TWO ESSAYS ON MANAGING INVENTORY AND PRODUCTION SYSTEMS WITH POISSON DEMAND

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Two Essays on Managing Inventory and Production Systems with Poisson Demand

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CERTIFICATE OF ORIGINALITY

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_________________________(Signed)

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ABSTRACT

Demand uncertainty usually increases the difficulty in managing inventory and production for system managers. Under a buy-and-sell environment, a desirable inventory replenishment policy, which achieves balanced trade-offs between inventory cost and order processing cost, can achieve a good system performance. Under a make-to-order environment, by contrast, no inventory is kept, and the production decisions play an important role in system control especially when the customers are delay-sensitive.

In this thesis, we study optimization models under a buy-and-sell setting and a make-to-order setting with a Poisson demand process. It comprises two essays. The first essay focuses on the inventory control in a buy-and-sell environment. Specifically, we consider the long-run average profit maximization problem in a single-item inventory system under continuous review. We assume constant replenishment lead time and partial backlogging of unmet demand. The special case with complete backlogging, which is a classical problem, is revisited first. The original objective function is transformed into a new one, namely "effective profit." By exploring the optimal policy under this new objective function, we show that the (s, S) policy is optimal to the original problem and can be obtained by binary search algorithms. Furthermore, this new approach enables us to obtain near closed-form solutions and nice economic interpretations of the optimal reorder point, which are obtained for the first time in the literature. This new method is then extended to the general model with partial backlogging, and a well-performed heuristic is developed to determine the (s, S) policy and the average profit.

The second essay focuses on the production control in a make-to-order environment. The two main features are fixed set-up cost and delay-sensitive customers. The latter is incorporated into the make-to-order system control for the first time. The production system is modeled as an $M/M/1$ queue with N-policy, where N is the waiting-customer-order threshold which triggers the production and is the control variable. Delay-sensitive customers make decisions on staying or leaving according to their expected waiting times, which depend on the information provided. Two information scenarios are considered, depending on whether the queue length of customer orders is observable or not. Customers' equilibrium strategies are obtained, which is a complement to the literature on queueing systems. The average cost function with strategic customers is obtained in closed form, which is either convex or piecewise convex in N when the waiting list of customer orders is not observable, but may not be convex when the waiting list is observable. The impact of strategic customer behavior on the average cost and the value of revealed information is demonstrated via numerical studies. We observe that if the customers are impatient, the system manager should be more cautious in making decisions on the choice of the threshold N and on whether revealing the information on the waiting list of customer orders.

PUBLICATIONS ARISING FROM THE THESIS

Guo, P., Q. Li. 2013. Strategic behavior and social optimization in partially-observable Markovian vacation queues. *Operations Research Letters* **41** 277–284.

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Background of Research

Managing inventory and production under demand uncertainty is a challenging task for any operations manager. Under a buy-and-sell environment, inventory replenishment decisions directly impact the overall performance of the system, where good performance comes from low inventory level, minimum order processing, and high service level. On the other hand, under a make-to-order environment where inventory holding is unnecessary, good control of the production process is important, especially when customers are delay-sensitive, as the production decisions directly affect the waiting time of the customers.

In this thesis, we study optimization models for a buy-and-sell setting and a make-to-order setting. In particular, our study focuses on "threshold policies" for Poisson demand processes. The thesis comprises two essays. The first essay studies a buy-and-sell environment via analyzing an inventory control model with Poisson demand, constant replenishment lead time, and partial backlogging. The focus of this study is to analyze the well-known (s, S) policy and its economic interpretations. The second essay studies a make-to-order environment via analyzing a production system, which is modeled as an $M/M/1$ queue, where the production is triggered when the number of waiting customer orders has accumulated to a certain threshold level.

In the first essay, we focus on a single-item continuous review inventory system with an objective of maximizing the long-run average profit. We assume Poisson demand and constant lead time, and we allow unsatisfied demand to be partially backlogged. We first analyze the special case with complete backlogging. This special case is a classical inventory problem, and various numerical algorithms have been developed to search for optimal inventory policies. We revisit this old problem with a new approach. Specifically, the given average-profit-maximization objective is first transformed into a new objective which we refer to as the "effective profit." The (s, S) policy is then shown to be optimal to the original problem and can be obtained through efficient binary search algorithms. Besides this computational advantage, our approach also generates near closed-form expressions for the optimal reorder point, which can be nicely explained by a marginal cost-benefit analysis. We then extend our analysis to the general partial backlogging case and provide a well-performed heuristic. This study makes the following contributions to the inventory literature:

1. For the case with zero lead time and partial backlogging, we obtain the optimal (s, S) policy in closed form, and provide economic interpretations of the optimal reorder point and the order-up-to level. The optimal reorder point balances the specific revenue rate, which we refer to as the "effective revenue rate," and the inventory cost rate for negative inventory levels, whereas the optimal order-up-to level balances the effective revenue rate and the inventory cost rate for positive inventory levels.

2. For the classical model with constant lead time and compete backlogging, we show that the optimal reorder point can be obtained via a marginal profit analysis. Such a marginal profit analysis has some similarities to the classical newsvendor model, but it takes a more complex form. Specifically, the optimal reorder point balances the effective revenue rate and the inventory cost rate at the end of the lead time period. Furthermore, the optimal reorder point can be expressed in near closed-form expressions. To the best of our knowledge, no such economic interpretation of the (s, S) policy or closed-form expressions

exist in the literature.

3. We develop a simple heuristic for determining the (s, S) policy and the average profit for the general partial backlogging model with constant lead time. We demonstrate that our heuristic is highly effective via a computational study.

In the second essay, we focus on a make-to-order production system with set-up cost and delay-sensitive customers. Due to the existence of the set-up cost, we assume production begins only when the number of waiting customers in the system reaches a threshold N, and once production starts, it will continue to fill customers' orders until all customers are served. We model such a make-to-order system as an $M/M/1$ queue with N-policy. Customers are delay-sensitive, and they decide to wait or leave according to the anticipated waiting time, which depends on the information level provided to the customers by the system. We consider two information scenarios depending on whether the queue length is observable or not. For each information scenario, we analyze the equilibrium strategies of the customers and develop the expected average cost of the production system. Numerical studies are also conducted to compare the system performance measures between the two information scenarios. This study makes the following contributions to the literature:

1. We conduct equilibrium analysis of customers' queueing decision with partial information provided to them. Specifically, when customers only know the server's status, we show that the avoid-the-crowd behavior exists when they see a busy server; that is, a customer's tendency of joining is decreasing with others' tendency. However, when incoming customers see an idle server, their tendency of joining is increasing with others' tendency, thus exhibiting the follow-the-crowd behavior. For the information scenario where the queue length is observable, we show that no matter the server status is observable or not, the equilibrium is the same.

2. Regarding the optimal control of a make-to-order system, our work differs from the literature by allowing customers to make decentralized purchasing

decisions, and we also consider the impact of information on the average cost. The expected average costs are obtained in closed forms. The expected average cost is shown to be either piecewise convex or convex in the threshold when the queue length is unobservable, but it may not be convex when the queue length is observable. We show by numerical tests that if the customers are impatient, the system manager needs to be more cautious in making decisions on the choice of the threshold N and on whether revealing the information on the queue length. Another interesting observation is that it is possible that when N is set to its optimal value, customers exhibit a paradoxical behavior: they stay if the server is idle but leave if the server is busy.

In the following subsections, we review the literature which is relevant to this thesis.

1.2 Literature on Stochastic Inventory Models and the (s, S) policy

There is a rich literature on stochastic inventory models and the (s, S) policy. For a comprehensive review, please refer to Zipkin (2000) and Porteus (2002). In the following, we only emphasize those works that are closely related to our research, and we focus on models with positive replenishment lead time. Note that an (r, Q) policy is equivalent to an (s, S) policy if the inventory position always hits the reorder point exactly, and therefore some of the works that we review involve (r, Q) policies.

We first give an overview on those works with complete backlogging. Those studies can be roughly divided into two types: (i) analyzing or proving the optimality of a policy, and (ii) analyzing the characteristics of a given policy and determining the policy parameters. For the first type, a number of classical papers such as Scarf (1960), Iglehart (1963), and Veinott (1966) study the optimality of the (s, S) policy for inventory models with backlogging and zero replenishment lead time, and their results that are extendible to the positive (constant) lead time case. Beckmann

(1961) shows the optimality of the (s, S) policy for a general demand arrival process with arbitrary intervals between demands and independently distributed demand quantities, where the replenishment lead time is constant, and order decision is made only after a demand occurs. Hordijk and Van der Duyn Schouten (1986) show the optimality of the (s, S) policy for the situation where the demand process comprises a compound Poisson process and a continuous process, the lead time is constant, and the order decision can be made at any time epoch. Song and Zipkin (1993) consider a model where the lead time is stochastic and the demand rate varies with an underlying state-of-the-world variable, and they show that a world-dependent (s, S) policy is optimal. Bensoussan *et al.* (2010) consider a model with a Poisson demand process and constant lead time, and they show that the (s, S) policy is optimal under the constraint that there is at most one order outstanding. For the second type, Sivazlian (1974), Richards (1975), and Sahin (1979) examine the steady-state distribution of the inventory position under an (s, S) policy. Archibald and Silver (1978), Sahin (1982), and Zheng and Federgruen (1991) discuss the determination of the optimal values of s and S for various models with backlogging. Feng and Xiao (2000) examine the model in Federgruen and Zheng (1992), and they develop a more efficient algorithm by introducing an auxiliary function and analyzing the properties of the optimal solution. Federgruen and Zheng (1992) consider an (r, Q) model with Poisson demand and develop a search algorithm for obtaining the optimal r and Q values. Zheng (1992) further analyzes this model and derives optimality conditions for the policy parameters.

Next, we review those works with lost sales. Again, some studies specifically address the optimality of the inventory policy. Karlin and Scarf (1958) study periodic review models with backlogging and lost sales. They develop some fundamental results and provide bounds on the optimal decisions and performance measures, which are further improved by Morton (1969). However, in general, the (s, S) policy need not be optimal for lost-sales inventory models with lead time; see, for example, Hill and Johansen (2006). Recently, Zipkin (2008a,b) studies inventory systems with lost sales and periodic review. Zipkin (2008a) tests some heuristics and shows that none of them are perfect, even though some of them work well for a backlogging system. Zipkin (2008b) studies some structural properties of the system. By using the concept of "L$^\natural$-convexity," new bounds on the optimal policy are developed. There are studies which focus on analyzing the characteristics of a given policy and the determination of the policy parameters. These include the classical work of Hadley and Whitin (1963), who derive the average cost function of an (r, Q) continuous review system with Poisson demand, and some more recent work such as Archibald (1981), Johansen and Thorstenson (1993), Hill and Johansen (2006), etc. For a comprehensive review on lost-sales inventory models, see Bijvank and Vis (2011). A common assumption in the lost-sales models with continuous review is that there is at most one outstanding order, which is also adopted in the first essay.

In contrast to the enormous literature on inventory models with either complete backlogging or complete lost sales, studies on models with partial backlogging are less common. There are a number of studies which focus on deterministic EOQ-type models with partial backlogging; see Taleizadeh *et al.* (2012) and the references therein. For inventory problems with stochastic demand, generally speaking, there are three types of models to describe partial backlogging. The first type simply assumes that a fraction of the unmet demands are backlogged; see, for example, Nahmias (1979), Kim and Park (1985), and Pang (2011). The second type assumes that there exists an upper bound on the backordered quantity, below which the excess demand is backlogged, and above which the excess demand is lost; see, for example, Rabinowitz *et al.* (1995) and Chu *et al.* (2001). The third type considers the partial backlogging case to be characterized by a choice probability associated with customers encountering stockouts as described in the introduction. Moinzadeh (1989) considers such a definition of partial backlogging and develops steady s-tate operating characteristics for an $(S-1, S)$ system which allows multiple orders outstanding during the lead time period. Ding *et al.* (2011) also consider such a definition of partial backlogging and develop an algorithm to determine the optimal rationing policy with multiple classes of customers and zero lead time. To the best of our knowledge, the questions on what the optimal policy is and how to calculate the optimal solution under our partial backlogging setting are not well addressed.

As mentioned above, Ding *et al.* (2011) study an inventory rationing problem with multiple-class demand and partial backlogging. They develop a dynamic program to determine the optimal rationing policy for the zero-lead-time case. Their rationing policy turns to the optimal inventory control policy if demand class reduces to one. Our study focuses on the inventory problem with single-class demand and positive lead time. The single-class demand assumption makes it possible to do theoretical analysis, but the positive lead time assumption makes the analysis more complicated. Our study resembles Ding *et al.*'s work on the aspect that the original average profit function is changed into another form. The new form allows us to obtain nicer expressions (closed-form in some special cases) for optimal decisions for the complete backlogging case and develop an effective algorithm for the general partial backlogging case.

It is worth mentioning that besides the study of the (s, S) and (r, Q) policies, there are also other streams of stochastic inventory research which study the optimality of other inventory policies. One such stream is about the joint inventory replenishment and pricing problems, where an (s, S, p) policy has been shown optimal to some inventory models; see Chen and Simchi-Levi (2012) for a comprehensive review of related literature. Another stream is about serial inventory models, where policies such as (r, nQ), (r, nQ, T), and (s, T) policies are considered; see, for example, Shang and Zhou (2010).

1.3 Literature on Queueing Systems and Optimal Control of Production

There are two related streams of literature closely related with the optimal control of the make-to-order production system with strategic customers.

The first stream is about the optimal control of production systems. An $M/G/1$ queueing modeled production system is presented by Heyman and Sobel (1984, p. 336), and the optimal threshold that triggers production is obtained in closed form. Note that queueing systems are usually used to model the production systems to analyze the steady-state distribution of the system, and that with the presence of set-up cost, the threshold policy and server vacations need to be considered. Thus, the works on the optimal control of queueing systems with threshold policy and server vacations are relevant to our study. Kella (1989, 1990) considers the $M/G/1$ queue with server vacations, in which the condition of the optimal threshold policy and the optimal control of the vacation scheme are obtained. Federgruen and So (1991) prove that the threshold policy is optimal in single server vacation queuing systems. Zhang (2006) shows that the average cost is convex in an $M/G/1$ queue with two-threshold policy vacations. Several searching algorithms have also been developed for determining the optimal control policy for some complicated queueing systems; see, for example, Lee and Srinivasan (1989), Zhang *et al.* (1997), and Ke (2003). The main difference between these works and our work is that we consider strategic customer behavior, whereas the foregoing works assume a stable demand process.

The second stream is about customers' strategic queueing behaviors in service systems. Such study is pioneered by Naor (1969), who shows that admission fee can be used to induce a socially optimal strategy in a fully observable $M/M/1$ queuing system. A similar model with an unobservable queue is studied by Edelson and Hildebrand (1975). For a comprehensive review of queueing models with strategic customers, see Hassin and Haviv (2003). Recently, Guo and Hassin (2011) consider a single-server vacation queueing model with N-policy, and they obtain the equilibrium and optimal strategies for identical customers. Guo and Hassin (2012) extends the study to the case with heterogeneous customers. These two studies consider the no-information and full-information scenarios. Our make-to-order production system is also modeled by a queueing system with N-policy, but we consider non-heterogeneous customers under two partial information scenarios. On the one hand, the equilibrium analysis in our work is a supplementary to that in Guo and Hassin (2011). On the other hand, the cost structure in our production system is more complicated, and we also further analyze the average cost function with respect to the threshold value N. There are also some studies on strategic queueing behaviors in the system where the service rates depend on the system congestion level, and such a system can be regarded as an extension of vacation queue with N-policy, since the service rate in the latter changes between 0 and a higher value. Economou *et* *al.* (2011) and Dimitrakopoulos and Burnetas (2011) study such systems, and both equilibrium strategies and socially optimal strategies are obtained and compared. A major difference between these systems and our system is that our system has a different cost structure, which includes a production set-up cost, a system operating cost, etc. Guo and Zhang (2012) consider a multi-server queueing system with congestion-based staffing policy with the presence of several kinds of costs incurred in switching the staff modes. They conduct an equilibrium analysis with an unobservable queue. Equilibrium analysis is also conducted on an observable queueing system with setup/closedown times by Sun *et al.* (2010), and on a clearing queueing system in alternating environment by Economou and Manou (2011).

PART I

Inventory Control in a Buy-and-Sell Environment

Chapter 2

The (s, S) Policy and the Model

Readers are probably all familiar with the (s, S) inventory policy. Under this policy, the manager places an order to increase the inventory position to S whenever the current inventory level drops below s. Research on this model can be roughly classified into two types. The first type concerns the optimality of this policy for a given setting, whereas the second type concerns how to determine the values of the policy parameters s and S. This policy has been proven to be optimal in many situations. For example, it is optimal to inventory systems with zero lead time under very general settings of demand process and cost parameters, regardless of whether the unmet demand is backlogged or lost (see, e.g., Cheng and Sethi 1999). If there exists a replenishment lead time, the (s, S) policy has also been shown to be optimal to some backlogging inventory systems. However, the optimal decisions have to be searched numerically, and the optimal decisions on s and S have never been expressed in closed forms mainly owing to the complexity of the cost functions. Our work, therefore, focuses on the following challenging issues: Is there an alternative way to determine the optimal (s, S) policy more efficiently? And, if so, is there any intuitive expression for the policy parameters?

In this essay, we will revisit the (s, S) policy by a new method. The setting that we consider is a single-product inventory model with continuous review, infinite horizon, Poisson demands, and constant lead time. A special feature of our model is that we consider partial backlogging. We model partial backlogging by modeling customers' behavior as independent Bernoulli trials: A customer encountering stock-out will choose to stay (i.e., be backlogged) with a probability γ or choose to leave (i.e., be lost) with a probability $1-\gamma$. This setting reflects many real-life situations, particularly in the retail business, where the customers make wait-or-leave decisions randomly and independently when the product is out of stock. Note that the complete backlogging case and the complete lost-sales case are two special cases of our model, with the choice probability γ being 1 and 0, respectively.

Specifically, we assume demand arrives according to a Poisson process with rate λ. Unmet demand is backlogged with probability γ and lost with probability $1-\gamma$, where $0 \leq \gamma \leq 1$. The replenishment lead time is constant. We say that a demand is *realized* if it is satisfied either from on-hand inventory or from a later replenishment. Backlogged demand is realized, whereas lost demand is not. The cost parameters are listed in Table 2.1.

Table 2.1: Notation.

System Parameters:	
λ	Demand rate
L	Replenishment lead time
γ	Probability that an unmet demand is backordered
Cost Parameters:	
K	Fixed set-up cost per order
p	Net profit per unit product
h	Holding cost per unit product per unit time
b	Backorder cost per unit product per unit time
ℓ	Lost sales penalty per unit product
$\tilde{p}=\gamma p-(1-\gamma)\ell$	Expected revenue from an arriving customer encountering stock-out

The replenishment decisions are made immediately after demand arrival epochs. Regarding the replenishment lead time, in order to ensure tractability of the model, we assume that there is at most one order outstanding at any point in time (i.e., the manager cannot place another order if he/she has not received the current order), which is a common assumption in the literature (see, e.g., Fricker and Goodhart 2000 and Pang and Chen 2009). We assume $p - \frac{h}{\lambda} > 0$, such that the largest expected profit of satisfying a unit of demand from on-hand inventory is positive (attained when the inventory level is one). The objective is to determine the optimal inventory policy such that the long-run average profit is maximized.

Because the partial backlogging model is a generalization of the lost-sales/ backlogging case, it therefore poses even more challenges in deriving the performance measures and finding the optimal policies. The difficulty can be explained as follows. When demands arrive according to a Poisson process, the stationary distribution of the inventory position, and, consequently, the expected profit, can be readily derived for the complete backlogging case, because the inventory position always drops by one unit whenever a demand occurs. In a partial backlogging model, however, the inventory position changes in a more complex way. When on-hand inventory is positive, the inventory position drops by one unit when a demand arrives, but when a demand faces a stock-out, the inventory position either drops by one unit owing to the backlogged demand or stays unchanged owing to the lost sales. Consequently, the stationary distribution function of the inventory position in the partial backlogging case takes a more complex form, and the expression for the average profit is even more complex. Therefore, we will fully demonstrate our new method with the special case with complete backlogging, and then we extend the analysis to the general partial backlogging model.

In our analysis, we let $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$ for any real number x, and we use $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ to denote the floor and ceiling functions, respectively.

Chapter 3

Preliminary Results

In this chapter, we first present an alternative formulation which enables us to conduct the analysis more easily, and we then apply the new formulation to solve the zero-lead-time case of our problem.

3.1 An Alternative Formulation

Let ξ be an arbitrary stationary inventory policy. Denote $\Pi(\xi, t)$ as the total profit accumulated during time period $[0, t]$ under inventory policy ξ. The long-run average profit is given by $\lim_{t\to+\infty} \frac{\Pi(\xi,t)}{t}$. According to the renewal theory, the long-run average profit equals the average profit in one order cycle (Ross 1996). Denote T_ξ as the expected length of an order cycle. Denote Π_ξ as the expected profit in the order cycle when the fixed ordering cost is ignored. Then,

$$\lim_{t\to+\infty} \frac{\Pi(\xi,t)}{t} = \frac{\Pi_\xi - K}{T_\xi}.$$

Our problem is to determine ξ so as to maximize this long-run average profit; that is,

$$\mathbf{P}_{\text{orig}}: \quad \max_{\xi} \left\{ \frac{\Pi_\xi - K}{T_\xi} \right\}. \tag{3.1}$$

Unfortunately, this objective function usually does not have nice structures. Typically, a numerical grid search over the policy parameters is needed.

Similar to Ding *et al.* (2011), we reformulate this average-profit-maximization problem as a new maximization problem with the objective function being an accumulative value. Let ϕ^*_{orig} be the optimal objective value to the original problem $\mathbf{P}_{\text{orig}}$. A new problem $\mathbf{P}_{\text{new}}$ is defined as follows:

$$\mathbf{P}_{\text{new}}: \quad \max_{\xi}\{\Pi_\xi - \phi^*_{\text{orig}} T_\xi\}. \tag{3.2}$$

We have the following theorem.

Theorem 1 *The optimal objective value of problem* $\mathbf{P}_{\text{new}}$ *is* K*. Furthermore, the optimal policy for problem* $\mathbf{P}_{\text{new}}$ *is also an optimal policy for problem* $\mathbf{P}_{\text{orig}}$*.*

Proof: Let ξ^*_{orig} be an optimal policy of problem $\mathbf{P}_{\text{orig}}$ and ξ^*_{new} be an optimal policy of problem $\mathbf{P}_{\text{new}}$. On the one hand, the optimal objective value of problem $\mathbf{P}_{\text{orig}}$, ϕ^*_{orig}, is achieved by policy ξ^*_{orig}. Thus,

$$\phi^*_{\text{orig}} = \frac{\Pi_{\xi^*_{\text{orig}}} - K}{T_{\xi^*_{\text{orig}}}},$$

or equivalently,

$$\Pi_{\xi^*_{\text{orig}}} - \phi^*_{\text{orig}} T_{\xi^*_{\text{orig}}} = K.$$

Because ξ^*_{orig} is a feasible policy for $\mathbf{P}_{\text{new}}$, we have $\Pi_{\xi^*_{\text{new}}} - \phi^*_{\text{orig}} T_{\xi^*_{\text{new}}} \geq \Pi_{\xi^*_{\text{orig}}} - \phi^*_{\text{orig}} T_{\xi^*_{\text{orig}}}$. Hence,

$$\Pi_{\xi^*_{\text{new}}} - \phi^*_{\text{orig}} T_{\xi^*_{\text{new}}} \geq K. \tag{3.3}$$

On the other hand, ξ^*_{new} is a feasible policy for $\mathbf{P}_{\text{orig}}$. The objective value satisfies

$$\frac{\Pi_{\xi^*_{\text{new}}} - K}{T_{\xi^*_{\text{new}}}} \leq \phi^*_{\text{orig}},$$

which implies

$$\Pi_{\xi^*_{\text{new}}} - \phi^*_{\text{orig}} T_{\xi^*_{\text{new}}} \leq K. \tag{3.4}$$

Inequalities (3.3) and (3.4) imply that

$$\Pi_{\xi^*_{\text{new}}} - \phi^*_{\text{orig}} T_{\xi^*_{\text{new}}} = K;$$

that is, the optimal objective value of problem $\mathbf{P}_{\text{new}}$ is K. Furthermore,

$$\frac{\Pi_{\xi^*_{\text{new}}} - K}{T_{\xi^*_{\text{new}}}} = \phi^*_{\text{orig}}.$$

Therefore, ξ^*_{new} is also an optimal policy for problem $\mathbf{P}_{\text{orig}}$. ∎

The input parameter ϕ^*_{orig} can be obtained by an iterative algorithm. This iterative algorithm solves a more general problem defined as $\mathbf{P}_\phi$:

$$\mathbf{P}_\phi : \quad \max_\xi \left\{ \Pi_\xi - \phi T_\xi \right\}. \tag{3.5}$$

Problem $\mathbf{P}_\phi$ differs from problem $\mathbf{P}_{\text{new}}$ only in that ϕ^*_{orig} is replaced by ϕ. The term ϕT_ξ can be viewed as the opportunity cost in time period T_ξ with profit rate ϕ. We refer to the term $\Pi_\xi - \phi T_\xi$ as the *effective profit* in this time period. The iterative algorithm is summarized in the following theorem. Its validity follows directly from Theorem 1 in Ding *et al.* (2011).

Theorem 2 *For any given ϕ, denote $\Pi(\phi)$ and $T(\phi)$ as the expected profit and length, respectively, of a cycle if the system is running under the optimal policy for problem $\mathbf{P}_\phi$. Define $J(\phi) = \Pi(\phi) - \phi T(\phi)$. Then, $J(\phi)$ is decreasing, and $J(\phi^*_{\text{orig}}) = K$.*

Proof: Consider any ϕ_1 and ϕ_2 such that $0 \leq \phi_1 < \phi_2$. We have

$$J(\phi_2) = \Pi(\phi_2) - \phi_2 T(\phi_2) < \Pi(\phi_2) - \phi_1 T(\phi_2) \leq \Pi(\phi_1) - \phi_1 T(\phi_1) = J(\phi_1),$$

where the first inequality holds because $\phi_1 < \phi_2$, and the second inequality holds because $\Pi(\phi_1) - \phi_1 T(\phi_1)$ is the optimal objective value of problem $\mathbf{P}_{\phi_1}$. Thus, $J(\phi)$ is decreasing. Furthermore, by Theorem 1, we have $J(\phi^*_{\text{orig}}) = \Pi(\phi^*_{\text{orig}}) - \phi^*_{\text{orig}} T(\phi^*_{\text{orig}}) = K$. ∎

According to this theorem, ϕ^*_{orig} can be obtained easily via a binary search.

Remark 1 *From Theorem 1, for any given profit rate ϕ, the optimal objective value of $\mathbf{P}_\phi$ can be regarded as the fixed set-up cost that the system can tolerate if the maximum profit rate we want to attain is ϕ.*

Remark 2 *Transforming the objective of problem* $\mathbf{P}_{\text{orig}}$ *to the objective function of problem* $\mathbf{P}_{\text{new}}$ *actually follows the basic idea of solving a fractional programming problem by using parametric approaches (see, e.g., Schaible and Ibaraki 1983, pp. 331-332). For our particular inventory problem, some specific properties can be developed. Thus, we introduce Theorems 1 and 2 in detail, and we structure them in a different way such that they can be easily applied in our later analysis.*

Remark 3 *The parametric approaches in fractional programming have been applied to the determination of inventory replenishment policies by Feng and Xiao (2000), in which an algorithm has been developed to calculate the optimal* (s, S) *policy parameters in a single item inventory system with complete backlogging. Lemma 1 there establishes a similar property as that in our Theorem 1, except that our Theorem 1 is more generally applicable. Feng and Xiao's study focuses on the average cost obtained from the limiting behavior of inventory positions in an* (s, S) *policy, whereas our work is on the new form of the objective function. With the new objective function, we do not need to restrict our attention to the average cost/profit in one order cycle. Instead, we can decompose the objective into effective profits in smaller time intervals. Such a nice property allows us to apply marginal analysis in determining the decision variables and to obtain economic interpretations, which are not available in Feng and Xiao (2000) or any other work in the literature.*

3.2 Solving Problem $\mathbf{P}_{\text{orig}}$ for the Zero-Lead-Time Case

In this section, we consider the special case of our model with zero lead time. According to Ding *et al.* (2011), the optimal policy is an (s, S) policy. Hence, we restrict our attention to (s, S) policies and focus on deriving the optimal s and S. Suppose the inventory level at time zero is I. We define an *initial cycle* as the time period from time zero to the time point when the inventory level first drops to s. Let $T_\xi(I)$ be the expected length of the initial cycle and $\Pi_\xi(I)$ be the expected profit accumulated in the initial cycle. Following the form of (3.5), define

$$g(I, \phi) = \max_{\xi} \left\{ \Pi_\xi(I) - \phi T_\xi(I) \right\},$$

which is the maximum effective profit in the initial cycle; see Figure 3.1.

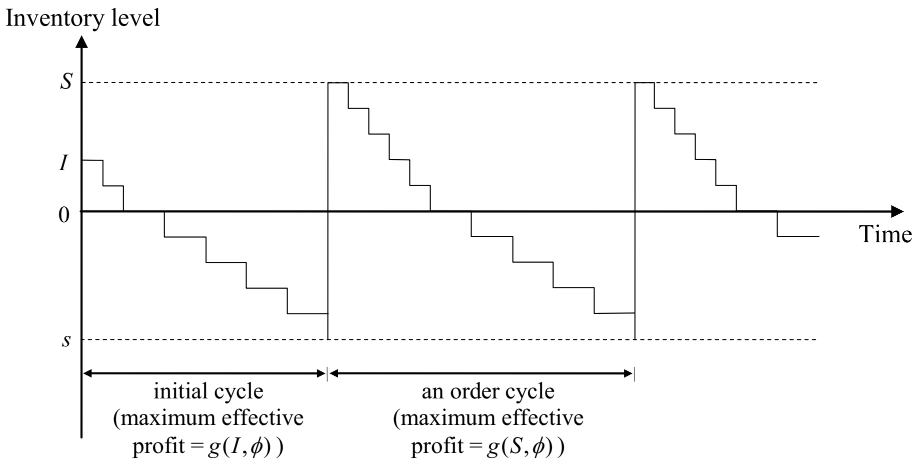


Figure 3.1: The inventory process and function $g(\cdot, \phi)$.

