THE HONG KONG POLYTECHNIC UNIVERSITY
Department of Mechanical Engineering

OPTIMIZATIONS OF DYNAMIC VIBRATION ABSORBERS FOR SUPPRESSING VIBRATIONS IN STRUCTURES

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A thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy

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CERTIFICATE OF ORIGINALITY

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ABSTRACT

H_∞ and H_2 optimization of the traditional dynamic vibration absorber (DVA) in single degree-of-freedom (SDOF) system are classical optimization problems and have been already solved for a long time. However, the H_∞ and H_2 optimization of the dynamic vibration absorbers in multi-degree-of-freedom (MDOF) or continuous systems have not been solved. Some researchers found out the optimum tuning conditions of MDOF or continuous systems but all the methods found in the literature are numerical optimizations and the results cannot provide physics insight on the effect of each tuning parameter to the performance of vibration suppression of the primary vibrating system. Optimization theories of the traditional DVA for suppressing vibration in beam and plate structures have been established and reported in this thesis, and better tuning conditions of the DVA have been found in comparison to those reported by other researchers.

Non-traditional designs of the DVA are some recent research topics. One of these designs has been proved to perform better than the traditional design in some applications and it is studied and reported in this thesis. Researchers in this area tend to use the fixed-points theory of Den Hartog (1985) in searching the optimum tuning conditions of DVAs. However, it has been shown in this thesis that the fixed-points theory may not applicable in some tuning conditions of a non-traditional DVA. A new theory is established for finding the optimum tuning condition of the non-traditional DVA.

PUBLICATIONS ARISING FROM THE THESIS

International Journals

1. Wong W.O., Tang S.L., Cheung Y.L., Cheng L., Design of a dynamic vibration absorber for vibration isolation of beams under point or distributed loading, *Journal of Sound and Vibration*, *Volume 301, Issues 3-5*, *3 April 2007*, *Pages 898-908;*
2. Wong W.O., Cheung Y.L., Optimal design of a damped dynamic vibration absorber for vibration control of structure excited by ground motion, *Engineering Structures*, *Volume 30, Issue 1*, *January 2008*, *Pages 282-286;*
3. Cheung Y.L., Wong W.O., H_∞ and H_2 optimizations of a dynamic vibration absorber for suppressing vibrations in plates, *Journal of Sound and Vibration*, *Volume 320, 2009*, *Pages 29-42.*
4. Cheung Y.L. and Wong W.O. Design of a non-traditional dynamic vibration absorber, *The Journal of the Acoustical Society of America, 126 (2), August 2009, Pages 564-567*.

Conference Paper

1. Cheung Y.L., Wong W.O., Optimization of Dynamic Vibration Absorbers for Vibration Suppression in Plates, *IMAC-XXVI: Conference & Exposition on Structural Dynamic.*

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CONTENTS

LIST OF FIGURES

NOTATIONS AND ABBREVIATIONS

A	Cross section area of the beam
c	Damping coefficient of the dynamic vibration absorber
C	Damping coefficient of the primary system
E	Young modulus of the structure
f	The external force
F	The amplitude of the external force
h	The thickness of the plate
$H(\cdot)$	The frequency response function
I	The second moment of area
j	$\sqrt{-1}$
k	Elastic stiffness of the dynamic vibration absorber
K	Elastic stiffness of the primary system
L	The length of the beam
L_x	The length of the plate
L_y	The length of the plate
m	Mass of the dynamic vibration absorber
M	Mass of the primary system
ε	$\mu\varphi_n^2(x_0)$ (Chapter 4) and $\mu\varphi_{\alpha\beta}^2(x_0, y_0)$ (Chapter 5)
$\delta(\cdot)$	Delta function
ϕ	The phase of the primary system
$\varphi_i(\cdot)$	i^{th} mode function of the structure
γ	The tuning ratio between the primary system and the DVA, i.e. ω_a / ω_n

λ	The frequency ratio ω/ω_n
μ	The mass ratio between the primary system and the DVA, m/M
ν	The Poisson ratio
ρ	The density of the structure
ω	The exciting frequency of the external source
ω_a	The natural frequency of the DVA, $\sqrt{k/m}$
ω_n	The natural frequency of the primary system, $\sqrt{K/M}$
ζ	The damping factor of the primary system, $C/2\sqrt{MK}$
ζ_a	The damping factor of the DVA, $c/2\sqrt{mk}$
DVA	Dynamic vibration absorber
SDOF	Single-degree of freedom
MDOF	Multi-degree of freedom

1. INTRODUCTION

1.1 Literature review

The traditional dynamic vibration absorber (DVA) is an auxiliary mass-spring system which, when correctly tuned and attached to a vibrating system subject to harmonic excitation, causes to cease the steady-state motion at the point to which it is attached (e.g. Korenev and Reznikov 1993, Den Hartog 1985, Hunt 1979). It has the advantage of providing a cheap and easy-to-maintain solution for suppressing vibration in vibrating systems with harmonic excitation. The first research conduced at the beginning of the twentieth century considered an undamped DVA tuned to the frequency of the disturbing force by Frahm (1911). Such an absorber is a narrow-band type as it is unable to eliminate structural vibration after a change in the disturbing frequency. Application of damping substantially widened the frequency band of the DVA's efficient operation.

Finding the optimum parameters of a viscous friction DVA in SDOF system drew the attention of many scholars. One of the optimization methods is H_∞ optimization. Ormondroyd and Den Hartog (1928) proposed the optimization principle of the damped DVA in terms of minimizing the maximum amplitude response of the primary system, which called H_∞ optimization of dynamic vibration absorber. Following this principle, Hahnkamm (1932) deduced the relationship for the optimum tuning of DVA using in the SDOF system. Brock (1946) developed the approximated optimum damping. This optimum design method of the dynamic vibration absorber is called the "fixed-points theory", which was well documented in the textbook by Den Hartog (1985). The exact solution of the H_∞ optimization for the DVA attached to undamped primary system was derived by Nishihara and Asami (1997). And the other important optimization method is H_2 optimization. Crandall and Mark (1963) proposed another optimization principle of the damped DVA in terms of minimizing the total vibration energy of the primary structure under white noise excitation, which called H_2 optimization of dynamic vibration absorber. The exact solution of the H_2 optimization for the DVA attached to undamped primary system was derived by Warburton (1980, 1981, 1982). Asami and Nishihara (2002) proposed the exact solution of the H_2 optimization for the DVA attached to damped primary system.

However, when applying dynamic vibration absorber to a continuous structure such as a beam, vibration can be eliminated only at the attachment point of the vibrating beam while amplification of vibration may occur in other parts of the beam. Research results on suppressing vibration in a region or the whole span of a beam structure by using the dynamic vibration absorber have been reported recently.

Many investigators discussed the optimum parameters of a viscous friction DVA in MDOF system. Rice (1993) reported the use of SIMPLEX nonlinear optimization method to determine the H_∞ optimum tuning of a DVA applied for suppressing the vibration of beam. Hadi and Arfiadi (1998) used a genetic algorithm to solve numerically the H_2 optimum tuning for a MDOF system. Jacquot (1976, 2000, 2001, 2003, 2004) provided the method to handle the problem when the system is with an additional sub-system and determined the H_2 optimum damping based on the transfer functions of a beam and a plate.

Brennan and Dayou (2000), Dayou and Kim (2005), and Dayou (2006) applied the fix-points theory and proposed a set of optimum tuning for global control of the kinetic energy of a continuous structure using DVA. Cha (2001, 2002, 2004, 2005, 2007), Cha and Ren (2006), and Cha and Zhou (2006) used multiple absorbers to isolate the vibration in a region of a vibrating structure.

Some investigators discussed the non-traditional DVAs. Ren (2001) and Liu and Liu (2005) proposed a new design of the DVA, which they connected the damper of the DVA to the ground rather than the primary structure, and derived the new optimum tuning which is better than the traditional one. Tang (2005) designed a rotational dynamic absorber (RVA) for absorber for absorbing rotational motion of a vibration structure. His simulation result showed that the vibration of the structure can be isolated in the forced region of a beam structure if both a translational DVA and a RVA are attached at a proper location on the beam.

Recent advances of the absorber designs with active controlled elements (Takita and Seto 1989, Moyka 1996, Tentor 2001, Jalili and Knowles, 2004, Chen, Fuh and Tung, 2005, Lin, 2005, Wu et al. 2007a, 2007b, 2007c) may be more flexible and powerful than the traditional spring mass absorber. One of the concepts of the absorber with active controlled elements is called Semi-Active dynamic vibration absorber. Semi-active DVAs allow the system parameters to be varied after implementation. The semi-active DVA may have variable inertia, variable damping, variable stiffness or variable initial conditions. A major advantage of semi-active system is the small energy expenditure needed to reduce vibration. Another important advance of the absorber is called active or hybrid dynamic vibration absorber. Active or hybrid DVAs have an arbitrary force actuator and controller in parallel with the spring and damper. This adds flexibility to incorporate control theory to provide counteracting force to the primary structure. This force has frequently been implemented with a voice coil actuator design. The majority of literature on this topic focuses on control methodology. However, these advanced absorbers require special knowledge in the design and operation, and their applications may be justified when more sophisticated vibration control solutions are required. On the other hand, the passive dynamic vibration absorber provides a cheaper and convenient solution for vibration suppression and isolation of vibrating systems with harmonic excitation.

The application of the DVA is also a research area for the researchers. One important application is the structural-borne noise attenuation using dynamic vibration absorbers. Fuller (1982, 1984) presented a technique for tuning absorbers applied to cylindrical shell to minimize radiated sound. Nagaya and Li (1997) presented a method using neural network procedure in solving non-linear equations in predicting tuning parameters of the absorber for higher mode noise absorber. Since absorbers can be made small and light and they can be installed conveniently, it finds widely application of DVA on attenuating the noise.

1.2 Motivation of the present study

The present study is concerned with the optimum tuning of DVA in SDOF, MDOF and continuous systems for vibration suppression. Research results may be applied in engineering applications such as bridge, building, naval structures and pressure vessel.

The optimum tuning conditions of the traditional DVA in SDOF system is well introduced in literature. However, most of the relevant methods found in literature about the optimum tunings of the traditional DVA in MDOF system are numerical ones such as those reported by Rice (1993), Hadi and Arfiadi (1998), and Jacquot (1976, 2000, 2001, 2003, 2004) etc. These numerical methods can only provide case by case solution to the problems and the effects of different parameters of the DVA such as its mass, damping, stiffness and its attachment point on the primary structure to the vibration suppression performance remain unclear. It has been shown by Wong et al (2007) that an improper location of the attachment point of a DVA on a beam can amplify the vibration in some region of the structure. Dayou (2006) proposed using the fixed-points theory to find the optimum tuning in MDOF system. However, his optimum tuning is not the same as those reported by Asami and Nishihara (2003), and Korenev and Reznikov (1993). In the present studies, the optimum tunings in beam and plate structures and a structure are presented and some new results are reported.

Another study is on a non-traditional design of DVA proposed by Ren (2001), and Liu and Liu (2005). They applied the fixed-points theory to find out the optimum tuning condition of the DVA. However, the fixed-point theory is suitable for the traditional DVA but it is no always correct for other DVA designs.

In the present studies, it has been the fixed-points theory cannot be used in some tuning conditions of this non-traditional DVA and a new method is proposed for finding the optimum tuning condition of this DVA and the performance of vibration suppression of the optimized DVA is tested.

1.3 Thesis contents and new results

The text is organized into six chapters and an appendix. This chapter has presented the literature review of the DVA. The second chapter outlines the DVA fundamentals.

Chapter 3 derives the vibration and optimization theories of using the traditional DVA for vibration suppression in SDOF system. H_∞ optimization and H_2 optimization theories have been derived. In H_∞ optimization, the fixed-points theory is introduced and the discussion in the fixed-points theory is presented in this chapter.

Chapter 4 establishes the optimization theories of the traditional DVA in beam structures. H_∞ optimization and the H_2 optimization theories are established for Euler-Bernoulli beams and an approximated optimum tuning condition of the DVA for suppressing vibrations in beam structures are presented in this chapter.

Chapter 5 establishes the optimization theories of the traditional DVA in plate structures. Plate structures can be commonly found in different types of engineering application. H_∞ optimization and the H_2 optimization theories are established for Kirchhoff plates and an approximated optimum tuning condition of the DVA for suppressing vibrations in plate structures are presented in this chapter.

Chapter 6 establishes the optimization theories of a non-traditional DVA. H_∞ optimization and the H_2 optimization theories are established and the optimum tuning conditions of this DVA for suppressing vibrations in SDOF systems are presented in this chapter.

Findings from the present research work are summed up in Chapter 7. Future work is also recommended and proposed, which of importance and needs to be conducted further.

2. DVA FUNDAMENTALS

Vibration theories of a single degree of freedom system (SDOF) with and without a DVA excited by an external force or ground motions are presented in this chapter,.

2.1 Vibration of a SDOF system excited by an external force

A SDOF, mass-spring-damper system excited by an external force *f* is illustrated in Figure 2.1.

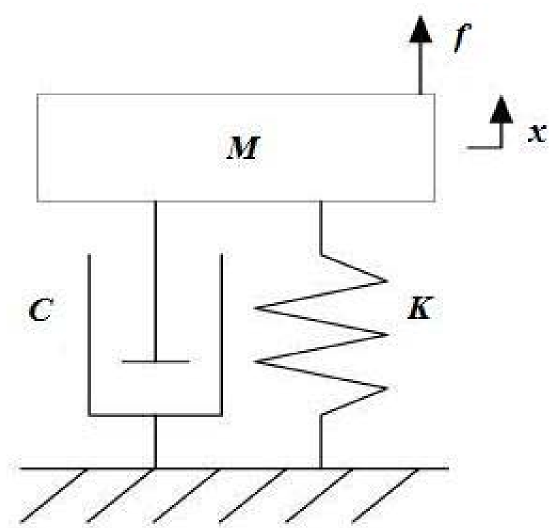


Figure 2.1 A SDOF vibrating system under an external force.

The equation of dynamic motions of mass *M* can be written as

$$M\ddot{x} + C\dot{x} + Kx = f \tag{2.1}$$

The frequency response function of mass *M* can be written as

$$H(\omega) = \frac{X}{F/K} = \frac{K}{K - M\omega^2 + jC\omega}, \tag{2.2}$$

The amplitude and the phase of the of mass M are

$$|H(\omega)| = \left|\frac{X}{F/K}\right| = \sqrt{\frac{K^2}{\left(K - M\omega^2\right)^2 + (C\omega)^2}}, \tag{2.3}$$

$$\text{and} \quad \tan\phi = \frac{C\omega}{K - M\omega^2} \tag{2.4}$$

where X is the amplitude of oscillation of the primary system, ϕ is the phase of the displacement with respect to the exciting force, F is the amplitude of the excitation, ω is the exciting frequency and $j = \sqrt{-1}$. Using equations 2.2, 2.3 and 2.4, the frequency response function may be written in dimensionless form as

$$H(\lambda) = \frac{X}{F/K} = \frac{1}{1 - \lambda^2 + 2j\zeta\lambda} \tag{2.5}$$

where λ is the frequency ratio or dimensionless frequency ω/ω_n, ζ is the damping factor of the primary system, i.e. $\zeta = C/2\sqrt{MK}$ and ω_n is the natural frequency of the primary system, i.e. $\omega_n = \sqrt{K/M}$.

The amplitude and the phase of the primary system are,

$$|H(\lambda)| = \left|\frac{X}{F/K}\right| = \sqrt{\frac{1}{\left(1 - \lambda^2\right)^2 + (2\zeta\lambda)^2}} \tag{2.6}$$

$$\text{and} \quad \tan\phi = \frac{2\zeta\lambda}{1 - \lambda^2} \tag{2.7}$$

The frequency response graph is shown in figure 2.2. It is seen that there is one resonance and the response is highest at the resonant frequency, $\omega_d = \omega_n \sqrt{1-\zeta^2}$.

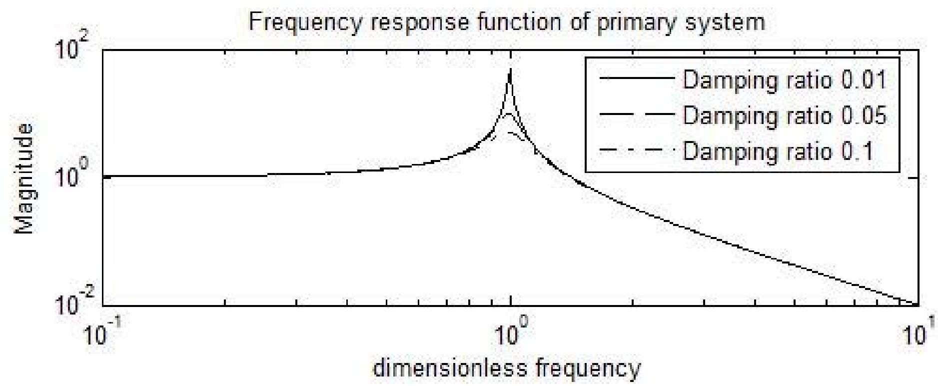


Figure 2.2 Frequency response function of a SDOF system under an external harmonic force.

2.2 The vibration in a SDOF system with a DVA excited by an external force

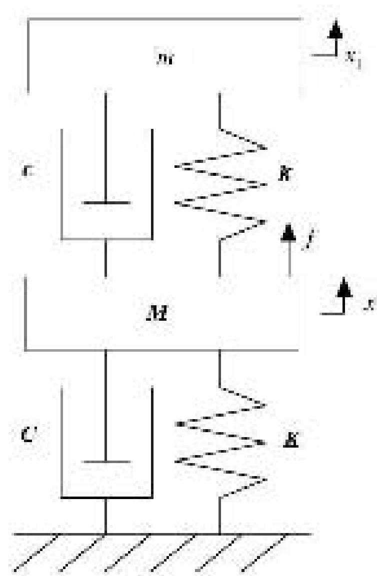


Figure 2.3 Primary system with DVA under an external force.

One of the reasons in using DVA is to eliminate the vibration of the primary system in a particular frequency such as the resonant frequency. The primary system with DVA excited by an external force *f* is shown in figure 2.3.

Assuming $C = 0$, the differential equations for the vibrations are

$$\begin{bmatrix} M & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{x}_1 \end{bmatrix}+\begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{x}_1 \end{bmatrix}+\begin{bmatrix} K+k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} x \\ x_1 \end{bmatrix}=\begin{bmatrix} f \\ 0 \end{bmatrix} \tag{2.8}$$

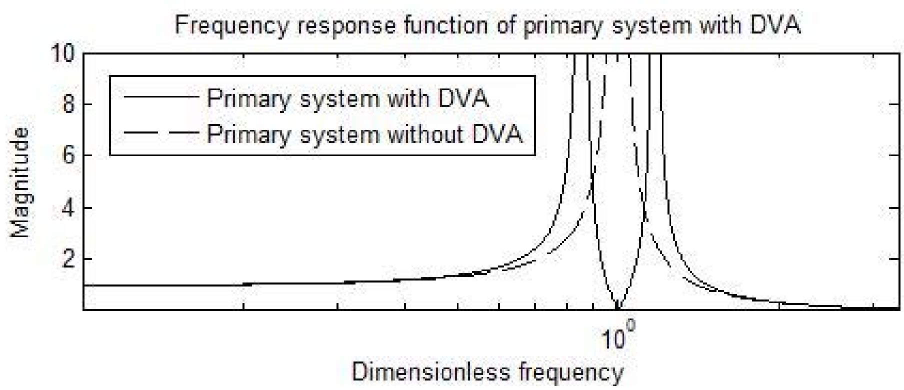
and the frequency response function can be written as

$$H(\omega)=\frac{X}{F}=\frac{k-m\omega^2+jc\omega}{\left[(K-M\omega^2)(k-m\omega^2)-m\omega^2 k\right]+jc\omega(K-M\omega^2-m\omega^2)} \tag{2.9}$$

Equation 2.9 can be rewritten in form of the dimensionless parameters as

$$H(\lambda) = \frac{X}{F/K} = \frac{\gamma^2 - \lambda^2 + 2j\zeta_a\gamma\lambda}{\left[(1-\lambda^2)(\gamma^2-\lambda^2) - \mu\lambda^2\gamma^2\right] + 2j\zeta_a\gamma\lambda(1-\lambda^2-\mu\lambda^2)} \tag{2.10}$$

where γ is the natural frequency ratio between the primary system and the DVA, i.e. $\gamma = \omega_a/\omega_n$, ω_a is the response frequency of the DVA, i.e. $\omega_a = \sqrt{k/m}$, ζ_a is the damping factor of the DVA, i.e. $\zeta_a = c/2\sqrt{mk}$ and μ is the mass ratio, i.e. $\mu = m/M$. Let $\gamma = 1$ and $\zeta = 0$ that the DVA can eliminate the highest response at the resonant frequency. As shown in figure 2.4, the frequency response of the primary system at the resonant frequency can be reduced to zero.



However, two resonant peaks appeared in the frequency spectrum of the mass *M*.

Figure 2.4 Frequency response function of a SDOF system with a DVA excited by an external force.

2.3 Vibration of a SDOF system excited by support motions

This is the case that the primary system is excited by support motions as illustrated in figure 2.5.

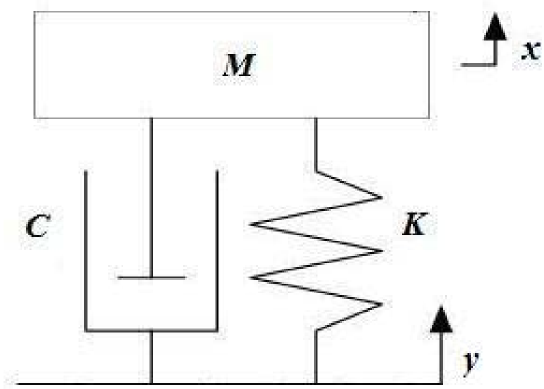


Figure 2.5 A SDOF system excited by support motion.

The differential equation for the vibrations of the main mass is

$$M\ddot{x} + C\dot{x} + Kx = C\dot{y} + Ky \tag{2.11}$$

And the frequency response function can be written as

$$H(\omega) = \frac{X}{Y} = \frac{K + jC\omega}{K - M\omega^2 + jC\omega} \tag{2.12}$$

The amplitude and the phase of the primary system are,

$$|H(\omega)| = \left|\frac{X}{Y}\right| = \sqrt{\frac{K^2 + (C\omega)^2}{\left(K - M\omega^2\right)^2 + (C\omega)^2}} \tag{2.13}$$

$$\text{and} \quad \tan\phi = \frac{MC\omega^3}{K\left(K - M\omega^2\right) + (\omega C)^2} \tag{2.14}$$

Y is the amplitude of the excitation. Using the dimensionless parameters into equations 2.13 and 2.14, the equations become,

$$H(\lambda) = \frac{X}{Y} = \frac{1 + 2j\zeta\lambda}{1 - \lambda^2 + 2j\zeta\lambda} \tag{2.15}$$

$$|H(\lambda)| = \left|\frac{X}{Y}\right| = \sqrt{\frac{1 + (2\zeta\lambda)^2}{(1 - \lambda^2)^2 + (2\zeta\lambda)^2}} \tag{2.16}$$

The frequency response graph is shown in figure 2.6. It is the same as the previous case that there is one resonance and the highest response appears at the resonant frequency. Magnitude of frequency response with any damping has the same value of $H(\lambda) = 1$ at frequency $\lambda = \sqrt{2}$.

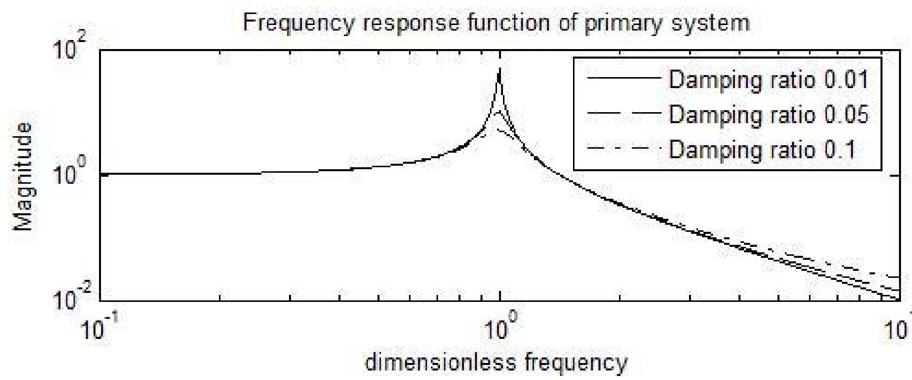


Figure 2.6 Frequency response function of a SDOF system excited by support motions.

2.4 Vibration of a SDOF system with a DVA under ground motions

Similarly to the previous case, the primary system with a DVA excited by ground motions is shown in Figure 2.7.

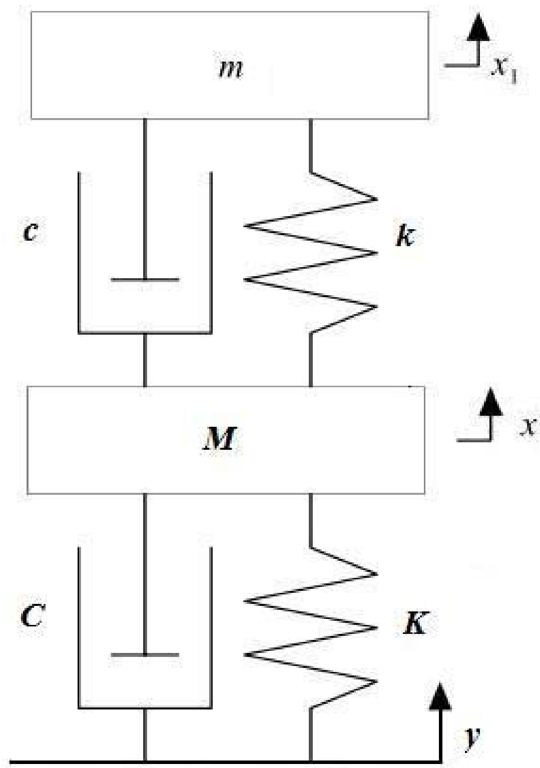


Figure 2.7 Primary system with DVA under a ground excitation

Assume $C = 0$, the differential equations for the vibrations are

$$\begin{bmatrix} M & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{x}_1 \end{bmatrix}+\begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{x}_1 \end{bmatrix}+\begin{bmatrix} K+k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} x \\ x_1 \end{bmatrix}=\begin{bmatrix} Ky \\ 0 \end{bmatrix} \tag{2.17}$$

and the frequency response function is found as

$$H(\omega)=\frac{X}{Y}=\frac{K\left(k-m\omega^2+jc\omega\right)}{\left[\left(K-M\omega^2\right)\left(k-m\omega^2\right)-m\omega^2 k\right]+jc\omega\left(K-M\omega^2-m\omega^2\right)} \tag{2.18}$$

Equation 2.18 can be rewritten using the dimensionless parameters as

$$H(\lambda)=\frac{X}{Y}=\frac{\gamma^2-\lambda^2+2j\zeta_a\gamma\lambda}{\left[\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right)-\mu\lambda^2\gamma^2\right]+2j\zeta_a\gamma\lambda\left(1-\lambda^2-\mu\lambda^2\right)} \tag{2.19}$$

Since equation 2.19 is identical to equation 2.10, the frequency response in this case is the same as the previous case.

2.5 Summary

In this chapter, the vibrations of a SDOF primary system exited by an external force or ground motions are presented. Different source of the excitation such as an external force at the primary system or a ground excitation can produce different frequency responses of the primary system. It means that the strategy of eliminating the vibration in the primary system should be different for different kinds of excitation even if the primary system is same.

One of the most common ideas to use the DVA is to set the natural frequency of the DVA to be the same as that of the primary system, i.e. $\gamma = 1$, so that the vibration at the resonant frequency becomes zero. However, some researchers such as Den Hartog (1985), Korenev and Reznikov (1993) pointed out the problems in adopting this idea. One of the problems is that the effective frequency range is very limited. Two new resonant frequencies appear after the DVA is attached and these two resonant frequencies are closed to the original resonant frequency. So the primary system becomes sensitive when the external force frequency is changed slightly. So, some researchers proposed other ideas to use the DVA in better ways which will be discussed in the following chapters.

