THE HONG KONG POLYTECHNIC UNIVERSITY

Department of Logistics and Maritime Studies

A STUDY OF NETWORK LOCATION PROBLEMS

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CERTIFICATE OF ORIGINALITY

I hereby declare that this thesis is my own work and that, to the best of my knowledge and belief, it reproduces no material previously published or written, nor material that has been accepted for the award of any other degree or diploma, except where due acknowledgement has been made in the text.

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Abstract

One of the most important branches of logistics management is to investigate where to locate new facilities such as transportation hubs, air and sea ports, retail outlets, and so on. There is a wide variety of applications of facility location models. These include, but not limited to, locating a given number of ambulances to minimize the maximum response time (the time between a demand point and the nearest ambulance), and locating warehouses within a supply chain to minimize the average transportation time to market. The above two problems have different objective functions: minimax (center) for the former and minisum (median) for the latter.

Facility location models differ not only in objective functions (center, median or others), but also in decision space (planar, discrete or networks), the number (singe-facility or multi-facility), capacity limit (uncapacitated or capacitated) and shape (isolated point or connected structure) of the facilities to locate, the nature of the inputs (static or dynamic; demands known with certainty or uncertainty), and other problem parameters.

This thesis focuses on studying the network location problems as follows: Demand points with certainty are taken to be at the nodes of the network, and to be served by their own nearest facilities, which are to be located anywhere in the given network. The facilities may be isolated points, or connected structures such as paths, trees, and so on. The objective is to locate a given number of uncapacitated facilities to minimize the ordered median function (OMf) and its special cases. The cost of satisfying the demand points depends on the distances between demands and facilities, which are measured by the shortest paths through the network. The organization of this thesis is as follows:

Chapter 1 introduces a taxonomy of location problems and presents the definition of the ordered median problems (OMP) proposed by Nickel and Puerto (2005), and describes two main methodologies applied in this thesis.

Chapter 2 presents a literature review of the network location models, including two main clues on the history and development of the ordered median problems.

Chapter 3 deals with the multi-facility ordered median problem in *undirected* networks, in which multiple isolated facilities are to be located. Multi-facility OMP in general networks are NP-hard, since the p-median and the p-center problems are special cases. In this chapter we use a finite dominating set (FDS) to study some special instances of the OMf in networks. FDS is a finite set of points to which some optimal solutions must belong, and is very useful for solving a variety of optimization problems, which enables one to restrict one's attention to a finite set of possible solutions. We first characterize an FDS for a special convex OMP in general networks, where the convex OMP is an important class in the OMP family. The FDS result generalizes some known results in the literature. Then, based on the FDS result, we obtain a polynomial size FDS and solve the problem confined to tree networks in polynomial time, which extends some results in the literature.

Chapter 4 is devoted to the multi-facility OMP in *directed* networks, since most of the networks in the real world are directed and not symmetric (undirected networks can be viewed as symmetric directed networks). For instance, routes are usually directed in a bus traffic system. In this chapter we again apply FDS to identify some possible solutions for a multi-facility OMP in a strongly connected directed network. We first prove that the OMP has an FDS in the node set, which generalizes the FDS result on the single-facility OMP in the literature. Furthermore, we show that the OMP can be solved efficiently based on the FDS result when the number of facilities is fixed and small. However, if the number of facilities is large, it is not practical for us to obtain an optimal solution in an efficient manner, since the OMP in directed networks is NP-hard. Hence, instead of finding an optimal solution, we resort to some approximation algorithms for some near-optimal solutions. At the end of Chapter 4, we present a $6\frac{2}{3}$-approximation algorithm for the p-median problem in directed networks.

Chapter 5 focuses on the OMP in *tree* networks, in which the facilities to locate are not isolated points but connected structures (e.g., paths, trees, etc). These problems are motivated by specific decision problems related to routing and network design. In this chapter we use the "nestedness property" to investigate subtree OMP in tree networks, where the nestedness property is the property that for any optimal solution x to the point OMP, there exists an optimal subtree to the corresponding subtree OMP including x. The nestedness property provides researchers with a powerful tool to develop some efficient algorithms. First, we prove the nestedness property for a special convex OMP in tree networks. This finding extends some classical results concerning the nestedness property. Second, we solve the problem in polynomial time by applying the nestedness property result. Finally, we provide one counter example to show that the nestedness property does not hold for the non-convex case.

Chapter 6, the last chapter, concludes the major findings of the thesis and suggests some directions for future research.

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Contents

List of Figures

Chapter 1

Introduction

One of the most important branches of supply chain management is facility location, which investigates where to locate new facilities such as transportation hubs, air and sea ports, retail outlets, etc, to minimize the cost of satisfying some set of demands. Almost every enterprise in each stage of a supply chain is faced with the problem of locating facilities at some time in its history. For instance, a manufacture company must identify a good location to provide raw materials as efficiently as possible. A production firm must determine the locations of its assembly plants, as well as warehouses. A retail outlet must locate stores. These location decisions are integral to a specific system's capability to satisfy demands efficiently, and can have lasting impacts on the system's flexibility.

There is a wide variety of applications of facility location models. These include, but not limited to, locating a given number of ambulances to minimize the maximum response time (the time between a demand point and the nearest ambulance), and locating warehouses within a supply chain to minimize the average transportation time to market. The above two problems have different objective functions: minimax (center) for the former and minisum (median) for the latter.

Facility location models differ not only in objective functions but also in other parameter indices. In the next section, we describe a taxonomy of location problems.

1.1 A Taxonomy of Location Problems

Location problems can be classified in a number of ways. The following classifications of location problems include, but are not limited to, the choices of decision spaces, objective functions, shape, number, and capacity limit of the facilities to locate, and the nature of the inputs.

First, most location problems focus on the choice of decision spaces. The basic types of problems with respect to decision spaces are listed below (see Huang, 2005).

- *Continuous location problems:* location problems in a general space endowed with some metric, e.g., the l_p norm. Facilities can be located anywhere in the given space.

- *Discrete location problems:* location problems where the sets of demand points and potential facility locations are finite.

- *Network location problems:* location problems that are connected to the links and nodes of an underlying network.

Second, different choices of objective functions lead to different types of location problems. The idea of making a facility placed at a location that is on average good for each demand leads to the median problem (also called the Weber or Fermat-Weber Problem) (see Wesolosky, 1993). Identifying a location that is as good as possible even for the most remote customer stimulates the idea of the center problem (see Plastria, 1995). The insight that both of the above points of view might be too extreme results in the cent-dian approach (see Halpern, 1978). Furthermore, considering a location that only satisfies the k farthest demand leads to the k-centrum criterion. The contrary to the k-centrum type is anti-k-centrum that only considers the k nearest demands. In addition, disregarding the k_1 nearest and the k_2 farthest weighted distances brings up the (k_1, k_2)-trimmed criterion. The aim to express all of the above ideas in a common format leads to a unified approach $-$ the Ordered Median function (OMf) (see Nickel and Puerto, 1999).

Third, different shapes of the facilities to locate lead to different location models. Traditionally, most research investigates location problems where a facility is represented by an isolated point in the metric space. Recently, the idea of locating a connected structure, e.g., a subway, stimulates the research on location problems where a facility should be connected, and cannot be represented by isolated points in the space.

In addition, facility location problems also differ in the number (singe-facility or multi-facility) and capacity limit (uncapacitated or capacitated) of the facilities to locate, the nature of the inputs (static or dynamic; demands known with certainty or uncertainty), and other problem parameters.

In this thesis we aim to study facility location problems in networks using the ordered median function and its special instances, where the facilities to locate have unlimited capacity, and demands in networks are known with certainty in advance. In the next section we formulate the multi-facility location problem to be studied in this thesis.

1.2 Problem Formulation

In our research problem throughout this thesis, demand points with certainty are taken to be at the nodes of the network, and to be served by their own nearest facilities, which are to be located anywhere in the given network. The facilities may be isolated points, or connected structures such as paths, trees, and so on. The objective is to locate a given number of uncapacitated facilities to minimize the ordered median function or its special cases. The cost of satisfying the demand points depends on the distances between demands and facilities, which are measured by the shortest paths through the network.

In the following we follow some notation from Nickel and Puerto (2005). First, we identify the elements that constitute the corresponding network model. Let $G = (V, E)$ be an undirected network with the node set $V = \{v_1, \cdots, v_n\}$ and the edge set $E = \{e_1, \cdots, e_m\}$. Each edge has a non-negative length and is assumed to be rectifiable. Thus, we will refer to the interior points on the edges. We let $A(G)$ denote the continuum set of points on the edges of G. Each edge $e \in E$ has an associated positive length by means of the function $l : E \to \mathbb{R}_+$. The edge lengths induce a distance function on $A(G)$. For any $x,\ y$ in $A(G)$, $d(x, y)$ denotes the length of a shortest path connecting x and y. Also, if Y is a subset of $A(G)$, then we define the distance from x to set Y by

$$d(x, Y) = d(Y, x) = \inf\{d(x, y) : y \in Y\}.$$

Through $w : V \to \mathbb{R}_{0+}$, every vertex is assigned a non-negative weight.

A point x on an edge $e = [v_i, v_j]$ can be written as a pair $x = (e, t),\ t \in [0, 1]$, with

$$d(v_k, x) = d(x, v_k) = \min\{d(v_k, v_i) + tl(e), d(v_k, v_j) + (1 - t)l(e)\},$$

where $v_k \in V$.

Let $p \geq 2$ be an integer. Then for $X_p = \{x_1, \cdots, x_p\} \subset A(G)$, the distance from a node $v_i \in V$ to the set X_p is

$$d(v_i, X_p) = d(X_p, v_i) = \min_{k=1,\cdots,p} d(v_i, x_k).$$

Now, for $X_p \subset A(G)$, we define

$$d(X_p) := (w_1 d(v_1, X_p), \cdots, w_n d(v_n, X_p)),$$

and

$$d_{\leq}(X_p) := sort_n(d(X_p)) = (w_{(1)} d(v_{(1)}, X_p), \cdots, w_{(n)} d(v_{(n)}, X_p)),$$

where $(\cdot)$ is a permutation of the set $\{1, \cdots, n\}$ satisfying

$$w_{(1)} d(v_{(1)}, X_p) \leq w_{(2)} d(v_{(2)}, X_p) \leq \cdots \leq w_{(n)} d(v_{(n)}, X_p).$$

To simplify notation, we denote the entries $w_i d(v_i, X_p)$ and $w_{(i)} d(v_{(i)}, X_p)$ in the above vectors by $d_i(X_p)$ and $d_{(i)}(X_p)$, respectively.

The p-facility ordered median problem (OMP) on $A(G)$ is defined as

$$OM_p(\lambda) \quad = \min_{X_p \subset A(G)} f_\lambda(X_p), \tag{1.1}$$

with

$$f_\lambda(X_p) := < \lambda, d_\leq(X_p) > = \sum_{i=1}^{n} \lambda_i d_{(i)}(X_p) \text{ and } \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n_{0+}.$$

Remark 1.1 The function $f_\lambda(X_p)$ is called the *Ordered Median function* (OMf). Note that the linear representation of this function is defined point-wise, since it changes when the order of the vector of distances is modified.

Remark 1.2 Different choices of λ in the OMf lead to different criteria: median, center, α-centdian, k-centrum, anti-k-centrum, (k_1, k_2)-trimmed, the convex and the concave cases. Specifically,

(1) The median criterion is a special case of the OMf with $\lambda = (1, 1, \cdots, 1)$, i.e., $f_\lambda(X_p) = \sum\limits_{i=1}^{n} d_i(X_p)$.

(2) The center criterion is a special case of the OMf with $\lambda = (0, \cdots, 0, 1)$, i.e., $f_\lambda(X_p) = \max\limits_{1\leq i\leq n} d_i(X_p)$.

(3) The α-centdian ($\alpha \in [0, 1]$, a convex combination of the median and center criteria) criterion is a special case of the OMf with $\lambda = (\alpha, \cdots, \alpha, 1)$, i.e., $f_\lambda(X_p) = \alpha \sum\limits_{i=1}^{n} d_i(X_p) + (1 - \alpha) \max\limits_{1\leq i\leq n} d_i(X_p)$.

(4) The k-centrum criterion is a special case of the OMf with $\lambda = (0, \cdots, 0, 1, \overset{k}{\cdots}, 1)$, i.e., $f_\lambda(X_p) = \sum\limits_{i=n-k+1}^{n} d_{(i)}(X_p)$.

(5) The anti-k-centrum criterion is a special case of the OMf with $\lambda = (1, \overset{k}{\cdots}, 1, 0, \cdots, 0)$, i.e., $f_\lambda(X_p) = \sum\limits_{i=1}^{k} d_{(i)}(X_p)$.

(6) The (k_1, k_2)-trimmed criterion is a special case of the OMf with
$\lambda = (0, \overset{k_1}{\cdots}, 0, 1, \cdots, 1, 0, \overset{k_2}{\cdots}, 0)$, i.e., $f_\lambda(X_p) = \sum_{i=k_1+1}^{n-k_2} d_{(i)}(X_p)$.

(7) The convex OMP criterion is a special case of the OMf with
$0 \leq \lambda_1 \leq \cdots \leq \lambda_n$.

(8) The concave OMP criterion is a special case of the OMf with
$\lambda_1 \geq \cdots \geq \lambda_n \geq 0$.

In this thesis we study location problems using the OMf and some of its special instances. In the next section, we introduce some research methodologies to study network location problems.

1.3 Research Methodologies

Network location problems are concerned with a general location model in which there are a number of demand points in a given network, such as transportation hubs, air and sea ports, retail outlets, and so on. These points are usually taken to be at the nodes of the network, and are to be served by facilities, which are to be located in the network. The facilities are usually isolated points, paths, trees, and so on, in the network. The objective is to locate the facilities in the network to attain some optimality, where the definition of optimality varies from problem to problem. However, the cost of satisfying the demand points depends, at least to some extent, on the distances between them and the facilities, which are measured by the shortest paths through the network (see Hooker, et al., 1991).

Since network location problems belong to combinatorial optimization problems, the techniques to deal with combinatorial optimization problems can be applied to solving network location problems. One most powerful approach to tackle location problems is to use the complexity theory to classify network location problems into polynomial solvable and NP-hard problems. The *complexity* of a problem is defined as the computational complexity of the best possible algorithm solving it. An algorithm is order $f(n)$ if there exists some constant a such that the number of basic operations required to run the algorithm for a problem of size n is less than or equal to $af(n)$ for all values of n. An algorithm is a polynomial time algorithm if its order $f(n)$ is a polynomial function of n.

A problem is polynomial solvable if it can be solved by a polynomial time algorithm, and no NP-hard problem can be solved by any known polynomial time algorithm. Most location problems in general networks are NP-hard, while location problems in special networks, such as tree networks, path networks, etc., are polynomial solvable. In this thesis we develop some polynomial time algorithms for location problems in tree networks, and use approximation algorithms to obtain near-optimal solutions for those in general networks with a reasonable performance guarantee.

Furthermore, to design efficient algorithms, we apply two methodologies to study network location problems with point-shaped facilities and connected facilities to locate, respectively. First, when the facilities to locate are isolated points, we mainly focus on the identification of a *finite dominating set* (FDS), a finite set of points to which some optimal solution for an OMP must belong. In the literature, there are several papers on the identification of an FDS. The seminal paper in this field was conducted by Hakimi (1964, 1965), who proved an FDS for the p-median problem. Several decades later, a distinguished research on FDS was provided by Hooker et al. (1991), who showed the characterizations of FDS for a number of location problems. Recently, much research focuses on obtaining an FDS for some instances of the OMP, as is evident in the studies by Nickel and Puerto (1999), Kalcsics et al. (2002), Kalcsics et al. (2003) and Puerto et al. (2005).

FDS is very useful in a variety of optimization problems, which enables one to restrict one's attention to a finite set of possible solutions. For instance, at the heart of the simplex algorithm for linear programming is the fact that the vertices of the feasible set form an FDS (see Hooker et al., 1991). Another classical example is the FDS result of the p-median problem, in which the nodes of the network comprise an FDS, as shown by Hakimi (1964, 1965). Furthermore, we can devise exact algorithms for network location models by taking advantage of a priori knowledge of an FDS when it is available. Moreover, obtaining an FDS allows the development of different types of algorithms to solve the problem concerned by enumerating a (finite) candidate set. In addition, we can establish the computational complexity of these algorithms, which depends strongly on the cardinality of the FDS.