Let $\mathcal{P}(I, \phi)$ be the net benefit from a realized demand at inventory level I, given that the profit rate is ϕ. Then,

$$\mathcal{P}(I, \phi) = \begin{cases} p - \frac{\phi + hI}{\lambda}, & \text{if } I \geq 1; \\ \frac{1}{\gamma}(\tilde{p} - \frac{\phi - bI}{\lambda}), & \text{if } I \leq 0. \end{cases} \tag{3.6}$$

This equation can be explained as follows. If $I \geq 1$, then the revenue brought by a realized demand is p. However, there is an expected cost of $\frac{\phi+hI}{\lambda}$ during the inter-arrival time period $\frac{1}{\lambda}$. Next, consider the case $I \leq 0$, in which an arriving demand could be lost before it becomes a realized demand. The realized demand and lost demand follow Poisson processes with rates $\gamma\lambda$ and $(1-\gamma)\lambda$, respectively. The expected inter-arrival time of realized demands is $\frac{1}{\gamma\lambda}$. Hence, the expected cost during the inter-arrival time period of two realized demands is $\frac{\phi - bI}{\gamma\lambda}$. The expected revenue rate per unit time is $\lambda\tilde{p}$, which implies that the expected revenue rate per realized demand is $\frac{\tilde{p}}{\gamma}$. Therefore, in this case, the expected net benefit is $\frac{1}{\gamma}(\tilde{p} - \frac{\phi - bI}{\lambda})$. Figure 3.2 depicts the function $\mathcal{P}(I, \phi)$. (Note: The domain of function $\mathcal{P}(I, \phi)$ should be discrete. For simplicity, Figure 3.2 treats the domain as continuous. The same simplification applies to some other figures in the thesis.)

Note that the effective profit in the initial cycle is equal to the total effective profit obtained from the realized demands when the inventory level is $I, I-1, \ldots, s+1$

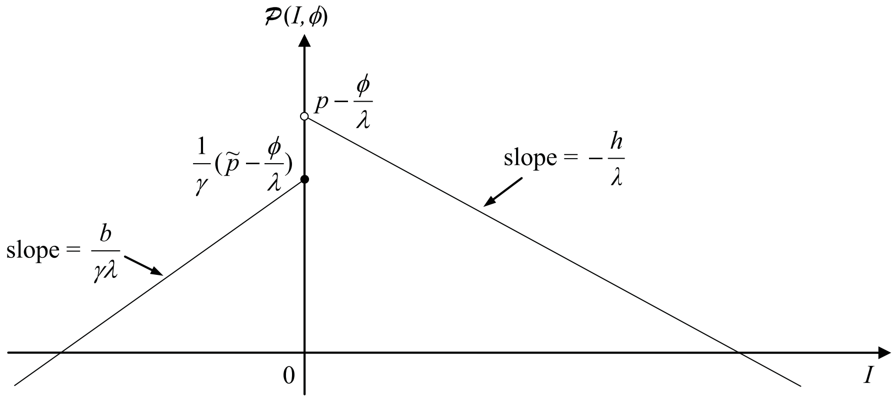


Figure 3.2: Function $\mathcal{P}(I, \phi)$.

(see Figure 3.1). The effective profit in the initial cycle is maximized if the reorder point s is optimally determined. Let $s^*(\phi)$ and $S^*(\phi)$ be the optimal reorder point and order-up-to level, respectively. Then,

$$g(I, \phi) = \sum_{k=s^*(\phi)+1}^{I} \mathcal{P}(k, \phi). \tag{3.7}$$

In this equation, the effective profit in the initial cycle is decomposed into the effective profits in smaller time intervals. At the end of an initial cycle, the inventory level is raised to the order-up-to level S. After the replenishment, the effective profit in the next order cycle is $g(S, \phi)$, which is maximized when $S = S^*(\phi)$. Hence, function $g(I, \phi)$ is maximized when I reaches $S^*(\phi)$, as depicted in Figure 3.3. Because $\mathcal{P}(I, \phi)$ is increasing when $I \leq 0$ and is decreasing when $I > 0$, function $g(I, \phi)$ is convex when $I \leq 0$ and concave when $I > 0$. Thus, $g(I, \phi)$ is quasi-concave in I.

The optimal reorder point and order-up-to level can be obtained as follows. Here, $\mathcal{P}(I, \phi)$ can be regarded as the marginal effective profit at inventory level I. The optimal order cycle includes all the inventory levels which are positively contributed to the total effective profit. If $\tilde{p} \geq \frac{\phi}{\lambda}$, then equation "$\mathcal{P}(I, \phi) = 0$" has two roots (see Figure 3.2). The optimal reorder point $s^*(\phi)$ is the largest integer which is no greater than the smaller root, and the optimal order-up-to level $S^*(\phi)$ is the largest integer which is no greater than the larger root. Thus, the optimal reorder

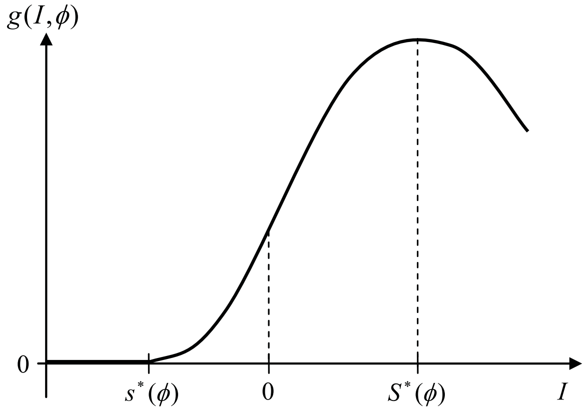


Figure 3.3: Function $g(I, \phi)$.

point and order-up-to level can be obtained by solving the equation "$\mathcal{P}(I, \phi) = 0$." If $\tilde{p} < \frac{\phi}{\lambda}$, then the optimal reorder point $s^*(\phi)$ is zero, and $S^*(\phi)$ can be obtained from the unique root of "$\mathcal{P}(I, \phi) = 0$." The following theorem provides closed-form expressions for these two quantities.

Theorem 3 *Consider any ϕ and γ such that $0 \leq \phi \leq \lambda p$ and $0 \leq \gamma \leq 1$. The optimal reorder point and optimal order-up-to level for problem $\mathbf{P}_\phi$ are*

$$s^*(\phi) = \min\left\{\left\lfloor \frac{\phi - \lambda\tilde{p}}{b} \right\rfloor, 0\right\} \tag{3.8}$$

and

$$S^*(\phi) = \left\lfloor \frac{\lambda p - \phi}{h} \right\rfloor. \tag{3.9}$$

Proof: We first show that the optimal reorder point must be non-positive. Suppose the inventory system is running under an (s, S) policy with $S > s \geq 1$. Consider the $(s-1, S-1)$ policy. Under the two inventory policies, the expected revenue earned and the expected length of an order cycle are the same, whereas the inventory holding cost is always lower by h per unit time under the second policy. Thus, the $(s-1, S-1)$ policy is better than the (s, S) policy. Therefore, the optimal reorder point satisfies $s^*(\phi) \leq 0$.

Note that the optimal reorder point should maximize $g(I, \phi)$ given by (3.7).

Thus, the optimal reorder point $s^*(\phi)$ satisfies

$$\begin{cases} \mathcal{P}(s^*(\phi)+1,\phi) > 0, \\ \mathcal{P}(s^*(\phi),\phi) \le 0. \end{cases} \tag{3.10}$$

Function $\mathcal{P}(k,\phi)$ is linearly increasing in k when $k \le 0$. Hence, if $\mathcal{P}(0,\phi) \le 0$, inequality set (3.10) has no negative solution. In this case, $s^*(\phi) = 0$. If $\mathcal{P}(0,\phi) > 0$, then $s^*(\phi) \le -1$. In this case, both $\mathcal{P}(s^*(\phi),\phi)$ and $\mathcal{P}(s^*(\phi)+1,\phi)$ belong to the second case of (3.6), and inequality set (3.10) implies that

$$\tilde{p} - \frac{\phi - b[s^*(\phi)+1]}{\lambda} > 0 \quad \text{and} \quad \tilde{p} - \frac{\phi - bs^*(\phi)}{\lambda} \le 0.$$

Thus, $\frac{\phi-\lambda\tilde{p}}{b} - 1 < s^*(\phi) \le \frac{\phi-\lambda\tilde{p}}{b}$, or equivalently, $s^*(\phi) = \left\lfloor \frac{\phi-\lambda\tilde{p}}{b} \right\rfloor$. Summarizing, we have $s^*(\phi) = \min\left\{\left\lfloor \frac{\phi-\lambda\tilde{p}}{b} \right\rfloor, 0\right\}$.

The maximizer of $g(I,\phi)$ is the optimal order-up-to level $S^*(\phi)$ (with tie broken by choosing the larger one). Specifically, it satisfies

$$g(S^*(\phi)-1,\phi) \le g(S^*(\phi),\phi) \quad \text{and} \quad g(S^*(\phi),\phi) > g(S^*(\phi)+1,\phi),$$

or equivalently,

$$\mathcal{P}(S^*(\phi),\phi) \ge 0 \quad \text{and} \quad \mathcal{P}(S^*(\phi)+1,\phi) < 0.$$

From these two inequalities, we obtain $\frac{\lambda p-\phi}{h} - 1 < S^*(\phi) \le \frac{\lambda p-\phi}{h}$. Therefore, $S^*(\phi) = \left\lfloor \frac{\lambda p-\phi}{h} \right\rfloor$. ∎

The optimal reorder point and order-up-to level can also be explained by the following marginal benefit analysis. Note that from (3.8), if the optimal reorder point $s^*(\phi)$ is negative, then it satisfies

$$b[-s^*(\phi)-1] < \lambda\tilde{p} - \phi \le b[-s^*(\phi)],$$

where $\lambda\tilde{p}$ is the revenue rate at negative inventory levels, and $b[-s^*(\phi)-1]$ and $b[-s^*(\phi)]$ are the inventory cost rates at inventory levels $s^*(\phi)+1$ and $s^*(\phi)$, respectively. We refer to the revenue rate less the given profit rate ϕ as the *effective revenue rate*. Then, the optimal reorder point balances the effective revenue rate and inventory cost rate for negative inventory levels. Similarly, from (3.9), the optimal order-up-to level $S^*(\phi)$ satisfies

$$hS^*(\phi) \leq \lambda p - \phi < h[S^*(\phi) + 1],$$

where λp is the revenue rate at positive inventory levels. Therefore, the optimal order-up-to level balances the effective revenue rate and inventory cost rate for positive inventory levels.

For any given ϕ, the optimal reorder point and order-up-to level of problem $\mathbf{P}_\phi$ can be calculated instantaneously from (3.8) and (3.9). By Theorem 2, the following binary search algorithm converges to ϕ^*_{orig}.

Algorithm $\mathbf{A}_{0,\gamma}$:

Initialization: Set $\underline{\phi} = 0$, $\bar{\phi} = \lambda p$, and the tolerance level ϵ.

Iteration:

Step 1. Let $\phi = (\underline{\phi} + \bar{\phi})/2$.

Step 2. Calculate $s^*(\phi)$ and $S^*(\phi)$ according to (3.8) and (3.9); determine $\Pi(\phi)$, $T(\phi)$, and $J(\phi)$.

Step 3. If $|J(\phi) - K| < \epsilon$, stop. If $J(\phi) < K$, then set $\bar{\phi} = \phi$, otherwise set $\underline{\phi} = \phi$; go to Step 1.

Optimal Policy: Calculate the optimal policy of problem $\mathbf{P}_\phi$ for the optimal ϕ obtained.

This binary search algorithm can be applied to the general models with positive lead time and partial backlogging. For notational convenience, we let $\mathbf{A}_{x,y}$ denote a binary search algorithm for the model with the lead time x and the backlogging probability y.

Chapter 4

The Complete Backlogging Case

In this chapter, we consider the situation with a constant lead time and complete backlogging, i.e., $\gamma = 1$. We start with an arbitrary inventory policy and then show that, under some mild conditions, the optimal policy is an (s, S) policy. A binary search algorithm is then used for determining the optimal s and S values. In the next chapter, we will consider the general partial backlogging case.

Throughout this chapter and the next chapter, ϕ is a default parameter. Unless otherwise noted, it is omitted for simplicity. For any function $f(x)$, let Δ be the difference operator, i.e., $\Delta f(x) = f(x) - f(x-1)$. For any two-variable function $f(x, y)$, we use Δ to denote the difference operator with respect to the second variable, i.e., $\Delta f(x, y) = f(x, y) - f(x, y-1)$. We also use Δ^2 to denote the second-order difference operator with respect to the second variable if it is applied to any two-variable function.

In Section 4.1, we define some easy-to-solve auxiliary problems with different numbers of ordering opportunities. In Section 4.2, the optimal solutions of the auxiliary problems are presented. In Section 4.3, the binary search algorithm is discussed.

4.1 The Auxiliary Problems

Consider a time period $[t, t_0]$, where t_0 is the time point at which the inventory level drops to s^* (i.e., the optimal reorder point for the zero-lead-time case) and the decision maker decides to finish the planning. For any integer $n \geq 1$, we define the auxiliary problem $\mathbf{AP}_n$ to be the inventory problem which maximizes the effective profit on $[t, t_0]$, with a constraint that the number of orders placed during this period must be exactly n. As n becomes large, the planning horizon of problem $\mathbf{AP}_n$ (i.e., $t_0 - t$) approaches infinity, and problem $\mathbf{AP}_n$ approaches the original problem where the effective profit on $[t, +\infty)$ is maximized.

For the $\mathbf{AP}_n$ problem, the time horizon is divided into time intervals by reordering and order arrival epochs. The index is numbered backwards. For $i = 1, 2, \ldots, n$, let t_i be the time point where the $(n - i + 1)$st order is placed. This order arrives at time $t_i + L$. By the assumption that there is at most one order outstanding at any time, the following inequalities hold:

$$t \leq t_n < t_n + L \leq \cdots \leq t_2 < t_2 + L \leq t_1 < t_1 + L \leq t_0.$$

For $i = 1, 2, \ldots, n$, let s_i and Q_i be the reorder point and order quantity for the $(n - i + 1)$st order (see Figure 4.1(a)). Let $s_0^* = s^*$ denote the inventory level at time t_0.

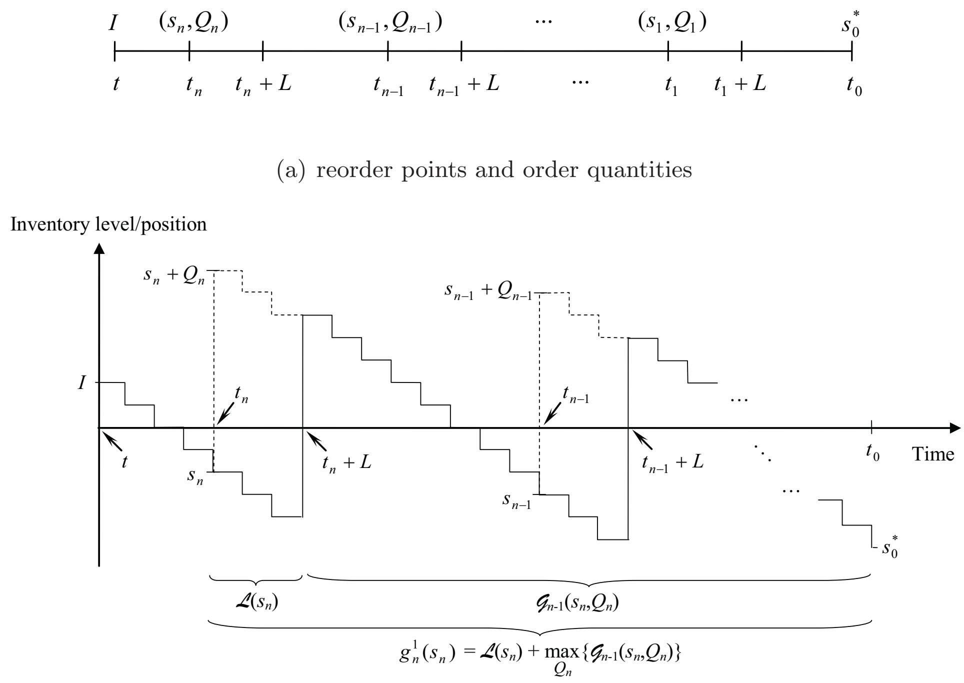
The objective of problem $\mathbf{AP}_n$ is the total effective profit on $[t, t_0]$, which is the sum of the one on the time period $[t, t_n]$ and those on the full order cycles $[t_i, t_{i-1}]$ ($i = 1, 2, \ldots, n$). Suppose the initial inventory level at time t is I. Let ν be an arbitrary (not necessarily stationary) inventory policy. Let $T_\nu(I) = t_0 - t$ be the length of the planning period $[t, t_0]$, and let $\Pi_\nu(I)$ be the total profit obtained in this period. Thus,

$$\mathbf{AP}_n: \quad \max_{\nu} \left\{ \Pi_\nu(I) - \phi T_\nu(I) \right\}.$$

To solve problem $\mathbf{AP}_n$, we need to determine the optimal s_i and Q_i for all $i = 1, 2, \ldots, n$. In the following, we will show that the optimal policy is stationary; that is, the s_i's are the same for $i = 1, 2, \ldots, n$, and so are the Q_i's (except for Q_1).

Let $g_n(I)$ be the optimal objective value of problem $\mathbf{AP}_n$. The initial value $g_0(I)$ is equal to $g(I)$, the maximum effective profit in the initial cycle of the zero-lead-time case studied in Chapter 3. The optimality equation for $g_n(I)$ is

$$g_n(I) = \max\{g_n^0(I), g_n^1(I)\}, \tag{4.1}$$



(a) reorder points and order quantities

(b) inventory level/position

Figure 4.1: Description of problem $\mathbf{AP}_n$.

where $g_n^0(I)$ is the maximum effective profit given that no order is placed at inventory level I, and $g_n^1(I)$ is the maximum effective profit given that an order is placed at inventory level I.

We now derive the formulas for $g_n^0(I)$ and $g_n^1(I)$. First, we consider the case where no order is placed at inventory level I, i.e., $t_n > t$. In this case,

$$g_n^0(I) = \mathcal{P}(I) + g_n(I-1), \tag{4.2}$$

where $\mathcal{P}(I)$ (which is the same as $\mathcal{P}(I, \phi)$ with parameter ϕ omitted) is the expected effective profit of fulfilling the first demand unit arrived at inventory level I, while $g_n(I-1)$ is the maximum effective profit-to-go after serving the first demand unit, which is the optimal objective value of the problem $\mathbf{AP}_n$ with initial inventory level $I-1$. Next, we consider the case where an order is placed at inventory level I, i.e.,

$t_n = t$ and $s_n = I$. In this case,

$$g_n^1(I) = \mathcal{L}(I) + \max_Q \{\mathcal{G}_{n-1}(I, Q)\}, \tag{4.3}$$

where $\mathcal{L}(I)$ is the expected effective profit during the lead time period, while $\mathcal{G}_{n-1}(I, Q)$ is the expected maximum effective profit-to-go in period $[t_n + L, t_0]$ if the order size is Q (see Figure 4.1(b)). In the complete backlogging case, all of the arriving demand will be realized. Let $\mathcal{D}$ be the realized demand during the lead time period. It is a random variable and has the same distribution function as the lead-time demand, which follows a Poisson distribution with mean λL. Then,

$$\mathcal{G}_{n-1}(I, Q) = \mathbb{E}_{\mathcal{D}}\big[g_{n-1}(I + Q - \mathcal{D})\big]. \tag{4.4}$$

Equations (4.1)–(4.4) provide the recursion to solve for $g_n(I)$.

Let $\psi(\cdot)$ and $\Psi(\cdot)$ be the probability mass function (pmf) and cumulative distribution function (cdf), respectively, of the lead-time demand, and let $\bar{\Psi}(\cdot) = 1 - \Psi(\cdot)$. Define $\delta(x) = \sum_{k=0}^{x^+}(x^+ - k)\psi(k)$, which is the expected leftover inventory if we attempt to satisfy the lead-time demand with x units of inventory. Define $\tilde{\delta}(x) = \sum_{k=x^+ + 1}^{+\infty}(k - x^+)\psi(k)$, which is the expected unmet demand incurred in the lead time period if we attempt to satisfy the lead-time demand with x units of inventory. In particular, if $x \leq 0$, then $\delta(x) = 0$ and $\tilde{\delta}(x) = \lambda L$. The expected effective profit in the lead time period can be obtained in closed form, as stated in the next lemma. Lemma 1 is a special case of Lemma 4. Thus, the proof of the former is incorporated into that of the latter.

Lemma 1 *Function $\mathcal{L}(I)$ satisfies*

$$\mathcal{L}(I) = \begin{cases} (\lambda p - \phi)L - (I - \frac{\lambda L}{2})hL - \frac{h+b}{2\lambda}\big[\lambda L\tilde{\delta}(I) - I\tilde{\delta}(I+1)\big], & \text{if } I \geq 1; \\ (\lambda p - \phi)L + (I - \frac{\lambda L}{2})bL, & \text{if } I \leq 0; \end{cases}$$

and

$$\Delta\mathcal{L}(I) = -\frac{I^+ - \delta(I)}{\lambda}h + \frac{\tilde{\delta}(I)}{\lambda}b. \tag{4.5}$$

The marginal effective profit in the lead time period, $\Delta\mathcal{L}(I)$, given by (4.5) can be explained as follows. If an order is placed at inventory level I rather than $I - 1$, one more unit of holding cost and one less unit of backordering cost will be charged in the in-stock and stock-out periods, respectively. From the definitions of $\delta(I)$ and $\tilde{\delta}(I)$, we can view the expected duration of these two periods as $[I^+ - \delta(I)]/\lambda$ and $\tilde{\delta}(I)/\lambda$, respectively.

Lemma 2 *When $I \geq 0$,*

$$\Delta^2 \mathcal{L}(I+1) = -\frac{h+b}{\lambda}\bar{\Psi}(I). \tag{4.6}$$

Furthermore, $\Delta\mathcal{L}(I)$ is convex decreasing on $[0,+\infty)$ and converges to $-hL$ as $I \to +\infty$.

Proof: Recall that $\tilde{\delta}(x-1) - \tilde{\delta}(x) = \bar{\Psi}(x-1)$ if $x \geq 1$. Also, from the definition of $\delta(\cdot)$, it is easy to check that $\delta(x) - \delta(x-1) = \Psi(x-1)$. From (4.5), when $I \geq 0$,

$$\begin{aligned}
&\Delta^2 \mathcal{L}(I+1) = \Delta\mathcal{L}(I+1) - \Delta\mathcal{L}(I) \\
&= -\frac{1-\delta(I+1)+\delta(I)}{\lambda}h + \frac{\tilde{\delta}(I+1)-\tilde{\delta}(I)}{\lambda}b \\
&= -\frac{h+b}{\lambda}\bar{\Psi}(I) < 0.
\end{aligned}$$

Thus, $\Delta\mathcal{L}(I)$ is decreasing. Note also that $\Psi(I)$ is increasing in I over $[0,+\infty)$, which implies that $\Delta^2\mathcal{L}(I+1)$ is increasing in I. Therefore, $\Delta\mathcal{L}(I)$ is convex decreasing in I over $[0,+\infty)$.

Next, we obtain the value of $\Delta\mathcal{L}(I)$ as $I \to +\infty$. Because $\tilde{\delta}(I) = \lambda L\bar{\Psi}(I-1) - I\bar{\Psi}(I)$ and $\delta(I) - \tilde{\delta}(I) = I - \lambda L$ when $I \geq 1$, we have

$$\lim_{I\to+\infty}\tilde{\delta}(I) = \lim_{I\to+\infty}\left[\lambda L\bar{\Psi}(I-1) - I\bar{\Psi}(I)\right] = 0$$

and

$$\lim_{I\to+\infty}\left[I-\delta(I)\right] = \lim_{I\to+\infty}\left[\lambda L - \tilde{\delta}(I)\right] = \lambda L.$$

Therefore,

$$\lim_{I\to+\infty}\Delta\mathcal{L}(I) = \lim_{I\to+\infty}\left[-\frac{I-\delta(I)}{\lambda}h + \frac{\tilde{\delta}(I)}{\lambda}b\right] = -\frac{\lambda L}{\lambda}h + \frac{0}{\lambda}b = -hL. \quad \blacksquare$$

4.2 Solving the Auxiliary Problems

In this section, we determine the optimal policy for the auxiliary problems. We first discuss the optimal policy for problem $\mathbf{AP}_1$. We then extend the result to problem $\mathbf{AP}_n$ for $n = 2, 3, \ldots$.

We first introduce a mild condition on the order quantity. For each order, we require the order size to be sufficiently large such that the inventory position immediately after ordering (i.e., the sum of inventory level immediately before ordering and the order size) is no less than the mean of lead-time demand. Under this condition, the expected inventory level immediately after an order arrival is nonnegative, which is reasonable in practice. Mathematically, given an arbitrary reorder point s, this condition requires the order quantity Q to satisfy

$$s + Q \geq \lambda L.$$

We now consider problem $\mathbf{AP}_1$. The decision problem is to determine s_1 and Q so as to maximize the expected effective profit on $[t, t_0]$. We adopt a sequential approach: First fix s_1 and find the optimal Q; then determine the optimal s_1. According to (4.3),

$$g_1^1(s_1) = \mathcal{L}(s_1) + \max_Q \{\mathcal{G}_0(s_1, Q)\}.$$

Recall that $\mathcal{G}_0(s_1, Q) = \mathbb{E}_{\mathcal{D}}[g_0(s_1 + Q - \mathcal{D})]$ and that $g_0(\cdot)$ is quasi-concave. Although the quasi-concavity is not preserved after taking expectation, function $\mathcal{G}_0(s_1, Q)$ has some nice properties which assure the uniqueness of a local maximizer, as summarized in the following theorem.

Theorem 4 *Under the condition that $s_1 + Q \geq \lambda L$, problem $\mathbf{AP}_1$ and its optimal order size $Q_1^*(\cdot)$ have the following properties:*

(i) $\Delta\mathcal{G}_0(s_1, Q)$ is concave in Q.

(ii) $s_1 + Q_1^*(s_1)$ is independent of s_1, where $Q_1^*(s_1)$ is the local maximizer of $\mathcal{G}_0(s_1, Q)$. (Note: We only consider integer values for $Q_1^*(s_1)$. Thus, if we treat $\mathcal{G}_0(s_1, Q)$ as a continuous function, then even when there is a unique

local maximizer $\hat{Q}$, the two integer points $\lfloor \hat{Q} \rfloor$ and $\lceil \hat{Q} \rceil$ may give the same function value. In such a case, we select $\lfloor \hat{Q} \rfloor$ to be the local maximizer of the function.)

Proof: (i) From (3.7), (4.4), and the definition of $\Delta\mathcal{G}_0(s_1, Q)$, we have

$$\begin{aligned}\Delta\mathcal{G}_0(s_1, Q) &= \mathbb{E}_{\mathcal{D}}\big[g_0(s_1 + Q - \mathcal{D}) - g_0(s_1 + Q - 1 - \mathcal{D})\big] \\ &= \sum_{d=0}^{s_1+Q-s_0^*-1} \mathcal{P}(s_1 + Q - d)\psi(d).\end{aligned}$$

To show that $\Delta\mathcal{G}_0(s_1, Q)$ is concave in Q for those values of Q such that $s_1+Q \geq \lambda L$, it is sufficient to show that $\Delta^2\mathcal{G}_0(s_1, Q+1)$ is decreasing in Q for $Q \geq \lambda L - s_1$. Note that

$$\begin{aligned}\Delta^2\mathcal{G}_0(s_1, Q+1) &= \sum_{d=0}^{s_1+Q-s_0^*-1} \Delta\mathcal{P}(s_1 + Q + 1 - d)\psi(d) + \mathcal{P}(s_0^* + 1)\psi(s_1 + Q - s_0^*) \\ &= -\frac{h}{\lambda}\sum_{d=0}^{s_1+Q}\psi(d) + \frac{b}{\lambda}\sum_{d=s_1+Q+1}^{s_1+Q-s_0^*-1}\psi(d) + \mathcal{P}(s_0^* + 1)\psi(s_1 + Q - s_0^*).\end{aligned}$$

Clearly, the first term in this expression is decreasing in Q. Because $\psi(\cdot)$ is the pmf of a Poisson distribution with mean λL, function $\psi(x)$ is decreasing in x if $x \geq \lambda L$. Note that $s_0^* \leq 0$. Thus, under the condition that $s_1 + Q \geq \lambda L$, both $\sum_{d=s_1+Q+1}^{s_1+Q-s_0^*-1} \psi(d)$ and $\psi(s_1+Q-s_0^*)$ are decreasing in Q. Note also that $\mathcal{P}(s_0^*+1) > 0$. Hence, all three terms in the above expression are decreasing in Q. We conclude that $\Delta^2\mathcal{G}_0(s_1, Q+1)$ is decreasing in Q.

(ii) Since $\Delta\mathcal{G}_0(s_1, Q)$ is concave in Q, for any given s_1, the local maximizer $Q_1^*(s_1)$ is unique. Let X be the value of $s_1 + Q$ that (locally) maximizes $\mathcal{G}_0(s_1, Q)$. From (4.4), $\mathcal{G}_0(s_1, Q)$ is a function of $s_1 + Q$. Thus, $s_1 + Q_1^*(s_1) = X$ for any given s_1. In other words, $s_1 + Q_1^*(s_1)$ is independent of s_1. ∎

From the concavity of $\Delta\mathcal{G}_0(s_1, Q)$, the equation "$\Delta\mathcal{G}_0(s_1, Q) = 0$" has at most two roots if Q is allowed to be any real number; see Figure 4.2. The larger root is the local maximizer of $\mathcal{G}_0(s_1, Q)$, and the smaller root is the local minimizer. We have conducted an extensive numerical study, and the results indicate that the objective

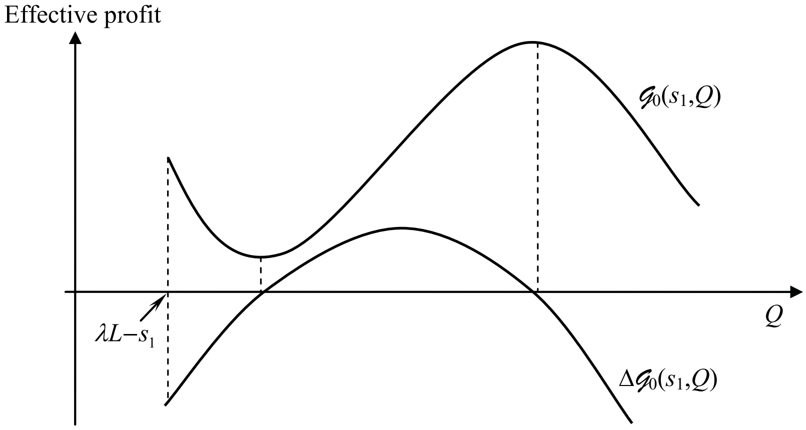


Figure 4.2: Functions $\mathcal{G}_0(s_1, Q)$ and $\Delta\mathcal{G}_0(s_1, Q)$.

value of the local maximizer is usually much larger than the that at the boundary point $\lambda L - s_1$. Thus, we restrict our attention to the local maximizer of $\mathcal{G}_0(s_1, Q)$. Let $\hat{Q}$ be the larger root of the equation "$\Delta\mathcal{G}_0(s_1, Q) = 0$," where

$$\Delta\mathcal{G}_0(s_1, Q) = \sum_{d=0}^{s_1+Q-s_0^*-1} \mathcal{P}(s_1 + Q - d)\psi(d)$$

(see the proof of Theorem 4). The optimal order quantity, $Q_1^*(s_1)$, equals either $\lfloor \hat{Q} \rfloor$ or $\lceil \hat{Q} \rceil$, whichever gives a higher $\mathcal{G}_0(s_1, \cdot)$ value (equals $\lfloor \hat{Q} \rfloor$ when there is a tie).

