoon and Hwang, 1995) is used with modification for the CNP framework.

Consider a typical *m*x*n* decision matrix $\bar{O}$ shown in Eq. 2.1. Let J^{+} be the set of benefit (or positive) criteria, i.e. more is better, and J^- be the set of negative criteria, less is better. The calculation consists of six steps.

Step 1: calculate the normalized decision matrix $\widehat{O} = \{\{w_j\}, \{\widehat{r}_{ij}\}\}$. The score r_{ij} is normalized as the normalized score $\widehat{r}_{ij}$ by the root-sum-square function of the form:

$$\widehat{r}_{ij} = \frac{r_{ij}}{\sum_{i=1}^{m} r_{ij}}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n. \tag{2.2.55}$$

Step 2: calculate the weighted normalized decision matrix $\hat{O} = \{\hat{r}_{ij}\}$. Each normalized score $\widehat{r}_{ij}$ is multiplied by its associated weight w_j. Thus, the weighted normalized score $\hat{r}_{ij}$ is of the form:

$$\hat{r}_{ij} = w_j \cdot \widehat{r}_{ij}, \quad j = 1, \ldots, n \tag{2.2.56}$$

Step 3: Determined the positive-ideal solution and the negative-ideal solution respectively by the following forms:

$$r^+ = \{r_1^+, \ldots, r_j^+, \ldots, r_n^+\}, \quad r_j^+ = \left\{\max_i\left(\hat{r}_{ij}\right) j \in J^+;\ \min_i\left(\hat{r}_{ij}\right) j \in J^-\right\}; \tag{2.2.57}$$

$$r^- = \{r_1^-, \ldots, r_j^-, \ldots, r_n^-\}, \quad r_j^- = \left\{\min_i\left(\hat{r}_{ij}\right) j \in J^+;\ \max_i\left(\hat{r}_{ij}\right) j \in J^-\right\}. \tag{2.2.58}$$

Step 4: Calculate the separation measures by m-dimensional Euclidean distance. The separation from the ideal alternative is of the form:

$$\phi_i^+ = \sqrt{\sum_{j=1}^{n}\left(r_j^+ - \hat{r}_{ij}\right)^2}, \quad i = 1, \ldots, m \tag{2.2.59}$$

Similarly, the separation from the negative ideal alternative is of the form:

$$\phi_i^- = \sqrt{\sum_{j=1}^{n}\left(r_j^- - \hat{r}_{ij}\right)^2}, \quad i = 1, \ldots, m \tag{2.2.60}$$

Step 5: calculate the relative closeness ϕ_i^* to the ideal solution

$$\phi_i^* = \frac{\phi_i^-}{\phi_i^+ + \phi_i^-}, \quad i = 1,\ldots,m \tag{2.2.61}$$

Usually the ideal option is with ϕ_i^* closest to 1.

Step 6: The ideal alternative is of the form:

$$T^* = T_k \text{, where } k = Arg\max\left[\left\{\phi_i^* : i = 1,\ldots,m\right\}\right] \tag{2.2.62}$$

Or rank alternatives according to ϕ_i^* in descending order.

In step 2, the weights can be determined by the cognitive prioritization of the pairwise opposite matrix (chapter 5) using the Compound Linguistic Ordinal Scale (chapter 4). TOPSIS can be fitted into the application of the CNP framework. However, this attempt is beyond the scope of the research.

2.2.8 Data Envelopment Analysis

Data Envelopment Analysis (DEA) was initially proposed by Charnes, Cooper, and Rhodes (CCR) in 1978. The major difference from the above decision model is that DEA considers the multiple output data and the multiple input data to measure the efficiency of the alternatives (or decision making units). DEA is optimization programming to determine the weights and efficiency by maximizing the ratio:

$$\frac{virtual\ output}{virtual\ input} \tag{2.2.63}$$

Usually the virtual output (or input) is the linear combination of the input (or output) variables multiplied by its weights, as follows:

$$\text{Virtual input} = \sum_{i=1}^{m} v_i x_{i0} \tag{2.2.64}$$

$$\text{Virtual output} = \sum_{r=1}^{s} u_r y_{r0} \tag{2.2.65}$$

The notions of the input matrix and the output matrix can respectively be presented as follows:

$$Input = \left[V \mid X \right] = \begin{array}{c} \\ c_1 \\ \vdots \\ c_i \\ \vdots \\ c_m \end{array} \begin{array}{cc} \begin{array}{ccccc} T_1 & \cdots & T_j & \cdots & T_n \end{array} & \begin{array}{ccccc} T_1 & \cdots & T_j & \cdots & T_n \end{array} \\ \begin{pmatrix} v_{11} & \cdots & v_{1j} & \cdots & v_{1n} \\ \vdots & \ddots & \vdots & \iddots & \vdots \\ v_{i1} & \cdots & v_{ij} & \cdots & v_{in} \\ \vdots & \iddots & \vdots & \ddots & \vdots \\ v_{m1} & \cdots & v_{mj} & \cdots & v_{mn} \end{pmatrix} & \begin{pmatrix} x_{11} & \cdots & x_{1j} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots & \iddots & \vdots \\ x_{i1} & \cdots & x_{ij} & \cdots & x_{in} \\ \vdots & \iddots & \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mj} & \cdots & x_{mn} \end{pmatrix} \end{array} \tag{2.2.66}$$

$$Output = \left[U \mid Y \right] = \begin{array}{c} \\ d_1 \\ \vdots \\ d_r \\ \vdots \\ d_s \end{array} \begin{array}{cc} \begin{array}{ccccc} T_1 & \cdots & T_j & \cdots & T_n \end{array} & \begin{array}{ccccc} T_1 & \cdots & T_j & \cdots & T_n \end{array} \\ \begin{pmatrix} u_{11} & \cdots & u_{1j} & \cdots & u_{1n} \\ \vdots & \ddots & \vdots & \iddots & \vdots \\ u_{r1} & \cdots & u_{rj} & \cdots & u_{rn} \\ \vdots & \iddots & \vdots & \ddots & \vdots \\ u_{s1} & \cdots & u_{sj} & \cdots & u_{sn} \end{pmatrix} & \begin{pmatrix} y_{11} & \cdots & y_{1j} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots & \iddots & \vdots \\ y_{r1} & \cdots & y_{rj} & \cdots & y_{rn} \\ \vdots & \iddots & \vdots & \ddots & \vdots \\ y_{s1} & \cdots & y_{sj} & \cdots & y_{sn} \end{pmatrix} \end{array} \tag{2.2.67}$$

X and Y represent the input data matrix and output data matrix respectively for a set of Decision Making Units (DMUs) denoted by $T = \{T_j\}$. V and U are the matrices of the corresponding weights with respect to X and Y respectively, and usually are solved by an optimization model.

In order to calculate the efficiency and weights of input data and output data, two essential models of DEA are introduced: CCR and BCC. The notations below are of minor modification.

CCR (Charnes et al., 1978) of a DMU_o or T_o , o=1,…,n, is of the form:

$$CCR\left(\left\{x_{ij}\right\},\left\{y_{rj}\right\},o\right)=$$

$$\begin{aligned} & max \quad h_o=\frac{\sum_{r=1}^{s} u_{ro} y_{ro}}{\sum_{i=1}^{m} v_{io} x_{io}} \\ & s.t. \quad \frac{\sum_{r=1}^{s} u_{ro} y_{rj}}{\sum_{i=1}^{m} v_{io} x_{ij}} \leq 1, \quad j=1,2,\cdots,n \\ & \qquad u_{ro} \geq 0, \quad r=1,2,\cdots,s \\ & \qquad v_{io} \geq 0, \quad i=1,2,\cdots,m \end{aligned} \tag{2.2.68}$$

For the DMU_o, if h_0=1, u_{ro}>0, and v_{io}>0, DMU_o is efficient. Otherwise, the DMU_o is inefficient. The above fractional program is equivalent to the linear program (Banker et al.,1984) as follows:

$$LPCCR\left(\left\{x_{ij}\right\},\left\{y_{rj}\right\},o\right)=$$

$$\begin{aligned} & max \quad h_0=\sum_{r=1}^{s} u_{ro} y_{ro} \\ & s.t. \quad \sum_{i=1}^{m} v_{io} x_{io}=1 \\ & \qquad \sum_{r=1}^{s} u_{ro} y_{rj} \leq \sum_{i=1}^{m} v_{io} x_{ij}, \quad j=1,2,\cdots,n \\ & \qquad u_r \geq \varepsilon>0, \quad r=1,2,\cdots,s \\ & \qquad v_i \geq \varepsilon>0, \quad i=1,2,\cdots,m \end{aligned} \tag{2.2.69}$$

ε is the sufficient small value, i.e. 0.0000001.

The CCR (Charnes et al., 1978) was extended by Banker et al. (1984) (BCC) as follows:

$$
\begin{aligned}
& BCC\left(\{x_{ij}\},\{y_{rj}\},o\right)= \\
& \qquad max \quad h_o=\frac{\sum_{r=1}^{s}u_{ro}y_{ro}-u_o}{\sum_{i=1}^{m}v_{io}x_{io}} \\
& \qquad s.t. \quad \frac{\sum_{r=1}^{s}u_r y_{rj}-u_o}{\sum_{i=1}^{m}v_i x_{ij}}\leq 1, \quad j=1,2,\cdots,n \\
& \qquad\qquad u_r\geq\varepsilon>0, \quad r=1,2,\cdots,s \\
& \qquad\qquad v_i\geq\varepsilon>0, \quad i=1,2,\cdots,m
\end{aligned}
\tag{2.2.70}
$$

The above fractional program is equivalent to the linear program (Banker et al., 1984) as follows:

$$
\begin{aligned}
& LPBCC\left(\{x_{ij}\},\{y_{rj}\},o\right)= \\
& \qquad max \quad h_o=\sum_{r=1}^{s}u_r y_{ro}-u_o \\
& \qquad s.t. \quad \sum_{i=1}^{m}v_i x_{io}=1 \\
& \qquad\qquad \sum_{r=1}^{s}u_r y_{rj}-\sum_{i=1}^{m}v_i x_{ij}-u_o\leq 0, \quad j=1,2,\cdots,n \\
& \qquad\qquad u_r\geq\varepsilon>0, \quad r=1,2,\cdots,s \\
& \qquad\qquad v_i\geq\varepsilon>0, \quad i=1,2,\cdots,m
\end{aligned}
\tag{2.2.71}
$$

On the basis of the above four DEA forms, if different aggregation operators are applied, the results may be different, as there is no critical reason that only weighted arithmetic mean is the choice. Thus this study proposes various general forms that different aggregation operators can be applied.

On the basis of CCR, the general form is as below:

$$
\begin{aligned}
& G_{CCR}\left(\{x_{ij}\},\{y_{rj}\},o\right)= \\
& \quad max \quad h_0=\frac{F\left(u_0,y_0\right)}{F\left(v_0,x_0\right)} \\
& \quad s.t. \quad \frac{F\left(u_j,y_j^T\right)}{F\left(v_j,x_j^T\right)}\le 1, \quad j=1,2,\cdots,n \\
& \qquad u_r\ge 0, \quad r=1,2,\cdots,s \\
& \qquad v_i\ge 0, \quad i=1,2,\cdots,m
\end{aligned}
\tag{2.2.72}
$$

F is an aggregation operator. The quasi-linear means (table 2.3) are the ideal choices for F. In the conventional DEA optimization, F is the weighted arithmetic mean. y_j^T and x_j^T are row vector j's of $\{y_{rj}\}^T$ and $\{x_{ij}\}^T$ respectively, which are the same as the transposition of the column vector j's of $\{x_{ij}\}^T$ and $\{y_{rj}\}^T$ respectively.

The above fractional general form is equivalent to the linear program as follows:

$$
\begin{aligned}
& G_{LPCCR}\left(\{x_{ij}\},\{y_{rj}\},o,F\right)= \\
& max \quad h_o=F\left(u_o,y_o\right) \\
& s.t. \quad F\left(v_0,x_0\right)=1 \\
& \qquad F\left(u_o,y_j^T\right)\le F\left(v_o,x_j^T\right), \quad j=1,2,\cdots,n \\
& \qquad u_{ro}\ge \varepsilon>0, \quad u_{ro}\in u_o,\ r=1,2,\cdots,s \\
& \qquad v_{io}\ge \varepsilon>0, \quad v_{io}\in v_o,\ i=1,2,\cdots,m
\end{aligned}
\tag{2.2.73}
$$

On the basis of BCC, the general form is of the form:

$$
G_{BCC}\left(\{x_{ij}\},\{y_{rj}\},o,F\right)=
$$

$$
\begin{aligned}
& max \quad h_o = \frac{F(u_o, y_o) - u_{o'}}{F(v_o, x_o)} \\
& s.t. \quad \frac{F\left(u_j, y_j^T\right) - u_{o'}}{F\left(v_j, x_j^T\right)} \le 1, \quad j = 1, 2, \cdots, n \\
& \qquad u_r \ge \varepsilon > 0, \quad r = 1, 2, \cdots, s \\
& \qquad v_i \ge \varepsilon > 0, \quad i = 1, 2, \cdots, m
\end{aligned}
\tag{2.2.70}
$$

The above fractional program is equivalent to the linear program as follows:

$$
\begin{aligned}
& G_{LPBCC}\left(\{x_{ij}\}, \{y_{rj}\}, o, F\right) = \\
& \quad max \quad h_o = F(u_o, y_o) - u_{o'} \\
& \quad s.t. \quad F\left(v_j, x_j\right) = 1 \\
& \qquad F\left(u_o, y_j^T\right) - F\left(v_o, x_j^T\right) - u_{o'} \le 0, \quad j = 1, 2, \cdots, n \\
& \qquad u_r \ge \varepsilon > 0, \quad r = 1, 2, \cdots, s \\
& \qquad v_i \ge \varepsilon > 0, \quad i = 1, 2, \cdots, m
\end{aligned}
\tag{2.2.71}
$$

Regarding DEA, the input matrix and the output matrix can be determined by the proposed CNP measurement. The input variables and output variables may consider the predefined priority set, which is derived by CNP, to represent the importance from the perceptions of the decision makers in addition to the weights determined by an optimization model.

2.3 Fuzzy soft computing

Zedah (2001) defined soft computing as follows:

"By design, soft computing is pluralistic in nature in the sense that it is a coalition of methodologies which are drawn together by a quest for accommodation with the pervasive imprecision of the real world. At this juncture, the principal members of the coalition are fuzzy logic, neuro-computing, evolutionary computing, probabilistic computing, chaotic computing and machine learning. What is important is that members of the coalition are, for the most part, complementary rather than competitive."

As the research of soft computing is vast, this section only reviews the key concepts of the fuzzy soft computing which are used by the CNP model. The key concepts are as follows.

2.3.1 Fuzzy linguistic variable and membership functions

Fuzzy linguistic labels are applied in most decision models. A fuzzy linguistic label can be represented by a fuzzy number which is represented by a fuzzy set (Zadeh 1965, 1975, 1996). Fuzzy sets capture the ability to handle uncertainty by approximate methods.

Let X be a universal set (or universal of discourse) of elements x's, and then a fuzzy set α in X is a set of ordered pairs, i.e. $\alpha = \left\{ \left(x, \mu_{\alpha}(x) \right) : x \in X \right\}$. μ_{α} is called the membership function (regarding its operation) or the grade of membership (regarding its output) which defined as $\mu_{\alpha} : X \to [0,1]$ (Zadeh 1965).

The following is a list of the types of memberships. (a) to (i) are from (Pedrycz, 1997; Bargiela and Pedrycz, 2003; Pedrycz and Gomide, 2007; Engelbrecht 2007) whilst (j) to (n) are defined by the author.

a) The triangular membership function

$$\mu_\alpha(x) = \begin{cases} \dfrac{x-l}{m-l}, & l \le x \le m \\ \dfrac{u-x}{u-m}, & m \le x \le u \quad ,l \le m \le u \\ 0, & otherwise \end{cases} , \tag{2.3.1}$$

l is the fuzzy up boundary, and u is the fuzzy low boundary and m is the modal value. The triangular membership is applied mostly in fuzzy theories and applications.

b) Γ membership function

$$\mu_\alpha(x) = \begin{cases} 0, & x \le l \\ 1-e^{-k(x-l)^2}, & x > l \end{cases} ,k > 0 \qquad \text{Or} \tag{2.3.2}$$

$$\mu_\alpha(x) = \begin{cases} 0 & , \quad x \le l \\ \dfrac{k(x-l)^2}{1+k(x-l)^2}, & x > l \end{cases} ,k > 0 \tag{2.3.3}$$

c) S membership function

$$\mu_\alpha(x) = \begin{cases} 0 & , \quad x \le l \\ 2\left(\dfrac{x-l}{u-l}\right)^2 & , \quad l \le x \le m \\ 1-2\left(\dfrac{x-u}{u-l}\right)^2 & , \quad m \le x \le u \\ 0 & , \quad x > u \end{cases} \tag{2.3.4}$$

d) Trapezoidal membership function

$$\mu_\alpha(x) = \begin{cases} 0 & , \quad x < l \\ \dfrac{x-l}{m-l} & , \ l \le x \le m \\ 1 & , \ \mathrm{m} \le x \le \pi \\ \dfrac{u-x}{u-\pi} & , \ \pi \le x \le u \\ 0 & , \quad x > u \end{cases} \tag{2.3.5}$$

e) Gaussian membership function

$$\mu_\alpha(x) = e^{-k(x-m)^2}, k > 0 \quad \text{Or} \tag{2.3.6}$$

$$\mu_\alpha(x) = e^{\frac{-(x-m)^2}{\sigma^2}} \tag{2.3.7}$$

f) Non-Symmetric Gaussian fuzzy sets

$$\mu_\alpha(x) = \begin{cases} e^{\frac{-(x-m)^2}{\sigma^2}} & , \ x \le m \\ e^{\frac{-(x-m)^2}{k^2}} & , \ x > m \end{cases} \tag{2.3.8}$$

g) Exponential-like function

$$\mu_\alpha(x) = \frac{1}{1+k(x-m)^2} \quad , k > 1 \quad \text{Or} \tag{2.3.9}$$

$$\mu_\alpha(x) = \frac{k(x-m)^2}{1+k(x-m)^2} \quad , k > 0 \tag{2.3.10}$$

h) Logistic function

$$\mu_\alpha(x) = \frac{1}{1+e^{-kx}} \tag{2.3.11}$$

i) quadratic function

$$\mu_{\alpha}\left(x\right)=\begin{cases}1\text{-p}^{2}\left(x-m\right)^{2}, & x\in\left[\frac{m-1}{p},\frac{m+1}{p}\right]\\ 0, & \text{otherwise}\end{cases} \quad (2.3.12)$$

The membership functions can be represented by the general form $\left(x;\gamma_{\alpha},d_{\alpha},\tau_{\alpha}\right)$: γ_{α} is the modal value of a fuzzy set α, d_{α} is the equal distance from γ_{α} to the boundary, and τ_{α} is the scale factor to shape the membership, and $\tau_{\alpha}>0$. The membership functions, which are newly proposed by the author, are shown as follows.

j) Triangular-based membership function

$$\mu_{\alpha}^{TbMF}=\begin{cases}\left(\dfrac{x+d_{\alpha}-\gamma_{\alpha}}{d_{\alpha}}\right)^{\tau_{\alpha}}, & x\in\left[\gamma_{\alpha}-d_{\alpha},\gamma_{\alpha}\right]\\ 1, & x=\gamma_{\alpha}\\ \left(\dfrac{-x+d_{\alpha}+\gamma_{\alpha}}{d_{\alpha}}\right)^{\tau_{\alpha}}, & x\in\left[\gamma_{\alpha},\gamma_{\alpha}+d_{\alpha}\right]\\ 0, & \text{otherwise}\end{cases} \quad (2.3.13)$$

k) Parabola-based membership function

$$\mu_{\alpha}^{PbMF}=\begin{cases}\left[1-\left(\dfrac{x-\gamma_{\alpha}}{d_{\alpha}}\right)^{2}\right]^{\tau_{\alpha}}, & x\in\left[\gamma_{\alpha}-d_{\alpha},\gamma_{\alpha}+d_{\alpha}\right]\\ 0, & \text{otherwise}\end{cases} \quad (2.3.14)$$

l) Linear Complex Cosine-based Membership function

$$\mu_{\alpha}^{LCCbMF}=\begin{cases}\left[\dfrac{1}{2}\left(1+Cos\left(\dfrac{\pi\left|d_{\alpha}-x\right|}{d_{\alpha}}\right)\right)\right]^{\tau_{\alpha}}, & x\in\left[\gamma_{\alpha}-d_{\alpha},\gamma_{\alpha}+d_{\alpha}\right]\\ 0, & \text{otherwise}\end{cases} \quad (2.3.15)$$

m) Linear Simplified Cosine-based Membership function

$$\mu_{\alpha}^{LCSbMF} = \begin{cases} \left[Cos\left(\dfrac{\pi\left(x-\gamma_{\alpha}\right)}{2\gamma_{\alpha}} \right) \right]^{\tau_{\alpha}} & , x \in \left[\gamma_{\alpha} - d_{\alpha}, \gamma_{\alpha} + d_{\alpha}\right] \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.3.16}$$

n) Parabolic Sine-based Membership function

$$\mu_{\alpha}^{PSbMF} = \begin{cases} \left[Sin\left(\dfrac{\pi}{2}\left(1-\left(\dfrac{x-\gamma_{\alpha}}{d_{\alpha}} \right)^{2} \right) \right) \right]^{\tau_{\alpha}} & , x \in \left[\gamma_{\alpha} - d_{\alpha}, \gamma_{\alpha} + d_{\alpha}\right] \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.3.17}$$

The Sine functions and Cosine functions can be interchanged by substituting $Sin(\theta) = Cos\left(\theta + \pi/2\right)$.

This research recommends the PbMF, type *k*.

2.3.2 Basic operations of triangular fuzzy sets

Consider two TFNs $\alpha_1 = (l_1, m_1, u_1)$ and $\alpha_2 = (l_2, m_2, u_2)$. The most essential operational axioms are as follows:

Addition:

$$\begin{aligned} \alpha_1 + \alpha_2 &= (l_1, m_1, u_1) + (l_2, m_2, u_2) \\ &= (l_1 + l_2, m_1 + m_2, u_1 + u_2) \end{aligned} \tag{2.3.18}$$

Subtraction:

$$\begin{aligned} \alpha_1 - \alpha_2 &= (l_1, m_1, u_1) - (l_2, m_2, u_2) \\ &= (l_1 - l_2, m_1 - m_2, u_1 - u_2) \end{aligned} \tag{2.3.19}$$

Multiplication:

$$\begin{aligned}\alpha_1 \bullet \alpha_2 &= (l_1, m_1, u_1) \bullet (l_2, m_2, u_2) \\ &= (l_1 \bullet l_2, m_1 \bullet m_2, u_1 \bullet u_2)\end{aligned} \tag{2.3.20}$$

Division:

$$\begin{aligned}\alpha_1 / \alpha_2 &= (l_1, m_1, u_1) / (l_2, m_2, u_2) \\ &= (l_1 / l_2, m_1 / m_2, u_1 / u_2)\end{aligned} \tag{2.3.21}$$

Inversion:

$$(l_1, m_1, u_1)^{-1} = \left(1/(u_1 \cup l_1), 1/m_1, 1/(u_1 \cap l_1)\right). \tag{2.3.22}$$

Example 2.1:

Let $\alpha_1 = (5,6,7)$, $\alpha_2 = \left(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right)$. Then their memberships are:

$$\mu_{\alpha_1}(x) = \begin{cases} x-5, & 5 \le x \le 6 \\ 7-x, & 6 \le x \le 7 \\ 0, & otherwise \end{cases}, \quad \mu_{\alpha_2}(x) = \begin{cases} x-5, & \frac{1}{4} \le x \le \frac{1}{3} \\ 7-x, & \frac{1}{2} \le x \le 7 \\ 0, & otherwise \end{cases}.$$

Addition of both is $\alpha_1 + \alpha_2 = \left(\frac{21}{4}, \frac{19}{3}, \frac{15}{2}\right)$, subtraction of both is $\alpha_1 - \alpha_2 = \left(\frac{19}{4}, \frac{17}{3}, \frac{13}{2}\right)$, multiplication of both is $\alpha_1 \bullet \alpha_2 = \left(\frac{5}{4}, 2, \frac{7}{2}\right)$, and division of both is $\frac{\alpha_1}{\alpha_2} = (20,18,14)$. Inversions of both are $\alpha_1^{-1} = \left(\frac{1}{7}, \frac{1}{6}, \frac{1}{5}\right)$ and $\alpha_2^{-1} = (2,3,4)$.

2.3.3 Aggregation operator properties

Aggregation Operators (AOs) are applied in many domains on problems concerning the fusion of a collection of information granules. These domains include mathematics, physics, engineering, economics, management, and social sciences.

According to (Fodor and Roubens,1994; M. Grabisch et al., 1995; Yager and Rybalov ,1998; Detyniecki, 2000; Calvo and Mesiar, 2003) , there are some properties for the aggregators:

1. Boundary conditions : $A(0,\ldots,0)=0$ and $A(1,\ldots,1)=1$;
2. Monotonicity: $A(x_1,\ldots,x_i,\ldots,x_n)\geq A(x_1,\ldots,x'_i,\ldots,x_n)$ if $x_i \geq x'_i$.
3. Continuity: A is continuous with respect to each of its variables.
4. Associativity: $A(x_1,x_2,x_3)=A(x_1,A(x_2,x_3))=A(A(x_1,x_2),x_3)$.
5. Symmetry: also known as commutativity or anonymity. For every permutation δ of $\{1,2,\ldots,n\}$, the operator satisfies: $A\left(x_{\delta(1)},x_{\delta(2)},\ldots,x_{\delta(n)}\right)=A(x_1,x_2,\ldots,x_n)$.
6. Bisymmetry: $A\left(A(x_{11},x_{12}),A(x_{21},x_{22})\right)=A\left(A(x_{11},x_{21}),A(x_{12},x_{22})\right)$
7. Absorbent Element: $A(x_1,\ldots,a,\ldots,x_n)=a$;
8. Neutral Element: $A^{(n)}(x_1,\ldots,e,\ldots,x_n)=A^{(n-1)}(x_1,\ldots,x_{n-1})$
9. Idempotence: $A(x,x,\ldots,x)=x$;
10. Compensation: $\min_{i=1}^{n}(x_i)\leq A(x_1,x_2,\ldots,x_n)\leq \max_{i=1}^{n}(x_i)$
11. Reinforcement: full, downward, and upward reinforcements (Yager and Rybalov ,1998).

Different operators are associated with different choices of the above properties. There are no absolute rules that associate what properties to what operators. The researchers usually define some properties, and then create their operators.

2.3.4 Categories of Aggregation Operators

Aggregation operators (AOs) can be classified as the non-weighted AO and weighted AO (Yuen, 2009d). As a non-weighted AO is the special case of a weighted AO such that all weights are equal, the weighted AOs are discussed. Aggregation operators have been contributed by many researchers. The followings introduce operators which are frequently used (Yuen, 2009d).

a) Quasi-linear means

The general form of quasi-linear means (Bullen et el. 1988; Marichal, 1998;, Smolikava and Wachowiak, 2002) is of the form:

$$qlm(W,C)=h^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\omega_i h(c_i)\right)\ ,\ c\in I^p\ . \qquad (2.3.23)$$

The function $h:I\rightarrow\Re$, called the generator of $qlm(w,c)$ is continuous and strictly monotonic. If $h(x)=x^{\alpha}$, qlm is the weighted root power (*wrp*) or weighted generalized mean, and other three types are extensions (table 2.3).

Table 2.3: Some forms of Quasi-linear means

1. Weighted Root Power $wrp(\alpha;W,C)=\left(\sum_{i=1}^{n} w_i c_i^{\alpha}\right)^{1/\alpha}$	2. Weighted Harmonic mean ($\alpha \to -1$): $whm(W,C)=\frac{1}{\sum_{i=1}^{n}\frac{w_i}{c_i}}$
3. Weighted Geometric mean ($\alpha \to 0$): $wgm(W,C)=\prod_{i=1}^{n} c_i^{w_i}$	4. Weighted Arithmetic mean ($\alpha \to 1$): $wam(W,C)=\sum_{i=1}^{n} w_i c_i$

b) Ordered Weighted Averaging

OWA (Yager, 1988, 2004) is the weighted arithmetic mean (*wam*) in which its weight values are related to the order position of C.

$$owa(W,C)=\sum_{i=1}^{n} w_i b_j\,, \tag{2.3.24}$$

, where b_j is the *j*th largest of the C, $w_i \in [0,1]$ and $\sum_{i\in\{1,\ldots,n\}} w_i = 1$. w_i can be generated from a regular non-decreasing quantifier *Q*, which is of the form:

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right),\quad i=1,\ldots,n\,, \tag{2.3.25}$$

where *Q* can be defined by $Q(\alpha;r)=r^{\alpha}$, $\alpha \ge 0$.

c) Weighted median

In weighted median aggregation (Yager, 1994; Smolikava and Wachowiak, 2002), each element c_i is replaced by two elements:

$$c_i^{+} = (1-w_i) + w_i \cdot c_i \tag{2.3.26}$$

$$c_i^- = w_i \cdot c_i. \qquad (2.3.27)$$

Then the median value is computed by

$$wmed(W,C) = Median\left(c_1^+, c_1^-, \ldots, c_i^+, c_i^-, \ldots, c_n^+, c_n^-\right). \qquad (2.3.28)$$

Alternatively, c_i^+ and c_i^- can be computed by T-conorm and T-norm, denoted as S and T respectively, having the forms

$$c_i^+ = S(1 - w_i, c_i) \qquad (2.3.29)$$

$$c_i^- = T(w_i, c_i) \qquad (2.3.30)$$

S and T are defined as follows.

d) T-norms and T-conorms

T-norms have the properties in which $T(x,1) = x$ and $T(x,y) \leq \min(x,y)$ whilst T-conorms have the properties in which $S(x,0) = x$ and $S(x,y) \leq \max(x,y)$ (Detyniecki, 2000). Different kinds of T-norms and T-connorms (Detyniecki, 2000; Smolikava and Wachowiak, 2002) are shown in table 2.4.

e) Weighted Gamma Operator

Zimmermann and Zysno (1980) proposed a gamma operator on the unit interval based on T-norms and T-conorms. Calvo and Mesiar (2003) modified the equation with a weighted assignment, which is of the form:

$$wgo(\alpha;C,W)=\left(\prod_{i=1}^{n}c_i^{w_i}\right)^{1-\alpha}\left(1-\prod_{i=1}^{n}(1-c_i)^{w_i}\right)^{\alpha}. \qquad (2.3.31)$$

Table 2.4: Forms of T-norms and T-connorms

1.	*Min-Max*	$Tm(a,b)=\min\{a,b\}$	$Sm(a,b)=\max\{a,b\}$
2.	*Lukasiewicz*	$Tl(a,b)=\max\{a+b-1,0\}$	$Sl(a,b)=\min\{a+b,1\}$
3.	*Product/ Probabilistic*	$Tp(a,b)=ab$	$Sp(a,b)=a+b-ab$
4.	*Dubois and Prade*	$Tdp(\alpha;a,b)=\dfrac{a\cdot b}{\max\{a,b,\alpha\}}$, $\alpha\in(0,1)$	$Sdp(\alpha;a,b)=$ $1-\dfrac{(1-a)(1-b)}{\max\{(1-a),(1-b),\alpha\}}$ $,\alpha\in(0,1)$
5.	*Yager*	$Ty(\alpha;a,b)=$ $\max\left\{0,1-\left[(1-a)^{\alpha}+(1-b)^{\alpha}\right]^{1/\alpha}\right\}$	$Sy(\alpha;a,b)=$ $\min\left\{1,\left(a^{\alpha}+b^{\alpha}\right)^{1/\alpha}\right\},\alpha>0$
6.	*Frank*	$Tf(\alpha;a,b)=$ $\log_{\alpha}\left[1+\dfrac{(\alpha^{a}-1)(\alpha^{b}-1)}{\alpha-1}\right]$ $,\alpha>0,\alpha\neq1$	$Sf(\alpha;a,b)=$ $1-\log_{\alpha}\left[1+\dfrac{(\alpha^{1-a}-1)(\alpha^{1-b}-1)}{\alpha-1}\right.$
7.	*Weber-Sugen o*	$Tws(\alpha_T;a,b)=$ $\max\left\{\dfrac{a+b-1+\alpha_T\cdot a\cdot b}{1+\alpha_T},0\right\}$ $\alpha_T>-1$	$Sws(\alpha_T;a,b)=$ $\min\{a+b+\alpha_S\cdot a\cdot b,1\}$ $\alpha_S=\dfrac{\alpha_T}{(1+\alpha_T)}$
8.	*Schweizer & Sklar*	$Tss(\alpha;a,b)=$ $1-\left[(1-a)^{\alpha}+(1-b)^{\alpha}-(1-a)^{\alpha}(1-\right.$	$Sss(\alpha;a,b)=\left[a^{\alpha}+b^{\alpha}-a^{\alpha}b^{\alpha}\right]$ $,\ \alpha>0$

f) OWMAX and OWMIN

Ordered weighted maximum (*owmax*) and ordered weighted minimum operators (*owmin*) were proposed by Dubois el at. (1988). Unlike OWA which deals with weighted arithmetic mean, owmax and owmin apply weighted maximum and minimum (Marichal, 1988).

For any weight vector $W = (w_1, \ldots, w_n) \in [0,1]^n$ such that $1 = w_1 \geq \ldots \geq w_n$, *owmax* is of the form:

$$\mathrm{owmax}(W, C) = \bigvee_{i=1}^{n} \left(w_i \wedge c_{(i)} \right), \quad C \in [0,1]^n. \tag{2.3.32}$$

For $W \in [0,1]^n$ such that $w_1 \geq \ldots \geq w_n = 0$, *owmin* is of the form:

$$\mathrm{owmin}(W, C) = \bigwedge_{i=1}^{n} \left(w_i \vee c_{(i)} \right), C \in [0,1]^n. \tag{2.3.33}$$

g) Leximin ordering

Leximin ordering was proposed by Dubois et al. (1996). Yager (1997) improved the Lexmin ordering, based on OWA weights. Let Δ denote a distention threshold between the values being aggregated, the Leximin is of the form:

$$\mathrm{leximin}(W, C) = \sum_{i=1}^{n} w_i b_i \tag{2.3.34}$$

, where b_i is a sorted $C \in I^n$ in descending order such that $b_1 > \ldots > b_n$. In addition,

$$w_j = LexW(\Delta, n) = \begin{cases} \dfrac{\Delta^{(n-j)}}{(1+\Delta)^{n-j}} & , j = 1 \\ \dfrac{\Delta^{(n-j)}}{(1+\Delta)^{n+1-j}} & , j = 2, \ldots, n \end{cases}, \quad w_j \in W \tag{2.3.35}$$

2.4 Rating scales and measurement

The details of the related topics of the rating scales and measurement are as follows.

2.4.1 Measurement and scales

Many fields of various activities involve assessment. Assessment takes on many forms such as interviews, examinations, multiple choices, diagnostic tests, and continuous assessments. A rating scale model is an essential component for these forms and attracts close attention from academics as applications in assessment are vast, including such fields as crime, economics, finance, education, politics, marketing, engineering, social science, and physiology.

Such assessments are usually referred to as psychometrics, which is the science of how to maximize the quality of assessment (Rust and Golombok, 1999). In psychometrics, the ordinal rating scales are usually referred to as soul searching for an appropriate category or statement in the set of alternatives representing the possible views of or possible perceptions toward the attributes of the objects. A good psychometric scale plays an essential role in determining the quality of the assessment.

Rating scales are often applied in measurement. Measurement consists of rules for assigning symbols to objects so as to (1) represent quantities of attributes numerically (scaling) or (2) define whether the objects fall into the same or into a different category with respect to a given attribute (classification) (Nunnally and Bernstein,1994). The meaning of "symbol" includes the logically defined definite objects, and the dynamic processes unifying language and cognition (Hadamard, 1996).

Regarding syntactic symbols, a study (Zorzi et al., 2002) that appeared in *Nature* concluded that "although most people focus on symbolic aspects of numbers, and few seem to be aware of the intimate relationship between numbers and visuo-spatial representations, thinking of numbers in spatial terms (as has been reported by great mathematicians (Hadamard,1996)) may be more efficient because it is grounded in the actual neural representation of numbers". This research suggests that the best actual neural representation of numbers or quantity concepts for most people is the linguistic terms or words which are the basis of the qualitative representation for quantity.

In the social sciences, most of the time the "objects" are people, a "rule" involves the explicitly stated assignment of numbers, and "attributes" concerns particular features of the objects (e.g. people) that are not themselves measured; It is their attributes that are measured (e.g. self-esteem) (Netemeyer et al. 2003).

Measurement includes evaluating numbers such that they reflect the differing degrees of the attribute being assessed (DeVellis,1991; Haynes et al. 1999; Nunnally and Bernstein, 1994; Netemeyer et al. 2003). Measurement is related to other forms of symbolic representation such as that involved in computer data representation and natural language (Finkelstein and Learning, 1984).

The types of measurement can be classified as nominal, ordinal, interval, and ratio (Stevens,1946). This classification outline is still used and discussed by many recent researchers. Nominal-level and Ordinal-level measurements are referred to as Categorical Measurement whilst Interval-level and ratio-level measurements are referred to as metric measurement (Blaikie,2003). Qualitative data and quantitative data are used to refer to data in words and numbers respectively (Blaikie,2003). Interval and ratio scales, represented by a numerical system, are usually used in psychophysics on the basis of quantitative data whilst nominal and ordinal scales, including linguistic representation and/or numerical enumeration, are usually used by psychologists or in psychometrics. When an observer estimates a measure, it is a subjective judgement, and estimates lie along a psychological continuum (Mcdonnell,1969). The fact is that the use of linguistic labels makes expert judgment more reliable and consistent (Ben-Arieh and Chen, 2006).

2.4.2 Likert-Like Scales

The ordinal rating scales are usually referred to as mind searching for an appropriate category or statement in the set of alternatives representing the possible views of our possible perceptions toward the attributes of the objects.

The Likert scale (1932), which is an the ordinal scale, is widely used in various studies. Others (or Likert-Like Scales), which can be regarded as minor variations of Likert scale, include numerical scale (fig.2.2) graphic rating scale (fig. 2.3), visual analogue scale (Wewers and Lowem,1990) (fig. 2.4), and semantic differential scale (Osgood et al., 1957) (fig. 2.5).

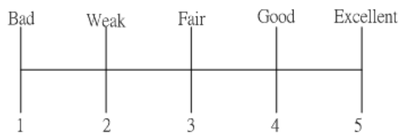


Figure 2.1: Likert scale

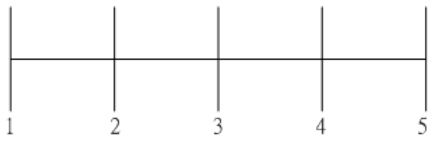


Figure 2.2: Numerical scale

Figure 2.3: Graphic rating scale

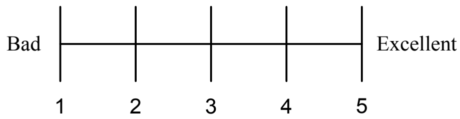


Figure 2.4: Visual analogue scale

Bad____ : ____ : ____ : ____ : ____Excellent

Figure 2.5: Semantic differential scale

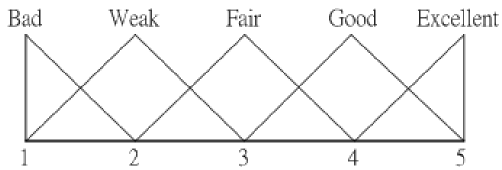


Figure 2.6: Fuzzy set scale

Thus, the general form of the Likert-like scales can be defined as follows.

Definition 2.2 $(RC, S, \wp)$: The general form of a qualitative rating scaling model is 3-tuple $(RC, S, \wp)$ where the scale for response RC is the function displaying response categories for raters, the scaling function S defines rating categories, and reference function $\wp$ is to apply S to RC.

For example, the Likert Scale is 3-tuple $\left(RC, \left(\aleph', \overline{X'}, f_{\overline{X'}}(\aleph')\right), \wp\right)$. RC is shown in fig. 2.1, $S = \left(\aleph', \overline{X'}, f_{\overline{X'}}(\aleph')\right)$ is the scale model. The response linguistic categories $\aleph' = \{\text{'Bad','Weak','Fair','Good','Excellent'}\}$ and the interval scale $\overline{X'} = \{1,2,3,4,5\}$ where $\overline{X} = f_{\overline{X'}}(\aleph')$ is the function of rules for assigning numbers to the linguistic labels in the categories. Natural and programming languages are usually considered in view of their syntactic, pragmatic (procedural, algorithmic, functional), and semantic aspects (Muravyov and Savolainen, 1997). The questions of the semantics of a language for the representation of measurement results are not yet clear (Muravyov and Savolainen, 1997). Thus, the assignment of $\overline{X} = f_{\overline{X'}}(\aleph')$ of a Likert Scale is open to discussion, especially the interval analysis and numerical representation analysis.

A fuzzy linguistic variable which consists of linguistic labels is also considered as a type of ordinal scale as it is a single step rating process in the rating interface, but its numerical representation methods are fuzzy numbers.

The use of linguistic labels makes expert judgment more reliable and consistent (Ben-Arieh and Chen, 2006). This is due to the fact that humans employ mostly words in computing and reasoning, arriving at conclusions expressed as words from premises expressed in a natural language or having the form of mental perceptions (Zadeh, 1996).

2.4.3 Cardinal number of rating scale

Regarding the cardinal number of the set of the terms, the qualitative scales, i.e. ordinal and nominal, are referred to as being 'soft' or 'weak scales' (Muravyov and Savolainen, 1997). This may be due to the reason that the number of the terms is very limited, as an expert could usually only manage a set with (7 ± 2) terms (Miller, 1956). The incomplete and ambiguous scale category descriptors may result in significant evaluation errors and may not reflect the facts. If fewer categories, i.e., 3 or 5, are applied, the descriptors are clearly insufficient. Increasing the number of ordinal categories will induce ambiguity. This can be referred to in the research from Hakel (1968) and a comparison with Simpson (1944) who investigated twenty modifying words like usually, often, sometimes, occasionally, seldom, rarely and commonly. He concluded that they do not mean the same thing to all people. The problem also appears in a similar study by Hoyt (1972) who investigated how people interpret quantifying adjectives such as a clear mandate, most, numerous, a substantial majority, a minority of, a large proportion of, a significant number of, many,…., hardly any, a couple, and a few.

In view of the number of rating options and computational methods, chapter 4 proposes a Syntactic Rule Algorithm (Algorithm 4.1) to produce a large scale of terms , e.g. $m(n-1)+1,\ \ m,n = 7 \pm 2$, to reduce the softness and weakness of the scales. The Deductive Rating Strategy Algorithm (Algorithm 4.3) as a rating interface is proposed to improve the rating process.

2.4.4 Numerical representation and computational rules

Regarding the computational methods to map the linguistic labels to numerical symbols, Zorzi et al. (2002) has alleged that research into numerical representation for linguistic terms is limited. Steven (1946) has implied that linearity of an ordinal scale is open to question. Blaikie (2003) has argued that the commonly used Likert categories are not necessarily evenly spaced along this level of agreement continuum, although researchers frequently assume that they are. There are many situations where observations cannot be described accurately when, for instance, they depend on environmental conditions or on individual responses (Urso and Gastaldi, 2002). For another reason, the findings of the studies (Simpson, 1944; Hakel, 1968) also supports the views that the ordinal position of the words is not well defined and universally applied, and intervals are not even. The studies (Simpson, 1944; Hakel, 1968) also imply that statistics from a population is not an effective method to get the representation value. Fuzzy theory may be the appropriate approach to address the problem.

However, there are limitations to bridge the classical measurement and fuzzy theory for the following three reasons.

Firstly, most fuzzy studies focus on physiological measurement, that is to fuzzify the crisp value of the measureable object on the basis of a physiological instrument, and then to make further calculation. Examples like the temporary, the numerical value, 5 degree, physiological measurement, may be fuzzifed as “cold”, with a certain membership, i.e. 0.8, for a term set. “Temperature=(Cold,0.8)” set associated with other parameters are defuzzified to a meaningful result with the dedicated aggregators. This process is “number in and number out” via words in the middle (NI-NO-WM).

For many measurement applications, after the surveys are completed, the data are collected and further processed to answer research questions about characteristics, relationships, patterns or influences in some social phenomenon. This is called data analysis which can be divided into four types: univariate descriptive, bivariate descriptive, explanatory, and inferential (Blaikie,2003). After data analysis, the study concludes their findings mainly by using words.

It seems there is a gap when applying fuzzy theory to such kinds of data analysis. A problem may be attributed to the properties of fuzzy numbers. If the data type is the fuzzy number, the data type will not be applied in the model. In this situation, a fuzzy number can be converted into a crisp number. A crisp number can be regarded as a special fuzzy number having no fuzziness associated with it (Wewers and Lowem, 1990). Typically a triangular number (a,b,c) is used to represent a linguistic label. Usually the modal value of the triangle number is used to represent the linguistic label. However, this approach leads to comes back to the same assignment method of Likert scales; for example, 1 is for “very disagree”, and 5 is for “very agree”. Due to a lack of theoretical approaches to develop a pattern using a collection of crisp numbers presenting a collection of linguistic labels with the fuzzy theory, chapter 4 proposes a Semantic Rule

Algorithm (algorithm 4.2) to address the problem.

Secondly, the literature on decision problems mainly discusses two main problems: the aggregation process and the exploitation process (Herrera and Martinez, 2001).

However, there is a lack of research discussing the problem of WI-WO-NM. For example, the 2-tuple fuzzy linguistic model (Herrera and Martinez, 2001), which claims to be superior to classic fuzzy linguistic representation, is an approximative computational model based on the Extension Principle (Degani and Bortolan 1988; Chang and Chen, 1994), and the Ordinal Linguistic Computation Model (Yager, 1995). However it ignores the process of rating and assessment in the beginning stage. Also the 2-tuple fuzzy linguistic term model is still only a numerical assessment with partial linguistic assessment.

A 2-tuple fuzzy linguistic term (s_i, α) is a numerical number in manner. For example, in fig. 2.6, assume the expert thinks that some attribute of an object is $\beta = 3.4$, then $s_3 = 'Fair'$ with $\alpha = 0.4$, i.e. $\Delta(3.4) = (s_3, 0.4)$. α and β still need a numerical justification from the expert. It is questionable how the data obtained from the expert's subjective measurement approximates to decimal accuracy, instead of fuzzy linguistic input. In the real world, the uncertainty, constraints, and even the vague knowledge of the experts imply that decision makers cannot provide exact numbers to express their opinions (Ben-Arieh and Chen, 2006). To address this problem, this study changes α into a linguistic symbol, which is transferred into a number though fuzzy normal distribution. Another barrier of 2-tuple is that this method needs a dedicated fuzzy aggregator and cannot correspond to classical statistical research for most psychometric research, especially the fields of social science and psychological measurement.

Thirdly, although there are many fuzzy set and fuzzy logic studies (e.g. Bilgic, and Turksen,2000; Türksen, 1991; Deschrijver and Kerre, 2007; Zadeh, 1975, 1996,2005,2008), the theoretical distribution of the collective patterns of the rating scales in the assessment processes does not seem to have received enough attention in the fuzzy set theory literature, which mainly considers the approaches of the elicitation of the membership. The problem of numerical representation for the linguistic terms is still unsolved.

It seems the field of psychometric measurement in assessment processes such as psychometric scales developed by fuzzy theory, have not received enough attention in the fuzzy theory literature, especially the problem of how to reduce the vagueness of the intervals of ordinal rating categories and the problem of numerical representation for the linguistic terms. The Compound Linguistic Ordinal Scale Model (chapter 4) is proposed to address this issue.

2.5 Analytic Hierarchy Process / Analytic Network Process (AHP/ANP)

The pairwise comparison method is originated from psychological research (Thurstone, 1927). Prof. Saaty further developed the concept in a mathematical way, and applied the concept in Analytic Hierarchy Process (AHP) (Saaty, 1980, 1990, 1994,2000, 2001) and Analytic Network Process (ANP) (Satty 2005). The pairwise comparison method has been widely studied extensively and applied in multi-criteria decision making (MCDM) domains about 30 years, e.g. the survey from Ho (2008). However, there are criticisms against this method (e.g. Belton and Gear, 1983; Dyer, 1990a and 1990b; Murphy, 1993; Perez, 1995; Forman and Gass, 2001; Wang and Elhag, 2006; Rozann, 2007). The related topics of Analytic Hierarchy Process / Analytic Network Process are illustrated as the following sections.

2.5.1 The processes

The AHP consists of four major operations: definition, assessment, prioritization, and synthesis. In definition, experts need to define an objective of the problem O, a set of alternatives of the solution $\overline{T}=\{t_1,t_2,\ldots,t_j,\ldots,t_m\}$, and a set of criteria to be evaluated $C=\{c_1,c_2,\ldots,c_i,\ldots,c_n\}$. In the assessment, decision makers need to set the verbal judgment for each pairwise comparison on the basis of their experience and knowledge. The verbal judgment is usually on a 9-point verbal scale with corresponding numerical representations illustrated in table 2.5. The pairwise comparison is performed by a pairwise comparison matrix, denoted by $A=\{a_{ij}\}$, $0<a_{ij}=a_{ji}^{-1}$, $i,j=1,2,\ldots,n$, where a_{ij} is a numeric point to estimate the relative importance of object i dominating object j. A pairwise comparison matrix is also called a reciprocal matrix due to the axiom of $0<a_{ij}=a_{ji}^{-1}$. The reciprocal matrices of all assessments are formed by transforming the linguistic labels to numerical values.

Table 2.5: Pairwise comparison scale schema

Verbal Scales	Numerical Representation
Equally important	1
Weakly important	2
Moderately important	3
Moderately plus	4
Strongly important	5
Strong Plus	6
Very Strongly	7
Very, very strongly	8
Extremely important	9
Reciprocals of Above	(from 1/2 to 1/9)

To determine the validity of a pairwise matrix, the concept of Consistency Ratio is applied. If CR>0.1, the pairwise matrix is not consistent, then the comparisons should be revised. Otherwise, the pairwise matrix is accepted. Consistency Ratio is obtained by dividing Consistency Index by Random Index (table 2.6) which is an average random consistency index derived from a sample of randomly generated reciprocal matrices using the scales in table 2.5. Consistency index is in the form $CI = \frac{\lambda_{\max} - n}{n-1}$, where a principal eigenvalue $\lambda_{\max}$ can be derived by the form $\lambda_{\max} = n + \frac{1}{n}\sum_{1 \le i < j \le n} \frac{\delta_{ij}^{\ 2}}{1+\delta_{ij}}$, $\delta_{ij} = \left(\frac{a_{ij}}{w_i / w_j} - 1\right)$. A is consistent if $\lambda_{\max} = n$, and it is not consistent if $\lambda_{\max} > n$.

Table 2.6: Random consistency index (R.I) (Saaty, 1980)

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
RI	0	0	.58	.90	1.12	1.24	1.32	1.41	1.45	1.49	1.51	1.48	1.56	1.57	1.59

In the prioritization process, a local priority vector $W = \{w_1, \ldots, w_n\}$, $\sum_{i=1}^{n} w_i = 1$ is generated from a reciprocal matrix A by a Prioritization Operator (PO), i.e. $PO: A \to W$. PO has been studied by many studies. This study reviews eleven important prioritization operators further stated in Chapter 2.5.2.

In the synthesis stage, the local priority vectors W's are aggregated as a global priority vector $V = \{v_1, \ldots, v_n\}$ by an aggregation operator $Agg: \{W\} \to V$. Usually the aggregation operator uses the weighted average summation operator.

These four operations are likely to induce four fundamental problems: (i) selection of criteria in stage one; (ii) selection of numerical scales in stage two; (iii) selection of prioritization operators (or methods) in stage three; and (iv) selection of aggregation operators in stage four. These problems probably create rank reversals. Problem (ii) has been addressed by Compound Linguistic Ordinal Scale (Yuen and Lau, 2009).

Concerning problem (iii), although there are many POs (illustrated in Chapter 2.5.2), actually the best prioritization operator relies on the particular content of a pairwise matrix, and none of prioritization methods performs better than the others in every inconsistent case. (Yuen, 2009b, 2009c, 2009g; Srdjevic, 2005; Mikhailov and Sing, 1999) also have verified this issue. Thus, it is appropriate to propose a framework to select the most appropriate prioritization operator for each reciprocal matrix among sufficient PO candidates with defined measurement methods. A Mixed Prioritization Operators Strategy using defined measurement methods (Yuen, 2009b, 2009c, 2009g; Srdjevic, 2005) can address this issue.

If a unique operator is proposed, the proof is needed to show that this unique operator performs better than others on the basis of some measurement criteria. The measurement method should be reasonable and convincible. Chapter 2.5.2 demonstrates the analytical prioritization operators whilst Chapter 2.5.3 presents the measurement methods, which are further illustrated by the graph theory in Chapter 2.5.4.

2.5.2 Analytical prioritization operators (APOs)

a) Eigenvector(EV)

Eigenvector operator is proposed by (Saaty, 1980). EV is to derive the principal eigenvector $\lambda_{\max}$ of A as the non-normalized priority vector w' by solving the following Eigen system.

$$Aw' = \lambda_{\max} w', \quad w' = \{w'_1, \ldots, w'_n\} \tag{2.5.1}$$

And the solution of w', which is normalized as $\{w_i\}$, is given by

$$w' = \lim_{x \to \infty} \left(\frac{A^k e^T}{eA^k e^T} \right), \quad e = (1, 1, \ldots, 1)$$

$$w_i = \frac{w'_i}{\sum_{i=1}^{n} w'_i}, \; i = 1, 2, \ldots, n \tag{2.5.2}$$

b) Normalization Operators

Normalization operator was introduced in (Saaty, 1980). The methods are named according to their calculation steps since Saaty (1980) has not given them appropriate names.

b-1) Normalization of the Row Sum (NRS)

NRS is to sum the elements in each row and normalize by dividing each sum by the total of all the sums, thus the results now add up to unity. NRS has the form:

$$a'_i = \sum_{j=1}^{n} a_{ij} \qquad i = 1, 2, \ldots, n$$

$$w_i = \frac{a'_i}{\sum_{i=1}^{n} a'_i} \qquad i = 1, 2, \ldots, n \tag{2.5.3}$$

b-2) Normalization of Reciprocals of Column Sum (NRCS)

NRCS is to take the sum of the elements in each column, form the reciprocals of these sums, and then normalize so that these numbers are added to unity, e.g. to divide each reciprocal by the sum of the reciprocals. It has the following form:

$$a'_i = \frac{1}{\sum_{i=1}^{n} a_{ij}} \qquad \mathrm{j} = 1, 2, \ldots, n$$

$$w_i = \frac{a'_i}{\sum_{i=1}^{n} a'_i} \qquad i = 1, 2, \ldots, n \tag{2.5.4}$$

b-3) Arithmetic Mean of Normalized Columns (AMNC)

Each element in *A* is divided by the sum of each column in *A*, and then the mean of each row is taken as the priority w_i. It has the following form:

$$a'_{ij} = \frac{a_{ij}}{\sum_{i=1}^{n} a_{ij}} \qquad i, j = 1, 2, \ldots, n \text{, and}$$

$$w_i = \frac{1}{n} \sum_{j=1}^{n} a'_{ij} \qquad i = 1, 2, \ldots, n \tag{2.5.5}$$

b-4) Normalization of Geometric Means of Rows (NGMR)

NHMR is to multiply the *n* elements in each row and take the *n*th root, and then normalize so that these numbers add to unity. It is the following form:

$$w'_i = \prod_{j=1}^{n} a_{ij}^{1/n}, \; i = 1, 2, \ldots, n$$

$$w_i = \frac{w'_i}{\sum_{i=1}^{n} w'_i}, \quad i = 1, 2, \ldots, n \tag{2.5.6}$$

c) Direct Least Squares / Weighted Least Squares (DLS/WLS)

This method is used to minimize the sum of errors of the differences between the judgments and their derived values. The Direct Least Squares proposed in (Chu et al, 1979) has the following form:

$$\text{Min} \quad \sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_{ij} - {w_i}/{w_j}\right)^2$$

$$\text{Subject to } \sum_{i=1}^{n} w_i = 1, \quad w_i > 0, i = 1, 2, \ldots, n \tag{2.5.7}$$

The above non-linear optimization problem has no special tractable form or closed form and is very difficult to be solved (Chu et al, 1979). For efficient computation with closed form, Chu et al (1979) modified the objective function and proposed the Weighted Least Squares (WLS) in the following form:

$$\text{Min} \quad \sum_{i=1}^{n}\sum_{j=1}^{n}\left(w_i - a_{ij}w_j\right)^2$$

$$\text{Subject to } \sum_{i=1}^{n} w_i = 1, \quad w_i > 0, i = 1, 2, \ldots, n \tag{2.5.8}$$

Although this method provides the closed form for the answer, the variance likely is larger than DLS.

d) Logarithm Least Squares Method (LLS)

LLS initially were proposed by Crawford and Williams (1985), and has been intensively studied by many authors (e.g. Saaty and Vagas, 1984; Budescu et al, 1986; Zahedi, 1986; Blankmeyer , 1987; Golany and Kress , 1993; Lootsma, 1996; Barzilai, 1997; Lin, 2006). The LLS has the following form:

$$\text{Min} \quad \sum_{i=1}^{n}\sum_{j>i}^{n}\left(\ln a_{ij}-\left(\ln w'_i-\ln w'_j\right)\right)^2$$

$$\text{Subject to } \prod_{i=1}^{n} w'_i = 1, \ \ w'_i > 0, i = 1,2,\ldots,n \tag{2.5.9}$$

The final result (w_i) is derived from normalization of (w'_i). Crawford and Williams (1985) have proved that the solution is unique, and is equivalent to NGMR, which is preferable due to its simplicity.

e) Fuzzy Programming (FP)

The FP is proposed by Mikhailov (2000), and has the form:

$$\begin{aligned}
&\max && \mu \\
&\text{Subject to} && \mu d_j^+ + R_j W^T \geq d_j^+, \\
& && \mu d_j^- - R_j W^T \geq d_j^-, \quad j = 1,2,\ldots,m, \ \ 1 \geq \mu \geq 0 \\
& && \sum_{i=1}^{n} w_i = 1, \quad w_i > 0, \ j = 1,2,\ldots,n
\end{aligned} \tag{2.5.10}$$

$R_j \in R^{m\times n} = \{a_{ij}\}$ is the row vector. The values of the left and right tolerance parameters d_j^- and d_j^+ represent the admissible interval of approximate satisfaction of the crisp equality $R_j W^T = 0$. The measure of intersection of μ is a natural consistency index of the FP. Its value however depends on the tolerance parameters. For the practical implementation of the FP, it is reasonable that all these parameters has equal values. Limitation of this method is that parameters d_j^- and d_j^+ are undermined by Mikhailov (2000). This leads to infinite candidate values for the parameters. Mikhailov (2000) sets $d_j^- = d_j^+ = 1$ in his example.

f) Goal Programming (GP)

Bryson (1995) proposed goal programming operator (GP), which uses relative deviations $\delta_{ij}^{+}/\delta_{ij}^{-}$ to measure the relationship between w_i/w_j and a_{ij}. The relationship has the following form:

$$\left(\frac{\delta_{ij}^{+}}{\delta_{ij}^{-}}\right)\left(\frac{w_i}{w_j}\right)=a_{ij},$$

,where $\left(\delta_{ij}^{+}\geq 1 \quad \& \quad \delta_{ij}^{-}=1\right)$ or $\left(\delta_{ij}^{-}\geq 1 \quad \& \quad \delta_{ij}^{-}=0\right)$ (2.5.11)

In Bryson's method, the aim is to minimize $\prod_{i}\prod_{j>i}\left(\delta_{ij}^{+}\delta_{ij}^{-}\right)$. To solve Eq. 2.5.11 , the non-linear programming problem is translated into the linear goal programming problem with the following form:

$$\text{Min} \qquad \ln\Theta=\sum_{i=1}^{n}\sum_{j>i}^{n}\left(\ln\delta_{ij}^{+}+\ln\delta_{ij}^{-}\right)$$

$$\text{Subject to } \ln w_i-\ln w_j+\ln\delta_{ij}^{+}-\ln\delta_{ij}^{-}=\ln a_{ij},\quad \forall(i,j)\in IJ$$

,where $IJ=\{(i,j):1\leq i<j\leq n\}$; $\ln\delta_{ij}^{+}$ and $\ln\delta_{ij}^{-}$ are non-negative. (2.5.12)

Ideally the objective value should be 0 when $\ln\delta_{ij}^{+}=\ln\delta_{ij}^{-}=0$, i.e. $\delta_{ij}^{+}=\delta_{ij}^{-}=1$. The computer tries to minimize the value as low as possible for the solution by many loops. In the simulation of this research, GP does not provide the unique results of the priorities, even if the same objective value is achieved by the functions *FindMinimum[.]* and *NMinimize[.]* in Mathematica. Also *Lingo* and *Mathematica* provide different priority results subject to the same objective values. Also the priority vector with adding $w_i>0,\forall i$ as the constraints is different from the one without adding $w_i>0,\forall i$, although the objective values are the same. These facts can be concluded that GP leads to various priority vectors for a same objective values. (The mathematical proof is left for readers as it is beyond the research purpose.)

g) Enhanced Goal Programming (EGP)

Lin (2006) has proposed Enhanced Goal Programming model, which is the combination of GP and LLS, and has the form:

$$
\begin{aligned}
&\text{Min} \quad \ln\Theta + \varepsilon\Delta \\
&\text{Subject to } \ln w_i - \ln w_j + \ln\delta_{ij}^{+} - \ln\delta_{ij}^{-} = \ln a_{ij}, \quad \forall(i,j)\in IJ \\
&\qquad \ln\Theta = \sum_{i=1}^{n}\sum_{j>i}^{n}\left(\ln\delta_{ij}^{+} + \ln\delta_{ij}^{-}\right) \\
&\qquad \Delta = \sum_{i=1}^{n}\sum_{j>i}^{n}\left(\left(\ln\delta_{ij}^{+}\right)^2 + \left(\ln\delta_{ij}^{-}\right)^2\right)
\end{aligned}
\tag{2.5.13}
$$

, where $IJ = \{(i,j): 1 \le i < j \le n\}$, $\ln\delta_{ij}^{+} \ge 0$, $\ln\delta_{ij}^{-} \ge 0$ and ε is a sufficient small positive number. The term sufficiently small means that any increase of ε will cause $\ln\Theta$ to lose its optimality. In his paper, ε is set to 10^{-10} as an example, which is approximate to 0. When $\ln\Theta$ reaches its optimum, a tradeoff rate exists between $\ln\Theta$ and Δ. The decrease of Δ leads to the increase of $\ln\Theta$. The effect of ε is to depress $\ln\Theta$ to increase. When ε is sufficiently small, any sacrifice of $\ln\Theta$ for reducing Δ would be fruitless. Thus the model forces the solution to minimize $\ln\Theta$ before minimizing Δ. However, this optimization method induces more computational effort than LLS and GP.

h) Least Penalty Optimization Operators

LPO operators are proposed in the research (Yuen, 2009h) after several POs are reviewed. Least Penalty Optimization (LPO) operators are the POs that the penalty is used in the optimization model. Two LPO operators are proposed: Least Product of Penalty and Direct Squares (LPPDS) and Least Product of Penalty and Weighted Squares (LPPWS).

The LPPDS Operator is to apply a set of penalties $\{B_{ij}\}$ in the Direct Least Squares, and it has the following form:

$$\text{Min} \qquad L=\sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij}\cdot\left(a_{ij}-{w_i}/{w_j}\right)^2$$

$$,\text{where } B_{ij}=\begin{cases}\beta_1, & w_i>w_j \ \& \ a_{ij}>1 \\ & \text{or } w_i<w_j \ \& \ a_{ij}<1 \\ \beta_2, & w_i=w_j \ \& \ a_{ji}\neq 1 \\ & \text{or } w_i\neq w_j \ \& \ a_{ji}=1 \\ \beta_3, & otherwise\end{cases},1=\beta_1\le\beta_2\le\beta_3 \qquad (2.5.14)$$

$$\text{Subject to } \sum_{i=1}^{n} w_i=1,\ \ w_i>0, i=1,2,\ldots,n$$

β_1,β_2,β_3 are the penalty weights.

The solution can be easily solved by some software tools such as *Excel, Mathlab, Lingo*, as well as *Mathematica* which is used in this research. Yuen (2009h) indicated that LPPDS performs better than DLS, and other POs on the basis of *Root Mean Penalty Weighted Square Variance (*RMPWSV) measurement, which is shown in the next section.

The LPPWS Operator is to apply a set of penalties $\{B_{ij}\}$ in the Weighted Least Squares, and it has the following form:

$$\text{Min} \quad G = \sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij}\cdot\left(w_i - a_{ij}w_j\right)^2$$

$$\text{, where } B_{ij} = \begin{cases} \beta_1, & w_i > w_j \ \& \ a_{ij} > 1 \\ & \text{or } w_i < w_j \ \& \ a_{ij} < 1 \\ \beta_2, & w_i = w_j \ \& \ a_{ji} \neq 1, \quad 1 = \beta_1 \le \beta_2 \le \beta_3 \\ & \text{or } w_i \neq w_j \ \& \ a_{ji} = 1 \\ \beta_3, & otherwise \end{cases} \tag{2.5.15}$$

$$\text{Subject to } \sum_{i=1}^{n} w_i = 1, \quad w_i > 0, i = 1,2,\ldots,n$$

Unlike Weighted Least Squares, it is very difficult to get the closed form of the above solutions. Thus this research suggests numerical methods to solve equations, which is performed by many software tools. Yuen (2009h) indicated that *LPPWS* performs better than WLS on the basis of the *Root Mean Penalty Weighted Square Variance (*RMPWSV) measurement model, which is shown in the next section.

Example 2.2

Consider a 3x3 pairwise matrix with the priority set $W = \{w_1, w_2, w_3\}$. The reciprocal matrix with CR is:

$$A = \begin{pmatrix} 1 & \frac{1}{4} & \frac{1}{2} \\ 4 & 1 & 5 \\ 2 & \frac{1}{5} & 1 \end{pmatrix}, \text{CR= } 0.081 \tag{2.5.16}$$

The results of the above eleven prioritization operators are shown in table 2.7 (The result of NGMR is the same as the one of LLS). P_{11} is the Least Product of Penalty and Direct Squares operator, and P_{12} is the Least Product of Penalty and Weighted Squares operator. As it can be observed that different prioritization operators produce different values, and possibly lead to different local rank. For the local rank, the larger number of the priority is defined as the higher rank. The question is which PO the most appropriate to represent the approximate value is. Thus this follows the discussion of Prioritization Operator Measurement Models in the next section.

Table 2.7: Prioritization results for various analytic prioritization operators

Notations	POs	w_1	w_2	w_3	Local Rank
P_1	*EV*	0.1265	0.6870	0.1865	(1,3,2)
P_2	*NRS*	0.1171	0.6689	0.2140	(1,3,2)
P_3	*NRCS*	0.1448	0.6992	0.1560	(1,3,2)
P_4	*AMNC*	0.1307	0.6768	0.1925	(1,3,2)
P_5	*NGMR/ LLS*	0.1265	0.6870	0.1865	(1,3,2)
P_6	*DLS*	0.1587	0.6926	0.1487	(2,3,1)
P_7	*WLS*	0.1535	0.6977	0.1487	(2,3,1)
P_8	*FP*	0.1282	0.7179	0.1538	(1,3,2)
P_9	*LGP*	0.0769	0.7692	0.1538	(1,3,2)
P_{10}	*EGP*	0.1265	0.6870	0.1864	(1,3,2)
P_{11}	*LPPDS*	0.1551	0.6898	0.1551	(1,3,1)
P_{12}	*LPPWS*	0.1552	0.6896	0.1552	(1,3,1)

2.5.3. Analytic prioritization operator measurement models

When a problem is introduced, many possible solutions are proposed. This leads to a question which solution model the best one is. Thus the study of the measurement models is introduced. This also leads to various measurement models. Then the next question is which measurement model the most appropriate is. The fittest measurement model must be supported by the convincible reasons that it performs better than other models.

The Analytic Prioritization Operator Measurement Models evaluate the fitness of the prioritization operators (Yuen, 2009g). Thus they can be used for selecting the fittest PO by the comparing different POs (Yuen, 2009g). Several important measurement models are reviewed. By taking the advantages and eliminating the disadvantages of these methods, a new variance model is developed.

Golany and Kress (1993) proposed the Total Deviation (TD) to measure the sum of the square deviations between ratio of weights and their corresponding entry in the matrix. Mikhailov and Sing (1999) took the square root of TD as Euclidean Distance (ED) as follows:

$$ED(A,W)=\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_{ij}-\frac{w_i}{w_j}\right)^2} \qquad (2.5.17)$$

As ED depends on the size, i.e. nxn, of the reciprocal matrix A, for easier interpretation of the result, it is more appropriate to use the average of the value. The

Root Mean Square Variance takes the root of the average of the sum of square deviations (yuen, 2009b, 2009c, 2009g), as follows:

$$RMSV(A,W)=\sqrt{\frac{1}{n\times n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_{ij}-\frac{w_i}{w_j}\right)^2} \quad (2.5.18)$$

However, a limitation of TD/ED/RMSV is that the penalty weights are not justified. For example, the penalty of the condition $\left(w_i > w_j \ \& \ a_{ij} < 1 \,\&\, a_{ij} \neq \frac{w_i}{w_j}\right) \equiv True$ is not the same as the one of the condition $\left(w_i > w_j \ \& \ a_{ij} > 1 \,\&\, a_{ji} \neq \frac{w_j}{w_i}\right) \equiv True$. In short, the weights of these two conditions should not be equal.

To determine the variance associated with penalty weights, Minimum Violation (Golany and Kress, 1993), which is used in (Mikhailov and Sing, 1999 ; Srdjevic , 2005), can be further developed as weight determination, as follows:

$$MV(A,W)=\sum_i\sum_j I_{ij}$$

$$I_{ij}=\begin{cases}1 & ,w_i > w_j \ \& \ a_{ij} < 1\\ 0.5 & ,w_i = w_j \ \& \ a_{ij} \neq 1\\ 0.5 & ,w_i \neq w_j \ \& \ a_{ij} = 1\\ 0 & ,Otherwise\end{cases} \quad (2.5.19)$$

A mistake of above function is that I_{ij} should be 1 if $w_i < w_j \ \& \ a_{ij} > 1$. In addition, as the value of MV depends on the size (n^2) of the matrix (usually a larger sized matrix leads to a higher value of MV), the mean value of MV is more appropriate for measuring POs. Thus, on the basis of these two improvements, the *Mean MV* (yuen, 2009b, 2009c, 2009g) has the form:

$$MMV(A,W) = \frac{1}{n^2}\left(\sum_i \sum_j I_{ij}\right)$$

$$\text{,where} \quad I_{ij} = \begin{cases} 1 & ,\left(w_i > w_j \ \ \& \ \ a_{ji} > 1\right) \\ 1 & \left(w_i < w_j \ \ \& \ \ a_{ji} < 1\right) \\ 0.5 & ,w_i = w_j \ \ \& \ \ a_{ji} \neq 1 \\ 0.5 & ,w_i \neq w_j \ \ \& \ \ a_{ji} = 1 \\ 0 & ,Otherwise \end{cases} ; \qquad (2.5.20)$$

A limitation of MMV is that it counts the penalty scores only, and ignores the actual variance values.

Finally, to combine the advantages of RMSV and MMV, and offset their shortages, the *Root Mean Penalty Weighted Square Variance* σ (which is revised from yuen (2009b, 2009c, 2009g)), as follows:

$$\sigma = RMPWSV(A,W) = \sqrt{\frac{1}{n \times n}\sum_i \sum_j Y_{ij}}$$

$$\text{, where} \quad Y_{ij} = \begin{cases} \beta_1\left(a_{ij} - \dfrac{w_i}{w_j}\right)^2 & ,w_i > w_j \ \ \& \ \ a_{ij} > 1 \\ & or\ w_i < w_j \ \ \& \ \ a_{ij} < 1 \\ \beta_2\left(a_{ij} - \dfrac{w_i}{w_j}\right)^2 & ,w_i = w_j \ \ \& \ \ a_{ij} \neq 1 \ ,1 = \beta_1 \leq \beta_2 \leq \beta_3 \\ & \text{or } w_i \neq w_j \ \ \& \ \ a_{ij} = 1 \\ \beta_3\left(a_{ij} - \dfrac{w_i}{w_j}\right)^2 & ,otherwise \end{cases} \qquad (2.5.21)$$

$\beta = \{\beta_1, \beta_2, \beta_3\}$ is the vector of penalty weights. RMSV is the special case of RMPWSV if $\beta_1 = \beta_2 = \beta_3 = 1$. In MMV, $\beta_1 = 0, \beta_2 = 0.5, \beta_3 = 1$. However, zero of β_1 can cancel the variance. By default settings of σ, $\beta_1 = 1$ is defined, and also $\beta_2 = 3, \beta_3 = 10$.

Remarks: PO Measurement models such as RMPWSV and RMSV for CNP are different from ones for AHP, although they have the same name. More PO measurement models of AHP can be referred to the study (Yuen, 2009g).

2.5.4. Graph theory interpretation

The conventional graph theory can show the prioritization problem of only three criteria. If more than three criteria, the graphical representation is impossible as it is a complex hyper dimensional problem. In fact, the visualization of the hyper dimensions is out of the human perception. Thus, 2D and 3D representations are applied for a prioritization problem of three criteria since the reciprocal matrix of two criteria is always consistent, as follows (Yuen, 2009h).

a) Two dimensional representation

Consider a 3x3 prioritization problem with the priority set $W = \{w_1, w_2, w_3\}$ which is of the form:

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} \\ 1/a_{12} & 1 & 5 \\ 1/a_{13} & 1/a_{23} & 1 \end{pmatrix} \tag{2.5.22}$$

As the axiom of the ratio scale is the form $a_{ij} = w_i/w_j$, a system of three linear equations is of the form:

$$\begin{cases} w_1 - a_{12}w_2 = 0 \\ w_1 - a_{13}w_3 = 0 \\ w_2 - a_{23}w_3 = 0 \end{cases} \tag{2.5.23}$$

Another axiom of the priorities is the form $\sum_{i=1}^{n} w_i = 1$, thus $w_1 + w_2 + w_3 = 1$. To eliminate w_3 and plot a plane of w_1 and w_2, the new form of the linear system is :

$$\begin{cases} w_2 = \dfrac{w_1}{a_{12}} \\ w_2 = a_{13} - \dfrac{w_1(a_{13}+1)}{a_{13}} \\ w_2 = \dfrac{a_{23}(1-w_1)}{1+a_{23}} \end{cases} \tag{2.5.24}$$

To illustrate the above linear system, the next step is to plot the lines in the 2D plane. Let $a_{12} = \dfrac{1}{4}$, $a_{13} = \dfrac{5}{4}$, and $a_{23} = 5$, the matrix is perfectly consistent. In fig. 2.7, the three equations have an intercept point (0.172,0.690), which is the unique solution of the priorities.

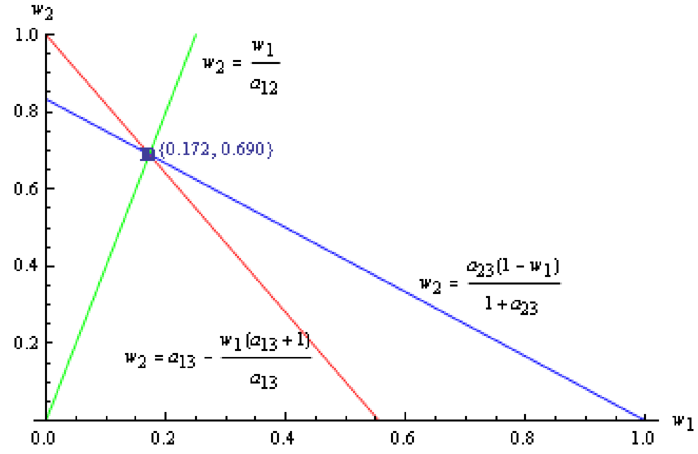


Figure 2.7: The feasible points of APOs in overview

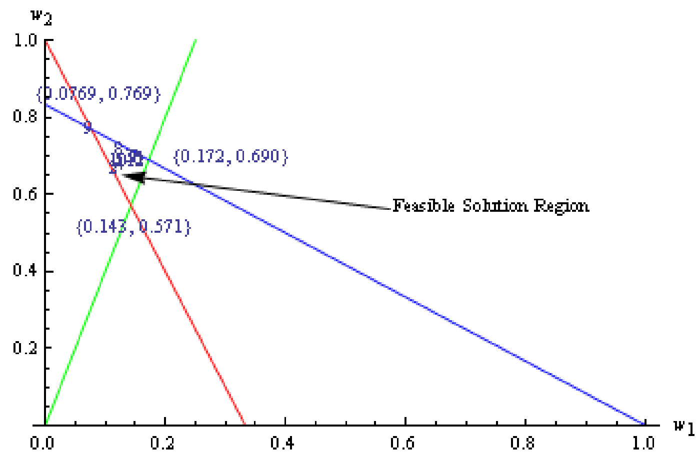


Figure 2.8: The feasible solution region of APOs

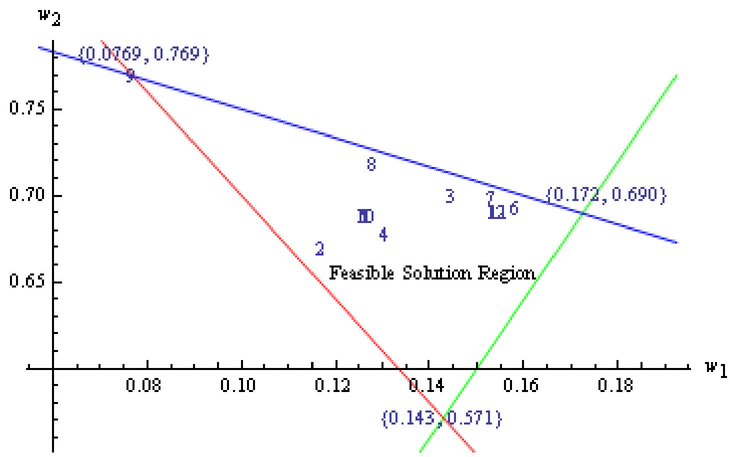


Figure 2.9: The feasible solution region of APOs in focus view

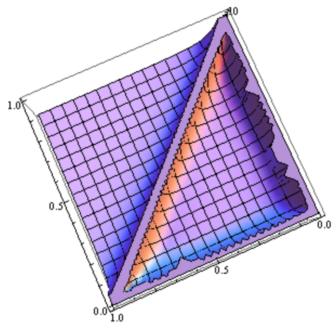
Consider the matrix in Example 1, i.e. $a_{13} = \frac{1}{2}$. In fig. 2.8, the Feasible Solution Region is constituted by three lines. Fig 3 shows the focus view of the region. It can be observed that all solutions of all prioritization operators proposed in this paper are located within this region. The index number indicates the index of the PO, which is defined in table 1. In fig. 2.9, it can be seen that the results of 5 and 10, as well as 11 and 12 are overlapped.

It might be suggested to draw some lines for the proposed Prioritization Operator Measurement Functions, e.g. ED $\sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_{ij} - \frac{w_i}{w_j}\right)^2 = 0$ or RMPWSV $\sum_{i=1}^{n}\sum_{j=1}^{n} Y_{ij} \cdot \left(a_{ij} - \frac{w_i}{w_j}\right)^2 = 0$, to elaborate the best approximate solution points. However, it is impossible to show the lines in the plane. The reason is that, for $w_1 \in [0,1]$, w_2 is the complex number after the algebraic elimination operation, or vice versa. Thus the third dimension is needed to be created for measurement functions.

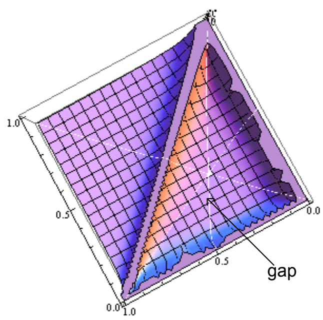
b) Three dimensional representation

In the 2D plane of w_1 and w_2, a dimension z is created for exploring the evaluation value by a measurement function. Two measurement functions are explored and compared in this section: *Root Mean Square Variance (RMSV) and Root Mean Penalty Weighted Square Variance (RMPWSV).*

Fig 2.10 shows the *RMSV* and *RMPWSV* for all $w_1, w_2 \in [0,1]$ from a top view. It shows that there are three white lines to separate the regions in the graph of RMPWSV whilst this does not happen in RMSV. The reason can be found in fig. 2.11 which shows the same content of fig. 2.10 but from a side view. It can be observed that some areas are leveled up accordingly. This is due to the fact that the penalty weights $B_{ij}(w_i, w_j)$ increase in 3 or 10 times. This is similar to the earthquake. If the intensity of the earthquake increases, the degree of the level increases also. For instance, $(\beta_1, \beta_2, \beta_3)$ is from $(1,3,10)$ to $(1,10,100)$. This is shown in fig. 2.12.

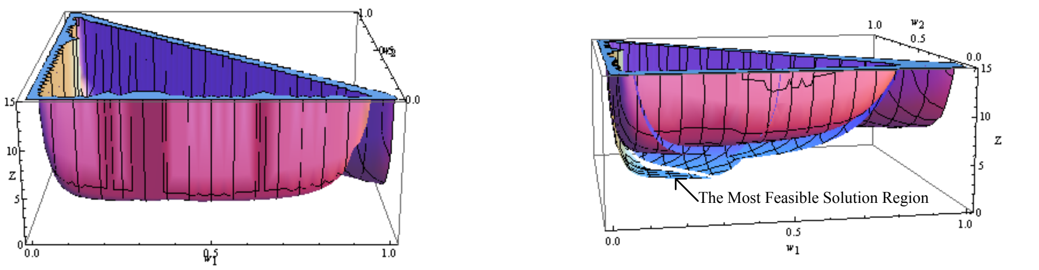


Graph of Root Mean Square Variance



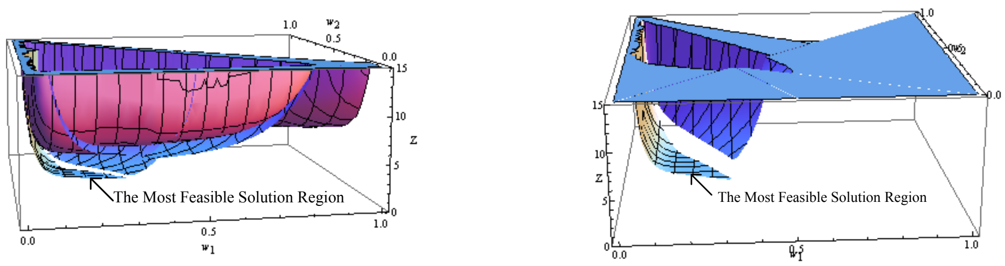
Graph of Root Mean Penalty Weighted Square Variance

Figure 2.10: Top view of the measurement values of APOs on plane (w_1 , w_2)



Root Mean Square Variance | *Root Mean Penalty Weighted Square Variance*

Figure 2.11: Side view of the measurement values of APOs on plane (w_1 , w_2)



RMPWSV with penalty set $(\beta_1, \beta_2, \beta_3) = (1,3,10)$ | *RMPWSV with penalty set* $(\beta_1, \beta_2, \beta_3) = (1,10,100)$

Figure 2.12: The Most Feasible Solution Region of APOs with respect to two β values

The least value of z , i.e. $z_{\min}$, implies the most appropriateness of the combination of the priorities. Thus $z_{\min}$ is in the lowest plane, which is called the Most Feasible Solution Region (MFSR). The MFSR in fig. 2.12 is also within the Feasible Solution Region (FSR) shown in figs. 2.8 and 2.9. $z_{\min}$ can be found by an optimization model discussed in the next section.

Example 2.3

This example continues Example 1. Firstly the Root Mean Square Variance, Mean Minimum Violation, and Root Mean Penalty Weighted Square Variances are found. Then the solution points of each POs on RMSV and RMPWSV in the three dimensional graphs respectively are shown. Finally the results are interpreted.

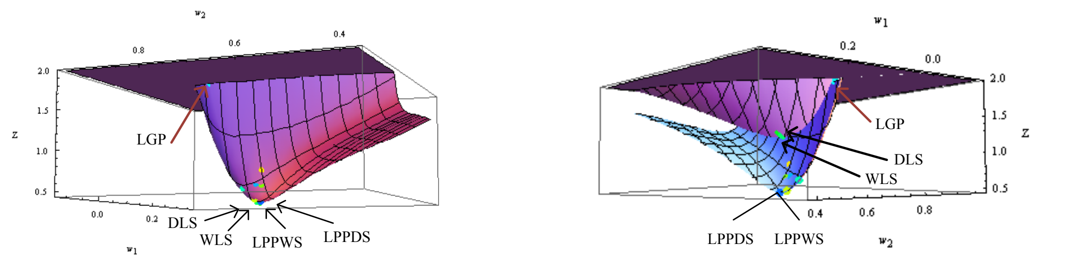
Table 2.8: Analytic prioritization operators' measurement results

Notations	POs	RMSV	MMV	RMPWSV	PO Ranks
P_1	*EV*	0.6743	0	0.6743	(8,1,6)
P_2	*NRS*	0.8502	0	0.8502	(11,1,9)
P_3	*NRCS*	0.4703	0	0.4703	(5,1,3)
P_4	*AMNC*	0.6590	0	0.6590	(7,1,5)
P_5	*NGMR*	0.6743	0	0.6743	(8,1,6)
P_6	*DLS*	0.4349	0.2222	1.2804	(1,11,11)
P_7	*WLS*	0.4396	0.2222	1.2411	(2,11,10)
P_8	*FP*	0.6171	0	0.6171	(6,1,4)
P_9	*LGP*	2.0006	0	2.0006	(12,1,12)
P_{10}	*EGP*	0.6744	0	0.6744	(10,1,8)
P_{11}	*LPPDS*	0.4418	0	0.4418	(3,1,1)
P_{12}	*LPPWS*	0.4419	0	0.4419	(4,1,2)

Table 2.8 shows RMSV, MMV, and RMPWSV. Although both DLS and WLS have the least two RMSV, they have the highest two violations. Thus RMPWSVs of only DLS and WLS increases (with respect to RMSV of them) and become the highest two. Others remain unchanged, e.g. their RMPWSV is the same as their RMSV.

When assigning the solution points of all POs in the 3D graphs shown in Fig. 2.13. It shows that DLS, WLs, LPPWS, LPPDS are close with respect to RMSV, and DLS is the best solution using RMSV as the measurement criterion. The related values can be found in Table 2.7. However, when RMPWSV is used, the results of the DLS and WLS are otherwise due to their volitions. In Fig. 2.13, it can be observed that DLS and WLS are not in the Most Feasible Solution Region (MFSR). Thus LPPWS and LPPDS are the best two POs. And LPPDS performs the best result.

LGP is not the appropriate method as it always in the intercept of two lines (Fig. 2.8), which means relatively high value of RMSV or RMPWSV. In order to investigate the validity of LPPWS and LPPDS with more cases, Yuen (2009h) performed various numerical analyses.



Solution points of POs using RMSV *Solution points of POs using RMPWSV*

Figure 2.13: Focus views of solution points of APOs in the 3D graph

2.6 Fuzzy Analytic Hierarchy Process (FAHP)

The Analytic Hierarchy Process (Saaty, 1980) is a popular model to aggregate multiple criteria for decision making. The limitation is that the measurement scale for the value of the utility function, which is basically numerical and probabilistically judgmental, induces evaluation problem. This introduces the studies on fuzzy AHP (e.g. LaarHoven and Pedrycz, 1983; Boender et al. 1989; Chang, 1996; Wang et al., 2006, 2008; Yuen, 2008; Yuen and Lau, 2008b, 2008c) to address the limitation. Applications of the Analytic Hierarchy Process (AHP) and the Fuzzy Analytic Hierarchy Process increasingly address the attentions of the industry applications and scholarly research.

2.6.1 FAHP classifications

This review, which is revised from (Yuen, 2008), classifies the Fuzzy AHP into two types of core processes. Type I includes fuzzy assessment, fuzzy prioritization, defuzzification, and crisp synthesis. Type II includes fuzzy assessment, fuzzy prioritization, and fuzzy synthesis. The Extent Analysis Method (EAM) (Chang, 1996) is Type I whilst modified Fuzzy LLSM (Wang et al., 2006, 2008) is Type II.

In the fuzzy assessment, a fuzzy comparison matrix is expressed by

$$\bar{A}=\left(\bar{a}_{ij}\right)_{nxn}=\begin{pmatrix} (1,1,1) & (l_{12},m_{12},u_{12}) & \cdots & (l_{1n},m_{1n},u_{1n}) \\ (l_{21},m_{21},u_{21}) & (1,1,1) & \cdots & (l_{2n},m_{2n},u_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (l_{n1},m_{n1},u_{n1}) & (l_{n2},m_{n2},u_{n2}) & \cdots & (1,1,1) \end{pmatrix} \tag{2.6.1}$$

,where $\bar{a}_{ij}=\left(l_{ij},m_{ij},u_{ij}\right)=\bar{a}_{ji}^{-1}=\left(1/u_{ji},1/m_{ji},1/l_{ji}\right)$,for $i,j=1,\ldots,n$ and $i\neq j$. $\bar{a}_{ij}=\left(1,1,1\right)$ if $i=j$.

The verbal judgment is usually on a 9 point verbal scale represented by fuzzy numbers: (1,1,1) for equal importance, (1.5, 2, 2.5) for weak importance, and finally (8.5,9,9.5) for extreme importance. In this study, $\bar{A}$ is further decomposed as follows:

$$A^{l}=\{l\}=\begin{pmatrix} 1 & l_{12} & \cdots & l_{1n} \\ u_{21} & 1 & \cdots & l_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \cdots & 1 \end{pmatrix} \tag{2.6.2}$$

$$A^{m}=\{m\}=\begin{pmatrix} 1 & m_{12} & \cdots & m_{1n} \\ m_{21} & 1 & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & 1 \end{pmatrix} \tag{2.6.3}$$

$$A^{u}=\{u\}=\begin{pmatrix} 1 & u_{12} & \cdots & u_{1n} \\ l_{21} & 1 & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & 1 \end{pmatrix} \tag{2.6.4}$$

The fuzzy consistence ratio (FCR) of $\bar{A}$, which is proposed by this study, is defined as

$$FCR=CR\left(A^{l}\right)^{0.25}\cdot CR\left(A^{m}\right)^{0.5}\cdot CR\left(A^{u}\right)^{0.25} \tag{2.6.5}$$

,where CR is the function of the crisp consistency ratio of the crisp AHP. If FCR>1, revision of $\bar{A}$ is needed.

In the fuzzy prioritization, $\bar{A}$ is derived as a vector of fuzzy priorities or fuzzy relative weights $\tilde{W} = \{\tilde{w}_i\}$ and $\tilde{w}_i = \left(w_i^U, w_i^M, w_i^L\right)$. These two steps are the same in the Type I and Type II methods, but the following steps are different.

In the Type I method, each $\tilde{w}_i$ is defuzzified as a crisp number, and then these crisp numbers are synthesized. This synthesized step is the same as the crisp AHP. The aggregation technique $Agg:\{W\} \rightarrow V$ is usually the weighted average method. Thus the final value is the crisp number.

In the Type II method, each $\tilde{w}_i$ is directly aggregated as a global fuzzy priority vector $\bar{V} = \{\bar{v}_1, \ldots, \bar{v}_n\}$, $\bar{v}_i = \left(\tilde{v}_i^U, \tilde{v}_i^M, \tilde{v}_i^L\right)$ by a fuzzy aggregation operator $FAgg:\{\bar{W}\} \rightarrow \bar{V}$.

The problems of Fuzzy AHP are similar to the generic AHP problems as the Fuzzy AHP is the extension of the AHP.

2.6.2 Extent analysis method

Chang (1996) proposed an Extent Analysis Method to derive the priority of a fuzzy comparison matrix, with five steps as follows:

Step 1: sum up each row of $\bar{A}$ by fuzzy addition:

$$RS_i = \sum_{j=1}^{n} a_{ij} = \left(\sum_{j=1}^{n} l_{ij}, \sum_{j=1}^{n} m_{ij}, \sum_{j=1}^{n} u_{ij}\right), \quad i = 1, \ldots, n \tag{2.6.6}$$

Step 2: normalize RS_i, $i = 1,\dots,n$ by

$$S_i = \frac{RS_i}{\sum_{j=1}^{n} RS_j} = \left(\frac{\sum_{j=1}^{n} l_{ij}}{\sum_{i=1}^{n}\sum_{j=1}^{n} u_{ij}}, \frac{\sum_{j=1}^{n} m_{ij}}{\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}}, \frac{\sum_{j=1}^{n} u_{ij}}{\sum_{i=1}^{n}\sum_{j=1}^{n} l_{ij}} \right), \quad i = 1,\dots,n \qquad (2.6.7)$$

Step 3: compute the degree of possibility of $S_i \geq S_j$ by

$$V\left(S_i \geq S_j\right) = \begin{cases} 1, & m_i \geq m_j \\ \dfrac{u_i - l_j}{\left(u_i - m_i\right) + \left(m_j - l_j\right)}, & 1_j \leq u_j \\ 0, & otherwise \end{cases}, \quad i, j = 1,\dots,n; j \neq i,$$

$$S_i = \left(l_i, m_i, u_i\right) \text{ and } S_j = \left(l_j, m_j, u_j\right). \qquad (2.6.8)$$

Step 4: calculate the degree of possibility of S_j over all the other (n-1) fuzzy numbers by

$$V'\left(S_i \geq S_j : j = 1,\dots,n; j \neq i\right) = \min_{j \in \{i,\dots,n\}, j \neq i} V'\left(S_i \geq S_j\right), i = 1,\dots,n \qquad (2.6.9)$$

Step 5: derive the priority vector $W = \left(w_1,\dots,w_n\right)$ of $\overline{A}$ by the following form:

$$w_i = \frac{V'\left(S_i \geq S_j : j = 1,\dots,n; j \neq i\right)}{\sum_{k=1}^{n} V'\left(S_k \geq S_j : j = 1,\dots,n; j \neq k\right)}, \quad i = 1,\dots,n \qquad (2.6.10)$$

One critical error of EAM is the steps 1 and 2 which can be regarded as the Fuzzy Normalized Row Sum Method (NRSM) in direct form. However, this normalization method is false as $\overline{a}_{ij} = \left(l_{ij}, m_{ij}, u_{ij}\right) = \overline{a}_{ji}^{-1} = \left(1/u_{ji}, 1/m_{ji}, 1/l_{ji}\right)$. To improve this error, $l_{ij} \in A^l$, $m_{ij} \in A^m$, $u_{ij} \in A^u$ is used in Eqs. 2.2.6 and 2.6.7. In other words, $l_{ij} \notin \overline{A}$, $m_{ij} \notin \overline{A}$, $u_{ij} \notin \overline{A}$.

In addition, as many fuzzy AHP applications are used in Chang's EAM (2006), Wang (2008) pointed out some other shortcomings of Chang's model and proposed a modified fuzzy LLSM on the basis of previous studies (LaarHoven and Pedrycz, 1983; Boender et al. 1989).

2.6.3 Modified fuzzy logarithmic least squares method (mf-LLSM)

The modified fuzzy LLSM (Wang et al., 2006, 2008) derives the priorities of the triangular fuzzy comparison matrix. The FPO of MF-LLSM has following form:

$$\begin{aligned} Min\ J = \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \Big(& \left(\ln w_i^L - \ln w_j^U - \ln l_{ij}\right)^2 \\ & + \left(\ln w_i^M - \ln w_j^M - \ln m_{ij}\right)^2 \\ & + \left(\ln w_i^U - \ln w_j^L - \ln u_{ij}\right)^2 \Big) \end{aligned}$$

Subject to

$$\begin{cases} w_i^L + \sum_{j=1, j\neq i}^{n} w_j^U \geq 1 & \\ w_i^U + \sum_{j=1, j\neq i}^{n} w_j^L \leq 1 & \\ \sum_{i=1}^{n} w_i^M = 1 & i = 1, \ldots, n \\ \sum_{i=1}^{n} \left(w_i^L + w_i^U\right) = 2 & \\ w_i^U \geq w_i^M \geq w_i^L > 0 & \end{cases} \qquad (2.6.11)$$

The optimum solution to the above model forms a normalized vector of triangular fuzzy weights $\tilde{w}_i = \left(w_i^U, w_i^M, w_i^L\right),\ i = 1,\ldots,n$.

Table 2.9: Synthesis of local fuzzy weights of the modified fuzzy LLSM

Alternatives	Criterion 1	$\cdots$	Criterion j	$\cdots$	Criterion m	Global fuzzy weights
	$\left(w_1^L, w_1^M, w_1^U\right)$	$\cdots$	$\left(w_j^L, w_j^M, w_j^U\right)$	$\cdots$	$\left(w_m^L, w_m^M, w_m^U\right)$	
A_1	$\left(w_{11}^L, w_{11}^M, w_{11}^U\right)$	$\cdots$	$\left(w_{1j}^L, w_{1j}^M, w_{1j}^U\right)$	$\cdots$	$\left(w_{1m}^L, w_{1m}^M, w_{1m}^U\right)$	$\left(w_{A_1}^L, w_{A_1}^M, w_{A_1}^U\right)$
$\vdots$	$\vdots$	$\cdots$		$\cdots$	$\vdots$	$\vdots$
A_k	$\left(w_{k1}^L, w_{k1}^M, w_{k1}^U\right)$	$\cdots$	$\left(w_{kj}^L, w_{kj}^M, w_{kj}^U\right)$	$\cdots$	$\left(w_{km}^L, w_{km}^M, w_{km}^U\right)$	$\left(w_{A_k}^L, w_{A_k}^M, w_{A_k}^U\right)$
$\vdots$	$\vdots$	$\cdots$		$\cdots$	$\vdots$	$\vdots$
A_n	$\left(w_{n1}^L, w_{n1}^M, w_{n1}^U\right)$	$\cdots$	$\left(w_{nj}^L, w_{nj}^M, w_{nj}^U\right)$	$\cdots$	$\left(w_{nm}^L, w_{nm}^M, w_{nm}^U\right)$	$\left(w_{A_n}^L, w_{A_n}^M, w_{A_n}^U\right)$

After the local fuzzy weights are obtained, a global fuzzy weight should then be calculated with the presentation in table 2.9. Global fuzzy weights can be obtained by solving the following two linear programming models and an equation for each decision alternative.

$$w_{A_k}^L = \underset{w \in \Omega w}{Min} \sum_{j=1}^{m} w_{kj}^L w_j\ ,\ k = 1,\ldots,K, \tag{2.6.12}$$

$$w_{A_k}^U = \underset{w \in \Omega w}{Max} \sum_{j=1}^{m} w_{kj}^U w_j\ ,\ k = 1,\ldots,K, \tag{2.6.13}$$

$$w_{A_k}^M = \underset{w \in \Omega w}{Max} \sum_{j=1}^{m} w_{kj}^M w_j^M\ ,\ k = 1,\ldots,K, \tag{2.6.14}$$

where $\Omega_w = \left\{ W = (w_1, \dots, w_m) \mid w_j^U \ge w_j^M \ge w_j^L, \sum_{j=1}^{m} w_j = 1, j = 1, \dots, m \right\}$ is the space of weights $\left(w_j^L, w_j^M, w_j^U\right)$ is the normalized triangular fuzzy weight of criterion j $(j = 1, \dots, m)$ and $\left(w_{kj}^L, w_{kj}^M, w_{kj}^U\right)$ is the normalized triangular fuzzy weight of alternative A_k with respect to the criterion j $(k = 1, \dots, K; j = 1, \dots, m)$.

Remarks: EAM produces fuzzy relative weights from the Fuzzy Normalized Row Sum Method (NRSM) in Steps 1-2, although the result is not correct. These fuzzy weights are finally converted to a crisp weight value. On the other hand, the MF-LLSM produces a fuzzy value by the fuzzy optimization method. It is more appropriate to compare MF-LLSM and NRSM of EAM.

In order to fairly critique both methods, the Fuzzy Prioritization Measurement (FPM) model is proposed to measure the fitness levels for a fuzzy comparison matrix in next section.

2.6.4 Fuzzy analytic prioritization measurement models

The Fuzzy Analytic Prioritization Measurement (FAPM) model evaluates the validity of the fuzzy analytic prioritization operators. The crisp Root Mean Penalty Weighted Square Variance (RMPWSV) (Chapter 2.5.3) is extended as a Fuzzy

RMPWSV (FPRMPWSV) by considering a modal value and two interval values of $\bar{A}$.

Thus the FPRMPWSV is of the form:

$$\bar{\sigma} = \left(\sigma^L, \sigma^M, \sigma^U\right), \text{ where} \tag{2.6.15}$$

$$\sigma^L = WRMSV\left(\{l\}, \{w_i^L\}\right), \tag{2.6.16}$$

$$\sigma^M = WRMSV\left(\{m\}, \{w_i^M\}\right), \tag{2.6.17}$$

$$\sigma^U = WRMSV\left(\{u\}, \{w_i^U\}\right). \tag{2.6.18}$$

FPM model is the aggregation of the 3 values of the FPRMPWSV and is defined as follows:

$$\hat{\sigma} = \alpha^L \bullet \sigma^L + \alpha^M \bullet \sigma^M + \alpha^U \bullet \sigma^U \tag{2.6.19}$$

, where $\alpha^M \geq \alpha^L$ or $\alpha^M \geq \alpha^U$, and $\alpha^L + \alpha^M + \alpha^U = 1$. By default $\alpha^M = 0.5$ and $\alpha^L = \alpha^U = 0.25$.

Example 2.4

Two fuzzy analytic comparison matrices are illustrated: one is from Wang et al. (2008), and another is proposed by the author.

Consider two decision criteria with their fuzzy relative weights (Wang et al.,2008): $\tilde{w}_1 = (0.65, 0.7, 0.75)$ and $\tilde{w}_2 = (0.25, 0.3, 0.35)$. Thus the fuzzy comparison matrix is:

$$\bar{A} = \begin{bmatrix} (1,1,1) & (1.8571, 2.333, 3) \\ (0.3333, 0.4286, 0.5385) & (1,1,1) \end{bmatrix}$$

By using the Fuzzy Normalized Row Sum Method (FNRS) of EAM, then $\tilde{w}_1 = (0.516, 0.700, 0.955)$, and $\tilde{w}_2 = (0.24, 0.3, 0.367)$. For the FWRMSV $\bar{\sigma}$=(0.147, 0, 0.2017). The aggregation of FWRMSV $\hat{\sigma}$ is 0.0827.

By using the modified fuzzy LLSM, $\tilde{w}_1 =$ (0.65, 0.700, 0.75), and $\tilde{w}_2 = (0.250, 0.300, 0.350)$. Then $\bar{\sigma}$= (0.379, 0, 0.438), and $\hat{\sigma} = 0.203$, which is larger than FNRS.

Another example is of the form:

$$\bar{A} = \begin{bmatrix} (1,1,1) & (1,2,3) & (3,4,5) \\ (1/3,1/2,1) & (1,1,1) & (1,2,3) \\ (1/5,1/4,1/3) & (1/3,1/2,1) & (1,1,1) \end{bmatrix}.$$

For FNRS, $\bar{\sigma}$= (0.8172, 0, 0.6282), and $\hat{\sigma}$ =0.361. For modified fuzzy LLSM, $\bar{\sigma}$= (0.9195, 0, 0.692), and $\hat{\sigma}$ =0.403, which is also larger than FNRS.

The above results can conclude that although Wang et al. (2008) has proved that EAM may produce wrong decision, in this study, the modified fuzzy LLSM produces a higher aggregated value of FWRMSV than FNRS of EAM, if only the fuzzy prioritization process is considered only. This means that the modified fuzzy LLSM may produce rank reversals due to the approximated fitness being lower than the FNRS. However, as the modified fuzzy LLSM produces the result in fuzzy number, it is selected for comparisons in Chapter 8.

2.7 Cognitive sciences

In the 1960s, cognitive psychology represented a new branch of psychology. Cognitive psychology is the scientific study of the mind. It has been developed in the other branches, such as behavioral psychology, and was heavily influenced by technological developments and the way they help us in understanding complex behaviours (Braisby and Gellatly, 2005). Cognitive psychology is committed to using computers as a tool for aiding understanding of the mind (Braisby and Gellatly, 2005). Some of the fundamental questions that cognitive psychology examines are: how does memory work? How do we perceive our environment? How do we infer from patterns of light or sound the presence of objects in our environment, and their properties? How do we reason, and solve problems? How do we think? (Braisby and Gellatly, 2005).

Cognitive science is a broad view of the cognition study of the mind, and is not limited to the psychological aspect. Cognitive Science can be roughly summed up as the scientific interdisciplinary study of the mind, including philosophy, psychology, linguistics, artificial intelligence, robotics, and neuroscience (Friedenber and Silverman, 2006). Cognitive science is not a unified field of study like each of the disciplines themselves, but a collaborative effort among researchers working in various fields (Friedenber and Silverman, 2006).

This research is interested in the decision making issues for CNP from cognitive aspects. Several essential development topics are reviewed as follows.

2.7.1 Cognitive decision making

In psychology, descriptive and normative theories are commonly used. Normative theories define the supposed ideal decision whilst descriptive theories attempt to characterize how people actually make decisions (Ayton, 2005). The subjective expected utility (SEU) theory (Von Neumann & Morgenstern ,1947), which is a normative theory, has been widely studied in choice problems using probability theory.

Regarding the cognitive theories, one obstacle for the development of the cognitive theories of judgment and decision making behavior is that there are considerable differences among the theories that (appropriately) are called cognitive (Hastie and Pennington, 1995). On the other hand, decision making is the transdisciplinary research in many areas such as mathematics, business, economics, industry engineering, computer science, psychology. In any one area, there are also many sub-areas. For example, in chapter 2.2, there are several models with different mathematical algorithms in the fields of mathematics. Thus it is difficult to unify the notion of cognitive decision making in a single field. Cognitive decision making should be studied in the interdisciplinary aspects of cognitive sciences and decision making. Among each, they are related to other disciplines that are also related. In the CNP model, the cognitive perspective of decision making, which is the symbolic mathematical system using process algebra representation, is proposed in chapter 3.

2.7.2 Perception and computational intelligence

According to Mather (2006), the methods used to study perception include Lesion experiments, clinical studies, single-unit recordings, brain imaging, psychophysics, and artificial intelligence. Artificial intelligence is the computational method to make a machine to function as a human brain. AI also has the name computational intelligence. The perception of the Cognitive Network Process is studied from the computational intelligence (or the alternative name, artificial intelligence) perspective.

The concept of computation lies in the concept of representation at the heart of most present-day theories of perception (Mather, 2006). Boring (1950) noted: “The immediate objects of the perception of our senses are merely particular states induces in the nerves” (P82). As the specific internal state of the brain, in the form of a particular pattern of neural activity, in some sense represents the state of the outside world. Perception must involve the formation of these *representations* in the brain (Mather,2006). Most modern theories of perception are in essence theories about how the brain builds and uses *representation of* the world (Mather, 2006).

There are two views of the representation in the brain: analogue representation and symbolic representation. In the concept of analogue representation, the values in one system, such as spatial position or response rate, vary proportionately with values in another system (Mather, 2006). Symbolic representation is a representation in which discrete symbols such as characters or words in one system act as tokens for the state of another system (Mather, 2006). The computation model of the CNP applies the symbolic representation notion as symbolic representation and computations have traditionally been associated with human cognition, such as problem solving (Newell and Simon, 1972; Mather, 2006) .

Computation can be defined as the manipulation of quantities or symbols according to a set of formal rules, which is called algorithms (Mather, 2006). The perception systems can be considered as representational systems- internal brain states representing the state of the outside world (Mather, 2006). Perceptual analysis proceeds through a series of representation, reflecting a series of neural processing stages (Mather, 2006). Then representation at each level is computed as a new representation at the next level.

Sensation is also the key notion of perception. According to Mather (2006), "simulation of the sense organ induces a conscious mental state. For example, we may sense "sound" when air pressure waves enter the ear. These mental states have particular qualitative, experiential and felt properties such as loudness, pain or color (sometimes are called sensations or **qualia**). By their very nature, sensations are private, and accessible only to the person who has them. Most researchers believe that sensations can be regarded as identical to specific brain states or functions of brain states. For example, there is a specific brain state associated with the sensation of the colour red. If one's sensation of colour changed to, say, green, there would be a corresponding change in brain state. The assumed link between sensations and brain states lies at the very foundation of modern theories of perception, as will become clear below. However, an "explanatory gap" (Levine, 1983, 1999) remains between the physical world (brain states) and the mental world (sensations). No one has been able to explain precisely how the qualitative nature of sensation can be explained by reference to neural activity."

From the above reviews, it can be concluded that in the decision model from a subjective aspect, the decision matrix usually is the output perception of the experts through the symbolic representation function of the external world and the sensation function of the symbolic representation. In other words, experts are essential for the accuracy of the decision result as they provide the inputs of the decision matrix.

2.7.3 Cognitive style and decision making

The term 'cognitive style', was used by Allport (1937), and has been described as a person's typical or habitual mode of problem solving, thinking, perceiving and remembering (Rading and Cheema, 1991). , a style is considered to be a fairly fixed characteristic of an individual (Rading and Cheema, 1991).

Studies in cognitive styles initially developed as a result of interest in individual differences, particularly during the 1960's (Rading and Cheema, 1991). Since the early 1970's, they have been more seriously considered by the teaching and training world (Rading and Cheema, 1991). Recently, they may be developed with artificial intelligence. The new field may be called cognitive styles of computation, which is to classify the individual styles of the algorithms' or functions within the same category. One example is in chapter 6.

Different researchers have used a variety of labels for the styles they have investigated. Rading and Cheema (1991) suggested that the labels be grouped into two principal cognitive styles. These were labeled the Wholist-Analytic and Verbialiser-Imager dimensions.

Most researchers apply a set of uni-dimensional labels, which are postulated in the individual preferences, for the statistical research of the cognitive style. This leads to not having a formal definition of the labels of the cognitive style. In this research, the cognitive style is described by a variable decision attitude which includes three basic labels: pessimistic, neural, and optimistic.

In decision making, decision attitudes can be applied. There are decision attitudes for the assessment (DAFA), decision attitudes for the information fusion (DAFA), decision attitudes for the volition (DAFV). However, this research is limited only to the scope of decision attitudes for the information fusion (chapter 6).

In chapters 2.3.4 and 2.3.5, the aggregation operator is a function or an algorithm to process information, analogous to the human's information process. Cognitive psychology deals with the human information process. Cognitive style is the individual differences of the information processes of the mind. As there are similar relationships between the attributes of the aggregation operators and the cognitive styles, chapter 6 proposes the Cognitive Style and Aggregation Operator (CSAO) model, which includes several algorithms to classify the individual style of the AOs using the linguistic approach.

2.7.4 Cognitive architecture and intelligent decision agent system

Newell (1973) argued that it was not sufficient to develop a collection of discrete models to describe a broad range of psychological phenomena. This leads to the development of cognitive architecture (Mulholland and Watt, 2005). According to Mulholland and Watt (2005), Cognitive Architecture is an overarching framework that can account for a number of phenomena using a fixed set of mechanisms.