3. OPTIMIZATIONS OF THE TRADITIONAL DVA FOR SUPPRESSION VIBRATIONS IN SDOF SYSTEM

The derivation of the optimization theories of the traditional DVA for suppressing vibrations in a SDOF system is presented in this chapter. Two different types of excitations, harmonic excitation with a varying frequency and stationary random excitation, are considered. Different optimization methods, H_∞ optimization and H_2 optimization, are introduced for different types of excitation.

3.1 H_∞ optimization of the traditional DVA for SDOF system

One of the disadvantages of using the undamped DVA is that the frequency range is very limited. The vibration of the primary system becomes large when the frequency of the excitation is changed. So, some researchers proposed other optimization schemes which can be more effective in using the DVA for vibration absorption.

In the previous chapter, the frequency response functions of the SDOF with a DVA are discussed. The frequency response functions of the system excited by an external force or due to ground motions are stated below for the ease of discussions.

$$H(\lambda)=\frac{X}{F/K}=\frac{\gamma^2-\lambda^2+2j\zeta_a\gamma\lambda}{\left[\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right)-\mu\lambda^2\gamma^2\right]+2j\zeta_a\gamma\lambda\left(1-\lambda^2-\mu\lambda^2\right)} \tag{2.16}$$

$$H(\lambda)=\frac{X}{Y}=\frac{\gamma^2-\lambda^2+2j\zeta_a\gamma\lambda}{\left[\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right)-\mu\lambda^2\gamma^2\right]+2j\zeta_a\gamma\lambda\left(1-\lambda^2-\mu\lambda^2\right)} \tag{2.19}$$

Ormondroyd and Den Hartog (1928) pointed out the damping of the DVA has an optimum value so as to minimize amplitude response of the SDOF system. Such optimization criterion is now known as H_∞ optimization. The objective is to minimize the maximum amplitude ratio of the response of the primary system to the excitation force or motion, i.e.

$$\max\left(\left|H\left(\lambda,\gamma_{H\infty},\zeta_{H\infty}\right)\right|\right)=\min\left(\max_{\gamma,\zeta_a}\left|H\left(\lambda\right)\right|\right) \tag{3.1}$$

The advantage of this optimization criterion is that the vibration of the system under a harmonic excitation with an unstable frequency. And all frequency responses obeyed the inequality,

$$\left|H\left(\lambda,\gamma_{H\infty},\zeta_{H\infty}\right)\right|\le\min\left(\max_{\gamma,\zeta_a}\left|H\left(\lambda\right)\right|\right),\ \text{where}\ \ \lambda\in\Re^{+} \tag{3.2}$$

The procedure to find the optimum tuning frequency of the absorber according to this H_∞ optimization criterion is shown in the following paragraphs. Using equation 2.10, the amplitude of the frequency response may be written as

$$\left|H\left(\lambda\right)\right|=\left|\frac{X}{F/K}\right|=\sqrt{\frac{\left(\gamma^2-\lambda^2\right)^2+\left(2\zeta_a\gamma\lambda\right)^2}{\left[\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right)-\mu\lambda^2\gamma^2\right]^2+\left[2\zeta_a\gamma\lambda\left(1-\lambda^2-\mu\lambda^2\right)\right]^2}} \tag{3.3}$$

Rewriting the equation into the form

$$\left|H\left(\lambda\right)\right|=\left|\frac{X}{F/K}\right|=\sqrt{\frac{A+B\zeta_a^2}{C+D\zeta_a^2}} \tag{3.4}$$

where $A=\left(\gamma^2-\lambda^2\right)^2$, $B=\left(2\gamma\lambda\right)^2$, $C=\left[\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right)-\mu\lambda^2\gamma^2\right]^2$,

and $D=\left[2\gamma\lambda\left(1-\lambda^2-\mu\lambda^2\right)\right]^2$

If the frequency responses, as shown in figure 3.1, are independent of damping, we may write

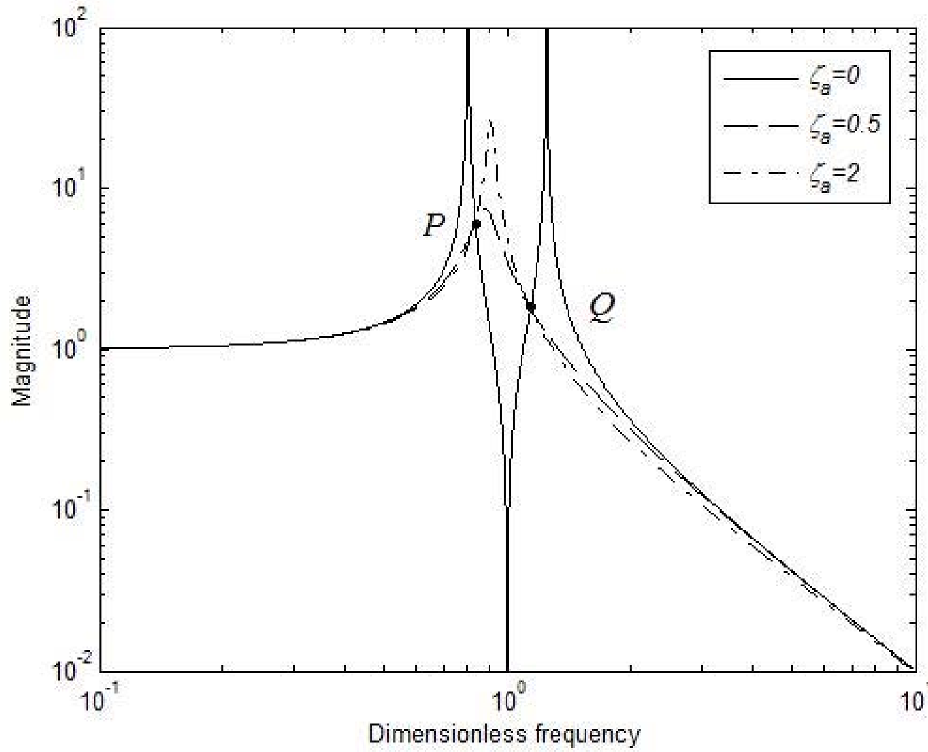


Figure 3.1 The frequency response of the primary system with DVA at $\gamma = 1$

$$\frac{A}{C} = \frac{B}{D} \tag{3.5a}$$

$$\text{or} \quad \left(\frac{\gamma^2 - \lambda^2}{(1-\lambda^2)(\gamma^2 - \lambda^2) - \mu\lambda^2\gamma^2} \right)^2 = \left(\frac{1}{1 - \lambda^2 - \mu\lambda^2} \right)^2 \tag{3.5b}$$

Taking square root on both sides of equation 3.5b, we have

$$\frac{\gamma^2 - \lambda^2}{(1-\lambda^2)(\gamma^2 - \lambda^2) - \mu\lambda^2\gamma^2} = \pm \frac{1}{1 - \lambda^2 - \mu\lambda^2} \tag{3.6}$$

It can be shown that $\lambda = 0$ when the positive sign is taken for the expression on the right hand side of equation 3.6. This result shows that all curves in Figure 3.1 meet at $\lambda = 0$. Now taking the negative sign of the expression on the right hand side of equation 3.6, we have

$$\frac{\gamma^2 - \lambda^2}{(1-\lambda^2)(\gamma^2 - \lambda^2) - \mu\lambda^2\gamma^2} = -\frac{1}{1-\lambda^2 - \mu\lambda^2} \tag{3.7}$$

Rewriting equation 3.7 as

$$\lambda^4 - \frac{2 + 2\gamma^2 + 2\mu\gamma^2}{2+\mu}\lambda^2 + \frac{2\gamma^2}{2+\mu} = 0 \tag{3.8}$$

Equation 3.8 is a quadratic equation in λ^2. Let the two roots of equation 3.8 be λ_a^2 and λ_b^2 where $0 < \lambda_a < \lambda_b$. The sum and the product of the roots are respectively written as

$$\lambda_a^2 + \lambda_b^2 = \frac{1 + \gamma^2 + \mu\gamma^2}{2+\mu} \tag{3.9a}$$

$$\text{and} \quad \lambda_a^2\lambda_b^2 = \frac{2\gamma^2}{2+\mu} \tag{3.9b}$$

The amplitudes of the frequency response at these two roots are independent of the damping ratio ζ_a, where these two points, *P* and *Q*, are called ‘fixed points’. The amplitudes of the frequency response at λ_a^2 and λ_b^2 are

$$|H(\lambda_a)| = \left|\frac{B}{D}\right| = \left|\frac{1}{1 - \lambda_a^2 - \mu\lambda_a^2}\right| \tag{3.10a}$$

$$\text{and} \quad |H(\lambda_b)| = \left|\frac{B}{D}\right| = \left|\frac{1}{1 - \lambda_b^2 - \mu\lambda_b^2}\right| \tag{3.10b}$$

At any damping ratio, the frequency response must pass through these two fixed points *P* and *Q*. So the optimum condition should obey the following equation:

$$\max(|H(\lambda_a, \gamma_{H\infty}, \zeta_{H\infty})|) = \min\left(\max_{\gamma,\zeta_a}(|H(\lambda_a)|, |H(\lambda_b)|)\right) \tag{3.11}$$

The relation between the fixed points and tuning ratio is shown in figures 3.2 and 3.3.

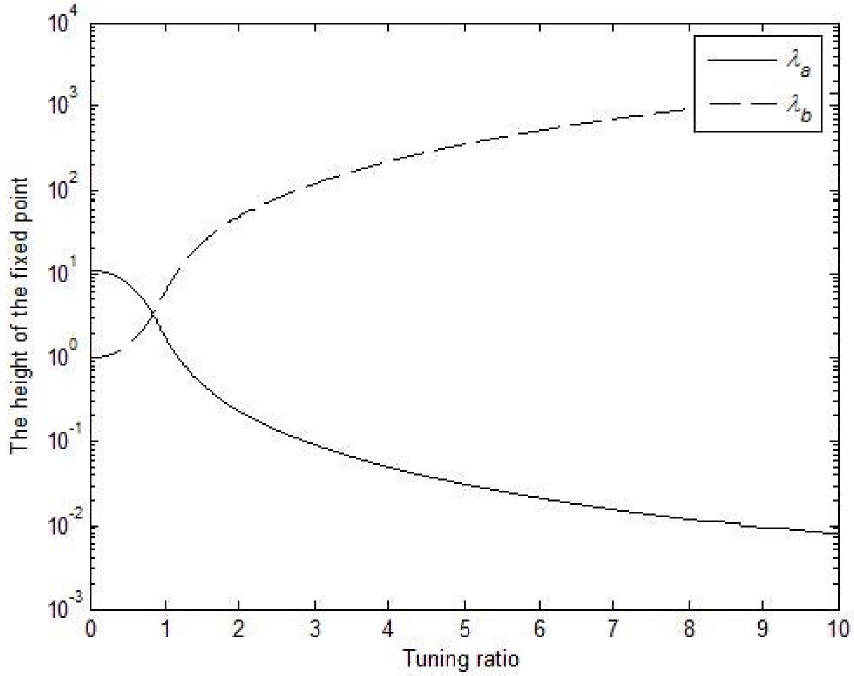


Figure 3.2 The height of fixed points in the frequency response spectrum of mass M versus tuning ratio γ at $\mu = 0.2$

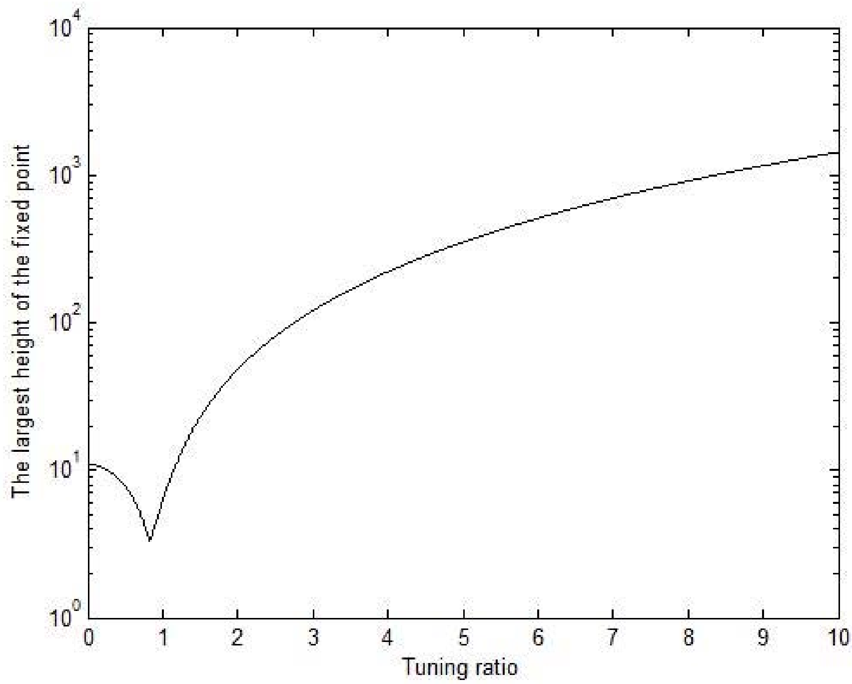


Figure 3.3 $\max_{\gamma,\zeta}\left(\left|H(\lambda_a)\right|,\left|H(\lambda_b)\right|\right)$ versus tuning ratio γ at $\mu = 0.2$

From figure 3.2 and figure 3.3, the minimum height of the fixed points is found when the height of the fixed point *P* is equal to the fixed point *Q*, i.e. $|H(\lambda_a)| = |H(\lambda_b)|$. So the fixed points of the frequency response are adjusted to the same, i.e.

$$\frac{1}{1-\lambda_a^2-\mu\lambda_a^2} = -\frac{1}{1-\lambda_b^2-\mu\lambda_b^2} \tag{3.12}$$

Using equations 3.9 and 3.10, the tuning ratio which let the amplitude of the fixed points be the same is

$$\gamma_{H\infty} = \frac{1}{1+\mu} \tag{3.13}$$

The frequency response using $\gamma = \frac{1}{1+\mu}$ is shown in figure 3.4.

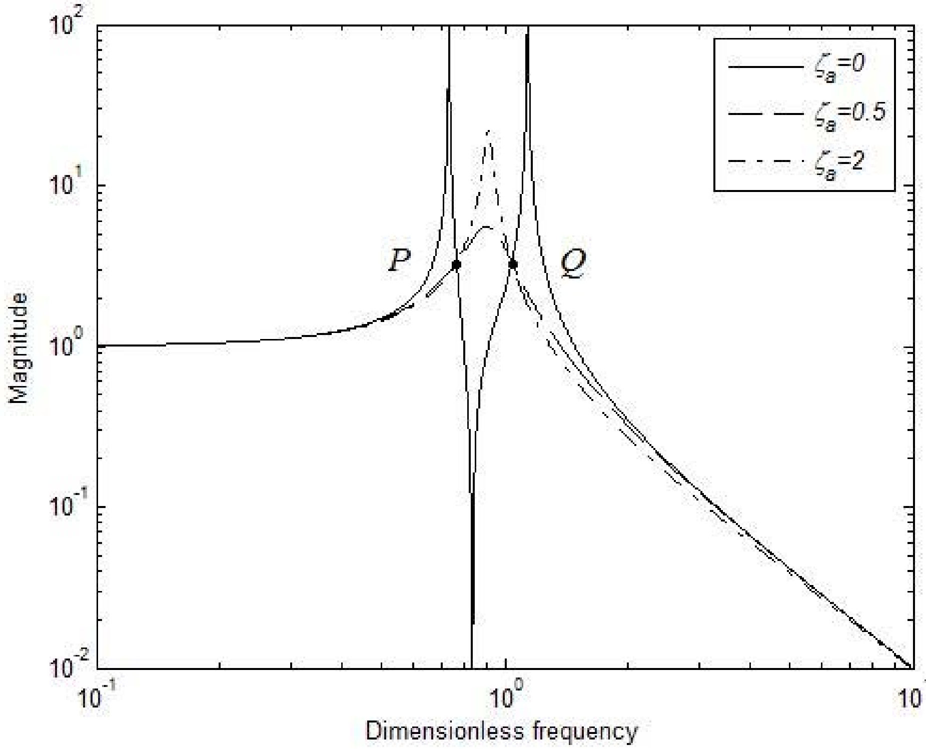


Figure 3.4 The frequency response of the primary mass *M* with DVA tuned at $\gamma_{H\infty}$

Substitute equation 3.13 into equation 3.8, we may write

$$\lambda^4 - \frac{2}{1+\mu}\lambda^2 + \frac{2}{(2+\mu)(1+\mu)^2} = 0 \tag{3.14}$$

The roots of equation 3.14 can be written as

$$\lambda_a^2 = \frac{1}{1+\mu}\left(1 - \sqrt{\frac{\mu}{2+\mu}}\right) \tag{3.15a}$$

and $$\lambda_b^2 = \frac{1}{1+\mu}\left(1 + \sqrt{\frac{\mu}{2+\mu}}\right) \tag{3.15b}$$

Substitute equation 3.15 into equation 3.12, the response amplitude at the fixed points is

$$G = |H(\lambda_a)| = |H(\lambda_b)| = \sqrt{\frac{2+\mu}{\mu}} \tag{3.16}$$

In the above, the optimum tuning condition was deduced. The next step is to determine the optimum damping in order to make the fixed points to become the peaks on the response curve. We may write

$$\left.\frac{\partial}{\partial\lambda^2}|H(\lambda)|^2\right|_{\lambda=\lambda_a} = \left.\frac{\partial}{\partial\lambda^2}|H(\lambda)|^2\right|_{\lambda=\lambda_b} = 0 \tag{3.17}$$

Let

$$|H(\lambda)|^2 = \frac{p}{q} \tag{3.18}$$

Where $$p = (\gamma^2 - \lambda^2)^2 + (2\zeta_a\gamma\lambda)^2 \tag{3.19}$$

$$q = \left[\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right)-\mu\lambda^2\gamma^2\right]^2 + \left[2\zeta_a\gamma\lambda\left(1-\lambda^2-\mu\lambda^2\right)\right]^2 \tag{3.20}$$

Substitute (3.18) into (3.17), we may write

$$\frac{\partial}{\partial\lambda^2}\left|H(\lambda)\right|^2 = \frac{\partial}{\partial\lambda^2}\left(\frac{p}{q}\right) = 0 \tag{3.21}$$

$$\frac{\dfrac{\partial p}{\partial\lambda^2}q - \dfrac{\partial q}{\partial\lambda^2}p}{q^2} = 0 \tag{3.22}$$

$$\frac{\partial p}{\partial\lambda^2}q - \frac{\partial q}{\partial\lambda^2}p = 0 \tag{3.23}$$

Using equations 3.19 and 3.20, we may write

$$\frac{\partial p}{\partial\lambda^2} = 2\left(\lambda^2-\gamma^2\right) + \left(2\zeta_a\gamma\right)^2 \tag{3.24}$$

$$\frac{\partial q}{\partial\lambda^2} = 2\left[\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right)-\mu\lambda^2\gamma^2\right]\left(-1+2\lambda^2-\gamma^2-\mu\gamma^2\right) + \left(2\zeta_a\gamma\right)^2\left(1-\lambda^2-\mu\lambda^2\right)\left(1-3\lambda^2-3\mu\lambda^2\right) \tag{3.25}$$

Under the optimum tuning condition, we have

$$G^2 = \frac{p}{q} = \frac{2+\mu}{\mu} \tag{3.26}$$

Therefore,

$$\frac{\partial p}{\partial\lambda^2} - \frac{\partial q}{\partial\lambda^2}G^2 = 0 \tag{3.27}$$

Solving equation 3.27 for ζ^2, we have

$$\zeta_a^2 = \frac{2\left[\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right)-\mu\lambda^2\gamma^2\right]\left(-1+2\lambda^2-\gamma^2-\mu\gamma^2\right)G^2 - 2\left(\lambda^2-\gamma^2\right)}{\left(2\gamma\right)^2 - \left(2\gamma\right)^2\left(1-\lambda^2-\mu\lambda^2\right)\left(1-2\lambda^2-2\mu\lambda^2\right)G^2} \tag{3.28}$$

Substituting equations 3.13, 3.15 and 3.16 into the equation above, we find the optimum damping written as

$$\zeta_a = \sqrt{\frac{\mu\left(3 \pm \sqrt{\frac{\mu}{2+\mu}}\right)}{8(1+\mu)}} \tag{3.29}$$

Taking an average of ζ_a and the optimum tuning becomes

$$\gamma_{H\infty} = \frac{1}{1+\mu} \quad \text{and} \quad \zeta_{H\infty} = \sqrt{\frac{\zeta_a^2 + \zeta_b^2}{2}} = \sqrt{\frac{3\mu}{8(1+\mu)}} \tag{3.30}$$

From equation 3.16, the resonant amplitude ratio is approximately

$$\left|\frac{X}{Y}\right|_{\max} = \sqrt{\frac{2+\mu}{\mu}} \tag{3.31}$$

The frequency response of the primary mass M at $\mu = 0.2$ with different damping ratios is shown in figure 3.5.

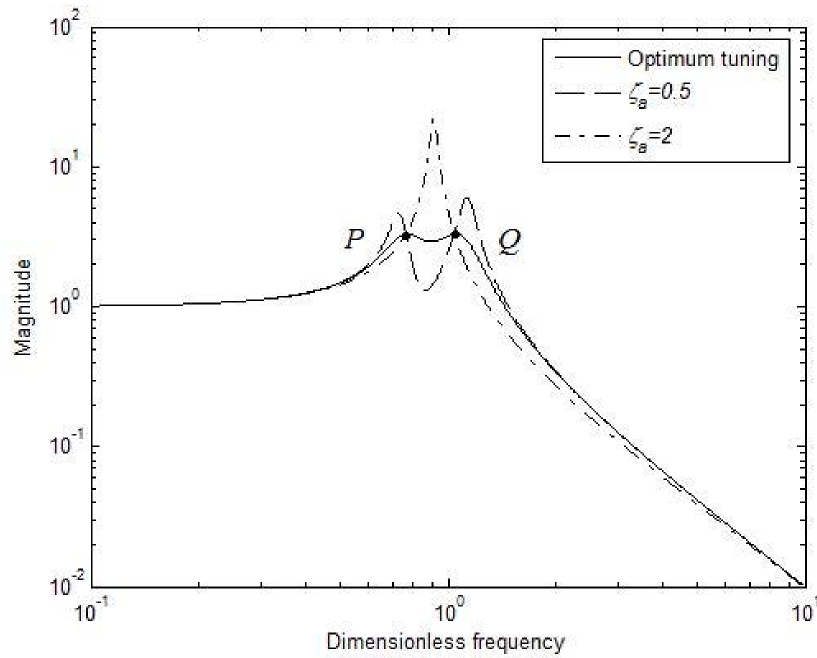


Figure 3.5 The frequency response of the primary system with DVA using the H_∞ optimum tuning

In figure 3.5, the fixed points P and Q are close to the resonant response of the primary system. Equation 3.2 can be satisfied and the primary system obey the optimization requirement expressed as

$$\left|H\left(\lambda,\gamma_{H\infty},\zeta_{H\infty}\right)\right| \le \min\left(\max_{\gamma,\zeta_a}\left|H(\lambda)\right|\right) \approx \sqrt{\frac{2+\mu}{\mu}}\text{, where } \lambda \in \Re^{+} \tag{3.32}$$

This optimization is called H_∞ optimization. With the same procedure, the optimum tuning for the velocity and acceleration responses of the primary system can also be found and they are tabulated in the following.

Table 3.1 The H_∞ optimum tuning in a SDOF system

Transfer function	Tuning ratio	Damping ratio	The height of the fixed point
$\frac{L\{x(t)\}}{L\{y(t)\}}$	$\frac{1}{1+\mu}$	$\sqrt{\frac{3\mu}{8(1+\mu)}}$	$\sqrt{\frac{2+\mu}{\mu}}$
$\frac{L\{\dot{x}(t)\}}{\omega_n L\{y(t)\}}$	$\frac{1}{1+\mu}\sqrt{\frac{2+\mu}{2}}$	$\frac{1}{4(2+\mu)}\sqrt{\frac{\mu\left(24+24\mu+5\mu^2\right)}{1+\mu}}$	$\sqrt{\frac{2+\mu}{\mu(1+\mu)}}$
$\frac{L\{\ddot{x}(t)\}}{\omega_n^2 L\{y(t)\}}$	$\sqrt{\frac{1}{1+\mu}}$	$\frac{1}{2}\sqrt{\frac{3\mu}{2+\mu}}$	$\sqrt{\frac{2}{\mu(1+\mu)}}$

In fact, these optimum tunings based on the fixed point are not the exact solutions. The exact H_∞ optimum tunings were found by Nishihara and Asami [2] at 1997. Nishihara and Asami provided different method to solve the H_∞ optimum tunings. However, based on their results [36], the error compared with the exact optimum tunings and the approximated optimum tunings is under 1% when $\mu < 1$. So these results still have the significance of the people using the damped DVA.

3.2 H_2 optimization of the traditional DVA for SDOF system

The dynamic absorber used to mitigate random excitation has very important practical application particularly for tower structures, whose vibrations are caused by wind and seismic loads. In this case, the single frequency excitation is not the case in these real situations. A different optimization scheme should be considered.

Crandall and Mark (1963) proposed another optimization principle of the damped DVA with the objective function on minimizing the total vibration energy of the primary structure under white noise excitation, i.e.

$$\min_{\gamma,\zeta_a}\left(E\left[x^2\right]\right) \tag{3.33}$$

which called H_2 optimization of dynamic vibration absorber. The exact solution of the H_2 optimization for the DVA attached to undamped primary system was derived by Warburton (1980).

