THE STUDY OF A CLASS OF ANALYTICAL SOLUTIONS FOR SIX FLUID DYNAMICAL SYSTEMS

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Certificate of Originality

I hereby declare that this submission is my own work and that, to the best of my knowledge and belief, it contains no material previously published or written by another person nor material which to a substantial extent has been accepted for the award of any other degree or diploma of a university or other institute of higher learning, except where due acknowledgment is made in the text.

Yuen Man Wai

To

My wife,

Janet Fan

Abstract

The systems of gas dynamics (Euler, Euler-Poisson, Navier-Stokes, Navier-Stokes-Poisson equations) and of shallow water (Camassa-Holm and Degasperis-Procesi equations) are important basic models in fluid mechanics and astrophysics. Constructing analytical or exact solutions for the partial differential equations is a vital part in nonlinear sciences. Indeed, scientists and mathematicians are eager to seek analytical solutions for better understanding of the evolution of these kinds of systems. In this PhD thesis, I consider the construction of analytical solutions for the above systems. As these systems share similar mathematical structures in some aspects, I will exhibit some common features among them, including certain blowup and stability phenomena. In detail, I attempt to employ the well-known separation method to its fullest extent and introduce a novel pertubational method to seek analytical solutions with free boundaries. The main idea is to reduce the nonlinear partial differential systems into several ordinary or functional differential equations, or to simpler partial differential equations under some suitable assumptions on the functional structures of the solutions. After proving the existence of solutions of the corresponding simpler differential equations, the analytical solutions for the original nonlinear systems are constructed. One of the applications of such analytical solutions is to test numerical methods designed for these systems. Another application is to provide samples of concrete solutions so as to affirm or support theoretical hypotheses or conjectures about these complicated systems.

A substantial percentage of the results presented in this thesis have appeared in print. In total, sixteen published papers (not counting preprints; see the lists in the next three pages) are the direct outcome of work done during my PhD study. The fact that these results are well-received by referees and editors attests to the great interest of others in these analytical solutions.

The most significant contributions of this thesis are as follows:

- I am the first to reduce the compressible density-dependent Navier-Stokes equations in R^N to new $1+N$ differential functional equations, which lead to solutions with elliptical symmetry and drift phenomena.
- I am the first to obtain self-similar solutions in explicit form for the 2-component shallow water systems.
- We construct the first rotational solutions in explicit form for the 2-dimensional Euler-Poisson equations and demonstrate the principle that rotation can prevent blowup.

The thesis is organized as follows:

- A brief introduction of the above six models is provided.
- The separation method is applied to construct solutions with free boundaries for the systems of gas dynamics and shallow water.

- In addition, some solutions with rotation are constructed for the 2-dimensional Euler-Poisson and 3-dimensional Euler equations.
- Based on the separation method, a novel pertubational method is used to obtain more general classes of analytical solutions for the 1-dimensional Euler and Camassa-Holm equations.
- Finally a summary is provided to conclude the works done and other related works in the PhD studies, together with some future research insights for the further development of this thesis is included.

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Publications Arising from the Thesis

Published Papers:

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3. Yeung L.H. and **Yuen M.W.** (2011), *Analytical Solutions to the Navier-Stokes-Poisson Equations with Density-dependent Viscosity and with Pressure*, Proc. Amer. Math. Soc. **139**, 3951-3960.

4. Yeung L.H. and **Yuen M.W.** (2012), *Note for "Some Exact Blowup Solutions to the Pressureless Euler Equations in R^N" [Commun. Nonlinear Sci. Numer. Simul. 16 (2011), 2993-2998]*, Commun. in Nonlinear Sci. and Numer. Simul. **17**, 485–487.

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Pre-prints:

1. Kwong M.K. and **Yuen M.W.**, *Periodical Self-similar Solutions to the Isothermal Euler-Poisson Equations for Spiral Galaxies*, Submitted.

2. Kwong M.K. and **Yuen M.W.**, *Pulsating Flows of the 2D Euler-Poisson Equations, Submitted.*

3. **Yuen M.W.**, *Self-similar Solutions with Elliptic Symmetry for the Compressible Euler and Navier-Stokes Equations in* R^N, arXiv:1104.3687, Submitted.

4. **Yuen M.W.**, *Self-similar Solutions with Elliptic Symmetry for the Density-Dependent Navier-Stokes Equations in* R^N, arXiv:1105.0582, Submitted.

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Yuen Man Wai

Contents

Chapter 1

Introduction

In the first chapter, four important models in gas dynamics – Euler, Euler-Poisson, Navier-Stokes, Navier-Stokes-Poisson equations – and two models in shallow water systems – Camassa-Holm and Degasperis-Procesi equations – are described.

1.1 Gas Dynamics

We are interested in four well-known isentropic models in fluid mechanics.

- ***Euler***: There is no interaction between the fluid particles due to gravitational or electrostatic types of forces and the fluid is not viscous. These assumptions are expressed as

  $$\delta = 0 \quad \text{and} \quad vis(\rho, \vec{u}) = 0$$

  in the mathematical formulation given below.

- ***Euler-Poisson***: Gravitational (attractive) or electrostatic (attractive or repulsive) type of interaction is present, but the fluid is not viscous. Mathematically,

  $$\delta = \pm 1 \quad \text{and} \quad vis(\rho, \vec{u}) = 0.$$

- ***Navier-Stokes***: The fluid is viscous, but there are no inter-particle forces, that is,

  $$\delta = 0 \quad \text{and} \quad vis(\rho, \vec{u}) \neq 0.$$

- ***Navier-Stokes-Poisson***: The fluid is viscous and inter-particle forces are present, that is,
$$\delta = \pm 1 \quad \text{and} \quad vis(\rho, \vec{u}) \neq 0.$$

The above models can be written in the unified form

$$\begin{cases} \rho_t + \nabla \cdot (\rho \vec{u}) = 0 \\ \rho \left[\vec{u}_t + (\vec{u} \cdot \nabla) \vec{u} \right] + \nabla P = \delta \rho \nabla \Phi + vis(\rho, \vec{u}) \\ \Delta \Phi(t, \vec{x}) = \alpha(N) \left(\rho - \Lambda \right). \end{cases} \tag{1.1}$$

The explanation of the various notations is given below.

- The independent variables are time t and position $\vec{x} = (x_1, x_2, x_3, ..., x_N) \in \mathbf{R}^N$. For lower dimensional cases, we also use the notation $x = x_1$, $y = x_2$ and $z = x_3$.

- The unknown functions $\rho = \rho(t, \vec{x}) \geq 0$ and $\vec{u} = \vec{u}(t, \vec{x}) = (u_1, u_2, ..., u_N) \in \mathbf{R}^N$ are the density and the velocity, respectively, at a given time t and position $\vec{x}$. In the one dimensional case, we will write u instead of $\vec{u}$ for convenience.

- $\alpha(N)$ is a constant related to the unit ball in R^N :
$$\alpha(1) = 2; \quad \alpha(2) = 2\pi$$
and for $N \geq 3$,
$$\alpha(N) = N(N-2)V(N) = N(N-2) \frac{\pi^{N/2}}{\Gamma(N/2+1)} \tag{1.2}$$
where $V(N)$ is the volume of the unit ball in R^N and Γ is the Gamma function.

- P is the pressure term. In some models, $P = 0$ and the system is said to be "pressureless". In the contrary case, the system is said to be "with pressure". A common hypothesis is that P satisfies the γ-law, namely
$$P(\rho) = K \rho^{\gamma} \tag{1.3}$$
with constants $K > 0$ and $\gamma \geq 1$. Note that P depends only on the density function, and not directly on t, $\vec{x}$ or $\vec{u}$. The fluid is said to be isothermal if $\gamma = 1$.

- The number Λ is called the background or cosmological constant. When Λ is positive, the space is said to be "open"; when it is negative, the space is said to be "closed"; when it is zero, the space is "flat".

- The term $vis(\rho, \vec{u})$ is the viscosity function

$$vis(\rho, \vec{u}) = \bigtriangledown(\mu(\rho) \bigtriangledown \cdot \vec{u}). \tag{1.4}$$

  A common assumption is that $\mu(\rho)$ satisfies

$$\mu(\rho) = \kappa \rho^{\theta} \tag{1.5}$$

  where κ and $\theta \geq 0$ are constants.

- The constant δ can take one of three values: -1, 1 or 0.

  - When $\delta = -1$, the system can model fluids that are self-gravitating, such as gaseous stars. In addition, the evolution of the simple cosmology can be modelled by the dust distribution without the pressure term. This describes the stellar systems of collisionless and gravitational n-body systems For detail of the modelling approach, see Fliche and Triay [28]. The pressureless Euler-Poisson equations can be derived from the Vlasov-Poisson-Boltzmann model with the zero mean free path [31]. For $N = 3$ and $\delta = -1$, the equations (1.1) are the classical (non-relativistic) descriptions of a galaxy in astrophysics. See Binney and Tremaine [4] and Chandrasekhar [8] for details about the systems.

  - When $\delta = 1$, the system is the compressible Euler-Poisson and Navier-Stokes-Poisson equations with repulsive forces. The equation $(1.1)_3$ is the Poisson equation through which the potential with repulsive forces is determined by the density distribution of the electrons. In this case, the system can be viewed as a semiconductor model. See Chen [9] and Lions [51] for the detailed analysis of the system.

  - For $\delta = 0$, the system is the classical Euler or Navier-Stokes equations for fluid mechanics. See Chen and Wang [10] and Lions [51].

For an introduction to the above systems, readers may refer to Landau and Lifshitz [46], and Nishida [62], in addition to [51] which has been mentioned above.

In the above systems, the self-gravitational potential field $\Phi = \Phi(t, \vec{x})$ is determined by the density ρ itself, by solving the Poisson equation $(1.1)_3$

$$\Phi(t, \vec{x}) = \int_{R^N} G(\vec{x} - \vec{y})(\rho(t, \vec{y}) - \Lambda)d\vec{y}$$

where G is the Green's function for the Poisson equation in the N-dimensional spaces defined by

$$G(\vec{x}) = \begin{cases} |\vec{x}| & \text{for } N = 1 \\ \log|\vec{x}| & \text{for } N = 2 \\ \dfrac{-1}{|\vec{x}|^{N-2}} & \text{for } N \geq 3. \end{cases}$$

If we seek solutions in radial symmetry with $r = \left(\sum_{i=1}^{N} x_i^2\right)^{1/2}$, the Poisson equation $(1.1)_3$ is transformed to

$$r^{N-1}\Phi_{rr} + (N-1)\, r^{N-2}\Phi_r = \alpha\,(N)\,(\rho - \Lambda) r^{N-1}$$

$$\Phi_r = \frac{\alpha\,(N)}{r^{N-1}} \int_0^r (\rho(t,s) - \Lambda) s^{N-1} ds.$$

In general, radial solutions without rotation can be written in the form

$$\rho(t, \vec{x}) = \rho(t, r)$$

and

$$\vec{u} = \frac{\vec{x}}{r} V(t, r) := \frac{\vec{x}}{r} V.$$

Standard computation, by substituting the above expressions into (1.1), leads to the radially symmetric form of our fluid dynamical systems

$$\begin{cases} \rho_t + V\rho_r + \rho V_r + \dfrac{N-1}{r}\rho V \ = \ 0 \\ \rho\,(V_t + VV_r) + P_r(\rho) \ = \ \delta\rho\Phi_r + \mu_r\,(\rho)\left(\dfrac{N-1}{r}V + V_r\right) \\ \qquad + \mu(\rho)\left[V_{rr} + \dfrac{N-1}{r}V_r - \left(\dfrac{N-1}{r^2}\right)V\right]. \end{cases} \tag{1.6}$$

For the 3-dimensional case, the hydrostatic equilibrium for the Euler-Poisson and Navier-Stokes-Poisson equations with attractive forces ($\delta = -1$), as specified by

$\vec{u} = 0$, has been known for a long time. According to Chandrasekhar [8], the ratio between the core density $\rho(0)$ and the mean density $\bar{\rho}$ for $6/5 < \gamma < 2$ is given by

$$\frac{\bar{\rho}}{\rho(0)} = \left(\frac{-3}{s}\dot{g}\left(s\right)\right)_{s=s_0}$$

where $g(s)$ is the solution of the following Lane-Emden equation with $n = 1/(\gamma - 1)$ with $\gamma > 1$,

$$\ddot{g}(s) + \frac{2}{s}\dot{g}(s) + g^n(s) = 0, \qquad g(0) = \beta > 0,\ \dot{g}(0) = 0$$

and s_0 is the first zero of $g(s_0) = 0$. We can solve the Lane-Emden equation analytically for

$$g_{anal}(s) = \begin{cases} 1 - \dfrac{1}{6}s^2 & \text{for } n = 0 \\ \dfrac{\sin s}{s} & \text{for } n = 1 \\ \dfrac{1}{\sqrt{1 + s^2/3}} & \text{for } n = 5. \end{cases}$$

For other values of n, only numerical solutions can be obtained. It can be shown that for $n < 5$, the radius of polytropic models is finite; for $n \geq 5$, the radius is infinite.

For the isothermal case ($\gamma = 1$), the corresponding stationary solution is the solution of the Liouville equation,

$$\ddot{g}(z) + \frac{2}{s}\dot{g}(s) + \frac{4\pi}{K}e^{g(s)} = 0.$$

Makino [57], Gambin [29] and Bezard [2] obtained local existence results for the compressible Euler-Poisson equations.

1.2 Shallow Water Systems

The 2-component ***Camassa-Holm*** equations can be expressed in the following form

$$\begin{cases} \rho_t + u\rho_x + \rho u_x = 0 \\ m_t + 2u_x m + u m_x + \sigma\rho\rho_x = 0 \end{cases} \tag{1.7}$$

with

$$m = u - \alpha^2 u_{xx}. \tag{1.8}$$

We are here confined to the one-dimensional case $x \in R$. The unknown functions are again the density $\rho = \rho(t,x) \geq 0$ and the velocity $u = u(t,x) \in R$ of the fluid under study. The constant σ is equal to 1 or -1. If $\sigma = -1$, the gravity acceleration points upwards. See, for example, Constantin's paper [12] and other's papers [11], [34], [35]. For $\sigma = 1$, the corresponding model has been investigated in [18], [27], [35] and [34].

When $\rho \equiv 0$, the system returns to the classical 1-component Camassa-Holm equation first introduced in Camassa and Holm [5].

The search for a model equation which can capture breaking waves and peaked traveling waves is a long-standing open problem; see Whitham [73]. The classical 1-component Camassa-Holm equation satisfies the above conditions as a model equation. The search for peaked traveling waves is motivated by the wish to discover waves replicating a characteristic for the wave of great height (waves of largest amplitude) that are exact traveling solutions of the shallow water equations, whether periodic or solitary; see Constantin [14], Constantin and Escher [17] and Toland [70]. The breaking waves can be interpreted as solutions which remain bounded but the slope at some point becomes unbounded at some finite time; see Constantin [15]. Meanwhile, there is an alternative derivation of the Camassa-Holm equation in Johnson [41] and Constantin and Lannes [19]. With $\sigma = 1$, the system (1.7) is integrable; see Ivanov [38] and [5].

Meanwhile, the 2-component ***Degasperis-Procesi*** shallow water system – see Popowicz [64] and Jin-Guo [40] – can be expressed in the following form

$$\begin{cases} \rho_t + k_2 u \rho_x + (k_1 + k_2)\rho u_x = 0, \ x \in R \\ u_t - u_{xxt} + 4uu_x - 3u_x u_{xx} - uu_{xxx} + k_3 \rho \rho_x = 0, \end{cases} \tag{1.9}$$

where k_1, k_2, k_3 are constants. For $\rho = 0$, the system (1.9) reverts to the classical 1-component Degasperis-Procesi equation [26], [52] and [97].

1.3 Outline of the Thesis

The six differential systems described in the previous two sections are very important basic models in fluid mechanics and astrophysics. Constructing analytical or exact solutions for the partial differential equations is a vital part to nonlinear sciences. Indeed, scientists and mathematicians are eager to seek analytical solutions for better understanding of the evolution of these kinds of systems. In this PhD thesis, we consider the construction of analytical solutions for the above systems, which share similar mathematical structures in some aspects. We would also like to exhibit some common features among them, including certain blowup and stability phenomena.

The separation method was first used by Goldreich and Weber in 1980 to construct radially symmetric solutions of the 3-dimensional Euler-Poisson equations. The result was later improved and generalized by Makino, Deng–Xiang–Yang and Yuen, among others to the N-dimensional case.

We then observed that the separation method is applicable to other similar systems. The author's PhD project is an attempt to fully exploit this method to include as many systems as possible. This thesis, as well as some published and accepted articles, represent the outcome of this effort.

On the surface, the method appears to be fairly routine. Based on experience, one guesses a possible functional structure of the solutions and tries to verify that they are indeed solutions by direct computation. Yet, much effort focuses on looking for ways to extend the functional structure by relaxing the symmetry requirements for the solutions. For instance, instead of requiring radial symmetry, we succeeded in finding solutions that are anisotropic (meaning that the solution evolves differently in different directions), leading to solutions with elliptic symmetry. As another example, the classical radially symmetric solutions are irrotational, and we constructed, for R^2, global rotational solutions that are in a more general sense also radially symmetric. We also found a novel pertubational method that gives solutions having a drift component. The work involves a lot of inductive reasoning and patience in trial and error.

The main idea of the separation method is to reduce the nonlinear partial differential systems into several ordinary differential equations, functional differential equations or simpler partial differential equations under some suitable assumptions about the functional structures of the solutions. After the existence of solutions of the corresponding simpler differential equations have been proven, the analytical solutions for the original nonlinear systems are constructed.

One application of constructing these analytical solutions is to verify the existence or non-existence of solutions having certain properties, such as blowing up, or being periodic. Another application is to test numerical methods designed for these systems.

Quite a number of the results presented in this thesis have appeared or will soon appear in print. In total, fifteen update published or accepted papers (not counting preprints; see the lists in the next two pages) are the direct outcome of work done during my PhD study. The fact that these results are well-received by referees and editors attests to the great interest of others in these analytical solutions.

The most significant contributions of this thesis are as follows:

- We are the first to exhibit periodic solutions of the Euler-Poisson equations when the cosmological background constant is negative.
- We are the first to reduce the compressible density-dependent Navier-Stokes equations in R^N to new $1 + N$ differential functional equations, which lead to solutions with elliptical symmetry and drift phenomena.
- We are the first to obtain self-similar solutions in explicit form for the 2-component shallow water systems.
- We construct the first rotational solutions in explicit form for the 2-dimensional Euler-Poisson equations and demonstrate the principle that rotation can prevent blowup.

The thesis is organized as follows:

In Chapter 2, we apply the separation method to construct particular solutions for the Euler, Euler-Poisson, Navier-Stokes, and Navier-Stokes-Poisson equations.

In Chapter 3, we further apply the separation method to seek solutions for the 2-component Camassa-Holm and Degasperis-Procesi equations.

In Chapter 4, we study the existence of pulsating flows for the 2D pressureless Euler-Poisson equations. The approach used in this chapter is markedly different from that in other chapters, being much more theoretical. The desired solutions cannot be expressed in explicit form and their existence can only be established theoretically. The result is a demonstration of the principle that rotation can prevent the onset of blowup.

In Chapter 5, we construct rotational solutions for the 2-dimensional Euler-Poisson and 3-dimensional Euler equations.

In Chapter 6, we introduce a perturbational method to obtain a new class of solutions for the 2-component Camassa-Holm and the compressible Euler and Navier-Stokes equations in the 1-dimensional case.

In the last chapter, we summarize the work reported in this thesis along with other related work done during the PhD studies. Future directions of research are indicated.

Chapter 2

The Separation Method for Gas Dynamics

In the second chapter, the separation method is applied to construct solutions for the systems in gas dynamics. Solutions with radial, elliptical, cylindrical and line symmetries are obtained. We are the first to exhibit periodic solutions of the Euler-Poisson equations with a negative cosmological constant and to reduce the density-dependent Navier-Stokes equations in R^N to $1+N$ differential functional equations leading to solutions with elliptical symmetry and with a drift velocity.

2.1 The Separation Method

Historically, Goldreich and Weber [32] were the first (in 1980 in the study of astrophysics) to construct a family of radially symmetric analytical blowup (collapsing) solutions of the 3-dimensional Euler-Poisson equations with $\Lambda = 0$ and $\gamma = 4/3$,

$$\begin{cases} \rho_t + V\rho_r + \rho V_r + \dfrac{2}{r}\rho V = 0 \\ \rho\left(V_t + VV_r\right) + K\dfrac{\partial}{\partial r}\rho^{\gamma} = -\dfrac{4\pi\rho}{r^2}\displaystyle\int_0^r \rho(t,s)s^2 ds. \end{cases} \tag{2.1}$$

The system models non-rotating gas spheres. Subsequently, in 1992, Makino [58] gave a rigorous mathematical proof of the existence of such solutions. In 2003, Deng, Xiang and Yang [24] extended the above result to yield blowup solutions for the general N-dimensional case ($N \geq 3$). The corresponding system of equations is

$$\begin{cases} \rho_t + V\rho_r + \rho V_r + \dfrac{m}{r}\rho V = 0 \\ \rho\left(V_t + VV_r\right) + K\dfrac{\partial}{\partial r}\rho^{\gamma} \; = \; -\dfrac{\alpha(N)\rho}{r^m}\displaystyle\int_0^r \rho(t,\tau)\tau^m d\tau \end{cases} \tag{2.2}$$

with $m = N-1$ and $\gamma = (2N-2)/N$.

In 2008, Yuen [80] completed the picture by finding the analogous solutions in R^2 with $\gamma = 1$. The work was part of the author's MPhil degree project, completed before the start of the current PhD programme.

This above-mentioned family of analytical solutions of (2.2) is given as follows.

For $N \geq 3$ and $\gamma = (2N-2)/N$ (in [24]),

$$\begin{cases} \rho(t,r) = \begin{cases} \dfrac{1}{a^N(t)} g^{\delta}(\frac{r}{a(t)}) & \text{for } \; r < a(t)S_{\mu} \\ \qquad 0 & \text{for } \; a(t)S_{\mu} \leq r \end{cases} \\ V(t,r) = \dfrac{\dot{a}(t)}{a(t)} r \\ \qquad \ddot{a}(t) = \dfrac{-\lambda}{a^m(t)}, \qquad\qquad a(0) = a_0 > 0, \; \dot{a}(0) = a_1 \\ \qquad \ddot{g}(s) + \dfrac{m}{s}\dot{g}(s) + \dfrac{\alpha(N)}{2mK} g^{\delta}(s) = \mu, \qquad g(0) = \beta > 0, \; \dot{g}(0) = 0, \end{cases}$$

where $\delta = N/(N-2)$, $\mu = [N(N-2)\lambda]/(2N-2)K$ and the finite number S_{μ} is the first zero of $g(s)$.

For $N = 2$ and $\gamma = 1$ (in [80]),

$$\begin{cases} \rho(t,r) = \dfrac{1}{a^2(t)} e^{g(r/a(t))} \\ V(t,r) = \dfrac{\dot{a}(t)}{a(t)} r \\ \qquad \ddot{a}(t) = \dfrac{-\lambda}{a(t)}, \qquad\qquad a(0) = a_0 > 0, \; \dot{a}(0) = a_1 \\ \qquad \ddot{g}(s) + \dfrac{1}{s}\dot{g}(s) + \dfrac{2\pi}{K} e^{g(s)} = \mu, \qquad g(0) = \beta, \dot{g}(0) = 0, \end{cases}$$

where $\mu = 2\lambda/K$ with a sufficiently small λ, and β are constants.

For other constructions of analytical solutions by the separation method for the Euler-Poisson equations with damping, interested readers can refer to Yuen [79]. Local existence results can be found in Makino [57], Bezard [2] and Gamblin [29].

2.2 The Navier-Stokes-Poisson Equations with Density-Dependent Viscosity

The first result in the thesis is the construction of analytical radially symmetric blowup solutions of the pressureless N-dimensional Navier-Stokes-Poisson equations with density-dependent viscosity and attractive forces ($\delta = -1$) with zero background ($\Lambda = 0$)

$$\begin{cases} \rho_t + V\rho_r + \rho V_r + \dfrac{m}{r}\rho V = 0 \\ \rho\left(V_t + VV_r\right) = -\dfrac{\alpha(N)\rho}{r^m}\displaystyle\int_0^r \rho\tau^m d\tau + [\kappa\rho^\theta]_r\left(\dfrac{m}{r}V + V_r\right) \\ \qquad + \left(\kappa\rho^\theta\right)\left(V_{rr} + \dfrac{m}{r}V_r + \dfrac{m}{r^2}V\right). \end{cases} \tag{2.3}$$

As reported in [83], we are the first to apply the separation method to a pressureless Navier-Stokes-Poisson system with a density-dependent viscosity term (2.3). The viscosity term (the last two terms on the right-hand side of (2.3)) replaces the pressure term in (2.2) (the last term on the left-hand side) to balance the equation.

Theorem 2.1 ([83]) *For the N-dimensional pressureless Navier-Stokes-Poisson equations (2.3), there exists a family of radially symmetric solutions:*
For $N \geq 2$ and $N \neq 3$, with $\xi = N/(N-3)$,

$$\begin{cases} \rho(t,r) = \begin{cases} \dfrac{1}{(T-Ct)^N}\, g^{\xi}\left(\dfrac{r}{T-Ct}\right) & \text{for} \quad \dfrac{r}{T-Ct} < S_0 \\ 0 & \text{for} \quad S_0 \leq \dfrac{r}{T-ct} \end{cases} \\ V(t,r) = \dfrac{-C}{T-Ct}r \\ \qquad \ddot{g}(s) + \dfrac{m}{s}\dot{g}(s) + \dfrac{\alpha(N)}{(2N-3)C\xi\kappa}\, g^{\xi}(s) = 0, \qquad g(0) = \beta > 0,\ \dot{g}(0) = 0; \end{cases} \tag{2.4}$$

For $N = 3$,

$$\begin{cases} \rho(t,r) = \dfrac{1}{(T-Ct)^3}\, e^{g(r/(T-Ct))}, \\ V(t,r) = \dfrac{-C}{T-Ct}r \\ \qquad \ddot{g}(s) + \dfrac{2}{s}\dot{g}(s) + \dfrac{4\pi}{CN\kappa}e^{g(s)} = 0, \quad g(0) = \beta,\ \dot{g}(0) = 0, \end{cases} \tag{2.5}$$

where $T > 0$, $\kappa > 0$, $C \neq 0$ and β are constants, and the finite number S_0 is the first zero of $g(s)$. In particular, for $C > 0$, the solutions blow up at the finite time T/C.

Remark 2.1 *Throughout this work, the solutions of a system is described usually in a form exemplified by (2.4) or (2.5). The first two equations express the density and velocity functions in terms of one or two auxiliary functions which satisfy the ordinary differential equations and initial conditions that are listed below. In this case, we have only one auxiliary function $g(s)$ and it satisfies the second-order initial value problem given as the last row in (2.4) or (2.5). Since we can get different $g(s)$ by choosing different initial value β, we have actually constructed a one-parameter family of solutions.*

Remark 2.2 *One must first establish the existence of the auxiliary function $g(s)$ before one can ensure the existence of solutions to the original system. On the other hand, properties of $g(s)$ can be used to derive properties of the solutions.*

In the separation method, the first step is often to seek solutions of some general form for the continuity equation of mass. The next Lemma provides one such family of solutions in the radially symmetric case. Similar results were known to the earliest researchers in the field, and are progressively refined by subsequent authors.

The Lemma is deliberately stated in a form more general than what is needed in this section, but the more general result will be used in the next section.

Lemma 2.1 (Extension of Lemma 6 of [82]) *The functions*

$$\rho(t,r) = \frac{f(r/a(t))}{a^N(t)}, \qquad V(t,r) = \frac{\dot{a}(t)}{a(t)}r \tag{2.6}$$

where $f \geq 0$ and $a(t) > 0$ are any two arbitrary C^1 functions, are solutions of the radially symmetric N-dimensional equation of conservation of mass (2.3)$_1$.

Proof. Substituting (2.6) into the first equation of (2.3), we obtain

$$\begin{aligned}
&\rho_t + V\rho_r + \rho V_r + \frac{m}{r}\rho V \\
&= \frac{-N\dot{a}(t)f(r/a(t))}{a^{N+1}(t)} - \frac{\dot{a}(t)r\dot{f}(r/a(t))}{a^{N+2}(t)} \\
&\quad + \frac{\dot{a}(t)r}{a(t)}\frac{\dot{f}(r/a(t))}{a^{N+1}(t)} + \frac{f(r/a(t))}{a^N(t)}\frac{\dot{a}(t)}{a(t)} + \frac{m}{r}\frac{f(r/a(t))}{a^N(t)}\frac{\dot{a}(t)}{a(t)}r \\
&= 0.
\end{aligned}$$

The equation of conservation of mass is thus satisfied. ■

We only need a special case of the Lemma in order to prove Theorem 2.1.

Corollary 2.1 *The functions*

$$\rho(t,r)=\frac{f(r/(T-Ct))}{(T-Ct)^N},\qquad V(t,r)=\frac{-C}{T-Ct}r,$$

where $f\geq 0\in C^1$, and T and C are positive constants, are solutions of the radially symmetric N-dimensional equation of conservation of mass (2.3)$_1$.

Proof. This is a special case of the above Lemma, with $a(t)=T-Ct$. ■

The next Lemma is essential to complete the proof of the Theorem in the 2-dimensional case (please refer to Remarks 2.1 and 2.2). It establishes the existence of the function $g(s)$ for all $s\geq 0$, and its asymptotic behaviour as $s\to\infty$. The result is not obviously covered by standard existence results of ordinary differential equations due to the singularity of the second term at $s=0$, and the singularity of the third term when $g(s)=0$. In fact, the singularity at $s=0$ mandates that the second initial condition $\dot{g}(0)=0$ must hold. The Lemma is actually implied by Lemmas 9 and 10 in our earlier work [80].

Lemma 2.2 *The ordinary differential equation (which is the third equation in (2.4) for the particular case $N=3$)*

$$\begin{cases}\ddot{g}(s)+\dfrac{1}{s}\dot{g}(s)-\dfrac{\sigma}{g^2(s)}=0\\ g(0)=\beta>0,\ \dot{g}(0)=0,\end{cases}\tag{2.7}$$

where σ is a positive constant, has a solution $g(s)\in C^2$ and $\lim_{s\to\infty}g(s)=\infty$ for all $s\geq 0$.

Proof. By integrating (2.7), we have

$$\dot{g}(s)=\frac{\sigma}{s}\int_0^s\frac{1}{g^2(\tau)}\tau d\tau\geq 0.\tag{2.8}$$

Thus, for $0<s<s_0$, $g(s)$ has a uniform lower bound

$$g(s)\geq g(0)=\beta>0.$$

Local existence of solution guarantees that $g(s)$ exists in some neighborhood of $s=0$. Therefore, there are two possibilities to consider when trying to extend the solution.

(1) $g(s)$ only exists in some finite interval $[0,s_0]$. We have two sub-cases.

(1a) $\lim\limits_{s \to s_{0-}} g(s) = \infty$;

(1b) $g(s)$ has a uniform upper bound, i.e. $g(s) \leq \beta_0$ for some constant β_0.

(2) $g(s)$ exists in $[0, \infty)$.

(2a) $\lim\limits_{s \to \infty} g(s) = \infty$;

(2b) $g(s)$ has a uniform upper bound, i.e. $g(s) \leq \beta$ for some constant $\beta > 0$.

We claim that possibility (1) does not exist. Let us consider (1b) first: If the statement (1b) is true, (2.8) becomes

$$\frac{\sigma s}{2\beta^2} = \frac{\sigma}{s} \int_0^s \frac{\tau}{\beta^2} d\tau \geq \dot{g}(s). \tag{2.9}$$

Thus, $\dot{g}(s)$ is bounded in $[0, s_0]$. Then we can use the local existence theorem again to obtain a larger domain of existence $[0, s_0 + \delta]$ for some positive number δ. This is a contradiction. Therefore, (1b) is rejected.

Next we consider the case (1a). From (2.9), $\dot{g}(s)$ has an upper bound in $[0, s_0]$

$$\frac{\sigma s_0}{2\beta^2} \geq \dot{g}(s).$$

It follows that

$$g(s_0) = g(0) + \int_0^{s_0} \dot{g}(\tau) d\tau \leq \beta + \int_0^{s_0} \frac{\sigma s_0}{2\beta^2} d\tau = \beta + \frac{\sigma s_0^2}{2\beta^2}.$$

Since $g(s)$ is bounded above in $[0, s_0]$, this contradicts (1a). So, we can exclude the possibility (1).

Now we claim that the possibility (2b) does not exist. This is because

$$\dot{g}(s) = \frac{\sigma}{s} \int_0^s \frac{\tau}{g^2(\tau)} d\tau \geq \frac{\sigma}{s} \int_0^s \frac{\tau}{\beta^2} d\tau = \frac{\sigma s}{2\beta^2}.$$

Then we have

$$g(s) \geq \beta + \frac{\sigma}{4\beta^2} s^2,$$

and so

$$\lim_{s \to \infty} g(s) = \infty,$$

which contradicts the case (2b). The only remaining case (2a) is our desired conclusion. ■

For $N \geq 4$, our blowup solutions involve solutions of the Lane-Emden equation,

$$\begin{cases} \ddot{g}(s) + \frac{m}{s}\dot{g}(s) + \sigma g^{\xi}(s) = 0 \\ g(0) = \beta > 0, \ \dot{g}(0) = 0, \end{cases} \tag{2.10}$$

where $\xi > 1$ and σ are positive constants, which is reducible to a particular case of the Emden-Fowler equation,

$$\ddot{h} + s^{1-n}h^n = 0,$$

with $n > 1$, using the transformation,

$$h = \frac{g(s)}{s}.$$

For the existence and uniqueness of the equation (2.10), readers may refer to the survey paper by Wong [74].

Proof of Theorem 2.1. Lemma 2.1 means that (2.4) satisfies $(2.3)_1$.

Here we denote $g(z) = g$. For the case of $N \geq 2$ and $N \neq 3$ with $\theta = (2N-3)/N$, we plug the solutions (2.4) into the momentum equation $(2.3)_2$,

$$\begin{aligned} &\rho(V_t + VV_r) + \frac{\alpha(N)\rho}{r^m}\int_0^r \rho\tau^m d\tau - [\kappa\rho^\theta]_r \left(\frac{m}{r}V + V_r\right) - \kappa\rho^\theta\left(V_{rr} + \frac{m}{r}V_r - \frac{m}{r^2}V\right) \\ &= \frac{\alpha(N)\rho}{r^m}\int_0^r \frac{g^{\xi}}{(T-Ct)^N}\tau^m d\tau + \frac{(2N-3)C\kappa}{\xi}\frac{g}{(T-Ct)^{N-3}}\frac{g^{\xi-1}\dot{g}}{(T-Ct)^{N+2}} \\ &= \frac{\rho}{(T-Ct)^m}Q\left(\frac{r}{T-Ct}\right). \end{aligned}$$

Denote

$$Q(s) := Q\left(\frac{r}{T-Ct}\right) = \frac{(2N-3)C\kappa}{\xi}\dot{g}(s) + \frac{\alpha(N)}{s^m}\int_0^s g^{\xi}\tau^m d\tau.$$

Differentiate $Q(s)$ with respect to s,

$$\begin{aligned} \dot{Q}(s) &= \frac{(2N-3)C\kappa}{\xi}\ddot{g} + \alpha(N)g^{\xi} - \frac{m\alpha(N)}{s^m}\int_0^s g^{\xi}\tau^m d\tau \\ &= -\frac{m}{s}Q(s). \end{aligned}$$

The above result holds due to the fact that we have chosen $g(s)$ to satisfy the Lane-Emden equation, namely, the third equation in (2.4). With $Q(0) = 0$, this implies that $Q(s) = 0$. Thus, the momentum equation $(2.3)_2$ is satisfied.

The proof for the case of $N = 3$ is similar and is thus omitted.

Finally, it is obvious that the solutions blow up at the finite time T/C if $C > 0$. This completes the proof. ■

Corollary 2.2 *The blowup rate of the solutions (2.4) and (2.5) is*

$$\lim_{t \to \frac{T}{C}^-} \rho(t,0)(T - Ct)^N \geq O(1).$$

2.3 Periodic Solutions of the Euler-Poisson Equations with a Negative Background Constant

The Goldreich and Weber's solutions of the Euler-Poisson equations (2.1) are for systems with $\Lambda = 0$ and are non-time-periodic. In [84], we were the first to observe that when the cosmological constant Λ is negative, some solutions of the system can exhibit a periodic nature. This new phenomenon lends itself to some interesting physical interpretation.

For simplicity, we consider here the 3-dimensional case, $N = 3$, and choose $\Lambda = -3/4\pi$. Under the assumption of radial symmetry, the system of equations is

$$\begin{cases} \rho_t + V\rho_r + \rho V_r + \dfrac{2}{r}\rho V = 0 \\ \rho\left(V_t + VV_r\right) + K\dfrac{\partial}{\partial r}\rho^\gamma = -\dfrac{4\pi\rho}{r^2}\displaystyle\int_0^r \left(\rho(t,\tau) + \frac{3}{4\pi}\right)\tau^2 d\tau. \end{cases} \tag{2.11}$$

Compare this with (2.1); the cosmological constant appears as the extra term $3/4\pi$ under the integral sign on the right-hand side of the second equation.

The following Theorem is a special case of the result reported in [84].

Theorem 2.2 ([84]) *Assume $\Lambda = -3/4\pi$ and $\gamma = 4/3$. The 3-dimensional Euler-*

Poisson equations in spherical symmetry (2.11) have the following family of solutions

$$
\begin{cases}
\rho(t,r) = \begin{cases} \dfrac{1}{a^3(t)} g^3\left(\dfrac{r}{a(t)}\right) & \text{for} \quad r < a(t)S_\mu \\ 0 & \text{for} \quad a(t)S_\mu \le r \end{cases} \\
V(t,r) = \dfrac{\dot{a}(t)}{a(t)} r \\
\qquad \ddot{a}(t) = \dfrac{\lambda}{a^2(t)} - a(t), \qquad\qquad a(0) = a_0 > 0,\ \dot{a}(0) = a_1 \\
\qquad \ddot{g}(s) + \dfrac{2}{s}\dot{g}(s) + \dfrac{\pi}{K} g^3(s) = \dfrac{3\lambda}{4K}, \quad g(0) = \beta > 0,\ \dot{g}(0) = 0
\end{cases}
\tag{2.12}
$$

where λ, $\alpha_0 > 0$, a_1 and $\beta > 0$ are arbitrary constants, and the finite number S_μ is the first zero of $g(s)$.

(1) When $\lambda \le 0$, the solutions collapse at a finite time T.

(2) When $\lambda > 0$, the solutions are non-trivially time-periodic, except in the case where $a_0 = \sqrt[3]{-\lambda}$ and $a_1 = 0$, when $a(t)$ is a constant and as a result, $\rho(t,r)$ is independent of t while $V(t,r) \equiv 0$.

Remark 2.3 *In this system, the solutions we constructed depend on two auxiliary functions $a(t)$ and $g(s)$, which can vary with the choices of four parameters, λ, $a_0 > 0$, a_1 and $\beta > 0$.*

Remark 2.4 *Both auxiliary functions satisfy a second-order differential equation. In the equation for $g(s)$, the independent variable s appears explicitly in the coefficient and it has a singularity at $s = 0$, just like the corresponding $g(s)$ in Theorem 2.1.*

Remark 2.5 *On the other hand, the equation for the new function $a(t)$ is autonomous, that is, the independent variable t does not appear explicitly in the coefficient. Indeed, it is a Hamiltonian system. This makes it possible to use the well-known tool of energy method to study $a(t)$.*

Remark 2.6 *If $a(t)$ vanishes at some value $t = T > 0$, then $\rho(t,r)$ blows up at $t = T$. On the other hand, if $a(t)$ is a periodic function of t, then $\rho(t,r)$ is also a periodic function of t, uniformly in r.*

The following two lemmas address the two properties of $a(t)$ mentioned in the above remark.

Lemma 2.3 *Suppose that $\lambda \leq 0$. For any given initial conditions a_0 and a_1, the solution $a(t)$ of (2.12)$_3$ must vanish at some $t = T > 0$.*

Proof. Suppose that $a(t)$ does not vanish at any $t > 0$. Since $a(0) > 0$, $a(t) > 0$ for all $t > 0$. It follows from $(2.12)_3$ that, since $\lambda \leq 0$,

$$\ddot{a}(t) + a(t) \leq 0.$$

By the well-known Sturm's comparison theorem in oscillation theory, we see that $a(t)$ must oscillate faster than the solution $y(t)$ of the comparison equation

$$\ddot{y}(t) + y(t) = 0.$$

But $y(t) = A\sin(t + t_0)$ for some constants A and t_0 and it must vanish at some $t = \tau > 0$. Since we know that $a(t)$ oscillates faster than $y(t)$, it must then vanish at some point earlier than τ, contradicting our earlier assumption that $a(t)$ does not vanish. ■

The following is a well-known result in the study of autonomous second-order equations. See, for example, Section 4.3 of Lakin and Sanchez [45]. We include its proof for the sake of completeness.

Lemma 2.4 *Let $K : (0, \infty) \to (0, \infty)$ be a positive C^1 function, such that*

$$\lim_{z \to 0} K(z) = \lim_{z \to \infty} K(z) = \infty \tag{2.13}$$

and its derivative $k(z) = K'(z)$ vanishes only at one value $z = z_0$. Suppose that $a(t)$ satisfies the differential equation

$$\ddot{a}(t) + k(a(t)) = 0. \tag{2.14}$$

Then $a(t)$ is a non-trivial periodic function, unless $a(0) = z_0$ and $\dot{a}(0) = 0$, in which case $a(t) \equiv z_0$ is a constant function.

Proof. Equation (2.14) represents a Hamiltonian system which has a first integral, obtained by multiplying the equation by $\dot{a}(t)$ and then integrating.

$$\frac{\dot{a}^2(t)}{2} + K(a(t)) = E := \frac{\dot{a}^2(0)}{2} + K(a(0)). \tag{2.15}$$

The first term on the left represents the kinetic energy, and the second term $K(a(t))$ the potential energy. The equality (2.15) says that the total energy is conserved, being a constant E. Together with (2.13), (2.15) implies that a solution $a(t)$ must be uniformly bounded from below, away from 0, and uniformly bounded from above.

By assumption, $k(z)$ must be of one sign in each of the intervals $(0, z_0)$ and (z_0, ∞). The condition (2.13) further requires that $k(z) < 0$ for $z < z_0$ and $k(z) > 0$ for $z > z_0$. As a consequence, at some point when $a(t) < z_0$, $\ddot{a}(t) = -k(a(t)) > 0$, and hence $a(t)$ concaves upwards. The same argument shows that, when $a(t) > z_0$, $a(t)$ concaves downwards.

We would like to show that $a(t)$ has an infinite number of local maxima and minima.

Suppose that at some point $t = t_0$, $\dot{a}(t_0) > 0$. We claim that there must exist a point $t_1 \in (t_0, \infty)$ at which $a(t)$ attains a local maximum, with $\dot{a}(t_1) = 0$, and $a(t_1) > z_0$.

We first prove that $\dot{a}(t)$ must vanish at some point beyond t_0. Suppose the contrary, that is, $\dot{a}(t) > 0$ for all $t > t_0$. Then $a(t)$ is an increasing function of t for $t > t_0$. Since $a(t)$ is bounded from above, we conclude that $\lim\limits_{t\to\infty} a(t) = z_1$ exists. From (2.14), we see that $\lim\limits_{t\to\infty} \ddot{a}(t) = -k(z_1)$ also exists. The only way that this will not contradict the previous statement (that $\lim\limits_{t\to a(t)}$ exists) is to require $k(z_1) = 0$, and we thus conclude that $z_1 = z_0$. Since $a(t)$ is increasing for $t > t_0$ and has the limit z_0, we see that $a(t) < z_0$ for all $t > t_0$. Thus, $a(t)$ concaves upward for $t > t_0$. In other words, $\dot{a}(t)$ is an increasing function in (t_0, ∞) and $\dot{a}(t) \geq \dot{a}(t_0) > 0$. This contradicts the earlier statement that $\lim\limits_{t\to\infty} a(t)$ exists. Hence, there must be points beyond t_0 at which $\dot{a}(t)$ vanishes.

