RISK MANAGEMENT IN FINANCE AND INSURANCE VIA STOCHASTIC OPTIMIZATION

LIU JINGZHEN

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The Hong Kong Polytechnic University

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Risk Management in Finance and Insurance via Stochastic Optimization

LIU JINGZHEN

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CERTIFICATE OF ORIGINALITY

I hereby declare that this submission is my own work and that, to the best of my knowledge and belief, it contains no material previously published or written by another person nor material which to a substantial extent has been accepted for the award of any other degree or diploma of a university or other institute of higher learning, except where due acknowledgment is made in the text.

JINGZHEN LIU

Abstract

This thesis is concerned with the study of the risk-constrained portfolio selection problem arising from an ordinary investor and the insurer being an investor.

We first consider the problem for an insurer who can invest her surplus into financial market. With value at risk (VaR) imposed as the dynamic risk constraint, the portfolio selection problem is considered with two objectives: the ruin probability minimization and wealth utility maximization. A closed-form solution is found by solving the associated Hamilton-Jacob-Bellman (HJB) equation for the first problem. By using the exponential utility function, we solve the second problem by transforming this stochastic optimal control problem into a deterministic optimal control one and using control parametrization method.

Second, we consider the risk-constrained utility maximizing problem with a jump diffusion model and a regime switching model for an ordinary investor. Conditional value at risk (CVaR) and maximal value at risk (MVaR) are used as the risk constraint in the two models, respectively. The associated HJB equations are treated with numerical techniques.

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Notations

N, R	set of integer, real numbers
R^n	n dimensional space
$M^\top, \alpha^\top$	the transpose of a matrix or a vector
μ	drift rate (or drift rate vector)
r	risk-free interest rate
σ	volatility (or volatility matrix)
t	time variables
T	terminal date
$\mathcal{T}$	time interval
$S(t)$	price of underlying asset or prices of n underlying assets
$B(t)$	prices of risk free asset
$X(t)$	the dynamics of wealth
$W^0(t)$	1-dimensional Brownian Motion
$W(t)$	1(or d)-dimensional Brownian Motion
$\widetilde{\mu}(t)$	n+1 -dimensional drift rates vector
$\widetilde{\sigma}(t)$	(n+1) $\times$(d+1) matrix
$\widetilde{W}(t)$	d+1-dimensional Brownian Motion
$(\Omega, \mathcal{F}, P)$	a complete probability space
ω	the element of Ω
EX	expectation of the random variable X
$VarX$	variance of the random variable X
$\mathcal{F}_t$	the σ-algebra generated by the process $W(t)$.
$\mathcal{F}_t^Y$	the σ-algebra generated by the process $Y(t)$.
$D(S(t))$	denotes the diagonal matrix $diag[S_1, ..., S_n]$.

τ	the ruin time variables
$q(t)$	the proportional reinsurance
π	invested in the risky asset (or risky assets)
$V(x,t)$	value function
$\boldsymbol{V}(x,t)$	N-dimensional value function vector
$\mathcal{B}(R^n)$	The Borel σ-algebra on R^n
$C^2(\mathcal{D})$	set of 2-times continuously differentiable functions on $\mathcal{D}$
$\psi(x)$	the ruin probability for an insurer
J	the compound Possion process
O	the state space
Q	$[0,T) \times O$
$\overline{Q}$	the closure of Q
λ	the mean arrival rate of jumps of the Poisson process
A	the generator of a process
$u(t,\cdot)$	the control process
U	the value space of $u(t,\cdot)$
$\mathcal{U}$	the set of admissible control process u
$U_1(\cdot), U_2(\cdot)$	utility function
$\boldsymbol{\lambda_1}, \boldsymbol{\lambda_2}$	Lagrangian multiplier

CVaR	conditional value at risk
DP	dynamic programming
HJB	Hamilton-Jacobi-Bellman
MVaR	maximal value at risk
MVO	mean-variance optimization
PDE	partial differential equation
VaR	value at risk

Chapter 1

Introduction

All investors, ranging from private individuals to banks and insurance companies, face the investment problem of how to allocate a certain amount of money in different assets and at what time instant. The pioneering work of Markowitz [94] first provided a mathematically elegant way to formulate the optimal portfolio allocation problem and developed the celebrated $mean - variance\ optimization$ (MVO) approach for optimal $portfolio\ allocation$. He considered a single-period model and adopted variance (or standard deviation) as a measure of the portfolio's risk. The novelty of his mean-variance approach is that it reduces the optimal portfolio allocation problem to the one in which only the mean and the variance of the rate of return are involved under the normality assumption for the rates of return of the risky assets. This greatly simplifies the problem of optimal portfolio allocation and makes a great leap forward to the development of the field. The mean-variance approach by Markowitz has also laid down solid theoretical foundation to the optimal portfolio allocation problem and opened up an important field, namely, the modern portfolio theory.

Merton [95, 98] pioneered the development of the optimal $consumption - portfolio$ allocation problem in a $continuous - time$ framework, which provides a more realistic setting to deal with the problem. His work has opened up an important field in modern finance, namely, the $continuous - time$ portfolio theory. Under the assumption that returns from the risky assets are stationary (i.e. the coefficients of the dynamics of the returns are constant), and some specific forms of the utility function, Merton derived closed-form solutions to the optimal portfolio allocation in a continuous-time setting.

Sparked by the work of Markowitz and Merton, portfolio selection problems have been extensively studied along the two main categories: *MVO problems* and *utility maximization*. Due to the appeal of utility maximization problems that they can incorporate the risk attributes of an individual investor through the investor's utility function, the literature is dominated by the work in Merton's direction. Merton's problem is later revisited with different kind of variation and constraint, for example, with the introduction of the transaction cost, with the constraint that the investor may also face regulatory requirements that it never has negative holdings in any stock. In this work, we will focus on the problem in Merton's direction and contribute to further extension.

Most of the work considers the portfolio selection problem for an ordinary investor, that is, how to allocate her wealth between the risky asset and risk free assets without an external risk process. In reality, a kind of investors, such as an insurance company, face the problem with receiving the stochastic cash flow, the risk from which can not be traded away in the marketplace. This problem arises from the merging of finance and insurance market, which is an important problem in actuarial fields. In other words, in order to increase the profit or decrease the ruin probability, the insurance company will invest the part of its reserve into financial market. Since the wealth can go negative due to the process of stochastic cash flow, the chance of *ruin* is highly possible to happen. Therefore, two important problems in insurance literature would be the *ruin minimization* problem and the *utility maximization* problem in finite time.

In a real market, the dynamics of stock prices, in addition to the geometry Brownian motion used in Merton [95], often encounter large movements and non-stationary return. The price evolution could cause a heavy tailed loss, and hence it would be of practical relevance and importance to consider the asset models which can capture actual stock market behavior. The analysis of price evolution which does reveal sudden and rare breaks logically accounted for by exogenous events on information is often captured by the $jump - diffusion$ process. And, recently, Markov-modulated regime switching models have received much attention among researchers and market practitioners, which incorporate the feature of non-stationary returns. From an economic perspective, the Markovian $regime - switching$ model can describe the stochastic evolution of investment opportunity sets due to structural changes in the state of the economy.

With the rapid development of the derivatives markets, together with margin trading on certain financial products, the exposure to losses of an investment can be many times more than the initial capital allocation for that investment. Without a careful analysis of the potential danger, the investment could cause catastrophic consequences when a shock occurs. The financial crisis of 2007-present triggered by a liquidity shortfall in the United States banking system has resulted in the collapse of large financial institutions, such as the Bankruptcy of the Bear Stearns, the failure of American International Group (AIG), the collapse of Lehman Brothers, the"bail out" of banks by national governments and downturns in stock markets around the world. It is regarded as the worst financial crisis since the Great Depression of the 1930s. Due to the current financial crisis, *risk management* appears to be a discipline that is being taken far more seriously these days.

The concern about risk is traced back in the original work of Markowitz [94], who has investigated the appropriate definition and measurement of risk. The basic approach in that work is to recognize that an investor faces a trade-off between risk and return and to develop the implications of that trade-off. The main innovation introduced by Markowitz is the use of *volatility* as a measure of risk to measure the risk of a portfolio via the joint distribution of returns of all assets. In recent years, the growth of trade activity and instances of financial market instability have prompted new studies underscoring the need for market participation to develop reliable risk measurement techniques. $Value\ at\ Risk(VaR)$ has emerged as an important risk management tool with the precise task of answering to the following very relevant and precise questions that how much the potential loss can be expected over a given period with a given probability. Since VaR received its first wide representation in July 1993 in the Group of Thirty report, the number of users and uses for VaR have increased dramatically. It has widely been adopted within banking, insurance and finance industries sectors for quantifying the market risk, portfolio optimization and setting capital adequacy(Jorion [76], Dowd[33]), which include Basle Committee on Banking Supervision, the SEC, the International Swap and Derivatives Association and the Derivatives Policy Group.

The portfolio selection problem, with both risk and return concerned, proceeds in two directions. One direction is the risk minimization with the return requirement, which starts from the work of Markowitz [94]. But as the name might imply, risk management, is not about eliminating or minimizing risk. In today's challenging economic climate, firms must balance risk with caution and the long-term interests of and returns to shareholders. That's according to the latest report, entitled Financial Reform: A Framework for Financial Stability released by The Group of Thirty (G30), an international body of leading financiers and academics. Therefore, the other direction is the utility maximization with risk constraint embedded, which appears more attractive as the utility maximization is the main objective for an investor. In the literature, several shortfalls with the study along the second direction include the inconsistence of the portfolio with the risk constraint, the static setting or the assumption that full knowledge in future is known. A reasonable treatment of this problem recently is imposing the risk constraint in a dynamic manner, i.e. reevaluating it daily (or at least weekly), which is based on the rules of Basel II Accord[1.1].

Our focus in this work is on the dynamic portfolio choice of an investor trader subject to a risk limit specified in terms of VaR ($CVaR, MVaR$). This problem has not yet received a complete treatment in the existing insurance literature and general model assumption for the dynamics of the risky asset.

1.1 Our Work and Outlines

Our focus is on the dynamic portfolio choice subject to certain risk constraint defined in terms of VaR ($CVaR, MVaR$), both for an ordinary investor and an insurer. In our work, the risk constraint is considered over a short duration of time with the assumption that portfolio allocations do not change within this interval. We apply VaR ($CVaR, MVaR$) as a risk constraint continuously over time. We consider this

[1.1] The current Basel Committee on Banking Supervision regulations introduce a risk control regulation, which imposes a minimal level of eligible capital that the agents must maintain at all times as a function of the portfolio VaR. This case is very important because it obligates financial institutions in all developed countries to maintain eligible capital as a function of their bi-weekly market-risk VaR. It is related to the capital reserve which the financial institution needs to hold, in order to prevent (at a sufficiently high confidence level) insolvency due to an adverse development in the market situation.

calculated VaR beyond the modeled horizon and constant investment/proportional reinsurance opportunity set throughout. At each instant, the VaR ($CVaR, MVaR$) is estimated and is applied to influence the strategy decision.

For the portfolio selection problem of an insurer, we first impose the dynamic risk constraint on the portfolio selection. To make the risk constraint dynamically-consistent with the portfolio, it makes sense to treat the problems (ruin probability minimization and utility maximization) with imposing the risk constraint dynamically. For the problem of ruin probability minimization, we derive the closed form solution for ruin probability. As for maximizing the utility of final wealth, we transform the stochastic control problem into the deterministic one, which appears much easier since there are many packages for solving such a problem.

The risk constrained problem has not yet received a wide concern in other models than normal distribution assumption in the existing literature. As an extension, we consider it in a jump diffusion model and a switching model to improve the volatility modeling.

When applying the dynamic risk constraint to the two models, the risk is stabilized, and the risky investment is cut to meet the risk management. To the best of our knowledge, we are the first to investigate the utility maximization problem when using $CVaR$ as a dynamic risk constraint in the jump diffusion model. Moreover, we are the first to use the $MVaR$ as the risk constraint in the switching model dynamically. This means that the optimal consumption and investment results developed here are uniformly optimal over different states of the economy described by the chain. In other words, our method here can provide a conservative and prudent approach to determine the optimal consumption and investment with risk constraints. In the literature, most researchers considered the case that the VaR does not depend on the states of the economy.

1.1.1 Outline of this work

In Chapter 2, the background of this work is presented, the model formulation and methodology for the stochastic optimization problem are introduced, and the risk measure and history of risk management in the literature are presented.

In Chapter 3, the problem of minimizing ruin probability with dynamical risk constraint is investigated firstly. If the insurer is allowed to invest his money in a risky market which is composed of risk free asset and risky asset, VaR in a short time horizon is analyzed for risk regulator and imposed as a constraint. Then the problem is how to minimize the probability of ruin with a dynamic risk constraint. By solving the corresponding Hamilton-Jacobi-Bellman equations, the explicit expressions for the optimal value function and the corresponding optimal strategies are obtained.

Secondly, with a constrained VaR, the problem that the expected exponential utility for the final wealth is to be maximized is considered. This can be posed as a stochastic optimal control problem, where the diffusion reserve model introduced by Promislow and Young [108] is used. Furthermore, proportional reinsurance, which is another important instrument for decreasing the risk or increasing the final wealth, is also included in this problem formulation. Then we transform the stochastic optimal control problem into a deterministic one. The solution of this deterministic problem can be approximated as an optimization problem by using existing optimization software, such as $NLPQLP$(see [89, 90, 125]).

In Chapter 4, we consider the optimal portfolio selection problem subject to the risk constraint with more realistic models in finance. In particular, we focus on using more sophistical stochastic models to overcome some of the drawbacks of the pure geometry Brownian motion process.

In the jump diffusion model, the asset price is assumed to be driven by a Brownian motion perturbed by a compound Poisson process. This resembles a price process perturbed by an exogenous factor which may cause large movements in price. The jump size of the Poisson process and the rate of jump define, respectively, a scenario and its occurrence probability. The stress testing is conducted to evaluate the performance and assess the resilience of the portfolio subject to exceptional but major events. We use $CVaR$ as the risk constraint for the jump diffusion model, since VaR is not coherent for this model. We examine how a conditional-value-at-risk constraint exerts influence on the portfolio composition.

In the regime switching model, the price dynamics of the risky asset are governed by a Markov-modulated geometric Brownian motion. In particular, the market parameters including the market interest rate of a bank account, the appreciation rate and the volatility of the risky asset switch over time according to a continuous-time Markov chain. The $MVaR$ is defined as the maximum value of the $VaRs$ of the portfolio in a short time duration over different states of the chain. We consider the optimal consumption and investment problem when both the regime-switching effect and the VaR constraint are present.

The problem is reduced to the solution of the associated HJB equation, and an efficient numerical method is proposed for the optimal constrained portfolio. We shall provide numerical results for the sensitivity analysis of the optimal portfolio, the optimal consumption and the $CVaR$ ($MVaR$) level with respect to model parameters. These results are also used to investigate the effect of the switching regimes.

Finally, concluding remarks are presented in Chapter 5, where some future problems and research directions are also included.

Chapter 2

Background

2.1 Market Model

Assume that the market consists of a risk-free asset and n risky assets (say stocks). First, we fix a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is a real-world probability. The stock prices are modeled as a geometric Brownian motion. Expressed mathematically, the dynamics of the risky assets and the risk-free asset evolve according to

$$
\begin{aligned}
dS(t) &= S(t)(\mu dt + \sigma dW(t)), \\
dB(t) &= rB(t)dt,
\end{aligned}
\tag{2.1.1}
$$

respectively[2.1]. Let "$\top$" denote the transpose of a vector or matrix throughout this paper. Here, $W = (W_1, ..., W_d)^\top$ is a d-dimensional standard Brownian motion, which stands in for any and all sources of uncertainty in the price history of n stocks. The vector $\mu = (\mu_1, ...\mu_n)^\top \in \mathcal{R}^n$ is the appreciation rates, matrix σ is the $n \times d$ volatility matrix, and $D(S(t))$ denotes the diagonal matrix $diag[S_1, ..., S_n]$. For one stock, Merton[96] first introduced this geometric Brownian motion model into the continuous time finance.

[2.1] Model (2.1.1) is the traditional log-normal asset price model corresponding to the classical Black-Scholes model.

2.1.1 The ordinary investor

We name the investor an *ordinary investor* if her trading strategy is $self-financing$, in other words, no other money is going in or out the market except the money generated by the trading strategy. Denote $\mathcal{T} \triangleq [0, T]$, we now assume that, in the financial market, a small investor with an initial capital $x(\geq 0)$ can decide, at each time $t \in \mathcal{T}$,

(i) what account of wealth, $\pi_i(t),\ i = 1, ...n$, she should invest in each of the available stocks, and $x(t) - \sum_{i=1}^{n} \pi_i(t)$ is the amount of wealth invested in the risk-free bond;

(ii) what her consumption $c(t)(\geq 0)$ should be.

Define the filtration $\mathcal{F}_t := \sigma\{W(s), 0 \leq s \leq t\}$, which represents the information available to the investor at each time t.

Definition 2.1. A real valued process $f(\cdot)$, defined on Q is *progressively measurable* if the map$(r, \omega) \to f(R, \omega)$ from Q into R is $\mathcal{B}_s \times \mathcal{F}_s$-measurable for each $s \in \mathcal{T}$.

Denote $\pi(t) = (\pi_1(t), \pi_2(t), ..., \pi_n(t))$. Here, $(\pi(t), c(t))$ is called a $portfolio$ or $strategy$, which is $\mathcal{F}_t$- progressively measurable and satisfies certain integral condition. The portfolio is assumed to be rebalanced instantly and free of cost. As a consequence, the wealth process $X(t)$ can be reformulated in terms of a process, which can be expressed as

$$\begin{aligned} dX(t) &= \sum_{i=1}^{n} \pi_i(t) dS(t) + (X(t) - \sum_{i=1}^{n} \pi_i(t))dt - c(t) \\ &= [rX(t) + (\mu_i - r)\pi_i(t)]dt + \pi(t)\sigma dW(t) - c(t). \end{aligned} \tag{2.1.2}$$

2.1.2 The insurer

In recent years, finance and insurance market has started to link. In order to increase profits or decrease risks, the insurance company will invest part of its surplus into financial market, which means that the insurance company, as an investor, will receive stochastic cash flow denoted by its insurance business. There has been increasing attention towards the utilization of stochastic control theory to investment-related problems.

Especially, maximizing the expected exponential utility and minimizing the probability of ruin as two important objective functions in the actuarial literature have attracted a great deal of interest.

In the actuarial science literature, there are two types of risk models to the surplus process, the classical Cramer-Lundberg model and the diffusion approximation. The Cramer-Lundberg model is used in the literature as an approximation to reality if the number of individual contracts is large. Many characteristics of the risk process cannot be calculated in closed form. One therefore often uses a diffusion approximation to the risk model which, hopefully, helps to take almost optimal decisions. For the theory of diffusion approximations, see Schmidli [114] for instance.

We use the surplus model introduced by Promislow and Young [108]. The accumulated claim process $C(t)$ is modeled as

$$dC(t) = kdt - bdW^0(t), \tag{2.1.3}$$

where $W^0(t)$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathrm{P})$, adapted with the filtration $\mathcal{F}_t$ [2.2]. The surplus process $\{X_t; t \geq 0\}$, which represents the liquid assets of the company (also called the risk or the surplus process) is taken as the state variable. With a safety loading $\theta > 0$, the continuously paid premium is assumed to be $c = (1+\theta)k$. Denote $a = \theta k$. In the absence of investment, the wealth is governed by

$$\begin{cases} dX(t) &= cdt - dC(t) \\ &= adt + bdW^0(t), \\ X_0 = x, & \end{cases} \tag{2.1.4}$$

where x denotes the initial reserve.

Strategy I $(\pi(t))$: Suppose that the insurer can invest its surplus in the financial market with $\pi(t) := \{\pi(t,\omega)\}_{t\in\mathcal{T}}$ being the amount invested in the risky assets. Let the assets prices follow (2.1.1), then

$$\begin{cases} dX_t &= (rX_t + \pi(t)(\mu - r\mathbf{1}) + a)dt + \pi(t)\sigma dW(t) + bdW^0(t), \\ X_0 &= x, \end{cases} \tag{2.1.5}$$

Also, the cedent can divert part of her risk to the reinsurer by purchasing the reinsurance. It is another increasingly important element in insurance business.

[2.2] Now $\mathcal{F}_t$ is the natural filtration of (W^0(t), W(t))

Strategy II ($q(t)$): The proportional reinsurance $q(t) := \{q(t,\omega)\}_{t\in\mathcal{T}}$ is a predictable process with $0 \leq q(t) \leq 1$, for each $t \in [0,T]$. If the risk exposure of the company is fixed, then the reinsurer pays $q(t)$ of each claim while the rest is paid by the cedent. To this end, the cedent diverts part of the premiums to the reinsurer at the rate of $(1+\eta)kq(t)$ with a proportional loading of $\eta > \theta$

$$\begin{cases} dX(t) &= (\theta - \eta q(t))adt + b(1-q(t))dW^0(t), \\ X_0 &= x. \end{cases} \tag{2.1.6}$$

Strategy III ($\pi(t), q(t)$): Incorporating the strategy $(\pi(t),\, q(t))$ in (2.1.4) and using the assets prices model (2.1.1) again, the dynamics of the resulting wealth process

$$\begin{cases} dX(t) &= (rX(t)+(\mu - r\mathbf{1})\pi(t))dt + (\theta - \eta q(t))adt + b(1-q(t))dW^0(t) \\ & \quad +\sigma\pi(t)dW(t), \\ X_0 &= x. \end{cases} \tag{2.1.7}$$

All of the above strategies at time t are $\mathcal{F}_t$- progressively measurable and satisfies certain integral conditions.

2.1.3 Model generalization

In a complementary paper [97], Merton modeled stock prices as stochastic processes with the market coefficients depending on the stock price, S, defined as

$$dS(t) = S(t)(\mu(t,S(t))dt + \sigma(t,S(t))dW(t)). \tag{2.1.8}$$

This model allows the market coefficients to vary with time and the stock price, which admits greater latitude when fitting actual market data to the model. For example, there is empirical evidence which suggests that a low stock price increases the stock price volatility more than a high stock price.

Harrison and Kreps [64] and Harrison and Pliska [65, 66] developed the mathematics of continuous-time finance and allowed the stock price processes to be general stochastic processes. This permits the market coefficients to be general random processes, such as in the following equation

$$dS(t) = S(t)(\mu(t)dt + \sigma(t)dW(t)). \tag{2.1.9}$$

Modeling the market coefficients as random processes increases the generality of the stock price model significantly. For example, it allows the use of stochastic volatility models, which provides a better fit of actual market data to the model.

2.1.3.1 Jump diffusion model

In addition to the continuous process based on geometric Brownian motion, the analysis of price risky asset evolution does reveal sudden and rare breaks which could be logically accounted for by exogenous events on information. Such a behavior from probabilistic point of view is naturally modeled by a point process. This process governed by Brownian motion and point process is called jump-diffusion process. The work includes [62, 63, 98] and so on, which have investigated the optimal investment problem. In other field, the jump-diffusion process is mainly introduced in the pricing of options, such as Merton [98], Jones [75] and Jarrow and Rudd [74]. For the mean-variance analysis approach, see Guo and Xu [59].

Here, we model the prices of the risky assets by a geometry Levy process, which satisfies the following stochastic differential equation

$$dS(t) \;=\; D(S(t))(\mu(t)dt + \sigma(t)dW(t)) + \int_{(-1,\infty)^k} \gamma(S(t^-), z)\overline{N}(dt, dz). \quad (2.1.10)$$

Here $W(t)$ is defined as , $\gamma \in R^{n\times l}$, and $\overline{N}(dt, dz) = N(dt, dz) - \lambda(dz)dt$ is the compensator of the homogeneous Poisson random measure N(dt,dz) on $R^+ \times (-1, \infty)^l$ with intensity measure $EN(1, dz) = \lambda(dz)$, where $\lambda(dz)$ is the Levy measure associated with N.

2.1.3.2 Regime switching model

Despite the elasticity of the model in (2.1.9), empirical evidence suggests that a modification of this model, called a regime-switching model, results in a more realistic model of the stock market (see empirical evidence by Gray [17] and Kalimipalli and Susmel [27] and the references therein).

To incorporate the feature of non-stationary returns, recently, regime-switching, or Markov-modulated, models have received much attention among both researchers and market practitioners. From an economic perspective, the Markovian regime-switching model can describe the stochastic evolution of investment opportunity sets due to structural changes in the state of the economy. This important economic feature cannot be captured by a constant-coefficient model. Hamilton [61] pioneered the econometric applications of regime-switching models by considering a discrete-time Markov-switching autoregressive time series model. Since then, regime-switching models, both discrete-time and continuous-time, have found a wide range of applications in economics and finance.

Let $Y := \{Y(t)\}_{t\in\mathcal{T}}$ be a continuous-time, finite-state Markov chain with state space $\mathcal{Y} := (Y_1, Y_2 \ldots, Y_N)$ defined on $(\Omega, \mathcal{F}, \mathcal{P})$. The states of Y are interpreted as different states of an economy. Following Elliott et al. (1994), we shall represent the state $\mathcal{Y}$ as a finite set of unit vectors $\mathcal{E} := \{e_1, e_2, \ldots, e_N\}$, where $e_i = (0, \ldots, 1, \ldots, 0)^\top \in \Re^N$, for each $i = 1, 2, \ldots, N$. This is called the canonical representation of the state space of Y. Suppose that Q denotes the rate matrix or the generator $[q_{ij}]_{i,j=1,2,\ldots,N}$ of Y, which specifies the statistical properties of Y. With the canonical representation of the state space of Y, Elliott *et al.* [38] provided the following semi-martingale decomposition for Y:

$$Y(t) = Y(0) + \int_0^t QY(s)ds + M(t) \ , \tag{2.1.11}$$

where $\{M(t)\}_{t\in I}$ is an R^N-valued martingale with respect to the filtration $\mathcal{F}_t^Y = \sigma\{Y(s), 0 \leq s \leq t\}$.

Assume that the appreciation rate and the volatility of the dynamics switch over time according to the state of a continuous time observed Markov chain, with the included control, the dynamics evolves as:

$$dS(t) \quad = \quad \mu(t, Y(t))dt + \sigma(t, Y(t))dW(t). \tag{2.1.12}$$

Some papers on the use of regime-switching models in finance include Elliott and van der Hoek [40] for asset allocation, Elliott, Hunter and Jamieson [42] and Elliott and Kopp [43] for short rate models, Elliott and Hinz [41] for portfolio analysis and chart analysis, Guo [59] and Elliott *et al.* [39] for option pricing under incomplete markets, Buffington and Elliott Elliott *et al.* [44] for volatility estimation, Elliott *et al.* [39] for valuing options under Markov-switching GARCH models and Elliott *et al.* [46] for pricing and hedging variance and volatility swaps, and others. Regime-switching models provide a natural and convenient way to describe the impact of the structural changes in (macro)-economic conditions and business cycles on the price dynamics. They provided a pertinent way to describe the non-stationary feature of returns of risky assets. More recently, Yin and Zhou [133, 134] established a mean-variance portfolio selection problem under Markovian regime-switching models in a continuous-time economy. They introduced the stochastic linear-quadratic control to deal with the problem and established closed-form solutions to mean-variance efficient portfolios and efficient frontiers.

Gabih, Sass and Wunderlich [54] considered the utility maximization problem with shortfall risk constraints when the dynamics of the stock return is modulated by a continuous-time, finite-state hidden Markov chain. They employed the separation principle to separate the control problem or the utility maximization problem and the filtering problem of the hidden Markov chain. Gundel and Weber [58] obtained closed-form solution to a utility maximization problem under a joint budget and downside risk constraint, where the risk constraint is specified by a class of convex risk measures proposed in Follmer and Schied [51] and Frittelli and Rosazza Gianin [53]. They considered a general semi-martingale framework for the asset price dynamics and developed the closed-form solution based on the martingale approach for constrained maximization problems. Sotomayor and Cadenillas [120] considered an optimal consumption and investment problem with bankruptcy constraint under a Markovian regime-switching model for the asset price dynamics.

2.2 The portfolio selection problem

2.2.1 Utility function

Both modern and classical theories of economic behavior use utility functions to describe the amount of satisfaction of financial agents from wealth or consumption.

A classical example about utility would be that a glass of water has a much higher utility for somebody who is lost in the desert than somebody in the civilization. Although the glass of water might be exactly the same and therefore its price, the two persons in the mentioned situation will perceive its value differently. For an investor given a range of investment choices, a utility function is used to measure the investor's utility of goods or services, by assigning it a numerical value.

In economic literature, for a utility function U, which is the function of wealth x, several properties are considered:

(a) $U'(x) > 0$, it reflects the fact that an investment with higher return has always a higher utility than an investment with a lower return.

(b)(i) $U''(x) > 0$, or (ii) $U''(x) = 0$, or (iii) $U''(x) < 0$, it shows that the investor's attitude to risk is risk-loving, risk-neutral, or risk-averse, respectively.

(c) Define the risk aversion measure *absolute risk aversion*

$$A(x) = -\frac{U'(x)}{U''(x)},$$

then (i) $A'(x) > 0$: increasing absolute risk aversion ($IARA$), (ii) $A'(x) = 0$: constant absolute risk aversion ($CARA$), (iii) $A'(x) < 0$: decreasing absolute risk aversion ($DARA$). This property shows that the investor will increase (keep, or decrease) the amount invested in risky assets when the wealth increases.

(c') The other risk aversion measure is the *relative risk aversion* (RRA)

$$R(x) = -\frac{xU'(x)}{U''(x)},$$

then (i) $A'(x) > 0$: increasing relative risk aversion ($IRRA$), (ii) $A'(x) = 0$: constant relative risk aversion ($CRRA$), (iii) $A'(x) < 0$: decreasing relative risk aversion ($DRRA$). This property is an assumption about the change of the percentage of wealth invested in risky assets as wealth changes.

The following utility functions appear to be frequently used in the literature of economics, insurance and finance. All of them satisfy (a) and (b)(iii).

- Quadratic Utility Function ($IARA$)

$$U(x) = ax - bx^2, a \geq 0, b \geq 0, 0 < x < \frac{a}{2b}.$$

It has the property of $IARA$, which violates the common thought that a plausible risk aversion should decrease, or at least should not increase with x, see [23].

- Power Utility Function ($CRRA$)

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \gamma \geq 0, \gamma \neq 1, x > 0.$$

The power utility has a constant relative risk aversion of γ, and belongs to the class of $CRRA$. It is commonly employed in the portfolio selection problem. As a special case when $\gamma \to 1$, it is reduced to the logarithmic utility function $ln(x)$.