Property (ii) of Theorem 4 indicates that, for different reorder points, the optimal decisions of the order quantity will increase the inventory position to the same level, which is the optimal order-up-to level of problem $\mathbf{AP}_1$. We denote this optimal order-up-to level as S_1^*. Note that the optimal effective profit-to-go in time period $[t_1 + L, t_0]$ depends on the order-up-to level and is independent of the reorder point. Hence, for different reorder points, the optimal effective profit-to-go are identical. Therefore, the optimal reorder point should be selected in such a way that the effective profit in period $[t, t_1 + L]$ is maximized.

We next present a marginal benefit analysis of the effective profit in period $[t, t_1 + L]$ to derive the optimal reorder point. Consider the ordering decision at the inventory level I. If an order is placed, an effective profit $\mathcal{L}(I)$ is obtained. If no order is placed, then an ordering decision is made at inventory level $I - 1$, and the effective profit is $\mathcal{P}(I) + \mathcal{L}(I - 1)$, where $\mathcal{P}(I)$ is the benefit due to a realized demand. Thus, the optimal reorder point can be determined by comparing the marginal effective profit in the lead time period, i.e., $\Delta\mathcal{L}(\cdot)$, with the effective profit from a realized demand, i.e., $\mathcal{P}(\cdot)$. Figure 4.3 depicts both functions $\Delta\mathcal{L}(\cdot)$ and $\mathcal{P}(\cdot)$. Function $\mathcal{P}(I)$ is piecewise linear in I (see Figure 3.2). Function $\Delta\mathcal{L}(I)$ is equal to bL on $(-\infty, 0]$, convex decreasing on $[0, +\infty)$, and converges to $-hL$ as $I \to +\infty$. The two curves have at most two intersections. The smaller intersection represents the reorder point, because ordering is more beneficial for all the inventory levels below it. It is also possible that, if L is very large, the curve of $\mathcal{P}(\cdot)$ lies below the curve of $\Delta\mathcal{L}(\cdot)$. In such a case, ordering is always better off, and the policy is reduced to be a periodic review policy (i.e., when receiving orders after L time units, place another order immediately). Hence, the optimal reorder point, denoted as s_1^*, satisfies

$$\Delta\mathcal{L}(s_1^*) \geq \mathcal{P}(s_1^*) \quad \text{and} \quad \Delta\mathcal{L}(s_1^*+1) < \mathcal{P}(s_1^*+1). \tag{4.7}$$

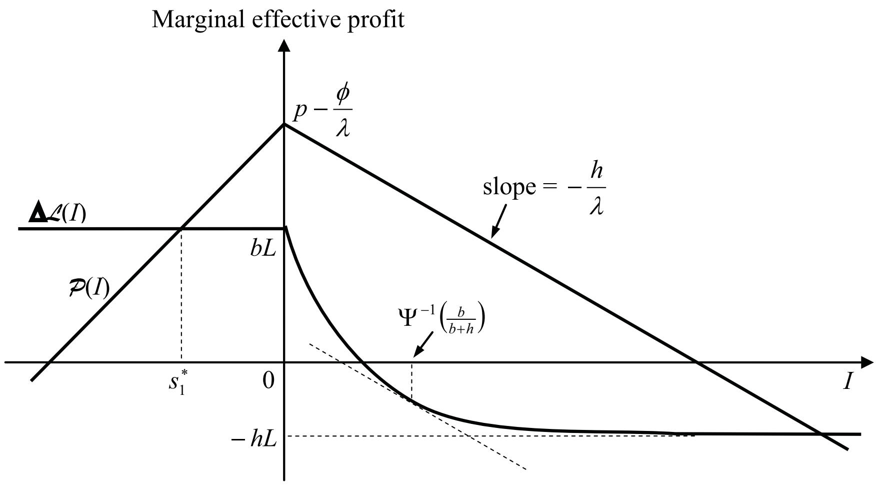


Figure 4.3: Marginal effective profit in the complete backlogging model.

Remark 4 *Figure 4.3 also provides an important observation on the reorder point. Note that when ϕ is sufficiently large, the two curves has one unique tangent point at $\Psi^{-1}(\frac{b}{b+h})$, which can be obtained by solving the equation $\Delta^2\mathcal{L}(I+1) = -\frac{h}{\lambda}$ (and ignoring the integrality requirement of I), where $\Delta^2\mathcal{L}(I+1)$ represents the slope of*

$\Delta\mathcal{L}(\cdot)$ between points I and $I+1$. The value $\Psi^{-1}(\frac{b}{b+h})$ can be regarded as the optimal order size of a newsvendor problem facing a lead-time demand. Since the reorder point always lies on the left hand side of the tangent point no matter what the value of ϕ is, this "newsvendor order size" is an upper bound on the optimal reorder point. In other words, it suggests that under no circumstances an order should be placed if the current inventory level exceeds this "newsvendor order size."

Based on inequalities (4.7), we can derive a simple condition/expression for the optimal reorder point. Suppose an order is placed at inventory level I, where I is an integer. Recall that $\mathcal{D}$ is the realized demand during the lead time period. Let

$$\mathcal{C}(I) = h\mathbb{E}_{\mathcal{D}}[(I-\mathcal{D})^+] + b\mathbb{E}_{\mathcal{D}}[(I-\mathcal{D})^-]$$

be the expected inventory cost rate at the end of the lead time period, where $\mathbb{E}_{\mathcal{D}}[(I-\mathcal{D})^+]$ and $\mathbb{E}_{\mathcal{D}}[(I-\mathcal{D})^-]$ are the expected leftover inventory level and expected backorder level, respectively, at the end of the lead time period. From the definitions of $\delta(x)$ and $\tilde{\delta}(x)$, it can be shown that

$$\mathcal{C}(I) = h\delta(I) + b[\tilde{\delta}(I) + I^-],$$

where I^- is the amount of backorder, if any, at the beginning of the lead time period. In particular, if $I \leq 0$, then $\mathcal{C}(I) = b(\lambda L - I)$. The optimal reorder point can be expressed in terms of $\mathcal{C}(\cdot)$, as stated in the next theorem.

Theorem 5 *For problem $\mathbf{AP}_1$, the optimal reorder point s_1^* satisfies*

$$\mathcal{C}(s_1^*+1) < \lambda p - \phi \leq \mathcal{C}(s_1^*). \tag{4.8}$$

In particular, if $\lambda L + \frac{\phi-\lambda p}{b} < 0$, then s_1^* can be expressed in the following closed form:

$$s_1^* = \left\lfloor \lambda L + \frac{\phi - \lambda p}{b} \right\rfloor \leq -1. \tag{4.9}$$

Proof: Because $\gamma = 1$, by (3.6),

$$\mathcal{P}(I) = \begin{cases} p - \frac{\phi+hI}{\lambda}, & \text{if } I \geq 1; \\ p - \frac{\phi-bI}{\lambda}, & \text{if } I \leq 0. \end{cases} \tag{4.10}$$

We consider two different cases.

Case 1: $p - \frac{\phi}{\lambda} > bL$ (i.e., $\lambda L + \frac{\phi - \lambda p}{b} < 0$). In this case, as shown in Figure 4.3, the optimal reorder point s_1^* is negative. Using (4.5) and (4.10), inequalities (4.7) can be rewritten as

$$b[\lambda L - (s_1^* + 1)] < \lambda p - \phi \leq b(\lambda L - s_1^*), \tag{4.11}$$

or equivalently,

$$\lambda L + \frac{\phi - \lambda p}{b} - 1 < s_1^* \leq \lambda L + \frac{\phi - \lambda p}{b}.$$

More specifically, we have $s_1^* = \left\lfloor \lambda L + \frac{\phi - \lambda p}{b} \right\rfloor \leq -1$. Because $\mathcal{C}(x) = b(\lambda L - x)$ when $x \leq 0$, inequalities (4.11) can be rewritten as $\mathcal{C}(s_1^* + 1) < \lambda p - \phi \leq \mathcal{C}(s_1^*)$.

Case 2: $p - \frac{\phi}{\lambda} \leq bL$. In this case, the optimal reorder point s_1^* is nonnegative (see Figure 4.3, where s_1^* becomes positive if the curve of $\Delta\mathcal{L}(I)$ is raised above the point $p - \frac{\phi}{\lambda}$). Using (4.5) and (4.10), inequalities (4.7) can be rewritten as

$$h\delta(s_1^* + 1) + b\tilde{\delta}(s_1^* + 1) < \lambda p - \phi \leq h\delta(s_1^*) + b\tilde{\delta}(s_1^*),$$

or equivalently, $\mathcal{C}(s_1^* + 1) < \lambda p - \phi \leq \mathcal{C}(s_1^*)$.

Summarizing the above, s_1^* satisfies (4.8), and it satisfies (4.9) if $\lambda L + \frac{\phi - \lambda p}{b} < 0$.

∎

In (4.8), $\lambda p - \phi$ is the effective revenue rate, and $\mathcal{C}(\cdot)$ is the expected inventory cost rate at the end of the lead time period. Thus, in this positive lead time case, the optimal reorder point s_1^* balances the expected inventory cost rate and the expected effective revenue rate.

Once the optimal reorder point s_1^* is determined, the optimal objective value of problem $\mathbf{AP}_1$ with any initial inventory level I, i.e., $g_1(I)$, can be specifically formulated by the recursive equations (4.2) and (4.3). We have

$$g_1(I) = \begin{cases} \sum_{k=s_1^*+1}^{I} \mathcal{P}(k) + g_1(s_1^*), & \text{if } I > s_1^*; \\ \mathcal{L}(I) + \mathcal{G}_0(I, S_1^* - I), & \text{if } I \leq s_1^*. \end{cases}$$

Based on $g_1(\cdot)$, we can then solve problem $\mathbf{AP}_2$, following the same approach as solving $\mathbf{AP}_1$. The above approach can be generalized into solving problem $\mathbf{AP}_n$ from $g_{n-1}(\cdot)$. The results are summarized in the next theorem.

Theorem 6 *For $n \geq 2$, under the condition that $s_i + Q \geq \lambda L$ ($i = 1, 2, \ldots, n$), problem $\mathbf{AP}_n$ and its optimal order size $Q_n^*(\cdot)$ have the following properties:*

(i) Either $\Delta\mathcal{G}_{n-1}(s_n, Q)$ or $\mathcal{G}_{n-1}(s_n, Q)$ (or both) is concave in Q.

(ii) $s_n + Q_n^*(s_n)$ is independent of s_n, where $Q_n^*(s_n)$ is the local maximizer of $\mathcal{G}_{n-1}(s_n, \cdot)$ (see Theorem 4 for tie-breaking rule).

(iii) $s_n^* = s_1^*$, which is determined by (4.8).

Proof: We prove properties (i)–(iii) by induction. Clearly, property (iii) is valid when $n = 1$. By Theorem 4, properties (i) and (ii) are also valid when $n = 1$. Consider any $n \geq 2$. Suppose that for $i = 1, 2, \ldots, n-1$, the optimal policy of problem $\mathbf{AP}_i$ satisfies the following:

(a) Either $\Delta\mathcal{G}_{i-1}(s_i, Q)$ or $\mathcal{G}_{i-1}(s_i, Q)$ (or both) is concave in Q;

(b) $s_i + Q_i^*(s_i)$ is independent of s_i;

(c) $s_i^* = s_1^*$.

We will show that properties (a), (b), and (c) also hold for $i = n$.

Given the optimal reorder point s_{n-1}^*, by (4.2) and (4.3), the optimal objective value of problem $\mathbf{AP}_{n-1}$ is given as

$$g_{n-1}(I) = \begin{cases} \sum_{k=s_{n-1}^*+1}^{I} \mathcal{P}(k) + g_{n-1}(s_{n-1}^*), & \text{if } I > s_{n-1}^*; \\ \mathcal{L}(I) + \mathcal{G}_{n-2}(I, Q_{n-1}^*(I)), & \text{if } I \leq s_{n-1}^*. \end{cases} \tag{4.12}$$

Note that according to (b), $I + Q_{n-1}^*(I)$ is independent of I. From (4.4), $\mathcal{G}_{n-2}(I, Q_{n-1}^*(I))$ is a function of $I + Q_{n-1}^*(I)$. Thus, in the second case of (4.12), the term "$\mathcal{G}_{n-2}(I, Q_{n-1}^*(I))$" is independent of I. Hence, for $I \leq s_{n-1}^*$, we have

$$\Delta g_{n-1}(I) = \Delta\mathcal{L}(I).$$

Next, consider the first case of (4.12). We have

$$\Delta g_{n-1}(I) = \mathcal{P}(I)$$

for $I \geq s_{n-1}^* + 1$. Using these two equations and equation (4.4), we obtain

$$\begin{aligned}
\Delta \mathcal{G}_{n-1}(s_n, Q) &= \mathbb{E}_{\mathcal{D}}[\Delta g_{n-1}(s_n + Q - \mathcal{D})] = \sum_{d=0}^{+\infty} \Delta g_{n-1}(s_n + Q - d)\psi(d) \\
&= \sum_{d=0}^{s_n+Q-s_{n-1}^*-1} \mathcal{P}(s_n + Q - d)\psi(d) + \sum_{d=s_n+Q-s_{n-1}^*}^{+\infty} \Delta\mathcal{L}(s_n + Q - d)\psi(d).
\end{aligned} \tag{4.13}$$

Denote $\tilde{Q} = s_n + Q - s_{n-1}^*$. We have

$$\begin{aligned}
&\Delta^2 \mathcal{G}_{n-1}(s_n, Q+1) \\
&= \sum_{d=0}^{\tilde{Q}-1} \Delta\mathcal{P}(s_n + Q + 1 - d)\psi(d) + \left[\mathcal{P}(s_{n-1}^* + 1) - \Delta\mathcal{L}(s_{n-1}^*)\right] \psi(\tilde{Q}) \\
&\quad + \sum_{d=\tilde{Q}+1}^{+\infty} \Delta^2\mathcal{L}(s_n + Q + 1 - d)\psi(d).
\end{aligned} \tag{4.14}$$

Consider those values of Q such that $s_n + Q \geq \lambda L$. We consider two different cases. Case 1: $s_{n-1}^* < 0$. In this case, we prove that $\Delta\mathcal{G}_{n-1}(s_n, Q)$ is concave in Q. It suffices to show that $\Delta^2\mathcal{G}_{n-1}(s_n, Q+1)$ is decreasing in Q for $Q \geq \lambda L - s_n$. When $s_{n-1}^* < 0$, we have $s_n + Q + 1 - d < 0$ for all $d \geq \tilde{Q} + 1$. From (4.5), $\Delta\mathcal{L}(I) = bL$ for $I \leq 0$, which implies that $\Delta^2\mathcal{L}(s_n + Q + 1 - d) = 0$ for all $d \geq \tilde{Q} + 1$. Hence, in this case, equation (4.14) can be rewritten as

$$\begin{aligned}
\Delta^2\mathcal{G}_{n-1}(s_n, Q+1) = &-\frac{h}{\lambda} \sum_{d=0}^{s_n+Q} \psi(d) + \frac{b}{\lambda} \sum_{d=s_n+Q+1}^{\tilde{Q}-1} \psi(d) \\
&+ \left[\mathcal{P}(s_{n-1}^* + 1) - \Delta\mathcal{L}(s_{n-1}^*)\right] \psi(\tilde{Q})
\end{aligned} \tag{4.15}$$

(as $\Delta\mathcal{P}(I) = -\frac{h}{\lambda}$ when $I \geq 1$ and $\Delta\mathcal{P}(I) = \frac{b}{\lambda}$ when $I \leq 0$; see Figure 3.2). The first term on the right hand side of (4.15) is decreasing in Q. Function $\psi(x)$ is decreasing in x if $x \geq \lambda L$. Thus, under the condition that $s_n + Q \geq \lambda L$, both $\sum_{d=s_n+Q+1}^{\tilde{Q}-1} \psi(d)$ and $\psi(\tilde{Q})$ are decreasing in Q. Note that

$$\begin{aligned}
\mathcal{P}(s_{n-1}^* + 1) - \Delta\mathcal{L}(s_{n-1}^*) &= \mathcal{P}(s_{n-1}^* + 1) - \Delta\mathcal{L}(s_{n-1}^* + 1) \\
&= \mathcal{P}(s_1^* + 1) - \Delta\mathcal{L}(s_1^* + 1) > 0,
\end{aligned}$$

where the inequality follows from (4.7). Hence, all three items on the right hand side of (4.15) are decreasing in Q. Therefore, $\Delta^2\mathcal{G}_{n-1}(s_n, Q+1)$ is decreasing in Q.

Case 2: $s^*_{n-1} \geq 0$. By Lemma 1, $\Delta\mathcal{L}(I)$ is decreasing when $I \geq 1$. From (4.5), $\Delta\mathcal{L}(I) = bL$ for $I \leq 0$. Moreover,

$$\begin{aligned}\Delta\mathcal{L}(1) &= -\frac{1-\delta(1)}{\lambda}h + \frac{\tilde{\delta}(1)}{\lambda}b \\ &= -\frac{1-\delta(1)}{\lambda}h - \frac{\lambda L - \tilde{\delta}(1)}{\lambda}b + bL < bL.\end{aligned}$$

Thus, $\Delta\mathcal{L}(I)$ is non-increasing for any I. Hence, $\Delta^2\mathcal{L}(s_n + Q + 1 - d) \leq 0$ for any d. Therefore, in this case, equation (4.14) implies that

$$\begin{aligned}&\Delta^2\mathcal{G}_{n-1}(s_n, Q{+}1) \\ &\leq \sum_{d=0}^{\tilde{Q}-1} \Delta\mathcal{P}(s_n + Q + 1 - d)\psi(d) + \left[\mathcal{P}(s^*_{n-1}+1) - \Delta\mathcal{L}(s^*_{n-1})\right]\psi(\tilde{Q}) \\ &= -\frac{h}{\lambda}\sum_{d=0}^{\tilde{Q}-1}\psi(d) - \left[\Delta\mathcal{L}(s^*_{n-1}) - \mathcal{P}(s^*_{n-1}+1)\right]\psi(\tilde{Q}).\end{aligned}$$

By (3.6), (4.7), and the induction assumption that $s^*_{n-1} = s^*_1$, we have

$$\Delta\mathcal{L}(s^*_{n-1}) - \mathcal{P}(s^*_{n-1}+1) = \Delta\mathcal{L}(s^*_1) - \mathcal{P}(s^*_1+1) \geq \mathcal{P}(s^*_1) - \mathcal{P}(s^*_1+1) = \frac{h}{\lambda} > 0.$$

This implies that $\Delta^2\mathcal{G}_{n-1}(s_n, Q) < 0$. Hence, $\mathcal{G}_{n-1}(s_n, Q)$ is concave in Q.

Summarizing Cases 1 and 2, we conclude that property (a) holds for $i = n$. Thus, the local maximizer $Q^*_n(s_n)$ is unique (see Theorem 4 for tie-breaking rule). Let X be the value of $s_n + Q$ that (locally) maximizes $\mathcal{G}_{n-1}(s_n, Q)$. From (4.4), $\mathcal{G}_{n-1}(s_n, Q)$ is a function of $s_n + Q$. Thus, $s_n + Q^*_n(s_n) = X$ for any given s_n. In other words, $s_n + Q^*_n(s_n)$ is independent of s_n; that is, property (b) holds for $i = n$.

The optimal reorder point s^*_n satisfies

$$g^1_n(s^*_n) \geq g^0_n(s^*_n) \quad \text{and} \quad g^1_n(s^*_n+1) < g^0_n(s^*_n+1); \tag{4.16}$$

that is, ordering at s^*_n should be no worse than not ordering, whereas not ordering at $s^*_n + 1$ should be better than ordering. Consider the first inequality in (4.16).

From (4.1), (4.2), and (4.3), we have

$$\begin{aligned} g_n^0(s_n^*) &= \mathcal{P}(s_n^*) + g_n(s_n^* - 1) \geq \mathcal{P}(s_n^*) + g_n^1(s_n^* - 1) \\ &= \mathcal{P}(s_n^*) + \mathcal{L}(s_n^* - 1) + \max_Q\{\mathcal{G}_{n-1}(s_n^* - 1, Q)\}. \end{aligned} \tag{4.17}$$

Note that we have already shown that property (b) holds for $i = n$; that is, $s_n^* + Q_n^*(s_n^*) = s_n^* - 1 + Q_n^*(s_n^* - 1)$. This, together with (4.4), implies that $\max_Q\{\mathcal{G}_{n-1}(s_n^*, Q)\} = \max_Q\{\mathcal{G}_{n-1}(s_n^* - 1, Q)\}$. This, together with (4.3), implies that

$$g_n^1(s_n^*) = \mathcal{L}(s_n^*) + \max_Q\{\mathcal{G}_{n-1}(s_n^* - 1, Q)\}. \tag{4.18}$$

From (4.17), (4.18), and the first inequality in (4.16), we have $\Delta\mathcal{L}(s_n^*) \geq \mathcal{P}(s_n^*)$. Similarly, from the second inequality in (4.16), together with the fact that $g_n(s_n^*) = g_n^1(s_n^*)$, we can show that $\Delta\mathcal{L}(s_n^* + 1) < \mathcal{P}(s_n^* + 1)$. Therefore, the optimal reorder point s_n^* satisfies

$$\Delta\mathcal{L}(s_n^*) \geq \mathcal{P}(s_n^*) \quad \text{and} \quad \Delta\mathcal{L}(s_n^* + 1) < \mathcal{P}(s_n^* + 1).$$

This condition is identical to that on s_1^* given by (4.7). This implies that $s_n^* = s_1^*$. Hence, property (c) also holds for $i = n$. ∎

From property (ii) of Theorem 6, the optimal order quantity $Q_n^*(s_n^*)$ maximizes the effective profit in time period $[t_n + L, t_0]$, see Figure 4.1(b). Note that the maximum effective profit in time period $[t_{n-1}, t_0]$ can be attained once the values of s_i and Q_i, $1 \leq i \leq n-1$, are optimally determined. Thus, $Q_n^*(s_n^*)$ actually maximizes the effective profit in time period $[t_n + L, t_{n-1}]$; that is, the period between the order arrival and the next order placement.

Properties (ii) and (iii) of Theorem 6 imply that the optimal replenishment policy is a stationary (s, S) policy. Regarding the optimal reorder point, it can be determined from a neat form (i.e., (4.8)) with intuitive economic interpretation; that is, the optimal reorder point balances the effective revenue rate (i.e., $\lambda p - \phi$) and the expected inventory cost rate at the end of the lead time period (i.e., $\mathcal{C}(\cdot)$). This economic interpretation of the optimal reorder point has some similarities to that of the classical newsvendor model. To the best of our knowledge, such an economic interpretation is obtained for the (s, S) inventory system for the first time. It can be obtained mainly due to the fact that we introduce the concept of the "effective profit," which is further taken as the new objective function. Regarding the order quantity, once the replenishment policy is stationary, function $\Delta\mathcal{G}_{n-1}(s_n^*, Q)$ can be simplified to

$$\Delta\mathcal{G}_{n-1}(s_n^*, Q) = \sum_{d=0}^{Q-1} \mathcal{P}(s_n^* + Q - d)\psi(d) + \sum_{d=Q}^{+\infty} \Delta\mathcal{L}(s_n^* + Q - d)\psi(d)$$

(see (4.13) in the proof of Theorem 6 and by the fact that $s_n^* = s_{n-1}^*$). With this expression, the optimal order size can be obtained by solving the following inequalities:

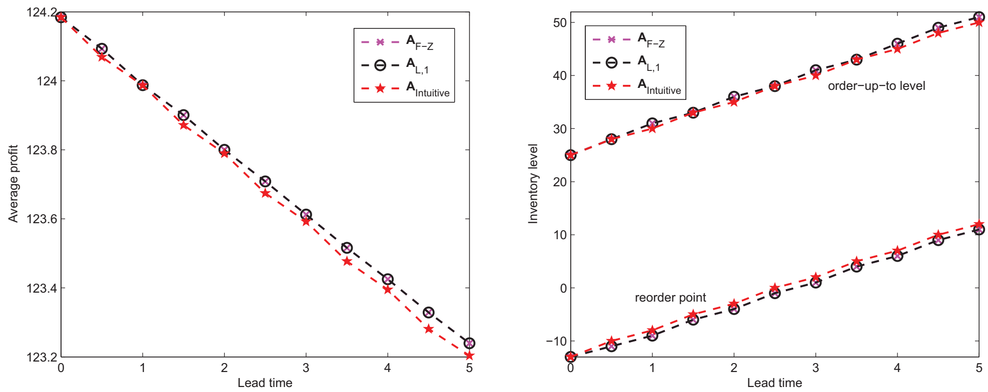
$$\Delta\mathcal{G}_{n-1}(s_n^*, Q) \geq 0 \quad \text{and} \quad \Delta\mathcal{G}_{n-1}(s_n^*, Q+1) < 0. \tag{4.19}$$

4.3 Determining the Optimal Policy

Theorem 6 implies that for a given ϕ, the optimal policy is a stationary (s, S) policy. Specifically, for any given profit rate ϕ, we can determine the reorder point from condition (4.8) and the order quantity from condition (4.19). Then, according to Theorem 2, the optimal ϕ can be obtained through a binary search algorithm $\mathbf{A}_{L,1}$, which is similar to algorithm $\mathbf{A}_{0,\gamma}$ presented in Section 3.2. Note that in algorithm $\mathbf{A}_{L,1}$, if λp is taken as an upper bound of the average profit rate ϕ, it is possible that ϕ is too large such that condition (4.8) has no solution. In such a case, we can simply let the upper bound $\Psi^{-1}(\frac{b}{b+h})$ (see Remark 4) be the reorder point. This would not affect the determination of the final optimal profit rate.

We have conducted numerical tests on the binary-search algorithm $\mathbf{A}_{L,1}$ by comparing its performance with that of an algorithm developed by Federgruen and Zheng (1992), which we denote as $\mathbf{A}_{\text{F-Z}}$. The model considered in Federgruen and Zheng (1992) is the same as the complete backlogging case of our model. Algorithm $\mathbf{A}_{\text{F-Z}}$ determines the optimal (r, Q) policy. It considers the objective function in the form of (3.1) for a given (r, Q) policy, which is a sum of some unimodal functions. It enumerates the order quantities until the criterion of determining the optimal reorder point is achieved.

In our numerical tests, we set the cost parameters to $p = 15$, $h = 0.5$, $b = 2$, $\lambda = 2$, and $K = 40$, and we vary the lead time from 0 to 5. The results obtained from the two methods are depicted in Figure 4.4. The results from our algorithm $\mathbf{A}_{L,1}$ fully coincide with those obtained from $\mathbf{A}_{\text{F-Z}}$, indicating that the restriction "$s + Q \geq \lambda L$" on the order quantity is a mild condition. We have also conducted extensive numerical studies with other parameter settings and did not find any instance in which $\mathbf{A}_{L,1}$ and $\mathbf{A}_{\text{F-Z}}$ generate different results.



(a) average profit versus lead time

(b) inventory level versus lead time

Figure 4.4: Comparison of algorithms $\mathbf{A}_{L,1}$, $\mathbf{A}_{\text{F-Z}}$, and $\mathbf{A}_{\text{Intuitive}}$.

Figure 4.4 also depicts the results generated by another heuristic, which we denote as $\mathbf{A}_{\text{Intuitive}}$. This heuristic applies the optimal policy of the zero-lead-time model but adjusts the reorder point and order-up-to level by the expected lead-time demand. Specifically, $s_0^* + \lambda L$ is taken as the reorder point, and $S_0^* + \lambda L$ is taken as the order-up-to level, where (s_0^*, S_0^*) is the optimal policy for the zero-lead-time system (with complete backlogging). The closed forms of s_0^* and S_0^* are given in Theorem 3 with $\gamma = 1$ (i.e., $\tilde{p} = p$).

We present this heuristic because of the interesting observation from the closed-form reorder points of the zero-lead-time model and the positive-lead-time model (i.e., (3.8) and (4.9)). Note that in a complete backlogging inventory system, no demand will be lost. Thus, it is intuitive to believe that we can apply the replenishment decision of a zero-lead-time model to a positive-lead-time model but make the ordering in advance by L time units. As the lead-time demand is stochastic, it seems that we can increase the reorder point and order-up-to level by the expected lead-time demand as an approximation. This seems to be consistent with the reorder points given in closed forms; that is, for any given profit rate ϕ (and ignoring the integrality requirement of the reorder point), the reorder point (4.9) is greater than the reorder point (3.8) by λL units, which is the expected lead-time demand. However, the optimal average profit should decrease as the lead time increases (see Figure 4.4(a)). Thus, the optimal value of ϕ in (4.9) should be smaller than that in (3.8). Therefore, the difference of the optimal reorder points given by (3.8) and (4.9) should be less than λL. This can be observed from Figure 4.4(b); that is, the non-positive reorder points of $\mathbf{A}_{L,0}$ are smaller than those of $\mathbf{A}_{\text{Intuitive}}$. From this figure, we can also observe that the relationship between the two reorder points holds even when the reorder points are positive. In addition, the order-up-to levels of $\mathbf{A}_{L,0}$ are larger than those of $\mathbf{A}_{\text{Intuitive}}$.

Summarizing the above observations, we conclude that $\mathbf{A}_{\text{Intuitive}}$ is a simple heuristic for the positive-lead-time model. As shown in Figure 4.4(a), it performs well, especially when the lead time is short. However, to achieve a better solution, we need to adjust the reorder point downward and adjust the order-up-to level upward.

Chapter 5

The Partial Backlogging Case

In this chapter, we provide a simple heuristic to determine the (s, S) policy for the partial backlogging model with constant lead time. The heuristic is developed from the insights of the optimal solution of the complete backlogging model. Numerical comparisons show that the heuristic performs well. Before presenting the heuristic, we first present some properties of the partial backlogging model.

5.1 Properties of the Model

Let $\psi_B(i|I)$ be the probability that the on-hand inventory drops to zero during the lead time period and that i demand units are backlogged, given that inventory level at beginning of the lead time period is I. Note that in the lead time period, the first I^+ arriving demand units are realized from on-hand inventory, and the following demand units are partially backlogged. The probability of k demand units facing stock out is $\psi(k+I^+)$. Each of these k units is either backlogged with probability γ or lost with probability $1-\gamma$. Thus, the probability of backlogging i units of them is $\binom{k}{i}\gamma^i(1-\gamma)^{k-i}$, which is the binomial probability of achieving i successes out of k trials with success rate γ. Hence,

$$\psi_B(i|I) = \sum_{k=i}^{+\infty} \binom{k}{i} \gamma^i (1-\gamma)^{k-i} \psi(k+I^+).$$

In particular, if $I \leq 0$, then

$$\psi_B(i|I) = \sum_{k=i}^{+\infty} \binom{k}{i} \gamma^i (1-\gamma)^{k-i} \psi(k) = e^{-\gamma\lambda L} \frac{(\gamma\lambda L)^i}{i!},$$

which is a Poisson probability with mean $\gamma\lambda L$. One key property of the pmf $\psi_B(\cdot|I)$ is summarized in the following lemma.