To extend the definition of the cognitive architecture, it can be regarded as a Multi

Agent System performing dedicated tasks though different activities performed by the agents. An agent is a type of hardware or (more usually) a software entity with some of these characteristics (Ferber, 1999; Shoham, 1999; He and Leung 2002): ongoing execution, environmented awareness, agent awareness, autonomy, adaptiveness, intelligence, mobility, anthropomorphism, and ability to reproduce. A Multi Agent System (MAS) is a group of the agents joined together to complete a task. A MAS has the following characteristics (He et al 2002; Jennings at el 1998): (1) each agent has partial information or limited capabilities (knowledge, information, or resources), thus each agent has a limited viewpoint; (2) there is no global system control; (3) data in an MAS are decentralized; (4) computation is asynchronous; and (5) different agents can be heterogeneous, for example, with respect to knowledge representation, data format, reasoning model, solution evaluation criteria, goal, architecture, algorithm, language, or hardware platform.

The Intelligent Decision Agent System is defined as a multi-agent-system, including an intelligent decision agent which is the agent embedded with computational algorithms that can “think and make a decision”.

The studies (Yuen 2009a; Yuen and Lau, 2006, 2008a) have proposed a framework to apply the decision algorithms in an enterprise system. The enterprise decision platform (Fig. 2.14), which is a kind of Intelligent Decision Agent System, contains seven tiers described below:

In the Presentation Service Agents Tier (PSAT), different roles of users can interact with the systems by different devices such as web browsers, window client applications, mobile devices, and office documents such as spread sheets and office Word documents. Each device can be regarded as an agent communicating with others by web services. Depending on the user requirements on a specific environment, the interface can be implemented only without implementation of the business logic in this layer. This layer is crucial to the user-friendly experiences such as delivery of right information to right place with right format.

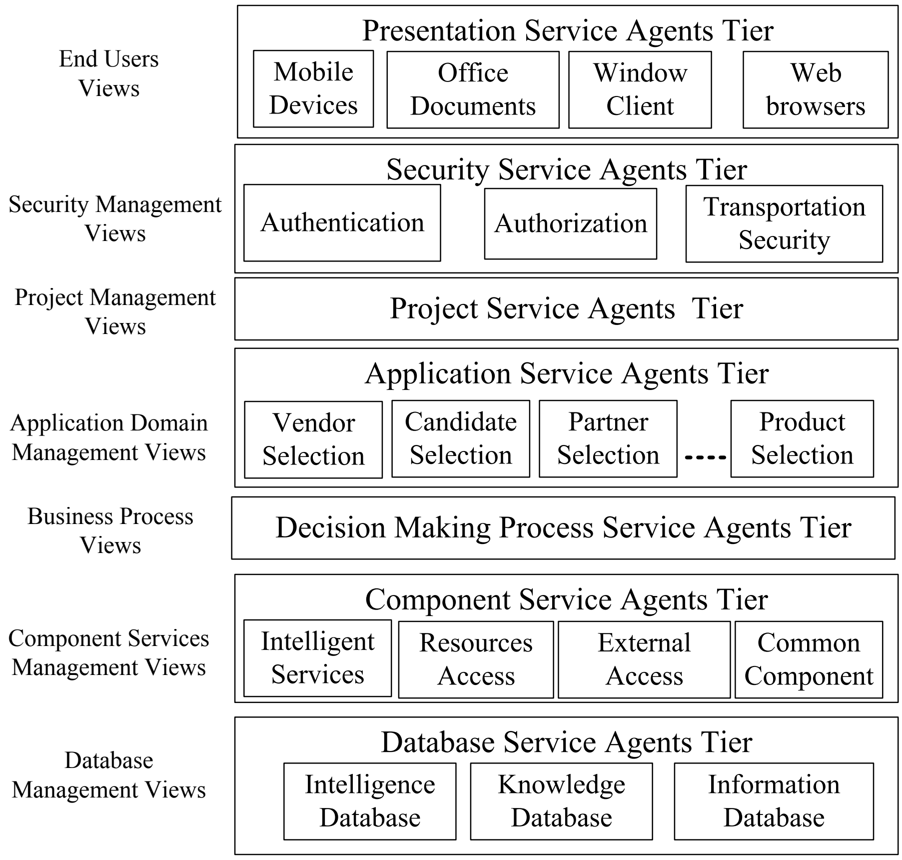


Figure 2.14: The cognitive framework of enterprise decision platform

The Security Service Agents Tier (SSAT) involves the protocols for remote access control, as well as authentication and authorization. Security is the top consideration in the enterprise system. If this function is ignored, no one would like to use the system due to the unreliability. Thus, this tier is the view of every one. The security management corresponds to controlling the delivery of the right resources to the right people. SSAT includes three service agents: Authentication Service Agent which identifies the user, Authorization Service Agent which manages the user's rights, and the Transportation Security Service Agent which responds by ensuring that no data is released in the transportation process.

The Project Service Agents Tier (PSAT) is designed in view of the project management that administers the corresponding projects. PSAT provides services for end users, such as auditors, domain experts and decision makers, so that they can interact with their dedicated projects. The agents in this tier are dynamically created or inherited from the decision template in the Application Service Agents Tier, and are further modified to meet the special requirements by the domain experts.

The Application Service Agents Tier (ASAT) is designed with reference to the application domain experts for the evaluation scope for their corresponding projects. ASAT provides a set of decision templates to meet different decision project requirements, thus this layer enables knowledge sharing of design project criteria, aggregation rules and other attributes. Each template can be regarded as a template service agent such as: Vendor Selection Service Agent, and Partner Selection Service Agent. The Application Service Agent Tier follows the business process managed in the Decision Making Process Service Agents Tier.

The Decision Making Process Services Agents Tier (DMPSA) includes various algorithms of the decision models. While an expert defines a certain decision model for a decision problem, the agents embedded the decision algorithms will perform the related jobs.

The Component Service Agents Tier (CSAT) is designed with reference to software developers to develop the corresponding components. CSA exposes the web interfaces for the decision components, including the decision algorithms, as the consumable web methods. CSAT consists of four component service agents. The Intelligence Service Agent comprises decision algorithms for linguistic modelling, linguistic representation, granular aggregation, rule reasoning, and result exploitation. The Resources Access Service Agents provide the services for data access among their upper and lower tiers. The External Access Agent provides services to access external agents that are out of this framework, such as email server agents and SMS server agent. The Common Library includes the general components for the application such as the graphic engine agent which is usually created from third party software obtained from vendors.

The Database Service Agents Tier (DSAT) is designed with reference to database management. DSAT contains three database systems which perform different roles. The Intelligent Database Service Agent provides the general patterns of parametric inputs from domain experts. The Knowledge Database Service Agent stores the general templates for each project and the templates are derived from the application services layer. The Information Database Service Agent organizes the data for each project which is queried and updated in the Project Service Agent Tier. Separation of these three databases services improves accessibility and manageability of the system.

Five advantages of the seven tier design are:

- Role separations: the new architecture meets different users' perspectives. Each user performs his job without heavy influence to others.
- Reusability: the components are implemented as web methods, which can share and distribute the services to other agents by the loose coupling. A new project can be created rapidly from the application template.
- Knowledge Sharing: The components are implemented as services. Each application class can consume the services with loose coupling. Each project can

inherit the design, criteria, and the schema from multiple application classes. Each user can query multiple projects in a single profile.

- Extensibility and Flexibility: the agents in each tier can be added and modified. Interfaces of the agents in the presentation tiers can be changed accordingly as the business logic is implemented in the components service tier, which are further propagated in ASAT and PSAT.
- Cross platforms and cross applications: end users use different devices to communicate with the systems regardless of the platforms or the applications they used. Development of the Application Service Agent does not rely on the vendors (e.g. MySQL, Oracle, or Microsoft) of the database system, as the Database Service Agents handle this issue.

Chapter 3 Cognitive Network Process

3.1 Introduction

The Cognitive Network Process $CNP = (PGP, CAP, AAP, IFP, DVP)$ is the cognitive architecture which comprises of five cognitive decision processes: the Problem Cognition Process (PCP), Cognitive Assessment Process (CAP), Cognitive Prioritization Process (CPP), Multiple Information Fusion Processes (MIP), and Decisional Volition Process (DVP). The cognitive architecture of CNP is shown in fig. 3.1.

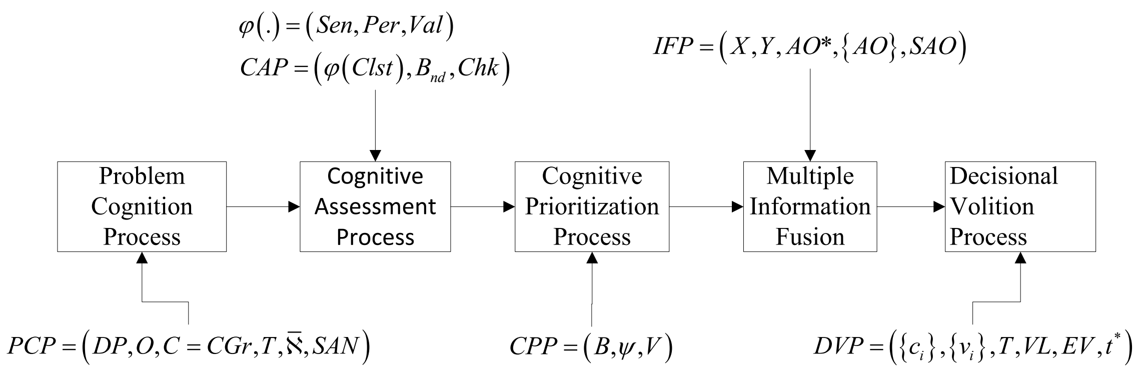


Figure 3.1: The cognitive architecture of the CNP

In view of the collaboration functions, the CNP is handled by two stages: the human cognition stage and the machine cognition stage. The human cognition process involves PCP and CAP, which are not handled by computer algorithm so far since they are of very complex psycho-intelligence. The Machine cognition stage comprises of CPP, MIP and DVP, which are associated with comprehensive algorithms to ensure the efficacy of action. Algorithm 3.1 shows the procedure of CNP.

Algorithm 3.1 (Cognitive Network Process):

Input: Feasible Decision Problem

// Human Cognition Stage

Step 1: Problem Cognition Process (*PCP*)

Step 2: Cognitive Assessment Process (CAP)

//Machine Cognition Stage

Step 3: Cognitive Prioritization Process (CPP)

Step 4: Multiple Information fusion process (MIP)

Step 5: Decisional Volition Process (DVP).

Output: Feasible Solution. #End

Each process is developed in this chapter, and several important functions are further investigated in the development and testing in chapters 4-6.

3.2 Problem cognition process

Problem Cognition Process $PCP = \left(DP, O, C = CGr, T, \left(\aleph, \bar{X}_{\aleph}\right), SAN\right)$ is to formulate the Decision Problem (*DP*) as the measurable Structural Assessment Network

(*SAN*) model, which comprises of four units: an objective *O*, a criteria structure (or structural criteria) C determined by criteria granulation process (*CGr*), a set of alternatives T, and a list of measurement scale schema $\left(\aleph, \bar{X}_{\aleph}\right)$.

The objective *O* of the decision problem (DP) is usually one sentence statement expressing the desired goal of the decision maker. A set of alternatives (or candidates) $T = \left[T_1, \ldots, T_i, \ldots, T_m\right]$ of the *DP* is the possible solutions for the desired goal. These alternatives are evaluated with respect to the structural criteria C.

3.2.1 Structural criteria

The structural criteria can be divided as several layers by a Criteria Granulation process (*CGr*). The number of the layers depends on the complexity of the SAN, and the usual number is two, and up to three layers.

The first layer criteria set is determined by the criteria granulation of the objective, $CGr(O) = \{c_i\}$, and $C = \{c_i\} = \{c_1, c_2, \ldots, c_i, \ldots, c_n\}$.

The second layer criteria set is determined by the criteria granulation of one first layer criterion, i.e. $CGr(c_i) = \{c_{i,j}\} = \{c_{i,1}, \ldots, c_{i,q_i}\}$, $q_i = |c_i|$ $, \forall i \in \{1, \ldots, n\}$;

The third layer criteria set is determined by criteria granulation of one second layer criterion, i.e. $CGr\left(c_{i,j}\right)=\left\{c_{i,j,k}\right\}=\left\{c_{i,j,1},\ldots,c_{i,j,q_{i,j}}\right\}$, $q_{i,j}=\left|c_{i,j}\right|$, $\forall\left(i,j\right)\in\left\{\left(i,j\right)\mid i\in\left\{1,\ldots,n\right\},j\in\left\{1,\ldots,q_i\right\}\right\}$.

To simplify the initial illustration of the CNP model, a criterion is assumed in upper layer which is dependent of its sub-criteria set (or attribute set) in the lower layer, and each criterion in each layer is independent.

Sometimes some elements in C are deductive or negative. To address this problem, the criteria granulation function can classify the elements into a positive set and negative set. The presentation is as follows.

The first layer criteria vector is in the form:

$$\bar{C}=\left\{C^{+},C^{-}\right\},\ C^{+}=\left\{c_1,\ldots,c_{n'}\right\},\ C^{-}=\left\{c_{n'+1},\ldots,c_n\right\};$$

or simply $CGr\left(O\right)=\left\{\left\{c_1,\ldots,c_{n'}\right\},\left\{c_{n'+1},\ldots,c_n\right\}\right\}$.

The second layer criteria vector is the form:

$$CGr\left(c_i\right)=\left\{\left\{c_{i,j}^{+}\right\},\left\{c_{i,j}^{-}\right\}\right\}=\left\{\left\{c_{i,1},\ldots,c_{i,q_i'}\right\},\left\{c_{i,q_i'+1},\ldots,c_{i,q_i}\right\}\right\}$$

, where $q_i'=\left|\left\{c_{i,j}^{+}\right\}\right|$ & $q_i=\left|c_i\right|$, $\forall i\in\left\{1,\ldots,n\right\}$.

The third layer criteria vector is of the form:

$$CGr\left(c_{i,j}\right)=\left\{c_{i,j,k}^{+},c_{i,j,k}^{-}\right\}=\left\{\left\{c_{i,j,1},\ldots,c_{i,j,q'_{i,j}}\right\},\left\{c_{i,j,q'_{i,j}+1},\ldots,c_{i,j,q_{i,j}}\right\}\right\}$$

, where $q'_{i,j} = \left|c^{+}_{i,j,k}\right|$ & $q_{i,j} = \left|c_{i,j}\right|,\ \forall(i,j) \in \{(i,j) \mid i \in \{1,\ldots,n\}, j \in \{1,\ldots,q_i\}\}$;

The above illustration shows a case of three layers. Mathematically, it can be expended into many layers by adding more dimensions in the subscript, e.g. $c_{i,j,k,\ldots,z}$. However, real practice usually applies one or two layer(s), and up to three layers. Four layers are rare. This happens in quantitative and qualitative research regarding the concept of the multidimensional scale.

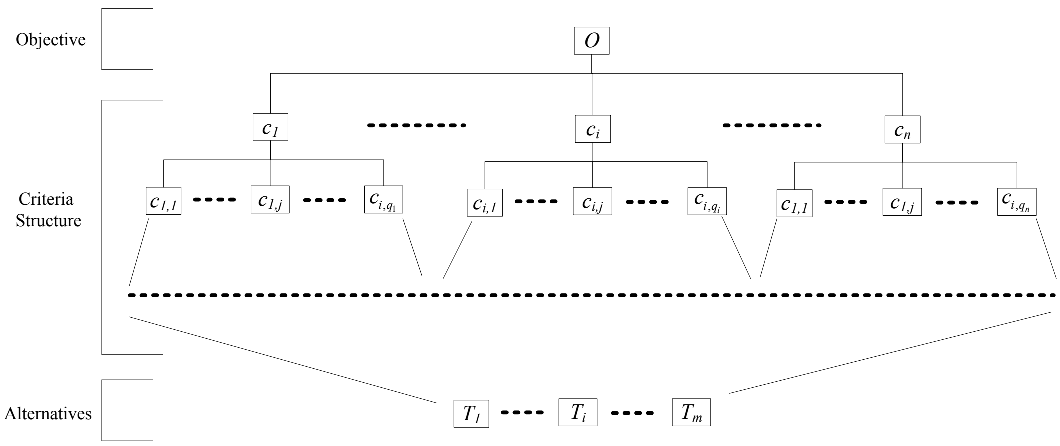


Figure 3.2: A structured assessment network of CNP with only positive criteria

3.2.2 Structure assessment network

If the structural criteria are of positive contribution to the objective, the model of this decision problem is named "Cognitive Hierarchy Process" (Fig. 3.2). Otherwise, if some members in the structural criteria are the positive contributions to the objective whilst some are the negative contributions, this model is called "Cognitive Network Process" (Fig. 3.3). In fact, the CHP is a special case of the CNP.

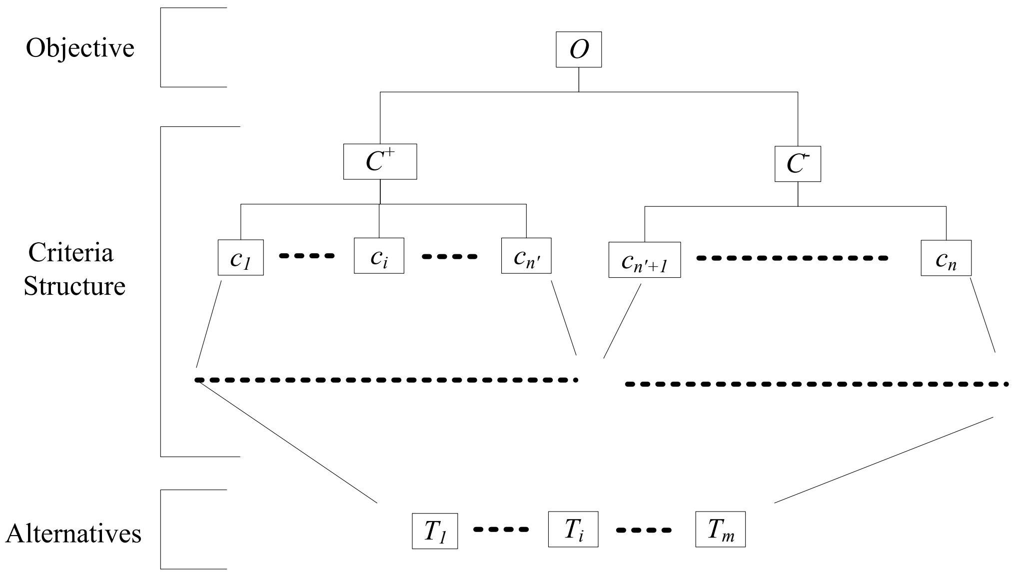


Figure 3.3: A structured assessment network of CNP with positive and negative criteria

3.2.3 Clusters of SAN

A Structured Assessment Network (*SAN*) comprises clusters joined by nodes. A cluster $Clst(nd, gn)$ is a node nd with a set of granules gn derived from the criteria granulation function $CGr(nd)$. Thus, a cluster is also denoted by $Clst(nd, CGr(nd))$. The clusters can be classified as three categories: objective cluster (or O cluster), structural criteria clusters, and measurable cluster (or $\breve{c}_i$ Cluster). Table 3.1 gives a description of cluster units.

Objective cluster $Clst\left(O,\{c_i\}\right)$ is the cluster with the node of the objective which is measured by the set of first layer criteria.

Structural criteria clusters can be classified as first layer criterion clusters (c_i cluster), second layer criterion clusters ($c_{i,j}$ cluster), and so on. Thus the n-order layer criterion cluster is the cluster with the node of n-order layer criterion, which is measured by its granules in the (n+1)-order layer.

A measurable cluster or $\breve{c}_i$ Cluster is defined as the cluster with the node of a criterion in the lowest layer of criteria structure, and the node is measured by the comparison of the set of alternatives, or by the direct measurement from the obsolete scale, without comparison. If the measurement is from the comparison of the set of alternatives, this measurement is called "relative measurement". If the measurement comes from a score of the absolute scales without mutual comparison, this measurement is called "absolute measurement".

The cardinal number of the set of the measurable criterion cluster set $\left|\{\breve{c}_i\}\right|$ relies on the number of the layer of the criteria structure, for example, $\left|\{\breve{c}_i\}\right| = \sum_{i=1}^{n}\sum_{j=1}^{q_i} q_{ij}$ in the three layer structural criteria.

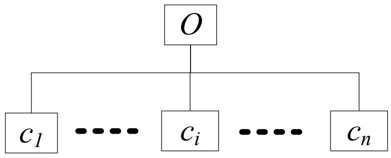
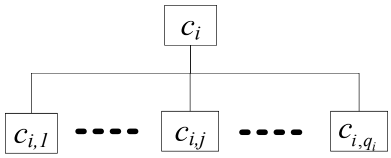
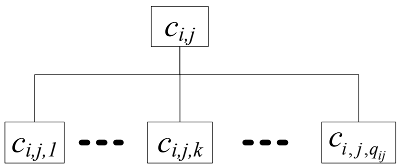
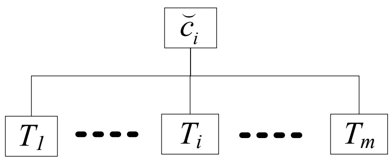
Cluster Categories	Diagram	Number of clusters
Objective Cluster (O Cluster) $Clst\left(O,\{c_i\}\right)$		1
First Layer Criterion Cluster (c_i Cluster) $Clst\left(c_i,\{c_{i,j}:j=1,\ldots,q_i\}\right)$		n
Second Layer Criterion Cluster ($c_{i,j}$ Cluster) $Clst\left(c_{i,j},\{c_{i,j,k}:k=1,\ldots,q_{i,j}\}\right)$		$\left\|\{c_{i,j}\}\right\|=\sum_{i=1}^{n}q_i$
Measurable Criterion cluster (Or Input Cluster) ($\breve{c}_i$ Cluster) $Clst\left(\breve{c}_i,\bar{T}\right)$		$\breve{n}=\left\|\{\breve{c}_i\}\right\|$

Table 3.1: Description of cluster categories

3.2.4 Compound rating scale

A list of measurement scale schemas $\overline{\aleph}$ is used to quantify or qualify the criteria. Let a measurement scale schema be $\aleph \in \overline{\aleph}$. $\aleph$ can be single space or multiple spaces of the rating scales. A classical schema uses a single space, i.e. $\aleph = \{\alpha_1, \alpha_2, \ldots, \alpha_i, \ldots, \alpha_p\}$. The rating interfaces of the classical rating scales, however, possibly lead to problems concerning the choices of linguistic terms, accuracy of linguistic representation of numbers and decisions in rating dilemmas (chapter 4). To address the above problems, this research (chapter 4) proposes a Compound Linguistic Ordinal Scale (CLOS) model , which is an ordinal-in-ordinal scale model, as a promised alternative for the classic Rating Scale Models, which provide 7±2 options usually. CLOS, which provides (7±2)((7±2)-1)+1=[21,73] options or more, is the Deductive Rating Strategy (DRS) of the Hedge-Direction-Atom Linguistic Representation Model (HAD LPM) with a cross reference relationship. The simulation result (which is shown in chapter 4) indicates that the proposed model helps to reduce the bias of the rating dilemma for a single rater and more accurately reflects consistency among raters.

Regarding the syntactic form, CLOS is established on a compound linguistic variable $\alpha \in \aleph_{mn}$ which comprises elements from the linguistic term vectors respectively: hedge vector $\overrightarrow{V_h}$, directional vector $\overrightarrow{V_d}$, and atomic vector $\overrightarrow{V_a}$. A matrix of compound linguistic variable $\aleph_{mn}$ is built on the Syntactic Rule Algorithm (algorithm 4.1) $\aleph_{mn} = G_{\aleph}\left(\overrightarrow{V_h}, \overrightarrow{V_d}, \overrightarrow{V_a}\right)$, has following form:

$$\aleph_{mn} = \begin{array}{c} \\ \alpha_1 \\ \vdots \\ \alpha_\eta \\ \alpha_{\eta+1} \\ \alpha_{\eta+2} \\ \vdots \\ \alpha_m \end{array} \begin{array}{c} \begin{array}{ccccc} \alpha^1 & \alpha^2 & \cdots & \alpha^{n-1} & \alpha^n \end{array} \\ \begin{bmatrix} \varnothing & v_{hd_1} \oplus v_{a_2} & \cdots & v_{hd_1} \oplus v_{a_{n-1}} & v_{hd_1} \oplus v_{a_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varnothing & v_{hd_\eta} \oplus v_{a_2} & \ddots & v_{hd_\eta} \oplus v_{a_{n-1}} & v_{hd_\eta} \oplus v_{a_n} \\ v_{a_1}{}^{\theta} & v_{a_2}{}^{\theta} & \ddots & v_{a_{n-1}}{}^{\theta} & v_{a_n}{}^{\theta} \\ v_{hd_{\eta+2}} \oplus v_{a_1} & v_{hd_{\eta+2}} \oplus v_{a_2} & \ddots & hd_{\eta+2} \oplus v_{a_{n-1}} & \varnothing \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{hd_m} \oplus v_{a_1} & v_{hd_m} \oplus v_{a_2} & \cdots & v_{hd_m} \oplus v_{a_{n-1}} & \varnothing \end{bmatrix} \end{array} \quad (3.1)$$

, where v_{hd} is the element of the combination of $\overrightarrow{V_h}$ and $\overrightarrow{V_d}$.

Regarding the rating process, CLOS is a dual rating scale. If triple spaces are applied, the evaluation effort must increase, whilst a single space is less representative. Thus double spaces are the middle way to improve the assessment quality. The measurement scale in fact is the psychometric scale. The CNP employs CLOS as the ideal interface to quantify the human perception of an object by a deductive rating strategy (algorithm 4.3).

Regarding the numerical representation, this research further develops a Semantic Rule Algorithm or a Fuzzy Normal Distribution $\overline{X} = f_{\overline{X}}(\aleph) = M(\aleph)$ (algorithm 4.2) in chapter 4.

If the data type of the rating scale is a fuzzy number, then the CNP model is classified as a fuzzy CNP problem. If the data type is a crisp number, then the CNP model is a crisp CNP model or a CNP model. The crisp CNP problem is the special case of the fuzzy CNP problem as the crisp CNP model does not need the interval computing, but relies on the modal value of the fuzzy number. The extension can be found in chapter 7.

To conclude the procedure of PCP, algorithm 3.2 is proposed as follows.

Algorithm 3.2 (Problem Cognition Process $PCP = (O, C, T, \aleph)$):

Input: Feasible Decision Problem

Step 1: Establish objective O;

Step 2: Search a list of the possible alternatives T;

Step 3: Define the strutral criteria C;

Step 4: Determine a list of measurement Scales $\bar{\aleph}$;

Output: Structured Assessment Network and $\bar{\aleph}$. **#End**

3.3 Cognitive assessment process

In the Cognitive Assessment Process $CAP = (\varphi(Clst), B_{nd}, Chk)$, a list of the Pairwise Opposite Matrices (POMs) is assessed by the Cognitive Assessment Function of the clusters $\varphi(Clst)$ performed by the decision makers (or raters) . The clusters are from the Structural Assessment Network. The Accordance check (Chk) is the mathematical algorithm or function to examine the Accordance Index of B_{nd}.

3.3.1 Pairwise opposite matrix

One of the innermost layers of the CNP onion is the Pairwise Opposite Matrix (Cognitive Pairwise Matrix, or Cognitive Comparison Matrix) B_{nd} of $Clst(nd, gn)$ which is of the form:

$$B_{nd} = \varphi\left(Clst(nd, gn)\right) = \begin{array}{c} \\ gn_1 \\ gn_2 \\ \vdots \\ gn_n \end{array} \begin{array}{c} \begin{array}{cccc} gn_1 & gn_2 & \cdots & gn_n \end{array} \\ \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{n} \end{bmatrix} \end{array},$$

$$n = |gn|, \; b_{ij} \cong V(gn_i) - V(gn_j). \tag{3.2}$$

$V(gn_i)$ is the utility value of gn_i. b_{ij} is from a scale schema $\aleph$. The calculation of $V(gn_i)$ can be found in chapter 3.4 and 5 in details.

In the primitive CNP, there are several types of POMs.

The POM of O cluster by the set of cognitive assessment functions is of the form $\varphi\left(Clst\left(O, \{c_i\}\right)\right)$. The Pairwise Opposite Matrix of one measurable cluster $Clst\left(O, \{c_i\}\right)$ is of the form.

$$B_O = \varphi\left(Clst\left(O,\{c_i\}\right)\right) = \begin{array}{c} \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{array} \begin{array}{c} \begin{array}{cccc} c_1 & c_2 & \cdots & c_n \end{array} \\ \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \end{array} \tag{3.3}$$

Similarly, the list of the POMs of $\breve{c}_i$ Cluster by the set of cognitive assessment functions is the form:

$$\varphi\left(Clst\left(\{\breve{c}_i\},\bar{T}\right)\right) = \left\langle \varphi\left(Clst\left(\breve{c}_i,\bar{T}\right)\right)\right\rangle_{i=1}^{n} = \left\{\varphi\left(\breve{c}_1,\bar{T}\right),\ldots,\varphi\left(\breve{c}_n,\bar{T}\right)\right\} \tag{3.4}$$

The Pairwise Opposite Matrix of one measurable cluster $Clst\left(\breve{c}_i,\bar{T}\right)$ is of the form.

$$B_{\breve{c}} = \varphi\left(Clst\left(\breve{c}_i,\bar{T}\right)\right) = \begin{array}{c} \\ T_1 \\ T_2 \\ \vdots \\ T_m \end{array} \begin{array}{c} \begin{array}{cccc} T_1 & T_2 & \cdots & T_m \end{array} \\ \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mm} \end{bmatrix} \end{array}, \text{ for all } i \tag{3.5}$$

This matrix construction can be deductive for applying in other clusters.

3.3.2 Cognitive assessment function

The Cognitive Assessment Function $\varphi(.) = (Sen, Per, Val)$ comprises the three expert psychological activities:

1. Sensation (Sen): The process to obtain sensory inputs of the information with respective to the Structured Assessment Network Model.

2. Perception (Per): the process to recognize and understand the sensory inputs.

3. Valuation (Val): the process to provide the psychological value for the perception of the sensory inputs. The value can be in number or in linguistic terms.

3.3.3 Accordance check

Let $B=\{b_{ij}:i,j\in(1,\ldots,n)\}$ be the pairwise opposite matrix and $D=\{d_{ij}:i,j\in(1,\ldots,n)\}$ be the contradiction matrix. The Accordant Index is in the form:

$$AI=\frac{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}d_{ij}}}{n^2},$$

$$d_{ij}=\sqrt{Mean\left(\left(\frac{1}{\kappa}\left(B_i+B_j^T-b_{ij}\right)\right)^2\right)},\quad i,j\in(1,\ldots,n). \tag{3.6}$$

where $AI\geq 0$, κ is the normal utility, and then $n\kappa$ is the population utility. If $AI=0$, then B is perfectly accordant; If $0<AI\leq 0.1$, then B is satisfactory. If $AI>0.1$, then B is unsatisfactory (The details of the mathematical development is in chapter 5).

To conclude the CAP, the following algorithm is proposed.

Algorithm 3.3 (Cognitive Assessment Process):

Input: Structured Network Model

Step 1: Sensation of the question forms and external information

Step 2: Perception of the sensation

Step 3: Valuation: measurement of their perception

Step 4: Check Accordance Index of the Assessment

Step 5: Loop until all VI's are feasible

Output: A set of opposite pairwise matrices #End

3.4 Cognitive prioritization process

One of the innermost layers of the CNP onion is the Cognitive Prioritization Process. The details are as follows.

The Cognitive Prioritization Process is of the form $CPP = (B, \psi, V)$. A pairwise opposite matrix B is prioritized to the absolute priority vector *W*, i.e. $\psi : B \to V$.

The POM is to interpret the utility values of the node's granules. Let a set of the real (ideal) utility be $V = \{v_1, \dots, v_n\}$, and the comparison score be $b_{ij} \cong v_i - v_j$. The ideal pairwise opposite matrix of $Clst(nd, gn)$ is $\tilde{B}_{nd} = [v_i - v_j]$. A subjective judgmental pairwise matrix is $B_{nd} = [b_{ij}]$. $\tilde{B}_{nd}$ is determined by B_{nd} as follows:

$$\tilde{B}_{nd} = [\tilde{b}_{ij}] = \begin{bmatrix} v_1 - v_1 & v_1 - v_2 & \dots & v_1 - v_n \\ v_2 - v_1 & v_2 - v_2 & \dots & v_2 - v_n \\ \vdots & \vdots & \ddots & \vdots \\ v_z - v_1 & v_z - v_2 & \cdots & v_n - v_n \end{bmatrix} \cong \begin{bmatrix} 0 & b_{12} & \dots & b_{1n} \\ b_{21} & 0 & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{z1} & b_{z2} & \cdots & 0 \end{bmatrix} = [b_{ij}] = B_{nd}$$

(3.7)

The value of $V = \{v_1, \ldots, v_n\}$ can be determined by the Cognitive Prioritization Operator (CPO). Various Cognitive Prioritization Operators (CPOs) are proposed in chapter 5. The Least Penalty Squares (LPS) or the Discrete Least Squares (DLS), which is the default setting for CPO, is of the form:

$$\text{Min} \quad \widehat{\Delta} = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \beta_{ij} \cdot \left(b_{ij} - v_i + v_j\right)^2$$

$$, \beta_{ij} = \begin{cases} \beta_1, & \mathrm{v}_i > v_j \;\&\; \mathrm{b}_{ij} > 0 \\ & \text{or } \mathrm{v}_i < v_j \;\&\; \mathrm{b}_{ij} < 0 \\ \beta_2, & \mathrm{v}_i = v_j \;\&\; \mathrm{b}_{ij} \neq 0 \\ & \text{or } v_i \neq v_j \;\&\; \mathrm{b}_{ij} = 0 \\ \beta_3, & \textit{otherwise} \end{cases}, \; 1 = \beta_1 \leq \beta_2 \leq \beta_3 \tag{3.8}$$

$$\text{s.t.} \quad \begin{array}{l} \sum_{i=1}^{n} v_i = n\kappa, \\ \mathrm{v}_i \geq 0, i = 1, 2, \ldots, n \end{array},$$

, where $n = |\{v_i\}|$, and κ is the normal utility, $\{\beta_1, \beta_2, \beta_3\}$ is a set of penalty indices.

The above optimization model is also named the optimization operator. The solution can be easily formed by some software tools such as *Excel, Mathlab, Lingo*, as well as *Mathematica,* which is used by this research.

Another alternative of the CPO is the Row Average plus the normal Utility (RAU), which is of the form:

$$v_i = \left(\frac{1}{n} \sum_{j=1}^{n} b_{ij}\right) + \kappa, \forall i \in \{1, \ldots, n\} \tag{3.9}$$

In most cases, if $\mathrm{AI} \leq 1$, the results of RAU and LPS are the same or very closed with measurements of the Cognitive Distortion Index (chapter 5.6.3)

To conclude the CPP, the following algorithm is proposed.

Algorithm 3.3 (**Cognitive Prioritization Process**):

Input: A set of opposite pairwise matrices;

Determination of output type: Absolute Weight or Relative Weight

Process: Cognitive Prioritization of each pairwise matrix

Output: A list of weight with respect the matrices #End

3.5 Multiple information fusion processes

Information Fusion Process $IFP = \left(X, Y, AO^*, \{AO\}, SAO\right)$ is the function to aggregate multiple sources of data granules X from each evaluated cluster of SAN to an overall result set Y to represent the attributes of the decision objective by the selection of the most appropriate aggregation operator (AO*) among a set of the AO candidates $\widetilde{AO} = \{AO\}$, i.e. $SAO: \{AO\} \rightarrow AO^*$, and $AO^*: X \rightarrow Y$.

In cognitive psychology, cognitive psychologists investigate how people (or animals) select sensory information, and choose a method to process information. In view of the computational intelligence aspect of cognitive psychology, chapter 6 proposes a Cognitive Style and Aggregation Operator (CSAO) model for the evaluation and selection of AOs on the basis of the decision attitudes.

Multiple Information Fusion Processes $MIF = (\{dg\}, \{IFP\}, Y)$ are the processes which apply a course of Information Fusion processes $\{IFP\}$ to digest the input of the set of data granules $\{dg\}$ to complete a final task, e.g. to calculate the overall score Y.

In CNP, there are three categories of *IFP* due to the sources of the data granules: fusion of decisional matrix, fusion of structural criteria, and fusion of collective judgment. The fusion of collective judgment will be discussed in chapter 7.

3.5.1 Decisional matrix

The decisional matrix O is constructed with respect to the objective cluster $Clst(O, \{c_j, v_j\}, T_i)$, and has the form:

$$O = \begin{array}{c c} & \begin{array}{ccccc} (v_1 & \cdots & v_j & \cdots & v_n) \\ c_1 & \cdots & c_j & \cdots & c_n \end{array} \\ \begin{array}{c} T_1 \\ \vdots \\ T_i \\ \vdots \\ T_m \end{array} & \left(\begin{array}{ccccc} & & & & \\ & & & & \\ & & a_{ij} & & \\ & & & & \\ & & & & \end{array} \right) \end{array} \tag{3.10}$$

,where v_j is the utility weight. In CNP, $\sum_j v_j$ is not necessary equal to 1. This is just the issue of the rescaling function, i.e. $f : v_j \to w_j$, $\forall j \in \{1, \ldots, n\}$ and $\sum_j w_j = 1$. In the CNP, $\{v_j\}$ is derived from the cognitive prioritization of the POM in CPP.

If the criteria include both positive and negative elements, they can be separated into two matrices: a positive decisional matrix and a negative decisional matrix.

The positive decisional matrix is built with respect to $Clst\left(O^{+},C^{+},T\right)$ as below:

$$O^{+}=\begin{array}{c}\\ \\ T_1\\ \vdots\\ T_i\\ \vdots\\ T_m\end{array}\begin{array}{c}\begin{matrix}(v_1 & \cdots & v_i & \cdots & v_{n'})\\ c_1 & \cdots & c_j & \cdots & c_{n'}\end{matrix}\\ \left(\begin{matrix} & & & & \\ & & & & \\ & & a_{ij} & & \\ & & & & \\ & & & & \end{matrix}\right)\end{array} \tag{3.11}$$

The set of positive weight $\{v_1,\ldots,v_{n'}\}$ is derived from the differential prioritization of the POM of C^{+}.

Negative Decisional matrix is built with respect to $Clst\left(O^{-},C^{-},T\right)$, and has the form:

$$O^{-}=\begin{array}{c}\\ \\ T_1\\ \vdots\\ T_i\\ \vdots\\ T_m\end{array}\begin{array}{c}\begin{matrix}(v_{n'+1} & \cdots & v_i & \cdots & v_n)\\ c_{n'+1} & \cdots & c_j & \cdots & c_n\end{matrix}\\ \left(\begin{matrix} & & & & \\ & & & & \\ & & a_{ij} & & \\ & & & & \\ & & & & \end{matrix}\right)\end{array} \tag{3.12}$$

The set of positive weight $\{v_{n'+1},\ldots,v_n\}$ is derived from the cognitive prioritization of the opposite comparison matrix of C^{-}.

For better presentation, both matrices can be formed in the partitions of a matrix, i.e. $\bar{O}=\left[\bar{O}^{+} \,\vdots\, \bar{O}^{-}\right]$.

3.5.2 Aggregation matrices of structural criteria

An aggregation matrix of structural criteria is used to aggregate elements in the lower layer criteria as the value of a node in the upper layer.

An aggregation matrix of the first layer criterion is built on the first layer criterion cluster $Clst\left(c_i,\left\{c_{i,j}:j=1,\ldots,q_i\right\}\right)$, $\forall i$, as follows:

$$\bar{c}_i = \begin{array}{c} \\ \\ T_1 \\ \vdots \\ T_{i'} \\ \vdots \\ T_m \end{array} \begin{array}{c} \begin{pmatrix} v_{i,1} & \cdots & v_{i,j} & \cdots & v_{i,n} \end{pmatrix} \\ \begin{array}{ccccc} c_{i,1} & \cdots & c_{i,j} & \cdots & c_{i,n} \end{array} \\ \left(\begin{array}{ccc} & & \\ & & \\ & a_{i'j} & \\ & & \\ & & \end{array} \right) \end{array}, \forall i \tag{3.13}$$

, where $$\left(v_{i,1},\ldots,v_{i,n}\right) = CPO\left(\begin{array}{c} \\ C_{i,1} \\ \vdots \\ C_{i,j'} \\ \vdots \\ C_{i,n} \end{array} \begin{array}{c} \begin{array}{ccccc} C_{i,1} & \cdots & C_{i,j} & \cdots & C_{i,n} \end{array} \\ \left(\begin{array}{ccc} & & \\ & & \\ & a_{j'j} & \\ & & \\ & & \end{array} \right) \end{array}, \left\{\beta_1,\beta_2,\beta_3\right\},\kappa \right) \quad ,\forall i . \tag{3.14}$$

Similarly, an aggregation matrix of the second layer criterion is built on the cluster $Clst\left(c_{i,j},\left\{c_{i,j,k}:k=1,\ldots,q_{i,j}\right\}\right)$, and has the form:

$$\bar{c}_{ij} = \begin{array}{c} \\ \\ T_1 \\ \vdots \\ T_{i'} \\ \vdots \\ T_m \end{array} \begin{array}{c} \begin{pmatrix} v_{i,j,1} & \cdots & v_{i,j,k} & \cdots & v_{i,j,q_{ij}} \end{pmatrix} \\ \begin{array}{ccccc} c_{i,j,1} & \cdots & c_{i,j,k} & \cdots & c_{i,j,q_{ij}} \end{array} \\ \left(\begin{array}{ccc} & & \\ & & \\ & a_{i'k} & \\ & & \\ & & \end{array} \right) \end{array}, \forall i,j \tag{3.15}$$

The set of the utility weights of the $c_{i,j}$ cluster, i.e. $\left(v_{i,j,1},\ldots,v_{i,j,k},\ldots,v_{i,j,q_{ij}}\right)$ is derived from the differential prioritization of the opposite comparison matrix of the node $c_{i,j}$.

The measurable criteria are the lowest level of the structural criteria. The measurement value is propagated into the upper levels, and finally reach the top level for constructing the decision matrix. There are two methods for setting the measurable criterion $\breve{c}_{\breve{i}}$: one is direct rating; and the other is getting the differential prioritization of the opposite matrix $\overline{\breve{c}_{\breve{i}}}$ from the differential pairwise rating. The former method is the **Absolute Measurement** which is useful whilst the measurable criteria are available. The latter method is the **Relative Measurement** which is useful whilst the measurable criteria are unavailable, and can be achieved by subjective measurement.

The matrix of the subjective comparison measurement for the measurable criterion is constructed with respect to the cluster $Clst\left(\breve{c}_{\breve{i}}, \overline{T}\right)$, and has the form:

$$\overline{\breve{c}_{\breve{i}}} = \begin{array}{c} \\ T_1 \\ \vdots \\ T_{i'} \\ \vdots \\ T_m \end{array} \begin{array}{c} \begin{array}{ccccc} T_1 & \cdots & T_{j'} & \cdots & T_m \end{array} \\ \left(\begin{array}{ccccc} & & & & \\ & & & & \\ & & b_{i',j'} & & \\ & & & & \\ & & & & \end{array} \right) \end{array}, \quad i \in \{i, \ldots, \breve{n}\} \tag{3.16}$$

3.5.3 Aggregation of clusters

Consider a cluster $Clst(nd, gn)$. The aggression function for a node nd is to combine the set of its data granules $\{ng_i\}$ and the set of the corresponding weights of the granules $\{wng_i\}$ into a meaningful or representative value for *nd*. The function has the form:

$$nd = Agg\left(\left[\left(ng_i, wng_i\right)\right]_{i=1}^{|ng_i|}\right) \tag{3.17}$$

, where $|ng_i|$ is the cardinal number of the nodes.

Consider a typical CNP structure comprising the structural criteria of two layers and one expert layer: the Objective cluster $Clst\left(O, \{c_i : i = 1, \ldots, q\}\right)$, c_i Cluster $Clst\left(c_i, \{c_{i,j} : j = 1, \ldots, q_i\}\right)$, and expert cluster, $Clst\left(\breve{c}_i, \{e_k\}\right)$. On the basis of a template clusters, the following forms are defined.

The aggression function for a criterion c is to combine its sub-criteria (or c_i) and weights of the sub-criteria w_i into a meaningful or representative value for *O*. The function has the form:

$$O = Oagg\left(\left[\left(c_i, w_i\right)\right]_{i=1}^{q}\right) \tag{3.18}$$

, where *q* is the cardinal number of the sub-criteria.

Likewise, aggregation of c_i has the form:

$$c_i = Cagg_i\left(\left[\left(c_{ij}, w_{ij}\right)\right]_{j=1}^{q_j}\right) \tag{3.19}$$

Note that the cardinal number q_i may not be equal to the cardinal number $q_{i'\neq i}$. Thus, C is a non-rectangular matrix or a jagged array, i.e. $C = \left\{\left(c_{1,1}, \ldots, c_{1,q_1}\right), \ldots, \left(c_{n,1}, \ldots, c_{n,q_n}\right)\right\}$.

Aggregation operators are essential in the information fusion. There are a number of AOs (see chapter 2.3.5), which produce different values, although many decision applications use the simple weighted average (or weighted arithmetic mean). The question is which aggregation operator the most suitable.

Interestingly, this study (chapter 6) finds that the selection of the aggregation operators can be determined by the decision attitudes. Thus a Cognitive Style and Aggregation Operator (CSAO) model is proposed to analyze the mapping relationship between aggregation operators and decision attitudes. The selection of the most appropriate AOs (SAO) can be determined by the $CSAO$ model of the decision attitude input, i.e. $SAO = CSAO$. The CSAO has the following form:

$$CSAO : (\{AO\}, X, D, da) \to AO^* \tag{3.20}$$

,where X is the set of data granules, da is a prefer decision attitude, AO* is the most preferable AO, and $\{AO\}$ is the set of AO candidates. The details of the development of CSAO algorithm are given in Chapter 6.

3.6 Decisional volition process

The Decisional Volition Process $DVP = (\{c_i\}, \{v_i\}, T, VL, EV, t^*)$ is the process to decide the final decision $t^* \in T = [T_1, \ldots, T_m]$ with the inputs of a set of first layer criteria $\{c_i\}$ and the corresponding weight set $\{v_i\}$ by the volition function VL. $VL : T \to t^*$, where $EV : (\{c_i\}, \{v_i\}) \to T$ is the evaluation function.

The evaluation function EV is the special case of information fusion, which discusses the fusion of the decisional matrix which comprises $\{c_i\}$ and $\{v_i\}$, in order to return the value of T. Usually, the best alternative is determined by the highest score $T^* = Max(T)$, and its position γ is returned by the argument of the maximum function $\arg\max$.

$$t^* = VL(T^*) = T_\gamma \text{, where } \gamma = \arg\max_{i\in\{1,2,\ldots,m\}}\left(\{T_1, T_2, \ldots, T_i, \ldots, T_m\}\right) \tag{3.21}$$

(In rare cases, if lowest score is applied, the argument of the minimum function $\arg\min$ is used.)

Sometimes, t^* may be a vector of chosen alternatives. This means that more than one candidate is chosen. Let γ' be the number of candidates to be selected. The ordering function returns the vector of ranking values with respect to T, and has the form:

$$Ordering(T) = \{I(i) \mid i = 1, \ldots, m\}$$

$$\text{, where } I(i) = \sum_{k=1}^{m} r_k(T_i),$$

$$r_k(T_i) = \begin{cases} 1 & , T_i > T_k \ \& \ \mathrm{i} \neq k \\ 0, & Otherwise \end{cases} \tag{3.22}$$

Thus t^* is the form:

$$t^* = \left\{T_i \mid \left(\gamma' \geq \left(m - I(i)\right)\right), I(i) \in Ordering\left(\overline{T}\right), \forall i \in \{1, \ldots, m\}\right\} \tag{3.23}$$

Another issue is to define the Evaluation function as the form: $PEV : C \to t^{\bullet}$, where the value of $t^{\bullet}$ is from the continuous scale and C is the aggregated overall result from MIP. The PEV is the parametric evaluation function of the decision problem. In other words, the CNP of the previous four steps can serve as the evaluation platform to measure the input parameters of the parametric (valuation) function of the decision problem. Examples are shown in Case 5 of chapter 8.

3.7 Extensions of cognitive network process

The Cognitive Network Process can be classified into sixteen types. Four types are classified from the viewpoint of measurements and another four types are classified from the viewpoint of experts and data types. A combination of both gives sixteen types. The details are given below.

3.7.1. Measurement views

Table 3.2 presents four types of CNP regarding measurements: relative positive (RP) measurement, absolute positive (AP) measurement, relative positive and negative measurement RPN, and absolute positive and negative (APN) measurement.

Relative Measurement means the value of the measurable criteria is derived by opposite comparison judgment. Absolute Measurement means the value of the measurable criteria is derived by direct rating. Positive Measurement means the values of all criteria are positive. Positive and negative measurement means some criteria are positive and the remainders are negative. In other words, PR is the special case of RPN where there are no negative criteria. Similarly AP is the special case of APN.

Table 3.2: Types of CNP in views of measurements

		Measurable Criteria	
		Relative Measurement	Absolute Measurement
Structural Criteria	Nature: Positive Measurement	RP	AP
	Nature: Positive and Negative Measurement	RPN	APN

In the applications chapter, absolute measurement of the CNP is illustrated in case 5 of chapter 8; Relative measurement of the CNP is illustrated in cases 1-4.

3.7.2 Expert and data type views

Table 3.3 shows different types of CNP due to the data types and number of experts in the CNP system. Data types are classified as a crisp number and a fuzzy number. The number of experts is referred to "collective", which is more than one expert, and "individual", for a single expert.

Table 3.3: Types of CNP model in views of experts and data types

		Number of Experts	
		Individual	Collective
Data Type	Crisp	Primitive Cognitive Network Process (P-CNP)	Collective Cognitive Network Process (C-CNP)
	Fuzzy	Fuzzy Cognitive Network Process (F-CNP)	Fuzzy Collective Cognitive Network Process (FC-CNP)

In the narrow view of the definition of the Cognitive Network Process, CNP is the crisp inputs of the individual decisions. In the broad definition, for the variations of CNP, the Collective Cognitive Network Process (C-CNP) is the crisp inputs of the collective decision; the Fuzzy Cognitive Network Process (F-CNP) is the fuzzy inputs of the individual decision; the Fuzzy Collective Cognitive Network Process (FC-CNP) is the fuzzy inputs of the collective decision.

In the applications chapter, P-CNP is demonstrated in cases 1-3 of chapter 8; F-CNP is illustrated in case 4 whilst F-CCNP is presented in case 5.

3.8 Summary

This chapter presents the cognitive structure of the cognitive network process, which comprises the formulations of the algorithms of the cognitive decision process represented by process algebra. It states various high motivations for further discussion in the coming chapters: Chapter 4 discusses the development of CLOS which is used in the Cognitive Assessment Process; Chapter 5 discusses the development of the Pairwise Opposite matrix and Cognitive Prioritization Operator which is used in the Cognitive Prioritization Process; Chapter 6 discusses the aggregation operator issue in the Multiple Information Fusion Process; Chapter 7 discusses the extension models of the CNP. The applicability and usability of the CNP is illustrated in Chapter 8, and conclusions are drawn in chapter 9.

Chapter 4 Compound Linguistic Ordinal Scale

4.1. Introduction

The organization of this chapter is structured as follows. Chapter 4.2 introduces the concept of Hedge-Direction-Atom Linguistic Representation Model (HDA-LRM) and the syntactic rules for Compound Linguistic Variable (CLV), HDA-LRM. Next, the semantic rules or "Computing with CLV" for HDA-LRM are discussed in chapter 4.3. These are how to map CLV into represented numbers in a matrix by Fuzzy Normal Distribution. Chapter 4.4 illustrates the Compound Linguistic Ordinal Scale (CLOS) Model, which is a rating scales model applying the Deductive Rating Strategy (DRS) and HDA-LRM. In chapter 4.5, the simulation analyses of the CLOS model are performed. The analysis includes comparisons of various scenarios with different parameters, and a comparison with a classical ordinal model. Finally, conclusions are drawn in chapter 4.6.

4.2. Hedge-direction-atom linguistic representation model (HDA-LRM)

A Hedge-Direction-Atom Linguistic Representation Model is 3-tuple $\left(\aleph, \overline{X}, f_{\overline{X}}(\aleph)\right)$. $\overline{X}$ or $\overline{X}_{\aleph}$ is the set of representation values of Compound Linguistic Variable $\aleph$ (definition 4.2) from the numeric-linguistic representation function $f_{\overline{X}}(\aleph):\aleph \to \overline{X}$.

If $\overline{X}$ is the set of pure numerical values, HDA-LRM can be applied to the classical statistics and probability theory directly. If $\overline{X}$ is the set of fuzzy numbers, then $\left(\aleph, \overline{X}, f_{\overline{X}}(\aleph)\right)$ can be integrated with a fuzzy reasoning model as a pure fuzzy model. Therefore, the dedicated fuzzy aggregators are discussed for further analysis. As classical rating scales with statistics and probability theory are successfully applied in different application domains, such as psychometrics and quantitative research in business management and social sciences as stated earlier, this research focuses discussion of $\overline{X}$ when it is a set of crisp values, and demonstrates how classical rating applications can improve with the introduction of the new theory (please refer to Example 4.3).

Defining appropriate schema for processing the meaning of the linguistic symbols of a fuzzy linguistic variable is challenging as it requires linguistic terms which are universally understood and accepted in a fuzzy quantitative sense. It seems that there is a lack of convenient rules in the literature to build a set of a large scale of linguistic terms and a lack of theory relating to computation of the distribution of the linguistic terms into numbers. Compound Linguistic Variable is the ideal term set formulation system for this problem. CLV is built on the following definition.

Definition 4.1 (Linguistic Variable) (Zadeh,1975)**:** A linguistic variable is characterized by a quintuple $(\aleph, T(\aleph), X, G, M)$ in which $\aleph$ is the name of a linguistic variable; $T(\aleph)$ (or simply T) denotes the term-set of $\aleph$, that is, the set of names of linguistic values of $\aleph$, with each value being a fuzzy variable denoted generally by α and ranging over a universe of discourse X which is associated with the base variable x; G is syntactic rule (which usually has the form of a grammar) for generating the names, α, of values of $\aleph$; and M is a semantic rule for associating with each α meaning, $M(\alpha)$, which is a fuzzy subset of X.

CLV extends above foundation, and is defined as follows:

Definition 4.2 (CLV**):** A Compound Linguistic Variable $(\aleph, T(\aleph), X, G, M)$ is a linguistic variable $\aleph$ whose term-set (T) in matrix form is governed by syntactic rules G and is characterized by semantic rules M *mapping* $\aleph$ *into* the universal of discourse X *(Real number domain)*. Each term α in T contains the continuous or discrete elements *x's* generated by M in X.

To illustrate CLV*,* term-set is described as follows.

Definition 4.3 ($T(\aleph)$)**:** The syntactic pattern of a linguistic term-set T for CLV is a quintuple $(\overline{V_h}, \overline{V_d}, \overline{V_a}, G_{\aleph})$. A term-set T of a linguistic variable $\aleph$ is syntactically mapped from $\overrightarrow{V_h},\overrightarrow{V_d},\overrightarrow{V_a}$ with the syntactic relation $G_{\aleph}$, i.e.:

$$G_{\aleph}:\left(\overrightarrow{V_h},\overrightarrow{V_d},\overrightarrow{V_a}\right)\mapsto T(\aleph)=\aleph \tag{4.1}$$

where "$\mapsto$" denotes "be linguistically mapped to", $G_{\aleph}$ is the syntactic relationship function for $\aleph$. $\overrightarrow{V_h},\overrightarrow{V_d},\overrightarrow{V_a}$ is defined in definitions 4.4-4.7. Notations of V_h, V_d, V_a, which are not vectorized or ordered, are different from $\overrightarrow{V_h},\overrightarrow{V_d},\overrightarrow{V_a}$. This difference is shown in example 4.1.

Definition 4.4 ($\overrightarrow{V_a}$): An atomic linguistic term v_{a} is used to roughly describe the statement or measures the objects in the initial sense. A vectorized atomic linguistic variable $\overrightarrow{V_a}$ is the form $\overrightarrow{V_{a_n}}=\left[v_{a_i}\right]_{j=1}^{n}=\left[v_{a_1},\cdots,v_{a_n}\right]$ where $\le_{j=1}^{n}\left(v_{a_j}\right)\equiv v_{a_1}\le\cdots\le v_{a_n}$ and $v_{\mathrm{a}}\in\overrightarrow{V_a}$. n is an odd number and larger than or equal to three. For example, $\overrightarrow{V_a}=\left[\text{Poor, Weak, Fair, Good, Excellent}\right]$ where Poor < Weak < Fair < Good < Excellent.

Definition 4.5 ($\overrightarrow{V_h}$): A Hedge term v_h is the communicative strategies for adjusting the linguistic quantity in v_{a}. A vectorized hedge linguistic variable $\overrightarrow{V_h}$ is the form: $\overrightarrow{V_h}=\left[v_{h_i}\right]_{i=1}^{\eta}$ where $\le_{i=1}^{\eta}\left(v_{h_i}\right)$ and $v_h\in\overrightarrow{V_h}$. For example, $\overrightarrow{V_h}=\left[\text{Little,Quite,Much}\right]$ where Little < Quite < Much.

Definition 4.6 ($\overrightarrow{V_d}$): A Directional term v_d is used for decision making strategies to define the direction of v_h's . A vectorized directional linguistic variable $\overrightarrow{V_d}$ consists of three ordinal directional terms v_{d}'s ,i.e. $\overrightarrow{V_d}=\left[v_{d_i}\right]_{i=1}^{3}=\left[v_{\mathrm{d}}^{-}, v_{\mathrm{d}}^{\theta}, v_{\mathrm{d}}^{+}\right]$ with $\leq_{i=1}^{3}\left(v_{d_i}\right)$ (or $v_{\mathrm{d}}^{-} \leq v_{\mathrm{d}}^{\theta} \leq v_{\mathrm{d}}^{+}$). A static point v_d^{θ} is the separation line to divide the positive domain and negative domain of a linguistic term. v_d^{+} modifies the v_h as a positive linguistic term while v_d^{-} does as a negative linguistic term since v_h is the pure "qualitative quantity" in nature. For Example, $\overrightarrow{V_d}=\left[\text{Below,Absolutely,Above}\right]$ where Below < Absolutely < Above.

v_h and v_d can form syntactic terms as a directional hedge linguistic term v_{hd}. Thus the following definition is introduced.

Definition 4.7 ($\overrightarrow{V_{hd}}$): Let $\overrightarrow{V_{hd}}=\left[v_{hd_i}\right]_{i=1}^{m}$ with $\leq_{i=1}^{m}\left(v_{hd_i}\right)$. To extend the notations for its properties, then

$$v_{hd}^{-} \triangleq v_h \oplus v_d^{-} \quad \text{, where } v_{hd}^{-} \in \overrightarrow{V_{hd}}^{-}=\left[v_{hd_i}\right]_{i=1}^{\eta}, \eta=\left|\overrightarrow{V_h}\right| \tag{4.2}$$

$$v_{hd}^{\theta} \triangleq v_h \oplus v_d^{\theta} \triangleq v_d^{\theta}=\overrightarrow{V_{hd}}^{\theta} \text{, where } \overrightarrow{V_{hd}}^{\theta}=\left\{v_{hd_{\eta+1}}\right\} \tag{4.3}$$

$$v_{hd}^{+} \triangleq v_h \oplus v_d^{+} \quad \text{, where } v_{hd}^{+} \in \overrightarrow{V_{hd}}^{+}=\left[v_{hd_i}\right]_{i=\eta+2}^{m} \tag{4.4}$$

v_{hd}^{-} is a negative hedge linguistic term, v_{hd}^{θ} is a static hedge linguistic term, and v_{hd}^{+} is a positive hedge linguistic term. "$\oplus$" is a "linguistic addition" which means the addition of two linguistic terms in the string format ("+" is the addition of two numbers).

"≜" is a linguistic equality, which means "is linguistically equal to", whilst "=" can be used for either numerical equality or for the assignment of terms for a variable.

To investigate the formulation of $\overrightarrow{V_{hd}}$, the syntactic function $G_{\overrightarrow{V_{hd}}} : \left(\overrightarrow{V_h}, \overrightarrow{V_d}\right) \mapsto \overrightarrow{V_{hd}}$, where $\overrightarrow{V_{hd}} = \left[\overrightarrow{V_{hd}}^{-}, \overrightarrow{V_{hd}}^{\theta}, \overrightarrow{V_{hd}}^{+}\right]$ is proposed. The following proposition is to illustrate $G_{\overrightarrow{V_{hd}}}$.

Proposition 4.1 ($\overrightarrow{V_{hd}} = G_{\overrightarrow{V_{hd}}}\left(\overrightarrow{V_h}, \overrightarrow{V_d}\right)$): If $\overrightarrow{V_h} = \left[v_{h_i}\right]_{i=1}^{\eta}$ and $\overrightarrow{V_d} = \left[v_d^{-}, v_d^{\theta}, v_d^{+}\right]$, then

$$\begin{aligned} \overrightarrow{V_{hd}} &= G_{\overrightarrow{V_{hd}}}\left(\overrightarrow{V_h}, \overrightarrow{V_d}\right) = \left[v_{hd_i}\right]_{i=1}^{m} \\ &= \left[\left(v_{h_\eta} \oplus v_d^{-}\right), \cdots, \left(v_{h_1} \oplus v_d^{-}\right), v_d^{\theta}, \left(v_{h_1} \oplus v_d^{+}\right), \cdots, \left(v_{h_\eta} \oplus v_d^{+}\right)\right] \end{aligned} \tag{4.6}$$

Proof:

As $\overrightarrow{V_{hd}}^{-} = \left[\left(v_{h_\eta} \oplus v_d^{-}\right), \cdots, \left(v_{h_1} \oplus v_d^{-}\right)\right]$ (Eq(2)) , $\overrightarrow{V_{hd}}^{\theta} = \left[v_{hd}^{\theta}\right] = \left[v_h \oplus v_d^{\theta}\right] = \left[v_d^{\theta}\right]$ (Eq(3)), and $\overrightarrow{V_{hd}}^{+} = \left[\left(v_{h_1} \oplus v_d^{+}\right), \cdots, \left(v_{h_{1\eta}} \oplus v_d^{+}\right)\right]$ (from Eq(4)) , then $\overrightarrow{V_{hd}} = \left[\overrightarrow{V_{hd}}^{-}, v_d^{\theta}, \overrightarrow{V_{hd}}^{+}\right]$.

□

For example, $\overrightarrow{V_{hd}}$ = ["Much Below", "Quite Below", "Little Below", "Absolutely", "Little Above", "Quite Above", "Much Above"].

v_{hd} can be used with v_a to form a new meaning. The concept of a compound linguistic term is given.

Definition 4.8 (α): A compound linguistic term α can be formed by a syntactic equation, and is the form: $\alpha = G_\alpha(v_h, v_d, v_a) = (v_h \oplus v_d) \oplus v_a = G_\alpha(v_{hd}, v_a) = v_{hd} \oplus v_a$, where v_h, v_d, v_a are any members from $\overrightarrow{V_d}, \overrightarrow{V_h}, \overrightarrow{V_a}$ respectively. To extend the notation, $\alpha_{ij} \triangleq v_{hd_i} \oplus v_{a_j}$, $\alpha_i \triangleq v_{hd_i} \oplus \overrightarrow{V_a} = \left[v_{hd_i} \oplus v_{a_j} \right]_{j=1}^{n}$, $\alpha^j \triangleq \overrightarrow{V_{hd}} \oplus v_{a_j} = \left[v_{hd_i} \oplus v_{a_j} \right]_{i=1}^{m}$, for all $i \in \{1, \cdots, m\}$, $j \in \{1, \cdots, n\}$ where cardinal numbers $m = \left| \overrightarrow{V_{hd}} \right|$ and $n = \left| \overrightarrow{V_a} \right|$.

For example, $\alpha^3 =$ [“Much Below Fair”, “Quite Below Fair”, “Little Below Fair”, “Absolutely Fair”, “Little Above Fair”, “Quite Above Fair”, “Much Above Fair”]. However, there are some unusual cases. if “Absolutely Excellent” is defined as the maximum value, there is no need for “Much Above Excellent” in the semantic view. Thus an exceptional case of the syntactic form $v_{hd} \oplus v_a$ is defined as follows.

Definition 4.9 ($\varnothing$): v_d^- cannot modify v_{a_1} whereas the v_d^+ cannot modify v_{a_n}. Thus, the syntactic forms are:

$$v_h \oplus v_d^- \oplus v_{a_1} \triangleq v_{hd}^- \oplus v_{a_1} \triangleq \varnothing \quad \text{,where } v_{hd}^- \in \overrightarrow{V_{hd}}^- \tag{4.7}$$

$$v_h \oplus v_d^+ \oplus v_{a_n} \triangleq v_{hd}^+ \oplus v_{a_n} \triangleq \varnothing \quad \text{,where } v_{hd}^+ \in \overrightarrow{V_{hd}}^+ \tag{4.8}$$

For all $v_h \in \overrightarrow{V_h}$, and $\varnothing$ is the null element.

There are sufficient conditions to derive a piecewise formula to calculate $G_\alpha(v_{hd}, v_a)$:

Proposition 4.2: In the semantic aspect, the Normalized Syntactic Function is:

$$G_{\alpha_{ij}}\left(v_{hd_i}, v_{a_j}\right) \triangleq \begin{cases} \varnothing & j=1 \ \& \ i \in \{1,\cdots,((m+1)/2)\} \\ v_{hd_i} \oplus v_{a_j} & j \neq 1, n \ \& \ \forall i \\ \varnothing & j=n \ \& \ i \in \{((m+1)/2),\cdots,m\} \end{cases} \tag{4.9}$$

Proof: definition 4.8 gives $\alpha_{ij} \triangleq v_{hd_i} \oplus v_{a_j}$ with the constraints (4.7) and (4.8) in definition 4.9 which imply that $\alpha_{ij} : \left(j=1 \ \& \ i \in \{1,\cdots,((m+1)/2)\}\right) \, or \left(j=n \ \& \ i \in \{((m+1)/2),\cdots,m\}\right)$ are null elements. Then the form is held. □

To extend proposition 2, the following theorem is satisfied.

Theorem 4.1 (CLV): The Linguistic Cartesian Product $G_{\aleph}$ of $\overrightarrow{V_a}$ and $\overrightarrow{V_{hd}}$ forms a term set $T(\aleph)$ of CLV as follows:

$$\aleph_{mn} = G_{\aleph}\left(\overrightarrow{V_{hd}}, \overrightarrow{V_a}\right) = \begin{array}{c} \\ \alpha_1 \\ \vdots \\ \alpha_\eta \\ \alpha_{\eta+1} \\ \alpha_{\eta+2} \\ \vdots \\ \alpha_m \end{array} \begin{array}{c} \begin{array}{ccccc} \alpha^1 & \alpha^2 & \cdots & \alpha^{n-1} & \alpha^n \end{array} \\ \begin{bmatrix} \varnothing & v_{hd_1} \oplus v_{a_2} & \cdots & v_{hd_1} \oplus v_{a_{n-1}} & v_{hd_1} \oplus v_{a_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varnothing & v_{hd_\eta} \oplus v_{a_2} & \ddots & v_{hd_\eta} \oplus v_{a_{n-1}} & v_{hd_\eta} \oplus v_{a_n} \\ v_{a_1}{}^{\theta} & v_{a_2}{}^{\theta} & \ddots & v_{a_{n-1}}{}^{\theta} & v_{a_n}{}^{\theta} \\ v_{hd_{\eta+2}} \oplus v_{a_1} & v_{hd_{\eta+2}} \oplus v_{a_2} & \ddots & hd_{\eta+2} \oplus v_{a_{n-1}} & \varnothing \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{hd_m} \oplus v_{a_1} & v_{hd_m} \oplus v_{a_2} & \cdots & v_{hd_m} \oplus v_{a_{n-1}} & \varnothing \end{bmatrix} \end{array}$$

(4.10)

,which can be briefly represented by

$$\aleph_{mn} = G_{\aleph}\left(\overrightarrow{V_{hd}}, \overrightarrow{V_a}\right) = \left[\!\left[\left[v_{hd_i} \oplus v_{a_j} \right]_{j=1}^{n} \right]\!\right]_{i=1}^{m} \tag{4.11}$$

, where double bracket $[\![\bullet]\!] = [\bullet]^T$ and T is the transposition.

Proof:

The Cartesian product of two linguistic sets $\overrightarrow{V_{hd}}$ and $\overrightarrow{V_a}$ is defined as $\overrightarrow{V_{hd}} \times \overrightarrow{V_a}$, which $\left(v_{hd_i}, v_{a_j}\right)$ is its order pair. Linguistic Cartesian product $\overrightarrow{V_{hd}} \otimes \overrightarrow{V_a}$ is defined as

$$\overrightarrow{V_{hd}} \otimes \overrightarrow{V_a} = G_\alpha\left(\overrightarrow{V_{hd}} \times \overrightarrow{V_a}\right) = \left[\!\left[\left[G_\alpha\left(v_{hd_i}, v_{a_j}\right) \right]_{j=1}^{n} \right]\!\right]_{i=1}^{m} \tag{4.12}$$

, where $G_\alpha\left(v_{hd_i}, v_{a_j}\right)$ is from Eq(4.9) is proposition 4.2. Next Calculate each entry $G_\alpha\left(v_{hd}, v_a\right)$ to get $G_\aleph\left(\overrightarrow{V_{hd}}, \overrightarrow{V_a}\right)$. □

Theorem 4.1 extends two Corollaries.

Corollary 4.1 (Cardinal number of $\aleph_{mn}$)**:**

As $\left|\aleph_{mn}\right| = m \times n - 2\eta$ and $m = 2\eta + 1$ where $m = \left|\overrightarrow{V_{hd}}\right|$ and $n = \left|\overrightarrow{V_a}\right|$, then $\left|\aleph_{mn}\right| = m(n-1)+1$.

Corollary 2 (Relationship of i and j):

$$j = \begin{cases} 1 & ,i = 1,\ldots,\eta \\ 2,\ldots,n\text{-}1 & ,i = 1,\ldots,m \\ n & ,i = \eta,\ldots,m \end{cases} \quad \text{or} \quad i = \begin{cases} 1,\ldots,\eta & ,\ \mathrm{j} = 1 \\ 1,\ldots,m & ,\ \mathrm{j} = 2,\ldots,n\text{-}1 \\ 2,\ldots,n\text{-}1 & ,\ \mathrm{j} = n \end{cases}$$

Algorithm 4.1 is to conclude this section.

Algorithm 4.1 (Syntactic Rule Algorithm $\aleph_{mn} = G_{\aleph}\left(\overrightarrow{V_h}, \overrightarrow{V_d}, \overrightarrow{V_a}\right)$):

Step 1: Input: Linguistic term sets ($\overrightarrow{V_h}$, $\overrightarrow{V_d}$, $\overrightarrow{V_a}$) //definitions 4.2-4.6

Step 2: Proceed $G_{\overrightarrow{V_{hd}}}\left(\overrightarrow{V_h}, \overrightarrow{V_d}\right) = \overrightarrow{V_{hd}}$ // proposition 4.1

Step 3: Proceed $G_{\aleph}\left(\overrightarrow{V_{hd}}, \overrightarrow{V_a}\right)$ //proposition 4.2 and theorem 4.1

Step 4: Return: $\aleph_{mn} = G_{\aleph}\left(\overrightarrow{V_{hd}}, \overrightarrow{V_a}\right)$ //End

Example 4.1:

Provided that $V_a = \{\text{Excellent,Good,Fair,Weak,Poor}\}$, $V_d = \{\text{Below,Absolutely,Above}\}$, and $V_h = \{\text{Little,Quite,Much}\}$. Find the CLV.

$CLV = \left(\overrightarrow{V_h}, \overrightarrow{V_d}, \overrightarrow{V_a}, G\right)$ where V_h, V_d, and V_a are vectorized as $\overrightarrow{V_h}$, $\overrightarrow{V_d}$, $\overrightarrow{V_a}$. From $\overrightarrow{V_{hd}} = G_{\overrightarrow{V_{hd}}}\left(\overrightarrow{V_h}, \overrightarrow{V_d}\right)$, thus,

$$\begin{aligned}\overrightarrow{V_{hd}} &= \left[v_{hd_1}, \cdots, v_{hd_7}\right] \\ &= \begin{bmatrix}\text{"Much Below","Quite Below","Little Below","Absolutely"} \sim \\ \text{"Little Above","Quite Above","Much Above"}\end{bmatrix}\end{aligned}$$

where $\sim$ denotes a line break as one line cannot contain all the members.

$\overrightarrow{V_a} = [\text{Poor, Weak, Fair, Good, Excellent}]$ where $\leq_{j=1}^{5}\left(v_{a_j}\right)$. As $\aleph_{\text{mn}} = G_{\aleph}\left(\overrightarrow{V_{hd}}, \overrightarrow{V_a}\right)$, then

$$\aleph_{7,5} = \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \end{array} \begin{array}{c} \begin{array}{ccccc} \alpha^1 & \alpha^2 & \alpha^3 & \alpha^4 & \alpha^5 \end{array} \\ \begin{bmatrix} 0 & MB\text{-}W & MB\text{-}F & MB\text{-}G & MB\text{-}E \\ 0 & QB\text{-}W & QB\text{-}F & QB\text{-}G & QB\text{-}E \\ 0 & LB\text{-}W & LB\text{-}F & LB\text{-}G & LB\text{-}E \\ A\text{-}P & A\text{-}W & A\text{-}F & A\text{-}G & A\text{-}E \\ LA\text{-}P & LA\text{-}W & LA\text{-}F & LA\text{-}G & 0 \\ QA\text{-}P & QA\text{-}W & QA\text{-}F & QA\text{-}G & 0 \\ MA\text{-}P & MA\text{-}W & MA\text{-}F & MA\text{-}G & 0 \end{bmatrix} \end{array}$$

Example 4.1 shows that three vectors of linguistic terms can produce a large scale of linguistic terms. These linguistic terms can be used for describing the attribute of an object in a more precise form. However, when several variables are combined, the matrix structure cannot be formed by Linguistic Cartesian Product in view of human understanding. Thus the syntactic form needs verification with experts. Experts should firstly determine an attribute of an object which should be measured, and then justify appropriate $(\overrightarrow{V_h}, \overrightarrow{V_d}, \overrightarrow{V_a})$ to form CLV which can be used for describing the attribute. However, the next question that is how the semantic forms can be represented for computational intelligence. This paper proposes Fuzzy Normal Distribution in next section to address the semantic issues for CLV.

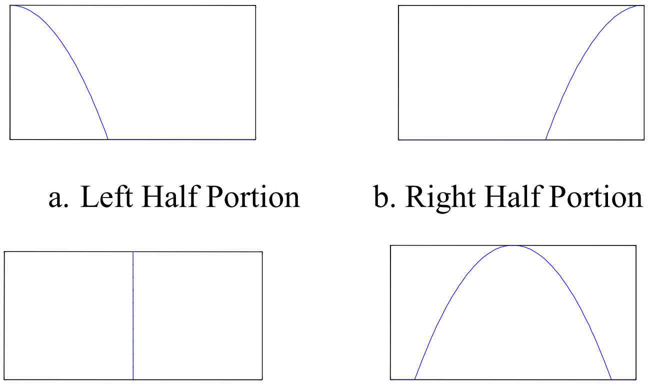
4.3. Fuzzy normal distribution (FND)

The syntactic rules of CLV are investigated in chapter 4.2. The semantic rules M are discussed in this section. Firstly Fuzzy Normal Distribution is defined, and then the properties of the fuzzy set are introduced.

Fuzzy Normal Distribution $\left(\left(\vec{\aleph}, X, \mu_{\alpha^{j=1,\cdots,n}}\right), \left(MFI, FI, \mu^{-1}_{\alpha^{j=1,\cdots,n}}\right)\right)$ is the semantic rule M which converts $T(\aleph)$ to X, i.e. $M_{\aleph}: \aleph \rightarrow X$. $\vec{\aleph}$ -Design Procedure $\left(\vec{\aleph}, X, \mu_{\alpha^{j=1,\cdots,n}}\right)$ is an atomic term v_{a_j} (or $\alpha^{j=1,\cdots,n} \in \vec{\aleph}$, which is different from $\alpha_{ij} \in \aleph$) associated with its Single Core Symmetric fuzzy set α^j of the membership function μ_{α^j} in the universal of discourse X. Equivalently, the name of the fuzzy set α^j can be used for the name of the linguistic label v_{a_j} and vice versa. On the basis of α^j Design Procedure , $\left(MFI, FI, \mu^{-1}_{\alpha^{j=1,\cdots,n}}\right)$ of $\vec{\aleph}$ is Fuzzy Interval Distribution *FID* which Membership Fuzzy Intervals *MFI* of $\overrightarrow{V_h}$ in $[0,1]$ are converted to Fuzzy Intervals *FI* of α_{ij} in X by inverse membership functions $\mu^{-1}_{\alpha^{j=1,\cdots,n}}$, i.e. $\mu_{\alpha^j}^{-1}: MFI\left(\alpha_{ij}\right) \rightarrow FI\left(\alpha_{ij}\right)$ where $j = 1,\cdots,n$ and $i = 1,\cdots,m$.

A Single Core Symmetric fuzzy set α^j is a 3-tuple fuzzy set $\left(\gamma_{\alpha^j}, d_{\alpha^j}, \mu_{\alpha^j}\right)$ that is convex, consisting of only a single core γ_{α^j} in the middle point of the fuzzy boundary of α^j *in X*, the tolerance distance d_{α^j} from γ_{α^j} to the boundary, and membership μ_{α^j} which symmetrically, continuously and gradually decreases from 1 to 0, along from γ_{α^j} to $\alpha_{l_j} = \gamma_{\alpha^j} + d_{\alpha^j}$ or/and $\alpha_{u_j} = \gamma_{\alpha^j} - d_{\alpha^j}$.

Regarding the shapes of the SCS fuzzy sets (Fig. 4.1), the Whole Shape (*WS*) of a SCS fuzzy set consists of a Left Half Portion (*LHP*) and a Right Half Portion (*RHP*) segmented by a singleton line. A singleton line exits where the membership is equal to one, as there is only one in a fuzzy variable (or a fuzzy set) x in X. A Half Portion (HP) means either LHP or RHP, i.e. $HP \in \{LHP, RHP\}$.



a. Left Half Portion b. Right Half Portion

c. Singleton Line d. Whole Shape

Figure 4.1: Membership shape description

The shape of α^1 is the LHP in the most left hand side in the universal of discourse X while α^n is the RHP in the most right hand side. $\alpha^{j=2,\cdots,n-1}$ is the whole shape distributed in X. Regarding the shape of the membership function, for a SCS Fuzzy Set $\left(\gamma_{\alpha^j}, d_{\alpha^j}, \mu_{\alpha^j}\right)$, if the membership is linear based, a triangular-based membership function is of the form:

$$\mu_{\alpha^j} = TbMF_{\alpha^j} = \begin{cases} \left(\dfrac{x + d_{\alpha^j} - \gamma_{\alpha^j}}{d_{\alpha^j}}\right)^{\tau_{\alpha^j}} & , x \in \left[\gamma_{\alpha^j} - d_{\alpha^j}, \gamma_{\alpha^j}\right] \\ 1 & , x = \gamma_{\alpha^j} \\ \left(\dfrac{-x + d_{\alpha^j} + \gamma_{\alpha^j}}{d_{\alpha^j}}\right)^{\tau_{\alpha^j}} & , x \in \left[\gamma_{\alpha^j}, \gamma_{\alpha^j} + d_{\alpha^j}\right] \end{cases}, \tau_{\alpha^j} \text{ is tuning level.} \quad (4.13)$$

This follows the triangular-based inversed membership function $TbMF_{\alpha^j}^{-1}$ for the base element x^{ϕ}, where $\phi = '-','\theta','+'$, which represent LHP, singleton, and RHP respectively.

$$x^{\phi} = \mu^{-1}_{\alpha^{j\phi}}\left(\mu_{\alpha^{j\phi}}\right) = TbMF_{\alpha^{j}}^{\ -1} = \begin{cases} \gamma_{\alpha^{j}} - d_{\alpha^{j}} + d_{\alpha^{j}}\left(\mu_{\alpha^{j\phi}}\right)^{1/\tau_{\alpha^{j}}} & ,\phi = '-' \\ \gamma_{\alpha^{j}} & ,\phi = '\theta' \\ \gamma_{\alpha^{j}} + d_{\alpha^{j}} - d_{\alpha^{j}}\left(\mu_{\alpha^{j\phi}}\right)^{1/\tau_{\alpha^{j}}} & ,\phi = '+' \end{cases} \quad (4.14)$$

Similarly, if the shape function is the Parabola-based Membership function, then

$$\mu_{\alpha^{j}} = PbMF_{\alpha^{j}} = \left(\frac{-1}{d_{\alpha^{j}}^{\ 2}} x^{2} + \frac{2\gamma_{\alpha^{j}}}{d_{\alpha^{j}}^{\ 2}} x + \frac{d_{\alpha^{j}}^{\ 2} - \gamma_{\alpha^{j}}^{\ 2}}{d_{\alpha^{j}}^{\ 2}} \right)^{\tau_{\alpha^{j}}}, \quad x \in \left[\alpha_{l_j}, \alpha_{u_j}\right] \quad .$$

(4.15)

Then the Parabola-based Membership Inversed Function $PbMF_{\alpha^{j}}^{\ -1}$ for the base element x^{ϕ} is the form:

$$x^{\phi} = \mu^{-1}_{\alpha^{j\phi}}\left(\mu_{\alpha^{j\phi}}\right) = PbMF_{\alpha^{j}}^{\ -1}\left(\mu_{\alpha^{j\phi}}\right) = \begin{cases} \gamma_{\alpha^{j}} - d_{\alpha^{j}}\sqrt{1-\left(\mu_{\alpha^{j\phi}}\right)^{1/\tau_{\alpha^{j}}}} & ,\phi = '-' \\ \gamma_{\alpha^{j}} & ,\phi = '\theta' \\ \gamma_{\alpha^{j}} + d_{\alpha^{j}}\sqrt{1-\left(\mu_{\alpha^{j\phi}}\right)^{1/\tau_{\alpha^{j}}}} & ,\phi = '+' \end{cases}$$

(4.16)

In selection of the membership functions, three criteria should be satisfied (Bargiela and Pedrycz, 2003): available domain knowledge, simplicity of the membership function, and possible parametric optimization of the fuzzy sets (calibration of the membership function). More approaches of elicitation of the membership can be found in Chapter 2.3.2.

4.3.1 Membership fuzziness distribution

Membership Fuzziness Distribution is of the form, $\left(\overrightarrow{V_h}^T, \overrightarrow{V_d}, \alpha, \mu, \varphi, dis, MCI, \lambda, \mu_{Norm}, \phi, invp, revp, MFI\right)$. The membership μ=[0, 1] can be fuzzily classified by $\overrightarrow{V_h}$. The membership utility measurement function $\varphi\left(\overrightarrow{V_h}\right)$ is the numerical judgments for $\overrightarrow{V_h}$. The *distance* of $v_{h_i} \in \overrightarrow{V_h}$, i.e. $dis\left(v_{h_i}\right)$, is characterized by the proportion of $\varphi\left(v_{h_i}\right)$ in $\sum_{\overrightarrow{V_h}} \varphi\left(v_{h_i}\right)$. On the basis of $dis\left(\overrightarrow{V_h}\right)$, Membership Crisp Interval *MCI* determines the close interval of $\overrightarrow{V_h}$ (e.g. $[\mu'_l, \mu'_u]$). And the Membership Fuzziness Factor λ, which is with the constraints by a membership normalization function μ_{Norm}, fuzzifies the $MCI\left(\overrightarrow{V_h}\right)$ into values of $MFI\left(\overrightarrow{V_h}\right)$. $\overrightarrow{V_h}^{\phi='-','\theta','+'}$ is determined by $\overrightarrow{V_d}$. $MFI\left(\overrightarrow{V_h}^{-}\right)$ is determined by the Inverse Position Function in*vp of* $MFI\left(\overrightarrow{V_h}\right)$ while $MFI\left(\overrightarrow{V_h}^{+}\right)$ is determined by Reversed Position Function *revp of* $MFI\left(\overrightarrow{V_h}\right)$. Finally, $MFI(\alpha)$ is determined by $MFI\left(\overrightarrow{V_h}^{\phi}\right)$. For a clear presentation, $\overrightarrow{V_h}^T$ or $\left[\!\left[\overrightarrow{V_h}\right]\!\right]$, transposition of $\overrightarrow{V_h}$, is preferred. Fig. 4.2 graphically summarizes the MFD method.

Let $MFI\left(\overrightarrow{V_h}\right)$ be discussed initially. If fuzzy boundaries from the classes needs to be justified, the crisp boundaries can be assumed first and then adjusted the interval into fuzzy boundaries. Conversely, if crisp boundaries among the classes need to be justified, the fuzzy boundaries initially can be assumed first, and then the intervals can be adjusted to derive the values. Therefore, *MFI* is determined by (MCI, λ).

Definition 4.10 ($MFI\left(\left[\!\left[\overrightarrow{V_h}\right]\!\right]\right)$): The *MFI* of $\left[\!\left[\overrightarrow{V_h}\right]\!\right]$ is determined by the lower boundary (μ_L) and upper boundary (μ_U) of the membership, and has the form:

$$MFI\left(\left[\!\left[\overrightarrow{V_h}\right]\!\right]\right)=\begin{bmatrix} MFI\left(v_{h_1}\right) \\ \vdots \\ MFI\left(v_{h_\eta}\right) \end{bmatrix}=\begin{array}{c} \begin{array}{cc} \mu_L & \mu_U \end{array} \\ \begin{bmatrix} \mu_{l_1} & \mu_{u_1} \\ \vdots & \vdots \\ \mu_{l_\eta} & \mu_{u_\eta} \end{bmatrix} \end{array}=\begin{array}{c} \begin{array}{cc} \mu_L & \mu_U \end{array} \\ \left[\!\left[\mu_{l_j}, \mu_{u_j}\right]\!\right]_{j=1}^{\eta} \end{array} \tag{4.17}$$

,where is $\eta=\left|\overrightarrow{V_h}\right|$, $\mu_L=\left[\!\left[\mu_{l_i}\right]\!\right]_{i=1}^{\eta}$, and $\mu_U=\left[\!\left[\mu_{u_i}\right]\!\right]_{i=1}^{\eta}$.

The next question is how to justify μ_{l_j} and μ_{u_j}. This follows the concept of Membership Crisp Interval *MCI*, which is determined by membership distance function dis and Membership Utility Measurement function φ initially.

Definition 4.11 (*dis*()): The membership distance function is of the form

$$dis\left(v_{h_i}\right)=\frac{\varphi\left(v_{h_i}\right)}{\sum_{\overrightarrow{V_h}}\varphi\left(v_{h_i}\right)}, \quad i\in\{1,2,\cdots,\eta\}. \tag{4.18}$$

The Membership Utility Measurement function of $\overrightarrow{V_h}$ is of the form

$$\varphi\left(\overrightarrow{V_h}\right)=\left[\varphi\left(v_{h_1}\right),\cdots,\varphi\left(v_{h_\eta}\right)\right].$$

On the basis of definitions 4.10 and 4.11, then $MCI(V_h)$ is formed as follows:

Proposition 4.3 ($MCI(V_h)$)**:** Let $MCI(v_{h_i}) = [\mu'_{l_i} \;\; \mu'_{u_i}]$. For $MCI(\overrightarrow{V_h})$ which a parameter $\overrightarrow{V_h}$ is in matrix form,

$$MCI\left(\left[\!\left[\overrightarrow{V_h}\right]\!\right]\right) = MCI\left(\left[\!\left[v_{h_1}\right]\!\right]_{\eta}^{1}\right) = \begin{array}{c} \mu'_L \quad \mu'_U \\ \left[\!\left[\mu'_{l_j}, \mu'_{u_j}\right]\!\right]_{j=1}^{\eta} \end{array} \tag{4.19}$$

, which is determined by:

$$\begin{array}{c} \mu'_L \quad \mu'_U \\ \left[\!\left[\mu'_{l_j}, \mu'_{u_j}\right]\!\right]_{j=1}^{\eta} \end{array} = \begin{array}{c} \mu'_L \qquad\qquad \mu'_U \\ \left[\!\left[\sum_{i=j+1}^{\eta} dis\left(v_{h_i}\right), \sum_{i=j}^{\eta} dis\left(v_{h_i}\right)\right]\!\right]_{j=1}^{\eta} \end{array} \tag{4.20}$$

where $\mu'_{l_\eta} = 0$ and $\mu'_{u_1} = 1$, $i \in \{1, 2, \cdots, \eta\}$.

Proof: The membership μ is [0,1]. μ'_{l_η} is the floor boundary, and thus $\mu'_{l_\eta} = 0$ whilst μ'_{u_1} is the ceiling boundary, and thus $\mu'_{u_1} = 1$. μ is separated to v_{h_i} by crisp points. For v_{h_i} and $v_{h_{i+1}}$, $i \in \{1, 2, \cdots, \eta\}$, μ'_{u_i} of v_{h_i} is equal to the lower boundary $\mu'_{l_{i+1}}$ of $v_{h_{i+1}}$, i.e. $\mu'_{l_{i+1}} = \mu'_{u_i}$, as they are at the same point in the continuous curve. As $\mu'_{u_i} = \mu'_{l_i} + dis(v_{h_i})$, then (4.20). □

The next step is to define a Membership Fuzziness Factor to fuzzify *MCI* to *MFI*.

Definition 4.12 (λ): The Membership Fuzziness Factors λ's of μ'_{u_i} and μ'_{l_i}, i.e. $\lambda_{\mu_{u_i}}$ and $\lambda_{\mu_{l_i}}$, are used for fuzzifying the *MCI* to *MFI*. Thus, they have the forms:

$$\lambda_{\mu_{u_i}} = \lambda_{\mu_{l_i}} = \frac{\lambda_0}{2}, \text{ where } \lambda_{\mu_{u_i}}, \lambda_{\mu_{l_i}} \in [0,1] \text{ (i.e. } \lambda_0 \in [0,2]) \tag{4.21}$$

$$\mu_{u_i} = \begin{cases} \mu'_{u_i} = 1 & ,i = 1 \\ \mu'_{u_i} + \lambda_{\mu_{u_i}} dis\left(v_{h_i}\right) & ,i = 2,\cdots,\eta \end{cases} \tag{4.22}$$

$$\mu_{l_i} = \begin{cases} \mu'_{l_i} - \lambda_{\mu_{l_i}} dis\left(v_{h_i}\right) & ,i = 1,\cdots,\eta - 1 \\ \mu'_{l_i} = 0 & ,i = \eta \end{cases} \tag{4.23}$$

(4.21) is the default definition. It can be changed in order to adjust the membership fuzziness with other functions for $\lambda_{\mu_{l_i}}$ or $\lambda_{\mu_{u_i}}$. In addition, if $\lambda_{\mu_{u_i}}$ or $\lambda_{\mu_{l_i}}$ is excessively large, μ_{u_i} or μ_{l_i} may be larger than 1 and less than 0. This makes the membership fuzziness process unstable. Thus, this situation follows lemma 4.1 as the constraints:

Lemma 4.1 ($\mu_{Norm}(\lambda)$): The Membership Normalization Function μ_{Norm} for validating the validity of $\lambda_{\mu_{l_i}}$ or $\lambda_{\mu_{u_i}}$, is the forms

$$\mu_{Norm}\left(\lambda_{\mu_{u_i}}\right) = \begin{cases} \lambda_{\mu_{u_i}} & ,0 \le \mu_{u_i} \le 1 \\ \text{"Error"} & ,\mu_{u_i} > 1 \text{ or } \mu_{l_i} < 0 \end{cases} ,\forall i \in \{2,\cdots,\eta\} \tag{4.24}$$

$$\mu_{Norm}\left(\lambda_{\mu_{l_i}}\right) = \begin{cases} \lambda_{\mu_{l_i}} & ,0 \le \mu_{l_i} \le 1 \\ \text{"Error"} & ,\mu_{l_i} > 1 \text{ or } \mu_{l_i} < 0 \end{cases} ,\forall i \in \{1,\cdots,\eta - 1\} \tag{4.25}$$

If there is an “error”, λ function has to be justified again.

Proof:

As $\lambda_{\mu_{u_i}} \in [0,1] : \mu_{u_i} \le 1, \forall i \in \{2,\cdots,\eta\}$ and $\lambda_{\mu_{l_i}} \in [0,1] : \mu_{l_i} \ge 0, \forall i \in \{1,\cdots,\eta - 1\}$, On the basis of definition 4.12. This proposition is held. □

Proposition 4.4 ($MFI\left(\left[\!\left[\overrightarrow{V_h}\right]\!\right]\right)$)**:** The membership [0,1] is fuzzified by $\overrightarrow{V_h}$. The Membership Fuzzy Intervals of $\overrightarrow{V_h}$, i.e. $MFI\left(\overrightarrow{V_h}\right)$ are determined by μ_L and μ_U . λ fuzzifies the *MCI*. Thus,

$$MFI\left(\left[\!\left[\overrightarrow{V_h}\right]\!\right]\right)=MFI\left(\left[\!\left[\begin{matrix}v_{h_1}\\ \vdots \\ v_{h_\eta}\end{matrix}\right]\!\right]\right)=\begin{cases}\left[\mu'_{l_i}-\lambda_{\mu_{l_i}}dis\left(v_{h_i}\right),\quad 1\right]_{i=1}\\ \left[\!\left[\mu'_{l_i}-\lambda_{\mu_{l_i}}dis\left(v_{h_i}\right),\ \mu'_{u_i}+\lambda_{\mu_{l_i}}dis\left(\sigma_i\right)\right]\!\right]_{i=2}^{\eta-1}\\ \left[\quad 0\quad ,\quad \mu'_{u_i}+\lambda_{\mu_{l_i}}dis\left(\sigma_i\right)\right]_{i=\eta}\end{cases} \tag{4.26}$$

, where $\lambda_{\mu_{l_i}}$ or $\lambda_{\mu_{u_i}}$ is confined to (4.24) and (4.25) in lemma 4.1.

Proof:

$$MCI\left(\left[\!\left[\overrightarrow{V_h}\right]\!\right]\right)=\overset{\mu'_L\quad\mu'_U}{\left[\!\left[\mu'_{l_j},\mu'_{u_j}\right]\!\right]_{j=1}^{\eta}}\xrightarrow[\forall i\in\{1,\cdots,\eta\}]{\mu_{u_i},\mu_{l_i}}\overset{\mu_L\quad\mu_U}{\left[\!\left[\mu_{l_j},\mu_{u_j}\right]\!\right]_{j=1}^{\eta}}=MFI\left(\left[\!\left[\overrightarrow{V_h}\right]\!\right]\right)$$ with $\mu_{Norm}\left(\lambda_{\mu_{l_i}}\right)$

and $\mu_{Norm}\left(\lambda_{\mu_{u_i}}\right)$ (lemma 4.1). □

4.3.2 Fuzzy interval distribution (FID)

A fuzzy interval FI of α_{ij} , i.e. $FI\left(\alpha_{ij}\right)$, is in the portion $\left[x_{l_i},x_{u_i}\right]$ of α^j in X that is

$$FI\left(\alpha_{ij}\right)=\alpha^j\left[x_{l_i},x_{u_i}\right] \tag{4.27}$$

A variable name before an interval specifies that a super set guides the diminution of the fuzzy set. It may ignore the variable name, but the levels continuously expand, the traceability becomes the problem. For listing the fuzzy intervals with the set of members, then

$$FI\left(\widehat{\alpha^j}\right)=FI\left(\left[\!\left[\alpha_{ij}\right]\!\right]_{j=1}^{m}\right)=\alpha^j\left[\!\left[x_{l_i},x_{u_i}\right]\!\right]_{i=1}^{m} \tag{4.28}$$

Similarly, to extend the notations $\vec{\aleph}=\left[\widehat{\alpha^j}\right]_{j=1}^{n}$, $\widehat{\alpha^j}=\left[\!\left[\alpha_{ij}\right]\!\right]_{i=1}^{m}$, $\widehat{\alpha^j}^{-}=\left[\!\left[\alpha_{ij}\right]\!\right]_{i=1}^{\eta}$, $\widehat{\alpha^j}^{+}=\left[\!\left[\alpha_{ij}\right]\!\right]_{i=\eta+2}^{m}$. The "^" on a linguistic variable (i.e. $\hat{\aleph}$ or $\widehat{\alpha^j}$) means the list of all variable members of the variable.

A Fuzzy Interval Distribution $\left(\aleph,MFI,FI,P,\mu_{\alpha^{j\phi}}^{-1}\right)$ is the process to map $\aleph$ to the fuzzy interval $FI(\aleph)$ in X by the corresponding Position function P of the corresponding inversed membership function $\mu_{\alpha^{j\phi}}^{-1}$ of $MFI(\aleph)$. *P=vip,hrp* is described in propositions 5, 6. In short, $FI(\aleph)=\mu_{\alpha^{j\phi}}^{-1}\left(P\left(MFI(\aleph)\right)\right)$.

$\aleph,MFI$ and $\mu_{\alpha^{j\phi}}^{-1}$ have been discussed in previous sections. Fig. 2 graphically summarizes the method for designing $MFI\left(\left[\!\left[\vec{V_h}\right]\!\right]\right)$ where $\eta=3$. In addition, FI is defined. P is not defined. To Investigate P, Fig. 3 shows the mapping from $MFI\left(\left[\!\left[\vec{V_h}\right]\!\right]\right)$ to $FI\left(\left[\!\left[\alpha_{ij}\right]\!\right]_{i=1}^{m}\right)$. Consider a single $MFI\left(v_{h_i}\right)$, there are two $FI\left(\alpha_{ij}\right)$'s in v_a^{+} and v_a^{-} respectively. This leads to two relationships for the discussion:

1. $MFI\left(\vec{V_h}\right)\rightarrow MFI\left(\vec{V_h}^{-}\right)$, where $\vec{V_h}^{-}$ is related to v_a^{-} (4.29)
2. $MFI\left(\vec{V_h}\right)\rightarrow MFI\left(\vec{V_h}^{+}\right)$, where $\vec{V_h}^{+}$ is related to v_a^{+}. (4.30)

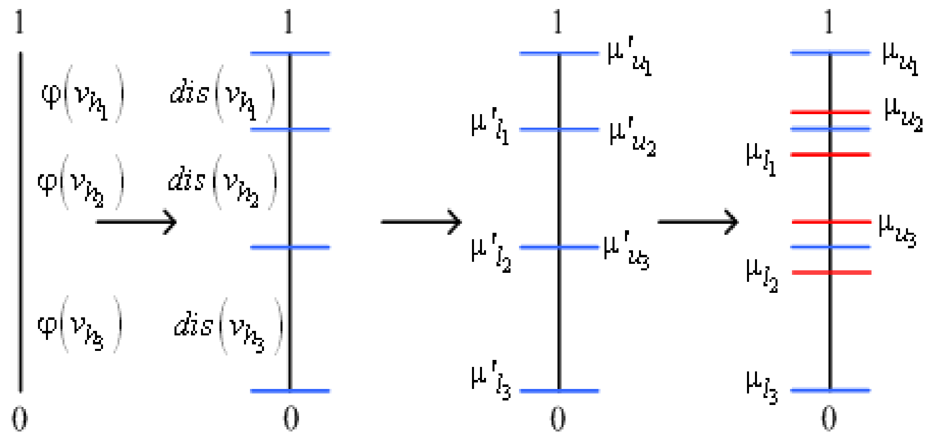


Figure 4.2: Calculation process for $MFI\left(\left[\!\left[\overrightarrow{V_h}\right]\!\right]\right)$

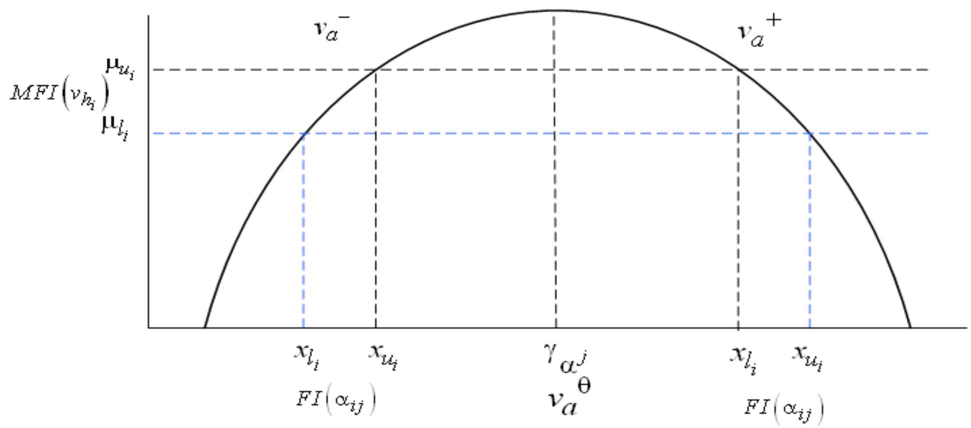


Figure 4.3: $MFI \xrightarrow{\mu^{-1}{}_{\alpha^j}} FI$

$\mu_\alpha(x)^- \in [0,1] : x \in x_{RHP}$, where x_{RHP} is the set of the base variables of RHP, or in v_a^-. Similarly, $\mu_\alpha(x)^+ \in [0,1] : x \in x_{LHP}$. Therefore,

for $MFI\left(\overrightarrow{V_h}\right) \rightarrow MFI\left(\overrightarrow{V_h}^-\right)$, $\left(\mu_{l_i}{}^-, \mu_{u_i}{}^-\right) : \left(\mu_{l_i}, \mu_{u_i}\right) \in MFI\left(\overrightarrow{V_h}^-\right)$.

For $MFI\left(\overrightarrow{V_h}\right) \rightarrow MFI\left(\overrightarrow{V_h}^+\right)$, $\left(\mu_{l_i}{}^+, \mu_{u_i}{}^+\right) : \left(\mu_{l_i}, \mu_{u_i}\right) \in MFI\left(\overrightarrow{V_h}^+\right)$.

To look better, the following forms are defined.

$$MFI\left(\left[\!\left[\overrightarrow{V_h}^{+}\right]\!\right]\right)== \begin{array}{c} \begin{array}{cc} \mu_L{}^{+} & \mu_U{}^{+} \end{array} \\ \begin{bmatrix} \mu_{l_1}{}^{+} & \mu_{u_1}{}^{+} \\ \vdots & \vdots \\ \mu_{l_\eta}{}^{+} & \mu_{u_\eta}{}^{+} \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{cc} \mu_L{}^{+} & \mu_U{}^{+} \end{array} \\ \begin{bmatrix} \mu_{l_1} & \mu_{u_1} \\ \vdots & \vdots \\ \mu_{l_\eta} & \mu_{u_\eta} \end{bmatrix}^{+} \end{array} = \begin{array}{c} \begin{array}{cc} \mu_L{}^{+} & \mu_U{}^{+} \end{array} \\ \left[\mu_{l_j}, \mu_{u_j}\right]_{j=1}^{\eta} \end{array} \tag{4.31}$$

$$MFI\left(\left[\!\left[\overrightarrow{V_h}^{-}\right]\!\right]\right)= \begin{array}{c} \begin{array}{cc} \mu_L{}^{-} & \mu_U{}^{-} \end{array} \\ \begin{bmatrix} \mu_{l_1}{}^{-} & \mu_{u_1}{}^{-} \\ \vdots & \vdots \\ \mu_{l_\eta}{}^{-} & \mu_{u_\eta}{}^{-} \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{cc} \mu_L{}^{-} & \mu_U{}^{-} \end{array} \\ \begin{bmatrix} \mu_{l_1} & \mu_{u_1} \\ \vdots & \vdots \\ \mu_{l_\eta} & \mu_{u_\eta} \end{bmatrix}^{-} \end{array} = \begin{array}{c} \begin{array}{cc} \mu_L{}^{-} & \mu_U{}^{-} \end{array} \\ \left[\mu_{l_j}, \mu_{u_j}\right]_{j=1}^{\eta} \end{array} \tag{4.32}$$

$$MFI\left(\left[\!\left[\overrightarrow{V_{hd}}\right]\!\right]\right)=\left[\!\left[\mu_{l_j}, \mu_{u_j}\right]\!\right]_{j=1}^{m} = \begin{bmatrix} MFI\left(\left[\!\left[\overrightarrow{V_h}^{-}\right]\!\right]\right) \\ \hdashline MFI\left(v_d{}^{\theta}\right) \\ \hdashline MFI\left(\left[\!\left[\overrightarrow{V_h}^{+}\right]\!\right]\right) \end{bmatrix} = \begin{bmatrix} \left[\mu_{l_j}, \mu_{u_j}\right] \\ \hdashline \left[\mu_{l_\theta}, \mu_{u_\theta}\right] \\ \hdashline \left[\mu_{l_j}, \mu_{u_j}\right] \end{bmatrix}_{j=1}^{\eta} \tag{4.33}$$

Propositions 4.5 and 4.6 are satisfied for answering (4.29) and (4.30). To have (4.31)-(4.33), lemma 4.2 is developed.

Lemma 4.2: For RHP, $v_a{}^{-}$, or $\phi='-'$, iif $x_{l_i} \le x_{u_i}$, then $\mu_{l_i} \le \mu_{u_i}$, or $\mu_{l_i}{}^{-} \le \mu_{u_i}{}^{-}$. For LHP, $v_a{}^{+}$, or $\phi='+'$, iif $x_{l_i} \le x_{u_i}$, then $\mu_{l_i} \ge \mu_{u_i}$, or $\mu_{l_i}{}^{+} \ge \mu_{u_i}{}^{+}$.

Proof:

For $\phi='-'$, or RHP, the larger μ_{α^j}, the larger x_{RHP} is, and vice versa. Therefore, iif $x_{l_i} \le x_{u_i}$, then $\mu_{l_i} \le \mu_{u_i}$. Similarly, for $\phi='+'$, or LHP, the larger μ_{α^j}, the less x_{RHP}.

□

Proposition 4.5 (vip): $MFI\left(\left[\!\left[\overrightarrow{V_h}^{-}\right]\!\right]\right)$ is vertically inversed, *vip*, of $MFI\left(\left[\!\left[\overrightarrow{V_h}\right]\!\right]\right)$,

$$MFI\left(\left[\!\left[\overrightarrow{V_h}^{-}\right]\!\right]\right)=vip\left(MFI\left(\left[\!\left[\overrightarrow{V_h}\right]\!\right]\right)\right)\equiv \begin{array}{c}\mu_L{}^{-}\ \ \mu_U{}^{-}\\ \left\lceil \mu_{l_j},\mu_{u_j}\right\rceil_{j=1}^{\eta}\end{array}=\begin{array}{c}\mu_L\ \ \ \mu_U\\ \left[\!\left[\mu_{l_j},\mu_{u_j}\right]\!\right]_{j=\eta}^{1}\end{array} \tag{4.34}$$

Proof:

$\mu_{\aleph^-}^{-1}\left(MFI\left(\left[\!\left[\overrightarrow{V_h}^{-}\right]\!\right]\right)\right)=\left[\!\left[\left[\mu_{l_1},\mu_{u_1}\right],\ldots,\left[\mu_{l_\eta},\mu_{u_\eta}\right]\right]\!\right]$, then $\left[\mu_{l_1},\mu_{u_1}\right]\geq\ldots\geq\left[\mu_{l_\eta},\mu_{u_\eta}\right]$.

To have $\left[\mu_{l_1},\mu_{u_1}\right]\leq\ldots\leq\left[\mu_{l_\eta},\mu_{u_\eta}\right]$, $vip(.)$ is applied □

Proposition 4.6 (hrp): $MFI\left(\left[\!\left[\overrightarrow{V_h}^{+}\right]\!\right]\right)$ is horizontally reversed, *revp*, of $MFI\left(\left[\!\left[\overrightarrow{V_h}\right]\!\right]\right)$,

$$MFI\left(\left[\!\left[\overrightarrow{V_h}^{+}\right]\!\right]\right)=hrp\left(MFI\left(\left[\!\left[\overrightarrow{V_h}\right]\!\right]\right)\right)\equiv \begin{array}{c}\mu_L{}^{+}\ \ \mu_U{}^{+}\\ \left\lfloor \mu_{l_j},\mu_{u_j}\right\rfloor_{j=1}^{\eta}\end{array}=\begin{array}{c}\mu_U\ \ \ \mu_L\\ \left[\!\left[\mu_{U_j},\mu_{l_j}\right]\!\right]_{j=1}^{\eta}\end{array} \tag{4.35}$$

Proof:

From lemma 4.2, iif $x_{l_i}\leq x_{u_i}$, then $\mu_{l_i}{}^{+}\geq\mu_{u_i}{}^{+}$. There is horizontal reverse relationship between μ_L and μ_U of $MFI\left(\left[\!\left[\overrightarrow{V_h}^{+}\right]\!\right]\right)$ and $MFI\left(\left[\!\left[\overrightarrow{V_h}\right]\!\right]\right)$. □

Proposition 4.7: ($MFI\left(\widehat{\alpha^j}\right)$):

$$MFI\left(\left[\!\left[\widehat{\alpha^j}\right]\!\right]\right)=MFI\left(\left[\!\left[\alpha_{ij}\right]\!\right]_{i=1}^{m}\right)=\begin{cases}MFI\left(\left[\!\left[\overrightarrow{V_h}^{-}\right]\!\right]\right), & 2\leq j<n\\ MFI\left(v_d{}^{\theta}\right) & ,\forall j\\ MFI\left(\left[\!\left[\overrightarrow{V_h}^{+}\right]\!\right]\right), & 1<j\leq n-1\end{cases} \tag{4.36}$$

Proof:

Firstly, $MFI\left(\alpha_{ij}\right)$ can be represented by $MFI\left(v_{hd_i}\right)$ in α^j as $\alpha_{ij}\triangleq v_{hd_i}\oplus v_{a_j}$.

Secondly, $MFI\left(\left[\!\left[\overrightarrow{V_h}^{-}\right]\!\right]\right)$ and $MFI\left(\left[\!\left[\overrightarrow{V_h}^{-}\right]\!\right]\right)$ can be determined by propositions 4.5 and 4.6. Thirdly α^1 is RHP Fuzzy set only, and α^n is LHP set only. $\alpha^{j\neq 1,n}$ is the whole set. □

Theorem 4.2 is satisfied.

