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The Hong Kong Polytechnic University
Department of Applied Mathematics

# Integrable Ermakov Structure in Nonlinear Continuum Mechanics and Optics 

Hongli An

A thesis submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy

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Hongli An

To my family

## Abstract

Ermakov-Ray-Reid systems have recently attracted much attention due to their novel invariant of motion, nonlinear superposition principles and extensive physical applications. In this thesis, our main concern is with integrable structure underlying certain models in nonlinear continuum mechanics and optics via reduction to such Ermakov-type systems.

The main contributions of this thesis are as follows :

In hydrodynamics, a shallow water system with a circular paraboloidal bottom topography is investigated via the elliptic vortex procedure. Key theorems analogous to those of Ball and Cushman-Roisin et al are generalised and used to construct the analytical vortex solutions in terms of an elliptic integral function. In particular, a class of typical pulsrodon solutions with a breather-type free boundary oscillation is isolated and its behaviour is simulated.

In nonlinear optics, a coupled 2+1-dimensional optics model is studied via a variational approach. Three distinct reductions to integrable Ermakov systems are set down. The underlying Hamiltonian structures render their complete integration. It is shown that integrable Hamiltonian Ermakov systems likewise arise in a 3+1-dimensional optics model. In particular, an Ovisannikov-Dyson type reduction is obtained wherein the eigenmode of the solution explains a remarkable flip-over effect observed experimentally.

Integrable Ermakov-Ray-Reid structure is shown to arise out of a $2+1$-dimensional
modulated Madelung system with logarithmic and Bohm quantum potentials via an exponential-type elliptic vortex ansatz. In addition, exact analytical solutions of the original system are obtained in terms of an elliptic integral representation.

In magnetogasdynamics, a power-type elliptic vortex ansatz and two-parameter pressure-density relation are introduced into a $2+1$-dimensioanl magnetogasdynamic system and a finite dimensional nonlinear dynamical system is thereby obtained. The latter admits integrable Hamiltonian Ermakov structure and a Lax pair formulation when the adiabatic index $\gamma=2$. Exact solutions of the magnetogasdynamic systems are constructed which describe a rotating elliptic plasma cylinder bounded by a vacuum state.

This thesis is based on the following papers written by the author during the period of stay at the Department of Applied Mathematics, The Hong Kong Polytechnic University as a graduate student:

1. An H.L., Rogers C., A 2+1-dimensional non-isothermal magnetogasdynamic system: Hamiltonian-Ermakov integrable reduction, (Submitted to) Symmetry, Integrability and Geometry: Methods and Applications (2012).
2. Rogers C., Malomed B. and An H.L., Ermakov-Ray-Reid reductions of variational approximations in nonlinear optics, Stud. Appl. Math. doi: 10.1111/j.14679590.2012.00557.x (2012).
3. An H.L., Numerical pulsrodons of the $2+1$-dimensional rotating shallow water system, Phys. Lett. A. 375 (2011) 1921-1925.
4. Rogers C. and An H.L., On a 2+1-dimensional Madelung system with logarithmic and with bohm quantum potentials. Ermakov reduction, Phys. Scr. 84 (2011) 045004.
5. Rogers C. and An H.L., Ermakov-Ray-Reid systems in 2+1-dimensional rotating shallow water theory, Stud. Appl. Math. 125 (2010) 275-299.
6. Rogers C., Malomed B., Chow K.W. and An H.L., Ermakov-Ray-Reid systems in nonlinear optics, J. Phys. A: Math. Theor. 43 (2010) 455214.

In addition, the following is a list of other paper written by the author during the period of her PhD study.

1. Rogers C. and An H.L., A non-isothermal spinning magnetogasdynamic cloud system: a Hamiltonian Ermakov integrable reduction, International Conference on Waves and Stability in Continuous Media, Italy (2011): Note Mat. 32 (2012) 175-191.

## Acknowledgments

This thesis would not have been possible without the support, encouragement, input and ideas of many people. Although it is impossible to thank everyone sufficiently for their assistance, it is my hope that this short acknowledgment will serve to recognize their contributions along my doctoral journey.

First and foremost, my sincere appreciation goes to my former chief supervisor Professor Colin Rogers, who has introduced me to the research topics of this thesis and guided me from February 2009 to June 2011. I formally thank him here for his patient guidance, kind support, generous encouragement and insightful suggestions during these years. His face to face communication, enlightening comments and entertaining my numerous questions are of great assistance for the preparation of the final thesis. His unique devotion to scholar and passion for research are worthy my lifelong learning.

Especially, my deep gratitude is extended to my current chief supervisor Professor Mankam Kwong, who has guided me since June 2011. Here, I formally thank him for his continuous encouragement, warm-hearted assistance, valuable comments and suggestions for my work as well as enthusiastic participation on the examining committee. His dedication to research and optimistic attitude towards life are an ideal worth striving for.

Moreover, I am much indebted to my co-supervisors, Professor Cheongki Chan and Professor Kwokwing Chow (The HongKong University), who have given me much assistance and inspiring encouragement in the past years. It would be my great honor
and precious experience to study for my PhD degree under the guidance of the above four famous professors.

I would like to thank Professor Boris Malomed (Tel Aviv University, Israel) and Professor Wolfgang Schief (The University of New South Wales, Sydney) for their helpful discussions and valuable suggestions. Special thanks to Professor Yong Chen (The East China Normal University), who gives me continuous encouragement and introduces me on how to do research work and write academic paper at the beginning of my research journey. Many thanks to Professor Senyue Lou (Shanghai Jiaotong University) for his encouragement, kind assistance and participation on the examining committee. Special thanks to Professor Yiuchung Hon (City University of HongKong) for his kind assistance and participation on the examining committee. Much gratitude to Professor Dexing Kong (Zhejing University), Professor Engui Fan (Fudan University), Professor Zhixiang Zhou (Fudan University), Professor Xingbiao Hu (Chinese Academy of Science), Professor Qingping Liu (China University of Mining and Technology), Professor Biao Li (Ningbo University) and Dr Yuqi Li (Ningbo University) for their kind help during my PhD study. I also thank Dr Zhiliang Li, Dr Junxiao Zhao, Dr Chao Song, Dr Yonghui Sun, Cuihua Lü, Jinhua Li, Lian Wang, Bingwei Zhang, and Songge Zhang for the precious friendship and continuous caring.

I wish to thank Professor Liqun Qi, Professor Xiaojun Chen, Professor Yanping Lin and Dr Xingqiu Zhao for their kind help and encouragement. I sincerely thank Manwai Yuen for his kind discussion and communication during at his spare time. I also thank Yi Xu and Chuanxin Bian for their kind help on teaching me computational software Matlab. Thanks are extended to the other friends in DE 405 and DE 407. They made my life in HongKong colorful and enjoyable. In addition, thanks also go to the Research Committee of The Hong Kong Polytechnic University for the financial support during the entire period of my study. I also express my gratitude to the supportive staffs in Department of Applied Mathematics for their kind help, particularly to Eva Yiu who is very helpful in typing the manuscripts written by Professor Coin Rogers and me.

Finally, my deepest gratitude goes to my dear family and my boyfriend Dr Haixing Zhu for their unconditional support, sacrifices, tolerance and encouragement over the years of graduate study. I dedicate this to all of you.

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## Chapter 1

## Introduction

Nonlinear coupled dynamical systems of Ermakov-Ray-Reid type introduced in 1979 [89] have been subsequently shown to arise in a variety of physical contexts, most notably in nonlinear optics (see, e.g. [22, 40-42, 127, 129]). In this thesis, our main goal is to analyse Ermakov-Ray-Reid structure not only in nonlinear optics but also in hydrodynamics and magnetogasdynamics. We commence the thesis with a short review of some relevant literature which has motivated this research.

### 1.1 Literature Review

The analysis of the coupled nonlinear ordinary equations known as Ermakov-Ray-Reid systems originated in the work of Steen in 1874 [124] together with independent results of Ermakov published in 1880 [30]. In the latter paper, a time-dependent oscillator with variable frequency

$$
\begin{equation*}
\ddot{q}+\omega^{2}(t) q=0 \tag{1.1.1}
\end{equation*}
$$

was considered in conjunction with the nonlinear oscillation equation [124]

$$
\begin{equation*}
\ddot{\rho}+\omega^{2}(t) \rho=\rho^{-3} . \tag{1.1.2}
\end{equation*}
$$

On elimination of $\omega^{2}(t)$ and integration, a first integral is obtained, namely

$$
\begin{equation*}
I=\frac{1}{2}\left[(\rho \dot{q}-\dot{\rho} q)^{2}+(q / \rho)^{2}\right] . \tag{1.1.3}
\end{equation*}
$$

This is commonly called the Ermakov-Lewis invariant after H Ralph Lewis who 'rediscovered' it in a study of the motion of charged particles moving in magnetic fields with Kruskal's asymptotic method in 1966 [64]. This invariant is associated with the conservation of angular momentum [29] and has subsequently been the subject of much literature (see e.g. [34, 45, 53, 65, 68]). It was shown in the work of Steen [124] that, remarkably, the general solution of the nonlinear oscillator equation (1.1.2) may be expressed as a nonlinear superposition of linearly independent solutions of the linear oscillator equation (1.1.1).

In 1979, Ray and Reid [89] introduced an important generalisation of the 'Ermakov pair' (1.1.1) and (1.1.2), namely the coupled nonlinear system

$$
\left\{\begin{array}{l}
\ddot{x}+\omega^{2}(t) x=\frac{1}{x^{2} y} f(y / x),  \tag{1.1.4}\\
\ddot{y}+\omega^{2}(t) y=\frac{1}{x y^{2}} g(x / y)
\end{array}\right.
$$

where $f$ and $g$ are arbitrary functions of their indicated arguments. This admits a novel integral of motion, namely the Ray-Reid invariant

$$
\begin{equation*}
I=\frac{1}{2}(x \dot{y}-\dot{x} y)^{2}+\int^{y / x} f(u) d u+\int^{x / y} g(v) d v \tag{1.1.5}
\end{equation*}
$$

Moreover, the system (1.1.4) admits a novel nonlinear superposition principles [90, 91] which may be regarded as generalising that of Ermakov pair (1.1.1) and (1.1.2). Linear structure underlying the Ermakov-Ray-Reid system (1.1.4) was isolated in [8] while stability and periodicity were subsequently discussed for a special subclass by Athorne [6]. Particular 2+1-dimensional Ermakov-Ray-Reid systems were constructed in [103] while Ermakov-Ray-Reid systems of arbitrary order and dimension which admit a Ray-Reid type invariant and associated nonlinear superposition principles were presented in [112]. Multi-component Ermakov systems were introduced in [108] and application made to an $N$-layer fluid model. The algebraic structure underlying these multi-component Ermakov systems was subsequently analysed in [7]. There are also a number of theo-
retical studies mainly on Lie symmetry structures of Ermakov-Ray-Reid systems (see, e.g. [38, 62, 92]).

In terms of the applications, Ermakov systems arise most notably in nonlinear optics where they occur in the description of elliptic Gaussian beams [22, 40-42, 44, 127, 129]. In nonlinear elasticity, the nonlinear oscillation equation (1.1.2) arises in the analysis of the radial oscillation of hyperelastic tubes [99, 120, 121]. In hydrodynamics, such systems arise out of both a single-layer and two-layer shallow water theory $[96,100]$. In molecular structures, Ermakov pairs occur in the description of soliton behaviour in the vicinity of impurities [37,46]. Ermakov pairs also arise in time-dependent quantum problems $[47,93,95,119]$ as well as in cosmology $[11,48,113]$.

The possession of a novel integral of motion and concomitant nonlinear superposition principles in combination with their established physical relevance has motivated the present extensive investigation into Ermakov-Ray-Reid systems in this thesis.

### 1.2 Motivation and Objectives

### 1.2.1 Hydrodynamics

The study of hydrodynamics is of great importance, in particular, in oceanography and geophysics, notably in the analysis of tidal oscillations and vortex structures in the upper ocean [61]. In the latter context, shallow water systems derived via the 3+1dimensional Euler equations [84] incorporating a Coriolis force play an important role and have received considerable attention in the literature (vide §2.1).

A fundamental contribution to the analysis of shallow water systems relevant to the present thesis was made by Ball [9]. Important theorems concerning the time evolution physical quantities such as moments of inertia were presented therein. It was subsequently shown, remarkably, that these theorems can be readily constructed by a

Lie-group analysis [97]. In [96], the $f$-plane shallow water model of Cushman-Roisin was investigated via introduction of an elliptic vortex ansatz. Thereby, an eight-dimensional dynamical system which admits a general analytical solution as well as particular pulsrodons corresponding to pulsating elliptic warm-core eddies was isolated. The application of the Ball-type theorems proved key to the construction of the complete solution. It is natural to inquire as to whether the Ball-type theorems can be similarly employed when the shallow water system has bottom topographies of a circular paraboloidal or elliptical paraboloidal type when there is also privileged underlying Lie-group structure [66]. This will be the subject of the opening chapter.

### 1.2.2 Nonlinear Optics

In a pioneering paper, Wagner et al [127], starting from Maxwell's equations, derived dynamical equations for the envelope of the self-trapping field via the paraxial approximation. Subsequently, a coupled pair of nonlinear dynamical equations for the evolution of the transverse radii of elliptical beams in a polarized medium was set down. This pair turns out to be an integrable Ermakov-Ray-Reid system [30,90,91]. Remarkably, such systems have subsequently been shown to arise in both self-trapping and self-focusing nonlinear optics contexts [22,40-42,44]. In particular, the evolution of the size and shape of a light spot and wave front in an elliptical Gaussian beam described by such Ermakov-Ray-Reid systems [22, 40]. The Ermakov-Ray-Reid invariant in conjunction with the Hamiltonian may be employed to reveal the underlying properties of such nonlinear optics.

The nonlinear Schrödinger equation (NLS) and its variants are ubiquitous as models in nonlinear optics $[2,56]$. Thus, the coupled NLS may be employed to study the copropagation of two optical pulses in a nonlinear planar waveguide [85]. On the other hand, the logarithmic NLS originally introduced by Bialynicki-Birula and Mycielski [12-14] in quantum mechanics and notable for admitting analytical Gaussons, may be applied to investigate the propagation of Gaussian beams in a saturable medium [123].

However, there has been but little investigation into nonlinear optical models based on their underlying Ermakov-Ray-Reid structure. The analysis of Ermakov-Ray-Reid systems that arise through variational approximation will be the subject of Chapter 3.

### 1.2.3 Madelung-Type Hydrodynamic Systems

The classical Madelung transformation [69] associates a nonlinear Schrödinger equation with a hydrodynamic system. Madelung systems involving logarithmic terms or Bohm quantum potentials have most notably arisen in plasma physics and nonlinear optics [27,63, $72,82,83,106,109,127]$.

In [100], Rogers and An showed that the rotating shallow water system subject to a circular paraboloidal basin admits an underlying integrable structure of Ermakov-RayReid type. In particular, in the case of irrotational motion and ignoring the Coriolis force, connection was made to a NLS equation with a Bohm quantum potential via the Madelung transformation. That is to say, the Ermakov structure was derived for the equivalent NLS system via the elliptic vortex ansatz introduced in [96].

In Chapter 4, we investigate nonlinear integrable reduction of Ermakov-Ray-Reid type which arises out of NLS equations incorporating both Bohm and logarithmic type quantum potentials. The construction of exponential-type elliptic vortex ansatz and concomitant Ball-type theorems proves crucial to the exact solution of these dynamical systems.

### 1.2.4 Magnetogasdynamics

The analysis of the motion of electrically conducting fluids and plasmas as described by Lundquist magnetogasdynamic system is of considerable importance in astrophysics, geophysics and engineering applications (see, e.g. [16, 88, 122]). In general, analytical
solutions are not available due to the inherent nonlinearity of the governing Lundquist system.

In a series of papers on magnetogasdynamics [75-77], Neukirch et al introduced a novel solution procedure wherein the nonlinear acceleration terms in the Lundquist momentum equation either vanish or, are assumed to be conservative. In recent work [111], Rogers and Schief showed that a 2+1-dimensional magnetogasdynamic system with a parabolic gas law admits exact elliptic vortex reduction to an integrable Hamiltonian Ermakov system. An important extension to more general gas laws and elliptic vortex ansatz is examined in Chapter 5.

## Chapter 2

## A Shallow Water System with Paraboloidal Bottom Topography: An Integrable Sub-System

### 2.1 Background

In recent years, much attention has been paid to oceanic warm-core eddies (rings), which consist of rotating isolated water masses (see, e.g. [23,31,58,59,117]). Such eddies play an important role in large scale oceanic circulation and can profoundly affect the transfer of physical, chemical, and biological properties across frontal zones and exert a substantial contribution to the horizontal and vertical mixing of different water masses [80]. The physical importance of warm-core eddies explains the number of observational, experimental, numerical and analytical investigations that have been carried out to elucidate different aspects of their dynamics (see, e.g. [24, 25, 54, 55, 73, 96, 114, 115]).

The reduced gravity shallow water model has been widely used in the theoretical study of warm-core eddies (see, e.g. [24, 25, 54, 96]). Although the model excludes
relevant oceanic processes such as wave radiation toward the exterior ocean or baroclinic instabilities, it has proved an model which produces many fundamental characteristics of eddy dynamics. Indeed, laboratory experiments on warm-core eddies evolution by Rubino and Brandt [114] have established that this evolution is consistent with known analytic solutions describing the dynamics of warm-core eddies of the nonlinear, reduced gravity shallow water equations evolving on an $f$-plane (see, e.g. [24, 96]).

Interest in shallow water systems goes back to work of Goldsbrough [39] who, in a study of tidal oscillations in an elliptic basin obtained a class of exact elliptical vortex solutions whose velocity components are linear and height field is a quadratic function of the horizontal coordinates. The latter work, in turn, is related to that of Kirchoff [60] on vortex structures in the classical 2+1-dimensional Euler system (in which the velocity components are linear and the pressure is quadratic). Subsequently, Ball [9,10] obtained key results on the time evolution of moments of inertia and invariance properties of the rotating shallow water system. Using this system, Thacker [126] investigated the tidal oscillations in elliptical basins whose depth profile is parabolic. With the oceanic warmcore eddies in mind, Cushman-Roisin et al [24,25] obtained two canonical solutions of the reduced gravity shallow water system. One solution corresponds to a steady clockwise rotation of an unchanging elliptical eddy and the other to a pulsating circular eddy. The former has been termed a rodon and the latter a pulson. Ripa [94] later showed that the rodon-type solution was Lyapunov stable to disturbances within the class of elliptical solutions.

Important contributions to the study of the rotating shallow water equation were made by Young [128], who employed the theorems of Ball and the invariants of the rotating shallow system to reduce the nonlinear elliptical vortex dynamics to quadratures. Rogers [97] and later Levi et al [66] subjected the rotating shallow water system with elliptic and circular cylindrical bottom topographies to a Lie-group analysis and a variety of group-invariant solutions were isolated. Importantly, it was shown in [97] that the invariance theorems of Ball have a group-theoretic origin. Rogers [96] subsequently obtained the complete solution of an eight-dimensional nonlinear dynamical
system resulting from the introduction of an elliptical-vortex ansatz into the reduced gravity shallow water system. A new subclass of so-called pulsrodon solutions which correspond to pulsating, rotating elliptical eddies was obtained. The application of a Ball-type moment of inertia theorem set down in [9] proved key to the construction of such solutions. The shallow water system in the case of plane bottom topography was subsequently investigated in an elegant Lagrangian treatment using Hamiltonian dynamics by Holm [52]. In particular, it was shown that the canonical exact solutions namely the rodon, pulson and pulsrodon (which rotates and pulses periodically) are orbitally Lyapunov stable to perturbations within the class of elliptical vortex solutions.

In this chapter, the elliptical vortex procedures of [96] are applied to the case of a rotating shallow water system with a circular paraboloidal bottom topography. Importantly, key theorems analogous to those of Cushman-Roisin et al [25] are generalised and used to solve the resulting eight-dimensional dynamical system. In particular, a class of pulsrodon solutions is derived and its behavior is displayed via numerical simulations. Additionally, such exact pulsrodons are also used to show the efficiency of numerical pulsrodons obtained in [3]. It was established by Rogers and An [100] that, remarkably, the nonlinear dynamical system admits underlying integrable structure of Ermakov-Ray-Reid type. This system, which describes the time evolution of the semi-axes of the elliptical moving shoreline on the paraboidal basin, is also Hamiltonian.

### 2.2 The Rotating Shallow Water Model

The rotating shallow water system governing the motion of an incompressible fluid and considered in the present chapter takes the form

$$
\begin{align*}
& \frac{\partial h}{\partial t}+\operatorname{div}(h \mathbf{q})=0 \\
& \frac{\partial \mathbf{q}}{\partial t}+\mathbf{q} \cdot \nabla \mathbf{q}+f \mathbf{k} \times \mathbf{q}+\nabla(Z+h)=0 \tag{2.2.1}
\end{align*}
$$

Here, $f$ is the Coriolis parameter, $\mathbf{q}=u \mathbf{i}+v \mathbf{j}$ is the velocity vector and $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ is the usual orthonormal basis. In the above,

$$
\begin{equation*}
z=Z(x, y) \tag{2.2.2}
\end{equation*}
$$

is the equation of the basin surface underlying the liquid while

$$
\begin{equation*}
z=\eta(x, y, t)=Z+h(x, y, t) \tag{2.2.3}
\end{equation*}
$$

denotes the upper free surface. The geometric configuration is as below in Fig 2.1.


Fig. 2.1: The Basin Geometry.

This classical rotating shallow system may be readily derived ab initio via a hydrostatic approximation and incorporates the relevant kinematic and boundary conditions in a manner reviewed, in detail, in the work of Rogers [97]. Therein, it was established that the system (2.2.1) admits privileged Lie group symmetries when the underlying basin is either a circular or an elliptical paraboloid. Accordingly, attention in the subsequent sections is restricted to the same case with

$$
\begin{equation*}
Z=A^{*} x^{2}+B^{*} y^{2} \tag{2.2.4}
\end{equation*}
$$

### 2.3 Elliptical Vortices in the Rotating Shallow Water Model

Here, the elliptical vortex procedures described in [96] in connection with the hydrodynamic $f$-plane system are extended to the case of a paraboloidal bottom topography
of the type (2.2.4). Key to this extension will be the generalisation of results originally obtained in the $f$-plane context concerning the time-evolution of important physical quantities.

### 2.3.1 An Integrable Nonlinear Dynamical Sub-system

Exact solutions of the rotating shallow system (2.2.1) are now sought via an ellipticalvortex ansatz (Rogers and An [100])

$$
\begin{align*}
& \mathbf{q}=\mathbf{L}(t) \mathbf{x}+\mathbf{M}(t),  \tag{2.3.1}\\
& h=\mathbf{x}^{T} \mathbf{E}(t) \mathbf{x}+h_{0}(t),
\end{align*} \quad \mathbf{x}=\binom{x-q(t)}{y-p(t)}
$$

where

$$
\begin{gather*}
\mathbf{L}=\left(\begin{array}{ll}
u_{1}(t) & u_{2}(t) \\
v_{1}(t) & v_{2}(t)
\end{array}\right), \mathbf{M}=\binom{\dot{q}(t)}{\dot{p}(t)},  \tag{2.3.2}\\
\mathbf{E}=\left(\begin{array}{ll}
a(t) & b(t) \\
b(t) & c(t)
\end{array}\right)
\end{gather*}
$$

The presence of the terms in $p(t), q(t), \dot{p}(t)$ and $\dot{q}(t)$ corresponds to an underlying Lie group invariance of the system (2.2.1) when augmented by the geometric constraint (2.2.4)(see [97]). It is required that the eddy edge $h=0$ be closed so that

$$
\begin{equation*}
\triangle=a c-b^{2}>0 \tag{2.3.3}
\end{equation*}
$$

In general, the eddy's rim is an ellipse, which degenerates to a circle if $a=c$ and $b=0$.

Insertion of the ansatz (2.3.1) into the continuity equation $(2.2 .1)_{1}$ produces

$$
\left(\begin{array}{c}
\dot{a}  \tag{2.3.4}\\
\dot{b} \\
\dot{c}
\end{array}\right)+\left(\begin{array}{ccc}
3 u_{1}+v_{2} & 2 v_{1} & 0 \\
u_{2} & 2\left(u_{1}+v_{2}\right) & v_{1} \\
0 & 2 u_{2} & u_{1}+3 v_{2}
\end{array}\right)\left(\begin{array}{l}
a \\
b \\
c
\end{array}\right)=0
$$

together with

$$
\begin{equation*}
\dot{h}_{0}=-\left(u_{1}+v_{2}\right) h_{0} . \tag{2.3.5}
\end{equation*}
$$

Substitution of (2.3.1) into the momentum equation $(2.2 .1)_{2}$ now gives

$$
\left(\begin{array}{c}
\dot{u}_{1}  \tag{2.3.6}\\
\dot{u}_{2} \\
\dot{v}_{1} \\
\dot{v}_{2}
\end{array}\right)+\left(\begin{array}{cc}
\mathbf{L}^{T} & -f \mathbf{I} \\
f \mathbf{I} & \mathbf{L}^{T}
\end{array}\right)\left(\begin{array}{c}
u_{1} \\
u_{2} \\
v_{1} \\
v_{2}
\end{array}\right)+2\left(\begin{array}{c}
a \\
b \\
b \\
c
\end{array}\right)+2\left(\begin{array}{c}
A^{*} \\
0 \\
0 \\
B^{*}
\end{array}\right)=\mathbf{0}
$$

augmented by

$$
\begin{align*}
& \ddot{p}+f \dot{q}+2 B^{*} p=0,  \tag{2.3.7}\\
& \ddot{q}-f \dot{p}+2 A^{*} q=0 .
\end{align*}
$$

It is noted below that when $A^{*}=B^{*}$, relations obtain which are key to the subsequent development. These may be established by direct appeal to the system (2.3.4)-(2.3.6) and are recorded in the following theorem.

Theorem 2.3.1 If

$$
\begin{align*}
& R^{*}=v_{1}-u_{2}+f, \quad \triangle=a c-b^{2}>0 \\
& M^{*}=a\left(u_{2}-\frac{f}{2}\right)+b\left(v_{2}-u_{1}\right)-c\left(v_{1}+\frac{f}{2}\right)  \tag{2.3.8}\\
& Q^{*}=-a\left(u_{2}^{2}+v_{2}^{2}\right)+2 b\left(u_{1} u_{2}+v_{1} v_{2}\right)-c\left(u_{1}^{2}+v_{1}^{2}\right)+4 \triangle-2 A^{*}(a+c)
\end{align*}
$$

then the following relations hold:

$$
\begin{align*}
& \dot{h}_{0}=-\left(u_{1}+v_{2}\right) h_{0} \\
& \dot{\triangle}=-4\left(u_{1}+v_{2}\right) \triangle \\
& \dot{R}^{*}=-\left(u_{1}+v_{2}\right) R^{*}  \tag{2.3.9}\\
& \dot{M}^{*}=-3\left(u_{1}+v_{2}\right) M^{*} \\
& \dot{Q}^{*}=-3\left(u_{1}+v_{2}\right) Q^{*}
\end{align*}
$$

These relations constitute analogous of those derived in the context of plane bottom topography in [25,96]. The quantities $h_{0}, \Delta, R^{*}, M^{*}$ and $Q^{*}$ are related to physical
invariants as follows:
(i) $\quad$ volume $=\iint h d x d y=\frac{1}{2} \pi h_{0}^{2} \triangle^{-\frac{1}{2}}$,
(ii) $\quad$ energy $=\iint \frac{1}{2} h\left[h+\left(u^{2}+v^{2}\right)\right] d x d y$

$$
=\frac{1}{24} \pi h_{0}^{3} \triangle^{-\frac{3}{2}} Q^{*},
$$

(iii) potential vorticity $=\iint h\left(\frac{v_{x}-u_{y}+f}{h}\right) d x d y$

$$
\begin{equation*}
=\pi h_{0} \triangle^{-\frac{1}{2}} R^{*}, \tag{2.3.10}
\end{equation*}
$$

(iv) angular momentum $=\iint h\left[(x v-y u)+\frac{f}{2}\left(x^{2}+y^{2}\right)\right] d x d y$

$$
=\frac{1}{12} \pi h_{0}^{3} \triangle^{-\frac{3}{2}} M^{*} .
$$

In the sequel, attention is restricted to the case $A^{*}=B^{*}$ corresponding to an underlying circular paraboloidal topography. Accordingly, the above relations (2.3.9) are valid.

It proves convenient to proceed in terms of new variables as employed in [96], namely

$$
\begin{align*}
& G=u_{1}+v_{2}, \quad G_{R}=\frac{1}{2}\left(v_{1}-u_{2}\right) \\
& G_{S}=\frac{1}{2}\left(v_{1}+u_{2}\right), \quad G_{N}=\frac{1}{2}\left(u_{1}-v_{2}\right)  \tag{2.3.11}\\
& B=a+c, \quad B_{S}=b, \quad B_{N}=\frac{1}{2}(a-c)
\end{align*}
$$

Here, $G$ and $G_{R}$ represent, in turn, the divergence and spin of the velocity field, while $G_{S}$ and $G_{N}$ denote shear and normal deformation rates. The geometry of the moving shoreline boundary $h=0$ is now determined by $B, B_{S}$ and $B_{N}$ together with $h_{0}$.

In terms of the new variables, the last two important relations in Theorem 2.3.1 adopt the form

$$
\begin{align*}
& \dot{M}^{*}=-3 G M^{*} \\
& \dot{Q}^{*}=-3 G Q^{*} \tag{2.3.12}
\end{align*}
$$

implying the integral of motion

$$
\begin{equation*}
M^{*} / Q^{*}=\text { const } \tag{2.3.13}
\end{equation*}
$$

The nonlinear dynamical equations (2.3.4)-(2.3.6) reduce in extenso to the equivalent generalized Kirwan-Liu system:

$$
\begin{align*}
& \dot{h}_{0}+h_{0} G=0, \\
& \dot{B}+2\left[B G+2\left(B_{N} G_{N}+B_{S} G_{S}\right)\right]=0, \\
& \dot{B}_{S}+2 B_{S} G+G_{S} B-2 B_{N} G_{R}=0, \\
& \dot{B}_{N}+2 B_{N} G+G_{N} B+2 B_{S} G_{R}=0, \\
& \dot{G}+\frac{1}{2} G^{2}+2\left(G_{N}^{2}+G_{S}^{2}-G_{R}^{2}\right)-2 f G_{R}+2 B+2\left(A^{*}+B^{*}\right)=0,  \tag{2.3.14}\\
& \dot{G}_{R}+G G_{R}+\frac{1}{2} f G=0, \\
& \dot{G}_{S}+G G_{S}+f G_{N}+2 B_{S}=0, \\
& \dot{G}_{N}+G G_{N}-f G_{S}+2 B_{N}+A^{*}-B^{*}=0 .
\end{align*}
$$

If we set

$$
\begin{equation*}
\Omega=1 /\left(G_{R}+\frac{f}{2}\right)^{1 / 2}, \quad G_{R}+\frac{f}{2} \neq 0 \tag{2.3.15}
\end{equation*}
$$

then $(2.3 .14)_{6}$ shows that

$$
\begin{equation*}
G=2 \dot{\Omega} / \Omega \tag{2.3.16}
\end{equation*}
$$

The precluded irrotational case $G_{R}+\frac{f}{2}=0$ is dealt with at the end of next section. Relation (2.3.14) ${ }_{1}$ together with (2.3.16) gives, in turn,

$$
\begin{equation*}
h_{0}=C_{I} / \Omega^{2} \tag{2.3.17}
\end{equation*}
$$

where $C_{I}$ is an arbitrary constant of integration.

New modulated variables are now introduced as in the $f$-plane study, viz

$$
\begin{gather*}
\bar{B}=\Omega^{4} B, \quad \bar{B}_{S}=\Omega^{4} B_{S}, \quad \bar{B}_{N}=\Omega^{4} B_{N},  \tag{2.3.18}\\
\bar{G}_{S}=\Omega^{2} G_{S}, \quad \bar{G}_{N}=\Omega^{2} G_{N}
\end{gather*}
$$

whence the system (2.3.14) reduces to

$$
\begin{align*}
& \dot{\bar{B}}+4\left(\bar{B}_{N} \bar{G}_{N}+\bar{B}_{S} \bar{G}_{S}\right) / \Omega^{2}=0 \\
& \dot{\bar{B}}_{S}+\left(\bar{B} \bar{G}_{S}-2 \bar{B}_{N}\right) / \Omega^{2}+f \bar{B}_{N}=0 \\
& \dot{\bar{B}}_{N}+\left(\bar{B} \bar{G}_{N}+2 \bar{B}_{S}\right) / \Omega^{2}-f \bar{B}_{S}=0  \tag{2.3.19}\\
& \dot{\bar{G}}_{N}-f \bar{G}_{S}+2 \bar{B}_{N} / \Omega^{2}+\left(A^{*}-B^{*}\right) \Omega^{2}=0 \\
& \dot{\bar{G}}_{S}+f \bar{G}_{N}+2 \bar{B}_{S} / \Omega^{2}=0
\end{align*}
$$

together with a nonlinear equation for $\Omega$, namely,

$$
\begin{equation*}
\Omega^{3} \ddot{\Omega}+f^{2} \Omega^{4} / 4-1+\bar{G}_{N}^{2}+\bar{G}_{S}^{2}+\bar{B}+\Omega^{4}\left(A^{*}+B^{*}\right)=0 . \tag{2.3.20}
\end{equation*}
$$

In what follows, we proceed only with the circular paraboloidal case when $A^{*}=B^{*}$. The system (2.3.19) remarkably reduces to exactly that of the $f$-plane analysis in [96]. The presence of a circular paraboloidal bottom topography is only made manifest in the nonlinear equation (2.3.20) obtained from the corresponding $f$-plane (plane bottom topography) equation via the replacement $f^{2} \rightarrow f^{2}+8 A^{*}$. Accordingly, the analytical procedures used in [96] are directly applied to solve the system (2.3.19)-(2.3.20). These are summarised in the next sub-section.