The frequency response functions of the SDOF system with a DVA are restated below for the ease of discussion. The frequency response functions of the system excited by an external force and the system excited by ground motions are written respectively as

$$H(\lambda)=\frac{X}{F/K}=\frac{\gamma^2-\lambda^2+2j\zeta_a\gamma\lambda}{\left[\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right)-\mu\lambda^2\gamma^2\right]+2j\zeta_a\gamma\lambda\left(1-\lambda^2-\mu\lambda^2\right)} \tag{2.16}$$

and $$H(\lambda)=\frac{X}{Y}=\frac{\gamma^2-\lambda^2+2j\zeta_a\gamma\lambda}{\left[\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right)-\mu\lambda^2\gamma^2\right]+2j\zeta_a\gamma\lambda\left(1-\lambda^2-\mu\lambda^2\right)} \tag{2.19}$$

The mean square motion $E\left[x^2\right]$ of the stationary response process $x(t)$ can be obtained when either the autocorrelation function $R_x(\tau)$ or the spectral density $S_x(\omega)$ of the response is known according to the following formulae:

$$R_x(0) = E[x^2] \tag{3.34}$$

$$E[x^2] = \int_{-\infty}^{\infty} S_x(\omega) d\omega \tag{3.35}$$

The autocorrelation function of the response and the spectral density are written as

$$R_x(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_y(\tau + \theta_1 - \theta_2) h(\theta_1) h(\theta_2) d\theta_1 d\theta_2 \tag{3.36}$$

$$S_x(\omega) = |H(\omega)|^2 S_y(\omega) \tag{3.37}$$

Thus the mean square motion can be written as

$$E[x^2] = R_x(0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_y(\theta_1 - \theta_2) h(\theta_1) h(\theta_2) d\theta_1 d\theta_2 \tag{3.38}$$

The mean square motion can be written in terms of the input mean square spectral density $S_y(\omega)$ as

$$E[x^2] = \int_{-\infty}^{\infty} S_x(\omega) d\omega = \int_{-\infty}^{\infty} |H(\omega)|^2 S_y(\omega) d\omega \tag{3.39}$$

If the input spectrum is assumed to be ideally white, i.e. $S_y(\omega) = S_0$, a constant for all frequencies, the integral of equation 3.39 can then be reduced to

$$E[x^2] = S_0 \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega \tag{3.40}$$

Using equation 3.40, the non-dimensional mean square motion can be defined as

$$E[x^2] = \frac{\omega_n S_0}{2\pi} \int_{-\infty}^{\infty} |H(\lambda)|^2 d\lambda \tag{3.41}$$

Substituting equation 2.19 into equation 3.41, the equation can be written as,

$$E\left[x^2\right] = \frac{\omega_n S_0}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\gamma^2 - \lambda^2 + 2j\zeta_a\gamma\lambda}{\left[\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right) - \mu\lambda^2\gamma^2\right] + 2j\zeta_a\gamma\lambda\left(1-\lambda^2-\mu\lambda^2\right)}\right|^2 d\lambda \tag{3.42}$$

Two useful formulae as shown below from Gradshteyn and Ryzhik (1994) are used to solve equation 3.41.

$$\text{If}\quad H(\omega) = \frac{-j\omega^3 B_3 - \omega^2 B_2 + j\omega B_1 + B_0}{\omega^4 A_4 - j\omega^3 A_3 - \omega^2 A_2 + j\omega A_1 + A_0} \tag{3.43a}$$

$$\text{then}\quad \int_{-\infty}^{\infty}\left|H(\omega)\right|^2 d\omega = \pi\frac{\left[\begin{array}{l}\frac{B_0^2}{A_0}\left(A_2A_3 - A_1A_4\right) + A_3\left(B_1^2 - 2B_0B_2\right)\\ \qquad + A_1\left(B_2^2 - 2B_1B_3\right) + \frac{B_3^2}{A_4}\left(A_1A_2 - A_0A_3\right)\end{array}\right]}{A_1\left(A_2A_3 - A_1A_4\right) - A_0A_3^2} \tag{3.43b}$$

Comparing equation 3.42 and equation 3.43, the equation can be written as

$$\begin{array}{l} A_0 = \gamma^2, \quad A_1 = 2\zeta_a\gamma, \quad A_2 = 1+\gamma^2+\mu\gamma^2, \quad A_3 = 2\zeta\gamma(1+\mu), \quad A_4 = 1 \\ B_0 = \gamma^2, \quad B_1 = 2\zeta_a\gamma, \quad B_2 = 1, \quad B_3 = 0 \end{array} \tag{3.44}$$

Using equations 3.43 and 3.44, the mean square motion in equation 3.42 can be written as

$$E\left[x^2\right] = \frac{\omega_n S_0}{4\mu\zeta_a\gamma}\left[(1+\mu)^2\gamma^4 + \left(4\zeta_a^2(1+\mu) - 2 - \mu\right)\gamma^2 + 1\right] \tag{3.45}$$

If $\frac{\partial}{\partial\gamma}E\left[x^2\right] = \frac{\partial}{\partial\zeta_a}E\left[x^2\right] = 0$ exists, the system will have an optimum tuning condition.

The derivatives of equation 3.45 may be written as

$$\frac{\partial}{\partial\gamma}E\left[x^2\right] = \frac{\omega_n S_0}{4\mu\zeta_a\gamma^2}\left[3(1+\mu)^2\gamma^4 + \left(4\zeta_a^2(1+\mu) - 2 - \mu\right)\gamma^2 - 1\right] \tag{3.46a}$$

$$\text{and}\quad \frac{\partial}{\partial\zeta}E\left[x^2\right] = \frac{\omega_n S_0}{4\mu\zeta_a^2\gamma^2}\left[-(1+\mu)^2\gamma^4 + \left(4\zeta_a^2(1+\mu) + 2 + \mu\right)\gamma^2 - 1\right] \tag{3.46b}$$

Using equations 3.46a and 3.46b, the optimum tuning can be found as

$$\gamma_{H2} = \sqrt{\frac{\mu+2}{2(\mu+1)^2}} \quad \text{and} \quad \zeta_{H2} = \frac{1}{2}\sqrt{\frac{\mu(3\mu+4)}{2(\mu+2)(\mu+1)}} \tag{3.47}$$

This optimization is called H_2 optimization. The optimum tuning of the velocity response can also be found using a similar procedure and the tuning frequency and damping ratios are shown in the following table.

Table 3.2 The H_2 optimum tuning in a SDOF system

Transfer function	Tuning frequency ratio	Damping ratio	Optimized value of performance index
$E[x^2]$	$\sqrt{\frac{\mu+2}{2(\mu+1)^2}}$	$\frac{1}{2}\sqrt{\frac{\mu(3\mu+4)}{2(\mu+2)(\mu+1)}}$	$\frac{1}{2}\sqrt{\frac{4+3\mu}{\mu(1+\mu)}}$
$E[\dot{x}^2]$	$\sqrt{\frac{1}{1+\mu}}$	$\sqrt{\frac{\mu}{2}}$	$\sqrt{\frac{1}{\mu(1+\mu)}}$

This H_2 optimum tuning can minimize the mean square motion of the undamped primary system under a white noise excitation and this result is an exact solution of the H_2 optimization problem.

3.3 Summary

H_∞ optimization and H_2 optimization are introduced and the analytical solutions including the tuning frequency and damping ratios of the DVA for these optimization problems in SDOF system are derived.

From table 3.1 and table 3.2, the tuning ratios and the damping ratios of the motion are

$$\gamma_{H\infty} = \frac{1}{1+\mu} \quad \text{and} \quad \zeta_{H\infty} = \sqrt{\frac{3\mu}{8(1+\mu)}} \tag{3.30}$$

$$\gamma_{H2} = \sqrt{\frac{\mu+2}{2(\mu+1)^2}} \quad \text{and} \quad \zeta_{H2} = \frac{1}{2}\sqrt{\frac{\mu(3\mu+4)}{2(\mu+2)(\mu+1)}} \tag{3.47}$$

The graphs of both tuning ratio and damping ratio versus mass ratio of the DVA are plotted in figures 3.6 and 3.7, respectively.

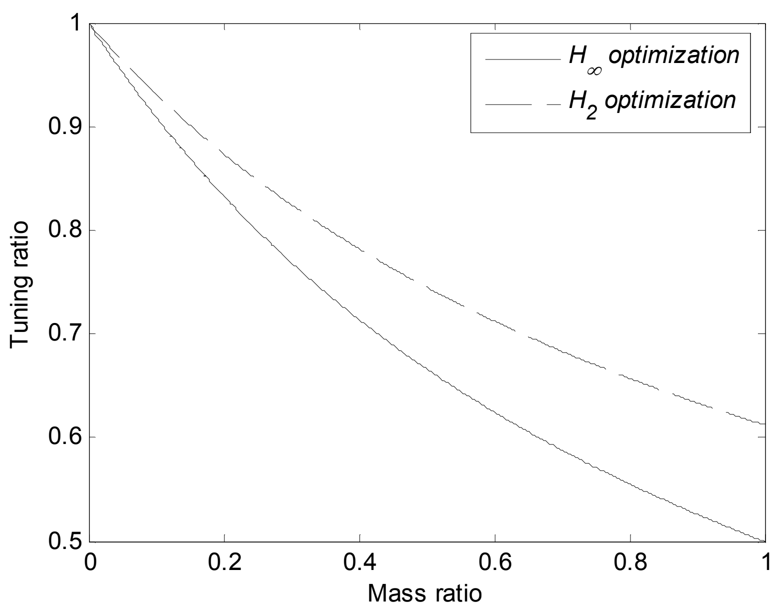


Figure 3.6 Mass ratio vs. tuning ratio in different optimization methods

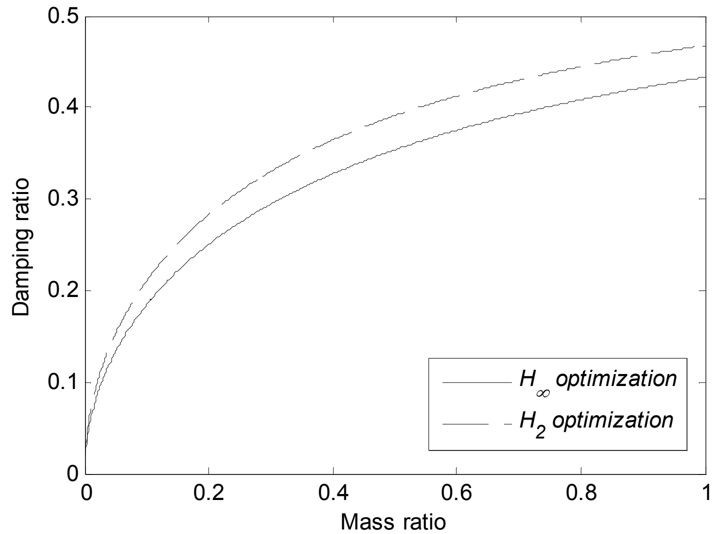


Figure 3.7 Mass ratio vs. damping ratio in different optimization methods

From studying these graphs, the following conclusions can be drawn.

1. In both optimization methods, the tuning ratios decrease when the mass ratio increases.
2. In both optimization methods, the damping ratios increase when the mass ratio increases.
3. Comparing the two optimization methods, the tuning ratio and the damping ratio of the H_∞ optimization are lower than those of the H_2 optimization. Moreover, the difference between these two tuning ratios increases when the mass ratio increases.

However, this is not the end of the story of the optimum tuning. In the previous sections, the optimum tunings of using DVA in an undamped SDOF are introduced. The optimum tuning of DVA for a MDOF or a continuous system has not been found in the literature. Solutions of these optimization problems will be presented in the following chapters.

4. OPTIMIZATIONS OF THE TRADITIONAL DVA FOR SUPPRESSION VIBRATIONS IN BEAM STRUCTURES

Optimization theory of the traditional DVA for suppressing vibrations in beam structures is presented in this chapter. Beam structure is a very common structure in engineering application such as building and bridge.

The beam structure is considered as a MDOF or continuous vibrating system. The beam has more than one resonant frequency. Another different consideration in applying a DVA between the SDOF and the beam vibrating systems is that the DVA can be attached at different location on the beam and the effect of vibration suppression can be very different. So new problems in this case are what is the good position of attaching DVA and what is the difference of the optimum tuning frequency and damping ratios when the attaching position is changed? In this part, the research of DVA optimizations includes the reduction of vibration at a particular point on the beam and the total kinetic energy of the beam structure.

All discussions are based on the assumptions listed below.

1. The beam is assumed to be an Euler-Bernoulli beam.
2. The dynamic response of the beam is due to the dominant mode only, i.e. single mode response only, and the responses of other modes may be ignored.
3. The modes can be well separated.

The reason of using the Euler-Bernoulli beam is that the dominant modes of the structure are always in the lower modes. In these modes, the effects of shear deformation and rotational inertia are not large. According to these reasons, the advance beam model, such as Timoshenko beam, is not used in my calculation.

An approximated H_∞ optimum tuning and H_2 optimum tuning conditions are derived analytically and compared to the results reported by another researcher (Dayou 2006) who has applied a different approach to the problem.

The error of the approximated optimum tunings are shown in section 4.4 and the effect of the non-tuned mode is discussed in section 4.5.

4.1 The frequency response function of the beam structure with the traditional DVA

Referring to figure 4.1, consider the motion of a cantilever beam due to the distributed force applying between $x = 0$ and $x = x_o$. A DVA is attached at $x = x_o$. The length of the beam is L, mass per unit length ρA, with bending stiffness EI. The boundary conditions are any combination of pinned, clamped or free supports. The Euler-Bernoulli equation can be written in equation 4.1 and the detail of derivation of this equation is shown in Appendix A.

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = p(t)g(x) + F(t)\delta(x - x_o) \tag{4.1}$$

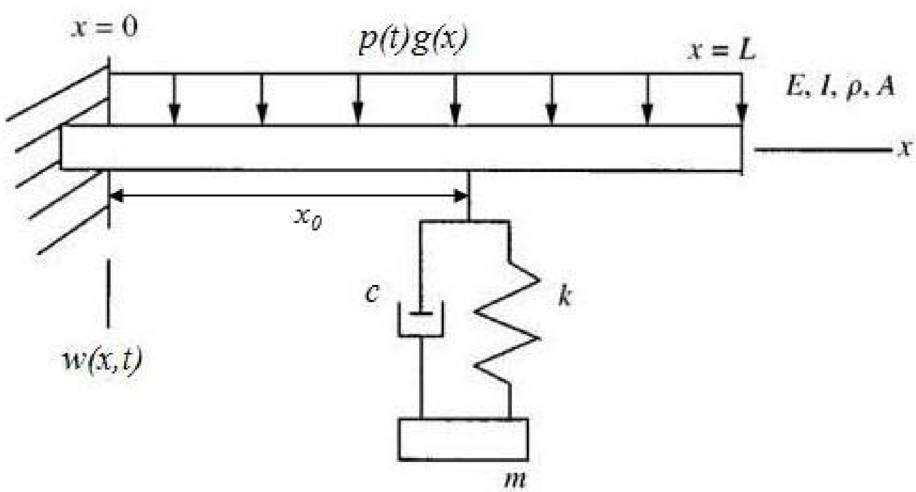


Figure 4.1 The cantilever beam with a DVA under a external force

Here it has been assumed that the externally applied forcing function can be expressed as $p(t)g(x)$, where $p(t)$ is a function of time and $g(x)$ is a deterministic function of x. The solution to equation 4.1 can be expanded in a Fourier series written as

$$w(x,t)=\sum_{i=1}^{\infty}q_i(t)\varphi_i(x) \tag{4.2}$$

$$\text{where } \int_0^L (\varphi_i(x))^2\,dx = L$$

where $\varphi_i(x)$ is the eigenfunction of the beam without the DVA. Similarly, the spatial part of the forcing function can be expanded as

$$g(x)=\sum_{i=1}^{\infty}a_i\varphi_i(x) \tag{4.3}$$

The Dirac delta functions can also be expanded as

$$\delta(x-x_o)=\sum_{i=1}^{\infty}b_i\varphi_i(x) \tag{4.4}$$

where the Fourier coefficients a_i and b_i are respectively

$$a_i=\frac{1}{L}\int_0^L g(x)\varphi_i(x)dx \quad \text{and} \quad b_i=\frac{\varphi_i(x_o)}{L} \tag{4.5}$$

Here a_i depends only on the spatial distribution of the forcing function *g(x)*. If the equations 4.2, 4.3 and 4.4 are substituted into equation 4.1 and Laplace transformation is taken on the resulting equation with respect to time, the transformed result is a set of algebraic equations written as

$$\rho As^2Q_i(s)+EI\beta_i^4Q_i(s)=a_iP(s)+b_iF(s) \quad i\in N \tag{4.6}$$

$$\text{where } \frac{\partial^4 w}{\partial x^4}=\sum_{i=1}^{\infty}\beta_i^4q_i(t)\varphi_i(x) \tag{4.7}$$

If this is solved for the generalized co-ordinates $Q_i(s)$, the result may be written as

$$Q_i(s) = \frac{a_i P(s) + b_i F(s)}{\rho A s^2 + EI\beta_i^4} \tag{4.8}$$

Then if *P(s)* and *F(s)* were known then the *s*-domain motion of any point on the beam could be given as

$$W(x,s) = \sum_{i=1}^{\infty} \frac{a_i P(s) + b_i F(s)}{\rho A s^2 + EI\beta_i^4} \varphi_i(x) \tag{4.9}$$

where $W(x,s)$ is the Laplace transform of $w(x,s)$ with respect to time. For a damped DVA, the transfer function between the motion at the attachment point with the DVA, $W(x_o,s)$, and the force transmitted to the beam, $F(s)$, is written as equation 4.10 below and the proof is shown in Appendix B.

$$\frac{F(s)}{W(x_o,s)} = -\frac{ms^2(cs+k)}{ms^2+cs+k} \tag{4.10}$$

Substitute equation 4.10 into 4.9, we have

$$W(x,s) = \sum_{i=1}^{\infty} \frac{a_i P(s) - b_i W(x_o,s)\dfrac{ms^2(cs+k)}{ms^2+cs+k}}{\rho A s^2 + EI\beta_i^4} \varphi_i(x) \tag{4.11}$$

When $x = x_o$, equation 4.11 becomes,

$$W(x_o,s) = \sum_{i=1}^{\infty} \frac{a_i \varphi_i(x_o) P(s)}{\rho A s^2 + EI\beta_i^4} - W(x_o,s)\frac{ms^2(cs+k)}{ms^2+cs+k} \sum_{i=1}^{\infty} \frac{b_i \varphi_i(x_o)}{\rho A s^2 + EI\beta_i^4} \tag{4.12}$$

$$\Rightarrow W(x_o,s) = \frac{\displaystyle\sum_{i=1}^{\infty} \frac{a_i \varphi_i(x_o) P(s)}{\rho A s^2 + EI\beta_i^4}}{1 + \dfrac{ms^2(cs+k)}{ms^2+cs+k} \displaystyle\sum_{i=1}^{\infty} \frac{b_i \varphi_i(x_o)}{\rho A s^2 + EI\beta_i^4}} \tag{4.13}$$

Substitute (4.13) into (4.11), we have the following transfer function:

$$\frac{W(x,s)}{P(s)} = \sum_{i=1}^{\infty} \frac{a_i - b_i \dfrac{\sum_{i=1}^{\infty} \dfrac{a_i \varphi_i(x_o)}{\rho A s^2 + EI\beta_i^4}}{\dfrac{ms^2 + cs + k}{ms^2(cs+k)} + \sum_{i=1}^{\infty} \dfrac{b_i \varphi_i(x_o)}{\rho A s^2 + EI\beta_i^4}}}{\rho A s^2 + EI\beta_i^4} \varphi_i(x) \tag{4.14}$$

Replacing $s = j\omega$ where $j = \sqrt{-1}$ in equation 4.14 for the steady-state response of the beam and rewrite the resulting equation in a non-dimensional form as

$$\frac{W(x,\lambda)}{P(\lambda)} = \frac{1}{\rho A \omega_n^2} \sum_{i=1}^{\infty} \frac{a_i - b_i \dfrac{\mu L \sum_{i=1}^{\infty} \dfrac{a_i \varphi_i(x_o)}{\gamma_i^2 - \lambda^2}}{-\dfrac{2j\zeta_a \gamma\lambda + \gamma^2 - \lambda^2}{\lambda^2 (2j\zeta_a \gamma\lambda + \gamma^2)} + \mu L \sum_{i=1}^{\infty} \dfrac{b_i \varphi_i(x_o)}{\gamma_i^2 - \lambda^2}}}{\gamma_i^2 - \lambda^2} \varphi_i(x) \tag{4.15}$$

Similarly, the transfer functions of the velocity and the acceleration response at the point x on the beam structure can be rewritten, respectively, as

$$\frac{L\{\dot{w}(x,t)\}}{\omega_n L\{p(t)\}} = j\lambda \frac{W(x,\lambda)}{P(\lambda)} \tag{4.16}$$

and $$\frac{L\{\ddot{w}(x,t)\}}{\omega_n L\{p(t)\}} = -\lambda^2 \frac{W(x,\lambda)}{P(\lambda)} \tag{4.17}$$

4.2 Optimization for minimizing the vibration at a point on the beam structure

For a beam structure with well-separated natural frequencies, the modal displacement response in the vicinity of the n^{th} natural frequency may be approximated by considering $i = n$ and ignoring other modes in the equation 4.15. Consider $i = n$ and ignoring other modes, equation 4.15 may be rewritten as

$$\frac{W(x,\lambda)}{P(\lambda)} = \frac{1}{\rho A\omega_n^2}\frac{a_n - b_n\dfrac{\mu L\dfrac{a_n\varphi_n(x_o)}{1-\lambda^2}}{-\dfrac{2j\zeta_a\gamma\lambda+\gamma^2-\lambda^2}{\lambda^2(2j\zeta_a\gamma\lambda+\gamma^2)}+\mu L\dfrac{b_n\varphi_n(x_o)}{1-\lambda^2}}}{1-\lambda^2}\varphi_n(x) \tag{4.18}$$

Equation 4.18 can be simplified as

$$\frac{W(x,\lambda)}{P(\lambda)} = \frac{a_n\varphi_n(x)}{\rho A\omega_n^2}\frac{\gamma^2-\lambda^2+2j\zeta_a\gamma\lambda}{\left[(1-\lambda^2)(\gamma^2-\lambda^2)-\varepsilon\lambda^2\gamma^2\right]+2j\zeta_a\gamma\lambda(1-\lambda^2-\varepsilon\lambda^2)} \tag{4.19}$$

where $\varepsilon = \mu\varphi_n^2(x_0)$

Equation 4.19 can be rewritten in a form as

$$\frac{W(x,\lambda)}{P(\lambda)} = \frac{a_n\varphi_n(x)}{\rho A\omega_n^2}H(\lambda) \tag{4.21}$$

where

$$H(\lambda) = \frac{\gamma^2-\lambda^2+2j\zeta_a\gamma\lambda}{\left[(1-\lambda^2)(\gamma^2-\lambda^2)-\varepsilon\lambda^2\gamma^2\right]+2j\zeta_a\gamma\lambda(1-\lambda^2-\varepsilon\lambda^2)} \tag{4.20}$$

In considering H_∞ optimization, the objective is to minimize the maximum amplitude ratio of the response of the primary system to the excitation force or motion, i.e.

$$\max\left(\left|\frac{W(x,\lambda,\gamma_{H\infty},\zeta_{H\infty})}{P(\lambda)}\right|\right) = \min\left(\max_{\gamma,\zeta_a}\left|\frac{W(x,\lambda)}{P(\lambda)}\right|\right) \tag{4.22}$$

It is noted that only the function $H(\lambda)$ is required to be considered in the optimization because $\frac{a_n\varphi_n(x)}{\rho A\omega_n^2}$ is a constant term. The objective function of the optimization may be written as

$$\max\left(\left|\frac{W(x,\lambda,\gamma_{H\infty},\zeta_{H\infty})}{P(\lambda)}\right|\right) \approx \frac{a_n\varphi_n(x)}{\rho A\omega_n^2}\max\left(\left|H(\lambda,\gamma_{H\infty},\zeta_{H\infty})\right|\right) \tag{4.23}$$

The equation 4.20 is equivalent to the equation 2.19 in the SDOF system attached with a DVA if the term ε in equation 4.20 is replaced by the mass ratio μ. Therefore, applying the fixed-points theory, the optimum tuning can be found in the same way as the case of SDOF system and the result are listed in Table 4.1

Table 4.1 The H_∞ optimum tuning at a point x in the beam structure

Transfer function	Tuning ratio	Damping ratio	The height of the fixed point
$\frac{L\{w(t)\}}{L\{p(t)\}}$	$\frac{1}{1+\varepsilon}$	$\sqrt{\frac{3\varepsilon}{8(1+\varepsilon)}}$	$\frac{a_n\varphi_n(x)}{\rho A\omega_n^2}\sqrt{\frac{2+\varepsilon}{\varepsilon}}$
$\frac{L\{\dot{w}(t)\}}{\omega_n L\{p(t)\}}$	$\frac{1}{1+\varepsilon}\sqrt{\frac{2+\varepsilon}{2}}$	$\frac{1}{4(2+\varepsilon)}\sqrt{\frac{\varepsilon(24+24\varepsilon+5\varepsilon^2)}{1+\varepsilon}}$	$\frac{a_n\varphi_n(x)}{\rho A\omega_n^2}\sqrt{\frac{2+\varepsilon}{\varepsilon(1+\varepsilon)}}$
$\frac{L\{\ddot{w}(t)\}}{\omega_n^2 L\{p(t)\}}$	$\sqrt{\frac{1}{1+\varepsilon}}$	$\frac{1}{2}\sqrt{\frac{3\varepsilon}{2+\varepsilon}}$	$\frac{a_n\varphi_n(x)}{\rho A\omega_n^2}\sqrt{\frac{2}{\varepsilon(1+\varepsilon)}}$

Similarly, for the H_2 optimization of searching the DVA parameters for suppressing vibration of a beam structure, the objective is to minimize the total vibration energy of the beam structure at the point x at all frequencies. The performance index can be defined as

$$\min_{\gamma,\zeta_a}\left(E\left[w^2(t)\right]\right) \tag{4.24}$$

The mean square motion can be written in terms of the input mean square spectral density $S_p(\omega)$ as

$$E\left[w^2(t)\right]=\int_{-\infty}^{\infty}S_w(\omega)d\omega=\int_{-\infty}^{\infty}\left|\frac{W(\omega)}{P(\omega)}\right|^2 S_p(\omega)d\omega \tag{4.25}$$

If the input spectrum is assumed to be ideally white, i.e. $S_p(\omega)=S_0$, a constant for all frequencies, the integral of equation 4.25 can then be reduced to

$$E\left[w^2\right]=S_0\int_{-\infty}^{\infty}\left|\frac{W(\omega)}{P(\omega)}\right|^2 d\omega \tag{4.26}$$

Using equation 4.26, the non-dimensional mean square motion can be defined as

$$E\left[x^2\right]=\frac{\omega_n S_0}{2\pi}\int_{-\infty}^{\infty}\left|\frac{W(\omega)}{P(\omega)}\right|^2 d\lambda\approx\frac{\omega_n S_0}{2\pi}\left(\frac{a_n\varphi_n(x)}{\rho A\omega_n^2}\right)^2\int_{-\infty}^{\infty}\left|H(\lambda)\right|^2 d\lambda \tag{4.27}$$

Similarly, equation 4.20 is equivalent to the equation 2.19 in the SDOF system attached with a DVA if the term ε in equation 4.20 is replaced by the mass ratio μ. Therefore the optimum DVA parameters can be found in the same way as in the case of the SDOF system and the result are listed in Table 4.2.