Theorem 4.2 (Fuzzy interval distribution)**:**

From $\mu_{\alpha^{j\phi}}{}^{-1}: MFI(\alpha_{ij}) \to FI(\alpha_{ij})$, i.e. $\mu_{\alpha^{j\phi}}{}^{-1}: \alpha^j\left[\mu_{l_i}{}^{\phi}, \mu_{u_i}{}^{\phi}\right] \to \alpha^j\left[x_{l_i}{}^{\phi}, x_{u_i}{}^{\phi}\right]$,

$x_{l_i}{}^{\phi} = \mu_{\alpha^{j\phi}}{}^{-1}\left(\mu_{l_i}{}^{\phi}\right)$ and $x_{u_i}{}^{\phi} = \mu_{\alpha^{j\phi}}{}^{-1}\left(\mu_{u_i}{}^{\phi}\right)$, $\phi = '-','\theta','+'$, then

$$FI\left(\left[\!\left[\widehat{\alpha^j}\right]\!\right]\right) = FI\left(\left[\!\left[\alpha_{ij}\right]\!\right]_{i=1}^{m}\right) = \mu_{\alpha^{j\phi}}{}^{-1}\left(P\left(MFI\left(\left[\!\left[\alpha_{ij}\right]\!\right]_{i=1}^{m}\right)\right)\right),\ P{=}vip,hrp; \tag{4.37}$$

Explicitly,

$$FI\left(\left[\!\left[\widehat{\alpha^j}\right]\!\right]\right) = \begin{bmatrix} \mu_{\alpha^{j-}}{}^{-1}\left(vip\left(MFI\left(\left[\!\left[\overrightarrow{V_h}^{-}\right]\!\right]\right)\right)\right) \\ \hdashline \mu_{\alpha^{j-}}{}^{-1}\left(MFI\left(\left[\!\left[\overrightarrow{V_h}^{\theta}\right]\!\right]\right)\right) \\ \hdashline \mu_{\alpha^{j+}}{}^{-1}\left(vip\left(MFI\left(\left[\!\left[\overrightarrow{V_h}^{+}\right]\!\right]\right)\right)\right) \end{bmatrix} = \begin{cases} \left[\mu_{\alpha^{j-}}{}^{-1}\left(\mu_{l_i}{}^{-}\right),\ \mu_{\alpha^{j-}}{}^{-1}\left(\mu_{l_i}{}^{-}\right)\right]_{i=1}^{\eta} & ,j\in\{2,\ldots,n\}\ \ \&\ \ \phi='-' \\ \left[\mu_{\alpha^{j}}{}^{-1}\left(\mu_{l_\theta}\right),\ \mu_{\alpha^{j}}{}^{-1}\left(\mu_{l_\theta}\right)\right]_{i=\eta+1} & ,\forall j\ \ \&\ \ \phi='\theta' \\ \left[\mu_{\alpha^{j+}}{}^{-1}\left(\mu_{l_i}{}^{+}\right),\ \mu_{\alpha^{j+}}{}^{-1}\left(\mu_{l_i}{}^{+}\right)\right]_{i=\eta+2}^{m} & ,j\in\{1,\ldots,n-1\}\ \&\ \phi='+' \end{cases} \tag{4.38}$$

Proof :

Firstly, $MFI\left(\left[\!\left[\overrightarrow{V_h}^{-}\right]\!\right]\right), MFI\left(v_h{}^{\theta}\right)$ and $MFI\left(\left[\!\left[\overrightarrow{V_h}^{+}\right]\!\right]\right)$ are determined in proposition 4.7.

Secondly,

If $j\in\{2,\cdots,m\}$, and $\phi='-'$, then $MFI\left(\left[\!\left[\overrightarrow{V_h}^{-}\right]\!\right]\right) \xrightarrow[\mu_{\alpha^{j-}}{}^{-1}\left(\mu_{u_i}{}^{-}\right)]{\mu_{\alpha^{j-}}{}^{-1}\left(\mu_{l_i}{}^{-}\right)} FI\left(\widehat{\alpha^j}^{-}\right) = FI\left(\left[\!\left[\alpha_{ij}\right]\!\right]_{i=1}^{\eta}\right);$

If $\forall j$, and $\phi = '\theta'$, then $MFI\left(v_h{}^{\theta}\right) \xrightarrow[x_{u_0} = \gamma_{\alpha^j}]{x_{l_0} = \gamma_{\alpha^j}} FI\left[\alpha_{ij}\right]$;

If $j \in \{1, \cdots, m-1\}$ and $\phi = '+'$, then

$$MFI\left(\left[\!\left[\overrightarrow{V_h}^{+}\right]\!\right]\right) \xrightarrow[\mu_{\alpha^{j+}}{}^{-1}\left(\mu_{u_i}{}^{+}\right)]{\mu_{\alpha^{j+}}{}^{-1}\left(\mu_{l_i}{}^{+}\right)} FI\left(\widehat{\alpha^{j}}^{+}\right) = FI\left(\left[\!\left[\alpha_{ij}\right]\!\right]_{i=\eta+2}^{m}\right).$$

These three conditions can form the piecewise equation in (4.38) which means (4.37). Examples of the inverse membership functions can be referred to (4.14) and (4.16). □

Theorem 4.2 returns interval representation values of a matrix of linguistic terms. The crisp representation values from the intervals are defined as follows.

Definition 4.13 ($f_{\overline{X}}(\aleph) = M(\aleph)$ in Crisp number): From $f_{\overline{X}}(\aleph): \aleph \to \overline{X}$, if $\overline{X}$ is a matrix of crisp numbers, then $\overline{X} = f_{\overline{X}}(\aleph) = M(\aleph) = mean\left(FI\left(\vec{\aleph}\right)\right)$ where $M(\aleph)$ is the semantic rules for $\aleph$. Similarly, for each entry in $\aleph$, $\overline{X}_{\alpha_{ij}} = f_{\overline{X}}\left(\alpha_{ij}\right) = mean\left(FI\left(\alpha_{ij}\right)\right)$.

Definition 4.13 returns a crisp number. A fuzzy number is presented by a model value and a pair of interval values. Thus on the basis of the theorem 4.2 and definition 4.13, definition 4.14 defines CLV represented by fuzzy numbers.

Definition 4.14 ($f_{\overline{X}}(\aleph) = M(\aleph)$ in Fuzzy Number**):** For $f'_{\overline{X}}(\aleph): \aleph \to \overline{X}'$, if $\overline{X}'$ is a matrix of representation values in fuzzy number, then $\overline{X}'_{\aleph} = f'_{\overline{X}}(\aleph) = M(\aleph) = \left[\left[\!\left[\left(\overline{x}_{l_{ij}}, \overline{x}_{\pi_{ij}}, \overline{x}_{u_{ij}}\right)\right]\!\right]_{i=1}^{m}\right]_{j=1}^{n}$, where the interval values $\left[\overline{x}_{l_{ij}}, \overline{x}_{u_{ij}}\right] \in FI\left(\vec{\aleph}\right)$ is from theorem 4.2, and the model value $x_{\pi_{ij}} \in mean\left(FI\left(\vec{\aleph}\right)\right)$ is from definition 4.13.

Some may regard $\alpha_{\eta,j} \triangleq v_{hd_\eta} \oplus v_{a_j}$, $\forall j \in \{1,\ldots,n\}$ should not be singleton. Thus

$$\overline{X}'_{\alpha_{\eta j}} = \begin{cases} \left(\overline{X}_{\alpha_{\eta,1}}, \overline{X}_{\alpha_{\eta,1}}, \overline{X}_{\alpha_{\eta,2}}\right) & , j=1 \\ \left(\overline{X}_{\alpha_{\eta-1,j}}, \overline{X}_{\alpha_{\eta,j}}, \overline{X}_{\alpha_{\eta+1,j}}\right) & , j \neq 1, n, \quad \forall j \in \{1,\ldots,n\} \\ \left(\overline{X}_{\alpha_{\eta,n-1}}, \overline{X}_{\alpha_{\eta,n-11}}, \overline{X}_{\alpha_{\eta,n}}\right) & , j \neq 1, n \end{cases} \tag{4.39}$$

,where $\overline{X}_{\alpha_{ij}} = f_{\overline{X}}\left(\alpha_{ij}\right) = mean\left(FI\left(\alpha_{ij}\right)\right)$ in definition 4.13.

Finally, algorithm 4.2 is to conclude this section.

Algorithm 4.2 (Semantic Rule Algorithm / Fuzzy Normal Distribution: $\overline{X} = f_{\overline{X}}(\aleph) = M(\aleph)$**):**

1. Get valid γ_{α^j}, d_{α^j}, τ_{α^j}, X for α^j, $\forall j$;
2. Construct $\mu_{\alpha^j \phi}{}^{-1}$; //e.g. eq(4.13) –eq(4.14)
3. Input valid $\varphi\left(\overrightarrow{V_h}\right)$, λ_0; //Definitions 4.11-4.12
4. Calculate $MCI\left(\left[\!\left[\overrightarrow{V_h}\right]\!\right]\right)$ and $MFI\left(\left[\!\left[\overrightarrow{V_h}\right]\!\right]\right)$ //Propositions 4.3 and 4.4
5. Calculate $MFI\left(\left[\!\left[\overrightarrow{V_h}^{+}\right]\!\right]\right)$ and $MFI\left(\left[\!\left[\overrightarrow{V_h}^{-}\right]\!\right]\right)$; // Propositions 4.5 and 4.6
6. Calculate $FI\left(\left[\!\left[\widehat{\alpha^j}\right]\!\right]\right) = \mu_{\alpha^j \phi}\left(P\left(MFI\left(\left[\!\left[\alpha_{ij}\right]\!\right]_{i=1}^{m}\right)\right)\right), \forall j$; // Theorem 4.2
7. If $\overline{X}$ is a matrix of crisp numbers, calculate $\overline{X} = f_{\overline{X}}(\aleph)$ by definition 4.13;
8. If $\overline{X}$ is a matrix of fuzzy numbers, calculate $\overline{X}' = f_{\overline{X}}(\aleph)$ by definitions 4.13-4.14, and/or Eq.4.39;
9. Return $\overline{X}$; **//END.**

4.4. Compound linguistic ordinal scale

Due to different interpretations of the term set by different experts, most researchers agree that an expert can handle the ranking with limited alternatives. Miller (1956) has indicated that an expert could manage a set with (7 ± 2) terms while Bonissine and Decker (1986) has suggested that one could manage up to 11 or 13 terms. This is open to discussion, especially the rating problem from the experts as there are an excessive number of options. In (Herrera and Martinez, 2001; Herrera-Viedma and Martinez 2005), a sound model of "computing with words" for managing 15 terms for a Basic Linguistic Term Set (BLTS) is used. However, the term set does not illustrate the linguistic label for each term, and it is unreasonable for the expert to rate using mathematical symbols only. In this case, it seems incredible that an expert can handle $\left|\aleph_{7\pm2,7\pm2}\right| = [21,73]$ (corollary 4.1) linguistic terms although CLV can produce a large scale of compound linguistic terms on the basis of their views. This incredibility can be addressed by the Compound Linguistic Ordinal Scale Model defined as follows:

Definition 4.15 (CLOS): A Compound Linguistic Ordinal Scale Model is 3-tuple $(DRS, HDA-LRM, \wp) = \left(\left(\overrightarrow{V_{hd}}_j, \overrightarrow{V_a}, Rs\right), \left(\aleph, \overline{X}, f_{\overline{X}}(\aleph)\right), \wp\right)$, which is the Deductive Rating Strategy $\left(\overrightarrow{V_{hd}}_j, \overrightarrow{V_a}, Rs\right)$ of a Hedge Direction Atom Linguistic Representation Model $\left(\aleph, \overline{X}, f_{\overline{X}}(\aleph)\right)$ with a cross reference relationship $\wp$. $\wp$ is used by a measurement function using DRS and HDA-LRM.

Definition 4.16(DRS **)**: In the Deductive Rating Strategy $\left(\overrightarrow{V_{hd\,j}}, \overrightarrow{V_a}, Rs\right)$, $\overrightarrow{V_a}$ is the set of first rating categories with atomic descriptors $\left[v_{a_j}\right]_{j=1}^{n}$, then $\overrightarrow{V_{hd\,j}}$ is the set of second rating categories derived from the Second Rating Alternatives Transformation Function Rs of $v_{a_j} \in \overrightarrow{V_a}$.

On the basis of above definition, proposition 4.8 is satisfied.

Proposition 4.8 ($\overrightarrow{V_{hd\,j}} = Rs\left(v_{a_j}\right)$):

$$\overrightarrow{V_{hd\,j}} = Rs\left(v_{a_j}\right) = \begin{cases} \left[v_{hd_i}\right]_{i=1}^{\eta} & \text{if } j = 1 \\ \left[v_{hd_i}\right]_{i=1}^{m} & \text{if } j \neq 1, n \\ \left[v_{hd_i}\right]_{i=\eta+2}^{m} & \text{if } j = n \end{cases} \tag{4.40}$$

Proof:

The result can be derived from proposition 4.2, theorem 4.1 and definition 4.16. □

There may be a possible suggestion to have more than two dimension spaces in the rating process. However, the question is whether they are necessary and practical. One concern is how to use linguistic terms to form such rich dimension spaces in a meaningful way. Another concern is how many times for a rater need to rate in a single survey. Classical approach is one time. It may be unclear to express the quantity and finally influence the accuracy of the research result. Thus the CLOS model suggests two times which are sufficient. However, if three times or more, raters may lose their temper to finish the survey, especially the marketing surveys. Thus two step rating process, which is the middle way, is applied for CLOS to maintain the consistence and accuracy of the approximate rating results.

Algorithm 4.3 concludes this section.

Algorithm 4.3: (Deductive Rating Strategy : $\left(\overrightarrow{V_{hd}}_j, \overrightarrow{V_a}, Rs\right)$

1. Observe external information;
2. Understand the problem which needs to be classified;
3. Understand the CLOS model $\left(\left(\overrightarrow{V_{hd}}_j, \overrightarrow{V_a}, Rs\right), \left(\aleph, \overline{X}, f_{\overline{X}}(\aleph)\right), \wp\right)$;
4. First rating step: choose v_{a_j} in $\overrightarrow{V_a} \equiv \left[v_{a_j}\right]_{j=1}^{n}$;
5. Computer shows second options: $\overrightarrow{V_{hd}}_j = Rs\left(v_{a_j}\right)$;
6. Second rating step: Select the second option with revision of first option;
   6.1. if first option is confirmed, then the rater chooses v_{hd_i} in $\overrightarrow{V_{hd}}_j$ as the second option;
   6.2. Else go to Step 3
7. Return $\alpha_{ij} = \left(v_{hd_i}, v_{a_j}\right)$ **//End**

Example 4.2:

Assume the $\left(\aleph, \overline{X}, f_{\overline{X}}(\aleph)\right)$ is applied with a cross reference relationship $\wp$. In the double categories $\left(\overrightarrow{V_{hd}}_j, \overrightarrow{V_a}\right)$ the rater initially chooses a v_{a_j} from the first Category $\overrightarrow{V_a}$ (fig. 4.4-1), then $\overrightarrow{V_{hd}}_j$ is a second option from the second Rating Category function, i.e. $Rs\left(v_{a_j}\right)$ (fig. 4.4-2 to 4.4-6), next the rater chooses a v_{hd} from $\overrightarrow{V_{hd}}_j$. Thus the decision (v_{hd}, v_a) is made. Therefore, instead of selecting one from

$\left|\aleph_{7,5}\right|=29$, which is difficult to achieve, the rater just selects one from $\left|\overrightarrow{V_a}\right|=5$, and then selects one from $\left|\overrightarrow{V_{hd}}_1\right|=\left|\overrightarrow{V_{hd}}_{j=n=5}\right|=4$ or $\left|\overrightarrow{V_{hd}}_{j\neq 1,n}\right|=7$ only.

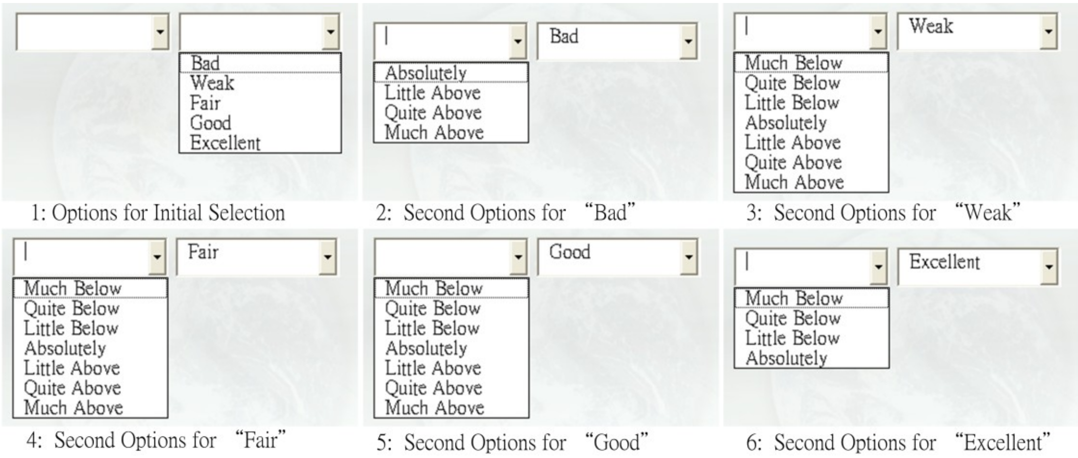


Figure 4.4: Deductive Strategy $\left(\overrightarrow{V_{hd}}_j,\overrightarrow{V_a},Rs\right)$ of CLOS

Before rating, the rater refers to $\wp$ from $\left(\aleph,\overline{X},f_{\overline{X}}(\aleph)\right)$ as table 4.1 or table 4.2 is derived from algorithm 4.2 with parameters $X=[1,5]$, $\vec{\gamma}=[1,2,3,4,5]$, $d_{\alpha^{1,\ldots,5}}=1$, $\tau_{\alpha^{1,\ldots,5}}=2$, $\mu_{\alpha^{j\phi}}{}^{-1}=PbMF^{-1}$, $\varphi\left(\overrightarrow{V_h}\right)=[1,2,3]$, $\lambda_0=0.5$. The operation of algorithm 4.2 is as follows:

As steps 1 to 3 are given, the calculation of step 4 is as follows:

$$dis\left(\overrightarrow{V_h}\right)=\left[dis(\text{"}little\text{"})\ \ dis(\text{"}quite\text{"})\ \ dis(\text{"}much\text{"})\right]=\begin{bmatrix}\frac{1}{6} & \frac{1}{3} & \frac{1}{2}\end{bmatrix}$$

and the cardinal number η of $\overrightarrow{V_h}$ is $\eta=3$,

then, $$MCI\left(\overrightarrow{V_h}^T\right)=MCI\left(\begin{bmatrix}v_{h_1}\\ v_{h_2}\\ v_{h_3}\end{bmatrix}\right)=\begin{array}{cc}\mu'_L & \mu'_U\end{array}\begin{bmatrix}0.8333 & 1\\ 0.5000 & 0.8333\\ 0 & 0.5000\end{bmatrix}.$$

As membership fuzziness factor $\lambda_0 = 0.5$, then $\lambda_{\mu_{u_i}} = \lambda_{\mu_{l_i}} = \frac{\lambda_0}{2} = 0.25$, thus the membership fuzziness is

$$MFI\left(\overrightarrow{V_h}^T, \lambda\right) = \begin{array}{c} \begin{array}{cc} \mu_L & \mu_U \end{array} \\ \begin{bmatrix} 0.7917 & 1 \\ 0.4167 & 0.9167 \\ 0 & 0.6250 \end{bmatrix} \end{array}.$$

For step 5,

$$MFI\left(\left[\!\left[\overrightarrow{V_h}^+\right]\!\right]\right) = rep\left(MFI\left(\left[\!\left[\overrightarrow{V_h}\right]\!\right]\right)\right) = \begin{array}{c} \begin{array}{cc} \mu_L & \mu_U \end{array} \\ \begin{bmatrix} 1 & 0.7917 \\ 0.9167 & 0.4167 \\ 0.6250 & 0 \end{bmatrix} \end{array}, \text{ and } MFI\left(\left[\!\left[\overrightarrow{V_h}^-\right]\!\right]\right) = \begin{array}{c} \begin{array}{cc} \mu_L & \mu_U \end{array} \\ \begin{bmatrix} 0 & 0.6250 \\ 0.4167 & 0.9167 \\ 0.7917 & 0 \end{bmatrix} \end{array}.$$

For step 6, $FI\left(\left[\!\left[\alpha^3\right]\!\right]\right)$ is taken as an example, then

$$FI\left(\left[\!\left[\alpha^3\right]\!\right]\right) = \mu_{\alpha^{j\phi}}\left(MFI\left(\left[\!\left[\alpha_{i,3}\right]\!\right]_{i=1}^{7}\right)\right) = FI\left(\begin{bmatrix} MFI\left(\left[\!\left[\overrightarrow{V_h}^+\right]\!\right]\right) \\ \text{-----------} \\ MFI\left(\left[\!\left[\overrightarrow{V_h}^\theta\right]\!\right]\right) \\ \text{-----------} \\ MFI\left(\left[\!\left[\overrightarrow{V_h}^-\right]\!\right]\right) \end{bmatrix}\right) = \begin{bmatrix} \begin{bmatrix} 2 & 2.5424 \\ 2.4046 & .279 \\ 2.6680 & .300 \end{bmatrix} \\ \begin{bmatrix} 3.0000 & 3.0000 \end{bmatrix} \\ \begin{bmatrix} 3.0000 & 3.332 \\ 3.2063 & 3.5954 \\ 3.4576 & 4.0000 \end{bmatrix} \end{bmatrix}.$$

Finally,

$$\overline{X}_{\widehat{\alpha^3}} = f_{\overline{X}}\left(\widehat{\alpha^3}\right) = mean\left(FI\left(\left[\!\left[\alpha^3\right]\!\right]\right)\right) = \begin{bmatrix} 2.27 & 2.60 & 2.84 & 3.00 & 3.16 & 3.40 & 3.73 \end{bmatrix}^T.$$

The similar calculations for $FI\left(\left[\!\left[\alpha^1\right]\!\right]\right), \ldots, FI\left(\left[\!\left[\alpha^5\right]\!\right]\right)$ are performed. Table 4.1 shows the final results in crisp number for $f_{\overline{X}}(\aleph)$. If the representation values are in fuzzy number, then table 4.2 is applied. Some may regard $\alpha_{\eta,j} \triangleq v_{hd_\eta} \oplus v_{a_j}$, $\forall j \in \{1,\ldots,n\}$ should not be singleton, thus $\overline{X}'_{\alpha_{\eta j}}, \forall j \in \{1,\ldots,n\}$ can be [(1,1,1.17), (1.83,2,2.17), (2.83,3,3.17), (3.8,4,4.17), (4.83,5,5)] using Eq. 4.39.

Table 4.1: $\wp$ from $\left(\aleph,\overline{X},f_{\overline{X}}(\aleph)\right)$ if $\overline{X}$ is a matrix of crisp numbers

$\wp$	Bad	Weak	Fair	Good	Excellent
Much Below	null	1.2712	2.2712	3.2712	4.2712
Quite Below	null	1.5991	2.5991	3.5991	4.5991
Little Below	null	1.8340	2.8340	3.8340	4.8340
Absolutely	1	2	3	4	5
Little Above	1.1660	2.1660	3.1660	4.1660	null
Quite Above	1.4009	2.4009	3.4009	4.4009	null
Much Above	1.7288	2.7288	3.7288	4.7288	null

Table 4.2: $\wp$ from $\left(\aleph,\overline{X}',f_{\overline{X}}(\aleph)\right)$ if $\overline{X}'$ is a matrix of fuzzy numbers

$\wp$	Bad	Weak	Fair	Good	Excellent
Much Below	null	(1.00,1.27,1.54)	(2.00,2.27,2.54)	(3.00,3.27,3.54)	(4.00,4.27,4.54)
Quite Below	null	(1.40, 1.60, 1.79)	(2.40,2.60,2.79)	(3.40,3.60,3.79)	(4.40, 4.60,4.79)
Little Below	null	(1.67,1.83,2.00)	(2.67,2.83,3.00)	(3.67,3.83,4.00)	(4.67,4.83,5.00)
Absolutely	(1,1,1)	(2,2,2)	(3,3,3)	(4,4,4)	(5,5,5)
Little Above	(1.00, 1.17, 1.33)	(2.00,2.17,2.33)	(3.00,3.17,3.33)	(4.00,4.17,4.33)	null
Quite Above	(1.20, 1.40, 1,59)	(2.20,2.40,2.59)	(3.20,3.40,3.59)	(4.20,4.40,4.59)	null
Much Above	(1.46,1.73,2.00)	(2.46,2.73,3.00)	(3.45,3.73,4.00)	(4.45,4.73,5.00)	null

In this example, $\vec{\gamma}=[1,2,3,4,5]$ is assumed on the basis of rank order which is widely used by the quantitative researchers. It is possible to use Fuzzy Normal Distribution to produce the vector representation numbers for $\vec{\gamma}$ and then make further calculation by FND. This case is beyond the scope of this paper. The next section discusses the patterns of using different values of other parameters.

4.5. **Numerical analyses and discussion**

In algorithm 4.2, it can be observed that ($\lambda_0, \varphi\left(\overline{V_h}\right), \tau_{\alpha^j}, \mu_{\alpha^{j\phi}}{}^{-1}, \gamma_{\alpha^j}, d_{\alpha^j}, X$) are the parameters to develop the representation values. In examples 4.1 and 4.2, ($\gamma_{\alpha^j}, d_{\alpha^j}$, X) can be directly defined for the case scenario. Other parameters may need a scaling procedure to find the most suitable values. In order to support the scaling procedure, cases 4.1 to 4.4 examine the influences of the four parameters ($\lambda_0, \varphi\left(\overline{V_h}\right), \tau_{\alpha^j}, \mu_{\alpha^{j\phi}}{}^{-1}$) for $\overline{X}$ respectively. In addition, case 4.5 investigates how this CLOS model is superior to the classical rating scales, e.g. the Likert Scale. In these cases, the default parameters apply to those specified in examples 4.1 and 4.2, except $\tau_{\alpha^{1,\ldots,5}} = 1$. The data for plotting the figures are generated by a Mathematica Program on the basis of Algorithm 4.2. Figures are plotted from the data in Appendix I.

Case 4.1: Relationship of λ_0 and $\overline{X}_{\alpha_{ij}}$

λ_0 is defined in definition 4.12. Its availability range is tested in this case. From fig. 4.5, if $\lambda_0 \le 1$ (or $\lambda_{\mu_{l_i}} = \lambda_{\mu_{u_i}} \le 0.5$), the relationship of λ_0 and $\overline{X}_{\alpha_{ij}}$ can be described by the linear regression function $\overline{X}_{\alpha_{ij}} = \bar{a}_{\alpha_{ij}} \lambda_0 + \bar{b}_{\alpha_{ij}}$. If $\lambda_{\mu_{l_i}}$ or $\lambda_{\mu_{u_i}}$ is excessively large, the overlap of the fuzzy interval is excessively large and “Error” is given according to lemma 4.1 , i.e. the upper boundary is not more than 1, and the lower boundary is not less than 0. The appropriate λ_0 depends on various situations. From the above analysis, $\lambda_0 \in [0,1]$ is recommended. By Default, $\lambda_0 = 0.5$ is chosen.

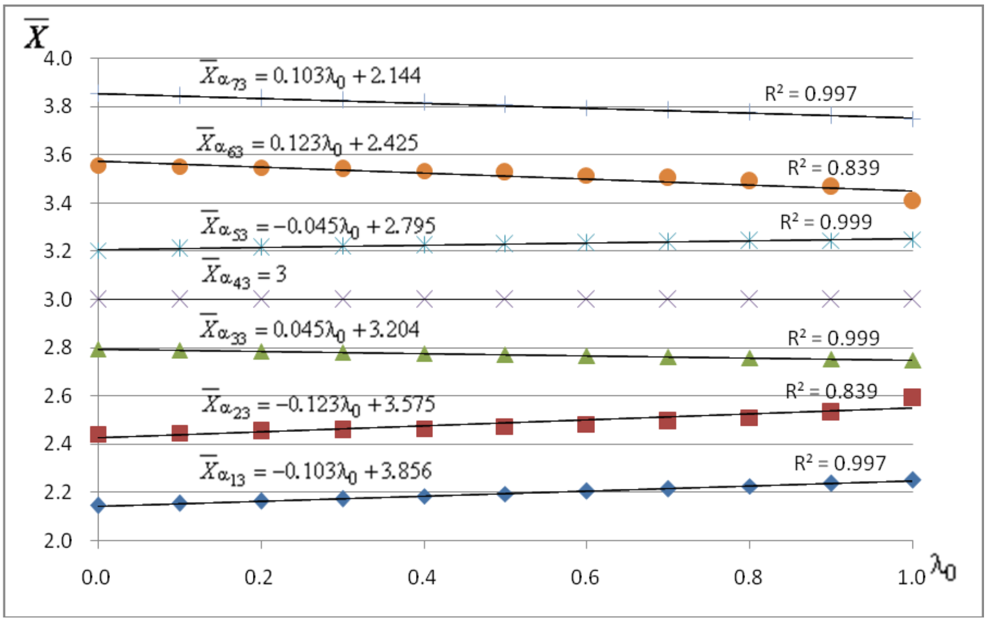


Figure 4.5: The relationship of $\overline{X}\alpha_{i,j=3}$ and λ_0

Case 4.2: Comparison of $\overline{X}_{\varphi_k}$ on the basis of $<_{i=1}^{3} \varphi\left(v_{h_i}\right), =_{i=1}^{3} \varphi\left(v_{h_i}\right)$, and $>_{i=1}^{3} \varphi\left(v_{h_i}\right)$

$\varphi\left(\overrightarrow{V_h}\right)$ is defined in definition 4.11. Some of its properties are tested in this case. Assume the set $\overline{\varphi}\left(\overrightarrow{V_h}\right) = \{[1,2,3],[1,1,1],[3,2,1]\}$. For this comparison, consider a plot of an ordinal number of $\alpha_{i,j=3}$, $i = 1,\ldots,7$, versus the representation real number $\overline{X}$ for each measure function $\varphi\left(\overrightarrow{V_h}\right)_k \in \overline{\varphi}\left(\overrightarrow{V_h}\right)$, i.e. $\overline{X}_{\varphi_k}$ in fig. 4.6. To conclude, if $\varphi_1 = \left[\varphi\left(v_{h_i}\right):\ <_{i=1}^{\eta} \varphi\left(v_{h_i}\right)\right]$, $\varphi_2 = \left[\varphi\left(v_{h_i}\right):\ =_{i=1}^{\eta} \varphi\left(v_{h_i}\right)\right]$, and $\varphi_3 = \left[\varphi\left(v_{h_i}\right):\ >_{i=1}^{\eta} \varphi\left(v_{h_i}\right)\right]$, then $\overline{X}_{\varphi_3} \geq \overline{X}_{\varphi_1} \geq \overline{X}_{\varphi_2}$ in $\widehat{\alpha^j}^{+}$, and conversely, $\overline{X}_{\varphi_3} \leq \overline{X}_{\varphi_1} \leq \overline{X}_{\varphi_2}$ in $\widehat{\alpha^j}^{-}$. This implies that a horizontal tendency of the slope means the representation numbers of

$\left[\alpha_{i,j=3}\right]_{i=1}^{7}$ are closer while a vertical one means the representation numbers are more dispersed.

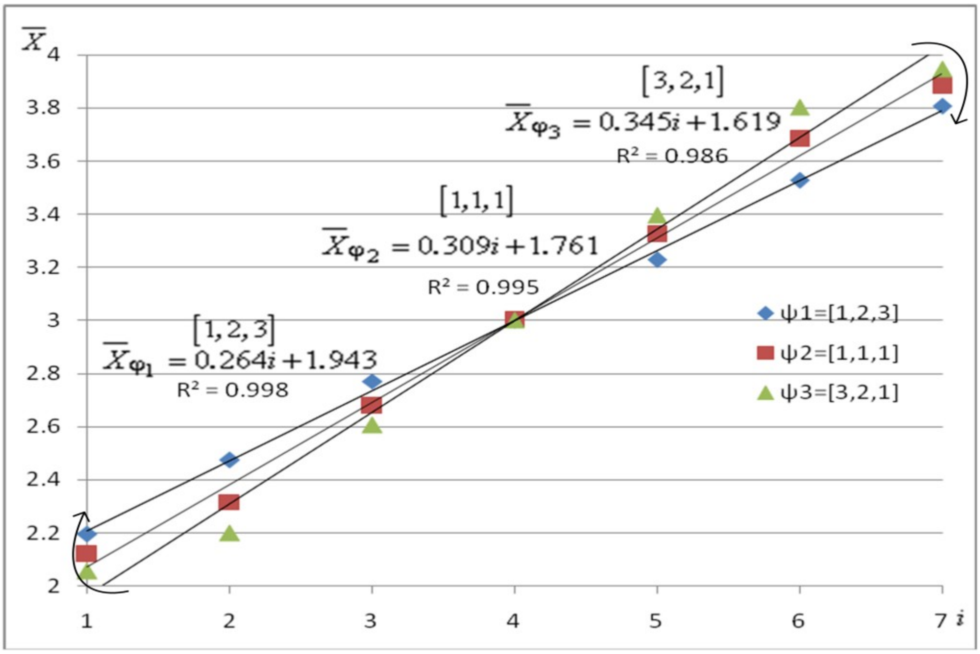


Figure 4.6: The relationship of $\overline{X}_{\varphi_k}$ and i

Case 4.3: Comparison of different τ_{α^j}

$\varphi\left(\overrightarrow{V_h}\right)$ is defined in Eqs. (4.13)-(4.16). Its numerical sensitivities are tested in this case. Assume the two sets: $\overline{\tau}_{\alpha^{j=3}} = \{0.1, 0.2, \cdots, 1\}$, which is used to smooth the shape, and $\overline{\tau}'_{\alpha^3} = \{1, 2, \cdots, 10\}$, which is used to sharpen the shape, and $\tau_{\alpha^3_k}, \tau_{\alpha^3} \in \overline{\tau}_{\alpha^3}$, and $\tau'_{\alpha^3_k}, \tau_{\alpha^3} \in \overline{\tau}'_{\alpha^3}$. Fig. 4.7 shows the relationship between $\overline{\tau}_{\alpha^3}$ and $\overline{X}_{\alpha_{i,j=3}}$, which can be presented by the regression lines $\overline{X}_{\alpha_{i,j=3}} = \overline{a}_{\alpha_{i,j=3}} \tau_{\alpha^{j=3}_k} + \overline{b}_{\alpha_{i,j=3}}$, and $i=1,\ldots,7$. Fig. 4.7 shows that the larger $\tau_{\alpha^j_k}$ leads to the larger convergence of $\overline{X}_{\alpha_{ij}}$. This is due to the fact that the smaller $\tau_{\alpha^j{}_k}$,i.e. $\tau_{\alpha^j{}_{k=1}} = 0.1$, which leads to the shape of the membership approximating to a rounded rectangular shape, and the larger $\tau_{\alpha^j{}_k}$,i.e. $\tau_{\alpha^j{}_{10}} = 1$, results in less smoothing in the parabola shape.

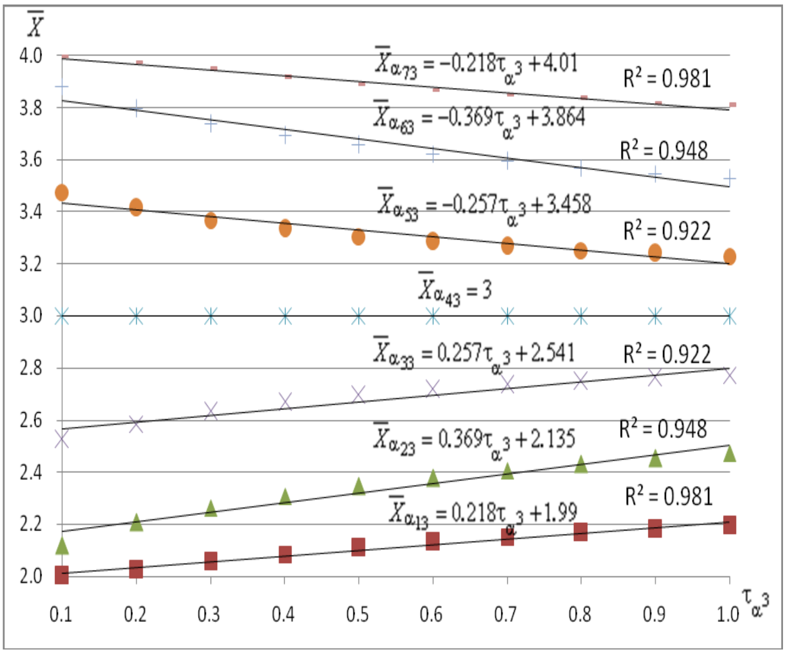


Figure 4.7: The relationship of $\overline{X}_{\alpha_{i,3}}$ and $\tau_{\alpha^3{}_k}$, where $\mu_{\alpha^j\phi}{}^{-1} = PbMF^{-1}$, $0.1 \le \tau_{\alpha^j} \le 1$

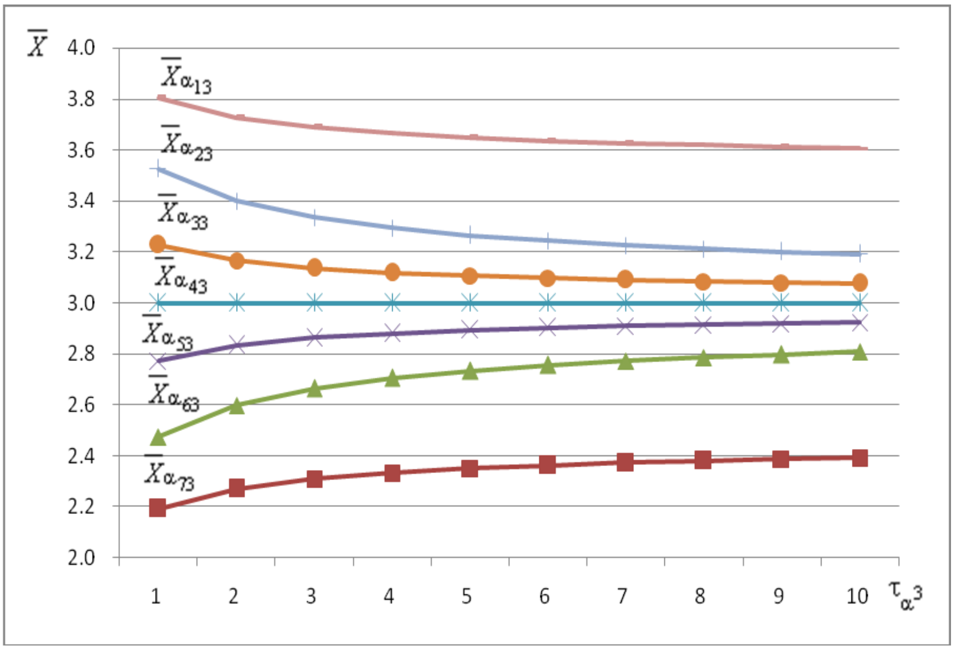


Figure 4.8: The relationship of $\overline{X}_{\alpha_{i,3}}$ and $\tau_{\alpha^3{}_k}$, where $\mu_{\alpha^j\phi}{}^{-1} = PbMF^{-1}$, $1 \le \tau_{\alpha^j} \le 10$

The result of comparison of $\overline{\tau'_{\alpha^3}}$ and $\overline{X}$ is presented in fig. 4.8. The larger $\tau'_{\alpha^j{}_k}$ leads to convergence to a constant but abnormal value. This abnormal situation is due to the fact that the larger $\tau'_{\alpha^j{}_k}$ produces greater sharpness of the shape which is finally approximated to a singleton line and never contains fuzzy overlap with adjacent fuzzy sets. In other words, the meaning of fuzziness loss if $\tau'_{\alpha^j{}_k}$ is excessively large. The convergent abnormal values are concluded as:

$$\text{for } \tau_{\alpha^j} \to \infty,\ \overline{X}_{\alpha_{i,j}} \to \left\{\begin{array}{ll} \gamma_{\alpha^j} - 0.5d_{\alpha^j} & ,i=1 \\ \overline{X}_{\alpha_{\eta+1,j}} = \gamma_{\alpha^j} & ,i=2,\cdots,m-1 \\ \gamma_{\alpha^j} + 0.5d_{\alpha^j} & ,i=m \end{array}\right\}, j = \begin{cases} 1 & ,i=1,\ldots,\eta \\ 2,\ldots,n\text{-}1 & ,i=1,\ldots,m \\ n & ,i=\eta,\ldots,m \end{cases},$$

In this scenario, $\overline{X}_{\alpha_{1,3}}$=2.5, $\overline{X}_{\alpha_{2,3}} = \overline{X}_{\alpha_{3,3}} = \overline{X}_{\alpha_{4,3}} = \overline{X}_{\alpha_{5,3}} = \overline{X}_{\alpha_{6,3}}$ =3.0, $= \overline{X}_{\alpha_{7,3}}$ =3.5.

These abnormal results are explained by each representation value not being the same. If some of them are the same, this means that some linguistic terms are redundant. Thus, each linguistic term should contain a suitable distance between two adjacent linguistic terms. In the above two analyses, $0.5 \le \tau_{\alpha^j} \le 4$ may meet the requirement of the suitable distance by observation. By default, $\tau_{\alpha^j} = 1$.

Case 4.4: Comparison with Triangular-based *MF* with different τ_{α^j}

$\mu_{\alpha^j\phi}^{-1}$ is defined in Eqs (4.13)-(4.16). Its numerical sensitivities by using different τ_{α^j} and the shapes of the membership are tested in this case. If the triangular membership function is applied, the result is different from case 3. As shown in fig. 4.9

and fig. 4.10, the lines of $\left[\overline{X}_{\alpha_{i,3}}\right]_{i=1}^{7}$ also converge when $\bar{\tau}_{\alpha^3}$ increases. The group of the lines converges more rapidly than the lines of PbMF. This can be explained by the sharpness of *TbMF*, which is larger than the one of *PbMF* giving the same value of τ_{α^j} .It can be concluded that the different membership functions applied contribute to the different speeds of convergent patterns of the group of lines. The speeds of convergence of the group of lines can be affected by the shape whether is concave or convex with different levels from the basic curves (e.g. triangular or parabola) after the different power indices τ 's are applied.

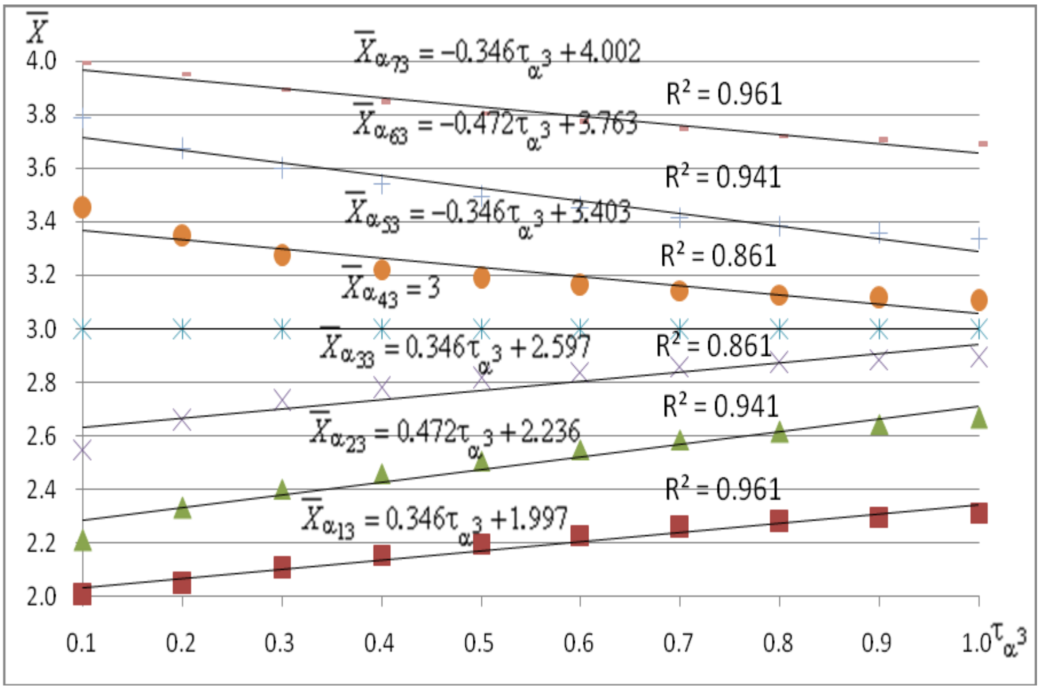


Figure 4.9: The relationship of $\overline{X}_{\alpha_{i,3}}$ and $\tau_{\alpha^3_k}$, where $\mu_{\alpha^j\phi}^{-1} = TbMF^{-1}$, $0.1 \le \tau_{\alpha^j} \le 1$

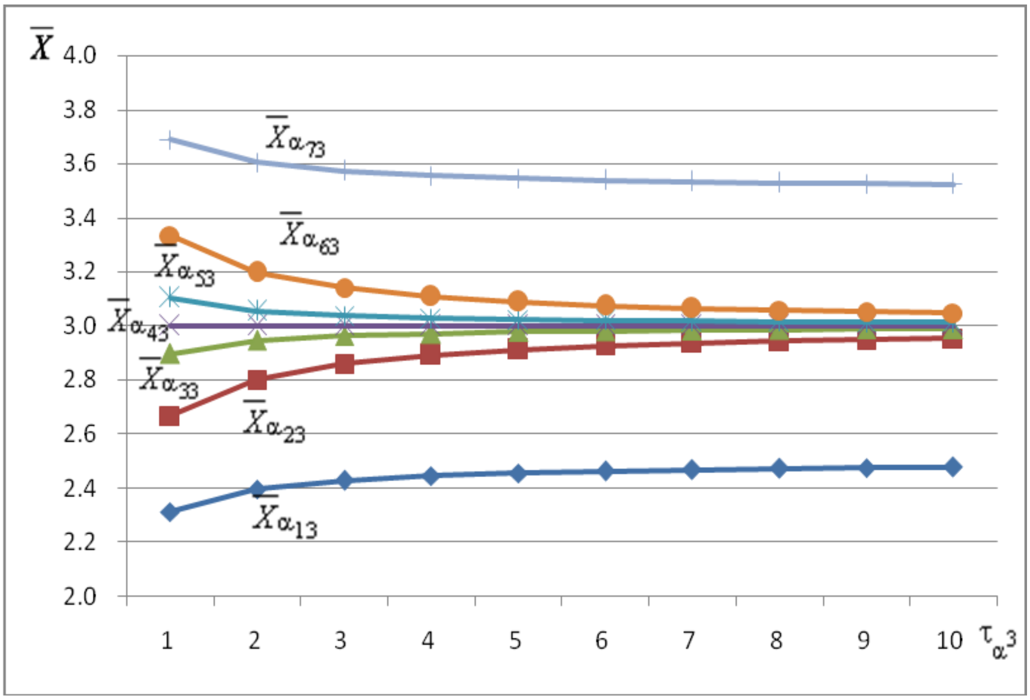


Figure 4.10: The relationship of $\overline{X}_{\alpha_{i,3}}$ and $\tau_{\alpha^3{}_k}$, where $\mu_{\alpha^{j\phi}}{}^{-1} = TbMF^{-1}$, $1 \le \tau_{\alpha^j} \le 10$

Case 4.5: Comparison with Likert-like Scales

This case is to compare CLOS and Likert-like Scales. Assumption 1 is that two raters face a rating dilemma to choose one between “fair” ,i.e. 3, and “good” ,i.e. 4, in the Likert-like rating scales. Assumption 2 is that two raters have similar views. Due to this rating dilemma, one chooses “fair” and another one chooses “good”. Bias of this scale is induced and is measured by the standard deviation $\sigma = \sqrt{\frac{1}{2}\left((3-3.5)^2 + (4-3.5)^2\right)} = 0.5$. If both rate “fair”, the result is underestimated while it is overestimated if they both choose “good”. The reason for these cases is that the true value should be located between 3 and 4 on the basis of the first assumption. As there is no extra option in the middle, neither a linguistic term nor a numerical value for the raters, CLOS can help to solve this problem of the extra option.

To discuss this problem, the linguistic inputs and their represented value matrix of CLOS is applied in table 4.1 in example 4.2. The rater has a rating dilemma in choosing either α^{j} and α^{j+1}, j=3,4 , initially. If α^{j} is chosen firstly, then v_{hd_i}, i=1,…,7 , is chosen from $\overrightarrow{Vhd}_j = Rs\left(\alpha^{j}\right)$. If α^{j+1} is chosen first, then $v_{hd_{i'}}$ is chosen from $\overrightarrow{Vhd}_{j+1}$. If one rater does this in the first case, and the other does it in the second cases, the combination of $\alpha_{i,j}$ and $\alpha_{i',j+1}$ is $\left|\hat{\alpha}^{j}\right| \times \left|\hat{\alpha}^{j+1}\right| = 49$, j=3. The bias of $\alpha_{i,j}$ and $\alpha_{i',j+1}$ is measured by the standard deviation $\sigma = \frac{1}{2} abs\left(\overline{X}\alpha_{i,j} - \overline{X}\alpha_{i',j'}\right)$ ($abs(.)$ returns the absolute value). Fig. 4.11 exhibits the distribution of σ of all possible combinations of $\overline{X}\alpha_{i,j}$ and $\overline{X}\alpha_{i',j+1}$.

If $\sigma \le 0.5$, it means that the raters have similar views. If the raters have similar views, there will be a rating tendency. For example, if one rater starts from $\alpha^{j=3}$, it is reasonable that $\alpha_{i=5,\dots,7\ ,j=3}$ is a possible option whilst $\alpha_{i=1,\dots,3\ ,j=4}$ is a possible options if the other rater starts from $\alpha^{j=4}$. Thus, by observing the lines $\overline{X}\alpha_{5,3}$, $\overline{X}\alpha_{6,3}$ $\overline{X}\alpha_{7,3}$ in the region of X-axis from $\overline{X}\alpha_{1,4}$ to $\overline{X}\alpha_{3,4}$, there is $\sigma < 0.42 < 0.5$ (Region A in fig. 4.11). It can be concluded that CLOS helps to reduce the bias from the raters, and is superior to the classical Likert-like scales as the new rating interface provides extra options for giving a value to a rating tendency.

If the raters choose $\alpha_{i=4\ ,j=3}$ and $\alpha_{i=4\ ,j=4}$ respectively, it means that their opinions are different since the probability of this combination is quite small i.e. $\frac{1}{49} = 0.0204$. Although the bias is the same as those from Likert-like Scales with $\alpha_{j=3}$ and $\alpha_{j=4}$, Likert-like Scales cannot reflect that raters' opinions are different while the probability of this combination is high, i.e. $|\alpha_{j=3}| \times |\alpha_{j=4}| = 1$. In other words, Likert-like Scales force raters to have bias opinions where there is the value in the middle due to no extra suitable choices for raters. Similarly, if the choices are $\alpha_{i=1,\ldots,3\ ,j=3}$ and $\alpha_{i=5,\ldots,7\ ,j=4}$ respectively, this certainly means that the raters have an obvious bias as the tendency is opposite. This violates the assumption 1- the rating dilemma. Thus, the CLOS Model can reflect and assert the bias from raters due to this tendency measure.

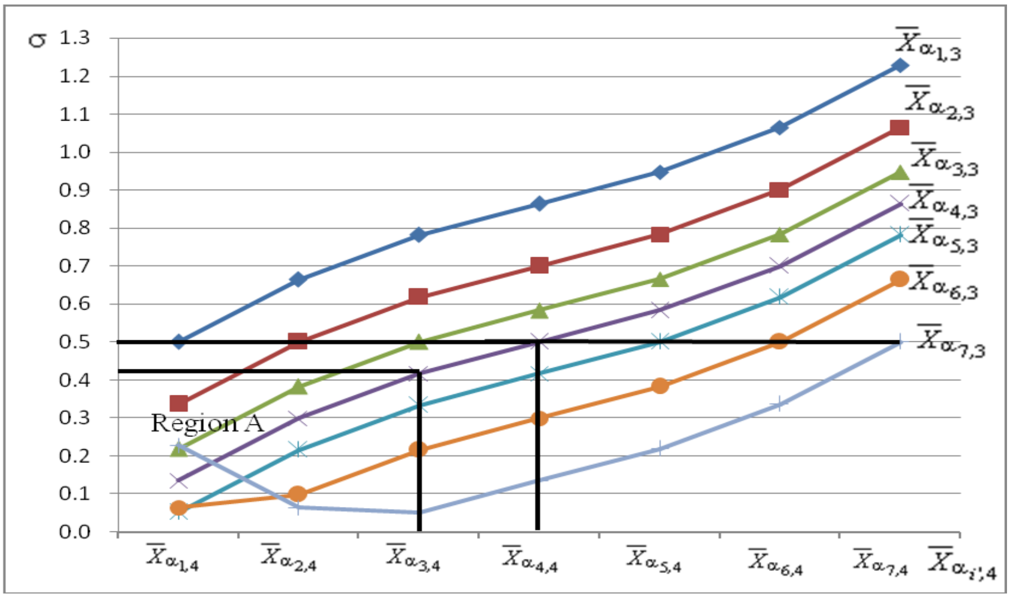


Figure 4.11: The bias of $\alpha_{i,j}$ and $\alpha_{i',j+1}$

Example 4.3

This example is the continuation of example 4.2. This example shows the improvement of a simple statistical application using CLOS. The early research by Stevens (1946) appeared in *Science* has indicated that means and standard deviations computed on an (classical) ordinal scale are in error to the extent that the successive intervals on the scale are unequal in size. This example attempts to interpret his idea using classical ordinal scales and improve his idea using CLOS. In this example, five raters, raters 1,2,…,5, evaluate scores for an object individually. Let the real scores of their perception be (3.3, 2.7, 2.8, 1.7, 3.8). Assume that the most appropriate scales are assigned to the object. Thus, the linguistic terms of Likert-like Scales are approximated to (F, F, F, W, G) whilst Compound Linguistic Ordinal Scale is approximated to (QA-F, MA-W, LB-F, MA-B, LB-G). The representation values, the notations, the conventional statistical results and the improvement are shown in table 4.3. "Improvement" means the level of improvements in the approximated values to the "real values" by using CLOS instead of using Likert scales.

This example shows how CLOS model improves the weakness of classical ordinal model. For individual rating improvement, the average improvement is 83.3% (*Average* (0.667, 0.9,0.85,0.9,0.85)) as an individual improvement is calculated by $P_k = \left(|s_k - s'_k| - |\bar{s}_k - s'_k|\right) / |s_k - s'_k|, k = 1,\ldots,8$. For statistical measurement improvements, values of the operators, sum, mean and standard deviation, improve 68.6%, 68.6%, and 86.3% respectively by $P_k, k = 1,\ldots,8$. For the error reduction, the difference of SD, average of obsolete errors, and root mean square errors reduce more than 80% by $P_k = (s_k - \bar{s}_k)/s_k, k = 9,10,11$.

Table 4.3: Improvements of using Compound Linguistic Ordinal Scale

Performance Index k	Raters (i)	Real values (s'_k)	Likert Scales (s_k)	CLOS ($\bar{s}_k$)	Improvement (P_k)
1	1	3.3	3	3.40	0.667
2	2	2.7	3	2.73	0.900
3	3	2.8	3	2.83	0.850
4	4	1.7	2	1.73	0.900
5	5	3.8	4	3.83	0.850
6	Sum	14.3	15	14.52	0.686
7	Mean	2.86	3	2.904	0.686
8	Standard Deviation	0.783	0.707	0.793	0.863
9	Difference of SD		0.076	0.010	0.863
10	Average of Obsolete Errors		0.260	0.044	0.831
11	Root Mean Square Errors		0.118	0.023	0.803

This simulated analysis provides a better understanding of the causality improvement of using CLOS with comparison of classical Likert-Like scales. In real world applications, some applications may be related to thousands of raters for measuring several objects with a collection of criteria for each object. This is usually conducted in the research of business management and social sciences. Some may be related to many criteria with a single rater, e.g. psychological tests. Using classical ordinal scales possibly increases the measurement errors and statistical errors due to its limited approximation capability of the sufficiency of the representation values. This example shows how the measurement values and statistical values can be improved by CLOS. For more application of CLOS, (Yuen 2009e, 2009f; Yuen and Lau, 2009) have applied the concept in the development of the decision making systems.

4.6. Summary and remarks

CLOS is a Deductive Rating Strategy $\left(\overrightarrow{V_{hd}}_j, \overrightarrow{V_a}, Rs\right)$ of the Hedge-Direction-Atom Linguistic Representation Model $\left(\aleph, \overline{X}, f_{\overline{X}}(\aleph)\right)$ with a cross reference relationship $\wp$. In the HDA-LRM, Compound Linguistic Variable $\aleph$ is produced by syntactic rule, i.e. $\aleph_{mn} = G_{\aleph}\left(\overrightarrow{V_h}, \overrightarrow{V_d}, \overrightarrow{V_a}\right)$, which produces a large number of linguistic descriptors. The semantic rule of "Computing with CLV" $M(\aleph)$ maps CLV into representation numbers in matrix $\overline{X}$ by Fuzzy Normal Distribution $f_{\overline{X}}(\aleph)$, and produces the numerical results meeting the different requirements of different scenario using few scalable descriptable user-defined parameters: $\left(\varphi\left(\overrightarrow{V_h}\right), \lambda_0, X, \left(\gamma_{\alpha^j}, d_{\alpha^j}, \tau_{\alpha^j}\right)\right)$. The Deductive Rating Strategy is the ideal rating interface for HDA-LRM as the cardinal number of $\aleph$ is large. Three algorithms are developed for CLOS model. Three examples are exhibited for this usability.

The Compound Linguistic Ordinal Scale Model, which is an ordinal in ordinal Scale Model, is a promising alternative for the classic rating scale models such as Likert scale and those minor variations of Likert scales (named Likert-Like Scales). Miller (1956) has indicated that an expert could manage a set with (7 ± 2) terms. And many rating scales including Likert-like scales and the choice of the fuzzy linguistic terms use this principle. By breakthrough of this principle, HDA-LRM can provide $(7\pm2)((7\pm2)-1)+1=[21,73]$ options which seem incredible for an expert being able to handle. Unlike the classical rating model which is the single step rating process, CLOS uses a DRS in which a rater chooses a 2-tuple option (v_{hd},v_a) in two steps with a rethink process. The advantage of CLOS is that CLOS is an ideal rating interface for addressing the problem of the rating dilemma.

The numerical analyses include comparisons of various scenarios with different parameters, and the comparison with classical models. One of the simulation results shows that the proposed model helps to reduce the bias of rating dilemma for a single rater, and rater bias among experts who hold similar opinions; The proposed model also accurately reflects raters' consistency and inconsistency, thus to improve the quality of the assessment due to high validity and reliability of the subjective measurement results. The significance of the model is that it can be used as the measurement instrument applied to large scale systems, surveys and questionnaire designs, psychometrics, rater statistics and multicriteria multiexpert decision problems in various fields using the deductive rating strategy of the breakthrough number of linguistic choices.

Chapter 5 Cognitive Pairwise Comparison

5.1 Introduction

This research investigates the appropriateness of the fundamental assumption $a_{ij} = w_i / w_j$ to represent the pairwise comparison, and proposed the better one with the axiom $b_{ij} = v_i - v_j$. To do this, the structure of this chapter is as follows. Chapter 5.2 states the cognitive representation problem of reciprocal comparison. Chapter 5.3 proposes the interval rating scale schema for pairwise opposite comparison whilst Chapter 5.4 proposes the pairwise opposite matrix (or cognitive matrix) applied to the interval rating scale schema. Chapter 5.5 proposes the various cognitive prioritization operators (CPOs) to derive the individual utility vector from the cognitive matrix. Chapter 6.6 proposes six Cognitive Prioritization Operator Measurement (CPOM) models to measure the properties of the CPOs. Chapter 6.7 shows the graph theory interpretations in 2D and 3D views for the CPOs. Chapter 6.8 performs and discusses the numerical analyses of CPOs on the basis of CPOM models, and finally chapter 6.9 concludes this chapter.

5.2 Cognitive representation problem of Saaty's reciprocal comparison

Consider some simple comparison cases using Saaty's ratio scale to represent the linguistic scales. In this case, the height of two persons, e.g. Peter and Jason, are compared. If Peter is 1.4 m, and Jason is 1.5 m, we may say that Jason is slightly taller than Peter by our observation. However, if Saaty's ratio scale is applied, this becomes another story. The fact that Jason is slightly taller than Peter will be interpreted as the statement that Jason is 2 times taller than Peter. It is ridiculous to change the original meaning to the exaggerated expression. If Peter is 1.4 m, and Jason is 1.7 m, by our perception, Jason is much taller than Peter. For numerical representation of Saaty's theory, Jason is 5 times or 7 times taller than Peter.

Another misleading case is to compare their weights. If Peter is 45 kg, Jason will be 90 kg if Jason is slightly heavier than Peter (Jason is 48kg in fact). Similarly, to compare their ages, providing that Peter is 10 years old, Jason will be 20 years old if Jason is a little older than Peter (Jason is 13 in fact). To illustrate one more example, providing that Peter's IQ is 120, Jason's will be 240 if Jason is slightly more intelligent than Peter (Jason's IQ is 123 in fact).

It can be concluded that Saaty's ratio scales for pairwise comparison do not represent the reality of the cognition, and usually produce exaggerated results beyond our common sense, although mathematically the operation of the reciprocal matrix seems to be useful.

Interval scales are more appropriate in above cases. For example, in the case of Peter and Jason who are 1.4 m and 1.5 m respectively, the statement "Jason is 0.1m taller than Peter" is matched with the statement "Jason slightly taller than Peter by our observation." Similarly, "Jason is 3kg heavier than Peter" can be represented by "Jason is moderately heavier than Peter".

The perception of interval (or difference) of two objects is relatively much simpler than the perception of the ratio of both. The reasons are that operations of addition and subtraction are easier than the operations of multiplication and division, which are based on addition and subtraction respectively. In primary school, we learn addition and subtraction prior to multiplication and division. If we do not know addition and subtraction, it is impossible to know multiplication and division. Also if we do not know the simple multiplication table, we do not know how to perform multiplication and division quickly. Due to the invention of the calculator, many adults may forget the multiplication table. In fact, we do not call the multiplication table, linking with linguistic labels, for making comparison. Thus, addition and subtraction are straightforward for the comparison of two objects.

The next section discusses the rating scale schema using interval scales.

5.3 Rating scale schema for pairwise opposite comparison

"How many times is the size of an adult to that of a baby?" Regarding subjective measurement, it is difficult to guess the right answer, as multiplication is needed. However, it is easier to judge on addition concept, for example, "what is the difference between the size of an adult and a baby?" We may answer "the size of an adult is much larger than a baby." Regarding the numerical representation of the intuitive cognition, it is questionable to use division (or ratio) to represent "much larger". In fact, in our perception, the most appropriate method is the straightforward method which uses the difference (or interval) between two objects.

Table 5.1: Terminology of categories of pairwise comparisons

		Analytic Network Process	Cognitive Network Process
Rating scales	Date type	ratio scales	interval scales
	Model type	analytic scales	cognitive scales
Pairwise matrix	Scale type	pairwise ratio matrix	pairwise interval matrix
	Comparison type	reciprocal comparison matrix	opposite comparison matrix
	Structure type	pairwise reciprocal matrix	pairwise opposite matrix
	Model type	analytic pairwise matrix	cognitive pairwise matrix
Prioritization operator	Scale type	multiple prioritization operator	differential prioritization operator
	Model type	analytic prioritization operator	cognitive prioritization operator

Table 5.1 clarifies different terminologies for the categories of the Analytic Network Process (Saaty, 1980) and the Cognitive Network Process. The Cognitive Hierarchy Process (CHP) is the special case of CNP, whilst the Analytic Hierarchy Process is special to ANP.

Rating scales are used for constructing a pairwise matrix, which is derived as a priority vector or utility vector by the prioritization operator, which is also referred to as the prioritization method or prioritization model. Since different scales are applied, ANP and CNP have different meanings, and thus the related terminologies are associated with their pairwise matrices and prioritization operators. In the rating scale of CNP, the interval scale or cognitive scale serves the same meanings, and do other categories.

The rating scale can be a single rating process or a double rating process (Chapter 4). To compare interval and ratio scales, the double rating process is beyond the topic of this paper although CNP uses double rating process. The interval scales are defined as follows.

Definition 5.1 (Interval Scale): Let $\aleph$ be the set of linguistic labels of the interval scales such as {equally, …, Extremely} and the opposite of the set. The numerical representation of the interval scales for differential comparison is in the form:

$$\overline{X} = \left\{ \alpha_i = {}^{i\kappa}\!/_{\tau} \mid \forall i \in \{-\tau, \ldots, -1, 0, 1, \ldots, \tau\}, \kappa > 0 \right\} \tag{5.1}$$

, where κ (read kappa) is the normal utility, which is the mean of the individual utility values of the comparison objects, and $\kappa > 0$; $2\tau + 1$ is the number of the intervals of the scale schema. $\tau + 1$ is usually 7 ± 2 in a single rating process.

An example of definition 5.1 is shown in table 5.2. On the basis of the concept of interval scales, the next section further develops the concept of the Pairwise Opposite Matrix.

Table 5.2: Scale schemas: pairwise reciprocal comparison and pairwise opposite comparison

i	Verbal scales $\aleph$	Numerical representation of PRC (Ratio Scales) $\overline{X}' = \{i+1, i = 0,\ldots,\tau\}$	Numerical representation of POC (Interval Scales) $\overline{X} = \left\{ {}^{i\kappa}\!/_{\tau}, i = 0,\ldots,\tau \right\}$
0	Equally	1	0
1	Weakly	2	$\kappa/8$
2	Moderately	3	$\kappa/4$
3	Moderately plus	4	$3\kappa/8$
4	Strongly	5	$\kappa/2$
5	Strong Plus	6	$5\kappa/8$
6	Very Strongly	7	$3\kappa/4$
7	Very, very strongly	8	$7\kappa/8$
8	Extremely	9	κ
{-i}	Reciprocals / opposites of Above	(from 1/2 to 1/9)	(from $-\kappa$ to 0)

5.4 Pairwise opposite matrix (POM)

A list of the comparisons using interval scales can form the Opposite Comparison Matrix, which is defined as follows.

Definition 5.2 (POM): The Pairwise Opposite Matrix (or Opposite Comparison Matrix) is used to interpret the individual utilities of the candidates. Let an ideal utility set be $V = \{v_1,\ldots,v_n\}$, and the comparison score is $b_{ij} \cong v_i - v_j$. The ideal pairwise opposite matrix is $\tilde{B} = \left[v_i - v_j \right]$. A subjective judgmental pairwise opposite matrix using interval scales is $B = \left[b_{ij} \right]$. $\tilde{B}$ is determined by B as follows:

$$\tilde{B}=\left[\tilde{b}_{ij}\right]=\begin{bmatrix} v_1-v_1 & v_1-v_2 & \dots & v_1-v_n \\ v_2-v_1 & v_2-v_2 & \dots & v_2-v_n \\ \vdots & \vdots & \ddots & \vdots \\ v_n-v_1 & v_n-v_2 & \cdots & v_n-v_n \end{bmatrix} \cong \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} = \left[b_{ij}\right] = B \quad (5.2)$$

If $i=j$, then $b_{ij}=v_i-v_j=0$. Thus the above matrix is in the form:

$$\tilde{B}=\left[\tilde{b}_{ij}\right]=\begin{bmatrix} 0 & v_1-v_2 & \dots & v_1-v_n \\ v_2-v_1 & 0 & \dots & v_2-v_n \\ \vdots & \vdots & \ddots & \vdots \\ v_n-v_1 & v_n-v_2 & \cdots & 0 \end{bmatrix} \cong \begin{bmatrix} 0 & b_{12} & \dots & b_{1n} \\ b_{21} & 0 & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & 0 \end{bmatrix} = \left[b_{ij}\right] = B \quad (5.3)$$

Usually, $\left[b_{ij}\right]$ is given through the rating process of an assessment expert, using a rating category from $\aleph$ which is numerically represented by $\overline{X}$. The decision maker only fills an upper triangular matrix of the form:

$$B^{+}=\begin{cases} b_{ij} & i<j \\ 0 & \text{otherwise} \end{cases}, \text{ written explicitly, } B^{+}=\begin{bmatrix} 0 & b_{12} & \dots & b_{1n} \\ 0 & 0 & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad (5.4)$$

The lower triangular matrix is given by the opposite of an upper triangular matrix of the form:

$$B^{-}=\begin{cases} b_{ij} & i>j \\ 0 & \text{Otherwise} \end{cases}, \text{ written explicitly, } B^{-}=\begin{bmatrix} 0 & 0 & \dots & 0 \\ b_{21} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & 0 \end{bmatrix} \quad (5.5)$$

$\left[b_{ij}\right]$ is achieved by $B = B^{+} + B^{-}$. For a complete comparison of a set of candidates, POM needs $n(n-1)/2$ ratings. Some properties of B are shown in the following propositions.

Proposition 5.1 (summation to zero property):

The summation of the elements in B is equal to 0. That is

$$\sum_{i}\sum_{j} b_{ij} = 0,\ b_{ij} \in B \tag{5.6}$$

Proof:

$\sum_{i}\sum_{j} b_{ij} = \sum_{i}\sum_{j>i}\left(b_{ij} + b_{ji}\right) + 0 \cdot I = 0$, where I is the identity matrix and $b_{ij} + b_{ji} = 0$.

□

The next issue is to discuss the Accordant Index of B. There are three propositions are developed as follows.

Proposition 5.2 (perfect accordant by transitivity):

An pairwise opposite matrix $B = \left\{b_{ij} : i, j \in (1,\ldots,n)\right\}$ is perfect accordant if

$$b_{ik} + b_{kj} = b_{ij},\ \ k \in (1,\ldots,n) \text{ or} \tag{5.7}$$

$$b_{ik} - b_{jk} = b_{ij},\ \ k \in (1,\ldots,n). \tag{5.8}$$

Proof:

The proof is trivial. Let $b_{ij} = v_i - v_j$, $b_{ik} = v_i - v_k$, $b_{jk} = v_j - v_k$, then

$b_{ik} + b_{jk} = (v_i - v_k) - (v_j - v_k) = v_i - v_j = b_{ij}$, $\forall k \in (1,\ldots,n)$. □

On the basis of proposition 5.2, the Accordant Index is formed in the proposition 5.3.

Proposition 5.3: (Accordant Index):

Let $B=\{b_{ij}:i,j\in(1,\ldots,n)\}$ be the pairwise opposite matrix and $D=\{d_{ij}:i,j\in(1,\ldots,n)\}$ be the contradiction matrix. The Accordant Index is in the form:

$$AI=\frac{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}d_{ij}}}{n^2},$$

$$d_{ij}=\sqrt{Mean\left(\left(\frac{1}{\kappa}\left(B_i+B_j^T-b_{ij}\right)\right)^2\right)},\ i,j\in(1,\ldots,n). \tag{5.9}$$

where $AI\geq 0$, κ is the normal utility, and then $n\kappa$ is the population utility. If $AI=0$, then B is perfectly accordant; If $0<AI\leq 0.1$, then B is satisfactory, then. If $AI>0.1$, then B is unsatisfactory .

Proof:

For $B_j=\{b_{1j},\ldots,b_{nj}\}^T$, then $B_j^T=\{b_{1j},\ldots,b_{nj}\}$.

Since $B_i=\{b_{i1},\ldots,b_{in}\}$ and $B_j^T=\{b_{1j},\ldots,b_{nj}\}$, then $B_i+B_j^T=\{b_{i1}+b_{1j},\ldots,b_{in}+b_{nj}\}$.

Case 1: B is perfectly accordant:

As $b_{ik}+b_{kj}=b_{ij}$, $k\in(1,\ldots,n)$, then $b_{ij}\in\{b_{i1}+b_{1j},\ldots,b_{in}+b_{nj}\}$.

Thus $B_i+B_j^T-b_{ij}=0\cdot e^n$, where n-identity row e^n is the row of n identity elements, e.g, $\overbrace{\{1,\ldots,1\}}^{n}$. Thus $\{d_{ij}\}=\{0\}$. AI=0.

Case 2: B is not accordant:

Let $B_i + B_j^T = \{b_{i1} + b_{1j}, \ldots, b_{in} + b_{nj}\} = \{b'_1, \ldots, b'_n\}$, and then

$B_i + B_j^T - b_{ij} = \{b'_1, \ldots, b'_n\} - b_{ij}$.

To normalize the above form, then $\frac{1}{\kappa}\left(B_i + B_j^T - b_{ij}\right) = \frac{1}{\kappa}\left(\{b'_1, \ldots, b'_n\} - b_{ij}\right)$.

Next,

$$d_{ij} = \sqrt{Mean\left(\left(\frac{1}{\kappa}\left(\{b'_1 - b_{ij}, \ldots, b'_n - b_{ij}\}\right)\right)^2\right)},$$

$$d_{ij} = \sqrt{\frac{1}{n\left(\kappa^2\right)}\sum_{k=1}^{n}\left(b'_k - b_{ij}\right)^2} \geq 0.$$

Thus, $AI = \frac{1}{n^2}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}} \geq 0$.

Under this condition, if $AI \leq 0.1$, B is defined to be unsatisfactory . If $AI > 0.1$, B is defined to be unsatisfactory . □

AI is directly derived from the POM. Thus the advantages of AI include avoiding the use of the random index and in calculating the utility weights in advance.

For a 3x3 pairwise opposite matrix, if $b_{12} = 0.6$ and $b_{23} = 0.5$, then $b_{13} = b_{12} + b_{23} = 1.1$, which is larger than $Max(\aleph) = \kappa = 1$. In this situation, discordance is induced due to the limited boundary of the rating scale. This issue is called an out-boundary problem which is illustrated as follows:

Proposition 5.4 (Out-boundary property):

A pairwise opposite matrix $B=\{b_{ij}: i,j \in (1,\ldots,n)\}$ is associated with an out-boundary error if

$$b_{ik}+b_{kj} > Max\left(\overline{X}\right)=\kappa \text{ or } b_{ik}+b_{kj} < Min\left(\overline{X}\right)=-\kappa,\ \ \forall k \in (1,\ldots,n). \tag{5.10}$$

Otherwise, the pairwise opposite matrix is within-boundary. If this happens, then

$$\left(b_{ik}+b_{kj}\right) \in \overline{X}=[-\kappa,\kappa]. \tag{5.11}$$

Proof:

Let a scale item of the rating scale be the form $\alpha_{i'} \in \overline{X}$, and $\alpha_{i'}={i'\kappa}/{\tau}$ $,i'=-\tau,\ldots,-1,0,1,\ldots,\tau$. If $b_{ik}=\alpha_s$ and $b_{jk}=\alpha_t$, then $\alpha_s-\alpha_t=\dfrac{(s-t)\kappa}{\tau}$.

Since $s,t \in \{-\tau,\ldots,-1,0,1,\ldots,\tau\}$ and $s \neq t$, then $\alpha_s-\alpha_t=\dfrac{(s-t)\kappa}{\tau}$.

This follows $Min\left(\{(s-t)\}\right)=-2\tau$ and $Max\left(\{(s-t)\}\right)=2\tau$.

In other words, $(s-t) \in \{-2\tau,-2\tau+1,\ldots,-\tau\ldots0,\ldots,\tau,\ldots,2\tau-1,2\tau\}$.

If $(s-t)>\tau$ or $(s-t)<\tau$, then $\left(b_{ik}-b_{jk}\right)=\left(\alpha_s-\alpha_t\right)>\kappa$ or $\left(b_{ik}-b_{jk}\right)=\left(\alpha_s-\alpha_t\right)<\kappa$.

As $Max\left(\overline{X}\right)=\alpha_\tau={\tau\kappa}/{\tau}=\kappa$ and $Min\left(\overline{X}\right)=\alpha_{-\tau}={-\tau\kappa}/{\tau}=-\kappa$, i.e. $\overline{X}=[-\kappa,\ldots,\kappa]$ and

$\left(b_{ik}-b_{jk}\right)=\left(\alpha_s-\alpha_t\right)>\kappa$ or $\left(b_{ik}-b_{jk}\right)=\left(\alpha_s-\alpha_t\right)<\kappa$, then an outbound error exists.

Otherwise, i.e. $-\tau \leq (s-t) \leq \tau$, the pairwise opposite matrix is within boundary.

If this happens, then

$$\left\{\left(b_{ik}+b_{kj}\right)=\alpha_s-\alpha_t=\frac{(s-t)\kappa}{\tau}:(s-t)\in\{-\tau,\ldots,-1,0,1,\ldots,\tau\}\right\}=\left\{-\kappa,\frac{-\tau+1}{\tau},\ldots,\kappa\right\}=\overline{X}.\ \square$$

The next section discusses how to derive the individual utility vector of the Opposite Comparison Matrix.

5.5 Cognitive prioritization operators

To interpret the utility weights of the pairwise opposite matrix, this research proposes five cognitive prioritization operators as follows.

5.5.1 Row Average plus normal Utility (RAU)

The Row Average plus the normal Utility (RAU) is the simplest method to derive the individual utility, which is derived in theorem 5.1.

Theorem 5.1 (RAU):

The vector of individual utilities can be derived by

$$V = Avg\left(B\right) + \kappa \tag{5.12}$$

,where $Avg\left(B\right)$ returns the average of each row of *B, i.e.*

$$Avg\left(B\right) = \left\{ \frac{1}{n} \sum_{j=1}^{n} b_{ij} : \forall i \in \left\{1, \ldots, n\right\} \right\}.$$

For the entry of V, the individual utility is of the form:

$$v_i = \left(\frac{1}{n} \sum_{j=1}^{n} b_{ij} \right) + \kappa, \forall i \in \left\{1, \ldots, n\right\} \tag{5.13}$$

Proof:

$$B \cong \tilde{B}$$

$B + V \cong \tilde{B} + V = n \bullet V$, which is written explicitly,

$$\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} + (v_1, \dots, v_n)$$

$$\cong \begin{bmatrix} v_1 - v_1 & v_1 - v_2 & \dots & v_1 - v_n \\ v_2 - v_1 & v_2 - v_2 & \dots & v_2 - v_n \\ \vdots & \vdots & \ddots & \vdots \\ v_n - v_1 & v_n - v_2 & \cdots & v_n - v_n \end{bmatrix} + (v_1, \dots, v_n)$$

$= n \bullet (v_1, \dots, v_n)$. Thus,

$$nv_i = \sum_{j=1}^{n} b_{ij} + \sum_{j=1}^{n} v_j$$

$$nv_i = \sum_{j=1}^{n} b_{ij} + n\kappa$$

$v_i = \dfrac{\sum_{j=1}^{n} b_{ij} + n\kappa}{n}$, $\forall i \in \{1, \dots, n\}$. Thus the solution is found. □

5.5.2 Aggregation of Solutions of Linear Systems

For each $b_{ij} \cong v_i - v_j$ in row i of B, the linear systems of B are of the form:

$$L_i = \begin{cases} \{v_i - v_j = b_{ij} : j = 1, \dots, n\}, & i \neq j \\ \sum_{k=1}^{n} v_k = n\kappa \ , & i = j \end{cases}, \ \forall i \in \{1, \dots, n\} \tag{5.14}$$

, which is written explicitly by

$$L_1 = \begin{Bmatrix} \sum_{k=1}^{n} v_k = n\kappa \\ v_1 - v_2 = b_{12} \\ v_1 - v_3 = b_{13} \\ \vdots \\ v_1 - v_n = b_{1n} \end{Bmatrix}, \ L_2 = \begin{Bmatrix} v_2 - v_1 = b_{21} \\ \sum_{k=1}^{n} v_k = n\kappa \\ v_2 - v_3 = b_{23} \\ \vdots \\ v_2 - v_n = b_{2n} \end{Bmatrix}, \dots, \ L_n = \begin{Bmatrix} v_n - v_1 = b_{n1} \\ v_n - v_2 = b_{n2} \\ \vdots \\ v_n - v_{n-1} = b_{n,n-1} \\ \sum_{k=1}^{n} v_k = n\kappa \end{Bmatrix}$$

If $\sum_{k=1}^{n} v_k = n\kappa$ is removed, L_i has at least one solution. Thus $\sum_{k=1}^{n} v_k = n\kappa$ has to be added to have a unique solution set. To solve the above equation systems, they are formed as augmented matrices $\left[E_i \mid b_i\right], \forall i$, written explicitly,

$$\left[E_1 \mid b_1\right] = \left[\begin{array}{cccccc|c} 1 & 1 & 1 & \dots & & 1 & n\kappa \\ 1 & -1 & 0 & \dots & & 0 & b_{12} \\ 1 & 0 & -1 & \dots & & 0 & b_{13} \\ \vdots & \vdots & \ddots & \ddots & & \vdots & \vdots \\ 1 & 0 & \dots & \dots -1 & & 0 & b_{1,n-1} \\ 1 & 0 & \cdots & \cdots & & -1 & b_{1n} \end{array}\right], \dots, \left[E_n \mid b_n\right] = \left[\begin{array}{ccccc|c} -1 & 0 & 0 & \dots 0 & 1 & b_{n1} \\ 0 & -1 & 0 & \dots 0 & 1 & b_{n2} \\ 0 & 0 & -1 & 0 \dots 0 & 1 & b_{n3} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots -1 & 1 & b_{n,n-1} \\ 1 & 1 & \cdots & \cdots 1 & 1 & n\kappa \end{array}\right]$$

The above matrix can be solved by Gaussian elimination. Then the individual utility vector for each row of B is found by using the reduced row echelon form of $\left[E_i \mid b_i\right]$, i.e. $\left[I \mid \omega_i\right]$, which is written explicitly by

$$\left[I \mid \omega_i\right] = \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & \omega_{i1} \\ 0 & 1 & \ddots & \vdots & \omega_{i2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & 1 & \omega_{in} \end{array}\right], \ i = 1, \dots, n \tag{5.15}$$

$\omega_i = \left(\omega_{i1}, \dots, \omega_{in}\right)$ is the set of constants which is the individual utility vector on the vector of ith row of B. (Another method to solve L_i is the Inverse Matrix method: $E_i \bullet \omega_i = b_i$, $\omega_i = b_i \bullet E_i^{-1}$, which is not discussed in this study.)

The following theorem shows the solution of $\omega_i = \left(\omega_{i1}, \dots, \omega_{in}\right)$ by the Gaussian elimination method.

Theorem 5.2 (ω_{ij}):

$$\omega_{ij} = \begin{cases} \kappa + \dfrac{1}{n} \sum\limits_{k \in \{1,\ldots,n\}} b_{ik} & j = i \\ \kappa + \dfrac{1}{n}\left((1-n) b_{ij} + \sum\limits_{k \neq i \ \& \ k \in \{1,\ldots,n\}} b_{ik} \right) & j \neq i \end{cases}, \quad \forall i \in \{1,\ldots,n\}. \tag{5.16}$$

Proof:

This theorem can be proved by the Gaussian elimination method in which the augmented matrix $[E_i \mid b_i]$ is transformed to the reduced row echelon form $[I \mid \omega_i]$. Explicitly,

$$[I \mid \omega_1] = \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & \omega_{11} = \kappa + \frac{1}{n}\sum_k b_{1k} \\ 0 & 1 & \ddots & \vdots & \omega_{12} = \kappa + \frac{1}{n}\left((1-n) b_{12} + \sum_{k \neq 2} b_{1k}\right) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & 1 & \omega_{1n} = \kappa + \frac{1}{n}\left((1-n) b_{1n} + \sum_{k \neq n} b_{1k}\right) \end{array}\right],$$

$$[I \mid \omega_2] = \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & \omega_{21} = \kappa + \frac{1}{n}\left((1-n) b_{21} + \sum_{k \neq 1} b_{2k}\right) \\ 0 & 1 & \ddots & \vdots & \omega_{22} = \kappa + \frac{1}{n}\sum_k b_{2k} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & 1 & \omega_{2n} = \kappa + \frac{1}{n}\left((1-n) b_{2n} + \sum_{k \neq n} b_{2k}\right) \end{array}\right],\ldots,$$

$$[I \mid \omega_n] = \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & \omega_{n1} = \kappa + \frac{1}{n}\left((1-n) b_{n1} + \sum_{k \neq 1} b_{nk}\right) \\ 0 & 1 & \ddots & \vdots & \omega_{n2} = \kappa + \frac{1}{n}\left((1-n) b_{nn} + \sum_{k \neq 2} b_{2k}\right) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & 1 & \omega_{nn} = \kappa + \frac{1}{n}\sum_k b_{nk} \end{array}\right]$$

Thus, the general form is as shown in eq.5.16. □

The $\{\omega_i\}$ is represented by following matrix:

$$\bar{\omega} = \{\omega_i\} = \begin{bmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1n} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{n1} & \omega_{n2} & \cdots & \omega_{nn} \end{bmatrix} \tag{5.17}$$

, where the row weight ω_{ij} is the weight of criteria *j* in row *i* of *B*. Row *i* of B means that criterion *i* is compared with other criteria in *B*.

From theorem 5.2, $\bar{\omega}$ is the form:

$$\bar{\omega} = \begin{bmatrix} \kappa + \frac{1}{n}\sum_k b_{1k} & \kappa + \frac{1}{n}\left((1-n)b_{12} + \sum_{k\neq 2} b_{1k}\right) & \cdots & \kappa + \frac{1}{n}\left((1-n)b_{1n} + \sum_{k\neq n} b_{1k}\right) \\ \kappa + \frac{1}{n}\left((1-n)b_{21} + \sum_{k\neq 1} b_{2k}\right) & \kappa + \frac{1}{n}\sum_k b_{2k} & \cdots & \kappa + \frac{1}{n}\left((1-n)b_{2n} + \sum_{k\neq n} b_{2k}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \kappa + \frac{1}{n}\left((1-n)b_{n1} + \sum_{k\neq 1} b_{nk}\right) & \kappa + \frac{1}{n}\left((1-n)b_{n2} + \sum_{k\neq 2} b_{nk}\right) & \cdots & \kappa + \frac{1}{n}\sum_k b_{nk} \end{bmatrix} \tag{5.18}$$

The accordant matrix is determined by following proposition.

Proposition 5.5: Let $\omega_i = (\omega_{i1}, \ldots, \omega_{in})$, $i = 1, \ldots, n$. For each row vector in B, if $\omega_1 = \omega_2 = \ldots = \omega_n$, B is accordant. If $\omega_i \neq \omega_j$, for any $i, j = 1, \ldots, n$, then B is discordant.

Proof:

For an accordant matrix, there should be a solution set in the linear systems set $L = \{L_1, \ldots, L_n\}$, which can be represented by the augmented matrices. The row reduction of L returns a row echelon form. If the row echelon form contains a vector $\left(0 \ \cdots \ 0 \mid c\right)$

where c is a constant, this means the matrix has no solution. Otherwise, L has a unique solution.

It can be proved that when $\omega_i \neq \omega_j$, $\forall i, j = 1,\ldots,n$, the row echelon form must contain a vector $\left(0 \quad \cdots \quad 0 \mid c\right)$. This means that L has no unique solution and the matrix is discordant. Otherwise, L has unique solution set and A is the accordant matrix.

□

If B is accordant, then $\omega_{i1} = \omega_{i2} = \ldots = \omega_{in}$, for any $i = 1,\ldots,n$. If B is accordant, any row in B can be computed as the individual utility vector. This means $v_i = \omega_{i1} = \omega_{i2} = \ldots = \omega_{in}$, $\forall i = 1,\ldots,n$, which is represented in the form:

$$\begin{bmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1n} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{n1} & \omega_{n2} & \cdots & \omega_{nn} \end{bmatrix} = \left.\begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ v_1 & v_2 & \cdots & v_n \\ \vdots & \vdots & \cdots & \vdots \\ v_1 & v_2 & \cdots & v_n \end{bmatrix}\right\} n \tag{5.19}$$

Table 5.3: Examples of non-weighted mean

1. generalized mean $gm(\omega_i) = \left(\frac{1}{n}\sum_{i=1}^{n}\omega_{ij}^{p}\right)^{1/p}, j = 1,\ldots,n$	2. Harmonic mean $hm(\omega_i) = \frac{n}{\sum_{i=1}^{n}\omega_{ij}}, j = 1,\ldots,n$
3. Geometric mean $geo(\omega_i) = \sqrt[n]{\prod_{i=1}^{n}\omega_{ij}}, j = 1,\ldots,n$	4. Arithmetic mean $am(\omega_i) = \frac{1}{n}\sum_{i=1}^{n}\omega_{ij}, j = 1,\ldots,n$

If A is discordant, then $\omega_{ij} \neq \omega_{i,k\neq j}$, for any $i, j = 1,\ldots,n$. Then in this new model, v_i can be computed by the aggregation operator $AO : \bar{\omega} \to V$. Some AO alternatives are shown in table 5.3. More options are suggested in chapter 2.3.5. The scope of the aggregation operators is beyond the topic in this paper. By default setting, the arithmetic mean is taken as it produces the same result as RAU, which is shown in theorem 5.3.

Theorem 5.3:

The closed form solution of the Arithmetic Mean of Solutions of Linear Systems (AMSLS) (or Arithmetic Mean in short) is in the form of the row average plus the normal utility (RAU).

Proof:

Let $\omega_i = (\omega_{i1}, \ldots, \omega_{in})$, $i = 1, \ldots, n$. Then

$$v_j = am(V) = \frac{1}{n}\sum_{i=1}^{n} \omega_{ij} \ , j = 1, \ldots, n$$

$$v_j = \frac{1}{n}\left(\left(\kappa + \frac{1}{n}\sum_{k} b_{1k}\right) + \left(\kappa + \frac{1}{n}\left((n-1)b_{2j} + \sum_{k\neq 1} b_{2k}\right)\right) + \ldots + \left(\kappa + \frac{1}{n}\left((n-1)b_{nj} + \sum_{k\neq 1} b_{nk}\right)\right)\right)$$

$$v_j = \frac{1}{n}\left(\kappa + \frac{1}{n}\sum_{k\in\{1,\ldots,n\}} b_{jk} + \sum_{i\neq j}\left(\kappa + \frac{1}{n}\left((n-1)b_{ij} + \sum_{k\neq j\ \&\ k\in\{1,\ldots,n\}} b_{ik}\right)\right)\right)$$

$$= \frac{1}{n}\left(\kappa + \frac{1}{n}\sum_{k\in\{1,\ldots,n\}} b_{jk} + \sum_{i\neq j}\left(\kappa + \frac{1}{n}\left((n-1)b_{ij} + \sum_{k\neq j\ \&\ k\in\{1,\ldots,n\}} b_{ik}\right)\right)\right)$$

$$= \frac{1}{n}\left(n\kappa + \frac{1}{n}\sum_{k\in\{1,\ldots,n\}} b_{jk} + \sum_{i\neq j}\left(\frac{1}{n}\left((n-1)b_{ij}\right)\right) + \sum_{i\neq j}\left(\frac{1}{n}\left(\sum_{k\neq j\ \&\ k\in\{1,\ldots,n\}} b_{ik}\right)\right)\right)$$

$$= \frac{1}{n^2}\left(\sum_{k\in\{1,\ldots,n\}} b_{jk} + \sum_{i\neq j}\left((n-1)b_{ij}\right) + \sum_{i\neq j}\left(\sum_{k\neq j\ \&\ k\in\{1,\ldots,n\}} b_{ik}\right)\right) + \kappa$$

, which is divided into two cases:

Case 1:

As $b_{ik} = -b_{ki}$, then

$$\sum_{i \neq j}\left(\sum_{k \neq j \ \& \ k \in \{1,\dots,n\}} b_{ik}\right) = \sum_{i \neq j}\left(\sum_{k \neq j \ \& \ k > i} \left(b_{ik} + b_{ki}\right)\right) = 0.$$

Case 2:

$$\begin{aligned} & \sum_{k \in \{1,\dots,n\}} b_{jk} + (1-n)\sum_{i \neq j}\left(b_{ij}\right) \\ &= \sum_{k \in \{1,\dots,n\}} b_{jk} - (1-n)\sum_{i \neq j}\left(b_{ji}\right) \\ &= (1+n-1)\sum_{k \in \{1,\dots,n\}} b_{jk} \\ &= n\sum_{k \in \{1,\dots,n\}} b_{jk} \end{aligned}$$

To combine both cases,

$$v_i = \frac{1}{n^2}\left(n\sum_{k \in \{1,\dots,n\}} b_{ik} + 0\right) + \kappa = \frac{1}{n}\left(\sum_{k \in \{1,\dots,n\}} b_{ik}\right) + \kappa.$$

□

The Aggregation of the Solutions of Linear Systems is summarized in algorithm 5.1.

Algorithm 5.1 (Aggregation of Solutions of Linear Systems):

Input: AO, B, κ

Step 1: Form the augmented Matrix $\left[E_i \mid b_i\right]$, $\forall i \in \{1,\dots,n\}$

Step 2: Find the reduced row echelon form $\left[I \mid \omega_i\right]$, $\forall i \in \{1,\dots,n\}$

Step 3: Form ϖ

Step 4: $v_i = AO\left(\omega_i\right)$, $\omega_i = \left(\omega_{i1},\dots,\omega_{in}\right) \in \varpi$, $i = 1,\dots,n$.

Output: $V = \{v_i\}$ **//END**

5.5.3 Primitive least squares (PLS) optimization

The utility linear system (or the system of utility equations) of the upper triangular matrix B^{+} has a matrix of equations, i.e. $E \cdot V^{T} = b$ where E is an l by n coefficient matrix, V^{T} is the column vector with n entries, and b is a column vector with l entries. A system of equations is formed as $L = \left\{ v_i - v_j = b_{ij} : i = 1, \ldots, n \;\; \& \;\; i = j+1, \ldots, n \right\}$, which is written explicitly by

$$L = \left\{ \begin{matrix} v_1 - v_2 = b_{12} \\ \vdots \\ v_1 - v_n = b_{1n} \\ v_2 - v_3 = b_{23} \\ \vdots \\ v_2 - v_n = b_{2n} \\ \vdots \\ v_{n-1} - v_n = b_{n-1,n} \end{matrix} \right\} \tag{5.20}$$

The above form has at least one solution. To have a unique solution set, the function of the summation of the individual utilities $\sum_{k=1}^{n} v_k = n\kappa$ is added to L. Finally, the system of matrix equations, $E \cdot W^{T} = b$, is of the form:

$$\overbrace{\begin{bmatrix} 1 & -1 & 0 & \ldots & 0 \\ 1 & 0 & -1 & \ldots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 0 & & \cdots & -1 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 1 & 0 & \ldots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots 1 & -1 \\ 1 & 1 & 1 & \ldots & 1 \end{bmatrix}}^{n} \bullet \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \left. \begin{bmatrix} b_{12} \\ b_{13} \\ \vdots \\ b_{n,n} \\ n\kappa \end{bmatrix} \right\} l+1 \tag{5.21}$$

, where n is the number of the criteria to be measured, l is the number of the equations, i.e., $l = \frac{n(n-1)}{2} + 1$.

There are several methods to solve such systems of linear equations. Gaussian elimination is a typical method using an augmented matrix $[E \mid b]$, which is written explicitly by

$$\overbrace{\left[\begin{array}{ccccc:c} 1 & -1 & 0 & \dots & 0 & b_{12} \\ 1 & 0 & -1 & \dots & 0 & b_{13} \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 1 & 0 & & \cdots & -1 & b_{1n} \\ 0 & 1 & -1 & \cdots & 0 & b_{23} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & 0 & \dots & -1 & b_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & 0 & \dots 1 & -1 & b_{n-1,n} \\ 1 & 1 & 1 & \dots & 1 & n\kappa \end{array}\right]}^{n+1} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\\1\\1\\1\end{array}}\right\} l \tag{5.22}$$

If the pairwise opposite matrix is accordant, the system of linear equations has a solution which can be represented by a row echelon form of the augmented matrices. Otherwise, if the opposite matrix is discordant, the row echelon form contains a vector $\left(0 \quad \cdots \quad 0 \mid c\right)$ where c is a constant, and means the matrix has no solution.

To handle the case of the discordant matrix, the system of equations L can be changed to the form:

$$\Delta = \left\{\Delta_{ij} = b_{ij} - v_i + v_j : i = 1,\dots,n \ \& \ i = j+1,\dots,n\right\} \tag{5.23}$$

Δ_{ij} is the difference (or error) between the reality b_{ij} and the ideality $\left(v_i - v_j\right)$. Δ is written explicitly by

$$\Delta = \left\{ \begin{array}{c} b_{12} - v_1 + v_2 = \Delta_{12} \\ \vdots \\ b_{1n} - v_1 + v_n = \Delta_{1n} \\ b_{23} - v_2 + v_3 = \Delta_{23} \\ \vdots \\ b_{2n} - v_2 + v_n = \Delta_{2n} \\ \vdots \\ b_{n-1,n} - v_{n-1} + v_n = \Delta_{n-1,n} \end{array} \right\} \tag{5.24}$$

To obtain the solution of V, $\{\Delta_{ij}\}$ is minimized. When minimization is performed, the elements of V have negative values. To prevent this, $\{\Delta_{ij}^2\}$ is minimized. Prior to forming the optimization model, following proposition holds.

Proposition 5.6 (B^+ for objective function):

To reduce the computation workload, the elements of the upper triangle opposite matrix B^+ (or the lower one) is sufficient for constructing the objective function.

Proof:

The objective function minimizes the sum of $\{\Delta_{ij}^2\}$, i.e. $Min\ \sum_{i=1}^{n}\sum_{j=i}^{n}\Delta_{ij}{}^2$. This gives

$\sum_{i=1}^{n}\sum_{j=i+1}^{n}\Delta_{ij}{}^2 = \sum_{i=1}^{n}\sum_{j=1}^{j}\Delta_{ij}{}^2$, i.e.

$\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left(b_{ij} - v_i + v_j\right)^2 = \sum_{i=1}^{n}\sum_{j=1}^{j}\left(b_{ij} - v_i + v_j\right)^2 = \overline{\Delta}$. Thus,

$\sum_{i=1}^{n}\sum_{j=i}^{n}\left(b_{ij} - v_i + v_j\right)^2 = \sum_{i=1}^{n}\sum_{j=i+1}^{n}\left(b_{ij} - v_i + v_j\right)^2 + \sum_{i=1}^{n}\sum_{j=1}^{j}\left(b_{ij} - v_i + v_j\right)^2 = 2\overline{\Delta}$. Hence, this proposition holds.

□

On the basis of (5.24), the Primitive Least Squares Optimization model is of the form:

$\mathrm{PLS}\left(B^{+},\kappa\right)=$

$$\text{Min} \quad \bar{\Delta}=\sum\nolimits_{i=1}^{n}\sum\nolimits_{j=i+1}^{n}\left(b_{ij}-v_i+v_j\right)^2 \tag{5.25}$$

$$\text{s.t.} \quad \sum\nolimits_{i=1}^{n} v_i = n\kappa ,$$

where $n=\left|\{v_i\}\right|$, and κ is the normal utility.

The solution of the closed form can be solved manually. This interesting finding is shown in the following theorem.

Theorem 5.4 (Closed form of PLS):

The closed form solution of the Primitive Least Squares Optimization model is the row average plus the normal utility (RAU), which is of the form:

$$v_i=\left(\frac{1}{n}\sum_{j=1}^{n}b_{ij}\right)+\kappa ,\forall i\in\{1,\ldots,n\} \tag{5.26}$$

Proof:

To have the closed form solution, the partial differentiation of $\bar{\Delta}$ with respective to all $v_k \in V$ is derived and is of the form:

$$L_k=\frac{\delta\bar{\Delta}}{\delta v_k}=0,\ k=1,2,\ldots,n.$$

Then the linear system $\{L_i\}$ is solved for V. Therefore,

$$\bar{\Delta}=\sum\nolimits_{i=1}^{n}\sum\nolimits_{j=i+1}^{n}\left(b_{ij}-v_i+v_j\right)^2$$

$$\bar{\Delta}=\sum_{i\neq k}\sum_{j>i,j\neq k}\Delta_{ij}+\sum_{i=k,j>i,j\neq k}\Delta_{kj}+\sum_{i\neq k,j>i,j=k}\Delta_{ik},$$

$$\bar{\Delta}=\sum_{i\neq k}\sum_{j>i,j\neq k}\Delta_{ij}+\sum_{n\geq j>k}\Delta_{kj}+\sum_{i<k}\Delta_{ik}$$

Let $\bar{\Delta}_1=\sum_{i\neq k}\sum_{j>i,j\neq k}\Delta_{ij}$, $\bar{\Delta}_2=\sum_{n\geq j>k}\Delta_{kj}$, and $\bar{\Delta}_3=\sum_{i<k}\Delta_{ik}$. Thus

$$\bar{\Delta}=\bar{\Delta}_1+\bar{\Delta}_2+\bar{\Delta}_3$$

$$\frac{\delta\bar{\Delta}}{\delta v_k}=\frac{\delta\bar{\Delta}_1}{\delta v_k}+\frac{\delta\bar{\Delta}_2}{\delta v_k}+\frac{\delta\bar{\Delta}_3}{\delta v_k}.$$

Case 1: $\frac{\delta\bar{\Delta}_1}{\delta v_k}$

$$\frac{\delta\bar{\Delta}_1}{\delta v_k}=\frac{\delta\left(\sum_{i\neq k}\sum_{j>i,j\neq k}\Delta_{ij}\right)}{\delta v_k}=0$$

Case 2: $\frac{\delta\bar{\Delta}_2}{\delta v_k}=\frac{\delta\left(\sum_{n\geq j>k}\Delta_{kj}\right)}{\delta v_k}$

$$=\frac{\delta\left(\sum_{n\geq j>k}\left(b_{kj}-v_k+v_j\right)^2\right)}{\delta v_k}$$

$$=-2\sum_{n\geq j>k}\left(b_{kj}-v_k+v_j\right)$$

Case 3: $\frac{\delta\bar{\Delta}_3}{\delta v_k}=\frac{\delta\left(\sum_{i<k}\Delta_{ik}\right)}{\delta v_k}$

$$=\frac{\delta\left(\sum_{i<k}\left(b_{ik}-v_i+v_k\right)^2\right)}{\delta v_k}$$

$$=2\sum_{i<k}\left(b_{ik}-v_i+v_k\right)$$

To combine the three cases,

$$\frac{\delta\bar{\Delta}}{\delta v_k}=\frac{\delta\bar{\Delta}_1}{\delta v_k}+\frac{\delta\bar{\Delta}_2}{\delta v_k}+\frac{\delta\bar{\Delta}_3}{\delta v_k}$$

$$=-2\sum_{n\geq j>i}\left(b_{kj}-v_k+v_j\right)+2\sum_{i<k}\left(b_{ik}-v_i+v_k\right)$$

$$=-2\left(\sum_{n\geq j>k}\left(b_{kj}\right)-\left(n-k\right)v_k+\sum_{n\geq j>k}\left(v_j\right)\right)+2\left(\sum_{i<k}\left(b_{ik}\right)-\sum_{i<k}\left(v_i\right)+\left(k-1\right)v_k\right)$$

$$=-2\left(\sum_{n\geq j>k}\left(b_{kj}\right)-\left(n-k\right)v_k+\sum_{n\geq j>k}\left(v_j\right)\right)+2\left(\sum_{i<k}\left(b_{ik}\right)-\sum_{i<k}\left(v_i\right)+\left(k-1\right)v_k\right)$$

$$=2\left(\left(\left(n-1\right)v_k\right)-\left(\sum_{n\geq j>k}\left(v_j\right)+\sum_{1\leq i<k}\left(v_i\right)\right)+\left(\sum_{1\leq i<k}\left(b_{ik}\right)-\sum_{n\geq j>k}\left(b_{kj}\right)\right)\right)$$

$$=2\left(\left(\left(n-1\right)v_k\right)-\left(\sum_{n\geq j>k}\left(v_j\right)+\sum_{i<k}\left(v_i\right)\right)+\left(-\sum_{i<k}\left(b_{ki}\right)-\sum_{n\geq j>k}\left(b_{kj}\right)\right)\right)$$

$$=2\left(\left(\left(n-1\right)v_k\right)-\left(\sum_{i\neq k}\left(v_i\right)\right)-\left(\sum_{j}\left(b_{kj}\right)\right)\right)$$

Since $\frac{\partial^2\bar{\Delta}}{\partial v_k}=2\left(n-1\right)>0$, $\bar{\Delta}$ is convex.

As $L_k=\frac{\delta\bar{\Delta}}{\delta v_k}=0$, $k=1,2,\ldots,n$, there exists a minimal v_k , $k=1,2,\ldots,n$.

The values can be solved by a linear system. Thus the augmented matrix $\left[E \;\vdots\; b\right]$ is of the form:

$$\left[E \mid b\right]=\left[\begin{array}{ccccc|c} n-1 & -1 & -1 & \ldots -1 & \sum_i (b_{1i}) \\ -1 & n-1 & -1 & \ldots -1 & \sum_i (b_{2i}) \\ -1 & -1 & n-1 & \ldots -1 & \sum_i (b_{3i}) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & -1 & \ldots & \ldots n-1 & \sum_i (b_{ni}) \\ 1 & 1 & \cdots & \cdots 1 & n\kappa \end{array}\right]$$

V is solved by Gaussian elimination of $\left[E \mid b\right]$, in which the final row is added to other rows. Thus,

$$\left[\begin{array}{cccc|c} n & 0 & 0 & \ldots 0 & n\kappa + \sum_i (b_{1i}) \\ 0 & n & 0 & \ldots 0 & n\kappa + \sum_i (b_{2i}) \\ 0 & 0 & n & \ldots 0 & n\kappa + \sum_i (b_{3i}) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & \ldots n & n\kappa + \sum_i (b_{ni}) \end{array}\right]$$

To divide the above system by n, the reduced row echelon form is

$$\left[I \mid V\right]=\left[\begin{array}{cccc|c} 1 & 0 & 0 & \ldots 0 & \frac{1}{n}\sum_i (b_{1i}) + \kappa \\ 0 & 1 & 0 & \ldots 0 & \frac{1}{n}\sum_i (b_{2i}) + \kappa \\ 0 & 0 & 1 & \ldots 0 & \frac{1}{n}\sum_i (b_{3i}) + \kappa \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & \ldots 1 & \frac{1}{n}\sum_i (b_{ni}) + \kappa \end{array}\right]$$

Interestingly, the result is the same as the row average plus the normal utility.

□

From sections 5.1 to 5.3, it seems the Row Average plus the normal Utility (RAU) is encouraging. The value of κ is critical for the solutions of RAU, PLS, and AMSLS. To rescale κ, following proposition holds.

Proposition 5.7 (κ):

If $Max(\aleph) \le \kappa$ or $-Max(\aleph) \le -\kappa$, then $v_i \ge 0$, and vice versa.

Proof:

For $\forall i \in \{1,\ldots,n\}$, if $v_i \ge 0$, then $\left(\frac{1}{n}\sum_{j=1}^{n} b_{ij}\right) \ge \kappa$. If $\left(\frac{1}{n}\sum_{j=1}^{n} b_{ij}\right) \ge \kappa$, then $Max(\aleph) \le \kappa$ since $b_{ij} \in \aleph$.

□

If $Max(\overline{X}) > \kappa$, it is possible that some negative individual utilities exist, i.e. $\exists v_i < 0$. One method is to increase the normal utility κ such that $Max(\overline{X}) \le \kappa$. Another method is to decrease $Max(\overline{X})$ such that $Max(\overline{X}) \le \kappa$, and rescale the numerical scale $\overline{X}$ for $\aleph$. If these two methods are not allowed, another method is to modify the operator. This is shown as follows.

5.5.4 Bounded least squares (BLS) optimization

If $Max(\overline{X}) > \kappa$ (i.e. κ is too small or $Max(\overline{X})$ is too large), PLS will produce a negative individual utility. To avoid the negative issue, the conditions $v_i > 0, i = 1, 2, \ldots, n$ are added in the primitive form. The new form is called Bounded Least Squares Optimization as below.

$$\text{BLS}(B,\kappa)=$$

$$\text{Min}\quad \bar{\Delta}=\sum\nolimits_{i=1}^{n}\sum\nolimits_{j=i+1}^{n}\left(b_{ij}-v_i+v_j\right)^2 \tag{5.27}$$

$$\text{s.t.}\quad \begin{array}{l}\sum\nolimits_{i=1}^{n} v_i = n\kappa,\\ v_i > 0, i=1,2,\ldots,n\end{array},$$

where $n=\left|\{v_i\}\right|$, and κ is the normal utility.

The above problem can be solved by Karush-Kuhn-Tucker (KKT) method (Karush, 1939; Araora,2004), which is built on the Lagrange theorem. The algorithm of ALS is as follows:

Algorithm 5.2 (KKT Solution OF BLS):

Input: (B, κ)

Step 1. Get the Lagrange Function:

$$Lg(V,\kappa,U,S)=\sum\nolimits_{i=1}^{n}\sum\nolimits_{j=i+1}^{n}\left(b_{ij}-v_i+v_j\right)^2+\sum_{i=1}^{n}u_i\left(-v_i+s_i^2\right)+u_{n+1}\left(\left(\sum_{i=1}^{n}v_i\right)-\kappa n\right),$$

, Where $U=\{u_1,\ldots,u_{n+1}\}$ and $s=\{s_1,\ldots,s_n\}$.

Step 2. Get differential equations:

$$\frac{\delta Lg}{\delta v_i}=\frac{\delta\sum\nolimits_{i=1}^{n}\sum\nolimits_{j=i+1}^{n}\left(b_{ij}-v_i+v_j\right)^2}{\delta v_i}+\frac{\delta\sum_{i=1}^{n}u_i\left(-v_i+s_i^2\right)}{\delta v_i}+\frac{\delta u_{n+1}\left(\left(\sum_{i=1}^{n}v_i\right)-\kappa n\right)}{\delta v_i}=0,$$

$i=1,\ldots,n$;

$$\frac{\delta Lg}{\delta u_i}=\left(-v_i+s_i^2\right)=0,\quad i=1,\ldots,n;$$

$$\frac{\delta Lg}{\delta u_{n+1}} = \left(\sum_{i=1}^{n} v_i \right) - \kappa n = 0 \text{, } i = 1, \ldots, n \text{;}$$

$$\frac{\delta Lg}{\delta s_i} = 2 s_i u_i = 0 \text{, } i = 1, \ldots, n \text{;}$$

Step 3. Solve the above equations. There are multiple solutions for $sol_1 = (V, U, S)$.

Explicitly,

$$Sol_1 = [V' \vdots S' \vdots U'] \left[\begin{array}{ccc|ccc|ccc} v'_{11} & \cdots & v'_{1n} & s_{11} & \cdots & s_{1n} & u_{11} & \cdots & u_{1,n+1} \\ v'_{21} & \cdots & v'_{2n} & s_{21} & \cdots & s_{2n} & u_{21} & \cdots & u_{1,n+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v'_{m1} & \cdots & v'_{nn} & s_{m1} & \cdots & s_{nn} & u_{m1} & \cdots & u_{m,n+1} \end{array} \right]$$

, where the local variable *m* is the number of solutions.

Step 4. Filter Sol_1 such that $\left\{ v_i : \left(\sum_{i=1}^{n} v_i \right) - \kappa n = 0, v_i > 0, i = 1, 2, \ldots, n \right\}$, and find

$sol_2 \subseteq sol_1$.

Step 5. Filter Sol_2 with the condition of the Non-negativity of Lagrange Multipliers for Inequalities, i.e. $u_i \geq 0$, $i = 1, \ldots, n+1$, and $sol_3 \subseteq sol_2$ is formed.

Step 6. Filter sol_3 by the Feasibility Check for Inequalities, i.e. $\left(-v_i + s_i^2 \right) = 0$ and $2 s_i u_i = 0$, $i = 1, \ldots, n$, and $sol_4 \subseteq sol_3$ is formed.

Step 7. If the individual utility vector v^* in each row of sol_4 is the same, return $v^* \in sol_4$.

Return: v^* **//END**

If $Max\left(\overline{X}\right) < \kappa$, Bounded Least Squares (NLS) optimization is one of the methods to return the approximate value without a negative value, however this method is not recommended. The most efficient way is to change the value of the normal utility κ. However, there are infinite values for κ. In this case, κ is defined subject to $Min\left(V\right) = 0$.

5.5.5 Least penalty squares (LPS or DLS) optimization

The individual utility vector $V = \{v_1, \ldots, v_n\}$ is determined by the Least Penalty Squares (LPS), which is also called Discrete Least Squares (DLS), and apply a set of penalties $\{\beta_{ij}\}$ in the Least Squares, and in the form:

$$\text{DLS}\left(B^+, \{\beta_1, \beta_2, \beta_3\}, \kappa\right) = \text{LPS}\left(B^+, \{\beta_1, \beta_2, \beta_3\}, \kappa\right)$$

$$\text{Min} \qquad \widehat{\Delta} = \sum\nolimits_{i=1}^{n} \sum\nolimits_{j=i+1}^{n} \beta_{ij} \cdot \left(b_{ij} - v_i + v_j\right)^2$$

$$, \ \beta_{ij} = \begin{cases} \beta_1, & \mathrm{v}_i > v_j \ \ \& \ \ \mathrm{b}_{ij} > 0 \\ & \text{or } \mathrm{v}_i < v_j \ \ \& \ \ \mathrm{b}_{ij} < 0 \\ \beta_2, & \mathrm{v}_i = v_j \ \ \& \ \ \mathrm{b}_{ij} \neq 0 \\ & \text{or } v_i \neq v_j \ \ \& \ \ \mathrm{b}_{ij} = 0 \\ \beta_3, & otherwise \end{cases}, \ 1 = \beta_1 \le \beta_2 \le \beta_3 \tag{5.28}$$

$$\text{s.t.} \qquad \begin{array}{l} \sum\nolimits_{i=1}^{n} v_i = n\kappa, \\ \mathrm{v}_i \ge 0, i = 1, 2, \ldots, n \end{array},$$

where $n = \left|\{v_i\}\right|$, and κ is the normal utility.