Let t_1 be the first of such points. Then $a(t)$ is increasing in (t_0, t_1). If $a(t_1) \leq z_0$, then $a(t) \leq z_0$ for all $t \in (t_0, t_1)$ and $\dot{a}(t)$ is an increasing function in the same interval. This contradicts the assumption that $\dot{a}(t_0) > 0$ and the fact that $\dot{a}(t_1) = 0$. Hence, we must have $a(t_1) > z_0$. Since $a(t)$ concaves downwards at t_1, it has a local maximum at t_1.

We can prove in the same way that if at some point $t = t_2$, $\dot{a}(t_2) > 0$, there must exist a point $t_3 \in (t_2, \infty)$ at which $a(t)$ attains a local minimum, with $\dot{a}(t_3) = 0$, and $a(t_3) < z_0$.

Combining these two claims, we can easily see that there exists a sequence of points $t_1 < t_2 < t_3 < . < t_n < ..$ of alternating local maxima and local minima of $a(t)$. At each t_n, the potential energy are the same as the initial energy

$$K(a(t_n)) = E.$$

Since $K(z)$ is decreasing in $(0, z_0)$ and increasing in (z_0, ∞), there is a unique value in each interval that corresponds to the value E. Hence,

$$a(t_1) = a(t_3) = ... \quad \text{and} \quad a(t_2) = a(t_4) = ...$$

In other words, all the local maxima (minima) have the same value.

It now follows from the uniqueness result of ordinary differential equation that $a(t)$ in (t_1, ∞) is identical to itself in (t_3, ∞) after translation. In other words, $a(t)$ is periodic. ■

Remark 2.7 *An alternative proof can be given using the phase plane technique of dynamical systems. A solution $a(t)$ corresponds to an orbit in the $(a, \dot{a})$ plane. Any closed orbit represents a periodic solution if and only if the time needed to traverse the trajectory is finite. Let $\underline{a} \in (0, z_0)$ and $\overline{a} \in (z_0, \infty)$ be the only points at which $K(\underline{a}) = K(\overline{a}) = E$. By explicitly integrating the energy equation (2.15), we obtain the time to traverse the trajectory, which is*

$$T = \sqrt{2} \int_{\underline{a}}^{\overline{a}} \frac{dz}{\sqrt{E - K(z)}}.$$

This is an improper integral because the denominator of the integrand vanishes at the endpoints $\underline{a}$ and $\overline{a}$. Under the hypotheses of the Lemma, it can be shown easily that the rate of divergence of the integrand at the endpoints is sufficiently tame (more precisely, it is $O((z - \underline{a})^{-1/2})$ and $O((\overline{a} - z)^{-1/2})$) to guarantee that T is finite, and hence the conclusion of the Lemma holds. We omit the details.

For the proof of Theorem 2.2, we need a special case of the Lemma, with $k(z) = z - \lambda/z^2$. Another special case will be used in Sections 4.1 and 5.1.

Corollary 2.3 *Suppose that $\lambda > 0$. For any given initial conditions a_0 and a_1, other than $(a_0 = \sqrt[3]{\lambda}, a_1 = 0)$, the solution $a(t)$ of (2.12)$_3$ is non-trivially periodic. In the exceptional case, $a(t) \equiv \sqrt[3]{\lambda}$ is a constant function.*

Proof of Theorem 2.2. By Lemma 2.1, the solutions given in (2.12) satisfy $(2.3)_1$. Next we show that the momentum equation is also satisfied. For convenience, denote $g(z) = g$. Plugging (2.12) into $(2.11)_2$), we obtain

$$
\begin{aligned}
&\rho(V_t + VV_r) + K\frac{\partial}{\partial r}\rho^{4/3} + \frac{4\pi\rho}{r^2}\int_0^r (\rho - \Lambda)\tau^2 d\tau + \kappa\left(V_{rr} + \frac{m}{r}V_r - \frac{m}{r^2}V\right) \\
&= \rho\frac{\ddot{a}(t)}{a(t)}r + 4K\left(\frac{g^3}{a(t)^3}\right)^{1/3}\frac{g^2\dot{g}}{a(t)^4} + \frac{4\pi\rho}{r^2}\int_0^r (\rho - \Lambda)\tau^2 d\tau \\
&= \rho\left[\frac{\frac{-\lambda}{a^2(t)} + a(t)}{a(t)}r\right] + 4K\frac{g^3}{a(t)^3}\frac{\dot{g}}{a(t)^2} + \frac{4\pi\rho}{r^2a^2(t)}\int_0^r g^3\tau^2 d\tau - \frac{3\rho}{r^2}\int_0^r \tau^2 d\tau \\
&= \rho\frac{-\lambda r}{a^3(t)} + \rho r + 4K\rho\frac{\dot{g}}{a^2(t)} + \frac{4\pi\rho}{r^2a^3(t)}\int_0^r g^3\tau^2 d\tau - \rho r \\
&= \frac{\rho}{a^2(t)}\left[-\frac{\lambda}{a(t)}r + 4K\dot{g} + \frac{4\pi}{r^2a(t)}\int_0^r g^3\tau^2 d\tau\right] \\
&= \frac{\rho}{a^2(t)}\left[-\frac{\lambda}{a(t)}r + 4K\dot{g} + \frac{\alpha(N)}{\left(\frac{r}{a(t)}\right)^2}\int_0^{r/a(t)} g^3\tau^2 d\tau\right] \\
&= \frac{\rho}{a^2(t)}Q\left(\frac{r}{a(t)}\right).
\end{aligned}
\tag{2.16}
$$

In the above, we have used the differential equation $(2.12)_3$ satisfied by $a(t)$. Let us denote

$$
Q\left(\frac{r}{a(t)}\right) = Q(s) = -\lambda s + 4K\dot{g}(s) + \frac{4\pi}{s^2}\int_0^s g^3\tau^2 d\tau
$$

and differentiate $Q(s)$ with respect to s to obtain

$$
\begin{aligned}
\dot{Q}(s) &= -\lambda + 4K\ddot{g}(s) + 4\pi g^3(s) - \frac{2\cdot 4\pi}{s^3}\int_0^s g^3\tau^2 d\tau \\
&= -\frac{2}{s}\left[\lambda s + 4K\dot{g}(s) - K\mu s + \frac{4\pi}{s^2}\int_0^s g^3\tau^2 d\tau\right] \\
&= -\frac{2}{s}Q(s).
\end{aligned}
\tag{2.17}
$$

The second inequality in the above holds due to $(2.12)_4$.

Since $\lim_{s\to 0^+} Q(s) = Q(0) = 0$, (2.17) implies that $Q(s) = 0$ and (2.16) implies that the momentum equation is satisfied.

The statements (1) and (2) are consequences of Lemma 2.3 and Corollary 2.3. ■

Remark 2.8 *The method described in this section can be easily extended to the general Euler-Poisson equations in* R^N*, by following the same approaches used in the previous works [24] and [80]. Please refer to [84] for details.*

Remark 2.9 *The existence of collapsing solutions (with suitable choices of initial values) is similar to the case when cosmological background is absent and is thus not surprising. However, the existence of a time-periodic pattern is characteristic of the negative constant* Λ*. This phenomenon is never seen before in the zero background Euler-Poisson equations, with or without frictional damping, as studied in [79] and [80].*

With suitable initial conditions, for example, $0 < a(0) = \epsilon << 1$ *and* $\dot{a}(0) = 0$*, the density becomes*

$$\rho(0,0) = \frac{\alpha^3}{\epsilon^3} >> M_0$$

where M_0 *is an arbitrary constant. Such solutions provide a possible explanation of how the universe can expand and then almost re-collapse. The density at the origin can be periodically greater than any given constant*

$$\rho(T,0) >> M_0.$$

Notice that this phenomena is not the same as $\rho(T,0) = \infty$ *with a finite time* T*. In the re-collapsing model with a negative cosmological constant* $\Lambda < 0$*, as described in [21], the solutions are periodic with time-singular points. Here, the solutions of the almost re-collapsing model are significantly different from the above situation, as the globally periodical solutions contain no time-singular point. As the time-periodic effect is due to the negative cosmological constant in our model, more solutions of this pattern are expected for other* γ *values. Further work may be done using numerical simulation to study the stability of the solutions. If some gaseous stars (a galaxy) obey the* γ*- law* ($\gamma = 4/3$)*, it may provide an alternative explanation about its evolution. The time-periodic solutions coincide with the expansion segment (the red-shift effect) in a short time. Therefore, it is extremely hard to detect which model is more accurate by the observation.*

2.4 Self-similar Solutions with Elliptical Symmetry

In physics, Sedov [66] in 1953 and Ovsiannikov [63] in 1965 first constructed self-similar solutions of gas dynamic systems having elliptic symmetry. Prior to that, the separation method pioneered by Goldreich, Weber and Makino had only been applied to obtain radially symmetric solutions.

In a radially symmetric solution, the velocities of all fluid particles point directly towards or away from the origin, and the magnitude of the velocities of particles equidistant from the origin are identical. In an elliptically symmetric solution, the velocity of a particle no longer points directly towards the origin. Instead, the components of the velocity along the direction of the coordinate axes vary independently from each other. As a matter of fact, it may happen that along the x-direction, particles move towards each other, while along the y-direction, they move away from each other.

In [95], Yuen, being unaware of the earlier work by Sedov and Ovsiannikov, proposed an even more general concept of elliptic symmetry *with a drift*, and constructed solutions of this type for the Euler equations in R^n. The results are presented in this section.

Theorem 2.3 ([95]) *To the Euler equations in* R^N

$$\begin{cases} \rho_t + \nabla \cdot (\rho \vec{u}) = 0 \\ \rho \left[\vec{u}_t + (\vec{u} \cdot \nabla) \vec{u} \right] + K\rho^\gamma = 0, \end{cases} \tag{2.18}$$

there exists a family of solutions

$$\begin{cases} \rho = \dfrac{f(s)}{\prod\limits_{k=1}^{N} a_k} \\ u_i = \dfrac{\dot{a}_i}{a_i} (x_i + d_i) \qquad \text{for } i = 1, 2, ..., N \end{cases} \tag{2.19}$$

where

$$f(s) = \begin{cases} \beta e^{-\frac{\xi}{2K} s} & \text{for } \gamma = 1 \\ \max \left(\left(-\dfrac{\xi(\gamma - 1)}{2K\gamma} s + \beta \right)^{\frac{1}{\gamma - 1}}, \ 0 \right) & \text{for } \gamma > 1 \end{cases} \tag{2.20}$$

with $s = \sum\limits_{k=1}^{N} \dfrac{(x_k + d_k)^2}{a_k^2(t)}$*, and arbitrary constants* $\beta \geq 0$*,* d_k*,* ξ*; and the auxiliary* *functions $a_i = a_i(t)$ satisfy the Emden dynamical system*

$$\begin{cases} \ddot{a}_i = \dfrac{\xi}{a_i \left(\prod\limits_{k=1}^{N} a_k\right)^{\gamma-1}}, & \text{for } i = 1, 2, ..., N \\ a_i(0) = a_{i0} > 0, \ \dot{a}_i(0) = a_{i1} \end{cases} \tag{2.21}$$

with arbitrary constants a_{i0} and a_{i1}.

Corollary 2.4 *With $\gamma = 1$,*

(1a) for $\xi < 0$, the solutions (2.19) blow up in finite time;

(1b) for $\xi > 0$, the solutions (2.19) exists globally.

With $\gamma > 1$,

(2a) for $\xi < 0$ and some $a_{i1} < 0$, the solutions (2.19) blow up at or before the finite time

$$T = \min(-a_{i0}/a_{i1} : a_{1i} < 0, i = 1, 2, ..., N);$$

(2b) for $\xi > 0$ and $a_{i1} \geq 0$ the solutions (2.19) exist globally.

We make use of a recent result of Yeung and Yuen [78], which gives a general algorithm to generate solutions for the mass equations $(2.19)_1$ using either an arbitrary implicit or explicit function.

Lemma 2.5 (Lemma 1 in [78]) *The conservation of mass equation $(2.18)_1$ has solutions of the following form*

$$\begin{cases} \rho = \dfrac{f\left(\dfrac{x_1 + d_1}{a_1(t)}, \dfrac{x_2 + d_2}{a_2(t)}, \cdots, \dfrac{x_N + d_N}{a_N(t)}\right)}{\prod\limits_{i=1}^{N} a_i(t)} \\ u_i = \dfrac{\dot{a}_i(t)}{a_i(t)}(x_i + d_i) \qquad \text{for } i = 1, 2, ..., N \end{cases} \tag{2.22}$$

with an arbitrary C^1 function $f \geq 0$ and $a_i(t) > 0$ and constants d_i.

Proof. We substitute the second expression involving u_i in (2.22), into $(2.18)_1$, while leaving ρ alone to get

$$\frac{\partial}{\partial t}\rho + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \rho \frac{\dot{a}_i(t)}{a_i(t)}(x_i + d_i) + \sum_{i=1}^{N} \frac{\rho \dot{a}_i(t)}{a_i(t)} = 0.$$

This can be regarded as a semi-linear partial differential equation in the unknown ρ, having the following general solution

$$F\left(\prod_{i=1}^{N} a_i(t)\rho\,,\ \frac{x_1+d_1}{a_1(t)}\,,\ \frac{x_2+d_2}{a_2(t)}\,,\ \cdots\,,\ \frac{x_N+d_N}{a_N,(t)}\right)=0$$

where F is an arbitrary C^1 function for $\rho \geq 0$. By solving the above equation explicitly for the first argument, we obtain the first equation in (2.22). ■

Proof of Theorem 2.3. By Lemma 2.5, the mass equation is satisfied.

Making use of the i-th momentum equation $(2.18)_2$, we obtain

$$\begin{aligned}
&\rho\left[\frac{\partial u_i}{\partial t}+\sum_{k=1}^{N} u_k\frac{\partial u_i}{\partial x_k}\right]+K\frac{\partial}{\partial x_i}\rho^{\gamma}\\
&=\rho\left[\frac{\partial}{\partial t}\left(\frac{\dot a_i}{a_i}(x_i+d_i)\right)+\left(\frac{\dot a_i}{a_i}(x_i+d_i)\right)\frac{\partial}{\partial x_i}\left(\frac{\dot a_i}{a_i}(x_i+d_i)\right)\right]+K\gamma\rho^{\gamma-1}\frac{\partial\rho}{\partial x_i}\\
&=\rho\left\{\left[\left(\frac{\ddot a_i}{a_i}-\frac{(\dot a_i)^2}{(a_i)^2}\right)(x_i+d_i)+\frac{(\dot a_i)^2}{(a_i)^2}(x_i+d_i)\right]+K\gamma\rho^{\gamma-2}\frac{\partial}{\partial x_i}\frac{f(s)}{\prod\limits_{k=1}^{N}a_k}\right\}\\
&=\rho\left\{\frac{\ddot a_i}{a_i}(x_i+d_i)+2K\gamma\frac{f^{\gamma-2}(s)}{\left(\prod\limits_{k=1}^{N}a_k\right)^{\gamma-2}}\frac{\dot f\,(s)}{\left(\prod\limits_{k=1}^{N}a_k\right)}\left(\frac{x_i+d_i}{a_i^2}\right)\right\}\\
&=\frac{(x_i+d_i)\,\rho}{a_i^2}\left\{\ddot a_i a_i+2K\gamma\frac{f^{\gamma-2}(s)\dot f\,(s)}{\left(\prod\limits_{k=1}^{N}a_k\right)^{\gamma-1}}\right\}\\
&=\frac{(x_i+d_i)\,\rho}{a_i^2\left(\prod\limits_{k=1}^{N}a_k\right)^{\gamma-1}}\left\{\xi+2K\gamma f^{\gamma-2}(s)\dot f\,(s)\right\}.
\end{aligned}$$

In the last step, we use the properties of the N-dimensional Emden dynamical system. Local existence of solutions for the Emden dynamical system can be guaranteed by the usual existence theory, established, for example, by using the Banach fixed point theorem.

To further simplify the above expression, we require either the first order ordinary differential equation

$$\xi+2K\gamma f^{\gamma-2}(s)\dot f(s)=0,\quad f(0)=\beta\geq 0,$$

to hold or $\rho=0$. By solving this differential equation, we arrive at the required expression (2.20) for f. ■

Remark 2.10 *Even though they all satisfy similar differential equations, the* N *functions* $a_i(t)$ *can differ from each other because different initial conditions can be imposed on them. Even if we start out with requiring all* a_{i0} *to be equal, the* a_{i1} *can also be chosen to be different so that at a subsequent time,* $a_i(t)$ *can still be different from each other.*

An interesting situation arises if we choose $a_{10} > 0$ *and* $a_{20} < 0$*. Along the* x_1 *axis direction, particles are moving towards each other, while along the* x_2 *axis direction, they move away from each other. It is natural to ask how these initial values can affect the blowup or global behaviour of the solutions. Corollary 2.4 answers this question partially. It will be nice if a complete answer can be achieved.*

Remark 2.11 *When all the initial conditions for* $a_i(t)$ *are chosen to be the same, all the functions* a_i *become identical and we fall back to the radial symmetric case. Hence, radial symmetry is subsumed under elliptic symmetry. The term "elliptic" is derived from the fact that at all points on the ellipsoid*

$$\sum_{k=1}^{N} \frac{(x_k + d_k)^2}{a_k^2(t)} = \text{ constant},$$

all the fluid particles have the same density ρ *and speed* $|\vec{u}|$.

Remark 2.12 *The vector* $(d_1, d_2, ., d_N)$ *represents the drift. When* $d_i = 0$ *for all* i*, our concept of elliptic symmetry reduces to that of Sedov [66] and Ovsiannikov [63].*

Remark 2.13 *There are physical applications of the fluids with elliptic symmetry. For detail, see Gornushkin et al [33] and Baxter-Shabanov [3].*

Proof of Corollary 2.4. With $\gamma = 1$, the Emden dynamical system (2.21) becomes N de-coupled conventional Emden equations, each one having the form

$$\ddot{a}_i(t) = \frac{\xi}{a_i(t)}, \qquad a_i(0) = a_{i0} > 0, \quad \dot{a}_i(0) = a_{i1}.$$

It is not difficult to see that if $\xi < 0$, $a_i(t)$ must vanish at some finite time. As a consequence, the corresponding solution (2.19) blows up. This proves (1a).

(1b) For $\xi > 0$, it is well-known that the functions $a_i(t)$ does not vanish at any $t > 0$ and hence the solutions (2.19) exist globally.

Next, we turn to the case of $\gamma > 1$.

(2a) For $\xi < 0$ and some $a_{i1} < 0$, we see that $\ddot{a}_i(t) \le 0$, implying that $\dot{a}_i(t) \le a_{i1} < 0$. Hence, $a_i(t)$ must vanish at or before the finite time

$$T = \min(-a_{i0}/a_{i1} : a_{i1} < 0,\ i = 1, 2, ..., N)$$

and the corresponding ρ blows up at the same time.

(2b) For $\xi > 0$ and all $a_{i1} \ge 0$, it is clear that $a_i(t)$ will not vanish for any $t > 0$ and the solution (2.19) exist globally. ■

In our paper [82], we reported that a result analogous to Theorem 2.3 also holds when the pressure term in (2.18) is replaced by a density-dependent viscosity term.

$$\begin{cases} \rho_t + \nabla \cdot (\rho \vec{u}) = 0 \\ \rho \left[\dfrac{\partial}{\partial t} u_i + \vec{u} \cdot \nabla u_i \right] = \kappa \dfrac{\partial}{\partial x_i} \left(\rho^\theta \nabla \cdot \vec{u} \right) \text{ for } i = 1, 2, ..., N \end{cases} . \tag{2.23}$$

This observation was pursued further in a subsequent paper [96] and the following general result was obtained.

Theorem 2.4 ([96]) *The pressureless density-dependent Navier-Stokes equations (2.23) with $\kappa > 0$ in R^N has the following family of solutions*

$$\begin{cases} \rho = \dfrac{f(s)}{\prod\limits_{k=1}^{N} a_k} \\ u_i = \dfrac{\dot{a}_i}{a_i} (x_i + d_i) \qquad \text{for } i = 1, 2, ..., N \end{cases} \tag{2.24}$$

where

$$f(s) = \begin{cases} \beta e^{-\frac{\xi}{2\kappa} s} & \text{for } \theta = 1 \\ \max \left(\left(-\dfrac{\xi(\theta - 1)}{2\kappa\gamma} s + \beta \right)^{\frac{1}{\theta - 1}},\ 0 \right) & \text{for } \theta \ne 1 \end{cases}$$

with $s = \sum\limits_{k=1}^{N} \dfrac{(x_k + d_k)^2}{a_k^2(t)}$, *arbitrary constants* $\beta \ge 0$, d_k *and* ξ*; and the auxiliary functions* $a_i = a_i(t)$ *satisfy the Emden dynamical system*

$$\begin{cases} \ddot{a}_i(t) = \dfrac{-\xi \left(\sum\limits_{k=1}^{N} \frac{\dot{a}_k(t)}{a_k(t)} \right)}{a_i(t) \left(\prod\limits_{k=1}^{N} a_k(t) \right)^{\theta - 1}} \qquad \text{for } i = 1, 2, ..., N \\ a_i(0) = a_{i0} > 0,\ \dot{a}_i(0) = a_{i1} \end{cases}$$

with arbitrary constants a_{i0} and a_{i1}.

In particular, for $\xi < 0$,

(1) if all $a_{i1} < 0$, the solutions (2.24) blow up at or before the finite time

$$T = \min(-a_{i0}/a_{i1} : a_{1i} < 0,\ i = 1, 2, ..., N);$$

(2) if all $a_{i1} \geq 0$ the solutions (2.24) exist globally.

The proof of this Theorem is similar to that of the previous one. We omit the details. Interested readers can consult the source material [96].

2.5 Qualitative Properties of the Emden System

The Emden system is the reduced system of the solutions with elliptical symmetry for the Euler equations in the last section. When the qualitative properties (globally positive or non-globally positive) of the system are known, the corresponding blowup or global existence of the analytical solutions with elliptical symmetry can be determined. In this section, we list some non-trivial qualitative properties for the Emden system.

A solution of the Emden system

$$\ddot{a}_i(t) = -\frac{1}{a_i(t)\left(\prod\limits_{k=1}^{N} a_k(t)\right)^{\gamma-1}},\ i = 1, 2, ..., N, \tag{2.25}$$

with the constant $\gamma > 1$, is said to be *globally positive* (GP) if $a_i(t) > 0$ for all $t > 0$ and for all i. A solution that is not GP is said to be non-GP (NGP).

If the Emden system is NGP, the corresponding solution with elliptical symmetry blows up at a finite time; If the Emden system is GP, the corresponding solution exists globally.

Proposition 2.5 *The system possesses both GP and NGP solutions.*

Proof: With the particular choice $a_1(0) = a_2(0) = ... = a_N(0)$ and $\dot{a}_1(0) = \dot{a}_2(0) = ...\dot{a}_N(0)$, the system becomes the classical Emden equation. The well known Emden equation possesses both GP and NGP solutions.

Proposition 2.6 *The Emden system (2.25) has the invariant*

$$E(t) = \sum_{k=i}^{N} \frac{\dot{a}_i^2(t)}{2} - \frac{1}{(\gamma-1)\left(\prod_{k=1}^{N} a_k(t)\right)^{\gamma-1}} = \theta. \tag{2.26}$$

Proof: By differentiating $E(t)$, we have

$$\begin{aligned}
&\frac{dE(t)}{dt} \\
&= \sum_{i=1}^{N} \dot{a}_i(t)\ddot{a}_i(t) + \sum_{i=1}^{N} \frac{(\gamma-1)\dot{a}_i(t)}{(\gamma-1)a_i(t)\left(\prod_{k=1}^{N} a_k(t)\right)^{\gamma-1}} \\
&= \sum_{i=1}^{N} \frac{-\dot{a}_i(t)}{a_i(t)\left(\prod_{k=1}^{N} a_k(t)\right)^{\gamma-1}} + \sum_{i=1}^{N} \frac{\dot{a}_i(t)}{a_i(t)\left(\prod_{k=1}^{N} a_k(t)\right)^{\gamma-1}} \\
&= 0.
\end{aligned}$$

Hence, $E(t)$ is a constant.

Corollary 2.5 *Let θ be defined as in (2.26). If $\theta < 0$, then the solution is NGP.*

Proof: From the definition of θ and (2.25), we see that, for any i,

$$\theta = \sum_{i=1}^{N} \frac{\dot{a}_i^2(t)}{2} + \frac{a_i(t)\ddot{a}_i(t)}{\gamma - 1}.$$

Hence,

$$\begin{aligned}
a_i(t)\ddot{a}_i(t) = (\gamma-1)\left(\theta - \sum_{i=1}^{N} \frac{\dot{a}_i^2(t)}{2}\right) &\le (\gamma-1)\theta \\
\ddot{a}_i(t) &\le \frac{(\gamma-1)\theta}{a_i(t)}.
\end{aligned}$$

We know that the solution of the above Emden equation blows up at a finite time t for $\theta < 0$. Therefore, the solution is NGP.

We give an alternative proof. Suppose the contrary, that is, $a_i(t)$ is GP. Then each $a_i(t)$ is increasing and so it either diverges to ∞ or converge to a positive constant. They cannot be all bounded. Otherwise the system (2.25) shows that $\ddot{a}_i(t)$ converges to a negative constant and that implies that $\dot{a}_i(t)$ will eventually become negative.

Hence,

$$\frac{1}{(\gamma-1)\left(\prod_{k=1}^{N} a_k(t)\right)^{\gamma-1}} \to 0$$

and

$$\sum_{i=1}^{N} \frac{\dot{a}_i^2(t)}{2} = \theta + \frac{1}{(\gamma-1)\left(\prod\limits_{k=1}^{N} a_k(t)\right)^{\gamma-1}} \to \theta < 0$$

and some $\dot{a}_k(t)$ must be negative for large t. Therefore, the solution is NGP.

When the total energy is non-negative, we have the following results:

Proposition 2.7 *(1) If for $N \geq 2$ and $\gamma \geq 2$, any one of the solutions of the following inequalities:*

$$\begin{cases} \ddot{A}_i(t) \leq \dfrac{-1}{A_i^{\gamma}(t)\prod\limits_{k\neq i}^{N}\left\{2(\gamma-1)\left[\theta t^2+(a_{j0}a_{j1}+a_{k0}a_{k1})t+\frac{\left(a_{j1}\right)^2}{2}+\frac{(a_{k1})^2}{2}\right]-A_i^2(t)\right\}^{\frac{\gamma-1}{2}}} \\ A_i(0) = a_{i0} > 0,\ \dot{A}_i(0) = a_{i1} > 0 \end{cases}$$

blows up, then the corresponding solution of the Emden system (2.25) blows up;
(2) If for $N = 2$ and $1 < \gamma \leq 2$, the solutions of the following inequalities:

$$\begin{cases} \ddot{A}_1(t) \geq \dfrac{-1}{A_1^{\gamma}(t)\left\{2(\gamma-1)\left[\theta t^2+(a_{10}a_{11}+a_{20}a_{21})t+\frac{(a_{11})^2}{2}+\frac{(a_{21})^2}{2}\right]-A_1^2(t)\right\}^{\frac{\gamma-1}{2}}} \\ A_1(0) = a_{10} > 0,\ \dot{A}_1(0) = a_{11} > 0 \end{cases}$$

and

$$\begin{cases} \ddot{A}_2(t) \geq \dfrac{-1}{A_2^{\gamma}(t)\left\{2(\gamma-1)\left[\theta t^2+(a_{10}a_{11}+a_{20}a_{21})t+\frac{(a_{11})^2}{2}+\frac{(a_{21})^2}{2}\right]-A_2^2(t)\right\}^{\frac{\gamma-1}{2}}} \\ A_2(0) = a_{20} > 0,\ \dot{A}_2(0) = a_{21} > 0 \end{cases}$$

exist globally, then the solutions of the Emden system exist globally.

Proof: We have

$$\frac{1}{\gamma-1} a_j(t)\ddot{a}_j(t) = \theta - \sum_{i=1}^{N} \frac{\dot{a}_i^2(t)}{2} \quad \text{for } j \in 1, 2, ...N.$$

We add the above equations to get

$$\frac{1}{\gamma-1} a_j(t)\ddot{a}_j(t) + a_i(t)\ddot{a}_i(t) = 2\theta - \sum_{k=1}^{N} \dot{a}_k^2(t) \text{ for } j \neq i$$

$$\frac{1}{\gamma-1}\left(a_j(t)\ddot{a}_j(t) + a_i(t)\ddot{a}_i(t) + \dot{a}_j^2(t) + \dot{a}_i^2(t)\right) = 2\theta - \frac{\gamma-2}{\gamma-1}\left(\sum_{k=1}^{N} \dot{a}_k^2(t)\right)$$

$$\frac{1}{\gamma-1}\frac{d}{dt}(a_j(t)\dot{a}_j(t) + a_i(t)\dot{a}_i(t)) = 2\theta - \frac{\gamma-2}{\gamma-1}\left(\sum_{k=1}^{N} \dot{a}_k^2(t)\right) \leq 2\theta$$

for $\gamma \geq 2$,

$$\frac{a_j(t)\dot{a}_j(t) + a_i(t)\dot{a}_i(t)}{\gamma - 1} \leq 2\theta t + a_{j0}a_{j1} + a_{i0}a_{i1}$$

$$\frac{1}{2(\gamma-1)}\frac{d}{dt}\left[a_j^2(t) + a_i^2(t)\right] \leq 2\theta t + a_{j0}a_{j1} + a_{i0}a_{i1}$$

$$\frac{1}{2(\gamma-1)}\left[a_j^2(t) + a_i^2(t)\right] \leq \theta t^2 + (a_{j0}a_{j1} + a_{i0}a_{i1})t + \frac{(a_{j1})^2}{2} + \frac{(a_{k1})^2}{2}$$

$$a_j^2(t) \leq 2(\gamma-1)\left[\theta t^2 + (a_{j0}a_{j1} + a_{k0}a_{k1})t + \frac{(a_{j1})^2}{2} + \frac{(a_{k1})^2}{2}\right] - a_k^2(t)$$

$$a_j(t) \leq \left\{2(\gamma-1)\left[\theta t^2 + (a_{j0}a_{j1} + a_{i0}a_{i1})t + \frac{(a_{j1})^2}{2} + \frac{(a_{i1})^2}{2}\right] - a_i^2(t)\right\}^{\frac{1}{2}}.$$

We can, therefore, estimate the Emden system by the N Emden inequalities as stated in the Proposition and arrive at the desired conclusion.

In the following, we restrict ourselves to the 2-dimensional case ($N = 2$). For convenience, we write $a(t)$ and $b(t)$ instead of $a_1(t)$ and $a_2(t)$, and denote

$$\alpha = a(0), \quad \alpha_1 = \dot{a}(0), \quad \beta = b(0) \quad \text{and} \quad \beta_1 = \dot{b}(0).$$

The following comparison properties is useful.

Proposition 2.8 *Suppose*

$$\alpha \leq \beta, \quad \frac{\alpha_1}{\alpha} < \frac{\beta_1}{\beta}$$

and $a(t) > 0$ *for* $t \in [0, T)$.

Then

$$\frac{\dot{a}(t)}{a(t)} < \frac{\dot{b}(t)}{b(t)}$$

and this implies that

$$a(t) < b(t), \quad \dot{a}(t) < \dot{b}(t).$$

Proof: We treat the Emden system like second order linear differential equations:

$$\begin{cases} \ddot{a}(t) + \dfrac{1}{a^{\gamma+1}(t)b^{\gamma-1}(t)}a(t) = 0, & a(0) = \alpha > 0,\ \dot{a}(0) = \alpha_1 \\ \ddot{b}(t) + \dfrac{1}{b^{\gamma+1}(t)a^{\gamma-1}(t)}b(t) = 0, & b(0) = \beta > 0,\ \dot{b}(0) = \beta_1. \end{cases} \tag{2.27}$$

Suppose the proposition is not true. That means there exists a first finite time T, such that $a(T) = b(T)$. We may apply the condition

$$a(t) < b(t) \quad \text{for } 0 < t < T$$

for the linear system (2.27) to get

$$\frac{1}{a^{\gamma+1}(t)b^{\gamma-1}(t)} > \frac{1}{b^{\gamma+1}(t)a^{\gamma-1}(t)} \quad \text{for } 0 < t < T.$$

Then we apply the well-known Sturm's comparison theorem (see, for example, Chapter 8 of Coddington and Levinson's book [19] for details) to obtain

$$\frac{\dot{a}(T)}{a(T)} < \frac{\dot{b}(T)}{b(T)}.$$

which contradicts our assumption, thus proving the proposition.

Proposition 2.9 *Suppose that the solutions $a_1(t)$ and $b_1(t)$ of*

$$\begin{cases} \ddot{a}_1(t) + \dfrac{1}{a_1(t)\left(a_1(t)b_1(t)\right)^{\gamma-1}} = 0, & a_1(0) = a_{10} > 0,\ \dot{a}_1(0) = a_{11} \\ \ddot{b}_1(t) + \dfrac{1}{b_1(t)\left(a_1(t)b_1(t)\right)^{\gamma-1}} = 0, & b_1(0) = b_{10} > 0,\ \dot{b}_1(0) = b_{11}, \end{cases} \tag{2.28}$$

are GP and $a_2(t)$ and $b_2(t)$ are solutions of

$$\begin{cases} \ddot{a}_2(t) + \dfrac{1}{a_2(t)\left(a_2(t)b_2(t)\right)^{\gamma-1}} = 0, & a_2(0) = a_{20} > 0,\ \dot{a}_2(0) = a_{21} \\ \ddot{b}_2(t) + \dfrac{1}{b_2(t)\left(a_2(t)b_2(t)\right)^{\gamma-1}} = 0, & b_2(0) = b_{20} > 0,\ \dot{b}_2(0) = b_{21}, \end{cases} \tag{2.29}$$

with

$$a_{10} < a_{20}, \quad b_{10} < b_{20}, \tag{2.30}$$

and

$$\frac{a_{11}}{a_{10}} < \frac{a_{21}}{a_{20}}, \quad \frac{b_{11}}{b_{10}} < \frac{b_{21}}{b_{20}}. \tag{2.31}$$

Then

$$a_1(t) < a_2(t) \quad , b_1(t) < b_2(t) \tag{2.32}$$

and $a_2(t)$ and $b_2(t)$ are also GP.

Proof: The proposition is proved by comparing equations $(2.28)_1$-$(2.29)_1$ with $(2.28)_2$-$(2.29)_2$.

Similarly, we have the following corollary:

Corollary 2.6 *Suppose that the solutions $a_1(t)$ and $b_1(t)$ of*

$$\begin{cases} \ddot{a}_1(t) + \dfrac{1}{a_1(t)\left(a_1(t)b_1(t)\right)^{\gamma-1}} = 0, & a_1(0) = a_{10} > 0,\ \dot{a}_1(0) = a_{11} \\ \ddot{b}_1(t) + \dfrac{1}{b_1(t)\left(a_1(t)b_1(t)\right)^{\gamma-1}} = 0, & b_1(0) = b_{10} > 0,\ \dot{b}_1(0) = b_{11}, \end{cases}$$

are NGP and $a_2(t)$ and $b_2(t)$ are solutions of

$$\begin{cases} \ddot{a}_2(t) + \dfrac{1}{a_2(t)\,(a_2(t)b_2(t))^{\gamma-1}} = 0, & a_2(0) = a_{20} > 0,\ \dot{a}_2(0) = a_{21} \\ \ddot{b}_2(t) + \dfrac{1}{b_2(t)\,(a_2(t)b_2(t))^{\gamma-1}} = 0, & b_2(0) = b_{20} > 0,\ \dot{b}_2(0) = b_{21}, \end{cases}$$

with

$$a_{10} > a_{20}, \quad b_{10} > b_{20}$$

and

$$\frac{a_{11}}{a_{10}} > \frac{a_{21}}{a_{20}}, \quad \frac{b_{11}}{b_{10}} > \frac{b_{21}}{b_{20}}.$$

Then

$$a_1(t) > a_2(t),\ b_1(t) > b_2(t)$$

and $a_2(t)$ and $b_2(t)$ are also NGP.

Proposition 2.10 *(a) Given $\alpha, \alpha_1, \beta > 0$, there exists $\beta_1 > 0$ such that the corresponding solution is GP.*

(b) Given $\alpha, \alpha_1, \beta_1 > 0$, there exists $\beta > 0$ such that the corresponding solution is GP.

Proof: (a) Integrating the Emden system over $[0, T]$, we get

$$\dot{a}(t) = \alpha_1 - \int_0^t \frac{ds}{a(s)\,(a(s)b(s))^{\gamma-1}} \tag{2.33}$$

and

$$\dot{b}(t) = \beta_1 - \int_0^t \frac{ds}{b(s)\,(a(s)b(s))^{\gamma-1}}. \tag{2.34}$$

We claim that if β_1 is sufficiently large, then

$$\dot{a}(t) \geq \frac{\alpha_1}{2} \quad \text{and} \quad \dot{b}(t) \geq \frac{\beta_1}{2} \quad \text{for } t \geq 0. \tag{2.35}$$

Suppose this is false. Then there exists a finite time $T > 0$ such that either

$$\dot{a}(T) = \frac{\alpha_1}{2} \quad \text{or} \quad \dot{b}(T) = \frac{\beta_1}{2}. \tag{2.36}$$

Then in $[0, T]$, we have

$$a(t) \geq \alpha + \frac{\alpha_1 t}{2}, \qquad b(t) \geq \beta + \frac{\beta_1 t}{2}. \tag{2.37}$$

We substitute the above inequalities into the equalities (2.33) to get

$$\begin{aligned}
\dot{b}(T) &= \beta_1 - \int_0^T \frac{ds}{b(s)\,(a(s)b(s))^{\gamma-1}} \\
&\geq \beta_1 - \int_0^T \frac{ds}{\left(\alpha + \frac{\alpha_1 s}{2}\right)^{\gamma-1}\left(\beta + \frac{\beta_1 s}{2}\right)^{\gamma}} \\
&= \beta_1 - \int_0^T \frac{ds}{\left(\frac{\alpha}{\beta}\right)^{\gamma-1}\left(\beta + \frac{\alpha_1 \beta s}{2\alpha}\right)^{\gamma-1}\left(\beta + \frac{\beta_1 s}{2}\right)^{\gamma}} \\
&\geq \beta_1 - \frac{1}{\left(\frac{\alpha}{\beta}\right)^{\gamma-1}} \int_0^T \frac{ds}{\left(\beta + \frac{\alpha_1 \beta s}{2\alpha}\right)^{2\gamma-1}}
\end{aligned}$$

By choosing $\beta_1 \geq \dfrac{\alpha_1 \beta}{\alpha}$, the last expression is

$$\begin{aligned}
&= \beta_1 - \left[\frac{-1}{(2\gamma-2)\left(\frac{\alpha}{\beta}\right)^{\gamma-1}} \frac{2\alpha}{\alpha_1 \beta} \frac{1}{\left(\beta + \frac{\alpha_1 \beta s}{2\alpha}\right)^{2\gamma-2}} \right]_{s=0}^{s=T} \\
&= \beta_1 + \frac{-1}{(2\gamma-2)\left(\frac{\alpha}{\beta}\right)^{\gamma-1}} \frac{2\alpha}{\alpha_1 \beta} \frac{1}{\left(\beta + \frac{\alpha_1 \beta T}{2\alpha}\right)^{2\gamma-2}} \\
&\quad - \frac{1}{(2\gamma-2)\left(\frac{\alpha}{\beta}\right)^{\gamma-1}} \frac{2\alpha}{\alpha_1 \beta} \frac{1}{\beta^{2\gamma-2}} \\
&\geq \beta_1 - \frac{1}{(2\gamma-2)\left(\frac{\alpha}{\beta}\right)^{\gamma-1}} \frac{2\alpha}{\alpha_1 \beta} \frac{1}{\beta^{2\gamma-2}} \\
&\geq \frac{2\beta_1}{3}.
\end{aligned}$$

The last inequality is obtained by choosing

$$\beta_1 - \frac{1}{(2\gamma-2)\left(\frac{\alpha}{\beta}\right)^{\gamma-1}} \frac{2\alpha}{\alpha_1 \beta} \frac{1}{\beta^{2\gamma-2}} \geq \frac{2\beta_1}{3}.$$

Similarly, for the equality (2.34), we have:

$$\begin{aligned}
\dot{a}(T) &= \alpha_1 - \int_0^T \frac{ds}{a(s)\,(a(s)b(s))^{\gamma-1}} \\
&\geq \alpha_1 - \int_0^T \frac{ds}{\left(\alpha + \frac{\alpha_1 s}{2}\right)^{\gamma}\left(\beta + \frac{\beta_1 s}{2}\right)^{\gamma-1}} \\
&= \alpha_1 - \int_0^T \frac{ds}{\left(\frac{\beta}{\alpha}\right)^{\gamma-1}\left(\alpha + \frac{\alpha_1 s}{2}\right)^{\gamma}\left(\beta + \frac{\alpha \beta_1 s}{2\beta}\right)^{\gamma-1}}.
\end{aligned} \tag{2.38}$$

We may control the above integral as the follows:

$$
\begin{aligned}
&\int_0^T \frac{ds}{\left(\frac{\beta}{\alpha}\right)^{\gamma-1}\left(\alpha+\frac{\alpha_1 s}{2}\right)^{\gamma}\left(\beta+\frac{\alpha\beta_1 s}{2\beta}\right)^{\gamma-1}} \\
&= \int_0^1 \frac{ds}{\left(\frac{\beta}{\alpha}\right)^{\gamma-1}\left(\alpha+\frac{\alpha_1 s}{2}\right)^{\gamma}\left(\alpha+\frac{\alpha\beta_1 s}{2\beta}\right)^{\gamma-1}} \\
&\quad + \int_1^T \frac{ds}{\left(\frac{\beta}{\alpha}\right)^{\gamma-1}\left(\alpha+\frac{\alpha_1 s}{2}\right)^{\gamma}\left(\alpha+\frac{\alpha\beta_1 s}{2\beta}\right)^{\gamma-1}} \\
&\le \int_0^1 \frac{ds}{\alpha\beta^{\gamma-1}\left(\alpha+\frac{\alpha\beta_1 s}{2\beta}\right)^{\gamma-1}} + \int_1^T \frac{ds}{\left(\frac{\beta}{\alpha}\right)^{\gamma-1}\left(\frac{\alpha\beta_1}{2\beta}\right)^{\gamma-1}\left(\alpha+\frac{\alpha_1 s}{2}\right)^{\gamma}}.
\end{aligned}
$$

We examine the above first integral in three cases:

Case 1: $1 < \gamma < 2$,

$$
\begin{aligned}
&\int_0^1 \frac{ds}{\alpha\beta^{\gamma-1}\left(\alpha+\frac{\alpha\beta_1 s}{2\beta}\right)^{\gamma-1}} \\
&\le \frac{1}{\alpha\beta^{\gamma-1}\left(\frac{\alpha\beta_1}{2\beta}\right)^{\gamma-1}} \int_0^1 \frac{ds}{s^{\gamma-1}} \\
&= \frac{1}{\alpha\beta^{\gamma-1}\left(\frac{\alpha\beta_1}{2\beta}\right)^{\gamma-1}(2-\gamma)}\; s^{2-\gamma}\Big|_{s=0}^{s=1} \\
&= \frac{1}{\alpha\beta^{\gamma-1}\left(\frac{\alpha\beta_1}{2\beta}\right)^{\gamma-1}(2-\gamma)}.
\end{aligned}
$$

Case 2: $\gamma = 2$,

$$
\begin{aligned}
&\int_0^1 \frac{ds}{\alpha\beta^{\gamma-1}\left(\alpha+\frac{\alpha\beta_1 s}{2\beta}\right)^{\gamma-1}} \\
&= \frac{1}{\alpha\beta}\frac{2\beta}{\alpha\beta_1}\ln\left(\alpha+\frac{\alpha\beta_1 s}{2\beta}\right)\Bigg|_{s=0}^{s=1} \\
&= \frac{2}{\alpha^2\beta_1}\left[\ln\left(\alpha+\frac{\alpha\beta_1}{2\beta}\right)-\ln\alpha\right].
\end{aligned}
$$

Case 3: $\gamma > 2$,

$$
\begin{aligned}
&\int_0^1 \frac{ds}{\alpha\beta^{\gamma-1}\left(\alpha+\frac{\alpha\beta_1 s}{2\beta}\right)^{\gamma-1}} \\
&= \frac{1}{\alpha\beta^{\gamma-1}(\gamma-2)\frac{2\beta}{\alpha\beta_1}}\frac{1}{\left(\alpha+\frac{\alpha\beta_1 s}{2\beta}\right)^{\gamma-2}}\Bigg|_{s=0}^{s=1}
\end{aligned}
$$

$$= \frac{1}{\alpha\beta^{\gamma-1}(\gamma-2)\frac{2\beta}{\alpha\beta_1}} \left[\frac{1}{\left(\alpha + \frac{\alpha\beta_1}{2\beta}\right)^{\gamma-2}} - \frac{1}{\alpha^{\gamma-2}} \right].$$

We require a sufficiently large β_1 such that

$$\begin{cases} \dfrac{1}{\alpha\beta^{\gamma-1}\left(\frac{\alpha\beta_1}{2\beta}\right)^{\gamma-1}(2-\gamma)} \le \dfrac{\alpha_1}{6} & \text{for } 1 < \gamma < 2 \\ \dfrac{2}{\alpha^2\beta_1}\left[\ln\left(\alpha + \dfrac{\alpha\beta_1}{2\beta}\right) - \ln\alpha\right] \le \dfrac{\alpha_1}{6} & \text{for } \gamma = 2 \\ \dfrac{1}{\alpha\beta^{\gamma-1}(\gamma-2)\frac{2\beta}{\alpha\beta_1}}\left[\dfrac{1}{\left(\alpha + \frac{\alpha\beta_1}{2\beta}\right)^{\gamma-2}} - \dfrac{1}{\alpha^{\gamma-2}}\right] \le \dfrac{\alpha_1}{6} & \text{for } \gamma > 2. \end{cases}$$

Then, we control the second term of the equation as follows:

$$\begin{aligned} &\int_1^T \frac{ds}{\left(\frac{\beta}{\alpha}\right)^{\gamma-1}\left(\frac{\alpha\beta_1}{2\beta}\right)^{\gamma-1}\left(\alpha + \frac{\alpha_1 s}{2}\right)^{\gamma}} \\ &= -\frac{1}{\left(\frac{\beta}{\alpha}\right)^{\gamma-1}\left(\frac{\alpha\beta_1}{2\beta}\right)^{\gamma-1}} \frac{1}{\left(\alpha + \frac{\alpha_1 s}{2}\right)^{\gamma-1}}\Bigg|_{s=1}^{s=T} \\ &= -\frac{1}{\left(\frac{\beta}{\alpha}\right)^{\gamma-1}\left(\frac{\alpha\beta_1}{2\beta}\right)^{\gamma-1}} \frac{1}{\left(\alpha + \frac{\alpha_1 T}{2}\right)^{\gamma-1}} + \frac{1}{\left(\frac{\beta}{\alpha}\right)^{\gamma-1}\left(\frac{\alpha\beta_1}{2\beta}\right)^{\gamma-1}} \frac{1}{\left(\alpha + \frac{\alpha_1}{2}\right)^{\gamma-1}} \\ &\le \frac{\alpha_1}{6} \end{aligned}$$

by choosing

$$\frac{1}{\left(\frac{\beta}{\alpha}\right)^{\gamma-1}\left(\frac{\alpha\beta_1}{2\beta}\right)^{\gamma-1}} \frac{1}{\left(\alpha + \frac{\alpha_1}{2}\right)^{\gamma-1}} \le \frac{\alpha_1}{6}.$$

After that, equation (2.38) becomes

$$\dot{a}(T) \ge \alpha_1 - \frac{\alpha_1}{3} = \frac{2\alpha_1}{3}.$$

This contradicts the equations in (2.36). Thus, given $\alpha, \alpha_1, \beta > 0$, there exists $\beta_1 > 0$ such that the corresponding solution is GP.