Second, the facilities we want to locate are connected structures, e.g., trees or paths, and cannot be represented by isolated points in the space, which is motivated by specific decision problems related to routing and network design. There are already several studies on the location of connected structures (see Morgan and Slater, 1980; Hedetniemi et al., 1981; Slater, 1982; Minieka and Patel, 1983; Minieka, 1985; Becker and Perl, 1985; Tamir et al., 2002, Puerto and Tamir, 2005).

A natural idea to study this type of location problems is to link its optimal connected facility with one optimal point-shaped facility such that the possible optimal connected facility can grow at the point-shaped facility. This idea leads to the *nestedness property*, the property that for any optimal solution x to the point OMP, there exists an optimal subtree to the corresponding OMP including x. The nestedness property provides researchers with a powerful tool to develop some efficient algorithms.

The seminal study on nestedness property was introduced by Minieka (1985), who presented some efficient algorithms for optimally locating a path or a tree in a tree network. Recently, there are some studies on efficient algorithms for location problems in networks, especially in tree networks, using the nestedness property, as is evident in Wang (2000), Tamir et al. (2002), and Puerto and Tamir (2005). In this thesis we apply the nestedness property to study subtree OMP.

1.4 Organization of the Thesis

The thesis begins with an introduction on facility location in networks, including its classification, problem formulation and research methodologies. The remain chapters in this thesis are organized as follows:

Chapter 2 presents a literature review of network location models, in which we introduce the history and the development of the network location problems under study in this thesis.

Chapter 3 deals with the multi-facility ordered median problems in undirected networks, in which multiple isolated facilities are to be located. Multi-facility OMP in general networks are NP-hard, since the p-median and the p-center problems are special cases. In this chapter we use a finite dominating set (FDS) to study some special cases of the OMP in networks. FDS is a finite set of points to which some optimal solutions must belong, and is very useful for solving a variety of optimization problems, which enables one to restrict one's attention to a finite set of possible solutions. We first characterize an FDS for a special convex OMP in general networks, where the convex OMP is an important class in the OMP family. The FDS result generalizes some known results in the literature. Then, based on the FDS result, we obtain a polynomial size FDS and solve the problem confined to tree networks in polynomial time, which extends some results in the literature.

Chapter 4 is devoted to the multi-facility OMP in directed networks, since most of the networks in the real world are directed and not symmetric. For instance, routes are usually directed in a bus traffic system. In this chapter we also use the FDS to identify some possible solutions for a multi-facility OMP in a strongly connected directed network. We first prove that the OMP has an FDS in the node set, which generalizes the FDS result on the single-facility OMP in the literature. Furthermore, we show that the OMP can be solved efficiently based on the FDS result when the number of facilities is fixed and small. However, if the number of facilities is large, it is not reasonable for us to find an optimal solution, since the OMP in directed networks is NP-hard. Hence, instead of finding an optimal solution, we resort to some approximation algorithms for near-optimal solutions. In the end of Chapter 4, we present a $6\frac{2}{3}$-approximation algorithm for the p-median problem in directed networks.

Chapter 5 presents the subtree OMP in tree networks, in which the facilities to locate are not isolated points but connected structures (e.g., path, tree, etc.) These problems are motivated by specific decision problems related to routing and network design. In this chapter we use the nestedness property to investigate the subtree OMP in tree networks, where the nestedness property is the property that for any optimal solution x to the point OMP, there exists an optimal subtree to the corresponding OMP including x. The nestedness property provides researchers with a powerful tool to develop some efficient algorithms. First, we prove the nestedness property for a special convex OMP in tree networks. This finding extends some classical results concerning the nestedness property. Second, we solve the problem in polynomial time by applying the nestedness property result. Finally, we provide one counter example to show that the nestedness property cannot hold for the non-convex case.

Chapter 6, the last chapter, concludes the major findings of the thesis and suggests some directions for future research.

Chapter 2

Literature Review

Location science is a field of analytical study that can arguably be traced back to Pierre de Fermat, Evagelistica Torricelli, and Battista Cavallieri early in the seventeenth century (see Hale and Moberg, 2003). Facility location in networks is one of the most important and well-developed branches in location science, which is evident in numerous surveys and textbooks (see Mirchandani and Francis, 1990; Daskin, 1995; Drezner, 1995; Labbé et al., 1995; Puerto, 1996; Drezner and Hamacher, 2002). The seminal work on network location problems was introduced by Hakimi in 1964.

As introduced in Chapter 1, network location models differ in a variety of parameter indices, such as objective functions, shape requirements of the facilities to locate, the nature of the inputs and so on. For the sake of clarity, we focus on the development of two main clues: objective functions and shape requirements of the facilities to locate along with the application of the research methodologies.

From the objective function point of view, in location science, there exist two predominant objective functions: *minisum* (also called *median*) and *minimax* (also called *center*).

The p-median problem on a network was introduced by Hakimi (1964), which finds the location of p facilities to minimize the demand-weighted total distance between demand nodes and the facilities to which they are assigned. The p-median problem was later investigated by Hakimi (1965), Goldman (1971), Minieka (1977), Kariv and Hakimi (1979b). Hakimi's original studies proposed the well-known property that bears his name, the Hakimi property, which suggests that for the p-median problem on a network, at least one optimal solution must be included in the node set of the network. Moreover, some algorithms for 1-median problem on a network were presented. For instance, Goldman (1971) provided simple algorithms for locating a single facility for both an acyclic network and a network containing exactly one cycle. Another well-known, albeit trivial, algorithm for the 1-median problem on an acyclic network is known as the Chinese Algorithm (Francis, McGinns and White, 1992). In addition, Dearing et al. (1976) provided a thorough treatment of convexity on a network.

The p-center problem in a network was also first formulated by Hakimi (1964), which identifies the location of p facilities to minimize the maximum distance that demand is from its closest facility. This problem was later addressed by Hakimi (1965), Minieka (1970, 1977), Kariv and Hakimi (1979b), Elzinga and Hearn (1972), and Tansel et al. (1982), respectively. The p-center problem was proved to be NP-hard by Kariv and Hakimi (1979b). Another location model involving the maximum distance is a maximal covering location problem, first formulated by Church and ReVelle (1974). The maximal covering location problem focuses on the location of p facilities to maximize the number of covered demands. Note that all the demands in the maximal covering location problem are not necessarily severed by facilities. Church and Meadows (1979) provided a pseudo-Hakimi property for the maximal covering problem, which stated that there exists a finite augmented set of the nodes containing at least one optimal solution to the problem in any network.

Except for the above models, there exist other important location models. For instance, Halpern (1976) introduced the α-centdian as a parametric solution concept based on the bicriteria center/median model in a tree network. He established the corresponding trade-off with a convex combination of the unweighted center and weighted median objectives. Another important model, the k-centrum problem was first introduced by Slater (1978), which minimizes the sum of the k farthest weighted distances. The other objective functions, such as anti-k-centrum (the sum of the k closest weighted distances), (k_1, k_2)-trimmed (disregarding the k_1 closest and the k_2 farthest weighted distances), the concave (the λ-weights in a decreasing order) and the convex (the λ-weights in an increasing order) cases have also been studied within the location science community.

Although modern location theory is now more than 90 years old, and there is a vast collection of papers and books on this topic, a group of researchers realized that a common theory is still missing. Thus Nickel and Puerto (1999) introduced a unified approach − the ordered median function − to express most of the relevant location objectives, which started their work on a unified framework. In this thesis, we focus on network location problems using the ordered median function and its special instances.

Along with the development of location objectives, the shapes of the facilities to locate varies according to different practical requirements. Location theory was traditionally concerned with the optimal location of point-shaped facilities (single-facility or multi-facility) at either vertices or along arcs of a network. The seminal work in this area was due to Hakimi (1964, 1965).

Recently, there is more attention on the location of connected structures, which cannot be represented by isolated points in a network, motivated by concrete decision problems related to routing and network design. For instance, in order to improve the mobility of the population and reduce traffic congestion, many existing rapid transit networks are being updated by extending or adding lines. These lines can be viewed as new facilities. In fact, studies on the location of connected structures (which are called *extensive* facilities) already appeared in the early 1980's. Slater (1981, 1982) extended the network location theory to include a facility that is not merely a single point but a path. His work was confined to tree networks. A path in a tree network with the median criterion is defined as a *core*. A linear time algorithm for computing a core of a tree network was proposed by Morgan and Slater (1980).

Slater imposed no constraints on the length of the path selected as a core. Minieka and Pate (1983) first studied the problem of finding in a tree network a core of a specified length, exploring some properties of the problem and concluding that it is difficult to design an efficient algorithm for the problem. Later, Minieka (1985) extended the Minieka and Pate's study (1983) to consider the problems of optimally locating in a tree network a path and a tree of a specified length.

Peng and Lo (1994) proposed an efficient parallel algorithm for computing a core of a tree network on the EREW PRAM model, in which a PRAM consists of a collection of autonomous processors, each having access to a common memory and performing the same instruction at each step, and a memory location cannot be simultaneously accessed by more than one processor. They also proposed efficient parallel algorithms on the EREW PRAM model for optimally locating in a tree network a path and a tree of specified length with the center and median criteria (Peng and Lo, 1994; 1996).

Since the introduction of the ordered median function, there have been increasing studies dealing with a single facility, multi-facility, or connected facilities using this new objective function, reproving a lot of known results in an easier way and generating more insight into the geometrical structure of the optimal solution sets with respect to different criteria (see e.g., Tamir, 1996; Tamir et al. 1998, Nickel and Puerto, 1999; Tamir, 2001; Kalcsics et al., 2002, Tamir et al., 2002; Puerto and Tamir, 2005; Puerto and Rodríguez-Chía, 2005). In this thesis, based on the known findings on the above studies, we continue on network location problems with the ordered median function, using some methodologies introduced in Section 1.3.

Chapter 3

Multi-facility OMP in Undirected Networks

3.1 Introduction

In this chapter we use the FDS methodology to study the multi-facility ordered median problems (OMP) in undirected networks, in which the multiple isolated facilities are to be located. Study on network location problems with FDS application was first introduced by Hakimi (1964). Afterwards, much research related to network location problems started to focus on the identification of FDS. FDS is very useful in a variety of optimization problems, which enables one to restrict one's attention to a finite set of possible solutions. For instance, at the heart of the simplex algorithm for linear programming is the fact that the vertices of the feasible set form an FDS (see Hooker et al., 1993). Another classical example is the FDS result for the p-median problem, in which the nodes of the network comprise an FDS, as shown by Hakimi (1964, 1965). Furthermore, we can devise exact algorithms for network location models by taking advantage of a priori knowledge of an FDS when it is available. Moreover, obtaining an FDS allows the development of different types of algorithms to solve the problem concerned by enumerating a (finite) candidate set. In addition, we can establish the computational complexity of these algorithms, which depends strongly on the cardinality of the FDS.

Multi-facility OMP in general networks are NP-hard, since the p-median and the p-center problems are special instances of the OMP. First, we review some known results on the FDS for the OMP in the literature. Second, we characterize an FDS for a special convex OMP with the λ-weights defined in 3.1 and 3.2 in general networks, where the convex OMP is an important class in the OMP family. The FDS result generalizes some known results in the literature. Finally, based on the FDS result, we obtain a polynomial size FDS and solve the problem confined to tree networks in polynomial time.

To characterize an FDS in this thesis, we cite some related notation from Nickel and Puerto (2005) as follows:

Definition 3.1

(1). A point x on an edge $e = [v_i, v_j] \in E$ is called a *bottleneck point* of node v_k, if $w_k \neq 0$, and

$$d(x, v_k) = d(x, v_i) + d(v_i, v_k) = d(x, v_j) + d(v_j, v_k).$$

(2). BN_i denote the set of all the bottleneck points of a node $v_i \in V$.

(3). $BN := \bigcup_{i=1}^{n} BN_i$ is the set of all the bottleneck points of the network.

(4). $NBN := \bigcup_{\substack{i=1 \\ w_i<0}}^{n} BN_i$. A point in NBN is called a *negative bottleneck point.*

Definition 3.2

(1). For all $v_i, v_j \in V,\ i \neq j$, define

$$EQ'_{ij} := \{x \in A(G) : w_i d(v_i, x) = w_j d(v_j, x)\}.$$

(2). Let EQ_{ij} be the relative boundary of EQ'_{ij}, i.e., the set of end points of the closed subedges forming the elements in EQ'_{ij}; $EQ_{ij}^{kl} \subseteq EQ_{ij}$ be the equilibrium points of nodes $v_i,\ v_j$ on the edge $[v_k, v_l]$ for any $i, j \in \{1, \cdots, n\}$, and $k,\ l$ such that $[v_k, v_l] \in E$.

(3). $EQ := \bigcup_{\substack{i,j \\ i\neq j}}^{n} EQ_{ij}$. A point $x \in EQ$ is called an *equilibrium point* of G.

(4). Two points $a,\ b \in EQ$ are called *consecutive* if there is no other $c \in EQ$ on a shortest path between a and b.

Definition 3.3

(1). Define the set of ranges (canonical set of distances) $R \in \mathbb{R}_+$ by

$$R := \{r \in \mathbb{R}_+ | \exists x \in EQ_{ij} : d_i(x) = r = d_j(x)$$

$$\text{or } \exists v_i,\ v_j \in V, v_i \neq v_j : r = w_i d(v_i, v_j)\}.$$

Ranges correspond to the weighted distance values between equilibrium points and nodes.

(2). A point x is called an $r-extreme$ point or $pseudo-equilibrium$ with range $r \in R$ if there exists a node $v_i \in V$ with $r = w_i d(x, v_i)$.

(3). Denote by PEQ the set of all the pseudo-equilibria with respect to all the ranges $r \in R$, i.e.,

$$PEQ(r) = \{y \in A(G) | w_i d(v_i, y) = r, v_i \in V\}$$

with $r \in R,\ PEQ = \cup_{r \in R} PEQ(r)$.

Example 3.1

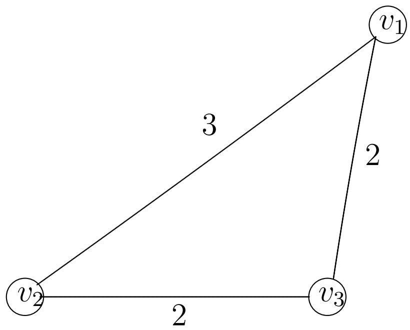

Figure 3.1. A cycle network of Example 3.1

Consider the cycle network in Figure 3.1 with the node set $V = \{v_1, v_2, v_3\}$ and the λ-weights: $w_1 = 1,\ w_2 = 2,\ w_3 = 2$. The distances between two nodes are shown in Figure 3.1. Based on the above definitions, we can compute the sets defined above as follows:

(1) The set of all the bottleneck points of node v_1 is $BN_1 = \{([v_2, v_3], 1/4)\}$. Similarly, $BN_2 = \{([v_1, v_3], 1/4)\}$, and $BN_3 = \{([v_1, v_2], 1/2)\}$. Hence, the set of the bottleneck points of the network in Figure 3.1 is $BN = \{([v_2, v_3], 1/4), [v_1, v_3], 1/4), ([v_1, v_2], 1/2)\}$.

(2) The set of the equilibrium points of nodes v_1 and v_2 is $EQ_{12} = EQ_{12}^{12} \cup EQ_{12}^{13} \cup EQ_{12}^{23}$. It is easy to compute that $EQ_{12}^{12} = \{([v_1, v_2], 2/3)\}$, $EQ_{12}^{13} = \emptyset$, and $EQ_{12}^{23} = \{([v_2, v_3], 2/3)\}$. Thus we obtain that $EQ_{12} = \{([v_1, v_2], 2/3), ([v_2, v_3], 2/3)\}$.

Similarly, we have that $EQ_{13} = \{([v_1, v_3], 2/3), ([v_2, v_3], 1/6)\}$, and $EQ_{23} = \{([v_1, v_2], 1/6), ([v_2, v_3], 1/2)\}$, which directly lead to the set EQ.
(3) According to the above EQ, we can compute the set of ranges

$$R = \{4/3, 2, 8/3, 3, 10/3, 4, 5, 6\}.$$

Then we can identify all the pseudo-equilibria with every $r \in R$. For instance, $([v_1, v_3], 2/3)$ is a pseudo-equilibrium with range 4/3.

In the following the FDS for some research on the single-facility or multi-facility OMP can be represented by a combination of the node set V, EQ, BN, NBN, and PEQ.

3.2 Previous Results

3.2.1 Single-facility OMP

The paper by Nickel and Puerto (1999) contained the first results concerning the FDS for some instances of the OMP. The first one is an FDS for the single-facility OMP.