- Exponential Utility Function ($CARA$)

$$U(x) = -e^{-\gamma x}, \gamma > 0,$$

its absolute risk aversion is constant and equal to γ. Exponential utility can produce simple results if asset returns are normally distributed. This utility function plays a prominent role in insurance mathematics and actuarial practice. It is the only utility function under which the principle of zero utility gives a fair premium that is independent of the level of reserve of an insurance company (see [55] page 68]).

2.2.2 The problem formulation

The problem of an ordinary investor

In the work of Merton[96], the situation of an investor is to decide how much to consume and how to allocate her wealth in a risky stock and a bank account of constant interest r, i.e., the objective is to find $(\pi(t), c(t))_{t\in\mathcal{T}}$ so that

$$E[\int_t^T U(c(s,x))dx + U(T, X(T)], \qquad (s,x) \in [0,T] \times R^+,$$

is maximized, with the dynamics following (2.1.2). Here $U(\cdot)$ is the utility function.

For the power utility function $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$, Merton has showed that both risky investment and consumption are a constant fraction of wealth:

$$\alpha(x) = \frac{1}{\gamma}\frac{\mu - r}{\sigma^2}, \qquad c(t,x) = g(t)x,$$

where $g(t)$ is a decreasing function of t.

For simplicity, we denote a strategy by u and the resulting wealth process by X_t^u. Define the corresponding ruin time as

$$\tau^u \triangleq \inf\{t \geq 0: \ X_t^u \leq 0\}, \tag{2.2.1}$$

and denote the probability of ruin given the initial reserve x by

$$\psi_u(x) = P(\tau_u < \infty | X_0 = x).$$

• One objective is to minimize the ruin probability, that is, the goal is to find an optimal strategy u^* such that

$$\psi(x) = \inf_u \psi_u(x). \tag{2.2.2}$$

• Another goal is the final wealth utility maximization problem

$$\sup_{u \in R} E_{t,x} U(T, X(T)), \qquad (t,x) \in [0,T] \times R^+.$$

In the literature, there have been many articles which consider the ruin probability minimization and/ or utility maximization problem by investing the capital in a Black-Scholes market and/ or reinsurance. The literature includes [16, 69, 70, 71, 92, 108, 115, 116, 123]. When the price dynamics of the risky asset are governed by a geometric Brownian motion, the problem has been studied extensively by many researchers, such as Browne [16, 115, 123, 133]. In most of these works, only one risky asset is considered. Bai and Guo [10] have extended it to the case with a multi-asset in a complete market with the constraint of no shortselling.

2.3 Risk measures and risk-related problem

Before introducing the risk measure, the concept of coherent measure is reviewed first.

Coherent measure:

Any acceptable risk measure $\rho: X \to R$ must satisfy the following properties:

• Positive homogeneity: $\rho(\lambda x) = \lambda \rho(x)$ for all random variables x and all positive real numbers λ.

• Subadditivity: $\rho(x + y) \leq \rho(x) + \rho(y)$ for all random variables x and y.

• Monotonicity: $x \leq y$ implies $\rho(x) \geq \rho(y)$ for all random variables x and y.

• Transitional invariance: for all random variables x and real numbers a, $\rho(x+ar_0) = \rho(x) - a$ and all riskless rates r_0.

Remark: It can be proved that any positively homogeneous functional ρ is convex if and only if it is subadditive.

Value at risk

The definitions of VaR and $CVaR$ are borrowed from Rockafellar and Uryasev [109]. Let $g(x)$ be the loss associated with a portfolio. Assume that $g(x)$ is the loss induced by x. Its occurrence probability is denoted by $p(x)$.

Definition of VaR

The probability of $g(x)$ not exceeding a threshold is given by

$$\Psi(a) = \int_{g(x) \leq a} p(x)dx. \tag{2.3.1}$$

Then, for a specified probability level k in (0, 1), the value of the VaR for the loss random variable is defined by

$$VaR \triangleq \min(a \mid \Psi(a) \geq k) \tag{2.3.2}$$

The existing VaR-related academic literature focuses mainly on measuring VaR from different estimation methods to various calculation models. The classical works in VaR methodology distinguish mainly three basic estimation concepts: historical, Monte-Carlo and scenario simulations. Theoretical research that takes the VaR as a risk measurement was initiated by Dowd [33] and Jorion [76] who applied the VaR approach based on risk management emerging as the industry standard by choice or by regulation.

Although VaR can be quite efficiently estimated and managed when underlying risk factors are normally (log-normally) distributed. However, for non-normal distributions, VaR may have undesirable properties and is not coherent [102, 109, 110].

First, it doesn't shed light on the size of extreme losses exceeding the VaR. As a result, the risk manager doesn't know the potential loss he may suffer when VaR is violated. VaR's failure to consider tail losses can then create some perverse outcomes. For instance, if a prospective investment has a high expected return but also involves the possibility of a very high loss, a VaR-based decision calculus might suggest that the investor should go ahead with the investment if the higher loss does not affect the VaR, regardless of the sizes of the higher expected return and possible higher losses.

Second, if the distribution of the return isn't elliptical, sub-additivity of VaR will be lost. This implies that portfolio diversification does not lead to an increase in risk and the $VaRs$ of different risk sources do not add up. As a result, a firm can create artificial subsidiaries in order to save capital, which is against the regular capital management. Moreover, from the *remark*, the absence of sub-additivity will result in the lack of convexity. And this makes it difficult to solve the optimization problems concerned.

Conditional Value-at-Risk

To overcome the drawbacks above, an alternative measure, which is known as the conditional value at risk ($CVaR$), also called Mean Excess Loss, Mean Shortfall, or Tail VaR, is introduced to the optimization modeling in [128].

Definition of CVaR

$$CVaR \triangleq \frac{1}{1-k}\int_{g(x)\geq VaR} g(x)p(x)dx, \tag{2.3.3}$$

By definition, $CVaR$ comes out as the conditional expectation of the loss exceeding VaR. It quantifies the loss beyondVaR effectively, which will be shown in the later application. Moreover, $CVaR$ is a coherent risk measure with desirable properties such as subadditivity and convexity ([109, 122]). $CVaR$ is gaining its application for risk management in the finance and insurance industry (see [47, 110, 124]).

VaR ($CVaR$) related problem

In the static (one-period) setting, Rockafellar and Uryasev [11] considered $CVaR$ minimization with the return requirement. Along the direction of the mean-VaR optimization, Klupperlberg and Korn [83], Alexander and Baptista [3] and Kast et al.[81] studied utility maximization with risk constraint embedded.

The formulation of the problem in a continuous-time was introduced by Luciano [91], Emmer, Kluppelberg and Korn [48] and Basak and Shapiro [11], all of them considered the optimal portfolio allocation problem by maximizing the utility function of an economic agent with the VaR *constraint with continuous trading*. Luciano [91] provided analysis on derivations from the VaR constraint instead of explicitly applying the constraint to the optimal portfolio allocation problem. In Emmer, Kluppelberg and Korn [48], VaR is derived under the assumption that the strategy should maintain the current portfolio weight. However, these conclusions are based on models that are either static or dynamically inconsistent. A dynamically-consistent model is proposed in Basak and Shapiro [11]. However they imposed the VaR constraint only at the initial time and VaR is based on possible portfolio revisions from now on. Moreover, unintended results are demonstrated in their work. They showed that a VaR-risk constrained manager often optimally chooses a larger exposure to risky assets than other managers and consequently incurs larger losses when losses occur. It assumes that the portfolio's VaR is never reevaluated after the initial date. In particular, VaR limits have been found to induce increased risk exposure in some states and an increased probability of extreme losses. Yiu [135] imposed the VaR in a dynamic-consistent manner. To make the calculations tractable, he calculated the constraints abstracting from within-interval trading and from considerations of backtesting, the constraint is $dynamically-consistent$ with the portfolio selection. His approach applied the VaR constraint over time and stresses the repeated recalculations of the $VaRs$. It also described how the VaR affects the investment decision dynamically. In Yiu [135], VaR is calculated under the assumption that the current portfolio is kept unchanged over the VaR horizon period. The measure of VaR in Yiu [135] only requires knowledge of the current portfolio value, the current portfolio composition and the conditional distribution of asset returns. It reflects the actual practice and the fact that financial institutions monitor their traders and do not typically know the traders' future portfolio choices over the VaR horizon.

Sparked by an unprecedented surge in the usage of various risk measures as risk management tools among banks and financial and insurance institutions, more recently, for insurance-related problem, minimizing the VaR and the $CVaR$ as the optimal standard criteria, Tan and Cai [21] and Cai *et al*. [22] study the optimal retentions for the reinsurance and optimal retention level of a stop-loss reinsurance. Zhang *et al*.[136] consider the utility maximization problem by using the $CVaR$ constraint as a risk constraint in the insurance company. They impose the constraint in the same manner as Basak and Shapiro [11]. To the best of our knowledge, the work of Zhang *et al*.[136] is the first one, which imbeds the risk constraint into the problem of utility maximization. However, the shortcoming with their work is similar to [11], since they impose the risk constraint in the same manner.

2.4 The methodology

Stochastic control is the study of dynamical systems subject to random perturbations which can be controlled in order to optimize some performance criterion. A stochastic control problem allows for decisions to be made at each time. These decisions will constitute the solution to the problem.

In the existing literature, there are two approaches for solving the stochastic control problem. The first is using the *dynamic programming* (DP) and the other is the *martingale method*. Merton [96] is the first who applied stochastic control formulation to the portfolio optimization problem with constant coefficients. Gradually the economic literature is dominated by the stochastic dynamic programming approach, which has the advantage that it identifies the optimal strategy automatically as a function of the underlying observables, which is sometimes called a feedback form. To use dynamic programming, a necessary assumption is that the wealth should be governed by Markovian dynamics.

2.4.1 The controlled Markov process and dynamic programming

We focus mainly in this work on finite horizon control problems. However, all the results can easily be extended to the infinite time.

For the finite time interval $\mathcal{T} = [0, T]$, the state of a dynamics system is denoted by $\{X(s)\}_{s\in I}$, valued in the state space O, which is defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a filtration $\mathcal{F}_t$. For later use, we let $Q = [0, T) \times O$ and $\overline{Q}$ denote the closure of Q.

Roughly speaking, the Markov property says that if the present state $X(t)$ is known, then the future is independent of the old. This can be expressed precisely as follows. Consider any finite set of times $s_1 < s_2 < ... < s_m < s \in \mathcal{T} = [0, T]$, then the Markovien property tells that

$$P(X(s) \in \mathcal{B}(\mathcal{R})|X(s_1), ..., X(s_m)) = P(X(s) \in \mathcal{B}(\mathcal{R})|X(s_m)).$$

Before introducing the dynamic programming, the following property is needed.

Let $\theta < \tau$ be a stopping time before T. Assume that $\Phi(t, x) \in C^2(Q) \cap C(\overline{Q})$, with suitable growth condition, and denote the operator

$$A\Phi(t, x) = \lim_{h\to 0^+} h^{-1}[E_{tx}\Phi(t + h, X(t + h)) - \Phi(t, x)]. \tag{2.4.1}$$

The Dynkin formula:

$$(D) \qquad E_{t,x}\Phi(\theta, X(\theta) - \Phi(t, X(t)) = E_{t,x} \int_t^\theta A\Phi(s, x(s))ds$$

2.4.1.1 The controlled Markov process

The control $u = \{u(s), s \in \mathcal{T}\}$ is a progressively measurable process valued in the control set U, a subset of R^n. For the following control models, the control u is called *admissible* if the corresponding stochastic differential equation of the controlled state dynamics has a strong solution and (2.4.3) is well defined. Let $\mathcal{U}$ denote the set of admissible control process u. The controller is assumed to know the history of states when control is chosen now. We denote the dynamics of $X(t)$ with the control included by $X^u(t)$, and let τ be the exit time of $(s, x(s))$ from $\overline{Q}$.

Given two Borelian real-valued functions f and g_τ defined on $[0, T] \times R^n \times U$ and R^n, respectively, define

$$h(t, X(t)) = \begin{cases} g(t, x), & \text{if } (t, x) \in [t_0, T) \times R^n, \\ \\ g_\tau(x), & \text{if } (t, x) \in T \times R^n, \end{cases} \tag{2.4.2}$$

and we define a performance function

$$J(t, x, u) = E\left[\int_0^\tau f(s, X(s), u(s))ds + h(\tau, X(\tau))\right]. \tag{2.4.3}$$

Here further (integrability) conditions on f and g are needed in order that the above expectation is well-defined.

2.4.1.2 The optimal control problem

The stochastic optimal control problem is to find an optimal control $u^* \in \mathcal{U}$ and the *value function* $v(t, x)$ which is defined by

$$v(t, x) = \sup_{u \in \mathcal{U}} J(t, x, u) = J(t, x, u^*). \tag{2.4.4}$$

Historically handled by Bellman's principles, the research on control theory considerably developed over the past few decades, inspired in particular by problems emerging from mathematical finance. Bellman's optimality principle, initiated by Bellman [12] and also called the *dynamic programming* (DP) principle, is a fundamental principle in control theory. It formally means that if one has followed an optimal control decision until some arbitrary observation time, say θ, then, given this information, it remains optimal to use it after θ:

$$(DP) \qquad V(t, x) = \sup_{u \in \mathcal{U}} E\left[\int_t^\theta f(s, X(s), u(s))ds + V(\theta, X(\theta))\right] \tag{2.4.5}$$

Although the (DP) has a clear intuitive meaning, its rigorous proof is technical and has been studied by several authors and by different methods. We refer it to [49, 87].

With the help of $dynkin\ formula$, the (DP) principle leads to solving the Hamilton-Jacobi-Bellman (HJB) equation associated to the stochastic control problem

$$(HJB) \qquad f(t, X(t), u(t)) + \max_{u \in \mathcal{U}} A^u \Phi(t, x) = 0, \tag{2.4.6}$$

where $A^u(x, t)$ denotes the generator of $X(t)$ with constant control u included.

The controlled Markov process is extensively studied in Flemming [49] and Krylov [87]. The verification theorem of controlled jump diffusion process can be found in [100] and the regime switching process in [120], respectively.

Remark 2.1: The Markovian nature of the problem implies that it should suffice to consider control process of the form $u(s) = \underline{u}(s, X(s))$, which is called a $Markov\ control\ policy$. Without loss of generality, the optimal control in the following verification theorem will be chosen as $Markov\ control\ \ policy$.

The classical verification approach consists of finding a smooth solution to the HJB equation, and checking that this candidate, under suitable sufficient conditions, coincides with the value function. This result is usually called a $verification\ theorem$ and provides the optimal control as a byproduct. The assertions of a verification theorem may slightly vary from problem to problem, depending on the required sufficient technical conditions. As the proof is standard, in the following section, we will simply state the verification theorems.

- *The controlled diffusion*

The controlled diffusion is defined by

$$\begin{aligned} dX(t) &= \mu(X(t), u(t))dt + \sigma(X(t), u(t))dW(t), \\ X(0) &= x \in R^n, \end{aligned} \tag{2.4.7}$$

where

$$\mu : R^n \times U \to R^n, \quad \sigma : R^n \times U \to R^{n \times d}.$$

The functions μ and σ are given functions. The control u is called *admissiable* if (2.4.7) has a strong solution and (2.4.3) is well defined. For each constant control $v \in U$, the generator of $X(t)$

$$A^v\Phi(t,x,v) \tag{2.4.8}$$
$$= \frac{\partial \Phi}{\partial t}(t,x,v) + \sum_{i=1}^{n} \mu_i(X(t),v)\frac{\partial \Phi}{\partial x_i}(t,x,v) + \frac{1}{2}\sum_{i,j=1}^{k}(\sigma\sigma^T)_{i,j}(x,v)\frac{\partial^2 \phi}{\partial x_i \partial x_j}(t,x,v).$$

The verification theorem for controlled diffusion model is

Verification Theorem 2.2: Let $\Phi \in C^2(Q) \cap C(\overline{Q})$ be a classical solution of (2.4.6), $E_{t,x}\Phi(\tau, x_\tau) \geq -\infty$ for all $(t,x) \in Q$, and

$$E_{t,x}\int_t^\tau A^{u(s)}\Phi(s,x_s)ds < \infty.$$

Then

(a) $\Phi(t,x) \geq J^u(t,x)$;

(b) Let $\underline{u}^* \in \mathcal{U}$ and satisfy

$$\underline{u}^*(s) \in argmax_{u\in\mathcal{U}}[A^u V(s, x^*(s)) + f(x, x(s), u(s))]$$

for almost all $(s,\omega) \in Q$, then

$$V(t,x) = \Phi(t,x) = J^{\underline{u}^*}(t,x).$$

The first and most famous application in finance of this verification theorem for stochastic control problem is Merton's portfolio selection problem.

- *Stochastic control of Jump diffusions*

The controlled jump diffusion is

$$\begin{aligned} dX(t) &= \mu(t,X(t),u(t))dt + \sigma(t,X(t),u(t))dW(t)) + \\ &\int_{(-1,\infty)^k} \gamma(X(t^-),z,u(t))\overline{N}(dt,dz). \end{aligned} \tag{2.4.9}$$

With the constant control v included, the generator of $X(t)$ is

$$A^v\Phi(x,t) = \sum_{i=1}^{n}\mu_i(x,v)\frac{\partial\Phi}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma^T_{i,j})(x,v)\frac{\partial^2\Phi}{\partial x_i\partial x_j}(x) +$$
$$\sum_{k=1}^{l}\int_R\{\Phi(x+\gamma^k(x,u(x),z_k)) - \Phi(x) - \nabla\cdot\Phi(x)\gamma^k(x,u(x),z_k)\}\gamma_k(dz_k). \tag{2.4.10}$$

Verification Theorem 2.3 **:** Suppose $\Phi \in C^2(Q) \cap C(\overline{Q})$ satisfying the following conditions

a) (i) $A^v\Phi(x) + f(x,v) \le 0$ for all $x \in O$, $v \in U$;

(ii) $E^x[|\Phi(x(\tau))| + \int_0^\tau \{|A\Phi(X(t))| + |\sigma^T(X(t))\nabla\Phi(X(t))|^2 + \sum_{i=1}^l \int_R |\phi(x+\gamma^j(x,u(x),z_j)) - \Phi(x)\}|^2 \gamma_j(dz_j dt)] < \infty$ for all $u \in \mathcal{U}$;

b) moreover, suppose that for each $x \in O$ there exists $u^*(x) \in U$ such that

(iii) $A^{u^*}\phi(x) + f(x, u^*(x)) = 0;$

then u^* is an optimal control and

$$V(x,t) = \Phi(x,t) = J^{u^*}(x,t) \quad \text{for} \quad \text{all } x \in O.$$

- *Controlled regime switching diffusions*

For each constant control u, the controlled switching regime diffusion is defined by

$$\begin{aligned} dX(t) &= \mu(X(t), Y(t), u(t))dt + \sigma(X(t), Y(t), u(t))dW(t), \\ X(0) &= x \in R^n. \end{aligned} \tag{2.4.11}$$

Let $\mathcal{G}_t$ denote the sigma field generated by $\{(X(s), Y(s)) : s \le t\}$. With a constant control policy v, (X(t),Y(t)) is a Markov process with generator

$$A^v\phi(x,t,i) = \frac{\partial \phi}{\partial t} + \mu(X(t), i, v)\frac{\partial \Phi}{\partial x} + \frac{1}{2}\sigma^2(X(t), i, v)\frac{\partial^2 \phi}{\partial x^2} + \langle \phi, QY(t) \rangle_i. \tag{2.4.12}$$

According to Ito formula:

$$\begin{aligned} \Phi(t_1, X(t_1), Y(t_1), u(t_1)) &= \Phi(t, X(t), Y(t), u(t)) + \\ &\int_t^{t_1} [A^{u(s)}\Phi(s, X(s), Y(s))ds + d\bar{M}(s)], \end{aligned} \tag{2.4.13}$$

where $d\bar{M}(s) = \mu(X(s), Y(s), u(s))dW(s) + dM(s)$. With some reasonable assumptions, *Dynkin formula* holds. The resulting HJB equations are the following system:

$$f(t, x(t), u(t), i) + \max_{u \in U} A^u\Phi(x, t, u(t), i) = 0, \quad i = 1, 2, ...n. \tag{2.4.14}$$

Verification Theorem 2.4: Let $\Phi(t,x,i) \in C^2(Q) \cap C(\overline{Q})$ be a classical solution of (2.4.6) for all $i = 1, 2, \ldots, n$, $E_{t,x}\Phi(\tau, x(\tau), y(\tau)) < \infty$ for all $(t,x) \in Q$, and

$$E_{t,x}\int_t^\tau |A^{u(s)}\Phi(s, x(s), y(s))|ds < \infty.$$

Then for all $(t,x) \in Q$:
(a) $\Phi(t,x,i) \geq J^u(t,x,i)$.
(b) Let $\underline{u}^*$ be measurable and satisfying

$$\underline{u}^*(s) \in argmax[A^v V(s,x^*(s),y(s),u(s))] + f(s,x(s),y(s),u(s))$$

for almost all $(s,\omega) \in Q$, then

$$\Phi(t,x,i) = V(t,x,i) = J^{u^*}(t,x,i).$$

Remark 2.5: Note that now the Markov control policy $u = \underline{u}(t, X(t), Y(t))$.

2.4.2 Martingale method

The classical optimal investment/consumption problem dealing with the *martingale method* is initially treated under the assumption of complete markets [2.3], which implies that the family of martingale measures is a singleton. This approach was developed by Cox and Huang [25, 26], Karatzas *et al.* [80] and Pliska [105]. With the help of the Girsanov Theorem, the original probability can be changed into an equivalent martingale measure under which all the stock prices discounted by the bond rate become martingales. The fact that every martingale relative to a Brownian filtration can be represented as a stochastic integral with respect to the underlying Brownian motion plays a key role in the proof. Moreover, the consumption process can also be financed, that is, there is a corresponding portfolio process which, together with the consumption process, results in a nonnegative wealth process. Then those authors found a simple expression for the optimal investment/consumption process.

Difficulties with this approach arise in incomplete markets. Fortunately, the introduction by Harrison and Kreps [64], Harrison and Pliska [65] and Ross [112] of the

[2.3] Incomplete markets in Mathematical Finance correspond to a setting, in which the controller has full information about many aspects of the system (the market), but various exogenously imposed constraints (taxation, transaction costs, bad credit rating, legislature etc.) prevent him/her from choosing the control (portfolio) outside a given constraint set. In fact, even without government-imposed portfolio constraints, financial markets will typically not offer tradable assets corresponding to a variety of sources of uncertainty (weather conditions, non-listed companies, etc.) The financial agent will still observe many of these sources, as their uncertainty evolves, but will typically not be able to trade in all of them notion of equivalent martingale measures has created the possibility of solving such problems by *convex duality* methods. One distinctive aspect of this approach is that it relates the original, or "primal", stochastic control problem to a certain"dual" one, in the sense that a solution to the primal problem induces a solution for the dual (and vice versa). This duality goes back to Bismut [14], and is exploited by many authors, such as [67, 68, 79], and more recently by Kramkov and Schachermayer [86]. They related the marginal utility from the terminal wealth of the optimal portfolio to the density of the martingale measure, using powerful $convex - duality$ techniques. In particular, the papers by Kramkov and Schachermayer [86] discussed the minimal conditions on the agent's utility function and the financial market model. Since then, stochastic duality theory proved to be remarkably successful as a method of solving portfolio selection problems, again because of its capability to exploit the underlying convexity.

Convex duality methods establish a connection between the original problem, called the *primal problem*, and another problem, called the *dual problem*. A common theme of all these papers is to take the original problem, which involves a maximization over a class of policies, and restate it in terms of the dual problem, which involves a minimization over some family of "constructed"measures. The hope is that the dual problem is easier to solve than the primal problem. The convexity properties of the primal problem are critical in establishing the connection between this problem and the corresponding dual problem. This connection allows us to construct the solution to the primal problem by using the solution to the dual problem.

The work mentioned above dealt with the application of martingale (duality) to problems in which there are no portfolio constraints, that is, at every instant the investor can freely distribute the wealth among all of the assets. However, there are a range of issues where the portfolio may be restricted in some way (see [29, 132]), or where the objective may be to super-replicate some contingent claim while observing a portfolio constraint. For example, the holding of the money-market account should never be below some fixed value (see Cvitanic and Karatzas [30], Karatzas and Kou [77]). Another example is an optimal investment/consumption problem in the presence of transaction costs (see Cvitanic and Karatzas [31]). Their method of solution involves a completion of the incomplete market. This is called a *fictitious completion*, since the market is completed with fictitious stocks. The fictitious stocks are carefully chosen so that the optimal portfolio will not be invested in them. The optimal portfolio process in the fictitious market will then be a potential solution in the original, incomplete market. The authors construct many fictitious markets and find the optimal portfolio process in each one. The optimal solution in the original and incomplete market is then the optimal portfolio process which maximizes the expected utility of terminal wealth.

Although it appears attractable to apply convex duality to portfolio selection problems, there has been a severe lack of transparency when applying convex duality methods, as pointed out in Rogers [111]. The method works only when the candidate dual problem can be produced as desired. However, little is known about when the dual problem is good enough, especially, for the problems with portfolio constraints, which are studied by the group of Cvitanic and Karatzas with fictitious markets. The introductory comments in Cvitanic and Karatzas [29] suggested that the construction of the fictitious markets is the outcome of a good deal of patient experimentation (the situation is not unlike that of solving a complicated differential equation, in which one might patiently experiment with different candidate solutions to eventually come up with the actual solution, the correctness of which is verified by substitution).

The comparison of DP method and the martingale method:

- For DP, the optimal control is first derived as a function of the $value\ function$ and then substituted into the HJB equation and the solution leads to the $value\ function$, while for the latter the procedure goes on in the inverse direction: the value function is first derived without referring to the control and the optimal strategy is then obtained from the martingale representation according to the value function.

- DP is a dynamic approach, while the latter is static.

- DP needs the Markovianity, while the latter needs a martingale measure.

2.4.3 The deterministic transform

Let the dynamics of the wealth be denoted by $X(t)$. Most of the work focuses on the models that $X(t)$ or $\ln X(t)$ is a semimartingale, therefore the decomposition of the utility function of $U(X(T))$ into the sum/product of a deterministic term and a martingale term can be expected. That is, $U(X(T)) = U_1(X_1(T)) + U_2(X_2(T))$ or $U(X(T)) = U_1(X_1(T)) \cdot U_2(X_2(T))$, where $X_1(T)$ is a deterministic process and $U_2(X_2(t))$ is martingale. If we succeed in this decomposition, the treatment of the expectation of the utility can be greatly simplified for the deterministic problem.

Flemming and Hernandez [50] deal with the investment problem, when $\ln X(t)$ has a decomposition of the deterministic part and Brownian martingale. The problem of the $HARA$ utility function is reduced to the deterministic problem by taking expectation. Later Flemming and Shiu [51] consider the same problem but with the stochastic volatility and consumption. Using the same argument, the problem is simplified by reducing an uncertainty.

In this work, we treat the problem of embedding the risk constraint into the utility maximization. Experiential utility, which is popular in actuary mathematics, is used here. The problem is reduced to a deterministic optimal control problem. Thus, it is easily solved by using some software packages for the deterministic control problem.

Chapter 3

Risk management in insurance company

In this chapter, we consider two important problems in insurance: minimizing the ruin probability and maximizing the expected utility of final wealth.

With investment and/or proportional reinsurance applied, the problems can be posed as stochastic optimal control problems. In the literature, they are dominated by the dynamic programming methods (see for example [10, 16, 133]). Wang *et al.*[131] is the only one who applies the martingale approach, when the risk process is modeled by the Levy process to investigate the optimal investment for an insurer.

In this chapter, VaR, which is dynamically-consistent with the control, is applied to influence the decision. We obtain the closed-form solution for the ruin probability minimization. Due to the appearance of the portfolio constraint, unfortunately, it is difficult to find the closed-form solution when treating the utility solution by either dynamic programming or martingale methods. However, for the exponential utility function, the problem can be transformed into the optimal deterministic problem and it can be solved by the numerical soft package.

3.1 Minimizing ruin probability

Suppose that the insurer is allowed to invest her surplus in a financial market consisting of a risk-free asset (bond or bank account) and a risky asset (stock or mutual fund). A strategy u is described by a stochastic process π_t, which represents the amount invested in the risky asset at time t. A restriction considered in this paper is the prohibition of short selling of the risky assets. We analyze VaR over a short time horizon and impose it as a risk constraint. Then the problem is how to minimize the probability of ruin with a VaR constraint.

In the absence of control we use the reserve model

$$\begin{cases} dX_t = adt + bdW^0(t), \\ X_0 = x, \end{cases} \tag{3.1.1}$$

where x is the initial reserve.

Specifically, the price process of the risk-free asset is given by

$$dB_t = rB_t dt, \qquad\qquad r > 0,$$

and the price process of the risky asset follows geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW(t), \qquad \mu > r,$$

where r, μ and σ are constants, $\rho_S (0 \leq \rho_S^2 \leq 1)$ is the correlation coefficient between $W(t)$ and $W^0(t)$. As pointed out in [16], the case that $\rho_S^2 = 1$ is uninteresting since there is only one source of randomness. We will not consider this uninteresting case. $W(t)$ can be rewritten as $\sqrt{1-\rho^2}W^1(t) + \rho W^0(t)$, where $W^0(t)$ and $W^1(t)$ are independent. Incorporating strategy u (*or* $\pi(t)$) into (3.1.1), the resulting reserver process X_t^u can be rewritten as

$$\begin{cases} dX_t^u &= (rX_t^u + \pi_t(\mu - r\mathbf{1}) + a)dt + \pi_t\sigma\sqrt{1-\rho^2}dW^1(t) + (\pi_t\sigma\rho + b)dW^0(t), \\ X_0 &= x, \end{cases} \tag{3.1.2}$$

We call a strategy π *admissible* if it is $\mathcal{F}_t$- progressively measurable, and satisfies $E\int_0^\infty \pi_t^2 dt < \infty$. Let $\mathcal{U}_{I_1}$ denote the class of all *admissible* strategies u.

For each admissible strategy π, the corresponding ruin time is

$$\tau^\pi := \inf\{t \geq 0 : \ X_t^\pi \leq 0\}, \tag{3.1.3}$$

the probability of ruin is

$$\psi_\pi(x) = P(\tau_\pi < \infty | X_0 = x),$$

and the minimal probability of ruin is

$$\psi(x) = \inf_\pi \psi_\pi(x). \tag{3.1.4}$$

By Fleming and Soner [49], the problem is reduced to solving the associated (HJB) equation

$$(rx+a)\psi^{'}(x) + \frac{1}{2}b^2\psi^{''}(x) + \min_\pi [(\mu - r)\pi\psi^{'}(x) + (\frac{1}{2}\sigma^2\pi^2 + \sigma b\rho_S\pi)\psi^{''}(x)] = 0, \tag{3.1.5}$$

with the boundary condition

$$\psi(0) = 1, \quad \psi(\infty) = 0. \tag{3.1.6}$$

The problem has been studied by Browne [16] by the following verification theorem of the HJB equation.

