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Optimal Coherent Feedback Control of Linear Quantum Stochastic Systems

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Certificate of Originality

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____________________________(Signed)

BIAN Chuanxin (Name of student)

Dedicate to my parents.

Abstract

The thesis is concerned with the coherent feedback control of linear quantum stochastic systems. Two topics are considered:

1. Squeezing enhancement of degenerate parametric amplifiers (DPAs) via coherent feedback control.

2. Coherent linear quadratic Gaussian (LQG) and H_∞ control of linear quantum stochastic systems.

For topic 1, the definition of squeezing ratio is first introduced by means of quadratures' variances. An in-depth investigation of squeezing performance of lossy DPAs is presented in the static case. A sufficient and necessary condition is proposed to guarantee the effectiveness for the scheme of feedback loop. To overcome the conservatism and achieve better squeezing performance, a coherent feedback control scheme is proposed. The problem is converted into a non-convex constrained programming, genetic algorithm (GA) and sequential quadratic programming (SQP) are applied in numerical examples to obtain the local optima. Detailed implementation procedure is also shown by using common optical instruments.

For topic 2, a class of open linear quantum systems is formulated in terms of quantum stochastic differential equations (QSDEs) on a quantum probability space. Physical realizability is also reviewed which guarantees the system to be a meaningful quantum system. Then the standard quantum LQG and H_∞ control problems are proposed based on the closed-loop plant-controller feedback control system. Under this framework, the mixed problem under consideration can be treated as a more general schematic which encompasses both quantum LQG control and quantum H_∞ control. To solve the matrix polynomial equality constraints of physical realizability, the problem is then reformulated into a rank constrained linear matrix inequality (LMI) problem which is solved by Matlab and the toolbox therein. Simulation examples illustrate the advantages of the proposed method.

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$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{R}^n$	set of n-dimensional real vectors
$\mathbb{R}^{m\times n}$	set of $m \times n$ real matrices
$\mathbb{C}^{m\times n}$	set of $m \times n$ complex matrices
x^T	transpose of matrix/vector x
$x^\#$	adjoint of each element of matrix/vector x
$x^\dagger$	$:= (x^\#)^T$
$\breve{x}$	$:= [x^T \; x^\dagger]^T$
I_n	identity matrix of dimension n
$\Delta(U, V)$	$:= \begin{bmatrix} U & V \\ V^\# & U^\# \end{bmatrix}$
J_n	$:= \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}$
$X^\flat$	$:= J_m X^\dagger J_n$

Chapter 1

Introduction

1.1 Background

Control theory studies how to influence the behavior of dynamical systems, which plays a significant role in the advance of human civilization as it provides a solid theoretical foundation for mechanical manufacturing, computer networks, industrial automation, intelligent vehicle systems, mobile robotics, aerospace engineering and national defense, etc.. Although the application of control theory in various types of systems can date back to antiquity, a more formal analysis of the field was pioneered by dynamics analysis of centrifugal governor systems by Maxwell (1867), which greatly promoted the development of control theory. In the past decades, the rapid development of microtechnology and precision technologies makes it possible to utilize control theory to manipulate systems at quantum scale, and thus the new subject quantum control theory came into being. As a new-rising branch of control theory, this challenging subject has been attracting extensive attention of scholars among the whole academia.

While the first quantum revolution revealed to us the physical essence of the world at micro-scale, the second quantum revolution impels scientists to make full use of these basic rules to develop more advanced technologies, see Dowling and Milburn (2003). Nowadays, miniaturization and high-performance demand motivate us to observe and control a complex system governed by the laws of quantum physics. However, what makes a control system quantum? So far as our current knowledge, all systems can be regarded as quantum but the non-classical phenomena could only be observed by using specific laboratory devices. For the purpose of describing the unique features of quantum systems, Hudson and Parthasarathy (1984) formulated stochastic systems in terms of quantum stochastic differential equations (QSDEs). Various objectives such as quantum entanglement preservation, coherence stabilization, state preparation and feedback error correction etc. could be achieved by means of traditional measurement-based control schemes, see Xiang and Xiong (2007), Bouten et al. (2007), Bouten et al. (2009), Doherty and Jacobs (1999), Gardiner and Zoller (2004), Wiseman and Milburn (2010), Yamamoto et al. (2007), Zhang et al. (2007), Lidar and Schneider (2005), van Handel et al. (2005), Wiseman et al. (2002).

In this traditional picture of quantum feedback control schemes, though the plant to be controlled is governed by quantum law, the controller processes measurement outcomes which are classical and thus could be described classically. In principle, this measurement-based feedback method for controlling a quantum system by a feedback controller is to make a measurement on the system and then to produce a control input whose value depends on measurement outcomes to drive the system to a desired state. Though laboratory results have shown that measurement-based feedback control can be applied effectively, limitation and drawbacks evidently exist. First of all, measuring a quantum system will inevitably destroy its coherence. As a result of measurement, the system jumps to one state or another probabilistically which significantly complicates the control process. The well-known quantum non-demolition (QND) procedure measures an external field, instead of directly on the system itself. However, the state of the joint system-field is changed. Thus, if the field is traced out, the state of the system will be changed, for more details, one can refer to Braginsky et al. (1980), Imoto et al. (1985), Holland et al. (1990), Kuzmich et al. (2000).

To overcome the drawbacks brought by measurement-based feedback control, an alternative control mechanism, coherent feedback control, has been theoretically proposed by Wiseman and Milburn (1994a), Lloyd (2000), James et al. (2008), Nurdin et al. (2009a), Zhang and James (2011), Zhang et al. (2012); and also experimently demonstrated by Nelson et al. (2000), Mabuchi (2008), Iida et al. (2012). Coherent feedback control is based on establishing a fully quantum feedback loop (in which the sensor, controller, and actuator are all quantum systems) which abandons the classic measurement, and the quantum coherent controller obtains quantum information by interacting with the quantum plant directly or indirectly, and then processes information coherently. Experiments demonstrate that coherent feedback controller can perform tasks that the controller in a measurement-based feedback loop cannot. Compared with measurement-based feedback control, coherent feedback control enjoys exceptional advantages: First of all, fully quantum devices will certainly preserve quantum coherence within the whole feedback loop, see Wiseman and Milburn (1994a), Lloyd (2000). Secondly, coherent feedback control makes each step in the feedback loop completely reversible, thus is deterministic, see Lloyd (2000). Moreover, in several coherent feedback schemes, it has a higher processing speed, see Igarashi and Kikuchi (2008).

Recently, substantial development has been done in coherent feedback control of quantum optical systems. For instance, quantum feedback network has been studied by Wiseman and Milburn (1994b), Yanagisawa and Kimura (2003a), Gough and James (2009a), Gough and James (2009b), Nurdin et al. (2009b), James and Gough (2010), Gough et al. (2010). James et al. (2008) discussed H_∞ control of linear stochastic quantum systems, in which they also proposed an algebraic criterion for physical realizability, by which one can design a coherent feedback controller to manipulate the system to achieve desired performance. Mabuchi (2008) designed an optical experiment to illustrate the scheme proposed by James et al. (2008). A quantum coherent LQG control problem was addressed by Nurdin et al. (2009a), but unlike the H_∞ control problem discussed by James et al. (2008), the specific character of LQG control makes the design of the coherent feedback controller a formidable challenge, the authors proposed a complex numerical algorithm to obtain the desired coherent feedback controller. Yanagisawa and Kimura (2003b) has formulated the purely nonclassical feedback system, in which information is transferred in quantum logic. As an important application, they also found that by the transfer function approach, a quantum feedback system can generate squeezed states. Gough and Wildfeuer (2009) proposed a detailed squeezing enhancement scheme based on a coherent feedback theory. Zhang and James (2011) applied direct and indirect couplings for coherent feedback control of linear quantum stochastic feedback systems, in which they established the quantum version of dissipation theory, positive real lemma, and bounded real lemma. Ideal squeezers and phase shifters were utilized by Zhang et al. (2012) to design a coherent feedback control scheme for linear quantum optical systems. The proposed optimization algorithm can be numerically verified to achieve a better LQG performance compared with former works. Harno and Petersen (2010a) proposed a differential evolution method to obtain an optimal quantum coherent controller. They have also studied the entanglement control problem as a case study.

1.2 Literature review

1.2.1 Quantum squeezing of light

In quantum mechanics, *Heisenberg's Uncertainty Principle* declares that the product of two conjugate quadratures' standard deviation is no less than a constant. Thus squeezing of a light field is essentially to suppress one quadrature's uncertainty at the expense of increasing that of the other. Such uncertainty is called quantum noise which is a direct consequence of the existence of photons. Squeezed states of light were first studied by scholars interested in the property such as generalized minimum-uncertainty states, see Stoler (1970), Yuen (1976), Hollenhorst (1979), Caves (1981) for details. And the first experimental realization of squeezed light was reported by Slusher et al. (1985) using four-wave mixing in sodium atoms. Loudon and Knight (1987) presented a detailed review on the research progress of squeezed light. Being nonclassical states of light, squeezed states hold promising applications in the fields of quantum information, quantum communication and high precision metrology, among others, see Loudon and Knight (1987).

Squeezed states can be used to improve signal-to-noise ratio (SNR), thus weak signals can be transmitted with the same SNR and the same light power. With this important property, squeezed light is widely used in optical communication, optical measurements and quantum teleportation. For details, one may refer to Caves (1981), Yamamoto and Haus (1986), Furusawa et al. (1998), Bowen et al. (2003) and Noh et al. (2009). Moreover, Kraus and Cirac (2004) proposed that squeezed light allows the generation of entanglement states of light, Ou et al. (1992) found that it also provides experimental facility for demonstrating Einstein-Podolsky-Rosen (EPR) paradox. It is worth mentioning that in the study of laser interferometric gravitational wave detection, squeezed light is regarded as a key technology for future generations of such detectors as it can be used to increase the detector sensitivity without increasing the laser power, see Vahlbruch et al. (2005), Vahlbruch et al. (2006). According to Schnabel (2008), squeezed vacuum states can correlate shot noise and photon radiation pressure noise, thereby reducing the overall noise level significantly. Additionally, several spectroscopic applications of squeezed light have been proposed. Gardiner (1986) calculated the effect on atomic decay rates by replacing the normal vacuum environment with a squeezed-vacuum-state light beam, and Carmichael et al. (1987) determined the resonance fluorescence spectrum of a driven atom under similar conditions. Finally, it has been shown that an order of magnitude improvement in sensitivity could in principle be achieved in frequency modulation laser spectroscopy by the use of currently available detectors in conjunction with squeezing of the frequency modulation sidebands, see Yurke and Whittaker (1987).

Due to these and other far-reaching potential applications of squeezed light, considerable efforts have been devoted to the study of generation and enhancement of squeezing in quantum optics. Yuen and Shapiro (1979) first proposed the theoretical result that degenerate backward four-wave mixing could generate squeezed light, which could be detected by homodyne beating. Milestone was established by Slusher et al. (1985), where they successfully produced squeezed light by means of four-wave mixing due to Na atoms in an optical cavity, the balanced homodyne detector demonstrated that the total noise level in the suppressed quadrature drops below the vacuum noise level. On the other hand, second-harmonic generation is also a candidate to generate squeezed light. From aspect of photon statistics, Kozierowski and Tana (1977) showed that the output light both at the fundamental and at the second-harmonic frequency has sub-Poissonian photon statistics. Hillery (1987) demonstrated that if the second harmonic is originally in a coherent state, the square of the amplitude of the fundamental is squeezed. Moreover, calculations therein provides a way to generate a squeezed second harmonic in a traveling wave configuration. As an experimental realization, Sizmann et al. (1990) have observed the intensity fluctuation of the second harmonic mode generated in a cavity pumped by a Nd: YAG laser. As a mathematical method, Li and Kumar (1994) have studied squeezing of both the fundamental and harmonic fields undergoing traveling waves, second-harmonic generation in $\chi^{(2)}$ nonlinear media by linearizing the nonlinear operator equations around the mean-field values. By studying the squeezing spectra of several intracavity nonlinear optical systems, Collett and Walls (1985) have found that the perfect squeezing is in principle possible. Especially, they have theoretically proposed that in order to approach zero fluctuation in one quadrature, one has to operate near a critical point where the fluctuations in the other quadrature approach infinity.

In principle, squeezing in a beam can also be generated by using another beam which has non-classical correlation with the beam of interest, with either feedback or feedforward. Pioneered by Jakeman and Walker (1985) and Machida and Yamamoto (1986), scientists began to use feedback mechanism to generate sub-possionian states. Wiseman and Milburn (1994b) have proposed a QND quadrature measurement scheme which could successfully generate perfect squeezed light via homodyne feedback. However, this theory only holds when the cavity dynamics can produce nonclassical light. By Wiseman et al. (1995), it is argued that difficulties of QND measurement have made it unpopular in feedback-based squeezing enhancement. The second-harmonic generation can produce nonclassical correlations between the two output beams, when one of which is directly detected and the resulting photo-current is used in the feedback loop, theoretical analysis shows that significant enhancement can be achieved. Another squeezing enhancement mechanism is investigated by Barchielli et al. (2009), where the authors have discussed a trapped two-level atom which is pumped by a coherent monochromatic laser. Based on the theory of homodyne detection feedback proposed by Wiseman and Milburn (1994b), they studied how feedback affects the spectrum of the fluorescence light and thus enhances its squeezing performance.

It is well recognized that all the above schemes involve classical measurement which inevitably disturbs the coherence of light in the process of measurement. It is therefore natural to seek for alternative ways to generate squeezing. Yanagisawa and

Kimura (2003b) have formulated a purely nonclassical feedback system, in which the information is processed in quantum logic. The transfer function setup of quantum version has provided a new standpoint of quantum mechanics. As an important application, they have shown that such feedback system could achieve field squeezing. The problem of squeezing enhancement of degenerate parametric amplifiers (DPAs) via coherent feedback control has been studied recently by Gough and Wildfeuer (2009), where they considered a dynamic DPA but only specialized to static lossless cases. By putting a DPA in a loop mediated by a beam splitter, they showed that arbitrary squeezing can be achieved provided that the DPA is lossless. Accordingly, Iida et al. (2012) presented an experimental demonstration of optical field squeezing, which agrees well with the theory taking into account time delays and losses in the coherent feedback loop. These studies reveal the effectiveness of squeezing enhancement via coherent feedback control. Nevertheless, coherent feedback-based squeezing enhancement is still at its preliminary stage; systematic, first-principles advanced feedback control methodologies have yet to be developed.

1.2.2 Coherent H_∞ and LQG control of linear stochastic quantum systems

In control theory, the objective of a feedback control system design is to achieve desired performance criterion in spite of external disturbances. Two foremost design tools developed are H^2 and H_∞ control, which are based on optimization theory aiming at minimizing specific norms of a transfer function from exogenous disturbance to a pertinent controlled output of the given plant. The performance objective of H_∞ control is usually specified as an L_2 disturbance attenuation problem with the standard interpretation such as guaranteeing robust stability or tracking. By using optimization techniques, a designer synthesizes controllers to achieve closed-loop robust performance or stabilization. On the other hand, assuming that the disturbance and measurement noise are Gaussian stochastic processes with known power spectral densities, the designer translates the design specifications into a quadratic performance criterion represented by some state variables and control signal inputs. The object of design then is to minimize the performance criterion by using an appropriate feedback controller while at the same time guaranteeing the closed-loop stability. That is known as linear quadratic Gaussian (LQG) control theory for linear systems, which is a stochastic interpretation of H_2 optimal control theory. Additionally, it is also meaningful and thus desirable to investigate mixed H_∞/H_2 (LQG) control problems, in which not only robustness specifications (in terms of an H_∞ constraint) but also performance specifications (measured in H_2 norm-like criteria or upper bounds thereof) can be taken into account. However, intrinsic conflicts often exist between achievable performance and system robustness, acceptable treatment is to make some suitable trade-offs between them.

As quantum control theory develops as an emerging discipline, scientists found that many research methodologies of classical control theory can be parallelly transplanted into quantum control theory. In principle, linear quantum stochastic systems can be represented by stochastic differential equations (SDEs) like classical stochastic systems do. Nevertheless, unique quantum characters have made that SDE with arbitrary parameter values may not correspond to a physically meaningful quantum system. Study shows that the quantum noncommutative relation will require the SDE's parameters to satisfy additional constraints, which are the so-called physical realizability constraints, see James et al. (2008). Such SDEs are defined as quantum stochastic differential equations (QSDEs), for details, refer to Gardiner and Collett (1985), Gardiner et al. (1992), James et al. (2008), Nurdin et al. (2009b), Nurdin et al. (2009a). James et al. (2008) have proposed a general framework of quantum H_∞ control for a class of linear QSDEs. Based on the analysis of dissipation properties, a quantum version of bounded real lemma is developed, with which they have also derived necessary and sufficient conditions for quantum H_∞ control problems in terms of a pair of Riccati equations. With such schematic established therein, a quantum optical experiment is successfully implemented by Mabuchi (2008). A coherent quantum LQG control problem is studied by Nurdin et al. (2009a), the authors have proposed an optimization algorithm, where the matrix polynomial equality constraints of physical realizability conditions are transformed into a rank constrained LMI problem. A numerical procedure is proposed to yield a coherent quantum controller to meet with the pre-specified LQG performance. Coherent control by direct couplings is studied by Zhang and James (2011), where they have set up a general open quantum feedback network by direct and indirect coupling theory. As an extended framework which encompasses both direct and indirect coupling schemes, coherent H_∞ and LQG synthesis problems are solved by a multi-step optimization algorithm. A dynamic game approach is proposed by Maalouf and Petersen (2010), where the result of an equivalent auxiliary classical stochastic problem can be used to solve the finite horizon H_∞ control problem for a class of linear quantum systems. A class of large-scale linear complex quantum stochastic systems with unstructured uncertainty is considered by Harno and Petersen (2010b). Instead of considering the uncertainty interconnection, the authors treat the off-diagonal blocks of the transfer function matrix of a non-decentralized linear coherent quantum controller as additive uncertainties. It can be verified that the resulting decentralized coherent quantum controller is not only robust against unstructured uncertainties and additive uncertainties, but also meet the specified disturbance attenuation. In addition, a class of linear passive quantum systems is discussed by Maalouf and Petersen (2011a), in which the system can be modeled purely in terms of annihilation operators. A complex quantum version of the bounded real lemma, the strict bounded real lemma and the lossless bounded real lemma are presented. Based on the results obtained, the H_∞ control problem for a class of linear passive quantum systems has been investigated by Maalouf and Petersen (2011b). And the coherent quantum controller can be obtained by solving a pair of complex algebraic Riccati equations. Finally, a couple of papers have presented detailed review on the H_∞ and LQG control of linear stochastic quantum systems, see Dong and Petersen (2010), Zhang and James (2012).