Theorem 3.1 (Nickel and Puerto, 1999) An optimal solution for the single-facility OMP with non-negative λ-weights can always be found in the set $V \cup EQ$.

When the λ-weights can be negative, Kalcsics et al. (2002) provided a more general FDS below.

Theorem 3.2 (Kalcsics et al., 2002) The set $V \cup EQ \cup NBN$ is a finite dominating set for the single-facility OMP with general node weights.

Meanwhile, Kalcsics et al. (2002) also dealt with the case of directed networks with non-negative node weights as follows:

Theorem 3.3 (Kalcsics et al., 2002) The single-facility OMP on directed networks with non-negative node weights always has an optimal solution in the node-set V. If in addition $\lambda_1 > 0$ and $w_i > 0$, $\forall i = 1, \ldots, n$, then any optimal solution is in V.

3.2.2 Multi-facility OMP

In the case of multi-facility location, the first well-known result was introduced by Hakimi (1964) below.

Theorem 3.4 (Hakimi, 1964) There exists an optimal solution for the p-median problem in the node set V.

After the introduction of the OMf, Nickel et al. (2003) identified an FDS for the OMP with $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n \geq 0$, which generalizes Theorem 3.4 as follows:

Theorem 3.5 (Nickel et al., 2003) The p-facility OMP with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$ always has an optimal solution X_p^* contained in V.

Remark 3.1
As we know, the problem stated in Theorem 3.5 is also called the concave OMP. We have seen that V is an FDS for the concave multi-facility OMP, and $V \cup EQ$ is an FDS for the single-facility OMP. Then a natural question arises: Is $V \cup EQ$ also an FDS for the multi-facility OMP? The answer is negative. In the following we provide a counter example (see Figure 3.2).

Example 3.2

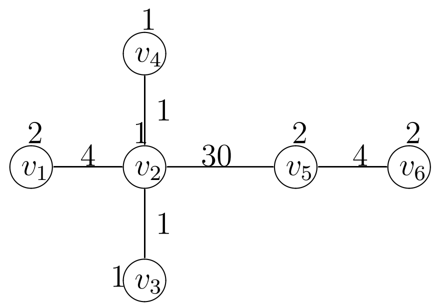

Figure 3.2. A tree $T = (V, E)$ of Example 3.2

Consider the 2-facility OMP with $\lambda = (1, 1, \ldots, 1, 2)$ in the tree network in Figure 3.2. If X_2 is restricted in $V \cup EQ$, the optimal solution is given by

$$X_2 = \{EQ_{13}^{12} = ([v_1, v_2], \frac{5}{12}), EQ_{56}^{56} = ([v_5, v_6], \frac{1}{2})\},$$

with the objective value $f_\lambda(X_2) = 24\frac{1}{3}$. If we disregard this restriction, then we have a better solution, i.e.,

$$X_2^* = \{x^* = ([v_1, v_2], \frac{1}{2}), EQ_{56}^{56} = ([v_5, v_6], \frac{1}{2})\},$$

with the objective function value of 24. Note that x^* is neither an equilibrium point nor a vertex.

Despite this negative result, Kalcsics et al. (2003) characterized a polynomial size FDS for an important class of the p-facility OMP. Let $1 \leq k < n,\ \lambda^k = (a, \ldots, a, b, \ldots, b) \in \mathbb{R}^n_{0+}$, where

$$a = \lambda_1 = \cdots = \lambda_k \neq \lambda_{k+1} = \cdots = \lambda_n = b.$$

Note that the λ-weights corresponding to the center, centdian or k-centrum problem are of this type.

Theorem 3.6 (Kalcsics et al., 2003) The p-facility OMP with non-negative node weights $p \geq 2$ and $\lambda^k \in \mathbb{R}^n_{0+}$, $1 \leq k \leq n-1$ always has an optimal solution $X_p^* \subseteq A(G)$ in the set PEQ. Moreover, $X_p^* \cap (V \cup EQ) \neq \emptyset$.

Theorem 3.6 not only proves the existence of an FDS, but also allows us to identify an FDS for any given problem. On the other hand, Puerto and Rodríguez-Chía (2005) proved that there is no polynomial size FDS for the multi-facility OMP even in path networks. Thus it is interesting to investigate the structure of the λ-weights such that there must exist a polynomial size FDS.

In the next section we characterize a polynomial size FDS for a special case of the convex p-facility OMP in general networks, in which the λ-weights are defined as follows:

$$\lambda_1 \geq 0, \qquad \lambda_{i+1} \geq \sum_{j=1}^{i} \lambda_j, \ (i \in \mathbb{Z}_+). \tag{3.1}$$

3.3 Special Convex OMP in Networks

Before we prove our theorem, we introduce some lemmas below.

Lemma 3.1 Let $G = (V, E)$ be an undirected network with non-negative node weights, $X_p = \{x_1, \cdots, x_p\} \subseteq A(G)$, $x = (e, t) \in X_p$ with $e \in E$ and $t \in [0, 1]$ an arbitrary solution point and $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n_{0+}$. Then there exists a point $x' = (e, t'),\ t' \in [0, 1]$, such that $f_\lambda(X'_p) \leq f_\lambda(X_p)$, where $X'_p = X_p \backslash \{x\} \cup \{x'\}$, and either

$$x' \in V \text{or } d_{(i)}(X'_p) = d_{(i+1)}(X'_p) \text{ for some } i \ (i \in \{1, \cdots, n-1\}) \tag{3.2}$$

holds.

Proof. Let $X_p = \{x_1, \cdots, x_p\} \subseteq A(G)$ with $x_l = (e_l, s_l), s_l \in [0,1], l = 1, \cdots, p$, such that X_p does not satisfy one of the relations in (3.2). Without loss of generality, let $x_1 = x \in A(G)$. Define $X_p(t) := \{x_1(t), x_2, \cdots, x_p\}$, where $x_1(t) := (e_1, s_1 + t), t \in [-s_1, 1 - s_1]$. In the following we prove the concavity of the objective function.

Let $T := [\underline{t}, \bar{t}]$ be an interval with $-s_1 \leq \underline{t} \leq 0 \leq \bar{t} \leq 1 - s_1$ such that $d_{(i)}(X_p(t)) < d_{(i+1)}(X_p(t))$ holds for all $i \in \{1, \cdots, n-1\}$ and all $t \in T$. This interval exists since $d_{(i)}(X_p(t)) < d_{(i+1)}(X_p(t))$ holds for $t = 0$ and all distance functions $d_i(\cdot)$ are continuous on an edge. Then we consider the following two cases.

Case 1. No re-allocation of the nodes $v_i = v_{(s)}$ $(s \in \{1, \cdots, n\})$ allocated to x_1 occurs with respect to $x_1(t)$ for all $t \in T_0 = [-s_1, 1 - s_1]$. We have $d_{(s)}(X_p(t)) = d_i(x_1(t))$ for all $t \in T_0$. Note that $d_i(x_1(t))$ is concave on e_1. For those nodes x_j allocated to $x_l \in X_p$ $(x_l \neq x_1)$, we have $d_{(t)}(X_p(t)) = d_j(x_l)$ $(t \neq s)$, which is constant. Thus we claim that $d_{(i)}(X_p(t))$ is a concave function of $t \in T_0$ $(i = 1, \cdots, n)$.

Case 2. If the re-allocation of nodes occurs, i.e., there exists at least a node $v_j \in V$ such that $d(v_j, x_1(t_0)) = d(v_j, x_l)$ $(x_l \neq x_1)$ for some $t_0 \in T_0$, then $x_1(t_0)$ is not the bottleneck point of this node on edge e_1, otherwise the re-allocation will not change with respect to $x_1(t)$, $t \in T_0$. Without loss of generality, assume that v_j is allocated to $x_1(t)$ for $t \leq 0$ and to $x_l, l \in \{2, \cdots, p\}$ for $t > 0$. Thus we have $d_j(X_p(t)) = d_j(x_1(t))$ for $t \leq 0$ on one edge e_1 and $d_j(X_p(t)) = d_j(x_l(t))$ for $t > 0$ on one edge e_l. In order to be re-allocated, the distance function of v_j on edge e_1, $d_j(x_1(t))$, has to be increasing for $t \leq 0$. After the change of allocations, we obtain $d_j(X_p(t)) = d_j(x_l)$ on e_l, which is constant with respect to t. Hence, for those nodes v_i re-allocated to other solution points with respect to $x_1(t)$, we claim that $d_i(X_p(t))$ is concave for $t \in T_0$. Moreover, the distance functions for the other nodes are independent of t. Hence, based on the assumption that $d_{(i)}(X_p(t)) < d_{(i+1)}(X_p(t))$ holds for all $i \in \{1, \cdots, n-1\}$ and all $t \in T$, we claim that the objective function $f_\lambda(X_p(t))$ is also concave in the interval $T \subseteq T_0$.

According to the definition of T and the above arguments, the concavity of $f_\lambda(X_p(t))$ holds during the process of extending T from $[\underline{t}, \overline{t}]$ to T_0 until $d_{(i)}(X_p(t)) = d_{(i+1)}(X_p(t))$ for some $i \in \{1, \cdots, n-1\}$ holds. Thus, without loss of generality, we may assume that the objective function is increasing for $t \in [-s_1, 0)$. Hence, we may decrease $\underline{t}$ until $x_1(t) \in V$ or $d_{(i)}(X_p(t)) = d_{(i+1)}(X_p(t))$ for some $i \in \{1, \cdots, n-1\}$ holds. $\square$

With Lemma 3.1 we can move an arbitrary solution point on its edge either to the left or to the right without increasing the objective function value until a point is attained for which (3.2) holds. Obviously, we can repeat this procedure for all the points in X_p. Hence we obtain the second lemma below.

Lemma 3.2 Let $X_p = \{x_1, \cdots, x_p\} \subseteq A(G)$, $p \geq 2$, be a solution for the p-facility OMP with all node weights equal to 1 and the λ-weights defined in (3.1). Then there exists a solution X'_p with $f_\lambda(X'_p) \leq f_\lambda(X_p)$ such that the following two cases must occur:
(1). At least one solution point of X'_p belongs to the set $V \cup EQ$;
(2). For each solution point in $X'_p \backslash (V \cup EQ)$, there exists another solution point in $X'_p \cap (V \cup EQ)$ and two nodes allocated to each of the two points such that the distances of these two nodes to their respective solution points are equal.

Proof. Let x_s ($s \in \{1, \cdots, p\}$) be the solution point related to the largest distance in $d_{\leq}(X_p)$. By Lemma 3.1, we can move x_s on its edge to the left or to the right without increasing the objective function value until $x_s(t) \in V$ or $d_{(i)}(X_p(t)) = d_{(i+1)}(X_p(t))$ holds for some $i \in \{1, \cdots, n-1\}$, where $X_p(t) = (X_p \backslash x_s) \cup \{x_s(t)\}$.

If $x_s(t) \in V$, we continue to identify the solution point related to the second largest distance in the updated inequality sequence. Otherwise, we have that $d_{(i)}(X_p(t)) = d_{(i+1)}(X_p(t)) = r_s$ holds for some $i \in \{1, \cdots, n-1\}$. In the following we consider two cases.

Case 1. At least two nodes, e.g., v_i and v_j, are allocated to $x_s(t)$ with the same distance r_s. In this case $x_s(t) \in EQ_{ij}$ is an equilibrium of the two nodes v_i and v_j with range r_s. As a result, $r_s \in R$ (R corresponds to the function values of equilibria or to the node-to-node distances), and those solution points with distance r_s are pseudo-equilibria with range r_s.

Case 2. Only one node, e.g., v_s, is allocated to $x_s(t)$ with distance r_s. According to the proof of Lemma 3.1, $x_s(t)$ is the solution point related to the largest element in the updated inequality sequence. We can continue to move $x_s(t)$ on its edge to the left or to the right with decreasing the objective function value until $x_s(t_0) \in V$ or at least two nodes, v_i and v_j, allocated to $x_s(t_0)$ with the same distance. This procedure is feasible as we claim below: Without loss of generality, assume that the largest distance in the updated inequality $d_{(n)}(x_s(t))$ decreases when we move x_s by a small δ on its edge to the left. Since $\lambda_1 \geq 0,\ \lambda_{i+1} \geq \sum_{j=1}^{i} \lambda_j,\ (i \in \mathbb{Z}_+)$ and all node weights are equal to 1, it is easy to check that the variation of the objective function, denoted as $Var(f)$, satisfies the following inequality

$$Var(f) \leq (\sum_{j=1}^{n-1} \lambda_j - \lambda_n)\delta \leq 0.$$

Hence, with the same argument as in Case 1, we obtain that $x_s(t_0) \in V$ or $x_s(t_0)$ is an equilibrium point.

Next we use the same procedure as above to deal with the solution point related to the largest distance in the updated inequality sequence, except those solution points that have been identified as PEQ points. Obviously, we can continue to move one solution point after another without increasing the objective function until the desired result follows. $\square$

As we know, the λ-weights in (3.1) has at most two elements with the same values. In fact, according to the above argument, it is easy to see that Lemma 2 also holds for the λ-weights taking at least two elements with the same values, i.e., $\lambda(k, m, t) = (k_1, \overset{m_1}{\cdots}, k_1, k_2, \overset{m_2}{\cdots}, k_2, \cdots, k_t \overset{m_t}{\cdots}, k_t) \in \mathbb{R}^n_{0+}$ $(m_i \in \mathbb{Z}_+, k_j \in \mathbb{R}_{0+}, t \geq 2, i, j = 1, \cdots, t)$ satisfying

$$k_1 \geq 0, \qquad k_{i+1} \geq \sum_{j=1}^{i} m_j k_j, \ (i \in \mathbb{Z}_+). \tag{3.3}$$

Hence, with the above argument and Lemma 3.2, we have the identification of the FDS as follows:

Theorem 3.7 The p-facility OMP with all node weights equal to 1, $p \geq 2$ and λ defined in (3.1) or (3.3) always has an optimal solution $X_p^* \subseteq A(G)$ in the set PEQ. Furthermore, $X_p^* \cap (V \cup EQ) \neq \emptyset$.

Proof. Let X_p $(p \geq 2)$ be an optimal solution. With Lemma 3.2 there exists another optimal solution X_p^* with $f_\lambda(X_p^*) = f_\lambda(X_p)$ such that $X_p^* \subseteq PEQ$ and $X_p^* \cap (V \cup EQ) \neq \emptyset$. $\square$

Remark 3.1

Since PEQ is an FDS for the p-facility OMP with the λ-weights defined in (3.1), as stated by Kalcsics et al. (2003), a natural question refers to the number of elements contained in the set PEQ. Denote $h = |EQ|$, then we have a range r for every equilibrium and every pair of nodes $u, v \in V, u \neq v$, which lead to $|R| = O(h + n^2)$ ranges. Since every distance function $d_i(\cdot)$ can assume a value $r \in R$ in at most two points on an edge $e \in E$, we obtain $O(nm)$ r-extreme points and hence $|PEQ| = O(nm(h+n^2))$. With the above results, it is possible to devise an algorithm to solve the ordered problem exactly.

Remark 3.2

The location model stated in Theorem 3.7 can be easily applied in the real world. For example, in the location of a distribution center perishable goods, in which the goal is for the longer distances and the total travel distance to be as small as possible, we might apply some penalty to a customer with the weighted distance in Position n more than the sum of the former $(n-1)$ penalties.

Remark 3.3

Since the λ-weights in Theorem 3.7 are a special one of the convex case, including a p-center case, our problem belongs to the set of the convex p-facility OMP.

In the following we solve the p-facility OMP stated in Theorem 3.7 on a general network. By Theorem 3.7, there always exists an optimal solution in which one of the points, e.g., x_p, is a node or an equilibrium point. According to the proof of Lemma 3.2, we see that all the other solution points are either nodes or pseudo-equilibria with respect to the range of the equilibrium or one of the ranges of the nodes. Therefore, we may first compute the set of equilibria

EQ, then the ranges R, and afterwards the pseudo-equilibria for every $r \in R$. The latter should be saved with reference to r in a set $PEQ[r]$.