### 2.3.2 Parametrisation and Solution

It is shown that when $A^{*}=B^{*}$, the dynamical system (2.3.19) admits the three integrals of motion (cf. Rogers and An [100])

$$
\begin{align*}
& \bar{B}_{S}^{2}+\bar{B}_{N}^{2}-\frac{1}{4} \bar{B}^{2}=C_{I I} \\
& \bar{G}_{S}^{2}+\bar{G}_{N}^{2}-\bar{B}=C_{I I I}  \tag{2.3.21}\\
& \bar{B}+2\left(\bar{B}_{S} \bar{G}_{N}-\bar{B}_{N} \bar{G}_{S}\right)=C_{I V}
\end{align*}
$$

the relevance of which are described below. Here $C_{I I}, C_{I I I}$ and $C_{I V}$ are constants of integration. It is noted that $(2.3 .21)_{1}$ shows

$$
\begin{equation*}
\triangle=-C_{I I} / \Omega^{8} \tag{2.3.22}
\end{equation*}
$$

whence the condition (2.3.3) requires that $C_{I I}<0$.
The integrals of motion $(2.3 .21)_{1,2}$ are parametrised, in turn, according to

$$
\begin{array}{ll}
\bar{B}_{S}=-\sqrt{C_{I I}+\frac{1}{4} \bar{B}^{2}} \cos \phi(t), & \bar{B}_{N}=-\sqrt{C_{I I}+\frac{1}{4} \bar{B}^{2}} \sin \phi(t),  \tag{2.3.23}\\
\bar{G}_{S}=-\sqrt{C_{I I I}+\bar{B}} \sin \theta(t), & \bar{G}_{N}=+\sqrt{C_{I I I}+\bar{B}} \cos \theta(t)
\end{array}
$$

where signs are adopted compatible with the subsequent construction of pulsrodon-type solutions.

Substitution of the parametrisation (2.3.23) into (2.3.19) ${ }_{1}$ yields

$$
\begin{equation*}
\dot{\bar{B}}+\frac{4}{\Omega^{2}} \sqrt{C_{I I}+\frac{1}{4} \bar{B}^{2}} \sqrt{C_{I I I}+\bar{B}} \sin (\theta-\phi)=0 \tag{2.3.24}
\end{equation*}
$$

Conditions (2.3.19) $)_{2,3}$ reduce to a single relation, viz.

$$
\begin{equation*}
\sqrt{C_{I I}+\frac{1}{4} \bar{B}^{2}}\left[\dot{\phi}+\frac{2}{\Omega^{2}}-f\right]-\frac{\bar{B}}{\Omega^{2}} \sqrt{C_{I I I}+\bar{B}} \cos (\theta-\phi)=0 . \tag{2.3.25}
\end{equation*}
$$

Similarly, conditions (2.3.19) $)_{4,5}$ produce another single requirement

$$
\begin{equation*}
\sqrt{C_{I I I}+\bar{B}}[f-\dot{\theta}]-\frac{2}{\Omega^{2}} \sqrt{C_{I I}+\frac{1}{4} \bar{B}^{2}} \cos (\theta-\phi)=0 . \tag{2.3.26}
\end{equation*}
$$

It is seen that insertion of (2.3.16) into (2.3.12) ${ }_{1}$ results, on integration, in:

$$
\begin{equation*}
M^{*}=\left[-\bar{B}+2\left(\bar{B}_{N} \bar{G}_{S}-\bar{B}_{S} \bar{G}_{N}\right)\right] \Omega^{-6}=C_{I V} \Omega^{-6} \tag{2.3.27}
\end{equation*}
$$

whence, by appeal to the particular parametrisation (2.3.23),

$$
\begin{equation*}
\bar{B}=-C_{I V}+2 \sqrt{\left(C_{I I}+\frac{1}{4} \bar{B}^{2}\right)\left(C_{I I I}+\bar{B}\right)} \cos (\theta-\phi) . \tag{2.3.28}
\end{equation*}
$$

It is readily validated that (2.3.28) satisfies (2.3.24). Elimination of $\theta-\phi$ between (2.3.28) and (2.3.25), (2.3.26), in turn, yields

$$
\begin{align*}
& \dot{\phi}=f+\frac{2}{\Omega^{2}}\left[-1+\bar{B}\left(\bar{B}+C_{I V}\right) /\left(4 C_{I I}+\bar{B}^{2}\right)\right],  \tag{2.3.29}\\
& \dot{\theta}=f-\frac{1}{\Omega^{2}}\left(\bar{B}+C_{I V}\right) /\left(\bar{B}+C_{I I I}\right) .
\end{align*}
$$

Bearing the construction of exact solutions in mind, we turn to consider the secondorder nonlinear equation (2.3.20), namely,

$$
\begin{equation*}
\Omega^{3} \ddot{\Omega}+\left(f^{2}+8 A^{*}\right) \Omega^{4} / 4+C_{I I I}+2 \bar{B}-1=0 . \tag{2.3.30}
\end{equation*}
$$

The latter, as it stands, is intractable unless $\bar{B}=\lambda+\mu \Omega^{4},(\lambda, \mu \in \mathbb{R})$ when it reduces to the classical Steen-Ermakov-Pinney equation [30, 86, 124]. However, in general, it readily leads to an important relation which is shown in the following theorem:

Theorem 2.3.2 If $M^{*}$ and $Q^{*}$ are given by (2.3.8), then

$$
\begin{align*}
\left(\Omega^{2} \bar{B}\right)+\left(f^{2}+8 A^{*}\right) \Omega^{2} \bar{B} & =-2\left(Q^{*}+f M^{*}\right) \Omega^{6}  \tag{2.3.31}\\
& =-2\left(C_{V}+f C_{I V}\right)
\end{align*}
$$

This result constitutes a generalisation of the well-known result which corresponds to plane bottom topography given in $[25,96]$ and states that

$$
\begin{equation*}
\ddot{I}_{\epsilon}+I_{\epsilon}=\frac{1}{6} \pi \triangle^{-3 / 2} h_{0}^{3}(Q+M) \tag{2.3.32}
\end{equation*}
$$

where $I_{\epsilon}$ is the moment of inertia given by

$$
\begin{equation*}
I_{\epsilon}=\iint_{\epsilon}\left(x^{2}+y^{2}\right) h d x d y=-\frac{\pi}{12} h_{0}^{3} \triangle^{-\frac{3}{2}}(a+c)=-\frac{\pi}{12}\left[\frac{C_{I}}{\left(-C_{I I}\right)^{1 / 2}}\right]^{3} \Omega^{2} \bar{B} \tag{2.3.33}
\end{equation*}
$$

with the double integral being taken over the eddy $\epsilon$. In the more general circular paraboloidal basin geometry under present consideration, the moment of inertia equation embodied in (2.3.31) corresponds to a result originally obtained by Ball [9] and subsequently derived by Rogers via a Lie group approach [97]. Here, it proves key to the completion of our construction of the general solution of the eight-dimensional dynamical system (2.3.14).

It is noted that in the present circular paraboloidal case, the integral of motion

$$
\begin{equation*}
\dot{I}_{\epsilon}^{2}+\left(f^{2}+8 A^{*}\right) I_{\epsilon}^{2}-\frac{1}{3} \pi \triangle^{-3 / 2} h_{0}^{3}\left(Q^{*}+f M^{*}\right) I_{\epsilon} \tag{2.3.34}
\end{equation*}
$$

involving the moment of inertia $I_{\epsilon}$ provides an additional invariant of the rotating shallow water system.

On integration of (2.3.31), we obtain

$$
\begin{gather*}
\Omega^{2} \bar{B}=C_{V I} \cos \sqrt{f^{2}+8 A^{*}} t+C_{V I I} \sin \sqrt{f^{2}+8 A^{*}} t-2\left(C_{V}+f C_{I V}\right) /\left(f^{2}+8 A^{*}\right), \\
\left(f^{2}+8 A^{*} \neq 0\right) \tag{2.3.35}
\end{gather*}
$$

where $C_{V I}$ and $C_{V I I}$ are additional integration constants. While on elimination of $\theta-\phi$ and $\Omega$ in (2.3.24) via the relations (2.3.28) and (2.3.35), it is seen that $\bar{B}$ is given by

$$
\begin{aligned}
& \int_{C_{V I I I}}^{\bar{B}} \frac{d \bar{B}^{*}}{\bar{B}^{*} \sqrt{\left(\bar{B}^{* 2}+4 C_{I I}\right)\left(\bar{B}^{*}+C_{I I I}\right)-\left(\bar{B}^{*}+C_{I V}\right)^{2}}} \\
& =-2 \int_{0}^{t} \frac{d t^{*}}{C_{V I} \cos \left(\sqrt{f^{2}+8 A^{*}} t^{*}\right)+C_{V I I} \sin \left(\sqrt{f^{2}+8 A^{*}} t^{*}\right)-2\left(C_{V}+f C_{I V}\right) /\left(f^{2}+8 A^{*}\right)}
\end{aligned}
$$

$$
\begin{equation*}
(\bar{B} \neq \text { const }) \tag{2.3.36}
\end{equation*}
$$

where $\left.\bar{B}\right|_{t=0}=C_{V I I I}$. The elliptic integral of $\bar{B}$ can be readily treated by classical methods described in Ref. [18].

The generalisation of the elliptic vortex approach of [96] to circular paraboloidal basins is now completed. Thus, the corresponding shallow water velocity components are given by

$$
\begin{array}{ll}
u_{1}=\frac{\dot{\Omega}}{\Omega}+\frac{1}{\Omega^{2}} \sqrt{C_{I I I}+\bar{B}} \cos \theta(t), & v_{1}=-\frac{1}{\Omega^{2}} \sqrt{C_{I I I}+\bar{B}} \sin \theta(t)+\frac{1}{\Omega^{2}}-\frac{f}{2}, \\
u_{2}=-\frac{1}{\Omega^{2}} \sqrt{C_{I I I}+\bar{B}} \sin \theta(t)-\frac{1}{\Omega^{2}}+\frac{f}{2}, & v_{2}=\frac{\dot{\Omega}}{\Omega}-\frac{1}{\Omega^{2}} \sqrt{C_{I I I}+\bar{B}} \cos \theta(t), \tag{2.3.37}
\end{array}
$$

while the elliptic moving shoreline parameters are

$$
\begin{gather*}
a=\frac{1}{\Omega^{4}}\left[\frac{1}{2} \bar{B}-\sqrt{C_{I I}+\frac{1}{4} \bar{B}^{2}} \sin \phi(t)\right], \quad b=-\frac{1}{\Omega^{4}} \sqrt{C_{I I}+\frac{1}{4} \bar{B}^{2}} \cos \phi(t), \\
c=\frac{1}{\Omega^{4}}\left[\frac{1}{2} \bar{B}+\sqrt{C_{I I}+\frac{1}{4} \bar{B}^{2}} \sin \phi(t)\right],  \tag{2.3.38}\\
h_{0}=\frac{C_{I}}{\Omega^{2}} .
\end{gather*}
$$

## The Precluded Case

$$
G_{R}+\frac{f}{2}=0
$$

It is recalled that the parametrisation in terms of $\Omega$ in the above proceeded provided (cf.(2.3.15))

$$
\begin{equation*}
G_{R}+\frac{f}{2} \neq 0 \tag{2.3.39}
\end{equation*}
$$

and thereby solutions of the rotating shallow water system in the generic case has been obtained via (2.3.37)-(2.3.38). If $G_{R}+\frac{f}{2}=0$, that is

$$
\begin{equation*}
v_{1}-u_{2}+f=0 \tag{2.3.40}
\end{equation*}
$$

and the parametrisation (2.3.23) is again introduced. Then it is readily shown that an analogous reduction applies mutatis mutandis. Therefore, the resulting parametrisation becomes (cf.(2.3.37))

$$
\begin{align*}
& u_{1}=\frac{\dot{\Omega}}{\Omega}+\frac{1}{\Omega^{2}} \sqrt{C_{I I I}+\bar{B}} \cos \theta(t), \quad v_{1}=-\frac{1}{\Omega^{2}} \sqrt{C_{I I I}+\bar{B}} \sin \theta(t)-\frac{f}{2} \\
& u_{2}=-\frac{1}{\Omega^{2}} \sqrt{C_{I I I}+\bar{B}} \sin \theta(t)+\frac{f}{2}, \quad v_{2}=-\frac{1}{\Omega^{2}} \sqrt{C_{I I I}+\bar{B}} \cos \theta(t)+\frac{\dot{\Omega}}{\Omega} \tag{2.3.41}
\end{align*}
$$

and the elliptic moving shoreline parameters are given by the relations (2.3.38).

It is interesting to remark that the constraint (2.3.39) corresponds to a privileged class of motions which may be connected to irrotational shallow water motions with $f=0$. Thus, Chesnokov [21] recently used Lie-group analysis to establish a novel connection between the rotating shallow water system (2.2.1) with $Z=0$ and an associated non-rotating system with $f=0$. The general Lie-group analysis of [66] may be readily adduced to extend this result to elliptic paraboloidal bottom topographies.

### 2.3.3 The Pulsrodon: Circular Paraboloidal Bottom Topography

It is noted that elliptical integral expression (2.3.36) is only valid if $\bar{B} \neq$ const. Here, we consider the reduction when $\bar{B}$ is constant. In this case, $(2.3 .19)_{1}$ reduces to

$$
\begin{equation*}
\bar{B}_{N} \bar{G}_{N}+\bar{B}_{S} \bar{G}_{S}=0 \tag{2.3.42}
\end{equation*}
$$

If we set

$$
\begin{equation*}
\bar{B}_{N}=\alpha \bar{G}_{S}, \quad \bar{B}_{S}=-\alpha \bar{G}_{N} . \tag{2.3.43}
\end{equation*}
$$

then the system (2.3.19) is reducible to the linear coupled equations

$$
\begin{equation*}
\dot{\bar{G}}_{N}=\left(f-\frac{2 \alpha}{\Omega^{2}}\right) \bar{G}_{S}, \quad \dot{\bar{G}}_{S}=\left(\frac{2 \alpha}{\Omega^{2}}-f\right) \bar{G}_{N} \tag{2.3.44}
\end{equation*}
$$

where

$$
\begin{equation*}
\bar{B}=2 \alpha(1-\alpha), \quad \dot{\alpha}=0 . \tag{2.3.45}
\end{equation*}
$$

The general solution of the system (2.3.44) is

$$
\begin{equation*}
\bar{G}_{N}=\tilde{G}_{0} \sin \eta, \quad \bar{G}_{S}=\tilde{G}_{0} \cos \eta \tag{2.3.46}
\end{equation*}
$$

where

$$
\begin{equation*}
\eta=f t-2 \alpha \int \frac{1}{\Omega^{2}} d t . \tag{2.3.47}
\end{equation*}
$$

It is emphasized that the three integrals of motion set down in (2.3.21) are valid, in particular, for the case of $\bar{B}=$ const.

Insertion of (2.3.45) and (2.3.46) into (2.3.20) results in the general Steen-Ermakov equation $[30,124]$

$$
\begin{equation*}
\ddot{\Omega}+\left(\frac{1}{4} f^{2}+2 A^{*}\right) \Omega=\frac{1-\tilde{G}_{0}^{2}-2 \bar{B}}{\Omega^{3}} . \tag{2.3.48}
\end{equation*}
$$

The latter, which originated in the work of Steen [124], arises in a wide range of areas of physical importance, most notably in quantum mechanics, optics, and nonlinear elasticity (see, e.g. [33,74,121]). Another avatar of this equation has appeared recently in a study of pulsons by Sutyrin [125]. It is distinguished by its admittance of a nonlinear
superposition principle which was derived by Lie group considerations [107]. Therein, the Steen-Ermakov equation was derived in another manner in the context of moving shoreline analysis in a rotating shallow water system with circular paraboloidal bottom topography.

The general solution of (2.3.48) is given by

$$
\begin{equation*}
\Omega=\sqrt{\delta_{1} \Omega_{1}^{2}+2 \delta_{2} \Omega_{1} \Omega_{2}+\delta_{3} \Omega_{2}^{2}} \tag{2.3.49}
\end{equation*}
$$

where $\Omega_{1}, \Omega_{2}$ are independent solutions of the associated linear oscillator equation

$$
\begin{equation*}
\ddot{\Omega}_{L}+\left(\frac{f^{2}}{4}+2 A^{*}\right) \Omega_{L}=0 \tag{2.3.50}
\end{equation*}
$$

with unit Wronskian, that is

$$
\begin{equation*}
W\left(\Omega_{1}, \Omega_{2}\right)=\Omega_{1} \dot{\Omega}_{2}-\Omega_{2} \dot{\Omega}_{1}=1 \tag{2.3.51}
\end{equation*}
$$

and the constants $\delta_{i}(i=1,2,3)$ are constrained by the relation

$$
\begin{equation*}
\delta_{1} \delta_{3}-\delta_{2}^{2}=\left(1-\tilde{G}_{0}^{2}-2 \bar{B}\right):=\delta_{0} . \tag{2.3.52}
\end{equation*}
$$

If we choose $\Omega_{1}, \Omega_{2}$ as

$$
\begin{equation*}
\Omega_{1}=\cos \omega t, \quad \Omega_{2}=\frac{1}{\omega} \sin \omega t, \quad \omega=\sqrt{f^{2} / 4+2 A^{*}} \tag{2.3.53}
\end{equation*}
$$

then, the general solution of the Steen-Ermakov equation (2.3.48) is determined by

$$
\begin{equation*}
\Omega=\sqrt{\delta_{4} \cos (2 \omega t+\theta)+\delta_{5}} \tag{2.3.54}
\end{equation*}
$$

where the constants $\delta_{4}, \delta_{5}$ and $\theta$ are related by

$$
\begin{equation*}
\omega^{2}\left(\delta_{4}^{2}-\delta_{5}^{2}\right)+\delta_{0}=0, \quad \theta=\arctan \frac{2 \delta_{2} \omega}{\delta_{3}-\delta_{1} \omega^{2}} . \tag{2.3.55}
\end{equation*}
$$

The reality constraints associated with the relations (2.3.45), (2.3.52) and (2.3.55) require that

$$
\begin{equation*}
\delta_{5}>\delta_{4} \geq 0, \quad \alpha(1-\alpha)<0, \quad 2 \alpha^{2}-2 \alpha+1>\tilde{G}_{0}^{2} \tag{2.3.56}
\end{equation*}
$$

without loss of generality.

A subclass of analytical solutions of the rotating shallow water system is now obtained, namely

$$
\begin{array}{ll}
u_{1}=\frac{\dot{\Omega}}{\Omega}+\frac{\tilde{G}_{0}}{\Omega^{2}} \sin \eta, & u_{2}=\frac{\tilde{G}_{0}}{\Omega^{2}} \cos \eta-\frac{1}{\Omega^{2}}+\frac{f}{2}, \\
v_{1}=\frac{\tilde{G}_{0}}{\Omega^{2}} \cos \eta+\frac{1}{\Omega^{2}}-\frac{f}{2}, & v_{2}=\frac{\dot{\Omega}}{\Omega}-\frac{\tilde{G}_{0}}{\Omega^{2}} \sin \eta,  \tag{2.3.57}\\
a=\frac{\alpha}{\Omega^{4}}\left(1-\alpha+\tilde{G}_{0} \cos \eta\right), & b=-\frac{\alpha \tilde{G}_{0}}{\Omega^{4}} \sin \eta, \\
c=\frac{\alpha}{\Omega^{4}}\left(1-\alpha-\tilde{G}_{0} \cos \eta\right), & h_{0}=\frac{C_{1}}{\Omega^{2}} .
\end{array}
$$

This subclass corresponds to the pulsrodons of elliptic warm-core eddy theory that were originally constructed in the case of plane bottom topography ( $A^{*}=0$ ) by Rogers [96]. Pulsrodons and their duals were later derived by Holm [52] via an elegant Lagrangian formulation.

Below, the exact solution for the moving shoreline $h=0$ is used to exhibit typical eddy boundary evolution. Fig 2.2 shows the time evolution of a small eccentricity elliptical eddy. From the figure, one can see that the clockwise rotation of the elliptical mode is successive but irregular, being faster when the eddy is expanded (wider rim) and slower when the eddy is contracted (smaller rim). A plausible explanation is as follows: for a given eccentricity, the larger the eddy, the greater the radius of curvature at the extremities compared to the radius of inertia, and the lesser the inertial tendency for a particle to overshoot the rim's curve at its point of maximum curvature. In Fig 2.3, the eccentricity of the eddy is increased and the same behavior is displayed. Interestingly, this evolution of the upper free surface for such a typical pulsrodon coincides with an oscillating "breather-type" motion.

### 2.3.4 Pulsrodons: Numerical Treatment

In this subsection, our main concern is with the construction of pulsrodons of the shallow water system via a numerical treatment [3]. The approach to be adopted here


Fig 2.2: The temporal evolution of a small eccentricity elliptical eddy.


Fig 2.3: The temporal evolution of a large eccentricity elliptical eddy.
is the homotopy perturbation method [50,67]. With this method, it is seen that series representations of pulsrodon which rapidly converge to the exact ones are obtained.

To employ the homotopy procedure, it proves convenient to rewrite the rotating
shallow system (2.2.1) with a circular paraboloidal basin in the operator form :

$$
\left\{\begin{array}{l}
L h+(u h)_{x}+(v h)_{y}=0,  \tag{2.3.58}\\
L u+u u_{x}+v u_{y}+h_{x}-f v+2 A^{*} x=0, \\
L v+u v_{x}+v v_{y}+h_{y}+f u+2 A^{*} y=0
\end{array}\right.
$$

where $L=\frac{\partial}{\partial t}$ with inverse $L^{-1}=\int_{0}^{t}(\cdot) d s$. With the construction of pulsrodons in mind, we set the initial values as (cf.(2.3.57)) :

$$
\begin{align*}
& u(x, y, 0)=\left(\frac{\tau_{0}}{\omega_{0}}+\frac{g_{0}}{\omega_{0}^{2}} \sin \eta_{0}\right) x+\left(\frac{f}{2}-\frac{1}{\omega_{0}^{2}}+\frac{g_{0}}{\omega_{0}^{2}} \cos \eta_{0}\right) y \\
& v(x, y, 0)=\left(\frac{1}{\omega_{0}^{2}}-\frac{f}{2}+\frac{g_{0}}{\omega_{0}^{2}} \cos \eta_{0}\right) x+\left(\frac{\tau_{0}}{\omega_{0}}-\frac{g_{0}}{\omega_{0}^{2}} \sin \eta_{0}\right) y  \tag{2.3.59}\\
& h(x, y, 0)=\frac{C_{1}}{\omega_{0}^{2}}+\frac{\alpha(1-\alpha)}{\omega_{0}^{4}}\left(x^{2}+y^{2}\right)+\frac{\alpha g_{0}}{\omega_{0}^{4}}\left(\cos \eta_{0} x^{2}-2 \sin \eta_{0} x y-\cos \eta_{0} y^{2}\right)
\end{align*}
$$

where $C_{1}, \alpha, \tau_{0}, \omega_{0}, g_{0}$ and $\eta_{0}$ are constants to be determined.

Now, a homotopy : $\Omega \times[0,1] \rightarrow R$ is constructed, which satisfies

$$
\left\{\begin{array}{l}
H_{1}(u, v, h, p)=L h+p\left[(u h)_{x}+(v h)_{y}\right]=0  \tag{2.3.60}\\
H_{2}(u, v, h, p)=L u+p\left(u u_{x}+v u_{y}+h_{x}-f v+2 A^{*} x\right)=0 \\
H_{3}(u, v, h, p)=L v+p\left(u v_{x}+v v_{y}+h_{y}+f u+2 A^{*} y\right)=0
\end{array}\right.
$$

where $p \in[0,1]$ is an embedding parameter. It is observed that when $p=0$, the homotopy model (2.3.60) reduces to a linear system

$$
\left\{\begin{array}{l}
H_{1}(u, v, h, 0)=L h=0  \tag{2.3.61}\\
H_{2}(u, v, h, 0)=L u=0 \\
H_{3}(u, v, h, 0)=L v=0
\end{array}\right.
$$

which may be readily solved. On the other hand, when $p=1$, this model corresponds to the original problem (2.3.58), namely

$$
\left\{\begin{array}{l}
H_{1}(u, v, h, 1)=L h+(u h)_{x}+(v h)_{y}=0,  \tag{2.3.62}\\
H_{2}(u, v, h, 1)=L u+u u_{x}+v u_{y}+h_{x}-f v+2 A^{*} x=0, \\
H_{3}(u, v, h, 1)=L v+u v_{x}+v v_{y}+h_{y}+f u+2 A^{*} y=0 .
\end{array}\right.
$$

In topology, the continuous deformation from $H_{i}(u, v, h, 0)$ to $H_{i}(u, v, h, 1),(i=1,2,3)$ is commonly termed homotopic.

According to the perturbation theory of $[50,67]$, the solution of (2.3.58) can be expressed by an infinite-series involving the homotopy parameter $p$, namely

$$
\begin{equation*}
u=\sum_{i=0}^{+\infty} p^{i} u_{i}, \quad v=\sum_{i=0}^{+\infty} p^{i} v_{i}, \quad h=\sum_{i=0}^{+\infty} p^{i} h_{i} . \tag{2.3.63}
\end{equation*}
$$

Substitution of the series solution ansatz (2.3.63) into (2.3.58), produces a set of algebraic equations of $p^{i}$. Balancing the terms of $p^{i}$, produces an over-determined differential system with the unknown variables $u_{i}, v_{i}$ and $h_{i}(i=0,1, \cdots)$ :

$$
\begin{align*}
& O\left(p^{0}\right): \quad L u_{0}=0, \quad L v_{0}=0, \quad L h_{0}=0, \\
& O\left(p^{1}\right): \quad L h_{1}+u_{0, x} h_{0}+u_{0} h_{0, x}+v_{0, y} h_{0}+v_{0} h_{0, y}=0, \\
& L u_{1}+u_{0} u_{0, x}+v_{0} u_{0, y}+h_{0, x}-f v_{0}+2 A^{*} x=0, \\
& L v_{1}+u_{0} v_{0, x}+v_{0} v_{0, y}+h_{0, y}+f u_{0}+2 A^{*} y=0, \\
& O\left(p^{2}\right): \quad L h_{2}+\left(u_{0} h_{1}+u_{1} h_{0}\right)_{x}+\left(v_{0} h_{1}+v_{1} h_{0}\right)_{y}=0, \\
& L u_{2}+\left(u_{0} u_{1}\right)_{x}+v_{0} u_{1, y}+v_{1} u_{0, y}+h_{1, x}-f v_{1}=0, \\
& L v_{2}+\left(v_{0} v_{1}\right)_{y}+u_{0} v_{1, x}+u_{1} v_{0, x}+h_{1, y}+f u_{1}=0,  \tag{2.3.64}\\
& O\left(p^{i}\right): \quad L h_{i}+\sum_{k=0}^{i-1}\left(u_{k} h_{i-k-1}\right)_{x}+\sum_{k=0}^{i-1}\left(v_{k} h_{i-k-1}\right)_{y}=0, \quad i>2, \\
& L u_{i}+\sum_{k=0}^{i-1} u_{k, x} u_{i-k-1}+\sum_{k=0}^{i-1} v_{k} u_{i-k-1, y}+h_{i-1, x}-f v_{i-1}=0, \\
& L v_{i}+\sum_{k=0}^{i-1} v_{k, y} v_{i-k-1}+\sum_{k=0}^{i-1} u_{k} v_{i-k-1, x}+h_{i-1, y}+f u_{i-1}=0,
\end{align*}
$$

With the aid of symbolic computation of Maple, one can readily obtain the solutions
of the above system :

$$
\begin{align*}
& u_{0}=F_{0}(x, y), \quad u_{1}=F_{1}(x, y)+t \bar{F}_{1}(x, y), \\
& v_{0}=G_{0}(x, y), \quad v_{1}=G_{1}(x, y)+t \bar{G}_{1}(x, y), \\
& h_{0}=H_{0}(x, y), \quad h_{1}=H_{1}(x, y)+t \bar{H}_{1}(x, y), \\
& u_{2}=F_{2}(x, y)+t \bar{F}_{2}(x, y)+\frac{1}{2} t^{2} \bar{F}_{3}(x, y),  \tag{2.3.65}\\
& v_{2}=G_{2}(x, y)+t \bar{G}_{2}(x, y)+\frac{1}{2} t^{2} \bar{G}_{3}(x, y), \\
& h_{2}=H_{2}(x, y)+t \bar{H}_{2}(x, y)+\frac{1}{2} t^{2} \bar{H}_{3}(x, y),
\end{align*}
$$

where

$$
\begin{align*}
& F_{i}(x, y)=u_{i}(x, y, 0), \quad \sum_{i=0}^{\infty} F_{i}(x, y)=u(x, y, 0), \\
& G_{i}(x, y)=v_{i}(x, y, 0), \quad \sum_{i=0}^{\infty} G_{i}(x, y)=v(x, y, 0), \\
& H_{i}(x, y)=h_{i}(x, y, 0), \quad \sum_{i=0}^{\infty} H_{i}(x, y)=h(x, y, 0), \\
& \bar{F}_{1}(x, y)=f F_{0}-F_{0} F_{0 x}-G_{0} F_{0 y}-H_{0 x}-2 A^{*} x, \\
& \bar{G}_{1}(x, y)=-f F_{0}-F_{0} G_{0 x}-G_{0} G_{0 y}-H_{0 y}-2 A^{*} y, \\
& \bar{H}_{1}(x, y)=-F_{0 x} H_{0}-F_{0} H_{0 x}-G_{0 y} H_{0}-G_{0} H_{0 y}, \\
& \bar{F}_{2}(x, y)=f G_{1}-\left(F_{0} F_{1}\right)_{x}-G_{0} F_{1 y}-G_{1} F_{0 y}-H_{1 x},  \tag{2.3.66}\\
& \bar{G}_{2}(x, y)=-f F_{1}-\left(G_{0} G_{1}\right)_{y}-F_{0} G_{1 x}-F_{1} G_{0 x}-H_{1 y}, \\
& \bar{H}_{2}(x, y)=-\left(F_{0} H_{1}+F_{1} H_{0}\right)_{x}-\left(G_{0} H_{1}+G_{1} H_{0}\right)_{y}, \\
& \bar{F}_{3}(x, y)=f \bar{G}_{1}-\left(F_{0} \bar{F}_{1}\right)_{x}-G_{0} \bar{F}_{1 y}-\bar{G}_{1} F_{0 y}-\bar{H}_{1 x}, \\
& \bar{G}_{3}(x, y)=-f \bar{F}_{1}-\left(G_{0} \bar{G}_{1}\right)_{y}-F_{0} \bar{G}_{1 x}-\bar{F}_{1} G_{0 x}-\bar{H}_{1 y}, \\
& \bar{H}_{3}(x, y)=-\left(F_{0} \bar{H}_{1}+\bar{F}_{1} H_{0}\right)_{x}-\left(G_{0} \bar{H}_{1}+\bar{G}_{1} H_{0}\right)_{y}
\end{align*}
$$

Therefore, the preceding gives series representations for $u, v$ and $h$ associated with
pulsrodons, namely,

$$
\begin{align*}
u(x, y, t)= & \lim _{p \rightarrow 1} \sum_{i=0}^{+\infty} p^{i} u_{i}=u_{0}+u_{1}+u_{2}+u_{3}+\cdots \\
= & \sum_{i=0}^{\infty} F_{i}(x, y)+\left(\bar{F}_{1}+\bar{F}_{2}\right) t+\frac{1}{2} \bar{F}_{3} t^{2}+\cdots \\
= & \left(\frac{\tau_{0}}{\omega_{0}}+\frac{g_{0}}{\omega_{0}^{2}} \sin \eta_{0}\right) x+\left(\frac{f}{2}-\frac{1}{\omega_{0}^{2}}+\frac{g_{0}}{\omega_{0}^{2}} \cos \eta_{0}\right) y+\left(\bar{F}_{1}+\bar{F}_{2}\right) t+\frac{1}{2} \bar{F}_{3} t^{2}+\cdots \\
v(x, y, t)= & \lim _{p \rightarrow 1} \sum_{i=0}^{+\infty} p^{i} v_{i}=v_{0}+v_{1}+v_{2}+v_{3}+\cdots \\
= & \sum_{i=0}^{\infty} G_{i}(x, y)+\left(\bar{G}_{1}+\bar{G}_{2}\right) t+\frac{1}{2} \bar{G}_{3} t^{2}+\cdots \\
= & \left(\frac{1}{\omega_{0}^{2}}-\frac{f}{2}+\frac{g_{0}}{\omega_{0}^{2}} \cos \eta_{0}\right) x+\left(\frac{\tau_{0}}{\omega_{0}}-\frac{g_{0}}{\omega_{0}^{2}} \sin \eta_{0}\right) y+\left(\bar{G}_{1}+\bar{G}_{2}\right) t+\frac{1}{2} \bar{G}_{3} t^{2}+\cdots \\
h(x, y, t)= & \lim _{p \rightarrow 1} \sum_{i=0}^{+\infty} p^{i} h_{i}=h_{0}+h_{1}+h_{2}+h_{3}+\cdots \\
= & \sum_{i=0}^{\infty} H_{i}(x, y)+\left(\bar{H}_{1}+\bar{H}_{2}\right) t+\frac{1}{2} \bar{H}_{3} t^{2}+\cdots  \tag{2.3.67}\\
= & \frac{\alpha(1-\alpha)}{\omega_{0}^{4}}\left(x^{2}+y^{2}\right)+\frac{\alpha g_{0}}{\omega_{0}^{4}}\left(\cos \eta_{0} x^{2}-2 \sin \eta_{0} x y-\cos \eta_{0} y^{2}\right) \\
& +\frac{C_{1}}{\omega_{0}^{2}}+\left(\bar{H}_{1}+\bar{H}_{2}\right) t+\frac{1}{2} \bar{H}_{3} t^{2}+\cdots
\end{align*}
$$

It is recalled that the exact pulsrodons derived previously (cf.(2.3.57)) are given by

$$
\begin{align*}
& u(x, y, t)=\left(\frac{\dot{\Omega}}{\Omega}+\frac{\tilde{G}_{0}}{\Omega^{2}} \sin \eta\right) x+\left(\frac{f}{2}-\frac{\tilde{G}_{0}}{\Omega^{2}} \cos \eta-\frac{1}{\Omega^{2}}\right) y, \\
& v(x, y, t)=\left(\frac{1}{\Omega^{2}}-\frac{\tilde{G}_{0}}{\Omega^{2}} \cos \eta-\frac{f}{2}\right) x+\left(\frac{\dot{\Omega}}{\Omega}-\frac{\tilde{G}_{0}}{\Omega^{2}} \sin \eta\right) y,  \tag{2.3.68}\\
& h(x, y, t)=\frac{\alpha}{\Omega^{4}}\left(1-\alpha+\tilde{G}_{0} \cos \eta\right) x^{2}-\frac{2 \alpha \tilde{G}_{0}}{\Omega^{4}} \sin \eta x y+\frac{\alpha}{\Omega^{4}}\left(1-\alpha-\tilde{G}_{0} \cos \eta\right) y^{2}+\frac{C_{I}}{\Omega^{2}}
\end{align*}
$$

where $g_{0}=\tilde{G}_{0},\left.\Omega\right|_{t=0}=\omega_{0},\left.\dot{\Omega}\right|_{t=0}=\tau_{0}$ and $\eta_{0}=\eta-f t+2 \alpha \int_{0}^{t} \frac{1}{\Omega^{2}} d t$.
Comparison of the approximate solution $\left\{\phi_{n}^{u}, \phi_{n}^{v}, \phi_{n}^{h}\right\}=\left\{\sum_{i=0}^{n} u_{i}, \sum_{i=0}^{n} v_{i}, \sum_{i=0}^{n} h_{i}\right\}$ $(n=2)$ and exact solution $\{u, v, h\}$ is presented in Table 1-3. There, the parameters are selected as $C_{1}=0.1, f=2, A^{*}=1.5, \alpha=2, g_{0}=1, \tau_{0}=1.2, \omega_{0}=\sqrt{2}$.