1.3 Summary of contributions of the thesis

The original contributions of this thesis are as follows:

- An in-depth analysis of squeezing performance of lossy DPAs are presented. Dynamical systems of lossy DPA are formulated in terms of QSDEs, numerical analysis of the squeezing ratios for the systems are given in terms of systems' transfer functions. To achieve better squeezing performance, an efficient coherent feedback control technique is proposed. The problem is then reformulated into a constrained nonlinear programming problem, which can be solved based on genetic algorithms (GA) and sequential quadratic programming (SQP). It is shown that the squeezing performance is significantly improved compared with former works. Finally, a detailed implementation scheme is presented by means of quantum optical devices. This study demonstrates that a systematic coherent feedback control design is promising in squeezing enhancement in quantum optics.

- Elaborate formulation of mixed LQG/H_∞ coherent feecdback control problems of linear quantum stochastic systems has been investigated, which can be viewed as a more general version of quantum LQG control problem or H_∞ control problem. An LMI based optimization approach is proposed to design a fully quantum controller to meet with both LQG and H_∞ performance. A cavity and a DPA are chosen as systems to be controlled in the numerical examples to illustrate the validity of the scheme, results show that some suitable trade-offs should be made between LQG performance and H_∞ performance as the intrinsic conflicts exist. Realization scheme of the coherent feedback controllers is also given by means of quantum optical devices based on the quantum network synthesis theory.

1.4 Organization of the thesis

The thesis is structured as follows.

- Chap. 2 reviews the preliminary knowledge of quantum control which includes some basic concepts of linear quantum systems, such as states and observables. Then the evolution of closed quantum systems is presented in terms of QSDEs. With the brief introduction of quantum fields, a general model of linear quantum stochastic systems is presented. By (S, L, H) parameterization, several kinds of quantum networks are reviewed.

- Chap. 3 focuses on squeezing performance analysis and squeezing enhancement of DPAs. First, the definition of squeezing ratio is proposed by using variance of quadratures. Then squeezing performance analysis of lossy DPAs is given for the cases of open loop and feedback loop. Then, a new coherent feedback control approach is presented to enhance squeezing performance. Finally, detailed implementation procedures of the resulting coherent feedback controllers are given in terms of quantum optical devices.

- Chap. 4 is devoted on mixed LQG/H_∞ control of linear quantum stochastic systems. A general model of open linear quantum systems is presented, and physical realizability of linear QSDEs is reviewed. The standard mixed LQG/H_∞

control problem is established in terms of the composite plant-controller system. To resolve the polynomial equality constraints due to physical realizability, the problem is then reformulated into a rank constrained optimization problem which can be solved by using Matlab and toolboxes therein. Finally, numerical examples are shown to demonstrate the effectiveness of the algorithm.

- Chap. 5 concludes the whole thesis and plans for the future work.

Chapter 2

Linear quantum stochastic systems

The purpose of this chapter is to discuss the formulation of linear quantum stochastic systems. We begin with the basic concepts of quantum states, observables and the Schrodinger's equation for closed systems. Of particular importance is the setup of linear quantum stochastic models of open linear quantum systems coupled to bosonic fields. Finally, we review the theory of quantum networks by (S, L, H) parameterization (refer to James et al. (2008), Nurdin et al. (2009b), Gough and James (2009b), Gough et al. (2010), Zhang and James (2012) for more details). The whole framework of linear quantum stochastic systems is based on quantum probability, which is the noncommutative counterpart of Kolmogorov's axiomatic characterization of classical probability theory, we refer readers to Accardi et al. (1982), Hudson and Parthasarathy (1984), Parthasarathy (1992), Meyer (1995), Bouten et al. (2007) for more details.

2.1 Basic concepts of linear quantum systems

2.1.1 States and observables

A state of a physical system usually contains the status of the system that enables the calculation of statistical quantities associated with observables. To each quantum mechanical system, there corresponds a Hilbert space $\mathcal{H}$, and each state of the quantum system is represented by an element of $\mathcal{H}$, written by $|\psi\rangle \in \mathbb{R}^n$. Following Dirac notation, $|\psi\rangle$ is called a *ket*, and its dual is denoted by $\langle\psi|$ which is called a *bra*. Algebraically, $|\psi\rangle$ and $\langle\psi|$ correspond to a column vector and a row vector, respectively. Thus, the inner product of $|\psi_1\rangle$ and $|\psi_2\rangle$ is defined by $\langle\psi_1|\psi_2\rangle$.

Generally, if we consider the situation in which the system is prepared with probabilities P_i in various states $|\psi_i\rangle$, $(i = 1, 2, 3, ...)$, we can define the *density operator* ρ, to be

$$\rho = \sum_i P_i |\psi_i\rangle\langle\psi_i| \tag{2.1}$$

with such characteristics: (i) self-adjoint on $\mathcal{H}$ that is positive $\rho \geqslant 0$; (ii) normalized as $\mathrm{Tr}[\rho] = 1$. It is obvious that all the measurable information is contained in the density operator.

Physical quantities such as position, momentum, spin, etc. are represented by self-adjoint operators on $\mathcal{H}$ which are called *observables* (see Yanagisawa and Kimura (2003a), Bouten et al. (2007)). Given a system in state $|\psi\rangle$, the expected value of an observable X is

$$\langle X\rangle = \langle\psi|X|\psi\rangle = \mathrm{Tr}\left[\rho X\right]. \tag{2.2}$$

According to the *spectral theorem*, if X is a normal operator $(XX^\dagger = X^\dagger X)$ on a finite dimensional Hilber space $\mathcal{H}$, X can be written as

$$X = \sum_{x \in \mathrm{spec}(X)} x P_x, \tag{2.3}$$

where $\mathrm{spec}(X)$ denotes the set of eigenvalues of X, and P_x is the projection operator on the associated eigenspace satisfying

$$P_x = \sum_{x_j = x} |x_j\rangle\langle x_j|, \tag{2.4}$$

$$I = \sum_{x \in \mathrm{spec}(X)} P_x, \tag{2.5}$$

where $\{|x_j\rangle\}$ $(j = 1, ...n)$ are orthonormal eigenvectors corresponding to eigenvalue x.

A linear quantum system can be viewed as a collection of n quantum harmonic oscillators which can be represented by *annihilation operators* $a = [a_1, ..., a_n]^T$ which are defined as:

$$a_j = \frac{1}{\sqrt{2}} (q_j + ip_j), \tag{2.6}$$

and its adjoint $a^\# = [a_1^*, ..., a_n^*]^T$ is called *creation operator* with

$$a_j^* = \frac{1}{\sqrt{2}} (q_j - ip_j), \tag{2.7}$$

where q and p are basic operators that correspond to the position and momentum observables of a system, respectively, defined by

$$\begin{aligned} (q\psi)(q) &= q\psi(q), \\ (p\psi)(q) &= -i\frac{\partial}{\partial q}\psi(q) \end{aligned} \tag{2.8}$$

for $\psi \in \mathcal{H}$, and the Planck's constant $\hbar$ is reduced to be 1 for simplicity.

It can be found that a and $a^\#$ are not self-adjoint operators, and satisfy the commutation relation

$$\left[a, a^\dagger\right] = I \tag{2.9}$$

where $[X, Y]$ denotes the commutator defined by $[X, Y] := XY - YX$.

2.1.2 Evolution of closed linear quantum systems

According to quantum mechanics the state vector $|\psi(t)\rangle$ evolves in time according to the schrödinger equation,

$$\frac{d}{dt}|\psi(t)\rangle = -iH|\psi(t)\rangle, \tag{2.10}$$

where H is the Hamiltonian of the system that corresponds to the observable of energy. According to Zhang and James (2011), H is given by

$$H = \frac{1}{2}\breve{a}^\dagger \begin{bmatrix} \Omega_- & \Omega_+ \\ \Omega_+^\# & \Omega_-^\# \end{bmatrix} \breve{a}, \tag{2.11}$$

where Ω_- and $\Omega_+ \in \mathbb{C}^{n\times n}$ satisfying $\Omega_- = \Omega_-^\dagger$ and $\Omega_+ = \Omega_+^T$.

The solution of the Schrödinger equation may be represented in terms of the unitary time-evolution operator $U(t)$ which transforms the state $|\psi(0)\rangle$ at an initial time $t_0 = 0$ to the state $|\psi(t)\rangle$ at time t,

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle. \tag{2.12}$$

Thus combining (2.10) and (2.12), we may get a Schrödinger equation for the time-evolution operator $U(t)$,

$$\frac{d}{dt}U(t) = -iHU(t), \tag{2.13}$$

with the initial condition $U(0) = I$.

In the Heisenberg picture, the annihilation operators evolve according to $a_j(t) = U^*(t)a_jU(t)$, which yields the quantum differential equation of a,

$$\dot{a}_j(t) = -i\left[a_j(t), H\right], \quad a_j(0) = a_j. \tag{2.14}$$

By substituting (2.11) into (2.14), we obtain the following lienar differential e-quation in the compact form

$$\dot{\breve{a}}(t) = A\breve{a}(t) \tag{2.15}$$

where $A = -\Delta(i\Omega_-, i\Omega_+)$.

2.2 Open linear quantum systems

Generally, a quantum system is never completely isolated from its environment. Such quantum systems are said to be *open quantum systems*. In this section, we turn to the notion of linear quantum stochastic models which are used to provide tractable models for how open linear quantum systems evolve by interacting with environments.

2.2.1 Quantum fields

In quantum field theory, a boson quantum field can be defined on a Fock space $\mathcal{F}$ consisting of a collection of m channels (light fields) each with annihilation operators $b_j(t)$ and creation operators $b_j^*(t)$ satisfying the singular commutation relations

$$\begin{aligned} &[b_j(t), b_k^*(t')] = \delta_{jk}\delta(t-t'), \\ &[b_j(t), b_k(t')] = [b_j^*(t), b_k^*(t')] = 0 \quad (j,k = 1, ..., m) \end{aligned} \tag{2.16}$$

for all t and t', where $\delta(t)$ is the Dirac delta function.

In this thesis, we consider the boson fields in vacuum state, which is a natural quantum extension of white noise, and can be described using the *quantum Itō calculus* (see Hudson and Parthasarathy (1984), Gardiner and Collett (1985), Parthasarathy (1992), Bouten et al. (2007) for details). Define three fundamental integrated field operators as follows

$$B_j(t) = \int_0^t b_j(s)ds, \quad B_j^*(t) = \int_0^t b_j^*(s)ds, \quad \Lambda_{jk}(t) = \int_0^t b_j^*(s)b_k(s)ds. \tag{2.17}$$

These field operators may be regarded as *quantum stochastic processes* (Gardiner and Zoller, 2004, Chapter 5) with non-zero Itō products

$$\begin{aligned} dB_j(t)dB_k^*(t) &= \delta_{jk}dt, \\ d\Lambda_{jk}dB_l^*(t) &= \delta_{kl}dB_j^*(t) \\ dB_j(t)d\Lambda_{kl}(t) &= \delta_{jk}dB_l(t), \\ d\lambda_{jk}(t)d\Lambda_{lm}(t) &= \delta_{kl}d\Lambda_{jm}(t), \quad (j,k,l = 1, ..., m). \end{aligned} \tag{2.18}$$

2.2.2 Linear quantum stochastic models with (S, L, H) parameterization

As studied above an open linear quantum system G can be viewed as a collection of n quantum harmonic oscillators $a(t)$ interacting with an m-channel quantum fields such a system can be specified by a triple of physical parameters (S, L, H) on Hilbert space $\mathcal{H}$ (see Gough and James (2009b)). In this triple S is a scattering matrix of dimension m which is unitary; and L is a vector of coupling operators defined by

$$L = C_{-}a + C_{+}a^{\#} \tag{2.19}$$

where C_{-} and $C_{+} \in \mathbb{C}^{m\times n}$; H is the Hamiltonian describing the self-energy of the system, satisfying (2.11).

In terms of the parameters S, L and H, the dynamics of the open linear quantum system G can be described by the unitary evolution operator $U(t)$ solving the stochastic Schrödinger equation

$$dU(t) = \left\{ dB^{\dagger}(t)L - L^{\dagger}SdB(t) - \left(\frac{1}{2}L^{\dagger}L + iH\right)dt + \mathrm{Tr}[(S - I_m)d\Lambda^T] \right\} \tag{2.20}$$

with initial value $U(0) = I$, and Λ is a m by m matrix with entries Λ_{jk}, c.f. (2.17).

An important feature of this type of open linear quantum model is the input-output behavior. $B_{out}(t)$ is used to denote the field after the interaction, which satisfies $B_{out}(t) = U^{*}(t)B(t)U(t)$. As the annihilation operators evolve following $a_j(t) = U^{*}(t)a_jU(t)$, the system can be represented by such QSDEs in Stratonovich form

$$\begin{aligned} \dot{\breve{a}}(t) &= A\breve{a}(t) + B\breve{b}(t), \quad \breve{a}(0) = \breve{a}, \\ \breve{b}_{out}(t) &= C\breve{a}(t) + D\breve{b}(t). \end{aligned} \tag{2.21}$$

As shown by Gough and Wildfeuer (2009), Gough et al. (2010), the matrix coefficients (A, B, C, D) of (2.21) and the physical parameters (S, L, H) have the following corresponding relations

$$\begin{aligned} A &= -\frac{1}{2}C^{\flat}C - iJ_n\Delta(\Omega_-, \Omega_+), \quad B = -C^{\flat}\Delta(S, 0), \\ C &= \Delta(C_-, C_+), \quad D = \Delta(S, 0). \end{aligned} \tag{2.22}$$

In the transfer function form (see Zhou et al. (1996)), G is written as $\Xi(s) = D + C(sI - A)^{-1}B$.

So far, we have represented linear quantum stochastic systems in terms of annihilation and creation operators, such form as (2.21) is usually called *annihilation-creation form*. In addition, there is an alternative representation which can be obtained by a simple linear transformation of annihilation-creation form, and is referred to as *quadrature form*.

Define a unitary transformation matrix

$$\Lambda_n = \frac{1}{\sqrt{2}} \begin{bmatrix} I_n & I_n \\ -iI_n & iI_n \end{bmatrix}. \tag{2.23}$$

Then define quadrature form operators

$$x = \Lambda_n \breve{a}, \;\; \tilde{b}_{out} = \Lambda_m \breve{b}_{out}, \;\; \tilde{b} = \Lambda_m \breve{b}. \tag{2.24}$$

The quadrature form of (2.21) may be written as

$$\begin{aligned} dx(t) &= \tilde{A}x(t)dt + \tilde{B}d\tilde{B}(t), \\ d\tilde{B}_{out}(t) &= \tilde{C}x(t)dt + \tilde{D}d\tilde{B}(t), \end{aligned} \tag{2.25}$$

where

$$\tilde{A} = \Lambda_n A \Lambda_n^{\dagger}, \quad \tilde{B} = \Lambda_n B \Lambda_m^{\dagger}, \quad \tilde{C} = \Lambda_m C \Lambda_n^{\dagger}, \quad \tilde{D} = \Lambda_m D \Lambda_m^{\dagger}, \tag{2.26}$$

and the quadrature form of the system transfer matrix can be obtained immediately, which is

$$\tilde{\Xi}(s) = \tilde{C}(sI_n - \tilde{A})^{-1}\tilde{B} + \tilde{D} = \Lambda_m \Xi(s) \Lambda_m^{\dagger}. \tag{2.27}$$

2.3 Quantum networks by interconnection

As discussed in the above section, the quantum harmonic oscillators can be described by the parameters (S, L, H). Moreover, these parameters also provide a powerful tool for analyzing the interconnection of linear quantum stochastic systems. In this section, we review several kinds of quantum networks which are established by using parameters (S, L, H). In the following, we begin with the definition of *concatenation product* and *series product*.

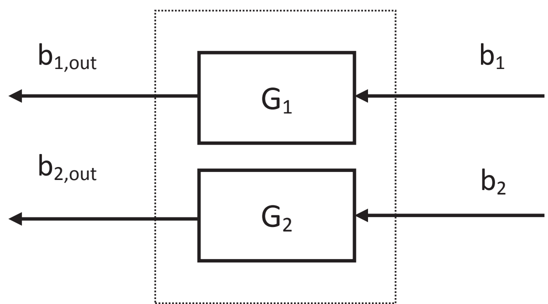

Figure 2.1: Concatenation product $G_1 \boxplus G_2$.

Definition 2.1 (Concatenation product proposed by Gough and James (2009b))**.** *Given two systems $G_1 = (S_1, L_1, H_1)$ and $G_2 = (S_2, L_2, H_2)$, see Fig. 2.1, we define their concatenation to be the system $G_1 \boxplus G_2$ by*

$$G_1 \boxplus G_2 = \left(\begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}, \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, H_1 + H_2 \right). \tag{2.28}$$

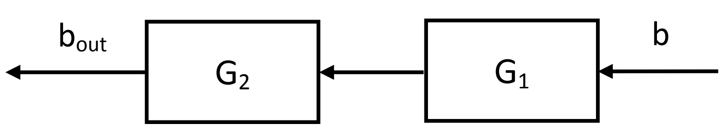

Figure 2.2: Series product $G_2 \lhd G_1$.