Lemma 3 *For any given I, there exists a mode, $\mathcal{M}_B(I)$, of the pmf $\psi_B(i|I)$ such that $\psi_B(i|I)$ is decreasing in i on $[\mathcal{M}_B(I), +\infty)$.*

Proof: If $I \leq 0$, $\psi_B(\cdot|I)$ is equal to the pmf of a Poisson distribution with mean $\gamma\lambda L$. In this case, let $\mathcal{M}_B(I) = \lfloor \gamma\lambda L \rfloor$. Then, $\psi_B(i|I)$ is decreasing in i on $[\mathcal{M}_B(I), +\infty)$.

Next, we consider the case where $I \geq 1$. The proof utilizes the *variation diminishing property* of a *totally positive* function introduced by Karlin (1968). Let $\mathcal{S}(\cdot)$ be the function of *number of sign changes* (see Song 1994, p. 608). Then, for any (weakly) unimodal function f and any real number x, $\mathcal{S}(f - x) \leq 2$. Furthermore, if $\mathcal{S}(f-x) = 2$, then the sign sequence is "$-, +, -$"; if f is decreasing and $\mathcal{S}(f-x) = 1$, then the sign sequence is "$+, -$".

For integers $k, i = 0, 1, 2, \ldots$, let

$$\omega(k,i) = \binom{k}{i} \gamma^i (1-\gamma)^{k-i} = \binom{k}{i} \left(\frac{\gamma}{1-\gamma}\right)^i (1-\gamma)^k.$$

Consider any integers $i_1 < i_2 < \cdots < i_r$ and $k_1 < k_2 < \cdots < k_r$, where r is an arbitrary integer. We have

$$\begin{vmatrix} \omega(k_1, i_1) & \cdots & \omega(k_r, i_1) \\ \vdots & & \vdots \\ \omega(k_1, i_r) & \cdots & \omega(k_r, i_r) \end{vmatrix} = \begin{vmatrix} \binom{k_1}{i_1} & \cdots & \binom{k_r}{i_1} \\ \vdots & & \vdots \\ \binom{k_1}{i_r} & \cdots & \binom{k_r}{i_r} \end{vmatrix} \cdot \left(\frac{\gamma}{1-\gamma}\right)^{\sum_{\ell=1}^r i_\ell} (1-\gamma)^{\sum_{\ell=1}^r k_\ell} \geq 0,$$

where the inequality holds because $\binom{k}{i}$ is a totally positive function (Karlin 1968, p. 137). Thus, $\omega(k,i)$ is a totally positive function. Because $I \geq 1$, we have

$$\psi_B(i|I) = \sum_{k=i}^{+\infty} \binom{k}{i} \gamma^i (1-\gamma)^{k-i} \psi(k + I^+) = \sum_{k=i}^{+\infty} \omega(k,i) \psi(k+I).$$

Hence,

$$\psi_B(i|I) - x = \sum_{k=i}^{+\infty} \omega(k,i)[\psi(k+I) - x],$$

for any real number x. Note that $\psi(k+I)$ is a truncated Poisson distribution which is either (weakly) unimodal or decreasing. Thus, $\mathcal{S}(\psi(k+I)-x) \leq 2$. Note also that $\omega(k,i)$ is totally positive. According to the *variation diminishing property*, we have $\mathcal{S}(\psi_B(i|I)-x) \leq 2$. If there exists a real number $\tilde{x}$ such that $\mathcal{S}(\psi_B(i|I)-\tilde{x}) = 2$, then the sign sequence of $\psi_B(i|I)-\tilde{x}$ must be "$-,+,-$", which is the same as that of $\psi(k+I)-\tilde{x}$. In this case, $\psi_B(i|I)$ is first increasing and then decreasing in i. If we let $\mathcal{M}_B(I)$ be the mode (or the larger mode if there are two) of the pmf $\psi_B(i|I)$, then $\psi_B(i|I)$ is decreasing in i on $[\mathcal{M}_B(I),+\infty)$. If no such $\tilde{x}$ exists, then $\psi_B(i|I)$ is decreasing in i (note: $\psi_B(\cdot|I)$ cannot be a constant function). If we let $\mathcal{M}_B(I)$ be the mode of the pmf $\psi_B(i|I)$, i.e., $\mathcal{M}_B(I) = 0$, then $\psi_B(i|I)$ is decreasing in i on $[\mathcal{M}_B(I),+\infty)$. ∎

We also generalize some notations defined in Section 4. Recall that $\mathcal{C}(I)$ is the expected inventory cost rate at the end of the lead time period, given that the order is placed at inventory level I. In the partial backlogging case, the expression for $\mathcal{C}(I)$ should be generalized to

$$\mathcal{C}(I) = h\delta(I) + b[\gamma\tilde{\delta}(I) + I^-], \tag{5.1}$$

where $\gamma\tilde{\delta}(I)$ is the expected backordered quantity out of the unmet lead-time demand. Consider $\mathcal{L}(I)$, i.e., the expected effective profit in the lead time period given that an order is placed at inventory level I. We have the following lemma.

Lemma 4 *Function $\mathcal{L}(I)$ satisfies*

$$\mathcal{L}(I) = \begin{cases} (\lambda p-\phi)L-(p-\tilde{p})\tilde{\delta}(I)-(I-\frac{\lambda L}{2})hL-\frac{h+\gamma b}{2\lambda}\left[\lambda L\tilde{\delta}(I)-I\tilde{\delta}(I+1)\right], & \text{if } I \geq 1; \\ (\lambda\tilde{p}-\phi)L+(I-\frac{\gamma\lambda L}{2})bL, & \text{if } I \leq 0; \end{cases}$$

and

$$\Delta\mathcal{L}(I) = \begin{cases} -\frac{I-\delta(I)}{\lambda}h+\frac{\tilde{\delta}(I)}{\lambda}\gamma b+(1-\gamma)(p+\ell)\bar{\Psi}(I-1), & \text{if } I \geq 1; \\ bL, & \text{if } I \leq 0. \end{cases} \tag{5.2}$$

Proof: Function $\mathcal{L}(I)$ is the expected effective profit in the lead time period. The expectation is taken on the lead-time demand. Let $\mathcal{L}(I|k)$ be the expected effective profit, given that the initial inventory level is I and a total of $k \geq 0$ demand units arrived during the lead time period. Then, $\mathcal{L}(I) = \sum_{k=0}^{+\infty} \mathcal{L}(I|k)\psi(k)$.

Suppose that there are k demand units arrived in the lead time. Let the starting point of the lead time be $\tau_0 = 0$; let the arrival time of the ith demand units be τ_i $(1 \leq i \leq k)$; and let $\tau_{k+1} = L$. Then,

$$0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_k < \tau_{k+1} = L.$$

Note that $\tau_1, \tau_2, \ldots, \tau_k$ have the same distribution as the order statistics corresponding to k independent random variables uniformly distributed on the interval $(0, L)$ (see Ross 2010, p. 66). Note also that the ith order statistic from a random sample of size n from the Uniform$(0,1)$ population has a Beta$(i, n-i+1)$ distribution (Arnold *et al.* 1992, sec. 4.7), the mean of which is $\frac{i}{n+1}$. Thus, for $i = 1, 2, \ldots, k$,

$$\mathbb{E}[\tau_i] = \frac{i}{k+1} L. \tag{5.3}$$

Next, we consider two different cases.

Case 1: $I \leq 0$. In this case, each arriving demand is realized with a probability γ. Consider the inventory level immediately after the arrival of the ith demand at time τ_i. For $0 \leq j \leq i$, the probability that j demands are realized in the time period $(0, \tau_i]$ is $\binom{i}{j}\gamma^j(1-\gamma)^{i-j}$. This is the binomial probability of achieving j success out of i trials with success rate γ. The expectation of this binomial distribution is γi. Thus, the expected inventory level immediately after time τ_i is $I - \gamma i$. Hence,

$$\begin{aligned}
\mathcal{L}(I|k) &= \mathbb{E}\left[bI\tau_1 + \sum_{i=1}^{k}\left[\tilde{p} + b(I-\gamma i)(\tau_{i+1} - \tau_i)\right] - \phi L\right] \\
&= k\tilde{p} + \mathbb{E}\left[\sum_{i=0}^{k} b(I-\gamma i)(\tau_{i+1}-\tau_i)\right] - \phi L \\
&= k\tilde{p} - \mathbb{E}\left[b\gamma\sum_{i=0}^{k} i(\tau_{i+1}-\tau_i)\right] + \mathbb{E}\left[bI\sum_{i=0}^{k}(\tau_{i+1}-\tau_i)\right] - \phi L \\
&= k\tilde{p} - b\gamma\left(kL - \sum_{i=1}^{k}\mathbb{E}[\tau_i]\right) + bIL - \phi L \\
&= k\tilde{p} + \left(I - \frac{\gamma k}{2}\right)bL - \phi L,
\end{aligned} \tag{5.4}$$

where the last equality follows from equation (5.3). Taking expectation on the lead-time demand k, we have

$$\mathcal{L}(I) = \lambda L\tilde{p} + \Big(I - \frac{\gamma\lambda L}{2}\Big)bL - \phi L. \tag{5.5}$$

Case 2: $I \geq 1$. In this case, the calculation of $\mathcal{L}(I|k)$ depends on whether $k \leq I$ or $k \geq I+1$. If $k \leq I$, all the demands are satisfied by on-hand inventory. For $i \leq k$, the inventory level immediately after time τ_i is $I - i$. Thus,

$$\begin{aligned}\mathcal{L}(I|k) =& \mathbb{E}\Big[-hI\tau_1 + \sum_{i=1}^{k}\Big[p - h(I-i)(\tau_{i+1}-\tau_i)\Big] - \phi L\Big] \\ =& kp - \Big(I - \frac{k}{2}\Big)hL - \phi L, \end{aligned} \tag{5.6}$$

where the derivation of the second equality follows the same steps as (5.4). If $k \geq I+1$, then the first I demands are satisfied by on-hand inventory, while each of the following $k - I$ demands is realized with a probability γ. Note that the Ith demand arrives at τ_I. During the period $(0, \tau_I)$, there are $I-1$ demand units which are all satisfied by on-hand inventory. Hence, by (5.6), the expected effective profit during this period is

$$\mathbb{E}\big[(I-1)p - (I - \frac{I-1}{2})h\tau_I - \phi\tau_I\big].$$

At time τ_I, a profit p is obtained from the Ith demand unit. During the period (τ_I, L), there are $k - I$ demand units each of which is realized with a probability γ. Hence, by (5.4), the expected effective profit during this period is

$$\mathbb{E}\big[(k-I)\tilde{p} + (0 - \frac{\gamma(k-I)}{2})b(L-\tau_I) - \phi(L-\tau_I)\big].$$

Therefore,

$$\begin{aligned}\mathcal{L}(I|k) = \mathbb{E}\Big[&(I-1)p - \Big(I - \frac{I-1}{2}\Big)h\tau_I - \phi\tau_I + p + (k-I)\tilde{p} \\ &+ \Big(0 - \frac{\gamma(k-I)}{2}\Big)b(L-\tau_I) - \phi(L-\tau_I)\Big] \\ = Ip &+ (k-I)\tilde{p} - hL\frac{I(I+1)}{2(k+1)} - \gamma bL\frac{(k-I)(k-I+1)}{2(k+1)} - \phi L.\end{aligned}$$

By taking expectation of $\mathcal{L}(I|k)$ on the lead-time demand k, we obtain

$$\begin{aligned}
\mathcal{L}(I) &= \sum_{k=0}^{I} \mathcal{L}(I|k)\psi(k) + \sum_{k=I+1}^{+\infty} \mathcal{L}(I|k)\psi(k) \\
&= \sum_{k=0}^{I} \Big[kp - \Big(I - \frac{k}{2}\Big)hL - \phi L\Big]\psi(k) \\
&\quad + \sum_{k=I+1}^{+\infty} \Big[Ip + (k-I)\tilde{p} - hL\frac{I(I+1)}{2(k+1)} - \gamma bL\frac{(k-I)(k-I+1)}{2(k+1)} - \phi L\Big]\psi(k)
\end{aligned} \tag{5.7}$$

Note that

$$\sum_{k=0}^{I} kp\psi(k) + \sum_{k=I+1}^{+\infty} Ip\psi(k) = \Big[\sum_{k=0}^{+\infty} k\psi(k) - \sum_{k=I+1}^{+\infty} (k-I)\psi(k)\Big]p = [\lambda L - \tilde{\delta}(I)]p,$$

$$\begin{aligned}
&\sum_{k=0}^{I} \Big(I - \frac{k}{2}\Big)hL\psi(k) + \sum_{k=I+1}^{+\infty} hL\frac{I(I+1)}{2(k+1)}\psi(k) \\
&\quad = hL\sum_{k=0}^{+\infty} \Big(I - \frac{k}{2}\Big)\psi(k) + hL\sum_{k=I+1}^{+\infty} \Big[\frac{I(I+1)}{2(k+1)} - \Big(I - \frac{k}{2}\Big)\Big]\frac{(\lambda L)^k}{k!}e^{-\lambda L} \\
&\quad = \Big(I - \frac{\lambda L}{2}\Big)hL + \frac{h}{2\lambda}\sum_{k=I+1}^{+\infty} \big[I(I+1) - 2I(k+1) + k(k+1)\big]\psi(k+1),
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=I+1}^{+\infty} \gamma bL\frac{(k-I)(k-I+1)}{2(k+1)}\psi(k) \\
&= \gamma bL\sum_{k=I+1}^{+\infty} \frac{(k-I)(k-I+1)}{2(k+1)} \cdot \frac{(\lambda L)^k}{k!}e^{-\lambda L} \\
&= \frac{\gamma b}{2\lambda}\sum_{k=I+1}^{+\infty} \big[I(I+1) - 2I(k+1) + k(k+1)\big]\psi(k+1).
\end{aligned}$$

Thus, equation (5.7) can be rewritten as

$$\begin{aligned}
\mathcal{L}(I) = [\lambda L - \tilde{\delta}(I)]p + \tilde{\delta}(I)\tilde{p} - \Big(I - \frac{\lambda L}{2}\Big)hL \\
- \frac{h+\gamma b}{2\lambda}\sum_{k=I+1}^{+\infty} \big[I(I+1) - 2I(k+1) + k(k+1)\big]\psi(k+1) - \phi L.
\end{aligned}$$

It is easy to check that $\lambda L\tilde{\delta}(I) - I\tilde{\delta}(I+1) = \sum_{k=I+1}^{+\infty} \big[I(I+1) - 2I(k+1) + k(k+1)\big]\psi(k+1)$. Hence,

$$\mathcal{L}(I) = (\lambda p - \phi)L - (p - \tilde{p})\tilde{\delta}(I) - \Big(I - \frac{\lambda L}{2}\Big)hL - \frac{h+\gamma b}{2\lambda}\big[\lambda L\tilde{\delta}(I) - I\tilde{\delta}(I+1)\big]. \quad (5.8)$$

It is easy to check that equation (5.8) is also valid when $I = 0$.

Next, we derive the expression for $\Delta\mathcal{L}(I)$. If $I \le 0$, then from (5.5), $\Delta\mathcal{L}(I) = bL$. For the case where $I \ge 1$, from (5.8), we have

$$\begin{aligned}\Delta\mathcal{L}(I) =& \big[\tilde{\delta}(I-1) - \tilde{\delta}(I)\big](p - \tilde{p}) - hL \\ & - \frac{h+\gamma b}{2\lambda}\Big\{\lambda L\big[\tilde{\delta}(I) - \tilde{\delta}(I-1)\big] - I\tilde{\delta}(I+1) + (I-1)\tilde{\delta}(I)\Big\}.\end{aligned}$$

From the definitions of $\delta(\cdot)$ and $\tilde{\delta}(\cdot)$, it is easy to check that, if $x \ge 1$, $\tilde{\delta}(x) = \lambda L\bar{\Psi}(x-1) - x\bar{\Psi}(x)$, $\tilde{\delta}(x-1) - \tilde{\delta}(x) = \bar{\Psi}(x-1)$, and $\delta(x) - \tilde{\delta}(x) = x - \lambda L$. Note also that $p - \tilde{p} = (1-\gamma)(p+\ell)$. Thus,

$$\begin{aligned}\Delta\mathcal{L}(I) &= (p-\tilde{p})\bar{\Psi}(I-1) - hL - \frac{h+\gamma b}{2\lambda}\Big[-\lambda L\bar{\Psi}(I-1) + I\bar{\Psi}(I) - \tilde{\delta}(I)\Big] \\ &= (p-\tilde{p})\bar{\Psi}(I-1) - hL + \frac{h+\gamma b}{\lambda}\tilde{\delta}(I) \\ &= -\frac{\lambda L - \tilde{\delta}(I)}{\lambda}h + \frac{\tilde{\delta}(I)}{\lambda}\gamma b + (p-\tilde{p})\bar{\Psi}(I-1) \\ &= -\frac{I - \delta(I)}{\lambda}h + \frac{\tilde{\delta}(I)}{\lambda}\gamma b + (1-\gamma)(p+\ell)\bar{\Psi}(I-1).\end{aligned}$$

Lemma 1 is a special case of Lemma 4 when $\gamma = 1$. Recall that $\delta(x) = 0$ and $\tilde{\delta}(x) = \lambda L$ if $x \le 0$. Hence, if $\gamma = 1$, then the function $\Delta\mathcal{L}(I)$ can be uniformly expressed as

$$\Delta\mathcal{L}(I) = -\frac{I^+ - \delta(I)}{\lambda}h + \frac{\tilde{\delta}(I)}{\lambda}b.$$

This completes the proof of Lemmas 1 and 4. ∎

The first case of the marginal effective profit in the lead time period, given by (5.2), can be explained as follows. Suppose an order is placed at inventory level I rather than $I-1$. Then, one more unit of holding cost will be charged in the in-stock period. The expectation of this additional cost is $\frac{I-\delta(I)}{\lambda}h$, where $\frac{I-\delta(I)}{\lambda}$ is the expected length of the in-stock period. In the stock-out period, either one less unit of backlogging cost will be charged (with probability γ) or one more unit demand will be realized (with probability $1-\gamma$). Thus, the expected cost saving is $\gamma\frac{\tilde{\delta}(I)}{\lambda}b + (1-\gamma)(p+\ell)\bar{\Psi}(I-1)$, where $\frac{\tilde{\delta}(I)}{\lambda}$ is the expected length of the stock-out period, $p+\ell$ is the benefit of earning the revenue from a realized demand plus the saving from one less lost demand, and $\Psi(I-1)$ is the probability that the stock-out occurs.

For any integer I and profit rate ϕ, let

$$\mathcal{R}(I) = \lambda p \Psi(I-1) + \lambda \tilde{p} \bar{\Psi}(I-1) - \phi. \tag{5.9}$$

This is the expected effective revenue rate at the end of the lead time period if the order is placed at inventory level I, given that the profit rate is ϕ. Specifically, the expected revenue rate is λp if the inventory level is positive, which happens with probability $\Psi(I-1)$; and the expected revenue rate is $\lambda\tilde{p}$ if the inventory level is non-positive, which happens with probability $\bar{\Psi}(I-1)$.

5.2 The Heuristic

Recall the main idea of the algorithm developed in Section 4 for determining the optimal (s,S) policy. For any given profit rate ϕ, the reorder point and order quantity, which maximize the effective profit, can be determined optimally. A binary search algorithm is then applied to search for the optimal profit rate. In the partial backlogging model, we also run a binary search to determine the final value of average profit. However, as the effective profit function with partial backlogging is more complicated, some properties in Section 4 fail to hold. As a result, for any given profit rate ϕ, the reorder point and order quantity are difficult to determine optimally. Therefore, we provide a heuristic to determine an approximate reorder point and an approximate order quantity.

Our heuristic considers the effective profit in a certain period rather than the infinite time horizon; see Figure 5.1 for the period $[t, t_0]$. Period $[t, t_1]$ is an initial cycle, while period $[t_1, t_0]$ is a full cycle, which begins and ends at the same inventory level s. For any given profit rate ϕ, the reorder point $\tilde{s}(\phi)$ and order quantity $\tilde{Q}(\phi)$

are determined sequentially: The reorder point is determined first by applying the economic interpretation of the optimal reorder point we obtained for the complete backlogging model in Section 4; that is, $\tilde{s}(\phi)$ balances the expected effective revenue rate and the expected inventory cost rate at the end of the lead time period. Then, given the reorder point $\tilde{s}(\phi)$, the order quantity is determined such that the effective profit during the time period $[t_1 + L, t_0]$ is maximized. Next, we present the details of the heuristic, followed by some properties.

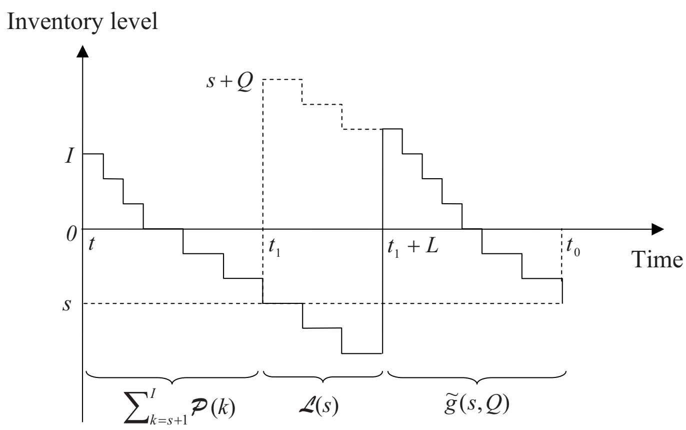


Figure 5.1: The planning horizon of the heuristic.

The heuristic works as follows. The main structure of the heuristic is to run a binary search to determine the final value of average profit, just like algorithms $\mathbf{A}_{0,\gamma}$ and $\mathbf{A}_{L,0}$. For any given profit rate ϕ such that $0 \leq \phi \leq \lambda p$, the reorder point and order-up-to level are sequentially obtained as follows:

(i) Reorder point: Let

$$\tilde{s}(\phi) = \Big\{ x \;\Big|\; \mathcal{C}(x) \geq \mathcal{R}(x) \text{ and } \mathcal{C}(x+1) < \mathcal{R}(x+1) \Big\} \tag{5.10}$$

be the reorder point if it exists. Otherwise, let $\tilde{s}(\phi) = \Psi^{-1}\big(\frac{\gamma b}{h+\gamma b}\big)$.

(ii) Order-up-to level: Let $\tilde{S}(\phi) = \tilde{s}(\phi) + \tilde{Q}(\phi)$ be the order-up-to level, where

$$\tilde{Q}(\phi) = \underset{Q}{\operatorname{argmax}} \{\tilde{\mathcal{G}}(\tilde{s}(\phi), Q)\} \tag{5.11}$$

and

$$\tilde{\mathcal{G}}(s,Q) = \sum_{d=0}^{s-1} \left[\sum_{k=s+1}^{s+Q-d} \mathcal{P}(k) \right] \psi(d) + \sum_{i=0}^{Q-s^- -s-1} \left[\sum_{k=s+1}^{-s^- +Q-i} \mathcal{P}(k) \right] \psi_B(i|s). \tag{5.12}$$

Regarding the reorder point, we have $\mathcal{R}(s) = \mathcal{R}(s+1) = \lambda\tilde{p} - \phi$ if $s \leq -1$. Thus, if $\tilde{s}(\phi)$ is negative, it can be expressed in the following closed form:

$$\tilde{s}(\phi) = \left\lfloor \gamma\lambda L + \frac{\phi - \lambda\tilde{p}}{b} \right\rfloor. \tag{5.13}$$

Furthermore, the property of the reorder point is summarized in the following theorem.

Theorem 7 *For any profit rate ϕ, if the reorder point is given by* (5.10)*, then it is unique and satisfies*

$$\Delta\mathcal{L}(\tilde{s}(\phi)) \geq \mathcal{P}(\tilde{s}(\phi)) \quad \textit{and} \quad \Delta\mathcal{L}(\tilde{s}(\phi)+1) < \mathcal{P}(\tilde{s}(\phi)+1). \tag{5.14}$$

Proof: To show that the reorder point in (5.10) is unique, we explore the properties of functions $\mathcal{C}(x)$ and $\mathcal{R}(x)$. First, consider the expected inventory cost rate given by (5.1). When $x \leq 0$, we have $\tilde{\delta}(x) = \lambda L$ and $\delta(x) = 0$, and therefore $\mathcal{C}(x) = (\gamma\lambda L - x)b$. When $x \geq 1$, it is easy to check that $\tilde{\delta}(x-1) - \tilde{\delta}(x) = \bar{\Psi}(x-1)$ and $\delta(x) - \delta(x-1) = \Psi(x-1)$, and therefore $\Delta\mathcal{C}(x) = -\gamma b + (h+\gamma b)\Psi(x-1)$, which implies that

$$\Delta^2\mathcal{C}(x+1) = (h+\gamma b)\psi(x) > 0.$$

Thus, $\mathcal{C}(x)$ is convex in $[1, +\infty)$. Next, consider the expected effective revenue rate defined in (5.9). When $x \leq 0$, $\mathcal{R}(x) = \lambda\tilde{p} - \phi$. When $x \geq 1$, we have

$$\Delta\mathcal{R}(x) = \lambda(1-\gamma)(p+\ell)\psi(x-1) > 0.$$

Note also that the pmf $\psi(x)$ is first increasing and then decreasing in x. Hence, $\mathcal{R}(x)$ is first convex increasing and then concave increasing in $[1, +\infty)$. Functions $\mathcal{C}(\cdot)$ and $\mathcal{R}(\cdot)$ are depicted in Figure 5.2. The two curves have at most two intersections. Only the smaller one satisfies the inequalities in (5.10). Therefore, the reorder point given by (5.10) is unique.

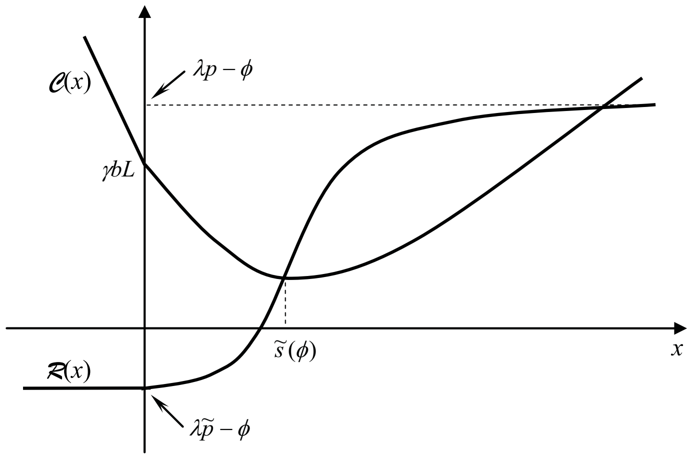


Figure 5.2: The inventory cost rate and revenue rate.

The inequalities in (5.14) can be verified as follows. For notational convenience, we denote $\tilde{s}(\phi)$ as $\tilde{s}$. We consider two different cases.

Case 1: $\tilde{s} \leq -1$. In this case, $\mathcal{C}(\tilde{s}) = (\gamma\lambda L - \tilde{s})b$, $\Delta\mathcal{L}(\tilde{s}) = bL$, and $\mathcal{R}(\tilde{s}) = \lambda\tilde{p} - \phi$. Thus, the inequality "$\mathcal{C}(\tilde{s}) \geq \mathcal{R}(\tilde{s})$" is equivalent to $(\gamma\lambda L - \tilde{s})b \geq \lambda\tilde{p} - \phi$, which implies that $bL \geq \frac{1}{\gamma}\left(\tilde{p} - \frac{\phi - b\tilde{s}}{\lambda}\right)$, or equivalently, $\Delta\mathcal{L}(\tilde{s}) \geq \mathcal{P}(\tilde{s})$. Similarly, the inequality "$\mathcal{C}(\tilde{s}+1) < \mathcal{R}(\tilde{s}+1)$" implies that $\Delta\mathcal{L}(\tilde{s}+1) < \mathcal{P}(\tilde{s}+1)$.

Case 2: $\tilde{s} \geq 0$. In this case, from (5.1) and (5.9), the inequality $\mathcal{C}(\tilde{s}) \geq \mathcal{R}(\tilde{s})$ is equivalent to

$$h\delta(\tilde{s}) + \gamma b\tilde{\delta}(\tilde{s}) \geq \lambda p\Psi(\tilde{s}-1) + \lambda\tilde{p}\bar{\Psi}(\tilde{s}-1) - \phi,$$

which can be rewritten as

$$-\frac{\tilde{s} - \delta(\tilde{s})}{\lambda}h + \frac{\tilde{\delta}(\tilde{s})}{\lambda}\gamma b + (1-\gamma)(p+\ell)\bar{\Psi}(\tilde{s}-1) \geq p - \frac{\phi + h\tilde{s}}{\lambda}.$$

From (3.6) and (5.2), the above inequality is equivalent to $\Delta\mathcal{L}(\tilde{s}) \geq \mathcal{P}(\tilde{s})$. Similarly, the inequality $\mathcal{C}(\tilde{s}+1) < \mathcal{R}(\tilde{s}+1)$ implies that $\Delta\mathcal{L}(\tilde{s}+1) < \mathcal{P}(\tilde{s}+1)$. ∎

Recall the interpretations of $\Delta\mathcal{L}(\cdot)$ and $\mathcal{P}(\cdot)$ in the marginal benefit analysis in Section 4.2. Having the reorder point satisfy the inequalities in (5.14) implies that the reorder point in our heuristic shall be consistent with the one obtained from the marginal benefit analysis of the effective profit in period $[t, t_1 + L]$. Thus, the reorder point $\tilde{s}(\phi)$ shall maximize the effective profit over the time period $[t, t_1 + L]$.

Next, we consider the property of the order quantity $\tilde{Q}(\phi)$. Consider Figure 5.1, and let $\tilde{I}$ be the inventory level immediately after the order arrival at time $t_1 + L$. The inventory level drops from $\tilde{I}$ to the reorder point s in $[t_1 + L, t_0]$. Thus, the effective profit in $[t_1 + L, t_0]$ is $\sum_{k=s+1}^{\tilde{I}} \mathcal{P}(k)$. Note also that $\tilde{I} = s + Q - \mathcal{D}$, where $\mathcal{D}$ is the realized demand in the lead time period. When the demand during lead time is less than s, we have $\tilde{I} = s + Q - d$ with probability $\psi(d)$ if $d \leq s - 1$. When the demand during lead time is at least s, we have $\tilde{I} = s + Q - (s^+ + i) = -s^- + Q - i$ with probability $\psi_B(i|s)$ if $i \geq 0$. However, we only consider the backordered demand i such that $0 \leq i \leq Q - s^- - s - 1$, since otherwise the inventory level drops to/below s during the lead time period and the time period $[t_1 + L, t_0]$ vanishes. Hence, $\tilde{\mathcal{G}}(s, Q)$ in (5.12) is the expected total effective profit in $[t_1 + L, t_0]$ given that the reorder point is s and the order quantity is Q. Therefore, given the reorder point $\tilde{s}(\phi)$, the order quantity in (5.11) maximizes the expected effective profit in $[t_1 + L, t_0]$. Furthermore, this order quantity can be uniquely determined (see Theorem 4 for tie-breaking rule). A property of function $\tilde{\mathcal{G}}(s, Q)$ is given in the following theorem.