Verification theorem I_1: Let $\Phi(x)$ be a convex smooth solution of (3.1.6) with the boundary condition (3.1.17), and

$$\pi^* = \arg min_\pi \{[(\mu - r)\pi\psi^{'}(x) + (\frac{1}{2}\sigma^2\pi^2 + \sigma b\rho_S\pi)\psi^{''}(x)]\} \tag{3.1.7}$$

then $\psi(x)$= $\Psi(x)$ and π^* is the optimal portfolio.

The proof of this theorem is standard (see Taksar and Markussen [123], Nrylov Chapter I, or Browne [16]). When there is no VaR constraint, the main result is Lemma 3.1.1 by Browne [16].

Lemma 3.1.1. The optimal strategy is given by

$$\pi^0(x) = \frac{1}{\mu - r}\left[-(rx+a) + \sqrt{(rx+X)^2 + Y^2}\right], \tag{3.1.8}$$

where $X = a - \frac{b\rho_S(\mu-r)}{\sigma}$ and $Y = \frac{b\sqrt{1-\rho_S^2}(\mu-r)}{\sigma}$.
Then, the optimal investment strategy without constraint is

$$\pi_t^* = \frac{1}{\mu - r}\left[-(rX_t^* + a) + \sqrt{(rX_t^* + a - \frac{b\rho_S(\mu - r)}{\sigma})^2 + \frac{b^2(1-\rho_S^2)(\mu - r)^2}{\sigma^2}}\right]. \tag{3.1.9}$$

It follows from (3.1.8) that $\pi(x)$ is decreasing with respect to x.

When short selling is forbidden, we use the superscript $\pi^0(x)$ to denote the optimal strategy. It can be seen from (3.1.9) that if $\rho \le 0$, then $\pi(0) > 0$ and $\pi(\infty) > 0$, and the optimal strategy follows that in Lemma 2.1. However, if $\rho > 0$, then $\pi(\infty) < 0$, the short selling constraint is active when x is larger than some value.

To facilitate our proof later, we summarize some results from Schmidli [114] here.

Lemma 3.1.2. When no short selling is imposed as the constraint, if $\pi(0) \le 0$, then the optimal strategyis $\pi^0(0) = 0$. If $\pi(0) > 0$, we denote

$$x_0 = \begin{cases} \frac{b\rho(\mu-r)}{2\sigma r} + \frac{b(1-\rho^2)(\mu-r)}{2\sigma\rho r} - \frac{a}{r}, & \rho > 0, \\ \\ \infty, & \rho \le 0, \end{cases} \tag{3.1.10}$$

then the optimal strategy is given by

$$\pi^0(x) = \begin{cases} \frac{-(rx+a)+\sqrt{(rx+X)^2+Y^2}}{\mu-r}, & 0 \le x < x_0, \\ \\ 0, & x \ge x_0, \end{cases} \tag{3.1.11}$$

which is also a decreasing function with respect to x.

3.1.1 VaR constraint

In order to regulate the market risk and for the market supervision, a portfolio should be able to control the level of risk. In this work, VaR is used as the risk measure. Denote

$$\widetilde{\pi}_t := (1, \pi_t),$$

$$\widetilde{\sigma} := \begin{pmatrix} b & 0 \\ \rho\sigma & \sqrt{(1-\rho^2)}\sigma \end{pmatrix},$$

and

$$\widetilde{W}(t) := (W^0(t), W^1(t))^\top,$$

which is a two- dimensional standard Brownian motion.

Let $\Delta t = s - t$ be the horizon period, integrating (3.1.2) leads to

$$e^{-rs}X_s^\pi - e^{-rt}X_t^\pi = \int_t^s e^{-ru}(\pi(u)(\mu - r) + a)du + \int_t^s e^{-ru}\widetilde{\pi}(u)\widetilde{\sigma}d\widetilde{W}(u). \quad (3.1.12)$$

Assuming that the portfolio is adjusted frequently and the interval $[t, s)$ is small, we can approximate $\pi(u)$ by $\pi(t)$. This means that there is no trading between constraint re-evaluation, and the investment is roughly constant in the given horizon period. It is a reasonable approximation since the portfolio can only be adjusted in discrete time and the decision made is based on the holdings at time t. Denote $e^{-rs}X_t^\pi$ by Y_t. From (3.1.12), we have

$$Y_s^\pi = Y_t^\pi + (\pi_t(\mu - r) + a)\frac{e^{-rs} - e^{-rt}}{-r} + \int_t^s e^{r(s-u)}\widetilde{\pi}(u)\sigma d\widetilde{W}(u).$$

Then, the conditional mean on time t is given by

$$E_t X_s^\pi = e^{r(s-t)}X^\pi(t) + (\pi_t(\mu - r) + a)\frac{1 - e^{r(s-t)}}{-r},$$

and the conditional covariance is given by

$$Cov_t[X_s^\pi, X_u^\pi] = \widetilde{\pi}_t\widetilde{\sigma}\widetilde{\sigma}^\top\widetilde{\pi}_t^\top e^{r|s-u|-e^{r(s+u-2t)}}.$$

The conditional variance is therefore given by

$$Var_t(X_s^\pi) = \frac{\widetilde{\pi}\widetilde{\sigma}\widetilde{\sigma}^\top\widetilde{\pi}^\top}{-2r}(1 - e^{2r(s-t)}).$$

We denote the discounted loss $e^{-r\Delta t}X_{t+\Delta t}^\pi - X_t^\pi$ by $\triangle X^\pi(t)$. Another definition of VaR

$$P(\triangle X_t^\pi \leq VaR_t) = k$$

implies that

$$VaR_t = \phi^{-1}(k)\sqrt{\frac{\widetilde{\pi}_t\widetilde{\sigma}\widetilde{\sigma}^\top\widetilde{\pi}_t^\top}{-2r}(1 - e^{2r\Delta t})} + (\pi_t(\mu - r) + a)\frac{1 - e^{r\Delta t}}{r}.$$

The risk constraint may now be imposed, i.e.,

$$VaR \leq \check{R}, \quad (3.1.13)$$

where $\check{R}$ denotes the predefined risk level that the investor can tolerate. Then the constraint of restricting VaR at level $\check{R}$ is

$$k_1\sqrt{\sigma^2\pi_t^2 + 2\sigma b\rho_S\pi_t + b^2} - k_2\pi_t \leq R,$$

where

$$k_1 = \phi^{-1}(k)\sqrt{\frac{e^{2r\Delta t}-1}{2r}}, \quad k_2 = (\mu - r)\frac{e^{r\Delta t}-1}{r}, \quad \text{and } R = \check{R} - \frac{e^{r\Delta t}-1}{r}a.$$

Let $\mathcal{U}_{I_1}$ denote the set of all admissible strategies which satisfy

$$k_1\sqrt{\sigma^2\pi_t^2 + 2\sigma b\rho_S\pi_t + b^2} - k_2\pi_t \le R. \tag{3.1.14}$$

Then the optimal portfolio problem with VaR constraint is to find the optimal strategy $\pi^* \in \mathcal{U}_{I_1}$ such that

$$\psi(x) = \inf_{\pi\in\mathcal{U}} \psi_\pi(x), \tag{3.1.15}$$

which is regarded as **Problem $\mathbf{I}_1$.**

3.1.2 The HJB equation and its solutions

To solve the problem with both short selling and VaR constraint, we use the dynamic programming approach described in Fleming and Soner (1993). With the help of dynamic programming, the problem is to solve the HJB equation with VaR constraint:

$$\begin{aligned}&(rx+a)\psi'(x) + \frac{1}{2}b^2\psi''(x) + \\ &\min_{k_1\sqrt{\sigma^2\pi^2+2\sigma b\rho_S\pi+b^2}-k_2\pi\le R}[(\mu-r)\pi\psi'(x) + (\frac{1}{2}\sigma^2\pi^2 + \sigma b\rho_S\pi)\psi''(x)] = 0,\end{aligned} \tag{3.1.16}$$

with the boundary condition

$$\psi(0) = 1, \quad \psi(\infty) = 0. \tag{3.1.17}$$

Rearranging the constraint condition

$$k_1\sqrt{\sigma^2\pi^2 + 2\sigma b\rho_S\pi + b^2} - k_2\pi \le R$$

yields

$$(k_1^2\sigma^2 - k_2^2)\pi^2 + 2(k_1^2\sigma b\rho_S - k_2R)\pi + k_1^2b^2 - R^2 \le 0. \tag{3.1.18}$$

As the risk arising from the insurance itself cannot be eliminated, we hope to adjust the risky investment π to satisfy the risk constraint due to the correlations and mutual effect of insurance and investment. Before discussing the problem with constraint, we investigate how VaR changes with π. Denote

$$V(\pi) = k_1\sqrt{\sigma^2\pi^2 + 2\sigma b\rho\pi + b^2} - k_2\pi,$$

then the constraint condition is

$$V(\pi) \leq R. \tag{3.1.19}$$

The derivative of $V(\pi)$

$$V'(\pi) \quad = \quad \frac{k_1\sigma(\sigma\pi + \rho b)}{\sqrt{(\sigma\pi + \rho b)^2 + (1-\rho^2)b^2}} - k_2. \tag{3.1.20}$$

From this expression, it is easily seen that the following properties hold.

(a) If σ is so large such that $\frac{k_1\sigma}{\sqrt{1+\frac{(1-\rho^2)}{\rho^2}}} > k_2$, we will differentiate the case $\rho \geq 0$ or $\rho < 0$. If $\rho \geq 0$, then VaR always increases with π. In this case, to decrease VaR we should decrease the risky investment. While if $\rho < 0$, then $V'(\pi) < 0$ in the interval $[0, \sqrt{\frac{(1-\rho^2)b^2k_2^2}{(k_1^2\sigma^2-k_2^2)\sigma^2}} - \frac{\rho b}{\sigma})$, and $V'(\pi) \geq 0$ in the interval $[\sqrt{\frac{(1-\rho^2)b^2k_2^2}{(k_1^2\sigma^2-k_2^2)\sigma^2}} - \frac{\rho b}{\sigma}, \infty)$. To decrease the risk, the adjustment of risky investment is similar to (b).

(b) If σ is large enough to make $k_1\sigma > k_2$ but $\frac{k_1\sigma}{\sqrt{1+\frac{(1-\rho^2)}{\rho^2}}} < k_2$ hold, then whether $\rho \geq 0$ or $\rho < 0$, we have $V'(\pi) < 0$ in $(0, \sqrt{\frac{(1-\rho^2)b^2k_2^2}{(k_1^2\sigma^2-k_2^2)\sigma^2}} - \frac{\rho b}{\sigma}]$ and $V'(\pi) \geq 0$ in $[\sqrt{\frac{(1-\rho^2)b^2k_2^2}{(k_1^2\sigma^2-k_2^2)\sigma^2}} - \frac{\rho b}{\sigma}, \infty)$, that is, VaR becomes smaller first and becomes larger later when π becomes larger. In this case, to decrease VaR, we hope to increase the risky investment if it is small and decrease it if it is too large.

(c) If σ is small, which leads to $k_1\sigma \leq k_2$, then whether $\rho \geq 0$ or not, $V'(\pi) < 0$ for all nonnegative π, that is, VaR decreases with π. In fact, $k_1\sigma \leq k_2$ means that the insurer has certain confidence (the given probability) that the risky investment will bring return. As a result, increasing risky investment can eliminate the risk.

From the analysis, to decrease the risk, we expect to improve risky investment when $V'(\pi) < 0$ and cut it if $V'(\pi) \geq 0$. In fact, this can be justified by the following results.

In this work, for the problem with constraint, we obtain the explicit expression of optimal risky investment and analytical solution of the ruin probability by solving the resulted HJB equation. To show the effect of risk constraint, we compare the investment strategies with those without constraint.

The complete results are listed in Table 4.1. The results justify the analysis about VaR above. In this table, ¶ denotes the strategy interval that is optimal for the no short selling constraint but violate the VaR constraint, π_1 and π_2 are defined by the roots of (3.1.21), x_1, x_2 and x_γ are defined by (3.1.27), (3.1.34) and (3.1.76), respectively.

The details are given in the following sections.

Case 3.1.1 *The case of* $k_1^2\sigma^2 - k_2^2 > 0$

Denote

$$\begin{aligned}\triangle &= 4(k_1^2\sigma b\rho_S - k_2 R)^2 - 4(k_1^2\sigma^2 - k_2^2)(k_1^2 b^2 - R^2)\\ &= 4k_1^2(k_1^2\sigma^2 b^2\rho_S^2 + \sigma^2 R^2 + k_2^2 b^2 - k_1^2\sigma^2 b^2 - 2k_2\sigma b\rho_S R).\end{aligned}$$

Then we solve the optimal problem in the following cases:
If $\triangle < 0$, there does not exist π which satisfies (3.1.18). It is a trivial case and makes no sense as a too small value is used for R.
If $\triangle \geq 0$, the equation

$$(k_1^2\sigma^2 - k_2^2)\pi^2 + 2(k_1^2\sigma b\rho_S - k_2 R)\pi + k_1^2 b^2 - R^2 = 0 \tag{3.1.21}$$

has two roots and we denote them by π_1 and π_2, respectively.

Case 3.1.1-(i) If $\pi_1 > 0$, then the solution to (3.1.18) is $[\pi_1, \pi_2]$, which coincides with that of (3.1.19). Thus (3.1.16) becomes

$$(rx+a)\psi^{'}(x) + \frac{1}{2}b^2\psi^{''}(x) + \min_{\pi_1\leq\pi\leq\pi_2}[(\mu - r)\pi\psi^{'}(x) + (\frac{1}{2}\sigma^2\pi^2 + \sigma b\rho\pi)\psi^{''}(x)]\} = 0. \tag{3.1.22}$$

with the boundary condition

$$\psi(0) = 1, \quad \psi(\infty) = 0. \tag{3.1.23}$$

From (3.1.8), we can see that $\pi(x)$ is strictly decreasing with respect to x in $[0, x_0)$, and be the constant 0 when $x \geq x_0$. Therefore, the maximizer is attained at $x = 0$, and

$k_1^2\sigma^2 - k_2^2 > 0$					
$\Delta < 0$, There is no admissible strategies					
$\Delta \geq 0$			x	$\pi^0(x)$	$\pi^*(x) =$
	$\pi_1 > 0$	$\pi_1 > \pi^0(0)$	$\forall x$	¶ $\pi^0(x) < \pi_1$	π_1
	admisssiable interval $[\pi_1, \pi_2]$	$\pi_1 \leq \pi^0(0) \leq \pi_2$	$x \in [0, x_1)$	$\pi^0(x) \in [\pi_1, \pi_2]$	$\pi^0(x)$
			$x \in [x_1, \infty]$	¶ $\pi^0(x) < \pi_1$	π_1
		$\pi_2 < \pi^0(0)$	$x \in [x_1, \infty)$	¶ $\pi^0(x) < \pi_1$	π_1
			$x \in [x_2, x_1)$	$\pi^0(x) \in (\pi_1, \pi_2]$	$\pi^0(x)$
			$x \in [0, x_2)$	¶ $\pi^0(x) > \pi_2$	π_2
	$\pi_1 \leq 0$ *admissiable interval* $[0, \pi_2]$	$\pi_2 \geq \pi^0(0)$	$\forall x$	$\pi^0(x) \in [0, \pi_2]$	$\pi^0(x)$
		$\pi_2 < \pi^0(0)$	$x \in [0, x_2)$	¶ $\pi^0(x) > \pi_2$	π_2
			$x \in [x_2, \infty)$	$\pi^0(x) \in [0, \pi_2]$	$\pi^0(x)$
$k_1^2\sigma^2 - k_2^2 < 0$					
$\Delta < 0$, the strategy is still optimal in the constrained case					
$\Delta \geq 0$	*admisssiable interval* : $[\pi_2, \infty)$		x	$\pi^0(x)$	$\pi^*(x) =$
		$\pi_2 > \pi^0(0)$	$\forall x$	¶ $\pi^0(x) < \pi_2$	π_2
		$\pi^0(0) \geq \pi_2$	$x \in [0, x_2)$	$\pi^0(x) \in (\pi_2, \infty)$	$\pi^0(x)$
			$x \in [x_2, \infty)$	¶ $\pi^0(x) \leq \pi_2$	π_2
$k_1^2\sigma^2 - k_2^2 = 0$					
$k_1^2\sigma b\rho - k_2 R > 0$, There is no admissible strategies					
			x	$\pi^0(x)$	$\pi^*(x) =$
$k_1^2\sigma b\rho - k_2 R < 0$	$R^2 - k_1^2 b^2 > 0$	$\pi^0(x)$ is still optimal in the constrained case			
	$R^2 - k_1^2 b^2 \leq 0$ *admisssiable interval* : $[\gamma, \infty)$	$\gamma > \pi^0(0)$	$\forall x$	¶ $\pi^0(x) < \gamma$	γ
		$\gamma \leq \pi^0(0)$	$[0, x_\gamma)$	$\pi^0(x) > \gamma$	$\pi^0(x)$
			$[x_\gamma, \infty)$	$\pi^0(x) \leq \gamma$	γ
$k_1^2\sigma b\rho - k_2 R = 0$	$R^2 - k_1^2 b^2 > 0$	$\pi^0(x)$ is still optimal in the constrained case			
	$R^2 - k_1^2 b^2 < 0$	There is no optimal strategy			

Table 3.1: The main results when VaR constraint is imposed.

equals $\frac{-a+\sqrt{a^2-2\frac{ab\rho(\mu-r)}{\sigma}+\frac{b^2(\mu-r)^2}{\sigma^2}}}{\mu-r}$.

Case 3.1.1-(i)-(a)

If $\pi_1 > \pi^0(0)$, the optimal strategy$\pi^0(x)$ given by (3.1.11) does not satisfy (3.1.22) for any $x \geq 0$. Let $\pi^*(x)$ denote the optimal strategywith constraint, we guess $\pi^*(x)$ as

$$\pi^*(x) = \pi_1, \quad x \geq 0,$$

and the corresponding solution of (3.1.22) is

$$f(x) = 1 - c_{1,1} \int_0^x e^{\int_0^y g_1(u)du} dy, \tag{3.1.24}$$

where

$$g_1(u) = -\frac{ru + a + (\mu - r)\pi_1}{\frac{1}{2}b^2 + \frac{1}{2}\sigma^2\pi_1^2 + \sigma\pi_1 b\rho}, \tag{3.1.25}$$

$$c_{1,1} = \frac{1}{\int_0^\infty e^{\int_0^y g_1(u)du} dy}. \tag{3.1.26}$$

In fact, the guess is true according to Theorem 3.1.1(a).

Case 3.1.1-(i)-(b)

If $\pi_2 \geq \pi^0(0) \geq \pi_1$, denote

$$x_1 = \begin{cases} \check{x}_1, & \text{if there exists } \check{x}_1 > 0 \text{ such that } \pi^0(\check{x}_1) = \pi_1, \\ \\ \infty, & \text{otherwise.} \end{cases} \tag{3.1.27}$$

For $x > x_1$, $\pi^0(x) < \pi_1$ due to the decreasing property of $\pi^0(x)$. Therefore $\pi^0(x)$ does not satisfy (3.1.18) on (x_1, ∞). In this case, we guess that

$$\pi^*(x) = \begin{cases} \frac{-(rx+a)+\sqrt{(rx+X)^2+Y^2}}{\mu-r}, & 0 < x < x_1, \\ \\ \pi_1, & x \geq x_1. \end{cases} \tag{3.1.28}$$

Plugging (3.1.28) into (3.1.22) and solving the resulting equation, we get that

$$f(x) = \begin{cases} 1 - c_{2,1} \int_0^x \exp[-(\frac{\mu-r}{\sigma})^2 \int_0^y \frac{du}{-(ru+X)+\sqrt{(ru+X)^2+Y^2}}]dy, & 0 < x < x_1, \\ \\ c_{2,2} + c_{2,3} \int_{x_1}^x e^{\int_{x_1}^y g_1(u)du} dy, & x \geq x_1, \end{cases} \tag{3.1.29}$$

where

$$c_{2,1} = \frac{1}{\int_0^{x_1} \exp[-(\frac{\mu-r}{\sigma})^2 \int_0^y \frac{du}{-(ru+X)+\sqrt{(ru+X)^2+Y^2}}]dy + M}, \tag{3.1.30}$$

$$c_{2,2} = 1 - c_{2,1} \int_0^{x_1} \exp[-(\frac{\mu-r}{\sigma})^2 \int_0^y \frac{du}{-(ru+X)+\sqrt{(ru+X)^2+Y^2}}]dy, \tag{3.1.31}$$

$$c_{2,3} = -c_{2,1} \exp[-(\frac{\mu-r}{\sigma})^2 \int_0^{x_1} \frac{du}{-(ru+X)+\sqrt{(ru+X)^2+Y^2}}], \tag{3.1.32}$$

and

$$M = \exp[-(\frac{\mu-r}{\sigma})^2 \int_0^{x_1} \frac{du}{-(ru+X)+\sqrt{(ru+X)^2+Y^2}}] \int_{x_1}^{\infty} e^{\int_{x_1}^y g_1(u)du} dy. \tag{3.1.33}$$

This guess is justified by Theorem 3.1.1.(b).

Case 3.1.1-(i)-(c)

If $\pi^0(0) > \pi_2 \geq \pi_1$, then $\pi^0(x) > \pi_2$ when $0 \leq x < x_2$ and $\pi^0(x) < \pi_1$ when $x > x_1$, which does not satisfy the constraint. Denote

$$x_2 = \begin{cases} \check{x}_2, & \text{if there exists } \check{x}_2 > 0 \text{ such that } \pi^0(\check{x}_2) = \pi_2, \\ \infty, & \text{otherwise.} \end{cases} \tag{3.1.34}$$

Similarly, we guess that

$$\pi^*(x) = \begin{cases} \pi_2, & 0 \leq x < x_2, \\ \frac{-(rx+a)+\sqrt{(rx+X)^2+Y^2}}{\mu-r}, & x_2 \leq x < x_1, \\ \pi_1, & x \geq x_1, \end{cases} \tag{3.1.35}$$

and

$$f(x) = \begin{cases} 1 - c_{3,1} \int_0^x e^{\int_0^y g_2(u)du} dy, & 0 \leq x < x_2, \\ c_{3,2} + c_{3,3} \int_{x_2}^x \exp[-(\frac{\mu-r}{\sigma})^2 \int_{x_2}^y \frac{du}{-(ru+X)+\sqrt{(ru+X)^2+Y^2}}]dy, & x_2 \leq x < x_1, \\ c_{3,4} + c_{3,5} \int_{x_1}^x e^{\int_{x_1}^y g_1(u)du} dy, & x \geq x_1, \end{cases} \tag{3.1.36}$$

where

$$g_2(u) = -\frac{ru + a + (\mu - r)\pi_2}{\frac{1}{2}b^2 + \frac{1}{2}\sigma^2\pi_2^2 + \sigma\pi_2 b\rho}, \quad (3.1.37)$$

$$c_{3,3} = \frac{-1}{e^{-\int_0^{x_2} g_2(u)du} + \int_{x_2}^{x_1} \exp[-(\frac{\mu-r}{\sigma})^2 \int_{x_2}^{y} \frac{du}{-(ru+X)+\sqrt{(ru+X)^2+Y^2}}]dy + N}, \quad (3.1.38)$$

$$c_{3,1} = -c_{3,3}e^{-\int_0^{x_2} g_2(u)du}, \quad (3.1.39)$$

$$c_{3,2} = 1 - c_{3,1}\int_0^{x_2} e^{\int_0^y g_2(u)du}dy, \quad (3.1.40)$$

$$c_{3,4} = c_{3,2} + c_{3,3}\int_{x_2}^{x_1} \exp[-(\frac{\mu - r}{\sigma})^2 \int_{x_2}^{y} \frac{du}{-(ru+X) + \sqrt{(ru+X)^2 + Y^2}}]dy, (3.1.41)$$

$$c_{3,4} + c_{3,5}\int_{x_1}^{\infty} e^{\int_{x_1}^{y} g_1(u)du}dy = 0, \quad (3.1.42)$$

$$N = \exp[-(\frac{\mu - r}{\sigma})^2 \int_{x_2}^{x_1} \frac{du}{-(ru+X) + \sqrt{(ru+X)^2 + Y^2}}]. \quad (3.1.43)$$

It is just the result of Theorem 3.1.1(c).

Theorem 3.1.1 *When $k_1^2\sigma^2 - k_2^2 > 0$, $\triangle \geq 0$ and $\pi_1 > 0$, the admissible interval is* $[\pi_1, \pi_2]$.

(a) If $\pi_1 > \pi^0(0)$, let $f(x)$ be given by (3.1.24), then $f(x)$ is a twice continuously differentiable concave solution of (3.1.16)-(3.1.17) and the corresponding optimal strategy with constraint is $\pi^*(x) \equiv \pi_1$.

(b) If $\pi_2 \geq \pi^0(0) \geq \pi_1$, let $f(x)$ be given by (3.1.29), then $f(x)$ is a twice continuously differentiable concave solution of (3.1.16)-(3.1.17) and the corresponding optimal strategy with constraint is given by (3.1.28).

(c) If $\pi^0(0) > \pi_2$, let $f(x)$ be given by (3.1.36), then $f(x)$ is a twice continuously differentiable concave solution of (3.1.16)-(3.1.17) and the corresponding optimal strategy with constraint is given by (3.1.35).

Proof. (a) Let $f(x)$ be given by (3.1.24), differentiating

$$(rx + a)f'(x) + \frac{1}{2}b^2 f''(x) + (\mu - r)\pi f'(x) + (\frac{1}{2}\sigma^2\pi^2 + \sigma b\rho\pi)f''(x)$$

with respect to π and setting the derivatives equal to zero, we get the optimal strategy

$$\pi_{\min}(x) = -\frac{\mu - r}{\sigma^2 g_1(x)} - \frac{b\rho}{\sigma}. \tag{3.1.44}$$

By $\pi^0(0) < \pi_1$, we have

$$\frac{-a + \sqrt{a^2 - 2ab\rho\frac{\mu - r}{\sigma} + b^2\frac{(\mu-r)^2}{\sigma^2}}}{\mu - r} < \pi_1, \tag{3.1.45}$$

which implies that

$$\frac{1}{2}(\mu - r)\sigma^2\pi_1^2 + a\sigma^2\pi_1 + ab\sigma\rho - \frac{1}{2}(\mu - r)b^2 > 0. \tag{3.1.46}$$

Then we can conclude that

$$\pi_{\min}(0) < \pi_1. \tag{3.1.47}$$

It is easy to verify that $\pi_{\min}(x)$ is decreasing. Therefore,

$$\pi_{\min}(x) = -\frac{\mu - r}{\sigma^2 g_1(x)} - \frac{b\rho}{\sigma} < \pi_1, \quad x \geq 0. \tag{3.1.48}$$

From $f^{''}(x) > 0$, we know that the minimum of the left side in (3.1.22) is attained at π_1, then (3.1.22) becomes

$$(rx + a)f^{'}(x) + \frac{1}{2}b^2 f^{''}(x) + (\mu - r)\pi_1 f^{'}(x) + (\frac{1}{2}\sigma^2\pi_1^2 + \sigma b\rho\pi_1)f^{''}(x) = 0. \tag{3.1.49}$$

By the construction of $f(x)$, it is easily seen that $f(x)$ solves (3.1.49).

(b) Let $f(x)$ be given by (3.1.29), differentiating

$$(rx + a)f^{'}(x) + \frac{1}{2}b^2 f^{''}(x) + (\mu - r)\pi f^{'}(x) + (\frac{1}{2}\sigma^2\pi^2 + \sigma b\rho\pi)f^{''}(x)$$

with respect to π and setting the derivatives equal to zero, we get the optimal strategy $\pi(x)$ given by

$$\pi_{\min}(x) = \begin{cases} \frac{-(rx+a) + \sqrt{(rx+X)^2 + Y^2}}{\mu - r}, & 0 \leq x < x_1, \\ \beta(x), & x \geq x_1, \end{cases} \tag{3.1.50}$$

where

$$\beta(x) = -\frac{\mu - r}{\sigma^2 g_1(x)} - \frac{b\rho}{\sigma}.$$

In order to prove that (3.1.28) is the optimal strategy with constraint, we need to verify that

$$\beta(x) \le \pi_1, \;\; for\ x \ge x_1.$$

This follows from the fact that the function $\beta(x)$ is decreasing and $\beta(x_1) = \pi^0(x_1) = \pi_1$. The rest of the proof is the same as that in Theorem 3.1.1(a).
The proof of (c) is just to copy the procedure above. □

Case 3.1.1-(ii)
If $\pi_1 \le 0$, then the solution to (3.1.19) is $[0, \pi_2]$, and (3.1.16) becomes

$$(rx+a)\psi'(x) + \frac{1}{2}b^2\psi''(x) + \min_{0\le\pi\le\pi_2}[(\mu - r)\pi\psi'(x) + (\frac{1}{2}\sigma^2\pi^2 + \sigma b\rho\pi)\psi''(x)]\} = 0. \tag{3.1.51}$$

with the boundary condition

$$\psi(0) = 1, \quad \psi(\infty) = 0. \tag{3.1.52}$$

If $\pi^0(0) \le \pi_2$, then VaR constraint is inactive and the optimal solution follows that without constraint.
If $\pi^0(0) > \pi_2$, denote

$$\pi^*(x) = \begin{cases} \pi_2, & 0 < x < x_2, \\ \pi^0(x), & x \ge x_2, \end{cases} \tag{3.1.53}$$

and

$$f(x) = \begin{cases} 1 - c_{4,1}\int_0^x e^{\int_0^y g_2(u)du}dy, & 0 \le x < x_2, \\ c_{4,2} + c_{4,3}\int_{x_2}^x \exp[-(\frac{\mu-r}{\sigma})^2\int_{x_2}^y \frac{du}{-(ru+X)+\sqrt{(ru+X)^2+Y^2}}]dy, & x_2 \le x < x_0, \\ c_{4,4} + c_{4,5}\int_{x_0}^x e^{\int_{x_0}^y g_1(u)du}dy, & x \ge x_0, \end{cases} \tag{3.1.54}$$

where

$$g_2(u) = -\frac{ru + a + (\mu - r)\pi_2}{\frac{1}{2}b^2 + \frac{1}{2}\sigma^2(\pi_2)^2 + \sigma\pi_2 b\rho}, \tag{3.1.55}$$

$$c_{4,3} = \frac{-1}{e^{-\int_0^{x_2} g_2(u)du} + \int_{x_2}^{x_0} \exp[-(\frac{\mu-r}{\sigma})^2 \int_{x_2}^{y} \frac{du}{-(ru+X)+\sqrt{(ru+X)^2+Y^2}}]dy + N}, \quad (3.1.56)$$

$$c_{4,1} = -c_{4,3} e^{-\int_0^{x_2} g_2(u)du}, \quad (3.1.57)$$

$$c_{4,2} = 1 - c_{4,1} \int_0^{x_2} e^{\int_0^y g_2(u)du} dy, \quad (3.1.58)$$

$$c_{4,4} = c_{4,2} + c_{4,3} \int_{x_2}^{x_0} \exp[-(\frac{\mu - r}{\sigma})^2 \int_{x_2}^{y} \frac{du}{-(ru+X)+\sqrt{(ru+X)^2+Y^2}}]dy, \quad (3.1.59)$$

$$c_{4,4} + c_{4,5} \int_{x_0}^{\infty} e^{\int_{x_0}^{y} g_0(u)du} dy = 0, \quad (3.1.60)$$

$$P = \exp[-(\frac{\mu - r}{\sigma})^2 \int_{x_2}^{x_0} \frac{du}{-(ru+X)+\sqrt{(ru+X)^2+Y^2}}]. \quad (3.1.61)$$

Theorem 3.1.2. *If $k_1^2\sigma^2 - k_2^2 > 0$, $\triangle \geq 0$ and $\pi_1 < 0$, let f be given by (3.1.54). Then $f(x)$ is a twice continuously differentiable concave solution of (3.1.16)-(3.1.17) and the corresponding optimal strategy with constraint is given by (3.1.53).*

Proof. The proof is similar to that of Theorem 3.1.1. Here we omit it.