Next, choose a candidate x_p from the set $V \cup EQ$. If $x_p \in EQ_{ij}$ is an equilibrium of range $r = d_i(x_p) = d_j(x_p)$, then the objective function value $f_\lambda(X_{p-1} \cup \{x_p\})$ is determined for all the $p-1$ subsets $X_{p-1} = \{x_1, \cdots, x_{p-1}\}$ of $V \cup PEQ[r]$. If $x_p = v$ is a node, then the set R_v can be obtained. Moreover, for all the subsets $X_{p-1} = \{x_1, \cdots, x_{p-1}\}$ of $V \cup \{PEQ[r] | r \in R_v\}$, the objective function value $f_\lambda(X_{p-1} \cup \{x_p\})$ should be determined. Thus we apply the algorithm introduced by Kalcsics et al. (2003) as follows:

Algorithm 3.1

Computation of an optimal solution set X_p^*

Input: Network $G = (V, E)$, distance-matrix $D, p \geq 2$, and a vector λ satisfying the conditions defined in (3.1) or (3.3)

Output: An optimal solution set X_p^*

1. Initialization

   Let $X_p^* := \emptyset, res := +\infty$.

2. First compute EQ, then the set of ranges R, and based on these sets, determine for every $r \in R$ the pseudo-equilibria and save them with reference to r in a set $PEQ[r]$.

3. **For** all equilibrium points in EQ **Do**

   Let $x_p \in EQ_{ij}$ and compute the range r of the equilibrium, i.e., $r := d_i(x_p) = d_j(x_p)$.

   for all $X_{p-1} = \{x_1, \cdots, x_{p-1}\} \subseteq V \cup PEQ[r]$ **do**

   Compute $f_\lambda(X_p)$, where $X_p := X_{p-1} \cup \{x_p\}$.

   if $f_\lambda(X_p) < res$ **then** $X_p^* := \{X_p\}, res := f_\lambda(X_p^*)$

4. **For** all $v_i \in V$ **Do**

   Let $x_p := v$ and compute the set R_v of all ranges of the node.

   for all $X_{p-1} = \{x_1, \cdots, x_{p-1}\} \subseteq V \cup \{PEQ[r] | r \in R_v\}$ **do**

   Compute $f_\lambda(X_p)$, where $X_p := X_{p-1} \cup \{x_p\}$.

   if $f_\lambda(X_p) < res$ **then** $X_p^* := \{X_p\}, res := f_\lambda(X_p^*)$

5. **Return** X_p^*.

As claimed by Kalcsics et al. (2003), the above algorithm has time complexity $O(pm^{p-1}n^p \log n(h+n^p))$. Note that the above problem is NP-hard, since one of its particular instances is p-center problems, which is NP-hard.

In the following we solve the p-facility OMP stated in Theorem 3.7 with the λ-weights defined as (3.3) in which $t = 3$ in polynomial time on a tree network, by adapting and modifying the algorithm provided by Kalcsics et al. (2003).

3.4 A Polynomial Algorithm for Multi-facility OMP

Before developing the algorithm, we introduce some related notation. Suppose that the given tree $T = (V, E), |V| = n$ and $|E| = n - 1$, is rooted at some distinguished node, say, v_1. For each pair of nodes v_i, v_j, we say that v_i is a *descendant* of v_j if v_j is on the unique path connecting v_i to the root v_1. If v_i is a descendant of v_j and v_i is connected to v_j with an edge, then v_i is a *child* of v_j and v_j is the unique *father* of v_i. If a node has no children, it is called a *leaf* of the tree. As shown in Tamir (1996), the original tree can be assumed to be a binary tree, where each non-leaf node v_j has exactly two children, $v_{j(1)}$ and $v_{j(2)}$.

The former is called the *left child*, and the latter, the *right child*. For each node v_j, V_j denotes the set of its descendants.

In the following we develop an algorithm, which consists of three phases. In the first phase, compute and augment an FDS into the node set of T. By Theorem 3.7, $Y = PEQ$ is an FDS for this problem and is of cardinality $O(n^4)$ since $|E| = n - 1$. This phase has complexity $O(n^4)$ by the procedure of Kim et al. (1996). Let T^a denote the augmented tree with the node set Y. Each point in Y is called a *seminode*. In particular, a node in V is also a seminode.

In the second phase, similar to the one in Tamir et al. (1998), for each node v_j, we compute and sort the distances from v_j to all seminodes in T^a. This sequence is denoted by $L_j = \{r_j^1, \cdots, r_j^m\}$, where $r_j^i \leq r_j^{i+1}, i = 1, \cdots, m-1$, and $r_j^1 = 0$. Without loss of generality, there is a one-to-one correspondence between the elements in L_j and the seminodes in Y (Tamir et al., 1998). The seminode corresponding to r_j^i is denoted by $y_j^i, i = 1, \cdots, m$. With the centroid decomposition approach given in Kim et al. (1996), the total computational effort of this phase is $O(n^6)$.

In the third phase, some functions are introduced as below. First, for each node v_j, an integer $q = 0, 1, \cdots, p, t_j^i \in L_j$, two integers $l_i = 0, 1, \cdots, m_i$, and c_i being a distance from any node to a seminode ($i = 2, 3, c_2 \leq c_3$), let $G(v_j, q, r_j^i, l_2, l_3, c_2, c_3)$ be the optimal value of the subproblem defined on the subtree T_j with the following conditions:

1. A total of at least one and at most q seminodes (service centers) can be selected in T_j, at least one of which is located in $\{y_j^1, \cdots, y_j^i\} \cap Y_j$;
2. There are exactly l_i vertices associated with k_i λ-weights, where

$l_i \leq \min\{m_i, |V_j|\}, i = 2, 3$, and V_j is the node set of T_j;

3. c_i are the minimum distances allowed for a distance with a k_i ($i = 2, 3$) λ-weights, respectively;

4. $G(v_j, 0, r, 0, 0, c_1, c_2) = +\infty$, and $G(v_j, q, r_j^i, l_2, l_3, c_2, c_3) = +\infty$ for any combination of parameters that leads to an infeasible configuration.

According to the definition, $G(v_j, q, r_j^i, l_2, l_3, c_2, c_3)$ is computed only for $q \leq |V_j|$, and if $l_i > 0$, then $c_i \leq \max\{w_k d(v_k, y)|v_k \in V_j \text{ and } y \in Y_j\},\ i = 2, 3$, where Y_j is the seminode set of the augmented subtree rooted at $y_j^1 = v_j$.

Similarly, for each node v_j, an integer $q = 0, 1, \cdots, p$, $r_j \in L_j$, two integers $l_i = 0, 1, \cdots, m_i$, and c_i ($i = 2, 3$) being a distance from any node to a seminode, define $F(v_j, q, r_j, l_2, l_3, c_2, c_3)$ as the optimal value of the subproblem defined in T_j satisfying the following conditions:

1. A total of q service centers can be located in T_j;

2. There are already some selected seminodes in $Y \backslash Y_j$ and the closest among them to v_j is at a distance r_j;

3. There are exactly l_i vertices associated with k_i λ-weights, where $l_i \leq \min\{m_i, |V_j|\}, i = 2, 3$;

4. c_i are the minimum distances allowed for a distance with a k_i ($i = 2, 3$) λ-weight, respectively.

Obviously, the function F is only computed for those r_j^i that correspond to $y_j^i \in Y \backslash Y_j$.

As in Kalcsics et al. (2003), the algorithm computes the function G and

F at all the leaves of T and then, recursively, proceeding from the leaves to the root, computes these functions at all nodes of T. Thus the optimal value of the problem is given by

$$\min_{\substack{c_2,c_3\\c_2\le c_3}} G(v_1,p,r_1^m,m_2,m_3,c_2,c_3),$$

where v_1 is the root of the tree.

Define

$$f_j(r,l_2,l_3,c_2,c_3)=\begin{cases} k_1r & \text{if } r<c_2,\\ k_2r & \text{if } c_2\le r\le c_3 \text{ and } 0<l_2<m_2,\\ k_3r & \text{if } r\ge c_3 \text{ and } 0<l_3<m_3,\\ +\infty & \text{otherwise,}\end{cases}$$

and

$$g_j(r,l_2,l_3,c_2,c_3)=\begin{cases} k_1r & \text{if } r<c_2,\\ k_2r & \text{if } c_2\le r\le c_3 \text{ and } l_2>0, l_3=0,\\ k_3r & \text{if } r\ge c_3 \text{ and } l_2=0, l_3>0,\\ +\infty & \text{otherwise.}\end{cases}$$

Let v_j be a leaf of T. Then,

$$\begin{aligned} &G(v_j,0,r_j^i,0,0,c_2,c_3)=+\infty, \quad i=1,\cdots,m;\ c_i\ne 0,\\ &G(v_j,1,r_j^i,0,0,c_2,c_3)=0, \qquad c_i\ne 0,\\ &G(v_j,1,r_j^i,l_2,l_3,c_2,c_3)=+\infty \quad \text{otherwise.}\end{aligned}$$

For each $i=1,\cdots,m$ such that $y_j^i\in Y\backslash Y_j$,

$$F_j(v_j,0,r_j^i,0,0,c_1,c_2)=\begin{cases} k_1r & \text{if } r_j^i<c_2,\\ +\infty & \text{if } r_j^i\ge c_2,\end{cases}$$

$$F_j(v_j, 0, r_j^i, 1, 0, c_1, c_2) = \begin{cases} k_2 r & \text{if } c_2 \leq r_j^i \leq c_3, \\ +\infty & \text{otherwise}, \end{cases}$$

$$F_j(v_j, 0, r_j^i, 0, 1, c_1, c_2) = \begin{cases} k_3 r & \text{if } r_j^i \geq c_2, \\ +\infty & \text{otherwise}, \end{cases}$$

$$F(v_j, 1, r_j^i, 0, 0, c_2, c_3) = 0,$$
$$F(v_j, 1, r_j^i, l_1, l_2, c_2, c_3) = +\infty.$$

Let v_j be a non-leaf node in V, and let $v_{j(1)}$ and $v_{j(2)}$ be its left child and right child, respectively. The r_j^1 corresponds to v_j. In addition, it corresponds to a pair of elements, say $r_{j(1)}^k \in L_{j(1)}$ and $r_{j(2)}^t \in L_{j(2)}$, respectively. Then denote

$$\begin{aligned} G(r_{j(1)}^k, r_{j(2)}^t) &= \min_{\substack{q_1+q_2=(q-1)^+ \\ l_{21}+l_{22}+l_{31}+l_{32}=l}} \Big\{ F(v_{j(1)}, q_1, r_{j(1)}^k, l_{21}, l_{31}, c_2, c_3) \\ &\quad + F(v_{j(2)}, q_2, r_{j(2)}^t, l_{22}, l_{32}, c_2, c_3) \Big\}; \end{aligned}$$

$$\begin{aligned} G(r_{j(1)}^i, r_{j(2)}^t) &= \min_{\substack{q_1+q_2=(q-2)^+ \\ l_{21}+l_{22}+l_{31}+l_{32}=l \\ y_{j(1)}^i \in (v_{j(1)}, v_j)}} \Big\{ F(v_{j(1)}, q_1, r_{j(1)}^i, l_{21}, l_{31}, c_2, c_3) \\ &\quad + F(v_{j(2)}, q_2, r_{j(2)}^t, l_{22}, l_{32}, c_2, c_3) \Big\}; \end{aligned}$$

$$\begin{aligned} G(r_{j(1)}^k, r_{j(2)}^i) &= \min_{\substack{q_1+q_2=(q-2)^+ \\ l_{21}+l_{22}+l_{31}+l_{32}=l \\ y_{j(2)}^i \in (v_{j(2)}, v_j)}} \Big\{ F(v_{j(1)}, q_1, r_{j(1)}^k, l_{21}, l_{31}, c_2, c_3) \\ &\quad + F(v_{j(2)}, q_2, r_{j(2)}^i, l_{22}, l_{32}, c_2, c_3) \Big\}, \end{aligned}$$

where for any number a, we denote by $a^+ = \max(0, a)$.

Thus we have

$$G(v_j, q, r_j^1, l_2, l_3, c_2, c_3) = \min \Big\{ G(r_{j(1)}^k, r_{j(2)}^t) + G(r_{j(1)}^i, r_{j(2)}^t) + G(r_{j(1)}^k, r_{j(2)}^i) \Big\}.$$

For $i = 2, \cdots, m$, consider r_j^i. If $y_j^i \in Y \backslash Y_j$, then $G(v_j, q, r_j^i, l_2, l_3, c_2, c_3) = G(v_j, q, r_j^{i-1}, l_2, l_3, c_2, c_3)$. If $y_j^i \in Y_{j(1)}$, then it corresponds to $r_{j(1)}^k \in L_{j(1)}$ and to $r_{j(2)}^t \in L_{j(2)}$. In this case, we can compute G in the following way. For the sake of clarity, denote

$$
\begin{aligned}
& H_1(v_j, q, r_j^i, l_2, l_3, c_2, c_3) \\
= & \min_{\substack{q_1+q_2=q \\ 1\le q_1 \le |V_{j(1)}| \\ q_2 \le |V_{j(2)}| \\ l_{21}+l_{22}+l_{31}+l_{32}=\begin{cases} l & \text{if } r_j^i < c_2 \\ (l-1)^+ & \text{if } r_j^i \ge c_2, \end{cases} \\ l_{hi} \le \{l_h, |V_{j(i)}|\},\ h=2,3,\ i=1,2}} \Big\{ G(v_{j(1)}, q_1, r_{j(1)}^k, l_{21}, l_{31}, c_2, c_3) \\
& \qquad + F(v_{j(2)}, q_2, r_{j(2)}^t, l_{22}, l_{32}, c_2, c_3) \Big\}.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
& G(v_j, q, r_j^i, l_2, l_3, c_2, c_3) \\
= \ & \min \Big\{ G(v_j, q, r_j^{(i-1)}, l_2, l_3, c_2, c_3); \\
& \quad g_j(r_j^i, l_2, l_3, c_2, c_3) + H_1(v_j, q, r_j^i, l_2, l_3, c_2, c_3) \Big\}.
\end{aligned}
$$

If $y_j^i \in (v_j, v_{j(1)})$ and $y_j^i \neq v_{j(1)}$, then define

$$
\begin{aligned}
& H_2(v_j, q, r_j^i, l_2, l_3, c_2, c_3) \\
= & \min_{\substack{q_1+q_2=q-1 \\ 1\le q_1 \le |V_{j(1)}| \\ q_2 \le |V_{j(2)}| \\ l_{21}+l_{22}+l_{31}+l_{32}=\begin{cases} l & \text{if } r_j^i < c_2 \\ (l-1)^+ & \text{if } r_j^i \ge c_2, \end{cases} \\ l_{hi} \le \{l_h, |V_{j(i)}|\},\ h=2,3,\ i=1,2}} \Big\{ F(v_{j(1)}, q_1, r_{j(1)}^k, l_{21}, l_{31}, c_2, c_3) \\
& \qquad + F(v_{j(2)}, q_2, r_{j(2)}^t, l_{22}, l_{32}, c_2, c_3) \Big\}.
\end{aligned}
$$

Hence we obtain

$$\begin{aligned}
& G(v_j, q, r_j^i, l_2, l_3, c_2, c_3) \\
= \ & \min\Big\{G(v_j, q, r_j^{(i-1)}, l_2, l_3, c_2, c_3); g_j(r_j^i, l_2, l_3, c_2, c_3) + H_2(v_j, q, r_j^i, l_2, l_3, c_2, c_3)\Big\}.
\end{aligned}$$

Analogous formulas can be derived for $y_j^i \in Y_{j(2)}$ with obvious changes.

Once the function G is obtained, we compute the function F. Let y_j^i be a seminode in $Y \backslash Y_j$. Thus, y_j^i corresponds to some elements, say $r_{j(1)}^k \in L_{j(1)}$ and $r_{j(2)}^t \in L_{j(2)}$. Similarly, define

$$\begin{aligned}
& H_3(v_j, q, r_j^i, l_2, l_3, c_2, c_3) \\
= \ & \min_{\substack{q_1+q_2=q \\ q_1 \le |V_{j(1)}| \\ q_2 \le |V_{j(2)}| \\ l_{21}+l_{22}+l_{31}+l_{32} = \begin{cases} l & \text{if } r_j^i < c_2 \\ (l-1)^+ & \text{if } r_j^i \ge c_2, \end{cases} \\ l_{hi} \le \{l_h, |V_{j(i)}|\},\ h=2,3,\ i=1,2}} \Big\{F(v_{j(1)}, q_1, r_{j(1)}^k, l_{21}, l_{31}, c_2, c_3) \\
& \qquad + F(v_{j(2)}, q_2, r_{j(2)}^t, l_{22}, l_{32}, c_2, c_3)\Big\}.
\end{aligned}$$

Hence, we also have

$$\begin{aligned}
& F(v_j, q, r_j^i, l_2, l_3, c_2, c_3) \\
= \ & \min\Big\{G(v_j, q, r_j^i, l_2, l_3, c_2, c_3); f_j(r_j^i, l_2, l_3, c_2, c_3) + H_3(v_j, q, r_j^i, l_2, l_3, c_2, c_3)\Big\}.
\end{aligned}$$

Complexity As claimed by Kalcsics et al. (2003), the computational effort required to evaluate the functions G and F depends on the cardinality of the FDS for this problem. According to Theorem 3.7, PEQ is an FDS with cardinality $O(n^4)$. Therefore, it follows directly from the recursive equations that the effort to compute the function G at a given node v_j, for all relevant values of q, r, l_2, l_3 and c_2, c_3, is $O(p^2(n^4)m_2^2m_3^2n^2(n^4)^2) = O(m_2^2m_3^2p^2n^{14})$.