Theorem 8 *For any profit rate ϕ, under the condition that $Q - |\tilde{s}(\phi)| \geq \mathcal{M}_B(\tilde{s}(\phi))$, the function $\tilde{\mathcal{G}}(\tilde{s}(\phi), \cdot)$ has an unique local maximizer (see Theorem 4 for tie-breaking rule).*

Proof: We consider two different cases. Case 1: $\tilde{s}(\phi) < 0$. In this case, from (5.13), $\tilde{s}(\phi) \geq \left\lfloor \frac{\phi - \lambda\tilde{p}}{b} \right\rfloor$ for any profit rate ϕ. Thus, to show that $\tilde{\mathcal{G}}(\tilde{s}(\phi), \cdot)$ has a unique local maximizer, it suffices to show that either $\Delta\tilde{\mathcal{G}}(s, Q)$ or $\tilde{\mathcal{G}}(s, Q)$ (or both) is concave in Q for any s such that $s \geq \left\lfloor \frac{\phi - \lambda\tilde{p}}{b} \right\rfloor$. We will show that $\Delta\tilde{\mathcal{G}}(s, Q)$ is concave in Q. To do so, it is sufficient to show that $\Delta^2\tilde{\mathcal{G}}(s, Q)$ is decreasing in Q. Note that when $s \leq 0$, equation (5.12) can be simplified to

$$\tilde{\mathcal{G}}(s, Q) = \sum_{i=0}^{Q-1} \left[\sum_{k=s+1}^{Q+s-i} \mathcal{P}(k) \right] \psi_B(i|s).$$

This implies that

$$\Delta\tilde{\mathcal{G}}(s, Q) = \tilde{\mathcal{G}}(s, Q) - \tilde{\mathcal{G}}(s, Q-1) = \sum_{i=0}^{Q-1} \mathcal{P}(Q + s - i)\psi_B(i|s).$$

This in turn implies that

$$
\begin{aligned}
&\Delta^2\tilde{\mathcal{G}}(s,Q+1) = \Delta\tilde{\mathcal{G}}(s,Q+1) - \Delta\tilde{\mathcal{G}}(s,Q)\\
&= \sum_{i=0}^{Q-1} \Delta\mathcal{P}(Q+1+s-i)\psi_B(i|s) + \mathcal{P}(s+1)\psi_B(Q|s)\\
&= -\frac{h}{\lambda}\sum_{i=0}^{Q+s}\psi_B(i|s) + \big[\mathcal{P}(0^+) - \mathcal{P}(0)\big]\psi_B(Q+s|s)\\
&\quad + \frac{b}{\gamma\lambda}\sum_{i=Q+s+1}^{Q-1}\psi_B(i|s) + \mathcal{P}(s+1)\psi_B(Q|s),
\end{aligned}
\tag{5.15}
$$

where $\mathcal{P}(0^+) = p - \frac{\phi}{\lambda}$. The first term in expression (5.15) is decreasing in Q. It is easy to check that

$$
\mathcal{P}(0^+) - \mathcal{P}(0) = \frac{1-\gamma}{\gamma}\Big(\frac{\phi}{\lambda} + \frac{\ell}{\gamma}\Big) \geq 0.
$$

According to Lemma 3, $\psi_B(i|s)$ is decreasing in i when $i \geq \mathcal{M}_B(s)$. Thus, under the condition "$Q + s \geq \mathcal{M}_B(s)$," the quantities $\sum_{i=Q+s+1}^{Q-1}\psi_B(i|s)$, $\psi_B(Q|s)$, and $\psi_B(Q+s|s)$ are decreasing in Q. Hence, the second and third terms in expression (5.15) are non-increasing in Q. Note that $\lfloor\frac{\phi-\lambda\tilde{p}}{b}\rfloor$ is the optimal reorder point for the zero-lead-time model (see Theorem 3). Thus, $\mathcal{P}(s+1) > 0$ for any $\lfloor\frac{\phi-\lambda\tilde{p}}{b}\rfloor \leq s < 0$. Hence, the fourth term in expression (5.15) is also decreasing in Q. Therefore, $\Delta^2\tilde{\mathcal{G}}(s,Q+1)$ is decreasing in Q.

Case 2: $\tilde{s}(\phi) \geq 0$. In this case, it suffices to show that either $\Delta\tilde{\mathcal{G}}(s,Q)$ or $\tilde{\mathcal{G}}(s,Q)$ (or both) is concave in Q for any s such that $s \geq 0$. When $s \geq 0$, equation (5.12) becomes

$$
\tilde{\mathcal{G}}(s,Q) = \sum_{d=0}^{s-1}\Bigg[\sum_{k=s+1}^{s+Q-d}\mathcal{P}(k)\Bigg]\psi(d) + \sum_{i=0}^{Q-s-1}\Bigg[\sum_{k=s+1}^{Q-i}\mathcal{P}(k)\Bigg]\psi_B(i|s).
$$

Hence,

$$
\begin{aligned}
\Delta\tilde{\mathcal{G}}(s,Q) = &\sum_{d=0}^{s-1}\mathcal{P}(s+Q-d)\psi(d) + \sum_{i=0}^{Q-s-2}\mathcal{P}(Q-i)\psi_B(i|s)\\
&+ \mathcal{P}(s+1)\psi_B(Q-s-1|s).
\end{aligned}
$$

This implies that

$$\begin{aligned}\Delta^2\tilde{\mathcal{G}}(s,Q+1) =& \sum_{d=0}^{s-1}\Delta\mathcal{P}(s+Q+1-d)\psi(d) + \sum_{i=0}^{Q-s-1}\Delta\mathcal{P}(Q+1-i)\psi_B(i|s) \\ &+ \mathcal{P}(s+1)\psi_B(Q-s|s) \\ =& -\frac{h}{\lambda}\left[\sum_{d=0}^{s-1}\psi(d) + \sum_{i=0}^{Q-s-1}\psi_B(i|s)\right] + \mathcal{P}(s+1)\psi_B(Q-s|s). \quad (5.16)\end{aligned}$$

If $\mathcal{P}(s+1) \le 0$, then $\Delta^2\tilde{\mathcal{G}}(s,Q+1) < 0$ and $\tilde{\mathcal{G}}(s,Q)$ is concave in Q. If $\mathcal{P}(s+1) > 0$, we will show that $\Delta\tilde{\mathcal{G}}(s,Q)$ is concave in Q. Note that the first term in expression (5.16) is decreasing in Q. According to Lemma 3, $\psi_B(i|s)$ is decreasing in i when $i \ge \mathcal{M}_B(s)$. Thus, under the condition "$Q-s \ge \mathcal{M}_B(s)$," $\psi_B(Q-s|s)$ is decreasing in Q. Hence, the second term in expression (5.16) is also decreasing in Q. Therefore, $\Delta^2\tilde{\mathcal{G}}(s,Q+1)$ is decreasing in Q, which implies that $\Delta\tilde{\mathcal{G}}(s,Q)$ is concave in Q. ▮

From Theorem 8, the order quantity $\tilde{Q}(\phi)$ is either the unique local maximizer of $\tilde{\mathcal{G}}(\tilde{s}(\phi),Q)$ or the boundary value $\mathcal{M}_B(\tilde{s}(\phi)) + |\tilde{s}(\phi)|$. Similar to the complete backlogging case in Section 4, we have conducted an extensive numerical study, and the results indicate that the local maximizer of $\tilde{\mathcal{G}}(\tilde{s}(\phi),Q)$ is also the global maximizer in most cases. Also, our numerical results indicate that the optimal order quantity always satisfy the condition "$Q - |\tilde{s}(\phi)| \ge \mathcal{M}_B(\tilde{s}(\phi))$," which is a mild condition on the order quantity similar to that stated in Theorem 6 for the complete backlogging model.

5.3 Numerical Results

In this section, we present the results of our numerical study on the partial backlogging model. We first compare our heuristic solutions with the optimal stationary (s,S) policy to show the effectiveness of our heuristic. We then compare the solutions of our heuristic with those of two simple heuristics to provide additional insights.

5.3.1 Effectiveness of the Heuristic

We demonstrate the effectiveness of our heuristic via a numerical study. In this study, we compare the numerical results of our heuristic with those of the optimal stationary (s, S) policy. To execute our heuristic, we use the reorder point and order quantity given by (5.10) and (5.11), respectively, and conduct a binary search to determine the final decisions on the profit rate, reorder point, and order quantity. Following the notation introduced in Section 3.2, we denote our heuristic as $\mathbf{A}_{L,\gamma}$, because it is applied to the model with lead time L and backlogging rate γ. The optimal stationary (s, S) policy is obtained via global search over the two-dimensional space, and we denote this algorithm as $\mathbf{A}_{\text{Optimal}}$.

In the numerical study, we set $p = 30$, $h = 1$, and $\lambda = 5$. We consider different values for each of b, ℓ, and K. Specifically, we set $b \in \{2, 5\}$, $\ell \in \{4, 10\}$, and $K \in \{100, 400, 1600\}$. Thus, we have 12 combinations of these parameters. We refer to the set of numerical tests for combination j as "Test j" for $j = 1, 2, \ldots, 12$. For Test j, we let the lead time L vary from 0 to 5 with step size 0.5 and let the backlogging probability γ vary from 0 to 1 with step size 0.1. Thus, Test j comprises $11 \times 11 = 121$ test instances. The computational study uses a total of $12 \times 121 = 1452$ test instances. For each test instance, we compare the results obtained by $\mathbf{A}_{L,\gamma}$ and $\mathbf{A}_{\text{Optimal}}$. The results are summarized in Table 5.1.

Each entry of the 5th column of Table 5.1 shows the number of test instances (among the 121 instances) in which $\mathbf{A}_{L,\gamma}$ generates a different solution from $\mathbf{A}_{\text{Optimal}}$. The maximum absolute difference in reorder point and order-up-to level generated by these two algorithms are shown in the 7th and 8th columns. The results indicate that our heuristic generates the best possible stationary (s, S) solution most of the time. For each test instance, we also compute the percentage difference in expected profit between the solution obtained by $\mathbf{A}_{L,\gamma}$ and the solution obtained by $\mathbf{A}_{\text{Optimal}}$; that is,

$$\frac{\text{Expected profit obtained by } \mathbf{A}_{\text{Optimal}} - \text{Expected profit obtained by } \mathbf{A}_{L,\gamma}}{\text{Expected profit obtained by } \mathbf{A}_{\text{Optimal}}} \times 100\%.$$

The average and maximum of the percentage differences are reported in the 6th and 9th columns, respectively, of the table. These percentage differences in expected

Table 5.1: Results of numerical tests.

Test set	b	ℓ	K	Number of instances with $\mathbf{A}_{L,\gamma} \neq \mathbf{A}_{\text{Optimal}}$	Average difference in expected profit	Maximum difference in		
						reorder point	order -up-to level	expected profit
Test 1	2	4	100	0	0.00%	0	0	0.00%
Test 2	2	4	400	9	0.01%	1	1	0.04%
Test 3	2	4	1600	16	0.70%	8	6	1.50%
Test 4	2	10	100	4	0.05%	1	3	0.15%
Test 5	2	10	400	6	0.02%	1	1	0.04%
Test 6	2	10	1600	20	1.07%	9	5	4.07%
Test 7	5	4	100	4	0.01%	1	1	0.01%
Test 8	5	4	400	3	0.01%	1	1	0.03%
Test 9	5	4	1600	36	1.57%	9	6	6.69%
Test 10	5	10	100	2	0.08%	1	3	0.15%
Test 11	5	10	400	0	0.00%	0	0	0.00%
Test 12	5	10	1600	32	1.17%	7	3	4.84%

profit are small, which indicate that our heuristic is highly effective. From Table 2, we observe that when K is large (i.e., $K = 1600$), the performance of our heuristic drops slightly. If K is further increased (say, $K \geq 2000$), the optimal profits of the test instances become either very low or even negative, and the parameter setting becomes unrealistic.

5.3.2 Comparison with Other Heuristics

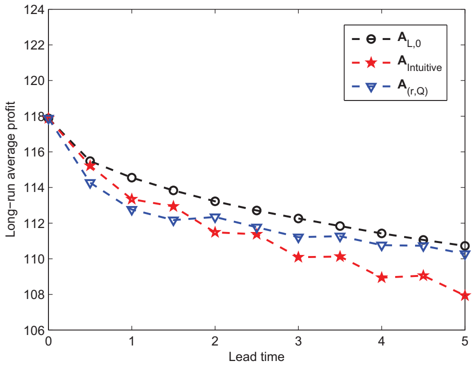
We also compare the performance of our heuristic with that of two other heuristics to see why ours has a good performance. The first heuristic we consider is $\mathbf{A}_{\text{Intuitive}}$, which is similar to heuristic $\mathbf{A}_{\text{Intuitive}}$ introduced in Section 4.3. This intuitive heuristic applies the optimal policy of the zero-lead-time model but adjusts the reorder point and order-up-to level by the expected lead-time demand. The second heuristic, denoted as $\mathbf{A}_{(r,Q)}$, determines the reorder point and order quantity independently. In this heuristic, the order quantity is determined by the EOQ formula with demand rate λ; that is, we set $Q = \sqrt{2K\lambda/h}$. The reorder point r is obtained by setting $\Psi(r)$ equal to the service level

$$\alpha = \frac{\gamma bL + (1-\gamma)(p+\ell)}{hL + \gamma bL + (1-\gamma)(p+\ell)}.$$

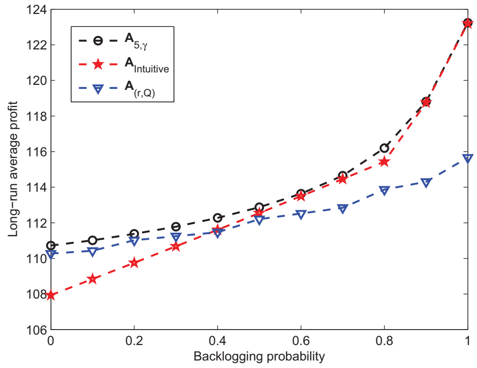
To interpret this service level, we consider the newsvendor problem defined on the lead time L, where the unit overage cost is hL and the unit underage cost is $\gamma bL+(1-\gamma)(p+\ell)$. Then, the service level α is the critical fractile solution of this newsvendor problem. Thus, with service level α, the reorder point $r = \Psi^{-1}(\alpha)$, which is also the order quantity of the newsvendor problem, shall maximize the expected profit in the lead time period. Algorithm $\mathbf{A}_{(r,Q)}$ can be regarded as an extension of the (r,Q) policy which is commonly used for the complete backlogging and lost-sales inventory systems (see Nahmias 2009, sec. 5.4–5.5).

Figure 5.3 depicts the numerical results of the three heuristics, where the cost parameters are set according to Test 1 in Table 5.1. Figures 5.3(a), (c), & (e) depict the impact of L on the heuristics' solutions when γ is fixed at 0, while Figures 5.3(b), (d), & (f) depict the impact of γ on the heuristics' solutions when L is fixed at 5. The numerical results obtained by $\mathbf{A}_{\text{Optimal}}$ are not shown in the figure, as they are identical to that of $\mathbf{A}_{L,\gamma}$ (see results of Test 1 in Table 5.1).

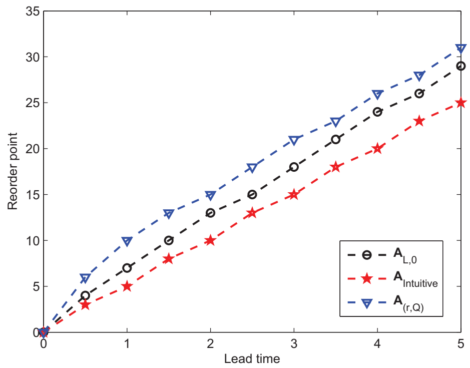
We first consider the lost-sales case (i.e., when $\gamma = 0$). In this case, the "safety stock," i.e., the reorder point less the expected lead-time demand, is commonly used to buffer the demands that exceed the expected quantity due to demand uncertainty. The reorder point of $\mathbf{A}_{\text{Intuitive}}$ is $s_0^* + \lambda L = \lambda L$, where $s_0^* = 0$ is the optimal reorder point for the zero-lead-time model (see Theorem 3, where $\tilde{p} < 0$ when $\gamma = 0$) and λL is the expected lead-time demand. Thus, the safety stock in $\mathbf{A}_{\text{Intuitive}}$ is zero. Hence, the difference between the reorder points of $\mathbf{A}_{\text{Intuitive}}$ and $\mathbf{A}_{L,0}$ ($\mathbf{A}_{(r,Q)}$) is the safety stock in $\mathbf{A}_{L,0}$ ($\mathbf{A}_{(r,Q)}$). Figure 5.3(a) shows that $\mathbf{A}_{(r,Q)}$ and $\mathbf{A}_{\text{Intuitive}}$ are outperformed by $\mathbf{A}_{L,\gamma}$. Algorithm $\mathbf{A}_{\text{Intuitive}}$ performs worse than $\mathbf{A}_{L,0}$ because the safety stock in $\mathbf{A}_{\text{Intuitive}}$ is low (i.e., zero), which keeps the inventory cost low but suffers from more lost sales in the lead time period. In addition, the profit lost in $\mathbf{A}_{\text{Intuitive}}$ becomes more significant as the lead time becomes longer. This is because as the lead time increases, the variance of the lead-time demand (i.e., λL) increases, and the probability that the lead-time demand exceeds its mean (i.e., λL) increases,



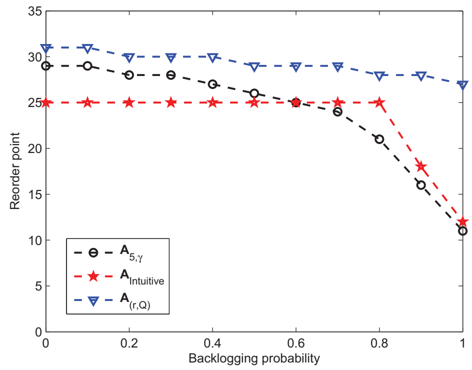
(a) long-run average profit versus L



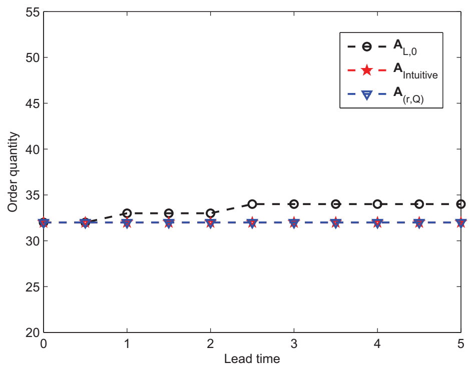
(b) long-run average profit versus γ



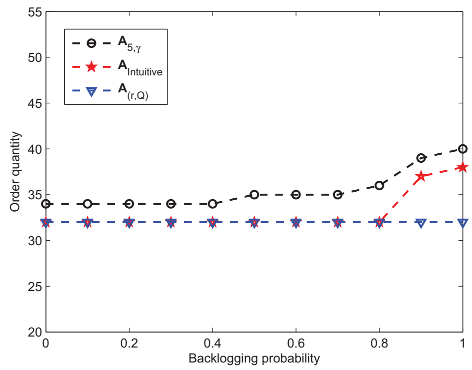
(c) reorder point versus L



(d) reorder point versus γ



(e) order quantity versus L



(f) order quantity versus γ

Figure 5.3: Numerical results of the partial backlogging case.

which causes more lost sales. Algorithm $\mathbf{A}_{(r,Q)}$ performs worse than $\mathbf{A}_{L,0}$ because the safety stock in $\mathbf{A}_{(r,Q)}$ is too high (see Figure 5.3(c)), which reduces the penalty caused by lost sales in the lead time period but incurs a higher inventory cost after the replenishment arrives.

We next consider the situation where the lead time is fixed but the backlogging probability varies. The numerical results are shown in Figures 5.3(b), (d), & (f). The results indicate that $\mathbf{A}_{\text{Intuitive}}$ is effective if the backlogging probability is high. However, the performance of $\mathbf{A}_{(r,Q)}$ drops as the backlogging probability increases. The reorder point of $\mathbf{A}_{(r,Q)}$ is determined by considering the demand in the lead time period as a newsvendor problem. It maximizes the profit in the lead time period without considering the profit after the order arrival. As shown in Figure 5.3(d), the reorder point of $\mathbf{A}_{(r,Q)}$ is kept at a relatively high level, i.e., greater than the expected lead-time demand $\lambda L = 25$. As the backlogging probability increases, less unmet demand will be lost, and it might become beneficial to use a lower reorder point to maintain some backorders in an order cycle. Thus, in Figure 5.3(d), as the backlogging probability increases, the reorder point in $\mathbf{A}_{\text{Intuitive}}$ eventually drops below λL. This occurs when the reorder point given by (3.8) becomes negative and decreasing with γ rather than staying at zero. However, the reorder point of $\mathbf{A}_{(r,Q)}$ is always kept at a higher level, and the difference between of the reorder points of $\mathbf{A}_{5,\gamma}$ and $\mathbf{A}_{(r,Q)}$ becomes significant when the backlogging probability is large (say, when $\gamma > 0.8$). Hence, the performance of $\mathbf{A}_{(r,Q)}$ drops significantly when γ is large.

According to the test instances, heuristic $\mathbf{A}_{L,\gamma}$ usually determines a more accurate reorder point than the other two heuristics. This is because in $\mathbf{A}_{L,\gamma}$, the profit rate ϕ is used both in determining the reorder point and in determining the order quantity. The profit rate ϕ establishes a connection between the effective profit in the lead time period and the effective profit after the order arrival. Thus, a reorder point which better balances the profits in these two parts of an order cycle can be obtained.

PART II

Optimal Control in a Make-to-Order Environment

Chapter 6

Model Description

With merits of perfectly matching supply with demand, make-to-order (MTO) production is broadly adopted by many companies today, where a production run is initiated only when customer orders are received. A drawback of an MTO system, however, is that customers have to spend time on waiting for delivery. The waiting time becomes more significant if a set-up cost is incurred when a production run is initiated. With the existence of set-up costs, the MTO system may not immediately start its production when an order arrives. Instead, to achieve economy of scale, a production run is started only when the number of accumulated customer orders reaches a certain threshold. Such a policy has been studied by many researchers; see, for example, Heyman and Sobel (1984, p. 336).

Clearly, the lead time of delivery is a key factor affecting customer purchase behavior when customers are delay-sensitive. Furthermore, depending on the length of the waiting line, a customer may decide to either wait for the order or leave without purchase. Such a decision also affects other customers' expected waiting time in the system. For example, when the system is idle and production has not been started, an arriving customer who decides to wait can trigger the production quicker, which benefits the later-coming customers. Therefore, customers' equilibrium strategies on waiting or leaving need to be taken into consideration. In other words, the effective demand arrival process is a process resulting from customers' decentralized decision. A feature of this essay is to study the make-to-order optimal control problem with the existence of such delay-sensitive customers and production set-up costs. We model such a system as an $M/M/1$ queue with N-policy; that is, demands arrive according to a Poisson process with rate Λ, the processing times of customer orders follow an independent and identically distributed exponential distribution with rate μ, and a production run starts when the number of waiting customer orders accumulates to N and ends when the system is empty, where $N \geq 1$. The production system adopts a first-come-first-served principle. Following the terminologies in queueing theory, we refer to the production processor as a server, which processes customer orders and can be either busy or idle. We refer to the waiting list of customer orders as the queue, and we refer to the exponential processing time as service time. We let $\rho = \Lambda/\mu$ be the system utility.

We use a utility function to model customers' decentralized decision on either waiting or leaving. An incoming customer is informed with some information on the system status. Specifically, we consider two information scenarios: unobservable queue length and observable queue length. In the former case, the queue length is unobservable but the status of the server is observable; that is, customers know whether production is in progress but do not know the number of waiting customer orders. The set of possible observable system states is $S = \{B, I\}$, where B and I represent the status where the server is "busy" and "idle," respectively. In the latter case, customers can observe the waiting list of customer orders. The set of possible observable system states is $S = \{0, 1, 2, \ldots\}$, where $s \in S$ represents the state with s waiting customer orders. Based on the information provided by the system, the customer can then estimate his/her own expected waiting time and consequently decides to stay or to leave.

On observing information state $s \in S$, the customer estimates the expected waiting time W_s (which includes the service time). We assume a linear utility function for each customer. Let θ be the customers' delay-sensitivity parameter. We assume identical customers; that is, all the customers have the same value of θ. The utility of an incoming customer who observes system state s is denoted as U_s, which satisfies

$$U_s = R - \theta W_s, \tag{6.1}$$

where R is the *reward* from receiving the service and θW_s is the *expected waiting cost*. The customer decides to stay if $U_s \geq 0$, and leave otherwise. Note that the customer's waiting time is also the amount of time that his/her order is backlogged. Thus, θW_s can also be interpreted as the backlogging cost. Define

$$\nu = \frac{\mu R}{\theta},$$

which represents the largest number of service intervals that an arriving customer is willing to wait. If $\nu = 1$, then all the customers who observe waiting customers will not join the system. In this case, to maintain an active production system, the value of N must be set to 1. Thus, in the following analysis, we assume that $\nu > 1$.

We consider four types of costs in this MTO production system. Whenever the server changes its status, a fixed set-up cost or shut-down cost is incurred. The fixed shut-down cost can be incorporated into the set-up cost. Thus, without loss of generality, we let K be the set-up cost, and normalize the shut-down cost to zero. For the customer orders that are waiting in the system, a waiting cost of h per unit time per order is incurred. When the server is busy, a system operating cost of c per unit time is incurred. In addition, for each strategic customer order that leaves the system, a lost-sales penalty ℓ is incurred. We assume that $\ell > \frac{c}{\mu}$, where $\frac{c}{\mu}$ is the expected production cost of serving a customer order. This assumption ensures that it is profitable to let the server work.

Chapter 7

System with Unobservable Queue Length

In this chapter, we consider the information scenario where incoming customers cannot observe the queue length but can observe the server status; that is $S = \{B, I\}$. Consider, for example, a computer repair shop where the repairman spends part of his work hours on repairing computers dropped off by walk-in customers and spends the rest of his time on other work. Walk-in customers visit the repair shop, drop off their malfunctioned computers, and leave the shop. The shop will phone the customers when the repair work is completed. It is not difficult for the walk-in customers to find out if the repairman is busy (i.e., doing repair work) or idle (i.e., doing other work), but the number of orders waiting in line is unobservable by the customers (as the customers do not wait inside the shop). Such an example fits into our information scenario. In the following sections, we will first conduct the equilibrium analysis with a given threshold value N, and then derive the expected average cost function and obtain the optimal value of N.

7.1 The equilibrium arrival rates

Since customers are identical, a customer's strategy can be represented by the probabilities of choosing different options. Let (α_B, α_I) be the strategy adopted by the potential customers, where $\alpha_s \in [0,1]$ is the probability that a customer will enter the queue after observing the state s, for $s = B, I$. Let $\lambda_s = \alpha_s \Lambda$ for $s = B, I$. Then, λ_s is the effective demand rate (or arrival rate) in state s. Alternatively, instead of working on the probabilities of choosing the two options, we can directly work on the effective arrival rates λ_B and λ_I. If $\lambda_I = 0$, the system can never be activated. Thus, in the following, we restrict our attention to the strategies with which the system is active for some time periods; that is, $\lambda_I > 0$.

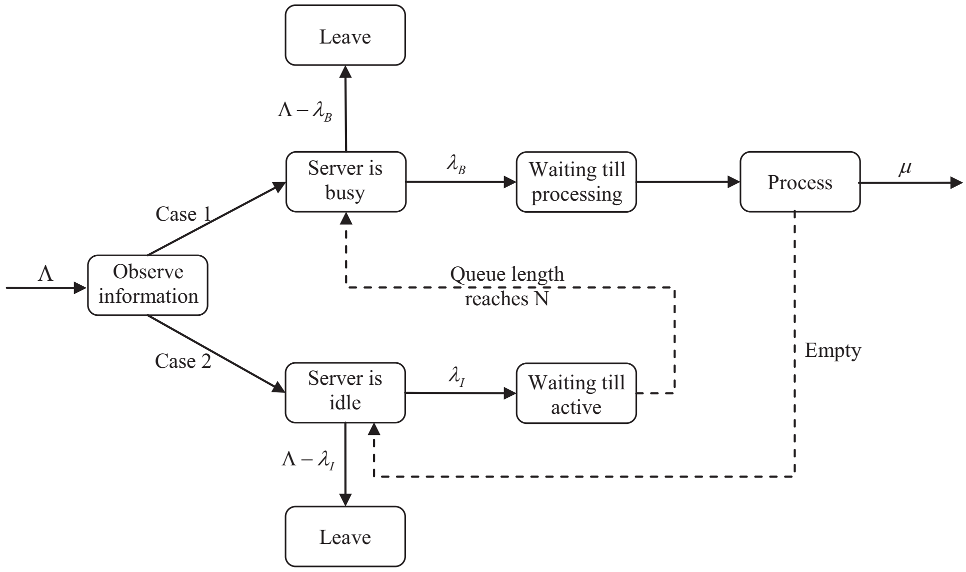


Figure 7.1: MTO system with unobservable queue length

Figure 7.1 depicts the way the strategic customers leave or stay in the production system. Upon observing the system state s, the staying customers enter the queue according to a Poisson process with rate λ_s, for $s = B, I$. Thus, our production system is a vacation queue with N-policy and a state-dependent arrival process. The following lemma provides the expected waiting time, conditional on the information state, for such a vacation queue.