Table 4.2 The H_2 optimum tuning at a point x in a beam structure

Transfer function	Tuning ratio	Damping ratio	Optimized value of performance index
$E\left[w^2(t)\right]$	$\sqrt{\dfrac{\varepsilon+2}{2(\varepsilon+1)^2}}$	$\dfrac{1}{2}\sqrt{\dfrac{\varepsilon(3\varepsilon+4)}{2(\varepsilon+2)(\varepsilon+1)}}$	$\dfrac{1}{2}\dfrac{a_n\varphi_n(x)}{\rho A\omega_n^2}\sqrt{\dfrac{4+3\varepsilon}{\varepsilon(1+\varepsilon)}}$
$E\left[\dot{w}^2(t)\right]$	$\sqrt{\dfrac{1}{1+\varepsilon}}$	$\sqrt{\dfrac{\varepsilon}{2}}$	$\dfrac{a_n\varphi_n(x)}{\rho A\omega_n^2}\sqrt{\dfrac{1}{\varepsilon(1+\varepsilon)}}$

4.3 Optimization for minimizing the root mean square motion over the whole domain of the beam structure

Equation 4.15 is restated below for the ease of discussion.

$$\frac{W(x,\lambda)}{P(\lambda)}=\frac{1}{\rho A\omega_n^2}\sum_{i=1}^{\infty}\frac{a_i-b_i\dfrac{\mu L\sum_{i=1}^{\infty}\dfrac{a_i\varphi_i(x_o)}{\gamma_i^2-\lambda^2}}{-\dfrac{2j\zeta_a\gamma f+\gamma^2-\lambda^2}{\lambda^2(2j\zeta_a\gamma\lambda+\gamma^2)}+\mu L\sum_{i=1}^{\infty}\dfrac{b_i\varphi_i(x_o)}{\gamma_i^2-\lambda^2}}}{\gamma_i^2-\lambda^2}\varphi_i(x) \tag{4.15}$$

The root mean square motion over the whole domain of the beam structure may be written as

$$\int_0^L\left(\frac{W(x,\lambda)}{P(\lambda)}\right)^2dx$$

$$=\int_0^L\left(\frac{1}{\rho A\omega_n^2}\right)^2\left(\sum_{i=1}^{\infty}\frac{a_i-b_i\dfrac{\mu L\sum_{i=1}^{\infty}\dfrac{a_i\varphi_i(x_o)}{\gamma_i^2-\lambda^2}}{-\dfrac{2j\zeta_a\gamma f+\gamma^2-\lambda^2}{\lambda^2(2j\zeta_a\gamma\lambda+\gamma^2)}+\mu L\sum_{i=1}^{\infty}\dfrac{b_i\varphi_i(x_o)}{\gamma_i^2-\lambda^2}}}{\gamma_i^2-\lambda^2}\varphi_i(x)\right)^2dx \tag{4.28}$$

Consider the orthogonality relations of the eigenfunctions, we may write

$$\int_0^L\varphi_m(x)\varphi_n(x)dx=0\quad\text{if}\quad m\neq n \tag{4.29a}$$

and $$\int_0^L\varphi_m(x)\varphi_n(x)dx=L\quad\text{if}\quad m=n \tag{4.29b}$$

Equation 4.28 can be simplified with the above orthogonality relations of the eigenfunctions as

$$\int_0^L \left(\frac{W(x,\lambda)}{P(\lambda)} \right)^2 dx = \left(\frac{\sqrt{L}}{\rho A \omega_n^2} \sum_{i=1}^{\infty} \frac{a_i - b_i \dfrac{\mu L \sum_{i=1}^{\infty} \dfrac{a_i \varphi_i(x_o)}{\gamma_i^2 - \lambda^2}}{-\dfrac{2 j \zeta_a \gamma f + \gamma^2 - \lambda^2}{\lambda^2 (2 j \zeta_a \gamma \lambda + \gamma^2)} + \mu L \sum_{i=1}^{\infty} \dfrac{b_i \varphi_i(x_o)}{\gamma_i^2 - \lambda^2}}}{\gamma_i^2 - \lambda^2} \right)^2 \tag{4.30}$$

For a beam structure with well-separated natural frequencies, the modal displacement response in the vicinity of the n^{th} natural frequency may be approximated by considering $i = n$ and ignoring other modes in equation 4.30 and written as

$$\int_0^L \left(\frac{W(x,\lambda)}{P(\lambda)} \right)^2 dx = \left(\frac{\sqrt{L}}{\rho A \omega_n^2} \frac{a_n - b_n \dfrac{\mu L \dfrac{a_n \varphi_n(x_o)}{1 - \lambda^2}}{-\dfrac{2 j \zeta_a \gamma f + \gamma^2 - \lambda^2}{\lambda^2 (2 j \zeta_a \gamma \lambda + \gamma^2)} + \mu L \dfrac{b_n \varphi_n(x_o)}{1 - \lambda^2}}}{1 - \lambda^2} \right)^2 \tag{4.31}$$

Equation 4.31 can be simplified as

$$\sqrt{\int_0^L \left(\frac{W(x,\lambda)}{P(\lambda)}\right)^2 dx} = \frac{a_n\sqrt{L}}{\rho A\omega_n^2}\frac{\gamma^2-\lambda^2+2j\zeta_a\gamma\lambda}{\left[(1-\lambda^2)(\gamma^2-\lambda^2)-\varepsilon\lambda^2\gamma^2\right]+2j\zeta_a\gamma\lambda(1-\lambda^2-\varepsilon\lambda^2)} \tag{4.32}$$

$$\text{where} \quad \varepsilon = \mu\varphi_n^2(x_0)$$

Equation 4.32 can be rewritten in a form as

$$\sqrt{\int_0^L \left(\frac{W(x,\lambda)}{P(\lambda)}\right)^2 dx} = \frac{a_n\sqrt{L}}{\rho A\omega_n^2}H(\lambda) \tag{4.34}$$

where

$$H(\lambda) = \frac{\gamma^2-\lambda^2+2j\zeta_a\gamma\lambda}{\left[(1-\lambda^2)(\gamma^2-\lambda^2)-\varepsilon\lambda^2\gamma^2\right]+2j\zeta_a\gamma\lambda(1-\lambda^2-\varepsilon\lambda^2)} \tag{4.33}$$

In solving the H_∞ optimization problem, the objective function is to minimize the maximum amplitude ratio between the response of the primary system relative and the excitation force or motion, i.e.

$$\max\left(\left|\sqrt{\int_0^L \left(\frac{W(x,\lambda,\gamma_{H\infty},\zeta_{H\infty})}{P(\lambda)}\right)^2 dx}\right|\right) = \min\left(\max_{\gamma,\zeta_a}\left|\sqrt{\int_0^L \left(\frac{W(x,\lambda)}{P(\lambda)}\right)^2 dx}\right|\right) \tag{4.35}$$

It is noted that only the function $H(\lambda)$ is required to be considered in the optimization because $\frac{a_n\sqrt{L}}{\rho A\omega_n^2}$ is a constant term. The objective function may be rewritten as

$$\max\left(\left|\sqrt{\int_0^L \left(\frac{W(x,\lambda,\gamma_{H\infty},\zeta_{H\infty})}{P(\lambda)}\right)^2 dx}\right|\right) = \frac{a_n\sqrt{L}}{\rho A\omega_n^2}\min\left(\max_{\gamma,\zeta_a}|H(\lambda)|\right) \tag{4.36}$$

Equation 4.33 is equivalent to equation 2.19 in the SDOF system attached with a DVA if the term ε in equation 4.33 is replaced by the mass ratio μ. Therefore, by applying the fixed-points theory, the optimum parameters of the DVA can be found in the same way as in the case of the SDOF system and the result are listed in Table 4.3.

Table 4.3 The H_∞ optimum tuning of the root mean square motion of the beam structure

Transfer function	Tuning ratio	Damping ratio	The height of the fixed point
$\sqrt{\int_0^L \left(\frac{L\{w(t)\}}{L\{p(t)\}}\right)^2 dx}$	$\frac{1}{1+\varepsilon}$	$\sqrt{\frac{3\varepsilon}{8(1+\varepsilon)}}$	$\frac{a_n\sqrt{L}}{\rho A\omega_n^2}\sqrt{\frac{2+\varepsilon}{\varepsilon}}$
$\sqrt{\int_0^L \left(\frac{L\{\dot{w}(t)\}}{\omega_n L\{p(t)\}}\right)^2 dx}$	$\frac{1}{1+\varepsilon}\sqrt{\frac{2+\varepsilon}{2}}$	$\frac{1}{4(2+\varepsilon)}\sqrt{\frac{\varepsilon(24+24\varepsilon+5\varepsilon^2)}{1+\varepsilon}}$	$\frac{a_n\sqrt{L}}{\rho A\omega_n^2}\sqrt{\frac{2+\varepsilon}{\varepsilon(1+\varepsilon)}}$
$\sqrt{\int_0^L \left(\frac{L\{\ddot{w}(t)\}}{\omega_n^2 L\{p(t)\}}\right)^2 dx}$	$\sqrt{\frac{1}{1+\varepsilon}}$	$\frac{1}{2}\sqrt{\frac{3\varepsilon}{2+\varepsilon}}$	$\frac{a_n\sqrt{L}}{\rho A\omega_n^2}\sqrt{\frac{2}{\varepsilon(1+\varepsilon)}}$

Similarly, in searching the H_2 optimization solution of the DVA for suppressing vibration in a beam structure, the performance index can be defined as

$$\min_{\gamma,\zeta_a}\left(E\left[\sqrt{\int_0^L \left(\frac{L\{w(t)\}}{L\{p(t)\}}\right)^2 dx}\right]\right) \tag{4.37}$$

Following equations 4.25, 4.26 and 4.27, the optimum DVA parameters can be found in the same way as in the case of the SDOF system and the result are listed in Table 4.4

Table 4.4 The H_2 optimum tuning of the root mean square motion of the beam structure

Transfer function	Tuning ratio	Damping ratio	Optimized value of performance index
$E\left[\sqrt{\int_0^L \left(\frac{L\{w(t)\}}{L\{p(t)\}}\right)^2 dx}\right]$	$\sqrt{\frac{\varepsilon+2}{2(\varepsilon+1)^2}}$	$\frac{1}{2}\sqrt{\frac{\varepsilon(3\varepsilon+4)}{2(\varepsilon+2)(\varepsilon+1)}}$	$\frac{1}{2}\frac{a_n\sqrt{L}}{\rho A\omega_n^2}\sqrt{\frac{4+3\varepsilon}{\varepsilon(1+\varepsilon)}}$
$E\left[\sqrt{\int_0^L \left(\frac{L\{\dot{w}(t)\}}{L\{p(t)\}}\right)^2 dx}\right]$	$\sqrt{\frac{1}{1+\varepsilon}}$	$\sqrt{\frac{\varepsilon}{2}}$	$\frac{a_n\sqrt{L}}{\rho A\omega_n^2}\sqrt{\frac{1}{\varepsilon(1+\varepsilon)}}$

4.4 Numerical Simulation

A simply supported Euler beam attached with a DVA as illustrated in Figure 4.2 is considered in the following numerical study. The eigenfunctions and the eigenvalues can be written as

$$\varphi(x) = \frac{2}{L}\sin(\beta_i x) \tag{4.38}$$

$$\text{where} \quad \beta_i = \frac{i\pi}{L} \quad i \in N$$

The reciprocal of the denominator integral is

$$K_i = \int_0^L \sin^2(\beta_i x)dx = \frac{L}{2} \tag{4.39}$$

$$a_i = \frac{2}{L}\sin\left(\frac{i\pi x_1}{L}\right) \quad \text{and} \quad b_i = \frac{2}{L}\sin\left(\frac{i\pi x_o}{L}\right) \tag{4.40}$$

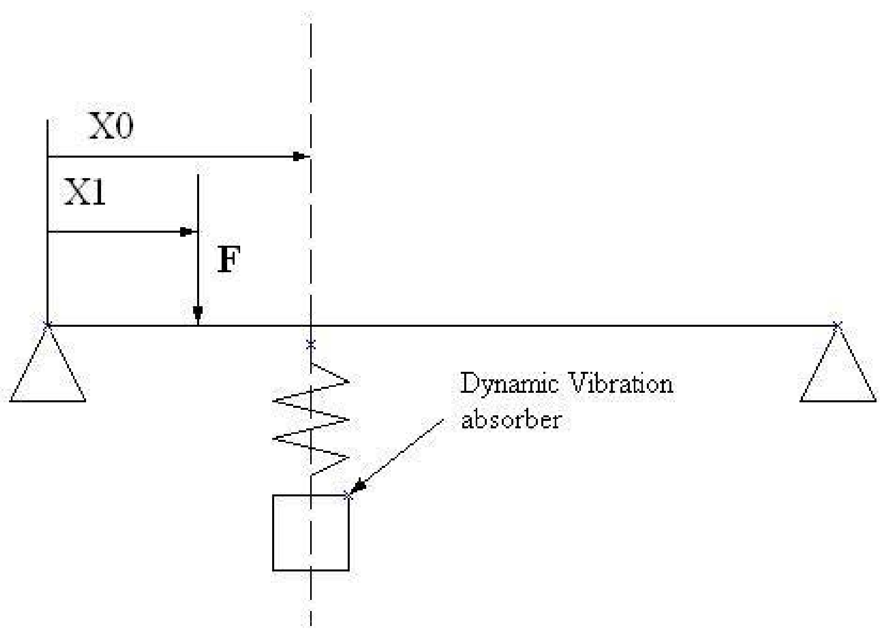


Figure 4.2 The simply supported beam with DVA under a concentrated force

The material of beam is assumed to be aluminum of density and Young's modulus 2710 kg/m^3 and 6.9GPa, respectively. The mass ratio μ is 0.2. The dimensions of the beam are 1m (length) X 0.025m (Width) X 0.0025m (height). x_0 is 0.3m and x_1 is 0.2m.

As shown in figure 4.3, the maximum response of the beam at x_0 without damping is much higher than the maximum response of the beam using the optimum tuning frequency and damping of the DVA. Moreover, the second resonance of the beam at x_0 can also be suppressed.

As shown in figure 4.4, the optimum tuning parameters presented in the previous section are applied and the resulting frequency response of the beam is compared to the one using the optimum DVA parameters suggested by Den Hartog (1985). The maximum response of the system can be reduced by more than 20% in this case.

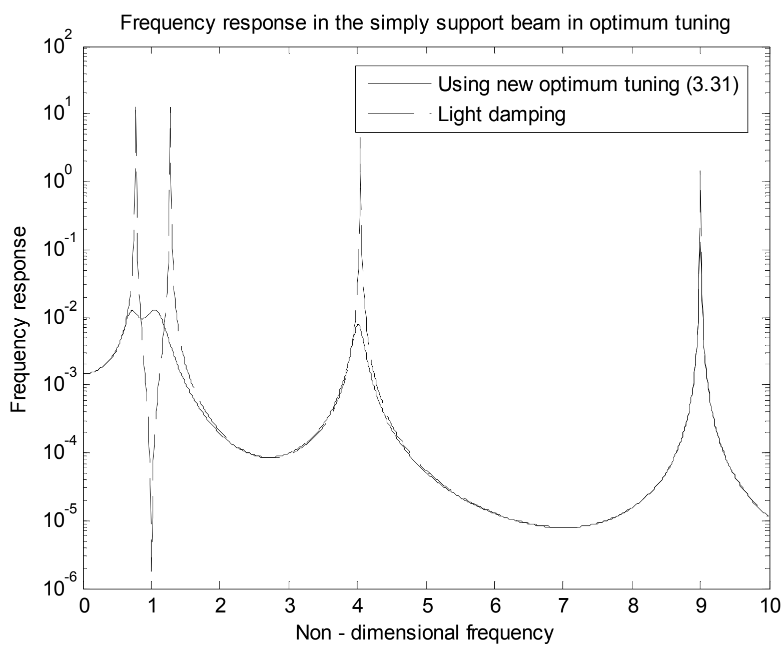


Figure 4.3 Frequency responses of the beam at $x = x_0$ with very light damping and optimum tuning frequency and damping in the DVA

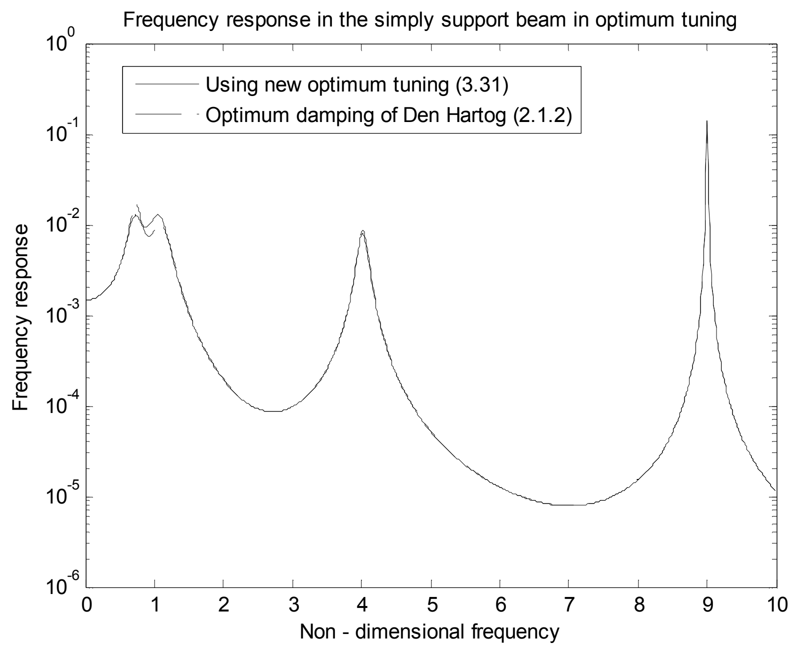


Figure 4.4 Frequency responses of the beam at $x = x_0$ with the proposed optimum parameters of the DVA and the ones made by Den Hartog.

To test the usefulness of the derived H_∞ optimization solution for suppressing vibrations a continuous vibrating system, the numerical testing case for the minimization of the maximum kinetic energy of a vibrating beam reported by Dayou (2006) was studied by applying the present theory and the result was compared to those obtained by Dayou. The vibrating beam considered by Dayou was a simply supported aluminum beam excited by a point force of unit amplitude at $0.1L$ as shown in figure 4.5. The eigenfunctions and eigenvalues of the beam could be written respectively as (Srinivasan 1982)

$$\varphi_p(x) = \sin(\frac{p\pi x}{L}), p = 1, 2, 3\ldots, \text{ and} \tag{4.41}$$

$$\omega_p{}^2 = \left(\frac{p\pi}{L}\right)^4 \left(\frac{EI}{\rho A}\right), p = 1, 2, 3\ldots \tag{4.42}$$

where L = 1 m, E = 207 GPa, I = 8.1295×10^{-10} m^4, ρ = 7870 kg/m^3 and A = 2.42 x 10^{-4} m^2. A dynamic vibration absorber was attached at $x_o = 0.5L$ and mass ratio, μ was 0.05.

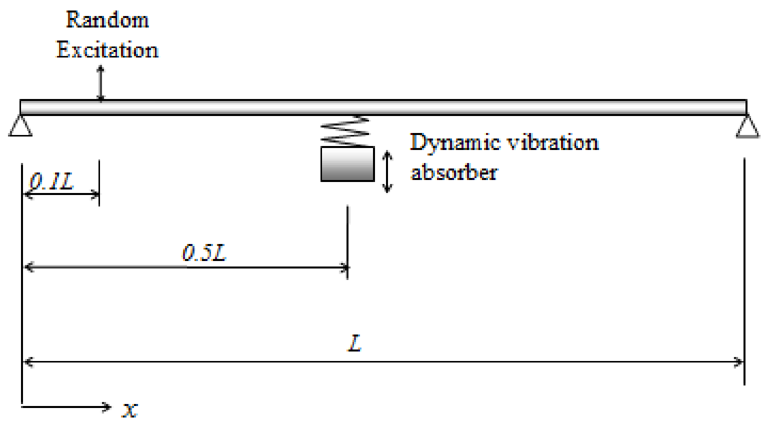


Figure 4.5 Schematics of a simply supported beam with a vibration absorber excited by a random force at x= 0.1L.

From equations (4.2) and (4.5), we have

$$a_p = \frac{\int_0^L \delta(x-x_1)\sin\left(\frac{p\pi x}{L}\right)dx}{\int_0^L \sin^2\left(\frac{p\pi x}{L}\right)dx} = \frac{2}{L}\sin\left(\frac{p\pi x_1}{L}\right) \tag{4.43a}$$

$$b_p = \frac{\sin\left(\frac{p\pi x_o}{L}\right)}{\int_0^L \sin^2\left(\frac{p\pi x}{L}\right)dx} = \frac{2}{L}\sin\left(\frac{p\pi x_o}{L}\right),\ p = 1, 2, 3\ldots \tag{4.43b}$$

The optimization problem could be expressed as

$$H_{\infty\_beam\_vel} = \inf_f\left(\sup_{\zeta,T}\left(\left|\int_0^L\left(\frac{\dot{U}(x,f)}{\omega_1 W(f)}\right)^2 dx\right|\right)\right) = \inf_f\left(\sup_{\zeta,T}\left(\left|\int_0^L\left(\frac{jfU(x,f)}{\omega_1 W(f)}\right)^2 dx\right|\right)\right) \tag{4.44}$$

where

$$\frac{U(x,f)}{W(f)} = \left(\frac{a_1 L}{\rho h\omega_1^2}\right)\left[\frac{T^2 - f^2 + 2j\zeta_a Tf}{(2j\zeta_a Tf + T^2 - f^2)(1-f^2) - \varepsilon f^2(2j\zeta_a Tf + T^2)}\right] \tag{4.45}$$

and $\varepsilon = \mu\varphi_1^2(x_o)$ which was the one-dimensional version of the ε used in the theory section.

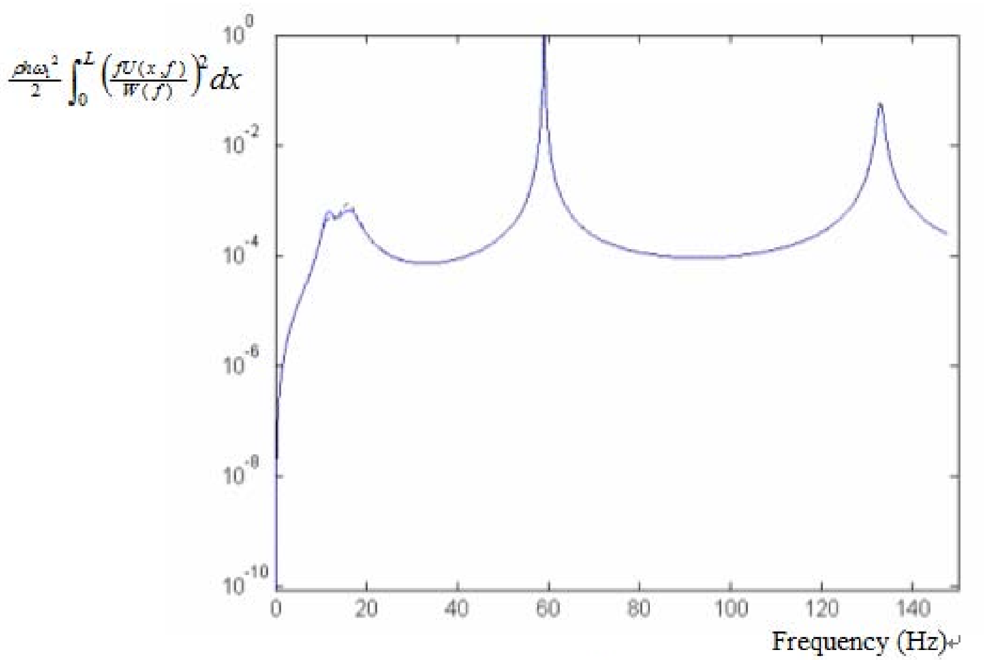
According to Dayou (2006), the optimum frequency and damping ratios were $\frac{1}{1+\varepsilon}$ = 0.8333 and $\sqrt{\frac{3\varepsilon}{8(1+\varepsilon)}}$ = 0.25 respectively. Based on the present theory and the derived expressions of the optimum frequency and damping ratios for H_∞ optimization with different types of transfer functions as shown in Table 4.3, the frequency and damping ratios for minimum kinetic energy amplitude of the plate were T = $\frac{1}{1+\varepsilon}\sqrt{\frac{2+\varepsilon}{2}}$ = 0.8740, and ζ_a = $\frac{1}{4(2+\varepsilon)}\sqrt{\frac{\varepsilon(24+24\varepsilon+5\varepsilon^2)}{1+\varepsilon}}$ = 0.2498 respectively. Kinetic energy amplitudes of the whole beam at steady state were calculated at different excitation frequencies according to equation:

$$\frac{\rho h\omega_1^2}{2}\int_0^L\left[\frac{U(x,f)}{W(f)}\right]^2 dx$$

$$=\frac{\rho h\omega_1^2}{2}\sum_{p=1}^{p_{\max}}\left(\frac{Lf}{\rho h\,\omega_1^2}\right)^2\frac{\left(a_p-\dfrac{b_p\mu L\sum\limits_{p=1}^{p_{\max}}\dfrac{a_p\varphi_p(x_o)}{\gamma_p^2-f^2}}{\left(\dfrac{f^2-2j\zeta_aTf-T}{f^2(2j\zeta_aTf+T^2)}+\mu L\sum\limits_{p=1}^{p_{\max}}\dfrac{b_p\varphi_p(x_o,y_o)}{\gamma_p^2-f^2}\right)}\right)^2}{\left(\gamma_p^2-f^2\right)^2} \quad (4.46)$$

A Matlab program is written to calculate these kinetic energy amplitudes and the results were plotted in figure 4.6(a). Ten vibration modes ($p_{\max}$ = 10) of the beam were used in the calculation. The kinetic energies of the beam calculated based on the present theory and that by Dayou were plotted in Figure 4.6(a) for comparison.

The amplitude of the kinetic energy at the first resonance of the beam was suppressed after adding the vibration absorber. However, it could be observed in Figure 4.6(b), a close-up of the spectrum around the first natural frequency of the beam, the heights of the two peaks of the curve of Dayou had a big difference indicating that the damping and frequency ratios of the absorber were not optimal based on the fixed-points theory. The maximum amplitude of the kinetic energy of the whole beam around the first natural frequency of the beam calculated with the proposed frequency and damping ratios was found to be about 32% smaller than that of the beam with the frequency ratio (T = 0.8333) and damping ratio (ζ = 0.25) used by Dayou.



(a)

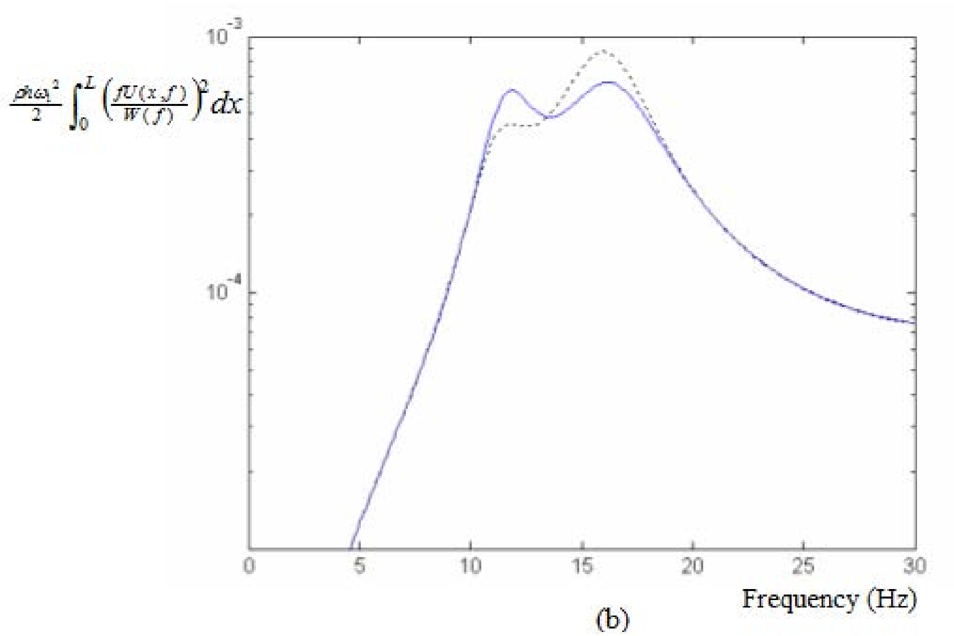


Figure 4.6 Kinetic energy (in J/N^2) of a simply supported with optimum vibration absorber fitted at x_o = 0.5L with the first natural frequency as the control target: (a) figure showing all three modes; (b) in the vicinity of the first mode. --------- T = 0.8333, ζ = 0.25 (Dayou 2006), ———— T = 0.8775, ζ = 0.2556 (present theory)

The exact values of tuning frequency and damping ratios to minimize the maximum amplitude of the kinetic energy of the whole beam around the first natural frequency of the beam were determined numerically with equation 4.15 as T = 0.8775 and ζ = 0.2556. The difference of this maximum amplitude of kinetic energy in using the proposed and the exact sets of T and ζ was found to be about 3%.

4.5 Effect of the tuned DVA in other modes

The approximated optimum tunings are derived for the dominant mode only. However, the DVA under the approximated optimum tunings also affects other modes. In this part, the effect of the attached DVA on other modes of the structure is discussed.

Consider the transfer function of the DVA represented by equation (4.10) and substituting $s = j\omega$ in equation (4.10), the steady state frequency response function of the beam at the attachment point can be written as

$$\frac{F(\omega)}{W(x_o,\omega)} = \frac{mk\omega^2 + jmc\omega^3}{k - m\omega^2 + jc\omega} \tag{4.47}$$

Equation (4.47) may be rewritten as

$$\frac{mk\omega^2 + jmc\omega^3}{k - m\omega^2 + jc\omega} \equiv -k_e(\omega) - j\omega c_e(\omega) \tag{4.48}$$

where $k_e(\omega) = -\dfrac{mk\omega^2\left(k - m\omega^2\right) + m\left(c\omega^2\right)^2}{\left(k - m\omega^2\right)^2 + (c\omega)^2}$ and

$$c_e(\omega) = \frac{m^2 c\omega^4}{\left(k - m\omega^2\right)^2 + (c\omega)^2}$$

or rewriting in non-dimensional form as

$$\frac{F(\lambda)}{m\omega_n^2 W(x_o,\lambda)} = \frac{\lambda^2\left(2j\zeta_a\gamma\lambda + \gamma^2\right)}{\gamma^2 - \lambda^2 + 2j\zeta_a\gamma\lambda} \equiv \gamma_e^2(\lambda) + 2j\zeta_e(\lambda)\gamma_e(\lambda)\lambda \tag{4.49}$$

where $\gamma_e(\lambda) = \sqrt{\dfrac{k_e(\lambda)}{m\omega_n^2}}$ and $\zeta_e(\lambda) = \dfrac{c_e(\lambda)}{2\sqrt{mk_e(\lambda)}}$

where $k_e(\omega)$ and $c_e(\omega)$ are equivalent spring stiffness and equivalent damping coefficient at the exciting frequency ω respectively. The physical meanings of these values are that the effect of the DVA attached to the structure at the exciting frequency ω is same as the structure with the spring and the damper, with the spring stiffness equals to k_e and the damping coefficient equals to c_e, at the same position of the DVA attached and under the same exciting frequency ω as illustrated in figure 4.7.