In the following we give one example to show the procedure of the above polynomial algorithm.

Example 3.3

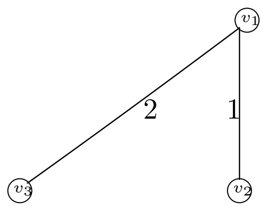

Figure 3.3 A tree network T

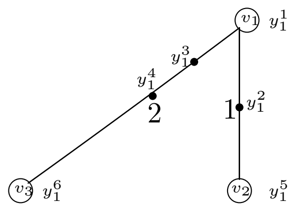

Figure 3.4 An augmented tree T^a

In the tree network T rooted at v_1 (see Figure 3.3) with the node set $V = \{v_1, v_2, v_3\}$ and all the node-weights equal to 1. We consider the 2-facility OMP with $\lambda = (1, 1, 2)$.

According to the above algorithm and the definition of PEQ, we first obtain that $PEQ = \{v_1, v_2, v_3, ([v_1, v_2], 1/2), ([v_1, v_3], 1/4), ([v_1, v_3], 1/2)\}$, and T^a denotes the augmented tree with the node set PEQ. Second, we compute and sort the distances from v_1 to all the seminodes in T^a as the sequence $L_1 = (r_1^1, r_1^2, r_1^3, r_1^4, r_1^5, r_1^6) = (0, 1/2, 1/2, 1, 1, 2)$, and the corresponding seminodes are denoted by $y_1^1, \cdots, y_1^6$, respectively (see Figure 3.4). Third, let C be the set of distances from any node to a seminode, and then we have $C = \{0, 1/2, 1, 3/2, 2, 5/2, 3\}$. Thus an optimal value of the problem is given as follows:

$$\min_{\substack{c_2, c_3 \in C \\ c_2 \leq c_3}} G(v_1, 2, r_1^6, 1, 1, c_2, c_3). \tag{3.4}$$

It follows that the optimal solution of (3.4) can be obtained by computing the functions F and G from its leaves, v_2 and v_3, to the root v_1. It is not difficult to check that v_3 and $([v_1, v_2], 1/2)$ are optimal locations of the two facilities with the optimal value $5/2$.

3.5 Conclusions

In this chapter we identified a polynomial size FDS for the multi-facility ordered median problem on networks, in which the set of λ parameters can take at least two different values. This FDS result not only includes the FDS research for the p-center problem, but also extends the case provided by Kalcsics et al. (2003) to some extent. Furthermore, we gave a polynomial time algorithm for the problem with at most three different values in the λ-weights on tree networks.

Although we have characterized a polynomial size FDS for a special case of the convex p-ordered median problem, the identification of a polynomial size FDS for the convex p-ordered median problem needs to be further investigated.

Furthermore, Puerto and Rodríguez-Chía (2005) proved that there is no FDS of polynomial size for the p-facility OMP with general λ-weights by constructing a path network with the λ-weights including two same elements. Thus another challenging research direction is to identify the characteristics of the λ-weights such that there exists a polynomial size FDS for the multi-facility OMP.

Chapter 4

Multi-facility OMP in Directed Networks

4.1 Introduction

Chapter 3 investigated the multi-facility OMP in undirected networks. In this chapter we focus on the multi-facility OMP in directed networks, since most of the networks in the real world are directed and not symmetric (undirected networks can be viewed as symmetric directed networks). For instance, routes are usually directed in a bus traffic system. In this chapter we also use the FDS to identify some possible solutions for a multi-facility OMP in a strongly connected directed network.

In the literature there are some known results identifying FDS for different instances of the multi-facility ordered median problem. For instance, Kalcsics et al. (2003) proved that the set of pseudo-equilibria (PEQ) is an FDS for the

p-facility OMP where the λ-weights take at most two different values. Puerto et al. (2005) showed that the set $F = ((V \cup EQ) \times PEQ) \cup T \subset A(G) \times A(G)$ is an FDS for the 2-facility OMP in any network for any choice of non-negative λ-weights. However, so far there is no study focusing on the multi-facility OMP in directed networks. In this chapter we show that the node set is an FDS of the p-facility OMP with non-negative λ-weights in a strongly connected directed network. Based on this result we design an exact algorithm and a constant-factor approximation algorithm for the p-median problem, respectively.

The multi-facility ordered median problem in a strongly connected directed network is defined similarly to that in Kalcsics et al. (2002) as follows: Let $N_D = (V, E)$ be a directed network with the node set $V = \{v_1, \cdots, v_n\}$ and the arc set $E = \{e_1, \cdots, e_m\}$. Each node v_i is associated with a non-negative weight w_i. Each arc $e_j = [v_s, v_t]$ is the edge assigned by a positive length $l(e_j)$, directed from v_s to v_t, and is assumed to be rectifiable, where v_s is called the *head* of e_j, and v_t is called the *tail* of e_j. A point x on an arc $e = [v_i, v_j]$ can be written as a pair $x = (e, t),\ t \in [0, 1]$, where the distance from v_i to x is $tl(e)$. We assume that N_D is embedded in the Euclidean plane. Let $A(N_D)$ be the continuum set of points on the arcs of N_D. For any $x, y \in A(N_D)$, a directed path $\overrightarrow{P}(x, y)$ is a path directed from x to y, and the distance from x to y, $\overrightarrow{d}(x, y)$, is the length of the shortest directed path $\overrightarrow{P}$ from x to y, i.e., the sum of the lengths of the partial arcs in $\overrightarrow{P}$. A directed network N_D is *strongly connected* if there exists a directed path from each vertex to another vertex. The distance between $v_i \in V$ and a point $x_j \in A(N_D)$ is denoted as $\overline{d}_j(v_i) = \overline{d}(x_j, v_i) = w_i(\overrightarrow{d}(x_j, v_i) + \overrightarrow{d}(v_i, x_j))$, and $\overline{d}_j(\cdot)$ is viewed as a distance function of $A(N_D)$ from the facility x_j. Let $X_p = \{x_1, \cdots, x_p\} \subset A(N_D)$ $(p \geq 2)$ be a set of p facilities, and the distance from a node $v_i \in V$ to the set X_p is defined as $\overline{d}(v_i, X_p) = \min_{k=1,\cdots,p} \overline{d}(v_i, x_k)$. We denote $\overline{d}_i(X_p) = \overline{d}(v_i, X_p)$ for $i = 1, \cdots, n$, $\overline{d}(X_p) = (\overline{d}_i(X_p), \cdots, \overline{d}_n(X_p))$, and $\overline{d}_{\leq}(X_p) = (\overline{d}_{(1)}(X_p), \cdots, \overline{d}_{(n)}(X_p))$ satisfying $\overline{d}_{(1)}(X_p) \leq \cdots \leq \overline{d}_{(n)}(X_p)$.

The p-facility ordered median problem (OMP) in a strongly connected directed network is defined as

$$OM_p(\lambda) = \min_{X_p \subset A(N_D)} f_\lambda(X_p) = \min_{X_p \subset A(N_D)} \sum_{i=1}^{n} \lambda_i \overline{d}_{(i)}(X_p),$$

where $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n_{0+}$.

4.2 FDS for Multi-facility OMP

Before we derive an FDS for the p-facility ordered median problem in a strongly connected directed network N_D, we present some lemmas below. First, we cite a result from Nickel and Puerto (2005) as follows:

Lemma 4.1 Let $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ be two vectors in $\mathbb{R}^n$. Suppose that $x \leq y$, then

$$x_{\leq} = (x_{(1)}, \cdots, x_{(n)}) \leq y_{\leq} = (y_{(1)}, \cdots, y_{(n)}).$$

Second, we follow one observation in Kalcsics et al. (2002) below.

Lemma 4.2 Let $e = [v_i, v_j] \in E$ be an arc of the directed network N_D, and x a point in the interior (v_i, v_j) of the arc e. Then for a node $v_k \in V$, the distance function $\overline{d}_k(\cdot)$ is constant on the interior (v_i, v_j), and $\overline{d}_k(v_i), \overline{d}_k(v_j) \leq \overline{d}_k(x)$.

From Lemma 4.2, it follows that we can move an arbitrary point located in the interior of one arc to its head or its tail without increasing the values of the elements in the distance $\overline{d}(X_p)$. This leads to the following lemma.

Lemma 4.3 Let $N_D = (V, E)$ be a strongly connected directed network with non-negative node weights, $X_p = \{x_1, \cdots, x_p\} \subseteq A(G)$ $(p \geq 2)$, $x = (e, t) \in X_p$ with $e = [v_i, v_j]$ and $t \in (0, 1)$ an arbitrary solution point in (v_i, v_j), and $X'_p = X_p \backslash \{x\} \cup \{v_i\}$ or $X_p \backslash \{x\} \cup \{v_j\}$. Then we have $\overline{d}(X'_p) \leq \overline{d}(X_p)$.

Obviously, based on Lemma 4.3, we can move those points in X_p, that are not located in the vertices of the network to their heads or tails, respectively. This leads to the following corollary.

Corollary 4.1 Let $X_p = \{x_1, \cdots, x_p\} \subseteq A(N_D)$, $p \geq 2$, be a solution for the p-facility ordered median problem with non-negative node weights. Then there exists another solution $X'_p \subseteq V$ with $\overline{d}(X'_p) \leq \overline{d}(X_p)$.

Proof. Assume that $X_p \nsubseteq V$. Then, according to Lemma 4.3, we start by moving one solution point after the other until we obtain a new solution X'_p such that $X'_p \subseteq V$ and $\overline{d}(X'_p) \leq \overline{d}(X_p)$. □

Thus we can derive a finite dominating set for the p-facility ordered median problem in a directed network below.

Theorem 4.1 The p-facility ordered median problem $(p \geq 2)$ in a strongly connected directed network, with non-negative node weights, has an optimal solution in the node set V.

Proof. Let X_p $(p \geq 2)$ be an optimal solution such that $X_p \nsubseteq V$. By Corollary 4.1, we claim that there exists another solution $X'_p = \{x_1, \cdots, x_p\} \subseteq V$ such that $\overline{d}(X'_p) \leq \overline{d}(X_p)$. Then we aim to prove that X'_p is also optimal, which is equivalent to prove that $f_\lambda(X'_p) \leq f_\lambda(X_p)$. According to Lemma 4.1, we have that $\overline{d}_{\leq}(X_p') \leq \overline{d}_{\leq}(X_p)$, which completes the proof since the objective function $f_\lambda(\cdot)$ is the scalar product of the non-negative λ-weights and the sorted weighted distances $\overline{d}_{\leq}(\cdot)$. □

In the next section, with the above FDS results, we develop an exact algorithm to solve OMP in a strongly connected directed network by evaluating the objective function at p points of the node set.

4.3 An Exact Algorithm for Multi-facility OMP

Algorithm 4.1

Computation of an optimal solution set X_p^*

Input: Strongly connected directed network $N_D = (V, E)$, distance-matrix $D, p \geq 2$, and non-negative λ-weights

Output: An optimal solution set X_p^*

1. Initialization

   Let $X_p^* := \emptyset, res := +\infty$

2. **For** all $v_i \in V$ **Do**

   Let $x_p := v_i$

   for all $X_{p-1} = \{x_1, \cdots, x_{p-1}\} \subseteq V$ **do**

   Compute $f_\lambda(X_p)$, where $X_p := X_{p-1} \cup \{x_p\}$.

   if $f_\lambda(X_p) < res$ **then** $X_p^* := \{X_p\}, res := f_\lambda(X_p^*)$

3. **Return** X_p^*.

In Step 2 we have to evaluate the objective function for a node for all the subsets of size $p-1$ of the node set V. Since we have $O(\binom{n}{p-1}) = O(n)^{p-1}$ different subsets and the evaluation of the objective function takes $O(pn \log n)$ time using a line-sweep algorithm (Bentley and Ottmann, 1979), the above algorithm has a total time complexity of $O(pn^{p+1} \log n)$.

Note that the above problem is easily solved based on the FDS result when p is fixed and small. In the following we provide one example to show how to solve the OMP efficiently with the FDS result.

Example 4.1

$N_D = (V, E)$ is a strongly connected directed network with six nodes (see Figure 4.1). Consider the 2-facility OMP with $\lambda = (1, 1, 2, 2, 3, 3)$ in which two facilities need to be located. The node weights are $w_1 = w_3 = w_4 = w_5 = 1$ and $w_2 = w_6 = 2$, and the distances between nodes are shown in Figure 4.1.

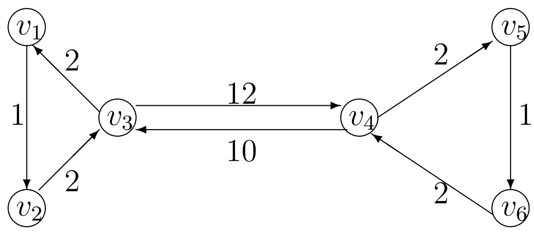

Figure 4.1. A strongly connected directed network of Example 4.1

Based on the above FDS result, the optimal location of the two facilities must be in the node set $\{v_1, \cdots, v_6\}$. Furthermore, according to the distance distribution in Figure 4.1, the one optimal facility must be in $\{v_1, v_2, v_3\}$, and the other optimal facility must be in $\{v_4, v_5, v_6\}$. It is easy to prove that $\{v_2, v_6\}$ is an optimal solution.

As shown in Example 4.1, Algorithm 4.1 is efficient when the size of p is fixed and small. However, when the numbers of demand points and facilities increase, the complexity grows in exponential time. Moreover, note that the p-center problem and the p-median problem are NP-hard, and these two problems are special instances of the OMP, the problem stated in Theorem 4.1 is also NP-hard. Due to this reason, efforts to obtain an optimal solution in an efficient manner becomes impossible. Thus we may resort to approximation algorithms to obtain near-optimal solutions with a reasonable relative error.

In the next section, based on the above FDS result, we present a constant-factor approximation algorithm for the multi-facility OMP with $\lambda = (1, \ldots, 1)$ confined to a directed network.

4.4 Approximation algorithms for the OMP

Currently, constructing approximation algorithms is one of the most successful approaches to treat NP-hard optimization problems. Since the introduction of the concept of NP-hardness, which is viewed as a concept for proving the intractability of optimization problems, increasing attention has been paid to the following question (see Hromkovič, 2003): If an optimization problem does not admit any efficient algorithm computing an optimal solution, is there a possibility of efficiently computing at least an approximation of the optimal solution? The answer is affirmative, which was already given in the middle of the 1970s. It is of great practical importance if we can reduce from exponential complexity to polynomial complexity when a small change is given on the conditions and the cost of a solution differs from the cost of an optimal solution by at most $\varepsilon\%$ of the cost of an optimal solution for some $\varepsilon > 0$. Hence, for NP-hard problems, it is more practical to investigate whether there exists a polynomial-time approximation algorithm that solves them with a reasonable relative error.

Recently there are already some studies on approximation algorithms for the p-median problem, a special instance of the OMP. For instance, Charikar and Guha (1999) provided a 4-approximation algorithm for the metric p-median problem in $O(n^2(L+n)\log n)$ time, where L is the number of bits needed to represent the metric distance. Jain and Vzairani (1999) provided an approximation algorithm with an approximation guarantee of 6 for the metric p-median problem. Thorup (2001) presented a $12+o(1)$ constant factor approximation algorithm for the p-median problem. Jain et al. (2002) again presented a lower bound on the approximation guarantee of the metric p-median problem, showing that it may not be approximated with a factor strictly smaller than $1+2/\epsilon$.

In this section, we give a constant-factor approximation algorithm for a special instance of the multi-facility OMP, the p-median problem, in directed networks. First, we introduce the concept of an approximation algorithm as follows:

Definition 4.1 Given a minimization problem, an algorithm is said to be a (polynomial) *ρ-approximation algorithm*, if for any instance of the problem, the algorithm runs in polynomial time and outputs a solution that has a cost at most $\rho \geq 1$ times the minimum cost, where ρ is called the *performance guarantee* or the *approximation ratio* of the algorithm.