## Table 1

Comparison of $\phi_{2}^{u}$ and $u(x, y, t)$ given by $(2.3 .67)_{1}$ and $(2.3 .68)_{1}$ at $y=20$.

| $x$ | $t$ | $\phi_{2}^{u}$ | $u(x, y, t)$ | $\left\|\frac{u(x, y, t)-\phi_{2}^{u}}{u(x, y, t)}\right\|$ |
| :---: | :---: | :---: | :---: | :---: |
| -10 | 0.001 | 3.859493490 | 3.860046388 | $1.432361025 \mathrm{e}-4$ |
| -10 | 0.002 | 3.862990452 | 3.864096535 | $2.862462130 \mathrm{e}-4$ |
| -10 | 0.003 | 3.866486986 | 3.868146548 | $4.290328661 \mathrm{e}-4$ |
| -5 | 0.001 | 5.907544349 | 5.907820757 | $4.678679523 \mathrm{e}-5$ |
| -5 | 0.002 | 5.910367379 | 5.910920281 | $9.353907238 \mathrm{e}-5$ |
| -5 | 0.003 | 5.913189881 | 5.914019358 | $1.402560509 \mathrm{e}-4$ |
| -1 | 0.001 | 7.545985034 | 7.546040254 | $7.317745220 \mathrm{e}-6$ |
| -1 | 0.002 | 7.548268919 | 7.548379279 | $1.462035702 \mathrm{e}-5$ |
| -1 | 0.003 | 7.550552193 | 7.550717607 | $2.190705687 \mathrm{e}-5$ |

## Table 2

Comparison of $\phi_{2}^{v}$ and $v(x, y, t)$ given by $(2.3 .67)_{2}$ and $(2.3 .68)_{2}$ at $y=10$.

| $x$ | $t$ | $\phi_{2}^{v}$ | $v(x, y, t)$ | $\left\|\frac{v(x, y, t)-\phi_{2}^{v}}{v(x, y, t)}\right\|$ |
| :---: | :---: | :---: | :---: | :---: |
| 10 | 0.001 | 16.14297293 | 16.14186708 | $6.850818400 \mathrm{e}-5$ |
| 10 | 0.002 | 16.13774623 | 16.13553390 | $1.371091910 \mathrm{e}-4$ |
| 10 | 0.003 | 16.13252045 | 16.12920102 | $2.058025066 \mathrm{e}-4$ |
| 15 | 0.001 | 12.38448015 | 12.38337430 | $8.930118506 \mathrm{e}-5$ |
| 15 | 0.002 | 12.37911357 | 12.37690128 | $1.787434472 \mathrm{e}-4$ |
| 15 | 0.003 | 12.37374804 | 12.37042867 | $2.683310408 \mathrm{e}-4$ |
| 20 | 0.001 | 8.625987356 | 8.624881528 | $1.282137032 \mathrm{e}-4$ |
| 20 | 0.002 | 8.620480922 | 8.618268656 | $2.566949452 \mathrm{e}-4$ |
| 20 | 0.003 | 8.614975643 | 8.611656335 | $3.854436209 \mathrm{e}-4$ |

## Table 3

Comparison of $\phi_{2}^{h}=\sum_{i=0}^{2} h_{n}$ and $h(x, y, t)$ given by $(2.3 .67)_{3}$ and $(2.3 .68)_{3}$ at $y=2$.

| $x$ | $t$ | $\phi_{2}^{h}$ | $h(x, y, t)$ | $\left\|\frac{h(x, y, t)-\phi_{2}^{h}}{h(x, y, t)}\right\|$ |
| :---: | :---: | :---: | :---: | :---: |
| 10 | 0.001 | -4.095304769 | -4.095304749 | $4.883641444 \mathrm{e}-9$ |
| 10 | 0.002 | -4.095143892 | -4.095144182 | $7.081557745 \mathrm{e}-8$ |
| 10 | 0.003 | -4.094984520 | -4.094984934 | $1.010992731 \mathrm{e}-7$ |
| 20 | 0.001 | -26.19910546 | -26.19910580 | $1.297754216 \mathrm{e}-8$ |
| 20 | 0.002 | -26.18715621 | -26.18715739 | $4.506025539 \mathrm{e}-8$ |
| 20 | 0.003 | -26.17521065 | -26.17521323 | $9.856653229 \mathrm{e}-9$ |
| 30 | 0.001 | -150.7431809 | -150.7431825 | $1.061407868 \mathrm{e}-8$ |
| 30 | 0.002 | -150.6848556 | -150.6848623 | $4.446365678 \mathrm{e}-8$ |
| 30 | 0.003 | -150.6265550 | -150.6265694 | $9.560066366 \mathrm{e}-8$ |

From Table 1-3, it can be seen that the numerical solutions provided by the homotopy method closely approximate the the exact solutions for the pulsrodons. General aspects of the convergence of series solutions derived the homotopy method to the exact, one can refer to [50].

## Chapter 3

## Ermakov-Ray-Reid Systems in Nonlinear Optics: A Variational Approach

### 3.1 Background

In this chapter, our concern is with Ermakov-Ray-Reid systems which arise in nonlinear optical models $[30,90,91,104,108,112]$. Firstly, we consider a coupled 2+1dimensional nonlinear Schrödinger (NLS) system which incorporates that investigated by Pietrzyk [85] in an analysis of the co-propagation of two optical pulses in a Kerr-type planar wave guide. A variational approach elucidated by Anderson and Bonnedal [5] is adopted and thereby a 4-component nonlinear dynamical system is isolated as in [106]. The latter admits three classes of reductions to Ermakov-Ray-Reid subsystems. The underlying Hamiltonian structure of these systems makes them completely integrable [106].

Secondly, we investigate a modulated 3+1-dimensional NLS equations incorporating both a de Broglie-Bohm and Bialynicki-Birula logarithmic potential term as well as a
harmonic trap [105]. It is remarked that a de Broglie-Bohm potential [17, 19] arises in the pioneering work of Wagner et al [127] while a NLS equation involving a logarithmic potential was originally studied in a nonlinear optics context by Snyder and Mitchell [123] in connection with the propagation of Gaussian beams in a saturable medium. The logarithmic Schrödinger equation was originally introduced by Bialynicki-Birula and Mycielski [12-14] in a quantum-mechanical context and is noteworthy for its admittance of Gausson-type solutions. In the present chapter, on application of the variational approximation to the 3+1-dimensional modulated NLS equations is shown to result in a coupled nonlinear system for the beam widths. A reduction to Hamiltonian Ermakov system is obtained. In particular, an Ovsiannikov-Dyson reduction $[28,35,36,81]$ is discussed wherein the eigenmode of the solution explains a kind of flip-over effect that was experimentally observed in a model descriptive of an asymmetric expansion of laser induced plasmas into vacuum [43].

### 3.2 A 2+1-Dimensional Coupled Nonlinear Schrödinger System

This section is devoted to the investigation of a coupled nonlinear Schrödinger system with a harmonic trap which incorporates that investigated in [85] in an analysis of two-pulse interaction in a Kerr medium. In what follows, we shall indicate how the variational approach is applied to the model as in [5], leading to a nonlinear dynamical system describing the evolution of the basic parameters of the beam. The latter admits three distinct reductions to Ermakov-Ray-Reid system of integrable Hamiltonian type [106].

### 3.2.1 The Nonlinear Optics System

The coupled 2+1-dimensional nonlinear Schrödinger system considered here adopts the form of (Rogers, Malomed, Chow and An [106]):

$$
\begin{align*}
& i \frac{\partial \Psi_{1}}{\partial \zeta}+\frac{\sigma_{1}}{2} \frac{\partial^{2} \Psi_{1}}{\partial \tau^{2}}+\frac{1}{2} \frac{\partial^{2} \Psi_{1}}{\partial \xi^{2}}+\left(\left|\Psi_{1}\right|^{2}+2\left|\Psi_{2}\right|^{2}\right) \Psi_{1}+\omega_{1}^{2}(\zeta)\left(\xi^{2}+\tau^{2}\right) \Psi_{1}=0 \\
& i \frac{\partial \Psi_{2}}{\partial \zeta}+\frac{\sigma_{2}}{2} \frac{\partial^{2} \Psi_{2}}{\partial \tau^{2}}+\frac{\mu}{2} \frac{\partial^{2} \Psi_{2}}{\partial \xi^{2}}+r\left(\left|\Psi_{2}\right|^{2}+2\left|\Psi_{1}\right|^{2}\right) \Psi_{2}+\omega_{2}^{2}(\zeta)\left(\xi^{2}+\tau^{2}\right) \Psi_{2}=0 \tag{3.2.1}
\end{align*}
$$

In the absence of the terms corresponding to a harmonic trap, namely those involves $\omega_{1}(\zeta)$ and $\omega_{2}(\zeta)$, the system is that investigated by Pietrzyk [85] in an analysis of the co-propagation of two optical pulses in a nonlinear Kerr-type planar wave guide.

It is observed that the coupled nonlinear Schrödinger system can be derived from the variational principle $\delta S=0$ with action ${ }^{1}$

$$
\begin{equation*}
S=\int \mathcal{L} d \xi d \tau d \zeta \tag{3.2.2}
\end{equation*}
$$

where $\mathcal{L}$ is the Lagrangian density given by

$$
\begin{align*}
& \mathcal{L}=\frac{i}{2}\left(\Psi_{1}^{*} \frac{\partial \Psi_{1}}{\partial \zeta}-\Psi_{1} \frac{\partial \Psi_{1}^{*}}{\partial \zeta}\right)+\frac{i}{2 r}\left(\Psi_{2}^{*} \frac{\partial \Psi_{2}}{\partial \zeta}-\Psi_{2} \frac{\partial \Psi_{2}^{*}}{\partial \zeta}\right)-\frac{1}{2}\left|\frac{\partial \Psi_{1}}{\partial \xi}\right|^{2} \\
&-\frac{\sigma_{1}}{2}\left|\frac{\partial \Psi_{1}}{\partial \tau}\right|^{2}-\frac{\mu}{2 r}\left|\frac{\partial \Psi_{2}}{\partial \xi}\right|^{2}-\frac{\sigma_{2}}{2 r}\left|\frac{\partial \Psi_{2}}{\partial \tau}\right|^{2}+\omega_{1}^{2}(\zeta)\left(\xi^{2}+\tau^{2}\right)\left|\Psi_{1}\right|^{2}  \tag{3.2.3}\\
&+\omega_{2}^{2}(\zeta)\left(\xi^{2}+\tau^{2}\right)\left|\Psi_{2}\right|^{2}+\frac{1}{2}\left|\Psi_{1}\right|^{4}+2\left|\Psi_{1}\right|^{2}\left|\Psi_{2}\right|^{2}+\frac{1}{2}\left|\Psi_{2}\right|^{4}
\end{align*}
$$

and the asterisk denotes the complex conjugation.

It is emphasized that variational formulations constitute an important approach to construct approximate solutions in certain nonlinear optics contexts (see, Malomed [70, 71]).

[^0]
### 3.2.2 The Variational Approach: Reduction to A Nonlinear Dynamical System

According to the variational principle [5, 70], one needs to find appropriate $\Psi_{j}$ $(j=1,2)$ such that the action $S$ an extremum within a set of trial functions. A natural choice for the trial functions $\Psi_{j}$ is the 12-parameter Gaussian ansatz, namely:

$$
\begin{array}{r}
\Psi_{j}(\zeta, \xi, \tau)=A_{j}(\zeta) \exp \left\{-\frac{1}{2} \frac{\tau^{2}}{\omega_{\tau j}(\zeta)}-\frac{1}{2} \frac{\xi^{2}}{\omega_{\xi j}(\zeta)}\right\} \exp \left\{\frac{i}{2} \tau^{2} C_{\tau j}(\zeta)+\frac{i}{2} \xi^{2} C_{\xi j}(\zeta)\right\}  \tag{3.2.4}\\
j=1,2
\end{array}
$$

In the nonlinear optics context of [85], $A_{j}^{*}$ are complex amplitudes, $\omega_{\tau j}, \omega_{\xi j}$ are temporal and spatial widths while $C_{\tau j}$ and $C_{\xi j}$ are temporal and spatial chirps. All these quantities are $\zeta$-dependent.

Substitution of the Gaussian function (3.2.4) into (3.2.3), we obtain :

$$
\begin{align*}
&\langle\mathcal{L}\rangle=\frac{1}{2}\left\{i\left(\dot{A}_{1} A_{1}^{*}-A_{1} \dot{A}_{1}^{*}\right)-\left|A_{1}\right|^{2}\left(\tau^{2} \dot{C}_{\tau 1}+\xi^{2} \dot{C}_{\xi 1}\right)-\xi^{2}\left|A_{1}\right|^{2}\left(C_{\xi 1}^{2}+\frac{1}{\omega_{\xi 1}^{2}}\right)\right. \\
&-\left.\sigma_{1} \tau^{2}\left|A_{1}\right|^{2}\left(C_{\tau 1}^{2}+\frac{1}{\omega_{\tau 1}^{2}}\right)+2 \omega_{1}^{2}\left|A_{1}\right|^{2}\left(\xi^{2}+\tau^{2}\right)\right\} \exp \left\{-\frac{\tau^{2}}{\omega_{\tau 1}}-\frac{\xi^{2}}{\omega_{\xi 1}}\right\} \\
&+\frac{1}{2 r}\left\{i\left(\dot{A}_{2} A_{2}^{*}-A_{2} \dot{A}_{2}^{*}\right)-\left|A_{2}\right|^{2}\left(\tau^{2} \dot{C}_{\tau 2}+\xi^{2} \dot{C}_{\xi 2}\right)-\mu \xi^{2}\left|A_{2}\right|^{2}\left(C_{\xi 2}^{2}+\frac{1}{\omega_{\xi 2}^{2}}\right)\right. \\
&\left.\quad-\sigma_{2} \tau^{2}\left|A_{2}\right|^{2}\left(C_{\tau 2}^{2}+\frac{1}{\omega_{\tau 2}^{2}}\right)+2 r \omega_{2}^{2}\left|A_{2}\right|^{2}\left(\xi^{2}+\tau^{2}\right)\right\} \exp \left\{-\frac{\tau^{2}}{\omega_{\tau 2}}-\frac{\xi^{2}}{\omega_{\xi 2}}\right\} \\
& \quad+\frac{1}{2}\left|A_{1}\right|^{4} \exp \left\{-\frac{2 \tau^{2}}{\omega_{\tau 1}}-\frac{2 \xi^{2}}{\omega_{\xi 1}}\right\}+\frac{1}{2}\left|A_{2}\right|^{4} \exp \left\{-\frac{2 \tau^{2}}{\omega_{\tau 2}}-\frac{2 \xi^{2}}{\omega_{\xi 2}}\right\} \\
& \quad+2\left|A_{1}\right|^{2}\left|A_{2}\right|^{2} \exp \left\{-\frac{\tau^{2}}{\omega_{\tau 1}}-\frac{\xi^{2}}{\omega_{\xi 1}}-\frac{2 \tau^{2}}{\omega_{\tau 2}}-\frac{2 \xi^{2}}{\omega_{\xi 2}}\right\} \tag{3.2.5}
\end{align*}
$$

In the above and the sequel, the dots denote derivatives with respect to $\zeta$.

Integration of (3.2.5) with respect to $\xi$ and $\tau$, produces the reduced Lagrangian

$$
\begin{align*}
\langle\mathcal{L}\rangle=\int_{-\infty}^{+\infty} \mathcal{L} d \xi d \tau= & \frac{\pi}{4} \sqrt{\omega_{\tau 1} \omega_{\xi 1}}\left\{\left|A_{1}\right|^{4}+2 i\left(\dot{A}_{1} A_{1}^{*}-A_{1} \dot{A}_{1}^{*}\right)-\left|A_{1}\right|^{2}\left(\dot{C}_{\tau 1} \omega_{\tau 1}+\dot{C}_{\xi 1} \omega_{\xi 1}\right)\right. \\
& \left.-\omega_{\xi 1}\left|A_{1}\right|^{2}\left(C_{\xi 1}^{2}+\frac{1}{\omega_{\xi 1}^{2}}\right)-\sigma_{1} \omega_{\tau 1}\left|A_{1}\right|^{2}\left(C_{\tau 1}^{2}+\frac{1}{\omega_{\tau 1}^{2}}\right)+2 \omega_{1}^{2}\left|A_{1}\right|^{2}\left(\omega_{\xi 1}+\omega_{\tau 1}\right)\right\} \\
& +\frac{\pi}{4 r} \sqrt{\omega_{\tau 2} \omega_{\xi 2}}\left\{\left|A_{2}\right|^{4}+2 i\left(\dot{A}_{2} A_{2}^{*}-A_{2} \dot{A}_{2}^{*}\right)-\left|A_{2}\right|^{2}\left(\dot{C}_{\tau 2} \omega_{\tau 2}+\dot{C}_{\xi 2} \omega_{\xi 2}\right)\right. \\
& \left.-\mu \omega_{\xi 2}\left|A_{2}\right|^{2}\left(C_{\xi 2}^{2}+\frac{1}{\omega_{\xi 2}^{2}}\right)-\sigma_{2} \omega_{\tau 2}\left|A_{2}\right|^{2}\left(C_{\tau 2}^{2}+\frac{1}{\omega_{\tau 2}^{2}}\right)+2 \omega_{2}^{2}\left|A_{2}\right|^{2}\left(\omega_{\xi 2}+\omega_{\tau 2}\right)\right\} \\
& +2\left|A_{1}\right|^{2}\left|A_{2}\right|^{2} \sqrt{\frac{\omega_{\xi 1} \omega_{\xi 2}}{\omega_{\xi 1}+\omega_{\xi 2}}} \sqrt{\frac{\omega_{\tau 1} \omega_{\tau 2}}{\omega_{\tau 1}+\omega_{\tau 2}}} . \tag{3.2.6}
\end{align*}
$$

The latter, in turn, results in the equivalent Euler-Lagrange equations :

$$
\begin{gather*}
\frac{\partial\langle\mathcal{L}\rangle}{\partial p}-\frac{d}{d \zeta} \frac{\partial\langle\mathcal{L}\rangle}{\partial \dot{p}}=0  \tag{3.2.7}\\
p=\left\{A_{j}, A_{j}^{*}, \omega_{\xi j}, \omega_{\tau j}, C_{\xi j}, C_{\tau j}, j=1,2\right\},
\end{gather*}
$$

which embodies the following 12-dimensional nonlinear dynamical system :

$$
\begin{align*}
& \frac{\partial\langle\mathcal{L}\rangle}{\partial A_{1}}-\frac{d}{d \zeta} \frac{\partial\langle\mathcal{L}\rangle}{\partial \dot{A}_{1}}=0: \\
& \Longrightarrow \\
& \quad-i \dot{A}_{1}^{*}-\frac{1}{4} A_{1}^{*}\left[\left(\dot{C}_{\tau 1} \omega_{\tau 1}+\dot{C}_{\xi 1} \omega_{\xi 1}\right)+\omega_{\xi 1}\left(C_{\xi 1}^{2}+\frac{1}{\omega_{\xi 1}^{2}}\right)+\sigma_{1} \omega_{\tau 1}\left(C_{\tau 1}^{2}+\frac{1}{\omega_{\tau 1}^{2}}\right)\right. \\
& \left.\quad-2 \omega_{1}^{2}\left(\omega_{\xi 1}+\omega_{\tau 1}\right)\right]+\frac{1}{2} A_{1}^{*}\left[\left|A_{1}\right|^{2}-\frac{i}{2} \frac{d}{d \zeta} \ln \left(\omega_{\xi 1} \omega_{\tau 1}\right)\right.  \tag{3.2.8}\\
& \left.\quad+4\left|A_{2}\right|^{2} \sqrt{\omega_{\xi 2} \omega_{\tau 2}} \sqrt{\frac{1}{\left(\omega_{\xi 1}+\omega_{\xi 2}\right)\left(\omega_{\tau 1}+\omega_{\tau 2}\right)}}\right]=0, \\
& \begin{array}{l}
\frac{\partial\langle\mathcal{L}\rangle}{\partial A_{1}^{*}}-\frac{d}{d \zeta} \frac{\partial\langle\mathcal{L}\rangle}{\partial \dot{A}_{1}^{*}}=0: \\
\Longrightarrow \\
\quad i \dot{A}_{1}-\frac{1}{4} A_{1}\left[\left(\dot{C}_{\tau 1} \omega_{\tau 1}+\dot{C}_{\xi 1} \omega_{\xi 1}\right)+\omega_{\xi 1}\left(C_{\xi 1}^{2}+\frac{1}{\omega_{\xi 1}^{2}}\right)+\sigma_{1} \omega_{\tau 1}\left(C_{\tau 1}^{2}+\frac{1}{\omega_{\tau 1}^{2}}\right)\right] \\
\left.\quad-2 \omega_{1}^{2}\left(\omega_{\xi 1}+\omega_{\tau 1}\right)\right]+\frac{1}{2} A_{1}\left[\left|A_{1}\right|^{2}+\frac{i}{2} \frac{d}{d \zeta} \ln \left(\omega_{\xi 1} \omega_{\tau 1}\right)\right. \\
\left.\quad+4\left|A_{2}\right|^{2} \sqrt{\omega_{\xi 2} \omega_{\tau 2}} \sqrt{\frac{1}{\left(\omega_{\xi 1}+\omega_{\xi 2}\right)\left(\omega_{\tau 1}+\omega_{\tau 2}\right)}}\right]=0,
\end{array}
\end{align*}
$$

$$
\begin{align*}
& \frac{\partial\langle\mathcal{L}\rangle}{\partial A_{2}}-\frac{d}{d \zeta} \frac{\partial\langle\mathcal{L}\rangle}{\partial \dot{A}_{2}}=0: \\
& \Longrightarrow-\frac{i}{r} \dot{A}_{2}^{*}-\frac{1}{4 r} A_{2}^{*}\left[\left(\dot{C}_{\tau 2} \omega_{\tau 2}+\dot{C}_{\xi 2} \omega_{\xi 2}\right)+\mu \omega_{\xi 2}\left(C_{\xi 2}^{2}+\frac{1}{\omega_{\xi 2}^{2}}\right)+\sigma_{2} \omega_{\tau 2}\left(C_{\tau 2}^{2}+\frac{1}{\omega_{\tau 2}^{2}}\right)\right] \\
& \quad-2 r \omega_{2}^{2}\left(\omega_{\xi 2}+\omega_{\tau 2}\right)+\frac{1}{2} A_{2}^{*}\left[\left|A_{2}\right|^{2}-\frac{i}{2 r} \frac{d}{d \zeta} \ln \left(\omega_{\xi 2} \omega_{\tau 2}\right)\right. \\
& \left.\quad+4\left|A_{1}\right|^{2} \sqrt{\omega_{\xi 1} \omega_{\tau 1}} \sqrt{\frac{1}{\left(\omega_{\xi 1}+\omega_{\xi 2}\right)\left(\omega_{\tau 1}+\omega_{\tau 2}\right)}}\right]=0 \tag{3.2.10}
\end{align*}
$$

$$
\frac{\partial\langle\mathcal{L}\rangle}{\partial A_{2}^{*}}-\frac{d}{d \zeta} \frac{\partial\langle\mathcal{L}\rangle}{\partial \dot{A}_{2}^{*}}=0:
$$

$$
\Longrightarrow \frac{i}{r} \dot{A}_{2}-\frac{1}{4 r} A_{2}\left[\left(\dot{C}_{\tau 2} \omega_{\tau 2}+\dot{C}_{\xi 2} \omega_{\xi 2}\right)+\mu \omega_{\xi 2}\left(C_{\xi 2}^{2}+\frac{1}{\omega_{\xi 2}^{2}}\right)+\sigma_{2} \omega_{\tau 2}\left(C_{\tau 2}^{2}+\frac{1}{\omega_{\tau 2}^{2}}\right)\right.
$$

$$
-2 r \omega_{2}^{2}\left(\omega_{\xi 2}+\omega_{\tau 2}\right)+\frac{1}{2} A_{2}\left[\left|A_{2}\right|^{2}+\frac{i}{2 r} \frac{d}{d \zeta} \ln \left(\omega_{\xi 2} \omega_{\tau 2}\right)\right.
$$

$$
\begin{equation*}
\left.+4\left|A_{1}\right|^{2} \sqrt{\omega_{\xi 1} \omega_{\tau 1}} \sqrt{\frac{1}{\left(\omega_{\xi 1}+\omega_{\xi 2}\right)\left(\omega_{\tau 1}+\omega_{\tau 2}\right)}}\right]=0 \tag{3.2.11}
\end{equation*}
$$

$$
\frac{\partial\langle\mathcal{L}\rangle}{\partial C_{\xi_{1}}}-\frac{d}{d \zeta} \frac{\partial\langle\mathcal{L}\rangle}{\partial \dot{C}_{\xi_{1}}}=0:
$$

$$
\begin{equation*}
\Longrightarrow \frac{d}{d \zeta}\left(\left|A_{1}\right|^{2} \omega_{\xi 1} \sqrt{\omega_{\xi 1} \omega_{\tau 1}}\right)-2\left|A_{1}\right|^{2} C_{\xi 1} \omega_{\xi 1} \sqrt{\omega_{\xi 1} \omega_{\tau 1}}=0 \tag{3.2.12}
\end{equation*}
$$

$$
\frac{\partial\langle\mathcal{L}\rangle}{\partial C_{\xi_{2}}}-\frac{d}{d \zeta} \frac{\partial\langle\mathcal{L}\rangle}{\partial \dot{C}_{\xi_{2}}}=0
$$

$$
\begin{equation*}
\Longrightarrow \frac{d}{d \zeta}\left(\left|A_{2}\right|^{2} \omega_{\xi 2} \sqrt{\omega_{\xi 2} \omega_{\tau 2}}\right)-2 \mu\left|A_{2}\right|^{2} C_{\xi 2} \omega_{\xi 2} \sqrt{\omega_{\xi 2} \omega_{\tau 2}}=0 \tag{3.2.13}
\end{equation*}
$$

$$
\frac{\partial\langle\mathcal{L}\rangle}{\partial C_{\tau_{1}}}-\frac{d}{d \zeta} \frac{\partial\langle\mathcal{L}\rangle}{\partial \dot{C}_{\tau_{1}}}=0:
$$

$$
\begin{equation*}
\Longrightarrow \frac{d}{d \zeta}\left(\left|A_{1}\right|^{2} \omega_{\tau 1} \sqrt{\omega_{\xi 1} \omega_{\tau 1}}\right)-2 \sigma_{1}\left|A_{1}\right|^{2} C_{\tau 1} \omega_{\tau 1} \sqrt{\omega_{\xi 1} \omega_{\tau 1}}=0 \tag{3.2.14}
\end{equation*}
$$

$$
\frac{\partial\langle\mathcal{L}\rangle}{\partial C_{\tau_{2}}}-\frac{d}{d \zeta} \frac{\partial\langle\mathcal{L}\rangle}{\partial \dot{C}_{\tau_{2}}}=0:
$$

$$
\begin{equation*}
\Longrightarrow \frac{d}{d \zeta}\left(\left|A_{2}\right|^{2} \omega_{\tau 2} \sqrt{\omega_{\xi 2} \omega_{\tau 2}}\right)-2 \sigma_{2}\left|A_{2}\right|^{2} C_{\tau 2} \omega_{\tau 2} \sqrt{\omega_{\xi 2} \omega_{\tau 2}}=0 \tag{3.2.15}
\end{equation*}
$$

$$
\begin{align*}
& \frac{\partial\langle\mathcal{L}\rangle}{\partial \omega_{\xi_{1}}}-\frac{d}{d \zeta} \frac{\partial\langle\mathcal{L}\rangle}{\partial \dot{\omega}_{\xi_{1}}}=0: \\
& \Longrightarrow i\left(\dot{A}_{1} A_{1}^{*}-A_{1} \dot{A}_{1}^{*}\right)-\frac{1}{2}\left|A_{1}\right|^{2}\left[\left(\dot{C}_{\tau 1} \omega_{\tau 1}+3 \dot{C}_{\xi 1} \omega_{\xi 1}\right)+\left(3 C_{\xi 1}^{2} \omega_{\xi 1}+\sigma_{1} C_{\tau 1}^{2} \omega_{\tau 1}\right)\right. \\
& \left.+\left(\frac{\sigma_{1}}{\omega_{\tau 1}}-\frac{1}{\omega_{\xi 1}}\right)-2 \omega_{1}^{2}\left(\omega_{\tau 1}+3 \omega_{\xi 1}\right)-\left|A_{1}\right|^{2}\right] \\
& +4\left|A_{1}\right|^{2}\left|A_{2}\right|^{2} \sqrt{\frac{\omega_{\tau 2}}{\omega_{\xi 2}}} \sqrt{\frac{\omega_{\xi 1}+\omega_{\xi 2}}{\omega_{\tau 1}+\omega_{\tau 2}}} \frac{\omega_{\xi 2}^{2}}{\left(\omega_{\xi 1}+\omega_{\xi 2}\right)^{2}}=0,  \tag{3.2.16}\\
& \frac{\partial\langle\mathcal{L}\rangle}{\partial \omega_{\xi_{2}}}-\frac{d}{d \zeta} \frac{\partial\langle\mathcal{L}\rangle}{\partial \dot{\omega}_{\xi_{2}}}=0: \\
& \Longrightarrow \frac{i}{r}\left(\dot{A}_{2} A_{2}^{*}-A_{2} \dot{A}_{2}^{*}\right)-\frac{1}{2 r}\left|A_{2}\right|^{2}\left[\left(\dot{C}_{\tau 2} \omega_{\tau 2}+3 \dot{C}_{\xi 2} \omega_{\xi 2}\right)+\left(3 \mu C_{\xi 1}^{2} \omega_{\xi 2}+\sigma_{2} C_{\tau 2}^{2} \omega_{\tau 2}\right)\right. \\
& \left.+\left(\frac{\sigma_{2}}{\omega_{\tau 2}}-\frac{\mu}{\omega_{\xi 2}}\right)-2 r \omega_{2}^{2}\left(\omega_{\tau 2}+3 \omega_{\xi 2}\right)-\left|A_{2}\right|^{2}\right] \\
& +4\left|A_{1}\right|^{2}\left|A_{2}\right|^{2} \sqrt{\frac{\omega_{\tau 1}}{\omega_{\xi 1}}} \sqrt{\frac{\omega_{\xi 1}+\omega_{\xi 2}}{\omega_{\tau 1}+\omega_{\tau 2}}} \frac{\omega_{\xi 1}^{2}}{\left(\omega_{\xi 1}+\omega_{\xi 2}\right)^{2}}=0,  \tag{3.2.17}\\
& \frac{\partial\langle\mathcal{L}\rangle}{\partial \omega_{\tau_{1}}}-\frac{d}{d \zeta} \frac{\partial\langle\mathcal{L}\rangle}{\partial \dot{\omega}_{\tau_{1}}}=0: \\
& \Longrightarrow i\left(\dot{A}_{1} A_{1}^{*}-A_{1} \dot{A}_{1}^{*}\right)-\frac{1}{2}\left|A_{1}\right|^{2}\left[\left(3 \dot{C}_{\tau 1} \omega_{\tau 1}+\dot{C}_{\xi 1} \omega_{\xi 1}\right)+\left(C_{\xi 1}^{2} \omega_{\xi 1}+3 \sigma_{1} C_{\tau 1}^{2} \omega_{\tau 1}\right)\right. \\
& \left.-\left(\frac{\sigma_{1}}{\omega_{\tau 1}}-\frac{1}{\omega_{\xi 1}}\right)-2 \omega_{1}^{2}\left(\omega_{\xi 1}+3 \omega_{\tau 1}\right)-\left|A_{1}\right|^{2}\right] \\
& +4\left|A_{1}\right|^{2}\left|A_{2}\right|^{2} \sqrt{\frac{\omega_{\xi 2}}{\omega_{\tau 2}}} \sqrt{\frac{\omega_{\tau 1}+\omega_{\tau 2}}{\omega_{\xi 1}+\omega_{\xi 2}}} \frac{\omega_{\tau 2}^{2}}{\left(\omega_{\tau 1}+\omega_{\tau 2}\right)^{2}}=0,  \tag{3.2.18}\\
& \frac{\partial\langle\mathcal{L}\rangle}{\partial \omega_{\tau_{2}}}-\frac{d}{d \zeta} \frac{\partial\langle\mathcal{L}\rangle}{\partial \dot{\omega}_{\tau_{2}}}=0: \\
& \Longrightarrow \frac{i}{r}\left(\dot{A}_{2} A_{2}^{*}-A_{2} \dot{A}_{2}^{*}\right)-\frac{1}{2 r}\left|A_{2}\right|^{2}\left[\left(3 \dot{C}_{\tau 2} \omega_{\tau 2}+\dot{C}_{\xi 2} \omega_{\xi 2}\right)+\left(\mu C_{\xi 2}^{2} \omega_{\xi 2}+3 \sigma_{2} C_{\tau 2}^{2} \omega_{\tau 2}\right)\right. \\
& \left.-\left(\frac{\sigma_{2}}{\omega_{\tau 2}}-\frac{\mu}{\omega_{\xi 2}}\right)-2 r \omega_{2}^{2}\left(\omega_{\xi 2}+3 \omega_{\tau 2}\right)-\left|A_{2}\right|^{2}\right] \\
& +4\left|A_{1}\right|^{2}\left|A_{2}\right|^{2} \sqrt{\frac{\omega_{\xi 1}}{\omega_{\tau 1}}} \sqrt{\frac{\omega_{\tau 1}+\omega_{\tau 2}}{\omega_{\xi 1}+\omega_{\xi 2}}} \frac{\omega_{\tau 1}^{2}}{\left(\omega_{\tau 1}+\omega_{\tau 2}\right)^{2}}=0 . \tag{3.2.19}
\end{align*}
$$