Consider the example in the previous section. The mass of the DVA is $0.3388kg$, the damping coefficient is $18.1429Nm^{-1}s$ and the spring stiffness is $1.885kNm^{-1}$. The equivalent stiffness and damping at different excitation frequency are calculated according to equation (4.48) and plotted in figure 4.8.

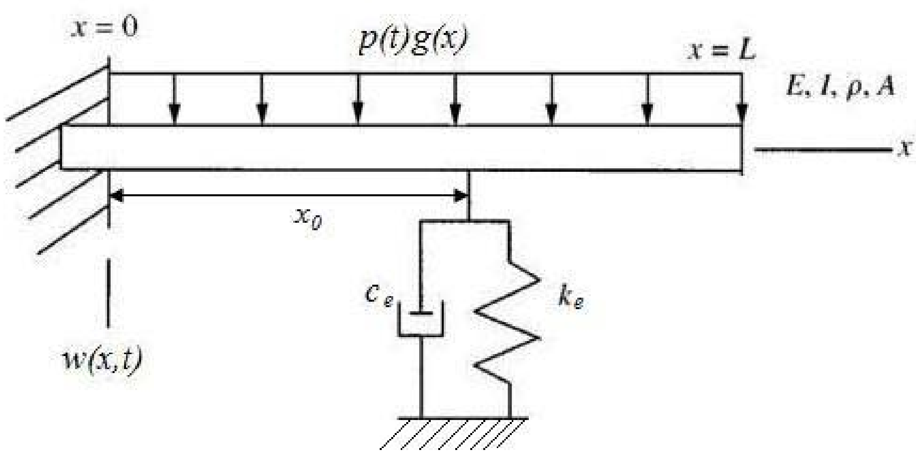


Figure 4.7 The cantilever beam with an equivalent spring and an equivalent damper under a external force

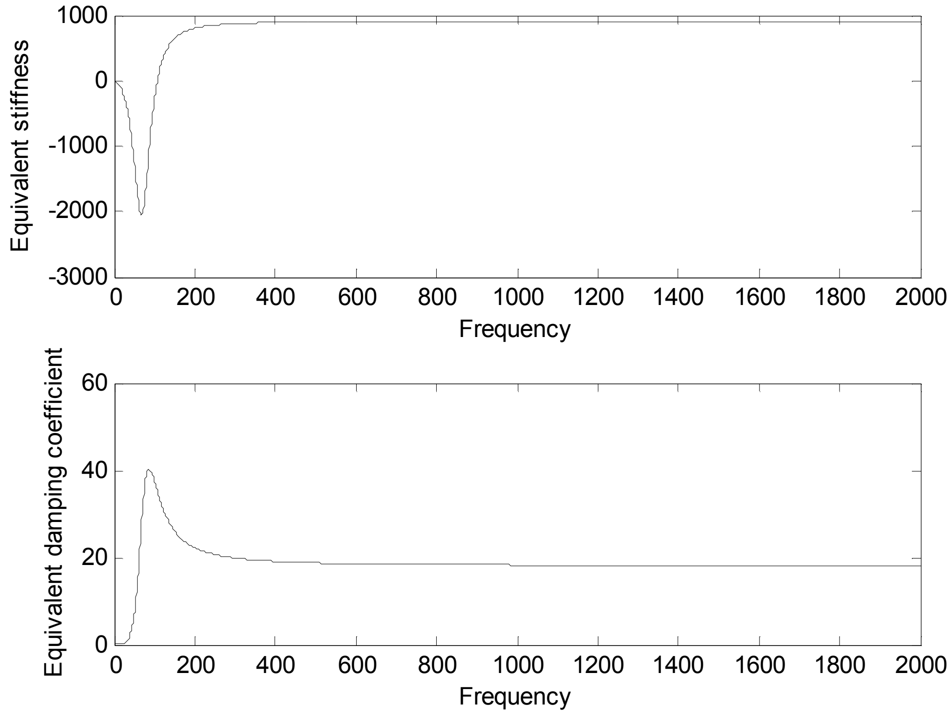


Figure 4.8 The equivalent stiffness and the equivalent damping coefficient of the DVA

In the example of the previous section, the natural frequencies of the beam without DVA are 113.7Hz, 454.6Hz, 1022.9Hz and 1818.5Hz. And the equivalent stiffness and the damping coefficient at these resonant frequencies are listed below.

Table 4.5 The equivalent stiffness and damping coefficient in the resonant frequencies

Mode	Natural frequency (*Hz*)	Equivalent stiffness (Nm^{-1})	Equivalent damping coefficient ($Nm^{-1}s$)
1	113.7	186.56	33.21
2	454.6	897.77	18.88
3	1022.9	910.43	18.28
4	1818.5	912.40	18.19

The frequency response function of the beam at the attachment point using the equivalent stiffness and equivalent damping coefficient are plotted in figure 4.9. Two cases are compared in the followings: one is the cantilever connected with a DVA at the free end while the other is the equivalent system as illustrated in figure 4.7. The frequency response function at the attachment point are calculated and plotted in figure 4.9 for comparison.

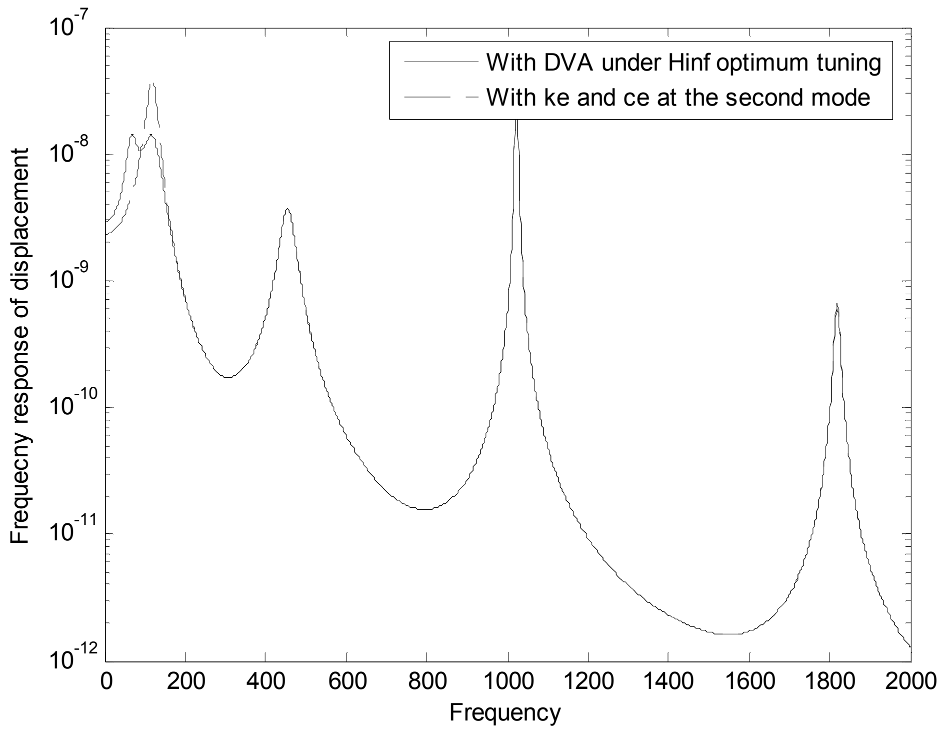


Figure 4.9 The primary system with the equivalent spring and damper

In figure 4.9, it can be seen that the response of the two curves are almost overlapped together at all modes except the first mode. This shows that the equivalent system in figure 4.7 can be used to represent the cantilever beam attached with the DVA for the frequency response at vibration modes other than the tuned mode of the DVA. In figure 4.8, it is observed that the equivalent damping coefficient is larger than the original damping coefficient in the DVA, i.e. $c = 18.1429 Nm^{-1}s$ when the natural frequency is higher than the natural frequency of the first mode and it approach to c as excitation frequency approaching infinity. That means the tuned DVA does not only have an effect in the dominant mode but it also has a damper effect in other higher order modes. The proposed equivalent vibrating system may be used to represent the effect of the DVA to the response of the higher order modes of the vibrating beam which is neglected in the analysis of the previous sections of this chapter.

4.6 Conclusion in the optimization of DVA for suppressing vibrations in beam structures

Based on the research as presented in the previous sections, the following conclusions are drawn.

i. The optimum tuning parameters of DVA for the minimization of root mean square motion of the beam are the same as those for the minimization of vibration at a point on the beam.

ii. The optimum tuning frequency and damping ratios of the DVA on the beam structure do not only depend on the mass ratio as the case of SDOF system in Chapter 3. It depends on the value of $\varepsilon = \mu\varphi_n^2(x_o)$ which in this case that means it depends on both the mass ratio and the modal response of the beam at the attachment point of the DVA.

iii. The optimized DVA can in general suppress only the modal response of the beam structure at the mode which the DVA is tuned on. The DVA has little effect on the response of the beam at other modes.

iv. The maximum response amplitude of the beam with the optimized DVA depends on the density of the beam, cross section area of the beam, the length of the beam, the tuning frequency of the DVA, the mode shape of the beam and the location of the excitation.

5. OPTIMZATIONS OF THE TRADITIONAL DVA FOR SUPPRESSING VIBRATIONS IN PLATE STRUCTURES

Jacquot (2001) proposed a transfer function of the plate attached with a dynamic vibration absorber. He set the frequency ratio to one and determined the optimum damping ratio of the absorber based on the transfer function. However, it is shown in the latter sections of this chapter that the optimum frequency ratio of the absorber is in general not equal to one and another set of optimum frequency and damping ratios have been derived based on the proposed analytical model.

In this chapter, a theory is established for describing the excitation-response relation leading to the H_∞ and H_2 optimum tuning of the dynamic vibration absorber attached onto a plate structure. The present case is much more complicated than a SDOF structure because an improper selection of attachment point for the absorber may lead to an amplification of vibration in other parts of the structure (Wong et al. 2007). The established theory improves our understanding of the effects of different parameters including the mass, damping and tuning ratios and also the point of attachment of the absorber on the vibration absorption by the absorber. The optimum tuning as derived in this article based on the fixed-points theory (Den Hartog 1985) includes tuning frequency and damping ratios of the absorber and also the position of the absorber on the vibrating structure. The objective of the optimum tuning is to minimize vibrational displacement, velocity and acceleration of a point on the plate as well as the minimization of root mean square motion over the whole domain of the plate. The numerical simulations are used to show the usefulness of the optimization solutions leading to better vibration control in continuous systems than those suggested by Jacquot (2001) based on another approach to the problem.

5.1 Theory

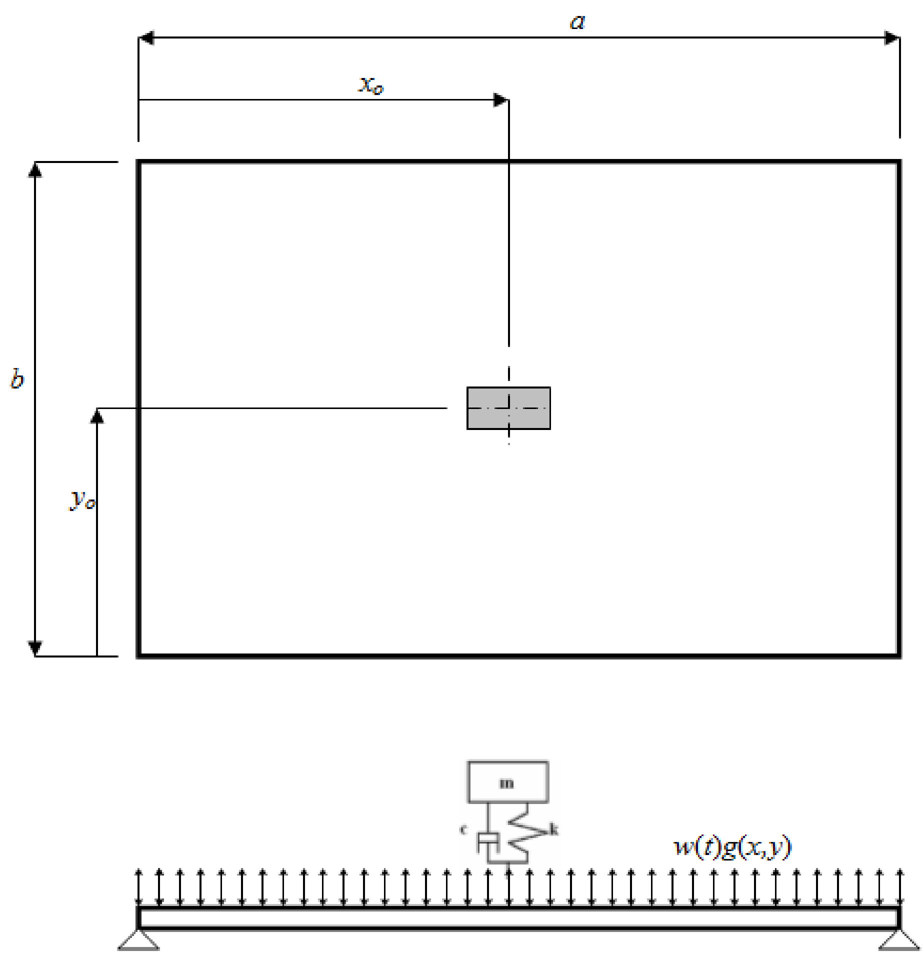


Figure 5.1 A simply-supported rectangular plate under external distributed force $w(t)g(x,y)$ and carrying a dynamic vibration absorber at point (x_o, y_o).

Consider a thin rectangular plate on the rectangular domain $0 \leq x \leq L_x$ and $0 \leq y \leq L_y$ which carries a dynamic vibration absorber at point (x_o, y_o) as shown in Figure 5.1. The plate is under external distributed force $w(t)g(x, y)$ and the point force $r(t)$ is transmitted to the plate by the attached dynamic absorber. The equation of motion for the plate may be written as

$$\nabla^4 u + \frac{\rho h}{D}\frac{\partial^2 u}{\partial t^2} = \frac{w(t)g(x,y)}{D} + \frac{r(t)}{D}\delta(x-x_o)\delta(y-y_o) \tag{5.1}$$

where the flexural rigidity of the plate D is defined as

$$D = \frac{Eh^3}{12(1-\nu^2)} \tag{5.2}$$

and where E is the modulus of elasticity, v is the Poisson ratio, h is the thickness of the plate and ρ is the material density.

It is assumed that the externally applied forcing function is $w(t)g(x,y)$, where $g(x,y)$ is a deterministic function of x and y, and $w(t)$ is a stationary random function of time. The equation of motion of the free vibration of the plate without the absorber may be written as

$$\nabla^4 \varphi_{pq}(x,y) = \frac{\rho h}{D}\omega_{pq}^2 \varphi_{pq}(x,y) \tag{5.3}$$

where ω_{pq} and $\varphi_{pq}(x,y)$ are the pq^{th} natural frequency and eigenfunction of the plate without absorber respectively. The solution to equation 5.1 can be expanded in a Fourier series written as

$$u(x,y,t) = \sum_{p=1,q=1}^{\infty} n_{pq}(t)\varphi_{pq}(x,y). \tag{5.4}$$

Similarly, the spatial part of the forcing function can be expanded as

$$g(x,y) = \sum_{p=1,q=1}^{\infty} a_{pq}\varphi_{pq}(x,y). \tag{5.5}$$

The Dirac delta functions can also be expanded as

$$\delta(x-x_o)\delta(y-y_o) = \sum_{p=1,q=1}^{\infty} b_{pq}\varphi_{pq}(x,y) \tag{5.6}$$

where the Fourier coefficients a_{pq} and b_{pq} are respectively

$$a_{pq} = \left(\tfrac{1}{L_x L_y}\right)\int_0^a \int_0^b g(x,y)\varphi_{pq}(x,y)dydx \text{ , and} \tag{5.7}$$

$$b_{pq} = \left(\tfrac{1}{L_x L_y}\right)\varphi_{pq}(x_o, y_o). \tag{5.8}$$

Substituting equations 5.4, 5.5 and 5.6 into equation 5.1 and performing Laplace transformation on the resulting equation with respect to time, the result may be written as

$$\sum_{p=1,q=1}^{\infty}\left[\frac{\rho h}{D}\omega_{pq}^2 N_{pq}(s) + \frac{\rho h}{D}s^2 N_{pq}(s) - \frac{a_{pq}}{D}W(s) - \frac{b_{pq}}{D}R(s)\right]\varphi_{ij}(x,y) = 0, \quad p,q = 1,2,3\ldots \tag{5.9}$$

where $N_{pq}(s)$, $R(s)$ and $W(s)$ are the Laplace transform of $n_{pq}(t)$, $r(t)$, and $w(t)$ respectively.

Since the eigenvectors $\varphi_{pq}(x)$ are linearly independent, we may write

$$\frac{\rho h}{D}\omega_{pq}^2 N_{pq}(s) + \frac{\rho h}{D}s^2 N_{pq}(s) - \frac{a_{pq}}{D}W(s) - \frac{b_{pq}}{D}R(s) \;= 0, \quad p,q = 1,\, 2,\, 3\ldots \tag{5.10}$$

From equation (10) above, the generalized co-ordinates $N_{pq}(s)$ may be written as

$$N_{pq}(s) = \left(\frac{1}{\rho h}\right)\left[\frac{a_{pq}W(s) + b_{pq}R(s)}{\omega_{pq}^2 + s^2}\right]. \tag{5.11}$$

Performing Laplace transformation on equation 5.4 and eliminating $N_{pq}(s)$ in the resulting equation with equation 5.11, the s-domain motion of any point on the plate could be written as

$$U(x,y,s)=\frac{1}{\rho h}\sum_{p=1,q=1}^{\infty}\left[\frac{a_{pq}W(s)+b_{pq}R(s)}{\omega_{pq}^{2}+s^{2}}\right]\varphi_{pq}(x,y) \tag{5.12}$$

where $U(x,y,s)$ is the Laplace transform of $u(x,y,t)$ with respect to time.

The force transmitted to the beam at the point of attachment may be written as

$$R(s)=-U(x_o,y_o,s)\left[\frac{ms^{2}(cs+k)}{ms^{2}+cs+k}\right]. \tag{5.13}$$

The functions $R(s)$ in equation 5.12 can be eliminated using equation 5.13 to give

$$U(x,y,s)=\frac{1}{\rho h}\sum_{p=1,q=1}^{\infty}\left[\frac{a_{pq}W(s)-b_{pq}\dfrac{ms^{2}(cs+k)}{ms^{2}+cs+k}U(x_o,y_o,s)}{\omega_{pq}^{2}+s^{2}}\right]\varphi_{pq}(x,y). \tag{5.14}$$

This expresses the motion of an arbitrary point (x,y) on the vibrating plate in terms of the forcing function $W(s)$ and the motion at the point of attachment (x_o,y_o). This relation would definitely be valid at the attachment point (x_o,y_o) leading to

$$U(x_o,y_o,s)=\frac{1}{\rho h}\sum_{p=1,q=1}^{\infty}\left[\frac{a_{pq}W(s)-b_{pq}\dfrac{ms^{2}(cs+k)}{ms^{2}+cs+k}U(x_o,y_o,s)}{\omega_{pq}^{2}+s^{2}}\right]\varphi_{pq}(x,y). \tag{5.15}$$

This can be rearranged to arrive at a transfer function between $W(s)$ and $U(x_o,y_o,s)$ written as

$$\frac{U(x_o,y_o,s)}{W(s)}=\frac{\dfrac{1}{\rho h}\displaystyle\sum_{p=1,q=1}^{\infty}\frac{a_{pq}\varphi_{pq}(x_o,y_o)}{\omega_{pq}^{2}+s^{2}}}{1+\dfrac{1}{\rho h}\left[\dfrac{ms^{2}(cs+k)}{ms^{2}+cs+k}\right]\displaystyle\sum_{p=1,q=1}^{\infty}\frac{b_{ij}\varphi_{ij}(x_o,y_o)}{\omega_{pq}^{2}+s^{2}}}. \tag{5.16}$$

The following non–dimensional parameters are used in the following derivations.

$\mu = \dfrac{m}{\rho h L_x L_y}$ is the mass ratio between the masses of the absorber and the plate;

$\zeta_a = \dfrac{c}{2\sqrt{mk}}$ is the damping ratio of the absorber;

$\omega_a = \sqrt{\dfrac{k}{m}}$ is the undamped natural frequency of the absorber;

$T = \dfrac{\omega_a}{\omega_{\alpha\beta}}$ is the ratio between the absorber frequency and a reference natural frequency $\omega_{\alpha\beta}$ of the plate;

$\gamma_{pq} = \dfrac{\omega_{pq}}{\omega_{\alpha\beta}}$ is the non-dimensional natural frequency of the plate referred to $\omega_{\alpha\beta}$;

$\lambda = \omega / \omega_{\alpha\beta}$ is the normalized frequency.

The frequency response function of the plate can be obtained by substituting equation 5.16 into equation 5.14 and replacing s by $j\omega$ in the resulting equation written in non-dimensional form as

$$\frac{U(x,y,\lambda)}{W(\lambda)} = \frac{1}{\rho h \omega_{\alpha\beta}^2} \sum_{p=1,q=1}^{\infty} \left\{ \frac{a_{pq} - b_{pq} \left[\dfrac{\mu ab \displaystyle\sum_{p=1,q=1}^{\infty} \frac{a_{pq}\varphi_{pq}(x_o, y_o)}{\gamma_{pq}^2 - \lambda^2}}{\dfrac{-\lambda^2 + 2\zeta_a T\lambda + T^2}{-\lambda^2(2\zeta_a T\lambda + T^2)} + \mu ab \displaystyle\sum_{i=1,j=1}^{\infty} \frac{b_{pq}\varphi_{pq}(x_o, y_o)}{\gamma_{pq}^2 - \lambda^2}} \right]}{\gamma_{pq}^2 - \lambda^2} \right\} \varphi_{pq}(x,y) \tag{5.17}$$

where $j = \sqrt{-1}$. The transfer functions of the velocity and the acceleration responses at point (x,y) on the plate surface may be written respectively as

$$\frac{\dot{U}(x,y,\lambda)}{\omega_{\alpha\beta}\,W(\lambda)} = jf\,\frac{U(x,y,\lambda)}{W(\lambda)}. \tag{5.18}$$

and $$\frac{\ddot{U}(x,y,\lambda)}{\omega_{\alpha\beta}^2\,W(\lambda)} = -f^2\,\frac{U(x,y,\lambda)}{W(\lambda)} \tag{5.19}$$

5.2 Optimization for minimizing the vibration at a point (x,y) on the plate

For a structure with well separated natural frequencies, the modal displacement response in the vicinity of the uv^{th} natural frequency may be approximated by considering $p = \alpha$ and $q = \beta$ and ignoring other modes in equation 5.17. Equation 5.17 may then be written as

$$\frac{U(x,y,\lambda)}{W(\lambda)} = \left(\frac{a_{\alpha\beta}\varphi_{\alpha\beta}(x,y)}{\rho h\,\omega_{\alpha\beta}^2}\right)\left[\frac{1 - b_{\alpha\beta}\dfrac{\mu L_x L_y \dfrac{\varphi_{\alpha\beta}(x_o,y_o)}{1-\lambda^2}}{\left(-\dfrac{-\lambda^2 + 2j\zeta_a T\lambda + T^2}{\lambda^2(2j\zeta_a T\lambda + T^2)} + \mu L_x L_y \dfrac{b_{\alpha\beta}\varphi_{\alpha\beta}(x_o,y_o)}{1-\lambda^2}\right)}}{1-\lambda^2}\right]$$

$$= \left(\frac{a_{\alpha\beta}\varphi_{\alpha\beta}(x,y)}{\rho h\,\omega_{\alpha\beta}^2}\right)\left[\frac{T^2 - \lambda^2 + 2j\zeta_a T\lambda}{(2j\zeta_a T\lambda + T^2 - \lambda^2)(1-\lambda^2) - \varepsilon\lambda^2(2j\zeta_a T\lambda + T^2)}\right] = \left(\frac{a_{\alpha\beta}\varphi_{\alpha\beta}(x,y)}{\rho h\,\omega_{\alpha\beta}^2}\right) Z(\lambda) \tag{5.20}$$

where $$Z(\lambda) = \frac{T^2 - \lambda^2 + 2j\zeta_a T\lambda}{(2j\zeta_a T\lambda + T^2 - \lambda^2)(1-\lambda^2) - \varepsilon\lambda^2(2j\zeta_a T\lambda + T^2)} = \left[\frac{\rho h\,\omega_{\alpha\beta}^2\, U(x,y,\lambda)}{a_{\alpha\beta}\varphi_{\alpha\beta}(x,y)\, W(\lambda)}\right] \tag{5.21a}$$

is the non-dimensional frequency response of the plate, and

$$\varepsilon = \mu\varphi_{\alpha\beta}^2(x_o,y_o)\,. \tag{5.21b}$$

The error of ignoring other modes is discussed in section 5.4. The objective of H_∞ optimization is to minimize the maximum vibration amplitude response at the point (x,y) and the performance index may be defined as

$$H_{\infty\_pt\_disp} = \inf_{\lambda}\left(\sup_{\zeta,T}\left(\left|\frac{U(x,y,\lambda)}{W(\lambda)}\right|\right)\right). \tag{5.22a}$$

Similarly, the performance indices of H_∞ optimization for minimizing the maximum velocity and acceleration amplitude responses at the point (x,y) may be defined respectively as

$$H_{\infty\_pt\_vel} = \inf_{\lambda}\left(\sup_{\zeta,T}\left(\left|\frac{\dot{U}(x,y,\lambda)}{\omega_{\alpha\beta}W(\lambda)}\right|\right)\right). \tag{5.22b}$$

and $$H_{\infty\_pt\_acc} = \inf_{\lambda}\left(\sup_{\zeta,T}\left(\left|\frac{\ddot{U}(x,y,\lambda)}{\omega_{\alpha\beta}^2 W(\lambda)}\right|\right)\right) \tag{5.22c}$$

The expression $Z(\lambda)$ in equation 5.20 is equivalent to the amplitude ratio as derived by Den Hartog (1985) in the SDOF system attached with a dynamic vibration absorber if the term ε in Equation 5.21a is replaced by the mass ratio μ. ε may therefore be considered as the equivalent mass ratio for applying vibration absorber to control vibrations in plate structures. The optimum frequency and damping ratios, and the height of the fixed points in the frequency spectrum of the primary system in equations (5.22a), (5.22b) and (5.22c) for H_∞ optimization can be derived based on the fixed-points theory in the same way as in the case of SDOF system (Asami et al. 2002; Asami and Nishihara 2003) and the results are listed in Table 5.1.

The objective of H_2 optimization of the absorber is to minimize the total vibration energy of the mass at point (x,y) of the plate of all frequencies in the system and the performance index may be defined as (Crandall and Mark 1963)

$$H_{2\_pt} = \inf_{\zeta,T}\left(\frac{E\left[U^2\right]}{2\pi S_F \omega_{\alpha\beta}/D^2}\right). \tag{5.23}$$

Table 5.1 The approximated H_∞ tuning of the plate for control of vibration at point (x,y) and the height of the fixed points in the response spectrum.