Lemma 5 *Given the state-dependent arrival rates* (λ_B, λ_I)*, the expected waiting times of seeing states* B *and* I *are*

$$W_B = \frac{1}{\mu - \lambda_B} + \frac{N+1}{2\mu} \tag{7.1}$$

and

$$W_I = \frac{N-1}{2\lambda_I} + \frac{N+1}{2\mu}. \tag{7.2}$$

Proof: We first derive the steady-state probabilities. Let $\rho_s = \lambda_s/\mu$ for $s = B, I$. Let P_0 be probability that the system is empty, and let $P_{k,s}$ be the probability that the queue length is k and the server's status is s ($k = 1, 2, \ldots$; $s = B, I$). Clearly, $P_{N+k,I} = 0$, where $k = 0, 1, \ldots$. Denote $P_{0,I} = P_0$. The balance equations are

$$\begin{aligned}
&\lambda_I P_0 = \mu P_{1,B};\\
&\lambda_I P_0 + \lambda_B P_{k,B} = \mu P_{k+1,B}, \quad k = 1, 2, \ldots, N-1;\\
&\lambda_I P_{k-1,I} = \lambda_I P_{k,I}, \quad k = 1, 2, \ldots, N-1;\\
&\lambda_B P_{N+k,B} = \mu P_{N+k+1,B}, \quad k = 0, 1, \ldots.
\end{aligned}$$

Solving this set of equations, we obtain

$$P_{k,I} = P_0, \quad k = 1, 2, \ldots, N-1; \tag{7.3}$$

$$P_{k,B} = \frac{1-\rho_B^k}{1-\rho_B}\rho_I P_0, \quad k = 1, 2, \ldots, N; \tag{7.4}$$

$$P_{N+k,B} = \rho_B^k \frac{1-\rho_B^N}{1-\rho_B}\rho_I P_0, \quad k = 0, 1, \ldots. \tag{7.5}$$

From the probability normalization condition, we have

$$P_0 + \sum_{k=1}^{N-1} P_{k,I} + \sum_{k=1}^{N-1} P_{k,B} + \sum_{k=0}^{+\infty} P_{N+k,B} = 1.$$

By (7.3)–(7.5),

$$P_0 + \sum_{k=1}^{N-1} P_0 + \sum_{k=1}^{N-1} \frac{1-\rho_B^k}{1-\rho_B}\rho_I P_0 + \sum_{k=0}^{+\infty} \rho_B^k \frac{1-\rho_B^N}{1-\rho_B}\rho_I P_0 = 1,$$

which implies that

$$P_0\left[N + \frac{N-1}{1-\rho_B}\rho_I - \frac{\rho_B(1-\rho_B^{N-1})}{(1-\rho_B)^2}\rho_I + \frac{1-\rho_B^N}{(1-\rho_B)^2}\rho_I\right] = 1.$$

Upon simplification, we have

$$P_0 = \frac{1-\rho_B}{N(1-\rho_B+\rho_I)}.$$

Substituting this into (7.3)–(7.5), we have

$$
\begin{aligned}
P_{k,I} &= \frac{1-\rho_B}{N(1-\rho_B+\rho_I)}, \quad k = 1, 2, \ldots, N-1; \\
P_{k,B} &= \frac{\rho_I(1-\rho_B^k)}{N(1-\rho_B+\rho_I)}, \quad k = 1, 2, \ldots, N-1; \\
P_{N+k,B} &= \frac{\rho_I \rho_B^k (1-\rho_B^N)}{N(1-\rho_B+\rho_I)}, \quad k = 0, 1, \ldots.
\end{aligned}
$$

Let P_s be the probability that an arriving customer observes state s, for $s = B, I$. We have

$$
\begin{aligned}
P_I &= P_0 + \sum_{k=1}^{N-1} P_{k,I} = NP_0 = \frac{1-\rho_B}{1-\rho_B+\rho_I}; \\
P_B &= 1 - P_I = \frac{\rho_I}{1-\rho_B+\rho_I}.
\end{aligned}
$$

Next, we derive the conditional expected waiting times. Let $P(k|s)$ be the probability that the queue length is k conditional on observing server's status s, for $s = B, I$ and $k = 0, 1, \ldots$. When the arriving customer observes a busy server, the conditional probabilities are

$$
\begin{aligned}
P(k|B) &= \frac{P_{k,B}}{P_B} = \frac{1-\rho_B^k}{N}, \quad k = 1, 2, \ldots, N-1; \\
P(N+k|B) &= \frac{P_{N+k,B}}{P_B} = \frac{\rho_B^k(1-\rho_B^N)}{N}, \quad k = 0, 1, \ldots.
\end{aligned}
$$

Note that if the queue length is k and the server is busy, the expected waiting time of the arriving customer, denoted as $W(k|B)$, is the expected service time of the $k+1$ customers in the system. Hence, $W(k|B) = \frac{k+1}{\mu}$. Thus, the expected waiting time in seeing a busy server is

$$
\begin{aligned}
W_B &= \sum_{k=1}^{+\infty} W(k|B)P(k|B) \\
&= \sum_{k=1}^{N-1} \frac{k+1}{\mu} \cdot \frac{1-\rho_B^k}{N} + \sum_{k=0}^{+\infty} \frac{N+k+1}{\mu} \cdot \frac{\rho_B^k(1-\rho_B^N)}{N} \\
&= \frac{1}{\mu}\left[1 + \frac{1}{N}\sum_{k=1}^{N-1} k(1-\rho_B^k) + \frac{1-\rho_B^N}{N}\sum_{k=0}^{+\infty}(N+k)\rho_B^k\right] \\
&= \frac{1}{\mu}\left\{1+\frac{1}{N}\left[\frac{N(N-1)}{2} - \frac{\rho_B(1-\rho_B^{N-1})}{(1-\rho_B)^2} + \frac{(N-1)\rho_B^N}{1-\rho_B}\right] + \frac{1-\rho_B^N}{N}\left[\frac{N}{1-\rho_B} + \frac{\rho_B}{(1-\rho_B)^2}\right]\right\} \\
&= \frac{1}{\mu}\left(\frac{N+1}{2} + \frac{1}{1-\rho_B}\right) \\
&= \frac{1}{\mu-\lambda_B} + \frac{N+1}{2\mu}.
\end{aligned}
$$

When the arriving customer observes an idle server, the conditional probabilities and expected waiting times are

$$P(k|I) = \frac{P_{k,I}}{P_I} = \frac{1}{N}, \quad k = 0, 1, \ldots, N-1$$

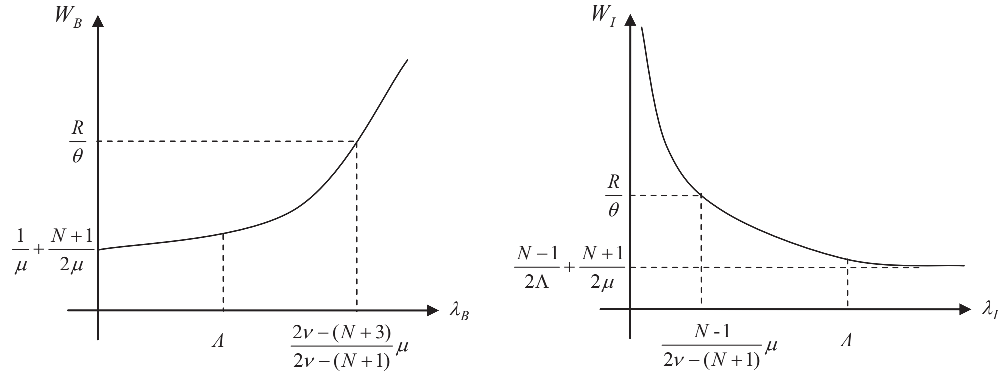
and

$$W(k|I) = \frac{N-(k+1)}{\lambda_I} + \frac{k+1}{\mu}, \quad k = 0, 1, \ldots, N-1,$$

where $\frac{N-(k+1)}{\lambda_I}$ is the expected time before the server becomes active and $\frac{k+1}{\mu}$ is the expected waiting time once the server starts working. Furthermore,

$$W_I = \sum_{k=0}^{N-1} W(k|I)P(k|I) = \sum_{k=0}^{N-1}\left[\frac{N-(k+1)}{\lambda_I} + \frac{k+1}{\mu}\right]\frac{1}{N} = \frac{N-1}{2\lambda_I} + \frac{N+1}{2\mu}. \quad \blacksquare$$

From (7.1), we can see that the expected waiting time upon seeing a busy server is the expected waiting time in a standard $M/M/1$ queue (i.e., $\frac{1}{\mu-\lambda_B}$) plus an extra amount (i.e., $\frac{N+1}{2\mu}$). In equation (7.2), the term $\frac{N-1}{2\lambda_I}$ measures the expected waiting time for the server to be activated, and the term $\frac{N+1}{2\mu}$ measures the expected waiting time once the server starts working. From (7.1) and (7.2), we can see that the expected waiting time W_B (W_I) is independent of the arrival rate λ_I (λ_B). Thus, we can determine the equilibrium arrival rate of seeing a busy and idle server separately.



(a) seeing a busy server

(b) seeing an idle server

Figure 7.2: Waiting time versus arrival rate

We first consider the equilibrium arrival rate of seeing a busy server, denoted as λ_B^e. From (7.1), the expected waiting time W_B is increasing with λ_B (see Figure 7.2(a)). This demonstrates the "avoid-the-crowd" customer behavior, and consequently at most one equilibrium arrival rate exists (see Hassin and Haviv 2003, pp. 6–7). If

$$\frac{R}{\theta} \leq \frac{1}{\mu} + \frac{N+1}{2\mu},$$

where the right-hand side is the smallest possible expected waiting time attained at $\lambda_B = 0$, then the customer's utility can never be positive no matter what the arrival rate λ_B is. Thus, if $N \geq 2\nu - 3$, there exists no positive equilibrium. Otherwise, there must exist an arrival rate which satisfies

$$\frac{R}{\theta} = \frac{1}{\mu - \lambda_B} + \frac{N+1}{2\mu}.$$

This equation has a unique solution of

$$\lambda_B = \mu - \frac{2\mu\theta}{2\mu R - (N+1)\theta} = \frac{2\nu - (N+3)}{2\nu - (N+1)}\mu.$$

Denote

$$\tilde{\Lambda} = \frac{2\nu - (N+3)}{2\nu - (N+1)}\mu.$$

Note that the effective demand rate λ_B will not exceed the potential arrival rate Λ. Thus, in this case, the equilibrium arrival rate is $\min\{\tilde{\Lambda}, \Lambda\}$. These results are summarized in the following theorem.

Theorem 9 *(i) If $N \geq 2\nu - 3$, there exists no positive equilibrium arrival rate, i.e., $\lambda_B^e = 0$. (ii) If $N < 2\nu - 3$, there exists a unique positive equilibrium arrival rate $\lambda_B^e = \min\{\tilde{\Lambda}, \Lambda\}$.*

Next, we consider the equilibrium arrival rate of seeing an idle server, denoted as λ_I^e. From (7.2), the expected waiting time W_I is decreasing in λ_I (see Figure 7.2(b)). This demonstrates the "follow-the-crowd" customer behavior, and consequently multiple equilibria could exist (see Hassin and Haviv 2003, pp. 6–7). Clearly, "all leaving upon seeing an idle server" is always a pure equilibrium strategy. This is because if all other customers choose to leave when observing an idle server, the system will never be active and the best choice for the tagged customer is "leaving upon seeing an idle server" too. In the following, we consider positive equilibrium arrival rates. If

$$\frac{R}{\theta} < \frac{N-1}{2\Lambda} + \frac{N+1}{2\mu},$$

where the right-hand side is the smallest possible expected waiting time attained at $\lambda_I = \Lambda$, then a customer's utility can never be positive no matter what the arrival rate λ_I is. Thus, if $N > \frac{2\rho\nu+1-\rho}{1+\rho}$, there exists no positive equilibrium, i.e., $\lambda_I^e = 0$. Otherwise, "all stay" is an equilibrium strategy, i.e., $\lambda_I^e = \Lambda$. In addition, if $N < \frac{2\rho\nu+1-\rho}{1+\rho}$, then there exists an equilibrium arrival rate $\lambda_I \in (0, \Lambda)$, which satisfies

$$\frac{R}{\theta} = \frac{N-1}{2\lambda_I} + \frac{N+1}{2\mu}.$$

This equation has a unique solution of

$$\lambda_I = \frac{(N-1)\mu\theta}{2\mu R - (N+1)\theta} = \frac{N-1}{2\nu - (N+1)}\mu,$$

which is also an equilibrium arrival rate. These results are summarized in the following theorem.

Theorem 10 *(i) If $N > \frac{2\rho\nu+1-\rho}{1+\rho}$, there exists no positive equilibrium arrival rate, i.e., $\lambda_I^e = 0$. (ii) If $N = \frac{2\rho\nu+1-\rho}{1+\rho}$, there exists a unique positive equilibrium arrival rate: $\lambda_I^e = \Lambda$. (iii) If $N < \frac{2\rho\nu+1-\rho}{1+\rho}$, there exist two positive equilibrium arrival rates: $\lambda_I^e = \frac{N-1}{2\nu-(N+1)}\mu < \Lambda$ and $\lambda_I^e = \Lambda$.*

Note that in property (iii) of Theorem 10, the larger equilibrium $\lambda_I^e = \Lambda$ is stable, whereas the smaller one $\lambda_I^e = \frac{N-1}{2\nu-(N+1)}\mu$ is not. The expected waiting time W_I is decreasing in λ_I (see Figure 7.2(b)). Thus, when an idle server is observed, the expected waiting time will be reduced if more customers choose to join the system, which in turn will attract even more customers to join the system until the larger equilibrium Λ is reached.

In Theorems 9 and 10, we obtain the equilibrium arrival rates of observing a busy server and an idle server, respectively. To maintain an active production system, we only consider the equilibrium strategies with $\lambda_I^e > 0$, i.e., $\lambda_I^e = \Lambda$. In addition, to execute the optimal control, we will restrict our attention to stable equilibrium strategies. There are three possible equilibrium strategies, namely $(\lambda_B^e, \lambda_I^e) = (\Lambda, \Lambda)$, $(\tilde{\Lambda}, \Lambda)$, and $(0, \Lambda)$, depending on the system parameters, where the equilibrium $(\tilde{\Lambda}, \Lambda)$ holds only when $0 < \tilde{\Lambda} < \Lambda$.

7.2 The expected average cost and optimal decision

In this section, we first derive an expression for the average cost of the production system. We then analyze its properties and determine the optimal value of the threshold N.

Suppose the customers' arrival rates are (λ_B, λ_I). We refer to the time point where the server's status changes from busy to idle as a *regeneration point*, and the time period between two consecutive regeneration points as a *production cycle*. We consider four types of costs in the production system: set-up cost, customer waiting cost, operation cost, and lost-sales penalty. Then, by the renewal theory (Ross 1996), the expected average cost of the system is given by

$$AC = \frac{K + \mathbb{E}[\text{operation cost in a cycle}] + \mathbb{E}[\text{waiting cost in a cycle}] + \mathbb{E}[\text{lost-sales penalty in a cycle}]}{\mathbb{E}[\text{cycle time}]}. \tag{7.6}$$

Let T be the expected time of a production cycle, and let T_I and T_B be the expected time of the server being idle and busy in the production cycle, respectively. Then, $T = T_I + T_B$. The expected operation cost in one production cycle is cT_B.

Note that the lost-sales demand units consist of two Poisson processes with rates $\Lambda - \lambda_B$ and $\Lambda - \lambda_I$ in the time periods when the server is busy and idle, respectively (see Figure 7.1). Thus, the expected lost-sales penalty in one production cycle is $\ell[(\Lambda - \lambda_B)T_B + (\Lambda - \lambda_I)T_I]$. We let $\mathcal{WC}$ be the expected customer waiting cost in one production cycle. Hence, from (7.6), the expected average cost of the system can be expressed as

$$AC = \frac{K + cT_B + \mathcal{WC} + \ell[(\Lambda - \lambda_B)T_B + (\Lambda - \lambda_I)T_I]}{T}, \tag{7.7}$$

which can be further specified, as stated in the following lemma.

Lemma 6 *Given the state-dependent arrival rates* (λ_B, λ_I)*, the expected average cost of the system is*

$$\begin{aligned} AC = & \frac{K\lambda_I(\mu - \lambda_B)}{N(\mu - \lambda_B + \lambda_I)} + \frac{c\lambda_I}{\mu - \lambda_B + \lambda_I} \\ & + h\left[\frac{\mu\lambda_I}{(\mu - \lambda_B + \lambda_I)(\mu - \lambda_B)} + \frac{N-1}{2}\right] + \ell\Big(\Lambda - \frac{\mu\lambda_I}{\mu - \lambda_B + \lambda_I}\Big). \end{aligned} \tag{7.8}$$

Proof: Note that T_I is the sum of N inter-arrival times of a Poisson process with rate λ_I. We call the time period from the moment when there are k customers in the system till the moment that the number of customers drops to $k-1$ for the first time a "1-busy period." Hence, T_B is the sum of N stochastically identical 1-busy periods of an $M/M/1$ queue with arrival rate λ_B and service rate μ. Thus,

$$T_I = \frac{N}{\lambda_I} \quad \text{and} \quad T_B = \frac{N}{\mu - \lambda_B}, \tag{7.9}$$

which imply that

$$T = \frac{N}{\lambda_I} + \frac{N}{\mu - \lambda_B} = \frac{N(\mu - \lambda_B + \lambda_I)}{\lambda_I(\mu - \lambda_B)}. \tag{7.10}$$

Next, we consider the expected customer waiting cost in one production cycle, $\mathcal{WC}$. Let $\mathcal{WC}_I$ and $\mathcal{WC}_B$ be the expected waiting cost incurred in the periods when the server is idle and busy, respectively. Then, $\mathcal{WC} = \mathcal{WC}_I + \mathcal{WC}_B$. It is easy to see that during the server's idle period, the total waiting cost incurred is

$$\mathcal{WC}_I = \frac{h}{\lambda_I} + \frac{2h}{\lambda_I} + \cdots + \frac{(N-1)h}{\lambda_I} = \frac{N(N-1)h}{2\lambda_I}.$$

To determine $\mathcal{WC}_B$, suppose that there are k customers in the system, where $1 \leq k \leq N$, and that the server has just started processing a customer order. Let τ_k be the expected total waiting cost incurred in a 1-busy period (i.e., the time period till the number of customers in the system drops to $k-1$ for the first time). Because the N 1-busy periods are stochastically identical, we have

$$\tau_k = \frac{h}{\mu - \lambda_B} + \tau_{k-1}$$

for $k = 2, 3, \ldots, N$, where $\frac{1}{\mu - \lambda_B}$ is the duration of a 1-busy period. The above recursive equation implies that

$$\tau_k = \frac{(k-1)h}{\mu - \lambda_B} + \tau_1$$

for $k = 1, 2, \ldots, N$. Note that τ_1 is the total waiting cost incurred in a 1-busy period of an $M/M/1$ queue with 1 customer in the system at the beginning. In this 1-busy period, the expected queue length is $\mu/(\mu - \lambda_B)$, and the expected duration of the 1-busy period is $1/(\mu - \lambda_B)$. Thus,

$$\tau_1 = \frac{\mu h}{(\mu - \lambda_B)^2}.$$

Hence,

$$\mathcal{WC}_B = \tau_N + \tau_{N-1} + \cdots + \tau_1 = \sum_{k=1}^{N} \left[\frac{(k-1)h}{\mu - \lambda_B} + \tau_1 \right] = \left[\frac{N(N-1)}{2(\mu - \lambda_B)} + \frac{N\mu}{(\mu - \lambda_B)^2} \right] h.$$

Therefore,

$$\mathcal{WC} = \mathcal{WC}_I + \mathcal{WC}_B = \left[\frac{N(N-1)}{2\lambda_I} + \frac{N(N-1)}{2(\mu - \lambda_B)} + \frac{N\mu}{(\mu - \lambda_B)^2} \right] h. \tag{7.11}$$

Substituting (7.9)–(7.11) into (7.7) and simplifying the expression, we obtain equation (7.8). ∎

The following theorem states that for any equilibrium arrival rates $(\lambda_B^e, \lambda_I^e)$, the expected average cost is always convex in N. Note that N is an integer. For simplicity, when we analyze the convexity and local minimum of the expected average cost function, we ignore the integrality of N and treat the domain of the function as continuous.

Theorem 11 *(i) If* $(\lambda_B^e, \lambda_I^e) = (0, \Lambda)$*, then* AC *is convex in* N*, and the local minimum is attained at* $\eta_1 = \sqrt{\frac{2K\Lambda}{(1+\rho)h}}$*. (ii) If* $(\lambda_B^e, \lambda_I^e) = (\tilde{\Lambda}, \Lambda)$*, which incurs only if* $0 < \tilde{\Lambda} < \Lambda$*, then* AC *is convex in* N*, and the local minimum is attained at* $\eta_2 = \frac{-K\Lambda+\sqrt{\delta}}{-c+(h/\rho)+\ell\mu}$*, where* $\delta = K^2\Lambda^2 + K[-c + (h/\rho) + \ell\mu][2\mu + \Lambda(2\nu - 1)]$*. (iii) If* $(\lambda_B^e, \lambda_I^e) = (\Lambda, \Lambda)$*, which incurs only if* $\rho < 1$*, then* AC *is convex in* N*, and the local minimum is attained at* $\eta_3 = \sqrt{\frac{2K\Lambda(1-\rho)}{h}}$*.*

Proof: (i) $(\lambda_B^e, \lambda_I^e) = (0, \Lambda)$. In this case, the expected average cost (7.8) can be simplified to

$$AC = \frac{K\Lambda\mu}{N(\mu+\Lambda)} + \frac{c\Lambda}{\mu+\Lambda} + h\Big(\frac{\Lambda}{\mu+\Lambda} + \frac{N-1}{2}\Big) + \ell\Big(\Lambda - \frac{\mu\Lambda}{\mu+\Lambda}\Big).$$

Thus,

$$\frac{dAC}{dN} = -\frac{K\Lambda\mu}{(\mu+\Lambda)N^2} + \frac{h}{2} \quad \text{and} \quad \frac{d^2AC}{dN^2} = \frac{2K\Lambda\mu}{(\mu+\Lambda)N^3} > 0.$$

Hence, AC is convex in N. It is easy to check that, by solving the equation "$dAC/dN = 0$," the minimum of AC is attained at $\eta_1 = \sqrt{\frac{2K\Lambda\mu}{(\mu+\Lambda)h}} = \sqrt{\frac{2K\Lambda}{(1+\rho)h}}$.

(ii) $(\lambda_B^e, \lambda_I^e) = (\tilde{\Lambda}, \Lambda)$, where $0 < \tilde{\Lambda} < \Lambda$. In this case, we consider the four terms in (7.8) separately. Let AC_i be the ith term on the right hand side of (7.8) for $i = 1, 2, 3, 4$. Let $\Omega = 2\nu - (N+1)$. Then, $\tilde{\Lambda} = \mu(\Omega - 2)/\Omega$. Consider the first term. We have

$$AC_1 = \frac{K\lambda_I(\mu - \lambda_B)}{N(\mu - \lambda_B + \lambda_I)} = \frac{K\Lambda(\mu - \tilde{\Lambda})}{N(\mu - \tilde{\Lambda} + \Lambda)} = \frac{2K\mu\Lambda}{N(2\mu + \Lambda\Omega)},$$

which implies that

$$\frac{dAC_1}{dN} = \frac{-2K\mu\Lambda(2\mu + \Lambda\Omega - \Lambda N)}{N^2(2\mu+\Lambda\Omega)^2} = \frac{-2K\mu\Lambda}{N^2(2\mu+\Lambda\Omega)^2}\Big\{2\mu + \Lambda\big[2\nu - (2N+1)\big]\Big\}$$

and

$$\begin{aligned}
&\frac{d^2AC_1}{dN^2} \\
&= -2K\mu\Lambda \cdot \frac{-2\Lambda N^2(2\mu+\Lambda\Omega)^2 - (2\mu+\Lambda\Omega-\Lambda N)[2N(2\mu+\Lambda\Omega)^2 + N^2 2(2\mu+\Lambda\Omega)(-\Lambda)]}{N^4(2\mu+\Lambda\Omega)^4} \\
&= 4K\mu\Lambda\left[\frac{\Lambda}{N^2(2\mu+\Lambda\Omega)^2} + \frac{(2\mu+\Lambda\Omega-\Lambda N)^2}{N^3(2\mu+\Lambda\Omega)^3}\right] > 0.
\end{aligned}$$

Consider the second term. We have

$$AC_2 = \frac{c\lambda_I}{\mu - \lambda_B + \lambda_I} = \frac{c\Lambda}{\mu - \tilde{\Lambda} + \Lambda} = \frac{c\Lambda\Omega}{2\mu + \Lambda\Omega},$$

which implies that

$$\frac{dAC_2}{dN} = \frac{c\Lambda[-(2\mu + \Lambda\Omega) - \Omega(-\Lambda)]}{(2\mu + \Lambda\Omega)^2} = \frac{-2c\mu\Lambda}{(2\mu + \Lambda\Omega)^2}$$

and

$$\frac{d^2AC_2}{dN^2} = \frac{-4c\mu\Lambda^2}{(2\mu + \Lambda\Omega)^3}. \tag{7.12}$$

Consider the third term. We have

$$\begin{aligned} AC_3 =& h\left[\frac{\mu\lambda_I}{(\mu - \lambda_B + \lambda_I)(\mu - \lambda_B)} + \frac{N-1}{2}\right] \\ =& h\left[\frac{\mu\Lambda}{(\mu - \tilde{\Lambda} + \Lambda)(\mu - \tilde{\Lambda})} + \frac{N-1}{2}\right] = h\left[\frac{\Lambda\Omega^2}{2(2\mu + \Lambda\Omega)} + \frac{N-1}{2}\right], \end{aligned}$$

which implies that

$$\frac{dAC_3}{dN} = h\left[\frac{(2\mu + \Lambda\Omega)(-2\Lambda\Omega) - \Lambda\Omega^2(-\Lambda)}{2(2\mu + \Lambda\Omega)^2} + \frac{1}{2}\right] = \frac{2\mu^2 h}{(2\mu + \Lambda\Omega)^2}$$

and

$$\frac{d^2AC_3}{dN^2} = \frac{4\mu^2\Lambda h}{(2\mu + \Lambda\Omega)^3}. \tag{7.13}$$

Consider the fourth term. We have

$$AC_4 = \ell\Big(\Lambda - \frac{\mu\lambda_I}{\mu - \lambda_B + \lambda_I}\Big) = \ell\Big(\Lambda - \frac{\mu\Lambda}{\mu - \tilde{\Lambda} + \Lambda}\Big) = \ell\Big(\Lambda - \frac{\mu\Lambda\Omega}{2\mu + \Lambda\Omega}\Big),$$

which implies that

$$\frac{dAC_4}{dN} = \frac{-\ell\mu\Lambda\left[-(2\mu + \Lambda\Omega) - \Omega(-\Lambda)\right]}{(2\mu + \Lambda\Omega)^2} = \frac{2\ell\mu^2\Lambda}{(2\mu + \Lambda\Omega)^2}$$

and

$$\frac{d^2AC_4}{dN^2} = \frac{4\ell\mu^2\Lambda^2}{(2\mu + \Lambda\Omega)^3}. \tag{7.14}$$

From (7.12)–(7.14), we have

$$\frac{d^2AC}{dN^2} = \sum_{i=1}^{4} \frac{d^2AC_i}{dN^2} = \frac{d^2AC_1}{dN^2} + \frac{4\mu^2\Lambda^2\big(-\frac{c}{\mu} + \frac{h}{\Lambda} + \ell\big)}{(2\mu + \Lambda\Omega)^3} > 0,$$

where the last inequality holds because $d^2AC_1/dN^2 > 0$ and $\ell > \frac{c}{\mu}$. Hence, AC is convex in N. It is easy to check that, by determining the positive root of the equation "$\sum_{i=1}^{4} dAC_i/dN = 0$," the minimum of AC is attained at $\eta_2 = \frac{-K\Lambda+\sqrt{\delta}}{-c+(h/\rho)+\ell\mu}$, where $\delta = K^2\Lambda^2 + K[-c+(h/\rho)+\ell\mu][2\mu+\Lambda(2\nu-1)]$.

(iii) $(\lambda_B^e, \lambda_I^e) = (\Lambda, \Lambda)$. From property (ii) of Theorem 9, $\lambda_B^e = \Lambda$ incurs if $N < 2\nu - 3$ and $\tilde{\Lambda} \geq \Lambda$. This implies that $\rho < 1$. In this case, the expected average cost (7.8) can be simplified to

$$AC = \frac{K\Lambda(\mu-\Lambda)}{N\mu} + \frac{c\Lambda}{\mu} + h\Big(\frac{\Lambda}{\mu-\Lambda} + \frac{N-1}{2}\Big),$$

which implies that

$$\frac{dAC}{dN} = -\frac{K\Lambda(\mu-\Lambda)}{N^2\mu} + \frac{h}{2} \quad \text{and} \quad \frac{d^2AC}{dN^2} = \frac{2K\Lambda(\mu-\Lambda)}{N^3\mu} > 0,$$

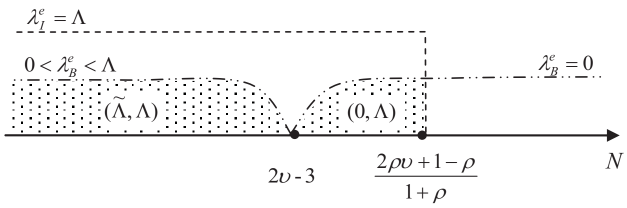
where the inequality holds because $\rho < 1$ (i.e., $\Lambda < \mu$). Hence, AC is convex in N. It is easy to check that, by solving the equation "$dAC/dN = 0$," the minimum of AC is attained at $\eta_3 = \sqrt{\frac{2K\Lambda(1-\rho)}{h}}$. ∎

Note that $(\lambda_B^e, \lambda_I^e)$ actually changes with N. To consider the cost function with different equilibrium arrival rates in different ranges of N, we consider four different cases depending on the values of ρ and ν. For each case, we derive the average cost function and discuss the convexity of the function. We explore the optimal decision of N numerically. We consider the following setting for our numerical studies: $\Lambda = 10$, $c = 10$, $h = 5$, $\ell = 50$, and $K = 1000$. The values of ρ and ν, however, are changeable to represent different cases. We also restrict our attention to $N \in \big[1, \frac{2\rho\nu+1-\rho}{1+\rho}\big]$, since otherwise $\lambda_I^e = 0$ and the system is never active.

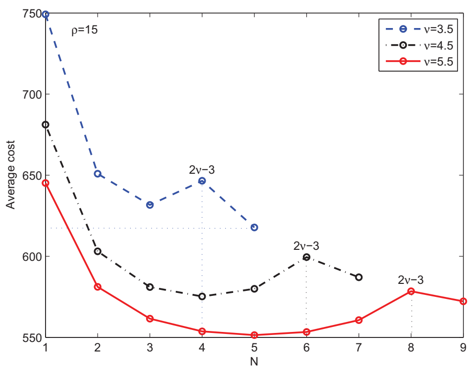
Case 1: $\rho \geq 1$ and $\nu \leq 2 + \rho$. In this case, it is easy to check that

$$2\nu - 3 \leq \frac{2\rho\nu + 1 - \rho}{1+\rho}.$$

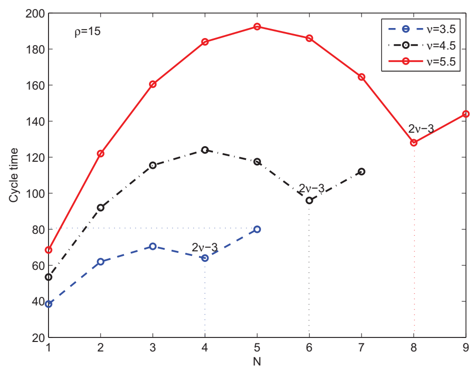
Thus, if $N \in [1, 2\nu - 3)$, the equilibrium arrival rates are $(\tilde{\Lambda}, \Lambda)$; and if $N \in \big[2\nu - 3, \frac{2\rho\nu+1-\rho}{1+\rho}\big]$, the equilibrium arrival rates are $(0, \Lambda)$; see Figure 7.3(a) for illustration. Once the optimal N is set within the interval $\big[2\nu - 3, \frac{2\rho\nu+1-\rho}{1+\rho}\big]$, the customer's strategic behavior appears to be paradoxical: if the server is not working, all customers will stay in the system and wait; once the server begins to work, every incoming customer will leave without placing order. Intuitively, such customer behavior will result in frequent set-ups of the production system and hurt the system performance. However, could it be possible that such behavior is desirable? We shall explore this possibility.