Multiplication of (3.2.8) and (3.2.10) by $A_{1}$ and $A_{2}$ respectively and subtracting the complex conjugates yields :

$$
\begin{equation*}
\frac{d}{d \zeta}\left(\left|A_{j}\right|^{2} \sqrt{\omega_{\xi j} \omega_{\tau j}}\right)=0, \quad j=1,2 \tag{3.2.20}
\end{equation*}
$$

whence

$$
\begin{equation*}
\left|A_{j}\right|^{2} \sqrt{\omega_{\xi j} \omega_{\tau j}}=T_{j}=\text { const } . \tag{3.2.21}
\end{equation*}
$$

It is interesting to notice that relations of (3.2.21) imply the energy conservation of motion which may be directly obtained via

$$
\begin{equation*}
\int_{-\infty}^{+\infty}|\Psi(\zeta, \xi, \tau)|^{2} d \xi d \tau=\pi\left|A_{j}\right|^{2} \sqrt{\omega_{\xi j} \omega_{\tau j}}=\text { const } \tag{3.2.22}
\end{equation*}
$$

Similarly, multiplication of (3.2.8) and (3.2.10) by $A_{1}$ and $A_{2}$, respectively, and addition of complex conjugates, produces :

$$
\begin{gather*}
i\left(\dot{A}_{1} A_{1}^{*}-A_{1} \dot{A}_{1}^{*}\right)-\frac{1}{2}\left|A_{1}\right|^{2}\left[\left(\dot{C}_{\tau 1} \omega_{\tau 1}+\dot{C}_{\xi 1} \omega_{\xi 1}\right)+\left(C_{\xi 1}^{2} \omega_{\xi 1}+\sigma_{1} C_{\tau 1}^{2} \omega_{\tau 1}\right)+\frac{\sigma_{1}}{\omega_{\tau 1}}+\frac{1}{\omega_{\xi 1}}\right. \\
\left.-2 \omega_{1}^{2}\left(\omega_{\xi 1}+\omega_{\tau 1}\right)\right]+\left|A_{1}\right|^{4}+4\left|A_{1}\right|^{2}\left|A_{2}\right|^{2} \sqrt{\omega_{\xi 2} \omega_{\tau 2}} \sqrt{\frac{1}{\left(\omega_{\tau 1}+\omega_{\tau 2}\right)\left(\omega_{\xi 1}+\omega_{\xi 2}\right)}}=0 \tag{3.2.23}
\end{gather*}
$$

and

$$
\begin{align*}
& \frac{i}{r}\left(\dot{A}_{2} A_{2}^{*}-A_{2} \dot{A}_{2}^{*}\right)-\frac{1}{2 r}\left|A_{2}\right|^{2}\left[\left(\dot{C}_{\tau 2} \omega_{\tau 2}+\dot{C}_{\xi 2} \omega_{\xi 2}\right)+\left(\mu C_{\xi 1}^{2} \omega_{\xi 2}+\sigma_{2} C_{\tau 2}^{2} \omega_{\tau 2}\right)+\frac{\sigma_{2}}{\omega_{\tau 2}}+\frac{\mu}{\omega_{\xi 2}}\right. \\
& \left.\quad-2 r \omega_{2}^{2}\left(\omega_{\xi 2}+\omega_{\tau 2}\right)\right]+\left|A_{2}\right|^{4}+4\left|A_{1}\right|^{2}\left|A_{2}\right|^{2} \sqrt{\frac{\omega_{\tau 1}}{\omega_{\xi 1}}} \sqrt{\frac{1}{\left(\omega_{\tau 1}+\omega_{\tau 2}\right)\left(\omega_{\xi 1}+\omega_{\xi 2}\right)}}=0 \tag{3.2.24}
\end{align*}
$$

Substitution of (3.2.21) into (3.2.12) and (3.2.14), delivers, in turn

$$
\begin{equation*}
C_{\xi 1}=\frac{1}{2} \frac{d}{d \zeta} \ln \left(\omega_{\xi 1}\right), \quad C_{\tau 1}=\frac{1}{2 \sigma_{1}} \frac{d}{d \zeta} \ln \left(\omega_{\tau 1}\right) \tag{3.2.25}
\end{equation*}
$$

while combination of (3.2.21) with (3.2.13) and (3.2.15), yields :

$$
\begin{equation*}
C_{\xi 2}=\frac{1}{2 \mu} \frac{d}{d \zeta} \ln \left(\omega_{\xi 2}\right), \quad C_{\tau 2}=\frac{1}{2 \sigma_{2}} \frac{d}{d \zeta} \ln \left(\omega_{\tau 2}\right) . \tag{3.2.26}
\end{equation*}
$$

On subtraction of (3.2.23) from (3.2.16) and (3.2.18), one can readily obtain two important relations:

$$
\begin{align*}
\frac{1}{\omega_{\xi 1}}-\dot{C}_{\xi 1} \omega_{\xi 1} & -C_{\xi 1}^{2} \omega_{\xi 1}+2 \omega_{1}^{2} \omega_{\xi 1}-\frac{1}{2}\left|A_{1}\right|^{2} \\
& -4\left|A_{2}\right|^{2} \frac{\omega_{\xi 1}}{\omega_{\xi 1}+\omega_{\xi 2}} \sqrt{\frac{\omega_{\xi 2} \omega_{\tau 2}}{\left(\omega_{\tau 1}+\omega_{\tau 2}\right)\left(\omega_{\xi 1}+\omega_{\xi 2}\right)}}=0 \tag{3.2.27}
\end{align*}
$$

and

$$
\begin{align*}
\frac{\sigma_{1}}{\omega_{\tau 1}}-\dot{C}_{\tau 1} \omega_{\tau 1} & -\sigma_{1} C_{\tau 1}^{2} \omega_{\tau 1}+2 \omega_{1}^{2} \omega_{\tau 1}-\frac{1}{2}\left|A_{1}\right|^{2} \\
& -4\left|A_{2}\right|^{2} \frac{\omega_{\tau 1}}{\omega_{\tau 1}+\omega_{\tau 2}} \sqrt{\frac{\omega_{\xi 2} \omega_{\tau 2}}{\left(\omega_{\tau 1}+\omega_{\tau 2}\right)\left(\omega_{\xi 1}+\omega_{\xi 2}\right)}}=0 \tag{3.2.28}
\end{align*}
$$

In a similar manner, from (3.2.17) and (3.2.19), one can derive the further two relations:

$$
\begin{align*}
\frac{\mu}{r} \frac{1}{\omega_{\xi 2}}-\frac{1}{r} \dot{C}_{\xi 2} \omega_{\xi 2} & -\frac{\mu}{r} C_{\xi 2}^{2} \omega_{\xi 2}+2 \omega_{2}^{2} \omega_{\xi 2}-\frac{1}{2}\left|A_{2}\right|^{2} \\
& -4\left|A_{1}\right|^{2} \frac{\omega_{\xi 2}}{\omega_{\xi 1}+\omega_{\xi 2}} \sqrt{\frac{\omega_{\xi 1} \omega_{\tau 1}}{\left(\omega_{\tau 1}+\omega_{\tau 2}\right)\left(\omega_{\xi 1}+\omega_{\xi 2}\right)}}=0 \tag{3.2.29}
\end{align*}
$$

and

$$
\begin{align*}
\frac{\sigma_{2}}{r} \frac{1}{\omega_{\tau 2}}-\frac{1}{r} \dot{C}_{\tau 2} \omega_{\tau 2} & -\frac{\sigma_{2}}{r} C_{\tau 2}^{2} \omega_{\tau 2}+2 \omega_{2}^{2} \omega_{\tau 2}-\frac{1}{2}\left|A_{2}\right|^{2} \\
& -4\left|A_{1}\right|^{2} \frac{\omega_{\tau 2}}{\omega_{\tau 1}+\omega_{\tau 2}} \sqrt{\frac{\omega_{\xi 1} \omega_{\tau 1}}{\left(\omega_{\tau 1}+\omega_{\tau 2}\right)\left(\omega_{\xi 1}+\omega_{\xi 2}\right)}}=0 . \tag{3.2.30}
\end{align*}
$$

Insertion of the relations of $T_{j}, C_{\xi j}, C_{\tau j}$ given by (3.2.21), (3.2.25) and (3.2.26), results in a 4 -component nonlinear system:

$$
\begin{align*}
& \ddot{\omega}_{\xi 1}=\frac{2}{\omega_{\xi 1}}+\frac{\dot{\omega}_{\xi 1}^{2}}{2 \omega_{\xi 1}}+4 \omega_{1}^{2} \omega_{\xi 1}-\frac{T_{1}}{\sqrt{\omega_{\xi 1} \omega_{\tau 1}}}-\frac{8 T_{2} \omega_{\xi 1}}{\omega_{\xi 1}+\omega_{\xi 2}} \sqrt{\frac{1}{\left(\omega_{\tau 1}+\omega_{\tau 2}\right)\left(\omega_{\xi 1}+\omega_{\xi 2}\right)}}, \\
& \ddot{\omega}_{\tau 1}=\frac{2 \sigma_{1}^{2}}{\omega_{\tau 1}}+\frac{\dot{\omega}_{\tau 1}^{2}}{2 \omega_{\tau 1}}+4 \sigma_{1} \omega_{1}^{2} \omega_{\tau 1}-\frac{\sigma_{1} T_{1}}{\sqrt{\omega_{\xi 1} \omega_{\tau 1}}}-\frac{8 \sigma_{1} T_{2} \omega_{\tau 1}}{\omega_{\tau 1}+\omega_{\tau 2}} \sqrt{\frac{1}{\left(\omega_{\tau 1}+\omega_{\tau 2}\right)\left(\omega_{\xi 1}+\omega_{\xi 2}\right)}}, \\
& \ddot{\omega}_{\xi 2}=\frac{2 \mu^{2}}{\omega_{\xi 2}}+\frac{\dot{\omega}_{\xi 2}^{2}}{2 \omega_{\xi 2}}+4 r \mu \omega_{2}^{2} \omega_{\xi 2}-\frac{r \mu T_{2}}{\sqrt{\omega_{\xi 1} \omega_{\tau 1}}}-\frac{8 r \mu T_{1} \omega_{\xi 2}}{\omega_{\xi 1}+\omega_{\xi 2}} \sqrt{\frac{1}{\left(\omega_{\tau 1}+\omega_{\tau 2}\right)\left(\omega_{\xi 1}+\omega_{\xi 2}\right)}}, \\
& \ddot{\omega}_{\tau 2}=\frac{2 \sigma_{2}^{2}}{\omega_{\tau 2}}+\frac{\dot{\omega}_{\tau 2}^{2}}{2 \omega_{\tau 2}}+4 r \sigma_{2} \omega_{2}^{2} \omega_{\tau 2}-\frac{r \sigma_{2} T_{2}}{\sqrt{\omega_{\xi 2} \omega_{\tau 2}}}-\frac{8 r \sigma_{2} T_{1} \omega_{\tau 2}}{\omega_{\tau 1}+\omega_{\tau 2}} \sqrt{\frac{1}{\left(\omega_{\tau 1}+\omega_{\tau 2}\right)\left(\omega_{\xi 1}+\omega_{\xi 2}\right)}} . \tag{3.2.31}
\end{align*}
$$

If we now set

$$
\begin{equation*}
\Omega_{1}=\sqrt{\omega_{\tau 1}}, \quad \Omega_{2}=\sqrt{\omega_{\xi 1}}, \quad \Omega_{3}=\sqrt{\omega_{\tau 2}}, \quad \Omega_{4}=\sqrt{\omega_{\xi 2}} \tag{3.2.32}
\end{equation*}
$$

then the system (3.2.31) reduces to the form

$$
\begin{align*}
& \ddot{\Omega}_{1}-2 \sigma_{1} \omega_{1}^{2} \Omega_{1}=\frac{\Omega_{1}}{\Omega_{2}^{4}}\left[\sigma_{1}^{2}\left(\frac{\Omega_{2}}{\Omega_{1}}\right)^{4}-\frac{\sigma_{1} T_{1}}{2}\left(\frac{\Omega_{2}}{\Omega_{1}}\right)^{3}\right]-\frac{\Omega_{1}}{\Omega_{4}^{4}} \frac{4 \sigma_{1} T_{2}}{\Delta_{13}^{3} \Delta_{24}}, \\
& \ddot{\Omega}_{2}-2 \omega_{1}^{2} \Omega_{2}=\frac{\Omega_{2}}{\Omega_{1}^{4}}\left[\left(\frac{\Omega_{1}}{\Omega_{2}}\right)^{4}-\frac{T_{1}}{2}\left(\frac{\Omega_{1}}{\Omega_{2}}\right)^{3}\right]-\frac{\Omega_{2}}{\Omega_{4}^{4}} \frac{4 T_{2}}{\Delta_{13} \Delta_{24}^{3}}, \\
& \ddot{\Omega}_{3}-2 r \sigma_{2} \omega_{2}^{2} \Omega_{3}=\frac{\sigma_{2}^{2}}{\Omega_{3}^{3}}+\frac{\Omega_{3}}{\Omega_{4}^{4}}\left[-\frac{r \sigma_{2} T_{2}}{2}\left(\frac{\Omega_{4}}{\Omega_{3}}\right)^{3}-\frac{4 r \sigma_{2} T_{1}}{\Delta_{13}^{3} \Delta_{24}}\right],  \tag{3.2.33}\\
& \ddot{\Omega}_{4}-2 r \mu \omega_{2}^{2} \Omega_{4}=\frac{1}{\Omega_{4}^{3}}\left[\mu^{2}-\frac{\mu r T_{2}}{2}\left(\frac{\Omega_{4}}{\Omega_{3}}\right)-\frac{4 T_{1} \mu r}{\Delta_{13} \Delta_{24}^{3}}\right], \\
& \Delta_{13}=\left[\left(\frac{\Omega_{1}}{\Omega_{4}}\right)^{2}+\left(\frac{\Omega_{3}}{\Omega_{4}}\right)^{2}\right]^{1 / 2} \quad \text { and } \quad \Delta_{24}=\left[1+\left(\frac{\Omega_{2}}{\Omega_{4}}\right)^{2}\right]^{1 / 2} .
\end{align*}
$$

It is this nonlinear dynamical system that will be the subject of the subsequent analysis. It is noted that in the absence of terms in $\omega_{1}, \omega_{2}$ corresponding to the harmonic traps, the system (3.2.33) coincides with that set down in [85].

### 3.2.3 Three Distinct Ermakov-Ray-Reid Reductions

Here, we return to the un-modulated optics system wherein $\omega_{1}^{2}=\omega_{2}^{2}=0$ as originally investigated in the context of two-pulse propagation in Kerr-type planar wave guides in [85]. Therein, it was asserted that it is only in the special case of $\sigma_{1}=\sigma_{2}=1, T_{2}=0$ will an analytic solution, namely

$$
\begin{equation*}
\omega_{\xi j}(\zeta)=\omega_{\tau j}(\zeta)=\left[1+\zeta^{2}\left(1-\frac{k_{j}}{2}\right)\right]^{1 / 2} \tag{3.2.34}
\end{equation*}
$$

be available. That is not actually the case to be established below, where three distinct situations are described which lead to complete analytic solution via integrable Hamiltonian Ermakov-Ray-Reid systems.

## Case I

$$
\Omega_{1}=\Omega_{3}, \quad \Omega_{2}=\Omega_{4}
$$

In this case, wherein

$$
\begin{equation*}
\omega_{\tau 1}=\omega_{\tau 2}, \quad \omega_{\xi 1}=\omega_{\xi 2} \tag{3.2.35}
\end{equation*}
$$

with the choice of parameters

$$
\begin{equation*}
\sigma_{1}=\sigma_{2}, \quad r=1, \quad \mu=1, \quad T_{1}=T_{2} \tag{3.2.36}
\end{equation*}
$$

the nonlinear optics system (3.2.33) reduces to the Ermakov-Ray-Reid system

$$
\begin{align*}
& \ddot{\Omega}_{1}=\frac{\sigma_{1}^{2}}{\Omega_{1}^{3}}-\left(\frac{3 T_{1} \sigma_{1}}{2}\right) \frac{1}{\Omega_{1}^{2} \Omega_{2}},  \tag{3.2.37}\\
& \ddot{\Omega}_{2}=\frac{1}{\Omega_{2}^{3}}-\left(\frac{3 T_{1} \sigma_{1}}{2}\right) \frac{1}{\Omega_{1} \Omega_{2}^{2}}
\end{align*}
$$

with the Ray-Reid invariant

$$
\begin{equation*}
I_{1}=\frac{1}{2}\left[\left(\dot{\Omega}_{1} \Omega_{2}-\Omega_{1} \dot{\Omega}_{2}\right)^{2}+\left(\frac{\Omega_{1}}{\Omega_{2}}\right)^{2}+\sigma_{1}^{2}\left(\frac{\Omega_{2}}{\Omega_{1}}\right)^{2}-3 T_{1} \sigma_{1}\left(\frac{\Omega_{1}}{\Omega_{2}}+\frac{\Omega_{2}}{\Omega_{1}}\right)\right] \tag{3.2.38}
\end{equation*}
$$

It is seen that the system (3.2.37) has previously been derived in other nonlinear optics contexts in [20, 22, 44, 129].

## Case II

$$
\Omega_{1}=\Omega_{4}, \quad \Omega_{2}=\Omega_{3}
$$

In this case

$$
\begin{equation*}
\omega_{\tau 1}=\omega_{\xi 2}, \quad \omega_{\tau 2}=\omega_{\xi 1} \tag{3.2.39}
\end{equation*}
$$

and if we set

$$
\begin{equation*}
\sigma_{1}=\mu r, \quad \sigma_{2}^{2}=1, \quad r=1 / \sigma_{2}, \quad T_{1}=T_{2} \tag{3.2.40}
\end{equation*}
$$

then the system (3.2.33) reduces to the Ermakov-Ray-Reid system

$$
\begin{align*}
& \ddot{\Omega}_{1}=\frac{1}{\Omega_{1}^{2} \Omega_{2}}\left[\sigma_{1}^{2}\left(\frac{\Omega_{2}}{\Omega_{1}}\right)-\frac{\sigma_{1} T_{1}}{2}-4 T_{1} \sigma_{1}\left(\frac{\Omega_{1}}{\Omega_{2}}\right)^{3} /\left(1+\left(\frac{\Omega_{1}}{\Omega_{2}}\right)^{2}\right)^{2}\right]  \tag{3.2.41}\\
& \ddot{\Omega}_{2}=\frac{1}{\Omega_{1} \Omega_{2}^{2}}\left[\frac{\Omega_{1}}{\Omega_{2}}-\frac{T_{1}}{2}-4 T_{1}\left(\frac{\Omega_{1}}{\Omega_{2}}\right) /\left(1+\left(\frac{\Omega_{1}}{\Omega_{2}}\right)^{2}\right)^{2}\right]
\end{align*}
$$

with the integral of motion

$$
\begin{align*}
I_{2}=\frac{1}{2}\left[\left(\dot{\Omega}_{1} \Omega_{2}-\right.\right. & \left.\Omega_{1} \dot{\Omega}_{2}\right)^{2}+\left(\frac{\Omega_{1}}{\Omega_{2}}\right)^{2}+\sigma_{1}^{2}\left(\frac{\Omega_{2}}{\Omega_{1}}\right)^{2} \\
& \left.-\frac{T_{1}}{2}\left(\frac{\Omega_{1}}{\Omega_{2}}\right)-\frac{\sigma_{1} T_{1}}{2}\left(\frac{\Omega_{2}}{\Omega_{1}}\right)-\frac{4 T_{1} \sigma_{1}}{1+\left(\frac{\Omega_{1}}{\Omega_{2}}\right)^{2}}-\frac{4 T_{1}}{1+\left(\frac{\Omega_{1}}{\Omega_{2}}\right)^{2}}\right] . \tag{3.2.42}
\end{align*}
$$

## Case III

$$
\Omega_{1}=\Omega_{2}, \quad \Omega_{3}=\Omega_{4}
$$

Here,

$$
\begin{equation*}
\omega_{\tau 1}=\omega_{\xi 1}, \quad \omega_{\tau 2}=\omega_{\xi 2} \tag{3.2.43}
\end{equation*}
$$

and if we set

$$
\begin{equation*}
\sigma_{1}=1, \quad \sigma_{2}=\mu \tag{3.2.44}
\end{equation*}
$$

then the system (3.2.33) reduces to the Ermakov-Ray-Reid system

$$
\begin{align*}
& \ddot{\Omega}_{1}=\frac{1}{\Omega_{1}^{2} \Omega_{4}}\left[\frac{\Omega_{4}}{\Omega_{1}}\left(1-\frac{T_{1}}{2}\right)-4 T_{2}\left(\frac{\Omega_{1}}{\Omega_{4}}\right)^{3} /\left(1+\left(\frac{\Omega_{1}}{\Omega_{4}}\right)^{2}\right)^{2}\right], \\
& \ddot{\Omega}_{4}=\frac{1}{\Omega_{4}^{2} \Omega_{1}}\left[\frac{\Omega_{1}}{\Omega_{4}}\left(\sigma_{2}^{2}-\frac{r \sigma_{2} T_{2}}{2}\right)-4 T_{1} \sigma_{2} r\left(\frac{\Omega_{1}}{\Omega_{4}}\right) /\left(1+\left(\frac{\Omega_{1}}{\Omega_{4}}\right)^{2}\right)^{2}\right] \tag{3.2.45}
\end{align*}
$$

and the Ray-Reid integral of motion is given by

$$
\begin{align*}
I_{3}=\frac{1}{2}\left[\left(\dot{\Omega}_{1} \Omega_{4}-\Omega_{1} \dot{\Omega}_{4}\right)^{2}\right. & -\frac{4 T_{1} r \sigma_{2}}{1+\left(\frac{\Omega_{1}}{\Omega_{4}}\right)^{2}}-\frac{4 T_{2}}{1+\left(\frac{\Omega_{1}}{\Omega_{4}}\right)^{2}} \\
& \left.+\left(1-\frac{T_{1}}{2}\right)\left(\frac{\Omega_{4}}{\Omega_{1}}\right)^{2}+\left(\sigma_{2}^{2}-\frac{r \sigma_{2} T_{2}}{2}\right)\left(\frac{\Omega_{1}}{\Omega_{4}}\right)^{2}\right] . \tag{3.2.46}
\end{align*}
$$

### 3.2.4 The Hamiltonian Ermakov Systems and Integrals of Motion

Hamiltonian Ermakov systems have been discussed by Goncharenko et al in the study of elliptic Gaussian beams in nonlinear optics [40, 41] and recently shown by

Rogers and An to arise in a rotating shallow water model with circular paraboloidal bottom topography [100]. Thus, if a Ermakov-Ray-Reid system

$$
\begin{align*}
& \ddot{\alpha}+\omega^{2} \alpha=\frac{1}{\alpha^{2} \beta} F(\beta / \alpha), \\
& \ddot{\beta}+\omega^{2} \beta=\frac{1}{\alpha \beta^{2}} G(\beta / \alpha) \tag{3.2.47}
\end{align*}
$$

adopts the form of

$$
\begin{equation*}
\ddot{\alpha}=-\frac{\partial H}{\partial \alpha}, \quad \ddot{\beta}=-\frac{\partial H}{\partial \beta} \tag{3.2.48}
\end{equation*}
$$

then, such system has associated Hamiltonian

$$
\begin{equation*}
H=\frac{1}{2}\left(\dot{\alpha}^{2}+\dot{\beta}^{2}\right)+\frac{1}{2} \omega^{2}\left(\alpha^{2}+\beta^{2}\right)+V(\alpha, \beta) \tag{3.2.49}
\end{equation*}
$$

where $V(\alpha, \beta)$ is the potential function of $\alpha$ and $\beta$.

Interestingly, it is noticed, in the present Cases I - III established above, that the Ermakov-Ray-Reid systems indeed possess the Hamiltonian form (3.2.48). The associated Hamiltonians are as follows :

Case I: $\left(\Omega_{1}=\Omega_{3}, \quad \Omega_{2}=\Omega_{4}\right)$

$$
\begin{equation*}
H_{1}=\frac{1}{2}\left[\dot{\Omega}_{1}^{2}+\dot{\Omega}_{2}^{2}+\frac{\sigma_{1}^{2}}{\Omega_{1}^{2}}+\frac{1}{\Omega_{2}^{2}}-\frac{3 T_{1} \sigma_{1}}{\Omega_{1} \Omega_{2}}\right] \tag{3.2.50}
\end{equation*}
$$

Case II : $\left(\Omega_{1}=\Omega_{4}, \quad \Omega_{2}=\Omega_{3}\right)$

$$
\begin{equation*}
H_{2}=\frac{1}{2}\left[\dot{\Omega}_{1}^{2}+\sigma_{1} \dot{\Omega}_{2}^{2}+\frac{\sigma_{1}^{2}}{\Omega_{1}^{2}}+\frac{\sigma_{1}}{\Omega_{2}^{2}}-\frac{\sigma_{1} T_{1}}{\Omega_{1} \Omega_{2}}-\frac{4 \sigma_{1} T_{1}}{\left(\Omega_{1}^{2}+\Omega_{2}^{2}\right)}\right] . \tag{3.2.51}
\end{equation*}
$$

Case III : $\left(\Omega_{1}=\Omega_{2}, \Omega_{3}=\Omega_{4}\right)$

$$
\begin{equation*}
H_{3}=\frac{1}{2}\left[r \sigma_{2} T_{1} \dot{\Omega}_{1}^{2}+T_{2} \dot{\Omega}_{4}^{2}+\frac{r \sigma_{2} T_{1}}{\Omega_{1}^{2}}\left(1-\frac{T_{1}}{2}\right)+\frac{\sigma_{2} T_{2}}{\Omega_{4}^{2}}\left(\sigma_{2}-\frac{r T_{2}}{2}\right)-\frac{4 r \sigma_{2} T_{1} T_{2}}{\left(\Omega_{1}^{2}+\Omega_{4}^{2}\right)}\right] . \tag{3.2.52}
\end{equation*}
$$

Additionally, it is observed that if $\omega_{1}^{2}$ and $\omega_{2}^{2}$ are constant, then the system (3.2.33)
shows that

$$
\begin{align*}
& \bar{\lambda} \dot{\Omega}_{1} \ddot{\Omega}_{1}+\bar{\mu} \dot{\Omega}_{2} \ddot{\Omega}_{2}+\bar{\nu} \dot{\Omega}_{3} \ddot{\Omega}_{3}+\bar{\zeta} \dot{\Omega}_{4} \ddot{\Omega}_{4} \\
&=\bar{\lambda}\left[2 \sigma_{1} \omega_{1}^{2} \Omega_{1} \dot{\Omega}_{1}+\frac{\sigma_{1}^{2} \dot{\Omega}_{1}}{\Omega_{1}^{3}}-\frac{\sigma_{1} T_{1}}{2} \frac{\dot{\Omega}_{1}}{\Omega_{2} \Omega_{1}^{2}}-\frac{4 T_{2} \sigma_{1} \Omega_{1} \dot{\Omega}_{1}}{\left(\Omega_{1}^{2}+\Omega_{3}^{2}\right)^{3 / 2}\left(\Omega_{2}^{2}+\Omega_{4}^{2}\right)^{1 / 2}}\right] \\
&+\bar{\mu}\left[2 \omega_{1}^{2} \Omega_{2} \dot{\Omega}_{2}+\frac{\dot{\Omega}_{2}}{\Omega_{2}^{3}}-\frac{T_{1}}{2} \frac{\dot{\Omega}_{2}}{\Omega_{1} \Omega_{2}^{2}}-\frac{4 T_{2} \Omega_{2} \dot{\Omega}_{2}}{\left(\Omega_{1}^{2}+\Omega_{3}^{2}\right)^{1 / 2}\left(\Omega_{2}^{2}+\Omega_{4}^{2}\right)^{3 / 2}}\right]  \tag{3.2.53}\\
&+\bar{\nu}\left[2 r \sigma_{2} \omega_{2}^{2} \Omega_{3} \dot{\Omega}_{3}+\frac{\sigma_{2}^{2} \dot{\Omega}_{3}}{\Omega_{3}^{3}}-\frac{\sigma_{2} T_{2} r \dot{\Omega}_{3}}{2 \Omega_{3}^{2} \Omega_{4}}-\frac{4 \sigma_{2} T_{1} r \Omega_{3} \dot{\Omega}_{3}}{\left(\Omega_{1}^{2}+\Omega_{3}^{2}\right)^{3 / 2}\left(\Omega_{2}^{2}+\Omega_{4}^{2}\right)^{1 / 2}}\right] \\
&+\bar{\zeta}\left[2 r \mu \omega_{2}^{2} \Omega_{4} \dot{\Omega}_{4}+\frac{\mu^{2} \dot{\Omega}_{4}}{\Omega_{4}^{3}}-\frac{1}{2} \frac{\mu T_{2} r \dot{\Omega}_{4}}{\Omega_{4}^{2} \Omega_{3}}-\frac{4 T_{1} \mu r \Omega_{4} \dot{\Omega}_{4}}{\left(\Omega_{1}^{2}+\Omega_{3}^{2}\right)^{1 / 2}\left(\Omega_{2}^{2}+\Omega_{4}^{2}\right)^{3 / 2}}\right] .
\end{align*}
$$

On setting

$$
\bar{\lambda}: \bar{\mu}: \bar{\nu}: \bar{\zeta}=\frac{r T_{1}}{\sigma_{1}}: r T_{1}: \frac{T_{2}}{\sigma_{2}}: \frac{T_{2}}{\mu}
$$

and integrating (3.2.53) with respect to time $t$, now produces the general Hamiltonian integral of motion

$$
\begin{align*}
H= & \frac{1}{2}\left[\frac{r T_{1}}{\sigma_{1}} \dot{\Omega}_{1}^{2}+r T_{1} \dot{\Omega}_{2}^{2}+\frac{T_{2}}{\sigma_{2}} \dot{\Omega}_{3}^{2}+\frac{T_{2}}{\mu} \dot{\Omega}_{4}^{2}\right] \\
& -r T_{1} \omega_{1}^{2} \Omega_{1}^{2}-r T_{1} \omega_{1}^{2} \Omega_{2}^{2}-r T_{2} \omega_{2}^{2} \Omega_{3}^{2}-r T_{2} \omega_{2}^{2} \Omega_{4}^{2} \\
& +\frac{1}{2}\left[\left(r \sigma_{1} T_{1}\right) \frac{1}{\Omega_{1}^{2}}+\left(r T_{1}\right) \frac{1}{\Omega_{2}^{2}}+\left(T_{2} \sigma_{2}\right) \frac{1}{\Omega_{3}^{2}}+\left(T_{2} \mu\right) \frac{1}{\Omega_{4}^{2}}\right]  \tag{3.2.54}\\
& +\frac{r}{2}\left(\frac{T_{1}^{2}}{\Omega_{1} \Omega_{2}}+\frac{T_{2}^{2}}{\Omega_{3} \Omega_{4}}\right)+\frac{4 r T_{1} T_{2}}{\left(\Omega_{1}^{2}+\Omega_{3}^{2}\right)^{1 / 2}\left(\Omega_{2}^{2}+\Omega_{4}^{2}\right)^{1 / 2}} .
\end{align*}
$$

It is emphasized that the existence of integrals of motion (namely the Ray-Reid invariant and Hamiltonian invariant) for the three particular reductions of (3.2.33) allows their complete integration. Analytical solutions of these Hamiltonian Ermakov systems may be readily constructed via the procedure of $[100,106]$.