Case 3.1.2 *The case of $k_1^2\sigma^2 - k_2^2 < 0$*

If $\triangle < 0$, all $\pi \geq 0$ satisfies (3.1.18). Thus, the investment strategy given by (3.1.11) is still optimal. If $\triangle \geq 0$, the equation

$$(k_1^2\sigma^2 - k_2^2)\pi^2 + 2(k_1^2\sigma b\rho - k_2 R)\pi + k_1^2 b^2 - R^2 = 0 \quad (3.1.62)$$

has two roots and we still denote them by π_1 and π_2, respectively. Due to the no short selling constraint $\pi \geq 0$ and $V'(\pi) < 0$ in $[0, \infty)$, the solution to (3.1.19) is $[\pi_2, \infty)$. Thus (3.1.16) becomes

$$(rx + a)\psi'(x) + \frac{1}{2}b^2\psi''(x) + \min_{\pi \geq \pi_2}[(\mu - r)\pi\psi'(x) + (\frac{1}{2}\sigma^2\pi^2 + \sigma b\rho\pi)\psi''(x)] = 0. \quad (3.1.63)$$

with the boundary conditions

$$\psi(0) = 1, \quad \psi(\infty) = 0. \tag{3.1.64}$$

Case 3.1.2-(i)

If $\pi_2 > \pi^0(0)$, then all the strategies defined by (3.1.9) are less than π_2. From the previous analysis, we should increase risky investment to satisfy the VaR constraint. Let $\pi^*(x) = \pi_2$ and

$$f(x) = 1 - c_{5,1} \int_0^x e^{\int_0^y g_2(u)du} dy, \tag{3.1.65}$$

where

$$c_{5,1} = \frac{1}{\int_0^\infty e^{\int_0^y g_2(u)du} dy}, \tag{3.1.66}$$

and $g_2(u)$ is defined by (3.1.55).

Then the optimal strategy is just $\pi^*(x)$ and $f(x)$ is the optimal solution, which is the conclusion of Theorem 3.1.3 (a).

Case 3.1.2-(ii)

If $\pi_2 \leq \pi^0(0)$, let x_2 be defined by (3.1.34), then the risk is above the VaR constraint when $x > x_2$. In order to satisfy the risk constraint, the risky investment should be increased. Denote

$$\pi^*(x) = \begin{cases} \frac{-(ru+a)+\sqrt{(rx+X)^2+Y^2}}{\mu-r}, & 0 \leq x < x_2, \\ \\ \pi_2, & x \geq x_2, \end{cases} \tag{3.1.67}$$

and

$$f(x) = \begin{cases} 1 - c_{6,1} \int_0^x \exp[-(\frac{\mu-r}{\sigma})^2] \frac{dy}{-(ru+X)+\sqrt{(ru+X)^2+Y^2}}] dy, & 0 < x < x_2, \\ \\ c_{6,2} + c_{6,3} \int_{x_2}^x e^{\int_{x_2}^y g_2(u)du} dy, & x \geq x_2, \end{cases} \tag{3.1.68}$$

where

$$c_{6,1} = \frac{1}{\int_0^{x_2} \exp[-(\frac{\mu-r}{\sigma})^2 \int_0^y \frac{du}{-(ru+X)+\sqrt{(ru+X)^2+Y^2}}] dy + M}, \tag{3.1.69}$$

$$c_{6,2} = 1 - c_{6,1} \int_0^{x_2} \exp[-(\frac{\mu - r}{\sigma})^2 \int_0^y \frac{du}{-(ru+X) + \sqrt{(ru+X)^2 + Y^2}}]dy, \quad (3.1.70)$$

$$c_{6,3} = -c_{6,1} \exp[-(\frac{\mu - r}{\sigma})^2 \int_0^{x_2} \frac{du}{-(ru+X) + \sqrt{(ru+X)^2 + Y^2}}]. \quad (3.1.71)$$

Then the optimal strategyand solution are optimal, which is justified by Theorem 3.1.3 (b).

Theorem 3.1.3. *When $k_1^2\sigma^2 - k_2^2 < 0$, $\triangle \geq 0$, the admissible interval is $[\pi_2, \infty)$.*
(a) If $\pi_2 \geq \pi^0(0)$, let f be given by (3.1.65). Then $f(x)$ is a twice continuously differentiable concave solution of (3.1.16)-(3.1.17) and the corresponding optimal strategy with constraint is given by π_2.
(b) If $\pi_2 < \pi^0(0)$, let f be given by (3.1.68). Then $f(x)$ is a twice continuously differentiable concave solution of (3.1.16)-(3.1.17) and the corresponding optimal strategywith constraint is given by (3.1.67).

Proof. The proof is similar to that of Theorem 3.1.1. Here we omit it.

Case 3.1.3 *The case of* $k_1^2\sigma^2 - k_2^2 = 0$

Case 3.1.3-(i) If $k_1^2\sigma b\rho - k_2 R > 0$, there is no solution for (3.1.18), which is a trivial case.
Case 3.1.3-(ii) If $k_1^2\sigma b\rho - k_2 R = 0$, we have two cases to discuss.
Case 3.1.3-(ii)-(a)
If $R^2 - k_1^2 b^2 > 0$, the investment strategy given by (3.1.9) satisfies (3.1.18), so it is still optimal with VaR imposed.

Case 3.1.3-(ii)-(b)
If $R^2 - k_1^2 b^2 \leq 0$, there is no solution to (3.1.18), that is, there is no optimal strategy in the constrained case.

Case 3.1.3-(iii) If $k_1^2\sigma b\rho - k_2 R < 0$, we denote

$$\begin{aligned} \check{\gamma} &= \frac{R^2 - k_1^2 b^2}{2(k_1^2\sigma b\rho - k_2 R)}, \\ \gamma &= \max(\check{\gamma}, 0), \end{aligned} \quad (3.1.72)$$

then the admissible interval is $[\gamma, \infty)$.

Case 3.1.3-(iii)-(a)

If $\gamma > \pi^0(0)$, then $\pi^0(x) < \gamma$ for each $x \geq 0$, which can not satisfy the constraint, then we should increase the amount in the risky asst. In fact, it is trivial to prove that the optimal strategy is $\pi^*(x) = \gamma$ and

$$f(x) = 1 - c_{7,1} \int_0^x e^{\int_0^y g_\gamma(u)du} dy, \tag{3.1.73}$$

where

$$g_\gamma(u) = -\frac{ru + a + (\mu - r)\check{\gamma}}{\frac{1}{2}b^2 + \frac{1}{2}\sigma^2\check{\gamma}^2 + \sigma\check{\gamma}b\rho}, \tag{3.1.74}$$

$$c_{7,1} = \frac{1}{\int_0^\infty e^{\int_0^y g_\gamma(u)du} dy}. \tag{3.1.75}$$

Case 3.1.3-(iii)-(b)

For the case $\gamma < \pi^0(0)$, denote

$$x_\gamma = \begin{cases} \check{x}_\gamma, & \text{if there exists } \check{x}_\gamma > 0 \text{ such that } \pi(\check{x}_\gamma) = \gamma, \\ \\ \infty, & \text{otherwise.} \end{cases} \tag{3.1.76}$$

Obviously, if $x \in (x_\gamma, \infty)$, then $\pi^0(x) < \gamma$, which does not satisfy the constraint. Just similar to the previous discussion, the optimal strategy and the optimal solution to (3.1.16) are

$$\pi^*(x) = \begin{cases} \frac{-(rx+a)+\sqrt{(rx+X)^2+Y^2}}{\mu-r}, & 0 < x < x_\gamma, \\ \\ \gamma, & x \geq x_\gamma, \end{cases} \tag{3.1.77}$$

and

$$f(x) = \begin{cases} 1 - c_{8,1} \int_0^x \exp[-(\frac{\mu-r}{\sigma})^2 \int_0^y \frac{du}{-(ru+X)+\sqrt{(ru+X)^2+Y^2}}]dy, & 0 \leq x < x_\gamma, \\ \\ c_{8,2} + c_{8,3} \int_{x_\gamma}^x e^{\int_{x_\gamma}^y g_\gamma(u)du} dy, & x \geq x_\gamma, \end{cases} \tag{3.1.78}$$

respectively, where

$$c_{8,1} = \frac{1}{M_\gamma + N_\gamma \int_{x_\gamma}^\infty e^{\int_{x_\gamma}^y g_\gamma(u)du} dy}, \tag{3.1.79}$$

$$c_{8,2} = 1 - c_1 M_\gamma, \tag{3.1.80}$$

$$c_{8,3} = -c_1 N_\gamma, \tag{3.1.81}$$

$$M_\gamma = \int_0^{x_\gamma} \exp[-(\frac{\mu - r}{\sigma})^2 \int_0^y \frac{du}{-(ru + X) + \sqrt{(ru + X)^2 + Y^2}}]dy, \tag{3.1.82}$$

$$N_\gamma = \exp[-(\frac{\mu - r}{\sigma})^2 \int_0^{x_\gamma} \frac{du}{-(ru + X) + \sqrt{(ru + X)^2 + Y^2}}]. \tag{3.1.83}$$

We omit the proof.

3.1.3 Conclusion for this section

The results in this section can be summarized as follows. The sensitivity analysis shows how VaR changes with the value of risky investment. To satisfy the risk constraint, risky investment is adjusted based on the sensitive analysis. By solving the associated HJB equation, we obtain the analytic solutions. If the ruin probability is minimized, the insurer will put less money into the risky asset if her surplus is large. Once the dynamic VaR is taken into account and imposed as a risk constraint, the results show that the risky investment coincides with that without constraint for the states that VaR is inactive. And for the states with an active VaR constraint, risky investment should be decreased/increased if VaR is increasing/ decreasing with the risky investment.

3.2 The wealth maximization with risk constraint

We investigate the wealth maximization problem in finite time interval in this subsection. Optimal investment and proportional reinsurance ($u \triangleq (q(t), \pi(t))$) are included in this problem formulation in both complete and incomplete market. For risk management, VaR will be imposed dynamically as the risk constraint. Assume that r(t), $\mu(t)$ and $\sigma(t)$ are deterministic function of t. Incorporating the strategy $u = (q(t), \pi(t))$ in (2.1.4), the dynamics of the resulting wealth process X_t^u follows

$$\begin{cases} dX^u(t) &= (rX^u(t) + \pi(t)(\mu(t) - r(t)\mathbf{1}))dt + (\theta - \eta q(t))a dt + b(1 - q(t))dW^0(t) \\ & \quad + \pi(t)\sigma(t)dW(t) \\ X_0 &= x. \end{cases} \tag{3.2.1}$$

The strategy $u = (q(t), \pi(t))$ (equivalently, $\widetilde{\pi}(t)$), $0 \le t \le T$, is said to be *admissible* if

(a) it is $\mathcal{F}_t$ -progressively measurable,

(b) $\int_0^T \|\widetilde{\sigma}^\top(t)\widetilde{\pi}(t)\|^2 dt < \infty \quad P$ - almost surely,

(c) $q(t) \in [0,1], \ \forall t \in [0,T]$.

Let $\mathcal{U}_{I_2}$ be the class of all *admissible* strategies u.

We assume that the following conditions are satisfied in this section.

Assumption 3.2.1. There exist M and $\epsilon \in R^+$ such that

$$
\begin{aligned}
&0 \le r(t) < \mu_i(t), \ b(t) \le M, \\
&\xi^\top \Sigma(t)\xi \ge \epsilon\|\xi\|^2, \qquad \forall (t,\xi) \in [0,T] \times \mathcal{R}^n \\
&b^2(t)x^2 \ge \epsilon x^2, \qquad \forall (t,x) \in [0,T] \times \mathcal{R},
\end{aligned}
\tag{3.2.2}
$$

where $\Sigma(t) = \sigma(t)\sigma^\top(t)$.

Problem (I_2). The problem is to find an admissible strategy $u^* \in \mathcal{U}_{I_2}$ such that the expectation of the final wealth utility defined by

$$
U(X^u(T)) = \lambda_0 - \frac{\gamma}{m} e^{-mX^u(T)} \tag{3.2.3}
$$

is maximized, where the constant m is the risk aversion parameter (see Pratt [107]).

When the market is *complete*[3.1], the problem has been solved by Bai and Guo [10]. While it is difficult to derive the smooth solution of the associated HJB equation in the incomplete market. Generally speaking, it should resort to numerical solution to this second order nonlinear PDE.

Although the problem presented here is a stochastic optimal control problem, however, for the exponential utility function, the problem can be reduced to a deterministic control problem. The solution of the former is significantly more difficult than the solution of the latter since there are many existing packages for the latter, such as control parametrization (see [126] for reference).

[3.1]See the footnote 2.3 in Chapter 2 about the meaning

In this section, we first show how the stochastic optimal control problem can be transformed into a deterministic optimal control problem. Then the solution of this deterministic problem is approximated by the existing optimization software, such as $NLPQLP$(see [89, 90, 125]). Some numerical examples are given to demonstrate the effectiveness of the proposed approach .

3.2.1 The deterministic problem

We first present a transform theorem which connects the original stochastic optimal control problem and the deterministic optimal control problem.

Rewrite the dynamics (3.2.1) as

$$\begin{cases} dX^u(t) &= (r(t)X^u(t)+\pi(t)(\mu(t)-r(t)))dt+(\theta-\eta q(t))adt+b(1-q(t))dW^0(t)+ \\ & \quad \pi(t)\sigma dW(t) \\ &= [rX^u(t)+(\theta-\eta)a+\widetilde{\pi}(t)(\widetilde{\mu}(t)-r(t)\widetilde{\mathbf{1}})]dt+\widetilde{\pi}(t)\widetilde{\sigma}(t)d\widetilde{W}(t), \\ X_0 &= x, \end{cases} \tag{3.2.4}$$

where

$$\widetilde{\sigma}(t) = \begin{pmatrix} b & 0 \\ 0 & \sigma(t) \end{pmatrix},$$

$$\begin{aligned} \widetilde{\pi}(t) &= (\pi_0(t),\pi_1(t),...,\pi_n(t)), \quad \pi_0(t)=1-q(t); \\ \widetilde{\mu}(t) &= (\mu_0(t),\mu_1(t),...,\mu_n(t))^\top, \quad \mu_0(t)=a\eta+r; \\ \widetilde{W} &= (W_0,W_1,...,W_d)^\top, \widetilde{\mathbf{1}}^\top = \underbrace{\{1,1,\cdots,1\}}_{n+1}. \end{aligned} \tag{3.2.5}$$

Let $\check{\mathcal{H}}$ denote the class of all $L^2[0,T]$ functions and $\check{u}(t)$ denote the control which is only dependent on the time t.

Theorem 3.2.2. For any path ω, denote $\check{u}_\omega(t) := u(t,\omega)$. Assume that $U(X^u(t))$ can be written as the product of $U_1(X_1^u(t))$ and $U_2(X_2^u(t))$, where $U_1(X_1^{\check{u}}(t))$ is deterministic for all $\check{u}\in\mathcal{L}^2[0,T]$, and $U_2(X_2^u(t))$ is nonnegative local martingale. Then,

$$\max_u EU(X^u(T)) \le \max_u EU_1(X_1^u(T)) \le \max_{\check{u}\in L^2[0,T]} U_1(X_1^{\check{u}}(T)). \tag{3.2.6}$$

Denote

$$\breve{u}^* = \arg\max_{\breve{u}\in L^2[0,T]} U_1(X_1^{\breve{u}}(T)). \tag{3.2.7}$$

For any $\omega \in \Omega$, let

$$u^*(t,\omega) = \breve{u}_\omega^*(t) := \breve{u}^*(t).$$

If $U(X_2^{u^*}(T))$ is a martingale, then we have

$$\max_u EU(X^u(T)) = U_1(X_1^{\breve{u}^*}(T)). \tag{3.2.8}$$

Proof. See the Appendices.

From (3.2.4), we have

$$X(t) = c(t) + \int_0^t e^{r(t-s)}(\widetilde{\pi}(s)(\widetilde{\mu}(s) - r(s)\widetilde{\mathbf{1}})ds + \widetilde{\pi}\widetilde{\sigma}d\widetilde{W}(s)), \tag{3.2.9}$$

where $c(t) = e^{rt}x + \frac{e^{rt}-1}{r}(\theta - \eta)a$. Thus,

$$\begin{aligned}
&\max_u E\{U(X^u(T))\} && (3.2.10)\\
\Leftrightarrow\ &\max_u\{-\frac{\gamma\exp(-mc(T) + \int_0^T e^{-r(s-T)}(-m\widetilde{\pi}(s)(\widetilde{\mu}(s) - r(s)\widetilde{\mathbf{1}})ds - m\widetilde{\pi}(s)\widetilde{\sigma}d\widetilde{W}(s)))}{m}\}\\
\Leftrightarrow\ &\max_u\{-E\exp(\int_0^T -me^{-r(s-T)}\widetilde{\pi}(s)(\widetilde{\mu}(s) - r(s)\widetilde{\mathbf{1}})ds + e^{-r(s-T)}m\widetilde{\pi}(s)\widetilde{\sigma}(s)d\widetilde{W}(s))\}
\end{aligned}$$

Denote $\tilde{\sigma}\tilde{\sigma}^\top(s)$ by $\widetilde{\Sigma}(s)$,

$$\begin{aligned}
X_1^u(T) &:= \int_0^T -me^{-r(s-T)}\widetilde{\pi}(s)(\widetilde{\mu}(s) - r(s)\widetilde{\mathbf{1}}) + \frac{m^2 e^{-2\int_s^T r(\tau)d\tau}\widetilde{\pi}(s)\widetilde{\Sigma}(s)\widetilde{\pi}^\top(s)}{2}ds,\\
X_2^u(T) &:= \int_0^T -me^{-r(s-T)}\widetilde{\pi}(s)\widetilde{\sigma(s)}d\widetilde{W}(s) - \frac{m^2 e^{-2\int_s^T r(\tau)d\tau}\widetilde{\pi}\widetilde{\Sigma}(s)\widetilde{\pi}^\top(s)}{2}ds, && (3.2.11)
\end{aligned}$$

and

$$U_1(X_1^u(T)) := -\exp(X_1^u(T)),\ U_2(X_2^u(T)) := \exp(X_2^u(T)).$$

Obviously, $U_2(X_2^u(T))$ is a martingale under Assumption 3.2.1. By Theorem 3.2.2, the primal problem (i.e. Problem (I_2)) is reduced to the deterministic problem

$$\max_{\breve{u}\in\breve{\mathcal{H}}} U_1(X_1^{\breve{u}}(T)). \tag{3.2.12}$$

This is equivalent to

Problem (I_2'). Find an admissible strategy $\breve{u} \in \mathcal{H}$ such that

$$\int_0^T (-me^{-r(s-t)}\widetilde{\pi}(s)(\widetilde{\mu}(s) - r(s)\widetilde{\mathbf{1}}) + \frac{m^2 e^{-2r(s-t)}\widetilde{\pi}\widetilde{\Sigma}(s)\widetilde{\pi}^\top}{2})ds \tag{3.2.13}$$

is minimized, subject to

$$0 \le q(t) \le 1. \tag{3.2.14}$$

In the next subsection, we will show that Problem (I_2') admits an optimal solution.

3.2.2 Existence of optimal solutions

As Theorem 3.2.2 tells us that optimal strategy is the same for almost all the paths, the problem is reduced to seeking a deterministic control strategy for Problem (I_2'). If $\check{u} \in \check{\mathcal{H}}$, define

$$\|\check{u}\|_2^2 = \int_0^T \|\check{u}\|^2 ds.$$

Note that $\widetilde{\mathcal{H}}$ is a Hilbert space if the inner product of $\check{u}^{(a)}$ and $\check{u}^{(b)}$ is defined by

$$\langle \check{u}^{(a)}, \check{u}^{(b)} \rangle = \int_0^T \check{u}^{(a)}(s)(\check{u}^{(b)})^\top(s) ds \tag{3.2.15}$$

Theorem 3.2.3. Let function $J(\widetilde{\pi})$ be defined by

$$J(\widetilde{\pi}) = \begin{cases} -m \int_0^T e^{\int_s^T r(\tau)d\tau} \widetilde{\pi}(s)(\widetilde{\mu}(s) - r(s)\widetilde{\mathbf{1}}) + \frac{m^2}{2} e^{2\int_s^T r(\tau)d\tau} \widetilde{\pi}(s)\widetilde{\Sigma}(s)\widetilde{\pi}^\top(s) ds, & \widetilde{\pi} \in \check{\mathcal{H}} \\ \infty, & \text{otherwise.} \end{cases}$$

Then, the following properties are satisfied.
(i) $J(\widetilde{\pi})$ is convex;
(ii) $J(\widetilde{\pi})$ is coercive, i.e.,

$$\lim_{\|\widetilde{\pi}\| \to \infty} J(\widetilde{\pi}) = \infty;$$

and
(iii) $J(\widetilde{\pi})$ is lower-semicontinuous, i.e., for every $\widetilde{\pi}$ and $\widetilde{\pi}^{(n)} \in \check{\mathcal{H}}$, with

$$\lim_{n \to \infty} \|\widetilde{\pi}^{(n)} - \pi\|_2 = 0,$$

we have

$$J(\widetilde{\pi}) \le \liminf_{n \to \infty} J(\widetilde{\pi}^{(n)}); \tag{3.2.16}$$

(iv) There exists a $\widetilde{\pi}^* \in \check{\mathcal{H}}$ such that $J(\widetilde{\pi}^*) = \inf_{\widetilde{\pi} \in \check{\mathcal{H}}} J(\widetilde{\pi})$.

Proof. The proof is given in the Appendices.

Suppose that the strategy is constrained in a closed and convex set $\mathcal{C}$. Denote $\mathcal{K} = \{\widetilde{\pi} | \widetilde{\pi}(t) \in \mathcal{C}, \widetilde{\pi} \in \check{\mathcal{H}}\}$. With the similar argument to Theorem 3.2.2, we can concentrate on the deterministic strategy $\mathcal{K}$.

Corollary 3.2.4. Define

$$J^K(\widetilde{\pi}) := \begin{cases} -m \int_0^T e^{\int_t^T r(\tau)d\tau} \widetilde{\pi}(t)(\tilde{\mu}(t) - r(t)\widetilde{\mathbf{1}}) + \frac{m^2}{2} e^{2\int_t^T r(\tau)d\tau} \widetilde{\pi}(t)\widetilde{\Sigma}(t)\widetilde{\pi}^\top(t)dt, u \in \mathcal{K} \\ \infty, \qquad \text{otherwise.} \end{cases}$$

Then there exists $\widetilde{\pi}^* \in \mathcal{K}$ such that $J(\widetilde{\pi}^*) = \inf_{\widetilde{\pi} \in \mathcal{U}} J(\widetilde{\pi})$.

Proof. The proof is just to repeat the procedure of Theorem 3.2.2.

3.2.3 Numerical examples

As it is often impossible to derive the analytical solution, numerical methods are unavoidable. We use the control parametrization method (see [89, 90, 125]). Let the time horizon [0,T] be partitioned into N subintervals, and let $\widetilde{\pi}(\in \check{\mathcal{H}})$ be approximated as piecewise constant function, given by

$$\widetilde{\pi}(t) = \sum_{j=0}^{N} \chi_{[t_j, t_{j+1})}(t)\widetilde{\pi}^j, \tag{3.2.17}$$

where $0 < t_0 < t_1 < t_2 < \cdots t_N < t_{N+1} = T$ and $\chi_I(t)$ is the indicator function defined by

$$\chi_I(t) = \begin{cases} 1, & \text{if } t \in I, \\ 0, & \text{elsewhere.} \end{cases}$$

For each $j = 1, \ldots, N$, $\widetilde{\pi}^j$ is a vector specifying the value of $\widetilde{\pi}$ on the sub-interval $[t_j, t_{j+1})$. Such a strategy should be chosen to minimize (3.2.13) subject to the dynamics (3.2.14). Then **Problem** (I_2') can be solved as an optimization problem, and various optimization software packages, such as $NLPGLP$ (see[89, 90, 125]), can be used for this purpose.

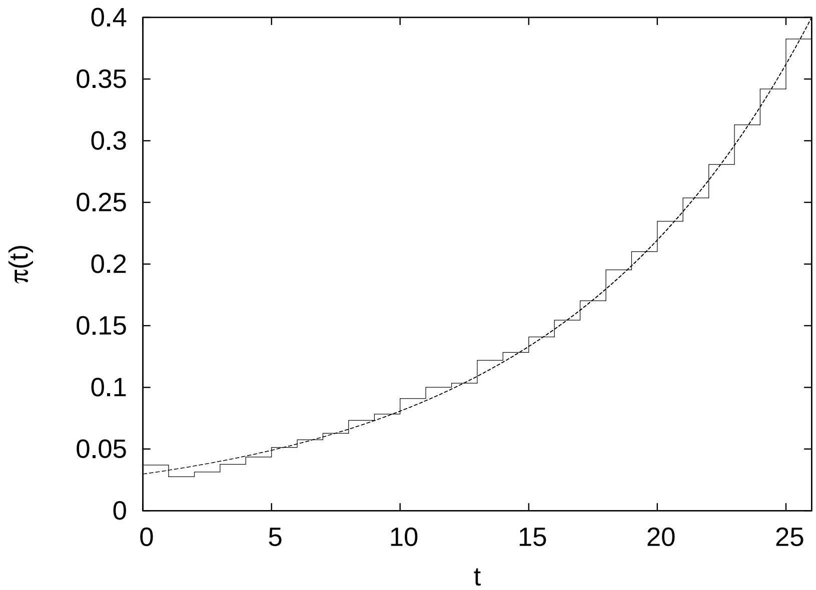

Figure 3.2.1: The risky investment in Case 1

3.2.3.1 The optimal problem in a complete market

Consider a complete market, i.e., the number of the stocks n is equal to the number of underlying of the market(the dimension of the $W(t)$). The proportional reinsurance $q(t) \in [0, 1]$ is the only constraint. In this market, Bai and Guo [10] have studied the problem by the stochastic dynamic programming. We conduct the numerical experiments with the help of $NLPQLP$ for our model. Two sets of parameters are given to show that when trading interval approaches to zero the solutions converge to those in [10].

Case I ($0 < q(t) < 1$ when the reinsurance constraint is inactive): the model parameters m=1.0, r=0.1, T=26, a=0.3, η=0.1, b=0.2, μ=0.2 and σ=0.5. Figure 3.2.1 and Figure 3.2.2 plot the risky investment and the proportional reinsurance, respectively.

Case II (q(t)=0 when the reinsurance constraint is active): a=0.7, η=1, m=1, r=0.05, T=26, b=0.3, μ=0.2, σ=0.5. Figure 3.2.3 and Figure 3.2.4 compare the risky investment and the proportional reinsurance under the risk control with those under no control, respectively.