(a) the equilibrium arrival rates



(b) AC versus N



(c) T versus N

Figure 7.3: Case 1 with $\rho \geq 1$ and $\nu \leq 2 + \rho$.

Actually, the second interval $\left[2\nu - 3, \frac{2\rho\nu+1-\rho}{1+\rho}\right]$ is quite narrow by noticing that

$$\frac{2\rho\nu + 1 - \rho}{1+\rho} - (2\nu - 3) = 2 + \frac{2}{1+\rho} - \frac{2\nu}{1+\rho} < 3.$$

Hence, the interval $\left[2\nu - 3, \frac{2\rho\nu+1-\rho}{1+\rho}\right]$ contains at most two integer values. However, according to Theorem 11, the expected average cost is convex in N in each of the two intervals $[1, 2\nu - 3)$ and $\left[2\nu - 3, \frac{2\rho\nu+1-\rho}{1+\rho}\right]$. Consequently, there could exist two local optima for N. Therefore, it is rare but still possible that the optimal N falls in the second interval.

We present three numerical examples to illustrate the optimal values of N by setting $\rho = 15$ and $\nu = 3.5, 4.5, 5.5$. The impacts of the threshold value N on the expected average cost and on the expected cycle time are depicted in Figures 7.3(b) and 7.3(c), respectively. Take the case with $\nu = 4.5$ as an example. We have $2\nu - 3 = 6$ and $\frac{2\rho\nu+1-\rho}{1+\rho} = 7.56$. Thus, the average cost is convex in the intervals $[1, 6)$ and $[6, 7.56]$ (see Figure 7.3(b)), where the second interval $[6, 7.56]$ contains two integers (hence in the figure we can only observe that the average cost is decreasing).

For the case with $\nu = 3.5$, the minimum average cost is attained at $N = 5$, which indeed falls inside the second interval and results in the interesting equilibrium arrival rates $(0, \Lambda)$ (see Figure 7.3(b)). This numerical result shows that it is indeed possible to induce customers to adopt such paradoxical behavior. We use the numbers in this example to illustrate the reason. When the system utilization is high and the customers are unwilling to wait, setting a smaller N might be less effective in enlarging the production cycle than setting a larger N. In our example with $\rho = 15$ and $\nu = 3.5$, we have $\mu = \frac{\Lambda}{\rho} = \frac{10}{15} = \frac{2}{3}$. Consider the cases with $N = 3$ and $N = 5$. When $N = 3$, because $N < 2\nu - 3$, by Theorem 9,

$$\lambda_B^e = \min\left\{\tilde{\Lambda}, \Lambda\right\} = \min\left\{\frac{2\nu - (3+3)}{2\nu - (3+1)}\mu, \Lambda\right\} = \min\left\{\frac{2}{9}, 10\right\} = \frac{2}{9},$$

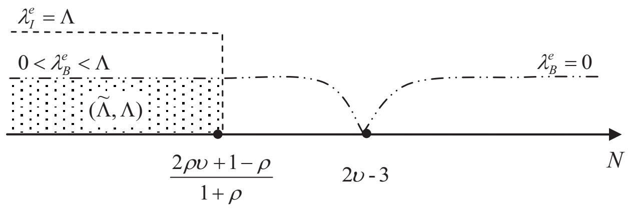
and the expected cycle time is $\frac{N}{\Lambda} + \frac{N}{\mu - \lambda_B^e} = \frac{3}{10} + \frac{3}{(2/3)-(2/9)} = \frac{141}{20}$. When $N = 5$, because $N \geq 2\nu - 3$, by Theorem 9, $\lambda_B^e = 0$, and the expected cycle time is $\frac{N}{\Lambda} + \frac{N}{\mu - \lambda_B^e} = \frac{5}{10} + \frac{5}{(2/3)-0} = 8$. Hence, the expected cycle time is shorter when $N = 3$ than when $N = 5$ (see Figure 7.3(c)). Therefore, although customers' strategic behavior can result in frequent set-ups, the larger N slows down the set-up frequencies and makes it smaller.

Case 2: $\rho \geq 1$ and $\nu > 2 + \rho$. In this case, it is easy to check that

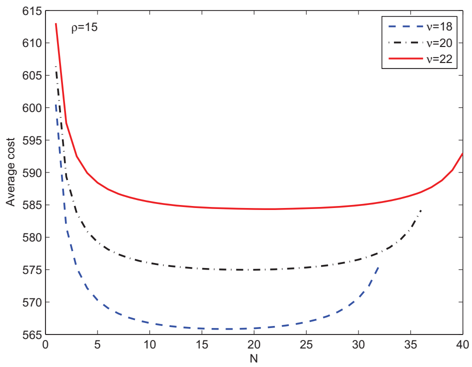
$$2\nu - 3 > \frac{2\rho\nu + 1 - \rho}{1 + \rho}.$$

Thus, for $N \in \left[1, \frac{2\rho\nu+1-\rho}{1+\rho}\right]$, the only possible equilibrium arrival rates are $(\tilde{\Lambda}, \Lambda)$ (see Figure 7.4(a)).

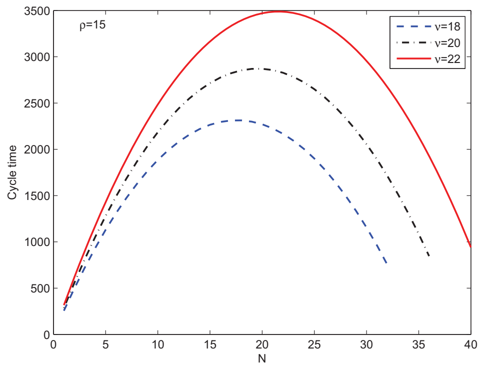
According to Theorem 11, with the equilibrium arrival rates $(\tilde{\Lambda}, \Lambda)$, the expected average cost is convex in N. We consider three numerical examples with $\rho = 15$ and $\nu = 18, 12, 22$. The impacts of the threshold value on the expected average cost and on the expected cycle time are depicted in Figures 7.4(b) and 7.4(c). Take the



(a) the equilibrium arrival rates



(b) AC versus N



(c) T versus N

Figure 7.4: Case 2 with $\rho \geq 1$ and $\nu > 2 + \rho$.

case with $\nu = 20$ as an example. When N is small (say, when $N \leq 5$), the average cost sharply decreases. This is because the cycle time is short and sensitive in N (see Figure 7.4(c)). The production system has frequent set-ups, which result in a significant average set-up cost. When N increases to a certain level, the average cost becomes insensitive in N (say, when $N \in [10, 25]$), mainly due to the strategic behavior of the customers. As N increases, fewer potential customers will stay in the system once the server is activated. Thus, the expected operation cost and expected customer waiting cost decrease, but the lost-sales penalty cost increases (see the proof of Theorem 11). Customers' strategic behavior will balance different costs to maintain a stable average cost. When N increases to the level near the upper bound (i.e., $\frac{2\rho\nu+1-\rho}{1+\rho}$), the average cost mildly increases. This is because N is too large such that only a small portion of the potential customers will stay in the system, which reduces the cycle time (see Figure 7.4(c)). Another observation is that the interval of N with an insensitive average cost becomes narrower as ν

decreases (see Figure 7.4(b)). This implies that if the customers are less willing to wait, then the decision maker of the production system should be more cautious about the decision on N. In general, the average cost is convex in N, and the global optimal value N^* is given as

$$N^* = \min\left\{\eta_2, \frac{2\rho\nu + 1 - \rho}{1+\rho}\right\}.$$

Case 3: we consider the situation where $\rho < 1$. *Case 3(i):* $\nu \leq 2 + \rho$. In this case, it is easy to check that

$$2\nu - 3 \leq \frac{2\rho\nu + 1 - \rho}{1+\rho}.$$

Note that

$$\frac{d}{d\rho}\frac{2\rho\nu + 1 - \rho}{1+\rho} = \frac{2(v-1)}{(1+\rho)^2} > 0,$$

which implies that $\frac{2\rho\nu+1-\rho}{1+\rho}$ is increasing in ρ. Thus,

$$\frac{2\rho\nu + 1 - \rho}{1+\rho} < \frac{2\nu + 1 - 1}{1+1} = \nu \leq 2 + \rho < 3;$$

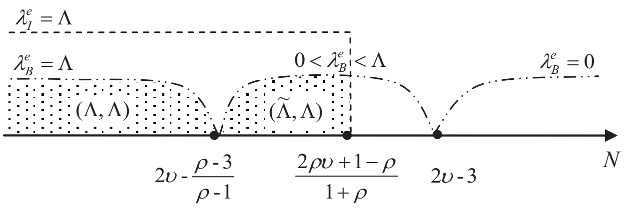
that is, there are only two possible values of N such that the system can be activated: $N = 1$ and $N = 2$. Hence, a simple comparison between the values of AC with $N = 1$ and $N = 2$ yields the optimal decision.

Case 3(ii): $2 + \rho < \nu \leq \frac{2}{1-\rho}$. In this case, it is easy to check that

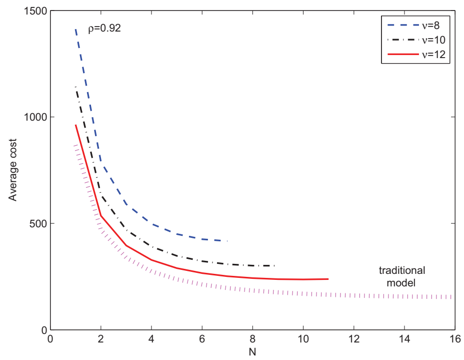
$$2\nu - \frac{\rho - 3}{\rho - 1} \leq \frac{2\rho\nu + 1 - \rho}{1+\rho} < 2\nu - 3.$$

Thus, the equilibrium arrival rates are (Λ, Λ) for $N \in \left[1, 2\nu - \frac{\rho-3}{\rho-1}\right]$ and $(\tilde{\Lambda}, \Lambda)$ for $N \in \left(2\nu - \frac{\rho-3}{\rho-1}, \frac{2\rho\nu+1-\rho}{1+\rho}\right]$; see Figure 7.5(a) for illustration. From Theorem 11, the expected average cost under the equilibrium strategy is convex in the two intervals.

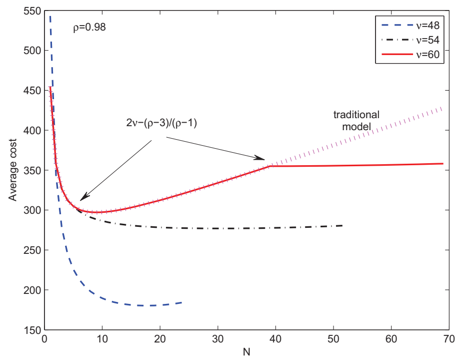
We now present six numerical examples to illustrate the optimal decisions and minimal cost. Since $\rho < 1$, we also plot the result of the traditional model with a constant arrival rate Λ. For the three examples depicted in Figures 7.5(b), we set $\rho = 0.92$ and $\nu = 8, 10, 12$. The value of ν is small such that $2\nu - \frac{\rho-3}{\rho-1} < 1$, and only one interval exists which allows the cost function to be convex. The average cost



(a) the equilibrium arrival rates



(b) AC versus N



(c) AC versus N

Figure 7.5: Case 3(ii) with $\rho < 1$ and $2 + \rho < \nu \leq \frac{2}{1-\rho}$.

with strategic customer behavior is larger than that of the traditional model, and the optimal value N^* is smaller than that in the traditional model.

However, this is not always the case. For the other three examples depicted in Figures 7.5(c), we set $\rho = 0.98$ and $\nu = 25, 53, 70$. In those examples, the optimal N is no less than that in the traditional model, and the optimal average cost is also no larger than that in the traditional model. The strategic customer behavior actually benefits the production system.

Case 3(iii): $\nu > 2 + \rho$ and $\nu > \frac{2}{1-\rho}$. In this case, it is easy to check that

$$\frac{2\rho\nu + 1 - \rho}{1 + \rho} < 2\nu - \frac{\rho - 3}{\rho - 1} < 2\nu - 3.$$

Thus, the equilibrium arrival rates are (Λ, Λ) for $N \in \left[1, \frac{2\rho\nu+1-\rho}{1+\rho}\right]$; see Figure 7.6 for illustration.

Note that the decentralized equilibrium arrival rate is equal to the total arrival rate. Hence, the impact of the threshold value on the average cost is the same

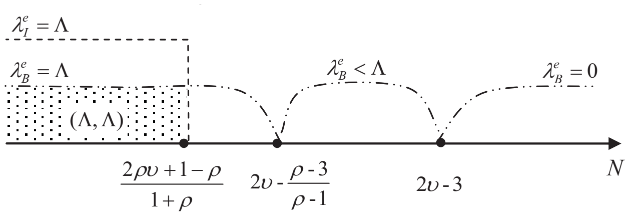


Figure 7.6: Case 3(iii) with $\rho < 1$, $\nu > 2 + \rho$, and $\nu > \frac{2}{1-\rho}$.

as that in the traditional model as long as $N \leq \frac{2\rho\nu+1-\rho}{1+\rho}$. From Theorem 11, the expected average cost is convex and the optimal value can be specified as

$$N^* = \min\left\{\eta_3, \frac{2\rho\nu + 1 - \rho}{1 + \rho}\right\},$$

where $\eta_3 = \sqrt{\frac{2K\Lambda(1-\rho)}{h}}$ is the optimal value of N for the traditional $M/M/1$ queue with N-policy and a constant arrival rate Λ (see Heyman and Sobel 1984, p. 336). In this case, the optimal average cost with strategic customers is no larger than that in the traditional model.

Chapter 8

System with Observable Queue Length

Trade show is a common marketing tool nowadays. For some commodities which cannot be stored in the trade show directly, sellers usually just accept customer orders and guarantee that their orders will be delivered once a certain number of orders have been accumulated. In this case, the existing customer orders are often shown to an incoming customer, or equivalently, the queue length is observable. As commodities are usually standard with unified price, the seller's profit-maximization problem is identical to a cost-minimization problem with lost-sales penalty being counted. In this chapter, we consider this information scenario where customers can observe the queue length. We will first show that no matter the status of the server is observable or not, the equilibrium strategies are the same. Consequently, this information scenario represents both almost observable case (queue is observable but no server status) and fully observable case (both queue and server status are observable). We then derive the expected average cost function to investigate the optimal decision on N.

8.1 The customer equilibrium strategy

We first consider the case where the server status is unobservable; that is, the observable system states is $S = \{0, 1, \ldots\}$. Since the queue length is observable, we consider a threshold strategy for customers, under which there exists a threshold $\bar{n}$, and customers will stay if the queue length is less than $\bar{n}$, and will leave otherwise.

Since the server's status is unobservable, to ensure that the server can be activated, any customers who observe states $0, 1, \ldots, N-1$ should stay in the system. Hence, two conditions must be satisfied to guarantee the system to be activated. The first condition is that the threshold value of the threshold strategy must be not less than N, i.e., $\bar{n} \geq N$. The second one is that the expected utilities of the customers who observe status $0, 1, \ldots, N-1$ should be nonnegative. The second condition need to be further specified from the expected waiting times. In the following, we analyze these two conditions in detail.

We let W_k be expected waiting times of seeing the queue length k, for $0 \leq k \leq \bar{n}-1$. The following lemma provides some properties of W_k.

Lemma 7 *For the $M/M/1$ queue with N-policy and partially observable information on the queue length, if the threshold strategy with threshold value $\bar{n}$ is applied, where $\bar{n} \geq N$, then (i)*

$$W_k = \begin{cases} \frac{k+1}{\mu} + \frac{(1-\rho)[N-(k+1)]}{\Lambda(1-\rho^{k+1})}, & \textit{if } k = 0, 1, \ldots, N-1; \\ \frac{k+1}{\mu}, & \textit{if } k = N, N+1, \ldots, \bar{n}-1; \end{cases} \tag{8.1}$$

and (ii) W_k is convex in k for $k \in [0, N-1]$.

Proof: (i) Suppose the threshold strategy $\bar{n}$, where $\bar{n} \geq N$, is adopted and an arriving customer observes state k. Then, $0 \leq k \leq \bar{n}-1$. If $N \leq k \leq \bar{n}-1$, the server must be busy and the expected waiting time is $W_k = \frac{k+1}{\mu}$; and if $0 \leq k \leq N-1$, the expected waiting time depends on the steady-state probabilities of the system. When only the queue length is observable and the threshold strategy is adopted, the steady-state probabilities are the same as those in the fully observable case which have been discussed in Guo and Hassin (2011) (see *Case 1* in Section 5.2 and corresponding content in the e-companion therein). Since we use different notations, we simply present the results for the sake of easy understanding. The balance equations are

$$\begin{aligned}
&\Lambda P_0 = \mu P_{1,B};\\
&\Lambda P_0 + \Lambda P_{k,B} = \mu P_{k+1,B}, \quad k = 1, 2, \ldots, N-1;\\
&\Lambda P_{k-1,I} = \Lambda P_{k,I}, \quad k = 1, 2, \ldots, N-1;\\
&\Lambda P_{N+k,B} = \mu P_{N+k+1,B}, \quad k = 0, 1, \ldots, \bar{n}-N-1;
\end{aligned}$$

and the probability normalization condition is

$$P_0 + \sum_{k=1}^{N-1} P_{k,I} + \sum_{k=1}^{N-1} P_{k,B} + \sum_{k=0}^{\bar{n}-N} P_{N+k,B} = 1. \tag{8.2}$$

We first restrict our attention to the general case with $\rho \neq 1$. Solving the set of balance equations and normalizing the probabilities by (8.2), we obtain

$$\begin{aligned}
&P_{k,B} = \frac{\rho-\rho^{k+1}}{1-\rho} P_0, \quad k = 1, 2, \ldots, N;\\
&P_{k,I} = P_0, \quad k = 1, 2, \ldots, N-1;\\
&P_{N+k,B} = \rho^j P_{N,B}, \quad k = 1, 2, \ldots, \bar{n}-N;
\end{aligned}$$

where

$$P_0 = \frac{(1-\rho)^2}{N - N\rho - \rho^{\bar{n}-N+2} + \rho^{\bar{n}+2}}.$$

Upon observing state k, where $k \le N-1$, the server might be busy with probability $\frac{P_{k,B}}{P_{k,B}+P_{k,I}}$, where the expected waiting time of the tagged customer is $\frac{k+1}{\mu}$; and the server might also be idle with probability $\frac{P_{k,I}}{P_{k,B}+P_{k,I}}$, when the expected waiting time of the tagged customer is $\frac{N-(k+1)}{\Lambda} + \frac{k+1}{\mu}$. Thus, the expected waiting time of an arriving customer who observes state k is

$$\begin{aligned}
W_k &= \frac{P_{k,B}}{P_{k,B}+P_{k,I}} \cdot \frac{k+1}{\mu} + \frac{P_{k,I}}{P_{k,B}+P_{k,I}} \Big[\frac{N-(k+1)}{\Lambda} + \frac{k+1}{\mu}\Big]\\
&= \frac{\frac{\rho-\rho^{k+1}}{1-\rho} P_0}{P_0 + \frac{\rho-\rho^{k+1}}{1-\rho} P_0} \cdot \frac{k+1}{\mu} + \frac{P_0}{P_0 + \frac{\rho-\rho^{k+1}}{1-\rho} P_0} \Big[\frac{N-(k+1)}{\Lambda} + \frac{k+1}{\mu}\Big]\\
&= \frac{k+1}{\mu} + \frac{1-\rho}{1-\rho^{k+1}} \cdot \frac{N-(k+1)}{\Lambda}.
\end{aligned} \tag{8.3}$$

If $\rho = 1$, then $\Lambda = \mu$. Solving the set of balance equations and normalizing the steady-state probabilities by (8.2), we have

$$\begin{aligned}
P_{k,I} &= \frac{2}{N(2\bar{n} - N + 3)}, \quad k = 1, 2, \ldots, N-1; \\
P_{k,B} &= \frac{2k}{N(2\bar{n} - N + 3)}, \quad k = 1, 2, \ldots, N-1; \\
P_{N+k,B} &= \frac{2}{2\bar{n} - N + 3}, \quad k = 0, 1, \ldots, \bar{n} - N. \qquad (8.4)
\end{aligned}$$

Similarly, the tagged customer who observes state k, where $0 \le k \le N-1$, satisfies

$$\begin{aligned}
W_k &= \frac{P_{k,B}}{P_{k,B} + P_{k,I}} \cdot \frac{k+1}{\mu} + \frac{P_{k,I}}{P_{k,B} + P_{k,I}} \left[\frac{N-(k+1)}{\Lambda} + \frac{k+1}{\mu} \right] \\
&= \frac{k}{k+1} \cdot \frac{k+1}{\mu} + \frac{1}{k+1} \left[\frac{N-(k+1)}{\Lambda} + \frac{k+1}{\mu} \right] \\
&= \frac{k+1}{\mu} + \frac{1}{k+1} \cdot \frac{N-(k+1)}{\Lambda} \\
&= \frac{k}{\Lambda} + \frac{N}{(k+1)\Lambda}. \qquad (8.5)
\end{aligned}$$

Compare (8.3) and (8.5), we can see that

$$\begin{aligned}
&\lim_{\rho \to 1} \left\{ \frac{k+1}{\mu} + \frac{1-\rho}{1-\rho^{k+1}} \cdot \frac{N-(k+1)}{\Lambda} \right\} \\
=& \lim_{\rho \to 1} \left\{ \frac{(k+1)\rho}{\Lambda} \right\} + \lim_{\rho \to 1} \left\{ \frac{-1}{-(k+1)\rho^k} \cdot \frac{N-(k+1)}{\Lambda} \right\} \\
=& \frac{k}{\Lambda} + \frac{N}{(k+1)\Lambda},
\end{aligned}$$

where l'Hôpital's rule is applied in the first equation. This completes the proof of property (i).

(ii) If $\rho = 1$, it is easy to check from (8.5) that $d^2 W_k / dk^2 = 2N/[(k+1)^3 \Lambda] > 0$, which implies that W_k is convex in k for $k \in [0, N-1]$. Next, we consider the case where $\rho \ne 1$. For any real number x, define $W(x) = \frac{x}{\mu} + \frac{1-\rho}{1-\rho^x} \cdot \frac{N-x}{\Lambda}$. Then, it suffices to show that $W(x)$ is convex on $[1, N]$. Taking the first and second derivatives of $W(x)$, we have

$$\frac{dW(x)}{dx} = \frac{1}{\mu} + \frac{1-\rho}{\Lambda} \left[\frac{-(1-\rho^x) + (N-x)\rho^x \ln \rho}{(1-\rho^x)^2} \right]$$

and

$$\frac{d^2 W(x)}{dx^2} = \frac{-2(1-\rho)\rho^x \ln \rho}{\Lambda(1-\rho^x)^2} + \frac{(1-\rho)\rho^x (N-x)(\rho^x + 1) \ln^2 \rho}{\Lambda(1-\rho^x)^3}. \qquad (8.6)$$

If $x \in [1, N]$, then $N - x \geq 0$, and it is easy to check that the two terms on the right hand side of (8.6) are positive, no matter $\rho > 1$ or $\rho < 1$. Thus, $d^2W(x)/dx^2 > 0$. Hence, $W(x)$ is convex on $[1, N]$. ∎

From property (ii) of Lemma 7, among the customers who observe queue length $0, 1, \ldots, N-1$, either the one observing 0 or the one observing $N-1$ has the longest expected waiting time. It is easy to check that

$$W_0 = \frac{1}{\mu} + \frac{N-1}{\Lambda} \quad \text{and} \quad W_{N-1} = \frac{N}{\mu}.$$

Thus, if $\Lambda \geq \mu$, then $W_{N-1} \geq W_0$, and the arriving customer who observes $N-1$ customers in the system has the longest expected waiting time. If $\Lambda \leq \mu$, then $W_{N-1} \leq W_0$, and the arriving customer who observes an empty system has the longest expected waiting time. Interestingly, the longest waiting times are the same as those in the case where both server's status and queue length are observable (see Guo and Hassin 2011). This is because when an empty system observed, the customer knows that the server must be idle; and when the queue length $N-1$ is observed, the customer knows that the server must be busy once he/she joins the system. Hence, the expected waiting times are identical in the queue-observable systems with or without information on the server status. Consequently, the conditions of the existence of the threshold strategy and the equilibrium threshold strategy are the same for both the almost observable and fully observable systems, as summarized in the following theorem (which are the same as Propositions 5 and 6 in Guo and Hassin 2011).

Theorem 12 *Consider the $M/M/1$ queue with N-policy and observable information on the queue length. (i) There exists an threshold equilibrium strategy with $\bar{n}_e \geq N$ if and only if (a) $\rho \geq 1$ and $N \leq \nu$ or (b) $\rho \leq 1$ and $N \leq \rho(\nu - 1) + 1$. (ii) If condition (a) or (b) in (i) holds, the unique equilibrium threshold is $\bar{n}_e = \lfloor \nu \rfloor$.*

Proof: We first show that condition (a) or (b) in (i) is necessary. Suppose there is an equilibrium threshold strategy $\bar{n}^e$, where $\bar{n}^e \geq N$. Since the server's status is not observable and N-policy is adopted, we must have

$$\frac{R}{\theta} \geq W_s, \tag{8.7}$$

for $s = 0, 1, \ldots, N-1$, such that the system can be activated. As $\bar{n}^e \geq N$, by Lemma 7, the expected waiting time W_k is convex in k if $k \in [0, N-1]$ (see Figure 8.1). Thus, the inequality set (8.7) is equivalent to

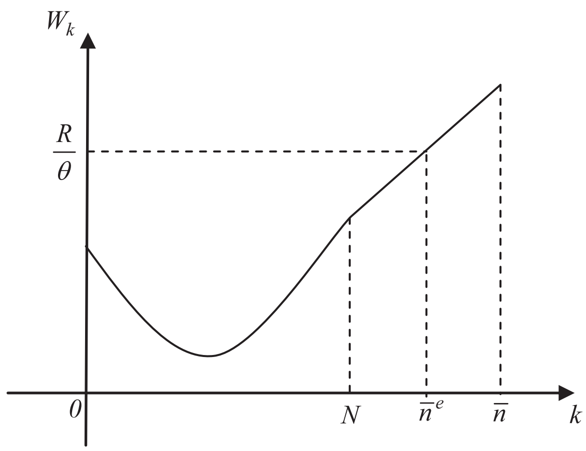


Figure 8.1: The expected waiting time.

$$\frac{R}{\theta} \geq W_0 \quad \text{and} \quad \frac{R}{\theta} \geq W_{N-1}. \tag{8.8}$$

Note also that from (8.1) we have

$$W_0 = \frac{1}{\mu} + \frac{N-1}{\Lambda} \quad \text{and} \quad W_{N-1} = \frac{N}{\mu}. \tag{8.9}$$

Therefore, if $\rho \geq 1$, then $W_{N-1} \geq W_0$, and (8.8) implies that $N \leq \nu$. In this case, we have condition (a). If $\rho \leq 1$, then $W_{N-1} \leq W_0$, and (8.8) implies that $N \leq \rho(\nu - 1) + 1$. In this case, we have condition (b). Hence, condition (a) or (b) in property (i) is necessary.

Next, we show condition (a) or (b) in property (i) is sufficient and specify the equilibrium threshold. Suppose condition (a) in (i) holds. From (8.9), it is easy to check that

$$\frac{R}{\theta} \geq W_{N-1} \geq W_0.$$

This, together with property (ii) of Lemma 7 implies that no matter what threshold strategy $\bar{n}$, where $\bar{n} \geq N$, is adopted, any arriving customers who observe state s, where $0 \leq s \leq N-1$, will stay in the system. That is, the system can be activated if any threshold strategy $\bar{n}$, where $\bar{n} \geq N$, is adopted. Note also that from (8.1), the expected waiting time W_k is linearly increasing in k if $k \in [N, \bar{n}-1]$; see also Figure 8.1. Thus, if we try the numbers $N, N+1, \ldots$ as the threshold $\bar{n}$ one by one, there must exist a threshold $\bar{n}^e$ such that

$$\frac{R}{\theta} \geq \frac{\bar{n}^e}{\mu} \quad \text{and} \quad \frac{R}{\theta} < \frac{\bar{n}^e+1}{\mu},$$

or equivalently, $\bar{n}^e = \lfloor \nu \rfloor$. Therefore, condition (a) in (i) is sufficient, and $\bar{n}^e = \lfloor \nu \rfloor$ is is an equilibrium threshold. Similar analysis can be conducted when condition (b) in (i) holds. This completes the proof of Theorem 12. ∎

Remark 5 *From Theorem 12, the equilibrium strategy and the corresponding conditions of the system with almost observable information are the same as those in the system with fully observable information. Thus, as long as the customers are informed with the N-policy the system adopted and the queue length, providing or not providing the information on the server status will generate the same effect. This observation can be applied to reduce the administrative costs on information spreading.*

In property (ii) of Theorem 12, the equilibrium $\bar{n}_e = \lfloor \nu \rfloor$ is independent of N. However, the value of N affects the steady-state probabilities of the system, the expected length of a production cycle, and the expected waiting times of the customers staying in the system. Thus, although the equilibrium strategy is fixed, the expected average cost depends on the value of N. In the next section, we will study the expected average cost function and the impact of N.

8.2 The expected average cost and optimal decision

Suppose the threshold strategy $\bar{n}$ is adopted. The expected average cost (7.6) is also valid. We adopt the same notations in Section 7.2. Also let T be the expected time length of one production cycle, let T_B and T_I be the expected time lengths of the server being busy and idle in one production cycle, respectively, and let $\mathcal{WC}$ be the expected total customer waiting cost in one production cycle. More complicated first-step analysis can be applied to calculate the cycle time T_B and the total waiting cost $\mathcal{WC}$. Note that when the threshold strategy $\bar{n}$ is adopted, the potential customers will leave the system if and only if there are $\bar{n}$ customers in the system. Let $P_{\bar{n}}$ be the probability that there are $\bar{n}$ customers in the system. Hence, the expected lost demand units in one production cycle is $P_{\bar{n}}\Lambda T$. Therefore, we have the expected average cost with observable queue be summarized in the following lemma.