The closed form solution of LPS (or DLS) is very difficult to solve since β_{ij} is the conditioned discrete variable and cannot be derived by partial differentiation. Thus the Lagrange theorem and the KKT theorem cannot be applied. One method is to use the Exhaustive Computing method, where all combinations of $\{v'_i \in [0, n\kappa] : i = 1, \ldots, n\}$ are computed, and then return to the individual utility V with respect to the minimized penalty weighted error sum $\vec{\Delta}^*$. The limitation of this algorithm is the computational workload, which however is trivial with today's powerful computers, and hence the exhaustive computing method deserves be utilized. In addition, the solution can be easily solved by powerful software tools such as *Excel, Mathlab, Lingo*, as well as *Mathematica,* which is used in this research. Chapter 5.7 further illustrates the concepts of PLS and LPS using graph theory.

5.6 Cognitive prioritization operator measurement (CPOM) models

When a problem is introduced, some possible solutions are proposed. This leads to the question as to which solution model is the best one, thus a study of the measurement models is introduced. This also leads to various measurement models. The next question is taken asking which measurement model is the most appropriate. The fittest measurement model must be supported by convincing evidence that it performs better than other models.

The Cognitive Prioritization Operator Measurement (CPOM) models evaluate the fitness of the prioritization operators. Thus they can be used for selecting the fittest CPO by comparing different CPOs. This research proposes six CPOM models as follows.

5.6.1 Worst absolute distance variance (WADV)

The Worst Absolute Distance Variance is to measure the greatest variance of B and V, and has the form:

$$WADV(B,V) = \underset{i,j}{Max}\left\{abs\left(b_{ij} - v_i + v_j\right)\right\} \tag{5.29}$$

, $abs(.)$ is the function to return obsolete value.

5.6.2 Mean absolute distance variance (MADV)

The Mean Absolute Square Variance is of the form:

$$MADV(B,V) = \frac{1}{n \times (n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} abs\left(b_{ij} - v_i + v_j\right). \tag{5.30}$$

5.6.3 Mean penalty weighted absolute distance variance (MPWADV)

The Mean Penalty Weighted Absolute Distance Variance is of the form:

$$MPWADV(B,V) = \sqrt{\frac{1}{n \times (n-1)} \sum_{i} \sum_{j} Y_{ij}}$$

$$\text{, where } Y_{ij} = \begin{cases} \beta_1 \bullet Abs\left(b_{ij} - v_i + v_j\right) & , v_i > v_j \;\; \& \;\; \mathrm{b}_{ij} > 0 \\ & or \;\; \mathrm{v}_i < v_j \;\; \& \;\; \mathrm{b}_{ij} < 0 \\ & or \;\; \mathrm{v}_i = v_j \;\; \& \, b_{ij} = 0 \\ \beta_2 \bullet Abs\left(b_{ij} - v_i + v_j\right) & , v_i = v_j \;\; \& \;\; \mathrm{b}_{ij} \neq 0 \\ & \text{or } \mathrm{v}_i \neq v_j \;\; \& \;\; \mathrm{b}_{ij} = 0 \\ \beta_3 \bullet Abs\left(b_{ij} - v_i + v_j\right) & , otherwise \end{cases} \text{, } 1 = \beta_1 \leq \beta_2 \leq \beta_3 \tag{5.31}$$

As it is more difficult than PMPWSV, which is introduced in chapter 6.6., to solve, when the above form is converted into an optimization model, this model is not selected as the Cognitive Distortion Index.

5.6.4 Root mean square variance (RMSV)

Euclidean Distance (ED) is the square root of the sum of the square deviations between the difference of two individual utilities and their corresponding entry in the matrix. It has the form:

$$ED(B,V)=\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(b_{ij}-v_i+v_j\right)^2} \tag{5.32}$$

As ED depends on the size, i.e. *nx(n-1)*, of the opposite matrix *B*, for the effective interpretation of the result, it is more appropriate to use the mean of the value. The Root Mean Square Variance which takes the root of the average of the sum of the square deviations is as follows:

$$RMSV(B,V)=\sqrt{\frac{1}{n\times(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(b_{ij}-v_i+v_j\right)^2} \tag{5.33}$$

However, a limitation of ED/RMSV is that the penalty weights are not justified. For example, the penalty of the condition $(v_i > v_j \ \& \ \mathrm{b}_{ij} < 0) \equiv True$ is not the same as the penalty of the condition $(v_i > v_j \ \& \ \mathrm{b}_{ij} > 0) \equiv True$. In short, the penalty weights of these two conditions should not be equal. This issue introduces the measurement of the mean contradiction.

5.6.5 Mean contradiction (MC)

The Mean Contradiction is of the form:

$$MC(A,V)=\frac{1}{n(n-1)}\left(\sum_i\sum_j I_{ij}\right)$$

$$,\text{where } I_{ij} = \begin{cases} I_1 & ,\left(v_i > v_j \ \ \& \ \ \mathrm{b}_{ij} < 0\right) \\ I_1 & \left(v_i < v_j \ \ \& \ \ \mathrm{b}_{ij} > 0\right) \\ I_2 & ,v_i = v_j \ \ \& \ \ \mathrm{b}_{ij} \neq 0 \\ I_2 & ,v_i \neq v_j \ \ \& \ \ \mathrm{b}_{ij} = 0 \\ I_3 & ,Otherwise \end{cases} \quad , I_1 \geq I_2 \geq I_3 = 0; \tag{5.34}$$

In the default setting of MC, $I_1 = 1$, $I_2 = 0.5$, $I_3 = 0$. A limitation of *MC* is that it counts the sum of penalty scores only, and ignores the actual variance values.

5.6.6 Root mean penalty weighted square variance (RMPWSV)

To combine the advantages of *RMSV* and *MC*, and offset their shortcomings, this paper proposes the *Root Mean Penalty Weighted Square Variance* σ, as follows:

$$\sigma = RMPWSV\left(B,V\right) = \sqrt{\frac{1}{n \times \left(n-1\right)} \sum_i \sum_j Y_{ij}}$$

$$, \text{where } \ Y_{ij} = \begin{cases} \beta_1\left(b_{ij} - v_i + v_j\right)^2 & ,v_i > v_j \ \ \& \ \ \mathrm{b}_{ij} > 0 \\ & or\ \mathrm{v}_i < v_j \ \ \& \ \ \mathrm{b}_{ij} < 0 \\ & or\ \mathrm{v}_i = v_j \ \ \& b_{ij} = 0 \\ \beta_2\left(b_{ij} - v_i + v_j\right)^2 & ,v_i = v_j \ \ \& \ \ \mathrm{b}_{ij} \neq 0 \\ & \text{or } \mathrm{v}_i \neq v_j \ \ \& \ \ \mathrm{b}_{ij} = 0 \\ \beta_3\left(b_{ij} - v_i + v_j\right)^2 & ,otherwise \end{cases} , \ 1 = \beta_1 \leq \beta_2 \leq \beta_3 \tag{5.35}$$

, $\beta = \left\{\beta_1, \beta_2, \beta_3\right\}$ is the vector of penalty weights. RMSV is a special case of RMPWSV if $\beta_1 = \beta_2 = \beta_3 = 1$.

In RMPWSV, $\beta_1 = 0$ can cancel the variance. By default settings of σ, $\beta_1 = 1$ is defined, and also $\beta_2 = 3, \beta_3 = 10$. This research regards *RMPWSV* as more significant than other measurement models which are used for reference only. PMPWSV is selected to be defined as the Cognitive Distortion Index (CDI) as its value is more appropriate in reflecting the error between *B* and *V*. The details of the relations of *RMPWSV* (or CDI) and other measurement models are given in the numerical analysis section.

5.7. Graph theory interpretation

Conventional graph theory can show the prioritization of only three criteria. If there are more than three criteria, graphical representation is impossible as it is a complex hyper dimensional problem. In fact, this visualization is beyond the human perception. Thus, this section applies 2D and 3D representations for the prioritization of three criteria, since the pairwise opposite matrix of two criteria is always accordant.

5.7.1 Two dimensional representation

Consider a 3x3 prioritization problem with a individual utility set $V=\{v_1,v_2,v_3\}$, which is of the form:

$$B=\begin{pmatrix} 0 & b_{12} & b_{13} \\ -b_{12} & 0 & b_{23} \\ -b_{13} & -b_{23} & 0 \end{pmatrix} \tag{5.36}$$

As the axiom of the ratio scale is $b_{ij}=v_i-v_j$, a system of three linear equations is formed:

$$\begin{cases} v_1-v_2=b_{12} \\ v_1-v_3=b_{13} \\ v_2-v_3=b_{23} \end{cases} \tag{5.37}$$

Another axiom of the population utility is the form $\sum_{i=1}^{n} v_i = n\kappa$, thus $v_1 + v_2 + v_3 = n\kappa$. To eliminate v_3 and plot a plane of v_1 and v_2, the new form of the linear system is :

$$\begin{cases} v_2 = v_1 - b_{12} \\ v_2 = b_{13} + n\kappa - 2v_1 \\ v_2 = \frac{1}{2}\left(b_{23} + n\kappa - v_1\right) \end{cases} \tag{5.38}$$

To illustrate the above linear system, the next step is to plot the lines in the 2D plane. Let $b_{12} = -0.1$, $b_{13} = -0.1$, $b_{23} = 0$, and $\kappa = 1$. The opposite matrix is perfectly accordant. In fig. 5.1, the three linear equations form an intercept point (0.933, 1.033), which is the unique solution of the individual utility vector.

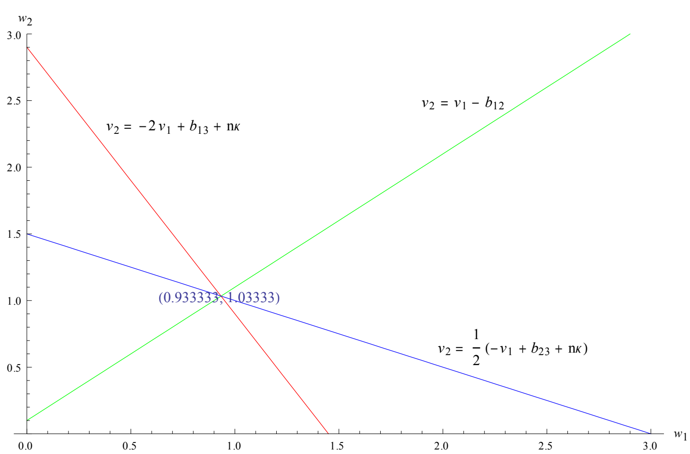


Figure 5.1: The feasible points of perfectly accordant opposite matrix

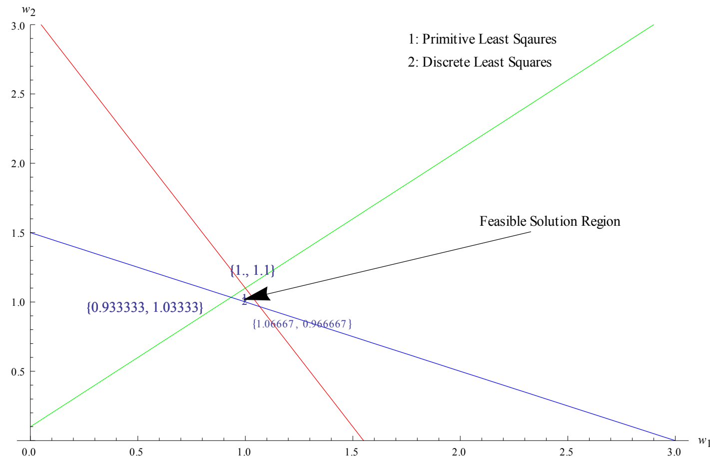


Figure 5.2: The feasible solution region of CPOs

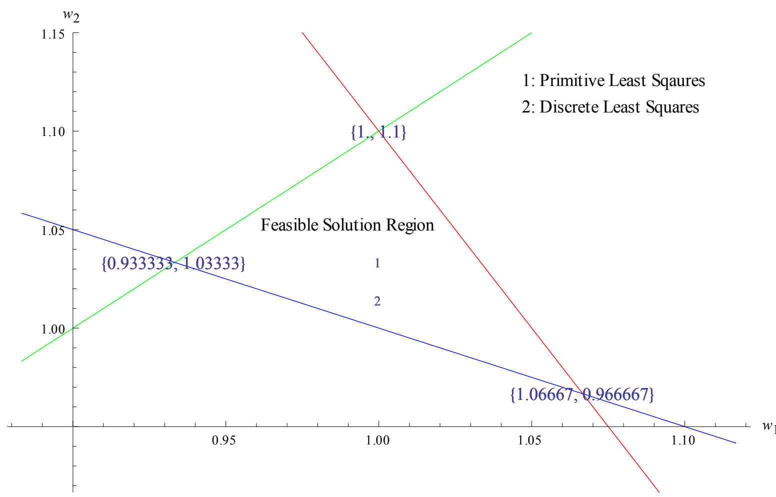


Figure 5.3: The feasible solution region of CPOs in focus view

Consider the matrix in example 5.1, i.e. $b_{12} = -0.1$, $b_{13} = 0.1$, $b_{23} = 0$. In fig. 5.2, the Feasible Solution Region is constituted by the three lines. Fig 5.3 shows the focus view of the region. It can be observed that the solutions of two prioritization operators proposed in this paper are located within this region. The index numbers indicate the indices of the CPOs: 1 is the PLS/RAU/AMSLS, and 2 is the DLS/LPS.

It might be suggested to draw some lines for the proposed Cognitive Prioritization Operator Measurement functions, e.g. *RMSV* or *RMPWSV*, to elaborate on the best approximate solution points. However, they are graphically impossible to show in this plane, and the reason is in the following proposition.

Proposition 5.8:

For an opposite matrix with three variables and $v_1 \in [0, n\kappa]$, v_2 is a complex number except for one case after the algebraic operation of *RMSV=0* or *RMPWSV* =0, and vice versa. Thus the measurement functions cannot be shown in the plane. For this exceptional case, there exists one real solution point (v_1, v_2), when the opposite matrix is perfectly accordant.

Proof:

$$RMSV(B,V) = \sqrt{\frac{1}{n \times (n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(b_{ij} - v_i + v_j\right)^2} = 0$$

For n=3, then

$$\left(b_{12} - v_1 + v_2\right)^2 + \left(b_{13} - v_1 + v_3\right)^2 + \left(b_{23} - v_2 + v_3\right)^2 = 0$$

To eliminate v_3 by using $v_1 + v_2 + v_3 = n\kappa$, then

$$\left(b_{12} - v_1 + v_2\right)^2 + \left(b_{13} + n\kappa - 2v_1 - v_2\right)^2 + \left(b_{23} + n\kappa - 2v_2 - v_1\right)^2 = 0$$

Solving the above equation, then

$$v_2 = \frac{1}{6}\left(3n\kappa - b_{12} + b_{13} + 2b_{23} - 3v_1 \pm \sqrt{\begin{matrix} -3n\kappa^2 - 5b_{12}^2 - 5b_{13}^2 - 2b_{23}^2 - 2b_{12}\left(b_{13} + 4b_{23} - 9v_1\right) \\ -6n\kappa\left(b_{12} + b_{13} - 3v_1\right) - 27v_1^2 + 2b_{13}\left(2b_{23} + 9v_1\right) \end{matrix}}\right).$$

Let $\sqrt{\begin{matrix} -3n\kappa^2 - 5b_{12}^2 - 5b_{13}^2 - 2b_{23}^2 - 2b_{12}\left(b_{13} + 4b_{23} - 9v_1\right) \\ -6n\kappa\left(b_{12} + b_{13} - 3v_1\right) - 27v_1^2 + 2b_{13}\left(2b_{23} + 9v_1\right) \end{matrix}} = \sqrt{\Delta''}$.

For $v_1 \in [0, n\kappa]$, $\Delta'' \le 0$. Hence v_2 is a complex number except when $\Delta''=0$. If $\Delta''=0$, v_1 is a real number and so is v_2. In addition, if $v_1 \in [0, n\kappa]$ when $\Delta''=0$, then the matrix must be perfectly accordant.

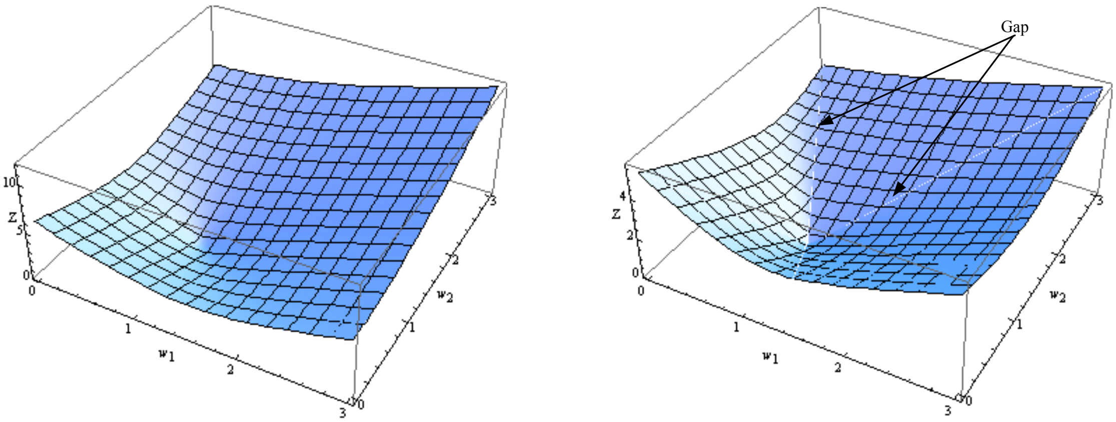
□

As a complex number cannot be drawn in the plane, in this case, a third dimension is needed to be created for measurement functions.

5.7.2 Three dimensional representation

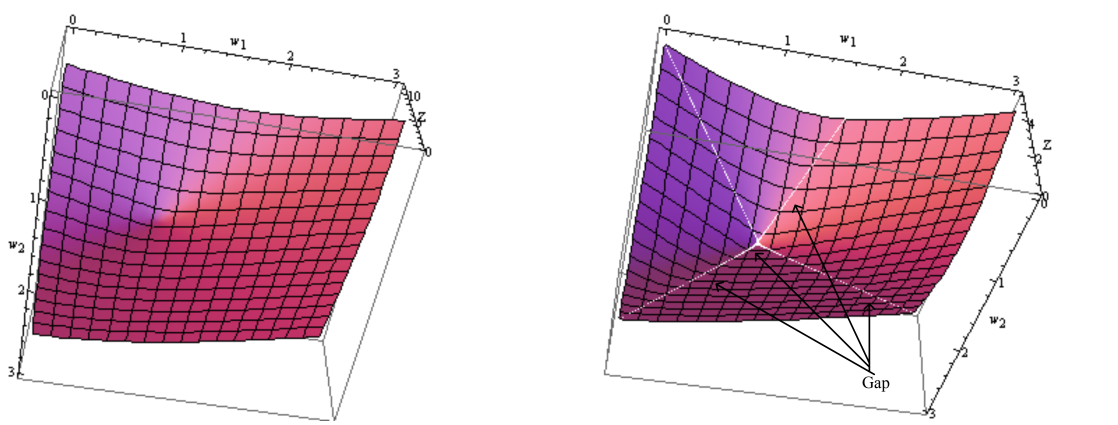
In the 2D plane of w_1 and w_2, a dimension z is created for exploring the evaluation value by a measurement function. Two measurement functions are explored and compared in this section: *Root Mean Square Variance (RMSV) and Root Mean Penalty Weighted Square Variance (RMPWSV).*

Fig 5.4 shows the *RMSV* and *RMPWSV* for all $v_1, v_2 \in [0, n\kappa]$, in the top view. It can be observed that there are three white lines to separate the regions in the RMPWSV graph whilst this does not happen in RMSV. The reason can be found in fig. 5.5 which shows the same content of fig. 5.4 but in a side view. It can be observed that some areas are leveled up accordingly. This is due to the penalty weights (β) increasing in 3 or 10 times. This is analogue to an earthquake. If the intensity of the earthquake increases, the degree of the ground level is raised. For instance, $(\beta_1, \beta_2, \beta_3)$ is from $(1,3,10)$ to $(1,10,100)$ (fig. 5.6).



Graph of Root Mean Square Variance *Graph of Root Mean Penalty Weighted Square Variance*

Figure 5.4: Top View of the measurement values of CPOs on plane (w_1 , w_2)

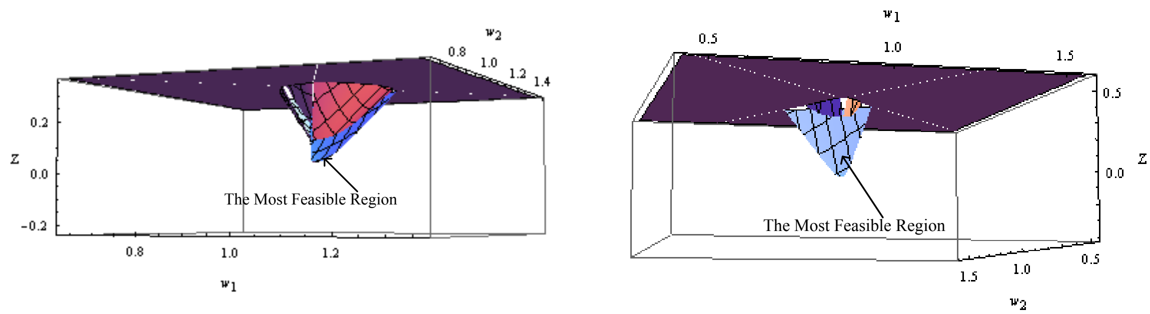


Root Mean Square Variance *Root Mean Penalty Weighted Square Variance*

Figure 5.5: Bottom View of the measurement values of CPOs on plane (w_1 , w_2)

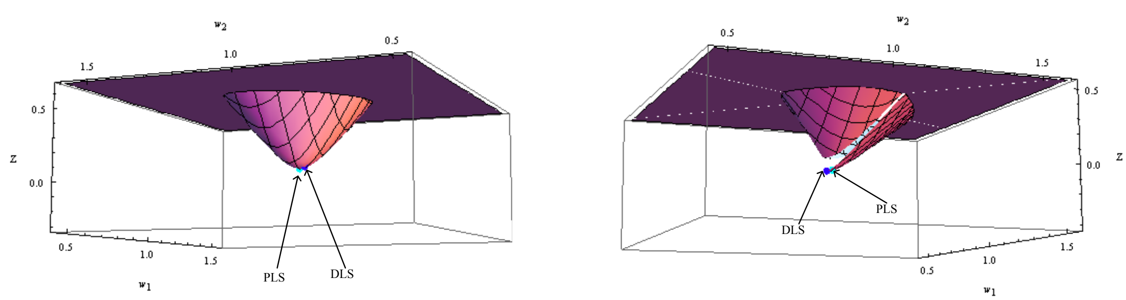
Fig.5.7 compares these two CPOs, PLS/RAU/AMSLS and LPS/DLS, by two 3D graphs. The least value of z , i.e. $z_{\min}$, indicates the most appropriate of the combination of the individual utilities. Thus $z_{\min}$ is in the lowest plane, which is called the Most Feasible Solution Region (MFSR). The MFSR in fig. 5.6 is also within the Feasible Solution Region

(FSR), as shown in figs. 5.2 and 5.3. The value of $z_{\min}$ can be found by the optimization model with respect to the measurement function.



RMPWSV with penalty set $(\beta_1, \beta_2, \beta_3) = (1,3,10)$ *RMPWSV with penalty set* $(\beta_1, \beta_2, \beta_3) = (1,10,100)$

Figure 5.6: The most feasible solution region of CPOs with two β values



Solution points of CPOs using RMSV *Solution points of CPOs using RMPWSV*

Figure 5.7: Focus views of solution points of CPOs in the 3D graph

5.8. Numerical analyses and discussion

Two major analyses are performed and discussed, as follows.

5.8.1 Stability and validity analysis

The analyses show the comparisons of the results of two prioritization operators, RAU (or PLS or AMSLS) and LPS (or DLS), on the basis of the CPOM models. The simulation includes 168 (21 x8) cases from eight template matrices of different dimensions. The rating scale is defined as $\overline{X} = \{\alpha \in [-1,1] : (\alpha = -1 + 0.1i),\ (i = 0,\ldots,20)\}$, and r is chosen from $\overline{X}$, i.e. $r \in \overline{X}$. The template matrices are shown as follows.

$$T3(r) = \begin{pmatrix} 0 & 0.2 & r \\ -0.2 & 0 & 0 \\ -r & 0 & 0 \end{pmatrix}$$

$$T4(r) = \begin{pmatrix} 0 & 0.2 & 0.3 & r \\ -0.2 & 0 & 0.1 & 0 \\ -0.3 & -0.1 & 0 & -0.1 \\ -r & 0 & 0.1 & 0 \end{pmatrix}$$

$$T5(r) = \begin{pmatrix} 0 & 0.2 & 0.3 & 0.1 & r \\ -0.2 & 0 & 0.1 & -0.1 & 0 \\ -0.3 & -0.1 & 0 & -0.2 & -0.1 \\ -0.1 & 0.1 & 0.2 & 0 & 0.1 \\ -r & 0 & 0.1 & -0.1 & 0 \end{pmatrix}$$

$$T6(r) = \begin{pmatrix} 0 & 0.2 & 0.3 & 0.5 & -0.4 & r \\ -0.2 & 0 & 0.1 & 0.3 & -0.6 & 0.1 \\ -0.3 & -0.1 & 0 & -0.2 & -0.7 & 0 \\ -0.5 & -0.3 & -0.2 & 0 & -0.9 & -0.2 \\ 0.4 & 0.6 & 0.7 & 0.9 & 0 & 0.7 \\ -r & -0.1 & 0 & 0.2 & -0.7 & 0 \end{pmatrix}$$

$$T7(r)=\begin{pmatrix} 0 & -0.3 & 0.2 & 0.4 & -0.6 & 0.3 & r \\ 0.3 & 0 & 0.5 & 0.7 & -0.3 & 0.6 & 0.7 \\ -0.2 & -0.5 & 0 & 0.2 & -0.8 & 0.1 & 0.2 \\ -0.4 & -0.7 & -0.2 & 0 & -1 & -0.1 & 0 \\ 0.6 & 0.3 & 0.8 & 1 & 0 & 0.9 & 1 \\ -0.3 & -0.6 & -0.1 & 0.1 & -0.9 & 0 & 0.1 \\ -r & -0.7 & -0.2 & 0 & -1 & -0.1 & 0 \end{pmatrix}$$

$$T8(r)=\begin{pmatrix} 0 & 0.4 & -0.5 & 0.3 & -0.6 & -0.3 & 0.4 & r \\ -0.4 & 0 & -0.9 & -0.1 & -1 & -0.7 & 0 & -0.6 \\ 0.5 & 0.9 & 0 & 0.8 & -0.1 & 0.2 & 0.9 & 0.3 \\ -0.3 & 0.1 & -0.8 & 0 & -0.9 & -0.6 & 0.1 & -0.5 \\ 0.6 & 1 & 0.1 & 0.9 & 0 & 0.3 & 1 & 0.4 \\ 0.3 & 0.7 & -0.2 & 0.6 & -0.3 & 0 & 0.7 & 0.1 \\ -0.4 & 0 & -0.9 & -0.1 & -1 & -0.7 & 0 & -0.6 \\ -r & 0.6 & -0.3 & 0.5 & -0.4 & -0.1 & 0.6 & 0 \end{pmatrix}$$

$$T9(r)=\begin{pmatrix} 0 & 0.2 & -0.4 & -0.5 & -0.6 & -0.3 & 0.4 & 0.1 & r \\ -0.2 & 0 & -0.6 & -0.7 & -0.8 & -0.5 & 0.2 & -0.1 & 0.1 \\ 0.4 & 0.6 & 0 & -0.1 & -0.2 & 0.1 & 0.8 & 0.5 & 0.7 \\ 0.5 & 0.7 & 0.1 & 0 & -0.1 & 0.2 & 0.9 & 0.6 & 0.8 \\ 0.6 & 0.8 & 0.2 & 0.1 & 0 & 0.3 & 1 & 0.7 & 0.9 \\ 0.3 & 0.5 & -0.1 & -0.2 & -0.3 & 0 & 0.7 & 0.4 & 0.6 \\ -0.4 & -0.2 & -0.8 & -0.9 & -1 & -0.7 & 0 & -0.3 & -0.1 \\ -0.1 & 0.1 & -0.5 & -0.6 & -0.7 & -0.4 & 0.3 & 0 & 0.2 \\ -r & -0.1 & -0.7 & -0.8 & -0.9 & -0.6 & 0.1 & -0.2 & 0 \end{pmatrix},$$

$$T10(r)=\begin{pmatrix} 0 & 0.2 & -0.4 & -0.5 & -0.6 & -0.3 & 0.4 & 0.1 & 0.3 & r \\ -0.2 & 0 & -0.6 & -0.7 & -0.8 & -0.5 & 0.2 & -0.1 & 0.1 & -0.3 \\ 0.4 & 0.6 & 0 & -0.1 & -0.2 & 0.1 & 0.8 & 0.5 & 0.7 & 0.3 \\ 0.5 & 0.7 & 0.1 & 0 & -0.1 & 0.2 & 0.9 & 0.6 & 0.8 & 0.4 \\ 0.6 & 0.8 & 0.2 & 0.1 & 0 & 0.3 & 1 & 0.7 & 0.9 & 0.5 \\ 0.3 & 0.5 & -0.1 & -0.2 & -0.3 & 0 & 0.7 & 0.4 & 0.6 & 0.2 \\ -0.4 & -0.2 & -0.8 & -0.9 & -1 & -0.7 & 0 & -0.3 & -0.1 & -0.5 \\ -0.1 & 0.1 & -0.5 & -0.6 & -0.7 & -0.4 & 0.3 & 0 & 0.2 & -0.2 \\ -0.3 & -0.1 & -0.7 & -0.8 & -0.9 & -0.6 & 0.1 & -0.2 & 0 & -0.4 \\ -r & 0.3 & -0.3 & -0.4 & -0.5 & -0.2 & 0.5 & 0.2 & 0.4 & 0 \end{pmatrix}.$$

The results are illustrated Appendix II and the essential results are further plotted in figures 5.8- 5.12, in which each figure includes eight sub-figures representing for the eight template matrices respectively. The impacts are as follows:

Fig. 5.8 shows the results of the RMPWSV of the template matrices. Fig. 5.9 shows the results of the mean contradiction of the template matrices. It indicates that $PMPWSV\left(RAU\right) \geq PMPWSV\left(LPS\right)$, especially $MC \neq 0$. Fig. 5.10 shows the results of the RMSV of the template matrices. It indicates that $RMSV\left(RAU\right) \leq RMSV\left(LPS\right)$, and also the fact that $RMSV\left(RAU\right) = RMSV\left(LPS\right)$ does not necessarily follow $PMPWSV\left(RAU\right) = PMPWSV\left(LPS\right)$. This issue is due to the existence of the contraction.

Fig. 5.11 shows the results of the $WADV$ of the eight template matrices. It indicates that the least of $PMPWSV$ or $RMSV$ between RAU and LPS does not follow the least of $WADV$ between RAU and LPS. Fig. 5.12 shows MPWADV of the template matrices. Theoretically, if $PMPWSV\left(RAU\right) \geq PMPWSV\left(LPS\right)$, then $MPWADV\left(RAU\right) \geq MPWADV\left(LPS\right)$, as both $PMPWSV$ and $MPWADV$ show the distances with penalties, and LPS minimizes the distance with penalty . However, sometimes $MPWADV\left(RAU\right) < MPWADV\left(LPS\right)$ happens, due to the rounding error of an individual utility which produces this abnormality. The reason is also applicable in the minor case $RMPWSV\left(RAU\right) < RMPWSV\left(LPS\right)$, shown in Fig. 5.8. However, this abnormal situation is not shown in the dash frames of figures 5.8 and 5.12.

If $MC = 0$, then

$PMPWSV\left(RAU\right) = PMPWSV\left(LPS\right)$,

$WADV(RAU) = WADV(LPS)$,

$MPWADV(RAU) = MPWADV(LPS)$.

The main reason is that each penalty weight of LPS is equal to one if $MC = 0$. In this case, LPS is PLS, which produces the same result of RAU.

If $AI \leq 0.1$, then

$PMPWSV(RAU) \cong PMPWSV(LPS)$,

$WADV(RAU) \cong WADV(LPS)$,

$RMSV(RAU) \cong RMSV(LPS)$, and $MPWADV(RAU) \cong MPWADV(LPS)$.

They are framed with dash lines in the figures.

On the basis of the above findings, the best practice for choosing the cognitive prioritization operators is as follows.

If $AI \leq 0.1$, especially $MC = 0$, RAU is recommended. For one thing, interestingly, it produces the same result as AMSLS and PLS. If no out-boundary problem exists, RAU also produces the same result as BLS and LPS. For another, its computational effort is the least. Therefore, when a pairwise opposite matrix is perfect accordant, or satisfactory without violation, RAU is more preferable.

If $MC \neq 0$, LPS is suggested . PLS is the basic form for developing BLS, which is further developed as LPS (or DLS). In view of the approximate accuracy of the discordant matrix with contradiction, LPS (or DLS) is more preferable as it minimizes the summation of the multiples of contradiction and distance errors.

If $AI \leq 0.1$ and $MC \neq 0$, and only the rank of the single matrix is considered, then RAU is suggested. If the individual utility values are significant, LPS is suggested.

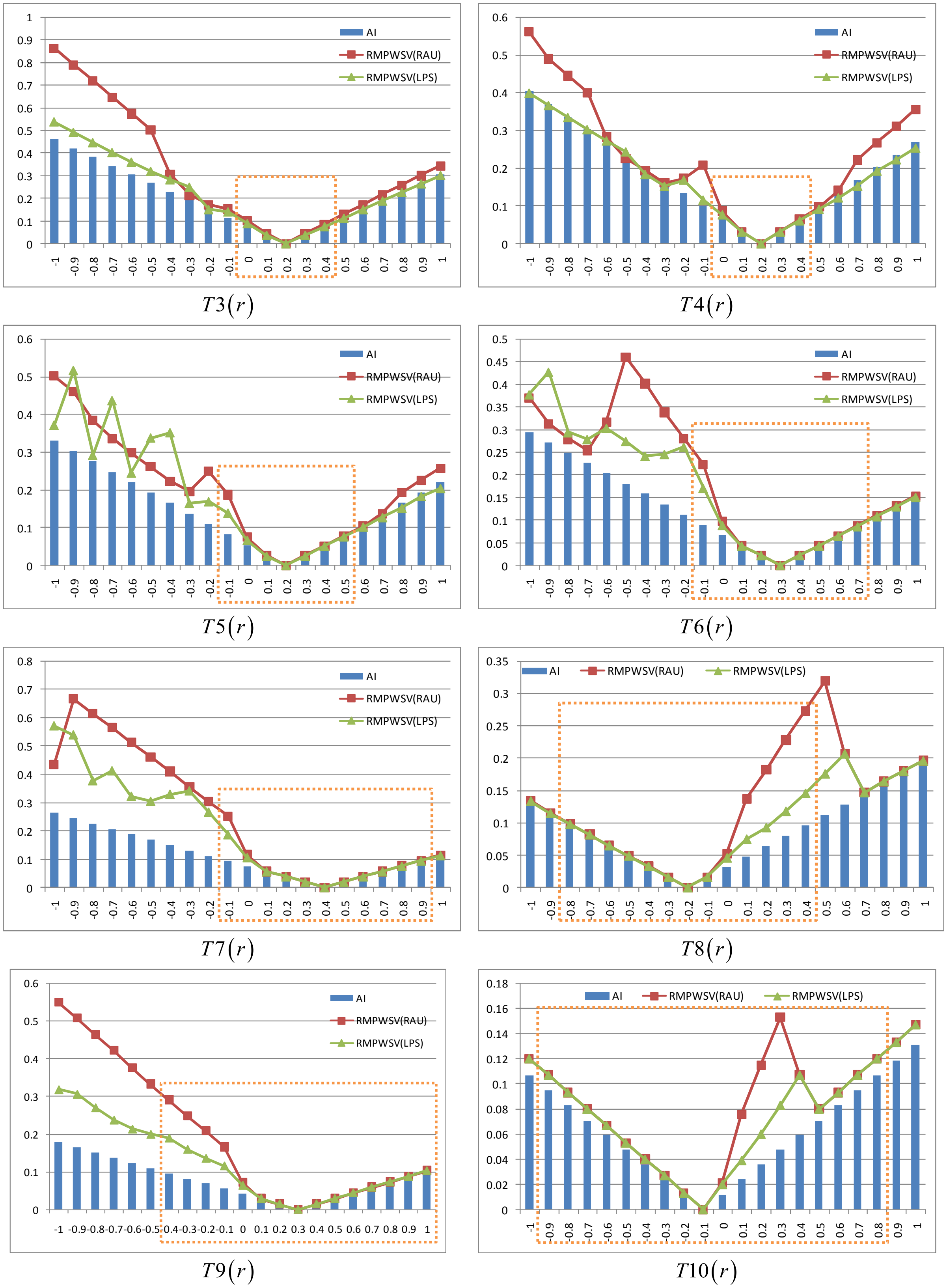


Figure 5.8: RMPWSV of the CPOs of the template matrices

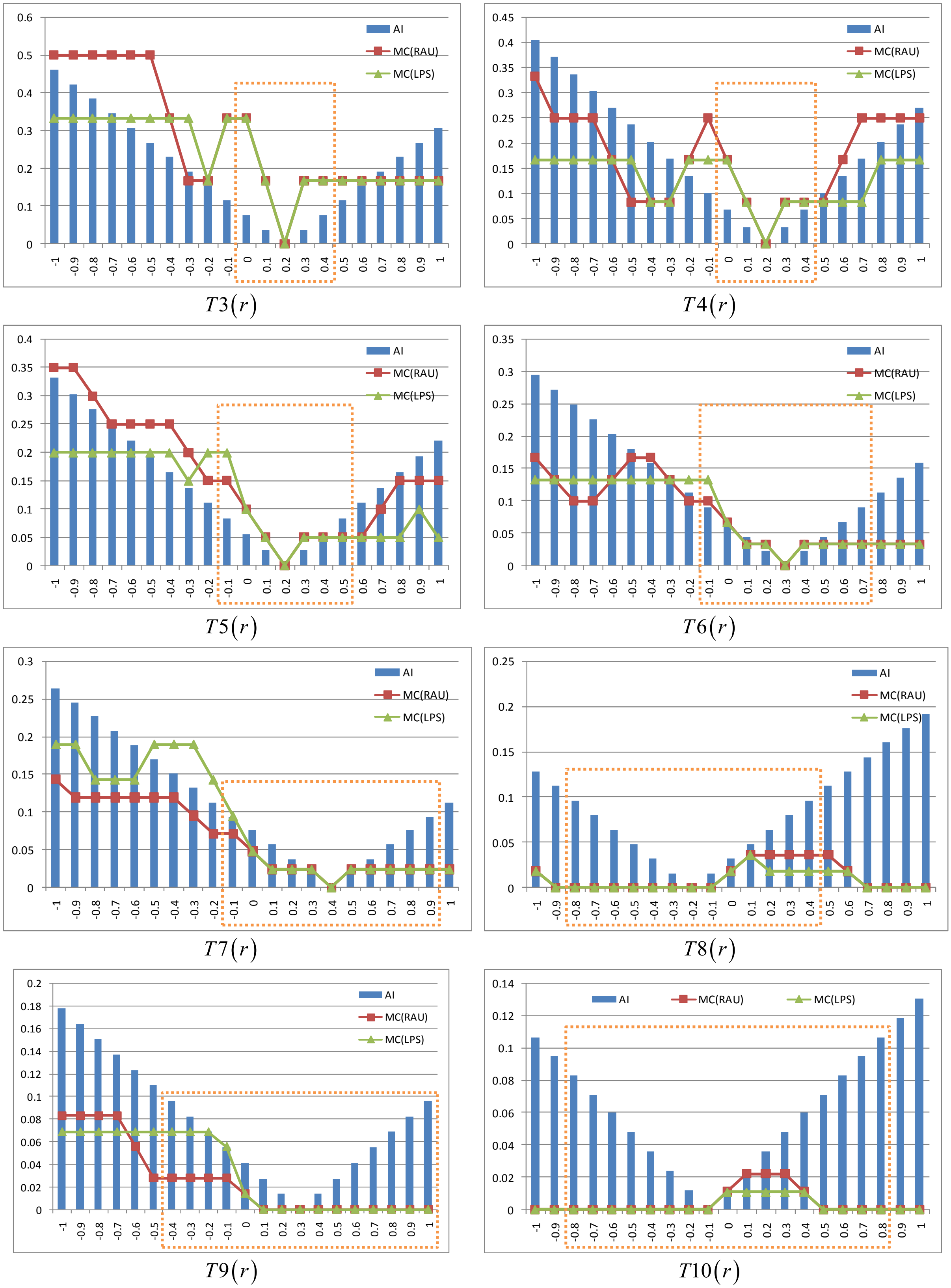


Figure 5.9: Mean contradiction index of the CPOs of template matrices

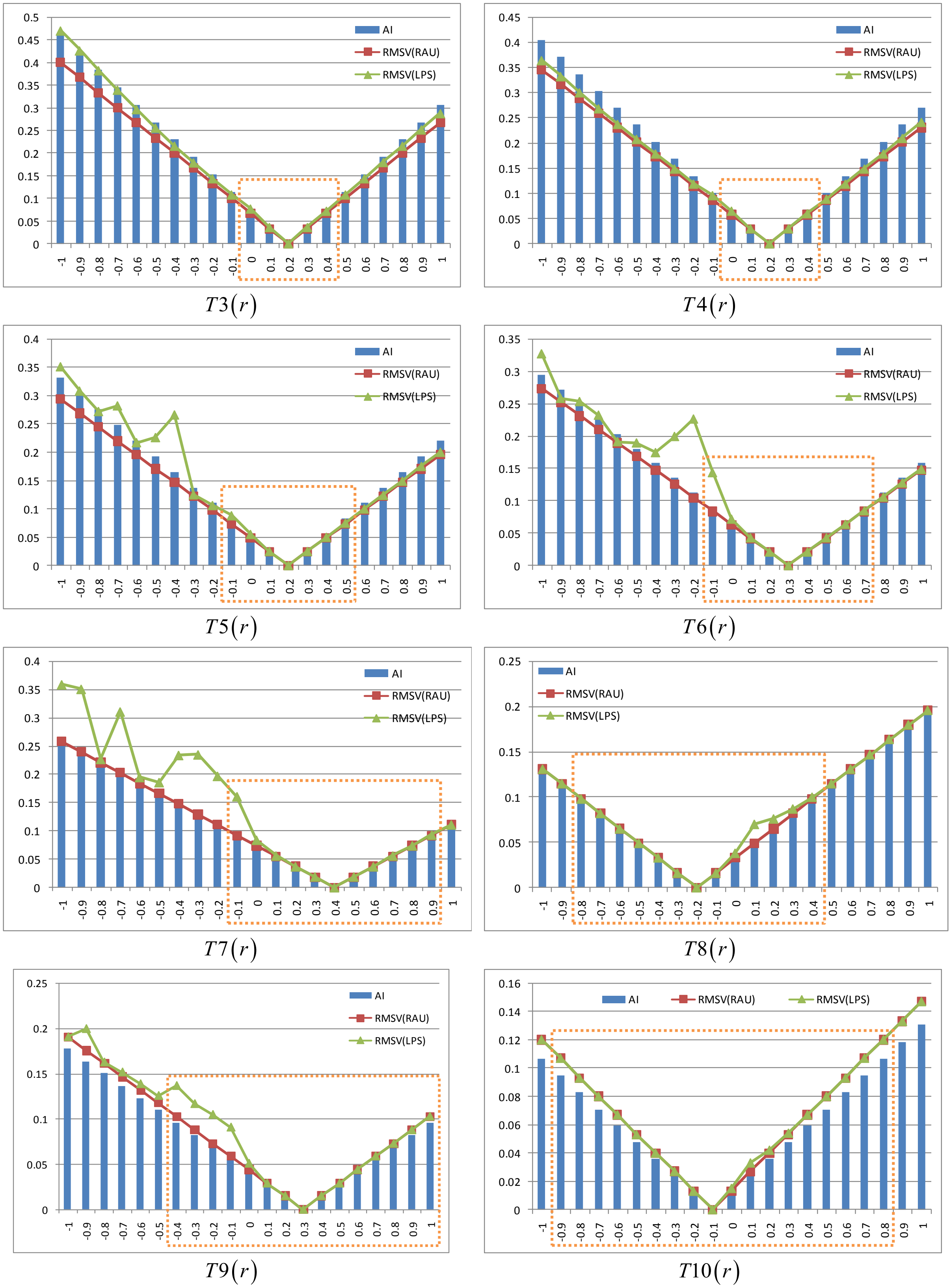


Figure 5.10: RMSV of the CPOs of template matrices

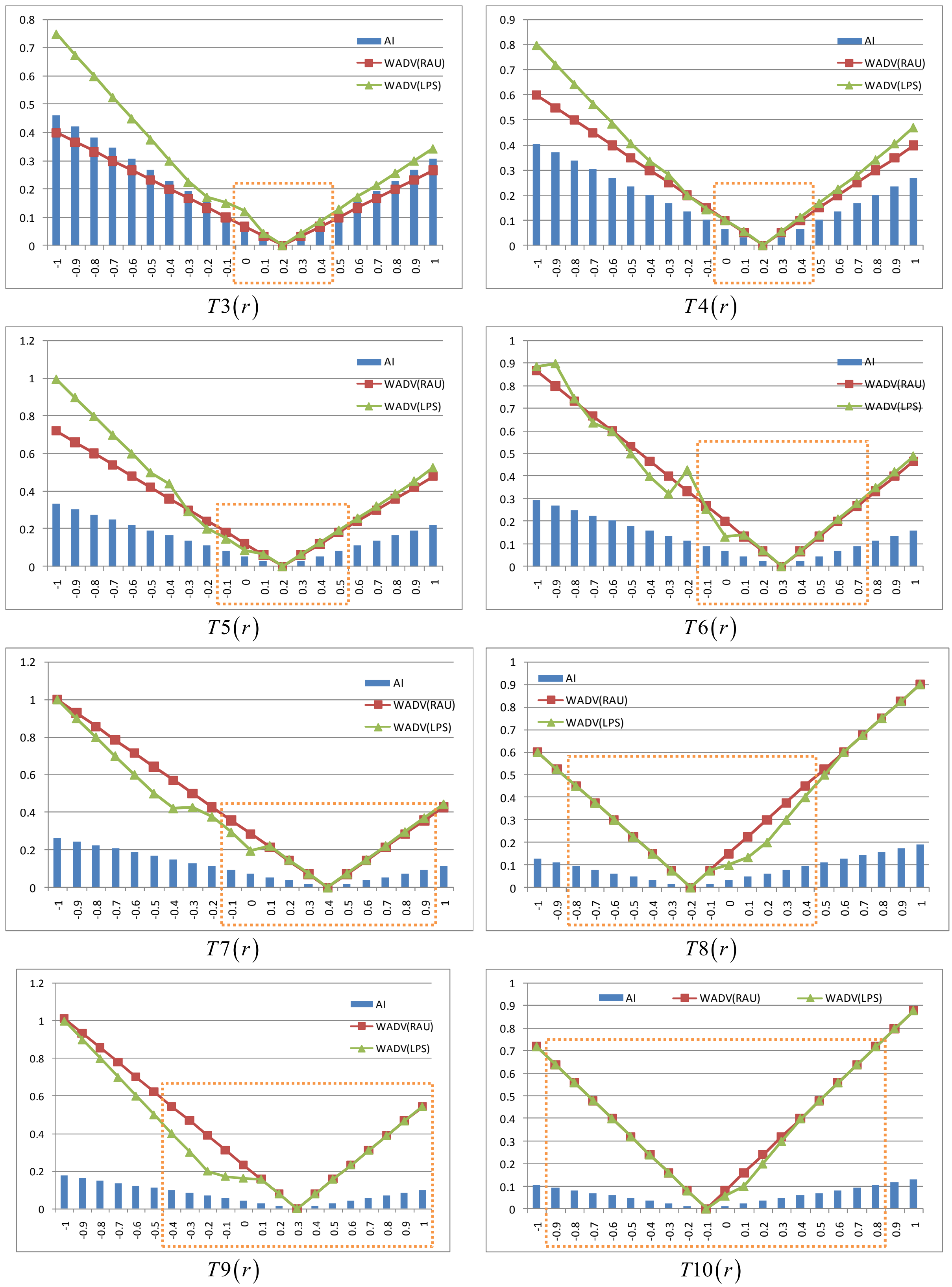


Figure 5.11: WADV of the CPOs of template matrices

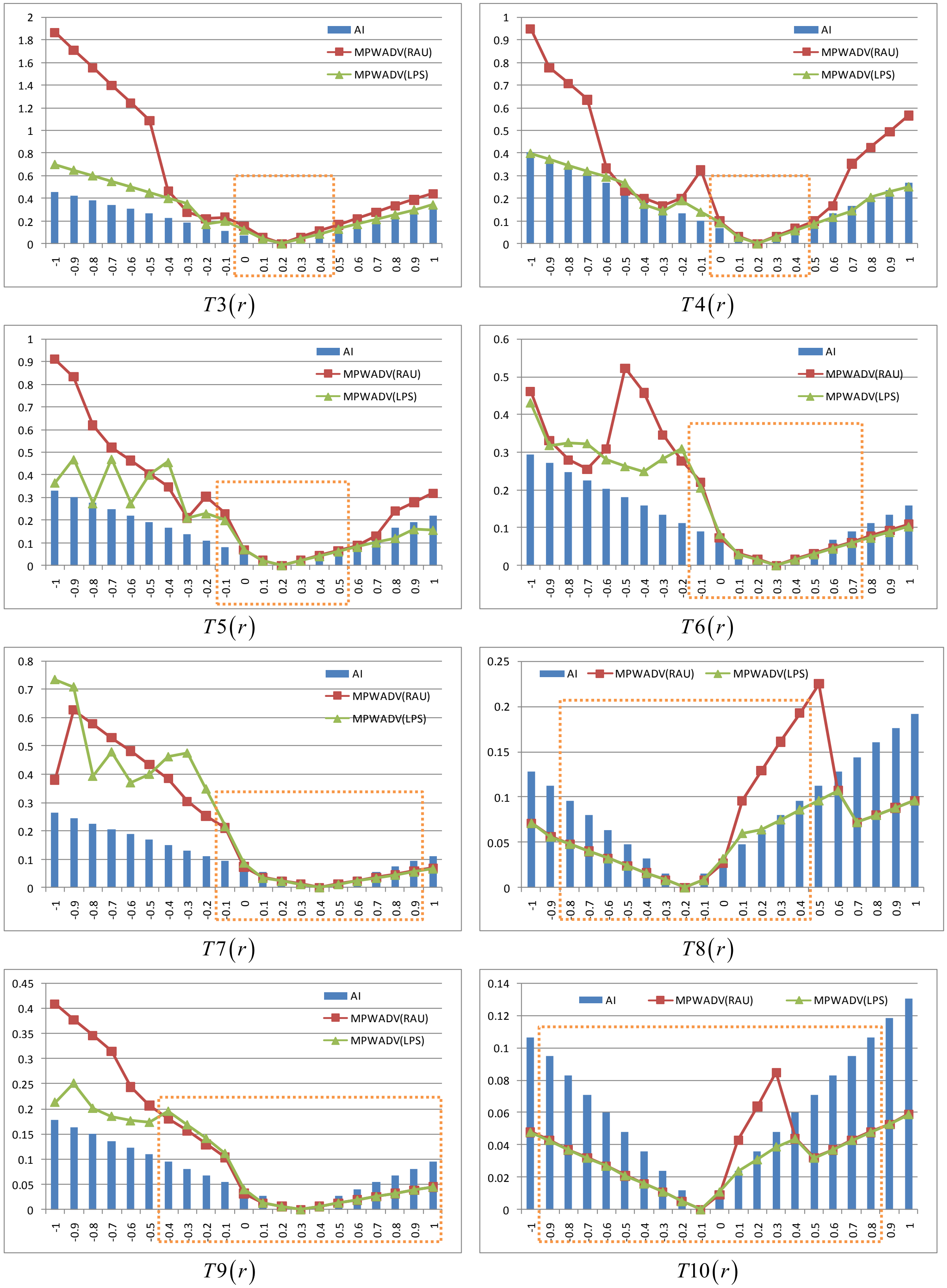


Figure 5.12: MPWADV of the CPOs of template matrices

5.8.2 Comparison with pairwise reciprocal matrix

To fairly compare the pairwise reciprocal matrix (PRM) and the pairwise opposite matrix (POM), PRM is perfectly consistent whilst POM is perfectly accordant. The rating scale schemas of POM and PRM are defined in table 5.4.

Table 5.4: Match references between ratio and interval scales

ℵ	iEx	iVVS	iVS	iSP	iS	iMP	iM	iW	E	W	M	MP	S	SP	VS	VVS	Ex
$\overline{X}(\kappa=1)$	-1	-7/8	-6/8	-5/8	-4/8	-3/8	-2/8	-1/8	0	1/8	2/8	3/8	4/8	5/8	6/8	7/8	1
$\overline{X}(\kappa=1/3)$	-1/3	-7/24	-6/24	-5/24	-4/24	-3/24	-2/24	-1/24	0	1/24	2/24	3/14	4/24	5/24	6/24	7/24	1/3
$\overline{X}'$	1/9	1/8	1/7	1/6	1/5	1/4	1/3	1/2	1	2	3	4	5	6	7	8	9

In PRM, summation of the priority vector $W=\{w_1,\ldots,w_n\}$ is equal to one, i.e. $\sum_{i=1}^{n} w_i = 1$. W is said to be a normalized priority vector (or a priority vector in short). In order for comparison, the individual utility from POM is rescaled (or normalized) as a normalized priority vector by the rescale function of the normalization function, and has the following form:

$$W=\left\{w_i : w_i=\frac{v_i}{n\kappa}, \forall i\in\{1,\ldots,n\}\right\}, \text{ which } \sum_{i\in\{1,\ldots,n\}} v_i = n\kappa \tag{5.39}$$

For the comparisons, four issues are discussed as follows.

a) Dependence issues

The accordant POM does not match the consistent PRM by directly switching the numerical reference values in table 5.4, and vice versa. The main reason is the axioms

$a_{ij} \cong w_i / w_j$ and $b_{ij} \cong v_i - v_j$. To explain further, consider the linguistic representation of an accordant POM ,

$$LM_1 = \begin{pmatrix} E & W & SP \\ iW & E & S \\ iSP & iS & E \end{pmatrix}$$, which is numerically represented by

$$POM_1 = \begin{pmatrix} 0 & 1/8 & 5/8 \\ -1/8 & 0 & 4/8 \\ -5/8 & -4/8 & 0 \end{pmatrix}.$$

The individual utility vector is $V_1 = \{1.25, 1.042, 0.708\}$, and the priority vector is $W'_1 = \{0.417, 0.375, 0.208\}$.

Regarding PRM, the linguistic matrix is numerically represented by

$$PRM_1 = \begin{pmatrix} 1 & 2 & 6 \\ 1/2 & 1 & 5 \\ 1/6 & 1/5 & 1 \end{pmatrix},$$

and its priority vector is $W_1 = \{0.577, 0.342, 0.081\}$.

To preserve the consistency, it is changed to

$$PRM_2 = \begin{pmatrix} 1 & 2 & 6 \\ 1/2 & 1 & 3 \\ 1/6 & 1/3 & 1 \end{pmatrix},$$

and its priority vector is $W_2 = \{0.6, 0.3, 0.1\}$.

Converting PRM_2 to the linguistic matrix

$$LM_2 = \begin{pmatrix} E & W & SP \\ iW & E & M \\ iSP & iM & E \end{pmatrix}$$

, which maps to POM numerically, follows

$$POM_2 = \begin{pmatrix} 0 & 1/8 & 5/8 \\ -1/8 & 0 & 2/8 \\ -5/8 & -2/8 & 0 \end{pmatrix}$$

, and its priority vector is $W'_2 = \{0.417, 0.347, 0.236\}$.

It can be observed that for the same linguistic matrix, both PRM and POM have different results. To compare with W'_1 and W_1, as well as W'_2 and W_2, the distance between the least and the highest priorities of PRMs are larger than the one of the POMs.

It also can be observed that the change of one entry of a single element of POM does not influence other unrelated elements. In this case, only b_{23} of the POMs are changed, w'_1 is kept to 0.417 and only w'_2 and w'_3 are changed accordantly. This issue can be proved by the RAU formula.

$$v_i = \left(\frac{1}{3} \sum_{j=1}^{3} b_{ij} \right) + \kappa, \forall i \in \{1, \ldots, 3\}$$

Substitute the values into the above formula. Change of b_{23} means change of b_{32} as they have the opposite relationship. Thus v_1 is unchanged, and v_2 and v_3 are changed due to $b_{23} \cong v_2 - v_3$. As the population utility is constant, thus only w'_2 and w'_3 are changed accordantly.

However, PRM does otherwise. Change of any one element will finally influence the population of the priorities, which is less reasonable as a subjective issue, especially in the PRM with little inconsistency.

Thus the cognition representation of PRM is problematic and produces exaggerate results, which are also mentioned in chapter 5.2.

b) Out-boundary problem

It is unavoidable that both PRM and POM have boundary problems. Regarding POM, for all $k \in (1,\ldots,n)$, if $b_{ik} + b_{kj} > Max(\overline{X})$ or $b_{ik} + b_{kj} < Min(\overline{X})$, then $b_{ik} + b_{kj} \neq b_{ij}$ (proposition 5.4). Regarding PRM, for all $k \in (1,\ldots,n)$, if $a_{ik} \bullet a_{kj} > Max(\overline{X}')$ or $a_{ik} \bullet a_{kj} < Min(\overline{X}')$, then $a_{ik} \bullet a_{kj} \neq a_{ij}$. In other words, POM is not accordant, and PRM is not consistent, due to a lack of suitable linguistic scales in the out-boundary cases.

However, the range of the out-boundary of PRM is much higher than POM. If $a_{ik} = a_{kj} = 9$, then $a_{ik} \bullet a_{kj} = 81$ and the reciprocal is $1/81$. The range of the out-boundary is $81/9 = 9$ times that of $Max(\overline{X}')$. However, in POM, $b_{ik} = b_{kj} = 1$, then $b_{ik} + b_{kj} = 2$. The range of the out-boundary is about $2/1 = 2$ times that of $Max(\overline{X})$ no matter what the values of κ are.

To conclude, although POM has an out-boundary problem, it is much more trivial than PRM.

c) Scale capability

Assume both POM and PRM are within-boundary, and regarding PRM, if $a_{ik}=2$ and $a_{kj}=\frac{1}{3}$, no item in the rating scales satisfies $a_{ik}\bullet a_{kj}=a_{ij}$, i.e. $\left(a_{ik}\bullet a_{kj}=2/3\right)\notin\left(\overline{X}'\right)$ where $a_{ij}\in\aleph'$. However, this issue never happens in POM as $Min\left(\overline{X}\right)\le\left(b_{ik}+b_{kj}\right)\le Max\left(\overline{X}\right)$ and $\left(b_{ik}+b_{kj}\right)\in\overline{X}$, which are proved in proposition 5.4. In this within-boundary situation, it is concluded that POM associated with interval scales can guarantee accordant comparison with the sufficient scale capability, but PRM do as otherwise. (Of course, the logic problem inducing the discordant comparison is another story.)

d) Representation Comparisons

Consider a simple case in which the real utility is $V=\{1.1,1,0.9\}$, which $n\kappa=\sum_{v_i\in V}v_i=3$, and thus $W=\{0.3667,0.3333,0.3\}$. The consistent PRM and accordant POM are

$$PRM_3=\begin{pmatrix}1 & 1.1 & 1.222\\ 0.9091 & 1 & 1.111\\ 0.8181 & 0.9 & 1\end{pmatrix}\text{ and }POM_3=\begin{pmatrix}0 & 0.1 & 0.2\\ -0.1 & 0 & 0.1\\ -0.2 & -0.1 & 0\end{pmatrix}\text{ respectively.}$$

The judgment matrices using linguistic scales should be

$$LM_3=\begin{pmatrix}E & E & E\\ E & E & E\\ E & E & E\end{pmatrix}\text{ and }LM_4=\begin{pmatrix}E & W & M\\ iW & E & W\\ iM & iW & E\end{pmatrix}\text{ respectively.}$$

Their numerical presentation for the judgment matrices should be

$$PRM_4 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \text{ and } POM_4 = \begin{pmatrix} 0 & 1/8 & 2/8 \\ -1/8 & 0 & 1/8 \\ -2/8 & -1/8 & 0 \end{pmatrix} \text{ respectively.}$$

Then the priority vector of PRM_4 is $W(PRM_4) = (0.3333, 0.3333, 0.3333)$ and its root mean square error (rmse) is 0.0272. The individual utility and the priority vectors of POM are $V(POM_4) = (1.125, 1, 0.875)$ and $W(POM_4) = (0.375, 0.3333, 0.2917)$. Its root mean square error is 0.0064 .

The alternative approach is to calculate the population utility equal to one, i.e. $n\kappa = \sum_{v_i \in V} v_i = 1$. Then the normal utility is κ = 1/3 for a 3x3 matrix. According to table 5.4, the new scale values of the linguistic judgment matrix should be:

$$POM_5 = \begin{pmatrix} 0 & 1/24 & 2/24 \\ -1/24 & 0 & 1/24 \\ -2/24 & -1/24 & 0 \end{pmatrix}$$

In this case, the individual utility and priority vectors have the same result $(0.375, 0.3333, 0.2917)$.

In this case, $w_1 < w_2 < w_3$ is the real rank. POM produces a rank exactly the same as the real rank. However, PRM produces equal rank, $w_1 = w_2 = w_3$. The approximate error (root mean square error) of PRM is 4 times more than POM (0.0272/0.0064). It can be concluded that representation of the ratio scales of the PRM is not appropriate as it produces a misleading rank and higher errors. This often happens when the differences among priorities are small (e.g. 0.05), especially the high dimensions of matrices (e.g. *n*>9). POM is more appropriate for handling small difference comparisons.

e) Zero utility/priority

Consider the case that one of the priorities is equal to zero, i.e. $W = \{0, 0.2, 0.8\}$. However, PRM does not produce zero priority since the PRM cannot be constructed due to $w_i/0 \to \infty$.

Regarding POM, the individual utility can produce a zero value if $\left(b_{ik} + b_{kj}\right) = \kappa$, negative numbers if $\left(b_{ik} + b_{kj}\right) > \kappa$ and positive numbers if $\left(b_{ik} + b_{kj}\right) < \kappa$. It is subject to the settings of the normal utility κ, in which the default setting is the maximum value of the interval scale schema, i.e. $Max\left(\overline{X}\right)$, and the scale range, in which the default setting is [-$Max\left(\overline{X}\right), Max\left(\overline{X}\right)$].

Thus the POM is not subject to comparing the positive values. The priority vector is one of the special cases of the conversion of the utility vector.

5.9 Summary and remarks

It is questionable that the cognitive comparison of two objects can be represented by their ratio as humans do not tend to calculate multiplication or division by subjective measurement. This research proposes a straightforward novel approach where the cognitive comparison of two objects is represented by the difference between them. The main reason is that the perception of the linguistic terms should be a difference concept rather than a ratio concept, as in our natural language.

The cognitive comparison can be represented by the pairwise opposite matrix filled by verbal judgment represented by numerical values from decision makers. The cognitive prioritization operator is the function to derive the individual utility vector from the POM. Although it is not necessary, the priority vector is the normalization of the individual utility vector. The priority vector can serve as weights for some decision models in which the weights of the criteria are not determined.

This study proposes five CPOs: Row Average plus normal Utility (RAU) , Aggregation of Solutions of Linear Systems (ASLS) which includes Arithmetic Mean of Solutions of Linear Systems (AMSLS), Primitive Least Squares (PLS) optimization, Bounded Least Squares (BLS) Optimization and Least Penalty Squares (LPS) Optimization. The closed form solution of AMSLS and PLS is RAU. BLS deals with the case which converts the negative utility into a non-negative one without change of the normal utility value and interval scale definition. LPS is suggested when the accordant index (AI) is less than one and the mean contradiction (MC) is not equal to zero. If MC is equal to zero and AI is less than 0.1, RAU should be used due to the computational effort of RAU being much less than the LPS. For the modern powerful computer, the computational effort becomes trivial. Thus LPS is recommended for CPO default setting.

Six Cognitive Prioritization Operator Measurement Models are proposed: Worst Absolute Distance Variance (WADV), Mean Absolute Distance Variance (MADV), Mean Penalty Weighted Absolute Distance Variance (MPWADV), Root Mean Square Variance (RMSV), Mean Contradiction (MC) and Root Mean Penalty Weighted Square Variance (PMPWSV). The PMPWSV, which combines the advantages and offsets the disadvantages of the improved MC and RMSV, is selected as the Cognitive Distortion Index (CDI) since its value is more appropriate in reflecting the error between *B* and *V*.

The graphical representation for a 3x3 pairwise matrix shows the advantages of RMPWSV over RMSV, as RMSV ignores the contradiction errors. In the 2D graph, three linear equations from the discordant pairwise opposite matrix can formulate the 3 lines and form the Feasible Solution Region. All values of the POs are located in this region. In the 3D graph, adding the RMPWSV as an extra dimension can formulate the Most Feasible Solution Region (MFSR) within the FSR. The solution point of LPS is always located in the lowest point in the MFSR (Example 5.2).

In numerical analyses, eight template matrices from 3x3 to 10x10 respectively formulate 168 cases to show the validity of RAU and LPS. The numerical result suggests the best practice of the CPOs. Various comparisons demonstrate how the proposed pairwise opposite matrix is superior to Saaty's pairwise reciprocal matrix.

The impact of PRM and its CPOs stresses the high motivation for various aspects. They also can be used for the parametric inputs of decision models in which the weights of the criteria are not determined. In addition, as they can also deal with the problems of selection, sorting, and ranking, they may be used in many domains such as such as material sciences, transportation sciences, psychometrics, social sciences, business research, decision sciences, computer sciences and engineering management.

Chapter 6 Cognitive Style and Aggregation Operator

6.1 Introduction

There are many aggregation operators, AOs, as seen in the literature review (chapter 2.3.5). Each aggregation operator can be regarded as an individual, however they produce different results, so they have individual differences. Thus it is possible to use cognitive style, which is a study of individual information processing (chapter 2.7.4), to describe the style of the aggregation operator. A style is considered to be a fairly fixed characteristic of an individual (Rading and Cheema, 1991).

Although the discussions of AOs are very broad, there is a lack of research for the best practice in choosing aggregation operators. The selection of the AOs can make use of the theory of cognitive style. However, no research has been found to investigate the relationship between aggregation operators and the cognitive styles. Cognitive styles can be used to select the best individual for the decision making.

Most researchers narrowed a set of uni-dimensional labels for research into cognitive style. As there is no universal rule applying which labels the cognitive style (chapter 2.7.4), in this research the cognitive style is a construct measured by a variable decision attitude which includes three basic members: pessimistic, neural, and optimistic. The proposed Cognitive Style and Aggregation Operator (CSAO) is extensively revised from (yuen, 2009d).

The remainder of this chapter is organized as follows. Chapter 6.2 defines the properties of aggregation operators. Chapter 6.3 proposes a CSAO I model, which is an algorithm. Chapter 6.4 proposes a CSAO I model which uses the compound linguistic ordinal scale. The numerical analyses are performed and discussed in Chapter 6.5. The conclusion is drawn in chapter 6.6.

6.2 Fundamental definitions of aggregation operators

There is much research on the techniques of aggregation operators, and details are in literature review (Chapter 2.7.4). The formal definitions of aggregation operators are as follows.

Definition 6.1: A generic aggregation operator *Agg* is a function which aggregates a set of granules $X=(x_1,\ldots,x_i,\ldots,x_n)$ into an Aggregated Value *y*. It has the form:

$$y = Agg_{(n)}^{(t)}\left(\alpha;\left(x_1,\ldots,x_i,\ldots,x_n\right)\right) = Agg_{(n)}^{(t)}\left(\alpha;X\right) \tag{6.1}$$

t is the length of tuple(s) of x_i and *n* is the number of the granules. α is a construct parameter or a bag of construct parameters to scale *Agg*.

Sometimes, α is not shown if the information of α is not important for discussion in some scenarios. Likewise, AO can be simplified as the notations such as

Agg, $Agg(\alpha;X)$, $Agg^{(t)}(\alpha;X)$ or $Agg^{(t)}_{(n)}(\alpha;X)$. This research is only interested in $t\in\{1,2\}$. To extend definition 6.1, the following definition is proposed.

Definition 6.2: Agg is a non-weighted AO such that $x_i = c_i$ where c_i is a single element, or 1-tuple, and $c_i \in C$. Thus,

$$Agg^{(1)}(\alpha;X) = Agg^{(1)}(\alpha;(c_1,\ldots,c_i,\ldots,c_n)) = Agg^{(1)}(\alpha;C) \qquad (6.2)$$

Definition 6.3: A is a weighted AO such that $x_i = \{c_i, v_i\}$ where $v_i \in V = (v_1,\ldots,v_n)$ is a utility weight. Thus x_i is a pair (or 2-tuple). The weighted AO is of the form:

$$Agg^{(2)}(\alpha;X) = Agg^{(2)}(\alpha;(\{c_1,v_1\},\ldots,\{c_i,v_i\},\ldots,\{c_n,v_n\})) \qquad (6.3)$$

Definition 6.4: If $w_i = \frac{v_i}{\sum_{i=1}^{n} v_i}$, then $w_i \in W = (w_1,\ldots,w_n)$ is the probability weight such that $\sum_{i\in\{1,\ldots,n\}} w_i = 1$. Thus A is a normalized weighted AO of the form:

$$Agg^{(2)}(\alpha;X) = Agg^{(2)}(\alpha;(\{c_1,w_1\},\ldots,\{c_i,w_i\},\ldots,\{c_n,w_n\})) \qquad (6.4)$$

This chapter focuses on the discussion of the normalized weighted AO.

Let y be the output of the AO of X. Usually y and c_i have a fix interval $I' = [a,b] \subseteq [-\infty,\infty]$. Many studies used the fix interval $I = [0,1]$ for discussion. This is a only mathematical matter of scaling or normalizing the I' into I. To merge the discussion with other studies, and to associate membership theory to the aggregation problems (as the membership value also belongs to $[0,1]$), this research uses a fix interval $I=[0,1]$. The scaling functions of I' into I are beyond the research topic here. Now let X and y be scaled, and the extension of definition 6.2 is as follows.

Definition 6.5: Let $I=[0,1]$, $c_i, y \in I$. A non-weighted aggregation operator is the function $Agg: I^n \to I$. A weighted aggregation operator is the function $Agg: V^T \times I^n \to I$, and a normalized weighted aggregation operator is the function $Agg: W^T \times I^n \to I$.

The next section discusses the categories of the information fusion on the basis of the usability of aggregation operators.

6.3 Decision attitude and aggregation operator 1 (DAAO-1, or CSAO-1)

Under uncertainty, different decision makers would have different decision attitudes since they have characteristics of cognitive style or individual difference. The decision attitudes (DAs) can be described by a collection of linguistic terms represented by a collection of DA atomic fuzzy sets, $D=\{d_1,\cdots,d_j,\cdots,d_p\}$, (or the 1st degree DA fuzzy variable) which is further classified as a collection of compound fuzzy sets $HD = \{d_{ij} : i = 1,\ldots,p; j = 1,\ldots,r\}$ with added directional hedge fuzzy sets $H = \{h_1,\ldots,h_r\}$, (The 2nd degree DA fuzzy variable). The details of compound fuzzy variable are in chapter 4.

The range of the membership of a decision attitude fuzzy set is in [0,1] and the aggregated value also belongs to [0,1]. The aggregated value of the membership (or the likelihood) of a decision attitude has the relationship, shown in the following definition.

Definition 6.7: A aggregated value y from a normalized aggregation operator *Agg* of the set of input parameters X belongs to a decision attitude fuzzy set d_j, with the membership value $d_j(y) \in [0,1]$ by the membership function $d_j : y \to I$, $I = [0,1]$.

As the fuzzy set is characterized by this membership function, the same notation d_j is used for a fuzzy set of membership. Usually, the membership function applies a triangular function $\mu_j(a,b,c)$ which is defined by three points.

Different input parameter sets, X's, result in different Effective Aggregation Ranges (EAR) from a collection of the aggregation operators. The effective aggregation range $[y_*, y^*]$ is defined as follows.

Definition 6.8: Let the set of the aggregated values from the set $\widetilde{Agg}$ of the aggregation operators be $Y=(y_1,\dots,y_k,\dots,y_m)$. The permutation of Y is $\vec{Y}=\{y_{(1)},\dots,y_{(k)},\cdots,y_{(m)}\}$, where $y_{(1)} \le y_{(2)} \cdots \le y_{(m)}$. Thus, the Effective Aggregation Range is $[y_*, y^*]$, where $y_* = y_{(1)} = \min(Y)$ is the low-boundary, $y^* = y_{(m)} = \max(Y)$ is the up-boundary.

The above definition follows two lemmas.

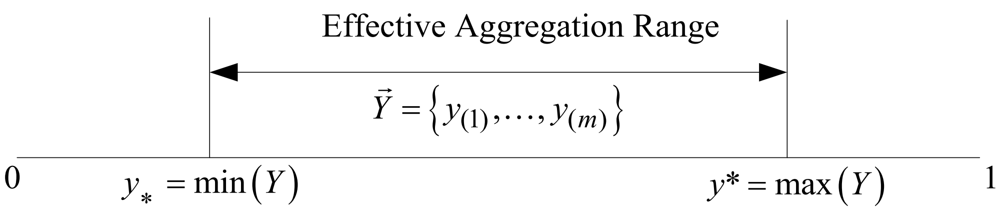


Figure 6.1: Effective Aggregation Range of AOs

Lemma 6.1: The EAR is the proper subset of *I*, *i.e.* $[y_*, y^*] \subseteq [0,1]$ (see fig. 6.1).

Proof:

As $y = A_{(n)}^{(t)}(\alpha; X) \in I = [0,1]$, $y_* = \min(Y) \ge 0$, and $y^* = \max(Y) \le 1$, the lemma holds. □

Lemma 6.2: The collection of AOs is the form $\widetilde{Agg}: X \to [y_*, y^*]^m$, where m is the dimension (the number) of the output set.

Proof:

This lemma is directly derived from definition 6.8. □

The CSAO model describes how the cognitive styles of the aggregation operators can be reflected by the decision attitudes. The CSAO can be represented by a collection of the DA fuzzy sets. Thus, following proposition holds.

Proposition 6.2 (D_{Agg}): The collection of decision attitude fuzzy sets for an aggregation operator A is $D_{Agg}=\left\{\left\{y,d_1\left(y\right)\right\},\ldots,\left\{y,d_j\left(y\right)\right\},\ldots,\left\{y,d_p\left(y\right)\right\}\right\}$, where $y\in\left[y_*,y^*\right]$.

Proof:

Let the collection of decision attitude fuzzy sets be $D=\left\{d_1,\cdots,d_j,\cdots,d_p\right\}$, and the discourse universal of D is the interval $\left[y_*,y^*\right]\subseteq[0,1]$ (lemma 6.1). Thus the collection of the memberships of the set of decision attitudes D for a aggregation operator is $D_A:\left[y_*,y^*\right]\to I^p$. As the fuzzy set is generally defined as a collection of pairs, the form is given above. □

Proposition 6.3 ($D_{\widetilde{Agg}}$): A collection of the 1st degree DA fuzzy sets $D_{\widetilde{Agg}}$ for a collection of aggregation operators $\widetilde{Agg}=\left(Agg_1,\ldots,Agg_k,\ldots,Agg_m\right)$ is of the form:

$$D_{\widetilde{Agg}}=\begin{pmatrix} \left\{\left\{y_{(1)},d_1\right\}\right. & ,\ldots, & \left\{y_{(1)},d_j\right\} & ,\ldots, & \left.\left\{y_{(1)},d_p\right\}\right\} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \left\{\left\{y_{(k)},d_1\right\}\right. & ,\ldots, & \left\{y_{(k)},d_j\right\} & ,\ldots, & \left.\left\{y_{(k)},d_p\right\}\right\} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \left\{\left\{y_{(m)},d_1\right\}\right. & ,\ldots, & \left\{y_{(m)},d_j\right\} & ,\ldots, & \left.\left\{y_{(m)},d_p\right\}\right\} \end{pmatrix} \tag{6.5}$$

, where $\left\{y_{(k)},d_j\right\}=\left\{y_{(k)},d_j\left(y_{(k)}\right)\right\},\forall k,\forall j$, and $y_{(1)}\le y_{(2)}\cdots\le y_{(m)}$.

Proof:

By using proposition 6.1 and lemma 6.2, then above proposition is formed. □

Definition of Information Fusion Process, proposed in Chapter 3, is recalled as follows:

Definition 6.9: The Information Fusion Process $IFP = \left(\bar{X}, Y, AO^*, \{AO\}, SAO\right)$ is the function to aggregate multiple sources of data granules $\bar{X}$ as a meaningful value Y to represent an object by selection of the most appropriate aggregation operator (AO*) among a set of the AO candidates $\widetilde{AO} = \{AO\}$, i.e. $SAO: \{AO\} \to AO^*$, and $AO^*: \bar{X} \to Y$.

The CSAO model is the ideal function for *SAO*. On the basis of the above definition, two definitions are proposed for the selection of AO in $D_{\widetilde{Agg}}$.

Definition 6.10: If an aggregation operator has more than one membership of DAs, the selection of DAs for the AO is of the form:

$$d^*(k) = ArgMax\left(\left\{\left\{y_{(k)}, d_1\right\}, \ldots, \left\{y_{(k)}, d_j\right\}, \ldots, \left\{y_{(k)}, d_p\right\}\right\}\right) \tag{6.6}$$

Definition 6.11: If a DA linguistic term includes more than one aggregation operator, the selection of AOs in a DA linguistic term is of the form:

$$d_j^* = ArgMax\left(\left\{\left\{y_{(1)},d_j\right\},\ldots,\left\{y_{(k)},d_j\right\},\ldots,\left\{y_{(m)},d_j\right\}\right\}\right) \tag{6.7}$$

The *DAAO-1*for CSAO is concluded in following algorithm.

Algorithm 6.1: $DAAO-1 = CSAO1\left(D,\widetilde{Agg},X\right)$

Input:

a. A collection of the membership functions of DA fuzzy sets

$D=\left\{d_1,\cdots,d_j,\cdots,d_p\right\}$;

b. A collection of AOs: $\widetilde{Agg}=\left(Agg_1,\ldots,Agg_k,\ldots,Agg_m\right)$;

c. A collection of information granules: $X=\left(x_1,\ldots,x_i,\ldots,x_n\right)$;

Process:

Step 1. Compute $\widetilde{Agg}\left(X\right)$, and then $Y=\left(y_1,\ldots,y_k,\ldots,y_m\right)$ is achieved;

Step 2. Get the permutation of *Y*: $\vec{Y}=\left\{y_{(1)},\ldots,y_{(k)},\cdots,y_{(m)}\right\}$;

Step 3. Get $\left[y_*,y^*\right]=\left[y_{(1)},y_{(m)}\right]$;

Step 4. Calculate intervals and modal values for D by equally dividing $\left[y_*,y^*\right]$;

i. $d_1=\left(y_*,y_*,y_*+\dfrac{y^*-y_*}{p-1}\right)$

ii. $d_{j\neq 1,p}=\left(y_*+\dfrac{y^*-y_*}{p-1}(j-2),y_*+\dfrac{y^*-y_*}{p-1}(j-1),y_*+\dfrac{y^*-y_*}{p-1}(j)\right)$

iii. $d_p=\left(y^*-\dfrac{y^*-y_*}{p-1},y^*,y^*\right)$

Step 5. Elicit memberships for *D* by interpolation of the three points (a,b,c):

Step 6. Calculate $D(\vec{Y})$, $D_{\widetilde{Agg}}$ and $d^*(k)$, $\forall k$.

Step 7. Get $d_j^*, \forall j$.

Output:. $\{d_j^*\}$. **//END**

This study focuses on discussion of the weighted aggregation operators of which $x_i = \{w_i, c_i\} \in X$ is the input.

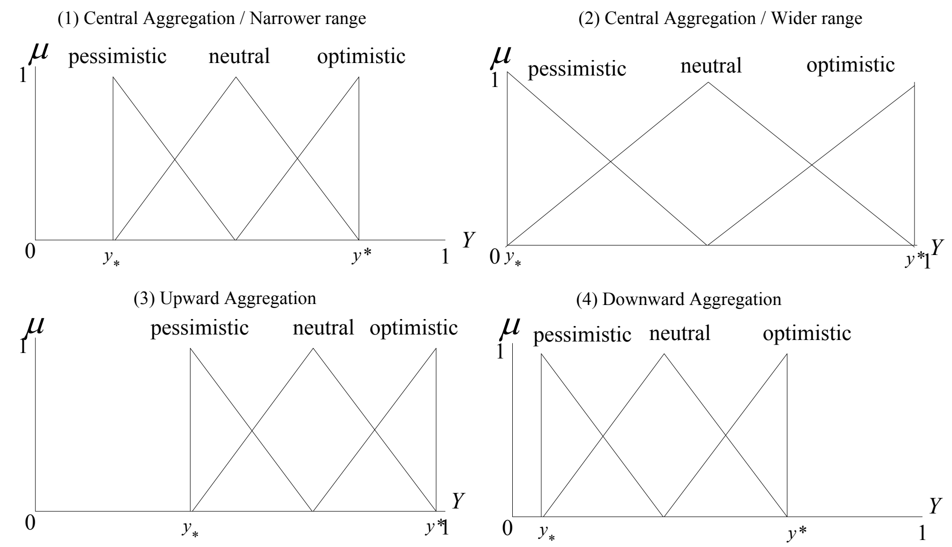


Figure 6.2: Properties of Effective Aggregation Range

To conclude, the CSAO description model is the function $g : X \to I$ or $g = \widetilde{Agg} \circ D = D(\widetilde{Agg}(X))$. It means that the function g maps the collection of information granules X with the set of the aggregators $\widetilde{Agg}$, to the membership interval [0,1] corresponding to the collection of decision attitude fuzzy sets D.

In most practice, the decision attitudes can be described by three linguistic terms: pessimistic, neutral and optimistic. Fig. 6.2 shows some properties of the DA fuzzy sets.

The properties of EAR can be summarized as followings.

Proposition 6.4: Let $y' = mean\left(y_*, y^*\right) = \frac{1}{2}\left(y_* + y^*\right)$, and then

1. Effective aggregation range (EAR) is of downward aggregation if $y' < 0.5$;.
2. EAR is of upward aggregation if $y' > 0.5$;
3. EAR is of central aggregation if $y' = 0.5$;
4. EAR 2 is more upward than EAR 1 if $y'_1 < y'_2$. Or EAR 1 is more downward than the EAR 2.
5. EAR 2 is wider than EAR 1 if $y^*_1 - y_{*_1} < y^*_2 - y_{*_2}$. Or EAR 1 is narrower than EAR 2.

Example 6.1

A numerical example analysis of the algorithm of the CSAO description model is illustrated as follows.

Input

a) Define the collection of decision attitude fuzzy sets:

Let $D=\{d_1,d_2,d_3\}$ represent the set of pessimistic, neural, and optimistic decision attitudes. $d_1=\mu\left(y_*,y_*,y'\right)$, $d_2=\mu\left(y_*,y',y^*\right)$, $d_2=\mu\left(y',y',y^*\right)$, where μ is the triangular membership function.

b) Define a collection of the Aggregation Operators:

$$\widetilde{Agg}=\left(Agg_1,\ldots,Agg_k,\ldots,Agg_{17}\right)=\left\{\begin{array}{l} wrp, whm, wgm, wam, owa, \mathrm{owmax}, \mathrm{owmin}, \\ \mathrm{Lexmin}, wgo, wmed, wmed_l, wmed_{mm}, \\ wmed_{dp}, wmed_y, wmed_f, wmed_{ws}, wmed_{ss} \end{array}\right\}$$

The aggregation operator can be found in the literature review (chapter 2.3.5). For the notation, $wmed_l$ is $wmed$ with Lukasiewicz T-norm and T-connorm. This naming convention is also applied to other $wmed$ s taking different T-norms and T-connorms. In addition, as α affects the aggregation result, then a different value of α can be regarded as a different operator. This example takes $\alpha=0.2$ for all parametric operators.

c) Get the collection of information granules:

Let $X=\left(x_1,\ldots,x_5\right)$ be weighted criteria; $C=\left(0.4,0.5,0.6,0.7,0.9\right)$, $W=owaW\left(0.6,5\right)=\left(0.3801,0.1964,0.1589,0.1387,0.1253\right)$, and thus $X=\left(\left(0.4,0.1978\right),\ldots,\left(0.9,0.6250\right)\right)$.

Process:

Step 1: Compute Y by $\tilde{A}(X)$:

$$Y = \tilde{A}(X) = \begin{Bmatrix} 0.5375, 0.5137, 0.5332, 0.5557, 0.6949, 0.3807, 0.4, 0.6939, \\ 0.5019, 0.4619, 0.5127, 0.5, 0.5, 0.5, 0.1193, 0.5199, 0.4868 \end{Bmatrix}$$

Step 2: Get the $\vec{Y}$, and $\left[y_*, y^*\right]$:

Get Ordering$(Y) = \{14,11,13,15,16,2,3,16,9,4,10,6,6,6,1,12,5\}$, then

$$\vec{Y} = \begin{Bmatrix} 0.1193, 0.3807, 0.4, 0.4619, 0.4868, 0.5, 0.5, 0.5, 0.5019, 0.5127, \\ 0.5137, 0.5199, 0.5332, 0.5375, 0.5557, 0.6939, 0.6949 \end{Bmatrix}.$$

Step 3: $\left[y_*, y^*\right] = \left[y_{(1)}, y_{(m)}\right] = [0.1193, 0.6949]$.

Steps 4 and 5: Assign intervals and interpolate memberships for D.

Let $(y_*, y', y^*) = [0.1193, 0.4071, 0.6949]$ be substituted to $\mu(a,b,c)$ in D, and then the CSAO-1 pattern is shown in fig.6.3. It can be observed that the proposed numerical integration is downward integration as $y' = 0.4071 < 0.5$.

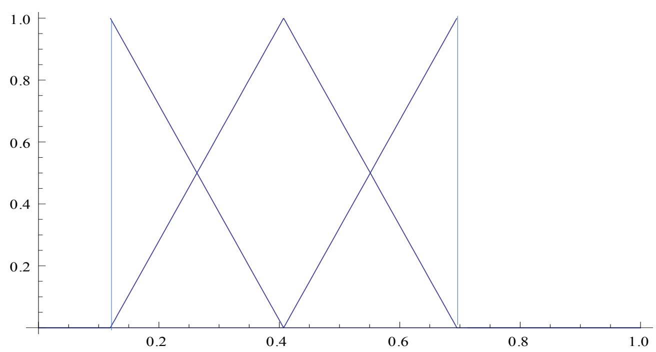


Figure 6.3: Fuzzy sets in CSAO-1 pattern

Table 6.1: The results for $D_{\widetilde{Agg}}$ of 17 AOs

(k)	Agg	$y_{(k)}$	$D\left(y_{(k)}\right)$	$d^*(k)$
1	$wmed_f$	0.1193	{1,0,0}	Pess
2	owmax	0.3807	{0.0917,0.9083,0}	Pess
3	owmin	0.4	{0.0247,0.9753,0}	Ntl
4	$wmed$	0.4619	{0,0.8095,0.1905}	Ntl
5	$wmed_{ss}$	0.4868	{0,0.7231,0.2769}	Ntl
6	$wmed_{mm}$	0.5	{0,0.6773,0.3227}	Ntl
7	$wmed_{dp}$	0.5	{0,0.6773,0.3227}	Ntl
8	$wmed_y$	0.5	{0,0.6773,0.3227}	Ntl
9	wgo	0.5019	{0,0.6707,0.3293}	Ntl
10	$wmed_l$	0.5127	{0,0.6333,0.3667}	Ntl
11	whm	0.5137	{0,0.6298,0.3703}	Ntl
12	$wmed_{ws}$	0.5199	{0,0.6080,0.3920}	Ntl
13	wgm	0.5332	{0,0.5618,0.4382}	Ntl
14	wrp	0.5375	{0,0.5470,0.4530}	Ntl
15	wam	0.5557	{0,0.4838,0.5162}	Opt
16	owa	0.6949	{0,0,1}	Opt
17	Leximin	0.6949	{0,0,1}	Opt

Step 6: Calculate $D\left(\vec{Y}\right)$, $D_{\widetilde{Agg}}$ and $d^*(k)$.

Table 6.1 summarizes the results for $D_{\widetilde{Agg}}$, $\left\{y_{(k)}, D_{Agg}\left(y_{(k)}\right)\right\} \in D_{\widetilde{Agg}}$, $\forall k \in \{1,\cdots,17\}$.

Step 7 and Output:

$\left\{d_j^*\right\} = \{1,3,17\}$, which means { $wmed_f$, owmin, owa / Leximin }

owa / Leximin produce the same result.

The interpretations of the above example are as follows.

The weighted median with other t-connorms and t-norms (Yager 1994; Smolikova and Wachowiak, 2002) is likely to produce questionable results. Firstly, t-conform and t-norm are initially designed for aggregation of two fuzzy sets, and are not suitable for the weighted criteria since $wmed(W,C)$ has different meanings to $wmed(C,W)$. Secondly, the definition of the tuning parameter α is infinitive since each α represents a new aggregation operator due to different output values. Thirdly, the more the criteria is to be aggregated, the lesser values in W *as* $\sum_{w_i \in W} w_i = 1$ are followed. As t-norms or t-conorms are mainly based on Min and Max of two sets, a misleading result will result.

owmax and owmin are not the effective AOs for the decision matrix. The third reason of the above description explains this issue. Lexmin and *owa* produce the same result as the weights used by them, and are not defined by their intrinsic functions.

If these aggregation operators are removed, the new result is shown in example 6.2. Further investigation for *owa* is concluded after illustration of example 6.2.

Example 6.2

Let be $\widetilde{Agg} = (Agg_1, \ldots, Agg_k, \ldots, Agg_7) = \{wrp, whm, wgm, wam, owa, wgo, wmed\}$. Others remain unchanged. The new results for $D_{\widetilde{Agg}}$ are shown in table 6.4, and finally,

$\{d_j^*\} = \{1,6,7\}$, which is $\{$ $wmed$, wam, owa $\}$.

Table 6.2: The results for $D_{\widetilde{Agg}}$ of seven AOs

(k)	Agg	$y_{(k)}$	$D_{\widetilde{Agg}}\left(y_{(k)}\right)$	$d^*(k)$
1	$wmed$	0.4619	{1,0,0}	Pess
2	wgo	0.5019	{0.6570,0.343,0}	Pess
3	whm	0.5137	{0.5558,0.4442,0}	Pess
4	wgm	0.5332	{0.3880,0.6120,0}	Ntl
5	wrp	0.5375	{0.3513,0.6487,0}	Ntl
6	wam	0.5557	{0.1953,0.8047,0}	Ntl
7	owa	0.6949	{0,0,1}	Opt

Examples 6.1 and 6.2 imply that not all AOs can be applied in DSAO (DAAO). It is similar to not all people being suitable for a single job, an interest, or a subject domain as they have different cognitive styles. People who are suitable for a job are pooled and selected accordingly with respect to the decision maker. Thus only the suitable AOs can be taken in DSAO, and then classified. The one which mostly reflects the decision maker's cognitive style is selected.

In addition, owa seems to produce exaggerate results in the above example. The main reason is that the order of the values of the criteria is sorted in descending order. This action is unnecessary. For one reason, the weight and the criterion are matched; for another reason, the different initial settings of the criteria order are very likely to produce different results. For the third reason, there is no point to mismatch the weight and the criterion pair.

The next section discusses DSAO-2 in detail.

6.4 Decision attitude and aggregation operator 2 (DAAO-2, or CSAO-2)

Usually a fuzzy set consists of several AOs. If a decision maker chooses a linguistic term for the decision attitudes, although the choices are narrowed, he still needs to choose the right one representing his cognitive style. Thus the DA atomic fuzzy set is further classified, defined as follows.

Definition 6.12: The membership of DA d_j can be described by the set of directional hedges of DA $H=\{h_1,\ldots,h_\eta,\ldots,h_r\}$ and $\leq_{i=1}^{r}(h)=h_1\leq\ldots\leq h_r$. DA is formed by a vector of hedge terms $\overrightarrow{V_h}=\left[v_{h_i}\right]_{i=1}^{\eta}$ with $\leq_{i=1}^{\eta}\left(v_{h_i}\right)$ and a vector of directional terms $\overrightarrow{V_d}=\left[v_{d_i}\right]_{i=1}^{3}=\left[v_{\mathrm{d}}^{-}, v_{\mathrm{d}}^{\theta}, v_{\mathrm{d}}^{+}\right]$ with $\leq_{i=1}^{3}\left(v_{d_i}\right)$. The formation of directional hedges can be referred to proposition 4.1. For example, H={"much below", "quite below", "little below", "absolutely", "little below", quite above", "much above"}.

Thus the following proposition holds.

Proposition 6.5 (*HD*): The Linguistic Cartesian Product $G_{\aleph}$ of D and H forms a collection of compound fuzzy sets $HD=\{h_i\oplus d_j : i=1,\ldots,r; j=1,\ldots,p\}$, which is of the form.

$$HD = G_{\aleph}\left(H, D\right) = \begin{bmatrix} \varnothing & h_1 \oplus d_2 & \cdots & h_1 \oplus d_p \\ \vdots & \vdots & \ddots & \vdots \\ \varnothing & h_\eta \oplus d_2 & \ddots & h_\eta \oplus d_p \\ d_1{}^{\theta} & d_2{}^{\theta} & \ddots & d_p{}^{\theta} \\ h_{\eta+2} \oplus d_1 & h_{\eta+2} \oplus d_2 & \ddots & \varnothing \\ \vdots & \vdots & \ddots & \vdots \\ h_r \oplus d_1 & h_r \oplus d_2 & \cdots & \varnothing \end{bmatrix} \tag{6.8}$$

Proof:

It is derived from theorem 4.1 and algorithm 4.1. □

The compound linguistic terms for the decision attitude are used by a deductive rating strategy which is the double step rating process (see chapter 4.4). The next issue discusses the patterns for the second degree decision attitudes which use the semantic rule algorithm (algorithm 4.2) to build up a matrix of fuzzy sets.

Definition 6.13 ($\left\{d_{ij}\right\} = f_{\overline{X}}\left(HD\right)$): Let $\left\{d_{ij}\right\}$ be the matrix of the fuzzy numbers of HD. $\left\{d_{ij}\right\}$ is determined by the semantic rule algorithm $f_{\overline{X}}\left(HD\right)$ (algorithm 4.2), which is of the form:

$$\left\{d_{ij} : i = 1, \ldots, r; j = 1, \ldots, p\right\} = f_{\overline{X}}\left(HD\right)$$

$$= f_{\overline{X}}\left(\left\{\left(\gamma_{d^j}, \Delta_{d^j}, \tau_{d^j}, \left\{\mu_{d^{j\phi}}{}^{-1}\right\}^{\phi}\right)\right\}, \left[y_*, y^*\right], \left(\varphi\left(\overrightarrow{V_h}\right), \lambda_0\right)\right) \tag{6.9}$$

,where $\left\{\left(\gamma_{d^j}, \Delta_{d^j}, \tau_{d^j}, \left\{\mu_{d^{j\phi}}{}^{-1}\right\}^{\phi}\right)\right\}$ is the 1st degree DA fuzzy sets which are the symmetric fuzzy set: γ_{d^j} is the modal value, Δ_{d^j} is symmetric distance (by default,

$\Delta_{d^1} = \Delta_{d^2} =, \ldots, = \Delta_{d^p}$), τ_{d^j} is the tuning parameter of the membership function, μ_{d^j} is the membership function of d_j or d^j, and $\mu_{d^j\phi}^{-1}$ is the inverse membership function. The collection of the 1st degree DA fuzzy sets is called the 1st degree DA fuzzy variable. The parameters of the membership fuzziness process $\left(\varphi\left(\overline{V_h}\right), \lambda_0\right)$ determine the distribution of the 2nd degree DA fuzzy variable with respect to the corresponding 1st degree DA fuzzy sets.

With the above definition, the following proposition holds.

Proposition 6.6 ($D''_{\widetilde{Agg}}$ **):** A collection of the 2nd degree DA fuzzy sets $D''_{\widetilde{Agg}}$ for a collection of aggregation operators $\widetilde{Agg} = \left(Agg_1, \ldots, Agg_k, \ldots, Agg_m\right)$ is of the form:

$$D''_{\widetilde{Agg}}\left(\vec{Y}\right) = \begin{bmatrix} \varnothing & \begin{Bmatrix} \{y_{(1)}, d_{1,2}\} \\ \vdots \\ \{y_{(m)}, d_{1,2}\} \end{Bmatrix} & \cdots & \begin{Bmatrix} \{y_{(1)}, d_{1,p}\} \\ \vdots \\ \{y_{(m)}, d_{1,p}\} \end{Bmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \varnothing & \begin{Bmatrix} \{y_{(1)}, d_{\eta,2}\} \\ \vdots \\ \{y_{(m)}, d_{\eta,2}\} \end{Bmatrix} & \ddots & \begin{Bmatrix} \{y_{(1)}, d_{\eta,p}\} \\ \vdots \\ \{y_{(m)}, d_{\eta,p}\} \end{Bmatrix} \\ \begin{Bmatrix} \{y_{(1)}, d_{\eta+1,1}\} \\ \vdots \\ \{y_{(m)}, d_{\eta+1,1}\} \end{Bmatrix} & \begin{Bmatrix} \{y_{(1)}, d_{\eta+1,2}\} \\ \vdots \\ \{y_{(m)}, d_{\eta+1,2}\} \end{Bmatrix} & \ddots & \begin{Bmatrix} \{y_{(1)}, d_{\eta+1,p}\} \\ \vdots \\ \{y_{(m)}, d_{\eta+1,p}\} \end{Bmatrix} \\ \begin{Bmatrix} \{y_{(1)}, d_{\eta+2,1}\} \\ \vdots \\ \{y_{(m)}, d_{\eta+2,1}\} \end{Bmatrix} & \begin{Bmatrix} \{y_{(1)}, d_{\eta+2,2}\} \\ \vdots \\ \{y_{(m)}, d_{\eta+2,2}\} \end{Bmatrix} & \ddots & \varnothing \\ \vdots & \vdots & \ddots & \vdots \\ \begin{Bmatrix} \{y_{(1)}, d_{r,1}\} \\ \vdots \\ \{y_{(m)}, d_{r,1}\} \end{Bmatrix} & \begin{Bmatrix} \{y_{(1)}, d_{r,2}\} \\ \vdots \\ \{y_{(m)}, d_{r,2}\} \end{Bmatrix} & \cdots & \varnothing \end{bmatrix} \tag{6.10}$$

Proof:

Proposition 6.2 indicates D_{Agg}, which further extends to $D_{\widetilde{Agg}}$ in proposition 6.3.

Proposition 6.5 develops HD in which its fuzzy number set

$\left\{d_{ij} : i = 1,\ldots,r; j = 1,\ldots,p\right\}$ is defined in definition 6.10. The D_{Agg} can be applied in HD.

Thus, the form of $D''_{\widetilde{Agg}}$ is derived. □

Regarding the final selection of the representation of the 2nd degree DA fuzzy sets and AOs, two definitions are formed.

Definition 6.14: If an aggregation operator has more than one of the 2nd degree DA fuzzy sets, the selection of DAs for the dedicated AO is of the form:

$$d''^{*}(k) = ArgMax\left(\left\{\left\{y_{(k)}, d_{i,j}\right\} : d_{i,j} \neq \varnothing\right\}\right) \tag{6.11}$$

$d''^{*}(k)$ returns the index of the linguistic label to describe the AO.

Definition 6.15: If the 2nd degree DA fuzzy set d_{ij} includes more than one aggregation operator, the selection of AOs of d_{ij} is of the form:

$$d_{ij}^{*} = ArgMax\left(\left\{\left\{y_{(1)}, d_{ij}\right\},\ldots,\left\{y_{(k)}, d_{ij}\right\},\ldots,\left\{y_{(m)}, d_{ij}\right\}\right\}\right) \tag{6.12}$$

d_{ij}^{*} returns the index in $\vec{Y}$ to represent the linguistic label d_{ij}.

Algorithm 6.2: $DAAO-2=CSAO2\left(D,\widetilde{Agg},X,\left(\overrightarrow{V_h},\overrightarrow{V_d}\right),\left(\varphi\left(\overrightarrow{V_h}\right),\lambda_0\right)\right)$.

Input:

a. A collection of the 1st degree DA linguistic variable: $D=\left\{d_1,\cdots,d_j,\cdots,d_p\right\}$ is comprised of the membership set $\left\{\mu_{d^j}\right\}$ and the corresponding inverse membership set $\left\{\mu_{d^j,\phi='-','\theta','+'}{}^{-1}\right\}$ with the tuning factor set $\left\{\tau_{d^j}\right\}$;

b. A vector of hedge terms $\overrightarrow{V_h}$ and A vector of directional terms $\overrightarrow{V_d}$;

c. A collection of AOs: $\widetilde{Agg}=\left(Agg_1,\ldots,Agg_k,\ldots,Agg_m\right)$;

d. A collection of information granules: $X=\left(x_1,\ldots,x_i,\ldots,x_n\right)$;

e. A collection of the parameters of the member fuzziness process: $\left(\varphi\left(\overrightarrow{V_h}\right),\lambda_0\right)$;

Process:

Step 1. Compute $\widetilde{Agg}\left(X\right)$, and then $Y=\left(y_1,\ldots,y_k,\ldots,y_m\right)$ is achieved;

Step 2. Get the permutation of Y: $\vec{Y}=\left\{y_{(1)},\ldots,y_{(k)},\cdots,y_{(m)}\right\}$;

Step 3. Get $\left[y_*,y^*\right]=\left[y_{(1)},y_{(m)}\right]$;

Step 4. Calculate intervals and $\left\{\left(\gamma_{d^j},\Delta_{d^j}\right)\right\}_{j=1}^{p}$ for D by equally diving $\left[y_*,y^*\right]$;

i. $$d_1=\left(y_*,y_*,y_*+\frac{y^*-y_*}{p-1}\right)=\left(\gamma_{d^1},\gamma_{d^1},\gamma_{d^1}+\Delta_{d^1}\right)$$

ii. $$d_{j\neq 1,p}=\left(y_*+\frac{y^*-y_*}{p-1}(j-2),y_*+\frac{y^*-y_*}{p-1}(j-1),y_*+\frac{y^*-y_*}{p-1}(j)\right)$$

$$=\left(\gamma_{d^j}-\Delta_{d^j},\gamma_{d^j},\gamma_{d^j}+\Delta_{d^j}\right)$$

iii. $d_p = \left(y^* - \dfrac{y^* - y_*}{p-1}, y^*, y^* \right) = \left(\gamma_{d^p} - \Delta_{d^p}, \gamma_{d^p}, \gamma_{d^p} \right)$

Step 5. Elicit memberships μ_{d^j} for D by interpolation of (a,b,c).

Step 6. Calculate $D\left(\vec{Y}\right)$, $D_{\widetilde{Agg}}$ and $d^*(k)$, $\forall k$.

Step 7. Form HD by algorithm 4.2.

Step 8. Calculate $\left\{ d_{ij} : i = 1, \ldots, r; j = 1, \ldots, p \right\}$ of HD by

$$f_{\widetilde{X}}\left(\left\{ \left(\gamma_{d^j}, \Delta_{d^j}, \tau_{d^j}, \mu_{d^j \phi}{}^{-1} \right) \right\}, \left[y_*, y^* \right], \left(\varphi\left(\overrightarrow{V_h} \right), \lambda_0 \right) \right) \quad \text{(algorithm 4.2)}$$

Step 9. Calculate $D''_{\widetilde{Agg}}\left(\vec{Y}\right)$

Step 10. Calculate $d''^*(k)$, $\forall k$ in $D''_{\widetilde{Agg}}\left(\vec{Y}\right)$.

Step 11. Calculate d_{ij}^*, $i = 1, \ldots, r; j = 1, \ldots, p$

Output:. $\left\{ d_{ij}^* \right\}$ **//END**

Example 6.3

This example is a continuation of example 6.1. DAAO-2 is illustrated as follows.

Input:

a. $D = \left\{ d_1, d_2, d_3 \right\} = \{\text{P,N,O}\}$;

μ_{d^j} is the symmetric triangular membership, $\forall j \in \left\{1,2,3\right\}$;

$\mu_{d^j, \phi = '-', '\theta', '+'}{}^{-1}$ is the inversed triangular membership set, $\forall j \in \left\{1,2,3\right\}$;

$\tau_{d^j} = 1$, $\forall j \in \left\{1,2,3\right\}$;

b. $\overrightarrow{V_h} = \left[\text{Little,Quite,Much}\right]$, and $\overrightarrow{V_d} = \left[\text{Below,Absolutely,Above}\right]$

c. A collection of AOs: $\widetilde{Agg} = (Agg_1, \ldots, Agg_k, \ldots, Agg_{17})$;

d. A collection of information granules:

$C = (0.4, 0.5, 0.6, 0.7, 0.9)$,

$W = owaW(0.6, 5) = (0.3801, 0.1964, 0.1589, 0.1387, 0.1253)$, and thus

$X = ((0.4, 0.1978), \ldots, (0.9, 0.6250))$;

e. A collection of the parameters of the member fuzziness process:

$(\varphi(\overrightarrow{V_h}), \lambda_0) = (\{1,2,3\}, 0.5)$;

Process:

Steps 1-3:

$$Y = \widetilde{Agg}(X) = \begin{Bmatrix} 0.5375, 0.5137, 0.5332, 0.5557, 0.6949, 0.3807, 0.4, 0.6939, \\ 0.5019, 0.4619, 0.5127, 0.5, 0.5, 0.5, 0.1193, 0.5199, 0.4868 \end{Bmatrix};$$
$$\vec{Y} = \begin{Bmatrix} 0.1193, 0.3807, 0.4, 0.4619, 0.4868, 0.5, 0.5, 0.5, 0.5019, 0.5127, \\ 0.5137, 0.5199, 0.5332, 0.5375, 0.5557, 0.6939, 0.6949 \end{Bmatrix};$$
$$[y_*, y^*] = [y_{(1)}, y_{(m)}] = [0.1193, 0.6949];$$

Step 4: Calculate intervals and $\left\{\left(\gamma_{d^j}, \Delta_{d^j}\right)\right\}_{j=1}^{3}$ for D:

i. $d_{\cdot} = (0.1193, 0.1193, 0.4071)$;

ii. $d_2 = (0.1193, 0.4071, 0.6949)$;

iii. $d_3 = (0.4071, 0.6949, 0.6949)$;

iv. $\{\gamma_{d^j}\} = \{0.1193, 0.4071, 0.6949\}$ and $\Delta_{d^j} = 1, \forall j \in \{1,2,3\}$;

Step 5: Elicit memberships μ_{d^j} for D. The results are shown in fig.6.3.

Step 6: Calculate $D(\vec{Y})=\left\{D\left(y_{(k)}\right)\right\}$, $D_{\widetilde{Agg}}$ and $d^*(k)$, $\forall k$. The results are shown in table 6.3.

Step 7: Form HD by algorithm 4.2.

$$\overrightarrow{V_{hd}}=\left[v_{hd_1},\cdots,v_{hd_7}\right] =\left[\begin{array}{l}\text{"Much Below","Quite Below","Little Below","Absolutely"} \sim \\ \text{"Little Above","Quite Above","Much Above"}\end{array}\right]$$

, thus

$$HD=G_{\aleph}(H,D)=\begin{bmatrix} \varnothing & MB\text{-}N & MB\text{-}O \\ \varnothing & QB\text{-}N & QB\text{-}O \\ \varnothing & LB\text{-}N & LB\text{-}O \\ A\text{-}P & A\text{-}N & A\text{-}O \\ LA\text{-}P & LA\text{-}N & \varnothing \\ QA\text{-}P & QA\text{-}N & \varnothing \\ MA\text{-}P & MA\text{-}N & \varnothing \end{bmatrix}.$$

Step 8: Calculate $\{d_{ij}\}$ of HD by

$$f_{\overline{X}}\left(\left\{\left(\gamma_{d^j},\Delta_{d^j},\tau_{d^j},\mu_{d^j\phi}{}^{-1}\right)\right\},\left[y_*,y^*\right],\left(\varphi\left(\overline{V_h}\right),\lambda_0\right)\right) \quad \text{(algorithm 4.2):}$$

$$\{d_{ij}\}=\begin{pmatrix} \varnothing & (0.1193,0.2092,0.2992) & (0.4071,0.4971,0.5870) \\ \varnothing & (0.2392,0.3112,0.3831) & (0.5270,0.5990,0.6709) \\ \varnothing & (0.3472,0.3771,0.4071) & (0.6350,0.6649,0.6949) \\ (0.1193,0.1193,0.1193) & (0.4071,0.4071,0.4071) & (0.6949,0.6949,0.6949) \\ (0.1193,0.1493,0.1793) & (0.4071,0.4371,0.4671) & \varnothing \\ (0.1433,0.2152,0.2872) & (0.4310,0.5030,0.5750) & \varnothing \\ (0.2272,0.3172,0.4071) & (0.5150,0.6050,0.6950) & \varnothing \end{pmatrix}$$

Step 9: Calculate $D''_{\widetilde{Agg}}\left(\vec{Y}\right)$

$$\begin{pmatrix} \varnothing & \left(0^{17}\right) & \begin{pmatrix} 0^3, 0.6095, 0.8860, 0.9672, 0.9672, \\ 0.9672, 0.9463, 0.8265, 0.8152, \\ 0.7455, 0.5978, 0.5503, 0.3482, 0^2 \end{pmatrix} \\ \varnothing & \left(0^1, 0.0332, 0^{16}\right) & \left(0^{12}, 0.0860, 0.1455, 0.3981, 0^2\right) \\ \varnothing & \left(0^1, 0.8799, 0.2372, 0^{14}\right) & \left(0^{17}\right) \\ \left(1, 0^{16}\right) & \left(0^{17}\right) & \left(0^{16}, 1, 1\right) \\ \left(0^{17}\right) & \left(0^3, 0.1716, 0^{13}\right) & \varnothing \\ \left(0^{17}\right) & \begin{pmatrix} 0^3, 0.4285, 0.7742, 0.9576, \\ 0.9576, 0.9576, 0.9838, \\ 0.8665, 0.8523, 0.7652, \\ 0.5806, 0.5212, 0.2686, 0^2 \end{pmatrix} & \varnothing \\ \left(0^1, 0.2933, 0.0791, 0^{14}\right) & \begin{pmatrix} 0^{11}, 0.0545, 0.2022, 0.2497, \\ 0.4518, 0^2 \end{pmatrix} & \varnothing \end{pmatrix}$$

“0” means that the membership of AO is equal to zero in this compound linguistic term.