Here, we perform some numerical simulations of the solutions (3.2.4) descriptive of beam propagation. Since, the Hamiltonian Ermakov system obtained in Case II is more generalised than that in Case I and III, so we take it as example to exhibit the numerical
results (see Fig 3.1-3.3). Fig 3.1 shows the time evolutions of the pulse widths. Fig 3.2 depicts the approximate theoretical shape of the optical beam obtained via the variational approach. Fig 3.3 exhibits the contour of corresponding beam propagation in Fig 3.2.


Fig 3.1: Time evolutions of pulse widths: (a) is for temporal width $\omega_{\tau_{1}}=\Omega_{1}^{2}$, (b) is for spacial with $\omega_{\xi_{1}}=\Omega_{2}^{2}$.


Fig 3.2: Theoretical eigenmodes $\left|\Psi_{j}\right|$ of the beams given by (3.2.4). Fig (a) is for $\left|\Psi_{1}\right|$ and Fig (b) for $\left|\Psi_{2}\right|$.


Fig 3.3: Contour plots of the theoretical modes $\left|\Psi_{j}\right|$ corresponding to Fig 3.2. Fig (a) is for $\left|\Psi_{1}\right|$ and Fig (b) for $\left|\Psi_{2}\right|$.

### 3.3 A 3+1- Dimensional Modulated Nonlinear Schrödinger Equation

In this section, the application of the variational approximation is extended to investigate a 3+1-dimensional modulated NLS equation. A multi-parameter Gaussian ansatz is introduced which results in a coupled nonlinear system for the beam widths. Particular reductions to integrable subsystems of Ermakov-Ray-Reid type and Ovsiannikov-Dyson type are obtained.

### 3.3.1 The Governing Equations

The modulated 3+1-dimensional nonlinear Schrödinger equation considered here adopts the form of (Rogers, Malomed and An [105])

$$
\begin{align*}
i \frac{\partial u}{\partial t}+\left[\frac { 1 } { 2 } \left(\frac{\partial^{2}}{\partial x^{2}}\right.\right. & \left.+\frac{\partial^{2}}{\partial y^{2}}+\zeta(t) \frac{\partial^{2}}{\partial z^{2}}\right)-s \frac{\nabla^{2}|u|}{|u|} \\
& \left.+\delta(t) \ln |u|+\epsilon(t)|u|^{2 n}-\frac{1}{2} \omega^{2}(t)\left(x^{2}+y^{2}+z^{2}\right)\right] u=0 \tag{3.3.1}
\end{align*}
$$

which incorporates a Bialynicki-Birula logarithmic and de-Broglie Bohm-type quantum potential term in $\nabla^{2}|u| /|u|$ together with a harmonic trap. It is noted that when $s<1 / 2$, the term of Bohm-type quantum potential may be removed via an analogous transformation introduced by Rogers et al in [106].

In the sequel, our task is to isolate the competing effects of modulation and nonlinearity via the variational approximation method of the type that has been employed in the preceding section.

It is seen that the NLS equation (3.3.1) can be readily derived via the Euler-Lagrange equation

$$
\begin{equation*}
\delta L=\frac{\partial \mathcal{L}}{\partial u^{*}}-\frac{\mathrm{d}}{\mathrm{dt}} \frac{\partial \mathcal{L}}{\partial u_{t}^{*}}-\frac{\mathrm{d}}{\mathrm{dx}} \frac{\partial \mathcal{L}}{\partial u_{x}^{*}}-\frac{\mathrm{d}}{\mathrm{dy}} \frac{\partial \mathcal{L}}{\partial u_{y}^{*}}-\frac{\mathrm{d}}{\mathrm{dz}} \frac{\partial \mathcal{L}}{\partial u_{z}^{*}}=0 \tag{3.3.2}
\end{equation*}
$$

where

$$
\begin{equation*}
L=\iiint \int \mathcal{L}\left(u, u^{*}, u_{t}, u_{t}^{*}, u_{x}, u_{x}^{*}, u_{y}, u_{y}^{*}, u_{z}, u_{z}^{*}\right) \text { dxdydz dt } \tag{3.3.3}
\end{equation*}
$$

and $\mathcal{L}$ is the Lagrangian density given by

$$
\begin{aligned}
\mathcal{L}=\frac{i}{2}\left(u^{*} u_{t}\right. & \left.-u u_{t}^{*}\right)-\frac{1}{2}\left|u_{x}\right|^{2}-\frac{1}{2}\left|u_{y}\right|^{2}-\frac{1}{2} \zeta(t)\left|u_{z}\right|^{2}-\frac{1}{2} \delta(t)|u|^{2} \\
& +\delta(t)|u|^{2} \ln |u|+\frac{\epsilon(t)}{n+1}|u|^{2 n+2}+\frac{s}{2}\left(\left|u_{x}\right|^{2}+\left|u_{y}\right|^{2}+\left|u_{z}\right|^{2}\right) \\
& +\frac{s}{4} \frac{u^{*}}{u}\left(u_{x}^{2}+u_{y}^{2}+u_{z}^{2}\right)+\frac{s}{4} \frac{u}{u^{*}}\left(u_{x}^{* 2}+u_{y}^{* 2}+u_{z}^{* 2}\right)-\frac{1}{2} \omega^{2}(t)\left(x^{2}+y^{2}+z^{2}\right)|u|^{2}, \\
& (n \neq-1) .
\end{aligned}
$$

The multi-parameter Gaussian wave ansatz

$$
\begin{equation*}
u(x, y, z, t)=A(t) \exp \left\{i \phi(t)-\frac{1}{2}\left[\frac{x^{2}}{W^{2}(t)}+\frac{y^{2}}{V^{2}(t)}+\frac{z^{2}}{T^{2}(t)}\right]+\frac{i}{2}\left[b(t) x^{2}+c(t) y^{2}+\beta(t) z^{2}\right]\right\} \tag{3.3.5}
\end{equation*}
$$

is introduced as a trial function. Substitution of (3.3.5) into (3.3.3) produces

$$
\begin{gather*}
\mathcal{L}=\frac{1}{2}\left\{i\left(A^{*} \dot{A}-A \dot{A}^{*}\right)-2|A|^{2} \dot{\phi}-|A|^{2}\left[x^{2}\left(\dot{b}+b^{2}\right)+y^{2}\left(\dot{c}+c^{2}\right)+z^{2}\left(\dot{\beta}+\zeta(t) \beta^{2}\right)\right]\right. \\
-\delta(t)|A|^{2}-\omega^{2}(t)|A|^{2}\left(x^{2}+y^{2}+z^{2}\right)-|A|^{2}(1-2 s)\left(\frac{x^{2}}{W^{4}}+\frac{y^{2}}{V^{4}}+\frac{\zeta(t) z^{2}}{T^{4}}\right) \\
\left.-\delta(t)|A|^{2}\left(\frac{x^{2}}{W^{2}}+\frac{y^{2}}{V^{2}}+\frac{z^{2}}{T^{2}}-2 \ln |A|\right)\right\} \exp \left(-\frac{x^{2}}{W^{2}}-\frac{y^{2}}{V^{2}}-\frac{z^{2}}{T^{2}}\right) \\
+\frac{\epsilon(t)}{n+1}|A|^{2 n+2} \exp \left[-(n+1)\left(\frac{x^{2}}{W^{2}}+\frac{y^{2}}{V^{2}}+\frac{z^{2}}{T^{2}}\right)\right] . \tag{3.3.6}
\end{gather*}
$$

with associated reduced Lagrangian

$$
\begin{array}{rl}
\langle\mathcal{L}\rangle= & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathcal{L} \text { dxdydz } \\
=\pi^{\frac{3}{2}} W & V T\left\{\frac{i}{2}\left(A^{*} \dot{A}-A \dot{A}^{*}\right)-|A|^{2} \dot{\phi}+\delta(t)|A|^{2} \ln |A|-\frac{5}{4} \delta(t)|A|^{2}\right.  \tag{3.3.7}\\
& -\frac{1}{4}|A|^{2}\left[\left(\dot{b}+b^{2}+\omega^{2}(t)\right) W^{2}+\left(\dot{c}+c^{2}+\omega^{2}(t)\right) V^{2}+\left(\dot{\beta}+\zeta(t) \beta^{2}+\omega^{2}(t)\right) T^{2}\right] \\
& \left.\quad+\frac{s}{2}|A|^{2}\left(\frac{1}{W^{2}}+\frac{1}{V^{2}}+\frac{1}{T^{2}}\right)-\frac{1}{4}|A|^{2}\left(\frac{1}{W^{2}}+\frac{1}{V^{2}}+\frac{\zeta(t)}{T^{2}}\right)+\frac{\epsilon(t)}{(n+1)^{\frac{5}{2}}}|A|^{2 n+2}\right\} .
\end{array}
$$

The corresponding Euler-Lagrange equations

$$
\begin{equation*}
\frac{\partial\langle\mathcal{L}\rangle}{\partial p}-\frac{\mathrm{d}}{\mathrm{dt}} \frac{\partial\langle\mathcal{L}\rangle}{\partial \dot{p}}=0, \quad p=\{A, W, V, T, b, c, \beta\} \tag{3.3.8}
\end{equation*}
$$

lead to the integral of motion

$$
\begin{equation*}
|A|^{2} W V T=\text { const } \equiv N \tag{3.3.9}
\end{equation*}
$$

together with the relations

$$
\begin{equation*}
b=\dot{W} / W, \quad c=\dot{V} / V, \quad \beta=\dot{T} / \zeta(t) T \tag{3.3.10}
\end{equation*}
$$

and ultimately the following nonlinear system for $\{W, V, T\}$ :

$$
\begin{align*}
\ddot{W}+\omega^{2}(t) W & =\frac{1-2 s}{W^{3}}-\frac{\delta(t)}{W}-\frac{2 n \epsilon(t)|A|^{2 n}}{W(n+1)^{\frac{5}{2}}} \\
& =\frac{1-2 s}{W^{3}}-\frac{\delta(t)}{W}-\frac{2 n}{(n+1)^{\frac{5}{2}}} \frac{\epsilon(t) N^{n}}{W^{n+1} V^{n} T^{n}} \\
\ddot{V}+\omega^{2}(t) V & =\frac{1-2 s}{V^{3}}-\frac{\delta(t)}{V}-\frac{2 n \epsilon(t)|A|^{2 n}}{V(n+1)^{\frac{5}{2}}} \\
= & \frac{1-2 s}{V^{3}}-\frac{\delta(t)}{V}-\frac{2 n}{(n+1)^{\frac{5}{2}}} \frac{\epsilon(t) N^{n}}{W^{n} V^{n+1} T^{n}},  \tag{3.3.11}\\
(\dot{T} / \zeta(t)) \dot{+}+\omega^{2}(t) T & =\frac{\zeta(t)-2 s}{T^{3}}-\frac{\delta(t)}{T}-\frac{2 n \epsilon(t)|A|^{2 n}}{T(n+1)^{\frac{5}{2}}} \\
\quad & \frac{\zeta(t)-2 s}{T^{3}}-\frac{\delta(t)}{T}-\frac{2 n}{(n+1)^{\frac{5}{2}}} \frac{\epsilon(t) N^{n}}{W^{n} V^{n} T^{n+1}} .
\end{align*}
$$

In general, the above system is analytically intractable. However, for certain classes of modulations $\{\delta(t), \epsilon(t)\}$, it can be reduced to consideration of an integrable Ermakov-Ray-Reid system [105].

### 3.3.2 Associated Reductions of the Nonlinear Dynamical System

In the sequel, two distinct associated reductions of the nonlinear dynamical system (3.3.11) are to be discussed. One is the Ermakov-Ray-Reid type and the other is the Ovsiannikov-Dyson type.

## I. A Ermakov-Ray-Reid Reduction

If the modulations $\{\delta(t), \epsilon(t)\}$ are chosen the form of

$$
\begin{equation*}
\delta(t)=\frac{1}{W V} \Omega\left(\frac{W}{V}\right), \quad \epsilon(t)=\frac{1}{W V|A|^{2 n}} \Lambda\left(\frac{W}{V}\right) \tag{3.3.12}
\end{equation*}
$$

then (3.3.11) reduces to a Ermakov-Ray-Reid type system

$$
\begin{align*}
& \ddot{W}+\omega^{2}(t) W=\frac{1}{W^{2} V}\left[(1-2 s) \frac{V}{W}+\Psi\left(\frac{W}{V}\right)\right],  \tag{3.3.13}\\
& \ddot{V}+\omega^{2}(t) V=\frac{1}{W V^{2}}\left[(1-2 s) \frac{W}{V}+\Psi\left(\frac{W}{V}\right)\right]
\end{align*}
$$

augmented by

$$
\begin{equation*}
\left(\frac{\dot{T}}{\zeta(t)}\right) \cdot \omega^{2}(t) T=\frac{\zeta(t)-2 s}{T^{3}}+\frac{1}{W V T} \Psi\left(\frac{W}{V}\right) \tag{3.3.14}
\end{equation*}
$$

where

$$
\begin{equation*}
\Psi\left(\frac{W}{V}\right)=-\Omega\left(\frac{W}{V}\right)-\frac{2 n}{(n+1)^{3 / 2}} \Lambda\left(\frac{W}{V}\right) . \tag{3.3.15}
\end{equation*}
$$

In the absence of a de-Broglie-Bohm potential so that $s=0$, then on introduction of the new independent variable $t^{*}$ according to

$$
\begin{equation*}
d t^{*}=\zeta(t) d t \tag{3.3.16}
\end{equation*}
$$

if the modulation $\zeta(t)$ adopts the form

$$
\begin{equation*}
\zeta(t)=k T^{-2}, \quad k \in \mathbb{R} \tag{3.3.17}
\end{equation*}
$$

and $\omega(t)=0$, then (3.3.14) reduces to the classical Steen-Ermakov equation $[30,124]$

$$
\begin{equation*}
T_{t^{*} t^{*}}-\frac{1}{k W V} \Psi\left(\frac{W}{V}\right) T=\frac{1}{T^{3}} . \tag{3.3.18}
\end{equation*}
$$

The latter has general solution constructed by the nonlinear superposition principle [90, 100, 107]

$$
\begin{equation*}
T=\sqrt{\lambda T_{1}^{2}+2 \mu T_{1} T_{2}+\nu T_{2}^{2}} \tag{3.3.19}
\end{equation*}
$$

where $T_{1}$ and $T_{2}$ are linearly independent solutions of

$$
\begin{equation*}
T_{t^{*} t^{*}}-\frac{1}{k W V} \Psi\left(\frac{W}{V}\right) T=0 \tag{3.3.20}
\end{equation*}
$$

with unit Wronskian and

$$
\begin{equation*}
\lambda \nu-\mu^{2}=1 \tag{3.3.21}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\zeta(t)=\frac{k}{\lambda T_{1}^{2}+2 \mu T_{1} T_{2}+\nu T_{2}^{2}} \tag{3.3.22}
\end{equation*}
$$

and

$$
\begin{equation*}
t=\frac{1}{k} \int\left[\lambda T_{1}^{2}+2 \mu T_{1} T_{2}+\nu T_{2}^{2}\right] d t^{*} \tag{3.3.23}
\end{equation*}
$$

so that $T(t)$ is given parametrically in terms of $t^{*}$ via the relations (3.3.22) and (3.3.23). Likewise $T(t)$ is given parametrically in terms of $t^{*}$ through (3.3.19) and (3.3.23).

In particular, the Ermakov-Ray-Reid system (3.3.13) adopts the Hamiltonian form

$$
\begin{equation*}
\ddot{W}=-\frac{\partial \bar{\Omega}}{\partial W}, \quad \ddot{V}=-\frac{\partial \bar{\Omega}}{\partial V} \tag{3.3.24}
\end{equation*}
$$

iff $\Psi=$ const $=\mathbb{C}$ in which case, the Ermakov-Ray-Reid system (3.3.13) reduces

$$
\begin{align*}
& \ddot{W}=\frac{(1-2 s)}{W^{3}}+\frac{\mathbb{C}}{W^{2} V}, \\
& \ddot{V}=\frac{(1-2 s)}{V^{3}}+\frac{\mathbb{C}}{V^{2} W} . \tag{3.3.25}
\end{align*}
$$

This system arises in a variety of other nonlinear optics contexts via the paraxial approximation [20, 22, 44, 129]. The availability of two integrals of motion, namely the Ray-Reid invariant and the Hamiltonian readily allows its integration $[100,106]$.

In light of the above analysis, we perform a numerical integration of the original problem described by (3.3.1). The initial conditions of governing equations (3.3.25) and (3.3.20) are chosen by $(W(0), V(0), T(0))=(0.1,0.2,0.3)$ and $\mathbb{C}=2$. The results are exhibited in Fig 3.4-3.6. Fig 3.4 shows the approximate theoretical mode shape obtained from the variational theory in $x y$-plane. Fig 3.5 and 3.6 depict the theoretical eigenmode shapes in $x z$ - and $y z$ - plane, respectively.


Fig 3.4: (a) Theoretic eigenmode $|u|$ at $x y$-plane. (b) Contour plot of the theoretic eigenmode $|u|$ in Fig 3.4(a).


Fig 3.5: (a) Theoretic eigenmode $|u|$ at $x z$-plane. (b) Contour plot of the theoretic eigenmode $|u|$ in Fig 3.5(a).


Fig 3.6: (a) Theoretic eigenmode $|u|$ at $y z$-plane. (b) Contour plot of the theoretic eigenmode $|u|$ in Fig 3.6(a).

## II. An Ovsiannikov-Dyson Type Reduction

If $\zeta(t)=1, \omega^{2}(t)=0, s=1 / 2$ and the modulations

$$
\begin{equation*}
\delta(t)=\frac{\lambda}{(W V T)^{\gamma-1}}, \quad \epsilon(t)=\frac{\mu}{(W V T)^{\gamma-1}} \tag{3.3.26}
\end{equation*}
$$

then, on appropriate scaling, the system (3.3.11) reduces to the form

$$
\begin{equation*}
W \ddot{W}=V \ddot{V}=T \ddot{T}=\frac{\mathbb{C}}{(W V T)^{\gamma-1}} . \tag{3.3.27}
\end{equation*}
$$

The latter has origin in work of Ovsiannikov [81] and Dyson [28] on spinning clouds in anisentropic gasdynamics. Importantly, it was established by Gaffet [35, 36] via a Painlevé test that the system (3.3.27) is integrable in the case of an ideal monoatomic gas with adiabatic index $\gamma=5 / 3$. It is of particular interest to notice that when $\gamma=5 / 3$, the theoretical eigenmode $\ln |u|$ (with $u$ given by (3.3.5)) exhibits a flip-over phenomena. In order to shed some light on the behaviours of the flip-over effect, we choose the initial data $(W(0), V(0), T(0))=(0.1,0.25,0.5)$ and $(W(0), V(0), T(0))=(0.5,0.25,0.1)$, respectively. Fig 3.7 shows that the eigenmode $\ln |u|$ changes its shape from vertically elongated (cigar-like shape) to horizontally elongated (pancake-like shape) as it expands and vice versa (see Fig 3.8). Such flip-over effect has recently been experimentally observed by Gornushkin in a model that describes an asymmetric expansion of laser induced plasma into a vacuum [43].


Fig 3.7: Time evolutions of the ellipsoid given by $\ln |u|=$ const. The initial asymmetry ratios is $0.1: 0.25: 0.5$.

It is noted that in [28], Dyson considered the large-time asymptotics of the special case of (3.3.27) with $W=V$ so that the system becomes

$$
\begin{align*}
\ddot{W} & =\frac{\mathbb{C} W T}{\left(W^{2} T\right)^{\gamma}},  \tag{3.3.28}\\
\ddot{T} & =\frac{\mathbb{C} W^{2}}{\left(W^{2} T\right)^{\gamma}} .
\end{align*}
$$



Fig 3.8: Time evolutions of the ellipsoid given by $\ln |u|=$ const. The initial asymmetry ratios is $0.5: 0.25: 0.1$.

Here, we observe that this system adopts Ermakov-Ray-Reid form iff $\gamma=5 / 3$ in which case it becomes

$$
\begin{align*}
\ddot{W} & =\frac{\mathbb{C}}{W^{2} T}\left(\frac{T}{W}\right)^{1 / 3}, \\
\ddot{T} & =\frac{\mathbb{C}}{W T^{2}}\left(\frac{T}{W}\right)^{1 / 3}, \tag{3.3.29}
\end{align*}
$$

with the Ray-Reid invariant

$$
\begin{equation*}
I=\frac{1}{2}(W \dot{T}-\dot{W} T)^{2}+\frac{3 \mathbb{C}}{4}\left[\left(\frac{T}{W}\right)^{4 / 3}+2\left(\frac{W}{T}\right)^{2 / 3}\right] \tag{3.3.30}
\end{equation*}
$$

## Chapter 4

## On A 2+1-Dimensional Madelung System with Logarithmic and Bohm Quantum Potentials

### 4.1 Introduction

Here, our concern will be with a $2+1$-dimensional modulated Madelung system with logarithmic and Bohm quantum potentials. Early work on systems with a logarithmic potential goes back to Bialynicki-Birula and Mycielski [14], who introduced a nonlinear Schrödinger (NLS) equation of the type

$$
i \frac{\partial \Psi}{\partial t}+\nabla^{2} \Psi-(\epsilon \ln |\Psi|+V(\mathbf{r}, t)) \Psi=0
$$

in the context of quantum mechanics. Since then there has been an extensive literature devoted to the analysis of this type equation (see, inter alia [13-15, 32, 49, 51, 57, 116]). Particular interest in such systems centres around its admittance of Gaussian shaped soliton-like solutions [14]. In [74], Nassar set down a general class of NLS equations incorporating both logarithmic and Bohm quantum potential terms and made connection to hydrodynamic type systems via a Madelung transformation. Madelung systems
containing Bohm quantum potential terms arise in both plasma physics theory and nonlinear optics $[74,127]$. In particular, in the case of a logarithmically nonlinear saturable medium, the Madelung system as derived via Maxwell's equations in a nonlinear optics context by Wagner et al in [127] involves both Bohm and Bialynicki-Birula quantum potential contributions.

In this chapter, a modulated versions of 2+1-dimensional Madelung system with logarithmic and Bohm quantum potentials is investigated. Introduction of an exponentialtype elliptic vortex ansatz as originally introduced in the context of elliptic warm core eddy theory in [96] into the Madelung system results in an eight-dimensional nonlinear dynamical system which admits exact analytical solutions in terms of an elliptic integral representation. This eight-dimensional dynamical system has an underlying integrable Hamiltonian structure. Novel integrals of motion render its complete integration.

### 4.2 The Governing Equations

The 2+1-dimensional Madelung system to be considered here adopts the form (Rogers and An [101])

$$
\begin{gather*}
\frac{\partial h}{\partial t}+\nabla(h \nabla \Phi)=0  \tag{4.2.1}\\
\frac{\partial \Phi}{\partial t}+\frac{1}{2}(\nabla \Phi)^{2}-2(s-1) \frac{\nabla^{2} h^{1 / 2}}{h^{1 / 2}}+\epsilon(t) \ln h+2 \omega^{2}(t)\left(x^{2}+y^{2}\right)=0 . \tag{4.2.2}
\end{gather*}
$$

This incorporates a modulated logarithmic quantum potential together with a Bohm quantum potential term and harmonic trap contribution. It is noted that a Madelung system of the above type with a Bohm quantum potential, but in the absence of a harmonic trap term and for a general nonlinear optical medium has been derived by Wagner et al in [127].

Introduction of the Madelung transformation [69]

$$
\begin{equation*}
\Psi=h^{1 / 2} \exp \left(\frac{i \Phi}{2}\right) \tag{4.2.3}
\end{equation*}
$$

shows that the nonlinear coupled system (4.2.1)-(4.2.2) may be encapsulated in the logarithmic NLS type equation

$$
\begin{equation*}
i \frac{\partial \Psi}{\partial t}+\nabla^{2} \Psi-\left[s \frac{\nabla^{2}|\Psi|}{|\Psi|}+\epsilon(t) \ln |\Psi|+\omega^{2}(t)\left(x^{2}+y^{2}\right)\right] \Psi=0 \tag{4.2.4}
\end{equation*}
$$

incorporating the Madelung-Bohm quantum potential type term $\nabla^{2}|\Psi| /|\Psi|$. The latter has its origin in de-Broglie-Bohm quantum theory $[17,19]$.

If we now set

$$
\begin{equation*}
\mathbf{q}=\nabla \Phi \tag{4.2.5}
\end{equation*}
$$

then the Madelung hydrodynamic system equivalent to (4.2.1)-(4.2.2) becomes

$$
\begin{gather*}
\frac{\partial h}{\partial t}+\operatorname{div}(h \mathbf{q})=0  \tag{4.2.6}\\
\frac{\partial \mathbf{q}}{\partial t}+\mathbf{q} \cdot \nabla \mathbf{q}-2(1-s) \frac{\nabla\left(\nabla^{2} h^{1 / 2}\right)}{h^{1 / 2}}+\epsilon(t) \nabla \ln h+2 \omega^{2}(t) \nabla\left(x^{2}+y^{2}\right)=\mathbf{0} \tag{4.2.7}
\end{gather*}
$$

augmented by the irrotational condition

$$
\begin{equation*}
\nabla \times \mathbf{q}=\mathbf{0} \tag{4.2.8}
\end{equation*}
$$

It is the Madelung hydrodynamic system (4.2.6)-(4.2.7) incorporating the harmonic trap that will be the subject of the present chapter.

### 4.3 A Class of Exact Analytical Solutions of the Madelung System

This section is devoted to investigations on the Madelung system incorporating the logarithmic and Bohm quantum potentials (4.2.6)-(4.2.7) via an exponential-type elliptic vortex ansatz wherein an eight-dimensional nonlinear dynamical system is isolated. Appropriate choice of the modulated physical variables and construction of a representation that analogous to Ball-type moment of inertia theorem in [96] proves key to the construction of solutions of the dynamical system.

### 4.3.1 An Exponential-Type Elliptic Vortex Ansatz

Integrable substructure of the Madelung system (4.2.6)-(4.2.7) is now sought via the exponential-type elliptic vortex ansatz (Rogers and An [101])

$$
\begin{array}{ll}
\mathbf{q}=\mathbf{L}(t) \mathbf{x}+\mathbf{M}(t),  \tag{4.3.1}\\
h=\sigma(t) \exp \left(\mathbf{x}^{T} \mathbf{E}(t) \mathbf{x}\right), & \mathbf{x}=\binom{x-q(t)}{y-p(t)}, ~
\end{array}
$$

where

$$
\begin{gather*}
\mathbf{L}=\left(\begin{array}{ll}
u_{1}(t) & u_{2}(t) \\
v_{1}(t) & v_{2}(t)
\end{array}\right), \mathbf{M}=\binom{\dot{q}(t)}{\dot{p}(t)},  \tag{4.3.2}\\
\mathbf{E}=\left(\begin{array}{ll}
a(t) & b(t) \\
b(t) & c(t)
\end{array}\right)
\end{gather*}
$$

At this stage, we proceed without the irrotationality constraint (4.2.8) which may be imposed 'a posteriori'.

Insertion of the ansatz (4.3.1)-(4.3.2) into (4.2.6)-(4.2.7) produces an eight-dimensional nonlinear dynamical system

$$
\begin{align*}
& \dot{u}_{1}=-u_{1}^{2}-u_{2} v_{1}+4(1-s)\left(a^{2}+b^{2}\right)-2 a \epsilon(t)-4 \omega^{2}, \\
& \dot{u}_{2}=-u_{1} u_{2}-u_{2} v_{2}+4(1-s)(a b+b c)-2 b \epsilon(t), \\
& \dot{v}_{1}=-u_{1} v_{1}-v_{1} v_{2}+4(1-s)(a b+b c)-2 b \epsilon(t), \\
& \dot{v}_{2}=-u_{2} v_{1}-v_{2}^{2}+4(1-s)\left(b^{2}+c^{2}\right)-2 c \epsilon(t)-4 \omega^{2},  \tag{4.3.3}\\
& \dot{a}=-2 u_{1} a-2 v_{1} b, \\
& \dot{b}=-u_{2} a-b\left(u_{1}+v_{2}\right)-v_{1} c, \\
& \dot{c}=-2 u_{2} b-2 v_{2} c \\
& \dot{\sigma}=-\left(u_{1}+v_{2}\right) \sigma
\end{align*}
$$

augmented by the linear oscillator equations

$$
\begin{equation*}
\ddot{p}+4 \omega^{2} p=0, \quad \ddot{q}+4 \omega^{2} q=0 . \tag{4.3.4}
\end{equation*}
$$

In what follows, it proves convenient to introduce new variables as previously adopted in the shallow water context in $[96,100]$, namely

$$
\begin{array}{ll}
G=u_{1}+v_{2}, & G_{R}=\frac{1}{2}\left(v_{1}-u_{2}\right), \\
G_{S}=\frac{1}{2}\left(v_{1}+u_{2}\right), & G_{N}=\frac{1}{2}\left(u_{1}-v_{2}\right),  \tag{4.3.5}\\
B & =a+c, \quad B_{S}=b,
\end{array} B_{N}=\frac{1}{2}(a-c) . ~ \$
$$

Thus, $G$ and $G_{R}$ represent, in turn, the divergence and spin of the Madelung velocity field, while $G_{S}$ and $G_{N}$ represent shear and normal deformation rates.

Two relations which are key to the subsequent development and which may be established by appeal to the original system (4.3.3) are now recorded. They may be validated by symbolic computation and are embodied in the following theorem:

## Theorem I

$$
\begin{align*}
& \dot{M}^{*}=-2 G M^{*}  \tag{4.3.6}\\
& \dot{Q}^{*}=-2 G Q^{*}
\end{align*}
$$

where

$$
\begin{align*}
M^{*}= & a u_{2}+b\left(v_{2}-u_{1}\right)-c v_{1}, \quad \triangle=a c-b^{2}, \\
Q^{*}= & -a\left(u_{2}^{2}+v_{2}^{2}\right)+2 b\left(u_{1} u_{2}+v_{1} v_{2}\right)-c\left(u_{1}^{2}+v_{1}^{2}\right)  \tag{4.3.7}\\
& -4 \omega^{2}(a+c)+4 \triangle-4(1-s) \triangle(a+c)-4 \triangle \int \epsilon(t) G d t .
\end{align*}
$$

## Corollary

It is seen that the relations (4.3.6) imply the integral of motion

$$
\begin{equation*}
M^{*} / Q^{*}=\text { const } \tag{4.3.8}
\end{equation*}
$$

Remarkably, the relations (4.3.6) are analogous to those obtained in the context of elliptic warm core eddy theory in [96] and in a rotating shallow water setting in [100].