Definition 2.2 (Series product proposed by Gough and James (2009b)). *Given two systems* $G_1 = (S_1, L_1, H_1)$ *and* $G_2 = (S_2, L_2, H_2)$ *with the same number of field channels, see Fig. 2.2, the series product* $G_2 \lhd G_1$ *is defined by*

$$G_2 \lhd G_1 = \left(S_2 S_1,\ L_2 + S_2 L_1,\ H_1 + H_2 + \frac{1}{2i} \left(L_2^\dagger S_2 L_1 - L_1^\dagger S_2^\dagger L_2 \right) \right). \tag{2.29}$$

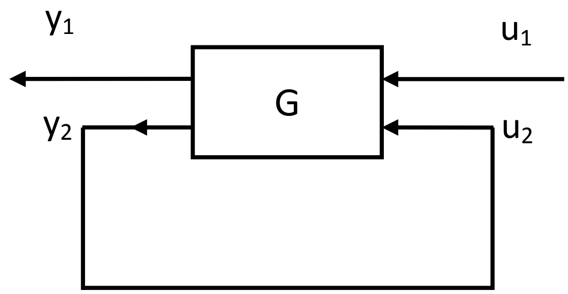

Figure 2.3: Linear fractional transformation $F(G)$.

Problem 2.1 (Linear fractional transformation studied by Gough and James (2009b), Gough et al. (2010) and Zhang and James (2012)). *Assume* G *is a four-port system with input fields* $u = (u_1, u_2)^T$ *and output fields* $y = (y_1, y_2)^T$ *as shown in Fig. 2.3, the parameters is given by* $\left(\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, H \right)$. *If the the system is put in a feedback setup with* $u_2 = y_2$*, then the feedback system* $F(G)$ *can be obtained by applying the linear fractional transformation*

$$\begin{aligned} F(G) = \Big(& S_{11} + S_{12}(I - S_{22})^{-1} S_{21},\ L_1 + S_{12}(I - S_{22})^{-1} L_2, \\ & H + \mathbf{Im}\{L_1^\dagger S_{12}(I - S_{22})^{-1} L_2\} + \mathbf{Im}\{L_2^\dagger S_{22}(I - S_{22})^{-1} L_2\} \Big) \end{aligned} \tag{2.30}$$

from u_1 *to* y_1*, where* $\mathbf{Im}\{\bullet\}$ *denotes the imaginary part of* $\bullet$*. Here we assume that* $I - S_{22}$ *is nonsingular.*

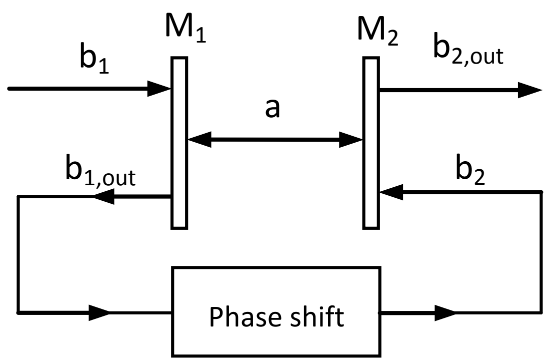

Figure 2.4: Cavity in a feedback loop.

In the following, we present an example as application of the products defined above, in which all the components involved are fully quantum.

Example 2.1 (Cavity in a feedback loop proposed by Wiseman and Milburn (1994a) and Gough and James (2009b))**.** *We reconsider the all-optical feedback example studied by Wiseman and Milburn (1994a) and Gough and James (2009b). The cavity consists of a pair of partially transmitting mirrors* M_1 *and* M_2*. Input light field* b_1 *first irradiates on* M_1 *and interacts with the internal cavity mode* a *and thus yields an output beam* $b_{1,out}$*. Then* $b_{1,out}$ *goes through a phase shifter* $S = e^{i\theta}$ *when reflected by* M_2*. Detailed schematic is shown in Fig. 2.4. Assume both* M_1 *and* M_2 *have the same transmittivity, and the coupling operator for the two field channels are* $L_1 = L_2 = \sqrt{\gamma}a$*, where* γ *is the damping rate.*

Before feedback, the cavity and the phase shifter are specified by $G_c = \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, 0\right)$*, and* $G_p = (S, 0, 0)$*, respectively. Therefore we can calculate the closed-loop feedback system as*

$$\begin{aligned} G_{cl} &= (I, L_2, 0,) \lhd (S, 0, 0) \lhd (I, L_1, 0) \\ &= \left(S, SL_1 + L_2, \frac{1}{2i}(L_2^* SL_1 - L_1^* S^* L_2)\right) \qquad (2.31) \\ &= \left(e^{i\theta}, \left(I + e^{i\theta}\right)\sqrt{\gamma}a, \gamma \sin\theta a^* a\right). \end{aligned}$$

As the cavity and the phase shifter of the system are all static components, we may derive the QSDE of the closed-loop system in terms of annihilation operators based on the corresponding relations given in (2.22)

$$\begin{aligned} \dot{a}(t) &= -(I + e^{i\theta})\gamma a(t) - (I + e^{i\theta})\sqrt{\gamma} b_1(t) \\ b_{2,out}(t) &= (I + e^{i\theta})\sqrt{\gamma} a(t) + e^{i\theta} b_1(t). \end{aligned} \tag{2.32}$$

Chapter 3

Squeezing enhancement of degenerate parametric amplifiers

The main purpose of this chapter is to present an in-depth investigation of squeezing performance of lossy DPAs and present a coherent feedback control approach to achieving better squeezing performance in the lossy case. Here we focus on the case that the DPA is lossy as the noise will affect the squeezing performance and *perfect squeezing* (see Collett and Walls (1985)) does not exist. Therefore the central problem is to design a coherent feedback control scheme to improve squeezing performance. It turns out that such problem can be formulated into a problem of constrained nonlinear programming, which is solved by combining genetic algorithms and sequential quadratic programming. We show that in the lossy case, in contrast to the scheme proposed by Gough and Wildfeuer (2009), the method proposed in this paper can enhance squeezing significantly, cf. Table 3.1. This study demonstrates that a systematic coherent feedback control design is promising in squeezing enhancement in quantum optics.

3.1 Squeezing ratio

We assume the quantum system G is an $m-$channel input-output squeezing component which is initialized in vacuum state. Interaction of G with the boson input field

$\tilde{b}_{in}(\omega)$ may produce the output field $\tilde{b}_{out}(\omega)$. For each $i = 1, \ldots, m$, denote

$$\tilde{b}_{in,i}(\omega) = \begin{bmatrix} Q_{in,i}(\omega) \\ P_{in,i}(\omega) \end{bmatrix}, \quad \tilde{b}_{out,i}(\omega) = \begin{bmatrix} Q_{out,i}(\omega) \\ P_{out,i}(\omega) \end{bmatrix}, \tag{3.1}$$

where the capital letters Q and P denote amplitude and phase quadratures, respectively. Assume the input fields are standard quantum white noise Shapiro and Wagner (1984), that is,

$$\langle Q_{in,i}(\omega)\rangle = \langle P_{in,i}(\omega)\rangle \equiv 0, \quad \forall \omega \in \mathbb{R},$$

and

$$\mathbf{Var}(Q_{in,i}(\omega)) = \mathbf{Var}(P_{in,i}(\omega)) \equiv \frac{1}{2}, \quad \forall \omega \in \mathbb{R}. \tag{3.2}$$

Under the assumption of initial vacuum state for the system involved, the output field has zero mean, that is,

$$\langle Q_{out,i}(\omega)\rangle = \langle P_{out,i}(\omega)\rangle \equiv 0, \quad \forall \omega \in \mathbb{R}.$$

Consequently, variances of its two quadratures are

$$\mathbf{Var}(Q_{out,i}(\omega)) = \langle Q_{out,i}(\omega)^2\rangle, \tag{3.3a}$$

$$\mathbf{Var}(P_{out,i}(\omega)) = \langle P_{out,i}(\omega)^2\rangle. \tag{3.3b}$$

Though the input fields are standard quantum white noise, interaction with the linear quantum optical system G may produce output field which does not satisfy (3.2), that is, one quadrature might be amplified whereas the other *squeezed*.

Definition 3.1 (squeezing ratio (Bian et al. (2012)))**.** *The squeezing ratio of quantum input-output system is given by the ratio of the variance of output quadrature and the corresponding input quadrature. To be more specific, the squeezing ratio of the amplitude quadrature is given by* $\mu_q(\omega) = \frac{\langle Q_{out,1}(\omega)^2\rangle}{\langle Q_{in,1}(\omega)^2\rangle}$*, and the squeezing ratio of the phase quadrature is given by* $\mu_p(\omega) = \frac{\langle P_{out,1}(\omega)^2\rangle}{\langle P_{in,1}(\omega)^2\rangle}$.

Remark 3.1. *It should be noted that all* $\langle Q_{out,1}(\omega)^2\rangle, \langle Q_{in,1}(\omega)^2\rangle, \langle P_{out,1}(\omega)^2\rangle$, *and* $\langle P_{in,1}(\omega)^2\rangle$ *are impulsive functions multiplied by scalars. Nevertheless, the ratios* $\mu_q(\omega)$ *and* $\mu_p(\omega)$ *are not impulsive functions, that is, the impulsive functions are canceled. This can be seen from the proof of Theorem 3.2.*

3.2 Analysis of squeezing performance

In this section, we present a detailed analysis of the squeezing performance of a lossy DPA.

3.2.1 Lossy DPA

A lossy DPA is a linear quantum optical system interacting with two independent boson fields ($m = 2$) (Leonhardt (2003), Gardiner and Zoller (2004)). The schematic is shown in Fig. 3.1.

Figure 3.1: Lossy DPA.

The DPA interacts with the input field $b_{in,1}$ and produces an output field $b_{out,1}$. Moreover, the DPA is assumed to be lossy, which is modeled as interaction with a vacuum input field $b_{in,2}$. In the (S, L, H) language, the lossy DPA has parameters

$$S = I,\ \Omega_- = 0,\ \Omega_+ = \frac{i\epsilon}{2},\ C_- = [\sqrt{\kappa}\ \sqrt{\gamma}]^T,\ C_+ = 0,$$

where $0 < \epsilon < \kappa + \gamma$ is assumed for Hurwitz stability (Gough et al. (2010), Zhou et al. (1996)). According to the corresponding relation (2.22), we may present the system in the QSDE form

$$\begin{aligned} d\breve{a}(t) &= \begin{bmatrix} -\frac{\kappa+\gamma}{2} & \frac{\epsilon}{2} \\ \frac{\epsilon}{2} & -\frac{\kappa+\gamma}{2} \end{bmatrix} \breve{a}(t)dt - \sqrt{\kappa}d\breve{B}_{in,1}(t) - \sqrt{\gamma}d\breve{B}_{in,2}(t), \\ d\breve{B}_{out,1}(t) &= \sqrt{\kappa}\breve{a}(t)dt + d\breve{B}_{in,1}(t), \\ d\breve{B}_{out,2}(t) &= \sqrt{\gamma}\breve{a}(t)dt + d\breve{B}_{in,2}(t). \end{aligned} \tag{3.4}$$

By using transformation matrix $\Lambda = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$, the system can be transferred in the quadrature form

$$\begin{aligned} dx(t) &= \begin{bmatrix} \frac{\epsilon-\kappa-\gamma}{2} & 0 \\ 0 & \frac{-\epsilon-\kappa-\gamma}{2} \end{bmatrix} x(t)dt - \sqrt{\kappa}d\tilde{B}_{in,1}(t) - \sqrt{\gamma}d\tilde{B}_{in,2}(t), \\ d\tilde{B}_{out,1}(t) &= \sqrt{\kappa}x(t)dt + d\tilde{B}_{in,1}(t), \\ d\tilde{B}_{out,2}(t) &= \sqrt{\gamma}x(t)dt + d\tilde{B}_{in,2}(t). \end{aligned} \tag{3.5}$$

Therefore we have the following input-output relations in the frequency domain

$$\begin{aligned} Q_{out,1}(s) &= \left(1 - \frac{2\kappa}{\kappa-\epsilon+\gamma+2s}\right) Q_{in,1}(s) + \frac{2\sqrt{\gamma\kappa}}{\epsilon-\kappa-\gamma-2s} Q_{in,2}(s), \\ P_{out,1}(s) &= \left(1 - \frac{2\kappa}{\epsilon+\kappa+\gamma+2s}\right) P_{in,1}(s) - \frac{2\sqrt{\gamma\kappa}}{\epsilon+\gamma+\kappa+2s} P_{in,2}(s). \end{aligned} \tag{3.6}$$

Assume that input fields $b_{in,1}$ and $b_{in,2}$ are independent standard quantum white noise. Then it can be found that

$$\mathbf{Var}(Q_{out,1}(\omega)) = \left(\left|1 - \frac{2\kappa}{\kappa-\epsilon+\gamma+2i\omega}\right|^2 + \left|\frac{2\sqrt{\gamma\kappa}}{\epsilon-\kappa-\gamma-2s}\right|^2 \right) \langle Q_{in,1}(\omega)^2 \rangle.$$

According to Definition 3.1, the squeezing ratio of the amplitude quadrature is

$$\mu_q(\omega) = \frac{\langle Q_{out,1}(\omega)^2 \rangle}{\langle Q_{in,1}(\omega)^2 \rangle} = \left|1 - \frac{2\kappa}{\kappa-\epsilon+\gamma+2i\omega}\right|^2 + \left|\frac{2\sqrt{\gamma\kappa}}{\epsilon-\kappa-\gamma-2i\omega}\right|^2 \tag{3.7}$$

Similarly, the squeezing ratio of the phase quadrature is

$$\mu_p(\omega) = \frac{\langle P_{out,1}(\omega)^2 \rangle}{\langle P_{in,1}(\omega)^2 \rangle} = \left|1 - \frac{2\kappa}{\epsilon+\kappa+\gamma+2i\omega}\right|^2 + \left|\frac{2\sqrt{\gamma\kappa}}{\epsilon+\gamma+\kappa+2i\omega}\right|^2. \tag{3.8}$$

The following example illustrates the squeezing performance of the lossy DPA.

Example 3.1. *Assume the physical parameters to be* $\gamma = 0.2$, $\kappa = 0.6$, $\epsilon = 0.2$, *and plot the graph of* μ_p *and* μ_q *versus frequency* ω *in Fig. 3.2. It can be easily seen that* μ_p *tends to* 0.5200 *and* μ_q *goes to* 2.333 *as* $\omega \to 0^+$. *That is, the phase quadrature is squeezed.*

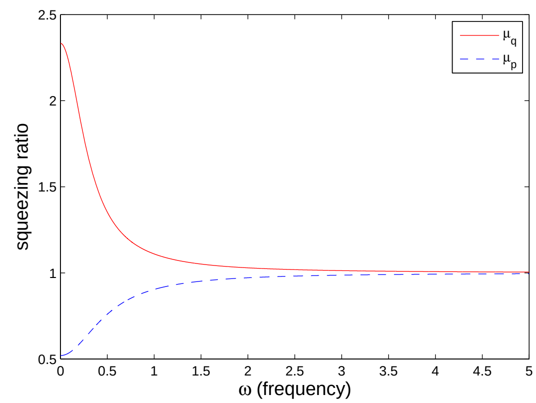

Figure 3.2: Squeezing ratios of quadratures of the lossy DPA.

3.2.2 Lossy DPA in feedback loop

To enhance squeezing performance of DPAs, Gough and Wildfeuer (2009) proposed to put a DPA into a feedback loop closed by a beam splitter. The setup is shown in Fig. 3.3.

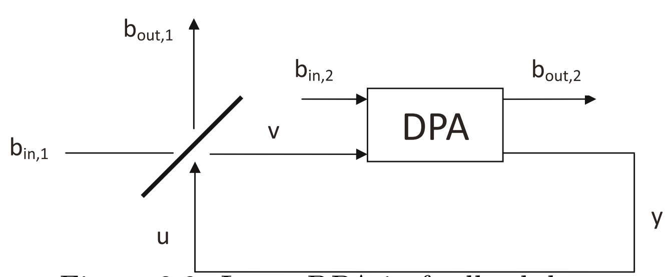

Figure 3.3: Lossy DPA in feedback loop.

The beam splitter in Fig. 3.3 is a static device which can be modeled as

$$\begin{bmatrix} b_{out,1} \\ v \end{bmatrix} = S_{BM} \begin{bmatrix} b_{in,1} \\ u \end{bmatrix}, \tag{3.9}$$

where

$$S_{BM} = \begin{bmatrix} \alpha & \sqrt{1-\alpha^2} \\ \sqrt{1-\alpha^2} & -\alpha \end{bmatrix} \tag{3.10}$$

with $0 \leqslant \alpha \leqslant 1$.

By means of linear fractional transformation (LFT) (James and Gough (2010), Zhou et al. (1996)), the closed-loop system in Fig. 3.3 can be written as

$$\begin{aligned} d\breve{a}(t) &= \begin{bmatrix} \frac{-\kappa+\alpha\kappa-\gamma-\gamma\alpha}{2+2\alpha} & \frac{\epsilon}{2} \\ \frac{\epsilon}{2} & \frac{-\kappa+\alpha\kappa-\gamma-\gamma\alpha}{2+2\alpha} \end{bmatrix} \breve{a}(t)dt \\ &\quad - \frac{\sqrt{\kappa(1-\alpha^2)}}{1+\alpha} d\breve{B}_{in,1}(t) - \sqrt{\gamma} d\breve{B}_{in,2}(t), \\ d\breve{B}_{out,1}(t) &= \frac{\sqrt{\kappa(1-\alpha^2)}}{1+\alpha} \breve{a}(t)dt + d\breve{B}_{in,1}(t), \\ d\breve{B}_{out,2}(t) &= \sqrt{\gamma}\breve{a}(t)dt + d\breve{B}_{in,2}(t). \end{aligned} \tag{3.11}$$

Clearly, when $\alpha = 0$, system (3.11) reduces to system (3.4).