(b) We claim that if β is sufficiently large, then

$$\dot{a}(t) \ge \frac{\alpha_1}{2} \quad \text{and} \quad \dot{b}(t) \ge \frac{\beta_1}{2} \qquad \text{for } t \ge 0. \tag{2.39}$$

Suppose this is false. Then there exists a finite time $T > 0$ such that either

$$\dot{a}(T) = \frac{\alpha_1}{2} \quad \text{or} \quad \dot{b}(T) = \frac{\beta_1}{2}. \tag{2.40}$$

Then in $[0, T]$, we have

$$a(t) \geq \alpha + \frac{\alpha_1 t}{2} \quad \text{and} \quad b(t) \geq \beta + \frac{\beta_1 t}{2}. \tag{2.41}$$

We substitute these inequalities into equation (2.33) to get

$$\begin{aligned}
\dot{a}(T) &= \alpha_1 - \int_0^T \frac{ds}{a^\gamma(s) b^{\gamma-1}(s)} \\
&\geq \alpha_1 - \int_0^T \frac{ds}{\left(\alpha + \frac{\alpha_1 s}{2}\right)^\gamma \left(\beta + \frac{\beta_1 s}{2}\right)^{\gamma-1}} \\
&\geq \alpha_1 - \int_0^T \frac{ds}{\beta^{\gamma-1} \left(\alpha + \frac{\alpha_1 s}{2}\right)^\gamma} \\
&\geq \alpha_1 + \left. \frac{1}{\frac{\alpha_1}{2} \beta^{\gamma-1} (\gamma - 1) \left(\alpha + \frac{\alpha_1 s}{2}\right)^{\gamma-1}} \right|_{s=0}^{s=T} \\
&\geq \alpha_1 + \frac{2}{\frac{\alpha_1}{2} \beta^{\gamma-1} (\gamma - 1) \left(\alpha + \frac{\alpha_1 T}{2}\right)^{\gamma-1}} - \frac{2}{\frac{\alpha_1}{2} \beta^{\gamma-1} (\gamma - 1) \alpha^{\gamma-1}} \\
&\geq \frac{2\alpha_1}{3}
\end{aligned}$$

by choosing

$$\alpha_1 - \frac{2}{\frac{\alpha_1}{2} \beta^{\gamma-1} (\gamma - 1) \alpha^{\gamma-1}} \geq \frac{2\alpha_1}{3}.$$

For equation (2.34), we have

$$\begin{aligned}
\dot{b}(T) &= \beta_1 - \int_0^T \frac{ds}{a^{\gamma-1}(s) b^\gamma(s)} \\
&\geq \beta_1 - \int_0^T \frac{ds}{\left(\alpha + \frac{\alpha_1 s}{2}\right)^{\gamma-1} \left(\beta + \frac{\beta_1 s}{2}\right)^\gamma} \\
&\geq \beta_1 - \int_0^T \frac{ds}{\alpha^{\gamma-1} \left(\beta + \frac{\beta_1 s}{2}\right)^\gamma} ds \\
&\geq \beta_1 + \left. \frac{1}{\frac{2}{\beta_1} \alpha^{\gamma-1} \left(\beta + \frac{\beta_1 s}{2}\right)^{\gamma-1}} \right|_{s=0}^{s=T} \\
&= \beta_1 + \frac{1}{\frac{2}{\beta_1} \alpha^{\gamma-1} \left(\beta + \frac{\beta_1 T}{2}\right)^{\gamma-1}} - \frac{1}{\frac{2}{\beta_1} \alpha^{\gamma-1} \beta^{\gamma-1}} \\
&\geq \frac{2\beta_1}{3}
\end{aligned}$$

by choosing

$$\beta_1 - \frac{1}{\frac{2}{\beta_1} \alpha^{\gamma-1} \beta^{\gamma-1}} \geq \frac{2\beta_1}{3}.$$

This contradicts equations (2.40). Thus, Given $\alpha, \alpha_1, \beta_1 > 0$, there exists $\beta > 0$ such that the corresponding solution is GP.

As a consequence of Proposition 2.7, we have:

Proposition 2.11 *Given α, α_1, $\beta > 0$, there exists a critical value $\beta_1^* > 0$, such that if $\beta \geq \beta_1^*$, the corresponding solution is GP, while if $\beta_1 < \beta_1^*$, the corresponding solution is NGP.*

Proof: A direct application of Proposition 2.8 gives the conclusion except for the situation $\inf \beta = \beta_1^*$. We claim that in this critical case, the solution is GP. Suppose this is not true. This means that for $\beta > \beta_1^*$, the solution is GP, but for β_1^* the solution is non-GP. We know that $\dot{a}^*(T) = 0$ for some finite time T. We can construct a sequence such that $\beta_n \to \beta_1^*+$ by the continuous dependence of solutions on the initial conditions, in the bounded domain where $a(t) \geq \epsilon$ and $b(t) \geq \epsilon$ with some constant $\epsilon > 0$ for $0 < t \leq T$. Then there exists a constant N such that $\dot{a}_n(T) = \dot{a}^*(T) = 0$ for $n \geq N$, by the continuous dependence of the initial values. It means there exist some β_n for $n \geq N$, such that the corresponding solution is non-GP. This contradiction proves our claim.

Proposition 2.12 *Suppose*

$$\alpha \leq \beta, \quad \frac{\alpha_1}{\alpha} < \frac{\beta_1}{\beta} \quad \textit{and} \quad \alpha_1\beta_1 \leq \frac{1}{(\gamma-1)\,(\alpha\beta)^{\gamma-1}} - \epsilon_0$$

for some positive constant $\epsilon_0 > 0$. Then the solution is NGP.

Proof: From the second proof of Corollary 1, we see that if $a(t)$ and $b(t)$ are GP, then when t is sufficiently large, we have

$$\frac{1}{(a(t)b(t))^{\gamma-1}} < \epsilon$$

with a sufficiently small constant $\epsilon > 0$. By applying Proposition 4 with $\alpha \leq \beta$, $\alpha_1/\alpha < \beta_1/\beta$, we obtain

$$b(t) > a(t)$$

$$\frac{1}{b(t)} < \frac{1}{a(t)}.$$

Suppose the conclusion is not true. Then $\dot{a}(t) > 0$ for all time $t > 0$.

By integrating the Emden system, we have

$$\dot{a}(0) - \dot{a}(t) = \int_0^t \frac{ds}{a^{\gamma}(s) b^{\gamma-1}(s)},$$

$$\dot{a}(0) = \int_0^t \frac{ds}{a^{\gamma}(s) b^{\gamma-1}(s)} + \dot{a}(t).$$

Hence,

$$\dot{a}(0) \geq \int_0^t \frac{ds}{a^{\gamma}(s) b^{\gamma-1}(s)}.$$

On the other hand, we have

$$\dot{b}(0) - \dot{b}(t) = \int_0^t \frac{ds}{a^{\gamma-1}(s) b^{\gamma}(s)} \leq \int_0^t \frac{ds}{a^{\gamma}(s) b^{\gamma-1}(s)} \leq \dot{a}(0).$$

$$\dot{b}(t) \geq \dot{b}(0) - \dot{a}(0) = \beta_1 - \alpha_1.$$

Hence,

$$\left[\dot{b}(t)\right]^2 \geq (\beta_1 - \alpha_1)^2. \tag{2.42}$$

From the total energy, we have

$$\frac{\dot{a}^2(t)}{2} + \frac{\dot{b}^2(t)}{2} - \frac{1}{(\gamma-1)\,(a(t)b(t))^{\gamma-1}} = \theta = \frac{(\alpha_1)^2}{2} + \frac{(\beta_1)^2}{2} - \frac{1}{(\gamma-1)\,(\alpha\beta)^{\gamma-1}}$$

$$\frac{(\beta_1-\alpha_1)^2}{2} \leq \frac{\dot{b}^2}{2} < \frac{(\alpha_1)^2}{2} + \frac{(\beta_1)^2}{2} - \frac{1}{(\gamma-1)\,(\alpha\beta)^{\gamma-1}} + \frac{\epsilon}{(\gamma-1)}.$$

With a sufficiently small constant $\epsilon > 0$, when t is sufficiently large,

$$(\beta_1-\alpha_1)^2 \leq (\alpha_1)^2 + (\beta_1)^2 - \frac{2}{(\gamma-1)\,(\alpha\beta)^{\gamma-1}} + \frac{2\epsilon}{(\gamma-1)}$$

$$\alpha_1\beta_1 > \frac{1}{(\gamma-1)\,(\alpha\beta)^{\gamma-1}} - \frac{\epsilon}{(\gamma-1)}. \tag{2.43}$$

Then, by choosing the constant $\frac{\epsilon}{(\gamma-1)} \leq \epsilon_0$, inequality (2.43) contradicts the second condition in the hypotheses. Therefore, the solution must be NGP.

Remark: Proposition 2.8 covers the example $\alpha = \beta = 1$, $\alpha_1 = \frac{1}{2}$ and $\beta_1 = \frac{199}{100}$, but Corollary 2.5 does not. This shows that Proposition 2.8 is stronger than Corollary 2.5.

For $\gamma = 2$, the system has a second invariant.

Proposition 2.13 *The Emden system*

$$\begin{cases} \ddot{a}(t) = \dfrac{-1}{a^2(t)b(t)}, \\ \ddot{b}(t) = \dfrac{-1}{a(t)b^2(t)}, \end{cases} \tag{2.44}$$

has a second invariant

$$\theta_2 = \frac{(\dot{a}(t)b(t) - a(t)\dot{b}(t))^2}{2} - \int_0^t \left(\frac{-\dot{a}(t)b(t)}{a^2(t)} + \frac{\dot{a}(t)b(t)}{b^2(t)} + \frac{a(t)\dot{b}(t)}{a^2(t)} - \frac{a(t)\dot{b}(t)}{b^2(t)} \right) ds.$$

Proof: The function θ_2 is conserved because

$$\begin{aligned}
\frac{d\theta_2}{dt} &= (\dot{a}(t)b(t) - a(t)\dot{b}(t))(\ddot{a}(t)b(t) + \dot{a}(t)\dot{b}(t) - \dot{a}(t)\dot{b}(t) - a(t)\ddot{b}(t)) \\
&- \left(\frac{-\dot{a}(t)b(t)}{a^2(t)} + \frac{\dot{a}(t)b(t)}{b^2(t)} + \frac{a(t)\dot{b}(t)}{a^2(t)} - \frac{a(t)\dot{b}(t)}{b^2(t)} \right) \\
&= (\dot{a}(t)b(t) - a(t)\dot{b}(t))(\ddot{a}(t)b(t) - a(t)\ddot{b}(t)) \\
&- \left(\frac{-\dot{a}(t)b(t)}{a^2(t)} + \frac{\dot{a}(t)b(t)}{b^2(t)} + \frac{a(t)\dot{b}(t)}{a^2(t)} - \frac{a(t)\dot{b}(t)}{b^2(t)} \right) \\
&= (\dot{a}(t)b(t) - a(t)\dot{b}(t)) \left(\frac{-b(t)}{a^2(t)b(t)} - \frac{-a(t)}{a(t)b^2(t)} \right) \\
&- \left(\frac{-\dot{a}(t)b(t)}{a^2(t)} + \frac{\dot{a}(t)b(t)}{b^2(t)} + \frac{a(t)\dot{b}(t)}{a^2(t)} - \frac{a(t)\dot{b}(t)}{b^2(t)} \right) \\
&= (\dot{a}(t)b(t) - a(t)\dot{b}(t)) \left(\frac{-1}{a^2(t)} + \frac{1}{b(t)^2} \right) \\
&- \left(\frac{-\dot{a}(t)b(t)}{a^2(t)} + \frac{\dot{a}(t)b(t)}{b^2(t)} + \frac{a(t)\dot{b}(t)}{a^2(t)} - \frac{a(t)\dot{b}(t)}{b^2(t)} \right) \\
&= \frac{-\dot{a}(t)b(t)}{a^2(t)} + \frac{\dot{a}(t)b(t)}{b^2(t)} + \frac{a(t)\dot{b}(t)}{a^2(t)} - \frac{a(t)\dot{b}(t)}{b^2(t)} \\
&- \left(\frac{-\dot{a}(t)b(t)}{a^2(t)} + \frac{\dot{a}(t)b(t)}{b^2(t)} + \frac{a(t)\dot{b}(t)}{a^2(t)} - \frac{a(t)\dot{b}(t)}{b^2(t)} \right) \\
&= 0.
\end{aligned}$$

When $N = \gamma = 2$, we have the following corollary:

Corollary 2.7 *A solution of the Emden system with $N = \gamma = 2$ is GP if and only if the solution of each of the following two initial value problems is GP:*

$$\begin{cases} \ddot{A}(t) = \frac{-1}{A^2(t)\sqrt{2\theta t^2 + 2(a_{10}a_{11} + a_{20}a_{21})t + \alpha^2 + \beta^2 - A^2(t)}} \\ A(0) = \alpha > 0, \ \dot{A}(0) = \alpha_1, \end{cases}$$

and

$$\begin{cases} \ddot{B}(t) = \frac{-1}{B^2(t)\sqrt{2\theta t^2 + 2(a_{10}a_{11} + a_{20}a_{21})t + \alpha^2 + \beta^2 - B^2(t)}} \\ B(0) = \beta > 0, \ \dot{B}(0) = \beta_1. \end{cases}$$

2.6 Cylindrical Solutions for the Euler-Poisson Equations

In this section, we construct cylindrical solutions for the N-dimensional Euler-Poisson equations,

$$\begin{cases} \rho_t + \nabla \cdot \vec{u}\rho + \nabla\rho \cdot \vec{u} = 0 \\ \rho\left(\frac{\partial u_i}{\partial t} + \sum_{k=1}^{N} u_k \frac{\partial u_i}{\partial x_k}\right) + \frac{\partial}{\partial x_i}P(\rho) = -\rho\frac{\partial}{\partial x_i}\Phi(\rho) \quad \text{for } i = 1, 2, ...N \\ \Delta\Phi(t, \vec{x}) = \alpha(N)\rho. \end{cases} \tag{2.45}$$

We observe that the functions for R^N $(N \geq 3)$

$$\begin{cases} \rho(t, r) = \rho(t, \bar{r}) = \frac{1}{a^2(t)} e^{y(\bar{r}/a(t))}, \\ \vec{u}(t, \vec{x}) = \frac{\dot{a}(t)}{a(t)}(x_1, x_2, 0, ..., 0) \\ \quad \ddot{a}(t) = \frac{-\lambda}{a(t)}, \qquad a(0) = a_1 > 0,\ \dot{a}(0) = a_2 \\ \quad \ddot{y}(z) + \frac{1}{z}\dot{y}(z) + \frac{\alpha(N)}{K}e^{y(z)} = \frac{2\lambda}{K}, \qquad y(0) = \alpha,\ \dot{y}(0) = 0 \end{cases} \tag{2.46}$$

with $\bar{r} = \sqrt{x_1^2 + x_2^2}$ are the trivial extended cylindrical solutions for Yuen's 2-dimensional solutions [80].

The following Theorem is the corresponding result for the cylindrical case with non-trivial i-th $(i \geq 3)$ component of the velocity $\vec{u}$.

Theorem 2.14 ([89]) *The isothermal Euler-Poisson equations (2.45) in* R^N $(N \geq 3)$ *has the following family of solutions,*

$$\begin{cases} \rho(t, \vec{x}) = \frac{C}{a^2(t)} e^{-\frac{\phi(s)}{K}}, \\ \vec{u}(t, \vec{x}) = \frac{\dot{a}(t)}{a(t)}(x_1, x_2, x_1 + x_2, ...x_1 + x_2) \\ \quad a(t) = a_0 + a_1 t \\ \quad s\ddot{\phi}(s) + \dot{\phi}(s) - \epsilon^* e^{-\frac{\phi(s)}{K}} = 0,\ \phi(0) = \alpha,\ \dot{\phi}(0) = \epsilon^* e^{-\frac{\alpha}{K}} \end{cases} \tag{2.47}$$

where $s = \frac{x_1^2 + x_2^2}{a^2(t)}$, $a_0 > 0$, a_1 , $C > 0$, $\frac{\alpha(N)C}{4} = \epsilon^*$ *and* α *are arbitrary constants.*

(1) If $a_1 < 0$, *the solutions (2.47) blow up at the finite time* $T = -a_0/a_1$.

(2) If $a_1 \geq 0$, the solutions (2.47) exist globally.

Remark 2.14 *Our 3-dimensional blowup solutions (2.47) and (2.46) could be used to interpret the evolution of cylindrical clouds for star formation in astrophysics. For details about the physical meaning of cylindrical solutions for the Euler-Poisson system (2.45), readers may refer to the related literature Shu-Adams-Lizano [67], Tomisaka [71], Cook-Shapiro-Stephens [20] and Holden-Hoppins-Baxter-Fatuzzo [37].*

Lemma 2.6 *For the continuity equation $(2.45)_1$ in R^N, there exist solutions with the following functional structure*

$$\rho(t,\vec{x}) = \frac{f(s)}{a^2(t)}, \qquad \vec{u}(t,\vec{x}) = \frac{\dot{a}(t)}{a(t)}(x_1, x_2, x_1+x_2, ..., x_1+x_2) \tag{2.48}$$

with arbitrary C^1 scalar functions $f(s) \geq 0$ and $s = \dfrac{x_1^2 + x_2^2}{a^2(t)}$, $a(t) \neq 0$.

Proof. We plug the solutions (2.48) into the continuity equation $(2.45)_1$,

$$\begin{aligned}
&\rho_t + \nabla \cdot \vec{u}\rho + \nabla\rho \cdot \vec{u} \\
&= \frac{\partial}{\partial t}\left[\frac{f(s)}{a^2(t)}\right] + \left[\nabla \cdot \frac{\dot{a}(t)}{a(t)}(x_1, x_2, x_1+x_2, ..., x_1+x_2)\right]\frac{f(s)}{a^2(t)} \\
&\quad + \left[\nabla \frac{f(s)}{a^2(t)}\right] \cdot \frac{\dot{a}(t)}{a(t)}(x_1, x_2, x_1+x_2, ..., x_1+x_2) \\
&= \frac{-2\dot{a}(t)}{a^3(t)} f(s) + \frac{1}{a^2(t)}\frac{\partial}{\partial t} f(s) \\
&\quad + \frac{\dot{a}(t)}{a(t)}\left(\frac{\partial}{\partial x_1}x_1 + \frac{\partial}{\partial x_2}x_2 + \sum_{i=3}^{N}\frac{\partial}{\partial x_i}(x_1+x_2)\right)\frac{f(s)}{a^2(t)} \\
&\quad + \frac{\dot{a}(t)}{a(t)}\left[\frac{\partial}{\partial x_1}\frac{f(s)}{a^2(t)} \cdot x_1 + \frac{\partial}{\partial x_2}\frac{f(s)}{a^2(t)} \cdot x_2 + \sum_{i=3}^{N}\frac{\partial}{\partial x_i}\frac{f(s)}{a^2(t)} \cdot (x_1+x_2)\right] \\
&= \frac{-2\dot{a}(t)}{a^3(t)} f(s) - \frac{2}{a^2(t)}\dot{f}(s)\frac{(x_1^2+x_2^2)\dot{a}(t)}{a^3(t)} + \frac{2\dot{a}(t)}{a(t)}\frac{f(s)}{a^2(t)} \\
&\quad + \frac{\dot{a}(t)}{a(t)}\left[\frac{\dot{f}(s)}{a^2(t)}\frac{2x_1^2}{a^2(t)} + \frac{\dot{f}(s)}{a^2(t)}\frac{2x_2^2}{a^2(t)}\right] \\
&= 0.
\end{aligned}$$

■

The following lemma takes care of the Poisson equation $(2.45)_3$ for our solutions (2.47).

Lemma 2.7 *The function*

$$\rho = \frac{C}{a^2(t)} e^{-\frac{\phi(s)}{K}} \tag{2.49}$$

satisfies the Poisson equation $(2.45)_3$ *in* R^N $(N \geq 3)$. *Here,* $\phi(s)$ *satisfies the second-order ordinary differential equation* $(2.47)_4$.

Proof. We can check that our potential functions $\phi(t, x_{1,} x_2)$ satisfy the Poisson equation $(2.45)_3$

$$\begin{aligned}
&\Delta\phi(t, \vec{x}) - \alpha(N)\rho \\
&\quad = \nabla \cdot \nabla\phi\,(s) - \frac{\alpha(N)C}{a^2(t)} e^{-\frac{\phi(s)}{K}} \\
&\quad = \nabla \cdot \left[\frac{\partial}{\partial x_1}\phi\,(s)\,, \frac{\partial}{\partial x_2}\phi\,(s)\,, \frac{\partial}{\partial x_3}\phi\,(s)\,, ..., \frac{\partial}{\partial x_N}\phi\,(s)\right] - \frac{\alpha(N)C}{a^2(t)} e^{-\frac{\phi(s)}{K}} \\
&\quad = \frac{\partial}{\partial x_1}\left[\dot{\phi}(s)\frac{2x_1}{a^2(t)}\right] + \frac{\partial}{\partial x_2}\left[\dot{\phi}(s)\frac{2x_2}{a^2(t)}\right] - \frac{\alpha(N)C}{a^2(t)} e^{-\frac{\phi(s)}{K}} \\
&\quad = \ddot{\phi}(s)\frac{4x_1^2}{a^4(t)} + \dot{\phi}(s)\frac{2}{a^2(t)} + \ddot{\phi}(s)\frac{4x_2^2}{a^4(t)} + \dot{\phi}(s)\frac{2}{a^2(t)} - \frac{\alpha(N)C}{a^2(t)} e^{-\frac{\phi(s)}{K}} \\
&\quad = \frac{4}{a^2(t)}\left(s\ddot{\phi}(s) + \dot{\phi}(s) - \frac{\alpha(N)C}{4} e^{-\frac{\phi(s)}{K}}\right).
\end{aligned}$$

In view of $(2.47)_4$, we see that the Poisson equation is satisfied. ■

The following lemma establishes the global existence of the function $\phi(s)$. Similar lemmas have been given as Lemmas 9 and 10, in [80].

Lemma 2.8 *Given* $K > 0$ *and arbitrary* α, *there exists a small constant* ε^*, *such that* $0 < \varepsilon^* < K$, *the ordinary differential equation* $(2.47)_4$ *has a unique solution* $\phi(s) \in C^1[0, \infty)$.

Proof. First, the equation $(2.47)_4$ can be rewritten as

$$\frac{d}{ds}\left(s\dot{\phi}(s)\right) = \varepsilon^* e^{-\frac{\phi(s)}{K}}.$$

With the initial conditions: $\phi(0) = \alpha$ and $\dot{\phi}(0) = \epsilon^* e^{-\frac{\alpha}{K}}$, we deduce that

$$\dot{\phi}(s) = \frac{\varepsilon^*}{s}\int_0^s e^{-\frac{\phi(\tau)}{K}} d\tau > 0. \tag{2.50}$$

Set

$$f(s,\phi(s)) = \frac{\varepsilon^*}{s}\int_0^s e^{-\frac{\phi(\tau)}{K}}d\tau.$$

For *any* $s_0 > 0$, $f \in C^1[0,\ s_0]$. For any $\phi_1,\ \phi_2 \in C^1[0,\ s_0]$, we have,

$$|f(s,\phi_1(s)) - f(s,\phi_2(s))| = \frac{\varepsilon^*\left|\int_0^s \left(e^{-\frac{\phi_2(\tau)}{K}} - e^{-\frac{\phi_1(\tau)}{K}}\right)d\tau\right|}{s}.$$

As e^ϕ is a C^1 function of ϕ, we can show that the function e^ϕ is Lipschitz-continuous. Then we get,

$$\begin{aligned}
&|f(s,\phi_1(s)) - f(s,\phi_2(s))| \\
&\quad = \frac{\varepsilon^*\int_0^s |\phi_2(\tau) - \phi_1(\tau)|\,d\tau}{Ks} \\
&\quad \le \frac{\varepsilon^*}{K}\left(\sup_{0\le s\le s_0}|\phi_1(s) - \phi_2(s)|\right).
\end{aligned}$$

Let

$$T\phi(s) = \alpha + \int_0^s f(\tau,\phi(\tau))d\tau.$$

We have $T\phi \in C[0,\ s_0]$ and

$$\begin{aligned}
&|T\phi_1(s) - T\phi_2(s)| \\
&\quad = \left|\int_0^s f(\tau,\phi_1(\tau))d\tau - \int_0^s f(\tau,\phi_2(\tau))d\tau\right| \\
&\quad \le \frac{\varepsilon^*}{K}\left(\sup_{0\le s\le s_0}|\phi_1(s) - \phi_2(s)|\right).
\end{aligned}$$

By choosing the small constant ε^*, such that $0 < \varepsilon^* < K$, this shows that the mapping $T : C[0,\ s_0] \to C[0,\ s_0]$, is a contraction with the sup-norm. By the fixed point theorem, there exists a unique $\phi(s) \in C[0,\ s_0]$, such that $T\phi(s) = \phi(s)$.

In addition, from the equation (2.50), we see that the function $\phi(s)$ is increasing

$$\phi(0) \le \phi(s), \text{ for } s \in [0,+\infty).$$

Then, we have

$$0 \le f(s,\phi(s)) = \frac{\varepsilon^*}{s}\int_0^s e^{-\frac{\phi(\tau)}{K}}d\tau \le \frac{\varepsilon^*}{s}\int_0^s e^{-\frac{\phi(0)}{K}}d\tau = \varepsilon^* e^{-\frac{\phi(0)}{K}}.$$

As the function $f(s,\phi(s))$ is bounded and Lipschitz for any finite time $s \in [0,+\infty)$, the conditions of the Picard's existence theorem are satisfied to have the global existence of the solutions. Therefore, given $K > 0$ and arbitrary α, there exists a small constant ε^*, such that $0 < \varepsilon^* < K$, the ordinary differential equation $(2.47)_4$ has a unique solution $\phi(s) \in C^1[0, \infty)$. ■

The figure below shows the shape of the solution for the particular equation with $\epsilon^* = K = 1$ and $\alpha = 0$

$$\begin{cases} s\ddot{\phi}(s)+\dot{\phi}(s) - e^{-\phi(s)} = 0 \\ \phi(0) = 0, \ \dot{\phi}(0) = 1. \end{cases} \tag{2.51}$$

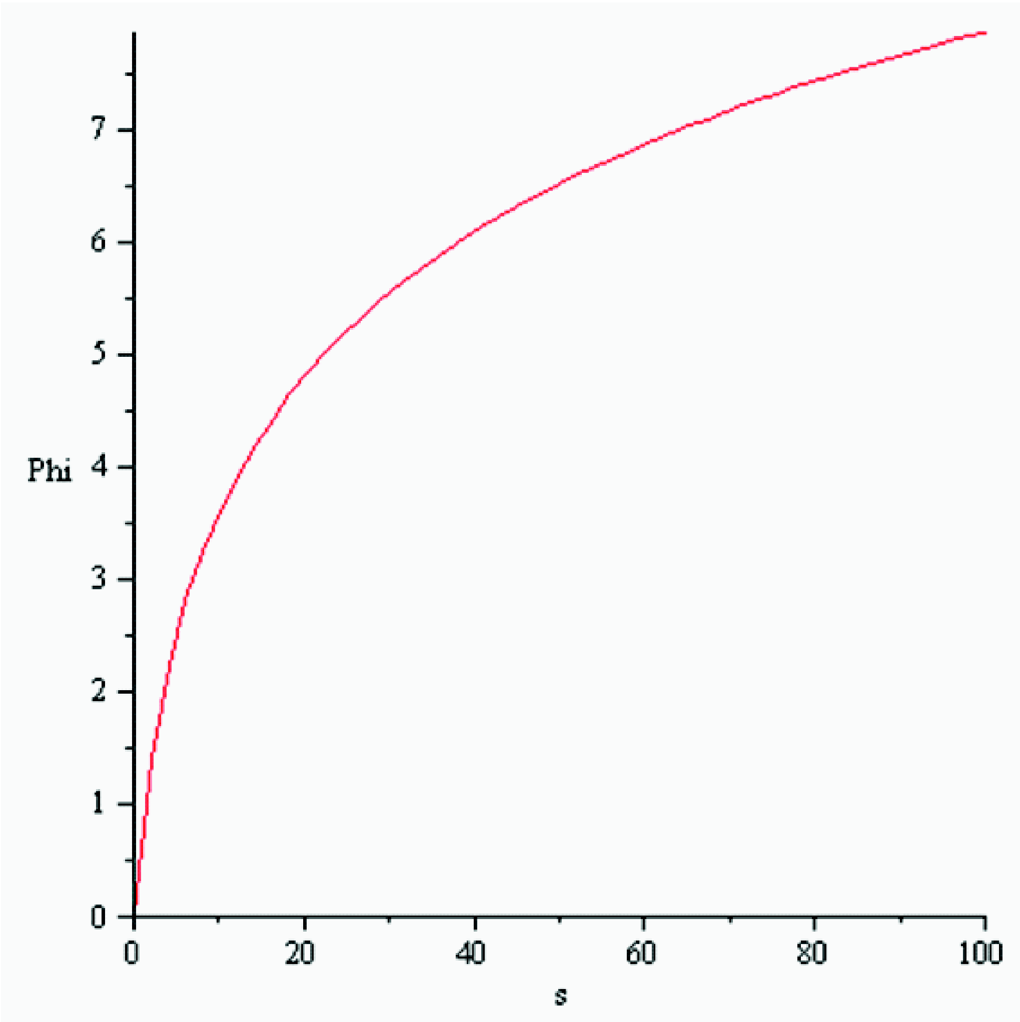

Fig. 2.1: $\Phi(s)$ in the equation (2.51)

We are now ready to check that the solutions (2.47) satisfy the Euler-Poisson equations (2.45).

Proof of Theorem 2.14. By Lemma 2.6 and Lemma 2.7, the functions (2.47) satisfy $(2.45)_1$ and $(2.45)_3$. For the first component of the isothermal momentum equations $(2.45)_2$ in R^N $(N \geq 3)$, we have

$$\begin{aligned}
&\rho\left(\frac{\partial u_1}{\partial t}+\sum_{k=1}^{N} u_k \frac{\partial u_1}{\partial x_k}\right)+\frac{\partial}{\partial x_1} K\rho+\rho\frac{\partial \phi}{\partial x_1} \\
&=\rho\left(\begin{array}{c}\frac{\partial}{\partial t}\frac{\dot{a}(t)}{a(t)}x_1+\frac{\dot{a}(t)x_1}{a(t)}\frac{\partial}{\partial x_1}\frac{\dot{a}(t)x_1}{a(t)}+\frac{\dot{a}(t)x_2}{a(t)}\frac{\partial}{\partial x_2}\frac{\dot{a}(t)x_1}{a(t)} \\ +\sum_{i=3}^{N}\frac{\dot{a}(t)(x_1+x_2)}{a(t)}\frac{\partial}{\partial x_i}\left(\frac{\dot{a}(t)x_1}{a(t)}\right)\end{array}\right)+K\frac{\partial}{\partial x_1}\frac{Ce^{-\frac{\phi(s)}{K}}}{a^2(t)}+\rho\frac{\partial \phi}{\partial x_1}(s) \\
&=\rho\left(\left(\frac{\ddot{a}(t)}{a(t)}-\frac{\dot{a}^2(t)}{a^2(t)}\right)x_1+\frac{\dot{a}(t)x_1}{a(t)}\frac{\dot{a}(t)}{a(t)}\right)-\frac{Ce^{-\frac{\phi\left(\frac{x_1^2+x_2^2}{a^2(t)}\right)}{K}}}{a^2(t)}\frac{\partial \phi}{\partial x_1}(s)+\rho\frac{\partial \phi}{\partial x_1}(s) \\
&=\rho\frac{\ddot{a}(t)}{a(t)}x_1-\rho\frac{\partial \phi(s)}{\partial x_1}+\rho\frac{\partial \phi(s)}{\partial x_1} \\
&=0
\end{aligned}$$

with

$$a(t)=a_0+a_1 t$$

for $a_0>0$.

A similar conclusion holds for the second component of the momentum equations.

For the i-th component $(i \geq 3)$ of the isothermal momentum equations $(2.45)_2$ in R^N, we verify that

$$\begin{aligned}
&\rho\left(\frac{\partial u_i}{\partial t}+\sum_{k=1}^{N} u_k \frac{\partial u_i}{\partial x_k}\right)+\frac{\partial}{\partial x_i} K\rho+\rho\frac{\partial \phi}{\partial x_i} \\
&=\rho\left(\begin{array}{c}\frac{\partial}{\partial t}\frac{\dot{a}(t)(x_1+x_2)}{a(t)}+\frac{\dot{a}(t)x_1}{a(t)}\frac{\partial}{\partial x_1}\frac{\dot{a}(t)x_1}{a(t)}+\frac{\dot{a}(t)x_2}{a(t)}\frac{\partial}{\partial x_2}\frac{\dot{a}(t)x_2}{a(t)} \\ +\sum_{k=3}^{N} u_k\frac{\partial}{\partial x_k}\frac{\dot{a}(t)(x_1+x_2)}{a(t)}\end{array}\right) \\
&\quad+K\frac{\partial}{\partial x_i}\frac{e^{-\frac{\phi(s)}{K}+C}}{a^2(t)}+\rho\frac{\partial}{\partial x_i}\phi(s) \\
&=\rho\left(\left(\frac{\ddot{a}(t)}{a(t)}-\frac{\dot{a}^2(t)}{a^2(t)}\right)(x_1+x_2)+\frac{\dot{a}^2(t)x_1}{a(t)^2}+\frac{\dot{a}^2(t)x_2}{a^2(t)}\right) \\
&=\rho\frac{\ddot{a}(t)}{a(t)}(x_1+x_2) \\
&=0.
\end{aligned}$$

Therefore, our functions (2.47) satisfy the Euler-Poisson equations.

The statements (1) and (2) are obviously true. ■

The blowup rate of the constructed solutions is easily obtained.

Corollary 2.8 *The blowup rate of the solutions (2.47) is*

$$\lim_{t\to T}\rho(t,\vec{0})\,(T-t)^2 \geq O(1).$$

Remark 2.15 *The functions*

$$\rho(t,\vec{x})=\frac{C}{a^2(t)}e^{-\frac{\phi(s)}{K}},\qquad \vec{u}(t,\vec{x})=\frac{\dot{a}(t)}{a(t)}(x_1,x_2,x_3,...,x_N)$$

are not the solutions for the Euler-Poisson equations (2.45) in R^N $(N\geq 3)$.

Remark 2.16 *Our solutions (2.47) and (2.46) also work for the isothermal Navier-Stokes-Poisson equations in R^N $(N\geq 3)$*

$$\begin{cases} \rho_t+\nabla\cdot(\rho\vec{u}) = 0 \\ \rho\left[\vec{u}+(\vec{u}\cdot\nabla)\vec{u}\right]+\nabla K\rho = \delta\rho\nabla\phi+\mu\Delta\vec{u} \\ \Delta\phi(t,\vec{x}) = \alpha(N)\rho \end{cases}$$

where $\mu>0$ is a positive constant.

2.7 Line Symmetric Solutions for the Euler-Poisson Equations

In this section, we obtain novel results in the line symmetric case for the 2-dimensional Euler-Poisson equations (2.52)

$$\begin{cases} \rho_t+\nabla\cdot(\rho\vec{u}) = 0 \\ \rho\left[\vec{u}+(\vec{u}\cdot\nabla)\vec{u}\right]+\nabla P = -\rho\nabla\phi\,\cdot \\ \Delta\phi(t,\vec{x}) = \alpha(N)\rho \end{cases} \tag{2.52}$$

as given in the following theorem.

Theorem 2.15 ([86]) *For the 2-dimensional isothermal Euler-Poisson equations (2.52), there exists a family of solutions,*

$$\begin{cases} \rho(t,x,y) = \frac{C}{a^2(t)} e^{-\frac{\phi\left(\frac{Ax+By}{a(t)}\right)}{K}} \\ \vec{u}(t,x,y) = \dfrac{\dot{a}(t)}{a(t)}(x,y) \\ \qquad a(t) = a_0 + a_1 t \\ \qquad \ddot{\phi}(s) - \epsilon^* e^{-\frac{\phi(s)}{K}} = 0, \ \phi(0) = \alpha, \ \dot{\phi}(0) = \beta \end{cases} \tag{2.53}$$

where A, $B \geq 0$, such that A and B are not both 0, $C > 0$, $a_0 \neq 0$, a_1, $\dfrac{2\pi C}{A^2 + B^2} = \epsilon^* > 0$, α and β are arbitrarily constants.

(1) If $a_0 > 0$ and $a_1 < 0$, the solutions (2.53) blow up at the finite time $T = -a_1/a_0$.

(2) If $a_0 > 0$ and $a_1 \geq 0$, the solutions (2.53) are global.

Lemma 2.9 *For the continuity equation $(2.52)_1$ in R^2, there exist solutions of the form*

$$\rho(t,x,y) = \frac{f\left(\dfrac{Ax+By}{a(t)}\right)}{a^2(t)}, \qquad \vec{u}(t,x,y) = \frac{\dot{a}(t)}{a(t)}(x,y) \tag{2.54}$$

where the scalar function $f(s) \geq 0 \in C^1$ and $a(t) \neq 0 \in C^1$.

Proof. We plug the solutions (2.53) into the continuity equation $(2.52)_1$,

$$\begin{aligned} &\rho_t + \nabla \cdot \vec{u}\rho + \nabla\rho \cdot \vec{u} \\ &= \frac{\partial}{\partial t}\left[\frac{f\left(\dfrac{Ax+By}{a(t)}\right)}{a^2(t)}\right] + \nabla \cdot \frac{\dot{a}(t)}{a(t)}(x,y)\frac{f\left(\dfrac{Ax+By}{a(t)}\right)}{a^2(t)} \\ &\quad + \nabla \frac{f\left(\dfrac{Ax+By}{a(t)}\right)}{a^2(t)} \cdot \frac{\dot{a}(t)}{a(t)}(x,y) \\ &= \frac{-2\dot{a}(t)}{a^3(t)} f\left(\frac{Ax+By}{a(t)}\right) + \frac{1}{a^2(t)}\frac{\partial}{\partial t} f\left(\frac{Ax+By}{a(t)}\right) \\ &\quad + \frac{\dot{a}(t)}{a(t)}\left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y\right)\frac{f\left(\dfrac{Ax+By}{a(t)}\right)}{a^2(t)} \end{aligned}$$

$$
\begin{aligned}
&+\frac{\dot a(t)}{a(t)}\left[\frac{\partial}{\partial x}\frac{f\left(\dfrac{Ax+By}{a(t)}\right)}{a^2(t)}\cdot x+\frac{\partial}{\partial y}\frac{f\left(\dfrac{Ax+By}{a(t)}\right)}{a^2(t)}\cdot y\right]\\
&=\frac{-2\dot a(t)}{a^3(t)}f\left(\frac{Ax+By}{a(t)}\right)-\frac{1}{a^2(t)}\dot f\left(\frac{Ax+By}{a(t)}\right)\frac{(Ax+By)\dot a(t)}{a^2(t)}\\
&+2\frac{\dot a(t)}{a(t)}\frac{f\left(\dfrac{Ax+By}{a(t)}\right)}{a^2(t)}\\
&+\frac{\dot a(t)}{a(t)}\left[\frac{\dot f\left(\dfrac{Ax+By}{a(t)}\right)}{a^2(t)}\frac{Ax}{a(t)}+\frac{\dot f\left(\dfrac{Ax+By}{a(t)}\right)}{a^2(t)}\frac{By}{a(t)}\right]\\
&=0.
\end{aligned}
$$

■

The global existence of solutions for the ordinary differential equation $(2.53)_3$ can be shown as follows.

Lemma 2.10 *The equation,*

$$
\begin{cases}
\ddot\phi(s)-\epsilon^* e^{-\frac{\phi(s)}{K}}=0\\
\phi(0)=\alpha,\ \dot\phi(0)=\beta
\end{cases}
\tag{2.55}
$$

where $\epsilon^*>0$, α and β are constants, has a solution in $\phi(s)\in C^2(-\infty,\infty)$ and $\lim\limits_{s\to\pm\infty}\phi(s)=\infty$.

Proof. The proof is similar to Lemma 3 in [84].

Multiply equation (2.55) by $\dot a(t)$ and then integrate it to obtain

$$
\begin{gathered}
(\ddot\phi(s)-\epsilon^* e^{-\frac{\phi(s)}{K}})\dot\phi(s)=0\\
\int_0^s\dot\phi(\tau)d\dot\phi(\tau)-\epsilon^*\int_0^s e^{-\frac{\phi(\tau)}{K}}d\phi(\tau)=0\\
\frac{1}{2}\dot\phi^2(s)+\epsilon^* e^{-\frac{\phi(s)}{K}}=\theta
\end{gathered}
$$

with the constant $\theta=\frac{1}{2}\dot\phi^2(0)+\epsilon^* e^{-\frac{\phi(0)}{K}}>0$.