However, here the parameter $s$ and time modulated variable $\epsilon(t)$ are involved. It proves that the selection of these two variables are crucial to the calculations established below.

Substitution of (4.3.5) into the system (4.3.3), delivers the nonlinear system

$$
\begin{align*}
& \dot{\sigma}+G \sigma=0 \\
& \dot{B}+B G+4\left(B_{N} G_{N}+B_{S} G_{S}\right)=0, \\
& \dot{B}_{S}+B_{S} G+G_{S} B-2 B_{N} G_{R}=0 \\
& \dot{B}_{N}+B_{N} G+B G_{N}+2 B_{S} G_{R}=0, \\
& \dot{G}_{R}+G G_{R}=0  \tag{4.3.9}\\
& \dot{G}_{S}+G G_{S}+2 \epsilon B_{S}-4(1-s) B B_{S}=0, \\
& \dot{G}_{N}+G G_{N}+2 \epsilon B_{N}-4(1-s) B B_{N}=0, \\
& \dot{G}+\frac{1}{2} G^{2}+2\left(G_{S}^{2}+G_{N}^{2}-G_{R}^{2}\right)+2 \epsilon B+8 \omega^{2}-8(1-s)\left(\frac{1}{4} B^{2}+B_{N}^{2}+B_{S}^{2}\right)=0 .
\end{align*}
$$

It is seen that the spacial structure of the original model has been removed and only time dependent terms remain. In the sequel, focus will be on the construction of the analytical solution of the above nonlinear dynamical system.

If we now introduce $\Omega$ via

$$
\begin{equation*}
G=\frac{2 \dot{\Omega}}{\Omega} \tag{4.3.10}
\end{equation*}
$$

then $(4.3 .9)_{6}$ and (4.3.9) $)_{1}$ yield, in turn

$$
\begin{equation*}
G_{R}=c_{0} \Omega^{-2}, \quad \sigma=c_{\mathrm{I}} \Omega^{-2} . \tag{4.3.11}
\end{equation*}
$$

The irrotational case corresponds to $G_{R}=0$.

New $\Omega$-modulated variables are now introduced via

$$
\begin{gather*}
\bar{B}=\Omega^{2} B, \quad \bar{B}_{S}=\Omega^{2} B_{S}, \quad \bar{B}_{N}=\Omega^{2} B_{N}, \\
\bar{G}_{S}=\Omega^{2} G_{S}, \quad \bar{G}_{N}=\Omega^{2} G_{N} \tag{4.3.12}
\end{gather*}
$$

whence, the residual six equations of the nonlinear system (4.3.9) reduce to

$$
\begin{align*}
& \dot{\bar{B}}+4\left(\bar{B}_{N} \bar{G}_{N}+\bar{B}_{S} \bar{G}_{S}\right) / \Omega^{2}=0 \\
& \dot{\bar{B}}_{S}+\left(\bar{B} \bar{G}_{S}-2 c_{0} \bar{B}_{N}\right) / \Omega^{2}=0 \\
& \dot{\bar{B}}_{N}+\left(\bar{B} \bar{G}_{N}+2 c_{0} \bar{B}_{S}\right) / \Omega^{2}=0  \tag{4.3.13}\\
& \dot{\bar{G}}_{S}+2 \epsilon \bar{B}_{S}-4(1-s) \bar{B} \bar{B}_{S} / \Omega^{2}=0 \\
& \dot{\bar{G}}_{N}+2 \epsilon \bar{B}_{N}-4(1-s) \bar{B} \bar{B}_{N} / \Omega^{2}=0
\end{align*}
$$

together with

$$
\begin{equation*}
\Omega^{3} \ddot{\Omega}-c_{0}^{2}+\bar{G}_{S}^{2}+\bar{G}_{N}^{2}+\epsilon \bar{B} \Omega^{2}-(1-s)\left(\bar{B}^{2}+4 \bar{B}_{S}^{2}+4 \bar{B}_{N}^{2}\right)+4 \omega^{2} \Omega^{4}=0 \tag{4.3.14}
\end{equation*}
$$

It is the seven-dimensional dynamical system (4.3.13) and (4.3.14) and its various avatars will be analysed in detail with a view to construct the explicit solutions to the original Madelung system.

### 4.3.2 First Integrals and Analytical Solutions

It is readily verified that combination of $(4.3 .13)_{2}$ and $(4.3 .13)_{3}$ together with use of $(4.3 .13)_{1}$ produces the integral of motion

$$
\begin{equation*}
\bar{B}_{S}^{2}+\bar{B}_{N}^{2}-\frac{\bar{B}^{2}}{4}=c_{\mathrm{II}}, \tag{4.3.15}
\end{equation*}
$$

while, similarly, combination of $(4.3 .13)_{4}$ and $(4.3 .13)_{5}$ yields

$$
\begin{equation*}
\bar{G}_{S}^{2}+\bar{G}_{N}^{2}+(1-s) \bar{B}^{2}-\int \epsilon(t) \dot{\bar{B}} \Omega^{2} d t=0 . \tag{4.3.16}
\end{equation*}
$$

It is seen that when $\bar{B}=$ const or $\epsilon(t)=\frac{k}{\Omega^{2}}$ a second explicit integral of motion is obtained. In the sequel, we shall proceed with the latter case first.

Case I: $\quad \bar{B} \neq 0$ and $\epsilon(\mathrm{t})=\frac{\mathrm{k}}{\Omega^{2}}$
It is observed that if $\bar{B} \neq 0$ and the modulation is chosen as

$$
\begin{equation*}
\epsilon(t)=\frac{k}{\Omega^{2}} \tag{4.3.17}
\end{equation*}
$$

then (4.3.16) produces the second integral of motion, namely

$$
\begin{equation*}
\bar{G}_{S}^{2}+\bar{G}_{N}^{2}+(1-s) \bar{B}^{2}-k \bar{B}=c_{\mathrm{III}} \tag{4.3.18}
\end{equation*}
$$

The integrals of motion (4.3.15) and (4.3.18) may be conveniently parametrised according to

$$
\begin{gather*}
\bar{B}_{S}= \pm \sqrt{c_{\mathrm{II}}+\frac{1}{4} \bar{B}^{2}} \cos \phi(t), \quad \bar{B}_{N}= \pm \sqrt{c_{\mathrm{II}}+\frac{1}{4} \bar{B}^{2}} \sin \phi(t) \\
\bar{G}_{S}= \pm \sqrt{c_{\mathrm{III}}+k \bar{B}-(1-s) \bar{B}^{2}} \sin \theta(t), \quad \bar{G}_{N}= \pm \sqrt{c_{\mathrm{III}}+k \bar{B}-(1-s) \bar{B}^{2}} \cos \theta(t) \tag{4.3.19}
\end{gather*}
$$

In order to illustrate the subsequent procedure, we proceed with the specific parametrisation

$$
\begin{gather*}
\bar{B}_{S}=-\sqrt{c_{\mathrm{II}}+\frac{1}{4} \bar{B}^{2}} \cos \phi(t), \bar{B}_{N}=-\sqrt{c_{\mathrm{II}}+\frac{1}{4} \bar{B}^{2}} \sin \phi(t), \\
\bar{G}_{S}=-\sqrt{c_{\mathrm{III}}+k \bar{B}-(1-s) \bar{B}^{2}} \sin \theta(t), \quad \bar{G}_{N}=+\sqrt{c_{\mathrm{III}}+k \bar{B}-(1-s) \bar{B}^{2}} \cos \theta(t), \tag{4.3.20}
\end{gather*}
$$

whence

$$
\begin{align*}
& \bar{B}_{S} \bar{G}_{S}+\bar{B}_{N} \bar{G}_{N}=\sqrt{c_{\mathrm{II}}+\frac{1}{4} \bar{B}^{2}} \sqrt{c_{\mathrm{III}}+k \bar{B}-(1-s) \bar{B}^{2}} \sin (\theta-\phi)  \tag{4.3.21}\\
& \bar{B}_{N} \bar{G}_{S}-\bar{B}_{N} \bar{G}_{S}=\sqrt{c_{\mathrm{II}}+\frac{1}{4} \bar{B}^{2}} \sqrt{c_{\mathrm{III}}+k \bar{B}-(1-s) \bar{B}^{2}} \cos (\theta-\phi) \tag{4.3.22}
\end{align*}
$$

Accordingly, (4.3.13) $)_{1}$ yields

$$
\begin{equation*}
\dot{\bar{B}}+\frac{4}{\Omega^{2}} \sqrt{c_{\mathrm{II}}+\frac{1}{4} \bar{B}^{2}} \sqrt{c_{\mathrm{III}}+k \bar{B}-(1-s) \bar{B}^{2}} \sin (\theta-\phi)=0 . \tag{4.3.23}
\end{equation*}
$$

Conditions (4.3.13) $)_{2,3}$ reduce to a single condition, namely

$$
\begin{equation*}
\sqrt{c_{\mathrm{II}}+\frac{\bar{B}^{2}}{4}}\left(\dot{\phi}+\frac{2 c_{0}}{\Omega^{2}}\right)-\frac{\bar{B}}{\Omega^{2}} \sqrt{c_{\mathrm{III}}+k \bar{B}-(1-s) \bar{B}^{2}} \cos (\theta-\phi)=0, \tag{4.3.24}
\end{equation*}
$$

while, in a similar manner, $(4.3 .13)_{4,5}$ reduce to another single condition

$$
\begin{equation*}
\dot{\theta} \sqrt{c_{\mathrm{III}}+k \bar{B}-(1-s) \bar{B}^{2}}+2\left[\epsilon-\frac{2 \bar{B}(1-s)}{\Omega^{2}}\right] \sqrt{c_{\mathrm{II}}+\frac{1}{4} \bar{B}^{2}} \cos (\theta-\phi)=0 \tag{4.3.25}
\end{equation*}
$$

that is, in the case of the modulation $\epsilon$ given by (4.3.17),

$$
\begin{equation*}
\dot{\theta} \sqrt{c_{\mathrm{III}}+k \bar{B}-(1-s) \bar{B}^{2}}+\frac{2}{\Omega^{2}}[k-2 \bar{B}(1-s)] \sqrt{c_{\mathrm{II}}+\frac{1}{4} \bar{B}^{2}} \cos (\theta-\phi)=0 . \tag{4.3.26}
\end{equation*}
$$

The relation (4.3.6) may now be used to obtain the explicit solution of (4.3.23). Thus, (4.3.6) together with (4.3.10) implies, on integration, that

$$
\begin{equation*}
M^{*}=\left[-c_{0} \bar{B}+2\left(\bar{B}_{N} \bar{G}_{S}-\bar{B}_{S} \bar{G}_{N}\right)\right] \Omega^{-4}=c_{\mathrm{IV}} \Omega^{-4} \tag{4.3.27}
\end{equation*}
$$

whence, on use of (4.3.22) it is seen that

$$
\begin{equation*}
c_{0} \bar{B}=-c_{\mathrm{IV}}+2 \sqrt{c_{\mathrm{II}}+\frac{1}{4} \bar{B}^{2}} \sqrt{c_{\mathrm{III}}+k \bar{B}-(1-s) \bar{B}^{2}} \cos (\theta-\phi) . \tag{4.3.28}
\end{equation*}
$$

Elimination of $\cos (\theta-\phi)$ in (4.3.24) and (4.3.26) by means of (4.3.28) now yields, in turn

$$
\begin{equation*}
\dot{\phi}=\frac{2}{\Omega^{2}}\left[\frac{\bar{B}\left(c_{0} \bar{B}+c_{\mathrm{IV}}\right)}{\bar{B}^{2}+4 c_{\mathrm{II}}}-c_{0}\right] \tag{4.3.29}
\end{equation*}
$$

and

$$
\begin{equation*}
\dot{\theta}=\frac{2}{\Omega^{2}}\left[\bar{B}(1-s)-\frac{k}{2}\right] \frac{c_{0} \bar{B}+c_{\mathrm{IV}}}{c_{\mathrm{III}}+k \bar{B}-(1-s) \bar{B}^{2}} . \tag{4.3.30}
\end{equation*}
$$

A second key result is embodied in :

## Theorem II

$$
\begin{equation*}
\left(\ddot{\Omega^{2}} \bar{B}\right)+16 \omega^{2}\left(\Omega^{2} \bar{B}\right)=-2\left(Q^{*}-4 \triangle\right) \Omega^{4} \tag{4.3.31}
\end{equation*}
$$

where, substitution of (4.3.10) into (4.3.6) and integration yields

$$
\begin{equation*}
Q^{*}=c_{\mathrm{v}} \Omega^{-4} \tag{4.3.32}
\end{equation*}
$$

so that

$$
\begin{equation*}
\left(\ddot{\Omega^{2}} \bar{B}\right)+16 \omega^{2}\left(\Omega^{2} \bar{B}\right)+2\left(c_{\mathrm{V}}+4 c_{\mathrm{II}}\right)=0 \tag{4.3.33}
\end{equation*}
$$

Interestingly, it is noticed that the Ball-type theorem established here is associated with $Q^{*}$ and $\triangle$. While the relation obtained in the hydrodynamic context in $[96,100]$ is represented by $Q^{*}$ and $M^{*}$.

If we proceed with $\omega=$ const $\neq 0$, then (4.3.33) yields

$$
\begin{equation*}
\Omega^{2} \bar{B}=c_{\mathrm{VI}} \cos (4 \omega t)+c_{\mathrm{VII}} \sin (4 \omega t)-\frac{\left(c_{\mathrm{V}}+4 c_{\mathrm{II}}\right)}{8 \omega^{2}} \tag{4.3.34}
\end{equation*}
$$

while elimination of $\theta-\phi$ between (4.3.23) and (4.3.28) together with use of (4.3.34) now delivers $\bar{B}$ via the elliptic integral relation

$$
\begin{align*}
\int_{c_{\mathrm{VIII}}}^{\bar{B}} & \frac{d \sigma}{\sigma \sqrt{\left(\sigma^{2}+4 c_{\mathrm{II}}\right)\left(c_{\mathrm{III}}+k \sigma-(1-s) \sigma^{2}\right)-\left(c_{0} \sigma+c_{\mathrm{IV}}\right)^{2}}} \\
& = \pm 2 \int_{0}^{t} \frac{d \tau}{c_{\mathrm{VI}} \cos 4 \omega \tau+c_{\mathrm{VII}} \sin 4 \omega \tau-\frac{\left(c_{\mathrm{V}}+4 c_{\mathrm{II}}\right)}{8 \omega^{2}}} \tag{4.3.35}
\end{align*}
$$

where $\left.\bar{B}\right|_{t=0}=c_{\text {VIII }}$. The elliptic integral in (4.3.35) can be treated by standard methods described in [18]. Once $\bar{B}$ has been obtained then $\Omega$ is given by (4.3.34), while $\phi$ and $\theta$ are then determined by integration, in turn, of (4.3.29) and (4.3.30). It is noted that, in the irrotational case pertinent to the construction via the Madelung transformation of exact wave packet solutions of the NLS equation (4.2.4), it is required to take $c_{0}=0$.

The original matrices $\mathbf{L}(t)$ and $\mathbf{M}(t)$ in (4.3.1) are given by $\mathbf{L}=\frac{1}{\Omega^{2}}\left(\begin{array}{ll}\Omega \dot{\Omega}+\sqrt{c_{\mathrm{III}}+k \bar{B}-(1-s) \bar{B}^{2}} & \cos \theta \\ -\left(c_{0}+\sqrt{c_{\mathrm{III}}+k \bar{B}-(1-s) \bar{B}^{2}} \sin \theta\right) \\ c_{0}-\sqrt{c_{\mathrm{III}}+k \bar{B}-(1-s) \bar{B}^{2}} \sin \theta & \Omega \dot{\Omega}-\sqrt{c_{\mathrm{III}}+k \bar{B}-(1-s) \bar{B}^{2}} \cos \theta\end{array}\right)$
and

$$
\mathbf{E}=\frac{1}{\Omega^{2}}\left(\begin{array}{ll}
\frac{\bar{B}}{2}-\sqrt{\frac{1}{4} \bar{B}^{2}+c_{\mathrm{II}}} \sin \phi & -\sqrt{c_{\mathrm{II}}+\frac{1}{4} \bar{B}^{2}} \cos \phi  \tag{4.3.37}\\
-\sqrt{c_{\mathrm{II}}+\frac{1}{4} \bar{B}^{2}} \cos \phi & \frac{\bar{B}}{2}+\sqrt{\frac{1}{4} \bar{B}^{2}+c_{\mathrm{II}}} \sin \phi
\end{array}\right)
$$

while $\sigma$ is given by (4.3.11) $)_{2}$ and $p, q$ by (4.3.4). The intruded terms $p, q, \dot{p}, \dot{q}$ in the class of exact solutions corresponds to invariance of the system (4.2.6) - (4.2.7) under a Lie group transformation. It is recalled that the preceding analysis has been carried out with the modulation (4.3.17).

## Case II: $\bar{B}=$ const

When $\bar{B}=$ const, then mutatis mutandis, an analogous parametrisation procedure may be employed. In the present case, the second integral of motion adopts the form

$$
\begin{equation*}
\bar{G}_{S}^{2}+\bar{G}_{N}^{2}=c_{\mathrm{III}}^{*} \tag{4.3.38}
\end{equation*}
$$

which may be parametrised via

$$
\begin{equation*}
\bar{G}_{S}=-\sqrt{c_{\mathrm{III}}^{*}} \sin \theta(t), \quad \bar{G}_{N}=+\sqrt{c_{\mathrm{III}}^{*}} \cos \theta(t) \tag{4.3.39}
\end{equation*}
$$

whence,

$$
\begin{equation*}
\bar{B}_{S} \bar{G}_{S}+\bar{B}_{N} \bar{G}_{N}=\sqrt{c_{\mathrm{III}}^{*}\left(c_{\mathrm{II}}+\frac{1}{4} \bar{B}^{2}\right)} \sin (\theta-\phi), \tag{4.3.40}
\end{equation*}
$$

and

$$
\begin{equation*}
\bar{B}_{N} \bar{G}_{S}-\bar{G}_{N} \bar{B}_{S}=\sqrt{c_{\mathrm{III}}^{*}\left(c_{\mathrm{II}}+\frac{1}{4} \bar{B}^{2}\right)} \cos (\theta-\phi) . \tag{4.3.41}
\end{equation*}
$$

Insertion of (4.3.40) in (4.3.13) $)_{1}$ shows that if $c_{\mathrm{III}}^{*} \neq 0, c_{\mathrm{II}}+\frac{1}{4} \bar{B}^{2} \neq 0$ then $\theta=\phi+n \pi$. If we proceed with $\theta=\phi$ then (4.3.41) reduces to

$$
\begin{equation*}
\bar{B}_{N} \bar{G}_{S}-\bar{G}_{N} \bar{B}_{S}=\sqrt{c_{\mathrm{III}}^{*}\left(c_{\mathrm{II}}+\frac{1}{4} \bar{B}^{2}\right)}=\text { const } . \tag{4.3.42}
\end{equation*}
$$

Conditions (4.3.13) $)_{2,3}$ reduce to the single condition

$$
\begin{equation*}
\dot{\phi}=\frac{1}{\Omega^{2}}\left[\bar{B} \sqrt{c_{\mathrm{III}}^{*} /\left(c_{\mathrm{II}}+\frac{1}{4} \bar{B}^{2}\right)}-2 c_{0}\right] \tag{4.3.43}
\end{equation*}
$$

while (4.3.13) $)_{4,5}$ lead to the single condition

$$
\begin{equation*}
\dot{\theta}=\frac{2}{\Omega^{2}}\left[2 \bar{B}(1-s)-\epsilon \Omega^{2}\right] \sqrt{\left(c_{\mathrm{II}}+\frac{1}{4} \bar{B}^{2}\right) / c_{\mathrm{III}}^{*}} . \tag{4.3.44}
\end{equation*}
$$

Interestingly, with $\theta=\phi$, consistency of (4.3.43) and (4.3.44) in the present case $\bar{B}=$ const again requires the modulation $\epsilon(t)$ to be of the type (4.3.17) in which case the constant $\bar{B}$ is determined by the relation

$$
\begin{equation*}
\bar{B} \sqrt{c_{\mathrm{III}}^{*} /\left(c_{\mathrm{II}}+\frac{1}{4} \bar{B}^{2}\right)}-2 c_{0}=2[2 \bar{B}(1-s)-k] \sqrt{\left(c_{\mathrm{II}}+\frac{1}{4} \bar{B}^{2}\right) / c_{\mathrm{III}}^{*}} \tag{4.3.45}
\end{equation*}
$$

Moreover, (4.3.14) reduces to a classical Steen-Ermakov equation [30, 124]

$$
\begin{equation*}
\ddot{\Omega}+4 \omega^{2} \Omega=\mathbb{K} / \Omega^{3}, \tag{4.3.46}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathbb{K}=c_{0}^{2}-c_{\mathrm{III}}^{*}+2(1-s)\left(\bar{B}^{2}+2 c_{\mathrm{II}}\right)-k \bar{B} . \tag{4.3.47}
\end{equation*}
$$

It is readily checked that (4.3.46) is consistent with (4.3.33) modulo a relation between $\bar{B}$ and the residual arbitrary constants.

The Madelung-type hydrodynamic variables $\mathbf{q}$ and $h$ are again given by (4.3.1)(4.3.2) but now with

$$
\mathbf{L}=\frac{1}{\Omega^{2}}\left(\begin{array}{cc}
\Omega \dot{\Omega}+\sqrt{c_{\mathrm{III}}^{*}} \cos \theta & -\left(c_{0}+\sqrt{c_{\mathrm{III}}^{*}} \sin \theta\right)  \tag{4.3.48}\\
c_{0}-\sqrt{c_{\mathrm{III}}^{*}} \sin \theta & \Omega \dot{\Omega}-\sqrt{c_{\mathrm{III}}^{*}} \cos \theta
\end{array}\right)
$$

and

$$
\mathbf{E}=\frac{1}{\Omega^{2}}\left(\begin{array}{ll}
\frac{\bar{B}}{2}-\sqrt{\frac{1}{4} \bar{B}^{2}+c_{\mathrm{II}}} \sin \theta & -\sqrt{c_{\mathrm{II}}+\frac{1}{4} \bar{B}^{2}} \cos \theta  \tag{4.3.49}\\
-\sqrt{c_{\mathrm{II}}+\frac{1}{4} \bar{B}^{2}} \cos \theta & \frac{\bar{B}}{2}+\sqrt{\frac{1}{4} \bar{B}^{2}+c_{\mathrm{II}}} \sin \theta
\end{array}\right)
$$

where

$$
\begin{equation*}
\Omega=\sqrt{\lambda \Omega_{1}^{2}+2 \mu \Omega_{1} \Omega_{2}+\nu \Omega_{2}^{2}}, \tag{4.3.50}
\end{equation*}
$$

with $\Omega_{1}, \Omega_{2}$ linearly independent solutions of

$$
\begin{equation*}
\ddot{\Omega}+4 \omega^{2} \Omega=0 \tag{4.3.51}
\end{equation*}
$$

with unit Wronskian and $\lambda \nu-\mu^{2}=\mathbb{K}$. In the above, the angle $\theta=\phi$ is given by integration of (4.3.43). In the irrotational case $c_{0}=0$, associated wave-packet solutions of periodically modulated $2+1$-dimensional NLS equations of the type (4.2.4) incorporating logarithmic and Bohm quantum potentials are now readily constructed via the Madelung transformation.

### 4.4 Ermakov-Ray-Reid Connection and Integrability

Here, it is shown that, remarkably, the nonlinear dynamical system (4.3.3), namely

$$
\begin{align*}
& \dot{\mathbf{E}}+2 \mathbf{E L}-\Lambda \sigma_{1}=\mathbf{0}  \tag{4.4.1}\\
& \dot{\mathbf{L}}+\mathbf{L}^{2}-4(1-s) \mathbf{E}^{2}+2 \epsilon \mathbf{E}+4 \omega^{2} \mathbf{I}=\mathbf{0}
\end{align*}
$$

where $\sigma_{1}$ is

$$
\sigma_{1}=\left(\begin{array}{cc}
0 & 1  \tag{4.4.2}\\
-1 & 0
\end{array}\right)
$$

and $\Lambda=a u_{2}+b v_{2}-b u_{1}-c v_{1}$, admits an underlying integrable structure of Ermakov-Ray-Reid type as obtained by Rogers and An in the context of a rotating shallow water equation with a circular paraboloidal bottom topography in [100].

With construction of Ermakov-Ray-Reid system in mind, we now turn to the relations (4.3.7) $)_{2}$ and (4.3.32). It is seen that combination of them leads to the expression

$$
\begin{align*}
-\frac{\bar{B}}{\Omega^{2}}\left[\bar{G}_{S}^{2}+\right. & \left.\bar{G}_{N}^{2}-4 c_{\mathrm{II}}(1-s)\right]-\bar{B} \Omega^{2}\left(G_{R}^{2}+4 \omega^{2}+G^{2} / 4\right)  \tag{4.4.3}\\
& +2 G\left(\bar{B}_{S} \bar{G}_{S}+\bar{B}_{N} \bar{G}_{N}\right)+2\left(G_{R}+c_{0} \bar{B}\right)-\frac{4 k c_{\mathrm{II}}}{\Omega^{2}}=c_{\mathrm{V}}+4 c_{\mathrm{II}}
\end{align*}
$$

While conditions (4.3.23) and (4.3.28), by virtue of the first integral (4.3.18), imply that

$$
\begin{equation*}
\bar{G}_{S}^{2}+\bar{G}_{N}^{2}=\frac{\Omega^{4} \dot{\bar{B}}^{2}+4\left(c_{0} \bar{B}+c_{\mathrm{IV}}\right)}{4\left(\bar{B}^{2}+4 c_{\mathrm{II}}\right)} \tag{4.4.4}
\end{equation*}
$$

whence, (4.4.3) is reformulated in terms of $\Omega$ and $\bar{B}$, namely

$$
\begin{align*}
& \frac{-\bar{B}}{\bar{B}^{2}+4 c_{\mathrm{II}}}\left[\frac{\dot{\bar{B}}^{2} \Omega^{2}}{4}+\frac{\bar{B}\left(c_{0} \bar{B}+c_{\mathrm{IV}}\right)^{2}}{\Omega^{2}}\right]+\frac{\bar{B}}{\Omega^{2}}\left[c_{0}^{2}+4(1-s) c_{\mathrm{II}}\right]  \tag{4.4.5}\\
&+\frac{2}{\Omega^{2}}\left[c_{0} c_{\mathrm{IV}}-2 k c_{\mathrm{II}}\right]-\dot{\Omega}(\bar{B} \dot{\Omega}+\dot{\bar{B}} \Omega)-4 \omega^{2} \bar{B} \Omega^{2}=c_{\mathrm{V}}+4 c_{\mathrm{II}}
\end{align*}
$$

Remarkably, the integral of motion (4.4.5) turns out to be the Hamiltonian invariant of the Ermakov-Ray-Reid system to be established blow. Indeed, if, for instance, in the
case of $c_{\mathrm{I}}>0$ and $\bar{B}<0<-4 c_{\mathrm{II}}<\bar{B}^{2}$, we adopt the physical variables

$$
\begin{equation*}
\Phi=\Omega \sqrt{\frac{c_{\mathrm{I}}}{2 c_{\mathrm{II}}}\left(\bar{B}-\sqrt{\bar{B}^{2}+4 c_{\mathrm{II}}}\right)}, \quad \Psi=\Omega \sqrt{\frac{c_{\mathrm{I}}}{2 c_{\mathrm{II}}}\left(\bar{B}+\sqrt{\bar{B}^{2}+4 c_{\mathrm{II}}}\right)} \tag{4.4.6}
\end{equation*}
$$

then, up to irrelevant constant, (4.4.5) assumes the 'symmetric form'

$$
\begin{equation*}
\mathcal{H}=\frac{1}{2}\left(\dot{\Phi}^{2}+\dot{\Psi}^{2}\right)+2 \omega^{2}\left(\Phi^{2}+\Psi^{2}\right)+\frac{1}{\Phi \Psi} J(\Phi / \Psi) \tag{4.4.7}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{H}=\frac{c_{\mathrm{I}}\left(c_{\mathrm{V}}+4 c_{\mathrm{II}}\right)}{2 c_{\mathrm{II}}} \tag{4.4.8}
\end{equation*}
$$

and

$$
\begin{align*}
J(\xi)=\frac{c_{\mathrm{I}}^{2}}{4 c_{\mathrm{II}}^{2}}\left[\left(c_{\mathrm{IV}}+2 c_{0} \sqrt{-c_{\mathrm{II}}}\right)^{2} \frac{\xi}{(\xi+1)^{2}}\right. & +\left(c_{\mathrm{IV}}-2 c_{0} \sqrt{-c_{\mathrm{II}}}\right)^{2} \frac{\xi}{(\xi-1)^{2}} \\
& \left.+8 c_{\mathrm{II}}^{2}(1-s)\left(\xi+\frac{1}{\xi}\right)-8 k c_{\mathrm{II}} \sqrt{-c_{\mathrm{II}}}\right] \tag{4.4.9}
\end{align*}
$$

It is established that, remarkably, the two physical variables $\Phi$ and $\Psi$ given by (4.4.6) are governed by the Ermakov-Ray-Reid system, namely (cf. Rogers and An [101])

$$
\begin{align*}
& \ddot{\Phi}+4 \omega^{2} \Phi=\frac{1}{\Phi^{2} \Psi} F(\Psi / \Phi)=\frac{1}{\Phi^{2} \Psi}\left[J(\Phi / \Psi)-\frac{\Phi}{\Psi} J^{\prime}(\Phi / \Psi)\right] \\
& \ddot{\Psi}+4 \omega^{2} \Psi=\frac{1}{\Phi \Psi^{2}} G(\Phi / \Psi)=\frac{1}{\Phi \Psi^{2}}\left[J(\Phi / \Psi)+\frac{\Phi}{\Psi} J^{\prime}(\Phi / \Psi)\right] \tag{4.4.10}
\end{align*}
$$

Moreover, one may readily verify that the associated Ermakov-Ray-Reid system has additional property of adopting a Hamiltonian form

$$
\begin{equation*}
\ddot{\Phi}=-\frac{\partial \mathcal{H}}{\partial \Phi}, \quad \ddot{\Psi}=-\frac{\partial \mathcal{H}}{\partial \Psi} \tag{4.4.11}
\end{equation*}
$$

that is

$$
\begin{align*}
& \ddot{\Phi}+4 \omega^{2} \Phi=\frac{1}{\Phi^{2} \Psi} \frac{d}{d(\Psi / \Phi)}\left[\frac{\Psi}{\Phi} J(\Phi / \Psi)\right] \\
& \ddot{\Psi}+4 \omega^{2} \Psi=\frac{1}{\Phi \Psi^{2}} \frac{d}{d(\Phi / \Psi)}\left[\frac{\Phi}{\Psi} J(\Phi / \Psi)\right] \tag{4.4.12}
\end{align*}
$$

Here $\mathcal{H}$ is the Hamiltonian invariant given by (4.4.7) and $J$ is determined by (4.4.9). Accordingly, the system (4.4.10) is integrable and analytical solution may be explicitly obtained via the procedure described in [100].

## Chapter 5

## A Rotating Magnetogasdynamic System: Integrable Hamiltonian Ermakov Reduction

### 5.1 Introduction

The purpose of this chapter is to study a $2+1$-dimensional magnetogasdynamic system incorporating rotation with a view of isolation integrable dynamical substructure. The Lundquist equations of magnetogasdynamics are intrinsically nonlinear and in general, are analytically intractable. However, under certain physically acceptable assumptions, analytical progress has been achieved. In [75-77], particular classes of time-dependent two-dimensional solutions were derived by Neukirch et al via a procedure in which the nonlinear acceleration terms in the Lundquist momentum equation either vanish or, are conservative. On the other hand, in [78, 79, 87], exact solutions were constructed by Lie group analysis and magnetogasdynamic substitution principles. By contrast, in recent work [98, 102, 111], an elliptic vortex-type ansatz has been introduced in 2+1-dimensional magnetogasdynamic contexts and underlying integrable

Ermakov-Ray-Reid structure has been isolated.

In this chapter, a novel two-parameter pressure-density ansatz which extends that assumed in $[77,98,111,118]$ is introduced to analyse a non-isothermal rotating magnetogasdynamic system. A novel power-type elliptic vortex procedure and two-parameter pressure-density ansatz are introduced and thereby, via elimination of the spatial dependence, reduction obtains to a finite-dimensional nonlinear dynamical system in time together with adjoined algebraic conditions. Time-modulated physical variables are introduced to derive analytical solutions of the original magnetogasdynamic system. It is demonstrated that the nonlinear dynamical system admits an underlying integrable Hamiltonian Ermakov structure and as well as a Lax pair representation analogous to that in [110] .