Remark 3.2. *To guarantee the Hurwitz stability of the system (3.11), it is required that* $0 < \epsilon < \gamma + \frac{1-\alpha}{1+\alpha}\kappa$.

Using a procedure similar to that in Sec. 3.2.1, we find the squeezing ratio

$$\begin{aligned} \mu_q(\omega) &= \frac{\langle Q_{out,1}(\omega)^2 \rangle}{\langle Q_{in,1}(\omega)^2 \rangle} \\ &= \left| \frac{\gamma - \epsilon - \kappa + 2i\omega - \alpha\epsilon + \alpha\gamma + \alpha\kappa + 2i\alpha\omega}{\gamma - \epsilon + \kappa + 2i\omega - \alpha\epsilon - \alpha\kappa + \alpha\gamma + 2i\alpha\omega} \right|^2 \\ &\quad + \left| \frac{2\sqrt{\gamma\kappa(1-\alpha^2)}}{\epsilon - \kappa - \gamma - 2i\omega + \alpha\epsilon + \alpha\kappa - \alpha\gamma - 2i\alpha\omega} \right|^2, \end{aligned} \tag{3.12}$$

and

$$\mu_p(\omega) = \frac{\langle P_{out,1}(\omega)^2\rangle}{\langle P_{in,1}(\omega)^2\rangle}$$
$$= \left|\frac{\epsilon+\gamma-\kappa+2i\omega+\alpha\epsilon+\alpha\kappa+\alpha\gamma+2i\alpha\omega}{\epsilon+\kappa+\gamma+2i\omega+\alpha\epsilon+\alpha\gamma-\alpha\kappa+2i\alpha\omega}\right|^2 \tag{3.13}$$
$$+\left|\frac{2\sqrt{\gamma\kappa(1-\alpha^2)}}{\epsilon+\kappa+\gamma+2i\omega+\alpha\epsilon-\alpha\kappa+\alpha\gamma+2i\alpha\omega}\right|^2.$$

When the DPA is lossless (namely, $\gamma = 0$), it can be seen that the second terms of μ_p and μ_q are both 0. Moreover, when $\alpha = \frac{\kappa-\epsilon}{\kappa+\epsilon}$, μ_p goes to 0 and μ_q tends to ∞ when $\omega \to 0$. This is the case of the so-called *infinite squeezing* (Gough and Wildfeuer, 2009, Sec. V). In this paper, we are interested in squeezing performance in the presence of loss, as is always the case in practice. In this case, the second terms of μ_p and μ_q are in general not 0 any more.

Theorem 3.1. *The feedback loop scheme proposed by Gough and Wildfeuer (2009) can improve squeezing performance over open lossy DPA shown in Fig. 3.1 in the static case if and only if* $0 < \epsilon < \kappa - \gamma$.

Proof. Above all, according to Remark 3.2, it is required $\alpha < \frac{\kappa-\epsilon+\gamma}{\kappa+\epsilon-\gamma} := \alpha^\star$ to preserve Hurwitz stability of the closed-loop system.

Moreover, in the static case where $\omega = 0$, (3.13) can be calculated as

$$\mu_p = \left(\frac{\epsilon+\gamma-\kappa+\alpha\epsilon+\alpha\kappa+\alpha\gamma}{\epsilon+\kappa+\gamma+\alpha\epsilon+\alpha\gamma-\alpha\kappa}\right)^2 + \left(\frac{2\sqrt{\gamma\kappa(1-\alpha^2)}}{\epsilon+\kappa+\gamma+\alpha\epsilon-\alpha\kappa+\alpha\gamma}\right)^2$$
$$= 1 - \frac{4\epsilon\kappa}{(\frac{\epsilon+\gamma}{\sqrt{\beta}}+\kappa\sqrt{\beta})^2}.$$

where we let $\beta = \frac{1-\alpha}{1+\alpha} \in [0,\ 1]$. To minimize (3.14), we need to consider two cases:

case 1 $\kappa > \epsilon + \gamma$. It is easy to see that $\mu_p \leqslant \frac{\gamma}{\epsilon+\gamma}$, where the equality holds if and only if $\frac{\epsilon+\gamma}{\sqrt{\beta}} = \kappa\sqrt{\beta}$. That is, $\alpha = \frac{\kappa-\epsilon-\gamma}{\kappa+\epsilon+\gamma} < \alpha^\star$.

case 2 $\kappa \leqslant \epsilon+\gamma$. In this case, the minimal value of μ_p is obtained when $\beta = 1$, $\alpha = 0$, and $\mu_p = 1 - \frac{4\epsilon\kappa}{(\epsilon+\gamma+\kappa)^2}$. Therefore in this case the feedback scheme proposed by Gough and Wildfeuer (2009) has no advantage over the lossy DPA in the open loop.

To summarize, the feedback loop scheme can achieve better squeezing performance compared with open lossy DPA in the static case if and only if $0 < \epsilon < \kappa - \gamma$. To be more specific, the minimal value of squeezing ratio $\mu_p = \frac{\gamma}{\epsilon+\gamma}$ is obtained when $\alpha = \frac{\kappa-\epsilon-\gamma}{\kappa+\epsilon+\gamma}$. This completes the proof. $\square$

Example 3.2. *Using the same parameter values as in Example 3.1, we find that the minimal value of μ_p is 0.500 obtained when $\alpha = 0.2$ as $\omega \to 0^+$. Compared with the case of the lossy DPA ($\mu_p = 0.5200$), the squeezing performance has been enhanced only slightly by this method. It can be seen that μ_q is amplified from 2.333 to 3, and the product of μ_p and μ_q is also amplified from 1.2132 to 1.5. The corresponding graph of μ_p and μ_q versus frequency ω is shown in Fig. 3.4.*

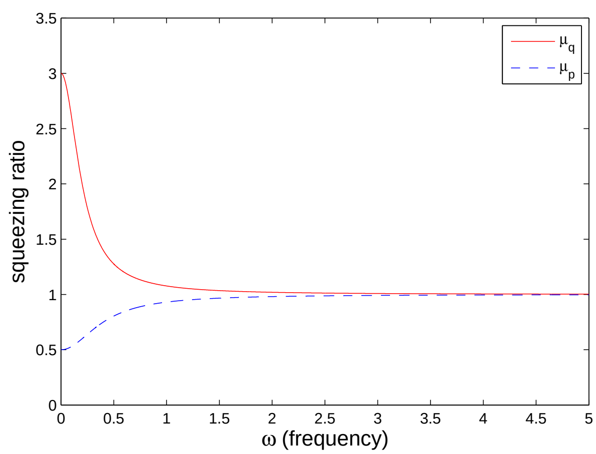

Figure 3.4: Squeezing ratios of quadratures by feedback scheme.

3.3 Coherent feedback controller design

In this section we present a new coherent feedback controller design method to enhance the DPA's squeezing performance. The controller is assumed to be another quantum oscillator which is series connected with a DPA, and closed by a beam splitter, see Fig. 3.5. Here the controller **K** is driven by a vacuum input field w_{in} which is used to describe the loss in the controller.

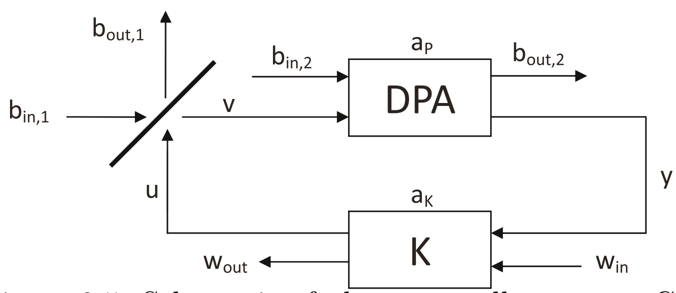

Figure 3.5: Schematic of plant-controller system G.

3.3.1 Formulation of the quantum coherent controller

Assume that the controller **K** in Fig. 3.5 to be designed is also a quantum system which can be viewed as a collection of k quantum harmonic oscillators coupling with l-channel quantum field, thus it can be parameterized by

$$\begin{aligned} S_K &= I, \\ L_K &= C_{K-}a_K + C_{K+}a_K^*, \\ H_K &= \frac{1}{2}\breve{a}_K^* \begin{bmatrix} \Omega_{K-} & \Omega_{K+} \\ \Omega_{K+}^\# & \Omega_{K-}^\# \end{bmatrix} \breve{a}_K, \end{aligned} \tag{3.14}$$

where C_{K-} and $C_{K+} \in \mathbb{C}^{l\times k}$, Ω_{K-} and $\Omega_{K+} \in \mathbb{C}^{k\times k}$ satisfying $\Omega_{K-} = \Omega_{K-}^\dagger$ and $\Omega_{K+} = \Omega_{K+}^T$.

3.3.2 The plant-controller closed-loop system

We consider the simple case where $k = 1$ and $l = 2$. Assume the controller to be designed is specified by the following parameters

$$\begin{aligned} S_K &= I, \\ L_K &= \begin{bmatrix} \sqrt{x_1} \\ \sqrt{x_2} \end{bmatrix} a_K + \begin{bmatrix} \sqrt{x_3} \\ \sqrt{x_4} \end{bmatrix} a_K^*, \\ H_K &= \frac{1}{2} \breve{a}_K^* \begin{bmatrix} x_5 & x_6 \\ x_6^* & x_5 \end{bmatrix} \breve{a}_K, \end{aligned} \tag{3.15}$$

where $x_1, x_2, x_3, x_4 \in \mathbb{R}^+$, $x_5 \in \mathbb{R}$ and $x_6 \in \mathbb{C}$ are scalar variables to be determined [1]. We also assume that this coherent feedback controller is initialized in vacuum state.

Based on the plant's equations obtained in Sec. 3.2.1, we may get the composite closed-loop system's dynamics and output equations

$$\begin{aligned} \dot{\breve{a}}_p(t) &= A_p \breve{a}_p(t) + \frac{\alpha\sqrt{\kappa}}{1+\alpha} M_1 \breve{a}_k(t) - \sqrt{\frac{\kappa(1-\alpha)}{1+\alpha}} \breve{b}_{in,1}(t) - \sqrt{\gamma} \breve{b}_{in,2}(t), \\ \dot{\breve{a}}_k(t) &= -\frac{\sqrt{\kappa}}{1+\alpha} M_2 \breve{a}_p(t) + A_k \breve{a}_k(t) - \sqrt{\frac{1-\alpha}{1+\alpha}} M_2 \breve{b}_{in,1}(t) - N \breve{w}_{in}(t), \\ \breve{b}_{out,1}(t) &= \sqrt{\frac{\kappa(1-\alpha)}{1+\alpha}} \breve{a}_p(t) + \sqrt{\frac{1-\alpha}{1+\alpha}} M_1 \breve{a}_k(t) + \breve{b}_{in,1}(t), \end{aligned} \tag{3.16}$$

where

$$A_p = A_{p0} + A_{pb} = \begin{bmatrix} -\frac{\kappa+\gamma}{2} & \frac{\epsilon}{2} \\ \frac{\epsilon}{2} & -\frac{\kappa+\gamma}{2} \end{bmatrix} + \begin{bmatrix} \frac{\alpha\kappa}{1+\alpha} & 0 \\ 0 & \frac{\alpha\kappa}{1+\alpha} \end{bmatrix},$$

$$\begin{aligned} A_k &= A_{k0} + A_{kb} \\ &= \begin{bmatrix} -\frac{(x_1+x_2-x_3-x_4)}{2} - ix_5 & -ix_6 \\ ix_6^* & -\frac{(x_1+x_2-x_3-x_4)}{2} + ix_5 \end{bmatrix} + \begin{bmatrix} \frac{\alpha}{1+\alpha}(x_1 - x_3) & 0 \\ 0 & \frac{\alpha}{1+\alpha}(x_1 - x_3) \end{bmatrix}, \end{aligned}$$

$$M_1 = \begin{bmatrix} \sqrt{x_1} & \sqrt{x_3} \\ \sqrt{x_3} & \sqrt{x_1} \end{bmatrix}, \quad M_2 = \begin{bmatrix} \sqrt{x_1} & -\sqrt{x_3} \\ -\sqrt{x_3} & \sqrt{x_1} \end{bmatrix}, \quad N = \begin{bmatrix} \sqrt{x_2} & -\sqrt{x_4} \\ -\sqrt{x_4} & \sqrt{x_2} \end{bmatrix}.$$

[1] In general, couplings matrices C_{K-} and C_{K+} are complex matrices. But here they are confined to real matrices for simplicity.

3.3.3 Control objectives and methodologies

Based on the relationship given by (2.26), the quadrature form of (3.16) can be easily obtained. Denoting the quadrature form system transfer matrix by $\tilde{\Xi}(s) = [\tilde{\xi}(x_i, s)_{j,k}]$, $i = 1, 2, ..., 6;\ j = 1, 2;\ k = 1, 2, ..., 6$, which can be derived via (2.27). In order to get the squeezing ratio, we need the following theorem.

Theorem 3.2. *If the quantum input-output system is described in transfer function form with quadrature variables*

$$\begin{bmatrix} Q_{out}(\omega) \\ P_{out}(\omega) \end{bmatrix} = \begin{bmatrix} a_{11}(\omega) & a_{12}(\omega) \\ a_{21}(\omega) & a_{22}(\omega) \end{bmatrix} \begin{bmatrix} Q_{in}(\omega) \\ P_{in}(\omega) \end{bmatrix}, \tag{3.17}$$

where a_{ij} $(i, j = 1, 2)$ are real functions of ω defined in the frequency domain. Then squeezing ratios of the quantum system can be written as

$$\begin{aligned} \mu_q(\omega) &= |a_{11}(\omega)|^2 + |a_{12}(\omega)|^2 + i(a_{11}^*(\omega)a_{12}(\omega) - a_{12}^*(\omega)a_{11}(\omega)), \\ \mu_p(\omega) &= |a_{21}(\omega)|^2 + |a_{22}(\omega)|^2 + i(a_{21}^*(\omega)a_{22}(\omega) - a_{22}^*(\omega)a_{21}(\omega)). \end{aligned} \tag{3.18}$$

Proof. Using the linear relations stated above, we have the variance expression of the output amplitude quadrature

$$\begin{aligned} \langle Q_{out}(\omega)^2 \rangle = |a_{11}(\omega)|^2 \langle Q_{in}(\omega)^2 \rangle &+ |a_{12}(\omega)|^2 \langle P_{in}(\omega)^2 \rangle \\ &+ a_{11}(\omega)a_{12}^*(\omega)\langle Q_{in}(\omega)P_{in}(\omega)\rangle \\ &+ a_{12}(\omega)a_{11}^*(\omega)\langle P_{in}(\omega)Q_{in}(\omega)\rangle. \end{aligned} \tag{3.19}$$

As $\tilde{b}_{in} = \Lambda_m \breve{b}_{in}$, we have

$$\begin{aligned} Q_{in}(\omega) &= \frac{1}{\sqrt{2}} b_{in}(\omega) + \frac{1}{\sqrt{2}} b_{in}^*(\omega), \\ P_{in}(\omega) &= -\frac{i}{\sqrt{2}} b_{in}(\omega) + \frac{i}{\sqrt{2}} b_{in}^*(\omega). \end{aligned} \tag{3.20}$$

Then, the covariance is

$$
\begin{aligned}
&\langle Q_{in}(\omega)P_{in}(\omega)\rangle \\
&= \frac{1}{\sqrt{2}}\langle b_{in}(\omega)+b_{in}^*(\omega)\rangle \times \frac{i}{\sqrt{2}}\langle b_{in}^*(\omega)-b_{in}(\omega)\rangle \\
&= \frac{i}{2}\left(b_{in}(\omega)b_{in}^*(\omega)-b_{in}(\omega)^2+b_{in}^*(\omega)^2-b_{in}^*(\omega)b_{in}(\omega)\right) \\
&= \frac{i}{2}\langle b_{in}(\omega)b_{in}^*(\omega)\rangle \\
&= \frac{i}{2}\delta(0).
\end{aligned} \tag{3.21}
$$

Similarly, we also have $\langle Q_{in}(\omega)P_{in}(\omega)\rangle = -\frac{i}{2}\delta(0)$. By substitution into (3.19), we obtain

$$
\langle Q_{out}(\omega)^2\rangle = \left(\frac{1}{2}|a_{11}(\omega)|^2+\frac{1}{2}|a_{12}(\omega)|^2+\frac{i}{2}a_{11}(\omega)a_{12}^*(\omega)-\frac{i}{2}a_{12}(\omega)a_{11}^*(\omega)\right)\delta(0).
$$

Together with equation (3.2), we my obtain the first equation in (3.18). By the same method, we can get derive the second equation in (3.18). This completes the proof. □

To this point, according to Theorem 3.2 given above, some algebra yields

$$
\begin{aligned}
\mu_q(\omega) &= \frac{\langle Q_{out,1}(\omega)^2\rangle}{\langle Q_{in,1}(\omega)^2\rangle} \\
&= \sum_{k=1}^{6}|\tilde{\xi}_{1,k}(\omega)|^2+\sum_{k=1}^{3} i\left(\tilde{\xi}_{1,2k-1}^*(\omega)\tilde{\xi}_{1,2k}(\omega)-\tilde{\xi}_{1,2k}^*(\omega)\tilde{\xi}_{1,2k-1}(\omega)\right),
\end{aligned} \tag{3.22}
$$

and

$$
\begin{aligned}
\mu_p(\omega) &= \frac{\langle P_{out,1}(\omega)^2\rangle}{\langle P_{in,1}(\omega)^2\rangle} \\
&= \sum_{k=1}^{6}|\tilde{\xi}_{2,k}(\omega)|^2+\sum_{k=1}^{3} i\left(\tilde{\xi}_{2,2k-1}^*(\omega)\tilde{\xi}_{2,2k}(\omega)-\tilde{\xi}_{2,2k}^*(\omega)\tilde{\xi}_{2,2k-1}(\omega)\right).
\end{aligned} \tag{3.23}
$$

Clearly, the squeezing ratios $\mu_q(\omega)$ and $\mu_p(\omega)$ can both be determined by decision variables x_i ($i = 1, \ldots, 6$) for a fixed frequency ω. In order to get better squeezing performance, we optimize $\mu_p(\omega)$ over these decision variables. Moreover, to guarantee certain degree of robustness of the controller stability and closed-loop stability of system (3.16) (Zhou et al. (1996)), two constraints are added to this optimization procedure. Specifically, we consider the following nonlinear programming problem

$$\begin{aligned} \min \quad & \mu_p(\omega) \\ s.t. \quad & \lambda(A_K) + (\lambda(A_K))^{\#} + 2\phi_K \leqslant 0 \\ & \lambda(A_{cl}) + (\lambda(A_{cl}))^{\#} + 2\phi_{cl} \leqslant 0 \end{aligned} \tag{3.24}$$

where A_{cl} is the system matrix of system (3.16), $\lambda(\bullet)$ denotes the vector composed of eigenvalues of matrix $\bullet$, and the positive vectors ϕ_{cl} and ϕ_K are used to bound the real part of eigenvalues of A_{cl} and A_K for robust stability.