The index of “0” means the number of zeros.

Step 10: Calculate $d''^{*}(k)$, $\forall k$ in $D''_{\widetilde{Agg}}\left(\vec{Y}\right)$. The results are shown in table 6.3.

Step 11 and Return: Calculate d_{ij}^{*}, $i = 1, \ldots, r; j = 1, \ldots, p$

$$\left\{d_{ij}^{*}\right\} = d_{ij}^{*}\left(\begin{bmatrix} \varnothing & MB\text{-}N & MB\text{-}O \\ \varnothing & QB\text{-}N & QB\text{-}O \\ \varnothing & LB\text{-}N & LB\text{-}O \\ A\text{-}P & A\text{-}N & A\text{-}O \\ LA\text{-}P & LA\text{-}N & \varnothing \\ QA\text{-}P & QA\text{-}N & \varnothing \\ MA\text{-}P & MA\text{-}N & \varnothing \end{bmatrix}\right) = \begin{bmatrix} \varnothing & 0 & 6,7,8 \\ \varnothing & 2 & 15 \\ \varnothing & 2 & 0 \\ 1 & 0 & 16,17 \\ 0 & 4 & \varnothing \\ 0 & 9 & \varnothing \\ 2 & 15 & \varnothing \end{bmatrix}$$

“0” means no AO is available in this compound linguistic term. Another number means the index in $\vec{Y}$.

If a linguistic term (e.g. MB-O, A-O) includes more than one AOs (e.g. (6,7,8) or (16,17)), either of the AOs can be used since the AOs produce the same result with respect to a compound fuzzy set.

Table 6.3: The results for $D''(\vec{Y})$ and $d''^{*}(k)$ of 17 AOs

(k)	*Agg*	$y_{(k)}$	$D''(y_{(k)})$	$d''^{*}(k)$
1	$wmed_f$	0.1193	{A-P(1)}	A-P
2	owmax	0.3807	{MA-P(0.293),QB-N(0.033),LB-N(0.880)}	LB-N
3	owmin	0.4	{MA-P(0.079),QB-N(0.237)}	QB-N
4	$wmed$	0.4619	{LA-N(0.172),QA-N(0.428),MB-P(0.609)}	MB-P
5	$wmed_{ss}$	0.4868	{QA-N(0.774),MB-O(0.886)}	MB-O
6	$wmed_{mm}$	0.5	{QA-N(0.958),MB-O(0.967)}	MB-O
7	$wmed_{dp}$	0.5	{QA-N(0.958),MB-O(0.967)}	MB-O
8	$wmed_y$	0.5	{QA-N(0.958),MB-O(0.967)}	MB-O
9	wgo	0.5019	{QA-N(0.984),MB-O(0.946)}	QA-N
10	$wmed_l$	0.5127	{QA-N(0.866),MB-O(0.827)}	QA-N
11	whm	0.5137	{QA-N(0.852),MB-O(0.815)}	QA-N
12	$wmed_{ws}$	0.5199	{QA-N(0.765),MA-N(0.054),MB-O(0.745)}	QA-N
13	wgm	0.5332	{QA-N(0.581),MA-N(0.202),MB-O(0.598),QB-O(0.086)}	MB-O
14	wrp	0.5375	{QA-N(0.521),MA-N(0.250),MB-O(0.550),QB-O(0.145)}	MB-O
15	wam	0.5557	{QA-N(0.269),MA-N(0.452),MB-O(0.348),QB-O(0.398)}	MA-N
16	owa	0.6949	{A-O(1)}	A-O
17	Leximin	0.6949	{A-O(1)}	A-O

Example 6.4

Using DAAO-2, this example considers only seven AOs used in example 6.2. Steps 1 and 7 are skipped. The remains of the steps are illustrated as follows:

Step 8: Calculate $\{d_{ij}\}$

$$\{d_{ij}\} = \begin{pmatrix} \varnothing & (0.4619, 0.4983, 0.5347) & (0.5784, 0.6148, 0.6512) \\ \varnothing & (0.5105, 0.5396, 0.5687) & (0.6270, 0.6561, 0.6852) \\ \varnothing & (0.5542, 0.5663, 0.5784) & (0.6707, 0.6828, 0.6949) \\ (0.4619, 0.4619, 0.4619) & (0.5784, 0.5784, 0.5784) & (0.6949, 0.6949, 0.6949) \\ (0.4619, 0.4741, 0.4862) & (0.5784, 0.5906, 0.6027) & \varnothing \\ (0.4716, 0.5008, 0.5299) & (0.5881, 0.6173, 0.6464) & \varnothing \\ (0.5056, 0.5402, 0.5784) & (0.6221, 0.6585, 0.6949) & \varnothing \end{pmatrix}$$

Step 9: Calculate $D''_{\widetilde{Agg}}(\vec{Y})$.

$$D''_{\widetilde{Agg}}(\vec{Y}) = \begin{pmatrix} \varnothing & \begin{pmatrix} 0^1, 0.9025, 0.5786, \\ 0.0415, 0^3 \end{pmatrix} & (0^7) \\ \varnothing & \begin{pmatrix} 0^2, 0.1101, 0.7813, \\ 0.9282, 0.4477, 0^1 \end{pmatrix} & (0^7) \\ \varnothing & (0^5, 0.1255, 0^1) & (0^7) \\ (1, 0^6) & (0^7) & (0^7, 1) \\ (0^7) & (0^7) & \varnothing \\ (0^2, 0.9615, 0.5566, 0^4) & (0^7) & \varnothing \\ \begin{pmatrix} 0^2, 0.2214, 0.7584, \\ 0.8759, 0.6248, 0^1 \end{pmatrix} & (0^7) & \varnothing \end{pmatrix}$$

“0” means that the membership of AO is equal to zero in this compound linguistic term. The index of “0” means the number of zeros.

Step 10: Calculate $d''^{*}(k)$, $\forall k$ in $D''_{\widetilde{Agg}}(\vec{Y})$.

$d^{*}(k)$ is shown in table 6.4.

Table 6.4: The results for $D''(\vec{Y})$ and $d''^{*}(k)$ of seven AOs

(k)	Agg	$y_{(k)}$	$D(y_{(k)})$	$d^{*}(k)$
1	*wmed*	0.4619	{A-P(1)}	A-P
2	*wgo*	0.5019	{QA-P(0.962),MB-N(0.903)}	QA-P
3	*whm*	0.5137	{QA-P(0.557),MA-P(0.221),MB-N(0.579),QB-N(0.110)}	MB-N
4	*wgm*	0.5332	{MA-P(0.758),MB-N(0.042),QB-N(0.781)}	QB-N
5	*wrp*	0.5375	{MA-P(0.876),QB-N(0.928)}	QB-N
6	*wam*	0.5557	{MA-P(0.625),QB-N(0.448),LB-N(0.126)}	MA-P
7	*owa*	0.6949	{A-O(1)}	A-O

Step 11 and Return : $\left\{d_{ij}^{*}\right\}$ is shown as follows.

$$\left\{d_{ij}^{*}\right\} = d_{ij}^{*}\left(\begin{bmatrix} \varnothing & MB\text{-}N & MB\text{-}O \\ \varnothing & QB\text{-}N & QB\text{-}O \\ \varnothing & LB\text{-}N & LB\text{-}O \\ A\text{-}P & A\text{-}N & A\text{-}O \\ LA\text{-}P & LA\text{-}N & \varnothing \\ QA\text{-}P & QA\text{-}N & \varnothing \\ MA\text{-}P & MA\text{-}N & \varnothing \end{bmatrix}\right) = \begin{bmatrix} \varnothing & 2 & 0 \\ \varnothing & 5 & 0 \\ \varnothing & 6 & 0 \\ 1 & 0 & 7 \\ 0 & 0 & \varnothing \\ 2 & 0 & \varnothing \\ 5 & 0 & \varnothing \end{bmatrix}$$

"0" means no AO is available in this compound linguistic term. Another number means the index in $\vec{Y}$.

One can purely use DAAO-1, or DAAO-2. However, the selection function by $ArgMax$ is excessively straightforward in DAAO-1 in many AO candidates for one DA linguistic term d_j , whilst DAAO-2 contains no AOs for some linguistic terms if insufficient AO candidates for the relatively large scale of the compound linguistic terms. Regarding the number of AO candidates, the selection strategy to combine DAAO-1 and DAAO-2 is of the following algorithm.

Algorithm 6.3 (Selection Strategy, $SAO\left(\left(d_j, h_i\right), \left(D_{\widetilde{Agg}}, d''^{*}(k)\right)\right)$)

Input: $D_{\widetilde{Agg}}$ of DAAO-1, and $d''^{*}(k)$ of DAAO-2.

Selection Process:

Step 1: Select an atomic term of DA d_j.

Step 2: Check if no AO return for the d_j in $D_{\widetilde{Agg}}$,

True: Return Empty message and go to Step 1.

False: Go to Step 3.

Step 3: Check if only one AO return for the d_j in $D_{\widetilde{Agg}}$,

True: Return $Agg_{(k)}$.

False: Go to Step 4.

Step 4: Select the directional hedge term h_i.

Step 5: Check if no AO return for the $d_{ij} = h_i \oplus d_j$ in $d''^{*}(k)$,

True: Return Empty message and go to Step 4 or 1.

False: Return $Agg_{(k)} = d''^{*}(k)$.

Return: $Agg_{(k)}$. **//End**

Example 6.5:

Consider Examples 6.1 and 6.4. Three cases are illustrated.

Case 1: d_3="Opt".

Input: $D_{\widetilde{Agg}}$ of DAAO-1 in table 6.2 and $d''^*(k)$ of DAAO-2 in table 6.4.

Selection Process:

Step 1: Select an atomic term of DA: d_3="Opt".

Step 2: owa return for the d_j in $D_{\widetilde{Agg}}$,

Step 3: Only one AO return for the d_j in $D_{\widetilde{Agg}}$,

Return: $Agg_{(7)} = owa$

Case 2: d_2="Ntl".

Input: $D_{\widetilde{Agg}}$ of DAAO-1 in table 6.2 and $d''^*(k)$ of DAAO-2 in table 6.4.

Selection Process:

Step 1: Select an atomic term of DA: d_2="Ntl".

Steps 2 and 3: wgm, wrp, and wam return for the d_j.

Step 4: Select the directional hedge term h_i.

Step 5: Check if no AO return for the $d_{ij} = h_i \oplus d_j$ in $d''^*(k)$,

True: Return Empty message and go to Step 4 (As d_j="Ntl" is assumed, Step 1 is skipped).

False: Return $Agg_{(k)} = d''^{*}(k)$.

Return: $Agg_{(k)} = wgo, wrp, wam$ depends on which valid h_i is firstly selected.

Case 3, which d_1 ="Pes", is similar to Case 2. $Agg_{(k)} = wmed, wgo, wrp$. depends on which valid h_i is firstly selected.

6.5 CSAO in decision matrix

In a decision matrix, more than one alternative is considered. This means different input value sets X 's possibly produce different $\{d^{*}(k)\}$, $\{d_j^{*}\}$, and $\{d_{ij}^{*}\}$. To address this issue, three definitions are created as follows:

Definition 6.14: In the decision matrix, the linguistic presentation of the style of the decision attitude for the AOs is computed by the form:

$$\{d_{\beta}^{*}(k)\}^{*} = Max\left(Mode\left(Join\left(\{d_{\beta}^{*}(k)\}\right)\right)\right) \tag{6.13}$$

,where β is the index of the alternative of the decision matrix. $Join$ is the function to combine the matrices, and $Mode$ is the value that occurs the most frequently in an entry of $Join\left(\left\{\{d_{ij}^{*}\}_{\beta}\right\}\right)$.

Definition 6.15: In a decision matrix, the AO of the style of the decision attitude for the linguistic terms is computed as:

$$\left\{\left\{d_j^*\right\}_\beta\right\}^* = Max\left(Mode\left(Join\left(\left\{\left\{d_j^*\right\}_\beta\right\}\right)\right)\right) \quad (6.14)$$

Definition 6.16: let $\left\{d_{ij}^*\right\}_\beta$ be the DAAO-2 pattern of the alternative β. Then, the pattern of the decision matrix is of the form:

$$\left\{\left\{d_{ij}^*\right\}_\beta\right\}^* = Max\left(Mode\left(Join\left(\left\{\left\{d_{ij}^*\right\}_\beta\right\}\right)\right)\right) \quad (6.15)$$

If more than one AO index is returned in the entry, the index number with the highest value is chosen since it is likely to produce higher value for each alternative of the decision matrix. Thus the Max is taken. Also Max can eliminate "0" values. The Selection Strategy in Decision matrix is illustrated in algorithm 6.4**.**

Algorithm 6.4 ($Agg_{(k)} = \overline{CSAO}\left(\left(h_i, d_j\right), \{X\}, \widetilde{Agg}, D, \left(\overrightarrow{V_h}, \overrightarrow{V_d}\right), \left(\varphi\left(\overrightarrow{V_h}\right), \lambda_0\right)\right)$ **)**

Input: $\left(h_i, d_j\right), D, \widetilde{Agg}, X, \left(\overrightarrow{V_h}, \overrightarrow{V_d}\right), \left(\varphi\left(\overrightarrow{V_h}\right), \lambda_0\right)$

Process:

Step 1: Calculated $d_\beta^*(k)$ in $CSAO1\left(D, \widetilde{Agg}, X_\beta\right)$ $\forall \beta \in \{1, \ldots, |\{X\}|\}$

(Algorithm 6.1)

Step 2: $\left\{d_j^*\right\}_\beta = CSAO1\left(D, \widetilde{Agg}, X_\beta\right)$, $\forall \beta \in \{1, \ldots, |\{X\}|\}$ (Algorithm 6.1)

Step 3: $\left\{d_{ij}^*\right\}_\beta = CSAO2\left(D, \widetilde{Agg}, X_\beta, \left(\overrightarrow{V_h}, \overrightarrow{V_d}\right), \left(\varphi\left(\overrightarrow{V_h}\right), \lambda_0\right)\right)$, $\forall \beta \in \{1, \ldots, |\{X\}|\}$

(Algorithm 6.2)

Step 4: $\left\{d_{\beta}^{*}(k)\right\}^{*}=Max\left(Mode\left(Join\left(\left\{d_{\beta}^{*}(k)\right\}\right)\right)\right)$

Step 5: $\left\{\left\{d_{j}^{*}\right\}_{\beta}\right\}^{*}=Max\left(Mode\left(Join\left(\left\{\left\{d_{j}^{*}\right\}_{\beta}\right\}\right)\right)\right)$

Step 6: $\left\{\left\{d_{ij}^{*}\right\}_{\beta}\right\}^{*}=Max\left(Mode\left(Join\left(\left\{\left\{d_{ij}^{*}\right\}_{\beta}\right\}\right)\right)\right)$

Step 7: Check if no AO return for the d_j in $\left\{d_{\beta}^{*}(k)\right\}^{*}$,

True: Return Empty message and go to Input to request another d_j.

False: Go to Step 4.

Step 8: Check the numbers of AO's return for the d_j in $\left\{d_{\beta}^{*}(k)\right\}^{*}$,

1: Return $Agg_{(k)}=\left\{d_{\beta}^{*}(k)\right\}^{*}$ without considering h_i.

2-3: Return $Agg_{(k)}=\left\{\left\{d_{j}^{*}\right\}_{\beta}\right\}^{*}$ without considering h_i.

≥ 4: Go to Step 9.

Step 9: Check if no AO return for the $d_{ij}=h_i \oplus d_j$ in $\left\{\left\{d_{ij}^{*}\right\}_{\beta}\right\}^{*}$,

True: Return Empty message and go to Input for new $\left(h_i, d_j\right)$.

False: Return $Agg_{(k)}=\left\{\left\{d_{ij}^{*}\right\}_{\beta}\right\}^{*}$.

Return: $Agg_{(k)}$. **//End**

The use of this algorithm is shown in the chapter 6.5.2. The next section performs the numerical analysis for the proposed DSAO model to validate its usability and validity.

6.6 Numerical analyses and discussion

Three major analyses are performed and discussed as follows.

6.6.1 Scenario

Consider a decision Matrix as follows,

$$\begin{array}{cc} W & (w_1 \quad w_2 \quad w_3 \quad w_4 \quad w_5) \\ C & \; c_1 \quad\; c_2 \quad\; c_3 \quad\; c_4 \quad\; c_5 \end{array}$$

$$\bar{O} = \begin{array}{c} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{array} \begin{pmatrix} 0.5 & 0.5 & 0.6 & 0.7 & 0.9 \\ 0.5 & 0.7 & 0.9 & 0.8 & 0.5 \\ 0.6 & 0.9 & 0.5 & 0.7 & 0.5 \\ 0.4 & 0.5 & 0.6 & 0.8 & 0.9 \\ 0.5 & 0.9 & 0.5 & 0.7 & 0.5 \end{pmatrix}$$

,where $W = owaW(\delta), \delta \in \{0.1, 0.2, \ldots, 1\}$, which is shown in table 6.5.

Table 6.5: W generated by $owaW(\delta), \delta \in \{0.1, 0.2, \ldots, 1\}$

δ	w_1	w_2	w_3	w_4	w_5
0.1	0.851	0.061	0.038	0.028	0.022
0.2	0.725	0.108	0.070	0.053	0.044
0.3	0.617	0.143	0.098	0.077	0.065
0.4	0.525	0.168	0.122	0.099	0.085
0.5	0.447	0.185	0.142	0.120	0.106
0.6	0.381	0.196	0.159	0.139	0.125
0.7	0.324	0.202	0.173	0.156	0.145
0.8	0.276	0.205	0.184	0.172	0.163
0.9	0.235	0.203	0.193	0.187	0.182
1	0.200	0.200	0.200	0.200	0.200

In this section, firstly, ten different decision matrices of the above form are created with 10 weight sets (table 6.5). The matrices are further aggregated by 10 aggregation operators defined as follows:

$$\widetilde{Agg} = (whm, wgm, wam, wmed, wrp01, wrp05, wrp20, wgo01, wgo05, wgo09).$$

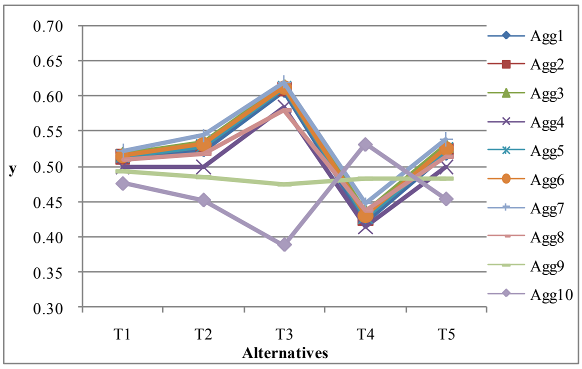
, where 01 means $\alpha = 0.1$, and so on, and the next results are produced.

Secondly, regarding the research values for discussion, the decision matrix with $\alpha = 0.9$ is selected for the application of DSAO-2.

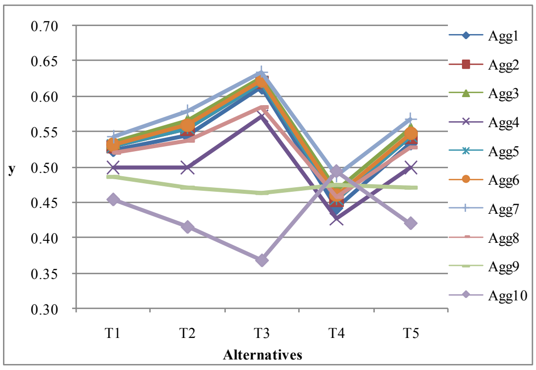
6.6.2 Properties of individual AOs

Ten decision matrices of the variation of weight sets are used for ten AOs. The weight sets are generated by $owaW(\delta), \delta \in \{0.1, 0.2, \ldots, 1\}$ and are shown in table 6.5. The larger δ means the less gap among the individual weights. When $\delta = 1$, all weights are of equal values. The numerical results are presented in Appendix III. The data are plotted in figures 6.4a-b.

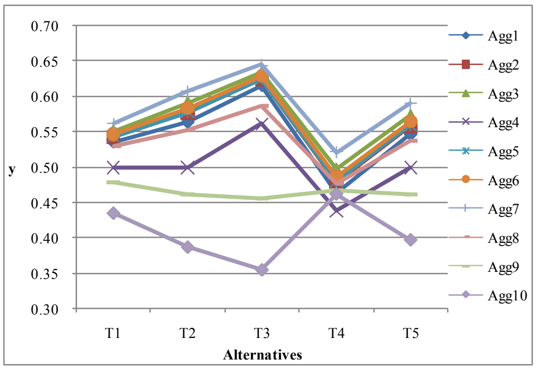
Figures 6.4a and 6.4b show that different AOs behave differently for different decision matrices. This means that each AO has a different style. *wrp and wgo* with different α produce different results and likely different ranks. This means that a different AO with different α can have its own style.



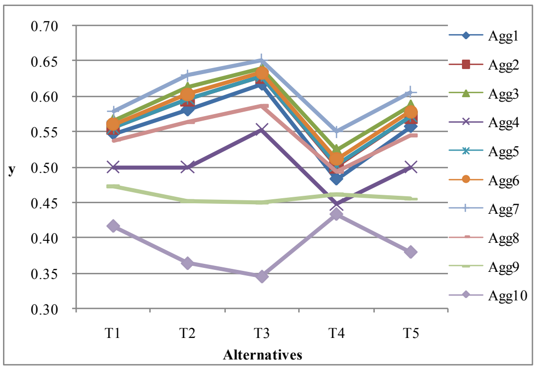
owaW(0.1)



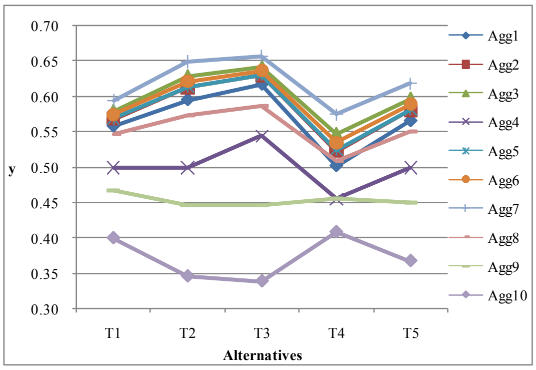
owaW(0.2)



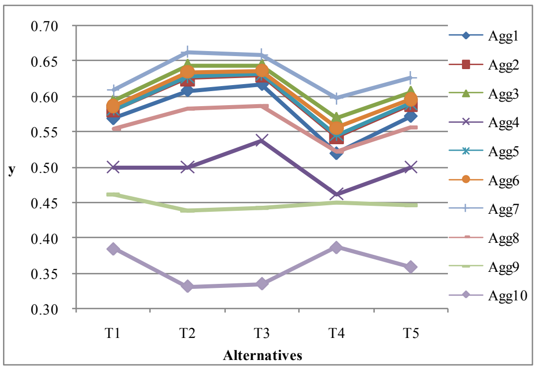
owaW(0.3)



owaW(0.4)



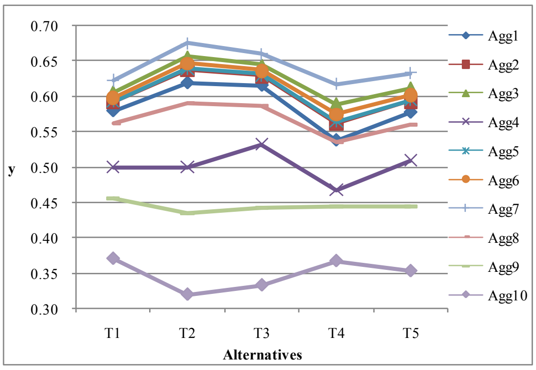
owaW(0.5)



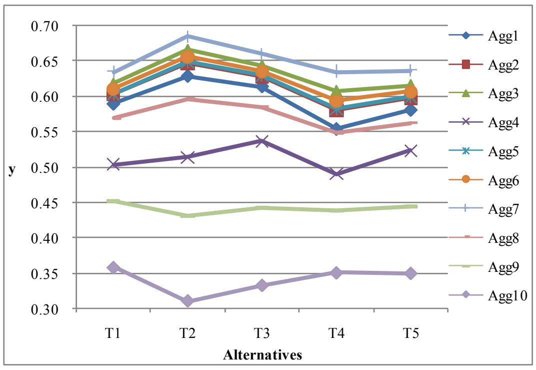
owaW(0.6)

Figure 6.4a: Results of individual aggregation operators (part I)

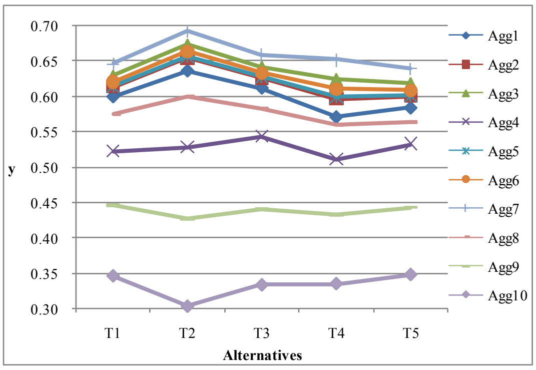
Although $w_1 > w_2 > \ldots > w_5$ except for $\delta = 1$, the distribution among the weights are narrowed whilst δ increases. The sensitivity of each AO for the changes of weight is different. When the difference among the weights get less (e.g. increase of δ), the outputs of *wgo05 and wgo09* decrease while the output of others AOs increase. In addition, *wmed* has relative sensitivity of the change of the values of weights.



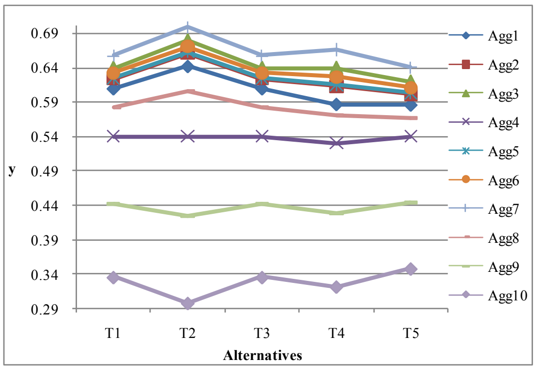
owaW(0.7)



owaW(0.8)



owaW(0.9)



owaW(1.0)

Figure 6.4b: Results of individual aggregation operators (part II)

Regarding the patterns of the AO population in the figures, the figures show that the lines of AOs are closer while δ is less. When δ increases, which means the gap of the weights of the criteria is reduced, the lines get farther apart. The main reason is that the criteria in a high index become more significant, and the values of the criteria in a higher index are more than the values of the criteria in a lower index.

Regarding the patterns of CSAO, the number of the AOs in Opt should be more than the number of the AOs in Pes. The main reason is that more lines are located in upper position of the y-axis. This issue is investigated in depth in the next sub-section.

6.6.3 Selection of AO by CSAO

What a decision maker finally feels of interest is not the properties of the aggregation operators, but which AO is the most suitable. In fact, there is likely no absolute answer. In the real world, no decision maker can always guarantee an absolutely accurate answer (except for those who are arrogant), but the best and the most appropriate answer which he thinks is correct (but others may not agree). Similarly, why do they make different decisions when the objective situation and background are the same? One of the explanations is that they have different cognitive styles or individual differences. Some make clever decisions whilst some do not. In the mathematician's view, how they make decision can be modeled by equations. In the CSAO model, each AO reflects a different cognitive style. CSAO is used to classify the cognitive styles. This research proposes that CSAO is represented by DAAO-1 and DAAO-2.

Table 6.6 shows the $d^*(k)$ of DAAO-1 of the proposed decision matrix where

$W = owaW(0.9)$. Interestingly, no matter which alternative input set of the decision matrix is used, the order of the AOs (k) is preserved to be the same.

If the decision maker chooses *Opt*, there are seven options to represent the Optimistic AO. It is too subjective to use ArgMax in eq.6.7, thus DAAO-2 is needed. From DAAO-2 (algorithm 6.2), $\{d_{ij}^*\}_1$, $\{d_{ij}^*\}_2, \{d_{ij}^*\}_3, \{d_{ij}^*\}_4, \{d_{ij}^*\}_5$ are respectively:

$$\begin{bmatrix} \varnothing & 0 & 3 \\ \varnothing & 2 & 5 \\ \varnothing & 0 & 9 \\ 1 & 0 & 10 \\ 0 & 3 & \varnothing \\ 0 & 3 & \varnothing \\ 2 & 5 & \varnothing \end{bmatrix}, \begin{bmatrix} \varnothing & 0 & 3 \\ \varnothing & 2 & 5 \\ \varnothing & 0 & 9 \\ 1 & 0 & 10 \\ 0 & 3 & \varnothing \\ 0 & 3 & \varnothing \\ 2 & 5 & \varnothing \end{bmatrix}, \begin{bmatrix} \varnothing & 0 & 3 \\ \varnothing & 2 & 5 \\ \varnothing & 0 & 9 \\ 1 & 0 & 10 \\ 0 & 0 & \varnothing \\ 0 & 3 & \varnothing \\ 2 & 5 & \varnothing \end{bmatrix}, \begin{bmatrix} \varnothing & 2 & 4 \\ \varnothing & 2 & 7 \\ \varnothing & 0 & 9 \\ 1 & 0 & 10 \\ 0 & 3 & \varnothing \\ 0 & 4 & \varnothing \\ 2 & 7 & \varnothing \end{bmatrix}, \begin{bmatrix} \varnothing & 0 & 3 \\ \varnothing & 2 & 5 \\ \varnothing & 0 & 9 \\ 1 & 0 & 10 \\ 0 & 0 & \varnothing \\ 0 & 3 & \varnothing \\ 2 & 6 & \varnothing \end{bmatrix}$$

From eq. (6. 15), then

$$\{d_{ij}^*\}^* = Max\left(Mode\left(Join\left(\left\{\{d_{ij}^*\}_\beta\right\}\right)\right)\right) = \begin{bmatrix} \varnothing & 0 & 3 \\ \varnothing & 2 & 5 \\ \varnothing & 0 & 9 \\ 1 & 0 & 10 \\ 0 & 3 & \varnothing \\ 0 & 3 & \varnothing \\ 2 & 5 & \varnothing \end{bmatrix}$$

If a decision maker chooses “Pes” for the AO in the decision system, in the first rating step, there is only one choice, $wrp01$, as it is indicated in table 6.6. The second rating category in V_{hd} is unnecessary.

If “Ntl” is chosen, for the representation of AO, $wrp05$ and $wgo05$ are the candidates, by using eq. 6.7, where $wgo05$ is for “Ntl”.

Table 6.6: The linguistic presentation of the style of the decision attitude for the AOs of the decision matrix $\{d^*(k)\}^*$

(k)	Agg	$d_1^*(k)$	$d_2^*(k)$	$d_3^*(k)$	$d_4^*(k)$	$d_5^*(k)$	$\{d_\beta^*(k)\}^*$
1	$wrp01$	Pes	Pes	Pes	Pes	Pes	Pes
2	$wrp05$	Ntl	Ntl	Ntl	Ntl	Ntl	Ntl
3	$wgo05$	Ntl	Ntl	Ntl	Ntl	Ntl	Ntl
4	wam	Opt	Opt	Opt	Ntl	Ntl	Opt
5	$wrp20$	Opt	Opt	Opt	Ntl	Opt	Opt
6	$wgo01$	Opt	Opt	Opt	Opt	Opt	Opt
7	$wgo09$	Opt	Opt	Opt	Opt	Opt	Opt
8	$wmed$	Opt	Opt	Opt	Opt	Opt	Opt
9	wgm	Opt	Opt	Opt	Opt	Opt	Opt
10	whm	Opt	Opt	Opt	Opt	Opt	Opt

Table 6.7: The AO of the style of the decision attitude for the linguistic terms of the decision matrix $\left\{\{d_j^*\}_\beta\right\}^*$

j	d	$\{d_j^*\}_1$	$\{d_j^*\}_2$	$\{d_j^*\}_3$	$\{d_j^*\}_4$	$\{d_j^*\}_5$	$\left\{\{d_j^*\}_\beta\right\}^*$
1	Pes	1	1	1	1	1	1
2	Ntl	3	3	3	3	3	3
3	Opt	10	10	10	10	10	10

When "Opt" is chosen, there are seven candidates. It is too straightforward to use E.q. 6.7. Thus the second rating category in V_{hd} is needed. The index of the AO can be found in $\{d_{ij}^{*}\}^{*}$. $wgo05, wrp20, wgo09$ and wgm are the options with respect to the choice of the second rating linguistic term.

6.7 Summary and remarks

As different aggregation operators produce different results, these results can be described by the possibility likelihoods of the decision attitudes. The selection of the aggregation operators is related to the likelihoods of the decision attitudes of the operators. To achieve the proposal, the Cognitive Style and Aggregation Operator (CSAO) model is proposed to analyze the mapping relationship between aggregation operators and decision attitudes on the basis of fuzzy set theory. The CSAO model has two types of Decision Attitude and Aggregation Operator (DAAO) model: DAAO-1, DAAO-2. The difference is that DAAO-1 applies classical single dimension linguistic terms whilst DAAO-2 applies the compound linguistic terms proposed in chapter 4. Three Algorithms for AO selection are developed in this chapter.

The appropriate operators will be chosen according to the linguistic terms of the decision attitudes in the CSAO model. The cognitive style is characterized by the decision attitude. The CSAO model is useful for measuring the distribution of the AOs.

Examples 6.1 and 6.3 test 17 AOs. On the basis of the result pattern, examples 6.2 and 6.4 select only 7 AOs. From the numerical examples, it can be concluded that the weighted median with other t-connorms and t-norms, owmax, owmin, and *owa* in the literature review (Chapter 2.3.5) (Yager ijufks 1994, Renata Smolikova and Wachowiak, 2002) is not appropriate for the Cognitive Network Process. The reasons are stated after the numerical examples 6.1-6.2.

In the detailed numerical analyses section, 10 AOs are tested for 10 decision matrices. The best practices of AO selection are illustrated using the combination of DAAO-1 and DAAO-2. The results can be found in chapter 6.5.

Limitation of the CSAO model is that the CSAO relies on the definitions of the candidates. If some candidates are abnormal, the CSAO pattern will be abnormal too. Usually the abnormal operators produce excessively optimistic or excessively positive results. In this case, the expert can remove the abnormal AO by his perception, and then recalculate the patterns again. After several refinements of the patterns, the appropriate CSAO model can be developed.

The CSAO is devoted to a proposal as how to map a collection of aggregation operators into a collection of decision attitudes by the CSAO model. This model is typically useful for those unsolved issues in the selection of aggregation operators. The OA candidates are determined by the decision maker with respect to cognitive style, which is characterized by decision attitudes. Thus the CSAO model is useful for the decision making applications with consideration of the cognitive styles (or decision attitudes) of the decision makers. If there is no specification of the selection of AO from experts, the weighted average is the default setting for the CNP model due to its computation efficiency, easy understanding, and wide acceptance.

Chapter 7 Fuzzy Collective Cognitive Network Process

7.1 Introduction

In chapter 3, the concept of the Cognitive Network Process is presented. Chapters 4, 5, and 6 develop the models of the compound linguistic ordinal scales, cognitive pairwise matrix and cognitive prioritization operator, as well as the cognitive style and aggregation operator (CSAO) respectively.

The narrow definition of the CNP is of the Structural Assessment Network (SAN) of a single decision maker, and the crisp value represented for the compound linguistic variable. This chapter extends the concept of CNP, and proposes a boarder definition of CNP, which is named Fuzzy Collective Cognitive Network Process (FCCNP). The FCCNP is of multiple decision makers with fuzzy inputs. FCCNP can be divided into Collective CNP (CCNP) and fuzzy CNP (FCNP). CCNP is of the SAN of multiple decision makers with crisp inputs, whilst FCNP is of the SAN of a single decision maker with fuzzy inputs.

The structure of this chapter is organized as follows. Chapter 7.2 discusses the concept of the collective cognitive network process. Chapter 7.3 discusses the concept of the fuzzy cognitive network process. To merge the concepts of CCNP and FCNP, chapter 7.4 discusses the concept of the fuzzy collective network process. The numerical analysis is performed in chapter 7.5 whilst chapter 7.6 concludes this chapter.

7.2 Collective cognitive network process (CCNP)

The collective cognitive network process (CCNP), which is also named as the group cognitive network process, is the CNP involved by a collection of experts $\{e\}$. Thus CCNP has the form:

$$CCNP = \left(CNP, \{(e, we)\}\right) \tag{7.1}$$

, where (e, we) is a 2-tuple in which an expert e has authority weight we.

The following shows the variation between CNP and CCNP in views of rating scales, pairwise opposite matrix, cognitive prioritization operator and information fusion.

7.2.1 Cognitive rating scales

CNP applies the Compound Linguistic Ordinal Scale (CLOS) model, which is proposed in chapter 4, for the pairwise opposite comparison. For the pairwise opposite comparison, the scales of object A to object B, which is denoted by $X_{\aleph}^{+}$ is the opposite relationship of the scales of object B to object A, which is denoted by $X_{\aleph}^{-}$.

Regarding the syntactic form, CLOS is established on a compound linguistic variable $\alpha \in \aleph_{mn}$ which is comprised of the element from the linguistic term vectors respectively: hedge vector $\overrightarrow{V_h}$, directional vector $\overrightarrow{V_d}$, and atomic vector $\overrightarrow{V_a}$. A matrix of compound linguistic variable $\aleph_{mn}$ is built on the syntactic rule algorithm (algorithm 4.1) $\aleph_{mn} = G_{\aleph}\left(\overrightarrow{V_h}, \overrightarrow{V_d}, \overrightarrow{V_a}\right)$, has the following form:

$$\aleph_{mn} = \begin{array}{c} \\ \alpha_1 \\ \vdots \\ \alpha_\eta \\ \alpha_{\eta+1} \\ \alpha_{\eta+2} \\ \vdots \\ \alpha_m \end{array} \begin{array}{c} \begin{array}{ccccc} \alpha^1 & \alpha^2 & \cdots & \alpha^{n-1} & \alpha^n \end{array} \\ \begin{bmatrix} \varnothing & v_{hd_1} \oplus v_{a_2} & \cdots & v_{hd_1} \oplus v_{a_{n-1}} & v_{hd_1} \oplus v_{a_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varnothing & v_{hd_\eta} \oplus v_{a_2} & \ddots & v_{hd_\eta} \oplus v_{a_{n-1}} & v_{hd_\eta} \oplus v_{a_n} \\ {v_{a_1}}^{\theta} & {v_{a_2}}^{\theta} & \ddots & {v_{a_{n-1}}}^{\theta} & {v_{a_n}}^{\theta} \\ v_{hd_{\eta+2}} \oplus v_{a_1} & v_{hd_{\eta+2}} \oplus v_{a_2} & \ddots & v_{hd_{\eta+2}} \oplus v_{a_{n-1}} & \varnothing \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{hd_m} \oplus v_{a_1} & v_{hd_m} \oplus v_{a_2} & \cdots & v_{hd_m} \oplus v_{a_{n-1}} & \varnothing \end{bmatrix} \end{array} \tag{7.2}$$

, where v_{hd} is the element of the combination of $\overrightarrow{V_h}$ and $\overrightarrow{V_d}$.

The numerical representation is derived by the semantic rule algorithm or fuzzy normal distribution (algorithm 4.2) in chapter 4.

$$\bar{X}_{\aleph}^{+} = f_{\overline{X}}(\aleph) = M(\aleph)$$

$$= f_{\overline{X}}\left(\left\{\left(\gamma_{\alpha^j}, d_{\alpha^j}, \tau_{\alpha^j}, \left\{{\mu_{d^j\phi}}^{-1}\right\}^{\phi}\right)\right\}, [X_{\min}, X_{\max}], \left(\varphi\left(\overrightarrow{V_h}\right), \lambda_0\right)\right)$$

$$= f_{\overline{X}}\left(\left\{\left(\gamma_{\alpha^j}, d_{\alpha^j}, \tau_{\alpha^j}, \left\{{\mu_{d^j\phi}}^{-1}\right\}^{\phi}\right)\right\}, [0, \kappa], \left(\varphi\left(\overrightarrow{V_h}\right), \lambda_0\right)\right) \tag{7.3}$$

$\bar{X}_{\aleph}^{+}$ is the crisp numerical representation of $\aleph$. κ is the normal utility, which is the mean of the individual utility values of the comparison objects, and $\kappa > 0$. γ_{α^j} is the modal value, d_{α^j} is symmetric distance (by default, $d_{\alpha^1} = d_{\alpha^2} =, \ldots, = d_{\alpha^n}$), τ_{α^j} is tuning parameter of the membership function, μ_{d^j} is the membership function of α^j, and $\mu_{d^j\phi}{}^{-1}$ is the inverse membership function. The parameters of membership fuzziness process $\left(\varphi\left(\overrightarrow{V_h}\right),\lambda_0\right)$ determine the distribution of the $\overrightarrow{V_{hd}}$ with respect to the corresponding atomic fuzzy sets.

Table 7.1: Pairwise comparison interval scale schema using the CLOS

Numerical Representation $\bar{X}_{\aleph}$	Atomic verbal Scale $\overrightarrow{V_a}$	Explanations
0	Equally	Two activities contribute equally to the objective.
2	Slightly	Experience and judgment slightly favour one activity over another.
4	Moderately	Experience and judgment moderately favour one activity over another.
6	Strongly	An activity is favored very strongly over another; its dominance demonstrated in practice.
8	Essentially	The evidence favoring one activity over another is of the highest possible order of affirmation
(0-2) (2-4) (4-6) (6-8)	$\overrightarrow{V_{hd}}=$ [“much below”, “quite below”, “little below”, “absolutely”, “little below”, “ quite above”, “much above”]	Intermediate values between adjacent scale values using $\overrightarrow{V_{hd}}$. The detailed values are shown in the matrix $X_{\aleph}^{+}$ which is calculated by $f_{\overline{X}}\left(\left\{\left(\gamma_{\alpha^j},d_{\alpha^j},\tau_{\alpha^j},\left\{\mu_{d^j\phi}{}^{-1}\right\}^{\phi}\right)\right\},[0,\kappa],\left(\varphi\left(\overrightarrow{V_h}\right),\lambda_0\right)\right)$.
(-8,0)	Opposite of the above	If object *i* is compared with object *j* with assignment of a value, then object *j* compared with object *i* has the opposite of the value. The details values are shown in $X_{\aleph}^{-}$.

In addition, $X_{\aleph}^{+}$ is the opposite linguistic representation of $X_{\aleph}^{-}$, and $\bar{X}_{\aleph}^{+} = -\bar{X}_{\aleph}^{-}$ where $\bar{X}_{\aleph}^{+} = [0,\kappa]$ and $\bar{X}_{\aleph}^{-} = [-\kappa,0]$. Thus $\bar{X}_{\aleph} = \{\bar{X}_{\aleph}^{-}, \bar{X}_{\aleph}^{+}\} = [-\kappa,\kappa]$.

Example 7.1

Let the comparison interval scale schema of the Hedge-Direction-Atom Linguistic Representation Model be $\left(\aleph, \bar{X}_{\aleph} = \{\bar{X}_{\aleph}^{-}, \bar{X}_{\aleph}^{+}\}, f_{\bar{X}}(\aleph)\right)$. To construct the labels of the comparison interval scale $\aleph$, let $\overrightarrow{V_a} = [\text{Equal,Slight,Moderate,Strong,Essential}]$, $\overrightarrow{V_h} = [\text{Little,Quite,Much}]$ and $\overrightarrow{V_d} = [\text{Below,Absolutely,Above}]$. By algorithm 4.1, then

$$\aleph = \begin{bmatrix} \varnothing & MB\text{-}Sl & MB\text{-}Mo & MB\text{-}St & MB\text{-}Es \\ \varnothing & QB\text{-}Sl & QB\text{-}Mo & QB\text{-}St & QB\text{-}Es \\ \varnothing & LB\text{-}Sl & LB\text{-}Mo & LB\text{-}St & LB\text{-}Es \\ A\text{-}Eq & A\text{-}Sl & A\text{-}Mo & A\text{-}St & A\text{-}Es \\ LA\text{-}Eq & LA-Sl & LA\text{-}Mo & LA\text{-}St & \varnothing \\ QA\text{-}Eq & QA\text{-}Sl & QA\text{-}Mo & QA\text{-}St & \varnothing \\ MA\text{-}Eq & MA-Sl & MA\text{-}Mo & MA\text{-}St & \varnothing \end{bmatrix}.$$

Regarding the representation values, let $[0,\kappa] = [0,8]$, $d_{\alpha^{1,\dots,5}} = 2$, $\vec{\gamma} = [0,2,4,6,8]$, $\tau_{\alpha^{1,\dots,5}} = 2$, $\{\mu_{\alpha^{j\phi}}{}^{-1}\} = PbMF^{-1}$, $\varphi(\overrightarrow{V_h}) = [1,1,1]$, $\lambda_0 = 0.5$. By using algorithm 4.2, $X_{\aleph}^{+}$ is of the form:

$$X_{\aleph}^{+} = \begin{bmatrix} 0 & 0.645 & 2.645 & 4.645 & 6.645 \\ 0 & 1.366 & 3.366 & 5.366 & 7.366 \\ 0 & 1.764 & 3.764 & 5.764 & 7.764 \\ 0 & 2.000 & 4.000 & 6.000 & 8.000 \\ 0.236 & 2.236 & 4.236 & 6.236 & 0 \\ 0.634 & 2.634 & 4.634 & 6.634 & 0 \\ 1.355 & 3.355 & 5.355 & 7.355 & 0 \end{bmatrix},$$

and the opposite of $X_{\aleph}^{+}$ is

$$X_{\aleph}^{-} = \begin{bmatrix} 0 & -0.645 & -2.645 & -4.645 & -6.645 \\ 0 & -1.366 & -3.366 & -5.366 & -7.366 \\ 0 & -1.764 & -3.764 & -5.764 & -7.764 \\ 0 & -2.000 & -4.000 & -6.000 & -8.000 \\ -0.236 & -2.236 & -4.236 & -6.236 & 0 \\ -0.634 & -2.634 & -4.634 & -6.634 & 0 \\ -1.355 & -3.355 & -5.355 & -7.355 & 0 \end{bmatrix}$$

The result is summarized in table 7.1. As there are many options, a Deducted Rating Strategy is needed. Fig. 7.1 illustrates the steps to show how the rating process is performed. The details are summarized in algorithm 4.3.

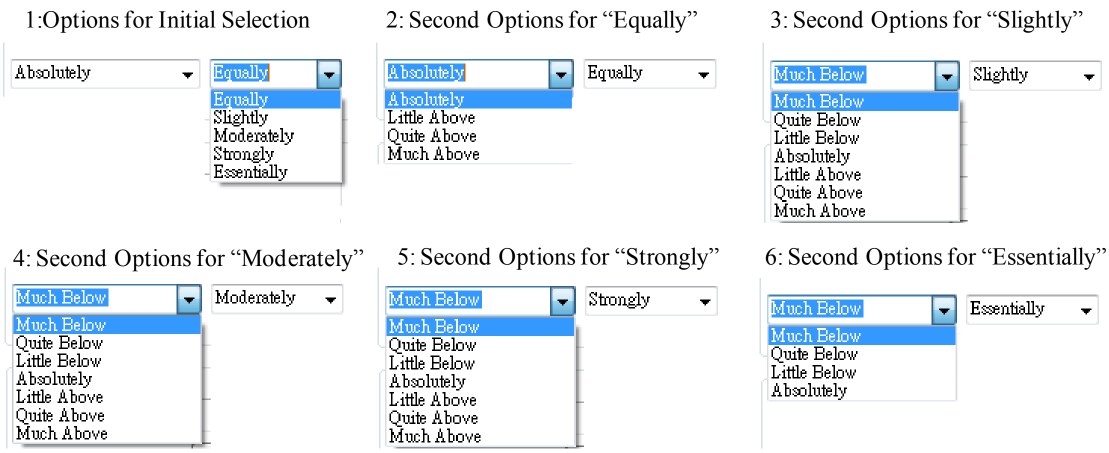


Figure 7.1: Deducted rating strategy for the compound interval scale

The atomic variable can be added to 9 terms. When a compound rating scale is used, more options are created. This issue is illustrated in following example.

Example 7.2:

Let the comparison interval scale schema of the Hedge-Direction-Atom Linguistic Representation Model be $\left(\aleph, \overline{X}_{\aleph} = \left\{\overline{X}_{\aleph}^{-}, \overline{X}_{\aleph}^{+}\right\}, f_{\overline{X}}(\aleph)\right)$. To construct the labels of the comparison interval scale $\aleph$, $\overrightarrow{V_a}$ and $\overrightarrow{V_{hd}}$ are shown in table 7.2. By algorithm 4.1, $\aleph$ is of the form:

$$\begin{bmatrix} \varnothing & MB\text{-}Wk & MB\text{-}Mo & MB\text{-}Mp & MB\text{-}St & MB\text{-}Sp & MB\text{-}VS & MB\text{-}VSp & MB\text{-}Es \\ \varnothing & QB\text{-}Wk & QB\text{-}Mo & QB\text{-}Mp & QB\text{-}St & QB\text{-}Sp & QB\text{-}VS & QB\text{-}VSp & QB\text{-}Es \\ \varnothing & LB\text{-}Wk & LB\text{-}Mo & LB\text{-}Mp & LB\text{-}St & LB\text{-}Sp & LB\text{-}VS & LB\text{-}VSp & LB\text{-}Es \\ A\text{-}Eq & A\text{-}Wk & A\text{-}Mo & A\text{-}Mp & A\text{-}St & A\text{-}Sp & A\text{-}VS & A\text{-}VSp & A\text{-}Es \\ LA\text{-}Eq & LA-Wk & LA\text{-}Mo & LA\text{-}Mp & LA\text{-}St & LA\text{-}Sp & LA\text{-}VS & LA\text{-}VSp & \varnothing \\ QA\text{-}Eq & QA\text{-}Wk & QA\text{-}Mo & QA\text{-}Mp & QA\text{-}St & QA\text{-}Sp & QA\text{-}VS & QA\text{-}VSp & \varnothing \\ MA\text{-}Eq & MA-Wk & MA\text{-}Mo & MA\text{-}Mp & MA\text{-}St & MA\text{-}Sp & MA\text{-}VS & MA\text{-}VSp & \varnothing \end{bmatrix}.$$

Regarding the representation values $\overline{X}$, let $[0,\kappa]=[0,8]$, $d_{\alpha^{1,\dots,9}}=1$, $\vec{\gamma}=[0,1,2,3,4,5,6,7,8]$, $\tau_{\alpha^{1,\dots,5}}=2$, $\left\{\mu_{\alpha^{j\phi}}{}^{-1}\right\}=PbMF^{-1}$, $\varphi\left(\overrightarrow{V_h}\right)=[1,1,1]$, $\lambda_0=0.5$. By algorithm 4.2, $X_{\aleph}^{+}$ is of the form:

$$X_{\aleph}^{+}=\begin{bmatrix} 0 & 0.323 & 1.323 & 2.323 & 3.323 & 4.323 & 5.323 & 6.323 & 7.323 \\ 0 & 0.683 & 1.683 & 2.683 & 3.683 & 4.683 & 5.683 & 6.683 & 7.683 \\ 0 & 0.882 & 1.882 & 2.882 & 3.882 & 4.882 & 5.882 & 6.882 & 7.882 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0.118 & 1.118 & 2.118 & 3.118 & 4.118 & 5.118 & 6.118 & 7.118 & 0 \\ 0.317 & 1.317 & 2.317 & 3.317 & 4.317 & 5.317 & 6.317 & 7.317 & 0 \\ 0.677 & 1.677 & 2.677 & 3.677 & 4.677 & 5.677 & 6.677 & 7.677 & 0 \end{bmatrix}$$

, and the opposite of $X_{\aleph}^{+}$ is of the form:

$$X_{\aleph}^{-}=\begin{bmatrix} 0 & -0.323 & -1.323 & -2.323 & -3.323 & -4.323 & -5.323 & -6.323 & -7.323 \\ 0 & -0.683 & -1.683 & -2.683 & -3.683 & -4.683 & -5.683 & -6.683 & -7.683 \\ 0 & -0.882 & -1.882 & -2.882 & -3.882 & -4.882 & -5.882 & -6.882 & -7.882 \\ 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 \\ -0.118 & -1.118 & -2.118 & -3.118 & -4.118 & -5.118 & -6.118 & -7.118 & 0 \\ -0.317 & -1.317 & -2.317 & -3.317 & -4.317 & -5.317 & -6.317 & -7.317 & 0 \\ -0.677 & -1.677 & -2.677 & -3.677 & -4.677 & -5.677 & -6.677 & -7.677 & 0 \end{bmatrix}$$

Table 7.2: Scale schema: pairwise opposite comparison

Index i	Atomic verbal scale $\overrightarrow{V_a}$	Numerical scales $X_{\aleph}^{+} = \left\{ {i\kappa}/{n}, i = 0, \ldots, n \right\}$
0	Equally	0
1	Weakly	$\kappa/8$
2	Moderately	$\kappa/4$
3	Moderately plus	$3\kappa/8$
4	Strongly	$\kappa/2$
5	Strong Plus	$5\kappa/8$
6	Very Strongly	$3\kappa/4$
7	Very strongly Plus	$7\kappa/8$
8	Extremely	κ
{(*i,j*)}	$\overrightarrow{V_{hd}} =$ ["much below", "quite below", "little below", "absolutely", "little below", "quite above", "much above"]	Intermediate values between adjacent scale values using the directional hedge variable $\overrightarrow{V_{hd}}$. The details are shown in $X_{\aleph}^{+}$ which is calculated by the form: $f_{\overline{X}}\left(\left\{\left(\gamma_{\alpha^j}, d_{\alpha^j}, \tau_{\alpha^j}, \left\{{\mu_{d^{j\phi}}}^{-1}\right\}^{\phi}\right)\right\}, [0,\kappa], \left(\varphi\left(\overrightarrow{V_h}\right), \lambda_0\right)\right).$
{*-i*}	opposites of Above	$X_{\aleph}^{-}$ in (from $-\kappa$ to 0)

The results are summarized in table 7.2 (In this example, κ=1) . As there are many options, the deducted rating strategy (algorithm 4.3) is needed. Fig. 7.2 illustrates the steps to show how the rating process is performed.

The single rating process is only a special case of this double step rating process as the directional hedge terms are ignored. Thus for the atomic linguistic terms and the representation values, one can refer to tables 7.1-7.2. This paper uses compound rating scales rather than single rating scales due to the computational accuracy.

Figure 7.2: Deducted rating strategy for the compound interval scale of 9 point atomic terms

7.2.2 Pairwise opposite matrix (POM)

The Pairwise Opposite Matrix is used to interpret the individual utilities of the candidates. Let an ideal utility set be $V=\{v_1,\ldots,v_n\}$, and the comparison score be $b_{ij}\cong v_i-v_j$. The ideal pairwise opposite matrix is $\tilde{B}=\left[v_i-v_j\right]$. A subjective judgmental pairwise opposite matrix using interval scales is $B=\left[b_{ij}\right]$. $\tilde{B}$ is determined by B as follows:

$$\tilde{B}=\left[\tilde{b}_{ij}\right]=\begin{bmatrix} v_1-v_1 & v_1-v_2 & \ldots & v_1-v_n \\ v_2-v_1 & v_2-v_2 & \ldots & v_2-v_n \\ \vdots & \vdots & \ddots & \vdots \\ v_n-v_1 & v_n-v_2 & \cdots & v_n-v_n \end{bmatrix}\cong\begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}=\left[b_{ij}\right]=B \quad (7.4)$$

If $i=j$, then $b_{ij}=v_i-v_j=0$. Thus the above matrix is in the form:

$$\tilde{B}=\left[\tilde{b}_{ij}\right]=\begin{bmatrix} 0 & v_1-v_2 & \ldots & v_1-v_n \\ v_2-v_1 & 0 & \ldots & v_2-v_n \\ \vdots & \vdots & \ddots & \vdots \\ v_n-v_1 & v_n-v_2 & \cdots & 0 \end{bmatrix}\cong\begin{bmatrix} 0 & b_{12} & \ldots & b_{1n} \\ b_{21} & 0 & \ldots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & 0 \end{bmatrix}=\left[b_{ij}\right]=B \quad (7.5)$$

Usually, $\left[b_{ij}\right]$ is given through the rating process of the expert, and $b_{ij}\in\bar{X}_{\aleph}=\left\{\bar{X}_{\aleph}^{-},\bar{X}_{\aleph}^{+}\right\}=[-\kappa,\kappa]$. The expert only fills an upper triangular matrix of the form:

$$B^{+}=\begin{cases} b_{ij} & i<j \\ 0 & \text{otherwise} \end{cases}\text{, or written explicitly, } B^{+}=\begin{bmatrix} 0 & b_{12} & \ldots & b_{1n} \\ 0 & 0 & \ldots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad (7.6)$$

The lower triangular matrix is given by the opposite of an upper triangular matrix of the form:

$$B^- = \begin{cases} b_{ij} & i > j \\ 0 & \text{Otherwise} \end{cases}, \text{ or written explicitly, } B^- = \begin{bmatrix} 0 & 0 & \dots & 0 \\ b_{21} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & 0 \end{bmatrix} \quad (7.7)$$

$\left[b_{ij}\right]$ is achieved by $B = B^+ + B^-$. For a complete comparison of a set of candidates, POM needs $n(n-1)/2$ ratings. B is validated by the Accordant Index of the form:

$$AI = \frac{1}{n^2}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}}\ ,\ \ d_{ij} = \sqrt{Mean\left(\left(\frac{1}{\kappa}\left(B_i + B_j^T - b_{ij}\right)\right)^2\right)},\ \ i,j \in (1,\dots,n)$$

, where $AI \geq 0$, and κ is the normal utility. (7.8)

If $AI = 0$, then B is perfectly accordant; If $0 < AI \leq 0.1$, then B is satisfactory, then. If $AI > 0.1$, then B is unsatisfactory .

After the set of POM's is assessed, they are converted into utility vectors by a cognitive prioritization operation is as the following section.

7.2.3 Cognitive prioritization operator (CPO)

Two methods are recommended in chapter 5: Primitive Least Squares (PLS) (or Row Average plus the normal Utility (RAU)) and Least Penalty Squares (LPS), as follows.

The vector of individual utilities can be derived by the Primitive Least Squares Optimization model which is of the form:

$\text{PLS}\left(B^{+},\kappa\right)=$

$$\text{Min} \quad \bar{\Delta}=\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left(b_{ij}-v_i+v_j\right)^2 \tag{7.9}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} v_i = n\kappa ,$$

where $n=\left|\{v_i\}\right|$, and κ is the normal utility.

The solution of the closed form can be solved manually and is **RAU**, given by:

$$RAU\left(B,\kappa\right)=\left[v_i : v_i=\left(\frac{1}{n}\sum_{j=1}^{n} b_{ij}\right)+\kappa, \forall i\in\{1,\ldots,n\}\right] \tag{7.10}$$

Regarding LPS, the individual utility vector can be derived by

$\text{LPS}\left(B^{+},\kappa\right)=$

$$\text{Min} \quad \hat{\Delta}=\sum_{i=1}^{n}\sum_{j=i+1}^{n}\beta_{ij}\cdot\left(b_{ij}-v_i+v_j\right)^2$$

$$, \ \beta_{ij}=\begin{cases}\beta_1, & \text{v}_i>v_j \ \& \ \text{b}_{ij}>0 \\ & \text{or v}_i<v_j \ \& \ \text{b}_{ij}<0 \\ \beta_2, & \text{v}_i=v_j \ \& \ \text{b}_{ij}\neq 0 \\ & \text{or } v_i\neq v_j \ \& \ \text{b}_{ij}=0 \\ \beta_3, & otherwise\end{cases}, \ 1=\beta_1\leq\beta_2\leq\beta_3 \tag{7.11}$$

$$\text{s.t.} \quad \begin{matrix}\sum_{i=1}^{n} v_i = n\kappa, \\ \text{v}_i\geq 0, i=1,2,\ldots,n\end{matrix},$$

where $n=\left|\{v_i\}\right|$, and κ is the normal utility.

For the most decision problems, summation of the priority vector $W=\{w_1,\ldots,w_n\}$ is equal to one, i.e. $\sum_{i=1}^{n} w_i = 1$. W is said to be a normalized priority vector (or a priority vector in short). In order to use the proposed methods, the individual utility from POM is rescaled (or normalized) as a normalized priority vector by the rescale function of the normalization function, and has the following form.

$$W = \left\{ w_i : w_i = \frac{v_i}{n\kappa}, \forall i \in \{1,\ldots,n\} \right\}, \text{ which } \sum_{i\in\{1,\ldots,n\}} v_i = n\kappa \qquad (7.12)$$

7.2.4 Crisp multi-experts multi-criteria aggregation model

There are three types of aggregation: aggregation of the structural criteria, aggregation of the structural experts, and aggregation of both, i.e. multi-experts multi-criteria aggregation (MEMC). The primitive CNP typically applies the first type whilst the Collective CNP applies the third.

Consider a cluster $Clst(nd, gn)$. The aggression function for a node *nd* is to combine the set of its data granules $\{ng_i\}$ and the set of the corresponding weights of the granules $\{wng_i\}$ into a meaningful or representative value for *nd*. The function has the form:

$$nd = Agg\left(\left[(ng_i, wng_i) \right]_{i=1}^{|ng_i|} \right) \qquad (7.13)$$

, where $|ng_i|$ is the cardinal number of the nodes.

Consider a typical CNP structure comprising of the structural criteria of two layers and one expert layer: Objective cluster $Clst(O, \{c_i : i = 1,\ldots,q\})$, c_i Cluster $Clst(c_i, \{c_{i,j} : j = 1,\ldots,q_i\})$, and expert cluster, $Clst(\breve{c}_i, \{e_k\})$. On the basis of a template clusters, following forms are defined.

The aggression function for a criterion c is to combine its sub-criteria c_i and weights of the sub-criteria w_i into a meaningful or representative value for *O*. The function has the form:

$$O = Oagg\left(\left[\left(c_i, w_i\right)\right]_{i=1}^{q}\right) \tag{7.14}$$

, where q is the cardinal number of the criteria.

Likewise, aggregation of c_i has the form:

$$c_i = Cagg_i\left(\left[\left(c_{ij}, w_{ij}\right)\right]_{j=1}^{q_j}\right) \tag{7.15}$$

, where q_j is the cardinal number of the sub-criteria of criteria j.

Note that the cardinal number q_i may not be equal to the cardinal number $q_{i'\neq i}$. Thus, C is a non-rectangular matrix or a jagged array, i.e. $C = \left\{\left(c_{1,1}, \ldots, c_{1,q_1}\right), \ldots, \left(c_{n,1}, \ldots, c_{n,q_n}\right)\right\}$.

The aggregation of the collective experts $\left\{e_{ij}\right\}$ evaluating a measurable criterion *i* $\breve{c}_i$ with the weights of experts $\left\{we_{ij}\right\}$ is of the form:

$$\breve{c}_i = Eagg_i\left(\left[\left(e_{ij}, we_{ij}\right)\right]_{j=1}^{\left|\left\{we_{ij}\right\}\right|}\right) \tag{7.16}$$

For a measurable criterion *i* of the collection of experts $Clst\left(\breve{c}_i, \left\{e_k\right\}\right)$, we_{ij} can be derived by the cognitive prioritization *(CP)* of a pairwise opposite matrix, and has the form:

$$\left\{we_{ij}\right\} = CP\begin{pmatrix} & e_1 & \cdots & e_j & \cdots & e_{|\{we_{ij}\}|} \\ e_1 & & & & & \\ \vdots & & & & & \\ e_{j'} & & & a_{j',j} & & \\ \vdots & & & & & \\ e_{|\{we_{ij}\}|} & & & & & \end{pmatrix}, \quad \forall i. \tag{7.17}$$

On the other hand, in the structural criteria of two layers, c_{ij} is the measurable criterion. The collective experts $Clst\left(\breve{c}_i, \{e_k\}\right)$ can be regarded as the sub-layer of $c_{i,j}$ Cluster, i.e. $Clst\left(c_{i,j}, \{c_{i,j,k}\}\right)$, and has the form:

$$c_{ij} = Eagg_i\left(\left[\left(c_{ijk}, w_{ijk}\right)\right]_{k=1}^{q_{i,j}}\right) \tag{7.18}$$

,where the cardinal number of experts for c_{ij} is $q_{i,j}$. c_{ijk} is the value of attribute *j* of criteria *i* evaluated by expert *k,* and w_{ijk} is the corresponding weight. In this model, each attribute may not be evaluated by the same expert, and the same cardinal number of experts, i.e. $r_{i,j}$ may not be equal to $r_{i,j'\neq j}$. Thus, $\breve{c}_i$ is a non-rectangular matrix (or jagged array).

On the basis of above definitions, proposition 7.1 holds.

Proposition 7.1 (Multiple positive Aggregations): In a typical CCNP structure comprising the positive structural criteria of two layers and one layer of expert, the aggregations for an objective *O* are of the form:

$$O = Oagg\left(\left[\left(c_i, w_i\right)\right]_{i=1}^{q}\right)$$

$$= Cagg\left(\left[\left(Cagg_i\left(\left[\left(Eagg_i\left(\left[\left(c_{ijk}, w_{ijk}\right)\right]_{k=1}^{q_{i,j}}\right), w_{ij}\right)\right]_{j=1}^{q_j}\right), w_i\right)\right]_{i=1}^{q}\right) \tag{7.19}$$

Proof:

For $Clst\left(\breve{c}_i, \{e_k\}\right)$,

$$c_{ij} = Eagg_i\left(\left[\left(c_{ijk}, w_{ijk}\right)\right]_{k=1}^{q_{i,j}}\right);$$

For $Clst\left(c_i, \{c_{i,j}\}\right)$,

$$c_i = Cagg_i\left(\left[\left(c_{ij}, w_{ij}\right)\right]_{j=1}^{q_j}\right)$$

$$= Cagg_i\left(\left[\left(Eagg_{ij}\left(\left[\left(c_{ijk}, w_{ijk}\right)\right]_{k=1}^{q_{i,j}}\right), w_{ij}\right)\right]_{j=1}^{q_j}\right);$$

For $Clst\left(O, \{c_i\}\right)$,

$$O = Cagg\left(\left[\left(c_i, w_i\right)\right]_{i=1}^{q}\right)$$

$$= Cagg\left(\left[\left(Cagg_i\left(\left[\left(c_{ij}, w_{ij}\right)\right]_{j=1}^{q_j}\right), w_i\right)\right]_{i=1}^{q}\right)$$

$$= Cagg\left(\left[\left(Cagg_i\left(\left[\left(Eagg_i\left(\left[\left(c_{ijk}, w_{ijk}\right)\right]_{k=1}^{q_{i,j}}\right), w_{ij}\right)\right]_{j=1}^{q_j}\right), w_i\right)\right]_{i=1}^{q}\right).$$

□

Likewise, this follows proposition 7.2.

Proposition 7.2(Multiple positive and negative Aggregations): In a typical CCNP structure comprising the structural criteria of two layers, which are positive and negative, and one layer of expert, the aggregations for an objective including $Clst\left(\bar{O}^{+},\{c_i : i=1,\ldots,q'\}\right)$ and $Clst\left(\bar{O}^{-},\{c_i : i=q'+1,\ldots,q\}\right)$ are of the forms

$$\bar{O}^{+} = Cagg\left(\left[\left(Cagg_i\left(\left[\left(Eagg_i\left(\left[\left(c_{ijk},w_{ijk}\right)\right]_{k=1}^{q_{i,j}}\right),w_{ij}\right)\right]_{j=1}^{q_j}\right),w_i\right)\right]_{i=1}^{q'}\right) \tag{7.20}$$

$$\bar{O}^{-} = Cagg\left(\left[\left(Cagg_i\left(\left[\left(Eagg_i\left(\left[\left(c_{ijk},w_{ijk}\right)\right]_{k=1}^{q_{i,j}}\right),w_{ij}\right)\right]_{j=1}^{q_j}\right),w_i\right)\right]_{i=q'+1}^{q}\right) \tag{7.21}$$

$$\bar{O} = Oagg\left(\left[\left(c_i,w_i\right)\right]_{i=1}^{q}\right) = \bar{O}^{+} - \bar{O}^{-} \tag{7.22}$$

Proof:

The proof is trivial. It further develops proposition 7.2. □

In the above CCNP model, at least three kinds of aggregation operators are needed: $Eagg_i$, $Cagg_i$, and $Cagg$. The choice of the aggregation operators can apply to the cognitive style and aggregation operator (CSAO) model, which is comprised of DAAO-1 and DAAO-2 (Chapter 6). The weighted average operator is the default setting for the CNP model due to its computation efficiency, easy understanding, and wide acceptance.

7.2.5 Crisp multi-experts multi-criteria decision matrix

The multi-criteria decision matrix of $Clst\left(nd,gn\right)$ with respect to T of a single expert has the following form:

$$dm\left(T,Clst\left(nd,gn\right)\right)=\begin{array}{c|c} & \begin{pmatrix} wgn_1 & \dots & wgn_j & \dots & wgn_n \\ xgn_1 & \cdots & xgn_j & \cdots & xgn_n \end{pmatrix} \\ \begin{array}{c} T_1 \\ \vdots \\ T_k \\ \vdots \\ T_m \end{array} & \begin{pmatrix} & & & & \\ & & & & \\ & & v_{k,j} & & \\ & & & & \\ & & & & \end{pmatrix} \end{array} \tag{7.23}$$

The aggregation problem of $dm\left(T,Clst\left(nd,gn\right)\right)$ is as follows:

$$T_{nd}=Agg\left(\begin{array}{c|c} & \begin{pmatrix} wgn_1 & \dots & wgn_j & \dots & wgn_n \\ xgn_1 & \cdots & xgn_j & \cdots & xgn_n \end{pmatrix} \\ \begin{array}{c} T_1 \\ \vdots \\ T_k \\ \vdots \\ T_m \end{array} & \begin{pmatrix} & & & & \\ & & & & \\ & & v_{k,j} & & \\ & & & & \\ & & & & \end{pmatrix} \end{array}\right) \tag{7.24}$$

, where $gn=\left(gn_1,\ldots,gn_j,\ldots,gn_n\right)$, $gn_j=\left(xgn_j,wgn_j\right)$; $T_{nd}=\left(T_1,\ldots,T_k,\ldots,T_m\right)$; Agg is a aggregation operator, $T_k=Agg\left(\left\{\left(v_{k,j},wgn_j\right)\right\}\right)$, $k=1,\ldots,m$, $v_{k,j}$ is the utility value with respect to T_k and gn_j. Thus Eq. 7.24 is also of the form:

$$T_{nd}=\left\{T_k:T_k=Agg\left(\left\{\left(v_{k,j},wgn_j\right)\right\}\right),k=1,\ldots,m\right\} \tag{7.25}$$

On the basis of the above definitions, the multi-experts multi-criteria decision matrix of $Clst\left(nd,\left\{gn_j\right\}\right)$ of the multi-experts $\left\{\left(e_{j'},we_{j'}\right)\right\}$ with respect to T_k is of the following form:

$$dm\left(\left\{\left(e_{j'},we_{j'}\right)\right\},T_k,Clst\left(nd,\left\{gn_j\right\}\right)\right)=\begin{array}{cc} & \\ & \\ e_1 & we_1 \\ \vdots & \vdots \\ e_{j'} & we_{j'} \\ \vdots & \vdots \\ e_{q_{nd}} & we_{q_{nd}} \end{array}\begin{array}{c} \begin{pmatrix} wgn_1 & \dots & wgn_j & \dots & wgn_n \end{pmatrix} \\ \begin{array}{ccccc} xgn_1 & \cdots & xgn_j & \cdots & xgn_n \end{array} \\ \begin{pmatrix} v_{1,1} & \cdots & v_{1,j} & \cdots & v_{1,n} \\ \vdots & \ddots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ v_{j',1} & \cdots & v_{j',j} & \cdots & v_{j',n} \\ \vdots & \cdot^{\cdot^{\cdot}} & \vdots & \ddots & \vdots \\ v_{q_{nd},1} & \cdots & v_{q_{nd},j} & \cdots & v_{q_{nd},n} \end{pmatrix} \end{array}$$

(7.26)

, where $we_{j'}$ is the relative weight of the expert $e_{j'}$; $v_{j',j}$ is the utility value with respect to $\left(e_{j'},we_{j'}\right)$ and xgn_j.

The aggregation problem of $dm\left(\left\{\left(e_{j'},we_{j'}\right)\right\},T_k,Clst\left(nd,\left\{gn_j\right\}\right)\right)$ is of the form:

$$nd_{T_k}=Cagg(Eagg(\begin{array}{cc} & \\ & \\ e_1 & we_1 \\ \vdots & \vdots \\ e_{j'} & we_{j'} \\ \vdots & \vdots \\ e_{q_{nd}} & we_{q_{nd}} \end{array}\begin{array}{c} \begin{pmatrix} wgn_1 & \dots & wgn_j & \dots & wgn_n \end{pmatrix} \\ \begin{array}{ccccc} xgn_1 & \cdots & xgn_j & \cdots & xgn_n \end{array} \\ \begin{pmatrix} v_{1,1} & \cdots & v_{1,j} & \cdots & v_{1,n} \\ \vdots & \ddots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ v_{j',1} & \cdots & v_{j',j} & \cdots & \vdots \\ \vdots & \cdot^{\cdot^{\cdot}} & \vdots & \ddots & \vdots \\ v_{q_{nd},1} & \cdots & v_{q_{nd},j} & \cdots & v_{q_{nd},n} \end{pmatrix} \end{array})) \qquad (7.27)$$

,where $Eagg$ is the aggregation function of the expert judgments for xgn_j, and has the form:

$$xgn_j=Eagg\left(\left[\left(v_{j',j},we_{j'}\right)\right]_{j'=1}^{q_{nd}}\right),\quad \forall j\in\{1,\ldots,n\} \qquad (7.28)$$

$Cagg$ is the function to aggregate the set of the criteria pairs $\left\{\left(xgn_j,wgn_j\right)\right\}$ in a node of alternative k, i.e. nd_{T_k}, of the form:

$$nd_{T_k}=Cagg\left(\left[\left(xgn_j,wgn_j\right)\right]_{j=1}^{n}\right),\quad \forall k\in\{1,\ldots,m\} \qquad (7.29)$$

$$nd_{T_k}=Cagg\left(\left[\left(Eagg\left(\left[\left(v_{j',j},we_{j'}\right)\right]_{k=1}^{q_{nd}}\right),wgn_j\right)\right]_{j=1}^{n}\right),\quad \forall k\in\{1,\ldots,m\} \qquad (7.30)$$

Propositions 7.2 and 7.3 are an application of nd_{T_k}, which is the usual case in CNP.

The set of the multi-experts multi-criteria decision matrix of all alternatives is of the form

$$dm\left(\left\{\left(e_{j'}, we_{j'}\right)\right\}, T, Clst\left(nd, \left\{gn_j\right\}\right)\right) = \left\{dm\left(\left\{\left(e_{j'}, we_{j'}\right)\right\}, T_k, Clst\left(nd, \left\{gn_j\right\}\right)\right)\right\} \quad (7.31)$$

and the set of the alternative for a node is

$$nd_T = \left\{nd_{T_k} : nd_{T_k} = Cagg\left(\left[\left(Eagg\left(\left[\left(v_{j',j}, we_{j'}\right)\right]_{k=1}^{q_{nd}}\right), wgn_j\right)\right]_{j=1}^{n}\right), \forall k \in \{1, \ldots, m\}\right\}$$

(7.32)

In this case, which is for the discussion of the propositions 7.2 and 7.3, the output value is for an alternative only. The calculation process is repeated for other alternatives with different assessment values. And finally the volition decision process (chapter 3) is applied.

7.3 Fuzzy cognitive network process (FCNP)

The fuzzy cognitive network process (FCNP), is the CNP with fuzzy input. The extension of CNP is shown as follows with respect to the rating scales, pairwise opposite matrix, cognitive prioritization operator and information fusion.

7.3.1 Fuzzy cognitive rating scales

Let the comparison interval scale schema of the Hedge-Direction-Atom Linguistic Representation Model be $\left(\aleph, \bar{X}'_{\aleph} = \left\{\bar{X}'^{-}_{\aleph}, \bar{X}'^{+}_{\aleph}\right\}, f_{\bar{X}'}(\aleph)\right)$. FCNP applies the same compound linguistic variable $\aleph$ for the rating scales as the CNP does. However, the representation numbers $\bar{X}'_{\aleph}$ of $\aleph$ are fuzzy numbers which are generated by the Semantic Rule Algorithm with fuzzy output $f_{\bar{X}'}(\aleph)$ (algorithm 4.2).

Let the fuzzy number of a compound linguistic term have the form

$$\alpha_{i'j'} = v_{hd_{i'}} \oplus v_{a_{j'}} = \left(\bar{x}_{l_{i'j'}}, \bar{x}_{\pi_{i'j'}}, \bar{x}_{u_{i'j'}}\right) \tag{7.33}$$

,where i' is the index of directional hedge term, and j' is the index of the atomic term.

Thus, $\bar{X}'^{+}_{\aleph}$ is the fuzzy numerical representation of $\aleph$, which is shown in (7.34), and has the form.

$$\bar{X}'^{+}_{\aleph} = \begin{bmatrix} \varnothing & \left(\bar{x}_{l_{1,2}}, \bar{x}_{\pi_{\eta,2}}, \bar{x}_{u_{\eta,2}}\right) & \cdots & \left(\bar{x}_{l_{1,n}}, \bar{x}_{\pi_{1,n}}, \bar{x}_{u_{1,n}}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \varnothing & \left(\bar{x}_{l_{\eta,2}}, \bar{x}_{\pi_{\eta,2}}, \bar{x}_{u_{\eta,2}}\right) & \ddots & \left(\bar{x}_{l_{\eta,n}}, \bar{x}_{\pi_{\eta,n}}, \bar{x}_{u_{\eta,n}}\right) \\ \left(\bar{x}_{l_{\eta+1,1}}, \bar{x}_{\pi_{\eta+1,1}}, \bar{x}_{u_{\eta+1,1}}\right) & \left(\bar{x}_{l_{\eta+1,2}}, \bar{x}_{\pi_{\eta+1,2}}, \bar{x}_{u_{\eta+1,2}}\right) & \ddots & \left(\bar{x}_{l_{\eta+1,n}}, \bar{x}_{\pi_{\eta+1,n}}, \bar{x}_{u_{\eta+1,n}}\right) \\ \left(\bar{x}_{l_{\eta+2,1}}, \bar{x}_{\pi_{\eta+2,1}}, \bar{x}_{u_{\eta+2,1}}\right) & \left(\bar{x}_{l_{\eta+2,2}}, \bar{x}_{\pi_{\eta+2,2}}, \bar{x}_{u_{\eta+2,2}}\right) & \ddots & \varnothing \\ \vdots & \vdots & \ddots & \vdots \\ \left(\bar{x}_{l_{m,1}}, \bar{x}_{\pi_{m,1}}, \bar{x}_{u_{m,1}}\right) & \left(\bar{x}_{l_{m,2}}, \bar{x}_{\pi_{m,2}}, \bar{x}_{u_{m,2}}\right) & \cdots & \varnothing \end{bmatrix}$$

(7.34)

The opposite matrix of the above form is of the following form:

$$\bar{X}'^{-}_{\aleph} = \begin{bmatrix} \varnothing & \left(-\bar{x}_{u_{\eta,2}}, -\bar{x}_{\pi_{\eta,2}}, -\bar{x}_{l_{1,2}}\right) & \cdots & \left(-\bar{x}_{u_{1,n}}, -\bar{x}_{\pi_{1,n}}, -\bar{x}_{l_{1,n}}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \varnothing & \left(-\bar{x}_{u_{\eta,2}}, -\bar{x}_{\pi_{\eta,2}}, -\bar{x}_{l_{\eta,2}}\right) & \ddots & \left(-\bar{x}_{u_{\eta,n}}, -\bar{x}_{\pi_{\eta,n}}, -\bar{x}_{l_{\eta,n}}\right) \\ \left(-\bar{x}_{u_{\eta+1,1}}, -\bar{x}_{\pi_{\eta+1,1}}, -\bar{x}_{l_{\eta+1,1}}\right) & \left(-\bar{x}_{u_{\eta+1,2}}, -\bar{x}_{\pi_{\eta+1,2}}, -\bar{x}_{l_{\eta+1,2}}\right) & \ddots & \left(-\bar{x}_{u_{\eta+1,n}}, -\bar{x}_{\pi_{\eta+1,n}}, -\bar{x}_{l_{\eta+1,n}}\right) \\ \left(-\bar{x}_{u_{\eta+2,1}}, -\bar{x}_{\pi_{\eta+2,1}}, -\bar{x}_{l_{\eta+2,1}}\right) & \left(-\bar{x}_{u_{\eta+2,2}}, -\bar{x}_{\pi_{\eta+2,2}}, -\bar{x}_{l_{\eta+2,2}}\right) & \ddots & \varnothing \\ \vdots & \vdots & \ddots & \vdots \\ \left(-\bar{x}_{u_{m,1}}, -\bar{x}_{\pi_{m,1}}, -\bar{x}_{l_{m,1}}\right) & \left(-\bar{x}_{u_{m,2}}, -\bar{x}_{\pi_{m,2}}, -\bar{x}_{l_{m,2}}\right) & \cdots & \varnothing \end{bmatrix} \tag{7.35}$$

Table 7.3: The result of $\bar{X}'^{+}_{\aleph}$ with five atomic terms

	Equally	Slightly	Moderately	Strongly	Essentially
Much Below	Null	(0, 0.645, 1.291)	(2., 2.645, 3.291)	(4., 4.645, 5.291)	(6., 6.645, 7.291)
Quite Below	Nul	(1., 1.366, 1.732)	(3., 3.366, 3.732)	(5., 5.366, 5.732)	(7., 7.366, 7.732)
Little Below	Null	(1.528, 1.764, 2.)	(3.528, 3.764, 4.)	(5.528, 5.764, 6.)	(7.528, 7.764, 8.)
Absolutely	(0, 0, 0)	(1.764, 2., 2.236)	(3.764, 4., 4.236)	(5.764, 6., 6.236)	(7.764, 8., 8.)
Little Above	(0, 0.236, 0.472)	(2., 2.236, 2.472)	(4., 4.236, 4.472)	(6., 6.236, 6.472)	Null
Quite Above	(0.268, 0.634, 1.)	(2.268, 2.634, 3.)	(4.268, 4.634, 5.)	(6.268, 6.634, 7.)	Null
Much Above	(0.709, 1.355, 2.)	(2.709, 3.355, 4.)	(4.709, 5.355, 6.)	(6.709, 7.355, 8.)	Null

Table 7.4: The result of $\bar{X}'^{-}_{\aleph}$ with five atomic terms

	Equally	Slightly	Moderately	Strongly	Essentially
Much Below	Null	(-1.291,- 0.645, 0)	(2., 2.645, 3.291)	(-5.291, -4.645, -4.)	(-7.291, -6.645, -6.)
Quite Below	Null	(-1.732, -1.366, -1)	(3., 3.366, 3.732)	(-5.732, -5.366, -5.)	(-7.732, -7.366, -7.)
Little Below	Null	(-2., -1.764, -1.528)	(-3.528,- 3.764,- 4.)	(-6.,- 5.764, -5.528)	(-8., -7.764, -7.528)
Absolutely	(-0.236, 0, 0)	(-2.236.,- 2., -1.764.)	(-4.236, -4., -3.764)	(-6.236,- 6.,- 5.764)	(-8.,- 8.,- 7.764)
Little Above	(- 0.472, -0.236, 0)	(- 2.472,- 2.236, -2.)	(-4.472, -4.236, -4.)	(-6.472, -6.236, -6.)	Null
Quite Above	(-1., -0.634, -0.268)	(-3., -2.634, -2.268)	(-5., -4.634, -4.268)	(-7., -6.634, -6.268)	Null
Much Above	(-2., -1.355, -0.709)	(-4., -3.355, -2.709)	(-6., -5.355, -4.709)	(-8., -7.355, -6.709)	Null

Example 7.2

This example is a continuation of example 7.1. Assume other settings remain unchanged, except for the output of algorithm 4.2, which is a fuzzy output. $\bar{X}'^{+}_{\aleph}$ is shown in table 7.3 whilst $\bar{X}'^{-}_{\aleph}$ is shown in table 7.4. However, "A-Eq" is (0,0,0), always.

Table 7.5: The results of $\bar{X}'^{+}_{\aleph}$ with nine atomic terms

	MB	QB	LB	A	LA	QA	MA
Eq	Null	Null	Null	(0, 0, 0)	(0, 0.12, 0.24)	(0.13, 0.32, 0.5)	(0.35, 0.68, 1.)
Wk	(0, 0.32, 0.65)	(0.5, 0.68, 0.87)	(0.76, 0.88, 1.)	(0.88, 1., 1.12)	(1., 1.12, 1.24)	(1.13, 1.32, 1.5)	(1.35, 1.68, 2.)
Mo	(1., 1.32, 1.65)	(1.5, 1.68, 1.87)	(1.76, 1.88, 2.)	(1.88, 2., 2.12)	(2., 2.12, 2.24)	(2.13, 2.32, 2.5)	(2.35, 2.68, 3.)
Mp	(2., 2.32, 2.65)	(2.5, 2.68, 2.87)	(2.76, 2.88, 3.)	(2.88, 3., 3.12)	(3., 3.12, 3.24)	(3.13, 3.32, 3.5)	(3.35, 3.68, 4.)
St	(3., 3.32, 3.65)	(3.5, 3.68, 3.87)	(3.76, 3.88, 4.)	(3.88, 4., 4.12)	(4., 4.12, 4.24)	(4.13, 4.32, 4.5)	(4.35, 4.68, 5.)
Sp	(4., 4.32, 4.65)	(4.5, 4.68, 4.87)	(4.76, 4.88, 5.)	(4.88, 5., 5.12)	(5., 5.12, 5.24)	(5.13, 5.32, 5.5)	(5.35, 5.68, 6.)
VS	(5., 5.32, 5.65)	(5.5, 5.68, 5.87)	(5.76, 5.88, 6.)	(5.88, 6., 6.12)	(6., 6.12, 6.24)	(6.13, 6.32, 6.5)	(6.35, 6.68, 7.)
VVS	(6., 6.32, 6.65)	(6.5, 6.68, 6.87)	(6.76, 6.88, 7.)	(6.88, 7., 7.12)	(7., 7.12, 7.24)	(7.13, 7.32, 7.5)	(7.35, 7.68, 8.)
Es	(7., 7.32, 7.65)	(7.5, 7.68, 7.87)	(7.76, 7.88, 8.)	(7.88, 8., 8.)	Null	Null	Null

Table 7.6: The results of $\bar{X}'^{-}_{\aleph}$ with nine atomic terms

	MB	QB	LB	A	LA	QA	MA
Eq	Null	Null	Null	(0, 0, 0)	(-0.24, -0.12, 0)	(-0.5, -0.32, -0.13)	(-1., -0.68, -0.35)
Wk	(-0.65, -0.32, 0)	(-0.87, -0.68, -0.5)	(-1, -0.88, -0.76.)	(-1.12, -1., -0.88.)	(-1.24, -1.12, -1.)	(,-1.5 -1.32, -1.13)	(-2., -1.68, -1.35)
Mo	(-1.65, -1.32, -1.)	(-1.87, -1.68, -1.5)	(-2, -1.88, -1.76.)	(-2.12, -2., -1.88)	(-2.24,-2.12, -2.)	(-2.5, -2.32, -2.13)	(-3., -2.68, -2.35)
Mp	(-2.65, -2.32, -2.)	(-2.87, -2.68, -2.5)	(-3., -2.88, -2.76)	(-3.12, -3., -2.88)	(-3.24, -3.12, -3.)	(-3.5, -3.32, -3.13)	(-4., -3.68, -3.35)
St	(-3.65, -3.32, -3.)	(-3.87, -3.68, -3.5)	(-4, -3.88, -3.76.)	(-412., -4., -3.88)	(-4.24, -4.12, -4.)	(-4.5, -4.32, -4.13)	(-5., -4.68, -4.35)
Sp	(-4.65, -4.32, -4.)	(-4.87, -4.68, -4.5)	(-5., -4.88, -4.76)	(-5.12, -5., -4.88)	(-5.24, -5.12, -5.)	(-5.5, -5.32, -5.13)	(-6., -5.68, -5.35)
VS	(-5.65, -5.32, -5.)	(-5.87, -5.68, -5.5)	(-6., -5.88, -5.76)	(-6.12, -6., -5.88)	(-6.24, -6.12, -6.)	(-6.5, -6.32, -6.13)	(-7., -6.68, -6.35)
VVS	(-6.65, -6.32, -6.)	(-6.87, -6.68, -6.5)	(-7., -6.88, -6.76)	(-7.12, -7., -6.88)	(-7.24, -7.12, -7.)	(-7.5, -7.32, -7.13)	(-8., -7.68, -7.35)
Es	(-7.65, -7.32, -7.)	(-7.87, -7.68, -7.5)	(-8., -7.88, -7.76)	(-8., -8., -7.88)	Null	Null	Null

Example 7.3

This example is a continuation of example 7.2. Assume other settings remain unchanged, except for the output of algorithm 4.2, which is a fuzzy output. $\bar{X}'^{+}_{\aleph}$ is shown in table 7.3 whilst $\bar{X}'^{-}_{\aleph}$ is shown in table 7.4. However, A-Eq is (0,0,0), always.

7.3.2 Fuzzy pairwise opposite matrix

The fuzzy pairwise opposite matrix (FPOM) is used to interpret the individual utilities of the candidates in fuzzy numbers. Let an ideal fuzzy utility set be $\widehat{V} = \{\widehat{v}_1, \ldots, \widehat{v}_n\}$, where the fuzzy utility is of the form $\widehat{v}_i = \left(v_i^l, v_i^\pi, v_i^u\right)$, and the comparison score in fuzzy number is $\widehat{b}_{ij} \cong \widehat{v}_i - \widehat{v}_j$. The ideal fuzzy pairwise opposite matrix is $\tilde{\tilde{B}} = \left[\widehat{v}_i - \widehat{v}_j\right]$. A subjective judgmental fuzzy pairwise opposite matrix using interval scales is $\widehat{B} = \left[\widehat{b}_{ij}\right]$. $\tilde{\tilde{B}}$ is determined by $\widehat{B}$ as follows:

$$\tilde{\tilde{B}} = \left[\tilde{\tilde{b}}_{ij}\right] = \begin{bmatrix} \widehat{v}_1 - \widehat{v}_1 & \widehat{v}_1 - \widehat{v}_2 & \cdots & \widehat{v}_1 - \widehat{v}_n \\ \widehat{v}_2 - \widehat{v}_1 & \widehat{v}_2 - \widehat{v}_2 & \cdots & \widehat{v}_2 - \widehat{v}_n \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{v}_n - \widehat{v}_1 & \widehat{v}_n - \widehat{v}_2 & \cdots & \widehat{v}_n - \widehat{v}_n \end{bmatrix} \cong \begin{bmatrix} \widehat{b}_{11} & \widehat{b}_{12} & \cdots & \widehat{b}_{1n} \\ \widehat{b}_{21} & \widehat{b}_{22} & \cdots & \widehat{b}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{b}_{n1} & \widehat{b}_{n2} & \cdots & \widehat{b}_{nn} \end{bmatrix} = \left[\widehat{b}_{ij}\right] = \widehat{B} \quad (7.36)$$

, where $\widehat{b}_{ij} = \left(b_{ij}^l, b_{ij}^\pi, b_{ij}^u\right) = -\widehat{b}_{ji} = \left(-b_{ji}^u, -b_{ji}^\pi, -b_{ji}^l\right)$, and for $i, j = 1, \ldots, n$ and $i \neq j$. If $i = j$, then $\widehat{b}_{ij} = \widehat{v}_i - \widehat{v}_j = (0,0,0)$. Thus the above matrix has the form:

$$\tilde{\tilde{B}} = \begin{bmatrix} (0,0,0) & \widehat{v}_1 - \widehat{v}_2 & \cdots & \widehat{v}_1 - \widehat{v}_n \\ \widehat{v}_2 - \widehat{v}_1 & (0,0,0) & \cdots & \widehat{v}_2 - \widehat{v}_n \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{v}_n - \widehat{v}_1 & \widehat{v}_n - \widehat{v}_2 & \cdots & (0,0,0) \end{bmatrix}$$

$$\cong \begin{bmatrix} (0,0,0) & \left(b_{12}^{l},b_{12}^{\pi},b_{12}^{u}\right) & \cdots & \left(b_{1n}^{l},b_{1n}^{\pi},b_{1n}^{u}\right) \\ \left(-b_{12}^{u},-b_{12}^{\pi},-b_{12}^{l}\right) & (0,0,0) & \cdots & \left(b_{2n}^{l},b_{2n}^{\pi},b_{2n}^{u}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(-b_{1n}^{u},-b_{1n}^{\pi},-b_{1n}^{l}\right) & \left(-b_{2n}^{u},-b_{2n}^{\pi},-b_{2n}^{l}\right) & \cdots & (0,0,0) \end{bmatrix} = \widehat{B} \tag{7.37}$$

$\widehat{B}$ can be decomposed as three matrices as follows:

$$B^{l} = \begin{bmatrix} 0 & b_{12}^{l} & \cdots & b_{1n}^{l} \\ b_{21}^{u} & 0 & \cdots & b_{2n}^{l} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}^{u} & b_{n2}^{u} & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & b_{12}^{l} & \cdots & b_{1n}^{l} \\ -b_{12}^{l} & 0 & \cdots & b_{2n}^{l} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{1n}^{l} & -b_{2n}^{l} & \cdots & 0 \end{bmatrix} \tag{7.38}$$

$$B^{\pi} = \begin{bmatrix} 0 & b_{12}^{\pi} & \cdots & b_{1n}^{\pi} \\ b_{21}^{\pi} & 0 & \cdots & b_{2n}^{\pi} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}^{\pi} & b_{n2}^{\pi} & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & b_{12}^{\pi} & \cdots & b_{1n}^{\pi} \\ -b_{12}^{\pi} & 0 & \cdots & b_{2n}^{\pi} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{1n}^{\pi} & -b_{2n}^{\pi} & \cdots & 0 \end{bmatrix} \tag{7.39}$$

$$B^{u} = \begin{bmatrix} 0 & b_{12}^{u} & \cdots & b_{1n}^{u} \\ b_{21}^{l} & 0 & \cdots & b_{2n}^{u} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}^{l} & b_{n2}^{l} & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & b_{12}^{u} & \cdots & b_{1n}^{u} \\ -b_{12}^{u} & 0 & \cdots & b_{2n}^{u} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{1n}^{u} & -b_{2n}^{u} & \cdots & 0 \end{bmatrix} \tag{7.40}$$

Usually, $\widehat{b}_{ij} \in \widehat{B}$ is given through the rating process of the expert in the compound scale in fuzzy number, i.e. $\widehat{b}_{ij} \in \bar{X}'_{\aleph} = \left\{\bar{X}'^{-}_{\aleph}, \bar{X}'^{+}_{\aleph}\right\}$. The expert only fills a fuzzy upper triangular matrix of the form:

$$\widehat{B}^{+} = \begin{cases} \widehat{b}_{ij} & i<j \\ (0,0,0) & \text{otherwise} \end{cases}$$, written explicitly,

$$\widehat{B}^{+} = \begin{bmatrix} (0,0,0) & \widehat{b}_{12} & \cdots & \widehat{b}_{1n} \\ (0,0,0) & (0,0,0) & \cdots & \widehat{b}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ (0,0,0) & (0,0,0) & \cdots & (0,0,0) \end{bmatrix} = \begin{bmatrix} (0,0,0) & \left(b_{12}^{l},b_{12}^{\pi},b_{12}^{u}\right) & \cdots & \left(b_{1n}^{l},b_{1n}^{\pi},b_{1n}^{u}\right) \\ (0,0,0) & (0,0,0) & \cdots & \left(b_{2n}^{l},b_{2n}^{\pi},b_{2n}^{u}\right) \\ \vdots & \vdots & \ddots & \vdots \\ (0,0,0) & (0,0,0) & \cdots & (0,0,0) \end{bmatrix}$$

(7.41)

The lower triangular matrix is given by the opposite of an upper triangular matrix of the form:

$$\widehat{B}^{-}=\begin{cases}\widehat{b}_{ij} & i>j\\ (0,0,0) & \text{Otherwise}\end{cases}\text{, written explicitly,}$$

$$\widehat{B}^{-}=\begin{bmatrix}(0,0,0) & (0,0,0) & \cdots & (0,0,0)\\ \widehat{b}_{21} & (0,0,0) & \cdots & (0,0,0)\\ \vdots & \vdots & \ddots & \vdots\\ \widehat{b}_{n1} & \widehat{b}_{n2} & \cdots & (0,0,0)\end{bmatrix}=\begin{bmatrix}(0,0,0) & (0,0,0) & \cdots & (0,0,0)\\ \left(-b_{12}^{u},-b_{12}^{\pi},-b_{12}^{l}\right) & (0,0,0) & \cdots & (0,0,0)\\ \vdots & \vdots & \ddots & \vdots\\ \left(-b_{1n}^{u},-b_{1n}^{\pi},-b_{1n}^{l}\right) & \left(-b_{2n}^{u},-b_{2n}^{\pi},-b_{2n}^{l}\right) & \cdots & (0,0,0)\end{bmatrix} \tag{7.42}$$

$\left[\widehat{b}_{ij}\right]$ is achieved by $\widehat{B}=\widehat{B}^{+}+\widehat{B}^{-}$. For a complete comparison of a set of candidates, FPOM needs $n(n-1)/2$ ratings. $\widehat{B}$ is validated by the Fuzzy Accordant Index $\widehat{AI}$ or FAI is of the form:

$\widehat{AI}=\left(AI^{l}\right)^{0.25}\times\left(AI^{\pi}\right)^{0.5}\times\left(AI^{u}\right)^{0.25}$, where

$$AI^{l}=\frac{1}{n^{2}}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_{ij}^{l}}\ ,\ \ \delta_{ij}^{l}=\sqrt{Mean\left(\left(\frac{1}{\kappa^{l}}\left(B_{i}^{l}+\left(B_{j}^{l}\right)^{T}-b_{ij}^{l}\right)\right)^{2}\right)},\forall i,\forall j\in(1,\ldots,n);$$

$$AI^{\pi}=\frac{1}{n^{2}}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_{ij}^{\pi}}\ ,\ \ \delta_{ij}^{\pi}=\sqrt{Mean\left(\left(\frac{1}{\kappa^{\pi}}\left(B_{i}^{\pi}+\left(B_{j}^{\pi}\right)^{T}-b_{ij}^{\pi}\right)\right)^{2}\right)},\forall i,\forall j\in(1,\ldots,n);$$

$$AI^{u}=\frac{1}{n^{2}}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_{ij}^{u}}\ ,\ \ \delta_{ij}^{u}=\sqrt{Mean\left(\left(\frac{1}{\kappa^{\pi}}\left(B_{i}^{u}+\left(B_{j}^{u}\right)^{T}-b_{ij}^{\pi}\right)\right)^{2}\right)},\forall i,\forall j\in(1,\ldots,n); \tag{7.43}$$

$\widehat{\kappa}=\left(\kappa^{l},\kappa^{\pi},\kappa^{u}\right)$ is the fuzzy normal utility. By default, $\widehat{\kappa}=\left(\kappa^{l},\kappa^{\pi},\kappa^{u}\right)=\left(Max\left(\bar{X}_{\aleph}\right)-\delta,Max\left(\bar{X}_{\aleph}\right),Max\left(\bar{X}_{\aleph}\right)+\delta\right)$, and δ is the average of the modal values of two adjacent atomic terms.

$\widehat{AI}$ is the normalized weighted geometric of $\left(AI^l, AI^\pi, AI^u\right)$, and $\widehat{AI} \geq 0$. If $\widehat{AI} = 0$, then $\widehat{B}$ is perfectly accordant; If $0 < \widehat{AI} \leq 0.1$, $\widehat{B}$ is then satisfactory. If $\widehat{AI} > 0.1$, $\widehat{B}$ is then unsatisfactory .

In a fuzzy POM $\widehat{B} = \left\{\widehat{b}_{ij} : \widehat{b}_{ij} = \left(b^l_{ij}, b^\pi_{ij}, b^u_{ij}\right); i, j = 1, \ldots, n\right\}$ with the fuzzy utilities $\widehat{v}_i = \left(v^l_i, v^\pi_i, v^u_i\right), i = 1, \ldots, n$, if $\left\{v^\pi_i\right\}$ is derived from accordant POM $\left\{b^\pi_{ij}\right\}$, then $\left\{v^l_i\right\}$ and $\left\{v^u_i\right\}$ are not derived from the accordant matrices since $\left\{b^l_{ij}\right\}$ and $\left\{b^u_{ij}\right\}$ are not accordant when $\left\{b^\pi_{ij}\right\}$ is accordant. This problem is invertible due to the rating scale in fuzzy number. To discuss this issue, assume $b^\pi_{ik} + b^\pi_{kj} = b^\pi_{ij}$ be preserved, and $b^\pi_{ik} = b^\pi_{kj} = 2$. Thus $b^\pi_{ij} = 4$. To apply in fuzzy case, let $\widehat{b}_{ik} = \widehat{b}_{kj} = (1, 2, 3)$. Although $b^\pi_{ik} + b^\pi_{kj} = b^\pi_{ij}$ is still valid, $b^l_{ik} + b^l_{kj} = b^l_{ij}$ and $b^u_{ik} + b^u_{kj} = b^u_{ij}$ are not valid. Thus it follows $\widehat{b}_{ik} + \widehat{b}_{kj} \neq \widehat{b}_{ij}$ as $\widehat{b}_{ij} = (3, 4, 5) \neq (2, 4, 6) = \widehat{b}_{ik} + \widehat{b}_{kj}$. Thus $\left\{v^l_i\right\}$ and $\left\{v^u_i\right\}$ are not derived from the accordant matrices.

The weighted geometric mean in FAI cancels the effect of this discordance. Any one of $\left\{b^l_{ij}\right\}$, $\left\{b^u_{ij}\right\}$ and $\left\{b^\pi_{ij}\right\}$ is accordant and will produce a fuzzy accordant matrix, i.e. FAI=0 since $\widehat{AI} = \left(AI^l\right)^{0.25} \times \left(AI^\pi\right)^{0.5} \times \left(AI^u\right)^{0.25}$.

7.3.3 Fuzzy cognitive prioritization operator (FCPO)

Regarding crisp cognitive prioritization, two methods are proposed (chapter 5): Primitive Least Squares (PLS) (or Row Average plus the normal Utility (RAU)) and

Least Penalty Squares (LPS). The development of the fuzzy cognitive prioritization operators is on the basis of the above two operators.

The vector of fuzzy individual utilities $\widehat{V}=\{\widehat{v}_1,\ldots,\widehat{v}_n\}$, $\widehat{v}_i=\left(v_i^l,v_i^\pi,v_i^u\right)$ can be derived by the fuzzy Primitive Least Squares (FPLS) optimization model which is of the form:

$$\text{Min}\quad \widehat{\Delta}=\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left[\left(b_{ij}^l-v_i^l+v_j^l\right)^2+\left(b_{ij}^\pi-v_i^\pi+v_j^\pi\right)^2+\left(b_{ij}^u-v_i^u+v_j^u\right)^2\right]$$

$$\text{s.t.}\quad \sum_{i=1}^{n}v_i^l=n\kappa^l$$

$$\sum_{i=1}^{n}v_i^\pi=n\kappa^\pi$$

$$\sum_{i=1}^{n}v_i^u=n\kappa^u \tag{7.44}$$

,where $n=\left|\{\widehat{v}_i\}\right|$ is the cardinal number of the fuzzy utility vector, $\left(b_{ij}^l,b_{ij}^\pi,b_{ij}^u\right)\in\widehat{B}$ is the fuzzy entry of $\widehat{B}$, $n\widehat{\kappa}=\left(n\kappa^l,n\kappa^\pi,n\kappa^u\right)$ is the fuzzy population utility, and $\widehat{\kappa}=\left(\kappa^l,\kappa^\pi,\kappa^u\right)$ is the fuzzy normal utility. By default, $\widehat{\kappa}=\left(\kappa^l,\kappa^\pi,\kappa^u\right)=\left(Max\left(\bar{X}_\aleph\right)-\delta,Max\left(\bar{X}_\aleph\right),Max\left(\bar{X}_\aleph\right)+\delta\right)$, and δ is the average of the modal values of two adjacent atomic terms.

The solution of the closed form of FPLS can be solved manually as follows.

Theorem 7.1: The solution of the closed form of FPLS is the Fuzzy Row Average plus the normal Utility *(FRAU)*, as follows:

$$FRAU\left(\widehat{B},\widehat{\kappa}\right)=$$

$$\left[\left(v_i^l, v_i^\pi, v_i^u\right) \mid \left(\begin{array}{c} v_i^l = \left(\dfrac{1}{n}\left(\sum_{j=1}^{i} b_{ij}^u + \sum_{j=i+1}^{n} b_{ij}^l\right)\right) + \kappa^l \\ v_i^\pi = \left(\dfrac{1}{n}\sum_{j=1}^{n} b_{ij}^\pi\right) + \kappa^\pi \\ v_i^u = \left(\dfrac{1}{n}\left(\sum_{j=1}^{i} b_{ij}^l + \sum_{j=i+1}^{n} b_{ij}^u\right)\right) + \kappa^u \end{array}\right), \forall i \in \{1,\ldots,n\}\right] \tag{7.45}$$

Proof:

As $\widehat{\Delta}^l$, $\widehat{\Delta}^\pi$, and $\widehat{\Delta}^u$ are independent, FPLS can be transformed into multiple objective programming,

$$\text{MPLS}\left(\widehat{B}^+,\widehat{\kappa}\right)=$$

$$\text{Min} \quad \widehat{\Delta}^l = \sum\nolimits_{i=1}^{n}\sum\nolimits_{j=i+1}^{n}\left(b_{ij}^l - v_i^l + v_j^l\right)^2$$

$$\text{Min} \quad \widehat{\Delta}^\pi = \sum\nolimits_{i=1}^{n}\sum\nolimits_{j=i+1}^{n}\left(b_{ij}^\pi - v_i^\pi + v_j^\pi\right)^2$$

$$\text{Min} \quad \widehat{\Delta}^u = \sum\nolimits_{i=1}^{n}\sum\nolimits_{j=i+1}^{n}\left(b_{ij}^u - v_i^u + v_j^u\right)^2$$

$$\text{s.t.} \quad \sum\nolimits_{i=1}^{n} v_i^l = n\kappa^l,$$

$$\sum\nolimits_{i=1}^{n} v_i^\pi = n\kappa^\pi,$$

$$\sum\nolimits_{i=1}^{n} v_i^u = n\kappa^u.$$

The multiple objective programming can be transformed into three optimization models, which are as follows:

$\mathrm{FPLS}\left(\widehat{B}^{+},\widehat{\kappa}\right)=\left(\mathrm{FPLS}_{l},\mathrm{FPLS}_{\pi},\mathrm{FPLS}_{u}\right)$, where

$\mathrm{FPLS}_{l}\left(\widehat{B}_{l}^{+},\kappa^{l}\right)=$

$$\text{Min}\quad \widehat{\Delta}^{l}=\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left(b_{ij}^{l}-v_{i}^{l}+v_{j}^{l}\right)^{2}$$

$$\text{s.t.}\quad \sum_{i=1}^{n}v_{i}^{l}=n\kappa^{l};$$

$\mathrm{FPLS}_{\pi}\left(\widehat{B}_{\pi}^{+},\kappa^{\pi}\right)=$

$$\text{Min}\quad \widehat{\Delta}^{\pi}=\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left(b_{ij}^{\pi}-v_{i}^{\pi}+v_{j}^{\pi}\right)^{2}$$

$$\text{s.t.}\quad \sum_{i=1}^{n}v_{i}^{\pi}=n\kappa^{\pi};$$

$\mathrm{FPLS}_{u}\left(\widehat{B}_{u}^{+},\kappa^{u}\right)=$

$$\text{Min}\quad \widehat{\Delta}^{u}=\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left(b_{ij}^{u}-v_{i}^{u}+v_{j}^{u}\right)^{2}$$

$$\text{s.t.}\quad \sum_{i=1}^{n}v_{i}^{u}=n\kappa^{u};$$

The solution is derived by solving the above three optimization models using theorem 5.4. □

Similarly, regarding the Fuzzy Least Penalty Squares (FLPS) operator, the vector of the individual utilities can be derived as follows:

$\mathrm{FLPS}\left(\widehat{B}^{+},\widehat{\kappa}\right)=$

$$\text{Min}\quad \widehat{\Delta}=\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left[\beta_{ij}^{l}\cdot\left(b_{ij}^{l}-v_{i}^{l}+v_{j}^{l}\right)^{2}+\beta_{ij}^{\pi}\cdot\left(b_{ij}^{\pi}-v_{i}^{\pi}+v_{j}^{\pi}\right)^{2}+\beta_{ij}^{u}\cdot\left(b_{ij}^{u}-v_{i}^{u}+v_{j}^{u}\right)^{2}\right]$$

where,

$$\beta_{ij}^{l} = \begin{cases} \beta_1, & \mathrm{v}_i^l > v_j^l \ \ \& \ \ \mathrm{b}_{ij}^l > 0 \\ & \text{or } \mathrm{v}_i^l < v_j^l \ \ \& \ \ \mathrm{b}_{ij}^l < 0 \\ \beta_2, & \mathrm{v}_i^l = v_j^l \ \ \& \ \ \mathrm{b}_{ij}^l \neq 0 \\ & \text{or } v_i^l \neq v_j^l \ \ \& \ \ \mathrm{b}_{ij}^l = 0 \\ \beta_3, & otherwise \end{cases} ,1 = \beta_1 \leq \beta_2 \leq \beta_3;$$

$$\beta_{ij}^{\pi} = \begin{cases} \beta_1, & \mathrm{v}_i^\pi > v_j^\pi \ \ \& \ \ \mathrm{b}_{ij}^\pi > 0 \\ & \text{or } \mathrm{v}_i^\pi < v_j^\pi \ \ \& \ \ \mathrm{b}_{ij}^\pi < 0 \\ \beta_2, & \mathrm{v}_i^\pi = v_j^\pi \ \ \& \ \ \mathrm{b}_{ij}^\pi \neq 0 \\ & \text{or } v_i^\pi \neq v_j^\pi \ \ \& \ \ \mathrm{b}_{ij}^\pi = 0 \\ \beta_3, & otherwise \end{cases}, \ 1 = \beta_1 \leq \beta_2 \leq \beta_3;$$

$$\beta_{ij}^{u} = \begin{cases} \beta_1, & v_i^u > v_j^u \ \ \& \ \ b_{ij}^u > 0 \\ & \text{or } v_i^u < v_j^u \ \ \& \ \ b_{ij}^u < 0 \\ \beta_2, & v_i^u = v_j^u \ \ \& \ \ b_{ij}^u \neq 0 \\ & \text{or } v_i^u \neq v_j^u \ \ \& \ \ b_{ij}^u = 0 \\ \beta_3, & otherwise \end{cases}, \ 1 = \beta_1 \leq \beta_2 \leq \beta_3;$$

s.t. $\sum_{i=1}^{n} v_i^l = n\kappa^l$; $\sum_{i=1}^{n} v_i^\pi = n\kappa^\pi$; $\sum_{i=1}^{n} v_i^u = n\kappa^u$;

$v_i^l, v_i^\pi, v_i^u \geq 0, i = 1, 2, \ldots, n$;

where $n = \left|\{v_i\}\right|$, and κ is the normal utility. (7.46)

Regarding some fuzzy decision problems, the fuzzy priority vector (or fuzzy normalized utility weighting vector) is denoted by $\widehat{W} = \{\widehat{w}_1, \ldots, \widehat{w}_i, \ldots, \widehat{w}_n\}$, where $\widehat{w}_i = \left(w_i^l, w_i^\pi, w_i^u\right)$, and the summation of the modal values, w_i^π, of $\widehat{W}$ is equal to one, i.e. $\sum_{i=1}^{n} w_i^\pi = 1$. Thus $\widehat{W}$ is said to be a fuzzy normalized priority vector (or a fuzzy priority vector in short). In order to use the proposed utility weighting vector, the fuzzy individual utility from the FPOM is rescaled (or normalized) as a fuzzy normalized priority vector by the rescale function of the normalization function, as follows.

$$W=\left\{\widehat{w}_i=\left(w_i^l,w_i^\pi,w_i^u\right):\left(w_i^l,w_i^\pi,w_i^u\right)=\left(\frac{v_i^l}{n\kappa^\pi},\frac{v_i^\pi}{n\kappa^\pi},\frac{v_i^u}{n\kappa^\pi}\right),\forall i\in\{1,\ldots,n\}\right\}$$

$$\text{,in which } \sum_{i\in\{1,\ldots,n\}} v_i^\pi=n\kappa^\pi \text{ and } \kappa^\pi=Max\left(\bar{X}_\aleph\right) \tag{7.47}$$

7.3.4 Fuzzy cognitive prioritization operator measurement (FCPOM)

$\widehat{B}$ can be decomposed as three matrices: B^l, B^π and B^u. Thus the crisp Cognitive Prioritization Operator Measurement (C-CPOM) Models can be reused. The Fuzzy Cognitive Prioritization Operator Measurement (F-CPOM) model extends C-CPOM (chapter 5.6) by considering a modal value and two interval values. Thus the general form of the value of the Fuzzy Cognitive Prioritization Operator Measurement (F-CPOM) model is

$$\bar{\sigma}=\left\{\left(\sigma^l,\sigma^\pi,\sigma^u\right):\begin{pmatrix}\sigma^l=fm\left(B^l,\left\{v_i^l\right\}\right)\\ \sigma^\pi=fm\left(B^\pi,\left\{v_i^\pi\right\}\right)\\ \sigma^u=fm\left(B^u,\left\{v_i^u\right\}\right)\end{pmatrix}\right\}. \tag{7.48}$$

$\bar{\sigma}$ is the fuzzy reference value of F-CPOM. fm is a function of C-CPOM which can be found in chapter 5.6. The crisp reference value of F-CPOM $\hat{\sigma}$ is the weighted average of $\bar{\sigma}=\left(\sigma^l,\sigma^\pi,\sigma^u\right)$ and is of the form:

$$\hat{\sigma}=\alpha^l\bullet\sigma^l+\alpha^\pi\bullet\sigma^\pi+\alpha^u\bullet\sigma^u \tag{7.49}$$

, where the coefficients of the conditions: $\alpha^u \geq \alpha^\pi \geq \alpha^l$, and $\alpha^l + \alpha^\pi + \alpha^u = 1$. By default $\alpha^m = 0.5$ and $\alpha^l = \alpha^u = 0.25$.