In general, the minimum cost is unknown. Hence, in order to obtain a performance guarantee of an algorithm, we have to compare the cost of the solution produced by the algorithm to a lower bound of the minimum cost. Thus we resort to linear programming relaxations, which can provide good lower bounds for the minimum cost.

In the following we apply the approximation algorithm for the metric p-median problem from Charikar et al. (2002) to the p-median problem in directed networks, a special case of the multi-facility OMP. Note that there is something different between the p-median problem in networks and the metric p-median problem in Charikar et al. (2002). Specifically, the facilities in the former model can be located anywhere in the given network, while the facilities in the latter model are only limited to a finite set of points.

Fortunately, for the OMP in a strongly connected directed network, we have proven that there must exist at least one optimal solution in the node set. Therefore, we can reduce the p-median problem in directed networks to the following location model: In a given strongly connected directed network $N_D = (V, E)$, the problem is to select p nodes as servers or facilities and assign each demand in the node set V to its nearest server, so as to minimize the p-median objective value.

Hence the problem can be described as the following integer linear program, which is modified from Charikar et al. (2002):

$$\text{Minimize} \qquad \sum_{i,j \in N} w_j \bar{d}_{ij} x_{ij} \tag{4.1}$$

subject to

$$\sum_{i \in N} x_{ij} = 1, \qquad \text{for each } j \in N, \tag{4.2}$$

$$x_{ij} \leq y_i, \qquad \text{for each } i, j \in N, \tag{4.3}$$

$$\sum_{i \in N} y_i \leq p, \tag{4.4}$$

$$x_{ij} \in \{0, 1\} \qquad \text{for each } i, j \in N, \tag{4.5}$$

$$y_i \in \{0, 1\} \qquad \text{for each } i \in N, \tag{4.6}$$

where $N = \{1, \cdots, n\}$.

In the above integer linear program model, there are several variables: w_j represents the demand for node v_j; $\bar{d}_{ij}$ represents the distance cost between v_i and v_j constraints; y_i is a 0-1 variable, which indicates if the location at v_i is selected as a facility; x_{ij} is a 0-1 variable, which indicates if demand node v_j is assigned to the facility at v_i. The constraints (4.2) make sure that each demand node v_j is assigned to some facility at v_i, the constraints (4.3) make sure that a facility must have been open at v_i, whenever a demand node v_j is assigned to a facility at v_i, and constraints (4.4) ensures that at most p facilities can be located in the node set V.

Then we convert the integer linear program (4.1)-(4.6) to a linear programming relaxation, by replacing the 0-1 constraints (4.5) and (4.6), respectively, by

$$x_{ij} \geq 0, \qquad \text{for each } i, j \in N, \tag{4.7}$$

$$y_i \geq 0, \qquad \text{for each } i \in N. \tag{4.8}$$

Let $(\bar{x}, \bar{y})$ denote a feasible solution to the LP relaxation and $\bar{f}_{LP}$ denote its objective function value. Thus we can apply the three-step algorithm introduced by Charikar et al. (2002) to solve the p-median problem in directed networks as follows:

Step 1. First, simplify the problem instance by consolidating nearby locations. Do not change the linear programming solution $(\bar{x}, \bar{y})$ but modify only the demands. Let $w' = \{w'_1, \cdots, w'_n\}$ denote the new set of demands, then the following property holds: each feasible integer solution for the modified instance with demands $w'_j,\ j \in N$, can be converted to a feasible integer solution for the original instance, at an added cost of at most $4\bar{f}_{LP}$.

Step 2. Next, simplify the structure of the solution by consolidating nearby fractional facilities. Specifically, modify the solution $(\bar{x}, \bar{y})$ to obtain a new solution (x', y') such that

$$y'_j = 0 \qquad \text{for each } j \in N, \text{ such that} w'_j = 0, \tag{4.9}$$

$$y'_i \geq 1/2 \qquad \text{for each } i \in N, \text{ such that} w'_i > 0. \tag{4.10}$$

Refer to such a solution as a $\frac{1}{2}$-restricted solution. Moreover, the cost of the $\frac{1}{2}$-restricted solution produced is at most $2\bar{f}_{LP}$.

Step 3. Finally, show how to convert a feasible $\{\frac{1}{2}, 1\}$-integral solution to the linear programming relaxation to a feasible integral solution of the cost at most $\frac{4}{3}$ times the cost of the $\{\frac{1}{2}, 1\}$-integral solution.

As proved by Charikar et al. (2002), the modified three-step algorithm provides a constant performance guarantee of $6\frac{2}{3}$.

4.5 Conclusions

In this chapter we also applied the FDS method to identify some possible solutions for a multi-facility OMP in a strongly connected directed network. We first proved that the OMP has an FDS in the node set, which generalizes the FDS result on the single-facility OMP in the literature. Then, based on this FDS result, we proposed an exact algorithm to solve the problem. Furthermore, we showed that the OMP can be solved efficiently based on the FDS result when the number of facilities is fixed and small. However, if the number of facilities is not small, it is not practical for us to find an optimal solution, since the OMP in directed networks is NP-hard. Finally we presented a $6\frac{2}{3}$-approximation algorithm for the p-median problem in directed networks.

However, there is no such work on developing approximation algorithms for the general OMP in networks. Since the OMf is non-linear, we can not directly reduce the general OMP to linear programming. Thus it is very interesting and challenging to develop some constant-factor approximation algorithms for the general OMP in the future.

Chapter 5

Subtree OMP in Tree Networks

5.1 Introduction

Since the seminal paper by Minieka (1985) presented some efficient algorithms for optimally locating a path (tree) in a tree network, by applying the property that an optimal solution to the point median or center problem must be included in some optimal solution to the path (tree) median or center problem, the property has provided researchers with a powerful tool to develop algorithms to handle extensive facility location problems. This property is called the *nestedness property*.

Recently, there are some studies on efficient algorithms for network location problems in networks, especially in tree networks, using the nestedness property, as is evident in Minieka (1985), Wang (2000), Tamir et al. (2002), and Puerto and Tamir (2005).

For the convenience of description, some related terminologies are given first.

5.1.1 Notation

We adopt some pertinent notation from Puerto and Tamir (2005). Suppose that $T = (V, E)$ is an undirected tree network with node set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set $E = \{e_2, e_3, \cdots, e_n\}$. Each node v_i is associated with a non-negative weight w_i. Each edge e_j is assigned a positive length l_j and is assumed to be rectifiable. In particular, an edge e_j is viewed as an interval of length l_j so that we can refer to its interior points. Let $e(x, y)$ be a *subedge* of e_j, where x and y are points on e_j. We assume that T is embedded in the Euclidean plane. Denote $A(T)$ as the continuum set of points on the edges of T, and $T(u)$ as the subtree rooted at the point u. For any $x, y \in A(T)$, let $P(x, y)$ be the unique simple path in $A(T)$ connecting x and y, and $d(x, y)$ be the length of $P(x, y)$. A subset $S \subset A(T)$ is called a *subtree* if it is closed and connected, and $d(x, S) = min\{d(x, y) | y \in S\}$ for any $x \in A(T)$. The length of S, $l(S)$, is defined as the sum of the lengths of its partial edges, and $V(S)$ is denoted as the set of the nodes in S.

Then we give the definition of the ordered median objective of a subtree S similar to that from Nickel and Puerto (2005, p.199). Let $d(S) = (w_1 d(v_1, S), \cdots, w_n d(v_n, S))$ and $d_{\leq}(S) = (w_{(1)} d(v_{(1)}, S), \cdots, w_{(n)} d(v_{(n)}, S))$ satisfying $w_{(1)} d(v_{(1)}, S) \leq w_{(2)} d(v_{(2)}, S) \leq \cdots \leq w_{(n)} d(v_{(n)}, S)$. The subtree ordered median problem (OMP) on $A(T)$ is defined as $f(\lambda) = \min_{S \subset A(T)} \sum_{i=1}^{n} \lambda_i w_{(i)} d(v_{(i)}, S)$, where $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n_{0+}$.

For different choices of λ, we obtain different types of objectives. Note that taking $\lambda = (1, 1, \cdots, 1)$ yields the median problem; taking $\lambda = (0, 0, \cdots, 0, 1)$ gives the center problem; taking $\lambda = (0, \overset{n-k}{\cdots}, 0, 1, \cdots, 1)$ and $\lambda = (\alpha, \overset{n-1}{\cdots}, \alpha, 1)$ ($\alpha \in [0, 1]$) leads to the $k-$centrum problem and α-centdian problem, respectively. In addition, taking $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ leads to the convex ordered median problem.

In addition, we also adopt some related terminologies from Puerto and Tamir (2005, p.316). Given a positive real number $l(\leq l(A(T)))$, the *tactical subtree* OMP is the problem of finding a subtree S with $l(S) \leq l$ that minimizes the ordered median objective. When $l = 0$, we refer to the problem as the *point* OMP instead of the subtree OMP, and call its solution a *point solution*. Given a positive real number α, the *strategic* OMP is the problem of finding a subtree S minimizing the sum of the ordered median objective and the setup cost $\alpha L(S)$ of the facility. If the endpoints of S may be anywhere (must be nodes) in $A(T)$, we call the model *continuous* (*discrete*). The *nestedness property* is the property that for any optimal solution x to the point OMP, there exists an optimal solution, with its length smaller than or equal to l, to the corresponding subtree OMP that contains x.

In the following we cite some previous results on the nestedness property, which are used to develop some efficient algorithms.

5.1.2 Previous Results

The first result concerning the nestedness property was presented by Minieka (1985). He proved that the nestedness property holds for both of the median and center network location problems.

Theorem 5.1 (Minieka, 1985) Every optimal tree-shaped facility of size L contains the unique center C for all L; for all feasible L, there is a minimum distancesum tree T_L^* of size L such that $T_L^* \cap M \neq \emptyset$ (M denotes the set of all the medians).

Tamir et al. (2002) proved that the nestedness property holds for the subtree cent-dian problem in a tree network.

Theorem 5.2 (Tamir et al., 2002) Let x_c be a cent-dian point of a tree network T. For each length L there is an optimal cent-dian subtree with length L containing x_c.

Recently, Puerto and Tamir (2005) proved that the nestedness property also holds for the continuous tactical, discrete strategic and continuous strategic k-centrum problems.

Theorem 5.3 (Puerto and Tamir, 2005)

(1). Let v' be an optimal solution for the continuous point k-centrum problem. If v' is a node, then there is a an optimal solution to the strategic discrete subtree k-centrum problem that contains v'. If v' is not a node, then there is an optimal solution that contains one of the two nodes of the edge containing v'.

(2). Let v' be an optimal solution for the continuous point k-centrum problem. There is an optimal solution to the strategic continuous subtree k-centrum problem that contains v'.

(3). Let v' be an optimal solutions for the continuous point k-centrum problem. Then there exists an optimal solution to the tactical continuous subtree k-centrum problem that contains v'.

In the following we prove that the nestedness property holds for the tactical continuous subtree OMP in a tree network with the λ-weights defined below:

$$\lambda(n,k) = (a, \overset{n-k}{\cdots}, a, b, \cdots, b) \in \mathbb{R}^n_{0+} (0 \le a \le b,\ 1 \le k \le n-1), \tag{5.1}$$

and apply the nestedness property to solve the problem in polynomial time.

5.2 Nestedness Property for Subtree OMP

Theorem 5.4 Let v^0 be an optimal solution to the continuous point OMP in a tree network T with $\lambda(n,k)$ in (5.1). There exists an optimal subtree to the corresponding tactical continuous subtree OMP that contains v^0.

Proof. We modify the technique introduced by Puerto and Tamir (2005). The tree network T can be considered as being rooted at v^0. Let $v^1, \cdots, v^t$ be the set of nodes that are neighbors of v^0, and $T^1, \cdots, T^t$ be the components rooted at $v^1, \cdots, v^t$, respectively.

Let T^0 be an optimal subtree to the corresponding tactical continuous OMP that does not satisfy the property stated in the theorem, and has the root u closest to v^0 among all the optimal subtrees. It is easy to obtain $l(T^0) = l$. Without loss of generality, we assume that T^0 is located in $T^1 \cup P[v^1, v^0)$. Note that u is the closest point to v^0 in T^0.

The logic of the proof is as follows: First, select an appropriate leaf u' of T^0. Second, perturb the subtree T^0 at u' and its root u by a small positive δ to generate a perturbed subtree T^p. That is, we decrease the length of the unique subedge of T^0 incident to u' by δ, and augment the path $P[u, u'']$ to T^0, where u'' is the point on $P[u, v^0]$ with $d(u, u'') = \delta$. Let $F(T^0)$ and $F(T^p)$ be the objective values at T^0 and T^p, respectively. Finally we prove that $F(T^p) - F(T^0) \le 0$, which contradicts the property that T^0 has the closest root to v^0 among all the optimal subtrees of length l.

Let $H_1(v^0)$ ($H_2(v^0)$) be the set of nodes corresponding to the $(n-k)$ smallest (k largest) weighted distances of nodes from v^0. $H_1(T^0)$ and ($H_2(T^0)$) are defined similarly. Suppose that k_{11} (k_{12}) of the nodes in $H_1(v^0)$ ($H_2(v^0)$) are in T^1, and k_{21} (k_{22}) are outside T^1 with $k_{11}+k_{21}=n-k$ ($k_{12}+k_{22}=k$). Denote W_{11}, W_{12}, W_{21} and W_{22} as the total weights of the k_{11}, k_{12}, k_{21} and k_{22} nodes, respectively. According to the optimality of v^0, we have $aW_{11}+bW_{12} \leq aW_{21}+bW_{22}$.

Define $M_u^1(v^0)=V(T(u)) \cap H_1(v^0)$, $M_u^1(T^0)=V(T(u)) \cap H_1(T^0)$, $N_u^1(v^0)=V(T \backslash T(u)) \cap H_1(v^0)$, $N_u^1(T^0)=V(T \backslash T(u)) \cap H_1(T^0)$, $M_u^2(v^0)=V(T(u)) \cap H_2(v^0)$, $M_u^2(T^0)=V(T(u)) \cap H_2(T^0)$, $N_u^2(v^0)=V(T \backslash T(u)) \cap H_2(v^0)$, and $N_u^2(T^0)=V(T \backslash T(u)) \cap H_2(T^0)$. According to the definitions related to v^0 (T^0), we observe that the following two properties hold for $c=v^0$ ($c=T^0$).
(1) $|M_u^1(c)|+|N_u^1(c)|=n-k$, and $|M_u^2(c)|+|N_u^2(c)|=k$;
(2) $M_u^1(c)$ ($N_u^1(c)$) and $M_u^2(c)$ ($N_u^2(c)$) are *node complementary* in $T(u)(T \backslash T(u))$, i.e., $M_u^1(c) \cup M_u^2(c)=V(T(u))$ ($N_u^1(c) \cup N_u^2(c)=V(T \backslash T(u))$).
Moreover, it is easy to see that $|M_u^2(T^0)| \leq |M_u^2(v^0)|$ and $|N_u^2(T^0)| \geq |N_u^2(v^0)|$.

In addition, considering that the sets of nodes $M_u^2(T^0)$ and $N_u^2(T^0)$ may not be uniquely defined, we make a similar non-degeneracy assumption on the two sets as that stated in Puerto and Tamir (2005, p. 330) as follows: Denote X as the set of leaves of T^0. For each $x \in X$, e_x is identified as the unique subedge of T^0 incident to x. Assume that there are $s_x \geq 0$ nodes in $V(T(x)) \cap H_2(T^0)$. If they are not uniquely defined, we select them according to the following procedure: If there exist at least two nodes with equal weighted distance values for any point in e_x, then we select them arbitrarily. Otherwise, because of the fact that all the weighted distance functions are piecewise linear, there exists a point $x'(\neq x)$ on e_x, sufficiently close to x, satisfying that no pair of functions in the collection of linear functions $\{g_i(y) = w_i d(v_i, y) : v_i \in V(T(x))\}$ has an interior intersection point in the path $P[x, x']$. In particular, the ordering of these linear functions is independent of y. Suppose that the s_x nodes in $V(T(x)) \cap H_2(T^0)$ correspond to the s_x largest weighted distance functions over $P[x, x']$ in the above collection. Moreover, assume that there are $t_u \geq 0$ nodes in $V(T \backslash T(u)) \cap H_2(T^0)$. Similarly, we suppose that the t_u nodes in $V(T \backslash T(u)) \cap H_2(T^0)$ correspond to the t_u largest weighted distance functions over $P[u, z]$, where $z \neq u$ is sufficiently close to u on $P[u, v^0]$.