(3.24) is a non-convex constrained optimization problem. As is known that non-convex optimization exhibits very perplexing characteristics and the dependence on initial values is the common failing factor of most existing global optimization algorithms. Among all algorithms, *genetic algorithm* (GA) (Goldberg (1989)) has its intrinsic hidden parallelism and better global optimization searching ability. Therefore, for the purpose of finding a good solution to the above optimization problem, we use GA to get the initial point at first, and then refine our results with *sequential quadratic programming* (SQP) (Nocedal and Wright (2006)). As one of the most effective methods for nonlinearly constrained optimization, SQP is based on solving a series of subproblems designed to minimize a quadratic model of the objective subject to a linearization of the constraints. The iteration generates a sequence of approximation results, it is hoped that the sequence may converge to a nearly optimal solution, say x^*. Both GA and SQP can be implemented by the MATLAB optimization toolbox easily.

3.3.4 Simulation results and comparisons

Here we choose the same DPA's parameters as those in the previous examples. Moreover, set $x_i \in [0.2, 0.8]$ $(i = 1, 2, \ldots, 4)$ to bound the couplings of the controller and let its Hamiltonian coefficients $x_5 \in \mathbb{R}$, $x_6 \in \mathbb{C}$ be free. Fix $\phi_K = \phi_{cl}$ with their elements of them are 0.25 in (3.24), denoted by ϕ in the sequel. Fig. 3.6 is obtained by solving the nonlinear programming problem (3.24).

The result shows that when $\omega \to 0^+$, the optimal value of the squeezing ratio μ_p goes to 0.2582, which is much more enhanced compared with the former ones. In addition, the optimal value of the decision variables are: $x_1 = 0.8000, x_2 = 0.5181, x_3 = 0.5450, x_4 = 0.2000, x_5 = -0.4318, x_6 = 0.3696 + 0.0705i$. Then we get the parameters of the coherent controller:

$$
\begin{aligned}
S_K &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
L_K &= \begin{bmatrix} L_{K1} \\ L_{K2} \end{bmatrix} = \begin{bmatrix} 0.8944 \\ 0.7198 \end{bmatrix} a_K + \begin{bmatrix} 0.7382 \\ 0.4472 \end{bmatrix} a_K^*, \\
H_K &= \frac{1}{2} \breve{a}_K^* \begin{bmatrix} -0.4318 & 0.3696 + 0.0705i \\ 0.3696 - 0.0705i & -0.4318 \end{bmatrix} \breve{a}_K.
\end{aligned}
\tag{3.25}
$$

Table 3.1: Squeezing performance comparison

μ_p	$\kappa = 0.6; \gamma = 0.2; \epsilon = 0.2$	$\kappa = 0.5; \gamma = 0.4; \epsilon = 0.2$
Case I: lossy DPA (open loop)	0.5200	0.6694
Case II: lossy DPA (feedback loop)	0.5000	0.6694
Case III: the proposed scheme	0.2582 ($\phi = 0.25$)	0.3292 ($\phi = 0.25$)

Table 3.1 illustrates the squeezing performance comparison among the three methods discussed above, where Case I, II and III stand for the scenarios of lossy DPA in Fig. 3.1, DPA in feedback loop in Fig. 3.3, and the scheme proposed in Fig. 3.5, respectively. All results obtained are under the condition that the DPA is

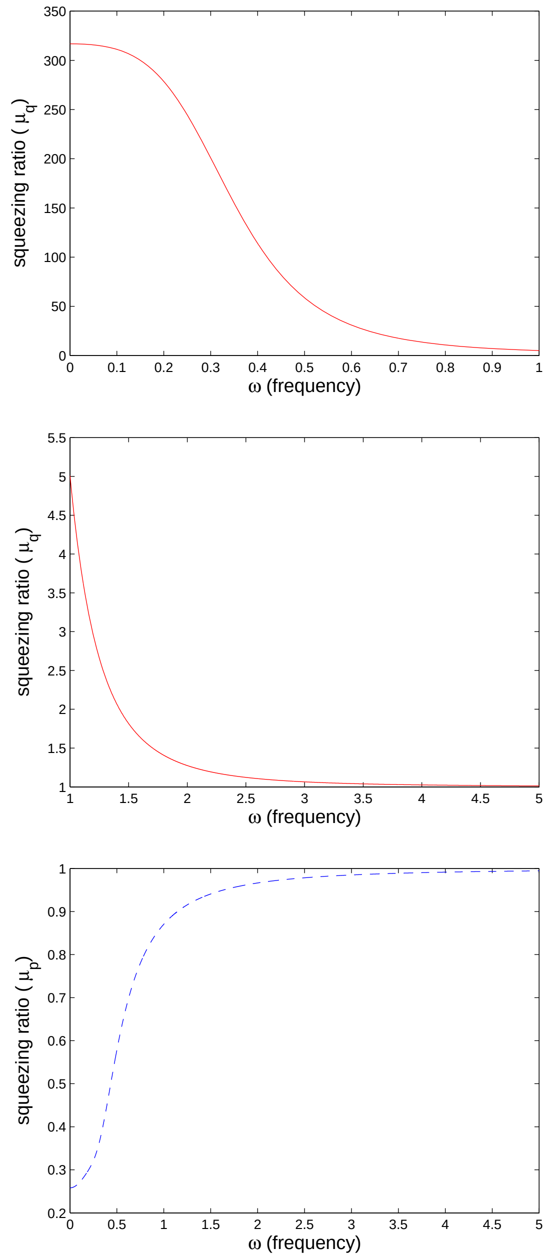

Figure 3.6: Squeezing ratios of quadratures by coherent feedback control scheme.

lossy. From the comparison shown in Table 3.1, it is easily found that the squeezing performance can be significantly enhanced with the proposed coherent feedback control method.

Finally, some words about the overall noise of the output fields. It is assumed the input field is in the vacuum state, hence the product of variances of its two quadratures is 1/4. On the other hand, the above studies show that the lossy DPA, the feedback scheme proposed by Gough and Wildfeuer (2009), and the proposed scheme all yield products of variances of two output field quadratures that are greater than 1/4. That is, the overall noise is amplified. The reason is clear, the environment-induced noises from vacuum fluctuations for the lossy DPA, the beam splitter, and the coherent feedback controller affect the measurement of the output electromagnetic field. In what follows we study what will happen if we are allowed to design the lossy DPA, the beam splitter and the coherent feedback controller simultaneously. Assuming $\kappa, \gamma \in [0.2, 0.8]$, $\epsilon, \alpha \in [0, 1]$, and adding an extra constraint $\mu_q \leqslant 100$ to the nonlinear programming problem (3.24), we find that when $\kappa = 0.8$, $\gamma = 0.2$, $\epsilon = 0.7549$, $\alpha = 0.09639$, $x_1 = 0.8$, $x_2 = 0.2$, $x_3 = 0.2$, $x_4 = 0.2$, $x_5 = 0.01818$, $x_6 = 0.01817 + 0.3203i$, the optimal squeezing ratio μ_p is 0.01005. Moreover, the effective value of μ_q is 99.9909. Their product is 1.0053, that is, the overall noise in the output electromagnetic field is merely very slightly amplified.

3.4 Implementation of quantum coherent feedback controllers

For the purpose of applications in quantum optical engineering, in this section we show how the controller proposed in Sec. 3.3 can be implemented with optical instruments. Nurdin et al. (2009b) first provided a systematic synthesis theory of linear quantum stochastic systems, in which they proposed the synthesis of linear quantum stochastic systems by assembling interconnection of one degree of freedom open quantum harmonic oscillators and, they also investigated how one degree of freedom open oscillators could be synthesized by quantum optical components. Therefore, any open quantum harmonic oscillator of arbitrary degrees of freedom can be synthesized and implemented.

In the last section, the quantum coherent feedback controller designed can be viewed as a one degree of freedom open quantum harmonic oscillator. Based on the synthesis theory proposed by Nurdin et al. (2009b), we can synthesize the controller with a couple of connections of beam splitters, two-mode squeezers and fully reflecting mirrors and partially transmitting mirrors (Leonhardt (2003), Bachor and Ralph (2004), Gardiner and Zoller (2004)). To implement the controller and get the detailed parameter values of the components, one needs to implement both H_K and L_K. An implement procedure is given in the sequel.

First, to implement H_K, we will use a DPA with frequency detuning. Recall the Hamiltonian expression (Nurdin et al., 2009b, Sec. 6.1.2) for a DPA:

$$\begin{aligned} H &= \Delta a^* a - \frac{i}{2}\left(\epsilon (a^*)^2 - \epsilon^* a^2\right) \\ &= \frac{1}{2}\breve{a}^\dagger \begin{bmatrix} \Delta & -i\epsilon \\ i\epsilon^\# & \Delta \end{bmatrix} \breve{a} + c, \end{aligned} \tag{3.26}$$

where Δ is the frequency detuning defined as $\Delta = \omega_{cav} - \omega_r$ with ω_{cav} being cavity frequency and ω_r being reference frequency. ϵ is a complex number representing the effective pump intensity. c is just a constant that has no effect on the dynamics of the controller, and thus can be ignored.

Comparing (3.26) with H_K in (3.25), it is easily found that the complex effective pump intensity parameter ϵ is $-0.0705+0.3696i$, while the cavity detuning parameter Δ is -0.4318.

Secondly, a coupling operator L can be realized by two-mode squeezers, beam splitters and auxiliary cavity modes. Write L as

$$L = \frac{1}{\sqrt{\zeta}}(-\epsilon_2^* a + \epsilon_1 a^*), \tag{3.27}$$

where ζ is the coupling coefficient of the partially transmitting mirror, ϵ_1 is the effective pump intensity of the two-mode squeezer and $\epsilon_2 = 2\Theta e^{-i\Phi}$, where Θ is the mixing angle of the beam splitter and Φ is the relative phase between the input fields by the beam splitter, for details, one can refer to Nurdin et al. (2009b), Leonhardt (2003), Bachor and Ralph (2004), and Gardiner and Zoller (2004).

According to the form the coupling operator L_K in (3.25), we see that the coherent feedback controller interacts with two independent field channels. In what follows the couplings to these two channels will be realized separately. For $L_{K1} = 0.8944a_K + 0.7382a_K^*$, by (3.27), we set the coupling coefficient of the mirror M_1 to be $\zeta = 100$, the two-mode squeezer's effective pump intensity ϵ_1 to be 7.382, and the mixing angle of the beam splitter Θ to be -4.472 with $\Phi = 0$. Similarly, for $L_{K2} = 0.7198a_K + 0.4472a_K^*$, suppose the coupling coefficient ζ of mirror M_2 is also 100, then other parameters can be computed as: $\epsilon_1 = 4.472$, $\Theta = -3.599$, and $\Phi = 0$.

Finally, the detailed physical implementation scheme is shown in Fig. 3.7, where P_1 and P_2 are 180° phase shifters, M_1 and M_2 are partially transmitting mirrors, S_1 and S_2 are beam splitters respectively, D_0 is the DPA, D_1 and D_2 are two-mode squeezers, and the black rectangles denote fully reflecting mirrors.

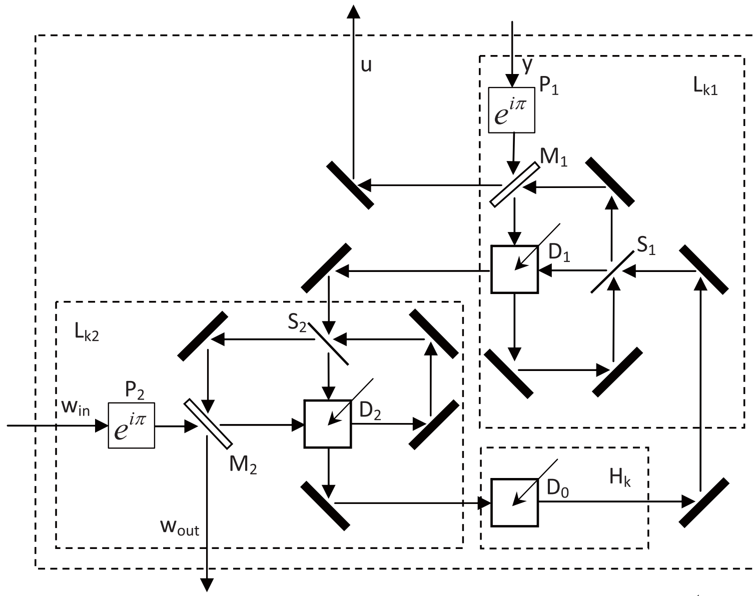

Figure 3.7: Realization of coherent controller K in (3.25).

Chapter 4

Coherent LQG/H_∞ feedback control of linear quantum stochastic systems

The purpose of this chapter is to investigate an efficient coherent feedback controller design scheme for the general mixed LQG/H_∞ quantum coherent feedback control problem of linear quantum stochastic systems. To start with, we present the general QSDEs for the open linear quantum systems, and review the physical realizability which is proposed by James et al. (2008). Then we integrate the quantum LQG control problem (Nurdin et al. (2009a)) and H_∞ control problem (James et al. (2008)) to formulate the standard mixed problem. To tackle the difficulty of the physical realizability constraints, the problem is reformulated as a rank constraint LMI problem, and numerical solution is attained by using Matlab and the toolboxes "Sedumi", "Yalmip" (Lofberg (2004)) and "Lmirank" (Orsi et al. (2006)) therein. The cavity and the DPA are used as plants in the examples to show the validity of the scheme. Simultaneously, numerical results demonstrate the perplexing characteristics of the non-convex optimization. Finally, detailed implementation procedures are also presented by means of quantum optical devices based on the quantum network synthesis theory.

4.1 Formulation of linear quantum stochastic systems

4.1.1 A general model of open linear quantum systems

The class of linear quantum systems under consideration can be described by using non-commutative or quantum probability theory (Bouten et al. (2007)). A general model of quantum stochastic system of N quantum harmonic oscillators can be described by N pairs of canonical position and momentum operators q, p, satisfying the canonical commutation relation (CCR) $[q_j, p_k] = i\delta_{jk}$, and $[q_j, q_k] = [p_j, p_k] = 0$ $(j, k = 1, 2, \cdots, N)$. Define $x(t) = [q_1(t), p_1(t), \cdots, q_N(t), p_N(t)]^T$, if the system interacts with an N_w-channel quantum fields, and yields N_y-channel quantum fields, it obeys the following quantum stochastic differential equations (QSDEs) (James et al. (2008))

$$\begin{aligned} dx(t) &= Ax(t)\,dt + Bdw(t), \quad x(0) = x_0, \\ dy(t) &= Cx(t)\,dt + Ddw(t), \end{aligned} \tag{4.1}$$

where A, B, C and D are real matrices in $\mathbb{R}^{n\times n}, \mathbb{R}^{n\times n_w}, \mathbb{R}^{n_y\times n}$ and $\mathbb{R}^{n_y\times n_w}$, respectively, here n, n_w, n_y are positive even integers with $n = 2N$, $n_w = 2N_w$, $n_y = 2N_y$. $x(t) = [x_1(t) \cdots x_n(t)]^T$ is a vector of self-adjoint possibly non-commutative system variables. $w(t) = [w_1(t) \cdots w_{n_w}(t)]^T$ is a vector of input signals, including noise, finite-energy signal and control input signal.

The initial system variables $x(0) = x_0$ are Gaussian with state ρ, and satisfy the commutation relations

$$[x_j(0), x_k(0)] = i\Theta_{jk}, \quad j, k = 1, \ldots, n, \tag{4.2}$$

where Θ is a real antisymmetric matrix with component Θ_{jk} and $i = \sqrt{-1}$. Here the symbol [,] is defined as $[A, B] = AB - BA$.

Here we follow James et al. (2008)'s results, the matrix Θ can be chosen as one of the following two forms:

1. Canonical if $\Theta = diag\,(J, ..., J)$;

2. Degenerate canonial if $\Theta = diag\,(0_{n_1 \times n_1}, J..., J)$, where $0 < n_1 \leqslant n$.

Where J denotes the real symmetric 2×2 matrix $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and the "diag" notation indicates a block diagonal matrix assembled from given entries.