Transfer function	Tuning ratio	Damping ratio	Height of the fixed points in the response spectrum
$\dfrac{U(x,y,\lambda)}{W(\lambda)}$	$\dfrac{1}{1+\varepsilon}$	$\sqrt{\dfrac{3\varepsilon}{8(1+\varepsilon)}}$	$\dfrac{a_{\alpha\beta}\varphi_{\alpha\beta}(x,y)}{\rho h\omega_{\alpha\beta}^2}\sqrt{\dfrac{2}{\varepsilon}+1}$
$\dfrac{\dot{U}(x,y,\lambda)}{\omega_{\alpha\beta}W(\lambda)}$	$\dfrac{1}{1+\varepsilon}\sqrt{\dfrac{2+\varepsilon}{2}}$	$\dfrac{1}{4(2+\varepsilon)}\sqrt{\dfrac{\varepsilon(24+24\varepsilon+5\varepsilon^2)}{1+\varepsilon}}$	$\dfrac{a_{\alpha\beta}\varphi_{\alpha\beta}(x,y)}{\rho h\omega_{\alpha\beta}^2}\sqrt{\dfrac{2}{\varepsilon}-\dfrac{1}{1+\varepsilon}}$
$\dfrac{\ddot{U}(x,y,\lambda)}{\omega_{\alpha\beta}^2 W(\lambda)}$	$\sqrt{\dfrac{1}{1+\varepsilon}}$	$\dfrac{1}{2}\sqrt{\dfrac{3\varepsilon}{2+\varepsilon}}$	$\dfrac{a_{\alpha\beta}\varphi_{\alpha\beta}(x,y)}{\rho h\omega_{\alpha\beta}^2}\sqrt{\dfrac{2}{\varepsilon}-\dfrac{2}{1+\varepsilon}}$

where $E[U^2]$ is the ensemble mean of U^2 and S_F is the spectral density of the excitation. The optimum tuning frequency ratio and damping ratio for H_2 optimization of the system can be derived based on the fixed-points theory as (Asami et al. 2002)

$$T_d = \frac{1}{(1+\varepsilon)}\sqrt{1+\frac{2}{\varepsilon}}\,8 \tag{5.24a}$$

$$\text{and} \quad \zeta_{H_2} = \frac{1}{2}\sqrt{\frac{\varepsilon(4+3\varepsilon)}{2(1+\varepsilon)(2+\varepsilon)}} \tag{5.24b}$$

If the forcing function $w(t)$ has power spectral density $S_F(f)$, the spectral density of the vibration response of the point (x,y) on the plate may be written as

$$S_u(x,y,f) = \left|\frac{U(x,y,\lambda)}{W(\lambda)}\right|^2 S_F(f) \tag{5.25}$$

With optimum frequency and damping ratios as expressed in equations 5.24a and 5.24b, the mean square motion at point (x,y) can be derived as

$$\sigma_u^2(x,y) = \frac{\omega_{\alpha\beta}}{2\pi}\int_{-\infty}^{\infty}\left|\frac{U(x,y,\lambda)}{W(\lambda)}\right|^2 S_F(f)df = \frac{\omega_{\alpha\beta}}{2}\left(\frac{a_{\alpha\beta}\varphi_{\alpha\beta}(x,y)}{\rho h\omega_{\alpha\beta}^2}\right)^2\sqrt{\frac{3\varepsilon+4}{\varepsilon(\varepsilon+1)}} \tag{5.26}$$

5.3 Optimization for minimizing the root mean square motion over the whole domain of the plate

Using equation 5.17 and integrating the square of amplitude response over the whole domain of the plate, we may write

$$\int_0^a \int_0^b \left(\frac{U(x,y,\lambda)}{W(\lambda)} \right)^2 dydx$$

$$= \int_0^{L_x} \int_0^{L_y} \left(\frac{1}{\rho h \omega_{\alpha\beta}^2} \right)^2 \left(\sum_{p=1,q=1}^{\infty} \frac{ \left(a_{pq} - \frac{ b_{pq} \mu L_x L_y \sum_{p=1,q=1}^{\infty} \frac{a_{pq} \varphi_{pq}(x_o, y_o)}{\gamma_{pq}^2 - \lambda^2} }{ \left(\frac{\lambda^2 - 2j\zeta_a T\lambda - T}{\lambda^2 (2j\zeta_a T\lambda + T^2)} + \mu L_x L_y \sum_{p=1,q=1}^{\infty} \frac{b_{pq} \varphi_{pq}(x_o, y_o)}{\gamma_{pq}^2 - \lambda^2} \right) } \right) }{\gamma_{pq}^2 - \lambda^2} \varphi_{pq}(x,y) \right)^2 dydx$$

(5.27)

Consider the orthogonality relations of the eigenfunctions, we may write

$$\begin{aligned} &\int_0^{L_x} \int_0^{L_y} \varphi_{pq}(x,y) \varphi_{\alpha\beta}(x,y) dydx = 0, \quad \text{if } p \neq \alpha \text{ or } q \neq \beta, \text{and} \\ &\int_0^{L_x} \int_0^{L_y} \varphi_{pq}(x,y) \varphi_{\alpha\beta}(x,y) dydx = L_x L_y, \quad \text{if } p = \alpha \text{ and } q = \beta. \end{aligned} \tag{5.28}$$

Equation 5.28 can be simplified with the above orthogonality relations of the eigenfunctions as

$$\int_0^{L_x}\int_0^{L_y}\left(\frac{U(x,y,\lambda)}{W(\lambda)}\right)^2 dydx \tag{5.29}$$
$$=\sum_{p=1,q=1}^{\infty}\left(\frac{L_xL_y}{\rho h\,\omega_{\alpha\beta}^2}\frac{\left(a_{pq}-\dfrac{b_{pq}\mu L_xL_y\displaystyle\sum_{p=1,q=1}^{\infty}\frac{a_{pq}\varphi_{pq}(x_o,y_o)}{\gamma_{pq}^2-\lambda^2}}{\left(\dfrac{\lambda^2-2j\zeta_aT\lambda-T}{\lambda^2(2j\zeta_aT\lambda+T^2)}+\mu L_xL_y\displaystyle\sum_{p=1,q=1}^{\infty}\frac{b_{pq}\varphi_{pq}(x_o,y_o)}{\gamma_{pq}^2-\lambda^2}\right)}\right)}{\gamma_{pq}^2-\lambda^2}\right)^2$$

For a structure with well separated natural frequencies, the mean square modal displacement response in the vicinity of the $\alpha\beta^{\text{th}}$ natural frequency may be approximated by considering $p = \alpha$ and $q = \beta$ and ignoring other modes. Equation 5.29 may be written as

$$\int_0^{L_x}\int_0^{L_y}\left(\frac{U(x,y,\lambda)}{W(\lambda)}\right)^2 dydx \tag{5.30}$$
$$=\left(\frac{L_xL_y}{\rho h\,\omega_{\alpha\beta}^2}\frac{\left(a_{\alpha\beta}-b_{\alpha\beta}\dfrac{\mu L_xL_y\dfrac{a_{\alpha\beta}\varphi_{\alpha\beta}(x_o,y_o)}{1-\lambda^2}}{\left(\dfrac{\lambda^2-2j\zeta_aT\lambda-T}{\lambda^2(2j\zeta_aT\lambda+T^2)}+\mu L_xL_y\dfrac{b_{\alpha\beta}\varphi_{\alpha\beta}(x_o,y_o)}{1-\lambda^2}\right)}\right)}{1-\lambda^2}\right)^2$$

The root mean square amplitude response of the vibrating plate with an absorber may be written as

$$\sqrt{\int_0^{L_x}\int_0^{L_y}\left(\frac{U(x,y,\lambda)}{W(\lambda}\right)^2 dydx}$$

$$=\left(\frac{a_{\alpha\beta}L_xL_y}{\rho h\ \omega_{\alpha\beta}^2}\right)\left(\frac{1-b_{\alpha\beta}\dfrac{\mu L_xL_y\dfrac{\varphi_{\alpha\beta}(x_o,y_o)}{1-\lambda^2}}{\left(\dfrac{\lambda^2-2j\zeta_a T\lambda-T}{\lambda^2(2j\zeta_a T\lambda+T^2)}+\mu L_xL_y\dfrac{b_{\alpha\beta}\varphi_{\alpha\beta}(x_o,y_o)}{1-\lambda^2}\right)}}{1-\lambda^2}\right)$$

$$=\left(\frac{a_{\alpha\beta}L_xL_y}{\rho h\omega_{\alpha\beta}^2}\right)\left[\frac{T^2-\lambda^2+2j\zeta_a T\lambda}{(2j\zeta_a T\lambda+T^2-\lambda^2)(1-\lambda^2)-\varepsilon f^2(2j\zeta_a T\lambda+T^2)}\right]$$

$$=\left(\frac{a_{\alpha\beta}L_xL_y}{\rho h\omega_{\alpha\beta}^2}\right)Z(\lambda) \tag{5.31}$$

The magnitude of the root-mean-square amplitude response of the plate may be written as

$$\left|\sqrt{\int_0^{L_x}\int_0^{L_y}\left(\frac{U(x,y,\lambda)}{W(\lambda)}\right)^2 dydx}\right|=\left|\frac{a_{\alpha\beta}}{\rho h\omega_{\alpha\beta}^2}\right|\left|Z(\lambda)\right|. \tag{5.32}$$

The H_∞ optimizations for minimizing root mean square motion, velocity and acceleration responses of the whole plate are written respectively as

$$H_{\infty\_plate\_disp}=\inf_{\lambda}\left(\sup_{\zeta,T}\left(\left|\sqrt{\int_0^{L_x}\int_0^{L_y}\left(\frac{U(x,y,\lambda)}{W(\lambda)}\right)^2 dydx}\right|\right)\right) \tag{5.33a}$$

$$=\inf_{\lambda}\left(\sup_{\zeta,T}\left(\left|\sqrt{\int_0^{L_x}\int_0^{L_y}\left(\frac{j\lambda U(x,y,\lambda)}{W(\lambda)}\right)^2 dydx}\right|\right)\right)$$

$$H_{\infty\_plate\_acc} = \inf_{\lambda}\left(\sup_{\zeta,T}\left(\left|\sqrt{\int_0^{L_x}\int_0^{L_y}\left(\frac{\ddot{U}(x,y,\lambda)}{W(\lambda)}\right)^2 dydx}\right|\right)\right) \tag{5.33b}$$

$$= \inf_{\lambda}\left(\sup_{\zeta,T}\left(\left|\sqrt{\int_0^{L_x}\int_0^{L_y}\left(\frac{-f^2U(x,y,\lambda)}{W(\lambda)}\right)^2 dydx}\right|\right)\right)$$

The optimum tuning frequency and damping ratios and the height of the fixed points in the frequency spectrum of the primary system in equations (33a), (33b) and (33c) for H_∞ optimization can be derived based on the fixed-points theory as in the case of SDOF system (Asami et al. 2002; Asami and Nishihara 2003) and the results are listed in Table 5.2.

Table 5.2 The approximated H_∞ tuning of the plate for control of vibration of the whole plate and the height of the fixed points in the response spectrum.

Transfer function	Tuning ratio	Damping ratio	Height of the fixed points in the response spectrum
$\left\vert\sqrt{\int_0^a\int_0^b\left(\frac{U(x,y,\lambda)}{W(\lambda)}\right)^2 dydx}\right\vert$	$\frac{1}{1+\varepsilon}$	$\sqrt{\frac{3\varepsilon}{8(1+\varepsilon)}}$	$\left(\frac{a_{\alpha\beta}}{\rho h\omega_{\alpha\beta}^2}\right)^2\left(\frac{2}{\varepsilon}+1\right)$
$\left\vert\sqrt{\int_0^a\int_0^b\left(\frac{\dot{U}(x,y,\lambda)}{\omega_{\alpha\beta}W(\lambda)}\right)^2 dydx}\right\vert$	$\frac{1}{1+\varepsilon}\sqrt{\frac{2+\varepsilon}{2}}$	$\frac{1}{4(2+\varepsilon)}\sqrt{\frac{\varepsilon(24+24\varepsilon+5\varepsilon^2)}{1+\varepsilon}}$	$\left(\frac{a_{\alpha\beta}}{\rho h\omega_{\alpha\beta}^2}\right)^2\left(\frac{2}{\varepsilon}-\frac{1}{1+\varepsilon}\right)$
$\left\vert\sqrt{\int_0^a\int_0^b\left(\frac{\ddot{U}(x,y,\lambda)}{\omega_{\alpha\beta}^2W(\lambda)}\right)^2 dydx}\right\vert$	$\sqrt{\frac{1}{1+\varepsilon}}$	$\frac{1}{2}\sqrt{\frac{3\varepsilon}{2+\varepsilon}}$	$\left(\frac{a_{\alpha\beta}}{\rho h\omega_{\alpha\beta}^2}\right)^2\left(\frac{2}{\varepsilon}-\frac{2}{1+\varepsilon}\right)$

The objective of H_2 optimization in this case is to minimize the vibration energy of the whole plate of all frequencies of the system. The performance index in this case may be defined as

$$H_{2\_plate} = \inf_{\zeta,T}\left(\frac{E\left[\int_0^{L_x}\int_0^{L_y}\left(\frac{U(x,y,\lambda)}{W(\lambda)}\right)^2 dydx\right]}{2\pi S_f \omega_{\alpha\beta}/D^2}\right). \tag{5.34}$$

Warburton (1980) had derived the frequency and damping rations for H_2 optimization of SDOF system as $T_{H2\_SDOF} = \sqrt{\frac{\mu+2}{\mu(1+\mu)^2}}$ and $\zeta_{H2\_SDOF} = \frac{1}{2}\sqrt{\frac{\mu(4+3\mu)}{2(1+\mu)(2+\mu)}}$. Using a similar approach, the frequency and damping ratios for H_2 optimization of the plate structure can be derived as

$$T_{H_2} = \sqrt{\frac{\varepsilon+2}{\varepsilon(1+\varepsilon)^2}} \quad . \tag{5.35a}$$

$$\text{and} \quad \zeta_{H_2} = \frac{1}{2}\sqrt{\frac{\varepsilon(4+3\varepsilon)}{2(1+\varepsilon)(2+\varepsilon)}} \tag{5.35b}$$

The H_2 optimization is the minimization of the root mean square motion response over the whole domain of the plate under wide-band random excitation. Warburton (1980) had derived the mean square motion of a H_2 optimized SDOF system to be $\frac{X}{Y} = \sqrt{\frac{3\mu+4}{\mu(\mu+1)}}$. Using a similar approach, the optimum frequency and damping ratios as expressed in equations 5.35a and 5.35b, the total mean square motion of the whole plate can be derived as

$$\frac{\omega_{\alpha\beta}}{2\pi}\int_{-\infty}^{\infty}\left|\sqrt{\int_0^{L_x}\int_0^{L_y}\left(\frac{U(x,y,\lambda)}{W(\lambda)}\right)^2 dydx}\right|^2 S_w(\lambda)d\lambda = \frac{\omega_{\alpha\beta}}{2}\left(\frac{a_{\alpha\beta}L_xL_y}{\rho h\,\omega_{\alpha\beta}^2}\right)^2\sqrt{\frac{3\varepsilon+4}{\varepsilon(\varepsilon+1)}}\,. \tag{5.36}$$

5.4 Simulation results and discussion

To test the usefulness of the derived H_2 optimization solution for suppressing vibrations in plates, the numerical case studied by Jacquot (2001) was analysed with the optimum tuning derived in the previous section and the results were compared to those obtained by Jacquot. The vibration of a square plate with four sides simply supported was considered. The eigenfunctions may be written as

$$\varphi_{pq} = 2\sin(p\pi x)\sin(q\pi y) \tag{5.37}$$

The excitation was stationary and random in time, i.e. $g(x,y)=1$, and it was uniformly applied on the plate. In this case,

$$a_{pq} = \frac{8}{pq\pi^2}, \quad p,q = 2n-1 \quad n \in N \\ \text{else} \quad a_{pq} = 0. \tag{5.38}$$

$$b_{pq} = \varphi_{pq}(x_o, y_o) = 2\sin p\pi x_o \sin q\pi y_o, \quad p,q \in N \tag{5.39}$$

The dimensions of the plate were a = 1m, b = 1m and h = 0.01m. The material of the plate was aluminum of $\rho = 2.71\times10^3 kgm^{-3}$, $E = 6.9\times10^9 Pa$ and $\nu = 0.33$. In the analysis made by Jacquot (2001), the frequency ratio was chosen as 1. The vibration mode required to be suppressed was $\alpha = \beta = 1$. The attachment position of the absorber on the plate was $x_o = y_o = 0.5$. The mass ratio and damping ratio for minimum mean square motion at the attachment point were found to be 0.275 and 0.45 respectively by Jacquot. In the current analysis, the same mass ratio was used so that the result of vibration suppression could be compared to that of Jacquot. The modal response amplitude at the point of attachment $\varphi_{11}(x_o, y_o)$ was 2 and therefore ε was 1.1 according to equation 5.21b. The optimum frequency and damping ratios in this case were calculated to be 0.5929 and 0.3927 respectively in applying equations 5.24a and 5.24b. The vibration amplitude response at point (x_o,y_o) of the plate was calculated according to equation 5.17. The spectral density of the vibration amplitude response at point (x_o,y_o) was calculated according to equation 5.25

and it was plotted in Figure 5.2 and compared to the corresponding curve by Jacquot (equations 25 and 28 of Jacquot 2001).

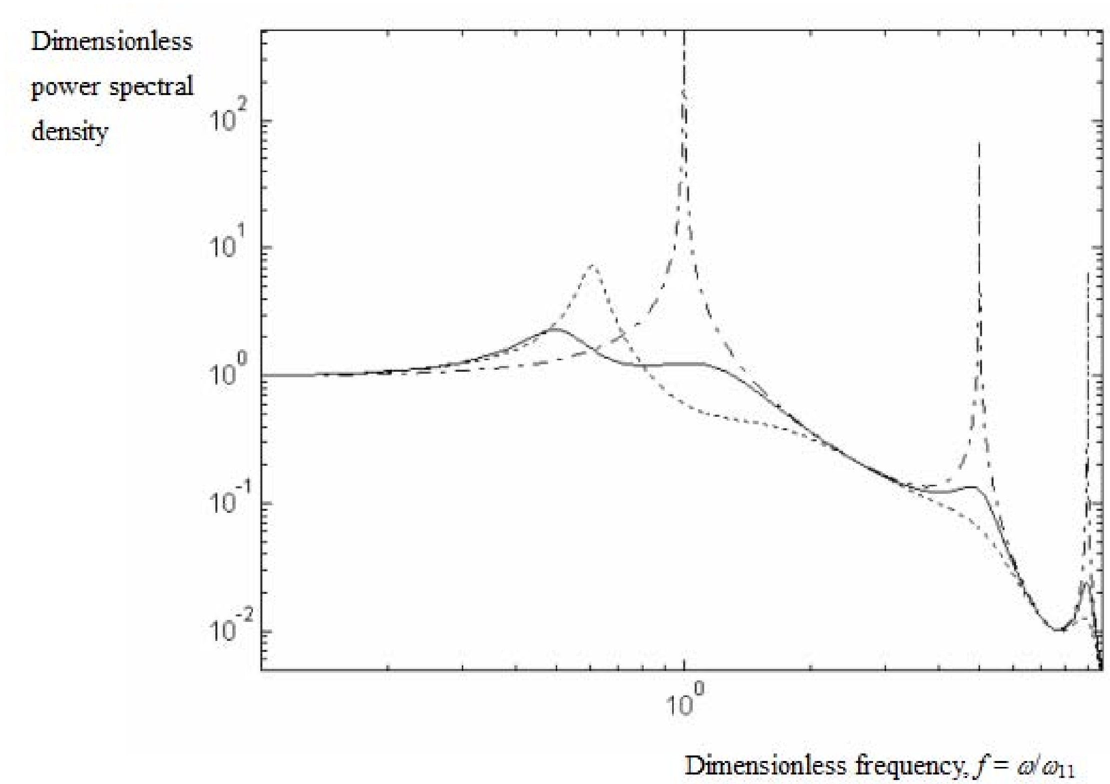


Figure 5.2 Dimensionless motion power spectral density of a square plate with $g(x,y) = 1$, $\mu = 0.275$, $x_o = y_o = a/2$. ------- Jacquot's result (2001); ——— Present theory, Equation (26); –·–·–·– No absorber added.

The spectral density of the vibration amplitude response at point (x_o,y_o) for the case of no absorber added was also plotted for comparison. It could be observed in Figure 5.2 that both Jacquot's result and the present result provided vibration control at point (x_o,y_o) of the plate. However, the mean square motion at point (x_o,y_o) of the plate with the proposed frequency and damping ratios was found to be 55.8% smaller than that obtained by Jacquot. Jacquot also reported that there was an optimum mass ratio leading to minimum mean square motion of the plate but no particular optimum mass ratio could be found in applying the present theory. Based on equations 5.21b and 5.26, it was observed that mass ratio should be as high as possible in order to reduce the mean square motion of the plate.

Mean square motions at the attachment point of the plate based on the response of 20 vibration modes were calculated numerically using equation (5.17) with different values of tuning frequency and damping ratios and the result is plotted in Figure 5.3 below. The values of tuning frequency and damping ratios for minimum mean square motions are found to be T = 0.595 and ζ_a = 0.459. The difference of mean square motion of the plate at point (x_o,y_o) using the exact and the proposed sets of T and ζ was found to be about 1%. This shows that the proposed optimum tuning frequency and damping ratios are quite accurate even though they are determined based on the vibration response of only one mode of the plate.

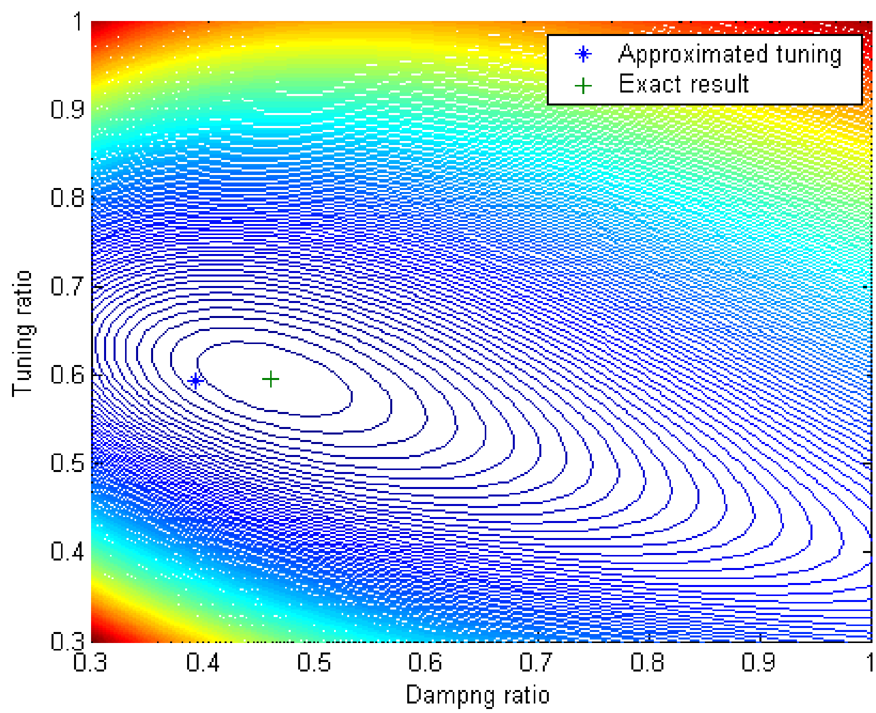


Figure 5.3 Contour plot of the mean square motion of the plate at the attachment point of DVA at different tuning and damping ratios

H_∞ optimization of motion control is also checked in this case. The exact tuning ratio and damping ratio are calculated numerically using equation 5.17 as T = 0.468 and ζ_a = 0.494

respectively and the approximated tuning ratio and damping ratio are calculated according to the formulae in Table 5.2 as T = 0.4762 and ζ_a = 0.4432 respectively. The global maximum response of the exact tuning and the approximated tuning are calculated to be 2.7322 and 2.8322, respectively. In this case, the global maximum using the approximated tuning is about 3% higher than the exact value.

5.5 Summary

In this chapter, analytical solutions to the H_∞ and H_2 optimization problems of dynamic vibration absorber attached to a vibrating plate under random excitation have been derived. Expressions of the optimum tuning frequency and damping ratios are derived for the absorber assuming single mode vibration of the plate. The error due to this assumption is discussed in section 5.4. The effects of the DVA on other higher order modes are discussed in section 4.5.

The optimum tuning frequency and damping ratios of the absorber derived in the present theory for solving the H_∞ and H_2 optimization problems applied to vibrating plate structures have similar forms to those of the SDOF system. However, the tuning equations are based on the equivalent mass ratio ε which is a function of both the mass ratio and the position of the absorber on the plate structure. Moreover, it is derived that both the optimum tuning frequency and the damping ratios for minimum vibration at a certain point are the same as those in the case of the minimum mean square motion for the whole plate. That means the mean square motion would be minimum when the vibration at a single point of the surface is minimum.

Secondly, the vibration response in H_∞ optimization and the mean square motion in H_2 optimization would be reduced when the equivalent mass ratio ε is increased under the optimum tuning condition. That means a higher mass ratio and an attachment point of the absorber having higher modal response should be chosen for the suppression of vibration for the whole plate or at one point of the plate. This finding is different from that of Jacquot (2001) who showed that there would be an optimum mass ratio for minimum mean square motion of a vibrating plate under random excitation. There is no optimum mass ratio found in the present analysis. Jacquot found an optimum mass ratio because he fixed the tuning ratio, $T = 1$. It is found in the present analysis that the mass ratio should be as high as possible.

Based on the example in section 5.4, the approximated optimum tunings are better than the Jacquot derived one.

Thirdly, based on the expressions as shown in tables 5.1 and 5.2 for the heights of the fixed points in the response spectrum for H_∞ optimization, it is found that the heights of the fixed points in the (dimensionless) displacement response spectrum are higher those of the (dimensionless) velocity response spectrum and in turn higher than those of the (dimensionless) acceleration response spectrum.

6. OPTIMIZATION OF A NON-TRADITIONAL DYNAMIC VIBRATION ABSORBER

A non-traditional dynamic vibration absorber is proposed for the minimization of maximum vibration response of a vibrating structure. Unlike the traditional damped absorber configuration, the proposed absorber has a linear viscous damper connecting the absorber mass directly to the ground instead of the main mass. Optimum parameters (for H_∞ optimization) of the proposed absorber are derived based on the fixed-points theory for minimizing the maximum vibration response of a single-degree-of-freedom system under harmonic excitation. The extent of reduction of maximum vibration response of the primary system when using the traditional dynamic absorber is compared with that using the proposed one. Under the optimum tuning condition of the absorbers, it is proved analytically that the proposed absorber provides both a greater reduction of maximum vibration and velocity responses of the primary system than the traditional absorber.

6.1 H_∞ optimization of the non-traditional DVA

The design of the non-traditional DVA is shown as Figure 6.1. The elements of the non-traditional DVA are totally the same as the traditional DVA. It also has a mass, a damper and a spring. The difference between the new one and the traditional one is that the damper is connecting to the ground instead of the primary system.