With the above non-degeneracy assumption, the change in the objective is linear in δ when we perturb T^0 at both the chosen leaf u' and the root u, by a sufficiently small δ. This change equals the sum of the variations at u and u' in the direction of v^0, which are denoted by $var(u)$ and $var(u')$, respectively. Thus we have $F(T^p) - F(T^0) = var(u) + var(u')$. Define

$$\begin{aligned} f(N,u) &= a\Big(\sum_{v_i \in N_u^1(T^0)\backslash N_u^1(v^0)} w_i + \sum_{v_j \in N_u^1(v^0),\ v_j \in V(T^1\backslash T(u))} w_j - \sum_{v_k \in N_u^1(v^0)\backslash N_u^1(T^0)} w_k\Big) \\ &\quad + b\Big(\sum_{v_i \in N_u^2(T^0)\backslash N_u^2(v^0)} w_i + \sum_{v_j \in N_u^2(v^0),\ v_j \in V(T^1\backslash T(u))} w_j - \sum_{v_k \in N_u^2(v^0)\backslash N_u^2(T^0)} w_k\Big), \end{aligned}$$

and

$$\begin{aligned} f(M,u') &= a\Big(\sum_{v_i \in M_u^1(T^0)\backslash M_u^1(v^0),\ v_i \in V(T(u'))} w_i - \sum_{v_j \in M_u^1(v^0)\backslash M_u^1(T^0),\ v_j \in V(T(u'))} w_j \\ &\quad - \sum_{v_k \in M_u^1(v^0),\ v_k \in V(T(u)\backslash T(u'))} w_k\Big) + b\Big(\sum_{v_i \in M_u^2(T^0)\backslash M_u^2(v^0),\ v_i \in V(T(u'))} w_i \\ &\quad - \sum_{v_j \in M_u^2(v^0)\backslash M_u^2(T^0),\ v_j \in V(T(u'))} w_j - \sum_{v_k \in M_u^2(v^0),\ v_k \in V(T(u)\backslash T(u'))} w_k\Big). \end{aligned}$$

Then the above two variations are calculated as follows:

$$\begin{aligned}
var(u) &= -\Big[a \sum_{v_i \in N_u^1(T^0)} w_i + b \sum_{v_j \in N_u^2(T^0)} w_j\Big]\delta \\
&= -\Big[a\Big(\sum_{v_i \in N_u^1(T^0)\cap N_u^1(v^0)} w_i + \sum_{v_j \in N_u^1(T^0)\setminus N_u^1(v^0)} w_j\Big) \\
&\quad +b\Big(\sum_{v_i \in N_u^2(T^0)\cap N_u^2(v^0)} w_i + \sum_{v_j \in N_u^2(T^0)\setminus N_u^2(v^0)} w_j\Big)\Big]\delta \\
&= -\Big[a \sum_{v_i \in N_u^1(v^0)} w_i + b \sum_{v_j \in N_u^2(v^0)} w_j\Big]\delta \\
&\quad -\Big[a\Big(\sum_{v_i \in N_u^1(T^0)\setminus N_u^1(v^0)} w_i - \sum_{v_j \in N_u^1(v^0)\setminus N_u^1(T^0)} w_j\Big) \\
&\quad +b\Big(\sum_{v_i \in N_u^2(T^0)\setminus N_u^2(v^0)} w_i - \sum_{v_j \in N_u^2(v^0)\setminus N_u^2(T^0)} w_j\Big)\Big]\delta \\
&= -\Big[aW_{21} + bW_{22} + f(N,u)\Big]\delta, \qquad (5.2)
\end{aligned}$$

and

$$\begin{aligned}
var(u') &= \Big[a \sum_{v_i \in M_u^1(T^0),\ v_i \in V(T(u'))} w_i + b \sum_{v_j \in M_u^2(T^0),\ v_j \in V(T(u'))} w_j\Big]\delta \\
&= \Big[a\Big(\sum_{v_i \in M_u^1(T^0)\cap M_u^1(v^0),\ v_i \in V(T(u'))} w_i + \sum_{v_j \in M_u^1(T^0)\setminus M_u^1(v^0),\ v_j \in V(T(u'))} w_j\Big) \\
&\quad +b\Big(\sum_{v_i \in M_u^2(T^0)\cap M_u^2(v^0),\ v_i \in V(T(u'))} w_i + \sum_{v_j \in M_u^2(T^0)\setminus M_u^2(v^0),\ v_j \in V(T(u'))} w_j\Big)\Big]\delta \\
&= \Big[a \sum_{v_i \in M_u^1(v^0)} w_i + b \sum_{v_j \in M_u^2(v^0)} w_j + f(M,u')\Big]\delta \\
&\le \Big[aW_{11} + bW_{12} + f(M,u')\Big]\delta. \qquad (5.3)
\end{aligned}$$

Thus we get

$$F(T^p) - F(T^0) \;\le\; [(aW_{11} + bW_{12}) - (aW_{21} + bW_{22})]\delta + (f(M,u') - f(N,u))\delta.$$

Since $aW_{11} + bW_{12} \le aW_{21} + bW_{22}$, we only need to show that

$$f(M,u') - f(N,u) \;\le\; 0, \qquad (5.4)$$

which implies that $F(T^p) - F(T^0) \leq 0$. In the following we show how to choose the point u' and prove Inequality (5.4).

With the node complementary property, we get

$$M_u^1(T^0)\backslash M_u^1(v^0) = M_u^2(v^0)\backslash M_u^2(T^0), M_u^1(v^0)\backslash M_u^1(T^0) = M_u^2(T^0)\backslash M_u^2(v^0),$$
$$N_u^1(T^0)\backslash N_u^1(v^0) = N_u^2(v^0)\backslash N_u^2(T^0), \text{ and } N_u^1(v^0)\backslash N_u^1(T^0) = N_u^2(T^0)\backslash N_u^2(v^0).$$

Thus, with $0 \leq a \leq b$, we have

$$f(N,u) \geq (b-a)\Big(\sum_{v_i \in N_u^2(T^0)\backslash N_u^2(v^0)} w_i - \sum_{v_j \in N_u^2(v^0)\backslash N_u^2(T^0),\ v_j \in V(T\backslash T^1)} w_j\Big).$$

Moreover, it is easy to check that

$$f(M,u') \leq (b-a)\Big(\sum_{v_i \in M_u^2(T^0)\backslash M_u^2(v^0),\ v_i \in V(T(u'))} w_i - \sum_{v_j \in M_u^2(v^0)\backslash M_u^2(T^0)} w_j\Big).$$

First, we aim to prove $f(N,u) \geq 0$. If $N_u^2(v^0)\backslash N_u^2(T^0) = \emptyset$, then we have $f(N,u) \geq 0$ since $b \geq a$. Otherwise, we have $N_u^2(v^0)\backslash N_u^2(T^0) \neq \emptyset$, which leads to $N_u^2(T^0)\backslash N_u^2(v^0) \neq \emptyset$ since $|N_u^2(T^0)| \geq |N_u^2(v^0)|$. Then we make the claim below: For each $v_i \in N_u^2(T^0)\backslash N_u^2(v^0)$, we have $w_i \geq \max\{w_j | v_j \in N_u^2(v^0)\backslash N_u^2(T^0),\ v_j \in V(T\backslash T^1)\}$ (for each node $v_j \in N_u^2(v^0)\backslash N_u^2(T^0)$, we have $w_i d(v_i, v^0) \leq w_j d(v_j, v^0)$ and $w_i d(v_i, u) \geq w_j d(v_j, u)$. Since $v_i \in V(T\backslash T(u))$ and $v_j \in V(T\backslash T^1)$, we get $d(v_i, u) \leq d(v_i, v^0) + d(v^0, u)$, and $d(v_j, u) = d(v_j, v^0) + d(v^0, u)$, and obtain $w_i \geq w_j$ by the above inequalities). With the above claim, and the facts that $|N_u^2(T^0)\backslash N_u^2(v^0)| \geq |N_u^2(v^0)\backslash N_u^2(T^0)|$ and $b \geq a \geq 0$, we obtain $f(N,u) \geq 0$.

Second, we identify the point u' and prove $f(M,u') \leq 0$. If $M_u^2(T^0)\backslash M_u^2(v^0) = \emptyset$, then it is easy to see that $f(M,u') \leq 0$ since $a \leq b$. Otherwise, we have $M_u^2(T^0)\backslash M_u^2(v^0) \neq \emptyset$, which leads to $M_u^2(v^0)\backslash M_u^2(T^0) \neq \emptyset$ since $|M_u^2(v^0)| \geq |M_u^2(T^0)|$. Denote $T_1, \cdots, T_m$ as the components of $T(u)\backslash T^0$, and $x_1, \cdots, x_m$ as their points of intersection with T^0, respectively (we have $f(M,u') \le 0$ when $T^0 = T(u)$, with the proof in Case 2.2 below).

Case 1. $|(M_u^2(v^0)\backslash M_u^2(T^0)) \cap V(T_i)| \ge |(M_u^2(T^0)\backslash M_u^2(v^0)) \cap V(T_i)|$ holds for some i.

In this case, we choose x_i, the intersection point between T_i and T^0, as u', decrease the length of the subedge incident to x_i by a small δ, and perturb T^0 at u by δ. Then we claim that $w_i \ge \max\{w_j | v_j \in (M_u^2(T^0)\backslash M_u^2(v^0)) \cap V(T_i)\}$ for each $v_i \in (M_u^2(v^0)\backslash M_u^2(T^0)) \cap V(T_i)$, since $w_i(d(v_i,u') + d(u',v^0)) \ge w_j(d(v_j,u') + d(u',v^0))$ and $w_i d(v_i,u') \le w_j d(v_j,T^0) = w_j d(v_j,u')$, which leads to $w_i \ge w_j$. Thus we have

$$f(M,u') \le (b-a)\Big(\sum_{v_m \in (M_u^2(T^0)\backslash M_u^2(v^0)) \cap V(T_i)} w_m - \sum_{v_h \in (M_u^2(v^0)\backslash M_u^2(T^0)) \cap V(T_i)} w_h\Big) \le 0.$$

Case 2. $|(M_u^2(v^0)\backslash M_u^2(T^0)) \cap V(T_i)| < |(M_u^2(T^0)\backslash M_u^2(v^0)) \cap V(T_i)|$ holds for all i.

Case 2.1. T^0 has no leaf node of $T(u)$. Then we decompose T^0 into m paths ending at $x_1, \cdots, x_m$, respectively, with the condition that each pair of these paths has no interior intersection points. Denote the m paths as $P_1, \cdots, P_m$, and define $T_i' = T_i \cup P_i$ $(i = 1, \cdots, m)$. Since $|M_u^2(v^0)| \ge |M_u^2(T^0)|$, there exists at least one component, without loss of generality say T_1', such that $|(M_u^2(v^0)\backslash M_u^2(T^0)) \cap V(T_1')| \ge |(M_u^2(T^0)\backslash M_u^2(v^0)) \cap V(T_1')|$. Thus x_1 is identified as u'. Then we claim that $w_i \ge \max\{w_j | v_j \in (M_u^2(T^0)\backslash M_u^2(v^0)) \cap V(T_1)\}$ for each $v_i \in M_u^2(v^0)\backslash M_u^2(T^0)) \cap V(T_1')$ (if $v_i \in P_1$, $w_i d(v_i,v^0) \ge w_j d(v_j,v^0)$, which leads to $w_i \ge w_j$ since $d(v_i,v^0) < d(v_j,v^0)$. Otherwise, we have $v_i \in V(T_1)$, which has been proved in Case 1). Hence we obtain

$$f(M,u') \le (b-a)\Big(\sum_{v_m \in (M_u^2(T^0)\backslash M_u^2(v^0)) \cap V(T_1)} w_m - \sum_{v_h \in (M_u^2(v^0)\backslash M_u^2(T^0)) \cap V(T_1')} w_h\Big) \le 0.$$

Case 2.2. T^0 has at least one leaf node of $T(u)$. Define $d_1 = \max\{d(v_i, u)|v_i \in V(T^0)\}$ and $d_2 = \max\{d(v_j, u)|v_j$ is a leaf node of $T^0\}$. Let v_s be the node such that $d(v_s, u) = d_1$, and v_t be the leaf node in T^0 such that $d(v_t, u) = d_2$. Obviously, $d_2 \le d_1$.

First, suppose $d_2 = d_1$, then select v_t as u'. If $v_t \in M_u^2(v^0)$, then it is obvious that $f(M, u') \le 0$. Otherwise, $v_t \in M_u^1(v^0)$. With the assumption of Case 2, there exists at least one node, say v_i, in T^0, such that $v_i \in M_u^2(v^0) \backslash M_u^2(T^0)$. Observing that $w_i d(v_i, v^0) \ge w_t d(v_t, v^0)$ and $d(v_i, v^0) \le d(v_t, v^0)$, we have $w_i \ge w_t$, which leads to $f(M, u') \le 0$. Second, suppose $d_2 < d_1$. Denote $\overline{T^0}$ as the union of the path connecting u and v_s, and the paths connecting u and each leaf node in T^0, respectively. Without loss of generality, let x_1 be one end point of $\overline{T^0}$. Then decompose $T^0 \backslash \overline{T^0}$ into other $(m-1)$ paths ending at $x_2, \cdots, x_m$, respectively, satisfying that each pair of the paths has no interior intersection points. Using the similar procedure in Case 2.1, we obtain $f_2(M, u') \le 0$, too.

Therefore, we conclude that

$$F(T^p) - F(T^0) \le [(aW_{11} + bW_{12}) - (aW_{21} + bW_{22})]\delta \le 0,$$

which completes the proof. $\square$

Remark 5.1 Theorem 5.4 includes some classic location problems as follows:

(1) The median problem: $a = b = 1$.

(2) The center problem: $a = 0$, $b = 1$ and $k = 1$ (Minieka, 1985).

(3) The 1/2-centdian problem: $a = 1/2$, $b = 1$ and $k = 1$ (Tamir et al., 2002).

(4) The k-centrum problem: $a = 0$ and $b = 1$ (Puerto and Tamir, 2005).

5.3 Complexity of Subtree OMP

Based on the above nestedness result, the tactical continuous subtree OMP with $\lambda(n,k)$ in (5.1) can be reduced to the model in which an optimal subtree contains one corresponding optimal point. Using the technique introduced by Puerto and Tamir (2005), we assume that the known point is v_1 and T is rooted at v_1. For each $e_j \in E(T)$ connecting v_j and its parent $p(v_j)$, assign a variable x_j such that $0 \leq x_j \leq l_j$, and denote $x_j(e_j)$ as the point on edge e_j such that its distance from $p(v_j)$ is x_j. For each node $v_i \in V(T)$, the weighted distance of v_i from a subtree is $y_i = w_i \sum\limits_{v_j \in P[v_i,v_1)} (l_j - x_j)$. Then we can apply the LP formulation for the convex OMP by Ogryczak and Tamir (2003) and obtain a reduced one as follows: min $(b-a)(kt+\sum\limits_{i=1}^{n} d_i^+)$, subject to $d_i^+ \geq y_i - t,\ d_i^+ \geq 0\ (i=1,\cdots,n)$, $y_i = w_i \sum\limits_{v_m \in P[v_i,v_1)} (l_m - x_m),\ (v_i \in V(T), 0 \leq x_j \leq l_j,\ j=2,\cdots,n)$, and $\sum_{j=2}^{n} x_j \leq l$. By the complexity result in Nickel and Puerto (2005, p. 274), we can find an optimal point solution in $O(n \log^2 n)$ time. Thus, using the algorithm of Vaidya (1990), we conclude that the tactical continuous subtree OMP stated in Theorem 1 can be solved in $O(n^3 + n^{2.5} I)$ time, where n is the number of demand points in a given tree network, and I denotes the total number of bits needed to represent the input.

On the other hand, there also exist some negative results for the nestedness property. For instance, Puerto and Tamir (2005) claimed that the discrete tactical k-centrum problem does not have the nestedness property with respect to the point solution, even for the regular median objective, n-centrum, by showing the following example on the line: $v_1 = 0, v_2 = 2, v_3 = v_2 + 1/4,$ and $v_4 = v_3 + 1$; $w_1 = 2$ and $w_i = f$ for $i = 2,3,4$. The unique solution for the tactical discrete problem with $L = 0$ is v_2, and the unique solution for the tactical discrete problem with $L = 1$ is the edge (v_3, v_4).