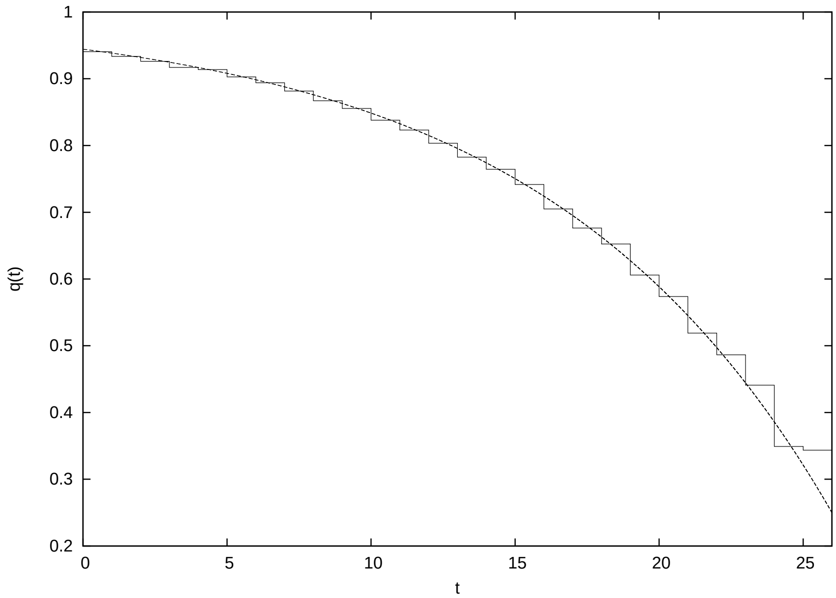

Figure 3.2.2: The proportional reinsurance in Case 1

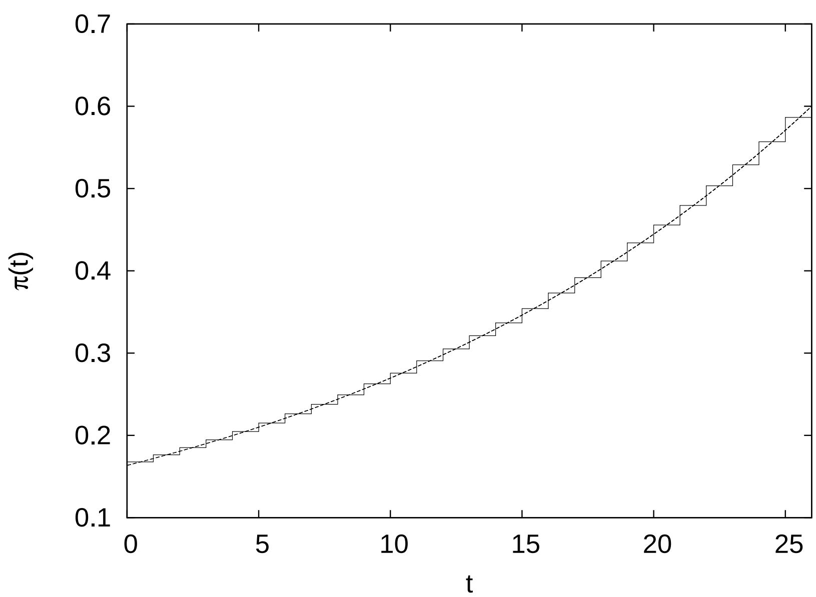

Figure 3.2.3: The risky investment in Case 2

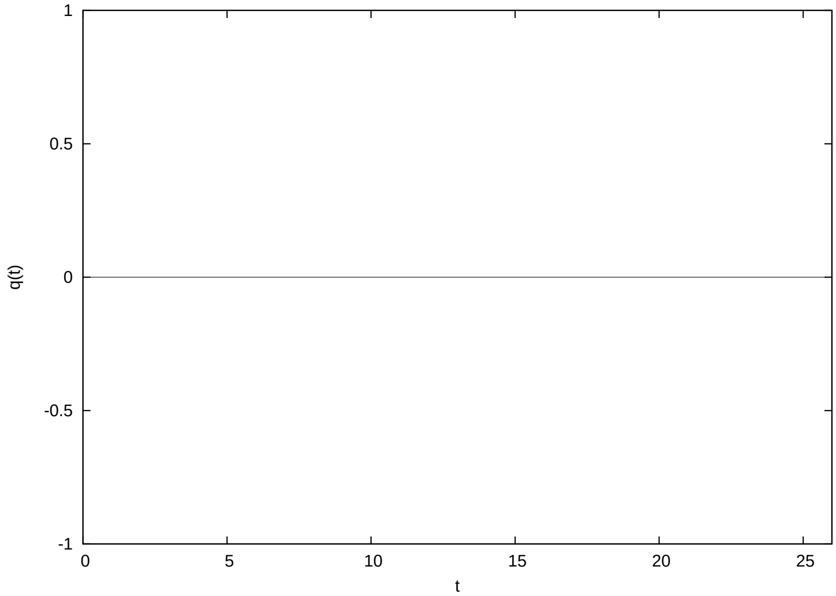

Figure 3.2.4: The propotional reinsurance in Case 2

3.2.3.2 The incomplete market

- *The number of stocks is less than the uncertainty* ($n < m$):

Incompleteness arises if n is assumed to be strictly smaller than m. To show the effectiveness of the method, we also solve the problem under this assumption ($n < m$). Here, one risky asset is supposed, and the Brownian Motion is two-dimensional. The following parameters are used: m=0.4, r=0.1, a=0.2, T=26, η=0.2, μ=0.2, b=0.2, σ_1=0.5, σ_2=0.2. Figure 3.2.5 and Figure 3.2.6 plot the risky investment and the proportional reinsurance, respectively.

- *Portfolio constraint: VaR is imposed as the risk constraint*:

In this example, parameters are assumed to be constants. We consider the loss from time t_n to t_{n+1}, for n=0, 1, ... , N. Let $\Delta t = t_{n+1} - t_n$, $Y^u(s) = e^{-rs}X^u(s)$ and $\Delta Y^u(t) = Y^u(t_{n+1}) - Y(t_n)$. Suppose that $(\pi(s), q(s))$ is unchanged in the interval $[t_n, t_{n+1})$, we have

$$\Delta Y^u(t) = \frac{e^{-r(t+\Delta t)} - e^{-rt}}{-r}\{\widetilde{\pi}(s)(\widetilde{\mu} - r\widetilde{\mathbf{1}}) + \frac{(\theta - \eta)a}{r}\} + e^{-r(t+\Delta t)}\int_t^{t+\Delta t} e^{-rs}\widetilde{\pi}\widetilde{\sigma}d\widetilde{W}(s).$$

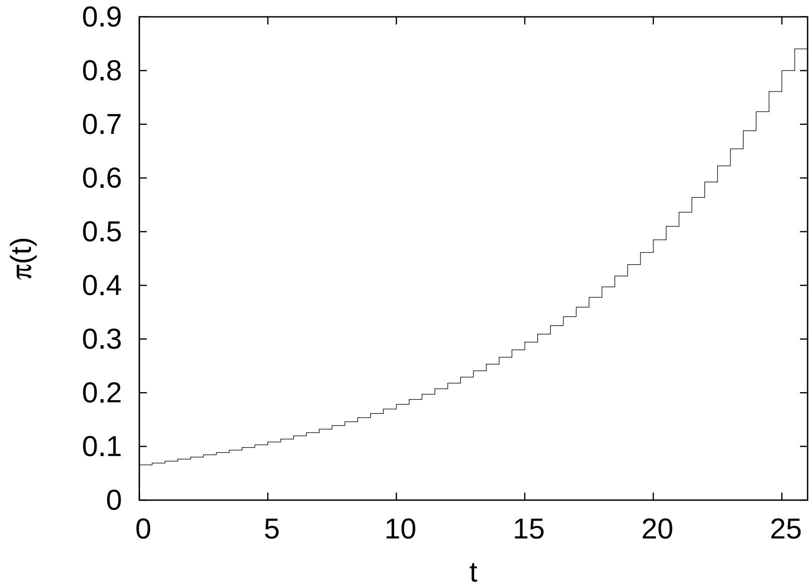

Figure 3.2.5: The optimal risky investment in the incomplete market ($n < m$)

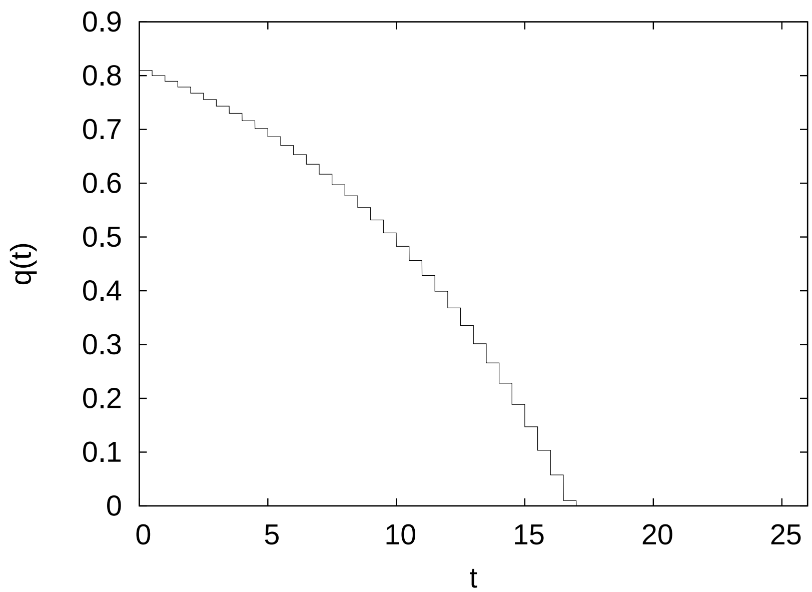

Figure 3.2.6: The optimal propotional reinsurance in the incomplete market ($n < m$)

It follows that

$$X^u(t+\Delta t) = e^{r\Delta t}(X^u(t) - N^u(t)) + N^u(t) + \int_t^{t+\Delta t} e^{r(t+\Delta t - s)}\widetilde{\pi}\widetilde{\sigma}d\widetilde{W}(s), \quad (3.2.18)$$

where

$$N^u(\Delta t) = \frac{\widetilde{\pi}(s)(\widetilde{\mu} - r\widetilde{\mathbf{1}}) + (\theta - \eta)a)}{-r}. \quad (3.2.19)$$

Denote the discounted loss $X^u(t+\Delta t) - e^{r\Delta t}X^u(t)$ by $\Delta X^u(t)$. Then

$$Var_t\{\Delta X^u(s)\} = \frac{1 - e^{2r(s-t)}}{-2r}\widetilde{\pi}\widetilde{\Sigma}\widetilde{\pi}^\top \quad (3.2.20)$$

and

$$E_t\{\Delta X^u(s)\} = N^u(t) + e^{r(s-t)}(X^u(t) - N^u(t)). \quad (3.2.21)$$

It follows from (3.2.20) and (3.2.21) that

$$VaR = -N^u(t)(1 - e^{r\Delta t}) + \Phi^{-1}(k)\sqrt{\frac{1 - e^{2r\Delta t}}{-2r}\widetilde{\pi}\widetilde{\Sigma}\widetilde{\pi}^\top}. \quad (3.2.22)$$

If we constrain the maximal risk to R, i.e.,

$$VaR \leq R, \quad (3.2.23)$$

then the portfolio constraint is

$$-(\widetilde{\pi}_t(\widetilde{\mu} - r\mathbf{1}) + (\theta - \eta)a)\frac{e^{r\Delta t} - 1}{r} + \Phi^{-1}(k)\sqrt{\frac{1 - e^{2r\Delta t}}{-2r}\widetilde{\pi}\widetilde{\Sigma}\widetilde{\pi}^\top} \leq R. \quad (3.2.24)$$

Remark 3.1. If k is larger than 0.5, this constraint set is closed and convex. In fact, it is nonempty for reasonable choices of the parameters.

In this experiment, we use the parameters: $a = 0.1$, $\eta = 0.2$, $\mu = 0.2$, $\sigma = 0.5$, $m = 0.5$, $r = 0.1$, $b = 0.2$, $\theta = 0.5$ and $R = 0.6$.

Figure 3.2.7 compares the risky investment with and without VaR constraint. It is easily seen from this figure that if VaR is active, the risky investment should be cut down to meet the risk management. Figure 3.2.8 plots the proportional reinsurance, which is also compared in both cases. If the constraint is active, the proportional reinsurance should be increased, compared with the case without constraint. Figure 3.2.9 reflects the VaR level and VaR is stabilized once it is active.

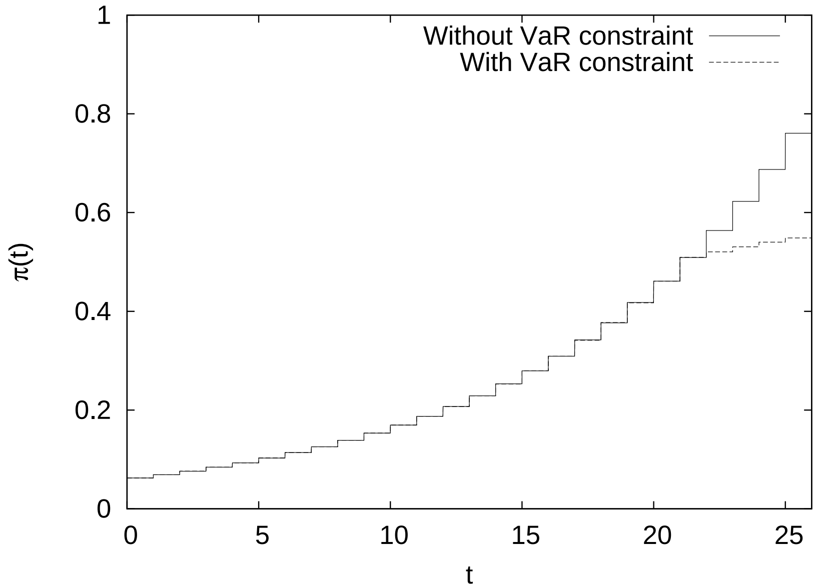

Figure 3.2.7: The risky investment compared in the cases: with and without constraint.

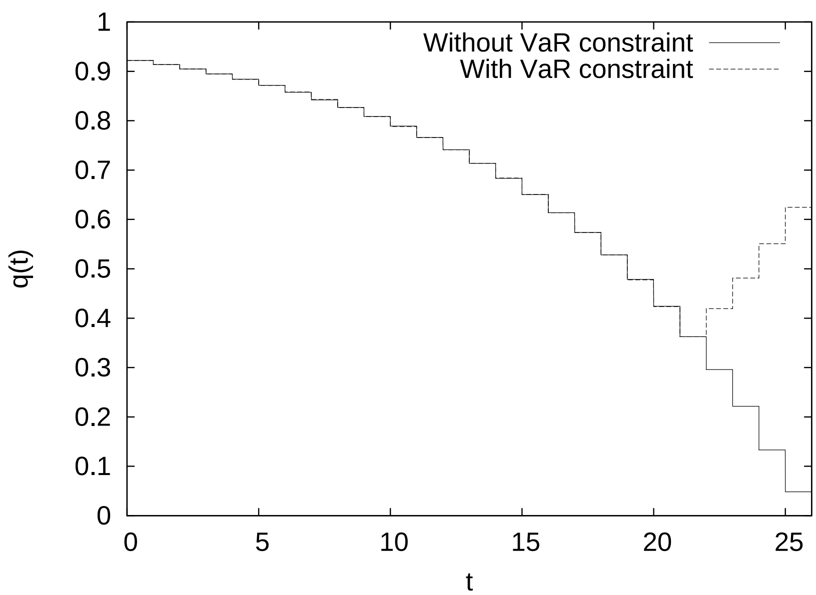

Figure 3.2.8: The propotional resinsurance compared in the cases: with and without constraint.

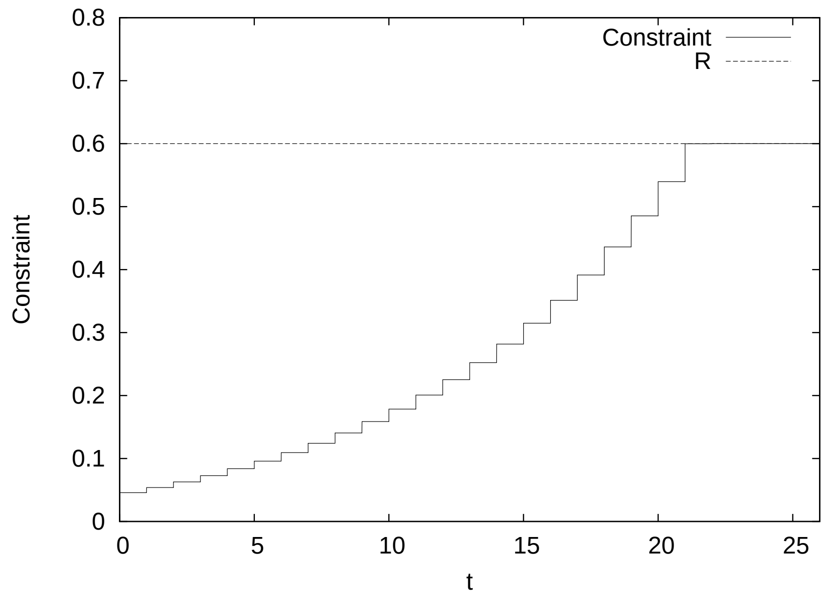

Figure 3.2.9: VaR level.

Chapter 4

The optimal consumption and investment problem

In Merton's problem, the value function can be separated into a function of t and of x: $V(t,x) = h(t)x^\gamma$. With this transformation, the HJB equation becomes an ordinary differential equation with respect to $h(t)$, which is explicitly solved. In this chapter, the optimal control problem is studied when the dynamics for risky asset follows the jump diffusion and regime switching model, and we will borrow the idea of Merton to solve the HJB equations.

In this chapter, suppose that an agent can invest her wealth into a continuous-time financial model consisting of a bank account B and a risky asset S, which are tradable continuously over a finite time horizon $[0,T]$, where $T \in (0,\infty)$. Let $\pi := \{\pi(t)\}_{t\in\mathcal{T}}$ denote the amount of wealth allocated to the risky asset S at time t, where $\pi(t) := \pi(t,\omega)$ is $\mathcal{G}_t$ [4.1]- progressively measurable.

Also suppose that an agent needs continuous consumption over this given period. Let $\{c(t)\}_{t\in\mathcal{T}}$ represent the consumption rate of an economic agent at time t, where $c(t) := c(t,\omega)$ is a non-negative $\mathcal{G}_t$- progressively measurable process. Define $u(t) := (c(t),\pi(t))$ to be our control process. We say $\{u(t)\}_{t\in\mathcal{T}}$ is *admissible* if

$$\int_0^t (\pi(s)^2 + c(s))ds < \infty \ , \quad t \geq 0 \ , \quad \mathcal{P} - \text{a.s.} \tag{4.0.1}$$

[4.1] It will be defined in a later section (page 60)

Suppose that $U_1(\cdot,\cdot), U_1(\cdot,\cdot) : \mathcal{T} \times [0,\infty) \to R$ are two utility functions. Let τ represent the time at which the agent's wealth reaches 0, i.e., $\tau = \inf\{t, |x(t) = 0\}$, and let $U_1(x, 0) = 0$. By selecting optimal admissible control, the investor wishes to maximize the total expected utility of consumption and wealth over a given time horizon $[0, T]$ with an initial wealth x_0,

$$\max_{u(t)} E\left[\int_0^\tau U_1(t, c(t))dt\right] + \Psi(\tau, X_\tau) \tag{4.0.2}$$

where

$$\Psi(t, x(t)) = \begin{cases} U_1(t, x), & \text{if}(t, x) \in [t_0, T] \times R^n \\ \\ U_2(x), & \text{if}(t, x) \in T \times R^n \end{cases} \tag{4.0.3}$$

Suppose that upon bankruptcy the agent should quit and the boundary condition is just $V(0, t) = 0$ from (4.0.3). By the numerical solution to the HJB equation, we investigate the effect of the $CVaR$ ($MVaR$) constraint.

4.1 The jump diffusion model

Risk-constrained allocation of risky assets in financial portfolios is particularly important in situations when asset returns appear to have large fluctuations. This problem is addressed here.

We will resort to stress testing to get useful information on a firm's risk exposure. Stress testing ([3, 93, 88, 124]) can be considered as a procedure that aims to identify possible losses which may accrue under extreme movements of asset prices by constructing scenarios. We implement it by the methods in Berkovwitz [13], who suggests assigning probabilities to scenarios that are identified by stress testing.

To construct scenarios which are reflecting real situations in a portfolio model, a compound Poisson process is incorporated into the stock price evolution. This resembles a price process perturbed by an exogenous factor which may cause large movements in price. The jump size of the Poisson process and the rate of jump define, respectively, a scenario and its occurrence probability.

The stress testing is conducted to evaluate the performance and assess the resilience of the portfolio subject to exceptional but major events. We examine how a conditional-value-at-risk constraint exerts an influence on the portfolio composition.

4.1.1 Continuous-time optimal portfolios

We proceed to formulate the investment model as follows:

i) Let $B(t)$ be the deterministic price process of the risk free asset, say a bond. It is written as

$$dB(t) = rB(t)dt \tag{4.1.1}$$

with a fixed interest rate r.

ii) Let $S(t)$ denote the price process of the risky asset. It is assumed that $S(t)$ evolves according to the jump diffusion model. Specially,

$$dS(t) = S(t)(\mu dt + \sigma dw(t) + dJ(t)) \tag{4.1.2}$$

where μ, σ and $W(t)$ are defined as previously, $J(t)$ is a compound Poisson process. We assume that $J(t)$ takes the form $\sum_{k=1}^{N(t)} Y_k$, where, $N(t)$ is a Poisson process with rate λ, which denotes the number of extreme events (sudden jump of the dynamics of price process)that have occurred up to time t. And $Y_i, i \geq 1$, are independent and identically distributed random variables which reflect that how severe the extreme event can be when it occurs. We also assume that for each $k = 1, \ldots, N(t)$, the value of Y_k is greater than or equal to -1. We denote the augmented σ-algebra generated by $W(t)$ and $J(t)$ by $\mathcal{G}_t$.

Then, the dynamics of a portfolio, which consists of $B(t)$, $S(t)$ and the consumption $c(t)$, is given by

$$\begin{aligned} dX(t) &= \frac{X(t) - \pi(t)}{B(t)} dB(t) + \pi(t) \frac{dS(t)}{S(t)} - c(t)dt \\ &= (\pi(t)(\mu - r) + rX(t) - c(t))dt + \pi(t)\sigma dW(t) + \pi(t)dJ(t). \end{aligned} \tag{4.1.3}$$

4.1.2 Stress testing of the loss and $CVaR$

Before conducting the stress testing and applying risk constraint, we first derive the return (loss) over $[t, t+\Delta t)$ when a compound Poisson process is added into the stock evolution. Assume that Δt is small and the portfolio is not adjusted in $[t, t+\Delta t)$. Let

$$\alpha = -r, \quad \theta(t) = \frac{\pi(t)(\mu - r) - c(t)}{-r}. \tag{4.1.4}$$

Then, the dynamics (4.1.3) becomes

$$dX(t) = \alpha(\theta(t) - X(t))dt + \pi(t)\sigma dW(t) + \pi(t)dJ(t). \tag{4.1.5}$$

Define

$$Y(t) := e^{\alpha t}X(t). \tag{4.1.6}$$

From (4.1.5), we have

$$dY(t) = \alpha\theta(t)e^{\alpha t}dt + e^{\alpha t}\pi(t)\sigma dW(t) + e^{\alpha t}\pi(t)dJ(t). \tag{4.1.7}$$

By integrating (4.1.7) over $[t, t+\Delta t)$, we obtain

$$\begin{aligned} & Y(t+\Delta t) - Y(t) \\ = \ & \alpha\int_t^{t+\Delta t}\theta(s)e^{\alpha s}ds + \int_t^{t+\Delta t}e^{\alpha s}\pi(s)\sigma dW(s) + \int_t^{t+\Delta t}e^{\alpha s}\pi(s)dJ(s). \end{aligned} \tag{4.1.8}$$

Assume that Δt is so small that $\pi(s)$, $c(s)$, $\theta(s)$ and $e^{\alpha s}$ can be approximated by $\pi(t)$, $c(t)$, $\theta(t)$ and $e^{\alpha t}$ (with errors less than $0(\Delta t^{\frac{3}{2}})$), respectively. Then, (4.1.8) becomes

$$\begin{aligned} & Y(t+\Delta t) - Y(t) \\ = \ & \theta(t)(e^{\alpha(t+\Delta t)} - e^{\alpha t}) + \int_t^{t+\Delta t}e^{\alpha t}\pi(t)\sigma dW(s) + \int_t^{t+\Delta t}e^{\alpha t}\pi(s)dJ(s). \end{aligned} \tag{4.1.9}$$

From (4.1.6), we obtain

$$\begin{aligned} & X(t+\Delta t) \\ = \ & e^{-\alpha\Delta t}(X(t) - \theta(t)) + \theta(t) + e^{-\alpha\Delta t}\omega(t)\int_t^{t+\Delta t}(\sigma dW(s) + dJ(s)). \end{aligned} \tag{4.1.10}$$

Let the return, adjusted for the future value of the current portfolio value consistent with (4.1.10),

$$\Delta X(t) = X(t+\Delta t) - e^{r\Delta t}X(t). \tag{4.1.11}$$

From (4.1.5) and (4.1.11), the loss is

$$
\begin{aligned}
-\Delta X(t) &= e^{-\alpha\Delta t}\theta(t) - \theta(t) - e^{-\alpha\Delta t}\pi(t)(\Delta P(t) + \Delta Q(t)) \\
&= a_1\pi(t)(\Delta P(t)) + \Delta Q(t)) + a_2\pi(t) + bc(t) \qquad (4.1.12)
\end{aligned}
$$

where

$$
a_1 = -e^{r\Delta t}, \quad a_2 = \frac{\mu - r}{r}(e^{r\Delta t} - 1), \quad b = -\frac{1}{r}(e^{r\Delta t} - 1), \qquad (4.1.13)
$$

$$
\Delta P(t) = \int_t^{t+\Delta t} \sigma dW(s) \qquad (4.1.14)
$$

and

$$
\Delta Q(t) = \int_t^{t+\Delta t} dJ(s). \qquad (4.1.15)
$$

It is clear from the Markov property of jump-diffusion process that, at time t, $\Delta P(t) \sim N(0, \Delta t\sigma^2)$, and $\Delta Q(t)$ has a compound Poisson distribution given by

$$
\Delta Q(t) = (\sum_{k=1}^{N(t+\Delta t)} Y_k - \sum_{k=1}^{N(t)} Y_k) \sim \sum_{k=1}^{N(\Delta t)} Y_k. \qquad (4.1.16)
$$

Let $f_t(x)$ denote the density function of the loss, and define

$$
Z = a_1\pi(t)\Delta P(t) + a_2\pi(t) + bc(t).
$$

Then, from the answer to Problem 14 of Section 1.8 given in [57], it follows that

$$
f_t(x) = e^{-\lambda\Delta t}\sum_{k=0}^{\infty}\frac{(\lambda\Delta t)^k}{k!}\int \Phi(z)\cdot\Phi^{k*}(x-z)dz. \qquad (4.1.17)
$$

Here $\Phi(x)$ is the density function of $a_2\pi(t)+bc(t)$, $\Phi^{k*}(x)(k \geq 1)$ is the density function of $\sum_{i=1}^{k} a_i\pi(t)Y_i$ and $\Phi^{0*}(x)$ is the dirac function.

4.1.2.1 Stress testing

In (4.1.14), for a given portfolio, b is a negative constant and a_2 is positive assuming that $\mu > r$. Therefore, the loss mainly comes from the movements of stock prices. In the dynamics of (4.1.13), $\Delta P(t)$ has a normal distribution and $\Delta Q(t)$ captures the extreme losses.

As we know, for a sufficiently small time interval Δt, the jumps with i ($i \geq 2$) jumps can be neglected. Then, the scenarios can be constructed as follows. Let the event with a jump in this interval be the scenario and its probability denoted by α. Thus, the loss can be tested as a normal model with a scenario being incorporated. It follows from (4.1.17) that the combined loss density function is approximated by

$$f_c \sim (1-\alpha)f + \alpha f_s = (1-\lambda\Delta t)f_s + \lambda\Delta t f_s,$$

where f and f_s are the first two terms of the right hand of (4.1.17).

The general VaR is

$$VaR = a_1'|\pi(t)| + a_2\omega(t) + bc(t), \tag{4.1.18}$$

where a_1'=$aF_c^{-1}(k)$ with

$$F_c^{-1}(k) = \min(a \mid \int_{-\infty}^{a} f_c(x)dx \geq k). \tag{4.1.19}$$

The general $CVaR$ is

$$CVaR = a_1''|\pi(t)| + a_2\pi(t) + bc(t), \tag{4.1.20}$$

where a_1''=$aH(k)$ and

$$H(k) = \frac{1}{1-k}\int_{x \geq F_c^{-1}(k)} x f_c(x)dx. \tag{4.1.21}$$

4.1.3 Optimal problem with risk constraint

Let $\mathcal{U}_J$ denote the class of all *admissible* strategies u. Suppose that $CVaR$ can not exceed a level R. Then, the final optimal portfolio problem with the $CVaR$ constraint can be formally stated as :

Problem F_1

$$\max_{(\pi(x,t),c(x,t))\in\mathcal{U}_J} E\left[\int_0^\tau U_1(t,c(t))dt\right] + \Psi(\tau, X_\tau) \tag{4.1.22}$$

subject to

$$dX(t) = (\pi(t)(\mu - re) + rX(t) - c(t))dt + \pi(t)(\sigma dW(t) + dU(t)), \tag{4.1.23}$$

$$a_1''|\pi(t)| + a_2\pi(t) + bc(t) \leq R. \tag{4.1.24}$$

To solve the optimal portfolio problem, we make use of the dynamic programming method introduced in Chapter 2. Define the value function as

$$V(x,t) := \sup_{(\pi(x,t),c(x,t))\in\mathcal{U}_J} E\left[\int_0^{\tau} U_1(t,c(t))dt\right] + \Psi(\tau, X_\tau) \tag{4.1.25}$$

where x is a possible state of X_t. Denote

$$G(x,\pi(x,t),c(x,t)) := \pi(x,t)(\mu - r) + rx - c(x,t)$$

and

$$H(\pi(x,t)) := \pi^2(x,t)\sigma^2.$$

Then, the corresponding HJB equation is given by

$$\begin{aligned}&\frac{\partial V}{\partial t} + \sup_{\pi(x,t),c(x,t)} (U_1(t,c(x,t)) + G(x,\pi(x,t),c(x,t))\frac{\partial V}{\partial x} + \frac{1}{2}H(\pi(x,t))\frac{\partial^2 V}{\partial x^2}\\ &+\lambda E(V(x+\pi(x,t)Y,t) - V(x,t))) = 0 \end{aligned}\tag{4.1.26}$$

with the boundary conditions

$$V(x,T) = U_2(T,X(T)), \quad V(0,t) = 0, \tag{4.1.27}$$

subject to the $CVaR$ constraint

$$a_1''|\pi(t)| + a_2\pi(t) + bc(t) \leq R. \tag{4.1.28}$$

For the optimal portfolio problem with the $CVaR$ constraint, it requires first to solve the following static optimization problem

$$\begin{aligned}&\max_{\pi(x,t),c(x,t)} (U_1(t,c(x,t)) + G(x,\pi(x,t),c(x,t))\frac{\partial V}{\partial x} + \frac{1}{2}H(\pi(x,t))\frac{\partial^2 V}{\partial x^2}\\ &+\lambda E(V(x+\pi(x,t)Y,t) - V(x,t)))\end{aligned}\tag{4.1.29}$$

subject to (4.1.28).