Usually, the vector quantity $w\,(t)$ describes the input signals and is assumed to admit the decomposition

$$dw\,(t) = \beta_w\,(t)\,dt + d\tilde{w}\,(t)\,, \tag{4.3}$$

where $\tilde{w}\,(t)$ is the noise part and $\beta_w\,(t)$ is a self-adjoint, adapted process, it is used to represent the signal passed to system (4.1) from other systems. For simplicity, here we assume the components of $\beta_w(t)$ commutes with those of $d\tilde{w}(t)$ and also those of $x(t)$ for all $t \geqslant 0$. The noise $\tilde{w}\,(t)$ is a vector of self-adjoint quantum noise with Ito table

$$d\tilde{w}\,(t)\,d\tilde{w}^T\,(t) = F_{\tilde{w}}dt, \tag{4.4}$$

where $F_{\tilde{w}}$ is a nonnegative Hermitian matrix. This determines the following commutation relations for the noise components

$$\begin{aligned} \left[d\tilde{w}\,(t)\,, d\tilde{w}^T\,(t)\right] &= d\tilde{w}\,(t)\,d\tilde{w}^T\,(t) - \left(d\tilde{w}\,(t)\,d\tilde{w}^T\,(t)\right)^T \\ &= 2T_{\tilde{w}}dt, \end{aligned} \tag{4.5}$$

where we use the notation $T_{\tilde{w}} = \frac{1}{2}\left(F_{\tilde{w}} - F_{\tilde{w}}^T\right)$. This noise processes can be represented as operators on an appropriate Fock space, for more details, one can refer to Parthasarathy (1992).

System (4.1) can also be written in the annihilation-creation form as described in section 2.2. Define $a = [a_1, \ldots, a_N]^T$ with entries $a_j = (1/\sqrt{2})(q_j + ip_j)$ being the annihilation operator of the jth quantum harmonic oscillator, and its complex conjugation $a_j^* = (1/\sqrt{2})(q_j - ip_j)$ being the creation operator, $j = 1, 2, ...N$; they satisfy the canonical commutation relation $[a_j, a_k^*] = \delta_{jk}$. Then the annihilation-creation form of (4.1) may be written as

$$\begin{aligned} d\breve{a}(t) &= \breve{A}\breve{a}(t)dt + \breve{B}d\breve{w}(t), \quad \breve{a}(0) = \breve{a}_0, \\ d\breve{y}(t) &= \breve{C}\breve{a}(t)dt + \breve{D}d\breve{w}(t), \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} \breve{A} &= \Psi_n A \Psi_n^\dagger, \quad \breve{B} = \Psi_n B \Psi_{n_w}^\dagger, \\ \breve{C} &= \Psi_{n_w} C \Psi_n^\dagger, \quad \breve{D} = \Psi_{n_w} D \Psi_{n_w}^\dagger, \end{aligned} \tag{4.7}$$

and

$$\Psi_n = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 1 & i & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 1 & i \\ 1 & -i & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 1 & -i & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 1 & -i \end{bmatrix}_{n\times n}.$$

4.1.2 Physical realizability of linear QSDEs

Quantum mechanics dictates that closed physical quantum systems evolve in a unitary manner which implies the preservation of the canonical commutation relations

$$x(t)\, x^T(t) - \left(x(t)\, x^T(t)\right)^T = i\Theta \quad \text{for all } t \geqslant 0. \tag{4.8}$$

Then the system descibed by (4.1) with arbitary system matrices may not to be a meaningful system. James et al. (2008) developed a precise notion of physical realizability based on the concept of an open quantum harmonic oscillator, which is the basic "dynamical unit" of physical realizable linear quantum systems. Such an oscillator can be completely described by a triple of physical parameters (S, L, H) as discussed in Sec. 2.2.2. From James et al. (2008), we can attain the following result.

Theorem 4.1. *System* (4.1) *is a physically realizable quantum system if and only if*

$$iA\Theta + i\Theta A^T + BT_{\tilde{w}}B^T = 0, \tag{4.9a}$$

$$B\begin{bmatrix} I_{2n_y \times 2n_y} \\ 0_{(2n_w - 2n_y)\times 2n_y} \end{bmatrix} = \Theta C^T diag_{n_y}(J), \tag{4.9b}$$

$$D = \begin{bmatrix} I_{2n_y \times 2n_y} & 0_{2n_y \times (2n_w - 2n_y)} \end{bmatrix}. \tag{4.9c}$$

Remark 4.1. *Relations* (4.9) *are called **physical realizability conditions** (James et al. (2008), Nurdin et al. (2009a)). These conditions guarantee that the dynamic system corresponds to a physical system. To be more specific, the above theorem is just for the case that* Θ *is canonical. When* Θ *is degenerate canonical, we may perform an augmentation in which* Θ *is embedded into a larger skew symmetric matrix* $\tilde{\Theta}$*, which is canonical up to permutation. Interested reader can refer James et al. (2008) and reference therein.*

4.2 LQG/H_∞ control of linear quantum stochastic systems

In this section, we formulate the QSDE of the closed-loop plant-controller system, and then formulate standard LQG and H_∞ problem. We use two distinct channels to describe LQG and H_∞ problems separately, then the mixed LQG/H_∞ problem can be viewed as the synthesis of both LQG and H_∞ problems.

4.2.1 Composite plant-controller system

The quantum plant $\mathbf{P}$ can be described by a system of QSDEs, for simplicity, the equations are given in quadrature form

$$\begin{aligned} dx(t) &= Ax(t)dt + B_0 dv(t) + B_1 dw(t) + B_2 du(t), \\ dy(t) &= C_2 x(t)dt + D_{20} dv(t) + D_{21} dw(t), \\ dz_\infty(t) &= C_1 x(t)dt + D_{12} du(t), \\ z_l(t) &= C_z x(t) + D_z \beta_u(t), \end{aligned} \tag{4.10}$$

where A, B_0, B_1, B_2, C_2, D_{20}, D_{21}, C_1, D_{12}, C_z and D_z are real matrices in $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times n_v}$, $\mathbb{R}^{n\times n_w}$, $\mathbb{R}^{n\times n_u}$, $\mathbb{R}^{n_y\times n}$, $\mathbb{R}^{n_y\times n_v}$, $\mathbb{R}^{n_y\times n_w}$, $\mathbb{R}^{n_\infty\times n}$, $\mathbb{R}^{n_\infty\times n_u}$, $\mathbb{R}^{n_2\times n}$, $\mathbb{R}^{n_2\times n_u}$, respectively. Furthermore, n, n_v, n_w, n_u, n_∞, n_l and n_y are positive integers, $x\,(t) = [x_1(t), ..., x_n(t)]^T$ is a vector of self-adjoint possibly non-commutative system variables. $v(t) = [v_1(t), ..., v_{n_v}(t)]^T$ is referred to as quantum noise. $w\,(t) = [w_1(t), ..., w_{n_w}(t)]^T$ is the exogenous input and $u(t) = [u_1\,(t)\,, ..., u_{n_u}(t)]^T$ is the controlled input. $z_\infty\,(t) = [z_{\infty_1}(t), ..., z_{\infty_{n_\infty}}(t)]^T$ and $z_l\,(t) = [z_{l_1}(t), ..., z_{l_{n_l}}(t)]^T$ are the controlled outputs which are referred to as H_∞ and LQG channels, respectively. The exogenous input and the control input have following decomposition

$$\begin{aligned} dw\,(t) &= \beta_w\,(t)\,dt + d\tilde{w}\,(t)\,, \\ du\,(t) &= \beta_u\,(t)\,dt + d\tilde{u}\,(t)\,, \end{aligned} \tag{4.11}$$

where $\tilde{w}\,(t)$ and $\tilde{u}\,(t)$ are noise part of $w\,(t)$ and $u\,(t)$, $\beta_w\,(t)$ and $\beta_u\,(t)$ are self-adjoint, adapted processes, respectively. The vectors $v\,(t)\,, \tilde{w}\,(t)$ and $\tilde{u}\,(t)$ are independent quantum noise (meaning that they live in distinct Fock space) with Ito matrices $F_v, F_{\tilde{w}}$ and $F_{\tilde{u}}$ that are all nonnegative Hermitian. We also assume that $x\,(0)\,x^T\,(0) - \left(x\,(0)\,x^T\,(0)\right)^T = \Theta$.

On the other hand, the coherent feedback controller $\mathbf{K}$ which is to be designed is assumed to be a non-commutative stochastic system of the form

$$d\xi(t) = A_K\xi(t)\,dt + B_{K1}db_{vK1}(t) + B_{K2}db_{vK2}(t) + B_{K3}dy(t),$$
$$du(t) = C_K\xi(t)\,dt + db_{vK1}, \tag{4.12}$$

where $\xi(t) = [\xi_1(t), ..., \xi_{n_K}(t)]^T$ is a vector of self-adjoint operators of the same dimension as $x(t)$ (that means the controller is of the same dimension as the plant), the coefficient matrix B_{K2} is a square matrix of the same dimension as A_K, and B_{K1} has the same number of columns as there are rows of C_K. The noise $b_{vKi}(t)$, $i = 1, 2$, are vectors of non-commutative Wiener process (in vaccum state) with non-zero Ito products and which are independent of $w(t)$. Also, we assume that $\xi(0)\xi^T(0) - \left(\xi(0)\xi^T(0)\right)^T = i\Theta_K$. Here Θ_K is the skew symmetric commutation matrix for the controller variables ξ that could be a canonical or degenerate canonical form.

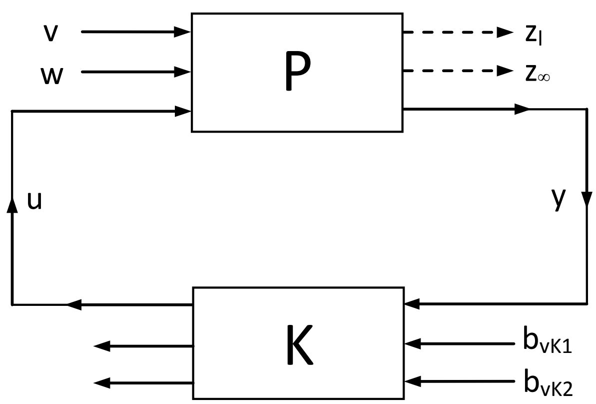

Figure 4.1: Schematic of closed-loop plant-controller system.

Assume further that $x(0)\xi^T(0) - \left(\xi(0)x^T(0)\right)^T = 0$. The closed-loop system is obtained by the identification $\beta_u(t) \equiv C_K\xi(t)$ and $\tilde{u}(t) \equiv b_{vK1}(t)$. The corresponding scheme is shown in Fig. 4.1. Interconnecting (4.10) and (4.12), the closed-loop system in the quadrature representation is given by

$$\begin{aligned} d\eta(t) &= M\eta(t)\,dt + Nd\tilde{w}_{cl}(t) + H\beta_w(t)\,dt, \\ dz_\infty(t) &= \Gamma\eta(t)\,dt + \Pi d\tilde{w}_{cl}(t), \\ z_l(t) &= \Psi\eta(t), \end{aligned} \tag{4.13}$$

where $\eta(t) = [x^T(t)\ \xi^T(t)]^T$, $\tilde{w}_{cl}(t) = [v^T(t)\ \tilde{w}^T(t)\ b_{vK1}^T(t)\ b_{vK2}^T(t)]^T$, and

$$M = \begin{bmatrix} A & B_2C_K \\ B_{K3}C_2 & A_K \end{bmatrix}, \quad N = \begin{bmatrix} B_0 & B_1 & B_2 & 0 \\ B_{K3}D_{20} & B_{K3}D_{21} & B_{K1} & B_{K2} \end{bmatrix},$$

$$H = \begin{bmatrix} B_1 \\ B_{K3}D_{21} \end{bmatrix}, \quad \Gamma = \begin{bmatrix} C_1 & D_{12}C_K \end{bmatrix},$$

$$\Pi = \begin{bmatrix} 0 & 0 & D_{12} & 0 \end{bmatrix}, \quad \Psi = \begin{bmatrix} C_z & D_zC_K \end{bmatrix},$$

$$T = \begin{bmatrix} C_2 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} D_{20} & D_{21} & 0 & 0 \end{bmatrix}.$$

4.2.2 LQG performance

LQG problem is one of the most fundamental optimal control problems. It concerns uncertain linear systems disturbed by additive white Gaussian noise, having incomplete state information and undergoing control subject to quadratic costs. LQG index can describe the transient performance response perfectly. In Nurdin et al. (2009a), the authors presented a detailed formulation of quantum LQG problem, here we briefly describe it as follows.

We interpret $\tilde{w}_{cl} \rightarrow z_l$ as the performance channel for measuring LQG performance, then associate a quadratic LQG performance index with the obtained closed-loop system (4.13), along the line of Nurdin et al. (2009a), the infinite-horizon LQG cost is

$$\begin{aligned} \mathcal{L}_\infty &= \lim_{t_f\to\infty} \frac{1}{t_f}\int_0^{t_f} \frac{1}{2}\left\langle z_l^T(t)z_l(t)\right\rangle dt \\ &= \lim_{t_f\to\infty} \frac{1}{t_f}\int_0^{t_f} \mathrm{Tr}\{\Psi S_L(t)\Psi^T\}dt \\ &= \mathrm{Tr}\{\Psi S_L\Psi^T\}, \end{aligned} \tag{4.14}$$

where S_L is the unique symmetric positive definite solution of the Lyapunov equation

$$MS_L + S_L M^T + NN^T = 0. \tag{4.15}$$

Problem 4.1. *The LQG control problem is to find a proper, real-rational controller* $\boldsymbol{K}$ *which satisfies the following properties.*

1. *There exists a symmetric matrix* $S_L > 0$ *satisfying (4.15).*

2. *Relaxed LQG performance. The cost function* $\mathcal{L}_\infty = \mathrm{Tr}\{\Psi S_L \Psi^T\} < \gamma_l$ *is satisfied for a prespecified constant* $\gamma_l > 0$.

3. $\boldsymbol{K}$ *is a quantum coherent feedback controller. That is,* $A_K, B_{K1}, B_{K2}, B_{K3}, C_K$ *satisfy the conditions of* ***Theorem 4.1*** *with the identification:* $A \equiv A_K,\ B \equiv [B_{K1}\ B_{K2}\ B_{K3}],\ C \equiv C_K,\ D \equiv [I_{n_u \times n_u}\ 0]$, *To be more specific,*

$$\begin{aligned} A_K\Theta_K + \Theta_K A_K^T + B_{K1} diag_{n_1/2}(J) B_{K1}^T + B_{K2} diag_{n_2/2}(J) B_{K2}^T & \\ + B_{K3} diag_{n_3/2}(J) B_{K3}^T = 0, & \quad (4.16a) \\ B_{K1} = \Theta_K C_K^T diag_{N_u}(J). & \quad (4.16b) \end{aligned}$$

Remark 4.2. *Note that in* ***Problem 4.1****-2, it is specified that the considered LQG problem is relaxed by considering the cost function* $\mathcal{L}_\infty < \gamma_l$ *rather than achieving its minimal value, since the physical realizability relations (4.16) are polynomial equality constraints on the controller matrices* A_k, B_{K1}, B_{K2}, B_{K3} *and* C_K, *which exhibits very perplexing characteristics and is difficult to solve numerically using general existing optimization algorithms. It has been more fruitful to consider the relaxed problem of finding a controller satisfies a pre-specified cost bound* $\gamma_l > 0$, *and then formulate it into a rank constrained LMI feasibility problem.*

Theorem 4.2. *(Nurdin et al. (2009a)) There exists a proper, real-rational controller* $\boldsymbol{K}$ *which solves* ***Problem 4.1*** *for given* Θ_K *and* $\gamma_l > 0$ *if and only if there exist* A_K*,* B_{K1}*,* B_{K2}*,* B_{K3}*,* C_K*, symmetric matrix* $P_L = S_L^{-1}$ *and* Q *such that the equality constraints (4.16) and inequality constraints (4.17) hold.*

$$\begin{aligned}\begin{bmatrix} M^T P_L + P_L M & P_L N \\ N^T P_L & -I \end{bmatrix} &< 0, \\ \begin{bmatrix} P_L & \Psi^T \\ \Psi & Q \end{bmatrix} &> 0, \\ \mathrm{Tr}(Q) &< \gamma_l. \end{aligned} \tag{4.17}$$

4.2.3 H_∞ performance

H_∞ control problem was originally introduced by Zames (1981) and has subsequently played a major role in the area of robust control theory. It mainly concerns the robustness of the system parameter uncertainty or perturbation of environment. In literature, classic H_∞ control problems are usually formulated by using Strict Bounded Real Lemma, see Zhou and Khargonekar (1988), Petersen et al. (1991), Gahinet and Apkarian (1994). James et al. (2008) have proposed a standard H_∞ coherent feedback control problem of linear quantum stochastic systems by using Bounded Real Lemma of quantum version, a solution to the quantum H_∞ coherent feedback control problem can be obtained in terms of a pair of algebraic Riccati equations. In this work, we apply LMIs to formulate the quantum H_∞ control problem.

The H_∞ norm is defined as

$$\|T\|_\infty = \sup_\omega \sigma_{\max}\left[T\left(j\omega\right)\right] = \sup_\omega \sqrt{\lambda_{\max}\left(T^*(j\omega)T(j\omega)\right)}, \tag{4.18}$$

where $\sigma_{\max}$ is the maximum singular value of a matrix, and $\lambda_{\max}$ is the maximum eigenvalue of a Hermitian matrix.

Lemma 4.1 (Strict Bounded Real Lemma (James et al. (2008)))**.** *For the linear* *quantum stochastic system G described by (4.1), the following statements are equivalent.*

(i) *The quantum system G defined by (4.1) is strictly bounded real with disturbance attenuation γ_∞.*

(ii) $\|D + C(sI - A)^{-1}B\|_\infty < \gamma_\infty$ *and A is Hurwitz stable.*

(iii) $\gamma_\infty^2 I - D^T D > 0$ *and there exists a symmetric positive definite solution P_H to the LMI*

$$\begin{bmatrix} A^T P_H + P_H A & P_H B & C^T \\ B^T P_H & -\gamma_\infty I & D^T \\ C & D & -\gamma_\infty I \end{bmatrix} < 0. \tag{4.19}$$

Refer to closed-loop system (4.13), we interpret $\beta_w \to z_\infty$ as the robustness channel for measuring H_∞ performance. Then we raise the following quantum H_∞ control problem.