We define the kinetic energy as

$$
F_{kin}(t):=\frac{\dot\phi^2(t)}{2}
\tag{2.56}
$$

and the potential energy as

$$F_{pot}(t) = \epsilon^* e^{-\frac{\phi(t)}{K}}. \tag{2.57}$$

The total energy is conserved

$$\frac{d}{dt}(F_{kin}(t) + F_{pot}(t)) = 0$$

$$F_{kin}(t) + F_{pot}(t) = \theta.$$

By the classical energy method for conservative systems (in Section 4.3 of [45]), the solutions have a trajectory. We may calculate the required time for traveling the whole orbit

$$\begin{aligned} T &= \int_0^T \frac{d\phi(\tau)}{\sqrt{2(\theta - \epsilon^* e^{-\frac{\phi(\tau)}{K}})}} \\ &= \int_\alpha^\infty \frac{d\tau}{\sqrt{2(\theta - \epsilon^* e^{-\frac{\tau}{K}})}} \\ &\geq \int_\alpha^\infty \frac{d\tau}{\sqrt{2\theta}} \\ &= \infty. \end{aligned}$$

Therefore, the solution $\phi(s)$ exists globally for $s \in [0, \infty)$ and $\lim_{s\to\infty} \phi(s) = \infty$.

For the interval $s \in (-\infty, 0]$, the proof is similar. ■

On the other hand, the following lemma handles the Poisson equation $(2.52)_3$ for our solutions (2.53).

Lemma 2.11 *The solutions*

$$\rho = \frac{C}{a^2(t)} e^{-\frac{\phi\left(\frac{Ax+By}{a(t)}\right)}{K}} \tag{2.58}$$

with the second-order ordinary differential equation

$$\ddot{\phi}(s) - \epsilon^* e^{-\frac{\phi(s)}{K}} = 0, \ \phi(0) = \alpha, \ \dot{\phi}(0) = \beta \tag{2.59}$$

where $s := (Ax + By)/a(t)$ *and* C, $\frac{2\pi C}{A^2+B^2} = \epsilon^* > 0$, α *and* β *are constants, satisfy the Poisson equation* $(2.52)_3$ *in* R^2.

Proof. We check that our potential function $\phi(t,x,y)$ satisfies the Poisson equation $(2.52)_3$

$$
\begin{aligned}
&\Delta\phi(t,x,y) - 2\pi\rho \\
&= \frac{\partial}{\partial x}\left[\dot{\phi}\left(\frac{Ax+By}{a(t)}\right)\frac{A}{a(t)}\right] + \frac{\partial}{\partial y}\left[\dot{\phi}\left(\frac{Ax+By}{a(t)}\right)\frac{B}{a(t)}\right] - \frac{2\pi C}{a^2(t)}e^{-\frac{\phi\left(\frac{Ax+By}{a(t)}\right)}{K}} \\
&= \frac{A^2+B^2}{a^2(t)}\left(\ddot{\phi}(s) - \frac{2\pi C}{A^2+B^2}e^{-\frac{\phi(s)}{K}}\right),
\end{aligned}
$$

where A and B are not both 0. By choosing $s := (Ax+By)/a(t)$ and requiring the ordinary differential equation

$$
\ddot{\phi}(s) - \epsilon^* e^{-\frac{\phi(s)}{K}} = 0,\ \phi(0)=\alpha,\ \dot{\phi}(0)=\beta,
$$

to hold with $\frac{2\pi C}{A^2+B^2} = \epsilon^*$, and α and β are the constants given in Lemma 2.10, we see that the solutions (2.58) satisfy the Poisson equation $(2.52)_3$. ■

With the above Lemmas, it is easy to check that the solutions satisfy the Euler-Poisson equations (2.52). The main technique exploited in our approach is to use the pressure term $\nabla K\rho$ to balance the potential term $-\rho\nabla\phi$ for the momentum equations. We omit the details.

Remark 2.17 *For the case of $A=0$ and $B=0$, the corresponding solutions (2.53) reduce to the special solutions in [79].*

Remark 2.18 *Denote $z = Ax+By$. From Lemma (2.10), before the blowup time T, we have*

$$
\lim_{z\to\pm\infty}\rho(t,x,y) = \lim_{z\to\pm\infty}\frac{C}{a^2(t)}e^{-\frac{\phi\left(\frac{z}{a(t)}\right)}{K}} = 0. \tag{2.60}
$$

Chapter 3

The Separation Method for Shallow Water Systems

In Chapter 2, we saw the successful application of the separation method to various versions of the Euler's equations. In this chapter, we further extend the method to construct self-similar solutions for the shallow water systems, modelled respectively by the 2-component Camassa-Holm and Degasperis-Procesi equations.

3.1 The 2-Component Camassa-Holm Equations

The 2-component Camassa-Holm equations refer to the system

$$\begin{cases} \rho_t + u\rho_x + \rho u_x = 0 \\ m_t + 2u_x m + u m_x + \sigma\rho\rho_x = 0 \end{cases} \tag{3.1}$$

with

$$m = u - \alpha^2 u_{xx}. \tag{3.2}$$

Here σ can assume the value of 1 or -1 depending on the physical properties of the system.

In [85], we made the observations that (3.1) is similar to the isentropic Euler system in some sense, and applied the separation method to obtain some exciting results. As before, we were able to reduce the nonlinear partial differential equations (3.1) to simpler ordinary differential equations, from which the solutions can be constructed. The result is reproduced below.

Theorem 3.1 ([85]) *Define the function $a(s)$ as the solution of the Emden equation*

$$\begin{cases} \ddot{a}(s) - \frac{\xi}{3a^{1/3}(s)} = 0 \\ a(0) = a_0 \neq 0, \ \dot{a}(0) = a_1 \end{cases} \tag{3.3}$$

and

$$f(\eta) = \frac{\xi}{\sigma}\sqrt{-\frac{\sigma}{\xi}\eta^2 + \left(\frac{\sigma\alpha}{\xi}\right)^2} \tag{3.4}$$

where $\eta = \frac{x}{a^{1/3}(s)}$, $\alpha \geq 0$, $\xi \neq 0$, a_0 and a_1 are arbitrary constants.
For the 2-component Camassa-Holm equations (3.1), there exists a family of solutions

(1) For $\sigma = -1$,

(1a) with $\xi < 0$ and $a_0 > 0$,

$$\rho(t,x) = \begin{cases} \frac{f(\eta)}{a^{1/3}(3t)}, \ \textit{for } \eta^2 < -\frac{\alpha^2}{\xi} \\ 0, \ \textit{for } \eta^2 \geq -\frac{\alpha^2}{\xi} \end{cases}, \quad u(t,x) = \frac{\dot{a}(3t)}{a(3t)}x. \tag{3.5}$$

The solution (3.5) blows up at a finite time T.

(1b) with $\xi > 0$ and $a_0 < 0$,

$$\rho(t,x) = \frac{f(\eta)}{a^{1/3}(3t)}, \quad u(t,x) = \frac{\dot{a}(3t)}{a(3t)}x. \tag{3.6}$$

The solution (3.6) may exist globally or blow up in finite time depending on the initial value $\dot{a}(0)$;

(2) for $\sigma = 1$,

(2a) with $\xi > 0$ and $a_0 > 0$,

$$\rho(t,x) = \begin{cases} \frac{f(\eta)}{a^{1/3}(3t)}, \ \textit{for } \eta^2 < \frac{\alpha^2}{\xi} \\ 0, \ \textit{for } \eta^2 \geq \frac{\alpha^2}{\xi} \end{cases}, \quad u(t,x) = \frac{\dot{a}(3t)}{a(3t)}x. \tag{3.7}$$

The solution (3.7) may exist globally or blow up in finite time depending on the initial value $\dot{a}(0)$;

(2b) with $\xi < 0$ and $a_0 < 0$,

$$\rho(t,x) = \frac{f(\eta)}{a^{1/3}(3t)}, \quad u(t,x) = \frac{\dot{a}(3t)}{a(3t)}x. \tag{3.8}$$

The solution (3.8) blows up at a finite time T.

As our first step, we design a nice functional structure for the solutions of the mass equation, which is similar to that for the Euler system.

Lemma 3.1 *For the* 1*-dimensional equation of mass (3.1)*$_1$

$$\rho_t + u\rho_x + \rho u_x = 0, \tag{3.9}$$

there exist solutions

$$\rho(t,x) = \frac{f(\eta)}{a^{1/3}(3t)}, \quad u(t,x) = \frac{\dot{a}(3t)}{a(3t)}x \tag{3.10}$$

for arbitrary $f(\eta) \geq 0 \in C^1$ *with* $\eta = \frac{x}{a^{1/3}(3t)}$*, and arbitrary* $a(3t) > 0 \in C^1$*.*

Proof. Substituting (3.10) into (3.9), we obtain

$$\begin{aligned}
&\rho_t + u\rho_x + \rho u_x \\
&= \frac{\partial}{\partial t}\left(\frac{f(\eta)}{a^{1/3}(3t)}\right) + \frac{\dot{a}(3t)}{a(3t)}x\frac{\partial}{\partial x}\left(\frac{f(\eta)}{a^{1/3}(3t)}\right) + \frac{f(\eta)}{a^{1/3}(3t)}\frac{\partial}{\partial x}\left(\frac{\dot{a}(3t)}{a(3t)}x\right) \\
&= \frac{1}{a^{1/3+1}(3t)}(-\frac{1}{3})\cdot\dot{a}(3t)\cdot 3\cdot f(\eta) + \frac{1}{a^{1/3}(3t)}\dot{f}(\eta)\frac{\partial}{\partial t}(\frac{x}{a^{1/3}(3t)}) \\
&\quad + \frac{\dot{a}(3t)x}{a(3t)}\frac{\dot{f}(\eta)}{a(3t)}\frac{\partial}{\partial x}\left(\frac{x}{a^{1/3}(3t)}\right) + \frac{f(\eta)}{a^{1/3}(3t)}\frac{\dot{a}(3t)}{a(3t)} \\
&= -\frac{\dot{a}(3t)f(\eta)}{a^{1/3+1}(3t)} + \frac{1}{a^{1/3}(3t)}\dot{f}(\eta)\frac{x}{a^{1/3+1}(3t)}\frac{1}{3}\dot{a}(3t)\cdot 3 \\
&\quad + \frac{\dot{a}(3t)x}{a(3t)}\frac{\dot{f}(\eta)}{a^{1/3}(3t)}\frac{1}{a^{1/3}(3t)} + \frac{f(\eta)}{a^{1/3}(3t)}\frac{\dot{a}(3t)}{a(3t)} \\
&= 0.
\end{aligned}$$

■

Equation (3.3)$_1$ is a particular case of the more general equation

$$\ddot{a}(s) - \frac{\xi\,\mathrm{sign}(a(s))}{|a(s)|^{\kappa}} = 0, \tag{3.11}$$

where $0 < \kappa < 1$, which in turn is a particular case of (2.14). The qualitative properties of (3.11) have been studied in [24] and [80]. Local existence of solutions is covered by standard theory. The blowup property of the time function $a(s)$ is given by the following lemma.

Lemma 3.2 *For the Emden equation (3.11), with initial conditions (3.3)*$_2$*,* *(1) if* $\xi < 0$*, there exists a finite time* S*, such that* $\lim_{s \to S^-} a(s) = 0$.

(2) if $\xi > 0$ *and* $a_0 > 0$ *there exists a finite time* S*, such that* $\lim_{s \to S^-} a(s) = 0$ *if and only if* $a_1 \leq -\sqrt{2\xi a_0^{1-\kappa}/(1-\kappa)}$*. In the contrary case, the solution* $a(s)$ *exists globally and*

$$\lim_{s \to \infty} a(s) = \infty. \tag{3.12}$$

(3) if $\xi > 0$ *and* $a_0 < 0$ *there exists a finite time* S*, such that* $\lim_{s \to S^-} a(s) = 0$ *if and only if* $a_1 \geq \sqrt{2\xi |a_0|^{1-\kappa}/(1-\kappa)}$*. In the contrary case, the solution* $a(s)$ *exists globally and*

$$\lim_{s \to \infty} a(s) = -\infty. \tag{3.13}$$

Proof. (1) By assumption $\xi < 0$, implying that the curve $a(s)$ concaves downwards. Suppose that $\dot{a}(\sigma) \leq 0$ at some finite time $\sigma > 0$. Then $\dot{a}(s) < 0$ for $s > \sigma$ and so the curve must intersect the s-axis at some finite $S > \sigma$.

It remains to show that the contrary case cannot happen. In that case, $\dot{a}(s) > 0$ for all $s > 0$ and $a(s)$ is an increasing function.

As in the proof of Lemma 2.4, (3.11) has a first integral given by the conservation identity

$$\frac{\dot{a}^2(s)}{2} - \frac{\xi a^{1-\kappa}(s)}{1-\kappa} = E \tag{3.14}$$

where the two terms on the left-hand side are the kinetic and potential energies, respectively, and E is a constant determined by the initial conditions.

Since $\xi < 0$, both terms on the left-hand side of (3.14) are positive. Therefore, each term is bounded above by E. As a consequence, $a(s)$ is bounded above. and so $a(s)$ converges to a finite limit a^*, which can be determined from (3.14) by noting that $\dot{a}(s) \to 0$ as $s \to \infty$.

The differential equation (3.14) leads to the solution

$$\int_0^{a(s)} \frac{\sqrt{1-\kappa}\, da}{\sqrt{2\theta(1-\kappa) + 2\xi a^{1-\kappa}}} = s. \tag{3.15}$$

As $s \to \infty$, the upper limit $a(s) \to a^*$. Since the denominator of the integrand vanishes at $a = a^*$, the left-hand side is an improper integral. It is easy to verify, however, that the integral converges because the power of a in the denominator is less than 1. However, this contradicts the fact that the right-hand side is unbounded as $s \to \infty$, and completes the proof of (1).

(2) for $\xi > 0$, $a_0 > 0$ and

(a) $E < 0$, as the infimum of $a(s)$ is

$$a_{\text{inf}} = \left[\frac{-E(1-\kappa)}{\xi}\right]^{\frac{1}{1-\kappa}} > 0,$$

the solution $a(s)$ is uniformly bounded below. The time for achieving the infimum is finite if $a_1 < 0$,

$$S_1 = \int_0^{S_1} ds = \int_{a_{\text{inf}}}^{a_0} \frac{\sqrt{1-\kappa}\, da}{\sqrt{2E(1-\kappa) + 2\xi a^{1-\kappa}}} < +\infty.$$

Therefore, for any constant a_1, the solution $a(s)$ must increase, after some finite time. Moreover, the time for traveling the interval $[a_{\text{inf}}, +\infty)$ or $[a_0, +\infty)$ is infinite,

$$S_2 = \int_{a_{\text{inf}}}^{+\infty} \frac{\sqrt{1-\kappa}\, da}{\sqrt{2E(1-\kappa) + 2\xi a^{1-\kappa}}} < +\infty.$$

Therefore, we showed that the solution $a(s)$ globally exists, such that

$$\lim_{s \to +\infty} a(s) = +\infty;$$

(b) $E \geq 0$, from the total energy (3.14), we have

$$\frac{\dot{a}^2(s)}{2} - \frac{\xi a^{1-\kappa}(s)}{1-\kappa} \geq 0$$

$$\dot{a}(s) \leq -\sqrt{2\xi a^{1-\kappa}(s)/(1-\kappa)} \text{ or } \dot{a}(s) \geq \sqrt{2\xi a^{1-\kappa}(s)/(1-\kappa)}$$

For the case $a_1 \geq \sqrt{2\xi a_0^{1-\kappa}/(1-\kappa)}$, we can repeat the above analysis to show that the solution $a(t)$ exists globally, such that

$$\lim_{s \to +\infty} a(s) = +\infty.$$

For the case, $a_1 \leq \sqrt{2\xi a_0^{1-\kappa}/(1-\kappa)}$, we have

$$\frac{1}{\frac{1-\kappa}{2}+1} a(s)^{\frac{1-\kappa}{2}+1} \leq -\sqrt{2\xi/(1-\kappa)}s + \frac{1}{\frac{1-\kappa}{2}+1} a_0^{\frac{1-\kappa}{2}+1}.$$

Then, we may see that there exists a sufficiently large S, such that $\lim_{s \to S^-} a(s) = 0$. By combining the above two cases, we have statement (2): if $\xi > 0$ and $a_0 > 0$ there exists a finite time S, such that $\lim_{s \to S^-} a(s) = 0$ if and only if $a_1 \leq -\sqrt{2\xi a_0^{1-\kappa}/(1-\kappa)}$. In the contrary case, the solution $a(s)$ exists globally and

$$\lim_{s \to \infty} a(s) = \infty.$$

We complete the proof of (2).

(3) We may let $b(t) = -a(t)$, for the Emden equation with $a_0 < 0$,

$$\begin{cases} \ddot{a}(s) - \frac{\xi \operatorname{sign}(a(s))}{|a(s)|^{\kappa}} = 0, \\ a(0) = a_0 < 0,\ \dot{a}(0) = a_1 \end{cases}$$

have

$$\begin{cases} \ddot{b}(s) + \frac{\xi \operatorname{sign}(a(s))}{|b(s)|^{\kappa}} = 0, \\ b(0) = b_0 > 0,\ \dot{b}(0) = b_1. \end{cases} \tag{3.16}$$

Then by applying the similar analysis in the proof of statement (2) for the equation (3.16), it is clear to have statement (3). ■

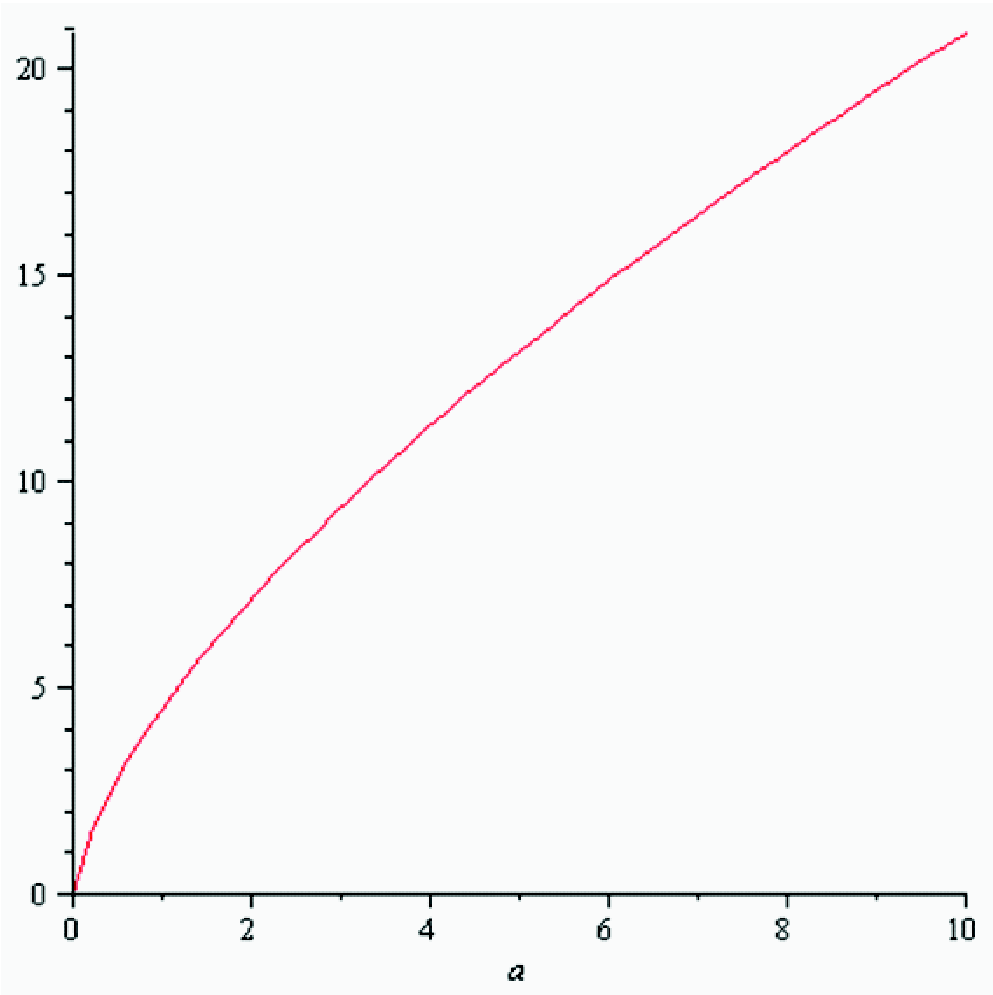

Fig. 3.1: curve for potential energy $-\frac{\xi a^{1-\kappa}(s)}{1-\kappa}$ with $\xi = -3$ and $\kappa = 1/3$

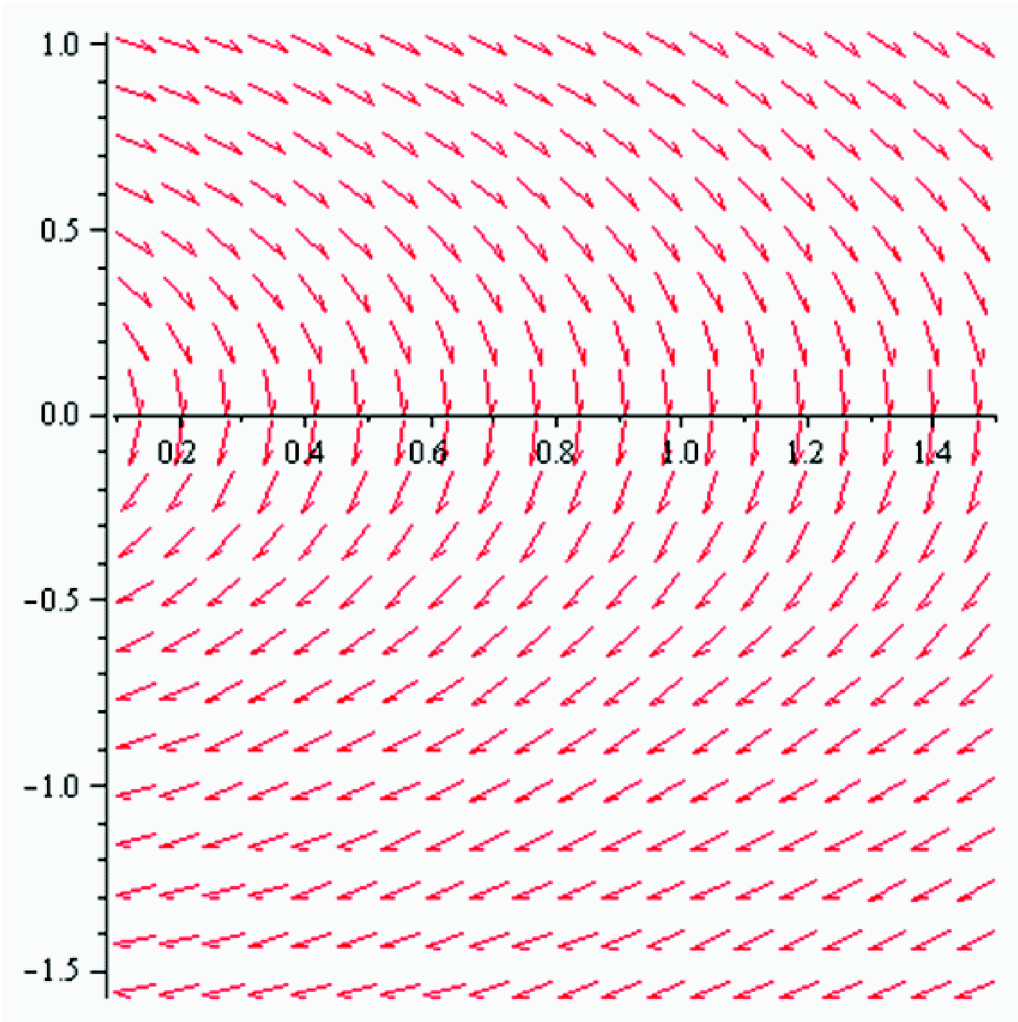

Fig. 3.2: phase diagram of the dynamical system $(a(s),\, \dot{a}(s))$ with $\xi = -3$ and $\kappa = 1/3$

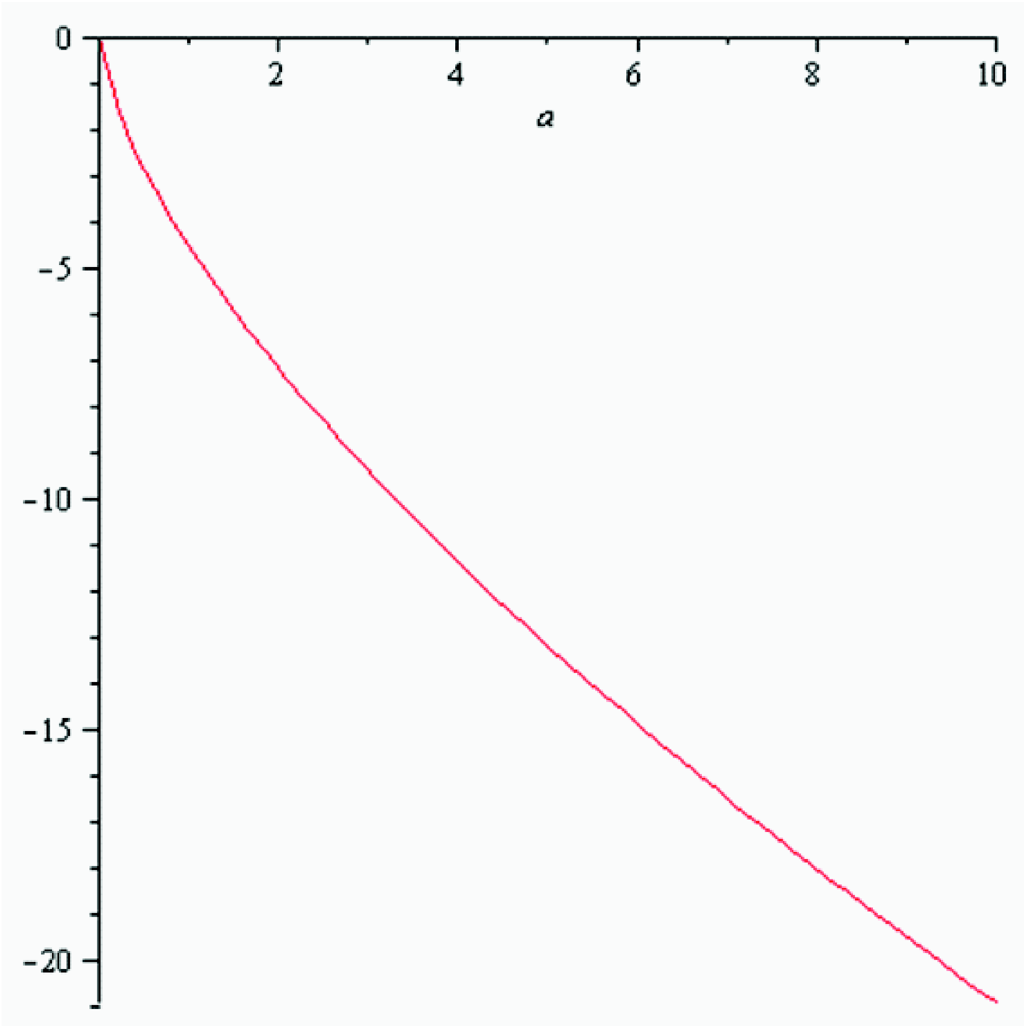

Fig. 3.3: curve for potential energy $-\frac{\xi a^{1-\kappa(s)}}{1-\kappa}$ with $\xi = 3$ and $\kappa = 1/3$.

Proof of Theorem 3.1. From Lemma 3.1, it is clear that functions (3.5), (3.6), (3.7) and (3.8) satisfy the mass equation, $(3.1)_1$, except for the two boundary points.

As the velocity u in the solutions (3.5)–(3.8) is a linear flow, $u_{xx} = 0$ and so $m = u$. The second equation of the $(3.1)_2$, becomes

$$\begin{aligned} &m_t + 2u_x m + u m_x + \sigma\rho\rho_x \\ &\quad = u_t + 3u_x u + \sigma\rho\rho_x \end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial}{\partial t}\left(\frac{\dot{a}(3t)}{a(3t)}\right)x + 3\left(\frac{\dot{a}(3t)}{a(3t)}\right)x\frac{\dot{a}(3t)}{a(3t)} + \%sigma\frac{f(\eta)}{a^{1/3}(3t)}\left(\frac{f(\eta)}{a^{1/3}(3t)}\right)_x \\
&= \left(3\frac{\ddot{a}(3t)}{a(3t)} - 3\frac{\dot{a}^2(3t)}{a^2(3t)}\right)x + 3\left(\frac{\dot{a}(3t)}{a(3t)}\right)\frac{\dot{a}(3t)}{a(3t)}x \\
&\quad + \sigma\frac{f(\eta)}{a^{1/3}(3t)}\frac{\dot{f}(\eta)}{a^{1/3}(3t)}\frac{1}{a^{1/3}(3t)} \\
&= 3\frac{\ddot{a}(3t)}{a(3t)}x + \sigma\frac{f(\eta)\dot{f}(\eta)}{a(3t)} \\
&= \frac{\sigma}{a(3t)}\left(\frac{\xi}{\sigma}\eta + f(\eta)\dot{f}(\eta)\right).
\end{aligned}
$$

We have used the fact that $a(s)$, with $s = 3t$ satisfies the Emden equation (3.3). Now if we choose $f(\eta)$ as in (3.4), we see that the last expression above vanishes and so $(3.1)_2$ is satisfied. ■

For the graphical illustration of the solution (3.7), for the integrable system with $\sigma = 1$, $\xi = 1$, $\alpha = 1$, and $a_0 = 1$, the initial shape of the self-similar solution can be found in Fig.3.4 below.

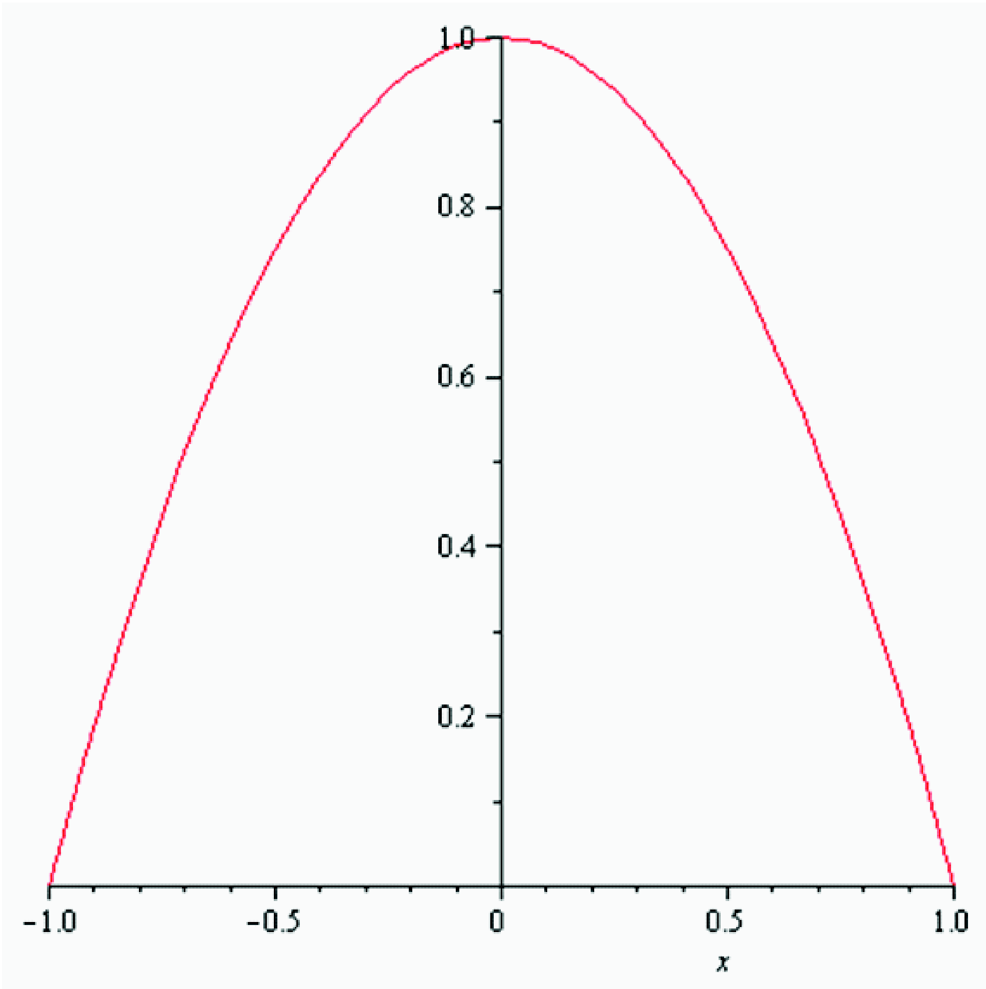

Fig. 3.4: $\rho(0,x) = \sqrt{-x^2+1}$

For the solution (3.8) with $\sigma = 1$, $\xi = -1$, $\alpha = 1$, and $a_0 = -1$, the corresponding graph is shown in Fig.3.5 below.

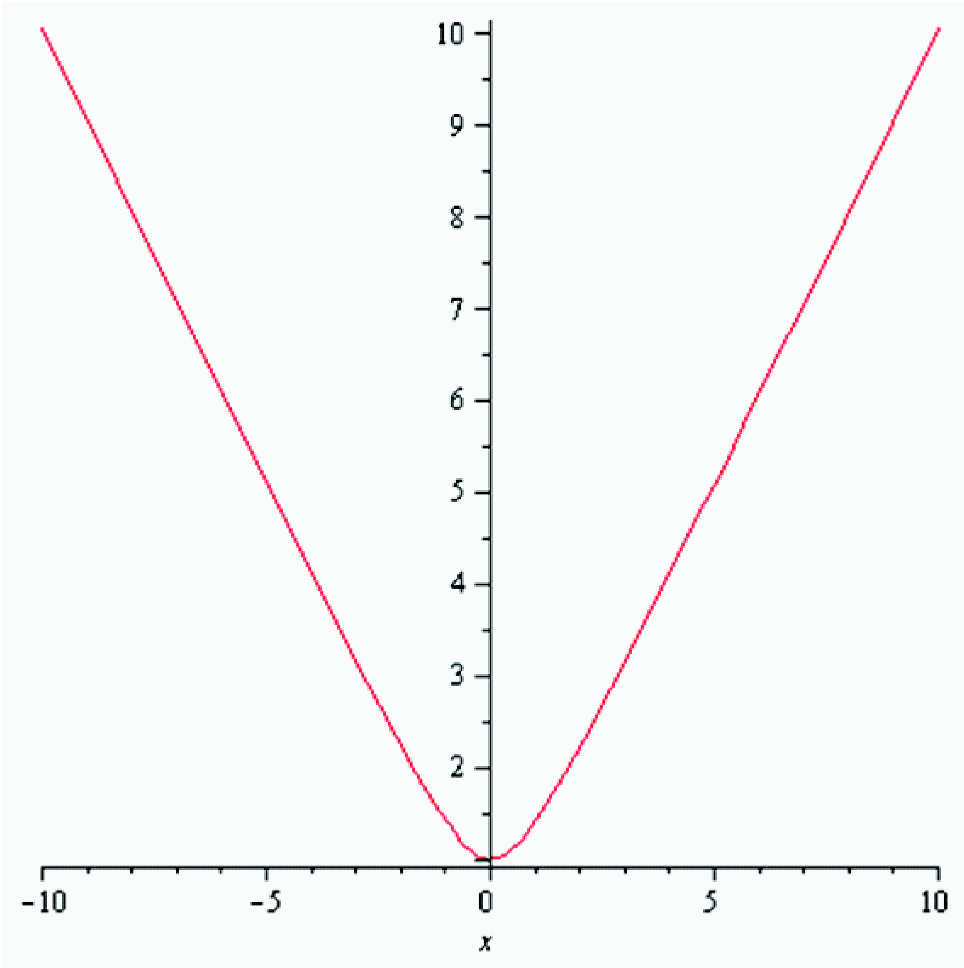

Fig. 3.5: $\rho(0,x) = \sqrt{x^2+1}$

Remark 3.1 *If the solution blows up at a finite time T, it collapses at the origin in the sense that*

$$\lim_{t \to T^-} \rho(t,0) = \infty.$$

If the solution exists globally, then at the origin,

$$\lim_{t \to \infty} \rho(t,0) = 0.$$

Remark 3.2 *The solutions (3.5) and (3.7) are only C^0 functions, as the function $f(\eta)$ is discontinuous at the two boundary points, for $\alpha > 0$,*

$$\lim_{\eta^2 \to \left|\frac{\alpha}{\xi}\right|} \dot{f}(\eta) \neq 0.$$

Remark 3.3 *For the integrable system with $\sigma = 1$, we can calculate the mass of*

(1) the blowup solution (3.8) (or (3.8)), $\xi < 0$ and $a_0 < 0$

$$\begin{aligned}
Mass &= \int_{-\infty}^{\infty} \rho(0,x)dx \\
&= \frac{\xi}{a_0^{1/3}} \int_{-\infty}^{\infty} \sqrt{-\frac{1}{\xi}\left(\frac{x}{a_0^{1/3}}\right)^2 + \left(\frac{1}{\xi}\alpha\right)^2} dx \\
&= \infty;
\end{aligned}$$

(2) the global solution (3.7), $\xi > 0$ and $a_0 > 0$

$$\begin{aligned}
Mass &= \int_{-\infty}^{\infty} \rho(0,x)dx \\
&= 2\int_0^{a_0^{1/3}\alpha\sqrt{\frac{1}{\xi}}} \frac{f\left(\frac{x}{a_0^{1/3}}\right)}{a_0^{1/3}} dx \\
&= \frac{2\xi}{a_0^{1/3}} \int_0^{a_o^{\frac{1}{3}}\alpha\sqrt{\frac{1}{\xi}}} \sqrt{-\frac{1}{\xi}\left(\frac{x}{a_0^{1/3}}\right)^2 + \left(\frac{\alpha}{\xi}\right)^2} dx \\
&= 2\xi \int_0^{\alpha\sqrt{\frac{1}{\xi}}} \sqrt{-\frac{1}{\xi}s^2 + \left(\frac{\alpha}{\xi}\right)^2} ds \\
&= \frac{2\xi}{\sqrt{\xi}} \int_0^{\alpha\sqrt{\frac{1}{\xi}}} \sqrt{\frac{\alpha^2}{\xi} - s^2 ds} \\
&= \frac{\alpha^2\pi}{2\sqrt{\xi}}.
\end{aligned}$$

We are also interested in how fast the blowup solutions tends to infinity as t approaches the critical time T.

Theorem 3.2 *The blowup rate of the solutions (3.5) and (3.8) for $\alpha > 0$, is*

$$\lim_{s \to S^-} \rho(s,0)(S-s)^{1/3} > 0.$$

Proof. We only need to study the blowup rate of the Emden equation (3.3) with $\xi < 0$.

Assume the total energy $E > 0$,.

$$\begin{aligned} S &= s + \int_s^S d\eta \\ &= s + \int_{a(s)}^0 \frac{d\eta}{da} da \\ &= s - \int_{a(s)}^0 \frac{1}{\sqrt{2E + 3\xi a^{2/3}(\eta)}} da \\ &\geq s + \int_0^{a(s)} \frac{da(\eta)}{\sqrt{2E}}. \end{aligned}$$

That is,

$$(S - s)^{1/3} \geq \delta a^{1/3}(s)$$

for some $\delta > 0$. It follows that

$$\lim_{s \to S^-} \rho(s, 0)(S - s)^{1/3} = \lim_{s \to S^-} \frac{\alpha}{a^{1/3}(s)} (S - s)^{1/3} \geq \alpha\delta > 0.$$

■

Note that our blowup rate in the above theorem is different from the results in Constantin and Escher [16] and [13].

3.2 The 2-Component Degasperis-Procesi Equations

It is natural to ask if the above result can be extended to the 2-component Degasperis-Procesi equations

$$\begin{cases} \rho_t + k_2 u \rho_x + (k_1 + k_2)\rho u_x = 0, \qquad x \in R \\ u_t - u_{xxt} + 4uu_x - 3u_x u_{xx} - uu_{xxx} + k_3 \rho\rho_x = 0. \end{cases} \tag{3.17}$$

The separation method has indeed been applied successfully again for this purpose in [92].

Theorem 3.3 ([92]) *Define the function $a(s)$ to be the solution of the Emden equation*

$$\begin{cases} \ddot{a}(s) - \frac{\xi}{4a^{\kappa}(s)} = 0 \\ a(0) = a_0 > 0, \ \dot{a}(0) = a_1 \end{cases} \tag{3.18}$$

and

$$f(\eta) = k_3\xi \sqrt{-\frac{\eta^2}{k_3\xi} + \left(\frac{\alpha}{k_3\xi}\right)^2} \tag{3.19}$$

where $\eta = \frac{x}{a^{k_2/4}(s)}$ *with* $s = 4t$; $\kappa = \frac{k_1}{2} + k_2 - 1$, $\alpha \geq 0$, $\xi \neq 0$, a_0 *and* a_1 *are constants. For the 2-component Degasperis-Procesi system (3.17), there exists a family of solutions.*

(1) For $k_3 = 0$, *and* $\xi = 0$, *we have*

$$\rho(t,x) = \frac{\rho_0(\eta)}{a^{(k_1+k_2)/4}(4t)}, \qquad u(t,x) = \frac{\dot{a}(4t)}{a(4t)}x \tag{3.20}$$

where $\rho_0 \geq 0$ *is an arbitrary* C^1 *function.*

(2) For $k_3 > 0$ *and* $\xi > 0$, *or*

(3) For $k_3 < 0$ *and* $\xi < 0$, *we have*

$$\rho(t,x) = \max\left(\frac{f(\eta)}{a^{(k_1+k_2)/4}(4t)},\ 0\right), \qquad u(t,x) = \frac{\dot{a}(4t)}{a(4t)}x. \tag{3.21}$$

Our solutions (3.21) can shed some light on the evolution of breaking waves. These are solutions that develop singularities due to unbounded derivatives while remaining bounded themselves.

Notice that the mass of a solution of $(3.17)_1$ is not conserved except when $k_1 = 0$ and $k_2 = 1$. Nevertheless, we can still obtain solutions with a nice functional structure for arbitrary k_1 and k_2.

Lemma 3.3 *For the* 1*-dimensional equation of mass* $(3.17)_1$

$$\rho_t + k_2 u\rho_x + (k_1 + k_2)\rho u_x = 0, \tag{3.22}$$

there exist solutions of the form

$$\rho(t,x) = \frac{f(\eta)}{a^{(k_1+k_2)/4}(4t)}, \qquad u(t,x) = \frac{\dot{a}(4t)}{a(4t)}x \tag{3.23}$$

with arbitrary $f(\eta) \geq 0 \in C^1$, $\eta = \frac{x}{a^{k_2/4}(4t)}$, *and* $a(4t) > 0 \in C^1$.

Proof. Direct verification gives

$$
\begin{aligned}
&\rho_t + k_2 u\rho_x + (k_1+k_2)\rho u_x \\
&\quad = \frac{\partial}{\partial t}\left(\frac{f(\eta)}{a^{(k_1+k_2)/4}(4t)}\right) + k_2\frac{\dot{a}(4t)}{a(4t)}x\frac{\partial}{\partial x}\left(\frac{f(\eta)}{a^{(k_1+k_2)/4}(4t)}\right) \\
&\quad\quad + (k_1+k_2)\frac{f(\eta)}{a^{(k_1+k_2)/4}(4t)}\frac{\partial}{\partial x}\left(\frac{\dot{a}(4t)}{a(4t)}x\right) \\
&\quad = \frac{1}{a^{(k_1+k_2)/4+1}(4t)}\left(-\frac{(k_1+k_2)}{4}\right)\cdot \dot{a}(4t)\cdot 4\cdot f(\eta) \\
&\quad\quad + \frac{1}{a^{(k_1+k_2)/4}(4t)}\dot{f}(\eta)\frac{\partial}{\partial t}\left(\frac{x}{a^{k_2/4}(4t)}\right) \\
&\quad\quad + \frac{k_2\dot{a}(4t)x}{a(4t)}\frac{\dot{f}(\eta)}{a^{(k_1+k_2)/4}(4t)}\frac{\partial}{\partial x}\left(\frac{x}{a^{k_2/4}(4t)}\right) + \frac{(k_1+k_2)f(\eta)}{a^{(k_1+k_2)/4}(4t)}\frac{\dot{a}(4t)}{a(4t)} \\
&\quad = -\frac{(k_1+k_2)\dot{a}(4t)f(\eta)}{a^{(k_1+k_2)/4+1}(4t)} - \frac{1}{a^{(k_1+k_2)/4}(4t)}\dot{f}(\eta)\frac{x}{a^{k_2/4+1}(4t)}\frac{k_2}{4}\dot{a}(4t)\cdot 4 \\
&\quad\quad + \frac{k_2\dot{a}(4t)x}{a(4t)}\frac{\dot{f}(\eta)}{a^{(k_1+k_2)/4}(4t)}\frac{1}{a^{k_2/4}(4t)} + (k_1+k_2)\frac{f(\eta)}{a^{(k_1+k_2)/4}(4t)}\frac{\dot{a}(4t)}{a(4t)} \\
&\quad = 0.
\end{aligned}
$$

■

Proof of Theorem 3.3.