Since the k-centrum objective is a special case of the convex OMP, we investigate whether the non-convex OMP has the nestedness property. In the following we provide a counter example (see Figure 5.1) that shows that the nestedness property does not hold for a non-convex OMP.

5.4 A Counter Example for Non-convex OMP

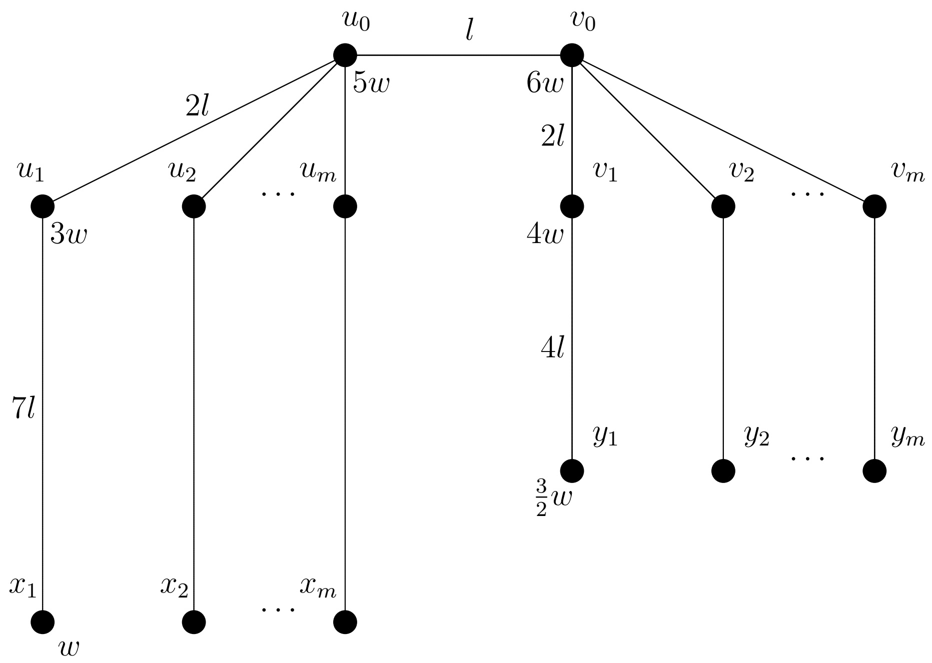

Figure 5.1. A counter example

In this section, we construct a tree network T and prove that T does not have the nestedness property with respect to the point solution for the continuous tactical model using a special case of the concave ordered median objective.

Let $T = (V, E)$ be a tree network with node set V and edge set E, where $V = \{u_0, v_0\} \cup \{u_1, \cdots, u_m\} \cup \{v_1, \cdots, v_m\} \cup \{x_1, \cdots, x_m\} \cup \{y_1, \cdots, y_m\}$, $E = \{u_0v_0, u_0u_i, i = 1, \cdots, m\} \cup \{v_0v_j, j = 1, \cdots, m\} \cup \{u_kx_k, k = 1, \cdots, m\} \cup \{v_ty_t, t = 1, \cdots, m\}$, and m is a sufficiently large positive integer. T is shown in Figure 5.1.

Let n be the order of V, then $n = 4m + 2$. Each node is associated with a weight. Specifically, $w(u_0) = 5w, w(v_0) = 6w, w(u_i) = 3w, w(v_i) = 4w, w(x_i) = w$, and $w(y_i) = \frac{3}{2}w$ $(i = 1, 2, \cdots, m)$, where w is a positive real number. Moreover, each edge has a length. In detail, $L(u_0v_0) = l, L(u_0u_i) = 2l, L(u_ix_i) = 7l, L(v_0v_i) = 2l$ and $L(v_iy_i) = 4l$ $(i = 1, \cdots, m)$, where l is a positive real number.

In the following we first resort to finding an optimal point solution for the continuous model that minimizes the sum of the $n - m$ smallest weighted distances, then we search for an optimal subtree of length $\frac{l}{2}$ for the continuous tactical model using the same objective type. For the sake of simplicity, the former and latter models are denoted as Model 1 and Model 2, respectively, and the objective f_λ is written as f.

To begin with, we need the following lemma.

Lemma 5.1 (Nickel and Puerto, 1999) There is at least one optimal solution for Model 1 in the node set V.

Lemma 5.2 u_0 is the unique optimal solution for Model 1 in T.

Proof. First, we prove that u_0 is optimal. By Lemma 5.1, to prove the optimality of u_0, it suffices to compute the objective value of Model 1 at each node.

Moreover, there exists some symmetry in T. That is, the objective values at u_i and u_j, v_i and v_j, x_i and x_j, and y_i and u_j $(i \neq j)$ are the same, respectively. Thus we only need to calculate the objective values at u_0, v_0, u_1, v_1, x_1 and y_1.

Let $U(v)(U(S))$ denote a set of m nodes corresponding to the m largest weighted distances from $v(S)$. It is easy to check that $U(u_0) = \{v_1, \cdots, v_m\}$. Then

$$\begin{aligned}
f(u_0) &= \sum_{v_i \in V \setminus U(u_0)} w_i d(v_i, u_0) \\
&= m \times w \times 9l + m \times 3w \times 2l + 6w + m \times \frac{3}{2} w \times 7l \\
&= (25.5m + 6)wl.
\end{aligned}$$

Similarly, $f(v_0) = (26m+5)wl$. However, if the facility is located at $u_1(x_1)$, then all the other nodes, except $x_1(u_1)$, arrive at $u_1(x_1)$ through u_0, which makes their weighted distances from $u_1(x_1)$ greater than those from u_0 with the same disregarding node set $\{v_1, v_2, \cdots, v_m\}$. Thus, $f(u_1) > f(u_0)$ and $f(x_1) > f(u_0)$. It is also evident that $f(v_1) > f(u_0)$ and $f(y_1) > f(u_0)$. Therefore, $f(u_0)$ is the smallest among the objective values at V, and thus u_0 is an optimal solution for Model 1.

In the following the uniqueness of u_0 is to be demonstrated. From the above discussion, it suffices to explore whether there is another optimal solution on the path $P(u_0, v_0)$.

Let x be the point on $P(u_0, v_0)$ such that $d(u_0, x) = xl (0 \leq x \leq 1)$. We distinguish the following two cases separately.

Case 1. $0 \le x < 0.6$

It is easy to check that $U(x) = \{v_1, v_2, \cdots, v_m\}$ and $f(x) = [(25.5 + 2.5x)m + 6 - x]wl$. So $min_{x \in [0,0.6)} f(x) = f(0) = (25.5m + 6)wl = f(u_0)$.

Case 2. $0.6 \le x \le 1$

Then $U(x) = \{x_1, x_2, \cdots, x_m\}$ and $f(x) = [(28.5 - 2.52.5x)m + 6 - x]wl$. In this case, $min_{x \in [0.6,1]} f(x) = f(1) = (26m + 5)wl = f(v_0) > f(u_0)$, since m is sufficiently large.

Hence, the uniqueness of u_0 holds. $\square$

Lemma 5.3 Let v be the point on the path $P(u_0, v_0)$ such that $d(v, v_0) = \frac{l}{2}$. Then $e(v, v_0)$ is the unique optimal subtree for Model 2, excluding u_0.

Proof. First, we check the local optimality of $e(v, v_0)$ with respect to the path $P(u_0, v_0)$. Let $e(y, u)$ be a subegde of length $\frac{l}{2}$ on $P(u_0, v_0)$ such that $d(u_0, y) = y$. It is obvious that $0 \le y \le 0.5$. There are three cases according to the range of y.

Case 1. $0 \le y < 0.1$

In this case, $U(e(y, u)) = \{v_1, v_2, \cdots, v_m\}$. Then $f(e(y, u)) = [(24.75 + 2.5y)m + 3 - y]wl$, the minimum of which is $(24.75m + 3)wl$.

Case 2. $0.1 \le y \le 0.3$

It is not hard to compute that $U(e(y, u)) = \{y_1, y_2, \cdots, y_m\}$, and $f(e(y, u)) = (25m + 3 - y)wl$, which yields the minimum value of $(25m + 2.7)wl$.

Case 3. $0.3 < y \le 0.5$

Then $U(e(y, u)) = \{x_1, x_2, \cdots, x_m\}$, and $f(e(y, u)) = [(25.75 - 2.5y)m + 3 - y]wl$.

It is evident that the minimum is $(24.5m+2.5)wl$ if and only if $e(y,u)=e(v,v_0)$, which leads to the local optimality of $e(v,v_0)$.

In the following the global optimality of $e(v,v_0)$ is to be demonstrated. On one hand, if the tree-shaped facility of length $\frac{l}{2}$, denoted as S, contains u_0 or lies in the left down subtree of u_0 in T, then $U(S)=\{v_1,v_2,\cdots,v_m\}$, which is same as the one corresponding to the case of $y=0$. Since m is a sufficiently large positive integer, an increase in distance from S to v_0 leads to a large increase in the sum of weighted distances from S to each $y_i(i=1,2,\cdots,m)$, and contributes to a small decrease in the sum of weighted distances from S to each of the elements in $\{u_1,\cdots,u_m,x_1,\cdots,x_m\}$. Hence, in this case, the corresponding objective value increases with the increase in the distance from S to v_0.

On the other hand, if S contains v_0 or locates in the right down subtree of v_0, the same result can be obtained.

In summary, $e(v,v_0)$ is the unique optimal subtree for Model 2. $\square$

By Lemmas 5.2 and 5.3, it is easy to establish the following theorem.

Theorem 5.5 The nestedness property of T in Figure 5.1 does not hold for the continuous tactical model minimizing the sum of the $(n-m)$ smallest weighted distances.

5.5 Conclusions

In this chapter we proved that the nestedness property holds for the tactical continuous subtree OMP with the λ-weights taking two different values. Due to this nestedness property result, the problem can be solved in polynomial time. Furthermore, we characterize a counter example to show that the nestedness property cannot hold for the non-convex OMP.

Our finding extends the results related to the nestedness property of Minieka (1985), Tamir et al. (2002) and Puerto and Tamir (2005), and fills a gap in the research on the nestedness property.

However, considering that our problem is a special case of the convex OMP, the problem of whether the nestedness property holds for the convex OMP remains open, which was posed by Puerto and Tamir (2005). On the other hand, considering that the (k_1, k_2)-trimmed problem is another important type in the OMP family, especially important in the statistics science, which disregards the k_1 smallest and the k_2 largest weighted distance functions, thus it is interesting and challenging to study the trimmed OMP in the future.

In addition, this chapter only deals with locating tree-shaped facilities using the OMf. In the real world, there are many different connected facilities, such as cycle-shaped, star-shaped, clique-shaped facilities, and so on. Hence study on the OMP with the location of different shape of facilities may be another future research direction.

Chapter 6

Summary and Suggestions for Future Research

In this thesis we investigated several network location problems with the ordered median function and its special instances.

First, we studied the multi-facility ordered median problems in undirected networks, in which the multiple isolated facilities are to be located. Since multi-facility OMP in general networks are NP-hard, we focused on some efficient polynomial algorithms for the OMP confined to tree networks. We adopted the FDS method to study some special instances of the OMP in networks. Specifically, we first characterized an FDS for a special convex OMP in general networks, where the convex OMP is an important class in the OMP family. The FDS result generalizes some known results in the literature, and leads to an exact algorithm for the OMP in general networks. Then, based on the FDS result, we obtained a polynomial size FDS and solved the problem confined to tree networks in polynomial time, which extends some results in the literature.

Second, we focused on the multi-facility OMP in directed networks, since most of the networks in the real world are directed and not symmetric (undirected networks can be viewed as symmetric directed networks). We made use of the FDS method to identify some possible solutions for a multi-facility OMP in a strongly connected directed network. We first proved that the OMP has an FDS in the node set, which generalizes the FDS result on the single-facility OMP in the literature. Then, based on this FDS result, we proposed an exact algorithm to solve the problem. Moreover, we showed that with the FDS result Algorithm 4.1 was efficient for the small number of the facilities to locate. However, when the number of the facilities is large, it is not practical to obtain an optimal solution in an efficient manner, since the OMP in networks is NP-hard. Thus we presented a constant-factor approximation algorithms for the OMP.

Third, we investigated the OMP in tree networks, in which the facilities to locate are not isolated points but connected structures (e.g., paths, trees, etc.). We adopted the nestedness property to study the subtree OMP in tree networks. Specifically, we proved the nestedness property for a special convex OMP in tree networks. This finding extends some classical results concerning the nestedness property. Then we solved the problem in polynomial time based on the nestedness property result. In addition, we provided one counter example to show that the nestedness property cannot hold for the non-convex case.

Finally, we propose some directions for future research with respect to the choices of objective functions and shape requirements of the facilities to locate with the corresponding methodology used in this thesis.

So far there are already some known results on FDS for the concave and some special convex instances of the OMf, the latter includes the center, the centdian and the k-centrum problems. As introduced in Chapter 1, the OMf is a big family of objective functions. There are three main types of objectives: concave OMP (the λ-weights in non-increasing order), convex OMP (the λ-weights in non-decreasing order), and (k_1, k_2)-trimmed OMP. Study on these three main types of problems may help us gain more insight into the inherent properties of the OMf and provide more efficient algorithms to solve them in practice. Hence, we should continue to investigate some unsolved problems in the future.

On the one hand, we should further study the identification of an FDS of polynomial size for the OMP as follows:

Question 6.1 Does an FDS of polynomial size exist for the convex or (k_1, k_2)-trimmed multi-facility OMP?

Currently there are already results of an FDS of polynomial size for concave case and some special instances of the convex OMP, and some efficient polynomial-time algorithms for the problems. However, many other special cases of the convex OMP remain unsolved. In this thesis, we used BN, NBN, EQ and PEQ to characterize an FDS successfully. We conjecture that Question 6.1 would hold for some special instances, and some sets of points, more general than PEQ, should be introduced to identify an FDS.

Furthermore, Puerto and Rodríguez-Chía (2005) proved that there is no FDS of polynomial size for the p-facility OMP with general λ-weights by constructing a path network with the λ-weights including two same elements. This counter example does not belong to the above three main types of the OMP.

Thus, from an algorithmic point of view, we may classify a type of problems with certain structures for the λ-weights, which can be solved by some FDS-based algorithms in polynomial time. Thus another open problem is posed as follows:

Problem 6.2 Identify the characteristics of the λ-weights such that there exists a polynomial size FDS for the OMP.

On the other hand, once we prove the nestedness property for some OMP, we can develop a powerful algorithm by growing from one point solution into a possible connected facility. However, to the best of my knowledge, only some special convex cases of the OMf (e.g., p-center, k-centrum, cent-dian, etc.) have the nestedness property. Furthermore, we proved that the nestedness property cannot hold for the non-convex OMP by constructing a counter example. Thus the nestedness property for the (k_1, k_2)-trimmed OMP remains to be studied further as follows:

Problem 6.3 Does the nestedness property hold for the (k_1, k_2)-trimmed OMP? If the answer is affirmative, develop an efficient algorithm based on the nestedness property result. Otherwise, construct a counter example.

Moreover, this thesis only deals with tree-shaped facilities. However, in the real word, the connected facility may be star-shaped, clique-shaped, and so on. Thus, to develop some efficient algorithms based on the nestedness property to solve those OMP, we should prove the nestedness property for them in advance.

The two methodologies, FDS and nestedness property, lead us to develop some efficient polynomial-time algorithms to solve the OMP in some special networks. Unfortunately, the two methodologies can only be limited to some special instances of the OMP. From the practical application point of view, some constant-factor approximation algorithms should to be developed, which is another direction for our future research.

Problem 6.4 Develop some constant-factor approximation algorithms for the OMP in networks. Furthermore, it is important to establish a performance guarantee as low as possible for approximation algorithms.

In addition, since our network location models are assumed that the demands of customers are known with certainty in advance, disregarding this assumption can provide many open challenging location problems.

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Appendix

Forthcoming Publication\Submitted Papers

(1) H.J. Tang, T.C.E. Cheng, and C.T. Ng, Finite dominating sets for the multi-facility ordered median problem in networks and algorithmic applications, accepted by *Computers & Industrial Engineering.*

(2) H.J. Tang, T.C.E. Cheng, and C.T. Ng, An extended result on the nestedness property for the tactical continuous subtree ordered median problem in networks, submitted to *European Journal of Operations Research.*

(3) H.J. Tang, T.C.E. Cheng, and C.T. Ng, Multi-facility ordered median problem in directed networks, submitted to *Applied Mathematical Modelling.*