The Fuzzy Cognitive Distortion Index, $\widehat{CDI}$, applies the *Root Mean Penalty Weighted Square Variance* (chapter 5.6.7) *,i.e.* $fm = RMPWSV$, to the above equations.

The next issue discusses the aggregation of the fuzzy utility sets.

7.3.5 Fuzzy aggregation operators (FPOs)

Let $\widehat{dm}\left(\widehat{T}, Clst\left(\widehat{nd}, \widehat{gn}\right)\right)$ be the fuzzy decision matrix of cluster $Clst\left(\widehat{nd}, \widehat{gn}\right)$ with respect to a vector of alternatives $\widehat{T} = \left\{\widehat{T}_k\right\}$. The form is shown in tables 7.7-7.8.

Regarding a cluster $Clst\left(\widehat{nd}, \widehat{gn}\right)$, $\widehat{W} = \left(\widehat{w}_1, \ldots \widehat{w}_j, \ldots, \widehat{w}_n\right)$, $\widehat{w}_j = \left(w_j^l, w_j^\pi, w_j^u\right)$ is the space of the normalized fuzzy weights with respect to a fuzzy node $\widehat{gn}_j$, $j = 1, \ldots, m$, and is of the form:

$$\widehat{W} = \left\{\left(\widehat{w}_1, \ldots \widehat{w}_j, \ldots, \widehat{w}_n\right) : \widehat{w}_j = \left(w_j^l, w_j^\pi, w_j^u\right), w_j^l \leq w_j^\pi \leq w_j^u, \sum_{j=1}^{n} w_j^\pi = 1, j = 1, \ldots, n\right\} \tag{7.50}$$

$\left(v_{kj}^l, v_{kj}^\pi, v_{kj}^u\right)$ is the fuzzy individual utility of an alternative T_k, $k = 1, \ldots, m$ with respect to the fuzzy node $\widehat{gn}_j$, $j = 1, \ldots, m$.

$\left(v_{kj}^l, v_{kj}^\pi, v_{kj}^u\right)$ can be the normalized fuzzy relative utility (or fuzzy relative weight) $\left(w_{kj}^L, w_{kj}^M, w_{kj}^U\right)$ on the basis of Eq. 7.34. However, FCNP is not based on the relative values of the decision as the relative values do not give the absolute scores. If the alternatives have low scores, the choice only reflects the highest one from the set of the alternatives with low performance. The highest one may not be suitable. Thus, the normalized fuzzy relative utility is only fit for the problem with the only alternatives identified.

Table 7.7: Aggregation of $\left(T, Clst\left(\widehat{nd}, \widehat{gn}\right)\right)$ with fuzzy priority utility vector

Alternatives	$\widehat{gn}_1$	$\cdots$	$\widehat{gn}_j$	$\cdots$	$\widehat{gn}_n$	Fuzzy node utilities $\widehat{nd}$
	$\left(w_1^l, w_1^\pi, w_1^u\right)$	$\cdots$	$\left(w_j^l, w_j^\pi, w_j^u\right)$	$\cdots$	$\left(w_n^l, w_n^\pi, w_n^u\right)$	
T_1	$\left(v_{11}^l, v_{11}^\pi, v_{11}^u\right)$	$\cdots$	$\left(v_{1j}^l, v_{1j}^\pi, v_{1j}^u\right)$	$\cdots$	$\left(v_{1n}^l, v_{1n}^\pi, v_{1n}^u\right)$	$\left(v_{T_1}^l, v_{T_1}^\pi, v_{T_1}^u\right)$
$\vdots$	$\vdots$	$\cdots$		$\cdots$	$\vdots$	$\vdots$
T_k	$\left(v_{k1}^l, v_{k1}^\pi, v_{k1}^u\right)$	$\cdots$	$\left(v_{kj}^l, v_{kj}^\pi, v_{kj}^u\right)$	$\cdots$	$\left(v_{kn}^l, v_{kn}^\pi, v_{kn}^u\right)$	$\left(v_{T_k}^l, v_{T_k}^\pi, v_{T_k}^u\right)$
$\vdots$	$\vdots$	$\cdots$		$\cdots$	$\vdots$	$\vdots$
T_m	$\left(v_{n1}^l, v_{n1}^\pi, v_{n1}^u\right)$	$\cdots$	$\left(v_{nj}^l, v_{nj}^\pi, v_{nj}^u\right)$	$\cdots$	$\left(v_{mn}^l, v_{mn}^\pi, v_{mn}^u\right)$	$\left(v_{T_m}^l, v_{T_m}^\pi, v_{T_m}^u\right)$

As FCNP does not use the relative utility since the relative utility is a special case of absolute utility, table 7.8 can be ignored in this research. In table 7.7, for each decision alternative, the fuzzy utility value of $\widehat{gn}_j$, i.e. $\widehat{xgn}_{kj} = \left(v_{kj}^l, v_{kj}^\pi, v_{kj}^u\right)$, and fuzzy weights of $\widehat{gn}_j$, i.e. $\widehat{wgn}_{kj} = \left(w_j^l, w_j^\pi, w_j^u\right)$, are obtained from the Cognitive Assessment Process. Next the fuzzy node scores of T_k, i.e. $\widehat{nd}_{T_k} = \left(v_{T_k}^l, v_{T_k}^\pi, v_{T_k}^u\right)$, are aggregated, as follows:

$$\widehat{nd}_{T_k}=\left\{\left(v_{T_k}^{l},v_{T_k}^{\pi},v_{T_k}^{u}\right):\begin{pmatrix} v_{T_k}^{l}=Agg\left(\left\{v_{kj}^{l}\right\},w_j^{u}\right)\\ v_{T_k}^{\pi}=Agg\left(\left\{v_{kj}^{\pi}\right\},w_j^{\pi}\right)\\ v_{T_k}^{u}=Agg\left(\left\{v_{kj}^{u}\right\},w_j^{u}\right)\end{pmatrix}\right\},k=1,\ldots,n. \tag{7.51}$$

Table 7.8: Aggregation of $\left(T,Clst\left(\widehat{nd},\widehat{gn}\right)\right)$ with normalized priority vector

Alternatives	$\widehat{gn}_1$	$\cdots$	$\widehat{gn}_j$	$\cdots$	$\widehat{gn}_n$	Fuzzy node normalized priorities
	$\left(w_1^l,w_1^{\pi},w_1^u\right)$	$\cdots$	$\left(w_j^l,w_j^{\pi},w_j^u\right)$	$\cdots$	$\left(w_n^l,w_n^{\pi},w_n^u\right)$	
T_1	$\left(w_{11}^l,w_{11}^{\pi},w_{11}^u\right)$	$\cdots$	$\left(w_{1j}^l,w_{1j}^{\pi},w_{1j}^u\right)$	$\cdots$	$\left(w_{1n}^l,w_{1n}^{\pi},w_{1n}^u\right)$	$\left(w_{T_1}^l,w_{T_1}^{\pi},w_{T_1}^u\right)$
$\vdots$	$\vdots$	$\cdots$		$\cdots$	$\vdots$	$\vdots$
T_k	$\left(w_{k1}^l,w_{k1}^{\pi},w_{k1}^u\right)$	$\cdots$	$\left(w_{kj}^l,w_{kj}^{\pi},w_{kj}^u\right)$	$\cdots$	$\left(w_{kn}^l,w_{kn}^{\pi},w_{kn}^u\right)$	$\left(w_{T_k}^l,w_{T_k}^{\pi},w_{T_k}^u\right)$
$\vdots$	$\vdots$	$\cdots$		$\cdots$	$\vdots$	$\vdots$
T_m	$\left(w_{n1}^l,w_{n1}^{\pi},w_{n1}^u\right)$	$\cdots$	$\left(w_{nj}^l,w_{nj}^{\pi},w_{nj}^u\right)$	$\cdots$	$\left(w_{mn}^l,w_{mn}^{\pi},w_{mn}^u\right)$	$\left(w_{T_m}^l,w_{T_m}^{\pi},w_{T_m}^u\right)$

The next question is which aggregation operator should be applied. In FCNP, there are three alternatives for aggregating the results.

Each aggregation operator can be analogous to the cognitive style, or individual difference in thinking style. Different decision makers make different decisions as they have different cognitive styles. This issue is investigated in chapter 6.

$$\widehat{nd}_{T_k}=\left\{\left(v_{T_k}^{l},v_{T_k}^{\pi},v_{T_k}^{u}\right):\begin{pmatrix} v_{T_k}^{l}=SAO\left(\left\{v_{kj}^{l}\right\},w_j^{u}\right)\\ v_{T_k}^{\pi}=SAO\left(\left\{v_{kj}^{\pi}\right\},w_j^{\pi}\right)\\ v_{T_k}^{u}=SAO\left(\left\{v_{kj}^{u}\right\},w_j^{u}\right)\end{pmatrix}\right\},k=1,\ldots,n \tag{7.52}$$

The selection of the most appropriate AOs (SAO) of $\left(T,Clst\left(\widehat{nd},\widehat{gn}\right)\right)$ can be determined by the selection strategy of the cognitive style and aggregation operator model $\overline{CSAO}$ (in Algorithm 6.4). SAO is given by

$$SAO\left(\left(h_i,d_j\right),\left\{X\right\}\right)=\overline{CSAO}\left(\left(h_i,d_j\right),\left\{X\right\},\widetilde{Agg},D,\left(\overrightarrow{V_h},\overrightarrow{V_d}\right),\left(\varphi\left(\overrightarrow{V_h}\right),\lambda_0\right)\right)$$

(7.53)

, where $\left\{X\right\}$ is the set of data granules of the $\left(T,Clst\left(\widehat{nd},\widehat{gn}\right)\right)$, $da=\left(h_i,d_j\right)$ is the preferred compound decision attitude of atomic decision attitude d_j and the directional hedge term h_i,d_j. $SAO\left(X\right)$ is the delegate function which means that the $\overline{CSAO}$ applies the same parameters: $\left(\widetilde{Agg},D,\left(\overrightarrow{V_h},\overrightarrow{V_d}\right),\left(\varphi\left(\overrightarrow{V_h}\right),\lambda_0\right)\right)$. DA is the vector of the atomic decision attitude with membership input in DAAO-1, $\widetilde{Agg}$ is a vector of aggregation operators, $\left(\overrightarrow{V_h},\overrightarrow{V_d}\right)$ is the parameters to construct a directional hedge vector, and $\left(\varphi\left(\overrightarrow{V_h}\right),\lambda_0\right)$ is the parameters to control the distribution of the a directional hedge vector.

However, $\overline{CSAO}$ is quite complicated in view of the algorithms. In fact, the patterns of cognitive styles are complicated in the real world too. If $\overline{CSAO}$ is not applied, the weighted average operator is applied in default settings due to its computational efficiency, and most decision models apply this straightforward method, where

$$\widehat{nd}_{T_k} = \left\{ \left(v_{T_k}^{l}, v_{T_k}^{\pi}, v_{T_k}^{u} \right) : \left(\begin{array}{l} v_{T_k}^{l} = \frac{1}{n} \sum_{j=1}^{m} v_{kj}^{l} w_j^{l} \\ v_{T_k}^{\pi} = \frac{1}{n} \sum_{j=1}^{n} v_{kj}^{\pi} w_j^{\pi} \\ v_{T_k}^{u} = \frac{1}{n} \sum_{j=1}^{m} v_{kj}^{u} w_j^{u} \end{array} \right) \right\}, k = 1, \ldots, n. \quad (7.54)$$

On the other hand, if the decision makers feel that the weighted average operator may be excessively straightforward, $\overline{CSAO}$ is recommended. CNP applied $\overline{CSAO}$ whilst FCNP applies fuzzy $\overline{CSAO}$ which is in Eqs. 7.39-7.40.

7.4 Fuzzy collective cognitive network process (FCCNP)

The fuzzy collective cognitive network process (FCCNP), which is also named as the fuzzy group cognitive network process, is the CNP involved by a collection of experts $\{\hat{e}\}$ using linguistic terms which are represented by fuzzy numbers. Thus FCCNP has the form:

$$FCCNP = \left(\widehat{CNP}, \left\{ \left(\hat{e}, \widehat{we} \right) \right\} \right) \quad (7.55)$$

, where $\left(\hat{e}, \widehat{we} \right)$ is a 2-tuple fuzzy input in which an expert $\hat{e}$, with fuzzy authority weight $\widehat{we}$ provides.

The fuzzy collective cognitive network process (FCCNP), is the CCNP with fuzzy input parameters. Thus it is the combination of FCNP and CCNP. The FCCNP applies the concepts of fuzzy rating scales, fuzzy pairwise opposite matrix, fuzzy cognitive prioritization operator, and fuzzy aggregation in FCNP, as well as the fuzzy multi-experts multi-criteria aggregation (f-MEMC) operator, which is and extension of the crisp multi-experts multi-criteria aggregation (c-MEMC) operator in CCNP. Thus this section only discusses f-MEMC regarding its fuzzy decision matrix, as other issues are discussed in previous sections.

The fuzzy multi-experts multi-criteria (f-MEMC) decision matrix of the fuzzy cluster $Clst\left(\widehat{nd},\left\{\widehat{gn}_j\right\}\right)$, measured by the set of the fuzzy inputs of the multi-experts $\left\{\left(\widehat{e}_{k'},\widehat{we}_{k'}\right)\right\}$ with respect to the fuzzy alternative $\widehat{T}_k$, is of the following forms:

$$\widehat{dm}\left(\left\{\left(\widehat{e}_{k'},\widehat{we}_{k'}\right)\right\},\widehat{T}_k,Clst\left(\widehat{nd},\left\{\widehat{gn}_j\right\}\right)\right)=\begin{array}{cc|c} & & \left(\widehat{wgn}_1 \;\; \cdots \;\; \widehat{wgn}_j \;\; \cdots \;\; \widehat{wgn}_n\right) \\ & & \widehat{xgn}_1 \;\; \cdots \;\; \widehat{xgn}_j \;\; \cdots \;\; \widehat{xgn}_n \\ \begin{matrix} \widehat{e}_1 \\ \vdots \\ \widehat{e}_{k'} \\ \vdots \\ \widehat{e}_{q_{nd}} \end{matrix} & \begin{matrix} \widehat{we}_1 \\ \vdots \\ \widehat{we}_{j'} \\ \vdots \\ \widehat{we}_{q_{nd}} \end{matrix} & \begin{pmatrix} \widehat{v}_{1,1} & \cdots & \widehat{v}_{1,j} & \cdots & \widehat{v}_{1,n} \\ \vdots & \ddots & \vdots & \iddots & \vdots \\ \widehat{v}_{k',1} & \cdots & \widehat{v}_{k',j} & \cdots & \widehat{v}_{k',n} \\ \vdots & \iddots & \vdots & \ddots & \vdots \\ \widehat{v}_{q_{nd},1} & \cdots & \widehat{v}_{q_{nd},j} & \cdots & \widehat{v}_{q_{nd},n} \end{pmatrix} \end{array} \tag{7.56}$$

, where $\widehat{we}_{k'}$ is the fuzzy relative weight of the expert $\widehat{e}_{k'}$; $\widehat{v}_{k',j}=\left(v_{k',j}^{l},v_{k',j}^{\pi},v_{k',j}^{u}\right)$ is the fuzzy utility value with respect to $\left(\widehat{e}_{k'},\widehat{we}_{k'}\right)$ and $\widehat{xgn}_j$.

The aggregation problem of $\widehat{dm}\left(\left\{\left(\widehat{e}_{j'},\widehat{we}_{j'}\right)\right\},\widehat{T}_k,Clst\left(\widehat{nd},\left\{\widehat{gn}_j\right\}\right)\right)$ is of the form:

$$\widehat{nd}_{T_k}=\widehat{Cagg}(\widehat{Eagg}(\begin{array}{cc|c} & & \left(\widehat{wgn}_1 \;\; \cdots \;\; \widehat{wgn}_j \;\; \cdots \;\; \widehat{wgn}_n\right) \\ & & \widehat{xgn}_1 \;\; \cdots \;\; \widehat{xgn}_j \;\; \cdots \;\; \widehat{xgn}_n \\ \begin{matrix} \widehat{e}_1 \\ \vdots \\ \widehat{e}_{j'} \\ \vdots \\ \widehat{e}_{q_{nd}} \end{matrix} & \begin{matrix} \widehat{we}_1 \\ \vdots \\ \widehat{we}_{j'} \\ \vdots \\ \widehat{we}_{q_{nd}} \end{matrix} & \begin{pmatrix} \widehat{v}_{1,1} & \cdots & \widehat{v}_{1,j} & \cdots & \widehat{v}_{1,n} \\ \vdots & \ddots & \vdots & \iddots & \vdots \\ \widehat{v}_{j',1} & \cdots & \widehat{v}_{j',j} & \cdots & \widehat{v}_{j,n} \\ \vdots & \iddots & \vdots & \ddots & \vdots \\ \widehat{v}_{q_{nd},1} & \cdots & \widehat{v}_{q_{nd},j} & \cdots & \widehat{v}_{q_{nd},n} \end{pmatrix} \end{array})) \tag{7.57}$$

Table 7.9: Aggregation of $\widehat{dm}\left(\left\{\left(\widehat{e}_{k'},\widehat{we}_{k'}\right)\right\},\widehat{T}_k,Clst\left(\widehat{nd},\left\{\widehat{gn}_j\right\}\right)\right)$

		$\widehat{gn}_1$	$\ldots$	$\widehat{gn}_j$	$\ldots$	$\widehat{gn}_n$
		$\left(w_1^l,w_1^\pi,w_1^u\right)$	$\ldots$	$\left(w_j^l,w_j^\pi,w_j^u\right)$	$\ldots$	$\left(w_n^l,w_n^\pi,w_n^u\right)$
e_1	$\left(we_1^l,we_1^\pi,we_1^u\right)$	$\left(v_{11}^l,v_{11}^\pi,v_{11}^u\right)$	$\ldots$	$\left(v_{1j}^l,v_{1j}^\pi,v_{1j}^u\right)$	$\ldots$	$\left(v_{1n}^l,v_{1n}^\pi,v_{1n}^u\right)$
$\vdots$	$\vdots$	$\vdots$	$\ldots$		$\ldots$	$\vdots$
$e_{k'}$	$\left(we_{k'}^l,we_{k'}^\pi,we_{k'}^u\right)$	$\left(v_{k1}^l,v_{k1}^\pi,v_{k1}^u\right)$	$\ldots$	$\left(v_{k'j}^l,v_{k'j}^\pi,v_{k'j}^u\right)$	$\ldots$	$\left(v_{kn}^l,v_{kn}^\pi,v_{kn}^u\right)$
$\vdots$	$\vdots$	$\vdots$	$\ldots$		$\ldots$	$\vdots$
$e_{q_{nd}}$	$\left(we_{q_{nd}}^l,we_{q_{nd}}^\pi,we_{q_{nd}}^u\right)$	$\left(v_{q_{nd},1}^l,v_{q_{nd},1}^\pi,v_{q_{nd},1}^u\right)$	$\ldots$	$\left(v_{q_{nd},j}^l,v_{q_{nd},j}^\pi,v_{q_{nd},j}^u\right)$	$\ldots$	$\left(v_{q_{nd},n}^l,v_{q_{nd},n}^\pi,v_{q_{nd},n}^u\right)$
Σ	$\left(we^l,we^\pi,we^u\right)$	$\left(v_1^l,v_1^\pi,v_1^u\right)$	$\ldots$	$\left(v_j^l,v_j^\pi,v_j^u\right)$	$\ldots$	$\left(v_n^l,v_n^\pi,v_n^u\right)$
$\widehat{nd}_{T_k}=\left(v_{nd}^l,v_{nd}^\pi,v_{nd}^u\right)$						

The tabular form of the above aggregation is explicitly shown in table 7.9. The details are as follows:

$Eagg$ is the fuzzy aggregation function of the expert judgments for $\widehat{xgn}_j$, and has the form:

$$\widehat{xgn}_j=\widehat{Eagg}\left(\left[\left(\widehat{v}_{k',j},\widehat{we}_{k'}\right)\right]_{k'=1}^{q_{nd}}\right),\quad \forall j\in\{1,\ldots,n\} \tag{7.58}$$

Explicitly,

$$\widehat{xgn}_j=\left(v_j^l,v_j^\pi,v_j^u\right)=\widehat{Eagg}\left(\left[\left(\left(v_{k'j}^l,v_{k'j}^\pi,v_{k'j}^u\right),\left(we_{k'}^l,we_{k'}^\pi,we_{k'}^u\right)\right)\right]_{k'=1}^{q_{nd}}\right) \tag{7.59}$$

$\widehat{Cagg}$ is the fuzzy aggregation function of the set of the criteria pairs $\left\{\left(\widehat{xgn}_j, \widehat{wgn}_j\right)\right\}$ in a fuzzy node of alternative k, i.e. $\widehat{nd}_{T_k}$, and has the form:

$$\widehat{nd}_{T_k} = \widehat{Cagg}\left(\left[\left(\widehat{xgn}_j, \widehat{wgn}_j\right)\right]_{j=1}^{n}\right), \quad \forall k \in \{1,\ldots,m\} \tag{7.60}$$

Explicitly,

$$\widehat{nd}_{T_k} = \left(v_{nd}^{l}, v_{nd}^{\pi}, v_{nd}^{u}\right) = \widehat{Cagg}\left(\left[\left(\left(v_j^{l}, v_j^{\pi}, v_j^{u}\right), \left(w_j^{l}, w_j^{\pi}, w_j^{u}\right)\right)\right]_{j=1}^{n}\right) \tag{7.61}$$

Alternatively by substitution of Eq 7.58 or Eq. 7.59, then

$$\widehat{nd}_{T_k} = \widehat{Cagg}\left(\left[\left(\widehat{Eagg}\left(\left[\left(\widehat{v}_{k',j}, \widehat{we}_{k'}\right)\right]_{k'=1}^{q_{nd}}\right), \widehat{wgn}_j\right)\right]_{j=1}^{n}\right), \quad \forall k \in \{1,\ldots,m\} \tag{7.62}$$

Explicitly,

$$\widehat{nd}_{T_k} = \widehat{Cagg}\left(\left[\left(\widehat{Eagg}\left(\left[\left(\left(v_{k'j}^{l}, v_{k'j}^{\pi}, v_{k'j}^{u}\right), \left(we_{k'}^{l}, we_{k'}^{\pi}, we_{k'}^{u}\right)\right)\right]_{k'=1}^{q_{nd}}\right), \left(w_j^{l}, w_j^{\pi}, w_j^{u}\right)\right)\right]_{j=1}^{n}\right) \tag{7.63}$$

Propositions 7.2 and 7.3 are an application of $\widehat{nd}_{T_k}$, which is the usual case (2 layers of structural criteria and one layer of expert) in CNP.

The set of the multi-experts multi-criteria decision matrix of all alternatives is of the form:

$$dm\left(\widehat{dm}\left(\left\{\left(\widehat{e}_{j'}, \widehat{we}_{j'}\right)\right\}, \widehat{T}, Clst\left(\widehat{nd}, \left\{\widehat{gn}_j\right\}\right)\right)\right) = \left\{\widehat{dm}\left(\left\{\left(\widehat{e}_{j'}, \widehat{we}_{j'}\right)\right\}, \widehat{T}_k, Clst\left(\widehat{nd}, \left\{\widehat{gn}_j\right\}\right)\right)\right\} \tag{7.64}$$

And the set of the alternative for a node $\widehat{nd}_T$ is the form:

$$\widehat{nd}_T = \left\{ \widehat{nd}_{T_k} : \widehat{nd}_{T_k} = \widehat{Cagg}\left(\left[\left(\widehat{Eagg}\left(\left[\left(\widehat{v}_{k',j}, \widehat{we}_{k'} \right) \right]_{k'=1}^{q_{nd}} \right), \widehat{wgn}_j \right) \right]_{j=1}^{n} \right), \forall k \in \{1,\ldots,m\} \right\}$$

(7.65)

Finally the volition decision process is used. The process is similar to the crisp one stated in chapter 3.

7.5 Numerical analyses and discussion

The analyses show the comparisons of the results of two fuzzy prioritization operators, FRAU (or FPLS or FAMSLS) and FLPS (or FDLS), on the basis of the FCPOM models. The simulation includes 68 (17 x4) cases from eight template matrices of different dimensions. The rating scale, which $\widehat{r}$ chooses from, is defined in table 7.10. The template matrices are shown as follows.

$$\widehat{T3}(\widehat{r}) = \begin{pmatrix} (0,0,0) & (1,2,3) & \widehat{r} \\ (-3,-2,-1) & (0,0,0) & (0,0,0) \\ -\widehat{r} & (0,0,0) & (0,0,0) \end{pmatrix}$$

$$\widehat{T4}(\widehat{r}) = \begin{pmatrix} (0,0,0) & (1,2,3) & (2,3,4) & \widehat{r} \\ (-3,-2,-1) & (0,0,0) & (0,1,2) & (0,0,0) \\ (-4,-3,-2) & (-2,-1,0) & (0,0,0) & (-2,-1,0) \\ -\widehat{r} & (0,0,0) & (0,1,2) & (0,0,0) \end{pmatrix}$$

$$\widehat{T5}(\hat{r}) = \begin{pmatrix} (0,0,0) & (1,2,3) & (2,3,4) & (0,1,2) & \hat{r} \\ (-3,-2,-1) & (0,0,0) & (0,1,2) & (-2,-1,0) & (0,0,0) \\ (-4,-3,-2) & (-2,-1,0) & (0,0,0) & (-3,-2,-1) & (-2,-1,0) \\ (-2,-1,0) & (0,1,2) & (1,2,3) & (0,0,0) & (0,1,2) \\ -\hat{r} & (0,0,0) & (0,1,2) & (-2,-1,0) & (0,0,0) \end{pmatrix}$$

$$\widehat{T6}(\hat{r}) =$$

$$\begin{pmatrix} (0,0,0) & (1,2,3) & (2,3,4) & (3,4,5) & (-4,-3,-2) & \hat{r} \\ (-3,-2,-1) & (0,0,0) & (0,1,2) & (1,2,3) & (-6,-5,-4) & (0,1,2) \\ (-4,-3,-2) & (-2,-1,0) & (0,0,0) & (0,1,2) & (-7,-6,-5) & (0,0,0) \\ (-5,-4,-3) & (-3,-2,-1) & (-2,-1,0) & (0,0,0) & (-8,-7,-6) & (-2,-1,0) \\ (2,3,4) & (4,5,6) & (5,6,7) & (6,7,8) & (0,0,0) & (5,6,7) \\ -\hat{r} & (-2,-1,0) & (0,0,0) & (0,1,2) & (-7,-6,-5) & (0,0,0) \end{pmatrix}$$

Table 7.10 : Fuzzy cognitive comparison scale

Atomic verbal scale $\overrightarrow{V_a}$	Fuzzy scale X'^{+}_{α}
Equally	(0,0,0)
Weakly	(0,1,2)
Moderately	(1,2,3)
Moderately plus	(2,3,4)
Strongly	(3,4,5)
Strong Plus	(4,5,6)
Very Strongly	(5,6,7)
Very strongly Plus	(6,7,8)
Extremely	(7,8,8)
opposites of Above	$X^{-}_{\aleph}$

Table 7.11: FAI, FRMSV, FMC and FRMPWSV of FRAU and FLPS for $\widehat{T3}$ and $\widehat{T4}$

		$\widehat{T3}$	FRMSV		FMC		FRMPWSV		$\widehat{T4}$	FRMSV		FMC		FRMPWSV	
Index	$\hat{r}$	FAI	FRAU	FLPS	FRAU	FLPS	FRAU	FLPS	FAI	FRAU	FLPS	FRAU	FLPS	FRAU	FLPS
1	(-8,-8,-7)	0.47	3.25	4.79	0.50	0.46	5.93	8.63	0.42	2.84	3.04	0.29	0.17	4.87	4.97
2	(-8,-7,-6)	0.44	3.00	4.03	0.46	0.38	5.64	7.27	0.39	2.62	2.76	0.29	0.17	4.50	4.50
3	(-7,-6,-5)	0.39	2.67	3.10	0.42	0.17	4.87	5.33	0.35	2.33	2.46	0.23	0.17	4.18	4.03
4	(-6,-5,-4)	0.34	2.33	2.57	0.42	0.33	4.26	4.41	0.31	2.05	2.12	0.17	0.17	3.54	3.69
5	(-5,-4,-3)	0.29	2.00	2.33	0.33	0.17	3.87	3.75	0.27	1.77	1.84	0.21	0.13	3.06	3.02
6	(-4,-3,-2)	0.24	1.67	1.77	0.25	0.29	3.04	3.26	0.22	1.48	1.58	0.19	0.13	2.60	2.61
7	(-3,-2,-1)	0.19	1.33	1.42	0.33	0.17	2.43	2.50	0.18	1.20	1.42	0.21	0.13	2.30	2.31
8	(-2,-1,0)	0.15	1.00	1.08	0.33	0.29	2.04	1.98	0.14	0.93	1.04	0.23	0.15	1.72	1.75
9	(0,0,0)	0.10	0.67	0.77	0.33	0.33	1.36	1.32	0.10	0.69	0.79	0.21	0.21	1.43	1.39
10	(0,1,2)	0.05	0.33	0.37	0.21	0.21	0.63	0.60	0.06	0.43	0.47	0.15	0.15	0.81	0.80
11	(1,2,3)	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.25	0.28	0.08	0.13	0.49	0.48
12	(2,3,4)	0.05	0.33	0.36	0.17	0.17	0.61	0.58	0.06	0.43	0.46	0.13	0.13	0.77	0.76
13	(3,4,5)	0.10	0.67	0.72	0.17	0.17	1.22	1.17	0.10	0.67	0.70	0.10	0.13	1.15	1.13
14	(4,5,6)	0.15	1.00	1.08	0.17	0.17	1.83	1.75	0.14	0.93	0.97	0.10	0.13	1.58	1.56
15	(5,6,7)	0.19	1.33	1.44	0.17	0.17	2.43	2.33	0.18	1.20	1.25	0.17	0.13	2.09	2.00
16	(6,7,8)	0.24	1.67	1.80	0.17	0.17	3.04	2.92	0.22	1.48	1.53	0.23	0.13	2.58	2.46
17	(7,8,8)	0.28	1.92	2.07	0.17	0.17	3.50	3.35	0.26	1.70	1.76	0.25	0.13	2.87	2.82

The detailed results are illustrated in appendix IV. The simulation results include the Fuzzy Accordant Index (FAI), Fuzzy Root Mean Square Variance (FRMSV), Fuzzy Mean Contradiction (FMC) and Fuzzy Root Mean Penalty Weighted Square Variance (FRMPWSV), for two boundary values, one modal value and the aggregation, as well as the fuzzy utility vectors of the 68 cases. The essential data are extracted and summarized in tables 7.11 and 7.12. In order to efficiently present the data, figures 7.3 and 7.5 are plotted. Each figure includes four sub-figures representing the four template matrices respectively. The impacts are as follows:

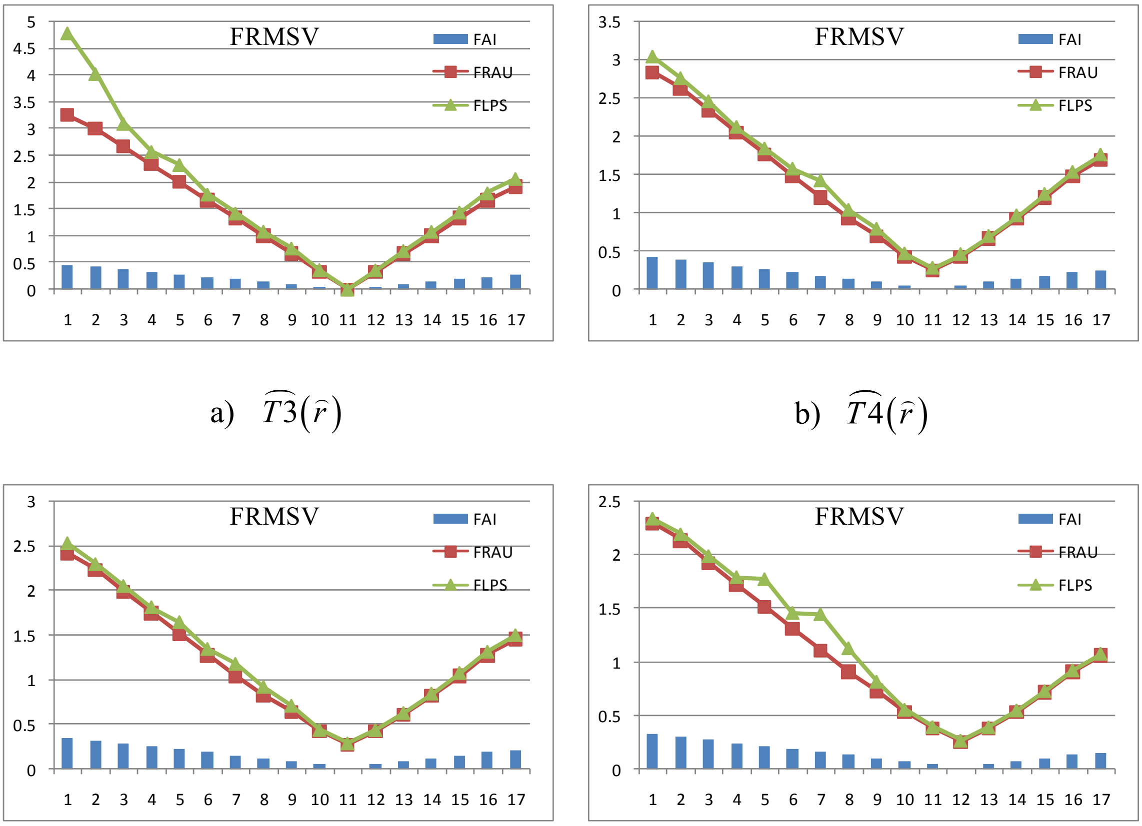
Table 7.12: FAI, FRMSV, FMC and FRMPWSV of FRAU and FLPS for $\widehat{T5}$ and $\widehat{T6}$

		$\widehat{T5}$	FRMSV		FMC		FRMPWSV		$\widehat{T6}$	FRMSV		FMC		FRMPWSV	
Index	$\hat{r}$	FAI	FRAU	FLPS	FRAU	FLPS	FRAU	FLPS	FAI	FRAU	FLPS	FRAU	FLPS	FRAU	FLPS
1	(-8,-8,-7)	0.36	2.42	2.53	0.29	0.24	4.11	4.08	0.33	2.29	2.34	0.13	0.12	3.77	3.77
2	(-8,-7,-6)	0.33	2.24	2.30	0.26	0.25	3.75	3.97	0.31	2.14	2.20	0.13	0.15	3.51	3.54
3	(-7,-6,-5)	0.30	1.99	2.05	0.29	0.25	3.35	3.52	0.28	1.93	1.99	0.13	0.15	3.60	3.23
4	(-6,-5,-4)	0.26	1.75	1.82	0.28	0.24	2.96	2.94	0.25	1.72	1.80	0.15	0.13	2.83	3.30
5	(-5,-4,-3)	0.23	1.52	1.65	0.25	0.19	2.60	2.71	0.22	1.52	1.78	0.15	0.13	2.50	2.98
6	(-4,-3,-2)	0.19	1.28	1.35	0.21	0.21	2.46	2.45	0.19	1.31	1.46	0.13	0.12	2.19	2.43
7	(-3,-2,-1)	0.16	1.05	1.18	0.16	0.19	1.78	1.95	0.16	1.11	1.45	0.12	0.13	1.84	2.34
8	(-2,-1,0)	0.13	0.82	0.93	0.18	0.20	1.49	1.61	0.14	0.91	1.13	0.12	0.11	1.62	1.91
9	(0,0,0)	0.09	0.64	0.71	0.16	0.16	1.29	1.27	0.11	0.74	0.82	0.09	0.10	1.49	1.47
10	(0,1,2)	0.06	0.42	0.44	0.13	0.13	0.77	0.76	0.08	0.54	0.56	0.08	0.08	0.94	0.93
11	(1,2,3)	0.00	0.27	0.29	0.09	0.09	0.53	0.52	0.05	0.38	0.40	0.07	0.07	0.67	0.66
12	(2,3,4)	0.06	0.42	0.44	0.10	0.11	0.77	0.76	0.00	0.26	0.27	0.05	0.05	0.46	0.46
13	(3,4,5)	0.09	0.61	0.63	0.10	0.11	1.07	1.06	0.05	0.38	0.40	0.06	0.06	0.65	0.65
14	(4,5,6)	0.13	0.82	0.85	0.11	0.11	1.42	1.40	0.08	0.54	0.55	0.07	0.07	0.90	0.90
15	(5,6,7)	0.16	1.05	1.08	0.10	0.11	1.79	1.77	0.11	0.72	0.73	0.06	0.07	1.19	1.18
16	(6,7,8)	0.19	1.28	1.32	0.14	0.11	2.20	2.15	0.14	0.91	0.93	0.06	0.07	1.50	1.49
17	(7,8,8)	0.22	1.46	1.50	0.16	0.11	2.47	2.44	0.16	1.07	1.08	0.07	0.07	1.74	1.73

Fig. 7.3 shows the results of the RMSV of the template matrices. The fact that the lines of FRAU just touch or are below the FLPS ones indicates that $FRMSV(FRAU) \le FRMSV(FLPS)$. This is the normal case as the objective

function of FPLS is FRMSV, and FRAU=FPLS, which has been proved in theorem 7.1.

Also the fact that $FRMSV(FRAU) = FRMSV(FLPS)$ does not necessarily follow $FPMPWSV(FRAU) = FPMPWSV(FLPS)$ when compared with fig. 7.4. This issue is related to the existence of the contraction which is shown in fig. 7.5.

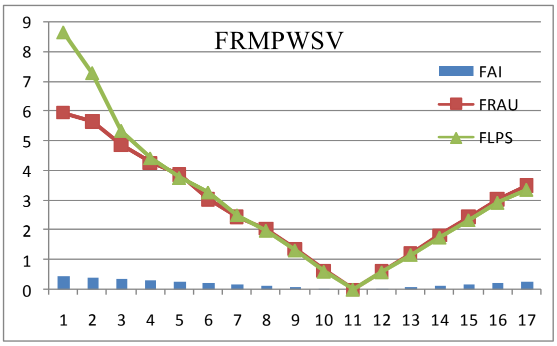


a) $\overset{\frown}{T3}(\hat{r})$ b) $\overset{\frown}{T4}(\hat{r})$

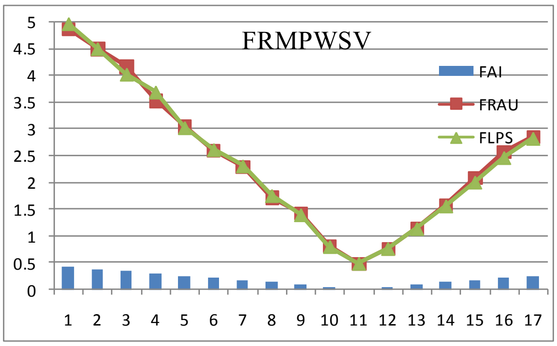
c) $\overset{\frown}{T5}(\hat{r})$ d) $\overset{\frown}{T6}(\hat{r})$

Figure 7.3 FRMSV of FRAU and FLPS for $\overset{\frown}{T3}(\hat{r})$, $\overset{\frown}{T4}(\hat{r})$, $\overset{\frown}{T5}(\hat{r})$, and $\overset{\frown}{T6}(\hat{r})$

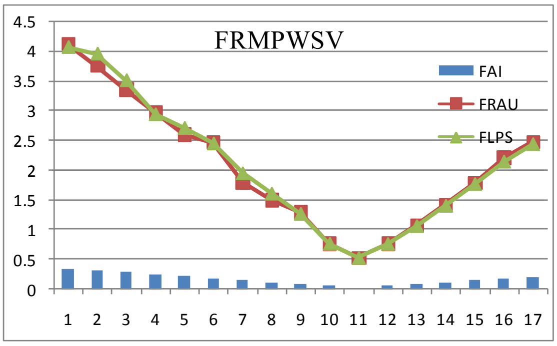
Fig. 7.4 shows the FRMPWSV of the four template matrices. The fact that the lines of FLPS just touch or are below the FRAU ones in the low FAI region indicates that $FPMPWSV(FRAU) \geq FPMPWSV(FLPS)$. This is the normal case as the objective function of FLPS is FPMPWSV. The abnormal cases of $FPMPWSV(FRAU) < FPMPWSV(FLPS)$ are due to the fact that the rounding results of FLPS are evaluated in FPMPWSV.



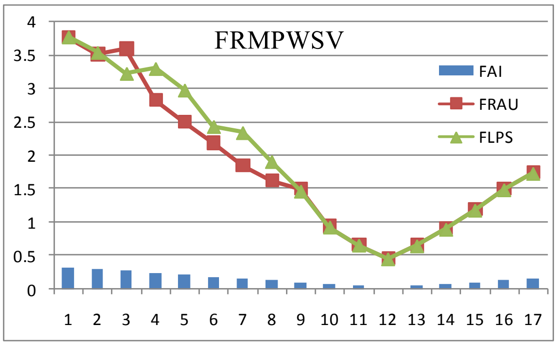
a) $\widehat{T3}(\hat{r})$



b) $\widehat{T4}(\hat{r})$



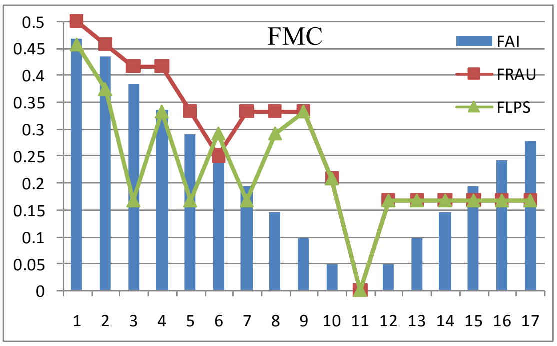
c) $\widehat{T5}(\hat{r})$



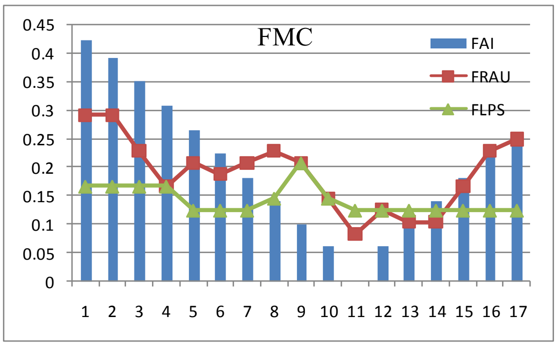
d) $\widehat{T6}(\hat{r})$

Figure 7.4: FRMPWSV of FRAU and FLPS for $\widehat{T3}(\hat{r})$, $\widehat{T4}(\hat{r})$, $\widehat{T5}(\hat{r})$, and $\widehat{T6}(\hat{r})$

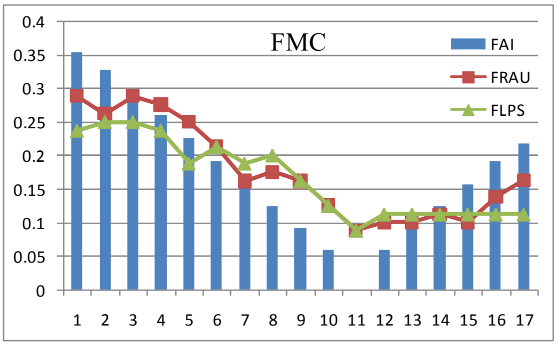
Fig. 7.5 shows the results of the fuzzy mean contradiction (FMC) of the four template matrices. In most cases, $FMC(FLPS) \le FMC(FRAU)$. The reasons for the exceptions are that, for one thing, FRAU produces a more accurate result of many digits in the computer program, however it is rounded to fewer digits, thus producing abnormal results. This issue can be seen in index 11 of fig. 7.5f. For another, the higher FMC can produce a lower FPMPWSV, but in rare cases, e.g indices 13 and 14 of $\widehat{T4}$, as seen in table 7.11.



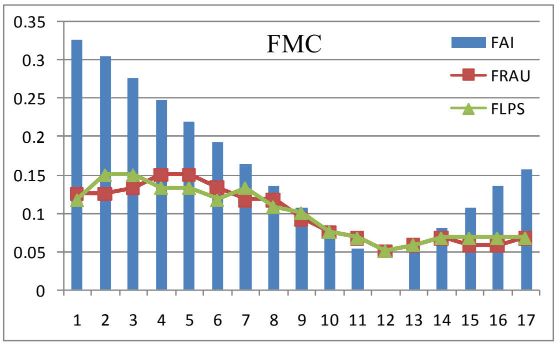
a) $\widehat{T3}(\hat{r})$



b) $\widehat{T4}(\hat{r})$



c) $\widehat{T5}(\hat{r})$



d) $\widehat{T6}(\hat{r})$

Figure 7.5: FMC of FRAU and FLPS for $\widehat{T3}(\hat{r})$, $\widehat{T4}(\hat{r}), \widehat{T5}(\hat{r})$, and $\widehat{T6}(\hat{r})$

If $FAI = 0$, it is not necessary that $FRAU = FLPS$. Thus it does not follow that $FPMPWSV(FRAU) = PMPWSV(FLPS)$ and $FRMSV(FRAU) = RMSV(FLPS)$, since FAI implies that only one of the modal or boundary values is accordant and others are not (chapter 7.3.2). The related data in detail can be referred to in appendix IV.

If $FAI \le 0.1$, then $FPMPWSV(FRAU) \cong FPMPWSV(FLPS)$, $FRMSV(FRAU) \cong FRMSV(FLPS)$, and $FMC(FRAU) \cong FMC(FLPS)$ since $FRAU \cong FLPS$.

With the above findings, the best practice for choosing the fuzzy cognitive prioritization operators is in the following orders.

1. If $FAI \le 0.1$, especially $FMC = 0$, FRAU is recommended. For one thing, interestingly, it produces the same result as FAMSLS and FPLS. For another, its computational effort is the least. Thus, when a fuzzy pairwise opposite matrix is fuzzy accordant, or satisfactory without violation, this method is more preferable.
2. If $AI \le 0.1$ and $FMC \ne 0$, and only the rank of the FPOM is considered, then FRAU is suggested. If the individual utility values are significant, FLPS is suggested.
3. If $FMC \ne 0$, FLPS is suggested . FPLS is the basic form for developing FLPS. In view of the approximate accuracy of the discordant matrix with contradiction, FLPS is more preferable as it minimizes the summation of the multiples of the contradiction and distance errors.

7.6 Summary and remarks

To conclude this chapter, the FCCNP from calculation viewpoint is briefly illustrated in algorithm 7.1.

Algorithm 7.1 (FC**CNP)**:

1. Define the project profile such as structural criteria, experts profiles for the domain;
2. Define the syntactic form and semantic form of the Compound Linguistic Ordinal Scales for cognitive comparison scale by algorithms 4.1 and 4.2;
3. Define the cognitive prioritization CP operator;
4. Define $Eagg$ and $Cagg$;
5. Evaluate all clusters, $\varphi\left(Clst\left(nd,gn\right)\right)$, by algorithm 4.3;
6. Use cognitive prioritization operator
7. Convert the relative fuzzy weight vector for a collection of experts.
8. Convert the relative fuzzy weight vectors for all clusters.
9. Form a set of decision matrices $dm\left(\widehat{dm}\left(\left\{\left(\hat{e}_{j'},\widehat{we}_{j'}\right)\right\},\hat{T},Clst\left(\widehat{nd},\left\{\widehat{gn}_j\right\}\right)\right)\right)$.
10. Unify $dm\left(\widehat{dm}\left(\left\{\left(\hat{e}_{j'},\widehat{we}_{j'}\right)\right\},\hat{T},Clst\left(\widehat{nd},\left\{\widehat{gn}_j\right\}\right)\right)\right)$ by $Eagg$ and $Cagg$;
11. Return O; **//End**

CNP, CCNP, and FCNP are the special cases of the above algorithms. If the number of experts is a single one with crisp inputs, the above algorithm is for CNP. If the number of experts is collective with crisp inputs, the above algorithm is for CCNP. If the number of experts is a single one with fuzzy inputs, the above algorithm is for FCNP.

The implication of CCNP is that it allows multiple expert decision making with different weights, and thus the decision is more convincible by group effort. The implication of FCNP is that it allows fuzzy inputs and produces fuzzy outputs. Thus imprecise inputs using fuzzy linguistic terms reduce the difficulty of the rating decision, as it is easier than the crisp input. The fuzzy output gives the fuzzy numbers for the decision makers to make the optimistic, neutral and pessimistic decisions, as a fuzzy number shows the range with membership. FCCNP takes all the advantages of CCNP and FCNP. Compared with the primitive CNP, FCCNP is of a more human workload in the administration process in handling the assessment, and a more computational workload in the prioritization and aggregation processes. However it has an efficient input for the comparison matrix, and more information from the fuzzy output.

Chapter 8 Applications

8.1 Introduction

In this chapter, five cases are illustrated and discussed with respect to various types of the CNP models. Each case comprises the case problem, ANP approach, CNP approach, and the discussion of the comparison.

Case 1 presents the high school selection problem (Saaty, 1980, p26-28) with comparisons of the primitive CNP model and the AHP model. This is one of the well-known examples in (Saaty, 1980).

Case 2 presents the transportation company selection problem (Kulak and Kahramna, 2005) with comparisons of the primitive CNP model and the AHP model with both prioritization measurement models being used. This case has analyzed by Yuen (2009b) using different AHP prioritization methods, and further revised by the primitive CNP.

Case 3 compares the CNP model and the improved ANP models for the R&D project selection problem (Yuen and Lau, 2009). The rating scales of ANP have been improved by CLOS in (Yuen and Lau, 2009). Thus the problem is further investigated by the CNP.

Case 4 compares the fuzzy CNP and Fuzzy AHP models for the software product selection problem which is studied by Yuen and Lau (2008c). The CNP solution is proposed for a local electronic company with comparison with the conventional method which is proposed previously.

Case 5 illustrates the use of the fuzzy collective CNP model as the evaluation tool for the problem of supplier number optimization (Berger et al, 2004). This case demonstrates how FCCNP functions as the parametric settings for the decision function.

Table 8.1 shows the default settings of the rating scales for AHP and CNP used in this chapter.

Table 8.1: Scale schemas conversion table for AHP and CNP

Verbal scale	Crisp number		Fuzzy Number	
	Ratio Scale (AHP)	Interval Scale (CNP, $\kappa = 8$)	Ratio Scale (FAHP)	Interval Scale (FCNP, $\kappa = (7,8,9)$)
Equally	1	0	(1,1,1)	(0,0,0)
Weakly	2	1	(1,2,3)	(0,1,2)
Moderately	3	2	(2,3,4)	(1,2,3)
Moderately plus	4	3	(3,4,5)	(2,3,4)
Strongly	5	4	(4,5,6)	(3,4,5)
Strong Plus	6	5	(5,6,7)	(4,5,6)
Very Strongly	7	6	(6,7,8)	(5,6,7)
Very, very strongly	8	7	(7,8,9)	(6,7,8)
Extremely	9	8	(8,9,9)	(7,8,8)
Reciprocals (AHP) / opposites (CNP) of the Above				

8.2 Case 1: High school selection

The case background, solutions using the AHP approach and the CNP approach respectively, and the discussion of the comparison are presented as follows.

8.2.1 Case 1 background

The intention of this case is to compare the results of AHP and CNP. The decision problem in the high school selection using AHP has been discussed by Saaty (1980, pp. 26-28). The definitions of the five criteria, and four alternatives are shown in table 8.2.1 and seven pairwise matrices are illustrated in table 8.2.2.

8.2.2 The AHP approach to case 1

The local priorities are derived by ten analytic prioritization operators (APOs), denoted by

$$P=\left[p_1,\ldots,p_{10}\right]=[EV,NRS,NRCS,AMNC,NGMR,DLS,WLS,FP,EGP,\text{ LPPDS}]$$

,where definitions of the analytic POs are illustrated in chapter 2.3.5.

The results of the prioritization and aggregation of the above APOs are illustrated in table 8.2.3. It can be observed that different analytic prioritization operators produce different priorities which likely lead to different preference orders or ranks. In this research, a higher score means a higher preference order.

8.2.3 The CNP approach to case 1

To apply CNP, the linguistic terms are assumed to be the same. Regarding the numerical representation, the Pairwise Reciprocal Matrices (PRMs) are changed to the Pairwise Opposite Matrices (POMs) with reference to table 8.1. The result of the POMs is shown in table 8.2.4. In order to compare with AHP, the results of the utility sets are normalized. The results of the normalized utility sets and their aggregation values using weighted average aggregation operator, which is set by default, is shown in table 8.2.5. Two cognitive prioritization operators, RAU (or PLS) and LPS, are used. It can be observed that their results are very close and the order is preserved.

8.2.4 Discussion of case 1

In table 8.2.2, the pairwise reciprocal matrices include three consistent matrices, i.e. A_3, A_4, A_6, and four non-consistent matrices, e.g. A_1, A_2, A_5, A_7. For the consistent matrices, any prioritization operator can be used as the value of the priority vector is the same. However, when the PRMs are converted to POMs in table 8.2.4, the pairwise opposite matrices include four accordant matrices, i.e. B_3, B_4, B_6, B_7, and three discordant matrices, e.g. B_1, B_2, B_5. In other words, after conversion, one more accordant matrix is established and one discordant matrix is removed, since B_7 is accordant and A_7 is inconsistent.

Regarding A_7 of table 8.2.2, if $a_{12} = 6$ and $a_{13} = 4$ are unchanged, there is no suitable rating scale for a_{23} to make A_7 consistent, since none of linguistic labels in table 8.1 represents $2/3$ due to the condition of $a_{23} = a_{13}/a_{12} = 4/6 = 2/3$. Thus this rating scale issue is a huge barrier in producing a consistent or satisfied matrix for human judgment. In fact, the ratio scale is ill-defined for pairwise comparisons, and is discussed in chapter 5.

A_1 is inconsistent and B_1 is discordant. Table 8.2.3 shows that W of A_1 has various results with respect to different APOs. On the other hand, table 8.2.5 shows that W of B_1 of the two CPOs are very close although B_1 is discordant. This implies that the choices of APOs remain unsettled as there are no widely accepted measurement methods. However, discordant matrices have relatively much less impact in CPOs. Since there are many APOs, the final result using a suitable APO for AHP is still uncertain. In table 8.2.2, some APOs support alternative 1 while some support alternative 2. The decision makers may become confused by the choice of APOs.

On the other hand, the CPOs of CNP are straightforward. Although there are only two recommended CPO candidates, they do not produce contract rankings. Particularly, their (normalized) utility values are very close, especially in low values of the accordant index. Thus, their final aggregation results are much close.

The CNP is more appropriate as the cognitive perception is more straightforward. The result of this model shows that Saaty's AHP method very likely produces misleading results as the ranks and priority vector variy very much from the CNP. Particularly, chapter 5 indicates that POM performs much better than PRM, as well as interval scale being more appropriate than ratio scale for cognitive comparisons.

Table 8.2.1: The criteria and alternatives of case 1 (Saaty 1980, p26-28)

Criteria	Alternatives
C_1: Learning	t_1: School A
C_2: Friends	t_2: School B
C_3: School Life	t_3: School C
C_4: Import substitution	
C_5: Regional gains	

Table 8.2.2: The Pairwise Reciprocal Matrices of the AHP of case 1 (Saaty 1980, p26-28)

A$_1$: Criteria

	C_1	C_2	C_3	C_4	C_5	C_6
C_1	1	4	3	1	3	4
C_2	$1/4$	1	7	3	$1/5$	1
C_3	$1/3$	$1/7$	1	1	1	$1/6$
C_4	1	$1/3$	5	1	1	$1/3$
C_5	$1/3$	5	5	1	1	3
C_6	$1/4$	1	6	3	$1/3$	1

C.I.=0.3, C.R.=0.24

Learning (A$_2$)

	t_1	t_2	t_3
t_1	1	$1/3$	$1/2$
t_2	3	1	3
t_3	2	$1/3$	1

C.I.=0.025, C.R.=0.04

A$_3$:Friends

	t_1	t_2	t_3
A_1	1	1	1
A_2	1	1	1
A_3	1	1	1

C.I.=C.R.=0

A$_4$: School Life

	t_1	t_2	t_3
t_1	1	5	1
t_2	$1/5$	1	$1/5$
t_3	1	5	1

C.I,=C.R.=0

A$_5$: Vocational Training

	t_1	t_2	t_3
t_1	1	9	7
t_2	$1/9$	1	$1/5$
t_3	$1/7$	5	1

C.I.=0.105, C.R.=0.18

A$_6$: College preparation

	t_1	t_2	t_3
t_1	1	$1/2$	1
t_2	2	1	2
t_3	1	$1/2$	1

C.I.=C.R.=0

A$_7$: Music classes

	t_1	t_2	t_3
t_1	1	6	4
t_2	$1/6$	1	$1/3$
t_3	$1/4$	3	1

C.I.=C.R.=0.04

Table 8.2.3: Priority vectors and synthesis results with various prioritization operators using AHP for case 1

			P_1									P_2					
	C_1	C_2	C_3	C_4	C_5	C_6	$\overline{W}$		C_1	C_2	C_3	C_4	C_5	C_6	$\overline{W}$	
W	0.321	0.140	0.035	0.128	0.237	0.139			0.242	0.188	0.031	0.131	0.232	0.175		
t_1	0.157	0.333	0.455	0.772	0.250	0.691	0.367	2	0.151	0.333	0.455	0.695	0.250	0.657	0.378	3
t_2	0.594	0.333	0.091	0.055	0.500	0.091	0.378	3	0.575	0.333	0.091	0.054	0.500	0.090	0.344	2
t_3	0.249	0.333	0.455	0.173	0.250	0.218	0.254	1	0.274	0.333	0.455	0.251	0.250	0.254	0.279	1
			P_3								P_4						
W	0.381	0.105	0.045	0.131	0.211	0.127			0.305	0.149	0.038	0.141	0.221	0.146		
t_1	0.169	0.333	0.455	0.809	0.250	0.711	0.369	2	0.159	0.333	0.455	0.750	0.250	0.685	0.377	3
t_2	0.607	0.333	0.091	0.068	0.500	0.101	0.397	3	0.589	0.333	0.091	0.060	0.500	0.093	0.365	2
t_3	0.225	0.333	0.455	0.124	0.250	0.189	0.234	1	0.252	0.333	0.455	0.190	0.250	0.221	0.258	1
			P_5								P_6						
W	0.316	0.139	0.036	0.125	0.236	0.148			0.184	0.220	0.037	0.150	0.210	0.197		
t_1	0.157	0.333	0.455	0.772	0.250	0.691	0.370	2	0.178	0.333	0.455	0.788	0.250	0.687	0.430	3
t_2	0.594	0.333	0.091	0.055	0.500	0.091	0.376	3	0.592	0.333	0.091	0.082	0.500	0.108	0.325	2
t_3	0.249	0.333	0.455	0.173	0.250	0.218	0.254	1	0.230	0.333	0.455	0.130	0.250	0.205	0.245	1
			P_7								P_8						
W	0.415	0.094	0.035	0.112	0.219	0.125			0.349	0.144	0.053	0.123	0.192	0.139		
t_1	0.174	0.333	0.455	0.804	0.250	0.707	0.353	2	0.159	0.333	0.455	0.796	0.250	0.703	0.371	2
t_2	0.606	0.333	0.091	0.074	0.500	0.107	0.417	3	0.619	0.333	0.091	0.082	0.500	0.109	0.390	3
t_3	0.221	0.333	0.455	0.122	0.250	0.187	0.230	1	0.222	0.333	0.455	0.122	0.250	0.188	0.239	1
			P_9									P_{10}					
W	0.431	0.131	0.027	0.137	0.144	0.131			0.203	0.203	0.037	0.157	0.203	0.198		
t_1	0.157	0.333	0.455	0.772	0.250	0.691	0.355	2	0.178	0.333	0.455	0.788	0.250	0.687	0.431	3
t_2	0.594	0.333	0.091	0.055	0.500	0.091	0.393	3	0.592	0.333	0.091	0.082	0.500	0.108	0.327	2
t_3	0.249	0.333	0.455	0.173	0.250	0.218	0.251	1	0.230	0.333	0.455	0.130	0.250	0.205	0.243	1

Table 8.2.4: The Pairwise Opposite Matrices of case 1 with reference to table 8.2.2

B_1: Criteria

	C_1	C_2	C_3	C_4	C_5	C_6
C_1	0	3	2	0	2	3
C_2	-3	0	6	2	4	0
C_3	-2	-6	0	0	0	-5
C_4	0	-2	4	0	0	-2
C_5	-2	4	4	0	0	2
C_6	-3	0	5	2	-2	0

AI=0.36

Learning (B_2)				B_3:Friends				B_4: School Life			
	t_1	t_2	t_3		t_1	t_2	t_3		t_1	t_2	t_3
t_1	0	-2	-1	t_1	0	0	0	t_1	0	4	0
t_2	2	0	2	t_2	0	0	0	t_2	-4	0	-4
t_3	1	-2	0	t_3	0	0	0	t_3	0	4	0
AI=0.048				AI=0				AI=0			

B_5: Vocational Training				B_6: College preparation				B_7: Music classes			
	t_1	t_2	t_3		t_1	t_2	t_3		t_1	t_2	t_3
t_1	0	8	6	t_1	0	-1	0	t_1	0	5	3
t_2	-8	0	-4	t_2	1	0	1	t_2	-5	0	-2
t_3	-6	4	0	t_3	0	-1	0	t_3	-3	2	0
AI=0.096				AI=0				AI=0			

Table 8.2.5: Cognitive prioritization and aggregation results of RAU and LPS (case 1)

	C_1	C_2	C_3	C_4	C_5	C_6	Result	Rank
RAU/PLS								
W	0.191	0.188	0.115	0.158	0.184	0.164		
t_1	0.292	0.333	0.389	0.528	0.319	0.444	0.378	3
t_2	0.389	0.333	0.222	0.167	0.361	0.236	0.294	1
t_3	0.319	0.333	0.389	0.306	0.319	0.319	0.328	2
LPS								
W	0.191	0.189	0.137	0.158	0.163	0.163		
t_1	0.292	0.333	0.389	0.528	0.319	0.444	0.380	3
t_2	0.389	0.333	0.222	0.167	0.361	0.236	0.291	1
t_3	0.319	0.333	0.389	0.306	0.319	0.319	0.329	2

8.3 Case 2: Transportation project selection

The case background, solutions using the AHP approach and the CNP approach respectively, and the discussion of the comparison are presented as follows.

8.3.1. Case 2 background

In this case, the transportation company selection problem using AHP, from Kulak and Kahramna (2005), is revised by the proposed CNP. One transportation company is selected from four candidates using five criteria: cost, defect rate, tardiness rate, flexibility and documentation ability. The notations of the problem are shown in the table 8.3.1. To further discuss the comparison of AHP and CNP, the measurement models for their POs are used for both approaches.

Remarks: CPO measurement models such as RMPWSV and RMSV for CNP are different from those for AHP, although some functions have the same names. To distinguish them, c-RMPWSV and c-RMSV are for CNP whilst a-RMPWSV and a-RMSV are for AHP. As this possible confusion only occurs when their PO measurement models are used for comparison, but seldom in application, thus the new name is not necessary as the name implies the structure of the functions.

In this case, only RMPWSV is discussed for the measurement model as it is more appropriate to the other models which are discussed in chapter 2.5 for AHP, and chapter 5 for CNP. The penalty weight vector of RMPWSV is $\beta = \{1, 1.5, 2\}$ in both cases.

Table 8.3.1: Criteria and alternatives of case 2 (Kulak and Kahramna, 2005)

Criteria	Description	Labels	Alternatives	Labels
TC	Transportation Cost	C_1	Transport Company 1	T_1
DR	Defective rate	C_2	Transport Company 2	T_2
TR	Tardiness Rate	C_3	Transport Company 3	T_3
F	Flexibility	C_4	Transport Company 4	T_4
DA	Documentation Ability	C_5		

Table 8.3.2: Pairwise reciprocal matrices of case 2 (Kulak and Kahramna, 2005)

B	TC	DR	TR	F	DA	A_1	T_1	T_2	T_3	T_4
TC	1.00	5.00	3.00	5.00	9.00	T_1	1.00	0.33	1.00	0.20
DR	0.20	1.00	0.50	0.50	7.00	T_2	3.00	1.00	3.00	0.50
TR	0.33	2.00	1.00	0.50	7.00	T_3	1.00	0.33	1.00	0.20
F	0.20	2.00	2.00	1.00	8.00	T_4	5.00	2.00	5.00	1.00
DA	0.11	0.14	0.14	0.13	1.00					
		CR=0.078						CR=0.002		
A_2	T_1	T_2	T_3	T_4		A_3	T_1	T_2	T_3	T_4
T_1	1.00	7.00	3.00	5.00		T_1	1.00	0.20	0.20	2.00
T_2	0.14	1.00	0.20	0.33		T_2	5.00	1.00	0.33	7.00
T_3	0.33	5.00	1.00	3.00		T_3	5.00	3.00	1.00	7.00
T_4	0.20	3.00	0.33	1.00		T_4	0.50	0.14	0.14	1.00
		CR=0.043						CR=0.063		
A_4	T_1	T_2	T_3	T_4		A_5	T_1	T_2	T_3	T_4
T_1	1.00	5.00	0.33	3.00		T_1	1.00	0.20	0.33	0.33
T_2	0.20	1.00	0.14	0.33		T_2	5.00	1.00	3.00	3.00
T_3	3.00	7.00	1.00	7.00		T_3	3.00	0.33	1.00	1.00
T_4	0.33	3.00	0.14	1.00		T_4	3.00	0.33	1.00	1.00
		CR=0.052						CR=0.016		

8.3.2. The AHP Approach to case 2

Six pairwise reciprocal matrices are shown in the table 8.3.2. All matrices are not perfect consistent, but satisfied. Regarding prioritization, a set of ten APOs are used as follows:

$P=\{p_1,\ldots,p_{10}\}=\{$EV, NRS, NRCS, AMNC, NGNR/LLS, WLS, FP, GP, LPPDS, LPPWS $\}$.

To define the best analytic prioritization operator, the best APO is defined as the one of the least values of the *Root Mean Penalty Weighted Square Variance* σ_a (a-RMPWSV) which is illustrated chapter 2.5. Table 8.3.3 shows the priorities and a-RMPWSV for six pairwise matrices using ten prioritization operators. The results using the method proposed by Kulak and Kahramna (2005) are the same as the results obtained when using P_4 (*AMNC*), although they used a different form of APO. The Saaty's Eigenvector method is P_1.

It can be observed that neither of these two APOs, in the six matrices, is selected as the best APO for the high values of a-RMPWSV. The best APO is always LPPDS with respect to taking a-RMPWSV as the measurement criterion.

Table 8.3.3: Priorities and WRMSV for six pairwise matrices of case 2 by using ten APOs

W_1	C_1	C_2	C_3	C_4	C_5	σ_a	C_1	T_1	T_2	T_3	T_4	σ_a
P_1	0.513	0.108	0.156	0.195	0.027	2.129	P_1	0.099	0.284	0.099	0.518	0.107
P_2	0.398	0.159	0.188	0.229	0.026	1.529	P_2	0.099	0.293	0.099	0.508	0.085
P_3	0.563	0.102	0.156	0.146	0.032	2.070	P_3	0.100	0.273	0.100	0.527	0.136
P_4	0.494	0.114	0.163	0.199	0.029	1.851	P_4	0.099	0.284	0.099	0.518	0.104
P_5	0.503	0.111	0.162	0.198	0.027	2.133	P_5	0.099	0.284	0.099	0.518	0.105
P_6	0.553	0.110	0.167	0.136	0.034	1.949	P_6	0.101	0.275	0.101	0.523	0.120
P_7	0.488	0.130	0.216	0.130	0.036	1.605	P_7	0.102	0.275	0.095	0.528	0.172
P_8	0.504	0.101	0.168	0.202	0.025	2.343	P_8	0.100	0.300	0.100	0.500	0.087
P_9	0.333	0.177	0.213	0.245	0.032	1.130	P_9	0.100	0.292	0.100	0.507	0.078
P_{10}	0.558	0.120	0.145	0.145	0.032	2.022	P_{10}	0.101	0.275	0.101	0.523	0.120
C_2	T_1	T_2	T_3	T_4		σ_a	C_3	T_1	T_2	T_3	T_4	σ_a
P_1	0.565	0.055	0.262	0.118		0.887	P_1	0.087	0.311	0.549	0.053	1.073
P_2	0.507	0.053	0.296	0.144		0.855	P_2	0.098	0.386	0.464	0.052	0.742
P_3	0.605	0.063	0.224	0.109		0.855	P_3	0.089	0.237	0.613	0.060	1.344
P_4	0.558	0.057	0.263	0.122		0.809	P_4	0.092	0.319	0.533	0.055	0.922
P_5	0.564	0.055	0.263	0.118		0.893	P_5	0.090	0.313	0.543	0.054	1.000
P_6	0.593	0.068	0.227	0.111		0.745	P_6	0.089	0.252	0.597	0.061	1.231
P_7	0.583	0.095	0.221	0.101		0.917	P_7	0.167	0.278	0.500	0.056	1.270
P_8	0.521	0.063	0.313	0.104		0.585	P_8	0.085	0.427	0.427	0.061	0.665
P_9	0.511	0.068	0.311	0.111		0.531	P_9	0.087	0.411	0.441	0.060	0.538
P_{10}	0.593	0.068	0.227	0.111		0.745	P_{10}	0.089	0.252	0.597	0.061	1.231
C_4	T_1	T_2	T_3	T_4		σ_a	C_5	T_1	T_2	T_3	T_4	σ_a
P_1	0.249	0.054	0.592	0.105		1.111	P_1	0.078	0.522	0.200	0.200	0.471
P_2	0.279	0.050	0.538	0.134		1.258	P_2	0.076	0.489	0.217	0.217	0.450
P_3	0.223	0.063	0.625	0.089		0.916	P_3	0.084	0.539	0.189	0.189	0.450
P_4	0.251	0.056	0.584	0.109		1.024	P_4	0.079	0.519	0.201	0.201	0.453
P_5	0.253	0.053	0.590	0.104		1.143	P_5	0.078	0.520	0.201	0.201	0.475
P_6	0.227	0.069	0.615	0.089		0.792	P_6	0.088	0.534	0.189	0.189	0.409
P_7	0.239	0.054	0.616	0.091		1.150	P_7	0.085	0.540	0.188	0.188	0.441
P_8	0.216	0.043	0.648	0.093		2.019	P_8	0.063	0.563	0.188	0.188	1.000
P_9	0.293	0.071	0.550	0.086		0.642	P_9	0.089	0.493	0.209	0.209	0.353
P_{10}	0.227	0.069	0.615	0.089		0.792	P_{10}	0.088	0.534	0.189	0.189	0.409

Table 8.3.4: Synthesis and measurement results of eight APOs (case 2)

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
T_1	0.176	0.204	0.167	0.18	0.179	0.17	0.162	0.196	0.217	0.176
T_2	0.225	0.222	0.224	0.225	0.224	0.229	0.252	0.233	0.230	0.225
T_3	0.286	0.302*	0.272	0.288	0.289	0.271	0.289	0.27	0.324*	0.265
T_4	0.313*	0.271	0.336*	0.306*	0.308*	0.33*	0.296*	0.301*	0.229	0.333*
$\sum(\sigma_a)$	5.777	4.919*	5.770	5.163	5.749	5.247	5.555	6.699	3.272*	5.319

Table 8.3.4 shows the results of a synthesis of eight APO methods. The result shows that $p_1, p_3, \ldots, p_8, p_{10}$ support the result that transportation company 4 is the best candidate, whilst p_2 and p_{10} show that company 3 is the best. To measure the results, the a-RMPWSVs of all pairwise matrices for all methods are summed up. It can be found that p_2 and p_{10}, which support company 3, have the least two summations of a-RMPWSVs, which are less than five. In this case, the APOs of the summation of a-RMPWSV, more than five may produce an inaccurate result, i.e. company 4.

8.3.3. The CNP Approach

To apply CNP, the linguistic terms are assumed to be the same. Regarding the numerical representation, the Pairwise Reciprocal Matrices (PRMs) are changed to the Pairwise Opposite Matrices (POMs) with reference to table 8.1. The result of the POMs is shown in table 8.3.5. In order to compare with AHP, the results of the utility sets are normalized.

The results of the normalized utility sets, *c-RMPWSVs*, and their aggregation values using a weighted average aggregation operator are shown in table 8.3.6. Two cognitive prioritization operators, RAU (or PLS) and LPS, are used. Interestingly, both CPOs produce the same result for the same matrix.

Table 8.3.5: Pairwise opposite matrices of case 2

B	TC	DR	TR	F	DA	B_1	T_1	T_2	T_3	T_4
TC	0	4	2	4	8	T_1	0	-2	0	-4
DR	-4	0	-1	-1	6	T_2	2	0	2	-1
TR	-2	1	0	-1	6	T_3	0	-2	0	-4
F	-4	1	1	0	7	T_4	4	1	4	0
DA	-8	-6	-6	-7	0					
AI=0.117						AI=0.042				
B_2	T_1	T_2	T_3	T_4		B_3	T_1	T_2	T_3	T_4
T_1	0	6	2	4		T_1	0	-4	-4	1
T_2	-6	0	-4	-2		T_2	4	0	-2	6
T_3	-2	4	0	2		T_3	4	2	0	6
T_4	-4	2	-2	0		T_4	-1	-6	-6	0
AI=0						AI=0.103				
B_4	T_1	T_2	T_3	T_4		B_5	T_1	T_2	T_3	T_4
T_1	0	4	-2	2		T_1	0	-4	-2	-2
T_2	-4	0	-6	-2		T_2	4	0	2	2
T_3	2	6	0	6		T_3	2	-2	0	0
T_4	-2	2	-6	0		T_4	2	-2	0	0
AI=0.085						AI=0				

Table 8.3.6: Cognitive prioritization and aggregation results of RAU and LPS (case 2)

		C_1	C_2	C_3	C_4	C_5	Result	Rank
	W	0.29	0.2	0.22	0.225	0.065		
T_1		0.203	0.344	0.195	0.281	0.188	0.246	3
T_2		0.273	0.156	0.313	0.156	0.313	0.235	2
T_3		0.203	0.281	0.344	0.359	0.250	0.288	4
T_4		0.320	0.219	0.148	0.203	0.250	0.231	1
σ_C	0.721	0.289	0	0.645	0.577	0	$\sum(\sigma_C)=2.233$	

8.2.4. Discussion of case 2

As AI and CR use different functions constructed from different perspectives, the levels of their values cannot be used for comparison of POM and PRM. However, if both are equal to zero, the PRM is consistent while the POM is accordant perfectly. The interesting finding is that the Accordant Indices of B_2 and B_5 are equal to zero, i.e. perfectly accordant, from conversions of inconsistent A_2 and A_5 respectively. This means that the accordant or consistent cognition for pairwise comparison in a matrix can be distorted as discordant or inconsistent by Saaty's PRM. On the other hand, POM is much more appropriate to reflect the accordant cognition for pairwise comparison in a matrix.

Regarding AHP, different APOs have different results which likely lead to the different ranks. On the other hand, regarding CNP, two CPOs interestingly produce the same results which are shown in table 8.3.6. Unlike the unsettled issues of APOs, while the result is the same for these two CPOs, the CNP reduces the hesitation of the decision maker, or increases the usability for the decision maker.

Regarding the rank, the best one using AHP, which most APOs support, is the worse one of using CNP. In addition, the rank of CNP is totally different from all possible cases of AHP. The finding of this case is worrying since so many applications apply the AHP method. The main reason, which also is illustrated in chapter 5, is that the PRM has misrepresentation of the cognitive paired comparisons, and the priority vector derived from PRM is questionable. If PRM is used in the hierarchical model, of course the result is misleading too. On the other hand, to solve the misrepresentation problem of PRM, POM is proposed as the ideal solution. The CNP is based on POMs in view of hierarchical clusters. More details are in chapter 5.

8.4. Case 3: R&D Project selection

The case background, solutions using the AHP approach and the CNP approach respectively, and the discussion of the comparison are presented as follows.

8.4.1. Case 3 background

This case discusses the comparisons of the primitive Cognitive Network Process (CNP) model and the improved Analytic Network Process (ANP) (Saaty 2005; Yuen and

Lau, 2009) model using recent research in the R&D project selection problem discussed by Yuen and Lau (2009).

In this case, the most preferable R&D project $T_i{*}$ is selected from $T=[T_1,T_2,T_3]$ measured by 13 attributes, i.e., $|c_{i,j}|=13$, belonging to four criteria $C=[c_1,c_2,c_3,c_4]$ (table 8.4.1). Each attribute $c_{i,j}$ is evaluated by two experts $e_{i,j,1}$, $e_{i,j,2}$ (table 8.4.2), who may be the same person but is irrelevant for the mathematical model. $e_{i,j,1}$ is from the R&D department, and $e_{i,j,2}$ is from the Marketing Department. The weights of these two experts are measured by the management board. In this case, the weights of these two experts are aggregated to be equal. The calculated method is based on the Linguistic Possibility-Probability Aggregation Model (LPPAM) proposed by Yuen and Lau (2009). LPPAM used multiple aggregation operators. However, to simplify the comparison, only the weighted arithmetic mean, *wam*, is applied in this case.

A valuation function $V=f$(Benefit, Opportunity, Cost, Risk), i.e. $V=f(c_1,c_2,c_3,c_4)$ is used in the volitional decision process. The set of positive factors $C^+=\{c_1,c_2\}$, and a set of deductive (or negative) factors $C^+=\{c_3,c_4\}$ are assigned by the sets of the normalized weights $W^+=\{w_1,w_2\}$ and $W^-=\{w_3,w_4\}$ respectively.

Table 8.4.1: Numerical rating scores of case 3 by two experts (Yuen and Lau, 2009)

Criteria	Attributes	Weight (w_{ij}) e_1	Weight (w_{ij}) e_2	Project 1 (A_1) e_1	Project 1 (A_1) e_2	Project 2 (A_2) e_1	Project 2 (A_2) e_2	Project 3 (A_3) e_1	Project 3 (A_3) e_2
Benefits (c_1)	Technology Merit (c_{11})	10	10	9.58	10	7.92	5.42	5.42	5.42
	Market Size (c_{12})	7.08	9.58	8.5	7.92	6	5.42	6	4.58
	Potential Return (c_{13})	7.5	10	9.58	8.5	6.82	7.08	6	4.58
	Market Growth (c_{14})	7.08	9.58	9	7.5	6	5.42	6	5.42
Opportunities (c_2)	Technology Leadership (c_{21})	10	10	10	7.92	7.5	5	7.08	5
	Sustain Development (c_{22})	7.92	10	8.5	7.92	5.42	5	5.42	5
Costs (c_3)	R&D Cost (c_{31})	10	9.58	9.58	8.5	7.08	6	8.5	7.92
	Implementation cost (c_{32})	10	9.58	9.58	7.08	7.08	5.42	9.58	7.92
Risks (c_4)	Marketing success (c_{41})	9.58	10	7.5	9.58	5	4.58	7.5	7.08
	R&D Success (c_{42})	9.58	9.58	7.5	9	5.42	6.5	7.5	9
	Competitors (c_{43})	9.58	10	7.5	10	5	6.5	7.5	7.08
	Financial Overrun (c_{44})	9.32	10	8.5	7.5	6	7.08	8.5	7.92
	Duration overrun (c_{45})	7.92	10	7.5	7.5	5.42	7.08	7.5	7.08

The valuation function for each project T_i is $V_i = C_i^{+} \bullet \left[w_i^{+} \right]^T - C_i^{-} \bullet \left[w_i^{-} \right]^T$ (T in this function is the transposition). C is determined by *wam* in table 8.4.3, and the set of relative weights are subject to $\omega_1+\omega_2=\omega_3+\omega_4=1$. The results of the aggregation and valuation are shown in tables 8.4.4 and 8.4.5. Project 1 (T_1) is preferable as the value of the valuation function is profitable and maximized.

Table 8.4.2: Numerical representation and the relative weights (case 3)

	d_1	d_2	d_3	d_4	Utility weight w_{e_k}	Relative Weight ω''_k for e_k
Weight $\psi_W(d_i)$	10.00	7.92	5.42	5.42	-	-
Relative Weight for ψ_W	0.35	0.28	0.19	0.19	-	-
Expert 1 (e_1)	10.00	7.92	9.32	9.58	9.22	0.50
Expert 2 (e_2)	7.92	10.00	9.58	9.32	9.07	0.50

Table 8.4.3: Criteria weights (case 3)

	c_1	c_2	c_3	c_4
ψ by e_1	IB-E	A-Ma	QA-Ma	IB-Ma
ψ by e_2	A-E	LA-Ma	LB-E	A-Ma
$f(\psi)$ by e_1	9.58	7.5	7.92	7.08
$f(\psi)$ by e_2	10	7.92	9.58	7.5
ω_i	0.56	0.44	0.54	0.46

Table 8.4.4: Aggregation values by weighted arithmetic mean (case 3)

Criteria	T_1		T_2		T_3		Aggregated Values		
	e_1	e_2	e_1	e_2	e_1	e_2	T_1	T_2	T_3
c_1	9.21	8.50	6.80	5.84	5.82	5.00	8.85	6.32	5.41
c_2	9.34	7.92	6.58	5.00	6.35	5.00	8.63	5.79	5.67
c_3	9.58	7.79	7.08	5.71	9.04	7.92	8.69	6.40	8.48
c_4	7.70	8.71	5.36	6.35	7.70	7.62	8.21	5.85	7.66

Table 8.4.5: Valuation values (case 3)

Criteria	Criteria Weight	Project 1	Project 2	Project 3
c_1	0.56	8.85	6.32	5.41
c_2	0.44	8.63	5.79	5.67
c_3	0.54	8.69	6.40	8.48
c_4	0.46	8.21	5.85	7.66
wam		0.29	-0.06	-2.58

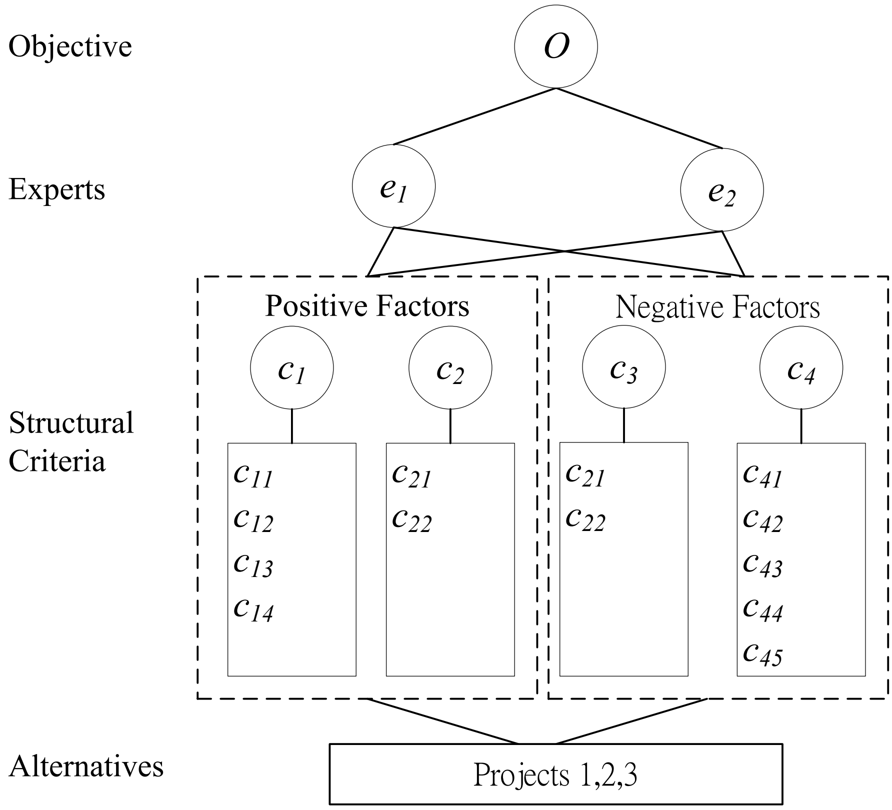


Figure 8.1: Network structures for the CNP and the ANP of the R&D Project selection problem

8.4.2. The improved ANP approach to case 3

Now the R&D project selection problem of direct rating structure (Yuen and Lau, 2009) is converted to the ANP selection structure, which is shown in Fig. 8.1. The classical AHP problem (Saaty 1980) does not deal with positive and negative factors. Saaty (2005) proposed ANP to address this problem. ANP is related to BOCR, which stands for Benefits, Opportunities, Costs and Risks denoted by c_1, c_2, c_3, c_4 respectively.

It is assumed that tables 8.4.1-8.4.5 reflect the real world situation, and the results have been collected. In order to convert the absolute rating scale to the comparison ratio scale, tables 8.4.6-8.4.8 have been prepared. These tables show that the relative weights between any two candidates are in the range [0.647, 2.092]. The conventional comparison scales for AHP are nine points. This means that the labels of the intensity of importance, which are more than or equal to 3, are redundant. Only three scales are useful: 0.5,1, and 2.

One limitation of ANP is that its nine point scale cannot produce certain patterns of priority distributions. This may finally lead to reversal of rank. For example, in Table 9.4.6, the "real" relative weights for *B*, *O*, *C*, and *R* (c_1, c_2, c_3, c_4) from expert 1 should be (0.561, 0.439, 0.558, 0.442). To achieve this weight vector, the pairwise matrix is the one shown in the e_1 column. However, in real practice, the numbers of the nine scales are all integers. The most approximate pairwise matrix should round the decimal digits to integers. Thus all elements of this new matrix are 1, and then the weights should be (0.5, 0.5, 0.5, 0.5). This produces errors corresponding to the "real" results. Such errors produce rank reversal in this situation.

Table 8.4.6: "Real" Pairwise Reciprocal Matrices of criterion comparisons (case 3)

	e_1	e_2	e_1				e_2			
			c_1	c_2	c_3	c_4	c_1	c_2	c_3	c_4
c_1	9.58	10	1.000	1.277			1.000	1.263		
c_2	7.5	7.92		1.000				1.000		
c_3	7.92	9.58			1.000	1.119			1.000	1.277
c_4	7.08	7.5				1.000				1.000

Table 8.4.7: "Real" Pairwise Reciprocal Matrices of attribute comparisons (case 3)

	e_1	e_2	e_1					e_2				
			c_{i1}	c_{i2}	c_{i3}	c_{i4}	c_{i5}	c_{i1}	c_{i2}	c_{i3}	c_{i4}	c_{i5}
c_{11}	10.00	10.00	1.000	1.412	1.333	1.412		1.000	1.044	1.000	1.044	
c_{12}	7.080	9.580		1.000	0.944	1.000			1.000	0.958	1.000	
c_{13}	7.500	10.000			1.000	1.059				1.000	1.044	
c_{14}	7.080	9.580				1.000					1.000	
c_{21}	10.00	10.000	1.000	1.263				1.000	1.000			
c_{22}	7.920	10.000		1.000					1.000			
c_{31}	10.00	9.580	1.000	1.000				1.000	1.000			
c_{32}	10.00	9.580		1.000					1.000			
c_{41}	9.580	10.000	1.000	1.000	1.000	1.028	1.210	1.000	1.044	1.000	1.000	1.000
c_{42}	9.580	9.580		1.000	1.000	1.028	1.210		1.000	0.958	0.958	0.958
c_{43}	9.580	10.000			1.000	1.028	1.210			1.000	1.000	1.000
c_{44}	9.320	10.000				1.000	1.177				1.000	1.000
c_{45}	7.920	10.000					1.000					1.000

Table 8.4.8: "Real" pairwise reciprocal matrices of alternative comparisons (case 3)

c_{ij}	A_1		A_2		A_3		E_1			E_2		
	e_1	e_2	e_1	e_2	e_1	e_2	T_1/T_2	T_1/T_3	T_2/T_3	T_1/T_2	T_1/T_3	T_2/T_3
c_{11}	9.58	10	7.92	5.42	5.42	5.42	1.210	1.768	1.461	1.845	1.845	1.000
c_{12}	8.5	7.92	6	5.42	6	4.58	1.417	1.417	1.000	1.461	1.729	1.183
c_{13}	9.58	8.5	6.82	7.08	6	4.58	1.405	1.597	1.137	1.201	1.856	1.546
c_{14}	9	7.5	6	5.42	6	5.42	1.500	1.500	1.000	1.384	1.384	1.000
c_{21}	10	7.92	7.5	5	7.08	5	1.333	1.412	1.059	1.584	1.584	1.000
c_{22}	8.5	7.92	5.42	5	5.42	5	1.568	1.568	1.000	1.584	1.584	1.000
c_{31}	9.58	8.5	7.08	6	8.5	7.92	1.353	1.127	0.833	1.417	1.073	0.758
c_{32}	9.58	7.08	7.08	5.42	9.58	7.92	1.353	1.000	0.739	1.306	0.894	0.684
c_{41}	7.5	9.58	5	4.58	7.5	7.08	1.500	1.000	0.667	2.092	1.353	0.647
c_{42}	7.5	9	5.42	6.5	7.5	9	1.384	1.000	0.723	1.385	1.000	0.722
c_{43}	7.5	10	5	6.5	7.5	7.08	1.500	1.000	0.667	1.538	1.412	0.918
c_{44}	8.5	7.5	6	7.08	8.5	7.92	1.417	1.000	0.706	1.059	0.947	0.894
c_{45}	7.5	7.5	5.42	7.08	7.5	7.08	1.384	1.000	0.723	1.059	1.059	1.000

To improve this situation by using the compound linguistic ordinal scale (CLOS), let HAD-LRM of the comparison scale schema be $\left(\aleph,\overline{X_{\aleph}},f_{\overline{X}}(\aleph)\right)_1$. To construct the labels of the comparison scale $\aleph_1$, let $\overrightarrow{V_{a_1}}=[\text{Equal,Weak,Moderate,Strong,Essential}]$. $\overrightarrow{V_h}=[\text{Little,Quite,Much}]$, $\overrightarrow{V_d}=[\text{Below,Absolutely,Above}]$, and thus $\overrightarrow{V_{hd}}=[\text{"MB","QB","LB","A","LA","QA","MA"}]$. Using algorithm 4.1, then

$$\aleph_1=\begin{bmatrix} 0 & MB\text{-}W & MB\text{-}M & MB\text{-}S & MB\text{-}Es \\ 0 & QB\text{-}W & QB\text{-}M & QB\text{-}S & QB\text{-}Es \\ 0 & LB\text{-}W & LB\text{-}M & LB\text{-}S & LB\text{-}Es \\ A\text{-}E & A\text{-}W & A\text{-}M & A\text{-}S & A\text{-}Es \\ LA\text{-}E & LA-W & LA\text{-}M & LA\text{-}S & 0 \\ QA\text{-}E & QA\text{-}W & QA\text{-}M & QA\text{-}S & 0 \\ MA\text{-}E & MA-W & MA\text{-}M & MA\text{-}S & 0 \end{bmatrix}.$$

For the calculation of the representation values $\overline{X_{\aleph_1}}$, let $X=[1,9]$, $d_{\alpha^{1,\dots,5}}=2$, $\vec{\gamma}=[1,3,5,7,9]$, $\tau_{\alpha^{1,\dots,5}}=2$, $\mu_{\alpha^{j\phi}}{}^{-1}=PbMF^{-1}$, $\varphi\left(\overrightarrow{V_h}\right)=[1,2,3]$, $\lambda_0=0.5$ for algorithm 4.3, this follows

$$\overline{X_{\aleph_1}}=f_{\overline{X}}(\aleph_1)=\begin{bmatrix} 0 & 1.542 & 3.542 & 5.542 & 7.542 \\ 0 & 2.198 & 4.198 & 6.198 & 8.198 \\ 0 & 2.668 & 4.668 & 6.668 & 8.668 \\ 1.000 & 3.000 & 5.000 & 7.000 & 9.000 \\ 1.332 & 3.332 & 5.332 & 7.332 & 0 \\ 1.802 & 3.802 & 5.802 & 7.802 & 0 \\ 2.458 & 4.448 & 6.458 & 8.458 & 0 \end{bmatrix}$$

, and its reciprocal matrix is:

$$\overline{X_{\aleph_1}}^{-1}=\begin{bmatrix} 0 & 0.648 & 0.282 & 0.180 & 0.133 \\ 0 & 0.455 & 0.238 & 0.161 & 0.122 \\ 0 & 0.375 & 0.214 & 0.150 & 0.115 \\ 1.000 & 0.335 & 0.200 & 0.143 & 0.111 \\ 0.751 & 0.300 & 0.188 & 0.136 & 0 \\ 0.555 & 0.263 & 0.172 & 0.128 & 0 \\ 0.407 & 0.224 & 0.155 & 0.118 & 0 \end{bmatrix}.$$

We now return to the problem of the nine point scale in table 8.4.6. The labels and approximate numbers of the new pairwise matrix for expert 1 are shown in table 8.4.9. The new relative weights are (0.571, 0.429, 0.571, 0.429). The ranks are shown correctly as the same as the “real one”. The sum of the absolute errors is reduced from 0.178 to 0.106. The error reduction is more than 40%. Thus it can be seen that the Compound Linguistic Ordinal Scales increase the chance of rank preservation of the ANP.

The real values of e_1 judgments are $\left[\varpi_i^{+}\right]=(0.561,0.439)$, $\left[\varpi_i^{-}\right]=(0.528,0.472)$ whilst the real one of e_2 judgments, $\left[\varpi_i^{+}\right]=(0.558,0.442)$,

$\left[\varpi_i^-\right]=(0.561,0.439)$. On the other hand, both experts of the improved ANP with CLOS are of the weight set: $\left[\varpi_i^+\right]=\left[\varpi_i^-\right]=(0.571,0.429)$. The ranks are preserved.

Table 8.4.9: Approximate PRMs of criterion comparison (case 3)

	e_1				e_2				e_1				e_2			
	c_1	c_2	c_3	c_4	c_1	c_2	c_3	c_4	c_1	c_2	c_3	c_4	c_1	c_2	c_3	c_4
c_1	E	LA-E			E	LA-E			1.000	1.332			1.000	1.332		
c_2		E				E				1.000				1.000		
c_3			E	LA-E			E	LA-E			1.000	1.332			1.000	1.332
c_4				E				E				1.000				1.000

Table 8.4.10: Approximate PRMs of attribute comparisons (case 3)

c_{ij}	e_1					e_2					e_1					e_2				
	c_{i1}	c_{i2}	c_{i3}	c_{i4}	c_{i5}	c_{i1}	c_{i2}	c_{i3}	c_{i4}	c_{i5}	c_{i1}	c_{i2}	c_{i3}	c_{i4}	c_{i5}	c_{i1}	c_{i2}	c_{i3}	c_{i4}	c_{i5}
c_{11}	E	MB-W	LA-E	MB-W		E	E	E	E		1.000	1.542	1.332	1.542		1.000	1.000	1.000	1.000	
c_{12}		E	E	E			E	E	E			1.000	1.000	1.000			1.000	1.000	1.000	
c_{13}			E	E				E	E				1.000	1.000				1.000	1.000	
c_{14}				E					E					1.000					1.000	
c_{21}	E	LA-E				E	E				1.000	1.332				1.000	1.000			
c_{22}		E					E					1.000					1.000			
c_{31}	E	E				E	E				1.000	1.000				1.000	1.000			
c_{32}		E					E					1.000					1.000			
c_{41}	E	E	E	E	LA-E	E	E	E	E	E	1.000	1.000	1.000	1.000	1.332	1.000	1.000	1.000	1.000	1.000
c_{42}		E	E	E	LA-E		E	E	E	E		1.000	1.000	1.000	1.332		1.000	1.000	1.000	1.000
c_{43}			E	E	LA-E			E	E	E			1.000	1.000	1.332			1.000	1.000	1.000
c_{44}				E	LA-E				E	E				1.000	1.332				1.000	1.000
c_{45}					E					E					1.000					1.000

Table 8.4.11: Approximate PRMs from Projects (case 3)

c_{ij}	E_1			E_2			E_1			E_2		
	T_1/T_2	T_1/T_3	T_2/T_3	T_1/T_2	T_1/T_3	T_2/T_3	T_1/T_2	T_1/T_3	T_2/T_3	T_1/T_2	T_1/T_3	T_2/T_3
c_{11}	LA-E	QA-E	MB-W	QA-E	QA-E	E	1.332	1.802	1.542	1.802	1.802	1
c_{12}	MB-W	MB-W	E	MB-W	QA-E	LA-E	1.542	1.542	1	1.542	1.802	1.332
c_{13}	MB-W	MB-W	LA-E	QA-E	QA-E	MB-W	1.542	1.542	1.332	1.802	1.802	1.542
c_{14}	MB-W	MB-W	E	QA-E	LA-E	E	1.542	1.542	1	1.802	1.332	1
c_{21}	LA-E	MB-W	E	MB-W	MB-W	E	1.332	1.542	1	1.542	1.542	1
c_{22}	MB-W	MB-W	E	MB-W	MB-W	E	1.542	1.542	1	1.542	1.542	1
c_{31}	LA-E	LA-E	I-LA-E	LA-E	LA-E	I-LA-E	1.332	1.332	0.751	1.332	1.332	0.751
c_{32}	LA-E	E	I-LA-E	LA-E	I-LA-E	I-LA-E	1.332	1	0.751	1.332	0.751	0.751
c_{41}	MB-W	E	I-LA-E	QB-W	E	I-LA-E	1.542	1	0.751	2.198	1	0.751
c_{42}	LA-E	E	I-LA-E	LA-E	E	I-LA-E	1.332	1	0.751	1.332	1	0.751
c_{43}	MB-W	E	I-LA-E	MB-W	E	I-LA-E	1.542	1	0.751	1.542	1	0.751
c_{44}	MB-W	E	I-LA-E	E	I-LA-E	I-LA-E	1.542	1	0.751	1	0.751	0.751
c_{45}	LA-E	E	I-LA-E	E	E	E	1.332	1	0.751	1	1	1

The next issue is the improvement of the synthesis function. Analytical prioritization means converting the pairwise matrices to local priorities. Synthesis means aggregation of these local priorities to global priorities. Summation of the priorities is equal to one. In ANP, the aggregation results for the priorities are obtained by using two formulae $V'_1 = bB + oO - cC - rR$, b+o+c+r=1 and $V'_2 = BO/CR$. As to which one is used in any particular case, this depends on which one is more appropriate to use for the interpretation of the outcome (Saaty, 2005). However, Yuen and Lau (2009) have indicated that both are not appropriate. The valuation function suggested is $V_i = C_i^{+} \bullet \left[\varpi_i^{+}\right]^T - C_i^{-} \bullet \left[\varpi_i^{-}\right]^T$, where $\sum\left[\varpi_i^{+}\right] = \sum\left[\varpi_i^{-}\right] = 1$.

The “real” weights are shown in tables 8.4.6-8.4.8, and the approximate pairwise matrices based on this data are shown in tables 8.4.9-8.4.11. Table 8.4.12 shows the priorities from pairwise matrices from tables 8.4.9 to 8.4.11, and the final results by using the valuation function, which is also used by CNP.

Table 8.4.12: Synthesis results of the improved ANP with respect to “real” reference values (case 3)

Criteria		*B* (c_1)		*O* (c_2)		*C* (c_3)		*R* (c_4)		*Val*		
Experts		e_1	e_2	e_1	e_2	e_1	e_2	e_1	e_2	e_1	e_2	*Final score*
Reference Values	w	0.561	0.558	0.439	0.442	0.528	0.561	0.472	0.439			
	T_1	0.422	0.439	0.419	0.442	0.373	0.364	0.371	0.384	0.049	0.068	***0.058***
	T_2	0.312	0.302	0.296	0.279	0.275	0.267	0.258	0.280	0.037	0.020	***0.028***
	T_3	0.267	0.259	0.285	0.279	0.352	0.370	0.371	0.336	-0.086	-0.087	***-0.087***
Improved ANP	w	0.571	0.571	0.429	0.429	0.571	0.571	0.429	0.429			
	T_1	0.433	0.459	0.425	0.435	0.381	0.364	0.374	0.359	0.052	0.087	***0.069***
	T_2	0.307	0.285	0.292	0.282	0.273	0.272	0.264	0.278	0.031	0.009	***0.020***
	T_3	0.259	0.257	0.283	0.282	0.346	0.364	0.362	0.363	-0.084	-0.096	***-0.090***

8.4.3. The CNP approach to case 3

Now the R&D project selection problem is converted to the CNP selection structure, which is also shown in Fig. 8.1. The CNP addresses the positive and negative factors. The calculation processes are similar to ANP, but CNP uses the reciprocal opposite matrices.

It is assumed that tables 8.4.1-8.4.5 (Yuen and Lau, 2009) reflect the real world situation. In order to approximate the real scales, tables 9.4.12-9.4.14 have been prepared.

The compound linguistic ordinal scale (CLOS) is used in CNP. Let HAD-LRM of the comparison interval scale schema be $\left(\aleph, \overline{X_{\aleph}}, f_{\overline{X}}(\aleph)\right)_2$. To construct the labels of the comparison interval scale $\aleph_2$, the linguistic representation labels are the same as $\aleph_1$

Table 8.4.13: “Real” pairwise opposite matrices of criterion comparisons (case 3)

	e_1	e_2	e_1				e_2			
			c_1	c_2	c_3	c_4	c_1	c_2	c_3	c_4
c_1	9.58	10	0	2.08			0	2.08		
c_2	7.5	7.92		0				0		
c_3	7.92	9.58			0	0.84			0	2.08
c_4	7.08	7.5				0				0

Table 8.4.14: “Real” pairwise opposite matrices of attribute comparisons (case 3)

	e_1	e_2	e_1					e_2				
			c_{i1}	c_{i2}	c_{i3}	c_{i4}	c_{i5}	c_{i1}	c_{i2}	c_{i3}	c_{i4}	c_{i5}
c_{11}	10.00	10.00	0	2.92	2.5	2.92		0	0.42	0	0.42	
c_{12}	7.080	9.580		0	-0.42	0			0	-0.42	0	
c_{13}	7.500	10.00			0	0.42				0	0.42	
c_{14}	7.080	9.580				0					0	
c_{21}	10.00	10.00	0	2.08				0	0			
c_{22}	7.920	10.00		0					0			
c_{31}	10.00	9.580	0	0				0	0			
c_{32}	10.00	9.580		0					0			
c_{41}	9.580	10.00	0	0	0	0.26	1.66	0	0.42	0	0	0
c_{42}	9.580	9.580		0	0	0.26	1.66		0	-0.42	-0.42	-0.42
c_{43}	9.580	10.00			0	0.26	1.66			0	0	0
c_{44}	9.320	10.00				0	1.4				0	0
c_{45}	7.920	10.00					0					0

Table 8.4.15: “Real” pairwise opposite matrices from alternative comparisons (case 3)

c_{ij}	T_1		T_2		T_3		e_1			e_2		
	e_1	e_2	e_1	e_2	e_1	e_2	T_1-T_2	T_1-T_3	T_2-T_3	T_1-T_2	T_1-T_3	T_2-T_3
c_{11}	9.58	10	7.92	5.42	5.42	5.42	1.66	4.16	2.5	4.58	4.58	0
c_{12}	8.5	7.92	6	5.42	6	4.58	2.5	2.5	0	2.5	3.34	0.84
c_{13}	9.58	8.5	6.82	7.08	6	4.58	2.76	3.58	0.82	1.42	3.92	2.5
c_{14}	9	7.5	6	5.42	6	5.42	3	3	0	2.08	2.08	0
c_{21}	10	7.92	7.5	5	7.08	5	2.5	2.92	0.42	2.92	2.92	0
c_{22}	8.5	7.92	5.42	5	5.42	5	3.08	3.08	0	2.92	2.92	0
c_{31}	9.58	8.5	7.08	6	8.5	7.92	2.5	1.08	-1.42	2.5	0.58	-1.92
c_{32}	9.58	7.08	7.08	5.42	9.58	7.92	2.5	0	-2.5	1.66	-0.84	-2.5
c_{41}	7.5	9.58	5	4.58	7.5	7.08	2.5	0	-2.5	5	2.5	-2.5
c_{42}	7.5	9	5.42	6.5	7.5	9	2.08	0	-2.08	2.5	0	-2.5
c_{43}	7.5	10	5	6.5	7.5	7.08	2.5	0	-2.5	3.5	2.92	-0.58
c_{44}	8.5	7.5	6	7.08	8.5	7.92	2.5	0	-2.5	0.42	-0.42	-0.84
c_{45}	7.5	7.5	5.42	7.08	7.5	7.08	2.08	0	-2.08	0.42	0.42	0

For the calculation of the representation values $\overline{X_{\aleph_2}}$, let $X=[0,10]$, $d_{\alpha^{1,\dots,5}}=2.5$, $\vec{\gamma}=[0,2.5,5,7.5,10]$, $\tau_{\alpha^{1,\dots,5}}=2$, $\mu_{\alpha^{j\phi}}{}^{-1}=PbMF^{-1}$, $\varphi(\vec{V_h})=[1,2,3]$, $\lambda_0=0.5$. By using algorithm 4.2, then

$$\overline{X_{\aleph_2}}=f_{\overline{X}}(\aleph_2)=\begin{bmatrix} 0 & 0.678 & 3.178 & 5.678 & 8.178 \\ 0 & 1.500 & 4.000 & 6.500 & 9.000 \\ 0 & 2.085 & 4.585 & 7.085 & 9.585 \\ 0.000 & 2.500 & 5.000 & 7.500 & 10.00 \\ 0.415 & 2.915 & 5.415 & 7.915 & 0 \\ 1.002 & 3.502 & 6.002 & 8.502 & 0 \\ 1.822 & 4.322 & 6.822 & 9.322 & 0 \end{bmatrix}$$

, and

$$-\overline{X_{\aleph_2}}=-f_{\overline{X}}(\aleph_2)=\begin{bmatrix} 0 & -0.678 & -3.178 & -5.678 & -8.178 \\ 0 & -1.500 & -4.000 & -6.500 & -9.000 \\ 0 & -2.085 & -4.585 & -7.085 & -9.585 \\ 0.000 & -2.500 & -5.000 & -7.500 & -10.00 \\ -0.415 & -2.915 & -5.415 & -7.915 & 0 \\ -1.002 & -3.502 & -6.002 & -8.502 & 0 \\ -1.822 & -4.322 & -6.822 & -9.322 & 0 \end{bmatrix}$$

,where $-\overline{X_{\aleph_2}}$ is the matrix of the values of the opposite of compound comparison scales, $\overline{X_{\aleph_2}}$.