Introducing the Lagrange function, we obtain

$$\begin{aligned}L(\pi(x,t),c(x,t),\lambda_1(x,t),\lambda_2(x,t)) &= U_1(t,c(x,t)) + G(x,\pi(x,t),c(x,t))\frac{\partial V}{\partial x}\\ &+\frac{1}{2}H(\pi(x,t))\frac{\partial^2 V}{\partial x^2} + \lambda E(V(x+\pi(x,t)Y,t) - V(x,t)) - \lambda_1(x,t)(R - a_1''\pi(x,t)\\ &-a_2\pi(x,t) - bc(x,t)) - \lambda_2(x,t)(R - (-a_1'')\pi(x,t) - a_2'\pi(x,t) - bc(x,t)).\end{aligned}\tag{4.1.30}$$

Then, the first order necessary conditions of optimality for the static optimization problem are given by

$$(\mu - r)\frac{\partial V}{\partial x} + \Sigma\pi(x,t)\frac{\partial^2 V}{\partial x^2} + \lambda E V_\pi(x,t)$$
$$+\lambda_1(x,t)\,(a_1'' + a_2) + \lambda_2(x,t)\,(-a_1' + a_2) = 0, \tag{4.1.31}$$

$$\frac{\partial U_1}{\partial c} = \frac{\partial V}{\partial x} - \lambda_1(x,t)b - \lambda_2(x,t)b, \tag{4.1.32}$$

$$\lambda_1(x,t)(R - a_1''\pi(x,t) - a_2\pi(x,t) - bc(x,t)) = 0, \tag{4.1.33}$$

$$\lambda_2(x,t)(R - (-a_1'')\pi(x,t) - a_2\pi(x,t) - bc(x,t)) = 0, \tag{4.1.34}$$

$$\lambda_1(x,t) \le 0, \tag{4.1.35}$$

$$\lambda_2(x,t) \le 0. \tag{4.1.36}$$

In addition, (4.1.31) is used for finding $\pi_{opt}(x,t)$, (4.1.32) is used to solve for $c_{opt}(x,t)$, (4.1.33) and (4.1.34) are applied to solve for $\lambda_1(x,t)$ and $\lambda_2(x,t)$ whenever $\lambda_1(x,t) \neq 0$ and $\lambda_2(x,t) \neq 0$. Substituting $\omega_{opt}(x,t)$ and $c_{opt}(x,t)$ into (4.1.26) gives

$$\frac{\partial V}{\partial t} + U_1(t, c_{opt}(x,t)) + G(x, \pi_{opt}(x,t), c_{opt}(x,t))\frac{\partial V}{\partial x} + \frac{1}{2}H(\pi_{opt}(x,t))\frac{\partial^2 V}{\partial x^2}$$
$$+\lambda E(V(x + \pi_{opt}(x,t)Y, t) - V(x,t)) = 0, \tag{4.1.37}$$

which can, in principle, be solved for the value function $V_{opt}(x,t)$. However, in view of the nonlinearity in $c_{opt}(x,t)$ and $\pi_{opt}(x,t)$, the HJB equation is highly nonlinear. Thus, numerical methods are required to be used for finding $\pi_{opt}(x,t)$, $c_{opt}(x,t)$, $\lambda_1(x,t)$, $\pi_2(x,t)$ and $V_{opt}(x,t)$ iteratively.

4.1.4 Numerical results

In this section, we consider the classical utility function

$$U_1(t,x) = U_2(t,x) = e^{-\delta t}x^\gamma \qquad \delta > 0\ ,\ 0 < \gamma < 1, \tag{4.1.38}$$

where δ is the discount factor. By using a trial function

$$V(x,t) = e^{-\delta t}h(t)x^\gamma \qquad \delta > 0\ ,\ 0 < \gamma < 1, \tag{4.1.39}$$

which separates the x and t variables, the HJB equation is reduced to a Bernoulli equation for $h(t)$ which is an ordinary differential equation.

When the $CVaR$ constraint is imposed, although the variation of $V(x,t)$ in x is still well modeled by the term x^γ, the function h will depend on x as well because of ω and c. However, we shall show later numerically that for some reasonable values of the parameters, h is a slow varying function of x and its derivatives with respect to x is therefore very small.

In the following sections, to illustrate the effect of the jump Y and simplify the calculations, Y is assumed to be a constant y. Let the utility function be defined by (4.1.39). Then the HJB equation for the value function is given by

$$\frac{\partial V}{\partial t} + e^{-\delta t} c_{opt}^{\gamma}(x,t) + (\omega_{opt}(x,t)(\mu - r) + rx - c_{opt}(x,t))\frac{\partial V}{\partial x} + \frac{1}{2}\pi_{opt}^2(x,t)\sigma^2 \frac{\partial^2 V}{\partial x^2}$$
$$+\lambda e^{-\delta t} h(x,t)((x + \pi(x,t)y)^\gamma - x^\gamma) = 0. \qquad (4.1.40)$$

Neglecting the derivatives of h with respect to x, we obtain

$$\frac{\partial V}{\partial x} = \gamma e^{-\delta t} h(t) x^{\gamma-1}, \quad \frac{\partial^2 V}{\partial x^2} = \gamma(\gamma-1)e^{-\delta t} h(t) x^{\gamma-2}, \qquad (4.1.41)$$

$$\frac{\partial V}{\partial t} = e^{-\delta t} h'(t) x^\gamma - \delta e^{-\delta t} h(t) x^\gamma. \qquad (4.1.42)$$

Then, substituting (4.1.41) and (4.1.42) into (4.1.40), dividing it by $e^{-\delta t}x^\gamma$ and rearranging the terms gives

$$h'(x,t) + A(\omega_{opt}(x,t), x)h(t) + B(c_{opt}(x,t), h(t)) = 0 \qquad (4.1.43)$$

with the boundary condition

$$h(x,T) = 1, \qquad (4.1.44)$$

where

$$A(\pi_{opt}(x,t), x) = \gamma\left(\frac{\pi_{opt}(x,t)}{x}(\mu - r) + r\right) + \frac{1}{2}\frac{\pi_{opt}^2(x,t)}{x^2}\sigma^2\gamma(\gamma-1) - \delta$$
$$+\lambda((1 + \frac{\omega_{opt}(x,t)}{x}y)^\gamma - 1), \qquad (4.1.45)$$
$$B(c_{opt}(x,t), h(t)) = \frac{c_{opt}^{\gamma}(x,t)}{x^\gamma} - \frac{\gamma h(x,t) c_{opt}(x,t)}{x}. \qquad (4.1.46)$$

We transform (4.1.43) into a convenient form

$$g'(t) + (1-\beta)A(\omega_{opt}(x,t), x)g(t) + (1-\beta)B(c_{opt}(x,t), g(t))g(t)^\gamma = 0,$$

with the boundary condition

$$g(T) = 1,$$

where

$$g(t) = h(t)^{1-\beta}, \quad \beta = -\frac{\gamma}{1-\gamma}. \qquad (4.1.47)$$

4.1.4.1 Algorithms

Dividing the computational domain into a grid of $N_x \times N_t$ mesh points, the final algorithm can be summarized as follows:

1) For the unconstrained case, $\lambda_1^{(0)}(x,t) = \lambda_2^{(0)}(x,t) = 0$. Thus (4.1.47) is reduced to

$$g'(t) + (1-\beta)Ag(t) + (1-\beta)B = 0, \tag{4.1.48}$$

where

$$A = \gamma\left(\frac{\pi_{opt}^{(0)}(x,t)}{x}(\mu - r) + r\right) + \frac{1}{2}\frac{\pi_{opt}^{(0)}(x,t)^2}{x^2}\sigma^2\gamma(\gamma-1) - \delta$$
$$+\lambda((1+\frac{\pi_{opt}^{(0)}}{x}y)^\gamma - 1) \tag{4.1.49}$$

and $B = 1-\gamma$.
From the optimality condition (4.1.31), $\pi_{opt}^{(0)}(x,t)$ is calculated from

$$(\mu-r)\gamma x^{\gamma-1}h(t) + \omega_{opt}^{(0)}(x,t)\sigma^2\gamma(\gamma-1)x^{\gamma-2}h(t) + \lambda h(t)\gamma(x+\pi_{opt}^{(0)}(x,t)y)^{\gamma-1}y = 0. \tag{4.1.50}$$

Dividing (4.1.50) by $\gamma h(t)x^{\gamma-1}$ yields

$$(\mu - r) + \frac{\pi_{opt}^{(0)}(x,t)}{x}\sigma^2(\gamma-1) + \lambda(1+\frac{\pi_{opt}^{(0)}(x,t)}{x}y)^{\gamma-1}y = 0. \tag{4.1.51}$$

By Newton's method, $\frac{\pi_{opt}^{(0)}(x,t)}{x}$ can be obtained, which is only dependent on time t. Substituting $\frac{\pi_{opt}^{(0)}(x,t)}{x}$ into (4.1.48), we get the explicit solution of (4.1.47)

$$g(t) = (\frac{B}{A}+1)e^{\frac{1}{B}A(T-t)} - \frac{B}{A}. \tag{4.1.52}$$

As a result, $c_{opt}^{(0)}(x,t)$ has a simple form of $xg^{-1}(t)$. The solution of the unconstrained problem will be used as an initial guess in the iterative algorithm. Set iterative index k=0.

2) For $x = [0, \Delta x, \cdots, N_x\Delta x]$ and $t = [(N_t-1)\Delta t, \cdots, \Delta t, 0]$, for notational simplicity, we omit (x,t) in all the functions involved. Then, we calculate $\pi_{opt}^{(k+1)}$, $\lambda_1^{(k+1)}$, $\lambda_2^{(k+1)}$ and $c_{opt}^{(k+1)}$ from

$$(\mu - r)V_x^{(k)}(x) + \sigma^2\pi_{opt}^{(k+1)}V_{xx}^{(k)}(x) + \lambda EV_\pi^{(k)}(x+\pi y, t)$$
$$+\lambda_1^{(k)}(a_1'' + a_2) + \lambda_2^{(k)}(-a_1'' + a_2) = 0, \tag{4.1.53}$$

$$\lambda_1^{(k+1)}(R - (a_1'' + a_2)\omega_{opt}^{(k+1)} - bc_{opt}^{(k)}) = 0, \tag{4.1.54}$$

$$\lambda_1^{(k+1)}(R - (-a_1'' + a_2)\omega_{opt}^{(k+1)} - bc_{opt}^{(k)}) = 0, \tag{4.1.55}$$

$$\gamma(c_{opt}^{(k+1)})^{\gamma-1} = (\gamma x^{\gamma-1}h^{(k)} - e^{\delta t})(\lambda_1^{(k+1)} + \lambda_2^{(k+1)})b \tag{4.1.56}$$

respectively.

3) For $x = [0, \Delta x, \cdots, N_x\Delta x]$ and $n = [N_t - 1, \cdots, 0]$, solve

$$g_n^{(k+1)} = g_{n+1}^{(k+1)} + \Delta t(1-\beta)A(\pi_{opt}^{(k+1)}, x)g_{n+1}^{(k+1)} + (1-\beta)B(c_{opt}^{(k+1)}, g_n^{(k)})(g_n^{(k)})^{\gamma}. \tag{4.1.57}$$

4) Return to 2) with $k =: k+1$ until convergence.

In the following subsections, we will first carry out a stress test to check how $CVaR$ catch the extreme loss with the constructed scenario. Then we explore the effect of the $CVaR$ constraint. Numerical experiments are carried out in the environment of Matlab 7 and Fortran 90.

In our calculations, unless otherwise specified, the following parameters are used: $\delta = 0.2$, $\gamma = 0.5$, the appreciation rate $\mu = 0.2$, $\sigma = 0.5$, $r = 0.1$, the discount factor δ = 0.2 , time T is fixed as 20 years, $N_t = 1000$ and Δt is equal to 1/52 year (i.e. about one week), $\Delta x = 2$ and $N_x = 500$.

4.1.4.2 Stress test

In this subsection, stress test is conducted to evaluate the risk exposure in a scenario. We consider a scenario, where $\Delta t = 0.1$ year, $\lambda = 1$ and the jump magnitude is -0.9 (i.e. the stock price may drop by 90 percent with 0.1 probability in 0.1 year). Figure 4.1.1 depicts density function of $a_1\omega(t)(\Delta P(t) + \Delta Q(\Delta t))$ in two situations: a normal economy (f) and a scenario characterized by a negative jump (f_c). It is easily seen from the heavy tail of f_c that bigger losses are more likely to occur in this situation. As a result, the investor will suffer a bigger mean loss above the same VaR, which will be shown by Figure 4.1.2. Let $CVaR$ (f) and $CVaR$ (f_c) denote, respectively, the $CVaRs$ based on f and f_c. Figure 4.1.2 shows that $CVaR$ (f_c) is much larger than $CVaR$ (f) for high confidence levels. Thus, to capture the loss exceeding the VaR, it is reasonable to use $CVaR$ as the risk measure.

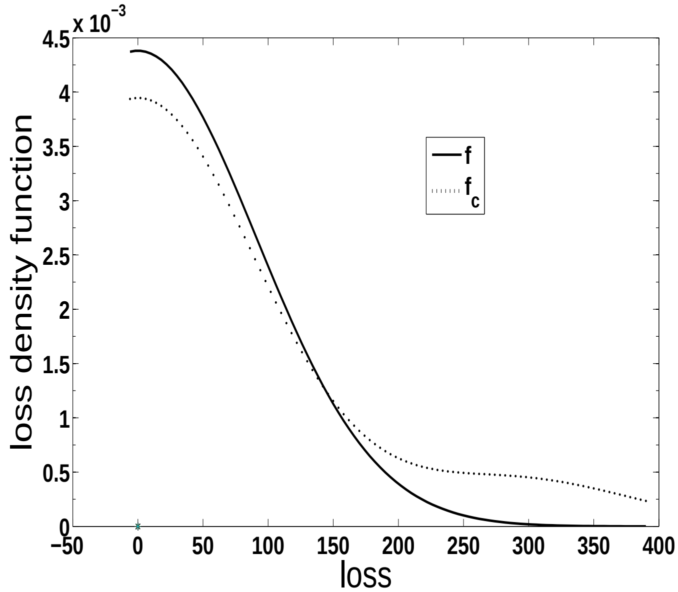

Figure 4.1.1: A typical loss distribution density with jumps

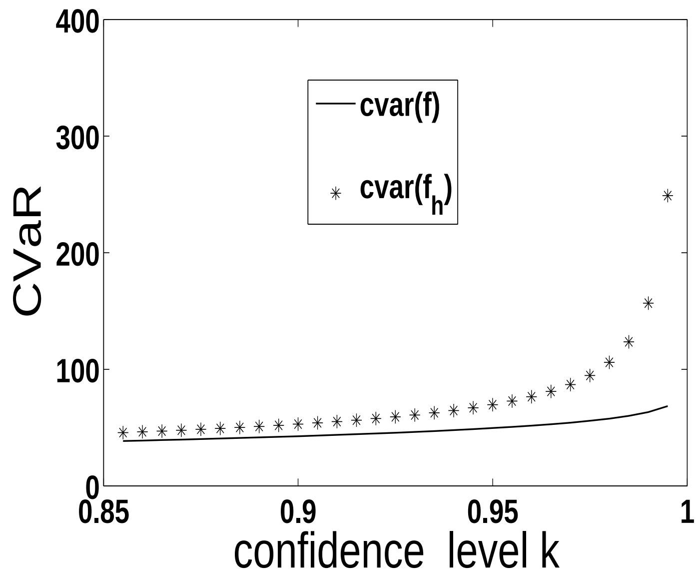

Figure 4.1.2: $CVaR$ for different confidence levels

4.1.4.3 The results with $CVaR$ constraint

In this subsection, $CVaRs$, optimal risky investments and the value functions are compared between the cases with and without constraint, respectively. Let $t = 10$, $x = 400$ and the risk level R is constrained to 120. The figures are plotted when the jump rate and jump magnitude vary between [0,0.4] and (-1,1], respectively, which denote different economic conditions.

Figure 4.1.3 describes the value of $CVaRs$. And Figure 4.1.4 shows that we have stabilized the $CVaRs$ once they are active.

Figure 4.1.5 and Figure 4.1.6 plot the risky investment in the cases without and with constraint, respectively. It is easily seen from the two figures that if the constraint is inactive, the allocation in risky assets is the same as that in the unconstrained case. However, once the $CVaR$ constraint is active, the risky investment should be cut back to meet the risk constraint.

From Figure 4.1.7 and Figure 4.1.8, it is clearly observed that $v(x,t)$ is decreased once the constraint is active. It is trivial since the investment is constrained.

Compared with the case without constraint, little difference can be seen for the consumption when $CVaR$ constraint is applied, and thus we will not plot the figure here. And this will be explained in the next numerical example.

For the given wealth and time, we examine closer the optimal portfolios with $CVaR$ constraint in the normal market and in bad economy market. A risk constraint $R = 70$ is assumed. Let the bad market parameters $\lambda = 0.12$, y=-0.3, t =10, and x varies from 0 to 1000. From Figure 4.1.9, similar investment trends are observed in both cases. Once the constraint becomes active, the investor should reduce the risky investment to decrease the risk exposure. The result is similar to that reported in ([135]), where the normal market with a VaR constraint is considered. Figure 4.1.10 depicts the consumption pattern at the time $t = 10$. With the parameters above, the investor should consume more in a bad economy. When the risk constraint is imposed, the investor would consume more than that without constraint. However, the effect of the risk constraint to the consumption pattern is negligible, which reflects that the risky investment has a big weight in the portfolio and it is enough to decrease the risk by cutting the risky investment. Figure 4.1.11 plots $h(x,t)$ over time t for two values of wealth. When the risk constraint is imposed, $h(x,t)$ is decreased little and little variation in x is observed with the approximation (4.1.41). In fact, the accuracy of the solution can be assessed by calculating the residual value of the HJB equation as that in Yiu [135]. We find that the discrete error measure of the HJB is less than

8.35557×10^{-4}.

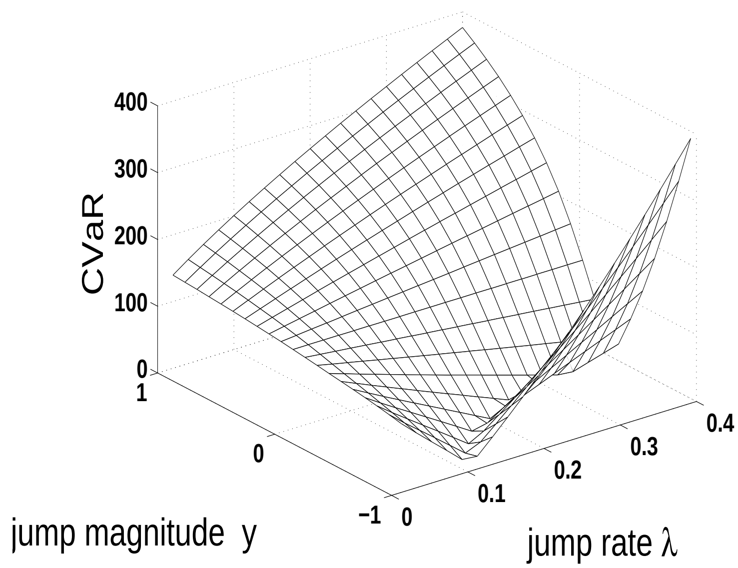

Figure 4.1.3: CVaR under different jump amplitude and jump rate

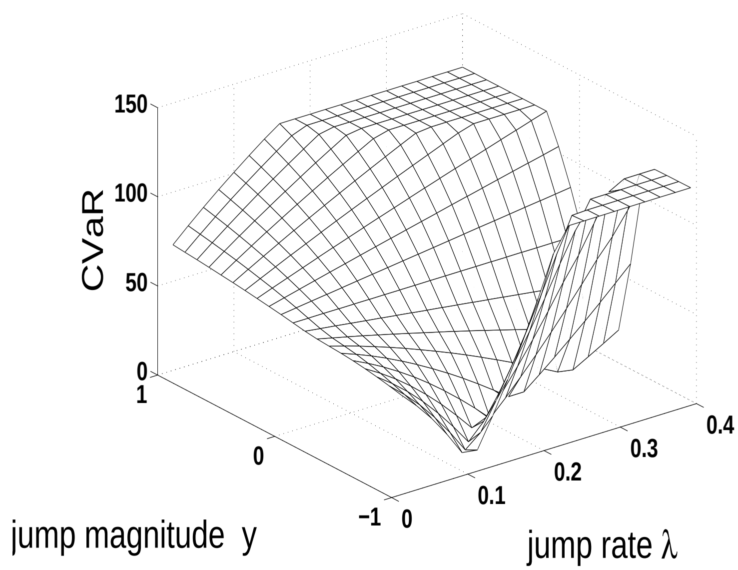

Figure 4.1.4: $CVaRs$ with stressed risk constraint under different jump magnitudes and jump rates for x=400, t = 10.

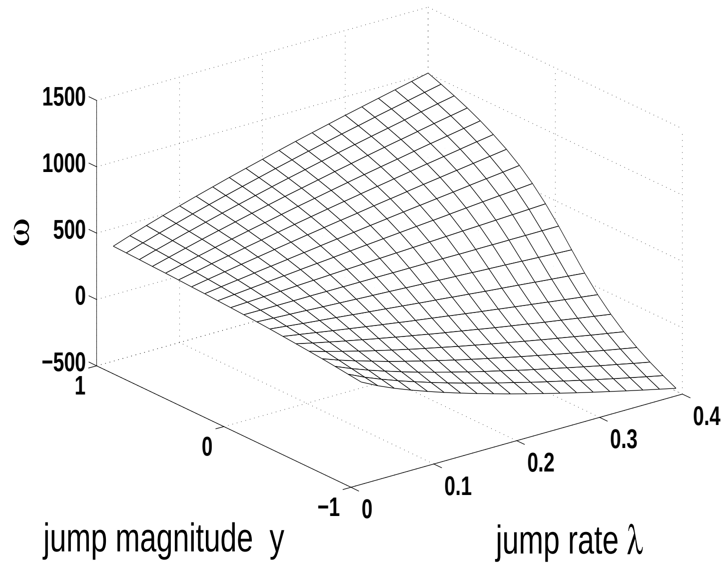

Figure 4.1.5: Risky investments under different jump magnitudes and jump rates

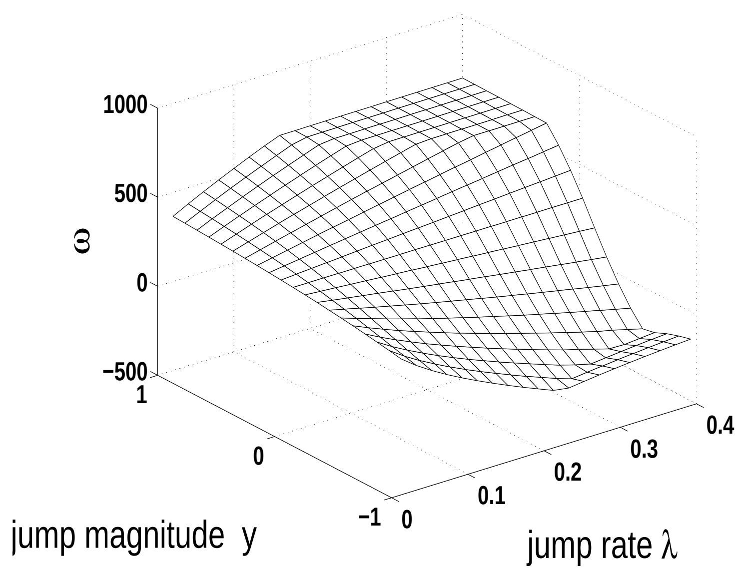

Figure 4.1.6: The optimal risky investment with stressed risk constraint under different jump magnitudes and jump rates for x=400, t=10.

4.2 The regime switching model

4.2.1 Price dynamics and the optimization problem

Suppose the instantaneous market interest rate $r(t)$ of the bank account B is:

$$r(t) = r(t, Y(t)) = \langle \mathbf{r}, Y(t) \rangle \ , \tag{4.2.1}$$

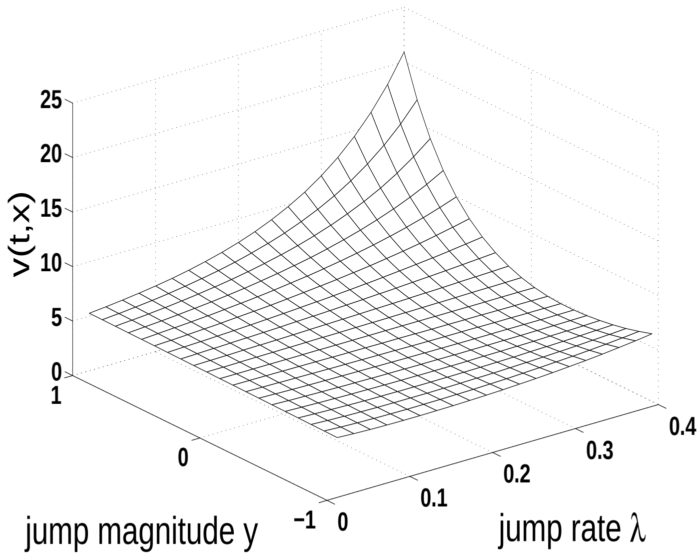

Figure 4.1.7: The value functions under different jump magnitudes and jump rates

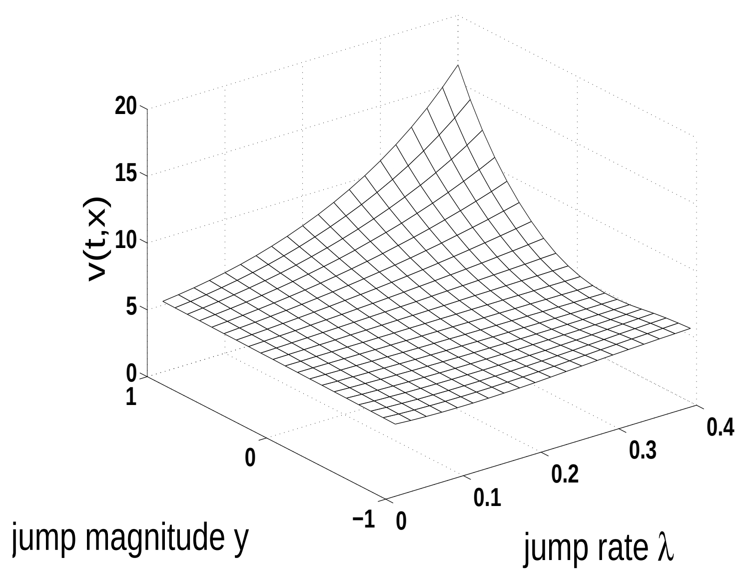

Figure 4.1.8: v(x,t) with stressed risk constraint under different jump magnitudes and jump rates for x=400, t=10.

where $Y(t)$ is defined by (2.1.11), $\langle \cdot, \cdot \rangle$ denote an inner product in $\Re^N$ and $\mathbf{r} := (r_1, r_2, \ldots, r_N)' \in \Re^N$ with $r_i > 0$ $(i = 1, 2, \ldots, N)$.

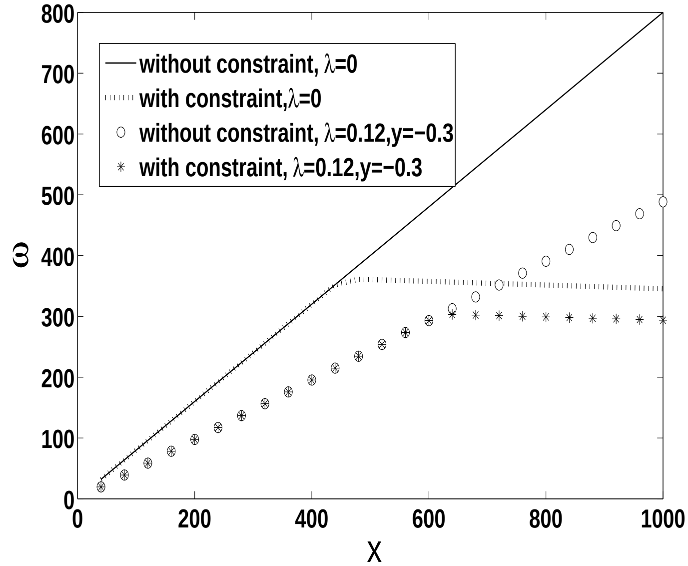

Figure 4.1.9: For λ=0.12, y=-0.3, t = 10, the optimal ω with stressed risk constraint

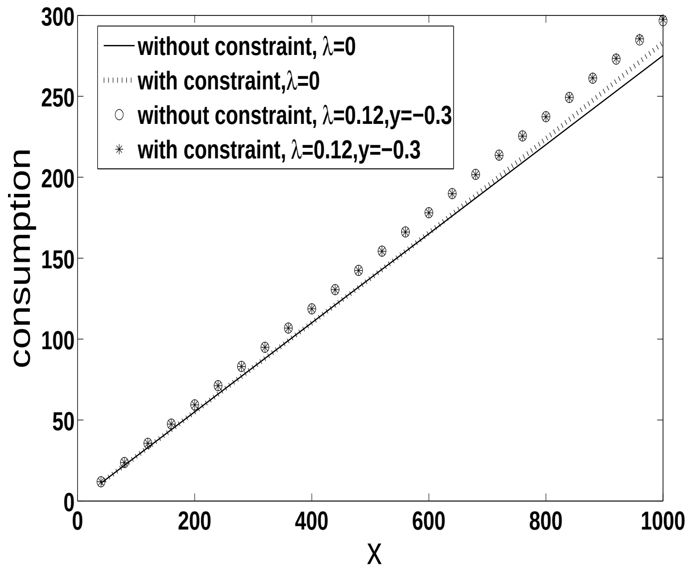

Figure 4.1.10: For λ=0.12, y=-0.3, t = 10, the optimal consumption

Then, the price process of the bank account B is governed by:

$$B(t) = \exp\left(\int_0^t r(s)ds\right) , \quad B(0) = 1 \ . \tag{4.2.2}$$

Let $\mu(t)$ and $\sigma(t)$ denote the appreciation rate and the volatility rate of the risky asset

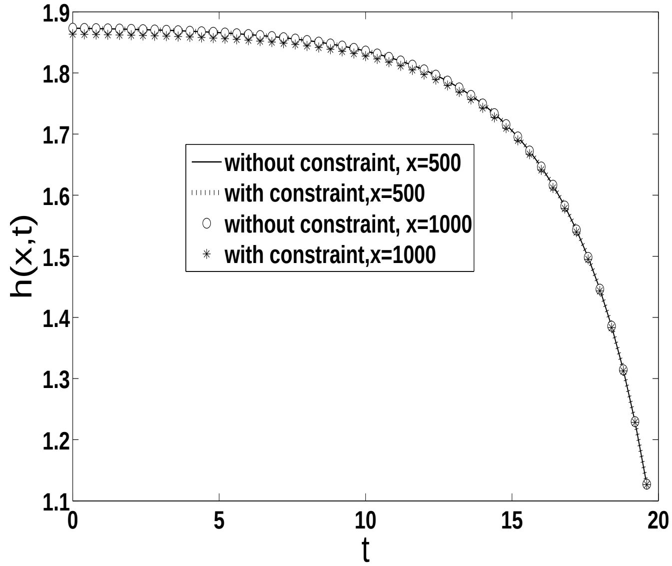

Figure 4.1.11: For λ=0.12, y=-0.3, the utility function

at time t, respectively. We suppose that

$$\begin{aligned} \mu(t) &:= \mu(t, Y(t)) = \langle \mu, Y(t) \rangle \ , \\ \sigma(t) &:= \sigma(t, Y(t)) = \langle \sigma, X(t) \rangle \ , \end{aligned} \tag{4.2.3}$$

where $\mu := (\mu_1, \mu_2, \ldots, \mu_N)^\top \in \Re^N$ and $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_N)^\top \in \Re^N$ with $\mu_i > r_i$ and $\sigma_i > 0$, for each $i = 1, 2, \ldots, N$.