The motion of the primary system and the DVA are governed by the following matrix equation.

$$\begin{bmatrix} M & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{x}_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{x}_1 \end{bmatrix} + \begin{bmatrix} K+k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} x \\ x_1 \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} \tag{6.1}$$

Taking Laplace transformation and replacing s by $j\omega$, the frequency response function may be written as

$$H(\omega) = \frac{X}{F} = \frac{k - m\omega^2 + jc\omega}{\left[\left(K + k - M\omega^2\right)\left(k - m\omega^2\right) - k^2\right] + jc\omega\left(K + k - M\omega^2\right)} \tag{6.2}$$

Rewriting equation 6.2 in dimensionless form, we have

$$H(\lambda) = \frac{X}{F/K} = \frac{\gamma^2 - \lambda^2 + 2j\zeta_a\gamma\lambda}{\left[\left(1-\lambda^2\right)\left(\gamma^2 - \lambda^2\right) - \mu\lambda^2\gamma^2\right] + 2j\zeta_a\gamma\lambda\left(1 - \lambda^2 + \mu\gamma^2\right)} \tag{6.3}$$

In the H_∞ optimization, the objective function is to minimize the maximum amplitude ratio of the response of the primary system to the excitation force or motion, i.e.

$$\max\left(\left|H\left(\lambda, \gamma_{H\infty}, \zeta_{H\infty}\right)\right|\right) = \min\left(\max_{\gamma,\zeta_a}\left|H(\lambda)\right|\right) \tag{6.4}$$

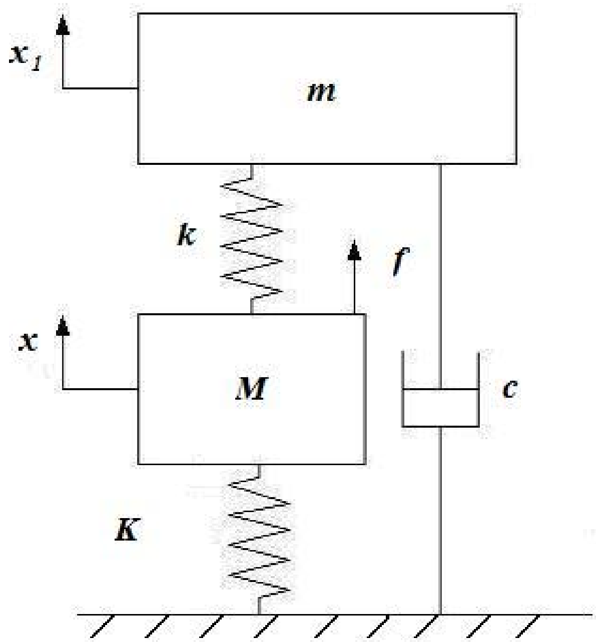


Figure 6.1 A SDOF system (*M*-*K*) mounted with the new DVA (*m*-*k*-*c*) excited by an external force

Using equation (6.3), the amplitude of the frequency response function is

$$|H(\lambda)| = \left|\frac{X}{F/K}\right| = \sqrt{\frac{\left(\gamma^2 - \lambda^2\right)^2 + \left(2\zeta_a \gamma\lambda\right)^2}{\left[\left(1-\lambda^2\right)\left(\gamma^2 - \lambda^2\right) - \mu\lambda^2\gamma^2\right]^2 + \left(2\zeta_a\gamma\lambda\right)^2\left(1-\lambda^2+\mu\gamma^2\right)^2}} \tag{6.5}$$

Equation 6.5 may be rewritten into the form as

$$|H(\lambda)| = \left|\frac{X}{F/K}\right| = \sqrt{\frac{A + B\zeta_a^2}{C + D\zeta_a^2}} \tag{6.6}$$

where $A = \left(\gamma^2 - \lambda^2\right)^2$, $B = (2\gamma\lambda)^2$, $C = \left[\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right) - \mu\lambda^2\gamma^2\right]^2$,

and $D = \left(2\gamma\lambda\left(1-\lambda^2+\mu\gamma^2\right)\right)^2$

Frequency responses of the primary mass *M* are calculated according to equation 6.6

with three damping ratios and the results are shown as figure 6.2. It can be observed that there are intersecting points *O*, *P* and *Q* which are independent of damping of the absorber. At those fixed points, we may write

$$\frac{A}{C} = \frac{B}{D} \tag{6.7a}$$

or $$\left(\frac{\gamma^2 - \lambda^2}{(1-\lambda^2)(\gamma^2 - \lambda^2) - \mu\lambda^2\gamma^2}\right)^2 = \left(\frac{1}{1-\lambda^2 + \mu\gamma^2}\right)^2 \tag{6.7b}$$

Taking square root of equation 6.7b, we have

$$\frac{\gamma^2 - \lambda^2}{(1-\lambda^2)(\gamma^2 - \lambda^2) - \mu\lambda^2\gamma^2} = \pm\frac{1}{1-\lambda^2 + \mu\gamma^2} \tag{6.8}$$

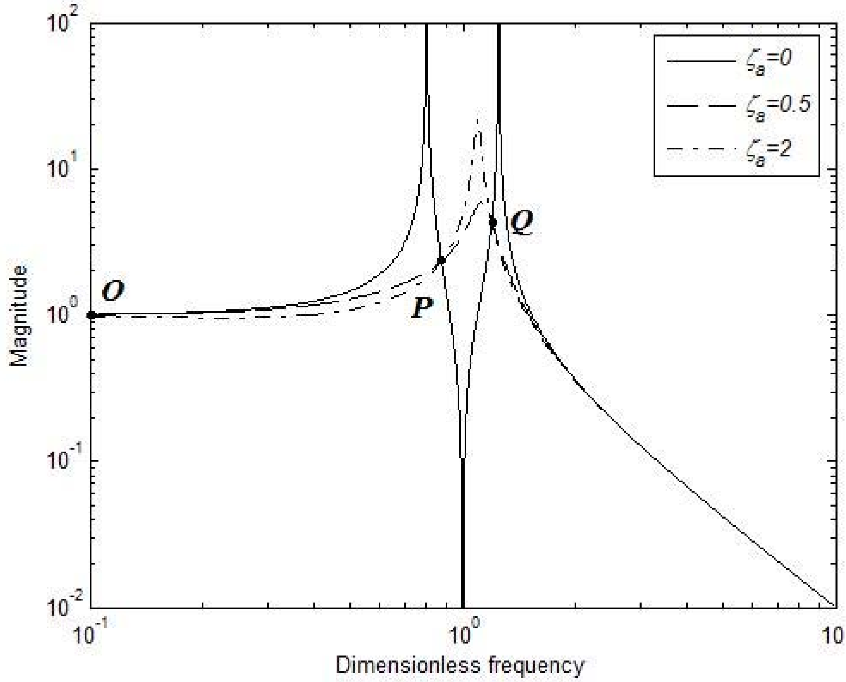


Figure 6.2 The frequency response of the primary system with the new DVA at $\gamma = 1$

It is found that $\lambda = 0$ and $H(0) = 1$ which corresponds to the fixed point *O* if we take the positive sign on the right hand side of equation 6.8. Taking the negative sign on the right hand side of equation 6.8, we have

$$\frac{\gamma^2-\lambda^2}{\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right)-\mu\lambda^2\gamma^2}=-\frac{1}{1-\lambda^2+\mu\gamma^2} \tag{6.9}$$

Equation (6.9) may be rewritten as

$$2\lambda^4-2\left(1+\gamma^2+\mu\gamma^2\right)\lambda^2+2\gamma^2+\mu\gamma^4=0 \tag{6.10}$$

The sum and product of the roots of equation 6.10 can be written respectively as

$$\lambda_a^2+\lambda_b^2=1+\gamma^2+\mu\gamma^2 \tag{6.11a}$$

$$\text{and}\quad \lambda_a^2\lambda_b^2=\frac{2\gamma^2+\mu\gamma^4}{2} \tag{6.11b}$$

Equation (6.10) is a quadratic equation in λ^2. Let λ_a^2 and λ_b^2 be the two roots of equation (6.10) and assume $0<\lambda_a<\lambda_b$. The amplitudes of the frequency response at these two roots are independent of the damping ratio ζ_a, where these two points, *P* and *Q*, are called 'fixed points'. The amplitudes of the frequency response at λ_a^2 and λ_b^2 are

$$\left|H(\lambda_a)\right|=\left|\frac{B}{D}\right|=\left|\frac{1}{1-\lambda_a^2+\mu\gamma^2}\right| \tag{6.12a}$$

$$\text{and}\quad \left|H(\lambda_b)\right|=\left|\frac{B}{D}\right|=\left|\frac{1}{1-\lambda_b^2+\mu\gamma^2}\right| \tag{6.12b}$$

At any damping ratio, the frequency response must include these three fixed points *O*, *P* and *Q*. So the H_∞ optimum condition of the DVA may be expressed as

$$\max\left(\left|H\left(\lambda_a, \gamma_{H\infty}, \zeta_{H\infty}\right)\right|\right) = \min\left(\max_{\gamma,\zeta_a}\left(\left|H\left(\lambda_a\right)\right|, \left|H\left(\lambda_b\right)\right|, \left|H\left(\lambda_0\right)\right|\right)\right) \tag{6.13}$$

$\left|H\left(\lambda_a\right)\right|$ and $\left|H\left(\lambda_b\right)\right|$ are calculated according to equations 6.12a and 6.12b at different values of γ and they are plotted together with $H(0)$ in figure 6.3. The response amplitudes at these three fixed points are compared and the maximum of these three responses at each value of γ, $\max_{\gamma,\zeta}\left(\left|H\left(\lambda_a\right)\right|, \left|H\left(\lambda_b\right)\right|, \left|H\left(\lambda_0\right)\right|\right)$, is plotted in figure 6.4. Point *A* in figure 6.3 is the intersecting point of the curves of $\left|H\left(\lambda_a\right)\right|$ and $\left|H\left(\lambda_b\right)\right|$. This is the case that the response amplitudes at the two fixed points *P* and *Q* to be the same and this condition is used in applying the fixed-points theory for searching the optimum tuning frequency of the traditional DVA. Optimum damping will then be found such that the response amplitude at the fixed points becomes local maxima of the response spectrum of mass *M*.

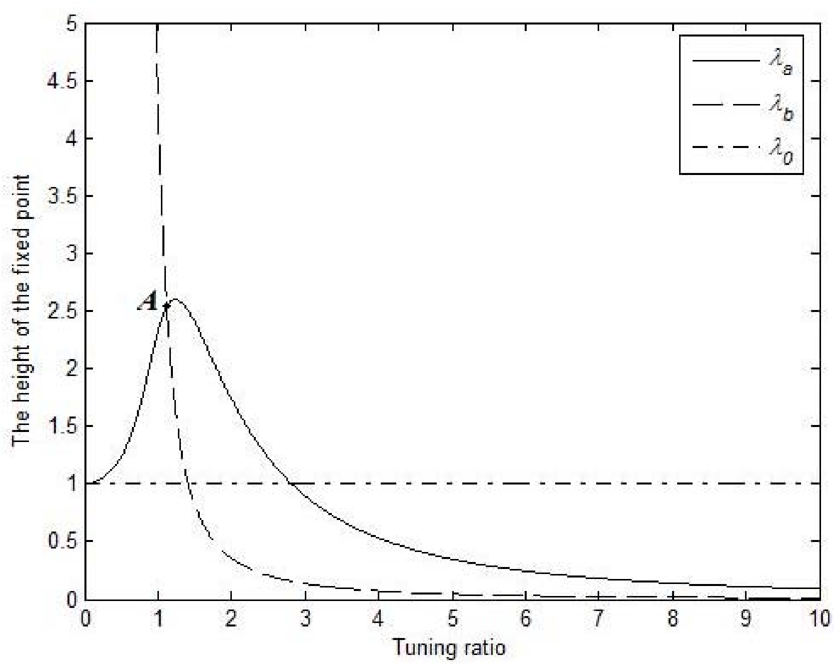


Figure 6.3 Response at the (height of) fixed points versus tuning ratio at $\mu = 0.2$

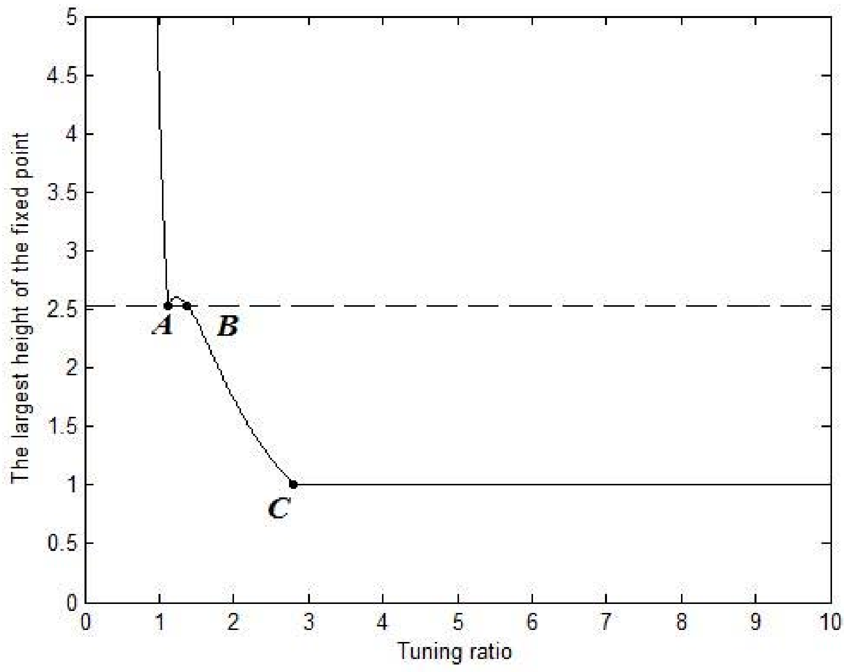


Figure 6.4 $\max_{\gamma,\zeta}\left(\left|H(\lambda_a)\right|,\left|H(\lambda_b)\right|,\left|H(\lambda_0)\right|\right)$ versus tuning ratio at $\mu = 0.2$

In figure 6.3, there are two ranges of frequency ratio γ that the two curves $\left|H(\lambda_a)\right|$ and $\left|H(\lambda_b)\right|$ require different methods for searching the optimized parameters of the DVA. In figure 6.3, the first frequency range of γ starts from zero to the frequency of point A while the second range starts from the frequency of point A to infinity. $\left|H(\lambda_a)\right|$ decreases and $\left|H(\lambda_b)\right|$ increases when γ increases in the first frequency range and therefore it satisfies the assumption of the fixed-points theory. However, both $\left|H(\lambda_a)\right|$ and $\left|H(\lambda_b)\right|$ decreases for all frequencies on the right hand side of the local maximum of $\left|H(\lambda_a)\right|$. Moreover, there is no intersecting point in the second frequency range and therefore the fixed-points theory cannot be used in searching the optimized parameters of the DVA. These two cases will be discussed separately in the following sections.

In figure 6.4, point *A* is a local minimum of the curve of $\max_{\gamma,\zeta}\left(\left|H(\lambda_a)\right|,\left|H(\lambda_b)\right|,\left|H(\lambda_0)\right|\right)$ and its corresponding tuning ratio was chosen by other researchers (Liu and Liu 2005; Ren 2001) as the optimum tuning frequency ratio. This is the optimum case only if the tuning frequency ratio can be chosen between zero and the tuning ratio of point *B* in figure 6.4. It will be shown in the following section that the damping and the tuning ratios are functions of the mass ratio. On the other hand, point C and the curve on the right of point C in figure 6.4 show that the response amplitude ratio at the fixed points are one or below. It will be shown in the following section that the optimum damping becomes a function of mass ratio and tuning ratio.

6.1.1 Local optimization of the non-traditional DVA using the fixed-points theory

The local minimum of the response amplitude at the fixed points can be found when $|H(\lambda_a)| = |H(\lambda_b)|$, i.e.

$$\frac{1}{1-\lambda_a^2+\mu\gamma^2} = -\frac{1}{1-\lambda_b^2+\mu\gamma^2} \tag{6.14}$$

Using equations (6.11) and (6.12), the tuning ratio leading to the same response amplitude of the fixed points is

$$\gamma_{opt\_local} = \sqrt{\frac{1}{1-\mu}} \tag{6.15}$$

The magnitude of frequency response of mass M using equation 6.5 and $\gamma_{opt\_local} = \sqrt{\frac{1}{1-\mu}}$ is calculated and the result is plotted as figure 6.5.

Substitute equation 6.15 into equation 6.10. Equation 6.10 can be rewritten as

$$\lambda^4 - \frac{2}{1-\mu}\lambda^2 + \frac{2-\mu}{2(1-\mu)^2} = 0 \tag{6.16}$$

The roots of the equation 6.16 are found as

$$\lambda_a^2 = \frac{1}{1-\mu}\left(1-\sqrt{\frac{\mu}{2}}\right) \tag{6.17a}$$

and $$\lambda_b^2 = \frac{1}{1-\mu}\left(1+\sqrt{\frac{\mu}{2}}\right) \tag{6.17b}$$

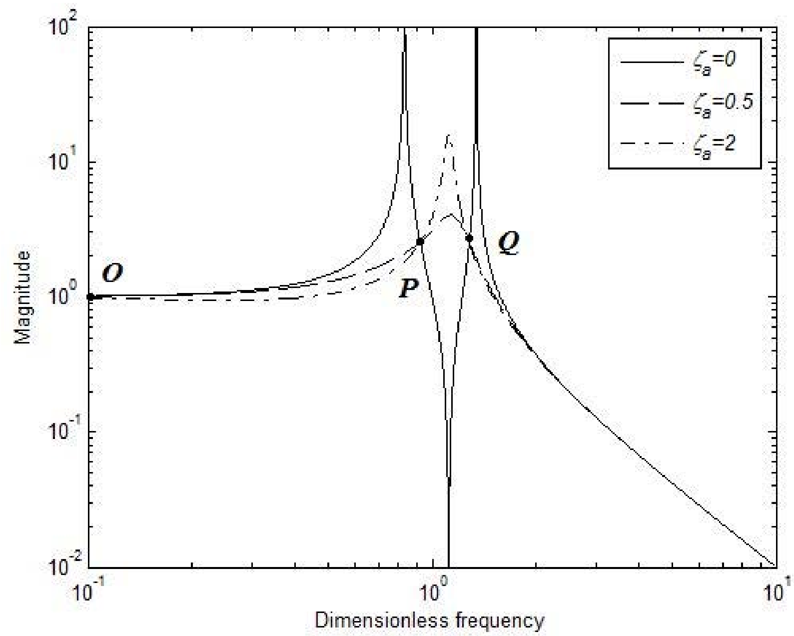


Figure 6.5 The frequency response of the primary system with a new DVA at γ_{opt_local}.

Substitute equations 6.17a and 6.17b into equations 6.12a and 6.12b, respectively. The response amplitude at the fixed points can be written as

$$G = \left|H(\lambda_a)\right| = \left|H(\lambda_b)\right| = \frac{2(1-\mu)}{\sqrt{2\mu}} \tag{6.18}$$

The next step is to determine the damping of the local minimum in order to make the fixed points to be the maximum points on the response curve. The condition of fixed point being the maximum means that the response curve would become local maxima at the fixed points, that is

$$\left.\frac{\partial}{\partial\lambda^2}\left|H(\lambda)\right|^2\right|_{\lambda=\lambda_a} = \left.\frac{\partial}{\partial\lambda^2}\left|H(\lambda)\right|^2\right|_{\lambda=\lambda_b} = 0 \tag{6.19}$$

$$\text{Let } \left|H(\lambda)\right|^2 = \frac{p}{q} \tag{6.20}$$

$$\text{where } p = \left(\gamma^2 - \lambda^2\right)^2 + \left(2\zeta_a\gamma\lambda\right)^2 \tag{6.21}$$

$$q = \left[\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right) - \mu\lambda^2\gamma^2\right]^2 + \left(2\zeta_a\gamma\lambda\right)^2\left(1-\lambda^2+\mu\gamma^2\right)^2 \tag{6.22}$$

Substitute equation 6.20 into equation 6.19, we have

$$\frac{\partial}{\partial\lambda^2}\left|H(\lambda)\right|^2 = \frac{\partial}{\partial\lambda^2}\left(\frac{p}{q}\right) = 0 \tag{6.23}$$

$$\frac{\dfrac{\partial p}{\partial\lambda^2}q - \dfrac{\partial q}{\partial\lambda^2}p}{q^2} = 0 \tag{6.24}$$

$$\frac{\partial p}{\partial\lambda^2}q - \frac{\partial q}{\partial\lambda^2}p = 0 \tag{6.25}$$

Using equations 6.21 and 6.22, we may write

$$\frac{\partial p}{\partial\lambda^2} = 2\left(\lambda^2-\gamma^2\right) + \left(2\zeta_a\gamma\right)^2 \tag{6.26}$$

$$\begin{aligned}\frac{\partial q}{\partial\lambda^2} = {} & 2\left[\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right) - \mu\lambda^2\gamma^2\right]\left(-1+2\lambda^2-\gamma^2-\mu\gamma^2\right) \\ & + \left(2\zeta_a\gamma\right)^2\left(1-\lambda^2+\mu\gamma^2\right)\left(1-3\lambda^2+\mu\gamma^2\right)\end{aligned} \tag{6.27}$$

Substitute $\gamma_{opt\_local} = \sqrt{\dfrac{1}{1-\mu}}$ into equation 6.20, we have

$$G^2 = \frac{p}{q} = \left(\frac{2(1-\mu)}{\sqrt{2\mu}}\right)^2 \tag{6.28}$$

Using equations 6.25 and 6.28, we may write

$$\frac{\partial p}{\partial\lambda^2} - \frac{\partial q}{\partial\lambda^2}G^2 = 0 \tag{6.29}$$

Solving $\zeta_a^{\ 2}$ from equations 6.26, 6.27 and 6.29, we have

$$\zeta_a^2 = \frac{G^2\left[\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right)-\mu\lambda^2\gamma^2\right]^2-\left(\gamma^2-\lambda^2\right)^2}{\left(2\gamma\lambda\right)^2\left(1-G^2\left(1-\lambda^2+\mu\gamma^2\right)^2\right)} \tag{6.30}$$

The optimum damping can be found by substituting equations (6.15), (6.17) and (6.18) into equation 6.30 and written as

$$\zeta_a = \sqrt{\frac{3\mu}{8\left(1\mp\sqrt{\frac{\mu}{2}}\right)}} \tag{6.31}$$

Taking a linear approximation of ζ_a^2, the optimum damping is chosen as,

$$\zeta_{opt\_local} = \sqrt{\frac{1}{2}\left(\frac{3\mu}{8\left(1-\sqrt{\frac{\mu}{2}}\right)}+\frac{3\mu}{8\left(1+\sqrt{\frac{\mu}{2}}\right)}\right)} = \frac{1}{2}\sqrt{\frac{3\mu}{2-\mu}} \tag{6.32}$$

From equation 6.18, the resonant amplitude ratio is approximately

$$\left|\frac{X}{Y}\right|_{local\_\max} = \frac{2(1-\mu)}{\sqrt{2\mu}} \tag{6.33}$$

The frequency response of primary mass M with $\mu = 0.2$, $\gamma = \gamma_{opt\_local}$ and damping ratio ζ_a = 0.1, 0.2 and ζ_{opt_local} are calculated respectively using equation 6.5 and the results are plotted in figure 6.6.

To compared the vibration suppression performance of this non-traditional DVA to the traditional DVA, frequency response curve of the primary mass M in figure 3.5 with optimum tuning condition is plotted together with the one in figure 6.6 with $\zeta_a =$ ζ_{opt_local} in figure 6.7. It is found that the maximum frequency response of mass *M* with the traditional DVA under optimum tuning is higher than the maximum frequency response of the primary system with the new DVA under a local optimum tuning by about 20% at $\mu = 0.2$.

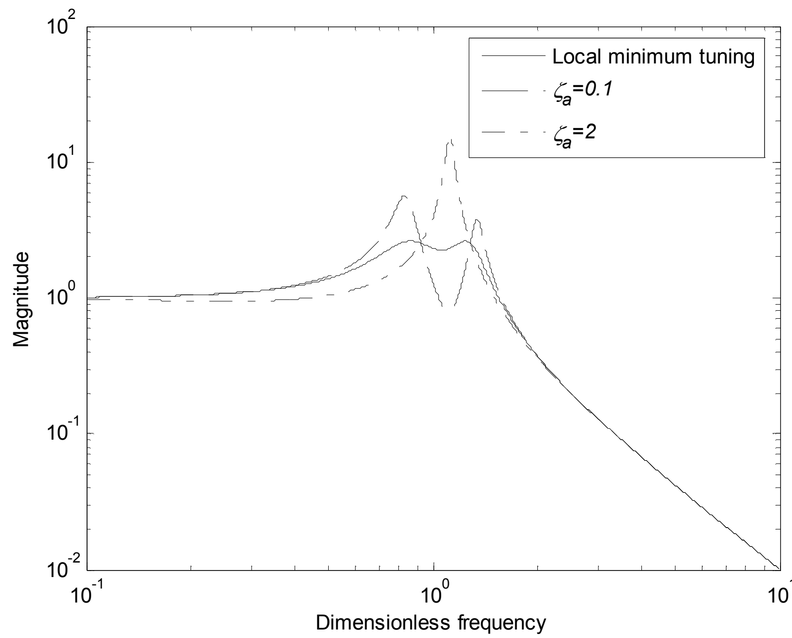


Figure 6.6 Frequency response of the primary mass *M* with the new DVA using the local optimum tuning

The height of the fixed point in the frequency response spectrum of the mass *M* using the optimized traditional DVA and the one using the non-traditional DVA with local optimum tuning are calculated at different mass ratio μ and plotted in figure 6.8 for comparison. The response amplitude of mass *M* using the traditional DVA is larger than that using the non-traditional DVA at any mass ratio between 0 and 1 and the difference increases with the increase of mass ratio μ.

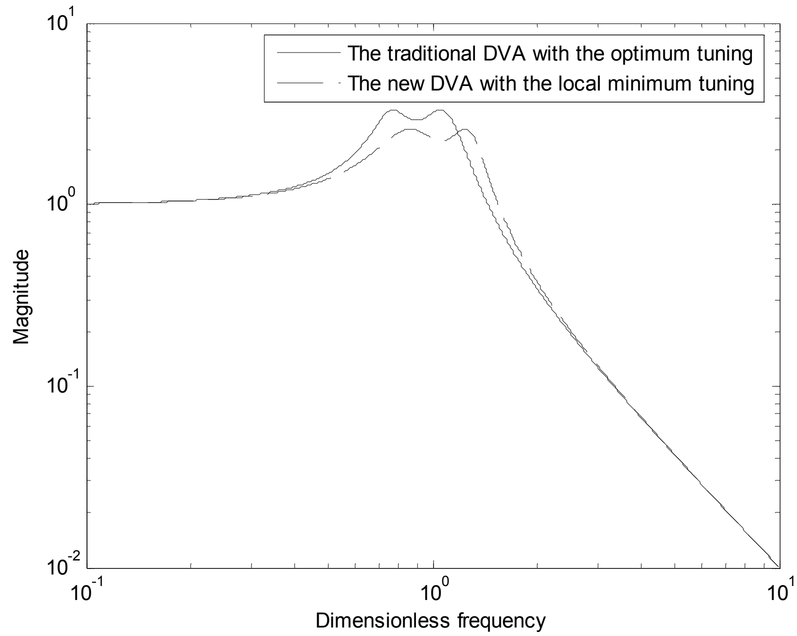


Figure 6.7 Frequency response of the primary mass M with the traditional DVA using the optimum tuning and with the new DVA using the local optimum tuning

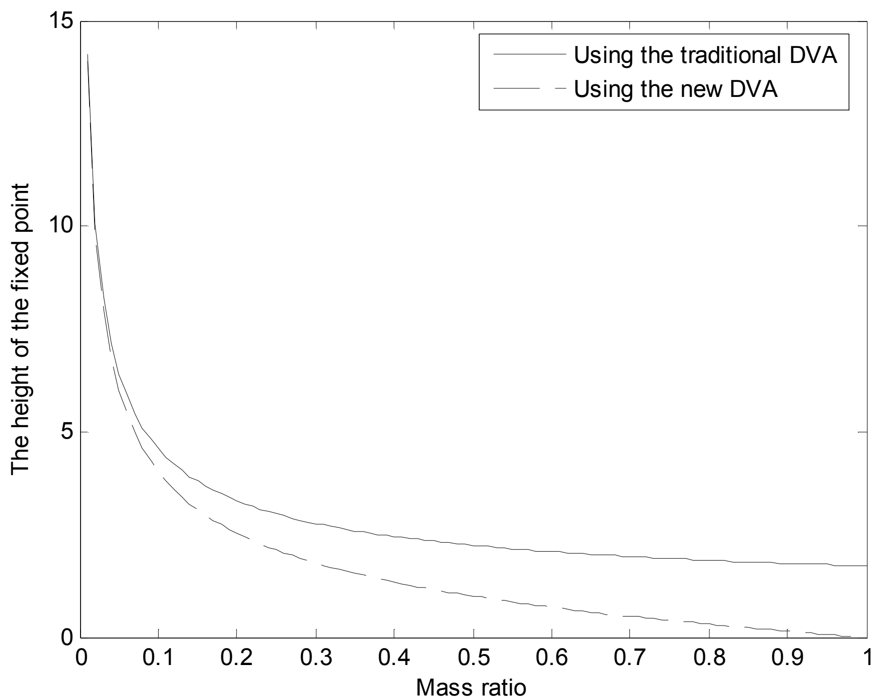


Figure 6.8 Mass ratio versus the height of the fixed point using different type of DVA.