As in the proof of Theorem 3.1, $u_{xx} = 0$ and the second equation in (3.17) becomes

$$
\begin{aligned}
&u_t - u_{xxt} + 4uu_x - 3u_x u_{xx} - uu_{xxx} + k_3\rho\rho_x \\
&\quad = u_t + 4uu_x + k_3\rho\rho_x \\
&\quad = \frac{\partial}{\partial t}\left(\frac{\dot{a}(4t)}{a(4t)}\right)x + 4\left(\frac{\dot{a}(4t)}{a(4t)}\right)x\frac{\dot{a}(4t)}{a(4t)} \\
&\quad\quad + k_3\frac{f(\eta)}{a^{(k_1+k_2)/4}(4t)}\left(\frac{f(\eta)}{a^{(k_1+k_2)/4}(4t)}\right)_x \\
&\quad = \left(4\frac{\ddot{a}(4t)}{a(4t)} - 4\frac{\dot{a}^2(4t)}{a^2(4t)}\right)x + 4\left(\frac{\dot{a}(4t)}{a(4t)}\right)\frac{\dot{a}(4t)}{a(4t)}x \\
&\quad\quad + k_3\frac{f(\eta)}{a^{(k_1+k_2)/4}(4t)}\frac{\dot{f}(\eta)}{a^{(k_1+k_2)/4}(4t)}\frac{1}{a^{k_2/4}(4t)} \\
&\quad = 4\frac{\ddot{a}(4t)}{a(4t)}x + k_3\frac{f(\eta)\dot{f}(\eta)}{a^{\frac{k_1}{2}+\frac{3k_2}{4}}(4t)} \\
&\quad = \frac{k_3}{a^{\frac{k_1}{2}+\frac{3k_2}{4}}(4t)}\left(\frac{\xi}{k_3}\ eta + f(\eta)\dot{f}(\eta)\right).
\end{aligned}
$$

For $k_3 \neq 0$, we use the Emden equation (3.18) together with the change of variables $s := 4t$, $\eta := x/a^{k_2/4}(s)$, and $\kappa = \frac{k_1}{2} + k_2 - 1$.

For $k_3 = 0$, we have $\xi = 0$ and

$$\rho(t,x) = \frac{\rho_0(\eta)}{a^{(k_1+k_2)/4}(4t)}, \qquad u(t,x) = \frac{\dot{a}(4t)}{a(4t)}x \tag{3.24}$$

where ρ_0 is an arbitrary C^1 function.

For the graphical illustration of the breaking wave solution (3.21), for the integrable system with $k_3 = 1$, $\xi = 1$, $\alpha = 2$, and $a_0 = 1$, we can see the initial shape of the self-similar solution:

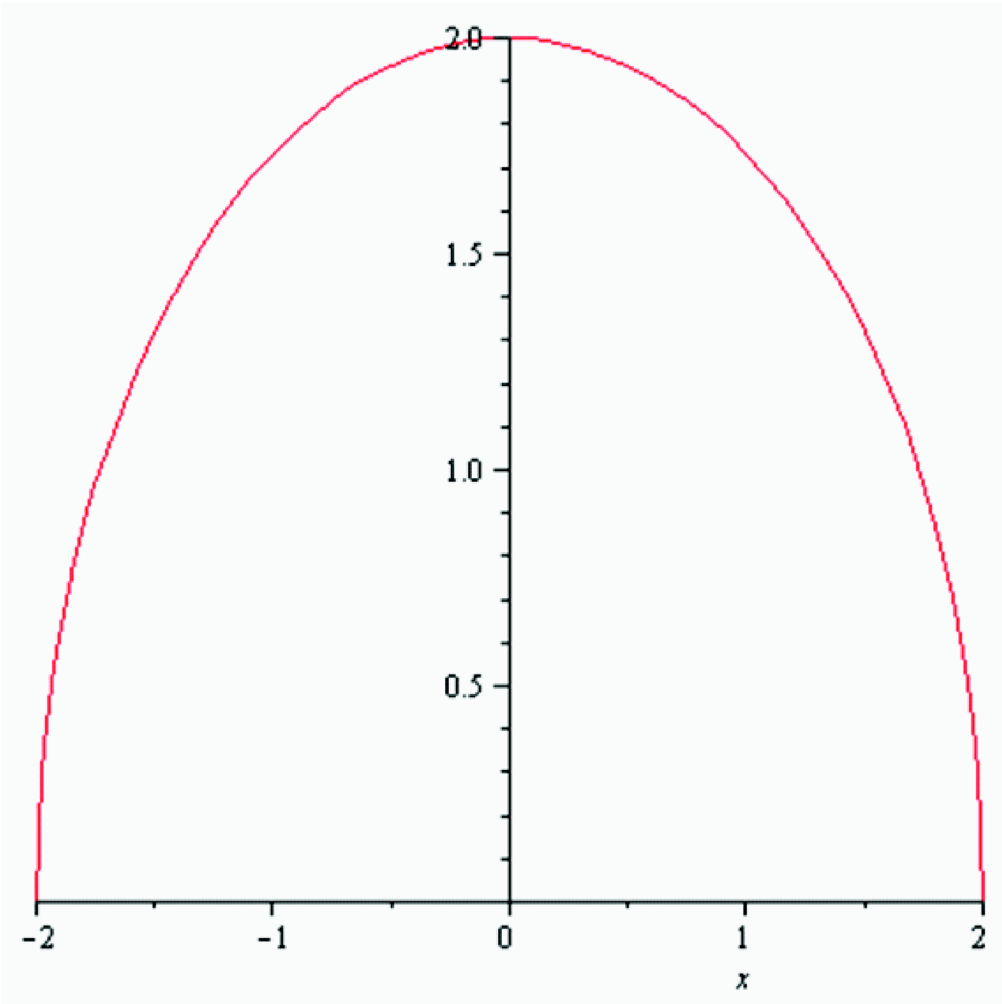

Fig. 3.6: $\rho(0,x) = \sqrt{-x^2 + 2^2}$

The rest of the proof is now straightforward. ■

Blowup properties of the solutions of (3.17), analogous to statements (1a) and (2a) of Theorem 3.1, Remarks 3.1–3.3, and Theorem 3.2 can be similarly established. We omit the details.

Chapter 4

Pulsating Flows to the 2D Euler-Poisson Equations

In this chapter, we show the existence of a class of "radially symmetric" rotational solutions to the 2-dimensional pressureless Euler-Poisson equations. The flows are global (i.e., exist for all $t > 0$), have compact support at all time, and pulsate periodically. Our result demonstrates that rotation can prevent blowup phenomena.

4.1 Introduction

The N-dimensional compressible Euler-Poisson equations

$$\begin{cases} \rho_t + \nabla\cdot(\rho\vec{u}) = 0 \\ \rho\left[\vec{u}_t + (\vec{u}\cdot\nabla)\vec{u}\right] + \nabla P = -\rho\nabla\phi \\ \Delta\phi(t,\vec{x}) = \alpha(N)\rho, \end{cases} \tag{4.1}$$

can be rewritten in the scalar form

$$\begin{cases} \dfrac{\partial\rho}{\partial t} + \displaystyle\sum_{k=1}^{N} u_k \frac{\partial\rho}{\partial x_k} + \rho\sum_{k=1}^{N}\frac{\partial u_k}{\partial x_k} = 0 \\ \rho\left(\dfrac{\partial u_i}{\partial t} + \displaystyle\sum_{k=1}^{N} u_k\frac{\partial u_i}{\partial x_k}\right) + \dfrac{\partial P}{\partial x_i} = -\rho\dfrac{\partial\phi}{\partial x_i} \qquad \text{for } i = 1, 2, ...N. \end{cases} \tag{4.2}$$

Existence and stability results can be found, for example, in [2], [7], [23], [25], [29], [39], [50], [54], [55], [56], [57], [60], [69] and [81].

For the most part we are concerned with classical solutions having compact support. In other words $\rho(0, \vec{x})$ has continuous first derivatives and vanishes outside a ball of finite radius, say, R. The velocity function $\vec{u}(0, \vec{x})$, on the other hand, is only assumed to be C^1 and is not required to have compact support. Local existence results guarantee that a C^1 solution exists in some time interval $[\,0, t^*)$, $t^* > 0$. Since the system is hyperbolic, the fluid will have compact support at any fixed future time $t > 0$, but, of course, the support may grow (and even become out of bound) with time. If t^* can be chosen as large as we please, the solution is said to be global. In the contrary case, there is an upper bound of all such t^*. Without loss of generality we may assume that t^* has already been chosen to be this upper bound. As $t \to t^*-$, the regularity of the solution is lost, either due to a blowup of ρ or $\vec{u}$ at some finite point $\vec{x}_0$, or a blowup of one of their first derivatives.

The simplest example of a break-down of solution regularity is the existence of shock waves in solutions of the Burgers' equation. The onset of turbulence in fluid motion is another. As a general rule, blowup solutions are more interesting and more difficult to study than global ones. In the case of one-dimensional or radially symmetric irrotational higher-dimensional flows, Engelberg, Liu, and Tadmor [22] and Liu and Tadmor [53] study the phenomenon of critical thresholds. A certain inequality involving the initial state of a flow determines whether finite-time blowup occurs later or not. Intuitively speaking, if initially the flow is quickly expanding, meaning that the flow particles all move away from each other with relative velocity exceeding a certain threshold, then the gravitational force will not be strong enough to pull the particles towards each other so as to cause collision in finite time.

Recently, Chae and Tadmor [7] demonstrate finite-time blowup for the pressureless Euler-Poisson system with attractive forces, under the condition that at the initial time $t = 0$, there exists a point x_0 for which the set

$$S := \{\, a \in R^N \big| \;\; \rho_0(a) > 0, \;\; \Omega(a) = 0, \;\; \nabla \cdot \vec{u}(0, x_0(0)) < 0\} \neq \emptyset, \tag{4.3}$$

is non-empty. The rescaled vorticity matrix Ω is defined as $(\Omega_{ij}) = \frac{1}{2}(\partial_i u^j - \partial_j u^i)$ with the notation $\vec{u} = (u^1, u^2, ..., u^N)$ in their paper. For $N = 2$, condition (4.3) alone is sufficient for blowup. For $N > 2$, an additional condition ensuring that the fluid velocity is not too large is needed.

They use the technique of spectral dynamics to derive the Riccati differential inequality,

$$\frac{D\,\mathrm{div}\,\vec{u}}{Dt} \leq -\frac{1}{N}(\mathrm{div}\,\vec{u})^2. \tag{4.4}$$

The inequality (4.4) blows up at or before $T = -N/(\nabla \cdot \vec{u}(x_0(0), 0))$.

In this chapter, we confine ourselves to the simpler particular class of two-dimensional pressureless systems, in which the term in ∇P $(4.1)_2$ is assumed to be 0. For the two-dimensional pressureless system, the Euler-Poisson equations (4.2) assumes the simpler form

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \displaystyle\sum_{k=1}^{2} u_k \frac{\partial \rho}{\partial x_k} + \rho \sum_{k=1}^{2} \frac{\partial u_k}{\partial x_k} = 0 \\ \dfrac{\partial u_i}{\partial t} + \displaystyle\sum_{k=1}^{2} u_k \frac{\partial u_i}{\partial x_k} = -\frac{\partial \phi}{\partial x_i} \text{ for } i = 1, 2. \end{cases} \tag{4.5}$$

We are interested in a special class of "radially symmetric" solutions of the form

$$\rho(t, \vec{x}) = \rho(t, r),\ \vec{u}(t, \vec{x}) = \frac{F(t, r)}{r}(x, y) + \frac{G(t, r)}{r}(-y, x) \tag{4.6}$$

For each fixed t, the functions ρ, F, and G take the same values for all points $\vec{x}$ that are equidistant from the origin. The function $F(t, r)$ represents the radial component of the flow velocity while $G(t, r)$ represents the rotational component.

When $G(t, r) \equiv 0$, the functions reduce to the classical radially symmetric solutions which have been well studied. See, for example, [50], [60], [79], [80] and [84]. These are irrotational flows for which the fluid particles are moving strictly in the radial direction. Such a flow satisfies the Chae-Tadmor condition (4.3) and hence by their result in [7], the solution cannot be global. Physically this means that the gravitational force, represented by the Poisson term in the flow equations, is so strong that it is always able to pull the flow particles closer and closer together and collision eventually occurs. The purpose of this chapter is to demonstrate that when rotational motion is present, collision can be circumvented to yield global solutions.

Theorem 4.1 ([44]) *There exists a global solution of the Euler-Poisson equations (4.5) of the form (4.6) for which $F \neq 0$. It is necessary that $G \neq 0$.*

The physical explanation is that part of the gravitational force is averted to balance the centrifugal force due to rotation. The principle that rotation can prevent the blowup of solutions was first suggested by Liu and Tadmor in [53], where they study the 2D Euler equation, with the Poisson term replaced by an external Coriolis force. Our result furnishes another example. In the system of Liu and Tadmor, the rotation is caused by an external Coriolis force, whereas in ours, the rotation is induced by the internal gravitational force.

Concerning the structure of global solutions of the type (4.6), we prove in Section 4.3 that

Theorem 4.2 ([44]) *For a global solution of the form in Theorem 4.1, the support of the solution must be decomposable into a sum of disjoint annular regions. At the boundary of the annulus, the flow particles have only rotational, but no radial, motion. Inside each annulus, the solution "pulsates" with uniform period.*

Here the term "pulsates" refers to the periodic change of the distance of a flow particle from the origin. See Section 4.2 for the detailed explanation. The pulsating periods in different regions can, however, be different.

For Theorem 4.3 and Section 4.4, we have to relax the smoothness requirement on $\rho(0, \vec{x})$ by requiring only that its first derivative exists everywhere except on the points of a particular circle, at which the two one-sided derivatives of ρ exist but are not identical. In this situation, the solution is no longer a classical solution and should be understood in a weaker sense. One approach is to solve the system of equations in the classical sense in the complement of the circle of singularity, and the solutions in the two regions, one inside the circle and one outside the circle, are then pieced together.

Our next result asserts that global solutions with non-trivial annular structures as described in Theorem 4.2 do exist in a weak sense.

Theorem 4.3 ([44]) *A given global solution of the form in Theorem 4.1 can be extended to a larger global (weak) solution by adding an annular pulsating solution, having any given period, outside the support of the original solution.*

First note that for any given function $f(r, t)$ of r and t, we have

$$f_t = \frac{\partial f(t, r)}{\partial t}, \tag{4.7}$$

$$f_x = \frac{\partial f}{\partial r} \cdot \frac{x}{r}, \; f_y = \frac{\partial f}{\partial r} \cdot \frac{y}{r}. \tag{4.8}$$

The following identities are also useful.

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\frac{x}{r^3}, \quad \frac{\partial}{\partial y}\left(\frac{1}{r}\right) = -\frac{y}{r^3}, \tag{4.9}$$

$$\frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \frac{y^2}{r^3}, \quad \frac{\partial}{\partial y}\left(\frac{x}{r}\right) = -\frac{xy}{r^3}, \tag{4.10}$$

$$\frac{\partial}{\partial x}\left(\frac{y}{r}\right) = -\frac{xy}{r^3}, \quad \frac{\partial}{\partial y}\left(\frac{x}{r}\right) = \frac{x^2}{r^3}. \tag{4.11}$$

From (4.6), the components of $\vec{u}$ have the form

$$u_1 = \frac{x}{r}F - \frac{y}{r}G, \tag{4.12}$$

$$u_2 = \frac{y}{r}F + \frac{x}{r}G. \tag{4.13}$$

Substituting this into the first momentum equation in $(4.5)_2$, we get

$$\frac{x}{r}F_t - \frac{y}{r}G_t + \left(\frac{x}{r}F - \frac{y}{r}G\right)\frac{\partial}{\partial x}\left(\frac{x}{r}F - \frac{y}{r}G\right) + \left(\frac{y}{r}F - \frac{x}{r}G\right)\frac{\partial}{\partial y}\left(\frac{y}{r}F - \frac{x}{r}G\right) = -\frac{x}{r}\phi_r. \tag{4.14}$$

Making use of the identities (4.9)–(4.11) above, and after quite a bit of algebraic simplification, we get

$$\frac{x}{r}\left(F_t + FF_r - \frac{G^2}{r} + \phi_r(\rho)\right) + \frac{y}{r}\left(G_t + FG_r + \frac{FG}{r}\right) = 0. \tag{4.15}$$

If we carry out the same calculations using the second momentum equation in $(4.5)_2$, we arrive at

$$\frac{y}{r}\left(F_t + FF_r - \frac{G^2}{r} + \phi_r(\rho)\right) + \frac{x}{r}\left(G_t + FG_r + \frac{FG}{r}\right) = 0. \tag{4.16}$$

Finally, it is easy to verify that the first equation of the Euler-Poisson equations (4.5), the equation of conservation of mass, is automatically satisfied by the class of solutions (4.6).

Then we need to have the two functions $F(t, r)$ and $G(t, r)$.to be

$$\begin{cases} F_t + FF_r - \dfrac{G^2}{r} + \dfrac{2\pi}{r}\displaystyle\int_0^r \rho(t,s)s\,ds = 0 \\ G_t + FG_r + \dfrac{FG}{r} = 0. \end{cases} \tag{4.17}$$

We employ the familiar technique of characteristic curves. Such curves are determined by the solution $r(t; a)$ of the initial value problem

$$\frac{dr(t;a)}{dt} = F(t, r), \quad r(0) = a. \tag{4.18}$$

Given any function $f(t, r)$ of r and t, we denote

$$f' := \frac{d}{dt} f(t, r(t;a)) = \left(\frac{\partial}{\partial t} + F(t,r) \frac{\partial}{\partial r} \right) f(t,r) \bigg|_{r=r(t;a)} . \tag{4.19}$$

With this notation, equations $(4.17)_1$ and $(4.17)_2$ reduce to the simpler form

$$\begin{cases} F' - \dfrac{G^2}{r} + \dfrac{M}{r} = 0, \\ G' + \dfrac{FG}{r} = 0, \end{cases} \tag{4.20}$$

where

$$M = M(a) = 2\pi \int_0^{r(t;a)} \rho(s,t) s \, ds$$

$$= 2\pi \int_0^a \rho(s,0) s \, ds \tag{4.21}$$

is a constant along a given characteristic curve. Knowing the initial density function $\rho(0, r)$, $M(a)$ can be determined from (4.21). Conversely, if $M(a)$ is known as a function of a, the initial density function is given by

$$\rho(0,a) = \frac{M'(a)}{2\pi a} . \tag{4.22}$$

Equations (4.18) and $(4.20)_2$ imply that

$$(rG)' = rG' + r'G = (-FG) + FG = 0. \tag{4.23}$$

Therefore,

$$r(t;a) G(t, r(t;a)) = c = c(a) \tag{4.24}$$

is a constant along any fixed characteristic curve. As a result

$$G(t, r(t;a)) = \frac{c}{r(t;a)} = \frac{c}{r} . \tag{4.25}$$

In the following, for the sake of convenience, we will suppress the explicit mentioning of the dependence of constants such as $M(a)$ and $c(a)$ on a, and simply write M and c instead, when there is no risk of confusion.

Substituting (4.25) into $(4.20)_1$ and making use of (4.18), we arrive at a second-order differential equation for $r(t;a)$

$$r'' + \frac{M}{r} - \frac{c^2}{r^3} = 0. \tag{4.26}$$

The initial conditions satisfied by r are

$$r(0) = a, \qquad r'(0) = a_1 = F(0, a). \tag{4.27}$$

Equation (4.26) describes a conservative dynamical system, having a constant energy

$$E = \frac{(r'(t))^2}{2} + M \ln(r(t)) + \frac{c^2}{2r^2(t)} \tag{4.28}$$

$$\equiv \frac{a_1^2}{2} + M \ln(a) + \frac{c^2}{2a^2}\,. \tag{4.29}$$

From this identity, all solutions can, in theory, be computed using a quadrature. It is obvious that all solutions oscillate periodically around the equilibrium point $r = c/\sqrt{M}$. The period and amplitude depend on the energy E (when M and c are given). More specifically, the maximum and minimum radii are given by the formulas

$$r_M = \exp\left(\frac{1}{2} M \mathcal{L}\left(-\frac{c^2 \exp(-2A)}{M}\right) + A\right) \tag{4.30}$$

and

$$r_M = \exp\left(\frac{1}{2} M \mathcal{L}\left(-1, -\frac{c^2 \exp(-2A)}{M}\right) + A\right), \tag{4.31}$$

respectively, where $A = E/M$, and $\mathcal{L}(\cdot)$ is the well-known Lambert W function (see [6]) defined by

$$\mathcal{L}(x) = w \Longleftrightarrow w e^w = x \tag{4.32}$$

and $\mathcal{L}(-1, \cdot)$ is the $(-1)^{\mathrm{st}}$ branch of the function. The period of the oscillation is given by

$$T = \int_{r_m}^{r_M} \frac{d\,r}{\sqrt{E - \ln(r) - 1/2r^2}}\,. \tag{4.33}$$

In the particular situation when

$$a^2 M = c^2 \qquad \text{and} \qquad a_1 = 0, \tag{4.34}$$

$r(t)$ becomes the stationary solution $r(t) \equiv a$ and $F(t, r(t; a)) \equiv 0$. This represents a degenerate oscillation with amplitude 0. The period of a stationary solution cannot be determined from the solution itself; a linear analysis in the neighborhood of the equilibrium point provides the needed tool. See Lemma 4.5 in the next section.

If we make (4.34) hold for all a, we arrive at a fairly trivial class of solutions of the 2D Euler-Poisson equations. Let us choose $M(r)$ to be an arbitrary non-decreasing C^1 function with compact support. Then it is easy to verify that the functions

$$\rho(t,r) = \frac{M'(r)}{r}, \quad F(t,r) \equiv 0, \quad G(t,r) = M(r) \tag{4.35}$$

satisfy (4.34) for all a and furnish a solution of the Euler-Poisson equations. It is a stationary solution, not because the flow particles are stationary, but in the sense that the solution functions do not change in time. This is a global solution that represents a purely rotational flow with no radial motion. The purpose of the rest of the article is to show the existence of global solutions with nontrivial radial motion.

4.2 Physical Interpretation

The moving fluid is interpreted as being made up of individual, infinitesimally small particles located at position $\vec{x}$, and traveling with velocity $\vec{u}$ at time t. The property of "radial symmetry" refers to the requirement that the "velocity" of particles having the same distance from the origin is in some sense the "same".

Up to now, the class of "radially symmetric" solutions that have been well studied are irrotational flows, that is, with $G \equiv 0$ in (4.6). Here we assume that the particles also revolve around the origin. In (4.6), $G(t,r)$, the rotational component, is perpendicular to the radial component $F(t,r)$. For all particles having the same distance from the origin, the magnitudes of these components are identical.

Normally, to follow a characteristic curve means to attach a coordinate reference frame to one particular particle; as the particle moves, the reference frame moves together with the particle. In our interpretation, we are not following just one individual particle, but instead the moving circular shell $S(t)$ which, for fixed time t, contains the particle we are following as well as all other particles with the same distance from the origin. In other words, the "characteristic curve" refers not to the trajectory of one particular particle, but rather the "size" (the radius) of the collective $S(t)$. Strictly speaking, the moving shell S refers to a circular shell plus its motion in time, while $S(t)$ refers to the circular shell at at particular time t. However, for the sake of convenience, we may sometimes use the symbol S to refer to a particular circular shell $S(t)$ at a fixed time t.

As we have seen in the previous section, the size of S varies periodically. In physical terms, the shell S expands and shrinks periodically as $r(t)$ increases and decreases. The special case when $r(t)$ is stationary can be regarded as a degenerate oscillation. If we follow a particular particle in S, we see that it revolves around the origin while its distance from the original varies periodically. This is similar to the motion of a planet in an elliptic trajectory around the sun, which is located at the origin. The shell S is like a continuum of identical small planets, which are assumed to be indistinguishable from each other, with identical distance from the sun. As each individual planet revolves, the collective observable effect is the pulsating motion of the shell. In the degenerate case when $r(t;a)$ is a constant, each particle in the shell moves in a circular path (instead of an oblong elliptic path) around the origin, and the shell appears to be stationary.

Now let us decompose the entire set of particles that make up the initial flow into circular shells, and denote by $S_a = S_a(t)$ the shell corresponding to the initial radius $a < R$. According to the above analysis, each shell S_a pulsates periodically with its own period and amplitude.

Suppose that initially a shell S_a is inside another shell S_b, that is, $a \leq b$. Our assumption that $\vec{u}$ is a C^1 solution implies that as time t increases, S_a remains inside S_b, lest the spatial derivatives of $\vec{u}$ and/or $\vec{\rho}$ blow up as S_a crashes into S_b, reminiscent of the formation of shock waves of the Burgers' equation when two characteristic curves intersect each other.

The quantity $M = M(a)$ in $(4.20)_1$ represents the total mass of the particles contained inside the circle bounded by S_a. From the above discussion, we see that particles that are initially inside a shell can never flow outward across the shell. Hence, M is a constant along a characteristic curve.[1]

Let

$$A_0 = \{a \in (0, R) \,|\, S_a\} \tag{4.36}$$

[1] It is interesting to see if a concept of 'weak" solutions can be formulated by allowing characteristic curves to intersect and then afterward each to continue its own way. After the intersection, the quantity M on a characteristic will not remain the same and we need to find the correct way to adjust it accordingly.

denote the set of all stationary shells, and

$$A = (0, R) \setminus A_0 \tag{4.37}$$

be its complement, the set of non-stationary shells. By continuity, A is an open subset of $(0, R)$ and hence is the union of a countable number of open intervals

$$A = \bigcup_{n=1}^{\infty} I_n. \tag{4.38}$$

In the next section, we shall see (Lemma 4.1) that under the hypothesis of having a global solution, all shells within each subinterval I_n must pulsate with the same period. This fact implies Theorem 4.2 .

4.3 Uniformly Pulsating Layer

As mentioned above, Theorem 4.2 is a consequence of the following Lemma.

Lemma 4.1 *Let I_n be one of the open subintervals of A in 4.38. Then the periods of all the pulsating shells S_{a_k}, $a \in I_n$ are the same.*

Proof. A mathematically rigorous proof can surely be written down, but it would be full of technical details and would be fairly dry and tedious reading. We prefer to describe the ideas of the proof in a pedagogical manner and let the readers fill in the analytic details themselves.

Take any $\bar{a} \in I_n$ and define the set

$$B = \{a \in I_n | S_a = S_{\bar{a}}\} . \tag{4.39}$$

By continuity, B is a closed subset of I_n. If we can show that B is also an open subset, then B must be the entire I_n. To that end, we need only to show that every $a \in B$ is an interior point of B.

Suppose the contrary. Then there must be a sequence of points $a_k \in I_n \setminus B$, $a_k \to a$, such that the period T_k of S_{a_k} differs from the period T of S_a, for all k, that is, $\varepsilon_k = |T_k - T| > 0$. As a consequence, a phase shift exists between the two "waves". Fig.4.1 depicts the graph of $r(t; a)$ (the radius of S_a) and $r(t; a_k)$ for one such k.

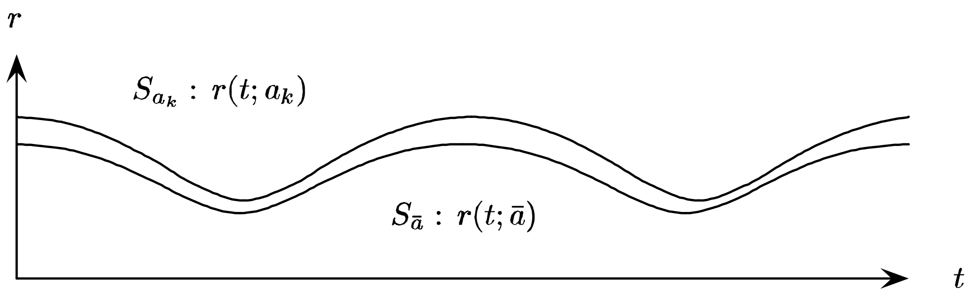

Fig. 4.1: Graphs of the radii of two shells

As seen in the last section, the hypothesis of having a global solution implies that the two characteristic curves cannot intersect each other. Suppose $a_k > a$. Then at $t = 0$, the graph of $r(t; a_k)$ starts above that of $r(t; a)$, and it should always remain on top. If $a_k < a$, then the opposite is true. Since the proof is similar in both cases, we can confine ourselves to the first case.

On the other hand, by choosing k large enough, we can make the minimum height of the upper curve strictly less than the maximum height of the lower curve, and we can also make the phase difference ε_k between the two shells as small as we please. As time evolves, the phase difference between the two shells will gradually build up. Depending on the relative sizes of the periods, the minimum point of the upper curve will move either towards the next or the previous maximum point of the lower curve. In Fig.4.1, the upper curve is chosen to have a slightly longer period. Suppose both curves assume their maximum at $t = 0$. The first minimum of the upper curve is then located slightly to the right of that of the lower curve. As we get to the second minimum point of the upper curve, it should have moved twice as much relative to the second minimum point of the lower curve. It is at least intuitively obvious that sooner or later, the minimum point of the upper curve will catch up with the maximum point of the lower curve. Since the former is strictly below the latter, we see a contradiction (the fact that the difference between the two periods can be made sufficiently small plays a role here). ■

For convenience, let us call two non-crashing shells compatible. Obviously, a larger shell will be compatible with a smaller one if the minimum radius of the former is larger than the maximum radius of the latter, no matter what the two periods of the shells are. On the other hand, if the minimum radius of the former is less than the maximum radius of the latter, then from the proof of Lemma 4.1, we see that the two shells must have the same period. It is interesting to note that in the critical case, when the minimum radius of the larger shell is equal to the maximum radius of the smaller shell, then there are several possibilities depending on the initial phase difference of the two shells, and may depend on whether the ratio of the two periods are rational or not.

Note that the uniformity of the periods of all the shells is only a necessary condition for the solution to be global and is far from being sufficient. Even if two shells have the same period, their oscillation amplitudes and/or phase difference may be such that the two shells can still crash into each other in finite time.

The period of a shell depends on the parameters M and c, as well as on the initial conditions a and a_1. Every one of these values has to be just right to give a specific period. Therefore, it is quite a rare exception that a solution does not blow up in finite time.

The strategy we adopt to construct our global solution is to first construct suitable individual shells and then piece them together. Any two shells of a global solution must be compatible.

Each shell $S(t)$ is the set of all points with distance $r(t)$ from the origin, where $r(t)$ is a solution of the initial value problem (4.26) (4.27). Note that we have four degrees of freedom in constructing a shell, namely M, c, a and a_1. To simplify things, we always choose $a_1 = 0$, but that still leaves us with three degrees of freedom. Such flexibility will be put to good use later.

Suppose that by some means we have constructed correctly all the shells S_a for $a \in [\alpha, \beta]$ for some numbers $0 < \alpha < \beta$, and they are all mutually compatible. Then the "solution" $\rho(t, r)$ and $\vec{u}(t, r)$ can be determined for all $r \in [r(t; \alpha), r(t; \beta)]$. More specifically, from the parameters M and c associated with each shell S_a, we can determine the functions ρ (using (4.22)) and G; and we can also find $F = r'$. Of course, one cannot simply piece together any arbitrary set of shells. The set must be chosen discretely so that the calculated functions ρ, F, and G are C^1 smooth with respect to the spatial coordinates. Furthermore, $M(a)$ must be a non-decreasing function because the density ρ, calculated using (4.22), ought to be nonnegative.

Let us call the appropriate functions calculated above a "partial solution" defined on $[\alpha, \beta]$. It may not be a complete solution yet unless $\alpha = 0$, and $\rho(0, \beta) = 0$. In case these two conditions are not both satisfied, we need to demonstrate that we can extend the partial solution into a complete solution with compact support.

The next Lemma shows how one class of partial solutions can be easily extended to complete solutions. The method generalizes the construction of the class of trivial rotational flows given by (4.35).

Lemma 4.2 *Suppose we have a partial solution defined on $[0, \beta]$ with S_β being a stationary shell. Then the partial solution can be extended to a complete solution.*

Proof. Starting with the known $M(r)$ defined on $[0, \beta]$, we can extend it to an increasing function on $[0, \infty)$ in such a way that $M(r)$ is a constant on $[\beta + 1, \infty)$. This will correspond to a $\rho(r, 0)$ that vanishes on $[\beta + 1, \infty)$. The shells S_a for $a \in (\beta, \beta + 1)$ are chosen to be stationary shells by letting $c(a) = \beta\sqrt{M(a)}$ as required in (4.34). ■

After that, we study (4.26) as a standalone differential equation, temporarily ignoring the background Euler-Poisson equations. The quantities M and c are simply regraded as constant parameters. We are interested in finding out how varying M and c can affect the solution, and in particular its period. The first two Lemmas involve the familiar methods of scaling the solution and the independent variable, respectively. The methods have been used in [61].

Lemma 4.3 *Let $r(t)$ be a solution of (4.26) and $\lambda > 0$ be any positive constant. Then $\bar{r}(t) = \lambda r(t)$ is a solution of*

$$\bar{r}'' + \frac{\bar{M}}{\bar{r}} - \frac{\bar{c}^2}{\bar{r}^3} = 0, \tag{4.40}$$

where

$$\bar{M} = \lambda^2 M, \ \ \bar{c} = \lambda^2 c. \tag{4.41}$$

Proof. The proof is by direct computation. ■

Note that $r(t)$ and $\bar{r}(t)$ have the same period and they do not intersect when $\lambda \neq 1$. Lemma 4.1 has several simple but useful applications.

Suppose we know a shell S_α with initial radius $\alpha > 0$. By scaling down the shell with a constant a/α for each $0 < a < \alpha$, we obtain the set of shells $\{S_a = aS_\alpha/\alpha \,|\, a \in [0, \alpha]\}$, which is a partial solution corresponding to a flow with constant density $\rho(t, r) = 2M(\alpha)/\alpha^2$.

Let $r_1(t)$ and $r_2(t)$ be two compatible shells, with $a_1 = r_1(0) < r_2(0) = a_2$, and $0 < \lambda_1 < \lambda_2$ be any two real constants; then $\lambda_1 r_1(t)$ and $\lambda_2 r_2(t)$ are again two compatible shells.

The following construction generalizes the above two observations. Suppose we are given a partial solution $r_a(t)$ over an interval $[\alpha, \beta]$. Take any smooth increasing function $\lambda : [\alpha, \beta] \to [0, \lambda(\beta)]$ such that $\lambda(\alpha) = 0$. Then the scaled shells $\lambda(a)r_a(t)$, $a \in [\alpha, \beta]$ yield a partial solution over $[0, \lambda(\beta)\beta]$.

Lemma 4.4 *Let $r(t)$ be a solution of (4.26) and $\mu > 0$ be any positive constant. Then $\hat{r}(t) = r(\mu t)$ is a solution of*

$$\hat{r}'' + \frac{\hat{M}}{\hat{r}} - \frac{\hat{c}^2}{\hat{r}^3} = 0, \tag{4.42}$$

where

$$\hat{M} = \mu^2 M, \;\; \hat{c} = \mu c. \tag{4.43}$$

Proof. Again the proof is by direct computation. ■

The period of r is μ times that of $\hat{r}$. This Lemma can be used to obtain a shell with a desired period from a known solution of (4.26) that may not have the correct period. This can then be coupled with Lemma 4.1 to scale the shell to have a desired initial radius.

As a corollary of Lemmas 4.3 and 4.4, by suitably choosing λ and μ, we can always transform equation (4.26) to an equation of the same form with any other choices of M and c.

For the rest of the section, we always assume that $r(t)$ takes the initial value $r'(0) = 0$. We shall write $r(t; a)$ if we want to emphasize the dependence of r on the initial radius $r(0) = a$.

Lemma 4.5 *As* $a \to c/\sqrt{M}$, $a \neq c/\sqrt{M}$, *the period of the solution* $r(t; a)$ *approaches* $\sqrt{2}c\pi/M$, *which can be adopted as the period of the stationary solution* $r(t) \equiv c/\sqrt{M}$.

Proof. The substitution $r(t) = s(t) + c/\sqrt{M}$ transforms (4.26) into

$$s''(t) + g(s) = 0, \tag{4.44}$$

where

$$\begin{aligned} g(s) & \\ &= \frac{M}{s + c/\sqrt{M}} - \frac{c^2}{(s + c/\sqrt{M})^3} && (4.45) \\ &= \frac{2M^2}{c^2} s + O(s^2). && (4.46) \end{aligned}$$

By retaining only the first order term of $g(s)$, we see that $s(t)$ has the asymptotic period of $\sqrt{2}M\pi/c$ as $s \to 0$. ∎

Applying Lemma 4.5 to the case $M = c = 1$, we see that the stationary solution $r(t) \equiv 1$ has the period $\sqrt{2}\pi$. It turns out that another choice, $M = 1/2$, $c = 1/\sqrt{2}$, may be more convenient because it helps to eliminate the factor $\sqrt{2}$ in some relevant computations. The stationary solution is still $r(t) \equiv 1$ but the period is 2π.

For the next stage of our discussion, we vary the initial radius a, increasing it from the stationary value, and would like to know how the period changes. Numerical experiments (illustrated in Fig.4.2 and Table 4.1), either by directly solving the initial value problem (4.26) (4.27) or by evaluating (4.33), strongly suggest

Conjecture *The period of the solution of the initial value problem (4.26),* $r(0) = a$, $r'(0) = 0$, *is an increasing function of* a *over* $[\,c/\sqrt{M}, \infty)$ *and a decreasing function over* $[\,0, c/\sqrt{M}\,]$.

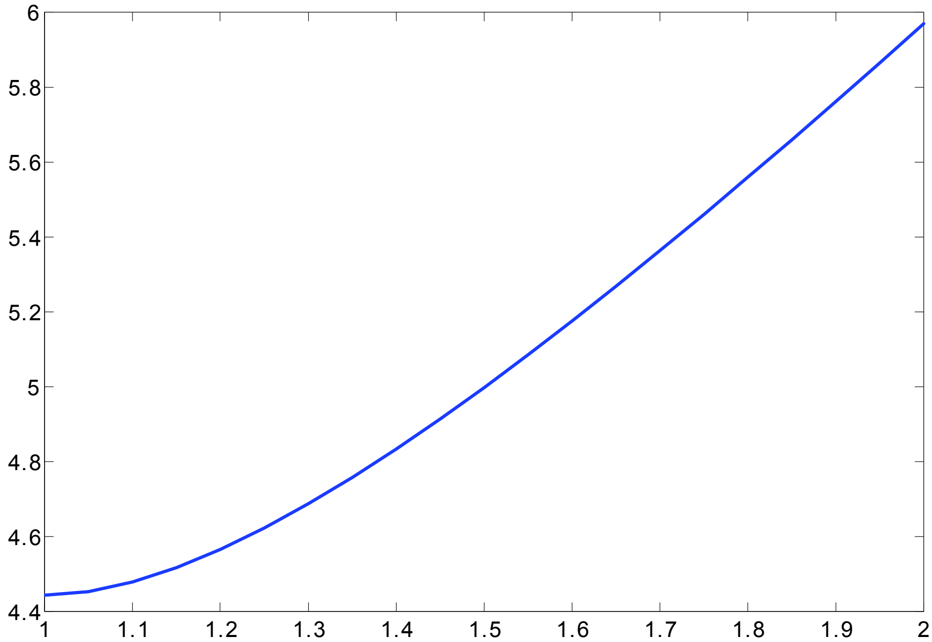

Fig. 4.2: $T(a)$ vs a, $M = c = 1$

a	$T(a)$	a	$T(a)$
1.02	2.2222162	1.22	2.2935196
1.04	2.2244910	1.24	2.3051739
1.06	2.2281031	1.26	2.3174210
1.08	2.2329380	1.28	2.3302202
1.10	2.2388907	1.30	2.3435345
1.12	2.2458704	1.32	2.3573310
1.14	2.2537925	1.34	2.3715788
1.16	2.2625819	1.36	2.3862496
1.18	2.2721720	1.38	2.4013178
1.20	2.2825025	1.40	2.4167597

Table 4.1: $T(a)$ vs a, $M = c = 1$

Supposedly the formula (4.33) contains all the information we need to establish this Conjecture, but we were unsuccessful in doing any useful analysis on it. We are only able to prove a partial result, which, fortunately, is all we need to construct our example of pulsating solution.

Lemma 4.6 *For given M and c, there exists a constant $\delta > c/\sqrt{M}$ such that the period of oscillation $T(a)$ of $r(t;a)$ is a C^∞ increasing function of a in the interval $[c/\sqrt{M}, \delta]$.*

Proof. In view of the scaling technique, we only have to prove the claim for a particular pair of (M, c). Let us choose $M = c^2 = 1/2$.

The fact that $T(a)$ is C^∞ follows from standard theory. As in the proof of Lemma 4.5, we first transform equation (4.26) to (4.44), and the initial conditions to

$$s(0) = a - 1, \quad s'(0) = 0. \tag{4.47}$$

We extend the method of small parameter analysis used in the proof of Lemma 4.5 to the third order, with one modification. Instead of determining T as the smallest positive time needed for $s(t)$ to attain $s(T) = a - 1$, we use the distance $\tau_3 - \tau_1$, where $\tau_1, \tau_2, \tau_3, ..$ are the points of intersection of the graph of $s(t)$ with the s-axis.

Let $b = s'(\tau_1)$. It follows from the conservation of energy that

$$b = \sqrt{2\ln(a) + \frac{1}{a^2} - 1}, \tag{4.48}$$

and it is easy to verify that b is an increasing function of a over $[1, \infty)$. Therefore, if we can show that T is an increasing function of b in some neighborhood of $b = 0$, then the conclusion of the Lemma follows.

By shifting τ_1 to the origin, we can further reduce the conclusion of the Lemma to the equivalent statement that the period T of the solution of the initial value problem (4.44) with initial conditions

$$s(0) = 0, \quad s'(0) = b \tag{4.49}$$

is an increasing function of b in some neighborhood of $b = 0$.

Assume that the solution has the expansion

$$s(t) = s_1(t)b + s_2(t)b^2 + s_3(t)b^3 + o(b^4). \tag{4.50}$$

We substitute this into (4.44), and expand the resulting equations into a power series of b. Note that (4.46) now becomes

$$g(s) = \frac{s(s+2)}{2(s+1)^3} = s - \frac{5}{2}s^2 + \frac{9}{2}s^3 + O(s^4). \tag{4.51}$$

By equating the coefficients of powers of b, we arrive at the following three initial value problems

$$s_1'' + s_1 = 0, \quad s_1(0) = 0,\ s_1'(0) = 1, \tag{4.52}$$

$$s_2'' + s_2 - \frac{5}{2}s_1 = 0, \quad s_2(0) = s_2'(0) = 0, \tag{4.53}$$

$$s_3'' + s_3 - 5s_1s_2 + \frac{9}{2}s_1^3 = 0, \quad s_3(0) = s_3'(0) = 0. \tag{4.54}$$

The first equation has the solution

$$s_1(t) = \sin(t). \tag{4.55}$$

Substituting this into (4.53) and solving the resulting initial value problem, we obtain

$$s_2(t) = \frac{5}{4} - \frac{5}{3}\cos(t) + \frac{5}{12}\cos(2t). \tag{4.56}$$

Finally, substituting (4.55) and (4.56) into (4.57) and solving, we obtain, with the help of MAPLE,

$$s_3(t) = -\frac{7}{9}\sin(t) - \frac{13}{12}\sin(t)(\cos(t))^2 + \frac{1}{144}(400\sin(t) - 132\,t)\cos(t). \tag{4.57}$$

Putting these solutions into (4.44) and substituting the special time $t = 2\pi$, we get

$$s(2\pi) = -\frac{11\pi}{6}b^3 + O(b^4), \tag{4.58}$$

and

$$s'(2\pi) = b + O(b^4). \tag{4.59}$$

From these two statements, we deduce that

$$T(b) = 2\pi + \frac{11\pi}{6}b^2 + O(b^3) \tag{4.60}$$

and the Lemma is proved. ■

Since we are interested in shells that don't crash into each other, the following comparison result turns out to be useful. In general, it is not easy to compare solutions of nonlinear differential equations. That is primarily why the Lemma starts with a lot of assumptions.