Table 8.4.16: Approximate POMs of criterion comparisons (case 3)

	e_1				e_2				e_1				e_2			
	c_1	c_2	c_3	c_4	c_1	c_2	c_3	c_4	c_1	c_2	c_3	c_4	c_1	c_2	c_3	c_4
c_1	AE	LBW			AE	LBW			0	2.085			0	2.085		
c_2		AE				AE				0				0		
c_3			AE	QAE			AE	LBW			0	1.002			0	2.085
c_4				AE				AE				0				0

Table 8.4.17: Approximate POMs of attribute comparisons (case 3)

c_{ij}	e_1					e_2					e_1					e_2				
	c_{i1}	c_{i2}	c_{i3}	c_{i4}	c_{i5}	c_{i1}	c_{i2}	c_{i3}	c_{i4}	c_{i5}	c_{i1}	c_{i2}	c_{i3}	c_{i4}	c_{i5}	c_{i1}	c_{i2}	c_{i3}	c_{i4}	c_{i5}
c_{11}	AE	LAW	AW	LAW		AE	LAE	AE	LAE		0	2.915	2.5	2.915		0	0.415	0	0.415	
c_{12}		AE	iLAE	AE			AE	iLAE	AE			0	-0.42	0			0	-0.415	0	
c_{13}			AE	LAE				AE	LAE				0	0.415				0	0.415	
c_{14}				AE					AE					0					0	
c_{21}	AE	LBW				AE	AE				0	2.085				0	0			
c_{22}		AE					AE					0					0			
c_{31}	AE	AE				AE	AE				0	0				0	0			
c_{32}		AE					AE					0					0			
c_{41}	AE	AE	AE	LAE	MAE	AE	LAE	AE	AE	AE	0	0	0	0.415	1.822	0	0.415	0	0	0
c_{42}		AE	AE	LAE	MAE		AE	iLAE	iLAE	iLAE		0	0	0.415	1.822		0	-0.415	-0.415	-0.415
c_{43}			AE	LAE	MAE			AE	AE	AE			0	0.415	1.822			0	0	0
c_{44}				AE	QBW				AE	AE				0	1.5				0	0
c_{45}					AE					AE					0					0

Table 8.4.18: Approximate POMs from alternative comparisons (case 3)

c_{ij}	E_1			E_2			E_1			E_2		
	T_1/T_2	T_1/T_3	T_2/T_3	T_1/T_2	T_1/T_3	T_2/T_3	T_1/T_2	T_1/T_3	T_2/T_3	T_1/T_2	T_1/T_3	T_2/T_3
c_{11}	MAE	QAW	AW	LBM	LBM	AE	1.822	3.502	2.5	4.585	4.585	0
c_{12}	AW	AW	AE	AW	QAW	QAE	2.5	2.5	0	2.5	3.502	1.002
c_{13}	LAW	QAW	QAE	QBW	QBW	AW	2.915	3.502	1.002	1.5	1.5	2.5
c_{14}	LAW	LAW	AE	LBW	LBW	AE	2.915	2.915	0	2.085	2.085	0
c_{21}	AW	LAW	LAE	LAW	LAW	AE	2.5	2.915	0.415	2.915	2.915	0
c_{22}	LAW	LAW	AE	LAW	LAW	AE	2.915	2.915	0	2.915	2.915	0
c_{31}	AW	QAE	iQBW	AW	LAE	iLBW	2.5	0	-1.5	2.5	0.415	-2.085
c_{32}	AW	AE	iAW	MAE	iQAE	iAW	2.5	0	-2.5	1.822	-1.002	-2.5
c_{41}	AW	AE	iAW	AM	AW	iAW	2.5	0	-2.5	5	2.5	-2.5
c_{42}	LBW	AE	iLBW	AW	AE	iAW	2.085	0	-2.085	2.5	0	-2.5
c_{43}	AW	AE	iAW	QAW	LAW	iLAE	2.5	0	-2.5	3.502	2.915	-0.415
c_{44}	AW	AE	iAW	LAE	iLAE	iQAE	2.5	0	-2.5	0.415	-0.415	-1.002
c_{45}	LBW	AE	iLBW	LAE	LAE	AE	2.085	0	-2.085	0.415	0.415	0

Table 8.4.19: Valuation results of the normalized CNP with respect to real reference values (case 3)

Criteria		*B* (c_1)		*O* (c_2)		*C* (c_3)		*R* (c_4)		*Val*		
Experts		e_1	e_2	e_1	e_2	e_1	e_2	e_1	e_2	e_1	e_2	*Final*
Real Reference Values	w	0.561	0.558	0.439	0.442	0.528	0.561	0.472	0.439			
	T_1	0.422	0.439	0.419	0.442	0.373	0.364	0.371	0.384	0.049	0.068	***0.058***
	T_2	0.312	0.302	0.296	0.279	0.275	0.267	0.258	0.280	0.037	0.020	***0.028***
	T_3	0.267	0.259	0.285	0.279	0.352	0.370	0.371	0.336	-0.086	-0.087	***-0.087***
CNP	w	0.552	0.552	0.448	0.448	0.525	0.552	0.475	0.448			
	T_1	0.396	0.396	0.382	0.398	0.361	0.354	0.359	0.372	0.030	0.035	***0.032***
	T_2	0.317	0.313	0.296	0.301	0.283	0.284	0.281	0.293	0.025	0.020	***0.022***
	T_3	0.287	0.291	0.322	0.301	0.356	0.362	0.359	0.335	-0.054	-0.055	***-0.055***

Table 8.4.20: Valuation results of the unnormalized CNP with respect to real reference values (case 3)

Criteria		*B* (c_1)		*O* (c_2)		*C* (c_3)		*R* (c_4)		*Val*		
Experts		e_1	e_2	e_1	e_2	e_1	e_2	e_1	e_2	e_1	e_2	*Final*
Reference Values	w	0.561	0.558	0.439	0.442	0.528	0.561	0.472	0.439			
	T_1	12.657	13.179	12.581	13.259	11.183	10.910	11.127	11.526	1.468	2.034	***1.751***
	T_2	9.347	9.065	8.867	8.371	8.265	7.997	7.746	8.395	1.117	0.586	***0.851***
	T_3	7.995	7.756	8.551	8.371	10.553	11.092	11.127	10.080	-2.584	-2.620	***-2.602***
CNP	w	0.552	0.552	0.448	0.448	0.525	0.552	0.475	0.448			
	T_1	11.875	11.865	11.867	11.943	10.833	10.623	10.781	11.152	1.063	1.040	***1.052***
	T_2	9.501	9.403	9.181	9.028	8.500	8.516	8.439	8.787	0.887	0.598	***0.742***
	T_3	8.624	8.732	10.025	9.028	10.667	10.862	10.781	10.060	-1.470	-1.638	***-1.554***

The valuation function of CNP is $V_i = C_i^+ \bullet \left[\varpi_i^+\right]^T - C_i^- \bullet \left[\varpi_i^-\right]^T$ where $\sum\left[\varpi_i^+\right] = \sum\left[\varpi_i^-\right] = 1$. In this case, from e_1's judgments, $\left[\varpi_i^+\right] = (0.552, 0.448)$, $\left[\varpi_i^-\right] = (0.525, 0.475)$; From e_2 's judgments, $\left[\varpi_i^+\right] = (0.552, 0.448)$, $\left[\varpi_i^-\right] = (0.552, 0.448)$.

The "real" weights are shown from tables 8.4.13-8.4.15, and the approximate

POMs based on this data are shown in tables 8.4.16-8.4.18. Tables 8.4.19- 8.4.20 show the priority weights and the utilities derived from POMs from tables 8.4.16 to 8.4.18, as well the results by the valuation function. Table 8.4.19 shows the normalized results whilst table 8.4.20 shows the unnormalized results.

8.4.4. Discussion of case 3

In the conventional assessment method, the rating scores are based on an individual single rating with the ordinal scales, such as Likert-like scales. Although this rating method may be too subjective, it is still popular in quantitative research and in industrial applications. Saaty (1980) proposed pairwise comparison ratio scale which attempts to reduce this subjectiveness. However, there are some significant problems. Firstly, extra effort is needed, as the number of ratings increase from n to $n(n-1)/2$. Secondly, the approximate methods, i.e. analytic prioritization methods, are still uncertain although many applications apply Saaty's Eigenvector method. Thirdly, it is not convenient to "make up" a judgmental matrix with a consistency ratio of less than 0.1, especially when there are five of more candidates to be compared. The "make up" is due to the two reasons: For one thing, nine-point scale has a mathematical limitation in forming a consistent PRM; for the other, ratio scales do not reflect the cognitive comparison of human judgment. The details are indicated in chapter 5.

To address the third problem, in the proposed improved ANP model, the Compound Linguistic Ordinal Scale is proposed in ANP. Thus more choices of rating scale lead to more chance to form a consistent matrix.

Secondly, the classical ANP model uses the synthesis function $V'_1 = bB + oO - cC - rR$, $b+o+c+r=1$. Mathematically, this definition implies that the lower sum of the negative weights (e.g. $c+r$) follows the higher sum of the positive weights (e.g. $b+o$). However, this is not necessarily true. In the real-world, it makes more sense that high risk is followed by high returns, no risk is followed by no return, and no pain is followed by no gain. Thus the improved ANP model refines the synthesis function, i.e. $b+o=c+r=1$, which is also used by CNP. This means that the coefficients are only distributed in positive factors or negative factors.

In CNP, extra effort is also needed. However, if the extra effort can produce the better approximate value, this effort is worthwhile. However, PRM seems not to produce the approximate value as the cognitive interpretation of the numerical representation is questionable (chapter 5), although mathematically it can produce the approximate value which is shown in chapter 8.4.2 by the improved ANP, but not the classical ANP.

In CNP, no cognitive operator problems exist. The main reason is that the pairwise opposite matrix (POM) uses the interval scale, which is much more straightforward to derive.

Regarding the results of the comparisons of the improved ANP and the primitive CNP (table 8.4.12 vs. table 8.4.19), mathematically both methods produce very acceptable approximate results. Although it seems that the improved ANP is much closer to the reference values than CNP, the differences between the approximate values to reference values are also subject to the scaling of CLOS. Their judgment matrices are of different labels and different cognitive perceptions of comparisons in numerical concept e.g. tables 8.4.6-8.4.11 vs. tables 8.4.13-8.4.18. Better representation of CLOS produces better results (i.e. much closer to the reference values).

As the expert uses straightforward comparison, i.e. interval scale rather than ratio scale, it is likely the labels are shown as tables 8.4.16-8.4.18. If matrix labels are converted by PRMs, which is derived by the analytic prioritization operator, the final result is likely misleading. In fact, POMs are appropriate for the pared comparison as they are straightforward, and easy to be understood.

This issue has been investigated in chapter 5. POMs using CLOS produce much more accurate results as there are more choices for an expert with rating in the deductive rating process (algorithm 4.3) of CLOS, which reduces the subjective rating dilemma problem (chapter 4).

8.5 Case 4: Software product selection

The case background, solutions using the AHP approach and the CNP approach respectively, and the discussion of the comparison are presented as follows.

8.5.1 Case 4 background

The case compares the results of Fuzzy CNP and Fuzzy AHP for software product selection considering the software quality model. The case is chosen from Yuen and Lau (2008). A company designing and manufacturing Smartphones includes software and hardware development. Recently the company would like to develop a new model of Smartphone. The company would like to add one accessory application into its product among three candidates T_1, T_2, T_3 with respect to the ISO six criteria of 27 sub-criteria (ISO/IEC9126-1: 2001) in fig. 8.2. The fuzzy rating scale, using triangular fuzzy numbers, is defined in table 8.5.1.

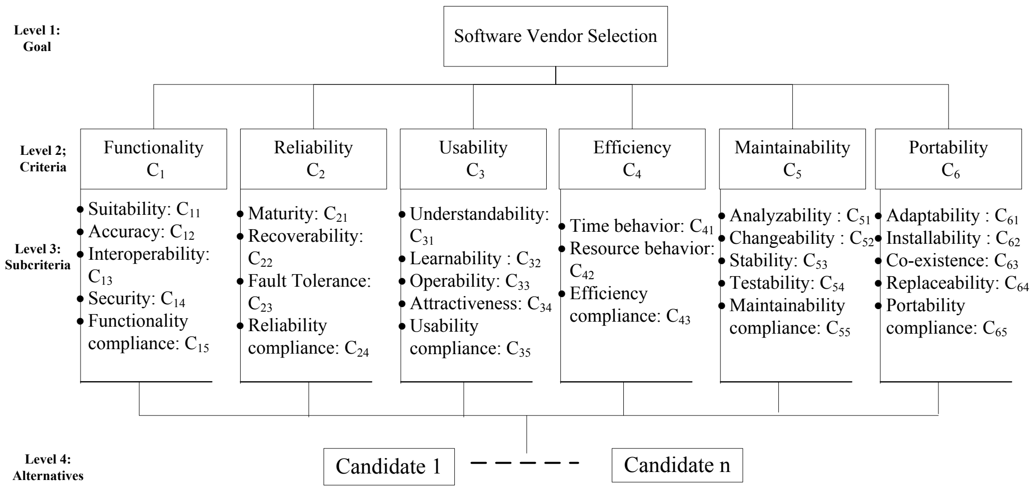


Figure 8.2: The network structure for software vendor section with 27 ISO sub-criteria

Table 8.5.1: References of fuzzy ratio scale of FAHP and fuzzy interval scale of FCNP (case 4)

Labels	Fuzzy AHP	Fuzzy CNP $\hat{\kappa}=(3.5,4,4.5)$
Equal	(1,1,1)	(0,0,0)
Low	(1.5, 2, 2.5)	(0.5,1,1.5)
Moderate	(2.5,3,3.5)	(1.5,2,2.5)
High	(3.5,4,4.5)	(2.5,3,3.5)
Low’	(0.4,0.5,0.67)	(-1.5,-1,-0.5)
Moderate'	(0.29,0.33,0.4)	(-2.5,-2,-1.5)
High’	(0.22,0.25,0.29)	(-3.5,-3,-2.5)

8.5.2 The Fuzzy AHP approach to case 4

Three steps are used in the fuzzy AHP approach to give the selection result.

Firstly, the fuzzy relative importance of the six quality attributes, with 27 subcriteria, is determined by using the fuzzy analytic prioritization operator, which is the modified Fuzzy LLSM (chapter 2.6.3). The input values and the results are shown in table 8.5.2. Fuzzy importance of the sub-criteria of C_1 to C_6 and the local fuzzy weights are shown in tables 8.5.3 to 8.5.8.

Secondly, the experts compare the three candidates: T_1, T_2, T_3 under each of six criteria separately. Tables 8.5.9-8.5.14 show the comparisons among candidates under each sub-criterion, and the prioritization results (or local fuzzy weights).

Thirdly, the prioritization results of six criteria and their fuzzy relative importance are aggregated by Eqs. (2.6.12)-(2.6.14) (Chapter 2.6.3), and the global fuzzy weights of the three candidates are determined. The details are shown in table 8.5.15. The result is also illustrated in Fig. 8.3 graphically. It is clear that T_2 is the best alternative, followed by T_1 and T_3.

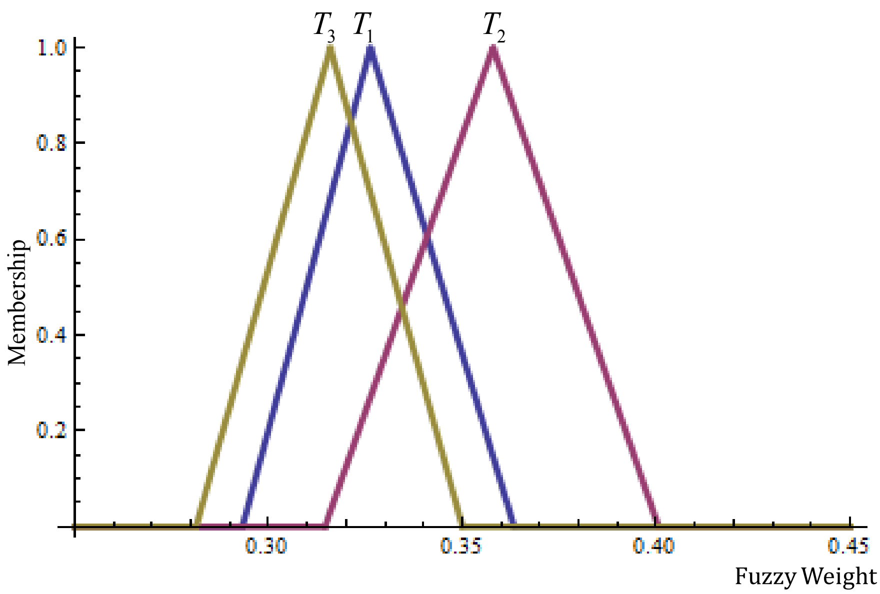


Figure 8.3: Global fuzzy weights of the three candidates by using FAHP

8.5.3 The Fuzzy CNP approach to case 4

To calculate the selection result by using the fuzzy CNP approach, three steps are used, as follows.

Firstly, the fuzzy relative importance of the six quality attributes, with 27 subcriteria, is determined by using the fuzzy cognitive prioritization operator, which is the Fuzzy Least Penalty Squares (FLPS) (Chapter 7). The choice of FLPS is due to the high value of the fuzzy accordant index, which means high discordance. Thus the Fuzzy

Row Average plus the normal Utility (FRAU) is not recommended for fuzzy cognitive prioritization. The fuzzy pairwise opposite matrices and the normalized fuzzy weights are shown in table 8.5.16. The POMs of the fuzzy importance (or normalized fuzzy weight) of the sub-criteria of C_1 to C_6 and the results of the fuzzy weights are shown in tables 8.5.17 to 8.5.22.

Secondly, the experts compare the three candidates: T_1, T_2, T_3 under each of six criteria separately. Tables 8.5.23-8.5.28 show the comparisons among the three candidates under each sub-criterion, and the prioritization results (or fuzzy weights).

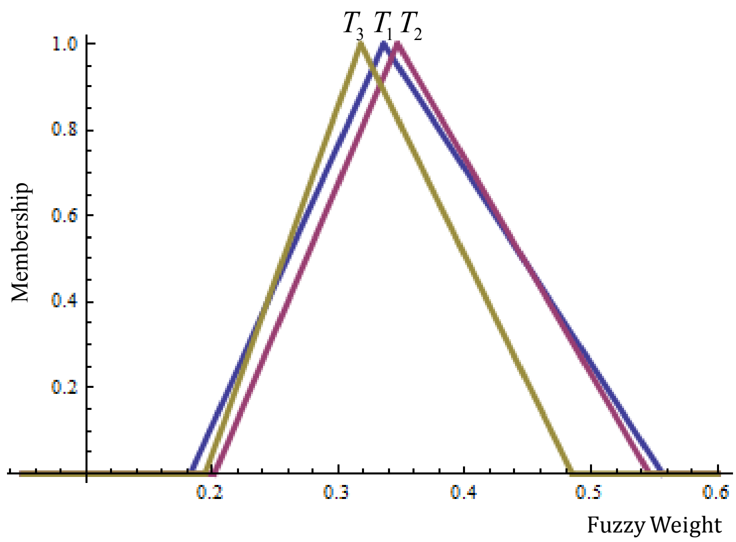


Figure 8.4: Aggregated fuzzy weights of the three candidates by using FCNP

Thirdly, the prioritization results of six criteria and their fuzzy relative importance are aggregated by the fuzzy weighted average or fuzzy weighted arithmetic mean (Chapter 7), and the final fuzzy weights of the three candidates are determined. The details are shown in table 8.5.29. The result is illustrated in Fig. 8.4 graphically. However, the result is interesting. The interval of T_1 embraces the intervals of T_2 and T_3. By comparing the modal values of the output fuzzy sets, it can be observed that T_2 is the best alternative, followed by T_1 and T_3. By comparing the up-boundary values, the order is $T_1 > T_2 > T_3$. By comparing the low-boundary values, the order is $T_2 > T_3 > T_1$. Usually, $T_2 > T_1 > T_3$ is selected as the modal value of the fuzzy set is relatively more important than the interval values. However, in this case, the confidence level is very low.

9.5.4 Discussion of case 4

This research proposes a fuzzy ANP model for software quality evaluation and software vendor selection under uncertainty, with comparison of the fuzzy AHP model of the modified fuzzy Logarithmic Least Squares Method (Wang et al., 2008). Six criteria of 27 subcriteria for the software quality measurement are adopted from ISO/IEC9126 (ISO, 2001). Two numerical examples illustrate the usability of the FANP and the FAHP.

In this comparison, both methods use the same scale labels, but different fuzzy representation values, due to the different axioms of pairwise comparisons, which are shown in table 8.5.1.

The result of FAHP is clear, i.e. $T_2 > T_1 > T_3$ in respect to comparing their intervals and modal values. However, regarding FCNP, the result is on a case-by-case basis. It can be observed that the interval of T_1 embraces the intervals of T_2 and T_3 in figure 8.4.

The fuzzy CNP can be used for the decision attitudes. If the decision maker is pessimist, low-boundary values are applied, and then the rank is $T_2 > T_3 > T_1$. If the decision maker is optimist, up-boundary values are applied, and then the rank is $T_1 > T_2 > T_3$. If the decision maker is neutral, modal values are applied, and then the rank is $T_2 > T_1 > T_3$. By default, neutral is applied. In FCNP, it is a special case of CNP. The advantage of judgment in using fuzzy numbers is for investigating the ranges of the final results, and applying the decision attitudes.

The original research of Yuen and Lau (2009) did not pay attention to fuzzy consistency ratio (FCR), as no previous research discussed the consistency issue in FAHP. FCR is defined in chapter 2.6. In this case, some fuzzy PRMs in FAHP are of FCR >0.1, which is unacceptable in the default setting. For improvement, those fuzzy PRMs of FCR >0.1 are required to be revised in the project.

The values of fuzzy accordant indices of the most of fuzzy POMs are high in this case. The main issue is that the fuzzy POMs are directly converted from fuzzy PRMs, using table 9.5.1, and assuming that they use the same labels. Thus the data in this case are only for discussion of the proposal. In real applications, the fuzzy accordant index

(FAI) is critical for the validity of the FCNP model. This discussion result is crucial to future applications and development using CNP.

Regarding the interval of the final output, the interval of the fuzzy weights of FCNP is larger than FAHP. This issue is related to the design of a fuzzy individual utility $\widehat{\kappa}$. If the interval of $\widehat{\kappa}$ is reduced, i.e. $\widehat{\kappa}=(3.75,4,4.25)$ in this case, the final result is narrowed, and is shown in fig 8.5. However, if $\widehat{\kappa}$ is excessively small, for example, its modal value is less than the modal value of the maximum of fuzzy scale, the fuzzy utility vector is likely to give negative results. This situation is also applied in the interval values. On the other hand, if $\widehat{\kappa}$ is excessively large, the interval of the final fuzzy outputs become large too.

By default, $\widehat{\kappa}=\left(\kappa^l,\kappa^\pi,\kappa^u\right)=\left(Max\left(\bar{X}_\aleph\right)-\delta, Max\left(\bar{X}_\aleph\right), Max\left(\bar{X}_\aleph\right)+\delta\right)$, and δ is the average of the modal values of two adjacent atomic terms (chapter 7).

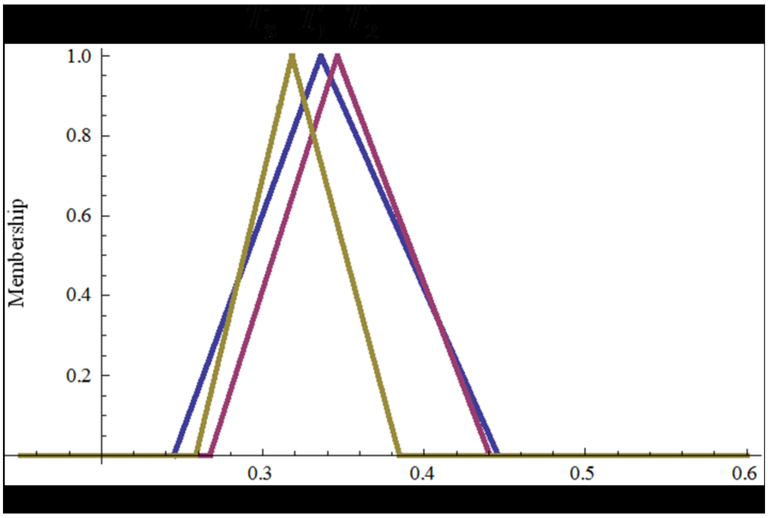


Figure 8.5: Aggregated fuzzy weights using FCNP with $\widehat{\kappa}=(3.75,4,4.25)$

Table 8.5.2: Fuzzy PRM (FCR=0.172) for the importance of six criteria and their fuzzy weights (case 4)

Criteria	C_1	C_2	C_3	C_4	C_5	C_6	FW
C_1	(1, 1, 1)	(1.5, 2, 2.5)	(2.5, 3, 3.5)	(1.5, 2, 2.5)	(2.5, 3, 3.5)	(2.5, 3, 3.5)	(0.27,0.29,0.31)
C_2	(0.4, 0.5, 0.67)	(1, 1, 1)	(1.5, 2, 2.5)	(1.5, 2, 2.5)	(1.5, 2, 2.5)	(1.5, 2, 2.5)	(0.18,0.21,0.24)
C_3	(0.29, 0.33, 0.4)	(0.4, 0.5, 0.67)	(1, 1, 1)	(2.5, 3, 3.5)	(1.5, 2, 2.5)	(1.5, 2, 2.5)	(0.15,0.17,0.19)
C_4	(1.5, 2, 2.5)	(0.4, 0.5, 0.67)	(0.29, 0.33, 0.4)	(1, 1, 1)	(1.5, 2, 2.5)	(1.5, 2, 2.5)	(0.12,0.14,0.16)
C_5	(0.29, 0.33, 0.4)	(0.4, 0.5, 0.67)	(0.4, 0.5, 0.67)	(0.4, 0.5, 0.67)	(1, 1, 1)	(0.22, 0.25, 0.29)	(0.07,0.07,0.08)
C_6	(0.29, 0.33, 0.4)	(0.4, 0.5, 0.67)	(0.4, 0.5, 0.67)	(0.4, 0.5, 0.67)	(3.5, 4, 4.5)	(1, 1, 1)	(0.1,0.11,0.13)

Table 8.5.3: Fuzzy PRM (FCR=0.04) for the fuzzy importance of five subcriteria of functionality and their fuzzy weights (case 4)

Criteria	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	FW
C_{11}	(1, 1, 1)	(1.5, 2, 2.5)	(0.29, 0.33, 0.4)	(0.4, 0.5, 0.67)	(0.22, 0.25, 0.29)	(0.09,0.1,0.11)
C_{12}	(0.4, 0.5, 0.67)	(1, 1, 1)	(0.22, 0.25, 0.29)	(0.29, 0.33, 0.4)	(0.22, 0.25, 0.29)	(0.06,0.07,0.07)
C_{13}	(2.5, 3, 3.5)	(3.5, 4, 4.5)	(1, 1, 1)	(0.4, 0.5, 0.67)	(0.4, 0.5, 0.67)	(0.19,0.21,0.23)
C_{14}	(1.5, 2, 2.5)	(2.5, 3, 3.5)	(1.5, 2, 2.5)	(1, 1, 1)	(0.4, 0.5, 0.67)	(0.2,0.24,0.27)
C_{15}	(3.5, 4, 4.5)	(3.5, 4, 4.5)	(1.5, 2, 2.5)	(1.5, 2, 2.5)	(1, 1, 1)	(0.34,0.38,0.41)

Table 8.5.4: Fuzzy PRM (FCR=0.027) for the fuzzy importance of four subcriteria of reliability and their fuzzy weights (case 4)

Criteria	C_{21}	C_{22}	C_{23}	C_{24}	FW
C_{21}	(1, 1, 1)	(2.5, 3, 3.5)	(2.5, 3, 3.5)	(0.4, 0.5, 0.67)	(0.27,0.3,0.33)
C_{22}	(0.29, 0.33, 0.4)	(1, 1, 1)	(1.5, 2, 2.5)	(0.22, 0.25, 0.29)	(0.12,0.13,0.14)
C_{23}	(0.29, 0.33, 0.4)	(0.4, 0.5, 0.67)	(1, 1, 1)	(0.22, 0.25, 0.29)	(0.09,0.09,0.1)
C_{24}	(1.5, 2, 2.5)	(3.5, 4, 4.5)	(3.5, 4, 4.5)	(1, 1, 1)	(0.44,0.48,0.51)

Table 8.5.5: Fuzzy PRM (FCR=0.044) for the fuzzy importance of five subcriteria of usability and their fuzzy weights (case 4)

Criteria	C_{31}	C_{32}	C_{33}	C_{34}	C_{35}	FW
C_{31}	(1, 1, 1)	(1.5, 2, 2.5)	(2.5, 3, 3.5)	(1.5, 2, 2.5)	(0.22, 0.25, 0.29)	(0.18,0.2,0.22)
C_{32}	(0.4, 0.5, 0.67)	(1, 1, 1)	(1.5, 2, 2.5)	(0.4, 0.5, 0.67)	(0.22, 0.25, 0.29)	(0.09,0.11,0.12)
C_{33}	(0.29, 0.33, 0.4)	(0.4, 0.5, 0.67)	(1, 1, 1)	(0.4, 0.5, 0.67)	(0.22, 0.25, 0.29)	(0.07,0.07,0.08)
C_{34}	(0.4, 0.5, 0.67)	(1.5, 2, 2.5)	(1.5, 2, 2.5)	(1, 1, 1)	(0.22, 0.25, 0.29)	(0.12,0.14,0.16)
C_{35}	(3.5, 4, 4.5)	(3.5, 4, 4.5)	(3.5, 4, 4.5)	(3.5, 4, 4.5)	(1, 1, 1)	(0.48,0.48,0.49)

Table 8.5.6: Fuzzy PRM (FCR=0.008) for the fuzzy importance of three subcriteria of efficiency and their fuzzy weights (case 4)

Criteria	C_{41}	C_{42}	C_{43}	FW
C_{41}	(1, 1, 1)	(1.5, 2, 2.5)	(0.4, 0.5, 0.67)	(0.25,0.3,0.35)
C_{42}	(0.4, 0.5, 0.67)	(1, 1, 1)	(0.29, 0.33, 0.4)	(0.15,0.16,0.18)
C_{43}	(1.5, 2, 2.5)	(2.5, 3, 3.5)	(1, 1, 1)	(0.49,0.54,0.58)

Table 8.5.7: Fuzzy PRM (FCR=0.033) for the fuzzy importance of five subcriteria of maintainability and their fuzzy weights (case 4)

Criteria	C_{51}	C_{52}	C_{53}	C_{54}	C_{55}	FW
C_{51}	(1, 1, 1)	(1.5, 2, 2.5)	(0.4, 0.5, 0.67)	(0.4, 0.5, 0.67)	(0.22, 0.25, 0.29)	(0.1,0.11,0.12)
C_{52}	(0.4, 0.5, 0.67)	(1, 1, 1)	(0.22, 0.25, 0.29)	(0.22, 0.25, 0.29)	(0.22, 0.25, 0.29)	(0.06,0.06,0.07)
C_{53}	(1.5, 2, 2.5)	(3.5, 4, 4.5)	(1, 1, 1)	(0.4, 0.5, 0.67)	(0.29, 0.33, 0.4)	(0.16,0.17,0.19)
C_{54}	(1.5, 2, 2.5)	(3.5, 4, 4.5)	(1.5, 2, 2.5)	(1, 1, 1)	(0.4, 0.5, 0.67)	(0.21,0.25,0.28)
C_{55}	(3.5, 4, 4.5)	(3.5, 4, 4.5)	(2.5, 3, 3.5)	(1.5, 2, 2.5)	(1, 1, 1)	(0.38,0.41,0.43)

Table 8.5.8: Fuzzy PRM (FCR=0.028) for the fuzzy importance of five subcriteria of portability and their fuzzy weights (case 4)

Criteria	C_{61}	C_{62}	C_{63}	C_{64}	C_{65}	FW
C_{61}	(1, 1, 1)	(1.5, 2, 2.5)	(0.4, 0.5, 0.67)	(0.4, 0.5, 0.67)	(0.22, 0.25, 0.29)	(0.1,0.11,0.13)
C_{62}	(0.4, 0.5, 0.67)	(1, 1, 1)	(0.22, 0.25, 0.29)	(0.22, 0.25, 0.29)	(0.22, 0.25, 0.29)	(0.06,0.06,0.07)
C_{63}	(1.5, 2, 2.5)	(3.5, 4, 4.5)	(1, 1, 1)	(0.4, 0.5, 0.67)	(0.4, 0.5, 0.67)	(0.17,0.19,0.22)
C_{64}	(1.5, 2, 2.5)	(3.5, 4, 4.5)	(1.5, 2, 2.5)	(1, 1, 1)	(0.4, 0.5, 0.67)	(0.22,0.25,0.28)
C_{65}	(3.5, 4, 4.5)	(3.5, 4, 4.5)	(1.5, 2, 2.5)	(1.5, 2, 2.5)	(1, 1, 1)	(0.35,0.38,0.41)

Table 8.5.9: Fuzzy PRMs of three candidates with respect to functionality C_1 and their fuzzy weights (case 4)

	C_{11} (FCR=0.018)				C_{12} (FCR=0.046)			
	T_1	T_2	T_3	FW	T_1	T_2	T_3	FW
T_1	(1, 1, 1)	(1, 1, 1)	(0.4, 0.5, 0.67)	(0.22,0.24,0.28)	(1, 1, 1)	(0.4, 0.5, 0.67)	(1, 1, 1)	(0.24,0.26,0.29)
T_2	(1, 1, 1)	(1, 1, 1)	(0.29, 0.33, 0.4)	(0.21,0.21,0.22)	(1.5, 2, 2.5)	(1, 1, 1)	(1, 1, 1)	(0.38,0.41,0.44)
T_3	(1.5, 2, 2.5)	(2.5, 3, 3.5)	(1, 1, 1)	(0.51,0.55,0.58)	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)	(0.33,0.33,0.33)
	C_{13} (FCR=0)				C_{14} (FCR=0)			
T_1	(1, 1, 1)	(0.4, 0.5, 0.67)	(1, 1, 1)	(0.22,0.24,0.28)	(1, 1, 1)	(0.4, 0.5, 0.67)	(1, 1, 1)	(0.23,0.25,0.28)
T_2	(1.5, 2, 2.5)	(1, 1, 1)	(2.5, 3, 3.5)	(0.51,0.55,0.58)	(1.5, 2, 2.5)	(1, 1, 1)	(1.5, 2, 2.5)	(0.45,0.5,0.53)
T_3	(1, 1, 1)	(0.29, 0.33, 0.4)	(1, 1, 1)	(0.21,0.21,0.22)	(1, 1, 1)	(0.4, 0.5, 0.67)	(1, 1, 1)	(0.23,0.25,0.28)
	C_{15} (FCR=0)							
T_1	(1, 1, 1)	(1.5, 2, 2.5)	(1, 1, 1)	(0.38,0.4,0.42)				
T_2	(0.4, 0.5, 0.67)	(1, 1, 1)	(0.4, 0.5, 0.67)	(0.17,0.2,0.25)				
T_3	(1, 1, 1)	(1.5, 2, 2.5)	(1, 1, 1)	(0.38,0.4,0.42)				

Table 8.5.10: Fuzzy PRMs of three candidates with respect to reliability C_2 and their fuzzy weights (case 4)

	C_{21} (FCR=0.823)				C_{22} (FCR=0.464)			
	T_1	T_2	T_3	FW	T_1	T_2	T_3	FW
T_1	(1, 1, 1)	(1, 1, 1)	(1.5, 2, 2.5)	(0.29,0.32,0.35)	(1, 1, 1)	(1.5, 2, 2.5)	(1, 1, 1)	(0.35,0.4,0.44)
T_2	(1, 1, 1)	(1, 1, 1)	(2.5, 3, 3.5)	(0.45,0.46,0.46)	(0.4, 0.5, 0.67)	(1, 1, 1)	(3.5, 4, 4.5)	(0.36,0.4,0.45)
T_3	(1.5, 2, 2.5)	(0.29, 0.33, 0.4)	(1, 1, 1)	(0.19,0.22,0.26)	(1, 1, 1)	(0.22, 0.25, 0.29)	(1, 1, 1)	(0.2,0.2,0.2)
	C_{23} (FCR=0.018)				C_{24} (FCR=0.455)			
T_1	(1, 1, 1)	(2.5, 3, 3.5)	(1.5, 2, 2.5)	(0.51,0.55,0.58)	(1, 1, 1)	(0.4, 0.5, 0.67)	(1.5, 2, 2.5)	(0.29,0.33,0.38)
T_2	(0.29, 0.33, 0.4)	(1, 1, 1)	(1, 1, 1)	(0.21,0.21,0.22)	(1.5, 2, 2.5)	(1, 1, 1)	(0.4, 0.5, 0.67)	(0.29,0.33,0.38)
T_3	(0.4, 0.5, 0.67)	(1, 1, 1)	(1, 1, 1)	(0.22,0.24,0.28)	(0.4, 0.5, 0.67)	(1.5, 2, 2.5)	(1, 1, 1)	(0.29,0.33,0.38)

Table 8.5.11: Fuzzy PRMs of three candidates with respect to usability C_3 and their fuzzy weights (case 4)

	C_{31} (FCR=0)				C_{32} (FCR=0.048)			
	T_1	T_2	T_3	FW	T_1	T_2	T_3	FW
T_1	(1, 1, 1)	(2.5, 3, 3.5)	(1, 1, 1)	(0.44,0.44,0.44)	(1, 1, 1)	(1.5, 2, 2.5)	(1.5, 2, 2.5)	(0.42,0.49,0.55)
T_2	(0.29, 0.33, 0.4)	(1, 1, 1)	(0.4, 0.5, 0.67)	(0.15,0.17,0.20)	(0.4, 0.5, 0.67)	(1, 1, 1)	(1.5, 2, 2.5)	(0.27,0.31,0.35)
T_3	(1, 1, 1)	(1.5, 2, 2.5)	(1, 1, 1)	(0.36,0.39,0.41)	(0.4, 0.5, 0.67)	(0.4, 0.5, 0.67)	(1, 1, 1)	(0.18,0.2,0.23)
	C_{33} (FCR=0.016)				C_{34} (FCR=0.127)			
T_1	(1, 1, 1)	(1.5, 2, 2.5)	(3.5, 4, 4.5)	(0.5,0.56,0.61)	(1, 1, 1)	(2.5, 3, 3.5)	(1, 1, 1)	(0.44,0.46,0.47)
T_2	(0.4, 0.5, 0.67)	(1, 1, 1)	(2.5, 3, 3.5)	(0.27,0.32,0.37)	(0.29, 0.33, 0.4)	(1, 1, 1)	(1, 1, 1)	(0.21,0.22,0.24)
T_3	(0.22, 0.25, 0.29)	(0.29, 0.33, 0.4)	(1, 1, 1)	(0.12,0.12,0.13)	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)	(0.32,0.32,0.32)
	C_{35} (FCR=0.047)							
T_1	(1, 1, 1)	(0.4, 0.5, 0.67)	(0.4, 0.5, 0.67)	(0.18,0.2,0.23)				
T_2	(1.5, 2, 2.5)	(1, 1, 1)	(1.5, 2, 2.5)	(0.42,0.49,0.55)				
T_3	(1.5, 2, 2.5)	(0.4, 0.5, 0.67)	(1, 1, 1)	(0.27,0.31,0.35)				

Table 8.5.12: Fuzzy PRMs of three candidates with respect to efficiency C_4 and their fuzzy weights (case 4)

	C_{41} (FCR=0.047)				C_{42} (FCR=0)			
	T_1	T_2	T_3	FW	T_1	T_2	T_3	FW
T_1	(1, 1, 1)	(1.5, 2, 2.5)	(1.5, 2, 2.5)	(0.42,0.49,0.55)	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)	(0.33,0.33,0.33)
T_2	(0.4, 0.5, 0.67)	(1, 1, 1)	(0.4, 0.5, 0.67)	(0.18,0.2,0.23)	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)	(0.33,0.33,0.33)
T_3	(0.4, 0.5, 0.67)	(1.5, 2, 2.5)	(1, 1, 1)	(0.27,0.31,0.35)	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)	(0.33,0.33,0.33)
	C_{43} (FCR=0.018)							
T_1	(1, 1, 1)	(1, 1, 1)	(0.4, 0.5, 0.67)	(0.22,0.24,0.28)				
T_2	(1, 1, 1)	(1, 1, 1)	(0.29, 0.33, 0.4)	(0.21,0.21,0.22)				
T_3	(1.5, 2, 2.5)	(2.5, 3, 3.5)	(1, 1, 1)	(0.51,0.55,0.58)				

Table 8.5.13: Fuzzy PRMs of three candidates with respect to maintainability C_5 and their fuzzy weights (case 4)

	C_{51} (FCR=0.046)				C_{52} (FCR=0)			
	T_1	T_2	T_3	FW	T_1	T_2	T_3	FW
T_1	(1, 1, 1)	(1.5, 2, 2.5)	(1, 1, 1)	(0.38,0.41,0.44)	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)	(0.33,0.33,0.33)
T_2	(0.4, 0.5, 0.67)	(1, 1, 1)	(1, 1, 1)	(0.24,0.26,0.29)	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)	(0.33,0.33,0.33)
T_3	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)	(0.33,0.33,0.33)	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)	(0.33,0.33,0.33)
	C_{53} (FCR=0)				C_{54} (FCR=0.049)			
T_3	(1, 1, 1)	(0.4, 0.5, 0.67)	(1, 1, 1)	(0.22,0.24,0.28)	(1, 1, 1)	(0.4, 0.5, 0.67)	(1, 1, 1)	(0.21,0.23,0.27)
T_2	(1.5, 2, 2.5)	(1, 1, 1)	(2.5, 3, 3.5)	(0.51,0.55,0.58)	(1.5, 2, 2.5)	(1, 1, 1)	(3.5, 4, 4.5)	(0.55,0.58,0.61)
T_3	(1, 1, 1)	(0.29, 0.33, 0.4)	(1, 1, 1)	(0.21,0.21,0.22)	(1, 1, 1)	(0.22, 0.25, 0.29)	(1, 1, 1)	(0.18,0.18,0.18)
	C_{55} (FCR=0)							
T_3	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)	(0.33,0.33,0.33)				
T_2	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)	(0.33,0.33,0.33)				
T_3	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)	(0.33,0.33,0.33)				

Table 8.5.14: Fuzzy PRMs of three candidates with respect to portability C_6 and their fuzzy weights (case 4)

	C_{61} (FCR=0)				C_{62} (FCR=0.455)			
	T_1	T_2	T_3	FW	T_1	T_2	T_3	FW
T_1	(1, 1, 1)	(0.29, 0.33, 0.4)	(1, 1, 1)	(0.21,0.21,0.22)	(1, 1, 1)	(0.4, 0.5, 0.67)	(1.5, 2, 2.5)	(0.29,0.33,0.38)
T_2	(2.5, 3, 3.5)	(1, 1, 1)	(1.5, 2, 2.5)	(0.51,0.55,0.58)	(1.5, 2, 2.5)	(1, 1, 1)	(0.4, 0.5, 0.67)	(0.29,0.33,0.38)
T_3	(1, 1, 1)	(0.4, 0.5, 0.67)	(1, 1, 1)	(0.22,0.24,0.28)	(0.4, 0.5, 0.67)	(1.5, 2, 2.5)	(1, 1, 1)	(0.29,0.33,0.38)
	C_{63} (FCR=0.046)				C_{64} (FCR=0.334)			
T_1	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)	(0.33,0.33,0.33)	(1, 1, 1)	(2.5, 3, 3.5)	(1, 1, 1)	(0.45,0.46,0.48)
T_2	(1, 1, 1)	(1, 1, 1)	(1.5, 2, 2.5)	(0.38,0.41,0.44)	(0.29, 0.33, 0.4)	(1, 1, 1)	(1.5, 2, 2.5)	(0.25,0.28,0.32)
T_3	(1, 1, 1)	(0.4, 0.5, 0.67)	(1, 1, 1)	(0.24,0.26,0.29)	(1, 1, 1)	(0.4, 0.5, 0.67)	(1, 1, 1)	(0.24,0.26,0.28)
	C_{65} (FCR=0.049)							
T_1	(1, 1, 1)	(0.4, 0.5, 0.67)	(1, 1, 1)	(0.21,0.23,0.27)				
T_2	(1.5, 2, 2.5)	(1, 1, 1)	(3.5, 4, 4.5)	(0.55,0.58,0.61)				
T_3	(1, 1, 1)	(0.22, 0.25, 0.29)	(1, 1, 1)	(0.18,0.18,0.18)				

Table 8.5.15: Aggregation Results for the fuzzy weights of the objectives (case 4)

	Fuzzy Importance	T_1	T_2	T_3
Criterion 1	(0.27,0.29,0.31)	(0.28,0.31,0.33)	(0.31,0.36,0.41)	(0.31,0.33,0.36)
Criterion 2	(0.18,0.21,0.24)	(0.32,0.36,0.4)	(0.33,0.37,0.4)	(0.24,0.27,0.31)
Criterion 3	(0.15,0.17,0.19)	(0.31,0.34,0.38)	(0.31,0.36,0.41)	(0.27,0.30,0.33)
Criterion 4	(0.12,0.14,0.16)	(0.29,0.33,0.38)	(0.22,0.23,0.24)	(0.4,0.44,0.48)
Criterion 5	(0.07,0.07,0.08)	(0.28,0.3,0.32)	(0.4,0.43,0.45)	(0.27,0.27,0.28)
Criterion6	(0.1,0.11,0.13)	(0.29,0.31,0.34)	(0.41,0.45,0.49)	(0.22,0.23,0.26)
Global fuzzy weights		(0.29,0.33,0.36)	(0.31,0.36,0.4)	(0.28,0.32,0.35)

(FCNP)

Table 8.5.16: Fuzzy POM (FAI=0.427) for the importance of six criteria and their fuzzy weights (case 4)

Criteria	C_1	C_2	C_3	C_4	C_5	C_6	FW
C_1	(0, 0, 0)	(0.5, 1, 1.5)	(1.5, 2, 2.5)	(0.5, 1, 1.5)	(1.5, 2, 2.5)	(1.5, 2, 2.5)	(0.19,0.24,0.29)
C_2	(-1.5, -1, -0.5)	(0, 0, 0)	(0.5, 1, 1.5)	(0.5, 1, 1.5)	(0.5, 1, 1.5)	(0.5, 1, 1.5)	(0.15,0.19,0.24)
C_3	(-2.5, -2, -1.5)	(-1.5, -1, -0.5)	(0, 0, 0)	(1.5, 2, 2.5)	(0.5, 1, 1.5)	(0.5, 1, 1.5)	(0.14,0.18,0.21)
C_4	(0.5, 1, 1.5)	(-1.5, -1, -0.5)	(-2.5, -2, -1.5)	(0, 0, 0)	(0.5, 1, 1.5)	(0.5, 1, 1.5)	(0.13,0.15,0.17)
C_5	(-2.5, -2, -1.5)	(-1.5, -1, -0.5)	(-1.5, -1, -0.5)	(-1.5, -1, -0.5)	(0, 0, 0)	(-3.5, -3, -2.5)	(0.08,0.09,0.11)
C_6	(-2.5, -2, -1.5)	(-1.5, -1, -0.5)	(-1.5, -1, -0.5)	(-1.5, -1, -0.5)	(2.5, 3, 3.5)	(0, 0, 0)	(0.13,0.15,0.15)

Table 8.5.17: Fuzzy POM (FAI=0.226) for the fuzzy importance of five subcriteria of functionality and their fuzzy weights (case 4)

Criteria	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	FW
C_{11}	(0, 0, 0)	(0.5, 1, 1.5)	(-2.5, -2, -1.5)	(-1.5, -1, -0.5)	(-3.5, -3, -2.5)	(0.07,0.13,0.19)
C_{12}	(-1.5, -1, -0.5)	(0, 0, 0)	(-3.5, -3, -2.5)	(-2.5, -2, -1.5)	(-3.5, -3, -2.5)	(0.03,0.08,0.13)
C_{13}	(1.5, 2, 2.5)	(2.5, 3, 3.5)	(0, 0, 0)	(-1.5, -1, -0.5)	(-1.5, -1, -0.5)	(0.21,0.24,0.27)
C_{14}	(0.5, 1, 1.5)	(1.5, 2, 2.5)	(0.5, 1, 1.5)	(0, 0, 0)	(-1.5, -1, -0.5)	(0.22,0.24,0.27)
C_{15}	(2.5, 3, 3.5)	(2.5, 3, 3.5)	(0.5, 1, 1.5)	(0.5, 1, 1.5)	(0, 0, 0)	(0.3,0.31,0.31)

Table 8.5.18: Fuzzy POM (FAI=0.112) for the fuzzy importance of four subcriteria of reliability and their fuzzy weights (case 4)

Criteria	C_{21}	C_{22}	C_{23}	C_{24}	FW
C_{21}	(0, 0, 0)	(1.5, 2, 2.5)	(1.5, 2, 2.5)	(-1.5, -1, -0.5)	(0.24,0.31,0.39)
C_{22}	(-2.5, -2, -1.5)	(0, 0, 0)	(0.5, 1, 1.5)	(-3.5, -3, -2.5)	(0.11,0.17,0.22)
C_{23}	(-2.5, -2, -1.5)	(-1.5, -1, -0.5)	(0, 0, 0)	(-3.5, -3, -2.5)	(0.09,0.13,0.16)
C_{24}	(0.5, 1, 1.5)	(2.5, 3, 3.5)	(2.5, 3, 3.5)	(0, 0, 0)	(0.39,0.4,0.41)

Table 8.5.19: Fuzzy POM (FAI=0.222) for the fuzzy importance of five subcriteria of usability and their fuzzy weights (case 4)

Criteria	C_{31}	C_{32}	C_{33}	C_{34}	C_{35}	FW
C_{31}	(0, 0, 0)	(0.5, 1, 1.5)	(1.5, 2, 2.5)	(0.5, 1, 1.5)	(-3.5, -3, -2.5)	(0.15,0.21,0.27)
C_{32}	(-1.5, -1, -0.5)	(0, 0, 0)	(0.5, 1, 1.5)	(-1.5, -1, -0.5)	(-3.5, -3, -2.5)	(0.1,0.15,0.19)
C_{33}	(-2.5, -2, -1.5)	(-1.5, -1, -0.5)	(0, 0, 0)	(-1.5, -1, -0.5)	(-3.5, -3, -2.5)	(0.07,0.11,0.14)
C_{34}	(-1.5, -1, -0.5)	(0.5, 1, 1.5)	(0.5, 1, 1.5)	(0, 0, 0)	(-3.5, -3, -2.5)	(0.15,0.17,0.19)
C_{35}	(2.5, 3, 3.5)	(2.5, 3, 3.5)	(2.5, 3, 3.5)	(2.5, 3, 3.5)	(0, 0, 0)	(0.35,0.36,0.37)

Table 8.5.20: Fuzzy POM (FAI=0) for the fuzzy importance of three subcriteria of efficiency and their fuzzy weights (case 4)

Criteria	C_{41}	C_{42}	C_{43}	FW
C_{41}	(0, 0, 0)	(0.5, 1, 1.5)	(-1.5, -1, -0.5)	(0.24,0.33,0.43)
C_{42}	(-1.5, -1, -0.5)	(0, 0, 0)	(-2.5, -2, -1.5)	(0.17,0.22,0.28)
C_{43}	(0.5, 1, 1.5)	(1.5, 2, 2.5)	(0, 0, 0)	(0.43,0.44,0.46)

Table 8.5.21: Fuzzy POM (FAI=0.218) for the fuzzy importance of five subcriteria of maintainability and their fuzzy weights (case 4)

Criteria	C_{51}	C_{52}	C_{53}	C_{54}	C_{55}	FW
C_{51}	(0, 0, 0)	(0.5, 1, 1.5)	(-1.5, -1, -0.5)	(-1.5, -1, -0.5)	(-3.5, -3, -2.5)	(0.09,0.15,0.21)
C_{52}	(-1.5, -1, -0.5)	(0, 0, 0)	(-3.5, -3, -2.5)	(-3.5, -3, -2.5)	(-3.5, -3, -2.5)	(0.02,0.07,0.11)
C_{53}	(0.5, 1, 1.5)	(2.5, 3, 3.5)	(0, 0, 0)	(-1.5, -1, -0.5)	(-2.5, -2, -1.5)	(0.18,0.21,0.25)
C_{54}	(0.5, 1, 1.5)	(2.5, 3, 3.5)	(0.5, 1, 1.5)	(0, 0, 0)	(-1.5, -1, -0.5)	(0.23,0.25,0.27)
C_{55}	(2.5, 3, 3.5)	(2.5, 3, 3.5)	(1.5, 2, 2.5)	(0.5, 1, 1.5)	(0, 0, 0)	(0.31,0.32,0.33)

Table 8.5.22: Fuzzy POM (FAI=0.213) for the fuzzy importance of five subcriteria of portability and their fuzzy weights (case 4)

Criteria	C_{61}	C_{62}	C_{63}	C_{64}	C_{65}	FW
C_{61}	(0, 0, 0)	(0.5, 1, 1.5)	(-1.5, -1, -0.5)	(-1.5, -1, -0.5)	(-3.5, -3, -2.5)	(0.09,0.15,0.21)
C_{62}	(-1.5, -1, -0.5)	(0, 0, 0)	(-3.5, -3, -2.5)	(-3.5, -3, -2.5)	(-3.5, -3, -2.5)	(0.02,0.07,0.11)
C_{63}	(0.5, 1, 1.5)	(2.5, 3, 3.5)	(0, 0, 0)	(-1.5, -1, -0.5)	(-1.5, -1, -0.5)	(0.19,0.23,0.26)
C_{64}	(0.5, 1, 1.5)	(2.5, 3, 3.5)	(0.5, 1, 1.5)	(0, 0, 0)	(-1.5, -1, -0.5)	(0.23,0.25,0.27)
C_{65}	(2.5, 3, 3.5)	(2.5, 3, 3.5)	(0.5, 1, 1.5)	(0.5, 1, 1.5)	(0, 0, 0)	(0.3,0.31,0.31)

Table 8.5.23: Fuzzy POMs of the three candidates with respect to functionality C_1 and their fuzzy weights (case 4)

	C_{11} (FAI=0.129)				C_{12} (FAI=0.12)			
	T_1	T_2	T_3	FW	T_1	T_2	T_3	FW
T_1	(0, 0, 0)	(0, 0, 0)	(-1.5, -1, -0.5)	(0.21,0.29,0.36)	(0, 0, 0)	(-1.5, -1, -0.5)	(0, 0, 0)	(0.24,0.31,0.38)
T_2	(0, 0, 0)	(0, 0, 0)	(-2.5, -2, -1.5)	(0.2,0.27,0.34)	(0.5, 1, 1.5)	(0, 0, 0)	(0, 0, 0)	(0.31,0.36,0.4)
T_3	(0.5, 1, 1.5)	(1.5, 2, 2.5)	(0, 0, 0)	(0.43,0.44,0.46)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0.28,0.33,0.39)
	C_{13} (FAI=0)				C_{14} (FAI=0)			
T_1	(0, 0, 0)	(-1.5, -1, -0.5)	(0, 0, 0)	(0.22,0.29,0.35)	(0, 0, 0)	(-1.5, -1, -0.5)	(0, 0, 0)	(0.23,0.3,0.36)
T_2	(0.5, 1, 1.5)	(0, 0, 0)	(1.5, 2, 2.5)	(0.39,0.44,0.5)	(0.5, 1, 1.5)	(0, 0, 0)	(0.5, 1, 1.5)	(0.35,0.41,0.46)
T_3	(0, 0, 0)	(-2.5, -2, -1.5)	(0, 0, 0)	(0.22,0.27,0.32)	(0, 0, 0)	(-1.5, -1, -0.5)	(0, 0, 0)	(0.25,0.3,0.34)
	C_{15} (FAI=0)							
T_1	(0, 0, 0)	(0.5, 1, 1.5)	(0, 0, 0)	(0.31,0.37,0.43)				
T_2	(-1.5, -1, -0.5)	(0, 0, 0)	(-1.5, -1, -0.5)	(0.2,0.26,0.31)				
T_3	(0, 0, 0)	(0.5, 1, 1.5)	(0, 0, 0)	(0.32,0.37,0.42)				

Table 8.5.24: Fuzzy POMs of the three candidates with respect to reliability C_2 and their fuzzy weights (case 4)

	C_{21} (FAI=0.211)				C_{22} (FAI=0.509)			
	T_1	T_2	T_3	FW	T_1	T_2	T_3	FW
T_1	(0, 0, 0)	(0, 0, 0)	(0.5, 1, 1.5)	(0.31,0.38,0.46)	(0, 0, 0)	(0.5, 1, 1.5)	(0, 0, 0)	(0.3,0.37,0.42)
T_2	(0, 0, 0)	(0, 0, 0)	(1.5, 2, 2.5)	(0.32,0.4,0.47)	(-1.5, -1, -0.5)	(0, 0, 0)	(2.5, 3, 3.5)	(0.29,0.37,0.41)
T_3	(0.5, 1, 1.5)	(-2.5, -2, -1.5)	(0, 0, 0)	(0.2,0.22,0.24)	(0, 0, 0)	(-3.5, -3, -2.5)	(0, 0, 0)	(0.24,0.26,0.33)
	C_{23} (FAI=0.129)				C_{24} (FAI=0.385)			
T_1	(0, 0, 0)	(1.5, 2, 2.5)	(0.5, 1, 1.5)	(0.35,0.44,0.54)	(0, 0, 0)	(-1.5, -1, -0.5)	(0.5, 1, 1.5)	(0.28,0.34,0.41)
T_2	(-2.5, -2, -1.5)	(0, 0, 0)	(0, 0, 0)	(0.23,0.27,0.31)	(0.5, 1, 1.5)	(0, 0, 0)	(-1.5, -1, -0.5)	(0.28,0.34,0.38)
T_3	(-1.5, -1, -0.5)	(0, 0, 0)	(0, 0, 0)	(0.25,0.29,0.32)	(-1.5, -1, -0.5)	(0.5, 1, 1.5)	(0, 0, 0)	(0.28,0.33,0.38)

Table 8.5.25: Fuzzy POMs of the three candidates with respect to usability C_3 and their fuzzy weights (case 4)

	C_{31} (FAI=0)				C_{32} (FAI=0.12)			
	T_1	T_2	T_3	FW	T_1	T_2	T_3	FW
T_1	(0, 0, 0)	(1.5, 2, 2.5)	(0, 0, 0)	(0.33,0.40,0.46)	(0, 0, 0)	(0.5, 1, 1.5)	(0.5, 1, 1.5)	(0.31,0.41,0.5)
T_2	(-2.5, -2, -1.5)	(0, 0, 0)	(-1.5, -1, -0.5)	(0.17,0.22,0.28)	(-1.5, -1, -0.5)	(0, 0, 0)	(0.5, 1, 1.5)	(0.28,0.33,0.39)
T_3	(0, 0, 0)	(0.5, 1, 1.5)	(0, 0, 0)	(0.33,0.38,0.43)	(-1.5, -1, -0.5)	(-1.5, -1, -0.5)	(0, 0, 0)	(0.24,0.26,0.28)
	C_{33} (FAI=0)				C_{34} (FAI=0.254)			
T_1	(0, 0, 0)	(0.5, 1, 1.5)	(2.5, 3, 3.5)	(0.39,0.48,0.57)	(0, 0, 0)	(1.5, 2, 2.5)	(0, 0, 0)	(0.31,0.38,0.44)
T_2	(-1.5, -1, -0.5)	(0, 0, 0)	(1.5, 2, 2.5)	(0.31,0.37,0.43)	(-2.5, -2, -1.5)	(0, 0, 0)	(0, 0, 0)	(0.24,0.29,0.33)
T_3	(-3.5, -3, -2.5)	(-2.5, -2, -1.5)	(0, 0, 0)	(0.13,0.15,0.17)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0.28,0.33,0.39)
	C_{35} (FAI=0.12)							
T_1	(0, 0, 0)	(-1.5, -1, -0.5)	(-1.5, -1, -0.5)	(0.17,0.26,0.35)				
T_2	(0.5, 1, 1.5)	(0, 0, 0)	(0.5, 1, 1.5)	(0.35,0.41,0.46)				
T_3	(0.5, 1, 1.5)	(-1.5, -1, -0.5)	(0, 0, 0)	(0.31,0.33,0.35)				

Table 8.5.26: Fuzzy POMs of the three candidates with respect to efficiency C_4 and their fuzzy weights (case 4)

	C_{41} (FAI=0.12)				C_{42} (FAI=0)			
	T_1	T_2	T_3	FW	T_1	T_2	T_3	FW
T_1	(0, 0, 0)	(0.5, 1, 1.5)	(0.5, 1, 1.5)	(0.31,0.41,0.5)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0.28,0.33,0.39)
T_2	(-1.5, -1, -0.5)	(0, 0, 0)	(-1.5, -1, -0.5)	(0.2,0.26,0.31)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0.28,0.33,0.39)
T_3	(-1.5, -1, -0.5)	(0.5, 1, 1.5)	(0, 0, 0)	(0.31,0.33,0.35)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0.28,0.33,0.39)
	C_{43} (FAI=0.129)							
T_1	(0, 0, 0)	(0, 0, 0)	(-1.5, -1, -0.5)	(0.21,0.29,0.36)				
T_2	(0, 0, 0)	(0, 0, 0)	(-2.5, -2, -1.5)	(0.2,0.27,0.34)				
T_3	(0.5, 1, 1.5)	(1.5, 2, 2.5)	(0, 0, 0)	(0.43,0.44,0.46)				

Table 8.5.27: Fuzzy POMs of the three candidates with respect to maintainability C_5 and their fuzzy weights (case 4)

	C_{51} (FAI=0.12)				C_{52} (FAI=0)			
	T_1	T_2	T_3	FW	T_1	T_2	T_3	FW
T_1	(0, 0, 0)	(0.5, 1, 1.5)	(0, 0, 0)	(0.29,0.36,0.42)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0.28,0.33,0.39)
T_2	(-1.5, -1, -0.5)	(0, 0, 0)	(0, 0, 0)	(0.27,0.31,0.36)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0.28,0.33,0.39)
T_3	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0.28,0.33,0.39)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0.28,0.33,0.39)
	C_{53} (FAI=0)				C_{54} (FAI=0.24)			
T_1	(0, 0, 0)	(-1.5, -1, -0.5)	(0, 0, 0)	(0.22,0.29,0.35)	(0, 0, 0)	(-1.5, -1, -0.5)	(0, 0, 0)	(0.21,0.28,0.34)
T_2	(0.5, 1, 1.5)	(0, 0, 0)	(1.5, 2, 2.5)	(0.39,0.44,0.5)	(0.5, 1, 1.5)	(0, 0, 0)	(2.5, 3, 3.5)	(0.43,0.48,0.54)
T_3	(0, 0, 0)	(-2.5, -2, -1.5)	(0, 0, 0)	(0.22,0.27,0.32)	(0, 0, 0)	(-3.5, -3, -2.5)	(0, 0, 0)	(0.2,0.24,0.29)
	C_{55} (FAI=0)							
T_1	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0.28,0.33,0.39)				
T_2	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0.28,0.33,0.39)				
T_3	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0.28,0.33,0.39)				

Table 8.5.28: Fuzzy POMs of the three candidates with respect to portability C_6 and their fuzzy weights (case 4)

	C_{61} (FAI=0.385)				C_{62} (FAI=0.385)			
	T_1	T_2	T_3	FW	T_1	T_2	T_3	FW
T_1	(0, 0, 0)	(-2.5, -2, -1.5)	(0, 0, 0)	(0.21,0.27,0.33)	(0, 0, 0)	(-1.5, -1, -0.5)	(0.5, 1, 1.5)	(0.28,0.34,0.41)
T_2	(1.5, 2, 2.5)	(0, 0, 0)	(0.5, 1, 1.5)	(0.39,0.44,0.5)	(0.5, 1, 1.5)	(0, 0, 0)	(-1.5, -1, -0.5)	(0.28,0.34,0.38)
T_3	(0, 0, 0)	(-1.5, -1, -0.5)	(0, 0, 0)	(0.24,0.29,0.33)	(-1.5, -1, -0.5)	(0.5, 1, 1.5)	(0, 0, 0)	(0.28,0.33,0.38)
	C_{63} (FAI=0.12)				C_{64} (FAI=0.376)			
T_1	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0.28,0.33,0.39)	(0, 0, 0)	(1.5, 2, 2.5)	(0, 0, 0)	(0.3,0.37,0.44)
T_2	(0, 0, 0)	(0, 0, 0)	(0.5, 1, 1.5)	(0.29,0.36,0.42)	(-2.5, -2, -1.5)	(0, 0, 0)	(0.5, 1, 1.5)	(0.27,0.32,0.37)
T_3	(0, 0, 0)	(-1.5, -1, -0.5)	(0, 0, 0)	(0.27,0.31,0.36)	(0, 0, 0)	(-1.5, -1, -0.5)	(0, 0, 0)	(0.26,0.31,0.37)
	C_{65} (FAI=0.24)							
T_1	(0, 0, 0)	(-1.5, -1, -0.5)	(0, 0, 0)	(0.21,0.28,0.34)				
T_2	(0.5, 1, 1.5)	(0, 0, 0)	(2.5, 3, 3.5)	(0.43,0.48,0.54)				
T_3	(0, 0, 0)	(-3.5, -3, -2.5)	(0, 0, 0)	(0.2,0.24,0.29)				

Table 8.5.29: Aggregation results for fuzzy weight of the final objective using FCNP (case 4)

	Fuzzy Importance	T_1	T_2	T_3
Criterion 1	(0.19,0.24,0.29)	(0.21,0.32,0.44)	(0.24,0.35,0.47)	(0.24,0.34,0.45)
Criterion 2	(0.15,0.19,0.24)	(0.25,0.37,0.52)	(0.24,0.35,0.47)	(0.21,0.28,0.37)
Criterion 3	(0.14,0.18,0.21)	(0.22,0.35,0.52)	(0.24,0.33,0.44)	(0.24,0.31,0.4)
Criterion 4	(0.13,0.15,0.17)	(0.21,0.34,0.49)	(0.18,0.28,0.4)	(0.3,0.38,0.47)
Criterion 5	(0.08,0.09,0.11)	(0.21,0.31,0.44)	(0.29,0.39,0.51)	(0.2,0.3,0.41)
Criterion 6	(0.13,0.15,0.15)	(0.21,0.32,0.44)	(0.29,0.4,0.52)	(0.2,0.29,0.4)
Final fuzzy weights		(0.18,0.34,0.56)	(0.2,0.35,0.55)	(0.19,0.32,0.49)

8.6 Case 5: Supplier optimization number- the FCCNP approach

The case background, solutions using the AHP approach and the CNP approach respectively, and discussion of the comparison are presented as follows.

8.6.1 Background of case 5

This case presents FCCNP as the evaluation solution to the problem of supplier number optimization. Berger et al (2004) proposed a supplier optimization model, which is the probability model, to select a suitable number N of suppliers, as follows:

$$N < 1 + \frac{\ln\left(\dfrac{B}{F(1-P)(1-S)}\right)}{\ln(S)} \tag{8.1}$$

P is "super-events," which cause many/all suppliers to be down; S is "unique events" that cause only a single supplier to be down, or an event uniquely associated with a particular supplier that puts it down during the supply cycle. F is the financial loss caused by disasters; B is the operating cost of working with multiple suppliers.

The problem with this model is that parameters (F, B, P, S) of Eq. 8.1 need to be evaluated, and it needs a complex administration process, including accounting activities and management group decision judgments, to achieve the knowledge. The following discusses how to use the Fuzzy Collective Network Process Approach (FCCNP) to address this parametric input problem.

8.6.2 The fuzzy collective cognitive network process (FCCNP) approach to case 5

The steps of the FCCNP are illustrated as follows.

a) Define Rating Scale Process

To evaluate the description of “Cost” for F and B, and the description of “probability” for P and S, the same schema of linguistic terms are applied. Thus $\overrightarrow{V_d}$ =[Below, Absolutely, Above], $\overrightarrow{V_h}$= [Little, Quite, Much], and $\overrightarrow{V_a}$ =[Low, Middle , High]. The fuzzy linguistic terms of F, B, P, and U are represented by a matrix as follows:

$$\begin{bmatrix} \varnothing & MB-M & MB-H \\ \varnothing & QB-M & QB-H \\ \varnothing & LB-M & LB-H \\ A-L & A-M & A-H \\ LA-L & LA-M & \varnothing \\ QA-L & QA-M & \varnothing \\ MA-L & MA-M & \varnothing \end{bmatrix}$$

The matrices of the fuzzy representation number of the compound linguistic terms of F, B, P, and S, are denoted as $\overline{X}_F$, $\overline{X}_B$, $\overline{X}_P$, $\overline{X}_S$. To find the fuzzy representation numbers, the settings of the parameters for algorithm 4.2 is shown in table 8.6.1. By using algorithm 4.2, the representation matrices in fuzzy numbers are shown in tables 8.6.2 to 8.6.4.

Table 8.6.1: Parameters for $\overline{X}_F$, $\overline{X}_B$, $\overline{X}_P$, $\overline{X}_S$, and $\overline{X}_w$ using algorithm 4.2 (case 5)

	$\overline{X}_F$	$\overline{X}_B$	$\overline{X}_P$	$\overline{X}_S$	$\overline{X}_w$
Min	1000	10	0	0	0
Max	10000	100	1	1	8
n	3	3	3	3	5
τ	1	1	1	1	2
γ	[1000,5500,10000]	[10,55,100]	[0,0.5,1]	[0,0.5,1]	[0,2,4,6,8]
d	4500	45	0.5	0.5	2
μ^{-1}	$PbMF^{-1}$				
$\varphi(\overrightarrow{V_h}), \lambda_0$	$\left[\varphi(little) \;\; \varphi(quite) \;\; \varphi(much)\right] = \left[1 \;\; 2 \;\; 3\right],\; \lambda_0 = 0.5$				

Table 8.6.2: The representation values for S and P, i.e. $\overline{X}_S$ and $\overline{X}_P$ (case 5)

$\overline{X}_S, \overline{X}_P$	L	A	H
MB	(0, 0, 0)	(0, 0.097, 0.194)	(0.5, 0.597, 0.694)
QB	(0, 0, 0)	(0.118, 0.237, 0.356)	(0.618, 0.737, 0.856)
LB	(0, 0, 0)	(0.272, 0.386, 0.5)	(0.772, 0.886, 1.)
A	(0, 0, 0)	(0.5, 0.5, 0.5)	(1., 1., 1.)
LA	(0, 0.114, 0.228)	(0.5, 0.614, 0.728)	(0, 0, 0)
QA	(0.144, 0.263, 0.382)	(0.644, 0.763, 0.882)	(0, 0, 0)
MA	(0.306, 0.403, 0.5)	(0.806, 0.903, 1.)	(0, 0, 0)

Table 8.6.3: The representation values for B, i.e. $\overline{X}_B$ (case 5)

$\overline{X}_B$	L	A	H
MB	(0, 0, 0)	(10., 18.7, 27.4)	(55., 63.7, 72.4)
QB	(0, 0, 0)	(20.6, 31.3, 42.)	(65.6, 76.3, 87.)
LB	(0, 0, 0)	(34.5, 44.7, 55.)	(79.5, 89.7, 100.)
A	(10., 10., 10.)	(55., 55., 55.)	(100., 100., 100.)
LA	(10., 20.3, 30.5)	(55., 65.3, 75.5)	(0, 0, 0)
QA	(23., 33.7, 44.4)	(68., 78.7, 89.4)	(0, 0, 0)
MA	(37.6, 46.3, 55.)	(82.6, 91.3, 100.)	(0, 0, 0)

Table 8.6.4: The matrix of the representation values for F, i.e. $\overline{X}_F$ (case 5)

$\overline{X}_F$	L	A	H
MB	(0, 0, 0)	(1000, 1872, 2744)	(5500, 6372, 7244)
QB	(0, 0, 0)	(2063, 3132, 4201)	(6563, 7632, 8701)
LB	(0, 0, 0)	(3446, 4473, 5500)	(7946, 8973, 10000)
A	(1000, 1000, 1000)	(5500, 5500, 5500)	(10000, 10000, 10000)
LA	(1000, 2027, 3054)	(5500, 6527, 7554)	(0, 0, 0)
QA	(2299, 3368, 4437)	(6799, 7868, 8937)	(0, 0, 0)
MA	(3756, 4628, 5500)	(8256, 9128, 10000)	(0, 0, 0)

For the weight measuring the criteria, the pairwise opposite matrix and cognitive prioritization operator are applied. The matrices of the representation values for the comparison interval scales $\overline{X}_w = \{\overline{X}_w^+, \overline{X}_w^-\}$ are shown in tables 8.6.5-8.6.6.

b) Assessment Process

The structural criteria are illustrated in Table 8.6.7. The structural criteria include four criteria of twelve sub-criteria. The fuzzy weights of the four criteria do not need to be determined as they do not fit the supplier optimization number function.

Three experts are involved in this evaluation project: e_1 is the product manager, e_2 is the supplier relationship manager, and e_3 is the marketing manager. Their fuzzy importance is derived by the matrix, which is determined by their senior, as follows:

$$\psi\left(\{e_k\}\right) = \begin{array}{c c} & \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} & \begin{bmatrix} AE & iLAMd & iLASl \\ LAMd & AE & LASl \\ LASl & iLASl & AE \end{bmatrix} \end{array}$$

Tables 8.6.8-8.6.9 show the cognitive pairwise comparisons of $\psi\left(cls\left(e_k, c_i, \{c_{ij}\}\right)\right)$, $\forall k \in \{1,2,3\}$ and $\forall i \in \{1,2,3,4\}$, which means a expert cluster k to evaluate a criterion i of sub-criterion j using cognitive pairwise comparisons. Table 8.6.10 includes the direct rating using CLOS. The details are discussed in the next step.

C) Information fusion process

Table 8.6.10 summarizes the results in evaluating F, B, P, S for a specific material. It shows the fuzzy weights of the experts $\{\widehat{we_k}\}$, fuzzy weights of the criteria with respect to the experts $\{\widehat{wc_{ijk}}\}$,and the rating scores for the criteria by the experts $\{\widehat{c_{ijk}}\}$. The fuzzy weighted arithmetic mean is taken for this aggregation. Finally the aggregation evaluation results of the four criteria are given. The results are further used in the decisional volition process.

d) Decisional volition Process

The parametric supplier number optimization function (Eq. 8.1) can be regarded as the decision volition function, stead of the typical aggregation operators such as the fuzzy weighted arithmetic mean. To calculate the optimum supplier number, the results of $\hat{C}_i$'s in table 8.6.10 are substituted by the parametric supplier number optimization function (Eq. 8.1). The fuzzy result is calculated as follows:

$$N_l < 1 + \frac{\ln\left(\dfrac{31.79}{2036(1-0.185)(1-0.259)}\right)}{\ln(0.259)} = 2.71$$

$$N_m < 1 + \frac{\ln\left(\dfrac{52.27}{3742(1-0.342)(1-0.443)}\right)}{\ln(0.443)} = 4.02$$

$$N^u < 1 + \frac{\ln\left(\dfrac{79.65}{6103(1-0.558)(1-0.693)}\right)}{\ln(0.693)} = 6.38$$

From the above calculations, it is suggested that **(2,4,6)** vendors be kept for this material A. If the up-boundary value 6 is taken as the result, someone may think 6 vendors is an excessive number. However, Chaudhry et al. (1993) have indicated that there are usually a maximum of 12 vendors for most practical cases. In this simulated case, when considering that the operation cost is (31.79,52.27,79.65), the disaster lost is (2036,3742,610), the super event is (0.185,0.342,0.558), and the risk/probability that the unique supplier will go to “down” is (0.259,0.443,0.693), so it is reasonable to use (2,4,6). Usually a modal value 4 is taken. Although this case is based solely on numerical data, with assumptions, the evaluation method definitely can describe and explain a real world situation in the similar way.

9.6.3 Discussion of case 5

This case illustrates how fuzzy collective cognitive network process functions as an evaluation platform for giving subjective data for group expert judgment to a parametric developed parametric function to estimate the number of suppliers.

In the previous four cases, the methods of relative measurement are shown. The limitation is that they only reflects the comparison score values, which usually belong to or are rescaled to [0,1]. For example, all candidates either have low scores or have equally high scores, but the sum of the relative scores is always one. If all candidates are below average performance, either the ANP or the CNP can only identify the best of the low performance candidates, but not how good the best one is.

This case illustrates the fuzzy weighted absolute measurement, which is fuzzy direct rating measurement associated with relative weights using fuzzy POMs and the fuzzy cognitive prioritization operator. In the selection problem, the evaluation values not only illustrate the best performance among candidates, but also the individual performance of each candidate. This can avoid selection from the low-performance candidates. In this case, the evaluation values from FCCNP are used for the operational parametric settings of the decision functions in the decisional volition process.

Table 8.6.5: The matrix of the representation values for the comparison interval scales $\overline{X}_w^+$ (case 5)

$\overline{X}_w^+$	Equally (E)	Slightly (Sl)	Moderately (Md)	Strongly (St)	Essentially (Es)
Much Below (MB)	Null	(0, 0.54, 1.08)	(2., 2.54, 3.08)	(4., 4.54, 5.08)	(6., 6.54, 7.08)
Quite Below (QB)	Null	(0.81, 1.2, 1.59)	(2.81, 3.2, 3.59)	(4.81, 5.2, 5.59)	(6.81, 7.2, 7.59)
Little Below (LB)	Null	(1.34, 1.67, 2.)	(3.34, 3.67, 4.)	(5.34, 5.67, 6.)	(7.34, 7.67, 8.)
Absolutely (A)	(0, 0, 0)	(1.67, 2., 2.33)	(3.67, 4., 4.33)	(5.67, 6., 6.33)	(7.67, 8., 8.)
Little Above (LA)	(0, 0.33, 0.66)	(2., 2.33, 2.66)	(4., 4.33, 4.66)	(6., 6.33, 6.66)	Null
Quite Above (QA)	(0.41, 0.8, 1.19)	(2.41, 2.80, 3.19)	(4.41, 4.8, 5.19)	(6.41, 6.8, 7.19)	Null
Much Above (MA)	(0.92, 1.46, 2.)	(2.92, 3.46, 4.)	(4.92, 5.46, 6.)	(6.92, 7.46, 8.)	Null

Table 8.6.6: The opposite of $\overline{X}_w^+$, i.e. $\overline{X}_w^-$ (case 5)

$\overline{X}_w^-$	Equally (E)	Slightly (Sl)	Moderately (Md)	Strongly (St)	Essentially (Es)
Much Below (MB)	Null	(1.08, -0.54, 0)	(-3.08, -2.54, -2.)	(-5.08, -4.54, -4.)	(-7.08, -6.54,-6)
Quite Below (QB)	Null	(-1.59, -1.2, -0.81)	(-3.59, -3.2, -2.81)	(-5.59, -5.2, -4.81)	(-7.59, -7.2,-6.81)
Little Below (LB)	Null	(-2., -1.67, -1.34)	(-4., -3.67, -3.34)	(-6., -5.67, -5.34)	(-8., -7.67,-7.34)
Absolutely (A)	(0, 0, 0)	(-2.33, -2., -1.67)	(-4.33, -4., -3.67)	(-6.33, -6., -5.67)	(-7.67, -8., -8.)
Little Above (LA)	(-0.66,-0.33,0)	(-2.66, -2.33,-2)	(-4.66, -4.33,-4.)	(-6.66, -6.33,-6.)	Null
Quite Above (QA)	(-1.19, -0.8, -0.41)	(-3.19, -2.80,-2.41)	(-5.19, -4.8, -4.41)	(-7.19, -6.8, -6.41)	(Null
Much Above (MA)	(-2., -1.46, -0.92)	(-4., -3.46,-2.92)	(-6., -5.46, -4.92)	(-8., -7.46, -6.92)	Null

Table 8.6.7: Structural criteria of case 5

Criteria	Sub-criteria	Criteria	Sub-criteria
F (C_1)	Impact on BOM? ($C_{1,1}$)	P(C_3)	Barriers to the Market? ($C_{3,1}$)
	Impact on business process? ($C_{1,2}$)		Market instability? ($C_{3,2}$)
	Impact on production Process? ($C_{1,3}$)		Uncertainty of General Economic? ($C_{3,3}$)
	Difficult to find alternatives? ($C_{1,4}$)		Policy Barrier? ($C_{3,4}$)
B(C_2)	Cost of monitoring the material market? ($C_{2,1}$)	S(C_4)	Competitive Relationship to Partner? ($C_{4,1}$)
	Cost of internal process? ($C_{2,2}$)		Unreliability of Partner's Financial Status? ($C_{4,2}$)
	Cost of relationship development? ($C_{2,3}$)		Unknown of the Brand Name? ($C_{4,3}$)
	Cost of external transaction process? ($C_{2,4}$)		Short period of Company History ($C_{4,3}$)

Table 8.6.8: $\psi\left(cls\left(e_k, c_i, \{c_{ij}\}\right)\right)$, $\forall k \in \{1,2,3\}$ and $\forall i \in \{1,2\}$ (case 5)

	e_1				e_2				e_3			
	c_{11}	c_{12}	c_{13}	c_{14}	c_{11}	c_{12}	c_{13}	c_{14}	c_{11}	c_{12}	c_{13}	c_{14}
c_{11}	AE	LASl	iLAMd	iLBSl	AE	iLAE	AE	iLBSt	AE	iLAMd	AE	iQASl
c_{12}	iLASl	AE	iQBSt	iLBMd	LAE	AE	LBSl	iLASt	LAMd	AE	LBMd	LBSl
c_{13}	LAMd	QBSt	AE	LASl	AE	iLBSl	AE	iQASt	AE	iLBMd	AE	iLBMd
c_{14}	LBSl	LBMd	iLASl	AE	LBst	LASt	QASt	AE	QASl	iLBSl	LBMd	AE
	FAI=0.054				FAI=0.079				FAI=0.064			
	c_{21}	c_{22}	c_{23}	c_{24}	c_{21}	c_{22}	c_{23}	c_{24}	c_{21}	c_{22}	c_{23}	c_{24}
c_{21}	AE	iMASl	LASl	LAE	AE	QAE	iMBMd	LASl	AE	QAE	iQASl	LASl
c_{22}	MASl	AE	LAMd	LBMd	iQAE	AE	iQASl	LBSl	iQAE	AE	iQASl	LBSl
c_{23}	iLASl	iLAMd	AE	iLBSl	iMBMd	QASl	AE	QBSt	iQASl	QASl	AE	QBSt
c_{24}	iLAE	iLBMd	LBSl	AE	iLASl	iLBSl	iQBSt	AE	iLASl	iLBSl	iQBSt	AE
	FAI=0.058				FAI=0.073				FAI=0.080			

Table 8.6.9: $\psi\left(cls\left(e_k, c_i, \left\{c_{ij}\right\}\right)\right)$, $\forall k \in \{1,2,3\}$ and $\forall i \in \{3,4\}$ (case 5)

	e_1				e_2				e_3			
	c_{31}	c_{32}	c_{33}	c_{34}	c_{31}	c_{32}	c_{33}	c_{34}	c_{31}	c_{32}	c_{33}	c_{34}
c_{31}	AE	AE	AE	iLAE	AE	LAE	LAE	LAE	AE	AE	LAE	LAE
c_{22}	AE	AE	AE	iLAE	iLAE	AE	AE	AE	AE	AE	LAE	LAE
c_{23}	AE	AE	AE	iLAE	iLAE	AE	AE	AE	iLAE	iLAE	AE	AE
c_{24}	LAE	LAE	LAE	AE	iLAE	AE	AE	AE	iLAE	iLAE	AE	AE
	FAI=0				FAI=0				FAI=0			
	c_{41}	c_{42}	c_{43}	c_{44}	c_{41}	c_{42}	c_{43}	c_{44}	c_{41}	c_{42}	c_{43}	c_{44}
c_{41}	AE	AE	QASl	MASl	AE	AE	ASl	QASl	AE	ASl	AMd	ASl
c_{42}	AE	AE	QASl	MASl	AE	AE	ASl	QASl	iASl	AE	LASl	LAE
c_{43}	iQASl	iQASl	AE	iQAE	iASl	iASl	AE	iLAE	iAMd	iLASl	AE	iMAE
c_{44}	iMASl	iMASl	QAE	AE	iQASl	iQASl	LAE	AE	iASl	iLAE	MAE	AE
	FAI=0.062				FAI=0.047				FAI=0			

Table 8.6.10: Evaluation result in fuzzy numbers for parameters of supplier number optimization function (case 5)

Criteria		$\widehat{we}_1 = (0.190,0.241,0.291)$		$\widehat{we}_2 = (0.384,0.426,0.468)$		$\widehat{we}_3 = (0.300,0.333,0.366)$		$\widehat{C}_i$
		$\widehat{wc}_{ij1}$	$\widehat{c}_{ij1}$	$\widehat{wc}_{ij2}$	$\widehat{c}_{ij2}$	$\widehat{wc}_{ij3}$	$\widehat{c}_{ij3}$	
F	$C_{1,1}$	(0.182, 0.221, 0.26)	(6799, 7868, 8937)	(0.163, 0.198, 0.237)	(3446, 4473, 5500)	(0.158, 0.194, 0.229)	(5500, 6527, 7554)	(2036,3742,6102)
	$C_{1,2}$	(0.204, 0.244, 0.284)	(2063, 3132, 4201)	(0.214, 0.255, 0.296)	(2299, 3368, 4437)	(0.213, 0.253, 0.292)	(3446, 4473, 5500)	
	$C_{1,3}$	(0.214, 0.247, 0.281)	(1000, 2027, 3054)	(0.219, 0.258, 0.297)	(2063, 3132, 4201)	(0.219, 0.255, 0.292)	(2063, 3132, 4201)	
	$C_{1,4}$	(0.259, 0.299, 0.337)	(1000, 2027, 3054)	(0.251, 0.288, 0.324)	(2063, 3132, 4201)	(0.274, 0.312, 0.351)	(2299, 3368, 4437)	
B	$C_{2,1}$	(0.128, 0.162, 0.197)	(55., 65.3, 75.5)	(0.182, 0.216, 0.246)	(65.6, 76.3, 87.)	(0.292, 0.326, 0.359)	(68., 78.7, 89.4)	(31.79,52.27,79.65)
	$C_{2,2}$	(0.307, 0.34, 0.372)	(34.5, 44.7, 55.)	(0.201, 0.235, 0.269)	(23., 33.7, 44.4)	(0.201, 0.235, 0.269)	(20.6, 31.3, 42.)	
	$C_{2,3}$	(0.214, 0.247, 0.281)	(23., 33.7, 44.4)	(0.219, 0.247, 0.276)	(20.6, 31.3, 42.)	(0.219, 0.255, 0.292)	(34.5, 44.7, 55.)	
	$C_{2,4}$	(0.258, 0.299, 0.332)	(55., 65.3, 75.5)	(0.251, 0.288, 0.324)	(55., 63.7, 72.4)	(0.221, 0.255, 0.289)	(68., 78.7, 89.4)	
P	$C_{3,1}$	(0.314, 0.343, 0.371)	(0.5, 0.614, 0.728)	(0.156, 0.189, 0.223)	(0.272, 0.386, 0.5)	(0.16, 0.193, 0.226)	(0.5, 0.5, 0.5)	(0.185,0.342,0.558)
	$C_{3,2}$	(0.156, 0.185, 0.214)	(0.118, 0.237, 0.356)	(0.305, 0.332, 0.359)	(0.272, 0.386, 0.5)	(0.306, 0.334, 0.363)	(0.5, 0.614, 0.728)	
	$C_{3,3}$	(0.214, 0.247, 0.281)	(0.5, 0.5, 0.5)	(0.219, 0.247, 0.276)	(0.144, 0.263, 0.382)	(0.219, 0.245, 0.271)	(0.118, 0.237, 0.356)	
	$C_{3,4}$	(0.172, 0.2, 0.228)	(0, 0.114, 0.228)	(0.183, 0.212, 0.241)	(0, 0.097, 0.194)	(0.159, 0.189, 0.219)	(0.118, 0.237, 0.356)	
S	$C_{4,1}$	(0.25, 0.274, 0.297)	(0.5, 0.614, 0.728)	(0.374, 0.397, 0.42)	(0.272, 0.386, 0.5)	(0.264, 0.288, 0.311)	(0.5, 0.5, 0.5)	(0.259,0.443,0.693)
	$C_{4,2}$	(0.207, 0.232, 0.255)	(0.618, 0.737, 0.856)	(0.155, 0.178, 0.201)	(0.644, 0.763, 0.882)	(0.155, 0.178, 0.201)	(0.618, 0.737, 0.856)	
	$C_{4,3}$	(0.234, 0.258, 0.281)	(0.272, 0.386, 0.5)	(0.219, 0.247, 0.276)	(0.118, 0.237, 0.356)	(0.219, 0.245, 0.271)	(0.272, 0.386, 0.5)	
	$C_{4,4}$	(0.187, 0.202, 0.228)	(0.118, 0.237, 0.356)	(0.19, 0.213, 0.235)	(0.272, 0.386, 0.5)	(0.221, 0.243, 0.265)	(0.5, 0.614, 0.728)	

Chapter 9 Conclusions

9.1 Summary of research work

Multi-criteria decision aiding models have been increasingly studied. Among these models, the Analytic Hierarchy Process is a popular model which has been widely studied and applied for thirty years. However, there is a misrepresentation in using the pairwise reciprocal matrix to derive the priority, as the ratio scale for cognition of the linguistic term is ill-defined (chapter 5.2). In addition, the numerical representation of the linguistic rating scale is insufficient, and the prioritization methods are still uncertain.

To address the above problems, the Cognitive Network Process, which is the symbolic mathematical system using process algebra representation, is proposed in Chapter 3. The CNP is the architecture for the interaction between the system and man. The cognitive architecture of the CNP comprises five processes: the Problem Cognition Process (PGP), the Cognitive Assessment Process (CAP), the Cognitive Prioritization Process (CPP), the Multiple Information Fusion Process (MIP), and the Decisional Volition Process (DVP). Each process is represented by a set of algorithms. Perception is performed by humans in the Human Cognition Process, including PGP and CAP. Computation is performed in CPP, MIP and DVP.

In the PGP, the decision problem is constructed as the measurable Structural

Assessment Network (SAN) model. The graphical representation of the SAN is similar to ANP. However, the algebraic representation is different. The CNP is based on clusters which are summed to the SAN. A cluster is defined as a node which contains a set of granular data. The node can be a criterion whilst the granular data can be the sub-criteria or attributes. The node could also be an expert, whilst granular data can be the attributes of the expert. Although the syntactic forms of the measurement scale schema are the same or similar, the semantic forms are different. CNP is based on the interval scale while ANP is based on the ratio scale. The cognitive scale uses the Compound Linguistic Ordinal Scale (CLOS) (chapter 4). In fact, the analytic scale (or ratio scale) of AHP/ANP also applies CLOS. CLOS are applied for the comparison of CNP and ANP in case 3 of chapter 8. One of the conclusions of this comparison is that CLOS can help both CNP and the improved ANP to approximate the real results.

In the CAP, the clusters of SAN are assessed by experts using pairwise opposite matrices. The expert can be regarded as a cognitive assessment function in the system. In this research, the expert is a human. Maybe in the future, the cognitive assessment function can be performed by a machine or through algorithms. The results of the evaluated clusters are measured by the accordance check function. If the accordance index (AI) is larger than 0.1, the cluster need to be re-assessed again.

In the CPP, the evaluated clusters of SAN, or the POMs, are derived for the individual utility set (or priority set) by the cognitive prioritization operator (chapter 5). The individual utility set can be either normalized or un-normalized. For the determination of the weights or relative measurement of CNP, normalization of the individual utility set is recommended. Five CPOs are proposed: Row Average plus normal Utility (RAU), Aggregation of Solutions of Linear Systems (ASLS) which includes Arithmetic Mean of Solutions of Linear Systems (AMSLS), Primitive Least Squares (PLS) optimization, Bounded Least Squares (BLS) Optimization and Least Penalty Squares (LPS) Optimization. The closed form solution of AMSLS and PLS is RAU. Six Cognitive Prioritization Operator Measurement (CPOMs) Models are also proposed to compare the CPOs: Worst Absolute Distance Variance (WADV), Mean Absolute Distance Variance (MADV), Mean Penalty Weighted Absolute Distance Variance (MPWADV), Root Mean Square Variance (RMSV), Mean Contradiction (MC) and Root Mean Penalty Weighted Square Variance (PMPWSV). The LPS is recommended when the mean contraction value is larger than one. If the contraction value is zero and the AI is not larger than 0.1, RAU is recommended as LPS and RAU also produce the same result, but the computation of RAU is much simpler than LPS.

In the MIP, the utility values of the evaluated clusters of the SAN are aggregated to an overall result set of the decision objective by the most appropriate aggregation operator. By default, the aggregation operator is the weighted arithmetic mean. If the cognitive style of the decision attitude is considered, the Cognitive Style and Aggregation Operator (CSAO) model (chapter 6) is applied. The CSAO model analyzes the mapping relationship between aggregation operators and decision attitudes. The POs are reviewed in chapter 2.3.5 whilst the cognitive style is reviewed in chapter 2.7.4.

In the DVP, the overall result set from the MIP is evaluated by the volition function. Usually, the function is used for selection though ranking (or ordering). Cases 1 to 4 of chapter 8 present the problems of selection in different areas. One can use a parametric function as the volition function to produce a result from a continuous scale. Case 5 of Chapter 8 shows the application of this usability.

The CNP family embraces the primitive CNP (or CNP in short), and the extent CNPs including Collective CNP, Fuzzy CNP, and Fuzzy CCNP. The primitive CNP is of the SAN of decision makers with crisp inputs. The CCNP is of the SAN of multiple decision makers with crisp inputs, whilst the FCNP is of the SAN of a single decision maker with fuzzy inputs. The FCCNP is of the SAN of multiple decision makers with fuzzy inputs. The Fuzzy Cognitive Prioritization Operator (FCPO) and its measurement models are also proposed and evaluated in chapter 7.

For the numerical evaluation of the components of CNP, the CLOS is tested and discussed in chapter 4.5, the CPOs and CPOMs are tested and discussed in chapter 5.8, the CSAO is tested and discussed in chapter 6.5, and the FCPOs and FCPOMs are tested and discussed in chapter 7.6.

For the evaluations of the applications of CNP, five cases are presented in chapter 8 with comparisons or complementation. Case 1 presents the high school selection (Saaty, 1980, p26-28) with comparisons of primitive CNP and AHP models. Case 2 presents the transportation company selection problem (Kulak and Kahramna, 2005) with comparisons of primitive CNP and AHP, and both prioritization measurement models are also used. Case 3 compares the CNP and the improved ANP models for the R&D project selection problem (Yuen and Lau, 2009). Case 4 compares the fuzzy CNP and Fuzzy AHP models for the software product selection problem (Yuen and Lau, 2008c). Case 5 illustrates the use of the fuzzy collective CNP model as the evaluation model for the problem of supplier number optimization (Berger et al, 2004). The significances of the comparisons and complementation are also discussed in chapter 8.

9.2 Contributions of this research

The contributions of this research can be classified in five categories.

9.2.1 The specific contributions of the primitive CNP

If a decision maker would like to compare at least two alternatives using at least two criteria without sufficient operational data on hand, the CNP is the ideal method. At least it is shown that the CNP performs better than the ANP in various aspects: such as rating scale, pairwise comparison method, aggregation method, and cognitive process.

The cognitive architecture of CNP, which consists of a list of algorithms, is developed. The presentation of the cognitive architecture can be implemented by various programming languages, and finally can be developed as commercial products like Expert Choice, which is based on ANP. The new programs may have marketing values as this research shows various advantages of CNP over ANP.

The cognitive decision process of CNP streamlines the business decision making activities. Each process of CNP is well-defined. The criteria can be organized as structural criteria which are decomposed as criterion clusters. The criterion clusters can be devaluated by cognitive pairwise comparison. The next step is to let the computer to do the analysis with well-developed and well-tested algorithms.

9.2.2 The specific contributions of the extent CNPs

The contributions of extent CNPs includes the contributions of the primitive CNP, and pluses one advantage, which supports decision attitudes of decision makers: e.g. optimists, neutralists, or pessimists. Decision attitudes can be determined by either CSAO, fuzzy inputs, or the both.

If only CSAO is considered, a list of aggregation operators from the decision makers' preferences is needed. CSAO-1 supports the one dimensional space of decision attitudes. CSAO-2 supports two dimensional spaces of decision attitudes, which are described by CLOS.

If only fuzzy inputs are defined, the CLOS for the cognitive scale should be fuzzy-type. The low-bound of the fuzzy output represents the pessimist, the modal value of the fuzzy output represents the neutralist, and the up-bound of the fuzzy output represents the optimist. It is potential to use CLOS to further the attitudes in two dimensional spaces. However, this is for future research.

If both CSAO and fuzzy inputs are considered, the cognitive scale should be fuzzy numbers and a list of the aggregation operators is needed.

9.2.3 The specific contributions of the CLOS

The Compound Linguistic Ordinal Scale (CLOS) model is a promising alternative for the classic rating scale models, such as Likert-like scales. Miller (1956) has indicated that an expert could manage a set with (7 ± 2) terms. Many rating scales include Likert-like scales and the choice of the fuzzy linguistic terms use this principle. By a breakthrough of this principle, CLOS can provide $(7 \pm 2)((7 \pm 2) - 1) + 1 = [21, 73]$ options which seem incredible for an expert to handle. Unlike the classical rating model which is a single step rating process, CLOS uses a Deductive Rating Strategy (algorithm

4.3) in which a rater chooses a 2-tuple option (v_{hd}, v_a) in two steps with a rethink process. The CLOS is an ideal rating interface for addressing the problem of the rating dilemma.

The proposed model also accurately reflects the raters' consistency and inconsistency, thus improving the quality of the assessment due to high validity and reliability of the subjective measurement results.

The output of CLOS can be fuzzy or crisp. As the crisp number is only the modal value of the fuzzy number, thus it can be used for both fuzzy and crisp decision making models.

9.2.4 The specific contributions of the POM and CPO

The Pairwise Opposite Matrix (POM) of CNP is the ideal alternative of the Pairwise Reciprocal Matrix (PRM) of AHP, as the solution of the PRM's problem is tested and presented in chapter 5. The Cognitive Prioritization Operator (CPO) is the function to derive the individual utility set from the POM.

The CPO is of high validity with various reasons. Firstly, the CPOs are well-developed. A number of definitions, propositions and theorems with proofs are presented in chapter 5. Secondly, the CPOs are well-tested. Six Cognitive Prioritization Operator Measurement (CPOM) Models are proposed for the potential improvement development of CPOs. In AHP, there are several APOs proposed after Saaty's EigenValue method. Many authors claimed their methods were superior to the Saaty's, and the details are reviewed in chapter 2.5. The author keeps an open mind, and proposes five CPOs evaluated by CPOMs (Chapter 5). And finally, the Row Average plus normal Utility (RAU) and the Least Penalty Squares (LPS) are suggested. In the numerical analysis and applications, it can be observed that their values are very close or the same, if the accordance index is not larger than 0.1.

9.2.5 The specific contributions of the CSAO

Although the discussions of AOs are very broad, there is a lack of research on the best practice in choosing the aggregation operators. The selection of the AOs can apply the theory of cognitive style. However, no research has yet investigated the relationship between aggregation operators and the cognitive styles. The cognitive styles can be used to select the best PO for decision making.

The CSAO is devoted to a proposal on how to map a collection of aggregation operators into a collection of cognitive styles (or decision attitudes) by the CSAO model. This model is typically useful for those unsolved issues in the selection of aggregation operators. The OA candidates are determined by the decision maker with respect to cognitive style, which is characterized by decision attitudes.

9.3 Future work

The future motivations of this research can be classified in four categories.

9.3.1 The specific motivations of the CNP

The CNP can handle all the problems which the AHP/ANP can handle. For the future applications of this model, the CNP can be applied in many domains such as transportation management, psychometrics, social sciences, business research, decision analyses, computer sciences, material management, and engineering management. For example, Chapter 8 shows five cases of the applications. In addition, the extent functions of the CNP including collective aspect and fuzzy soft-computing aspect can apply to the decision application models.

9.3.2 The specific motivations of the CLOS

The future applications of the Compound Linguistic Ordinal Scale model are that CLOS can be used as the measurement instrument applied to large scale systems, surveys and questionnaire designs, psychometrics, rater statistics, quantitative research (e.g. example 4.3), decision attitudes (e.g. CSAO), and multi-criteria multi-expert decision problems (e.g. AHP/ANP in case 4 of chapter 8) in various fields such as engineering sciences and social sciences, by using the deductive rating strategy of the breakthrough number of linguistic choices.

In addition, the Fuzzy normal distribution (FND) (algorithm 4.2) is the most significant for the cognitive computation of CLOS. FND addresses the high motivations for future development for classifying the interval and modal value of either the fuzzy set, compound fuzzy set, or even the type-II fuzzy set.

9.3.3 The specific motivations of the POM and CPO

One limitation of the cognitive pairwise comparison is the out-boundary problem, which nevertheless is less trivial than the out-boundary problem of analytic pairwise comparison. Some algorithms will be proposed for the improvement of the judgment of the POM. The value of normal utility κ will be studied in details for various scenarios due to the different perception of the paired difference.

The mathematical transformation between analytic pairwise comparison and cognitive pairwise comparison will be investigated. Thus the applications with analytic pairwise comparisons can be revised without further re-evaluations.

Apart from the five CPOs proposed in chapter 5, some better POs, which the author has not yet observed, may be proposed in the future research. Thus the CPOMs are the foundation to test the new comers' validity. If the new comers produce a better result than LPS using the proposed CPOMs, they would also be recommended, although the author cannot find any at the moment.

The other future implication is the usability of an individual utility set derived from POM by the dedicated CPO. The utility set can be normalized as normalized weights, which can be used for the operational parameter settings of other decision models such as SMART, generalized means, ELECTRE, PROMETHEE, DEA and TOPSIS (chapter 2.2). For example, Yuen (2009e) shows the case applying the POMs and CPO to TOPSIS whilst Yuen (2009f) demonstrates the case applying to ELECTRE I. In the future study, more decision models can be enhanced using the POMs and CPO.

9.3.4 The specific motivations of the CSAO

For future implications, the Cognitive Style and Aggregation Operator (CSAO) model is useful for the selection of the aggregation operator in the decision making models, such as AHP, CNP, ELECTRE, PROMETHEE, DEA and TOPSIS, with consideration of the cognitive styles (or decision attitudes) of the decision makers. Thus the CSAO model can apply to many decision applications.

9.3.5 Implementation Plan of the CNP

In view of the managerial implementation, the roadmap for using the CNP will be developed in the future study. With respect to the implementations of the existing cases, more cases to apply the CNP to a wider variety of situations will be studied.

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Appendices

Contents

Appendix I Tables for figures 4.5-4.11 of chapter 4

Table A4.1: Computational results for fig. 4.5

Parameters: $X=[1,5]$, $\gamma_{\alpha^3}=3$, $d_{\alpha^3}=1$, $\tau_{\alpha^3}=1$, $\mu_{\alpha^{j\phi}}{}^{-1}=PbMF^{-1}$, $\varphi(\overrightarrow{V_h})=[1,2,3]$, $0\le\lambda_0\le1$.

λ_0	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\overline{X}_{\alpha_{1,3}}$	2.1464	2.1554	2.1646	2.1740	2.1838	2.1938	2.2042	2.2150	2.2261	2.2378	2.2500
$\overline{X}_{\alpha_{2,3}}$	2.4423	2.4470	2.4523	2.4584	2.4655	2.4738	2.4836	2.4956	2.5108	2.5323	2.5918
$\overline{X}_{\alpha_{3,3}}$	2.7959	2.7908	2.7859	2.7811	2.7764	2.7718	2.7673	2.7628	2.7585	2.7542	2.7500
$\overline{X}_{\alpha_{4,3}}$	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000
$\overline{X}_{\alpha_{5,3}}$	3.2041	3.2092	3.2141	3.2189	3.2236	3.2282	3.2327	3.2372	3.2415	3.2458	3.2500
$\overline{X}_{\alpha_{6,3}}$	3.5577	3.5530	3.5477	3.5416	3.5345	3.5262	3.5164	3.5044	3.4892	3.4677	3.4082
$\overline{X}_{\alpha_{7,3}}$	3.8536	3.8446	3.8354	3.8260	3.8162	3.8062	3.7958	3.7850	3.7739	3.7622	3.7500

Table A4.2: Computational results for fig. 4.6.

Parameters: $X=[1,5]$, $\gamma_{\alpha^3}=3$, $d_{\alpha^3}=1$, $\tau_{\alpha^3}=1$, $\mu_{\alpha^{j\phi}}{}^{-1}=PbMF^{-1}$, $\overline{\varphi}(\overrightarrow{V_h})=\{[1,2,3],[1,1,1],[3,2,1]\}$, $\lambda_0=0.5$.

φ	α_{13}	α_{23}	α_{33}	α_{43}	α_{53}	α_{63}	α_{73}
$\varphi_1:1,2,3$	2.1938	2.4738	2.7718	3.0000	3.2282	3.5262	3.8062
$\varphi_2:1,1,1$	2.1181	2.3170	2.6773	3.0000	3.3227	3.6830	3.8819
$\varphi_3:3,2,1$	2.0551	2.1985	2.6047	3.0000	3.3953	3.8015	3.9449

Table A4.3: Computational results for fig. 4.7.

Parameters: $X=[1,5]$, $\gamma_{\alpha^3}=3$, $d_{\alpha^3}=1$, $\tau_{\alpha^3}=0.1,\ldots,1$, $\mu_{\alpha^{j\phi}}{}^{-1}=PbMF^{-1}$,

$\varphi\left(\overrightarrow{V_h}\right)=[1,2,3]$, $\lambda_0=0.5$.

τ_{α^3}	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\overline{X}_{\alpha_{1,3}}$	2.0023	2.0244	2.0552	2.0843	2.1097	2.1315	2.1503	2.1667	2.1811	2.1938
$\overline{X}_{\alpha_{2,3}}$	2.1189	2.2062	2.2628	2.3078	2.3456	2.3782	2.4067	2.4317	2.4539	2.4738
$\overline{X}_{\alpha_{3,3}}$	2.5248	2.5850	2.6322	2.6675	2.6945	2.7161	2.7337	2.7484	2.7609	2.7718
$\overline{X}_{\alpha_{4,3}}$	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000
$\overline{X}_{\alpha_{5,3}}$	3.4752	3.4150	3.3678	3.3325	3.3055	3.2839	3.2663	3.2516	3.2391	3.2282
$\overline{X}_{\alpha_{6,3}}$	3.8811	3.7938	3.7372	3.6922	3.6544	3.6218	3.5933	3.5683	3.5461	3.5262
$\overline{X}_{\alpha_{7,3}}$	3.9977	3.9756	3.9448	3.9157	3.8903	3.8685	3.8497	3.8333	3.8189	3.8062

Table A4: Computational Results for fig. 4.8.

Parameters: $X=[1,5]$, $\gamma_{\alpha^3}=3$, $d_{\alpha^3}=1$, $\tau_{\alpha^3}=1,\ldots,10$, $\mu_{\alpha^{j\phi}}{}^{-1}=PbMF^{-1}$,

$\varphi\left(\overrightarrow{V_h}\right)=[1,2,3]$, $\lambda_0=0.5$.

τ_{α^3}	1	2	3	4	5	6	7	8	9	10	10000
$\overline{X}_{\alpha_{1,3}}$	2.1938	2.2712	2.3096	2.3335	2.3502	2.3628	2.3726	2.3806	2.3872	2.3929	2.4966
$\overline{X}_{\alpha_{2,3}}$	2.4738	2.5991	2.6639	2.7050	2.7339	2.7558	2.7730	2.7870	2.7987	2.8087	2.9938
$\overline{X}_{\alpha_{3,3}}$	2.7718	2.8340	2.8631	2.8809	2.8932	2.9023	2.9094	2.9152	2.9200	2.9240	2.9976
$\overline{X}_{\alpha_{4,3}}$	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000
$\overline{X}_{\alpha_{5,3}}$	3.2282	3.1660	3.1369	3.1191	3.1068	3.0977	3.0906	3.0848	3.0800	3.0760	3.0024
$\overline{X}_{\alpha_{6,3}}$	3.5262	3.4009	3.3361	3.2950	3.2661	3.2442	3.2270	3.2130	3.2013	3.1913	3.0062
$\overline{X}_{\alpha_{7,3}}$	3.8062	3.7288	3.6904	3.6665	3.6498	3.6372	3.6274	3.6194	3.6128	3.6071	3.5034

Table A4.5: Computational results for fig. 4.9.

Parameters: $X=[1,5]$, $\gamma_{\alpha^3}=3$, $d_{\alpha^3}=1$, $\tau_{\alpha^3}=0.1,\ldots,1$, $\mu_{\alpha^{j\phi}}{}^{-1}=TbMF^{-1}$,

$\varphi(\overrightarrow{V_h})=[1,2,3]$, $\lambda_0=0.5$.

τ_{α^3}	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\overline{X}_{\alpha_{1,3}}$	2.0045	2.0477	2.1044	2.1544	2.1953	2.2284	2.2555	2.2779	2.2966	2.3125
$\overline{X}_{\alpha_{2,3}}$	2.2095	2.3299	2.4011	2.4583	2.5069	2.5487	2.5847	2.6159	2.6429	2.6667
$\overline{X}_{\alpha_{3,3}}$	2.5483	2.6555	2.7295	2.7788	2.8134	2.8387	2.8581	2.8734	2.8857	2.8958
$\overline{X}_{\alpha_{4,3}}$	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000
$\overline{X}_{\alpha_{5,3}}$	3.4517	3.3445	3.2705	3.2212	3.1866	3.1613	3.1419	3.1266	3.1143	3.1042
$\overline{X}_{\alpha_{6,3}}$	3.7905	3.6701	3.5989	3.5417	3.4931	3.4513	3.4153	3.3841	3.3571	3.3333
$\overline{X}_{\alpha_{7,3}}$	3.9955	3.9523	3.8956	3.8456	3.8047	3.7716	3.7445	3.7221	3.7034	3.6875

Table A4.6: Computational Results for fig. 4.10.

Parameters : $X=[1,5]$, $\gamma_{\alpha^3}=3$, $d_{\alpha^3}=1$, $\tau_{\alpha^3}=1,\ldots,10$, $\mu_{\alpha^{j\phi}}{}^{-1}=TbMF^{-1}$,

$\varphi(\overrightarrow{V_h})=[1,2,3]$, $\lambda_0=0.5$.

τ_{α^3}	1	2	3	4	5	6	7	8	9	10	10000
$\overline{X}_{\alpha_{1,3}}$	2.3125	2.3953	2.4275	2.4446	2.4551	2.4623	2.4675	2.4715	2.4746	2.4770	2.5000
$\overline{X}_{\alpha_{2,3}}$	2.6667	2.8015	2.8592	2.8910	2.9111	2.9249	2.9350	2.9428	2.9488	2.9538	3.0000
$\overline{X}_{\alpha_{3,3}}$	2.8958	2.9449	2.9625	2.9716	2.9772	2.9809	2.9836	2.9856	2.9872	2.9885	3.0000
$\overline{X}_{\alpha_{4,3}}$	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000
$\overline{X}_{\alpha_{5,3}}$	3.1042	3.0551	3.0375	3.0284	3.0228	3.0191	3.0164	3.0144	3.0128	3.0115	3.0000
$\overline{X}_{\alpha_{6,3}}$	3.3333	3.1985	3.1408	3.1090	3.0889	3.0751	3.0650	3.0572	3.0512	3.0462	3.0000
$\overline{X}_{\alpha_{7,3}}$	3.6875	3.6047	3.5725	3.5554	3.5449	3.5377	3.5325	3.5285	3.5254	3.5230	3.5000

Table A7: Computational Results for fig.11.

Parameters: $X = [1,5]$, $\gamma_{\alpha^3} = 3,4$, $d_{\alpha^3} = 1$, $\tau_{\alpha^3} = 2$, ${\mu_{\alpha^{j\phi}}}^{-1} = PbMF^{-1}$,

$\varphi\left(\overrightarrow{V_h}\right) = [1,2,3]$, $\lambda_0 = 0.5$ for calculation of σ.

	$\overline{X}_{\alpha_{1,4}}$	$\overline{X}_{\alpha_{2,4}}$	$\overline{X}_{\alpha_{3,4}}$	$\overline{X}_{\alpha_{4,4}}$	$\overline{X}_{\alpha_{5,4}}$	$\overline{X}_{\alpha_{6,4}}$	$\overline{X}_{\alpha_{7,4}}$
$\overline{X}_{\alpha_{1,3}}$	0.5000	0.6640	0.7814	0.8644	0.9474	1.0648	1.2288
$\overline{X}_{\alpha_{2,3}}$	0.3360	0.5000	0.6174	0.7004	0.7834	0.9009	1.0648
$\overline{X}_{\alpha_{3,3}}$	0.2186	0.3826	0.5000	0.5830	0.6660	0.7834	0.9474
$\overline{X}_{\alpha_{4,3}}$	0.1356	0.2996	0.4170	0.5000	0.5830	0.7004	0.8644
$\overline{X}_{\alpha_{5,3}}$	0.0526	0.2166	0.3340	0.4170	0.5000	0.6174	0.7814
$\overline{X}_{\alpha_{6,3}}$	0.0648	0.0991	0.2166	0.2996	0.3826	0.5000	0.6640
$\overline{X}_{\alpha_{7,3}}$	0.2288	0.0648	0.0526	0.1356	0.2186	0.3360	0.5000

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Appendix II

Tables for plotting the figures 5.8-5.12 of chapter 5.8.1

Table A5.1: The priority vectors of RAU and LPS for *T3(r)*.