### 5.2 The Magnetogasdynamic System

Here, we consider a anisentropic magnetogasdynamic system incorporating rotation, namely,

$$
\begin{gather*}
\frac{\partial \rho}{\partial t}+\operatorname{div}(\rho \mathbf{q})=0  \tag{5.2.1}\\
\rho\left[\frac{\partial \mathbf{q}}{\partial t}+(\mathbf{q} \cdot \nabla) \mathbf{q}+f \mathbf{k} \times \mathbf{q}\right]-\mu \operatorname{curl} \mathbf{H} \times \mathbf{H}+\nabla p=0,  \tag{5.2.2}\\
\operatorname{div} \mathbf{H}=0,  \tag{5.2.3}\\
\frac{\partial \mathbf{H}}{\partial t}=\operatorname{curl}(\mathbf{q} \times \mathbf{H}),  \tag{5.2.4}\\
\frac{\partial S}{\partial t}+\mathbf{q} \cdot \nabla S=0 \tag{5.2.5}
\end{gather*}
$$

where the velocity $\mathbf{q}$ and magnetic field $\mathbf{H}$ are given by

$$
\begin{gather*}
\mathbf{q}=u \mathbf{i}+v \mathbf{j},  \tag{5.2.6}\\
\mathbf{H}=\nabla A \times \mathbf{k}+h \mathbf{k} \tag{5.2.7}
\end{gather*}
$$

and a polytropic gas law is assumed, namely

$$
\begin{equation*}
S=-\ln \rho+\frac{1}{\gamma-1} \ln T, \quad \gamma \neq 1 . \tag{5.2.8}
\end{equation*}
$$

together with

$$
\begin{equation*}
p=\rho R T . \tag{5.2.9}
\end{equation*}
$$

In the above, the magneto-gas density $\rho(\mathbf{x}, t)$, pressure $p(\mathbf{x}, t)$, entropy $S(\mathbf{x}, t)$, temperature $T(\mathbf{x}, t)$ and magnetic flux $A(\mathbf{x}, t)$ are all assumed to be dependent only on $\mathbf{x}=x \mathbf{i}+y \mathbf{j}$ and time $t$.

Insertion of the representation (5.2.7) into Faraday's law (5.2.4) produces the convective constraint

$$
\begin{equation*}
\frac{\partial A}{\partial t}+\mathbf{q} \cdot \nabla A=0 \tag{5.2.10}
\end{equation*}
$$

together with

$$
\begin{equation*}
\frac{\partial h}{\partial t}+\operatorname{div}(h \mathbf{q})=0 \tag{5.2.11}
\end{equation*}
$$

By virtue of the continuity equation (5.2.1), the latter holds automatically if we set

$$
\begin{equation*}
h=\lambda \rho, \quad \lambda \in \mathbb{R} . \tag{5.2.12}
\end{equation*}
$$

In the sequel, a novel two-parameter $(m, n)$ pressure-density ansatz is introduced via

$$
\begin{equation*}
p=\varepsilon_{0}(t) \rho^{2}+\varepsilon_{1}(t) \rho^{n}+\varepsilon_{2}(t) \rho^{m} \tag{5.2.13}
\end{equation*}
$$

It is recalled that the the relation $p \sim \rho$ was previously adopted in [77] and $p \sim \rho^{2}$ arisen in astrophysical contexts in [118], while recently, the parabolic pressure-density law $p=p_{0}+p_{1} \rho+p_{2} \rho^{2}$ has been employed in [111].

Insertion of the two-parameter $(m, n)$ ansatz (5.2.13) into (5.2.9), it is seen that the compatibility of $(p, \rho, T)$ relation yields

$$
\begin{equation*}
T=\varepsilon_{0}(t) \rho+\varepsilon_{1}(t) \rho^{n-1}+\varepsilon_{2}(t) \rho^{m-1} \tag{5.2.14}
\end{equation*}
$$

whence, on use of (5.2.8), it is required that the entropy distribution adopt the form

$$
\begin{equation*}
S=-\ln \rho+\frac{1}{\gamma-1} \ln \left(\varepsilon_{0}(t) \rho+\varepsilon_{1}(t) \rho^{n-1}+\varepsilon_{2}(t) \rho^{m-1}\right) \tag{5.2.15}
\end{equation*}
$$

The energy equation (5.2.5) now requires that

$$
\left(\rho_{t}+\mathbf{q} \cdot \nabla \rho\right)\left[\frac{\varepsilon_{0}+(n-1) \varepsilon_{1} \rho^{n-2}+(m-1) \varepsilon_{2} \rho^{m-2}}{(\gamma-1)\left(\varepsilon_{0} \rho+\varepsilon_{1} \rho^{n-1}+\varepsilon_{2} \rho^{m-1}\right)}-\frac{1}{\rho}\right]+\frac{\dot{\varepsilon}_{0} \rho+\dot{\varepsilon}_{1} \rho^{n-1}+\dot{\varepsilon}_{2} \rho^{m-1}}{(\gamma-1)\left(\varepsilon_{0} \rho+\varepsilon_{1} \rho^{n-1}+\varepsilon_{2} \rho^{m-1}\right)}=0
$$

whence, on use of the continuity equation (5.2.1)

$$
\begin{equation*}
-\operatorname{div} \mathbf{q}\left[(2-\gamma) \varepsilon_{0} \rho+(n-\gamma) \varepsilon_{1} \rho^{n-1}+(m-\gamma) \varepsilon_{2} \rho^{m-1}\right]+\dot{\varepsilon}_{0} \rho+\dot{\varepsilon}_{1} \rho^{n-1}+\dot{\varepsilon}_{2} \rho^{m-1}=0 \tag{5.2.16}
\end{equation*}
$$

that is

$$
\begin{equation*}
\frac{1}{\gamma-1} \frac{\dot{T}}{T}\left[(2-\gamma) \varepsilon_{0} \rho+(n-\gamma) \varepsilon_{1} \rho^{n-1}+(m-\gamma) \varepsilon_{2} \rho^{m-1}\right]+\dot{\varepsilon}_{0} \rho+\dot{\varepsilon}_{1} \rho^{n-1}+\dot{\varepsilon}_{2} \rho^{m-1}=0 \tag{5.2.17}
\end{equation*}
$$

On substitution of (5.2.7) and (5.2.13) into the momentum equation (5.2.2), it is seen that
$\frac{\partial \mathbf{q}}{\partial t}+(\mathbf{q} \cdot \nabla) \mathbf{q}+f \mathbf{k} \times \mathbf{q}+\frac{1}{\rho}\left[\mu\left(\nabla^{2} A\right)(\nabla A)+\varepsilon_{1} \nabla \rho^{n}\right]+\left(\mu \lambda^{2}+2 \varepsilon_{0}\right) \nabla \rho+\frac{\varepsilon_{2}}{\rho} \nabla \rho^{m}=0$
together with

$$
\begin{equation*}
A_{y} \rho_{x}-A_{x} \rho_{y}=0 \tag{5.2.19}
\end{equation*}
$$

whence

$$
\begin{equation*}
A=A(\rho, t) . \tag{5.2.20}
\end{equation*}
$$

Attention here is restricted to the separable case

$$
\begin{equation*}
A=\Phi(\rho) \Psi(t) \tag{5.2.21}
\end{equation*}
$$

whence, on substitution into (5.2.10) and use of the continuity equation (5.2.1), yields

$$
\begin{equation*}
\dot{\Psi}(t)=\frac{\rho \Phi^{\prime}(\rho)}{\Phi(\rho)} \Psi(t) \operatorname{div} \mathbf{q} . \tag{5.2.22}
\end{equation*}
$$

Here, we proceed with

$$
\begin{equation*}
\Phi=\rho^{n} \tag{5.2.23}
\end{equation*}
$$

where $n$ is the parameter involving in the relation (5.2.14), so that

$$
\begin{equation*}
\dot{\Psi}=n \Psi \operatorname{div} \mathbf{q} \tag{5.2.24}
\end{equation*}
$$

and

$$
\begin{equation*}
A=\rho^{n} \Psi(t) . \tag{5.2.25}
\end{equation*}
$$

Hence, as in the case of the spinning non-conducting gas cloud analysis of Ovsiannikov [81] and Dyson [28], the divergence of the velocity is dependent only on time. Moveover, the relation (5.2.16) shows that

$$
-\frac{1}{n} \frac{\dot{\Psi}}{\Psi}\left[(2-\gamma) \varepsilon_{0} \rho+(n-\gamma) \varepsilon_{1} \rho^{n-1}+(m-\gamma) \varepsilon_{2} \rho^{m-1}\right]+\dot{\varepsilon}_{0} \rho+\dot{\varepsilon}_{1} \rho^{n-1}+\dot{\varepsilon}_{2} \rho^{m-1}=0
$$

and it is observed that this condition holds identically with

$$
\begin{equation*}
\varepsilon_{0}=\alpha_{0} \Psi^{\frac{2-\gamma}{n}}, \quad \varepsilon_{1}=\alpha_{1} \Psi^{\frac{n-\gamma}{n}}, \quad \varepsilon_{2}=\alpha_{2} \Psi^{\frac{m-\gamma}{n}} \tag{5.2.26}
\end{equation*}
$$

where $\alpha_{i}(i=1,2,3)$ are arbitrary constants of integration.

In addition, the isentropic condition (5.2.5) together with the polytropic gas law (5.2.8) and the continuity equation (5.2.1) show that

$$
\begin{equation*}
\operatorname{div} \mathbf{q}=\frac{1}{1-\gamma} \frac{\dot{T}}{T} \tag{5.2.27}
\end{equation*}
$$

whence, on use of (5.2.14),

$$
\begin{equation*}
(2-\gamma) \operatorname{div} \mathbf{q}=\frac{\dot{\varepsilon}_{0}}{\varepsilon_{0}}, \quad(n-\gamma) \operatorname{div} \mathbf{q}=\frac{\dot{\varepsilon}_{1}}{\varepsilon_{1}}, \quad(m-\gamma) \operatorname{div} \mathbf{q}=\frac{\dot{\varepsilon}_{2}}{\varepsilon_{2}} . \tag{5.2.28}
\end{equation*}
$$

It is seen that in view of (5.2.24), the final results of $\varepsilon_{i}$ given by (5.2.28) coincide with that given by (5.2.26).

In summary, the magnetogasdynamic system now reduces to consideration of the nonlinear coupled system

$$
\begin{gather*}
\frac{\partial \rho}{\partial t}+\operatorname{div}(\rho \mathbf{q})=0 \\
\dot{\Psi}=n \Psi \operatorname{div} \mathbf{q} \tag{5.2.29}
\end{gather*}
$$

$\frac{\partial \mathbf{q}}{\partial t}+(\mathbf{q} \cdot \nabla) \mathbf{q}+f \mathbf{k} \times \mathbf{q}+\frac{1}{\rho}\left(\mu \Psi^{2} \nabla^{2} \rho^{n}+\varepsilon_{1}\right) \nabla \rho^{n}+\left(\mu \lambda^{2}+2 \varepsilon_{0}\right) \nabla \rho+\frac{\varepsilon_{2} m}{m-1} \nabla \rho^{m-1}=0$. together with the additional algebraic conditions (5.2.26). It is this reduced system (5.2.29) that will be the subject of the subsequent sections. The inherent nonlinearity of the system (5.2.29) remains a major impediment to analytic progress. It is noted that this system is overdetermined since $(5.2 .29)_{3}$ is implicitly constrained by the requirement that $\operatorname{div} \mathbf{q}$ be a function of $t$ only.

### 5.3 Analytical Solutions of the Magnetogasdynamic System

In this section, an important extension of the elliptic vortex procedure of [96] which involves the pressure-density parameter $m$ is introduced into the reduced magnetogasdynamic system (5.2.29). The choices of the two pressure-density parameters involved in the pressure term and time modulated variables are key to the construction of the exact analytical solution.

### 5.3.1 Removal of Spacial Dependence

An integrable nonlinear dynamical sub-system is now sought via a power-type elliptic vortex ansatz involving a pressure-density parameter $m$ (An, Rogers and Schief [4])

$$
\begin{array}{ll}
\mathbf{q}=\mathbf{L}(t) \mathbf{x}+\mathbf{M}(t),  \tag{5.3.1}\\
\rho=\left(\mathbf{x}^{T} \mathbf{E}(t) \mathbf{x}+\rho_{0}\right)^{m-1},(m \neq 1) & \mathbf{x}=\binom{x-\bar{q}(t)}{y-\bar{p}(t)}
\end{array}
$$

where

$$
\mathbf{L}(t)=\left(\begin{array}{cc}
u_{1}(t) & u_{2}(t)  \tag{5.3.2}\\
v_{1}(t) & v_{2}(t)
\end{array}\right), \quad \mathbf{E}(t)=\left(\begin{array}{cc}
a(t) & b(t) \\
b(t) & c(t)
\end{array}\right), \quad \mathbf{M}(t)=\binom{\dot{q}(t)}{\dot{p}(t)} .
$$

It is emphasized that the precluded case with the parameters $m=1$ and $n=0$ coincides with what has been discussed in [111]. Here, we proceed with the general case $m \neq 1$ and $n \neq 0$.

Insertion of (5.3.1) into the continuity equation (5.2.29) ${ }_{1}$ yields

$$
\left(\begin{array}{c}
\dot{a}  \tag{5.3.3}\\
\dot{b} \\
\dot{c}
\end{array}\right)+\left(\begin{array}{ccc}
2 u_{1}+(m-1)\left(u_{1}+v_{2}\right) & 2 v_{1} & 0 \\
u_{2} & m\left(u_{1}+v_{2}\right) & v_{1} \\
0 & 2 u_{2} & 2 v_{2}+(m-1)\left(u_{1}+v_{2}\right)
\end{array}\right)\left(\begin{array}{l}
a \\
b \\
c
\end{array}\right)=\mathbf{0}
$$

together with

$$
\begin{equation*}
\dot{\rho}_{0}+\rho_{0}(m-1)\left(u_{1}+v_{2}\right)=0 . \tag{5.3.4}
\end{equation*}
$$

In order to reduce the system (5.2.29) to a amenable form, we proceed with

$$
\begin{equation*}
n=m-1 \tag{5.3.5}
\end{equation*}
$$

and

$$
\begin{align*}
& 2 \varepsilon_{0}+\mu \lambda^{2}=0,  \tag{5.3.6}\\
& \varepsilon_{1}+2 \mu \Psi^{2}(a+c)=0,
\end{align*}
$$

so that the two terms $\nabla \rho$ and $\nabla \rho^{n}$ vanish and the momentum equation $(5.2 .29)_{3}$ is reducible to

$$
\begin{equation*}
\frac{\partial \mathbf{q}}{\partial t}+(\mathbf{q} \cdot \nabla) \mathbf{q}+f \mathbf{k} \times \mathbf{q}+\frac{\varepsilon_{2} m}{m-1} \nabla \rho^{m-1}=0 \tag{5.3.7}
\end{equation*}
$$

It is observed that the consistency of $(5.3 .6)_{1}$ with $(5.2 .26)_{1}$ requires the adiabatic index $\gamma=2$.

Substitution of (5.3.1) into (5.3.7) now gives

$$
\left(\begin{array}{c}
\dot{u}_{1}  \tag{5.3.8}\\
\dot{u}_{2} \\
\dot{v}_{1} \\
\dot{v}_{2}
\end{array}\right)+\left(\begin{array}{cc}
\mathbf{L}^{T} & -f \mathbf{I} \\
f \mathbf{I} & \mathbf{L}^{T}
\end{array}\right)\left(\begin{array}{c}
u_{1} \\
u_{2} \\
v_{1} \\
v_{2}
\end{array}\right)+2 \frac{\varepsilon_{2} m}{m-1}\left(\begin{array}{l}
a \\
b \\
b \\
c
\end{array}\right)=\mathbf{0}
$$

augmented by the linear auxiliary equations

$$
\begin{equation*}
\ddot{\bar{p}}+f \dot{\bar{q}}=0, \quad \ddot{\bar{q}}-f \dot{\bar{p}}=0 . \tag{5.3.9}
\end{equation*}
$$

At this stage, it is noted that the spacial dependence of the original magnetogasdynamic system has been removed. Hence, the solution of the magnetogasdynamic system is encoded in the seven-dimensional time-dependent nonlinear system (5.3.3) and (5.3.8). Once the solution of the latter is known, the quantities $\rho_{0}$ and $\Psi$ are obtained via integration of (5.3.4) and $(5.2 .29)_{2}$, that is

$$
\begin{equation*}
\dot{\Psi}=(m-1)\left(u_{1}+v_{2}\right) \Psi . \tag{5.3.10}
\end{equation*}
$$

However, the conditions (5.2.26) and (5.3.6) remain. The admissibility of these constraints on the dynamical system will be examined in the sequel.

### 5.3.2 Canonical Variables

In what follows, it proves convenient to proceed in terms of new variables as previously employed in a hydrodynamics context in [96,100], namely

$$
\begin{align*}
& G=u_{1}+v_{2}, \quad G_{R}=\frac{1}{2}\left(v_{1}-u_{2}\right), \\
& G_{S}=\frac{1}{2}\left(v_{1}+u_{2}\right), \quad G_{N}=\frac{1}{2}\left(u_{1}-v_{2}\right),  \tag{5.3.11}\\
& B=a+c, \quad B_{S}=b, \quad B_{N}=\frac{1}{2}(a-c) .
\end{align*}
$$

Here, $G$ and $G_{R}$ correspond, in turn, to the divergence and spin of the velocity field, while $G_{S}$ and $G_{N}$ represent shear and normal deformation rates.

On use of the expressions (5.3.11), the system (5.3.3)-(5.3.4) together with (5.3.8) produce the eight-dimensional nonlinear dynamical system

$$
\begin{align*}
& \dot{\rho}_{0}+(m-1) \rho_{0} G=0, \\
& \dot{B}+m B G+4\left(B_{N} G_{N}+B_{S} G_{S}\right)=0, \\
& \dot{B}_{S}+m B_{S} G+B G_{S}-2 B_{N} G_{R}=0, \\
& \dot{B}_{N}+m B_{N} G+B G_{N}+2 B_{S} G_{R}=0, \\
& \dot{G}+\frac{1}{2} G^{2}+2\left(G_{N}^{2}+G_{S}^{2}-G_{R}^{2}\right)-2 f G_{R}+2 \frac{\varepsilon_{2} m}{m-1} B=0,  \tag{5.3.12}\\
& \dot{G}_{N}+G G_{N}-f G_{S}+2 \frac{\varepsilon_{2} m}{m-1} B_{N}=0, \\
& \dot{G}_{S}+G G_{S}+f G_{N}+2 \frac{\varepsilon_{2} m}{m-1} B_{S}=0, \\
& \dot{G}_{R}+G G_{R}+\frac{1}{2} f G=0
\end{align*}
$$

together with

$$
\begin{equation*}
\dot{\Psi}=(m-1) \Psi G . \tag{5.3.13}
\end{equation*}
$$

It is observed that the introduction of the pressure-density parameters $m$ and $\varepsilon_{2}$ leads to a generalisation of the nonlinear dynamical systems obtained in other various contexts in $[96,100-102,110,111]$.

The form of (5.3.12) ${ }_{4}$ suggests introducing a function $\Omega$ via

$$
\begin{equation*}
G=\frac{2 \dot{\Omega}}{\Omega} \tag{5.3.14}
\end{equation*}
$$

so that $(5.3 .12)_{1}$ and $(5.3 .12)_{8}$ show that

$$
\begin{equation*}
\rho_{0}=\frac{c_{\mathrm{I}}}{\Omega^{2(m-1)}} \tag{5.3.15}
\end{equation*}
$$

and

$$
\begin{equation*}
G_{R}=\frac{c_{0}}{\Omega^{2}}-\frac{1}{2} f \tag{5.3.16}
\end{equation*}
$$

The relation (5.3.13) produces

$$
\begin{equation*}
\Psi=\nu \Omega^{2(m-1)} \tag{5.3.17}
\end{equation*}
$$

where, in the above, $c_{0}, c_{\mathrm{I}}$ and $\nu$ are arbitrary constants of integration.

Two conditions which are key to the subsequent development and which may be established by appeal to the original system (5.3.12) are now recorded. They are readily validated by symbolic computation and are embodied in the following theorem:

## Theorem I

$$
\begin{align*}
& \dot{M}^{*}+(m+1) G M^{*}=0  \tag{5.3.18}\\
& \dot{Q}^{*}+(m+1) G Q^{*}=0
\end{align*}
$$

where

$$
\begin{align*}
M^{*}= & 2\left(B_{N} G_{S}-B_{S} G_{N}\right)-B\left(G_{R}+\frac{f}{2}\right), \quad \triangle=\frac{1}{4} B^{2}-B_{S}^{2}-B_{N}^{2} \\
Q^{*}= & -B\left(G_{S}^{2}+G_{N}^{2}+G_{R}^{2}+\frac{1}{4} G^{2}\right)+4 G_{R}\left(B_{N} G_{S}-B_{S} G_{N}\right)  \tag{5.3.19}\\
& +2 G\left(B_{S} G_{S}+B_{N} G_{N}\right)+4 \frac{\varepsilon_{2} m}{m-1} \triangle-4 \frac{m}{m-1} \triangle \Omega^{m-1} \int \dot{\varepsilon}_{2} \Omega^{1-m} d t
\end{align*}
$$

It is seen that the relation (5.3.19) in Theorem I generalises the results obtained in [96, 100-102].

New $\Omega$-modulated variables involving the pressure-density parameter $m$ are now introduced according to

$$
\begin{gather*}
\bar{B}=\Omega^{2 m} B, \quad \bar{B}_{S}=\Omega^{2 m} B_{S}, \quad \bar{B}_{N}=\Omega^{2 m} B_{N}  \tag{5.3.20}\\
\bar{G}_{S}=\Omega^{2} G_{S}, \quad \bar{G}_{N}=\Omega^{2} G_{N}
\end{gather*}
$$

whence the system (5.3.12) reduces to

$$
\begin{align*}
& \dot{\bar{B}}+\frac{4\left(\bar{B}_{N} \bar{G}_{N}+\bar{B}_{S} \bar{G}_{S}\right)}{\Omega^{2}}=0 \\
& \dot{\bar{B}}_{S}+f \bar{B}_{N}+\frac{\bar{B} \bar{G}_{S}-2 c_{0} \bar{B}_{N}}{\Omega^{2}}=0 \\
& \dot{\bar{B}}_{N}-f \bar{B}_{S}+\frac{\bar{B} \bar{G}_{N}+2 c_{0} \bar{B}_{S}}{\Omega^{2}}=0  \tag{5.3.21}\\
& \dot{\bar{G}}_{S}+f \bar{G}_{N}+\frac{2 \varepsilon_{2} m}{m-1} \frac{\bar{B}_{S}}{\Omega^{2(m-1)}}=0 \\
& \dot{\bar{G}}_{N}-f \bar{G}_{S}+\frac{2 \varepsilon_{2} m}{m-1} \frac{\bar{B}_{N}}{\Omega^{2(m-1)}}=0
\end{align*}
$$

augmented by the relations (5.3.15) and (5.3.16) together with a nonlinear equation for $\Omega$, namely

$$
\begin{equation*}
\Omega^{3} \ddot{\Omega}+\frac{1}{4} f^{2} \Omega^{4}+\bar{G}_{N}^{2}+\bar{G}_{S}^{2}-c_{0}^{2}+\frac{\varepsilon_{2} m}{m-1} \frac{\bar{B}}{\Omega^{2(m-2)}}=0 . \tag{5.3.22}
\end{equation*}
$$

The seven-dimensional dynamical system (5.3.21), (5.3.22) together with the timemodulated constraints given by (5.2.26) and (5.3.6) will be analysed in detail in the following.

### 5.3.3 The Constraints and First integrals

We now consider the algebraic conditions of $\varepsilon_{i},(i=1,2,3)$ given by (5.2.26) and (5.3.6). It is seen that the consistency of $\varepsilon_{0}$ requires the adiabatic index $\gamma=2$. Comparison of the two expressions for $\varepsilon_{1}$ in $(5.2 .26)_{2}$ with $(5.3 .6)_{2}$, namely

$$
\begin{equation*}
\alpha_{1} \Psi^{\frac{n-2}{n}}+2 \mu \Psi^{2}(a+c)=0 \tag{5.3.23}
\end{equation*}
$$

from the relations (5.3.5), (5.3.17) and $(5.3 .20)_{1}$, shows that

$$
\begin{equation*}
\alpha_{1} \nu^{\frac{m-3}{m-1}}+2 \mu \nu^{2} \Omega^{2} \bar{B}=0 \tag{5.3.24}
\end{equation*}
$$

whence

$$
\begin{equation*}
\nu=0 \quad \text { or } \quad \Omega^{2} \bar{B}=-\frac{\alpha_{1}}{2 \mu} \nu^{\frac{1+m}{1-m}}:=\delta . \tag{5.3.25}
\end{equation*}
$$

In the former case, the magnetic flux $A$ vanishes so that the magnetic field $H$ is purely transverse and the nonlinear dynamical system (5.3.21)-(5.3.22) is unconstrained. In the latter case, the equation $(5.3 .21)_{1}$ implies that the dynamical system (5.3.21)-(5.3.22) is constrained by

$$
\begin{equation*}
\bar{B}_{N} \bar{G}_{N}+\bar{B}_{S} \bar{G}_{S}-\delta \dot{\Omega} / \Omega=0 \tag{5.3.26}
\end{equation*}
$$

When it comes to consider the condition $\varepsilon_{2}$, on use of $(5.3 .17)$, the relation $(5.2 .26)_{3}$ shows that

$$
\begin{equation*}
\varepsilon_{2}=\alpha_{2} \nu^{\frac{m-2}{m-1}} \Omega^{2(m-2)} \tag{5.3.27}
\end{equation*}
$$

that is

$$
\begin{equation*}
\varepsilon_{2}=\frac{\alpha(m-1)}{m} \Omega^{2(m-2)} . \tag{5.3.28}
\end{equation*}
$$

In summary, under the constraint of (5.3.28), the nonlinear dynamical system (5.3.21) admits four integrals of motion, namely

$$
\begin{gather*}
\bar{B}_{S}^{2}+\bar{B}_{N}^{2}-\frac{\bar{B}^{2}}{4}=c_{\mathrm{II}},  \tag{5.3.29}\\
\bar{G}_{S}^{2}+\bar{G}_{N}^{2}-\alpha \bar{B}=c_{\mathrm{III}},  \tag{5.3.30}\\
2\left(\bar{B}_{N} \bar{G}_{S}-\bar{B}_{S} \bar{G}_{N}\right)-c_{0} \bar{B}=c_{\mathrm{IV}},  \tag{5.3.31}\\
2\left(G_{R}+c_{0} \bar{B}\right)+2 G\left(\bar{B}_{S} \bar{G}_{S}+\bar{B}_{N} \bar{G}_{N}\right)+4 \alpha c_{\mathrm{II}} \Omega^{-2} \frac{m-1}{m-3}-\bar{B} \Omega^{2}\left(\frac{\bar{G}_{N}^{2}}{\Omega^{4}}+\frac{G^{2}}{4}+G_{R}^{2}\right)=c_{\mathrm{V}}, \tag{5.3.32}
\end{gather*}
$$

where $c_{\mathrm{II}}, c_{\mathrm{III}}, c_{\mathrm{IV}}$ and $c_{\mathrm{V}}$ are constants of integration.

### 5.3.4 A Parametrisation

The integrals of motion (5.3.29) and (5.3.30) may be conveniently parametrised, in turn, according to

$$
\begin{array}{ll}
\bar{B}_{S}=-\sqrt{c_{\mathrm{II}}+\frac{1}{4} \bar{B}^{2}} \cos \phi(t), & \bar{B}_{N}=-\sqrt{c_{\mathrm{II}}+\frac{1}{4} \bar{B}^{2}} \sin \phi(t),  \tag{5.3.33}\\
\bar{G}_{S}=-\sqrt{c_{\mathrm{III}}+\bar{B}} \sin \theta(t), \quad \bar{G}_{N}=+\sqrt{c_{\mathrm{III}}+\bar{B}} \cos \theta(t) .
\end{array}
$$

Substitution of the parametrisation (5.3.33) into (5.3.21) $)_{1}$ yields

$$
\begin{equation*}
\dot{\bar{B}}+\frac{4}{\Omega^{2}} \sqrt{\left(c_{\mathrm{II}}+\bar{B}^{2} / 4\right)\left(c_{\mathrm{III}}+\alpha \bar{B}\right)} \sin (\theta-\phi)=0 \tag{5.3.34}
\end{equation*}
$$

while conditions (5.3.21) $)_{2,3}$ reduce to a single relation, namely

$$
\begin{equation*}
\left(\dot{\phi}-f+\frac{2 c_{0}}{\Omega^{2}}\right) \sqrt{c_{\mathrm{II}}+\frac{\bar{B}^{2}}{4}}-\frac{\bar{B}}{\Omega^{2}} \sqrt{c_{\mathrm{III}}+\alpha \bar{B}} \cos (\theta-\phi)=0 \tag{5.3.35}
\end{equation*}
$$

and similarly, $(5.3 .21)_{4,5}$ produce another single requirement

$$
\begin{equation*}
(f-\dot{\theta}) \sqrt{c_{\mathrm{III}}+\alpha \bar{B}}+\frac{2 \alpha}{\Omega^{2}} \sqrt{c_{\mathrm{II}}+\frac{\bar{B}^{2}}{4}} \cos (\theta-\phi)=0 . \tag{5.3.36}
\end{equation*}
$$

Substitution of (5.3.33) into (5.3.31) yields

$$
\begin{equation*}
c_{0} \bar{B}=-c_{\mathrm{IV}}+2 \sqrt{\left(c_{\mathrm{II}}+\bar{B}^{2} / 4\right)\left(c_{\mathrm{III}}+\alpha \bar{B}\right)} \cos (\theta-\phi) . \tag{5.3.37}
\end{equation*}
$$

Elimination of $\theta-\phi$ in (5.3.35) and (5.3.36) respectively, yields

$$
\begin{equation*}
\dot{\phi}=f+\frac{2}{\Omega^{2}}\left(\frac{\delta\left(c_{0} \delta+c_{\mathrm{IV}} \Omega^{2}\right)}{\delta^{2}+4 c_{\mathrm{II}} \Omega^{4}}-c_{0}\right) \tag{5.3.38}
\end{equation*}
$$

and

$$
\begin{equation*}
\dot{\theta}=f-\frac{\alpha}{\Omega^{2}}\left(\frac{c_{0} \delta+c_{\mathrm{IV}} \Omega^{2}}{\alpha \delta^{2}+c_{\mathrm{III}} \Omega^{2}}\right) . \tag{5.3.39}
\end{equation*}
$$

It remains to consider the nonlinear equation (5.3.22) for $\Omega$, namely

$$
\Omega^{3} \ddot{\Omega}+\frac{f^{2}}{4} \Omega^{4}+c_{\mathrm{III}}-c_{0}^{2}+\frac{\varepsilon_{2} m}{m-1} \frac{\bar{B}}{\Omega^{2(m-2)}}=0
$$

whence, by virtue of $(5.3 .25)_{2}$ and (5.3.28), produces to a generalised Steen-Ermakov equation

$$
\begin{equation*}
\ddot{\Omega}+\frac{1}{4} f^{2} \Omega=\frac{c_{0}^{2}-c_{\mathrm{III}}}{\Omega^{3}}-\frac{2 \alpha \delta}{\Omega^{5}} . \tag{5.3.40}
\end{equation*}
$$

On use of Theorem I, it is readily shown that

$$
\begin{equation*}
\left(\ddot{\Omega^{2}} \bar{B}\right)+f^{2} \Omega^{2} \bar{B}=-2\left(Q^{*}+f M^{*}\right) \Omega^{2(m+1)}=-2\left(c_{V}+f c_{\mathrm{IV}}\right) \tag{5.3.41}
\end{equation*}
$$

whence

$$
\Omega^{2} \bar{B}= \begin{cases}c_{\mathrm{VI}} \cos f t+c_{\mathrm{VII}} \sin f t-2\left(c_{\mathrm{V}}+f c_{\mathrm{IV}}\right) / f^{2}, & f \neq 0  \tag{5.3.42}\\ -c_{\mathrm{V}} t^{2}+c_{\mathrm{VI}} t+c_{\mathrm{VII}}, & f=0 .\end{cases}
$$

On elimination of $\theta-\phi$ and $\Omega$ in (5.3.34) via the relations (5.3.37) and (5.3.42) it is seen that, if $\bar{B} \neq$ const then $\bar{B}$ obeys the elliptic integral relation

$$
\begin{align*}
& \int_{c_{\mathrm{VIII}}}^{\bar{B}} \frac{d \bar{B}^{*}}{\bar{B}^{*} \sqrt{\left(\bar{B}^{* 2}+4 c_{\mathrm{II}}\right)\left(c_{\mathrm{III}}+\alpha \bar{B}^{*}\right)-\left(c_{0} \bar{B}^{*}+c_{\mathrm{IV}}\right)^{2}}} \\
& = \begin{cases}-2 \int_{0}^{t} \frac{d t^{*}}{c_{\mathrm{VI}} \cos f t^{*}+c_{\mathrm{VII}} \sin f t^{*}-2\left(c_{\mathrm{V}}+f c_{\mathrm{IV}}\right) / f^{2}}, & \text { if } f \neq 0 \\
-2 \int_{0}^{t} \frac{d t^{*}}{-c_{\mathrm{V}} t^{* 2}+c_{\mathrm{VI}} t^{*}+c_{\mathrm{VII}}}, & \text { if } f=0\end{cases} \tag{5.3.43}
\end{align*}
$$

where $\left.\bar{B}\right|_{t=0}=c_{\text {VIII }}$. In the present case, when the gas law (5.2.9) prevails, it is seen that (5.3.41) holds automatically by virtue of $(5.3 .25)_{2}$ with

$$
\Omega^{2} \bar{B}=\delta=\left\{\begin{array}{cl}
-2\left(c_{\mathrm{V}}+f c_{\mathrm{IV}}\right) / f^{2}, & f \neq 0  \tag{5.3.44}\\
c_{\mathrm{VII}}, & f=0
\end{array}\right.
$$

We now turn to consider the compatibility of the nonlinear equation (5.3.40) and (5.3.44) with the elliptic integral expression involving $\bar{B}$.