Lemma 4.7 *Assume that $a > c/\sqrt{M}$. Let $r(t)$ be the solution of (4.26) (4.27) and $\underline{r}(t)$ be the solution of*

$$\underline{r}'' + \frac{\underline{M}}{\underline{r}} - \frac{c^2}{\underline{r}^3} = 0 \tag{4.61}$$

under the same initial condition (4.27), and suppose that the following comparison conditions are satisfied

$$\underline{M} \leq M, \quad \frac{\underline{M}}{a} - \frac{c^2}{a^3} \geq \frac{M}{a} - \frac{c^2}{a^3}. \tag{4.62}$$

Let $[0, \tau]$ be an interval in which both $r(t)$ and $\underline{r}(t)$ are decreasing. Then

$$r(t) \geq \underline{r}(t) \quad \textit{for } t \in [0, \tau]. \tag{4.63}$$

Proof. Suppose that $r(t)$ decreases to a value a_1 at time t_1. The energy method provides an elementary way to compute t_1. Let $\underline{r}(t)$ decrease to the same height a_1 at time $\underline{t}_1$. If we can show that $\underline{t}_1 \leq t_1$, then the conclusion of the Lemma follows.

Define

$$f(r) = \frac{M}{r} - \frac{c^2}{r^3}, \qquad F(r) = \int_r^a f(s)\, ds \tag{4.64}$$

and, likewise,

$$\underline{f}(r) = \frac{\underline{M}}{r} - \frac{c^2}{r^3}, \qquad \underline{F}(r) = \int_r^1 \underline{f}(s)\, ds. \tag{4.65}$$

Using (4.62), it is easy to verify that, for all $r \in (0, a]$,

$$\underline{f}(r) \geq f(r). \tag{4.66}$$

This in turn implies that

$$\underline{F}(r) \geq F(r). \tag{4.67}$$

From the conservation of energy, we obtain

$$|\underline{r}'(t)| = \sqrt{2\underline{F}(t)} \geq \sqrt{2F(t)} = |r'(t)| \tag{4.68}$$

and

$$\underline{t}_1 = \int_{a_1}^a \frac{dr}{|\underline{r}'(t)|} \leq \int_{a_1}^a \frac{dr}{|r'(t)|} = t_1, \tag{4.69}$$

as desired. ∎

4.4 Main Result – Pulsating Solution

The results in the last section provide us with a way to construct compatible shells that have the same period.

We start with equation (4.26) having the particular choice $M = c = 1$. Let δ be the number determined in Lemma 4.6, and denote by J the interval

$$J = [1, \beta], \quad \beta = \min\{\delta, \sqrt{2}\}, \tag{4.70}$$

For each $a \in J$, let $r(t; a)$ be the corresponding pulsating shell. Initially, this family of shells $\{r(t;a) \,|\, a \in J\}$ does not have uniform periods and they intersect each other. We are going to perform a sequence of scalings on the shells to end up with a partial solution of the Euler-Poisson equations.

We illustrate how this is done to two particular shells. Choose two numbers, $a_1 < a_2$ in J, and let $r_1(t) = r(t; a_1)$ and $r_2(t) = r(t; a_2)$ be the two corresponding shells. First we scale them down to have initial radius 1 each,

$$\bar{r}_1(t) = \frac{r_1(t)}{a_1}, \quad \bar{r}_2(t) = \frac{r_2(t)}{a_2}. \tag{4.71}$$

By Lemma 4.3, the scaled-down equations have the parameters

$$M_1 = c_1 = \frac{1}{a_1^2}, \qquad \text{and} \qquad M_2 = c_2 = \frac{1}{a_2^2}, \tag{4.72}$$

respectively. Let T_1 and T_2 be the respective periods of $\bar{r}_1$ and $\bar{r}_2$, and T_0 the period of the stationary solution. By Lemma 4.6, $\sqrt{2}\pi = T_0 < T_1 < T_2$. In the interval $[0, T_1/2]$, both functions $\bar{r}_1$ and $\bar{r}_2$ are decreasing.

We now apply Lemma 4.7 to these two solutions, with $a = 1$, $M = M_1$, $c = c_1$, $\underline{M} = M_2$, and $\underline{c} = c_2$. It is easy to check that all the hypotheses are satisfied. The second inequality in (4.62) becomes

$$\frac{1}{a_2^2} - \frac{1}{a_2^4} \geq \frac{1}{a_1^2} - \frac{1}{a_1^4} \tag{4.73}$$

which is a consequence of the fact that the function $1/x^2 - 1/x^4$ is increasing for $x \in [1, \sqrt{2}]$. We point out that (4.73) may not hold if $a_2 > \sqrt{2}$. That is why we need to impose an upper bound on J, and hence on a_2, in (4.70). We conclude, from Lemma 4.7, that

$$\bar{r}_2(t) \leq \bar{r}_1(t) \quad \text{for all } t \in [0, T_1/2]. \tag{4.74}$$

Recall that $\bar{r}_2$ has a longer period than that of $\bar{r}_1$. Our next step is to scale the independent variable t so as to shrink each of the periods down to T_0. More specifically, we define

$$\hat{r}_1(t) = \bar{r}_1\left(\frac{T_1}{T_0}\,t\right), \quad \hat{r}_2(t) = \bar{r}_2\left(\frac{T_2}{T_0}\,t\right). \tag{4.75}$$

It is now easy to deduce from (4.74) that

$$\hat{r}_2(t) \leq \hat{r}_1(t) \tag{4.76}$$

for all $t \in [0, T_0/2]$. Knowing that $\hat{r}_2$ has the same period as that of $\hat{r}_2$, we conclude that (4.76) actually holds for all $t > 0$. Physically this means that the pulsating shells corresponding to $\hat{r}_2$ and $\hat{r}_1$ do not crash into each other except at those times when they both attain the maximum radius.

In addition, scaling $\bar{r}_2$ to $\hat{r}_2$ increases the M-parameter for the new shell to $\hat{M}_2 = T_2^2 M_2/T_0^2$, and there is no guarantee that it remains larger than the corresponding M-parameter for $\hat{r}_1$. For a physically realizable system, the M-parameter for an inner shell ought to be smaller. In reality, it can be shown that this last scaling does not increase $\hat{M}_2$ enough to overtake $\hat{M}_1$. Instead of proving this fact, we will coerce the shells to have obviously compatible physically parameters, by performing one more scaling

$$\check{r}_1(t) = \frac{T_0}{T_1}\hat{r}_2(t). \quad \check{r}_2(t) = \frac{T_0}{T_2}\hat{r}_2(t). \tag{4.77}$$

Then $\check{r}_1$ and $\check{r}_2$ become two compatible shells, the former being the outside shell.

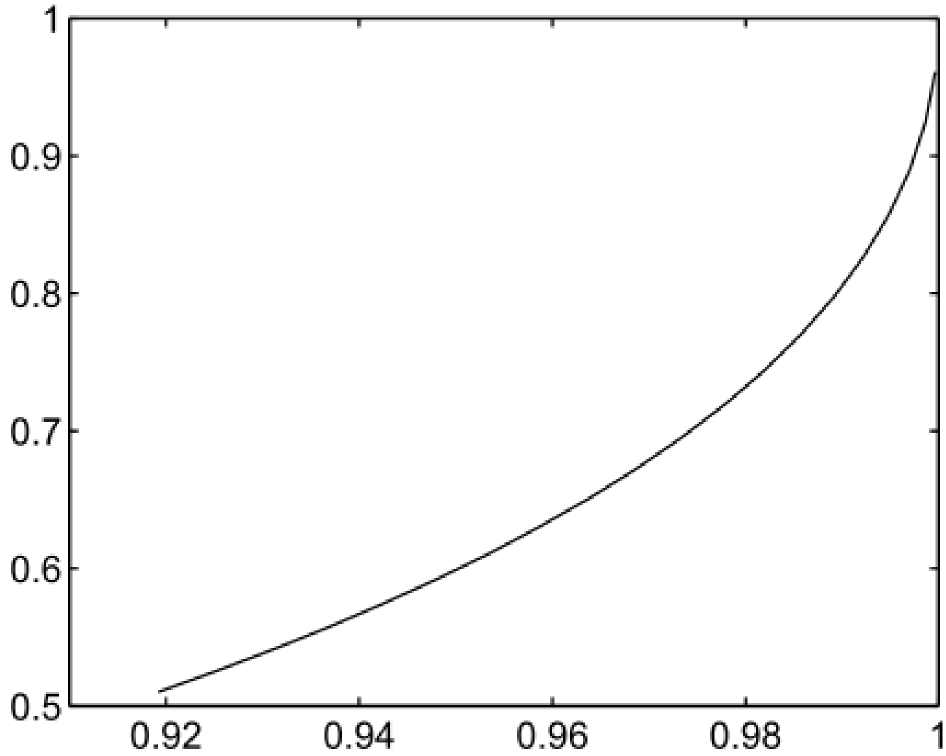

Fig. 4.3: Graph of $\check{M}$ vs $\check{r}(0)$

$\check{r}(0)$	$\check{M}$	$\check{r}(0)$	$\check{M}$
0.99965137	0.96116878	0.96857314	0.67186240
0.99862910	0.92455621	0.96367629	0.65036420
0.99701019	0.88999644	0.95858349	0.62988158
0.99485137	0.85733882	0.95331826	0.61035156
0.99220632	0.82644628	0.94790217	0.59171598
0.98912274	0.79719388	0.94235451	0.57392103
0.98564597	0.76946753	0.93669310	0.55691691
0.98181702	0.74316290	0.93093423	0.54065744
0.97767311	0.71818443	0.92509265	0.52509977
0.97324819	0.69444444	0.91918178	0.51020408

Table 4.2: Table of $\check{M}$ vs $\check{r}(0)$.

To summarize, after performing the above transformations on the functions $r(t)$ for every $a \in J$, we obtain a partial solution over the interval $[\alpha, 1] = [T_0/\beta T(\beta), 1]$, composed of the shells $\check{r}(t)$. More specifically, corresponding to every $a \in J$, there is a shell $\check{r}(t)$ with the following initial radius and parameters

$$\check{r}(0) = \frac{T_0}{aT(a)}\,, \quad \check{M} = \frac{1}{a^2}\,, \quad \check{c} = \frac{T_0^2}{a^4 T^2(a)}\,, \tag{4.78}$$

where $T(a)$ is the period function as in Lemma 4.6. The graph of $\check{M}$ versus $\check{r}(0)$ for the shells of the partial solution is shown in Fig.4.3 for the choice of $\beta = 1.4$, with the corresponding numerical data compiled in Table 4.2. The graphs of 20 particular shells are shown in Fig.4.4.

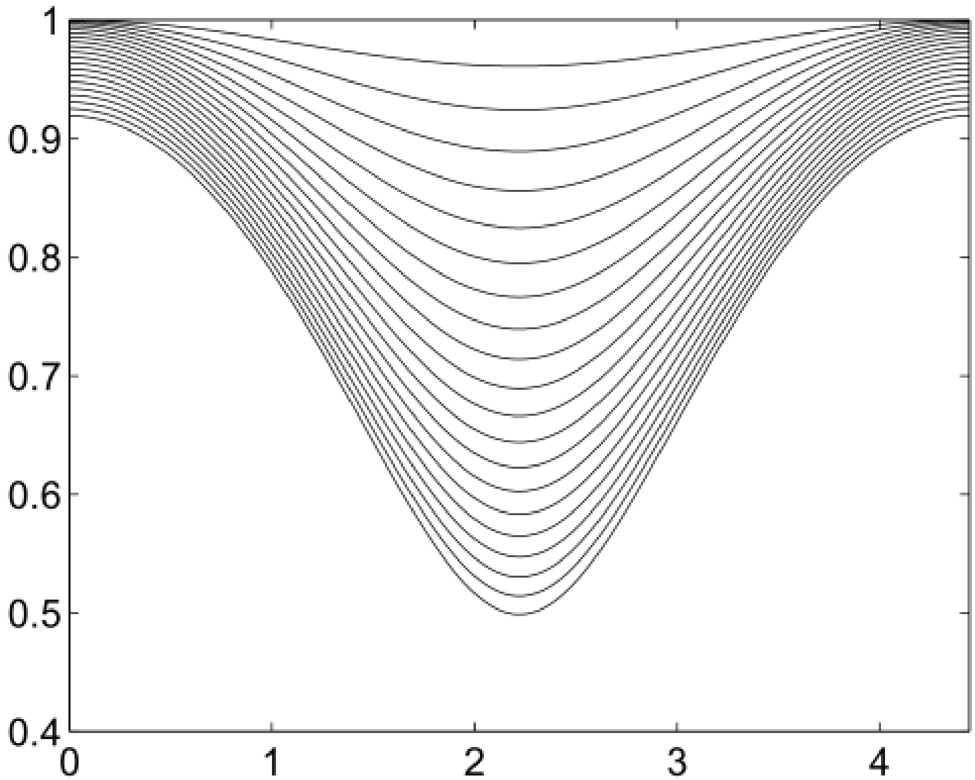

Fig. 4.4: Partial solution constructed using $\beta = 1.4$

Using the method presented in the discussion after Lemma 4.3, we can stretch this partial solution to give a partial solution over $[0, 1]$. For instance, the shells in Fig.4.5 are obtained by scaling the n-th innermost shell in Fig.4.4 by a factor of $(n-1)/20$. Finally, this last partial solution can be further extended to a full solution by using Lemma 4.2. This completes the proof of Theorem 4.1.

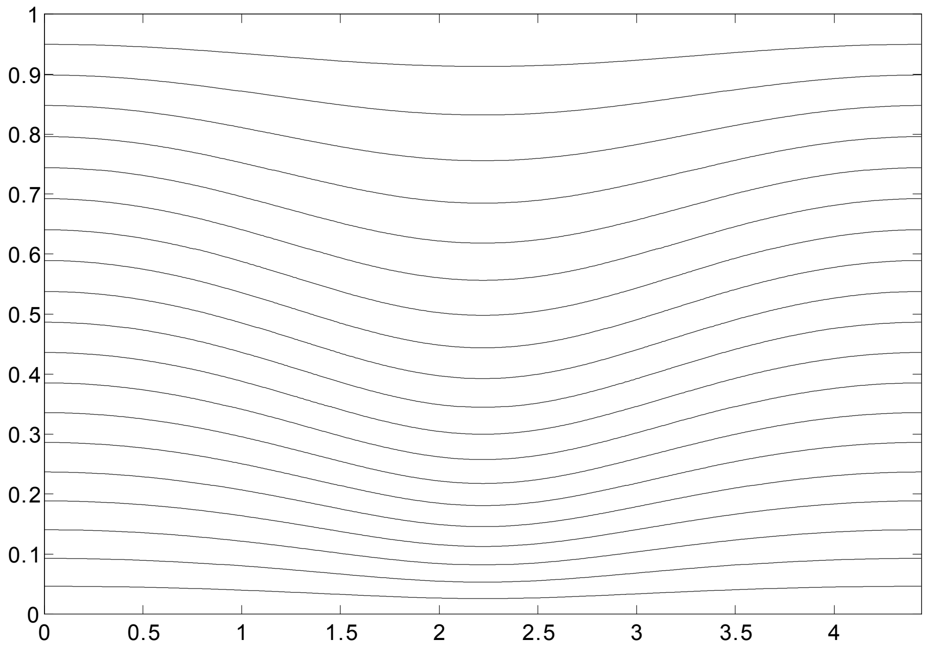

Fig. 4.5: Pulsating shells constructed using $\beta = 1.4$.

Global pulsating solutions are by no means unique. For example, we can use any subinterval of J instead of J in our construction to obtain a slightly different pulsating solution. Different scaling functions can also be used in the various steps in the construction.

All the shells in our example pulsate in phase, in the sense that they attain their maximum/minimum radii at the same time. Numerical experiments show that solutions in which the shells pulsate slightly out of phase can also be constructed. This fact can be established rigorously.

Theorem 4.3 stipulates the existence of an annular pulsating solution with stationary shells at the inner and outer boundaries. Let us first show that a (weak) solution of this type exists.

We start with the partial solution represented by the shells $\check{r}$ constructed in the last section and depicted in Fig.4.4. The outermost shell is stationary. As we decrease the initial radius, the inner shells pulsate with increasing amplitudes, and we stop at the radius $\alpha = T_0/aT(\alpha) \approx 0.92$, the last entry of $\check{r}(0)$ in Table 2.

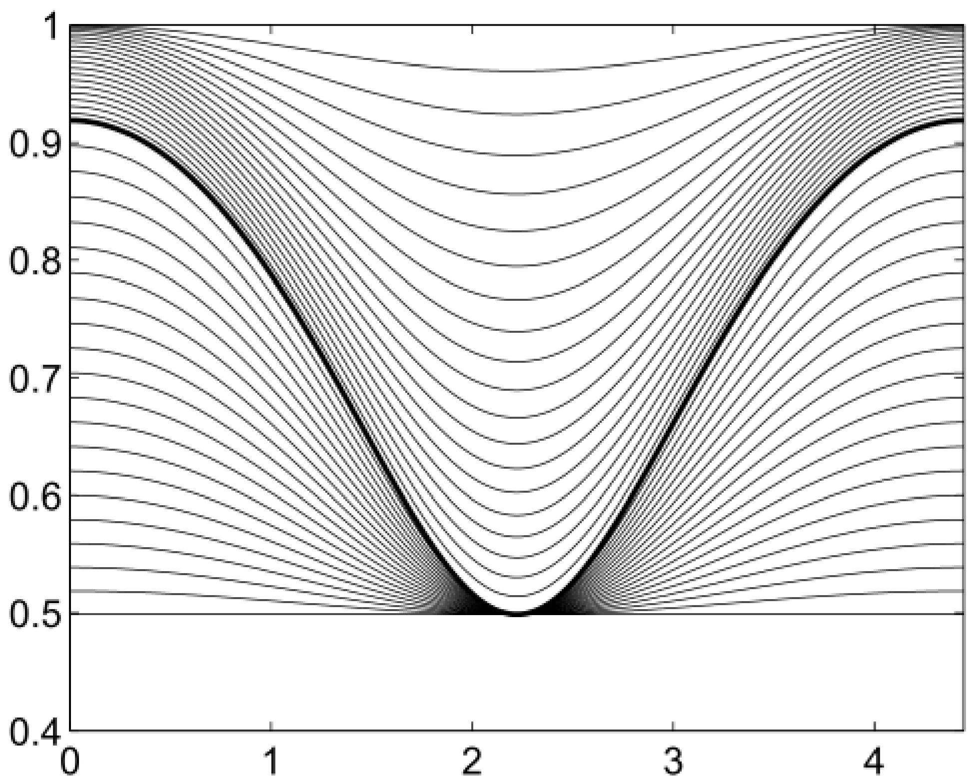

Fig. 4.6: Initial set of shells

Let us continue to decrease the initial radii of the shells, but reverse the process; namely, to make the shells pulsate with decreasing amplitudes, until we reach a stationary shell at some point. This is achieved as follows: for each of the shells above the innermost shell in Fig.4.4, we scale it down so that its minimum radius is exactly the same as that of the latter. The result is shown in Fig.4.6. The thick curve in the middle of all the shells is the same as the innermost shell in Fig.4.4, with initial radius α. Each of the shells above it has a “reflected image” below it. Numerical computation indicates that the new shells collapse together at their common minimum radius, which has the value 0.498739, but they do not intersect at any other points. Fig.4.6 clearly bears out this fact. A rigorous proof can easily be shown and we omit the details.

Does the augmented set of shells qualify as the annular solution that we are looking for? At first sight, the collapsing of the new shells seems to disqualify it. However, that can be easily circumvented by progressively shrinking each of the new shells a bit more. A more subtle and serious problem is revealed by examining the M-parameter of the shells, as a function of the initial radius.

Before proceeding any further, let us streamline the functional notations. The many scalings used in the construction in the previous section have led to different symbols for the many functions involved with the various shells. For convenience, let us rename the functions associated with the last set of shells so that the M-parameter is again denoted by $M(a)$, and the shells by $r(t)$.

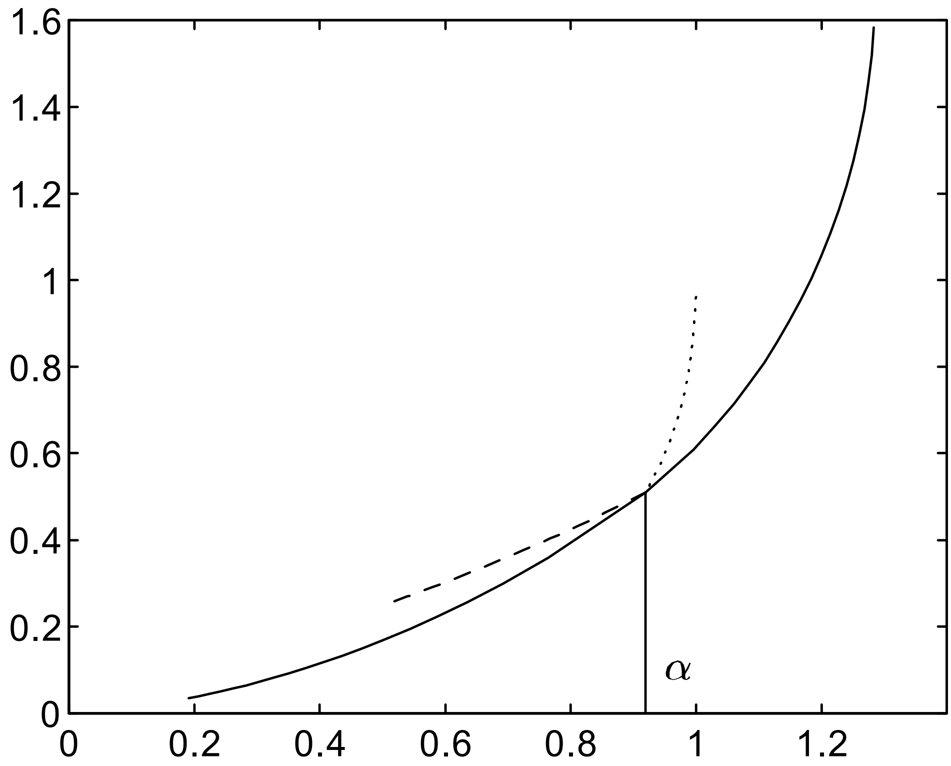

Fig. 4.7: graph of M vs $r(0)$ for the annular solution

When the old and new sets of shells are considered separately (that is, on the two subintervals $a < \alpha$ and $a > \alpha$, respectively) each of these functions is smooth, but the combined graph has a kink at the junction point α. This is illustrated in Fig.4.7 in which the dashed curve is the plot of $M(a)$ versus a for $a < \alpha$ and the dotted curve is its counterpart for $a > \alpha$. The derivative of M has a jump discontinuity, leading to a jump discontinuity of the solution function $\rho(t, r)$ across the dividing shell between the two sets.

To remedy this problem, we apply two scalings, one to each of the sets of shells. For the shells with $a < \alpha$, we use the scale

$$\bar{a} = \bar{r}(0) = \lambda_1(a)a, \quad \bar{M}(a) = \lambda_1^2(a)M(a) \quad \lambda_1(a) = (1 - \sqrt{\alpha - a}). \tag{4.79}$$

For those with $a > \alpha$, we use

$$\bar{a} = \bar{r}(0) = \lambda_2(a)a, \quad \bar{M}(a) = \lambda_2^2(a)M(a) \quad \lambda_2(a) = (1 + \sqrt{a - \alpha}). \tag{4.80}$$

In fact, any function $\lambda(a)$ that is smooth in $(0, \alpha) \cup (\alpha, 1)$ and satisfies

$$\lim_{a\to\alpha} \lambda(a) = 1 \quad \text{and} \quad \lim_{a\to\alpha\pm} \lambda'(a) = \infty \tag{4.81}$$

will do the trick. After the scaling, $\bar{M}$ becomes a function of $\bar{a}$ and is connected to it via a parametric relationship given by (4.79) and (4.80). The solid curve in Fig. 4.7 is the plot of $\check{M}(\check{a})$ versus $\check{a}$. The point α remains the same under the transformation. As long as we stay away from α, the scaling functions are smooth and hence, $\bar{M}$ is a smooth function of $\bar{a}$. For $a \neq \alpha$,

$$\frac{d\bar{M}}{d\bar{a}} = \frac{d\bar{M}}{da} \Big/ \frac{d\bar{a}}{da} = \frac{2\lambda(a)\lambda'^2(a)M'(a)}{\lambda'(a)a + \lambda(a)}, \tag{4.82}$$

where λ is λ_1 or λ_2 depending on whether $a < \alpha$ or not. It follows from (4.81) and the above identity that

$$\lim_{a\to\alpha\pm} \frac{d\bar{M}}{d\bar{a}} = \frac{2M(\alpha)}{\alpha}. \tag{4.83}$$

Since the two one-sided limits agree, $\bar{M}'(\bar{a})$ can be defined and $\bar{M}(\bar{a})$ can be considered a C^1 function, and it will not lead to a jump discontinuity of the density function $\rho(r, t)$. Unfortunately, a further analysis shows that the second derivative

$$\frac{d^2\bar{M}}{d\bar{a}^2} = \frac{d}{da}\left(\frac{d\bar{M}}{d\bar{a}}\right) \Big/ \frac{d\bar{a}}{da} \tag{4.84}$$

of $\bar{M}(\bar{a})$ has a jump discontinuity induced by that of the first derivative of $M(a)$ at α. Hence, $\rho(r, t)$ is only piecewise C^1 and it only satisfies the Euler-Poisson equations in a weak sense.

Now that we have the desired annular solution, we can scale it to have any size and period if necessary, so that the given global solution is contained inside the innermost shell of the annular solution. The gap between the global solution and the annular can be filled in by stationary shells in the obvious way and Theorem 4.3 is proved.

Chapter 5

Rotational Flows

In this chapter, we construct rotational and periodical solutions in explicit form for the 2-dimensional Euler-Poisson equations. Some special blowup solutions with infinite energy for the 3-dimensional compressible or incompressible Euler equations are also given.

5.1 The 2D Isothermal Euler-Poisson Equations

We continue our study of the system (4.1), this time by exploiting the separation method. The main idea is to apply the isothermal pressure term to balance the potential force to produce novel solutions with rotation.

Theorem 5.1 ([43]) *For the isothermal ($\gamma = 1$) Euler-Poisson equations (4.1) in R^2, there exists a family of global solutions with rotation in radial symmetry,*

$$\begin{cases} \rho(t,\vec{x}) = \dfrac{1}{a^2(t)} e^{g(r/a(t))} \\ \vec{u}(t,\vec{x}) = \dfrac{\dot{a}(t)}{a(t)}(x,y) + \dfrac{\xi}{a^2(t)}(-y,x) \\ \qquad \ddot{a}(t) = \dfrac{-\lambda}{a(t)} + \dfrac{\xi^2}{a^3(t)} & a(0) = a_0 > 0,\ \dot{a}(0) = a_1 \\ \qquad \ddot{g}(s) + \dfrac{1}{s}\dot{g}(s) + \dfrac{2\pi}{K} e^{g(s)} = \dfrac{2\lambda}{K}, & g(0) = \beta,\ \dot{g}(0) = 0 \end{cases} \tag{5.1}$$

with arbitrary constants $\xi \neq 0$, a_0, a_1 and β.

In particular, with $\lambda > 0$,

(1) the solutions (5.1) are time-periodic, except for the case $a_0 = \frac{|\xi|}{\sqrt{\lambda}}$ and $a_1 = 0$;

(2) when $a_0 = \frac{|\xi|}{\sqrt{\lambda}}$ and $a_1 = 0$, the solutions (5.1) are steady.

Remark 5.1 *Equation (5.1)$_3$ satisfied by $a(t)$ is, in fact, identical to the Emden equation (4.26) satisfied by r in the last chapter. The global existence and periodic nature of the solution is discussed near the end of Section 4.1.*

Equation (5.1)$_4$ satisfied by $g(s)$ is the Liouville equation which has been extensively studied in differential geometry. The global existence of solutions is well-known.

Remark 5.2 *Choosing $\xi = 0$ reduces (5.1) to Yuen's irrotational solutions in [80]. For $\lambda > 0$, these irrotational solutions blow up at a finite time T. Hence, we again see that rotational motion prevents blowup. The solutions (5.1) can be used to simulate the evolution of some rotational stars or galaxies.*

Our main effort is to uncover self-similar rotational solutions ($G(t,r) \neq 0$) of the mass equation (4.1)$_1$.

Lemma 5.1 *For the equation of conservation of mass*

$$\rho_t + \nabla \cdot (\rho \vec{u}) = 0, \tag{5.2}$$

there exist solutions of the form

$$\rho(t,\vec{x}) = \rho(t,r) = \frac{f(\frac{r}{a(t)})}{a^2(t)}, \qquad \vec{u}(t,\vec{x}) = \frac{F(t,r)}{r}(x,y) + \frac{G(t,r)}{r}(-y,x) \tag{5.3}$$

where

$$F(t,r) = \frac{\dot{a}(t)}{a(t)} r \tag{5.4}$$

with arbitrary functions $f \geq 0 \in C^1$ and $a(t) > 0 \in C^1$.

Proof. *Direct substitution gives*

$$\begin{aligned}
&\rho_t + \nabla \cdot (\rho \vec{u}) \\
&= \rho_t + \frac{\partial}{\partial x}\left[\rho \frac{Fx}{r} - \rho \frac{Gy}{r}\right] + \frac{\partial}{\partial y}\left[\rho \frac{Fy}{r} + \rho \frac{Gx}{r}\right] \\
&= \rho_t + \frac{\partial}{\partial x}\rho \frac{Fx}{r} + \rho\left(\frac{\partial}{\partial x}\frac{Fx}{r}\right) - (\frac{\partial}{\partial x}\rho)\frac{Gy}{r} - \rho(\frac{\partial}{\partial x}\frac{Gy}{r}) \\
&\quad + \frac{\partial}{\partial y}\rho \frac{Fy}{r} + \rho\left(\frac{\partial}{\partial y}\frac{Fy}{r}\right) + (\frac{\partial}{\partial y}\rho)\frac{Gx}{r} + \rho(\frac{\partial}{\partial y}\frac{Gx}{r}) \\
&= \rho_t + \rho_r \frac{x}{r}\frac{Fx}{r} + \rho\left(F_r \frac{x}{r}\right)\frac{x}{r} + \rho\frac{F}{r} - \rho F x \frac{x}{r^3} \\
&\quad - \rho_r \frac{x}{r}\frac{Gy}{r} - \rho G_r \frac{x}{r}\frac{y}{r} + \rho G y \frac{x}{r^3} + \rho_r \frac{y}{r}\frac{Fy}{r} + \rho\left(F_r \frac{y}{r}\right)\frac{y}{r}
\end{aligned}$$

$$
\begin{aligned}
&+\rho\frac{F}{r}-\rho Fy\frac{y}{r^3}+\rho_r\frac{y}{r}\frac{Gx}{r}+\rho\left(G_r\frac{y}{r}\right)\frac{x}{r}-\rho Gx\frac{y}{r^3}\\
=\;&\rho_t+\rho_r\frac{x}{r}\frac{Fx}{r}+\rho\left(F_r\frac{x}{r}\right)\frac{x}{r}+\rho\frac{F}{r}-\rho Fx\frac{x}{r^3}\\
&+\rho_r\frac{y}{r}\frac{Fy}{r}+\rho\left(F_r\frac{y}{r}\right)\frac{y}{r}+\rho\frac{F}{r}-\rho Fy\frac{y}{r^3}\\
=\;&\rho_t+\rho_rF+\rho F_r+\rho F\frac{1}{r}.
\end{aligned}
\tag{5.5}
$$

Now plug in the relations (5.3) and (5.4) to get

$$
\begin{aligned}
&=\frac{\partial}{\partial t}\frac{f(\frac{r}{a(t)})}{a^2(t)}+\frac{\partial}{\partial r}\frac{f(\frac{r}{a(t)})}{a^2(t)}\frac{\dot a(t)r}{a(t)}+\frac{f(\frac{r}{a(t)})}{a^2(t)}\frac{\dot a(t)}{a(t)}+\frac{f(\frac{r}{a(t)})}{a^2(t)}\frac{\dot a(t)}{a(t)}\\
&=\frac{-2\dot a(t)f(r/a(t))}{a^3(t)}-\frac{\dot a(t)r\dot f(r/a(t))}{a^4(t)}\\
&\quad+\frac{\dot f(r/a(t))}{a^3(t)}\frac{\dot a(t)r}{a(t)}+\frac{f(r/a(t))}{a^2(t)}\frac{\dot a(t)}{a(t)}+\frac{f(r/a(t))}{a^2(t)}\frac{\dot a(t)}{a(t)}\\
&=0.
\end{aligned}
$$

∎

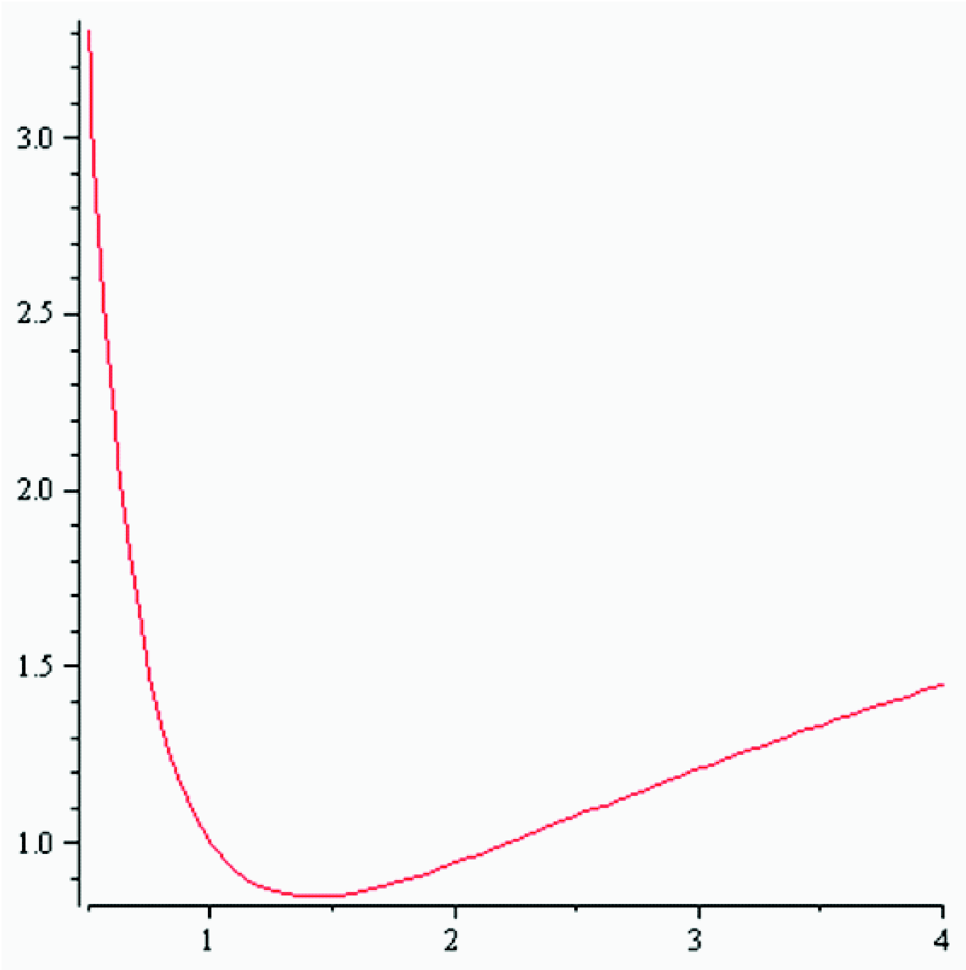

Fig. 5.1: F_pot with $\lambda = 1$ and $\xi^2 = 2$

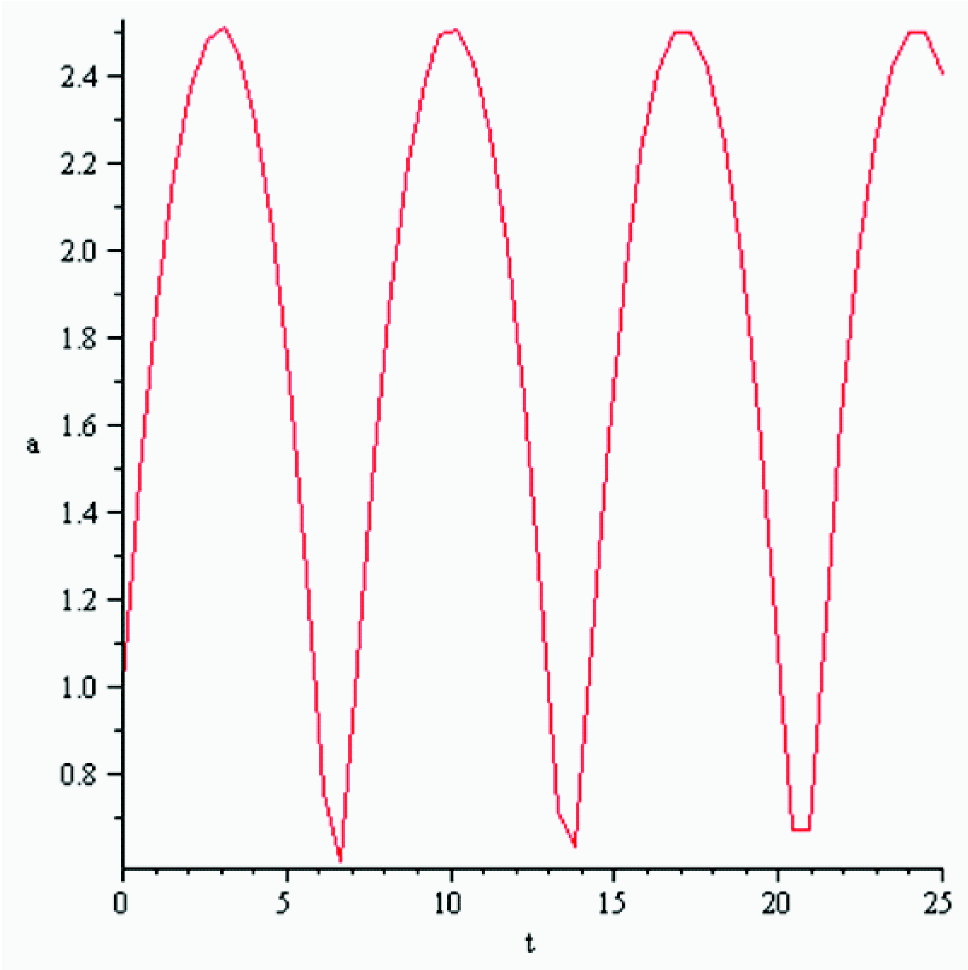

Fig. 5.2: $a(t)$ of the equation $\ddot{a}(t) = \frac{-1}{a(t)} + \frac{1}{a^3(t)}$, $a(0) = 1$, $\dot{a}(0) = 1$.

Proof of Thoerem 5.1. The proof for rotational fluids is similar to that for non-rotational ones. Lemma (5.1) takes care of the mass equation. For the first momentum equation $(4.1)_{21}$, we carry out the following computation

$$\begin{aligned}
&\rho\left[\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y}\right] + \frac{\partial}{\partial x}P + \rho\frac{\partial}{\partial x}\phi \\
&\quad = \rho\left(\frac{\partial}{\partial t}(F\frac{x}{r} - G\frac{y}{r}) + u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y}\right) + \frac{\partial}{\partial x}Ke^{f(\frac{r}{a(t)})} + \rho\frac{\partial\phi\left(\frac{r}{a(t)}\right)}{\partial x} \\
&\quad = \rho\left(\frac{\partial}{\partial t}(F\frac{x}{r} - G\frac{y}{r}) + u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y}\right) \\
&\qquad + Ke^{f(s)}\frac{x}{r}\frac{\partial}{a(t)\partial\left(\frac{r}{a(t)}\right)}f(s) + \frac{x}{r}\frac{\partial}{\partial r}\phi(s)
\end{aligned}$$

$$= \rho \left\{ \begin{array}{c} \frac{\partial}{\partial t}(F\frac{x}{r} - G\frac{y}{r}) + u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y} \\ +K\frac{1}{a(t)}\frac{x}{r}\frac{\partial}{\partial s}g(s) + \frac{x}{r}\frac{2\pi}{r}\int\limits_0^r \frac{e^{g(\frac{\eta}{a(t)})}}{a(t)^2}\eta d\eta \end{array} \right\}$$

$$= \rho \left\{ \begin{array}{l} \frac{\partial}{\partial t}(F\frac{x}{r} - G\frac{y}{r}) \\ +(F\frac{x}{r} - G\frac{y}{r})\frac{\partial}{\partial x}(F\frac{x}{r} - G\frac{y}{r}) + \left(F\frac{y}{r} + G\frac{x}{r}\right)\frac{\partial}{\partial y}(F\frac{x}{r} - G\frac{y}{r}) \\ +\frac{x}{a(t)r}\left[K\dot{g}(s) + \frac{2\pi}{\frac{r}{a(t)}}\int\limits_0^r \frac{e^{g(\frac{\eta}{a(t)})}}{a(t)^2}\eta d\eta\right] \end{array} \right\}$$

$$= \rho \left\{ \begin{array}{l} F_t\frac{x}{r} - G_t\frac{y}{r} \\ +(F\frac{x}{r} - G\frac{y}{r})[(F_r\frac{x}{r}\frac{x}{r} + F\frac{1}{r} + Fx\frac{\partial}{\partial x}\frac{1}{r}) - (G_r\frac{x}{r}\frac{y}{r} + Gy\frac{\partial}{\partial x}\frac{1}{r})] \\ +\left(F\frac{y}{r} + G\frac{x}{r}\right)\left[(F_r\frac{y}{r}\frac{x}{r} + Fx\frac{\partial}{\partial y}\frac{1}{r}) - (G_r\frac{y}{r}\frac{y}{r} + \frac{G}{r} + Gy\frac{\partial}{\partial y}\frac{1}{r}\right] \\ +\frac{x}{a(t)r}\left[K\dot{g}(s) + \frac{2\pi}{\frac{r}{a(t)}}\int\limits_0^r e^{g(\frac{\eta}{a(t)})}\left(\frac{\eta}{a(t)}\right)d\left(\frac{\eta}{a(t)}\right)\right] \end{array} \right\}$$

$$= \rho \left\{ \begin{array}{l} F_t\frac{x}{r} - G_t\frac{y}{r} \\ +(F\frac{x}{r} - G\frac{y}{r})[(F_r\frac{x}{r}\frac{x}{r} + F\frac{1}{r} - Fx\frac{x}{r^3}) - (G_r\frac{x}{r}\frac{y}{r} - Gy\frac{x}{r^3})] \\ +\left(F\frac{y}{r} + G\frac{x}{r}\right)\left[F_r\frac{y}{r}\frac{x}{r} - Fx\frac{y}{r^3} - (G_r\frac{y}{r}\frac{y}{r} + \frac{G}{r} - Gy\frac{y}{r^3})\right] \\ +\frac{x}{a(t)r}\left[K\dot{g}(s) + \frac{2\pi}{s}\int\limits_0^s e^{g(\tau)}\tau d\tau\right] \end{array} \right\}.$$

After simplification, we have

$$= \frac{x\rho}{r}\left(\begin{array}{l} F_t + FF_r - \frac{G^2}{r} + \frac{1}{a(t)}\left[K\dot{f}(s) + \frac{2\pi}{s}\int\limits_0^s e^{f(\tau)}\tau d\tau\right] \\ -\frac{y}{x}(G_t + FG_r + FG\frac{1}{r}) \end{array} \right).$$

Similarly, we obtain the corresponding result for the second momentum equation $(4.1)_{22}$

$$= \frac{y\rho}{r}\left(\begin{array}{l} F_t + FF_r - \dfrac{G^2}{r} + \dfrac{1}{a(t)}\left[K\dot{f}(s) + \dfrac{2\pi}{s}\int\limits_0^s e^{f(\tau)}\tau d\tau\right] \\ -\frac{x}{y}(G_t + FG_r + FG\frac{1}{r}) \end{array} \right).$$

It means that the isothermal Euler-Poisson system can be reduced to the simpler partial differential equations

$$\begin{cases} F_t + FF_r - \dfrac{G^2}{r} + \dfrac{1}{a(t)}\left[K\dot{g}(s) + \dfrac{2\pi}{s}\int\limits_0^s e^{g(\tau)}\tau d\tau\right] = 0 \\ G_t + FG_r + FG\dfrac{1}{r} = 0. \end{cases} \tag{5.6}$$

By taking

$$F(t,r) = \frac{\dot{a}(t)}{a(t)}r \quad \text{and} \quad G = \frac{\xi}{a^2(t)}r, \tag{5.7}$$

we can immediately verify $(5.6)_2$.

$$\begin{aligned} & G_t + FG_r + FG\frac{1}{r} \\ &= \frac{d}{dt}\frac{\xi}{a^2(t)}r + \frac{\dot{a}(t)}{a(t)}r\frac{\xi}{a^2(t)} + \frac{\dot{a}(t)}{a(t)}r\frac{\xi}{a^2(t)}r\frac{1}{r} \\ &= -\frac{2\xi\dot{a}(t)}{a^3(t)}r + \frac{2\xi\dot{a}(t)}{a^3(t)}r \\ &= 0. \end{aligned}$$

Then the equation $(5.6)_1$ becomes

$$\begin{aligned} & F_t + FF_r - \frac{G^2}{r} + \frac{1}{a(t)}\left[K\dot{g}(s) + \frac{2\pi}{s}\int_0^s e^{g(\tau)}\tau d\tau\right] \\ &= \frac{\ddot{a}(t)}{a(t)}r - \frac{\left(\frac{\xi}{a(t)^2}r\right)^2}{r} + \frac{1}{a(t)}\left[K\dot{g}(s) + \frac{2\pi}{s}\int_0^s e^{g(\tau)}\tau d\tau\right] \\ &= \frac{1}{a(t)}\left\{-\lambda s + \left[K\dot{g}(s) + \frac{2\pi}{s}\int_0^s e^{g(\tau)}\tau d\tau\right]\right\} \\ &= 0. \end{aligned} \tag{5.8}$$

We have made use of the facts that $a(t)$ and $g(s)$ satisfy the last two differential equations of 5.1, respectively.