6.1.2 Global optimization of the non-traditional DVA

Even if the local optimum tuning of the non-traditional DVA can provide a better result than the traditional DVA for vibration suppression of the primary system, it is not the best tuning as discussed in section 6.1. When the tuning ratio γ is higher than that of point B in figure 6.4, the height of the fixed point can be lower than the local optimum one (the height of point A in figure 6.4). There is no intersecting point between the curves of $|H(\lambda_a)|$ and $|H(\lambda_b)|$ in when the tuning ratio γ is higher than that of point A in figure 6.3 and therefore the fixed-points theory cannot be used for finding the global optimum tuning of DVA. As shown in figure 6.3, the frequency response at λ_a is always higher than the frequency response at λ_b , i.e. $H(\lambda_a) > H(\lambda_b)$, when the tuning ratio γ is higher than that of point A in figure 6.3. According to the objective function of the optimization as stated by equation 6.13, $|H(\lambda_a)|$ is used in searching the global optimum tuning of DVA. Derivation of the global optimum tuning parameters of the DVA is shown in the following.

Equation 6.3 is restated below for the ease of discussion.

$$H(\lambda) = \frac{X}{F/K} = \frac{\gamma^2 - \lambda^2 + 2j\zeta_a\gamma\lambda}{\left[(1-\lambda^2)(\gamma^2-\lambda^2) - \mu\lambda^2\gamma^2\right] + 2j\zeta_a\gamma\lambda(1-\lambda^2+\mu\gamma^2)} \tag{6.3}$$

Rewriting the equation into the form as

$$|H(\lambda)| = \left|\frac{X}{F/K}\right| = \sqrt{\frac{A + B\zeta_a^2}{C + D\zeta_a^2}} \tag{6.34}$$

where $A = (\gamma^2 - \lambda^2)^2$, $B = (2\gamma\lambda)^2$, $C = \left[(1-\lambda^2)(\gamma^2-\lambda^2) - \mu\lambda^2\gamma^2\right]^2$, and $D = \left(2\gamma\lambda\left(1-\lambda^2+\mu\gamma^2\right)\right)^2$

If the frequency responses are independent of the damping, we may write

$$\frac{A}{C} = \frac{B}{D} \quad \text{or} \quad \left(\frac{\gamma^2-\lambda^2}{\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right)-\mu\lambda^2\gamma^2}\right)^2 = \left(\frac{1}{1-\lambda^2+\mu\gamma^2}\right)^2 \tag{6.35}$$

Equation (6.35) can be rewritten after taking square root as

$$\frac{\gamma^2-\lambda^2}{\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right)-\mu\lambda^2\gamma^2} = \pm\frac{1}{1-\lambda^2+\mu\gamma^2} \tag{6.36}$$

Taking the negative sign of the right hand side of equation 6.36, we have

$$\frac{\gamma^2-\lambda^2}{\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right)-\mu\lambda^2\gamma^2} = -\frac{1}{1-\lambda^2+\mu\gamma^2} \tag{6.37}$$

Equation 6.37 can be rewritten as

$$2\lambda^4 - 2\left(1+\gamma^2+\mu\gamma^2\right)\lambda^2 + 2\gamma^2 + \mu\gamma^4 = 0 \tag{6.38}$$

Solutions of equation 6.38 can be found as

$$\lambda_{a,b}^2 = \frac{1+\left(1+\mu\right)\gamma^2 \mp \sqrt{1+2\left(\mu-1\right)\gamma^2+\left(1+\mu^2\right)\gamma^4}}{2} \tag{6.39}$$

As shown in figures 6.3 and 6.4, $\left|H\left(\lambda_a\right)\right| > \left|H\left(\lambda_b\right)\right|$ if the tuning frequency ratio γ is larger than that of point *B* or *C* in figure 6.4. Global optimum damping may be found by considering $\left|H\left(\lambda_a\right)\right|$ to be a maximum of the frequency response curve of mass *M*. We may write

$$\frac{\partial}{\partial \lambda^2}\left|H(\lambda)\right|^2\Big|_{\lambda=\lambda_a} = 0 \tag{6.40}$$

Using equations 6.3 and 6.40, it can be derived that

$$\zeta_{opt\_global} = \sqrt{\frac{\left(\gamma^2 - {\lambda_a}^2\right)\left[\left(2 - 3{\lambda_a}^2\right) + (1+2\mu)\gamma^2\right]}{\left(2\gamma\lambda_a\right)^2}} \tag{6.41}$$

To find the tuning ratio of point C in figure 6.4, we use equation 6.12a and let $\left|H(\lambda_a)\right|$ = 1. We can therefore write

$$\left|H(\lambda_a)\right| = \left|\frac{B}{D}\right| = \left|\frac{1}{1-\lambda_a^2 + \mu\gamma^2}\right| = 1 \tag{6.42}$$

Solving equation 6.40 for γ, we have the tuning frequency ratio at point C in figure 6.4 to be

$$\gamma_c = \frac{\lambda_a}{\sqrt{\mu}} \tag{6.43}$$

Referring to figure 6.4, the height of the fixed point become minimum if tuning ratio $\gamma > \gamma_c$. We may therefore state that H_∞ optimization of the non-traditional DVA is achieved if we select a tuning ratio $\gamma > \gamma_c$ and select the damping ratio of DVA according to equation 6.41.

The response amplitude of mass M are calculated according to equation 6.3 using the global optimum damping ζ_{opt_global} with mass ratio = 0.2 and tuning ratio = 2 and 3 and plotted in figures 6.9 and 6.10. The fixed points P and Q are marked in the figures for checking. The maximum response amplitude of mass M in these two figures are smaller than the maximum response amplitude of mass M in figure 6.7 which use the local optimum tuning and damping ratios.

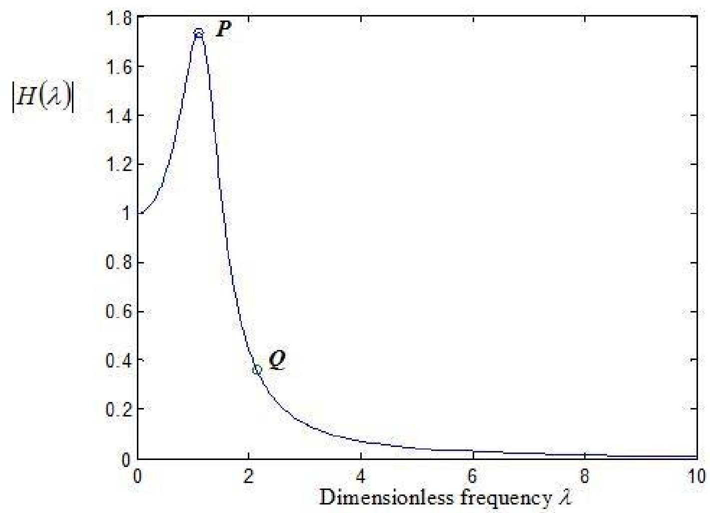


Figure 6.9 The frequency response of the primary system with the new DVA using optimum damping ratio ζ_{opt_global}, $\mu = 0.2$ and tuning ratio $\gamma = 2$.

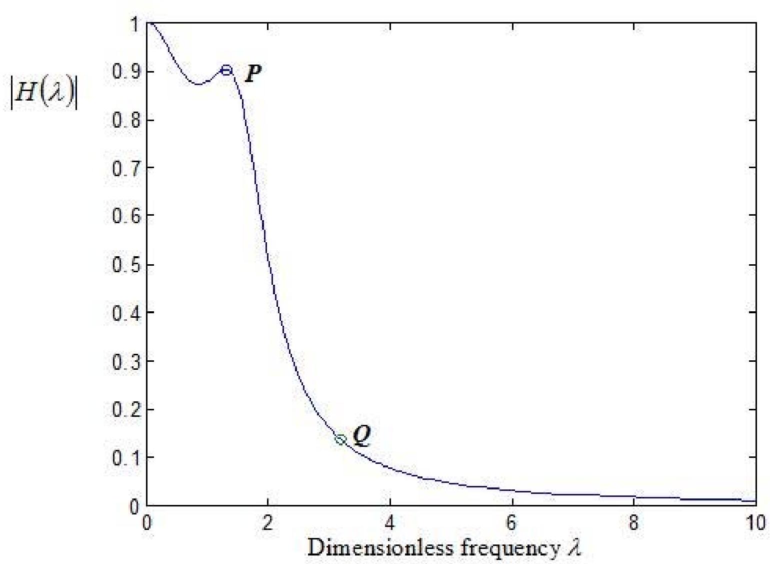


Figure 6.10 The frequency response of the primary system with the new DVA using optimum damping ratio ζ_{opt_global}, $\mu = 0.2$ and tuning ratio $\gamma = 3$.

6.2 H_2 optimization of the non-traditional DVA

The optimum tuning of the non-traditional DVA in H_2 optimization is derived in this section. The objective function of the H_2 optimization is the minimization of the total vibration energy of the primary structure under white noise excitation, i.e.

$$\min_{\gamma,\zeta_a}\left(E\left[x^2\right]\right) \tag{6.44}$$

The frequency response functions of a SDOF system attached with the non-traditional DVA as shown in figure 6.1 is restated below for the ease of discussion.

$$H(\lambda)=\frac{X}{F/K}=\frac{\gamma^2-\lambda^2+2j\zeta_a\gamma\lambda}{\left[\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right)-\mu\lambda^2\gamma^2\right]+2j\zeta_a\gamma\lambda\left(1-\lambda^2+\mu\gamma^2\right)} \tag{6.3}$$

Following the procedure from equations (3.34) to (3.41), the mean square motion in this case can be found as

$$E\left[x^2\right]=\frac{\omega_n S_0}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\gamma^2-\lambda^2+2j\zeta_a\gamma\lambda}{\left[\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right)-\mu\lambda^2\gamma^2\right]+2j\zeta_a\gamma\lambda\left(1-\lambda^2+\mu\gamma^2\right)}\right|^2 d\lambda \tag{6.45}$$

A useful formula of Gradshteyn and Ryzhik (1994) written as equation 6.46 below is used for solving equation 6.45.

$$\text{If}\quad H(\omega)=\frac{-j\omega^3B_3-\omega^2B_2+j\omega B_1+B_0}{\omega^4A_4-j\omega^3A_3-\omega^2A_2+j\omega A_1+A_0} \tag{6.46a}$$

$$\text{then}\quad \int_{-\infty}^{\infty}\left|H(\omega)\right|^2 d\omega=\pi\frac{\left[\begin{array}{l}\dfrac{B_0^2}{A_0}\left(A_2A_3-A_1A_4\right)+A_3\left(B_1^2-2B_0B_2\right)\\ \qquad +A_1\left(B_2^2-2B_1B_3\right)+\dfrac{B_3^2}{A_4}\left(A_1A_2-A_0A_3\right)\end{array}\right]}{A_1\left(A_2A_3-A_1A_4\right)-A_0A_3^2} \tag{6.46b}$$

Comparing equations (6.45) and (6.46), we may write

$$A_0 = \gamma^2, \quad A_1 = 2\zeta_a\gamma\left(1+\mu\gamma^2\right), \quad A_2 = 1+\gamma^2+\mu\gamma^2, \quad A_3 = 2\zeta_a\gamma, \quad A_4 = 1 \tag{6.47}$$
$$B_0 = \gamma^2, \quad B_1 = 2\zeta_a\gamma, \quad B_2 = 1, \quad B_3 = 0$$

Using equations 6.46 and 6.47, the mean square motion in equation 6.45 can be written as

$$E\left[x^2\right] = \frac{\omega_n S_0}{4\mu\zeta_a\gamma^5}\left[\gamma^4 + \left(\mu + 4\zeta_a^2 - 2\right)\gamma^2 + 1\right] \tag{6.48}$$

If $\frac{\partial}{\partial\gamma}E\left[x^2\right] = \frac{\partial}{\partial\zeta_a}E\left[x^2\right] = 0$, the system has a optimum tuning condition. The derivatives of equation 6.48 are

$$\frac{\partial}{\partial\gamma}E\left[x^2\right] = -\frac{\omega_n S_0}{4\mu\zeta_a\gamma^6}\left[\gamma^4 + \left(3\mu + 12\zeta_a^2 - 6\right)\gamma^2 + 5\right] \tag{6.49a}$$

and $$\frac{\partial}{\partial\zeta}E\left[x^2\right] = -\frac{\omega_n S_0}{4\mu\zeta_a^2\gamma^5}\left[\gamma^4 + \left(\mu - 4\zeta_a^2 - 2\right)\gamma^2 + 1\right] \tag{6.49b}$$

Using equation 6.49, the maximum/minimum solutions can be found respectively as

$$\gamma = \frac{1}{2}\sqrt{6 - 3\mu \pm \sqrt{(6-3\mu)^2 - 32}} \tag{6.50a}$$

and $$\zeta_a = \sqrt{\frac{(-2+\mu)\gamma^2 + 1 + \gamma^4}{4\gamma^2}} \tag{6.50b}$$

The mean square motions of the primary mass M with $\mu = 0.11$ at different tuning frequency and damping ratios are calculated using equation 6.48 and the contours of $E\left[x^2\right]$ versus γ and ζ_a are plotted in figure 6.11. It can be seen in figure 6.11 that there is a local minimum as well as a local maximum.

After checking figure 6.11 and using equation 6.50, the frequency ratio γ of the

DVA at the local minimum can be written as

$$\gamma_{H2\_opt\_local} = \frac{1}{2}\sqrt{6-3\mu-\sqrt{(6-3\mu)^2-32}} \tag{6.55}$$

and that at the local maximum is written as

$$\gamma = \frac{1}{2}\sqrt{6-3\mu+\sqrt{(6-3\mu)^2-32}} \tag{6.56}$$

In figure 6.11, the local optimum is found at $\gamma = 1.1494$ and $\zeta_a = 0.2168$, and the local maximum is found at $\gamma = 1.2304$ and $\zeta_a = 0.2667$. The contours on the top right corner of figure 6.11 show a decreasing trend of the mean square motion.

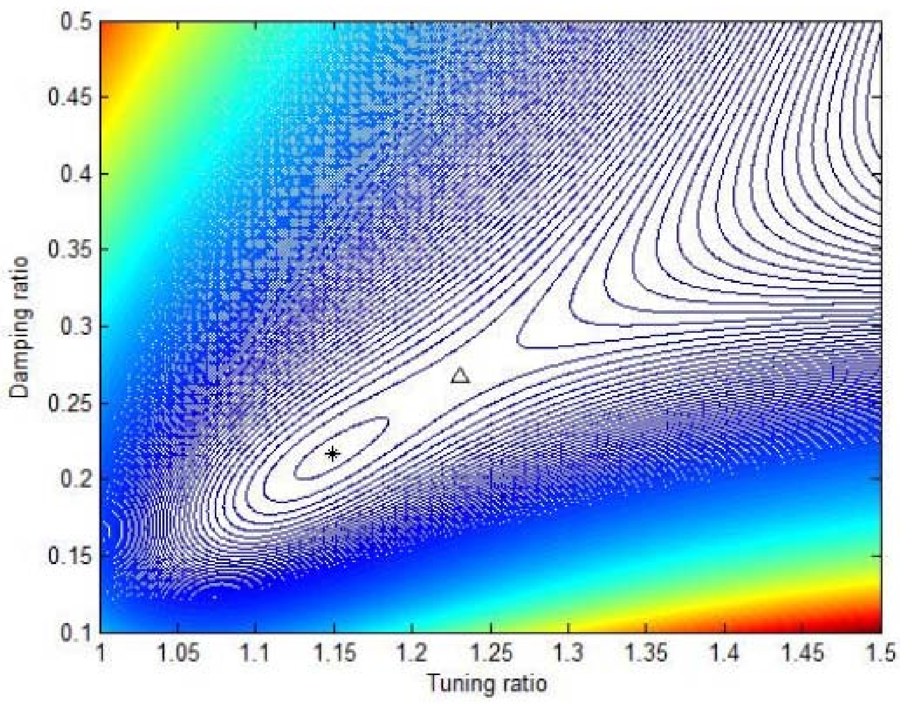


Figure 6.11 The contour plot of the mean square motion at $\mu = 0.11$. (*) – local minimum; (Δ) – local maximum

Equations 6.55 and 6.56 shows that it requires $(6-3\mu)^2 - 32 \geq 0$ in order to have a local minimum or maximum mean square motion to exist. This requirement leads to $\mu \leq 0.1144$ and therefore no local minimum and maximum mean square motion can be found if the mass ratio $\mu > 0.1144$. The mean square motions of the primary mass M with $\mu = 0.2$ at different tuning frequency and damping ratios are calculated using equation 6.48 and the contours of $E\left[x^2\right]$ versus γ and ζ_a are plotted in figure 6.12. The contours along the top right direction show a decreasing trend of the mean square motion and there is no local minimum or local maximum in figure 6.12.

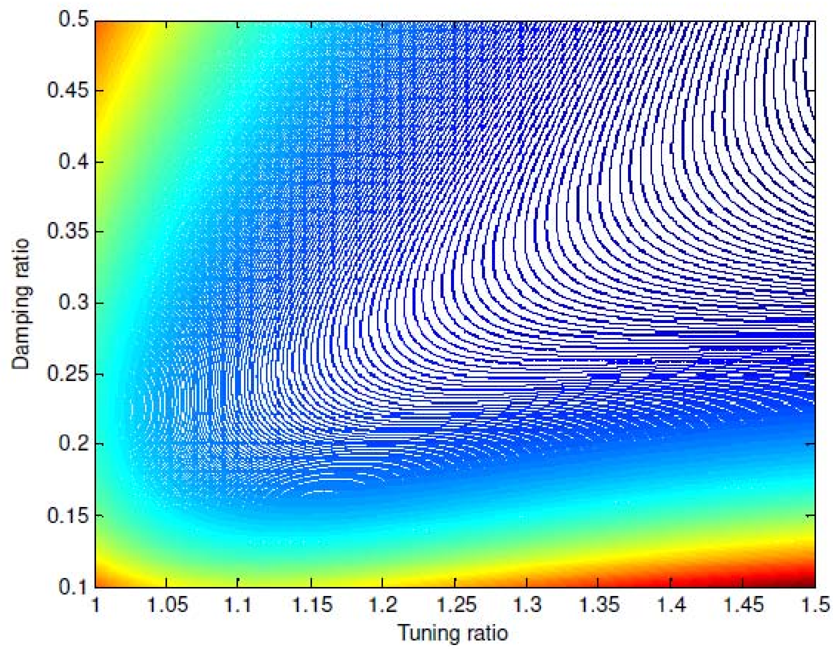


Figure 6.12 The contour plot of the mean square motion at $\mu = 0.2$

Since there is no global optimum tuning frequency exist in the H2 optimization of the non-traditional DVA for SDOF system, it is recommended that the local optimum tuning frequency (equation 6.55) or a much higher value of tuning frequency should be used. The best or optimum damping ratio after we select the tuning frequency and the mass ratio can be calculated according to equation 6.53b it is shown as the curve cutting across the contours of mean square motion in figure 6.13.

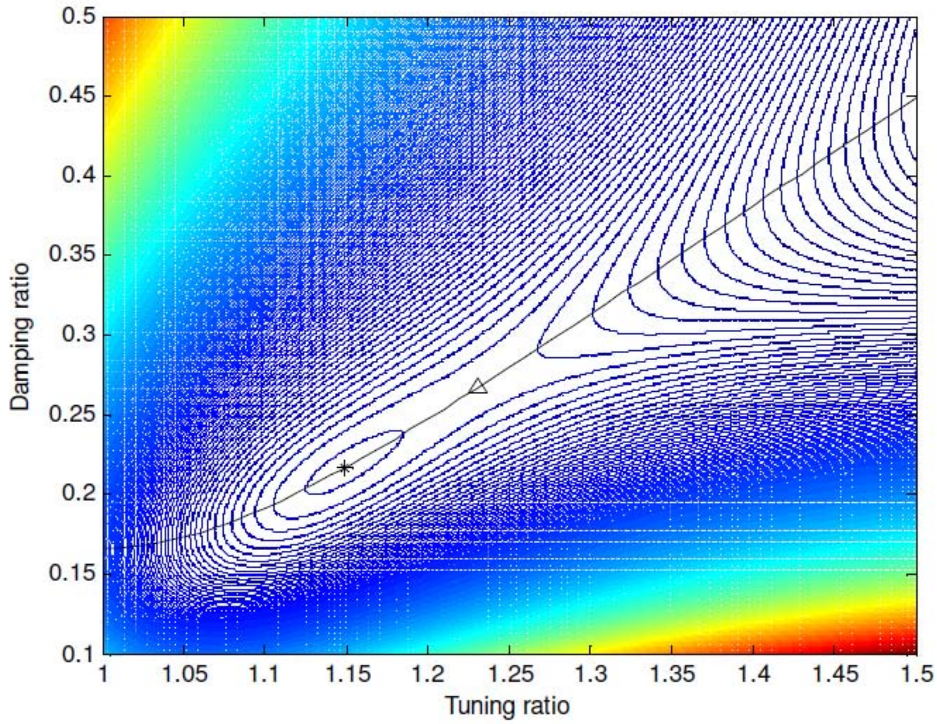


Figure 6.13 The contour plot of the mean square motion at $\mu = 0.11$. (*) – local minimum; (Δ) – local maximum; (-) – optimum damping

6.3 Summary

In this chapter, a non-traditional DVA is proposed and compared to the traditional DVA for suppressing vibration of SDOF system. It has been proved analytically that the performance of the non-traditional DVA is better than the traditional one. H_∞ and H_2 optimizations of the non-traditional DVA are solved and tested for this DVA. Relevant research reports found in literature are only about the H_∞ optimization of this DVA but they have only found the local optimum condition. As derived and proved in the section 6.1, global optimum conditions of the DVA exist and the vibration suppression performance of the DVA can be greatly improved if the global optimum tuning is applied instead of the local optimum tuning. This new finding improves our understanding of the dynamics of this DVA and it helps us to improve the vibration suppression performance of this non-traditional DVA.

On the other hand, H_2 optimization problem of this DVA has been solved analytically. To the author's knowledge, there is no research report found in literature on this topic. As derived in section 6.2, local H_2 optimum tuning condition of the DVA exists if mass ratio is 0.1144 or less. No global optimum tuning condition exists if mass ratio is higher than 0.1144 are found and it is recommended to use a high tuning frequency ratio if possible. The best value of damping ratio after one select the tuning frequency ratio is derived and stated in equation 6.53b.

7. CONCLUSIONS

Optimization theories of the traditional DVA applied suppressing vibrations in beam and plate structures are established in chapters 4 and 5, respectively. The optimum tuning conditions of the traditional DVA in beams and plates are proved to be very similar. It is found that the optimum tunings including the tuning frequency and damping ratios as well as the attachment location of the DVA in both the beam and plate structures depend on the mass ratio and modal response at the attachment point of the DVA. New and better results than those reported by other researchers (Dayou 2006, Jacquot 2001) have been found. These new results have been reported to and published in an international journal.

In chapter 6, optimization theory of a non-traditional DVA for suppressing vibration in a SDOF system is established. As shown in figure 6.3, the fixed-point theory is not suitable to this new DVA because the response amplitudes at both fixed-points decrease when the tuning ratio increases. Only the local optimum tuning condition of this DVA can be found if one uses the standard fixed-points theory. On the other hand, it has been found that global optimum tuning condition of the DVA exists and a new theory has been established for finding this global optimum tuning condition of the DVA. It is proved that the results reported by Ren (2001), and Liu and Liu (2005) are based on the local optimum condition of this DVA and there are global optimum conditions of the DVA that can produce much better vibration suppression of the primary systems. Moreover, H_2 optimization theory for this new DVA in reducing the kinetic energy of the primary mass under white noise excitation is established. This research provides results which are never found in the relevant literature. Some of the results have been reported to and published in an international journal and some other will be submitted for publication as well.

Future research can be focused on the new DVA applied to suppress vibrations in MDOF or continuous systems under different types of excitations.

APPENDIX A. EULER-BERNILLI BEAM WITH A FORCE

Consider figure A.1,

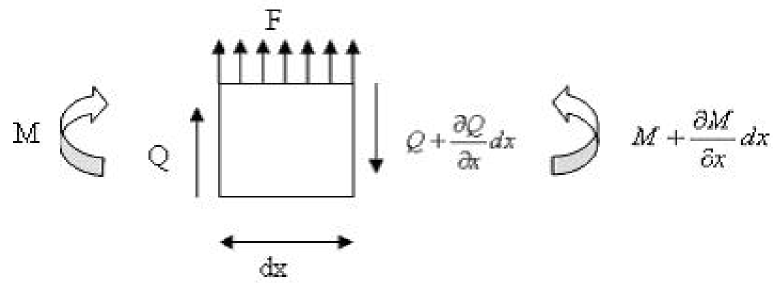


Figure A.1 Free body diagram of the beam element

Consider the free body diagram and applying Newton second law, we may write

$$\sum F_n = \rho A dx \frac{\partial^2 w}{\partial t^2} \tag{A.1}$$

$$\sum M_n = \rho I \frac{\partial^2 \theta}{\partial t^2} dx \tag{A.2}$$

From equation A.1, we have

$$Q - Q - \frac{\partial Q}{\partial x} dx + F dx = \rho A dx \frac{\partial^2 w}{\partial t^2} \tag{A.3}$$

$$\Rightarrow \rho A \frac{\partial^2 w}{\partial t^2} + \frac{\partial Q}{\partial x} = F \tag{A.4}$$

From equation A.2, we have

$$M - M - \frac{\partial M}{\partial x} dx + Q dx = 0 \tag{A.5}$$

$$\Rightarrow \frac{\partial M}{\partial x} = Q \tag{A.6}$$

The bending moment is related to the curvature of the beam element by the flexure equation written as

$$M = EI\frac{\partial^2 w}{\partial x^2} \tag{A.7}$$

Using equations A.6 and A.7, we may write

$$\frac{\partial}{\partial x}(EI\frac{\partial^2 w}{\partial x^2}) = Q \tag{A.8}$$

$$\Rightarrow Q = EI\frac{\partial^3 w}{\partial x^3} \tag{A.9}$$

Substitute equation A.9 into A.4, we have

$$\rho A\frac{\partial^2 w}{\partial t^2} + EI\frac{\partial^4 w}{\partial x^4} = F \tag{A.10}$$

APPENDIX B. THE FORCE DUE TO DVA APPLIED ON THE BEAM

Consider the figure A.2,

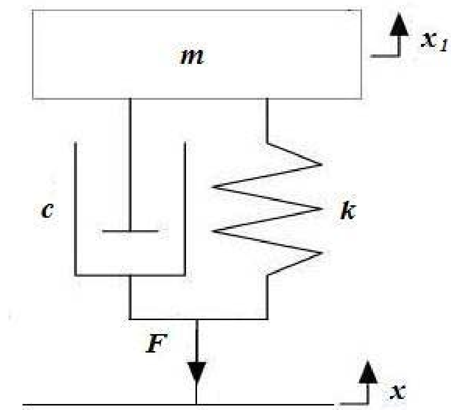


Figure A.2 Force due to DVA acting on the primary mass *M*

Considering force balance, we may write

$$m\frac{\partial^2 x}{\partial t^2} = -k(x_1 - x) - c(\dot{x}_1 - \dot{x}) \tag{A.11}$$

$$\text{and} \quad F = -k(x - x_1) - c(\dot{x} - \dot{x}_1) \tag{A.12}$$

Applying the Laplace transform into (A.11) and (A.12), we have

$$mX_1 s^2 = -k(X_1 - X) - cs(X_1 - X) \tag{A.13}$$

$$\text{and} \quad F = -k(X - X_1) - cs(X - X_1) \tag{A.14}$$

Using equations (A.13) and (A.14), we may write

$$\frac{X_1}{X} = \frac{cs + k}{ms^2 + cs + k} \tag{A.15}$$

$$\frac{X_1}{F} = -\frac{1}{ms^2} \tag{A.16}$$

Multiplying equations (A.15) and (A.16), we have

$$F = -\frac{ms^2(cs + k)}{ms^2 + cs + k} X \tag{A.17}$$

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