Problem 4.2. *The quantum H_∞ control problem is to find a proper, real-rational controller $\boldsymbol{K}$ which satisfies the following properties.*

1. *Internal stability. $\boldsymbol{K}$ stabilizes $\boldsymbol{P}$ exponentially in the closed loop.*

2. *H_∞ performance. Given $\gamma_\infty > 0$, the H_∞ norm of the closed-loop system $\|G_{\beta_w \to z_\infty}\|_\infty < \gamma_\infty$ is satisfied.*

3. *$\boldsymbol{K}$ is a quantum coherent feedback controller as stated in **Problem 4.1**-3.*

Theorem 4.3. *(Zhang and James (2011)) There exists a proper, real-rational controller $\boldsymbol{K}$ which solves **Problem 4.2** for given Θ_K and $\gamma_\infty > 0$ if and only if there exist A_K, B_{K1}, B_{K2}, B_{K3}, C_K, and symmetric matrix P_H such that the equality constraints (4.16) and inequality constraints (4.20) hold.*

$$
\begin{bmatrix} M^T P_H + P_H M & P_H H & \Gamma^T \\ H^T P_H & -\gamma_\infty I & 0 \\ \Gamma & 0 & -\gamma_\infty I \end{bmatrix} < 0, \qquad (4.20)
$$

$$
P_H > 0.
$$

4.2.4 Mixed LQG/H_∞ performance

In the above analysis, we have investigated the standard coherent quantum LQG and H_∞ control problem, and then transformed them into matrix inequalities problem. To sum up, we synthesize our results to briefly describe the mixed LQG/H_∞ problem.

Problem 4.3. *The mixed LQG/H_∞ control problem is to find a proper, real-rational controller* $\boldsymbol{K}$ *which solves* ***Problem 4.1*** *and* ***Problem 4.2*** *simultaneously.*

Theorem 4.4. *There exists a proper, real-rational controller* $\boldsymbol{K}$ *which solves* ***Problem 4.3*** *for given Θ_K, $\gamma_l > 0$ and $\gamma_\infty > 0$ if and only if there exist A_K, B_{K1}, B_{K2}, B_{K3}, C_K, symmetric matrix* [1] *$P = P_H = P_L = S_L^{-1}$ and Q such that the equality constraints (4.16), and inequality constraints (4.17), (4.20) hold.*

4.3 Algorithm based on rank constrained LMI

From the above analysis of the LQG and H_∞ feedback control problem, we seek to find an coherent feedback controller **K** which solves **Problem 4.3**. Nevertheless, it is easy to find that (4.17) and (4.20) are both non-linear matrix inequalities. Moreover, the physical realizability constraint (4.16) is a non-convex constraint. Therefore it is difficult to obtain the optimal solution by existing optimization algorithms. To this point, we shall discuss how to translate the above non-convex and non-linear

[1] In general, P_H and P_L are independent symmetric matrices, but here we let them be identical for computational simplicity.

constraints to linear ones. First, we redefine the plant (4.10) as follows

$$\begin{aligned} dx\left(t\right) &= Ax\left(t\right)dt + B_w d\tilde{w}_{cl}\left(t\right) + B_1\beta_w\left(t\right)dt + B_2\beta_u\left(t\right)dt, \\ dz_\infty\left(t\right) &= C_1 x\left(t\right)dt + D_\infty d\tilde{w}_{cl}\left(t\right) + D_{12}\beta_u\left(t\right)dt, \\ z_l &= C_z x\left(t\right) + D_z\beta_u\left(t\right), \\ dy'\left(t\right) &= Cx\left(t\right)dt + D_w d\tilde{w}_{cl}\left(t\right) + D\beta_w\left(t\right)dt, \end{aligned} \tag{4.21}$$

where $B_w = \begin{bmatrix} B_0 & B_1 & B_2 & 0 \end{bmatrix}$, $D_\infty = \begin{bmatrix} 0 & 0 & D_{12} & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 & C_2^T \end{bmatrix}^T$, $D = \begin{bmatrix} 0 & 0 & D_{21}^T \end{bmatrix}^T$ and $D_w = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ D_{20} & D_{21} & 0 & 0 \end{bmatrix}$.

Here y' is the output equation for the modified plant that now includes the quantum noise b_{vK1} and b_{vK2} that enter the controller, but not in the original plant. In this way all noise can now be thought of as coming from the modified plant. Then, redefine our controller equations as:

$$\begin{aligned} d\xi\left(t\right) &= A_K\xi\left(t\right)dt + B_{wK}dy'\left(t\right), \\ \beta_u\left(t\right) &= C_K\xi\left(t\right), \end{aligned} \tag{4.22}$$

with $B_{wK} = \begin{bmatrix} B_{K1} & B_{K2} & B_{K3} \end{bmatrix}$. And the closed-loop system has the same form as (4.13). For simplicity, in the following, we assume the controller the same order as the plant.

We now follow Scherer et al. (1997) by partitioning P and P^{-1} as

$$P = \begin{bmatrix} Y & \Xi \\ \Xi^T & * \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} X & \Sigma \\ \Sigma^T & * \end{bmatrix},$$

where X and Y are $n \times n$ symmetric matrices. And define

$$\Pi_1 := \begin{bmatrix} X & I \\ \Sigma^T & 0 \end{bmatrix}, \qquad \Pi_2 := \begin{bmatrix} I & Y \\ 0 & \Xi^T \end{bmatrix},$$

then we obtain

$$P\Pi_1 = \Pi_2.$$

And necessarily, we have

$$\Sigma \Xi^T = I - XY. \tag{4.23}$$

Now define the change of controller variables as follows:

$$\begin{aligned}
\hat{A} &:= \Xi A_K \Sigma^T + \Xi B_{wK} C X + Y B_2 C_K \Sigma^T + Y A X, \\
\hat{B} &:= \Xi B_{wK}, \\
\hat{C} &:= C_K \Sigma^T.
\end{aligned} \tag{4.24}$$

Then we can derive

$$\begin{aligned}
\Pi_1^T P M \Pi_1 = \Pi_2^T M \Pi_1 &= \begin{bmatrix} AX + B_2 \hat{C} & A \\ \hat{A} & YA + \hat{B} C \end{bmatrix}, \\
\Pi_1^T P N = \Pi_2^T N &= \begin{bmatrix} B_w \\ Y B_w + \hat{B} D_w \end{bmatrix}, \\
\Gamma \Pi_1 &= \begin{bmatrix} C_1 X + D_{12} \hat{C} & C_1 \end{bmatrix}, \\
\Pi_1^T P \Pi_1 = \Pi_1^T \Pi_2 &= \begin{bmatrix} X & I \\ I & Y \end{bmatrix}, \\
\Pi_1^T P H = \Pi_2^T H &= \begin{bmatrix} B_1 \\ Y B_1 + \hat{B} D \end{bmatrix}.
\end{aligned}$$

Therefore the LQG performance inequality constrains (4.17) can be transformed as

$$\begin{aligned}
\begin{bmatrix} AX + XA^T + B_2 \hat{C} + \left(B_2 \hat{C}\right)^T & \hat{A}^T + A & B_w \\ \hat{A} + A^T & A^T Y + YA + \hat{B} C + \left(\hat{B} C\right)^T & Y B_w + \hat{B} D_w \\ B_w^T & \left(Y B_w + \hat{B} D_w\right)^T & -I \end{bmatrix} &< 0, \\
\begin{bmatrix} X & I & \left(C_z X + D_z \hat{C}\right)^T \\ I & Y & C_z^T \\ \left(C_z X + D_z \hat{C}\right) & C_z & Q \end{bmatrix} &> 0, \\
\mathrm{Tr}(Q) &< \gamma_l.
\end{aligned} \tag{4.25}$$

Similarly, the H_∞ inequality constraint (4.20) turns into

$$\begin{bmatrix} AX + XA^T + B_2\hat{C} + \left(B_2\hat{C}\right)^T & \hat{A}^T + A & * & * \\ \hat{A} + A^T & A^TY + YA + \hat{B}C + \left(\hat{B}C\right)^T & * & * \\ B_1^T & \left(YB_1 + \hat{B}D\right)^T & -\gamma_\infty I & * \\ C_1X + D_{12}\hat{C} & C_1 & 0 & -\gamma_\infty I \end{bmatrix} < 0. \quad (4.26)$$

It is obvious that the above matrix inequalities are linear which are easy to solve by Matlab.

Through (4.24), we can obtain

$$\begin{aligned} C_K &= \hat{C}\Sigma^{-T}, \\ B_{wK} &= \Xi^{-1}\hat{B}, \\ A_K &= \Xi^{-1}\left(\hat{A} - \Xi B_{wK}CX - YB_2C_K\Sigma^T - YAX\right)\Sigma^{-T}. \end{aligned} \quad (4.27)$$

By instituting (4.27) into (4.16) and introduce new variables $\tilde{\Xi} = \Xi J_{N_\varsigma}$, $\tilde{A}_K = \Xi A_K$, $\tilde{B}_{Ki} = \Xi B_{Ki}$, $i = 1, 2, 3$. Then the physical realizability constraints (4.16) turn into

$$\begin{aligned} &\left(-\hat{A}\Sigma^{-T} + \left(\tilde{B}_{K3}C_2 + YA\right)X\Sigma^{-T} + YB_2C_K\right)\tilde{\Xi}^T \\ &\quad + \tilde{\Xi}\left(\hat{A}\Sigma^{-T} - \left(\tilde{B}_{K3}C_2 + YA\right)X\Sigma^{-T} - YB_2C_K\right)^T \\ &\quad + \sum_{i=1}^{3}\tilde{B}_{Ki}J_{N_{vKi}}\tilde{B}_{Ki}^T = 0, \end{aligned} \quad (4.28a)$$

$$\tilde{B}_{K1} - \tilde{\Xi}C_K^TJ_{N_{vK1}} = 0. \quad (4.28b)$$

If LMIs (4.25) (4.26) and the constraints (4.23), (4.28) yield a feasible solution $\hat{A}$, $\hat{B}$, $\hat{C}$, X, Y, Ξ, Σ and Q, then the controller coefficient matrices A_K, B_{wK}, C_K could be derived from (4.27). We summarize our results in the following theorem:

Theorem 4.5. *There exists a proper, real-rational controller K which solves **Problem 4.3** for given Θ_k, $\gamma_l > 0$ and $\gamma_\infty > 0$ if and only if there exist matrices $\hat{A}$,*

$\tilde{B}_{K1}$, $\tilde{B}_{K2}$, $\tilde{B}_{K3}$, $\hat{C}$, X, Y, $\tilde{\Xi}$, Σ, Ξ, C_K *satisfying the LMIs (4.25) (4.26) and the constraints (4.23), (4.28),* $\tilde{\Xi} = \Xi J_{N_\zeta}$, $\hat{C} = C_K \Sigma^T$.

It is clear that (4.28a) is polynomial matrix equality constraints. Following Nurdin et al. (2009a), we now proceed to linearize it by introducing appropriate matrix lifting variables and the associated equality constraints, which may reformulate the problem into a rank constrained LMI problem.

At first, we introduce 13 basic matrix variables: $M_1 = \hat{A}$, $M_2 = \tilde{B}_{K1}$, $M_3 = \tilde{B}_{K2}$, $M_4 = \tilde{B}_{K3}$, $M_5 = \hat{C}$, $M_6 = X$, $M_7 = Y$, $M_8 = \tilde{\Xi}$, $M_9 = \Sigma$, $M_{10} = \Xi$, $M_{11} = C_K$, $M_{12} = \check{A} = \hat{A}\Sigma^{-T}$, $M_{13} = \check{X} = X\Sigma^{-T}$. And define 18 matrix lifting variables as follows: $W_i = \tilde{B}_{Ki} J_{N_{vKi}}$, $(i = 1, 2, 3)$. $W_4 = YB_2$, $W_5 = \tilde{B}_{K3}C_2 + YA$, $W_6 = \tilde{\Xi}C_K^T$, $W_7 = \tilde{\Xi}\check{X}^T$, $W_8 = \check{A}\tilde{\Xi}^T$, $W_9 = YX$, $W_{10} = W_4 W_6^T$, $W_{11} = W_5 W_7^T$, $W_{12} = W_1 \tilde{B}_{K1}^T$, $W_{13} = W_2 \tilde{B}_{K2}^T$, $W_{14} = W_3 \tilde{B}_{K3}^T$, $W_{15} = \Xi\Sigma^T = I - YX$, $W_{16} = \check{A}\Sigma^T = \hat{A}$, $W_{17} = \check{X}\Sigma^T = X$, $W_{18} = C_K\Sigma^T = \hat{C}$.

Then we denote

$$\begin{aligned} V &= [I, Z_{x_1,1}^T, \cdots, Z_{x_{13},1}^T, Z_{v_1,1}^T, Z_{v_2,1}^T, \cdots, Z_{v_{18},1}^T]^T \\ &= [I, M_1^T, \cdots, M_{13}^T, W_1^T, \cdots, W_{18}^T]^T, \end{aligned} \tag{4.29}$$

and introduce a $32n$ by $32n$ symmetric matrix $Z = VV^T$ with the components $Z_{i,j} = [Z_{kl}]_{k=in+1,(i+1)n, l=jn+1,(j+1)n}$. Obviously, terms of the form $Z_{a,b}$ with $a, b \in \{x_1, ..., x_{13}\} \cup \{v_1, v_2, ..., v_{18}\} = \{1, ..., 13\} \cup \{14, 15, ..., 31\}$ could be identified with $Z_{a,1}\left(Z_{b,1}\right)^T$.

Necessarily, we require the components of matrix Z satisfy the following constraints

$$
\begin{aligned}
Z &\geqslant 0, & & \\
Z_{0,0} - I_{n\times n} &= 0, & Z_{v_7,1} - Z_{x_8,x_{13}} &= 0,\\
Z_{1,x_6} - Z_{x_6,1} &= 0, & Z_{v_8,1} - Z_{x_{12},x_8} &= 0\\
Z_{1,x_7} - Z_{x_7,1} &= 0, & Z_{v_9,1} - Z_{x_7,x_6} &= 0,\\
Z_{v_1,1} - Z_{x_2,1}J_{N_{vK1}} &= 0, & Z_{v_{10},1} - Z_{v_4,v_6} &= 0,\\
Z_{v_2,1} - Z_{x_3,1}J_{N_{vK2}} &= 0, & Z_{v_{11},1} - Z_{v_5,v_7} &= 0,\\
Z_{v_3,1} - Z_{x_4,1}J_{N_{vK3}} &= 0, & Z_{v_{12},1} - Z_{v_1,x_2} &= 0,\\
Z_{v_4,1} - Z_{x_7,1}B_2 &= 0, & Z_{v_{13},1} - Z_{v_2,x_3} &= 0,\\
Z_{v_5,1} - Z_{x_4,1}C_2 - Z_{x_7,1}A &= 0, & Z_{v_{14},1} - Z_{v_3,x_4} &= 0,\\
Z_{v_6,1} - Z_{x_8,x_{11}} &= 0, & Z_{v_{15},1} - Z_{x_{10},x_9} &= 0,\\
Z_{v_{16},1} - Z_{x_{12},x_9} &= 0, & Z_{v_{17},1} - Z_{x_{13},x_9} &= 0,\\
Z_{v_{18},1} - Z_{x_{11},x_9} &= 0, & Z_{v_{15},1} - I + Z_{v_9,1} &= 0,\\
Z_{x_1,1} - Z_{v_{16},1} &= 0, & Z_{x_6,1} - Z_{v_{17},1} &= 0,\\
Z_{x_8,1} - Z_{x_{10},1}J_{N_\xi} &= 0, & Z_{x_5,1} - Z_{v_{18},1} &= 0.
\end{aligned}
\tag{4.30}
$$

Then (4.28a) and (4.28b) become the following linear equality constraints

$$
-Z_{v_8,1} + Z_{v_8,1}^T + Z_{v_{11},1} - Z_{v_{11},1}^T + Z_{v_{10},1} - Z_{v_{10},1}^T + Z_{v_{12},1} + Z_{v_{13},1} + Z_{v_{14},1} = 0, \tag{4.31a}
$$

$$
Z_{x_2,1} - Z_{v_6,1}J_{N_{vK1}} = 0. \tag{4.31b}
$$

Moreover, we also require that Z satisfies a rank n constraint:

$$
rank(Z) \leqslant n. \tag{4.32}
$$

On the other hand, terms of LMI constraints (4.25) and (4.26) can also be expressed in terms of Z by replacing $\hat{A}$, $\hat{B}$, $\hat{C}$, X, Y with $Z_{x_1,1}$, $[Z_{x_2,1},\ Z_{x_3,1},\ Z_{x_4,1}]$, $Z_{x_5,1}$, $Z_{6_1,1}$, $Z_{x_7,1}$, respectively.

To this point, we have converted the polynomial matrix programming problem into a rank constrained problem. And based on the method proposed by Nurdin et al. (2009a), we can get a local optimal solution by using LMIRank (Orsi et al. (2006)), SeDuMi, and Yalmip (Lofberg (2004)).

4.4 Simulation examples and comparisons

In this section, we will provide two examples to illustrate the method proposed in the last section of designing the quantum controller which satisfies the LQG and H_∞ performance simultaneously.

Example 1: This example is taken from Section 6 of James et al. (2008) and the plant is an optical cavity resonantly coupled to three optical chanels as shown in Figure 1 of James et al. (2008). This will be solved by using Yalmip via the solver LMIRank through Matlab and the initial value will be solved through SeDuMi.