The statements (1) and (2) are obvious. ■

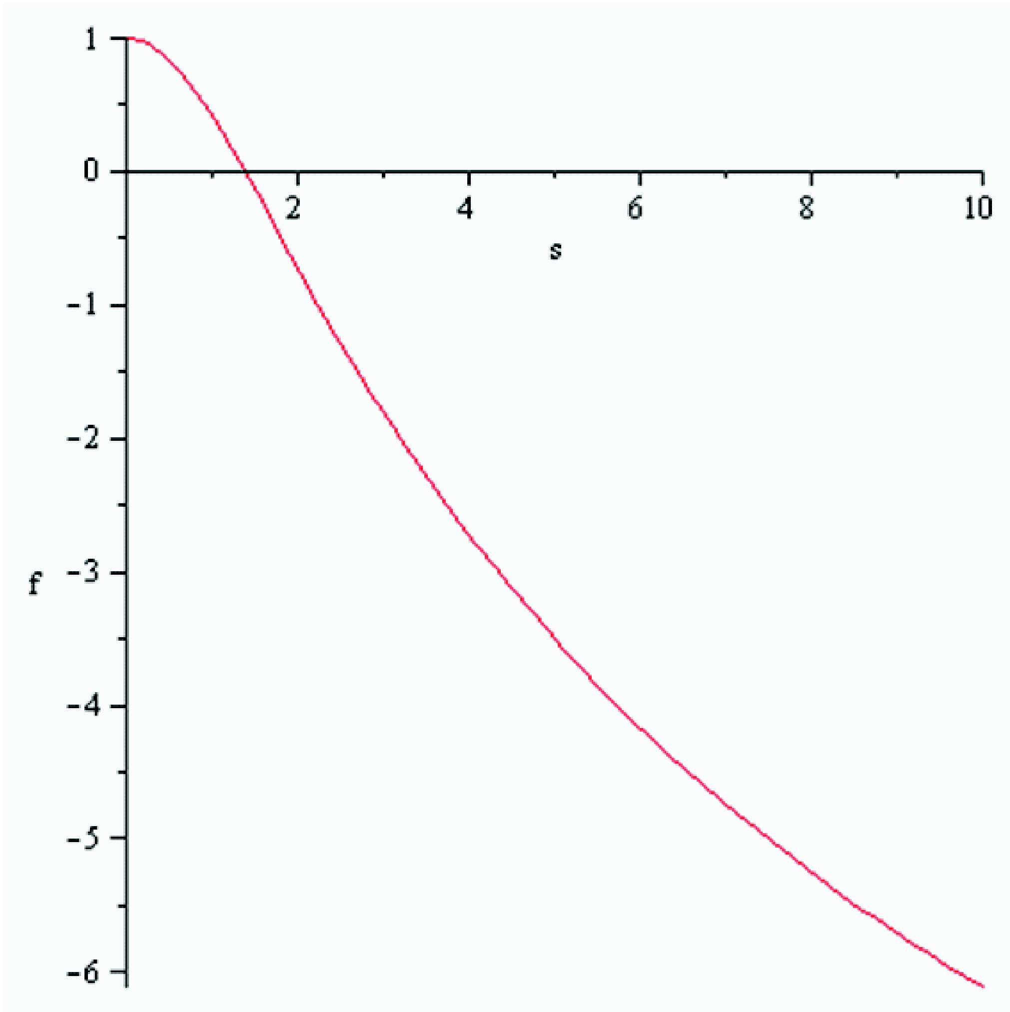

Fig. 5.3: Graph of $g(s)$ with $K = 2\pi$, $\lambda = 0$ and $\beta = 1$

Two graphs for the velocity $\vec{u}$ (5.1) are illustrated below:

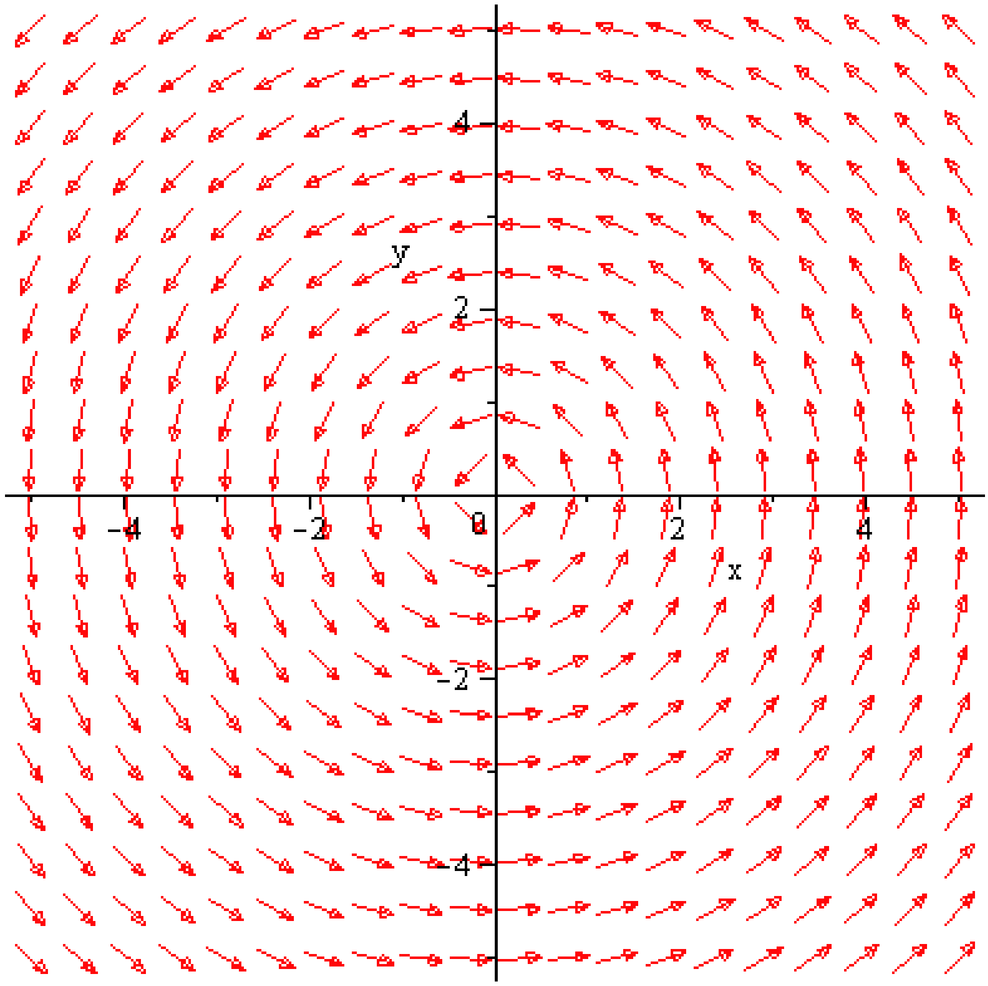

Fig. 5.4: steady rotational velocity $\vec{u}(t, \vec{x}) = (u_1, u_2) = (-y, x)$ with $a_0 = 1$, $a_1 = 0$, $\lambda = 1$ and $\xi = 1$

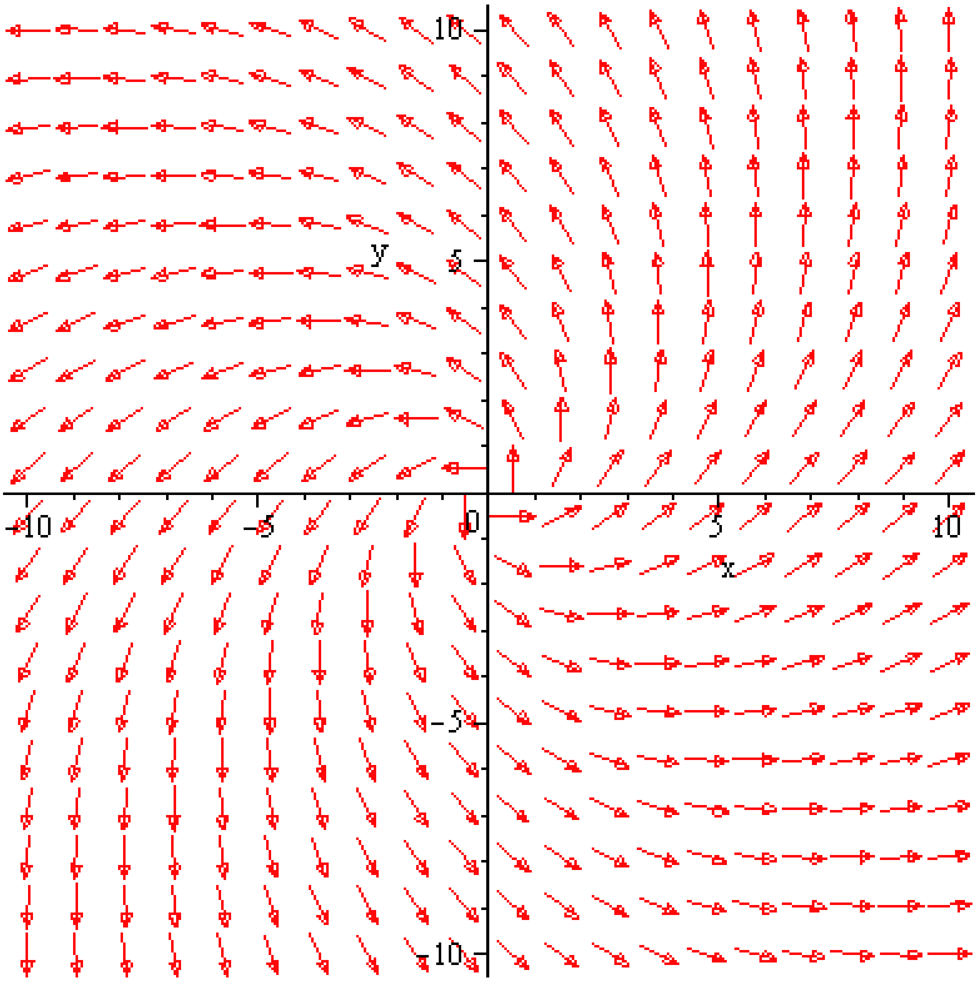

Fig. 5.5: rotational velocity $\vec{u}(0, \vec{x}) = (u_1, u_2) = (x + y, -x + y)$ with $a_0 = 1$, $a_1 = 1$ and $\xi = 1$

5.2 Exact 3D Solutions for the Euler Equations

In this section, we construct some exact rotational solutions for the 3-dimensional compressible or incompressible Euler equations using elementary functions.

Theorem 5.2 ([90]) *For the 3-dimensional compressible Euler equations*

$$\begin{cases} \rho_t + \nabla \cdot (\rho \vec{u}) = 0 \\ \rho \left[\vec{u}_t + (\vec{u} \cdot \nabla) \vec{u} \right] + \nabla P = 0, \end{cases}$$

there exists a class of rotational solutions:

for $\gamma > 1$,

$$\begin{cases} \rho = \max\left(\frac{\gamma-1}{K\gamma}s,\ 0\right)^{\frac{1}{\gamma-1}} \\ u_1 = a(t) + C\,(y-z) \\ u_2 = a(t) + C\,(-x+z) \\ u_3 = a(t) + C\,(x-y) \end{cases} \tag{5.9}$$

where

$$s = C^2\left[x^2+y^2+z^2-(xy+yz+xz)\right] - \dot{a}(t)(x+y+z) + b(t) \tag{5.10}$$

$$a(t) = c_0 + c_1 t \tag{5.11}$$

and

$$b(t) = 3c_0c_1t + \frac{3}{2}c_1^2t^2 + c_2 \tag{5.12}$$

with arbitrary constants C, c_0, c_1 *and* c_2;

for $\gamma = 1$,

$$\begin{cases} \rho = e^{\frac{s}{K}} \\ u_1 = a(t) + C\,(y-z) \\ u_2 = a(t) + C\,(-x+z) \\ u_3 = a(t) + C\,(x-y)\,. \end{cases} \tag{5.13}$$

The solutions (5.9) and (5.13) exist globally.

We omit the proof which involves checking the validity of the differential equations by direct computation. Interested readers can refer to the original article [90].

Note that the masses of the solutions are infinite. The rational functional form with $a(t) = 0$ and $C = 1$ for the velocity $\vec{u}$ has been given in Senba and Suzuki's book [65]. The velocities $\vec{u}$ are not spherically symmetric, but they are similar to the irrotational blowup or global solutions in Li [47], which have spherically symmetric velocities $\vec{u}$. The exact solutions can blow up on account of the infinite energy from the boundary. Another class of (cylindrical) blowup solutions with infinite energy has been found in [30].

Next we turn to the incompressible case.

Theorem 5.3 ([90]) *For the 3-dimensional incompressible Euler equations*

$$\begin{cases} \operatorname{div}\vec{u} \;=\; 0 \\ \rho\left[\vec{u}_t + (\vec{u}\cdot\nabla)\,\vec{u}\right] + \nabla P \;=\; 0, \end{cases}$$

there exists a class of rotational solutions

$$\begin{cases} \frac{P(\rho)}{\rho} = s_1 \\ u_1 = a_1(t) + C\,(y-z) \\ u_2 = a_2(t) + C\,(-x+z) \\ u_3 = a_3(t) + C\,(x-y) \end{cases} \tag{5.14}$$

with

$$s_1 = \begin{Bmatrix} C^2\left[x^2+y^2+z^2-(xy+yz+xz)\right] \\ -\left[\dot{a}_1(t)+C(a_2(t)+a_3(t))\right]x \\ -\left[\dot{a}_2(t)+C(a_1(t)+a_3(t))\right]y \\ -\left[\dot{a}_3(t)+C(a_1(t)+a_2(t))\right]z+b(t) \end{Bmatrix} \tag{5.15}$$

and

$$\begin{cases} \frac{P(\rho)}{\rho} = s_2 \\ u_1 = a_1(t) + C\,(y+z) \\ u_2 = a_2(t) + C\,(x+z) \\ u_3 = a_3(t) + C\,(x+y) \end{cases} \tag{5.16}$$

with

$$s_2 = \begin{Bmatrix} -C^2\left(x^2+y^2+z^2+xy+yz+xz\right) \\ -\left[\dot{a}_1(t)+C(a_2(t)+a_3(t))\right]x \\ -\left[\dot{a}_2(t)+C(a_1(t)+a_3(t))\right]y \\ -\left[\dot{a}_3(t)+C(a_1(t)+a_2(t))\right]z+b(t) \end{Bmatrix} \tag{5.17}$$

where $a_i(t)$ is an arbitrary local C^1 function for $i = 1$, 2 or 3, $b(t)$ is an arbitrary function, and C is an arbitrary constant.

In particular,

1. if $|a_i(T)| = \infty$ or $|\dot{a}_i(T)| = \infty$ with the first finite positive constant T, the solutions (5.14) and (5.16) blow up at a finite time T;

2. For global C^1 functions $a_i(t)$, the solutions (5.14) and (5.16) exist globally.

Again, interested readers can refer to the same article for the detailed proof.

Remark 5.3 *Here the kinetic energy of the solutions (5.14) and (5.16) for the incompressible fluids are infinite*

$$\frac{1}{2}\int_{R^3} \vec{u}^2 d\vec{x} = \infty.$$

Earlier, Gibbon, Moore and Stuart [30] gave another example of (cylindrical) blowup solutions with infinite energy for the incompressible Euler equations.

Remark 5.4 *In 1965, Arnold first introduced the well-known Arnold-Beltrami-Childress (ABC) flow*

$$\begin{cases} u_1 = A\sin z + C\cos y \\ u_2 = B\sin x + A\cos z \\ u_3 = C\sin y + B\cos x \end{cases} \tag{5.18}$$

with constants A*,* B*,* C *and a suitable pressure function* P *for the incompressible Euler equations in [1]. We observe that our solutions (5.14) and (5.16) are similar to the ABC flow.*

Remark 5.5 *Since* $\Delta\vec{u} = \vec{0}$*, the solutions (5.9) and (5.13) are also solutions of the 3-dimensional compressible Navier-Stokes equations, obtained by adding a term* $\mu\Delta\vec{u}$*,* $\mu \geq 0$*. to the right-hand side of the momentum equation of the Euler system.*

Likewise, the solutions (5.14) and (5.16) are also solutions of the corresponding 3-dimensional incompressible Navier-Stokes equations.

Chapter 6

The Perturbational Method

In this chapter, we introduce a novel perturbational method to augment the conventional separation method and apply it to the 2-component Camassa-Holm equations and 1-dimensional compressible Euler equations to obtain new solutions with drift phenomenon.

6.1 The 2-Component Camassa-Holm Equations

In Chapter 3, we used the separation method to obtain a class of blowup or global solutions for the Camassa-Holm equations (3.1) and the Degasperis-Procesi equations (3.17). It is natural to consider the more general linear velocity

$$u(t,x) = c(t)x + b(t) \tag{6.1}$$

to look for new solutions. The new idea is to substitute (6.1) into the Camassa-Holm equations and compare the coefficient of different polynomial degrees to deduce the necessary functional differential equations $(c(t), b(t), \rho^2(t,0))$. Then we apply the Hubble's transformation

$$c(t) = \frac{\dot{a}(3t)}{a(3t)} \tag{6.2}$$

with $\dot{a}(3t) := \dfrac{da(3t)}{d(3t)}$, to simplify the equations involving $(a(3t), b(t), \rho^2(0,t))$. After proving the local existence of the corresponding dynamical system, we can show the following results.

Theorem 6.1 ([94]) *For the 2-component Camassa-Holm equations (3.1), there exists a family of solutions*

$$\begin{cases} \rho^2(t,x) = \max\left\{R(t) - \frac{2}{\sigma}\left[\dot{b}(t) + 3b(t)\frac{\dot{a}(3t)}{a(3t)}\right]x - \frac{3\xi}{\sigma a^{\frac{4}{3}}(3t)}x^2,\ 0\right\} \\ u(t,x) = \frac{\dot{a}(3t)}{a(3t)}x + b(t) \end{cases} \tag{6.3}$$

where $a(t)$, $b(t)$ and $c(t)$ satisfy the following functional differential equations

$$\begin{cases} \frac{d^2}{dt^2}a(3t) = \frac{\xi}{a^{\frac{1}{3}}(3t)}, & a(0) = a_0 > 0,\ \dot{a}(0) = a_1 \\ \frac{d^2}{dt^2}b(t) + \frac{6\dot{a}(3t)}{a(3t)}\frac{d}{dt}b(t) + \frac{12\xi}{a^{\frac{4}{3}}(3t)}b(t) = 0, & b(0) = b_0,\ \dot{b}(0) = b_1 \\ \frac{dR(t)}{dt} + \frac{2\dot{a}(3t)}{a(3t)}R(t) = \frac{2b(t)}{\sigma}\left[\dot{b}(t) + 3b(t)\frac{\dot{a}(3t)}{a(3t)}\right], & R(0) = \alpha^2 \end{cases} \tag{6.4}$$

where $R(t) = \rho^2(t,0)$, ξ, a_0, a_1, b_0, b_1 and α are arbitrary constants.

Proof. In view of (3.2) and (6.1), the momentum equation $(3.1)_2$ becomes

$$u_t + 3uu_x + \sigma\rho\rho_x = 0$$

$$\dot{c}(t)x + \dot{b}(t) + 3[c(t)x + b(t)]c(t) + \frac{\sigma}{2}\frac{\partial}{\partial x}\rho^2 = 0$$

$$\frac{\sigma}{2}\frac{\partial}{\partial x}\rho^2 = -[\dot{b}(t) + 3b(t)c(t)] - [\dot{c}(t) + 3c^2(t)]x.$$

Integrating from $[0,x]$, we get

$$\frac{\sigma}{2}\int_0^x \frac{\partial}{\partial s}\rho^2 ds = -[\dot{b}(t) + 3b(t)c(t)]\int_0^x ds - [\dot{c}(t) + 3c^2(t)]\int_0^x s ds,$$

$$\frac{\sigma}{2}\left[\rho^2(t,x) - \rho^2(t,0)\right] = -[\dot{b}(t) + 3b(t)c(t)]x - \frac{[\dot{c}(t) + 3c^2(t)]}{2}x^2,$$

and, finally,

$$\rho^2(t,x) = \rho^2(t,0) - \frac{2}{\sigma}[\dot{b}(t) + 3b(t)c(t)]x - \frac{[\dot{c}(t) + 3c^2(t)]}{\sigma}x^2.$$

On the other hand, for the mass equation $(3.1)_1$, we obtain

$$\rho_t + [c(t)x + b(t)]\,\rho_x + \rho c(t) = 0.$$

Next, we multiply ρ on both sides to get

$$\frac{1}{2}\left(\rho^2\right)_t + \frac{[c(t)x + b(t)]}{2}\left(\rho^2\right)_x + \rho^2 c(t) = 0. \tag{6.5}$$

After that, we substitute equation (6.1) into equation (6.5):

$$\frac{1}{2}\left(\frac{\partial}{\partial t}\left[\rho^2(t,0)\right]-\frac{2}{\sigma}\frac{\partial}{\partial t}[\dot{b}(t)+3b(t)c(t)]x-\frac{\partial}{\partial t}\frac{[\dot{c}(t)+3c^2(t)]}{\sigma}x^2\right)$$
$$+[c(t)x+b(t)]\left(-\frac{1}{\sigma}[\dot{b}(t)+3b(t)c(t)]-\frac{1}{\sigma}[\dot{c}(t)+3c^2(t)]x\right)$$
$$+c(t)\left[\rho^2(t,0)-\frac{2}{\sigma}[\dot{b}(t)+3b(t)c(t)]x-\frac{[\dot{c}(t)+3c^2(t)]}{\sigma}x^2\right]$$
$$=\frac{1}{2}\frac{\partial}{\partial t}\left[\rho^2(t,0)\right]+c(t)\rho^2(t,0)-\frac{b(t)}{\sigma}[\dot{b}(t)+3b(t)c(t)]$$
$$+\left\{\begin{array}{c}-\frac{1}{\sigma}\frac{\partial}{\partial t}[\dot{b}(t)+3b(t)c(t)]-\frac{c(t)}{\sigma}[\dot{b}(t)+3b(t)c(t)]\\ -\frac{b(t)}{\sigma}[\dot{c}(t)+3c^2(t)]-\frac{2c(t)}{\sigma}[\dot{b}(t)+3b(t)c(t)]\end{array}\right\}x$$
$$+\left\{\begin{array}{c}-\frac{1}{2\sigma}\frac{\partial}{\partial t}[\dot{c}(t)+3c^2(t)]-\frac{1}{\sigma}[\dot{c}(t)+3c^2(t)]c(t)\\ -\frac{c(t)[\dot{c}(t)+3c^2(t)]}{\sigma}\end{array}\right\}x^2.$$

By comparing the coefficients of the polynomial, we arrive at the differential equations

$$\left\{\begin{array}{c}\frac{d}{dt}\left[\rho^2(t,0)\right]+2c(t)\rho^2(t,0)-\frac{2}{\sigma}b(t)[\dot{b}(t)+3b(t)c(t)]=0\\ \frac{d}{dt}[\dot{b}(t)+3b(t)c(t)]+3c(t)[\dot{b}(t)+3b(t)c(t)]+b(t)[\dot{c}(t)+3c^2(t)]=0\\ \frac{d}{dt}[\dot{c}(t)+3c^2(t)]+4[\dot{c}(t)+3c^2(t)]c(t)=0.\end{array}\right. \tag{6.6}$$

Let us first solve equation $(6.6)_3$. Using the Hubble's transformation, therefore (6.2), the equation is transformed to

$$\frac{d}{dt}\left[\frac{3\ddot{a}(3t)}{a(3t)}-\frac{3\dot{a}^2(3t)}{a^2(3t)}+\frac{3\dot{a}^2(3t)}{a^2(3t)}\right]+4\left[\frac{3\ddot{a}(3t)}{a(3t)}-\frac{3\dot{a}^2(3t)}{a^2(3t)}+\frac{3\dot{a}^2(3t)}{a^2(3t)}\right]\frac{\dot{a}(3t)}{a(3t)}=0$$

$$\left\{\begin{array}{l}\frac{d}{dt}\left(\frac{\ddot{a}(3t)}{a(3t)}\right)+\frac{4\ddot{a}(3t)}{a(3t)}\frac{\dot{a}(3t)}{a(3t)}=0\\ a(0)=a_0>0,\ \dot{a}(0)=a_1,\ \ddot{a}(0)=a_2\end{array}\right.$$

$$\frac{3\dddot{a}(3t)}{a(3t)}-\frac{3\dot{a}(3t)\ddot{a}(3t)}{a^2(3t)}+\frac{4\dot{a}(3t)\ddot{a}(3t)}{a^2(3t)}=0$$

$$\frac{\dddot{a}(3t)}{a(3t)}+\frac{\dot{a}(3t)\ddot{a}(3t)}{3a^2(3t)}=0.$$

Then we multiply $a^2(3t)$ on both sides to have

$$a(3t)\dddot{a}(3t)+\frac{\dot{a}(3t)\ddot{a}(3t)}{3}=0,$$

which reduces to $(6.4)_1$.

It is now easy to show that the second equation $(6.6)_2$ can be simplified to $(6.4)_2$ using our knowledge of the function $a(3t)$. From standard theory, $b(t)$ exists as long as $a(3t)$ and $\dot{a}(3t)$ exist.

Lastly, by denoting

$$H(t) = \frac{2\dot{a}(3t)}{a(3t)}, \quad G(t) = \frac{2b(t)}{\sigma}\left[\dot{b}(t) + 3b(t)\frac{\dot{a}(3t)}{a(3t)}\right], \tag{6.7}$$

the first equation $(6.6)_1$ becomes

$$\begin{cases} \dfrac{d}{dt}\left[\rho^2(t,0)\right] + \rho^2(t,0)H(t) = G(t) \\ \qquad\qquad \rho^2(0,0) = \alpha^2. \end{cases} \tag{6.8}$$

The solution of this first-order ordinary differential equation (6.8) is

$$\rho^2(t,0) = \frac{\int_0^t \mu(s)G(s)ds + k}{\mu(t)}$$

where

$$\mu(t) = e^{\int_0^t H(s)ds}.$$

Therefore, we obtain the density function from equation (6.1)

$$\rho^2(t,x) = \rho^2(t,0) - \frac{2}{\sigma}\left[\dot{b}(t) + 3b(t)\frac{\dot{a}(3t)}{a(3t)}\right]x - \frac{3\xi}{\sigma a^{\frac{4}{3}}(3t)}x^2.$$

For $\rho(x,t) \geq 0$, we may set

$$\rho^2(t,x) = \max\left\{\rho^2(t,0) - \frac{2}{\sigma}[\dot{b}(t) + 3b(t)\frac{\dot{a}(3t)}{a(3t)}]x - \frac{3\xi}{\sigma a^{\frac{4}{3}}(3t)}x^2,\ 0\right\}.$$

We thus conclude that the class of function (6.3) solves the Camassa-Holm equations.

■

We remark that the above solutions (6.3) fully cover previous known results in [85] by the separation method if we choose $b_0 = b_1 = 0$.

Notice that the above solutions are not radially symmetric for the density function ρ with $b(t) \neq 0$. Thus, the above solutions cannot be obtained by the conventional separation method of the self-similar functional.

Theorem 6.2 ([94]) *For the 2-component Camassa-Holm equations with radial symmetry*

$$\begin{cases} \rho_t + V\rho_r + \rho V_r = 0 \\ V_t + 3VV_r + \sigma\rho\rho_r = 0, \end{cases}$$

there exists a family of solutions

$$\begin{cases} \rho^2(t,r) = \max\left\{\rho^2(t,0) - \dfrac{2}{\sigma}\left[\dot{b}(t) + 3b(t)\dfrac{\dot{a}(3t)}{a(3t)}\right] r - \dfrac{3\xi}{\sigma a^{\frac{4}{3}}(3t)} r^2,\ 0\right\} \\ V(t,r) = \dfrac{\dot{a}(3t)}{a(3t)} r + b(t) \end{cases} \tag{6.9}$$

where $a(t)$, $b(t)$ and $c(t)$ are the auxiliary functions in the equations (6.4).

For the graphical illustration of the classical blowup solution (6.3) with the infinitive mass by choosing the parameters $\alpha = 1$, $b_1 = \frac{\sigma}{2}$, $a_0 = 1$, $a_1 = 0$ and $\xi = \frac{-\sigma}{3}$, we can see the initial shape of the non-radially symmetric solution:

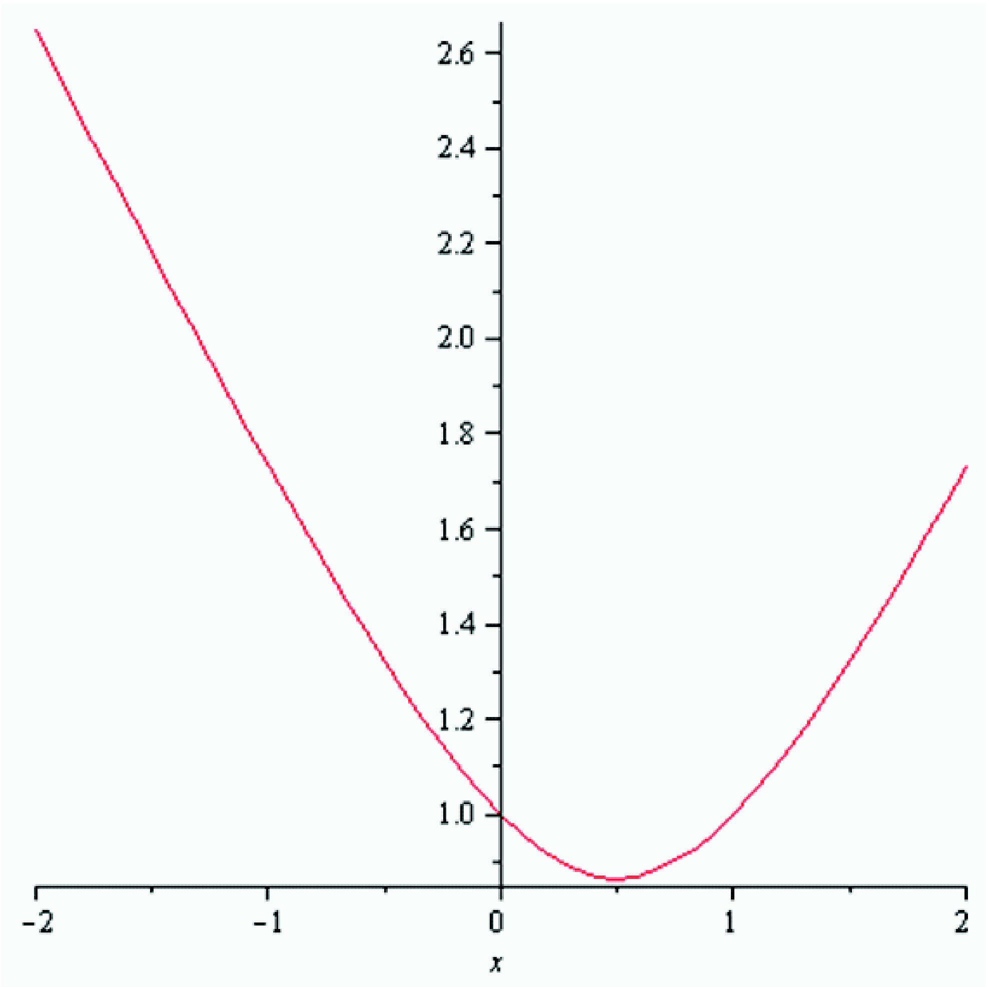

Fig. 6.1: $\rho_0(x) = \sqrt{1 - x + x^2}$

For the global breaking solutions (6.3), we choose the parameters $\alpha = 1$, $b_1 = \frac{\sigma}{2}$, $a_0 = 1$, $a_1 = 0$ and $\xi = \frac{\sigma}{3}$ to have the graph:

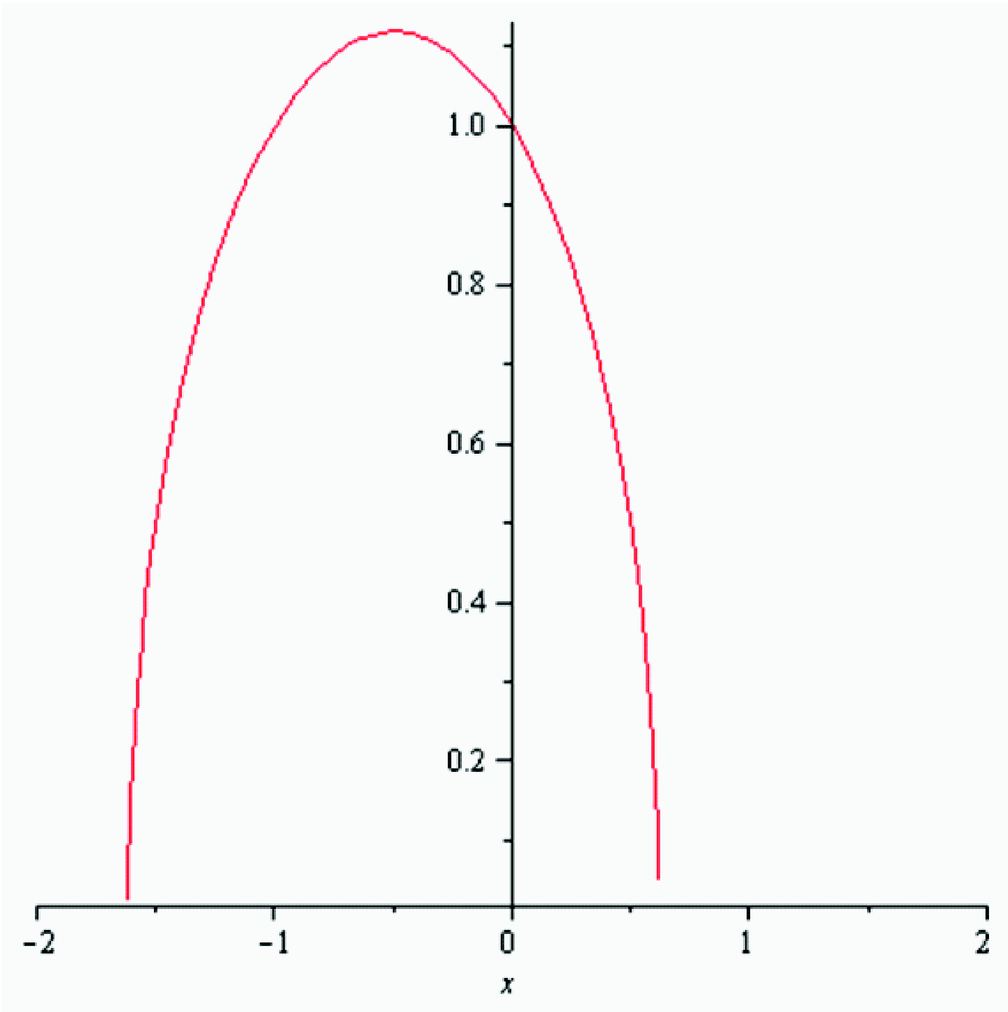

Fig. 6.2: $\rho_0(x) = \max(\sqrt{1 - x - x^2}, 0)$

6.2 The 1D Compressible Euler Equations

In a similar way, we can use the perturbational method to handle the 1-dimensional compressible adiabatic Euler equations

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0 \\ (u_t + u u_x) + K \dfrac{1}{\rho} \dfrac{\partial \rho^\gamma}{\partial x} = 0 \end{cases} \tag{6.10}$$

with $\gamma > 1$.

Theorem 6.3 ([91]) *There exists a family of solutions to the* 1*-dimensional compressible Euler equations (6.10)*

$$\begin{cases} \rho^{\gamma-1}(t,x) = \max\left\{\rho^{\gamma-1}(t,0) - \frac{\gamma-1}{K\gamma}\left[\dot{b}(t) + b(t)\frac{\dot{a}(t)}{a(t)}\right]x - \frac{(\gamma-1)\xi}{2K\gamma a^{\gamma+1}(t)}x^2,\ 0\right\} \\ u(t,x) = \frac{\dot{a}(t)}{a(t)}x + b(t) \end{cases} \tag{6.11}$$

where $a(t)$, $b(t)$ and $c(t)$ satisfy the following functional differential equations

$$\begin{cases} \ddot{a}(t) = \frac{\xi}{a^\gamma(t)}, \qquad a(0) = a_0 > 0,\ \dot{a}(0) = a_1 \\ \ddot{b}(t) + \frac{(1+\gamma)\dot{a}(t)}{a(t)}\dot{b}(t) + \left[\frac{2\xi}{a^{\gamma+1}(t)} + (\gamma-1)\frac{\dot{a}^2(t)}{a^2(t)}\right]b(t) = 0, \\ \qquad b(0) = b_0,\ \dot{b}(0) = b_1 \\ \frac{\partial}{\partial t}\rho^{\gamma-1}(t,0) + \rho^{\gamma-1}(t,0)\frac{\dot{a}(t)}{a(t)} - \frac{\gamma-1}{K\gamma}\left[\dot{b}(t) + b(t)\frac{\dot{a}(t)}{a(t)}\right]b(t) = 0, \\ \qquad \rho(0,0) = \alpha \end{cases} \tag{6.12}$$

where a_0, a_1, b_0, b_1 and α are arbitrary constants.

Remark 6.1 *As we can choose the two free constants b_0 and b_1 in Theorem 6.3, Theorem 2.3 in Section 2.3 (for which we can only choose one constant d_1 for the 1-dimensional case) cannot cover the theorem in this section.*

Remark 6.2 *Our solutions (6.11) fully cover the previous known ones with radial symmetry, for the 1-dimensional case in [59] and [48] with $b_0 = b_1 = 0$.*

Proof. First, we perturb the velocity as this form

$$u(t,x) = c(t)x + b(t). \tag{6.13}$$

The 1-dimensional momentum equation $(6.10)_2$ becomes, for the non-trivial solutions

$$(u_t + uu_x) + K\frac{1}{\rho}\frac{\partial\rho^\gamma}{\partial x} = 0$$

for $\gamma > 1$,

$$\dot{c}(t)x + \dot{b}(t) + [c(t)x + b(t)]c(t) + \frac{K\gamma}{\gamma-1}\frac{\partial}{\partial x}\rho^{\gamma-1} = 0$$

$$\frac{K\gamma}{\gamma-1}\frac{\partial}{\partial x}\rho^{\gamma-1} = -[\dot{b}(t) + b(t)c(t)] - [\dot{c}(t) + c^2(t)]x.$$

We take integration from $[0,x]$ to have

$$\frac{K\gamma}{\gamma-1}\int_0^x \frac{\partial}{\partial s}\rho^{\gamma-1}ds = -[\dot{b}(t)+b(t)c(t)]\int_0^x ds - [\dot{c}(t)+c^2(t)]\int_0^x sds \tag{6.14}$$

$$\frac{K\gamma}{\gamma-1}\left[\rho^{\gamma-1}(t,x)-\rho^{\gamma-1}(t,0)\right] = -[\dot{b}(t)+b(t)c(t)]x - \frac{\dot{c}(t)+c^2(t)}{2}x^2$$

$$\rho^{\gamma-1}(t,x) = \rho^{\gamma-1}(t,0) - \frac{\gamma-1}{K\gamma}[\dot{b}(t)+b(t)c(t)]x - \frac{\gamma-1}{2K\gamma}[\dot{c}(t)+c^2(t)]x^2. \tag{6.15}$$

On the other hand, for the 1-dimensional mass equation $(6.10)_1$, we obtain

$$\rho_t + \rho_x u + \rho u_x = 0$$

$$\rho_t + [c(t)x+b(t)]\,\rho_x + \rho c(t) = 0.$$

We multiply by $\rho^{\gamma-2}$ on both sides to have

$$\left(\frac{\rho^{\gamma-1}}{\gamma-1}\right)_t + [c(t)x+b(t)]\left(\frac{\rho^{\gamma-1}}{\gamma-1}\right)_x + \rho^{\gamma-1}c(t) = 0.$$

We substitute the above equation back into equation (6.15)

$$\begin{aligned}
&\left(\frac{\rho^{\gamma-1}}{\gamma-1}\right)_t + [c(t)x+b(t)]\left(\frac{\rho^{\gamma-1}}{\gamma-1}\right)_x + \rho^{\gamma-1}c(t)\\
&= \frac{1}{\gamma-1}\begin{pmatrix} \frac{\partial}{\partial t}\rho^{\gamma-1}(t,0) - \frac{\gamma-1}{K\gamma}\frac{\partial}{\partial t}[\dot{b}(t)+b(t)c(t)]x \\ -\frac{\gamma-1}{2K\gamma}\frac{\partial}{\partial t}[\dot{c}(t)+c^2(t)]x^2 \end{pmatrix}\\
&\quad + [c(t)x+b(t)]\cdot\frac{1}{\gamma-1}\begin{pmatrix} -\frac{\gamma-1}{K\gamma}\frac{\partial}{\partial x}[\dot{b}(t)+b(t)c(t)]x \\ -\frac{\gamma-1}{2K\gamma}\frac{\partial}{\partial x}[\dot{c}(t)+c^2(t)]x^2 \end{pmatrix}\\
&\quad + c(t)\left[\rho^{\gamma-1}(t,0) - \frac{\gamma-1}{K\gamma}[\dot{b}(t)+b(t)c(t)]x - \frac{\gamma-1}{K\gamma}\frac{[\dot{c}(t)+c^2(t)]}{2}x^2\right]\\
&= \frac{1}{\gamma-1}\begin{pmatrix} \frac{\partial}{\partial t}\rho^{\gamma-1}(t,0) - \frac{\gamma-1}{K\gamma}\frac{\partial}{\partial t}[\dot{b}(t)+b(t)c(t)]x \\ -\frac{\gamma-1}{2K\gamma}\frac{\partial}{\partial t}[\dot{c}(t)+c^2(t)]x^2 \end{pmatrix}\\
&\quad + [c(t)x+b(t)]\left(-\frac{1}{K\gamma}[\dot{b}(t)+b(t)c(t)] - \frac{1}{K\gamma}[\dot{c}(t)+c^2(t)]x\right)\\
&\quad + c(t)\left[\rho^{\gamma-1}(t,0) - \frac{\gamma-1}{K\gamma}[\dot{b}(t)+b(t)c(t)]x - \frac{\gamma-1}{2K\gamma}[\dot{c}(t)+c^2(t)]x^2\right]\\
&= \frac{1}{\gamma-1}\frac{\partial}{\partial t}\rho^{\gamma-1}(t,0) + c(t)\rho^{\gamma-1}(t,0) - \frac{b(t)}{K\gamma}[\dot{b}(t)+b(t)c(t)]\\
&\quad + \left\{\begin{array}{c} -\frac{1}{K\gamma}\frac{\partial}{\partial t}\left[\dot{b}(t)+b(t)c(t)\right] - \frac{c(t)}{K\gamma}\left[\dot{b}(t)+b(t)c(t)\right] \\ -\frac{b(t)}{K\gamma}[\dot{c}(t)+c^2(t)] - \frac{(\gamma-1)c(t)}{K\gamma}[\dot{b}(t)+b(t)c(t)] \end{array}\right\}x\\
&\quad + \left\{\begin{array}{c} -\frac{1}{2K\gamma}\frac{\partial}{\partial t}[\dot{c}(t)+c^2(t)] - \frac{c(t)}{K\gamma}[\dot{c}(t)+c^2(t)] \\ -\frac{(\gamma-1)c(t)}{2K\gamma}[\dot{c}(t)+c^2(t)] \end{array}\right\}x^2.
\end{aligned}$$

By comparing the coefficients of the polynomial, we require the functional differential equations involving $(c(t), b(t), \rho^{\gamma-1}(t,0))$

$$\begin{cases} \frac{d}{dt}\rho^{\gamma-1}(t,0) + (\gamma-1)c(t)\rho^{\gamma-1}(t,0) - \frac{\gamma-1}{K\gamma}b(t)[\dot{b}(t) + b(t)c(t)] = 0 \\ \frac{d}{dt}[\dot{b}(t) + b(t)c(t)] + \gamma c(t)[\dot{b}(t) + b(t)c(t)] + b(t)[\dot{c}(t) + c^2(t)] = 0 \\ \frac{d}{dt}[\dot{c}(t) + c^2(t)] + (\gamma+1)[\dot{c}(t) + c^2(t)]c(t) = 0 \end{cases} \tag{6.16}$$

to solve the compressible 1-dimensional Euler system.