B13	AI	V (RAU)	V(LPS)
-1	0.462	(0.733, 0.933, 1.333)	(0.917, 0.917, 1.167)
-0.9	0.423	(0.767, 0.933, 1.3)	(0.925, 0.925, 1.150)
-0.8	0.385	(0.8, 0.933, 1.267)	(0.933, 0.933, 1.133)
-0.7	0.346	(0.833, 0.933, 1.233)	(0.942, 0.942, 1.117)
-0.6	0.308	(0.867, 0.933, 1.2)	(0.950, 0.950, 1.1)
-0.5	0.269	(0.9, 0.933, 1.167)	(0.958, 0.958, 1.083)
-0.4	0.231	(0.933, 0.933, 1.133)	(0.967, 0.967, 1.067)
-0.3	0.192	(0.967, 0.933, 1.1)	(0.975, 0.975, 1.05)
-0.2	0.154	(1., 0.933, 1.067)	(1., 0.971, 1.029)
-0.1	0.115	(1.033, 0.933, 1.033)	(1.017, 0.967, 1.017)
0	0.077	(1.067, 0.933, 1.)	(1.04, 0.96, 1.)
0.1	0.038	(1.1, 0.933, 0.967)	(1.1, 0.943, 0.957)
0.2	0	(1.133, 0.933, 0.933)	(1.133, 0.933, 0.933)
0.3	0.038	(1.167, 0.933, 0.9)	(1.167, 0.924, 0.91)
0.4	0.077	(1.2, 0.933, 0.867)	(1.2, 0.914, 0.886)
0.5	0.115	(1.233, 0.933, 0.833)	(1.233, 0.905, 0.862)
0.6	0.154	(1.267, 0.933, 0.8)	(1.267, 0.895, 0.838)
0.7	0.192	(1.3, 0.933, 0.767)	(1.3, 0.886, 0.814)
0.8	0.231	(1.333, 0.933, 0.733)	(1.333, 0.876, 0.791)
0.9	0.269	(1.367, 0.933, 0.700)	(1.367, 0.867, 0.767)
1	0.308	(1.400, 0.933, 0.667)	(1.400, 0.857, 0.743)

Table A5.2: The priority vectors of RAU and LPS for *T4(r)*.

B14	AI	V (RAU)	V(LPS)
-1	0.406	(0.875, 0.975, 0.875, 1.275)	(0.975, 0.975, 0.875, 1.175)
-0.9	0.372	(0.9, 0.975, 0.875, 1.25)	(0.982, 0.982, 0.875, 1.161)
-0.8	0.338	(0.925, 0.975, 0.875, 1.225)	(0.989, 0.989, 0.875, 1.146)
-0.7	0.305	(0.950, 0.975, 0.875, 1.2)	(0.996, 0.996, 0.875, 1.132)
-0.6	0.271	(0.975, 0.975, 0.875, 1.175)	(1.003, 1.003, 0.875, 1.118)
-0.5	0.237	(1., 0.975, 0.875, 1.150)	(1.011, 1.011, 0.875, 1.104)
-0.4	0.203	(1.025, 0.975, 0.875, 1.125)	(1.025, 1.012, 0.875, 1.088)
-0.3	0.169	(1.05, 0.975, 0.875, 1.1)	(1.05, 1.006, 0.875, 1.069)
-0.2	0.135	(1.075, 0.975, 0.875, 1.075)	(1.061, 1.004, 0.875, 1.061)
-0.1	0.102	(1.1, 0.975, 0.875, 1.05)	(1.061, 1.004, 0.875, 1.061)
0	0.068	(1.125, 0.975, 0.875, 1.025)	(1.095, 0.995, 0.875, 1.035)
0.1	0.034	(1.150, 0.975, 0.875, 1.)	(1.150, 0.981, 0.875, 0.994)
0.2	0	(1.175, 0.975, 0.875, 0.975)	(1.175, 0.975, 0.875, 0.975)
0.3	0.034	(1.2, 0.975, 0.875, 0.950)	(1.2, 0.969, 0.875, 0.956)
0.4	0.068	(1.225, 0.975, 0.875, 0.925)	(1.225, 0.962, 0.875, 0.938)
0.5	0.102	(1.25, 0.975, 0.875, 0.9)	(1.25, 0.956, 0.875, 0.919)
0.6	0.135	(1.275, 0.975, 0.875, 0.875)	(1.275, 0.950, 0.875, 0.9)
0.7	0.169	(1.3, 0.975, 0.875, 0.85)	(1.3, 0.944, 0.875, 0.881)
0.8	0.203	(1.325, 0.975, 0.875, 0.825)	(1.325, 0.941, 0.867, 0.867)
0.9	0.237	(1.35, 0.975, 0.875, 0.8)	(1.35, 0.936, 0.857, 0.857)
1	0.271	(1.375, 0.975, 0.875, 0.775)	(1.375, 0.932, 0.846, 0.846)

Table A5.3: The priority vectors of RAU and LPS for *T5(r)*

B15	AI	V (RAU)	V(LPS)
-1	0.332	(0.92, 0.96, 0.86, 1.06, 1.2)	(1.107, 0.979, 0.692, 1.111, 1.111)
-0.9	0.304	(0.940, 0.96, 0.86, 1.06, 1.18)	(1.073, 1.025, 0.754, 1.075, 1.073)
-0.8	0.277	(0.96, 0.96, 0.86, 1.06, 1.16)	(1.045, 1.039, 0.824, 1.045, 1.046)
-0.7	0.249	(0.98, 0.96, 0.86, 1.06, 1.140)	(1.109, 1., 0.662, 1.119, 1.109)
-0.6	0.221	(1., 0.96, 0.86, 1.06, 1.12)	(1.081, 0.985, 0.77, 1.081, 1.082)
-0.5	0.194	(1.02, 0.96, 0.86, 1.06, 1.1)	(1.112, 0.977, 0.684, 1.115, 1.112)
-0.4	0.166	(1.04, 0.96, 0.86, 1.06, 1.08)	(1.126, 0.994, 0.585, 1.167, 1.126)
-0.3	0.138	(1.06, 0.96, 0.86, 1.06, 1.06)	(1.049, 0.988, 0.86, 1.047, 1.056)
-0.2	0.111	(1.08, 0.96, 0.86, 1.06, 1.04)	(1.044, 1.001, 0.866, 1.045, 1.044)
-0.1	0.083	(1.1, 0.96, 0.86, 1.06, 1.02)	(1.05, 0.996, 0.854, 1.05, 1.05)
0	0.055	(1.12, 0.96, 0.86, 1.06, 1.)	(1.09, 0.976, 0.86, 1.06, 1.014)
0.1	0.028	(1.140, 0.96, 0.86, 1.06, 0.98)	(1.140, 0.964, 0.86, 1.06, 0.976)
0.2	0	(1.16, 0.96, 0.86, 1.06, 0.96)	(1.16, 0.96, 0.86, 1.06, 0.96)
0.3	0.028	(1.18, 0.96, 0.86, 1.06, 0.940)	(1.18, 0.956, 0.86, 1.06, 0.944)
0.4	0.055	(1.2, 0.96, 0.86, 1.06, 0.92)	(1.2, 0.951, 0.86, 1.06, 0.929)
0.5	0.083	(1.22, 0.96, 0.86, 1.06, 0.9)	(1.22, 0.947, 0.86, 1.06, 0.913)
0.6	0.111	(1.24, 0.96, 0.86, 1.06, 0.88)	(1.24, 0.942, 0.86, 1.06, 0.898)
0.7	0.138	(1.26, 0.96, 0.86, 1.06, 0.86)	(1.26, 0.938, 0.86, 1.06, 0.882)
0.8	0.166	(1.28, 0.96, 0.86, 1.06, 0.84)	(1.28, 0.933, 0.86, 1.06, 0.867)
0.9	0.194	(1.3, 0.96, 0.86, 1.06, 0.820)	(1.3, 0.93, 0.855, 1.06, 0.855)
1	0.221	(1.32, 0.96, 0.86, 1.06, 0.8)	(1.32, 0.928, 0.846, 1.06, 0.846)

Table A5.4: The priority vectors of RAU and LPS for *T6(r)*.

B16	AI	V (RAU)	V(LPS)
-1	0.295	(0.933, 0.950, 0.85, 0.65, 1.55, 1.067)	(1.003, 1.003, 0.968, 0.594, 1.317, 1.116)
-0.9	0.272	(0.950, 0.950, 0.85, 0.65, 1.55, 1.05)	(0.974, 0.974, 0.872, 0.652, 1.554, 0.974)
-0.8	0.249	(0.967, 0.950, 0.85, 0.65, 1.55, 1.033)	(0.99, 0.99, 0.917, 0.646, 1.41, 1.046)
-0.7	0.227	(0.983, 0.950, 0.85, 0.65, 1.55, 1.017)	(0.986, 0.986, 0.913, 0.652, 1.414, 1.049)
-0.6	0.204	(1., 0.950, 0.85, 0.65, 1.55, 1.)	(0.977, 0.977, 0.869, 0.65, 1.55, 0.977)
-0.5	0.181	(1.017, 0.950, 0.85, 0.65, 1.55, 0.983)	(0.932, 0.932, 0.903, 0.683, 1.618, 0.932)
-0.4	0.159	(1.033, 0.950, 0.85, 0.65, 1.55, 0.967)	(0.937, 0.934, 0.918, 0.639, 1.635, 0.937)
-0.3	0.136	(1.05, 0.950, 0.85, 0.65, 1.55, 0.950)	(0.986, 0.972, 0.923, 0.769, 1.364, 0.986)
-0.2	0.113	(1.067, 0.950, 0.85, 0.65, 1.55, 0.933)	(1.015, 0.98, 0.93, 0.774, 1.287, 1.015)
-0.1	0.091	(1.083, 0.950, 0.85, 0.65, 1.55, 0.917)	(0.989, 0.988, 0.903, 0.698, 1.433, 0.988)
0	0.068	(1.1, 0.950, 0.85, 0.65, 1.55, 0.9)	(1.056, 0.950, 0.869, 0.65, 1.55, 0.925)
0.1	0.045	(1.117, 0.950, 0.85, 0.65, 1.55, 0.883)	(1.117, 0.950, 0.857, 0.65, 1.55, 0.877)
0.2	0.023	(1.133, 0.950, 0.85, 0.65, 1.55, 0.867)	(1.133, 0.950, 0.853, 0.65, 1.55, 0.863)
0.3	0	(1.150, 0.950, 0.85, 0.65, 1.55, 0.85)	(1.150, 0.950, 0.85, 0.65, 1.55, 0.85)
0.4	0.023	(1.167, 0.950, 0.85, 0.65, 1.55, 0.833)	(1.167, 0.950, 0.847, 0.65, 1.55, 0.837)
0.5	0.045	(1.183, 0.950, 0.85, 0.65, 1.55, 0.817)	(1.183, 0.950, 0.843, 0.65, 1.55, 0.823)
0.6	0.068	(1.2, 0.950, 0.85, 0.65, 1.55, 0.8)	(1.2, 0.950, 0.84, 0.65, 1.55, 0.81)
0.7	0.091	(1.217, 0.950, 0.85, 0.65, 1.55, 0.783)	(1.217, 0.950, 0.837, 0.65, 1.55, 0.797)
0.8	0.113	(1.233, 0.950, 0.85, 0.65, 1.55, 0.767)	(1.233, 0.950, 0.833, 0.65, 1.55, 0.783)
0.9	0.136	(1.25, 0.950, 0.85, 0.65, 1.55, 0.75)	(1.25, 0.950, 0.830, 0.65, 1.55, 0.77)
1	0.159	(1.267, 0.950, 0.85, 0.65, 1.55, 0.733)	(1.267, 0.950, 0.827, 0.65, 1.55, 0.757)

Table A5.5: The priority vectors of RAU and LPS for *T7(r)*.

B17	AI	V (RAU)	V(LPS)
-1	0.264	(0.857, 1.357, 0.857, 0.657, 1.657, 0.757, 0.857)	(0.922, 1.052, 0.923, 0.919, 1.558, 0.704, 0.922)
-0.9	0.245	(0.871, 1.357, 0.857, 0.657, 1.657, 0.757, 0.843)	(0.904, 0.988, 0.895, 0.837, 1.788, 0.684, 0.904)
-0.8	0.227	(0.886, 1.357, 0.857, 0.657, 1.657, 0.757, 0.829)	(0.836, 1.357, 0.815, 0.695, 1.657, 0.805, 0.836)
-0.7	0.208	(0.9, 1.357, 0.857, 0.657, 1.657, 0.757, 0.814)	(0.903, 1.07, 0.91, 0.854, 1.502, 0.859, 0.903)
-0.6	0.189	(0.914, 1.357, 0.857, 0.657, 1.657, 0.757, 0.8)	(0.838, 1.357, 0.815, 0.699, 1.657, 0.795, 0.838)
-0.5	0.17	(0.929, 1.357, 0.857, 0.657, 1.657, 0.757, 0.786)	(0.830, 1.355, 0.804, 0.698, 1.656, 0.827, 0.830)
-0.4	0.151	(0.943, 1.357, 0.857, 0.657, 1.657, 0.757, 0.771)	(0.873, 1.1521, 0.882, 0.839, 1.576, 0.805, 0.873)
-0.3	0.132	(0.957, 1.357, 0.857, 0.657, 1.657, 0.757, 0.757)	(0.882, 1.144, 0.882, 0.847, 1.553, 0.822, 0.871)
-0.2	0.113	(0.971, 1.357, 0.857, 0.657, 1.657, 0.757, 0.743)	(0.886, 1.19, 0.886, 0.771, 1.563, 0.836, 0.868)
-0.1	0.094	(0.986, 1.357, 0.857, 0.657, 1.657, 0.757, 0.729)	(0.86, 1.249, 0.859, 0.754, 1.626, 0.826, 0.826)
0	0.076	(1., 1.357, 0.857, 0.657, 1.657, 0.757, 0.714)	(0.944, 1.357, 0.857, 0.678, 1.657, 0.757, 0.75)
0.1	0.057	(1.014, 1.357, 0.857, 0.657, 1.657, 0.757, 0.700)	(1.014, 1.357, 0.857, 0.665, 1.657, 0.757, 0.692)
0.2	0.038	(1.029, 1.357, 0.857, 0.657, 1.657, 0.757, 0.686)	(1.029, 1.357, 0.857, 0.662, 1.657, 0.757, 0.68)
0.3	0.019	(1.043, 1.357, 0.857, 0.657, 1.657, 0.757, 0.671)	(1.043, 1.357, 0.857, 0.66, 1.657, 0.757, 0.669)
0.4	0	(1.057, 1.357, 0.857, 0.657, 1.657, 0.757, 0.657)	(1.057, 1.357, 0.857, 0.657, 1.657, 0.757, 0.657)
0.5	0.019	(1.071, 1.357, 0.857, 0.657, 1.657, 0.757, 0.643)	(1.071, 1.357, 0.857, 0.655, 1.657, 0.757, 0.646)
0.6	0.038	(1.086, 1.357, 0.857, 0.657, 1.657, 0.757, 0.629)	(1.086, 1.357, 0.857, 0.652, 1.657, 0.757, 0.634)
0.7	0.057	(1.1, 1.357, 0.857, 0.657, 1.657, 0.757, 0.614)	(1.1, 1.357, 0.857, 0.649, 1.657, 0.757, 0.622)
0.8	0.076	(1.114, 1.357, 0.857, 0.657, 1.657, 0.757, 0.6)	(1.114, 1.357, 0.857, 0.647, 1.657, 0.757, 0.61)
0.9	0.094	(1.129, 1.357, 0.857, 0.657, 1.657, 0.757, 0.586)	(1.129, 1.357, 0.857, 0.644, 1.657, 0.757, 0.599)
1	0.113	(1.143, 1.357, 0.857, 0.657, 1.657, 0.757, 0.571)	(1.143, 1.357, 0.857, 0.642, 1.657, 0.757, 0.587)

Table A5.6: The priority vectors of RAU and LPS for *T8(r)*.

B18	AI	V (RAU)	V(LPS)
-1	0.128	(0.838, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.238)	(0.838, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.238)
-0.9	0.112	(0.85, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.225)	(0.85, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.225)
-0.8	0.096	(0.862, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.212)	(0.862, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.212)
-0.7	0.08	(0.875, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.2)	(0.875, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.2)
-0.6	0.064	(0.888, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.188)	(0.888, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.188)
-0.5	0.048	(0.9, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.175)	(0.9, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.175)
-0.4	0.032	(0.913, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.162)	(0.913, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.162)
-0.3	0.016	(0.925, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.150)	(0.925, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.150)
-0.2	0	(0.938, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.138)	(0.938, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.138)
-0.1	0.016	(0.950, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.125)	(0.950, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.125)
0	0.032	(0.962, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.112)	(0.988, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.088)
0.1	0.048	(0.975, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.1)	(1.026, 0.54, 1.444, 0.629, 1.534, 1.26, 0.54, 1.026)
0.2	0.064	(0.988, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.088)	(1.038, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.038)
0.3	0.08	(1., 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.075)	(1.038, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.038)
0.4	0.096	(1.012, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.062)	(1.038, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.038)
0.5	0.112	(1.025, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.05)	(1.038, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.038)
0.6	0.128	(1.038, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.038)	(1.038, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.038)
0.7	0.144	(1.05, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.025)	(1.05, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.025)
0.8	0.16	(1.062, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.012)	(1.062, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.012)
0.9	0.176	(1.075, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.)	(1.075, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 1.)
1	0.192	(1.088, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 0.988)	(1.088, 0.538, 1.438, 0.638, 1.538, 1.238, 0.538, 0.988)

Table A5.7: The priority vectors of RAU and LPS for *T9(r)*.

B19	AI	V (RAU)	V(LPS)
-1	0.178	(0.767, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.756)	(0.761, 0.725, 1.311, 1.411, 1.511, 1.211, 0.511, 0.797, 0.761)
-0.9	0.164	(0.778, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.744)	(0.733, 0.667, 1.380, 1.416, 1.464, 1.346, 0.527, 0.734, 0.733)
-0.8	0.151	(0.789, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.733)	(0.761, 0.709, 1.311, 1.411, 1.511, 1.211, 0.511, 0.814, 0.761)
-0.7	0.137	(0.8, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.722)	(0.761, 0.748, 1.311, 1.411, 1.511, 1.211, 0.511, 0.774, 0.761)
-0.6	0.123	(0.811, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.711)	(0.76, 0.748, 1.311, 1.411, 1.511, 1.211, 0.511, 0.776, 0.76)
-0.5	0.11	(0.822, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.700)	(0.776, 0.708, 1.311, 1.411, 1.511, 1.211, 0.511, 0.785, 0.776)
-0.4	0.096	(0.833, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.689)	(0.723, 0.72, 1.342, 1.389, 1.446, 1.302, 0.576, 0.778, 0.723)
-0.3	0.082	(0.844, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.678)	(0.775, 0.748, 1.296, 1.368, 1.463, 1.239, 0.553, 0.781, 0.775)
-0.2	0.069	(0.856, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.667)	(0.761, 0.759, 1.311, 1.411, 1.511, 1.211, 0.511, 0.764, 0.761)
-0.1	0.055	(0.867, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.656)	(0.775, 0.747, 1.311, 1.411, 1.511, 1.211, 0.511, 0.775, 0.747)
0	0.041	(0.878, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.644)	(0.842, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.68)
0.1	0.027	(0.889, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.633)	(0.889, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.633)
0.2	0.014	(0.9, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.622)	(0.9, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.622)
0.3	0	(0.911, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.611)	(0.911, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.611)
0.4	0.014	(0.922, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.6)	(0.922, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.6)
0.5	0.027	(0.933, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.589)	(0.933, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.589)
0.6	0.041	(0.944, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.578)	(0.944, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.578)
0.7	0.055	(0.956, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.567)	(0.956, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.567)
0.8	0.069	(0.967, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.556)	(0.967, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.556)
0.9	0.082	(0.978, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.544)	(0.978, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.544)
1	0.096	(0.989, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.533)	(0.989, 0.711, 1.311, 1.411, 1.511, 1.211, 0.511, 0.811, 0.533)

Table A5.8: The priority vectors of RAU and LPS for *T10(r)*

B1,10	AI	V (RAU)	V(LPS)
-1	0.107	(0.820, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 1.1)	(0.820, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 1.1)
-0.9	0.095	(0.830, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 1.09)	(0.830, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 1.09)
-0.8	0.083	(0.84, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 1.08)	(0.84, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 1.08)
-0.7	0.071	(0.85, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 1.07)	(0.85, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 1.07)
-0.6	0.06	(0.86, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 1.06)	(0.86, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 1.06)
-0.5	0.048	(0.87, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 1.05)	(0.87, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 1.05)
-0.4	0.036	(0.88, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 1.04)	(0.88, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 1.04)
-0.3	0.024	(0.89, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 1.03)	(0.89, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 1.03)
-0.2	0.012	(0.9, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 1.02)	(0.9, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 1.02)
-0.1	0	(0.91, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 1.01)	(0.91, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 1.01)
0	0.012	(0.92, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 1.)	(0.931, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 0.989)
0.1	0.024	(0.93, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 0.99)	(0.96, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 0.96)
0.2	0.036	(0.940, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 0.98)	(0.96, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 0.96)
0.3	0.048	(0.950, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 0.97)	(0.959, 0.711, 1.31, 1.411, 1.513, 1.208, 0.513, 0.808, 0.609, 0.959)
0.4	0.06	(0.96, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 0.96)	(0.96, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 0.96)
0.5	0.071	(0.97, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 0.950)	(0.97, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 0.950)
0.6	0.083	(0.98, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 0.940	(0.98, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 0.940)
0.7	0.095	(0.99, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 0.93)	(0.99, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 0.93)
0.8	0.107	(1., 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 0.92)	(1., 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 0.92)
0.9	0.119	(1.01, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 0.91)	(1.01, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 0.91)
1	0.131	(1.02, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 0.9)	(1.02, 0.71, 1.31, 1.41, 1.51, 1.21, 0.51, 0.81, 0.61, 0.9)

Table A5.9: Measurements of RAU and LPS for $T3(r)$

b13	AI	RMPWSV (RAU)	RMPWSV (LPS)	MPWADV (RAU)	MPWADV (LPS)	MC (RAU)	MC (LPS)	WADV (RAU)	WADV (LPS)	RMSV (RAU)	RMSV (LPS)
-1	0.462	0.864	0.539	1.867	0.7	0.5	0.333	0.4	0.75	0.4	0.471
-0.9	0.423	0.792	0.492	1.711	0.65	0.5	0.333	0.367	0.675	0.367	0.427
-0.8	0.385	0.72	0.447	1.556	0.6	0.5	0.333	0.333	0.6	0.333	0.383
-0.7	0.346	0.648	0.403	1.4	0.55	0.5	0.333	0.3	0.525	0.3	0.34
-0.6	0.308	0.576	0.361	1.244	0.5	0.5	0.333	0.267	0.45	0.267	0.297
-0.5	0.269	0.504	0.32	1.089	0.45	0.5	0.333	0.233	0.375	0.233	0.256
-0.4	0.231	0.306	0.283	0.467	0.4	0.333	0.333	0.2	0.3	0.2	0.216
-0.3	0.192	0.215	0.25	0.278	0.35	0.167	0.333	0.167	0.225	0.167	0.179
-0.2	0.154	0.172	0.151	0.222	0.171	0.167	0.167	0.133	0.171	0.133	0.144
-0.1	0.115	0.153	0.141	0.233	0.2	0.333	0.333	0.1	0.15	0.1	0.108
0	0.077	0.102	0.089	0.156	0.12	0.333	0.333	0.067	0.12	0.067	0.077
0.1	0.038	0.043	0.038	0.056	0.043	0.167	0.167	0.033	0.043	0.033	0.036
0.2	0	0	0	0	0	0	0	0	0	0	0
0.3	0.038	0.043	0.038	0.056	0.043	0.167	0.167	0.033	0.043	0.033	0.036
0.4	0.077	0.086	0.076	0.111	0.086	0.167	0.167	0.067	0.086	0.067	0.072
0.5	0.115	0.129	0.113	0.167	0.129	0.167	0.167	0.1	0.129	0.1	0.108
0.6	0.154	0.172	0.151	0.222	0.171	0.167	0.167	0.133	0.172	0.133	0.144
0.7	0.192	0.215	0.189	0.278	0.214	0.167	0.167	0.167	0.214	0.167	0.18
0.8	0.231	0.258	0.227	0.333	0.257	0.167	0.167	0.2	0.257	0.2	0.216
0.9	0.269	0.301	0.265	0.389	0.3	0.167	0.167	0.233	0.3	0.233	0.252
1	0.308	0.344	0.302	0.444	0.343	0.167	0.167	0.267	0.343	0.267	0.288

Table A5.10: Measurements of RAU and LPS for $T4(r)$

b14	AI	RMPWSV (RAU)	RMPWSV (LPS)	MPWADV (RAU)	MPWADV (LPS)	MC (RAU)	MC (LPS)	WADV (RAU)	WADV (LPS)	RMSV (RAU)	RMSV (LPS)
-1	0.406	0.561	0.4	0.95	0.4	0.333	0.167	0.6	0.8	0.346	0.365
-0.9	0.372	0.489	0.367	0.779	0.374	0.25	0.167	0.55	0.721	0.318	0.333
-0.8	0.338	0.445	0.335	0.708	0.348	0.25	0.167	0.5	0.643	0.289	0.301
-0.7	0.305	0.4	0.303	0.637	0.321	0.25	0.167	0.45	0.564	0.26	0.269
-0.6	0.271	0.283	0.273	0.333	0.295	0.167	0.167	0.4	0.486	0.231	0.238
-0.5	0.237	0.226	0.244	0.233	0.269	0.083	0.167	0.35	0.407	0.202	0.208
-0.4	0.203	0.194	0.184	0.2	0.175	0.083	0.083	0.3	0.337	0.173	0.179
-0.3	0.169	0.161	0.153	0.167	0.146	0.083	0.083	0.25	0.281	0.144	0.149
-0.2	0.135	0.173	0.169	0.2	0.191	0.167	0.167	0.2	0.2	0.115	0.119
-0.1	0.102	0.208	0.116	0.325	0.14	0.25	0.167	0.15	0.143	0.087	0.096
0	0.068	0.087	0.077	0.1	0.093	0.167	0.167	0.1	0.1	0.058	0.065
0.1	0.034	0.032	0.031	0.033	0.029	0.083	0.083	0.05	0.056	0.029	0.03
0.2	0	0	0	0	0	0	0	0	0	0	0
0.3	0.034	0.032	0.031	0.033	0.029	0.083	0.083	0.05	0.056	0.029	0.03
0.4	0.068	0.065	0.061	0.067	0.058	0.083	0.083	0.1	0.113	0.058	0.06
0.5	0.102	0.097	0.092	0.1	0.087	0.083	0.083	0.15	0.169	0.087	0.089
0.6	0.135	0.141	0.122	0.167	0.117	0.167	0.083	0.2	0.225	0.115	0.119
0.7	0.169	0.222	0.153	0.354	0.146	0.25	0.083	0.25	0.281	0.144	0.149
0.8	0.203	0.267	0.193	0.425	0.205	0.25	0.167	0.3	0.342	0.173	0.179
0.9	0.237	0.311	0.223	0.496	0.229	0.25	0.167	0.35	0.407	0.202	0.21
1	0.271	0.356	0.254	0.567	0.252	0.25	0.167	0.4	0.471	0.231	0.242

Table A5.11: Measurements of RAU and LPS for $T5(r)$

b15	AI	RMPWSV (RAU)	RMPWSV (LPS)	MPWADV (RAU)	MPWADV (LPS)	MC (RAU)	MC (LPS)	WADV (RAU)	WADV (LPS)	RMSV (RAU)	RMSV (LPS)
-1	0.332	0.503	0.372	0.912	0.367	0.35	0.2	0.72	0.997	0.294	0.351
-0.9	0.304	0.461	0.517	0.836	0.47	0.35	0.2	0.66	0.9	0.269	0.308
-0.8	0.277	0.385	0.292	0.62	0.275	0.3	0.2	0.6	0.799	0.245	0.272
-0.7	0.249	0.337	0.437	0.522	0.472	0.25	0.2	0.54	0.7	0.22	0.282
-0.6	0.221	0.299	0.245	0.464	0.275	0.25	0.2	0.48	0.6	0.196	0.217
-0.5	0.194	0.262	0.338	0.406	0.402	0.25	0.2	0.42	0.5	0.171	0.226
-0.4	0.166	0.224	0.352	0.348	0.458	0.25	0.2	0.36	0.441	0.147	0.266
-0.3	0.138	0.197	0.165	0.21	0.211	0.2	0.15	0.3	0.293	0.122	0.125
-0.2	0.111	0.25	0.169	0.304	0.23	0.15	0.2	0.24	0.2	0.098	0.106
-0.1	0.083	0.188	0.139	0.228	0.2	0.15	0.2	0.18	0.146	0.073	0.088
0	0.055	0.075	0.067	0.068	0.071	0.1	0.1	0.12	0.086	0.049	0.055
0.1	0.028	0.026	0.025	0.022	0.02	0.05	0.05	0.06	0.064	0.024	0.025
0.2	0	0	0	0	0	0	0	0	0	0	0
0.3	0.028	0.026	0.025	0.022	0.02	0.05	0.05	0.06	0.064	0.024	0.025
0.4	0.055	0.052	0.051	0.044	0.04	0.05	0.05	0.12	0.129	0.049	0.05
0.5	0.083	0.078	0.076	0.066	0.061	0.05	0.05	0.18	0.193	0.073	0.075
0.6	0.111	0.104	0.102	0.088	0.081	0.05	0.05	0.24	0.258	0.098	0.1
0.7	0.138	0.138	0.127	0.13	0.101	0.1	0.05	0.3	0.322	0.122	0.124
0.8	0.166	0.193	0.152	0.24	0.121	0.15	0.05	0.36	0.387	0.147	0.149
0.9	0.194	0.226	0.183	0.28	0.16	0.15	0.1	0.42	0.455	0.171	0.175
1	0.221	0.258	0.204	0.32	0.157	0.15	0.05	0.48	0.526	0.196	0.2

Table A5.12: Measurements of RAU and LPS for $T6(r)$

b16	AI	RMPWSV (RAU)	RMPWSV (LPS)	MPWADV (RAU)	MPWADV (LPS)	MC (RAU)	MC (LPS)	WADV (RAU)	WADV (LPS)	RMSV (RAU)	RMSV (LPS)
-1	0.295	0.371	0.378	0.462	0.432	0.167	0.133	0.867	0.887	0.274	0.328
-0.9	0.272	0.314	0.428	0.333	0.319	0.133	0.133	0.8	0.9	0.253	0.259
-0.8	0.249	0.28	0.294	0.281	0.327	0.1	0.133	0.733	0.744	0.232	0.254
-0.7	0.227	0.255	0.279	0.256	0.324	0.1	0.133	0.667	0.637	0.211	0.233
-0.6	0.204	0.317	0.304	0.31	0.281	0.133	0.133	0.6	0.6	0.19	0.192
-0.5	0.181	0.461	0.275	0.524	0.263	0.167	0.133	0.533	0.5	0.169	0.19
-0.4	0.159	0.403	0.242	0.459	0.25	0.167	0.133	0.467	0.4	0.148	0.175
-0.3	0.136	0.339	0.246	0.347	0.284	0.133	0.133	0.4	0.322	0.126	0.2
-0.2	0.113	0.281	0.262	0.278	0.31	0.1	0.133	0.333	0.428	0.105	0.227
-0.1	0.091	0.224	0.171	0.222	0.206	0.1	0.133	0.267	0.255	0.084	0.144
0	0.068	0.098	0.089	0.073	0.083	0.067	0.067	0.2	0.131	0.063	0.072
0.1	0.045	0.044	0.043	0.031	0.029	0.033	0.033	0.133	0.14	0.042	0.043
0.2	0.023	0.022	0.022	0.016	0.015	0.033	0.033	0.067	0.07	0.021	0.021
0.3	0	0	0	0	0	0	0	0	0	0	0
0.4	0.023	0.022	0.022	0.016	0.015	0.033	0.033	0.067	0.07	0.021	0.021
0.5	0.045	0.044	0.043	0.031	0.029	0.033	0.033	0.133	0.14	0.042	0.043
0.6	0.068	0.066	0.065	0.047	0.044	0.033	0.033	0.2	0.21	0.063	0.064
0.7	0.091	0.088	0.086	0.062	0.059	0.033	0.033	0.267	0.28	0.084	0.085
0.8	0.113	0.11	0.108	0.078	0.073	0.033	0.033	0.333	0.35	0.105	0.106
0.9	0.136	0.132	0.13	0.093	0.088	0.033	0.033	0.4	0.42	0.126	0.128
1	0.159	0.154	0.151	0.109	0.103	0.033	0.033	0.467	0.49	0.148	0.149

Table A5.13: Measurements of RAU and LPS for $T7(r)$

b17	AI	RMPWSV (RAU)	RMPWSV (LPS)	MPWADV (RAU)	MPWADV (LPS)	MC (RAU)	MC (LPS)	WADV (RAU)	WADV (LPS)	RMSV (RAU)	RMSV (LPS)
-1	0.264	0.436	0.572	0.381	0.736	0.143	0.19	1	1	0.258	0.359
-0.9	0.245	0.667	0.54	0.628	0.71	0.119	0.19	0.928	0.9	0.24	0.351
-0.8	0.227	0.616	0.378	0.58	0.394	0.119	0.143	0.857	0.8	0.221	0.227
-0.7	0.208	0.565	0.413	0.531	0.48	0.119	0.143	0.786	0.7	0.203	0.311
-0.6	0.189	0.513	0.323	0.483	0.371	0.119	0.143	0.714	0.6	0.184	0.195
-0.5	0.17	0.462	0.305	0.435	0.401	0.119	0.19	0.643	0.5	0.166	0.186
-0.4	0.151	0.411	0.33	0.386	0.463	0.119	0.19	0.571	0.421	0.148	0.234
-0.3	0.132	0.355	0.342	0.305	0.476	0.095	0.19	0.5	0.427	0.129	0.235
-0.2	0.113	0.303	0.267	0.253	0.349	0.071	0.143	0.428	0.377	0.111	0.197
-0.1	0.094	0.252	0.187	0.211	0.216	0.071	0.095	0.357	0.294	0.092	0.161
0	0.076	0.116	0.106	0.073	0.087	0.048	0.048	0.286	0.195	0.074	0.084
0.1	0.057	0.057	0.056	0.035	0.033	0.024	0.024	0.214	0.222	0.055	0.056
0.2	0.038	0.038	0.038	0.023	0.022	0.024	0.024	0.143	0.148	0.037	0.037
0.3	0.019	0.019	0.019	0.012	0.011	0.024	0.024	0.072	0.074	0.018	0.019
0.4	0	0	0	0	0	0	0	0	0	0	0
0.5	0.019	0.019	0.019	0.012	0.011	0.024	0.024	0.071	0.074	0.018	0.019
0.6	0.038	0.038	0.038	0.023	0.022	0.024	0.024	0.143	0.148	0.037	0.037
0.7	0.057	0.057	0.056	0.035	0.033	0.024	0.024	0.214	0.222	0.055	0.056
0.8	0.076	0.076	0.075	0.046	0.044	0.024	0.024	0.286	0.296	0.074	0.074
0.9	0.094	0.095	0.094	0.058	0.055	0.024	0.024	0.357	0.37	0.092	0.093
1	0.113	0.114	0.113	0.069	0.066	0.024	0.024	0.428	0.444	0.111	0.111

Table A5.14: Measurements of RAU and LPS for $T8(r)$

b18	AI	RMPWSV (RAU)	RMPWSV (LPS)	MPWADV (RAU)	MPWADV (LPS)	MC (RAU)	MC (LPS)	WADV (RAU)	WADV (LPS)	RMSV (RAU)	RMSV (LPS)
-1	0.128	0.134	0.134	0.071	0.071	0.018	0.018	0.6	0.6	0.131	0.131
-0.9	0.112	0.115	0.115	0.056	0.056	0	0	0.525	0.525	0.115	0.115
-0.8	0.096	0.098	0.098	0.048	0.048	0	0	0.45	0.45	0.098	0.098
-0.7	0.08	0.082	0.082	0.04	0.04	0	0	0.375	0.375	0.082	0.082
-0.6	0.064	0.065	0.065	0.032	0.032	0	0	0.3	0.3	0.065	0.065
-0.5	0.048	0.049	0.049	0.024	0.024	0	0	0.225	0.225	0.049	0.049
-0.4	0.032	0.033	0.033	0.016	0.016	0	0	0.15	0.15	0.033	0.033
-0.3	0.016	0.016	0.016	0.008	0.008	0	0	0.075	0.075	0.016	0.016
-0.2	0	0	0	0	0	0	0	0	0	0	0
-0.1	0.016	0.016	0.016	0.008	0.008	0	0	0.075	0.075	0.016	0.016
0	0.032	0.052	0.046	0.027	0.032	0.018	0.018	0.15	0.1	0.033	0.038
0.1	0.048	0.137	0.075	0.096	0.06	0.036	0.036	0.225	0.134	0.049	0.07
0.2	0.064	0.182	0.093	0.129	0.064	0.036	0.018	0.3	0.2	0.065	0.076
0.3	0.08	0.228	0.118	0.161	0.075	0.036	0.018	0.375	0.3	0.082	0.087
0.4	0.096	0.273	0.146	0.193	0.086	0.036	0.018	0.45	0.4	0.098	0.1
0.5	0.112	0.319	0.176	0.225	0.096	0.036	0.018	0.525	0.5	0.115	0.115
0.6	0.128	0.207	0.207	0.107	0.107	0.018	0.018	0.6	0.6	0.131	0.131
0.7	0.144	0.147	0.147	0.072	0.072	0	0	0.675	0.675	0.147	0.147
0.8	0.16	0.164	0.164	0.08	0.08	0	0	0.75	0.75	0.164	0.164
0.9	0.176	0.18	0.18	0.088	0.088	0	0	0.825	0.825	0.18	0.18
1	0.192	0.196	0.196	0.096	0.096	0	0	0.9	0.9	0.196	0.196

Table A5.15: Measurements of RAU and LPS for $T9(r)$

b19	AI	RMPWSV (RAU)	RMPWSV (LPS)	MPWADV (RAU)	MPWADV (LPS)	MC (RAU)	MC (LPS)	WADV (RAU)	WADV (LPS)	RMSV (RAU)	RMSV (LPS)
-1	0.178	0.55	0.319	0.409	0.214	0.083	0.069	1.011	1	0.191	0.191
-0.9	0.164	0.508	0.307	0.378	0.252	0.083	0.069	0.933	0.9	0.176	0.2
-0.8	0.151	0.465	0.271	0.346	0.202	0.083	0.069	0.856	0.8	0.162	0.163
-0.7	0.137	0.423	0.238	0.315	0.186	0.083	0.069	0.778	0.7	0.147	0.152
-0.6	0.123	0.376	0.214	0.244	0.177	0.056	0.069	0.7	0.6	0.132	0.139
-0.5	0.11	0.333	0.2	0.207	0.174	0.028	0.069	0.622	0.5	0.118	0.126
-0.4	0.096	0.291	0.19	0.181	0.196	0.028	0.069	0.544	0.4	0.103	0.137
-0.3	0.082	0.249	0.16	0.156	0.169	0.028	0.069	0.467	0.3	0.088	0.117
-0.2	0.069	0.208	0.136	0.13	0.142	0.028	0.069	0.389	0.2	0.073	0.105
-0.1	0.055	0.166	0.116	0.104	0.112	0.028	0.056	0.311	0.172	0.059	0.091
0	0.041	0.071	0.064	0.032	0.04	0.014	0.014	0.233	0.162	0.044	0.051
0.1	0.027	0.029	0.029	0.013	0.013	0	0	0.156	0.156	0.029	0.029
0.2	0.014	0.015	0.015	0.006	0.006	0	0	0.078	0.078	0.015	0.015
0.3	0	0	0	0	0	0	0	0	0	0	0
0.4	0.014	0.015	0.015	0.006	0.006	0	0	0.078	0.078	0.015	0.015
0.5	0.027	0.029	0.029	0.013	0.013	0	0	0.156	0.156	0.029	0.029
0.6	0.041	0.044	0.044	0.019	0.019	0	0	0.233	0.233	0.044	0.044
0.7	0.055	0.059	0.059	0.026	0.026	0	0	0.311	0.311	0.059	0.059
0.8	0.069	0.073	0.073	0.032	0.032	0	0	0.389	0.389	0.073	0.073
0.9	0.082	0.088	0.088	0.039	0.039	0	0	0.467	0.467	0.088	0.088
1	0.096	0.103	0.103	0.045	0.045	0	0	0.544	0.544	0.103	0.103

Table A5.16: Measurements of RAU and LPS for $T10(r)$

b1,10	AI	RMPWSV (RAU)	RMPWSV (LPS)	MPWADV (RAU)	MPWADV (LPS)	MC (RAU)	MC (LPS)	WADV (RAU)	WADV (LPS)	RMSV (RAU)	RMSV (LPS)
-1	0.107	0.12	0.12	0.048	0.048	0	0	0.72	0.72	0.12	0.12
-0.9	0.095	0.107	0.107	0.043	0.043	0	0	0.64	0.64	0.107	0.107
-0.8	0.083	0.093	0.093	0.037	0.037	0	0	0.56	0.56	0.093	0.093
-0.7	0.071	0.08	0.08	0.032	0.032	0	0	0.48	0.48	0.08	0.08
-0.6	0.06	0.067	0.067	0.027	0.027	0	0	0.4	0.4	0.067	0.067
-0.5	0.048	0.053	0.053	0.021	0.021	0	0	0.32	0.32	0.053	0.053
-0.4	0.036	0.04	0.04	0.016	0.016	0	0	0.24	0.24	0.04	0.04
-0.3	0.024	0.027	0.027	0.011	0.011	0	0	0.16	0.16	0.027	0.027
-0.2	0.012	0.013	0.013	0.005	0.005	0	0	0.08	0.08	0.013	0.013
-0.1	0	0	0	0	0	0	0	0	0	0	0
0	0.012	0.021	0.02	0.009	0.011	0.011	0.011	0.08	0.057	0.013	0.015
0.1	0.024	0.076	0.039	0.043	0.024	0.022	0.011	0.16	0.1	0.027	0.033
0.2	0.036	0.115	0.06	0.064	0.031	0.022	0.011	0.24	0.2	0.04	0.042
0.3	0.048	0.153	0.083	0.085	0.039	0.022	0.011	0.32	0.3	0.053	0.054
0.4	0.06	0.107	0.107	0.044	0.044	0.011	0.011	0.4	0.4	0.067	0.067
0.5	0.071	0.08	0.08	0.032	0.032	0	0	0.48	0.48	0.08	0.08
0.6	0.083	0.093	0.093	0.037	0.037	0	0	0.56	0.56	0.093	0.093
0.7	0.095	0.107	0.107	0.043	0.043	0	0	0.64	0.64	0.107	0.107
0.8	0.107	0.12	0.12	0.048	0.048	0	0	0.72	0.72	0.12	0.12
0.9	0.119	0.133	0.133	0.053	0.053	0	0	0.8	0.8	0.133	0.133
1	0.131	0.147	0.147	0.059	0.059	0	0	0.88	0.88	0.147	0.147

Appendix III

Tables for figures 6.3a-6.3b of chapter 6

Table A6.1: Aggregation values of each AO with respect to owaW(0.1)

	T_1	T_2	T_3	T_4	T_5	Rank				
W	0.851	0.061	0.038	0.028	0.022					
Agg_1	0.512	0.523	0.608	0.421	0.518	2	4	**5**	1	3
Agg_2	0.515	0.529	0.611	0.427	0.523	2	4	**5**	1	3
Agg_3	0.518	0.536	0.615	0.436	0.530	2	4	**5**	1	3
Agg_4	0.500	0.500	0.585	0.415	0.500	2	2	**5**	1	2
Agg_5	0.515	0.529	0.611	0.428	0.524	2	4	**5**	1	3
Agg_6	0.516	0.532	0.613	0.431	0.526	2	4	**5**	1	3
Agg_7	0.523	0.545	0.620	0.448	0.539	2	4	**5**	1	3
Agg_8	0.510	0.520	0.581	0.438	0.515	2	4	**5**	1	3
Agg_9	0.493	0.485	0.475	0.483	0.483	**5**	4	1	2	3
Agg_{10}	0.476	0.452	0.389	0.532	0.454	4	2	1	**5**	3

Table A6.2: Aggregation values of each AO with respect to owaW(0.2)

	T_1	T_2	T_3	T_4	T_5	Rank				
W	0.725	0.108	0.070	0.053	0.044					
Agg_1	0.524	0.545	0.613	0.442	0.534	2	4	**5**	1	3
Agg_2	0.529	0.554	0.619	0.453	0.542	2	4	**5**	1	3
Agg_3	0.535	0.566	0.626	0.468	0.554	2	4	**5**	1	3
Agg_4	0.500	0.500	0.572	0.428	0.500	2	2	**5**	1	2
Agg_5	0.530	0.555	0.620	0.455	0.543	2	4	**5**	1	3
Agg_6	0.532	0.560	0.622	0.460	0.548	2	4	**5**	1	3
Agg_7	0.543	0.580	0.635	0.488	0.568	2	4	**5**	1	3
Agg_8	0.520	0.537	0.584	0.458	0.527	2	4	**5**	1	3
Agg_9	0.486	0.472	0.464	0.475	0.471	**5**	3	1	4	2
Agg_{10}	0.454	0.415	0.369	0.494	0.421	4	2	1	**5**	3

Table A6.3: Aggregation values of each AO with respect to owaW(0.3)

	T_1	T_2	T_3	T_4	T_5	Rank				
W	0.617	0.143	0.098	0.077	0.065					
Agg_1	0.536	0.564	0.616	0.463	0.547	2	4	**5**	1	3
Agg_2	0.543	0.576	0.625	0.478	0.558	2	4	**5**	1	3
Agg_3	0.551	0.591	0.634	0.497	0.573	2	4	**5**	1	3
Agg_4	0.500	0.500	0.562	0.438	0.500	2	2	**5**	1	2
Agg_5	0.543	0.578	0.625	0.480	0.559	2	4	**5**	1	3
Agg_6	0.547	0.583	0.629	0.487	0.565	2	4	**5**	1	3
Agg_7	0.562	0.608	0.645	0.521	0.590	2	4	**5**	1	3
Agg_8	0.529	0.551	0.586	0.476	0.537	2	4	**5**	1	3
Agg_9	0.480	0.462	0.456	0.469	0.462	**5**	2	1	4	3
Agg_{10}	0.434	0.387	0.355	0.461	0.397	4	2	1	**5**	3

Table A6.4: Aggregation values of each AO with respect to owaW(0.4)

	T_1	T_2	T_3	T_4	T_5	Rank				
W	0.525	0.168	0.122	0.099	0.085					
Agg_1	0.547	0.581	0.618	0.483	0.557	2	4	**5**	1	3
Agg_2	0.556	0.596	0.628	0.501	0.571	2	4	**5**	1	3
Agg_3	0.566	0.612	0.640	0.524	0.587	2	4	**5**	1	3
Agg_4	0.500	0.500	0.553	0.447	0.500	2	2	**5**	1	2
Agg_5	0.557	0.597	0.629	0.503	0.572	2	4	**5**	1	3
Agg_6	0.561	0.604	0.634	0.512	0.578	2	4	**5**	1	3
Agg_7	0.579	0.630	0.652	0.550	0.607	2	4	**5**	1	3
Agg_8	0.538	0.564	0.588	0.493	0.545	2	4	**5**	1	3
Agg_9	0.473	0.453	0.450	0.462	0.455	**5**	2	1	4	3
Agg_{10}	0.416	0.364	0.345	0.433	0.380	4	2	1	**5**	3

Table A6.5: Aggregation values of each AO with respect to owaW(0.5)

	T_1	T_2	T_3	T_4	T_5	Rank				
W	0.447	0.185	0.142	0.120	0.106					
Agg_1	0.559	0.596	0.618	0.502	0.566	2	4	5	1	3
Agg_2	0.568	0.612	0.630	0.523	0.580	2	4	5	1	3
Agg_3	0.580	0.630	0.643	0.548	0.598	2	4	5	1	3
Agg_4	0.500	0.500	0.545	0.455	0.500	2	2	5	1	2
Agg_5	0.570	0.614	0.631	0.525	0.582	2	4	5	1	3
Agg_6	0.574	0.621	0.636	0.535	0.589	2	4	5	1	3
Agg_7	0.595	0.649	0.657	0.576	0.618	2	4	5	1	3
Agg_8	0.547	0.574	0.588	0.509	0.552	2	4	5	1	3
Agg_9	0.467	0.445	0.446	0.456	0.450	5	1	2	4	3
Agg_{10}	0.400	0.345	0.338	0.408	0.367	4	2	1	5	3

Table A6.6: Aggregation values of each AO with respect to owaW(0.6)

	T_1	T_2	T_3	T_4	T_5	Rank				
W	0.381	0.196	0.159	0.139	0.125					
Agg_1	0.569	0.609	0.618	0.520	0.573	2	4	5	1	3
Agg_2	0.581	0.626	0.630	0.543	0.588	2	4	5	1	3
Agg_3	0.594	0.644	0.644	0.570	0.606	2	5	4	1	3
Agg_4	0.500	0.500	0.538	0.462	0.500	2	2	5	1	2
Agg_5	0.582	0.628	0.632	0.546	0.590	2	4	5	1	3
Agg_6	0.587	0.635	0.637	0.556	0.597	2	4	5	1	3
Agg_7	0.609	0.663	0.660	0.598	0.627	2	5	4	1	3
Agg_8	0.555	0.583	0.587	0.523	0.557	2	4	5	1	3
Agg_9	0.462	0.439	0.443	0.450	0.447	5	1	2	4	3
Agg_{10}	0.385	0.331	0.335	0.387	0.359	4	1	2	5	3

Table A6.7: Aggregation values of each AO with respect to owaW(0.7)

	T_1	T_2	T_3	T_4	T_5	Rank				
W	0.324	0.202	0.173	0.156	0.145					
Agg_1	0.580	0.620	0.616	0.538	0.578	3	5	4	1	2
Agg_2	0.592	0.638	0.630	0.562	0.594	2	5	4	1	3
Agg_3	0.606	0.656	0.645	0.590	0.612	2	5	4	1	3
Agg_4	0.500	0.500	0.532	0.468	0.510	2	2	5	1	4
Agg_5	0.593	0.639	0.631	0.565	0.595	2	5	4	1	3
Agg_6	0.599	0.647	0.637	0.576	0.603	2	5	4	1	3
Agg_7	0.622	0.675	0.661	0.618	0.633	2	5	4	1	3
Agg_8	0.562	0.590	0.587	0.536	0.560	3	5	4	1	2
Agg_9	0.457	0.434	0.442	0.444	0.445	5	1	2	3	4
Agg_{10}	0.371	0.320	0.333	0.367	0.353	5	1	2	4	3

Table A6.8: Aggregation values of each AO with respect to owaW(0.8)

	T_1	T_2	T_3	T_4	T_5	Rank				
W	0.276	0.205	0.184	0.172	0.163					
Agg_1	0.590	0.629	0.614	0.555	0.581	3	5	4	1	2
Agg_2	0.603	0.647	0.628	0.580	0.597	3	5	4	1	2
Agg_3	0.618	0.666	0.644	0.608	0.616	3	5	4	1	2
Agg_4	0.505	0.514	0.537	0.491	0.523	2	3	5	1	4
Agg_5	0.605	0.649	0.630	0.583	0.599	3	5	4	1	2
Agg_6	0.610	0.657	0.636	0.594	0.607	3	5	4	1	2
Agg_7	0.635	0.685	0.660	0.636	0.637	1	5	4	2	3
Agg_8	0.569	0.596	0.585	0.549	0.563	3	5	4	1	2
Agg_9	0.451	0.430	0.441	0.438	0.444	5	1	3	2	4
Agg_{10}	0.358	0.310	0.333	0.350	0.350	5	1	2	4	3

Table A6.9: Aggregation values of each AO with respect to owaW(0.9)

	T_1	T_2	T_3	T_4	T_5	Rank				
W	0.235	0.203	0.193	0.187	0.182					
Agg_1	0.600	0.636	0.612	0.571	0.584	3	5	4	1	2
Agg_2	0.614	0.655	0.626	0.597	0.600	3	5	4	1	2
Agg_3	0.629	0.674	0.642	0.625	0.619	3	5	4	2	1
Agg_4	0.523	0.528	0.543	0.511	0.533	2	3	5	1	4
Agg_5	0.615	0.657	0.628	0.600	0.602	3	5	4	1	2
Agg_6	0.621	0.664	0.634	0.611	0.609	3	5	4	2	1
Agg_7	0.646	0.693	0.659	0.652	0.639	2	5	4	3	1
Agg_8	0.576	0.601	0.584	0.560	0.565	3	5	4	1	2
Agg_9	0.447	0.427	0.441	0.433	0.443	5	1	3	2	4
Agg_{10}	0.346	0.303	0.333	0.335	0.348	4	1	2	3	5

Table A6.10: Aggregation values of each AO with respect to owaW(1.0)

	T_1	T_2	T_3	T_4	T_5	Rank				
W	0.200	0.200	0.200	0.200	0.200					
Agg_1	0.609	0.642	0.609	0.586	0.586	3	5	3	2	1
Agg_2	0.624	0.661	0.624	0.613	0.602	3	5	3	2	1
Agg_3	0.640	0.680	0.640	0.640	0.620	2	5	2	2	1
Agg_4	0.540	0.540	0.540	0.530	0.540	2	2	2	1	2
Agg_5	0.625	0.663	0.625	0.615	0.603	3	5	3	2	1
Agg_6	0.632	0.670	0.632	0.626	0.610	3	5	3	2	1
Agg_7	0.657	0.699	0.657	0.666	0.640	2	5	2	4	1
Agg_8	0.582	0.605	0.582	0.570	0.566	3	5	3	2	1
Agg_9	0.442	0.424	0.442	0.428	0.444	3	1	3	2	5
Agg_{10}	0.335	0.298	0.335	0.321	0.348	3	1	3	2	5

Appendix IV

Tables for figures 7.3-7.5 of chapter 7

Table A7.1a: Fuzzy Accordant Index of $\widehat{T3}(\hat{r})$

Index	$\hat{r}$	FAI	(AI^l, AI^π, AI^u)
1	(-8,-8,-7)	0.47	(0.495, 0.481, 0.428)
2	(-8,-7,-6)	0.435	(0.495, 0.433, 0.385)
3	(-7,-6,-5)	0.386	(0.44, 0.385, 0.342)
4	(-6,-5,-4)	0.338	(0.385, 0.337, 0.299)
5	(-5,-4,-3)	0.29	(0.33, 0.289, 0.257)
6	(-4,-3,-2)	0.242	(0.275, 0.241, 0.214)
7	(-3,-2,-1)	0.193	(0.22, 0.192, 0.171)
8	(-2,-1,0)	0.145	(0.165, 0.144, 0.128)
9	(0,0,0)	0.097	(0.055, 0.096, 0.128)
10	(0,1,2)	0.048	(0.055, 0.048, 0.043)
11	(1,2,3)	0	(0, 0, 0)
12	(2,3,4)	0.048	(0.055, 0.048, 0.043)
13	(3,4,5)	0.097	(0.11, 0.096, 0.086)
14	(4,5,6)	0.145	(0.165, 0.144, 0.128)
15	(5,6,7)	0.193	(0.22, 0.192, 0.171)
16	(6,7,8)	0.242	(0.275, 0.241, 0.214)
17	(7,8,8)	0.277	(0.33, 0.289, 0.214)

Table A7.2a: Fuzzy Accordant Index of $\widehat{T4}(\hat{r})$

Index	$\hat{r}$	FAI	(AI^l, AI^π, AI^u)
1	(-8,-8,-7)	0.424	(0.436, 0.423, 0.414)
2	(-8,-7,-6)	0.393	(0.436, 0.381, 0.376)
3	(-7,-6,-5)	0.351	(0.389, 0.338, 0.339)
4	(-6,-5,-4)	0.308	(0.341, 0.296, 0.302)
5	(-5,-4,-3)	0.266	(0.294, 0.254, 0.265)
6	(-4,-3,-2)	0.224	(0.247, 0.211, 0.229)
7	(-3,-2,-1)	0.183	(0.202, 0.169, 0.192)
8	(-2,-1,0)	0.141	(0.158, 0.127, 0.157)
9	(0,0,0)	0.1	(0.088, 0.085, 0.157)
10	(0,1,2)	0.062	(0.088, 0.042, 0.092)
11	(1,2,3)	0	(0.088, 0, 0.069)
12	(2,3,4)	0.062	(0.118, 0.042, 0.069)
13	(3,4,5)	0.1	(0.158, 0.085, 0.092)
14	(4,5,6)	0.141	(0.202, 0.127, 0.123)
15	(5,6,7)	0.183	(0.247, 0.169, 0.157)
16	(6,7,8)	0.224	(0.294, 0.211, 0.192)
17	(7,8,8)	0.255	(0.341, 0.254, 0.192)

Table A7.3a: Fuzzy Accordant Index of $\widehat{T5}(\hat{r})$

Index	$\hat{r}$	FAI	$\left(AI^{l}, AI^{\pi}, AI^{u}\right)$
1	(-8,-8,-7)	0.355	(0.371, 0.346, 0.359)
2	(-8,-7,-6)	0.329	(0.371, 0.311, 0.328)
3	(-7,-6,-5)	0.295	(0.332, 0.277, 0.298)
4	(-6,-5,-4)	0.261	(0.294, 0.242, 0.268)
5	(-5,-4,-3)	0.226	(0.256, 0.207, 0.238)
6	(-4,-3,-2)	0.192	(0.219, 0.173, 0.209)
7	(-3,-2,-1)	0.158	(0.183, 0.138, 0.18)
8	(-2,-1,0)	0.125	(0.149, 0.104, 0.152)
9	(0,0,0)	0.092	(0.101, 0.069, 0.152)
10	(0,1,2)	0.059	(0.101, 0.035, 0.1)
11	(1,2,3)	0	(0.106, 0, 0.082)
12	(2,3,4)	0.059	(0.129, 0.035, 0.078)
13	(3,4,5)	0.092	(0.16, 0.069, 0.093)
14	(4,5,6)	0.125	(0.195, 0.104, 0.116)
15	(5,6,7)	0.158	(0.231, 0.138, 0.142)
16	(6,7,8)	0.192	(0.269, 0.173, 0.17)
17	(7,8,8)	0.218	(0.307, 0.207, 0.17)

Table A7.4a: Fuzzy Accordant Index of $\widehat{T6}(\hat{r})$

Index	$\hat{r}$	FAI	$\left(AI^{l}, AI^{\pi}, AI^{u}\right)$
1	(-8,-8,-7)	0.326	(0.348, 0.312, 0.333)
2	(-8,-7,-6)	0.305	(0.348, 0.284, 0.308)
3	(-7,-6,-5)	0.276	(0.316, 0.255, 0.283)
4	(-6,-5,-4)	0.248	(0.285, 0.227, 0.258)
5	(-5,-4,-3)	0.22	(0.253, 0.198, 0.234)
6	(-4,-3,-2)	0.192	(0.222, 0.17, 0.209)
7	(-3,-2,-1)	0.164	(0.192, 0.142, 0.185)
8	(-2,-1,0)	0.136	(0.163, 0.113, 0.161)
9	(0,0,0)	0.107	(0.113, 0.085, 0.161)
10	(0,1,2)	0.081	(0.113, 0.057, 0.116)
11	(1,2,3)	0.053	(0.099, 0.028, 0.097)
12	(2,3,4)	0	(0.105, 0, 0.081)
13	(3,4,5)	0.053	(0.124, 0.028, 0.077)
14	(4,5,6)	0.081	(0.15, 0.057, 0.088)
15	(5,6,7)	0.107	(0.178, 0.085, 0.106)
16	(6,7,8)	0.136	(0.207, 0.113, 0.127)
17	(7,8,8)	0.157	(0.238, 0.142, 0.127)

Table A7.1b: Fuzzy utility of FRAU and FLPS for $\widehat{T3}(\hat{r})$

Index	FAI	FRAU	FLPS
1	0.47	((4.667, 6., 7.667), (6.667, 7.333, 8.), (9.667, 10.667, 11.333))	((3.688, 7.822, 8.758), (3.736, 8.08, 8.758), (13.576, 8.098, 9.484))
2	0.435	((4.667, 6.333, 8.), (6.667, 7.333, 8.), (9.667, 10.333, 11.))	((4.857, 7.824, 8.676), (4.244, 8.098, 8.676), (11.899, 8.078, 9.648))
3	0.386	((5., 6.667, 8.333), (6.667, 7.333, 8.), (9.333, 10., 10.667))	((5.560, 7.733, 8.689), (5.505, 7.733, 8.478), (9.935, 8.533, 9.833))
4	0.338	((5.333, 7., 8.667), (6.667, 7.333, 8.), (9., 9.667, 10.333))	((6.215, 7.583, 8.790), (6.703, 7.583, 8.578), (8.081, 8.833, 9.631))
5	0.29	((5.667, 7.333, 9.), (6.667, 7.333, 8.), (8.667, 9.333, 10.))	((6.729, 7.826, 9.01), (6.602, 7.826, 8.576), (7.669, 8.349, 9.414))
6	0.242	((6., 7.667, 9.333), (6.667, 7.333, 8.), (8.333, 9., 9.667))	((6.464, 7.745, 9.235), (6.464, 7.745, 8.446), (8.073, 8.511, 9.319))
7	0.193	((6.333, 8., 9.667), (6.667, 7.333, 8.), (8., 8.667, 9.333))	((6.338, 7.991, 9.404), (6.338, 7.737, 8.192), (8.324, 8.272, 9.404))
8	0.145	((6.667, 8.333, 10.), (6.667, 7.333, 8.), (7.667, 8.333, 9.))	((6.667, 8.167, 9.6), (6.544, 7.667, 8.4), (7.789, 8.167, 9.))
9	0.097	((7.333, 8.667, 10.), (6.667, 7.333, 8.), (7., 8., 9.))	((7.2, 8.4, 9.6), (6.8, 7.60, 8.4), (7., 8., 9.))
10	0.048	((7.333, 9., 10.667), (6.667, 7.333, 8.), (7., 7.667, 8.333))	((7.2, 9., 10.667), (6.8, 7.429, 8.095), (7., 7.571, 8.238))
11	0	((7.667, 9.333, 11.), (6.667, 7.333, 8.), (6.667, 7.333, 8.))	((7.667, 9.333, 11.), (6.667, 7.333, 8.), (6.667, 7.333, 8.))
12	0.048	((8., 9.667, 11.333), (6.667, 7.333, 8.), (6.333, 7., 7.667))	((8., 9.667, 11.333), (6.571, 7.238, 7.905), (6.429, 7.095, 7.762))
13	0.097	((8.333, 10., 11.667), (6.667, 7.333, 8.), (6., 6.667, 7.333))	((8.333, 10., 11.667), (6.476, 7.143, 7.810), (6.19, 6.857, 7.524))
14	0.145	((8.667, 10.333, 12.), (6.667, 7.333, 8.), (5.667, 6.333, 7.))	((8.667, 10.333, 12.), (6.381, 7.048, 7.714), (5.952, 6.619, 7.286))
15	0.193	((9., 10.667, 12.333), (6.667, 7.333, 8.), (5.333, 6., 6.667))	((9., 10.667, 12.333), (6.286, 6.952, 7.619), (5.714, 6.381, 7.048))
16	0.242	((9.333, 11., 12.667), (6.667, 7.333, 8.), (5., 5.667, 6.333))	((9.333, 11., 12.667), (6.19, 6.857, 7.524), (5.476, 6.143, 6.810))
17	0.277	((9.667, 11.333, 12.667), (6.667, 7.333, 8.), (4.667, 5.333, 6.333))	((9.667, 11.333, 12.667), (6.095, 6.762, 7.524), (5.238, 5.905, 6.810))

Table A7.2b: Fuzzy utility vector of FRAU and FLPS for $\widehat{T4}(\tilde{r})$

	FAI	FRAU	FLPS
1	0.424	((5.75, 7.25, 9.), (6.75, 7.75, 8.75), (6., 6.75, 7.5), (9.5, 10.25, 10.75))	((6.764, 8.398, 9.532), (6.764, 7.785, 8.734), (5.884, 6.388, 7.834), (8.588, 9.428, 9.9))
2	0.393	((5.75, 7.5, 9.25), (6.75, 7.75, 8.75), (6., 6.75, 7.5), (9.5, 10., 10.5))	((6.715, 8.169, 9.248), (6.715, 7.968, 8.887), (5.998, 6.539, 8.01), (8.572, 9.323, 9.855))
3	0.351	((6., 7.75, 9.5), (6.75, 7.75, 8.75), (6., 6.75, 7.5), (9.25, 9.75, 10.25))	((6.802, 8.418, 9.498), (6.802, 7.92, 8.809), (5.961, 6.522, 7.916), (8.434, 9.14, 9.776))
4	0.308	((6.25, 8., 9.75), (6.75, 7.75, 8.75), (6., 6.75, 7.5), (9., 9.5, 10.))	((6.810, 8.153, 9.555), (6.810, 8.042, 8.879), (6.11, 6.622, 8.012), (8.27, 9.183, 9.555))
5	0.266	((6.5, 8.25, 10.), (6.75, 7.75, 8.75), (6., 6.75, 7.5), (8.75, 9.25, 9.75))	((6.813, 8.429, 9.506), (6.812, 7.964, 8.771), (6.271, 6.612, 8.181), (8.104, 8.995, 9.542))
6	0.224	((6.75, 8.5, 10.25), (6.75, 7.75, 8.75), (6., 6.75, 7.5), (8.5, 9., 9.5))	((7.012, 8.19, 9.504), (6.793, 8.009, 8.825), (6.264, 6.75, 8.168), (7.931, 9.051, 9.504))
7	0.183	((7., 8.75, 10.5), (6.75, 7.75, 8.75), (6., 6.75, 7.5), (8.25, 8.75, 9.25))	((6.978, 8.553, 9.403), (6.907, 8.335, 8.945), (6.447, 6.278, 8.248), (7.667, 8.834, 9.403))
8	0.141	((7.25, 9., 10.75), (6.75, 7.75, 8.75), (6., 6.75, 7.5), (8., 8.5, 9.))	((7.25, 8.607, 10.167), (6.850, 8.036, 8.833), (6.283, 6.75, 8.), (7.617, 8.607, 9.))
9	0.1	((7.75, 9.25, 10.75), (6.75, 7.75, 8.75), (6., 6.75, 7.5), (7.5, 8.25, 9.))	((7.616, 8.95, 10.167), (6.777, 7.95, 8.833), (6.259, 6.75, 8.), (7.348, 8.35, 9.))
10	0.062	((7.75, 9.5, 11.25), (6.75, 7.75, 8.75), (6., 6.75, 7.5), (7.5, 8., 8.5))	((7.616, 9.5, 11.25), (6.777, 7.813, 8.617), (6.259, 6.75, 7.783), (7.348, 7.937, 8.35))
11	0	((8., 9.75, 11.5), (6.75, 7.75, 8.75), (6., 6.75, 7.5), (7.25, 7.75, 8.25))	((8., 9.75, 11.5), (6.7, 7.75, 8.567), (6.233, 6.75, 7.733), (7.067, 7.75, 8.2))
12	0.062	((8.25, 10., 11.75), (6.75, 7.75, 8.75), (6., 6.75, 7.5), (7., 7.5, 8.))	((8.25, 10., 11.75), (6.65, 7.688, 8.517), (6.217, 6.75, 7.683), (6.883, 7.562, 8.05))
13	0.1	((8.5, 10.25, 12.), (6.75, 7.75, 8.75), (6., 6.75, 7.5), (6.75, 7.25, 7.75))	((8.5, 10.25, 12.), (6.600, 7.625, 8.467), (6.2, 6.75, 7.633), (6.7, 7.375, 7.9))
14	0.141	((8.75, 10.5, 12.25), (6.75, 7.75, 8.75), (6., 6.75, 7.5), (6.5, 7., 7.5))	((8.75, 10.5, 12.25), (6.55, 7.563, 8.417), (6.183, 6.75, 7.583), (6.517, 7.187, 7.75))
15	0.183	((9., 10.75, 12.5), (6.75, 7.75, 8.75), (6., 6.75, 7.5), (6.25, 6.75, 7.25))	((9., 10.75, 12.5), (6.5, 7.5, 8.367), (6.167, 6.75, 7.533), (6.333, 7., 7.60))
16	0.224	((9.25, 11., 12.75), (6.75, 7.75, 8.75), (6., 6.75, 7.5), (6., 6.5, 7.))	((9.25, 11., 12.75), (6.45, 7.438, 8.317), (6.15, 6.75, 7.483), (6.15, 6.812, 7.45))
17	0.255	((9.5, 11.25, 12.75), (6.75, 7.75, 8.75), (6., 6.75, 7.5), (5.75, 6.25, 7.))	((9.524, 11.248, 12.748), (6.384, 7.411, 8.363), (6.046, 6.614, 7.438), (6.046, 6.726, 7.451))

Table A7.3b1: Fuzzy utility vector of FRAU and FLPS for $\widehat{T5}(\hat{r})$ Part 1

Index	FAI	FRAU	FLPS
1	0.355	((6., 7.6, 9.4), (6.4, 7.6, 8.8), (5.6, 6.6, 7.6), (8., 8.6, 9.2), (9., 9.6, 10.))	((6.818, 8.302, 9.386), (6.536, 7.743, 8.895), (5.688, 6.6, 7.939), (7.777, 8.61, 9.386), (8.181, 8.745, 9.394))
2	0.329	((6., 7.8, 9.6), (6.4, 7.6, 8.8), (5.6, 6.6, 7.6), (8., 8.6, 9.2), (9., 9.4, 9.8))	((6.685, 8.043, 9.385), (6.599, 7.792, 8.943), (5.778, 6.545, 8.015), (7.75, 8.639, 9.273), (8.189, 8.981, 9.385))
3	0.295	((6.2, 8., 9.8), (6.4, 7.6, 8.8), (5.6, 6.6, 7.6), (8., 8.6, 9.2), (8.8, 9.2, 9.6))	((6.732, 8.177, 9.359), (6.583, 7.822, 8.953), (5.881, 6.551, 8.016), (7.734, 8.621, 9.312), (8.07, 8.829, 9.359))
4	0.261	((6.4, 8.2, 10.), (6.4, 7.6, 8.8), (5.6, 6.6, 7.6), (8., 8.6, 9.2), (8.6, 9., 9.4))	((6.874, 8.357, 9.366), (6.557, 7.808, 8.956), (5.874, 6.558, 8.028), (7.74, 8.613, 9.283), (7.955, 8.664, 9.366))
5	0.226	((6.6, 8.4, 10.2), (6.4, 7.6, 8.8), (5.6, 6.6, 7.6), (8., 8.6, 9.2), (8.4, 8.8, 9.2))	((7.588, 8.629, 9.286), (6.007, 7.799, 9.042), (5.222, 6.054, 8.237), (8.079, 8.629, 9.149), (8.104, 8.889, 9.286))
6	0.192	((6.8, 8.6, 10.4), (6.4, 7.6, 8.8), (5.6, 6.6, 7.6), (8., 8.6, 9.2), (8.2, 8.6, 9.))	((7.168, 8.516, 9.575), (6.559, 7.852, 8.904), (5.78, 6.613, 8.056), (7.757, 8.504, 9.271), (7.737, 8.516, 9.195))
7	0.158	((7., 8.8, 10.6), (6.4, 7.6, 8.8), (5.6, 6.6, 7.6), (8., 8.6, 9.2), (8., 8.4, 8.8))	((7.265, 8.533, 9.303), (6.61, 7.856, 9.036), (5.598, 6.488, 8.162), (7.919, 8.584, 9.25), (7.607, 8.539, 9.25))
8	0.125	((7.2, 9., 10.8), (6.4, 7.6, 8.8), (5.6, 6.6, 7.6), (8., 8.6, 9.2), (7.8, 8.2, 8.6))	((7.356, 8.54, 10.237), (6.527, 7.849, 8.875), (5.865, 6.6, 7.951), (7.747, 8.54, 9.107), (7.505, 8.472, 8.83))

Table A7.3b2: Fuzzy utility vector of FRAU and FLPS for $\widehat{T5}(\hat{r})$ Part 2

Index	FAI	FRAU	FLPS
9	0.092	((7.6, 9.2, 10.8), (6.4, 7.6, 8.8), (5.6, 6.6, 7.6), (8., 8.6, 9.2), (7.4, 8., 8.6))	((7.575, 8.899, 10.237), (6.485, 7.756, 8.875), (5.853, 6.6, 7.951), (7.757, 8.6, 9.107), (7.33, 8.145, 8.83))
10	0.059	((7.6, 9.4, 11.2), (6.4, 7.6, 8.8), (5.6, 6.6, 7.6), (8., 8.6, 9.2), (7.4, 7.8, 8.2))	((7.575, 9.4, 11.2), (6.485, 7.644, 8.731), (5.853, 6.6, 7.778), (7.757, 8.6, 9.066), (7.33, 7.756, 8.224))
11	0	((7.8, 9.6, 11.4), (6.4, 7.6, 8.8), (5.6, 6.6, 7.6), (8., 8.6, 9.2), (7.2, 7.6, 8.))	((7.79, 9.6, 11.4), (6.444, 7.6, 8.701), (5.841, 6.6, 7.742), (7.766, 8.6, 9.058), (7.158, 7.6, 8.099))
12	0.059	((8., 9.8, 11.6), (6.4, 7.6, 8.8), (5.6, 6.6, 7.6), (8., 8.6, 9.2), (7., 7.4, 7.8))	((7.935, 9.8, 11.6), (6.417, 7.556, 8.672), (5.833, 6.6, 7.707), (7.773, 8.6, 9.049), (7.042, 7.444, 7.973))
13	0.092	((8.2, 10., 11.8), (6.4, 7.6, 8.8), (5.6, 6.6, 7.6), (8., 8.6, 9.2), (6.8, 7.2, 7.6))	((8.08, 10., 11.8), (6.389, 7.511, 8.642), (5.826, 6.6, 7.671), (7.779, 8.6, 9.04), (6.926, 7.289, 7.847))
14	0.125	((8.4, 10.2, 12.), (6.4, 7.6, 8.8), (5.6, 6.6, 7.6), (8., 8.6, 9.2), (6.6, 7., 7.4))	((8.224, 10.2, 12.), (6.362, 7.467, 8.612), (5.818, 6.6, 7.635), (7.786, 8.6, 9.032), (6.811, 7.133, 7.721))
15	0.158	((8.6, 10.4, 12.2), (6.4, 7.6, 8.8), (5.6, 6.6, 7.6), (8., 8.6, 9.2), (6.4, 6.8, 7.2))	((8.369, 10.4, 12.2), (6.334, 7.422, 8.582), (5.81, 6.6, 7.599), (7.792, 8.6, 9.023), (6.695, 6.978, 7.596))
16	0.192	((8.8, 10.6, 12.4), (6.4, 7.6, 8.8), (5.6, 6.6, 7.6), (8., 8.6, 9.2), (6.2, 6.6, 7.))	((8.514, 10.6, 12.4), (6.307, 7.378, 8.552), (5.802, 6.6, 7.563), (7.798, 8.6, 9.015), (6.579, 6.822, 7.47))
17	0.218	((9., 10.8, 12.4), (6.4, 7.6, 8.8), (5.6, 6.6, 7.6), (8., 8.6, 9.2), (6., 6.4, 7.))	((8.659, 10.8, 12.4), (6.279, 7.333, 8.552), (5.794, 6.6, 7.563), (7.805, 8.6, 9.015), (6.463, 6.667, 7.47))

Table A7.4b1: Fuzzy utility vector of FRAU and FLPS for $\widehat{T6}(\hat{r})$ Part 1

Ind.	FAI	FRAU	FLPS
1	0.326	((6., 7.667, 9.5), (6., 7.5, 9.), (5.5, 6.5, 7.5), (4.667, 5.5, 6.333), (12., 12.5, 13.), (7.833, 8.333, 8.667))	((6.529, 7.558, 8.741), (6.062, 7.537, 8.877), (5.459, 6.783, 7.72), (4.865, 5.5, 6.935), (12.045, 12.5, 12.986), (7.04, 8.121, 8.741))
2	0.305	((6., 7.833, 9.667), (6., 7.5, 9.), (5.5, 6.5, 7.5), (4.667, 5.5, 6.333), (12., 12.5, 13.), (7.833, 8.167, 8.5))	((6.378, 7.704, 8.811), (6.09, 7.805, 8.664), (5.599, 6.61, 7.767), (4.9, 5.5, 6.953), (11.992, 12.5, 12.995), (7.041, 7.881, 8.811))
3	0.276	((6.167, 8., 9.833), (6., 7.5, 9.), (5.5, 6.5, 7.5), (4.667, 5.5, 6.333), (12., 12.5, 13.), (7.667, 8., 8.333))	((6.11, 7.716, 8.787), (6.095, 7.818, 8.624), (5.63, 6.642, 7.856), (4.908, 5.5, 6.947), (11.998, 12.5, 13.), (7.259, 7.823, 8.787))
4	0.248	((6.333, 8.167, 10.), (6., 7.5, 9.), (5.5, 6.5, 7.5), (4.667, 5.5, 6.333), (12., 12.5, 13.), (7.5, 7.833, 8.167))	((6.034, 7.786, 8.843), (5.914, 7.671, 8.783), (5.373, 6.757, 7.833), (4.843, 5.5, 6.907), (11.938, 12.5, 13.006), (7.898, 7.786, 8.628))
5	0.22	((6.5, 8.333, 10.167), (6., 7.5, 9.), (5.5, 6.5, 7.5), (4.667, 5.5, 6.333), (12., 12.5, 13.), (7.333, 7.667, 8.))	((6.483, 7.746, 8.77), (6.083, 7.724, 8.748), (5.695, 6.814, 7.813), (4.924, 5.5, 6.934), (12., 12.47, 13.001), (6.815, 7.746, 8.735))
6	0.192	((6.667, 8.5, 10.333), (6., 7.5, 9.), (5.5, 6.5, 7.5), (4.667, 5.5, 6.333), (12., 12.5, 13.), (7.167, 7.5, 7.833))	((6.397, 7.777, 8.975), (5.932, 7.769, 8.667), (5.374, 6.677, 7.855), (4.843, 5.5, 6.898), (11.979, 12.5, 13.013), (7.475, 7.777, 8.592))
7	0.164	((6.833, 8.667, 10.5), (6., 7.5, 9.), (5.5, 6.5, 7.5), (4.667, 5.5, 6.333), (12., 12.5, 13.), (7., 7.333, 7.667))	((6.54, 7.747, 8.897), (6.067, 7.733, 8.897), (5.656, 6.788, 7.765), (4.914, 5.5, 6.887), (12.001, 12.5, 13.004), (6.823, 7.733, 8.55))
8	0.136	((7., 8.833, 10.667), (6., 7.5, 9.), (5.5, 6.5, 7.5), (4.667, 5.5, 6.333), (12., 12.5, 13.), (6.833, 7.167, 7.5))	((6.57, 7.829, 9.991), (5.828, 7.77, 9.128), (5.74, 6.704, 7.539), (5.097, 5.258, 6.733), (12.194, 12.608, 13.105), (6.57, 7.831, 7.504))

Table A7.4b2: Fuzzy utility vector of FRAU and FLPS for $\widehat{T6}(\hat{r})$ Part 2

Index	FAI	FRAU	FLPS
9	0.107	((7.333, 9., 10.667), (6., 7.5, 9.), (5.5, 6.5, 7.5), (4.667, 5.5, 6.333), (12., 12.5, 13.), (6.5, 7., 7.5))	((7.093, 8.562, 9.946), (6., 7.5, 9.), (5.63, 6.688, 7.571), (4.907, 5.5, 6.696), (12., 12.5, 13.), (6.37, 7.25, 7.786))
10	0.081	((7.333, 9.167, 11.), (6., 7.5, 9.), (5.5, 6.5, 7.5), (4.667, 5.5, 6.333), (12., 12.5, 13.), (6.5, 6.833, 7.167))	((7.093, 9.167, 11.), (6., 7.5, 9.), (5.63, 6.567, 7.396), (4.907, 5.5, 6.521), (12., 12.5, 13.), (6.37, 6.767, 7.083))
11	0.053	((7.5, 9.333, 11.167), (6., 7.5, 9.), (5.5, 6.5, 7.5), (4.667, 5.5, 6.333), (12., 12.5, 13.), (6.333, 6.667, 7.))	((7.5, 9.333, 11.167), (5.935, 7.5, 9.), (5.571, 6.533, 7.368), (4.893, 5.5, 6.493), (12., 12.5, 13.), (6.101, 6.633, 6.972))
12	0	((7.667, 9.5, 11.333), (6., 7.5, 9.), (5.5, 6.5, 7.5), (4.667, 5.5, 6.333), (12., 12.5, 13.), (6.167, 6.5, 6.833))	((7.667, 9.5, 11.333), (5.908, 7.5, 9.), (5.548, 6.5, 7.34), (4.887, 5.5, 6.465), (12., 12.5, 13.), (5.991, 6.5, 6.861))
13	0.053	((7.833, 9.667, 11.5), (6., 7.5, 9.), (5.5, 6.5, 7.5), (4.667, 5.5, 6.333), (12., 12.5, 13.), (6., 6.333, 6.667))	((7.833, 9.667, 11.5), (5.881, 7.5, 9.), (5.524, 6.467, 7.312), (4.881, 5.5, 6.438), (12., 12.5, 13.), (5.881, 6.367, 6.75))
14	0.081	((8., 9.833, 11.667), (6., 7.5, 9.), (5.5, 6.5, 7.5), (4.667, 5.5, 6.333), (12., 12.5, 13.), (5.833, 6.167, 6.5))	((8., 9.833, 11.667), (5.854, 7.5, 9.), (5.5, 6.433, 7.285), (4.875, 5.5, 6.41), (12., 12.5, 13.), (5.771, 6.233, 6.639))
15	0.107	((8.167, 10., 11.833), (6., 7.5, 9.), (5.5, 6.5, 7.5), (4.667, 5.5, 6.333), (12., 12.5, 13.), (5.667, 6., 6.333))	((8.167, 10., 11.833), (5.827, 7.5, 9.), (5.476, 6.4, 7.257), (4.869, 5.5, 6.382), (12., 12.5, 13.), (5.661, 6.1, 6.528))
16	0.136	((8.333, 10.167, 12.), (6., 7.5, 9.), (5.5, 6.5, 7.5), (4.667, 5.5, 6.333), (12., 12.5, 13.), (5.5, 5.833, 6.167))	((8.333, 10.167, 12.), (5.801, 7.5, 9.), (5.452, 6.367, 7.229), (4.863, 5.5, 6.354), (12., 12.5, 13.), (5.551, 5.967, 6.417))
17	0.157	((8.5, 10.333, 12.), (6., 7.5, 9.), (5.5, 6.5, 7.5), (4.667, 5.5, 6.333), (12., 12.5, 13.), (5.333, 5.667, 6.167))	((8.5, 10.333, 12.), (5.774, 7.5, 9.), (5.429, 6.333, 7.229), (4.857, 5.5, 6.354), (12., 12.5, 13.), (5.44, 5.833, 6.417))

Table A7.1c: Fuzzy Root Mean Square Variances of FRAU and FLPS for $\widehat{T3}(\hat{r})$

Index	FAI	FRMSV(FRAU) $\left(\hat{\sigma},\left(\sigma^{l},\sigma^{\pi},\sigma^{u}\right)\right)$	FRMSV(FLPS) $\left(\hat{\sigma},\left(\sigma^{l},\sigma^{\pi},\sigma^{u}\right)\right)$
1	0.47	(3.25, (3., 3.333, 3.333))	(4.787, (5.817, 4.646, 4.037))
2	0.435	(3., (3., 3., 3.))	(4.027, (4.46, 4.11, 3.427))
3	0.386	(2.667, (2.667, 2.667, 2.667))	(3.095, (3.023, 3.25, 2.857))
4	0.338	(2.333, (2.333, 2.333, 2.333))	(2.57, (2.658, 2.558, 2.507))
5	0.29	(2., (2., 2., 2.))	(2.327, (2.475, 2.335, 2.162))
6	0.242	(1.667, (1.667, 1.667, 1.667))	(1.774, (1.761, 1.787, 1.763))
7	0.193	(1.333, (1.333, 1.333, 1.333))	(1.42, (1.411, 1.448, 1.374))
8	0.145	(1., (1., 1., 1.))	(1.081, (1.015, 1.08, 1.149))
9	0.097	(0.667, (0.333, 0.667, 1.))	(0.766, (0.383, 0.766, 1.149))
10	0.048	(0.333, (0.333, 0.333, 0.333))	(0.365, (0.383, 0.36, 0.36))
11	0	(0, (0, 0, 0))	(0, (0, 0, 0))
12	0.048	(0.333, (0.333, 0.333, 0.333))	(0.36, (0.36, 0.36, 0.36))
13	0.097	(0.667, (0.667, 0.667, 0.667))	(0.719, (0.719, 0.719, 0.719))
14	0.145	(1., (1., 1., 1.))	(1.079, (1.079, 1.079, 1.079))
15	0.193	(1.333, (1.333, 1.333, 1.333))	(1.438, (1.438, 1.438, 1.438))
16	0.242	(1.667, (1.667, 1.667, 1.667))	(1.798, (1.798, 1.798, 1.798))
17	0.277	(1.917, (2., 2., 1.667))	(2.067, (2.157, 2.157, 1.798))

Table A7.2c: Fuzzy Root Mean Square Variances of FRAU and FLPS for $\widehat{T4}(\hat{r})$

Index	FAI	FRMSV(FRAU) $\left(\hat{\sigma},\left(\sigma^{l},\sigma^{\pi},\sigma^{u}\right)\right)$	FRMSV(FLPS) $\left(\hat{\sigma},\left(\sigma^{l},\sigma^{\pi},\sigma^{u}\right)\right)$
1	0.424	(2.836, (2.5, 2.887, 3.069))	(3.043, (2.738, 3.123, 3.188))
2	0.393	(2.62, (2.5, 2.598, 2.784))	(2.76, (2.729, 2.723, 2.866))
3	0.351	(2.334, (2.217, 2.309, 2.5))	(2.458, (2.407, 2.436, 2.553))
4	0.308	(2.049, (1.936, 2.021, 2.217))	(2.122, (2.08, 2.058, 2.293))
5	0.266	(1.765, (1.658, 1.732, 1.936))	(1.84, (1.773, 1.763, 2.062))
6	0.224	(1.482, (1.384, 1.443, 1.658))	(1.576, (1.492, 1.481, 1.85))
7	0.183	(1.203, (1.118, 1.155, 1.384))	(1.421, (1.275, 1.319, 1.77))
8	0.141	(0.929, (0.866, 0.866, 1.118))	(1.037, (0.953, 0.957, 1.284))
9	0.1	(0.693, (0.5, 0.577, 1.118))	(0.79, (0.568, 0.653, 1.284))
10	0.062	(0.431, (0.5, 0.289, 0.645))	(0.467, (0.568, 0.298, 0.705))
11	0	(0.25, (0.5, 0, 0.5))	(0.279, (0.557, 0, 0.557))
12	0.062	(0.431, (0.645, 0.289, 0.5))	(0.458, (0.681, 0.298, 0.557))
13	0.1	(0.667, (0.866, 0.577, 0.645))	(0.696, (0.891, 0.595, 0.705))
14	0.141	(0.929, (1.118, 0.866, 0.866))	(0.965, (1.14, 0.893, 0.933))
15	0.183	(1.203, (1.384, 1.155, 1.118))	(1.246, (1.408, 1.19, 1.196))
16	0.224	(1.482, (1.658, 1.443, 1.384))	(1.534, (1.685, 1.488, 1.475))
17	0.255	(1.696, (1.936, 1.732, 1.384))	(1.761, (1.975, 1.8, 1.468))

Table A7.3c: Fuzzy Root Mean Square Variances of FRAU and FLPS for $\widehat{T5}(\hat{r})$

Index	FAI	FRMSV(FRAU) $\left(\hat{\sigma},\left(\sigma^{l},\sigma^{\pi},\sigma^{u}\right)\right)$	FRMSV(FLPS) $\left(\hat{\sigma},\left(\sigma^{l},\sigma^{\pi},\sigma^{u}\right)\right)$
1	0.355	(2.418, (2.107, 2.449, 2.665))	(2.532, (2.269, 2.573, 2.713))
2	0.329	(2.235, (2.107, 2.205, 2.425))	(2.298, (2.252, 2.236, 2.467))
3	0.295	(1.994, (1.871, 1.96, 2.186))	(2.053, (2., 1.988, 2.238))
4	0.261	(1.754, (1.637, 1.715, 1.949))	(1.816, (1.756, 1.741, 2.027))
5	0.226	(1.515, (1.407, 1.47, 1.715))	(1.65, (1.632, 1.536, 1.896))
6	0.192	(1.279, (1.183, 1.225, 1.483))	(1.349, (1.278, 1.242, 1.634))
7	0.158	(1.047, (0.97, 0.98, 1.257))	(1.182, (1.038, 1.022, 1.646))
8	0.125	(0.821, (0.775, 0.735, 1.039))	(0.925, (0.855, 0.846, 1.155))
9	0.092	(0.637, (0.529, 0.49, 1.039))	(0.714, (0.59, 0.555, 1.155))
10	0.059	(0.421, (0.529, 0.245, 0.663))	(0.443, (0.59, 0.249, 0.684))
11	0	(0.274, (0.548, 0, 0.548))	(0.293, (0.599, 0, 0.574))
12	0.059	(0.421, (0.663, 0.245, 0.529))	(0.442, (0.704, 0.249, 0.566))
13	0.092	(0.608, (0.837, 0.49, 0.616))	(0.633, (0.875, 0.498, 0.662))
14	0.125	(0.821, (1.039, 0.735, 0.775))	(0.85, (1.079, 0.747, 0.827))
15	0.158	(1.047, (1.257, 0.98, 0.97))	(1.081, (1.302, 0.996, 1.028))
16	0.192	(1.279, (1.483, 1.225, 1.183))	(1.318, (1.536, 1.245, 1.249))
17	0.218	(1.459, (1.715, 1.47, 1.183))	(1.503, (1.775, 1.494, 1.249))

Table A7.4c: Fuzzy Root Mean Square Variances of FRAU and FLPS for $\widehat{T6}(\hat{r})$

Index	FAI	FRMSV(FRAU) $\left(\hat{\sigma},\left(\sigma^{l},\sigma^{\pi},\sigma^{u}\right)\right)$	FRMSV(FLPS) $\left(\hat{\sigma},\left(\sigma^{l},\sigma^{\pi},\sigma^{u}\right)\right)$
1	0.326	(2.294, (2.011, 2.319, 2.525))	(2.342, (2.104, 2.331, 2.604))
2	0.305	(2.137, (2.011, 2.108, 2.319))	(2.196, (2.093, 2.127, 2.437))
3	0.276	(1.929, (1.807, 1.897, 2.113))	(1.992, (1.835, 1.922, 2.29))
4	0.248	(1.722, (1.606, 1.687, 1.909))	(1.796, (1.643, 1.715, 2.112))
5	0.22	(1.516, (1.406, 1.476, 1.706))	(1.777, (1.499, 1.777, 2.056))
6	0.192	(1.312, (1.211, 1.265, 1.506))	(1.462, (1.247, 1.372, 1.859))
7	0.164	(1.11, (1.022, 1.054, 1.308))	(1.451, (1.258, 1.351, 1.846))
8	0.136	(0.911, (0.843, 0.843, 1.116))	(1.133, (0.968, 1.168, 1.226))
9	0.107	(0.735, (0.558, 0.632, 1.116))	(0.821, (0.609, 0.718, 1.241))
10	0.081	(0.54, (0.558, 0.422, 0.76))	(0.559, (0.609, 0.426, 0.774))
11	0.053	(0.383, (0.494, 0.211, 0.615))	(0.398, (0.539, 0.213, 0.629))
12	0	(0.258, (0.516, 0, 0.516))	(0.271, (0.55, 0, 0.533))
13	0.053	(0.383, (0.615, 0.211, 0.494))	(0.395, (0.639, 0.213, 0.515))
14	0.081	(0.54, (0.76, 0.422, 0.558))	(0.553, (0.778, 0.426, 0.583))
15	0.107	(0.72, (0.931, 0.632, 0.683))	(0.734, (0.946, 0.639, 0.712))
16	0.136	(0.911, (1.116, 0.843, 0.843))	(0.927, (1.13, 0.852, 0.875))
17	0.157	(1.065, (1.308, 1.054, 0.843))	(1.082, (1.324, 1.065, 0.875))

Table A7.1d: Fuzzy Root Mean Penalty Weighted Square Variances of FRAU and FLPS for $\widehat{T3}(\hat{r})$

Index	FAI	FRMPWSV(FRAU) $\left(\hat{\sigma},\left(\sigma^{l},\sigma^{\pi},\sigma^{u}\right)\right)$	FRMPWSV(FLPS) $\left(\hat{\sigma},\left(\sigma^{l},\sigma^{\pi},\sigma^{u}\right)\right)$
1	0.47	(5.934, (5.477, 6.086, 6.086))	(8.632, (12.856, 7.346, 6.977))
2	0.435	(5.639, (5.477, 5.477, 6.124))	(7.265, (9.928, 6.499, 6.135))
3	0.386	(4.869, (4.869, 4.869, 4.869))	(5.331, (6.26, 5.19, 4.683))
4	0.338	(4.26, (4.26, 4.26, 4.26))	(4.407, (4.387, 4.581, 4.079))
5	0.29	(3.867, (3.651, 4.082, 3.651))	(3.746, (4.033, 3.723, 3.503))
6	0.242	(3.043, (3.043, 3.043, 3.043))	(3.262, (3.278, 3.436, 2.899))
7	0.193	(2.434, (2.434, 2.434, 2.434))	(2.499, (2.875, 2.341, 2.438))
8	0.145	(2.041, (2.041, 2.041, 2.041))	(1.98, (1.966, 1.99, 1.975))
9	0.097	(1.361, (0.68, 1.361, 2.041))	(1.317, (0.658, 1.317, 1.975))
10	0.048	(0.627, (0.68, 0.609, 0.609))	(0.602, (0.658, 0.583, 0.583))
11	0	(0, (0, 0, 0))	(0, (0, 0, 0))
12	0.048	(0.609, (0.609, 0.609, 0.609))	(0.583, (0.583, 0.583, 0.583))
13	0.097	(1.217, (1.217, 1.217, 1.217))	(1.166, (1.166, 1.166, 1.166))
14	0.145	(1.826, (1.826, 1.826, 1.826))	(1.75, (1.75, 1.75, 1.75))
15	0.193	(2.434, (2.434, 2.434, 2.434))	(2.333, (2.333, 2.333, 2.333))
16	0.242	(3.043, (3.043, 3.043, 3.043))	(2.916, (2.916, 2.916, 2.916))
17	0.277	(3.499, (3.651, 3.651, 3.043))	(3.353, (3.499, 3.499, 2.916))

Table A7.2d: Fuzzy Root Mean Penalty Weighted Square Variance s of FRAU and FLPS for $\widehat{T4}(\hat{r})$

Index	FAI	FRMPWSV(FRAU) $\left(\hat{\sigma},\left(\sigma^{l},\sigma^{\pi},\sigma^{u}\right)\right)$	FRMPWSV(FLPS) $\left(\hat{\sigma},\left(\sigma^{l},\sigma^{\pi},\sigma^{u}\right)\right)$
1	0.424	(4.871, (4.36, 4.841, 5.441))	(4.973, (4.523, 5.05, 5.268))
2	0.393	(4.504, (4.36, 4.357, 4.94))	(4.504, (4.502, 4.394, 4.727))
3	0.351	(4.176, (4.098, 4.082, 4.44))	(4.026, (3.986, 3.931, 4.257))
4	0.308	(3.536, (3.423, 3.389, 3.943))	(3.691, (3.51, 3.336, 4.582))
5	0.266	(3.055, (2.962, 2.905, 3.45))	(3.023, (2.946, 2.866, 3.413))
6	0.224	(2.599, (2.592, 2.421, 2.962))	(2.612, (2.494, 2.437, 3.081))
7	0.183	(2.303, (2.073, 2.327, 2.484))	(2.307, (2.096, 2.111, 2.911))
8	0.141	(1.721, (1.661, 1.452, 2.316))	(1.751, (1.628, 1.557, 2.262))
9	0.1	(1.425, (1.058, 1.164, 2.316))	(1.392, (1.041, 1.133, 2.262))
10	0.062	(0.811, (1.058, 0.484, 1.218))	(0.795, (1.041, 0.477, 1.186))
11	0	(0.491, (0.982, 0, 0.982))	(0.48, (0.961, 0, 0.961))
12	0.062	(0.773, (1.141, 0.484, 0.982))	(0.759, (1.123, 0.477, 0.961))
13	0.1	(1.152, (1.452, 0.968, 1.218))	(1.132, (1.433, 0.955, 1.186))
14	0.141	(1.584, (1.84, 1.452, 1.589))	(1.556, (1.818, 1.432, 1.54))
15	0.183	(2.092, (2.265, 2.041, 2.022))	(2.003, (2.239, 1.909, 1.955))
16	0.224	(2.582, (3.002, 2.421, 2.484))	(2.463, (2.679, 2.387, 2.399))
17	0.255	(2.865, (3.166, 2.905, 2.484))	(2.823, (3.138, 2.88, 2.395))

Table A7.3d: Fuzzy Root Mean Penalty Weighted Square Variances of FRAU and FLPS for $\widehat{T5}(\hat{r})$

Index	FAI	FRMPWSV(FRAU) $\left(\hat{\sigma},\left(\sigma^{l},\sigma^{\pi},\sigma^{u}\right)\right)$	FRMPWSV(FLPS) $\left(\hat{\sigma},\left(\sigma^{l},\sigma^{\pi},\sigma^{u}\right)\right)$
1	0.355	(4.11, (3.768, 4.123, 4.426))	(4.076, (3.741, 4.099, 4.366))
2	0.329	(3.748, (3.768, 3.6, 4.025))	(3.965, (3.716, 3.584, 4.977))
3	0.295	(3.348, (3.365, 3.2, 3.626))	(3.517, (3.31, 3.183, 4.393))
4	0.261	(2.96, (3.008, 2.8, 3.231))	(2.944, (2.918, 2.786, 3.285))
5	0.226	(2.597, (2.577, 2.4, 3.012))	(2.712, (2.824, 2.489, 3.047))
6	0.192	(2.463, (2.2, 2.598, 2.458))	(2.451, (2.161, 2.494, 2.657))
7	0.158	(1.782, (1.844, 1.6, 2.086))	(1.948, (1.825, 1.651, 2.663))
8	0.125	(1.494, (1.523, 1.2, 2.052))	(1.606, (1.493, 1.462, 2.009))
9	0.092	(1.292, (1.118, 1., 2.052))	(1.265, (1.103, 0.975, 2.009))
10	0.059	(0.767, (1.118, 0.4, 1.149))	(0.761, (1.103, 0.398, 1.145))
11	0	(0.527, (1.114, 0, 0.995))	(0.522, (1.099, 0, 0.99))
12	0.059	(0.766, (1.265, 0.4, 1.))	(0.758, (1.25, 0.398, 0.988))
13	0.092	(1.071, (1.523, 0.8, 1.162))	(1.059, (1.506, 0.795, 1.141))
14	0.125	(1.418, (1.844, 1.2, 1.428))	(1.401, (1.823, 1.193, 1.398))
15	0.158	(1.788, (2.2, 1.6, 1.752))	(1.767, (2.174, 1.59, 1.713))
16	0.192	(2.202, (2.577, 2.062, 2.107))	(2.145, (2.545, 1.988, 2.061))
17	0.218	(2.468, (2.966, 2.4, 2.107))	(2.44, (2.929, 2.385, 2.061))

Table A7.4d: Fuzzy Root Mean Penalty Weighted Square Variances of FRAU and FLPS for $\widetilde{T6}(\hat{r})$

Index	FAI	FRMPWSV(FRAU) $\left(\hat{\sigma},\left(\sigma^{l},\sigma^{\pi},\sigma^{u}\right)\right)$	FRMPWSV(FLPS) $\left(\hat{\sigma},\left(\sigma^{l},\sigma^{\pi},\sigma^{u}\right)\right)$
1	0.326	(3.767, (3.45, 3.742, 4.133))	(3.771, (3.43, 3.726, 4.203))
2	0.305	(3.512, (3.45, 3.402, 3.794))	(3.54, (3.402, 3.403, 3.95))
3	0.276	(3.598, (3.093, 3.921, 3.457))	(3.226, (3.035, 3.077, 3.718))
4	0.248	(2.832, (2.765, 2.722, 3.122))	(3.299, (2.927, 3.42, 3.428))
5	0.22	(2.498, (2.441, 2.381, 2.79))	(2.977, (2.774, 2.888, 3.355))
6	0.192	(2.187, (2.125, 2.082, 2.462))	(2.427, (2.089, 2.216, 3.186))
7	0.164	(1.84, (1.819, 1.701, 2.14))	(2.341, (2.069, 2.153, 2.988))
8	0.136	(1.623, (1.532, 1.361, 2.238))	(1.909, (1.618, 1.904, 2.21))
9	0.107	(1.494, (1.122, 1.307, 2.238))	(1.466, (1.113, 1.276, 2.198))
10	0.081	(0.935, (1.122, 0.68, 1.256))	(0.93, (1.113, 0.678, 1.252))
11	0.053	(0.665, (0.95, 0.34, 1.03))	(0.66, (0.936, 0.339, 1.026))
12	0	(0.459, (0.95, 0, 0.885))	(0.455, (0.941, 0, 0.88))
13	0.053	(0.653, (1.069, 0.34, 0.863))	(0.649, (1.063, 0.339, 0.856))
14	0.081	(0.902, (1.275, 0.68, 0.974))	(0.897, (1.27, 0.678, 0.963))
15	0.107	(1.188, (1.532, 1.021, 1.18))	(1.182, (1.527, 1.017, 1.165))
16	0.136	(1.496, (1.819, 1.361, 1.442))	(1.487, (1.812, 1.356, 1.423))
17	0.157	(1.742, (2.125, 1.701, 1.442))	(1.732, (2.115, 1.696, 1.423))

Table A7.1e: Fuzzy Mean Contradiction of FRAU and FLPS for $\widehat{T3}(\hat{r})$

Index	FAI	FMC(FRAU) $\left(\hat{\sigma},\left(\sigma^{l},\sigma^{\pi},\sigma^{u}\right)\right)$	FMC(FLPS) $\left(\hat{\sigma},\left(\sigma^{l},\sigma^{\pi},\sigma^{u}\right)\right)$
1	0.47	(0.5, (0.5, 0.5, 0.5))	(0.458, (0.5, 0.5, 0.333))
2	0.435	(0.458, (0.5, 0.5, 0.333))	(0.375, (0.167, 0.5, 0.333))
3	0.386	(0.417, (0.5, 0.5, 0.167))	(0.167, (0.167, 0.167, 0.167))
4	0.338	(0.417, (0.5, 0.5, 0.167))	(0.333, (0.5, 0.333, 0.167))
5	0.29	(0.333, (0.5, 0.333, 0.167))	(0.167, (0.167, 0.167, 0.167))
6	0.242	(0.25, (0.5, 0.167, 0.167))	(0.292, (0.333, 0.333, 0.167))
7	0.193	(0.333, (0.5, 0.167, 0.5))	(0.167, (0.167, 0.167, 0.167))
8	0.145	(0.333, (0.333, 0.333, 0.333))	(0.292, (0.167, 0.333, 0.333))
9	0.097	(0.333, (0.333, 0.333, 0.333))	(0.333, (0.333, 0.333, 0.333))
10	0.048	(0.208, (0.333, 0.167, 0.167))	(0.208, (0.333, 0.167, 0.167))
11	0	(0, (0, 0, 0))	(0, (0, 0, 0))
12	0.048	(0.167, (0.167, 0.167, 0.167))	(0.167, (0.167, 0.167, 0.167))
13	0.097	(0.167, (0.167, 0.167, 0.167))	(0.167, (0.167, 0.167, 0.167))
14	0.145	(0.167, (0.167, 0.167, 0.167))	(0.167, (0.167, 0.167, 0.167))
15	0.193	(0.167, (0.167, 0.167, 0.167))	(0.167, (0.167, 0.167, 0.167))
16	0.242	(0.167, (0.167, 0.167, 0.167))	(0.167, (0.167, 0.167, 0.167))
17	0.277	(0.167, (0.167, 0.167, 0.167))	(0.167, (0.167, 0.167, 0.167))

Table A7.2e: Fuzzy Mean Contradiction of FRAU and FLPS for $\widehat{T4}(\hat{r})$

Index	FAI	FMC(FRAU) $\left(\hat{\sigma},\left(\sigma^{l},\sigma^{\pi},\sigma^{u}\right)\right)$	FMC(FLPS) $\left(\hat{\sigma},\left(\sigma^{l},\sigma^{\pi},\sigma^{u}\right)\right)$
1	0.424	(0.292, (0.5, 0.25, 0.167))	(0.167, (0.333, 0.083, 0.167))
2	0.393	(0.292, (0.5, 0.25, 0.167))	(0.167, (0.333, 0.083, 0.167))
3	0.351	(0.229, (0.417, 0.167, 0.167))	(0.167, (0.333, 0.083, 0.167))
4	0.308	(0.167, (0.333, 0.083, 0.167))	(0.167, (0.25, 0.083, 0.25))
5	0.266	(0.208, (0.333, 0.083, 0.333))	(0.125, (0.167, 0.083, 0.167))
6	0.224	(0.188, (0.25, 0.083, 0.333))	(0.125, (0.167, 0.083, 0.167))
7	0.183	(0.208, (0.167, 0.167, 0.333))	(0.125, (0.167, 0.083, 0.167))
8	0.141	(0.229, (0.167, 0.25, 0.25))	(0.146, (0.167, 0.083, 0.25))
9	0.1	(0.208, (0.25, 0.167, 0.25))	(0.208, (0.25, 0.167, 0.25))
10	0.062	(0.146, (0.25, 0.083, 0.167))	(0.146, (0.25, 0.083, 0.167))
11	0	(0.083, (0.167, 0, 0.167))	(0.125, (0.167, 0.083, 0.167))
12	0.062	(0.125, (0.167, 0.083, 0.167))	(0.125, (0.167, 0.083, 0.167))
13	0.1	(0.104, (0.083, 0.083, 0.167))	(0.125, (0.167, 0.083, 0.167))
14	0.141	(0.104, (0.167, 0.083, 0.083))	(0.125, (0.167, 0.083, 0.167))
15	0.183	(0.167, (0.167, 0.167, 0.167))	(0.125, (0.167, 0.083, 0.167))
16	0.224	(0.229, (0.25, 0.25, 0.167))	(0.125, (0.167, 0.083, 0.167))
17	0.255	(0.25, (0.333, 0.25, 0.167))	(0.125, (0.167, 0.083, 0.167))

Table A7.3e: Fuzzy Mean Contradiction of FRAU and FLPS for $\widehat{T5}(\hat{r})$

Index	FAI	FMC(FRAU) $\left(\hat{\sigma},\left(\sigma^{l},\sigma^{\pi},\sigma^{u}\right)\right)$	FMC(FLPS) $\left(\hat{\sigma},\left(\sigma^{l},\sigma^{\pi},\sigma^{u}\right)\right)$
1	0.355	(0.288, (0.3, 0.3, 0.25))	(0.237, (0.2, 0.25, 0.25))
2	0.329	(0.262, (0.3, 0.25, 0.25))	(0.25, (0.2, 0.25, 0.3))
3	0.295	(0.288, (0.3, 0.25, 0.35))	(0.25, (0.2, 0.25, 0.3))
4	0.261	(0.275, (0.25, 0.25, 0.35))	(0.237, (0.2, 0.25, 0.25))
5	0.226	(0.25, (0.2, 0.25, 0.3))	(0.188, (0.2, 0.15, 0.25))
6	0.192	(0.213, (0.2, 0.2, 0.25))	(0.213, (0.2, 0.2, 0.25))
7	0.158	(0.162, (0.15, 0.15, 0.2))	(0.188, (0.2, 0.15, 0.25))
8	0.125	(0.175, (0.2, 0.15, 0.2))	(0.2, (0.2, 0.2, 0.2))
9	0.092	(0.162, (0.25, 0.1, 0.2))	(0.162, (0.25, 0.1, 0.2))
10	0.059	(0.125, (0.25, 0.05, 0.15))	(0.125, (0.25, 0.05, 0.15))
11	0	(0.088, (0.2, 0, 0.15))	(0.088, (0.2, 0, 0.15))
12	0.059	(0.1, (0.15, 0.05, 0.15))	(0.112, (0.2, 0.05, 0.15))
13	0.092	(0.1, (0.2, 0.05, 0.1))	(0.112, (0.2, 0.05, 0.15))
14	0.125	(0.112, (0.2, 0.05, 0.15))	(0.112, (0.2, 0.05, 0.15))
15	0.158	(0.1, (0.15, 0.05, 0.15))	(0.112, (0.2, 0.05, 0.15))
16	0.192	(0.138, (0.2, 0.1, 0.15))	(0.112, (0.2, 0.05, 0.15))
17	0.218	(0.163, (0.2, 0.15, 0.15))	(0.112, (0.2, 0.05, 0.15))

Table A7.4e: Fuzzy Mean Contradiction of FRAU and FLPS for $\widehat{T6}(\hat{r})$

Index	FAI	FMC(FRAU) $\left(\hat{\sigma},\left(\sigma^{l},\sigma^{\pi},\sigma^{u}\right)\right)$	FMC(FLPS) $\left(\hat{\sigma},\left(\sigma^{l},\sigma^{\pi},\sigma^{u}\right)\right)$
1	0.326	(0.125, (0.167, 0.1, 0.133))	(0.117, (0.133, 0.1, 0.133))
2	0.305	(0.125, (0.167, 0.1, 0.133))	(0.15, (0.133, 0.167, 0.133))
3	0.276	(0.133, (0.133, 0.133, 0.133))	(0.15, (0.133, 0.167, 0.133))
4	0.248	(0.15, (0.133, 0.167, 0.133))	(0.133, (0.133, 0.133, 0.133))
5	0.22	(0.15, (0.133, 0.167, 0.133))	(0.133, (0.133, 0.1, 0.2))
6	0.192	(0.133, (0.133, 0.133, 0.133))	(0.117, (0.133, 0.1, 0.133))
7	0.164	(0.117, (0.133, 0.1, 0.133))	(0.133, (0.2, 0.1, 0.133))
8	0.136	(0.117, (0.2, 0.1, 0.067))	(0.108, (0.133, 0.1, 0.1))
9	0.107	(0.092, (0.167, 0.067, 0.067))	(0.1, (0.167, 0.067, 0.1))
10	0.081	(0.075, (0.167, 0.033, 0.067))	(0.075, (0.167, 0.033, 0.067))
11	0.053	(0.067, (0.133, 0.033, 0.067))	(0.067, (0.133, 0.033, 0.067))
12	0	(0.05, (0.133, 0, 0.067))	(0.05, (0.133, 0, 0.067))
13	0.053	(0.058, (0.1, 0.033, 0.067))	(0.058, (0.1, 0.033, 0.067))
14	0.081	(0.067, (0.133, 0.033, 0.067))	(0.067, (0.133, 0.033, 0.067))
15	0.107	(0.058, (0.133, 0.033, 0.033))	(0.067, (0.133, 0.033, 0.067))
16	0.136	(0.058, (0.1, 0.033, 0.067))	(0.067, (0.133, 0.033, 0.067))
17	0.157	(0.067, (0.133, 0.033, 0.067))	(0.067, (0.133, 0.033, 0.067))