The price process $\{S(t)\}_{t \in \mathcal{T}}$ of the risky asset S is assumed to be:

$$dS(t) = S(t)\bigg(\mu(t)dt + \sigma(t)dW(t)\bigg) \ , \quad S(0) = s \ . \tag{4.2.4}$$

For each $t \in \mathcal{T}$, write $\mathcal{G}(t) := \mathcal{F}^Y(t) \vee \mathcal{F}(t)$, an enlarged σ field generated by $\mathcal{F}^Y(t)$ and $\mathcal{F}(t)$.

Let $\{X(t)\}_{t \in \mathcal{T}}$ denote the wealth process of the economic agent with initial wealth $X(0) = x > 0$ and initial state $Y(0) = e_i \in \mathcal{E}$. Then,

$$dX(t) = [\pi(t)(\mu(t) - r(t)) + r(t)X(t) - c(t)]dt + \pi(t)\sigma(t)dW(t) \ . \tag{4.2.5}$$

The set of all admissible control processes is denoted by $\mathcal{U}_R$. Then, the expected discounted utility of the agent is defined as:

$$J(x_0, y; u(\cdot)) := E\bigg[\int_0^\tau U_1(t, c(t))dt + \Psi(\tau, X_\tau)|Y_0 = y, X(0) = x_0\bigg] \ , \tag{4.2.6}$$

In the sequel, we shall present the $MVaR$ constraint of the problem. First, as previously, for a short time interval h, let $\pi(\tau) = \pi(t)$ and $c(\tau) = c(t)$, for all $\tau \in [t, t+h]$. We also assume that there is no regime-switching in the small time interval. In other words, $X(\tau) = X(t)$, for all $\tau \in [t, t+h]$.

First, we define the following three quantities:

$$\begin{aligned} \alpha_i &:= -r_i \ , \\ \theta_i(\tau) &:= \theta_i(t) = \frac{\pi(t)(\mu_i - r_i) - c(t)}{-r_i} \ , \\ Y(\tau) &:= e^{\alpha_i \tau} X(\tau) \ , \quad i = 1, 2, \ldots, N \ , \quad \tau \in [t, t+h] \ . \end{aligned} \tag{4.2.7}$$

Then, in the small time interval $[t, t+h]$,

$$Y(t+h) - Y(t) = \theta_i(t)(e^{\alpha_i(t+h)} - e^{\alpha_i t}) + \int_t^{t+h} e^{\alpha_i \tau} \pi(t) \sigma_i dW(\tau) \ . \tag{4.2.8}$$

This implies that

$$X(t+h) = e^{-\alpha_i h}(X(t) - \theta_i(t)) + \theta_i(t) + \int_t^{t+h} e^{-\alpha_i(t+h-\tau)} \pi(t) \sigma_i dW(\tau) \ , \tag{4.2.9}$$

which is an Ornstein-Uhlenbeck process with a negative mean-reverting parameter α_i.

The conditional mean of $V(t+h)$ given $\mathcal{G}(t)$ and $X(t) = e_i$ under the measure $\mathcal{P}$ is:

$$E[X(t+h)|\mathcal{G}(t), X(t) = e_i] = \theta(t) + e^{-\alpha_i h}(X(t) - \theta_i(t)) \ , \tag{4.2.10}$$

and the conditional variance of $V(t+h)$ given $\mathcal{G}(t)$ and $Y(t) = e_i$ is:

$$Var[X(t+h)|\mathcal{G}(t), Y(t) = e_i] = \frac{\sigma_i^2 \pi^2(t)}{2\alpha_i}(1 - e^{-2\alpha_i h}) \ . \tag{4.2.11}$$

Now, we define the discounted net loss of the portfolio over the time interval $[t, t+h]$ as below:

$$\Delta_i X(t, h) := X(t) - e^{-r_i h} X(t+h) \ , \quad \forall i = 1, 2, \ldots, N \tag{4.2.12}$$

which, conditional on $\mathcal{G}(t)$, can be viewed as a function of the random variable $X(t+h)$.

Then, under $\mathcal{P}$, the VaR of the portfolio $VaR(t, h, i, \beta)$ with probability level β over the time interval $[t, t+h]$ given $\mathcal{G}(t)$ and $Y(t) = e_i$ is defined as:

$$\mathcal{P}(\Delta_i V(t, h) \leq VaR(t, h, i, k)|\mathcal{G}(t), Y(t) = e_i) = k \ . \tag{4.2.13}$$

Since $\Delta_i X(t,h)$ is normally distributed conditional on $\mathcal{G}(t)$ and $Y(t) = e_i$, it can be shown that

$$VaR(t,h,i,k) = -\theta_i(t)(1-e^{r_i h}) + \Phi^{-1}(k)\sqrt{\frac{\sigma_i^2 \pi^2(t)}{2\alpha_i}(1-e^{-2\alpha_i h})} \ , \qquad (4.2.14)$$

which depends on the portfolio decision $\pi(t)$ we made at time t.

Let

$$\begin{aligned} a_{1i} &:= \Phi^{-1}(1-k)\sqrt{\frac{e^{2r_i h}-1}{2r_i}} \ , \\ a_{2i} &:= -\left(\frac{\mu_i - r_i}{r_i}\right)(e^{r_i h}-1) \ , \\ b_i &:= \frac{1}{r_i}(e^{r_i h}-1) \ . \end{aligned} \qquad (4.2.15)$$

Then,

$$VaR(t,h,i,\beta) = a_{1i}|\pi(t)|\sigma_i + a_{2i}\pi(t) + b_i c(t) \ , \quad i = 1,2,\ldots,N \ . \qquad (4.2.16)$$

We define the $MVaR$ of the portfolio with probability level k over the time horizon $[t, t+h]$ given $\mathcal{G}(t)$ as:

$$MVaR(t,h,k) = \max_{i=1,2,\ldots,N} VaR(t,h,i,k) \ . \qquad (4.2.17)$$

Then, the constraint of restricting $MVaR$ at the level R is:

$$MVaR(t,h,k) \leq R \ . \qquad (4.2.18)$$

This is equivalent to the following N constraints:

$$a_{1i}|\pi(t)|\sigma_i + a_{2i}\pi(t) + b_i c(t) \leq R \ , \quad i = 1,2,\ldots,N \ . \qquad (4.2.19)$$

For simplicity, assume that there is no short selling. We have

$$(a_{1i}\sigma_i + a_{2i})\pi(t) + b_i c(t) \leq R \ , \quad i = 1,2,\ldots,N \ . \qquad (4.2.20)$$

We shall have an upper bound to constrain the investment in the risky asset S. That is, $\pi(t)$ is bounded above. The constraint on the consumption will be imposed if R is small.

The portfolio optimization problem with the $MVaR$ constraint is then formulated below:

Problem F_2:

$$\sup_{(\pi(x,t),c(x,t))\in\mathcal{U}_R} E\left[\int_0^\tau U_1(t,c(t))dt + \Psi(\tau, X_\tau)\right] \tag{4.2.21}$$

subject to:

$$dX(t) = [\pi(t)(\mu(t) - r(t)) + r(t)X(t) - c(t)]dt + \pi(t)\sigma(t)dW(t) \ ,$$
$$(a_{1i}\sigma_i + a_{2i})\pi(t) + b_i c(t) \le R \ , \quad i = 1, 2, \ldots, N \ . \tag{4.2.22}$$

4.2.2 Regime-switching HJB equation and the optimality conditions

Given $X(0) = x$ and $Y(0) = y$, denote the value function

$$V(t,x,y) \ := \ \sup_{(\pi,c)\in\mathcal{U}_R} E\left[\int_t^T U_1(\tau, c(\tau))d\tau | X(t) = x, Y(t) = y\right] . \tag{4.2.23}$$

We shall derive a system of regime-switching HJB equation for the value function.

Assume that the control process u is Markovian with respect to $\mathcal{G}$. That is,

$$u(t) = u(t, X(t), Y(t)) \ .$$

Then we write $\pi(t)$ by $\pi(t,x,y)$ and c(t) by $c(t,x,y)$, with $X(t) = x$ and $Y(t) = y$.

Let

$$F(t,x,y,\pi(t,x,y),c(t,x,y)) = \pi(t,x,y)(\mu(t) - r(t)) + r(t)x - c(t,x,y) \ , \tag{4.2.24}$$

and

$$G(t,x,y,\pi(t,x,y),c(t,x,y)) = \pi^2(t,x,y)\sigma^2(t) \ . \tag{4.2.25}$$

Let $V_i := V(t,x,e_i)$, for each $i = 1, 2, \ldots, N$, and $\mathbf{V} := (V_1, V_2, \ldots, V_N)$. Then, by the principle of dynamic programming in the stochastic optimal control, it can be shown that the value function V satisfies the following regime-switching HJB equation:

$$\frac{\partial V}{\partial t} + \sup_{\pi,c}\Bigg[U(t, c(t,x,Y(t))) + F(t,x,Y(t),\pi(t,x,y(t)),c(t,x,Y(t)))\frac{\partial V}{\partial x}$$
$$+\frac{1}{2}G(t,v,X(t),\pi(t,v,X(t)),c(t,v,X(t)))\frac{\partial^2 V}{\partial x^2} + \langle \mathbf{V}, QY(t)\rangle\Bigg] = 0 \ , \tag{4.2.26}$$

with terminal and boundary conditions:

$$V(T, x, y) = 0 \ ,$$

and

$$V(t, 0, y) = 0 \ ,$$

subject to a set of N VaR constraints:

$$(a_{1i}\sigma_i + a_{2i})\pi(t) + b_i c(t) \leq R \ , \quad i = 1, 2, \ldots, N \ .$$

Hence, the vector $\mathbf{V}$ of the value functions at different regimes satisfies the following system of coupled HJB equations:

$$\begin{aligned} &\frac{\partial V_i}{\partial t} + \sup_{\pi, c}\Bigg[U(t, c(t, x, e_i)) + F(t, x, e_i, \pi(t, x, e_i), c(t, x, e_i))\frac{\partial V_i}{\partial v} \\ &+\frac{1}{2}G(t, x, e_i, \pi(t, x, e_i), c(t, x, e_i))\frac{\partial^2 V_i}{\partial x^2} + \langle \mathbf{V}, Qe_i\rangle\Bigg] = 0 \ , \end{aligned} \tag{4.2.27}$$

with terminal and boundary conditions:

$$V_i(T, x, e_i) = 0 \ ,$$

and

$$V_i(t, 0, e_i) = 0 \ , \quad i = 1, 2, \ldots, N \ ,$$

subject to a set of N VaR constraints:

$$(a_{1i}\sigma_i + a_{2i})\pi(t) + b_i c(t) \leq R \ , \quad i = 1, 2, \ldots, N \ .$$

In the sequel, we shall consider the situation when there are two states in the Markov chain and the agent has a power utility function. We shall illustrate how to simplify the above system of HJB equations in this situation. First, we assume that the rate matrix Q of the chain is:

$$Q = \begin{pmatrix} -p & p \\ p & -p \end{pmatrix} \ , \tag{4.2.28}$$

where p is a positive real constant. We then consider the following power utility function for consumption:

$$U(t, c(t, x, y)) := e^{-\delta t} c(t, x, y)^{\gamma} \ , \quad \delta > 0 \ , \quad 0 < \gamma < 1 \ . \tag{4.2.29}$$

Here, δ represents an impatient factor for consumption and it is assumed to be a positive constant. In this case, the value functions for the two economic states satisfy the following pair of coupled HJB equations:

$$\frac{\partial V_1}{\partial t} + e^{-\delta t} c_{opt}^{\gamma}(t, x, e_1) + [\pi_{opt}(t, x, e_1)(\mu_1 - r_1) + r_1 x - c_{opt}(t, x, e_1)] \frac{\partial V_1}{\partial x}$$
$$+ \frac{1}{2}\pi_{opt}^2(t, x, e_1)\sigma_1^2 \frac{\partial^2 V_1}{\partial x^2} - pV_1 + pV_2 = 0 \ , \tag{4.2.30}$$

and

$$\frac{\partial V_2}{\partial t} + e^{-\delta t} c_{opt}^{\gamma}(t, x, e_2) + [\pi_{opt}(t, x, e_2)(\mu_2 - r_2) + r_2 x - c_{opt}(t, x, e_2)] \frac{\partial V_2}{\partial v}$$
$$+ \frac{1}{2}\pi_{opt}^2(t, x, e_2)\sigma_2^2 \frac{\partial^2 V_2}{\partial v^2} + pV_1 - pV_2 = 0 \ . \tag{4.2.31}$$

Following the approach in Merton [97], we assume that the value function is of the following form:

$$V_i = V(t, x, e_i) = e^{-\delta t} h_i(t, x) x^{\gamma} \ , \quad i = 1, 2 \ . \tag{4.2.32}$$

This form is in line with the form of the power utility function. As in Yiu [135], we neglect the derivatives of h_i $(i = 1, 2)$ with respect to v and obtain:

$$\frac{\partial V_i}{\partial x} = \gamma e^{-\delta t} h_i(t, x) x^{\gamma - 1} \ , \tag{4.2.33}$$

$$\frac{\partial^2 V_i}{\partial x^2} = \gamma(\gamma - 1) e^{-\delta t} h_i(t, x) x^{\gamma - 2} \ , \tag{4.2.34}$$

and

$$\frac{\partial V_i}{\partial t} = e^{-\delta t} h_i'(t, x) x^{\gamma} - \delta e^{-\delta t} h_i(t, x) v^{\gamma} \ , \quad i = 1, 2 \ , \tag{4.2.35}$$

where h_i' represents the derivative of h_i with respect to t. For each $i = 1, 2$, we define:

$$A_i(\pi_{opt}(t, x, e_i), v) h_i(t, x) :=$$
$$\gamma \left[\frac{\pi_{opt}(t, x, e_i)}{v}(\mu_i - r_i) + r_i \right] + \frac{1}{2}\left(\frac{\pi_{opt}^2(t, x, e_i)}{x^2} \right) \sigma_i^2 \gamma(\gamma - 1) - \delta - p \ , \tag{4.2.36}$$

and

$$B_i(\pi_{opt}(t, x, e_i), h_i(t, x)) := \frac{c_{opt}(t, x, e_i)}{v^{\gamma}} - \frac{\gamma h_i(t, x) c_{opt}(t, x, e_i)}{v} \ . \tag{4.2.37}$$

Substituting (4.2.36)-(4.2.37) into (4.2.30) and (4.2.31), we get

$$h_1'(t, x) + A_1(\pi_{opt}(t, x, e_1), v) h_1(t, x) + B_1(c_{opt}(t, x, e_1), h_1(t, x)) + p h_2(t, x) = 0 \tag{4.2.38}$$

and

$$h_2'(t,x) + A_2(\pi_{opt}(t,x,e_2),v)h_2(t,x) + B_2(c_{opt}(t,x,e_2),h_2(t,x)) + ph_1(t,x) = 0, \quad (4.2.39)$$

with terminal conditions $h_i(T,x) = 0$, for $i = 1, 2$.

Let $\psi := -\frac{\gamma}{1-\gamma}$ and $g_i(t,x) := (h_i(t,x))^{1-\psi}$, for each $i = 1, 2$. Then, we obtain the following system of coupled ordinary differential equations (O.D.E.s) for g_i $(i = 1, 2)$:

$$\begin{aligned} & g_1'(t,x) + A_1(\pi_{opt}(t,x,e_1),v)g_1(t,x) + (1-\psi)B_1(c_{opt}(t,x,e_1), g_1(t,x)^{1-\gamma}) \\ & +(1-\psi)pg_2(t,x)^{1-\gamma}g_1(t,x)^{\gamma} = 0, \end{aligned} \quad (4.2.40)$$

$$\begin{aligned} & g_2'(t,x) + A_2(\pi_{opt}(t,x,e_2),v)g_2(t,x) + (1-\psi)B_2(c_{opt}(t,x,e_2), g_2(t,x)^{1-\gamma}) \\ & +(1-\psi)pg_1(t,x)^{1-\gamma}g_2(t,x)^{\gamma} = 0 \ , \end{aligned} \quad (4.2.41)$$

with the terminal conditions $g_i(T,x) = 0$, for each $i = 1, 2$.

For the case when there is no $MVaR$ constraint, g_i $(i = 1, 2)$ satisfy the following system of coupled ordinary differential equations :

$$g_1'(t) + (1-\psi)A_1 g_1(t) + (1-\psi)B_1 + (1-\psi)pg_2(t)^{1-\gamma}g_1(t)^{\gamma} = 0 \ , \quad (4.2.42)$$

and

$$g_2'(t) + (1-\psi)A_2 g_2(t) + (1-\psi)B_2 + (1-\psi)pg_1(t)^{1-\gamma}g_2(t)^{\gamma} = 0 \ , \quad (4.2.43)$$

where

$$A_i = \gamma\left(\frac{(\mu_i - r_i)^2}{\sigma_i^2(1-\gamma)} + r + \frac{1}{2}\frac{(\mu^i - r_i)^2}{\sigma_i^2(1-\gamma)}\right) - \delta - p \ , \quad (4.2.44)$$

and $B_i = 1 - \gamma$, for $i = 1, 2$.

4.2.3 Numerical experiments and discussions

In this section, we shall conduct numerical experiments to provide sensitivity analysis for the optimal portfolio, the optimal consumption and the VaR level arising from the Markov-modulated model when the model parameters vary. We shall identify the model parameters that have significant effects on the optimal portfolio, the optimal consumption and the VaR level. Here, we also make comparisons of the qualitative behaviors of the optimal portfolio, the optimal consumption and the VaR level obtained from our model (Model I) to those arising from the model without switching regimes (Model II). For each of the comparisons, we vary the specimen value of one parameter of Model I and keep the other parameters of Model I unchanged. We also assume that the specimen values of Model II are the same as those of Model I, except for the varying parameter in Model I. If the value of the varying parameter in Model I is identical to the value of its counterpart in Model II, Model I and Model II are identical to each other. In this fashion, we can perform both the sensitivity analysis and the comparison between Model I and Model II at the same time. For illustration, we consider the situation when there are two states in the Markov chain in Model I, and, so we have a pair of coupled HJB equations for the optimal investment and consumption problem under Model I. Here, we assume that State 1 and State 2 of the chain X represent a Economy 1 (E1) and Economy 2 (E2), respectively. We shall solve this pair of coupled HJB equations numerically by employing an iterative algorithm.

4.2.3.1 The iterative algorithm

In this subsection, we shall present the iterative algorithm and specify some specimen values of the model parameters in our implementation. In the iterative algorithm, we use the unconstrained solution as an initial guess. We divide the domain of the computation into a grid of $N_t \times N_v$ mesh points, where N_t and N_v represent the number of mesh points in the space and the time domains, respectively. The steps in the iterative algorithm are presented as follows:

For each $v = [0, \Delta v, \cdots, N_v \Delta v]$, $t = [(N_t - 1)\Delta t, \cdots, \Delta t, 0]$ and $n = N_t - 1, \cdots, 0$,
Step I: For each $i = 1, 2$, set the initial values $\pi^{(0)}_{opt}(t, x, e_i) = \frac{(\mu_i - r_i)x}{\sigma_i^2(1-\gamma)}$ and $c^{(k)}_{opt}(t, x, e_i) = vh_i^{-1/(1-\gamma)}(0, x)$(i=1, 2).
$g^{(0)}_{1,n}$ and $g^{(0)}_{2,n}$ are computed from the following two equations:

$$g^{(0)}_{1,n} = g^{(0)}_{1,n+1} + \Delta t(1-\psi)A_1 g^{(0)}_{1,n+1} + \Delta t(1-\psi)B_1 + (1-\psi)p(g^{(0)}_{2,n+1})^{1-\gamma}(g^{(0)}_{1,n+1})^{\gamma}, \quad (4.2.45)$$

and

$$g^{(0)}_{2,n} = g^{(0)}_{2,n+1} + \Delta t(1-\psi)A_2 g^{(0)}_{2,n+1} + \Delta t(1-\psi)B_2 + (1-\psi)p(g^{(0)}_{1,n+1})^{1-\gamma}(g^{(0)}_{2,n+1})^{\gamma}. \quad (4.2.46)$$

Step II: With the risk constraint, according to (3.5), we look for π and c from

$$\max_{\pi,c}\Bigg[U(t,c(t,x,e_i))+F(t,x,e_i,\pi(t,x,e_i),c(t,x,e_i))\frac{\partial V_i}{\partial x} +\frac{1}{2}G(t,x,e_i,\pi(t,x,e_i),c(t,x,e_i))\frac{\partial^2 V_i}{\partial v^2}+\langle \mathbf{V},Qe_i\rangle\Bigg]. \tag{4.2.47}$$

That is, for $k>1$, $\pi^{(k)}_{opt}(t,x,e_i)$ and $c^{(k)}_{opt}(t,x,e_i)$ (i=1,2) are solved from

$$\sup_{\pi_1^{(k)},c_1^{(k)}}\Bigg[U_n(c_1^{(k)})+F_n(v,\pi_1^{(k)},c_1^{(k)})\frac{\partial V_1^{(k-1)}}{\partial x}+\frac{1}{2}G_n(\pi_1^{(k)},c_1^{(k)})\frac{\partial^2 V_{1,n}^{(k-1)}}{\partial v^2} -pV_{1,n}^{(k-1)}+pV_{2,n}^{(k-1)}\Bigg], \tag{4.2.48}$$

$$\sup_{\pi_2^{(k)},c_2^{(k)}}\Bigg[U_n(c_2^{(k)})+F_n(v,\pi_2^{(k)},c_2^{(k)})\frac{\partial V_2^{(k-1)}}{\partial x}+\frac{1}{2}G_n(\pi_2^{(k)},c_2^{(k)})\frac{\partial^2 V_{2,n}^{(k-1)}}{\partial v^2} -pV_{2,n}^{(k-1)}+pV_{1,n}^{(k-1)}\Bigg]. \tag{4.2.49}$$

Here

$$V_{i,n}^{(k-1)}=e^{-\delta t}h_{i,n}^{(k-1)}(t,x)x^{\gamma}=e^{-\delta t}g_{i,n}^{(k-1)}(t,x)^{1-\gamma}v^{\gamma}\ ,\quad i=1,2\ . \tag{4.2.50}$$

Also, we compute $g^{(k)}_{1,n}$ and $g^{(k)}_{2,n}$ from the following equations recursively.

$$g_{1,n}^{(k)}=g_{1,n+1}^{(k)}+\Delta t(1-\psi)A_1(\pi^{(k)}_{opt}(t,x,e_1),x)g_{1,n+1}^{(k)}+ \Delta t(1-\psi)B_1(c^{(k)}_{opt}(t,x,e_1),h_{1,n+1}^{(k-1)})+\Delta tp(1-\psi)(g_{2,n+1}^{(k-1)})^{1-\gamma}(g_{1,n+1}^{(k-1)})^{\gamma}\ , \tag{4.2.51}$$

and

$$g_{2,n}^{(k)}=g_{2,n+1}^{(k)}+\Delta t(1-\psi)A_1(\pi^{(k)}_{opt}(t,x,e_i)g_{2,n+1}^{(k)}+ \Delta t(1-\psi)B_2(c^{(k)}_{opt}(t,x,e_2),h_{2,n+1}^{(k-1)})+\Delta tp(1-\psi)(g_{1,n+1}^{(k-1)})^{1-\gamma}(g_{2,n+1}^{(k-1)})^{\gamma}\ . \tag{4.2.52}$$

Step III: Return to Step II with $k=k+1$ until $\|V^{(k-1)}-V^{(k)}\|_1<\epsilon$.

Here, we implement the above iterative algorithm by Matlab. We consider some hypothetical values for the model parameters and assume that $T=20$ years, $v=1000$, $N_v=500$, $N_t=1000$, $\delta=0.2$, $\gamma=0.5$, $r_1=0.1$, $\mu_1=0.2$ and $\sigma_1=0.5$. The maximum loss is limited to $R=100$ with probability k = 0.99. Indeed, we find that the optimal results are robust with respect to R. When the interest rate r_2 of the bond, the appreciation rate μ_2 and the volatility rate σ_2 of the stock in E_2 are identical to their corresponding values in E_1, Model I and Model II are identical to each other. In this case, the numerical results for the optimal investment, the optimal consumption and the VaR level obtained from Model I are the same as those arising from Model II no matter what the value of the parameter p in the rate matrix of the chain is. We find that the optimal results are robust with respect to the change in the value of p. By varying one parameter at one time, we can perform the sensitivity analysis for the optimal investment, the optimal consumption and the VaR level with respect to that particular parameter and also make comparison between Model I and Model II. Here, we further assume that the parameter p in the rate matrix is 0.5 and focus on how the optimal investment, the optimal consumption and the VaR level change with the parameters r_2, μ_2 and σ_2.

4.2.4 The effect of σ_2

Here, we shall focus on the effect of the volatility σ_2 in E_2 on the optimal investment, the optimal consumption and the VaR level. When $\sigma_1 > (<)\sigma_2$, E_1 is said to be a "Bad" ("Good") economy relative to E_2. In this case, Model I and Model II are different from each other. When $\sigma_1 = \sigma_2$ (E_1 and E_2 coincide), there is no switching regime and Model I is identical to model II.

Figure 4.2.1 plots π_1 against the volatility σ_2 from the potential regime E_2. It can be seen that when the $MVaR$ constraint is active, the current π_1 decreases as σ_2 increases. Figures 4.2.2 and 4.2.3 depict the plots of the optimal proportion of investment in the stock, the VaR level, respectively, against the portfolio value and $t = 14$ years. The plots are compared when there is no switching ($\sigma_2 = 0.5$) and when the state switches to E_2. From Figure 4.2.2, we see that when the Markov chain is changed to E_2, the optimal investment in the stock under Model II decreases significantly as σ_2 increases. The result here also reflects that the regime-switching in volatility has significant impact on the optimal investment. The VaR level decreases substantially when σ_2 increases. This means that the agent does not prefer to take the risk by investing in the stock when the volatility σ_2 is high.

Figures 4.2.4 and 4.2.5 plot the consumption level in E_1 and E_2, respectively, when the volatility in E_2 varies. Figure 4.2.4 shows that the current consumption level c_1 in E_1 is affected by σ_2 which is in the other regime. Figure 4.2.5 shows how c_2 is affected by σ_2 in the same regime. From both Figure 4.2.4 and 4.2.5 we see that the change in volatility in one regime has a common effect to both regimes. That is, the agent will consume more to increase his utility when the volatility in either regime increases.

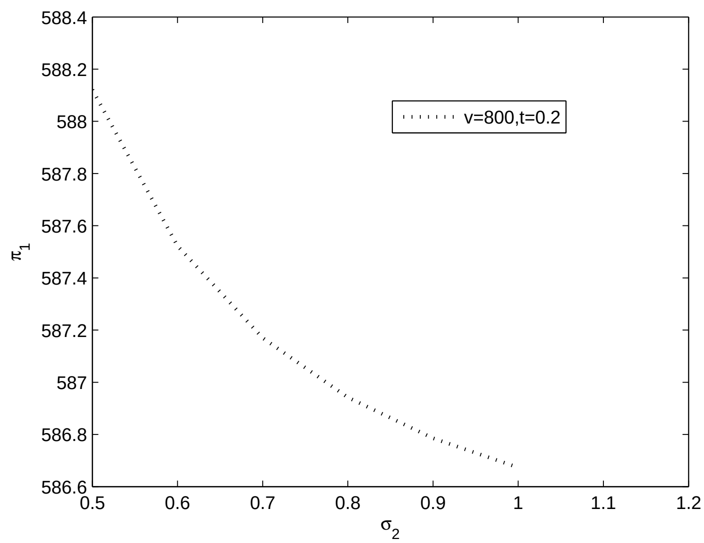

Figure 4.2.1: The optimal investment (E_1) against the volatility σ_2.

4.2.5 The effect of μ_2

In this subsection, we investigate the impact of the appreciation rate μ_2 of the stock in State 2 on the optimal investment, the optimal consumption and the VaR level. When $\mu_1 > (<)\mu_2$, E_1 is said to be a "Good" ("Bad") economy relative to E_2. In this case, Model I and Model II are different from each other. When $\mu_1 = \mu_2$, E_1 and E_2 coincide, and Model I and Model II are identical.