The dynamics of this optical cavity system can be described by the following equations

$$\begin{aligned}
dx\,(t) &= -\frac{\gamma}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}x\,(t)\,dt - \sqrt{\kappa_1}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}dv(t) - \sqrt{\kappa_2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}dw\,(t) - \sqrt{\kappa_3}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}du(t),\\
dy(t) &= \sqrt{\kappa_2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}x(t)dt + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}dw(t),\\
dz_\infty(t) &= \sqrt{\kappa_3}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}x(t)dt + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}du(t),\\
z_l(t) &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}x(t) + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\beta_u(t)
\end{aligned} \tag{4.33}$$

with parameters $\gamma = \kappa_1 + \kappa_2 + \kappa_3$, $\kappa_1 = 2.6$, $\kappa_2 = \kappa_3 = 0.2$.

At first, we set the performance parameter $\gamma_l = 3$, $\gamma_\infty = 2.8$. Numerically solving the mixed problem yields the following coherent feedback controller after 11 iterations of LMIRank.

$$
\begin{aligned}
d\xi(t) &= \begin{bmatrix} -0.4880 & -0.0002 \\ -0.0000 & -0.4888 \end{bmatrix} \xi(t)\,dt + \begin{bmatrix} 0.6158 & 0.0000 \\ 0.0000 & 0.6175 \end{bmatrix} db_{vK1}(t) \\
&\quad + \begin{bmatrix} 0.0061 & 0.0007 \\ 0.0005 & -0.0059 \end{bmatrix} db_{vK2}(t) + \begin{bmatrix} -0.7709 & -0.0001 \\ -0.0005 & -0.7738 \end{bmatrix} dy(t), \qquad (4.34) \\
du(t) &= \begin{bmatrix} -0.6175 & 0.0000 \\ 0.0000 & -0.6158 \end{bmatrix} \xi(t)\,dt + db_{vK1}.
\end{aligned}
$$

Additionally, the actual closed-loop LQG performance achieved by the controller is 2.7607 and the H_∞ performance is 0.5151.

If we set $\gamma_l = 2.5$, $\gamma_\infty = 0.1$, after 53 iterations, the LQG cost is 2.0288 while H_∞ performance is 0.0403. However, if we restrict $\gamma_l = 2$ and remove H_∞ constraints, the problem degenerates to pure LQG control problem. And we need to run 1000 iterations to get the results. It is found that the LQG performance can not be less than 2.0 numerically, and the residual of physical realizability can only achieve $0.1356e-005$.

Example 2: In this example, we choose the degenerate parametric amplifier (D-PA) as our plant. For more details about DPA, one may refer to Leonhardt (2003). Following the treatment of Gough and Wildfeuer (2009), we now consider the DPA with three input fields coupled to a single cavity mode a, the coupling strength are $\sqrt{\kappa_1}$, $\sqrt{\kappa_2}$, $\sqrt{\kappa_3}$ and Hamiltonian is $H_{DPA} = \frac{i\epsilon}{4}(a^{*2} - a^2)$.

Based on the correspondence given in (2.22), we derive the QSDE of the DPA:

$$
\begin{aligned}
dx\left(t\right) &= -\frac{1}{2}\begin{bmatrix}\gamma-\epsilon & 0\\ 0 & \gamma+\epsilon\end{bmatrix}x\left(t\right)dt-\sqrt{\kappa_3}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}dv(t)-\sqrt{\kappa_1}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}dw\left(t\right)\\
&\quad -\sqrt{\kappa_2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}du(t),\\
dy(t) &= \sqrt{\kappa_3}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}x(t)dt+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}dv(t),\\
dz_\infty(t) &= \sqrt{\kappa_2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}x(t)dt+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}du(t),\\
z_l(t) &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}x(t)+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\beta_u(t)
\end{aligned}
\tag{4.35}
$$

with parameters $\gamma=\kappa_1+\kappa_2+\kappa_3$, $\kappa_1=\kappa_2=0.2$, $\kappa_3=0.5$, $\epsilon=0.01$.

Firstly, set $\gamma_l=3$, $\gamma_\infty=0.5$, we have to run 81 iterations of LMIRank to yield the following coherent feedback controller.

$$
\begin{aligned}
d\xi\left(t\right) &= \begin{bmatrix}-0.0800 & 0.0532\\ -0.0863 & -0.1289\end{bmatrix}\xi\left(t\right)dt+\begin{bmatrix}-0.1628 & -0.0912\\ 0.0995 & -0.1250\end{bmatrix}db_{vK1}\left(t\right)\\
&\quad +\begin{bmatrix}-0.0002 & -0.0008\\ -0.0003 & 0.0013\end{bmatrix}db_{vK2}\left(t\right)+\begin{bmatrix}-0.3017 & -0.1772\\ 0.2362 & -0.4560\end{bmatrix}dy\left(t\right),\\
du\left(t\right) &= \begin{bmatrix}0.1250 & -0.0912\\ 0.0995 & 0.1628\end{bmatrix}\xi\left(t\right)dt+db_{vK1}.
\end{aligned}
\tag{4.36}
$$

However, once we choose $\gamma_l=5$ and $\gamma_\infty=1$, we have to run 350 iterations to achieve the result. Moreover, if we let $\gamma_l=3$ and remove H_∞ constraints, this pure LQG problem also need 406 iterations of LMIRank. To this point, we can see the perplexing characteristics of the non-convex optimization. Table 4.1 sums up the numerical optimization results obtained by running LMIRank for both examples.

Table 4.1: Optimization results comparison under different constraints

plant	constraints		results			
	γ_∞	γ_l	$\Vert G_{\beta_w \to z_\infty} \Vert_\infty$	$\mathcal{L}_\infty$	physical realizability	iterations
Cavity ($\gamma = \kappa_1 + \kappa_2 + \kappa_3$, $\kappa_1 = 2.6$, $\kappa_2 = \kappa_3 = 0.2$.)	2.8	3	0.5151	2.7607	0.1361e-010	11
	0.1	2.5	0.0403	2.0288	0.8993e-013	53
	N/A	3	0.4249	2.7629	0.1005e-009	11
	N/A	2	0.1365	2.0000	0.1356e-005	1000
	0.1	N/A	0.0143	2.0302	0.6795e-013	14
DPA ($\gamma = \kappa_1 + \kappa_2 + \kappa_3$, $\kappa_1 = \kappa_2 = 0.2$, $\kappa_3 = 0.5, \epsilon = 0.01$.)	1	5	0.5235	2.0178	0.2885e-011	350
	0.5	3	0.3839	2.0611	0.3255e-010	81
	N/A	3	0.4039	2.0324	0.6160e-010	406
	0.5	N/A	0.5906	2.0037	0.3001e-012	88

4.5 Implementation of quantum coherent feedback controllers

In this section we show how the controller proposed in Sec. 4.4 can be realized with optical instruments. In Mabuchi (2008), the author demonstrated an experimental realization of a fully quantum controller. Subsequently, Nurdin et al. (2009b) provided a systematic synthesis theory of linear quantum stochastic systems, where they proposed a specific scheme that arbitrarily complex linear quantum stochastic system can be constructed by a series of one degree of freedom open quantum harmonic oscillators. Moreover, they also investigated how one degree of freedom open oscillators could be synthesized by quantum optical components. Therefore, any open quantum harmonic oscillator of arbitrary degrees of freedom can be synthesized or implemented. In the following, we will implement the coherent feedback controller (4.36) for illustration.

Note that the quantum coherent feedback controller (4.36) designed in the last section is in the quadrature form. For convenience, we transform it into the annihilation-creation form. Based on the method proposed in Sec. 4.1, we rewrite (4.36) as:

$$
\begin{aligned}
d\breve{a}_{\xi}(t) = & \begin{bmatrix} -0.1044-0.0697i & 0.0244-0.0165i \\ 0.0244+0.0165i & -0.1044+0.0697i \end{bmatrix} \breve{a}_{\xi}(t)\, dt \\
& + \begin{bmatrix} -0.1439+0.0953i & -0.0189+0.0042i \\ -0.0189-0.0042i & -0.1439-0.0953i \end{bmatrix} d\breve{b}_{vK1}(t) \\
& + 10^{-3} * \begin{bmatrix} 0.5500+0.2500i & -0.7500-0.5500i \\ -0.7500+0.5500i & 0.5500-0.2500i \end{bmatrix} d\breve{b}_{vK2}(t) \\
& + \begin{bmatrix} -0.3788+0.2067i & 0.0771+0.0295i \\ 0.0771-0.0295i & -0.3788-0.2067i \end{bmatrix} d\breve{y}(t), \\
d\breve{u}(t) = & \begin{bmatrix} 0.1439+0.0953i & -0.0189+0.0042i \\ -0.0189-0.0042i & 0.1439-0.0953i \end{bmatrix} \breve{a}_{\xi}(t)\, dt + d\breve{b}_{vK1}.
\end{aligned}
\tag{4.37}
$$

In terms of the corresponding relationship (2.22), the controller has the following parameters in the (S, L, H) language.

$$
\begin{aligned}
S_K &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
L_K &= \begin{bmatrix} L_{K1} \\ L_{K2} \\ L_{K3} \end{bmatrix} = \begin{bmatrix} 0.1439+0.0953i \\ -0.00055+0.00025i \\ 0.3788+0.2067i \end{bmatrix} a_{\xi} + \begin{bmatrix} -0.0189+0.0042i \\ -0.00075-0.00055i \\ 0.0771+0.0295i \end{bmatrix} a_{\xi}^{*}, \\
H_K &= \frac{1}{2}\breve{a}_{\xi}^{*} \begin{bmatrix} 0.0697 & 0.0165+0.0244i \\ 0.0165-0.0244i & 0.0697 \end{bmatrix} \breve{a}_{\xi}.
\end{aligned}
\tag{4.38}
$$

Based on the synthesis theory proposed by Nurdin et al. (2009b), we can synthesize (4.38) with a couple of connections of beam splitters, two-mode squeezers and fully reflecting mirrors and partially transmitting mirrors (Leonhardt (2003), Bachor and Ralph (2004), Gardiner and Zoller (2004)). To implement the controller and get the detailed parameter values of the components, one needs to implement both H_K and L_K. A detailed procedure is given in the sequel.

Firstly, to implement H_K, we will use a DPA with frequency detuning. Recall the Hamiltonian expression (Nurdin et al., 2009b, Sec. 6.1.2) for a DPA:

$$\begin{aligned} H &= \Delta a^* a - \frac{i}{2}\left(\epsilon (a^*)^2 - \epsilon^* a^2\right) \\ &= \frac{1}{2}\breve{a}^\dagger \begin{bmatrix} \Delta & -i\epsilon \\ i\epsilon^\# & \Delta \end{bmatrix} \breve{a} + c, \end{aligned} \tag{4.39}$$

where Δ is the frequency detuning defined as $\Delta = \omega_{cav} - \omega_r$ with ω_{cav} being cavity frequency and ω_r being reference frequency. ϵ is a complex number representing the effective pump intensity. c is just a constant that has no effect on the dynamics of the controller, and thus can be ignored.

Comparing (4.39) with H_K in (4.38), it is easily found that the complex effective pump intensity parameter ϵ is $-0.0244 - 0.0165i$, while the cavity detuning parameter Δ is 0.0697.

Secondly, a coupling operator L can be realized by two-mode squeezers, beam splitters and auxiliary cavity modes. Write L as

$$L = \frac{1}{\sqrt{\zeta}}(-\epsilon_2^* a + \epsilon_1 a^*), \tag{4.40}$$

where ζ is the coupling coefficient of the partially transmitting mirror, ϵ_1 is the effective pump intensity of the two-mode squeezer and $\epsilon_2 = 2\Theta e^{-i\Phi}$, where Θ is the mixing angle of the beam splitter and Φ is the relative phase between the input fields by the beam splitter, for details, one can refer to Nurdin et al. (2009b), Leonhardt (2003), Bachor and Ralph (2004), and Gardiner and Zoller (2004).

According to the form of the coupling operator L_K in (4.38), we see that the coherent feedback controller interacts with three independent field channels. In what follows the couplings to these three channels will be realized separately.

For $L_{K1} = (0.1439 + 0.0953i)a_\xi + (-0.0189 + 0.0042i)a_\xi^*$, by (4.40), we set the coupling coefficient of the mirror M_1 to be $\zeta = 100$, the two-mode squeezer's effective pump intensity ϵ_1 to be $-0.189 + 0.042i$, and the mixing angle of the beam splitter Θ to be -0.8630 with $\Phi = 0.1862\pi$.

Similarly, for $L_{K2} = (-0.00055 + 0.00025i)a_\xi + (-0.00075 - 0.00055i)a_\xi^*$, suppose the coupling coefficient ζ of mirror M_2 is 10000, then other parameters can be computed as: $\epsilon_1 = -0.075 - 0.055i$, $\Theta = 0.0302$, and $\Phi = 1.8644\pi$.

At last, for $L_{K3} = (0.3788 + 0.2067i)a_\xi + (0.0771 + 0.0295i)a_\xi^*$, we set $\zeta = 100$, then we obtain: $\epsilon_1 = 0.771 + 0.295i$, $\Theta = -2.1576$, and $\Phi = 0.1590\pi$.

Finally, the detailed physical implementation scheme is shown in Fig. 4.2, where P_1, P_2 and P_3 are 180° phase shifters, M_1, M_2 and M_3 are partially transmitting mirrors, S_1, S_2 and S_3 are beam splitters respectively, D_0 is the DPA, D_1, D_2 and D_3 are two-mode squeezers, and the black rectangles denote fully reflecting mirrors.

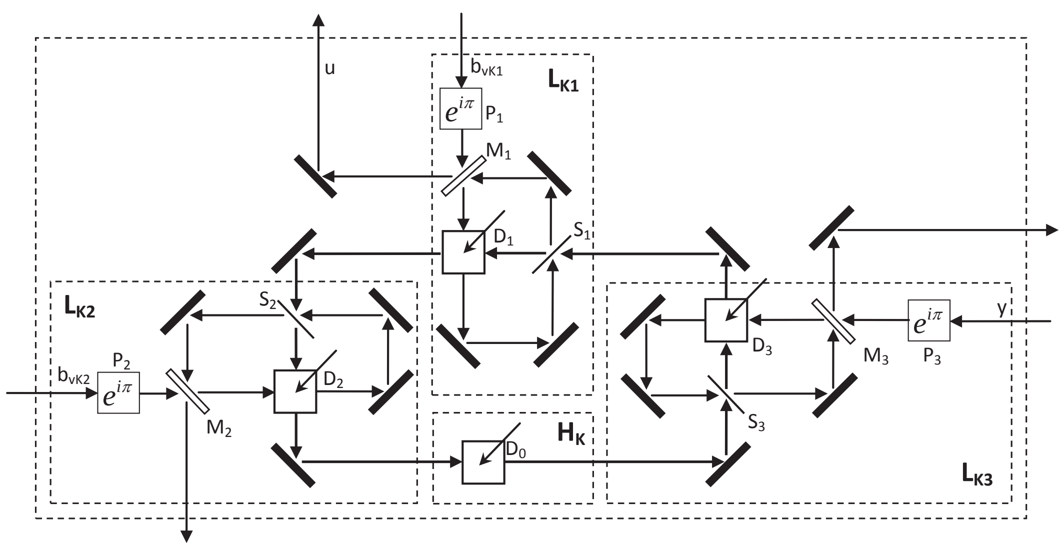

Figure 4.2: Realization of coherent controller K in (4.38).

Chapter 5

Conclusions and future work

This chapter draws conclusions on the thesis, and points out some possible research directions related to the work done in this thesis.

5.1 Conclusions

The focus of the thesis has been placed on various optimization methods for linear quantum stochastic systems to meet with specific state or performance objectives. Specifically, two research problems have been investigated in detail.

1. Squeezing ratio of quantum input-output system is proposed based on the analysis of variance of quadratures in vacuum state. Accordingly, squeezing performance of lossy DPA is investigated for the cases of open loop and feedback loop. Necessary and sufficient conditions have been derived to guarantee that the feedback loop scheme can enhance squeezing performance over open loop case. As an improved methodology, a coherent feedback control scheme has been proposed for squeezing enhancement of DPAs, in which the problem is a non-convex optimization problem which is solved by combined GA and SQP. Simulation results illustrate the advantages of the proposed method in contrast with previous approaches. Based on the recently developed quantum network synthesis theory, a detailed scheme has been proposed illustrating how

to implement the resulting coherent feedback controllers by means of quantum optical devices.

2. A class of linear quantum stochastic systems is formulated in terms of QSDEs in quadrature form. Then the standard quantum LQG and H^∞ control problems are proposed based on the closed-loop plant-controller feedback control system. The unique property of quantum LQG control problem makes it difficult to compute coherent feedback controllers analytically and H^∞ performance constraints also restrict the feasible region and indeed complicate the problem. In order to solve the intricate polynomial matrix equality constraints brought by the physical realizability condition, the problem is converted into a rank constraint LMI problem, a numerical algorithm is presented to attain a feasible solution. Several numerical examples are given to illustrate the effectiveness of the method. A detailed realization scheme of the resulting coherent feedback controller is also shown by means of quantum optical devices.

5.2 Future Work

Related topics for the future research work are listed below.

1. Although the combined GA and SQP are computationally efficient techniques in solving non-convex programming problems studied in Chap. 3, the computational cost is often an exponential function of the number of decision variables, and the optimized result can only approach one local optima. Thus it is not a theoretically efficient methodology. In the future, seeking for more efficient schemes to enhance DPA's squeezing performance is worthwhile and challenging.

2. For Quantum mixed LQG/H^∞ control problem raised in Chap. 4, due to the

unique characters of LQG coherent feedback control, the problem can not be easily solved analytically like the quantum H^∞ feedback control problem. The only effective method up to now is to introduce plenty of matrix lifting variables and transform the problem into rank constrained LMI problem. Solving such problem requires an immense amount of time, however feasible solution could not achieved regularly. Therefore, in the future work, trying to figure out a more effective way to quantum LQG control problem is indeed meaningful.

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