It is observed that the first integral of (5.3.40) is

$$
\begin{equation*}
\dot{\Omega}^{2}+\frac{1}{4} f^{2} \Omega^{2}+\frac{\left(c_{0}^{2}-c_{\mathrm{III}}\right)}{\Omega^{2}}-\frac{\alpha \delta}{\Omega^{4}}+k=0 \tag{5.3.45}
\end{equation*}
$$

where $k$ is a constant of integration. While the elliptic integral relation (5.3.43) shows

$$
\begin{equation*}
\dot{\bar{B}}^{2}+\frac{4}{\Omega^{4}}\left(c_{0} \bar{B}+c_{\mathrm{IV}}\right)^{2}=\frac{4}{\Omega^{4}}\left(4 c_{\mathrm{II}}+\bar{B}^{2}\right)\left(c_{\mathrm{III}}+\alpha \bar{B}\right) \tag{5.3.46}
\end{equation*}
$$

whence, on use of (5.3.44),

$$
\begin{equation*}
\delta^{2} \dot{\Omega}^{2}+\frac{\left(c_{0}^{2}-c_{\mathrm{III}}\right) \delta^{2}}{\Omega^{2}}+\left(c_{\mathrm{IV}}^{2}-4 c_{\mathrm{II}} c_{\mathrm{III}}\right) \Omega^{2}-\frac{\alpha \delta^{3}}{\Omega^{4}}+2 c_{0} c_{\mathrm{IV}}-4 \alpha c_{\mathrm{II}} \delta=0 . \tag{5.3.47}
\end{equation*}
$$

Thus, compatibility requires that

$$
\begin{equation*}
\delta^{2} f^{2}=4\left(c_{\mathrm{IV}}^{2}-4 c_{\mathrm{II}} c_{\mathrm{III}}\right) \tag{5.3.48}
\end{equation*}
$$

and

$$
\begin{equation*}
k \delta^{2}=2 c_{0} c_{\mathrm{IV}}-4 \alpha c_{\mathrm{II}} \delta . \tag{5.3.49}
\end{equation*}
$$

To conclude, if $\bar{B}=\delta \Omega^{-2}$ is determined explicitly by the elliptic integral representation (5.3.43), then $\Omega$ is given by the relation (5.3.44) and the angles $\phi$ and $\theta$ are obtained by integration, in turn, of (5.3.38) and (5.3.39). Thus, a multi-parameter class of exact solutions of the original 2+1-dimensional magnetogasdynamic system is generated with the velocity components $u_{1}, u_{2}, v_{1}, v_{2}$ and the quantities $a, b, c$ in the density relation given, in turn, by

$$
\begin{array}{ll}
u_{1}=\frac{\dot{\Omega}}{\Omega}+\frac{1}{\Omega^{3}} \sqrt{\alpha \delta+c_{\mathrm{III}}^{\Omega^{2}} \cos \theta(t),} & v_{1}=\frac{c_{0}}{\Omega^{2}}-\frac{f}{2}-\frac{1}{\Omega^{3}} \sqrt{\alpha \delta+c_{\mathrm{III}} \Omega^{2}} \sin \theta(t), \\
u_{2}=-\frac{c_{0}}{\Omega^{2}}+\frac{f}{2}-\frac{1}{\Omega^{3}} \sqrt{\alpha \delta+c_{\mathrm{III}} \Omega^{2}} \sin \theta(t), & v_{2}=\frac{\dot{\Omega}}{\Omega}-\frac{1}{\Omega^{2}} \sqrt{\alpha \delta+c_{\mathrm{III}} \Omega^{2}} \cos \theta(t) \tag{5.3.50}
\end{array}
$$

together with

$$
\begin{gather*}
a=\frac{1}{2 \Omega^{2(m+1)}}\left[\delta-\sqrt{4 c_{\mathrm{II}} \Omega^{4}+\delta^{2}} \sin \phi(t)\right], \quad b=\frac{1}{2 \Omega^{2(m+1)}} \sqrt{4 c_{\mathrm{II}} \Omega^{4}+\delta^{2}} \cos \phi(t), \\
c=\frac{1}{\Omega^{2(m+1)}}\left[\delta+\sqrt{4 c_{\mathrm{II}} \Omega^{4}+\delta^{2}} \sin \phi(t)\right], \\
\rho_{0}=\frac{c_{\mathrm{I}}}{\Omega^{2(m-1)}} . \tag{5.3.51}
\end{gather*}
$$

The magnetic flux $A$ is given by

$$
\begin{equation*}
A=\nu \rho^{m-1} \Omega^{2(m-1)}=\nu\left[a(x-\bar{q})^{2}+2 b(x-\bar{q})(y-\bar{p})+c(y-\bar{p})^{2}+\rho_{0}\right] \Omega^{2(m-1)} \tag{5.3.52}
\end{equation*}
$$

and the entropy distribution is given by

$$
\begin{equation*}
S=\ln (T / \rho) \tag{5.3.53}
\end{equation*}
$$

where $T$ is determined via

$$
\begin{equation*}
T=\varepsilon_{0} \rho^{2}+\varepsilon_{1} \rho^{m-1}+\varepsilon_{2} \rho^{m} . \tag{5.3.54}
\end{equation*}
$$

### 5.4 Hamiltonian Ermakov Structure

It is now demonstrated that the nonlinear dynamical system (5.3.12) may also be reformulated in terms of a Ermakov-Ray-Reid system which turns out to be Hamiltonian,
leading to an additional hidden first integral.

Here, it proves convenient to proceed with $\bar{p}(t)=\bar{q}(t)=0$ in the ansatz (5.3.1). However, the terms are readily re-introduced by use of a Lie group invariance of the magneto-gasdynamic system.

The semi-axes of the time-modulated ellipse

$$
\begin{gather*}
a(t) x^{2}+2 b(t) x y+c y^{2}+h_{0}(t)=0  \tag{5.4.1}\\
\left(a c-b^{2}>0\right)
\end{gather*}
$$

are given by

$$
\begin{equation*}
\Phi=\sqrt{\frac{2 \rho_{0}}{\sqrt{(a-c)^{2}+4 b^{2}}-(a+c)}} \tag{5.4.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\Psi=\sqrt{\frac{2 \rho_{0}}{-\sqrt{(a-c)^{2}+4 b^{2}}-(a+c)}} \tag{5.4.3}
\end{equation*}
$$

Substitution of the relation (5.3.51) delivers

$$
\begin{align*}
& \Phi=\sqrt{-\frac{c_{\mathrm{I}}}{2 c_{\mathrm{II}}}} \sqrt{-\delta-\sqrt{4 c_{\mathrm{II}} \Omega^{4}+\delta^{2}}}  \tag{5.4.4}\\
& \Psi=\sqrt{-\frac{c_{\mathrm{I}}}{2 c_{\mathrm{II}}}} \sqrt{-\delta+\sqrt{4 c_{\mathrm{II}} \Omega^{4}+\delta^{2}}} \tag{5.4.5}
\end{align*}
$$

where it is required that

$$
\begin{equation*}
c_{\mathrm{I}}>0, \quad c_{\mathrm{II}}<0, \quad \delta<0, \quad \delta^{2}+4 c_{\mathrm{II}} \Omega^{4}>0 . \tag{5.4.6}
\end{equation*}
$$

It is readily established that the semi-axes $\Phi, \Psi$ of the ellipse (5.4.1) are governed by a Ermakov-Ray-Reid system, namely

$$
\begin{align*}
& \ddot{\Phi}+\frac{1}{4} f^{2} \Phi=\frac{1}{\Phi^{2} \Psi}\left[\frac{Z Z^{\prime}}{1+(\Psi / \Phi)^{2}}-\left(\frac{\Psi}{\Phi}\right) \frac{\left(Z^{2}+k / 4\right)}{\left[1+(\Psi / \Phi)^{2}\right]^{2}}\right]  \tag{5.4.7}\\
& \ddot{\Psi}+\frac{1}{4} f^{2} \Psi=\frac{1}{\Phi \Psi^{2}}\left[-\frac{Z Z^{\prime}}{1+(\Psi / \Phi)^{2}}-\left(\frac{\Phi}{\Psi}\right) \frac{\left(Z^{2}+k / 4\right)}{\left[1+(\Phi / \Psi)^{2}\right]^{2}}\right]
\end{align*}
$$

where

$$
\begin{equation*}
Z=Z(\Phi / \Psi)=\Psi \dot{\Phi}-\dot{\Psi} \Phi=\frac{2 c_{\mathrm{I}}}{\Omega \sqrt{-c_{\mathrm{II}}}} \sqrt{\frac{\left(\delta^{2}+4 c_{\mathrm{II}} \Omega^{4}\right)\left(\alpha \delta+c_{\mathrm{III}} \Omega^{2}\right)-\Omega^{2}\left(c_{0} \delta+c_{\mathrm{IV}} \Omega^{2}\right)^{2}}{\delta^{2}+4 c_{\mathrm{II}} \Omega^{4}}} \tag{5.4.8}
\end{equation*}
$$

and $\Omega$ is given in terms of the ratio of the semi-axes via the relation

$$
\begin{equation*}
\Omega=\left(-\frac{\delta^{2}}{c_{\mathrm{II}}}\right)^{1 / 4}\left(\frac{\Psi}{\Phi}+\frac{\Phi}{\Psi}\right)^{-1 / 2} \tag{5.4.9}
\end{equation*}
$$

In the above, $k$ is the constant of integration given by

$$
\begin{equation*}
k=\left(\frac{c_{\mathrm{I}}}{c_{\mathrm{II}}}\right)^{2}\left(f^{2}\left(c_{\mathrm{IV}}^{2}+c_{\mathrm{VII}}^{2}\right)-\frac{4}{f^{2}}\left(c_{\mathrm{V}}+f c_{\mathrm{IV}}\right)^{2}\right), \quad(f \neq 0) \tag{5.4.10}
\end{equation*}
$$

In addition, the Ermakov-Ray-Reid system (5.4.7) is seen to be Hamiltonian with invariant

$$
\begin{equation*}
H=\frac{1}{2}\left(\dot{\Phi}^{2}+\dot{\Psi}^{2}\right)-\frac{1}{2\left(\Phi^{2}+\Psi^{2}\right)}\left[Z^{2}-\frac{f^{2}}{4}\left(\Phi^{2}+\Psi^{2}\right)^{2}+\frac{k}{4}\right] \tag{5.4.11}
\end{equation*}
$$

and accordingly, integrable in the manner described in [100].

It is established, in the above, that semi-axes $\Phi$ and $\Psi$ of the time modulated ellipse associated with the density in (5.3.1) are governed by a Ermakov-Ray-Reid system, albeit of some complexity. In the sequel, it is shown that a Ermakov-Ray-Reid system may also be associated with the velocity components, at least, in a particular reduction. Attention is restricted, as in the work of Dyson [28] on non-conducting gas clouds, to irrotational motions in the absence of a Coriolis term.

Thus, we set

$$
\mathbf{L}=\left(\begin{array}{cc}
\dot{\bar{\alpha}}(t) / \bar{\alpha}(t) & 0  \tag{5.4.12}\\
0 & \dot{\bar{\beta}}(t) / \bar{\beta}(t)
\end{array}\right), \quad \mathbf{E}=\left(\begin{array}{cc}
\bar{a}(t) & 0 \\
0 & \bar{c}(t)
\end{array}\right)
$$

in (5.3.2) corresponding to the subclass of exact solutions in (5.3.50) with $\theta=0, \phi=$ $\pi / 2$ and

$$
\begin{array}{ll}
\frac{\dot{\bar{\alpha}}}{\bar{\alpha}}=\frac{\dot{\Omega}}{\Omega}+\frac{1}{\Omega^{2}} \sqrt{c_{\mathrm{III}}+\frac{\alpha \delta}{\Omega^{2}}}, & \frac{\dot{\bar{\beta}}}{\bar{\beta}}=\frac{\dot{\Omega}}{\Omega}-\frac{1}{\Omega^{2}} \sqrt{c_{\mathrm{III}}+\frac{\alpha \delta}{\Omega^{2}}},  \tag{5.4.13}\\
\bar{a}=\frac{1}{2 \Omega^{2(m+1)}}\left[\delta-\sqrt{4 c_{\mathrm{II}} \Omega^{4}+\delta^{2}}\right], & \bar{c}=\frac{1}{2 \Omega^{2(m+1)}}\left[\delta+\sqrt{4 c_{\mathrm{II}} \Omega^{4}+\delta^{2}}\right] .
\end{array}
$$

In the present case, the continuity equation, via (5.3.3), yields

$$
\begin{equation*}
\frac{\dot{\bar{a}}}{\overline{\bar{a}}}+\frac{\dot{\bar{\alpha}}}{\bar{\alpha}}(m+1)+\frac{\dot{\bar{\beta}}}{\bar{\beta}}(m-1)=0, \quad \frac{\dot{\bar{c}}}{\bar{c}}+\frac{\dot{\bar{\alpha}}}{\overline{\bar{\alpha}}}(m-1)+\frac{\dot{\bar{\beta}}}{\bar{\beta}}(m+1)=0 \tag{5.4.14}
\end{equation*}
$$

whence

$$
\begin{equation*}
\bar{a}=\bar{c}_{\mathrm{I}} \bar{\alpha}^{-(m+1)} \bar{\beta}^{1-m}, \quad \bar{c}=\bar{c}_{\mathrm{II}} \bar{\alpha}^{1-m} \bar{\beta}^{-(m+1)} . \tag{5.4.15}
\end{equation*}
$$

Moreover, (5.3.4) shows that

$$
\begin{equation*}
\rho_{0}=\bar{c}_{\mathrm{III}}(\bar{\alpha} \bar{\beta})^{1-m}=\bar{c}_{\mathrm{III}}^{*} \Omega^{2(1-m)} . \tag{5.4.16}
\end{equation*}
$$

In the above, $\bar{c}_{\mathrm{I}}, \bar{c}_{\mathrm{II}}, \bar{c}_{\mathrm{III}}$ and $\bar{c}_{\text {III }}^{*}$ are arbitrary non-zero constants of integration. The momentum equation gives

$$
\begin{equation*}
\ddot{\bar{\alpha}}+2 \varepsilon_{2}(t) \frac{m}{m-1} \bar{a} \bar{\alpha}=0, \quad \ddot{\bar{\beta}}+2 \varepsilon_{2}(t) \frac{m}{m-1} \bar{c} \bar{\beta}=0 . \tag{5.4.17}
\end{equation*}
$$

together with

$$
\begin{equation*}
\ddot{\bar{p}}=0, \quad \ddot{\vec{q}}=0 . \tag{5.4.18}
\end{equation*}
$$

Insertion of the expressions (5.4.15) into (5.4.17) gives

$$
\begin{align*}
& \ddot{\bar{\alpha}}+2 \varepsilon_{2}(t) \frac{m}{m-1} \frac{\bar{c}_{\mathrm{I}}}{\bar{\alpha}^{2} \bar{\beta}}(\bar{\alpha} \bar{\beta})^{2-m}=0,  \tag{5.4.19}\\
& \ddot{\bar{\beta}}+2 \varepsilon_{2}(t) \frac{m}{m-1} \frac{\bar{c}_{\mathrm{I}}}{\bar{\alpha} \bar{\beta}^{2}}(\bar{\alpha} \bar{\beta})^{2-m}=0 \tag{5.4.20}
\end{align*}
$$

whence, in view of the constraint (5.3.28), we obtain the canonical Ermakov-Ray-Reid system

$$
\begin{equation*}
\ddot{\bar{\alpha}}=\frac{\bar{c}_{\mathrm{I}}^{*}}{\bar{\alpha}^{2} \bar{\beta}}, \quad \ddot{\bar{\beta}}=\frac{\bar{c}_{\mathrm{II}}^{*}}{\bar{\alpha} \bar{\beta}^{2}} . \tag{5.4.21}
\end{equation*}
$$

with the Ray-Reid invariant

$$
\begin{equation*}
I=\frac{1}{2}(\dot{\bar{\alpha}} \bar{\beta}-\bar{\alpha} \dot{\bar{\beta}})^{2}+\bar{c}_{\mathrm{I}}^{*} \frac{\bar{\beta}}{\bar{\alpha}}+\bar{c}_{\mathrm{II}}^{*} \overline{\bar{\beta}} \tag{5.4.22}
\end{equation*}
$$

where

$$
\begin{equation*}
\bar{c}_{\mathrm{I}}^{*}=-2 \alpha \bar{c}_{\mathrm{I}}\left(\frac{\bar{c}_{\mathrm{III}}^{*}}{\bar{c}_{\mathrm{III}}}\right)^{\frac{m-2}{m-1}}, \quad \bar{c}_{\mathrm{II}}^{*}=-2 \alpha \bar{c}_{\mathrm{II}}\left(\frac{\bar{c}_{\mathrm{III}}^{*}}{\bar{c}_{\mathrm{III}}}\right)^{\frac{m-2}{m-1}} \tag{5.4.23}
\end{equation*}
$$

It is observed that the system (5.4.21) is also Hamiltonian with the associated integral of motion

$$
\begin{equation*}
H=\frac{1}{2}\left(\bar{c}_{\mathrm{I}} \dot{\bar{\beta}}^{2}+\bar{c}_{\mathrm{II}} \dot{\bar{\alpha}}^{2}\right)+\frac{\bar{c}_{\mathrm{C}}^{*} \bar{c}_{\mathrm{II}}^{*}}{\bar{\alpha} \bar{\beta}} \tag{5.4.24}
\end{equation*}
$$

In conclusion, the existence of Ermakov-Ray-Reid systems, not only for density parameters but also for velocity components, emphasizes the importance of such systems in the study of the 2+1-dimensional magnetogasdynamic system.

### 5.5 A Lax Pair Formulation

In light of the above analysis, we now turn to the original magnetogasdynamic system and find, in the manner of [110], that the nonlinear dynamical system admits an associated Lax pair representation [1].

It is seen that the eight-dimensional nonlinear dynamical equations (5.3.3) together with (5.3.8) arising from the ansatz (5.3.1) and (5.3.2) may be rewritten into the compact matrix form as :

$$
\begin{align*}
& \dot{\mathbf{E}}+\mathbf{E L}+\mathbf{L}^{T} \mathbf{E}+(m-1) \mathbf{E} \operatorname{tr} \mathbf{L}=\mathbf{0} \\
& \dot{\mathbf{L}}+\mathbf{L}^{2}+f \mathbf{P L}+2 \varepsilon_{2}(t) \frac{m}{m-1} \mathbf{E}=\mathbf{0} \tag{5.5.1}
\end{align*}
$$

where $\mathbf{L}, \mathbf{E}$ are given by $(5.3 .2)_{1,2}$ and

$$
\mathbf{P}=\left(\begin{array}{cc}
0 & -1  \tag{5.5.2}\\
1 & 0
\end{array}\right)
$$

Moreover, the relations (5.3.4) and (5.3.9) yield

$$
\begin{equation*}
\dot{\rho}_{0}+(m-1) \rho_{0} \operatorname{tr} \mathbf{L}=0 \tag{5.5.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\dot{\mathbf{M}}+f \mathbf{P M}=\mathbf{0} . \tag{5.5.4}
\end{equation*}
$$

Here, it proves convenient to proceed with the gauge transformation according to [110]

$$
\begin{equation*}
\tilde{\mathbf{L}}=\mathbf{D L D}^{-1}+\frac{1}{2} f \mathbf{P}, \quad \tilde{\mathbf{E}}=\mathbf{D E D}^{-1} \tag{5.5.5}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathbf{D}=\exp \left(\frac{1}{2} \mathbf{P} f t\right) \tag{5.5.6}
\end{equation*}
$$

whence (5.5.1) yields

$$
\begin{align*}
& \dot{\tilde{\mathbf{E}}}+\tilde{\mathbf{E}} \tilde{\mathbf{L}}+\tilde{\mathbf{L}}^{T} \tilde{\mathbf{E}}+(m-1) \tilde{\mathbf{E}} \operatorname{tr} \tilde{\mathbf{L}}=0, \\
& \dot{\tilde{\mathbf{L}}}+\tilde{\mathbf{L}}^{2}+\frac{1}{4} f^{2} \mathbf{I}+2 \varepsilon_{2}(t) \frac{m}{m-1} \tilde{\mathbf{E}}=0 . \tag{5.5.7}
\end{align*}
$$

On application of the Cayley-Hamilton identity

$$
\begin{equation*}
\tilde{\mathbf{L}}^{2}-(\operatorname{tr} \tilde{\mathbf{L}}) \tilde{\mathbf{L}}+(\operatorname{det} \tilde{\mathbf{L}}) \mathbf{I}=0 \tag{5.5.8}
\end{equation*}
$$

the matrix equation $(5.5 .7)_{2}$ becomes

$$
\begin{equation*}
\dot{\tilde{\mathbf{L}}}+(\operatorname{tr} \tilde{\mathbf{L}}) \tilde{\mathbf{L}}-(\operatorname{det} \tilde{\mathbf{L}}) \mathbf{I}+\frac{1}{4} f^{2} \mathbf{I}+2 \varepsilon_{2}(t) \frac{m}{m-1} \tilde{\mathbf{E}}=0 . \tag{5.5.9}
\end{equation*}
$$

Moreover, on introduction of a new trace-free matrix $\tilde{\mathbf{Q}}$ via

$$
\begin{equation*}
\tilde{\mathbf{Q}}=\mathbf{P} \tilde{\mathbf{E}} \tag{5.5.10}
\end{equation*}
$$

and on use of the identity

$$
\begin{equation*}
\mathbf{P H P}=\mathbf{H}^{T}-(\operatorname{tr} \mathbf{H}) \mathbf{I} \tag{5.5.11}
\end{equation*}
$$

valid for any matrix $\mathbf{H}$, the system $(5.5 .7)_{2}$ results in

$$
\begin{equation*}
\dot{\tilde{\mathbf{Q}}}+[\tilde{\mathbf{Q}}, \tilde{\mathbf{L}}]+m(\operatorname{tr} \tilde{\mathbf{L}}) \tilde{\mathbf{Q}}=0 . \tag{5.5.12}
\end{equation*}
$$

Since $\operatorname{tr} \mathbf{L}=\operatorname{tr} \tilde{\mathbf{L}}=2 \dot{\Omega} / \Omega$, it is natural to introduce the scaling

$$
\begin{equation*}
\overline{\mathbf{L}}=\tilde{\mathbf{L}} \Omega^{2}, \quad \overline{\mathbf{E}}=\tilde{\mathbf{E}} \Omega^{2 m}, \quad \overline{\mathbf{Q}}=\tilde{\mathbf{Q}} \Omega^{2 m} \tag{5.5.13}
\end{equation*}
$$

so that (5.5.9) and (5.5.12) reduce, in turn, to

$$
\begin{align*}
& \dot{\overline{\mathbf{Q}}}+\Omega^{-2}[\overline{\mathbf{Q}}, \overline{\mathbf{L}}]=0, \\
& \dot{\overline{\mathbf{L}}}-\Omega^{-2}(\operatorname{det} \overline{\mathbf{L}}) \mathbf{I}+\frac{f^{2}}{4} \Omega^{2} \mathbf{I}+2 \varepsilon_{2}(t) \frac{m}{m-1} \Omega^{2(1-m)} \overline{\mathbf{E}}=0 . \tag{5.5.14}
\end{align*}
$$

At this stage, it is noticed that $(5.5 .14)_{1}$ may be reformulated in terms of two trace-free matrixes $\overline{\mathbf{Q}}$ and $\overline{\mathbf{L}}^{*}$, namely

$$
\begin{equation*}
\dot{\overline{\mathbf{Q}}}+\Omega^{-2}\left[\overline{\mathbf{Q}}, \overline{\mathbf{L}}^{*}\right]=0, \tag{5.5.15}
\end{equation*}
$$

where $\overline{\mathbf{L}}^{*}$ denotes the trace-free part of $\overline{\mathbf{L}}$. Further, (5.5.14) ${ }_{2}$ may be decomposed into the trace-free part

$$
\begin{equation*}
\dot{\overline{\mathbf{L}}}^{*}+\varepsilon_{2}(t) \frac{m}{m-1} \Omega^{2(1-m)}[\overline{\mathbf{Q}}, \mathbf{P}]=0 \tag{5.5.16}
\end{equation*}
$$

together with the trace part

$$
\begin{equation*}
\operatorname{tr} \dot{\overline{\mathbf{L}}}-2 \Omega^{-2}\left(\operatorname{det} \overline{\mathbf{L}}^{*}\right)-\frac{1}{2} \Omega^{-2}(\operatorname{tr} \overline{\mathbf{L}})^{2}+\frac{1}{2} f^{2} \Omega^{2}+2 \varepsilon_{2}(t) \frac{m}{m-1} \Omega^{2(1-m)}(\operatorname{tr} \overline{\mathbf{E}})=0 . \tag{5.5.17}
\end{equation*}
$$

In view of the constraint (5.3.28) with $\alpha$ scaling to unity, the systems (5.5.16) and (5.5.17) become

$$
\begin{align*}
& \dot{\overline{\mathbf{L}}}^{*}+\Omega^{-2}[\overline{\mathbf{Q}}, \mathbf{P}]=0, \\
& \operatorname{tr} \dot{\overline{\mathbf{L}}}-2 \Omega^{-2}\left(\operatorname{det} \overline{\mathbf{L}}^{*}\right)-\frac{1}{2} \Omega^{-2}(\operatorname{tr} \overline{\mathbf{L}})^{2}+\frac{1}{2} f^{2} \Omega^{2}+2 \Omega^{-2}(\operatorname{tr} \overline{\mathbf{E}})=0 . \tag{5.5.18}
\end{align*}
$$

In general, the matrix system (5.5.15) and (5.5.18) are coupled via the relation

$$
\begin{equation*}
\dot{\rho}_{0}+(m-1) \rho_{0} \operatorname{tr} \tilde{\mathbf{L}}=0 \tag{5.5.19}
\end{equation*}
$$

A new time variable $\tau$ is now introduced via

$$
\begin{equation*}
d \tau=\Omega^{-2} d t \tag{5.5.20}
\end{equation*}
$$

whence the equations (5.5.15) and (5.5.18) ${ }_{1}$ reduce to

$$
\begin{equation*}
\overline{\mathbf{Q}}^{\prime}+\left[\overline{\mathbf{Q}}, \overline{\mathbf{L}}^{*}\right]=0, \quad \overline{\mathbf{L}}^{*^{\prime}}+[\overline{\mathbf{Q}}, \mathbf{P}]=0 \tag{5.5.21}
\end{equation*}
$$

It is now seen that the matrix system (5.5.21) constitutes the compatibility condition

$$
\begin{equation*}
\mathcal{M}^{\prime}(\lambda)+[\mathcal{M}(\lambda), \mathcal{L}(\lambda)]=0 \tag{5.5.22}
\end{equation*}
$$

associated with the linear pair

$$
\begin{equation*}
\Psi^{\prime}=\mathcal{L}(\lambda) \Psi, \quad \mu \Psi=\mathcal{M}(\lambda) \Psi \tag{5.5.23}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{L}(\lambda)=\overline{\mathbf{L}}^{*}+\lambda \mathbf{P}, \quad \mathcal{M}(\lambda)=\overline{\mathbf{Q}}+\lambda \overline{\mathbf{L}}^{*}+\lambda^{2} \mathbf{P} \tag{5.5.24}
\end{equation*}
$$

and $\mu$ is an arbitrary constant. $\mathcal{L}$ and $\mathcal{M}$ represent Lax matrices for the nonlinear matrix system (5.5.21).

In addition, it is observed that if we set

$$
\begin{equation*}
\Sigma=\Omega^{-1} \tag{5.5.25}
\end{equation*}
$$

then the relation $(5.5 .18)_{2}$ reduces to a classical Steen-Ermakov type equation $[30,124]$

$$
\begin{equation*}
\Sigma^{\prime \prime}+\left(\operatorname{det} \overline{\mathbf{L}}^{*}-\operatorname{tr} \overline{\mathbf{E}}\right) \Sigma=\frac{f^{2}}{4 \Sigma^{3}} . \tag{5.5.26}
\end{equation*}
$$

Analogous results have been obtained in the case of non-isothermal spinning magnetogasdynamic system by Rogers and An in [102] and rotating gas clouds by Rogers and Schief in [110], respectively.

## Chapter 6

## Conclusions and Suggestions for Future Research

In this thesis, integrable structure underlying certain models in nonlinear continuum mechanics and optics has been sought which is associated with reduction to Ermakov-Ray-Reid systems. The latter admit a novel integral of motion. Numerical simulations have been performed to depict the physical behaviours exhibited by solutions obtained by means of the reductions. The details are as follows:

In Chapter 2, we investigated a $2+1$-dimensional rotating shallow water system with a circular paraboloidal bottom topography via an elliptic vortex. Key theorems of Ball-type concerning the evolution of moment of inertia and invariants of the shallow water model were generalised and employed to construct exact solutions. In particular, important pulsrodon-type solutions were isolated and their behaviour was simulated.

In Chapter 3, we studied two nonlinear optical models via a variational approach : one was a coupled $2+1$-dimensional NLS system and the other was a $3+1$-dimensional NLS equation incorporating both logarithmic and Bohm quantum potential terms. Three distinct reductions to Ermakov systems of Hamiltonian-type were set down. The Ray-Reid invariant and associated Hamiltonian invariant combined to allow their
complete integration. Such integrable Ermakov structure also arose in the latter 3+1dimensional optical model. Aspects of an Ovisannikov-Dyson type reduction were investigated wherein the eigenmode of the solution exhibits a remarkable flip-over effect that has been experimentally observed by Gornushkin et al in the model descriptive of an asymmetric expansion of laser induced plasmas into vacuum.

In Chapter 4, we discussed a Madelung-type hydrodynamic system with logarithmic and Bohm quantum potential terms. Appropriate choice of exponential-type elliptic vortex ansatz and modulated physical variables results in the generalised eightdimensional nonlinear dynamical system, which admits exact analytical solutions in terms of elliptic integrals. The latter, again, possess integrable Ermakov structure of Hamiltonian-type.

In Chapter 5, we considered a magnetogasdynamic system with a polytropic gas law. Introduction of a power-type elliptic vortex ansatz and two-parameter pressuredensity relation was shown to lead to a finite dimensional nonlinear dynamical system. The latter admits an integrable Hamiltonian Ermakov structure when the adiabatic index $\gamma=2$ and a Lax pair formulation may be constructed. Exact solutions of the magnetogasdynamic systems were thereby obtained which describe a rotating elliptic plasma cylinder bounded by a vacuum state.

Nevertheless, there is still a series of interesting and challenging problems that need consideration:

1. The generalised theorems of Ball-type established in Chapter 2 were obtained when the shallow water system has a circular paraboloidal bottom topgraphy $\left(A^{*}=B^{*}\right)$. In view of the importance of these theorems for construction of analytical solutions and associated Ermakov systems, it is natural to enquire whether analogous theorems exist for other geometries $\left(A^{*} \neq B^{*}\right)$, in particular, for elliptical paraboloidal basins?
2. In light of the limitations inherent in the variational approximation employed in Chapter 3, it would be of interest to investigate whether alternative approaches result
in a reduction to integrable Ermakov-Ray-Reid structure?
3. Based on the different forms of elliptic vortex ansatz adopted in this thesis, it is conjectured that more general forms may exist which lead to Ermakov-Ray-Reid reduction.
4. It is noted that the general $N$-component Ermakov system introduced by Rogers and Schief in an $N$-layer hydrodynamic context admits an iterative reduction to a system of $N-2$ linear equations augmented by the canonical 2-component Ermakov system. The latter has been widely used in various physical areas. Hence, it is anticipated that $N$-component Ermakov-type systems might also have such extensive physical applications.

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[^0]:    ${ }^{1}$ It is important to note that there are limits to the applicability of variational procedures in nonlinear optics (see e.g. [26])