With the imposed $MVaR$ constraint, Figure 4.2.6 plots π_1 against μ_2 in regime E_2. It can be seen that when the $MVaR$ constraint is active, π_1 increases with μ_2. Figures 4.2.7 and 4.2.8 depict the plots of the optimal proportion of investment in the stock and the VaR level, respectively, against the optimal portfolio value for different

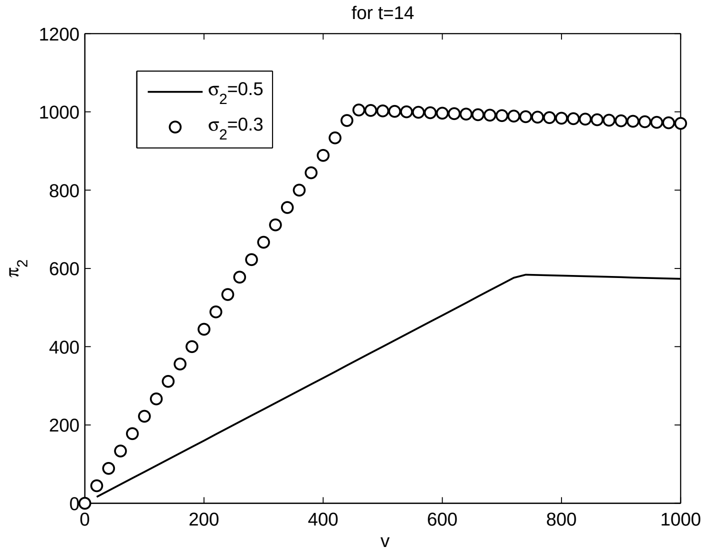

Figure 4.2.2: The optimal investment (E_2) against the portfolio value v for different σ_2

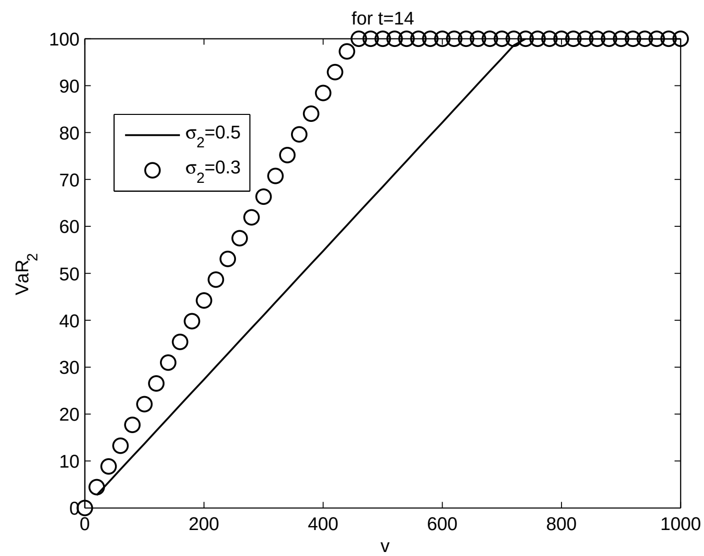

Figure 4.2.3: The VaR level (E_2) against the portfolio value v for different σ_2

value of μ_2 and $t = 14$ years. In these figures, we compare the cases when there is no regime switching and when the state switches to E_2. From Figure 4.2.7, the effect of μ_2 on the qualitative behavior of the optimal investment against the portfolio value x is significant. The optimal investment in the stock increases as μ_2 increases. Also, in the case that $\mu_2 = 0.2$ and $\mu_2 = 0.5$, there is a critical point at which there is a reversal of the optimal investment behavior from increasing to decreasing. From Figure 4.2.8, the

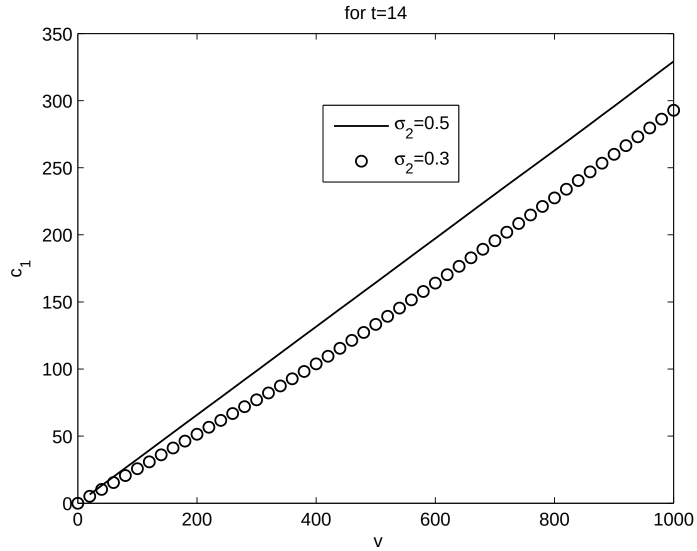

Figure 4.2.4: The optimal consumption (E_1) against the portfolio value v for different σ_2

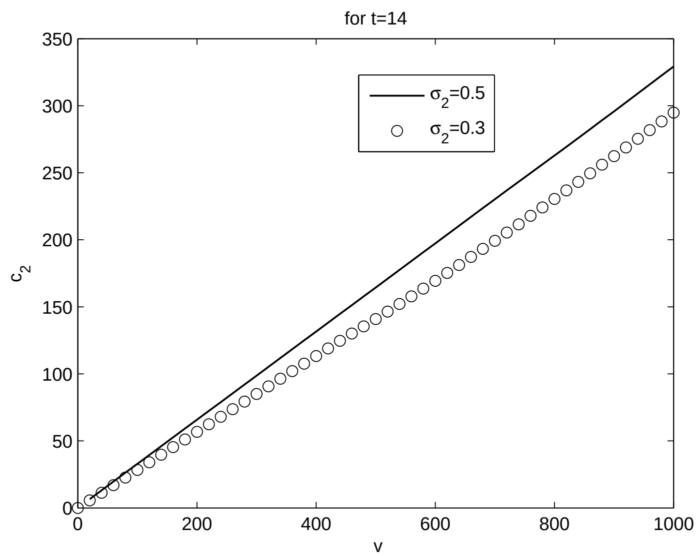

Figure 4.2.5: The optimal consumption (E_2) against the portfolio value v for different σ_2

qualitative behavior of the VaR level changes significantly with μ_2. In particular, an increase in μ_2 shifts the curve of the VaR level against the portfolio value x upwards. This reflects that when the appreciation rate μ_2 of the stock is higher, the economic agent is more willing to take higher risk by investing more in the stock.

Figures 4.2.9 and 4.2.10 plot the consumption level in E_1 and E_2, respectively, when

μ_2 in E_2 varies. Both figures show that the optimal consumption against the portfolio value x shifts downward when μ_2 increases. This reveals that the agent consumes less and invests more in the stock when the appreciation rate of the stock is higher.

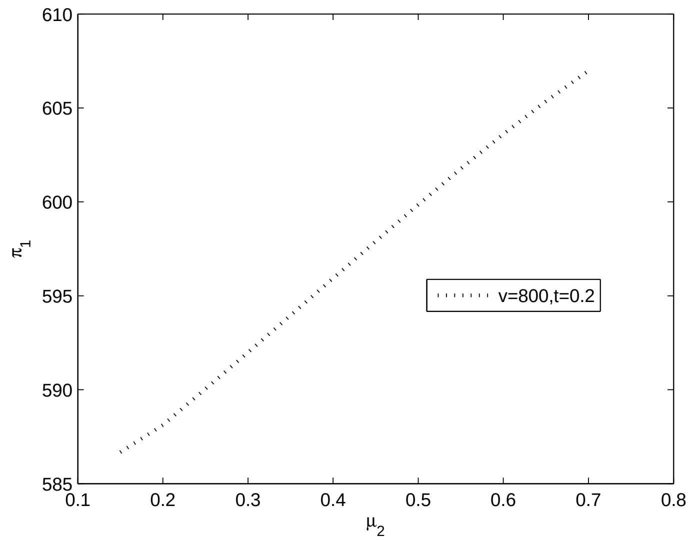

Figure 4.2.6: The optimal investment (E_1) against μ_2.

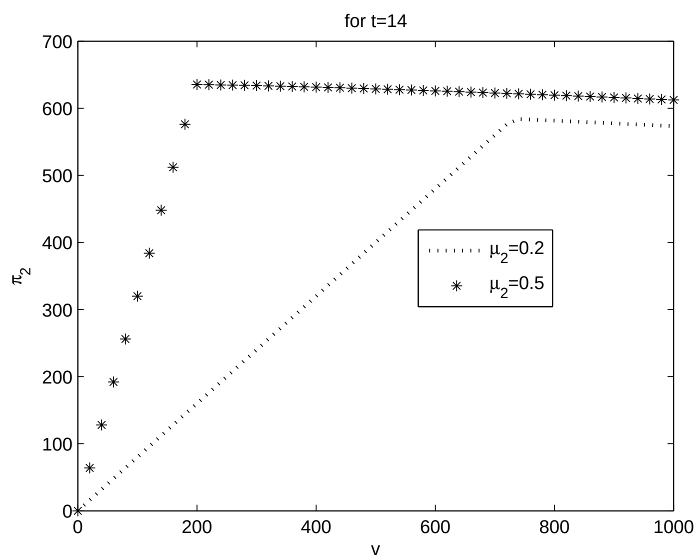

Figure 4.2.7: The optimal investment(E_2) against the portfolio value v for different μ_2

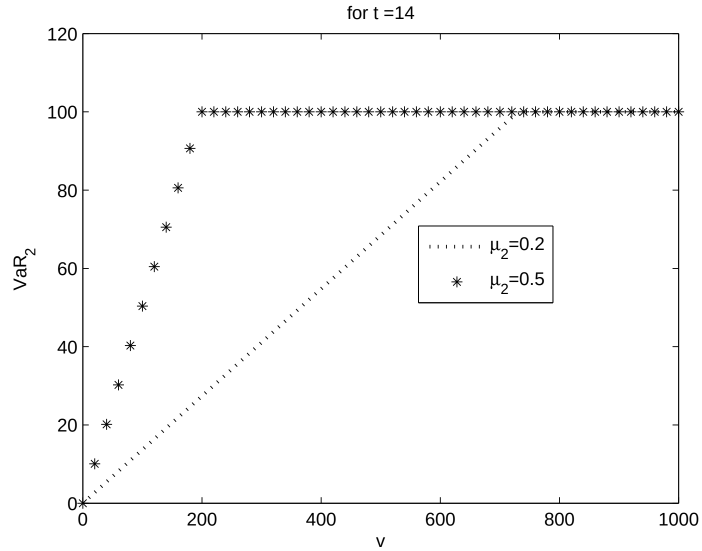

Figure 4.2.8: The optimal VaR (E_2) against the portfolio value v for different μ_2

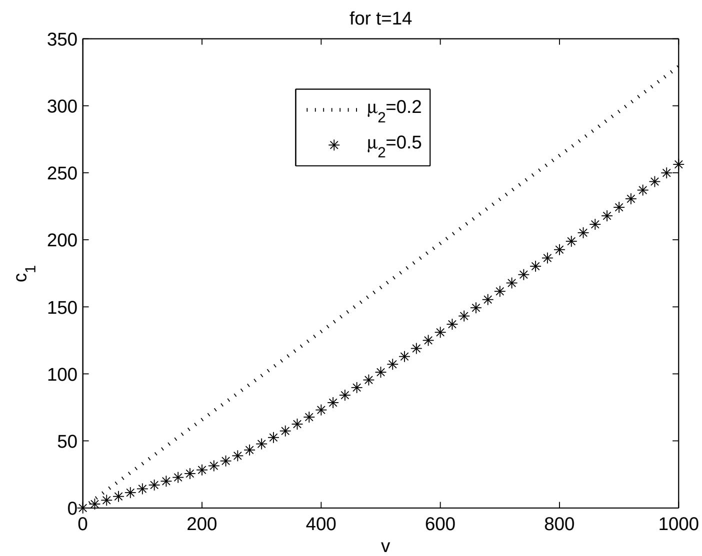

Figure 4.2.9: The optimal consumption (E_1) against the portfolio value v for different μ_2

4.2.6 The effect of r_2

We shall consider the impact of the interest rate r_2 of the bond on the optimal investment, the optimal consumption and the VaR level. When $r_1 > (<) r_2$, E1 is said to be a "Good" ("Bad") economy relative to E2. In this case, Model I and Model II are different from each other. When $r_1 = r_2$, E_1 and E_2 coincide, i.e., Model I and Model

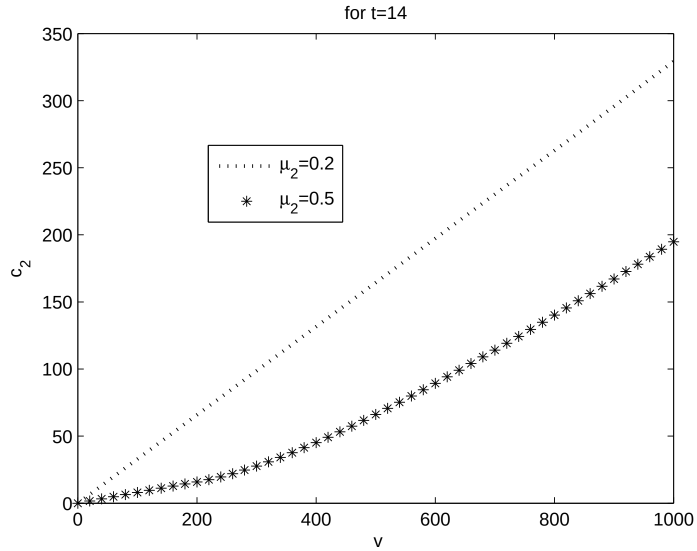

Figure 4.2.10: The optimal consumption (E_2) against the portfolio value v for different μ_2

II are identical. With the imposed $MVaR$ constraint, Figure 4.2.11 plots π_1 against r_2. The figure shows that when the $MVaR$ constraint is active, the current π_1 is smaller if the interest r_2 in E_2 gets higher. Figures 4.2.12 and 4.2.13 depict the plots of the optimal investment in the stock and the VaR level, respectively, against the optimal portfolio value for different value of r_2 and $t = 14$ years. We compare the cases in which there is that no regime switching and the cases where the state changes to E_2.

In Figure 4.2.12, the impact of r_2 on the qualitative behavior of the optimal investment against the portfolio value x is significant. The curve of the optimal investment against the portfolio value shifts upwards as r_2 decreases. Also, when r_2 decreases to a certain level, say $r_2 = 0.1$, the qualitative behavior of the optimal investment against the portfolio value changes. There is a critical point at which the optimal investment against the portfolio value x changes from increasing to decreasing. From Figure 4.2.13, the effect of r_2 on the VaR level is significant. A decrease in r_2 shifts the curve of the VaR level against the portfolio value upwards. It also results in the change in the qualitative behavior of the VaR level. For example, when r_2 decreases to 0.1, there is a critical point or a threshold level for the portfolio value, below which the VaR level increases as x does and above which the VaR level stays constant at a saturated level no matter what the value of r_2 is. This might be attributed to the presence of the $MVaR$ constraint, which limits the amount of risk taken by the agent. Figure 4.2.14

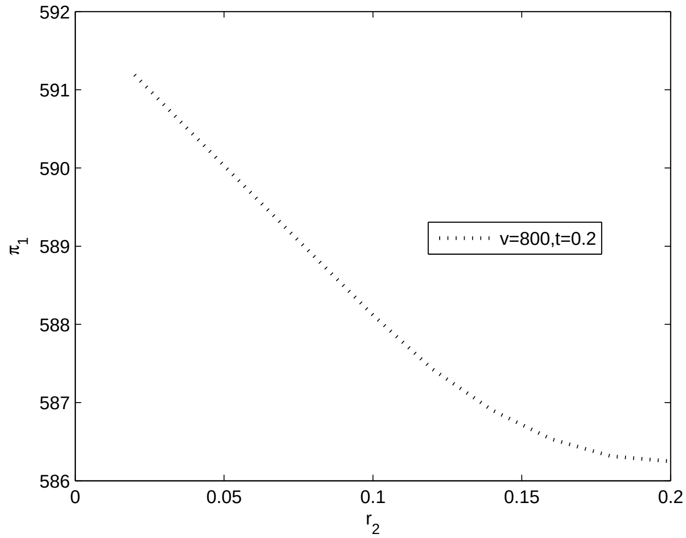

Figure 4.2.11: The optimal investment (E_1) against the volatility σ_2

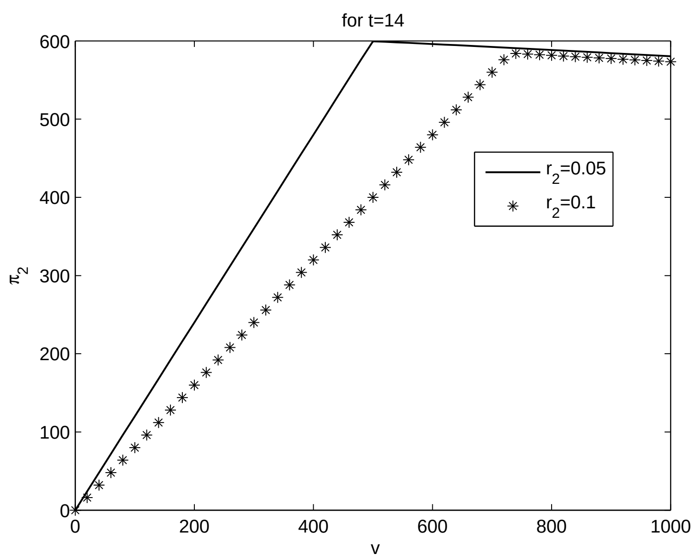

Figure 4.2.12: The optimal investment (E_2) against the portfolio value v for different r_2

and 4.2.15 plot the consumption level in E_1 and E_2, respectively, against the optimal portfolio value for different values of r_2. Figure 4.2.14 shows that current consumption is affected by μ_2 in the other regime. From Figure 4.2.14 and Figure 4.2.15, we see that the curve of the optimal consumptions against the portfolio value shift downwards as r_2 decreases in general.

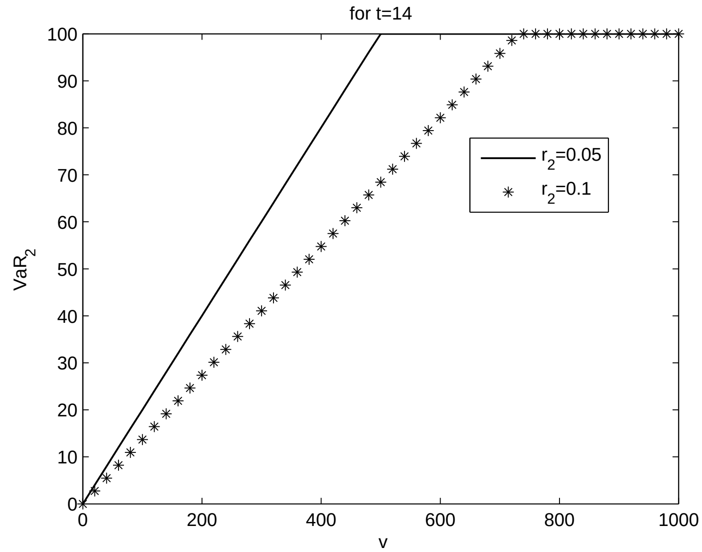

Figure 4.2.13: The VaR level (E_2) against the portfolio value v for different r_2

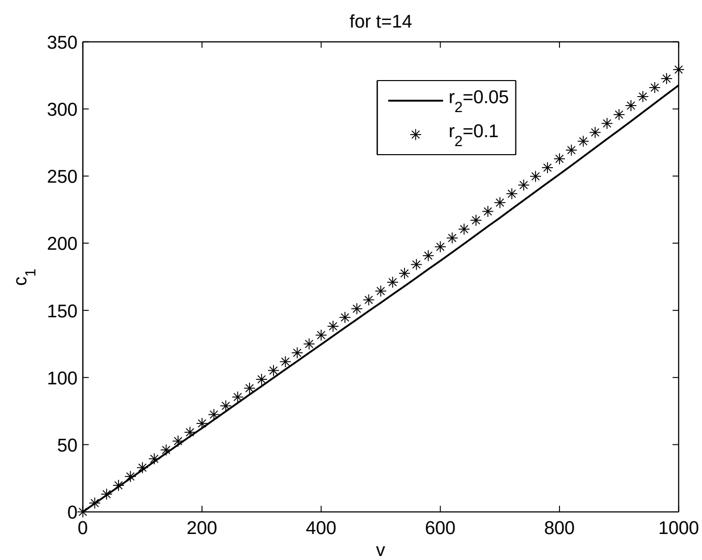

Figure 4.2.14: The optimal consumption (E_1) against the portfolio value v for different r_2

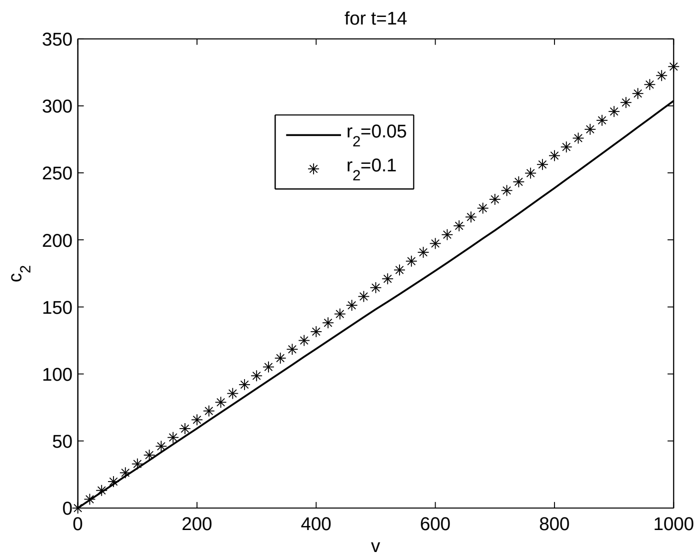

Figure 4.2.15: The optimal consumption (E_2) against the portfolio value v for different r_2

Chapter 5

Conclusions and future research directions

5.1 Conclusions

In this thesis, we have solved the risk constrained problem in finance and insurance. The dynamic risk is measured by VaR, $CVaR$ or $MVaR$, and imposed as a constraint, which is consistent with the strategy decision in the sense of dynamics. In the actuarial science literature, we are the first to consider this risk management model. Also we consider the risk constrained problem from an ordinary investor under the jump diffusion model and regime switching model. The dynamic risk constraint applied in this work overcomes the shortcomings of those mentioned in the introduction. The results can be summarized as follows:

The insurer being an investor

With VaR applied as the dynamic risk, ruin probability minimization and final wealth utility maximization have been studied in this thesis.

• For the ruin probability minimization, we obtain the closed-form solutions for the ruin probability and optimal investment in the risky asset. The results shows that if the constraint is inactive, the risky investment is the same as that without constraint. Once the constraint is active, the optimal risky investment should be decreased/ increased if

VaR is increasing/ decreasing with the risky investment.

• If the insurer's objective is to maximize the wealth utility in the finite time, we transform the primal stochastic control problem into a deterministic one. With the help of optimization software, numerical experiments are conducted to show the effect of the VaR constraint. The results show that the risky investment is smaller and the proportional reinsurance is larger than those without risk constraint. Also to show the effectiveness of this transformation, a numerical example is given, which indicates that our results coincide with the existing results when there is no constraint. Moreover, our methods can work on the incomplete market when the number of stocks is less than the number of uncertainty.

For an ordinary investor

We have investigated the portfolio selection problem with the jump diffusion model and the regime switching model.

• In this thesis, $CVaR$ is first embedded in the utility maximization problem under the jump diffusion model. With $CVaR$ as a risk constraint: (i) the stress test of the loss shows how severe an extreme event appears and $CVaR$ is more sensitive to catch the heavy-tailed loss; (ii) the numerical results shows that when $CVaR$ is active as the risk constraint, the risky investment should be cut to meet the risk management.

• We considered the optimal portfolio selection problem under the $MVaR$ constraint when the price dynamics of the risky asset are governed by a Markov-modulated GBM. The optimal portfolio selection problem was formulated as a constrained utility maximization problem over a finite time horizon. A system of coupled HJB equations are derived for the problem. We define $MVaR$ as the risk constraint and examine the sensitivity of the model parameters. These results are also used to investigate the effect of the switching regimes. The results show that the effect of the same model parameter from different regime (current or potential) on the portfolio selection is similar whether with the risk constraint or not. That is, for both cases the risky investment should be cut once the risk constraint is active.

5.2 Future research directions

To study optimal control problems, a powerful tool is dynamic programming. When *value function* is smooth enough, it solves *dynamic programm equation*. In this work, numerical algorithms to the HJB equations are used to investigate the effect of the risk constraint, which is implemented under the assumptions that there exists a sufficiently smooth solution to the HJB equation.

In general, however, the value function is not smooth enough to satisfy the *dynamic programm equations* in the classical or usual sense. Also there are many other functions other than the value function with satisfy the equation almost everywhere. Therefore a weak formulation of solutions to these equations is necessary if we are to pursue the method of dynamic programming.

Crandall and Lions have provided such a weak formulation which they called the viscosity solutions in their celebrated paper [27]. And a great deal of interest is arising in designing numerical scheme of approximating the solution of HJB equations, such as Markov chain approximation (see [49] Chapter IX). We will study the risk constrained problem with the scheme of viscosity solution in future work.

To study the problem, another popular method is to use the martingale method. It works well in more models, such as the Ito setting, than the Markov ones. The optimal portfolio is constructed by the martingale representation in a risk neutral martingale measure. For the risk constrained problem, we have obtained the existence of the solution and constructed it for the optimal consumption-investment problem in the diffusion model by the martingale method together with the convex technique, which is not included in this dissertation to keep consistence. We will investigate other models with this method.

In the existing literature, the transaction is assumed to be rebalanced instantly and free of cost. In fact, the costs incur when setting up a new portfolio or rebalancing an existing portfolio, thus it must be considered in any realistic analysis. Transaction costs can be used to model a number of costs, such as brokerage fees, bidask spreads, taxes, or even fund loads. The introduction of the proportional transaction cost is first accomplished by Magill and Cosntantinides [93]. Later the problem is widely studied by incorporating proportional transaction and/or fixed costs (See for example [9, 2, 32, 84, 93, 100, 119]). It is well recognized that transaction costs affect the investor's holding period of a particular asset, and investors accommodate transaction costs by drastically reducing the frequency and volume of trades. For the risk management, it should make sense to consider the effect of VaR constraint.

Also, intervention lag and execution delay always take place in decision-making problems in economics and finance. In many situations, firms or investors face regulatory delays (delivery lag), which may be significant, and thus need to be taken into account when management strategies are decided in an uncertain environment. In financial market context, execution delay is related to liquidity risk (see e.g. [121]) and occurs with the transaction cost. Indeed, hedge funds frequently hold illiquid assets, and need some time to find a counterpart to buy or sell them. Furthermore, this notice period gives the hedge fund manager a reasonable investment horizon. The decision involves optimally exercising a real option or optimally manipulating (with some associated cost) a state variable, which is the source of uncertainty. Several problems that fit into this framework can be found in the literature (see [20, 82] for reference). However, little is known about how the decision can be made with the risk constraint.

Chapter 6

Appendices

The proof of Theorem 2.2: The first part is obvious according to the definition and properties of nonnegative local martingale.
By assumption,

$$\max_{\check{u}\in L^2[0,T]} U_1(X_1^{\check{u}}(T)) = U_1(X_1^{\check{u}^*}(T)) \tag{A.1}$$

and

$$EU(X^{u^*}(T)) = EU_1(X_1^{u^*}(T)), \tag{A.2}$$

where (A.2) follows the fact that $U_2(X_2^{u^*}(T))$ is a martingale.
From the properties of nonnegative local martingale, we have

$$\begin{aligned} &\max_u EU(X^u(T)) \le \max_u EU_1(X_1^u(T)) \\ &\le E\max_u U_1(X_1^u(T)) \le U_1(X_1^{\check{u}^*}(T)). \end{aligned} \tag{A.3}$$

On the other hand, from the definition of $\check{u}$ and (3.2.7), obviously,

$$U_1(X_1^{\check{u}^*}(T)) = EU_1(X_1^{u^*}(T)) \le \max_u EU(X^u(T)). \tag{A.4}$$

Therefore,

$$\max_u EU(X^u(T)) = U_1(X_1^{\check{u}^*}(T)) \tag{A.5}$$

holds.

The proof of Theorem 2.3:

(i) The convexity is obvious, since

$$e^{\int_t^T r(\tau)d\tau}\widetilde{\pi}(t)(\tilde{\mu}(t)-r(t)\tilde{\mathbf{1}}) \text{ and } \frac{m^2}{2}e^{2\int_t^T r(\tau)d\tau}\widetilde{\pi}(t)\tilde{\Sigma}(t)\widetilde{\pi}^\top(t)$$

are concave.

(ii) From Assumption 3.2.1, we obtain

$$\begin{aligned}\lim_{\|\widetilde{\pi}\|_2\to\infty} J(\widetilde{\pi}) &\geq \lim_{\|\widetilde{\pi}\|_2\to\infty}(-mMe^{MT}\|\widetilde{\pi}\| + \frac{m^2}{2}\varepsilon\|\widetilde{\pi}\|^2) \\ &= \infty.\end{aligned} \tag{A.6}$$

(iii)

$$\begin{aligned}&\lim_{\|\widetilde{\pi}^{(n)}-\pi\|_2=0}\left|\int_0^T e^{\int_t^T r(\tau)d\tau}\widetilde{\pi}^{(n)}(t)(\tilde{\mu}(t)-r(t)\tilde{\mathbf{1}})dt - \int_0^T e^{\int_t^T r(\tau)d\tau}\widetilde{\pi}(t)(\tilde{\mu}(t)-r(t)\tilde{\mathbf{1}})dt\right| \\ &\leq \lim_{\|\widetilde{\pi}^{(n)}-\widetilde{\pi}\|_2=0}(Me^{MT}\|\widetilde{\pi}^{(n)}-\widetilde{\pi}\|_2) = 0.\end{aligned} \tag{A.7}$$

Also,

$$\begin{aligned}&\lim_{\|\widetilde{\pi}^{(n)}-\widetilde{\pi}\|_2=0}\left|\int_0^T \frac{m^2}{2}e^{2\int_t^T r(\tau)d\tau}(\widetilde{\pi}^{(n)}(t)\tilde{\Sigma}(t)(\widetilde{\pi}^{(n)})^\top(t)dt - \pi(t)\tilde{\Sigma}(t)\pi^\top(t))dt\right| \\ =& \lim_{\|\widetilde{\pi}^{(n)}-\widetilde{\pi}\|_2=0}\int_0^T \left|\|e^{\int_t^T r(\tau)d\tau}\widetilde{\pi}^{(n)}(t)\tilde{\sigma}\|^2 - \|e^{\int_t^T r(\tau)d\tau}\widetilde{\pi}(t)\tilde{\sigma}\|^2\right|dt \\ =& \lim_{\|\widetilde{\pi}^{(n)}-\widetilde{\pi}\|_2=0}\int_0^T e^{2\int_t^T r(\tau)d\tau}\left|(\widetilde{\pi}^{(n)}(t)\tilde{\sigma}-\widetilde{\pi}(t)\tilde{\sigma})(\widetilde{\pi}^{(n)}(t)\tilde{\sigma}+\widetilde{\pi}(t)\tilde{\sigma})^\top\right|dt \\ \leq& \lim_{\|\widetilde{\pi}^{(n)}-\widetilde{\pi}\|_2=0}\int_0^T e^{2MT}\|\widetilde{\pi}^{(n)}(t)\tilde{\sigma}-\widetilde{\pi}(t)\tilde{\sigma}\|_2\|\widetilde{\pi}^{(n)}(t)\tilde{\sigma}+\widetilde{\pi}(t)\tilde{\sigma}\|_2dt \\ =& \; 0.\end{aligned} \tag{A.8}$$

Thus, (iii) holds.

(iv) From Ekeland and Temam [37], $\exists$ a $\widetilde{\pi}^* \in \check{\mathcal{H}}$ such that $J(\widetilde{\pi}^*) = \min_{\widetilde{\pi}\in\mathcal{H}} J(\widetilde{\pi})$.

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