Lemma 8 *Suppose the threshold strategy $\bar{n}$, where $\bar{n} \geq N$, is adopted. Then, the expected average cost is given as follows. If $\rho \neq 1$, we have*

$$AC = \frac{K + cT_B + \mathcal{WC} + \ell P_{\bar{n}}\Lambda T}{T}, \tag{8.10}$$

where

$$T_B = \frac{1}{\mu(1-\rho)}\left[N - \frac{\rho^{\bar{n}-N+1}(1-\rho^N)}{1-\rho}\right],$$

$$T = \frac{1}{\mu(1-\rho)}\left[\frac{N}{\rho} - \frac{\rho^{\bar{n}-N+1}(1-\rho^N)}{1-\rho}\right],$$

$$\mathcal{WC} = \frac{N(N-1)h}{2\Lambda} + \frac{h}{\mu(1-\rho)}\left[\frac{\rho N}{1-\rho} + \frac{N(N+1)}{2} - \left(\bar{n} + \frac{1}{1-\rho}\right)\frac{\rho^{\bar{n}-N+1}(1-\rho^N)}{1-\rho}\right],$$

and

$$P_{\bar{n}} = \frac{\rho^{\bar{n}-N+1}(1-\rho^N)(1-\rho)}{N - N\rho - \rho^{\bar{n}-N+2} + \rho^{\bar{n}+2}}.$$

If $\rho = 1$, we have

$$AC = \frac{2K\Lambda}{(2\bar{n}-N+3)N} + \frac{c(2\bar{n}-N+1) + h[\bar{n}(\bar{n}+1) - \frac{1}{3}(N^2-3N+2)] + 2\ell\Lambda}{2\bar{n}-N+3}. \tag{8.11}$$

Proof: We first restrict our attention to the general case where $\rho \neq 1$, the special case $\rho = 1$ will be analyzed at the end. The validity of (8.10) follows directly from (7.6). Thus, it suffices to obtain the expressions for T_B, T, $\mathcal{WC}$, and $P_{\bar{n}}$.

Consider the cycle times T_B and T. It is easy to see that the time period when the server is idle is the sum of N Poisson inter-arrival times with rate Λ; that is, $T_I = \frac{N}{\Lambda}$. To determine T_B, suppose there are k customers in the system, where $1 \leq k \leq \bar{n}$, and the server has just started processing a customer order. Let τ_k be the first time that the system becomes empty. Then, $T_B = \tau_N$.

By first-step analysis, we can obtain the recursive equations

$$\tau_k = \frac{1}{\mu+\Lambda} + \frac{\Lambda}{\mu+\Lambda}\tau_{k+1} + \frac{\mu}{\mu+\Lambda}\tau_{k-1}, \quad k = 1, 2, \ldots, \bar{n}-1, \tag{8.12}$$

and the boundary condition

$$\tau_{\bar{n}} = \frac{1}{\mu+\Lambda} + \frac{\Lambda}{\mu+\Lambda}\tau_{\bar{n}} + \frac{\mu}{\mu+\Lambda}\tau_{\bar{n}-1}. \tag{8.13}$$

Define $\tau_0 = 0$, and let $\Delta\tau_k = \tau_k - \tau_{k-1}$ for $k = 1, 2, \ldots, \bar{n}$. Equations (8.12) and (8.13) imply that

$$\Lambda\Big(\Delta\tau_{k+1} - \frac{1}{\mu-\Lambda}\Big) = \mu\Big(\Delta\tau_k - \frac{1}{\mu-\Lambda}\Big), \quad k = 1, 2, \ldots, \bar{n}-1,$$

and

$$\Delta\tau_{\bar{n}} = \frac{1}{\mu},$$

respectively. Let $a_k = \Delta\tau_k - \frac{1}{\mu-\Lambda}$ for $k = 1, 2, \ldots, \bar{n}$. We have

$$a_{\bar{n}} = \frac{\rho}{\Lambda-\mu} \quad \text{and} \quad a_k = \rho a_{k+1},$$

for $1 \leq k \leq \bar{n}-1$, which imply that

$$a_k = \rho^{\bar{n}-k}a_{\bar{n}} = \frac{\rho^{\bar{n}-k+1}}{\Lambda-\mu},$$

for $1 \leq k \leq \bar{n}-1$. This, together with the definitions of τ_k, $\Delta\tau_k$, and a_k, implies that

$$\begin{aligned} T_B = \tau_N &= \sum_{k=1}^{N} \Delta\tau_k = \sum_{k=1}^{N}\Big(a_k + \frac{1}{\mu-\Lambda}\Big) \\ &= \frac{1}{\Lambda-\mu}\sum_{k=1}^{N}\rho^{\bar{n}-k+1} + \frac{N}{\mu-\Lambda} \\ &= \frac{1}{\mu(1-\rho)}\Big[N - \frac{\rho^{\bar{n}-N+1}(1-\rho^N)}{1-\rho}\Big]. \end{aligned}$$

Therefore,

$$T = T_B + T_I = \frac{1}{\mu(1-\rho)}\left[\frac{N}{\rho} - \frac{\rho^{\bar{n}-N+1}(1-\rho^N)}{1-\rho}\right].$$

Next, consider the total customer waiting cost in one production cycle, $\mathcal{WC}$. Let $\mathcal{WC}_I$ and $\mathcal{WC}_B$ be the expected waiting cost incurred in the periods when the server is idle and busy, respectively. Then, $\mathcal{WC} = \mathcal{WC}_I + \mathcal{WC}_B$. It is easy to see that

$$\mathcal{WC}_I = \frac{h}{\Lambda} + \frac{2h}{\Lambda} + \cdots + \frac{(N-1)h}{\Lambda} = \frac{N(N-1)h}{2\Lambda}. \tag{8.14}$$

To determine $\mathcal{WC}_B$, suppose there are k customers in the system, where $1 \le k \le \bar{n}$, and the server has just started processing a customer order. Let ζ_k be the total customer waiting cost in the time period until the server finishes processing all of the customers in the system. Then, $\mathcal{WC}_B = \zeta_N$.

By first-step analysis, we have the recursion equations

$$\zeta_k = \frac{kh}{\mu+\Lambda} + \frac{\Lambda}{\mu+\Lambda}\zeta_{k+1} + \frac{\mu}{\mu+\Lambda}\zeta_{k-1}, \quad k = 1, 2, \ldots, \bar{n}-1,$$

and the boundary equation

$$\zeta_{\bar{n}} = \frac{\bar{n}h}{\mu+\Lambda} + \frac{\Lambda}{\mu+\Lambda}\zeta_{\bar{n}} + \frac{\mu}{\mu+\Lambda}\zeta_{\bar{n}-1}.$$

Define $\zeta_0 = 0$, and let $\Delta\zeta_k = \zeta_k - \zeta_{k-1}$ for $k = 1, 2, \ldots, \bar{n}$. Then, the above equations imply that

$$\Delta\zeta_k = \rho\Delta\zeta_{k+1} + \frac{kh}{\mu}, \quad k = 1, 2, \ldots, \bar{n}-1,$$

and

$$\Delta\zeta_{\bar{n}} = \frac{\bar{n}h}{\mu},$$

respectively. Thus, by mathematical induction, it is easy to check that

$$\Delta\zeta_k = h\sum_{i=0}^{\bar{n}-k}\frac{(k+i)\rho^i}{\mu}, \quad 1 \le k \le \bar{n}-1.$$

This, together with the definitions of $\Delta\zeta_k$, implies that

$$\zeta_N = \sum_{k=1}^{N} \Delta\zeta_k = h \sum_{k=1}^{N} \left[\sum_{i=0}^{\bar{n}-k} \rho^i \frac{k+i}{\mu} \right] = h \sum_{k=1}^{N} \left[\frac{k}{\mu} \sum_{i=0}^{\bar{n}-k} \rho^i + \frac{1}{\mu} \sum_{i=0}^{\bar{n}-k} i\rho^i \right] \tag{8.15}$$

$$= h \sum_{k=1}^{N} \left\{ \frac{k}{\mu} \frac{1-\rho^{\bar{n}-k+1}}{1-\rho} + \frac{1}{\mu} \left[\frac{\rho(1-\rho^{\bar{n}-k})}{(1-\rho)^2} - \frac{(\bar{n}-k)\rho^{\bar{n}-k+1}}{1-\rho} \right] \right\}$$

$$= \frac{h}{\mu(1-\rho)} \sum_{k=1}^{N} \left[k(1-\rho^{\bar{n}-k+1}) + \frac{\rho(1-\rho^{\bar{n}-k})}{1-\rho} - (\bar{n}-k)\rho^{\bar{n}-k+1} \right]$$

$$= \frac{h}{\mu(1-\rho)} \sum_{k=1}^{N} \left[\frac{\rho}{1-\rho} + k - \Big(\bar{n} + \frac{1}{1-\rho}\Big) \rho^{\bar{n}-k+1} \right]$$

$$= \frac{h}{\mu(1-\rho)} \left[\frac{\rho N}{1-\rho} + \frac{N(N+1)}{2} - \Big(\bar{n} + \frac{1}{1-\rho}\Big) \frac{\rho^{\bar{n}-N+1}(1-\rho^N)}{1-\rho} \right]. \tag{8.16}$$

Therefore, from (8.14) and (8.16), we have

$$\mathcal{WC} = \frac{N(N-1)h}{2\Lambda} + \frac{h}{\mu(1-\rho)} \left[\frac{\rho N}{1-\rho} + \frac{N(N+1)}{2} - \Big(\bar{n} + \frac{1}{1-\rho}\Big) \frac{\rho^{\bar{n}-N+1}(1-\rho^N)}{1-\rho} \right],$$

where the first term is the total customer waiting cost in the time period when the server is idle, and the second term is the one for the time period with busy server.

Regarding the lost-sales probability $P_{\bar{n}}$, from the steady-state probabilities in the proof in Lemma 7, we have

$$P_{\bar{n}} = P_{\bar{n},B} = \rho^{\bar{n}-N} P_{N,B} = \frac{\rho^{\bar{n}-N}(\rho - \rho^{N+1})}{1-\rho} P_0 = \frac{\rho^{\bar{n}-N+1}(1-\rho^N)(1-\rho)}{N - N\rho - \rho^{\bar{n}-N+2} + \rho^{\bar{n}+2}}.$$

This completes the proof of the lemma with $\rho \neq 1$.

We next obtain the average cost for the special case $\rho = 1$, i.e., $\Lambda = \mu$. Regarding the cycle times, we also have $T_I = \frac{N}{\Lambda}$ and the validity of (8.12) and (8.13), which imply that

$$\Delta\tau_{k+1} = \frac{1}{\Lambda} + \Delta\tau_k, \quad k = 1, 2, \ldots, \bar{n}-1,$$

and

$$\Delta\tau_{\bar{n}} = \frac{1}{\Lambda}.$$

Thus,

$$\Delta\tau_k = \frac{\bar{n}+1-k}{\Lambda}, \quad k = 1, 2, \ldots, \bar{n},$$

from which we can obtain that

$$T_B = \tau_N = \sum_{k=1}^{N} \Delta\tau_k = \sum_{k=1}^{N} \frac{\bar{n}+1-k}{\Lambda} = \frac{(2\bar{n}-N+1)N}{2\Lambda}, \tag{8.17}$$

and

$$T = T_I + T_B = \frac{N}{\Lambda} + \frac{(2\bar{n}-N+1)N}{2\Lambda} = \frac{(2\bar{n}-N+3)N}{2\Lambda}. \tag{8.18}$$

Regarding the total waiting cost in one cycle, equations (8.14) and (8.15) hold, and the latter implies that

$$\begin{aligned}
\mathcal{WC}_B =& h\sum_{k=1}^{N}\left[\frac{k}{\Lambda}\sum_{i=0}^{\bar{n}-k} 1 + \frac{1}{\Lambda}\sum_{i=0}^{\bar{n}-k} i\right] = \frac{h}{\Lambda}\sum_{k=1}^{N}\left[k(\bar{n}-k+1) + \frac{(\bar{n}-k)(\bar{n}-k+1)}{2}\right] \\
=& \frac{h}{2\Lambda}\sum_{k=1}^{N}(\bar{n}-k+1)(\bar{n}+k) = \frac{h}{2\Lambda}\sum_{k=1}^{N}(\bar{n}^2 - k^2 + \bar{n} + k) \\
=& \frac{h}{2\Lambda}\left[\bar{n}(\bar{n}+1)N - \frac{N(N+1)(2N+1)}{6} + \frac{N(N+1)}{2}\right] \\
=& \frac{hN}{2\Lambda}\left[\bar{n}(\bar{n}+1) - \frac{N^2-1}{3}\right].
\end{aligned}$$

This, together with (8.14), implies that

$$\mathcal{WC} = \frac{hN}{2\Lambda}\left[\bar{n}(\bar{n}+1) - \frac{N^2-3N+2}{3}\right]. \tag{8.19}$$

Regarding the lost-sales probability $P_{\bar{n}}$, from (8.4), we have

$$P_{\bar{n}} = \frac{2}{2\bar{n}-N+3}. \tag{8.20}$$

Substituting (8.17)–(8.20) into (8.10) and simplifying the expression, we obtain equation (8.11). ∎

Unfortunately, the average cost function (8.10) is not necessarily convex (in our numerical study, there are cases where AC is not convex in N).

The impact of N on the expected average cost is studied numerically. We restrict our numerical studies to $\rho < 1$, as it allows us to compare the model with the traditional one without strategic customers. Figure 8.2 shows the numerical results with $\rho = 0.98$. There are two main observations from this numerical study. First, when N is small, the cost function is steeper than the one without strategic customers,

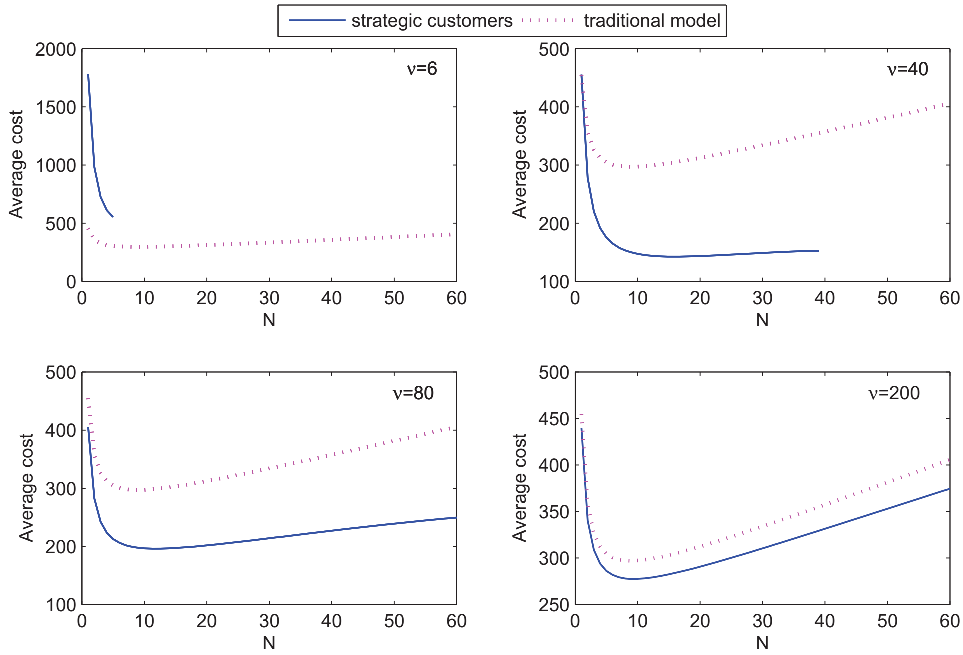


Figure 8.2: AC versus N for the system with observable queue length.

implying that the system performance with strategic customers is very sensitive to a small N. However, when N is large, the cost function with strategic customers become even more flat than the one without strategic customers, implying that the system performance with strategic customers is insensitive to a large N. Second, as ν becomes very large, the system performance with strategic customers converges to that of the traditional model. The system performance has similar patterns as what depicted in Figure 8.2 when ρ varies but less than 1, except that when the system utilization ρ is smaller, the system performance converges to that of the traditional model faster as ν increases.

Chapter 9

The Impact of Information

In this chapter, we investigate the difference on the minimal costs between the cases with unobservable and observable queue lengths. In Chapter 8 we show that if the queue length is observable, it has no effect on the system performance regardless whether the information on the server status is made available to the incoming customers. Thus, the information scenario where the queue length is observable is also the information scenario where both the queue length and the server status are observable. Hence, compared with the information scenario where the queue is not observable (i.e., the case discussed in Chapter 7), the information scenario where the queue length is observable can be regarded as having more information on the queue length. Therefore, the increase/decrease of the average costs in the two information scenarios is the contribution, either positive or negative, of providing information on the queue length.

Let N_u^* (N_o^*) and AC_u^* (AC_o^*) be the optimal values of N and the corresponding optimal average cost when the queue is unobservable (observable). We define

$$\Delta AC^* = \frac{AC_u^* - AC_o^*}{AC_o^*} \times 100\%,$$

which is the percentage difference between the optimal average costs under the two information scenarios. It reflects the percentage increase in the optimal average cost when less information is provided to the customers, and it can be regarded as the value of the information (i.e., queue length) on the system performance. If

$\Delta AC^* > 0$, then more information is beneficial to the production system. However, if $\Delta AC^* < 0$, then information hurts the system performance.

Next, we numerically study the impact of information on the expected average cost by using ΔAC^*. Our numerical examples have the same settings as those in Sections 7 and 8; that is, $\Lambda = 10$, $c = 10$, $h = 5$, $\ell = 50$, and $K = 1000$. We consider different values of ρ and ν. The optimal N is obtained through one-dimensional numerical search.

We first consider the impact of information on the optimal average cost by varying the value of ρ. We let $\nu = 40$ and 80. The numerical results are summarized in Table 9.1. A plot of the percentage difference in the expected average cost against ρ is shown in Figure 9.1.

Table 9.1: The impact of information for different values of ρ.

	$\nu = 40$					$\nu = 80$				
ρ	N_u^*	N_o^*	AC_u^*	AC_o^*	ΔAC^*	N_u^*	N_o^*	AC_u^*	AC_o^*	ΔAC^*
0.1	8	4	1144.1	2259.1	-49%	15	8	636.6	1144.1	-44%
0.2	14	8	607.2	1020.8	-41%	27	16	364.5	540.8	-33%
0.3	19	12	418.6	616.0	-32%	37	24	284.3	354.3	-20%
0.4	23	16	323.2	419.8	-23%	46	32	250.3	272.3	-8%
0.5	27	20	260.2	307.5	-15%	45	40	231.1	232.5	-1%
0.6	30	24	219.3	237.7	-8%	40	40	211.0	211.0	0%
0.7	33	28	189.6	193.3	-2%	35	35	189.4	189.4	0%
0.8	$\underline{28}$	$\underline{29}$	166.9	166.4	0%	28	28	166.9	166.9	0%
0.9	$\underline{20}$	$\underline{22}$	151.5	146.6	3%	20	20	151.5	151.3	0%
1	25	13	232.0	160.0	45%	40	10	412.3	240.5	71%
1.2	26	9	308.4	193.8	59%	43	6	493.3	393.3	25%
1.4	27	7	363.6	211.8	72%	45	6	551.4	411.8	34%
1.6	28	6	405.2	220.4	84%	47	13	595.1	420.4	42%
1.8	29	6	437.8	226.0	94%	48	11	629.2	426.0	48%
2	29	5	464.0	230.0	102%	50	53	656.5	430.0	53%
4	33	24	583.3	245.8	137%	59	5	780.2	445.8	75%
6	35	3	623.5	250.7	149%	64	18	821.6	450.7	82%
8	36	18	643.8	253.0	154%	68	2	842.4	453.0	86%
10	37	9	656.0	254.4	158%	70	10	854.9	454.4	88%

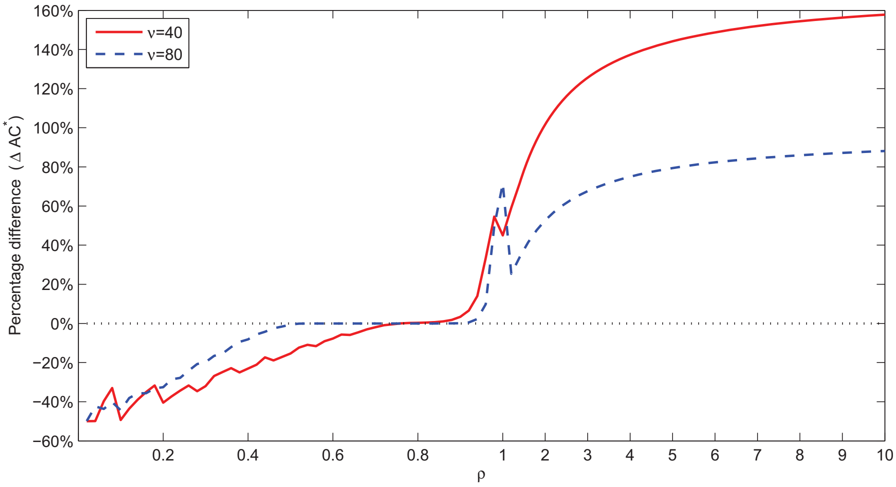


Figure 9.1: The impact of information, ΔAC^*, versus ρ.

Clearly, the value of information strongly depends on the utilization of the system, i.e., the value of ρ. When the customer traffic is light to moderate, e.g., $\rho < 0.5$, the value of information is negative, which means that providing more information to customers actually increases the average cost of the system. When the customer traffic is heavy to overloaded, e.g., $\rho > 0.9$, the value of information is positive, which means that providing more information to customers reduces the system average cost. Another important observation is that the optimal N under less information is typically larger than the one under full information, except for two cases which we underline in Table 9.1.

Table 9.2 summarizes the numerical results with varied values of ν. Recall that ν measures the maximal number of service cycles a customer is willing to wait. Hence, a larger ν represents more patient customers. We let ν vary from 2 to 200 with 19 different values, and we let $\rho = 0.3$ and 1.5. A plot of the percentage difference in the expected average cost against ν is also shown in Figure 9.2.

The main observation is that the absolute value of the percentage difference is large when when ν is small which represents impatient customers. However, as ν is large, the percentage difference converges to zero, which means that for very patient customers, the value of information is 0. Consequently, when customers are

Table 9.2: The impact of information for different values of ν.

	$\rho = 0.3$					$\rho = 1.5$				
ν	N_u^*	N_o^*	AC_u^*	AC_o^*	ΔAC^*	N_u^*	N_o^*	AC_u^*	AC_o^*	ΔAC^*
2	1	1	7811.2	7202.0	8%	2	1	2311.5	2143.2	8%
4	2	1	3507.6	7022.4	-50%	4	3	1058.1	427.6	147%
6	3	2	2343.5	3509.3	-33%	6	5	618.0	177.9	247%
8	4	3	1762.6	2343.7	-25%	7	7	459.1	108.8	322%
10	5	3	1415.1	2343.5	-40%	9	9	387.5	88.9	336%
12	6	4	1184.3	1762.6	-33%	10	10	351.7	86.5	307%
14	7	4	1020.1	1762.6	-42%	12	10	333.0	91.1	266%
16	7	5	1020.1	1415.1	-28%	13	9	323.9	98.6	228%
18	8	6	897.6	1184.3	-24%	15	9	320.4	107.6	198%
20	9	6	802.9	1184.3	-32%	16	9	320.5	117.1	174%
40	19	12	418.6	616.0	-32%	28	7	385.8	216.7	78%
60	28	18	322.6	436.5	-26%	37	6	477.9	316.7	51%
80	37	24	284.3	354.3	-20%	46	5	574.7	416.7	38%
100	46	30	269.8	311.0	-13%	53	5	673.0	516.7	30%
120	53	36	267.2	287.1	-7%	60	10	772.0	616.7	25%
140	53	42	267.2	274.3	-3%	67	2	871.3	716.7	22%
160	53	48	267.2	268.5	0%	73	9	970.8	816.7	19%
180	53	53	267.2	267.2	0%	79	20	1070.5	916.7	17%
200	53	53	267.2	267.2	0%	84	23	1170.2	1016.7	15%

impatient, managers should be cautious on deciding whether to hide or reveal the delay information.

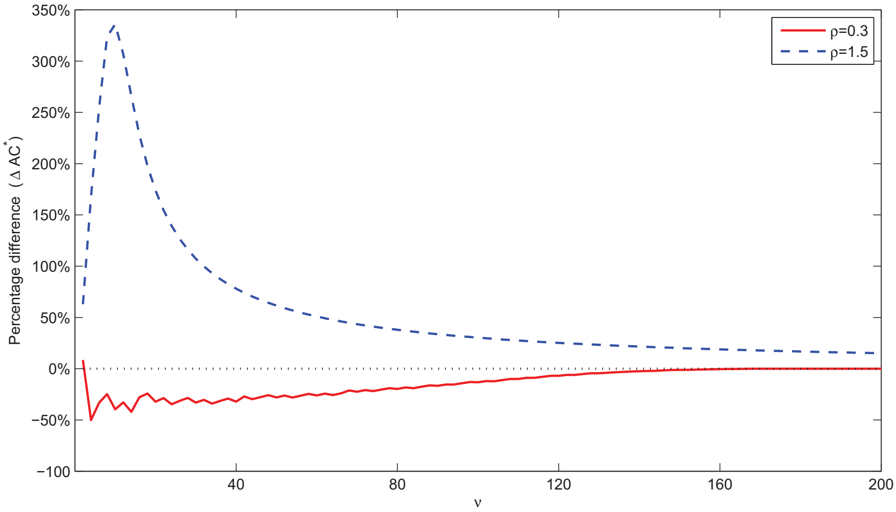


Figure 9.2: The impact of information, ΔAC^*, versus ν.

CONCLUSIONS

In this dissertation, we have studied two optimization models with Poisson demand: the inventory control model in a buy-and-sell environment and the production control model in a make-to-order environment.

In the first essay, we introduce a new approach for analyzing the traditional (s, S) policy for a continuous review inventory problem. We transform the original average-profit-maximization objective into a new effective profit objective. The problem is solved by applying a binary search to the transformed problem. Using this method, we obtain several interesting results. For the classical model with complete backlogging, we show that the optimal policy to maximize the new objective function is a stationary (s, S) policy. The optimal reorder point is obtained by balancing the effective revenue rate with the expected inventory cost rate at the end of the lead time period, while the optimal order-up-to level is independent of the reorder point. By applying such an economic interpretation of the optimal reorder point to the general partial backlogging model, we obtain a simple but effective heuristic. The effectiveness of our heuristic is tested via a numerical study. We further compare the performance of our heuristic with two other intuitive heuristics, and we demonstrate that the safety stock plays an important role on the performance of those heuristics. This insight can potentially be applied to other inventory models with more general settings, such as compound Poisson demand process or stochastic lead times, to help design effective heuristics.

From the analysis conducted in the first essay, we observe that the structural characteristics of the new effective profit objective function simplifies the analysis of the inventory problem and enables us to interpret the economic reasoning of the optimal reorder point. It is quite possible that this transformed objective function and similar analysis can be applied to other stochastic inventory problems. One possible future research direction is to apply this method to inventory systems with supply uncertainty; that is, when the order quantity received is dependent on but no greater than the quantity ordered. Both stochastic lead time demand and supply uncertainty can cause uncertainty on the inventory level immediately after the order arrival. However, the latter imposes a bigger challenge because the inventory level immediately after the order arrival is stochastically dependent on the order quantity, whereas the former is not. The new effective profit objective function might be applied to such models such that insightful results can be obtained.

Another possible application of the new objective function is to provide an alternative approach to deal with joint pricing-inventory problems. The effective profit objective has a dynamic structure, namely the current profit plus the profit-to-go. For state-dependent pricing problems, the price of the current stage only affects the demand and the effective profit in the current stage, while the effective profit-to-go depends only on future decisions. The structure of the effective profit function can be incorporated into the pricing problem with state-dependent pricing decisions. Some literature has considered this kind of inventory-pricing problems using the traditional objective (see Chen and Simchi-Levi 2012). However, we expect that some new structural results and insightful economic interpretations might be explored by using our new approach. For state-independent pricing problems (i.e., the pricing decision is made at the beginning of the planning horizon and applied to the entire planning horizon), closed-form optimal policies (i.e., replenishment policies similar to that presented in Theorem 3) might be developed and applied to the global optimal pricing decisions. To the best of our knowledge, it is an unsolved problem in the literature, although a lot of work has addressed the counterpart problems under the newsvendor framework.

We also conjecture that the new objective function might provide certain opportunities to develop a unified method to deal with periodic-review and continuous-review models. This is because the effective profit function is decomposable into effective profits of different time intervals, and this decomposition can be applied to both periodic-review and continuous-review models. The order decisions in periodic review models can only be made at the review points. Thus, in the periodic-review model, we can apply our method under the constraint that the order can only be placed at some pre-determined time points. In the special case where the inter-order time is required to be identical, the order policy becomes the "fixed inter-review period" replenishment policy. Furthermore, the "fixed inter-review period" might also be taken as a decision variable, and the performance of the best possible "fixed inter-review period" model and the performance of the continuous-review model can be compared analytically.

In the second essay, we investigate the optimal control of an MTO production system with strategic customers. Customers decide to stay in the system or leave without purchase according to their expected waiting time, which is affected by the information made available to them. We consider two scenarios depending on whether the queue length is observable or not. The production system is modeled as an $M/M/1$ queue with N-policy, which means that the production is triggered to start when N customer orders are received, and once the production starts it will keep on working until all of the waiting customer orders are processed.

We first derive the equilibrium strategies for customers' decision on staying versus leaving. Based on that, we derive the average cost function for the whole system which consists of fixed set-up cost, operation cost, waiting cost, and lost-sales penalty. When the queue length is unobservable but the server's status is observable, the average cost function is either convex or piecewise convex in the decision variable N. When the queue length is observable, such a function is not convex in general.

Numerical study is conducted to investigate the impact of customer strategic behavior on the system performance. In general, under strategic customer behavior, the average cost is very sensitive in N when N is small, but it becomes insensitive in N when N gets larger. Furthermore, the interval of N with insensitive average cost is wider for the customers with higher patience level. Thus, the system manager should be more cautious in choosing N if the customers are less willing to wait.

There are also some interesting observations in the case where the queue length is not observable. For the production system with high system utilization (i.e., $\rho \gg 1$) and very impatient customers, it is possible that optimal system control parameter N will induce the customers to adopt the paradoxical strategy $(0, 1)$, i.e., staying if the server is idle but leaving if it is busy. For the production system with heavy traffic (i.e., $\rho \approx 1$ but $\rho < 1$), it is possible that strategic customer behavior benefits the system performance with an optimal threshold being greater than that in the traditional model without strategic customers.

We also compare the system performance with observable versus unobservable queue length. Our numerical study shows that, for light to moderate customer traffic, hiding queue length information is more beneficial, whereas for heavy customer traffic, revealing queue length information is more beneficial. We show that the impact of information is large when customers are impatient, and not so when customers are very patient. Consequently, managers should be cautious in providing information to impatient customers.

In our analysis, we have modeled the MTO production system by an $M/M/1$ queue, since this model is one of the most commonly used queueing models. However, one limitation of our model is that in a real-life production system, the service time may not be exponentially distributed. Hence, generalizing our model to an $M/G/1$ setting will enhance its practicality. However, if our problem is modeled by an $M/G/1$ queue, the embedded Markov chain and the probability transition matrix need to be analyzed in order to obtain the steady-state probabilities. Such an extension is an interesting future research direction.

Another possible extension of the current topic is to consider two production systems and incorporate the competition between them. Under competition environment, customers strategically choose the one which gives them higher utility. The impact of the strategic behavior on the system performance might be greatly different from what have discussed in this essay. This is also an interesting problem to investigate. Future research may also consider a variant of our model which adopts a make-to-stock production mode. That is, the production starts when some customer orders are received, and stops only when a certain inventory level is reached.

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