For details (existence, uniqueness and continuous dependence) on theories of functional differential equations, the interested reader can see the classical literature [36] and [72].

Here, we solve equation $(6.16)_3$ with the Hubble's expression for $c(t)$

$$c(t) = \frac{\dot{a}(t)}{a(t)} \tag{6.17}$$

$$\frac{d}{dt}\left[\frac{d}{dt}\left(\frac{\dot{a}(t)}{a(t)}\right) + \frac{\dot{a}^2(t)}{a^2(t)}\right] + (\gamma+1)\left[\frac{d}{dt}\left(\frac{\dot{a}(t)}{a(t)}\right) + \frac{\dot{a}^2(t)}{a^2(t)}\right]\frac{\dot{a}(t)}{a(t)} = 0$$

$$\frac{d}{dt}\left[\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)} + \frac{\dot{a}^2(t)}{a^2(t)}\right] + (\gamma+1)\left[\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)} + \frac{\dot{a}^2(t)}{a^2(t)}\right]\frac{\dot{a}(t)}{a(t)} = 0$$

$$\begin{cases} \frac{d}{dt}\left[\frac{\ddot{a}(t)}{a(t)}\right] + (\gamma+1)\frac{\ddot{a}(t)}{a(t)}\frac{\dot{a}(t)}{a(t)} = 0 \\ a(0) = a_0 > 0,\ \dot{a}(0) = a_1,\ \ddot{a}(0) = a_2 \end{cases}$$

$$\frac{\dddot{a}(t)}{a(t)} - \frac{\dot{a}(t)\ddot{a}(t)}{a^2(t)} + (\gamma+1)\frac{\dot{a}(t)\ddot{a}(t)}{a^2(t)} = 0$$

$$\frac{\dddot{a}(t)}{a(t)} + \gamma\frac{\dot{a}(t)\ddot{a}(t)}{a^2(t)} = 0.$$

We multiply by $a^{\gamma+1}(t)$ on both sides to obtain

$$a^{\gamma}(t)\dddot{a}(t) + \gamma a^{\gamma-1}(t)\dot{a}(t)\ddot{a}(t) = 0.$$

Here, we can observe that this can be reduced to the Emden equation

$$\begin{cases} \ddot{a}(t) = \frac{\xi}{a^{\gamma}(t)} \\ a(0) = a_0 > 0,\ \dot{a}(0) = a_1 \end{cases} \tag{6.18}$$

where $\xi = a_0^{\gamma}a_2$ are an arbitrary constant by freely choosing a_2.

We remark that the Emden equation (6.18) was well-studied in the literature of astrophysics and mathematics. The local existence of the Emden equation (6.18) can be shown by the fixed point theorem.

For the second equation $(6.16)_2$ of the dynamic system, we have

$$\frac{d}{dt}\left[\dot{b}(t)+b(t)\frac{\dot{a}(t)}{a(t)}\right]+\gamma\left[\dot{b}(t)+b(t)\frac{\dot{a}(t)}{a(t)}\right]\frac{\dot{a}(t)}{a(t)}+\frac{\ddot{a}(t)}{a(t)}b(t)=0$$

$$\ddot{b}(t)+\dot{b}(t)\frac{\dot{a}(t)}{a(t)}+b(t)\frac{d}{dt}\frac{\dot{a}(t)}{a(t)}+\gamma\frac{\dot{a}(t)}{a(t)}\dot{b}(t)+\gamma b(t)\frac{\dot{a}^2(t)}{a^2(t)}+\frac{\ddot{a}(t)}{a(t)}b(t)=0$$

$$\ddot{b}(t)+\frac{(1+\gamma)\dot{a}(t)}{a(t)}\dot{b}(t)+\left[\frac{\ddot{a}(t)}{a(t)}-\frac{\dot{a}^2(t)}{a^2(t)}+\frac{\gamma\dot{a}^2(t)}{a^2(t)}+\frac{\ddot{a}(t)}{a(t)}\right]b(t)=0$$

$$\begin{cases}\ddot{b}(t)+\frac{(1+\gamma)\dot{a}(t)}{a(t)}\dot{b}(t)+\left[\frac{2\xi}{a^{\gamma+1}(t)}+(\gamma-1)\frac{\dot{a}^2(t)}{a^2(t)}\right]b(t)=0\\ b(0)=b_0,\ \dot{b}(0)=b_1\end{cases}$$

with the Emden equation (6.18) for $a(t)$.

We denote

$$f_1(t)=\frac{(1+\gamma)\,\dot{a}(t)}{a(t)},\ f_2(t)=\left[\frac{2\xi}{a^{\gamma+1}(t)}+(\gamma-1)\,\frac{\dot{a}^2(t)}{a^2(t)}\right] \tag{6.19}$$

to get

$$\begin{cases}\ddot{b}(t)+f_1(t)\dot{b}(t)+f_2(t)b(t)=0\\ b(0)=b_0,\ \dot{b}(0)=b_1.\end{cases} \tag{6.20}$$

Therefore, when the functions $f_1(t)$ and $f_2(t)$ are bounded, that is

$$|f_1(t)|<f_1,\ |f_2(t)|<f_2 \tag{6.21}$$

with the constants f_1 and f_2, provided that $a(t)\neq 0$ and $\dot{a}(t)$ exist for $0\leq t<T$. Then, the functions $b(t)$ and $\dot{b}(t)$ exist and are bounded by the comparison theorem of ordinary differential equations.

For the first equation $(6.16)_1$, as it is a first order ordinary differential equations only, we can directly solve the following:

$$\frac{d}{dt}\rho^{\gamma-1}(t,0)+(\gamma-1)\rho^{\gamma-1}(t,0)\frac{\dot{a}(t)}{a(t)}-\frac{\gamma-1}{K\gamma}\left[\dot{b}(t)+b(t)\frac{\dot{a}(t)}{a(t)}\right]b(t)=0.$$

Denote

$$H(t)=(\gamma-1)\frac{\dot{a}(t)}{a(t)},\ G(t)=\frac{\gamma-1}{K\gamma}\left[\dot{b}(t)+b(t)\frac{\dot{a}(t)}{a(t)}\right]b(t)$$

to solve

$$\frac{d}{dt}\rho^{\gamma-1}(t,0)+\rho^{\gamma-1}(t,0)H(t)=G(t)$$

with bounded $a(t)\neq 0$ and $\dot{a}(t)$ for $0\leq t<T$.

The formula of the first order ordinary differential equation is

$$\rho^{\gamma-1}(t,0)=\frac{\int_0^t\mu(s)G(s)ds+k}{\mu(t)}$$

with

$$\mu(t) = e^{\int_0^t H(s)ds}$$

and a constant k.

Therefore, we have the density function by equation (6.15)

$$\rho^{\gamma-1}(t,x) = \rho^{\gamma-1}(t,0) - \frac{\gamma-1}{K\gamma}\left[\dot{b}(t) + b(t)\frac{\dot{a}(t)}{a(t)}\right]x - \frac{\gamma-1}{2K\gamma}\frac{\ddot{a}(t)}{a(t)}x^2$$

$$\rho^{\gamma-1}(t,x) = \rho^{\gamma-1}(t,0) - \frac{\gamma-1}{K\gamma}\left[\dot{b}(t) + b(t)\frac{\dot{a}(t)}{a(t)}\right]x - \frac{(\gamma-1)\xi}{2K\gamma a^{\gamma+1}(t)}x^2.$$

For the non-negative density solutions $\rho(t,x)$, we must set

$$\rho^{\gamma-1}(t,x) = \max\left\{\rho^{\gamma-1}(t,0) - \frac{\gamma-1}{K\gamma}\left[\dot{b}(t) + b(t)\frac{\dot{a}(t)}{a(t)}\right]x - \frac{(\gamma-1)\xi}{2K\gamma a^{\gamma+1}(t)}x^2,\ 0\right\}.$$

■

We notice that the above solutions are not radially symmetric for the function $b(t) \neq 0$. Therefore, the above density solutions ρ cannot be obtained by the separation method of the self-similar functional, as

$$\rho(t,x) \neq f(\frac{x^2}{a(t)})g(a(t)) \text{ and } u(t,x) = \frac{\dot{a}(t)}{a(t)}x + b(t). \tag{6.22}$$

On the other hand, for the compressible 1-dimensional Euler system in radial symmetry (2.3), we may replace equation (6.14) to have the corresponding equation by taking the integration from $[0,\, r]$:

$$\frac{K\gamma}{\gamma-1}\int_0^r \frac{\partial}{\partial s}\rho^{\gamma-1}ds = -[\dot{b}(t) + b(t)c(t)]\int_0^r ds - [c(t) + c^2(t)]\int_0^r sds.$$

The corresponding result in radial symmetry obviously holds.

Theorem 6.4 ([91]) *There exists a family of solutions for the* 1*-dimensional compressible Euler equations with radial symmetry (6.10)*

$$\begin{cases} \rho^{\gamma-1}(t,r) = \max\left\{\rho^{\gamma-1}(t,0) - \frac{\gamma-1}{K\gamma}\left[\dot{b}(t) + b(t)\frac{\dot{a}(t)}{a(t)}\right]r - \frac{(\gamma-1)\xi}{2K\gamma a^{\gamma+1}(t)}r^2,\ 0\right\} \\ V(t,r) = \frac{\dot{a}(t)}{a(t)}r + b(t), \end{cases} \tag{6.23}$$

where $a(t)$, $b(t)$ and $c(t)$ are the auxiliary functions in the equations (6.12).

We note that the original solutions

$$u(t,x)=\frac{V(t,r)}{r} \tag{6.24}$$

in (6.10), with the functions $V(r,t)$ in the equation (6.23) are with the singular point at the origin 0, for $b(t)\neq 0$.

It is clear to see that the solutions (6.11) and (6.23) are also the solutions of the compressible 1-dimensional Navier-Stokes equations

$$\begin{cases} \rho_t+\nabla\cdot(\rho u)=0 \\ \rho\left[u+(u\cdot\nabla)\,u\right]+\nabla P=\mu\Delta u \end{cases}$$

with a positive constant μ.

To determine if the solutions are global or local only, we could use the following lemma about the Emden equation $(6.11)_3$.

Lemma 6.1 *For the Emden equation*

$$\begin{cases} \ddot{a}(t)=\dfrac{\xi}{a^{\kappa}(t)} \\ a(0)=a_0>0,\ \dot{a}(0)=a_1 \end{cases}$$

with the constant $\kappa>1$,

(1) if $\xi<0$

$$a_1<\sqrt{\frac{-2\xi}{\kappa-1}}a_0^{\frac{(-\kappa+1)}{2}};$$

there exists a finite time T*, such that*

$$\lim_{t\to T^-}a(t)=0,$$

otherwise, the solution $a(t)$ *exists globally, such that*

$$\lim_{t\to\infty}a(t)=\infty.$$

(2) if $\xi=0$*, with* $a_1<0$*, the solution* $a(t)$ *blows up in*

$$T=\frac{-a_0}{a_1};$$

otherwise, the solution $a(t)$ *exists globally.*

(3) if $\xi>0$*, the solution* $a(t)$ *exists globally, such that*

$$\lim_{t\to\infty}a(t)=\infty.$$

All the proofs can be shown by the standard energy method of classical mechanics. The particular proofs can be found in [24] with $\kappa > 1$ for blowup cases. Therefore, we can omit the proof here.

We observe that the gradient of the velocity is

$$\frac{\partial}{\partial x}u(t,x) = \frac{\dot{a}(t)}{a(t)}.$$

When the function $a(T) = 0$ with a finite time T, $\frac{\partial}{\partial x}u(T,x)$ blows up at every space-point x. Based on the above lemma about the Emden equation for $a(t)$, $(6.11)_3$, it is clear to have the corollary

Corollary 6.1 *(1a) For $\xi < 0$ and*

$$a_1 < \sqrt{\frac{-2\xi}{\kappa - 1}a_0^{\frac{(-\kappa+1)}{2}}},$$

the solutions (6.11) and (6.23) blow up at a finite time T;

(1b) For $\xi = 0$, with $a_1 < 0$, the solutions (6.11) blow up in

$$T = \frac{-a_0}{a_1}.$$

(2) otherwise, the solutions (6.11) and (6.23) exist globally.

For the graphical illustration of the blowup solution (6.11) with the infinite mass, by choosing the parameters $\gamma = 2$, $\alpha = 1$, $b_0 = 2$, $b_1 = 0$, $a_0 = 1$, $a_1 = 0 < a_0$, $K = 1$ and $\xi = -4$, we can see the initial shape of the non-self-similar and non-radially symmetric solution:

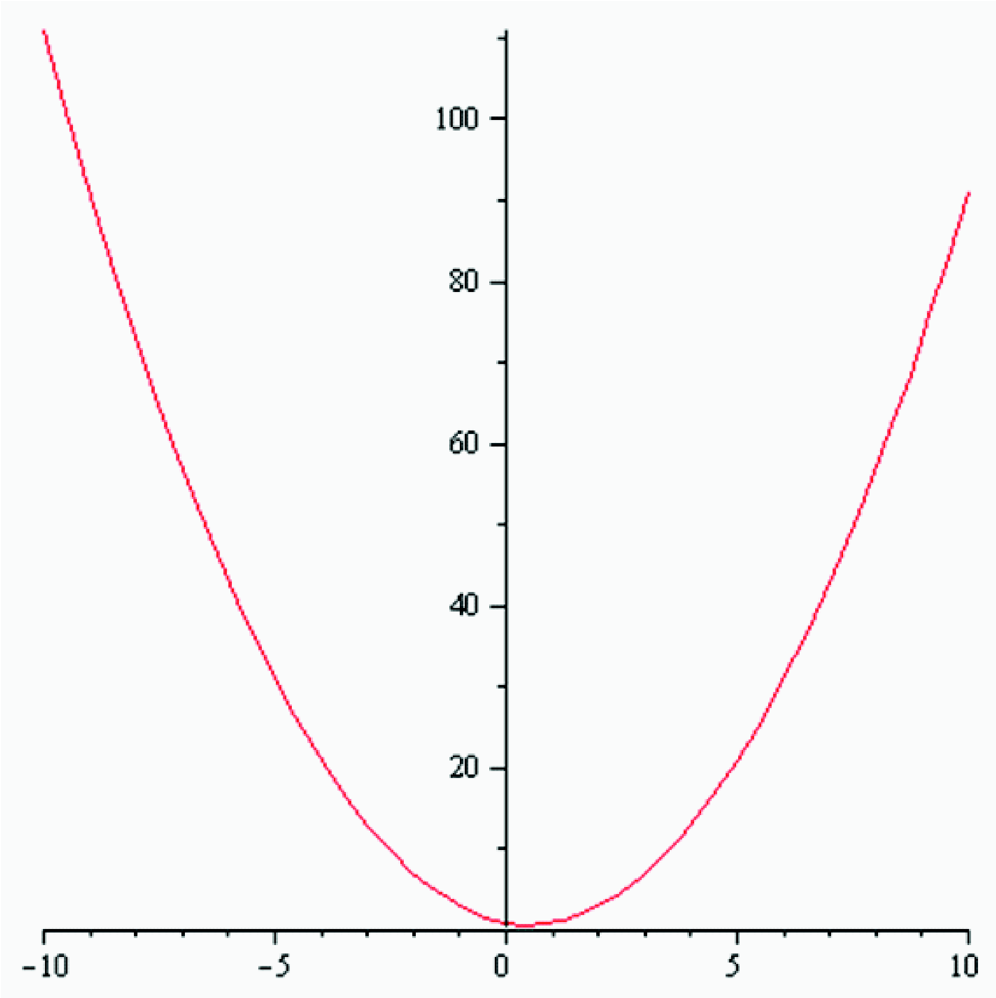

Fig. 6.3: $\rho_0(x) = 1 - x + x^2$

For the global solutions, we can see the initial shape of the corresponding solution by choosing the parameters $\gamma = 2$, $\alpha = 1$, $b_0 = 2$, $b_1 = 0$, $a_0 = 1$, $a_1 = 0$, $K = 1$ and $\xi = 4$ in Fig.6.4.

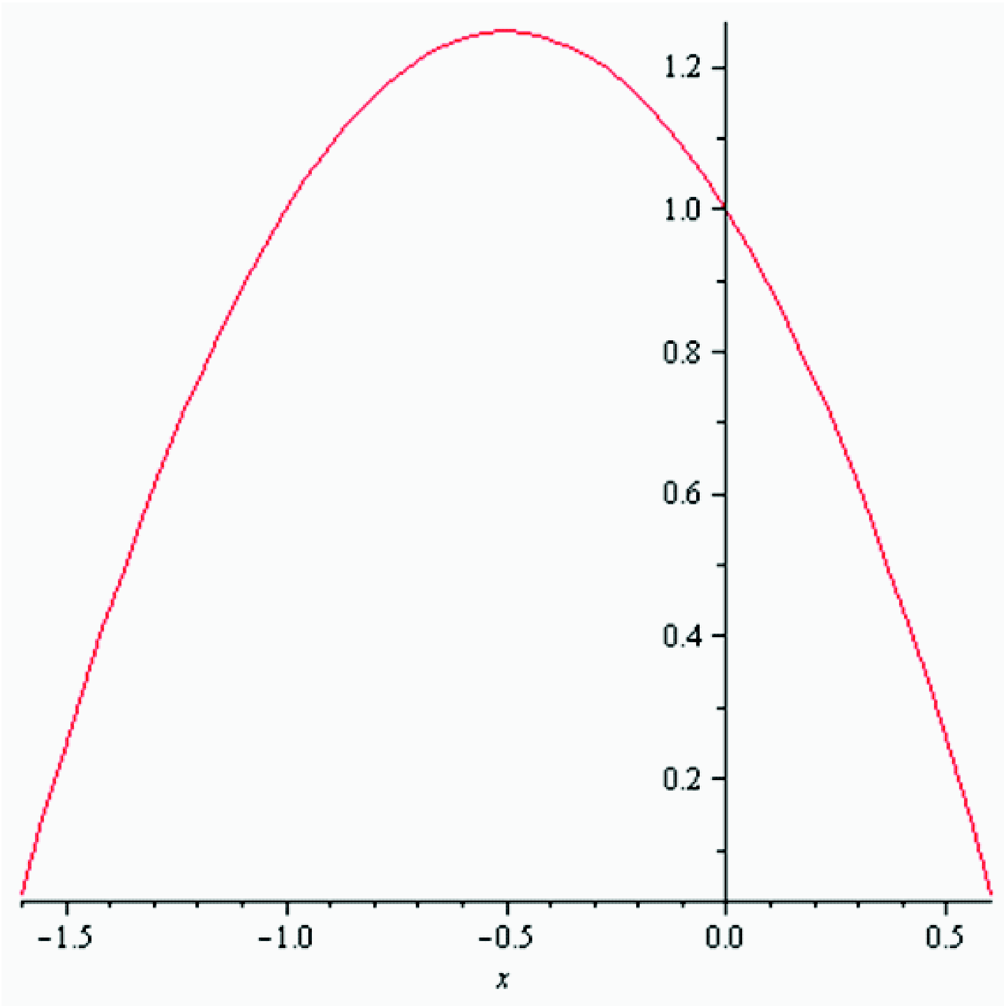

Fig. 6.4: $\rho_0(x) = 1 - x - x^2$

Chapter 7

Conclusions and Suggestions for Future Research

7.1 Summary and Other Related Work

In this thesis, I present the computational techniques that have enabled me to construct new analytical/exact solutions for six fluid dynamical systems: the Euler, Euler-Poisson, Navier-Stokes, Navier-Stokes-Poisson, Camassa-Holm and Degasperis-Procesi equations. More precisely, I have fully exploited the well-known separation method and devised a new perturbational method. The main principle behind the success in obtaining new solutions is to reduce the original nonlinear partial differential systems to several ordinary or functional differential equations, or to simpler partial differential equations by imposing some suitable functional structures on the solutions. A substantial portion of my effort is spent on searching for such suitable functional structures and on verifying the validity of the solutions. After that, existence and other qualitative results of the reduced simpler equations must then be established in order to complete the construction of explicit solutions for the original nonlinear systems.

My work in constructing solutions with elliptic symmetry and drift phenomena is particularly exciting since for many years authors have paid attention only to radially symmetric solutions. Although elliptically symmetric solutions have been known to some physicists for a while, their work is not well-known to mathematicians working on fluid dynamics. In my work, I have independently rediscovered and generalized the concept of elliptic symmetry in fluid dynamical systems. I have obtained $1 + N$ differential functional equations for the density-dependent Navier-Stokes equations in R^N, leading to the new Emden systems of differential equations. The qualitative study of the system yields some very interesting results concerning the global existence of positive solutions and their oscillatory properties.

For the 2-component shallow water systems, I am the first person to obtain the self-similar solutions in explicit form. For the 2-dimensional Euler-Poisson equations, we construct the first rotational and periodic solutions in explicit form.

The study of pulsating rotational solutions of the 2-dimensional Euler-Poisson equations represents a more analytical and theoretical diversion from the other part of my thesis work. Instead of having an explicit formula for the pulsating solution, its existence is only established through a long chain of nontrivial theoretical arguments.

Finally, I would like to point out that some other related work have also been accomplished during my Ph.D. study. These have not been described in the thesis. These include the following papers

[75] We constructed a class of self-similar solutions for the pressureless Navier-Stokes equations with density-dependent viscosity;

[76] We constructed a class of similar solutions with drifting term for the non-isentropic pressure Euler equations;

[77] We constructed a class of self-similar solutions for the Navier-Stokes-Poisson equations with density-dependent viscosity and with pressure;

[78] We gave a class of the implicit and explicit solutions for the pressureless Euler equations;

[88] I showed some blowup phenomena for the Euler equations with repulsive force and with pressure under some initial-boundary conditions;

[93] I constructed the class of self-similar solutions for the isentropic pressureless Euler equations and

[87] I showed some blowup phenomena for the Euler-Poisson equations with attractive forces under some initial-boundary conditions.

7.2 Future Studies

Finally, I would like to indicate some directions of further research based on the thesis in the following three remarks.

Remark 7.1 *Most of the constructed solutions assume linear functional forms for the velocity $\vec{u}$. Known solutions of the 3-dimensional systems in gas dynamics (Euler, Euler-Poisson, Navier-Stokes, Navier-Stokes-Poisson equations) suggest the more general functional form in Cartesian coordinate*

$$\begin{cases} u_1 = a_1(t) + b_{11}(t)x + b_{12}(t)y + b_{13}(t)z \\ u_2 = a_2(t) + b_{21}(t)x + b_{22}(t)y + b_{23}(t)z \\ u_3 = a_3(t) + b_{31}(t)x + b_{32}(t)y + b_{33}(t)z \end{cases} \tag{7.1}$$

with smooth C^1 functions $a_i(t)$ and $b_{ij}(t)$ for $i,\ j = 1,\ 2$ and 3.

Can we construct a blowup example with rotation having this more general form for the Euler system with finite energy?

In principle, we could adopt the trial-and-error approach to search for suitable density functions. Under the assumed functional form, the system is reducible to $1 + 12$ functional differential equations. Direct verification will be computational extremely intensive if it is to be carried out by hand. Symbolic or numerical computations are possible and indeed very promising tools that come in handy and we hope to investigate more systematically in this direction.

Remark 7.2 *In 2008, Jang [39] showed the existence of perturbed solutions of the stationary solutions with spherical symmetry for the Euler-Poisson equations with $\gamma = 6/5$. And in Section 6.2, the simple time-dependent pertubational method is developed for the Camassa-Holm and Euler equations. Can we further perturb the known exact or analytical solutions for the systems under some suitable norms to establish the weak existence results?*

Remark 7.3 *In Chapter 4, we introduce a class of rotational radially symmetric solutions of the two-dimensional Euler-Poisson equations, and demonstrate the existence of radially symmetric global pulsating solutions of the two-dimensional Euler-Poisson equations. All the shells pulsate with the same period. In a sense, such solutions are a rarity rather than the rule. A slight perturbation of the motion of* *one of the shells can destroy the global property of the solution. This is in contrast to the principle of critical thresholds.*

We would like to know if Theorem 4.3 can be improved to yield classical global solutions having annular structures. It is plausible that such solutions may not exist. In other words, global solutions of this type must possess some sort of singularity. It is interesting to ask whether non-radially symmetric global solutions exist. On the other hand, a more interesting question is, in the case when a global solution fails and some sort of collision of the characteristic curves does happen, is there a way to extend the definition of solution to a weaker sense to enable us to study what happens beyond the collision. The class of rotational radially symmetric solutions provides a concrete framework on which such a theory can be tested.

It is also of interest to see if there are any analogs of similar radially symmetric rotational solutions in higher dimensions.

Finally, the Conjecture in Section 4.3 seems so intuitively true, but its proof is tantalizingly out of reach. We would like to further investigate this Conjecture.

References

[1] Arnold V.I. (1965), *Sur la Topologie des Ecoulements Stationnaires des Fluides Parfaits*, (French) C. R. Acad. Sci. Paris **261**, 17–20.

[2] Bezard M. (1993), *Existence Locale de Solutions pour les Equations d'Euler-Poisson. (French) [Local Existence of Solutions for Euler-Poisson Equations]* Japan J. Indust. Appl. Math. **10**, 431–450.

[3] Baxter, N.P. and Shabanov, S.V. (2008), *The Flip-over Effect in Self-Similar Laser-Induced Plasma Expansion*, J. Math. Phys. **49**, 093101, 8 pp.

[4] Binney J. and Tremaine S. (1994), Galactic Dynamics, Princeton Univ. Press.

[5] Camassa R. and Holm D.D. (1993), *Integrable Shallow Water Equation with Peaked Solitons*, Phys. Rev. Lett. **71**, 1661–1664.

[6] Corless R.M., Gonnet G.H., Hare D.E.G., Jeffrey D.J. and Knuth D.E. (1996), *On the Lambert W Function*, Advances in Computational Mathematics **5**, 329-359.

[7] Chae D.H. and Tadmor E. (2008), *On the Finite Time Blow-up of the Euler-Poisson Equations in R^N*, Commun. Math. Sci. **6**, 785–789.

[8] Chandrasekhar S. (1939), An Introduction to the Study of Stellar Structure, Univ. of Chicago Press.

[9] Chen C.F. (1984), Introduction to Plasma Physics and Controlled Fusion, Plenum, New York.

[10] Chen G.Q. and Wang D.H. (2002), *The Cauchy Problem for the Euler Equations for Compressible Fluids*, Handbook of Mathematical Fluid Dynamics, Vol. I, 421-543, North-Holland, Amsterdam.

[11] Chen M., Liu S.Q. and Zhang Y. (2006), *A 2-component Generalization of the Camassa–Holm Equation and Its Solutions*, Lett. Math. Phys. **75**, 1–15.

[12] Constantin A. (2000a), *On the Blow-up Solutions of a Periodic Shallow Water Equation*, J. Nonlinear Sci. **10**, 391–399.

[13] Constantin A. (2000b), *Existence of Permanent and Breaking Waves for a Shallow Water Equation: a Geometric Approach,* Ann. Inst. Fourier (Grenoble) **50**, 321–362.

[14] Constantin A. (2006), *The Trajectories of Particles in Stokes Waves*, Invent. Math. **166**, 523–535.

[15] Constantin A. and Escher J. (1998), *Wave Breaking for Nonlinear Nonlocal Shallow Water Equations*, Acta Math. **181**, 229–243.

[16] Constantin A. and Escher J. (2000), *On the Blow-up Rate and the Blow-up Set of Breaking Waves for a Shallow Water Equation*, Math. Z. **233**, 75–91.

[17] Constantin A. and Escher J. (2007), *Particle Trajectories in Solitary Water Waves*, Bull. Amer. Math. Soc. (N.S.) **44**, 423–431.

[18] Constantin A. and Ivanov R. (2008), *On an Integrable Two-component Camassa-Holm Shallow Water System*, Phys. Lett. A **372**, 7129–7132.

[19] Constantin A. and Lannes D. (2009), *The Hydrodynamical Relevance of the Camassa-Holm and Degasperis-Procesi Equations*, Arch. Ration. Mech. Anal. **192**, 165–186.

[20] Cook J.N., Shapiro S.L. and Stephens B.C. (2003), *Magnetic Braking and Viscous Damping of Differential Rotation in Cylindrical Stars,* Astrophys. J. **599**, 1272–1292.

[21] Dąbrowski M.P. (2006), *Future State of the Universe*, Annalen der Physik (Leipzig) **15**, 352–363.

[22] Engelberg S., Liu H.L., and Tadmor E. (2001), *Critical Thresholds in Euler-Poisson Equations*, Indiana Univ. Math. J. **50**, 109–157.

[23] Deng Y.B., Liu T.P., Yang T. and Yao Z.A. (2002), *Solutions of Euler-Poisson Equations for Gaseous Stars*, Arch. Ration. Mech. Anal. **164**, 261–285.

[24] Deng Y.B., Xiang J.L. and Yang T. (2003), *Blowup Phenomena of Solutions to Euler-Poisson Equations*, J. Math. Anal. Appl. **286**, 295–306.

[25] Engelberg S. (1996), *Formation of Singularities in the Euler and Euler-Poisson Equations*, Phys. D **98**, 67–74.

[26] Escher J., Liu Y. and Yin Z. (2006), *Global Weak Solution and Blow-up Structure for the Degasperis-Procesi Equation*, J. Funct. Anal. **241** 457–485.

[27] Escher J., Lechtenfeld O. and Yin Z. (2007), *Well-posedness and Blow-up Phenomena for the 2-component Camassa–Holm Equation*, Discrete Contin. Dyn. Syst. Ser. A **19**, 493–513.

[28] Fliche H.H. and Triay R. (2007), *Euler-Poisson-Newton Approach in Cosmology*, Cosmology and Gravitation, 346–360, AIP Conf. Proc. **910**, Amer. Inst. Phys., Melville, New York.

[29] Gamblin P. (1993), *Solution reguliere a temps petit pour l'equation d'Euler-Poisson. (French) [Small-time Regular Solution for the Euler-Poisson Equation]* Comm. Partial Differential Equations **18**, 731–745.

[30] Gibbon J.D., Moore D.R. and Stuart J.T. (2003), *Exact, Infinite Energy, Blow-up Solutions of the Three-dimensional Euler Equations*, Nonlinearity **16**, 1823–1831.

[31] Glassey R.T. (1996), *The Cauchy Problem in Kinetic Theory*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA.

[32] Goldreich P. and Weber S. (1980), *Homologously Collapsing Stellar Cores*, Astrophys. J. **238**, 991–996.

[33] Gornushkin I.B., Shabanov S.V., Omenetto N., Winefordner J.D. (2006), *Non-isothermal Asymmetric Expansion of Laser Induced Plasmas into Vacuum*, Journal of Applied Physics **100**, 073304.

[34] Guan C.X. and Yin Z.Y. (2010), *Global Existence and Blow-up Phenomena for an Integrable Two-component Camassa–Holm Shallow Water System*, J. Differential Equations **248**, 2003–2014.

[35] Guo Z.G. and Zhou Y. (2010), *On Solutions to a Two-component Generalized Camassa-Holm System*, Stud. Appl. Math. **124**, 307–322.

[36] Hale J. (1977), Theory of Functional Differential Equations, 2nd Edition, Applied Mathematical Sciences **3**, Springer-Verlag, New York-Heidelberg, x+365 pp.

[37] Holden L., Hoppins K., Baxter B. and Fatuzzo M. (2009), *Self-similar Collapse Solutions for Cylindrical Cloud Geometries and Dynamic Equations of State*, Publications of the Astronomical Society of the Pacific **121**, 485–497.

[38] Ivanov R.I. (2006), *Extended Camassa–Holm Hierarchy and Conserved Quantities*, Z. Naturforsch. A **61**, 133–138.

[39] Jang J. (2008), *Nonlinear Instability in Gravitational Euler-Poisson Systems for* $\gamma = 6/5$, Arch. Ration. Mech. Anal. **188**, 265–307.

[40] Jin L. and Guo, Z.G. (2010) *On a Two-component Degasperis-Procesi Shallow Water System*, Nonlinear Anal. Real World Appl. **11**, 4164–4173.

[41] Johnson R.S. (2002), *Camassa-Holm, Korteweg-de Vries and Related Models for Water Waves*, J. Fluid Mech. **455**, 63–82.

[42] Kippenhahn R. and Weigert A. (1990), Stellar Structure and Evolution, Springer-Verlag.

[43] Kwong M.K. and Yuen M.W., *Periodical Self-similar Solutions to the Isothermal Euler-Poisson Equations for Spiral Galaxies*, Submitted.

[44] Kwong M.K. and Yuen M.W., *Pulsating Flows of the 2D Euler-Poisson Equations*, Submitted.

[45] Lakin W.D. and Sanchez D.A. (1982), Topics in Ordinary Differential Equations, Dover Pub. Inc., New York.

[46] Landau L.D. and Lifshitz E.M. (1959), *Fluid Mechanics*. Translated from the Russian by J. B. Sykes and W. H. Reid, Course of Theoretical Physics **6**, London: Pergamon Press.

[47] Li T.H. (2005), *Some Special Solutions of the Multidimensional Euler Equations in* R^N, Comm. Pure Appl. Anal. **4**, 757–762.

[48] Li T.H. and Wang D.H. (2006), *Blowup Phenomena of Solutions to the Euler equations for Compressible Fluid Flow*, J. Differential Equations **221**, 91–101.

[49] Liang Z.L. (2010), *Blowup Phenomena of the Compressible Euler Equations*, J. Math. Anal. Appl. **379**, 506–510.

[50] Lin S.S. (1997), *Stability of Gaseous Stars in Spherically Symmetric Motions*, SIAM J. Math. Analysis **28**, 539–569.

[51] Lions P.L. (1996, 1998), *Mathematical Topics in Fluid Mechanics*, **1**, **2**, Oxford Lecture Series in Mathematics and its Applications, **3**, **10**, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York.

[52] Liu Y. and Yin Z. (2006), *Global Existence and Blow-up Phenomena for the Degasperis-Procesi Equation*, Comm. Math. Phys. **267**, 801–820.

[53] Liu H.L. and Tadmor E. (2004), *Rotation Prevents Finite-time Breakdown*, Phys. D **188**, 262–276.

[54] Luo T. and Smoller J. (2004), *Rotating Fluids with Self-gravitation in Bounded Domains*, Arch. Ration. Mech. Anal. **173**, 345–377.

[55] Luo T. and Smoller J. (2008), *Nonlinear Dynamical Stability of Newtonian Rotating and Non-rotating White Dwarfs and Rotating Supermassive Stars*, Comm. Math. Phys. **284**, 425–457.

[56] Luo T. and Smoller J. (2009), *Existence and Non-linear Stability of Rotating Star Solutions of the Compressible Euler-Poisson Equations*, Arch. Ration. Mech. Anal. **191**, 447–496.

[57] Makino T. (1986), *On a Local Existence Theorem for the Evolution Equation of Gaseous Stars*, Patterns and Waves, 459–479, Stud. Math. Appl. **18**, North-Holland, Amsterdam.

[58] Makino T. (1992), *Blowing up Solutions of the Euler-Poission Equation for the Evolution of the Gaseous Stars*, Transport Theory and Statistical Physics **21**, 615–624.

[59] Makino T. (1993), *Exact Solutions for the Compressible Eluer Equation*, Journal of Osaka Sangyo University Natural Sciences **95**, 21–35.

[60] Makino T. and Perthame B. (1990), *Sur les Solutions a symmetric spherique de lequation d'Euler-Poisson Pour levolution d'etoiles gazeuses, (French) [On Radially Symmetric Solutions of the Euler-Poisson Equation for the Evolution of Gaseous Stars],* Japan J. Appl. Math. **7**, 165–170.

[61] Neukirch T. and Cheung D.L.G. (2001), *A Class of Accelerated Solutions of the Two-dimensional Ideal Magnetodynamic Equations*, Proc. R. Soc. Lond. A **457**, 2547–2566.

[62] Nishida T. (1986), *Equations of Fluid Dynamics-free Surface Problems*, Comm. Pure Appl. Math. **XXXIX**, 221–238.

[63] Ovsiannikov L.V. (1965), Dokl. Akad. Nauk SSSR **111**, 47.

[64] Popowicz Z. (2006), *A Two-component Generalization of the Degasperis-Procesi Equation*, J. Phys. A **39**, 13717.

[65] Senba T. and Suzuki T. (2004), Applied Analysis, Mathematical Methods In Natural Science, Imperial College Press, London.

[66] Sedov L.I. (1953), Dokl. Akad. Nauk SSSR **40**, 753.

[67] Shu F.H., Adams F.C. and Lizano S. (1987), *Star Formation in Molecular Clouds - Observation and Theory*, Annual Review of Astronomy and Astrophysics **25**, 23–81.

[68] Spiegel M.R., Lipschutz S. and Liu J. (2008), Mathematical Handbook of Formulas and Tables, 3nd ed., New York: McGraw-Hill.

[69] Tassoul J.L. (1978), Theory of Rotating Stars, Princeton University Press, Princeton.

[70] Toland J.F. (1996), *Stokes Waves*, Topol. Methods Nonlinear Anal. **7**, 1–48.

[71] Tomisaka K. (2001), Star Formation, Department of Astronomy, Institute Technology Bandung, Indonesia in August 7–9, 13–15, 2001.
http://th.nao.ac.jp/~tomisaka/Lecture_Notes/StarFormation/4.pdf

[72] Walter W. (1998), Ordinary Differential Equations, Translated from the 6th German (1996) Edition by Russell Thompson, Graduate Texts in Mathematics **182** Readings in Mathematics, Springer-Verlag, New York, xii+380 pp.

[73] Whitham G.B. (1974), Linear and Nonlinear Waves, Pure and Applied Mathematics, Wiley-Interscience, New York-London-Sydney.

[74] Wong J.S.W. (1975), *On the Generalized Emden-Fowler Equation*, SIAM Rev. **17**, 339–360.

[75] Yeung L.H. and Yuen M.W. (2009), *Analytical Solutions to the Navier-Stokes Equations with Density-dependent Viscosity and with Pressure*, J. Math. Phys. **50**, 083101, 6pp.

[76] Yeung L.H. and Yuen M.W. (2010), *Some Exact Blowup Solutions to Simple Cosmology Models*, Appl. Math. Sci. (Ruse) **4**, 3317–3326.

[77] Yeung L.H. and Yuen M.W. (2011), *Analytical Solutions to the Navier-Stokes-Poisson Equations with Density-dependent Viscosity and with Pressure*, Proc. Amer. Math. Soc. **139**, 3951-3960.

[78] Yeung L.H. and Yuen M.W. (2012), *Note for "Some Exact Blowup Solutions to the Pressureless Euler Equations in R^N Commun. Nonlinear Sci. Numer. Simul. 16 (2011), 2993-2998*, Commun. in Nonlinear Sci. and Numer. Simul. **17**, 485–487.

[79] Yuen M.W. (2007), *Blowup Solutions for a Class of Fluid Dynamical Equations in R^N*, J. Math. Anal. Appl. **329**, 1064–1079.

[80] Yuen M.W. (2008a), *Analytical Blowup Solutions to the 2-dimensional Isothermal Euler-Poisson Equations of Gaseous Stars*, J. Math. Anal. Appl. **341,** 445–456.

[81] Yuen M.W. (2008b), *Stabilities for Euler-Poisson Equations in Some Special Dimensions*, J. Math. Anal. Appl. **344**, 145–156.

[82] Yuen M.W. (2008c, 2009), *Analyitcal Solutions to the Navier-Stokes Equations*, J. Math. Phys. **49**, 113102, 10pp. With Erratum: Analytical Solutions to the Navier-Stokes Equations [J. Math. Phys. 49, 113102 (2008)], J. Math. Phys. **50**, 019901, 2pp.

[83] Yuen M.W. (2009a), *Analytical Blowup Solutions to the Pressureless Navier-Stokes-Poisson Equations with Density-dependent Viscosity in R^N*, Nonlinearity **22**, 2261–2268.

[84] Yuen M.W. (2009b), *Analytically Periodic Solutions to the 3-dimensional Euler-Poisson Equations of Gaseous Stars with a Negative Cosmological Constant,* Class. Quantum Grav. **26**, 235011, 8pp.

[85] Yuen M.W. (2010, 2011), *Self-similar Blowup Solutions to the 2-component Camassa-Holm Equations,* J. Math. Phys. **51**, 093524, 14pp. With *Erratum: "Self-similar Blowup Solutions to the 2-component Camassa-Holm Equations" [J. Mah. Phys. 51, 093524 (2010)]*, J. Math. Phys. **52**, 079901, 1p.

[86] Yuen M.W. (2011a), *Analytical Blowup Solutions to the* 2*-dimensional Isothermal Euler-Poisson Equations of Gaseous Stars II*, J. Math. Phys. **52**, 073512, 7pp.

[87] Yuen M.W. (2011b), *Blowup for the C^1 Solutions of the Euler-Poisson Equations of Gaseous Stars in R^N*, J. Math. Anal. Appl. **383**, 627–633.

[88] Yuen M.W. (2011c), *Blowup for the Euler and Euler-Poisson Equations with Repulsive Forces*, Nonlinear Anal. TMA **74**, 1465–1470.

[89] Yuen M.W. (2011d), *Cylindrical Blowup Solutions to the Isothermal Euler-Poisson Equations*, Accepted by Discrete Contin. Dyn. Syst. **2011**, Dynamical

Systems, Differential Equations and Applications 8th AIMS Conference, Suppl. 1448–1456. ISBN-13: 978-1-60133-007-9, Springfield.

[90] Yuen M.W. (2011e), *Exact, Rotational, Infinite Energy, Blowup Solutions to the 3-dimensional Euler Equations*, Phys. Lett. A **375**, 3107–3113.

[91] Yuen M.W. (2011f), *Perturbational Blowup Solutions to the 1-dimensional Compressible Euler Equations,* Phys. Lett. A **375**, 3821–3825.

[92] Yuen M.W. (2011g), *Self-similar Blowup Solutions to the 2-component Degasperis-Procesi Shallow Water System,* Commun. Nonlinear Sci. Numer. Simul. **16**, 3463–3469.

[93] Yuen M.W. (2011h), *Some Exact Solutions to the Pressureless Euler Equations in R^N*, Commun. Nonlinear Sci. Numer. Simul. **16**, 2993–2998.

[94] Yuen M.W. (2012), *Perturbational Blowup Solutions to the 2-component Camassa-Holm Equations*, J. Math. Anal. Appl. **390**, 596–602.

[95] Yuen M.W., *Self-similar Solutions with Elliptic Symmetry for the Compressible Euler and Navier-Stokes Equations in R^N*, arXiv:1104.3687, Submitted.

[96] Yuen M.W., *Self-similar Solutions with Elliptic Symmetry for the Density-dependent Navier-Stokes Equations in R^N*, arXiv:1105.0582, Submitted.

[97] Zhou Y. (2004), *Blow up Phenomena for the Integrable Degasperis-Procesi Equation*, Phys. Lett. A **328**, 157–162.